MATHEMATICAL CENTRE TRACTS 49

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CONNECTED ORDERABLE SPACES

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Survey of definitions and some results from:
H. Kok, Connected orderable spaces.
Amsterdam, 1973.

A topological space X is said to satisfy

- (*O*) – if X is orderable.
- (*CO*) – if X is cyclically orderable.
- (*S'*) – if among every three distinct points of X, there is one which separates the other two.
- (*K*) – if among every three distinct, connected, proper subsets of X, there are two which together do not cover the space X.
- (*E*) – if the subset $(X \times X) \setminus \Delta$ of the product space $X \times X$ is not connected (where Δ is the diagonal in $X \times X$).
- (*P*) – if for every two connected subsets A and B of X with a common endpoint p the following holds: $A \cap B = \{p\}$ or $A \subset B$ or $B \subset A$.
- (*H*) – if every connected subset of X has at most two endpoints.
- (*Hp*) – if every connected proper subset of X has at most two endpoints.
- (*Hd*) – if for every connected subset C of X such that C has at least three distinct endpoints, $C \setminus \{p,q\}$ is disconnected for every pair of distinct endpoints p, q of C.
- (*Ht*) – if no connected subset C of X has an endpoint triple.
- (*B*) – if there do not exist three mutually disjoint segments in X.
- (*B'*) – if every cut point of X is a strong cut point.
- (*B"*) – if for every $p \in X$: $X \setminus p$ has finitely many components.
- (*B'O*) – if every segment is open.
- (*B'C*) – if for every $p \in X$ and for every component C of $X \setminus p$: $\bar{C} = C \cup p$.
- (*Int*) – if the intersection of an arbitrary collection of connected subsets of X is connected.
- (*Int**) – if the closure of the intersection of an arbitrary collection of connected subsets of X is connected.
- (*Int'*) – if the intersection of an arbitrary collection of closed connected subsets of X is connected.
- (*Int 2*) – if the intersection of two connected subsets of X is connected.
- (*W*) – if for every two disjoint connected sets $A, B \subset X$ it is true that $|\bar{A} \cap \bar{B}| \leq 1$.

All spaces are assumed to be connected T_1.

Lemma 3.6. $(Ht) + (B'C) \Longrightarrow (B')$.

Lemma 3.7. $(Ht) + (B') +$ (at least one cut point) $\Longrightarrow (H)$.

Lemma 3.8. $(H) + (B') \Longrightarrow (O)$.

Theorem 3.9. $(H) + (B'C) \Longrightarrow (O)$.

Theorem 3.10. $(Ht) + (B'C) +$ (at least one cut point) $\Longrightarrow (O)$.

Theorem 3.11. $(CO) +$ (at least one cut point) $\Longrightarrow (O)$.

Theorem 3.12. $\neg(O) + (CO) \Longleftrightarrow$ (no cut points) + (no endpoint pairs).

Theorem 3.18. $\neg(O) + (CO) \Longleftrightarrow (Hp) + \neg(H)$.

Theorem 3.19. $\neg(O) + (CO) \Longleftrightarrow (Ht) +$ (no cut points).

Theorem 3.20. $\neg(O) + (CO) \Longleftrightarrow$ (the complement of each connected subset is connected).

Proposition 3.21. $(Ht) + (B'C) \Longrightarrow (Hp)$.

Theorem 4.1. $(S) + (B') \Longleftrightarrow (O)$.

Theorem 4.6. $(Ht) + (S) \Longrightarrow (O)$.

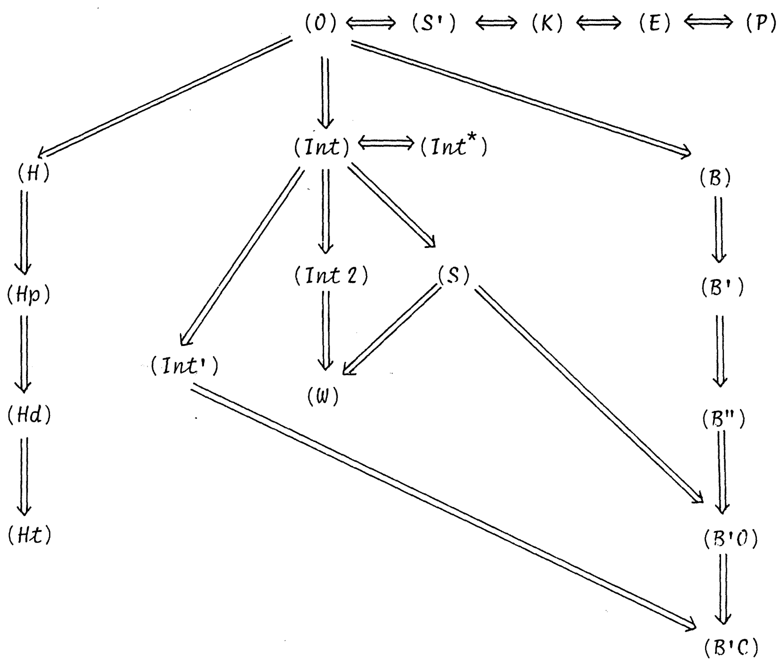
Theorem 4.12. $(W) + (B'C) \Longrightarrow (B'O)$.

Theorem 4.13. $(Ht) + (W) \Longrightarrow (H)$.

Theorem 4.21. $(Ht) + (Int') \Longrightarrow (O)$.

Theorem 4.22. $(Int') + (W) \Longrightarrow (B'O)$.

Theorem 4.25. $(Int') + (S) \Longleftrightarrow (Int)$.


(O)
(S')
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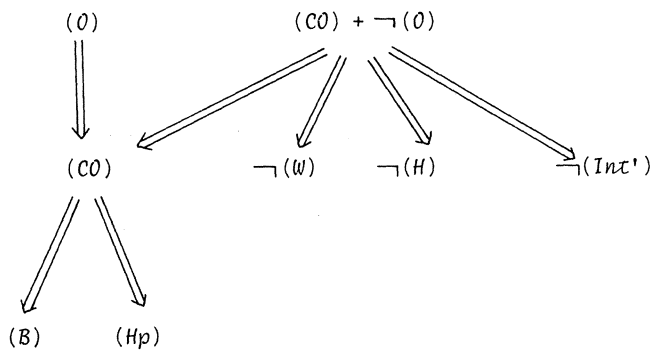
(B'C)


(O)
(CO) + ¬(O)
(CO)
¬(W)
¬(H)
¬(Int')
(B)
(Hp)

ACKNOWLEDGEMENTS

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CONTENTS

INTRODUCTION

This tract deals with connected orderable topological spaces. A topological space (X,I) is called orderable if there exists an ordering < on X such that the interval-topology $I_<$ is contained in I. If, moreover, $I_< = I$ then the space is called strictly orderable. In this tract we consider a number of properties of connected orderable spaces. The relations between these properties are investigated in the wider class of connected T_1-spaces. Some of these properties have already been studied by other authors; mostly, however, under the additional assumption that the space under consideration is locally connected.

In Chapter I besides the orderable and the strictly orderable spaces the cyclically orderable and the strictly cyclically orderable spaces are introduced. A number of lemmas is proved, which are frequently used. This Chapter ends with the treatment of the first collection of properties. These properties all concern segments.

The properties of connected T_1-spaces considered in Chapter II are all equivalent to the orderability of such spaces. A similar property is discussed at the end of Chapter III.

The set of properties discussed in Chapter III deals with the notion "randendlich", introduced by Herrlich. After investigating the relations between these properties it is examined under which extra conditions they are equivalent to the orderability of the connected T_1-space. Next we again pay attention to the cyclically orderable spaces. It turns out that these spaces can be characterized in terms of the properties treated in this Chapter.

Chapter IV deals with tree-like spaces and a number of properties concerning the intersection of connected subsets of a connected T_1-space. Of the results from this Chapter we mention:

(i) A tree-like space in which every cut point is a strong cut point is orderable.

(ii) In a tree-like space in which the intersection of closed connected subsets is connected, the intersection of arbitrary connected subsets is also connected.

In Chapter V, it is examined which are the relations between all these properties if the space under consideration is not only connected and T_1, but also locally connected.

In Chapter VI, several counterexamples are described. Together with the implications derived in the previous Chapters, they give a complete picture of all internal relations between the discussed properties - except for some unsolved problems.

The system of internal references is explained by the following examples:

Theorem 2 in Chapter IV is referred to as Theorem 4.2 if the reference is made outside Chapter IV, and as Theorem 2 otherwise.
Corollary 2.2 in Chapter IV (the second Corollary of Theorem 2 in Chapter IV) is referred to as Corollary 4.2.2 outside Chapter IV and as Corollary 2.2 otherwise.

CHAPTER I

PRELIMINARIES AND NOTATIONS

1.1. STRICTLY ORDERABLE SPACES

Let $(X,<)$ be a totally ordered set; let $a \in X$, $b \in X$ and $a < b$.
We use the following notation:

$$(a,b) = \{x \in X \mid a < x < b\};$$
$$[a,b] = \{x \in X \mid a \leq x \leq b\};$$

in the latter case we also allow a and b to be equal;

$$[a,b) = \{x \in X \mid a \leq x < b\};$$
$$(a,b] = \{x \in X \mid a < x \leq b\};$$
$$(a,\ ) = \{x \in X \mid a < x\};$$
$$(\ ,b) = \{x \in X \mid x < b\};$$
$$[a,\ ) = \{x \in X \mid a \leq x\};$$
$$(\ ,b] = \{x \in X \mid x \leq b\}.$$

A subset J of X is called an *open interval* if J is of the form $J = (a,b)$ or $J = (a,\ )$ or $J = (\ ,b)$ or $J = X$.
J is called a *closed interval* if J is of the form $J = [a,b]$ or $J = [a,\ )$ or $J = (\ ,b]$ or $J = X$. A closed interval $[a,b]$ is called *degenerate* if $a = b$. If $[a,b] = \{a,b\}$ where a and b are distinct points of X, then we call a and b *neighbours* in X; a is the *left neighbour* of b and b is the *right neighbour* of a. The set $\{a,b\}$ is called a *jump*.

A pair (A,B) of subsets of an ordered set $(X,<)$ is called a *cut*, if $X = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and if $a < b$ for all $a \in A$, $b \in B$.
A *gap* of a totally ordered set $(X,<)$ is a cut (A,B) of X, such that A has no largest element and B has no smallest element.

A totally ordered set $(X,<)$ is called *order-complete* if each non-void subset of X which is bounded above has a supremum in X. It is clear that an ordered set $(X,<)$ is order-complete iff each non-void subset which is bounded below has an infimum in X. Moreover, $(X,<)$ is order-complete if and only if there are no gaps.

A topological space (X,I) is called *strictly orderable* if there exists a total ordering $<$ on X, such that the sets of the form $\{x \in X \mid x < a\}$, $\{x \in X \mid a < x\}$, (where a runs through X) form a subbase for the topology I in X. In other words: X is strictly orderable iff there exists an ordering $<$ on X such that $I_< = I$, where $I_<$ is the interval topology.

THEOREM 1. *A strictly orderable space* (X,I) *is connected if and only if* $(X,<)$ *has no jumps and no gaps, where* $<$ *is a total ordering inducing the topology* I *of* X.

PROOF. See e.g. Kelley [18], Ch. I, Problem I.

1.2. ORDERABLE SPACES

A space X is called *orderable* if there exists a total ordering $<$ on X, such that the sets of the form $\{x \in X \mid x < a\}$, $\{x \in X \mid a < x\}$, (where a runs through X) are open in X. In other words: a space (X,I) is orderable iff there exists a total ordering $<$ for X such that $I_< \subset I$. The ordering $<$ is called *compatible* with the topology I.

REMARK. Frequently a space is called orderable if it is strictly orderable in our terminology. It is easy to see that our definition of orderability is the same as the definition given by Eilenberg [8].

THEOREM 2. *A subspace* A *of an orderable space* X *is orderable.*

PROOF. Let (X,I) be an orderable space. Let $<$ be a total ordering for X, such that $I_< \subset I$. Let A be a subset of X. By $<$ a total ordering $<_A$ is induced in A. The relative topology of A in (X,I) will be denoted by $I^{(A)}$, and the relative topology of A in $(X,I_<)$ by $I_<^{(A)}$. It is well-known and easy to see that $I_{<_A} \subset I_<^{(A)}$, and, as $I_< \subset I$, we have $I_<^{(A)} \subset I^{(A)}$. Hence

$$I_{<_A} \subset I^{(A)}.$$

REMARK. Observe that a subspace of a strictly orderable space need not be strictly orderable.

In a strictly orderable connected space the intervals are the only connected subspaces. In an orderable connected space the same is true:

THEOREM 3. *In an orderable connected space the intervals are the only connected subspaces.*

PROOF. Let (X,I) be an orderable connected space, and let $<$ be a total ordering on X compatible with I.
We first show that intervals in X are I-connected.
For that purpose, suppose that J is an open I-disconnected interval in X.
Then $J = A \cup B$, where A and B are open in (X,I), $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$. Let $p \in A$ and $q \in B$. We may assume $p < q$.
Let $C = \{x \in X \mid x < p\} \cup \{x \in A \mid x < q\}$ and
$D = \{x \in X \mid q < x\} \cup \{x \in B \mid p < x\}$.
Then C and D are open in (X,I), $p \in C$, $q \in D$, $X = C \cup D$ and $C \cap D = \emptyset$, which contradicts the connectedness of (X,I). By the connectedness of (X,I) it follows that the closure of an open interval in (X,I) is a closed interval, and hence every interval is connected in (X,I).

Since $I_< \subset I$, (X,I) cannot have more connected subsets than $(X,I_<)$, which completes the proof.

THEOREM 4. (cf. Eilenberg [8]). *Let* (X,I) *be an orderable connected space. Let* $<_1$ *and* $<_2$ *be two total orderings on* X *compatible with* I.
Then $<_1 = <_2$ *or* $<_1 = <_2^{-1}$.

PROOF. Suppose $<_1 \neq <_2$ and $<_1 \neq <_2^{-1}$.
Then we may assume without loss of generality, that there exist three distinct points p, a and b in X such that

$$a <_1 p <_1 b \text{ and } p <_2 a,\ p <_2 b.$$

By Theorem 3 it follows that
$A = \{x \in X \mid x <_1 p\} \cup \{x \in X \mid p <_2 x\} \cup \{x \in X \mid p <_1 x\}$ is connected in (X,I).
However, $A = X \setminus \{p\}$ and $X \setminus \{p\}$ is not connected in $(X,I_{<_1})$, so certainly not connected in (X,I).

COROLLARY 4.1. (cf. e.g. Herrlich [12]). *The total ordering for a strictly orderable connected space is unique up to inversion.*

THEOREM 5. *An orderable connected space* X *is strictly orderable if and only* *if* X *is locally connected.*

PROOF.

(i) $\Longrightarrow$: Let X be connected and strictly orderable. Then the collection of all open intervals is a base for X consisting of open connected sets. Hence X is locally connected.

(ii) $\Longleftarrow$: Let X be connected, locally connected and orderable. Then there exists a base for X consisting of open connected sets. By Theorem 3, these sets are open intervals. Hence the interval topology coincides with the topology of X.

1.3. CYCLIC ORDERABILITY

Let X be a set. A subset $R \subset X^3$ is called a *cyclic ordering* on X if:

(i) $\left.\begin{array}{l} a \neq b \neq c \neq a \\ (a,b,c) \notin R \end{array}\right\} \Longleftrightarrow (c,b,a) \in R.$

(ii) $(a,b,c) \in R \Longrightarrow (b,c,a) \in R.$

(iii) $\left.\begin{array}{l} (a,b,c) \in R \\ (a,c,d) \in R \end{array}\right\} \Longrightarrow (a,b,d) \in R.$

REMARK. For a detailed discussion of the concept of cyclic orderability we refer the reader to Čech [6], Ch. I, §5 and Huntington [14].

Let (X,I) be a topological space. X is called *strictly cyclically orderable* if there exists a cyclic ordering R on X such that the sets of the form $\{x \in X \mid (a,x,b) \in R\}$, $(a,b \in X)$ form a base for the topology I on X (or, which amounts to the same, form a subbase for the topology I on X). X is called *cyclically orderable* if there exists a cyclic ordering R on X such that the sets of the form $\{x \in X \mid (a,x,b) \in R\}$, $(a,b \in X)$ are open in X. The cyclic ordering R is called *compatible* with I.

PROPOSITION 6. *Let* X *be an orderable space. Then* X *is cyclically orderable.*

PROOF. Define a cyclic ordering R on X as follows:

$$(a,b,c) \in R \Longleftrightarrow \begin{cases} a \neq b \neq c \neq a \\ (a < b < c) \vee (c < a < b) \vee (b < c < a). \end{cases}$$

It is easily verified that R is indeed a cyclic ordering on X.
Since $\{x \in X \mid (a,x,b) \in R\} = \{x \in X \mid a < x < b\}$ if $a < b$,
and $\{x \in X \mid (a,x,b) \in R\} = \{x \in X \mid a < x\} \cup \{x \in X \mid x < b\}$ if $b < a$
the compatibility of R with the topology on X is an easy consequence of the orderability of X.

REMARK. 1. We will denote the cyclic ordering R obtained from the ordering $<$ as in Proposition 6 by $R_<$, and we say that $R_<$ is induced by $<$.

2. A strictly orderable space is not necessarily strictly cyclically orderable. One can take the half-open interval $[0,1)$ for a counterexample.

PROPOSITION 7. *Let* X *be a cyclically orderable space, and let* $p \in X$. *Then* $X \setminus \{p\}$ *is orderable.*

PROOF. Define a total ordering $<$ on $X \setminus \{p\}$ as follows:

$$a < b \Longleftrightarrow (p,a,b) \in R,$$

where R is a cyclic ordering compatible with the topology on X. It is easy to see that $<$ is indeed a total ordering on $X \setminus \{p\}$.
Since $\{x \in X \setminus \{p\} \mid x < a\} = \{x \in X \mid (p,x,a) \in R\}$
and $\{x \in X \setminus \{p\} \mid b < x\} = \{x \in X \mid (b,x,p) \in R\}$,
$X \setminus \{p\}$ is an orderable space.

REMARK. 1. We will denote the total ordering $<$ on $X \setminus \{p\}$ obtained from the cyclic ordering R as in Proposition 7 by $<_R^{(p)}$, and we say that $<_R^{(p)}$ is induced by R.

2. If X is a strictly cyclically orderable connected T_1-space and if $p \in X$, then $X \setminus \{p\}$ is strictly orderable. This will be shown in Chapter III.

3. From Proposition 7 and Theorem 2 it follows that every *proper* subset of a cyclically orderable space is orderable.

4. Let (X,I) be an orderable space. Let $<$ be a total ordering on X compatible with I. Let $R = R_<$ be the cyclic ordering on X induced by $<$. If $p \in X$, then R induces a total ordering $<_R = <_R^{(p)}$ on $X \setminus \{p\}$. The total orderings $<$ and $<_R$ coincide if and only if p is the smallest or the largest element of X. For, we have

$$a < p < b \Longrightarrow (a,p,b) \in R \Longleftrightarrow (p,b,a) \in R \Longleftrightarrow b <_R a,$$
$$a < b < p \Longrightarrow (a,b,p) \in R \Longleftrightarrow (p,a,b) \in R \Longleftrightarrow a <_R b,$$
$$p < a < b \Longrightarrow (p,a,b) \in R \Longleftrightarrow a <_R b.$$

Let X be a set and $R \subset X^3$ a cyclic ordering on X. Define a subset $R^{-1} \subset X^3$ as follows:

$$(a,b,c) \in R^{-1} \Longleftrightarrow (c,b,a) \in R.$$

It is easy to see that R^{-1} is also a cyclic ordering on X. R^{-1} is called the *inverse* of R.

REMARK. 1. Let $p \in X$ and let there be given an ordering < of the set $X \setminus \{p\}$. Then there exists precisely one cyclic ordering R on the set X such that the given ordering < of the set $X \setminus \{p\}$ coincides with $<_R^{(p)}$. For a proof of this assertion we refer to Čech [6], Theorem 5.2.1. As a consequence we have:
Let R_1 and R_2 be cyclic orderings on X. Let $p \in X$. Let $<_i = <_{R_i}^{(p)}$ be the ordering on $X \setminus \{p\}$ induced by R_i $(i = 1,2)$. Suppose $<_1 = <_2$ or $<_1 = <_2^{-1}$. Then $R_1 = R_2$ or $R_1 = R_2^{-1}$ respectively.
2. In a cyclically orderable, connected T_1-space the cyclic ordering compatible with the topology is unique up to inversion. The proof of this theorem will be given in Chapter III.

Let X be a non-orderable, cyclically orderable space. Let an *interval* in X be any set of one of the following forms (where p, a and b run through X): X, $X \setminus \{p\}$, $\{x \in X \mid (a,x,b) \in R\} = J(a,b)$, $J(a,b) \cup \{a\}$, $J(a,b) \cup \{b\}$, $J(a,b) \cup \{a,b\}$.

REMARK. In a non-orderable, cyclically orderable, connected T_1-space the connected subsets of X are precisely the intervals. The proof of this fact will be given in Chapter III. In that Chapter we will also give proofs of the following theorems:

(i) A strictly cyclically orderable, connected T_1-space is locally connected.

(ii) A non-orderable, cyclically orderable, locally connected, connected T_1-space is strictly cyclically orderable.

1.4. FURTHER DEFINITIONS AND NOTATIONS

From now on we shall deal only with connected T_1-spaces with more than one point.

A point $p \in C$ is called a *cut point* of the connected set $C \subset X$ if $C \setminus \{p\}$ is not connected.
A point $p \in C$ is called a *non-cut point* or an *endpoint* of the connected set $C \subset X$ if $C \setminus \{p\}$ is connected.
A subset C of X is called a *segment* if C is a component of $X \setminus \{p\}$, for some $p \in X$; in this case we also say that C is a segment of p in X.
When $A \subset X$, $B \subset X$, $A \cap B = \emptyset$ and both A and B are clopen (= closed-and-open) in $A \cup B$, we frequently write $A + B$ instead of $A \cup B$.
The pair (A,B) of subsets of X is called a *separation* (of $A \cup B$) if $A \cup B = A + B$, $A \neq \emptyset$ and $B \neq \emptyset$.
We say that $S \subset X$ *separates* $y \in X$ and $z \in X$ if there exists a separation (A,B) of $X \setminus S$ such that $y \in A$ and $z \in B$. In such a case we often write

$$X \setminus S = \underset{y}{A} + \underset{z}{B}.$$

The pair (y,z) of points of X is called *conjugated*, when there does not exist a point $x \in X$ such that x separates y and z.
A point $p \in C$ is called a *strong cut point* of the connected set $C \subset X$ if $C \setminus \{p\}$ has exactly two components (then there exists a unique separation of $C \setminus \{p\}$).
If $S \subset X$ is connected and $C \subset S$, C is called an *endset* of S if $S \setminus C$ is connected. In the special case when C consists of two or three points, we often call C an *endpoint pair*, *endpoint triple* respectively. Observe that a set of endpoints is not necessarily an endset.

We often write $X \setminus p$ instead of $X \setminus \{p\}$. An analogous abbreviation is used in similar cases.

Let (C) be a topological property and X be a topological space satisfying property (C). Then we often say: X is a (C)-space, instead of: X satisfies property (C).

For some special subsets of a connected T_1-space X we use the following notation (where a and b are distinct points of X):

$C(a,b) = \cap \{S \subset X \mid a,b \in S \text{ and } S \text{ is connected}\}$;

$K(a,b) = \cap \{S \subset X \mid a,b \in S \text{ and } S \text{ is connected and closed}\}$;

$E(a,b) = \{x \in X \mid x \text{ separates } a \text{ and } b\}$;

$S(a,b) = E(a,b) \cup \{a,b\}$.

It is well-known that $S(a,b)$ is an orderable subspace of X. The ordering $<$ compatible with the relative topology of $S(a,b)$, is the so-called *separation ordering*. (cf. e.g. Hocking and Young [13], p.49-53 or Moore [24], p.158-160).
For the sake of completeness we will recall the definition and some properties of the separation ordering:
For every $x \in E(a,b)$ let (A_x,B_x) be an arbitrary separation of $X \setminus x$ such that $a \in A_x$ and $b \in B_x$.
The separation ordering for $S(a,b)$ is defined as follows: a is the smallest and b is the largest element in the ordering, and for $x,y \in E(a,b)$ we have

$$x < y \iff x \text{ separates } a \text{ and } y \text{ in } X \iff x \in A_y \iff$$
$$\iff y \text{ separates } x \text{ and } b \text{ in } X \iff y \in B_x.$$

1.5. SOME LEMMAS

In this section we list some useful lemmas. Several elementary lemmas are probably well-known, although exact references in these cases are difficult to find.

