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One-dimensional random polymers

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One-dimensional random polymers

R.W. van der Hofstad

Summary. Polymers are long molecules consisting of many building blocks. Polymers have two characteristic properties. The first is that they are irregular, because there are different possibilities for the angles between the building blocks. The second is that they try to avoid self-intersections because of polarization of the building blocks or the excluded-volume-effect.

Probabilistic polymer models are based on lattice random walks or Brownian motion with a self-repellent interaction. The paths of these processes model the configuration of the polymer in space. The random walk or the Brownian motion models the irregularity, the self-repellence penalizes self-intersections. We study the Domb-Joyce model based on simple random walk, and the Edwards model based on Brownian motion. The Domb-Joyce model is a generalization of the self-avoiding walk where self-intersections are unlikely, but not impossible. The Edwards model is the continuous space-time analogue of the Domb-Joyce model. We are interested in the behavior of the end-to-end distance of the polymer as the number of building blocks increases. The end-to-end distance gives an indication of what the spatial extent of the polymer is.

Polymer models are prototypes of models with a global interaction: any piece of the polymer interacts with all the other pieces, even the ones that are far apart when measured along the polymer chain. This makes polymer models different from most of the mathematical literature, where Markovian models are studied with a local interaction in space and/or time. The global interaction makes these models mathematically hard.

In Chapter 1, we will give a general introduction to polymers and probabilistic polymer models, and we will give insight as to what kind of behavior we expect. Then we will describe the known results, thereby focussing on the one-dimensional case. In the other chapters we will extend these results. Among other things, we will describe the links between the two models mentioned above in dimension one.

Contents

Chapter 1

Polymers

1.1 Introduction

Polymers are long molecules consisting of building blocks called monomers. Polymers can consist of a few thousand monomers. Examples are plastics (like polyester and PVC), biopolymers (like cellulose, DNA, starch and certain proteins) and rubber. An important feature of a polymer is its functionality: the number of connections the monomers can make. If this functionality is two, then the polymer is called linear, otherwise it is called a branching polymer. In this work we will consider linear polymers only. An example of a linear polymer is polyethylene, where the monomers are CH_2, except at the endpoints, where they are CH_3.

Polymers have two characteristic properties.[1] The first is that they are irregular: there are different possibilities for the angles between the monomers. The second is that they try to avoid self-intersections. The reason for the latter can be either polarization or the excluded-volume-effect. Polarization means that the monomers have an electrical charge. The excluded-volume-effect means that once there is a monomer in a certain position, this position is full and there can be no other monomer there.[2]

Realistic dimensions for polymers are two (for a polymer on a surface) and three (e.g. for a polymer in a solvent). However, it turns out that two- and three-dimensional polymers are mathematically too difficult and therefore we will mainly consider one-dimensional polymers. We can think of a one-dimensional polymer as a polymer in a thin tube. Although dimension one is highly special, the investigation of one-dimensional polymers brings up general features that are shared by two- and three-dimensional models. The same is true for dimensions greater than or equal to four. Therefore, the 'folklore' of polymers also includes such 'non-realistic' dimensions.

Polymers are important chemical objects. Freed (1981) estimates that 60% of research

[1]More properties of polymers or the interaction of polymers with their environment will be mentioned in Section 1.8.

[2]Strictly speaking, a polymer cannot have any self-intersection since no two pieces of any molecule can occupy the same space. However, if we associate extra space with the polymer induced by the vibrations or the electrical fields of the monomers, then this associated space can have self-intersections.

in the chemical industry is related to polymers. Polymers are also interesting physical objects. De Gennes (1979) contains a description of many fascinating microscopic and macroscopic properties that are induced by the typical polymer characteristics. Finally, and most importantly for this work, polymers turn out to be mathematically interesting as well, as we will see later on. They are prototypes of models with a global path interaction.

Probabilistic polymer models are based on lattice random walks or Brownian motion with a self-repellent interaction. The paths of these processes model the configuration of the polymer in space. The random walk or Brownian motion models the irregularity, the self-repellence models the tendency to avoid self-intersections. An example of a probabilistic polymer model is the self-avoiding walk, where all paths of a fixed length with no self-intersections have equal probability. In Sections 1.2-1.3 we will introduce two probabilistic polymer models, namely, the Domb-Joyce model based on the simple random walk and the Edwards model based on Brownian motion. The Domb-Joyce model is a generalization of the self-avoiding walk, where self-intersections are penalized but not impossible. The Edwards model is the continuous space-time analogue of the Domb-Joyce model.

We will be interested in the shape of the polymer as the number of monomers increases. An interesting quantity that measures the typical size of the polymer is the mean-squared-displacement, i.e., the expectation of the square of the end-to-end distance of the polymer. The investigation of this quantity turns out to be difficult. What makes polymers hard to analyze is that the interaction acts globally: all pieces of the path interact with each other. This is different from models with a local interaction, like Markovian models in space and/or time.

Polymers have become increasingly popular among chemists, physicists and mathematicians. Most of the rigorous results known today are from the last decade. For a mathematical introduction to polymer models, see Freed (1981) and Madras and Slade (1993). For an introduction to polymers and their heuristics from the chemist's or physicist's point of view, see Flory (1971) and de Gennes (1979). The Domb-Joyce and the Edwards model are toy polymer models (see also footnote 1). Chemists and physicists are interested in more realistic models. However, at present these are mathematically too difficult to investigate. Even the Domb-Joyce and the Edwards model have not been analyzed in the realistic dimensions two and three.

1.2 The Domb-Joyce model

Let X be a random variable on $\mathbb{Z}^d$ such that $X = e$ with probability $1/2d$ for each of the $2d$ neighbors e of the origin. Let $S_i = \sum_{j=1}^{i} X_j$ $(i \in \mathbb{N})$, where $\{X_j\}_{j\in\mathbb{N}}$ are independent random variables having the same distribution as X, and let $S_0 = 0$. The process $(S_i)_{i\in\mathbb{N}_0}$ is called simple random walk on $\mathbb{Z}^d$, starting at the origin. $(S_i)_{i=0}^{n}$ is the n-step path of a random walker, who at times $0, \dots, n-1$ makes a step to one of his/her neighboring lattice points with equal probability. We can think of $(S_i)_{i=0}^{n}$ as modeling a linear polymer in space. In fact, this is the so-called ideal polymer (see Madras and Slade (1993) Section 2.2), where there is no interaction between the monomers. The process $(S_i)_{i\in\mathbb{N}_0}$ is very irregular, since at every step the walker changes his/her direction with probability $\frac{2d-1}{2d}$. However, the ideal polymer has a tendency to have many self-intersections. To model the polymer better we therefore need to introduce a penalty for paths having many self-intersections.

Let P be the law of simple random walk and let E be expectation with respect to P. For $n \in \mathbb{N}$, define a new probability measure Q_n^β on n-step paths by setting

$$Q_n^\beta\Big((S_i)_{i=0}^n\Big) = \frac{1}{Z_n^\beta} \exp\Big[-\beta \sum_{\substack{i,j=0\\ i\neq j}}^n 1_{\{S_i=S_j\}}\Big] P\Big((S_i)_{i=0}^n\Big), \tag{1.2.1}$$

where Z_n^β is the normalizing constant

$$Z_n^\beta = E\Big(\exp\Big[-\beta \sum_{\substack{i,j=0\\ i\neq j}}^n 1_{\{S_i=S_j\}}\Big]\Big) \tag{1.2.2}$$

and $\beta \in \mathbb{R}_0^+$ is a parameter. The law Q_n^β is called the *Domb-Joyce n-polymer measure with strength of repellence β*. A linear polymer consisting of n monomers is modeled by the distribution of $(S_i)_{i=0}^n$ under the measure Q_n^β, the ith step of the path playing the role of the orientation of the ith monomer of the polymer.

Q_n^β gives a penalty $e^{-2\beta}$ to every self-intersection of the path. This has a tendency to reduce the number of self-intersections. Furthermore, the irregularity of a typical path under Q_n^β is inherited from P. Special cases of the Domb-Joyce measure are $\beta = 0$, where $Q_n^0 = P$ is the law of simple random walk, and $\beta = \infty$, where Q_n^∞ is the law of the self-avoiding walk. The n-step self-avoiding walk measure gives equal probability to all n-step paths having no self-intersections. (Note that the sum in the exponent in (1.2.1) is zero if and only if $(S_i)_{i=0}^n$ is self-avoiding.)

The exponent in (1.2.1) depends in a sensitive way on the exact shape of the path. Self-intersections can occur between any two pieces of the path, even those that are far away when measured along the polymer chain. This makes it difficult to prove mathematical statements for the distribution of $(S_i)_{i=0}^n$ under Q_n^β as $n \to \infty$. Furthermore, $(Q_n^\beta)_{n\in\mathbb{N}}$ is *not* a consistent family of measures, in the sense that Q_n^β is not the projection on n-step paths of a stochastic process, like the distribution of $(S_i)_{i=0}^n$ under P. The reason is that for $m < n$

$$\frac{1}{Z_n^\beta} E\Big(\exp\Big[-\beta \sum_{\substack{i,j=0\\ i\neq j}}^n 1_{\{S_i=S_j\}}\Big] \Big| (S_i)_{i=0}^m\Big) \tag{1.2.3}$$

is not equal to

$$\frac{1}{Z_m^\beta} \exp\Big[-\beta \sum_{\substack{i,j=0\\ i\neq j}}^m 1_{\{S_i=S_j\}}\Big]. \tag{1.2.4}$$

In terms of the polymer, the inconsistency means that we cannot think of $(Q_n^\beta)_{n\in\mathbb{N}}$ as modeling a growing polymer. There is no 'time' relation that makes a sensible connection between Q_m^β an Q_n^β for $m \neq n$. So the Domb-Joyce model describes polymers of a fixed length in space.

There are various interesting questions one can ask about the shape of the polymer under the Domb-Joyce measure Q_n^β as $n \to \infty$. How much space does it occupy? Is it clumped or not? How large are the holes it leaves open? An indicator of the typical size of the polymer is the mean-squared displacement. The mean-squared displacement for a polymer consisting of n monomers is the expected square of the end-to-end distance $E_{Q_n^\beta}\left[|S_n|^2\right]$. The following conjecture on the behavior of this quantity as $n \to \infty$ is folklore:

Conjecture 1.1 *For every $\beta \in \mathbb{R}^+$*

$$E_{Q_n^\beta}\left[|S_n|^2\right] \sim Dn^{2\nu} \quad (n \to \infty), \tag{1.2.5}$$

where $D = D(\beta, d) > 0$ is some amplitude and $\nu = \nu(d)$ is some critical exponent. The latter is believed to be independent of β and to assume the values

$$\nu = \begin{cases} 1 & d = 1 \\ \frac{3}{4} & d = 2 \\ 0.588\ldots & d = 3 \\ \frac{1}{2} & d \geq 4. \end{cases} \tag{1.2.6}$$

For $d = 4$ it is believed that there are logarithmic corrections to the above behavior, i.e.,

$$E_{Q_n^\beta}\left[|S_n|^2\right] \sim Dn \log n^{\frac{1}{4}} \quad (n \to \infty). \tag{1.2.7}$$

The fact that $\beta = \infty$ is included in Conjecture 1.1 means that the Domb-Joyce model is in the same universality class as the self-avoiding walk. Therefore the Domb-Joyce model is often called the weakly self-avoiding walk.

For $\beta = 0$, we have

$$E_{Q_n^0}\left[|S_n|^2\right] = E_P\left[|S_n|^2\right] = n, \tag{1.2.8}$$

corresponding to $D = 1, \nu = \frac{1}{2}$ for all d. Comparing this with (1.2.5-1.2.7) in $d = 1, 2, 3$ and 4, we see that the exponential in (1.2.1) has a tendency to spread out the path in order to reduce the number of self-intersections, as was perhaps to be expected. Apparently, in $d = 1, 2, 3$ and 4 the self-repellence changes the scale of the polymer. This is also visible in Figure 1, where we see realizations of a 200-step simple random walk and a 200-step self-avoiding walk in $d = 2$. The simple random walk forms a spaghetti-like clump around the origin, while the self-avoiding walk moves away from the origin more.

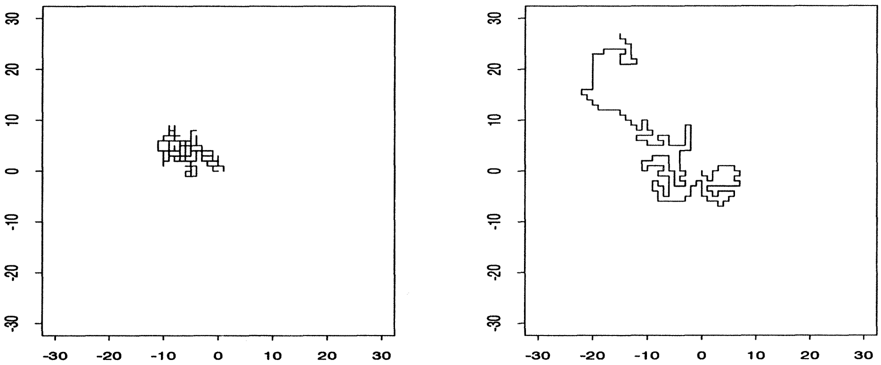


Figure 1

The fact that the exponents in (1.2.5) and (1.2.8) are the same for $d \geq 5$ means that the weakly self-avoiding walk in $d \geq 5$ is only a small perturbation of simple random walk, in the sense that there is no change in the qualitative behavior.

Monte Carlo simulations confirm the behavior in (1.2.5-1.2.7). For a reference to these methods for self-avoiding walks, see Madras and Slade (1993) Chapter 9 and Hughes (1995) Chapter 7 (where also references are given to the relevant literature). Flory developed a heuristic calculation predicting that the exponents in (1.2.6) are $\max\{\frac{3}{2+d}, \frac{1}{2}\}$ (see Madras and Slade (1993) Section 2.2). This calculation computes the probability of not having any self-intersections if we throw n points uniformly in a box of size n^ν and compares that to the probability for the endpoint of simple random walk to be n^ν. Maximizing over ν gives the value $\max\{\frac{3}{2+d}, \frac{1}{2}\}$.[3] However, numerical methods indicate that $\nu(3) < \frac{3}{5}$. The precise value for $\nu(3)$ is not known.

Conjecture 1.1 has been proved in a number of dimensions. For $d = 1$, Greven and den Hollander (1993) proved (1.2.5-1.2.6) using large deviation theory, and computed $D(\beta, 1)$. The ideas of their proof will be explained in Section 1.4. For $d \geq 5$, Hara and Slade (1992a,b) used the so-called lace expansion technique to prove that

$$\left\{S_{\lceil tn \rceil}/\sqrt{n}\right\}_{t \in [0,1]} \tag{1.2.9}$$

under the self-avoiding walk measure Q_n^∞ converges weakly to Brownian motion with a variance larger than one. The lace expansion is a method to compare a model with interaction to the model with no interaction (the so-called mean-field model). Hence, $D(\infty, d) > 1, \nu(d) = \frac{1}{2}$ for $d \geq 5$. Their result easily implies (1.2.5) for all $\beta \in \mathbb{R}^+$. For an explanation of the lace expansion in high dimensions, see Madras and Slade (1993)

[3]Freed (1981) derives a recursive formula for $\nu(d)$, predicting the same values.

Chapters 5-6. Brydges and Spencer (1985) had earlier used the lace expansion to prove (1.2.5) for $d \geq 5$ and β sufficiently small.

How can we understand these results? Heuristically, for $d = 1$ the geometry of $\mathbb{Z}$ is so easy that it is possible to use a Markovian-like description in space (see Section 1.4 for a more detailed explanation). On the other hand, for d large enough the self-intersections of simple random walk are so close that they only act locally, i.e., the behavior under the Domb-Joyce measure is not structurally different from the behavior under the simple random walk measure.

Remarkably, nothing is known for the realistic dimensions $d = 2, 3$. These are intermediate dimensions, where we lack the simplifying properties described above. Even seemingly innocent statements like

$$\begin{aligned} \lim_{n\to\infty} \tfrac{1}{n^2} E_{Q_n^\beta}\left[|S_n|^2\right] &= 0 \\ \lim_{n\to\infty} \tfrac{1}{n} E_{Q_n^\beta}\left[|S_n|^2\right] &= \infty \end{aligned} \qquad (1.2.10)$$

are unproved. Nature is giving us a hard time!

The investigation of the two-dimensional Domb-Joyce model is a real challenge. On the one hand, this problem is expected to be difficult, on the other hand it is expected to be easier than the three-dimensional model, where even the precise value of $\nu(3)$ is not heuristically known. Bounds for $\nu(d)$ like those following from (1.2.10) would be a great mathematical improvement in dimensions two and three.

1.3 The Edwards model

Let $(B_t)_{t\geq 0}$ be standard Brownian motion on $\mathbb{R}$, starting at the origin. Let $\widehat{P}$ denote its law, the Wiener measure. Originally, Brownian motion was used as a model to describe the irregular behavior of a dust particle in water due to collisions by the surrounding water molecules. This dust particle performs a highly irregular motion, which is reflected by the fact that the paths of Brownian motion are $\widehat{P}$-a.s. continuous, but nowhere differentiable. Brownian motion is the weak limit of simple random walk and therefore inherits many of its properties.

Let $\widehat{E}$ be expectation with respect to $\widehat{P}$. For $\beta \in \mathbb{R}_0^+$, define a new law $\widehat{Q}_T^\beta$ on paths of length T by specifying its Radon-Nikodym derivative with respect to $\widehat{P}$

$$\frac{d\widehat{Q}_T^\beta}{d\widehat{P}}\left((B_s)_{s\in[0,T]}\right) = \frac{1}{\widehat{Z}_T^\beta} \exp\left[-\beta \int_0^T ds \int_0^T dt\ \delta(B_s - B_t)\right] \qquad (T \geq 0). \qquad (1.3.1)$$

Here δ is the Dirac function, $\beta \in \mathbb{R}_0^+$ is the *strength of self-repellence*, and $\widehat{Z}_T^\beta$ is the normalizing constant

$$\widehat{Z}_T^\beta = \widehat{E}\left(\exp\left[-\beta \int_0^T ds \int_0^T dt\ \delta(B_s - B_t)\right]\right). \qquad (1.3.2)$$

Equation (1.3.1) means that for all Borel sets A

$$\widehat{Q}_T^\beta\left((B_s)_{s\in[0,T]}\in A\right)=\frac{1}{\widehat{Z}_T^\beta}\widehat{E}\left(\exp\left[-\beta\int_0^T ds\int_0^T dt\;\delta(B_s-B_t)\right]1_{\{(B_s)_{s\in[0,T]}\in A\}}\right). \quad (1.3.3)$$

$\widehat{Q}_T^\beta$ is called the *Edwards T-polymer measure with strength of repellence* β. For a rigorous definition of $\widehat{Q}_T^\beta$ in terms of Brownian local times, see Chapter 3.

The Edwards model is the continuous space-time analogue of the Domb-Joyce model. The exponent in (1.3.1) punishes self-intersections. Since Brownian motion is the weak limit of simple random walk, we can think of the Edwards model as 'looking at the Domb-Joyce model with a small interaction parameter from a distance'. This can in fact be made rigorous (see Chapter 2, Theorem 2.5).

We have defined the Edwards model only in dimension one. It can also be defined in dimensions two (see e.g. Cadre (1996), Le Gall (preprint 1996), Stoll (1989), Varadhan (1969), and Westwater (1980)) and three (see e.g. Bolthausen (1993), Rosen (1983), Westwater (1981) and Zhou (1992)). However, the difficulty in these dimensions is that the random variable in the exponent in (1.3.1) is infinite $\widehat{P}$-a.s. This is because most of the self-intersections of the Brownian path are for times that are close together. One way to solve this problem is to put in the indicator that $|t-s|\geq\epsilon$ in the exponents in (1.3.1-1.3.2). This truncation gives a well-defined measure $\widehat{Q}_{T,\epsilon}^\beta$ for every $\epsilon>0$. It can be shown that, as $\epsilon\downarrow 0$, $\widehat{Q}_{T,\epsilon}^\beta$ converges to some measure $\widehat{Q}_T^\beta$. This is defined to be the Edwards measure. Unfortunately, the above limiting procedure to obtain $\widehat{Q}_T^\beta$ makes it more difficult to investigate the behavior of $(B_t)_{t\in[0,T]}$ under $\widehat{Q}_T^\beta$. This is illustrated by the fact that in dimension three $\widehat{Q}_T^\beta$ turns out to be mutually singular with respect to $\widehat{P}$. Hence, there is no way of writing down a formula like (1.3.1) using a Radon-Nikodym derivative with respect to $\widehat{P}$. In dimension two $\widehat{Q}_T^\beta$ is absolutely continuous with respect to $\widehat{P}$, but still not much is known about the $T\to\infty$ behavior.

The advantage of the Edwards model over the Domb-Joyce model is that Brownian motion is somewhat easier to work with than simple random walk. The disadvantage is that the Edwards model is more difficult to define.

It is expected that a similar behavior to that in (1.2.5) holds, i.e., for $d=1,2,3$

$$E_{\widehat{Q}_T^\beta}\left[|B_T|^2\right]\sim\widehat{D}T^{2\hat{\nu}}\qquad(T\to\infty), \quad (1.3.4)$$

where $\widehat{D}=\widehat{D}(\beta,d)$ is some amplitude and $\hat{\nu}=\hat{\nu}(d)$ is a critical exponent that is believed to be the same as $\nu(d)$ in Conjecture 1.1. However, there is only a proof for this fact in $d=1$ (see Theorems 1.1 and 1.3 below). Because of the exponential factors in (1.2.1-1.2.2) and (1.3.1-1.3.2) it is very difficult to compare the Edwards model and the Domb-Joyce model directly. Since the typical paths under the Domb-Joyce measure and under the Edwards measure are atypical under P and $\widehat{P}$, respectively, it is not possible to use the invariance principle to prove convergence of Q_n^β towards $\widehat{Q}_1^\beta$ as $n\to\infty$. This is reflected by the fact that in dimension one, the behavior under the Domb-Joyce measure Q_n^β for n

large is different from the behavior under the Edwards measure $\widehat{Q}_T^\beta$ for T large. However, in Chapters 2-4 we will describe a link between the two models as $\beta \downarrow 0$.

In dimension one, much is now known about the behavior of the n-polymer under the Domb-Joyce measure and the T-polymer under the Edwards measure. There are laws of large numbers (LLN) and central limit theorems (CLT) for the end-points of the paths. We know what the underlying variational principles are. We have results explaining the scaling in the interaction parameter β and the links between the Domb-Joyce model and the Edwards model. All this, and the remaining open problems, will be explained in the rest of this work. The part of these results that is new is presented in Chapters 2-5. Together with the results previous to this work as described in Chapter 1 they give a clear picture of the behavior in the one-dimensional Domb-Joyce and Edwards model.

1.4 Law of large numbers for the Domb-Joyce model

Fix $d = 1$. In this section we will explain the LLN for the endpoint of the path in the Domb-Joyce model (Theorems 1.1 and 1.2 below). This implies (1.2.5-1.2.6).

Theorem 1.1 *(Greven and den Hollander (1993)) For every $\beta \in \mathbb{R}^+$ there exists a $\theta^*(\beta) \in (0,1)$ such that*

$$\lim_{n\to\infty} Q_n^\beta\Big(\big|\frac{1}{n}|S_n| - \theta^*(\beta)\big| \leq \varepsilon\Big) = 1 \quad \textit{for every} \quad \varepsilon > 0. \tag{1.4.1}$$

Theorem 1.1 proves Conjecture 1.1 for $d = 1$ with $\nu(1) = 1, D(\beta, 1) = \theta^*(\beta)^2$. Theorem 1.1 says that the self-repellence causes the path to have a ballistic behavior no matter how small the interaction parameter.[4] The quantity $\theta^*(\beta)$ in (1.4.1) is called the *speed of the polymer*.[5]

We will give a brief explanation of the proof of Theorem 1.1. Define

$$J_\beta(\theta) = \lim_{n\to\infty} \frac{1}{n} \log E\Big(\exp\big[-\beta \sum_{\substack{i,j=0 \\ i\neq j}}^{n} 1_{\{S_i=S_j\}}\big] 1_{\{S_n=\lceil\theta n\rceil\}}\Big). \tag{1.4.2}$$

Greven and den Hollander (1993) prove that this limit exists and that $J_\beta(\theta)$ is strictly negative with a unique maximum at the value $\theta = \theta^*(\beta)$. This implies that

$$J_\beta(\theta^*(\beta)) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^\beta, \tag{1.4.3}$$

[4]The term ballistic behavior refers to the fact that the end-to-end distance of the polymer grows proportionally to the number of monomers.

[5]By symmetry, the LLN implies that $Q_n^\beta(S_n/n)^{-1} \Longrightarrow^w \frac{1}{2}[\delta_{\theta^*(\beta)} + \delta_{-\theta^*(\beta)}]$ as $n \to \infty$, where δ_θ is the Dirac point measure in θ, $\Longrightarrow^w$ denotes weak convergence, and $\mu(X)^{-1}$ denotes the distribution of a random variable X under a measure μ.

i.e., $J_\beta(\theta^*(\beta))$ is the exponential growth rate of the normalizing constant Z_n^β. Furthermore, from (1.4.2-1.4.3) and (1.2.1-1.2.2) it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log Q_n^\beta (S_n = \lceil \theta n \rceil) = J_\beta(\theta) - J_\beta(\theta^*(\beta)), \tag{1.4.4}$$

which is strictly negative for $\theta \neq \theta^*(\beta)$.

To explain how (1.4.2-1.4.3) arise, we rewrite

$$\sum_{\substack{i,j=0\\ i\neq j}}^{n} 1_{\{S_i=S_j\}} = \sum_{x\in\mathbb{Z}} \ell_n^2(x) - (n+1), \tag{1.4.5}$$

where

$$\ell_n(x) = \#\{0 \le i \le n : S_i = x\} \qquad (n \in \mathbb{N}_0, x \in \mathbb{Z}) \tag{1.4.6}$$

are the local times. Substituting (1.4.5) into (1.4.2), we see that we have to understand

$$J_\beta(\theta) = -\beta + \lim_{n\to\infty} \frac{1}{n} \log E\Big(\exp\big[-\beta \sum_{x\in\mathbb{Z}} \ell_n(x)^2\big] 1_{\{S_n=\lceil\theta n\rceil\}}\Big). \tag{1.4.7}$$

We next rewrite (1.4.7) as

$$\begin{aligned} J_\beta(\theta) \;&=\; -\beta + \lim_{n\to\infty} \tfrac{1}{n} \log P(S_n = \lceil\theta n\rceil) \\ &\quad + \lim_{n\to\infty} \tfrac{1}{n} \log E\Big(\exp\big[-\beta \textstyle\sum_{x\in\mathbb{Z}} \ell_n(x)^2\big] \Big| S_n = \lceil\theta n\rceil\Big). \end{aligned} \tag{1.4.8}$$

From large deviation theory, it is well known that

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n = \lceil\theta n\rceil) = -\frac{1}{2}(1+\theta)\log(1+\theta) - \frac{1}{2}(1-\theta)\log(1-\theta). \tag{1.4.9}$$

From (1.4.8-1.4.9) we thus see that a balancing takes place: the first term in (1.4.8) is a decreasing function in θ, while the second term is expected to be increasing in θ, since if the walk is spread out more then there will be less self-intersections. The first key step in the proof of the LLN of Greven and den Hollander is

$$\begin{aligned} &\lim_{n\to\infty} \tfrac{1}{n} \log E\Big(\exp\big[-\beta \textstyle\sum_{x\in\mathbb{Z}} \ell_n(x)^2\big] \Big| S_n = \lceil\theta n\rceil\Big) \\ &\quad = \lim_{n\to\infty} \tfrac{1}{n} \log E\Big(\exp\big[-\beta \textstyle\sum_{x=1}^{\lceil\theta n\rceil} \ell_n(x)^2\big] \Big| S_n = \lceil\theta n\rceil\Big), \end{aligned} \tag{1.4.10}$$

which means that the contributions coming from local times below 0 and above $S_n = \lceil\theta n\rceil$ are negligible on an exponential scale.[6]

[6]In fact, Greven and den Hollander prove a slightly weaker statement, namely, (1.4.10) only for $\theta = \theta^*(\beta)$.

Let P_θ be the law of the random walk with drift $\theta \in (0,1)$. Under P_θ, the walker makes a step to the right with probability $\frac{1}{2}(1+\theta)$ and to the left with probability $\frac{1}{2}(1-\theta)$, causing an effective drift θ to the right. The second key step in the proof of Greven and den Hollander is

$$\begin{aligned}\lim_{n\to\infty} \tfrac{1}{n} \log E\Big(\exp\big[-\beta \textstyle\sum_{x=1}^{\lceil\theta n\rceil} \ell_n^2(x)\big]\Big| S_n = \lceil\theta n\rceil\Big) \\ = \lim_{n\to\infty} \tfrac{1}{n} \log E_{P_\theta}\Big(\exp\big[-\beta \textstyle\sum_{x=1}^{\lceil\theta n\rceil} \ell_\infty^2(x)\big]\Big),\end{aligned} \tag{1.4.11}$$

where

$$\ell_\infty(x) = \#\{i \geq 0 : S_i = x\} \qquad (x \in \mathbb{Z}) \tag{1.4.12}$$

are the infinite time local times, which are finite P_θ-a.s. (Note that, under P_θ, S_n is approximately $\lceil\theta n\rceil$ by the law of large numbers, which explains why the condition $S_n = \lceil\theta n\rceil$ drops out.) Next, $\{\ell_\infty(x)\}_{x\in\mathbb{Z}}$ turns out to have a Markov representation, which follows from Knight's description of the local times of simple random walk (see Section 4.2.2). This Markov property is a direct consequence of the simple geometric structure of $\mathbb{Z}$. Then, the stage is set to use large deviation techniques. These techniques are well suited to express limits of expectations of exponential functionals of Markov processes, like in the r.h.s. of (1.4.11), in terms of variational problems. After some more analysis, this procedure gives a recipe for $\theta^*(\beta)$, which we will now describe.

For $r \in \mathbb{R}, \beta \in \mathbb{R}^+$ define the matrix $A_{r,\beta}$ by

$$A_{r,\beta}(i,j) = e^{r(i+j-1)-\beta(i+j-1)^2} P(i,j) \qquad (i,j \in \mathbb{N}), \tag{1.4.13}$$

where P is the stochastic matrix

$$P(i,j) = \binom{i+j-2}{i-1} \left(\frac{1}{2}\right)^{i+j-1}. \tag{1.4.14}$$

Define $\lambda(r,\beta)$ to be the unique largest eigenvalue of $A_{r,\beta}$ in $l^2(\mathbb{N})$.[7] Then $\lambda(r,\beta)$ can be written as a variational problem by the Rayleigh representation for eigenvalues of compact self-adjoint operators. P is the transition matrix of the Markov process underlying the local time process, β is the self-repellence parameter as in (1.2.1-1.2.2) and r is a Lagrange parameter that plays the role of Lagrange multiplier in the underlying variational problem.

For fixed β, let $r^*(\beta)$ be the unique solution of the equation $\lambda(r,\beta) = 1$, i.e.,

$$\lambda(r^*(\beta), \beta) = 1. \tag{1.4.15}$$

In terms of these objects we have the following representation for $\theta^*(\beta)$ and the exponential rate $J_\beta(\theta^*(\beta))$:

[7] $A_{r,\beta} : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is positive, self-adjoint and compact for all $r \in \mathbb{R}, \beta \in \mathbb{R}^+$. Moreover, $(r,\beta) \mapsto \lambda(r,\beta)$ is analytic. Furthermore, $r \mapsto \lambda(r,\beta)$ is strictly increasing and log-convex, $\lambda(0,\beta) < 1$ and $\lim_{r\to\infty} \lambda(r,\beta) = \infty$ for every $\beta \in \mathbb{R}^+$ (see Greven and den Hollander (1993)).

Theorem 1.2 *(Greven and den Hollander (1993)) For every $\beta \in \mathbb{R}^+$*

$$J_\beta(\theta^*(\beta)) = -r^*(\beta). \tag{1.4.16}$$

$$\theta^*(\beta) = \left[\frac{\partial}{\partial r}\lambda(r,\beta)\right]^{-1}_{r=r^*(\beta)}. \tag{1.4.17}$$

Theorem 1.2 gives us a functional analytic description of the key quantities in Theorem 1.1, which turns out to be extremely useful (see Chapters 2 and 4).

Remark: The following conjecture is appealing:

Conjecture 1.2 *$\beta \mapsto \theta^*(\beta)$ is increasing.*

Indeed, if the penalty for self-intersections is higher, then we expect the walk to spread out more. However, Conjecture 1.2 turns out to be equivalent to a second-order differential inequality for $\lambda(r,\beta)$ that is highly non-trivial. Conjecture 1.2 is one of the open questions remaining in the one-dimensional Domb-Joyce model. It appears that the monotonicity of $\theta^*(\beta)$ is deeply hidden. This is related to the fact that $(Q_n^\beta)_{n\in\mathbb{N}}$ is not a consistent family of measures (see also Section 1.2). Furthermore, we have not been able to couple Q_n^β and $Q_n^{\beta'}$ for $\beta \neq \beta'$. Therefore we have no hope to give a coupling argument for Conjecture 1.2.

The following correlation inequality implies Conjecture 1.2:

$$Cov_{Q_n^\beta}\big(|S_n|, \sum_{x\in\mathbb{Z}} \ell_n(x)^2\big) \leq 0 \text{ for all } n \text{ large enough.} \tag{1.4.18}$$

This can be seen by differentiating $E_{Q_n^\beta}[|S_n|]$ with respect to β and using Theorem 1.1 and (1.2.1-1.2.2). It is not hard to check (1.4.18) for $\beta = 0$, where $Q_n^0 = P$. It has been shown to hold for $n \leq 15$ and all $\beta > 0$ by a computer enumeration. However, (1.4.18) is still open for $n > 15$ and $\beta > 0$.

1.5 Law of large numbers for the Edwards model

Westwater (1984) proved a law of large numbers for the Edwards model similar to Theorems 1.1-1.2.

Theorem 1.3 *(Westwater (1984)) For every $\beta \in \mathbb{R}^+$ there exists a $\hat{\theta}^*(\beta) \in (0,\infty)$ such that*

$$\lim_{T\to\infty} \widehat{Q}_T^\beta\Big(\Big|\frac{1}{T}|B_T| - \hat{\theta}^*(\beta)\Big| \leq \epsilon\Big) = 1 \text{ for every } \epsilon > 0. \tag{1.5.1}$$

The quantity $\hat{\theta}^*(\beta)$ is again called the *speed of the polymer*.[8] Westwater (1984) gives a description of $\hat{\theta}^*(\beta)$ in terms of a derivative of an eigenvalue of a differential operator on

[8] By symmetry, (1.5.1) says that the distribution of B_T/T under $\widehat{Q}_T^\beta$ converges weakly to $\frac{1}{2}(\delta_{\hat{\theta}^*(\beta)} + \delta_{-\hat{\theta}^*(\beta)})$ as $T \to \infty$.

$L^2(\mathbb{R}^+)$ similar to Theorem 1.2. The proof of Theorem 1.3 uses large deviation theory together with the Ray-Knight description for Brownian local times and is in this sense close to the proof of Theorems 1.1-1.2. From the proof it follows that

$$\hat{r}^*(\beta) = -\lim_{T\to\infty} \frac{1}{T} \log \widehat{Z}_T^\beta \tag{1.5.2}$$

exists. It turns out that $\hat{\theta}^*(\beta) = \hat{\theta}^*(1)\beta^{\frac{1}{3}}$ by Brownian scaling (see Chapter 2 and Section 3.1.2). Hence $\beta \mapsto \hat{\theta}^*(\beta)$ *is* increasing.

1.6 Central limit theorem for the Domb-Joyce model

The LLN for the Domb-Joyce model in Theorem 1.1 has been supplemented by a central limit theorem (CLT). In the CLT the fluctuations around the leading order behavior are investigated.

Theorem 1.4 *(König (1996)) For every $\beta \in \mathbb{R}^+$ there exists a $\sigma^*(\beta) \in \mathbb{R}^+$ such that*

$$\lim_{n\to\infty} Q_n^\beta\Big(\frac{|S_n| - \theta^*(\beta)n}{\sigma^*(\beta)\sqrt{n}} \le C\Big) = \mathcal{N}((-\infty, C]) \text{ for every } C \in \overline{\mathbb{R}}, \tag{1.6.1}$$

where $\mathcal{N}$ is the standard normal distribution and $\sigma^*(\beta)$ is given by

$$\sigma^*(\beta)^2 = \theta^*(\beta)^3 \Big[\frac{\partial^2}{\partial r^2}\lambda(r,\beta) - \frac{1}{\theta^{*2}(\beta)}\Big]_{r=r^*(\beta)}. \tag{1.6.2}$$

The quantity $\sigma^*(\beta)$ is the *spread of the polymer.* Theorem 1.4 was proved via a higher order large deviation analysis of (1.2.1). This proof includes a more precise investigation of the normalizing constant than in (1.4.3).

Remark: The following conjecture is again appealing:

Conjecture 1.3 *$\beta \mapsto \sigma^*(\beta)$ is decreasing.*

It is expected that the spread $\sigma^*(\beta)$ is less than one, the value for simple random walk. This means that as the path moves upwards (or downwards) to reach approximately the value $\pm\theta^*(\beta)n$ at time n, the fluctuations around the endpoint are squeezed compared to simple random walk. Note that $\lim_{\beta\to\infty} \sigma^*(\beta) = 0$ should hold, since the variance of the endpoint of the path equals zero for $\beta = \infty$.

1.7 Organization of this work

The rest of this work is organized as follows.

In Section 1.8 we describe some related models, each modeling different aspects of the polymer interacting with its environment.

In Chapter 2 we investigate the behavior of $\theta^*(\beta)$ and $r^*(\beta)$ as $\beta \downarrow 0$, the so-called weak interaction limit. The proofs rely heavily on the explicit forms given in (1.4.15) and (1.4.17). We investigate the scaling of the eigenvalue $\lambda(r,\beta)$ as $\beta \downarrow 0$ and $r \sim a\beta^{\frac{2}{3}}$ for any $a \in \mathbb{R}$. Furthermore, we give a link between these results and $\hat{\theta}^*(\beta)$ and $\hat{r}^*(\beta)$.

In Chapter 3 we prove a central limit theorem for the end-point of the path in the Edwards model, thereby extending the law of large numbers in Theorem 1.3. The proof uses a higher order large deviation analysis and the Ray-Knight theorems for Brownian local times. It turns out that the scaled mean $\hat{\theta}^*(\beta)$ and the scaled standard deviation $\hat{\sigma}^*(\beta)$ can be identified in terms of the one-parameter family of Sturm-Liouville differential operators that is analyzed in Chapter 2.

In Chapter 4 we prove a central limit theorem for the end-point of the path under $Q_n^{\beta_n}$ when $\beta_n \to 0$ and $\beta_n n^{\frac{3}{2}} \to \infty$ as $n \to \infty$. This means that the interaction parameter β depends on the number of monomers and decreases to zero as the number of monomers increases. We explain why this regime is interesting and also explain the links and differences with Chapters 2 and 3. Along the way, we extend the scaling results in Chapter 2 to the full spectrum of $A_{a\beta^{\frac{2}{3}},\beta}$ $(a \in \mathbb{R})$.

In Chapter 5 we give bounds for the constants involved in the LLNs and CLTs in Chapters 2-4 and explain what these bounds mean for the polymer. Furthermore, we prove that for small β the spread of the polymer $\sigma^*(\beta)$ in the Domb-Joyce model is indeed less than one.

In Chapters 1-5 we find many features that we also expect to occur in the Domb-Joyce model and the Edwards model in the realistic dimensions two and three, namely:

- There is a link between the Domb-Joyce model and the Edwards model in the sense that the scale of the polymer is the same in both models and is larger than the scale of the model without interaction.

- There is scaling in the interaction parameter for both models. For the Domb-Joyce model this scaling is asymptotic when the interaction parameter tends to zero, while it is exact for the Edwards model.

- The weak interaction limit is singular for both models, and the limits are equal.

- There are difficulties in proving monotonicity in the interaction parameter in the Domb-Joyce model, while this follows from the exact scaling in the Edwards model.

- Proofs are technical and often ad hoc methods in combination with functional analysis have to be used. Due to the global nature of the interaction we cannot prove results in a more standard way.

1.8 Related models: an outlook on the future

The models in (1.2.1-1.2.2) and (1.3.1-1.3.2) are caricatures of reality, since only the barest features of the polymer interaction are taken into account. There are more realistic models for polymers around. We will describe some of these models here.

1.8.1 Repulsion and attraction

For $n \in \mathbb{N}$, define the measure $Q_n^{\beta,\gamma}$ on n-step paths by setting

$$Q_n^{\beta,\gamma}\Big((S_i)_{i=0}^n\Big) = \frac{1}{Z_n^{\beta,\gamma}} \exp\Big[-\beta \sum_{\substack{i,j=0 \\ i\neq j}}^n 1_{\{S_i=S_j\}} + \frac{\gamma}{2d}\sum_{i,j=0}^n 1_{\{|S_i-S_j|=1\}}\Big] P\Big((S_i)_{i=0}^n\Big), \qquad (1.8.1)$$

where $Z_n^{\beta,\gamma}$ is the normalizing constant

$$Z_n^{\beta,\gamma} = E\Big(\exp\Big[-\beta \sum_{\substack{i,j=0 \\ i\neq j}}^n 1_{\{S_i=S_j\}} + \frac{\gamma}{2d}\sum_{i,j=0}^n 1_{\{|S_i-S_j|=1\}}\Big]\Big) \qquad (1.8.2)$$

and $\beta, \gamma \in \mathbb{R}_0^+$ are parameters. Equations (1.8.1-1.8.2) give a model for a polymer in a repulsive solution. The first term in the exponent models the polarization or the excluded-volume-effect as described in Section 1.2, the second term models the fact that the polymer wants to be close to itself because of the repulsion of the solution.

It is believed that there is a critical curve $\beta \mapsto \gamma(\beta)$ such that the following behavior holds:

(*i*) For $\gamma < \gamma(\beta)$ the polymer behaves as in Conjecture 1.1.
(*ii*) For $\gamma > \gamma(\beta)$ the polymer localizes in the sense that the leading order behavior is

$$E_{Q_n^{\beta,\gamma}}\left[|S_n|^2\right] \sim D^+\nu^+ \quad (n \to \infty), \qquad (1.8.3)$$

with $D^+ = D^+(\beta,\gamma,d) > 0$, $\nu^+ = \nu^+(d) < \frac{1}{2}$ for all d.

We have the following conjecture:

Conjecture 1.4 *For all* $d \geq 1$
(i) $\gamma(\beta) = \beta$.
(ii) $\nu^+(d) = 0$.

In van der Hofstad and Klenke (preprint 1998) there are heuristic arguments for this conjecture. At $\gamma = \beta$, there is a transition where on the one hand the normalizing constant is exponentially small for $\gamma < \beta$, while on the other hand we have that

$$Z_n^{\beta,\gamma} \geq e^{cn^2} \qquad (1.8.4)$$

for $\gamma > \beta$ and some positive constant $c = c(\beta,\gamma,d)$. This suggests the behaviour as in Conjecture 1,4.

1.8.2 Elasticity

For $n \in \mathbb{N}$, define the measure $Q_n^{\beta,p}$ on n-step paths by setting

$$Q_n^{\beta,p}\Big((S_i)_{i=0}^n\Big) = \frac{1}{Z_n^{\beta,p}} \exp\Big[-\beta \sum_{\substack{i,j=0 \\ i\neq j}}^{n} \frac{1_{\{S_i=S_j\}}}{|i-j|^p}\Big] P\Big((S_i)_{i=0}^n\Big), \tag{1.8.5}$$

where $Z_n^{\beta,p}$ is the normalizing constant

$$Z_n^{\beta,p} = E\Big(\exp\Big[-\beta \sum_{\substack{i,j=0 \\ i\neq j}}^{n} \frac{1_{\{S_i=S_j\}}}{|i-j|^p}\Big]\Big) \tag{1.8.6}$$

and $\beta \in \mathbb{R}_0^+, p \in \mathbb{R}_0^+$ are parameters. Equations (1.8.5-1.8.6) give a model for an elastic polymer, where the penalty oof a loop of length k is $e^{-2\frac{\beta}{k^p}}$ (so, the penalty for a self-intersection is small when the time difference is large). For $p = 0$ we get the Domb-Joyce model. Kennedy (1994) proves that in $d = 1$, for $0 \leq p < 1$ and for β sufficiently large, the behavior is ballistic: for any $\epsilon > 0$, there exists a $\beta_0 = \beta_0(\epsilon)$ such that for all $\beta \geq \beta_0$

$$\lim_{n\to\infty} Q_n^{\beta,p}\left(\frac{1}{n}|S_n| \geq 1-\epsilon\right) = 1. \tag{1.8.7}$$

The proof is based on a renormalization type argument. In Caracciolo et al. (1994) there is the following conjecture:

Conjecture 1.5 *For all $\beta \in \mathbb{R}^+$ and $p \in \mathbb{R}_0^+$,*

$$E_{Q_n^{\beta,p}}[S_n^2] \sim D_e n^{2\nu_e} \quad (n\to\infty), \tag{1.8.8}$$

where $D_e = D_e(\beta,p)$ is some amplitude and

$$\nu_e(p) = \begin{cases} \min\{\frac{2-p}{d},\nu\} & \text{for } p \leq \frac{4-d}{2} \\ \frac{1}{2} & \text{for } p \geq \frac{4-d}{2}. \end{cases} \tag{1.8.9}$$

Here $\nu = \nu(d)$ is the critical exponent for the Domb-Joyce model as in (1.2.6). Furthermore, for $p = \frac{4-d}{2}$ it is believed that there are logarithmic corrections to the above behavior.

The claim in Conjecture 1.5 means the following. If p is small, then the elastic polymer has the same critical exponent as self-avoiding walk, since there is a considerable penalty for self-intersections that are far apart. If p is intermediate, then the elastic polymer has a critical exponent that linearly interpolates between the self-avoiding walk exponent ν and $\frac{1}{2}$. If p is large, then the elastic polymer behaves like a small perturbation of simple random walk, since the intersections that give a contribution in (1.8.5) have to be close. In fact, this has been proved using the lace expansion in van der Hofstad, den Hollander and Slade (preprint 1998) for all $p > \frac{d-4}{2}$ and β small enough.
(For $d = 1$ and $p = \frac{3}{2}$, this conjecture fits nicely with the result that $\{S_{\lceil tn\rceil}/\sqrt{n}\}_{t\in[0,1]}$ under $Q_n^{\beta n^{-\frac{3}{2}}}$ converges to $\{B_t\}_{t\in[0,1]}$ under $\widehat{Q}_1^{\beta}$ (see Sections 1.2-1.3 and Chapters 2 and 4).)

It is hoped that the model for an elastic polymer in dimension one is related to the Domb-Joyce model in dimension two. This relation is being investigated.

1.8.3 Inhomogeneity

Instead of changing the interaction, we can also change the reference measure in the definition of the polymer model. We will do so in the next example.

Let $(S_i)_{i\in\mathbb{N}_0}$ be the inhomogeneous random walk where the steps are uniform in $[-R, R]^d \subset \mathbb{Z}^d$ ($R \in \mathbb{N}$), starting at the origin. This is a model for an inhomogeneous polymer, where there are monomers of different sizes. (It is possible to make the measure different from uniform, to incorporate different densities of the respective monomers.) The measure is now defined in the same manner as in (1.2.1-1.2.2). Note that this measure is also interesting in $d = 1$ for $\beta = \infty$ if $R > 1$.

König proves a LLN and a CLT for the end-point of the path in $d = 1$ as in Theorem 1.1 and Theorem 1.4 (see König (1994) and (1996)). Furthermore, for $\beta = \infty$ the speed is in $(0, R)$ (see König (1993)).

1.8.4 A polymer near an interface

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of random variables taking values ± 1 with probability $\frac{1}{2}$. For $n \in \mathbb{N}$, define the (random) measure $Q_n^{\lambda,h,\omega}$ on n-step paths by setting

$$Q_n^{\lambda,h,\omega}\Big((S_i)_{i=0}^n\Big) = \frac{1}{Z_n^{\lambda,h,\omega}} \exp\Big[\lambda \sum_{i=1}^n (\omega_i + h)\text{sign}(S_i)\Big] P\Big((S_i)_{i=0}^n\Big), \tag{1.8.10}$$

where $h \in [0, 1), \lambda \in [0, \infty)$ and $Z_n^{\lambda,h,\omega}$ is the (random) normalizing constant

$$Z_n^{\lambda,h,\omega} = E\Big(\exp\Big[\lambda \sum_{i=1}^n (\omega_i + h)\text{sign}(S_i)\Big]\Big). \tag{1.8.11}$$

Equations (1.8.10-1.8.11) model a polymer near an interface in the following way. Think of $(i, S_i)_{i=0}^n$ as a polymer consisting of n-monomers, where time is viewed as an extra spatial dimension, namely the direction in which the interface lies. Think of the upper half plane as 'oil' and the lower half plane as 'water'. There are two types of monomers, occurring randomly with equal probability and indexed by ω: $\omega_i = +1$ means that monomer i is attracted by 'water', while $\omega_i = -1$ means that monomer i is attracted by 'oil'. For $h \in (0, 1)$, the polymer has an overall tendency to prefer 'oil'.

Bolthausen and den Hollander (1998) investigate this model in dimension one. They define the polymer to be *localized* if

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n^{\lambda,h,\omega} > \lambda h, \tag{1.8.12}$$

and *delocalized* otherwise. (The limit exists ω-a.s. and is non-random.) They prove that there is a phase transition, i.e., there exists an $h_c = h_c(\lambda)$ such that there is localization for $0 \le h < h_c$, while there is delocalization for $h \ge h_c$. Intuitively, in the localized regime the polymer remains in a finite region close to the interface with high probability. In the delocalized regime, on the other hand, the frequency of time the polymer spends in the 'oil' converges to 1 (see Biskup and den Hollander (preprint 1997)). Note that if the polymer is always in the upper half plane, then we have equality in (1.8.12), since in that case $\text{sign}(S_i) = 1$ for all i and $\sum_{i=1}^{n} \omega_i = o(n)$ a.s. by the law of large numbers. Furthermore, Biskup and den Hollander (preprint 1997) prove that in the localized regime the tails of the distribution of the position of the path at any time in between 0 and n are exponentially small and the dependence on the boundary conditions (i.e., ω) is small.

1.8.5 Branching polymers

A lattice tree is a nearest neighbour connected graph on $\mathbb{Z}^d$ containing no loops. Lattice trees having no loops is analogous to self-avoiding walks having no self-intersections (see Section 1.1). We think of the uniform measure on all lattice trees as modeling the distribution in space of a branching polymer.

It turns out that for d large enough, the distribution of lattice trees converges to the distribution of super-Brownian motion conditioned on having mass one (so-called integrated super-Brownian excursion (see Derbez and Slade (1998a,b). Super-Brownian motion is a process that at any fixed time is a random measure, having Gaussian properties (see Aldous (1993)). Super-Brownian motion is the weak limit of critical branching random walks. The proof of the scaling of lattice trees uses a lace expansion.[9] This result is similar to (1.2.5-1.2.6) for $d \geq 5$ proved by Hara and Slade (1992a,b).

1.8.6 Conclusion: the scale of a random polymer

We can summarize the scaling behavior of the models in Sections 1.8.1.-1.8.5 in the following way. The scale of a polymer

- is different from the scale of the model without interaction if the dimension is low and the interaction is sufficiently global.
- is larger than the scale of the free model if the interaction is self-repellent, i.e., there is a penalty for self-intersections.
- is smaller than the scale of the free model if the interaction is self-attracting, i.e., if there is a reward for self-intersections (like in the model in (1.8.1-1.8.2) for $\gamma > \gamma(\beta)$).
- depends on the precise interaction in a very sensitive way.

For an expository text on probabilistic polymer models, see den Hollander (1996) and Slade (1996).

[9]They prove the same results for lattice trees in $d \geq 9$ that can take sufficiently large steps.

Chapter 2

Scaling for a random polymer

2.1 Introduction and main results

In this chapter we investigate the scaling behavior of the exponential rate $r^*(\beta)$ and the speed $\theta^*(\beta)$ in the Domb-Joyce model as $\beta \downarrow 0$ (recall Theorem 1.2). This scaling theory turns out to follow from a scaling analysis of $\lambda(a\beta^{\frac{2}{3}}, \beta)$ for $a \in \mathbb{R}$ as $\beta \downarrow 0$ described in Theorem 2.3 below. This analysis makes use of the notion of epi-convergence of functionals on $L^2(\mathbb{R}_0^+)$.

The rest of this section is organized as follows. In Section 2.1.1 we give an extension of Theorems 1.1–1.2. In Section 2.1.2 we present numerical evidence for the scaling of the key quantities $r^*(\beta)$ and $\theta^*(\beta)$. In Section 2.1.3 we give our main scaling theorem and present a handwaving argument how it implies the scaling of $r^*(\beta)$ and $\theta^*(\beta)$. In Section 2.1.4 we give some figures illustrating the scaling results in Section 2.1.3. In Section 2.1.5 we relate our results to the LLN for the Edwards model (recall Theorem 1.3).

2.1.1 Law of large numbers in Greven and den Hollander (1993): Theorem 2.1

Recall the definitions of $A_{r,\beta}$ and P:

$$A_{r,\beta}(i,j) = e^{r(i+j-1)-\beta(i+j-1)^2} P(i,j) \tag{2.1.1}$$

with

$$P(i,j) = \binom{i+j-2}{i-1} \left(\frac{1}{2}\right)^{i+j-1}. \tag{2.1.2}$$

Let $\tau_{r,\beta}$ be the unique normalized and positive eigenvector corresponding to $\lambda(r,\beta)$, i.e., the pair $(\lambda(r,\beta), \tau_{r,\beta})$ satisfies

$$A_{r,\beta}\tau_{r,\beta} = \lambda(r,\beta)\tau_{r,\beta}, \qquad (\tau_{r,\beta} \geq 0, \|\tau_{r,\beta}\|_{l^2(\mathbb{N})} = 1). \tag{2.1.3}$$

Recall that $r^*(\beta)$ is the unique solution of $\lambda(r,\beta) = 1$, i.e.,

$$\lambda(r^*(\beta),\beta) = 1 \tag{2.1.4}$$

Define μ_n to be the *empirical distribution of local times*, i.e.,

$$\mu_n = \frac{1}{|R_n|} \sum_{x \in R_n} \delta_{\ell_n(x)}, \tag{2.1.5}$$

where R_n is the range of the random walk

$$R_n = \big(\min_{0\leq i\leq n} S_i, \max_{0\leq i\leq n} S_i\big). \tag{2.1.6}$$

Theorem 2.1 *(Greven and den Hollander (1993))*
(*i*) *For every $\beta \in \mathbb{R}^+$ there exists $\theta^*(\beta) \in (0,1)$ such that*

$$\lim_{n\to\infty} Q_n^\beta\Big(\Big|\frac{1}{n}|S_n| - \theta^*(\beta)\Big| \leq \varepsilon\Big) = 1 \quad \textit{for every} \quad \varepsilon > 0, \tag{2.1.7}$$

where

$$\theta^*(\beta) = \Big[\frac{\partial}{\partial r}\lambda(r,\beta)\Big]^{-1}_{r=r^*(\beta)}. \tag{2.1.8}$$

Furthermore, $\beta \mapsto \theta^*(\beta)$ is analytic, $\lim_{\beta\downarrow 0}\theta^*(\beta) = 0$ and $\lim_{\beta\to\infty}\theta^*(\beta) = 1$.

(*ii*) *For every $\beta \in \mathbb{R}^+$ there exists a probability distribution μ_β^* on $\mathbb{N}$ such that*

$$\lim_{n\to\infty} Q_n^\beta\Big(\|\mu_n - \mu_\beta^*\|_{l^1} \leq \epsilon\Big) = 1 \quad \textit{for every} \quad \epsilon > 0, \tag{2.1.9}$$

where

$$\mu_\beta^*(k) = \Big[\sum_{\substack{i,j\in\mathbb{N} \\ i+j-1=k}} \tau_{r,\beta}(i) A_{r,\beta}(i,j) \tau_{r,\beta}(j)\Big]_{r=r^*(\beta)}. \tag{2.1.10}$$

Furthermore, $\beta \mapsto \mu_\beta^*$ is analytic, $\lim_{\beta\downarrow 0}\mu_\beta^* = 0$ and $\lim_{\beta\to\infty}\mu_\beta^* = \delta_1$ pointwise.

Recall that $\theta^*(\beta)$ is called the speed of the polymer.

Theorem 2.1 extends the results in Theorem 1.1–1.2 and states that the empirical distribution of the local times converges to μ_β^* given by the r.h.s. of (2.1.10). We can think of μ_β^* as the stationary distribution of $\{\ell_n(x)\}_{x=1}^{S_n}$ under $Q_n^\beta(\cdot|S_n > 0)$. Furthermore, Theorem 2.1 states that $\theta^*(\beta)$ converges to 0 as $\beta \downarrow 0$. In the sequel we will investigate at what rate this convergence takes place.

2.1.2 Numerical estimates of $r^*(\beta)$ and $\theta^*(\beta)$

Table 1 below lists numerical estimates of $r^*(\beta)$ and $\theta^*(\beta)$ obtained from (2.1.4-2.1.8), based on a 300×300 truncation of $A_{r,\beta}$ defined in (2.1.1). We have used a standard iteration method to estimate the largest eigenvalue and corresponding eigenvector for a range of r, β-values.

β	$\beta^{-\frac{2}{3}} r^*(\beta)$	$\beta^{-\frac{1}{3}} \theta^*(\beta)$
2	1.696	0.793
0.5	1.730	1.055
10^{-2}	2.011	1.10938
10^{-3}	2.098	1.10930
10^{-4}	2.144	1.10886
10^{-5}	2.168	1.10910
10^{-6}	2.179	1.10924

Table 1

There is ample evidence for the asymptotic behavior $r^*(\beta) \sim a^*\beta^{\frac{2}{3}}$ and $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$ ($\beta \downarrow 0$), with estimates $a^* = 2.19 \pm 0.01$ and $b^* = 1.109 \pm 0.001$.

The value of $\theta^*(\beta)$ has been computed by making use of the identity

$$\frac{1}{\theta^*(\beta)} = \sum_{k\in\mathbb{N}} k\mu^*_\beta(k) = 2\left[\sum_{i\in\mathbb{N}} i\tau^2_{r^*(\beta),\beta}(i)\right] - 1, \tag{2.1.11}$$

which follows by differentiating the relation $\langle \tau_{r,\beta}, A_{r,\beta}\tau_{r,\beta}\rangle_{l^2(\mathbb{N})}$ with respect to r, using the symmetry of $A_{r,\beta}$, (2.1.3) and $\langle \tau_{r,\beta}, \frac{\partial}{\partial r}\tau_{r,\beta}\rangle_{l^2(\mathbb{N})} = 0$. Since $\tau_{r,\beta}$ is easier to estimate than $\frac{\partial}{\partial r}\lambda(r,\beta)$, the relation in (2.1.11) allows for better accuracy than (2.1.8).

2.1.3 Main results for the weak interaction limit: Theorems 2.2–2.4

The goal of this chapter is to turn the numerical observations in Section 2.1.2 into mathematical statements. Our results are formulated in Theorems 2.2–2.4 below.

1. Our main scaling theorem reads:

Theorem 2.2 *There exist $a^*, b^* \in (0,\infty)$ and $\eta^* \in \{\eta \in L^1(\mathbb{R}^+) : \|\eta\|_{L^1} = 1, \eta > 0\}$ such that as $\beta \downarrow 0$*

$$\begin{array}{lll} \beta^{-\frac{2}{3}} r^*(\beta) & \longrightarrow & a^* \\ \beta^{-\frac{1}{3}} \theta^*(\beta) & \longrightarrow & b^* \\ \beta^{-\frac{1}{3}} \mu^*_\beta(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) & \longrightarrow^{L^1} & \eta^*(\cdot). \end{array} \tag{2.1.12}$$

2. The limits a^*, b^* and η^* in Theorem 2.2 can be identified in terms of the following *Sturm-Liouville problem.* For $a \in \mathbb{R}$, let $\mathcal{L}^a$ be the differential operator defined by

$$(\mathcal{L}^a x)(u) = (2au - 4u^2)x(u) + x'(u) + ux''(u) \qquad (x \in C^\infty(\mathbb{R}^+)). \tag{2.1.13}$$

In Section 2.5 we will show that the largest eigenvalue problem

$$\begin{array}{ll} \mathcal{L}^a x & = \rho x \ (\rho \in \mathbb{R}, x \in L^2(\mathbb{R}^+) \cap C^\infty(\mathbb{R}^+)) \\ (i) & \|x\|_{L^2} = 1, x > 0 \\ (ii) & \int_0^\infty \{u^2[x(u)]^2 + u[x'(u)]^2\} du < \infty \end{array} \tag{2.1.14}$$

has a unique solution $(\rho(a), x_a)$ with the following properties:

$$\begin{array}{ll} (i) & a \mapsto \rho(a) \text{ is analytic, strictly increasing and strictly convex on } \mathbb{R} \\ (ii) & \rho(0) < 0, \ \lim_{a \uparrow \infty} \rho(a) = \infty \text{ and } \lim_{a \downarrow -\infty} \rho(a) = -\infty \\ (iii) & a \mapsto x_a \text{ is analytic as a map from } \mathbb{R} \text{ to } L^2(\mathbb{R}^+). \end{array} \tag{2.1.15}$$

The main part of our analysis to prove Theorem 2.2 will revolve around the following theorem:

Theorem 2.3 *Fix $a \in \mathbb{R}$. As $\beta \downarrow 0$, uniformly in a on compacts in $\mathbb{R}$,*

$$\beta^{-\frac{1}{3}} \left[\lambda(a\beta^{\frac{2}{3}}, \beta) - 1\right] \quad \longrightarrow \quad \rho(a) \tag{2.1.16}$$

$$\beta^{-\frac{1}{6}} \tau_{a\beta^{\frac{2}{3}}, \beta}(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \quad \longrightarrow^{L^2} \quad x_a(\cdot). \tag{2.1.17}$$

Consequently, a^*, b^* and η^* are given by

$$\begin{array}{l} a^* \textit{ is the unique solution of } \rho(a) = 0 \\ b^* = [\rho'(a^*)]^{-1} \\ \eta^*(\cdot) = \frac{1}{2}[x_{a^*}(\frac{1}{2}\cdot)]^2. \end{array} \tag{2.1.18}$$

The scaling of the eigenvalue and the eigenfunction in (2.1.16-2.1.17) will be proved in Sections 2.2-2.5. We will show that (2.1.18) follows from (2.1.16-2.1.17) in Section 2.7. Using (2.1.16), we now give a *heuristic* explanation for the scaling of $r^*(\beta)$ and $\theta^*(\beta)$ in Theorem 2.2 and the values of the limits in (2.1.18). From (2.1.16), (1.4.15) and the monotonicity of $r \mapsto \lambda(r, \beta)$ and $a \mapsto \rho(a)$ it is clear that $r^*(\beta) \sim a^* \beta^{\frac{2}{3}}$. Then use (1.4.17) to get

$$
\begin{aligned}
[\theta^*(\beta)]^{-1} &= \left[\tfrac{\partial}{\partial r}\lambda(r,\beta)\right]_{r=r^*(\beta)} \\
&= \left[\beta^{-\frac{2}{3}}\tfrac{\partial}{\partial a}\lambda(a\beta^{\frac{2}{3}},\beta)\right]_{a=\beta^{-\frac{2}{3}}r^*(\beta)} \\
&\sim \left[\beta^{-\frac{2}{3}}\tfrac{\partial}{\partial a}(1+\beta^{\frac{1}{3}}\rho(a))\right]_{a=a^*} \\
&= \beta^{-\frac{1}{3}}\rho'(a^*).
\end{aligned}
\tag{2.1.19}
$$

3. The analysis in Section 2.5 of the Sturm-Liouville problem will lead to the following additional properties:

Theorem 2.4 *(i)* $u \mapsto x_{a^*}(u)$ *is analytic and strictly decreasing on* $\mathbb{R}_0^+ = [0,\infty)$.
(ii) $u \mapsto u\frac{d}{du}x_{a^*}(u)$ *is unimodal with a minimum at* $u = \frac{1}{2}a^*$.
(iii)

$$\lim_{u\to\infty} u^{-\frac{3}{2}} \log x_{a^*}(u) = -\frac{4}{3}. \tag{2.1.20}$$

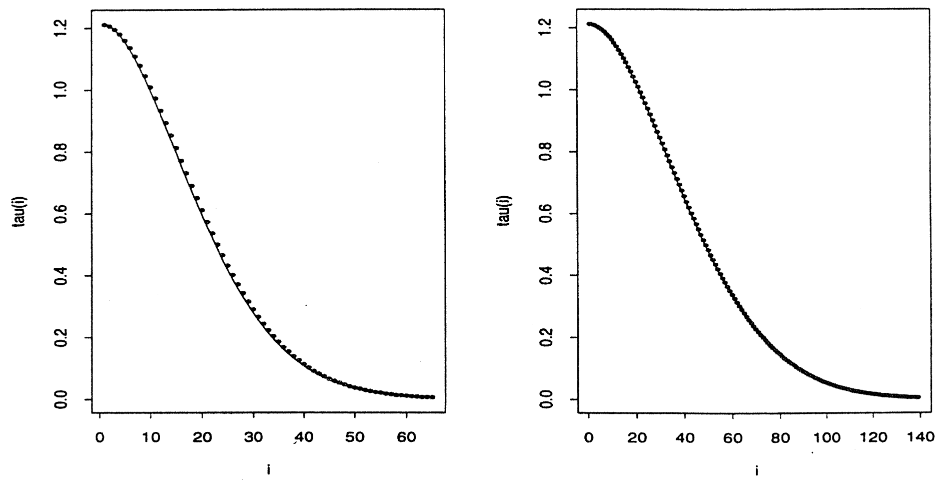
(iv)

$$\frac{1}{b^*} = 2\int_0^\infty u[x_{a^*}(u)]^2 du. \tag{2.1.21}$$

Theorems 2.2–2.4 are proved in Sections 2.3-2.7. Section 2.2 contains preparations.

