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TESTS FOR PREFERENCE

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CONTENTS

INTRODUCTION

This tract contains the results of investigations we made into a practical statistical situation. Our aim has been to make the results actually accessible to practicing statisticians. Therefore we give for instance several methods to compute (approximate) critical values for the test-statistics that occur.

The practical situation concerns the detection of differences of preferences or aversions between individuals when the observations are the (repeated) choices they have made. Suppose for instance that n persons may choose from k brands of chocolate. All persons may have the same absolute preferences, possibly changing in time, for special brands, but it is the *difference* between the persons with respect to these preferences that we wish to detect. (The title of this tract might thus have been "Tests for differences of preference").

The practical problem and the statistical solution of it are outlined in chapter 1, which gives the practicing statistician all the information he needs to apply the test.

The basis of the solution of the problem is a vector of observable random variables, $\vec{\underline{t}}_*$, of which the asymptotic normality is established under certain conditions. (Section 4.2). The class of quadratic forms in $\vec{\underline{t}}_*$

$$\mathcal{T} = \{\vec{\underline{t}}_*' Q \vec{\underline{t}}_* \mid Q \text{ non-negative definite}\}$$

is then considered as a possible class of, in practice, useful test-statistics. The use of quadratic forms is given extensive intuitive (section 2.1) and theoretical (chapter 8) motivation.

Two problems arose in the determination of the asymptotic distribution of $\vec{\underline{t}}_*' Q \vec{\underline{t}}_*$. The first problem was the singularity of the dispersion-matrix of $\vec{\underline{t}}_*$ (also asymptotically) and the second problem was that the matrix Q can be chosen more or less arbitrarily. It could be expected that only for some special choices of Q the test-statistic would (asymptotically) have a

χ^2-distribution. Both problems are solved by a theorem (theorem 3.2.1) which gives the distribution of a non-negative definite quadratic form in normal variables, also in the case that the dispersion matrix of the normal variables is singular. Using this theorem, asymptotic distributions of $\underline{\vec{t}}_*' Q \underline{\vec{t}}_*$ are determined (both under the null-hypothesis and under alternatives) (section 4.3).

A usual method to deal with singularity is to define a transformation to a lower dimensional space in which the dispersion matrix of the (transformed) variables is non-singular. This usually leads to complicated statistics and obscures the working of the tests. Using Rao's theory on g-inverses of matrices (RAO (1973)) it is shown that such a transformation is unnecessary (chapter 6). MADANSKY (1963) used the method of transformation to a lower dimension when he proposed a generalisation to Cochran's Q - test (COCHRAN (1950)). Both Madansky's and Cochran's test are a special case of the tests we investigated (chapter 6).

Consistency properties and power of the tests from $\mathcal{T}$ are considered in chapter 4. The asymptotic relative efficiencies of pairs of tests from $\mathcal{T}$, according to Pitman and Bahadur are established in chapter 5. Neither of these efficiency concepts gives a clear indication which Q to use when an overall type test is desired.

Therefore, again mostly motivated by intuitive arguments, a χ^2-type statistic is recommended for practical use (section 6.4). The recommendations are supported by the results of simulation which we give in chapter 9. It is shown there also that the tests can effectively be directed towards a special alternative by a suitable choice of the matrix Q.

Finally, in order to find a good and simple approximation for the distribution under H_0, the expectation and variance of the test-statistics are established for some special choices of Q (chapter 7).

CHAPTER 1

PRACTICE: A RECOMMENDED STATISTIC

1.1. A TESTING PROBLEM: GENERAL REMARKS

In this study, we investigate the properties of a class of statistical tests for a certain testing problem. As a result of the investigations a member of this class can be recommended for practical use. In this first chapter, we give the possible user all the information he needs when he wants to apply the recommended test.

A more formal approach is started in chapter 2 and the problem is developed further in chapters 3-8. Finally some numerical results are given as illustration in chapter 9.

In this section we begin with the statement of the problem. Although the proposed testing procedure is applicable in many other situations, it is convenient to adopt the terminology of the following example. This not only makes the description of the situation easier, but it is also natural, because this investigation was motivated by this example.

In 1975 KNEEPKENS (1975) wrote a report "De voornaamste kop op de voorpagina's van een vijftal landelijke Nederlandse dagbladen in de eerste twee maanden van 1964 en 1974", (in Dutch), in which he compared 5 Dutch newspapers in 1964 and 1974. In this report, Kneepkens asks the question if there exists a statistical test for the following situation.

'Newspaper' Problem

Each day, every one of n newspapers chooses a subject for its 'front-page' article from a category of subjects. The different categories are elements of a given categorical system

(1.1.1) $\quad \mathcal{C} = \{c_1,\ldots,c_k\}.$

On the basis of the observed choices made by the n newspapers on m different days, we want to find out if there are one or more newspapers differing, more or less persistently from the others with respect to their preferences for categories from $\mathcal{C}$.

Formulated as a testing problem, we would want to test the null-hypothesis that the newspapers do not differ among themselves with respect to their choices, against the alternative that at least one newspaper has a persistent preference or aversion for at least one subject-category.

Although this is still somewhat loosely formulated, it follows that we need to construct an overall-test not unlike Friedman's m rankings test. As in Friedman's case, the test that we shall construct is not likely to work well against *all* deviations from the null-hypothesis.

Now, let's be more specific. The mathematical model that we make for the 'newspaper'-problem will be based on the following assumptions

(1.1.2) the newspapers make their choices independently of each other;

(1.1.3) the choices which are made on different days are independent.

Let

(1.1.4) $(\underline{C}_i^{(1)}, \underline{C}_i^{(2)}, \ldots, \underline{C}_i^{(n)}) \in \mathcal{C}^n$

denote an observation on the i'th day, i.e.

(1.1.5) $\underline{C}_i^{(\nu)}$ is the category that is chosen on the i'th day by the ν'th newspaper.

Introduce the following random variables

(1.1.6) $\underline{x}_{ij}^{(\nu)} = \begin{cases} 1 \text{ if the } \nu\text{'th newspaper chooses } C_j \text{ on the i'th day;} \\ 0 \text{ otherwise.} \end{cases}$

and probabilities *)

(1.1.7) $p_{ij}^{(\nu)} \stackrel{d}{=} P(\underline{x}_{ij}^{(\nu)} = 1) = P(\underline{C}_i^{(\nu)} = C_j)$,

---

*) "$\stackrel{d}{=}$" indicates a definition.

with

$$(1.1.8) \qquad p_{i+}^{(\nu)} = 1.$$

(When an index is replaced by a "+" sign, we mean that the indexed quantity has been summed over that index, i.e.

$$p_{i+}^{(\nu)} = \sum_{j=1}^{k} p_{ij}^{(\nu)} \quad).$$

We define

$$(1.1.9) \qquad \underline{a}_{ij} \stackrel{d}{=} \underline{x}_{ij}^{(+)},$$

i.e. $\underline{a}_{ij}$ is the number of times that the category C_j has been chosen on the i'th day, and

$$(1.1.10) \qquad \underline{h}_j^{(\nu)} \stackrel{d}{=} \underline{x}_{+j}^{(\nu)},$$

i.e. $\underline{h}_j^{(\nu)}$ is the number of times that the ν'th newspaper has chosen category C_j. Note that

$$(1.1.11) \qquad \underline{h}_j^{(+)} \equiv \underline{a}_{+j}.$$

Because every newspaper chooses one category at a time, we have

$$(1.1.12) \qquad \underline{x}_{i+}^{(\nu)} \equiv 1.$$

The assumptions (1.1.2) and (1.1.3) mean that in our model we must take

(1.1.13) $\underline{x}_{ij}^{(\nu)}$ and $\underline{x}_{hl}^{(\mu)}$ to be independent whenever $i \neq h$ or $\nu \neq \mu$.

This completes our basic mathematical model. We can now formulate the null-hypothesis in terms of this model

$$(1.1.14) \qquad H_0\colon p_{ij}^{(1)} = p_{ij}^{(2)} = \ldots = p_{ij}^{(n)} \qquad \text{for all i and j.}$$

Denote the common value of $p_{ij}^{(\nu)}$, under H_0, by p_{ij}. It is then clear that the model still contains the m×k unknown parameters p_{ij}.

All the unknown (nuisance) parameters p_{ij}, however, can be eliminated when we condition on the event

(1.1.15) $\underline{A} = A$

where

(1.1.16) $$\underline{A} \equiv \begin{pmatrix} \underline{a}_{11} & \underline{a}_{12} & \cdot & \cdot & \cdot & \underline{a}_{1k} \\ \underline{a}_{21} & \underline{a}_{22} & \cdot & \cdot & \cdot & \underline{a}_{2k} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \underline{a}_{m1} & \underline{a}_{m2} & \cdot & \cdot & \cdot & \underline{a}_{mk} \end{pmatrix}$$

and A is a similarly defined $m \times k$ matrix which contains the observed values of $\underline{A}$.

When we consider the random variables $\underline{x}_{ij}^{(\nu)}$, but now conditioned on $\underline{A} = A$, i.e.

(1.1.17) $\underline{t}_{ij}^{(\nu)} \stackrel{d}{=} (\underline{x}_{ij}^{(\nu)} | \underline{A} = A)$

then the joint distribution of the $\underline{t}_{ij}^{(\nu)}$ contains, under H_0, no more unknown parameters. In fact, it is evident that, under H_0, given $\underline{A} = A$, all the ('generalised') permutations of

(1.1.18) $$\underbrace{C_1 \ldots C_1}_{a_{i1} \times} \; \underbrace{C_2 \ldots C_2}_{a_{i2} \times} \; \ldots \; \underbrace{C_k \ldots C_k}_{a_{ik} \times}$$

are equally likely to occur as outcomes of (1.1.4). That is, each generalised permutation has (conditional) probability

(1.1.19) $$\frac{a_{i1}!\, a_{i2}! \ldots a_{ik}!}{n!} .$$

Therefore we can use the observable random variables $\underline{t}_{ij}^{(\nu)}$ as building-stones for possible test-statistics.

Finally we define, analogous to (1.1.10)

(1.1.20) $\underline{f}_j^{(\nu)} \stackrel{d}{=} \underline{t}_{+j}^{(\nu)}$.

1.2. PRESENTATION OF THE DATA

Assume that the matrix A contains the observed values of $\underline{a}_{ij}$. Then under the condition of the event $\underline{A} = A$, we may represent the data as follows.

Table 1.2.1. Presentation of the data.

	newspaper					category				
	$\nu =$					$j =$				
	1	2	. . .	n		1	2	. . .	k	
1	$\underline{c}_1^{(1)}$	$\underline{c}_1^{(2)}$	. . .	$\underline{c}_1^{(n)}$		a_{11}	a_{12}	. . .	a_{1k}	n
2	$\underline{c}_2^{(1)}$	$\underline{c}_2^{(2)}$	. . .	$\underline{c}_2^{(n)}$		a_{21}	a_{22}	. . .	a_{2k}	n
.	.	.	.	.		.	.	.	.	.
.	.	.	.	.		.	.	.	.	.
.	.	.	.	.		.	.	.	.	.
m	$\underline{c}_m^{(1)}$	$\underline{c}_m^{(2)}$	. . .	$\underline{c}_m^{(n)}$		a_{m1}	a_{m2}	. . .	a_{mk}	n
						a_{+1}	a_{+2}	. . .	a_{+k}	mn
1	$\underline{f}_1^{(1)}$	$\underline{f}_1^{(2)}$	. . .	$\underline{f}_1^{(n)}$	a_{+1}					
2	$\underline{f}_2^{(1)}$	$\underline{f}_2^{(2)}$	. . .	$\underline{f}_2^{(n)}$	a_{+2}					
.	.	.	.	.	.					
.	.	.	.	.	.					
.	.	.	.	.	.					
k	$\underline{f}_k^{(1)}$	$\underline{f}_k^{(2)}$	. . .	$\underline{f}_k^{(n)}$	a_{+k}					
	m	m	. . .	m	nm					

EXAMPLE 1.2.1. In this example we present the data of 6 Dutch newspapers in 1964.

1. De Telegraaf
2. De Volkskrant
3. Het Parool
4. Algemeen Handelsblad
5. N.R.C.
6. De Waarheid.

We consider the following categories

C_1. Dutch economy
C_2. Dutch politics
C_3. remaining Dutch news
C_4. America
C_5. Africa & Middle East
C_6. remaining foreign news.

The source of the following table of data is KNEEPKENS (1975).

<u>Table 1.2.2</u>. Example of data of 6 Dutch newspapers.

	newspaper							category						
date	1	2	3	4	5	6		1	2	3	4	5	6	
2-1-'64	C_5	C_1	C_4	C_5	C_4	C_1		2	0	0	2	2	0	6
7-1-'64	C_6	C_6	C_6	C_6	C_1	C_1		2	0	0	0	0	4	6
11-1-'64	C_4	C_4	C_4	C_4	C_4	C_4		0	0	0	6	0	0	6
17-1-'64	C_6	C_1	C_5	C_5	C_6	C_1		2	0	0	0	2	2	6
23-1-'64	C_1	C_5	C_6	C_6	C_6	C_3		1	0	1	0	1	3	6
29-1-'64	C_5	C_5	C_6	C_5	C_6	C_5		0	0	0	0	4	2	6
4-2-'64	C_3	C_3	C_3	C_3	C_1	C_1		2	0	4	0	0	0	6
10-2-'64	C_3	C_3	C_3	C_3	C_3	C_3		0	0	6	0	0	0	6
15-2-'64	C_3	C_5	C_5	C_5	C_6	C_1		1	0	1	0	3	1	6
21-2-'64	C_2	C_6	C_6	C_6	C_6	C_2		0	2	0	0	0	4	6
27-2-'64	C_2	C_2	C_6	C_2	C_2	C_1		1	4	0	0	0	1	6
category								11	6	12	8	12	17	66
1	1	2	0	0	2	6	11							
2	2	1	0	1	1	1	6							
3	3	2	2	2	1	2	12							
4	1	1	2	1	2	1	8							
5	2	3	2	4	0	1	12							
6	2	2	5	3	5	0	17							
	11	11	11	11	11	11	66							

Inspection of the data of example 1.2.1. leads to the following remarks.

i. Between two subsequent days of observation, there are each time four days on which no observation was made (not counting sundays). This is done to satisfy as good as possible the assumption (1.1.3).

ii. On 11-1-'64 and 10-2-'64 all the newspapers chose a subject from the same category (C_4 and C_3 resp.). Because these observations cannot contribute to the detection of deviations from H_0, they are "useless", and they should play no role in our procedure. In section 2.5 we shall show that we may delete such observations when we use one of the tests that we have developed for this problem.

After deletion of these observations we have

Table 1.2.3. Data of example 1.2.1 after deletion of "useless" observations.

	newspaper							category						
date	1	2	3	4	5	6		1	2	3	4	5	6	
2-1-'64	C_5	C_1	C_4	C_5	C_4	C_1		2	0	0	2	2	0	6
7-1-'64	C_6	C_6	C_6	C_6	C_1	C_1		2	0	0	0	0	4	6
17-1-'64	C_6	C_1	C_5	C_5	C_6	C_1		2	0	0	0	2	2	6
23-1-'64	C_1	C_5	C_6	C_6	C_6	C_3		1	0	1	0	1	3	6
29-1-'64	C_5	C_5	C_6	C_5	C_6	C_5		0	0	0	0	4	2	6
4-2-'64	C_3	C_3	C_3	C_3	C_1	C_1		2	0	4	0	0	0	6
15-2-'64	C_3	C_5	C_5	C_5	C_6	C_1		1	0	1	0	3	1	6
21-2-'64	C_2	C_6	C_6	C_6	C_6	C_2		0	2	0	0	0	4	6
27-2-'64	C_2	C_2	C_6	C_2	C_2	C_1		1	4	0	0	0	1	6
category								11	6	6	2	12	17	54
1	1	2	0	0	2	6	11							
2	2	1	0	1	1	1	6							
3	2	1	1	1	0	1	6							
4	0	0	1	0	1	0	2							
5	2	3	2	4	0	1	12							
6	2	2	5	3	5	0	17							
	9	9	9	9	9	9	54							

1.3. THE 'CONDITIONAL' SITUATION

Sometimes an experimental setup leads directly to the situation which we have in section 1.1 after the conditioning on the event $\underline{A} = A$. We mean that a researcher may determine the elements of A in advance and perform an experiment in which the outcomes are of the type (1.1.18), which have each, under a 'H_0', the same probability.

Without changing the notation, the random variable $\underline{t}_{ij}^{(\nu)}$ would now mean

$$(1.3.1) \qquad \underline{t}_{ij}^{(\nu)} = \begin{cases} 1 & \text{if in the outcome on the i'th day } C_j \text{ is in the } \nu\text{'th place;} \\ 0 & \text{otherwise.} \end{cases}$$

The random variables $\underline{f}_j^{(\nu)}$ may be defined as in (1.1.20).

In fact, once we have conditioned on the event $\underline{A} = A$, it is not possible to discriminate between the two kinds of experiments and the two testing problems anymore, apart from the fact that the alternatives we are interested in may be chosen differently. This 'conditional' situation will be the starting point of the theory in chapter 2.

We give an example of this situation.

EXAMPLE 1.3.1. Suppose that a foreman distributes each morning n jobs among n workers. Among the n jobs are a_{ij} of the kind C_j, on the i'th day. We would now want to test the hypothesis that the foreman distributes the jobs at random, for instance against the alternative that some worker gets jobs assigned to him that are persistently of the same kind.

1.4. THE PROPOSED TEST

For the testing problems described in sections 1.1 and 1.2 we propose the following test-statistic

$$(1.4.1) \qquad \underline{v} \overset{d}{=} \sum_{j=1}^{k} \sum_{\nu=1}^{n} \frac{(\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n})^2}{\frac{a_{+j}}{n}} .$$

If, under the experimental situation of section 1.1 some category has not been chosen, we have $a_{+j} = 0$ and $(\underline{f}_j^{(\nu)} - a_{+j}/n)^2 \equiv 0$. In those cases we define $(\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n})^2 / \frac{a_{+j}}{n} \equiv 0$.

It is easily shown that, under H_0, $E\underline{f}_j^{(\nu)} = \frac{a_{+j}}{n}$, so our test-statistic has the form of the traditional chi-squared statistic.

In chapter 4 and 6 we shall show that the asymptotic distribution of $\frac{n-1}{n} \underline{v}|H_0$ is in a special case a χ^2-distribution with (n-1)(k-1) degrees of freedom, and in general the distribution of a linear combination of independent χ^2-variables. In both cases the following approximation is an improvement on these asymptotic distributions.

Approximate $c\underline{v}$ by a χ^2-distribution with η degrees of freedom, where c and η are determined such that, under H_0,

$$E c\underline{v} = cE\underline{v} = E\underline{\chi}^2[\eta] = \eta \tag{1.4.2}$$

and

$$\sigma^2(c\underline{v}) = c^2\sigma^2(\underline{v}) = \sigma^2(\underline{\chi}^2[\eta]) = 2\eta, \tag{1.4.3}$$

thus equating the first two moments of $c\underline{v}$ to those of $\chi^2[\eta]$. Hence

$$c = 2E\underline{v}/\sigma^2(\underline{v}), \tag{1.4.4}$$

$$\eta = 2(E\underline{v})^2/\sigma^2(\underline{v}). \tag{1.4.5}$$

$E\underline{v}$ and $\sigma^2(\underline{v})$ are given, under H_0, by

$$E\underline{v} = n \sum_{j=1}^{k} \frac{S_j}{E_j} \tag{1.4.6}$$

$$\sigma^2(\underline{v}) = \frac{2n^2}{n-1}\{\sum_{j=1}^{k} \frac{S_j^2 - T_j}{E_j} + \sum_{j\neq l}\sum \frac{S_{jl}^2 - T_{jl}}{E_j E_l}\}, \tag{1.4.7}$$

where

$$E_j = \frac{a_{+j}}{n} = \sum_{i=1}^{m} \frac{a_{ij}}{n}, \tag{1.4.8}$$

$$S_j = n^{-2} \sum_{i=1}^{m} a_{ij}(n-a_{ij}), \tag{1.4.9}$$

$$T_j = n^{-4} \sum_{i=1}^{m} a_{ij}^2(n-a_{ij})^2, \tag{1.4.10}$$

$$S_{jl} = n^{-2} \sum_{i=1}^{m} a_{ij}a_{il}, \tag{1.4.11}$$

$$T_{jl} = n^{-4} \sum_{i=1}^{m} a_{ij}^2 a_{il}^2. \tag{1.4.12}$$

Critical values and tail-probabilities of the distribution of $\underline{v}$ may be approximated using this method.

For practical calculations of $\underline{v}$, we may use the following formula

$$(1.4.13) \qquad \underline{v} \equiv n \sum_{j=1}^{k} \frac{\sum_{\nu=1}^{n} \{\underline{f}_j^{(\nu)}\}^2}{a_{+j}} - nm \; .$$

Finally we apply the test to the data of example 1.2.1.

<u>EXAMPLE 1.4.1</u>. For the data of example 1.2.1, after the deletion of "useless" observations (see table 1.2.3), our test-statistic takes the value $v = 33.19$. In this case we have $E(\underline{v}|H_0) = 20.11$ and $\sigma^2(\underline{v}|H_0) = 19.59$, so that $c = 2E/\sigma^2 = 2.0526$, $\eta = 41.27$ and $cv = 68.13$. The critical value of the $\chi^2[41.27]$ distribution for $\alpha = 0.05$ is equal to 57.26, so H_0 is rejected. The right tail-probability of 68.13 for the $\chi^2[41.27]$ distribution is equal to 0.0053.

An estimate of the right tail-probability of 33.19 in the exact distribution is equal to 0.007, indicating a close fit of the approximation. The estimate was obtained from 1000 simulated drawings from the exact distribution of $\underline{v}$.

CHAPTER 2

THEORY: PRELIMINARIES

2.1. THE PROBLEM

We consider a sequence $E_1,\dots,E_m$ of m independent experiments. The possible outcomes of E_i $(i=1,\dots,m)$ are the permutations of the n characters

(2.1.1) $$\underbrace{C_1\dots C_1}_{a_{i1}\times}\ \underbrace{C_2\dots C_2}_{a_{i2}\times}\ \dots\ \underbrace{C_k\dots C_k}_{a_{ik}\times}$$

where C_j $(j=1,\dots,k)$ occurs a_{ij} times, with $0 \le a_{ij} < n$ and $\sum_{j=1}^{k} a_{ij} = n$. Because of the repetitions of the characters in the permutations, we shall call such a permutation a 'word'.

As indicated by the notation, the characters C_j $(j=1,\dots,k)$ and the length n of the words will be the same for all experiments, but the numbers a_{ij} may differ from experiment to experiment. In asymptotic considerations we shall let $m\to\infty$ with n and k fixed.

The indices i,h will be used throughout this work to index the experiments, the indices j and l to index the characters and $\nu,\mu \in \{1,\dots,n\}$ to indicate the ν'th and μ'th place in a word. So we shall always have

(2.1.2) $$i,h \in \{1,\dots,m\};\ j,l \in \{1,\dots,k\};\ \nu,\mu \in \{1,\dots,n\}.$$

By this convention we can use these symbols without further explanation. Subject to this convention each given formula will hold for each value that the indices occurring in it can take, unless otherwise is indicated.

The properties of the numbers a_{ij} may be summarized by

(2.1.3) $$\vec{a}_i \overset{d}{=} (a_{i1},\dots,a_{ik})' \in \{(a_1,\dots,a_k)' \,|\, a_j \in \{0,\dots,n-1\},\ \sum_{j=1}^{k} a_j = n\}.$$

The number N_i of different words for E_i is

$$N_i = \frac{n!}{a_{i1}!\ a_{i2}!\ \ldots\ a_{ik}!}\ . \qquad (2.1.4)$$

Let

$$R_i \overset{d}{=} \{1,\ldots,N_i\} \qquad (2.1.5)$$

and let

$$\Omega_i \overset{d}{=} \{\pi_{i1},\ldots,\pi_{iN_i}\} \qquad (2.1.6)$$

denote the set of words pertaining to E_i, in lexicographical order. Then

$$\Omega \overset{d}{=} \Omega_1 \times \ldots \times \Omega_m \qquad (2.1.7)$$

is the set of all possible outcomes of the composite experiment

$$E \overset{d}{=} (E_1,\ldots,E_m). \qquad (2.1.8)$$

To complete the mathematical model we shall use, we look at the class of all probability distributions on Ω, with $E_1,\ldots,E_m$ independent. Let P_i be the class of probability distributions on Ω_i

$$P_i \overset{d}{=} \{(p_1,\ldots,p_{N_i})' | p_r \geq 0,\ r \in R_i,\ \sum_{r=1}^{N_i} p_r = 1\} \qquad (2.1.9)$$

and let $\underline{\omega}_i$ be random on Ω_i with

$$P(\underline{\omega}_i = \pi_{ir}) = p_{ir},\ r \in R_i,\ \vec{p}_i \overset{d}{=} (p_{i1},\ldots,p_{iN_i})' \in P_i. \qquad (2.1.10)$$

Then, according to the independence of $E_1,\ldots,E_m$ we have

$$P((\underline{\omega}_1,\ldots,\underline{\omega}_m) = (\pi_{1r_1},\ldots,\pi_{mr_m})) = \prod_{i=1}^{m} p_{ir_i} \qquad (2.1.11)$$

with $r_i \in R_i$ and $\vec{p}_i \in P_i$, and this is the class of probability distributions we consider. It will be indicated by

$$P = P_1 \times \ldots \times P_m. \qquad (2.1.12)$$

In order to formulate the hypotheses about $\mathcal{P}$ which we want to consider, we introduce parameters Δ_{ir} as follows

$$(2.1.13)\qquad \Delta_{ir} \overset{d}{=} p_{ir} - \frac{1}{N_i} \qquad (r \in \mathcal{R}_i),$$

with, obviously

$$(2.1.14)\qquad -\frac{1}{N_i} \le \Delta_{ir} \le 1 - \frac{1}{N_i}\ ; \qquad \sum_{r=1}^{N_i} \Delta_{ir} = 0.$$

Let

$$(2.1.15)\qquad \mathcal{D}_i \overset{d}{=} \{(\Delta_1,\ldots,\Delta_{N_i})' \,|\, -\frac{1}{N_i} \le \Delta_r \le 1 - \frac{1}{N_i},\ r \in \mathcal{R}_i,\ \sum_{r=1}^{N_i} \Delta_r = 0\},$$

and

$$(2.1.16)\qquad \vec{\Delta}_i \overset{d}{=} (\Delta_{i1},\ldots,\Delta_{iN_i})' \in \mathcal{D}_i,$$

then every element of

$$(2.1.17)\qquad \mathcal{D} \overset{d}{=} \mathcal{D}_1 \times \ldots \times \mathcal{D}_m$$

specifies a distribution from $\mathcal{P}$ and v.v. .

The hypothesis to be tested is

$$(2.1.18)\qquad H_0:\quad \forall_i \quad \vec{\Delta}_i = \vec{0}\ .$$

The widest class of alternative hypotheses is of course

$$(2.1.19)\qquad H_1:\quad \exists_i \quad \vec{\Delta}_i \neq \vec{0}\ ,$$

but this class is too amorphous for our purposes. In order to formulate a useful subclass of H_1, we first introduce elementary random variables on Ω_i, which are used for building up test-statistics.

Let, for all i, j and ν, $t_{ij}^{(\nu)}: \Omega_i \to \{0,1\}$ be defined as

$$(2.1.20)\qquad t_{ij}^{(\nu)}(\pi) \overset{d}{=} \begin{cases} 1 \text{ if in } \pi\ C_j \text{ occurs in the } \nu\text{'th place;} \\ 0 \text{ otherwise.} \end{cases}$$

The following relations are then easily proved

$$(2.1.21)\qquad \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir}) = N_i \frac{a_{ij}}{n},$$

$$(2.1.22)\qquad \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir})\ t_{ij}^{(\mu)}(\pi_{ir}) = N_i \frac{a_{ij}(a_{ij}-1)}{n(n-1)} \qquad (\nu\neq\mu),$$

$$(2.1.23)\qquad \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir})\ t_{il}^{(\mu)}(\pi_{ir}) = N_i \frac{a_{ij}a_{il}}{n(n-1)} \qquad (\nu\neq\mu, j\neq l).$$

Now let

$$(2.1.24)\qquad \delta_{ijl}^{(\nu,\mu)} \overset{d}{=} \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir})\ t_{il}^{(\mu)}(\pi_{ir})\ \Delta_{ir}$$

and

$$(2.1.25)\qquad \delta_{ij}^{(\nu,\mu)} \overset{d}{=} \delta_{ijj}^{(\nu,\mu)}; \qquad \delta_{ij}^{(\nu)} \overset{d}{=} \delta_{ijj}^{(\nu,\nu)}.$$

Let the random variables induced by $\mathcal{P}_i$ and (2.1.20) be denoted by $\underline{t}_{ij}^{(\nu)}$, then we have, under $\mathcal{P}_i$,

$$(2.1.26)\qquad P(\underline{t}_{ij}^{(\nu)} = 1) = \frac{a_{ij}}{n} + \delta_{ij}^{(\nu)},$$

$$(2.1.27)\qquad P(\underline{t}_{ij}^{(\nu)} = \underline{t}_{ij}^{(\mu)} = 1) = \frac{a_{ij}(a_{ij}-1)}{n(n-1)} + \delta_{ij}^{(\nu,\mu)} \qquad (\nu\neq\mu),$$

$$(2.1.28)\qquad P(\underline{t}_{ij}^{(\nu)} = \underline{t}_{il}^{(\mu)} = 1) = \frac{a_{ij}a_{il}}{n(n-1)} + \delta_{ijl}^{(\nu,\mu)} \qquad (\nu\neq\mu, j\neq l).$$

Notice that

$$(2.1.29)\qquad P(\underline{t}_{ij}^{(\nu)} = \underline{t}_{il}^{(\nu)} = 1) = 0, \qquad (j\neq l).$$

The proof of these relations follows from the fact that the left-hand members are equal to the expected value of the product of the r.v.'s occurring in these expressions. For instance for (2.1.28) we have, using (2.1.13), (2.1.23) and (2.1.24)

$$P(\underline{t}_{ij}^{(\nu)} = \underline{t}_{il}^{(\mu)} = 1) = E\underline{t}_{ij}^{(\nu)}\underline{t}_{il}^{(\mu)} = \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir})\ t_{il}^{(\mu)}(\pi_{ir})\ p_{ir} =$$

$$= \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir})\ t_{il}^{(\mu)}(\pi_{ir})\ \left(\frac{1}{N_i} + \Delta_{ir}\right) = \frac{a_{ij}a_{il}}{n(n-1)} + \delta_{ijl}^{(\nu,\mu)}.$$

The other two relations are special cases.

Note that under H_0 all δ's are equal to 0. Thus if

$$(2.1.30) \qquad H_0': \quad \forall_{ijl\nu\mu} \quad \delta_{ijl}^{(\nu,\mu)} = 0,$$

then

$$(2.1.31) \qquad H_0 \Rightarrow H_0'$$

but not the other way around. In our setup it is not possible to discriminate between the elements of H_0' and the elements of H_0. In fact, the asymptotic distributions of our test-statistics for elements of $H_0' \setminus H_0$ are the same as the distributions under H_0. The "alternatives" in $H_0' \setminus H_0$ therefore cannot be detected by our methods.

We can now tentatively formulate the alternative hypotheses we wish to consider.

We shall say that place ν has a *preference* for or an *aversion* from character C_j respectively if

$$(2.1.32) \qquad \sum_{i=1}^{m} \delta_{ij}^{(\nu)} > 0 \qquad \text{or} \qquad < 0.$$

An aversion thus is the same as a negative preference. Now it is easily verified by means of (2.1.26) that

$$(2.1.33) \qquad \sum_{\nu=1}^{n} \delta_{ij}^{(\nu)} = 0 \ ; \qquad \sum_{j=1}^{k} \delta_{ij}^{(\nu)} = 0 \ ;$$

thus preferences cannot occur in one place only, they are automatically balanced by aversions in other places and vice versa. In fact, preferences as defined above are relative preferences of the places with respect to the preferences of other places, not preferences in an absolute sense.

These considerations lead us to consider the following class of alternative hypotheses

$$(2.1.34) \qquad H_1': \quad \exists_{j\nu} \quad \left| \sum_{i=1}^{m} \delta_{ij}^{(\nu)} \right| > 0.$$

The statistic

$$(2.1.35) \qquad \sum_{i=1}^{m} (\underline{t}_{ij}^{(\nu)} - \frac{a_{ij}}{n})$$

has according to (2.1.26) the expected value $\sum_{i=1}^{m} \delta_{ij}^{(\nu)}$ and is thus obviously a good building stone for a test-statistic.

Defining

$$(2.1.36) \qquad \underline{f}_j^{(\nu)} \overset{d}{=} \sum_{i=1}^{m} \underline{t}_{ij}^{(\nu)} \; ; \qquad a_{+j} \overset{d}{=} \sum_{i=1}^{m} a_{ij} \, ,$$

we have

$$(2.1.37) \qquad E(\underline{f}_j^{(\nu)} | H_0) = E(\underline{f}_j^{(\nu)} | H_0') = \frac{a_{+j}}{n}$$

and an intuitively attractive test-statistic is

$$(2.1.38) \qquad \sum_{\nu=1}^{n} \sum_{j=1}^{k} (\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n})^2 / \frac{a_{+j}}{n} \, .$$

This statistic has the form of the traditional chi-squared statistic: it will assume large values under H_1' and large terms will indicate the preferences and aversions which cause the sum to be large.

It would be too much, however, to expect this statistic to have a chi-squared distribution and it will be shown later that it has a more complicated one (under H_0 as well as under H_1'), which can nevertheless be approximated by means of a modified chi-squared distribution.

The choice of a quadratic form in the $\underline{f}_j^{(\nu)}$ will be shown in chapter 4 to be indicated by the simultaneous asymptotic normality of the $\underline{f}_j^{(\nu)}$ and other theoretical considerations. Several degrees of generalisation of (2.1.38) can then be considered. The most promising one is

$$(2.1.39) \qquad \sum_{\nu=1}^{n} \sum_{j=1}^{k} g_j (\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n})^2 ,$$

where the g_j are weighing coefficients for the categories, which will generally (as in (2.1.38)) depend on the a_{+j}/n, but which may also express the experimenter's emphasis on certain characters as compared to others. One might also choose the weights dependent, not only on the characters, but also on the places: $g_j^{(\nu)}$ instead of g_j. We do not, however, elaborate this case in this thesis. In every trial every place occurs exactly once, but the frequencies of the characters may be different from trial to trial. In the applications which led to the development of our tests the places were equivalent: changing their order should have no influence on the experimental situation. Therefore, although the general theory developed later also covers this case, we do not, at this moment, aim at generalisations where different weights are attached to the places.

A further generalisation is to allow cross-terms in the test-statistic

$$(2.1.40)\qquad \sum_{\nu=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{k} g_{jl}(\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n})(\underline{f}_l^{(\nu)} - \frac{a_{+l}}{n}) .$$

The behaviour of such a test-statistic is far more complicated than that of (2.1.39) and as the result of theoretical considerations (mostly of an asymptotic character) the form (2.1.39) will emerge as the most useful one. The choice of weighing coefficients will be considered in chapters 4 and 5. Some special cases are treated in chapter 6.

The most general quadratic form is, of course

$$(2.1.41)\qquad \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{k} g_{jl}^{(\nu,\mu)}(\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n})(\underline{f}_l^{(\mu)} - \frac{a_{+l}}{n}) .$$

Although test-statistics of this generality are difficult to interpret and therefore of little practical use, the theory which will be developed in later chapters will completely cover this general case. For practical purposes specialisation to the form (2.1.39) is recommended and special attention is paid to this test-statistic and to (2.1.36).

As will appear later, tests based on (2.1.39) will, under acceptable conditions for the g_j, be consistent for $m \to \infty$ if

$$(2.1.42)\qquad \exists_{j\nu} \quad \left|\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \delta_{ij}^{(\nu)}\right| \to \infty \qquad \text{as} \qquad m \to \infty.$$

This holds e.g. for (2.1.38).

2.2. NOTATION AND SIMPLE RESULTS

Notational conventions.

If X_i denotes any quantity (scalar, vector, r.v., matrix etc.) indexed by the variable i which ranges (for instance) over the index set $\{1,\ldots,m\}$, we shall frequently use the derived quantities X_+, X_* and $X_.$, defined by

$$(2.2.1)\qquad X_+ = \sum_{i=1}^{m} X_i ,$$

$$(2.2.2)\qquad X_* = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} X_i = X_+/\sqrt{m} ,$$

$$(2.2.3)\qquad X_{\bullet} = \frac{1}{m}\sum_{i=1}^{m} X_i = X_+/m.$$

Note that X_+, X_* and $X_{\bullet}$ all depend on m, though this is not apparent from this notation.

Furthermore, I_n denotes the identity matrix of order n, $0_{n,k}$ is the $n \times k$ matrix consisting of zero's ($0_n \stackrel{d}{=} 0_{n,n}$), and $1_{n,k}$ the $n \times k$ matrix of one's ($1_n \stackrel{d}{=} 1_{n,n}$). We use the symbol $\otimes$ to denote the Kronecker product of a $p \times q$ matrix $A = (a_{\alpha\beta})$ and an $r \times s$ matrix B,

$$(2.2.4)\qquad A \otimes B \stackrel{d}{=} (a_{\alpha\beta}B),$$

i.e. $A \otimes B$ is a $pr \times qs$ matrix, expressible as a partitioned matrix with $a_{\alpha\beta}B$ as the (α,β)th partition, $\alpha = 1,\dots,p$, $\beta = 1,\dots,q$.