X will denote a connected T_1-space, and C a connected subset of X.

LEMMA 8. *If* A *is clopen in* $X \setminus C$, *then* $A \cup C$ *is connected.*

PROOF. Let $X \setminus C = A + B$. Suppose $A \cup C = S + T$ where $C \subset S$. Then $X = (B \cup S) + T$; hence $T = \emptyset$.

COROLLARY 8.1. *If* A *is clopen in* $X \setminus C$, *then* $X \setminus A$ *is connected.*

PROOF. $X \setminus A = C \cup B$ is connected by Lemma 8.

LEMMA 9. *If* T *is a component of* $X \setminus C$, *then* $X \setminus T$ *is connected.*

PROOF. Suppose $X \setminus T = A + B$ where $C \subset A$. Then, by Lemma 8, $B \cup T$ is connected in $X \setminus C$; hence $B \cup T = T$ and $B = \emptyset$.

COROLLARY 9.1. *If* T_i $(i = 1,2,\dots,n)$ *are finitely many components of* $X \setminus C$, *then* $X \setminus \bigcup_{i=1}^{n} T_i$ *is connected.*

PROOF. T_2 is a component of $(X \setminus T_1) \setminus C$. Hence, by Lemma 9, $(X \setminus T_1) \setminus T_2$ is connected; and so on.

LEMMA 10. *If* Q *is the intersection of an arbitrary collection of clopen subsets of* $X \setminus C$, *then* $X \setminus Q$ *is connected.*

PROOF. Let $Q = \bigcap_{\alpha\in A} \{H_\alpha \mid H_\alpha \text{ clopen in } X \setminus C\}$ for some indexed collection $\{H_\alpha \mid \alpha \in A\}$.
Consequently, $X \setminus Q = \bigcup_{\alpha\in A} \{X \setminus H_\alpha \mid H_\alpha \text{ clopen in } X \setminus C\}$.
By Lemma 8, $X \setminus H_\alpha$ is connected for every $\alpha \in A$.
Since every $X \setminus H_\alpha$ contains C, (and since without loss of generality we may assume that $C \neq \emptyset$) $X \setminus Q$ is connected.

COROLLARY 10.1. *If* Q *is a quasicomponent of* $X \setminus C$, *then* $X \setminus Q$ *is connected.*

REMARK. Most often these lemmas will be applied in the case when $C = \{p\}$, for some $p \in X$. For example: lemma 9 implies that the complement of a segment is connected, and lemma 8 implies: if $X \setminus p = A + B$ and $A \neq \emptyset$ then $A \cup p$ $(= \bar{A})$ is connected. References to these lemmas will usually not be made explicitly.

LEMMA 11. *Let* X *be a connected* T_1*-space;* $x_1 \in X$. *Let* B *be a non-void subset of* $X \setminus x_1$ *which satisfies at least one of the following conditions:*

a) B *is a clopen subset of* $X \setminus x_1$;

b) B *is a component of* $X \setminus x_1$;

c) B *is a quasicomponent of* $X \setminus x_1$

then, if $Y = X \setminus B$, *the following holds:*

(i) Y *is a connected* T_1*-space.*

(ii) If x_1 *is an endpoint of* X *or if* x_1 *is a strong cut point of* X, *then* x_1 *is an endpoint of* Y. *Conversely, if* x_1 *is an endpoint of* Y, *then in the cases b) and c)* x_1 *is either an endpoint or a strong cut point of* X; *this is no longer necessarily true in case a).*

(iii) If x_2 *is an endpoint of* X *and if* $x_2 \in Y$, *then* x_2 *is an endpoint of* Y.

(iv) If x_2 *is a cut point (strong cut point) of* X, *and if* $x_2 \in Y$, *then* x_2 *is a cut point (strong cut point) of* Y.

PROOF.

(i) See Corollary 8.1, Lemma 9, Corollary 10.1, respectively.

(ii) α) Let x_1 be an endpoint of X:
Then $B = X \setminus x_1$, hence $Y = \{x_1\}$. So the assertion is trivial.
β) Let x_1 be a strong cut point of X:
Then $X \setminus x_1 = A + B$, where both A and B are connected, hence $Y \setminus x_1 = A$ is connected.
γ) Conversely, let x_1 be an endpoint of Y.
Suppose now that x_1 is a cut point of X.
case b): B is a component of $X \setminus x_1$.
Then $X \setminus x_1 = (Y \setminus x_1) \cup B$, where both $Y \setminus x_1$ and B are connected; so x_1 is a strong cut point of X.
case c): B is a quasicomponent of $X \setminus x_1$.
If $\bar{Y} \cap B = \emptyset$, then $X \setminus x_1 = (Y \setminus x_1) + B$; hence B is an open quasicomponent, and consequently a component in $X \setminus x_1$ and we are back in case b).
So suppose $\bar{Y} \cap B \neq \emptyset$. Then the connected set $\bar{Y} \setminus x_1$ meets B, and hence B is that quasicomponent of $X \setminus x_1$, that contains $Y \setminus x_1$.
Since $Y = X \setminus B$ we have $Y = \{x_1\}$ and $B = X \setminus x_1$, contradicting $\bar{Y} \cap B \neq \emptyset$.
case a): B is clopen in $X \setminus x_1$.
In this case it is possible that x_1 is not a strong cut point of X, although it is an endpoint of Y.

Example:

$$X = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \vee y = 0\}; \; x_1 = (0,0);$$

$$B = \{(x,y) \in X \setminus x_1 \mid y \geq 0\}.$$

(iii)

If B is $\begin{cases} \text{a) a clopen subset of } X \setminus x_1, \\ \text{b) a component of } X \setminus x_1, \\ \text{c) a quasicomponent of } X \setminus x_1, \end{cases}$

then B is also $\begin{cases} \text{a) a clopen subset of } (X \setminus x_2) \setminus x_1, \\ \text{b) a component of } (X \setminus x_2) \setminus x_1, \\ \text{c) an intersection of clopen subsets of } (X \setminus x_2) \setminus x_1, \end{cases}$

respectively.

Consequently, by Corollary 8.1, Lemma 9 and Lemma 10 resp.

$$Y \setminus x_2 = (X \setminus x_2) \setminus B \text{ is connected.}$$

(iv) Let $X \setminus x_2 = A_1 \underset{x_1}{+} A_2$, where A_1 and A_2 are non-void.

Then $\bar{A}_2 = A_2 \cup x_2$ is connected in $X \setminus x_1$ and consequently $A_2 \cup x_2 \subset Y$, $A_1 \supset B$. Hence $Y \setminus x_2 = A_2 + (Y \cap A_1)$, and so x_2 is a cut point of Y. If x_2 is a strong cut point of X then, moreover, both A_1 and A_2 are connected.

Since $Y \setminus x_2 = A_2 + (Y \cap A_1)$, the only thing left to prove in this case is that $Y \cap A_1$ is connected.

Since $Y \cap A_1 = A_1 \setminus B$ and since B is a clopen subset of $A_1 \setminus x_1$, a component of $A_1 \setminus x_1$, or an intersection of clopen subsets of $A_1 \setminus x_1$ respectively, the connectedness of $Y \cap A_1$ is an immediate consequence of Corollary 8.1, Lemma 9 and Lemma 10 respectively.

LEMMA 12. *Let* X *be a connected* T_1*-space, and* $p \in X$. *Let* (A,B) *be a separation of* $X \setminus p$, *and* $x \in A$. *If* C *is the component of* p *in* $X \setminus x$, *and* P *is the component of* p *in* $\bar{A} \setminus x$, *then* $C = P \cup B$.

PROOF. $P \cup B = P \cup \bar{B} = P \cup (B \cup p)$, hence $P \cup B$ is connected in $X \setminus x$, so $P \cup B \subset C$.

It remains to show that $C \subset P \cup B$:

Suppose $C \setminus B = E + F$, and $p \in E$.

Then $C = (E \cup B) + F$, hence $F = \emptyset$, which means that $C \setminus B$ is connected in $\bar{A} \setminus x$.

So $C \setminus B \subset P$ and consequently $C \subset P \cup B$.

1.6. PROPERTIES CONCERNING SEGMENTS

We list the following abbreviations for properties of a connected T_1-space X:

(B) - There do not exist three mutually disjoint segments in X.

(B') - $\forall p \in X : X \setminus p$ has at most two components. (Every cut point is a strong cut point).

(B") - $\forall p \in X : X \setminus p$ has finitely many components.

(B'O) - Every segment is open.

(B'C) - $\forall p \in X : \forall$ component C of $X \setminus p : \bar{C} = C \cup p$.

THEOREM 13. *In a connected* T_1*-space* X:

$$(\mathcal{B}) \Longrightarrow (\mathcal{B}') \Longrightarrow (\mathcal{B}'') \Longrightarrow (\mathcal{B}'\mathcal{O}) \Longrightarrow (\mathcal{B}'\mathcal{C}).$$

PROOF.

$(\mathcal{B}) \Longrightarrow (\mathcal{B}')$: obvious.

$(\mathcal{B}') \Longrightarrow (\mathcal{B}'')$: obvious.

$(\mathcal{B}'') \Longrightarrow (\mathcal{B}'\mathcal{O})$: Let C be a component of $X \setminus p$ for some $p \in X$.

By $(\mathcal{B}'')$, $X \setminus p$ has finitely many components. Since every component of $X \setminus p$ is closed in $X \setminus p$, C is open in $X \setminus p$. Since X is a T_1-space, C is open in X.

$(\mathcal{B}'\mathcal{O}) \Longrightarrow (\mathcal{B}'\mathcal{C})$: Let C be a component of $X \setminus p$ for some $p \in X$.
C is closed in $X \setminus p$ and open in X. Since X is connected: $\bar{C} = C \cup p$.

REMARK. None of the above implications can be reversed. For counterexamples we refer to Chapter VI.
Property $(\mathcal{B})$ occurs in a paper of Buch [5]. For the relation between $(\mathcal{B})$ and the orderability of a connected T_1-space see Theorem 4 of Chapter II and Theorem 2 of Chapter IV.
Finally, we remark that in a locally connected, connected T_1-space property $(\mathcal{B}'\mathcal{O})$ holds, since local connectedness is equivalent to: components of open subsets are open. $(\mathcal{B}'\mathcal{O})$ does not imply the local connectedness of the space.
In some Theorems the properties $(\mathcal{B}'\mathcal{O})$ and $(\mathcal{B}'\mathcal{C})$ play the role of very weak substitutes for the local connectedness of a space.

CHAPTER II

SOME PROPERTIES EQUIVALENT TO THE ORDERABILITY OF A CONNECTED T_1-SPACE

2.1. INTRODUCTION AND DEFINITIONS

In this chapter we deal with more conditions on a connected T_1-space X which are equivalent to the orderability of X. These conditions have already been studied in other papers; in some cases however only under the additional assumption that the space under consideration is locally connected.

DEFINITION 1. A topological space X is said to satisfy

(*E*) - if the subset $(X \times X) \setminus \Delta$ of the product space $X \times X$ is not connected (where Δ is the diagonal in $X \times X$).

(*K*) - if among every three distinct, connected, proper subsets of X, there are two which together do not cover the space X.

(*S*') - if among every three distinct points of X, there is one which separates the other two.

(*O*) - if X is orderable.

Condition (*E*) occurs in a paper of Eilenberg [8], in which he proves that (*E*) is equivalent to the orderability of X, provided that X is a connected T_2-space. In [21] and [22] Kowalsky showed that in a connected, locally connected T_1-space X condition (*K*) is equivalent to the strict orderability of X. In a footnote of a paper of Duda [7] it is mentioned that Mrs. Zaremba observed that connected orderable spaces can be characterized by property (*S*'). In Theorem 3 we will prove this equivalence and the equivalence of (*K*) and (*O*) in connected T_1-spaces.

2.2. EQUIVALENCE OF (*O*), (*E*), (*S*') and (*K*)

LEMMA 1. *Let* X *be a connected* T_1*-space and let* x_1, x_2 *and* x_3 *be three distinct points of* X *such that* x_1 *separates the other two. Then, neither* x_2 *nor* x_3 *separates the other two points.*

PROOF. Since x_2 and x_3 belong to different components of $X \setminus x_1$ and since the complement of a segment is connected it follows that there is a connected subset of $X \setminus x_2$ containing x_1 and x_3 and a connected subset of $X \setminus x_3$ containing x_1 and x_2.

LEMMA 2. *In a connected* T_1*-space* X *the following holds*:

$$(S') \Longrightarrow (B).$$

PROOF. Suppose C_1, C_2 and C_3 are three mutually disjoint segments in X. Let $x_i \in C_i$, $(i = 1,2,3)$. Since $X \setminus C_i$ is connected and $x_j \in X \setminus C_i$ if $j \neq i$ it follows that x_i does not separate the other two points.

THEOREM 3. *Let* X *be a connected* T_1*-space. Then the following holds:*

$$(O) \Longleftrightarrow (E) \Longleftrightarrow (S') \Longleftrightarrow (K).$$

PROOF.

$(O) \Longleftrightarrow (E)$: see Eilenberg [8], Theorem I.
$(O) \Longrightarrow (S')$: evident.
$(S') \Longrightarrow (O)$:

(i) By condition (S'), the space X can have at most two endpoints. Since a connected T_1-space consisting of more than one point has infinitely many points, we can choose a cut point p in X.

(ii) By Lemma 2 and the fact that $(B) \Longrightarrow (B')$, X satisfies property (B'). Hence there exist connected, non-void subsets A_p and B_p of X such that $X \setminus p = A_p + B_p$.
For every $x \in A_p$ we can choose connected subsets A_x and B_x of X such that $X \setminus x = A_x + B_x$, where possibly A_x is empty (this is the case if x is an endpoint of X). For every $y \in B_p$ we choose connected subsets A_y and B_y of X such that $X \setminus y = A_y + B_y$. (Again, B_y may be empty).

(iii) Let $x \in A_p$ and $y \in B_p$. Then p separates x and y, and hence, by Lemma 1, $y \in B_x$ and $x \in A_y$. Since $A_x \cup x$ is connected in $X \setminus p$, and $A_p \cup p$ is connected in $X \setminus y$, it follows that $A_x \subset A_p \subset A_y$ (where $\subset$ means *proper* inclusion). Similarly, we can prove that $B_y \subset B_p \subset B_x$.

(iv) Now we will show that for every two distinct points x and y in X precisely one of the following two relations holds:

$$A_x \Subset A_y \text{ or } A_y \Subset A_x.$$

If $x = p$ or $y = p$ or if p separates x and y this is a consequence of the previous observation. So we need only handle the case when x and y are both in (for instance) A_p:
When $x \in A_y$ it follows by Lemma 1 that $y \in B_x$, and hence $A_x \cup x$ is a connected subset of $X \setminus y$. So $A_x \cup x \subset A_y$ and consequently $A_x \Subset A_y$.
When $x \in B_y$ it follows by property (S') that $y \in A_x$; since moreover $A_y \cup y$ is connected in $X \setminus x$ we have $A_y \cup y \subset A_x$, $A_y \Subset A_x$.

(v) Next we will prove the following equivalence:

$$A_x \Subset A_y \Longleftrightarrow x \in A_y.$$

α) $\Longrightarrow$: If $A_x \neq \emptyset$ then $A_x \cup x$, being a connected subset of $X \setminus y$, is a subset of A_y; hence $x \in A_y$.
If $A_x = \emptyset$, then $x \in A_p$ and so we may assume henceforth that $y \neq p$.
If p separates x and y, then $p \in A_y$ and $x \in A_y$.
If y separates p and x, then $y \in A_p$ and hence $p \in B_y$, so that $x \in A_y$.
β) $\Longleftarrow$: Since $x \in A_y$, it is clear that $A_y \not\subset A_x$, so we have $A_x \Subset A_y$.

(vi) Let us now define a total ordering on X as follows:

$$x < y \Longleftrightarrow A_x \Subset A_y.$$

By (iv) it is clear that $<$ is indeed a total ordering and from (v) that for every $a \in X$: $\{x \in X \mid x < a\} = A_a$ and $\{x \in X \mid a < x\} = B_a$, hence X is orderable.

(S') $\Longrightarrow$ (K) : Suppose that C_1, C_2 and C_3 are three distinct, connected, proper subsets of X such that

$$C_i \cup C_j = X, \text{ whenever } i \neq j.$$

Let

$$x_i \in X \setminus C_i \quad (i = 1,2,3)$$

then

$$x_i \in C_j \text{ if } i \neq j.$$

So x_j and x_k belong to a connected subset of $X \setminus x_i$ and hence x_i does not separate x_j and x_k $(i \neq j \neq k \neq i)$.

$(K) \Longrightarrow (S')$: Suppose that x_1, x_2 and x_3 are three distinct points such that no one of them separates the other two.
If x_i is an endpoint, then let $C_i = X \setminus x_i$.
If x_i is a cut point, then let $C_i = \bar{A}_i = A_i \cup x_i$, where A_i is that element of a separation of $X \setminus x_i$ which contains the other two points x_j $(j \neq i)$.
Then C_i is a connected, proper subset of X $(i = 1,2,3)$.
When x_i is an endpoint, then $C_i \cup C_j = X$ because $x_i \in C_j$ $(i \neq j)$.
When $i \neq j$ and both x_i and x_j are cut points, then $A_i \cup A_j$ is open in X and also closed $(\overline{A_i \cup A_j} = \bar{A}_i \cup \bar{A}_j = A_i \cup x_i \cup A_j \cup x_j = A_i \cup A_j)$. Since X is connected, $A_i \cup A_j = X$.
So also in this case

$$C_i \cup C_j = X \quad (i \neq j).$$

A (B)-space need not be orderable (every connected T_1-space consisting of more than one point and having no cut points is a counterexample). However, in the next Theorem we will prove that a (B)-space is orderable if the space has no endpoints.

THEOREM 4. *Let* X *be a connected* T_1*-space having no endpoints and satisfying property* (B). *Then* X *is orderable.*

PROOF. Suppose x_1, x_2 and x_3 are three distinct points of X such that no one of them separates the other two.
Then we have the following separations:

$$X \setminus x_1 = \underset{\substack{x_2 \\ x_3}}{A_1} + B_1 \; ; \; X \setminus x_2 = \underset{\substack{x_3 \\ x_1}}{A_2} + B_2 \; ; \; X \setminus x_3 = \underset{\substack{x_1 \\ x_2}}{A_3} + B_3,$$

where both A_i and B_i are non-empty and connected in X.
$\bar{B}_i = B_i \cup x_i$ is connected in $X \setminus x_j$ $(i \neq j)$, hence $\bar{B}_i \subset A_j$ $(i \neq j)$ and consequently $B_i \cap B_j = \emptyset$ $(i \neq j)$, which means that B_1, B_2 and B_3 are three mutually disjoint segments.

REMARK. 1. In Theorem 4.2 we will generalize the above result.

2. At the end of the next chapter we will introduce another condition, denoted by (P), which is also equivalent to the orderability of a connected T_1-space. Since, for the proof of this equivalence, we need some results concerning so-called (V)-spaces and (H)-spaces, we postpone this proof to the next chapter. Here we will confine ourselves to the definition:

DEFINITION 2. A topological space X is said to satisfy (P) if for every two connected subsets A and B of X with a common endpoint p the following holds: $A \cap B = \{p\}$ or $A \subset B$ or $B \subset A$.

Added in proof:

Van Dalen and Wattel [*"A topological characterization of ordered spaces"*, to be published in Gen. Topology Appl.] have given an interesting characterization of the orderability of a topological space, which of course in particular yields another characterization of the orderability of a connected T_1-space.

CHAPTER III

ON A PROPERTY OF ORDERED SPACES DUE TO HERRLICH AND SOME RELATED PROPERTIES

3.1. INTRODUCTION AND DEFINITIONS

The main purpose of this chapter is to discuss a property of ordered spaces, introduced by Herrlich in [11], and some related properties. In fact, these related conditions are weaker forms of Herrlich's condition. With the help of these conditions we are able to characterize non-orderable, cyclically orderable, connected T_1-spaces. The last two sections of this chapter are devoted to property (*V*) and property (*P*), respectively. Property (*V*) was studied by Hursch and Verbeek in [15] and [16], and generalized by Brouwer [3]. Property (*P*), which was mentioned already in the previous chapter turns out to be equivalent to (*O*) in connected T_1-spaces.

DEFINITION 1. A (connected) T_1-space X is said to satisfy

(*H*) - if every connected subset of X has at most two endpoints (in particular X has at most two endpoints).

(*Hp*) - if every connected *proper* subset of X has at most two endpoints.

(*Hd*) - if for every connected subset C of X such that C has at least three distinct endpoints, $C \setminus \{p,q\}$ is disconnected for every pair of distinct endpoints p, q of C.

(*Ht*) - if for every connected subset C of X such that p, q and r are three distinct endpoints of C, the set $C \setminus \{p,q,r\}$ is disconnected, (i.e. C cannot have an endpoint triple).

Condition (*H*) appeared in the doctoral dissertation [11] of Herrlich. Herrlich called spaces satisfying (*H*) "randendlich", and he proved the following theorem: A connected space X is strictly orderable if and only if X satisfies the following conditions:

(i) X is a T_1-space.
(ii) X is "randendlich", i.e. X satisfies (*H*).
(iii) X is locally connected.

This theorem was also published by Herrlich in [12].

3.2. RELATIONS BETWEEN (*H*), (*Hp*), (*Hd*) AND (*Ht*)

In this section we define for temporary use the following conditions on a connected T_1-space X:

(*Hdd*) - If $C \subset X$ is connected and p, q and r are distinct endpoints of C, and $C \setminus \{p,q\}$ is connected, then $C \setminus \{p,r\}$ is disconnected.

(*Hddd*) - If $C \subset X$ is connected and p, q and r are distinct endpoints of C, and $C \setminus \{p,q\}$ and $C \setminus \{p,r\}$ are connected, then $C \setminus \{q,r\}$ is disconnected.

PROPOSITION 1. *In a connected* T_1*-space* X *the conditions* (*Hd*) *and* (*Hdd*) *are equivalent.*

PROOF.

(i) (*Hd*) $\Longrightarrow$ (*Hdd*) : trivial.

(ii) (*Hdd*) $\Longrightarrow$ (*Hd*) : Suppose, contrary to (*Hd*), that there exists a connected set C in X with distinct endpoints p, q and r, such that $C \setminus \{p,q\}$ is connected. By (*Hdd*), $C \setminus \{p,r\}$ and $C \setminus \{q,r\}$ are disconnected.

Take an arbitrary separation (U,T) of $C \setminus \{p,r\}$ and assume $q \in U$. Then $U \cup p$, $U \cup r$, $T \cup p$ and $T \cup r$ are connected (since for instance $C \setminus r$ is connected, $(C \setminus r) \setminus p = U + T$, etc.).
Consequently $U \cup \{p,r\}$ and $T \cup \{p,r\}$ are connected and also $\bar{U} \supset U \cup \{p,r\}$, $\bar{T} \supset T \cup \{p,r\}$.
Now, $C \setminus \{p,q,r\} = S + T$, where $S = U \setminus q$. Then $S \cup \{p,q,r\} = U \cup \{p,r\} \subset \bar{U}$ is connected, and p and r are endpoints of this set. $C \setminus \{p,q\}$ is connected, so $S \cup r$ is also connected and since $p \in \bar{S}$ the set $S \cup \{p,r\}$ is connected. Hence the connected set $S \cup \{p,q,r\} \subset \bar{U}$ has endpoints p,q and r. Since $S \cup r$ is connected, we find that $S \cup q = U$ is <u>not</u> connected, by (*Hdd*). Mark that this holds for <u>any</u> separation (U,T) of $C \setminus \{p,r\}$ with $q \in U$.
Let Q be the component of $C \setminus \{p,r\}$ which contains q. Then Q is not open in C (otherwise Q is clopen in $C \setminus \{p,r\}$, so there would exist a separation (U',T') of $C \setminus \{p,r\}$ with $q \in U' = Q$ and U' connected, contrary to the observation above). Hence, there exists an element $x \in Q \setminus Q^{\circ}$, where Q° is the interior of Q in C. Q is a segment of r in $C \setminus p$, and hence $(C \setminus Q) \setminus p$ is connected. As Q is also a segment of p in $C \setminus r$, $(C \setminus Q) \setminus r$ is connected. It follows that $C \setminus Q$ and $(C \setminus Q) \cup x$ are connected too.

Hence the connected set $(C \setminus Q) \cup x$ has endpoints p, r and x. However, the connectedness of both $(C \setminus Q) \setminus p$ and $(C \setminus Q) \setminus r$ is a contradiction to (*Hdd*).