Our result $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$ implies that the speed is not right-differentiable at $\beta = 0$. Thus the limit of weak repellence cannot be treated by perturbation type arguments (i.e., by doing an expansion of (1.2.1-1.2.2) for small β). This is a general feature of one-dimensional polymer models that we will also encounter in Chapters 3-5.

2.1.4 Figures illustrating Theorem 2.3



Figures 2a and 2b

Figures 2a-b compare x_{a^*} with the numerical estimates in Section 2.1.2. The solid line is a power series approximation of x_a for $a = 2.19$. The dots are the values of $\beta^{-\frac{1}{6}}\tau_{r^*(\beta),\beta}\big(\lceil u\beta^{-\frac{1}{3}}\rceil\big)$ for $\beta = 10^{-4}$, respectively, 10^{-5} and $u \in [0,3]$. The agreement is excellent. (For $\beta = 10^{-6}$ all dots were found to lie on the solid line within printing precision.) Rigorous bounds on the constants a^* and b^* will be given in Chapter 5, where we prove that $a^* \in [2.188, 2.189]$ and $b^* \in [1.10, 1.124]$ (see Theorem 5.1$(i-ii)$).

2.1.5 Relation to the Edwards model: Theorems 2.5–2.7

First we will show how the Edwards model arises as a limit of the Domb-Joyce model where the interaction parameter decreases with n in a certain way. Then we will clarify the link between the weak interaction limit of the LLN for the Domb-Joyce model and the LLN for the Edwards model.

Theorem 2.5 *For every* $\hat\beta \in \mathbb{R}_0^+$

$$Q_n^{\hat\beta n^{-\frac{3}{2}}}\Big((n^{-\frac{1}{2}}S_{\lceil tn\rceil})_{0\le t\le 1} \in \cdot\Big) \Longrightarrow^{w} \widehat{Q}_1^{\hat\beta}\Big((B_t)_{0\le t\le 1} \in \cdot\Big) \quad as\ n \to \infty. \tag{2.1.22}$$

Proof. See Brydges and Slade (1995) Theorem 1.3. The double sum in (1.2.1) equals $\sum_x \ell_n^2(x) - (n+1)$ (recall (1.4.5)), of which the first term may be absorbed into the normalizing constant Z_n^β in (1.2.2). The key point is that $n^{-\frac{3}{2}} \sum_x \ell_n^2(x)$ under the law P converges to $\int_0^T dt \int_0^T ds\, \delta(B_t = B_s)$ under the law $\widehat{P}$ (recall Section 1.3). This immediately implies Theorem 2.5. The analogue for $T \neq 1$ is obvious. □

We can think of Theorem 2.5 in the following way. If we look at the polymer under the Domb-Joyce measure from a large distance and the interaction is small, then it looks like the polymer under the Edwards measure.

Westwater (1984) proves the following result, which is a recipe for $\hat{\theta}^*(\hat{\beta})$ and $\hat{r}^*(\hat{\beta})$ analogous to Theorem 1.2:

Theorem 2.6 *(Westwater (1984)) For every $\hat{\beta} \in \mathbb{R}^+$*

$$\begin{aligned} \hat{r}^*(\hat{\beta}) &= E(\hat{\beta}, 0) && (2.1.23)\\ \hat{\theta}^*(\hat{\beta}) &= \Big[\frac{\partial}{\partial \lambda} E(\hat{\beta}, \lambda)\Big]_{\lambda=0}, && (2.1.24) \end{aligned}$$

where $E(\hat{\beta}, \lambda)$ is the smallest eigenvalue in $L^2(\mathbb{R}^+)$ of the operator $\hat{\mathcal{L}}^{\hat{\beta},\lambda}$ given by

$$\Big(\hat{\mathcal{L}}^{\hat{\beta},\lambda} y\Big)(v) = \Big[\hat{\beta} v^2 + \lambda v^{-2} - \frac{1}{2} v^{-1}\big(\frac{d^2}{dv^2} + \frac{1}{4} v^{-2}\big) v^{-1}\Big] y(v). \qquad (2.1.25)$$

(The term between round brackets equals $v^{\frac{1}{2}} \Delta_{rad}^{(2)} v^{-\frac{1}{2}}$ with $\Delta_{rad}^{(2)}$ the 2-dimensional Laplace operator.)

Theorem 2.7 below links the scaling in Theorem 2.2 with the LLN in Theorem 1.3 and with the recipe for $\hat{\theta}^*(\hat{\beta})$ in Theorem 2.6:

Theorem 2.7 *For every $\hat{\beta} \in \mathbb{R}_0^+$*

$$\begin{aligned} E(\hat{\beta}, 0) &= a^* \hat{\beta}^{\frac{2}{3}} \\ \Big[\tfrac{\partial}{\partial \lambda} E(\hat{\beta}, \lambda)\Big]_{\lambda=0} &= b^* \hat{\beta}^{\frac{1}{3}}, \end{aligned} \qquad (2.1.26)$$

where a^*, b^* are the same constants as in Theorem 2.3.

Moreover, it turns out that η^* in (2.1.18) is the stationary distribution of the local times in between 0 and B_T under the measure $\widehat{Q}_T^1(\cdot|B_T > 0)$ (see Chapter 3).

Theorems 2.6–2.7 imply that the characteristic quantities of the Edwards model have exact scaling, while for the Domb-Joyce model there is only asymptotic scaling as $\beta \downarrow 0$. The powers $\frac{1}{3}$ and $\frac{2}{3}$ in Theorem 2.7 turn out to follow from Brownian scaling (see Chapter 3).

The link established in Theorem 2.7 is not unexpected, but far from trivial. On the one hand, if β is small, then the path has many self-intersections like Brownian motion.

On the other hand, the exponents in (1.2.1) and (1.3.1) make the random walk and the Brownian motion behave atypically. This makes it difficult to compare the two models directly and to use the weak convergence of random walk towards Brownian motion. Note that it is *not* possible that $\hat{\theta}^*(\beta) = \theta^*(\beta)$ for all β, since $\hat{\theta}^*(\beta)$ becomes arbitrarily large as $\beta \to \infty$, while obviously $\theta^*(\beta) \leq 1$. Furthermore, the proof of Theorem 2.7 given below is completely functional analytic and compares the respective recipes in Theorems 2.1 and 2.6. It therefore does not provide any probabilistic link between the two models.

Proof. Take the eigenvalue problem

$$\left(\hat{\mathcal{L}}^{\hat{\beta},\lambda} y\right)(v) = E(\hat{\beta},\lambda)y(v). \tag{2.1.27}$$

Substitute the following change of variables into (2.1.25):

$$\begin{aligned} y(v) &= v^{\frac{1}{2}} x(\tfrac{1}{2}\hat{\beta}^{\frac{1}{3}} v^2) \\ u &= \tfrac{1}{2}\hat{\beta}^{\frac{1}{3}} v^2. \end{aligned} \tag{2.1.28}$$

Then, after a small computation, we obtain the Sturm-Liouville problem in (2.1.13-2.1.14)

$$\left(\mathcal{L}^a x\right)(u) = \rho x(u), \tag{2.1.29}$$

with

$$\begin{aligned} a &= \hat{\beta}^{-\frac{2}{3}} E(\hat{\beta},\lambda) \\ \rho &= \hat{\beta}^{-\frac{1}{3}} \lambda. \end{aligned} \tag{2.1.30}$$

Think of (2.1.30) as a parameterization of the curve $a \mapsto \rho(a)$ in terms of λ. Recalling the definition of a^*, b^* in (2.1.18), we now get from (2.1.29-2.1.30) that

$$\rho(a^*) = 0 \Leftrightarrow a^* = \hat{\beta}^{-\frac{2}{3}} E(\hat{\beta}, 0) \tag{2.1.31}$$

and

$$\begin{aligned} \left[\tfrac{\partial}{\partial \lambda} E(\hat{\beta},\lambda)\right]_{\lambda=0} &= \hat{\beta}^{-\frac{1}{3}} \left[\tfrac{\partial}{\partial \rho} E(\hat{\beta},\rho\hat{\beta}^{\frac{1}{3}})\right]_{\rho=0} \\ &= \hat{\beta}^{-\frac{1}{3}} \left[\tfrac{\partial}{\partial \rho}\left(a(\rho)\hat{\beta}^{\frac{2}{3}}\right)\right]_{\rho=0} \\ &= a'(0)\hat{\beta}^{\frac{1}{3}} \\ &= \frac{1}{\rho'(a^*)}\hat{\beta}^{\frac{1}{3}} \\ &= b^*\hat{\beta}^{\frac{1}{3}}, \end{aligned} \tag{2.1.32}$$

where $\rho \mapsto a(\rho)$ is the inverse function of $a \mapsto \rho(a)$. □

Finally, we give a *heuristic* explanation of the power $\frac{1}{3}$ in our result $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$ $(\beta \downarrow 0)$. First, by Brownian scaling

$$E_{\widehat{Q}_1^{\hat{\beta}T^{\frac{3}{2}}}}(B_1^2) = \frac{1}{T} E_{\widehat{Q}_T^{\hat{\beta}}}(B_T^2). \tag{2.1.33}$$

Since, according to Theorem 1.3,

$$[\hat{\theta}^*(\hat{\beta})]^2 = \lim_{T\to\infty} \frac{1}{T^2} E_{\widehat{Q}_T^{\hat{\beta}}}(B_T^2), \tag{2.1.34}$$

it follows that

$$\hat{\theta}^*(\hat{\beta}) = \hat{\theta}^*(1)\hat{\beta}^{\frac{1}{3}} \tag{2.1.35}$$

(see also Section 3.1.2 where we give the argument in more detail). Next, according to Theorem 2.1,

$$[\theta^*(\hat{\beta})]^2 = \lim_{n\to\infty} \frac{1}{n^2} E_{Q_n^{\hat{\beta}}}(S_n^2). \tag{2.1.36}$$

Moreover, by Theorem 2.6 we know that for $\hat{\beta}, T$ fixed

$$\frac{1}{n} E_{Q_n^{\hat{\beta}(\frac{T}{n})^{\frac{3}{2}}}}(S_n^2) \sim E_{\widehat{Q}_1^{\hat{\beta}T^{\frac{3}{2}}}}(B_1^2) \quad (n \to \infty). \tag{2.1.37}$$

Now, if we assume that (2.1.37) continues to hold for $\hat{\beta}$ fixed and $T = n$, then by using (2.1.36-2.1.37), respectively, (2.1.33-2.1.34) we arrive at

$$\begin{aligned}
[\theta^*(\hat{\beta})]^2 &\sim \tfrac{1}{n^2} E_{Q_n^{\hat{\beta}}}(S_n^2) \\
&\sim \tfrac{1}{T} E_{\widehat{Q}_1^{\hat{\beta}T^{\frac{3}{2}}}}(B_1^2) \\
&= \tfrac{1}{T^2} E_{\widehat{Q}_T^{\hat{\beta}}}(B_T^2) \\
&\sim [\hat{\theta}^*(\hat{\beta})]^2 \qquad (T = n \to \infty).
\end{aligned} \tag{2.1.38}$$

The above argument clearly has uniformity problems because (2.1.35) and (2.1.38) would imply $\theta^*(\hat{\beta}) = \theta^*(1)\hat{\beta}^{\frac{1}{3}}$ for all $\hat{\beta}$. However, this cannot be true because $\theta^*(\beta) \leq 1$ for all β. Nevertheless, it explains the power $\frac{1}{3}$ without using the explicit solution.

2.1.6 Outline of the proof

The rest of this chapter is organized as follows.

In Section 2.2 we give a variational formula for the scaled eigenvalues using the Rayleigh representation and introduce the notion of epi-convergence.

In Section 2.3 we prove that the functional appearing in this representation epi-converges to a functional that is the quadratic form corresponding to the Sturm-Liouville differential operator $\mathcal{L}^a$ (recall 2.1.13).

In Section 2.4 we investigate the scaled eigenvectors of $A_{a\beta^{\frac{2}{3}},\beta}$ (which are the maximizers of the variational problem in the Rayleigh representation) as $\beta \downarrow 0$.

Section 2.5 contains the proof of an auxiliary lemma that is needed in Section 2.4.

In Section 2.6 we investigate the limiting variational problem. Then the stage is set to prove our main results in Section 2.7.

2.2 Preparations

In this section we formulate the functional analytic framework in which we are going to approach our scaling theorem. Section 2.2.1 shows that our key result, Theorem 2.2 in Section 2.1.3, is equivalent to convergence of a variational problem involving a certain functional F^a_β to a variational problem involving a certain limit functional F^a (Lemma 2.1 and Proposition 2.1 below). Section 2.2.2 shows that this convergence of variational problems holds when F^a_β epi-converges to F^a and certain compactness properties for the maximizers are satisfied (Proposition 2.2 below). In this section we also formulate the main steps that have to be checked in order to prove these facts (Proposition 2.3 below). In Section 2.2.3 we collect some properties of the matrix P, defined in (2.1.2), that will be needed in the proofs.

2.2.1 A variational representation: Lemma 2.1

Rayleigh's formula for the pair $(\lambda(r,\beta), \tau_{r,\beta})$ defined in (2.1.3) reads

$$\begin{aligned} &(i) \quad \lambda(r,\beta) = \max_{\substack{y\in l^2(\mathbb{N}),y\geq 0,\\ \|y\|_{l^2}\leq 1}} \langle y, A_{r,\beta} y\rangle_{l^2} \\ &(ii) \quad \tau_{r,\beta} \text{ is the unique maximizer.} \end{aligned} \tag{2.2.1}$$

In anticipation of the scaling suggested by Table 1, we pick $r = a\beta^{\frac{2}{3}}$ for some $a \in \mathbb{R}$ and rewrite (2.2.1) in the following form. Define the functional $F^a_\beta : L^2(\mathbb{R}^+) \mapsto \mathbb{R}$ as

$$F^a_\beta(x) = \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv\ x(u)x(v) A_{a\beta^{\frac{2}{3}},\beta}(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil) - \beta^{-\frac{1}{3}} \|x\|^2_{L^2}. \tag{2.2.2}$$

Lemma 2.1 *For all $\beta \in \mathbb{R}^+$*

$$\begin{aligned} &(i) \quad \beta^{-\frac{1}{3}}\left[\lambda(a\beta^{\frac{2}{3}},\beta) - 1\right] = \max_{\substack{x\in L^2(\mathbb{R}^+),x\geq 0,\\ \|x\|_{L^2}=1}} F^a_\beta(x) \\ &(ii) \quad \beta^{-\frac{1}{6}}\tau_{a\beta^{\frac{2}{3}},\beta}(\lceil \cdot\beta^{-\frac{1}{3}}\rceil) \textit{ is the unique maximizer.} \end{aligned} \tag{2.2.3}$$

Proof. (*i*) Fix $\beta \in \mathbb{R}^+$. For $x \in L^2(\mathbb{R}^+)$ define

$$\hat{x}(i) = \beta^{-\frac{1}{6}} \int_{(i-1)\beta^{\frac{1}{3}}}^{i\beta^{\frac{1}{3}}} x(u)du \qquad (i \in \mathbb{N}). \tag{2.2.4}$$

Then the first term in (2.2.2) equals $\beta^{-\frac{1}{3}}\langle \hat{x}, A_{a\beta^{\frac{2}{3}},\beta}\hat{x}\rangle_{l^2}$. Hence, using (2.2.1)(*i*), we may write

$$\beta^{-\frac{1}{3}}\left[\lambda(a\beta^{\frac{2}{3}},\beta) - 1\right] = \max_{\substack{y\in l^2(\mathbb{N}), y\geq 0,\\ \|y\|_{l^2}\leq 1}} \max_{\substack{x\in L^2(\mathbb{R}^+), x\geq 0,\\ \|x\|_{L^2}=1, \hat{x}=y}} F^a_\beta(x). \tag{2.2.5}$$

Note that, by Cauchy-Schwarz, we have $\|\hat{x}\|_{l^2} \leq \|x\|_{L^2}$ and so the restrictions $\|y\|_{l^2} \leq 1, \|x\|_{L^2} = 1, \hat{x} = y$ in (2.2.5) are compatible. Interchange the two maxima in (2.2.5) to get the claim.
(*ii*) Use that $\|\hat{x}\|_{l^2} = \|x\|_{L^2}$ if and only if $x(u) = \beta^{-\frac{1}{6}}\hat{x}(i)$ for $u \in ((i-1)\beta^{\frac{1}{3}}, i\beta^{\frac{1}{3}}]$. □

2.2.2 A key proposition: Proposition 2.1

In Sections 2.3-2.6 we will prove:

Proposition 2.1 *As* $\beta \downarrow 0$

$$\begin{aligned} &(i) \quad \max_{\substack{x\in L^2(\mathbb{R}^+), x\geq 0,\\ \|x\|_{L^2}=1}} F^a_\beta(x) \to \max_{\substack{x\in L^2(\mathbb{R}^+), x\geq 0,\\ \|x\|_{L^2}=1}} F^a(x) \\ &(ii) \quad \textit{unique maximizer l.h.s.} \to^{L^2} \textit{unique maximizer r.h.s.}, \end{aligned} \tag{2.2.6}$$

where the limit functional $F^a : L^2(\mathbb{R}^+) \mapsto \overline{\mathbb{R}}$ *is given by*

$$F^a(x) = \int_0^\infty \left\{(2au - 4u^2)[x(u)]^2 - u[x'(u)]^2\right\}du, \tag{2.2.7}$$

with the understanding that $F^a(x) = -\infty$ *if* $x \notin C^1(\mathbb{R}^+_0)$ *or if the integral is not defined.*

Note that by partial integration $F^a(x) = \langle x, \mathcal{L}^a x\rangle_{L^2}$ for all $x \in L^2(\mathbb{R}^+_0) \cap C^2(\mathbb{R}^+_0)$ satisfying (2.1.14)(*ii*), with $\mathcal{L}^a$ as defined in (2.1.13).

Lemma 2.1 and Proposition 2.1 imply (2.1.16-2.1.17) in Theorem 2.3 pointwise in a. It will later be strengthened to uniform convergence in a on compacts (see Lemma 2.19). To prove Proposition 2.1, we will need the notion of epi-convergence, which we next explain.

2.2.3 Epi-convergence: Propositions 2.2–2.3

Let (X, τ) be a metrizable topological space and let $Y \subset X$ be dense in X. Let

$$\begin{aligned} G_\beta &: X \mapsto \mathbb{R} \qquad (\beta > 0) \\ G &: X \mapsto \overline{\mathbb{R}}. \end{aligned} \tag{2.2.8}$$

Definition 2.1 *The family $(G_\beta)_{\beta>0}$ is said to be epi-convergent to G on Y, written*

$$\text{e-lim}_{\beta\downarrow 0} G_\beta = G \text{ on } Y, \tag{2.2.9}$$

if the following properties hold:

$$\begin{array}{ll} (i) & \forall x_\beta \to^\tau x \in Y : \limsup_{\beta\downarrow 0} \; G_\beta(x_\beta) \le G(x) \\ (ii) & \exists x_\beta \to^\tau x \in Y : \liminf_{\beta\downarrow 0} \; G_\beta(x_\beta) \ge G(x). \end{array} \tag{2.2.10}$$

The importance of the notion of epi-convergence is contained in the following proposition:

Proposition 2.2 *Suppose that*
(1) $\text{e-lim}_{\beta\downarrow 0} G_\beta = G$ *on* Y
(2) $\forall \beta > 0 : G_\beta$ *is continuous on* X *and has a unique maximizer* $\bar{x}_\beta \in X$
(3) $\exists K \subset Y$ *such that*
  (*i*) K *is* τ*-relatively compact in* X
  (*ii*) G *has a unique maximizer* $\bar{x} \in \overline{K}$
  (*iii*) $\exists (x_\beta)_{\beta>0} \subset \overline{K}$ *such that* $x_\beta - \bar{x}_\beta \to^\tau 0$ *and* $G_\beta(x_\beta) - G_\beta(\bar{x}_\beta) \to 0$ *as* $\beta \downarrow 0$.
Then as $\beta \downarrow 0$

$$\sup_{x\in X} G_\beta(x) \;\to\; \sup_{x\in X} G(x) \tag{2.2.11}$$

$$\bar{x}_\beta \;\to^\tau\; \bar{x}. \tag{2.2.12}$$

Proof. See Attouch (1984) Theorem 1.10 and Proposition 1.14. □

Remark: Epi-convergence differs from pointwise convergence: $\lim_{\beta\downarrow 0} G_\beta(x) = G(x)$ for all $x \in Y$. Namely, (2.2.10)(i),(ii) are weaker in the sense that they require only inequalities, but stronger in the sense that they involve limits in neighborhoods rather than single points. Epi-convergence is a unilateral notion. We have chosen the direction that is suitable for suprema rather than infima.

Fix $a \in \mathbb{R}$. We are going to apply Proposition 2.2 with the following choices:

$$\begin{array}{rcl} X &=& \{x \in L^2(\mathbb{R}^+) : x \ge 0, \|x\|_{L^2} = 1\} \\ Y &=& X \cap C^1(\mathbb{R}_0^+) \\ \tau &=& \text{topology induced by } \|\cdot\|_{L^2} \\ K &=& K_C^a = \{x \in Y : F^a(x) \ge -C\} \\ G_\beta &=& F_\beta^a \\ G &=& F^a, \end{array} \tag{2.2.13}$$

with F_β^a and F^a defined in (2.2.2) and (2.2.7) and with C large enough so that $K_C^a \neq \emptyset$. Our main result is:

Proposition 2.3 *Assumptions* $(1)-(3)$ *in Proposition 2.2 hold for the choice in (2.2.13).*

We prove Assumption (1) in Section 2.3, (3)(i),(ii) in Section 2.5 and (3)(iii) in Section 2.4. We already know (2) to be true because of Lemma 2.1(ii).

Proposition 2.3 proves Proposition 2.1 in Section 2.2.2.

2.2.4 Properties of P: Lemmas 2.2–2.4

We list a few identities and estimates for the matrix P, defined in (2.1.2), that will be needed later on.

Lemma 2.2 *For every* $i \geq 1, k \geq 0$

$$\sum_{j\geq 1} \frac{(i+j+k-2)!}{(i+j-2)!} P(i,j) = 2^k \frac{(i+k-1)!}{(i-1)!}. \tag{2.2.14}$$

Proof. Elementary. Use that the summands in the l.h.s. can be rewritten as $P(i+k,j)$ times the r.h.s. Then use that $\sum_{j\geq 1} P(i+k,j) = 1$. □

Lemma 2.3 (i) *For* $i,j \to \infty$ *such that* $i - j = o((i+j)^{\frac{2}{3}})$

$$P(i,j) = \left\{ \frac{1}{\sqrt{2\pi(i+j)}} \exp\left[-\frac{(i-j)^2}{2(i+j)} \right] \right\} \left[1 + \mathcal{O}((i+j)^{-\frac{1}{3}}) \right]. \tag{2.2.15}$$

(ii) *There exist* $0 < c_1 < c_2 < \infty$ *such that*

$$\exp\left[-c_2 \frac{(i-j)^2}{(i+j)} \right] \leq P(i,j) \leq \exp\left[-c_1 \frac{(i-j)^2}{(i+j)} \right] \text{ for all } i,j \geq 1. \tag{2.2.16}$$

Proof. Via Stirling's formula. See also Révész (1990) Theorem 2.8. □

Lemma 2.2 allows us to compute the following moments, which we will need in Section 2.3:

$$\sum_{j\geq 1}(i+j-1)^n P(i,j) = \begin{array}{ll} 2i & (n=1) \\ 4i^2 + 2i & (n=2) \\ 8i^3 + 12i^2 + 6i & (n=3) \\ 16i^4 + 48i^3 + 72i^2 + 32i & (n=4). \end{array} \tag{2.2.17}$$

Lemma 2.3(i) is a Gaussian approximation of P, while Lemma 2.3(ii) shows that $P(i,j)$ is small away from the diagonal.

Lemma 2.4 *For all* $i,j \geq 0$ *with* $(i,j) \neq (0,0)$

$$P(i+1,j) + P(i,j+1) - 2P(i+1,j+1) = 0, \tag{2.2.18}$$

with the convention $P(i,0) = P(0,j) = 0$.

Proof. Elementary. □

Lemma 2.4 will be needed in Section 2.4 to obtain estimates of $\tau_{\alpha\beta^{\frac{2}{3}},\beta}$, the eigenvector of $A_{\alpha\beta^{\frac{2}{3}},\beta}$.

2.3 $(F^a_\beta)_{\beta>0}$ is epi-convergent to F^a: Lemmas 2.5–2.8

In this section we prove Assumption (1) in Proposition 2.2 for the choice in (2.2.13).

This section is technically involved, as it consists of a chain of estimates and inequalities that are needed to handle the epi-convergence. The proof is contained in Lemmas 2.5–2.8 below.

Throughout Sections 2.3 and 2.4 we fix $a \in \mathbb{R}$ and we write the abbreviations $F_\beta = F^a_\beta$, $F = F^a$, $A_\beta = A_{a\beta^{\frac{2}{3}},\beta}$, $\lambda(\beta) = \lambda(a\beta^{\frac{2}{3}},\beta)$, $\tau_\beta = \tau_{a\beta^{\frac{2}{3}},\beta}$.

We begin by splitting F_β, F into two parts, namely (recall (2.2.2) and (2.2.7))

$$\begin{aligned} F_\beta &= F^1_\beta + F^2_\beta \\ F &= F^1 + F^2, \end{aligned} \tag{2.3.1}$$

with

$$\begin{aligned} F^1_\beta(x) &= \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv\ x^2(u)[A_\beta - P]\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \\ F^2_\beta(x) &= -\tfrac{1}{2}\beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv\ [x(u)-x(v)]^2 A_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned} F^1(x) &= \int_0^\infty du\ (2au - 4u^2)\ x^2(u) \\ F^2(x) &= -\int_0^\infty du\ u[x'(u)]^2. \end{aligned} \tag{2.3.3}$$

Lemma 2.5 $\forall (x_\beta)_{\beta>0}, x_\beta \to^{L^2} x \in X : \limsup_{\beta\downarrow 0} F^1_\beta(x_\beta) \leq F^1(x)$.

Proof. Abbreviate

$$e_\beta(i,j) = a\beta^{\frac{2}{3}}(i+j-1) - \beta(i+j-1)^2, \tag{2.3.4}$$

which is the exponent appearing in $A_\beta(i,j)$, i.e., $A_\beta = e^{e_\beta}P$ (see (2.1.1-2.1.2)). We note that e_β has the following properties:

$$\begin{aligned} &(i)\ e_\beta(i,j) \leq 0 \text{ for } i \geq a\beta^{-\frac{1}{3}}, j \geq 1 \\ &(ii)\ e_\beta(i,j) \leq \tfrac{1}{4}a^2\beta^{\frac{1}{3}} \text{ for } i,j \geq 1. \end{aligned} \tag{2.3.5}$$

Hence, for small enough β and large enough N

$$\begin{aligned} F^1_\beta(x_\beta) \leq\ & \beta^{-\frac{2}{3}} \int_0^N du \int_0^\infty dv\ x^2_\beta(u) \\ & \times\big\{e_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) + e^2_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big)\big\} P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \end{aligned} \tag{2.3.6}$$

(use that $e^t \le 1+t+t^2$ for $t \ll 1$ and $t \le 0$). The integral over v can be transformed into the following sum:

$$\beta^{\frac{1}{3}} \sum_{j\ge1} \{e_\beta(i,j) + e_\beta^2(i,j)\} P(i,j) \text{ with } i = \lceil u\beta^{-\frac{1}{3}} \rceil. \tag{2.3.7}$$

Using (2.2.17), we can carry out the summation. Namely,

$$\begin{aligned} \textstyle\sum_{j\ge1} e_\beta(i,j)P(i,j) &= a\beta^{\frac{2}{3}}(2i) - \beta(4i^2+2i) \\ \textstyle\sum_{j\ge1} e_\beta^2(i,j)P(i,j) &= a^2\beta^{\frac{4}{3}}(4i^2+2i) - 2a\beta^{\frac{5}{3}}(8i^3+12i^2+6i) \\ &\quad +\beta^2(16i^4+48i^3+72i^2+32i). \end{aligned} \tag{2.3.8}$$

Since $i = \lceil u\beta^{-\frac{1}{3}} \rceil \le (N+1)\beta^{-\frac{1}{3}}$, the contribution to (2.3.6) of the second sum can be estimated above by

$$\beta^{\frac{1}{3}}(6a^2(N+1)^2 + 168(N+1)^4) \int_0^N du\ x_\beta^2(u) = \mathcal{O}(\beta^{\frac{1}{3}}), \tag{2.3.9}$$

where we use that $\|x_\beta\|_{L^2} = 1$. The error term is uniform in x_β for fixed N. Hence we get

$$\begin{aligned} F_\beta^1(x_\beta) &\le \beta^{-\frac{1}{3}} \textstyle\int_0^N du\ x_\beta^2(u) \\ &\quad \times\left\{a\beta^{\frac{2}{3}}(2\lceil u\beta^{-\frac{1}{3}}\rceil) - \beta(4\lceil u\beta^{-\frac{1}{3}}\rceil^2 + 2\lceil u\beta^{-\frac{1}{3}}\rceil)\right\} + \mathcal{O}(\beta^{\frac{1}{3}}) \\ &= \textstyle\int_0^N du\ x_\beta^2(u)(2au - 4u^2) + \mathcal{O}(\beta^{\frac{1}{3}}). \end{aligned} \tag{2.3.10}$$

Now let $\beta \downarrow 0$. Then we obtain, recalling that $x_\beta \to^{L^2} x$,

$$\begin{aligned} \limsup_{\beta\downarrow0} F_\beta^1(x_\beta) &\le \limsup_{\beta\downarrow0} \textstyle\int_0^N du\ x_\beta^2(u)(2au-4u^2) \\ &= \textstyle\int_0^N du\ x^2(u)(2au-4u^2). \end{aligned} \tag{2.3.11}$$

Finally, let $N \to \infty$ and note that the r.h.s. of (2.3.11) converges to $F^1(x)$. □

Lemma 2.6 $\forall x \in X:\quad \liminf_{\beta\downarrow0} F_\beta^1(x) \ge F^1(x)$.

Proof. Estimate

$$F_\beta^1(x) \ge \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv\ x^2(u) e_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \tag{2.3.12}$$

(use that $e^t \ge 1+t$ for all t). The integral over v is $\beta^{\frac{1}{3}}$ times the first sum computed in (2.3.8) with $i = \lceil u\beta^{-\frac{1}{3}}\rceil$. Hence

$$\begin{aligned} F_\beta^1(x) &\ge \beta^{-\frac{1}{3}} \textstyle\int_0^\infty du\ x^2(u) \\ &\quad \times \left\{a\beta^{\frac{2}{3}}(2u\beta^{-\frac{1}{3}}) - \beta\big(4(u\beta^{-\frac{1}{3}}+1)^2 + 2(u\beta^{-\frac{1}{3}}+1)\big)\right\} \\ &= \textstyle\int_0^\infty du\ x^2(u)(2au-4u^2) + \mathcal{O}(\beta^{\frac{1}{3}}). \end{aligned} \tag{2.3.13}$$

Now let $\beta \downarrow 0$. Then the claim follows. □

Lemma 2.7 $\forall (x_\beta)_{\beta>0}, x_\beta \to^{L^2} x \in X \cap Y : \limsup_{\beta\downarrow 0} F^2_\beta(x_\beta) \le F^2(x)$.

Proof. The proof is in Steps 1-3 below.

STEP 1 *For every $\epsilon > 0$ and N, M finite*

$$F^2_\beta(x_\beta) \le -\frac{1}{2}(1+\mathcal{O}(\beta^{\frac{1}{9}})) \int_\epsilon^N du \int_{-M}^M dw \, \Big[\frac{1}{\beta^{\frac{1}{6}}}\{x_\beta(u) - x_\beta(u+w\beta^{\frac{1}{6}})\}\Big]^2 \phi_{2u}(w), \quad (2.3.14)$$

where ϕ_{2u} is the Gaussian density with mean zero and variance $2u$.

Proof. Pick $\epsilon > 0$ and N, M finite and N sufficiently large. Then

$$F^2_\beta(x_\beta) \le -\frac{1}{2}\beta^{-\frac{2}{3}} e^{-9N^2\beta^{\frac{1}{3}}} \int_\epsilon^N du \int_{u-M\beta^{\frac{1}{6}}}^{u+M\beta^{\frac{1}{6}}} dv \, [x_\beta(u) - x_\beta(v)]^2 P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big), \quad (2.3.15)$$

where we use that $A_\beta = e^{e_\beta} P$ and $e_\beta\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big) \ge -9N^2\beta^{\frac{1}{3}}$ on the region of integration (since N is sufficiently large, see (2.3.4)). Put $w = \beta^{-\frac{1}{6}}(v-u)$. Then by Lemma 2.3($i$)

$$\begin{aligned} F^2_\beta(x_\beta) \;&\le\; -\tfrac{1}{2}\beta^{-\frac{2}{3}} e^{-9N^2\beta^{\frac{1}{3}}} \textstyle\int_\epsilon^N du \int_{-M}^M dw \; \beta^{\frac{1}{6}} \, [x_\beta(u) - x_\beta(u+w\beta^{\frac{1}{6}})]^2 \\ &\quad \times\Big\{\tfrac{1}{\sqrt{2\pi 2u\beta^{-\frac{1}{3}}}} \exp\big[-\tfrac{w^2}{4u}\big]\Big\}(1+o(\beta^{\frac{1}{9}})), \end{aligned} \quad (2.3.16)$$

where the error term is uniform on the region of integration. Collecting all the powers of β, we get the claim. □

To investigate the limit of the integral in (2.3.14) as $\beta \downarrow 0$, we proceed with a technical fact contained in Steps 2 and 3 below. Let T_h be the translation operator defined by $T_h x_\beta(\cdot) = x_\beta(\cdot + h)$.

STEP 2 *For every $0 < b_1 < b_2 < \infty$*

$$\liminf_{h\to 0, \beta\downarrow 0} \int_{b_1}^{b_2} \Big\{\frac{1}{h}[T_h x_\beta - x_\beta](u)\Big\}^2 \ge \int_a^b [x'(u)]^2 \, du. \quad (2.3.17)$$

Proof. Since (2.3.17) is trivial when the liminf is infinite, we may assume that the liminf is finite, say L. Pick any subsequence h_n, β_n along which the liminf is reached, and put $y_n = \frac{1}{h_n}[T_{h_n} x_{\beta_n} - x_{\beta_n}]$. Then, because $\|y_n\|_{L^2[b_1,b_2]} \le L+1 < \infty$ for n large enough, it follows from the Banach-Alaoglu theorem (Rudin (1991) Theorem 3.15) that there exists a subsequence (y_{n_k}) and a $y \in L^2[b_1, b_2]$ such that

$$y_{n_k} \to y \text{ weakly in } L^2[b_1, b_2] \qquad (k \to \infty). \quad (2.3.18)$$

Thus, for any $\phi \in C_c^1(b_1,b_2) = \{\phi \in C^1(b_1,b_2) : \mathrm{supp}(\phi) \subset (b_1,b_2)\}$

$$\int_{b_1}^{b_2} y_{n_k}(u)\phi(u)du \to \int_{b_1}^{b_2} y(u)\phi(u)du \qquad (k\to\infty). \tag{2.3.19}$$

Next, the l.h.s. of (2.3.19) can be rewritten as

$$\begin{aligned} \int_{b_1}^{b_2} y_{n_k}(u)\phi(u)du &= \int_{b_1}^{b_2} \tfrac{1}{h_n}[T_{h_n}x_{\beta_n} - x_{\beta_n}](u)\phi(u)du \\ &= \int_{b_1+h_n 1_{\{h_n<0\}}}^{b_2+h_n 1_{\{h_n>0\}}} x_{\beta_n}(u)\tfrac{1}{h_n}[T_{-h_n}\phi - \phi](u)du \\ &= \int_{b_1}^{b_2} x_{\beta_n}(u)\tfrac{1}{h_n}[T_{-h_n}\phi - \phi](u)du + o(1) \qquad (n\to\infty). \end{aligned} \tag{2.3.20}$$

The last equality holds because $\|x_{\beta_n}\|_{L^2(\mathbb{R}^+)} = 1$ and $|\frac{1}{h_n}[T_{-h_n}\phi - \phi]| \leq \max_{u\in\mathbb{R}^+}|\phi'(u)| < \infty$. Let $n\to\infty$ and note that by the latter property

$$\frac{1}{h_n}[T_{-h_n}\phi - \phi] \to -\phi' \text{ pointwise and weakly in } L^2[b_1,b_2]. \tag{2.3.21}$$

Together with $x_{\beta_n} \to^{L^2} x$, (2.3.21) implies that the last integral in (2.3.20) tends to $\int_{b_1}^{b_2} x(u)[-\phi'(u)]du = \int_{b_1}^{b_2} x'(u)\phi(u)du$ (recall from (2.2.13) that $x \in Y \subset C^1(\mathbb{R}_0^+)$). Since $C_c^1(b_1,b_2)$ is dense in $L^2[b_1,b_2]$ in the weak topology, we thus have from (2.3.19)

$$y = x' \ a.e. \ on \ [b_1,b_2]. \tag{2.3.22}$$

The claim now follows by combining (2.3.18) and (2.3.22), and noting that $\|\cdot\|_{L^2[b_1,b_2]}$ is lower semicontinuous in the weak topology: $L = \lim_{k\to\infty}\|y_{n_k}\|_{L^2[b_1,b_2]} \geq \|y\|_{L^2[b_1,b_2]} = \|x'\|_{L^2[b_1,b_2]}$. □

STEP 3 *For every $\epsilon > 0$ and N finite, every $f:\mathbb{R}^+\to\mathbb{R}^+$ bounded and continuous, and every $w \in \mathbb{R}$*

$$\liminf_{\beta\downarrow 0}\int_\epsilon^N du\ f(u)\Big[\frac{1}{\beta^{\frac{1}{6}}}\{x_\beta(u) - x_\beta(u + w\beta^{\frac{1}{6}})\}\Big]^2 \geq \int_\epsilon^N du\ f(u)\big[wx'(u)\big]^2. \tag{2.3.23}$$

Proof. Pick any sequence $(f_n)_{n\in\mathbb{N}}$ of functions on $\mathbb{R}^+$ such that

$$\begin{aligned} &(i) \quad f_n(u) = f_{n,k} \text{ for } c_{n,k-1} < u \leq c_{n,k} \ (k = 1,\ldots,n; c_{n,0} = \epsilon, c_{n,n} = N) \\ &(ii) \quad f_n \leq f \\ &(iii) \quad f_n \uparrow f \text{ in sup-norm on } [\epsilon, N] \text{ as } n\to\infty. \end{aligned} \tag{2.3.24}$$

Then, by (i) and (ii),

$$\begin{aligned}
l.h.s.\ (2.3.23) &\geq \liminf_{\beta\downarrow 0} \int_\epsilon^N du\ f_n(u)\Big[\tfrac{1}{\beta^{\frac{1}{6}}}\{x_\beta(u) - x_\beta(u + w\beta^{\frac{1}{6}})\}\Big]^2 \\
&\geq \sum_{k=1}^n f_{n,k} \liminf_{\beta\downarrow 0} \int_{c_{n,k-1}}^{c_{n,k}} du\ \Big[\tfrac{1}{\beta^{\frac{1}{6}}}\{x_\beta(u) - x_\beta(u + w\beta^{\frac{1}{6}})\}\Big]^2 \\
&\geq \sum_{k=1}^n f_{n,k} \int_{c_{n,k-1}}^{c_{n,k}} du\ \big[wx'(u)\big]^2 \\
&= \int_\epsilon^N du\ f_n(u)\big[wx'(u)\big]^2,
\end{aligned} \tag{2.3.25}$$

where in the third inequality we use (2.3.17) with $h = w\beta^{\frac{1}{6}}$ and $b_1 = c_{n,k-1}, b_2 = c_{n,k}$ $(k = 1, \ldots, n)$. Now let $n \to \infty$ and use (iii) together with Fatou to get the claim in (2.3.23). □

Using (2.3.23) we can now finish the proof of Lemma 2.7. Indeed, continuing with (2.3.14), we get

$$\begin{aligned}
\limsup_{\beta\downarrow 0} F_\beta^2(x_\beta) &\leq -\tfrac{1}{2} \int_{-M}^M dw\ \int_\epsilon^N du\ \phi_{2u}(w)\big[wx'(u)\big]^2 \\
&= -\tfrac{1}{2} \int_\epsilon^N du\ \big[x'(u)\big]^2 \int_{-M}^M dw\ w^2\phi_{2u}(w).
\end{aligned} \tag{2.3.26}$$

Finally, let $M \to \infty$ and note that $\int_{-\infty}^\infty dw\ w^2\phi_{2u}(w) = 2u$. Then let $N \to \infty$ and $\epsilon \downarrow 0$ to get the claim in Lemma 2.7. □

Lemma 2.8 $\forall x \in Y : \liminf_{\beta\downarrow 0} F_\beta^2(x) \geq F^2(x)$.

Proof. Recall that

$$F_\beta^2(x) = -\beta^{-\frac{2}{3}} \int_0^\infty du \int_u^\infty dv\ [x(u) - x(v)]^2 A_\beta(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil). \tag{2.3.27}$$

Now, substitute

$$x(u) - x(v) = \int_u^v x'(s)ds \tag{2.3.28}$$

into (2.3.27) and use Cauchy-Schwarz and Fubini to obtain

$$F_\beta^2(x) \geq -\beta^{-\frac{2}{3}} \int_0^\infty ds\ [x'(s)]^2 \int_0^s du \int_s^\infty dv\ (v-u)A_\beta\ bigl(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil). \tag{2.3.29}$$

Then use (2.3.5) (ii) to get

$$F_\beta^2(x) \geq -e^{\frac{1}{4}a^*\beta^{\frac{1}{3}}}\beta^{-\frac{2}{3}} \int_0^\infty ds\ [x'(s)]^2 \int_0^s du \int_s^\infty dv\ (v-u)P\big(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil\big). \tag{2.3.30}$$

Perform the change of variables $w_1 = \beta^{-\frac{1}{6}}(u-s), w_2 = \beta^{-\frac{1}{6}}(v-s)$ to arrive at

$$\begin{aligned} F_\beta^2(x) \;\geq\; & -e^{\frac{1}{4}a^*\beta^{\frac{1}{3}}} \int_0^\infty ds\; [x'(s)]^2 \int_{-\infty}^0 dw_1 \int_0^\infty dw_2\; (w_2-w_1) \\ & \times \beta^{-\frac{1}{6}} P\big(\lceil s\beta^{-\frac{1}{3}} - w_1\beta^{-\frac{1}{6}}\rceil, \lceil s\beta^{-\frac{1}{3}} + w_2\beta^{-\frac{1}{6}}\rceil\big). \end{aligned} \tag{2.3.31}$$

Finally, use Lemma 2.3(i) to get

$$\limsup_{\beta\downarrow 0} F_\beta^2(x) \geq -\int_0^\infty ds[x'(s)]^2 \int_{-\infty}^0 dw_1 \int_0^\infty dw_2\; (w_2-w_1)\phi_{2s}(w_1-w_2) = F^2(x), \tag{2.3.32}$$

where the last equality follows by integration over the Gaussian density and (2.3.2). □

Lemmas 2.5–2.8 show that F_β epi-converges to F on Y. This proves Assumption (1) in Proposition 2.2 as was claimed in Proposition 2.3.

2.4 An approximate maximizer of F_β^a: Proposition 2.4

Again we fix $a \in \mathbb{R}$ and suppress it from the notation. Like Section 2.3, this section is technically somewhat involved, as it consists of a chain of estimates and inequalities that is needed to handle the approximation.

Define the scaled form of the eigenvector τ_β of A_β as

$$\bar{\tau}_\beta(u) = \beta^{-\frac{1}{6}}\tau_\beta(i) \text{ for } (i-1)\beta^{\frac{1}{3}} < u \leq i\beta^{\frac{1}{3}} \qquad (i \geq 1). \tag{2.4.1}$$

By Lemma 2.1, $\bar{\tau}_\beta$ is the unique maximizer of F_β. However, $\bar{\tau}_\beta$ is a step function and therefore $F(\bar{\tau}_\beta)$ is not defined, i.e., $\bar{\tau}_\beta \notin \overline{K} = \{x \in X : F(x) \geq -C\}$ (recall (2.2.13)). Thus, to apply Proposition 2.2, we must find an approximation of $\bar{\tau}_\beta$ that lies in $\overline{K}$ and approximates $F_\beta(\bar{\tau}_\beta)$ (i.e., we must prove Assumption (3)(iii) in Proposition 2.2).

Proposition 2.4 *There exists* $(\tilde{\tau}_\beta) \subset \overline{K}$ *such that as* $\beta \downarrow 0$

$$\begin{aligned} &(i)\;\; \|\bar{\tau}_\beta - \tilde{\tau}_\beta\|_{L^2} \to 0 \\ &(ii)\;\; 0 \leq F_\beta(\bar{\tau}_\beta) - F_\beta(\tilde{\tau}_\beta) \to 0. \end{aligned} \tag{2.4.2}$$

2.4.1 Proof of Proposition 2.4: Lemmas 2.9–2.12

The proof of Proposition 2.4 is contained in Lemmas 2.9–2.12 below. We prove Lemmas 2.9–2.11 in this section. The proof of Lemma 2.12 is in Section 2.5.

We will see that it suffices to pick for $\tilde{\tau}_\beta$ the following linear and renormed interpolation of $\bar{\tau}_\beta$:

$$\begin{aligned}\tilde{\tau}_\beta &= \hat{\tau}_\beta\|\hat{\tau}_\beta\|_{L^2}^{-1}\\ \hat{\tau}_\beta(u) &= \beta^{-\frac{1}{6}}\{\tau_\beta(i)+(u\beta^{-\frac{1}{3}}-i)(\tau_\beta(i)-\tau_\beta(i-1))\}\\ &\qquad \text{for } (i-1)\beta^{\frac{1}{3}}<u\le i\beta^{\frac{1}{3}} \qquad (i\ge 1)\end{aligned} \tag{2.4.3}$$

(put $\tau_\beta(0)=\tau_\beta(1)$).

We begin with two lemmas showing what is needed about τ_β in order to prove Proposition 2.4. Abbreviate $\Delta\tau_\beta(i)=\tau_\beta(i)-\tau_\beta(i-1)$ $(i\ge 1)$.

Lemma 2.9 *(i)* $\|\bar{\tau}_\beta-\tilde{\tau}_\beta\|_{L^2}\le\|\Delta\tau_\beta\|_{l^2}+\tau_\beta^2(0)$.
(ii) $0\le F_\beta(\bar{\tau}_\beta)-F_\beta(\tilde{\tau}_\beta)\le\lambda(\beta)\beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2}^2[1-\frac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2+\frac{1}{2}\tau_\beta^2(0)]^{-1}$.

Proof. *(i)* From (2.4.1) and (2.4.3) we compute

$$\|\bar{\tau}_\beta-\hat{\tau}_\beta\|_{L^2}^2 = \frac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2 \tag{2.4.4}$$

$$\|\hat{\tau}_\beta\|_{L^2}^2 = \|\tau_\beta\|_{l^2}^2-\langle\tau_\beta,\Delta\tau_\beta\rangle_{l^2}+\frac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2. \tag{2.4.5}$$

Using the relation $\langle\tau_\beta,\Delta\tau_\beta\rangle_{l^2}=\frac{1}{2}\|\Delta\tau_\beta\|_{l^2}^2-\frac{1}{2}\tau_\beta^2(0)$, together with (2.4.4-2.4.5) and $\|\tau_\beta\|_{l^2}=1$, we get

$$\begin{aligned}\|\bar{\tau}_\beta-\tilde{\tau}_\beta\|_{L^2} &\le \|\bar{\tau}_\beta-\hat{\tau}_\beta\|_{L^2}+\|\hat{\tau}_\beta-\tilde{\tau}_\beta\|_{L^2}\\ &=\|\bar{\tau}_\beta-\hat{\tau}_\beta\|_{L^2}+|\|\hat{\tau}_\beta\|_{L^2}-1|\\ &=(\tfrac{1}{3}\|\Delta\tau_\beta\|_{l^2}^2)^{1/2}+|[1-\tfrac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2+\tfrac{1}{2}\tau_\beta^2(0)]^{\frac{1}{2}}-1|\\ &\le(\tfrac{1}{3})^{1/2}\|\Delta\tau_\beta\|_{l^2}+\tfrac{1}{6}\|\Delta\tau_\beta\|_{l^2}^2+\tfrac{1}{2}\tau_\beta^2(0)\\ &\le((\tfrac{1}{3})^{1/2}+\tfrac{1}{3})\|\Delta\tau_\beta\|_{l^2}+\tfrac{1}{2}\tau_\beta^2(0),\end{aligned} \tag{2.4.6}$$

where we use that $\|\Delta\tau_\beta\|_{l^2}\le 2,\tau_\beta^2(0)\le 1$.
(ii) From the definition of F_β in (2.2.2) we get, after substitution of (2.4.1) and (2.4.3),

$$\begin{aligned}F_\beta(\bar{\tau}_\beta) &=\beta^{-\frac{1}{3}}\langle\tau_\beta,A_\beta\tau_\beta\rangle_{l^2}-\beta^{-\frac{1}{3}}\|\tau_\beta\|_{l^2}^2\\ F_\beta(\hat{\tau}_\beta) &=\beta^{-\frac{1}{3}}\langle(\tau_\beta-\tfrac{1}{2}\Delta\tau_\beta),A_\beta(\tau_\beta-\tfrac{1}{2}\Delta\tau_\beta)\rangle_{l^2}-\beta^{-\frac{1}{3}}\|\hat{\tau}_\beta\|_{L^2}^2.\end{aligned} \tag{2.4.7}$$

It follows from (2.4.7) that

$$\begin{aligned} F_\beta(\bar\tau_\beta) - F_\beta(\tilde\tau_\beta) &= F_\beta(\bar\tau_\beta) - \tfrac{1}{\|\hat\tau_\beta\|^2_{L^2}} F_\beta(\hat\tau_\beta) \\ &= \beta^{-\frac13} \tfrac{1}{\|\hat\tau_\beta\|^2_{L^2}} \Big\{ \tfrac13 \lambda(\beta) \|\Delta\tau_\beta\|^2_{l^2} - \tfrac14 \langle \Delta\tau_\beta, A_\beta \Delta\tau_\beta\rangle_{l^2} \Big\}, \end{aligned} \tag{2.4.8}$$

where in the second equality we use the symmetry of A_β and the relations $A_\beta \tau_\beta = \lambda(\beta)\tau_\beta$ and (2.4.5). Finally, observe that $|\langle \Delta\tau_\beta, A_\beta\Delta\tau_\beta\rangle_{l^2}| \le \langle |\Delta\tau_\beta|, A_\beta|\Delta\tau_\beta|\rangle_{l^2} \le \lambda(\beta)\|\Delta\tau_\beta\|^2_{l^2}$ to get the claim. □

Lemma 2.10

$$\begin{aligned} F(\tilde\tau_\beta) \quad &\ge -2\sqrt5 |a| \big(\beta^{\frac23} \textstyle\sum_{i\ge1} i^2 \tau^2_\beta(i)\big)^{1/2} [1 - \tfrac16 \|\Delta\tau_\beta\|^2_{l^2} + \tfrac12 \tau^2_\beta(0)]^{-\frac12} \\ &+ \Big\{ 20\beta^{\frac23} \textstyle\sum_{i\ge1} i^2\tau^2_\beta(i) + \beta^{-\frac13} \sum_{i\ge1} i\Delta\tau^2_\beta(i) \Big\} [1 - \tfrac16 \|\Delta\tau_\beta\|^2_{l^2} + \tfrac12 \tau^2_\beta(0)]^{-1}. \end{aligned} \tag{2.4.9}$$

Proof. According to (2.3.1) and (2.3.3)

$$F(\hat\tau_\beta) = \int_0^\infty du \ = \Big\{ (2au - 4u^2)\ \hat\tau^2_\beta(u) - u[\hat\tau'_\beta(u)]^2 \Big\}. \tag{2.4.10}$$

Use (3.3) to obtain the estimates

$$\begin{aligned} &\textstyle\int_0^\infty du\ u^2 \hat\tau^2_\beta(u) \le \beta^{\frac23} \sum_{i\ge1} i^2 \max\{\tau^2_\beta(i), \tau^2_\beta(i-1)\} \\ &\textstyle\int_0^\infty du\ u[\hat\tau'_\beta(u)]^2 \le \beta^{-\frac13} \sum_{i\ge1} i\Delta\tau^2_\beta(i). \end{aligned} \tag{2.4.11}$$

Since $\int_0^\infty du\ u\hat\tau^2_\beta(u) \le (\int_0^\infty du\ u^2\hat\tau^2_\beta(u))^{1/2} \|\hat\tau_\beta\|_{L^2}$, we get the claim because $F(\tilde\tau_\beta) = \frac{1}{\|\hat\tau_\beta\|^2_{L^2}} F_\beta(\hat\tau_\beta)$. □

Lemmas 2.9 and 2.10 set the stage for the proof of Proposition 2.4. Namely, we now see that it suffices to prove the following estimates:

Lemma 2.11 *There exists C such that for β small enough*

$$\begin{array}{llll} (i) & \sum_{i\ge1} i^2\tau^2_\beta(i) \le C\beta^{-\frac23} & (iii) & \tau^2_\beta(0) \le C\beta^{\frac13} \log\frac1\beta \\ (ii) & \sum_{i\ge1} i\Delta\tau^2_\beta(i) \le C\beta^{\frac13} & (iv) & \|\Delta\tau_\beta\|^2_{l^2} \le C\beta^{\frac23}\log\frac1\beta. \end{array} \tag{2.4.12}$$

Indeed, Lemmas 2.11(*iii-iv*) and 2.9(*i-ii*) imply (2.4.2), while Lemmas 2.11(*i-ii*) and Lemma 2.10 imply that $F(\tilde\tau_\beta) \ge -C$ for β small enough and C sufficiently large, which guarantees that $\tilde\tau_\beta \in \overline{K} = \overline{K}^a_C$. In the sequel, C will be a generic constant, possibly changing from line to line.

In the proof of Lemma 2.11 we will make use of the following additional lemma, the proof of which is deferred to Section 2.5:

Lemma 2.12

$$\begin{array}{lcll} \limsup_{\beta\downarrow 0} \beta^{-\frac{1}{3}}\big[\lambda(\beta)-1\big] & \leq & \frac{1}{4}a^2 & \\ \liminf_{\beta\downarrow 0} \beta^{-\frac{1}{3}}\big[\lambda(\beta)-1\big] & \geq & \frac{1}{2\pi}a^2-\frac{2}{a} & (a>1) \\ & & \frac{1}{\pi}a-\frac{1}{2\pi}-2 & (a\leq 1). \end{array} \tag{2.4.13}$$

Proof of Lemma 2.11(i). First estimate

$$\lambda(\beta)\sum_i i^2\tau_\beta^2(i) = \sum_{i,j} i^2\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \leq \sum_{i,j}(i+j-1)^2\tau_\beta(i)A_\beta(i,j)\tau_\beta(j). \tag{2.4.14}$$

Then use $1+t\leq e^t$ and (2.1.1), (2.1.3) to get

$$\begin{aligned} &\lambda(\beta)+\tfrac{1}{2}\beta\textstyle\sum_{i,j}(i+j-1)^2\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \\ &= \textstyle\sum_{i,j}[1+\frac{1}{2}\beta(i+j-1)^2]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \\ &\leq \textstyle\sum_{i,j}\tau_\beta(i)A_{a\beta^{\frac{2}{3}},\frac{1}{2}\beta}(i,j)\tau_\beta(j) \\ &\leq \lambda(a\beta^{\frac{2}{3}},\tfrac{1}{2}\beta). \end{aligned} \tag{2.4.15}$$

Rewrite (2.4.15) as

$$\beta^{\frac{2}{3}}\sum_{i,j}(i+j-1)^2\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \leq 2\beta^{-\frac{1}{3}}\Big[\lambda\Big(2^{\frac{2}{3}}a\Big(\frac{\beta}{2}\Big)^{\frac{2}{3}},\frac{\beta}{2}\Big)-\lambda(a\beta^{\frac{2}{3}},\beta)\Big]. \tag{2.4.16}$$

Finally, use Lemma 2.12 to get that

$$\limsup_{\beta\downarrow 0}\beta^{\frac{2}{3}}\sum_{i,j}(i+j-1)^2\tau_\beta(i)A_\beta(i,j)\tau_\beta(j)<\infty. \tag{2.4.17}$$

□

Proof of Lemma 2.11(ii). The proof is divided into 2 steps.

STEP 1 *For all* β

$$\sum_i i\Delta\tau_\beta^2(i+1) = \frac{1}{\lambda(\beta)}\sum_{i,j}(i+j-1)[1-e^{e_\beta(i+1,j)-e_\beta(i,j)}]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j). \tag{2.4.18}$$

Proof. Write out

$$\begin{aligned} \textstyle\sum_i i\Delta\tau_\beta^2(i+1) &= \textstyle\sum_i i[\tau_\beta(i+1)-\tau_\beta(i)]^2 \\ &= \textstyle\sum_i i[\tau_\beta^2(i+1)+\tau_\beta^2(i)] \\ &\quad -\tfrac{1}{\lambda(\beta)}\textstyle\sum_{i,j}2i\tau_\beta(i)A_\beta(i+1,j)\tau_\beta(j). \end{aligned} \tag{2.4.19}$$

Now substitute the relation (see (2.1.1-2.1.2))

$$\begin{aligned} A_\beta(i+1,j) &= e^{e_\beta(i+1,j)}P(i+1,j) \\ &= e^{e_\beta(i+1,j)}\tfrac{i+j-1}{2i}P(i,j) \\ &= e^{e_\beta(i+1,j)-e_\beta(i,j)}\tfrac{i+j-1}{2i}A_\beta(i,j). \end{aligned} \tag{2.4.20}$$

This gives

$$\begin{aligned} \textstyle\sum_i i\Delta\tau^2_\beta(i+1) &= \; r.h.s.\ (2.4.18) + \textstyle\sum_i i[\tau^2_\beta(i+1)+\tau^2_\beta(i)] \\ & \quad -\tfrac{1}{\lambda(\beta)}\textstyle\sum_{i,j}(i+j-1)\tau_\beta(i)A_\beta(i,j)\tau_\beta(j). \end{aligned} \tag{2.4.21}$$

Both sums in the r.h.s. are equal to $\sum_i(2i-1)\tau^2_\beta(i)$ and therefore cancel out. □

STEP 2 *For β small enough*

$$\frac{1}{\lambda(\beta)}\sum_{i,j}(i+j-1)[1-e^{e_\beta(i+1,j)-e_\beta(i,j)}]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \le C\beta^{\frac{1}{3}}. \tag{2.4.22}$$

Proof. By (2.3.4) we have $e_\beta(i+1,j)-e_\beta(i,j) = a\beta^{\frac{2}{3}} - \beta(2i+2j-1)$. Hence

$$\begin{aligned} l.h.s.\ (2.4.22) &\le \tfrac{1}{\lambda(\beta)}\textstyle\sum_{i,j}(i+j-1)[e_\beta(i,j)-e_\beta(i+1,j)]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \\ &\le \tfrac{1}{\lambda(\beta)}2\beta\textstyle\sum_{i,j}(i+j)^2\tau_\beta(i)A_\beta(i,j)\tau_\beta(j) \\ &\le 8\beta\textstyle\sum_i i^2\tau^2_\beta(i) \end{aligned} \tag{2.4.23}$$

(use that $e^t \ge 1+t$ for all t). In the third inequality we use the symmetry of A_β and the fact that $\|A_\beta\|_{l^2} = \lambda(\beta)$. The claim now follows from Lemma 2.11(i). □

Steps 1-2 complete the proof of Lemma 2.11(ii). □

Proof of Lemma 2.11(iii).