We consider the r.v.'s defined by (2.1.20) as

$$(2.2.5)\qquad \underline{t}_{ij}^{(\nu)} = \begin{cases} 1 & \text{if in the word obtained at the i'th trial, the character } C_j \text{ occurs in the } \nu\text{'th place;} \\ 0 & \text{otherwise.} \end{cases}$$

Let, for all i, j, ν,

$$(2.2.6)\qquad \tilde{\underline{t}}_{ij}^{(\nu)} \stackrel{d}{=} \underline{t}_{ij}^{(\nu)} - \frac{a_{ij}}{n},$$

and let

$$(2.2.7)\qquad \vec{\underline{t}}_i \stackrel{d}{=} (\tilde{\underline{t}}_{i1}^{(1)},\dots,\tilde{\underline{t}}_{ik}^{(1)};\tilde{\underline{t}}_{i1}^{(2)},\dots,\tilde{\underline{t}}_{ik}^{(2)};\dots;\tilde{\underline{t}}_{i1}^{(n)},\dots,\tilde{\underline{t}}_{ik}^{(n)})'.$$

Furthermore, we shall consider

$$(2.2.8)\qquad \vec{\underline{t}}_* \stackrel{d}{=} \frac{1}{\sqrt{m}}\sum_{i=1}^{m} \vec{\underline{t}}_i$$

We shall use the following real, symmetric, $k \times k$ matrix of weighing factors

$$(2.2.9)\qquad G \stackrel{d}{=} \begin{pmatrix} g_{11} & g_{12} & \cdot & \cdot & \cdot & g_{1k} \\ g_{21} & g_{22} & \cdot & \cdot & \cdot & g_{2k} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ g_{k1} & g_{k2} & \cdot & \cdot & \cdot & g_{kk} \end{pmatrix}, \qquad (g_{jl}=g_{lj}).$$

Let

(2.2.10) $\quad Q \overset{d}{=} I_n \otimes G.$

The test-statistic, defined in (2.1.40), may then, apart from a factor m^{-1}, be written as

(2.2.11) $$\underline{v} \equiv \underline{v}(G) \equiv \vec{\underline{t}}_*' Q \vec{\underline{t}}_* \equiv \frac{1}{m} \sum_{\nu=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{k} g_{jl} (\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n}) (\underline{f}_l^{(\nu)} - \frac{a_{+l}}{n}).$$

Though it is not essential w.r.t. our problem, we shall wish, for practical reasons, that

(2.2.12) $\quad \vec{\underline{t}}_*' Q \vec{\underline{t}}_* \geq 0,$

with probability one. This means that we have to choose G such that Q is non-negative definite (n.n.d). (A $k \times k$ matrix Q is n.n.d iff $\vec{x}' Q \vec{x} \geq 0$ for each $\vec{x} \in \mathbb{R}^k$).

In most of the theory it is irrelevant whether Q has the structure as in (2.2.10) or is an arbitrary n.n.d, real symmetric matrix. So from now on we shall suppose that Q is an arbitrary real, symmetric, n.n.d matrix.

We define the test function $\underline{\varphi}_{m,Q}$:

(2.2.13) $$\underline{\varphi}_{m,Q} = \begin{cases} 1 \text{ if } \vec{\underline{t}}_*' Q \vec{\underline{t}}_* \geq k_{1-\alpha}(m,Q); \\ 0 \text{ otherwise,} \end{cases}$$

where $k_{1-\alpha}(m,Q)$ is determined as the smallest value such that

(2.2.14) $\quad E(\underline{\varphi}_{m,Q} | H_0) \leq \alpha.$

Sometimes we shall randomise the test for theoretical purposes, i.e.

(2.2.15) $$\underline{\varphi}_{m,Q} = \begin{cases} 1 & \\ \gamma(m,Q) & \text{if } \vec{\underline{t}}_*' Q \vec{\underline{t}}_* \begin{matrix} > \\ = \\ < \end{matrix} k_{1-\alpha}'(m,Q) \\ 0 & \end{cases} ,$$

where $k_{1-\alpha}'(m,Q)$ is the highest possible outcome of $\vec{\underline{t}}_*' Q \vec{\underline{t}}_*$ which is smaller than $k_{1-\alpha}(m,Q)$, and $\gamma(m,Q)$ is determined such that in (2.2.14) the equality sign holds. It will be clear from the context whether we use $\underline{\varphi}_{m,Q}$ defined by (2.2.15) or by (2.2.13). The decision rule for the resulting level-α test is derived from $\underline{\varphi}_{m,Q}$ in the usual way.

We define the vector of expectations of $\underline{t}_{ij}^{(\nu)}$ in the same way as (2.2.7)

(2.2.16) $\quad \vec{\delta}_i \overset{d}{=} (\delta_{i1}^{(1)},\ldots,\delta_{ik}^{(1)};\delta_{i1}^{(2)},\ldots,\delta_{ik}^{(2)};\ldots;\delta_{i1}^{(n)},\ldots,\delta_{ik}^{(n)})'$.

It follows directly from (2.1.26), that

(2.2.17) $\quad E\vec{\underline{t}}_* = \vec{\delta}_*$,

which reduces, under H_0, to

(2.2.18) $\quad E(\vec{\underline{t}}_*|H_0) = \vec{0}$.

It is useful to define

(2.2.19) $\quad \vec{\underline{u}}_i \overset{d}{=} \vec{\underline{t}}_i - \vec{\delta}_i$, $\qquad (\vec{\underline{u}}_* \equiv \vec{\underline{t}}_* - \vec{\delta}_*)$.

The matrix of variances and covariances of the components of a vector of r.v.'s $\vec{\underline{x}}$ will be called the dispersion matrix of $\vec{\underline{x}}$, and will be denoted by

(2.2.20) $\quad D(\vec{\underline{x}})$.

In particular we define for each i

(2.2.21) $\quad \Sigma_{1i} \overset{d}{=} D(\vec{\underline{t}}_i)$,

(2.2.22) $\quad \Sigma_{0i} \overset{d}{=} D(\vec{\underline{t}}_i|H_0)$.

The entries of Σ_{1i} and Σ_{0i} may be found from the following moments, which may be derived from (2.1.26),...,(2.1.28), together with the obvious relation (cf.(2.1.29))

(2.2.23) $\quad P(\underline{t}_{ij}^{(\nu)} = \underline{t}_{il}^{(\nu)} = 1) = 0,$ $\qquad (j\neq l)$.

The moments are

(2.2.24) $\quad \sigma^2(\underline{t}_{ij}^{(\nu)}) = \frac{a_{ij}}{n} - \frac{a_{ij}^2}{n^2} - 2\,\delta_{ij}^{(\nu)}\frac{a_{ij}}{n} + \delta_{ij}^{(\nu)} - (\delta_{ij}^{(\nu)})^2$, (2.2.25) $\operatorname{cov}(\underline{t}_{ij}^{(\nu)},\underline{t}_{ij}^{(\mu)}) =$ $(\nu\neq\mu)$

$$= -\frac{a_{ij}(n-a_{ij})}{n^2(n-1)} + \delta_{ij}^{(\nu,\mu)} - \frac{a_{ij}}{n}(\delta_{ij}^{(\nu)}+\delta_{ij}^{(\mu)}) - \delta_{ij}^{(\nu)}\delta_{ij}^{(\mu)} ,$$

(2.2.26) $\operatorname{cov}(\underline{t}_{ij}^{(\nu)},\underline{t}_{il}^{(\nu)}) =$ $(j\neq l)$

$$= -\frac{a_{ij}a_{il}}{n^2} - \frac{a_{ij}}{n}\delta_{il}^{(\nu)} - \frac{a_{il}}{n}\delta_{ij}^{(\nu)} - \delta_{ij}^{(\nu)}\delta_{il}^{(\nu)} ,$$

(2.2.27) $\operatorname{cov}(\underline{t}_{ij}^{(\nu)},\underline{t}_{il}^{(\mu)}) =$ $(j\neq l,\nu\neq\mu)$

$$= \frac{a_{ij}a_{il}}{n^2(n-1)} + \delta_{ijl}^{(\nu,\mu)} - \frac{a_{ij}}{n}\delta_{il}^{(\mu)} - \frac{a_{il}}{n}\delta_{ij}^{(\nu)} - \delta_{ij}^{(\nu)}\delta_{il}^{(\mu)} .$$

Σ_{0i} follows from Σ_{1i} by omitting all terms containing a δ (cf.(2.1.18)). Let, for $n \geq 2$,

$$\text{(2.2.28)} \quad N \overset{d}{=} \begin{pmatrix} 1 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & 1 & \cdots & -\frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & 1 \end{pmatrix} ,$$

an $n \times n$ matrix, then

$$\text{(2.2.29)} \quad N^2 = \frac{n}{n-1} N,$$

and N is of rank n-1, with eigenvalues 0, and $\frac{n}{n-1}$ with multiplicity n-1. Furthermore, let

$$\text{(2.2.30)} \quad K_i = \frac{1}{n^2} \begin{pmatrix} a_{i1}(n-a_{i1}) & -a_{i1}a_{i2} & \cdots & -a_{i1}a_{ik} \\ -a_{i2}a_{i1} & a_{i2}(n-a_{i2}) & \cdots & -a_{i2}a_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{ik}a_{i1} & -a_{ik}a_{i2} & \cdots & a_{ik}(n-a_{ik}) \end{pmatrix}.$$

We then have, as can easily be verified

(2.2.31) $\quad D(\vec{\underline{t}}_i | H_0) = \Sigma_{0i} = N \otimes K_i$,

(2.2.32) $\quad D(\vec{\underline{t}}_* | H_0) = \Sigma_{0\bullet} = N \otimes K_\bullet$.

Notice that $\Sigma_{0\bullet}$ is singular, because the sums over rows and column's of $K_\bullet$ and N are zero. More generally, Σ_{0i} and Σ_{1i} are singular because, for each fixed i, the following $n+k-1$ linear relationships hold true for the $n \times k$ random variables $\underline{t}_{ij}^{(\nu)}$, both under H_0 and H_1,

(2.2.33) $\quad \underline{t}_{i+}^{(\nu)} \equiv 1, \qquad \underline{t}_{ij}^{(+)} \equiv a_{ij}, \qquad \nu=1,\dots,n;\ j=1,\dots,k.$

<u>REMARK 2.2.1</u>. Not only the singularity of $\Sigma_{0\bullet}$, but also its rank will play a part in the considerations. What can be said about the rank of $\Sigma_{0\bullet}$? Let's first consider the determinant of a matrix which has the same structure as K_i,

(2.2.34)
$$\begin{vmatrix} a_1(n-a_1) & -a_1a_2 & \cdot\ \cdot\ \cdot & -a_1a_j \\ -a_2a_1 & a_2(n-a_2) & \cdot\ \cdot\ \cdot & -a_2a_j \\ \vdots & \vdots & \ddots & \vdots \\ -a_ja_1 & -a_ja_2 & \cdot\ \cdot\ \cdot & a_j(n-a_j) \end{vmatrix} = a_1a_2\dots a_j\, n^{j-1}\,(n - \sum_{l=1}^{j} a_l).$$

Using this relation, it can easily be shown that K_i is singular. Also the rank of K_i may now be found easily. Let k_i be the number of positive a_{ij}'s at the i'th experiment. Then, again using (2.2.34) it follows that

(2.2.35) $\quad \mathrm{rank}(K_i) = k_i - 1.$

Moreover, because in our case we have $n - \sum_{l=1}^{j} a_l \geq 0$ for $j=1,\dots,k$, (2.2.34) is non-negative for each j, $j=1,\dots,k$. It follows that the matrix K_i is non-negative definite.

For $\mathrm{rank}(\Sigma_{0i})$ we find

(2.2.36) $\quad \text{rank}(\Sigma_{0i}) = \text{rank}(N \otimes K_i) = (\text{rank } N)(\text{rank } K_i) = (n-1)(k_i-1).$

Now consider the matrix K_+. We have

(2.2.37) $\quad \vec{x}' K_+ \vec{x} = \vec{x}' K_1 \vec{x} + \vec{x}' K_2 \vec{x} + \ldots + \vec{x}' K_m \vec{x}.$

If r denotes the rank of K_+, there exist k-r linearly independent vectors $\vec{x}$, such that $K_+ \vec{x} = 0$, or $\vec{x}' K_+ \vec{x} = 0$. But this implies, because the matrices K_i are non-negative definite that $\vec{x}' K_i \vec{x} = 0$ for all i. This means that

(2.2.38) $\quad r = \text{rank}(K_+) \geq \max_i \text{rank}(K_i) = \max_i (k_i-1).$

Because in any case $\text{rank}(K_+) \leq k-1$ we find

(2.2.39) $\quad \max_i (k_i-1) \leq \text{rank}(K_+) \leq k-1$

and similar bounds for $\text{rank}(\Sigma_{0.})$. It follows that $\text{rank}(\Sigma_{0.})$ is not a fixed number, but has to be determined in each separate case.

For the expectation of $\underline{v}$ we find

(2.2.40) $\quad E\underline{v}(Q) = \text{trace } Q\Sigma_{1.} + \vec{\delta}_*' Q \vec{\delta}_*$,

which reduces, under H_0, to

(2.2.41) $\quad E(\underline{v}(Q)|H_0) = \text{trace } Q\Sigma_{0.}$.

And when $Q = I_n \otimes G$,

(2.2.42) $\quad E(\underline{v}(Q)|H_0) = \text{trace}(I_n \otimes G)(N \otimes K_.) = \frac{n}{m} \sum_{i=1}^{m} \text{trace } GK_i.$

2.3. SOME ASYMPTOTIC CONSIDERATIONS

The choice of the weighing coefficients g_{jl} in the matrix G, or, more generally, the choice of the matrix Q, will largely be determined by the nature of the asymptotic distribution of $\underline{v}(Q)$ as $m \to \infty$.

Because the vectors $\vec{\underline{t}}_1$, $\vec{\underline{t}}_2$, ... are not identically distributed, it is necessary to impose some asymptotic conditions. Therefore we only consider infinite sequences of experiments for which the following 2 assumptions hold.

ASSUMPTION 1. For all j and l the following finite limits exist,

(2.3.1) $$\lim_{m\to\infty} a_{\bullet j} = a_j > 0, \qquad \text{(say)},$$

(2.3.2) $$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} (a_{ij} - \frac{a_{+j}}{m})(a_{il} - \frac{a_{+l}}{m}) = d_{jl}, \qquad \text{(say)}.$$

The alternatives we consider satisfy

ASSUMPTION 2. For all j,l,ν and μ the following limits exist,

(2.3.3) $$\lim_{m\to\infty} \delta_{\bullet jl}^{(\nu,\mu)} = \zeta_{jl}^{(\nu,\mu)}, \qquad \text{(say)},$$

(2.3.4) $$\lim_{m\to\infty} \delta_{*j}^{(\nu)} = \delta_j^{(\nu)}, \qquad \text{(say)}.$$

In (2.3.4), but not in (2.3.3), we accept $|\delta_j^{(\nu)}| = \infty$. As a special case of (2.3.3) we have

(2.3.5) $$\zeta_j^{(\nu)} \overset{d}{=} \zeta_{jj}^{(\nu,\nu)} = \lim_{m\to\infty} \delta_{\bullet j}^{(\nu)}.$$

The vectors $\vec{\delta}$ and $\vec{\zeta}$ with components $\delta_j^{(\nu)}$ and $\zeta_j^{(\nu)}$ are constructed as in (2.2.16).

At first sight, these conditions may seem to be very strong. It is indeed very easy to construct examples of infinite sequences of experiments that do not satisfy these assumptions. However, we have to bear in mind that for statistical purposes, the asymptotic distributions are only necessary to provide a good approximation for the finite situation. Furthermore, there exist situations for which the conditions are trivially fulfilled. For instance in the case that $\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \ldots$ and $\vec{\Delta}_1 = \vec{\Delta}_2 = \vec{\Delta}_3 = \ldots$. Or, more generally, if

$$(\vec{a}_1,\vec{a}_2,\ldots,\vec{a}_m) = (\vec{a}_{m+1},\vec{a}_{m+2},\ldots,\vec{a}_{2m}) = (\vec{a}_{2m+1},\vec{a}_{2m+2},\ldots,\vec{a}_{3m}) = \ldots$$

and

$$(\vec{\Delta}_1,\vec{\Delta}_2,\ldots,\vec{\Delta}_m) = (\vec{\Delta}_{m+1},\vec{\Delta}_{m+2},\ldots,\vec{\Delta}_{2m}) = (\vec{\Delta}_{2m+1},\vec{\Delta}_{2m+2},\ldots,\vec{\Delta}_{3m}) = \ldots$$

So if we have m_0 observations, we could think of an infinite sequence of experiments, where the whole block of m_0 experiments is repeated infinitely many times. Then we would have, for instance,

$$(2.3.6) \qquad \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} a_{ij} = \frac{1}{m_0} \sum_{i=1}^{m_0} a_{ij}$$

etcetera. In this way, all the limiting values are equal to the values in the finite case. It may be expected that the approximations derived from the asymptotic distributions are then fairly good.

We leave it to the reader to verify that under assumption 1,

$$(2.3.7) \qquad \lim_{m\to\infty} D(\vec{\underline{t}}_*|H_0) = \lim_{m\to\infty} \Sigma_{0\cdot} = \Sigma_0 , \qquad \text{(say)},$$

exists. Under assumptions 1 & 2

$$(2.3.8) \qquad \lim_{m\to\infty} D(\vec{\underline{t}}_*) = \lim_{m\to\infty} \Sigma_{1\cdot} = \Sigma_1 , \qquad \text{(say)},$$

exists as well as

$$(2.3.9) \qquad \lim_{m\to\infty} E(\vec{\underline{t}}_*) = \lim_{m\to\infty} \vec{\delta}_* = \vec{\delta} ,$$

where the components of $\vec{\delta}$ may be $+\infty$ or $-\infty$. Also

$$(2.3.10) \qquad \lim_{m\to\infty} K_{\bullet} = K , \qquad \text{(say)},$$

exists, and we have

$$(2.3.11) \qquad \Sigma_0 = N \otimes K.$$

The alternatives satisfying assumption 2 determine a subset $\mathcal{A}$ of $\mathcal{D}$, the set of all possible alternatives, with (cf.(2.1.17)),

$$(2.3.12) \qquad \mathcal{D} = \bigtimes_{i=1}^{m} \mathcal{D}_i ,$$

for infinite sequences of experiments. So we have

$$(2.3.13) \qquad \mathcal{A} = \{d \in \mathcal{D} | d \text{ satisfies assumption 2}\}.$$

It is convenient to split up $\mathcal{A}$ still further. Define

(2.3.14) $\quad A_1 \overset{d}{=} \{a \in A \mid \vec{\delta}\cdot\vec{\delta} = \infty\}$,

(2.3.15) $\quad A_2 \overset{d}{=} \{a \in A \mid 0 < \vec{\delta}\cdot\vec{\delta} < \infty\}$,

(2.3.16) $\quad A_3 \overset{d}{=} \{a \in A \mid \vec{\delta}\cdot\vec{\delta} = 0\}$.

Note that for alternatives in $A_2 \cup A_3$ we have

(2.3.17) $\quad \vec{\zeta} = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \vec{\delta}_i = \vec{0}$.

When an expectation is taken with respect to a particular alternative $a \in A$, we shall sometimes write E_a, and we shall write E_0 for the expectation under H_0.

We shall establish the following results.

i. When Q is non-singular, $\underline{v}(Q)$ is consistent against each alternative in A_1. When Q is singular, $\underline{v}(Q)$ may, or may not, be consistent against each alternative in A_1. In a number of cases (depending on the structure of Q and the particular alternative) it can be shown that the asymptotic distribution of $\underline{v}(Q)$ is the (standard) normal distribution (after a proper transformation).

ii. For alternatives in A_2, $\underline{v}(Q)$ has asymptotically a non-central χ^2-distribution, or the distribution of a linear combination of independent non-central χ^2-variables. The test based on $\underline{v}(Q)$ is not consistent, but its asymptotic power may still considerably exceed the level of significance.

iii. For alternatives in A_3, $\underline{v}(Q)$ has asymptotically a central χ^2-distribution or the distribution of a linear combination of central χ^2-variables. The test is not consistent and the asymptotic power remains close to the level of significance.

2.4. A SPECIAL CLASS OF ALTERNATIVES

In testing problems that are comparable to our situation, the concept of contiguous alternatives is often introduced, mostly for power considerations. As we do not need this kind of alternatives to obtain a workable approximation to the power function in our case (cf. section 4.4), it is not necessary for this purpose to pursue this subject.

However, in order to obtain expressions for the asymptotic relative Pitman efficiencies for pairs of tests from the class of tests we consider, we need a class of special alternatives which is a subset of the class of contiguous alternatives. Therefore, we shall give a short exposition of contiguous alternatives as they are in our situation and then specialize to the subclass we really use.

Contiguous alternatives were introduced by Le Cam in 1960 and have been widely used since then. We refer the interested reader to short introductions to the subject, which can be found in the standard books of WITTING & NOLLE (1970) and HAJEK & SIDAK (1967).

Contiguity is defined in terms of triangular arrays of probability distributions. Consider therefore the triangular array of random vectors

$$(\vec{\underline{t}}_{1(m)},\vec{\underline{t}}_{2(m)},\ldots,\vec{\underline{t}}_{m(m)}), \qquad m = 1,2,\ldots \tag{2.4.1}$$

where each $\vec{\underline{t}}_{i(m)}$ is defined as in (2.2.7). The distribution of the $\vec{\underline{t}}_{i(m)}$ is supposed to be governed by the triangular array of parameters

$$(\vec{a}_{1(m)},\vec{a}_{2(m)},\ldots,\vec{a}_{m(m)}), \qquad m = 1,2,\ldots \tag{2.4.2}$$

with each $\vec{a}_{i(m)}$ as in (2.1.3), and by the triangular array of parameters

$$(\vec{\Delta}_{1(m)},\vec{\Delta}_{2(m)},\ldots,\vec{\Delta}_{m(m)}), \qquad m = 1,2,\ldots \tag{2.4.3}$$

Let $P_{i(m)}$ denote the distribution of $\vec{\underline{t}}_{i(m)}$ under one choice of (2.4.2) and (2.4.3), and let $Q_{i(m)}$ be the distribution of $\vec{\underline{t}}_{i(m)}$ under another. The contiguity of the sequences $\{P_{i(m)}\}$ and $\{Q_{i(m)}\}$ may then be investigated using the characterizations of OOSTERHOFF & VAN ZWET (1975).

Contiguous alternatives are obtained when we consider the contiguity of a sequence $\{Q_{i(m)}\}$ with respect to a sequence $\{P^0_{i(m)}\}$ of distributions of $\vec{\underline{t}}_{i(m)}$ under H_0. Notice that, under H_0, only the parameters of (2.4.3) are completely determined. It follows that sequences of distributions $\{Q_{i(m)}\}$ which are contiguous to H_0 may be obtained with different choices of (2.4.2). In order not to get away too far from a practical interpretation, it is not unreasonable to limit our attention to the comparison of triangular sequences of experiments for which the triangular array (2.4.2) is the same. Adopting this point of view, we obtain from the characterizations of Oosterhoff & Van Zwet that a sequence of alternatives (or the sequence $\{Q_{i(m)}\}$ under these alternatives) is contiguous to H_0 iff

(2.4.4) $$\sum_{i=1}^{m} \sum_{r=1}^{N_{i(m)}} \Delta_{ir(m)}^2 = \mathcal{O}(1).$$

A formal proof of this fact may be found in DE GUNST & VAN DE GEER (1982).

We shall now proceed to define a subclass of such contiguous alternatives. The class chosen is needed in order to apply theorem 5.1.1 of section 5.1. Because this is the only instance where we use this class, the notation of this subclass is adapted to this application.

Consider a fixed sequence

(2.4.5) $$\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots$$

and define the triangular array (2.4.2) by means of

(2.4.6) $$\vec{a}_{i(m)} = \vec{a}_i, \qquad i = 1,\ldots,m.$$

Furthermore, let $\mathcal{a} \in \mathcal{A}_1$ be a fixed alternative which is determined by the sequence of vectors

(2.4.7) $$\vec{\Delta}_1, \vec{\Delta}_2, \vec{\Delta}_3, \ldots$$

Let

(2.4.8) $$\Theta \overset{d}{=} [0,1], \qquad \Theta' \overset{d}{=} (0,1].$$

Consider a sequence $\{\theta_m\}_{m=1}^{\infty}$ of values in Θ' such that

(2.4.9) $$m\theta_m^2 \to \eta \geq 0, \qquad \text{as} \qquad m \to \infty.$$

Define the triangular array (2.4.3) by means of

(2.4.10) $$\vec{\Delta}_{i(m)} = \theta_m \vec{\Delta}_i, \qquad i = 1,\ldots,m.$$

An easy calculation shows that the so defined alternative is indeed a contiguous alternative. To distinguish such alternatives clearly from the more general contiguous alternatives, we shall call our alternatives "*close*" alternatives. We shall denote the triangular array corresponding to close alternatives symbolically as

(2.4.11) $\{a_{\theta_m}\}$.

It is now interesting to investigate the behaviour of some already defined quantities under close alternatives. To distinguish quantities associated with close alternatives from their "normal" counterparts, we write the former with an extra "(m)". Doing so we have the following straightforward results.

(2.4.12) $$\lim_{m\to\infty} \delta_{\bullet jl}^{(m)\ (\nu,\mu)} =$$

$$= \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir})\, t_{il}^{(\mu)}(\pi_{ir})\, \theta_m\, \Delta_{ir} =$$

$$= \lim_{m\to\infty} \theta_m\, \delta_{\bullet jl}^{(\nu,\mu)} = 0.$$

Furthermore,

(2.4.13) $$\lim_{m\to\infty} \delta_{*j}^{(m)(\nu)} =$$

$$= \lim_{m\to\infty} \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir})\, \theta_m\, \Delta_{ir} =$$

$$= \lim_{m\to\infty} \theta_m \sqrt{m}\, \delta_{\bullet j}^{(\nu)} = \sqrt{\eta}\, \zeta_j^{(\nu)}.$$

It follows that

(2.4.14) $$\lim_{m\to\infty} \vec{\delta}_{*}^{(m)} = \sqrt{\eta}\, \vec{\zeta}\ .$$

For the expectation under close alternatives we therefore find

(2.4.15) $$E\vec{\underline{t}}_{*}^{(m)} \to \sqrt{\eta}\, \vec{\zeta}.$$

For the dispersion matrix under close alternatives we obtain

(2.4.16) $$D(\vec{\underline{t}}_{*}^{(m)}) \to \Sigma_0.$$

2.5. AN IMPORTANT PRACTICAL CASE (THE 'UNCONDITIONAL' SITUATION)

In this section we describe another class of experiments which are important for practical purposes. We refer to this class as 'the unconditional situation', because the 'unconditional situation' can be reduced to the 'conditional situation' of section 2.1 by the imposition of a suitable condition. The proposed test-statistics of section 2.1 act in the 'unconditional situation' as conditional test-statistics.

We consider, again, a sequence

$$E' \overset{d}{=} (E_1', E_2', \dots, E_m') \tag{2.5.1}$$

of m independent experiments. The result of each experiment is a word of length n consisting of characters from the fixed set $\{C_1, \dots, C_k\}$, each of these characters being available for each place. The number of possible words thus is k^n and the set Ω_i' of these words is the same for all i. The set of possible outcomes for E' then is

$$\Omega' \overset{d}{=} \Omega_1' \times \dots \times \Omega_m' \, . \tag{2.5.2}$$

For each i, $n \times k$ random variables $\underline{x}_{ij}^{(\nu)}$ are defined by means of the functions

$$x_{ij}^{(\nu)}(\omega') \overset{d}{=} \begin{cases} 1 \text{ if in } \omega' \; C_j \text{ occurs in the } \nu\text{'th place;} \\ 0 \text{ otherwise,} \end{cases} \qquad (\omega' \in \Omega_i) \tag{2.5.3}$$

and probabilities

$$p_{ij}^{(\nu)} \overset{d}{=} P(\underline{x}_{ij}^{(\nu)} = 1). \tag{2.5.4}$$

Thus in the random vector $(\underline{x}_{i1}^{(\nu)}, \dots, \underline{x}_{ik}^{(\nu)})'$ one of the components assumes the value 1 (indicating the ν'th character of the word) and the others the value 0.

Now, moreover, the experimental situation considered implies that the characters of each of the words are chosen independently: denoting the random element of Ω_i' by $\underline{\omega}_i'$ this means that the probability of the word $(C_{j_1},\ldots,C_{j_n})$ is equal to

$$(2.5.5)\qquad P(\underline{\omega}_i' = (C_{j_1},\ldots,C_{j_n})) = \prod_{\nu=1}^{n} p_{ij_\nu}^{(\nu)} = \prod_{\nu=1}^{n}\prod_{j=1}^{k} \{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\omega_i')}$$

with

$$(2.5.6)\qquad \omega_i' = (C_{j_1},\ldots,C_{j_n}).$$

For the whole sequence E' we get, with $\omega' = (\omega_1',\ldots,\omega_m') \in \Omega'$

$$(2.5.7)\qquad P(\underline{\omega}' = \omega') = \prod_{i=1}^{m}\prod_{\nu=1}^{n}\prod_{j=1}^{k} \{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\omega_i')}.$$

The number of parameters in this model is so large that reduction is imperative. This can be achieved by imposing a condition of the following character. Let

$$(2.5.8)\qquad \underline{a}_{ij} \overset{d}{=} \sum_{\nu=1}^{n} \underline{x}_{ij}^{(\nu)}$$

then the condition is

$$(2.5.9)\qquad A \overset{d}{=} \forall_{ij} \quad \underline{a}_{ij} = a_{ij},$$

where, in applications, the a_{ij} are the values assumed by $\underline{a}_{ij}$. (See remark 2.5.1). Applying A, the set Ω' is reduced to its subset Ω given by (2.1.6) and (2.1.7) and we have for all i and $\pi_{ir} \in \Omega_i$

$$(2.5.10)\qquad P(\underline{\omega}_i' = \pi_{ir}|A) = \frac{\prod_{j=1}^{k}\prod_{\nu=1}^{n} \{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\pi_{ir})}}{\sum_{s=1}^{N_i}\prod_{j=1}^{k}\prod_{\nu=1}^{n} \{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\pi_{is})}}$$

If we call this p_{ir}, as in (2.1.10), the conditional situation is identical to the situation in section 2.1, with

$$(2.5.11)\qquad \underline{t}_{ij}^{(\nu)} \equiv (\underline{x}_{ij}^{(\nu)}|A),$$

and the parameters Δ_{ir} and $\delta_{ijl}^{(\nu,\mu)}$ are functions of the $p_{ij}^{(\nu)}$ and of the a_{ij} from (2.5.9). We can now apply the methods of section 2.1; in particular we can test the hypothesis

$$(2.5.12) \qquad H_0^*: \quad \forall_{ij} \quad p_{ij}^{(1)} = \dots = p_{ij}^{(n)}$$

against the alternative

$$(2.5.13) \qquad H_1^*: \quad \exists_{j\nu} \quad \left| \sum_{i=1}^{m} (p_{ij}^{(\nu)} - \frac{1}{n} \sum_{\nu=1}^{n} p_{ij}^{(\nu)}) \right| > 0 \ ,$$

using one of the test-statistics from section 2.1. This is possible because evidently $H_0^* \Rightarrow H_0$: under A and H_0^* all words in Ω_i have the same probability. It is less evident that $H_1^{*}{}'$ corresponds to H_1' of (2.1.34), because of the complicated character of the $\delta_{ij}^{(\nu)}$. They depend not only, in a rather complicated way, on the $p_{ij}^{(\nu)}$, but also on the a_{ij} and are, therefore, in the unconditional situation random variables. As a matter of fact, these random variables have not even been defined yet. To remedy this omission, we start from (2.1.26), which can now be written as

$$(2.5.14) \qquad \delta_{ij}^{(\nu)} = P(\underline{t}_{ij}^{(\nu)} = 1) - \frac{a_{ij}}{n} = P(\underline{x}_{ij}^{(\nu)} = 1 | A) - \frac{a_{ij}}{n} =$$
$$= E(\underline{x}_{ij}^{(\nu)} | a_{ij}) - \frac{a_{ij}}{n} \ .$$

So if we define

$$(2.5.15) \qquad \underline{\delta}_{ij}^{(\nu)} \overset{d}{=} E(\underline{x}_{ij}^{(\nu)} | \underline{a}_{ij}) - \frac{\underline{a}_{ij}}{n} \ ,$$

we have, as we should

$$(2.5.16) \qquad \underline{\delta}_{ij}^{(\nu)} | A \equiv \delta_{ij}^{(\nu)} \ ,$$

while moreover

$$(2.5.17) \qquad E\underline{\delta}_{ij}^{(\nu)} = EE(\underline{x}_{ij}^{(\nu)} | \underline{a}_{ij}) - \frac{E\underline{a}_{ij}}{n} = E\underline{x}_{ij}^{(\nu)} - \frac{E\underline{a}_{ij}}{n} =$$
$$= p_{ij}^{(\nu)} - \frac{1}{n} \sum_{\nu=1}^{n} p_{ij}^{(\nu)} \ .$$

Now formula (2.1.34) for H_1' is later justified by the consistency of the test for (2.1.42). The analogon of (2.1.42) is now

$$(2.5.18)\qquad \exists_{j\nu}\quad \left|\frac{1}{\sqrt{m}}\sum_{i=1}^{m}(p_{ij}^{(\nu)}-\frac{1}{n}\sum_{\nu=1}^{n}p_{ij}^{(\nu)})\right|\to\infty \quad \text{for } m\to\infty$$

and if this condition holds we have according to (2.5.17)

$$(2.5.19)\qquad \exists_{j\nu}\quad \left|\frac{1}{\sqrt{m}}\sum_{i=1}^{m}E\underline{\delta}_{ij}^{(\nu)}\right|\to\infty .$$

Since, however, the $\underline{\delta}_{ij}^{(\nu)}$ only assume values between -1 and 1 and since, for any fixed j and ν, $\underline{\delta}_{1j}^{(\nu)}$, $\underline{\delta}_{2j}^{(\nu)}$, ... are independent this means that (see theorem 4.5.6)

$$(2.5.20)\qquad \exists_{j\nu}\quad \left|\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\underline{\delta}_{ij}^{(\nu)}\right|\overset{P}{\to}\infty \quad \text{for } m\to\infty.$$

Thus, the consistency of the test based on test-statistic (2.1.39) for the unconditional situation when (2.5.18) is satisfied, follows from the consistency in the conditional situation based on (2.1.42).

For future use we now give analogous definitions for $\underline{\delta}_{ij}^{(\nu,\mu)}$ and $\underline{\delta}_{ijl}^{(\nu,\mu)}$, based on (2.1.27) and (2.1.28),

$$(2.5.21)\qquad \underline{\delta}_{ij}^{(\nu,\mu)} \overset{d}{=} E(\underline{x}_{ij}^{(\nu)}\underline{x}_{ij}^{(\mu)}|\underline{a}_{ij}) - \frac{\underline{a}_{ij}(\underline{a}_{ij}-1)}{n(n-1)} \qquad (\nu\neq\mu),$$

$$(2.5.22)\qquad \underline{\delta}_{ijl}^{(\nu,\mu)} \overset{d}{=} E(\underline{x}_{ij}^{(\nu)}\underline{x}_{il}^{(\mu)}|\underline{a}_{ij},\underline{a}_{il}) - \frac{\underline{a}_{ij}\underline{a}_{il}}{n(n-1)} \qquad (j\neq l,\nu\neq\mu).$$

We have

$$(2.5.23)\qquad E\underline{\delta}_{ij}^{(\nu,\mu)} = \qquad (\nu\neq\mu),$$

$$= p_{ij}^{(\nu)}p_{ij}^{(\mu)} - \frac{1}{n(n-1)}\{(\sum_{\mu=1}^{n}p_{ij}^{(\mu)})^2-\sum_{\mu=1}^{n}(p_{ij}^{(\mu)})^2\},$$

$$(2.5.24)\qquad E\underline{\delta}_{ijl}^{(\nu,\mu)} = \qquad (j\neq l,\nu\neq\mu),$$

$$= p_{ij}^{(\nu)}p_{il}^{(\mu)} - \frac{1}{n(n-1)}\{(\sum_{\nu=1}^{n}p_{ij}^{(\nu)})(\sum_{\mu=1}^{n}p_{il}^{(\mu)}) - \sum_{\nu=1}^{n}p_{ij}^{(\nu)}p_{il}^{(\nu)}\}.$$

with

$$(2.5.25)\qquad E\underline{\delta}_{ij}^{(\nu,\mu)} = E\underline{\delta}_{ijj}^{(\nu,\mu)}.$$

Sometimes the unconditional analogon of (2.1.39), namely

$$\frac{1}{m}\sum_{j=1}^{k} g_j \sum_{\nu=1}^{n}\{\sum_{i=1}^{m}(\underline{x}_{ij}^{(\nu)} - \frac{\underline{a}_{ij}}{n})\}^2, \qquad (2.5.26)$$

which we shall call $\underline{w}(G)$, or more generally $\underline{w}(Q)$ (cf.(2.2)), will be considered. (The factor $\frac{1}{m}$, as in (2.2.11), serves asymptotic purposes; for a test-statistic such a factor is irrelevant). The distribution of $\underline{w}(Q)$ depends on the nuisance parameters $p_{ij}^{(\nu)}$ and is therefore unknown, also under H_0^*. This makes (2.5.26) unfit to be used as an unconditional test-statistic. The asymptotic distribution of $\underline{w}(Q)$ is, under certain conditions, nevertheless the same as that of $\underline{v}(Q)$.