LEMMA 2. *In a connected* T_1*-space* X: $(Hp) \Longrightarrow (Ht)$.

PROOF. It is very easy to see that a connected T_1-space X has property (Ht) if and only if for every connected subset $S \subset X : |\bar{S} \setminus S| < 3$. So suppose $S \subset X$ is connected and $|\bar{S} \setminus S| \geq 3$.
By (Hp) we clearly have $\bar{S} = X$.
Also, by (Hp), it is impossible that $|\bar{S} \setminus S| > 3$.
So we may assume: $\bar{S} = X$ and $\bar{S} \setminus S = \{r_1, r_2, r_3\}$ with distinct r_1, r_2 and r_3.
We consider the following cases:

a) If S has an endpoint p, then $X \setminus p$ is a connected proper subset of X with at least three endpoints.

b) Let S have a strong cut point p:
$S \setminus p = A + B$, where A and B are connected. We may assume $r_1, r_2 \in \bar{A}$. Moreover $p \in \bar{A}$. Hence $A \cup \{r_1, r_2, p\}$ is a connected proper subset of X with three endpoints.

c) Suppose that for some $p \in S$, $S \setminus p$ has at least 3 components, and that for one of these components, say A, it is true that $r_1, r_2, r_3 \in \overline{(S \setminus p) \setminus A}$. Then $(S \setminus A) \cup \{r_1, r_2, r_3\}$ is a connected proper subset of X with three endpoints.

d) Consequently, it remains to consider the following case:
$\forall p \in S : S \setminus p = A_p + B_p + C_p$ and $r_1 \in \bar{A}_p$, $r_2 \in \bar{B}_p$, $r_3 \in \bar{C}_p$.
Take a point $q \in A_p$, then

$$S \setminus q = A_q + B_q + C_q.$$

$B_p \cup C_p \cup p$ is connected in $S \setminus q$, so let $B_p \cup C_p \cup p \subset A_q$. But then $r_2, r_3 \in \bar{A}_q$, hence there exists a component B of $S \setminus q$ such that $r_1, r_2, r_3 \in \overline{(S \setminus q) \setminus B}$ which leads to a contradiction by c).

PROPOSITION 3. *In a connected* T_1*-space* $X : (Hp) \Longrightarrow (Hd)$.

PROOF. Suppose X satisfies (Hp), but does not satisfy (Hd).
By Proposition 1, X does not satisfy (Hdd) either.

Hence X has distinct endpoints p, q and r such that $X \setminus \{p,q\}$ and $X \setminus \{p,r\}$ are connected.
By Lemma 2, $X \setminus \{p,q,r\}$ is disconnected. We write

$$X \setminus \{p,q,r\} = S + T, \quad S \neq \emptyset, \quad T \neq \emptyset.$$

Observe that $S \cup \{q,r\}$ and $T \cup \{q,r\}$ are connected, and that consequently

$$q,\ r \in \bar{S} \quad \text{and} \quad q,\ r \in \bar{T}.$$

(i) If $X \setminus \{q,r\}$ is connected, then also $S \cup p$ (and $T \cup p$) is connected, and hence $p \in \bar{S}$ (and $p \in \bar{T}$). But then $\bar{S} = S \cup \{p,q,r\}$ is a connected proper subset of X with (at least) three endpoints. This contradicts (*Hp*).

(ii) Let $X \setminus \{q,r\}$ be disconnected. Since X is a connected T_1-space, we have

$$p \in \bar{S} \quad \text{or} \quad p \in \bar{T}.$$

Say $p \in \bar{S}$.
Then $S \cup \{p,q,r\}$ is a connected proper subset of X with (at least) three endpoints. This again contradicts (*Hp*).

PROPOSITION 4. *In a connected* T_1*-space* X : (*Hddd*) $\Longleftrightarrow$ (*Ht*).

PROOF.

a) (*Hddd*) $\Longrightarrow$ (*Ht*) follows immediately from the definitions.

b) Conversely, suppose (*Ht*) is satisfied and let $C \subset X$ be connected. Suppose p, q and r are distinct endpoints of C such that $C \setminus \{p,q\}$, $C \setminus \{p,r\}$ and $C \setminus \{q,r\}$ are connected. (*Ht*) implies that the set $C \setminus \{p,q,r\}$ is not connected. Let $C \setminus \{p,q,r\} = \bigcup_{\alpha \in A} C_\alpha$ be its decomposition into components.

It follows from (*Ht*) that $\bar{C}_\alpha$ is a proper subset of $C_\alpha \cup \{p,q,r\}$ for each $\alpha \in A$.
As a consequence, there are infinitely many components C_α (if there are only finitely many components C_α then $C \setminus \{p,q,r\} = C_\alpha + D_\alpha$; since $C \setminus \{p,q\}$, $C \setminus \{p,r\}$ and $C \setminus \{q,r\}$ are connected it follows that r, q and $p \in \bar{C}_\alpha$).

So we may assume that $p \notin \bar{C}_\alpha$ for three distinct elements $\alpha = \alpha_1, \alpha_2, \alpha_3 \in A$.
C_{α_i} $(i = 1,2,3)$ is closed in the connected set $C \setminus \{q,r\}$, and hence not open. Consequently there exist $d_{\alpha_i} \in C_{\alpha_i} \setminus \overset{\circ}{C}_{\alpha_i}$ $(i = 1,2,3)$, where $\overset{\circ}{C}_{\alpha_i}$ is the interior of C_{α_i} in $C \setminus \{q,r\}$.
By a repeated application of Lemma 1.9 we see that the set $S = (C \setminus \{q,r\}) \setminus (C_{\alpha_1} \cup C_{\alpha_2} \cup C_{\alpha_3})$ is connected. Moreover, $d_{\alpha_1}, d_{\alpha_2}, d_{\alpha_3} \in \bar{S}$. Consequently, $S \cup \{d_{\alpha_1}, d_{\alpha_2}, d_{\alpha_3}\}$ is connected and has an endpoint triple. This contradicts (Ht).

THEOREM 5. *In a connected* T_1*-space* X:

$$(O) \Longrightarrow (H) \Longrightarrow (Hp) \Longrightarrow (Hd) \Longrightarrow (Ht).$$

PROOF. From the foregoing Propositions follows in fact that

$$(O) \Longrightarrow (H) \Longrightarrow (Hp) \Longrightarrow (Hd) \Longleftrightarrow (Hdd) \Longrightarrow (Hddd) \Longleftrightarrow (Ht).$$

(since $(O) \Longrightarrow (H) \Longrightarrow (Hp)$ and $(Hd) \Longrightarrow (Ht)$ are trivial).

REMARK. No one of the above implications can be reversed. For counter-examples we refer to Chapter VI. However, in the present Chapter we will prove that cyclically orderable, non-orderable connected T_1-spaces are precisely those connected T_1-spaces which satisfy (Hp) but not (H).

3.3. ORDERABILITY OF (H) - SPACES

As we mentioned in the introduction of this chapter, Herrlich proved in [11] that a connected, locally connected T_1-space satisfying (H) is strictly orderable. In [19] the question was raised whether a connected T_1-space satisfying (H) is orderable or not. This question was answered in the negative by Hursch and Verbeek [15]. However, it turns out that in a connected T_1-space satisfying $(B'C)$ the conditions (H) and (O) are equivalent. Moreover, in a connected T_1-space satisfying $(B'C)$ which has at least one cut point also the conditions (Ht) and (O) are equivalent.

LEMMA 6. *In a connected* T_1*-space* X:

$$(Ht) + (B'C) \Longrightarrow (B').$$

PROOF. Suppose that for some $p \in X$:
$X \setminus p = A_1 + A_2 + A_3$, with $A_i \neq \emptyset$ $(i = 1,2,3)$.
Choose $x_i \in A_i$ $(i = 1,2,3)$.
Let P_i be the component of p in $\bar{A}_i \setminus x_i$ $(i = 1,2,3)$.
Then, by Lemma 1.12, $P_i \cup A_j \cup A_k$ is the component of p in $X \setminus x_i$ $(i \neq j \neq k \neq i)$. By $(B'C)$ we have $x_i \in \overline{P_i \cup A_j \cup A_k}$; hence $x_i \in \bar{P}_i$.
Consequently, $P_1 \cup P_2 \cup P_3 \cup \{x_1,x_2,x_3\}$ is a connected set which has an endpoint triple. This contradicts (Ht).

LEMMA 7. *Let* X *be a connected* T_1*-space with at least one cut point. Then* $(Ht) + (B') \Longrightarrow (H)$.

PROOF. Suppose there exists a connected set $C \subset X$ which has at least three distinct endpoints p, q and r.

I. First we show that then p, q and r are also endpoints of X.
Let $X \setminus p = A_p + B_p$, $X \setminus q = A_q + B_q$, $C \setminus p \subset A_p$, $C \setminus q \subset A_q$. We will prove $B_p = \emptyset$ and, by symmetry, the assertion will follow.
Suppose that $B_p \neq \emptyset$.
Let $s \in B_p$ and let P be the component of $\bar{B}_p \setminus s$ containing p. Then $s \in \bar{P}$.
If also $B_q \neq \emptyset$, then let $t \in B_q$ and let Q be the component of $\bar{B}_q \setminus t$ containing q. Then $t \in \bar{Q}$.
Now $C \cup P \cup Q \cup \{s,t\}$ is connected and has the endpoint triple $\{r,s,t\}$, which contradicts (Ht).
Hence $B_q = \emptyset$, which means that q is an endpoint of X. In the same way we can prove that r is an endpoint of X (again under the assumption $B_p \neq \emptyset$).

Now, $X \setminus p = A_p + B_p$ and $A_p \cup p = \bar{A}_p$ has the three endpoints p, q and r (see Lemma 1.11).
Consider $\bar{A}_p \setminus \{q,r\}$. If the component C_1 of $\bar{A}_p \setminus \{q,r\}$ containing p contains both q and r in its closure, then $P \cup C_1 \cup \{s,q,r\}$ is connected with endpoint triple $\{s,q,r\}$; which contradicts (Ht).
Thus let $q \notin \bar{C}_1$. Since $\bar{A}_p \setminus r$ is connected, $\bar{A}_p \setminus \{q,r\}$ must have infinitely many components. Let $\bar{A}_p \setminus \{q,r\} = \bigcup_{\alpha \in I} C_\alpha$ be the decomposition into components.

1. If for at least three distinct components C_α, say C_{α_1}, C_{α_2} and C_{α_3}, C_{α_i} is not open in $\bar{A}_p \setminus r$ then we can choose a point $d_{\alpha_i} \in C_{\alpha_i} \setminus \overset{\circ}{C}_{\alpha_i}$ $(i = 1,2,3)$, where $\overset{\circ}{C}_{\alpha_i}$ is the interior of C_{α_i} in $\bar{A}_p \setminus r$.
But then $[(\bar{A}_p \setminus r) \setminus (C_{\alpha_1} \cup C_{\alpha_2} \cup C_{\alpha_3})] \cup \{d_{\alpha_1}, d_{\alpha_2}, d_{\alpha_3}\}$ is connected with endpoint triple $\{d_{\alpha_1}, d_{\alpha_2}, d_{\alpha_3}\}$; contradicting $(\mathcal{H}t)$.
Hence $C_\alpha \subset A_p$ and C_α open in X (since A_p is open in X) for all but finitely many $\alpha \in I$.
Let $I_1 = \{\alpha \in I \mid C_\alpha \subset A_p$ and C_α is open in $X\}$; then I_1 is an infinite set.
Notice that $\bar{C}_\alpha = C_\alpha \cup \{q,r\}$ for each $\alpha \in I_1$.

2. Take a point $x_\alpha \in C_\alpha$, for every $\alpha \in I_1$.
If $\bar{C}_\alpha \setminus x_\alpha$ is connected for at least three elements $\alpha \in I_1$, say α_1, α_2 and α_3, then $C_{\alpha_1} \cup C_{\alpha_2} \cup C_{\alpha_3} \cup \{q,r\}$ is connected with endpoint triple $\{x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}\}$; which contradicts $(\mathcal{H}t)$.
Consequently, if $I_2 = \{\alpha \in I_1 \mid \bar{C}_\alpha \setminus x_\alpha$ is disconnected$\}$, then I_2 is an infinite set.

3. For every $\alpha \in I_2$ let $\bar{C}_\alpha \setminus x_\alpha = S_\alpha + T_\alpha$, where $q \in S_\alpha$, $T_\alpha \neq \emptyset$.
If $r \in S_\alpha$ for at least three elements $\alpha \in I_2$, say α_1, α_2 and α_3, then choose a point $t_{\alpha_i} \in T_{\alpha_i}$ and let V_{α_i} be the component of $\bar{T}_{\alpha_i} \setminus t_{\alpha_i}$ containing x_{α_i} $(i = 1,2,3)$. C_{α_i} is open in X, so T_{α_i} is clopen in $X \setminus x_{\alpha_i}$ since $q, r \notin T_{\alpha_i}$. Hence for some R_{α_i} : $X \setminus x_{\alpha_i} = R_{\alpha_i} + T_{\alpha_i}$. Since X satisfies $(\mathcal{B}')$ it follows from Lemma 1.11 that t_{α_i} is an endpoint or a strong cut point of $\bar{T}_{\alpha_i} = X \setminus R_{\alpha_i}$ $(i = 1,2,3)$. Therefore $t_{\alpha_i} \in \bar{V}_{\alpha_i}$ $(i = 1,2,3)$.
Now $\bigcup_{i=1}^{3} S_{\alpha_i} \cup V_{\alpha_i} \cup t_{\alpha_i}$ is connected with endpoint triple $\{t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_3}\}$; which again contradicts $(\mathcal{H}t)$.
Hence $I_3 = \{\alpha \in I_2 \mid r \in T_\alpha\}$ is an infinite set.

4. Let D_α be the component of q in $\bar{C}_\alpha \setminus x_\alpha$ for each $\alpha \in I_3$. If $x_\alpha \in \bar{D}_\alpha$ for at least three elements $\alpha \in I_3$, say α_1, α_2 and α_3, then $D_{\alpha_1} \cup D_{\alpha_2} \cup D_{\alpha_3} \cup \{x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}\}$ is connected with endpoint triple $\{x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}\}$; contradiction.
So $I_4 = \{\alpha \in I_3 \mid D_\alpha$ is closed$\}$ is infinite.

5. For each $\alpha \in I_4$, D_α is closed and hence not open in $\bar{C}_\alpha = C_\alpha \cup \{q,r\}$. So we can choose a point $d_\alpha \in D_\alpha \setminus D^\circ_\alpha$, where D°_α is the interior of D_α in $\bar{C}_\alpha$. Since $r \in \bar{C}_\alpha \setminus D_\alpha$ for every $\alpha \in I_4$ the set $(\bar{C}_{\alpha_1} \setminus D_{\alpha_1}) \cup (\bar{C}_{\alpha_2} \setminus D_{\alpha_2}) \cup (\bar{C}_{\alpha_3} \setminus D_{\alpha_3}) \cup \{d_{\alpha_1}, d_{\alpha_2}, d_{\alpha_3}\}$ is connected and has the endpoint triple $\{d_{\alpha_1}, d_{\alpha_2}, d_{\alpha_3}\}$, where α_1, α_2, α_3 are arbitrary distinct elements if I_4. Contradiction.
We conclude that $B_p = \emptyset$.
This proves I.

II. Now, let b be a cut point of X.
Then $X \setminus b = A_b + B_b$, where A_b and B_b are both non-empty and connected.
We may assume: $p, q \in A_b$. But then $\bar{A}_b$ is a connected subset of X having three endpoints b, p and q.
From I it follows that b is an endpoint of X. Contradiction.
This proves the theorem.

LEMMA 8. *In a connected* T_1*-space* X:

$$(H) + (B') \Longrightarrow (O).$$

PROOF. In fact we will prove: $(H) + (B') \Longrightarrow (S')$.
Let x_1, x_2 and x_3 be three distinct points of X.
Suppose x_1 does not separate x_2 and x_3, and x_2 does not separate x_1 and x_3.
Then we have

$$X \setminus x_1 = \underset{\substack{x_2\\x_3}}{A_1} + B_1 \quad ; \quad X \setminus x_2 = \underset{\substack{x_3\\x_1}}{A_2} + B_2 ,$$

where A_i, B_i $(i = 1,2)$ are connected.
Since $B_1 \cup x_1$ is connected in $X \setminus x_2$, we have $B_1 \subset A_2$ and consequently $B_1 \cap B_2 = \emptyset$, $B_2 \subset X \setminus B_1$.
By Lemma 1.11, $(X \setminus B_1) \setminus B_2$ is connected, satisfies (B') and has the points x_1 and x_2 as endpoints. Moreover, since condition (H) is clearly hereditary for connected subspaces, $Y = (X \setminus B_1) \setminus B_2$ is an (H)-space.

(i) Suppose x_3 does not separate x_1 and x_2 in Y. Then

$$Y \setminus x_3 = \underset{\substack{x_1\\ x_2}}{P_3} + Q_3 ,$$

and again by Lemma 1.11, $Y \setminus Q_3 = P_3 \cup x_3$ is connected and has at least three endpoints, namely x_1, x_2 and x_3, which contradicts (H).
Hence

$$Y \setminus x_3 = \underset{x_1}{P_3} + \underset{x_2}{Q_3}.$$

(ii) Since $B_1 \cup P_3$ and $B_2 \cup Q_3$ both are closed in $X \setminus x_3$ we have

$$X \setminus x_3 = \underset{x_1}{(B_1 \cup P_3)} + \underset{x_2}{(B_2 \cup Q_3)},$$

which means that x_3 separates x_1 and x_2 in X.

As a consequence of the previous lemmas we have:

THEOREM 9. *In a connected* T_1*-space* X:

$$(H) + (B'C) \Longrightarrow (O),$$

and

THEOREM 10. *In a connected* T_1*-space* X *having at least one cut point:*

$$(Ht) + (B'C) \Longrightarrow (O).$$

REMARK. A plane circle is a connected T_1-space without cut points, satisfying (Hp) and (B') and which is not orderable.

3.4. CYCLIC ORDERABILITY

In section 1.3 we introduced the notion of cyclic orderability. The next two sections are devoted to the study of this concept. We will show some theorems already announced in section 1.3 and we will prove that cyclic orderability is closely related to some of the conditions studied in the previous sections of the present chapter.

THEOREM 11. *Let* X *be a cyclically orderable connected* T_1*-space having at least one cut point. Then* X *is orderable.*

PROOF.

(i) Every connected proper subset of X is orderable, and hence satisfies (H). So X satisfies condition (Hp), and hence condition (Ht).

(ii) We now show that X satisfies (B').
For that purpose suppose that p is a cut point of X having at least three segments.
Then there exist non-empty sets A, B and C such that

$$X \setminus p = A + B + C.$$

Let $a \in A$, $b \in B$, $c \in C$ and assume $a < b < c$, where $< = <_R^{(p)}$ is the total ordering in $X \setminus p$, induced by the cyclic ordering R compatible with the topology on X.
Then $(a,c) = \{x \in X \mid (a,x,c) \in R\}$ is open in $X \setminus p$ and
$[a,c] = X \setminus \{x \in X \mid (c,x,a) \in R\}$ is closed in $X \setminus p$.
It follows that $D = B \cap (a,c) = B \cap [a,c]$ is a clopen non-void subset of $X \setminus p$. Hence $p \in \bar{D}$.
However, this is impossible, since $[a,c]$ is closed in X.
Consequently, X satisfies (B').

(iii) Since $(B') \Longrightarrow (B'C)$ and X has at least one cut point, we conclude, by Theorem 10, that X is orderable.

THEOREM 12. *Let* X *be a connected* T_1*-space. Then* X *is a non-orderable cyclically orderable space if and only if*:

(i) $\forall x \in X : X \setminus x$ *is connected.*
(ii) $\forall x, y \in X \ (x \neq y) : X \setminus \{x,y\}$ *is disconnected.*

PROOF.

1. Let X be a connected T_1-space, and let X be cyclically orderable, but not orderable. From Theorem 11 it follows that every point of X is an end-point of X, i.e. condition (i) holds.
Since a cyclically orderable space satisfies (Hp), X certainly satisfies (Hd). Since X contains no cut points, this means that $X \setminus \{x,y\}$ is disconnected for every pair of distinct points x and y in X, i.e. X also fulfils

condition (ii).

2. Let X be a connected T_1-space, satisfying conditions (i) and (ii). Then it is clear that X is not orderable.
So we have to prove that X is cyclically orderable.
Let $x_1 \in X$.

a) $X \setminus x_1$ satisfies property $(\mathcal{B}')$:
For, suppose

$$X \setminus \{x_1,x_2\} = \underset{a}{A} + \underset{b}{B} + C,\ C \neq \emptyset \text{ then } \bar{C} = C \cup \{x_1,x_2\} \text{ and } \bar{C}$$

is connected in $X \setminus \{a,b\}$. . . . . . (1)
Now let $X \setminus \{a,b\} = \underset{x_1}{P} + Q$, $Q \neq \emptyset$. Then $\bar{Q} = Q \cup \{a,b\}$ is connected, and so $x_2 \in Q$ (if $x_2 \notin Q$, then $\bar{Q} \subset X \setminus \{x_1,x_2\}$ which is impossible). This contradicts (1).

b) $X \setminus x_1$ satisfies property $(\mathcal{B})$:
For, suppose there exist three distinct points p_1, p_2, $p_3 \in X \setminus x_1$, such that

$$X \setminus \{x_1,p_i\} = A_i + B_i \ (i = 1,2,3),$$

with A_i and B_i non-empty and connected (i = 1,2,3) and with $B_i \cap B_j = \emptyset$ for $i \neq j$.
Let $b_1 \in B_1$ and $b_2 \in B_2$ and $X \setminus \{b_1,b_2\} = \underset{x_1}{P} + Q$, $Q \neq \emptyset$.

Now, $(X \setminus x_1) \setminus (B_1 \cup B_2 \cup B_3) = Y$ is connected in $X \setminus \{b_1,b_2\}$ by Corollary 1.9.1.
If $Y \subset P = X \setminus \bar{Q}$, then $\bar{Q} = Q \cup \{b_1,b_2\}$ is a connected set in $B_1 \cup B_2 \cup B_3$, meeting B_1 and B_2. However, B_1, B_2 and B_3 are separated sets since $B_i \subset A_j$ $(i \neq j)$.
If $Y \subset Q$, then $\bar{Y} \cup \bar{B}_3 = Y \cup B_3 \cup x_1$ is a connected set in $X \setminus \{b_1,b_2\}$, meeting both P and Q.
So we arrive at a contradiction.

c) $X \setminus x_1$ is a connected T_1-space, having no endpoints, and satisfying property $(\mathcal{B})$. By Theorem 2.4 such a space is orderable.
Let < be an ordering on $X \setminus x_1$.

Let $p, q \in X \setminus x_1$ be such that $p < q$.
Then

$$X \setminus \{x_1,p,q\} = \{z \in X \setminus x_1 \mid z < p\} \cup \{z \in X \setminus x_1 \mid p < z < q\} \cup$$

$$\cup \{z \in X \setminus x_1 \mid q < z\} = (\ ,p) \cup (p,q) \cup (q,\ ).$$

Since $X \setminus \{p,q\}$ is disconnected, x_1 cannot be a limitpoint of each of these three connected intervals. If $x_1 \notin \overline{(\ ,p)}$ then $(\ ,p)$ is a proper clopen subset of the connected space $X \setminus p$ which is impossible. Thus $x_1 \in \overline{(\ ,p)}$ and similarly $x_1 \in \overline{(q,\ )}$; hence $x_1 \notin \overline{(p,q)}$ . . . . . . (*)
Now we can define a cyclic ordering R on X as follows:

$$(a,b,x_1) \in R \Longleftrightarrow (x_1,a,b) \in R \Longleftrightarrow (b,x_1,a) \in R \Longleftrightarrow$$

$$\Longleftrightarrow (a < b,\ a \neq b \neq x_1 \neq a).$$

If a, b and c are elements of $X \setminus x_1$ then

$$(a,b,c) \in R \Longleftrightarrow (a < b < c) \vee (c < a < b) \vee (b < c < a).$$

It is easily verified that R is indeed a well-defined cyclic ordering on X. From (*) it follows that R is compatible with the topology on X, which means that X is cyclically orderable.

LEMMA 13. *Let* X *be an orderable space, having exactly two components, say* A *and* B. *Then either*

$$\forall x \in A : \forall y \in B : x < y \quad \textit{or}$$
$$\forall x \in A : \forall y \in B : y < x.$$

(*where* < *is a total ordering on* X *compatible with the topology on* X).