By Cauchy-Schwarz, we have for every N

$$\begin{aligned} \tau_\beta(0) &= \tau_\beta(N) - \textstyle\sum_{i=1}^N \Delta\tau_\beta(i) \\ &\le \tau_\beta(N) + (\textstyle\sum_{i=1}^N \frac{1}{i})^{1/2}(\textstyle\sum_{i=1}^N i\Delta\tau^2_\beta(i))^{1/2}. \end{aligned} \tag{2.4.24}$$

Pick $N = \lceil\beta^{-\frac{1}{2}}\rceil$. Lemma 2.11($i$) gives $\tau_\beta(\lceil\beta^{-\frac{1}{2}}\rceil) \le C\beta^{\frac{1}{3}}$. Together with Lemma 2.11(ii) and the estimate $\sum_{i=1}^{\lceil\beta^{-\frac{1}{2}}\rceil}\frac{1}{i} \le \log\frac{1}{\beta}$, the claim follows. □

Proof of Lemma 2.11(iv).
The proof is divided into 2 steps.

STEP 3 *For all* β

$$\begin{aligned}\sum_{i\geq 1}\Delta\tau_\beta^2(i+1) &= \frac{2}{\lambda(\beta)}\sum_{\substack{i,j\\(i,j)\neq(1,1)}}[1-e^{e_\beta(i-1,j)-e_\beta(i,j)}]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j)\\ &\quad -\tau_\beta^2(1)[1-\frac{2}{\lambda(\beta)}A_\beta(1,1)].\end{aligned} \tag{2.4.25}$$

Proof. By Lemma 2.4 we have the following relation:

$$A_\beta(i,j)-A_\beta(i-1,j)=A_\beta(i,j-1)-A_\beta(i,j)+2A_\beta(i,j)[1-e^{e_\beta(i-1,j)-e_\beta(i,j)}] \tag{2.4.26}$$

(note that $e_\beta(i-1,j)=e_\beta(i,j-1)$). Hence

$$\begin{aligned}\sum_i\Delta\tau_\beta^2(i+1) &= \sum_i[\tau_\beta(i+1)-\tau_\beta(i)]^2\\ &= \tau_\beta^2(1)+2\sum_{i\geq 2}\tau_\beta(i)[\tau_\beta(i)-\tau_\beta(i-1)]\\ &= \tau_\beta^2(1)+\frac{2}{\lambda(\beta)}\sum_{i\geq 2}\sum_j\tau_\beta(i)[A_\beta(i,j)-A_\beta(i-1,j)]\tau_\beta(j)\\ &= \tau_\beta^2(1)+\frac{2}{\lambda(\beta)}\sum_{i\geq 2}\sum_j\tau_\beta(i)[A_\beta(i,j-1)-A_\beta(i,j)]\tau_\beta(j)\\ &\quad+\frac{4}{\lambda(\beta)}\sum_{i\geq 2}\sum_j[1-e^{e_\beta(i-1,j)-e_\beta(i,j)}]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j).\end{aligned} \tag{2.4.27}$$

The third term in the last expression is twice the sum in the r.h.s. of (2.4.25) except for the part with $i=1, j\geq 2$. The second term can be rewritten by carrying out the sum over i, namely (use that $A(i,0)=0$)

$$\begin{aligned}j=1:\ & -\frac{2}{\lambda(\beta)}\sum_{i\geq 2}\tau_\beta(i)A_\beta(i,1)\tau_\beta(1)\\ & =-2\tau_\beta^2(1)+\frac{2}{\lambda(\beta)}\tau_\beta^2(1)A_\beta(1,1)\\ j\geq 2:\ & \frac{2}{\lambda(\beta)}\sum_{i\geq 2}\tau_\beta(i)[A_\beta(i,j-1)-A_\beta(i,j)]\tau_\beta(j)\\ & =2\tau_\beta(j)[\tau_\beta(j-1)-\tau_\beta(j)]-\frac{2}{\lambda(\beta)}\tau_\beta(1)[A_\beta(1,j-1)-A_\beta(1,j)]\tau_\beta(j).\end{aligned} \tag{2.4.28}$$

Thus, after carrying out the sum over j, we see that (2.4.27) becomes

$$\begin{aligned}\sum_i\Delta\tau_\beta^2(i+1) &= -\sum_j\Delta\tau_\beta^2(j+1)\\ &\quad+2\Big\{r.h.s.\ (2.4.25)-\frac{2}{\lambda(\beta)}\sum_{j\geq 2}[1-e^{e_\beta(0,j)-e_\beta(1,j)}]\tau_\beta(1)A_\beta(1,j)\tau_\beta(j)\\ &\quad+\tau_\beta^2(1)[1-\frac{2}{\lambda(\beta)}A_\beta(1,1)]\Big\}\\ &\quad+\frac{2}{\lambda(\beta)}\tau_\beta^2(1)A_\beta(1,1)-\frac{2}{\lambda(\beta)}\sum_{j\geq 2}\tau_\beta(1)[A_\beta(1,j-1)-A_\beta(1,j)]\tau_\beta(j).\end{aligned} \tag{2.4.29}$$

Now, by (2.4.26) for $i = 1$,

$$2[1 - e^{e_\beta(0,j)-e_\beta(1,j)}]A_\beta(1,j) = -[A_\beta(1,j-1) - A_\beta(1,j)] + A_\beta(1,j). \tag{2.4.30}$$

Hence (2.4.29) simplifies to

$$\begin{aligned}\textstyle\sum_i \Delta\tau_\beta^2(i+1) &= -\textstyle\sum_j \Delta\tau_\beta^2(j+1) + \ 2\ r.h.s.\ (3.35)\\ &+\Big\{2\tau_\beta^2(1) - \tfrac{2}{\lambda(\beta)}\tau_\beta^2(1)A_\beta(1,1) - \tfrac{2}{\lambda(\beta)}\textstyle\sum_{j\geq 2}\tau_\beta(1)A_\beta(1,j)\tau_\beta(j)\Big\}.\end{aligned} \tag{2.4.31}$$

But the term between braces is zero. □

STEP 4 *For β small enough*

$$r.h.s.\ (2.4.25) \leq C\beta^{\frac{2}{3}}\log\frac{1}{\beta}. \tag{2.4.32}$$

Proof. The first term in (2.4.25) is easy to bound. Indeed, we have $e_\beta(i-1,j) - e_\beta(i,j) = -a\beta^{\frac{2}{3}} + \beta(2i+2j-3)$ and hence we get

$$\begin{aligned}1^{st}\text{ term in (2.4.25)} &\leq \tfrac{2}{\lambda(\beta)}\textstyle\sum_{\substack{i,j\\(i,j)\neq(1,1)}} a\beta^{\frac{2}{3}}\tau_\beta(i)A_\beta(i,j)\tau_\beta(j)\\ &\leq 2a\beta^{\frac{2}{3}}\end{aligned} \tag{2.4.33}$$

(in the first inequality use that $e^t \geq 1+t$ for all t). For the second term in (2.4.25), use that $P(1,1) = \frac{1}{2}$ and $e_\beta(1,1) = a\beta^{\frac{2}{3}} - \beta$. Together with $\lambda(\beta) \geq 1 - C\beta^{\frac{1}{3}}$ (see Lemma 2.12) we get

$$2^{nd}\text{ term in (2.4.25)} \leq 2\tau_\beta^2(1)C\beta^{\frac{1}{3}}\ \text{for } \beta \text{ small enough.} \tag{2.4.34}$$

Finally, use Lemma 2.11(iii) to get the claim (recall that $\tau_\beta(0) = \tau_\beta(1)$ in (2.4.3)). □

Steps 3-4 complete the proof of Lemma 2.11(iv). □
Lemma 2.11 completes the proof of Proposition 2.4. Lemma 2.12 will be proved in Section 2.4.

Proposition 2.4 shows that Assumption (3)(iii) in Proposition 2.2 holds. We will prove Assumptions (3)(i),(ii) in Section 2.6.

2.5 Proof of Lemma 2.12

To prove the upper bound in (2.4.13), use (2.3.5)(ii) to get

$$\begin{aligned} \lambda(\beta) &= \sum_{i,j} \tau_\beta(i) A_\beta(i,j) \tau_\beta(j) \\ &= \sum_{i,j} \tau_\beta(i) e^{e_\beta(i,j)} P(i,j) \tau_\beta(j) \\ &\leq e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}} \sum_{i,j} \tau_\beta(i) P(i,j) \tau_\beta(j) \\ &\leq e^{\frac{1}{4}a^2\beta^{\frac{1}{3}}}, \end{aligned} \tag{2.5.1}$$

where the last inequality follows from $\|P\|_{l^2} \leq 1$. This immediately gives the claim.

To prove the lower bound in (2.4.13), use (2.2.3)(i) to get that for any $x \in L^2(\mathbb{R}^+)$ with $\|x\|_{L^2} = 1$

$$\beta^{-\frac{1}{3}}[\lambda(\beta) - 1] \geq F^a_\beta(x). \tag{2.5.2}$$

Pick for x

$$x_\sigma(u) = \Big(\frac{2}{\pi\sigma^2}\Big)^{\frac{1}{4}} e^{-\frac{u^2}{4\sigma^2}} \ (\sigma > 0). \tag{2.5.3}$$

Now, we know from Lemmas 2.5–2.8 that

$$\lim_{\beta\downarrow 0} F^a_\beta(x_\sigma) = F^a(x_\sigma). \tag{2.5.4}$$

Hence $\liminf_{\beta\downarrow 0} \beta^{-\frac{1}{3}}[\lambda(\beta) - 1] \geq F^a(x_\sigma)$. Compute

$$\begin{aligned} F^a(x_\sigma) &= \int_0^\infty \Big\{(2au - 4u^2)[x_\sigma(u)]^2 - u[x'_\sigma(u)]^2\Big\} du \\ &= \big(\tfrac{2}{\pi\sigma^2}\big)^{1/2} \int_0^\infty (2au - 4u^2 - \tfrac{u^3}{4\sigma^4}) e^{-\frac{u^2}{2\sigma^2}} du \\ &= \big(\tfrac{8}{\pi}\big)^{1/2} a\sigma - 4\sigma^2 - \tfrac{1}{(2\pi)^{1/2}\sigma}. \end{aligned} \tag{2.5.5}$$

Pick $\sigma = \sigma(a) = \frac{a\vee 1}{(8\pi)^{1/2}}$ to get the claim. □

2.6 Analysis of the limit variational problem

Recall the notation in (2.2.13)

$$\begin{aligned} X &= \{x \in L^2(\mathbb{R}^+) : x \geq 0, \|x\|_{L^2} = 1\} \\ Y &= X \cap C^1(\mathbb{R}^+_0) \\ K &= K^a_C = \{x \in Y : F^a(x) \geq -C\}. \end{aligned} \tag{2.6.1}$$

In this section we analyze the limit variational problem appearing in (2.2.6), i.e.,

$$\sup_{x \in X} F^a(x). \tag{2.6.2}$$

In Section 2.6.1 we show that $x \mapsto F^a(x)$ is upper semicontinuous and K_C^a is relatively compact in X (in the L^2-topology). This implies that F^a achieves a maximum in $\overline{K}_C^a = K_C^a = \{x \in X : F^a(x) \geq -C\}$ ($\neq \emptyset$ for C large enough). In Section 2.6.2 we show that all maxima of F^a in X are solutions of the Sturm-Liouville problem

$$\mathcal{L}^a x = \rho x \qquad (\rho \in \mathbb{R}, x \in X \cap C^\infty(\mathbb{R}^+)), \tag{2.6.3}$$

where $\mathcal{L}^a$ is defined in (2.1.13). In Section 2.6.3 we analyze (2.6.3) and show that it has a unique solution x_a satisfying $F^a(x_a) > -\infty$ and $x_a > 0$, with corresponding eigenvalue $\rho(a)$. This identifies x_a as the unique maximizer of (2.6.2) and $\rho(a)$ as the maximum. We also study $a \mapsto x_a$ and $a \mapsto \rho(a)$ to prove the claims that were made in (2.1.15).

2.6.1 Existence of a maximizer of F^a in $\overline{K}_C^a$: Lemma 2.13

It will be convenient to transform $F^a, \mathcal{L}^a, K_C^a$ as follows. Define (recall (2.2.7))

$$\begin{aligned} \hat{F}^a(x) &= -F^a(x) + (\tfrac{a^2}{4}+1)\|x\|_{L^2}^2 \\ &= \textstyle\int_0^\infty \{q(u)[x(u)]^2 + p(u)[x'(u)]^2\}du \end{aligned} \tag{2.6.4}$$

with

$$\begin{aligned} p(u) &= u \\ q(u) &= (2u - \tfrac{1}{2}a)^2 + 1. \end{aligned} \tag{2.6.5}$$

$\hat{F}^a$ is the 'energy' functional corresponding to the Sturm-Liouville differential operator $\hat{\mathcal{L}}^a$ defined by (recall (2.1.13))

$$\begin{aligned} (\hat{\mathcal{L}}^a x)(u) &= -(\mathcal{L}^a x)(u) + (\tfrac{a^2}{4}+1)x(u) \\ &= q(u)x(u) - [p(u)x'(u)]'. \end{aligned} \tag{2.6.6}$$

Define (recall (2.2.13))

$$\begin{aligned} \hat{K}_C^a &= K^a_{C-(\frac{a^2}{4}+1)} \\ &= \{x \in Y : \hat{F}^a(x) \leq C\}. \end{aligned} \tag{2.6.7}$$

Lemma 2.13 *For every $a \in \mathbb{R}$*
(*i*) $\hat{K}_C^a \neq \emptyset$ *for C large enough*
(*ii*) $\hat{K}_C^a$ *is relatively compact in $L^2(\mathbb{R}^+)$ for all $C \in \mathbb{R}$*
(*iii*) $x \mapsto \hat{F}^a(x)$ *is lower semicontinuous on X.*

Proof. Standard.
(i) Trivial.
(ii) We check the conditions in Dunford and Schwartz (1964) Theorem IV.8.20.
(a) $\hat{K}_C^a$ is bounded in $L^2(\mathbb{R}^+)$.
(b) By Cauchy-Schwarz

$$\begin{aligned}\int_0^\infty (x(u+v)-x(u))^2 du &= \int_0^\infty (\int_u^{u+v} x'(t)dt)^2 \\ &\leq \int_0^\infty du \; [\log(u+v) - \log u] \int_u^{u+v} dt \; t[x'(t)]^2 \\ &= \int_0^\infty dt \; t[x'(t)]^2 I(t,v) 1_{\{t\geq v\}},\end{aligned} \tag{2.6.8}$$

where

$$I(t,v) = (t+v)\log(1+\frac{v}{t}) + (t-v)\log(1-\frac{v}{t}). \tag{2.6.9}$$

Since $t \mapsto I(t,v)$ is decreasing and $I(v,v) = 2v\log 2$, it follows that

$$\lim_{v\downarrow 0} \int_0^\infty (x(u+v)-x(u))^2 du = 0 \text{ uniformly for } x \in \hat{K}_C^a. \tag{2.6.10}$$

(c) From $p(u) \geq 0$ and $\lim_{u\to\infty} q(u) = \infty$ follows

$$\lim_{N\to\infty} \int_N^\infty x^2(u)du = 0 \text{ uniformly for } x \in \hat{K}_C^a. \tag{2.6.11}$$

Conditions (a)-(c) imply that $\hat{K}_C^a$ is relatively compact.
(iii) Define

$$V^a = \{x \in L^2(\mathbb{R}^+) : \hat{F}^a(x) < \infty\}. \tag{2.6.12}$$

On V^a define the inner product

$$\langle x,y\rangle_{V^a} = \int_0^\infty \{q(u)x(u)y(u) + p(u)x'(u)y'(u)\}du. \tag{2.6.13}$$

Then $(V^a, \langle\cdot,\cdot\rangle_{V^a})$ is a Hilbert space, $\|x\|_{V^a} \geq \|x\|_{L^2}$ and

$$\hat{F}^a(x) = \langle x,x\rangle_{V^a} = \|x\|_{V^a}^2. \tag{2.6.14}$$

Thus we must prove that $\liminf_{n\to\infty} \|x_n\|_{V^a} \geq \|x\|_{V^a}$ for any $x_n \to^{L^2} x$.

Let $L = \liminf_{n\to\infty} \|x_n\|_{V^a}$. The case $L = \infty$ being trivial, assume $L < \infty$. Then, by the Banach-Alaoglu theorem (Rudin (1991) Theorem 3.15), there exists a subsequence (x_{n_k}) and a $y \in V^a$ such that $L = \lim_{k\to\infty} \|x_{n_k}\|_{V^a}$ and $x_{n_k} \to y$ weakly in V^a $(k \to \infty)$. Hence $L \geq \|y\|_{V^a}$ by Fatou. But, by (ii), weak convergence in V^a implies strong convergence in $L^2(\mathbb{R}^+)$. Hence $x_{n_k} \to^{L^2} y$. Together with $x_n \to^{L^2} x$ this implies $y = x$ and hence the claim follows.

Incidentally, note from (2.6.4-2.6.5) that V^a does not depend on a because it is nothing other than the collection of $x \in L^2(\mathbb{R}^+)$ for which $\int_0^\infty \{u^2[x(u)]^2 + u[x'(u)]^2\}du < \infty$ (recall (2.1.14)). □

Lemma 2.13 implies that $\hat{F}^a$ achieves a minimum in $\overline{\hat{K}}^a_C$ (for C large enough).

2.6.2 Characterization of the minimizer(s) of $\hat{F}^a$: Lemma 2.14

Lemma 2.14 *Any minimizer $\bar{x}$ of $\hat{F}^a$ in X is a solution of $\hat{\mathcal{L}}^a x = \rho x$ for $\rho = \hat{\rho}(a) \in \mathbb{R}$, the minimal eigenvalue of $\hat{\mathcal{L}}^a$ in V^a.*

Proof. Standard. Define $\hat{\rho}(a)$ by

$$\hat{\rho}(a) = \min_{x \in X} \hat{F}^a(x). \tag{2.6.15}$$

Let $\bar{x} \in V^a$ be any minimizer. Then for any $h \in L^2(\mathbb{R}^+)$ and $\epsilon > 0$

$$\hat{F}^a(\bar{x} + \epsilon h) \geq \hat{\rho}(a) \|\bar{x} + \epsilon h\|^2_{L^2}. \tag{2.6.16}$$

Writing out both sides of (2.6.16) and using that $\hat{F}^a(\bar{x}) = \hat{\rho}(a)$, we obtain (see (2.6.13-2.6.14))

$$2\epsilon \langle \bar{x}, h \rangle_{V^a} + \epsilon^2 \|h\|^2_{V^a} \geq \hat{\rho}(a) \{2\epsilon \langle \bar{x}, h \rangle_{L^2} + \epsilon^2 \|h\|^2_{L^2}\}. \tag{2.6.17}$$

Let $\epsilon \downarrow 0$ to obtain

$$\langle \bar{x}, h \rangle_{V^a} \geq \hat{\rho}(a) \langle \bar{x}, h \rangle_{L^2} \text{ for all } h \in V^a. \tag{2.6.18}$$

Replace h by $-h$ to get the reverse inequality. Thus

$$\langle \bar{x}, h \rangle_{V^a} = \hat{\rho}(a) \langle \bar{x}, h \rangle_{L^2} \text{ for all } h \in V^a. \tag{2.6.19}$$

Now note that we have from (2.6.6) and (2.6.13) by partial integration

$$\langle \bar{x}, h \rangle_{V^a} = \langle \bar{x}, \hat{\mathcal{L}}^a h \rangle_{L^2} \text{ for all } h \in C^2_c(\mathbb{R}^+). \tag{2.6.20}$$

It follows from (2.6.19-2.6.20) and the symmetry of $\hat{\mathcal{L}}^a$ that $\bar{x}$ is a weak solution of $\hat{\mathcal{L}}^a x = \hat{\rho}(a) x$. This in turn implies that $\bar{x}$ is a strong solution.

To see that $\hat{\rho}(a)$ is the minimal eigenvalue of $\hat{\mathcal{L}}^a$ in V^a, note that if $\hat{\mathcal{L}}^a x = \rho x$, then by (2.6.6), (2.6.13-2.6.14) and integration by parts

$$\hat{F}^a(x) = \langle x, x \rangle_{V^a} = \langle x, \hat{\mathcal{L}}^a x \rangle_{L^2} = \rho \|x\|_{L^2} = \rho. \tag{2.6.21}$$

□

2.6.3 Analysis of the Sturm-Liouville problem: Lemmas 2.15–2.17

Lemmas 2.13–2.14 show that F^a has a maximizer in $\overline{K}^a_C$ and that each maximizer is a solution of $\mathcal{L}^a x = \rho x$ for $\rho = \rho(a)$, the maximal eigenvalue of $\mathcal{L}^a$ in V^a (recall (2.6.4-2.6.7)).

Lemma 2.15 *(i) All solutions of $\mathcal{L}^a x = \rho x$ are of the form*

$$x_{a,\rho}(u) = f_{a,\rho}(u) + g_{a,\rho}(u)\log u, \tag{2.6.22}$$

where $f_{a,\rho}$ and $g_{a,\rho}$ are power series with infinite radius of convergence.
(ii) $F^a(x_{a,\rho}) = -\infty$ if $g_{a,\rho} \neq 0$.

Proof. (*i*) Formally substitute $f_{a,\rho}(u) = \sum_{n\geq 0} f_n u^n$ and $g_{a,\rho}(u) = \sum_{n\geq 0} g_n u^n$. Then the coefficients are found to satisfy the recurrence relations

$$\begin{aligned} g_n &= \tfrac{1}{n^2}(\rho g_{n-1} - 2a g_{n-2} + 4 g_{n-3}) \\ f_n &= \tfrac{1}{n^2}(\rho f_{n-1} - 2a f_{n-2} + 4 f_{n-3} - 2n g_n)\ (n \geq 1) \end{aligned} \tag{2.6.23}$$

(with $f_{-1} = f_{-2} = g_{-1} = g_{-2} = 0$). Note that $g_{a,\rho}$ is a solution of (2.6.3) and that $f_{a,\rho}$ depends on $g_{a,\rho}$. By induction on n, (2.6.23) is easily shown to give the following bounds:

$$\begin{aligned} |f_n| &\leq K_1^n (n!)^{-\frac{2}{3}} \\ |g_n| &\leq K_2^n (n!)^{-\frac{2}{3}}\ (n \geq 1), \end{aligned} \tag{2.6.24}$$

with K_1, K_2 large enough (depending on ρ, a and f_0, g_0). This implies that the formal solution exists everywhere.
(*ii*) Trivial, since $\frac{d}{du}x_{a,\rho}(u) \sim g_0 u^{-1}$ ($u \downarrow 0$) with $g_0 \neq 0$ implies $F^a(x_{a,\rho}) = -\infty$, while $g_0 = 0$ implies that $g_n \equiv 0$. □

At this stage we know from Lemma 2.15 that all maximizers of F^a are of the form $x_{a,\rho}(u) = f_{a,\rho}(u)$ and, in particular, are analytic on $\mathbb{R}_0^+$.

Our next step is to find the asymptotic behavior of the solutions of (2.6.3) as $u \to \infty$. This will be needed to get uniqueness of the maximizer.

Lemma 2.16 *$\mathcal{L}^a x = \rho x$ has two independent solutions $x^-_{a,\rho}$ and $x^+_{a,\rho}$ satisfying*

$$\lim_{u\to\infty} u^{-\frac{3}{2}} \log x^{\pm}_{a,\rho}(u) = \pm\frac{4}{3}. \tag{2.6.25}$$

Proof. We use Coddington and Levinson (1955) Theorem 2.1 page 143-144. Define

$$\begin{aligned} w_1(u) &= x(u^2) \\ w_2(u) &= u^{-2} w_1'(u). \end{aligned} \tag{2.6.26}$$

Then (2.6.3) can be written as

$$w'(u) = u^{-r} B(u) w(u), \tag{2.6.27}$$

where $r = 2$ and

$$\begin{aligned} w(u) &= \begin{pmatrix} w_1(u) \\ w_2(u) \end{pmatrix} \\ B(u) &= \begin{pmatrix} 0 & 1 \\ 16 - \frac{8a}{u^2} + \frac{4\rho}{u^4} & -\frac{3}{u^3} \end{pmatrix}. \end{aligned} \tag{2.6.28}$$

Note that $B(u) = \sum_{n\geq 0} u^{-n} B_n$ ($B_0 \neq 0$) is a convergent power series in u^{-1}, with B_0 having eigenvalues $\lambda_{1,2} = \pm 4$. Therefore (2.6.27) has a formal solution of the form

$$w(u) = P(u) u^R e^{Q(u)}, \tag{2.6.29}$$

where $P(u) = \sum_{n=0}^{\infty} u^{-n} P_n$ ($\det P_0 \neq 0$) is a formal power series in u^{-1}, R is a complex diagonal matrix and $Q = \frac{u^{r+1}}{r+1} Q_0 + \cdots + u Q_r$ is a matrix polynomial with Q_i diagonal and $Q_0 = \mathrm{diag}\{\lambda_1, \lambda_2\}$. From the proof of the theorem it follows that P, Q, R can be chosen to be real because $B, \lambda_{1,2}$ are real. On p.151 of Coddington and Levinson (1955) there is the further remark that for every formal solution there exists an actual solution with the same asymptotics. □

We see from Lemma 2.16 that $x_{a,\rho}^+ \notin L^2(\mathbb{R}^+)$ and so (2.6.3) has a unique solution in $L^2(\mathbb{R}^+)$ up to multiplicative constants.

Lemma 2.17 *Define*

$$\mathcal{S}_a = \{\rho \in \mathbb{R} : f_{a,\rho} \in L^2(\mathbb{R}^+), f_{a,\rho}(0) = 1\}. \tag{2.6.30}$$

Then

(*i*) $\mathcal{S}_a$ *is countable, bounded from above and has a maximum*

(*ii*) $\rho(a) = \max\ \mathcal{S}_a$ *is geometrically simple*

(*iii*) $f_{a,\rho(a)} > 0$

(*iv*) $\forall \rho \in \mathcal{S}_a, \rho < \max\ \mathcal{S}_a : f_{a,\rho}$ *changes sign in* $\mathbb{R}^+$.

Proof. Standard Sturm-Liouville theory.

(*i*),(*ii*) By Lemma 2.13(*ii*), V^a is compactly embedded in $L^2(\mathbb{R}^+)$ (compare (2.6.7) and (2.6.12)). Therefore the eigenfunctions of $\mathcal{L}^a$ in V^a form an orthogonal basis of V^a. Since V^a is separable, this in turns implies that $\mathcal{S}_a$ is countable. We know from Lemmas 2.15–2.16 that $\mathcal{L}^a$ has a unique eigenvector in V^a with eigenvalue $\rho(a)$, i.e., $\rho(a)$ is geometrically simple. Since $\rho(a) = \max_{x\in V^a} F^a(x) = \max \mathcal{S}_a$ by Lemma 2.14, we also know that $\mathcal{S}_a$ is bounded from above and has a maximum.

(*iii*) From (2.2.7) we see that $F^a(|f_{a,\rho(a)}|) = F^a(f_{a,\rho(a)})$. Therefore it follows from the uniqueness of the maximizer that $f_{a,\rho} = |f_{a,\rho}| \geq 0$. Let $u_0 = \inf\{u > 0 : \ f_{a,\rho(a)}(u) = 0\} > 0$. If $u_0 < \infty$, then we must have $\frac{d}{du} f_{a,\rho(a)}(u_0) = 0$ and $\frac{d^2}{du^2} f_{a,\rho(a)}(u_0) > 0$. However, this contradicts $(\mathcal{L}^a f_{a,\rho(a)})(u) = \rho(a) f_{a,\rho(a)}(u)$ at the point $u = u_0$ (see (2.1.13)).

(*iv*) This follows from (*iii*) and the fact that the eigenfunctions of $\mathcal{L}^a$ in V^a form an orthogonal basis. □

Lemmas 2.13–2.14 and 2.17 show that Assumptions (3)($i - ii$) in Proposition 2.2 hold.

2.6.4 Dependence on a: Lemma 2.18

The maximal eigenvalue and eigenvector of (2.1.13-2.1.14) are

$$\begin{aligned} \rho(a) &= \max\ \mathcal{S}_a \\ x_a &= \frac{f_{a,\rho(a)}}{\|f_{a,\rho(a)}\|_{L^2}}. \end{aligned} \tag{2.6.31}$$

We can now prove the following properties:

Lemma 2.18 *(i) $a \mapsto \rho(a)$ and $a \mapsto x_a$ are analytic*
(ii) $a \mapsto \rho(a)$ is strictly increasing and strictly convex on $\mathbb{R}$
(iii) $\rho(0) < 0$, $\lim_{a\uparrow\infty} \rho(a) = \infty$ and $\lim_{a\downarrow-\infty} \rho(a) = -\infty$.

Proof. (i) We give the proof by applying Crandall and Rabinowitz (1973) Lemma 1.3 in the following setting. Pick $a \in \mathbb{R}$ and consider the Hilbert space $(V, \langle\cdot,\cdot\rangle_V)$ with $V = V^0$. Then, from (2.6.5-2.6.6) and (2.6.13),

$$\begin{aligned} \langle x_a, y\rangle_{V^a} &= \langle \mathcal{L}^a x_a, y\rangle_{L^2} = \rho(a)\langle x_a, y\rangle_{L^2} \\ \langle x_a, y\rangle_{V^a} &= \langle x_a, y\rangle_V - 2ab(x_a, y) + \tfrac{a^2}{4}\langle x_a, y\rangle_{L^2}, \end{aligned} \tag{2.6.32}$$

where $b : V \times V \mapsto \mathbb{R}$ is the bilinear form defined by

$$b(x,y) = \int_0^\infty u x(u) y(u) du. \tag{2.6.33}$$

For every $x \in V$ the functional $y \mapsto b(x,y)$ is continuous and linear. Hence it follows from the Riesz representation theorem (see Rudin (1987) Theorem 6.19) that there exists a unique linear operator $B : V \mapsto V$ such that

$$b(x,y) = \langle Bx, y\rangle_V \text{ for all } x, y \in V. \tag{2.6.34}$$

B is symmetric because b is. B is bounded because

$$\begin{aligned} \|Bx\|_V^2 &= b(x, Bx) \\ &\le \left(\textstyle\int_0^\infty u^2 x^2(u) du\right)^{1/2} \|Bx\|_{L^2} \\ &\le \tfrac{1}{2}\|x\|_V \|Bx\|_{L^2} \\ &\le \tfrac{1}{2}\|x\|_V \|Bx\|_V \end{aligned} \tag{2.6.35}$$

(see (2.6.5) and (2.6.13)), so that $\|Bx\|_V \le \frac{1}{2}\|x\|_V$. To see that B is compact, let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in V. Then, by Lemma 2.13(ii), there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$

and an $x \in V$ such that $x_{n_k} \to^{L^2} x$ as $k \to \infty$. Hence, as in (2.6.35),

$$\begin{aligned} \|Bx_{n_k} - Bx\|_V^2 &= b(x_{n_k} - x, B(x_{n_k} - x)) \\ &\le \|x_{n_k} - x\|_{L^2} \tfrac{1}{2} \|B(x_{n_k} - x)\|_V \\ &\le \|x_{n_k} - x\|_{L^2} \tfrac{1}{4} \|x_{n_k} - x\|_V \\ &\to 0 \qquad (k \to \infty). \end{aligned} \tag{2.6.36}$$

In the same manner we can prove that there exists a unique linear, symmetric and compact operator $C : V \mapsto V$ such that

$$\langle x, y\rangle_{L^2} = \langle Cx, y\rangle_V \text{ for all } x, y \in V. \tag{2.6.37}$$

Now rewrite (2.6.32) as follows

$$\langle \big[Id - 2aB - \big(\rho(a) - \frac{a^2}{4}\big)C\big]x_a, y\rangle_V = 0 \text{ for all } y \in V. \tag{2.6.38}$$

Hence, $(V, \langle\cdot,\cdot\rangle_V)$ being a Hilbert space, we have

$$\begin{array}{l} x_a \text{ is a } C\text{-eigenfunction of } Id - 2aB \\ \text{with (largest) eigenvalue } \rho(a) - \frac{a^2}{4}. \end{array} \tag{2.6.39}$$

Next note that $a \mapsto Id - 2aB$ is analytic in the operator norm. Therefore, to get the claim from Crandall and Rabinowitz (1973) Lemma 1.3, it suffices to check that $\rho(a) - \frac{a^2}{4}$ is a C-simple eigenvalue of $Id - 2aB$, i.e.,
(a) $\dim(N(A^a)) = \text{codim}(R(A^a)) = 1$
(b) $Cx_a \notin R(A^a)$,
where $A^a = Id - 2aB - (\rho(a) - \frac{a^2}{4})C$ and $N(A^a), R(A^a)$ denote the null space, respectively, the range of A^a.

We have $\dim(N(A^a)) = 1$ because of Lemma 2.17(ii). Moreover, because $2aB + (\rho(a) - \frac{a^2}{4})C$ is compact, we have $\dim(N(A^a)) = \text{codim}(R(A^a))$ (see Rudin (1991) Theorem 4.25). This proves (a). To prove (b), first use that A^a is symmetric and bounded to get that $N(A^a) = R(A^a)^\perp$ (the orthogonal complement of $R(A^a)$) and $R(A^a) = \overline{R(A^a)}$ (see Rudin (1991) Theorems 4.12 and 4.23). Since $\overline{R(A^a)} = R(A^a)^{\perp\perp}$, it follows that $N(A^a)^\perp = R(A^a)$. Hence (b) is equivalent to $\langle Cx_a, x_a\rangle_V \neq 0$. But $\langle Cx_a, x_a\rangle_V = \langle x_a, x_a\rangle_{L^2} = 1$ by (2.6.37).
(ii) Because

$$\rho(a) = \sup_{x \in X} F^a(x) \tag{2.6.40}$$

with unique maximizer $x = f_{a,\rho(a)}$, we immediately see from (2.2.7) that

$$\frac{\rho(a+\epsilon) - \rho(a)}{\epsilon} \ge \int_0^\infty 2u[f_{a,\rho(a)}(u)]^2 du > 0 \tag{2.6.41}$$

(pick $\|f_{a,\rho(a)}\|_{L^2} = 1$). This demonstrates that $\rho'(a)$ is everywhere strictly positive. Moreover, since $a \mapsto F^a(x)$ is affine for every x we have from (2.6.40) that $a \mapsto \rho(a)$ is convex. Because of analyticity, it follows that either $a \mapsto \rho(a)$ is strictly convex or $\rho(a) = C_1 a + C_2$. However, the latter is impossible because of Lemma 2.12.
(*iii*) Trivial. Let $\epsilon \to \pm\infty$ in (2.6.41) or else see (2.2.7). □

2.7 Proof of Theorems 2.2– 2.4

We can now collect the results from Sections 2.2-2.6 and give the proofs of our theorems in Section 2.1.4.

Proof of Theorem 2.3. Combine Proposition 2.1 with Lemma 2.1. The proof of the uniform convergence in a on compacts in $\mathbb{R}$ is deferred to later (see Lemma 2.19). We will prove the equalities in (2.1.18) together with Theorem 2.2. □

Proof of Theorem 2.2.

1. $r^*(\beta) \sim a^*\beta^{\frac{2}{3}}$.

According to (1.4.15), $r^*(\beta)$ is defined as the unique solution of

$$\lambda(r,\beta) = 1. \tag{2.7.1}$$

From (2.1.16) we know that for every $a \in \mathbb{R}$

$$\beta^{-\frac{1}{3}}\left[\lambda(a\beta^{\frac{2}{3}},\beta) - 1\right] \to \rho(a). \tag{2.7.2}$$

Let $a^* > 0$ be the solution of $\rho(a) = 0$ (recall Lemma 2.18). Now, because $r \mapsto \lambda(r,\beta)$ is increasing (as is obvious from (2.1.1)), we have for every $\epsilon > 0$

$$\begin{array}{rcl} \lambda(r,\beta) & \geq & 1 + \beta^{\frac{1}{3}}\rho(a^*+\epsilon) + o(\beta^{\frac{1}{3}}) \text{ for } r \geq (a^*+\epsilon)\beta^{\frac{2}{3}} \\ \lambda(r,\beta) & \leq & 1 + \beta^{\frac{1}{3}}\rho(a^*-\epsilon) + o(\beta^{\frac{1}{3}}) \text{ for } r \leq (a^*-\epsilon)\beta^{\frac{2}{3}}. \end{array} \tag{2.7.3}$$

Since $\rho(a^*-\epsilon) < 0 < \rho(a^*+\epsilon)$ for every $\epsilon > 0$ (see Lemma 2.18(*ii*)), (2.7.1) combined with (2.7.3) implies

$$(a^*-\epsilon)\beta^{\frac{2}{3}} \leq r^*(\beta) \leq (a^*+\epsilon)\beta^{\frac{2}{3}} \text{ for } \beta \text{ small enough.} \tag{2.7.4}$$

Let $\epsilon \downarrow 0$ to get the claim.

2. $\theta^*(\beta) \sim b^*\beta^{\frac{1}{3}}$.

According to (1.4.17), $\theta^*(\beta)$ is defined as

$$\frac{1}{\theta^*(\beta)} = \left[\frac{\partial}{\partial r}\lambda(r,\beta)\right]_{r=r^*(\beta)}. \tag{2.7.5}$$

Define

$$\xi(r,\beta) = \frac{\frac{\partial}{\partial r}\lambda(r,\beta)}{\lambda(r,\beta)} = \frac{\partial}{\partial r}\log\lambda(r,\beta). \tag{2.7.6}$$

Because $r \mapsto \lambda(r,\beta)$ is increasing and log-convex (see footnote 7 in Section 1.4), we have that for all $h,\beta > 0$ and $a \in \mathbb{R}$

$$\begin{aligned} \xi(a\beta^{\frac{2}{3}},\beta) &\le \frac{1}{h\beta^{\frac{2}{3}}}\Big[\log\lambda((a+h)\beta^{\frac{2}{3}},\beta) - \log\lambda(a\beta^{\frac{2}{3}},\beta)\Big] \\ \xi(a\beta^{\frac{2}{3}},\beta) &\ge \frac{1}{h\beta^{\frac{2}{3}}}\Big[\log\lambda(a\beta^{\frac{2}{3}},\beta) - \log\lambda((a-h)\beta^{\frac{2}{3}},\beta)\Big]. \end{aligned} \tag{2.7.7}$$

Together with (2.7.2) this gives

$$\begin{aligned} \limsup_{\beta\downarrow 0}\beta^{\frac{1}{3}}\xi(a\beta^{\frac{2}{3}},\beta) &\le \frac{\rho(a+h)-\rho(a)}{h} \\ \liminf_{\beta\downarrow 0}\beta^{\frac{1}{3}}\xi(a\beta^{\frac{2}{3}},\beta) &\ge \frac{\rho(a)-\rho(a-h)}{h}. \end{aligned} \tag{2.7.8}$$

Let $h \downarrow 0$ to get (use Lemma 2.18)

$$\lim_{\beta\downarrow 0}\beta^{\frac{1}{3}}\xi(a\beta^{\frac{2}{3}},\beta) = \rho'(a). \tag{2.7.9}$$

Next, because $r \mapsto \xi(r,\beta)$ is increasing we have, via (2.7.4), for β small enough

$$\begin{aligned} \xi(r^*(\beta),\beta) &\le \xi((a^*+\epsilon)\beta^{\frac{2}{3}},\beta) = \beta^{-\frac{1}{3}}\rho'(a^*+\epsilon) + o(\beta^{-\frac{1}{3}}) \\ \xi(r^*(\beta),\beta) &\ge \xi((a^*-\epsilon)\beta^{\frac{2}{3}},\beta) = \beta^{-\frac{1}{3}}\rho'(a^*-\epsilon) + o(\beta^{-\frac{1}{3}}). \end{aligned} \tag{2.7.10}$$

Since (recall that $\lambda(r^*(\beta),\beta) = 1$)

$$\frac{1}{\theta^*(\beta)} = \xi(r^*(\beta),\beta), \tag{2.7.11}$$

it follows that

$$\rho'(a^*-\epsilon) \le \frac{1}{\beta^{-\frac{1}{3}}\theta^*(\beta)} \le \rho'(a^*+\epsilon) \text{ for } \beta \text{ small enough.} \tag{2.7.12}$$

Let $\epsilon \downarrow 0$ to get the claim with $\frac{1}{b^*} = \rho'(a^*)$.

3. $\beta^{-\frac{1}{6}}\tau_{r(\beta),\beta}(\lceil\cdot\beta^{-\frac{1}{3}}\rceil) \xrightarrow{L^2} x_a(\cdot)$ if $\lim_{\beta\downarrow 0}\beta^{-\frac{2}{3}}r(\beta) = a$.

Fix $a \in \mathbb{R}$. Let $r(\beta)$ be such that $\lim_{\beta\downarrow 0}\beta^{-\frac{2}{3}}r(\beta) = a$. Put $a(\beta) = \beta^{-\frac{2}{3}}r(\beta)$. Then, similarly as in Lemma 2.1,

$$\beta^{-\frac{1}{6}}\tau_{r(\beta),\beta}(\lceil\cdot\beta^{-\frac{1}{3}}\rceil) \text{ is the unique maximizer of } F_\beta^{a(\beta)}, \tag{2.7.13}$$

where the parameter a is replaced by $a(\beta)$.

Lemma 2.19 *Assumptions* (1) − (3) *in Proposition 2.2 hold for the following choice replacing (2.2.13):*

$$\begin{aligned} K &= K_C^a \ (C \text{ sufficiently large}) \\ G_\beta &= F_\beta^{a^*(\beta)} \\ G &= F^a. \end{aligned} \tag{2.7.14}$$

Proof. The point is that $\lim_{\beta\downarrow 0} a(\beta) = a$. It is trivial to check that all estimates in Sections 2.3 and 2.4 remain valid when the fixed parameter a is replaced by $a + o(1)$ ($\beta \downarrow 0$). See, in particular, the proofs of Lemmas 2.5, 2.6, 2.11–2.12. □

The claim in **3** now follows from Proposition 2.2.

4. $\beta^{-\frac{1}{3}}\mu_\beta^*(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) \to^{L^1} \frac{1}{2}x_{a^*}^2(\frac{1}{2}\cdot)$.

Abbreviate $A_\beta = A_{r^*(\beta),\beta}$ and $\tau_\beta = \tau_{r^*(\beta),\beta}$. According to (2.1.10)

$$\mu_\beta^*(k) = \sum_{\substack{i,j \\ i+j-1=k}} \tau_\beta(i) A_\beta(i,j) \tau_\beta(j). \tag{2.7.15}$$

Because of **3**, it suffices to prove that $\beta^{-\frac{1}{3}}\mu_\beta^*(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) - \frac{1}{2}\bar{\tau}_\beta^2(\lceil \frac{1}{2}\cdot \rceil) \to^{L^1} 0$. Now, use that

$$A_\beta(i, k-1-i) = e^{r^*(\beta)k - \beta k^2} \frac{1}{2} p_{k-1}(i), \tag{2.7.16}$$

where p_l is the probability mass function corresponding to a binomial random variable with parameters $\frac{1}{2}$ and $l \in \mathbb{N}$. Hence, write out and estimate

$$\|\beta^{-\frac{1}{3}}\mu_\beta^*(\lceil \cdot \beta^{-\frac{1}{3}} \rceil) - \frac{1}{2}\bar{\tau}_\beta^2(\frac{1}{2}\cdot)\|_{L^1} \le I_1(\beta) + I_2(\beta) + I_3(\beta), \tag{2.7.17}$$

where

$$\begin{aligned} I_1(\beta) &= \textstyle\sum_{k\in\mathbb{N}} \tau_\beta^2(\lceil \frac{1}{2}k \rceil) | \sum_{i,j:i+j-1=k} [A_\beta - P](i,j)| \\ I_2(\beta) &= \textstyle\sum_{k\in\mathbb{N}} \sum_{i,j:i+j-1=k} \tau_\beta(i) A_\beta(i,j) |\tau_\beta(j) - \tau_\beta(\lceil \frac{1}{2}k \rceil)| \\ I_3(\beta) &= \textstyle\sum_{k\in\mathbb{N}} \sum_{i,j:i+j-1=k} \tau_\beta(\lceil \frac{1}{2}k \rceil) A_\beta(i,j) |\tau_\beta(i) - \tau_\beta(\lceil \frac{1}{2}k \rceil)|. \end{aligned} \tag{2.7.18}$$

Use (2.7.16) to estimate

$$I_1(\beta) = \frac{1}{2}\sum_{k\in\mathbb{N}} \tau_\beta^2(\lceil \frac{1}{2}k \rceil) | e^{r^*(\beta)k - \beta k^2} - 1| \to 0 \tag{2.7.19}$$

by Lemma 2.11(i).

For $I_2(\beta)$, use (2.4.5) and Cauchy-Schwarz to estimate

$$\begin{aligned} I_2(\beta) &\leq \sum_{k\in\mathbb{N}} \sum_{i,j:i+j-1=k} \tau_\beta(i) A_\beta(i,j) \sum_{l=\frac{1}{2}k+1}^{j} |\Delta\tau_\beta(l)| \\ &\leq \tfrac{1}{2} e^{\frac{1}{4}a^{*2}\beta^{\frac{1}{3}}} \|\Delta\tau_\beta\|_{l^2} \sum_{i,j\in\mathbb{N}} \tau_\beta(i)|i-j|^{\frac{1}{2}} P(i,j) \\ &\leq \tfrac{1}{2} e^{\frac{1}{4}a^{*2}\beta^{\frac{1}{3}}} \|\Delta\tau_\beta\|_{l^2} \sum_i \tau_\beta(i)|i|^{\frac{1}{2}}, \end{aligned} \tag{2.7.20}$$

where the last inequality follows from Lemma 2.2 and Cauchy-Schwarz. Finally, use Cauchy-Schwarz once more together with Lemma 2.11(i),(iv) to get

$$I_2(\beta) \leq \frac{1}{2} e^{\frac{1}{4}a^{*2}\beta^{\frac{1}{3}}} \|\Delta\tau_\beta\|_{l^2} \Big(\sum_i |i|^{\frac{7}{4}} \tau_\beta^2(i)\Big)^{1/2} \Big(\sum_i |i|^{-\frac{5}{4}}\Big)^{1/2} \to 0. \tag{2.7.21}$$

The estimate for $I_3(\beta)$ is similar.

Results **1-4** complete the proofs of Theorems 2.2 and 2.3. □

Proof of Theorem 2.4. The asymptotic behavior of x_{a^*} in (iii) was proved in Lemma 2.16 (pick $a = a^*$ and $\rho = 0$). To prove (i) and (ii), we recall that x_{a^*} solves (see (2.1.13) and Theorem 2.2)

$$0 = (\mathcal{L}^{a^*}x)(u) = (2a^*u - 4u^2)x(u) + [ux']'(u) \tag{2.7.22}$$

and has a power series representation (see (2.6.23))

$$\begin{aligned} x_{a^*}(u) &= \sum_{n\geq 0} x_n u^n \\ x_n &= \tfrac{1}{n^2}(-2a^*x_{n-2} + 4x_{n-3}) \ (n \geq 1) \\ x_{-1} &= x_{-2} = 0. \end{aligned} \tag{2.7.23}$$

We observe that $u \mapsto 2a^*u - 4u^2$ changes sign from positive to negative at $u = \frac{1}{2}a^*$. Since $x_{a^*}(u) > 0$ for all $u \geq 0$, it follows from (2.7.22) that $u \mapsto u\frac{d}{du}x_{a^*}(u)$ is unimodal with a minimum at $u = \frac{1}{2}a^*$. It is clear that $u\frac{d}{du}x_{a^*}(u) \to 0$ as $u \downarrow 0$. By the unimodality we must have that $u\frac{d}{du}x_{a^*}(u) \to c$ as $u \to \infty$. However, c must be 0 otherwise $\int_0^\infty u[\frac{d}{du}x_{a^*}(u)]^2 du = \infty$, which is impossible since $F^{a^*}(x_{a^*}) = \rho(a^*) = 0 > -\infty$ (see (2.2.7)). Thus we conclude that $u\frac{d}{du}x_{a^*}(u) < 0$ for all $u > 0$, which implies that $u \mapsto x_{a^*}(u)$ is strictly decreasing.

To prove (iv), use the eigenvalue relation $\mathcal{L}^a x_a = \rho(a)x_a$, the symmetry of $\mathcal{L}^a$ and $\langle x_a, \frac{d}{da}x_a\rangle_{L^2} = 0$, to obtain

$$\begin{aligned} \rho'(a) &= \tfrac{d}{da}\langle x_a, \mathcal{L}^a x_a\rangle_{L^2} \\ &= \int_0^\infty 2u x_a^2(u) du. \end{aligned} \tag{2.7.24}$$

Now, use $\frac{1}{b^*} = \rho'(a^*)$.

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Chapter 3

Central limit theorem for the Edwards model

3.1 The Edwards model in terms of Brownian local times

As in Chapter 1, let $(B_t)_{t \geq 0}$ be standard one-dimensional Brownian motion, starting at 0. Let $\widehat{P}$ denote its distribution on path space and $\widehat{E}$ the corresponding expectation. In this chapter we will prove a CLT for the end-point of the path, extending the LLN proved by Westwater (see Theorem 1.3 in Section 1.5). First we will give a rigorous definition of the Edwards measure $\widehat{Q}_T^\beta$ in terms of Brownian local times.

There exists a jointly continuous version of the Brownian local time process $\{L(t,x)\}_{t \geq 0, x \in \mathbb{R}}$ satisfying the occupation times formula

$$\int_0^t f(B_s)\, ds = \int_{\mathbb{R}} L(t,x) f(x)\, dx \quad \widehat{P}\text{-a.s.} \qquad (f : \mathbb{R} \to \mathbb{R}^+ \text{ Borel}, t \geq 0) \tag{3.1.1}$$

(see Revuz and Yor (1991), Sect. VI.1). Think of $x \mapsto L(t,x)$ as the density of time the Brownian motion spends in x until time t. The Edwards measure $\widehat{Q}_T^\beta$ in (1.3.1-1.3.3) may now be defined as the Radon-Nikodym derivative with respect to $\widehat{P}$

$$\frac{d\widehat{Q}_T^\beta}{d\widehat{P}} = \frac{1}{Z_T^\beta} \exp\left[-\beta \int_{\mathbb{R}} L(T,x)^2\, dx\right], \tag{3.1.2}$$

where $Z_T^\beta = \widehat{E}(\exp[-\beta \int_{\mathbb{R}} L(T,x)^2\, dx])$ is the normalizing constant. The random variable $\int_{\mathbb{R}} L(T,x)^2\, dx$ is called the *self-intersection local time*. Think of this as the amount of time the Brownian motion spends in self-intersection points until time T.

The path measure $\widehat{Q}_T^\beta$ defined in (3.1.2) is the continuous space-time analogue of the Domb-Joyce measure defined in (1.2.1), which is clear from the fact that the exponent in (1.2.1) can also be rewritten in terms of the sum of squares of the local times of simple random walk (recall (1.4.5)).

The effect of the self-repellence is of particular interest. This effect is known to spread out the path on a linear scale (i.e., B_T is of order T under the law $\widehat{Q}_T^\beta$ as $T \to \infty$, see Theorem 1.3 in Section 1.5). We have investigated the β dependence of the exponential growth rate of the normalizing constant $\hat{r}^*(\beta)$ and the speed of the polymer $\hat{\theta}^*(\beta)$ in Chapter 2. It is the aim of the present chapter to study the fluctuations of B_T around the linear asymptotics. Our main result appears in Theorem 3.1 below.

3.1.1 Main theorem: Theorem 3.1

The speed $\hat{\theta}^*(\beta)$ in Theorem 1.3 was characterized by Westwater in terms of the smallest eigenvalue of a certain differential operator (see also Theorem 2.6). In the present paper, however, we prefer to work with a different operator, introduced and analyzed in Chapter 2. For $a \in \mathbb{R}$, define $\mathcal{K}^a : L^2(\mathbb{R}_0^+) \cap C^2(\mathbb{R}_0^+) \to C(\mathbb{R}_0^+)$ by

$$\big(\mathcal{K}^a x\big)(u) = 2ux''(u) + 2x'(u) + (au - u^2)x(u) \tag{3.1.3}$$

for $u \in \mathbb{R}_0^+$. The Sturm-Liouville operator $\mathcal{K}^a$ will play a key role in the present paper.[1] It is symmetric and has a largest eigenvalue $\rho(a)$ with multiplicity 1. The map $a \mapsto \rho(a)$ is real-analytic, strictly convex and strictly increasing, with $\rho(0) < 0$, $\lim_{a\to-\infty} \rho(a) = -\infty$ and $\lim_{a\to\infty} \rho(a) = \infty$.

Similar to (2.1.18), define $c^* \in \mathbb{R}^+$ by

$$c^{*2} = \frac{\rho''(a^*)}{\rho'(a^*)^3}. \tag{3.1.4}$$

Our main result is the following central limit theorem:

Theorem 3.1 *For every $\beta \in \mathbb{R}^+$ there exists $\hat{\sigma}^*(\beta) \in \mathbb{R}^+$ such that*

$$\lim_{T\to\infty} \widehat{Q}_T^\beta\Big(\frac{|B_T| - \hat{\theta}^*(\beta)T}{\hat{\sigma}^*(\beta)\sqrt{T}} \le C\Big) = \mathcal{N}((-\infty, C]) \text{ for all } C \in \mathbb{R}, \tag{3.1.5}$$

where $\mathcal{N}$ denotes the normal distribution with mean 0 and variance 1. The scaled mean and variance are given by

$$\hat{\theta}^*(\beta) = b^*\beta^{\frac{1}{3}}, \qquad \hat{\sigma}^*(\beta) = c^*. \tag{3.1.6}$$

Theorem 3.1 says that the fluctuations around the asymptotic mean have the classical order $\sqrt{T}$, are symmetric, and even do not depend on the interaction strength.

The numerical values of the constants in (3.1.6) are

$$a^* = 2.189 \pm 0.001, \qquad b^* = 1.11 \pm 0.01, \qquad c^* = 0.7 \pm 0.1. \tag{3.1.7}$$

For bounds on a^*, b^* and c^*, see Theorem 5.1(i-iii). Note that $c^* < 1$. Apparently, as the path is pushed out to infinity its fluctuations are squeezed compared to those of the free Brownian motion with $\hat{\theta}^*(0) = 0$, $\hat{\sigma}^*(0) = 1$.

[1] The operator $\mathcal{K}^a$ is a scaled version of the operator $\mathcal{L}^a$ analyzed in Chapter 2, Section 2.6, namely $(\mathcal{K}^a x)(u) = (\mathcal{L}^a \overline{x})(u/2)$ where $\overline{x}(u) = x(2u)$. Of course, this scaling does not change the spectrum of the operator.

3.1.2 Scaling in β

It is noteworthy that the scaled mean depends on β in such a simple manner and that the scaled variance does not depend on β at all. We have derived the β-dependence of the scaled mean and the exponential growth rate in an analytic manner in Section 2.1.5. However, these facts are direct consequences of the Brownian scaling property as we will see now. Namely, we will deduce from (3.1.5) that for every $\beta \in \mathbb{R}^+$

$$\hat{\theta}^*(\beta) = \hat{\theta}^*(1)\beta^{\frac{1}{3}}, \qquad \hat{\sigma}^*(\beta) = \hat{\sigma}^*(1). \tag{3.1.8}$$

Indeed, for $a, T > 0$

$$\big(B_T, \{L(T,x)\}_{x\in\mathbb{R}}\big) \stackrel{\mathcal{D}}{=} \big(a^{-\frac{1}{2}}B_{aT}, \{a^{-\frac{1}{2}}L(aT, a^{\frac{1}{2}}x)\}_{x\in\mathbb{R}}\big) \tag{3.1.9}$$

where $\stackrel{\mathcal{D}}{=}$ means equality in distribution (see Revuz and Yor (1991), Ch. VI, Ex. (2.11), 1°)). Apply this to $a = \beta^{\frac{2}{3}}$ to obtain, via (3.1.2), that

$$\widehat{Q}^\beta_T(B_T)^{-1} = \widehat{Q}^1_{\beta^{\frac{2}{3}}T}\big(\beta^{-\frac{1}{3}}B_{\beta^{\frac{2}{3}}T}\big)^{-1}, \tag{3.1.10}$$

where we write $\mu(X)^{-1}$ for the distribution of a random variable X under a measure μ. In particular, we have for all $C \in \mathbb{R}$

$$\begin{aligned} &\widehat{Q}^\beta_T\Big(\frac{B_T - \hat{\theta}^*(1)\beta^{\frac{1}{3}}T}{\hat{\sigma}^*(1)\sqrt{T}} \le C \,\Big|\, B_T > 0\Big) \\ &\quad = \widehat{Q}^1_{\beta^{\frac{2}{3}}T}\Big(\frac{B_{\beta^{\frac{2}{3}}T} - \hat{\theta}^*(1)\beta^{\frac{2}{3}}T}{\hat{\sigma}^*(1)\sqrt{\beta^{\frac{2}{3}}T}} \le C \,\Big|\, B_{\beta^{\frac{2}{3}}T} > 0\Big). \end{aligned} \tag{3.1.11}$$

The r.h.s. tends to $\mathcal{N}((-\infty, C])$ as $T \to \infty$ if $\beta \in \mathbb{R}^+$ (in (3.1.5) pick $\beta = 1$ and replace T by $\beta^{\frac{2}{3}}T$). Since the pair $(\hat{\theta}^*(\beta), \hat{\sigma}^*(\beta))$ is uniquely determined by (3.1.5), we arrive at (3.1.8).

3.1.3 Outline of the proof

Theorem 3.1 is the continuous analogue of the central limit theorem for the Domb-Joyce model proved by König (1996) (recall Theorem 1.4). We will be able to use the skeleton of that paper, but the Brownian context will require new ideas and methods. The remaining sections are devoted to the proof of Theorem 3.1. We give a short outline.

In Section 3.2, we use the well-known Ray-Knight theorems for the local times of Brownian motion to express the l.h.s. of (3.1.5) in terms of two- and zero-dimensional squared Bessel processes. The former describes the local times in the area $[0, B_T]$, the latter describe the local times in $(-\infty, 0]$, respectively, $[B_T, \infty)$.

In Section 3.3, with the help of some analytical properties of the operator $\mathcal{K}^a$ proved in Section 2.6, we introduce a Girsanov transformation of the two-dimensional squared Bessel process. The goal of this transformation is to absorb the exponential random variable $\exp\left(-\beta \int_0^{B_T} L(T,x)^2\,dx\right)$ into the transition probabilities. The transformed process turns out to have strong recurrence properties. The Gaussian behavior of $(B_T - \hat{\theta}^*(\beta)T)/\sqrt{T}$ is traced back to the asymptotic normality of the *inverse* of a certain additive functional of this transformed process. Thus, the central limit behavior is determined by those parts of the Brownian path that fall in the area $[0, B_T]$.

In Section 3.4, we prove a central limit theorem for the inverse process. Furthermore, as a second important ingredient in the proof, we derive a limit law and a rate of convergence result for the composition of the transformed process with the inverse process.

In Section 3.5, we finish the proof of Theorem 3.1 by showing that the contribution of the local times in $(-\infty, 0] \cup [B_T, \infty)$ remains bounded as $T \to \infty$ and is therefore cancelled by the normalization in the definition of the transformed path measure in (3.1.2).

3.2 Reformulation of the problem

3.2.1 The main proposition: Proposition 3.1

Since the dependence on β has already been isolated (see (3.1.11)), we may and will restrict to the case $\beta = 1$.

Throughout the sequel we will frequently refer to Revuz and Yor (1991), Karatzas and Shreve (1991). We will therefore adopt the abbreviations RY, respectively, KS for these references.

The remainder of this paper is devoted to the proof of the following key proposition:

Proposition 3.1 *There exists an $\widehat{L} \in \mathbb{R}^+$ such that for all $C \in \overline{\mathbb{R}}$*

$$\lim_{T\to\infty} e^{a^*T}\widehat{E}\left(e^{-\int_{\mathbb{R}} L(T,x)^2\,dx} 1_{0<B_T\le b^*T+C\sqrt{T}}\right) = \widehat{L}\mathcal{N}_{c^{*2}}((-\infty, C]), \tag{3.2.1}$$

where a^*, b^* and c^* are defined in (2.1.18) and (3.1.6), and $\mathcal{N}_{\sigma^2}$ denotes the normal distribution with mean 0 and variance σ^2.

Theorem 3.1 follows from Proposition 3.1, since it implies that the distribution of $(B_T - b^*T)/\sqrt{T}$ converges to $\mathcal{N}_{c^{*2}}$ (divide the l.h.s. of (3.2.1) by the same expression with $C = \infty$ and recall (3.1.2)).

Sections 3.2.2 and 3.2.3 contain preparatory material. Section 3.2.4 contains the key representation in terms of squared Bessel processes on which the proof of Proposition 3.1 will be based.

3.2.2 Ray-Knight theorems for Brownian local times

This subsection contains a description of the *time-changed* local time process in terms of squared Bessel processes. The material being fairly standard, our main purpose is to introduce appropriate notations and to prepare for Lemma 3.1 in Section 3.2.3 and Lemma 3.2 in Section 3.2.4.

For $u \in \mathbb{R}$ and $h \geq 0$, let τ_h^u denote the time change associated with $L(t,u)$, i.e.,

$$\tau_h^u = \inf\{\, t > 0 : L(t,u) > h \,\}. \tag{3.2.2}$$

Obviously, the map $h \mapsto \tau_h^u$ is right-continuous and increasing, and therefore makes at most countably many jumps for each $u \in \mathbb{R}$. Moreover, $\widehat{P}(L(\tau_h^u, u) = h \text{ for all } u \geq 0) = 1$ (see RY, Ch. VI). The following lemma contains the well-known Ray-Knight theorems. It identifies the distribution of the local times at the random time τ_h^u as a process in the spatial variable running forwards, respectively, backwards from u. We write $C_c^2(\mathbb{R}^+)$ to denote the set of twice continuously differentiable functions on $\mathbb{R}^+$ with compact support.

RK theorems *Fix $u, h \geq 0$. The random processes $\{L(\tau_h^u, u+v)\}_{v\geq 0}$ and $\{L(\tau_h^u, u-v)\}_{v\geq 0}$ are independent Markov processes, both starting at h.*

(*i*) *$\{L(\tau_h^u, u+v)\}_{v\geq 0}$ is a zero-dimensional squared Bessel process (BESQ^0) with generator*

$$(G^* f)(v) = 2v f''(v) \qquad (f \in C_c^2(\mathbb{R}^+)). \tag{3.2.3}$$

(*ii*) *$\{L(\tau_h^u, u-v)\}_{v\in[0,u]}$ is the restriction to the interval $[0,u]$ of a two-dimensional squared Bessel process (BESQ^2) with generator*

$$(Gf)(v) = 2v f''(v) + 2f'(v) \qquad (f \in C_c^2(\mathbb{R}^+)). \tag{3.2.4}$$

(*iii*) *$\{L(\tau_h^u, -v)\}_{v\geq 0}$ has the same transition probabilities as the process in (i).*

Proof. See RY, Sects. XI.1-2 and KS, Sects. 6.3-4. □

3.2.3 The distribution of $(\{L(T,x)\}_{x\in\mathbb{R}}, B_T)$: Lemma 3.1

The RK theorems give us a nice description of the local time process at certain stopping times. In order to apply them to (3.1.2), we need to go back to the fixed time T. This causes some complications (e.g. we must handle the global restriction $\int_{\mathbb{R}} L(T,x)\,dx = T$), but these may be overcome by an appropriate conditioning.