<u>REMARK 2.5.1</u>. In section 2.1 the conditions in (2.1.3) imply that $a_{ij} = n$ never occurs. In the unconditional experimental situation the probability of such an occurrence is equal to

$$\sum_{j=1}^{k}\prod_{\nu=1}^{n} p_{ij}^{(\nu)} \qquad (2.5.27)$$

and is thus positive.

It is clear that such an experiment, where all characters of a word are the same, cannot contribute to finding differences between the $p_{ij}^{(\nu)}$ and that the experiment is then useless and had better be left out of consideration.

What is the effect of the deletion of such observations? To obtain m 'useful' observations a random number $\underline{\ell}_m$ of observations will have to be taken, i.e. a sequence

$$E' = (E_1', E_2', \ldots, E_{\underline{\ell}_m}') \qquad (2.5.28)$$

of experiments has to be performed. Let

$$\underline{\ell}_i \stackrel{d}{=} \text{number of the i'th 'useful' experiment}, \qquad (2.5.29)$$

where a 'useful' experiment is an experiment which does not result in an outcome where all characters are equal.

Deletion of "useless" experiments yields the sequence

$$E'' = (E_{\underline{\ell}_1}', E_{\underline{\ell}_2}', \ldots, E_{\underline{\ell}_m}') \qquad (2.5.30)$$

of m independent experiments. Let

$$(2.5.31)\qquad I \overset{d}{=} (\underline{\ell}_1 = \ell_1 \wedge \underline{\ell}_2 = \ell_2 \wedge \ldots \wedge \underline{\ell}_m = \ell_m).$$

We shall now impose the condition I, where in applications, as in (2.5.9), the ℓ_i are the values assumed by the $\underline{\ell}_i$. Thus, given I, we consider the sequence

$$(2.5.32)\qquad E''(I) = (E'_{\ell_1}, E'_{\ell_2}, \ldots, E'_{\ell_m})$$

of m independent experiments.

The i'th experiment of $E''(I)$, $E''_i(I)$, has as set of possible outcomes

$$(2.5.33)\qquad \Omega''_i \overset{d}{=} \Omega'_{\ell_i} \setminus \{(C_1,\ldots,C_1), (C_2,\ldots,C_2),\ldots, (C_k,\ldots,C_k)\},$$

i.e. Ω''_i is the same set for each i. The set of possible outcomes for $E''(I)$ is then of course

$$(2.5.34)\qquad \Omega'' = \Omega''_1 \times \Omega''_2 \times \ldots \times \Omega''_m \;,$$

while the conditional probabilities are of course proportional to the unconditional ones (cf.(2.5.5)),

$$(2.5.35)\qquad P(\underline{\omega}''_i = (C_{j_1},\ldots,C_{j_n})) = P(\underline{\omega}'_{\ell_i} = (C_{j_1},\ldots,C_{j_n}) \mid I) = \frac{\prod_{\nu=1}^{n} p_{\ell_i j_\nu}^{(\nu)}}{1 - \sum_{j=1}^{k} \prod_{\nu=1}^{n} p_{\ell_i j}^{(\nu)}} .$$

Because, after conditioning on I, the probabilities of the 'deleted' experiments have become irrelevant, we may as well renumber the sequence of experiments, i.e. we shall replace ℓ_i by i throughout, in particular

$$(2.5.36)\qquad p_{\ell_i j}^{(\nu)} \rightarrow p_{ij}^{(\nu)} .$$

If we then write

$$(2.5.37)\qquad p_i \overset{d}{=} 1 - \sum_{j=1}^{k} \prod_{\nu=1}^{n} p_{ij}^{(\nu)},$$

and define $x_{ij}^{(\nu)}: \Omega_i'' \to \{0,1\}$ as in (2.5.3), we can write (2.5.35) as

$$(2.5.38)\qquad P(\underline{\omega}_i'' = \omega_i) = \frac{1}{p_i} \sum_{\nu=1}^{n} \sum_{j=1}^{k} \{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\omega_i)}.$$

A further conditioning on A reduces the set Ω'' to its subset Ω and we have for all i and $\pi_{ir} \in \Omega_i$

$$(2.5.39)\qquad P(\underline{\omega}_i'' = \pi_{ir}|A) = \frac{\frac{1}{p_i}\prod_{\nu=1}^{n}\prod_{j=1}^{k}\{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\pi_{ir})}}{\sum_{s=1}^{N_i}\frac{1}{p_i}\prod_{\nu=1}^{n}\prod_{j=1}^{k}\{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\pi_{is})}} =$$

$$= \frac{\prod_{\nu=1}^{n}\prod_{j=1}^{k}\{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\pi_{ir})}}{\sum_{s=1}^{N_i}\prod_{\nu=1}^{n}\prod_{j=1}^{k}\{p_{ij}^{(\nu)}\}^{x_{ij}^{(\nu)}(\pi_{is})}}.$$

This is quite the same as (2.5.10), so from here on we may proceed as from (2.5.10) on, the only difference being that the $p_{ij}^{(\nu)}$ are now not the original ones, because some experiments have been deleted. This has no influence, however, on the consistency of the test, because when (2.5.18) holds, it also holds for a sub-sequence. Furthermore it would be nice if for each finite m,

$$(2.5.40)\qquad P(\underline{\ell}_m < \infty) = 1.$$

The reader is referred to existing probability theory on this problem.

We conclude that we may safely delete observations for which $\underline{a}_{ij} = n$.

<u>REMARK 2.5.2</u>. Similarly, a category which does not occur in any of the experiments should (and can) be left out of consideration.

CHAPTER 3

SURVEY OF THEOREMS USED

3.1. DEFINITIONS AND THEOREMS ABOUT MATRICES

Consider a $q \times s$ matrix A of any rank r. A *generalised inverse* (or a *g-inverse*) of A is an $s \times q$ matrix, denoted by A^-, that satisfies

(3.1.1) $\quad A A^- A = A.$

If A^- furthermore satisfies

(3.1.2) $\quad A^- A A^- = A^-$

then A^- is called a *reflexive generalised inverse* of A. The notion of g-inverse is discussed extensively in RAO (1973).

We shall apply the notion of g-inverse in particular to real symmetric matrices. Let A be a real symmetric matrix of order q. By

(3.1.3) $\quad A = P \Lambda P'$

we denote the *canonical reduction* of A. That is, Λ is the diagonal matrix of eigenvalues of A (denoted by $\Lambda = \text{diag}(\lambda_1,\dots,\lambda_q)$) and the i'th column of the $q \times q$ matrix P consists of the eigenvector of A which corresponds to the eigenvalue λ_i $(i = 1,\dots,q)$. Because A is real and symmetric, the eigenvalues are all real, and we shall always suppose that they occur in Λ in decreasing order $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q)$. Furthermore, we shall always take the eigenvectors orthonormal, i.e.

(3.1.4) $\quad P'P = PP' = I_q.$

Now let A be moreover non-negative definite (n.n.d) and let it have rank r. (Dispersion matrices are non-negative definite). Then A has precisely r positive eigenvalues and zero as eigenvalue with multiplicity $q-r$. Let Λ_+ be the $r \times r$ diagonal matrix of the first r (positive) eigenvalues of A, and P_+ the $q \times r$ matrix of corresponding eigenvectors. Partition the matrices Λ and P as follows

$$(3.1.5) \qquad \Lambda = \begin{pmatrix} \Lambda_+ & 0_{r,q-r} \\ 0_{q-r,r} & 0_{q-r,q-r} \end{pmatrix},$$

$$(3.1.6) \qquad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where P_{11} has order r etc., so

$$(3.1.7) \qquad P_+ = \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix}.$$

It can easily be verified, using these partitionings, that

$$(3.1.8) \qquad A = P\Lambda P' = P_+\Lambda_+P_+'.$$

We shall call $P_+\Lambda_+P_+'$ the *positive canonical reduction* of A.

REMARK 3.1.1. Although we speak about "the" canonical reduction of A, this decomposition is not entirely unique. Any eigenvector $\vec{p}$ from P may, for instance, be replaced by $-\vec{p}$, without affecting the identity $A = P\Lambda P'$. Or, when A has two equal eigenvalues, the corresponding eigenvectors may be differently orthogonalised, so that P is changed but not $P\Lambda P'$. The decomposition is unique when all the eigenvalues of A are distinct and positive, and when we make the diagonal elements of P positive (SRIVASTAVA & KHATRI (1979), p.19).

Because the implications of this non-uniqueness are minor, we shall maintain the terminology. (The question arises again in remark 3.2.1 and example 3.2.1 of the next section).

The linear space spanned by the columns of a matrix X will be denoted by $M(X)$, and the linear space of solutions of the equation $X\vec{z} = \vec{0}$, the null-space of X, will be denoted by $N(X)$. For any matrix X we have

$$(3.1.9) \qquad M(X) = M(XX'),$$

a result which follows easily from the equivalence $\vec{y} \perp M(X) \iff \vec{y} \perp M(XX')$.

Because in the positive canonical reduction of A, the column vectors in P_+ are still orthonormal, we have

(3.1.10) $$P_+'P_+ = I_r,$$

but $P_+P_+' = I_q$ does not hold, unless A is non-singular. However we have the following

<u>LEMMA 3.1.1</u>. *Let* A *be a real, symmetric n.n.d matrix of order* q, *with rank* $r \leq q$. *Let* $A = P_+\Lambda_+P_+'$ *be the positive canonical reduction of* A. *Then*

(3.1.11) $$P_+P_+'\vec{x} = \vec{x}$$

for each $\vec{x} \in M(A)$.

<u>PROOF</u>. $\vec{x} \in M(A)$ implies $\vec{x} = A\vec{y}$ for some q-vector $\vec{y}$. Then

$$P_+P_+'\vec{x} = P_+P_+'A\vec{y} = P_+P_+'P_+\Lambda_+P_+'\vec{y} = P_+\Lambda_+P_+'\vec{y} = A\vec{y} = \vec{x}. \quad \square$$

P_+P_+' is the (uniquely determined) orthogonal projector on the linear space $M(A)$. See RAO (1973) for further details.

A natural way to obtain a g-inverse of A is to take

(3.1.12) $$A^- = P_+\Lambda_+^{-1}P_+'$$

because it apparently satisfies (3.1.1). A^- is even reflexive because it also satisfies (3.1.2). We shall call g-inverses defined as in (3.1.12) *natural generalised inverses* or *ng-inverses*.

Any $q \times s$ matrix B such that

(3.1.13) $$A = BB'$$

will be called a *square-root* of A. The set of square-roots of A is non-empty as follows from the following lemma.

<u>LEMMA 3.1.2</u>. *Any real, symmetric, n.n.d matrix* A *has at least one square-root.*

<u>PROOF</u>. Let $A = P_+ \Lambda_+ P_+'$ be the positive canonical reduction of A. Let

$$L \overset{d}{=} \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_r}) \tag{3.1.14}$$

and

$$B \overset{d}{=} P_+ L. \tag{3.1.15}$$

Then $BB' = P_+ LLP_+' = P_+ \Lambda_+ P_+' = A$, so B is the required square-root. □

The square-root of A defined by (3.1.15) will be called a *natural square-root* of A; it is a $q \times r$ matrix; r is the smallest number of columns that a square root of A with rank r can have. Square roots are uniquely determined, up to an orthonormal transformation:

<u>LEMMA 3.1.3</u>. *Let* A *be a real, symmetric, n.n.d matrix of order* q. *Let* $B: q \times s_1$ *and* $C: q \times s_2$ *be two square-roots of* A, *with* $s_2 \geq s_1$. *Then there exists an orthonormal matrix* $U: s_2 \times s_2$, *such that*

$$C = (B | 0_{q, s_2 - s_1}) U. \tag{3.1.16}$$

<u>PROOF</u>. SRIVASTAVA & KHATRI (1979), p.20. □

<u>LEMMA 3.1.4</u>. *Let* G *be an* $n \times m$ *matrix and* H *an* $m \times n$ *matrix. Then the non-zero eigenvalues of* GH *and* HG *are identical, that is, the same non-zero eigenvalues occur with the same multiplicities in* GH *and* HG.

<u>PROOF</u>. WILKINSON (1965), p.54. □

<u>COROLLARY 3.1.1</u>. *Let* Q *and* A *be real, symmetric, n.n.d matrices of order* q. *Let* B *and* C *be arbitrary square-roots of* A. *Then*

i. *The non-zero eigenvalues of* $B'QB$, $C'QC$ *and* QA *are identical in the sense of lemma 3.1.4.*

ii. $\text{rank}(B'QB) = \text{rank}(C'QC) = r$, *where* r = *number of non-zero eigenvalues of* QA.

<u>PROOF</u>. *i.* When we apply lemma 3.1.4 to $QA = QBB'$, with $G = QB$ and $H = B'$, it follows that the non-zero eigenvalues of QA and $B'QB$ are identical in

the above sense. The same is true for QA and C'QC and hence the non-zero eigenvalues of B'QB and C'QC are identical.

ii. Because B'QB and C'QC are real, symmetric and n.n.d, we have:

rank(B'QB) = number of non-zero eigenvalues of B'QB =
= number of non-zero eigenvalues of QA = r =
= number of non-zero eigenvalues of C'QC = rank(C'QC). □

REMARK 3.1.1. It follows from corollary 3.1.1 that, when the eigenvalues of QA have to be calculated, we can always take the natural square-root of A, B, and calculate the eigenvalues of B'QB. When A is not of full rank, B'QB is of smaller order than QA. It may then be easier to compute the eigenvalues of B'QB instead of those of QA. Moreover, it may be an advantadge that B'QB is symmetric while QA is not.

In section 4.1 we use the following $(n+k) \times (n \cdot k)$ matrix of rank $n+k-1$*),

$$(3.1.17) \qquad F = \begin{pmatrix} 11\ldots1 & 00\ldots0 & \ldots & 00\ldots0 \\ 00\ldots0 & 11\ldots1 & \ldots & 00\ldots0 \\ \vdots & \vdots & \ddots & \vdots \\ 00\ldots0 & 00\ldots0 & \ldots & 11\ldots1 \\ I_k & I_k & \ldots & I_k \end{pmatrix}$$

Notice that

$$(3.1.18) \qquad \vec{\delta}_i \in N(F), \qquad \vec{\delta}_* \in N(F),$$

where $\vec{\delta}_i$ and $\vec{\delta}_*$ were defined in section 2.2.

LEMMA 3.1.5. *Let* A *be a* $k \times k$ *matrix and* B *an* $n \times n$ *matrix. Let the* ρ*'th eigenvalue of* A *be* λ_ρ *and a corresponding eigenvector* $(p_{1\rho},\ldots,p_{k\rho})'$*; let the* τ*'th eigenvalue of* B *be* μ_τ *and a corresponding eigenvector* $\vec{q}_\tau$*. Then the set of eigenvalues of the matrix* $A \otimes B$ *is equal to*

$$\{x \mid x = \lambda_\rho \mu_\tau,\ \rho = 1,\ldots,k,\ \tau = 1,\ldots,n\}.$$

---

*) The symbols n and k have, in later applications, the same meaning as in chapter 2; the symbol q will usually be $n \cdot k$.

An eigenvector corresponding to the eigenvalue $\lambda_\rho \mu_\tau$ *is*

$$(p_{1\rho}\vec{q}_\tau', p_{2\rho}\vec{q}_\tau', \ldots, p_{k\rho}\vec{q}_\tau').$$

PROOF. ANDERSON (1958), p.348. □

Note that in particular it follows from lemma 3.1.5 that the eigenvalues of $A \otimes B$ and $B \otimes A$ are the same.

LEMMA 3.1.6. *Let* A *be a real, symmetric, n.n.d matrix of order* q *and rank* r, *and let* $\lambda_1,\ldots,\lambda_r$ *be the positive eigenvalues of* $\frac{n}{n-1}A$. *With* N *as defined by (2.2.28), the eigenvalues of* $A \otimes N$ *then are* $\lambda_1,\ldots,\lambda_r$, *all* >0, *each with multiplicity* n-1, *and 0 with multiplicity* $q+(n-1)(q-r)$.

PROOF. The eigenvalues of N are 0 and $\frac{n}{n-1}$ with multiplicity n-1. The result now follows from lemma 3.1.5. □

LEMMA 3.1.7. *Let* A *be a real, symmetric, n.n.d matrix of order* q *and rank* r. *Let* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$ *be the positive eigenvalues of* A. *Then*

$$(3.1.19) \qquad \lambda_r \vec{x}'\vec{x} \leq \vec{x}'A\vec{x} \leq \lambda_1 \vec{x}'\vec{x}$$

for each $\vec{x} \in M(A)$.

PROOF. Let $A = P_+ \Lambda_+ P_+'$ be the positive canonical reduction of A. Let $\vec{x} \in M(A)$. Then we have

$$\vec{x}'A\vec{x} = \vec{x}'P_+\Lambda_+P_+'\vec{x} \leq \lambda_1 \vec{x}'P_+P_+'\vec{x} = \lambda_1 \vec{x}'\vec{x},$$

because $P_+P_+'\vec{x} = \vec{x}$ for $\vec{x} \in M(A)$ by lemma 3.1.1. The other part of the inequality is proved in the same way. □

COROLLARY 3.1.2. *Let* A *be a real, symmetric, n.n.d matrix of order* q, *with rank* q, *and eigenvalues* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q \geq 0$. *Then*

$$(3.1.20) \qquad \lambda_q \vec{x}'\vec{x} \leq \vec{x}'A\vec{x} \leq \lambda_1 \vec{x}'\vec{x}$$

for each $\vec{x} \in \mathbb{R}^q$.

PROOF. This follows as a special case of lemma 3.1.7. □

LEMMA 3.1.8. *Let* A *be a real, symmetric, n.n.d matrix of order* q. *If* $(\vec{x}_m)_{m=1}^{\infty}$ *is a sequence of vectors of* q *components such that* $\vec{x}_m'\vec{x}_m \to \infty$ *as* $m \to \infty$, *then*

$$(3.1.21) \qquad \lim_{m\to\infty} \frac{\vec{x}_m'\vec{x}_m}{(\vec{x}_m' A \vec{x}_m)^{\frac{1}{2}}} = \infty .$$

PROOF. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q \geq 0$ be the eigenvalues of A. If $\lambda_1 = \lambda_2 = \ldots = \lambda_q = 0$ then $\vec{x}_m' A \vec{x}_m = 0$ for each m and (3.1.21) follows immediately. So suppose that $\lambda_1 > 0$. From lemma 3.1.7 it follows that, for each m,

$$\vec{x}_m' A \vec{x}_m \leq \lambda_1 \vec{x}_m'\vec{x}_m, \qquad \text{or,}$$

$$(\lambda_1 \vec{x}_m'\vec{x}_m)^{-\frac{1}{2}} \leq (\vec{x}_m' A \vec{x}_m)^{-\frac{1}{2}}, \qquad \text{or,}$$

$$\frac{(\vec{x}_m'\vec{x}_m)^{\frac{1}{2}}}{\sqrt{\lambda_1}} \leq \frac{\vec{x}_m'\vec{x}_m}{(\vec{x}_m' A \vec{x}_m)^{\frac{1}{2}}} .$$

Because the lefthand-side of this inequality diverges to ∞, the righthand-side necessarily also diverges to ∞. □

3.2. DISTRIBUTION OF QUADRATIC FORMS IN NORMAL VARIATES

Consider a random vector $\underline{\vec{x}}$ which has a q-dimensional normal distribution with expectation vector $\vec{\mu}$ and dispersion matrix Σ (we denote this as $\underline{\vec{x}} \sim N_q(\vec{\mu},\Sigma)$). Various theorems are known about the distribution of quadratic forms $\underline{\vec{x}}'Q\underline{\vec{x}}$ in such normal variates. However, most of the theorems concern necessary and sufficient conditions under which the quadratic form has a (non-)central χ^2-distribution. The following theorem gives a representation of the quadratic form in terms of independent standard normal variables for the case that Q is n.n.d and $\underline{\vec{x}}$ has an arbitrary normal distribution. This theorem is known for non-singular dipersion matrix Σ (cf. JOHNSON & KOTZ (1972)). We give a simple proof that includes the case of a singular Σ.

THEOREM 3.2.1. *Let* $\underline{\vec{x}} \sim N_q(\vec{\mu},\Sigma)$. *Let* Q *be a real, symmetric, n.n.d matrix of order* q. *Then there exist numbers* $r \in \mathbb{N}$, $c \in \mathbb{R}$ *and vectors* $\vec{\lambda} \in \mathbb{R}^r$, $\vec{\omega} \in \mathbb{R}^r$ *such that*

$$(3.2.1) \qquad \underline{\vec{x}}' Q \underline{\vec{x}} \overset{\sim}{=} c + \sum_{\tau=1}^{r} \lambda_\tau (\underline{u}_\tau + \omega_\tau)^2$$

with $\underline{\vec{u}} \sim N_r(\vec{0},I_r)$, *i.e.* $\underline{u}_1,\dots,\underline{u}_r$ *are independent and each has a standard normal distribution.*

Let B *be an arbitrary square-root of* Σ. *Explicit values of* r, c, $\vec{\lambda}$ *and* $\vec{\omega}$ *may then be calculated from*

$$(3.2.2) \qquad r = \operatorname{rank}(B'QB),$$

$$(3.2.3) \qquad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \text{ are the positive eigenvalues of } B'QB.$$

If we furthermore denote the positive canonical reduction of $B'QB$ *by* $P_+ \Lambda_+ P_+'$, *we have*

$$(3.2.4) \qquad \vec{\omega} = \Lambda_+^{-1} P_+' B' Q \vec{\mu} \ ,$$

$$(3.2.5) \qquad c = \vec{\mu}'(Q - QBP_+\Lambda_+^{-1}P_+'B'Q)\vec{\mu} \ .$$

PROOF. Let B be any $q \times s$ square-root of Σ $(BB' = \Sigma)$. Then for every random vector $\underline{\vec{y}} \sim N_s(\vec{0},I_s)$, we have

$$(3.2.6) \qquad \underline{\vec{x}} \overset{\sim}{=} \vec{\mu} + B\underline{\vec{y}}.$$

So

$$(3.2.7) \qquad \underline{\vec{x}}' Q \underline{\vec{x}} = (\vec{\mu} + B\underline{\vec{y}})'Q(\vec{\mu} + B\underline{\vec{y}}) \equiv \underline{\vec{y}}'B'QB\underline{\vec{y}} + 2\vec{\mu}'QB\underline{\vec{y}} + \vec{\mu}'Q\vec{\mu}.$$

The matrix $B'QB$ is a real, symmetric, n.n.d matrix, being the dispersion matrix of $B'\underline{\vec{s}}$, when $\underline{\vec{s}} \sim N_q(\vec{0},Q)$. Taking $r = \operatorname{rank}(B'QB)$, it follows that $B'QB$ has exactly r positive eigenvalues and zero as eigenvalue of multiplicity $s-r$. Let the eigenvalues of $B'QB$ be $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > \lambda_{r+1} = \dots = \lambda_s = 0$. Let $P_+\Lambda_+P_+'$ be the positive canonical reduction of $B'QB$. Let T be a square-root of Q, i.e. $TT' = Q$.

Notice that P_+P_+' projects on the linear space $M(B'QB)$ (lemma 3.1.1). By (3.1.9) we have

(3.2.8) $$M(B'QB) = M(B'TT'B) = M(B'T).$$

This means that

$$P_+P_+'B'T = B'T \Rightarrow T'B = T'BP_+P_+' \Rightarrow TT'B = TT'BP_+P_+' \Rightarrow$$

(3.2.9) $$QB = QBP_+P_+' .$$

Substitution in (3.2.7) gives

(3.2.10) $$\underline{\vec{x}}'Q\underline{\vec{x}} \stackrel{\sim}{=} \underline{\vec{y}}'P_+\Lambda_+P_+'\underline{\vec{y}} + 2\vec{\mu}'QBP_+P_+'\underline{\vec{y}} + \vec{\mu}'Q\vec{\mu}.$$

Now define $\underline{\vec{u}} \stackrel{d}{\equiv} P_+\underline{\vec{y}}$, so that $\underline{\vec{u}} \sim N_r(\vec{0},I_r)$. $(D(\underline{\vec{u}}) = D(P_+\underline{\vec{y}}) = P_+'D(\underline{\vec{y}})P_+ = P_+'I_sP_+ = P_+'P_+ = I_r)$. Then we obtain from (3.2.10),

(3.2.11) $$\underline{\vec{x}}'Q\underline{\vec{x}} \stackrel{\sim}{=} \underline{\vec{u}}'\Lambda_+\underline{\vec{u}} + 2\vec{\mu}'QBP_+\underline{\vec{u}} + \vec{\mu}'Q\vec{\mu} \equiv$$

$$\equiv \underline{\vec{u}}'\Lambda_+\underline{\vec{u}} + 2\vec{\omega}'\Lambda_+\underline{\vec{u}} + \vec{\omega}'\Lambda_+\vec{\omega} + c \equiv c + \sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + \omega_\tau)^2,$$

where

(3.2.12) $$\vec{\omega} = \Lambda_+^{-1}P_+'B'Q\vec{\mu} ,$$

and

(3.2.13) $$c = \vec{\mu}'Q\vec{\mu} - \vec{\omega}'\Lambda_+\vec{\omega} = \vec{\mu}'Q\vec{\mu} - \vec{\mu}'QBP_+\Lambda_+^{-1}\Lambda_+\Lambda_+^{-1}P_+'B'Q\vec{\mu} =$$

$$= \vec{\mu}'(Q - QBP_+\Lambda_+^{-1}P_+'B'Q)\vec{\mu}.$$

This establishes the existence of the quantities c, r, $\vec{\lambda}$ and $\vec{\omega}$, and the formulae (3.2.2) - (3.2.5). □

COROLLARY 3.2.1. *Let* $\underline{\vec{x}} \sim N_q(\vec{0},\Sigma)$. *Let* Q *be a real, symmetric, n.n.d matrix of order* q. *Then*

$$(3.2.14) \qquad \underline{\vec{x}}'Q\underline{\vec{x}} \stackrel{\sim}{=} \sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2$$

with $\underline{\vec{u}} \sim N_r(\vec{0},I_r)$, $\lambda_1,\ldots,\lambda_r$ *the positive eigenvalues of* $Q\Sigma$.

PROOF. This follows from theorem 3.2.1 and corollary 3.1.1. □

For practical purposes, KOTZ, JOHNSON & BOYD (1967a,b) give series representations for the distribution-functions of the random variables as given by the right-hand sides of (3.2.1) and (3.2.14). These representations, which can be used for numerical calculations, are given in section 3.3.

Notice that for $\lambda_1 = \lambda_2 = \ldots = \lambda_r = 1$, the distribution of (3.2.1) reduces to the non-central χ^2 - distribution:

$$(3.2.15) \qquad \sum_{\tau=1}^{r} (\underline{u}_\tau + \omega_\tau)^2 \sim \chi^2[r,\delta],$$

where r is the number of degrees of freedom and δ is the non-centrality parameter

$$(3.2.16) \qquad \delta = \sum_{\tau=1}^{r} \omega_\tau^2.$$

Similarly, the distribution of (3.2.14) reduces to the central χ^2 - distribution with r degrees of freedom, when each λ_τ is equal to 1:

$$(3.2.17) \qquad \sum_{\tau=1}^{r} \underline{u}_\tau^2 \sim \chi^2[r].$$

REMARK 3.2.1. Notice that theorem 3.2.1 is a representation theorem. That is, the random variable $\underline{\vec{x}}'Q\underline{\vec{x}}$ is represented by the random variable on the right-hand side of (3.2.1), which has the same probability distribution. However, because there is some freedom of choice with respect to B and the positive canonical reduction of B'QB, different choices of B or P_+ may result in different random variables on the right-hand side of (3.2.1). We illustrate this by the following example.

EXAMPLE 3.2.1. Take $\vec{\underline{x}} \sim N_3(\vec{\mu},\Sigma)$, with $\vec{\mu}' = (2,6,8)$,

$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, and $Q = I_3$. Choose $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Then $B'QB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = P_+\Lambda_+P_+'$, with $P_+ = I_3$, $\Lambda_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

We find

$$\vec{\omega} = \Lambda_+^{-1}P_+'B'Q\vec{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix},$$

i.e. (check that c = 0),

$$\underline{x}_1^2 + \underline{x}_2^2 + \underline{x}_3^2 \equiv \vec{\underline{x}}'\vec{\underline{x}} \cong (\underline{u}_1 + 2)^2 + 4(\underline{u}_2 + 3)^2 + 4(\underline{u}_3 + 4)^2$$

However, when we take $P_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}$, (check that $B'QB = P_+\Lambda_+P_+'$),

$$\vec{\omega} = \Lambda_+^{-1}P_+'B'Q\vec{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix},$$

i.e. (again c = 0)

$$\underline{x}_1^2 + \underline{x}_2^2 + \underline{x}_3^2 \equiv \vec{\underline{x}}'\vec{\underline{x}} \cong (\underline{u}_1 + 2)^2 + 4(\underline{u}_2 + 5)^2 + 4\underline{u}_3^2.$$

REMARK 3.2.2. Let us now consider in greater detail what happens when we use another square-root C ($q \times t$) of Σ, in our construction. From corollary 3.1.1 it follows that $Q\Sigma$, $B'QB$ and $C'QC$ have identical non-zero eigenvalues and that $r = \text{rank}(B'QB) = \text{rank}(C'QC)$. Furthermore, because $\vec{\underline{x}}'Q\vec{\underline{x}} \geq 0$ and $\sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + \omega_\tau)^2 \geq 0$ (with probability one), c is a non-negative constant. Moreover, c is the "minimum value" that $\vec{\underline{x}}'Q\vec{\underline{x}}$ can assume ($\vec{\underline{u}}$ has a positive density in $-\vec{\omega}$), which is of course independent of the choice of square-root of Σ. It follows that, whatever the choice of square-root of Σ is, we always obtain the same r, c and $\vec{\lambda}$. Differences in $\vec{\omega}$ may however occur as is illustrated in example 3.2.1.

Certain functions of $\vec{\omega}$, however, are not affected by different choices for the square-root of Σ or different choices for P_+. For instance, from (3.2.13) it follows that

$$\sum_{\tau=1}^{r} \lambda_\tau \omega_\tau^2 = \mu'Q\mu - c \qquad (3.2.18)$$

a quantity which is clearly independent of the choice of square-root of Σ or of P_+.

Another interesting identity is, using (3.2.9),

$$\sum_{\tau=1}^{r} \lambda_\tau^2 \omega_\tau^2 = \vec{\omega}'\Lambda_+\Lambda_+\vec{\omega} = \vec{\mu}'QBP_+\Lambda_+^{-1}\Lambda_+\Lambda_+\Lambda_+^{-1}P_+'B'Q\vec{\mu} = \tag{3.2.19}$$

$$= \vec{\mu}'QBP_+P_+'B'Q\vec{\mu} = \vec{\mu}'QBB'Q\vec{\mu} = \vec{\mu}'Q\Sigma Q\vec{\mu}.$$

Furthermore, using

$$P_+\Lambda_+P_+' = B'QB \;\Rightarrow\; \Lambda_+ = P_+'B'QBP_+ \tag{3.2.20}$$

and (3.2.9), we find

$$\sum_{\tau=1}^{r} \lambda_\tau^3 \omega_\tau^2 = \vec{\omega}'\Lambda_+^3\vec{\omega} = \vec{\mu}'QBP_+\Lambda_+^{-1}\Lambda_+^3\Lambda_+^{-1}P_+'B'Q\vec{\mu} = \vec{\mu}'QBP_+\Lambda_+P_+'B'Q\vec{\mu} = \tag{3.2.21}$$

$$= \vec{\mu}'QBP_+P_+'B'QBP_+P_+'B'Q\vec{\mu} = \vec{\mu}'QBB'QBB'Q\vec{\mu} = \vec{\mu}'Q\Sigma Q\Sigma Q\vec{\mu}.$$

In general, for $k \geq 2$,

$$\sum_{\tau=1}^{r} \lambda_\tau^k \omega_\tau^2 = \vec{\mu}'(Q\Sigma)^{k-1}Q\vec{\mu}. \tag{3.2.22}$$

<u>REMARK 3.2.3</u>. For the constant c of theorem 3.2.1, the following holds.

$$\vec{\mu} \in M(\Sigma) \;\Rightarrow\; c = 0. \tag{3.2.23}$$

This can be proved as follows. Suppose that $\vec{\mu} \in M(\Sigma)$. Then, by definition of B and by (3.1.9), $\vec{\mu} \in M(\Sigma) = M(BB') = M(B)$. This implies that there exists a vector $\vec{y}$ such that $\vec{\mu} = B\vec{y}$. It then follows that

$$c = \vec{\mu}'(Q - QBP_+\Lambda_+^{-1}P_+'B'Q)\vec{\mu} = \vec{y}'B'(Q - QBP_+\Lambda_+^{-1}P_+'B'Q)B\vec{y} =$$

$$= \vec{y}'B'QB\vec{y} - \vec{y}'B'QBP_+\Lambda_+^{-1}P_+'B'QB\vec{y} = \vec{y}'B'QB\vec{y} - \vec{y}'B'QB\vec{y} = 0,$$

because $P_+\Lambda_+^{-1}P_+'$ is the natural g-inverse of $B'QB$.

Thus, $\vec{\mu} \in M(\Sigma)$ is a sufficient condition for $c = 0$ to hold. It is however not necessary as the reader may check by considering a suitable example.

Necessary and sufficient conditions are given by

LEMMA 3.2.1. *For the constant* c *of theorem 3.2.1 we have*

(3.2.24) $$c = 0 \iff Q\vec{\mu} \in M(Q\Sigma).$$

PROOF. "$\Leftarrow$". $Q\vec{\mu} \in M(Q\Sigma)$ implies that there exists a vector $\vec{z}$ such that $Q\vec{\mu} = Q\Sigma\vec{z}$. Then we have

$$\vec{\mu}'Q\vec{\mu} = \vec{\mu}'Q\Sigma\vec{z} = \vec{z}'\Sigma Q\Sigma\vec{z};$$

$$\vec{\mu}'QBP_+\Lambda_+^{-1}P_+'B'Q\vec{\mu} = \vec{z}'\Sigma QBP_+\Lambda_+^{-1}P_+'B'Q\Sigma\vec{z} = \vec{z}'BB'QBP_+\Lambda_+^{-1}P_+'B'QBB'\vec{z} =$$

$$= \vec{z}'BB'QBB'\vec{z} = \vec{z}'\Sigma Q\Sigma\vec{z},$$

so $c = 0$ follows.

"$\Rightarrow$". First recall that $c \geq 0$, because c is the "minimum value" of a n.n.d quadratic form. This means that, for all $\vec{\mu}$

$$c = \vec{\mu}'(Q - QBP_+\Lambda_+^{-1}P_+'B'Q)\vec{\mu} \geq 0.$$

Thus the matrix

$$D \overset{d}{=} Q - QBP_+\Lambda_+^{-1}P_+'B'Q$$

is non-negative definite. Let E be a square-root of D ($D = EE'$). Then $c = 0$ implies

$$\vec{\mu}'D\vec{\mu} = \vec{\mu}'EE'\vec{\mu} = 0 \quad \Rightarrow \quad \|E'\vec{\mu}\| = 0 \quad \Rightarrow \quad E'\vec{\mu} = \vec{0} \quad \Rightarrow$$

$$\Rightarrow \quad EE'\vec{\mu} = \vec{0} \quad \Rightarrow \quad D\vec{\mu} = \vec{0} \quad \Rightarrow \quad (Q - QBP_+\Lambda_+^{-1}P_+'B'Q)\vec{\mu} = \vec{0} \quad \Rightarrow$$

$$\Rightarrow \quad Q\vec{\mu} = QBP_+\Lambda_+^{-1}P_+'B'Q\vec{\mu} \quad \Rightarrow \quad Q\vec{\mu} \in M(QB).$$

The result now follows, because $M(QB) = M(QBB') = M(Q\Sigma)$ as the reader may verify. □

Next, we prove a theorem which is closely related to theorem 3.2.1. It gives criteria for (3.2.1) to have a (non-)central χ^2-distribution, which can be verified without explicit calculation of eigenvalues and eigenvectors. This theorem is a special case of an already known theorem, which we reproduce here as theorem 3.2.3, but it can now be derived easily from theorem 3.2.1, which is the reason why we treat it here. First we prove

LEMMA 3.2.2. *Let* $\underline{\vec{x}} \sim N_q(\vec{\mu},\Sigma)$. *Let* Q *be a real, symmetric, n.n.d matrix of order* q. *The following two statements*

(3.2.25) $Q\Sigma$ is idempotent

(3.2.26) $Q\vec{\mu} \in M(Q\Sigma)$

are then equivalent to

(3.2.27) $Q\Sigma$ is idempotent

(3.2.28) $\vec{\mu}'Q\vec{\mu} = \vec{\mu}'Q\Sigma Q\vec{\mu}$.