PROOF. Suppose there exist points $p, q \in A$ and $r \in B$ such that $p < r < q$.
Then:

$$X \setminus r = \underset{p}{\{x \in X \mid x < r\}} + \underset{q}{\{x \in X \mid r < x\}},$$

while A is a connected subset of $X \setminus r$ containing both p and q, which is a contradiction.

THEOREM 14. *Let* X *be a cyclically orderable, connected* T_1*-space. Let* S_1 *and* S_2 *be two cyclic orderings on* X *compatible with the topology on* X*. Then* $S_1 = S_2$ *or* $S_1 = S_2^{-1}$.
In other words: in a cyclically orderable, connected T_1*-space the compatible cyclic ordering is unique up to inversion.*

PROOF.

(i) Suppose X has at least one endpoint p. By Proposition 1.7, $X \setminus p$ is an orderable, connected space. It follows from Theorem 1.4 that the compatible ordering in $X \setminus p$ is unique apart from inversion. The assertion now is a consequence of Čech [6], Theorem 5.2.1 (cf. the second Remark following Proposition 1.7).

(ii) Suppose every point of X is a cut point. Hence, by Theorem 11, X is an orderable space. We denote the compatible ordering on X by <. Let $R_1 = R_<$ be the induced cyclic ordering on X, and suppose that R_2 is another compatible cyclic ordering on X. We have to show: $R_1 = R_2$ or $R_1 = R_2^{-1}$.
Let $p \in X$.
Then $X \setminus p = A + B$, with both A and B non-empty and connected. We may assume: $\forall x \in A : \forall y \in B : x < y$.
Let $<_i \; = \; <^{(p)}_{R_i}$ be the ordering in $X \setminus p$ induced by R_i $(i = 1,2)$. On A and on B separately the orderings $<$ and $<_1$ coincide, while $\forall x \in A : \forall y \in B : y <_1 x$ (cf. part 4 of the first Remark following Proposition 1.7).
Since both A and B are orderable connected spaces, we have by Theorem 1.4:

on A : $(< \,=)\; <_1 \; = \; <_2$ or $(< \,=)\; <_1 \; = \; <_2^{-1}$,

and on B : $(< \,=)\; <_1 \; = \; <_2$ or $(< \,=)\; <_1 \; = \; <_2^{-1}$.

a) Suppose that $(< \,=)\; <_1 \; = \; <_2$ on A, and that $(< \,=)\; <_1 \; = \; <_2^{-1}$ on B. Take $a, b \in A$ such that $a <_1 b$ and $c, d \in B$ such that $c <_1 d$.
Then $a <_2 b$, which means $(p,a,b) \in R_2$, and $d <_2 c$, which means $(p,d,c) \in R_2$.
Let $O_1 = \{x \in X \mid p < x < c\}$ and $O_2 = \{x \in X \mid b < x < p\}$.
Then $p \in \bar{O}_1 \cap \bar{O}_2$ since X is connected.
We now consider the following two cases:

1) $b <_2 c$.
2) $c <_2 b$.

In both cases we will derive a contradiction.

Case 1): as $a <_2 b$ and $b <_2 c$ we have $a <_2 c$, which means $(p,a,c) \in R_2$.
Let $U_1 = \{x \in X \mid (c,x,a) \in R_2\}$.
Then U_1 is open in X and $p \in U_1$.
Suppose $U_1 \cap O_2 \neq \emptyset$ and let $x \in U_1 \cap O_2$. Then $(c,x,a) \in R_2$. Moreover:
$x \in O_2 \Longrightarrow a <_1 b <_1 x \Longrightarrow a <_2 x$, which means $(p,a,x) \in R_2$.
$(x,a,c) \in R_2$ and $(x,p,a) \in R_2$ imply $(x,p,c) \in R_2$. Hence $c <_2 x$. By Lemma 13, however, it follows from $x \in A$ and $c \in B$ that $x <_2 c$.
Consequently $U_1 \cap O_2 = \emptyset$, which contradicts the fact that $p \in \bar{O}_2$.

Case 2): as $d <_2 c$ and $c <_2 b$ we have $d <_2 b$, which means $(p,d,b) \in R_2$.
Let $U_2 = \{x \in X \mid (b,x,d) \in R_2\}$.
Then U_2 is open in X and $p \in U_2$.
Analogous to the previous case we can derive that $U_2 \cap O_1 = \emptyset$, which contradicts the fact that $p \in \bar{O}_1$.

b) Now we may assume that the orderings $<_1$ and $<_2$ coincide both on A and on B. We want to show that $<_1$ and $<_2$ coincide on $A \cup B$.
We know already:

$$\forall x \in A : \forall y \in B : y <_1 x.$$

Suppose that $<_1$ and $<_2$ do not coincide on $A \cup B$. Then we have

$$\forall x \in A : \forall y \in B : x <_2 y.$$

Take $s \in A$ and $t \in B$. Then

$$s < p < t \ ; \ t <_1 s \ ; \ s <_2 t.$$

Let $O = \{x \in X \mid p < x < t\}$. Then $p \in \bar{O}$.
Let $U_p = \{x \in X \mid (t,x,s) \in R_2\}$.
Then U_p is open in X.
$s <_2 t$ implies $(p,s,t) \in R_2$, hence $p \in U_p$.

1. Suppose $U_p \cap O \neq \emptyset$ and let $x \in U_p \cap O$. Then $(t,x,s) \in R_2$. Moreover,
$x \in O \Longrightarrow x < t \Longrightarrow x <_1 t \Longrightarrow x <_2 t \longrightarrow (p,x,t) \in R_2$.
$(x,s,t) \in R_2$ and $(x,t,p) \in R_2$ imply $(x,s,p) \in R_2$. Hence $x <_2 s$. However, $x \in B$ and $s \in A$ imply $s <_2 x$.

2. Consequently, $U_p \cap O = \emptyset$. This, however, contradicts $p \in \bar{O}$.

c) From the foregoing it follows that $<_1 = <_2$ or $<_1 = <_2^{-1}$ on $A \cup B$. Hence $R_1 = R_2$, resp. $R_1 = R_2^{-1}$ and the theorem is proved.

Recall that an interval in a non-orderable, cyclically orderable connected T_1-space X is any set of one of the following forms (where p,a and b run through X):

$$X,\ X \setminus p,\ \{x \in X \mid (a,x,b) \in R\} = J(a,b),\ J(a,b) \cup a,$$

$$J(a,b) \cup b,\ J(a,b) \cup \{a,b\}.$$

Now we will prove the following

LEMMA 15. *In a non-orderable, cyclically orderable connected* T_1*-space* X *the connected subsets are precisely the intervals.*

PROOF. The connectedness of every interval in X is an immediate consequence of the fact that $J(a,b) = \{x \in X \mid a < x < b\}$, where $< = <_R^{(p)}$ is the ordering induced by the compatible cyclic ordering R in $X \setminus p$, for some p with $(a,p,b) \notin R$.
Conversely, let C be a connected subset of X, such that $X \setminus C$ contains at least two points.
Let $p, q \in X \setminus C$ $(p \neq q)$.
$X \setminus \{p,q\} = \{x \in X \mid (p,x,q) \in R\} + \{x \in X \mid (q,x,p) \in R\}$.
So we may assume $C \subset \{x \mid (p,x,q) \in R\}$.
Let $r \in X$ be such that $(r,p,q) \in R$.
Let $< = <_R^{(r)}$ be the ordering in $X \setminus r$ induced by R.
Then $C \subset \{x \mid p < x < q\} = (p,q)$.
Hence, there exist $a, b \in X \setminus r$ such that

$$C = (a,b) \text{ or } C = [a,b) \text{ or } C = (a,b] \text{ or } C = [a,b].$$

Consequently,

$$C = J(a,b),\ C = J(a,b) \cup a,\ C = J(a,b) \cup b \text{ or}$$

$$C = J(a,b) \cup \{a,b\}.$$

THEOREM 16. *Let* X *be a strictly cyclically orderable, connected* T_1*-space. Then* X *is locally connected.*

PROOF.

a) Let X be an orderable space. Let R be the compatible cyclic ordering on X and < the compatible ordering on X. By Theorem 14 we may assume $R = R_<$. This means that the relation between < and R is as in Proposition 1.6. Hence the open intervals, with respect to the ordering <, are connected and form a base for the topology in X. Consequently, X is locally connected, and therefore strictly orderable.

b) Let X be a non-orderable space. Since, by Lemma 15, the intervals with respect to the compatible cyclic ordering form a base consisting of connected subsets of X, we conclude that X is locally connected.

COROLLARY 16.1. *Let* X *be a strictly cyclically orderable connected* T_1*-space. Then*:

$$\forall p \in X : X \setminus p \text{ is strictly orderable.}$$

THEOREM 17. *Let* X *be a non-orderable, cyclically orderable, locally connected, connected* T_1*-space.*
Then X *is strictly cyclically orderable.*

PROOF. Let U open in X and $p \in U$.
We have to show that there exist $a, b \in X$ such that

$$p \in \{x \in X \mid (a,x,b) \in R\} \subset U.$$

So we may assume $U \neq X$. Let $q \in X \setminus U$.
$X \setminus q$ is an orderable, locally connected, connected space, and consequently $X \setminus q$ is strictly orderable.
Since, by Theorem 12, $X \setminus q$ has no endpoints, there exist $a, b \in X \setminus q$ such that

$$p \in \{x \in X \mid a < x < b\} \subset U, \text{ where } < \; = \; <_R^{(q)}.$$

From $\{x \in X \mid a < x < b\} = \{x \in X \mid (a,x,b) \in R\}$ we now conclude that X is strictly cyclically orderable.

REMARK. It is not possible to omit the non-orderability of X in Theorem 17. The half-open interval [0,1) is a counterexample.

3.5. CHARACTERIZATION OF NON-ORDERABLE, CYCLICALLY ORDERABLE CONNECTED T_1-SPACES

In this section we want to characterize non-orderable, cyclically orderable connected T_1-spaces in terms of the conditions introduced in the first sections of this chapter.

THEOREM 18. *Let* X *be a connected* T_1*-space. Then* X *is a non-orderable cyclically orderable space if and only if* X *satisfies* (Hp) *but not* (H).

PROOF.

1. Let X be a connected T_1-space, which is cyclically orderable but not orderable. Then every point of X is an endpoint. Hence X does not satisfy condition (H). Since X is cyclically orderable, every connected proper subset of X is orderable, which means that X satisfies (Hp).

2. Let X be a connected T_1-space satisfying (Hp) but not (H). Then clearly X is not orderable and moreover X has at least three distinct endpoints p, q and r.

(i) We shall first show that X does not have cut points.
Suppose to the contrary that $s \in X$ is a cut point of X.
If $X \setminus s$ has exactly two components, then one of them must contain at least two of the points p, q and r. The closure of that component is a connected proper subset of X having at least three endpoints, which is impossible.
If $X \setminus s$ has at least four components there is a component C which does not contain any of the points p, q and r. But then $X \setminus C$ is connected and has p, q and r for endpoints (Lemma 1.11), which again is impossible.
It remains to consider the case that $X \setminus s$ has exactly three components, each of them containing precisely <u>one</u> of the points p, q and r.
Let $X \setminus s = \underset{p}{A} + \underset{q}{B} + \underset{r}{C}$.

Take any point $a \in A$ ($a \neq p$).
If a is an endpoint of X, $A \cup s$ has three endpoints a, p and s.
If a is a cut point of X, a must separate p, q and r (otherwise there is a proper subset of X having at least three endpoints). But this contradicts the fact that $B \cup C \cup s$ is connected in $X \setminus a$.

(ii) Since consequently every point of X is an endpoint of X, and since X satisfies (Hp), (and hence (Hd)), $X \setminus \{x,y\}$ is disconnected for every pair of distinct points x, y in X.

From Theorem 12 it now follows that X is cyclically orderable.

THEOREM 19. *Let* X *be a connected* T_1*-space. Then* X *is a non-orderable cyclically orderable space if and only if* X *satisfies property* (*Ht*) *and every point of* X *is an endpoint of* X.

PROOF.

1. The necessity of the condition follows immediately from Theorems 12 and 13.

2. To prove the sufficiency, let X be a connected T_1-space satisfying condition (*Ht*) and having no cut points. Then it is clear that X is not orderable. We will prove the cyclic orderability of X from Theorem 12 by showing that $X \setminus \{p,q\}$ is disconnected for every $p, q \in X$ $(p \neq q)$.

a) $\forall a, b, c \in X$ $(a \neq b \neq c \neq a)$: $X \setminus \{a,b,c\} = Y$ is disconnected by condition (*Ht*).

b) If $X \setminus \{a,b\}$ is connected, then $X \setminus \{a,b,c\}$ has at least three components $(a \neq b \neq c \neq a)$. For suppose $X \setminus \{a,b,c\} = C_1 + C_2$, where C_1 and C_2 are connected.
Then $c \in \bar{C}_1 \cap \bar{C}_2$, and $a, b \in \bar{C}_1 \cup \bar{C}_2 = X$.
If $a, b \in \bar{C}_1$, then $(\bar{C}_1 =)$ $C_1 \cup \{a,b,c\}$ is connected and has an endpoint triple, contradicting (*Ht*).
If $a \in \bar{C}_1 \setminus \bar{C}_2$ and $b \in \bar{C}_2 \setminus \bar{C}_1$, then $X \setminus c = (C_1 \cup a) + (C_2 \cup b)$, which is impossible, since c is an endpoint of X. . . . . . (*)

c) 1. Now suppose that $X \setminus \{p,q\}$ is connected for some $p, q \in X$ $(p \neq q)$. Then it follows from b) that the set $X \setminus \{p,q,r\}$ has at least three components, for every $r \in X \setminus \{p,q\}$.
Let

$$X \setminus \{p,q,r\} = \underset{s}{A} + \underset{t}{B} + \underset{u}{C},$$

then $A \cup r$, $B \cup r$, $C \cup r$ are connected, $r \in \bar{A} \cap \bar{B} \cap \bar{C}$, and [cf. (*)] we may assume $p, q \in \bar{A}$.

c) 2. Suppose $X \setminus \{t,u\}$ is disconnected.

Then $X \setminus \{t,u\} = D + E$, $D \neq \emptyset$, $E \neq \emptyset$.
If p, q, r $\notin$ E then the connected set $E \cup \{t,u\}$ would be contained in $X \setminus \{p,q,r\}$, which is impossible.
If $p \in D$ and $q, r \in E$ then the connected set $A \cup \{p,q,r\}$ is contained in $X \setminus \{t,u\}$, which is also impossible.
Hence $X \setminus \{t,u\}$ is connected.

c) 3. From b) it follows that:

$$X \setminus \{s,t,u\} = P + Q + R, \ (P \neq \emptyset, \ Q \neq \emptyset, \ R \neq \emptyset),$$

$s \in \bar{P} \cap \bar{Q} \cap \bar{R}$ and $P \cup s$, $Q \cup s$, $R \cup s$ are connected.
$\bar{P} \cap \{t,u\} \neq \emptyset$, since otherwise P would be clopen in $X \setminus s$.
Let $t \in \bar{P}$.
If p, q, r $\notin$ P then the connected set $P \cup s \cup t$ is contained in $X \setminus \{p,q,r\}$, which is impossible.
So we may assume that $p \in P$, $q \in Q$, $r \in R$. Moreover, as we observed already, the closure of each of these three sets contains at least one of the two points t, u.
Put $W = (A \cup r) \cup (R \cup s)$. Then W is connected and p, q, t, u $\notin$ W.
But $\bar{W}$ contains at least three of these four points, which contradicts (*Ht*).

THEOREM 20. *A connected* T_1*-space* X *is a non-orderable cyclically orderable space if and only if the complement of each connected subset of* X *is connected.*

PROOF.

1. The condition is necessary: follows immediately from Lemma 15, since the complement of an interval is again an interval or an empty set.

2. The condition is sufficient: $X \setminus p$ is connected, since $\{p\}$ is connected ($\forall p \in X$) and $X \setminus \{p,q\}$ is disconnected, since $\{p,q\}$ is disconnected ($\forall p,q \in X$, $p \neq q$). The assertion now follows from Theorem 12.

PROPOSITION 21. *In a connected* T_1*-space* X:

$$(B'C) + (Ht) \Longrightarrow (Hp).$$

PROOF.

(i) If X has at least one cut point, X is orderable by Theorem 10. Hence X satisfies (*Hp*).

(ii) If X has no cut points, X is a non-orderable, cyclically orderable space by Theorem 19. Hence, by Theorem 18, X satisfies (*Hp*).

3.6. ON (*V*) - SPACES

DEFINITION 2. A (connected) T_1-space X is said to satisfy

(*V*) - if X contains a point x_0 such that every connected subset of X containing x_0 is closed.

Condition (*V*) was studied by Hursch and Verbeek in [15] and [16]. They constructed a connected T_2-space, satisfying (*V*) and consequently (as they showed) satisfying (*H*), but not satisfying (*O*). So they settled a problem, raised in [19], in the negative. A generalization of condition (*V*) was introduced and discussed by Brouwer in [3].

In this section we only investigate those properties of (*V*)-spaces (i.e. spaces satisfying (*V*)) which we need for our purposes. For a more detailed discussion of (*V*)-spaces we refer to [15] and [3].

Recall that all spaces under consideration are assumed to be connected T_1-spaces containing at least two points.

Let X be a connected T_1-space satisfying (*V*).
Let x_0 be a point of X such that every connected subset of X containing x_0 is closed.

Let C be a component of $X \setminus x_0$. Since $X \setminus C$ is connected and $x_0 \in X \setminus C$, $X \setminus C$ is closed in X and therefore C is an open subset of X. Hence $\bar{C} = C \cup x_0$.
It follows also that no other point $x_1 \in X$ can have the property that every connected subset of X containing x_1 is closed. Hence x_0 is uniquely determined and x_0 is called the *base point* of X.

Let $x \in X$ and $x \neq x_0$. Let C_0 be that component of $X \setminus x$ containing x_0. Then C_0 is closed in X. This means that $X \setminus x$ consists of infinitely many components, since otherwise (every component of $X \setminus x$ and in particular) C_0 is

an open subset of X, which contradicts the connectedness of X.
Let C be a component of $X \setminus x$ not containing x_0. Since $X \setminus C$ is connected and $x_0 \in X \setminus C$, $X \setminus C$ is closed in X and therefore C is open in X. Hence $\bar{C} = C \cup x$.

So in a (V)-space X with base point x_0 the following holds:

Every component of $X \setminus x_0$ is open. If $x \neq x_0$ then $X \setminus x$ has infinitely many components. The component of $X \setminus x$ containing x_0 is closed and all other components of $X \setminus x$ are open.

3.7. ON CONDITION (P)

As we already announced at the end of Chapter II, we shall prove in this section that for connected T_1-spaces the orderability is equivalent to yet another property, called (P).

Recall that a space X is said to possess property (P), (or is said to be a (P)-space), iff for every pair of connected subsets A, B of X having a common endpoint p the following holds:

$$A \cap B = \{p\} \text{ or } A \subset B \text{ or } B \subset A.$$

THEOREM 22. *In a connected* T_1*-space* X: $(P) \Longleftrightarrow (O)$.

PROOF.

1. $\Longleftarrow$: trivial, since the only connected subsets of an orderable space are the intervals.

2. $\Longrightarrow$:

(i) It is clear from the definition that condition (P) is hereditary for connected subspaces. Hence, in order to show that a (P)-space is also an (H)-space it suffices to prove that a (P)-space cannot have more than two endpoints.
In order to do that, we suppose that, to the contrary, there exist three distinct endpoints p_1, p_2 and p_3 of the (P)-space X.

α) Suppose first that at least two of the three sets: $X \setminus \{p_1,p_2\}$, $X \setminus \{p_2,p_3\}$ and $X \setminus \{p_3,p_1\}$ are disconnected. (This will lead to a contradiction).

For instance, let

$$X \setminus \{p_1,p_2\} = \underset{p_3}{A} + B, \text{ where } B \neq \emptyset, \text{ and}$$

$$X \setminus \{p_2,p_3\} = \underset{p_1}{C} + D, \text{ where } D \neq \emptyset.$$

Since $B \cup p_1$ is connected in $X \setminus \{p_2,p_3\}$ we have $B \cup p_1 \subset C$, and hence $B \cap D = \emptyset$ and $A \cup C = X \setminus p_2$.
Since $X \setminus p_2$ is connected and since A and C are both open in $X \setminus p_2$ we have $A \cap C \neq \emptyset$. Let $x \in A \cap C$.
$A \cup \{p_1,p_2\}$ is connected and has p_2 as an endpoint.
$C \cup \{p_2,p_3\}$ is connected and has p_2 as an endpoint.
However, $(A \cup \{p_1,p_2\}) \cap (C \cup \{p_2,p_3\}) \supset \{p_2,x\}$, and (since $B \subset C$) neither $A \cup \{p_1,p_2\} \subset C \cup \{p_2,p_3\}$ nor $C \cup \{p_2,p_3\} \subset A \cup \{p_1,p_2\}$.
This contradicts (P).
We conclude that at least two of the three sets: $X \setminus \{p_1,p_2\}$, $X \setminus \{p_2,p_3\}$ and $X \setminus \{p_3,p_1\}$ are connected.

β) For instance, let $X \setminus \{p_1,p_2\}$ and $X \setminus \{p_2,p_3\}$ be connected. Then p_2 is an endpoint both of $X \setminus p_1$ and of $X \setminus p_3$.
But $(X \setminus p_1) \cap (X \setminus p_3) \neq \{p_2\}$ and neither $X \setminus p_1 \subset X \setminus p_3$ nor $X \setminus p_3 \subset X \setminus p_1$.
This contradiction proves that a (P)-space is an (H)-space.

(ii) Now we will show that a (P)-space cannot be a (V)-space.
Suppose X is a (V)-space with base point x_0.
Let $p \in X$ and $p \neq x_0$.
Then $X \setminus p$ has infinitely many components. If C_0 is the component of $X \setminus p$ containing x_0, then C_0 is closed in X, and the other components C_α $(\alpha \in A)$ of $X \setminus p$ are open in X.
Let $S = X \setminus C_0$ then S is open in X and connected.
Hence, there exists an element $q \in C_0 \cap \bar{S}$.
Choose any two α_1, $\alpha_2 \in A$ $(\alpha_1 \neq \alpha_2)$ and let $p_i \in C_{\alpha_i}$ $(i = 1,2)$.
Since C_{α_1} and C_{α_2} are clopen in $X \setminus p$, we can write

$$X \setminus p = \underset{p_1}{C_{\alpha_1}} + \underset{p_2}{C_{\alpha_2}} + D,$$

where $C_0 \subset D$.

Replacing p by p_1 or p_2 respectively, we may also conclude that there exist non-void connected sets S_1 and S_2 such that

$$X \setminus p_1 = S_1 + E_{x_0} \quad \text{and} \quad X \setminus p_2 = S_2 + F_{x_0}.$$

Since $S_i \cup p_i = \bar{S}_i$ is connected, it follows that

$$S_i \cup p_i \subset C_{\alpha_i} \quad (i = 1,2).$$

In particular $p \notin S_i$ and hence $p \in E \cap F$.
Since S_i is a component of $S \setminus p_i$, it follows that $S \setminus S_i$ is connected $(i = 1,2)$.
Since $\bar{S}_i = S_i \cup p_i$ and $q \in C_0$ we have $q \notin \bar{S}_i$; from $q \in \bar{S} = \bar{S}_i \cup \overline{(S \setminus S_i)}$ it then follows that $q \in \overline{S \setminus S_i}$ $(i = 1,2)$.
However, the sets $(S \setminus S_1) \cup q$ and $(S \setminus S_2) \cup q$ yield a contradiction to property (P).

(iii) The (P)-space X satisfies property (B').
Suppose, to the contrary, that for some $p \in X$ we have

$$X \setminus p = A_1 + A_2 + A_3, \text{ with non-void } A_i \ (i = 1,2,3).$$

$\bar{A}_i = A_i \cup p$ is connected, hence a (P)-space and consequently not a (V)-space. $(i = 1,2,3)$.
This means that there exist connected sets $B_i \subset \bar{A}_i$ such that $p \in B_i$ and distinct points $b_i \in A_i$ such that $b_i \in \bar{B}_i \setminus B_i$. $(i = 1,2,3)$.
It follows that the set $B_1 \cup B_2 \cup B_3 \cup \{b_1,b_2,b_3\}$ is connected and has an endpoint triple, which contradicts the fact that a (P)-space is an (H)-space.

(iv) Since a (P)-space is an (H)-space and satisfies property (B'), the orderability of X follows from Lemma 8.