This subsection contains a formal description of the joint distribution of the three random processes

$$\{L(T, B_T + x)\}_{x\geq 0}, \quad \{L(T, B_T - x)\}_{x\in[0,B_T]}, \quad \{L(T, -x)\}_{x\geq 0}, \tag{3.2.5}$$

in terms of the squared Bessel processes.[2] The main intuitive idea is that, up to a $\widehat{P}$-null-set (recall (3.2.2)),

$$\{\, \tau_h^u = T \,\} = \{\, B_T = u, L(T, B_T) = h \,\} \text{ for all } u, h \geq 0. \tag{3.2.6}$$

This has two consequences:

[2]This description has recently been improved by C. Leuridan (preprint 1996).

(*i*) Conditioned on $\{B_T = u, L(T, B_T) = h\}$, the three processes in (3.2.5) are the squared Bessel processes from the RK theorems conditioned on having total integral equal to T.

(*ii*) The distribution of $(B_T, L(T, B_T))$ can be expressed in terms of the squared Bessel processes.

We will make this precise in Lemma 3.1 below.

Let us first mention some earlier works on the distribution of $\{L(T,x)\}_{x\in\mathbb{R}}$ with $T \geq 0$ independent of the motion. Perkins (1982) proves that $\{L(1,x)\}_{x\in\mathbb{R}}$ is a semi-martingale. Jeulin (1985) uses stochastic calculus, in particular Tanaka's formula, to recover the RK theorems and Perkins' result and to prove the conditioned Markov property of the triple $(L(1,x), x \wedge B_1, \int_{-\infty}^{x} L(1,u)\,du)$ in x, given $\inf_{s\leq 1} B_s$. In Biane and Yor (1988), the RK theorems are extended to the case where T is an exponential time, independent of the Brownian motion, under $\widehat{P}(\cdot\,|L(T,0) = s, B_T = a)$ for any fixed $s, a > 0$. Finally, Biane, Le Gall and Yor (1987) also deal with the intuitive idea (3.2.6) when identifying the law of the process $\{\frac{1}{\sqrt{\tau_h^0}} B_{u\tau_h^0}\}_{u\in[0,1]}$.

Let us return to our identification of the law of the process $(\{L(T,x)\}_{x\in\mathbb{R}}, B_T)$. In order to formulate the details, we must first introduce some notation. For the remainder of this paper, let

$$\{X_v\}_{v\geq 0} = \mathrm{BESQ}^2, \qquad \{X_v^\star\}_{v\geq 0} = \mathrm{BESQ}^0. \tag{3.2.7}$$

Note that $\{X_v\}_{v\geq 0}$ is recurrent and has 0 as an entrance boundary, while $\{X_v^\star\}_{v\geq 0}$ is transient and has 0 as an absorbing boundary (see RY, Sect. XI.1). Denote by $\mathbb{P}_h$ and $\mathbb{P}_h^\star$ the distributions of the respective processes conditioned on starting at $h \geq 0$. Denote the corresponding expectations by $\mathbb{E}_h$ and $\mathbb{E}_h^\star$, respectively. Furthermore, define the following additive functional of BESQ^2 and its time change:

$$A(u) = \int_0^u X_v\,dv \qquad (u \geq 0), \tag{3.2.8}$$

$$\tag{3.2.9}$$

$$A^{-1}(t) = \inf\{u > 0 : A(u) > t\} \qquad (t \geq 0). \tag{3.2.10}$$

Note that both $u \mapsto A(u)$ and $t \mapsto A^{-1}(t)$ are continuous and strictly increasing towards infinity, $\mathbb{P}_h$-a.s. So A and A^{-1} are in fact inverse functions of each other. We also need the analogous functional for BESQ^0:

$$\begin{aligned} A^\star(u) &= \textstyle\int_0^u X_v^\star\,dv && (u \in [0,\infty]), \\ A^{\star-1}(t) &= \inf\{u \geq 0 : A^\star(u) > t\} && (t \geq 0). \end{aligned} \tag{3.2.11}$$

Note that, $\mathbb{P}_h^\star$-a.s., $u \mapsto A^\star(u)$ is strictly increasing on the time interval $[0, \xi_0]$, where $\xi_0 = \inf\{v \geq 0 : X_v^\star = 0\} < \infty$ denotes the absorption time of BESQ^0. Define Lebesgue densities ψ_h and $\phi_{h_1,t}$ by

$$\begin{aligned} \psi_h(t)\,dt &= \mathbb{P}_h^\star\,(A^\star(\infty) \in dt)\,, \\ \phi_{h_1,t}(u,h_2)\,du\,dh_2 &= \mathbb{P}_{h_1}(A^{-1}(t) \in du, X_u \in dh_2) \end{aligned} \tag{3.2.12}$$

for a.e. $h, t, h_1, u, h_2 \geq 0$. (The function ψ_h is explicitly identified in Lemma 3.6 in Section 3.5.2.) Put the quantities defined in (3.2.8-3.2.12) equal to zero if any of the variables is negative. Now the joint distribution of the three processes in (3.2.5) can be described as follows:

Lemma 3.1 *Fix $T > 0$. For all nonnegative Borel functions Φ_1, Φ_2 and Φ_3 on $C(\mathbb{R}_0^+)$ and for any interval $I \subset [0,\infty)$*

$$\begin{aligned} &\widehat{E}\Big(\Phi_1\big(\{L(T,B_T+x)\}_{x\geq 0}\big)\Phi_2\big(\{L(T,-x)\}_{x\geq 0}\big)\Phi_3\big(\{L(T,B_T-x)\}_{x\in[0,B_T]}\big)\,1_{B_T\in I}\Big) \\ &= \int_I du \int_{[0,\infty)^4} dt_1\,dh_1\,dt_2\,dh_2 \prod_{i=1}^{2} \mathbb{E}_{h_i}^\star\Big(\Phi_i\big(\{X_v^\star\}_{v\geq 0}\big)\,\Big|\,A^\star(\infty)=t_i\Big)\,\psi_{h_i}(t_i) \\ &\quad\times \mathbb{E}_{h_1}\Big(\Phi_3\big(\{X_v\}_{v\in[0,u]}\big)\,\Big|\,A^{-1}(T-t_1-t_2)=u, X_u=h_2\Big)\phi_{h_1,T-t_1-t_2}(u,h_2). \end{aligned} \tag{3.2.13}$$

Proof. Essentially, Lemma 3.1 is a formal rewrite using (3.2.8), (3.2.12) and the RK-theorems, which say that under $\mathbb{P}_h$ and $\mathbb{P}_h^\star$, respectively,

$$\begin{aligned} \{X_v\}_{v\in[0,u]} &\stackrel{\mathcal{D}}{=} \{L(\tau_h^u, u-v)\}_{v\in[0,u]} \\ \{X_v^\star\}_{v\geq 0} &\stackrel{\mathcal{D}}{=} \{L(\tau_h^u, u+v)\}_{v\geq 0}. \end{aligned} \tag{3.2.14}$$

However, the details are far from trivial.

We proceed in four steps, the first of which makes (3.2.6) precise and is the most technical.

STEP 1 $\widehat{P}(\tau_h^u \in dT)\,du\,dh = \widehat{P}(B_T \in du, L(T,B_T) \in dh)\,dT$ *for a.e.* $u, h, T \geq 0$.

Proof. From the occupation times formula (3.1.1) we get for every $t \geq 0$

$$\int_0^t 1_{B_s\in du}\,ds = L(t,u)\,du. \tag{3.2.15}$$

Thus, we obtain for every bounded and measurable functions $f : (\mathbb{R}^+)^2 \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}$

with compact support:

$$\begin{aligned}
&\int_0^\infty du \int_0^\infty dh\, f(u,h)\, \widehat{E}(g(\tau_h^u)) \\
&= \int_0^\infty du\, \widehat{E}\Big(\int_0^\infty d_t(L(t,u))\, f(u, L(t,u)) g(t)\Big) \\
&= \int_0^\infty du\, \widehat{E}\Big(\int_0^\infty d_t(L(t,u))\, g(t) E[f(u, L(t,u)) \mid B_t = u]\Big) \\
&= \int_0^\infty du \int_0^\infty dt\, \frac{d\widehat{E}(L(t,u))}{dt} g(t) \widehat{E}[f(u, L(t,u)) \mid B_t = u] \\
&= \int_0^\infty du \int_0^\infty dt\, \frac{\widehat{P}(B_t \in du)}{du}\, g(t) \widehat{E}[f(u, L(t,u)) \mid B_t = u] \\
&= \int_0^\infty dt\, g(t) \widehat{E}[f(B_t, L(t, B_t))].
\end{aligned} \tag{3.2.16}$$

(The second equality follows from Proposition 3 in Fitzsimmons, Pitman and Yor (1993), the fourth from (3.2.15).) □

Next, abbreviate for $u, h \geq 0$,

$$\mathcal{Z}_h^u = \Big(\tau_h^u, \int_0^\infty L(\tau_h^u, u+v)\, dv, L(\tau_h^u, 0), \int_0^\infty L(\tau_h^u, -v)\, dv\Big). \tag{3.2.17}$$

Then the distribution of $\mathcal{Z}_h^u$ is identified as:

STEP 2 *For every $u, h \geq 0$ and a.e. T, t_1, h_2, t_2,*

$$\widehat{P}(\mathcal{Z}_h^u \in d(T, t_1, h_2, t_2)) = \psi_h(t_1) \phi_{h, T-t_1-t_2}(u, h_2) \psi_{h_2}(t_2)\, dT\, dt_1\, dh_2\, dt_2. \tag{3.2.18}$$

Proof. According to the RK theorems, $\{L(\tau_h^u, -x)\}_{x\geq 0}$ is BESQ^0 starting at $L(\tau_h^u, 0)$. Moreover, $L(\tau_h^u, 0)$ itself has distribution $\mathbb{P}_h(X_u)^{-1}$. Furthermore, from (3.1.1) we have

$$\tau_h^u = \int_0^\infty L(\tau_h^u, u+v)\, dv + \int_0^u L(\tau_h^u, u-v)\, dv + \int_0^\infty L(\tau_h^u, -v)\, dv. \tag{3.2.19}$$

Combining these statements with the RK theorems and (3.2.14), we obtain

$$\begin{aligned}
\widehat{P}(\mathcal{Z}_h^u \in d(T, t_1, h_2, t_2)) = &\,\mathbb{P}_h^\star\Big(\int_0^\infty X_v^\star\, dv \in dt_1\Big)\, \mathbb{P}_{h_2}^\star\Big(\int_0^\infty X_v^\star\, dv \in dt_2\Big) \\
&\times \mathbb{P}_h\Big(\int_0^u X_v\, dv \in d(T - t_1 - t_2), X_u \in dh_2\Big).
\end{aligned} \tag{3.2.20}$$

But the r.h.s. of (3.2.20) equals the r.h.s. of (3.2.18), because of (3.2.12) and the identity $\{A(u) < T - t_1 - t_2\} = \{A^{-1}(T - t_1 - t_2) > u\}$ implied by (3.2.8). □

STEP 3 $\widehat{P}(\tau^{B_T}_{L(T,B_T)} = T) = 1$.

Proof. Simply note that $\tau^{B_T}_{L(T,B_T)} - T$ is distributed as the time change τ^0_0 for the process $(B_{T+t} - B_T)_{t\geq 0}$ (recall (3.2.2)). But $\widehat{P}(\tau^0_0 = 0) = 1$ (see RY, Remark 1°) following Prop. VI.2.5). □

STEP 4 *Proof of Lemma 3.1.*

Proof. First condition and integrate the l.h.s. of (3.2.13) with respect to the distribution of
$(B_T, L(T, B_T))$, which is identified in Step 1. According to Step 3, we may then replace T by $\tau^u_{h_1}$ on $\{B_T = u, L(T, B_T) = h_1\}$. Next, condition and integrate with respect to the conditional distribution of $\mathcal{Z}^u_{h_1}$ given $\{\tau^u_{h_1} = T\}$. Then the l.h.s. of (3.2.13) becomes

$$\begin{aligned}\int_I du \int_0^\infty dh_1 \frac{\widehat{P}(\tau^u_{h_1} \in dT)}{dT} \int_{[0,\infty)^3} &\frac{\widehat{P}(\mathcal{Z}^u_{h_1} \in d(T, t_1, h_2, t_2))}{\widehat{P}(\tau^u_{h_1} \in dT)} \\ &\times \widehat{E}\Big(\Phi_1\big(\{L(\tau^u_{h_1}, u + x)\}_{x\geq 0}\big)\Phi_2\big(\{L(\tau^u_{h_1}, -x)\}_{x\geq 0}\big) \\ &\quad \times \Phi_3\big(\{L(\tau^u_{h_1}, u - x)\}_{x\in[0,u]}\big) \,\big|\, \mathcal{Z}^u_{h_1} = (T, t_1, h_2, t_2)\Big).\end{aligned} \tag{3.2.21}$$

Now use Step 2, apply the description of the local time processes provided by the RK theorems in combination with (3.2.14) and (3.2.17), and again use the elementary relation between A and A^{-1} stated at the end of the proof of Step 2. Then we obtain that (3.2.21) is equal to the r.h.s. of (3.2.13). □

□

In Lemma 3.1, note that $A^*(\infty) = t_1$, respectively, t_2 corresponds to the Brownian motion spending t_1, respectively, t_2 time units is the boundary areas $[B_T, \infty)$, respectively, $(-\infty, 0]$, while $A^{-1}(T - t_1 - t_2)$ corresponds to the size of the middle area $[0, B_T]$ when the Brownian motion spends $T - t_1 - t_2$ time units there.

3.2.4 Application to the Edwards model: Lemma 3.2

We are now ready to formulate the key representation of the expectation appearing in the l.h.s. of (3.2.1). This representation will be the starting point for the proof of Proposition 3.1 in Sections 3.3-3.5. Abbreviate

$$C_T = b^*T + C\sqrt{T}. \tag{3.2.22}$$

Lemma 3.2 *For all $T > 0$,*

$$\begin{aligned}&E\Big(e^{-\int_{\mathbb{R}} L(T,x)^2\,dx} 1_{0<B_T\leq C_T}\Big) \\ &= \int_0^{C_T} du \int_{[0,\infty)^4} dt_1\, dh_1\, dt_2\, dh_2 \prod_{i=1}^2 \mathbb{E}^\star_{h_i}\Big(e^{-\int_0^\infty X^{\star 2}_v\,dv} \,\Big|\, A^\star(\infty) = t_i\Big)\psi_{h_i}(t_i) \\ &\quad \times \mathbb{E}_{h_1}\Big(e^{-\int_0^u X_v^2\,dv} \,\Big|\, A^{-1}(T - t_1 - t_2) = u, X_u = h_2\Big)\phi_{h_1, T-t_1-t_2}(u, h_2).\end{aligned} \tag{3.2.23}$$

Proof. This follows from Lemma 3.1. □

Thus, we have expressed the expectation in the l.h.s. of (3.2.1) in terms of integrals over BESQ^0 and BESQ^2 and their additive functionals. Henceforth we can forget about the underlying Brownian motion and focus on these processes using their generators given in (3.2.3) and (3.2.4).

The importance of Lemma 3.2 is the decomposition into a *product* of three expectations. The main reason to introduce the densities ψ_h and $\phi_{h_1,t}$ is the fact that the last factor in (3.2.23) depends on t_1 and t_2. This dependence will vanish in the limit as $T \to \infty$, as we will see in the sequel. After that the densities ψ_h and $\phi_{h_1,t}$ can again be absorbed into the expectations (recall (3.2.12)). In Section 3.5.2 we will identify ψ_h. Of $\phi_{h_1,t}$ we need little more than its existence.

3.3 Structure of the proof of Proposition 3.1

All we have done so far is to rewrite the key object of Proposition 3.1 in terms of expectations involving squared Bessel processes. We are now ready for our main attack.

In Section 3.3.1 we use Girsanov's formula to transform BESQ^2 into a new Markov process. The purpose of this transformation is to absorb the exponential factor appearing under the expectation in the last line of (3.2.23) into the transition probabilities of the new process. In Section 3.3.2 we list some properties of the transformed process. These are used in Section 3.3.3 to obtain a final reformulation of (3.2.23) on which the proof of Proposition 3.1 will be based. In Section 3.3.4 we formulate three main propositions, the proof of which is deferred to Sections 3.4-3.5. In Section 3.3.5 the proof of Proposition 3.1 is completed subject to these propositions.

3.3.1 A transformed Markov process: Lemma 3.3

Fix $a \in \mathbb{R}$ (later we will pick $a = a^*$). Recall from Section 3.1.1 that $\rho(a) \in \mathbb{R}$ is the largest eigenvalue of the operator $\mathcal{K}^a$ defined in (3.1.3). We denote the corresponding strictly positive and L^2-normalized eigenvector by x_a.[3] From Lemmas 2.16 and 2.18, we know that $x_a : \mathbb{R}_0^+ \to \mathbb{R}^+$ is real-analytic with $\lim_{u\to\infty} u^{-\frac{3}{2}} \log x_a(u) < 0$, and that $a \mapsto x_a$ is analytic as a map from $\mathbb{R}$ to $L^2(\mathbb{R}_0^+)$. Define

$$F_a(u) = u^2 - au + \rho(a) \qquad (u \in \mathbb{R}_0^+). \tag{3.3.1}$$

The following lemma defines the Girsanov transformation of BESQ^2 that we will need later:

Lemma 3.3 *For $t, h_1, h_2 \geq 0$, let $P_t(h_1, dh_2)$ denote the transition probability function of BESQ^2. Then*

$$\widehat{P}_t^a(h_1, dh_2) = \frac{x_a(h_2)}{x_a(h_1)} \mathbb{E}_{h_1}\left(e^{-\int_0^t F_a(X_v)\,dv} \,\Big|\, X_t = h_2\right) P_t(h_1, dh_2) \tag{3.3.2}$$

[3] This is a slight abuse of notation because of the scaling factor 2 (see footnote 1).

defines the transition probability function of a diffusion $\{X_v\}_{v\geq 0}$ *on* $\mathbb{R}_0^+$.

Proof. Recall the definition of the generator G of BESQ^2 given in (3.2.4). According to RY, Sect. VIII.3, if $f \in C^2(\mathbb{R}_0^+)$ satisfies the equation

$$G(f) + \frac{1}{2}G(f^2) - f\,G(f) = F_a, \tag{3.3.3}$$

then

$$\{D_t^{f,a}\}_{t\geq 0} = \left\{e^{f(X_t)-f(X_0)-\int_0^t F_a(X_s)\,ds}\right\}_{t\geq 0} \tag{3.3.4}$$

is a local martingale under $\mathbb{P}_h$ for any $h \geq 0$. Substitute $f = \log x$ in the l.h.s. of (3.3.3). Then an elementary calculation yields that for all $u \geq 0$

$$\begin{aligned}\big(G(f) + \frac{1}{2}G(f^2) - f\,G(f)\big)(u) &= 2uf''(u) + 2f'(u) + 2uf'(u)^2\\ &= \frac{2ux''(u) + 2x'(u)}{x(u)}.\end{aligned} \tag{3.3.5}$$

We now easily derive from the eigenvalue relation $\mathcal{K}_a x_a = \rho(a)x_a$ (recall (3.1.3)) that (3.3.3) is satisfied for $f = f_a = \log x_a$. Hence, $\{D_t^{f_a,a}\}_{t\geq 0}$ is a local martingale under $\mathbb{P}_h$. Since F_a is bounded from below and x_a is bounded from above, each $D_t^{f_a,a}$ is bounded $\mathbb{P}_h$-a.s. Hence $\{D_t^{f_a,a}\}_{t\geq 0}$ is a martingale under $\mathbb{P}_h$. The lemma now follows from RY, Prop. VIII.3.1. □

We will denote the distribution of the transformed process, conditioned on starting at $h \geq 0$, by $\widehat{\mathbb{P}}_h^a$ and the corresponding expectation by $\widehat{\mathbb{E}}_h^a$. Note that we have

$$\widehat{\mathbb{E}}_h^a(g(X_t)) = \mathbb{E}_h(D_t^{f_a,a}g(X_t)) \qquad (t \geq 0, g : \mathbb{R}_0^+ \to \mathbb{R} \text{ measurable}). \tag{3.3.6}$$

3.3.2 Properties of the transformed Markov process

We are going to list some properties of the process constructed in the preceding subsection.

1. The process introduced in Lemma 3.3 is a Feller process. According to RY, Prop. VIII.3.4, its generator is given by (recall $f_a = \log x_a$)

$$\begin{aligned}(\widehat{G}^a f)(u) &= (Gf)(u) + \big(G(f_a f) - f_a G(f) - fG(f_a)\big)(u)\\ &= (Gf)(u) + 4uf_a'(u)f'(u)\\ &= 2uf''(u) + 2f'(u)\Big(1 + 2u\frac{x_a'(u)}{x_a(u)}\Big) \qquad (f \in C_c^2(\mathbb{R}^+)).\end{aligned} \tag{3.3.7}$$

2. According to KS, Ch. 5, Eq. (5.42), the scale function for the process is given (up to an affine transformation) by

$$s_a(u) = \int_c^u \frac{dv}{v\, x_a^2(v)} \qquad (c > 0 \text{ arbitrary}). \tag{3.3.8}$$

Since x_a is continuous in zero and has a sub-exponential tail at infinity (see the remarks at the beginning of Section 3.3.1), the scale function satisfies

$$\lim_{u \downarrow 0} s_a(u) = -\infty \qquad \text{and} \qquad \lim_{u \to \infty} s_a(u) = \infty. \tag{3.3.9}$$

3. The probability measure on $\mathbb{R}_0^+$ given by

$$\mu_a(du) = x_a(u)^2\, du \tag{3.3.10}$$

is the normalized speed measure for the process (see KS, Ch. 5, Eq. (5.51)). Since it has finite mass, and because (3.3.9) holds, the process converges weakly towards μ_a from any starting point $h > 0$ (see KS, Ch. 5, Ex. 5.40), i.e.,

$$\lim_{t \to \infty} \widehat{\mathbb{E}}_h^a(f(X_t)) = \int_0^\infty f(u)\, \mu_a(du) \text{ for all bounded } f \in C(\mathbb{R}_0^+). \tag{3.3.11}$$

Using this convergence and the Feller property, one derives in a standard way that μ_a is the invariant distribution for the process. We write

$$\widehat{\mathbb{P}}^a = \int_0^\infty \widehat{\mathbb{P}}_h^a\, \mu_a(dh) \tag{3.3.12}$$

to denote the distribution of the process starting in the invariant distribution and write $\widehat{\mathbb{E}}^a$ for the corresponding expectation.

4. According to Ethier and Kurtz (1986), Theorem 6.1.4, the process $\{Y_t\}_{t \geq 0}$ given by

$$Y_t = X_{A^{-1}(t)} \qquad (t \geq 0) \tag{3.3.13}$$

is a diffusion under $\widehat{\mathbb{P}}^a$ with generator

$$\left(\widetilde{G}^a f\right)(u) = \frac{1}{u}\left(\widehat{G}^a f\right)(u) \qquad (u > 0, f \in C_c^2(\mathbb{R}^+)) \tag{3.3.14}$$

(see (3.3.7)). This process has the same scale function s_a as $\{X_t\}_{t \geq 0}$ (see (3.3.8)), and its normalized speed measure is given by

$$\nu_a(du) = \frac{u}{\rho'(a)} x_a^2(u)\, du. \tag{3.3.15}$$

(In order to see that $\nu_a(\mathbb{R}^+) = 1$, recall (2.7.24) and footnote 1). Similarly as in (3.3.11), for any starting point $h > 0$

$$\lim_{t \to \infty} \widehat{\mathbb{E}}_h^a(f(Y_t)) = \int_0^\infty f(u)\, \nu_a(du) \text{ for all bounded } f \in C(\mathbb{R}_0^+) \tag{3.3.16}$$

and hence ν_a is the invariant distribution of the process $\{Y_t\}_{t\geq 0}$. We write

$$\widetilde{\mathbb{P}}^a = \int_0^\infty \widehat{\mathbb{P}}_h^a\, \nu_a(dh) \tag{3.3.17}$$

to denote the distribution of the process $\{X_t\}_{t\geq 0}$ starting in the invariant distribution ν_a of the process $\{Y_t\}_{t\geq 0}$ and we write $\widetilde{\mathbb{E}}^a$ for the corresponding expectation.

3.3.3 Final reformulation: Lemma 3.4

Using the representation in Lemma 3.2, we will rewrite the l.h.s. of (3.2.1) in terms of the transformed process introduced in Lemma 3.3. This will be the final reformulation in terms of which the proof of Proposition 3.1 will be finished in Sections 3.3.4-3.3.5.

For $h, t \geq 0$ and $a \in \mathbb{R}$, introduce the abbreviation (recall (3.2.11) and (3.2.12))

$$\begin{aligned} F_a^\star(u) &= u^2 - au \qquad (u \in \mathbb{R}_0^+) \\ w_a(h,t) &= \mathbb{E}_h^\star\Big(e^{-\int_0^\infty F_a^\star(X_v^\star)\,dv} \,\Big|\, A^\star(\infty) = t\Big)\psi_h(t) = e^{at} w_0(h,t). \end{aligned} \tag{3.3.18}$$

Recall that $\widehat{\mathbb{E}}^a$ denotes the expectation for the transformed process $\{X_t\}_{t\geq 0}$ starting in the invariant starting distribution μ_a given by (3.3.10).

Lemma 3.4 *For every $T > 0$,*

$$\begin{aligned} &e^{a^* T} \widehat{E}\Big(e^{-\int_{\mathbb{R}} L(T,x)^2\,dx} 1_{0<B_T\leq C_T}\Big) \\ &= \int_0^\infty dt_1 \int_0^\infty dt_2\, \widehat{\mathbb{E}}^{a^*}\Big(\frac{w_{a^*}(X_0, t_1)}{x_{a^*}(X_0)} 1_{A^{-1}(T-t_1-t_2)\leq C_T} \frac{w_{a^*}(X_{A^{-1}(T-t_1-t_2)}, t_2)}{x_{a^*}(X_{A^{-1}(T-t_1-t_2)})}\Big). \end{aligned} \tag{3.3.19}$$

Proof. First, from (3.2.8), (3.3.1) and from $\rho(a^*) = 0$ it follows that on $\{A^{-1}(t) = u\}$

$$a^* t - \int_0^u X_v^2\,dv = -\int_0^u F_{a^*}(X_v)\,dv \qquad (t, u \geq 0). \tag{3.3.20}$$

By an absolute continuous transformation from $\mathbb{P}_h$ to $\widehat{\mathbb{P}}_h^{a^*}$, we therefore obtain via (3.3.2) the identity (recall (3.2.12))

$$\begin{aligned} &e^{a^* t}\, \mathbb{E}_{h_1}\Big(e^{-\int_0^u X_v^2\,dv} \,|\, A^{-1}(t) = u, X_u = h_2\Big)\phi_{h_1,t}(u, h_2)\,du\,dh_2 \\ &= \widehat{\mathbb{P}}_{h_1}^{a^*}\big(A^{-1}(t) \in du, X_u \in dh_2\big)\frac{x_{a^*}(h_1)}{x_{a^*}(h_2)} \end{aligned} \tag{3.3.21}$$

for a.e. $u, h_1, h_2, t \geq 0$. Similarly to (3.3.20), we have on $\{\int_0^\infty X_v^\star\,dv = t\}$

$$a^* t - \int_0^\infty (X_v^\star)^2\,dv = -\int_0^\infty F_{a^*}(X_v^\star)\,dv \qquad (t \geq 0) \tag{3.3.22}$$

and hence

$$e^{a^* t_i}\mathbb{E}^*_{h_i}\big(e^{-\int_0^\infty (X^*_v)^2\,dv} \mid A^*(\infty) = t_i\big)\psi_{h_i}(t_i) = w_{a^*}(h_i, t_i) \qquad (i = 1, 2). \tag{3.3.23}$$

Next, note that the l.h.s. of (3.3.19) is equal to the l.h.s. of (3.2.23) times the factor e^{a^*T}. We divide this factor into three parts, according to the identity $T = t_1 + (T - t_1 - t_2) + t_2$, and assign them to each of the three expectations in the r.h.s. of (3.2.23). Substitute (3.3.21) with $t = T - t_1 - t_2$ and (3.3.23) into (3.2.23). Then we obtain that the l.h.s. of (3.3.19) is equal to

$$\begin{aligned}\int_{[0,\infty)^4} & dh_1\, dh_2\, dt_1\, dt_2\, w_{a^*}(h_1, t_1) w_{a^*}(h_2, t_2)\frac{x_{a^*}(h_1)}{x_{a^*}(h_2)} \\ & \times \widehat{\mathbb{P}}^{a^*}_{h_1}\big(A^{-1}(T - t_1 - t_2) \le C_T, X_{A^{-1}(T-t_1-t_2)} \in dh_2\big).\end{aligned} \tag{3.3.24}$$

Now formally carry out the integration over h_1, h_2, recalling (3.3.10) and (3.3.12), to arrive at the r.h.s. of (3.3.19). □

Roughly speaking, the function w_{a^*} in the r.h.s. of (3.3.19) describes the contribution to the random variable $\exp[-\int_{\mathbb{R}} L(T,x)^2\,dx]$ coming from the boundary pieces (i.e., the parts of the path in $(-\infty, 0] \cup [B_T, \infty)$), while $A^{-1}(T - t_1 - t_2)$ gives the size of the area over which the middle piece (i.e., the parts of the path in $[0, B_T]$) spreads out.

3.3.4 Key steps in the proof of Proposition 3.1: Propositions 3.2–3.4

The proof of Proposition 3.1 now basically requires the following three ingredients:

(1) A CLT for $\{A^{-1}(t)\}_{t\ge 0}$ under $\widehat{\mathbb{P}}^{a^*}$.

(2) An extension of the weak convergence of $\{Y_t\}_{t\ge 0} = \{X_{A^{-1}(t)}\}_{t\ge 0}$ stated in (3.3.16).

(3) Some integrability properties of w_{a^*}.

The precise statements that we will need are formulated in Propositions 3.2–3.4 below. The proof of these propositions is deferred to Sections 3.4 and 3.5.

We need some more notation. Let $\langle \cdot\,, \cdot\rangle_{L^2}$ denote the standard inner product on $L^2(\mathbb{R}_0^+)$. Let $\langle \cdot\,, \cdot\rangle^\circ_{L^2}$ denote the weighted inner product

$$\langle f, g\rangle^\circ_{L^2} = \int_0^\infty dh\ hf(h)g(h) \tag{3.3.25}$$

on $L^{2,\circ}(\mathbb{R}_0^+) = \{\, f : \mathbb{R}_0^+ \to \mathbb{R} \text{ measurable} \mid \int_0^\infty dh\ hf^2(h) < \infty\,\}$. We write $||\cdot||_{L^2}$, respectively, $||\cdot||^\circ_{L^2}$ for the corresponding norms.

For bounded and measurable $f, g : \mathbb{R}_0^+ \to \mathbb{R}$, $T \geq 0$ and $a \in \mathbb{R}$, abbreviate (recall Lemma 3.3, (3.3.10) and (3.3.12))

$$N_{T,a}^{f,g} = \widehat{\mathbb{E}}^a\left(\frac{f}{x_a}(Y_0)\,\frac{g}{x_a}(Y_T)\right) = \int_0^\infty dh\ f(h)\mathbb{E}_h\left(e^{-\int_0^{A^{-1}(T)} F_a(X_s)\,ds} g(X_{A^{-1}(T)})\right). \tag{3.3.26}$$

Furthermore, define

$$\sigma^2(a) = \frac{\rho''(a)}{\rho'(a)^3} \tag{3.3.27}$$

and note that $\sigma^2(a^*) = c^{*2}$ defined in (3.1.4). Denote by $\rho^{-1} : \mathbb{R} \to \mathbb{R}$ the inverse function of $\rho : \mathbb{R} \to \mathbb{R}$.

Proposition 3.2 *For all bounded and measurable $f, g : \mathbb{R}_0^+ \to \mathbb{R}$ and for every $a, \lambda \in \mathbb{R}$ and all T, $T' \geq 0$,*

$$\widehat{\mathbb{E}}^a\left(\frac{f}{x_a}(Y_0)e^{\frac{\lambda}{\sqrt{T}}\left(A^{-1}(T')-\frac{T}{\rho'(a)}\right)}\,\frac{g}{x_a}(Y_{T'})\right) = e^{\frac{\lambda^2}{2}\sigma^2(\xi_T)} N_{T',a_{\lambda,T}}^{f,g} e^{(T-T')(a_{\lambda,T}-a)}, \tag{3.3.28}$$

where

$$a_{\lambda,T} = \rho^{-1}\left(\rho(a) - \frac{\lambda}{\sqrt{T}}\right) \tag{3.3.29}$$

and $\xi_T \in [a, a_{\lambda,T}] \cup [a_{\lambda,T}, a]$.

Proposition 3.3 *Let $f, g : \mathbb{R}_0^+ \to \mathbb{R}$ be measurable such that $f/\mathrm{id}, g \in L^{2,\circ}$. Then for every $a \in \mathbb{R}$ and $a_T \to a$,*

$$\lim_{T\to\infty} N_{T,a_T}^{f,g} = \frac{1}{\rho'(a)}\langle f, x_a\rangle_{L^2}\langle g, x_a\rangle_{L^2}^\circ. \tag{3.3.30}$$

Recall (3.3.18). For $a \in \mathbb{R}$, define $y_a : \mathbb{R}_0^+ \to [0, \infty]$ by

$$y_a(h) = \int_0^\infty w_a(h,t)\,dt = \mathbb{E}_h^*\left(e^{-\int_0^\infty F_a^*(X_v^*)\,dv}\right). \tag{3.3.31}$$

Furthermore, define, for $p \in (1,2)$, respectively, $q \in (2,\infty)$, and $t \geq 0$,

$$\begin{aligned} W_p^{(1)}(t) &= \left(\textstyle\int_0^\infty h^{1-p} x_{a^*}(h)^{2-p} w_{a^*}(h,t)^p\,dh\right)^{1/p}, \\ W_q^{(2)}(t) &= \left(\textstyle\int_0^\infty h x_{a^*}(h)^{2-q} w_{a^*}(h,t)^q\,dh\right)^{1/q}. \end{aligned} \tag{3.3.32}$$

Proposition 3.4

(i) y_{a^*} is bounded and measurable.

(ii) For any $p \in (1,2)$, $W_p^{(1)}$ is integrable on $\mathbb{R}^+$.

(iii) For any $q \in (2,\infty)$ that is sufficiently close to 2, $W_q^{(2)}$ is integrable on $\mathbb{R}^+$.

3.3.5 Proof of Proposition 3.1

In this subsection we finish the proof of Proposition 3.1 subject to Propositions 3.2–3.4. We will show that (3.2.1) follows from (3.3.19), with $\widehat{L}$ identified as

$$\widehat{L} = b^*\langle y_{a^*}, x_{a^*}\rangle_{L^2}\langle y_{a^*}, x_{a^*}\rangle^{\circ}_{L^2}. \tag{3.3.33}$$

STEP 1 *For all $t_1, t_2 > 0$, as $T \to \infty$ the integrand on the r.h.s. of (3.3.19) tends to*

$$b^*\langle w_{a^*}(\cdot, t_1), x_{a^*}\rangle_{L^2}\langle w_{a^*}(\cdot, t_2), x_{a^*}\rangle^{\circ}_{L^2}\mathcal{N}_{c^{*2}}((-\infty, C]).$$

Proof. By Proposition 3.4(*ii*), for all $t_1, t_2 > 0$ the functions $f = w_{a^*}(\cdot, t_1)$ and $g = w_{a^*}(\cdot, t_2)$ satisfy the assumptions of Proposition 3.3. Define a (non-Markovian) path measure $\mathbb{P}^{f,g}_{T,a}$ by

$$\frac{d\mathbb{P}^{f,g}_{T,a}}{d\widehat{\mathbb{P}}^a} = \frac{1}{N^{f,g}_{T,a}}\frac{f}{x_a}(Y_0)\frac{g}{x_a}(Y_T). \tag{3.3.34}$$

Write $\mathbb{E}^{f,g}_{T,a}$ for the corresponding expectation. Apply Proposition 3.2 for $a = a^*$ and $T' = T - t_1 - t_2$ to obtain that for every $\lambda \in \mathbb{R}$ and $T \geq t_1 + t_2$,

$$\mathbb{E}^{f,g}_{T-t_1-t_2,a^*}\left(e^{\frac{\lambda}{\sqrt{T}}[A^{-1}(T-t_1-t_2)-b^*T]}\right) = e^{\frac{\lambda^2}{2}\sigma^2(\xi^*_T)}\frac{N^{f,g}_{T-t_1-t_2,a^*_{\lambda,T}}}{N^{f,g}_{T-t_1-t_2,a^*}}e^{(t_1+t_2)(a^*_{\lambda,T}-a^*)}, \tag{3.3.35}$$

where $\rho(a^*) = 0$, $b^* = \frac{1}{\rho'(a^*)}$ (recall (3.1.6)), $a^*_{\lambda,T} = \rho^{-1}(-\frac{\lambda}{\sqrt{T}})$ and $\xi^*_T \in [a^*, a^*_{\lambda,T}] \cup [a^*_{\lambda,T}, a^*]$. Since ρ', ρ'' and ρ^{-1} are continuous, we have $a^*_{\lambda,T} \to a^*$ and $\sigma^2(\xi^*_T) \to c^{*2}$ as $T \to \infty$. Therefore, by Proposition 3.3, the r.h.s. of (3.3.35) tends to $e^{\frac{\lambda^2}{2}c^{*2}}$ as $T \to \infty$. Thus, the distribution of $\frac{1}{\sqrt{T}}[A^{-1}(T-t_1-t_2) - b^*T]$ under $\mathbb{P}^{f,g}_{T-t_1-t_2,a^*}$ converges weakly towards $\mathcal{N}_{c^{*2}}$. Via (3.3.34), this in turn implies that (recall (3.2.22))

$$\begin{aligned}
&\lim_{T\to\infty}\widehat{\mathbb{E}}^{a^*}\Big(\frac{w_{a^*}(X_0, t_1)}{x_{a^*}(X_0)}1_{A^{-1}(T-t_1-t_2)\leq C_T}\frac{w_{a^*}(X_{A^{-1}(T-t_1-t_2)}, t_2)}{x_{a^*}(X_{A^{-1}(T-t_1-t_2)})}\Big)\\
&= \lim_{T\to\infty} N^{f,g}_{T-t_1-t_2,a^*}\mathbb{P}^{f,g}_{T-t_1-t_2,a^*}\left(A^{-1}(T-t_1-t_2) - b^*T \leq C\sqrt{T}\right)\\
&= b^*\langle f, x_{a^*}\rangle_{L^2}\langle g, x_{a^*}\rangle^{\circ}_{L^2}\mathcal{N}_{c^{*2}}((-\infty, C]),
\end{aligned} \tag{3.3.36}$$

again according to Proposition 3.3. □

STEP 2 *For all $t_1, t_2 > 0$, and any $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, the integrand on the r.h.s. of (3.3.19) is bounded uniformly in $T > 0$ by $W^{(1)}_p(t_1)W^{(2)}_q(t_2)$ defined in (3.3.32).*

Proof. Recall 3. and 4. in Section 3.3.2. Make a change of measure from $\widehat{\mathbb{E}}^{a^*}$ to $\widetilde{\mathbb{E}}^{a^*}$, use the Hölder inequality and the stationarity of $\{Y_t\}_{t\geq 0}$ under $\widetilde{\mathbb{P}}^{a^*}$ (recall (3.3.15) and (3.3.17)), to obtain

$$\begin{aligned}
&\widehat{\mathbb{E}}^{a^*}\left(\frac{w_{a^*}(X_0,t_1)}{x_{a^*}(X_0)}\, 1_{A^{-1}(T-t_1-t_2)\leq C_T}\, \frac{w_{a^*}(X_{A^{-1}(T-t_1-t_2)},t_2)}{x_{a^*}(X_{A^{-1}(T-t_1-t_2)})}\right)\\
&\leq \rho'(a^*)\widetilde{\mathbb{E}}^{a^*}\left(\frac{w_{a^*}(Y_0,t_1)}{Y_0 x_{a^*}(Y_0)}\,\frac{w_{a^*}(Y_{T-t_1-t_2},t_2)}{x_{a^*}(Y_{T-t_1-t_2})}\right)\\
&\leq \rho'(a^*)\left(\widetilde{\mathbb{E}}^{a^*}\left(\left[\frac{w_{a^*}(Y_0,t_1)}{Y_0 x_{a^*}(Y_0)}\right]^p\right)\right)^{1/p}\left(\widetilde{\mathbb{E}}^{a^*}\left(\left[\frac{w_{a^*}(Y_{T-t_1-t_2},t_2)}{x_{a^*}(Y_{T-t_1-t_2})}\right]^q\right)\right)^{1/q}\\
&= W_p^{(1)}(t_1)\, W_q^{(2)}(t_2).
\end{aligned} \tag{3.3.37}$$

□

STEP 3 *Conclusion of the proof.*

Proof. Let $T\to\infty$ in (3.3.19) and note that, for some $p,q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$, the bound in Step 2 is integrable in $(t_1,t_2)\in(\mathbb{R}^+)^2$ by Proposition 3.4(ii) and (iii). By Steps 1-2 and the dominated convergence theorem we may interchange $T\to\infty$ and $\int_0^\infty dt_1\,\int_0^\infty dt_2$, to obtain

$$\begin{aligned}
&\lim\nolimits_{T\to\infty} \text{l.h.s. of (3.3.19)}\\
&= b^*\int_0^\infty dt_1\,\int_0^\infty dt_2\,\langle w_{a^*}(\cdot,t_1),x_{a^*}\rangle_{L^2}\langle w_{a^*}(\cdot,t_2),x_{a^*}\rangle^\circ_{L^2}\mathcal{N}_{c^{*2}}((-\infty,C]).
\end{aligned} \tag{3.3.38}$$

Use (3.3.31), Proposition 3.4(i) and Fubini's theorem to identify the r.h.s. of (3.3.38) as $\widehat{L}\mathcal{N}_{c^{*2}}((-\infty,C])$, with $\widehat{L}$ given in (3.3.33). □

3.4 CLT for the middle piece

This section contains the proofs of Propositions 3.2 and 3.3.

3.4.1 Proof of Proposition 3.2

Recall Lemma 3.3 and (3.3.26) to see that the l.h.s. of (3.3.28) is equal to

$$e^{-\frac{\lambda\sqrt{T}}{\rho'(a)}}\int_0^\infty dh\; f(h)\mathbb{E}_h\left(e^{-\int_0^{A^{-1}(T')}\left(F_a(X_s)-\frac{\lambda}{\sqrt{T}}\right)ds}g(X_{A^{-1}(T')})\right). \tag{3.4.1}$$

According to (3.3.29), $\rho(a_{\lambda,T}) = \rho(a) - \frac{\lambda}{\sqrt{T}}$. Since $T' = \int_0^{A^{-1}(T')} X_s\,ds$ (see (3.2.8)) and $F_a(u) = u^2 - au + \rho(a)$ (see (3.3.1)), we may write the exponents in (3.4.1) as

$$\begin{aligned} &-\int_0^{A^{-1}(T')} F_{a_{\lambda,T}}(X_s)\,ds + (a - a_{\lambda,T})\int_0^{A^{-1}(T')} X_s\,ds - \frac{\lambda\sqrt{T}}{\rho'(a)} \\ &= -\int_0^{A^{-1}(T')} F_{a_{\lambda,T}}(X_s)\,ds + T\left(a - a_{\lambda,T} - \frac{\lambda}{\sqrt{T}\rho'(a)}\right) + (T - T')(a_{\lambda,T} - a). \end{aligned} \tag{3.4.2}$$

Substitute this into (3.4.1) and use (3.3.26) to get that

$$\text{l.h.s. of (3.3.28)} = e^{T\left(a - a_{\lambda,T} - \frac{\lambda}{\sqrt{T}\rho'(a)}\right)} N^{f,g}_{T',a_{\lambda,T}} e^{(T-T')(a_{\lambda,T}-a)}. \tag{3.4.3}$$

Next, expand the inverse function ρ^{-1} of ρ as a Taylor series around $\rho(a)$ up to second order. It follows that there is an r_T in between $\rho(a)$ and $\rho(a) - \frac{\lambda}{\sqrt{T}}$ such that

$$\begin{aligned} a_{\lambda,T} &= \rho^{-1}(\rho(a) - \tfrac{\lambda}{\sqrt{T}}) = \rho^{-1}(\rho(a)) - \tfrac{\lambda}{\sqrt{T}}(\rho^{-1})'(\rho(a)) + \tfrac{\lambda^2}{2T}(\rho^{-1})''(r_T) \\ &= a - \tfrac{\lambda}{\sqrt{T}\rho'(a)} - \tfrac{\lambda^2}{2T}\tfrac{\rho''}{(\rho')^3}(\rho^{-1}(r_T)) = a - \tfrac{\lambda}{\sqrt{T}\rho'(a)} - \tfrac{\lambda^2}{2T}\sigma^2(\xi_T) \end{aligned} \tag{3.4.4}$$

(see (3.3.27)) with $\xi_T = \rho^{-1}(r_T)$. Observe that ξ_T is in between a and $a_{\lambda,T}$ by monotonicity of ρ. Now substitute (3.4.4) into (3.4.3) to arrive at (3.3.28).

3.4.2 Proof of Proposition 3.3

We will use an expansion in terms of the eigenfunctions of the operator $\mathcal{M}^a : L^{2,\circ}(\mathbb{R}_0^+) \cap C^2(\mathbb{R}_0^+) \to C(\mathbb{R}^+)$ defined by

$$(\mathcal{M}^a x)(u) = \frac{(\mathcal{K}^a x)(u) - \rho(a)x(u)}{u} \tag{3.4.5}$$

(recall (3.1.3)). Obviously, $\mathcal{M}^a$ is a symmetric operator with respect to $\langle\cdot,\cdot\rangle^\circ_{L^2}$ because $\mathcal{K}^a$ is a symmetric operator with respect to $\langle\cdot,\cdot\rangle_{L^2}$. It is also a Sturm-Liouville operator. We are going to identify its eigenvalues and eigenvectors in terms of the ones of $\mathcal{K}^a$.

For $l \in \mathbb{N}_0$, let $\rho^{(l)}(a)$ denote the lth largest eigenvalue of $\mathcal{K}^a$ and $x_a^{(l)} \in L^2(\mathbb{R}^+)$ the corresponding eigenfunction, normalized such that $||x_a^{(l)}||_{L^2} = 1$ (all eigenspaces are one-dimensional by Lemma 2.16). Then $\rho^{(0)} = \rho$, and each $\rho^{(l)}$ is continuous and strictly increasing (differentiate the formula $\rho^{(l)}(a) = \langle x_a^{(l)}, \mathcal{K}^a x_a^{(l)}\rangle_{L^2}$ to obtain $\frac{d}{da}\rho^{(l)}(a) = ||x_a^{(l)}||^{\circ 2}_{L^2}$ via (3.1.3)). Moreover, $\lim_{a\to\pm\infty}\rho^{(l)}(a) = \pm\infty$. Since $x_a^{(l)}$ has a subexponentially small tail at infinity (see Lemma 2.16), it is also an element of $L^{2,\circ}(\mathbb{R}_0^+)$.

Next, define $\alpha^{(l)}(a) \in \mathbb{R}$ and $y_a^{(l)} \in L^{2,\circ}(\mathbb{R}_0^+)$ by

$$\rho^{(l)}(a - \alpha^{(l)}(a)) = \rho(a) \qquad \text{and} \qquad y_a^{(l)} = \frac{x^{(l)}_{a-\alpha^{(l)}(a)}}{||x^{(l)}_{a-\alpha^{(l)}(a)}||^\circ_{L^2}} \qquad (l \in \mathbb{N}_0). \tag{3.4.6}$$

Note that $\alpha^{(0)}(a) = 0$, $y_a^{(0)} = x_a/\sqrt{\rho'(a)}$, and $\alpha^{(l+1)}(a) < \alpha^{(l)}(a)$ for all $l \in \mathbb{N}_0$ since $\rho^{(l)}(a)$ is strictly decreasing in l and strictly increasing in a.

STEP 1 *For each $a \in \mathbb{R}$, the sequence $(y_a^{(l)})_{l\in\mathbb{N}_0}$ is an orthonormal basis in $L^{2,\circ}(\mathbb{R}^+)$.*

Proof. Since $\mathcal{M}^a$ is a symmetric Sturm-Liouville operator, all the eigenspaces are orthogonal to each other and one-dimensional, and they span the space $L^{2,\circ}(\mathbb{R}^+)$. Thus, it suffices to show that the functions $y_a^{(0)}, y_a^{(1)}, \dots$ are all the eigenfunctions of $\mathcal{M}^a$. Now, from (3.1.3) and (3.4.5) we easily derive the equivalence

$$\mathcal{M}^a x = \alpha x \qquad \Longleftrightarrow \qquad \mathcal{K}^{a-\alpha} x = \rho(a) x, \tag{3.4.7}$$

which is valid for every $a, \alpha \in \mathbb{R}$ and $x \in C^2(\mathbb{R}_0^+)$. From (3.4.6) and (3.4.7) we see that $(\alpha^{(l)}(a))_{l\in\mathbb{N}_0}$ is the sequence of all the eigenvalues of $\mathcal{M}^a$ with corresponding eigenfunctions $(y_a^{(l)})_{l\in\mathbb{N}_0}$, since (3.4.7) implies that for every eigenvalue α of $\mathcal{M}^a$, there is an $l \in \mathbb{N}_0$ such that $\rho^{(l)}(a-\alpha) = \rho(a)$. □

STEP 2 *For every $h, T \geq 0$, $l \in \mathbb{N}_0$ and $a \in \mathbb{R}$,*

$$\widehat{\mathbb{E}}_h^a\left(\frac{y_a^{(l)}}{x_a}(Y_T)\right) = e^{\alpha^{(l)}(a)T}\frac{y_a^{(l)}}{x_a}(h). \tag{3.4.8}$$

Proof. Use (3.3.7) and (3.3.14) to compute, for $f \in C^2(\mathbb{R}^+)$,

$$\left(\widetilde{G}^a\left(\frac{f}{x_a}\right)\right)(u) = \frac{f(u)}{u x_a(u)}\left(\frac{2uf''(u) + 2f'(u)}{f(u)} - \frac{2u x_a''(u) + 2x_a'(u)}{x_a(u)}\right). \tag{3.4.9}$$

Apply this for $f = y_a^{(l)}$, use (3.1.3) and the eigenvalue relation $\mathcal{K}^{a'} x_{a'}^{(l)} = \rho^{(l)}(a') x_{a'}^{(l)}$ for $(a', l) = (a, 0)$ and for $(a', l) = (a - \alpha^{(l)}(a), l)$ to obtain

$$\widetilde{G}^a\left(\frac{y_a^{(l)}}{x_a}\right) = \alpha^{(l)}(a)\frac{y_a^{(l)}}{x_a}. \tag{3.4.10}$$

Thus, $\widetilde{G}^a$ being the generator of the process $\{Y_t\}_{t\geq 0}$, the function $f(T) = \widehat{\mathbb{E}}_h^a\left(\frac{y_a^{(l)}}{x_a}(Y_T)\right)$ satisfies the differential equation $f' = \alpha^{(l)}(a) f$. Therefore $f(T) = e^{\alpha^{(l)}(a)T} f(0)$, which is our assertion. □

STEP 3 *Conclusion of the proof.*

Proof. According to Step 1, we may expand $g \in L^{2,\circ}(\mathbb{R}_0^+)$ as

$$g = \sum_{l=0}^{\infty} y_{a_T}^{(l)} \langle g, y_{a_T}^{(l)}\rangle_{L^2}^{\circ} = \frac{x_{a_T}}{\rho'(a_T)}\langle g, x_{a_T}\rangle_{L^2}^{\circ} + \sum_{l=1}^{\infty} y_{a_T}^{(l)}\langle g, y_{a_T}^{(l)}\rangle_{L^2}^{\circ} \qquad (T \geq 0). \tag{3.4.11}$$

Substitute this into (3.3.26) to obtain (recall (3.3.10) and (3.3.12))

$$\begin{aligned}
&\big|N^{f,g}_{T,a_T} - \tfrac{1}{\rho'(a)}\langle f, x_a\rangle_{L^2}\langle g, x_a\rangle^\circ_{L^2}\big| \\
&\le \big|\tfrac{1}{\rho'(a_T)}\langle f, x_{a_T}\rangle_{L^2}\langle g, x_{a_T}\rangle^\circ_{L^2} - \tfrac{1}{\rho'(a)}\langle f, x_a\rangle_{L^2}\langle g, x_a\rangle^\circ_{L^2}\big| \\
&\quad + \textstyle\sum_{l=1}^\infty \Big| \Big(\int_0^\infty dh\ f(h)x_{a_T}(h)\widehat{\mathbb{E}}^a_h\Big(\tfrac{y^{(l)}_{a_T}}{x_{a_T}}(Y_T)\Big)\Big)\langle g, y^{(l)}_{a_T}\rangle^\circ_{L^2}\Big|.
\end{aligned} \tag{3.4.12}$$

With the help of Step 2, the second term on the r.h.s. of (3.4.12) equals

$$\begin{aligned}
&\textstyle\sum_{l=1}^\infty e^{\alpha^{(l)}(a_T)T}\Big| \Big(\int_0^\infty dh\ f(h)x_{a_T}(h)\tfrac{y^{(l)}_{a_T}}{x_{a_T}}(h)\Big)\langle g, y^{(l)}_{a_T}\rangle^\circ_{L^2}\Big| \\
&\le e^{\alpha^{(1)}(a_T)T}\textstyle\sum_{l=0}^\infty \big|\langle \tfrac{f}{\mathrm{id}}, y^{(l)}_{a_T}\rangle^\circ_{L^2}\langle g, y^{(l)}_{a_T}\rangle^\circ_{L^2}\big| \\
&\le e^{\alpha^{(1)}(a_T)T}\sqrt{\textstyle\sum_{l=0}^\infty \Big(\langle \tfrac{f}{\mathrm{id}}, y^{(l)}_{a_T}\rangle^\circ_{L^2}\Big)^2}\sqrt{\textstyle\sum_{l=0}^\infty \Big(\langle g, y^{(l)}_{a_T}\rangle^\circ_{L^2}\Big)^2} \\
&= e^{\alpha^{(1)}(a_T)T}\|\tfrac{f}{\mathrm{id}}\|^\circ_{L^2}\|g\|^\circ_{L^2}.
\end{aligned} \tag{3.4.13}$$

This tends to zero as $T \to \infty$ since $\lim_{T\to\infty}\alpha^{(1)}(a_T) = \alpha^{(1)}(a) < 0$. The first term on the r.h.s. of (3.4.12) vanishes as $T \to \infty$ because of the continuity of $a \mapsto x_a \in L^2(\mathbb{R}^+)$ and $a \mapsto \rho'(a)$ (see Lemma 2.18). □

3.5 Integrability for the boundary pieces

This section contains the proof of Proposition 3.4. It turns out that the functions w_a (in (3.3.18)) and y_a (in (3.3.31)) have a nice representation in terms of standard one-dimensional Brownian motion, and that y_a is a transformation of the Airy function. This will be explored in Section 3.5.2. Section 3.5.1 contains some preparations.

3.5.1 Preparations: Lemma 3.5

Let $\mathrm{Ai} : \mathbb{R} \to \mathbb{R}$ denote the Airy function, i.e., the unique (modulo a constant multiple) solution of the Airy equation

$$x''(u) - ux(u) = 0 \qquad (u \in \mathbb{R}) \tag{3.5.1}$$

that is bounded on $\mathbb{R}^+_0$. Let $u_1 = \sup\{u \in \mathbb{R} \mid \mathrm{Ai}(u) = 0\}$ be its largest zero. From Abramowitz and Stegun (1970), Table 10.13 and p. 450, it is known that $u_1 = -2,3381\ldots$. For $a < -2^{\frac{1}{3}}u_1$, define $z_a : \mathbb{R}^+_0 \to \mathbb{R}^+$ by

$$z_a(u) = \frac{\mathrm{Ai}\Big(2^{-\frac{1}{3}}(u-a)\Big)}{\mathrm{Ai}\Big(-2^{-\frac{1}{3}}a\Big)} \qquad (u \ge 0). \tag{3.5.2}$$

In Lemma 3.7 in Section 3.5.2, z_a will turn out to be equal to y_a. Some of its properties are given in the following lemma.

Lemma 3.5 *For all $a < -2^{\frac{1}{3}}u_1$, the function z_a is real-analytic, strictly positive on $\mathbb{R}_0^+$ with $z_a(0) = 1$, and satisfies*

$$2z_a''(u) + (a - u)\, z_a(u) = 0 \qquad (u \geq 0). \tag{3.5.3}$$

Moreover,

$$\lim_{u \to \infty} u^{-\frac{3}{2}} \log z_a(u) < 0. \tag{3.5.4}$$

Proof. It is well known that Ai is analytic. From (3.5.2) and the definition of u_1 it is clear that $z_a(0) = 1$ and that $z_a(u) > 0$ for $u \geq 0$. Equation (3.5.3) follows easily from (3.5.1). The asymptotics in (3.5.4) follows from Abramowitz and Stegun (1970), 10.4.59. □

Finally, Theorem 5.1(i) below shows in particular that Lemma 3.5 can be used for $a = a^*$. □

3.5.2 Proof of Proposition 3.4

Let $\widehat{P}_h$ be the distribution of standard one-dimensional Brownian motion $\{B_t\}_{t\geq 0}$ conditioned on starting at h and let $\widehat{E}_h$ be the corresponding expectation. Define

$$T_u = \inf\{\, t \geq 0 : B_t = u\,\} \qquad (u \in \mathbb{R}). \tag{3.5.5}$$

Lemma 3.6 *For every $a \in \mathbb{R}$ and $h, t > 0$,*

$$\begin{aligned} w_a(h,t) &= e^{at}\widehat{E}_{\frac{h}{2}}\left(e^{-\int_0^t 2B_s\,ds} \,\middle|\, T_0 = t\right)\, \psi_h(t), \\ \psi_h(t) &= \frac{\widehat{P}_{\frac{h}{2}}(T_0 \in dt)}{dt} = \frac{h}{2\sqrt{2\pi t^3}}\, e^{-\frac{h^2}{8t}}. \end{aligned} \tag{3.5.6}$$

Consequently,

$$y_a(h) = \widehat{E}_{\frac{h}{2}}\left(e^{\int_0^{T_0}(a-2B_s)\,ds}\right) \in [0, \infty]. \tag{3.5.7}$$

Proof. Recall (3.2.11). According to Ethier and Kurtz (1986), Theorem 6.1.4, the process $\{Y_t^*\}_{t\geq 0} = \{X^*_{A^{*-1}(t)}\}_{t\geq 0}$ is a diffusion with generator (see (3.2.4))

$$\left(\widetilde{G}^* f\right)(u) = \frac{1}{u}\left(G^* f\right)(u) = 2f''(u) \qquad (f \in C_c^2(\mathbb{R}^+)). \tag{3.5.8}$$

In other words, the distribution of $\{Y_t^\star\}_{t\geq 0}$ under $\mathbb{P}_h^\star$ is equal to that of $\{B_{4t\wedge T_0}\}_{t\geq 0}$ under $\widehat{P}_h$, which in turn is equal to that of $\{2B_{t\wedge T_0}\}_{t\geq 0}$ under $\widehat{P}_{\frac{h}{2}}$. Thus, noting that $\frac{d}{dt}A^{\star-1}(t) = 1/X^\star_{A^{\star-1}(t)}$ and hence $\int_0^{A^{\star-1}(t)} X_v^{\star 2}\,dv = \int_0^t X^\star_{A^{\star-1}(s)}\,ds$, we have

$$\begin{aligned}\mathbb{E}_h^\star\left(e^{-\int_0^\infty X_v^{\star 2}\,dv}\,\Big|\,A^\star(\infty)=t\right) &= \mathbb{E}_h^\star\left(e^{-\int_0^{\xi_0} X_v^{\star 2}\,dv}\,\Big|\,A^\star(\xi_0)=t\right)\\ &= \mathbb{E}_h^\star\left(e^{-\int_0^{A^{\star-1}(t)} X_v^{\star 2}\,dv}\,\Big|\,A^{\star-1}(t)=\xi_0\right) = \widehat{E}_{\frac{h}{2}}\left(e^{-\int_0^t 2B_s\,ds}\,\Big|\,T_0=t\right),\end{aligned} \tag{3.5.9}$$

which proves the first formula in (3.5.6) (see (3.3.18)). In the same way, we see that ψ_h defined in (3.2.12) equals the Lebesgue density of T_0 under $\widehat{P}_{\frac{h}{2}}$, and its explicit shape is found in RY, p. 102. Finally, the representation (3.5.7) is a direct consequence of (3.3.31). □

Proof of Proposition 3.4(*i*)**.** In view of Lemma 3.5 and Theorem 5.1(*i*), the following lemma implies Proposition 3.4(*i*).

Lemma 3.7 *$z_a = y_a$ for all $a < -2^{\frac{1}{3}}u_1$.*

Proof. Since $y_a(0) = z_a(0) = 1$ and since z_a is bounded on $\mathbb{R}_0^+$, it suffices to show that y_a satisfies the same differential equation as z_a (see (3.5.3)). But this easily follows from the argument in the proof of KS, Theorem 4.6.4.3, picking (in the notation used there) $\alpha = a < -2^{\frac{1}{3}}u_1$, $k(u) = u$, $\gamma_l = 0$, $b = 0$, and $c = \infty$. □

□

Proof of Proposition 3.4(*ii*) **and** (*iii*)**.** Fix $p \in (1,2)$ and $q \in (2,\infty)$. In the following, we use c as a generic positive constant, possibly varying from line to line.