PROOF. Suppose that (3.2.25) & (3.2.26) hold. Clearly (3.2.25) ⇒ (3.2.27). From (3.2.26) it follows that there exists a vector $\vec{y}$ such that $Q\vec{\mu} = Q\Sigma\vec{y}$. Then, using the idempotency of $Q\Sigma$,

$$\vec{\mu}'Q\Sigma Q\vec{\mu} = \vec{y}'\Sigma Q\Sigma Q\Sigma\vec{y} = \vec{y}'\Sigma Q\Sigma\vec{y} = \vec{\mu}'Q\Sigma\vec{y} = \vec{\mu}'Q\vec{\mu}.$$

so (3.2.28) follows.

Next, suppose that (3.2.27) & (3.2.28) hold. Clearly (3.2.27) ⇒ (3.2.25). Let B be a square-root of Σ, then the non-zero eigenvalues of $B'QB$ and $Q\Sigma$ are identical (cf. corollary 3.1.1). From (3.2.27) it then follows that the non-zero eigenvalues of $B'QB$ are all equal to 1. The positive canonical reduction of $B'QB$ then reduces to P_+P_+'. Consider now the distribution of $\underline{\vec{x}}'Q\underline{\vec{x}}$ as given by theorem 3.2.1, in particular the constant c, as given by (3.2.5). We have, using (3.2.9),

$$c = \vec{\mu}'Q\vec{\mu} - \vec{\mu}'QBP_+\Lambda_+^{-1}P_+'B'Q\vec{\mu} = \vec{\mu}'Q\vec{\mu} - \vec{\mu}'QBP_+P_+'B'Q\vec{\mu} =$$

$$= \vec{\mu}'Q\vec{\mu} - \vec{\mu}'QBB'Q\vec{\mu} = \vec{\mu}'Q\vec{\mu} - \vec{\mu}'Q\Sigma Q\vec{\mu} = 0$$

when (3.2.28) holds. So (3.2.27) & (3.2.28) $\Rightarrow$ $c = 0$. By lemma 3.2.1, $c = 0 \Rightarrow Q\vec{\mu} \in M(Q\Sigma)$. So (3.2.27) & (3.2.28) $\Rightarrow Q\vec{\mu} \in M(Q\Sigma)$. □

We can now formulate the second main theorem.

THEOREM 3.2.2. *Let* $\vec{\underline{x}} \sim N_q(\vec{\mu},\Sigma)$. *Let* Q *be a real, symmetric, n.n.d matrix of order* q. *The quadratic form* $\vec{\underline{x}}'Q\vec{\underline{x}}$ *has a non-central* χ^2*-distribution iff*

(3.2.29) $Q\Sigma$ is idempotent,

(3.2.30) $\vec{\mu}'Q\vec{\mu} = \vec{\mu}'Q\Sigma Q\vec{\mu}$.

The number of degrees of freedom is then

(3.2.31) $\text{trace}(Q\Sigma)$

and the non-centrality parameter is

(3.2.32) $\vec{\mu}'Q\vec{\mu}$.

PROOF. The distribution of $\vec{\underline{x}}'Q\vec{\underline{x}}$ is given by (3.2.1)

$$\vec{\underline{x}}'Q\vec{\underline{x}} = c + \sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + \omega_\tau)^2,$$

where $\lambda_1,\ldots,\lambda_r$ are the positive eigenvalues of $B'QB$ (cf. theorem 3.2.1) and $\vec{\omega}$ and c are given by (3.2.4) and (3.2.5) respectively.

Clearly, $\vec{\underline{x}}'Q\vec{\underline{x}}$ has a non-central χ^2-distribution iff

(3.2.33) $\lambda_1 = \lambda_2 = \ldots = \lambda_r = 1$,

(3.2.34) $c = 0$.

The equivalence between (3.2.34) and (3.2.26) is proved in lemma 3.2.2. We shall now prove the equivalence of (3.2.33) and (3.2.25). In view of lemma 3.2.2, the necessary and sufficient conditions are then given by (3.2.29) and (3.2.30).

"(3.2.25) $\Rightarrow$ (3.2.33)". When $Q\Sigma$ is idempotent, then the positive eigenvalues of $Q\Sigma$ and hence of $B'QB$ (cf. corollary 3.1.1) are all equal to 1.

"(3.2.33) ⇒ (3.2.25)". When (3.2.33) holds, the positive canonical reduction of B'QB reduces to P_+P_+'. Then, using (3.2.9),

$$Q\Sigma Q\Sigma = QBB'QBB' = QBP_+P_+'B' = QBB' = Q\Sigma$$

so that $Q\Sigma$ is idempotent.
This proves the equivalence of (3.2.25) and (3.2.33).

Furthermore,

number of degrees of freedom =

= number of non-zero eigenvalues of B'QB =

= trace(B'QB) = trace($Q\Sigma$),

using the fact that the positive eigenvalues of B'QB are 1, that for a square matrix the sum of its eigenvalues is equal to its trace and the results of corollary 3.1.1.

For the non-centrality parameter (3.2.16) we find, using (3.2.19) and (3.2.30),

$$\delta = \sum_{\tau=1}^{r} \omega_\tau^2 = \vec{\mu}'Q\Sigma Q\vec{\mu} = \vec{\mu}'Q\vec{\mu}.$$

This completes the proof. □

REMARK 3.2.4. In theorem 3.2.2 we have not excluded the case that $Q\Sigma = 0_q$, in which case the distribution of $\vec{\underline{x}}'Q\vec{\underline{x}}$ reduces to

(3.2.35) $P(\vec{\underline{x}}'Q\vec{\underline{x}} = 0) = 1$,

but this does not lead to problems, when we consider such a distribution as a member of the class of non-central χ^2 - distributions.

Sometimes, the condition that Q is n.n.d is dropped. For completeness, we also quote the theorem concerning this case.

THEOREM 3.2.3. *Let* $\vec{\underline{x}} \sim N_q(\vec{\mu},\Sigma)$. *Let* A *be a real, symmetric matrix of order* q. *The quadratic form* $\vec{\underline{x}}'A\vec{\underline{x}}$ *has a non-central* χ^2 *- distribution iff*

(3.2.36) $\Sigma A \Sigma A \Sigma = \Sigma A \Sigma,$

(3.2.37) $\vec{\mu}'A\Sigma A\vec{\mu} = \vec{\mu}'A\vec{\mu},$

(3.2.38) $\Sigma A \Sigma A\vec{\mu} = \Sigma A\vec{\mu}.$

The number of degrees of freedom is

(3.2.39) $\mathrm{trace}(A\Sigma)$

and the non-centrality parameter is

(3.2.40) $\vec{\mu}'A\vec{\mu}.$

PROOF. RAYNER & LIVINGSTONE (1965). □

3.3. THE DISTRIBUTIONS OF $\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2$ AND $\sum_{\tau=1}^{r} \lambda_\tau (\underline{u}_\tau + \omega_\tau)^2$.

The distributions of $\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2$ and $\sum_{\tau=1}^{r} \lambda_\tau (\underline{u}_\tau + \omega_\tau)^2$, with $\vec{\underline{u}} \sim N(\vec{0}, I_r)$, $\lambda_1,\ldots,\lambda_r \in \mathbb{R}_+$, $\omega_1,\ldots,\omega_r \in \mathbb{R}$ have been studied extensively. The reader is referred to JOHNSON & KOTZ (1970), and KOTZ, JOHNSON & BOYD (1967a,b).

We list here some of the facts that we used. Let

(3.3.1) $\underline{Q} \overset{d}{=} Q(\vec{\underline{u}},\vec{\lambda},\vec{\omega}) \overset{d}{=} \sum_{\tau=1}^{r} \lambda_\tau (\underline{u}_\tau + \omega_\tau)^2,$

(3.3.2) $\underline{Q}_0 \overset{d}{=} Q(\vec{\underline{u}},\vec{\lambda},\vec{0}).$

Moments.

(3.3.3) $E\underline{Q} = \sum_{\tau=1}^{r} \lambda_\tau + \sum_{\tau=1}^{r} \lambda_\tau \omega_\tau^2 ,$

(3.3.4) $$\sigma^2(\underline{Q}) = 2 \sum_{\tau=1}^{r} \lambda_\tau^2 + 4 \sum_{\tau=1}^{r} \lambda_\tau^2 \omega_\tau^2 ,$$

(3.3.5) $$\mu_3(\underline{Q}) = 8 \sum_{\tau=1}^{r} \lambda_\tau^3 + 24 \sum_{\tau=1}^{r} \lambda_\tau^3 \omega_\tau^2 .$$

The moments of $\underline{Q}_0$ follow by omitting the terms containing ω_τ's. If $\lambda_1 = \lambda_2 = \ldots = \lambda_r = 1$, the moments reduce to those of the (non-) central χ^2-distribution.

<u>Asymptotic expansions.</u>

The distribution function of $\underline{Q} \equiv \sum_{\tau=1}^{r} \lambda_\tau (\underline{u}_\tau + \omega_\tau)^2$,

(3.3.6) $$F(z;\vec{\lambda};\vec{\omega}) \stackrel{d}{=} P(\underline{Q} \le z)$$

may be represented in an infinite series of central χ^2-distributions

(3.3.7) $$F(z;\vec{\lambda};\vec{\omega}) = \sum_{k=0}^{\infty} a_k P(\underline{\chi}^2[r+2k] \le \tfrac{z}{\beta}),$$

with coefficients a_k, recursively defined by

(3.3.8) $$a_0 \stackrel{d}{=} \exp\{-\tfrac{1}{2} \sum_{\tau=1}^{r} \omega_\tau^2\} \cdot \prod_{\tau=1}^{r} (\tfrac{\beta}{\lambda_\tau})^{\frac{1}{2}},$$

(3.3.9) $$a_k \stackrel{d}{=} \frac{1}{k} \sum_{\ell=0}^{k-1} b_{k-\ell} a_\ell, \qquad k = 1,2,\ldots ,$$

(3.3.10) $$\gamma_\tau \stackrel{d}{=} 1 - \beta/\lambda_\tau, \qquad \tau = 1,2,\ldots,r,$$

(3.3.11) $$b_k \stackrel{d}{=} \tfrac{1}{2}k \sum_{\tau=1}^{r} \omega_\tau^2 \gamma_\tau^{k-1} + \tfrac{1}{2} \sum_{\tau=1}^{r} (1-k\omega_\tau^2)\gamma_\tau^k, \qquad k = 1,2,\ldots ,$$

$\underline{\chi}^2[\nu]$ a r.v. with a χ^2-distribution with ν degrees of freedom, and $\beta > 0$ a suitably chosen constant. The choice of β leaves some room to influence the rate of convergence of the series. A good choice is

(3.3.12) $$\beta = \frac{2\lambda_1\lambda_r}{\lambda_1+\lambda_r} ,$$

so that $|\gamma_\tau| < 1$ for $\tau = 1,\ldots,r$. In our calculations in chapter 9 we always take β as defined by (3.3.12).

The series is uniformly convergent for any bounded interval of $z > 0$.

A bound for the error ($E_N(z)$) resulting from the truncation of (3.3.7) after the N'th term is given by

$$(3.3.13)\qquad E_N(z) \le a_0 \left\{ \frac{\Gamma(\frac{r}{2}+N+1)}{\Gamma(\frac{r}{2})(N+1)!} \right\} \left\{ \frac{\mu^{N+1}}{(1-\mu)^{r/2+N}} \right\} P(\underline{\chi}^2[r+2N+2] \le \frac{(1-\mu)z}{\beta}),$$

for $0 < \mu < 1$, and

$$(3.3.14)\qquad E_N(z) \le a_0 \frac{z}{\beta} f_{[r]}(\frac{z}{\beta}) \exp\{\frac{z\mu}{2\beta}\} (\frac{z\mu}{2\beta})^{N+1} \frac{1}{(N+1)!}$$

for $\mu > 1$, $f_{[\nu]}$ is the density of the χ^2-distribution with ν degrees of freedom, while

$$(3.3.15)\qquad \mu \overset{d}{=} \frac{1}{2} \sum_{\tau=1}^{r} \frac{\omega_\tau^2 \beta}{\lambda_\tau} + \max \left|1 - \frac{\beta}{\lambda_\tau}\right|.$$

This expansion can fruitfully be used for computer calculation, when a subroutine program for the χ^2-distribution is available. (See also the examples in chapter 9).

Approximations.

When no computer is available, the distribution of $\underline{Q}_0$ may be approximated, using the same method as proposed in chapter 1, by an adapted χ^2-distribution, i.e. the distribution of $b\underline{\chi}^2[\nu]$, where b and ν are chosen to make the first two moments agree with those of $\underline{Q}_0$, i.e.

$$(3.3.15)\qquad b = (\sum_{\tau=1}^{r} \lambda_\tau^2)/(\sum_{\tau=1}^{r} \lambda_\tau),$$

$$(3.3.16)\qquad \nu = (\sum_{\tau=1}^{r} \lambda_\tau)^2/(\sum_{\tau=1}^{r} \lambda_\tau^2).$$

An improvement is possible if we use $a + b\underline{\chi}^2[\nu]$ instead of $b\underline{\chi}^2[\nu]$. We have in that case,

$$(3.3.17)\qquad a = \sum_{\tau=1}^{r} \lambda_\tau - (\sum_{\tau=1}^{r} \lambda_\tau^2)^2/(\sum_{\tau=1}^{r} \lambda_\tau^3) \; ; \; b = (\sum_{\tau=1}^{r} \lambda_\tau^3)/(\sum_{\tau=1}^{r} \lambda_\tau^2),$$

$$(3.3.18)\qquad \nu = (\sum_{\tau=1}^{r} \lambda_\tau^2)^3/(\sum_{\tau=1}^{r} \lambda_\tau^3)^2.$$

However, for positive a, this approximation assigns the value 0 to $P(\underline{Q}_0 \le z)$, for all $0 \le z \le a$, so this approximation does not work well for small z.

The most simple approximation to the distribution of $\underline{Q}$ is the distribution of $\underline{\chi}^2[\nu,\delta^2]$ with

$$(3.3.19) \qquad \nu = \sum_{\tau=1}^{r} \lambda_\tau \ ; \qquad \delta^2 = \sum_{\tau=1}^{r} \lambda_\tau \omega_\tau^2 \ ,$$

so that only the first moments of $\underline{Q}$ and $\underline{\chi}^2[\nu,\delta^2]$ agree. When we take

$$(3.3.20) \qquad \nu = -\frac{1}{2}\sigma^2(\underline{Q}) + 2E(\underline{Q}),$$

$$(3.3.21) \qquad \delta^2 = \frac{1}{2}\sigma^2(\underline{Q}) - E(\underline{Q}),$$

the first two moments of $\underline{Q}$ and $\underline{\chi}^2[\nu,\delta^2]$ agree.

And when we take $c\underline{\chi}^2[\nu,\delta^2]$ to approximate $\underline{Q}$, with

$$(3.3.22) \qquad c = \frac{2\sigma^2 + \sqrt{4\sigma^4 - 2\mu_3 E}}{4E} \ ,$$

$$(3.3.23) \qquad \delta^2 = \frac{\sigma^2 - 2cE}{2c^2} \ ,$$

$$(3.3.24) \qquad \nu = \frac{E}{c} - \delta^2,$$

then the first three moments of $c\underline{\chi}^2[\nu,\delta^2]$ agree with those of $\underline{Q}$. ($E = E(\underline{Q})$, $\sigma^2 = \sigma^2(\underline{Q})$, $\mu_3 = \mu_3(\underline{Q})$).

The last approximation, however, is only possible when

$$(3.3.25) \qquad 4\sigma^4(\underline{Q}) - 2\mu_3(\underline{Q})E(\underline{Q}) > 0 \ .$$

All these approximations necessitate the use of a table of the non-central χ^2-distribution. The reader is referred to JOHNSON & KOTZ (1970), p.137, for a survey of existing tables and approximations of the non-central χ^2-distribution.

We shall now prove a lemma concerning a slightly more general situation, but which we shall apply to the distribution of $\underline{Q}$.

LEMMA 3.3.1. *Let* $\underline{u}$ *be a r.v. with an absolutely continuous distribution. Suppose that the density* f *of* $\underline{u}$ *is symmetric with respect to 0 and that* f *is strictly decreasing, continuously differentiable and positive on* $[0,\infty)$. *Let* $\underline{u}_1,\ldots,\underline{u}_r$ *be independent and identically distributed as* $\underline{u}$. *Let*

$\lambda_1,\ldots,\lambda_r \in \mathbb{R}_+$, $\omega_1,\ldots,\omega_r \in \mathbb{R}$, $\exists_{\tau \in \{1,\ldots,r\}}: \omega_\tau \neq 0$.

The function

$$H_r(t,z) \overset{d}{=} P(\sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + t\omega_\tau)^2 \le z), \qquad t \ge 0,$$

is then strictly decreasing in t *on* $[0,\infty)$ *for each* $z \in \mathbb{R}_+$.

<u>PROOF</u>. Without loss of generality we may suppose that $\omega_\tau \ge 0$ for $\tau = 1,\ldots,r$, where at least one of the inequalities is strict. Let, for $\tau = 1,\ldots,r$,

$$\underline{y}_\tau \overset{d}{=} \lambda_\tau(\underline{u}_\tau + t\omega_\tau)^2$$

with density $g_\tau(t,y)$ and distribution function $G_\tau(t,y)$, and

$$\underline{z}_r \overset{d}{=} \sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + t\omega_\tau)^2$$

with density $h_r(t,z)$ and distribution function $H_r(t,z)$.

We shall prove the lemma by induction. First consider the case $r = 1$. We have

$$H_1(t,z) = G_1(t,z) = P(\lambda_1(\underline{u}_1 + t\omega_1)^2 \le z) =$$

$$= P(-\sqrt{\frac{z}{\lambda_1}} \le \underline{u}_1 + t\omega_1 \le \sqrt{\frac{z}{\lambda_1}}) = \int_{-\sqrt{\frac{z}{\lambda_1}} - t\omega_1}^{\sqrt{\frac{z}{\lambda_1}} - t\omega_1} f(u)\,du \,.$$

So

$$\frac{\partial H_1(t,z)}{\partial t} = -\omega_1 f(\sqrt{\frac{z}{\lambda_1}} - t\omega_1) + \omega_1 f(-\sqrt{\frac{z}{\lambda_1}} - t\omega_1) < 0, \qquad \text{for } z > 0,$$

because f is symmetric w.r.t. 0 and strictly decreasing on $[0,\infty)$ and $\omega_1 \neq 0$. It follows that $H_1(t,z)$ is strictly decreasing in t for each fixed $z > 0$.

Next, suppose that $H_r(t,z)$ is strictly decreasing in t for each $z > 0$. Then we have for $r+1$,

$$H_{r+1}(t,z) = P(\underline{z}_{r+1} \le z) = P(\underline{z}_r + \underline{y}_{r+1} \le z) = P(\underline{z}_r + \underline{y}_1 \le z) =$$

$$= \int_0^z g_1(t,x)H_r(t,z-x)\,dx.$$

Now we have

$$\frac{\partial H_{r+1}(t,z)}{\partial t} = \int_0^z H_r(t,z-x)\ \frac{\partial}{\partial t}\, g_1(t,x)\,dx + \int_0^z g_1(t,x)\ \frac{\partial}{\partial t}\, H_r(t,z-x)\,dx.$$

Now $g_1(t,x) > 0$ for $x \in [0,z]$ and $\frac{\partial}{\partial t} H_r(t,z-x) < 0$ for fixed z-x by the induction hypothesis. It follows that the second term is negative.

For the first term we can write, using partial integration,

$$\int_0^z H_r(t,z-x)\ \frac{\partial}{\partial t}\, g_1(t,x)\,dx =$$

$$= \left[\ H_r(t,z-x)\ \frac{\partial}{\partial t}\, G_1(t,x)\ \right]_0^z + \int_0^z h_r(t,z-x)\ \frac{\partial}{\partial t}\, G_1(t,x)\,dx =$$

$$= \qquad 0 \qquad + \int_0^z \underbrace{h_r(t,z-x)}_{>\,0}\ \underbrace{\frac{\partial}{\partial t}\, G_1(t,x)}_{<\,0}\,dx < 0$$

So $H_{r+1}(t,z)$ is also strictly decreasing in t for each $z > 0$. It is left to the reader to verify that all operations used were permissable. □

3.4. MULTIVARIATE CENTRAL LIMIT THEOREM

Because of the fact that in our problem the vectors $\underline{\vec{t}}_i$ do not have identical distributions, we need a multivariate C.L.T. for unequal components. The most general form of C.L.T. that we need is a theorem for triangular array's. Because we have not been able to find a suitable reference in the literature, we give this theorem here with its proof. The proof is based on theorem's 27.2 and 29.4 of BILLINGSLEY (1979). The former is a Lindeberg-type C.L.T. for triangular array's of random variables, while the latter provides a standard way in which limit theorems for random vectors can be derived from corresponding theorems about random variables. (The Cramer-Wold device).

THEOREM 3.4.1. *Let* $(\underline{\vec{x}}_{m,1},\ldots,\underline{\vec{x}}_{m,m})$ *be a triangular array of* q*-dimensional random vectors such that for each* m*, the vectors* $\underline{\vec{x}}_{m,1},\ldots,\underline{\vec{x}}_{m,m}$ *are independent, and,*

$$(3.4.1) \qquad E\underline{\vec{x}}_{m,i} = \vec{0} \qquad\qquad i = 1,\ldots,m ,$$

$$(3.4.2) \qquad D(\underline{\vec{x}}_{m,i}) = \Sigma_{m,i} \qquad\qquad i = 1,\ldots,m .$$

Suppose that

$$(3.4.3) \qquad \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \Sigma_{m,i} = \Sigma \neq 0_q ,$$

and that for every $\varepsilon > 0$*,*

$$(3.4.4) \qquad \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \int_{\|\vec{x}\| > \varepsilon\sqrt{m}} \|\vec{x}\|^{2}\, dF_{m,i}(\vec{x}) = 0$$

where $F_{m,i}$ *is the distribution function of* $\underline{\vec{x}}_{m,i}$.

Then

$$(3.4.5) \qquad \frac{1}{\sqrt{m}}\sum_{i=1}^{m} \underline{\vec{x}}_{m,i} \overset{\sim}{\to} N(\vec{0},\Sigma), \qquad \text{as } m\to\infty.$$

PROOF. Let $\vec{\lambda} = (\lambda_1,\ldots,\lambda_q)'$ be an arbitrary q-dimensional vector. Define the following random variables

$$\underline{y}_{m,i} \overset{d}{=} \vec{\lambda}'\underline{\vec{x}}_{m,i} .$$

Then $(\underline{y}_{m,1},\ldots,\underline{y}_{m,m})$, $m = 1,2,\ldots$ is a triangular array of one-dimensional random variables with the following properties for each m and $i = 1,\ldots,m$:

i. the variables $\underline{y}_{m,1},\ldots,\underline{y}_{m,m}$ are independent;

ii. $E\underline{y}_{m,i} = E\vec{\lambda}'\underline{\vec{x}}_{m,i} = 0$;

iii. $\sigma^2(\underline{y}_{m,i}) = \sigma^2(\vec{\lambda}'\underline{\vec{x}}_{m,i}) = \vec{\lambda}'\Sigma_{m,i}\vec{\lambda}$.

Define

$$s_m^2 \overset{d}{=} \sum_{i=1}^{m} \sigma^2(\underline{y}_{m,i}) = \vec{\lambda}'(\sum_{i=1}^{m} \Sigma_{m,i})\vec{\lambda} .$$

Note that (3.4.3) gives

$$\lim_{m\to\infty} \frac{s_m^2}{m} = \vec{\lambda}'(\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \Sigma_{m,i})\vec{\lambda} = \vec{\lambda}'\Sigma\vec{\lambda}.$$

We shall proceed to show that the Lindeberg-condition of theorem 27.2 of BILLINGSLEY (1979) is satisfied for the just defined variables $\underline{y}_{m,i}$. Let $G_{m,i}$ be the distribution function of $\underline{y}_{m,i}$. Then we have, by the Cauchy-Schwartz inequality, for every $\varepsilon > 0$,

$$\frac{1}{s_m^2} \sum_{i=1}^{m} \int_{|y| > \varepsilon s_m} y^2 dG_{m,i}(y) =$$

$$= \frac{1}{s_m^2} \sum_{i=1}^{m} \int_{|\vec{\lambda}'\vec{x}| > \varepsilon s_m} (\vec{\lambda}'\vec{x})^2 dF_{m,i}(\vec{x}) \leq$$

$$\leq \frac{1}{s_m^2} \vec{\lambda}'\vec{\lambda} \sum_{i=1}^{m} \int_{|\vec{\lambda}'\vec{x}| > \varepsilon s_m} (\vec{x}'\vec{x}) dF_{m,i}(\vec{x}) =$$

$$= \frac{||\vec{\lambda}||^2}{s_m^2} \sum_{i=1}^{m} \int_{|\vec{\lambda}'\vec{x}| > \varepsilon s_m} ||\vec{x}||^2 dF_{m,i}(\vec{x}).$$

For every $\varepsilon > 0$, there exists a $\varepsilon' > 0$, such that for sufficiently large m $(m \geq m'(\varepsilon))$,

$$\{\vec{x} \in \mathbb{R}^q | \ |\vec{\lambda}'\vec{x}| > \varepsilon s_m\} \subset \{\vec{x} \in \mathbb{R}^q | \ ||\vec{x}||^2 > (\varepsilon'\sqrt{m})^2\} .$$

So we have, for $m \geq m'(\varepsilon)$,

$$\frac{||\vec{\lambda}||^2}{s_m^2/m} \frac{1}{m} \sum_{i=1}^{m} \int_{|\vec{\lambda}'\vec{x}| > \varepsilon s_m} ||\vec{x}||^2 dF_{m,i}(\vec{x}) \leq$$

$$\leq \frac{||\vec{\lambda}||^2}{s_m^2/m} \frac{1}{m} \sum_{i=1}^{m} \int_{||\vec{x}|| > \varepsilon'\sqrt{m}} ||\vec{x}||^2 dF_{m,i}(\vec{x}).$$

Now, $\vec{\lambda}$ is fixed, s_m^2/m converges to a constant which may be taken to be unequal to zero (treat the case $\vec{\lambda}'\Sigma\vec{\lambda} = 0$ separately). With (3.4.4) it then follows that, for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{s_m^2} \sum_{i=1}^{m} \int_{|y| > \varepsilon s_m} y^2 dG_{m,i}(y) = 0 \;.$$

Applying the above mentioned theorem 27.2, it follows that

$$\frac{1}{s_m^2} \sum_{i=1}^{m} \underline{y}_{m,i} \overset{\sim}{\to} N(0,1)$$

as $m \to \infty$, for every $\vec{\lambda} \in \mathbb{R}^q$. If we take $\vec{\underline{x}} \sim N(\vec{0},\Sigma)$, this result may be written as

$$\frac{1}{s_m^2} \vec{\lambda}'(\sum_{i=1}^{m} \vec{\underline{x}}_{m,i}) \overset{L}{\to} \frac{1}{(\vec{\lambda}'\Sigma\vec{\lambda})^{\frac{1}{2}}} \vec{\lambda}'\vec{\underline{x}}$$

as $m \to \infty$, for every $\vec{\lambda} \in \mathbb{R}^q$.

Because $s_m/\sqrt{m} \to (\vec{\lambda}'\Sigma\vec{\lambda})^{\frac{1}{2}}$, as $m \to \infty$, we have also

$$\vec{\lambda}'(\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \vec{\underline{x}}_{m,i}) \overset{L}{\to} \vec{\lambda}'\vec{\underline{x}} \;,$$

as $m \to \infty$, for every $\vec{\lambda} \in \mathbb{R}^q$.

The proof is now completed with theorem 29.4 of BILLINGSLEY (1979), which states that $\vec{\underline{x}}_m \overset{L}{\to} \vec{\underline{x}}$ iff $\vec{\lambda}'\vec{\underline{x}}_m \overset{L}{\to} \vec{\lambda}'\vec{\underline{x}}$ for each $\vec{\lambda} \in \mathbb{R}^q$. □

CHAPTER 4

CONSISTENCY, ASYMPTOTIC DISTRIBUTIONS AND POWER

4.1. CONSISTENCY

A sequence of level-α tests $\{\underline{\varphi}_{m,Q}\}$ is consistent against a fixed alternative $\mathcal{a} \in \mathcal{A}$ (cf. section 2.3.), iff for $m \to \infty$

(4.1.1) $\quad E_{\mathcal{a}}\underline{\varphi}_{m,Q} \to 1.$

It is desirable that (4.1.1) holds for each $\alpha \in (0,1)$, so we shall call $\{\underline{\varphi}_{m,Q}\}$ consistent only if this is the case.

The class of alternatives against which $\{\underline{\varphi}_{m,Q}\}$, based on $\underline{v}(Q) \equiv \vec{\underline{t}}_*'Q\vec{\underline{t}}_*$, is consistent depends on the choice of Q. In this section we shall determine this class.

In section 4.3 we shall prove that, under H_0 and assumption 1, the distribution of $\vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ converges to a fixed distribution. It follows that the sequence of critical values $\{k_{1-\alpha}(m,Q)\}$ is at least bounded. Therefore (4.1.1) holds for every $\alpha \in (0,1)$ iff

(4.1.2) $\quad P_{\mathcal{a}}(\vec{\underline{t}}_*'Q\vec{\underline{t}}_* \geq M) \to 1 \qquad$ for each $M \in \mathbb{R}$.

THEOREM 4.1.1. *A necessary and sufficient condition for $\{\underline{\varphi}_{m,Q}\}$ to be consistent against a fixed alternative $\mathcal{a} \in \mathcal{A}$, is that*

(4.1.3) $\quad \lim_{m\to\infty} \vec{\delta}_*'Q\vec{\delta}_* = \infty$

for this alternative. The class of alternatives for which $\{\underline{\varphi}_{m,Q}\}$ is consistent forms a subclass of $\mathcal{A}_1$.

PROOF. The test $\{\underline{\varphi}_{m,Q}\}$ is consistent against $\mathcal{a} \in \mathcal{A}$ iff (4.1.2) holds. From the Cantelli inequality (RAO (1973)) it follows that for a sequence of random variables $\{\underline{y}_m\}$, (4.1.4) $\quad P(\underline{y}_m \geq M) \to 1 \qquad$ for each $M \in \mathbb{R}$

iff

(4.1.5) $\quad E\underline{y}_m \to \infty \qquad$ and $\qquad E\underline{y}_m/\sigma(\underline{y}_m) \to \infty$.

First suppose that *not* $\lim_{m\to\infty} \vec{\delta}_*'Q\vec{\delta}_* = \infty$. This means that $\vec{\delta}_*'Q\vec{\delta}_*$ has a finite limit point, $d \geq 0$, say. If we take $\underline{y}_m \overset{d}{=} \vec{\underline{t}}_*'Q\vec{\underline{t}}_*$, with $E_a\underline{y}_m =$ trace $Q\Sigma_{1.} + \vec{\delta}_*'Q\vec{\delta}_*$, there exists a subsequence $\{\underline{y}_{m_k}\}_{k=1}^{\infty}$, such that

(4.1.6) $\quad E\underline{y}_{m_k} \to \text{trace } Q\Sigma_1 + d < \infty.$

So (4.1.5) is not satisfied for this subsequence, and therefore the limit $\lim_{m\to\infty} P(\underline{y}_m \geq M)$ does not exist. It follows that $\{\underline{\varphi}_{m,Q}\}$ is not consistent.

Next, suppose that $\lim_{m\to\infty} \vec{\delta}_*'Q\vec{\delta}_* = \infty$. With $\vec{\underline{u}}_* \equiv \vec{\underline{t}}_* - \vec{\delta}_*$, we have

(4.1.7) $\quad \vec{\underline{t}}_*'Q\vec{\underline{t}}_* \equiv \vec{\underline{u}}_*'Q\vec{\underline{u}}_* + 2\vec{\delta}_*'Q\vec{\underline{u}}_* + \vec{\delta}_*'Q\vec{\delta}_*.$

Take this time $\underline{y}_m \overset{d}{=} 2\vec{\delta}_*'Q\vec{\underline{u}}_* + \vec{\delta}_*'Q\vec{\delta}_*$, then $E\underline{y}_m = \vec{\delta}_*'Q\vec{\delta}_*$, $\sigma^2(\underline{y}_m) = 4\vec{\delta}_*'Q\Sigma_{1.}Q\vec{\delta}_*$, and it follows from lemma 3.1.8 that (4.1.5) is satisfied. The consistency now follows from

(4.1.8) $\quad P_a(\vec{\underline{t}}_*'Q\vec{\underline{t}}_* \geq M) \geq P_a(\underline{y}_m \geq M)$

which is true because Q is n.n.d, and therefore $\vec{\underline{u}}_*'Q\vec{\underline{u}}_* \geq 0$ with probability 1.

The last statement of the theorem follows because $\vec{\delta}_*'Q\vec{\delta}_* \to \infty$ implies $\vec{\delta}_*'\vec{\delta}_* \to \infty$ by lemma 3.1.7. This completes the proof. □

THEOREM 4.1.2. *A sufficient condition for* $\{\underline{\varphi}_{m,Q}\}$ *to be consistent against each alternative in* A_1, *is that* $\vec{\delta}_* \in M(Q)$ *for each alternative in* A_1.

PROOF. Assume $\vec{\delta}_* \in M(Q)$ for each alternative in A_1. It follows from lemma 3.1.7 that then $\vec{\delta}_*'\vec{\delta}_* \to \infty \Longleftrightarrow \vec{\delta}_*'Q\vec{\delta}_* \to \infty$. Apply now theorem 4.1.1. □

REMARK 4.1.1. It follows in particular from theorem 4.1.2 that a sufficient condition for the consistency of $\{\underline{\varphi}_{m,Q}\}$ is that Q is non-singular.

The question naturally arises now if there exists a Q and an $a \in A_1$ for which $\vec{\delta}_*'Q\vec{\delta}_* \not\to \infty$. The above remark shows that Q has to be singular if this is to be true. A trivial example is furnished if we take $Q \stackrel{d}{=} I_n \otimes 1_k$, with rank n, because $\vec{\delta}_*' Q \vec{\delta}_* = 0$ ($\underline{\vec{t}}_*' Q \underline{\vec{t}}_* \equiv 0$) for each m and each $\alpha \in A_1$ for this choice of Q.

A more important question is the following. Does there exist, given a singular Q, an alternative $\alpha \in A_1$ for which $\vec{\delta}_*' Q \vec{\delta}_* \not\to \infty$, i.e. an alternative for which $\{\underline{\varphi}_{m,Q}\}$ is not consistent. The answer is: not always; there are singular matrices Q such that $\{\underline{\varphi}_{m,Q}\}$ is consistent against all alternatives in A_1.

Recall that the vectors $\vec{\delta}_i$ and $\vec{\delta}_*$ are elements of the null-space $N(F)$ of the matrix F defined in section 3.1. Conversely, under certain circumstances, any element of $N(F)$ may, apart from a constant factor, occur as a vector $\vec{\delta}_i$, as follows from the following lemma.

LEMMA 4.1.1. *Let* $n \geq k \geq 4$. *Let* $\vec{\eta}$ *be an arbitrary element from* $N(F)$. *Then for any experiment* E_i, *with* $a_{i1} > 0$, $a_{i2} > 0, \ldots, a_{ik} > 0$, *there exist constants* $\Delta_{i1}, \ldots, \Delta_{iN_i}$ *, and a constant* c, *such that*

$$\vec{\delta} \stackrel{d}{=} \vec{\eta}/c = E\underline{\vec{t}}_i . \tag{4.1.9}$$

PROOF. For the experiment E_i, with $a_{i1} > 0, \ldots, a_{ik} > 0$, we have if $n \geq k \geq 4$,

$$N_i = \frac{n!}{a_{i1}!\, a_{i2}! \ldots a_{ik}!} \geq \frac{n!}{(n-(k-1))!} = n(n-1)\ldots(n-k+2) \geq nk+1 .$$

Introduce the variables $x_1, \ldots, x_{N_i}$ and solve the following $nk+1$ equations (writing $\vec{\eta} = (\eta_1^{(1)}, \ldots, \eta_k^{(1)}, \eta_1^{(2)}, \ldots, \eta_k^{(2)}, \ldots, \eta_1^{(n)}, \ldots, \eta_k^{(n)})'$).

$$\eta_j^{(\nu)} = \sum_{r=1}^{N_i} t_{ij}^{(\nu)}(\pi_{ir}) x_r , \qquad \sum_{r=1}^{N_i} x_r = 0 .$$

Because the number of variables (N_i) exceeds the number of equations ($nk+1$), there is always a solution.

Next, choose a constant c such that the solutions x_r satisfy

$$-\frac{1}{N_i} \leq \frac{x_r}{c} \leq 1 - \frac{1}{N_i} .$$

Then we may take $\Delta_{ir} = x_r/c$, from which follows that $E\underline{\vec{t}}_i = \vec{\eta}/c = \vec{\delta}$. □

We then have the following theorem.