REMARK. Observe that in the proof of Theorem 22 we do not need to know that (V)-spaces really exist.

CHAPTER IV

ON TREE-LIKE SPACES AND THE INTERSECTION OF CONNECTED SUBSETS OF A CONNECTED T_1-SPACE

4.1. INTRODUCTION AND DEFINITIONS

In this chapter our attention is mainly focussed on property (*S*) and property (*Int*). A space having property (*S*) is sometimes called a *"tree-like"* space. Tree-like spaces have been studied by G.T. Whyburn in [28], G.L. Gurin in [10] and V.V. Proizvolov in [25] under the additional assumption that X is locally connected and peripherally bicompact respectively. Property (*Int*), the *"connected intersection property"* occurs also in Whyburn [28] for locally connected spaces. Some modifications of this condition, the properties (*Int 2*), (Int^*) and (*Int'*), will also be discussed in this chapter, where (*Int 2*) is again a property occurring in the paper of Whyburn [28].
As remarked already at the end of Chapter I the relation between (*B*) and (*O*) will be the subject of Theorem 2 of this chapter.
Finally, a property (*W*) will be studied. An equivalent form of this property is discussed by A.E. Brouwer [2]; some of the propositions and theorems in which condition (*W*) is occurring have already been proved in a slightly different way by him in [2].

DEFINITION 1. A topological space X is said to satisfy

- (*S*) - if $\forall x,y \in X$, $(x \neq y)$: $\exists z \in X$: z separates x and y. (no two points of X are conjugated). (A space satisfying (*S*) is called a tree-like space).
- (*Int*) - if the intersection of an arbitrary collection of connected subsets of X is connected.
- (Int^*) - if the *closure* of the intersection of an arbitrary collection of connected subsets of X is connected.
- (*Int'*) - if the intersection of an arbitrary collection of *closed* connected subsets of X is connected.
- (*Int 2*) - if the intersection of two connected subsets of X is connected.
- (*W*) - if for every two disjoint connected sets $A, B \subset X$ it is true that $|\bar{A} \cap \bar{B}| \leq 1$.

4.2. PROPERTIES OF TREE-LIKE SPACES

In this section we investigate several properties of tree-like spaces. First of all, we shall prove that a tree-like space is orderable if and only if every cut point is a strong cut point. As always, we only consider connected T_1-spaces.

THEOREM 1. *In a connected* T_1*-space* X:

$$(\mathcal{S}) + (\mathcal{B}') \Longleftrightarrow (\mathcal{O}).$$

PROOF.

(i) $\Longleftarrow$: trivial.

(ii) $\Longrightarrow$: In fact we will prove: $(\mathcal{S}) + (\mathcal{B}') \Longrightarrow (\mathcal{S}')$.
Suppose, to the contrary, that X satisfies $(\mathcal{S})$ and $(\mathcal{B}')$, but does not satisfy $(\mathcal{S}')$. Let p_1, p_2 and p_3 be three distinct points of X, such that no one of them separates the other two.
Then we have

$$X \setminus p_1 = A_{p_1} + B_{p_1} \;;\; X \setminus p_2 = A_{p_2} + B_{p_2} \;;\; X \setminus p_3 = A_{p_3} + B_{p_3},$$
$$\begin{matrix} & p_2 & & p_3 & & p_1 \\ & p_3 & & p_1 & & p_2 \end{matrix}$$

where (A_{p_i}, B_{p_i}) is a (unique) separation of $X \setminus p_i$ when p_i is a cut point, and where $A_{p_i} = \emptyset$ when p_i is an endpoint of X.

In both cases A_{p_i} and B_{p_i} are connected and open in X.
It is clear that

$$A_{p_i} \subset B_{p_j} \qquad (i \neq j)$$
$$A_{p_i} \cap A_{p_j} = \emptyset \; (i \neq j)$$
$$B_{p_i} \cup B_{p_j} = X \; (i \neq j).$$

Now, let

$$S(p_1,p_2) = E(p_1,p_2) \cup \{p_1,p_2\} = \{x \in X \mid x \text{ separates } p_1 \text{ and } p_2\} \cup \{p_1,p_2\};$$
$$S(p_2,p_3) = E(p_2,p_3) \cup \{p_2,p_3\} = \{x \in X \mid x \text{ separates } p_2 \text{ and } p_3\} \cup \{p_2,p_3\};$$
$$S(p_3,p_1) = E(p_3,p_1) \cup \{p_3,p_1\} = \{x \in X \mid x \text{ separates } p_3 \text{ and } p_1\} \cup \{p_3,p_1\}.$$

In the same way $S(p_2,p_1)$, $S(p_3,p_2)$ and $S(p_1,p_3)$ can be defined.
It is clear that $S(p_2,p_1) = S(p_1,p_2)$ etc.
Clearly, $p_1,p_2 \in S(p_1,p_2)$ and $p_3 \notin S(p_1,p_2)$ etc.
Moreover,

$$S(p_1,p_2) \cap S(p_2,p_3) \cap S(p_3,p_1) = \emptyset.$$

For, suppose $x \in S(p_1,p_2) \cap S(p_2,p_3) \cap S(p_3,p_1)$. Then $x \neq p_i$ $(i = 1,2,3)$ and hence x is a strong cut point, i.e.

$$X \setminus x = A_x + B_x,$$

and this separation is unique. However, this contradicts the fact that x must separate each two of the three points p_1, p_2 and p_3.

Also,

$$S(p_1,p_2) \subset S(p_2,p_3) \cup S(p_3,p_1) \text{ etc.}$$

For, let $x \in S(p_1,p_2)$.
If $x = p_i$ $(i = 1$ or $2)$, then certainly $x \in S(p_2,p_3) \cup S(p_3,p_1)$.
If $x \neq p_i$ $(i = 1,2,3)$, then x is a strong cut point which separates p_1 and p_2; i.e.

$$X \setminus x = \underset{p_1}{A_x} + \underset{p_2}{B_x}.$$

Since $p_3 \in A_x$ implies that $x \in S(p_2,p_3)$ and since $p_3 \in B_x$ implies that $x \in S(p_1,p_3)$, the assertion follows.
(Notice that $x \neq p_3$, since $p_3 \notin S(p_1,p_2)$).

Now, let

$$S = S(p_1,p_2) \cup S(p_2,p_3) \cup S(p_3,p_1).$$

Thus every point of S is contained in exactly two of the three subsets $S(p_1,p_2)$, $S(p_2,p_3)$ and $S(p_3,p_1)$.

Let

$$S_1 = S \setminus S(p_2,p_3),$$
$$S_2 = S \setminus S(p_3,p_1),$$
$$S_3 = S \setminus S(p_1,p_2).$$

Then

$$S_1 = S(p_1,p_2) \cap S(p_3,p_1) \qquad S(p_1,p_2) = S_1 \cup S_2$$
$$S_2 = S(p_2,p_3) \cap S(p_1,p_2) \quad \text{and} \quad S(p_2,p_3) = S_2 \cup S_3$$
$$S_3 = S(p_3,p_1) \cap S(p_2,p_3) \qquad S(p_3,p_1) = S_3 \cup S_1.$$

Moreover, $S = S_1 \cup S_2 \cup S_3$ and the sets S_1, S_2 and S_3 are mutually disjoint.
Since $S_i (i = 1,2,3)$ is a subset of $S(p_i,p_j)$ $(i \neq j)$, in each of the sets S_1, S_2 and S_3 we can introduce a total ordering, namely the separation ordering. We recall the definition and some properties of the separation ordering, for example in S_1. (It will then be clear that the separation orderings in $S(p_1,p_2)$ and in $S(p_3,p_1)$ coincide on S_1).
For every $x \in S_1$, $x \neq p_1$, let A_x be that component of $X \setminus x$ which contains the point p_1.
We define the separation ordering in S_1 as follows:

(i) $p_i < x$ for each $x \in S_1 \setminus p_1$.

(ii) if $x, y \in S_1 \setminus p_1$, then $x < y$ iff $x \in A_y$.

It is well-known, that

$$x < y \iff x \text{ separates } p_1 \text{ and } y \text{ in } X \iff y \text{ separates } x \text{ and } p_2 \text{ in } X$$
$$\iff y \text{ separates } x \text{ and } p_3 \text{ in } X \iff y \notin A_x \cup x \iff A_x \cup x \subset A_y.$$

Now, let

$$L_i = \bigcup_{x \in S_i} A_x \ (i = 1,2,3).$$

L_i $(i = 1,2,3)$ is open in X, because each A_x is open in X.

We shall prove

$$L_i \cap L_j = \emptyset \quad (i \neq j).$$

Suppose, to the contrary, that, for example, there exists a point y, such that $y \in L_1 \cap L_2$.
Then $y \in A_{x_1}$ for some $x_1 \in S_1$ and $y \in A_{x_2}$ for some $x_2 \in S_2$.
Since $A_{p_1} \cap A_{p_2} = \emptyset$, it is impossible that both $x_1 = p_1$ and $x_2 = p_2$.
If $x_1 = p_1$ and $x_2 \neq p_2$ we have the separations

$$X \setminus p_1 = \underset{y}{A_{p_1}} + \underset{\substack{p_2 \\ p_3}}{B_{p_1}} \quad ; \quad X \setminus x_2 = \underset{\substack{p_2 \\ y}}{A_{x_2}} + \underset{\substack{p_3 \\ p_1}}{B_{x_2}}.$$

$\bar{A}_{x_2} = A_{x_2} \cup x_2$ is connected in $X \setminus p_1$, but contains both y and p_2, which is impossible.
In a similar way it can be shown that the remaining case, $x_1 \neq p_1$ and $x_2 \neq p_2$, also yields a contradiction.

We now want to show that at most one of the sets S_1, S_2 and S_3 has a largest element in its ordering.
For this purpose we first recall that it is well-known that

$$x, y \in S_i \text{ and } x < y \text{ imply that } A_x \cup x \subset A_y.$$

Now suppose, to the contrary, that for instance S_1 and S_2 both have a largest element, say x_1 and x_2, respectively.
Then it follows from the fact that $x < y \Longleftrightarrow A_x \cup x \subset A_y$ that

$$L_1 = A_{x_1} \text{ (and consequently } x_1 \notin L_1)$$

and

$$L_2 = A_{x_2} \text{ (and consequently } x_2 \notin L_2).$$

Since $L_1 \cap L_2 = \emptyset$ and since both L_1 and L_2 are open, we clearly have

$\bar{L}_1 \cap L_2 = L_1 \cap \bar{L}_2 = \emptyset$.

If both x_1 and x_2 are cut points, then

$$\bar{L}_1 = \bar{A}_{x_1} = A_{x_1} \cup x_1, \text{ so } x_1 \notin L_2 \text{ and hence } x_1 \in B_{x_2}$$

and also

$$\bar{L}_2 = \bar{A}_{x_2} = A_{x_2} \cup x_2, \text{ so } x_2 \notin L_1 \text{ and hence } x_2 \in B_{x_1}.$$

If x_1 is a cut point and $x_2 = p_2$ then in the same way we can prove: $x_1 \in B_{x_2}$, while $x_2 = p_2 \in B_{x_1}$, because $x_1 \in S_1$.

If $x_1 = p_1$ and $x_2 = p_2$ then

$$x_1 = p_1 \in B_{p_2} \text{ and } x_2 = p_2 \in B_{p_1}.$$

So, in all cases, we have:

$$x_1 \in B_{x_2} \text{ and } x_2 \in B_{x_1}.$$

By (S), there exists a point $y \in X$ such that y separates x_1 and x_2.

So we have:

$$X \setminus y = A^{y}_{x_1} + B^{y}_{x_2}.$$

It follows that

$$\bar{B}_y = B_y \cup y \subset B_{x_1}, \text{ so } p_1 \in A_y \text{ and } y \in B_{x_1}$$

and

$$\bar{A}_y = A_y \cup y \subset B_{x_2}, \text{ so } p_2 \in B_y \text{ and } y \in B_{x_2}.$$

This means that y separates p_1 and p_2 and consequently $y \in S(p_1,p_2)$.

Hence $y \in S_1$ or $y \in S_2$.

If $y \in S_1$ it follows from $y \in B_{x_1}$ that $x_1 < y$.

If $y \in S_2$ it follows from $y \in B_{x_2}$ that $x_2 < y$.

In both cases we have a contradiction.

So we may assume that for instance S_1 and S_2 have no largest element. Then

$$L_i \supset A_{p_i} \cup S_i \quad (i = 1,2).$$

(If $x_i \in S_i$ there exists $y_i \in S_i$ such that $x_i < y_i$; then $A_{x_i} \cup x_i \subset A_{y_i} \subset L_i$, hence $x_i \in L_i$ $(i = 1,2)$.)

From this it follows that certainly $L_1 \neq \emptyset$ and $L_2 \neq \emptyset$.
Since X is connected, L_1 cannot be closed, and hence $\bar{L}_1 \setminus L_1 \neq \emptyset$.(Recall that $\bar{L}_1 \cap L_2 = \bar{L}_2 \cap L_1 = \emptyset$).

Now there are two possibilities:

a) $\bar{L}_1 \setminus L_1 = \{q\}$

b) $q_1, q_2 \in \bar{L}_1 \setminus L_1$ for two distinct points q_1 and q_2.

In both cases we shall derive a contradiction, thus finishing the proof of the theorem:

a) $\bar{L}_1 \setminus L_1 = \{q\}$:

Then $q \notin L_1$ and $q \notin L_2$.
Clearly, $L_1 = \bigcup_{x \in S_1} A_x$ is connected.
Moreover, L_1 is open (in X and hence open) in $X \setminus q$, and also $L_1 = \bar{L}_1 \cap (X \setminus q)$ is closed in $X \setminus q$.
Hence, L_1 is connected and clopen in $X \setminus q$.
Since $L_1 \cap L_2 = \emptyset$, $q \notin L_2$, $L_2 \neq \emptyset$ it follows that q is a cut point of X and that

$$X \setminus q = A_q + B_q, \text{ where } A_q = L_1.$$

Moreover $L_2 \subset B_q$, hence $p_1 \in A_q$ and $p_2 \in B_q$.
But this means that q is a point separating p_1 and p_2, so $q \in S(p_1,p_2) = S_1 \cup S_2 \subset L_1 \cup L_2$, which is a contradiction.

b) $q_1, q_2 \in \bar{L}_1 \setminus L_1$ and $q_1 \neq q_2$:

From (S) it follows that there exists a point $z \in X$, such that z separates

q_1 and q_2.
Since $L_1 \cup \{q_1,q_2\}$ is connected, z has to be a point of L_1.
Thus $z \in A_x$ for some $x \in S_1 \setminus p_1$.
Then we have the separations:

$$X \setminus z = A_z + B_z \quad ; \quad X \setminus x = \begin{matrix} A_x & + & B_x. \\ p_1 & & p_2 \\ z & & p_3 \end{matrix}$$

We may assume: $x \in A_z$.
Then $\bar{B}_z = B_z \cup z$ is connected in $X \setminus x$, and hence

$$\bar{B}_z \subset A_x \subset L_1;$$

since $q_1, q_2 \notin L_1$ it follows next that $q_1, q_2 \in A_z$.
Since (A_z,B_z) is a unique separation of $X \setminus z$, this contradicts the fact that z separates q_1 and q_2.

We are now able to prove the following theorem concerning the relation between properties $(\mathcal{B})$ and $(\mathcal{O})$:

THEOREM 2. *Let* X *be a connected* T_1*-space satisfying condition* $(\mathcal{B})$. *Let* $E = \{x \in X \mid x$ *is an endpoint of* $X\}$. *Let* E *be an endset of* X, *i.e.* $X \setminus E$ *is connected. Then* $X \setminus E$ *is orderable.*

PROOF.

(i) $X \setminus E$ satisfies condition $(\mathcal{B}')$:
Let $p \in X \setminus E$. Since X satisfies $(\mathcal{B})$ and hence $(\mathcal{B}')$ it follows that p is a strong cut point of X, i.e. $X \setminus p = A_p + B_p$, where both A_p and B_p are non-void and connected.
Then

$$(X \setminus E) \setminus p = (A_p \setminus E) + (B_p \setminus E), \text{ where possibly}$$

$$A_p \setminus E = \emptyset \text{ or } B_p \setminus E = \emptyset.$$

We have to prove that both $A_p \setminus E$ and $B_p \setminus E$ are connected.
Suppose, to the contrary, that $A_p \setminus E = R + S$, and hence

$$(X \setminus E) \setminus p = R + S + (B_p \setminus E), \text{ with } R \neq \emptyset \text{ and } S \neq \emptyset.$$

Then $R \cup p$ and $S \cup p$ are connected.
Let $r \in R$ and $s \in S$.
Let $X \setminus r = \underset{p}{A_r} + B_r \quad (B_r \neq \emptyset)$ and

$X \setminus s = \underset{p}{A_s} + B_s \quad (B_s \neq \emptyset)$.

$R \cup p$ is connected in $X \setminus s$, hence $r \in A_s$.
$S \cup p$ is connected in $X \setminus r$, hence $s \in A_r$.
Since $R \cup S = A_p \setminus E$, we have $r,s \in A_p$.
Consequently,

$$B_p \subset A_r,\ B_p \subset A_s \text{ and } B_r \subset A_s,$$

and hence

$$B_p \cap B_r = B_p \cap B_s = B_r \cap B_s = \emptyset.$$

This means that B_p, B_r and B_s are three mutually disjoint segments in X, which contradicts property $(\mathcal{B})$.

(ii) $X \setminus E$ satisfies condition (S):
Let $p_1, p_2 \in X \setminus E$ $(p_1 \neq p_2)$.
Then

$$\begin{cases} X \setminus p_1 = \underset{p_2}{A_1} + B_1 & (A_1 \neq \emptyset,\ B_1 \neq \emptyset) \\ X \setminus p_2 = \underset{p_1}{A_2} + B_2 & (A_2 \neq \emptyset,\ B_2 \neq \emptyset) \end{cases}$$

and hence

$$\begin{cases} (X \setminus E) \setminus p_1 = \underset{p_2}{(A_1 \setminus E)} + (B_1 \setminus E) \\ (X \setminus E) \setminus p_2 = \underset{p_1}{(A_2 \setminus E)} + (B_2 \setminus E) \end{cases}$$

where possibly $B_1 \setminus E = \emptyset$ or $B_2 \setminus E = \emptyset$.
Since $B_1 \setminus E \subset A_2 \setminus E$ it follows that $(A_1 \setminus E) \cup (A_2 \setminus E) = X \setminus E$ and since

$X \setminus E$ is connected we have $(A_1 \setminus E) \cap (A_2 \setminus E) \neq \emptyset$.
Let $p_3 \in (A_1 \setminus E) \cap (A_2 \setminus E)$.
Then $X \setminus p_3 = \underset{p_1}{A_3} + B_3$.
Suppose that also $p_2 \in A_3$. Then, since $p_3 \in A_1 \cap A_2$, it follows that $B_1 \cap B_2 = B_2 \cap B_3 = B_3 \cap B_1 = \emptyset$, which is impossible by condition ($\mathcal{B}$).

Hence $X \setminus p_3 = \underset{p_1}{A_3} + \underset{p_2}{B_3}$, and consequently

$$(X \setminus E) \setminus p_3 = \underset{p_1}{(A_3 \setminus E)} + \underset{p_2}{(B_3 \setminus E)}.$$

This means that p_3 separates p_1 and p_2 in $X \setminus E$.

(iii) The theorem now follows from (i), (ii) and Theorem 1.

COROLLARY 2.1. *Let* X *be a connected* T_1*-space, satisfying condition* ($\mathcal{B}$). *If* X *has exactly one endpoint* p, *then* $X \setminus p$ *is orderable.*

PROOF. Using the notation of Theorem 2, we have $E = \{p\}$, and $X \setminus E = X \setminus p$ is connected.

COROLLARY 2.2. (cf. Theorem 2.4 and Kok [19], Theorem 1). *Let* X *be a connected* T_1*-space satisfying condition* ($\mathcal{B}$) *and having no endpoints. Then* X *is orderable.*

PROOF. $E = \emptyset$ and $X \setminus E = X$ is connected.

Although all spaces under consideration are assumed to be connected T_1-spaces consisting of at least two points it is possible to prove that <u>every</u> tree-like space is a T_1-space. The following proposition even states that every tree-like space is Hausdorff.

PROPOSITION 3. *A tree-like space* X *is Hausdorff.*

PROOF.

(i) X is a T_1-space:
If $p \in X$ is such that $\{p\}$ is not closed, then there exists a point $q \in X \setminus p$ such that $q \in \overline{\{p\}}$. However, then $\{p,q\}$ is a connected subset of X, which means that p and q cannot be separated by a third point.

(ii) X is a T_2-space:
Let p, q $\in$ X such that p $\neq$ q. Then there exists a point r $\in$ X separating p and q. So we have

$$X \setminus r = A_r^{\;p} + B_r^{\;q},$$

where A_r and B_r are open in X, since X is a T_1-space.

THEOREM 4. *In a connected* T_1*-space* X:

$$(S) \Longrightarrow (B'O).$$

PROOF. Suppose C is a component of X \ p, which is not open. Then there exists a point r $\in$ C such that $r \in \overline{X \setminus C}$. Let q be a point separating p and r. Then we have

$$X \setminus q = A_q^{\;p} + B_q^{\;r}.$$

$\bar{B}_q = B_q \cup q$ is connected in X \ p, hence $\bar{B}_q \subset C$ and $X \setminus C \subset A_q$.
Then $r \in \overline{X \setminus C} \subset \bar{A}_q = A_q \cup q$. Contradiction.

THEOREM 5. *In a connected* T_1*-space* X:

$$(S) \Longrightarrow (W).$$

PROOF. Let A and B be disjoint connected subsets of X.
Let p, q $\in \bar{A} \cap \bar{B}$ (p $\neq$ q).
Since both $A_1 = A \cup \{p,q\}$ and $B_1 = B \cup \{p,q\}$ are connected, p and q clearly cannot be separated by a third point.

THEOREM 6. *In a connected* T_1*-space* X:

$$(Ht) + (S) \Longrightarrow (O).$$

PROOF. Since $(S) \Longrightarrow (B'O) \Longrightarrow (B'C)$ and $(Ht) + (B'C) \Longrightarrow (B')$ (Lemma 3.6), the assertion follows from Theorem 1.

PROPOSITION 7. *Let* X *be a connected* T_1*-space satisfying* (*S*).
Let $a, b \in X$ $(a \neq b)$.
Let $S(a,b) = E(a,b) \cup \{a,b\} = \{x \in X \mid x$ *separates* a *and* $b\} \cup \{a,b\}$.
Then $S(a,b)$ *is closed in* X.

PROOF. Suppose that $S(a,b)$ is not closed then there exists a point $p \in X$ such that $p \in \overline{S(a,b)} \setminus S(a,b)$.
For $x \in E(a,b)$ let $X \setminus x = \underset{a}{A_x} + \underset{b}{B_x}$ be a fixed separation between a and b.
Remember that in the separation ordering $<$ we have that a (resp. b) is the smallest (resp. greatest) element, while for all $x, y \in E(a,b)$ we have

$$x < y \iff x \in A_y \iff y \in B_x \iff \bar{A}_x = A_x \cup x \subset A_y \iff$$

$$\iff \bar{B}_y = B_y \cup y \subset B_x.$$

Let $A = \{x \in E(a,b) \mid p \in B_x\}$ and $B = \{x \in E(a,b) \mid p \in A_x\}$.
Now $A \cup B = E(a,b)$, $A \cap B = \emptyset$.
Hence $p \in \bar{A}$ or $p \in \bar{B}$. Suppose for instance that $p \in \bar{A}$. Then A cannot have a last element. (If z would be the last element of A, then $A \subset A_z \cup z = \bar{A}_z$ and hence, since $p \in \bar{A}$, $p \in \bar{A}_z$. Then $p \in A_z$, since certainly $p \neq z$. But this means that $z \in B$, which is a contradiction.)
Let $R = \bigcup_{x \in A} A_x$. Then also $R = \bigcup_{x \in A} \bar{A}_x$. So R is an open and connected subset of X. Moreover, since $A \subset R$ and $p \notin R$, $p \in \partial R = \bar{R} \setminus R$.
$a \in R$ and $b \in X \setminus R$ so either $b \in \partial R$ or ∂R separates a and b.
Since $p \neq b$ and since p does not separate a and b, ∂R must contain a point q different from p. (In the first case one may always take $q = b$).
As $R \cup \{p,q\} \subset \bar{R}$ is connected a point r separating p and q must belong to R. However, if $r \in R$, then $r \in A_x$ for some $x \in A$; thus $B_x \cup x$ is connected in $X \setminus r$ and contains p and q. This contradicts (*S*).