STEP 1 *$W_p^{(1)}$ is integrable at zero.*

Proof. Use (3.5.6) to estimate $w_{a^*}(h,t) \leq ct^{-\frac{3}{2}}he^{-\frac{h^2}{8t}}$ for any $h \geq 0$ and $t \in (0,1]$. Using the boundedness of $x_{a^*}^{2-p}$ on $\mathbb{R}^+$, this gives

$$\begin{aligned}W_p^{(1)}(t) &\leq c\left(\int_0^\infty h^{1-p}h^p t^{-\frac{3p}{2}} e^{-\frac{ph^2}{8t}}\,dh\right)^{1/p}\\ &= ct^{-\frac{3}{2}}\left(\int_0^\infty he^{-\frac{ph^2}{8t}}\,dh\right)^{1/p}\\ &= ct^{\frac{1}{p}-\frac{3}{2}},\end{aligned} \tag{3.5.10}$$

which is integrable at zero. □

STEP 2 *$W_q^{(2)}$ is integrable at zero.*

Proof. Use $h^{1+q}e^{-\frac{qh^2}{16t}} \leq ct^{\frac{1+q}{2}}$ for $t \in (0,1]$ and (as in Step 1) use (3.5.6) to estimate $w_{a^*}(h,t) \leq ct^{-\frac{3}{2}}he^{-\frac{h^2}{8t}}$ for any $h \geq 0$ and $t \in (0,1]$. This gives

$$\begin{aligned} W_q^{(2)}(t) &\leq ct^{-\frac{3}{2}}\Big(\int_0^\infty hx_{a^*}(h)^{2-q}h^q e^{-\frac{qh^2}{8t}}\,dh\Big)^{1/q} \\ &= ct^{-\frac{3}{2}}\Big(\int_0^\infty x_{a^*}(h)^{2-q}t^{\frac{1+q}{2}}e^{-\frac{qh^2}{16t}}\,dh\Big)^{1/q} \\ &\leq ct^{\frac{1}{2q}-1}\Big(\int_0^\infty x_{a^*}(h)^{2-q}e^{-\frac{qh^2}{16}}\,dh\Big)^{1/q}. \end{aligned} \tag{3.5.11}$$

The integral is finite for any $q > 2$ since $\lim_{h\to\infty} h^{-\frac{3}{2}} \log x_{a^*}(h)$ is finite (see Lemma 2.16). Thus, the r.h.s. of (3.5.11) is integrable in t at zero. □

STEP 3 $W_p^{(1)}$ *is integrable at infinity.*

Proof. Since $t \mapsto t^{-\frac{3}{2}}$ is a probability density on $[4,\infty)$, Jensen's inequality (and the boundedness of $x_{a^*}^{2-p}$ on $\mathbb{R}^+$) give

$$\begin{aligned} \int_4^\infty W_p^{(1)}(t)\,dt &\leq c\int_4^\infty \Big(\int_0^\infty h^{1-p}t^{\frac{3p}{2}}w_{a^*}(h,t)^p\,dh\Big)^{1/p} t^{-\frac{3}{2}}\,dt \\ &\leq c\Big(\int_0^\infty \int_0^\infty h^{1-p}t^{\frac{3}{2}(p-1)}w_{a^*}(h,t)^p\,dt\,dh\Big)^{1/p}. \end{aligned} \tag{3.5.12}$$

Use (3.5.6), Jensen's inequality for the conditioned expectation, and the Brownian scaling property to estimate

$$\begin{aligned} w_{a^*}(h,t)^p &\leq \psi_h(t)^{p-1}\psi_h(t)\,E_{\frac{h}{2}}\Big(e^{a^*pt-p\int_0^t 2B_s\,ds}\,\Big|\,T_0 = t\Big) \\ &\leq ch^{p-1}t^{\frac{3}{2}(1-p)}\psi_{hp^{\frac{1}{3}}}(tp^{\frac{2}{3}})\,E_{\frac{hp^{\frac{1}{3}}}{2}}\Big(e^{a^*p^{\frac{1}{3}}tp^{\frac{2}{3}}-\int_0^{tp^{\frac{2}{3}}}2B_s\,ds}\,\Big|\,T_0 = tp^{\frac{2}{3}}\Big) \\ &= ch^{p-1}t^{\frac{3}{2}(1-p)}w_{a^*p^{\frac{1}{3}}}(hp^{\frac{1}{3}},tp^{\frac{2}{3}}). \end{aligned} \tag{3.5.13}$$

Substitute this into (3.5.12) to get

$$\Big(\int_4^\infty W_p^{(1)}(t)\,dt\Big)^p \leq c\int_0^\infty z_{a^*p^{\frac{1}{3}}}(hp^{\frac{1}{3}})\,dh. \tag{3.5.14}$$

This is finite by (3.5.4) (see Theorem 5.1(i)). □

STEP 4 $W_q^{(2)}$ *is integrable at infinity if* $q \in (2,\infty)$ *is sufficiently close to 2.*

Proof. If we estimate in the same way as in (3.5.12) and in (3.5.13) but do not estimate $x_{a^*}(h)^{2-q}$, then we end up with

$$\left(\int_4^\infty W_q^{(2)}(t)\, dt\right)^q \leq c \int_0^\infty h^q x_{a^*}(h)^{2-q} z_{a^* q^{\frac{1}{3}}}(hq^{\frac{1}{3}})\, dh. \tag{3.5.15}$$

For q sufficiently close to 2, we have $a^* q^{\frac{1}{3}} < -2^{\frac{1}{3}} u_1$ (see Theorem 5.1(i)) and may apply (3.5.4). Finally, use that $\lim_{h\to\infty} h^{-\frac{3}{2}} \log x_{a^*}(h)$ is finite and Lemmas 3.5–3.7 to deduce that the r.h.s. of (3.5.15) is finite for q sufficiently close to 2. □

□

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Chapter 4

Central limit theorem for a weakly interacting random polymer

4.1 Introduction and main results

In this chapter we will be concerned with the end-to-end distance S_n under the measure

$$Q_n^{\beta_n} \text{ with } \beta_n \to 0 \text{ and } n^{\frac{3}{2}}\beta_n \to \infty \text{ as } n \to \infty. \tag{4.1.1}$$

This is a weak interaction limit where the strength of repellence decreases with the length of the path. Our main results are Theorems 4.1 a nd 4.2 in Sections 4.1.1 and 4.1.2. The regime in (4.1.1) will turn out to be mathematically interesting because the weak interaction limit is singular, in the sense that it cannot be obtained as a small perturbation of simple random walk. It is also physically interesting as a family of models describing a polymer with fixed n, β lying on some curve $n \mapsto \beta_n$ fitting the constraints.

Earlier work concerned the following cases:

(a) ('ballistic') $\beta_n \equiv \beta \in \mathbb{R}^+$: Greven and den Hollander (1993) and König (1996), Section 1.4 and Section 1.6,

(b) ('weakly ballistic') $\beta_n \equiv \beta \downarrow 0$: Chapter 2,

(c) ('diffusive') $\beta_n = \beta n^{-\frac{3}{2}}$ with $\beta \in \mathbb{R}^+$: Brydges and Slade (1995), Theorem 2.5 in Section 2.1.5.

Technically the regime in (4.1.1) is difficult because the polymer has intricate scaling properties.

The problem addressed in this paper was brought to our attention by David Brydges and Gordon Slade. They were interested in whether a law of large numbers (LLN) could be proved when $\beta_n = \beta n^{-p}$ ($\beta \in \mathbb{R}^+, p \in (0, \frac{3}{2})$) and in what way the result would be related to what was known for cases (b) and (c). This point will be clarified in Section 4.1.2, where we state a central limit theorem (CLT) for $|S_n|$ under the law $Q_n^{\beta_n}$ in (4.1.1).

4.1.1 Case $\beta_n \equiv \beta \downarrow 0$: Theorem 4.1

Recall that in Chapter 2 we proved that as $\beta \downarrow 0$, uniformly in a on compacts in $\mathbb{R}$,

$$\beta^{-\frac{1}{3}}\left[\lambda(a\beta^{\frac{2}{3}},\beta)-1\right] \to \rho(a), \tag{4.1.2}$$

$$\beta^{-\frac{1}{3}}\frac{\partial}{\partial a}\lambda(a\beta^{\frac{2}{3}},\beta) \to \rho'(a). \tag{4.1.3}$$

We will need that also the second derivative of λ has a scaling as in (4.1.2-4.1.3). This is formulated in the next theorem, which identifies the behavior for small β of the quantity $\sigma^*(\beta)$ introduced in (1.6.2).

Theorem 4.1 *As $\beta \downarrow 0$, uniformly in a on compacts in $\mathbb{R}$,*

$$\beta^{-\frac{1}{3}}\frac{\partial^2}{\partial a^2}\lambda(a\beta^{\frac{2}{3}},\beta) \to \rho''(a). \tag{4.1.4}$$

Consequently, as $\beta \downarrow 0$,

$$\sigma^*(\beta) \to c^*, \tag{4.1.5}$$

with c^* as in (3.1.4)

Note that (1.6.2), (4.1.4) together with Theorem 2.2 explain (4.1.5) and (3.1.4).

We will see that (4.1.2-4.1.3) and Theorem 4.1 are the key technical results underlying our central limit theorem, Theorem 4.2 in Section 4.1.2. The proof of Theorem 4.1 is given in Section 4.8. The proof uses an extension of (4.1.2), Proposition 4.3 in Section 4.4.2, which states that all eigenvalues of $A_{r,\beta}$ have a scaling as in (4.1.2). The proof of Proposition 4.3 is in Section 4.4.9.

Rigorous bounds on a^*, b^*, c^* will be given in Chapter 5.

4.1.2 Case $\beta_n \to 0$ and $n^{\frac{3}{2}}\beta_n \to \infty$: Theorem 4.2

In terms of the objects defined in Section 1.4, we can state the main result of this chapter. Recall (1.4.17) and (1.6.2). Define

$$\theta_n = \theta^*(\beta_n) \tag{4.1.6}$$

$$\sigma_n = \sigma^*(\beta_n). \tag{4.1.7}$$

Theorem 4.2 *If $\beta_n \to 0$ and $\beta_n n^{\frac{3}{2}} \to \infty$, then*

$$\lim_{n\to\infty} Q_n^{\beta_n}\Big(\frac{|S_n|-\theta_n n}{\sigma_n\sqrt{n}} \le C\Big) = \mathcal{N}((-\infty,C]) \text{ for every } C \in \mathbb{R}. \tag{4.1.8}$$

The proof of Theorem 4.2 will be given in Sections 4.2-4.9. The essence of Theorem 4.2 is that the CLT holds with $\theta_n = \theta^*(\beta_n)$ and $\sigma_n = \sigma^*(\beta_n)$, i.e., the interaction parameter β_n is simply substituted into the quantities $\theta^*(\beta)$ and $\sigma^*(\beta)$ appearing in Theorem 1.1 and 1.4 for $\beta_n \equiv \beta$. Thus the weak interaction behavior is *uniform* in the regime under consideration.

4.1.3 Discussion

Using Theorem 2.2 and (4.1.5), we see that the key quantities in Theorem 4.2 behave as

$$\theta_n n \sim b^* \beta_n^{\frac{1}{3}} n \tag{4.1.9}$$
$$\sigma_n \sqrt{n} \sim c^* \sqrt{n}. \tag{4.1.10}$$

If $\beta_n = \beta n^{-p}$ then $\theta_n n \sim b^* \beta^{\frac{1}{3}} n^{1-\frac{p}{3}}$. The exponent $1 - \frac{p}{3}$ is seen to be a linear interpolation between the boundary cases 1 ($p = 0$) and $\frac{1}{2}$ ($p = \frac{3}{2}$), corresponding to ballistic, respectively, diffusive behavior (compare with Theorems 1.1 and 2.5). This exponent was recently conjectured by Brydges and Slade (1995).

A heuristic explanation of the leading order behavior is the following. If the mean of S_n conditioned on $S_n > 0$ is α_n and if the local times below the starting point 0 and above the endpoint S_n are negligible (as in (1.4.10)), then the local times in between 0 and α_n are of order $n\alpha_n^{-1}$. (Note that the local times have to sum up to n.) Hence, the exponential in (1.2.1) is of order $\exp\{-\beta_n \alpha_n^{-1} n^2\}$. The probability for S_n to be of order α_n is of order $\exp\{-\alpha_n^2 n^{-1}\}$. The two exponentials have to balance each other, so α_n is of order $\beta_n^{\frac{1}{3}} n$.

The fact that the fluctuations of S_n are asymptotically Gaussian and are of order $\sqrt{n}$ means that the CLT is robust under the weak interaction limit, as was perhaps to be expected. However, the fact that the standard deviation $\sigma_n \sqrt{n} \sim c^* \sqrt{n}$ is asymptotically *independent* of the parameter β_n is rather striking. This has to do with scaling properties of Brownian motion. Indeed, the constants b^*, c^* also appear in the CLT for the Edwards model proved in Chapter 3. Namely, the speed $\hat{\theta}^*(\beta)$ and the spread $\hat{\sigma}^*(\beta)$ of the T-polymer in (1.3.1) in the limit as $T \to \infty$ are (recall Theorem 3.1)

$$\begin{aligned} \hat{\theta}^*(\beta) &= b^* \beta^{\frac{1}{3}} \\ \hat{\sigma}^*(\beta) &= c^* \qquad (\beta \in \mathbb{R}^+). \end{aligned} \tag{4.1.11}$$

Thus the weak interaction limit of the Domb-Joyce model connects up nicely with the Edwards model.

Despite this connection, the proof of our CLT is rather involved. In fact, we will need to develop the full scaling picture of the polymer measure, which is difficult because of the global nature of the path interaction. Unfortunately the weak interaction Domb-Joyce model and the Edwards model cannot be coupled nicely on one probability space. Therefore we will be able to benefit very little from what we know for the Brownian case.

Our proof uses a higher order large deviation analysis of $Z_n^{\beta_n}$ (see (1.2.2)) as $n \to \infty$, namely up to and including $\mathcal{O}(1)$. It turns out that the $\mathcal{O}(1)$-term is structurally *different* from the one appearing in the analysis of $\widehat{Z}_T^{\beta}$ as $T \to \infty$ (see (1.3.1)) given in Chapter 3. This is another indication that the models are hard to compare directly. For a comparison of the $\mathcal{O}(1)$-terms appearing in Chapters 3 and 4, see Theorem 5.1(*iv*).

We have $\theta^*(0) = 0, \sigma^*(0) = 1$ since $Q_n^0 = P$ for all n. Hence the speed $\theta^*(\beta)$ is continuous at $\beta = 0$ by Theorem 2.2. However, the spread $\sigma^*(\beta)$ is not continuous at $\beta = 0$, because of (4.1.5) and Theorem 5.1(*iii*). This once more shows that the weak interaction limit is singular.

Finally, we have no doubt that $\{(S_{\lfloor nt \rfloor} - \theta_n nt)/\sigma_n\sqrt{n}\}_{0\le t\le 1}$ under $Q_n^{\beta_n}(\,\cdot\,|S_n > 0)$ converges to Brownian motion. The convergence of the finite-dimensional distributions should run along the lines of the present paper, but will be much more involved. The tightness should be easy and should run along the lines of Madras and Slade (1993) Proof of Lemma 6.6.3.

4.1.4 Outline of the proof

Section 4.2 gives a Markovian description of the local times of simple random walk (Knight's theorem). We use this description to write the moment generating function of S_n under $Q_n^{\beta_n}(\,\cdot\,|S_n > 0)$ as the expectation of an exponential functional of three Markov chains. These Markov chains correspond to the local times in the areas $(-\infty, 0)$, $[0, S_n]$ and (S_n, ∞).

In Section 4.3 we absorb the exponential functional into the transition kernels of the Markov chains and rewrite the moment generating function as a correlation function involving three scaled continuous-time processes.

In Sections 4.5-4.7 we show that, in the limit as $n \to \infty$, the correlation function factorizes into a product of three parts. The part corresponding to $[0, S_n]$ gives the CLT in Theorem 4.2, the parts corresponding to $(-\infty, 0)$ and (S_n, ∞) give rise to constants that drop out in the normalization.

In Section 4.4 we formulate an important tool used in Sections 4.5-4.7: a scaling limit assertion for the spectrum of the transition kernels introduced in Section 4.3. The limit is the spectrum of the operator $\mathcal{L}^a$ defined in (3.1.3), which determines the constants in our CLT. The proof of the limit assertion appears in Section 4.9, the proof of Theorem 4.1 in Section 4.8.

4.2 Reformulation of the problem

In Section 4.2.1 we formulate our main proposition, Proposition 4.1 below, implying Theorem 4.2. In Sections 4.2.2 and 4.2.3 we apply Knight's description of the local times of simple random walk to get a convenient representation, Lemma 4.1 below, of the key quantity appearing in Proposition 4.1.

4.2.1 The main proposition: Proposition 4.1

Define the n-step local times

$$\ell_n(x) = \#\{0 \le i \le n : S_i = x\} \qquad (n \in \mathbb{N}_0, x \in \mathbb{Z}). \tag{4.2.1}$$

Then

$$\sum_{\substack{i,j=0\\ i\ne j}}^{n} 1_{\{S_i=S_j\}} = \sum_{x\in\mathbb{Z}} \ell_n^2(x) - (n+1), \tag{4.2.2}$$

and so (1.2.1) can be rewritten as

$$\frac{dQ_n^\beta}{dP} = \frac{1}{\widetilde{Z}_n^\beta} \exp\left[-\beta \sum_{x\in\mathbb{Z}} \ell_n^2(x)\right] \tag{4.2.3}$$

with $\widetilde{Z}_n^\beta = Z_n^\beta \exp[-\beta(n+1)]$.

Next, in addition to $\theta_n = \theta^*(\beta_n), \sigma_n = \sigma^*(\beta_n)$ introduced in (4.1.6-4.1.7), define (see (1.4.15))

$$r_n = r^*(\beta_n). \tag{4.2.4}$$

For future reference, we recall here the limiting behavior of r_n, θ_n and σ_n (see Theorem 2.2 and (4.1.5)):

$$\beta_n^{-\frac{2}{3}} r_n \to a^*, \qquad \beta_n^{-\frac{1}{3}} \theta_n \to b^*, \qquad \sigma_n \to c^*. \tag{4.2.5}$$

The rest of this paper is devoted to the proof of the following main proposition.

Proposition 4.1 *There is an $L \in \mathbb{R}^+$ such that for every $\mu \in \mathbb{R}$,*

$$\lim_{n\to\infty} e^{r_n n} E\left(e^{-\beta_n \sum_x \ell_n^2(x)} e^{\mu \frac{S_n - \theta_n n}{\sigma_n \sqrt{n}}} 1_{\{S_n>0\}}\right) = L e^{\frac{\mu^2}{2}}. \tag{4.2.6}$$

Proposition 4.1 implies that under the law $Q_n^{\beta_n}(\,\cdot\,|\,S_n > 0)$ the moment generating function of $(S_n - \theta_n n)/\sigma_n\sqrt{n}$ converges pointwise to the one of $\mathcal{N}$ as $n \to \infty$ (divide the l.h.s. of (4.2.6) by the same expression for $\mu = 0$ and use (4.2.3)). By symmetry, it therefore implies the central limit theorem as stated in Theorem 4.2.[1]

4.2.2 Knight's description of the local times

This subsection provides an important tool in the proof of Proposition 4.1, namely, a family of Markov chains that describes the local times of simple random walk (recall (4.2.1)) at certain stopping times, viewed as a process in the spatial parameter. The following material is taken from Knight (1963) and is the discrete space-time analogue of the Ray-Knight theorems for local times of Brownian motion (recall Section 3.2.2).

Fix $d \in \mathbb{N}_0$. Define the successive times at which the walker makes steps $d \to d+1$ and $d+1 \to d$, by putting $T_{0,d}^\uparrow = T_{0,d}^\downarrow = 0$ and for $k \in \mathbb{N}$,

$$\begin{aligned} T_{k,d}^\uparrow &= \inf\{i > T_{k-1,d}^\uparrow : S_{i-1} = d, S_i = d+1\}, \\ T_{k,d}^\downarrow &= \inf\{i > T_{k-1,d}^\downarrow : S_{i-1} = d+1, S_i = d\}. \end{aligned} \tag{4.2.7}$$

By discarding null sets we may and will assume that all these stopping times are finite (simple random walk is recurrent). Note that $T_{k,d}^\uparrow < T_{k,d}^\downarrow < T_{k+1,d}^\uparrow$. Recall the definition of

[1]In the original paper (van der Hofstad, den Hollander and König (1997b)) there was an erronous $\beta^{-\frac{1}{3}}$ in the statement of the main proposition Proposition 1.

the stochastic $\mathbb{N} \times \mathbb{N}$ matrix P in (1.4.14), and introduce a stochastic $\mathbb{N}_0 \times \mathbb{N}_0$ matrix $P^\star$ by putting

$$P^\star(i,j) = 1_{\{i \neq 0\}} P(i, j+1) + 1_{\{i=0\}} 1_{\{j=0\}} \qquad (i, j \in \mathbb{N}_0). \tag{4.2.8}$$

Let

$$\{m(x)\}_{x \in \mathbb{N}_0} \qquad \text{and} \qquad \{m^\star(x)\}_{x \in \mathbb{N}_0} \tag{4.2.9}$$

be the Markov chains with transition kernel P and $P^\star$ respectively. Later we will need that both $\{m(x)\}_{x \in \mathbb{N}_0}$ and $\{m^\star(x)\}_{x \in \mathbb{N}_0}$ are critical branching processes with a geometric offspring distribution with parameter $\frac{1}{2}$, where $\{m(x)\}_{x \in \mathbb{N}_0}$ has one immigrant per time unit and $\{m^\star(x)\}_{x \in \mathbb{N}_0}$ has none. The point 0 is therefore absorbing for $\{m^\star(x)\}_{x \in \mathbb{N}_0}$.

In terms of these Markov chains, we can describe the distribution of the local times of simple random walk at the stopping times $T^\uparrow_{k,d}$, respectively, $T^\downarrow_{k,d}$ as follows. (Here $\stackrel{\mathcal{L}}{=}$ means equality in law.)

Knight's Theorem *Fix $k, d \in \mathbb{N}$. Let $\{m(x)\}_{x \in \mathbb{N}_0}$ start at $m(0) = k$. Let $\{m_1^\star(x)\}_{x \in \mathbb{N}_0}$ and $\{m_2^\star(x)\}_{x \in \mathbb{N}_0}$ be two independent copies of $\{m^\star(x)\}_{x \in \mathbb{N}_0}$ starting at $m_1^\star(0) = m(0)$ and $m_2^\star(0) = m(d)$* respectively. *Then*

$$\begin{aligned}
\left\{\ell_{T^\uparrow_{k,d}}(d+1-x)\right\}_{x=1,\dots,d} &\stackrel{\mathcal{L}}{=} \{m(x) + m(x-1) - 1\}_{x=1,\dots,d}, \\
\left\{\ell_{T^\uparrow_{k,d}}(d+x)\right\}_{x \in \mathbb{N}} &\stackrel{\mathcal{L}}{=} \{m_1^\star(x) + m_1^\star(x-1)\}_{x \in \mathbb{N}}, \\
\left\{\ell_{T^\uparrow_{k,d}}(1-x)\right\}_{x \in \mathbb{N}} &\stackrel{\mathcal{L}}{=} \{m_2^\star(x) + m_2^\star(x-1)\}_{x \in \mathbb{N}}.
\end{aligned} \tag{4.2.10}$$

The three processes in the l.h.s. of (4.2.10) are conditionally independent given $m(0)$ and $m(d)$. Furthermore,

$$\ell_{T^\downarrow_{k,d}}(x) = \begin{cases} \ell_{T^\uparrow_{k,d}}(x) + 1_{\{x=d\}} & \textit{if } x \leq d, \\ \ell_{T^\uparrow_{k+1,d}}(x) - 1_{\{x=d+1\}} & \textit{otherwise.} \end{cases} \tag{4.2.11}$$

Proof. Fix $k, d \in \mathbb{N}$. Define the number of steps $x \to x+1$ until time $T^\uparrow_{k,d}$ by

$$m_{k,d}(x) = \#\{0 < i \leq T^\uparrow_{k,d} : S_{i-1} = x, S_i = x+1\} \qquad (x \in \mathbb{Z}). \tag{4.2.12}$$

From Theorem 1.1 and Corollary 1.1 in Knight (1963) it follows that the random processes $\{m_{k,d}(d+x)\}_{x \in \mathbb{N}_0}$ and $\{m_{k,d}(d-x)\}_{x \in \mathbb{N}_0}$ are independent Markov chains, both starting at k. Furthermore, $\{m_{k,d}(d-x)\}_{x \in \{0,\dots,d\}}$ is homogeneous and $\mathbb{N}$-valued with transition kernel P, while $\{m_{k,d}(d+x)\}_{x \in \mathbb{N}_0}$ and $\{m_{k,d}(-x)\}_{x \in \mathbb{N}_0}$ are homogeneous and $\mathbb{N}_0$-valued both with transition kernel $P^\star$.

Use the relation

$$\#\{0 < i \le T^{\dagger}_{k,d} : S_{i-1} = x, S_i = x-1\}$$
$$= \begin{cases} m_{k,d}(x-1) - 1 & \text{if } x \in \{1, \dots, d\} \\ m_{k,d}(x-1) & \text{otherwise} \end{cases} \tag{4.2.13}$$

and remember (4.2.1) to get

$$\ell_{T^{\dagger}_{k,d}}(x) = \begin{cases} m_{k,d}(x) + m_{k,d}(x-1) - 1 & \text{if } x \in \{1, \dots, d\}, \\ m_{k,d}(x) + m_{k,d}(x-1) & \text{otherwise.} \end{cases} \tag{4.2.14}$$

Hence (4.2.10) and the conditional independence assertion follow from the previous remarks. The reader easily verifies (4.2.11). □

In the sequel $\mathbb{P}_k$ and $\mathbb{P}^\star_k$ will denote the laws of the two Markov chains in (4.2.9) starting in $k \in \mathbb{N}$ and $k \in \mathbb{N}_0$ respectively. We write $\mathbb{E}_k$ and $\mathbb{E}^\star_k$ for expectation with respect to $\mathbb{P}_k$ and $\mathbb{P}^\star_k$ respectively.

4.2.3 The distribution of $(\{\ell_n(x)\}_{x\in\mathbb{Z}}, S_n)$: Lemma 4.1

The description of the local times given in Knight's theorem is very powerful, but has the disadvantage that the local times are observed at certain stopping times. For the description of the polymer we need to go back to the fixed time n. One of the problems we consequently have to deal with is the global restriction $\sum_{x\in\mathbb{Z}} \ell_n(x) = n+1$.

Fix d, $n \in \mathbb{N}$. In this subsection we derive a representation for the expression

$$E(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_n^2(x)} 1_{\{S_n = d\}}) \tag{4.2.15}$$

in terms of the Markov chains introduced in the preceding subsection. The idea is to sum over the number of steps $1 \to 0$, $d+1 \to d$ (respectively $d \to d+1$), and over the amount of time the walker spends in the three areas $-\mathbb{N}_0$, $\{1, \dots, d\}$ and $\{d+1, d+2, \dots\}$ until time n.

Define the functionals

$$U_d = \sum_{x=1}^{d} [m(x) + m(x-1) - 1], \tag{4.2.16}$$

$$V_d = \sum_{x=1}^{d} [m(x) + m(x-1) - 1]^2, \tag{4.2.17}$$

$$U^\star = \sum_{x=1}^{\infty} [m^\star(x) + m^\star(x-1)], \tag{4.2.18}$$

$$V^\star = \sum_{x=1}^{\infty} [m^\star(x) + m^\star(x-1)]^2. \tag{4.2.19}$$

In terms of these new objects we may write:

Lemma 4.1 *For all n, $d \in \mathbb{N}$,*

$$E\Big(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_n^2(x)} 1_{\{S_n=d+1,S_{n-1}=d\}}\Big)$$
$$= \sum_{k_1,n_1\in\mathbb{N}} \sum_{k_2,n_2\in\mathbb{N}} \prod_{i=1}^{2} \mathbb{E}^{\star}_{k_i}\Big(e^{-\beta_n V^{\star}} 1_{\{U^{\star}=n_i\}}\Big) \tag{4.2.20}$$
$$\times \mathbb{E}_{k_1}\Big(e^{-\beta_n V_d} 1_{\{U_d=n-n_1-n_2+1\}} 1_{\{m(d)=k_2\}}\Big)$$

and

$$E\Big(e^{-\beta_n \sum_{x\in\mathbb{Z}} \ell_n^2(x)} 1_{\{S_n=d,S_{n-1}=d+1\}}\Big)$$
$$= \sum_{k_1\in\mathbb{N}\setminus\{1\},n_1\in\mathbb{N}_0} \sum_{k_2,n_2\in\mathbb{N}} \prod_{i=1}^{2} \mathbb{E}^{\star}_{k_i}\Big(e^{-\beta_n [V^{\star}-\delta_i]} 1_{\{U^{\star}=n_i\}}\Big) \tag{4.2.21}$$
$$\times \mathbb{E}_{k_1-1}\Big(e^{-\beta_n [V_d+\delta_3]} 1_{\{U_d=n-n_1-n_2+1\}} 1_{\{m(d)=k_2\}}\Big),$$

with

$$\delta_1 = 2m^{\star}(1), \qquad \delta_2 = 0, \qquad \delta_3 = 2m(1). \tag{4.2.22}$$

Proof. Observe that $\sum_{x\in\mathbb{Z}} \ell_t(x) = t+1$ for any (random or fixed) $t \in \mathbb{N}_0$. Split the class of paths under the indicator in the l.h.s. of (4.2.20) according to the amount of time the walker spends in the three areas $-\mathbb{N}_0$, $\{1,\dots,d\}$ and $\{d+1,d+2,\dots\}$ and to the number of steps $0 \to 1$, $d \to d+1$ until time n :

$$\{S_n = d+1, S_{n-1} = d\} = \bigcup_{k\in\mathbb{N}} \Big\{T^{\uparrow}_{k,d} = n\Big\} = \bigcup_{k_1,k_2,n_1,n_2\in\mathbb{N}} \Bigg\{\sum_{x=1}^{\infty} \ell_{T^{\uparrow}_{k_1,d}}(d+x) = n_1,$$
$$\sum_{x=1}^{d} \ell_{T^{\uparrow}_{k_1,d}}(d+1-x) = n - n_1 - n_2 + 1, \sum_{x=1}^{\infty} \ell_{T^{\uparrow}_{k_1,d}}(1-x) = n_2, m_{k_1,d}(0) = k_2\Bigg\}. \tag{4.2.23}$$

Furthermore, write the exponent in the l.h.s. of (4.2.20) as

$$\sum_{x\in\mathbb{Z}} \ell_n^2(x) = \sum_{x=1}^{\infty} \ell_n^2(d+x) + \sum_{x=1}^{d} \ell_n^2(d+1-x) + \sum_{x=1}^{\infty} \ell_n^2(1-x). \tag{4.2.24}$$

Combine (4.2.23) and (4.2.24), use Knight's theorem and substitute (4.2.16–4.2.19) to arrive at (4.2.20).

In order to prove (4.2.21), split

$$\{S_n = d, S_{n-1} = d+1\} = \bigcup_{k\in\mathbb{N}} \{T^{\downarrow}_{k,d} = n\} = \bigcup_{k_1,k_2,n_1,n_2\in\mathbb{N}} \Bigg\{ \sum_{x=1}^{\infty} \ell_{T^{\downarrow}_{k,d}}(d+x) = n_1,$$
$$\sum_{x=1}^{d} \ell_{T^{\downarrow}_{k_1,d}}(d+x-1) = n - n_1 - n_2 + 1, \sum_{x=1}^{\infty} \ell_{T^{\downarrow}_{k_1,d}}(1-x) = n_2, m_{k_1,d}(0) = k_2 \Bigg\}. \tag{4.2.25}$$

Now substitute (4.2.11) and proceed analogously. Along the way, use that $\{\ell_{T^{\uparrow}_{k_1+1,d}}(d+x)\}_{x\in\mathbb{N}}$ and $\{\ell_{T^{\uparrow}_{k_1,d}}(d-x)\}_{x\in\mathbb{N}_0}$ are conditionally independent given $m_{k_1,d}(0)$, and shift the sums over k_1 and n_1 by one. □

In the proof of Proposition 4.1 we will focus on the contribution coming from the r.h.s. of (4.2.20). It will be argued at the end of Section 4.3.6 that (4.2.21) behaves in the same manner as (4.2.20) as $n \to \infty$, i.e., the small perturbations are harmless.

The role of Lemma 4.1 is that we have rewritten the key quantity of Proposition 4.1 in terms of expectations of exponential functionals of the two Markov chains defined in (4.2.9). We can henceforth forget about the underlying random walk. It is important that in Lemma 4.1 we have *products* of expectations.

4.3 Structure of the proof of Proposition 4.1

In this section we explain the main steps in the proof of Proposition 4.1. Our approach is a variation on the method used in Chapter 3. In Sections 4.3.1 and 4.3.2 we introduce transformed and time-changed Markov chains that are specially adapted to our problem. In Sections 4.3.3 and 4.3.4 we introduce several quantities that are needed to rewrite Lemma 4.1 in a more appropriate form, to be found in Lemmas 4.2 and 4.3 below. In Section 4.3.5 this leads to a key proposition, Proposition 4.2 below, that is the technical core of the argument. In Section 4.3.6 we finish the proof of Proposition 4.1 subject to Proposition 4.2. The proof of Proposition 4.2 follows in Sections 4.5-4.9.

4.3.1 A transformed Markov chain

In this subsection we define a transformation of the Markov chain $\{m(x)\}_{x\in\mathbb{N}_0}$ introduced in Section 4.2.2. The goal of this transformation is to absorb the random variable $e^{-\beta_n V_d}$ (see (4.2.17)) into the new transition probabilities.

Recall (1.4.13-1.4.15) and fix $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. As was pointed out in Section 1.4, the matrix $A_{r,\beta}$ has a unique largest eigenvalue $\lambda(r,\beta)$. We will denote the associated positive $l^2(\mathbb{N})$-normalized eigenvector by $\tau_{r,\beta}$. Consequently,

$$P_{r,\beta}(i,j) = \frac{A_{r,\beta}(i,j)}{\lambda(r,\beta)} \frac{\tau_{r,\beta}(j)}{\tau_{r,\beta}(i)} \qquad (i,j\in\mathbb{N}) \tag{4.3.1}$$

defines a stochastic matrix $P_{r,\beta}$. We will write $\mathbb{P}_k^{r,\beta}$ to denote the law of the Markov chain $\{m(x)\}_{x\in\mathbb{N}_0}$ starting at $k \in \mathbb{N}$ and having $P_{r,\beta}$ as its transition kernel. We write $\mathbb{E}_k^{r,\beta}$ for the corresponding expectation. Note that this chain is positive recurrent with invariant distribution $\{\tau_{r,\beta}^2(i)\}_{i\in\mathbb{N}}$. We write $\mathbb{P}^{r,\beta}, \mathbb{E}^{r,\beta}$ when the chain starts in its invariant distribution.

4.3.2 A time-changed Markov chain

Since it will turn out that the transformed Markov chain $\{m(x)\}_{x\in\mathbb{N}_0}$ needs to be evaluated at the random times at which the additive functional $\{U_d\}_{d\in\mathbb{N}}$ in (4.2.16) exceeds certain values, we must introduce some more notation. For $l \in \mathbb{N}$ define

$$T_l = \inf\{\, d \in \mathbb{N}_0 : U_d \geq l \,\} \tag{4.3.2}$$

and

$$X_l = U_{T_l} - l, \qquad Y_l = m(T_l), \qquad Z_l = m(T_l - 1). \tag{4.3.3}$$

The triple

$$\Gamma_l = (X_l, Y_l, Z_l) \tag{4.3.4}$$

is a random member of the set

$$\Sigma = \{\, (i,j,k) \in \mathbb{N}_0 \times \mathbb{N}^2 : i \leq j + k - 2 \,\}. \tag{4.3.5}$$

Fix $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. For any $k \in \mathbb{N}$, under the law $\mathbb{P}_k^{r,\beta}$ the process $\{\Gamma_l\}_{l\in\mathbb{N}_0}$ is a Markov renewal process with transition kernel $Q_{r,\beta}$ on Σ given by

$$\begin{aligned} &Q_{r,\beta}\left((i_1,j_1,k_1),(i_2,j_2,k_2)\right) \\ &\quad = 1_{\{i_1=0, i_2=j_2+k_2-2, k_2=j_1\}} P_{r,\beta}(j_1,j_2) + 1_{\{i_2=i_1-1, j_2=j_1, k_2=k_1\}} \end{aligned} \tag{4.3.6}$$

and starting at $\Gamma_0 = (0,k,k)$. It is easily checked that the probability distribution $\nu_{r,\beta}$ on Σ defined by

$$\nu_{r,\beta}(i,j,k) = \tau_{r,\beta}(j) \frac{A_{r,\beta}(j,k)}{\partial_r \lambda(r,\beta)} \tau_{r,\beta}(k) \tag{4.3.7}$$

is the associated invariant distribution (∂_r denotes the partial derivative with respect to r).[2]

We write $\widetilde{\mathbb{P}}^{r,\beta}$ and $\widetilde{\mathbb{E}}^{r,\beta}$ to denote probability and expectation with respect to the Markov chain $\{\Gamma_l\}_{l\in\mathbb{N}_0}$ starting in its invariant distribution $\nu_{r,\beta}$.

[2] To see that $\nu_{r,\beta}$ is normalized, differentiate the relation $\lambda(r,\beta) = \langle \tau_{r,\beta}, A_{r,\beta}\tau_{r,\beta}\rangle_{l^2}$ with respect to r and use that $\partial_r A_{r,\beta}(j,k) = (j+k-1)A_{r,\beta}(j,k)$, $A_{r,\beta}\tau_{r,\beta} = \lambda(r,\beta)\tau_{r,\beta}$ and $\partial_r\langle \tau_{r,\beta}, \tau_{r,\beta}\rangle_{l^2} = 0$.

4.3.3 Unscaled representation: Lemma 4.2

We are going to reformulate the r.h.s. of (4.2.20) in terms of $\{\Gamma_l\}_{l\in\mathbb{N}_0}$, since this is the natural object for our analysis. First we need some more notation. For $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, define the function $w_{r,\beta} : \mathbb{N}_0^2 \to \mathbb{R}_0^+$ by (see (4.2.18-4.2.19))

$$w_{r,\beta}(k,l) = \mathbb{E}_k^\star\left(e^{-\beta V^\star + rU^\star} 1_{\{U^\star = l\}}\right) = e^{rl} w_{0,\beta}(k,l) \tag{4.3.8}$$

and the functions $f_{r,\beta}^+$ and $f_{r,\beta}^- : \Sigma \times \mathbb{N}_0 \to \mathbb{R}^+$ by

$$f_{r,\beta}^\pm((i,j,k);l) = \frac{w_{r,\beta}(j, l\pm i)}{(j+k-1)\tau_{r,\beta}(j)}. \tag{4.3.9}$$

Our reformulation of the l.h.s. of (4.2.6) in Proposition 4.1 (up to some factors and an indicator) now reads as follows.

Lemma 4.2 *For $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$\begin{aligned} &e^{(n+1)r_n^*} E\left(e^{-\beta_n \sum_x \ell_n^2(x)} e^{\mu \frac{S_n}{\sigma_n \sqrt{n}}} 1_{\{0 \le S_{n-1} < S_n\}}\right) \\ &= \frac{1}{\theta_n^*} \sum_{n_1,n_2 \in \mathbb{Z}} \widetilde{\mathbb{E}}^{r_n^*,\beta_n}\left(f_{r_n^*,\beta_n}^+(\Gamma_0; n_1) 1_{\{X_0 \le n-n_1-n_2+1\}} f_{r_n^*,\beta_n}^-(\Gamma_{n-n_1-n_2+1}; n_2)\right), \end{aligned} \tag{4.3.10}$$

where $r_n^* = r_n^*(\mu)$ and $\theta_n^* = \theta_n^*(\mu)$ are given by

$$\lambda(r_n^*, \beta_n) = e^{-\frac{\mu}{\sigma_n\sqrt{n}}}, \tag{4.3.11}$$

$$\theta_n^* = \frac{\lambda(r_n^*, \beta_n)}{\partial_r \lambda(r_n^*, \beta_n)}. \tag{4.3.12}$$

Proof. Begin by observing that for every $k_1, k_2, n_1, n_2, d \in \mathbb{N}, r \in \mathbb{R}$ and r_n^* as in (4.3.11),

$$\begin{aligned} &e^{(n+1)r}[\lambda(r_n^*, \beta_n)]^d \mathbb{E}_{k_1}\left(e^{-\beta_n V_d} e^{\mu \frac{d}{\sigma_n\sqrt{n}}} 1_{\{U_d = n-n_1-n_2+1\}} 1_{\{m(d)=k_2\}}\right) \\ &= e^{(n_1+n_2)r}\,[\lambda(r,\beta_n)]^d\, \mathbb{P}_{k_1}^{r,\beta_n}\left(U_d = n-n_1-n_2+1, m(d) = k_2\right) \frac{\tau_{r,\beta_n}(k_1)}{\tau_{r,\beta_n}(k_2)}. \end{aligned} \tag{4.3.13}$$

This identity is a straightforward consequence of (1.4.13-1.4.14), (4.2.16-4.2.17), (4.3.1) and (4.3.11). Insert (4.3.13) for $r = r_n^*$ into (4.2.20), use (1.4.15) and (4.3.8), and write the abbreviations $w_n = w_{r_n^*,\beta_n}$, $\tau_n = \tau_{r_n^*,\beta_n}$, $\mathbb{P}_k^n = \mathbb{P}_k^{r_n^*,\beta_n}$ and $\mathbb{E}_k^n = \mathbb{E}_k^{r_n^*,\beta_n}$, to obtain

$$\begin{aligned}
&\text{l.h.s. of (4.3.10)}\\
&= \sum_{k_1,k_2\in\mathbb{N}} \sum_{n_1,n_2\in\mathbb{N}} w_n(k_1,n_1)w_n(k_2,n_2)\\
&\quad\times \sum_{d\in\mathbb{N}} \mathbb{P}_{k_1}^n\big(U_d = n-n_1-n_2+1, m(d)=k_2\big)\tfrac{\tau_n(k_1)}{\tau_n(k_2)}\\
&= \sum_{k_1,k_2\in\mathbb{N}} \sum_{n_1\in\mathbb{N}} w_n(k_1,n_1)\tfrac{\tau_n(k_1)}{\tau_n(k_2)}\\
&\quad\times \sum_{n_2\in\mathbb{N}} \sum_{d\in\mathbb{N}} \mathbb{E}_{k_1}^n\big(1_{\{U_d=n-n_1-n_2+1\}}1_{\{m(d)=k_2\}}w_n(k_2,n-n_1-U_d+1)\big).
\end{aligned} \tag{4.3.14}$$

Interchange the sum over d and n_2 and carry out the sum over n_2 to see that

$$\begin{aligned}
&\text{last line of (4.3.14)}\\
&= \sum_{d\in\mathbb{N}} \mathbb{E}_{k_1}^n\big(1_{\{U_d\le n-n_1\}}1_{\{m(d)=k_2\}}w_n(k_2,n-n_1-U_d+1)\big)\\
&= \mathbb{E}_{k_1}^n\left(\sum_{d=1}^{T_{n-n_1}} 1_{\{m(d)=k_2\}}w_n(k_2,n-n_1-U_d+1)\right)\\
&= \mathbb{E}_{k_1}^n\left(\sum_{k=1}^{n-n_1} 1_{\{m(T_k)=k_2\}}\frac{w_n\big(k_2,n-n_1-U_{T_k}+1\big)}{m(T_k)+m(T_k-1)-1}\right),
\end{aligned} \tag{4.3.15}$$

where the last equality holds because for every $d\in\mathbb{N}$ there are precisely $m(d)+m(d-1)-1$ numbers k such that $T_k = d$ (recall (4.2.16) and (4.3.2)). Now write $n_2 = n-n_1-k$ in (4.3.15) and use the notation in (4.3.3), to get

$$\begin{aligned}
&\text{r.h.s. of (4.3.15)}\\
&= \sum_{n_2=0}^{n-n_1-1} \mathbb{E}_{k_1}^n\left(1_{\{m(T_{n-n_1-n_2})=k_2\}}\frac{w_n\big(k_2,n-n_1-U_{T_{n-n_1-n_2}}+1\big)}{m(T_{n-n_1-n_2})+m(T_{n-n_1-n_2}-1)-1}\right)\\
&= \sum_{n_2=0}^{n-n_1-1} \mathbb{E}_{k_1}^n\left(1_{\{Y_{n-n_1-n_2}=k_2\}}\frac{w_n(k_2,n_2-X_{n-n_1-n_2}+1)}{Y_{n-n_1-n_2}+Z_{n-n_1-n_2}-1}\right).
\end{aligned} \tag{4.3.16}$$

Substitute (4.3.16) into (4.3.14), change the starting measure into $\{\tau_n(k_1)^2\}_{k_1\in\mathbb{N}}$, use that $Z_1 = m(0)$ since $T_1 = 1$, carry out the sums over k_1 and k_2 and recall (4.3.9), to arrive at

$$\text{l.h.s. of (4.3.10)} = \sum_{n_1\in\mathbb{N}} \sum_{n_2=0}^{n-n_1-1} \mathbb{E}^n\left(\frac{w_n(Z_1,n_1)}{\tau_n(Z_1)} f_n^-\left(\Gamma_{n-n_1-n_2}, n_2+1\right)\right), \tag{4.3.17}$$

where we abbreviate $\mathbb{E}^n = \mathbb{E}^{r_n^*,\beta_n}$ and $f_n^{\pm} = f_{r_n^*,\beta_n}^{\pm}$.

Now let $\overline{\mathbb{P}}^n$ be the distribution of the Markov chain $\{\Gamma_l\}_{l\in\mathbb{N}_0}$ on Σ with transition kernel $Q_{r_n^*,\beta_n}$ and initial distribution

$$\overline{\mathbb{P}}^n\left(\Gamma_0 = (i,j,k)\right) = 1_{\{i=0\}}[\theta_n^*]^{-1}\nu_{r_n^*,\beta_n}(i,j,k). \tag{4.3.18}$$

Since the distribution of Γ_1 is the same under $\overline{\mathbb{P}}^n$ as under $\mathbb{P}^n$, we can write $\overline{\mathbb{E}}^n$ instead of $\mathbb{E}^n$ in (4.3.17). Moreover, $Z_1 = Y_0$ under $\overline{\mathbb{P}}^n$. Therefore (4.3.18) allows us to change the starting measure from $\overline{\mathbb{P}}^n$ to $\widetilde{\mathbb{P}}^n = \widetilde{\mathbb{P}}^{r_n^*,\beta_n}$ and obtain

$$\text{l.h.s. of (4.3.10)} = \frac{1}{\theta_n^*}\sum_{n_1\in\mathbb{N}}\sum_{n_2=0}^{n-n_1-1}\widetilde{\mathbb{E}}^n\left(\frac{w_n(Y_0,n_1)}{\tau_n(Y_0)}1_{\{X_0=0\}}\,f_n^-\left(\Gamma_{n-n_1-n_2},n_2+1\right)\right). \tag{4.3.19}$$

Next, formally extend the time range of the Markov chain $\{\Gamma_l\}_{l\in\mathbb{N}_0}$ to the negative integers by putting

$$(X_l,Y_l,Z_l) = (-l,Y_0,Z_0) \qquad (l = -(Y_0+Z_0-2),\dots,0), \tag{4.3.20}$$

on $\{X_0 = 0\}$. Note that $\{\Gamma_l\}_{l\geq-(Y_0+Z_0-2)}$ is still a Markov chain with transition kernel $Q_{r_n^*,\beta_n}$. In (4.3.19) we can now use (4.3.20) to replace

$$\frac{1}{\theta_n^*}\frac{w_n(Y_0,n_1)}{\tau_n(Y_0)}1_{\{X_0=0\}} \tag{4.3.21}$$

by

$$\frac{1}{\theta_n^*}\sum_{i\in\mathbb{N}_0}\frac{w_n(Y_{-i},n_1)}{(Y_{-i}+Z_{-i}-1)\tau_n(Z_{-i})}1_{\{X_{-i}=i\}}. \tag{4.3.22}$$

Substitute this into the r.h.s. of (4.3.17) to obtain

$$\begin{aligned}&\text{l.h.s. of (4.3.10)}\\ &= \frac{1}{\theta_n^*}\sum_{n_1\in\mathbb{N}}\sum_{n_2=0}^{n-n_1-1}\sum_{i\in\mathbb{N}_0}\widetilde{\mathbb{E}}^n\left(\frac{w_n(Y_{-i},n_1)}{(Y_{-i}+Z_{-i}-1)\tau_n(Z_{-i})}1_{\{X_{-i}=i\}}f_n^-(\Gamma_{n-n_1-n_2},n_2+1)\right).\end{aligned} \tag{4.3.23}$$

Finally, use that $\{\Gamma_l\}_{l\geq-(Y_0+Z_0-2)}$ is stationary under $\widetilde{\mathbb{P}}^n$ to shift the time by i. Then shift the sum over n_1 by i. Carry out the sum over i and shift the sum over n_2 by one to obtain the r.h.s. of (4.3.10). □

In the r.h.s. of (4.3.10) appears a *correlation function.* In the sequel we will prove that the first and the last factor in this correlation function are asymptotically independent as $n\to\infty$. The indicator on $\{X_0 \leq n-n_1-n_2+1\}$ will be harmless, as X_0 will turn out to be of order $\beta_n^{-\frac{1}{3}} = o(\sqrt{n})$ and n_1 and n_2 of order $\beta_n^{-\frac{2}{3}} = o(n)$ (see (4.3.26) below).

The sums over n_1 and n_2 in (4.3.10) range only formally over $\mathbb{Z}$, since they are restricted by the conditions $n-n_1-n_2+1\geq 0$, $n_1+X_0\geq 0$ and $n_2-X_{n-n_1-n_2+1}\geq 0$ (see (4.3.8-4.3.9)).

4.3.4 Scaled representation: Lemma 4.3

The limiting behavior of the r.h.s. of (4.3.10) will come out of a scaling analysis. We will turn the sums over $(i,j,k) \in \Sigma$ and $n_1, n_2 \in \mathbb{Z}$ into integrals over

$$(u,v,w) \in S = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R} \quad \text{and} \quad t_1, t_2 \in \mathbb{R} \tag{4.3.24}$$

using the substitutions

$$(i,j,k) = (\lceil u\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} \rceil, \lceil v\beta^{-\frac{1}{3}} + w\beta^{-\frac{1}{6}} \rceil) = (u,v,w)_\beta \tag{4.3.25}$$

and

$$n_1 = \lceil t_1\beta^{-\frac{2}{3}} \rceil, \; n_2 = \lceil t_2\beta^{-\frac{2}{3}} \rceil. \tag{4.3.26}$$

Fix $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. Define a scaled version of the measure $\nu_{r,\beta}$ defined in (4.3.7) by

$$\overline{\nu}_{r,\beta}\,(u,v,w) = \beta^{-\frac{5}{6}} 1_{\{u \le 2v + w\beta^{\frac{1}{6}}\}} \nu_{r,\beta}\,((u,v,w)_\beta)\,. \tag{4.3.27}$$

Here the power of β is chosen so that $\overline{\nu}_{r,\beta}$ is a Lebesgue probability density on S.

Next, we need scaled versions of the functions $f^{\pm}_{r,\beta}$ defined in (4.3.9). Let

$$R = S \times \mathbb{R}^2. \tag{4.3.28}$$

Define $\overline{f}^{-}_{r,\beta,\delta}$, $\overline{f}^{+}_{r,\beta,\delta}$ and $\overline{g}^{n,-}_{r,\beta,\delta} : R \to \mathbb{R}_0^+$ by

$$\overline{f}^{+}_{r,\beta,\delta}((u,v,w),t_1,t_2) \;=\; e^{\delta(t_1-t_2)}\beta^{-\frac{5}{6}} f^{+}_{r,\beta}((u,v,w)_\beta; \lceil t_1\beta^{-\frac{2}{3}} \rceil) 1_{\{u \le n\beta^{\frac{2}{3}} - t_1 - t_2\}} \tag{4.3.29}$$

$$\overline{f}^{-}_{r,\beta,\delta}((u,v,w),t_1,t_2) \;=\; e^{\delta(t_1-t_2)}\beta^{-\frac{5}{6}} f^{-}_{r,\beta}((u,v,w)_\beta; \lceil t_1\beta^{-\frac{2}{3}} \rceil) \tag{4.3.30}$$

$$\begin{aligned} \overline{g}^{n,-}_{r,\beta,\delta}((u,v,w),t_1,t_2) &= e^{\delta(t_2-t_1)}\beta^{-\frac{5}{6}} \widetilde{\mathbb{E}}^{r,\beta}\left(f^{-}_{r,\beta}(\Gamma_{t_n^\beta(t_1,t_2)}; \lceil t_2\beta^{-\frac{2}{3}} \rceil) \,\Big|\, \Gamma_0 = (u,v,w)_\beta \right) \\ &= e^{\delta(t_2-t_1)}\beta^{-\frac{5}{6}} \left(Q^{t_n^\beta(t_1,t_2)}_{r,\beta} f^{-}_{r,\beta}(\;\cdot\;; \lceil t_2\beta^{-\frac{2}{3}} \rceil) \right) ((u,v,w)_\beta) \end{aligned} \tag{4.3.31}$$

for $(u,v,w) \in S$ and $t_1, t_2 \ge 0$, where

$$t_n^\beta(t_1,t_2) = n - \lceil t_1\beta^{-\frac{2}{3}} \rceil - \lceil t_2\beta^{-\frac{2}{3}} \rceil, \tag{4.3.32}$$

$Q^t_{r,\beta}$ is the tth power of the transition kernel $Q_{r,\beta}$ defined in (4.3.6), and $\delta > 0$ is an auxiliary parameter that will turn out to be convenient.

We will regard $\overline{\nu}_{r,\beta}$ as a function on R that does not depend on the last two coordinates.

In terms of the scaled objects introduced above we have the following representation for the l.h.s. of (4.2.6) appearing in Proposition 4.1 (up to an indicator).

Lemma 4.3 *For $\delta > 0$, $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$e^{r_n n} E\Big(e^{-\beta_n \sum_x \ell_n^2(x)} e^{\mu \frac{S_n - \theta_n n}{\sigma_n \sqrt{n}}} 1_{\{0 \le S_{n-1} < S_n\}}\Big)$$
$$= e^{n\left(r_n - r_n^* - \mu \frac{\theta_n}{\sigma_n \sqrt{n}}\right)} [\beta_n^{-\frac{1}{3}} \theta_n^*]^{-1} \Big\langle \sqrt{\overline{\nu}_{r_n^*,\beta_n}}\, \overline{f}^{+}_{r_n^*,\beta_n,\delta}, \sqrt{\overline{\nu}_{r_n^*,\beta_n}}\, \overline{g}^{n,-}_{r_n^*,\beta_n,\delta} \Big\rangle_{L^2(R)} \tag{4.3.33}$$

where $r_n^* = r_n^*(\mu)$ and $\theta_n^* = \theta_n^*(\mu)$ are given in (4.3.11-4.3.12).

Proof. Substitute (4.3.27) and (4.3.29-4.3.31) for $r = r_n^*$ and $\beta = \beta_n$ into (4.3.10). □

4.3.5 A key proposition: Proposition 4.2

Lemma 4.3 gives us the final representation for the quantity appearing in Proposition 4.1. We are now ready to state the main technical ingredient needed for the proof of Proposition 4.1.

Proposition 4.2 *There exists an integrable function $\gamma : (\mathbb{R}_0^+)^2 \to \mathbb{R}^+$ such that for $\delta > 0$ sufficiently small and any sequence $r_n' = \beta_n^{\frac{2}{3}}(a^* + o(1))$,*

$$\lim_{n\to\infty} \Big\langle \sqrt{\overline{\nu}_{r_n',\beta_n}}\, \overline{f}^{+}_{r_n',\beta_n,\delta}, \sqrt{\overline{\nu}_{r_n',\beta_n}}\, \overline{g}^{n,-}_{r_n',\beta_n,\delta} \Big\rangle_{L^2(R)} = b^* \int_0^\infty dt_1 \int_0^\infty dt_2\, \gamma(t_1,t_2). \tag{4.3.34}$$

The function γ will be identified in Section 4.4.3. The proof of Proposition 4.2 is given in Sections 4.5-4.9.

4.3.6 Proof of Proposition 4.1

In this subsection we finish the proof of Proposition 4.1 subject to Proposition 4.2. In fact, we show that Proposition 4.1 holds with

$$L = 2 \int_0^\infty dt_1 \int_0^\infty dt_2\, \gamma(t_1,t_2). \tag{4.3.35}$$

Fix $\mu \in \mathbb{R}$. First we analyze the asymptotics of the exponential in the r.h.s. of (4.3.33).

STEP 1 $\lim_{n\to\infty} \left\{ n \left(r_n - r_n^*(\mu) - \mu \frac{\theta_n}{\sigma_n \sqrt{n}} \right) \right\} = \frac{\mu^2}{2}$.

Proof. We write $s \mapsto \lambda^{-1}(s,\beta)$ for the inverse of $r \mapsto \lambda(r,\beta)$ for fixed β, and we write $\partial_s \lambda^{-1}$ for the partial derivative of λ^{-1} with respect to its first argument. Expand $\lambda^{-1}(s,\beta_n)$ in a Taylor series around $s = 1$. Abbreviate $\mu_n = \frac{\mu}{\sigma_n \sqrt{n}}$. Then, from (4.3.11), we obtain the existence of some number ξ_n in between 1 and $e^{-\mu_n}$ such that

$$\begin{aligned} r_n^*(\mu) &= \lambda^{-1}(e^{-\mu_n}, \beta_n) \\ &= \lambda^{-1}(1,\beta_n) + (e^{-\mu_n} - 1)\, \partial_s \lambda^{-1}(1,\beta_n) + \tfrac{1}{2} (e^{-\mu_n} - 1)^2\, \partial_s^2 \lambda^{-1}(\xi_n,\beta_n) \\ &= r_n + (e^{-\mu_n} - 1)\, \theta_n + \tfrac{1}{2} (e^{-\mu_n} - 1)^2\, \partial_s^2 \lambda^{-1}(\xi_n, \beta_n). \end{aligned} \tag{4.3.36}$$

Here the last equality follows from (1.4.15-1.4.17) and (4.1.6-4.1.7).

Next, we calculate

$$\begin{aligned}\partial_s^2\lambda^{-1}(\xi_n,\beta_n) &= \left[\partial_s\left[\partial_r\lambda(r,\beta_n)\right]^{-1}_{r=\lambda^{-1}(s,\beta_n)}\right]_{s=\xi_n}\\ &= -\left[\frac{\partial_r^2\lambda(r,\beta_n)}{\{\partial_r\lambda(r,\beta_n)\}^3}\right]_{r=\lambda^{-1}(\xi_n,\beta_n)}\\ &= -\left[\frac{\beta_n^{-\frac{1}{3}}\partial_a^2\lambda\left(a\beta_n^{\frac{2}{3}},\beta_n\right)}{\left\{\beta_n^{-\frac{1}{3}}\partial_a\lambda\left(a\beta_n^{\frac{2}{3}},\beta_n\right)\right\}^3}\right]_{a=\beta_n^{-\frac{2}{3}}\lambda^{-1}(\xi_n,\beta_n)}.\end{aligned} \tag{4.3.37}$$

Equation (4.1.2), together with the fact that $\mu_n = o(\beta_n^{\frac{1}{3}})$ (recall (4.1.1) and (4.2.5)), implies that

$$\lambda^{-1}(\xi_n,\beta_n) - \lambda^{-1}(1,\beta_n) = o(\beta_n^{\frac{2}{3}}). \tag{4.3.38}$$

Hence, (4.1.3) and (4.1.4) give that the numerator in the r.h.s. of (4.3.37) converges to $\rho''(a^*)$ and the denominator to $\rho'(a^*)^3$. Thus we obtain $\lim_{n\to\infty}\partial_s^2\lambda^{-1}(\xi_n,\beta_n) = c^{*2}$ with c^{*2} as in (3.1.4). Substituting (4.3.37) into (4.3.36), and noting that $e^{-\mu_n} - 1 = -\frac{\mu}{\sigma_n\sqrt{n}} + \mathcal{O}(\frac{1}{n})$, we get

$$\begin{aligned} r_n^*(\mu) &= r_n + \left(-\tfrac{\mu}{\sigma_n\sqrt{n}} + \mathcal{O}(\tfrac{1}{n})\right)\theta_n + \tfrac{1}{2}\left(\tfrac{\mu}{\sigma_n\sqrt{n}} + \mathcal{O}(\tfrac{1}{n})\right)^2 c^{*2}(1+o(1))\\ &= r_n - \mu\tfrac{\theta_n}{\sigma_n\sqrt{n}} + \tfrac{1}{2}\mu^2\tfrac{c^{*2}}{\sigma_n^2 n}(1+o(1)).\end{aligned} \tag{4.3.39}$$

This together with (4.2.5) implies the claim. □

STEP 2 $\lim_{n\to\infty} r_n^*\beta_n^{-\frac{2}{3}} = a^*$.

Proof. From Step 1 we have

$$\lim_{n\to\infty}(r_n - r_n^*)\beta_n^{-\frac{2}{3}} = \lim_{n\to\infty}\left(\mu\frac{\theta_n}{\sigma_n\sqrt{n}\beta_n^{\frac{2}{3}}} + \mu^2\frac{1}{2n\beta_n^{\frac{2}{3}}}\right). \tag{4.3.40}$$

Use (4.1.1) and (4.2.5) to obtain that the r.h.s. of (4.3.40) vanishes as $n\to\infty$. Now use (4.2.5) once more to get the claim. □

STEP 3 *Conclusion of the proof.*

Proof. Because of Step 2, we may apply Proposition 4.2 for $r_n' = r_n^*$ and $\theta_n' = \theta_n^*$ and obtain that the inner product in the r.h.s. of (4.3.33) tends to $b^*\frac{L}{2}$, where L is given in

(4.3.35). Furthermore, Step 1 says that the exponential in the r.h.s. of (4.3.33) converges towards $e^{\frac{\mu^2}{2}}$ as $n \to \infty$, while (4.1.3), (4.2.5) and (4.3.12) yield that $\beta_n^{-\frac{1}{3}}\theta_n^* \to b^*$ as $n \to \infty$.

Summarizing, we have now proved (4.2.6) with the additional indicator on the event $\{0 \leq S_{n-1} < S_n\}$ in the l.h.s. and the additional factor $\frac{1}{2}$ in the r.h.s. However, the limit assertion remains true with $1_{\{0 \leq S_{n-1} < S_n\}}$ replaced by $1_{\{0 \leq S_n < S_{n-1}\}}$, since (4.2.21) is only a small perturbation of (4.2.20). Indeed, $m^*(1)$ and $m(1)$ are of order $\beta_n^{-\frac{1}{3}} = o(\beta_n^{-1})$ (see also (4.4.24) and (4.4.28) below). The details are left to the reader.

Adding the two limit assertions, we end up with (4.2.6). □

4.4 Preparatory tools for the proof of Proposition 4.2

In this section we collect some tools that will be needed in the remaining sections for the proofs of Theorem 4.1 and Proposition 4.2. The quantities appearing below require some patience of the reader, as their full meaning will only become clear later on.

4.4.1 Spectral properties

In this subsection we describe spectral properties of some operators involved in the proof of Proposition 4.2, since we later will need to do some eigenvalue expansions. We are able to characterize the spectra of $\mathcal{L}^a$, $A_{r,\beta}$ and $\frac{1}{\mathrm{id}}\mathcal{L}^{a^*}$ completely, as well as a large part of the spectrum of $Q_{r,\beta}$. The latter will be needed in Section 4.6 to identify the l.h.s. of (4.3.34), and will turn out to approximate the spectrum of $\frac{1}{\mathrm{id}}\mathcal{L}^{a^*}$ in a certain sense.

$\mathcal{L}^a$: For any $a \in \mathbb{R}$, the differential operator $\mathcal{L}^a$ defined in (2.1.13) is a Sturm-Liouville operator on $L^2(\mathbb{R}_0^+)$. For $l \in \mathbb{N}_0$, let $\rho^{(l)}(a)$ be the l'th eigenvalue of $\mathcal{L}^a$ (arranged in decreasing order) with corresponding eigenfunction $x_a^{(l)} \in L^2(\mathbb{R}_0^+)$, normed such that $\|x_a^{(l)}\|_{L^2(\mathbb{R}_0^+)} = 1$. From general Sturm-Liouville theory it follows that $\rho^{(l)}(a)$ is simple, $x_a^{(l)}$ is a real-analytic function on $\mathbb{R}_0^+$ and

$$\{x_a^{(l)}\}_{l \in \mathbb{N}_0} \text{ is an orthonormal basis of } L^2(\mathbb{R}_0^+). \tag{4.4.1}$$

From Lemma 2.16 it follows that $x_a^{(l)}$ has a sub-exponentially small tail at infinity. The principal eigenvalue $\rho^{(0)}(a) = \rho(a)$ and corresponding eigenvector $x_a^{(0)} = x_a$ will play a key role in the sequel. Since x_a has no zeroes on $\mathbb{R}_0^+$, we may and will pick the sign such that $x_a(u) > 0$ for all $u \geq 0$. Note that $\mathcal{L}^{a^*} x_{a^*} = 0$, because $\rho(a^*) = 0$ (see Theorem 2.3).