THEOREM 4.1.3. *Let* $n \geq k \geq 4$. *Let* Q *be a singular (real, symmetric and n.n.d)* $nk \times nk$ *matrix, with* rank $Q < (n-1)(k-1)$. *Then there exists an alternative* $a \in A_1$ *such that* $\lim_{m\to\infty} \vec{\delta}_*'\vec{\delta}_* = \infty$, *but* $\lim_{m\to\infty} \vec{\delta}_*'Q\vec{\delta}_* = 0$.

PROOF. Because rank $Q < (n-1)(k-1)$, $N(Q) \cap N(F) \neq \{\vec{0}\}$. So take $\vec{\eta} \in N(Q) \cap N(F)$, $\vec{\eta} \neq \vec{0}$. By lemma 4.1.1, there now exist constants c and $\Delta_{i1},\ldots,\Delta_{iN_i}$ and an associated experiment E_i such that $E\underline{\vec{t}}_i = \vec{\eta}/c \overset{d}{=} \vec{\delta}_i$.
Also $\vec{\delta}_i \in N(Q) \cap N(F)$. It follows that

$$\vec{\delta}_*'\vec{\delta}_* = \left(\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\vec{\delta}_i\right)'\left(\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\vec{\delta}_i\right) = \frac{m}{c^2}\vec{\eta}'\vec{\eta} \to \infty,$$

but

$$\vec{\delta}_*'Q\vec{\delta}_* = \frac{m}{c^2}\vec{\eta}'Q\vec{\eta} = 0$$

for each m, because $\vec{\eta} \in N(Q)$. □

This theorem shows that, if the test is not directed against a very specialised alternative, test-statistics with a matrix Q with rank smaller than (n-1)(k-1) should be avoided, because there are then always alternatives that cannot be detected.

When rank Q is larger than or equal to (n-1)(k-1), two different situations can occur. We show, by examples, that it is possible that there still is an alternative for which $\{\underline{\varphi}_{m,Q}\}$ is not consistent (example 4.1.1) and it is also possible that, even though Q is singular, $\{\underline{\varphi}_{m,Q}\}$ is consistent against all alternatives in A_1 (example 4.1.2).

EXAMPLE 4.1.1. Take n = k = 4. Choose a sequence E_1, E_2, ... with $a_{i1} > 0$, $a_{i2} > 0,\ldots,\ a_{ik} > 0$. Define a c and $\Delta_{i1},\ldots,\Delta_{iN_i}$ such that

$$\vec{\delta}_i \overset{d}{=} E\underline{\vec{t}}_i = c(1,-1,0,0;-1,1,0,0;0,0,0,0;0,0,0,0)'$$

This is possible by lemma 4.1.1. This defines an alternative a to H_0 with

$$\vec{\delta}_*'\vec{\delta}_* = mc^2 \cdot 4 \to \infty, \qquad \text{as } m \to \infty,$$

so $a \in A_1$.

Take

$$Q \stackrel{d}{=} \left(\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 1 & 1 & \\ 1 & 1 & 0 & 0 & 1 & 1 & \\ 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0_{6,10} \\ 1 & 1 & 0 & 0 & 1 & 1 & \\ 1 & 1 & 0 & 0 & 1 & 1 & \\ \hline & & & 0_{10,6} & & & I_{10} \end{array}\right)$$

Q is a real, symmetric, n.n.d matrix of order 16, with rank 13, which is larger than $(n-1)(k-1) = 3 \cdot 3 = 9$. Obviously $\vec{\delta}_*' Q \vec{\delta}_* = 0$ for each m. It follows that for this particular alternative $\{\underline{\varphi}_{m,Q}\}$ is not consistent.

<u>EXAMPLE 4.1.2</u>. Take $n \geq 2$ and $k \geq 2$ arbitrary. Let

$$Q \stackrel{d}{=} \left(\begin{array}{c|c} I_{nk-k} & 0_{nk-k,k} \\ \hline 0_{k,nk-k} & 0_{k,k} \end{array}\right)$$

Obviously, Q is a real, symmetric, n.n.d matrix of order nk, with rank $nk-k > (n-1)(k-1)$. Consider an arbitrary alternative $a \in A_1$. Now

$$(4.1.10) \qquad \vec{\delta}_*' Q \vec{\delta}_* = \sum_{\nu=1}^{n-1} \sum_{j=1}^{k} \{\delta_{*j}^{(\nu)}\}^2 .$$

Because $a \in A_1$, we have $\vec{\delta}_*' \vec{\delta}_* \to \infty$, so at least one component of $\vec{\delta}_*$ must (in absolute value) tend to infinity. Due to the linear relationships $\delta_{*+}^{(\nu)} = 0$ for all ν, and $\delta_{*j}^{(+)} = 0$ for all j, this means that

i. at least four components must in absolute value tend to infinity;

ii. those four components cannot all be among $\delta_{*1}^{(n)}, \delta_{*2}^{(n)}, \ldots, \delta_{*k}^{(n)}$.

It follows that at least two terms of the sum (4.1.10) must tend to infinity, and so $\vec{\delta}_*' Q \vec{\delta}_* \to \infty$.

Because $a \in A_1$ was arbitrary, we have

$$\vec{\delta}_*' \vec{\delta}_* \to \infty \Rightarrow \vec{\delta}_*' Q \vec{\delta}_* \to \infty \qquad \text{for each } a \in A_1 .$$

From theorem 4.1.1 it then follows that $\{\underline{\varphi}_{m,Q}\}$ is consistent against each $a \in A_1$.

REMARK 4.1.2. Notice that, so far, Q is assumed to be a fixed, real, symmetric and n.n.d matrix, and that such a Q is chosen irrespective of the number of experiments m. This applies in particular to the examples in this section. However, in order to avoid problems like those outlined in theorem 4.1.3, or simply to obtain better results, the choice of Q is allowed to depend on, for instance, the values $\vec{a}_1,\ldots,\vec{a}_m$ obtained in the sequence of experiments $E_1,\ldots,E_m$. In the analysis of the asymptotic behaviour of our tests, this means that Q may depend on m, Q_m say. This does not affect our results, provided that

(4.1.11) $$\lim_{m\to\infty} Q_m = Q$$

exists and Q is a real, symmetric and n.n.d matrix. In order to avoid intricate notation we continue to write Q for Q_m, unless we want to stress the dependence on m.

REMARK 4.1.3. In section 6.1 we consider the following special choice for Q:

(4.1.12) $$Q = Q_m = \Sigma_{0\cdot}^{-},$$

where $\Sigma_{0\cdot}^{-}$ is any g-inverse of $\Sigma_{0\cdot}$ (Because the elements of a g-inverse of a matrix depend in a continuous way on the elements of the matrix, and the limit $\lim_{m\to\infty} \Sigma_{0\cdot}$ exists by assumption, also the limit $\lim_{m\to\infty} \Sigma_{0\cdot}^{-}$ exists.) Suppose that rank $\Sigma_{0\cdot} = (n-1)(k-1)$ (this rank is usually obtained for a sufficiently large value of m). Depending on the choice of particular g-inverse, rank $\Sigma_{0\cdot}^{-}$ may then vary from $(n-1)(k-1)$ to nk. In section 6.1 it is established that the test with such a Q is consistent against each alternative in $\mathcal{A}_1$. This is therefore a second example of the situation of example 4.1.2.

4.2. ASYMPTOTIC MULTIVARIATE NORMALITY

In this section we shall establish the asymptotic normality of $\underline{\vec{t}}_*$, under the assumptions 1 & 2 of section 2.3. Under these conditions, the dispersion matrix of $\underline{\vec{t}}_*$ converges to a limit as $m \to \infty$.

Recall that

$$(4.2.1) \qquad \Sigma_1 \overset{d}{\equiv} \lim_{m\to\infty} \Sigma_{1\cdot} = \lim_{m\to\infty} D(\underline{\vec{t}}_*),$$

$$(4.2.2) \qquad \Sigma_0 \overset{d}{\equiv} \lim_{m\to\infty} \Sigma_{0\cdot} = \lim_{m\to\infty} D(\underline{\vec{t}}_*|H_0),$$

$$(4.2.3) \qquad \vec{\delta} \overset{d}{\equiv} \lim_{m\to\infty} \vec{\delta}_* = \lim_{m\to\infty} E\underline{\vec{t}}_* ,$$

$$(4.2.4) \qquad \vec{\zeta} \overset{d}{\equiv} \lim_{m\to\infty} \vec{\delta}_\cdot .$$

<u>THEOREM 4.2.1</u>. *Under the assumptions 1 & 2 of section 2.3, we have for* $m \to \infty$,

$$(4.2.5) \qquad \underline{\vec{u}}_* \equiv \underline{\vec{t}}_* - \vec{\delta}_* \overset{\sim}{\to} N(\vec{0},\Sigma_1).$$

<u>PROOF</u>. $\underline{\vec{u}}_1, \underline{\vec{u}}_2, \ldots$ is a sequence of independent $n \times k$ dimensional random vectors, with $E\underline{\vec{u}}_i = \vec{0}$ and dispersion matrix $D(\underline{\vec{u}}_i) = D(\underline{\vec{t}}_i)$. Under the assumptions 1 & 2 we have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} D(\underline{\vec{u}}_i) = \lim_{m\to\infty} D(\underline{\vec{u}}_*) = \lim_{m\to\infty} D(\underline{\vec{t}}_*) = \Sigma_1.$$

Because there exists a constant c, such that $||\underline{\vec{u}}_i||^2 \le c$ with probability one, for $i = 1,2,\ldots$, the 'Lindeberg' condition of theorem 3.4.1 is trivially fulfilled. (Adaptation of theorem 3.4.1 to ordinary sequences of random vectors is straightforward). Therefore, all the conditions of theorem 3.4.1 are fulfilled and the result follows. □

<u>COROLLARY 4.2.1</u>. *Under the assumptions 1 & 2 of section 2.3 and* H_0, *we have for* $m \to \infty$,

$$(4.2.6) \qquad \underline{\vec{t}}_* \overset{\sim}{\to} N(\vec{0},\Sigma_0).$$

<u>PROOF</u>. Under H_0, $\vec{\delta} = \vec{0}$ and $D(\underline{\vec{t}}_*) \to \Sigma_0$. □

COROLLARY 4.2.2. *Under the assumptions 1 & 2, we have for alternatives in* $A_2 \cup A_3$,

$$(4.2.7) \qquad \vec{\underline{t}}_* \overset{\sim}{\to} N(\vec{\delta},\Sigma_1).$$

PROOF. Because $\vec{\underline{t}}_* \equiv \vec{\underline{u}}_* + \vec{\delta}_*$, $\vec{\underline{u}}_* \overset{\sim}{\to} N(\vec{0},\Sigma_1)$ by theorem 4.2.1, $\vec{\delta}_* \to \vec{\delta}$ by assumption 2, the result follows from a Cramer-type theorem. □

For "close" alternatives (cf. section 2.4) we have the following theorem.

THEOREM 4.2.2. *Under close alternatives we have*

$$(4.2.8) \qquad \vec{\underline{t}}_* \overset{\sim}{\to} N(\sqrt{\eta}\,\vec{\zeta},\Sigma_0).$$

PROOF. The proof is analogous to the proof of theorem 4.2.1, this time using the triangular-array method explicitly. □

4.3. ASYMPTOTIC DISTRIBUTION OF THE TEST-STATISTIC

In this section we derive the asymptotic distribution of $\underline{v}(Q) \equiv \vec{\underline{t}}_*'Q\vec{\underline{t}}_*$.

THEOREM 4.3.1. *Under* H_0, *under alternatives from* $A_2 \cup A_3$ *and under close alternatives, we have*

$$(4.3.1) \qquad \underline{v}(\vec{Q}) \equiv \vec{\underline{t}}_*'Q\vec{\underline{t}}_* \overset{L}{\to} \vec{\underline{x}}'Q\vec{\underline{x}}$$

where

$$(4.3.2) \qquad \vec{\underline{x}} \sim N(\vec{0},\Sigma_0), \text{ under } H_0;$$

$$(4.3.3) \qquad \vec{\underline{x}} \sim N(\vec{\delta},\Sigma_1), \text{ under alternatives from } A_2 \cup A_3;$$

$$(4.3.4) \qquad \vec{\underline{x}} \sim N(\sqrt{\eta}\,\vec{\zeta},\Sigma_0), \text{ under close alternatives.}$$

PROOF. In all three cases we have $\vec{\underline{t}}_* \overset{L}{\to} \vec{\underline{x}}$, where $\vec{\underline{x}}$ has one of the distributions (4.3.2)-(4.3.4). Because $(\cdot)'Q(\cdot)$ is a continuous function, the result follows. [When Q depends on m, Q_m say, with $Q_m \to Q$, write $\vec{\underline{t}}_*'Q_m\vec{\underline{t}}_* \equiv \vec{\underline{t}}_*'Q\vec{\underline{t}}_* + \vec{\underline{t}}_*'(Q_m - Q)\vec{\underline{t}}_*$, then $\vec{\underline{t}}_*'(Q_m - Q)\vec{\underline{t}}_* \overset{P}{\to} 0$ and the same result follows.] □

Note that the distributions of $\underline{\vec{x}}'Q\underline{\vec{x}}$ follow from theorem 3.2.1.

For alternatives in A_1, the situation is somewhat different. Although $\{\underline{\varphi}_{m,Q}\}$ is consistent against $a \in A_1$ when $\vec{\delta}_*'Q\vec{\delta}_* \to \infty$, the asymptotic distribution (a.d.) does not exist in all cases. We shall give some examples where the a.d. does exist.

<u>THEOREM 4.3.2</u>. *For alternatives in* A_1, $\underline{v}(Q) \equiv \underline{\vec{t}}_*'Q\underline{\vec{t}}_*$ *has the following a.d.*

$$(4.3.5) \qquad \frac{\underline{\vec{t}}_*'Q\underline{\vec{t}}_* - \vec{\delta}_*'Q\vec{\delta}_*}{\sqrt{m}} \overset{\sim}{\to} N(0,\sigma^2),$$

where

$$(4.3.6) \qquad \sigma^2 = 4\vec{\zeta}'Q\Sigma_1 Q\vec{\zeta}.$$

<u>PROOF</u>. From (4.1.7) it follows that

$$\frac{\underline{\vec{t}}_*'Q\underline{\vec{t}}_* - \vec{\delta}_*'Q\vec{\delta}_*}{\sqrt{m}} \equiv \frac{\underline{\vec{u}}_*'Q\underline{\vec{u}}_*}{\sqrt{m}} + 2\vec{\delta}_{\bullet}'Q\underline{\vec{u}}_*.$$

From theorem 4.2.1 it follows that $\underline{\vec{u}}_* \overset{\sim}{\to} N(\vec{0},\Sigma_1)$. As in theorem 4.3.1 $\underline{\vec{u}}_*'Q\underline{\vec{u}}_*$ then converges to a fixed distribution. Therefore $m^{-\frac{1}{2}}(\underline{\vec{u}}_*'Q\underline{\vec{u}}_*) \overset{P}{\to} 0$. Furthermore, from $\vec{\delta}_{\bullet} \to \vec{\zeta}$ it follows that $2\vec{\delta}_{\bullet}'Q\underline{\vec{u}}_* \overset{L}{\to} 2\vec{\zeta}'Q\underline{\vec{x}}$, where $\underline{\vec{x}} \sim N(\vec{0},\Sigma_1)$. The result follows from $E(2\vec{\zeta}'Q\underline{\vec{x}}) = 0$ and $\sigma^2(2\vec{\zeta}'Q\underline{\vec{x}}) = 4\vec{\zeta}'Q\Sigma_1 Q\vec{\zeta}$. □

Note that it may occur that σ^2, defined by (4.3.6), is equal to zero. The a.d. of $\underline{v}(Q)$ is then degenerate. Other transformations may still yield a proper a.d., as is illustrated by the following theorems. Let

$$(4.3.7) \qquad \sigma_m^2 \overset{d}{=} \sigma^2(2\vec{\delta}_*'Q\underline{\vec{u}}_*) = 4\vec{\delta}_*'Q\Sigma_{1\bullet}Q\vec{\delta}_*.$$

<u>THEOREM 4.3.3</u>. *For alternatives in* A_1, *and matrices* Q *such that* $\sigma_m^2 \to 0$, $\underline{v}(Q) \equiv \underline{\vec{t}}_*'Q\underline{\vec{t}}_*$ *has the following a.d.*

$$(4.3.8) \qquad \underline{\vec{t}}_*'Q\underline{\vec{t}}_* - \vec{\delta}_*'Q\vec{\delta}_* \overset{L}{\to} \underline{\vec{x}}'Q\underline{\vec{x}} \ ,$$

where $\underline{\vec{x}} \sim N(\vec{0},\Sigma_1)$.

<u>PROOF</u>. Because $\sigma_m^2 \to 0$ and $\underline{\vec{u}}_* \overset{\sim}{\to} N(\vec{0},\Sigma_1)$, the result follows from the identity (4.1.7). □

Note that $\sigma_m^2 \to 0$ implies that $4\vec{\zeta}'Q\Sigma_1 Q\vec{\zeta} = 0$, so the situation of theorem 4.3.3 is not covered by theorem 4.3.2.

THEOREM 4.3.4. *For alternatives in* A_1, *and matrices* Q *such that* $\sigma_m^2 \to \infty$ *and* $\lim_{m\to\infty} \vec{\delta}_*/\sigma_m$ *exists,* $\underline{v}(Q) \equiv \underline{\vec{t}}_*'Q\underline{\vec{t}}_*$ *has the following a.d.*

$$(4.3.9) \qquad \sigma_m^{-1}(\underline{\vec{t}}_*'Q\underline{\vec{t}}_* - \vec{\delta}_*'Q\vec{\delta}_*) \stackrel{\sim}{\to} N(0,1).$$

PROOF. From (4.1.7) it follows that

$$\sigma_m^{-1}(\underline{\vec{t}}_*'Q\underline{\vec{t}}_* - \vec{\delta}_*'Q\vec{\delta}_*) \equiv \sigma_m^{-1}\underline{\vec{u}}_*'Q\underline{\vec{u}}_* + 2\sigma_m^{-1}\vec{\delta}_*'Q\underline{\vec{u}}_*.$$

Because $\underline{\vec{u}}_*'Q\underline{\vec{u}}_*$ converges to a fixed distribution, $\sigma_m^{-1}\underline{\vec{u}}_*'Q\underline{\vec{u}}_* \stackrel{P}{\to} 0$. Furthermore $E(2\sigma_m^{-1}\vec{\delta}_*'Q\underline{\vec{u}}_*) = 0$ and $\sigma^2(2\sigma_m^{-1}\vec{\delta}_*'Q\underline{\vec{u}}_*) = 1$. The random variable $2\sigma_m^{-1}\vec{\delta}_*'Q\underline{\vec{u}}_*$ converges because $\lim_{m\to\infty} \sigma_m^{-1}\vec{\delta}_*$ exists and $\underline{\vec{u}}_* \stackrel{\sim}{\to} N(\vec{0},\Sigma_1)$. □

THEOREM 4.3.5. *For those alternatives in* A_1, *for which* $\lim_{m\to\infty} \vec{\delta}_*/\|\vec{\delta}_*\|$ *exists,* $\underline{v}(Q) \equiv \underline{\vec{t}}_*'Q\underline{\vec{t}}_*$ *has the following a.d.*

$$(4.3.10) \qquad \|\vec{\delta}_*\|^{-1}(\underline{\vec{t}}_*'Q\underline{\vec{t}}_* - \vec{\delta}_*'Q\vec{\delta}_*) \stackrel{\sim}{\to} N(0,\tau^2)$$

where

$$(4.3.11) \qquad \tau^2 = \lim_{m\to\infty} \|\vec{\delta}_*\|^{-2}\sigma_m^2 .$$

PROOF. The proof is analogous to the proof of theorem 4.3.4. □

4.4 APPROXIMATE CRITICAL VALUES AND APPROXIMATE POWER.

In section 4.2 we showed

$$(4.4.1) \qquad \vec{\underline{u}}_* \equiv \vec{\underline{t}}_* - \vec{\delta}_* \ \tilde{\to} \ N(\vec{0},\Sigma_1), \qquad m \to \infty,$$

where $\vec{\delta}_*$ and Σ_1 are calculated under a particular alternative $a \in A$. Under H_0 we have

$$(4.4.2) \qquad \vec{\underline{t}}_* \ \tilde{\to} \ N(\vec{0},\Sigma_0), \qquad m \to \infty$$

Now consider the number m of experiments as fixed and suppose that these m experiments are either performed under H_0 or under the (fixed) alternative $a \in A$. Obviously, only the first m components of a are now of interest. In view of (4.4.1) and (4.4.2), we put for sufficiently large m

$$(4.4.3) \qquad \vec{\underline{t}}_* \approx N(\vec{\delta}_*,\Sigma_{1.}),$$

and under H_0,

$$(4.4.4) \qquad \vec{\underline{t}}_* \approx N(0,\Sigma_{0.}),$$

i.e. we adopt the form of the asymptotic distribution, while using the actual moments of $\vec{\underline{t}}_*$.

Recall that the (exact) critical value of the test was defined by (2.2.13) and (2.2.14). As it is impossible, in practice, to determine the exact distribution of $\vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ under H_0 (except for very small m), the approximation (4.4.4) is used to obtain an approximate critical value. Using the approximation (4.4.4), the approximate distribution of $\vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ is obtained using theorem 3.2.1. In practice, further approximations to the distribution of $\vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ may be used, as described in section 3.3. Examples of the determination of critical values are given in chapter 9.

Let k be any critical value determined in one of those ways. The power of the test, against $a \in A$,

$$(4.4.5) \qquad P_a(\vec{\underline{t}}_*'Q\vec{\underline{t}}_* \geq k)$$

still cannot be determined exactly, except for very small m. However it may be estimated through the simulation of the distribution of $\vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ under $a \in A$, with the aid of a computer. Some examples can be found in chapter 9. The asymptotic power is equal to 1 for consistent tests, and this is apparently not a good approximation to the power for finite m. Therefore, we adopt the same method as under H_0, i.e. we use (4.4.3) as an approximation, thus

$$(4.4.6) \qquad P_a(\vec{\underline{t}}_*'Q\vec{\underline{t}}_* \geq k) \approx P(\vec{\underline{x}}'Q\vec{\underline{x}} \geq k \,|\, \vec{\underline{x}} \sim N(\vec{\delta}_*, \Sigma_{1.})).$$

Theorem 3.2.1 gives that (4.4.6) is equal to (we have c = 0, cf. remark 6.1.1)

$$(4.4.7) \qquad P(\sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + \omega_\tau)^2 \geq k),$$

with $\vec{\underline{u}} \sim N(\vec{0}, I_r)$, $\Sigma_{1.} = BB'$, $r = \text{rank } B'QB$, $\lambda_1, \ldots, \lambda_r$ the (positive) eigenvalues of $B'QB$ and $\vec{\omega} = \Lambda_+^{-1}P_+'B'Q\vec{\delta}_*$. The reader is referred to section 3.2 for details. The asymptotic expansions of KOTZ, JOHNSON & BOYD (1967b), as described in section 3.3, may be used for the actual calculation of (4.4.7). Examples are given in chapter 9.

For all these calculations a table of the non-central χ^2-distribution or the aid of a computer is necessary (the approximations involve the non-central χ^2-distribution).

However, there exists a lower bound for (4.4.7) that can be calculated easily using only a table of the standard normal distribution. It follows from the following theorem.

THEOREM 4.4.1. *Let* $\vec{\underline{u}} \sim N(\vec{0}, I_r)$, $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_+$, $\omega_1, \ldots, \omega_r \in \mathbb{R}_+ \cup \{0\}$, *then*

$$(4.4.8) \qquad P(\sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + \omega_\tau)^2 \leq z) \leq P(\frac{-\upsilon\sqrt{z} - \upsilon^2}{w} \leq \underline{u} \leq \frac{\upsilon\sqrt{z} - \upsilon^2}{w}), \qquad \forall \; z \geq 0.$$

where

$$\upsilon \stackrel{d}{=} \{\sum_{\tau=1}^{r} \lambda_\tau\omega_\tau^2\}^{\frac{1}{2}}, \qquad w \stackrel{d}{=} \{\sum_{\tau=1}^{r} \lambda_\tau^2\omega_\tau^2\}^{\frac{1}{2}}$$

and $\underline{u} \sim N(0,1)$.

PROOF.

$$P(\sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + \omega_\tau)^2 \leq z) = P(\sum_{\tau=1}^{r} \lambda_\tau\underline{u}_\tau^2 + 2\sum_{\tau=1}^{r} \lambda_\tau\omega_\tau\underline{u}_\tau + \sum_{\tau=1}^{r} \lambda_\tau\omega_\tau^2 \leq z).$$

By the Cauchy-Schwartz inequality we have

$$(\sum_{\tau=1}^{r} \lambda_\tau \omega_\tau \underline{u}_\tau)^2 \equiv (\sum_{\tau=1}^{r} \sqrt{\lambda_\tau}\underline{u}_\tau \sqrt{\lambda_\tau}\omega_\tau)^2 \leq (\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2)(\sum_{\tau=1}^{r} \lambda_\tau \omega_\tau^2).$$

Writing $\nu = \{\sum_{\tau=1}^{r} \lambda_\tau \omega_\tau^2\}^{\frac{1}{2}}$ and $\omega = \{\sum_{\tau=1}^{r} \lambda_\tau^2 \omega_\tau^2\}^{\frac{1}{2}}$, we have

$$P(\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2 + 2\sum_{\tau=1}^{r} \lambda_\tau \omega_\tau \underline{u}_\tau + \sum_{\tau=1}^{r} \lambda_\tau \omega_\tau^2 \leq z) =$$

$$= P(\nu^2 \sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2 + 2\nu^2 \sum_{\tau=1}^{r} \lambda_\tau \omega_\tau \underline{u}_\tau + \nu^4 \leq z\nu^2) \leq$$

$$\leq P((\sum_{\tau=1}^{r} \lambda_\tau \omega_\tau \underline{u}_\tau)^2 + 2\nu^2 \sum_{\tau=1}^{r} \lambda_\tau \omega_\tau \underline{u}_\tau + \nu^4 \leq z\nu^2) =$$

$$= P(\{\sum_{\tau=1}^{r} \lambda_\tau \omega_\tau \underline{u}_\tau + \nu^2\}^2 \leq z\nu^2) =$$

$$= P(-\nu\sqrt{z} \leq \sum_{\tau=1}^{r} \lambda_\tau \omega_\tau \underline{u}_\tau + \nu^2 \leq \nu\sqrt{z}) =$$

$$= P(-\nu\sqrt{z} - \nu^2 \leq \sum_{\tau=1}^{r} \lambda_\tau \omega_\tau \underline{u}_\tau \leq \nu\sqrt{z} - \nu^2) =$$

$$= P(\frac{-\nu\sqrt{z} - \nu^2}{\omega} \leq \underline{u} \leq \frac{\nu\sqrt{z} - \nu^2}{\omega}),$$

where $\underline{u} \sim N(0,1)$. □

When we apply theorem 4.4.1 to (4.4.7) we obtain

(4.4.9) $$P(\sum_{\tau=1}^{r} \lambda_\tau (\underline{u}_\tau + \omega_\tau)^2 \geq k) \geq 1 - P(\frac{-\nu\sqrt{k} - \nu^2}{\omega} \leq \underline{u} \leq \frac{\nu\sqrt{k} - \nu^2}{\omega}).$$

From (3.2.18), (3.2.19) and remark 3.2.3 it follows moreover that

(4.4.10) $$\nu^2 = \vec{\delta}_*' \Omega \vec{\delta}_*,$$

(4.4.11) $$\omega^2 = \vec{\delta}_*' \Omega \Sigma_{1.} \Omega \vec{\delta}_*.$$

Therefore, it is not even necessary to calculate the λ_τ's and ω_τ's explicitly to determine this lower bound (the right-hand side of (4.4.9)) of the power of the test for finite m.

Notice that this lower bound may also be used to make a quick estimate of the number of observations required to achieve a given power against a given alternative $a \in A$.

The last question that remains to be answered is whether we can use the approximation to the power function to select an 'optimal' Q-matrix. There are two approaches to this problem. Suppose that we have a fixed alternative $a \in A$ and a fixed m, so that $\vec{\delta}_*$ and $\Sigma_{1.}$ are fixed and completely known. We could then try to select the matrix Q which maximises the right-hand side of (4.4.6). Alternatively, suppose that the number of observations is not fixed in advance, and that we want to achieve a given power against a. We would then select a matrix Q that would need the least observations to do this. However, this would mean the recalculation of $\vec{\delta}_*$ and of $\Sigma_{1.}$ for several values of m. The situation is then rather complicated. Both approaches may not be equivalent. Furthermore it does not answer the question of optimality when we consider our test as an 'overall' test.

We do not pursue this problem here any further, because we return to it in the next chapter.

However, one step towards simplification of matters can already be made. In order to keep the critical value k (exact, or resulting from some approximation) in the neighbourhood of the critical values of the χ^2-distribution, we shall choose Q, without loss of generality, in such a way that for the a.d.,

$$(4.4.12) \qquad \sum_{\tau=1}^{r} \lambda_\tau = r = \text{rank } B'QB$$

where $BB' = \Sigma_0$.

4.5. ASYMPTOTIC DISTRIBUTIONS IN THE UNCONDITIONAL CASE

Although the unconditional version of our test-statistic, previously defined (in (2.5.26)) as

$$(4.5.1) \qquad \underline{w}(G) \stackrel{d}{=} \frac{1}{m} \sum_{j=1}^{k} g_j \sum_{\nu=1}^{n} \{ \sum_{i=1}^{m} (\underline{x}_{ij}^{(\nu)} - \frac{\underline{a}_{ij}}{n}) \}^2$$

is unfit to be used as a test-statistic, it is nevertheless interesting to investigate its asymptotic distribution. To this end we introduce similar notation as in section 2.2 (See also section 2.5). Let

$$(4.5.2) \qquad \dot{\underline{x}}_{ij}^{(\nu)} \stackrel{d}{=} \underline{x}_{ij}^{(\nu)} - \frac{\underline{a}_{ij}}{n},$$

$$(4.5.3)\qquad \vec{\underline{x}}_i \overset{d}{=} (\dot{\underline{x}}_{i1}^{(1)},\ldots,\dot{\underline{x}}_{ik}^{(1)};\dot{x}_{i1}^{(2)},\ldots,\dot{x}_{ik}^{(2)};\ldots;\dot{\underline{x}}_{i1}^{(n)},\ldots,\dot{\underline{x}}_{ik}^{(n)})'.$$

With $Q = I_n \otimes G$, or a more general Q, $\underline{w}(G)$ may be written as

$$(4.5.4)\qquad \underline{w}(Q) \equiv \vec{\underline{x}}_*'Q\vec{\underline{x}}_*.$$

Let

$$(4.5.5)\qquad \varepsilon_{ij}^{(\nu)} \overset{d}{=} E\dot{\underline{x}}_{ij}^{(\nu)} = p_{ij}^{(\nu)} - \frac{1}{n}\sum_{\mu=1}^{n} p_{ij}^{(\mu)} = p_{ij}^{(\nu)} - p_{ij}^{(\cdot)} .$$

Define $\vec{\varepsilon}_i$ as $\vec{\delta}_i$ in (2.2.16). Then

$$(4.5.6)\qquad E\vec{\underline{x}}_* = \vec{\varepsilon}_*$$

which reduces, under H_0^*, to $E(\vec{\underline{x}}_*|H_0^*) = \vec{0}$. Let

$$(4.5.7)\qquad \Sigma_{1i} \overset{d}{=} D(\vec{\underline{x}}_i) ,$$

$$(4.5.8)\qquad \Sigma_{0i} \overset{d}{=} D(\vec{\underline{x}}_i|H_0^*)$$

The entries of Σ_{1i} and Σ_{0i} may be found from

$$(4.5.9)\qquad \sigma^2(\dot{\underline{x}}_{ij}^{(\nu)}) = \frac{n-2}{n}(p_{ij}^{(\nu)} - \{p_{ij}^{(\nu)}\}^2) + \frac{1}{n^2}\sum_{\mu=1}^{n}(p_{ij}^{(\mu)} - \{p_{ij}^{(\mu)}\}^2),$$

$$(4.5.10)\qquad \sigma^2(\dot{\underline{x}}_{ij}^{(\nu)}|H_0^*) = \frac{n-1}{n}(p_{ij} - \{p_{ij}\}^2),$$

$$(4.5.11)\qquad \mathrm{cov}(\dot{\underline{x}}_{ij}^{(\nu)},\dot{\underline{x}}_{il}^{(\nu)}) = \frac{n-2}{n}(-p_{ij}^{(\nu)}p_{il}^{(\nu)}) + \frac{1}{n^2}\sum_{\mu=1}^{n}(-p_{ij}^{(\mu)}p_{il}^{(\mu)}),$$

$$(4.5.12)\qquad \mathrm{cov}(\dot{\underline{x}}_{ij}^{(\nu)},\dot{\underline{x}}_{il}^{(\nu)}|H_0^*) = \frac{n-1}{n}(-p_{ij}p_{il}), \qquad (j\neq l),$$

and the independence of the variables $\underline{x}_{ij}^{(\nu)}$ (with respect to the indices i and ν).

Furthermore, let

$$\text{(4.5.13)}\quad L_i \overset{d}{=} \frac{n-1}{n}\begin{pmatrix} p_{i1}(1-p_{i1}) & -p_{i1}p_{i2} & \cdots & -p_{i1}p_{ik} \\ -p_{i2}p_{i1} & p_{i2}(1-p_{i2}) & \cdots & -p_{i2}p_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{ik}p_{i1} & -p_{ik}p_{i2} & \cdots & p_{ik}(1-p_{ik}) \end{pmatrix},$$

then

$$\text{(4.5.14)}\quad \Sigma_{0i} = D(\vec{\underline{x}}_i | H_0^*) = N \otimes L_i,$$

$$\text{(4.5.15)}\quad \Sigma_{0\cdot} = D(\vec{\underline{x}}_* | H_0^*) = N \otimes L_{\cdot}.$$

We shall obtain the a.d. of $\underline{w}(Q)$, under H_0^*, under the following assumption.

<u>ASSUMPTION 3</u>.

$$\text{(4.5.16)}\quad \lim_{m\to\infty} p_{\cdot j} = p_j \ , \qquad \text{(say)},$$

$$\text{(4.5.17)}\quad \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} (p_{ij}-p_{\cdot j})(p_{il}-p_{\cdot l}) = e_{jl} \ , \qquad \text{(say)}.$$

Furthermore, we shall consider alternatives that satisfy

<u>ASSUMPTION 4</u>.

$$\text{(4.5.18)}\quad \lim_{m\to\infty} p_{\cdot j}^{(\nu)} = p_j^{(\nu)}, \qquad \text{(say)},$$

$$\text{(4.5.19)}\quad \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} (p_{ij}^{(\nu)}-p_{\cdot j}^{(\nu)})(p_{il}^{(\nu)}-p_{\cdot l}^{(\nu)}) = e_{jl}^{(\nu)}, \qquad \text{(say)},$$

$$\text{(4.5.20)}\quad \lim_{m\to\infty} p_{*j}^{(\nu)} - p_{*j}^{(\cdot)} = \varepsilon_j^{(\nu)}, \qquad \text{(say)},$$

where $|\varepsilon_j^{(\nu)}|$ may be ∞.

Let the vector $\vec{\varepsilon}$, with components $\varepsilon_j^{(\nu)}$ be constructed as in (2.2.16).

<u>REMARK 4.5.1</u>. The remarks that we have made on the plausability of the assumptions 1 & 2 in section 2.3 apply here also. Notice that assumption 4 & H_0^* imply assumption 3.

From (4.5.18) it follows that $\lim_{m\to\infty} p_{\cdot j}^{(\nu)} - p_{\cdot j}^{(\cdot)} = p_j^{(\nu)} - p_j^{(\cdot)}$. For those j and ν for which $p_j^{(\nu)} - p_j^{(\cdot)} \neq 0$, the $\varepsilon_j^{(\nu)}$ of (4.5.19) satisfy $|\varepsilon_j^{(\nu)}| = \infty$ and hence $\vec{\varepsilon}'\vec{\varepsilon} = \infty$. Only when $p_j^{(\nu)} - p_j^{(\cdot)} = 0$ for each j and ν we can have $|\varepsilon_j^{(\nu)}| < \infty$ for each j and ν, or $\vec{\varepsilon}'\vec{\varepsilon} < \infty$. However in that case the alternative clearly converges to H_0^*, and is not very interesting to us. So we shall consider mostly alternatives for which $\vec{\varepsilon}'\vec{\varepsilon} = \infty$, being the equivalent of the class of alternatives for which $\vec{\delta}'\vec{\delta} = \infty$ in the conditional case. For the sake of completeness, some results are also given for the case that $\vec{\varepsilon}'\vec{\varepsilon} < \infty$.

