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Computation of risk contribution in the Vasicek portfolio credit loss model

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For the calculation of the VaR contribution in the Vasicek one-factor portfolio credit loss model, two methods, namely the saddlepoint approximation and importance sampling, are very attractive in terms of accuracy, speed and robustness. We explore their connection with the Esscher transform.

1 The Vasicek portfolio credit loss model

Consider a credit portfolio consisting of n obligors with exposure w_i, $i = 1, \dots, n$. Assume that obligor i defaults if its standardized log asset value X_i is less than some default threshold γ_i after a fixed time horizon. The event of default can be modeled as a Bernoulli random variable $D_i = 1_{\{X_i<\gamma_i\}}$ with known default probability $p_i = P(X_i < \gamma_i)$. It follows that the loss L_i due to obligor i is simply $w_i D_i$ and the portfolio loss is given by $L = \sum_{i=1}^n L_i = \sum_{i=1}^n w_i D_i$.

A certain amount of capital reserve is required as a cushion for potential large losses in the portfolio. The most popular risk measure for this purpose is the *Value at Risk* (VaR). Let α be some given confidence level, the VaR is simply the α-quantile of the loss distribution of L. Thus,

$$\text{VaR}_\alpha = \inf\{x : \mathbb{P}(L \le x) \ge \alpha\}. \tag{1}$$

Along with the VaR, the *VaR Contribution* (VaRC) measures how much each obligor contributes to the total VaR of a portfolio. The determination of the risk contribution is of practical importance because it is necessary for loan pricing and it can provide limits on large credit exposures. It may also be useful for profitability assessment, asset allocation and portfolio optimization. Under some continuity conditions, the VaRC coincides with the conditional expectation of L_i given that the portfolio loss L is equal to $\text{VaR}_\alpha(L)$, i.e.,

$$\text{VaRC}_{i,\alpha} = w_i \frac{\partial \text{VaR}_\alpha}{\partial w_i}(L) = w_i E[D_i | L = \text{VaR}_\alpha(L)], \tag{2}$$

For more details see [1] and [2]. It must be recognized that the computation of the VaR contribution is a challenging computational problem because the conditional expectation is based on an extremely rare event.

The key issue in the portfolio credit loss modeling is the specification of the default dependence among obligors. In the Vasicek model [3], the dependence structure among counterparties in the portfolio is simplified by the introduction of a common factor that affects all counterparties. It is assumed that the standardized asset log-return X_i of obligor i can be decomposed into a systematic part Y and an idiosyncratic part Z_i such that

$$X_i = \sqrt{\rho_i} Y + \sqrt{1-\rho_i} Z_i, \tag{3}$$

where Y and all Z_i are independent standard normal random variables. In case $\rho_i = \rho$ for all i the parameter ρ is called the common asset correlation. One can derive that the VaR and the VaRC at the $\alpha-$percentile for an infinitely large portfolio without exposure concentration are as follows:

$$\text{VaR}_\alpha = \sum_i w_i \Phi\Big(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}}\Big), \tag{4}$$

$$\text{VaRC}_{i,\alpha} = w_i \Phi\Big(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}}\Big), \tag{5}$$

where Φ denotes the CDF of the standard normal distribution.

Note that the VaR contribution (5) is a portfolio-invariant linear function of w_i, which implies that the capital contributions of individual exposures only depend on the characteristics of the particular exposure and not on the rest of the portfolio. This cannot properly account the effect of exposure concentration. These formulas work well for portfolios consisting of a large number of small obligors but are less suitable for and tend to underestimate risks for portfolios with few obligors or portfolios dominated by a few large exposures much larger than the others.

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2 Computation of the VaR contribution

Various numerical methods have been proposed for the purpose of calculating the credit portfolio VaR and VaRC under the Vasicek one-factor model. Two comprehensive reviews are [4, 5]. Two methods seem to be especially attractive for the calculation of the VaR contribution in terms of accuracy, speed and robustness: the saddlepoint approximation [6] and importance sampling [7]. The semi-analytic saddlepoint approximation method is well known to provide good approximations to very small tail probabilities, and importance sampling is able to effectively cluster portfolio losses around the level of interest, making a rare event less rare. Both methods work well for small sized portfolios and portfolios with exposure concentration, where Vasicek's asymptotic formulas fail.

The secret behind the success of the saddlepoint approximation and importance sampling as in [7] is the Esschcer transform, also known as exponential tilting. Esscher transform is named after Esscher [8] where he considers the approximation of the aggregate claim amount distribution around a point of interest $\tilde{x}$ and has long played a fundamental role in actuarial science, It applies the Edgeworth expansion to a transformed distribution with a parameter chosen such that the new mean is equal to $\tilde{x}$. The Esscher transform was brought into finance in [9] which shows that it is also an efficient technique for valuing derivative securities.

Conditional on a realization of the common factor Y, the portfolio loss $L(Y) = \sum w_i D_i(Y)$ becomes a sum of independent Bernoulli random variables with $\mathbb{P}(D_i(Y) = 1) = p_i(Y)$. From now on we suppress Y in the formulas for notational convenience. According to the Central Limit Theorem, we have

$$L \sim N\left(\sum w_i p_i, \sum w_i^2 p_i(1-p_i)\right),$$

where $N(\mu, \sigma^2)$ denotes the Gaussian distribution with mean μ and variance σ^2. However it is known that if we are interested in the density of L in the tail, this normal approximation is not accurate.

For a particular loss level $\tilde{x}$, choosing a suitable Esscher transform is equivalent to finding a probability measure $\mathbb{Q}$ such that (i) $\mathbb{Q}$ has minimal relative entropy with respect to $\mathbb{P}$ and (ii) $E^{\mathbb{Q}}[L] = \tilde{x}$. It can be derived that under the new probability measure $\mathbb{Q}$, obligor i defaults with the probability

$$q_i(\lambda) = \frac{p_i e^{-\lambda w_i}}{1 - p_i + p_i e^{-\lambda w_i}}, \tag{6}$$

where λ solves the equation $\sum w_i q_i(\lambda) = \tilde{x}$. The amount

$$\pi(L_i, \lambda) = w_i q_i(\lambda) \tag{7}$$

is know as the *Esscher premium principle* for L_i. The equivalence of the Esscher premium principle and the minimization of the total relative entropy can be found in [10, 11].

It turns out that the VaR contribution is actually a kind of *mixture of Esscher premium principles* in the sense of [12]. Replacing $\tilde{x}$ by VaR and rewriting eq. (2), we obtain

$$\text{VaRC}_i = \frac{\int_{-\infty}^{+\infty} \pi(L_i, \lambda) q(\sum_{j \neq i} L_j = \text{VaR} - w_i | \lambda) d\lambda}{p(L = \text{VaR})}, \tag{8}$$

where p and q denote, respectively, probability density functions under the original probability measure $\mathbb{P}$ and the Esscher transformed (with parameter λ) probability measure $\mathbb{Q}$.

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