$A_{r,\beta}$: For any $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$, the matrix $A_{r,\beta}$ defined in (1.4.13) is a symmetric Hilbert-Schmidt operator on $l^2(\mathbb{N})$. For $l \in \mathbb{N}_0$, let $\lambda^{(l)}(r,\beta)$ be the l'th eigenvalue of $A_{r,\beta}$ (arranged in decreasing order of absolute values) with corresponding eigenvector $\tau_{r,\beta}^{(l)}$, normed such that $\|\tau_{r,\beta}^{(l)}\|_{l^2(\mathbb{N})} = 1$. Note that $\lambda^{(0)}(r,\beta) = \lambda(r,\beta)$ and $\tau_{r,\beta}^{(0)} = \tau_{r,\beta}$ as defined in Section 4.3.1. Differentiate the formula $\lambda^{(l)}(r,\beta) = \langle \tau^{(l)}_{r,\beta}, A_{r,\beta}\tau^{(l)}_{r,\beta}\rangle_{l^2(\mathbb{N})}$ with respect to r to obtain that

$$\partial_r \lambda^{(l)}(r,\beta) = \lambda^{(l)}(r,\beta) \sum_{i\in\mathbb{N}} (2i-1)\, \tau^{(l)}_{r,\beta}(i)^2. \tag{4.4.2}$$

Thus, $\lambda^{(l)}(\,\cdot\,,\beta)$ maps $\mathbb{R}$ either onto $-\mathbb{R}^+$, $\{0\}$ or $\mathbb{R}^+$. Since $\lambda^{(l)}(r,0) > 0$ for all $r < 0$,[3] the continuity of $\beta \mapsto \lambda^{(l)}(-1,\beta)$ in zero and (4.4.2) imply that $\lambda^{(l)}(r,\beta) > 0$ for all $r \in \mathbb{R}$ and all $\beta \in (0, \beta_0(l))$ for some $\beta_0(l) > 0$. Thus, the map $r \mapsto \lambda^{(l)}(r,\beta)$ is strictly increasing and has limits 0, respectively, ∞ as $r \to -\infty$, respectively, $r \to \infty$ for those β.

$\frac{1}{\mathrm{id}}\mathcal{L}^{a^*}$: Recall the definitions of $L^{2,\circ}(\mathbb{R}_0^+)$ and $\mathcal{M}^{a^*}$ in (3.3.25) and (3.4.5):

$$L^{2,\circ}(\mathbb{R}_0^+) = \left\{ f : \mathbb{R}_0^+ \to \mathbb{R} \text{ measurable} : \int_0^\infty dh\, h f(h)^2 < \infty \right\} \tag{4.4.3}$$

$$\left(\mathcal{M}^{a^*} x\right)(u) = \frac{\left(\mathcal{L}^{a^*} x\right)(u)}{u} \qquad (u > 0). \tag{4.4.4}$$

From the symmetry of $\mathcal{L}^{a^*}$ on $L^2(\mathbb{R}_0^+)$ it follows that $\mathcal{M}^{a^*}$ is symmetric with respect to the natural inner product $\langle f, g\rangle_{L^{2,\circ}(\mathbb{R}_0^+)} = \int_0^\infty dh\, h f(h) g(h)$ on $L^{2,\circ}(\mathbb{R}_0^+)$. Differentiate the formula $\rho^{(l)}(a) = \langle x_a^{(l)}, \mathcal{L}^a x_a^{(l)}\rangle_{L^2(\mathbb{R}_0^+)}$ with respect to a to obtain

$$\frac{d}{da}\rho^{(l)}(a) = 2\|x_a^{(l)}\|^2_{L^{2,\circ}(\mathbb{R}_0^+)} > 0. \tag{4.4.5}$$

Thus, $\rho^{(l)} : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing. Moreover, $\lim_{a\to\pm\infty} \rho^{(l)}(a) = \pm\infty$. Therefore, for every $l \in \mathbb{N}_0$ we may define $\alpha^{(l)} \in \mathbb{R}$ by

$$\rho^{(l)}(a^* - \alpha^{(l)}) = \rho(a^*) = 0. \tag{4.4.6}$$

Let

$$y^{(l)}(v) = \frac{x^{(l)}_{a^*-\alpha^{(l)}}(v)}{\|x^{(l)}_{a^*-\alpha^{(l)}}\|_{L^{2,\circ}(\mathbb{R}_0^+)}}. \tag{4.4.7}$$

Then $y^{(l)}$ is a normed element of $L^{2,\circ}(\mathbb{R}_0^+)$. As explained in Step 1 in the proof of Proposition 3.3, $\{y^{(l)}\}_{l\in\mathbb{N}_0}$ is the set of *all* eigenvectors of $\mathcal{M}^{a^*}$ (up to multiples) and

$$\{y^{(l)}\}_{l\in\mathbb{N}_0} \text{ is an orthonormal basis of } L^{2,\circ}(\mathbb{R}_0^+). \tag{4.4.8}$$

[3]To see why, use (1.4.13-1.4.14) and the Gamma-integral representation for $(i+j-2)!$ to write $2\langle \tau, A_{r,0}\tau\rangle_{l^2} = e^r \int_0^\infty dt\, e^{-t} \left(\sum_{i\in\mathbb{N}} \frac{\tau(i)}{(i-1)!}\left(\frac{te^r}{2}\right)^{i-1}\right)^2 > 0$ for any $\tau \in l^2(\mathbb{N}), \tau \neq 0$.

$Q_{r,\beta}$: Fix $l \in \mathbb{N}$ and $\beta \in \mathbb{R}^+$ so small that $\lambda^{(l)}(r,\beta) > 0$ for all $r \in \mathbb{R}$. Define $\alpha^{(l)}(r,\beta) \in \mathbb{R}$ by

$$\lambda^{(l)}(r - \alpha^{(l)}(r,\beta), \beta) = \lambda^{(0)}(r,\beta). \tag{4.4.9}$$

Note that $\alpha^{(l)}(r,\beta) < \alpha^{(0)}(r,\beta) = 0$ because the map $r \mapsto \lambda^{(l)}(r,\beta)$ is strictly increasing. Define a vector $\nu^{(l)}_{r,\beta}$ by

$$\begin{aligned} \nu^{(l)}_{r,\beta}(i,j,k) &= e^{-\alpha^{(l)}(r,\beta)(j+k-i)} \tau_{r,\beta}(j) A_{r,\beta}(j,k) \tau^{(l)}_{r-\alpha^{(l)}(r,\beta),\beta}(k) \\ &\quad \times \left(\partial_r \lambda(r,\beta)\, \partial_r \lambda^{(l)}(r - \alpha^{(l)}(r,\beta),\beta)\right)^{-\frac{1}{2}} \quad ((i,j,k) \in \Sigma). \end{aligned} \tag{4.4.10}$$

Note that $\nu^{(0)}_{r,\beta} = \nu_{r,\beta}$ defined in (4.3.7). A straightforward calculation shows that

$$\left(\nu^{(l)}_{r,\beta} Q_{r,\beta}\right)(i,j,k) = e^{\alpha^{(l)}(r,\beta)} \nu^{(l)}_{r,\beta}(i,j,k) \qquad ((i,j,k) \in \Sigma), \tag{4.4.11}$$

i.e., $\nu^{(l)}_{r,\beta}$ is a left-eigenvector of $Q_{r,\beta}$ with eigenvalue $e^{\alpha^{(l)}(r,\beta)}$. (In order to derive (4.4.11), we distinguish between the cases $i = j + k - 2$ and $i < j + k - 2$, use the eigenvalue property of $\tau^{(l)}_{r,\beta}$ for $A_{r,\beta}$ and the symmetry of $A_{r,\beta}$, and observe that $A_{r,\beta}(j,k)e^{-\alpha(j+k-1)} = A_{r-\alpha,\beta}(j,k)$ by (1.4.13).)

Next, introduce

$$y^{(l)}_{r,\beta} = \frac{\nu^{(l)}_{r,\beta}}{\sqrt{\nu^{(0)}_{r,\beta}}} : \Sigma \to \mathbb{R}. \tag{4.4.12}$$

This quantity will later turn out to play an analogous role as $y^{(l)}$ defined in (4.4.7). However, $Q_{r,\beta}$ is not reversible, so we cannot expect that $\{y^{(l)}_{r,\beta}\}_{l \in \mathbb{N}_0}$ is a basis of $l^2(\Sigma)$.

In the sequel we will suppress $\mathbb{R}^+_0$ and $\mathbb{N}$ from the notation for the spaces L^2 and $L^{2,\circ}$ and l^2 respectively.

4.4.2 Eigenvector scaling limits: Proposition 4.3

Proposition 4.3 below relates the eigenvalues and the eigenvectors of $\mathcal{L}^a$ and $A_{r,\beta}$. For $\beta \in \mathbb{R}^+$, define scaled L^2-versions of vectors $\tau_\beta \in l^2$ by putting

$$\overline{\tau}_\beta(h) = \beta^{-\frac{1}{6}} \tau_\beta(\lceil h\beta^{-\frac{1}{3}} \rceil) \qquad (h > 0) \tag{4.4.13}$$

and $\overline{\tau}_\beta(0) = \overline{\tau}_\beta(0+)$. Here the power of β is chosen in such a way that $\|\tau_\beta\|_{l^2} = \|\overline{\tau}_\beta\|_{L^2}$. We have the following scaling limit result extending (2.1.16-2.1.17).

Proposition 4.3 *As $\beta \downarrow 0$, uniformly in a on compacts in $\mathbb{R}$,* (*i*) *for all* $l \in \mathbb{N}_0$,

$$\begin{aligned} \beta^{-\frac{1}{3}}\left[\lambda^{(l)}(a\beta^{\frac{2}{3}},\beta)-1\right] &\to \rho^{(l)}(a) \\ \overline{\tau}^{(l)}_{a\beta^{\frac{2}{3}},\beta} &\to x^{(l)}_a \quad \text{in } L^{2,\circ} \text{ and in } L^2. \end{aligned} \tag{4.4.14}$$

(*ii*) $\overline{\tau}^{(0)}_{a\beta^{\frac{2}{3}},\beta} = \overline{\tau}_{a\beta^{\frac{2}{3}},\beta}$ *converges to* $x^{(0)}_a = x_a$ *uniformly on* $\mathbb{R}^+_0$, *provided* $|\rho(a)| < 1$.

(*iii*) *For* $r(\beta) = a^*\beta^{\frac{2}{3}}(1+o(1))$, *there exists* $C \in \mathbb{R}^+$ *such that for every* $u, v \in \mathbb{R}^+$

$$\overline{\tau}^{(0)}_{r(\beta),\beta}(u+v) \le C\overline{\tau}^{(0)}_{r(\beta),\beta}(u). \tag{4.4.15}$$

The proof of Proposition 4.3 is given in Section 4.9.

4.4.3 The function γ

In this subsection we introduce the function $\gamma : (\mathbb{R}^+_0)^2 \to \mathbb{R}^+$ that appears in the formulation of Proposition 4.2.

Denote by $X^\star = \{X^\star(\sigma)\}_{\sigma\ge 0}$ the zero-dimensional squared Bessel process with generator

$$(G^\star f)(u) = 2uf''(u). \tag{4.4.16}$$

With a slight abuse of notation (see the end of Section 4.2.2), we denote the distribution of $X^\star$ conditioned on starting at $v \ge 0$ by $\mathbb{P}^\star_v$ and the corresponding expectation by $\mathbb{E}^\star_v$. The point 0 is absorbing for $X^\star$ and is reached almost surely in finite time.

Recall that $F^\star_a(u) = u^2 - au$ for $a, u \in \mathbb{R}$. For $v, t \ge 0$ and $a \in \mathbb{R}$ define

$$w_a(v,t) = \mathbb{E}^\star_v\left(e^{-\int_0^\infty F^\star_a(X^\star(\sigma))\,d\sigma} \,\middle|\, \int_0^\infty X^\star(\sigma)\,d\sigma = t\right)\psi_v(t) = e^{at}w_0(v,t), \tag{4.4.17}$$

where (see Lemma 3.7)

$$\psi_v(t) = \frac{\mathbb{P}^\star_v\left(\int_0^\infty X^\star(\sigma)\,d\sigma \in dt\right)}{dt} = \frac{v}{\sqrt{2\pi t^3}}e^{-\frac{v^2}{2t}}. \tag{4.4.18}$$

It is shown in Lemmas 3.5–3.6 that there is a critical $a_c \in (2^{\frac{1}{3}}a^*, \infty)$ such that for every $a < a_c$ the function z_a defined by

$$z_a(v) = \int_0^\infty w_a(v,t)\,dt = \mathbb{E}^\star_v\left(e^{-\int_0^\infty F^\star_a(X^\star(\sigma))\,d\sigma}\right) \qquad (v \ge 0) \tag{4.4.19}$$

is real-analytic on $\mathbb{R}^+_0$ and has a sub-exponentially small tail at infinity.

Define a function $\gamma : (\mathbb{R}_0^+)^2 \to \mathbb{R}^+$ by

$$\gamma(t_1,t_2) = \frac{b^*}{2} \langle w_{a^*}(\cdot,t_1), x_{a^*}\rangle_{L^2} \langle w_{a^*}(\cdot,t_2), x_{a^*}\rangle_{L^2} . \qquad (4.4.20)$$

Note that

$$e^{\delta(t_1+t_2)}\gamma(t_1,t_2) = \frac{b^*}{2} \langle w_{a^*+\delta}(\cdot,t_1), x_{a^*}\rangle_{L^2} \langle w_{a^*+\delta}(\cdot,t_2), x_{a^*}\rangle_{L^2} . \qquad (4.4.21)$$

Hence, since $a^* < a_c$, the r.h.s. of (4.4.21) is integrable for $\delta > 0$ small enough. This implies in particular that $\gamma(t_1,t_2)$ decays exponentially fast towards zero as $t_1 \to \infty$ or $t_2 \to \infty$. From (4.4.18) it can be easily deduced that $\gamma(t_1,t_2)$ is of order $\mathcal{O}(t_i^{-\frac{1}{2}})$ for $t_i \downarrow 0$ $(i = 1,2)$. Consequently, γ is integrable on $(\mathbb{R}_0^+)^2$.

4.4.4 Convergence of the function $w_{r,\beta}$: Lemmas 4.4–4.6

For the proof of Proposition 4.2 we next isolate the appropriate convergence assertion for the function $w_{r,\beta}$ defined in (4.3.8). Recall that $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ is the Markov chain on $\mathbb{N}_0$ with transition kernel $P^\star$ that was introduced in Section 4.2.2, and $\mathbb{P}_k^\star$ is its distribution when started at $k \in \mathbb{N}_0$.

Fix $r \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. Define a scaled version of $w_{r,\beta}$ by putting

$$\overline{w}_{r,\beta}(v,t) = \beta^{-\frac{2}{3}} w_{r,\beta}(\lceil v\beta^{-\frac{1}{3}}\rceil, \lceil t\beta^{-\frac{2}{3}}\rceil) \qquad (v,t \geq 0). \qquad (4.4.22)$$

We also need to introduce the sum of $w_{r,\beta}$ over its second argument and its scaled version, namely

$$\begin{aligned} z_{r,\beta}(k) &= \sum_{l\in\mathbb{N}_0} w_{r,\beta}(k,l) = \mathbb{E}_k^\star\left(e^{rU^\star-\beta V^\star}\right) && (k\in\mathbb{N}_0) \\ \overline{z}_{r,\beta}(v) &= \int_0^\infty \overline{w}_{r,\beta}(v,t)\,dt = z_{r,\beta}(\lceil v\beta^{-\frac{1}{3}}\rceil) && (v \geq 0). \end{aligned} \qquad (4.4.23)$$

The reader gains more insight into these quantities once they are expressed in terms of the scaled continuous-time process

$$\left\{X_\beta^\star(\sigma)\right\}_{\sigma\geq 0} = \left\{\beta^{\frac{1}{3}}\left(m^\star(\lceil\sigma\beta^{-\frac{1}{3}}\rceil) + m^\star(\lceil\sigma\beta^{-\frac{1}{3}}\rceil - 1)\right)\right\}_{\sigma\geq 0}. \qquad (4.4.24)$$

Indeed, for any $v \geq 0$, denote the distribution of the process $X_\beta^\star = \{X_\beta^\star(\sigma)\}_{\sigma\geq 0}$ under $\mathbb{P}^\star_{\lceil v\beta^{-\frac{1}{3}}\rceil}$ by $\mathbb{P}_v^{\star,\beta}$ and the corresponding expectation by $\mathbb{E}_v^{\star,\beta}$. Then (see (4.2.18-4.2.19))

$$\begin{aligned} U^\star &= \beta^{-\frac{2}{3}} \int_0^\infty X_\beta^\star(\sigma)\,d\sigma, \\ V^\star &= \beta^{-1} \int_0^\infty X_\beta^\star(\sigma)^2\,d\sigma. \end{aligned} \qquad (4.4.25)$$

Thus, with the abbreviation $F_r^{(\beta)}(u) = u^2 - r\beta^{-\frac{2}{3}}u$, we have

$$\overline{w}_{r,\beta}(v,t) = \mathbb{E}_v^{\star,\beta}\left(e^{-\int_0^\infty F_r^{(\beta)}(X_\beta^\star(\sigma))\,d\sigma} \,\middle|\, \int_0^\infty X_\beta^\star(\sigma)\,d\sigma = \lceil t\beta^{-\frac{2}{3}}\rceil\beta^{\frac{2}{3}}\right)\psi_v^{(\beta)}(t), \tag{4.4.26}$$

where (see (4.4.25))

$$\psi_v^{(\beta)}(t) = \beta^{-\frac{2}{3}}\mathbb{P}_v^{\star,\beta}\left(U^\star = \lceil t\beta^{-\frac{2}{3}}\rceil\right). \tag{4.4.27}$$

In Section 4.5.2 we will identify $\psi_v^{(\beta)}$.

Recall that $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ is a branching process whose offspring distribution has mean one and variance two. From Ethier and Kurtz (1986) Theorem 9.1.3 it therefore follows that

$$\mathbb{P}_{v_\beta}^{\star,\beta} \Longrightarrow \mathbb{P}_v^\star \quad \text{if} \quad v_\beta \to v \in \mathbb{R}_0^+ \quad \text{and} \quad \beta \downarrow 0. \tag{4.4.28}$$

In view of this, the following assertions are plausible. Their proofs are deferred to Section 4.7.

Lemma 4.4 *For every $a < a_c$, $r_n' = \beta_n^{\frac{2}{3}}(a + o(1))$ and compact interval $I \subset \mathbb{R}^+$,*

$$\limsup_{n\to\infty} \int_I dt \int_0^\infty dv\, \frac{\overline{w}_{r_n',\beta_n}(v,t)^2}{v} \le \int_I dt \int_0^\infty dv\, \frac{w_a(v,t)^2}{v}. \tag{4.4.29}$$

Lemma 4.5 *For every $a < a_c$ and $r_n' = \beta_n^{\frac{2}{3}}(a + o(1))$ there are $q \in (0,1)$ and $C \in \mathbb{R}^+$ such that for sufficiently large n,*

$$\overline{z}_{r_n',\beta_n}(v) \le Cq^v \qquad (v \in \mathbb{R}_0^+). \tag{4.4.30}$$

Lemma 4.6 *For every $a < a_c$, $r_n' = \beta_n^{\frac{2}{3}}(a + o(1))$ and any interval $I \subset \mathbb{R}_0^+$,*

$$\int_I dt\, \overline{w}_{r_n',\beta_n}(\,\cdot\,,t) \longrightarrow^{L^2} \int_I dt\, w_a(\,\cdot\,,t) \quad \text{as} \quad n \to \infty. \tag{4.4.31}$$

4.5 Proof of Proposition 4.2

In this section we begin the proof of Proposition 4.2. The assertion we have to prove is of the form $\int_\mathbb{R}\int_\mathbb{R} \gamma_n \to \int_0^\infty \int_0^\infty \gamma$ for certain functions γ_n, γ. We will prove this assertion by splitting the integrals into the boundary pieces near 0, respectively, ∞ and the main piece in the middle, and showing that the boundary pieces give small contributions. In Section 4.5.1 we formulate the program in three lemmas. In Sections 4.5.2 and 4.5.3 we deal with the boundary pieces. The convergence of the main piece is proved in Section 4.6.

4.5.1 Splitting the integrals: Lemmas 4.7 –4.9

Fix some sequence $r'_n = \beta_n^{\frac{2}{3}}(a^* + o(1))$, put $\theta'_n = \lambda(r'_n, \beta_n)/\partial_r\lambda(r'_n, \beta_n)$ and observe from (4.1.3) that

$$b'_n = \beta_n^{-\frac{1}{3}}\theta'_n \to b^* \qquad (n \to \infty). \tag{4.5.1}$$

Furthermore, fix $\delta \in (0, a_c 2^{-\frac{1}{3}} - a^*)$, abbreviate $\overline{f}^{\pm}_{n,\delta} = \overline{f}^{\pm}_{r'_n,\beta_n,\delta}$, $\overline{g}^{-}_{n,\delta} = \overline{g}^{n,-}_{r'_n,\beta_n,\delta}$ and $\overline{\nu}_n = \overline{\nu}_{r'_n,\beta_n}$, and introduce the abbreviation (see the l.h.s. of (4.3.33))

$$\gamma_n(t_1, t_2) = \frac{1}{b'_n}\left\langle \sqrt{\overline{\nu}_n}\,\overline{f}^{+}_{n,\delta}(\cdot, t_1, t_2), \sqrt{\overline{\nu}_n}\,\overline{g}^{-}_{n,\delta}(\cdot, t_1, t_2)\right\rangle_{L^2(S)} \qquad (t_1, t_2 \in \mathbb{R}). \tag{4.5.2}$$

Observe from (4.3.29-4.3.31) that $\overline{f}^{+}_{n,\delta}(u, v, w, t_1, t_2) = 0$ for $t_1 < -u\beta_n^{\frac{1}{3}}$ and $\overline{g}^{-}_{n,\delta}(\cdot, t_1, t_2) = 0$ for $t_2 < 0$ (see also the end of Section 4.3.3) . Thus $\lim_{n\to\infty}\gamma_n(t_1, t_2) = 0$ for $t_1 < 0$ and $\gamma_n(t_1, t_2) = 0$ for $t_2 < 0$.

According to Lemma 4.3, Proposition 4.2 states that

$$\lim_{n\to\infty}\int_{\mathbb{R}} dt_1 \int_{\mathbb{R}} dt_2\, \gamma_n(t_1, t_2) = \int_0^\infty dt_1 \int_0^\infty dt_2\, \gamma(t_1, t_2), \tag{4.5.3}$$

where γ has been introduced in Section 4.4.3.

We split each of the two integrals in (4.5.3) into $\int_{-\infty}^{\varepsilon} + \int_{\varepsilon}^{N} + \int_N^\infty$. Lemmas 4.7–4.8 below state that the mixed contributions coming from the first and the third integral are small, uniformly in n, when $\varepsilon > 0$ is small and $N < \infty$ is large. The precise assertions are the following:

Lemma 4.7 *For any* $0 < \varepsilon < \infty$,

$$\lim_{N\to\infty}\limsup_{n\to\infty}\int_\varepsilon^\infty dt_1 \int_N^\infty dt_2\, \gamma_n(t_1, t_2) = 0 = \lim_{N\to\infty}\limsup_{n\to\infty}\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2\, \gamma_n(t_1, t_2). \tag{4.5.4}$$

Lemma 4.8

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{-\infty}^{\varepsilon} dt_1 \int_{-\infty}^{\infty} dt_2\, \gamma_n(t_1, t_2) = 0 = \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{-\infty}^{\infty} dt_1 \int_0^{\varepsilon} dt_2\, \gamma_n(t_1, t_2). \tag{4.5.5}$$

Since γ is integrable on $(\mathbb{R}_0^+)^2$, Proposition 4.2 directly follows from Lemmas 4.7–4.8 and the following lemma, which states the convergence of the main piece of the double integral.

Lemma 4.9 *For any* $0 < \varepsilon < N < \infty$,

$$\lim_{n\to\infty}\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma_n(t_1, t_2) = \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma(t_1, t_2). \tag{4.5.6}$$

4.5.2 Proof of Lemma 4.7: cutting away large t_1, t_2

We will give the proof for the second equality in (4.5.4) only, since the proof for the first is similar.

Recall (4.5.2) and use the Cauchy-Schwarz inequality to estimate

$$\begin{aligned}
&\textstyle\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2\, \gamma_n(t_1,t_2)\\
&\quad= \textstyle\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2 \int_S ds\ \overline{v}_n(s)\overline{f}^+_{n,\delta}(s,t_1,t_2)\overline{g}^-_{n,\delta}(s,t_1,t_2)\\
&\quad\le \left(\textstyle\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2 \int_S ds\ \overline{v}_n(s)\overline{f}^+_{n,\delta}(s,t_1,t_2)^2\right)^{1/2}\\
&\qquad\times \left(\textstyle\int_N^\infty dt_1 \int_\varepsilon^\infty dt_2 \int_S ds\ \overline{v}_n(s)\overline{g}^-_{n,\delta}(s,t_1,t_2)^2\right)^{1/2}.
\end{aligned} \tag{4.5.7}$$

In Steps 3 and 4 below we give the respective estimates for the two factors in the r.h.s. of (4.5.7). First we make two intermediate steps, the first of which identifies the function $\psi_v^{(\beta)}$ defined in (4.4.27). Recall that P denotes the distribution of simple random walk $(S_k)_{k\in\mathbb{N}_0}$.

STEP 1 *For any $v, t, \beta \in \mathbb{R}^+$,*

$$\psi_v^{(\beta)}(t) = \frac{\lceil v\beta^{-\frac13}\rceil}{\lceil t\beta^{-\frac23}\rceil\beta^{\frac13}}\beta^{-\frac13}P\left(S_{\lceil t\beta^{-\frac23}\rceil} = \lceil v\beta^{-\frac13}\rceil\right). \tag{4.5.8}$$

Proof. Since $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ is a critical branching process, Theorem (2.11.2) in Jagers (1975) implies that

$$\mathbb{P}_k^\star\left(\sum_{x=0}^\infty m^\star(x) = j\right) = \frac{k}{j}P^\star(j, j-k) \qquad (j \ge k). \tag{4.5.9}$$

Note that $U^\star = 2\sum_{x=0}^\infty m^\star(x) - k$ (recall (4.2.18)) $\mathbb{P}_k^\star$-a.s., and so (4.5.9) implies for $l \ge k$,

$$\mathbb{P}_k^\star(U^\star = l) = \frac{2k}{l+k}P^\star\left(\frac{l+k}{2}, \frac{l-k}{2}\right) = \frac{k}{l}\left(\frac12\right)^l \binom{l}{\frac12(l+k)} = \frac{k}{l}P(S_l = k). \tag{4.5.10}$$

Substitute $k = \lceil v\beta^{-\frac13}\rceil$ and $l = \lceil t\beta^{-\frac23}\rceil$ and recall (4.4.27) to arrive at (4.5.8). □

STEP 2 *There is a $C > 0$ such that for sufficiently small $\beta \in \mathbb{R}^+$,*

$$\frac{\overline{w}_{r,\beta}(v,t)^2}{\beta^{\frac13}\lceil v\beta^{-\frac13}\rceil} \le \frac{C}{t^{\frac32}}\overline{w}_{2r,2\beta}(2^{\frac13}v, 2^{\frac23}t) \qquad (r \in \mathbb{R}, v > 0, t > 0). \tag{4.5.11}$$

Proof. Use (4.4.22) and (4.4.26) to rewrite

$$\overline{w}_{r,\beta}(v,t) = e^{r\beta^{-\frac23}t}\mathbb{E}^\star_{\lfloor v\beta^{-\frac13}\rfloor}\left(e^{-\beta V^\star}\,\middle|\, U^\star = \lceil t\beta^{-\frac23}\rceil\right)\psi_v^{(\beta)}(t). \tag{4.5.12}$$

Use the Cauchy-Schwarz inequality and Step 1 to find

$$\begin{aligned}\overline{w}_{r,\beta}(v,t)^2 &\le e^{2r(2\beta)^{-\frac{2}{3}}t2^{\frac{2}{3}}}\mathbb{E}^\star_{\lfloor 2^{\frac{1}{3}}v(2\beta)^{-\frac{1}{3}}\rfloor}\left(e^{-2\beta V^\star}\,\middle|\,U^\star=\lceil t2^{\frac{2}{3}}(2\beta)^{-\frac{2}{3}}\rceil\right)\psi_v^{(\beta)}(t)^2\\ &=\overline{w}_{2r,2\beta}(2^{\frac{1}{3}}v,2^{\frac{2}{3}}t)\frac{\psi_v^{(\beta)}(t)^2}{\psi^{(2\beta)}_{2^{\frac{1}{3}}v}(2^{\frac{2}{3}}t)}=\overline{w}_{2r,2\beta}(2^{\frac{1}{3}}v,2^{\frac{2}{3}}t)2^{\frac{1}{3}}\psi_v^{(\beta)}(t).\end{aligned}\tag{4.5.13}$$

Use Step 1 and Stirling's formula to arrive at (4.5.11). □

STEP 3 *There is a $C>0$ such that for all $N,\epsilon>0$,*

$$\limsup_{n\to\infty}\int_N^\infty dt_1\int_\epsilon^\infty dt_2\int_S ds\;\overline{\nu}_n(s)\overline{f}^{\pm}_{n,\delta}(s,t_1,t_2)^2\le\frac{C}{N^{\frac{3}{2}}}e^{-\epsilon\delta}.\tag{4.5.14}$$

Proof. Abbreviate $\overline{\tau}_n=\overline{\tau}_{r_n',\beta_n}$ and $\overline{w}_n^{(\delta)}=\overline{w}_{r_n'+\delta\beta_n^{\frac{2}{3}},\beta_n}$. Use (4.3.29-4.3.30), (4.4.22), (4.3.25) and (4.3.9) to see that for $(u,v,w)\in S$,

$$\overline{f}^{\pm}_{n,\delta}((u,v,w),t_1,t_2)\le\frac{\overline{w}_n^{(\delta)}(v,t_1\pm u\beta_n^{\frac{1}{3}})}{\beta^{\frac{1}{3}}\lceil(2v+w\beta_n^{\frac{1}{6}})\beta^{-\frac{1}{3}}\rceil\overline{\tau}_n(v)}e^{\delta u\beta_n^{\frac{1}{3}}}e^{-\delta t_2}.\tag{4.5.15}$$

Introduce for $v,\widetilde{v}\in\mathbb{R}$ the notation

$$\overline{A}_n(v,\widetilde{v})=\begin{cases}\beta_n^{-\frac{1}{6}}A_{r_n',\beta_n}(\lceil v\beta_n^{-\frac{1}{3}}\rceil,\lceil\widetilde{v}\beta_n^{-\frac{1}{3}}\rceil) & \text{if } v,\widetilde{v}>0\\ 0 & \text{otherwise.}\end{cases}\tag{4.5.16}$$

Then for any $(u,v,w)\in S$,

$$\overline{\nu}_n(u,v,w)=b_n'\overline{\tau}_n(v)\overline{A}_n(v,v+w\beta_n^{\frac{1}{6}})\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}}).\tag{4.5.17}$$

Next write $\int_S ds$ as $\int_0^\infty dv\int_{\mathbb{R}}dw\int_0^{2v+w\beta_n^{\frac{1}{6}}}du$, substitute (4.5.15) and (4.5.17) into the l.h.s. of (4.5.14) and carry out the integral over t_2, to get

$$\begin{aligned}&\int_N^\infty dt_1\int_\epsilon^\infty dt_2\int_S ds\;\overline{\nu}_n(s)\overline{f}^{\pm}_{n,\delta}(s,t_1,t_2)^2\\ &\le\tfrac{b_n'}{2\delta}e^{-\epsilon\delta}\int_N^\infty dt_1\int_0^\infty dv\int_{\mathbb{R}}dw\,\overline{A}_n(v,v+w\beta_n^{\frac{1}{6}})\tfrac{\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)}\\ &\times\int_0^{2v+w\beta_n^{\frac{1}{6}}}du\,\tfrac{e^{2\delta u\beta_n^{\frac{1}{3}}}}{2v+w\beta_n^{\frac{1}{6}}}\,\tfrac{\overline{w}_n^{(\delta)}(v,t_1\pm u\beta_n^{\frac{1}{3}})^2}{\beta^{\frac{1}{3}}\lceil(2v+w\beta_n^{\frac{1}{6}})\beta^{-\frac{1}{3}}\rceil}.\end{aligned}\tag{4.5.18}$$

By (4.5.16) we may let the w-integral range over $[-v\beta_n^{-\frac{1}{6}},\infty)$ only, because of the estimate $2v+w\beta_n^{\frac{1}{6}}\ge v$. Now use Step 2 for $\beta=\beta_n, r=r_n'+\delta\beta_n^{\frac{2}{3}}$ and $t=t_1\pm u\beta_n^{\frac{1}{3}}$, estimate

$(t_1 + u\beta_n^{\frac{1}{3}})^{\frac{3}{2}} \geq N^{\frac{3}{2}}$ (the case "$-$" requires a further standard cutting argument for the u-integral which is left to the reader), carry out the t_1-integral and the u-integral, to obtain

$$\text{l.h.s. of (4.5.18)} \leq \frac{C}{N^{\frac{3}{2}}}\frac{b_n'}{2\delta}e^{-\epsilon\delta}\int_0^\infty dv \int_{\mathbb{R}} dw\, \overline{A}_n(v, v + w\beta_n^{\frac{1}{6}})\frac{\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)}e^{2v+w\beta_n^{\frac{1}{6}}}\overline{z}_n^{(\delta)}(v2^{\frac{1}{3}}), \tag{4.5.19}$$

where we abbreviated $\overline{z}_n^{(\delta)} = \overline{z}_{2(r_n'+\delta\beta_n^{\frac{2}{3}}),2\beta_n}$ (see (4.4.23)).

Split the w-integral into $\int_{-v\beta_n^{-\frac{1}{6}}}^{v\beta_n^{-\frac{1}{6}}} + \int_{v\beta_n^{-\frac{1}{6}}}^{\infty}$. In the first part, estimate $e^{\beta_n^{\frac{1}{3}}(2v+w\beta_n^{\frac{1}{6}})} \leq e^{3v\beta_n^{\frac{1}{3}}}$ and use the following scaled form of the eigenvector relation:

$$\int_{-v\beta_n^{-\frac{1}{6}}}^{\infty} dw\, \overline{A}_n(v, v + w\beta_n^{\frac{1}{6}})\frac{\overline{\tau}_n(v + w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} = \lambda(r_n', \beta_n) = 1 + o(1), \tag{4.5.20}$$

where the last equality follows from Proposition 4.3(i). In the second part, use Proposition 4.3(iii) to get that

$$c_1 = \sup_{n\in\mathbb{N}}\sup_{\widetilde{v}\geq v\geq 0}\frac{\overline{\tau}_n(v + \widetilde{v})}{\overline{\tau}_n(v)} < \infty \tag{4.5.21}$$

Furthermore, from Lemma 2.3(i) it follows that there exists some $c_2 > 0$ such that

$$\overline{A}_n(v, v + w\beta_n^{\frac{1}{6}}) \leq \beta_n^{-\frac{1}{6}}c_2\exp\left[-c_2\frac{w^2}{2v + w\beta_n^{\frac{1}{6}}}\right] \qquad (v, w > 0, n \in \mathbb{N}). \tag{4.5.22}$$

This bound is smaller than $c_2\beta_n^{-\frac{1}{6}}e^{-\frac{1}{3}c_2 w\beta_n^{-\frac{1}{6}}}$ for $v < w\beta_n^{\frac{1}{6}}$. Therefore we can estimate

$$\begin{aligned} &\int_{v\beta_n^{-\frac{1}{6}}}^{\infty} dw\, \overline{A}_n(v, v + w\beta_n^{\frac{1}{6}})\frac{\overline{\tau}_n(v+w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)}e^{\beta_n^{\frac{1}{3}}(2v+w\beta_n^{\frac{1}{6}})} \\ &\leq c_1 e^{2v\beta_n^{\frac{1}{3}}}\int_{v\beta_n^{-\frac{1}{3}}}^{\infty} d\widetilde{w}\, e^{-\widetilde{w}(\frac{c_2}{3}-\beta_n^{\frac{2}{3}})} \\ &\leq c_3 e^{-c_3 v\beta_n^{-\frac{1}{3}}} \end{aligned} \tag{4.5.23}$$

for large n and some $c_3 > 0$.

Collecting all these estimates and substituting them into the r.h.s. of (4.5.19), we get that, for some $\widetilde{C} > 0$,

$$\text{l.h.s. of (4.5.19)} \leq \frac{\widetilde{C}}{N^{\frac{3}{2}}}b_n' e^{-\epsilon\delta}\int_0^\infty dv\, e^{3v\beta_n^{\frac{1}{3}}}\overline{z}_n^{(\delta)}(v2^{\frac{1}{3}}) + o(1). \tag{4.5.24}$$

Now use (4.5.1) and Lemma 4.5 for $2\beta_n$ instead of β_n and for $a = 2^{\frac{1}{3}}a^* + \delta$. $\square$

STEP 4 *There is a $C > 0$ such that for all $N, \epsilon > 0$ and $k \in \mathbb{N}$,*

$$\limsup_{n\to\infty}\sup_{k\in\mathbb{N}}\int_N^\infty dt_1\int_\epsilon^\infty dt_2\int_S ds\ \overline{\nu}_n(s)\overline{g}^{k,-}_{r_n',\beta_n,\delta}(s,t_1,t_2)^2 \le \frac{C}{\epsilon^{\frac{3}{2}}}e^{-N\delta}. \tag{4.5.25}$$

Proof. Fix $t_1, t_2 \ge 0$, recall (4.3.32) and abbreviate $t_k^{\beta_n} = t_k^{\beta_n}(t_1,t_2)$. Apply the Cauchy-Schwarz inequality to the expectation with respect to the stochastic matrix Q_{r_n',β_n} in (4.3.31) (recall (4.3.6)), use (4.3.25) and (4.3.30-4.3.31) and recall that ν_n is invariant for Q_{r_n',β_n}, to see that

$$\begin{aligned}
&\textstyle\int_S ds\ \overline{\nu}_n(s)\overline{g}^{n,-}_{r_n',\beta_n,\delta}(s,t_1,t_2)^2\\
&\le e^{2\delta(t_2-t_1)}\textstyle\int_S ds\ \overline{\nu}_n(s)\beta_n^{-\frac{5}{3}}\left(Q^{t_k^{\beta_n}}_{r_n',\beta_n}f^-_{r_n',\beta_n}(\,\cdot\,;\lceil t_2\beta_n^{-\frac{2}{3}}\rceil)^2\right)((u,v,w)_{\beta_n}) \qquad (4.5.26)\\
&= \textstyle\int_S ds\ \overline{\nu}_n(s)\overline{f}^-_{r_n',\beta_n,\delta}(s,t_2,t_1)^2.
\end{aligned}$$

Now use Step 3 with the roles of N and ϵ reversed. □

4.5.3 Proof of Lemma 4.8: cutting away small t_1, t_2

In this subsection we prove Lemma 4.8 subject to Lemma 4.9. We will prove the second assertion in (4.5.5) only, the proof of the first is similar. For the proof it will be expedient to return to the underlying random walk picture that we left behind at the end of Section 4.2.

First we need some abbreviations. For $k, n \in \mathbb{N}$ and $\varepsilon > 0$, let

$$\begin{aligned}
K^{(n)}_{k,\epsilon} &= e^{r_n'k}E\left(e^{-\beta_n\sum_{x\in\mathbb{Z}}\ell_k^2(x)}1_{\left\{\sum_{x>S_k}\ell_k(x)\le\epsilon\beta_n^{-\frac{2}{3}}\right\}}1_{\{S_k>0\}}\right)\\
&= e^{r_n'k}E\left(e^{-\beta_n\sum_{x\in\mathbb{Z}}\ell_k^2(x)}1_{\left\{\sum_{x<0}\ell_k(x)\le\epsilon\beta_n^{-\frac{2}{3}}\right\}}1_{\{S_k>0\}}\right)
\end{aligned} \tag{4.5.27}$$

(the last equality holds by reversibility of the random walk). Let

$$L^{(n)}_{k,\epsilon} = e^{r_n'k}E\left(e^{-\beta_n\sum_{x\in\mathbb{Z}}\ell_k^2(x)}1_{\left\{\sum_{x>S_k}\ell_k(x)>\epsilon\beta_n^{-\frac{2}{3}},\sum_{x<0}\ell_k(x)>\epsilon\beta_n^{-\frac{2}{3}}\right\}}1_{\{S_k>0\}}\right) \tag{4.5.28}$$

and

$$Z^{(n)}_k = e^{r_n'k}E\left(e^{-\beta_n\sum_{x\in\mathbb{Z}}\ell_k^2(x)}1_{\{S_k>0\}}\right). \tag{4.5.29}$$

The proof of Lemma 4.8 is now divided into five steps. In the first step we will estimate $\int_{\mathbb{R}} dt_1\int_0^\epsilon dt_2\,\gamma_n(t_1,t_2)$ above by $K^{(n)}_{n,\epsilon}$. Then we will prove that $\limsup_{n\to\infty}K^{(n)}_{n,\epsilon}$ vanishes as $\epsilon\downarrow 0$.

STEP 1 *For all $n \in \mathbb{N}$, $\varepsilon > 0$,*

$$\int_{\mathbb{R}} dt_1 \int_0^{\varepsilon} dt_2\, \gamma_n(t_1, t_2) \le K_{n,\varepsilon}^{(n)}. \tag{4.5.30}$$

Proof. Tracing back the steps from Lemma 4.1 to Lemma 4.3, it is seen that t_2 plays the role of the scaled amount of time the random walk $(S_0, \dots, S_n)$ spends below 0. Indeed, recall (4.5.2), (4.3.27) and (4.3.9) and use Lemma 4.2, to see that

$$\begin{aligned} &\int_{\mathbb{R}} dt_1 \int_0^{\varepsilon} dt_2\, \gamma_n(t_1, t_2) \\ &\quad \le \sum_{n_1 \in \mathbb{N}} \sum_{n_2=1}^{\lceil \varepsilon \beta_n^{-\frac{2}{3}} \rceil} \widetilde{\mathbb{E}}^{r'_n, \beta_n} \left(f^+_{r'_n, \beta_n}(\Gamma_0; n_1)\, f^-_{r'_n, \beta_n}(\Gamma_{n-n_1-n_2+1}; n_2) \right). \end{aligned} \tag{4.5.31}$$

Then use Knight's Theorem to see that

$$\begin{aligned} &\int_0^{\infty} dt_1 \int_0^{\varepsilon} dt_2\, \gamma_n(t_1, t_2) \\ &\quad \le e^{r'_n n} E\left(e^{-\beta_n \sum_{x \in \mathbb{Z}} \ell_n^2(x)} 1_{\left\{\sum_{x<0} \ell_n(x) \le \varepsilon \beta_n^{-\frac{2}{3}}\right\}} 1_{\{0 \le S_{n-1} < S_n\}} \right) \le K_{n,\varepsilon}^{(n)}. \end{aligned} \tag{4.5.32}$$

□

Define

$$p_\varepsilon^{(n)} = P\left(\sum_{x<0} \ell_{\lfloor \beta_n^{-\frac{2}{3}} \rfloor}(x) < \varepsilon \beta_n^{-\frac{2}{3}} \right). \tag{4.5.33}$$

In terms of this quantity we have the following bound for $K_{m,\varepsilon}^{(n)}$:

STEP 2 *For all $m \ge \beta_n^{-\frac{2}{3}}$,*

$$K_{m,\varepsilon}^{(n)} \le 2 e^{r'_n \beta_n^{-\frac{2}{3}}} p_\varepsilon^{(n)} Z^{(n)}_{m - \lfloor \beta_n^{-\frac{2}{3}} \rfloor}. \tag{4.5.34}$$

Proof. Obviously, for all $k \le m$,

$$1_{\left\{ \sum\limits_{x > S_m} \ell_m(x) \le \varepsilon \beta_n^{-\frac{2}{3}} \right\}} e^{-\beta_n \sum\limits_{x\in\mathbb{Z}} \ell_m^2(x)} 1_{\{S_m > 0\}} \le 1_{\left\{ \sum\limits_{x > S_m} [\ell_m - \ell_k](x) \le \varepsilon \beta_n^{-\frac{2}{3}} \right\}} e^{-\beta_n \sum\limits_{x\in\mathbb{Z}} \ell_k^2(x)}. \tag{4.5.35}$$

But, $\{[\ell_m - \ell_k](x + S_k)\}_{x\in\mathbb{Z}}$ is independent of $\{\ell_k(x)\}_{x\in\mathbb{Z}}$ and has the same distribution as $\{\ell_{m-k}(x)\}_{x\in\mathbb{Z}}$. Multiply both sides with $e^{r'_n m}$, take expectations on both sides of (4.5.35) and pick $k = m - \lfloor \beta_n^{-\frac{2}{3}} \rfloor$, to arrive at

$$\begin{aligned} K_{m,\epsilon}^{(n)} &\le e^{r'_n m} P\left(\sum_{x > S_{m-k}} \ell_{m-k}(x) \le \varepsilon \beta_n^{-\frac{2}{3}} \right) E\left(e^{-\beta_n \sum\limits_{x\in\mathbb{Z}} \ell_k^2(x)} \right) \\ &= 2 e^{r'_n \lfloor \beta_n^{-\frac{2}{3}} \rfloor} p_\varepsilon^{(n)}\, Z^{(n)}_{m - \lfloor \beta_n^{-\frac{2}{3}} \rfloor}. \end{aligned} \tag{4.5.36}$$

□

STEP 3

$$\limsup_{n\to\infty} p_\varepsilon^{(n)} = \mathcal{O}(\sqrt{\varepsilon}) \quad as \quad \varepsilon \downarrow 0. \tag{4.5.37}$$

Proof. From the Arcsine law (see Spitzer (1976) Section 20) it follows that

$$\lim_{n\to\infty} p_\varepsilon^{(n)} = \frac{2}{\pi}\int_0^\varepsilon \frac{dx}{\sqrt{x(1-x)}}. \tag{4.5.38}$$

□

In view of (4.5.34) in order to prove Lemma 4.8 it suffices to prove that $\{Z^{(n)}_{n-\lfloor\beta_n^{-\frac{2}{3}}\rfloor}\}_{n\in\mathbb{N}}$ is bounded. We will do so by using a recursive chain of estimates on $Z_k^{(n)}$.

STEP 4 *For sufficiently small $\epsilon' > 0$ and all $\lfloor\beta_n^{-\frac{2}{3}}\rfloor < m \le n$,*

$$Z_m^{(n)} \le \frac{1}{2} Z^{(n)}_{m-\lfloor\beta_n^{-\frac{2}{3}}\rfloor} + L^{(n)}_{m,\epsilon'}. \tag{4.5.39}$$

Proof. Use that

$$Z_m^{(n)} \le 2K^{(n)}_{m,\varepsilon'} + L^{(n)}_{m,\varepsilon'} \tag{4.5.40}$$

(recall (4.5.27-4.5.28)). Then use Steps 2-3 and $r_n' = a^*\beta_n^{\frac{2}{3}}(1+o(1))$. □

STEP 5 *Proof of Lemma 4.8.*

Proof. Use Steps 1 - 2 for $m = n$ and Step 3 to get that $\int_{\mathbb{R}} dt_1 \int_0^\varepsilon dt_2\, \gamma_n(t_1,t_2) \le C\sqrt{\epsilon} Z^{(n)}_{n-\lfloor\beta_n^{-\frac{2}{3}}\rfloor}$ for some $C > 0$, all $n \in \mathbb{N}$ and all $\epsilon \in (0, \frac{1}{2})$, say. Define

$$C^{(n)}_{\varepsilon'} = \sup_{k\in\mathbb{N}} L^{(n)}_{k,\varepsilon'}. \tag{4.5.41}$$

Choose ε' according to Step 4 and apply (4.5.39) repeatedly to obtain

$$Z^{(n)}_{n-\lfloor\beta_n^{-\frac{2}{3}}\rfloor} \le C^{(n)}_{\varepsilon'} \sum_{l=0}^{\lfloor n\beta_n^{\frac{2}{3}}\rfloor-1} \left(\frac{1}{2}\right)^l + \left(\frac{1}{2}\right)^{\lfloor n\beta_n^{\frac{2}{3}}\rfloor} \sup_{1\le k\le\lfloor\beta_n^{-\frac{2}{3}}\rfloor} Z_k^{(n)} \le 2C^{(n)}_{\varepsilon'} + e^{r_n'\beta_n^{-\frac{2}{3}}}. \tag{4.5.42}$$

Like in Step 1, for all $\epsilon' > 0$,

$$L^{(n)}_{k,\epsilon'} \le 2\int_{\epsilon'}^\infty dt_1 \int_{\epsilon'}^\infty dt_2 \int_S ds\, \nu_n(s)\, f^+_{n,\delta}(s,t_1,t_2) g^{k,-}_{r_n',\beta_n,\delta}(s,t_1,t_2) + o(1). \tag{4.5.43}$$

(Here we use that the expressions in (4.2.20) and (4.2.21) have the same limiting behavior.) From (4.5.43), (4.5.7) and Steps 3-4 in Section 4.5.2 it follows that

$$\limsup_{n\to\infty} C^{(n)}_{\varepsilon'} < \infty \text{ for all } \varepsilon' > 0. \tag{4.5.44}$$

Finally, use (4.2.5) to get the boundedness of $\{Z^{(n)}_{n-\lfloor\beta_n^{-\frac{2}{3}}\rfloor}\}_{n\in\mathbb{N}}$. □

4.6 Proof of Lemma 4.9: intermediate t_1, t_2

In this section we give the proof of Lemma 4.9 subject to Lemmas 4.4–4.6. The latter will be proved in Section 4.7.

Our strategy is the following. In Section 4.6.1 we show strong convergence of the left argument of the inner product in (4.5.2), defining the function γ_n, and weak relative compactness of the right argument. Consequently, $\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma_n(t_1,t_2)$ converges along certain subsequences. The limit turns out to be independent of the subsequence and is identified in Section 4.6.2 with the help of a certain eigenvalue expansion.

For the remainder of this section, fix some sequence $r_n' = \beta_n^{\frac{2}{3}}(a^* + o(1))$, some $0 < \varepsilon < N < \infty$ and some $\delta > 0$ sufficiently small. Abbreviate

$$R' = S \times [\varepsilon, N] \times [\varepsilon, N] \subset R \tag{4.6.1}$$

and write $L^2(R')$ for the space of the square integrable functions on R'. Regard this as a subspace of $L^2(R)$. Recall the notations and abbreviations introduced at the beginning of Section 4.5.1. We introduce an operator Φ mapping functions $x : \mathbb{R}^+ \to \mathbb{R}$ to functions $\Phi x : S \to \mathbb{R}$ as

$$(\Phi x)(u,v,w) = x(v) 1_{\{u \le 2v\}} \sqrt{\phi_{2v}(w)}, \tag{4.6.2}$$

where ϕ_{2v} is the normal density with mean 0 and variance $2v$. Note that $\|\Phi x\|^2_{L^2(S)} = 2\|x\|^2_{L^{2,\circ}}$ for $x \in L^{2,\circ}$.

4.6.1 Convergence along subsequences: Lemma 4.10

This subsection is devoted to the proof of the following lemma:

Lemma 4.10 *The sequence $\{\sqrt{\overline{\nu}_n}\overline{g}_{n,\delta}\}_{n\in\mathbb{N}}$ is weakly relatively compact in $L^2(R')$. Furthermore, for every subsequence in n there exists a further subsequence along which*

$$\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma_n(t_1,t_2) \to \langle \mu_\delta, g_\delta \rangle_{L^2(R')}, \tag{4.6.3}$$

where g_δ is the weak limit of $\sqrt{\overline{\nu}_n}\overline{g}_{n,\delta}$ along this subsequence, and

$$\mu_\delta(s,t_1,t_2) = e^{\delta(t_1-t_2)}\sqrt{b^*}\left(\Phi \frac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}\right)(s). \tag{4.6.4}$$

Proof. The proof is divided into four steps. First we formulate a general functional analytic statement that will be needed in the proof.

STEP 1 *Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be measurable. If $x, x_1, x_2, \ldots \in L^2(\Omega)$ satisfy*
(i) $x_n \to x$ pointwise on Ω
(ii) $\limsup_{n\to\infty} \|x_n\|_{L^2(\Omega)} \le \|x\|_{L^2(\Omega)}$,
then $x_n \to x$ strongly in $L^2(\Omega)$.

Proof. By Condition (ii), every subsequence has a further subsequence that weakly converges. The weak limit must be x by Condition (i). Since strong convergence is equivalent to weak convergence and Condition (ii), the claim follows. □

For $(u,v,w) \in S$ and $n \in \mathbb{N}$ define

$$\begin{aligned} h_n(u,v,w) &= \tfrac{x_{a^*}(v)}{\overline{\tau}_n(v)}\sqrt{\tfrac{\overline{\nu}_n(u,v,w)}{2v+\beta_n^{\frac{1}{6}}w}}, \\ h(u,v,w) &= \tfrac{\sqrt{b^*}}{2}\left(\Phi\tfrac{x_{a^*}}{\sqrt{\mathrm{id}}}\right)(u,v,w). \end{aligned} \tag{4.6.5}$$

STEP 2 $h_n \to h$ *strongly in* $L^2(S)$.

Proof. Recall (4.5.16) and (4.5.17). Apply Step 1 for $\Omega = S$, $x_n = h_n$ and $x = h$. Condition (i) is satisfied by the uniform convergence of $\overline{\tau}_n$ to x_{a^*} (see Proposition 4.3(ii)) and by the fact that

$$\lim_{n\to\infty} \overline{A}_n(v, v + w\beta_n^{\frac{1}{6}}) = \phi_{2v}(w) \qquad (v > 0, w \in \mathbb{R}) \tag{4.6.6}$$

(which follows from Lemma 2.3(i), (1.4.13) and (4.2.5)). In order to show that Condition (ii) is satisfied, use the scaled eigenvalue relation (4.5.20) to calculate

$$\|h_n\|^2_{L^2(S)} = b'_n \int_0^\infty dv\, x^2_{a^*}(v) \int_{-\infty}^\infty dw\, \overline{A}_n(v, v + w\beta_n^{\frac{1}{6}}) \frac{\overline{\tau}_n(v + w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} = b'_n \lambda(r'_n, \beta_n), \tag{4.6.7}$$

where we use that x_{a^*} is L^2-normalized. Since $\|h\|_{L^2(S)} = b^* \|\frac{x_{a^*}}{\sqrt{\mathrm{id}}}\|_{L^{2,\circ}} = b^*$, the proof is finished via (4.5.1) and Proposition 4.3(i). □

Next abbreviate $\overline{w}_n = \overline{w}_{r'_n,\beta_n}$ and define

$$q_{n,\delta}((u,v,w),t_1,t_2) = \frac{\overline{w}_n(v, t + u\beta_n^{\frac{1}{3}})}{x_{a^*}(v)\sqrt{2v + \beta_n^{\frac{1}{6}}w}} e^{-\delta(t_2-t_1)} \tag{4.6.8}$$

and $\mu_{n,\delta} = q_{n,\delta}h_n \in L^2(R')$ (where h_n is regarded as an element of $L^2(R')$).

STEP 3 $\mu_{n,\delta} \to \mu_\delta$ *strongly in* $L^2(R')$.

Proof. It suffices to handle the case $\delta = 0$, since the dependence of $\mu_{n,\delta}$ and μ_δ on δ is very simple. Since h_n is uniformly bounded on R', in view of Step 2 it is enough to show that $q_{n,0}h \to \mu_0$ strongly in $L^2(R')$.

First we prove the weak convergence. To that end we want to show that $\langle q_{n,0}h, z\rangle_{L^2(R')} \to \langle \mu_0, z\rangle_{L^2(R')}$ for any $z \in L^2(R')$. We need to do this for functions $z(s,t_1,t_2) = y(s,t_2)\, 1_{a\le t_1\le b}$

with $y \in L^2(S \times [\epsilon, N])$ and $[a,b] \subset [\varepsilon, N]$ only, since the class of these functions is dense in $L^2(R')$. For such z we have (abbreviating $s = (u,v,w)$),

$$\begin{aligned}
\langle q_{n,0}h, z\rangle_{L^2(R')} &= \int_\varepsilon^N dt_2 \int_0^\infty dv \int_{-\infty}^\infty dw \int_0^{2v+w\beta_n^{\frac{1}{6}}} du \frac{\int_a^b dt_1\, \overline{w}_n(v, t_1 + u\beta_n^{\frac{1}{3}})}{x_{a^*}(v)\sqrt{2v+\beta_n^{\frac{1}{6}} w}} h(s) y(s,t_2) \\
&\overset{n\to\infty}{\longrightarrow} \int_\varepsilon^N dt_2 \int_0^\infty dv \int_{-\infty}^\infty dw \int_0^{2v} du \frac{\int_a^b dt_1\, w_{a^*}(v,t_1)}{x_{a^*}(v)\sqrt{2v}} h(s) y(s,t_2) \qquad (4.6.9)\\
&= \langle \mu_0, z\rangle_{L^2(R')},
\end{aligned}$$

where we used Lemmas 4.5–4.6 and the dominated convergence theorem.

In order to show the strong convergence, we estimate with the help of Lemma 4.4,

$$\begin{aligned}
&\limsup_{n\to\infty} \|q_{n,0}h\|^2_{L^2(R')} \\
&= (N-\varepsilon) \limsup_{n\to\infty} \int_\varepsilon^N dt_1 \int_0^\infty dv \int_{-\infty}^\infty dw \int_0^{2v+w\beta_n^{\frac{1}{6}}} du \frac{\overline{w}_n(v, t_1+u\beta_n^{\frac{1}{3}})^2}{2v+w\beta_n^{\frac{1}{6}}} \frac{b^*}{2v} \phi_{2v}(w) \\
&\le b^*(N-\varepsilon) \limsup_{n\to\infty} \int_\varepsilon^N dt_1 \int_0^\infty dv \frac{\overline{w}_n(v,t_1)^2}{2v} \qquad (4.6.10)\\
&\le b^*(N-\varepsilon) \int_\varepsilon^N dt_1 \int_0^\infty dv \frac{w_{a^*}(v,t_1)^2}{2v} \\
&= \|\mu_0\|^2_{L^2(R')}.
\end{aligned}$$

Fatou's lemma, together with the weak convergence, implies that $\|q_{n,0}h\|^2_{L^2(R')} \to \|\mu_0\|^2_{L^2(R')}$. Weak convergence together with convergence of norms implies strong convergence. □

STEP 4 *Proof of Lemma 4.10.*

Proof. From (4.5.25) it follows that $\sup_{n\in\mathbb{N}} \|\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}\|_{L^2(R')} < \infty$, and so the first assertion in Lemma 4.10 follows. Given any subsequence in n, choose some further subsequence along which $\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}$ converges weakly towards some $g_\delta \in L^2(R')$. Note that $\mu_{n,\delta} = \sqrt{\overline{\nu}_n}\overline{f}^+_{n,\delta}$. Then the second assertion follows from Step 3 by

$$\text{l.h.s. of (4.6.3)} = \left\langle \mu_{n,\delta}, \overline{g}^-_{n,\delta}\sqrt{\overline{\nu}_n}\right\rangle_{L^2(R')} \overset{n\to\infty}{\longrightarrow} \langle \mu_\delta, g_\delta\rangle_{L^2(R')}. \qquad (4.6.11)$$

□

□

4.6.2 Identification of the limit: Lemma 4.11

The proof of Lemma 4.9 is finished by the following assertion.

Lemma 4.11 *For all weak accumulation points $g_\delta \in L^2(R')$ of $\{\sqrt{\overline{\nu}_n}\,\overline{g}^-_{n,\delta}\}_n$,*

$$\langle \mu_\delta, g_\delta \rangle_{L^2(R')} = b^* \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\ \gamma(t_1, t_2). \tag{4.6.12}$$

Proof. The proof is divided into four steps. Recall Sections 4.4.1 and 4.4.2. For $l \in \mathbb{N}_0$, let

$$w^{(l)}(v,t) = \frac{1}{2} \left\langle y^{(l)}, w_{a^*}(\cdot, t_1) \right\rangle_{L^2} y^{(l)}(v). \tag{4.6.13}$$

Recall (4.6.2).

STEP 1 *For all $g \in L^2(R')$,*

$$\langle \mu_\delta, g \rangle_{L^2(R')} = \sum_{l=0}^{\infty} \sqrt{b^*} \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, e^{\delta(t_1-t_2)} \left\langle \left(\Phi w^{(l)}(\cdot, t_1)\right)(\cdot), g(\cdot, t_1, t_2) \right\rangle_{L^2(S)}. \tag{4.6.14}$$

Proof. Observe from (4.6.13) and (4.4.8) that $\sum_{l=0}^{k} w^{(l)}(\cdot, t_1)$ is the $L^{2,\circ}$- projection of $\frac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}$ onto the subspace spanned by $y^{(0)}, \dots, y^{(k)}$. According to (4.4.8), it therefore converges in $L^{2,\circ}$ to $\frac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}$ as $k \to \infty$, for any $t_1 \in [\varepsilon, N]$. Thus we also have $\sum_{l=0}^{k} \left(\Phi w^{(l)}(\cdot, t_1)\right) \to \left(\Phi \frac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}}\right)$ in $L^2(S)$. In view of (4.6.4), the dominated convergence theorem yields the assertion since we have the following integrable majorant:

$$\begin{aligned} &\left| e^{\delta(t_1-t_2)} \textstyle\sum_{l=0}^{k} \left\langle \left(\Phi w^{(l)}(\cdot, t_1)\right)(\cdot), g(\cdot, t_1, t_2) \right\rangle_{L^2(S)} \right| \\ &\le \|g(\cdot, t_1, t_2)\|_{L^2(S)} e^{\delta N} \left\| \textstyle\sum_{l=0}^{k} w^{(l)}(\cdot, t_1) \right\|_{L^{2,\circ}} \\ &\le \|g(\cdot, t_1, t_2)\|_{L^2(S)} e^{\delta N} \left\| \tfrac{w_{a^*}(\cdot,t_1)}{2\mathrm{id}} \right\|_{L^{2,\circ}}, \end{aligned} \tag{4.6.15}$$

where the second inequality is Bessel's inequality. Now use the estimate $w_{a^*}(v,t) \le e^{a^* t}\psi_v(t)$ (see (4.4.17)) to get the bound. □

Later we will apply Step 1 for g a weak accumulation point of $\{\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}\}_{n\in\mathbb{N}}$ to identify each summand in the r.h.s. of (4.6.14). For this we will use an approximation of $\Phi y^{(l)}$, where $y^{(l)}$ appears in the definition of $w^{(l)}$ in (4.6.13), in terms of the left-eigenvectors of $Q_{r'_n,\beta_n}$ introduced in Section 4.4.1. It will in fact turn out that every summand in the r.h.s. of (4.6.14) is equal to zero with the exception of the 0'th one, which is equal to the r.h.s. of (4.6.12). As we already mentioned in Section 4.4.1, we suspect that it is not possible to expand $\sqrt{\overline{\nu}_n}\overline{g}^-_{n,\delta}$ directly in terms of $\{y^{(l)}_{r'_n,\beta_n}\}_{l\in\mathbb{N}_0}$ (see (4.4.12)).