Under the assumption 3, we have

$$\lim_{m\to\infty} D(\vec{\underline{x}}_*|H_0^*) = \lim_{m\to\infty} \Sigma_{0\cdot} = \Sigma_0 , \qquad \text{(say)}, \tag{4.5.21}$$

while under assumption 4, the following limits exist

$$\lim_{m\to\infty} D(\vec{\underline{x}}_*) = \lim_{m\to\infty} \Sigma_{1\cdot} = \Sigma_1 , \qquad \text{(say)}, \tag{4.5.22}$$

$$\lim_{m\to\infty} E\vec{\underline{x}}_* = \lim_{m\to\infty} \vec{\varepsilon}_* = \vec{\varepsilon}. \tag{4.5.23}$$

It is again useful to define

$$\vec{\underline{z}}_i \overset{d}{=} \vec{\underline{x}}_i - \vec{\varepsilon}_i . \tag{4.5.24}$$

<u>THEOREM 4.5.1</u>. *Under the assumption 4, we have for* $m\to\infty$,

$$\vec{\underline{z}}_* \overset{\sim}{\to} N(\vec{0},\Sigma_1). \tag{4.5.25}$$

<u>PROOF</u>. The proof is analogous to the proof of theorem 4.2.1. □

<u>COROLLARY 4.5.1</u>. *Under assumption 4 & H_0^*, we have for* $m\to\infty$,

$$\vec{\underline{x}}_* \overset{\sim}{\to} N(\vec{0},\Sigma_0). \tag{4.5.26}$$

<u>PROOF</u>. As in corollary 4.2.1. □

<u>COROLLARY 4.5.2</u>. *Under the assumption 4, we have for* $\vec{\varepsilon}'\vec{\varepsilon} < \infty$,

$$\vec{\underline{x}}_* \overset{\sim}{\to} N(\vec{\varepsilon},\Sigma_1). \tag{4.5.27}$$

PROOF. As in corollary 4.2.2. □

THEOREM 4.5.2. *Under assumption 4 we have, under* H_0^* *and under alternatives such that* $\vec{\varepsilon}'\vec{\varepsilon} < \infty$,

$$\underline{w}(Q) \xrightarrow{L} \vec{\underline{x}}'Q\vec{\underline{x}} \tag{4.5.28}$$

where

$$\vec{\underline{x}} \sim N(\vec{0},\Sigma_0), \qquad \text{under } H_0^*, \tag{4.5.29}$$

$$\vec{\underline{x}} \sim N(\vec{\varepsilon},\Sigma_1), \qquad \text{under alternatives such that } \vec{\varepsilon}'\vec{\varepsilon} < \infty. \tag{4.5.30}$$

PROOF. As in theorem 4.3.1. □

When we consider a sequence of experiments E_1', E_2', ... with probabilities $p_{ij}^{(\nu)}$ that satisfy assumption 4, it is a natural question to investigate whether from assumption 4 it follows that the limits of the quantities of assumption 1 & 2 exist (or at least almost surely, because these quantities are now random variables). While assumption 4 is sufficient to ensure the convergence of the distributions in the unconditional case, it is not sufficient to ensure convergence in the conditional case. We then need an additional assumption, which, however, is as plausible as the others, because it also involves the convergence of the arithmetic mean of a sequence. See also the remarks in section 2.3.

ASSUMPTION 5. For all j, l, ν and μ the following limits exist

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} (p_{ij}^{(\nu)} - p_{\cdot j}^{(\nu)})(p_{il}^{(\mu)} - p_{\cdot l}^{(\mu)}) = e_{jl}^{(\nu,\mu)}, \qquad \text{(say)}. \tag{4.5.31}$$

Assumption 1 is implied by the assumptions 4 & 5 in the sense of the following theorem.

THEOREM 4.5.3. *Consider a sequence* E_1', E_2',... *satisfying assumptions 4 & 5. Then*

$$\underline{a}_{\cdot j} \xrightarrow{a.s.} p_j^{(+)}, \tag{4.5.32}$$

$$\frac{1}{m} \sum_{i=1}^{m} (\underline{a}_{ij} - \underline{a}_{\cdot j})^2 \xrightarrow{a.s.} \sum_{\nu=1}^{n} p_j^{(\nu)}(1 - p_j^{(\nu)}) + \sum_{\nu\neq\mu}\sum e_{jj}^{(\nu,\mu)}, \tag{4.5.33}$$

$$(4.5.34)\qquad \frac{1}{m}\sum_{i=1}^{m}(\underline{a}_{ij}-\underline{a}_{\cdot j})(\underline{a}_{il}-\underline{a}_{\cdot l}) \xrightarrow{a.s.} -\sum_{\nu=1}^{n} p_j^{(\nu)}p_l^{(\nu)} + \sum_{\nu\neq\mu}\sum e_{jl}^{(\nu,\mu)} \qquad (j\neq l).$$

PROOF. The results follow with the strong law of the large numbers (RAO (1973), p.114) and the fact that $\sigma^2(\underline{a}_{ij})$, $\sigma^2(\underline{a}_{ij}^2)$ and $\sigma^2(\underline{a}_{ij}\underline{a}_{il})$ are all bounded, uniformly in i. □

For the first part of assumption 2 we have an equivalent theorem.

THEOREM 4.5.4. *Let* E_1', E_2', ... *be a sequence satisfying assumptions 4 & 5. Then*

$$(4.5.35)\qquad \underline{\delta}_{\cdot jl}^{(\nu,\mu)} \xrightarrow{a.s.} e_{jl}^{(\nu,\mu)} + p_j^{(\nu)}p_l^{(\mu)} - \sum_{\nu\neq\mu}\sum(e_{jl}^{(\nu,\mu)} + p_j^{(\nu)}p_l^{(\mu)}) \qquad (j\neq l, \nu\neq\mu);$$

$$(4.5.36)\qquad \underline{\delta}_{\cdot j}^{(\nu,\mu)} \xrightarrow{a.s.} e_{jj}^{(\nu,\mu)} + p_j^{(\nu)}p_j^{(\mu)} - \sum_{\nu\neq\mu}\sum(e_{jj}^{(\nu,\mu)} + p_j^{(\nu)}p_j^{(\mu)}) \qquad (\nu\neq\mu);$$

$$(4.5.37)\qquad \underline{\delta}_{\cdot j}^{(\nu)} \xrightarrow{a.s.} p_j^{(\nu)} - \frac{1}{n}\sum_{\nu=1}^{n} p_j^{(\nu)}.$$

PROOF. As in theorem 4.5.3. □

In the second part of assumption 2, (2.3.4), it is assumed that $\lim_{m\to\infty} \delta_{*j}^{(\nu)} = \delta_j^{(\nu)}$, where $|\delta_j^{(\nu)}|$ may be ∞. We have

THEOREM 4.5.5. *Let* E_1', E_2', ... *be a sequence satisfying assumptions 4 & 5. For each pair* (j,ν) *such that*

$$(4.5.38)\qquad p_j^{(\nu)} - p_j^{(\cdot)} \neq 0,$$

we have

$$(4.5.39)\qquad |\underline{\delta}_{*j}^{(\nu)}| \xrightarrow{a.s.} \infty.$$

PROOF. The result follows directly from (4.5.37) and (4.5.38). □

It is, however, possible that $p_j^{(\nu)} = p_j^{(\cdot)}$ for some j and ν. The a.s. convergence of $\underline{\delta}_{*j}^{(\nu)}$ is then not guaranteed. But, for the consistency of the test in the conditional situation it is sufficient that $\vec{\delta}'\vec{\delta} = \infty$. We shall now show that (2.5.18) is sufficient to ensure the consistency of the test.

THEOREM 4.5.6. *A sufficient condition such that*

$$\vec{\underline{\delta}}_*'\vec{\underline{\delta}}_* \xrightarrow{\text{a.s.}} \infty \qquad (4.5.40)$$

is that

$$\exists_{j\nu} \quad \left|\frac{1}{\sqrt{m}} \sum_{i=1}^{m} (p_{ij}^{(\nu)} - p_{ij}^{(\cdot)})\right| \to \infty \qquad \textit{for } m \to \infty \qquad (4.5.41)$$

PROOF. Consider a j and ν for which (4.5.41) holds. Take $\underline{x}_i \overset{d}{=} \underline{\delta}_{ij}^{(\nu)}$, $\mu_i \overset{d}{=} p_{ij}^{(\nu)} - p_{ij}^{(\cdot)}$ and $\tilde{\underline{x}}_i \overset{d}{=} \underline{x}_i - \mu_i$. Then $E\underline{x}_i = \mu_i$ and $E\tilde{\underline{x}}_i = 0$. Furthermore, the variables $\tilde{\underline{x}}_1, \tilde{\underline{x}}_2, \ldots$ are independent. Because $\underline{\delta}_{ij}^{(\nu)}$ can only assume values between -1 and 1 for each i, $\sigma^2(\underline{\delta}_{ij}^{(\nu)}) = \sigma^2(\tilde{\underline{x}}_i)$ is bounded, uniformly in i. We have

$$\underline{\delta}_{*j}^{(\nu)} = \underline{x}_* = \frac{1}{\sqrt{m}}(\tilde{\underline{x}}_1+\ldots+\tilde{\underline{x}}_m) + \frac{1}{\sqrt{m}}(\mu_1+\ldots+\mu_m).$$

From assumption 4 (4.5.18) it follows that $\lim_{m\to\infty} m^{-1}(\mu_1+\ldots+\mu_m)$ exists and therefore

$$\frac{1}{\sqrt{m}}(\mu_1+\ldots+\mu_m) = \mathcal{O}(\sqrt{m}) \qquad \text{for } m \to \infty$$

From the strong law of the large numbers (RAO (1973), p.114) it follows that

$$\frac{1}{m}(\tilde{\underline{x}}_1+\ldots+\tilde{\underline{x}}_m) \xrightarrow{\text{a.s.}} 0 \qquad \text{as } m \to \infty$$

and therefore

$$\frac{1}{\sqrt{m}}(\tilde{\underline{x}}_1+\ldots+\tilde{\underline{x}}_m) = o(\sqrt{m}) \quad \text{a.s.} \quad \text{as } m \to \infty$$

It follows that $m^{-\frac{1}{2}}(\tilde{\underline{x}}_1+\ldots+\tilde{\underline{x}}_m)$ is negligible with respect to $m^{-\frac{1}{2}}(\mu_1+\ldots+\mu_m)$ (a.s.). So (4.5.41) gives

$$|\underline{\delta}_{*j}^{(\nu)}| \xrightarrow{a.s.} \infty$$

and therefore

$$\vec{\underline{\delta}}_*' \vec{\underline{\delta}}_* \xrightarrow{a.s.} \infty.$$

□

Finally, some remarks must be made concerning the power of the test in the unconditional case. Recall that the proposed test is carried out conditionally. For the interesting alternatives in the unconditional case, (4.5.41) holds. Theorem 4.5.6 then gives that the (conditional) test is consistent (a.s.). This means also that the unconditional asymptotic power is equal to 1. In section 4.4 we have described for the conditional case what to do to approximate the power for finite m. In the unconditional case, the situation is even more complicated. Not only are the $\underline{a}_{ij}$ now random variables, but also the critical values are now random. However, a possibly crude approximation can be obtained by computing $E\underline{\delta}_{ij}^{(\nu)}$ and $E\underline{a}_{ij}$ from the $p_{ij}^{(\nu)}$ of the alternative considered and substituting these values into the formulas of section 4.4 for the conditional power.

CHAPTER 5

ASYMPTOTIC RELATIVE EFFICIENCIES

5.1. PITMAN EFFICIENCIES

To make a comparison possible between the different tests that we get for different choices of the matrix Q, we shall investigate the asymptotic relative Pitman efficiency (ARPE) and the asymptotic relative Bahadur efficiency (ARBE), for two different consistent tests based on $\underline{v}(Q_1)$ and $\underline{v}(Q_2)$.

We shall start in this section to give a definition of ARPE as this is given by ROTHE (1979). Then we shall give some theorems of Rothe and in the next section we shall apply his theory to our situation.

We quote from ROTHE (1979).

Let $\{P_\theta,\ \theta \in \Theta\}$ be a family of probability distributions on a space (Ω, F), where Θ is an interval (finite or infinite) on the real line containing zero. Furthermore, $\{\underline{\varphi}_m\}$ is a sequence of level-α tests $(\alpha > 0)$ for H_0: $\theta = 0$ against H_1: $\theta \in \Theta \setminus \{0\}$. Take $\Theta' = \Theta \setminus \{0\}$. We shall assume that for every $\theta \neq 0$,

(5.1.1) $\quad E_\theta(\underline{\varphi}_m) \geq \alpha,$

(5.1.2) $\quad \lim_{m\to\infty} E_\theta(\underline{\varphi}_m) = 1.$

Usually, $\underline{\varphi}_m$ is a test based on m observations. Now the question arises how many observations are necessary to achieve a given power $\beta \in (\alpha, 1)$. So for $0 < \alpha < \beta < 1$, we define a function $N\colon \Theta' \to \mathbb{N}$, which is called a Pitman efficiency function for β (β-PEF), if

(5.1.3) $\quad E_\theta(\underline{\varphi}_{N(\theta)}) \geq \beta,$

(5.1.4) $\quad E_\theta(\underline{\varphi}_{N(\theta)-1}) < \beta,$

where $\underline{\varphi}_0 \equiv \alpha$.

Further, let

$$(5.1.5) \qquad \underline{N_\beta}(\theta) = \inf\{n \in \mathbb{N} \mid E_\theta(\underline{\varphi}_n) \geq \beta\},$$

$$(5.1.6) \qquad \overline{N_\beta}(\theta) = \inf\{n \in \mathbb{N} \mid E_\theta(\underline{\varphi}_m) \geq \beta \text{ for all } m \geq n\}.$$

Now let Π denote the collection of all sequences $\{\theta_m\}$, with $\theta_m \in \Theta'$, $\theta_m \to 0$. Let $\{\underline{\varphi}_m^{(i)}\}$, $i = 1,2$ be two sequences of level-α tests with β-PEF $\underline{N_\beta}^{(i)}$, $\overline{N_\beta}^{(i)}$, respectively. Then

$$(5.1.7) \qquad e_{12}^{-} \overset{d}{=} \inf_\Pi \liminf_{m \to \infty} \frac{\underline{N_\beta}^{(2)}(\theta_m)}{\overline{N_\beta}^{(1)}(\theta_m)}$$

resp.

$$(5.1.8) \qquad e_{12}^{+} \overset{d}{=} \sup_\Pi \limsup_{m \to \infty} \frac{\overline{N_\beta}^{(2)}(\theta_m)}{\underline{N_\beta}^{(1)}(\theta_m)}$$

are the lower (resp. upper) ARPE. If $e_{12}^{-} = e_{12}^{+} = e_{12}$ (say) then e_{12} is the ARPE of $\{\underline{\varphi}_m^{(1)}\}$ with respect to $\{\underline{\varphi}_m^{(2)}\}$.

Now if the following three conditions are satisfied, a general theorem about e_{12} is applicable.

<u>CONDITION A</u>. There is a strictly increasing and bijective function $H: [0,\infty) \to [\alpha,1)$ such that for sequences $\{\theta_m\}$ in Θ satisfying $m\theta_m^2 \to \eta \geq 0$, as $m \to \infty$, we have

$$\lim_{m\to\infty} E_{\theta_m}(\underline{\varphi}_m) = H(\eta).$$

<u>CONDITION B</u>. For every $m \in \mathbb{N}$, the function $\Psi_m: \theta \to E_\theta(\underline{\varphi}_m)$ is continuous at $\theta = 0$.

<u>CONDITION C</u>. For every sequence $\{\theta_m\} \in \Pi$ such that $m\theta_m^2 \to \infty$, we have

$$E_{\theta_m}(\underline{\varphi}_m) \to 1.$$

THEOREM 5.1.1. *Let* $\{\underline{\varphi}_m^{(i)}\}$, $i = 1,2$ *be level-*α *test-sequences satisfying conditions* A *(with functions* H_i*, respectively),* B *and* C. *Then the* ARPE *of* $\{\underline{\varphi}_m^{(1)}\}$ *with respect to* $\{\underline{\varphi}_m^{(2)}\}$ *exists and is equal to*

$$(5.1.9) \qquad e_{12}(\beta) = \frac{H_2^{-1}(\beta)}{H_1^{-1}(\beta)}, \qquad \beta \in (\alpha,1).$$

PROOF. ROTHE (1979). □

5.2. DETERMINATION OF "ARPE" IN OUR CASE

We apply the theory of section 5.1 in our situation using the 'close' alternatives as defined in section 2.4. So $a \in A_1$ is a fixed alternative, $\{\theta_m\}_{m=1}^{\infty}$ is a sequence in Θ' such that $m\theta_m^2 \to \eta \geq 0$, and $\{a_{\theta_m}\}_{m=1}^{\infty}$ is the associated close alternative.

We consider matrices Q such that (4.1.3) is satisfied for $a \in A_1$, i.e. $\underline{v}(Q)$ is consistent against a. We use $\underline{\varphi}_{m,Q}$ as defined by (2.2.15).

In view of (4.1.3), (5.1.1) holds at least from a certain index m_1 on. When $\{\underline{\varphi}_{m,Q}\}$ is consistent against $a \in A_1$, it is also consistent against a_θ, for each θ, $0 \leq \theta \leq 1$, as follows easily from theorem 4.1.1. Therefore (5.1.2) is also satisfied. We proceed now to verify the conditions A, B and C of the preceding section.

Condition A.

Let $\{\theta_m\}$ be a sequence in Θ such that $m\theta_m^2 \to \eta$. It follows from theorem 4.2.2 that $\underline{\vec{t}}_* \overset{\sim}{\to} N(\sqrt{\eta}\,\vec{\zeta},\Sigma_0)$ and from theorem 4.3.1 that $\underline{\vec{t}}_*'Q\underline{\vec{t}}_* \overset{L}{\to} \underline{\vec{x}}'Q\underline{\vec{x}}$, where $\underline{\vec{x}} \sim N(\sqrt{\eta}\,\vec{\zeta},\Sigma_0)$, under the close alternative $\{a_{\theta_m}\}$. Then

$$\lim_{m\to\infty} E_{\theta_m}(\underline{\varphi}_{m,Q}) = P(\underline{\vec{x}}'Q\underline{\vec{x}} \geq k_{1-\alpha}(Q)),$$

where $k_{1-\alpha}(Q)$ is a critical value as defined in section 4.4.

The question is now whether

$$(5.2.1) \qquad H(\eta) \overset{d}{=} P(\underline{\vec{x}}'Q\underline{\vec{x}} \geq k_{1-\alpha}(Q))$$

is a strictly increasing function of η.

We have

$$H(\eta) = P(\vec{\underline{x}}'Q\vec{\underline{x}} \geq k_{1-\alpha}(Q)) = P(\sum_{\tau=1}^{r} \lambda_\tau(\underline{u}_\tau + \sqrt{\eta}\,\omega_\tau)^2 \geq k_{1-\alpha}(Q)),$$

for some $r \in \mathbb{N}$, $\lambda_1 \geq \ldots \geq \lambda_r > 0$ and $\omega_1,\ldots,\omega_r \in \mathbb{R}$, $\vec{\underline{u}} \sim N(\vec{0},I_r)$. Clearly $H(0) = \alpha$, $\lim_{\eta\to\infty} H(\eta) = 1$.

The fact that $H(\eta)$ is strictly increasing follows from lemma 3.3.1. So condition A is fulfilled, and $H(\eta)$ has a unique inverse $H^{-1}(\beta)$ for each $\beta \in (\alpha,1)$.

Condition B.

This follows from the fact that the exact distribution of $\vec{\underline{t}}_*$ depends on θ in a continuous way, when θ is in a sufficiently small neighbourhood of 0.

Condition C.

Condition C follows from the fact that for sequences $\{\theta_m\}$ such that $m\theta_m^2 \to \infty$, $E_{\theta_m} \underline{y}_m \to \infty$ and $E_{\theta_m} \underline{y}_m/\sigma_{\theta_m}(\underline{y}_m) \to \infty$, with $\underline{y}_m \stackrel{d}{=} 2\vec{\delta}_*'(a_{\theta_m})Q\vec{\underline{u}}_* + \vec{\delta}_*'(a_{\theta_m})Q\vec{\delta}_*(a_{\theta_m})$. The rest of the arguments are similar to those of theorem 4.1.1.

It follows that the conditions of theorem 5.1.1 are satisfied. The ARPE of $\{\underline{\varphi}_{m,Q_1}\}$ with respect to $\{\underline{\varphi}_{m,Q_2}\}$ is thus given by

$$(5.2.2) \qquad e_{12}(\beta) = \frac{H_2^{-1}(\beta)}{H_1^{-1}(\beta)}, \qquad \beta \in (\alpha,1).$$

With

$$(5.2.3) \qquad H_i(\eta) \stackrel{d}{=} P(\vec{\underline{x}}'Q_i\vec{\underline{x}} \geq k_{1-\alpha}(Q_i)), \qquad i = 1,2,$$

and $\vec{\underline{x}} \sim N(\sqrt{\eta}\,\vec{\zeta},\Sigma_0)$.

No explicit formula can be given for $H_i^{-1}(\beta)$, though the inverse may be determined numerically. See chapter 9. In general, $e_{12}(\beta)$ will be dependent on α, β, Q_1, Q_2 and the particular alternative $a \in A_1$.

When two matrices Q_1 and Q_2 are compared, the criterion (5.2.2) selects as best test the one that first reaches β, as $\sqrt{\eta}\,\vec{\zeta}$ tends away from $\vec{0}$. When $Q_1 \neq Q_2$ would imply $H_1(\eta) \leq H_2(\eta)$ for all η, or $H_1(\eta) \geq H_2(\eta)$ for all η, then a 'best' Q could be selected and it would be independent of β. However, the implication considered does not always hold and so the selection of a 'best' Q does depend on β. The Pitman-efficiency is therefore not a helpful tool in selecting a 'good' Q.

When, however, we approximate $H(\eta)$ by a function $H^*(\eta)$, using the approximation (3.3.19), i.e.

$$(5.2.4) \qquad H^*(\eta) \overset{d}{=} P(\underline{\chi}^2[\nu,\delta^2] \geq \chi^2_{1-\alpha}[\nu]),$$

with

$$(5.2.5) \qquad \nu = \sum_{\tau=1}^{r} \lambda_\tau, \qquad \delta^2 = \eta \sum_{\tau=1}^{r} \lambda_\tau \omega_\tau^2,$$

we have (ROTHE (1979))

$$(5.2.6) \qquad H^{*-1}(\beta) = \frac{c^2(\nu,\alpha,\beta)}{\delta^2},$$

where $c^2(\nu,\alpha,\beta)$ is the (uniquely determined) non-centrality parameter such that the β-fractile of $\underline{\chi}^2[\nu,c^2(\nu,\alpha,\beta)]$ and the α-fractile of $\chi^2[\nu]$ coincide.

We now compare two matrices Q_1 and Q_2, for which

$$(5.2.7) \qquad \text{rank}(B'Q_1B) = \text{rank}(B'Q_2B) = r,$$

and

$$(5.2.8) \qquad \nu_1 = \nu_2 = \sum_{\tau=1}^{r} \lambda_\tau^{(1)} = \sum_{\tau=1}^{r} \lambda_\tau^{(2)} = r.$$

Then it follows from (3.2.18) that

$$(5.2.9) \qquad \delta_1^2 = \eta \sum_{\tau=1}^{r} \lambda_\tau^{(1)} (\omega_\tau^{(1)})^2 = \eta\, \vec{\zeta}'Q_1\vec{\zeta}\ ;$$

$$(5.2.10) \qquad \delta_2^2 = \eta \sum_{\tau=1}^{r} \lambda_\tau^{(2)} (\omega_\tau^{(2)})^2 = \eta\, \vec{\zeta}'Q_2\vec{\zeta}\ .$$

Then we may approximate the ARPE as follows

$$(5.2.11) \qquad e_{12}(\beta) = \frac{H_2^{-1}(\beta)}{H_1^{-1}(\beta)} \approx \frac{H_2^{*-1}(\beta)}{H_1^{*-1}(\beta)} = \frac{c^2(r,\alpha,\beta)}{\delta_2^2} \times \frac{\delta_1^2}{c^2(r,\alpha,\beta)} = $$

$$= \frac{\delta_1^2}{\delta_2^2} = \frac{\vec{\zeta}' Q_1 \vec{\zeta}}{\vec{\zeta}' Q_2 \vec{\zeta}},$$

which is independent of r, α and β. Moreover this approximate value of $e_{12}(\beta)$ corresponds to the usual ARPE in the case of χ^2-distributions. Note that this approximate value (e_{12}^* (say)) may be calculated directly, without prior calculation of eigenvalues etc. (cf. (3.2.18)).

When we would use e_{12}^* as a criterion to select a Q-matrix, we would choose the one that maximises the "non-centrality parameter", $\vec{\zeta}' Q \vec{\zeta}$, in accordance with usual practice.

5.3 BAHADUR EFFICIENCIES

In this section, we treat the so-called "approximate" - Bahadur asymptotic relative efficiency (ARBE). The reader is referred to BAHADUR (1960) for the definition of this concept. It is to be noted that the more interesting "exact" - Bahadur asymptotic relative efficiency requires knowledge of the exact distribution function of the test statistics, which is unavailable in our situation. Although the ARBE has certain serious disadvantages (cf. GROENEBOOM & OOSTERHOFF (1977)), it leads to interesting results in our case.

We now give short definitions of the "standard sequence" and the "Bahadur slope" needed for the calculation of ARBE.

In order to compute the "Bahadur slope" for a sequence of test-statistics $\{\underline{T}_m\}_{m=1}^{\infty}$, this sequence has to be a "standard sequence", i.e. it has to satisfy the following three conditions.

i. There exists a continuous probability distribution function F such that, under H_0,

$$(5.3.1) \qquad \lim_{m\to\infty} P(\underline{T}_m < x \mid H_0) = F(x).$$

ii. There exists a constant a, $0 < a < \infty$, such that

$$(5.3.2) \qquad \log(1 - F(x)) = -\frac{ax^2}{2}(1 + o(1)), \qquad \text{as } x \to \infty.$$

iii. There exists a function b on H_1, with $0 \le b < \infty$, such that for each $\theta \in H_1$

$$(5.3.3) \qquad \lim_{m\to\infty} P_\theta(|\frac{\underline{T}_m}{\sqrt{m}} - b(\theta)| > \varepsilon) = 0 \qquad \text{for every } \varepsilon > 0.$$

The asymptotic Bahadur slope is now defined to be

$$(5.3.4) \qquad G(\theta) = a\{b(\theta)\}^2.$$

The approximate relative Bahadur efficiency (ARBE) for two standard sequences $\{\underline{T}_m^{(1)}\}_{m=1}^{\infty}$ and $\{\underline{T}_m^{(2)}\}_{m=1}^{\infty}$ is then defined as

$$(5.3.5) \qquad E_{12}(\theta) = \frac{G_1(\theta)}{G_2(\theta)} = \frac{a_1\{b_1(\theta)\}^2}{a_2\{b_2(\theta)\}^2}.$$

5.4. DETERMINATION OF "ARBE" IN OUR CASE

We apply the theory of ARBE for tests against alternatives in A_1, i.e. instead of $\theta \in H_1$, we shall write $a \in A_1$.

<u>THEOREM 5.4.1</u>. *When Q is chosen such that* $\vec{\delta}_*'Q\vec{\delta}_* \to \infty$ *for each* $a \in A_1$, *then* $\{(\underline{v}(Q))^{\frac{1}{2}}\}_{m=1}^{\infty} = \{(\vec{\underline{t}}_*'Q\vec{\underline{t}}_*)^{\frac{1}{2}}\}_{m=1}^{\infty}$ *is a standard sequence for testing* H_0.

<u>PROOF</u>. We verify the three conditions for a standard sequence.

i. By theorem 4.3.1, we have, under H_0, $\underline{v}(Q) \xrightarrow{L} \vec{\underline{x}}'Q\vec{\underline{x}}$, with $\vec{\underline{x}} \sim N(\vec{0},\Sigma_0)$, and so $(\underline{v}(Q))^{\frac{1}{2}} \xrightarrow{L} (\vec{\underline{x}}'Q\vec{\underline{x}})^{\frac{1}{2}}$, and $(\vec{\underline{x}}'Q\vec{\underline{x}})^{\frac{1}{2}}$ has a continuous distribution. This proves *i*.

ii. Using theorem 3.2.1, we have

$$1 - F(x) = 1 - P((\vec{\underline{x}}'Q\vec{\underline{x}})^{\frac{1}{2}} < x) = 1 - P(\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2 < x^2),$$

with $\vec{\underline{u}} \sim N(\vec{0},I_r)$ and $\lambda_1 \ge \ldots \ge \lambda_r > 0$.
Note that

$$P(\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2 > x^2) \le P(\lambda_1 \sum_{\tau=1}^{r} \underline{u}_\tau^2 > x^2) = P((\underline{\chi}^2[r])^{\frac{1}{2}} > \frac{x}{\sqrt{\lambda_1}}) \ ;$$

$$P(\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2 > x^2) \ge P(\lambda_1 \underline{u}_1^2 > x^2) = P((\underline{\chi}^2[1])^{\frac{1}{2}} > \frac{x}{\sqrt{\lambda_1}}).$$

So

$$\log P((\underline{\chi}^2[1])^{\frac{1}{2}} > \frac{x}{\sqrt{\lambda_1}}) \le \log(1-F(x)) \le \log P((\underline{\chi}^2[r])^{\frac{1}{2}} > \frac{x}{\sqrt{\lambda_1}}).$$

BAHADUR (1960) showed for χ-distributions that (5.3.2) is satisfied with $a = 1$, regardless of the number of degrees of freedom. It follows that

$$-\frac{x^2}{2\lambda_1}(1 + o(1)) \le \log(1-F(x)) \le -\frac{x^2}{2\lambda_1}(1 + o(1)),$$

and so

$$\log(1-F(x)) = -\frac{x^2}{2\lambda_1}(1 + o(1)).$$

Hence (5.3.2) is satisfied in this case, with $a = \frac{1}{\lambda_1}$. This proves *ii*.

iii.

$$\frac{1}{m}\,\vec{\underline{t}}_*'Q\vec{\underline{t}}_* \equiv \frac{1}{m}\,\vec{\underline{u}}_*'Q\vec{\underline{u}}_* + \frac{2}{m}\,\vec{\delta}_*'Q\vec{\underline{u}}_* + \frac{1}{m}\,\vec{\delta}_*'Q\vec{\delta}_*.$$

By theorem 4.2.1 and 4.3.1, $\vec{\underline{u}}_*'Q\vec{\underline{u}}_* \overset{L}{\to} \vec{\underline{x}}'Q\vec{\underline{x}}$, so

$$\frac{1}{m}\,\vec{\underline{u}}_*'Q\vec{\underline{u}}_* \overset{P}{\to} 0.$$

The expectation of $\frac{2}{m}\,\vec{\delta}_*'Q\vec{\underline{u}}_*$ is zero, and its variance is equal to

$$\frac{4}{m^2}\,\vec{\delta}_*'Q\Sigma_{1.}Q\vec{\delta}_* = \frac{4}{m}\,\vec{\delta}_.'Q\Sigma_{1.}Q\vec{\delta}_. \;.$$

By assumption 2, $\vec{\delta}_.$ and $\Sigma_{1.}$ converge to a finite limit. Therefore

$$\lim_{m\to\infty} \frac{4}{m}\,\vec{\delta}_.'Q\Sigma_{1.}Q\vec{\delta}_. = 0.$$

It follows that

$$\frac{2}{m}\,\vec{\delta}_*'Q\vec{\underline{u}}_* \overset{P}{\to} 0.$$

Furthermore,

$$\lim_{m\to\infty} \frac{1}{m} \vec{\delta}_*'Q\vec{\delta}_* = \vec{\delta}_\bullet'Q\vec{\delta}_\bullet = \vec{\zeta}'Q\vec{\zeta} .$$

It follows that,

$$\frac{1}{m} \vec{\underline{t}}_*'Q\vec{\underline{t}}_* \overset{P}{\to} \vec{\zeta}'Q\vec{\zeta} ,$$

and with Slutzky's theorem,

$$(\frac{1}{m} \vec{\underline{t}}_*'Q\vec{\underline{t}}_*)^{\frac{1}{2}} \overset{P}{\to} (\vec{\zeta}'Q\vec{\zeta})^{\frac{1}{2}}.$$

This proves *iii.*

It follows from *i.*, *ii.* and *iii.* that $(\vec{\underline{t}}_*'Q\vec{\underline{t}}_*)^{\frac{1}{2}}$ is a standard sequence. □

We find that the ARBE of $\underline{v}(Q_1)$ w.r.t. $\underline{v}(Q_2)$ is equal to

$$E_{12}(a) = \frac{\frac{1}{\lambda_1^{(1)}} \vec{\zeta}'Q_1\vec{\zeta}}{\frac{1}{\lambda_1^{(2)}} \vec{\zeta}'Q_2\vec{\zeta}} . \tag{5.4.1}$$

This is almost equal to e_{12}^*. In the case that $\lambda_1^{(1)} = \lambda_1^{(2)}$ we even have $E_{12} = e_{12}^*$. This supports the use of e_{12}^* as a measure of relative efficiency. Notice that it is not surprising that the largest eigenvalue of $Q_1\Sigma_0$ and $Q_2\Sigma_0$ occur in E_{12}, because of their influence on the distribution of $\underline{v}(Q_l)$. When rank $B'Q_lB = r$, and Q_1 and Q_2 are chosen such that (4.4.12) holds, then $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$ will not differ very much, so in that case $e_{12}^* \approx E_{12}$. We conclude that the measures of relative efficiency ARPE and ARBE are, in our situation, not essentially different.

CHAPTER 6

SPECIAL CASES & PRACTICE

6.1 TEST-STATISTICS WITH A CHI-SQUARED ASYMPTOTIC DISTRIBUTION.

There are several (obvious) reasons to consider the cases for which the test-statistic $\vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ has a chi-squared a.d. under H_0. It can be shown that there always exists a matrix Q such that the test-statistic has this property. Such a matrix can be constructed as follows. Suppose that $\vec{\underline{x}} \sim N(\vec{0},\Sigma)$. Let Σ^- be any g-inverse of Σ. Then it follows from theorem 3.2.2 that the quadratic form $\vec{\underline{x}}'\Sigma^-\vec{\underline{x}}$ has a (central) χ^2-distribution, because $\Sigma\Sigma^-\Sigma\Sigma^- = \Sigma\Sigma^-$ by the properties of g-inverses, so that $\Sigma\Sigma^-$ is idempotent. The second condition of theorem 3.2.2 is automatically satisfied when $\vec{\mu} = \vec{0}$. The number of degrees of freedom is then

$$\text{trace}(\Sigma\Sigma^-) = \text{rank } \Sigma \tag{6.1.1}$$

because $\Sigma\Sigma^-$ is idempotent (RAO (1973)).

By corollary 4.2.1, we have, under assumption 1 & H_0,

$$\vec{\underline{t}}_* \overset{\sim}{\rightarrow} N(\vec{0},\Sigma_0), \qquad \Sigma_0 = N \otimes K. \tag{6.1.2}$$

So if we choose $Q = \Sigma_0^-$, it follows that $\vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ has an asymptotic χ^2-distribution. Of course, in practice, Σ_0 is replaced by $\Sigma_{0\bullet}$, i.e. in fact we consider the statistic

$$\underline{v}(\Sigma_{0\bullet}^-) \equiv \vec{\underline{t}}_*'\Sigma_{0\bullet}^-\vec{\underline{t}}_* \tag{6.1.3}$$

We now show that the distribution of this statistic is the same, whatever the choice of g-inverse $\Sigma_{0\bullet}^-$; in fact $\underline{v}(\Sigma_{0\bullet}^-)$ assumes the same numerical values for different choices of g-inverse $\Sigma_{0\bullet}^-$, for each $\vec{\omega} \in \Omega$. (Ω as defined in (2.1.7).) Furthermore, we shall show that this statistic is consistent against all alternatives in A_1.

The results in this section are, for a large part, due to DE GUNST & V.D. GEER (1982).

LEMMA 6.1.1. *For each* $\vec{\omega} \in \Omega$, *we have*

$$\vec{t}_*(\vec{\omega}) \in M(\Sigma_{0\cdot}). \tag{6.1.4}$$

PROOF. Let $\vec{x}$ be a vector which is perpendicular to the column's of $\Sigma_{0\cdot}$, i.e. $\vec{x}'\Sigma_{0\cdot} = \vec{0}$. Then we have $\sigma^2(x'\vec{\underline{t}}_*|H_0) = \vec{x}'\Sigma_{0\cdot}\vec{x} = 0$, and thus, because $E(\vec{\underline{t}}_*|H_0) = \vec{0}$, $P(\vec{x}'\vec{\underline{t}}_* = 0|H_0) = 1$. Because Ω contains a finite number of elements, all with positive probability under H_0, we have $\vec{x}'\vec{t}_*(\vec{\omega}) = 0$ for each $\vec{\omega} \in \Omega$, and hence $\vec{t}_*(\vec{\omega}) \in M(\Sigma_{0\cdot})$ for each $\vec{\omega} \in \Omega$. □

COROLLARY 6.1.1. *For each alternative* a, *we have*

$$\vec{\delta}_* \in M(\Sigma_{0\cdot}). \tag{6.1.5}$$

PROOF. $\vec{\delta}_* = E_a\vec{\underline{t}}_*$ is a linear combination of the quantities $\vec{t}_*(\vec{\omega})$. Because $M(\Sigma_{0\cdot})$ is a linear space, the result follows with lemma 6.1.1. □

REMARK 6.1.1. From corollary 6.1.1 and the implication (3.2.23) it follows that the constant c of theorem 3.2.1 as defined by (3.2.5) is equal to zero, for each value of m, when we apply the definition of c formally to our test-statistic. This also means that this constant is equal to zero in the limit.