PROPOSITION 8. *Let* X *be a connected* T_1*-space satisfying* (*S*).
Let $a, b \in X$ $(a \neq b)$. *Then the closed set* $S(a,b) = E(a,b) \cup \{a,b\}$ *has no jumps and no gaps in the usual separation ordering.*

PROOF.

(i) We first show that $S(a,b)$ has no jumps:

Let $x, y \in E(a,b)$, $x < y$.
Let z be a point in X separating x and y.
Then we have the following separations:

$$X \setminus x = \underset{a}{A_x} + \underset{\substack{b\\ \\y}}{B_x}\ ;\ X \setminus y = \underset{\substack{a\\ \\x}}{A_y} + \underset{b}{B_y}\ ;\ X \setminus z = \underset{x}{A_z} + \underset{y}{B_z}.$$

$\bar{A}_z = A_z \cup z$ is connected in $X \setminus y$, so $b \in B_z$.
$\bar{B}_z = B_z \cup z$ is connected in $X \setminus x$, so $a \in A_z$.

Hence z separates a and b, and $x < z < y$.

If $x = a$ or $y = b$, the assertion is proved in a similar way.

(ii) Secondly we show that $S(a,b)$ has no gaps:
Suppose, to the contrary, that there exist non-empty subsets A and B of $S(a,b)$ such that:
$S(a,b) = A \cup B$; $x \in A$ and $y \in B$ implies $x < y$; A has no last element and B has no first.
Let $P = \bigcup_{x\in A} A_x = \bigcup_{x\in A} \bar{A}_x$ and $Q = \bigcup_{y\in B} B_y = \bigcup_{y\in B} \bar{B}_y$, where A_x and B_y have the usual meaning.
Then P and Q are disjoint, non-empty, open, connected subsets of X; $A \subset P$, $B \subset Q$.

a) Suppose $\bar{P} \setminus P$ contains two distinct points p_1 and p_2.
Any point q, separating p_1 and p_2, must be contained in P, since $P \cup \{p_1,p_2\}$ is connected. Hence $q \in A_x$ for some $x \in A$. However, $\bar{B}_x$ is connected in $X \setminus q$ and contains p_1 and p_2. Contradiction.

b) Suppose $\bar{P} \setminus P = \{p\}$ for some $p \in X$.
Then P is a clopen subset of $X \setminus p$. Since $A \subset P$ and $B \subset Q$ this means that p separates a and b. However, $p \notin A \cup B = S(a,b)$. Contradiction.

THEOREM 9. *In a connected* T_1*-space* X *satisfying* (S) *the intersection of a segment and a connected set is connected.*

PROOF. Suppose C is a component of $X \setminus p$ and D is a connected subset of X.
By Theorem 4, C is a clopen subset of $X \setminus p$, so $X \setminus p = C + Q$.
Now, suppose $C \cap D = S + T$, with $S \neq \emptyset$ and $T \neq \emptyset$.
Then $D \setminus p = (C \cap D) + (Q \cap D) = S + T + (Q \cap D)$, (so that p is a cut point of D), hence $S \cup p$, $T \cup p$ and $S \cup T \cup p$ are connected.
Let $a \in S$ and $b \in T$, then $S(a,b) = E(a,b) \cup \{a,b\}$ is contained in $C \cap D$,

thus $S(a,b) = A + B$, where $a \in A$, $b \in B$, $A = S(a,b) \cap S$ and $B = S(a,b) \cap T$. If $x \in A$ and $y \in B$ it follows from the fact that $T \cup p$ is connected that x does not separate y and b, and hence $x < y$. (where $<$ again denotes the separation ordering). So, by Proposition 8, either A has no last element or B has no first element. Assume for instance that A has no last element. Then $V = \bigcup_{x \in A} A_x = \bigcup_{x \in A} \bar{A}_x$ is open and connected and $A \subset V$; also $V \cap T = \emptyset$ (for, $X \setminus x = A_x^a + B_x^b$ and if $x \in A$, then $T \cup p$ is connected in $X \setminus x$ and contains b).

By (S), $\partial V = \bar{V} \setminus V$ contains at most one point; on the other hand, since X is connected, ∂V cannot be empty; hence $\partial V = \{q\}$.

We observe first, that $q \in T$. For, if $q = b$, then $q \in T$.
If $q \neq b$, then, since V is clopen in $X \setminus q$, q separates a and b, and therefore $q \in B$, hence $q \in T$.

Now, let r be any point separating p and q. Since $T \cup p$ is connected and $q \in T$ we have $r \in T$. Since $V \cap T = \emptyset$, $\bar{V} \cup S \cup p$ is a connected set not containing r, but containing p and q. This is a contradiction.

PROPOSITION 10. *If a connected space* X *(with more than one point) satisfies property* (S), *then the space* X *is uncountable.*

PROOF. Let $a, b \in X$ $(a \neq b)$. By Proposition 8, the set $S(a,b)$ is continuously ordered, i.e. it has no jumps and no gaps in its (separation) ordering. Hence there is a subset of $S(a,b)$ with the ordertype of the real numbers. (cf. e.g. A.A. Fraenkel [9], p. 174).

4.3. ON CONDITION (W)

PROPOSITION 11. *Let* X *be a connected* T_1*-space. Then* X *is a* (W)*-space if and only if the boundary of each component of the complement of any non-empty connected proper subset of* X *consists of exactly one point.*

PROOF.

(i) Let S be a non-empty connected proper subset of X. Let C be a component of $X \setminus S$. Then, by Lemma 1.9, $X \setminus C$ is connected. Since X is a (W)-space $\overline{X \setminus C} \cap \bar{C}$ contains at most one point, and hence, by the connectedness of X, $\overline{X \setminus C} \cap \bar{C}$ contains exactly <u>one</u> point. But $\overline{X \setminus C} \cap \bar{C}$ is precisely the boundary of C.

(ii) Let A, B be connected disjoint subsets of X. Let C be a component of $X \setminus B$ such that $A \subset C$. Since $\bar{C} \cap \overline{X \setminus C} = \{p\}$ for some $p \in X$, it follows that $\bar{C} \cap \bar{B} \subset \{p\}$, and consequently $\bar{A} \cap \bar{B} \subset \{p\}$.

REMARK. Proposition 11 shows that condition (W) is equivalent to a condition studied by A.E. Brouwer [2], as we observed already before.

THEOREM 12. *In a connected* T_1*-space* X:

$$(W) + (B'C) \Longrightarrow (B'O).$$

PROOF. Let C be a segment of p in X. By condition $(B'C)$ and Proposition 11: $(C \cup p) \setminus C^{\circ} = \bar{C} \setminus C^{\circ} = \{q\}$ for some $q \in X$.
Hence, ($p = q$ and) $C = C^{\circ}$, i.e. C is open in X.

THEOREM 13. *In a connected* T_1*-space* X:

$$(Ht) + (W) \Longrightarrow (H).$$

PROOF. Let $C \subset X$ be connected and let p, q and r be three distinct endpoints of C.

1. Suppose first that $C \setminus \{p,q\}$ is not connected, hence $C \setminus \{p,q\} = S + T$.
Here, $S \cup p$ and $T \cup q$ are disjoint and connected.
However, $\overline{S \cup p} \cap \overline{T \cup q} \supset \{p,q\}$, which contradicts (W).

2. Thus we may assume that $C \setminus \{p,q\}$, $C \setminus \{q,r\}$ and $C \setminus \{r,p\}$ are connected.
By (Ht) we have that $C \setminus \{p,q,r\}$ is not connected. Hence $C \setminus \{p,q,r\} =$
$= U + V$.
Now, $U \cup p$ and $V \cup q$ are disjoint and connected.
However, $\overline{U \cup p} \cap \overline{V \cup q} \supset \{p,q,r\}$, which again contradicts (W).

4.4. CONNECTED INTERSECTION PROPERTIES

THEOREM 14. *In any topological space* X:

a) $(Int) \Longrightarrow (Int\,2)$

b) $(Int) \Longrightarrow (Int^{*}) \Longrightarrow (Int')$.

PROOF. Immediate from the definitions.

THEOREM 15. *In a connected* T_1*-space* X:

a) $(Int\,2) \Longrightarrow (W)$

b) $(Int^{*}) \Longrightarrow (W)$.

PROOF. Let A and B be disjoint connected subsets of X.
Let $p, q \in \bar{A} \cap \bar{B}$ $(p \neq q)$.
Then $A_1 = A \cup \{p,q\}$ and $B_1 = B \cup \{p,q\}$ are connected.
However, $A_1 \cap B_1 = \{p,q\}$ is closed and not connected, which contradicts both $(Int\,2)$ and (Int^{*}).

LEMMA 16. *In a connected* T_1*-space* X *satisfying* $(B'0)$ the following holds:

$$\forall a,b \in X \ (a \neq b) \quad : \quad C(a,b) = S(a,b).$$

PROOF. Recall that $C(a,b)$ denotes the intersection of all connected subsets of X, containing both a and b.

(i) If $p \notin C(a,b)$ then there exists a connected subset A of X such that $a,b \in A$ and $p \notin A$. Then clearly p does not separate a and b in X. Hence $p \notin S(a,b)$.

(ii) If $p \in C(a,b)$ and $p \notin \{a,b\}$ then a and b certainly do not belong to the same component of $X \setminus p$. Since, by $(B'0)$, components of $X \setminus p$ are clopen in $X \setminus p$, this means that p separates a and b.
Hence $p \in E(a,b)$.

REMARK. If X is a connected T_1-space and if $a,b \in X$ $(a \neq b)$ then we will use the following notation:

$$K(a,b) = \cap \{S \subset X \mid a,b \in S \ ; \ S \text{ connected and closed}\};$$

$$L(a,b) = K(a,b) \setminus \{a,b\}.$$

LEMMA 17. *Let* X *be a connected* T_1*-space satisfying* (Int'). *Then the following holds:* $\forall a,b \in X \ (a \neq b)$:

(i) $K(a,b)$ *is connected.*

(ii) $L(a,b)$ *is connected.*

(iii) $\overline{L(a,b)} = K(a,b)$.

PROOF.

(i) Immediate from the definition of (Int').

(ii) First, suppose that a is a cut point of $K(a,b)$; i.e.

$$K(a,b) \setminus a = \underset{b}{P} + Q, \text{ where } Q \neq \emptyset.$$

Then $\bar{P} = P \cup a$ is a closed and connected subset of X, which contains a and b and which moreover is a <u>proper</u> subset of $K(a,b)$. This is impossible.
Consequently, a and b are both endpoints of $K(a,b)$.
Suppose $K(a,b) \setminus \{a,b\} = U + V$, where $U \neq \emptyset$ and $V \neq \emptyset$.
Then $\bar{U} = U \cup \{a,b\}$ and $\bar{V} = V \cup \{a,b\}$.
Moreover, both $\bar{U}$ and $\bar{V}$ are connected.
However, $\bar{U} \cap \bar{V} = \{a,b\}$ is not connected. Contradiction.

(iii) Since X is T_1, this assertion follows immediately from the fact that $K(a,b) = L(a,b) \cup \{a,b\}$ is closed and connected.

THEOREM 18. *In a connected* T_1*-space* X:

$$(Int') \Longrightarrow (B'C).$$

PROOF. Let C be a component of $X \setminus p$ and let $r \in C$.
Then $L(r,p) \cup r$ is a connected subset of $X \setminus p$. Thus $L(r,p) \cup r \subset C$. Hence, $p \in \overline{L(r,p)} \subset \bar{C}$.

THEOREM 19. *In a connected* T_1*-space* X:

$$(Int^*) \Longrightarrow (B'O).$$

PROOF. Let C be a component of $X \setminus p$ and suppose that C is not open in X.
Since $(Int^*) \Longrightarrow (W)$ it follows from Prosition 11 that C is closed in X.
Since $(Int^*) \Longrightarrow (Int')$ this contradicts Theorem 18.

THEOREM 20. *In a connected* T_1*-space* X:

$$(Int^*) \Longrightarrow (S).$$

PROOF. Let a,b $\in$ X (a $\neq$ b).
Since a,b $\in$ C(a,b) and since $\overline{C(a,b)}$ is connected, it follows that C(a,b) has infinitely many points. From $(Int^*) \Longrightarrow (B'O)$, and from Lemma 16 we conclude that S(a,b) contains infinitely many points. Hence E(a,b) $\neq \emptyset$, which means that there exists a point c $\in$ X separating a and b.

THEOREM 21. *In a connected* T_1*-space* X:

$$(Ht) + (Int') \Longrightarrow (O).$$

PROOF. Since $(Int') \Longrightarrow (B'C)$ (Theorem 18), $(B'C) + (Ht) \Longrightarrow (Hp)$ (Proposition 3.21) and $(H) + (B'C) \Longrightarrow (O)$ (Theorem 3.9) it suffices to show that $(Hp) + (Int') \Longrightarrow (H)$. Suppose, to the contrary, that X does not satisfy property (H). Then, by Theorem 3.18, X is a non-orderable cyclically orderable space. Hence, by Theorem 3.12, every point of X is an endpoint, and X \ {x,y} is disconnected for all x,y $\in$ X (x $\neq$ y).
Let p,q $\in$ X. Then X \ {p,q} = A + B, where A $\neq \emptyset$ and B $\neq \emptyset$. $\bar{A}$ = A $\cup$ {p,q} is connected and $\bar{B}$ = B $\cup$ {p,q} is connected. However, $\bar{A} \cap \bar{B}$ = {p,q}, which contradicts (Int').

REMARK. Since a cyclically orderable space satisfies property (Ht) it follows from Theorem 13 and Theorem 21 that a non-orderable cyclically orderable connected T_1-space does not satisfy condition (W) or condition (Int').

THEOREM 22. *In a connected* T_1*-space* X:

$$(Int') + (W) \Longrightarrow (B'O).$$

PROOF. $(Int') \Longrightarrow (B'C)$ (Theorem 18) and $(W) + (B'C) \Longrightarrow (B'O)$ (Theorem 12).

PROPOSITION 23. *A connected* T_1*-space* X *satisfies* (Int) *if and only if for every* a,b $\in$ X (a $\neq$ b): S(a,b) *is connected.*

PROOF.

(i) Let X satisfy (*Int*). Then C(a,b) is connected. Applying Theorem 14, Lemma 16 and Theorem 19 we conclude that S(a,b) is connected.

(ii) Let S(a,b) be connected for every $a,b \in X$ $(a \neq b)$.
Let $\{C_\alpha\}_{\alpha\in A}$ be a collection of connected subsets of X. Suppose that $\bigcap_{\alpha\in A} C_\alpha$ is not connected. Then we have $\bigcap_{\alpha\in A} C_\alpha = A + B$, where $A \neq \emptyset$ and $B \neq \emptyset$. Let $a \in A$ and $b \in B$. A point p separating a and b is contained in every connected subset of X containing both a and b. Hence :
$S(a,b) \subset \bigcap_{\alpha\in A} C_\alpha$.
Consequently, $S(a,b) = (S(a,b) \cap A)_a + (S(a,b) \cap B)_b$, which contradicts the connectedness of S(a,b).

LEMMA 24. *Let* X *be a connected* T_1*-space, satisfying the conditions* (*Int'*) *and* (*W*). *Let* a *be an endpoint of* X. *Let* C *be a closed connected subset of* X, *such that* $a \in C$. *Then* a *is also an endpoint of* C.

PROOF. Suppose, to the contrary, that $C \setminus a = P + Q$, where $P \neq \emptyset$ and $Q \neq \emptyset$. Then $\bar{P} = P \cup a$ and $\bar{Q} = Q \cup a$ are closed connected subsets of X.
Let $b \in P$. Since $K(a,b) \subset \bar{P}$, we have $L(a,b) \subset P$. Let S be that component of $X \setminus L(a,b)$, which contains $\bar{Q} = Q \cup a$.
S is closed in X. (Otherwise, there exists some $c \in L(a,b)$ such that $c \in \bar{S}$; since $\bar{S} \cap \overline{L(a,b)} \supset \{a,c\}$, this contradicts (*W*).)
Hence $X \setminus S$ is non-empty, open and connected, and by (*W*), $\partial(X \setminus S) = \overline{X \setminus S} \cap S$ consists of precisely one point. Then clearly $\partial(X \setminus S) = \{a\}$. Consequently, $X \setminus S$ is clopen in $X \setminus a$. Since $S \setminus a \supset Q \neq \emptyset$, this implies that a is a cut point of X. Contradiction.

THEOREM 25. *In a connected* T_1*-space* X:

$$(Int') + (S) \Longleftrightarrow (Int).$$

PROOF.

(i) $\Longleftarrow$: $(Int) \Longrightarrow (Int^*) \Longrightarrow (Int')$ (Theorem 14b) and

$$(Int^*) \Longrightarrow (S) \qquad \text{(Theorem 20)}.$$

(ii) $\Longrightarrow$: Let $\{C_\alpha\}_{\alpha\in A}$ be a collection of connected subsets of X.

Suppose $\bigcap_{\alpha\in A} C_\alpha = P + Q$, where $P \neq \emptyset$ and $Q \neq \emptyset$.
Let $p \in P$ and $q \in Q$.
By (*Int'*), $C = \bigcap_{\alpha\in A} \bar{C}_\alpha$ is a connected closed subset of X.
By the definition of $K(p,q)$: $K(p,q) \subset C$.
However, by the connectedness of $K(p,q)$, we have:

$$K(p,q) \not\subset P \cup Q.$$

We now consider the following three cases:

a) $K(p,q) \setminus (P \cup Q) = \{r\}$ for some $r \in X$.

b) $K(p,q) \setminus (P \cup Q) = \{s,t\}$ for two distinct points $s,t \in X$.

c) $K(p,q) \setminus (P \cup Q)$ contains at least three distinct points of X.

In all three cases we shall derive a contradiction:

a) Suppose that $K(p,q) \setminus (P \cup Q) = \{r\}$ for some $r \in X$:
Then there exists an element $\alpha_0 \in A$ such that $r \notin C_{\alpha_0}$.
Let $S = \bar{C}_{\alpha_0} \setminus r$, then S is connected and $\bar{S} \setminus S = \{r\}$.
Moreover, $K(p,q) \subset \bar{S} = \bar{C}_{\alpha_0}$ and

$$K(p,q) \setminus r = \underset{p}{(K(p,q) \cap P)} + \underset{q}{(K(p,q) \cap Q)}.$$

Since it is clear that (*Int'*) is a hereditary property for closed connected subspaces and (*S*) is a hereditary property for connected subspaces, we now have the following situation:
$\bar{S}$ is a connected T_1-space, satisfying (*Int'*) and (*S*). The point r is an endpoint of $\bar{S}$. $K(p,q)$ is a closed connected subset of $\bar{S}$ and r is a cut point of $K(p,q)$. This contradicts Lemma 24.

b) Suppose that $K(p,q) \setminus (P \cup Q) = \{s,t\}$ for two distinct points $s,t \in X$:
Suppose that for instance s is a cut point of $K(p,q)$.
Let $\alpha_0 \in A$ be such that $s \notin C_{\alpha_0}$. Let $S = \bar{C}_{\alpha_0} \setminus s$. Then S is connected and $\bar{S} \setminus S = \{s\}$. Moreover, $K(p,q) \subset \bar{S}$ and $K(p,q) \setminus s$ is not connected. This contradicts Lemma 24.
Thus we may assume that s and t are endpoints of $K(p,q)$.
If we put $P_1 = K(p,q) \cap P$ and $Q_1 = K(p,q) \cap Q$ then we have:

$$K(p,q) \setminus \{s,t\} = \underset{p}{P_1} + \underset{q}{Q_1}.$$

$P_1 \cup s$ and $Q_1 \cup t$ are connected disjoint subsets of X.
However, $\overline{P_1 \cup s} \cap \overline{Q_1 \cup t} = \{s,t\}$, which contradicts condition (W).

c) Suppose that $K(p,q) \setminus (P \cup Q)$ contains at least three distinct points u, v, w of X:
If at least one of these three points is a cut point of $K(p,q)$, then we may derive a contradiction to Lemma 24 in a similar way to that in case a) and b). Thus we may assume that u, v and w are endpoints of $K(p,q)$.
By (S), there exists a point $s_1 \in X$ separating u and v; and $s_1 \neq p,q$ (see Lemma 17).
Therefore, we have $X \setminus s_1 = \underset{u,\,w}{A_1} + \underset{v}{B_1}$, where we may assume $w \in A_1$.

The point s_1 also separates p and q, since otherwise it easily follows that either $K(p,q) \subset A_1 \cup s_1$ or $K(p,q) \subset B_1 \cup s_1$, contradicting the fact that both points u and v belong to $K(p,q)$. We may assume $p \in A_1$ and $q \in B_1$. Since clearly $K(u,w) \subset A_1 \cup s_1$, it follows that $v \notin K(u,w)$.

Now we shall show that also $w \notin K(u,v)$ and $u \notin K(v,w)$:
Suppose, to the contrary, that $w \in K(u,v)$.
Then $K(u,w) \subset K(u,v)$ and $K(v,w) \subset K(u,v)$.
Let $s_2 \in X$ be a point separating u and w. Then:

$$X \setminus s_2 = \underset{u}{A_2} + \underset{w}{B_2},$$

and it follows that $v \in B_2$, since otherwise $K(u,v) \subset A_2 \cup s_2$, $w \notin K(u,v)$.
Hence $u \notin K(v,w) \subset B_2 \cup s_2$.

Now, suppose that there exists a point $r \in K(u,w) \cap K(v,w)$ such that $r \neq w$.
Then $K(u,r) \cup K(r,v) \supset K(u,v)$.
Let $s_3 \in X$ be a point separating r and w. Then:

$$X \setminus s_3 = \underset{r}{A_3} + \underset{w}{B_3}.$$

Since $r \in K(u,w) \cap K(v,w)$ we then have $u \notin B_3$ and $v \notin B_3$ and consequently $w \notin K(u,r) \cup K(v,r)$, which contradicts the assumption that $w \in K(u,v)$.

Hence $K(u,w) \cap K(v,w) = \{w\}$.
This means that w is a cut point of the closed connected subset $K(u,w) \cup K(v,w)$ of $K(p,q)$.
Since $K(p,q)$ is a closed connected subset of X and w is an endpoint of $K(p,q)$, this leads to a contradiction to Lemma 24.

Thus we have shown that $w \notin K(u,v)$. In the same way it can be proved that $u \notin K(v,w)$.

Let $s_4 \in X$ be a point separating u and w and such that $s_4 \notin \{p,q,v,s_1\}$. Such a point exists, since, by (S), the set $E(u,w)$ is infinite.
So we have:

$$X \setminus s_4 = \underset{u}{A_4} + \underset{w}{B_4}.$$

The point s_4 also separates p and q (as is seen by a reasoning analoguous to the one given above for s_1).
Suppose first that $p \in A_4$ and $q \in B_4$.
Since $K(u,p) \subset \bar{A}_4 = A_4 \cup s_4$ and $w \notin \bar{A}_4$ we have $w \notin K(u,p)$.
Since $K(v,q) \subset \bar{B}_1 = B_1 \cup s_1$ and $w \notin \bar{B}_1$ we have $w \notin K(v,q)$.

But then, by $w \notin K(u,v)$, $K(u,p) \cup K(v,q) \cup K(u,v)$ is a closed connected subset of X, containing p and q but not containing w, which is a contradiction to $w \in K(p,q)$.

Next, when we suppose $q \in A_4$ and $p \in B_4$ we can derive a contradiction to $u \in K(p,q)$ in a similar way (using $u \notin K(v,w)$.).

This completes the proof of the theorem.

THEOREM 26. *In a connected* T_1*-space* X:

$$(Int) \Longleftrightarrow (Int^*).$$

PROOF.

(i) $\Longrightarrow$: Theorem 14b.

(ii) $\Longleftarrow$: $(Int^*) \Longrightarrow (Int')$ (Theorem 14b)
$(Int^*) \Longrightarrow (S)$ (Theorem 20)
$(Int') + (S) \Longrightarrow (Int)$ (Theorem 25).