Fix $l \in \mathbb{N}_0$ and choose n so large that $\lambda^{(l)}(r, \beta_n) > 0$ for all $r \in \mathbb{R}$. In the sequel we will abbreviate $\alpha_n^{(l)} = \alpha^{(l)}(r'_n, \beta_n)$ and $\nu_n^{(l)} = \nu^{(l)}_{r'_n,\beta_n}$, $y_n^{(l)} = y^{(l)}_{r'_n,\beta_n}$, and we introduce $b_n^{(l)} = \beta_n^{\frac{1}{3}}[\partial_r \lambda^{(l)}(r'_n - \alpha_n^{(l)}, \beta_n)]^{-1}$. Note that from Proposition 4.3 and the monotonicity of $r \mapsto \lambda^{(l)}(r, \beta_n)$ and $a \mapsto \rho^{(l)}(a)$ we have

$$\lim_{n\to\infty} \alpha_n^{(l)} \beta_n^{-\frac{2}{3}} = \alpha^{(l)} \tag{4.6.16}$$

(see also (2.7.3-2.7.4)). The eigenvector property of $\nu_n^{(l)}$, defined in (4.4.10), leads to the following. Recall (4.4.12) and (4.3.25) and define $\overline{y}_n^{(l)}(u,v,w) = \beta_n^{-\frac{5}{12}} y_n^{(l)}((u,v,w)_{\beta_n})$ for $(u, v, w) \in S$.

STEP 2 *For large $n \in \mathbb{N}$ and all $t_1, t_2 \in [\varepsilon, N]$,*

$$\left\langle \overline{y}_n^{(l)}, \sqrt{\overline{\nu}_n} \overline{g}^-_{n,\delta}(\cdot, t_1, t_2) \right\rangle_{L^2(S)} = e^{\alpha_n^{(l)} t_n(t_1,t_2)} \left\langle \overline{y}_n^{(l)}, \sqrt{\overline{\nu}_n} \overline{f}^-_{n,\delta}(\cdot, t_1, t_2) \right\rangle_{L^2(S)}, \tag{4.6.17}$$

where $t_n(t_1, t_2) = n - \lceil t_1 \beta_n^{-\frac{2}{3}} \rceil - \lceil t_2 \beta_n^{-\frac{2}{3}} \rceil$.

Proof. Note that the term $\sqrt{\overline{\nu}_n}$ cancels in both inner products. From (4.3.30-4.3.31) and (4.3.9) it can be seen that (4.6.17) is nothing but the inner product of the scaled version of (4.4.11) with $f^-_{n,\delta}(\cdot, \lceil t_1 \beta_n^{-\frac{2}{3}} \rceil, \lceil t_2 \beta_n^{-\frac{2}{3}} \rceil)$. □

Recall the notation in (4.6.2).

STEP 3 *For any $l \in \mathbb{N}_0$,*

$$\lim_{n\to\infty} \| \overline{y}_n^{(l)} - \Phi y^{(l)} \|_{L^2(S)} = 0. \tag{4.6.18}$$

Proof. Recall that $\overline{\tau}_n = \overline{\tau}_{r'_n,\beta_n}$ (see also (4.5.16)). Define for $(u, v, w) \in S$

$$\begin{aligned} \Phi_n(u,v,w) &= 1_{\{u \le 2v + w\beta_n^{\frac{1}{6}}\}} e^{-\alpha_n^{(l)} \beta_n^{-\frac{1}{3}} (2v + w\beta_n^{\frac{1}{6}} - u)} \\ &\quad \times \sqrt{\overline{A}_n(v, v + w\beta_n^{\frac{1}{6}}) \frac{\overline{\tau}_n(v)}{\overline{\tau}_n(v + w\beta_n^{\frac{1}{6}})}}. \end{aligned} \tag{4.6.19}$$

Then it is clear that

$$\overline{y}_n^{(l)}(u,v,w) = \Phi_n(u,v,w) \sqrt{b_n^{(l)}}\, \overline{\tau}^{(l)}_{r'_n - \alpha_n^{(l)}, \beta_n}(v + w\beta_n^{\frac{1}{6}}) \qquad ((u,v,w) \in S). \tag{4.6.20}$$

As an intermediate step, we show first that

$$\lim_{n\to\infty} \| \widehat{y}_n^{(l)} - y_n^{(l)} \|^2_{L^2(S)} = 0, \tag{4.6.21}$$

where

$$\widehat{y}_n^{(l)}(u,v,w) = \Phi_n(u,v,w)y^{(l)}(v + w\beta_n^{\frac{1}{6}}) \qquad ((u,v,w) \in S). \tag{4.6.22}$$

To this end, we write $\int_S ds = \int_{\mathbb{R}} dw \int_0^\infty dv \int_0^{2v+w\beta_n^{\frac{1}{6}}} du$, shift the v-integral by $w\beta_n^{\frac{1}{6}}$ and evaluate the u-integral, to get

$$\begin{aligned}
&\|\widehat{y}_n^{(l)} - \overline{y}_n^{(l)}\|_{L^2(S)}^2 \\
&= \int_{\mathbb{R}} dw \int_{w\beta_n^{\frac{1}{6}}}^\infty dv \int_0^{2v-w\beta_n^{\frac{1}{6}}} du\, \Phi_n(u, v - w\beta_n^{\frac{1}{6}}, w)^2 \left(\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n}(v) - y^{(l)}(v)\right)^2 \\
&= \int_{\mathbb{R}} dw \int_{w\beta_n^{\frac{1}{6}}}^\infty dv\, \frac{e^{-\alpha_n^{(l)}\beta_n^{-\frac{1}{3}}(2v-w\beta_n^{\frac{1}{6}})}-1}{-2\alpha_n^{(l)}\beta_n^{-\frac{1}{3}}} \overline{A}_n(v - w\beta_n^{\frac{1}{6}}, v) \frac{\overline{\tau}_n(v-w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} \\
&\quad \times \left(\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n}(v) - y^{(l)}(v)\right)^2 \\
&\leq \int_0^\infty dv \int_{-\infty}^{v\beta_n^{-\frac{1}{6}}} dw\, \overline{A}_n(v - w\beta_n^{\frac{1}{6}}, v) \frac{\overline{\tau}_n(v-w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} \\
&\quad \times (v - \tfrac{w}{2}\beta_n^{\frac{1}{6}}) \left(\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n}(v) - y^{(l)}(v)\right)^2.
\end{aligned} \tag{4.6.23}$$

Similarly as in the proof of Step 3 in Lemma 4.7 in Section 4.5.2, split the w-integral into $\int_{-\infty}^{-v\beta_n^{-\frac{1}{6}}} + \int_{-v\beta_n^{-\frac{1}{6}}}^{v\beta_n^{-\frac{1}{6}}}$. Use (4.5.22) to see that the first part vanishes as $n \to \infty$. In the second part, estimate $v - \frac{w}{2}\beta_n^{\frac{1}{6}} \leq 3v$, carry out the w-integral and use (4.5.20), to see that

$$\|\widehat{y}_n^{(l)} - \overline{y}_n^{(l)}\|_{L^2(S)}^2 \leq o(1) + 3\|\sqrt{b_n^{(l)}}\overline{\tau}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n} - y^{(l)}\|_{L^{2,\circ}}. \tag{4.6.24}$$

Now (4.6.21) follows from Proposition 4.3(i) together with (4.6.16).

In order to show (4.6.18), it is now enough to show that $\widehat{y}_n^{(l)} \to \Phi y^{(l)}$ in $L^2(S)$. To do this, we will apply Step 1 in Section 4.6.1. First, in (4.6.19) we see that Φ_n converges pointwise towards $\Phi = \Phi 1$ on S. Indeed, use (4.6.16), (4.1.3) and (4.6.6) as well as the uniform convergence of $\overline{\tau}_n$ (see Proposition 4.3(ii)) to derive the pointwise convergence of Φ_n to Φ. Since $y^{(l)}$ is continuous on $\mathbb{R}^+$, clearly $\widehat{y}_n^{(l)}$ converges towards $\Phi y^{(l)}$ pointwise on S (see (4.6.22)). Thus, Condition (i) of Step 1 in Section 4.6.1 holds. Next, in order to show that Condition (ii) is satisfied, we derive as in (4.6.23),

$$\|\widehat{y}_n^{(l)}\|_{L^2(S)}^2 = \int_0^\infty dv \int_{-\infty}^{v\beta_n^{-\frac{1}{6}}} dw\, \overline{A}_n(v - w\beta_n^{\frac{1}{6}}, v) \frac{\overline{\tau}_n(v - w\beta_n^{\frac{1}{6}})}{\overline{\tau}_n(v)} (v - \frac{w}{2}\beta_n^{\frac{1}{6}}) y^{(l)}(v)^2. \tag{4.6.25}$$

This time, split the w-integral into $\int_{-\infty}^{-2pv\beta_n^{-\frac{1}{6}}} + \int_{-2pv\beta_n^{-\frac{1}{6}}}^{v\beta_n^{-\frac{1}{6}}}$ for some small $p > 0$. Proceed as in (4.6.23-4.6.24) to arrive at

$$\limsup_{n\to\infty} \|\widehat{y}_n^{(l)}\|_{L^2(S)}^2 \leq (1+p)\|y^{(l)}\|_{L^{2,\circ}}^2 = 1+p. \tag{4.6.26}$$

Letting $p \downarrow 0$, we see that also Condition (ii) holds. □

STEP 4 *Proof of Lemma 4.11.*

Proof. Let g_δ be any accumulation point of $\{\sqrt{\overline{\nu}_n}\overline{g}_{n,\delta}^{-}\}_{n\in\mathbb{N}}$ in $L^2(R')$. We may and will assume that $\sqrt{\overline{\nu}_n}\overline{g}_{n,\delta}^{-} \to g_\delta$ weakly in $L^2(R')$. We apply Step 1 to $g = g_\delta$. We will show that the lth summand in the r.h.s. of (4.6.14) is equal to zero for $l \geq 1$ and equal to $\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, \gamma(t_1,t_2)$ for $l = 0$. To this end, we recall that $(\Phi w^{(l)}(\cdot,t_1))(\cdot) = \frac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2} (\Phi y^{(l)})(\cdot)$ and use Step 3 to see that the l-th summand is equal to

$$\begin{aligned}
&\sqrt{b^*}\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, e^{\delta(t_1-t_2)}\tfrac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2} \left\langle (\Phi y^{(l)})(\cdot), g_\delta(\cdot,t_1,t_2)\right\rangle_{L^2(S)} \\
&= \sqrt{b^*}\lim_{n\to\infty}\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, e^{\delta(t_1-t_2)}\tfrac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2} \left\langle y_n^{(l)}, \sqrt{\overline{\nu}_n}\overline{g}_{n,\delta}^{-}(\cdot,t_1,t_2)\right\rangle_{L^2(S)},
\end{aligned} \tag{4.6.27}$$

since the map $(t_1,t_2) \mapsto e^{\delta(t_1-t_2)}\frac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2}$ is bounded on $[\varepsilon,N]^2$. Use Step 2 to get

$$\begin{aligned}
\text{r.h.s. of (4.6.27)} = \sqrt{b^*}\lim_{n\to\infty} &\int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2\, e^{\delta(t_1-t_2)}\tfrac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2} \\
&\times e^{\alpha_n^{(l)} t_n(t_1,t_2)}\left\langle \overline{y}_n^{(l)}, \sqrt{\overline{\nu}_n}\overline{f}_{n,\delta}^{-}(\cdot,t_1,t_2)\right\rangle_{L^2(S)}.
\end{aligned} \tag{4.6.28}$$

For $l \geq 1$, we estimate

$$\begin{aligned}
&|\text{r.h.s. of (4.6.28)}| \\
&\leq \sqrt{b^*}\limsup_{n\to\infty}\quad e^{\alpha_n^{(l)}(n-N\beta_n^{-\frac{2}{3}})}\left|\int_0^\infty dt_1\, \tfrac{1}{2}\left\langle y^{(l)}, w_{a^*}(\cdot,t_1)\right\rangle_{L^2}\right| \\
&\qquad\times\left|\int_0^\infty dt_2\left\langle \overline{y}_n^{(l)}, \sqrt{\overline{\nu}_n}\overline{f}_{r_n',n}^{-}(\cdot,t_2)\right\rangle_{L^2(S)}\right| \\
&\leq \sqrt{b^*}\limsup_{n\to\infty}\quad e^{\alpha_n^{(l)}(n-N\beta_n^{-\frac{2}{3}})}\tfrac{1}{2}\left|\langle y^{(l)}, z_{a^*}\rangle_{L^2}\left\langle \sqrt{b_n^{(l)}}\overline{T}^{(l)}_{r_n'-\alpha_n^{(l)},\beta_n}, \overline{z}_{r_n',\beta_n}\right\rangle_{L^{2,\circ}}\right|\sqrt{b_n^{(0)}}.
\end{aligned} \tag{4.6.29}$$

The r.h.s. is zero by (4.6.16), because $\alpha^{(l)} < 0$, $n\beta_n^{\frac{2}{3}} \to \infty$ and $(\|\overline{\tau}^{(l)}_{r'_n,\beta_n}\|_{L^{2,\circ}} \|\overline{z}_{r'_n,\beta_n}\|_{L^{2,\circ}})_n$ is bounded (see Proposition 4.3 and Lemma 4.5).

For $l = 0$, recall $y^{(0)} = \sqrt{b^*} x_{a^*}$ and use Proposition 4.3 and Lemma 4.6 to get

$$
\begin{aligned}
&\text{r.h.s. of (4.6.28)} \\
&= \sqrt{b^*} \lim_{n\to\infty} \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2 \, \tfrac{1}{2} \left\langle y^{(0)}, w_{a^*}(\cdot, t_1)\right\rangle_{L^2} \langle \sqrt{\overline{\nu}_n}, \sqrt{\overline{\nu}_n} \overline{f}_{n,0}(\cdot, t_2)\rangle_{L^2(S)} \\
&= \sqrt{b^*} \lim_{n\to\infty} b_n^{(0)} \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2 \, \tfrac{1}{2} \left\langle y^{(0)}, w_{a^*}(\cdot, t_1)\right\rangle_{L^2} \langle \overline{\tau}_n, \overline{w}_{r'_n,\beta_n}(\cdot, t_2)\rangle_{L^2} \qquad (4.6.30) \\
&= \tfrac{b^{*2}}{2} \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2 \, \langle x_{a^*}, w_{a^*}(\cdot, t_1)\rangle_{L^2} \langle x_{a^*}, w_{a^*}(\cdot, t_2)\rangle_{L^2} \\
&= b^* \int_\varepsilon^N dt_1 \int_\varepsilon^N dt_2 \, \gamma(t_1, t_2).
\end{aligned}
$$

□
□

4.7 Proof of Lemmas 4.4–4.6

4.7.1 Proof of Lemma 4.4: properties of $\overline{w}_{r_n,\beta_n}$

In this subsection we prove Lemma 4.4. Our first step is a pointwise asymptotic estimate. Fix $a < a_c, r'_n = a\beta_n^{\frac{2}{3}}(1 + o(1))$ and $v, t \in \mathbb{R}^+$. Pick a sequence $t_n \to t$ such that $\lceil t_n\beta_n^{-\frac{2}{3}}\rceil + \lceil v\beta_n^{-\frac{1}{3}}\rceil$ is even (otherwise $w_{r'_n,\beta_n}(v, t_n) = 0$).

STEP 1 $\limsup_{n\to\infty} \overline{w}_{r'_n,\beta_n}(v, t_n) \leq w_a(v, t)$.

Proof. Note that (4.5.8) implies, with the help of Stirling's formula, for every $\eta > 0$,

$$
\lim_{n\to\infty} \sup \left\{ \left|\psi_w^{(\beta_n)}(s) - \psi_w(s)\right| : w \in \mathbb{R}^+, s \geq \eta, \lceil s\beta_n^{-\frac{2}{3}}\rceil + \lceil w\beta_n^{-\frac{1}{3}}\rceil \text{ even} \right\} = 0. \qquad (4.7.1)
$$

Therefore it suffices to show that (see (4.5.12))

$$
\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}_v^{\star,\beta_n} \left(e^{-\beta_n V^\star} \,\middle|\, U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil \right) \\
&\leq \mathbb{E}_v^\star \left(e^{-\int_0^\infty X^\star(\sigma)^2 \, d\sigma} \,\middle|\, \int_0^\infty X^\star(\sigma) \, d\sigma = t \right).
\end{aligned}
\qquad (4.7.2)
$$

To this end, first note that for every $N \in \mathbb{N}$ and $\delta > 0$,

$$\mathbb{E}_v^{\star,\beta_n}\left(e^{-\beta_n V^\star}\,\Big|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil\right) \le \mathbb{E}_v^{\star,\beta_n}\left(e^{-\frac{1}{N}\sum_{k=1}^{N^2\wedge\xi_\delta^n} Y_n^\star(k)^2}\,\Big|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil\right), \tag{4.7.3}$$

where

$$\begin{aligned} \xi_\delta^n &= \beta_n^{\frac{1}{3}}\inf\{m\in\mathbb{N} : \int_0^{m\beta_n^{\frac{1}{3}}} X_{\beta_n}^\star(\sigma)\,d\sigma \ge t_n - \delta\} \\ Y_n^\star(k) &= \inf\left\{ X_{\beta_n}^\star(\sigma) \,\middle|\, \tfrac{k-1}{N} \le \sigma \le \tfrac{k}{N} \right\}. \end{aligned} \tag{4.7.4}$$

Our next aim is to show that the distribution of $\{Y_n^\star(k\wedge(N\xi_\delta^n))\}_{k=1}^{N^2}$ under $\mathbb{P}_v^{\star,\beta_n}(\,\cdot\,|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil)$ converges towards the one of $\{Y^\star(k\wedge(N\xi_\delta))\}_{k=1}^{N^2}$ under $\mathbb{P}_v^\star(\,\cdot\,|\int_0^\infty X^\star(\sigma)\,d\sigma = t)$, where

$$\begin{aligned} \xi_\delta &= \inf\{s>0 : \textstyle\int_0^s X^\star(\sigma)\,d\sigma \ge t-\delta\} \\ Y^\star(k) &= \inf\left\{ X^\star(\sigma) \,\middle|\, \tfrac{k-1}{N} \le \sigma \le \tfrac{k}{N} \right\}. \end{aligned} \tag{4.7.5}$$

To this end, pick Borel sets $A_1,\dots,A_{N^2}\subset\mathbb{R}_0^+$ and use the strong Markov property for the Markov chain $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ at time $(N_n\wedge\xi_\delta^n)\beta_n^{\frac{1}{3}}$ where $N_n = \beta_n^{\frac{1}{3}}\lceil N\beta_n^{-\frac{1}{3}}\rceil$, to obtain

$$\begin{aligned} &\mathbb{P}_v^{\star,\beta_n}\left(\cap_{k=1}^{N^2}\{Y_n^\star(k\wedge(N\xi_\delta^n))\in A_k\}\,\Big|\,U^\star = \lceil t_n\beta_n^{-\frac{2}{3}}\rceil\right)\psi_v^{(\beta_n)}(t_n) \\ &= \mathbb{E}_v^{\star,\beta_n}\left(\left[\textstyle\prod_{k=1}^{N^2} 1_{\{Y_n^\star(k\wedge(N\xi_\delta^n))\in A_k\}}\right]\psi^{(\beta_n)}_{X^\star_{\beta_n}(N_n\wedge\xi_\delta^n)}\left(t_n - \displaystyle\int_0^{N_n\wedge\xi_\delta^n} X_{\beta_n}^\star(\sigma)\,d\sigma\right)\right) \end{aligned} \tag{4.7.6}$$

(recall (4.2.18)). Since, by (4.7.4),

$$t_n - \int_0^{N_n\wedge\xi_\delta^n} X_{\beta_n}^\star(\sigma)\,d\sigma \ge \delta - \beta_n^{\frac{1}{3}} X_{\beta_n}^\star(N\wedge\xi_\delta^n), \tag{4.7.7}$$

we may insert the indicator on the event $\{t_n - \int_0^{N_n\wedge\xi_\delta^n} X_{\beta_n}^\star(\sigma)\,d\sigma \ge \frac{1}{2}\delta\}$ in the expectation in the r.h.s. of (4.7.6). Indeed, on the complement of this set, we have $X_{\beta_n}^\star(N_n\wedge\xi_\delta^n) \ge \frac{1}{2}\delta\beta_n^{-\frac{1}{3}}$, hence the ψ-term in the expectation in the r.h.s. of (4.7.6) is equal to zero by (4.5.8).

Then use (4.7.1) to see that

$$\begin{aligned} &\lim_{n\to\infty} \text{ r.h.s. of (4.7.6)} \\ &= \lim_{n\to\infty}\mathbb{E}_v^{\star,\beta_n}\left(\left[\textstyle\prod_{k=1}^{N^2} 1_{\{Y_n^\star(k\wedge\xi_\delta^n)\in A_k\}}\right]\psi_{X^\star_{\beta_n}(N\wedge\xi_\delta^n)}\left(t - \textstyle\int_0^{N\wedge\xi_\delta^n} X_{\beta_n}^\star(\sigma)\,d\sigma\right)\right) \end{aligned} \tag{4.7.8}$$

Since $(v,t) \mapsto \psi_v(t)$ is a bounded continuous function on $\mathbb{R}_0^+ \times [\frac{1}{2}\delta, \infty)$ and $X^\star \mapsto \xi_\delta$ is a continuous functional, the map

$$X^\star \mapsto \psi_{X^\star(N\wedge\xi_\delta)}\left(t - \int_0^{N\wedge\xi_\delta} X^\star(\sigma)\, d\sigma\right) \tag{4.7.9}$$

is a bounded continuous functional. Hence, we get from (4.4.28) that

$$\begin{aligned} &\lim_{n\to\infty} \text{l.h.s. of (4.7.6)} \\ &\quad = \mathbb{E}_v^\star\left(\left[\prod_{k=1}^{N^2} 1_{\{Y^\star(k\wedge(N\xi_\delta))\in A_k\}}\right] \psi_{X^\star(N\wedge\xi_\delta)}\left(t - \int_0^{N\wedge\xi_\delta} X^\star(\sigma)\, d\sigma\right)\right) \\ &\quad = \mathbb{P}_v^\star\left(\cap_{k=1}^{N^2}\{Y^\star(k\wedge(N\xi_\delta)) \in A_k\} \mid \int_0^\infty X^\star(\sigma)\, d\sigma = t\right) \psi_v(t), \end{aligned} \tag{4.7.10}$$

where we used the strong Markov property at time $N \wedge \xi_\delta$. Therefore, the distribution of $\{Y_n^\star(k \wedge (N\xi_\delta^n))\}_{k=1}^{N^2}$ under $\mathbb{P}_v^{\star,\beta_n}(\cdot \mid U^\star = \lceil t_n \beta_n^{-\frac{2}{3}} \rceil)$ converges towards that of $\{Y^\star(k \wedge (N\xi_\delta))\}_{k=1}^{N^2}$ under $\mathbb{P}_v^\star(\cdot \mid \int_0^\infty X^\star(\sigma)\, d\sigma = t)$. So we obtain for every $N \in \mathbb{N}$ and $\delta > 0$ that

$$\text{l.h.s. of (4.7.2)} \le \mathbb{E}_v^\star\left(e^{-\frac{1}{N}\sum_{k=1}^{N^2\wedge\xi_\delta} Y^\star(k)^2} \,\middle|\, \int_0^\infty X^\star(\sigma)\, d\sigma = t\right). \tag{4.7.11}$$

Now, let $\delta \to 0$ and $N \to \infty$ to get (4.7.2) by the dominated convergence theorem. □

STEP 2 *Conclusion of the proof of Lemma 4.4*

Proof. Let $I \subset \mathbb{R}^+$ be a compact interval. Since

$$\int_I dt \int_0^\infty dv \frac{\sup_n \overline{w}_{r_n', \beta_n}(v,t)^2}{v} < \infty \tag{4.7.12}$$

(use (4.4.26) and (4.5.8)), we may apply the reversed Fatou inequality, and so by Step 1 the assertion follows. □

4.7.2 Preparations for the proof of Lemmas 4.5–4.6

In this subsection we will start the proof of Lemmas 4.5–4.6. Their proofs will be finished in the next subsection. Recall Section 4.4.4.

We need some more notation. Let

$$\xi_n = \inf\{\sigma > 0 : X_{\beta_n}^\star(\sigma) = 0\} \tag{4.7.13}$$

be the absorption time of $X_{\beta_n}^\star$. Furthermore, define for $l > 0$

$$K_l^{(n)} = \sup_{v\in\mathbb{R}_0^+} \overline{z}_{r_n', \beta_n, l}(v), \tag{4.7.14}$$

where

$$\overline{z}_{r'_n,\beta_n,l}(v) = \mathbb{E}_v^{\star,\beta_n}\left(e^{-\int_0^\infty F_{r'_n}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma}1_{\{\xi_n\le l\}}\right). \tag{4.7.15}$$

Clearly, $K_l^{(n)}$ is finite for all l and n.

STEP 1 *There exists $N>0$ such that for all $l>0$ and $n\in\mathbb{N}$,*

$$K_l^{(n)} = \sup_{v\in[0,N]} \overline{z}_{r'_n,\beta_n,l}(v). \tag{4.7.16}$$

Proof. Pick N so large that $F_{r'_n}^{(\beta_n)}$ is positive and increasing on $[\frac{1}{2}N,\infty)$ for all $n\in\mathbb{N}$. Use the strong Markov property for the Markov chain $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ at time $\beta_n^{-\frac{1}{3}}\tau_N^{(n)}$ where

$$\tau_N^{(n)} = \beta_n^{\frac{1}{3}} \inf\left\{t\in\mathbb{N} : X_{\beta_n}^\star(t\beta_n^{\frac{1}{3}}) \le \frac{1}{2}N\right\} \tag{4.7.17}$$

to estimate

$$\begin{aligned}
\overline{z}_{r'_n,\beta_n,l}(v) &\le \mathbb{E}_v^{\star,\beta_n}\left(e^{-\int_0^{\tau_N^{(n)}} F_{r'_n}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma}1_{\{\xi_n\le l\}}\right)\sup_{u\in[0,\frac{1}{2}N]}\overline{z}_{r'_n,\beta_n,l}(u)\\
&\le \sup_{u\in[0,N]}\mathbb{E}_v^{\star,\beta_n}\left(e^{-\tau_N^{(n)}F_{r'_n}^{(\beta_n)}(\frac{1}{2}N)}\right)\overline{z}_{r'_n,\beta_n,l}(u)\\
&\le \sup_{u\in[0,N]}\overline{z}_{r'_n,\beta_n,l}(u),
\end{aligned} \tag{4.7.18}$$

where in the first inequality we use the monotonicity of $\overline{z}_{r'_n,\beta_n,l}(u)$ in l. □

In Step 2 below, we derive a recursive upper bound for $K_l^{(n)}$. For $\epsilon>0$, define

$$\eta_{k,N}^{(n)} = \sup_{v\in[0,N]}\mathbb{P}_v^{\star,\beta_n}(\xi_n>k) \tag{4.7.19}$$

$$C_{\epsilon,k,N}^{(n)} = \sup_{v\in[0,N]}\mathbb{E}_v^{\star,\beta_n}\left(e^{-(1+\epsilon)\int_0^k F_{r'_n}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma}\right). \tag{4.7.20}$$

STEP 2 *For all l,k,ϵ, $N>0$ and $n\in\mathbb{N}$,*

$$K_l^{(n)} \le C_{0,k,N}^{(n)} + K_l^{(n)}\left[C_{\epsilon,k,N}^{(n)}\right]^{\frac{\epsilon}{1+\epsilon}}\left[\eta_{k,N}^{(n)}\right]^{\frac{1}{1+\epsilon}}. \tag{4.7.21}$$

Proof. Use the monotonicity of $K_l^{(n)}$ in l, the Markov property at time k, and Hölder's inequality to obtain

$$\begin{aligned}
K_l^{(n)} &\leq K_{l+k}^{(n)} \\
&= \sup_{v\in[0,N]} \Bigg\{ \mathbb{E}_v^{\star,\beta_n}\left(e^{-\int_0^k F_{r_n'}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma} 1_{\{\xi_n\leq k\}} \right) \\
&\quad + \int_{\mathbb{R}^+} d\tilde{v}\, \mathbb{E}_v^{\star,\beta_n}\left(e^{-\int_0^k F_{r_n'}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma} 1_{\{\xi_n> k\}}; X_{\beta_n}^\star(k)\in d\tilde{v} \right) \overline{z}_{r_n',\beta_n,l}(\tilde{v}) \Bigg\} \\
&\leq C_{0,k,N}^{(n)} + \sup_{v\in[0,N]} \mathbb{E}_v^{\star,\beta_n}\left(e^{-\int_0^k F_{r_n'}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma} 1_{\{\xi_n> k\}} \right) K_l^{(n)} \\
&\leq C_{0,k,N}^{(n)} + K_l^{(n)} \left[\sup_{v\in[0,N]} \mathbb{E}_v^{\star,\beta_n}\left(e^{-(1+\epsilon)\int_0^k F_{r_n'}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma} \right) \right]^{\frac{\epsilon}{1+\epsilon}} \\
&\quad \times \left[\sup_{v\in[0,N]} \mathbb{P}_v^{\star,\beta_n}(\xi_n>k) \right]^{\frac{1}{1+\epsilon}} \\
&= C_{0,k,N}^{(n)} + K_l^{(n)} \left[C_{\epsilon,k,N}^{(n)} \right]^{\frac{\epsilon}{1+\epsilon}} \left[\eta_{k,N}^{(n)} \right]^{\frac{1}{1+\epsilon}} .
\end{aligned} \tag{4.7.22}$$

□

STEP 3 *For any* $N>0$, $\limsup_{k\to\infty}\limsup_{n\to\infty}\eta_{k,N}^{(n)}=0$.

Proof. For all $n,k\in\mathbb{N}$,

$$\begin{aligned}
\eta_{k,N}^{(n)} &= \sup_{v\in[0,N]} \left[1-\mathbb{P}_v^{\star,\beta_n}(X_{\beta_n}^\star(k)=0) \right] \\
&= \sup_{v\in[0,N]} \left[1-\left(1+\frac{1}{\lceil k\beta_n^{-\frac{1}{3}}\rceil} \right)^{-\lceil v\beta_n^{-\frac{1}{3}}\rceil} \right] \\
&= 1-\left(1+\frac{1}{\lceil k\beta_n^{-\frac{1}{3}}\rceil} \right)^{-\lceil N\beta_n^{-\frac{1}{3}}\rceil} .
\end{aligned} \tag{4.7.23}$$

The second equality is taken from Knight (1963) Theorem 1.2. Since $\beta\mapsto(1+\frac{1}{k\beta})^{-N\beta}$ decreases towards $e^{-\frac{N}{k}}$, the claim follows. □

STEP 4 *For every $N > 0$ and any $\epsilon \geq 0$ such that $a + \epsilon < a_c$,*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} C^{(n)}_{\epsilon,k,N} < \infty. \tag{4.7.24}$$

Proof. For all k,

$$\limsup_{n\to\infty} C^{(n)}_{\epsilon,k,N} \leq \sup_{v\in[0,N]} \mathbb{E}^\star_v \left(e^{-\int_0^k F_{(a+\epsilon)}(X^\star_\sigma)\, d\sigma}\right) \tag{4.7.25}$$

by the following argument. Pick $v_n \in [0, N]$ to be the maximizer in (4.7.20). Choose a subsequence (n_l) along which v_{n_l} converges towards some $v \in [0, N]$ and

$$\limsup_{n\to\infty} C^{(n)}_{\epsilon,k,N} = \lim_{l\to\infty} \mathbb{E}^{\star,\beta_{n_l}}_{v_{n_l}} \left(e^{-(1+\epsilon)\int_0^k F^{(n_l)}_{a_{n_l}}(X^\star_{n_l}(\sigma))\, d\sigma}\right). \tag{4.7.26}$$

Then, by (4.4.28) and the convergence of $F^{(n_l)}_{a_{n_l}}$ towards F_a,

$$\limsup_{n\to\infty} C^{(n)}_{\epsilon,k,N} \leq \mathbb{E}^\star_v \left(e^{-\int_0^k F_{a+\epsilon}(X^\star(\sigma))\, d\sigma}\right). \tag{4.7.27}$$

The r.h.s. of (4.7.27) converges to $z_{a+\epsilon}(v)$ as $k \to \infty$ (see Lemmas 3.5 and 3.7). □

4.7.3 Proof of Lemmas 4.5 and 4.6

STEP 5 *For all $r \in \mathbb{R}$, $\beta > 0$ and $k_1, k_2 \in \mathbb{N}_0$,*

$$z_{r,\beta}(k_1 + k_2) \leq z_{r,\beta}(k_1) z_{r,\beta}(k_2). \tag{4.7.28}$$

Proof. Let $\{m^\star_1(x)\}_{x\in\mathbb{N}_0}$ and $\{m^\star_2(x)\}_{x\in\mathbb{N}_0}$ be two independent copies of the Markov chain $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ in (4.2.9) starting at k_1 and k_2 respectively. Then the distribution of $\{m^\star(x)\}_{x\in\mathbb{N}_0} = \{m^\star_1(x) + m^\star_2(x)\}_{x\in\mathbb{N}_0}$ is equal to $\mathbb{P}^\star_{k_1+k_2}$ (since they are branching processes). Now recall (4.2.19) and estimate

$$-\beta V^\star \leq -\beta \sum_{x\in\mathbb{N}_0} [m^\star_1(x) + m^\star_1(x-1)]^2 - \beta \sum_{x\in\mathbb{N}_0} [m^\star_2(x) + m^\star_2(x-1)]^2, \tag{4.7.29}$$

recall (4.2.19) and (4.4.23), and use the independence. □

Let $a < a_c$ and choose some sequence $r'_n = \beta_n(a + o(1))$. First we derive a uniform bound on the function $\overline{z}_{r'_n,\beta_n}$ defined in (4.4.23).

STEP 6 $\sup_{n\in\mathbb{N}} \sup_{v\geq 0} \overline{z}_{r'_n,\beta_n}(v) < \infty$.

Proof. Pick N as in Step 1 and $\varepsilon > 0$ with $a + \varepsilon < a_c$. Then, according to Steps 3 and 4, for all sufficiently large k and sufficiently large n (depending on k) we may conclude from Step 2 that

$$K_l^{(n)} \leq \frac{C_{0,k,N}^{(n)}}{1 - \left[C_{\epsilon,k,N}^{(n)}\right]^{\frac{\epsilon}{1+\epsilon}} \left[\eta_{k,N}^{(n)}\right]^{\frac{1}{1+\epsilon}}}. \tag{4.7.30}$$

Letting $l \to \infty$ and using the monotone convergence theorem, we obtain that $\sup_{v\geq 0} \overline{z}_{r'_n,\beta_n}(v)$ is bounded above by the r.h.s. of (4.7.30). Letting first $n \to \infty$ and then $k \to \infty$, we get the claim via Step 4. □

STEP 7 *Conclusion of the Proof of Lemma 4.5*

Proof. Pick some $\varepsilon > 0$ and choose $N > 0$ such that

$$z_a(N) \leq 1 - 2\varepsilon. \tag{4.7.31}$$

Lemma 4.6 with $I = \mathbb{R}_0^+$ states

$$\lim_{n\to\infty} \overline{z}_{r'_n,\beta_n}(v) = z_a(v) \qquad (v \geq 0). \tag{4.7.32}$$

Consequently, for all sufficiently large $n \in \mathbb{N}$,

$$\overline{z}_{r'_n,\beta_n}(N) < 1 - \varepsilon. \tag{4.7.33}$$

Recall (4.4.23) and use Step 5 repeatedly to conclude that for all $v \geq 0$,

$$\overline{z}_{r'_n,\beta_n}(v) \leq (1-\varepsilon)^{\lfloor \frac{v}{N} \rfloor} \sup_{u\geq 0} \overline{z}_{r'_n,\beta_n}(u). \tag{4.7.34}$$

Now the assertion follows with $q = (1-\varepsilon)^{\frac{1}{N}}$ and some C chosen according to Step 4. □

STEP 8 *Proof of Lemma 4.6.*

Proof. First we derive the distributional convergence of

$$D_{r'_n}^{(n)} = \exp\left(-\int_0^\infty F_{r'_n}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\, d\sigma\right) 1_{\{\int_{\mathbb{R}_0^+} X_{\beta_n}^\star(\sigma)\, d\sigma \in I\}} \tag{4.7.35}$$

under $\mathbb{P}_v^{\star,\beta_n}$ where $I \subset \mathbb{R}_0^+$ is any interval. We do this for $I = \mathbb{R}_0^+$ only; the general case is similar. To this end, pick $\delta > 0$ and choose k so large such that $\mathbb{P}_v^\star(\xi > k) \leq \delta$ (where $\xi = \inf\{t > 0 : X^\star(t) = 0\}$ denotes the absorption time of $X^\star$) and such that

$\mathbb{P}_v^{\star,\beta_n}(\xi_n > k) \leq \delta$ for all $n \in \mathbb{N}$ (this is possible by Step 3). Then, for every $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}_v^{\star,\beta_n}\left(\int_0^\infty F_{r_n'}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma \geq \gamma\right) \\
&\leq \limsup_{n\to\infty}\left\{\mathbb{P}_v^{\star,\beta_n}\left(\int_0^k F_{r_n'}^{(\beta_n)}(X_{\beta_n}^\star(\sigma))\,d\sigma \geq \gamma\right) + \mathbb{P}_v^{\star,\beta_n}(\xi_n > k)\right\} \\
&\leq \mathbb{P}_v^\star\left(\int_0^k F_a^\star(X^\star(\sigma))\,d\sigma \geq \gamma\right) + \delta \\
&\leq \mathbb{P}_v^\star\left(\int_0^\infty F_a^\star(X^\star(\sigma))\,d\sigma \geq \gamma\right) + 2\delta.
\end{aligned} \tag{4.7.36}$$

Let $\delta \downarrow 0$ to get the upper bound. The lower bound is derived in a similar way.

Thus, we have derived the distributional convergence of $D_{r_n'}^{(n)}$ under $\mathbb{P}_v^{\star,\beta_n}$ towards that of $\exp(-\int_0^\infty F_a^\star(X^\star(\sigma))\,d\sigma)1_{\{\int_{\mathbb{R}_0^+} X^\star(\sigma)\,d\sigma \in I\}}$ under $\mathbb{P}_v^\star$. In order to derive the convergence in the L^1-norm, it is sufficient to prove uniform integrability of the sequence $\{D_{r_n'}^{(n)}\}$. This is done by simply noting that $(D_{r_n'}^{(n)})^{1+\epsilon} \leq D_{b_n}^{(n)}$ with $b_n = (1+\epsilon)r_n' = (1+\epsilon)a\beta_n^{\frac{2}{3}}$ and by applying Step 6 to b_n instead of r_n' for some $\epsilon > 0$ sufficiently small.

Summarizing, we have proved the pointwise convergence in (4.4.31). Using the bound in Lemma 4.5, we conclude that Lemma 4.6 holds, with the help of the dominated convergence theorem. □

4.8 Proof of Theorem 4.1

In this section, we prove Theorem 4.1 subject to Proposition 4.3. Fix $a \in \mathbb{R}$. In the sequel, when $r = a\beta^{\frac{2}{3}}$, we will only indicate the β-dependence and will write $\tau_\beta = \tau_{a\beta^{\frac{2}{3}},\beta}$, $A_\beta = A_{a\beta^{\frac{2}{3}},\beta}$ and $\lambda(\beta) = \lambda(a\beta^{\frac{2}{3}},\beta)$ and so on. Recall Sections 4.4.1-4.4.2. We will make repeated use of the scaling notation (4.4.13). The proof of Theorem 4.1 is divided into four steps.

STEP 1 *For all* $\beta \in \mathbb{R}^+$,

$$\beta^{-\frac{1}{3}}\partial_a^2\lambda(\beta) = \beta^{-\frac{1}{3}}[\partial_a\lambda(\beta)]^2 + \langle\partial_a\overline{\tau}_\beta, \overline{g}_\beta\rangle_{L^2}, \tag{4.8.1}$$

where

$$g_\beta(i) = \beta^{\frac{1}{3}}(2i-1)\tau_\beta(i) \qquad (i \in \mathbb{N}). \tag{4.8.2}$$

Proof. Differentiate (4.4.2) for $l = 0$ with respect to a. □

STEP 2 $\limsup_{\beta\downarrow 0} \|\partial_a \overline{\tau}_\beta\|_{L^2} < \infty$.

Proof. By differentiating the relation $\tau_\beta = \frac{1}{\lambda(\beta)} A_\beta \tau_\beta$ componentwise with respect to a, we have

$$\partial_a \tau_\beta(i) = -\frac{\partial_a \lambda(\beta)}{\lambda(\beta)^2}\tau_\beta(i) + \frac{\beta^{\frac{1}{3}} h_\beta(i)}{\lambda(\beta)} + \frac{(A_\beta \partial_a \tau_\beta)(i)}{\lambda(\beta)}, \tag{4.8.3}$$

where

$$h_\beta(i) = \beta^{-\frac{1}{3}}\left((\partial_a A_\beta)\tau_\beta\right)(i) \qquad (i \in \mathbb{N}). \tag{4.8.4}$$

Multiply (4.8.3) by $\beta^{-\frac{1}{3}}\partial_a \overline{\tau}_\beta(i)$, sum over $i \in \mathbb{N}$ and use the notation (2.3.2) and the fact that $\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta\rangle_{L^2} = 0$, to obtain

$$F^a_\beta(\partial_a \overline{\tau}_\beta) = \beta^{-\frac{1}{3}}[\lambda(\beta) - 1]\|\partial_a \overline{\tau}_\beta\|^2_{L^2} - \langle \overline{h}_\beta, \partial_a \overline{\tau}_\beta\rangle_{L^2}, \tag{4.8.5}$$

Use (4.9.3) below for $y = \partial_a \overline{\tau}_\beta / \|\partial_a \overline{\tau}_\beta\|_{L^2}$ and note that F^a_β is homogeneous of order two and that $\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta\rangle_{L^2} = 0$, to get

$$F^a_\beta(\partial_a \overline{\tau}_\beta) \le \beta^{-\frac{1}{3}}[\lambda^{(1)}(\beta) - 1]\|\partial_a \overline{\tau}_\beta\|^2_{L^2}. \tag{4.8.6}$$

Combine this with (4.8.5) and use the Cauchy-Schwarz inequality to obtain

$$\|\partial_a \overline{\tau}_\beta\|_{L^2} \le \frac{\|\overline{h}_\beta\|_{L^2}}{\beta^{-\frac{1}{3}}[\lambda(\beta) - \lambda^{(1)}(\beta)]}. \tag{4.8.7}$$

Now use that $\limsup_{\beta\downarrow 0} \|\overline{h}_\beta\|_{L^2} < \infty$ (see Lemma 2.11(i)) and that $\beta^{-\frac{1}{3}}[\lambda(\beta) - \lambda^{(1)}(\beta)] \to \rho(a) - \rho^{(1)}(a) > 0$ by Proposition 4.3(*i*) to get the claim. □

STEP 3 *$\partial_a \overline{\tau}_\beta$ converges to $\partial_a x_a$ weakly in L^2 as $\beta \downarrow 0$.*

Proof. By Step 2, every subsequence of $\{\partial_a \overline{\tau}_\beta\}_{\beta>0}$ has a further subsequence that converges weakly in L^2 as $\beta \downarrow 0$. Denote the weak limit along such a subsequence by y_a. We will prove that $y_a = \partial_a x_a$ independently of the subsequence involved. By (4.4.1), it suffices to prove for all $l \in \mathbb{N}_0$ that

$$\langle y_a, x^{(l)}_a\rangle_{L^2} = \langle \partial_a x_a, x^{(l)}_a\rangle_{L^2}. \tag{4.8.8}$$

This is easily derived for $l = 0$, because

$$\langle \partial_a x_a, x^{(0)}_a\rangle_{L^2} = 0 = \langle \partial_a \overline{\tau}_\beta, \overline{\tau}^{(0)}_\beta\rangle_{L^2} \to \langle y_a, x^{(0)}_a\rangle_{L^2} \tag{4.8.9}$$

along this subsequence, since $\overline{\tau}^{(0)}_\beta \to^{L^2} x_a$ according to Theorem 2.3.

In order to derive (4.8.8) for $l \geq 1$, differentiate the relation $\mathcal{L}^a x_a = \rho(a) x_a$ with respect to a to get with $f_a(u) = 2u x_a(u)$

$$f_a + \mathcal{L}^a \partial_a x_a = \rho(a) \partial_a x_a + \rho'(a) x_a. \tag{4.8.10}$$

Now, take the inner product with $x_a^{(l)}$ and use the L^2-symmetry of $\mathcal{L}^a$, $\langle x_a, x_a^{(l)} \rangle_{L^2} = 0$ and the eigenvalue relation $\mathcal{L}^a x_a^{(l)} = \rho^{(l)}(a) x_a^{(l)}$ to get for $l \geq 1$

$$\langle \partial_a x_a, x_a^{(l)} \rangle_{L^2} = \frac{\langle f_a, x_a^{(l)} \rangle_{L^2}}{\rho(a) - \rho^{(l)}(a)}. \tag{4.8.11}$$

By Proposition 4.3(i), $\overline{\tau}_\beta^{(l)}$ converges converges strongly to $x_a^{(l)}$ and therefore

$$\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta^{(l)} \rangle_{L^2} \to \langle y_a, x_a^{(l)} \rangle_{L^2} \tag{4.8.12}$$

along the subsequence. To investigate the l.h.s. of (4.8.12) for $l \geq 1$, multiply (4.8.3) by $\tau_\beta^{(l)}(i)$ and sum over $i \in \mathbb{N}$, to get

$$\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta^{(l)} \rangle_{L^2} = \lambda^{(l)}(\beta) \frac{\langle \overline{h}_\beta, \overline{\tau}_\beta^{(l)} \rangle_{L^2}}{\beta^{-\frac{1}{3}}[\lambda(\beta) - \lambda^{(l)}(\beta)]} \tag{4.8.13}$$

(recall (4.8.4)). Now use Proposition 4.3(i) and the fact that $\overline{h}_\beta \to^{L^2} f_a$ to see that

$$\langle \partial_a \overline{\tau}_\beta, \overline{\tau}_\beta^{(l)} \rangle_{L^2} \to \text{r.h.s. of (4.8.11)}. \tag{4.8.14}$$

Finally, combine (4.8.11), (4.8.12) and (4.8.14) to arrive at (4.8.8) for $l \geq 1$. □

STEP 4 *Conclusion of the proof of Theorem 4.1.*

Proof. Use (4.1.2-4.1.3) to see that the first summand in the r.h.s. of (4.8.1) vanishes as $\beta \downarrow 0$. Since $\overline{g}_\beta \to^{L^2} f_a$, we therefore conclude from Step 1 that

$$\lim_{\beta \downarrow 0} \beta^{-\frac{1}{3}} \partial_a^2 \lambda(\beta) = \langle \partial_a x_a, f_a \rangle_{L^2}. \tag{4.8.15}$$

Differentiate (4.4.5) with respect to a to see that the r.h.s. of (4.8.15) is equal to $\rho''(a)$. □

4.9 Proof of Proposition 4.3

In this section we prove Proposition 4.3. Recall the notion of of epi-convergence introduced in Section 2.2. In Section 4.9.1 we use the Rayleigh formula to derive a variational representation for the spectral gap of $A_{a\beta^{\frac{2}{3}},\beta}$. In Section 4.9.2 we will use the notion of epi-convergence to prove the convergence of eigenvalues and eigenfunctions as stated in Proposition 4.3(i). In Section 4.9.3 we will prove uniform convergence of the scaled largest eigenvector as stated in Proposition 4.3(ii).

4.9.1 Proof of Proposition 4.3(i): variational representations

In this subsection we derive a variational formula for the spectral gap of $A_{a\beta^{\frac{2}{3}},\beta}$. As in Section 4.8, we suppress the dependence on a in the notations of various objects we are dealing with. Fix $\beta \in \mathbb{R}^+$ so small that $\lambda^{(1)}(\beta) > 0$ (see below (4.4.2)).

Rayleigh's formula for $(\lambda^{(1)}(\beta), \overline{\tau}_\beta^{(1)})$ reads as follows. Recall the definition of $F_\beta^a : L^2 \to \overline{\mathbb{R}}$ in (2.2.2).

Lemma 4.12

$$\beta^{-\frac{1}{3}}\big[\lambda^{(1)}(\beta) - 1\big] = \max_{x\in L^2, \|x\|_{L^2}=1, \langle x, \overline{\tau}_\beta^{(0)}\rangle_{L^2}=0} F_\beta^a(x) \tag{4.9.1}$$

$\overline{\tau}_\beta^{(1)}$ is a maximizer.

Proof. Note that by the positivity of $\lambda^{(1)}(\beta)$ (see Section 4.4.1) and Rayleigh's formula, we have

$$\lambda^{(1)}(\beta) = \max_{x\in l^2, \|x\|_{l^2}=1, \langle x, \tau_\beta^{(0)}\rangle_{l^2}=0} \langle x, A_\beta x\rangle_{l^2}. \tag{4.9.2}$$

Now see the proof of Lemma 2.1. □

We now want to follow the same scheme as in Chapter 2 for the representation in Lemma 4.12 replacing the representation in Lemma 2.1. In order to prepare for this, we first rewrite the maximum in (4.9.1) in such a way as to remove the β-dependence from the set over which the maximum is taken. After that we subtract the maximum in (2.3.1) to get the spectral gap.

Lemma 4.13

$$-\beta^{-\frac{1}{3}}\big[\lambda^{(0)}(\beta) - \lambda^{(1)}(\beta)\big] = \max_{y\in L^2, \|y\|_{L^2}=1} \frac{F_\beta^a(y) - \beta^{-\frac{1}{3}}\big[\lambda^{(0)}(\beta) - 1\big]}{1 - \langle y, \overline{\tau}_\beta^{(0)}\rangle_{L^2}^2} \tag{4.9.3}$$

$\overline{\tau}_\beta^{(1)}$ is a maximizer.

Proof. First, since F_β^a is quadratic and $x \mapsto x - \langle x, \overline{\tau}_\beta^{(0)}\rangle_{L^2}\overline{\tau}_\beta^{(0)}$ is surjective from L^2 to $\{x \in L^2 : \langle x, \overline{\tau}_\beta^{(0)}\rangle_{L^2} = 0\}$, we may write

$$\text{r.h.s. of (4.9.1)} = \max_{y\in L^2, \|y\|_{L^2}=1} \frac{F_\beta^a\Big(y - \langle y, \overline{\tau}_\beta^{(0)}\rangle_{L^2}\overline{\tau}_\beta^{(0)}\Big)}{1 - \langle y, \overline{\tau}_\beta^{(0)}\rangle_{L^2}^2} \tag{4.9.4}$$

(where the functional is defined to be $-\infty$ when $y = \overline{\tau}_\beta^{(0)}$). Next, define the bilinear form

$$F_\beta^{(2)}(x,y) = \beta^{-\frac{2}{3}} \int_0^\infty du \int_0^\infty dv\ x(u)y(v)A_\beta(\lceil u\beta^{-\frac{1}{3}}\rceil, \lceil v\beta^{-\frac{1}{3}}\rceil) - \beta^{-\frac{1}{3}}\langle x,y\rangle_{L^2}. \tag{4.9.5}$$

Note that $F_\beta^{(2)}(x,x) = F_\beta^a(x)$. Moreover,

$$F_\beta^{(2)}(x,\overline{\tau}_\beta^{(0)}) = \beta^{-\frac{1}{3}}\left[\lambda^{(0)}(\beta) - 1\right]\langle x,\overline{\tau}_\beta^{(0)}\rangle_{L^2} \tag{4.9.6}$$

because $\overline{\tau}_\beta^{(0)}$ is the scaled eigenvector of A_β associated with $\lambda(\beta)$. Hence

$$\begin{aligned} &F_\beta^a\left(y - \langle y,\overline{\tau}_\beta^{(0)}\rangle_{L^2}\overline{\tau}_\beta^{(0)}\right) \\ &\quad = F_\beta^{(2)}(y,y) - 2\langle y,\overline{\tau}_\beta^{(0)}\rangle_{L^2}F_\beta^{(2)}(y,\overline{\tau}_\beta^{(0)}) + \langle y,\overline{\tau}_\beta^{(0)}\rangle_{L^2}^2 F_\beta^{(2)}(\overline{\tau}_\beta^{(0)},\overline{\tau}_\beta^{(0)}). \end{aligned} \tag{4.9.7}$$

Combine Lemma 4.12 with (4.9.4) and (4.9.6-4.9.7) to get the claim. □

Note that the numerator in the r.h.s. of (4.9.3) is negative for any $y \neq \overline{\tau}_\beta^{(0)}$ with $\|y\|_{L^2} = 1$ by Lemma 2.1(i), and that the denominator is maximal for $y = \overline{\tau}_\beta^{(1)}$ because $\langle\overline{\tau}_\beta^{(1)},\overline{\tau}_\beta^{(0)}\rangle_{L^2} = 0$.

4.9.2 Proof of Proposition 4.3(i): convergence of $\overline{\tau}_\beta^{(l)}$ and $\lambda^{(l)}(\beta)$

In this subsection we use the variational representation in Lemma 4.13 to prove Proposition 4.3(i) for $l = 1$ following the patterns of the proof of Proposition 4.3(i) for $l = 0$, given in Chapter 2. The proof for general l can then easily be completed using induction on l, as is explained in the end of this subsection.

In what follows convergence is studied for $\beta \downarrow 0$ and fixed $a \in \mathbb{R}$. However, all arguments remain valid when a is replaced by $a(\beta)$ with $\lim_{\beta\downarrow 0} a(\beta) = a$, i.e., the convergence is uniform for a in compacts. As in Section 4.8, a is suppressed from the notation when $r = a\beta^{\frac{2}{3}}$ and we only indicate the β-dependence.

We will apply Proposition 2.1 to the maximum in the r.h.s. of (4.9.3), this time with the following choices replacing (2.2.13):

$$\begin{aligned} X &= \{x \in L^2 : \|x\|_{L^2} = 1\} \\ Y &= X \cap C^1(\mathbb{R}_0^+) \\ \tau &= \text{topology induced by } \|\cdot\|_{L^2} \\ G_\beta(x) &= \frac{F_\beta^a(x) - \beta^{-\frac{1}{3}}\left[\lambda^{(0)}(\beta) - 1\right]}{1 - \langle x,\overline{\tau}_\beta^{(0)}\rangle_{L^2}^2} \\ G(x) &= \frac{F(x) - \rho^{(0)}(a)}{1 - \langle x,x_a^{(0)}\rangle_{L^2}^2} \\ K &= K_C = \{x \in Y : F(x) \geq -C\} \text{ for some } C \text{ large enough.} \end{aligned} \tag{4.9.8}$$

STEP 1 *If Assumptions* (1) – (3) *in Proposition 2.1 hold for the choice in (4.9.8), then Proposition 4.3(i) for $l = 1$ follows.*

Proof. Proposition 2.1 then implies that as $\beta \downarrow 0$

$$\begin{aligned} \beta^{-\frac{1}{3}}\left[\lambda^{(0)}(\beta) - \lambda^{(1)}(\beta)\right] &\rightarrow \max_{x\in X} G(x) \\ \overline{\tau}_\beta^{(1)} &\rightarrow^{L^2} \text{unique maximizer of } G. \end{aligned} \tag{4.9.9}$$

Repeat the argument in the proof of Lemma 4.13 to see that

$$\begin{aligned} \rho^{(1)}(a) = \max_{x\in X, \langle x, x_a^{(0)}\rangle_{L^2}=0} F(x) = \rho^{(0)}(a) + \max_{x\in X} G(x) \\ \text{with unique maximizer } x_a^{(1)} \end{aligned} \tag{4.9.10}$$

(recall Section 4.4.1 for the definition of $x_a^{(1)}$). This completes the proof of the scaling of $\lambda^{(1)}(\beta)$ and the L^2-convergence of $\overline{\tau}_\beta^{(1)}$. The $L^{2,\circ}$-convergence of $\overline{\tau}_\beta^{(1)}$ follows from the L^2-convergence and Lemma 4.15(i) below. □

STEP 2 *Proof of Assumptions* (1) – (3) *for the choice in (4.9.8).*

Proof. Proof of Assumption (1)

We know from Lemmas 2.5–2.8 that e-$\lim_{\beta\downarrow 0} F_\beta^a = F$. Moreover, $\overline{\tau}_\beta^{(0)} \rightarrow^{L^2} x_a^{(0)}$ by Proposition 4.3(i) for $l = 0$. Hence, for all $x_\beta \rightarrow^{L^2} x$ we have

$$\langle x_\beta, \overline{\tau}_\beta^{(0)}\rangle_{L^2} \rightarrow \langle x, x_a^{(0)}\rangle_{L^2}. \tag{4.9.11}$$

Since $\beta^{-\frac{1}{3}}\left[\lambda^{(0)}(\beta) - 1\right] \rightarrow \rho^{(0)}(a)$ by Proposition 4.3(i) for $l = 0$, the claim follows.

Proof of Assumption (2)

See Lemma 4.12(ii).

Proof of Assumption (3)

Assumption (3)(i) is proved in Lemma 2.13.

Assumption (3)(ii) follows from (4.9.10).

The proof of Assumption (3)(iii) requires a minor adaptation of the proof of the corresponding statement for $\overline{\tau}_\beta^{(0)}$ in Lemmas 2.9–2.11. The point is to construct a relatively compact sequence of approximate maximizers of G_β approximating $\overline{\tau}_\beta^{(1)}$ in L^2. For this sequence we will pick the following linear and renormalized interpolation of $\tau_\beta^{(1)}$. For sequences $\{\tau(i)\}_{i\in\mathbb{N}}$ introduce the notation

$$\Delta\tau(i) = \tau(i+1) - \tau(i) \qquad (i \in \mathbb{N}) \tag{4.9.12}$$

and define

$$\begin{aligned} \widetilde{\tau}_\beta^{(1)} &= \widehat{\tau}_\beta^{(1)} \|\widehat{\tau}_\beta^{(1)}\|_{L^2}^{-1} \\ \widehat{\tau}_\beta^{(1)}(u) &= \overline{\tau}_\beta^{(1)}(u) + \beta^{-\frac{1}{6}}(u\beta^{-\frac{1}{3}} - i)\Delta\tau_\beta^{(1)}(i-1) \qquad (i-1 < u\beta^{-\frac{1}{3}} \le i) \end{aligned} \tag{4.9.13}$$

(put $\tau_\beta^{(1)}(0) = \tau_\beta^{(1)}(1)$ and compare with (2.4.3)). We see from (4.9.8) and (4.9.11) that Assumption (3)(iii) is implied by the following lemma:

Lemma 4.14

$$\begin{array}{ll} (i) & \widetilde{\tau}_\beta^{(1)} - \overline{\tau}_\beta^{(1)} \to^{L^2} 0 \ \textit{as}\ \beta \downarrow 0 \\ (ii) & F_\beta^a(\overline{\tau}_\beta^{(1)}) - F_\beta^a(\widetilde{\tau}_\beta^{(1)}) \to 0 \ \textit{as}\ \beta \downarrow 0 \\ (iii) & \liminf_{\beta \downarrow 0} F(\widetilde{\tau}_\beta^{(1)}) > -\infty. \end{array}$$

Proof. The estimates in Lemmas 2.9–2.10 for $(\lambda^{(0)}(\beta), \overline{\tau}_\beta^{(0)})$ carry over to $(\lambda^{(1)}(\beta), \overline{\tau}_\beta^{(1)})$ because they only use the eigenvalue/eigenvector relation. Hence, Lemma 4.14 will be proved once we check that the following estimates in Lemma 2.11 carry over as well. In the following, we use C as a generic positive constant.

Lemma 4.15 *For small* $\beta > 0$,

$$\begin{array}{llll} (i) & \sum_{i\in\mathbb{N}} i^2[\tau_\beta^{(1)}(i)]^2 \le C\beta^{-\frac{2}{3}}, & (ii) & \sum_{i\in\mathbb{N}} i\Delta[\tau_\beta^{(1)}(i)]^2 \le C\beta^{\frac{1}{3}}, \\ (iii) & \tau_\beta^{(1)}(0)^2 \le C\beta^{\frac{1}{3}} \log\frac{1}{\beta}, & (iv) & \|\Delta\tau_\beta^{(1)}\|_{l^2}^2 \le C\beta^{\frac{2}{3}} \log\frac{1}{\beta}. \end{array}$$

Proof. (i) We give here a similar proof as for Lemma 2.11(i). First note that

$$\frac{1}{C} \le \beta^{-\frac{1}{3}}\left[\lambda^{(1)}(\beta) - 1\right] \le C \text{ for all } \beta \text{ small enough.} \tag{4.9.14}$$

Indeed, the upper bound is implied by $\lambda^{(1)}(\beta) \le \lambda^{(0)}(\beta)$ and Proposition 4.3(i) for $l = 0$, the lower bound can be obtained from Lemma 4.13 by an approximate test function, namely $x = x_a^{(1)} - \langle x_a^{(1)}, \overline{\tau}_\beta^{(0)}\rangle_{L^2}\overline{\tau}_\beta^{(0)}$, using (4.9.6) and Lemmas 2.5–2.8.

Fix $\beta > 0$ so small that $\lambda^{(1)}(\beta) > 0$. Use the eigenvector property of $\tau_\beta^{(1)}$ and use $1 + \frac{1}{2}\beta i^2 \le e^{\frac{1}{2}\beta(i+j-1)^2}$ for $i, j \in \mathbb{N}$ to estimate

$$\begin{aligned} \lambda^{(1)}(\beta)\left(1 + \tfrac{\beta}{2}\sum_{i\in\mathbb{N}} i^2\tau_\beta^{(1)}(i)^2\right) &= \sum_{i,j\in\mathbb{N}} \left(1 + \tfrac{\beta}{2}i^2\right)\tau_\beta^{(1)}(i) A_{a\beta^{\frac{2}{3}},\beta}(i,j)\tau_\beta^{(1)}(j) \\ &\le \sum_{i,j\in\mathbb{N}} |\tau_\beta^{(1)}(i)| A_{a\beta^{\frac{2}{3}},\frac{1}{2}\beta}(i,j)|\tau_\beta^{(1)}(j)| \le \lambda(a\beta^{\frac{2}{3}}, \tfrac{1}{2}\beta), \end{aligned} \tag{4.9.15}$$

where the last inequality uses the Rayleigh formula.