LEMMA 6.1.2. *Let* G *be the natural g-inverse, as defined in (3.1.11), of* $\Sigma_{0\cdot}$. *Then*

$$M(G) = M(\Sigma_{0\cdot}). \tag{6.1.6}$$

PROOF. Observe that

$$\vec{y}'G = \vec{0}' \iff \vec{y}'P_+\Lambda_+^{-1}P_+' = \vec{0}' \iff \vec{y}'P_+\Lambda_+^{-1}P_+'P_+\Lambda_+ = \vec{0}' \iff$$

$$\vec{y}'P_+ = \vec{0}' \iff \vec{y}'P_+\Lambda_+P_+' = \vec{0}' \iff \vec{y}'\Sigma_{0\cdot} = 0',$$

where $\Sigma_{0\cdot} = P_+\Lambda_+P_+'$ is the positive canonical reduction of $\Sigma_{0\cdot}$. Any vector that is orthogonal to G is therefore also orthogonal to $\Sigma_{0\cdot}$ and vice versa. Thus $M(G) = M(\Sigma_{0\cdot})$. □

LEMMA 6.1.3. *Let* G_1 *and* G_2 *be two different versions of* $\Sigma_{0\bullet}^{-}$. *Then*

$$\vec{t}_*'(\vec{\omega})G_1\vec{t}_*(\vec{\omega}) = \vec{t}_*'(\vec{\omega})G_2\vec{t}_*(\vec{\omega}) \tag{6.1.6}$$

for each $\vec{\omega} \in \Omega$.

PROOF. From lemma 6.1.1 it follows that $\vec{t}_*(\vec{\omega}) \in M(\Sigma_{0\bullet})$ for each $\vec{\omega} \in \Omega$. This means that for any $\vec{\omega} \in \Omega$ there exists a vector $\vec{x}$ such that $\vec{t}_*(\vec{\omega}) = \Sigma_{0\bullet}\vec{x}$. We then have, by the properties of g-inverses,

$$\vec{t}_*'(\vec{\omega})G_1\vec{t}_*(\vec{\omega}) = \vec{x}'\Sigma_{0\bullet}G_1\Sigma_{0\bullet}\vec{x} = \vec{x}'\Sigma_{0\bullet}\vec{x},$$

$$\vec{t}_*'(\vec{\omega})G_2\vec{t}_*(\omega) = \vec{x}'\Sigma_{0\bullet}G_2\Sigma_{0\bullet}\vec{x} = \vec{x}'\Sigma_{0\bullet}\vec{x}.$$

□

REMARK 6.1.2. The test-statistic proposed along different lines in MADANSKY (1963) as a generalisation of Cochran's Q-test can be shown to be of the type (6.1.3), with a specific choice for $\Sigma_{0\bullet}^{-}$. The above lemma then shows that (6.1.3) is identical with Madansky's statistic.

The consistency of the test now follows from

THEOREM 6.1.1. *The test* $\{\underline{\varphi}_{m,Q}\}$, *with* $Q = \Sigma_{0\bullet}^{-}$ *is for any choice of g-inverse* $\Sigma_{0\bullet}^{-}$ *consistent against each alternative in* A_1.

PROOF. First take $Q = G$, where G is the natural g-inverse of $\Sigma_{0\bullet}$. From (6.1.5) it follows that $\vec{\delta}_* \in M(\Sigma_{0\bullet})$. Then (6.1.6) implies that $\vec{\delta}_* \in M(G)$. By theorem 4.1.2, the test $\{\underline{\varphi}_{m,G}\}$ is then consistent against each alternative in A_1. Lemma 6.1.3 shows that the same is true for $\{\underline{\varphi}_{m,Q}\}$ when Q is any g-inverse of $\Sigma_{0\bullet}$. □

We have thus established that the test $\{\underline{\varphi}_{m,Q}\}$ with Q any g-inverse of $\Sigma_{0\bullet}$ is consistent against each alternative in A_1, and that the a.d. of the test-statistic $\vec{\underline{t}}_*'\Sigma_{0\bullet}^{-}\vec{\underline{t}}_*$ is chi-squared. Besides the advantages that such a test has, there are also disadvantages. Firstly, the calculation of $\Sigma_{0\bullet}^{-}$ is not always easy without a computer. A reduction of the calculations is possible, however, when we use the special structure of $\Sigma_{0\bullet} = N \otimes K_{\bullet}$. A g-inverse of $\Sigma_{0\bullet}$ is then $Q = I_n \otimes \frac{n-1}{n} K_{\bullet}^{-}$. Because the order of $K_{\bullet}$ is much smaller than the order of $\Sigma_{0\bullet}$, the determination of a g-inverse of $K_{\bullet}$ is more practicable. Of course, the natural g-inverse of $K_{\bullet}$ may be taken. Also, except for the cases k = 2 and k = 3, the matrix $\Sigma_{0\bullet}^{-}$ cannot be brought into diagonal form. This complicates the interpretation of practical results (cf. section 2.1). Moreover it means that the exact moments of this statistic are not available. This affects the accuracy of the approximations.

In the cases k = 2 and k = 3, however, matrices Q can be found, such that Q is non-singular, diagonal, and such that the a.d. of $\underline{v}(Q) \equiv \vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ is chi-squared.

6.2. THE CASE k = 2

In the case that k = 2, the eigenvalues of $Q\Sigma_0$, necessary for the calculation of the asymptotic distribution of $v(Q) \equiv \vec{\underline{t}}_*'Q\vec{\underline{t}}_*$, under H_0, may be found by an elementary calculation, at least when we take Q of the form $Q = I_n \otimes G$. It is even more simple when we take G diagonal, i.e.

$$G = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} . \tag{6.2.1}$$

Recall that

$$\Sigma_0 = N \otimes K, \tag{6.2.2}$$

with, because k = 2,

$$K = \begin{pmatrix} a & -a \\ -a & a \end{pmatrix} , \tag{6.2.3}$$

in which a is equal to the following limit (cf. (2.2.30) and (3.3.10))

$$a = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \frac{a_{i1}a_{i2}}{n^2} . \tag{6.2.4}$$

The existence of this limit is assumed in assumption 1.

To find the a.d. of $\vec{\underline{t}}_*'Q\vec{\underline{t}}_*$, we have to calculate the eigenvalues of

$$(6.2.5)\qquad Q\Sigma_0 = (I_n \otimes G)(N \otimes K) = N \otimes GK.$$

By lemma 3.1.5, the eigenvalues of $N \otimes GK$ can be found from the eigenvalues of $\frac{n}{n-1}$ GK. Let

$$(6.2.6)\qquad A \overset{d}{=} \frac{n}{n-1}\, a.$$

Then we have

$$\left|\frac{n}{n-1}GK - \lambda I_2\right| = \begin{vmatrix} g_1A-\lambda & -g_1A \\ -g_2A & g_2A-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -g_1A \\ -\lambda & g_2A-\lambda \end{vmatrix} =$$

$$= -\lambda \begin{vmatrix} 1 & -g_1A \\ 1 & g_2A-\lambda \end{vmatrix} = -\lambda(g_2A-\lambda+g_1A).$$

It follows that the eigenvalues of $\frac{n}{n-1}$ GK are equal to

$$(6.2.7)\qquad \lambda = 0 \;\vee\; \lambda = (g_1+g_2)A = \frac{n}{n-1}(g_1+g_2)a.$$

The eigenvalues of $Q\Sigma_0$ are therefore $\lambda = (g_1+g_2)A$ with multiplicity n-1 and 0 with multiplicity n+1.

To obtain an a.d. which is chi-squared, we only have to choose g_1 and g_2 such that

$$(6.2.8)\qquad g_1 + g_2 = \frac{n-1}{n}\,\frac{1}{a}\,.$$

If we furthermore take

$$(6.2.9)\qquad g_1 > 0 \wedge g_2 > 0,$$

G, and hence Q, has full rank. The resulting test is then consistent against all alternatives in $\mathcal{A}_1$.

In practical cases we take G = G(m) with

$$(6.2.10)\qquad g_1 + g_2 = \frac{n-1}{n}\,\frac{1}{a_m}$$

where

$$(6.2.11)\qquad a_m \overset{d}{=} \frac{1}{m}\sum_{i=1}^{m} \frac{a_{i1}a_{i2}}{n^2} .$$

From (6.2.8) or (6.2.10) it would appear that we still have a choice for g_1 and g_2. However for $k=2$, in this case

$$\underline{v}(Q) \equiv \frac{1}{m}\sum_{\nu=1}^{n} g_1(\underline{f}_1^{(\nu)} - \frac{a_{+1}}{n})^2 + \frac{1}{m}\sum_{\nu=1}^{n} g_2(\underline{f}_2^{(\nu)} - \frac{a_{+2}}{n})^2 \equiv$$

$$\equiv \frac{1}{m}\sum_{\nu=1}^{n} g_1(\underline{f}_1^{(\nu)} - \frac{a_{+1}}{n})^2 + \frac{1}{m}\sum_{\nu=1}^{n} g_2(m - \underline{f}_1^{(\nu)} - \frac{nm-a_{+1}}{n})^2 \equiv$$

$$\equiv \frac{1}{m}(g_1+g_2)\sum_{\nu=1}^{n} (\underline{f}_1^{(\nu)} - \frac{a_{+1}}{n})^2 .$$

So with (6.2.10) this gives

$$(6.2.12)\qquad \underline{v}(Q) \equiv \frac{n-1}{n}\,\frac{\sum_{\nu=1}^{n} (\underline{f}_1^{(\nu)} - \frac{a_{+1}}{n})^2}{\frac{1}{n^2}\sum_{i=1}^{m} a_{i1}(n-a_{i1})} \equiv \frac{n(n-1)\sum_{\nu=1}^{n} (\underline{f}_1^{(\nu)} - \frac{a_{+1}}{n})^2}{n\sum_{i=1}^{m} a_{i1} - \sum_{i=1}^{m} a_{i1}^2} .$$

Hence in this case we obtain Cochran's Q-statistic (COCHRAN (1950)). The asymptotic distribution, under H_0, is then $\chi^2[n-1]$.

For the diagonal matrix G of (6.2.1) with g_1 and g_2 given by (6.2.10) we shall write G_2 and the corresponding statistic as $\underline{v}(G_2)$.

6.3. THE CASE k = 3

In the case that $k=3$, we may proceed in the same way as for $k=2$. Let this time,

$$(6.3.1)\qquad G = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}$$

and

$$(6.3.2)\qquad K = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

with

$$(6.3.3) \qquad a = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \frac{a_{i1}(n-a_{i1})}{n^2}, \qquad \text{etc.,}$$

and

$$(6.3.4) \qquad A \overset{d}{=} \frac{n}{n-1} a, \qquad \text{etc.}$$

Note that

$$(6.3.5) \qquad A + D + E = D + B + F = E + F + C = 0.$$

It follows that

$$(6.3.6) \qquad AB - D^2 = BC - F^2 = DF - BE = AC - E^2 = -DC + EF = -AF + DE =$$

$$= -DC + FE = C_K \qquad \text{(say).}$$

C_K may also be written as

$$(6.3.7) \qquad C_K = -\frac{1}{2}(AF + BE + CD).$$

Imposing the condition that the sum of the eigenvalues of $Q\Sigma_0$ must be equal to $(n-1)(k-1) = (n-1)\cdot 2$, we find for the eigenvalues of $\frac{n}{n-1}GK$,

$$(6.3.8) \qquad \lambda = 0 \;\vee\; \lambda = 1 \pm \frac{1}{2}\sqrt{4-4(k_1k_2-k_3)},$$

with

$$(6.3.9) \qquad k_1 \overset{d}{=} g_2B - g_1D; \quad k_2 \overset{d}{=} g_3C - g_1E; \quad k_3 \overset{d}{=} (g_3F-g_1D)(g_2F-g_1E).$$

Both non-zero eigenvalues are now equal to 1 if

$$(6.3.10) \qquad k_1k_2 - k_3 = 1.$$

It may be verified that this is the case only when

$$(6.3.11) \qquad g_1 = -F/C_K; \quad g_2 = -E/C_K; \quad g_3 = -D/C_K.$$

It follows that G, and hence Q, is non-singular, and the resulting test is therefore consistent against each alternative in A_1.

In practical cases we shall determine $G = G(m)$ from $K_{\bullet}$ instead of K in the same way, and we shall call it $G_3 = G_3(m)$ and we shall write the corresponding statistic as $\underline{v}(G_3)$.

6.4. RECOMMENDATIONS

In section 4.1 it has been shown that the test based on the test-statistic $\underline{v}(Q) \equiv \vec{\underline{t}}_*'Q\vec{\underline{t}}_*$ is consistent against each alternative in A_1, the class of alternatives that we wish to detect, when Q is non-singular. When Q is singular, the test may, or may not, be consistent against each alternative in A_1. Because our aim was to design an overall test which is consistent against each alternative in A_1, we recommend the most simple form of test-statistic, i.e. with Q of the form $Q = I_n \otimes G$, with G diagonal with non-zero diagonal elements, so that Q is non-singular. The interpretation of the observations is easier when only quadratic terms occur in the test-statistic, because, when H_0 is rejected, it is possible to see from the term(s) which caused the rejection, where the preferences or aversions occurred. The drawback on the use of a diagonal G is, that for $k > 3$ it is not possible to define a G such that $\underline{v}(G)$ has a χ^2-distribution under all circumstances. But this disadvantage may be overcome by the application of a modified χ^2-approximation to the distribution of $\underline{v}(G)$.

Therefore, if there is no special interest in interaction between preferences, we recommend the use of a diagonal G. If the user attaches special weight to some categories he can adjust the weights accordingly. If there is no outside reason to weigh one category differently from others, the most "natural" weights, dependent on the number of occurrences of the categories, seem to be

$$(6.4.1) \qquad g_j = \{\frac{1}{m}\sum_{i=1}^{m}\frac{a_{ij}}{n}\}^{-1} = \{\frac{1}{m}\frac{a_{+j}}{n}\}^{-1}.$$

We shall call the diagonal matrix with these weights: G_g. The test-statistic then has the following form

$$(6.4.2) \qquad \underline{v}(G_g) \equiv \sum_{j=1}^{k}\sum_{\nu=1}^{n}\frac{(\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n})^2}{\frac{a_{+j}}{n}},$$

i.e. the form of the usual "goodness-of-fit" statistic.

This choice of g_j has the advantage that - as is apparent from the numerical results of chapter 9 - the approximations by means of an adapted χ^2 - distribution seems to be somewhat better in this case than with other weights.

The a.d. of $\underline{v}(G_g)$, under H_0, may be determined using the methods of chapter 4, and when rank $Q\Sigma_0 = r$, with $Q = I_n \otimes G_g$, we can use a correction factor $C = C(m)$ to make sure that $C\underline{v}(G_g)$ has

$$(6.4.3) \qquad \sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2 , \qquad \text{with} \qquad \sum_{\tau=1}^{r} \lambda_\tau = r,$$

as asymptotic distribution. Then (4.4.12) is also satisfied.

In a special case (see section 6.5) we have $\lambda_1 = \ldots = \lambda_r = 1$, so that in that case the a.d. is $\chi^2[r]$.

In general, when $\sum_{\tau=1}^{r} \lambda_\tau = r$, the λ_τ's will not be very far away from 1, and the distribution of (6.4.3) will then closely resemble a $\chi^2[r]$ distribution. (The asymptotic expansion (3.3.7) for the distribution of $\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2$ seems to work best when the λ_τ's are not too far away from 1.)

Of course, for the actual calculation of the λ_τ's of the a.d., the matrix Σ_0. is used. This makes the expectations of $C\underline{v}(G_g)$ and its a.d. equal to each other, but the variances are in general still different. Because the shape of the distribution of $C\underline{v}(G_g)$ will resemble a χ^2 - distribution with r degrees of freedom, we can use the above mentioned approximation by means of a modified χ^2 - distribution. The first two moments of $C\underline{v}(G_g)$ are then equal to the first two moments of its approximating distribution. The reader is referred to chapter 1 for a description of this approximation.

Because the expectation and variance have also been determined for $\underline{v}(G_1)$ and $\underline{v}(G_2)$, and in general for $\underline{v}(G)$ with diagonal G, see chapter 7, this method may also be applied to these variables.

To conclude, we recapitulate the reasons for the choice of $\underline{v}(G_g)$ as recommended test-statistic.

i. The test based on $\underline{v}(G_g)$ is consistent against each alternative in A_1;

ii. the test-statistic has a well-known, simple form, is easy to calculate and lends itself well for interpretation;

iii. in a special case, the a.d. is χ^2 and in general the a.d. will resemble a χ^2-distribution;

iv. the exact expectation and variance, under H_0, are known and a useful approximation exists, from which critical values may be determined.

6.5. ONE MORE SPECIAL CASE

In the special case that

$$(6.5.1)\qquad a_{1j} = a_{2j} = \dots = a_{mj} \neq 0 \quad \text{for each } j,$$

the a.d. of $\frac{n-1}{n}\,\underline{v}(G_g)$, under H_0, is $\chi^2[(n-1)(k-1)]$.

This can be shown as follows. Notice that

$$(6.5.2)\qquad a_j = \lim_{m\to\infty} a_{\bullet j} = a_{ij} \quad \text{for each } i \text{ and } j.$$

Let

$$(6.5.3)\qquad H \overset{d}{=} \frac{1}{n}\begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_k \\ a_1 & a_2 & \cdot & \cdot & \cdot & a_k \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot & a_k \end{pmatrix}.$$

Then from $a_+ = n$ it follows that H is idempotent, i.e.

$$(6.5.4)\qquad H^2 = H.$$

The diagonal elements of G_g, defined in (6.4.1), reduce under (6.5.1) to

$$(6.5.5)\qquad g_j = \frac{n}{a_j}$$

and K (defined in (2.3.10)) reduces to

$$(6.5.6)\qquad K = \frac{1}{n^2}\begin{pmatrix} a_1(n-a_1) & -a_1a_2 & \cdot & \cdot & \cdot & -a_1a_k \\ -a_2a_1 & a_2(n-a_2) & \cdot & \cdot & \cdot & -a_2a_k \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ -a_ka_1 & -a_ka_2 & \cdot & \cdot & \cdot & a_k(n-a_k) \end{pmatrix}.$$

The a.d. of $\frac{n-1}{n}\,\underline{v}(G_g)\,|\,H_0$ is determined by the eigenvalues of

$$Q\Sigma_0 = \frac{n-1}{n}(I_n \otimes G_g)(N \otimes K) = N \otimes \frac{n-1}{n}\,G_gK.$$

The eigenvalues may be found from the eigenvalues of $\frac{n}{n-1}(\frac{n-1}{n} G_g K) = G_g K$. We have

$$G_g K = \frac{n}{n^2}\begin{pmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_k} \end{pmatrix}\begin{pmatrix} a_1(n-a_1) & -a_1a_2 & \cdots & -a_1a_k \\ -a_2a_1 & a_2(n-a_2) & \cdots & -a_2a_k \\ \vdots & \vdots & \ddots & \vdots \\ -a_ka_1 & -a_ka_2 & \cdots & a_k(n-a_k) \end{pmatrix} =$$

$$= \frac{1}{n}\begin{pmatrix} n-a_1 & -a_2 & \cdots & -a_k \\ -a_1 & n-a_2 & \cdots & -a_k \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & \cdots & n-a_k \end{pmatrix} = I_k - H.$$

Now from (6.5.4) it follows that

$$(6.5.7) \qquad (I_k-H)(I_k-H) = I_k - H - H + H^2 = I_k - H,$$

so that $I_k - H$ is also idempotent. It follows that the eigenvalues of $G_g K = I_k - H$ are either 0 or 1 (RAO (1973)). Therefore the eigenvalues of $Q\Sigma_0$ are also either 0 or 1.

The a.d. of $\frac{n-1}{n}\underline{v}(G_g)$ is then chi-squared with

$$(6.5.8) \qquad \text{trace }(Q\Sigma_0) = n \text{ trace}(\frac{n-1}{n} G_g K) = (n-1)\sum_{j=1}^{k}(1-\frac{a_j}{n}) = (n-1)(k-1)$$

as number of degrees of freedom.

CHAPTER 7

EXPECTATION AND VARIANCE

7.1. NOTATION

In the exact expectation and variance of $\underline{v}(Q)$, which we shall derive for some special cases, the following quantities occur.

(7.1.1) $$E_j \overset{d}{=} n^{-1} \sum_{i=1}^{m} a_{ij},$$

(7.1.2) $$S_j \overset{d}{=} n^{-2} \sum_{i=1}^{m} a_{ij}(n-a_{ij}),$$

(7.1.3) $$T_j \overset{d}{=} n^{-4} \sum_{i=1}^{m} a_{ij}^2(n-a_{ij})^2,$$

(7.1.4) $$S_{jl} \overset{d}{=} n^{-2} \sum_{i=1}^{m} a_{ij}a_{il},$$

(7.1.5) $$T_{jl} \overset{d}{=} n^{-4} \sum_{i=1}^{m} a_{ij}^2 a_{il}^2.$$

7.2. EXPECTATION

The expectation of $\underline{v}(Q)$, which we already mentioned in (2.2.40),

(7.2.1) $$E\underline{v}(Q) = \text{trace}(Q\Sigma_{1\bullet}) + \vec{\delta}_*'Q\vec{\delta}_*,$$

may be found as an application of the general formula for the expectation of a quadratic form (RAO (1973)).

Under H_0, it reduces to

(7.2.2) $$E(\underline{v}(Q)|H_0) = \text{trace}(Q\Sigma_{0\bullet}),$$

and when $Q = I_n \otimes G$,

(7.2.3) $$E(\underline{v}(G)|H_0) = \frac{n}{m}\sum_{i=1}^{m} \text{trace}(GK_i).$$

When G is moreover diagonal, we have

(7.2.4) $$E(\underline{v}(G)|H_0) = \frac{n}{m}\sum_{j=1}^{k} g_j S_j.$$

For $\underline{v}(G_2)$ we obtain

(7.2.5) $$E(\underline{v}(G_2)|H_0) = n-1,$$

which is thus also the expectation of Cochran's Q-statistic. For $\underline{v}(G_3)$ we have, using (6.3.7),

(7.2.6) $$E(\underline{v}(G_3)|H_0) = \frac{n}{m}g_1S_1 + \frac{n}{m}g_2S_2 + \frac{n}{m}g_3S_3 = -\frac{n}{m}\frac{1}{C_K}\{FS_1+ES_2+DS_3\} =$$

$$= -n\frac{1}{C_K}\frac{n-1}{n}\{FA+EB+DC\} = -(n-1)\cdot\frac{1}{C_K}\cdot -2C_K = 2(n-1).$$

Notice that for $\underline{v}(G_2)$ and $\underline{v}(G_3)$ it is not necessary to apply a correction-factor to make the test-statistic satisfy (4.4.24).

Finally we have for $\underline{v}(G_g)$

(7.2.7) $$E(\underline{v}(G_g)|H_0) = n\sum_{j=1}^{k}\frac{S_j}{E_j}.$$

In the special case that

(7.2.8) $$a_{1j} = a_{2j} = \ldots = a_{mj} \quad \text{for each } j,$$

formula (7.2.7) reduces to

(7.2.9) $$E(\underline{v}(G_g)|H_0) = n(k-1).$$

We have

LEMMA 7.2.1.

(7.2.10) $$E(\underline{v}(G_g)|H_0) \le n(k-1)$$

with equality iff (7.2.8) *holds.*

PROOF. The proof is left to the reader. □

7.3. VARIANCE

The variance of $\underline{v}(Q)$ is hard to determine in general and therefore we limit ourselves to a special case. Let $Q = I_n \otimes G$, with G diagonal. We shall determine $\sigma^2(\underline{v}(G)|H_0)$. The determination of $\sigma^2(\underline{v}(G))$ under alternatives is completely analogous, but would take too much space to reproduce here. We shall therefore suppose, throughout this section, that H_0 holds. Let

$$\underline{s}_{ij}^{(\nu)} \overset{d}{\equiv} \underline{\tilde{t}}_{ij}^{(\nu)} \equiv \underline{t}_{ij}^{(\nu)} - \frac{a_{ij}}{n} , \tag{7.3.1}$$

$$\underline{e}_j^{(\nu)} \overset{d}{\equiv} \underline{\tilde{f}}_j^{(\nu)} \equiv \sum_{i=1}^{m} \underline{s}_{ij}^{(\nu)}, \tag{7.3.2}$$

$$\pi_{ij} \overset{d}{\equiv} \frac{a_{ij}}{n} . \tag{7.3.3}$$

LEMMA 7.3.1. *We have for all* i, i_1, i_2, j *and* l, *except where otherwise indicated.*

$$E\{\underline{s}_{ij}^{(1)}\}^2 = \sigma^2(\underline{t}_{ij}^{(1)}) = \pi_{ij} - \pi_{ij}^2 , \tag{7.3.4}$$

$$E\, \underline{s}_{ij}^{(1)}\underline{s}_{il}^{(1)} = \text{cov}(\underline{t}_{ij}^{(1)},\underline{t}_{il}^{(1)}) = -\pi_{ij}\pi_{il}, \qquad j\neq l, \tag{7.3.5}$$

$$E\, \underline{s}_{ij}^{(1)}\underline{s}_{ij}^{(2)} = \text{cov}(\underline{t}_{ij}^{(1)},\underline{t}_{ij}^{(2)}) = -\frac{1}{n-1}\{\pi_{ij}-\pi_{ij}^2\}, \tag{7.3.6}$$

$$E\, \underline{s}_{ij}^{(1)}\underline{s}_{il}^{(2)} = \text{cov}(\underline{t}_{ij}^{(1)},\underline{t}_{il}^{(2)}) = \frac{1}{n-1}\pi_{ij}\pi_{il} , \qquad j\neq l, \tag{7.3.7}$$

$$\text{cov}(\{\underline{s}_{ij}^{(1)}\}^2,\{\underline{s}_{il}^{(2)}\}^2) = -\frac{1}{n-1}\text{cov}(\{\underline{s}_{ij}^{(1)}\}^2,\{\underline{s}_{il}^{(1)}\}^2) , \tag{7.3.8}$$

$$\text{cov}(\{\underline{s}_{ij}^{(1)}\}^2,\{\underline{s}_{i_1l}^{(2)}\}\{\underline{s}_{i_2l}^{(2)}\}) = -\frac{1}{n-1}\text{cov}(\{\underline{s}_{ij}^{(1)}\}^2,\{\underline{s}_{i_1l}^{(1)}\}\{\underline{s}_{i_2l}^{(1)}\}) , \tag{7.3.9}$$

$$\text{cov}(\{\underline{s}_{i_1j}^{(1)}\}\{\underline{s}_{i_2j}^{(1)}\},\{\underline{s}_{il}^{(2)}\}^2) = -\frac{1}{n-1}\text{cov}(\{\underline{s}_{i_1j}^{(1)}\}\{\underline{s}_{i_2j}^{(1)}\},\{\underline{s}_{il}^{(1)}\}^2) . \tag{7.3.10}$$

<u>PROOF</u>. (7.3.4)-(7.3.7) follow from (2.2.24)-(2.2.27). We next prove (7.3.10). Write

$$(7.3.11) \qquad \underline{x}^{(\nu)} \stackrel{d}{=} \{\underline{s}_{i_1j}{}^{(\nu)}\}\{\underline{s}_{i_2j}{}^{(\nu)}\}.$$

Then, for each i and l,

$$\mathrm{cov}(\underline{x}^{(1)},\{\underline{s}_{il}{}^{(2)}\}^2) = \mathrm{cov}(\underline{x}^{(1)},\underline{t}_{il}{}^{(2)}(1-2\pi_{il})) =$$

$$= (1-2\pi_{il})\mathrm{cov}(\underline{x}^{(1)},\underline{t}_{il}{}^{(2)}) .$$

Observe that

$$\sum_{\nu=1}^{n} \mathrm{cov}(\underline{x}^{(1)},\underline{t}_{il}{}^{(\nu)}) = \mathrm{cov}(\underline{x}^{(1)}, \sum_{\nu=1}^{n} \underline{t}_{il}{}^{(\nu)}) = \mathrm{cov}(\underline{x}^{(1)},a_{ij}) = 0.$$

Thus, using the fact that, due to symmetry, the joint distributions of the pairs $(\underline{x}^{(1)},\underline{t}_{il}{}^{(2)}),\ldots,(\underline{x}^{(1)},\underline{t}_{il}{}^{(n)})$ are the same,

$$\mathrm{cov}(\underline{x}^{(1)},\underline{t}_{il}{}^{(1)}) + (n-1)\mathrm{cov}(\underline{x}^{(1)},\underline{t}_{il}{}^{(2)}) = 0.$$

Therefore

$$\mathrm{cov}(\underline{x}^{(1)},\underline{t}_{il}{}^{(2)}) = -\frac{1}{n-1}\mathrm{cov}(\underline{x}^{(1)},\underline{t}_{il}{}^{(1)}).$$

And thus also

$$\mathrm{cov}(\underline{x}^{(1)},\{\underline{s}_{il}{}^{(2)}\}^2) = -\frac{1}{n-1}\mathrm{cov}(\underline{x}^{(1)},\{\underline{s}_{il}{}^{(1)}\}^2).$$

This proves (7.3.10). Simultaneous interchanging of j and l and (1) and (2) in (7.3.10) gives (7.3.9). (7.3.8) follows if we take

$$\underline{x}_j{}^{(\nu)} \stackrel{d}{=} \{\underline{s}_{ij}{}^{(\nu)}\}^2$$

instead of (7.3.11). □

We use the following notation

$$(7.3.12) \qquad \sum{}^{*} = \sum_{i_1=1}^{m} \sum_{\substack{i_2=1 \\ i_1 \neq i_2}}^{m}$$

LEMMA 7.3.2.

$$(7.3.13) \qquad n \operatorname{cov}(\sum{}^{*}\{\underline{s}_{i_1 j}^{(1)}\}\{\underline{s}_{i_2 j}^{(1)}\}, \sum{}^{*}\{\underline{s}_{i_1 l}^{(1)}\}\{\underline{s}_{i_2 l}^{(1)}\}) +$$

$$+ n(n-1)\operatorname{cov}(\sum{}^{*}\{\underline{s}_{i_1 j}^{(1)}\}\{\underline{s}_{i_2 j}^{(1)}\}, \sum{}^{*}\{\underline{s}_{i_1 l}^{(2)}\}\{\underline{s}_{i_2 l}^{(2)}\}) =$$

$$= \begin{cases} \dfrac{2n^2}{n-1} \sum{}^{*} \pi_{i_1 j}\pi_{i_1 l}\pi_{i_2 j}\pi_{i_2 l} & \textit{if } j \neq l, \\ \dfrac{2n^2}{n-1} \sum{}^{*} \pi_{i_1 j}(1-\pi_{i_1 j})\pi_{i_2 j}(1-\pi_{i_2 j}) & \textit{if } j = l. \end{cases}$$

PROOF. Using independence when $i_1 \neq i_2$, $E\underline{s}_{ij}^{(\nu)} = 0$, and (7.3.5) we have for $j \neq l$,

$$\operatorname{cov}(\sum{}^{*}\{\underline{s}_{i_1 j}^{(1)}\}\{\underline{s}_{i_2 j}^{(1)}\}, \sum{}^{*}\{\underline{s}_{i_1 l}^{(1)}\}\{\underline{s}_{i_2 l}^{(1)}\}) =$$

$$= 2 \sum{}^{*} E\, \underline{s}_{i_1 j}^{(1)}\, \underline{s}_{i_2 j}^{(1)}\, \underline{s}_{i_1 l}^{(1)}\, \underline{s}_{i_2 l}^{(1)} =$$

$$= 2 \sum{}^{*} E\, \underline{s}_{i_1 j}^{(1)}\, \underline{s}_{i_1 l}^{(1)}\, E\, \underline{s}_{i_2 j}^{(1)}\, \underline{s}_{i_2 l}^{(1)} =$$

$$= 2 \sum{}^{*} (-\pi_{i_1 j}\pi_{i_1 l})(-\pi_{i_2 j}\pi_{i_2 l}) = 2 \sum{}^{*} \pi_{i_1 j}\pi_{i_1 l}\pi_{i_2 j}\pi_{i_2 l} .$$

Also, now using (7.3.7),

$$\operatorname{cov}(\sum{}^{*}\{\underline{s}_{i_1 j}^{(1)}\}\{\underline{s}_{i_2 j}^{(1)}\}, \sum{}^{*}\{\underline{s}_{i_1 l}^{(2)}\}\{\underline{s}_{i_2 l}^{(2)}\}) =$$

$$= \frac{2}{(n-1)^2} \sum{}^{*} \pi_{i_1 j}\pi_{i_1 l}\pi_{i_2 j}\pi_{i_2 l} .$$

These two results together give (7.3.13) if $j \neq l$. The case $j = l$ is proved analogously. □

THEOREM 7.3.1. *For diagonal* G, *we have*

$$(7.3.14)\qquad \sigma^2(\underline{v}(G)|H_0) = \frac{1}{m^2}\frac{2n^2}{n-1}\{\sum_{j=1}^{k} g_j^2(S_j^2-T_j) + \sum_{j\neq l}\sum g_j g_l(S_{jl}^2-T_{jl})\}.$$

PROOF. We have

$$\underline{v}(G) \equiv \frac{1}{m}\sum_{j=1}^{k}\sum_{\nu=1}^{n} g_j(\underline{f}_j^{(\nu)} - \frac{a_{+j}}{n})^2 \equiv \frac{1}{m}\sum_{j=1}^{k}\sum_{\nu=1}^{n} g_j\{\underline{e}_j^{(\nu)}\}^2.$$

Using permutability over the index ν we have, under H_0,

$$\begin{aligned}\sigma^2(\underline{v}(G)) &= \frac{1}{m^2}\sum_{j=1}^{k}\sum_{l=1}^{k} g_j g_l \mathrm{cov}(\sum_{\nu=1}^{n}\{\underline{e}_j^{(\nu)}\}^2, \sum_{\mu=1}^{n}\{\underline{e}_l^{(\mu)}\}^2) = \\ &= \frac{1}{m^2}\sum_{j=1}^{k}\sum_{l=1}^{k} g_j g_l\{\sum_{\nu=1}^{n}\sum_{\mu=1}^{n}\mathrm{cov}(\{\underline{e}_j^{(\nu)}\}^2,\{\underline{e}_l^{(\mu)}\}^2)\} = \\ &= \frac{1}{m^2}\sum_{j=1}^{k}\sum_{l=1}^{k} g_j g_l\{n\ \mathrm{cov}(\{\underline{e}_j^{(1)}\}^2,\{\underline{e}_l^{(1)}\}^2) + \\ &\qquad + n(n-1)\mathrm{cov}(\{\underline{e}_j^{(1)}\}^2,\{\underline{e}_l^{(2)}\}^2)\}.\end{aligned}$$

Next, observe that, using (7.3.8), (7.3.9) and (7.3.10),

$$\begin{aligned} &n\ \mathrm{cov}(\{\underline{e}_j^{(1)}\}^2, \{\underline{e}_l^{(1)}\}^2) + n(n-1)\mathrm{cov}(\{\underline{e}_j^{(1)}\}^2,\{\underline{e}_l^{(2)}\}^2) = \\ =\ & n\ \mathrm{cov}(\sum_{i=1}^{m}\{\underline{s}_{ij}^{(1)}\}^2 + \sum{}^{*}\underline{s}_{i_1j}^{(1)}\underline{s}_{i_2j}^{(1)}, \sum_{i=1}^{m}\{\underline{s}_{il}^{(1)}\}^2 + \sum{}^{*}\underline{s}_{i_1l}^{(1)}\underline{s}_{i_2l}^{(1)}) + \\ &+ n(n-1)\mathrm{cov}(\sum_{i=1}^{m}\{\underline{s}_{ij}^{(1)}\}^2 + \sum{}^{*}\underline{s}_{i_1j}^{(1)}\underline{s}_{i_2j}^{(1)}, \sum_{i=1}^{m}\{\underline{s}_{il}^{(2)}\}^2 + \sum{}^{*}\underline{s}_{i_1l}^{(2)}\underline{s}_{i_2l}^{(2)}) = \\ &= n\ \mathrm{cov}(\sum{}^{*}\underline{s}_{i_1j}^{(1)}\underline{s}_{i_2j}^{(1)}, \sum{}^{*}\underline{s}_{i_1l}^{(1)}\underline{s}_{i_2l}^{(1)}) + \\ &\qquad + n(n-1)\mathrm{cov}(\sum{}^{*}\underline{s}_{i_1j}^{(1)}\underline{s}_{i_2j}^{(1)}, \sum{}^{*}\underline{s}_{i_1l}^{(2)}\underline{s}_{i_2l}^{(2)}) .\end{aligned}$$

This is just the expression of lemma 7.3.2; the rest of the proof is simple calculation. □

For $\underline{v}(G_2)$ we obtain

$$(7.3.15)\qquad \sigma^2(\underline{v}(G_2)|H_0) = 2(n-1)(\frac{S_1^2 - T_1}{S_1^2}) ,$$

the variance of Cochran's Q-statistic (under H_0).

Substitution of (6.3.11) in (7.3.14) does not lead to a simpler form for $\sigma^2(\underline{v}(G_3)|H_0)$.