4.5. SOME REMARKS

1. Some conditions studied in the previous four chapters are hereditary for connected subspaces, some others are not.
In fact:

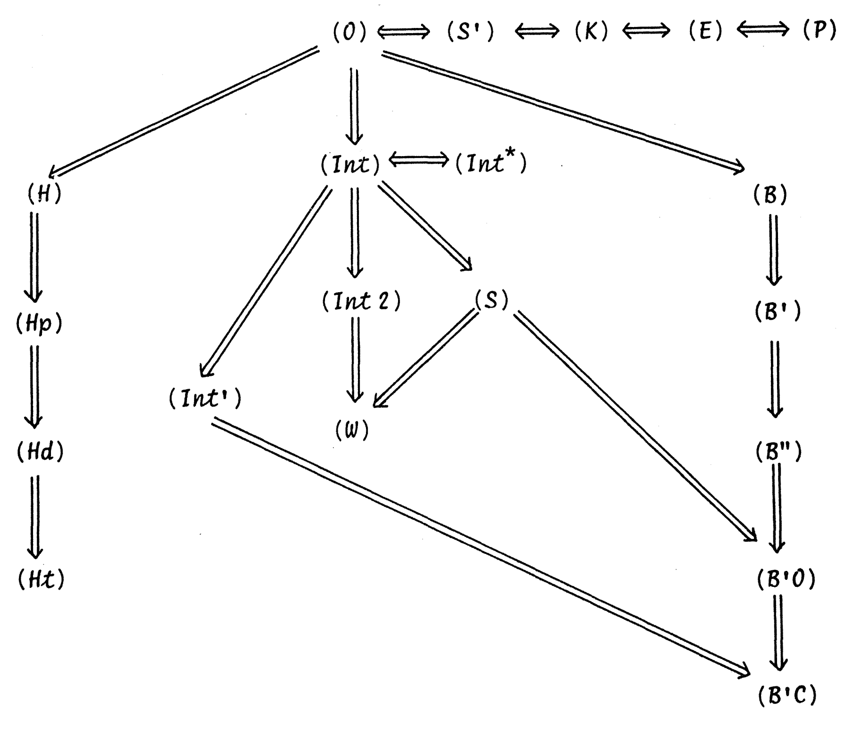
(i) The following properties are hereditary for connected subspaces: (O), (S'), (K), (E), (P), (H), (Hp), (Hd), (Ht), (Int), (Int^*), $(Int\,2)$, (S) and (W).

(ii) The following properties are <u>not</u> hereditary for connected subspaces: (B), (B'), (B''), $(B'O)$, $(B'C)$ and (Int').

(iii) (Int') is hereditary for *closed* connected subspaces.

2. Although it is not explicitly stated in all relevant places, it is clear that a connected orderable space satisfies all conditions (except (V)) occurring in this thesis (while the exceptional condition (V) is <u>never</u> satisfied in a connected orderable space with more than one point).

3. For convenience we list in the following scheme the implications of the type "$\alpha \Longrightarrow \beta$", where both α and β stand for precisely <u>one</u> condition treated in the foregoing four chapters:



CHAPTER V

THE LOCALLY CONNECTED CASE

5.1. INTRODUCTION

In the introductary sections of the previous chapters we have already mentioned that some of the notions we are studying in this thesis also have been investigated by several other authors, mostly, however, under additional assumptions, like local connectedness or peripheral compactness.
In this chapter we suppose that the space X under consideration is not only connected and T_1, but also locally connected. Hence it is no surprise, that many of the following results are well-known. The purpose of this chapter is to derive these results from the more general theorems obtained in the previous chapters.

5.2. THE LOCALLY CONNECTED CASE

In this section all spaces are assumed to be locally connected, connected T_1-spaces, which consist of at least two points.
The most important tools in proving the theorems of this section are firstly Theorem 1.5, which states that a connected orderable space X is strictly orderable if and only if X is locally connected, and secondly the implication: local connectedness $\Longrightarrow$ $(B'O)$, which is obvious, since in a locally connected space components of open subsets are open and since all spaces are assumed to be T_1.

THEOREM 1. *In a connected, locally connected* T_1*-space* X *the following six assertions are equivalent:*

(i) X *is strictly orderable.*
(ii) X *satisfies* (S').
(iii) X *is a* (K)*-space.*
(iv) X *is an* (H)*-space.*
(v) X *is an* (E)*-space.*
(vi) X *is a* (P)*-space.*

PROOF. Immediate from Theorem 1.5, and Theorem 2.3, Theorem 3.9 and Theorem 3.22.

REMARK. The following parts of Theorem 1 are well-known:

a) (i) $\Longleftrightarrow$ (iii), (see H.-J. Kowalsky [21] for the separable case and Kowalsky [22], Satz 15.5 for the general case).

b) (i) $\Longleftrightarrow$ (iv) , (see H. Herrlich [11], p. 42 and Herrlich [12], Satz 1).

c) (i) $\Longleftrightarrow$ (v) , (see S. Eilenberg [8] and the Introduction of the paper by B. Banaschewski [1]).

(The equivalence of (O) and (S') in connected T_1-spaces was observed in a paper by R. Duda [7]).

THEOREM 2. *In a connected, locally connected* T_1*-space* X *having at least one cut point*:

$$X \text{ satisfies } (Ht) \Longleftrightarrow X \text{ is strictly orderable.}$$

PROOF. Theorem 1.5 and Theorem 3.10.

THEOREM 3. *In a connected, locally connected* T_1*-space* X *the following five assertions are equivalent*:

(i) X *is non-orderable, and strictly cyclically orderable.*

(ii) X *is an* (Hp)*-space, but not an* (H)*-space.*

(iii) $\forall x \in X : X \setminus x$ *is connected and* $\forall x,y \in X\ (x \neq y) : X \setminus \{x,y\}$ *is disconnected.*

(iv) X *is an* (Ht)*-space, such that* : $\forall x \in X : X \setminus x$ *is connected.*

(v) *The complement of every connected subset of* X *is connected.*

PROOF. Theorem 3.17, Theorem 3.12, Theorem 3.18, Theorem 3.19 and Theorem 3.20.

THEOREM 4. *In a connected, locally connected* T_1*-space* X:

$$(Int) \Longleftrightarrow (S).$$

PROOF. See Whyburn [28], Theorem 9.3.

THEOREM 5. *In a connected, locally connected* T_2*-space* X:

$$(Int) \Longleftrightarrow (S) \Longleftrightarrow (Int') \Longleftrightarrow (Int\,2) \Longleftrightarrow (W).$$

PROOF.

(i) $(Int\,2) \Longleftrightarrow (S)$: see Whyburn [28], Theorem 9.1.

(ii) $(W) \Longrightarrow (S)$: (see also Brouwer [2]):
Let p and q be two distinct points in X.
Let U and V be two disjoint open connected neighbourhoods of p, resp. q.
Let A be the component of $X \setminus \bar{U}$ that contains q (and hence V). Then A is open in X. Since $X \setminus A$ is connected there exists, by property (W), exactly one point $r \in \bar{A} \setminus A$. Hence A is clopen in $X \setminus r$, which means that r separates p and q.

(iii) $(Int') \Longrightarrow (Int)$:
Let p and q be two distinct points in X.
Recall that K(p,q) denotes the intersection of all closed connected subsets of X containing p and q, while C(p,q) denotes the intersection of all connected subsets of X containing p and q. Then K(p,q) is closed and connected. Moreover, $C(p,q) \subset K(p,q)$. We have to prove the connectedness of C(p,q). In fact we will show that $C(p,q) = K(p,q)$.
Suppose, to the contrary, that there exists a point $r \in K(p,q)$ such that $r \notin C(p,q)$. Then there exists a connected subset $S \subset X$ such that $p,q \in S$, but $r \notin S$.
For every $x \in S$ let U_x be an open connected neighbourhood of x such that $r \notin \bar{U}_x$.
Then $\{U_x\}_{x \in S}$ is an open covering of the connected set S, hence there exists a simple chain $U_{x_1}, \ldots, U_{x_n}$ from p to q.
The union of the members of that chain is connected, contains p and q, but its closure does not contain r. Hence $r \notin K(p,q)$.

REMARK. It is not possible to replace "T_2" by "T_1" in the previous theorem. In fact, in connected, locally connected T_1-spaces none of the following implications is true:

$(Int') \Longrightarrow (Int\,2)$	(see example 28)
$(Int\,2) \Longrightarrow (Int')$	(see example 27)
$(Int') \Longrightarrow (W)$	(see example 28)
$(W) \Longrightarrow (Int')$	(see example 27).

However, we were not able to solve the following problems:

(i) Is it true that in a connected, locally connected T_1-space X property (W) implies $(Int\,2)$?

(ii) Is it true that in a connected, locally connected T_1-space X properties (W) and (Int') together imply $(Int\,2)$?

Even if we drop the condition that X be locally connected we could not find an example of a connected T_1-space X which satisfies (W) and (Int'), but which does not satisfy $(Int\,2)$. We conjecture that the answer to the last problem is negative.

THEOREM 6 (cf. V.B. Buch [5], Theorem 1). *Let* X *be a connected, locally connected* T_1*-space, satisfying condition* (B) *and having no endpoints. Then* X *is strictly orderable.*

PROOF. Theorem 1.5 and Corollary 4.2.2.

5.3. JONES' CONDITION OF LINEARITY

In 1939 F.B. Jones introduced in [17] the concept of linearity for Hausdorff spaces. We recall his definition:
A topological space X is called *linear* if every point of X has a local base of open sets, each of which has at most two boundary points. In this section we will show that a linear, connected T_2-space is strictly orderable or strictly cyclically orderable. This generalizes Theorem 11 of [17], which asserts that a nondegenerate connected linear Moore space is a simple continuous curve.
For the proof of our theorem we need the following results from Jones' paper:

(a) A linear, connected T_2-space is locally connected (cf. [17], Theorem 4).

(b) If C is an open connected subset of a connected, linear T_2-space, then C has at most two boundary points (cf. [17], Theorem 5).

PROPOSITION 7. *In a connected* T_2*-space* X:

$$\textit{linear} \Longleftrightarrow (Ht) + \textit{local connectedness}.$$

PROOF.

(i) $\Longrightarrow$: Suppose, that C is a connected subset of X with endpoint triple $\{p,q,r\}$. Let S be the component of $X \setminus \{p,q,r\}$ containing the connected set $C \setminus \{p,q,r\}$. By (a), S is open in X. However, since $p,q,r \in \bar{C}$, we have $p,q,r \in \bar{S} \setminus S$, which contradicts (b).

(ii) $\Longleftarrow$: Let $p \in X$ and let U be an open subset of X containing p. Since X is locally connected there exists an open connected subset of X such that $p \in S \subset U$. By (Ht), S can have at most two boundary points.

THEOREM 8. *A connected* T_2*-space* X *is linear if and only if* X *is strictly orderable or strictly cyclically orderable.*

PROOF.

(i) Let X be strictly orderable. Then, by Theorem 1.5, X is locally connected and X is certainly an (Ht)-space.
Let X be a non-orderable strictly cyclically orderable space. By Theorem 3.16 and 3.19 it follows that X is again a locally connected (Ht)-space.

(ii) Let X be a linear connected T_2-space. Then, by Proposition 7, X is a locally connected (Ht)-space.
If X has at least one cut point, then X is strictly orderable, by Theorem 2.
If X has no cut points, then, by Theorem 3, X is strictly cyclically orderable.

REMARK. Theorem 8 does not hold for connected T_1-spaces. See example 27.

CHAPTER VI

COUNTEREXAMPLES

6.1. INTRODUCTION

In this chapter we describe a number of counterexamples. Each of these examples is accompanied with two sets of properties (out of those studied in the previous chapters). The first set consists of properties which are satisfied by the example under consideration; the second set consists of properties which are not satisfied. (Only in non-trivial cases we include a proof of the fact that a specific property is satisfied or not).
The list of counterexamples given in section 6.3 is almost complete, in the following sense: except for a few cases all possible combinations of the studied conditions are investigated, and all implications which have not been proved in the foregoing chapters are refuted by a counterexample. Only a few questions remain, namely the questions mentioned in the Remark following Theorem 5.5 and related questions, such as: is a connected T_1-space satisfying $(\mathcal{W})$, (Int') and $(\mathcal{B})$ an $(\mathit{Int}\,2)$-space or not?

6.2. BICONNECTED AND WIDELY CONNECTED SPACES

A topological space X is said to be *biconnected* if X is connected and if X is not the union of two disjoint connected subsets consisting of more than one point (see [20], p. 214). A topological space X is said to be *widely connected* if X is connected and if every connected subset consisting of more than one point is dense in X (see [27], p. 254).
It is easy to see that a space X is biconnected if and only if X is connected and does not contain two disjoint connected subsets consisting of more than one point.
Now it is clear that a biconnected T_1-space is a $(\mathcal{W})$-space and that a widely connected T_1-space satisfies condition (Int').

6.3. LIST OF COUNTEREXAMPLES

All spaces X_i $(i = 1,2,\dots,50)$ listed below are connected T_1-spaces.

1. $X_1 = \{(x,y) \in I^2 \mid (\exists n \in \mathbb{N}_0 : x = ny) \vee (y = 0)\}$ with the subspace topology of $\mathbb{R}^2$. (I is the closed unit-interval and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).
X_1 is a T_2-space and satisfies (Int'), but satisfies neither $(\mathcal{B}'\mathcal{O})$, $(\mathcal{W})$ nor $(\mathcal{H}t)$.

2. $X_2 = \{(x,y) \in I^2 \mid \exists n \in \mathbb{N}_0 : x = ny\}$ with the subspace topology of $\mathbb{R}^2$.
X_2 is a T_2-space and satisfies (Int), but satisfies neither $(\mathcal{B}'')$ nor (Ht).

3. $X_3 = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ with the subspace topology of $\mathbb{R}^2$.
X_3 is a T_2-space and satisfies (Int) and $(\mathcal{B}'')$, but satisfies neither $(\mathcal{B}')$ nor (Ht).

4. $X_4 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ with the subspace topology of $\mathbb{R}^2$.
X_4 is a T_2-space and satisfies $(\mathcal{B})$ and (Hp), but satisfies neither $(\mathcal{W})$, (Int') nor (H).

5. $X_5 = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \wedge |y| \le 1\} \cap (X_3 \cup X_4)$.
X_5 is a T_2-space and satisfies $(\mathcal{B})$, but satisfies neither $(\mathcal{W})$, (Int') nor (Ht).

6. $X_6 = X_4 \cup \{(x,y) \in \mathbb{R}^2 \mid x = 1\}$ with the subspace topology of $\mathbb{R}^2$.
X_6 is a T_2-space and satisfies $(\mathcal{B}'')$, but satisfies neither $(\mathcal{B}')$, $(\mathcal{W})$, (Int') nor (Ht).

7. X_7 is the space obtained by identifying the point $(0,0)$ of X_2 with a point of X_4.
X_7 is a T_2-space and satisfies $(\mathcal{B}'\mathcal{O})$, but satisfies neither $(\mathcal{B}'')$, $(\mathcal{W})$, (Int') nor (Ht).

8. X_8 is the space obtained by identifying the point $(0,0)$ of X_1 with a point of X_4.
X_8 is a T_2-space and satisfies $(\mathcal{B}'\mathcal{C})$, but satisfies neither $(\mathcal{B}'\mathcal{O})$, $(\mathcal{W})$, (Int') nor (Ht).

9. $X_9 = X_2 \cup \{(\frac{1}{2},0)\}$ with the following topology:
Every $z \in X_9$ with $z \neq (\frac{1}{2},0)$ has the usual relativized Euclidean neighbourhoodsystem. For $z_0 = (\frac{1}{2},0)$ we define a local base $\mathcal{B}$ as follows:
Let $O_m = \{z \in \mathbb{R}^2 \mid d(z,z_0) < \frac{1}{m}\}$, where $m \in \mathbb{N}$ and d is the usual Euclidean metric.
Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$.
Let $S_n = \{(x,y) \in I^2 \mid x = ny\}$, where $n \in \mathbb{N}$.
For $F \in \mathcal{F}$ and $m \in \mathbb{N}$ let

$$B_{Fm} = \bigcup_{n \in F} (S_n \cap O_m) \cup \{(\tfrac{1}{2},0)\}.$$

Then put

$$\mathcal{B} = \{B_{Fm} \mid F \in \mathcal{F} \wedge m \in \mathbb{N}\}.$$

X_9 is a T_2-space and satisfies $(\mathit{Int}\,2)$, but satisfies neither $(B'C)$ nor (Ht).

10. $X_{10} = X_2 \cup \{(\frac{1}{2},0)\}$ with the subspace topology of $\mathbb{R}^2$.
X_{10} is a T_2-space and satisfies (W), but satisfies neither $(B'C)$, $(\mathit{Int}\,2)$ nor (Ht).

11. X_{11} is the biconnected space of Knaster and Kuratowski [20], p.241.
X_{11} is a T_2-space and satisfies (W), but satisfies neither $(B'C)$, $(\mathit{Int}\,2)$ nor (Ht).

12. X_{12} is the (V)-space, constructed in [15].
X_{12} is a T_2-space and satisfies (H) and (W), but satisfies neither $(B'C)$ nor $(\mathit{Int}\,2)$.

REMARK. Every (V)-space satisfies (H) and (W), as was proved by Brouwer in [2] and [3].

13. X_{13} is the space X constructed below. It is a modification of the space X_{12}. The construction of this space is due to A.E. Brouwer. Let $\mathbb{N}$ be the set of natural numbers, and let $P \subset \mathbb{N}$ be the set of prime numbers.
Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in A}$ be an ultrafilter on $\mathbb{N}$, containing the sets of the form $\{n \in \mathbb{N} \mid n \geq n_0\}$ for every $n_0 \in \mathbb{N}$.
Put $X = \{0\} \cup (\bigcup_{n\in\mathbb{N}} \mathbb{N}^n)$.

For $x \in X$ we define : $\text{length } x = \begin{cases} 2 \text{ if } x = 0 \\ n+2 \text{ if } x \in \mathbb{N}^n . \end{cases}$

We define a partial order $\leq$ on X by taking $0 \leq x$ for all $x \in X$ and $x \leq y$ if x is an initial segment of y, i.e. if $x \in \mathbb{N}^n$, $y \in \mathbb{N}^m$, $n \leq m$ and $x = (a_1,\ldots\ldots,a_n)$, $y = (a_1,\ldots\ldots,a_n,\ldots\ldots,a_m)$, where $a_1,\ldots\ldots,a_m \in \mathbb{N}$.*)
We write $x < y$ if $x \leq y$ and $x \neq y$.
If $n \geq 2$ and if $x = (a_1,\ldots\ldots,a_n) \in \mathbb{N}^n$, then let $x' = (a_1,\ldots\ldots,a_{n-1})$; if $x = (a)$, then let $x' = 0$. ($0'$ is not defined).
If $x = (a_1,\ldots\ldots,a_n) \in \mathbb{N}^n$ then $\ulcorner xt \urcorner \in \mathbb{N}^{n+1}$ is defined by $\ulcorner xt \urcorner = (a_1,\ldots\ldots,a_n,t)$.
In the same way $\ulcorner xtk \urcorner \in \mathbb{N}^{n+2}$ is defined by $\ulcorner xtk \urcorner = (a_1,\ldots\ldots,a_n,t,k)$.

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*) for typographical reasons we use the same symbol $\leq$ both for the usual ordering of the natural numbers and this partial ordering. Confusion seems unlikely.

We introduce a topology in X by taking as a subbase for the open sets all sets

(i) $\{z \in X \mid \exists k \in B_\alpha : \exists t \in \mathbb{N} : \overline{xtk} \le z\} \cup \{x\}$ for each $x \in X$ and each $\alpha \in A$.

(ii) $\{z \in X \mid x \nleq z \wedge z \ne x'\}$ for each $x \in X \setminus 0$.

(iii) $\{z \in X \mid p \in P \text{ divides length } z \Rightarrow p \in \{p_1,\dots,p_n\}\}$ for each finite set of primes $p_1,\dots,p_n$.

(It follows easily from (i) that for instance each set of the form $\{z \in X \mid x \le z\}$ is open.)

PROPOSITION 1. X *is a Hausdorffspace.*

PROOF. Let $u,v \in X$. We consider three cases:

(a) $u < v'$.

(b) $u = v'$.

(c) u and v are not comparable.

(a) In this case $\{z \in X \mid v \nleq z \wedge z \ne v'\}$ and $\{z \in X \mid v \le z\}$ are disjoint open neighbourhoods of u and v, respectively.

(b) In this case length v = length u + 1, so

$\{z \in X \mid p \in P \text{ divides length } z \Rightarrow p \text{ divides length } u\}$

and $\{z \in X \mid p \in P \text{ divides length } z \Rightarrow p \text{ divides length } v\}$

are disjoint neighbourhoods of u and v, respectively.

(c) Here $\{z \in X \mid u \le z\}$ and $\{z \in X \mid v \le z\}$ are disjoint neighbourhoods of u and v, respectively.

PROPOSITION 2. *Each connected set* $C \subset X$ *containing* 0 *is closed.*

PROOF. Let $u \in X \setminus C$ and suppose that $\{z \in X \mid u \le z\} \not\subset X \setminus C$. Then $u < y$ for some $y \in C$. But then

$$u = (a_1,\dots,a_n),\ y = (a_1,\dots,a_n,\dots,a_m) \text{ and}$$

$$C = \underset{y}{(C \cap \{z \in X \mid \overline{ua}_{n+1} \le z\})} + \underset{0}{(C \cap \{z \in X \mid \overline{ua}_{n+1} \nleq z \wedge z \ne u\})}$$

which contradicts the connectedness of C.

PROPOSITION 3. *Let* C *be a connected subset of* X *consisting of more than one point.*

(i) *If* $u \in C$ *and* $u' \notin C$ *for some* $u \in X \setminus 0$, *then* $C \subset \{z \in X \mid u \leq z\}$.

(ii) *If* $\{p,q\} \subset C$ *and* $p < q$, *then* $\{z \in X \mid p \leq z \leq q\} \subset C$.

(iii) C *contains at least two comparable points (and hence at least a pair* (u',u)*).*

(iv) C *does not have maximal members.*

PROOF.

(i) Let $u \in C$ and $u' \notin C$, then

$$C = (C \cap \{z \underset{u}{\mid} u \leq z\}) + (C \cap \{z \mid u \nleq z \wedge z \neq u'\}),$$

hence $C \cap \{z \mid u \leq z\} = C$.

(ii) Let $\{p,q\} \subset C$ and $p < q$. Suppose $A = \{z \mid p \leq z \leq q\} \not\subset C$. Then there exists an element $u \in A \cap C$ such that $u' \in A \cap (X \setminus C)$. Consequently, by (i), we have $C \subset \{z \mid u \leq z\}$, which contradicts $p \in C$.

(iii) Let $u \in C$ and suppose $u' \notin C$.
Then, by (i), $C \subset \{z \mid u \leq z\}$. Hence, there exists an element $v \in C$ such that $u < v$. (From (ii) it follows that the pair (v',v) belongs to C.)

(iv) Suppose $w \in C$ is a maximal member of C.
Then $C \cap \{z \mid w \leq z\} = \{w\}$ is a clopen subset of C, which contradicts the connectedness of C.

PROPOSITION 4. *For every* $u \in X \setminus 0$, u *and its predecessor* u' *do not have disjoint closed neighbourhoods. That is,* X *is a non-Urysohn space.*

PROOF. Let $u = (a_1,\dots,a_l)$.
For each $x \in X$, each $\alpha \in A$ and each finite family $\{x_1,\dots,x_n\}$ such that $x_i \nleq x$ and $x \neq x_i'$ we define the following neighbourhood of x:

$$U(x;\alpha;x_1,\dots,x_n) = \{z \mid z = x \vee (\exists k \in B_\alpha : \exists t \in \mathbb{N} : \widehat{xtk} \leq z)\} \cap$$

$$\cap \{z \mid \forall p \in P : (p \text{ divides length } z \Longrightarrow p \text{ divides length } x)\} \cap$$

$$\cap \bigcap_{i=1}^{n} \{z \mid x_i \nleq z \vee z \neq x_i'\}.$$

It is clear that if the $x_1,\ldots,x_n$ and α vary we obtain a neighbourhoodbase of x.

For $x = (b_1,\ldots,b_n)$ we put $\max x = \max\{b_1,\ldots,b_n\}$.

Now, let $U(u';\alpha;x_1,\ldots,x_n)$ and $U(u;\beta;x_{n+1},\ldots,x_m\}$ be two basic neighbourhoods of u' and u, respectively.

Choose $N \geq \max\{\max x_i \mid i = 1,\ldots,m\} + 1$ such that $N \in B_\alpha \cap B_\beta$.

Put $L = (\text{length } u) \times (\text{length } u') - 2 = (l+2)(l+1)-2 = l(l+3) \geq l + 3$.

$v = (a_1,\ldots,a_l,N,N,\ldots,N) \in \mathbb{N}^L$.

We will show that

$$v \in U(u';\alpha;x_1,\ldots,x_n)^- \cap U(u;\beta;x_{n+1},\ldots,x_m)^-.$$

Let $U(v;\gamma;x_{m+1},\ldots,x_k)$ be an arbitrary basic neighbourhood of v.