Now subtract $\lambda^{(1)}(\beta)$ on both sides of (4.9.15) and divide by $\frac{1}{2}\beta^{\frac{1}{3}}\lambda^{(1)}(\beta)$ to arrive at

$$\beta^{\frac{2}{3}}\sum_{i\in\mathbb{N}} i^2\tau_\beta^{(1)}(i)^2 \le \frac{2}{\lambda^{(1)}(\beta)}\beta^{-\frac{1}{3}}\left[\lambda(a\beta^{\frac{2}{3}},\frac{1}{2}\beta)-\lambda^{(1)}(\beta)\right]. \tag{4.9.16}$$

Use Proposition 4.3(i) for $l=0$ and (4.9.14) to see that the r.h.s. of (4.9.16) is bounded as $\beta\downarrow 0$.

(ii) In the proof of Lemma 2.11(ii), Step 1 is an equality for $\tau_\beta^{(1)}$ and Step 2 should be proved with absolute value signs (since $\tau_\beta^{(1)}$ is not nonnegative). This causes no further problems.

(iii) The proof of Lemma 2.11(iii) only uses Chapter 2 Lemma 2.11($i-ii$) and therefore remains valid with the help of Lemma 4.15($i-ii$).

(iv) In the proof of Lemma 2.11(iv), Step 3 is again an equality. In Step 4, again absolute value signs have to be introduced.

This completes the proof of Lemma 4.15. □

Lemma 4.15 completes the proof of Lemma 4.14. □

Lemma 4.14 finishes Step 2. □

STEP 3 *Proof of Proposition 4.3(i) for $l\ge 2$.*

Proof. The extension of the proof to $l\ge 2$ is made via induction on l. We describe the main line of thought, the details are left to the reader.

Indeed, using the Rayleigh representation of $\lambda^{(l)}(\beta)$ for $l\ge 2$ we have

$$\beta^{-\frac{1}{3}}\left[\lambda^{(l)}(\beta)-1\right]=\max_{\substack{x\in L^2,\|x\|_{L^2}=1\\ \langle x,\overline{\tau}_\beta^{(j)}\rangle_{L^2}=0,j=0,\dots,l-1}} F_\beta^a(x). \tag{4.9.17}$$

The β-dependence of the set can be removed as in Lemma 4.13. Using the induction hypothesis and the bounds in Lemma 4.15 for $\overline{\tau}_\beta^{(l)}$ instead of $\tau_\beta^{(1)}$, we can deduce the assertion for l from the one for $0,\dots,l-1$ in the same way as we derived the one for $l=1$ from the one for $l=0$ in Steps 1-2. □

4.9.3 Proof of Proposition 4.3(ii): uniform convergence of $\overline{\tau}_\beta$

In this subsection we prove uniform convergence of $\overline{\tau}_\beta$ by applying the Arzela-Ascoli theorem. The proof is divided into five steps. Fix $a\in\mathbb{R}$ satisfying $|\rho(a)|<1$ and recall (4.9.12).

STEP 1 $\overline{\tau}_\beta(0)\le 1+\beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2}$.

Proof. Pick $k \in \mathbb{N}, k \leq \beta^{-\frac{1}{3}}$ such that $\tau_\beta(k) \leq \beta^{\frac{1}{6}}$. This is possible, since $\|\tau_\beta\|_{l^2} = 1$. Write

$$\overline{\tau}_\beta(0) = \beta^{-\frac{1}{6}}\tau_\beta(1) = \beta^{-\frac{1}{6}}\tau_\beta(k) - \beta^{-\frac{1}{6}}\sum_{i=1}^{k}\Delta\tau_\beta(i) \leq 1 + \beta^{-\frac{1}{6}}\sum_{i=1}^{k}|\Delta\tau_\beta(i)|. \tag{4.9.18}$$

Now use the Cauchy-Schwarz inequality. □

STEP 2 $\limsup_{\beta\downarrow 0}\overline{\tau}_\beta(0) < \infty$.

Proof. Eq. (2.4.25) says that

$$\begin{aligned}\|\Delta\tau_\beta\|_{l^2}^2 \;\leq\; & \tfrac{2}{\lambda(\beta)}\sum_{(i,j)\in\mathbb{N}^2\setminus\{(1,1)\}}\left[1-e^{e_\beta(i-1,j)-e_\beta(i,j)}\right]\tau_\beta(i)A_\beta(i,j)\tau_\beta(j)\\ & -\tau_\beta^2(1)\left[1-\tfrac{2}{\lambda(\beta)}A_\beta(1,1)\right],\end{aligned} \tag{4.9.19}$$

where e_β is the exponent in (1.4.13)

$$e_\beta(i,j) = a\beta^{\frac{2}{3}}(i+j-1) - \beta(i+j-1)^2. \tag{4.9.20}$$

Use $1-e^t \leq t$ for all $t \in \mathbb{R}$ and $e_\beta(i-1,j) - e_\beta(i,j) \geq -a\beta^{\frac{2}{3}}$ and (1.4.13-1.4.14) and (4.1.2) to estimate in (4.9.19)

$$\|\Delta\tau_\beta\|_{l^2}^2 \leq 2|a|\beta^{\frac{2}{3}} - \tau_\beta^2(1)\frac{\lambda(\beta) - e^{a\beta^{\frac{2}{3}}-\beta}}{\lambda(\beta)} = \beta^{\frac{2}{3}}\left(2|a| - \overline{\tau}_\beta^2(0)\rho(a)(1+o(1))\right). \tag{4.9.21}$$

Substitute this in Step 1 and use the triangle inequality to obtain

$$\overline{\tau}_\beta(0) \leq 1 + \sqrt{2|a|} + \overline{\tau}_\beta(0)\sqrt{|\rho(a)|}(1+o(1)). \tag{4.9.22}$$

Now use that $|\rho(a)| < 1$ to get the claim. □

STEP 3 $\limsup_{\beta\downarrow 0}\beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2} < \infty$.

Proof. This is an easy consequence of (4.9.21) and Step 2. □

STEP 4 *As* $\beta \downarrow 0$, $\overline{\tau}_\beta$ *converges uniformly to* x_a *on* $[0, N]$ *for all* $N > 0$.

Proof. Define $\widehat{\tau}_\beta : \mathbb{R}_0^+ \to \mathbb{R}^+$ to be the scaled linear interpolation of $\{\tau_\beta(i)\}_{i\in\mathbb{N}}$, i.e.,

$$\widehat{\tau}_\beta(u) = \overline{\tau}_\beta(u) + \beta^{-\frac{1}{6}}(u\beta^{-\frac{1}{3}} - i)\Delta\tau_\beta(i-1) \qquad (i-1 < u\beta^{-\frac{1}{3}} \leq i, i \in \mathbb{N}). \tag{4.9.23}$$

With the help of Step 3, one obtains that

$$\|\widehat{\tau}_\beta - \overline{\tau}_\beta\|_\infty \leq \beta^{-\frac{1}{6}}\|\Delta\tau\|_{l^2} \to 0 \qquad (\beta \downarrow 0). \tag{4.9.24}$$

Similarly as the proof of Step 1, one obtains that

$$|\widehat{\tau}_\beta(u)-\widehat{\tau}_\beta(v)| = \left|\int_v^u \overline{\Delta\tau_\beta}(s)\,ds\right| \le |u-v|^{\frac{1}{2}}\beta^{-\frac{1}{3}}\|\Delta\tau_\beta\|_{l^2} \tag{4.9.25}$$

using the Cauchy-Schwarz inequality. Then use the Arzela-Ascoli theorem and Steps 2-3 to see that $\{\widehat{\tau}_\beta\}_{\beta\in\mathbb{R}^+}$ is relatively compact in the uniform norm on $[0,N]$. Use (4.9.24) and Proposition 4.3(i) for $l=0$ to finish the proof. □

STEP 5 *Conclusion of the proof of Proposition 4.3(ii).*

Proof. Let $\epsilon>0$ be given. We can choose N so large that $\max\{\overline{\tau}_\beta(v), x_a(v)\} < \frac{\epsilon}{2}$ for all $v\ge N$, β sufficiently small. This is possible since for all $u\in[v,v+1]$

$$\overline{\tau}_\beta(v) \le \overline{\tau}_\beta(u) + \left(\int_v^\infty \overline{\Delta\tau}^2_\beta(u)du\right)^{1/2} \tag{4.9.26}$$

by a similar argument as in Step 1. Hence, by Lemma 2.11($i-ii$) and the Cauchy-Schwarz inequality, for all $v\ge N$

$$\begin{aligned}
\overline{\tau}_\beta(v) &\le \textstyle\int_v^{v+1}\overline{\tau}_\beta(u)du + \left(\int_N^\infty \overline{\Delta\tau}^2_\beta(u)du\right)^{1/2}\\
&\le \tfrac{1}{N}\left(\textstyle\int_N^\infty u^2\overline{\tau}^2_\beta(u)du\right)^{1/2} + \tfrac{1}{\sqrt{N}}\left(\int_N^\infty u\overline{\Delta\tau}^2_\beta(u)du\right)^{1/2}\\
&= \mathcal{O}\left(\tfrac{1}{\sqrt{N}}\right).
\end{aligned} \tag{4.9.27}$$

From Step 4 we then have $\sup_{v\in[0,N]}|\overline{\tau}_\beta(v)-x_a(v)|<\epsilon$ for sufficiently small β. Thus, for those β, we have $\|\overline{\tau}_\beta - x_a\|_\infty \le \epsilon$.

4.9.4 Proof of Proposition 4.3(iii)

The proof presented here is similar to the proof of Step 5 in Section 4.7.3.

Note that

$$\begin{aligned}
\tau_{r,\beta}(k_1+k_2)\tau_{r,\beta}(1) &= \lim_{d\to\infty}\frac{1}{\lambda^d(r,\beta)}A^d_{r,\beta}(k_1+k_2,1)\\
&= \lim_{d\to\infty}\frac{1}{\lambda^d(r,\beta)}E_{k_1+k_2}\left(e^{rU_d+V_d}1_{m(d)=1}\right).
\end{aligned} \tag{4.9.28}$$

Here the first equality follows from the fact that $A_{r,\beta}$ has a positive spectral gap, while the second equality follows from (1.4.13-1.4.14), (4.2.9) and (4.2.16-4.2.17). Now, let $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ and $\{m(x)\}_{x\in\mathbb{N}_0}$ be two independent Markov chains as in (4.2.9) starting at k_1 and k_2 respectively. Then the distribution of $\{m^\star(x)+m(x)\}_{x\in\mathbb{N}_0}$ is equal to $\mathbb{P}_{k_1+k_2}$

(since they are branching processes with geometric offspring distribution and $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ has no immigrant, while $\{m(x)\}_{x\in\mathbb{N}_0}$ has one immigrant). Furthermore, use (4.2.16) to estimate

$$\begin{aligned} -\beta V_d &= -\beta\sum_{x=1}^{d}\big(m^\star(x)+m(x)+m^\star(x-1)+m(x-1)-1\big)^2 \\ &\le -\beta\sum_{x=1}^{d}\big(m^\star(x)+m^\star(x-1)\big)^2-\beta\sum_{x=1}^{d}\big(m(x)+m(x-1)-1\big)^2. \end{aligned} \tag{4.9.29}$$

Then use (4.2.16-4.2.19), (4.9.29) and the independence of $\{m^\star(x)\}_{x\in\mathbb{N}_0}$ and $\{m(x)\}_{x\in\mathbb{N}_0}$ to bound the expectation in the r.h.s. of (4.9.28) by

$$\tau_{r,\beta}(k_1+k_2)\tau_{r,\beta}(1)\le\lim_{d\to\infty}E^\star_{k_1}\left(e^{rU^\star-\beta V^\star}1_{m^\star(d)=0}\right)\frac{1}{\lambda^d(r,\beta)}E_{k_2}\left(e^{rU_d-\beta V_d}1_{m(d)=1}\right). \tag{4.9.30}$$

Recall (4.9.28) and (4.4.23) to arrive at

$$\tau_{r,\beta}(k_1+k_2)\tau_{r,\beta}(1)\le z_{r,\beta}(k_1)\tau_{r,\beta}(k_2)\tau_{r,\beta}(1). \tag{4.9.31}$$

Dividing by $\tau_{r,\beta}(1)$ and Lemma 4.5 now prove the claim. □

Acknowledgment: We thank Hans Duistermaat for many useful discussions during all stages of this chapter.

Chapter 5

The constants in the central limit theorems

In this chapter we will investigate the constants a^*, b^*, c^*, $\widehat{L}$ and L appearing in Chapters 2 to 4 (recall (2.1.18), (3.1.4)), (4.4.20) and (4.3.35)). Our main results are that the constant c^*, giving the spread of the polymer in both the Edwards model and the weakly interacting Domb-Joyce model, is strictly smaller than 1 and that the $\mathcal{O}(1)$-term of the normalizing constant in Chapter 3 is larger than the one appearing in Chapter 4.

The first result means that the variances in the CLT's for the Domb-Joyce model and the Edwards model are discontinuous at $\beta = 0$ and that the fluctuations around the asymptotic means are squeezed compared to the fluctuations of simple random walk and free Brownian motion. Intuitively, this is because the endpoint of the path lives on a larger scale than free Brownian motion and simple random walk, respectively. Therefore, we can think of the law of the endpoint of being "less random", which implies that the variance is smaller.

The second statement means that the normalizing constant in the Edwards model is larger than the normalizing constant in the Domb-Joyce model. This is intuitively reasonable: simple random walk is restricted to the integers, while Brownian motion is free to move over the real line. Brownian motion can therefore optimize the partition function better.

5.1 Main theorem: Theorem 5.1

The following is our main theorem:

Theorem 5.1

(i) $a^* \in [2.188, 2.189]$
(ii) $b^* \in [1.104, 1.124]$
(iii) $c^* \in [0.60, 0.66]$
(iv) $\widehat{L} = \frac{a^*}{2} L > L$.

The proof of Theorem 5.1 is given in Sections 5.2-5.6 and is based on estimates of the eigenvalues of the differential operator $\mathcal{K}^a$ (recall (3.1.3)). Section 5.2 describes the Sturm-Liouville theory with which we can estimate the constants. In Sections 5.3-5.6 we derive the estimates for a^*, b^*, c^*, $\widehat{L}$ and L respectively. The above estimates are computer assisted and we give exact error estimates.

The bounds in Theorem 5.1($i - ii$) can be made arbitrarily sharp by making the estimates of the eigenvalues sharper. For the bound in Theorem 5.1(iii) this is *not* the case, which is due to the fact that c^* in (3.1.4) is a more complicated object.

5.2 Preparations: Lemmas 5.1– 5.4

In this section we will analyze the zeroes of the eigenfunctions of the Sturm-Liouville differential operator $\mathcal{K}^a$ (recall (3.1.3)).

5.2.1 Sturm-Liouville theory: Lemmas 5.1– 5.3

Let $u \mapsto x_{a,\rho}(u)$ be the solution of

$$\big(\mathcal{K}^a x\big)(u) = 2x''(u) + 2x'(u) + (au - u^2)x(u) = \rho x(u), \tag{5.2.1}$$

with

$$x_{a,\rho}(0) = 1, x'_{a,\rho}(0) = \rho \tag{5.2.2}$$

(see also Section 2.6). This solution is unique by Lemma 2.15, but by Lemma 2.16 it need not be in $L^2(\mathbb{R}_0^+)$! In fact, the only values of ρ for which $x_{a,\rho}$ is in $L^2(\mathbb{R}_0^+)$ are the eigenvalues $\rho^{(k)}(a)$ (recall Section 3.3.2). In the sequel we will use the extreme sensitivity of the tails of $x_{a,\rho}$ with respect to a and ρ to get sharp numerical estimates.

The method described here is not new and can be found e.g. in Coddington and Levinson (1955) Theorem 2.1 on page 212 for regular differential operators. The case where the differential operator is regular singular, but $x(0) = 1$ (as in (5.2.1)), can easily be derived from this result. However, we will derscribe the method below, since it gives a good understanding of the behavior in our case.

Suppose that $u(a, \rho) < \infty$ is a zero of $x_{a,\rho}$. The starting point of our investigation is the following lemma:

Lemma 5.1 *For all $a, \rho \in \mathbb{R}$ and $u(a, \rho) < \infty$,*

$$\begin{aligned} \tfrac{\partial}{\partial \rho} u(a,\rho) &\geq 0, \\ \tfrac{\partial}{\partial a} u(a,\rho) &\leq 0. \end{aligned} \tag{5.2.3}$$

Proof. We will prove the first statement only. The proof of the second statement is analogous.

Fix a and suppose $u(a,\rho) < \infty$ is a zero of $x_{a,\rho}$. Then, by the implicit function theorem and the fact that $x'_{a,\rho}(u(a,\rho)) \neq 0$, $\rho \mapsto u(a,\rho)$ is a differentiable function. By (5.2.17) below, $x_{a,\rho}$ can be represented as a power series with coefficients that are differentiable in a and ρ. Hence

$$y_{a,\rho}(u) = \frac{d}{d\rho} x_{a,\rho}(u) \tag{5.2.4}$$

exists. Differentiate $x_{a,\rho}(u(a,\rho)) = 0$ with respect to ρ to get

$$0 = x'_{a,\rho}(u(a,\rho)) \frac{\partial}{\partial \rho} u(a,\rho) + y_{a,\rho}(u(a,\rho)). \tag{5.2.5}$$

Thus, to prove Lemma 5.1 it is sufficient to prove that $x'_{a,\rho}(u(a,\rho))$ and $y_{a,\rho}(u(a,\rho))$ have opposite sign.

To that end, note that $y_{a,\rho}$ satisfies the inhomogeneous differential equation

$$\big(\mathcal{K}^a y_{a,\rho}\big)(u) - \rho y_{a,\rho}(u) = x_{a,\rho}(u), \tag{5.2.6}$$

with

$$y_{a,\rho}(0) = 0, y'_{a,\rho}(0) = 1, \tag{5.2.7}$$

which is obtained by differentiating (5.2.1-5.2.2) with respect to ρ. Now, let $u \mapsto \hat{x}_{a,\rho}(u)$ be any solution of (5.2.1-5.2.2) such that $\lim_{u \downarrow 0} \frac{\hat{x}_{a,\rho}(u)}{\ln u} = -1$ (recall (2.6.22)) and note that $x_{a,\rho}$ and $\hat{x}_{a,\rho}$ are a basis of solutions for the homogeneous equation $\mathcal{K}^a x = \rho x$. Since the Wronskian of the differential equation (5.2.1-5.2.2) equals

$$u x'_{a,\rho}(u) \hat{x}_{a,\rho}(u) - u \hat{x}'_{a,\rho}(u) x_{a,\rho}(u) \equiv 1, \tag{5.2.8}$$

the solution to (5.2.6-5.2.7) is given by

$$y_{a,\rho}(u) = -\hat{x}_{a,\rho}(u) \int_0^u \xi x^2_{a,\rho}(\xi) d\xi + x_{a,\rho}(u) \int_0^u \xi x_{a,\rho}(\xi) \hat{x}_{a,\rho}(\xi) d\xi. \tag{5.2.9}$$

Since $u = u(a,\rho)$ is a zero of $x_{a,\rho}$, we obtain

$$y_{a,\rho}(u(a,\rho)) = -\hat{x}_{a,\rho}(u(a,\rho)) \int_0^{u(a,\rho)} \xi x^2_{a,\rho}(\xi) d\xi, \tag{5.2.10}$$

so $y_{a,\rho}(u(a,\rho))$ has opposite sign from $\hat{x}_{a,\rho}(u(a,\rho))$. Finally, substitution of $u = u(a,\rho)$ into (5.2.8) gives

$$u(a,\rho) x'_{a,\rho}(u(a,\rho)) \hat{x}_{a,\rho}(u(a,\rho)) = 1, \tag{5.2.11}$$

which together with (5.2.10) proves that $x'_{a,\rho}(u(a,\rho))$ and $y_{a,\rho}(u(a,\rho))$ indeed have opposite sign. □

Lemma 5.1 states that if there is a zero for $x_{a,\rho}$, then this zero will move to the left as ρ decreases or a increases and vice versa. Furthermore, $x_{a,\rho}(0) = 1$ prevents zeroes from moving to the negative axis. Hence, $x_{a,\rho}$ can only get more zeroes as ρ decreases or a increases.

Using Lemma 5.1, we will prove the following stronger statement:

Lemma 5.2 *Let $n = n(a,\rho)$ be defined by*

$$n(a,\rho) = \#\{\textit{zeroes of } x_{a,\rho}\}. \tag{5.2.12}$$

Then, for every $a \in \mathbb{R}$, $\rho \mapsto n(a,\rho)$ is a step function that makes a jump precisely at the eigenvalues $\rho^{(k)}(a)$, i.e., $n(a,\rho) = k$ for $\rho \in (\rho^{(k)}(a), \rho^{(k-1)}(a)]$ $(k \geq 1)$.

Proof. Fix $a \in \mathbb{R}$. For $k \in \mathbb{N}$, define

$$A_k = \{\rho : \exists I \subseteq \mathbb{R}_0^+ \text{ bounded such that } x_{a,\rho} \text{ has at least } k \text{ zeroes in } I\}. \tag{5.2.13}$$

Then A_k is an open interval, unbounded to the left by Lemma 5.1 and the fact that $x_{a,\rho}(0) = 1$. Consequently, A_k^c is a closed interval and has a smallest element $\overline{\rho}^{(k)}$. We will show that $\overline{\rho}^{(k)} = \rho^{(k)}(a)$.

To that end, let $u_k(a,\rho)$ be the kth zero of $x_{a,\rho}$. Then

$$\lim_{\rho \uparrow \overline{\rho}^{(k)}} u_k(a,\rho) = \infty. \tag{5.2.14}$$

To see why, suppose that $\lim_{\rho \uparrow \overline{\rho}^{(k)}} u_k(a,\rho) = v < \infty$. Then, by continuity of $\rho \mapsto x_{a,\rho}(u)$, $v = u_k(a, \overline{\rho}^{(k)})$ is the kth (finite) zero of $x_{a,\overline{\rho}^{(k)}}(u)$. Eq. (5.2.5), together with $x'_{a,\overline{\rho}^{(k)}}(v) \neq 0$ and $y_{a,\overline{\rho}^{(k)}}(v) \neq 0$ (recall (5.2.8-5.2.10)), give that $\frac{\partial}{\partial \rho} u_k(a, \overline{\rho}^{(k)}) > 0$, which is a contradiction.

Furthermore, since $\rho \mapsto x_{a,\rho}(u)$, $\rho \mapsto x'_{a,\rho}(u)$ and $\rho \mapsto x''_{a,\rho}(u)$ are continuous for all $u \in \mathbb{R}_0^+$ (see (5.2.17- 5.2.18), $x_{a,\overline{\rho}^{(k)}}(u)$ and $x'_{a,\overline{\rho}^{(k)}}(u)$ have opposite sign for large u by the following reasoning. Let

$$c(a,\rho) = \frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4\rho} \tag{5.2.15}$$

be the last zero of $p_{a,\rho}(u) = u^2 - au + \rho$. Take $\rho < \overline{\rho}^{(k)}$ such that $u_k(a,\rho) > c(a,\rho)$ (recall (5.2.14)). Then $x_{a,\rho}$ has a zero larger than $c(a,\rho)$. Next, rewrite (5.2.1) as

$$[u x'_{a,\rho}(u)]' = p_{a,\rho}(u) x_{a,\rho}(u), \tag{5.2.16}$$

where the $'$ stands for differentiation with respect to u. Then, for all $u \in [c(a,\rho), u_k(a,\rho))$, $x_{a,\rho}(u)$ and $x'_{a,\rho}(u)$ have opposite sign, since otherwise these signs would remain to be the same by (5.2.16) and hence $x_{a,\rho}$ cannot have a zero larger than u. (Note that $p_{a,\rho}(u) \geq 0$ for all $u \geq c(a,\rho)$.) Now, let $\rho \uparrow \overline{\rho}^{(k)}$ and use (5.2.14) and the continuity of $c(a,\rho)$, to see that $x_{a,\overline{\rho}^{(k)}}(u)$ and $x'_{a,\overline{\rho}^{(k)}}(u)$ have opposite sign for $u > c(a, \overline{\rho}^{(k)})$. The only way this is possible is when $\lim_{u \to \infty} x_{a,\overline{\rho}^{(k)}}(u)$ exists and is bounded. Use Lemma 2.16 to see that then $x_{a,\overline{\rho}^{(k)}}$ is in $L^2(\mathbb{R}_0^+)$. Hence, $\overline{\rho}^{(k)}$ has to be an eigenvalue of $\mathcal{K}^a$ in $L^2(\mathbb{R}_0^+)$. It is now easy to see that $\overline{\rho}^{(k)} = \rho^{(k)}(a)$ by counting the number of finite zeroes of $x_{a,\overline{\rho}^{(k)}}$, which has to be exactly $k - 1$. $\square$

Lemma 5.3 *If $v \geq c(a,\rho)$ and if $x_{a,\rho}(v)$ and $x'_{a,\rho}(v)$ have the same sign, then $x_{a,\rho}(u)$ and $x'_{a,\rho}(u)$ have the same sign for all $u \geq v$.*

Proof. Easy. See (5.2.16). □

Lemma 5.3 will be useful in order to determine the number of zeroes of $x_{a,\rho}$ from a computer plot of $x_{a,\rho}(u)$ for u in a bounded interval.

5.2.2 Power series approximation: Lemma 5.4

We end this preparatory section by explaining how we can determine the number of zeroes of $x_{a,\rho}$ in a bounded interval.

Use (2.6.23) to write $x_{a,\rho}(u)$ as a power series

$$x_{a,\rho}(u) = \sum_{n=0}^{\infty} g_n u^n, \tag{5.2.17}$$

where the g_n's satisfy the recurrence relation

$$g_n = \frac{1}{2n^2}(\rho g_{n-1} - a g_{n-2} + g_{n-3}) \quad (n \geq 1), \tag{5.2.18}$$

with $g_0 = 1, g_{-1} = g_{-2} = 0$. By induction on n, it is easy to derive the following bounds:

$$g_n \leq \frac{K(a,\rho)^n}{(n!)^{\frac{2}{3}}} \qquad (n \geq 1), \tag{5.2.19}$$

where $K(a,\rho)$ satisfies

$$\frac{|\rho|}{2^{\frac{5}{3}} K(a,\rho)} + \frac{|a|}{2^{\frac{4}{3}} K(a,\rho)^2} + \frac{1}{2K(a,\rho)^3} \leq 1. \tag{5.2.20}$$

In the sequel we will take

$$K(a,\rho) = \max\{2^{-\frac{2}{3}}|\rho|, \sqrt{\frac{3|a|}{2^{\frac{4}{3}}}}, \sqrt[3]{3}\}. \tag{5.2.21}$$

(This corresponds to bounding the first term in (5.2.20) by $\frac{1}{2}$, the second by $\frac{1}{3}$ and the third by $\frac{1}{6}$. This choice turns out to be good enough for the choices of a and ρ that we will use in the sequel.)

In order to estimate how well the power series with a finite number of terms approximates $x_{a,\rho}(u)$ on a bounded u-interval, we have to know what the contribution is of the remote summands in (5.2.17).

Lemma 5.4 *For every $k \in \mathbb{N}$, $\rho, a \in \mathbb{R}$ and $N \in \mathbb{R}^+$,*

$$|\sum_{n=k}^{\infty} g_n u^n| \leq \frac{[NC_k]^k}{(1-NC_k)\sqrt[3]{2\pi k}} \text{ uniformly for } u \in [0,N], \tag{5.2.22}$$

where C_k is given by

$$C_k = C_k(a,\rho) = \frac{K(a,\rho)e^{\frac{2}{3}}}{k^{\frac{2}{3}}}. \tag{5.2.23}$$

Proof. Use Stirling's inequality

$$n! \geq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{5.2.24}$$

and (5.2.19), to get

$$\begin{aligned} l.h.s.\ (5.2.22) &\leq \sum_{n=k}^{\infty} \frac{[NC_n]^n}{\sqrt[3]{2\pi n}} \\ &= \sum_{n=0}^{\infty} \frac{[NC_{n+k}]^{n+k}}{\sqrt[3]{2\pi(n+k)}} \\ &\leq \frac{[NC_k]^k}{\sqrt[3]{2\pi k}} \sum_{n=0}^{\infty} [NC_k]^n. \end{aligned} \tag{5.2.25}$$

□

We have now completed our preparation and can start with the proof of Theorem 5.1.

5.3 Proof of Theorem 5.1(*i*)

Fix $\rho = 0$ and $N = 8$, $k = 350$. Use Lemma 5.2 to see that if $x_{a,0}$ has a zero then $a > a^*$, while if $x_{a,0}$ has no zero then $a \leq a^*$. Next, (5.2.21) gives that

$$K(a,0) \leq 1.7 \text{ uniformly for } a \leq 2.2. \tag{5.3.1}$$

Hence, in (5.2.23),

$$C_k \leq 0.07. \tag{5.3.2}$$

Thus, by (5.2.22), the difference of $x_{a,\rho}$ and the power series approximation of $x_{a,\rho}(u)$ with 350 terms is smaller than or equal to 2×10^{-89} (for these values of N, a, ρ and k). The proof now follows from Figure 3, Lemma 5.3 and the fact that $c(a,0) = a < N = 8$ for $a \leq 2.2$ (recall (5.2.15)).

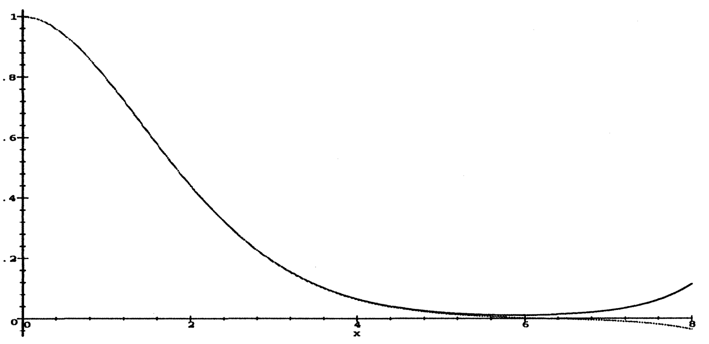


Figure 3a-b: The power series approximation of $x_{a,0}$ with $a = 2.187$, respectively, $a = 2.188$ and $N = 8, k = 350$.

5.4 Proof of Theorem 5.1(ii)

5.4.1 The lower bound for b^*

First we derive an equality, (5.4.2) below, that we will need later on to prove the lower bound for b^*. After that we use the results in Sections 5.2 and 5.3 to prove the upper bound for b^*.

Compute

$$\begin{aligned} 0 = [ux'_{a^*}(u)]^2 \,|_0^\infty &= 2\int_0^\infty (ux'_{a^*}(u))' ux'_{a^*}(u)du \\ &= -\int_0^\infty (a^*u - u^2)ux'_{a^*}(u)x_{a^*}(u)du, \end{aligned} \tag{5.4.1}$$

where we used (5.2.1) and $\rho(a^*) = 0$. Now, integrate by parts to get

$$0 = a^* \int_0^\infty ux^2_{a^*}(u)du - \frac{3}{2}\int_0^\infty u^2x^2_{a^*}(u)du = \frac{a^*}{b^*} - \frac{3}{2}\int_0^\infty u^2x^2_{a^*}(u)du, \tag{5.4.2}$$

since

$$\rho'(a) = \int_0^\infty ux_a^2(u)du. \tag{5.4.3}$$

(Note that the boundary terms at infinity disappear by the super exponential decay of x_{a^*} in Lemma 2.16.)

To prove the lower bound for b^*, use $\frac{1}{b^*} = \rho'(a^*)$ and recall (5.4.3) and write out using partial integration:

$$\begin{aligned} a^* - \tfrac{2}{b^*} &= \int_0^\infty du\ (a^*u - u^2)' x_{a^*}^2(u) \\ &= -2\int_0^\infty du\ (a^*u - u^2) x_{a^*}(u) x'_{a^*}(u) \\ &= 4\int_0^\infty du\ \left[x'_{a^*}(u)^2 + u x''_{a^*}(u) x'_{a^*}(u)\right] \\ &= 2\|x'\|^2_{L^2(\mathbb{R}_0^+)}, \end{aligned} \tag{5.4.4}$$

Here the second equality uses (3.1.3), while the third equality again follows from partial integration. Therefore, a rough lower bound for b^* is

$$a^* - \frac{2}{b^*} \geq 0 \text{ or } b^* \geq \frac{2}{a^*}, \tag{5.4.5}$$

which together with Theorem 5.1(i) gives

$$b^* \geq 0.91. \tag{5.4.6}$$

However, (5.4.6) can be improved using (5.4.2), partial integration and the Cauchy-Schwarz inequality:

$$\begin{aligned} 1 &= -2\int_0^\infty du\ u x_{a^*}(u) x'_{a^*}(u) \\ &\leq 2\|x'\|_{L^2(\mathbb{R}_0^+)} \sqrt{\int_0^\infty du\ u^2 x_{a^*}^2(u)} \\ &= \sqrt{2}\|x'\|_{L^2(\mathbb{R}_0^+)} \sqrt{\tfrac{4}{3}\tfrac{a^*}{b^*}} \\ &= \tfrac{1}{b^*}\sqrt{a^*b^* - 2}\sqrt{\tfrac{4}{3}a^*}. \end{aligned} \tag{5.4.7}$$

Rewrite this to get

$$a^*b^* - 2 \geq b^{*2}\frac{3}{4}\frac{1}{a^*} \tag{5.4.8}$$

or

$$b^* \geq \frac{2}{a^*} + b^{*2}\frac{3}{4}\frac{1}{a^{*2}}. \tag{5.4.9}$$

Now, insert (5.4.6) into the r.h.s. of (5.4.9) and use Theorem 5.1(i) to get

$$b^* \geq 1.043. \tag{5.4.10}$$

Iterating (5.4.9) seven times, each time with the improved lower bound, we arrive at the lower bound in Theorem 5.1(ii).

5.4.2 The upper bound for b^*

To prove the upper bound for b^*, use the monotonicity of $a \mapsto \rho'(a)$ and the relation $b^* = [\rho'(a^*)]^{-1}$ (see Theorems 2.3– 2.4), the mean value theorem and Theorem 5.1(i) to get that

$$b^* \leq \frac{1}{100[\rho(2.188) - \rho(2.178)]}. \tag{5.4.11}$$

Furthermore, $c(a, \rho) \leq 3 < N = 9$ for these values of a, ρ (recall (5.2.15)), so that Lemma 5.3 applies. Recall (5.2.21) to get

$$K(a, \rho) \leq 1.7 \text{ uniformly for } \rho \in [-0.0096, 0], a \in [2.178.2.188]. \tag{5.4.12}$$

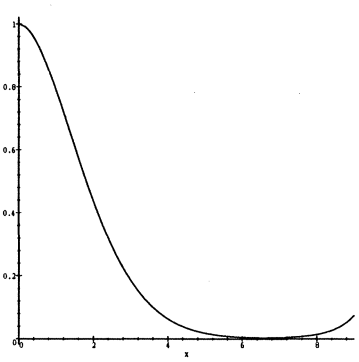
Hence (5.2.23) gives

$$C_k(a, \rho) \leq 0.07 \text{ uniformly for } \rho \in [-0.0096, 0], a \in [2.178.2.188]. \tag{5.4.13}$$

Thus, by (5.2.22), the difference between $x_{a,\rho}(u)$ and the power series approximation of $x_{a,\rho}(u)$ with 350 terms is smaller than or equal to 2×10^{-71} (for these values of N, a, ρ and k). Now use Lemma 5.2 and Figure 4 to get that

$$\begin{aligned} \rho(2.178) &\leq -0.0096 \\ \rho(2.188) &\geq -0.0007, \end{aligned} \tag{5.4.14}$$

since $x_{a,\rho}$ has one zero for $(a, \rho) = (2.187, -0,0096)$ (note that $x_{a,\rho}(N) < 0$ for $(a, \rho) = (2.187, -0,0096)$), while $x_{a,\rho}$ has no zero for $(a, \rho) = (2.188, -0,0007)$ (note that $x_{a,\rho}(N) > 0$ for $(a, \rho) = (2.188, -0,0007)$).



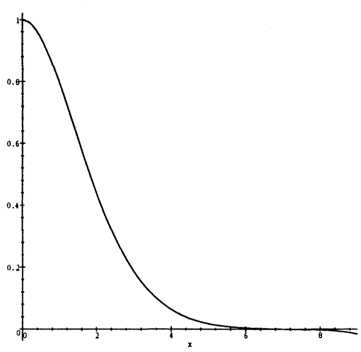


Figure 4a-b: The power series approximation of $x_{a,\rho}$ with $(a,\rho) = (2.188, -0.0007)$, respectively, $(a,\rho) = (2.187, -0.0096)$ and $N = 9, k = 350$.

5.5 Proof of Theorem 5.1(iii)

In Sections 5.5.1-5.5.2 we prove the upper bound for c^*, in Section 5.5.3 the lower bound for c^*.

5.5.1 The upper bound for c^*: Lemmas 5.5–5.6

By differentiating (5.4.3) with respect to a, we get

$$\rho''(a) = 2\int_0^\infty du\ u x_a(u) y_a(u), \tag{5.5.1}$$

where $y_a \in L^2(\mathbb{R}_0^+)$ is the function

$$y_a(u) = \frac{d}{da} x_a(u). \tag{5.5.2}$$

Differentiating the relation $||x_a||^2_{L^2(\mathbb{R}_0^+)} = 1$ with respect to a, we get

$$\langle x_a, y_a \rangle_{L^2(\mathbb{R}_0^+)} = 0. \tag{5.5.3}$$

Hence, we can rewrite (5.5.1) as

$$\rho''(a) = 2\int_0^\infty du\ (u - \rho'(a)) x_a(u)\ y_a(u). \tag{5.5.4}$$

Note that, by (5.4.3), also $u \mapsto (u - \rho'(a))x_a(u)$ is orthogonal to x_a. Furthermore, differentiating the eigenvalue relation $\mathcal{K}^a x_a = \rho(a) x_a$ with respect to a, we get that y_a satisfies the inhomogeneous differential equation

$$-\big(\mathcal{K}^a y\big)(u) + \rho(a) y(u) = f_a(u), \tag{5.5.5}$$

where

$$f_a(u) = (u - \rho'(a)) x_a(u). \tag{5.5.6}$$

Lemma 2.16 gives that all the $\rho^{(k)}(a)$'s have multiplicity one. The Rayleigh representation for $\rho^{(1)}(a)$ reads

$$\rho^{(1)}(a) = \sup_{y: ||y||_{L^2}=1, \langle x_a, y \rangle_{L^2}=0} \langle y, \mathcal{K}^a y \rangle_{L^2}. \tag{5.5.7}$$

Hence, we have that for all $x \in L^2(\mathbb{R}_0^+)$ such that $\langle x_a, x\rangle_{L^2} = 0$,

$$\langle x, (\rho(a) - \mathcal{K}^a)\, x\rangle_{L^2} \geq [\rho(a) - \rho^{(1)}(a)]||x||^2_{L^2(\mathbb{R}_0^+)}. \tag{5.5.8}$$

Therefore, we are in the situation of Griffel (1988) Proposition 10.31. Apply this proposition to get

$$\langle y, f_a\rangle_{L^2} \leq \frac{1}{[\rho(a) - \rho^{(1)}(a)]}||f_a||^2_{L^2}. \tag{5.5.9}$$

Substitute (5.5.6) and use (5.5.3) to get

$$\rho''(a) \leq \frac{2}{[\rho(a) - \rho^{(1)}(a)]}\int_0^\infty du\ (u - \rho'(a))^2\ x_a^2(u). \tag{5.5.10}$$

Because of (5.4.14) and (5.5.11) below, the following two inequalities suffice for the upper bound in Theorem 5.1(*iii*):

Lemma 5.5 $b^{*3}\int_0^\infty du\ (u - \rho'(a^*))^2\ x^2_{a^*}(u) = b^*(\frac{2}{3}a^*b^* - 1) \leq 0.72.$

Proof. See (5.4.2) and Theorem 5.1$(i - ii)$. □

Lemma 5.6 $-\rho^{(1)}(2.2) \in [3.3, 3.4]$.

Proof. See Section 5.5.2 below. □

5.5.2 Proof of Lemma 5.6: Spectral analysis of $\mathcal{K}^{a^*}$

In this section we will prove bounds for $-\rho^{(1)}(2.2)$, using computer plots of $x_{a,\rho}$ for $a = 2.2$ and suitable values of ρ, Lemma 5.2 and the error estimates in Lemma 5.4. Lemma 5.3 guarantees that there are exactly as many zeroes as seen in the plot.

In the same way as in (5.4.3) above, we have

$$\frac{d}{da}\rho^{(k)}(a) = \int_0^\infty du\ u x_a^{(k)}(u)^2 \geq 0, \tag{5.5.11}$$

where $x_a^{(k)}$ is the eigenfunction corresponding to the eigenvalue $\rho^{(k)}(a)$ (recall Section 4.4.1). Hence, all the eigenvalues are increasing in a. Therefore we can take $a = 2.2$ for an upper bound on $\rho^{(1)}(a^*)$. By (5.2.20)

$$K(2.2, \rho) \leq 2.15 \text{ uniformly for } \rho \in [-3.4, 0]. \tag{5.5.12}$$

Again we pick $N = 8$ and $k = 350$. Then by (5.2.23)

$$C_k(2.2, \rho) \leq 0.085 \text{ uniformly for } \rho \in [-3.4, 0]. \tag{5.5.13}$$

Therefore, by (5.2.22), the difference between $x_{a,\rho}(u)$ and the sum of the first 350 terms of the power series of $x_{a,\rho}(u)$ is smaller than or equal to 6×10^{-60}. In Figure 5 the sum of the first 350 terms of the power series of $x_{a,\rho}(u)$ is plotted for $a = 2.2$ and $\rho = -3.3$, respectively, $\rho = -3.4$. Since $c(2.2, -3.4) \leq 8$ and $c(2.2, -3.3) \leq 8$ (recall (5.2.15)), the number of zeroes of $x_{2.2,-3.4}$ is 1 and the number of zeroes of $x_{2.2,-3.3}$ is 2 by Lemma 5.3. This proves that $\rho^{(1)}(2.2) \in [-3.4, -3.3]$.

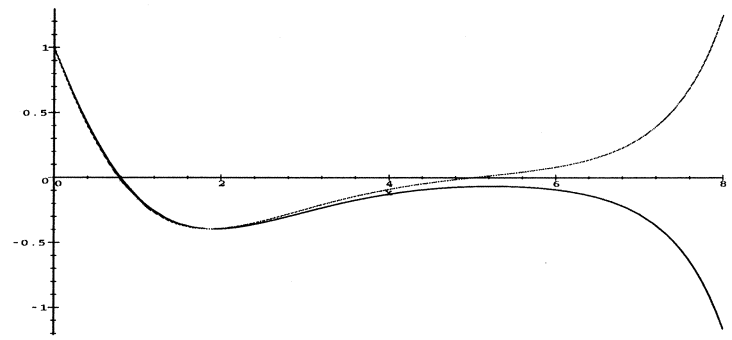


Figure 5a-b: The power series approximation of $x_{a,\rho}$ with $(a, \rho) = (2.2, -3.4)$, respectively, $(2.2, -3.3)$ and $k = 350, N = 8$.

5.5.3 The lower bound for c^*

For some $s > 0$, let

$$y(u) = s(u - \rho'(a))x_a(u). \tag{5.5.14}$$

Then y is orthogonal to x_a (see (5.4.3)).

By (5.5.5) and Griffel (1988) Proposition 10.31, it follows that

$$\frac{1}{2}\rho''(a) = \sup_{x:\|x\|_{L^2}=1, \langle x_a, x\rangle_{L^2}=0} [2\langle x, f_a\rangle_{L^2} + \langle x, (\rho(a) - \mathcal{K}^a)\, x\rangle_{L^2}] \tag{5.5.15}$$

(recall (5.5.6)). Substitution of $x = y$ (see (5.5.14)) gives

$$\frac{1}{2}\rho''(a) \geq \frac{2}{s}\|y\|^2_{L^2} + \langle y, (\rho(a) - \mathcal{K}^a)\, y\rangle_{L^2}. \tag{5.5.16}$$

Next, compute for $a = a^*$,

$$\begin{aligned} (\mathcal{K}^{a^*} y)(u) &= s(u - \rho'(a^*))(\mathcal{K}^{a^*} x_{a^*})(u) + 4sux'_{a^*}(u) + 2sx_{a^*}(u) \\ &= s(4ux'_{a^*}(u) + 2x_{a^*}(u)), \end{aligned} \tag{5.5.17}$$

where we use that $\rho(a^*) = 0$ (see Theorem 2.3). Hence, by partial integration

$$\begin{aligned} \langle y, \mathcal{K}^{a^*} y\rangle_{L^2} &= s^2 \int_0^\infty 4u(u - \rho'(a^*))x'_{a^*}(u)x_{a^*}(u) \\ &= -2s^2\rho'(a^*). \end{aligned} \tag{5.5.18}$$

Furthermore, use (5.4.3) and (5.4.2) to compute

$$\begin{aligned}\|y\|_{L^2}^2 &= s^2(\int_0^\infty u^2 x_{a^*}^2(u)du - \rho'(a^*)^2) \\ &= s^2\rho'(a^*)(\tfrac{2}{3}a^* - \rho'(a^*)).\end{aligned} \tag{5.5.19}$$

Substituting (5.5.17-5.5.19) into (5.5.16) and maximizing over s, we get

$$\rho''(a^*) \geq \rho'(a^*)^3 \left(\frac{2}{3}a^*b^* - 1\right)^2. \tag{5.5.20}$$

The lower bound now follows from the definition $c^{*2} = \frac{\rho''(a^*)}{\rho'(a^*)^3}$ (recall (3.1.4)) and Theorem 5.1$(i-ii)$.

5.6 Proof of Theorem 5.1(iv)

Recall Propositions 3.1 and 4.1 where L and $\widehat{L}$ appear. In this section we use the representations of $\widehat{L}$ and L to prove that $\widehat{L} = \frac{a^*}{2}L$. Since $a^* > 2$ by Theorem 5.1(i), this will prove that $\widehat{L} > L$ as claimed.

Recall from (3.3.33), Lemmas 3.5 and 3.7, (4.4.20) and (4.3.34-4.3.35) that

$$\widehat{L} = b^*\langle x_{a^*}, z_{a^*}\rangle_{L^2}\langle x_{a^*}, z_{a^*}\rangle_{L^2}^{\circ} \tag{5.6.1}$$

$$L = \frac{b^*}{2}\langle x_{a^*}, z_{a^*}\rangle_{L^2}^2, \tag{5.6.2}$$

(see Chapter 3 footnote 1 for the factor $\frac{1}{2}$) where z_{a^*} satisfies the Airy equation

$$2z''(u) + (a^* - u)z(u) = 0 \qquad (u \in \mathbb{R}). \tag{5.6.3}$$

Hence we have to prove that

$$\langle x_{a^*}, z_{a^*}\rangle_{L^2}^{\circ} = \frac{a^*}{2}\langle x_{a^*}, z_{a^*}\rangle_{L^2}. \tag{5.6.4}$$

The proof relies on the fact that the differential equations for x_{a^*} (recall (2.1.13)), respectively, z_{a^*} (recall (5.6.3)) are similar.

Since $\mathcal{K}^{a^*}x_{a^*} = \rho(a^*)x_{a^*} = 0$ (recall Theorem 2.3) and $\mathcal{K}^{a^*}$ is symmetric in $L^2(\mathbb{R}^+)$, we have

$$0 = \langle(\mathcal{K}^{a^*})^i x_{a^*}, z_{a^*}\rangle_{L^2} = \langle x_{a^*}, (\mathcal{K}^{a^*})^i z_{a^*}\rangle_{L^2} \qquad (i = 1, 2). \tag{5.6.5}$$

Compute (for $i = 1$)

$$\left(\mathcal{K}^{a^*} z_{a^*}\right)(u) = 2z'_{a^*}(u) + u\left[2z''_{a^*}(u) + (a^* - u)z_{a^*}(u)\right] = 2z'_{a^*}(u). \tag{5.6.6}$$

This gives (for $i = 2$)

$$\begin{aligned} \left((\mathcal{K}^{a^*})^2 z_{a^*}\right)(u) &= 2\left(\mathcal{K}^{a^*} z'_{a^*}\right)(u) \\ &= 4z''_{a^*}(u) + 2u\left[2z'''_{a^*}(u) + (a^* - u)z'_{a^*}(u)\right] \\ &= 4z''_{a^*}(u) + 2uz_{a^*}(u). \end{aligned} \tag{5.6.7}$$

Substitution of (5.6.3) into (5.6.7) gives

$$\left((\mathcal{K}^{a^*})^2 z_{a^*}\right)(u) = (4u - 2a^*)z_{a^*}(u). \tag{5.6.8}$$

After substituting (5.6.8) into (5.6.5) for $i = 2$, we end up with (5.6.4).

Note: Just prior to completion of this work, we received a letter from John Westwater explaining a different functional analytic method to obtain sharp numerical estimates on a^*, b^*, c^*. Rather than working with the Sturm-Liouville differential equation (5.2.1), he uses the variational problem in Westwater (1984) and a truncation of an expansion into Laguerre polynomials of the minimizer of this variational problem. His method gives rigorous upper bounds on a^*. All other estimates are non-rigorous for lack of error estimates. The values found are fully in agreement with the bounds in Theorem 5.1$(i - iii)$.

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Author index

Parts of this book have appeared in the following papers:

[1] R. van der Hofstad and F. den Hollander, *Scaling for a random polymer*, Communications of Mathematical Physics 169, 397–440, 1995.

[2] R. van der Hofstad, F. den Hollander and W. König, *Central limit theorem for the Edwards model*, Annals of Probability, 25, 573–597, 1997.

[3] R. van der Hofstad, F. den Hollander and W. König, *Central limit theorem for a weakly interacting random polymer*, Markov Processes and Related Fields, 3, 1–63, 1997.

[4] R. van der Hofstad, *The constants in the central limit theorem for the Edwards model.* Preprint 1997. To appear in the Journal of Statistical Physics.

Chapters 2-5 contain the same results as the respective papers above. They are in fact in a form very close to the original papers, but, to avoid repetition of definitions and results, the introductions of Chapters 2-5 have been shortened. Furthermore, the notation has been changed to make the whole consistent. A few proofs in Chapter 2 have been simplified.

Summary

Polymers are long molecules consisting of building blocks called monomers. Polymers can consist of a few thousand monomers. Polymers have two characteristic properties. The first is that they are irregular, because there are different possibilities for the angles between the monomers. The second is that they try to avoid self-intersections because of polarization of the monomers or the excluded-volume-effect. Polarization means that the monomers have an electrical charge. The excluded-volume-effect means that once there is a monomer in a certain position, this position is full and there can be no other monomer there. Realistic dimensions for polymers are two (for a polymer on a surface) and three (e.g. for a polymer in a solvent). However, it turns out that two- and three-dimensional polymers are mathematically too difficult and therefore we mainly consider one-dimensional polymers. We can think of a one-dimensional polymer as a polymer in a thin tube.

Probabilistic polymer models are based on lattice random walks or Brownian motion with a self-repellent interaction. The paths of these processes model the configuration of the polymer in space. The random walk or the Brownian motion models the irregularity, the self-repellence models the tendency to avoid self-intersections. We study the Domb-Joyce model based on simple random walk, and the Edwards model based on Brownian motion. The Domb-Joyce model is a generalization of the self-avoiding walk, namely where self-intersections are made unlikely but not impossible. The Edwards model is the continuous space-time analogue of the Domb-Joyce model. We are interested in the behavior of the end-to-end distance of the polymer as the number of monomers increases. The end-to-end distance gives an indication of what the spatial extent of the polymer is.

Polymer models are prototypes of models with a global interaction: any piece of the polymer interacts with all the other pieces, even the ones that are far apart when measured along the polymer chain. This makes polymer models different from most of the mathematical literature, where Markovian models are studied with a local interaction in space and/or time. The global interaction makes these models difficult, which is reflected by the fact that the proofs in this work are technical. The basic tools used in this work are the theory of large deviations, Markov chains and diffusions for the local time processes of simple random walk and Brownian motion, and functional analysis.

In Chapter 1 we give an introduction to polymers, define the above models and state the main results and conjectures. We formulate the law of large numbers for the one-dimensional Domb-Joyce model and give a sketch of the proof by Greven and den Hollander (1993), because this proof gives a clear picture of the techniques used in this field. We also

state the law of large numbers for the one-dimensional Edwards model proved by Westwater (1984) and the central limit theorem for the one-dimensional Domb-Joyce model proved by König (1996). A law of large numbers states that the end-to-end distance of the polymer divided by the number of monomers converges to a positive and finite number. This number is called the speed of the polymer and depends on the self-repellence parameter. A central limit theorem describes the fluctuations of the end-to-end distance around this linear behavior. Furthermore, we give an outlook on the future by describing a few related polymer models and some results and conjectures for these models.

In Chapter 2 we investigate how the key quantities in the law of large numbers for the Domb-Joyce model behave as the self-repellence parameter tends to zero, and we relate this behavior to the law of large numbers for the Edwards model. It follows from these results that the behavior in the Domb-Joyce model with a very small, but positive, self-repellence parameter is not comparable in any way to the random walk without self-repellence. This means that the weak interaction limit is singular.

In Chapter 3 we prove a central limit theorem for the Edwards model, in Chapter 4 a central limit theorem for the Domb-Joyce model when the self-repellence parameter tends to zero with the number of monomers. Furthermore, we explain the relation and the differences between the central limit theorems in Chapter 3 and 4. It turns out that the variances in these CLTs as a function of the self-repellence parameter are discontinuous at zero and are smaller than 1, the value for simple random walk and Brownian motion. This once again shows that the weak interaction limit is singular. The results in Chapter 4 show that the behavior of the end-to-end distance in the Domb-Joyce model is uniform in the interaction parameter.

In Chapter 5 we give bounds for the constants in the central limit theorems in Chapters 2-4 and we explain what the bounds mean for the polymers.

Chapters 2-5 give a clear picture of the behavior of the polymer in the Domb-Joyce and in the Edwards model in dimension one. The remaining open problems are related to monotonicity properties of both the mean and the variance in the CLT for the Domb-Joyce model. A challenge are the realistic dimensions $d = 2, 3$, where virtually nothing is known. The results in dimension one do give insight in these realistic dimensions and prove results that we also expect to hold in $d = 2, 3$.

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Glossary

Below we list the mathematical symbols that occur frequently in this work and refer to the formula or section in which they are introduced. Furthermore, we have included a brief explanation in words.

a^*	(2.1.18)	exponential rate normalizing constant Edwards model
$A_{r,\beta}(i,j)$	(1.4.13)	$l^2(\mathbb{N})$-matrix that describes distribution of the polymer under the Domb-Joyce measure
$\overline{A}_n$	(4.5.16)	scaled version of A_{r_n,β_n}
$A(u)$	(3.2.8)	additive functional of the two-dimensional squared Bessel process
$A^{-1}(t)$	(3.2.8)	inverse process of $u \mapsto A(u)$
$A^\star(u)$	(3.2.11)	additive functional of the zero-dimensional squared Bessel process
$A^{\star-1}(t)$	(3.2.11)	inverse process of $u \mapsto A^\star(u)$
$\alpha^{(l)}(a)$	(3.4.6)	eigenvalue of $\mathcal{M}^a$
$\alpha^{(l)}(r,\beta)$	(4.4.9)	eigenvalue of $Q_{r,\beta}$
b^*	(2.1.18), (2.1.21)	speed of the polymer in the Edwards model for $\beta = 1$
$(B_t)_{t\geq 0}$	Section 1.3	Brownian motion
c^*	(3.1.4)	standard deviation in the CLT for the Edwards model
C_k	(5.2.23)	constant in bound on approximation of $x_{a,\rho}$ by its power series with k terms
$c(a,\rho)$	(5.2.15)	last zero of $f_{a,\rho}$
F^a	(2.2.7)	limiting functional of F^a_β
$F_a(u)$	(3.3.1)	$u^2 - au + \rho(a)$
$F^\star_a(u)$	(3.3.18)	$u^2 - au$
F^1	(2.3.3)	first part of functional F^a
F^2	(2.3.3)	second part of functional F^a
F^a_β	(2.2.2)	functional to describe variational problem corresponding to the scaled eigenvalue of $A_{a\beta^{\frac{2}{3}},\beta}$
F^1_β	(2.3.1)	first part of functional F^a_β
F^2_β	(2.3.1)	second part of functional F^a
Γ_l	(4.3.4)	Markov renewal process

$f^{\pm}_{r,\beta}$	(4.3.9)	functions describing contribution of local times from the boundary pieces
$\overline{f}^{\pm}_{r,\beta,\delta}$	(4.3.29 – 4.3.30)	scaled $f^{\pm}_{r,\beta}$
$\overline{f}^{\pm}_{n,\delta}$	above (4.5.2)	abbreviation of $\overline{f}^{\pm}_{r^*_n,\beta_n,\delta}$
$\overline{g}^{n,-}_{r,\beta,\delta}$	(4.3.31)	matrix product of t_nth power of Q_{r,β_n} and $f^{-}_{r,\beta}$
$\overline{g}^{-}_{n,\delta}$	above (4.5.2)	abbreviation of $\overline{g}^{n,-}_{r^*_n,\beta_n,\delta}$
γ	(4.4.20)	integrand of integral representation of L
γ_n	(4.5.2)	discrete version of γ
$\mathcal{K}^a$	(3.1.3)	Sturm-Louville differential operator, scaled version of $\mathcal{L}^a$
$K(a,\rho)$	(5.2.19 – 5.2.21)	constant in bound of the coefficients of the power series representation of $x_{a,\rho}$
L	(4.2.6), (4.3.35)	corresponds to the limit of the normalizing constant in the weakly interacting Domb-Joyce model
$\{L(T,x)\}_{x\in\mathbb{R}}$	(3.1.1), (3.2.15)	Brownian local time process
$\widehat{L}$	(3.1), (3.3.33)	corresponds to the limit of the normalizing constant in the Edwards model
$\mathcal{L}^a$	(2.1.13)	Sturm-Louville differential operator
$\lambda(r,\beta)$	(2.2.1)	largest eigenvalue of $A_{r,\beta}$
$\lambda^{(l)}(r,\beta)$	Section 4.4.1	lth largest eigenvalue of $A_{r,\beta}$
$\{\ell_n(x)\}_{x\in\mathbb{Z}}$	(1.4.6)	local time process of simple random walk
$\mathcal{M}^a$	(3.4.5)	Sturm-Liouville operator $\frac{1}{\mathrm{id}}\mathcal{K}^a$
$\{m(x)\}_{x\in\mathbb{N}_0}$	(4.2.9)	Markov chain describing local times inbetween 0 and S_n
$\{m^*(x)\}_{x\in\mathbb{N}_0}$	(4.2.9)	Markov chain describing local times below 0 and above S_n
P	Section 1.2	simple random walk measure
$\widehat{P}$	Section 1.3	Wiener measure
$P(i,j)$	(1.4.14)	Markov transition matrix corresponding to $\{m(x)\}_{x\in\mathbb{N}_0}$
P^*	(4.2.8)	Markov transition matrix corresponding to $\{m(x)^*\}_{x\in\mathbb{N}_0}$
$\widehat{P}^a$	(3.3.2)	transition probability function of transformed two-dimensional Bessel process
$P_{r,\beta}$	(4.3.1)	transition matrix of transformed Markov chain
$p_{a,\rho}(u)$	below (5.2.15)	$u^2 - au + \rho$
ψ_h	(3.2.12)	density of total local time of Brownian motion below 0 or above B_T
$\psi^{(\beta)}_v$	(4.4.27), (4.5.8)	discrete version of ψ_h
μ^*_β	(2.1.10)	stationary distribution of local times under the Domb-Joyce measure inbetween 0 and the endpoint
μ_n	(2.1.5)	empirical distribution of local times
$\nu_{r,\beta}$	(4.3.7)	stationary distribution of $\{\Gamma_l\}_{l\in\mathbb{N}}$
$\nu^{(l)}_{r,\beta}$	(4.4.10)	lth real eigenvector of $Q_{r,\beta}$
$\overline{\nu}_{r,\beta}$	(4.3.27)	scaled version of $\nu_{r,\beta}$

Q_n^β	(1.2.1)	Domb-Joyce measure
$\widehat{Q}_T^\beta$	(1.3.1), (3.1.2)	Edwards measure
$Q_{r,\beta}$	Section 4.4.1	Markov transition matrix of $\{\Gamma_l\}_{l\in\mathbb{N}}$
$r^*(\beta)$	(1.4.15)	exponential rate of the normalizing constant in the Domb-Joyce model
r_n	(4.2.4)	$r^*(\beta_n)$
$\rho(a)$	below (2.1.14)	largest eigenvalue of $\mathcal{L}^a$
$(S_i)_{i\in\mathbb{N}_0}$	Section 1.2	simple random walk
$\sigma^*(\beta)$	(1.6.2)	standard deviation in the CLT for the Domb-Joyce model
$\hat{\sigma}^*(\beta)$	(3.1.6)	standard deviation in the CLT for the Edwards model
σ_n	(4.1.7)	$\sigma^*(\beta_n)$
τ_h^u	(3.2.2)	random time change of Brownian motion in the Ray-Knight theorems
$\tau_{r,\beta}$	(2.1.3), (2.2.1)	eigenvector of $A_{r,\beta}$ corresponding to $\lambda(r,\beta)$
$\theta^*(\beta)$	(1.4.17), (2.1.11)	speed of the polymer in the Domb-Joyce model
θ_n	(4.1.6)	$\theta^*(\beta_n)$
$\hat{\theta}^*(\beta)$	(3.1.6)	speed of the polymer in the Edwards model
$\{U_d\}_{d\in\mathbb{N}}$	(4.2.16)	additive functional of local times
U^*	(4.2.18)	total local time below 0 or above S_n
$\{V_d\}_{d\in\mathbb{N}}$	(4.2.17)	additive functional of squares of local times
V^*	(4.2.19)	total sum of squares of local time below 0 or above S_n
w_a	(3.3.18)	function describing contribution of local times below 0 and above B_T
$w_{r,\beta}$	(4.3.8)	discrete analogue of w_a
$\overline{w}_{r,\beta}$	(4.4.22)	scaled version of $w_{r,\beta}$
x_a	below (2.1.14)	largest eigenfunction of $\mathcal{L}^a$
$x_a^{(l)}$	Section 4.4.1	lth eigenfunction of $\mathcal{L}^a$
$x_{a,\rho}$	(5.2.1)	eigenfunction of $\mathcal{L}^a$ with eigenvalue ρ
$\{X_v\}_{v\geq 0}$	(3.2.7)	two-dimensional squared Bessel process
$\{X_v^*\}_{v\geq 0}$	(3.2.7)	zero-dimensional squared Bessel process
$y^{(l)}$	(4.4.7)	lth eigenfunction of $\mathcal{M}^a$
$y_{r,\beta}^{(l)}$	(4.4.12)	discrete version of $y^{(l)}$
$\{Y_t\}_{t\geq 0}$	(3.3.13)	time changed $\{X_v\}_{v\geq 0}$ process
z_a	(3.5.2)	scaled and shifted Airy function
Z_n^β	(1.2.2)	normalizing constant for the Domb-Joyce model
$\widehat{Z}_T^\beta$	(1.3.2)	normalizing constant for the Edwards model
$\langle f,g\rangle_{L^2}^\circ$	(3.3.25)	weighted L^2 inner product

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