Choose $N' \geq \max\{\max x_i \mid i = 1,\ldots,k\} + 1$ such that $N' \in B_\gamma$.

Let $p,q \in P$ be such that p divides length u', q divides length u and choose $r \in \mathbb{N}$ such that $p^r > L + 1$ and $q^r > L + 1$.

Then, if

$$s_1 = (\underbrace{a_1,\ldots,a_l,N,N,\ldots,N}_{L \text{ numbers}},\ N',\ldots,N') \in \mathbb{N}^{p^r-2},$$

and

$$s_2 = (\underbrace{a_1,\ldots,a_l,N,N,\ldots,N}_{L \text{ numbers}},N',\ldots,N') \in \mathbb{N}^{q^r-2},$$

we have

$$s_1 \in U(u';\alpha;x_1,\ldots,x_n) \cap U(v;\gamma;x_{m+1},\ldots,x_k)$$

and

$$s_2 \in U(u;\beta;x_{n+1},\ldots,x_m) \cap U(v;\gamma;x_{m+1},\ldots,x_k),$$

proving the assertion.

PROPOSITION 5. X *is connected.*

PROOF. Suppose $X = A + B$, where $0 \in A$ and $B \neq \emptyset$. Let $y \in B$ be such that length y is minimal in B.
Then $y' \in A$, and A and B are disjoint closed neighbourhoods of y and y', which contradicts Proposition 4.

PROPOSITION 6. *Each connected subset of* X *has at most one endpoint.*

PROOF. Let D be any connected subset of X. Let $u \in D$ and suppose that $C = D \setminus u$ is connected.
For each $a \in \mathbb{N}$ it then follows from

$$C = (C \cap \{z \mid \overline{ua} \leq z\}) + (C \cap \{z \mid \overline{ua} \nleq z \wedge z \neq u\})$$

that at least one of both summands is empty.
Hence, for all $a \in \mathbb{N}$,

$$C \subset \{z \mid \overline{ua} \leq z\} \; \ldots\ldots \; (1)$$

or

$$C \subset \{z \mid \overline{ua} \nleq z \wedge z \neq u\} \; \ldots\ldots \; (2).$$

If (2) applies for all $a \in \mathbb{N}$, it follows that $\{z \mid u \leq z\} \subset X \setminus C$, so that $u \notin \bar{C}$, which contradicts the connectedness of D.
Hence it follows that (1) applies for at least one $a \in \mathbb{N}$.
If v is another endpoint of D, it follows similarly that

$$(C \cup u) \setminus v = D \setminus v \subset \{z \mid \overline{vb} \leq z\} \text{ for some } b \in \mathbb{N}.$$

This is a contradiction, since $\overline{vb} \nleq u$.

PROPOSITION 7. *A subset* $C \subset X$ *is connected iff*

a) $(x \in C \wedge \overline{xt} \in C \Longrightarrow \exists \alpha \in A : B_\alpha \subset \{k \in \mathbb{N} \mid \overline{xtk} \in C\})$

and

b) $(x \notin C \wedge \overline{xt} \in C \Longrightarrow \forall z \in C : \overline{xt} \leq z)$.

PROOF.

(i) Let $C \subset X$ be connected.

Suppose $x \notin C$ and $\overline{xt} \in C$. Then, by Proposition 3, (i),

$$C \subset \{z \mid \overline{xt} \leq z\}.$$

Now, suppose $x \in C$ and $\overline{xt} \in C$.
Let $A = \{k \in \mathbb{N} \mid \overline{xtk} \notin C\}$.
If $A \notin \mathcal{B}$ then $\mathbb{N} \setminus A = B \in \mathcal{B}$ and the proof is finished.
If $A \in \mathcal{B}$ then put

$$U(x;\alpha) = \{x\} \cup \{z \mid \exists k \in A = B_\alpha : \exists s \in \mathbb{N} : \overline{xsk} \leq z\};$$

$$V = \{z \mid \overline{xt} \leq z\};$$

$$O = \{z \mid \overline{xt} \not\leq z \wedge z \neq x\}.$$

For every $k \in B_\alpha$ we have $\overline{xtk} \notin C$.
Hence, by Proposition 3, (ii), for every $z \in C : \overline{xtk} \not\leq z$.
Consequently, $U \cap V = \emptyset$.
But then we have

$$C = \underbrace{((U \cup O) \cap C)}_{x} + \underbrace{(V \cap C)}_{\overline{xt}},$$

which contradicts the connectedness of C.

(ii) Let $C \subset X$ be a set satisfying a) and b).
We may assume that C contains at least two points. From b) it easily follows that there exist two elements x and $\overline{xt}$ in X such that $x \in C$ and $\overline{xt} \in C$.
By choosing the element v in the proof of Proposition 4 such that $v \in C$ (which is possible by a)), it follows that x and $\overline{xt}$ do not have disjoint closed neighbourhoods in the relative topology of C.
Now, suppose $C = A + B$, where both A and B are non-empty.
Let $y \in A$ be such that length y is minimal in A.
Let $z \in B$ be such that length z is minimal in B.
If length y = length z it follows that $y' \notin C$ and that y and z are not comparable, which contradicts b).
Hence we may assume length $y <$ length z.
Again by b) we have $z' \in A$. This means that z and z' have disjoint closed neighbourhoods in C. Contradiction.

PROPOSITION 8. X *satisfies* (*Int 2*).

PROOF. Let C_1 and C_2 be connected subsets of X.
Suppose $\overline{xt} \in C_1 \cap C_2$ for some $x \in X$ and some $t \in \mathbb{N}$.
If $x \in C_1 \cap C_2$ then $\exists\alpha : B_\alpha \subset \{k \in \mathbb{N} \mid \overline{xtk} \in C_1\}$

$\exists\beta : B_\beta \subset \{k \in \mathbb{N} \mid \overline{xtk} \in C_2\}$

$\exists\gamma : B_\gamma \subset B_\alpha \cap B_\beta$.

Hence, $\exists\gamma : B_\gamma \subset \{k \in \mathbb{N} \mid \overline{xtk} \in C_1 \cap C_2\}$.
If $x \notin C_1 \cap C_2$ then for instance $x \notin C_2$ and $\forall z \in C_2 : \overline{xt} \le z$.
Hence, for every $z \in C_1 \cap C_2 : \overline{xt} \le z$.
By Proposition 7 it follows that $C_1 \cap C_2$ is connected.

From the foregoing propositions it follows:

X_{13} is a T_2-space and satisfies (*Int 2*) and (*H*), but does not satisfy (*B'C*).

14. X_{14} is another modification of example 12.
Let X_{14} be the disjoint union

$$X_{14} = \{0'\} \cup X_{12} \cup \{p\},$$

with topology determined by the following requirements:
As a subspace, X_{12} has its own topology and X_{12} is an open subset of X_{14}.
The sets $(U(0) \setminus \{0\}) \cup \{0'\}$ form an open neighbourhood-basis for 0',
where the sets U(0) are taken from an open neighbourhood-basis of 0 in X_{12}.
If $X_{12} \setminus \{0\} = \bigcup_{i \in \mathbb{N}} C_i$ is the decomposition of $X_{12} \setminus \{0\}$ into components,
then $\bigcup_{i \ge n} C_i \cup \{p\}$ is a basic open neighbourhood of p $(n = 1,2,3,\ldots)$.
(Notice that each C_i is open in X_{12}.)
X_{14} is a T_1-space and satisfies (*Ht*), but satisfies neither (*B'C*), (*Hd*) nor (*W*).

15. $X_{15} = \{(x,y) \in \mathbb{R}^2 \mid y = \sin\frac{1}{x} \text{ and } x > 0\} \cup \{(0,1),(0,-1)\}$ with the subspace topology of $\mathbb{R}^2$.
X_{15} is a T_2-space and satisfies (*B*) and (*Int 2*), but satisfies neither (*Ht*), (*Int'*) nor (*S*).

16. $X_{16} = X_{15} \cup \{(x,y) \in \mathbb{R}^2 \mid x = 1\}$ with the subspace topology of $\mathbb{R}^2$.
X_{16} is a T_2-space and satisfies (*B"*) and (*Int 2*), but satisfies neither (*Ht*), (*Int'*), (*S*) nor (*B'*).

17. X_{17} is obtained by identifying the point (0,0) of X_2 with a point of X_{15}.
X_{17} is a T_2-space and satisfies (*Int 2*) and (*B'O*), but satisfies neither (*Ht*), (*Int'*), (*S*) nor (*B"*).

18. $X_{18} = \{(x,y) \in I^2 \mid (\frac{1}{x} \in \mathbb{N}) \vee (y = 1) \vee (x = y = 0)\}$ with the subspace topology of $\mathbb{R}^2$.
X_{18} is a T_2-space and satisfies (*B"*) and (*W*), but satisfies neither (*Ht*), (*Int 2*), (*S*), (*Int'*) nor (*B'*).

19. X_{19} is obtained by identifying the point (0,0) of X_2 with a point of X_{18}.
X_{19} is a T_2-space and satisfies (*B'O*) and (*W*), but satisfies neither (*Ht*), (*Int 2*), (*S*), (*Int'*) nor (*B"*).

20. $X_{20} = X_{18} \setminus \{(0,1)\}$.
X_{20} is a T_2-space and satisfies (*B"*) and (*S*), but satisfies neither (*Ht*), (*Int 2*), (*Int'*) nor (*B'*).

21. X_{21} is the space of example 20, but with ultrafilterbasistopology at the point (0,0). (cf. the point z_0 of example 9.)
X_{21} is a T_2-space and satisfies (*B"*), (*S*) and (*Int 2*), but satisfies neither (*Ht*), (*Int'*) nor (*B'*).

22. X_{22} is obtained by identifying the point (0,0) of X_2 with a point of X_{21}.
X_{22} is a T_2-space and satisfies (*S*) and (*Int 2*), but satisfies neither (*Ht*), (*Int'*) nor (*B"*).

23. X_{23} is obtained by identifying the point (0,0) of X_2 with a point of X_{20}.
X_{23} is a T_2-space and satisfies (*S*), but satisfies neither (*Ht*), (*Int'*), (*Int 2*) not (*B"*).

24. $X_{24} = X_{14} \setminus \{p\}$.
X_{24} is a T_1-space and satisfies (*H*), but satisfies neither (*W*) nor (*B'C*).

25. X_{25} is obtained by identifying the basic point of X_{12} with a point of X_4.
X_{25} is a T_2-space and satisfies (*Hd*), but satisfies neither (*Hp*), (*B'C*) nor (*W*).

26. Let I_1, I_2 and I_3 be three copies of the unit-interval I. Identify the

left-endpoint of I_1, I_2 and I_3, respectively with the point (1,0), the point (0,1) and the point (-1,0) of X_4, respectively. Let X_{26} be the space thus obtained.

X_{26} is a T_2-space and satisfies $(\mathcal{B}')$, but satisfies neither $(\mathcal{H}t)$, $(\mathcal{W})$, $(\mathcal{I}nt')$ nor $(\mathcal{B})$.

27. Let I be the closed unit-interval.

Let X_{27} be the disjoint union $\{0'\} \dot\cup I$ with the following topology:
As a subspace I has its own topology and I is an open subspace of X_{27}.
If $0' \in U(0') \subset X_{27}$, then $U(0')$ is an (open) neighbourhood of 0' iff $(U(0') \setminus \{0'\}) \cup \{0\}$ is an (open) neighbourhood of 0 in I.

X_{27} is a locally connected T_1-space and satisfies $(\mathcal{B})$ and $(\mathcal{I}nt\,2)$, but satisfies neither $(\mathcal{H}t)$, $(\mathcal{I}nt')$ nor $(\mathcal{S})$.

28. $X_{28} = \mathbb{N}$ with the cofinite topology. Then X_{28} is a widely connected, locally connected space.

X_{28} is a T_1-space and satisfies $(\mathcal{B})$ and $(\mathcal{I}nt')$, but satisfies neither $(\mathcal{H}t)$ nor $(\mathcal{W})$.

29. $X_{29} = \{(x,y) \in \mathbb{R}^2 \mid (y = \sin \frac{1}{x} \wedge x > 0) \vee (-1 \leq y \leq +1 \wedge x = 0)\}$ with the subspace topology of $\mathbb{R}^2$.

X_{29} is a T_2-space and satisfies $(\mathcal{B})$ and $(\mathcal{I}nt')$, but satisfies neither $(\mathcal{H}t)$ nor $(\mathcal{W})$.

30. $X_{30} = \mathbb{N}$ with the following topology:

If $\mathcal{B} = \{B_\alpha\}_{\alpha\in A}$ is a free ultrafilter on $\mathbb{N}$ we take for open sets the empty set and the elements of $\mathcal{B}$.

For each $U \subset X_{30}$ the following four conditions are equivalent:

(i) $\overset{\circ}{U} \neq \emptyset$

(ii) $U = \overset{\circ}{U} \neq \emptyset$

(iii) $\bar{U} = X_{30}$

(iv) U is connected and U contains at least two points.

Hence, X_{30} is a locally connected, widely connected and biconnected space.
X_{30} is a T_1-space and satisfies $(\mathcal{I}nt\,2)$, $(\mathcal{I}nt')$ and $(\mathcal{B})$, but satisfies neither $(\mathcal{H}t)$ nor $(\mathcal{S})$.

31. X_{31} is the subset of the plane constructed by E.W. Miller in [23]. (For a short description of this example see Steen and Seebach [26], example 131.) X_{31} is a biconnected space without *dispersion point* (i.e. a

point p such that $X_{31} \setminus p$ is totally disconnected) which is also widely connected. Since a biconnected space without dispersion point clearly cannot contain any cut point it is easily seen that:
X_{31} is a T_2-space and satisfies $(\mathcal{W})$, (Int') and $(\mathcal{B})$, but satisfies neither $(\mathcal{H}t)$ nor $(\mathcal{S})$.

REMARK. We do not know whether or not X_{31} satisfies $(\mathit{Int}\,2)$. (When it is true that X_{31} does not satisfy $(\mathit{Int}\,2)$, this answers in the negative the last question in the Remark following Theorem 5.5.)

32. Let $X_{32} = \{(x,y) \in \mathbb{R}^2 \mid (y = 0 \wedge 0 < x \leq 1) \vee$

$$\vee (y = 1 \wedge 0 \leq x < 1) \vee (y = 2 \wedge 0 < x < 1)\}$$

with the following basic neighbourhood system:

$$U_i(a,0) = \{(a,0)\} \cup ([a - \tfrac{1}{i},a) \times \{0,1\})$$

$$U_i(a,1) = \{(a,1)\} \cup ((a,a + \tfrac{1}{i}] \times \{0,1\})$$

$$U_i(a,2) = \{(a,2)\} \cup (([a - \tfrac{1}{i},a) \cup (a,a + \tfrac{1}{i}]) \times \{0,1\})$$

$$(i = 1,2,3,\dots)$$

X_{32} is a T_1-space and satisfies $(\mathcal{B})$ and $(\mathcal{W})$, but satisfies neither $(\mathcal{H}t)$, $(\mathit{Int}\,2)$, (Int') nor $(\mathcal{S})$.

33. X_{33} is obtained by identifying the left-endpoints of three copies of the unit-interval I with three distinct endpoints of X_{32}, respectively. X_{33} is a T_1-space and satisfies $(\mathcal{B}')$ and $(\mathcal{W})$, but satisfies neither $(\mathcal{H}t)$, $(\mathit{Int}\,2)$, (Int'), $(\mathcal{S})$ nor $(\mathcal{B})$.

34. X_{34} is obtained by identifying the left-endpoints of three copies of the unit-interval I with three distinct points of X_{28}, respectively. X_{34} is a locally connected T_1-space and satisfies (Int') and $(\mathcal{B}')$, but satisfies neither $(\mathcal{H}t)$, $(\mathcal{W})$ nor $(\mathcal{B})$.

35. X_{35} is obtained by identifying the left-endpoints of three copies of the unit-interval I with three distinct endpoints of X_{29}, respectively. X_{35} is a T_2-space and satisfies (Int') and $(\mathcal{B}')$, but satisfies neither $(\mathcal{H}t)$, $(\mathcal{W})$ nor $(\mathcal{B})$.

36. X_{36} is obtained by identifying the left-endpoints of two copies of the unit-interval I with the points 0 and 0' of X_{27}, respectively.
X_{36} is a locally connected T_1-space and satisfies (*Int 2*) and (*B*'), but satisfies neither (*Ht*), (*Int*'), (*S*) nor (*B*).

37. X_{37} is obtained by identifying the left-endpoints of two copies of the unit-interval I with the points (0,1) and (0,-1) of X_{15}, respectively.
X_{37} is a T_2-space and satisfies (*Int 2*) and (*B*'), but satisfies neither (*Ht*), (*Int*'), (*S*) nor (*B*).

38. X_{38} is obtained by identifying the left-endpoints of three copies of the unit-interval I with three distinct points of X_{30}, respectively.
X_{38} is a locally connected T_1-space and satisfies (*Int 2*), (*Int*') and (*B*'), but satisfies neither (*Ht*), (*S*) nor (*B*).

39. X_{39} is obtained in an analogous way from X_{31}.
X_{39} is a T_2-space and satisfies (*W*), (*Int*') and (*B*'), but satisfies neither (*Ht*), (*S*) nor (*B*).

40. X_{40} is obtained by identifying the point (0,0) of X_3 with a point of X_{28}.
X_{40} is a locally connected T_1-space and satisfies (*B*") and (*Int*'), but satisfies neither (*Ht*), (*W*) nor (*B*').

41. X_{41} is obtained in an analogous way from X_{29}.
X_{41} is a T_2-space and satisfies (*B*") and (*Int*'), but satisfies neither (*Ht*), (*W*) nor (*B*').

42. X_{42} is obtained in an analogous way from X_{30}.
X_{42} is a locally connected T_1-space and satisfies (*B*"), (*Int 2*) and (*Int*'), but satisfies neither (*Ht*), (*S*) nor (*B*').

43. X_{43} is obtained is an analogous way from X_{31}.
X_{43} is a T_2-space and satisfies (*W*), (*Int*') and (*B*"), but satisfies neither (*B*'), (*Ht*) nor (*S*).

44. X_{44} is obtained by identifying the point (0,0) of X_2 with a point of X_{28}.
X_{44} is a locally connected T_1-space and satisfies (*B'0*) and (*Int*'), but satisfies neither (*Ht*), (*W*) nor (*B*").

45. X_{45} is obtained in an analogous way from X_{29}.
X_{45} is a T_2-space and satisfies (*B'0*) and (*Int*'), but satisfies neither

(*Ht*), (*W*) nor (*B*").

46. X_{46} is obtained in an analogous way from X_{30}.
X_{46} is a locally connected T_1-space and satisfies (*Int*') and (*Int 2*), but satisfies neither (*Ht*), (*S*) nor (*B*").

47. X_{47} is obtained in an analogous way from X_{31}.
X_{47} is a T_2-space and satisfies (*W*) and (*Int*'), but satisfies neither (*Ht*), (*S*) nor (*B*").

48. X_{48} is obtained by identifying the point (0,0) of X_3 with a point of X_{27}.
X_{48} is a locally connected T_1-space and satisfies (*Int 2*) and (*B*"), but satisfies neither (*Ht*), (*Int*'), (*S*) nor (*B*').

49. X_{49} is obtained by identifying the point (0,0) of X_2 with a point of X_{27}.
X_{49} is a locally connected T_1-space and satisfies (*Int 2*) and (*B'0*), but satisfies neither (*Ht*), (*Int*'), (*S*) nor (*B*").

50. $X_{50} = \{(x,y) \in \mathbb{R}^2 \mid y = \sin\frac{1}{x} \wedge x > 0\} \cup (-1,0] \times \{0\}$ with the relative topology of the plane.
X_{50} is an orderable space which is not strictly orderable.

In the following table we indicate schematically which properties are satisfied (+) and which are not satisfied (-) by each of the counterexamples listed above:

	(O)	*(H)*	*(Hp)*	*(Hd)*	*(Ht)*	*(B)*	*(B')*	*(B")*	*(B'O)*	*(B'C)*	*(Int)*	*(S)*	*(W)*	*(Int 2)*	*(Int')*	T_2	loc.conn.
1.	−	−	−	−	−	−	−	−	−	+	−	−	−	−	+	+	−
2.	−	−	−	−	−	−	−	−	+	+	+	+	+	+	+	+	+
3.	−	−	−	−	−	−	−	+	+	+	+	+	+	+	+	+	+
4.	−	−	+	+	+	+	+	+	+	+	−	−	−	−	−	+	+
5.	−	−	−	−	−	+	+	+	+	+	−	−	−	−	−	+	+
6.	−	−	−	−	−	−	−	+	+	+	−	−	−	−	−	+	+
7.	−	−	−	−	−	−	−	−	+	+	−	−	−	−	−	+	+
8.	−	−	−	−	−	−	−	−	−	+	−	−	−	−	−	+	−
9.	−	−	−	−	−	−	−	−	−	−	−	−	+	+	−	+	−
10.	−	−	−	−	−	−	−	−	−	−	−	−	+	−	−	+	−
11.	−	−	−	−	−	−	−	−	−	−	−	−	+	−	−	+	−
12.	−	+	+	+	+	−	−	−	−	−	−	−	+	−	−	+	−
13.	−	+	+	+	+	−	−	−	−	−	−	−	+	+	−	+	−
14.	−	−	−	−	+	−	−	−	−	−	−	−	−	−	−	−	−
15.	−	−	−	−	−	+	+	+	+	+	−	−	+	+	−	+	−
16.	−	−	−	−	−	−	−	+	+	+	−	−	+	+	−	+	−
17.	−	−	−	−	−	−	−	−	+	+	−	−	+	+	−	+	−
18.	−	−	−	−	−	−	−	+	+	+	−	−	+	−	−	+	−
19.	−	−	−	−	−	−	−	−	+	+	−	−	+	−	−	+	−
20.	−	−	−	−	−	−	−	+	+	+	−	+	+	−	−	+	−
21.	−	−	−	−	−	−	−	+	+	+	−	+	+	+	−	+	−
22.	−	−	−	−	−	−	−	−	+	+	−	+	+	+	−	+	−
23.	−	−	−	−	−	−	−	−	+	+	−	+	+	−	−	+	−
24.	−	+	+	+	+	−	−	−	−	−	−	−	−	−	−	−	−
25.	−	−	−	+	+	−	−	−	−	−	−	−	−	−	−	+	−
26.	−	−	−	−	−	−	+	+	+	+	−	−	−	−	−	+	+
27.	−	−	−	−	−	+	+	+	+	+	−	−	+	+	−	−	+
28.	−	−	−	−	−	+	+	+	+	+	−	−	−	−	+	−	+
29.	−	−	−	−	−	+	+	+	+	+	−	−	−	−	+	+	−
30.	−	−	−	−	−	+	+	+	+	+	−	−	+	+	+	−	+
31.	−	−	−	−	−	+	+	+	+	+	−	−	+	?	+	+	−
32.	−	−	−	−	−	+	+	+	+	+	−	−	+	−	−	−	−
33.	−	−	−	−	−	−	+	+	+	+	−	−	+	−	−	−	−
34.	−	−	−	−	−	−	+	+	+	+	−	−	−	−	+	−	+
35.	−	−	−	−	−	−	+	+	+	+	−	−	−	−	+	+	−
36.	−	−	−	−	−	−	+	+	+	+	−	−	+	+	−	−	+
37.	−	−	−	−	−	−	+	+	+	+	−	−	+	+	−	+	−
38.	−	−	−	−	−	−	+	+	+	+	−	−	+	+	+	−	+
39.	−	−	−	−	−	−	+	+	+	+	−	−	+	?	+	+	−
40.	−	−	−	−	−	−	−	+	+	+	−	−	−	−	+	−	+
41.	−	−	−	−	−	−	−	+	+	+	−	−	−	−	+	+	−
42.	−	−	−	−	−	−	−	+	+	+	−	−	+	+	+	−	+
43.	−	−	−	−	−	−	−	+	+	+	−	−	+	?	+	+	−
44.	−	−	−	−	−	−	−	−	+	+	−	−	−	−	+	−	+
45.	−	−	−	−	−	−	−	−	+	+	−	−	−	−	+	+	−
46.	−	−	−	−	−	−	−	−	+	+	−	−	+	+	+	−	+
47.	−	−	−	−	−	−	−	−	+	+	−	−	+	?	+	+	−
48.	−	−	−	−	−	−	−	+	+	+	−	−	+	+	−	−	+
49.	−	−	−	−	−	−	−	−	+	+	−	−	+	+	−	−	+
50.	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	−

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SUBJECT INDEX