For $\underline{v}(G_g)$ we obtain

$$(7.3.16) \qquad \sigma^2(\underline{v}(G_g)|H_0) = \frac{2n^2}{n-1}\{\sum_{j=1}^{k} \frac{S_j^2-T_j}{E_j} + \sum_{j\neq l}\sum \frac{S_{jl}^2-T_{jl}}{E_jE_l}\},$$

which reduces in the special case (7.2.8) to

$$(7.3.17) \qquad \sigma^2(\underline{v}(G_g)|H_0) = \frac{2n^2}{n-1}(k-1)\frac{m-1}{m} .$$

CHAPTER 8

MOTIVATION OF THE CHOICE OF QUADRATIC FORMS

To derive tests for the simple hypothesis H_0: $\forall_i \ \vec{\Delta}_i = \vec{0}$, against the composite hypothesis H_1: $\exists_i \ \vec{\Delta}_i \neq \vec{0}$, with certain optimality properties, there are basically only a few methods. We shall show why two standard methods fail in our situation, and why we have therefore chosen for a third method based on asymptotic distributions.

8.1. THE NEYMAN & PEARSON FUNDAMENTAL LEMMA - METHOD

Consider the problem of testing the simple hypothesis

(8.1.1) $$H_0: \quad \forall_i \quad \vec{\Delta}_i = \vec{0},$$

against the simple alternative

(8.1.2) $$H_1: \quad \vec{\Delta}_i = \vec{\Delta}_i^* ,$$

where $\vec{\Delta}_1^*,\dots,\vec{\Delta}_m^*$ are fixed elements from $\mathcal{D}_1,\dots,\mathcal{D}_m$. We have

(8.1.3) $$P(\underline{\omega}_i = \pi_{ir} | H_0) = \frac{1}{N_i} , \qquad r \in \mathcal{R}_i,$$

(8.1.4) $$P(\underline{\omega}_1 = \pi_{1r_1} \wedge \dots \wedge \underline{\omega}_m = \pi_{mr_m} | H_0) = \prod_{i=1}^{m} \frac{1}{N_i} , \qquad r_i \in \mathcal{R}_i,$$

(8.1.5) $$P(\underline{\omega}_i = \pi_{ir} | H_1) = \frac{1}{N_i} + \Delta_{ir}^*, \qquad r \in \mathcal{R}_i,$$

(8.1.6) $$P(\underline{\omega}_1 = \pi_{1r_1} \wedge \dots \wedge \underline{\omega}_m = \pi_{mr_m} | H_1) = \prod_{i=1}^{m} (\frac{1}{N_i} + \Delta_{ir_i}^*), \qquad r_i \in \mathcal{R}_i.$$

According to Neyman & Pearson's fundamental lemma, the most powerful test rejects H_0 for large values of the quotient of (8.1.6) and (8.1.4), i.e.

$$(8.1.7)\qquad \frac{\prod_{i=1}^{m}\left(\frac{1}{N_i}+\Delta^*_{ir_i}\right)}{\prod_{i=1}^{m}\frac{1}{N_i}} = \prod_{i=1}^{m}(1+N_i\Delta^*_{ir_i}),$$

leading to the test-statistic

$$(8.1.8)\qquad \underline{T}_1 \equiv \prod_{i=1}^{m}(1+N_i\Delta^*_i(\underline{\omega}_i)),$$

where $\Delta^*_i(.)$ is a function $\Omega_i \to \mathbb{R}$, with

$$(8.1.9)\qquad \Delta^*_i(\pi_{ir}) = \Delta^*_{ir}, \qquad r \in R_i.$$

For given $\vec{\Delta}^*_i$, this may lead to a useful test, with, after taking the logarithm,

$$(8.1.10)\qquad E \log \underline{T}_1 = \sum_{i=1}^{m}\sum_{r=1}^{N_i}\left(\frac{1}{N_i}+\Delta^*_{ir}\right)\log(1+N_i\Delta^*_{ir}),$$

$$(8.1.11)\qquad E(\log \underline{T}_1 | H_0) = \sum_{i=1}^{m}\sum_{r=1}^{N_i}\frac{1}{N_i}\log(1+N_i\Delta^*_{ir}).$$

By the Central Limit Theorem, log $\underline{T}_1$ is asymptotically normal. Critical values could be determined by enumerating the exact distribution, or can be based on the asymptotic distribution, after having computed the variance of log $\underline{T}_1$.

If, however, we are interested in the behaviour of this test also for other (or even all) alternatives from A_1, then it is not at all clear how E log $\underline{T}_1$ behaves under these alternatives. Moreover, we see no way to adapt it to work against other alternatives in A_1 too. Therefore, we do not pursue this method any further.

8.2. THE LIKELIHOOD-RATIO METHOD

Consider the problem of testing the simple hypothesis

$$(8.2.1)\qquad H_0: \quad \forall_i \quad \vec{\Delta}_i = \vec{0}$$

against

(8.2.2) $H_1: \quad \exists_i \quad \vec{\Delta}_i \neq \vec{0}.$

The likelihood-ratio test rejects H_0 for large values of

$$(8.2.3) \quad \Lambda = \frac{\sup_{\vec{\Delta}_i} P(\underline{\omega}_1 = \pi_{1r_1} \wedge \ldots \wedge \underline{\omega}_m = \pi_{mr_m} | H_1)}{P(\underline{\omega}_1 = \pi_{1r_1} \wedge \ldots \wedge \underline{\omega}_m = \pi_{mr_m} | H_0)} =$$

$$= \frac{\sup_{\vec{\Delta}_i} \prod_{i=1}^{m} (\frac{1}{N_i} + \Delta_{ir_i})}{\prod_{i=1}^{m} \frac{1}{N_i}} = \frac{1}{\prod_{i=1}^{m} \frac{1}{N_i}} = \prod_{i=1}^{m} N_i .$$

The likelihood-ratio is in this case apparently a constant, and is therefore also unfit to produce a useful test-statistic, to test H_0 against this wide class of alternatives.

Now suppose that we only wish to consider alternatives from H_1 for which words beginning with the character C_1 have, for each i, a higher probability than the other words. This restricts the possibilities considerably and so

$$\sup_{\vec{\Delta}_i} \prod_{i=1}^{m} (\frac{1}{N_i} + \Delta_{ir_i})$$

will take a lower value if the $\vec{\Delta}_i$ may only range over these restricted alternatives. Define

$$(8.2.4) \quad N_i' \stackrel{d}{=} \frac{(n-1)!}{(a_{i1}-1)!a_{i2}! \ldots a_{ik}!} ,$$

i.e. N_i' is the number of words with C_1 in the first position in the set of outcomes Ω_i of the i'th trial. Let

$$(8.2.5) \quad R_i' \stackrel{d}{=} \{1,\ldots,N_i'\}.$$

Then restricted alternatives may be formulated as follows

$$(8.2.6) \quad H_1^{**}: \forall_i \quad \Delta_{ir} > 0 \quad \text{for } r \in R_i' \quad \text{and} \quad \Delta_{ir} \leq 0 \quad \text{for } r \notin R_i' .$$

Notice that $H_1^{**} \Rightarrow H_1'$ but not conversely, so that a possible test derived in this way only works against this much smaller class of alternatives. We have

$$(8.2.7) \qquad \sup_{\vec{\Delta}_i \in H_1^{**}} \left(\frac{1}{N_i} + \Delta_{ir_i}\right) = \begin{cases} 1 \text{ if } r_i \in R_i' , \\ \frac{1}{N_i} \text{ if } r_i \notin R_i' . \end{cases}$$

This may be written as

$$(8.2.8) \qquad \sup_{\vec{\Delta}_i \in H_1^{**}} \left(\frac{1}{N_i} + \Delta_{ir_i}\right) = \left(\frac{1}{N_i}\right)^{1-t_{i1}^{(1)}(\pi_{ir_i})} .$$

So the likelihood-ratio becomes

$$(8.2.9) \qquad \Lambda = \frac{\prod_{i=1}^{m} \left(\frac{1}{N_i}\right)^{1-t_{i1}^{(1)}(\pi_{ir_i})}}{\prod_{i=1}^{m} \left(\frac{1}{N_i}\right)}$$

and the likelihood-ratio test rejects H_0 for large values of

$$(8.2.10) \qquad \prod_{i=1}^{m} \left(\frac{1}{N_i}\right)^{1-\underline{t}_{i1}^{(1)}}$$

or, equivalently, for large values of

$$(8.2.11) \qquad \underline{T}_1^{(1)} \stackrel{d}{=} \sum_{i=1}^{m} \underline{t}_{i1}^{(1)} \log N_i .$$

Analogously we may define

$$(8.2.12) \qquad \underline{T}_j^{(\nu)} \stackrel{d}{=} \sum_{i=1}^{m} \underline{t}_{ij}^{(\nu)} \log N_i .$$

Aiming at an overall test, as we do, we do not know in advance where possible preferences occur, so we might combine these statistics to, for instance

$$(8.2.13) \qquad \underline{T}_2 \stackrel{d}{=} \max_{j,\nu} \underline{T}_j^{(\nu)} ,$$

to get an overall test-statistic for H_0. This is possible, because the

$\underline{T}_j^{(\nu)}$ do not depend on any particular alternative. This would lead to a test for an outlier among the characters

Let's now study the r.v.'s $\underline{T}_j^{(\nu)}$ in greater detail. We have, using the results of chapter 2,

$$(8.2.14) \qquad E\underline{T}_j^{(\nu)} = \sum_{i=1}^{m} \frac{a_{ij}}{n} \log N_i + \sum_{i=1}^{m} \delta_{ij}^{(\nu)} \log N_i ,$$

$$(8.2.15) \qquad \sigma^2(\underline{T}_j^{(\nu)}) = \sum_{i=1}^{m} \{\frac{a_{ij}}{n} - \frac{a_{ij}^2}{n^2}\} \log^2 N_i \; + $$

$$+ \sum_{i=1}^{m} \{-2\delta_{ij}^{(\nu)}\frac{a_{ij}}{n} + \delta_{ij}^{(\nu)} - (\delta_{ij}^{(\nu)})^2\} \log^2 N_i \; .$$

The expectation and variance under H_0 are found by deleting the terms containing δ's.

The variables $\underline{T}_j^{(\nu)}$ are not independent, and their covariances may be found using (2.1.27) and (2.1.28). The (marginal) a.d.'s are normal by the C.L.T. The joint a.d. of the $\underline{T}_j^{(\nu)}$ may be found using the methods of chapter 4.

The distribution of $\underline{T}_2$, however, is difficult to obtain, the exact distribution as well as the asymptotic distribution (JOHNSON & KOTZ (1972), p.44). The development of this outlier-test would be an interesting subject for further research.

The result (8.2.14) suggests the use of a test-statistic similar to the one defined in (2.1.39)

$$(8.2.16) \qquad \underline{T}_3 \overset{d}{=} \frac{1}{m} \sum_{j=1}^{k} g_j \sum_{\nu=1}^{n} \{ \sum_{i=1}^{m} (\underline{t}_{ij}^{(\nu)} - \frac{a_{ij}}{n}) \log N_i \}^2 \; .$$

This statistic, which gives trials with a high number of possible words more weight than $\underline{v}(G)$ does, may be treated in the same way as the statistic $\underline{v}(G)$. Its a.d. may be determined in a similar way as that of $\underline{v}(G)$, both under H_0 and under alternatives.

By these considerations we could be led to consider a class of test-statistics which is even more general than (2.1.41), of the form

$$(8.2.17) \qquad \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{k} g_{jl}^{(\nu,\mu)} \{ \sum_{i=1}^{m} G_i(\underline{t}_{ij}^{(\nu)} - \frac{a_{ij}}{n})\}\{ \sum_{i=1}^{m} G_i(\underline{t}_{il}^{(\mu)} - \frac{a_{il}}{n})\}.$$

with

(8.2.18) $\quad G_i = \log N_i$

as suggested weights.

Although the analysis of the behaviour of such an extensive class of test-statistics would lead to a new and major enterprise, the suggestion that the use of $G_i = \log N_i$ would possibly increase the power of our tests in an adapted form is worthy of future consideration.

8.3. AN APPROXIMATE LIKELIHOOD-RATIO METHOD

Let $\underline{\vec{x}} \sim N_q(\vec{\mu},\Sigma)$. Suppose that $\vec{\mu}$ is unknown, but that Σ is a known, fixed, positive definite matrix. For the problem of testing H_0: $\vec{\mu} = \vec{\mu}_0$ against H_1: $\vec{\mu} \neq \vec{\mu}_0$, the likelihood-ratio is equal to

(8.3.1) $$\Lambda(\vec{x}) = \frac{\sup_{H_1} (2\pi)^{-q/2}|\Sigma|^{-\frac{1}{2}}\exp\{-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})\}}{\sup_{H_0} (2\pi)^{-q/2}|\Sigma|^{-\frac{1}{2}}\exp\{-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})\}}.$$

Clearly,

(8.3.2) $$\sup_{H_0} \exp\{-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})\} = \exp\{-\frac{1}{2}(\vec{x}-\vec{\mu}_0)'\Sigma^{-1}(\vec{x}-\vec{\mu}_0)\}.$$

Furthermore,

(8.3.3) $$\sup_{H_1} \exp\{-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})\} = 1,$$

because the infimum of $(\vec{x}-\vec{\mu})'\Sigma^{-1}(\vec{x}-\vec{\mu})$ is equal to 0, at $\vec{\mu} = \vec{x}$.

So the likelihood-ratio test rejects H_0 for large values of the statistic,

(8.3.4) $$(\underline{\vec{x}}-\vec{\mu}_0)'\Sigma^{-1}(\underline{\vec{x}}-\vec{\mu}_0),$$

i.e. a quadratic form in $\underline{\vec{x}}$, where the weighing coefficients are elements of the inverse of the covariance matrix Σ. It follows easily from theorem 3.2.1 that the distribution of this statistic is a central χ^2 - distribution with q degrees of freedom under H_0, and a non-central χ^2 - distribution with q degrees of freedom and non-centrality parameter $(\vec{\mu}-\vec{\mu}_0)'\Sigma^{-1}(\vec{\mu}-\vec{\mu}_0)$ under H_1.

The situation is not essentially changed when the dispersion matrix of $\underline{\vec{x}}|H_0$, Σ_0 (say), differs from the dispersion matrix of $\underline{\vec{x}}|H_1$, Σ_1 (say). The likelihood-ratio test-statistic would become

$$(8.3.5) \qquad (\underline{\vec{x}}-\vec{\mu}_0)'\Sigma_0^{-1}(\underline{\vec{x}}-\vec{\mu}_0),$$

with a central χ^2-distribution under H_0, but, in general, *not* a non-central χ^2-distribution under H_1. The distribution of (8.3.5) under H_1 may be determined with theorem 3.2.1.

When Σ is singular, for instance with rank $r < q$, a straightforward generalisation is possible. The density of $\underline{\vec{x}}$ can then be represented as (RAO (1973), p.528),

$$(2\pi)^{-r/2}(\lambda_1\cdot\ldots\cdot\lambda_r)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-}(\vec{x}-\vec{\mu})\}$$

where the density is concentrated on the hyperplane

$$N'\vec{x} = N'\vec{\mu}$$

with probability one. Σ^{-} is any g-inverse of Σ, $\lambda_1,\ldots,\lambda_r$ are the non-zero eigenvalues of Σ and N is a $q\times(q-r)$ matrix of rank $(q-r)$ such that $N'\Sigma = 0$.

Now suppose again that $\vec{\mu}$ is unknown, but that Σ is a known, fixed, non-negative definite matrix, and that we test the same H_0 as above. Because Σ is fixed, the matrix N is fixed and both $\underline{\vec{x}}$ and $\vec{\mu}$ satisfy the same $(q-r)$ linear constraints, under H_0 and under H_1. This means that the distribution of $\underline{\vec{x}}$ is concentrated on the same hyperplane under H_0 and under H_1.

The likelihood-ratio for this testing problem is then equal to

$$(8.3.6) \qquad \Lambda(\vec{x}) = \frac{\sup_{H_0}\ (2\pi)^{-r/2}(\lambda_1\cdot\ldots\cdot\lambda_r)^{-\frac{1}{2}}\exp\{-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-}(\vec{x}-\vec{\mu})\}}{\sup_{H_1}\ (2\pi)^{-r/2}(\lambda_1\cdot\ldots\cdot\lambda_r)^{-\frac{1}{2}}\exp\{-\frac{1}{2}(\vec{x}-\vec{\mu})'\Sigma^{-}(\vec{x}-\vec{\mu})\}}$$

for each $\vec{x} \in \{\vec{x}|N'\vec{x} = N'\vec{\mu}_0\}$.

It is the again clear that the likelihood-ratio test-statistic is equal to

$$(8.3.7) \qquad (\underline{\vec{x}}-\vec{\mu}_0)'\Sigma^{-}(\underline{\vec{x}}-\vec{\mu}_0).$$

Now choose a version of Σ^- which is real and symmetric. Then it follows from theorem 3.2.3 that this statistic has, under H_0, a central χ^2 - distribution with trace$(\Sigma\Sigma^-)$ = rank(Σ) = r as number of degrees of freedom. This is then also the case for arbitrary Σ^-. However, under H_1, even when we choose a n.n.d. version of Σ^-, (8.3.7) has not necessarily a non-central χ^2 - distribution, because (3.2.36) is not necessarily satisfied. Moreover, the distribution of (8.3.7) under H_1 will depend on the specific choice of Σ^-. Of course, the distribution may be determined with theorem 3.2.1.

In our testing problem we have according to (4.4.4) and (4.4.3),

(8.3.8) $\quad \underline{\vec{t}}_* \approx N(\vec{0},\Sigma_{0\cdot}) \quad$ under H_0

and

(8.3.9) $\quad \underline{\vec{t}}_* \approx N(\vec{\delta}_*,\Sigma_{1\cdot}) \quad$ under H_1,

where "≈" means "is approximately distributed as" (in (8.3.8) and (8.3.9)). When we make furthermore the crude assumption that $\Sigma_{1\cdot} \approx \Sigma_{0\cdot}$ for all alternatives, then it follows from the preceding theory that

(8.3.10) $\quad \underline{\vec{t}}_*'\Sigma_{0\cdot}^-\underline{\vec{t}}_*$

is an "approximate" likelihood-ratio test-statistic. This statistic has been considered in section 6.1 as a special case of the general class of test-statistics,

(8.3.11) $\quad \underline{\vec{t}}_*'Q\underline{\vec{t}}_*$

that we consider in this tract.

CHAPTER 9

NUMERICAL RESULTS

This research would not be complete without illustrative examples of a numerical kind. Because of the huge number of parameters in our problem, it is hardly possible to cover all the situations that can occur, and therefore the results of the numerical computations that we give must merely be seen as illustrations of the theory.

There are two kinds of numerical computations that have been made. The first kind concerns the elaboration of most of the formula's that occur in the theory, for a typical practical case, like the computation of exact moments, eigenvalues etc. The second kind concerns the numerical simulation of the exact probability distributions of the test-statistics involved.

All calculations were performed on the CDC - CYBER 73 computer of SARA ("Stichting Academisch Rekencentrum Amsterdam"). Several procedures were used from the library STATAL of statistical procedures, developed by the "Mathematisch Centrum", Amsterdam, and from the library NUMAL of numerical procedures developed by the University of Amsterdam.

We start with the definition of a typical practical case, in the conditional situation.

9.1. A TYPICAL CASE

Suppose we have the following tableau of observations, i.e. with $m = 10$, $n = 5$ and $k = 3$. The table has the same structure as table 1.2.1. For shortness, the categories chosen are indicated by their numbers instead of by their names.

Table 9.1.1. Example of an observation for $m = 10$, $n = 5$ and $k = 3$.

i \ ν	1	2	3	4	5		j=1	j=2	j=3	
1	1	1	2	3	3		2	1	2	5
2	1	1	2	3	2		2	2	1	5
3	3	3	2	2	3		0	2	3	5
4	1	3	3	3	1		2	0	3	5
5	2	2	3	2	2		0	4	1	5
6	1	2	3	2	2		1	3	1	5
7	1	3	3	2	3		1	1	3	5
8	1	2	3	1	1		3	1	1	5
9	1	3	3	2	1		2	1	2	5
10	2	1	3	2	3		1	2	2	5
							14	17	19	50
j=1	7	3	0	1	3	14				
j=2	2	3	3	6	3	17				
j=3	1	4	7	3	4	19				
	10	10	10	10	10	50				

We shall test our null-hypothesis on the basis of these observations. The very first thing to do is to select a Q matrix for the test-statistic $\underline{v}(Q)$.

We shall consider four different statistics, with Q of the form

$$Q = I_n \otimes G, \tag{9.1.1}$$

and G diagonal, i.e.

$$G = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix} . \tag{9.1.2}$$

The test-statistic that we recommend has weighing factors as given in (6.4.1). We obtain in this case

(9.1.3) $g_1 = \frac{50}{14} = 3.5714$; $g_2 = \frac{50}{17} = 2.9412$; $g_3 = \frac{50}{19} = 2.6316$.

In order to satisfy (4.4.12), we shall modify these weighing factors a little by multiplying each with the same constant factor (0.9610), giving

(9.1.4) $g_1 = 3.4320$; $g_2 = 2.8263$; $g_3 = 2.5288$.

This has of course no influence on the performance of the test. For ease of reference, we shall call the matrix Q defined by (9.1.1), (9.1.2) and (9.1.3), Q_1, and the associated test-statistic $\underline{v}(Q_1)$ or simply $\underline{v}_1$.

A second possible choice is to take the weighing factors equal:

(9.1.5) $g_1 = g_2 = g_3 = 2.8986$,

giving a matrix Q_2 and a statistic $\underline{v}_2$. The value 2.8986 is again the result of a modification (we could otherwise have taken $g_1 = g_2 = g_3 = 1$).

A third statistic $\underline{v}_3$ may be obtained when we have the impression (before the actual observations were made) that there is a preference for C_1 in the first position. We can then give more weight to the first character by choosing (for instance)

(9.1.6) $g_1 = 4$; $g_2 = 1$; $g_3 = 0.5$,

or, after modification,

(9.1.7) $g_1 = 6.7227$; $g_2 = 1.6807$; $g_3 = 0.8403$.

The fourth and last statistic that we consider, $\underline{v}_4$, has weighing factors given by (6.3.11),

(9.1.8) $g_1 = 3.4638$; $g_2 = 3.2072$; $g_3 = 2.1809$.

This gives also a g-inverse type, or Madansky-type statistic. The Q-matrices of $\underline{v}_3$ and $\underline{v}_4$ are called Q_3 and Q_4 resp.

Recapitulating, we shall consider the following four test-statistics.

Table 9.1.2. Weighing factors of four possible test-statistics.

	weighing factors			
statistic	g_1	g_2	g_3	type
$\underline{v}_1 \equiv \vec{\underline{t}}_*' Q_1 \vec{\underline{t}}_*$	3.4320	2.8263	2.5288	"χ^2"
$\underline{v}_2 \equiv \vec{\underline{t}}_*' Q_2 \vec{\underline{t}}_*$	2.8986	2.8986	2.8986	"equal weights"
$\underline{v}_3 \equiv \vec{\underline{t}}_*' Q_3 \vec{\underline{t}}_*$	6.7227	1.6807	0.8403	"directed"
$\underline{v}_4 \equiv \vec{\underline{t}}_*' Q_4 \vec{\underline{t}}_*$	3.4638	3.2072	2.1809	"asymptotic χ^2"

To investigate the performance of the test, we have constructed 2 alternatives, which we shall call $a_{(1)}$ and $a_{(2)}$.

Alternative $a_{(1)}$. Because we are in the conditional situation, an alternative is defined by the assignment of (unequal) probabilities to each of the possible words in each of the experiments E_i. We have

Table 9.1.3. Number of possible words per experiment.

i	$\vec{a}_i$			number of possible words N_i
1	2	1	2	30
2	2	2	1	30
3	0	2	3	10
4	2	0	3	10
5	0	4	1	5
6	1	3	1	20
7	1	1	3	20
8	3	1	1	20
9	2	1	2	30
10	1	2	2	30

(Notice that there are $30 \times 30 \times 10 \times \ldots \times 30 \times 30 = 3.24 \times 10^{12}$ possible ways of obtaining a table of observations like the one in table 9.1.1.).

We have constructed an alternative in which a preference for C_1 in the first position is reflected in the fact that

$$(9.1.9) \qquad P(\underline{t}_{i1}{}^{(1)} = 1 | \alpha_{(1)}) = 0.8$$

for those experiments for which $a_{i1} = 2$, and

$$(9.1.10) \qquad P(\underline{t}_{i1}{}^{(1)} = 1 | \alpha_{(1)}) = 0.4$$

in the cases that $a_{i1} = 1$. For the rest the probabilities are spread evenly over the words. For instance, in the first experiment, the 12 words commencing with C_1 have probability $0.8/12 = 0.0\not{6}$, while the other 18 of the 30 possible words have probability $0.2/18 = 0.0\not{1}$. In the sixth experiment, the 4 words beginning with C_1 have probability $0.4/4 = 0.1$ and the other 16 $0.6/16 = 0.0375$. In the third experiment the words have the same probability as under H_0, namely 0.1. The probabilities in the other experiments were determined likewise.

<u>Alternative $\alpha_{(2)}$</u>. This alternative is more intricate, because it has been constructed to represent three relative preferences, a preference of C_1 for the first position, a preference of C_2 for the second and a preference of C_3 for the third position.

Probabilities have been assigned in the following way. Probability 0.4 has been divided evenly over all the words of the type

$$(9.1.11) \qquad C_1 \quad C_2 \quad C_3 \quad \times \quad \times$$

where $\times$ stands for an arbitrary character, i.e. words which are completely in accordance with the presumed preferences. Probability 0.3 has been distributed over all the words of one of the following types

$$(9.1.12) \qquad \begin{array}{ccccc} C_1 & C_2 & \overline{C}_3 & \times & \times \\ C_1 & \overline{C}_2 & C_3 & \times & \times \\ \overline{C}_1 & C_2 & C_3 & \times & \times \end{array}$$

where $\overline{C}_j$ means: not the character C_j. Words of the type

$$(9.1.13)\quad \begin{matrix} c_1 & \bar{c}_2 & \bar{c}_3 & \times & \times \\ \bar{c}_1 & c_2 & \bar{c}_3 & \times & \times \\ \bar{c}_1 & \bar{c}_2 & c_3 & \times & \times \end{matrix}$$

together have probability 0.2, while words of the type

$$(9.1.14)\quad \bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3 \quad \times \quad \times$$

get together probability 0.1. We shall call the types of words given by (9.1.11) - (9.1.14), type A, B, C and D respectively. When words of a certain type do not occur, types have been taken together. The assignment of probabilities is illustrated in the following tables. For the first experiment we have

Table 9.1.4. Assignment of probabilities in the first experiment, under the alternative $a_{(2)}$.

word	type	probability
c_1 c_1 c_2 c_3 c_3	C	0.2/12 = 0.0166..
c_1 c_1 c_3 c_2 c_3	B	0.3/6 = 0.05
c_1 c_1 c_3 c_3 c_2	B	0.3/6 = 0.05
c_1 c_2 c_1 c_3 c_3	B	0.3/6 = 0.05
c_1 c_2 c_3 c_1 c_3	A	0.4/2 = 0.2
c_1 c_2 c_3 c_3 c_1	A	0.4/2 = 0.2
c_1 c_3 c_1 c_2 c_3	C	0.2/12 = 0.0166..
c_1 c_3 c_1 c_3 c_2	C	0.2/12 = 0.0166..
c_1 c_3 c_2 c_1 c_3	C	0.2/12 = 0.0166..
c_1 c_3 c_2 c_3 c_1	C	0.2/12 = 0.0166..
c_1 c_3 c_3 c_1 c_2	B	0.3/6 = 0.05
c_1 c_3 c_3 c_2 c_1	B	0.3/6 = 0.05
c_2 c_1 c_1 c_3 c_3	D	0.1/10 = 0.01
c_2 c_1 c_3 c_1 c_3	C	0.2/12 = 0.0166..
c_2 c_1 c_3 c_3 c_1	C	0.2/12 = 0.0166..
c_2 c_3 c_1 c_1 c_3	D	0.1/10 = 0.01
c_2 c_3 c_1 c_3 c_1	D	0.1/10 = 0.01
c_2 c_3 c_3 c_1 c_1	C	0.2/12 = 0.0166..

C_3	C_1	C_1	C_2	C_3	D	0.1/10 = 0.01
C_3	C_1	C_1	C_3	C_2	D	0.1/10 = 0.01
C_3	C_1	C_2	C_1	C_3	D	0.1/10 = 0.01
C_3	C_1	C_2	C_3	C_1	D	0.1/10 = 0.01
C_3	C_1	C_3	C_1	C_2	C	0.2/12 = 0.0166..
C_3	C_1	C_3	C_2	C_1	C	0.2/12 = 0.0166..
C_3	C_2	C_1	C_1	C_3	C	0.2/12 = 0.0166..
C_3	C_2	C_1	C_3	C_1	C	0.2/12 = 0.0166..
C_3	C_2	C_3	C_1	C_1	B	0.3/6 = 0.05
C_3	C_3	C_1	C_1	C_2	D	0.1/10 = 0.01
C_3	C_3	C_1	C_2	C_1	D	0.1/10 = 0.01
C_3	C_3	C_2	C_1	C_1	D	0.1/10 = 0.01

In the third experiment, the character C_1 does not occur, so words of the type A do not occur. In such cases we have given probability 0.7 to the words of type B, as is illustrated in the following table

Table 9.1.5. Assignment of probabilities in the third experiment, under the alternative $a_{(2)}$.

word					type	probability
C_2	C_2	C_3	C_3	C_3	B	0.7/3 = 0.233..
C_2	C_3	C_2	C_3	C_3	D	0.1/3 = 0.033..
C_2	C_3	C_3	C_2	C_3	C	0.2/4 = 0.05
C_2	C_3	C_3	C_3	C_2	C	0.2/4 = 0.05
C_3	C_2	C_2	C_3	C_3	C	0.2/4 = 0.05
C_3	C_2	C_3	C_2	C_3	B	0.7/3 = 0.233..
C_3	C_2	C_3	C_3	C_2	B	0.7/3 = 0.233..
C_3	C_3	C_2	C_2	C_3	D	0.1/3 = 0.033..
C_3	C_3	C_2	C_3	C_2	D	0.1/3 = 0.033..
C_3	C_3	C_3	C_2	C_2	C	0.2/4 = 0.05

The probabilities in the other experiments were determined in the same way.

9.2. ASYMPTOTIC DISTRIBUTIONS UNDER H_0 AND CRITICAL VALUES

The a.d. of $\underline{v}_i$, under H_0, is given by

$$\sum_{\tau=1}^{r} \lambda_\tau \underline{u}_\tau^2, \tag{9.2.1}$$

where $\lambda_1,\ldots,\lambda_r$ are the non-zero eigenvalues of $Q_i\Sigma_0$. For the actual calculations we work with $\Sigma_{0\cdot}$. Because $Q = I_n \otimes G$, $\Sigma_{0\cdot} = N \otimes K_\cdot$, the non-zero eigenvalues of $Q\Sigma_{0\cdot}$ are equal to the non-zero eigenvalues of $\frac{n}{n-1}\, GK_\cdot$, each of which must be taken with multiplicity (n-1). We have in our example

$$K_\cdot = \begin{pmatrix} 0.168 & -0.068 & -0.100 \\ -0.068 & 0.176 & -0.108 \\ -0.100 & -0.108 & 0.208 \end{pmatrix}. \tag{9.2.2}$$

The eigenvalues calculated for the four statistics are given in the following table.

Table 9.2.1. Eigenvalues for the a.d. of $\underline{v}_i$, $i = 1,\ldots,4$.

	eigenvalues	
statistic	$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$	$\lambda_5 = \lambda_6 = \lambda_7 = \lambda_8$
$\underline{v}_1$	1.0716	0.9284
$\underline{v}_2$	1.1328	0.8672
$\underline{v}_3$	1.5333	0.4667
$\underline{v}_4$	1.0000	1.0000

Let $\underline{Q}_0^{(1)},\ldots,\underline{Q}_0^{(4)}$ be random variables of the type (3.3.2), with eigenvalues as in table 9.2.1., i.e. their distributions are the a.d.'s of $\underline{v}_1,\ldots,\underline{v}_4$.

The distribution-functions of the a.d. of $\underline{v}_i$ can be calculated using (3.3.7). Some results are given in the following table.

Table 9.2.2. Distribution functions of $\underset{\sim}{Q}_0^{(1)},\ldots,\underset{\sim}{Q}_0^{(4)}$, the a.d.'s of $\underline{v}_1,\ldots,\underline{v}_4$, under H_0.

z	$P(\underset{\sim}{Q}_0^{(1)} \le z)$	$P(\underset{\sim}{Q}_0^{(2)} \le z)$	$P(\underset{\sim}{Q}_0^{(3)} \le z)$	$P(\underset{\sim}{Q}_0^{(4)} \le z)$
0	0.0000	0.0000	0.0000	0.0000
2	0.0191	0.0194	0.0283	0.0190
4	0.1434	0.1448	0.1770	0.1429
6	0.3535	0.3551	0.3888	0.3528
8	0.5669	0.5679	0.5836	0.5665
10	0.7350	0.7350	0.7316	0.7350
12	0.8485	0.8478	0.8333	0.8488
14	0.9179	0.9170	0.8991	0.9182
16	0.9573	0.9564	0.9402	0.9576
18	0.9785	0.9778	0.9650	0.9788
20	0.9895	0.9890	0.9798	0.9897
22	0.9950	0.9947	0.9884	0.9951
24	0.9976	0.9974	0.9934	0.9977

Notice that the last column of table 9.2.2 gives the distribution-function of the χ^2-distribution with 8 degrees of freedom.

Using an iterative zero-searching procedure, critical values of $\underset{\sim}{Q}_0^{(1)},\ldots,\underset{\sim}{Q}_0^{(4)}$ were obtained (we shall call this method of obtaining critical values: "method A"). The results are given in the following table.

Table 9.2.3. Critical values of the distributions of $\underset{\sim}{Q}_0^{(1)},\ldots,\underset{\sim}{Q}_0^{(4)}$. (Method A).

α	$Q^{(1)}_{0,1-\alpha}$	$Q^{(2)}_{0,1-\alpha}$	$Q^{(3)}_{0,1-\alpha}$	$Q^{(4)}_{0,1-\alpha}$
0.1000	13.3730	13.4016	14.0336	13.3616
0.0500	15.5293	15.5824	16.6753	15.5073
0.0250	17.5685	17.6505	19.2285	17.5345
0.0100	20.1432	20.2682	22.5142	20.0902
0.0050	22.0240	22.1850	24.9520	21.9550
0.0025	23.8545	24.0545	27.3545	23.7745
0.0010	26.2245	26.4745	30.4745	26.1245

Notice again that the last column contains the critical values of the $\chi^2[8]$ - distribution.

Most program-libraries of numerical methods contain procedures to calculate eigenvalues. They will, however, probably not contain a procedure to calculate the distribution of $\underset{=}{Q}_0^{(i)}$. The possible user of our methods thus has to write a program for these distributions and critical values himself.

To avoid this, he can use the approximation to the distribution of $\underset{=}{Q}_0$, which we described in section 3.3 and use a table of the χ^2-distribution. We have done this ("method B") for the approximation using two adapted moments. The correction factor (b) and the degrees of freedom (ν) are given by (3.3.15) and (3.3.16) respectively. We found

Table 9.2.4. Approximate critical values for the distributions of $\underset{=}{Q}_0^{(1)},\ldots,\underset{=}{Q}_0^{(3)}$, obtained from an approximation with two adapted moments. (Method B).

		$\underset{=}{Q}_0^{(1)}$	$\underset{=}{Q}_0^{(2)}$	$\underset{=}{Q}_0^{(3)}$
	b	1.0051	1.0176	1.2844
	ν	7.9592	7.8613	6.2286
		critical values		
α		$k_{B,1-\alpha}^{(1)}$	$k_{B,1-\alpha}^{(2)}$	$k_{B,1-\alpha}^{(3)}$
0.1000		13.3754	13.4090	14.0786
0.0500		15.5283	15.5795	16.6105
0.0250		17.5627	17.6313	19.0232
0.0100		20.1278	20.2193	22.0879
0.0050		21.9996	22.1084	24.3374
0.0025		23.8262	23.9522	26.5416
0.0010		26.1855	26.3342	29.4001

The determination of critical values may also be based on the exact moments of $\underset{-}{v}_1,\ldots,\underset{-}{v}_4$, which can be calculated from (7.2.2) and (7.3.14). ("Method C"). We have

Table 9.2.5. Exact moments of $\underline{v}_1,\ldots,\underline{v}_4$, under H_0.

i	$E\underline{v}_i \mid H_0$	$\sigma^2(\underline{v}_i \mid H_0)$	c	η
1	8.0000	14.0337	1.1401	9.1209
2	8.0000	14.2424	1.1234	8.9873
3	8.0000	17.8405	0.8968	7.1747
4	8.0000	13.9529	1.1467	9.1737

The last two column's contain c as defined by (1.4.4) and η, defined by (1.4.5). Using the method described in section 1.4 we find the following approximate critical values of the distribution of $\underline{v}_i$.

Table 9.2.6. Approximate critical values for the distributions of $\underline{v}_1,\ldots,\underline{v}_4$, using the exact moments. (Method C).

α	$k^{(1)}_{C,1-\alpha}$	$k^{(2)}_{C,1-\alpha}$	$k^{(3)}_{C,1-\alpha}$	$k^{(4)}_{C,1-\alpha}$
0.1000	13.0182	13.0558	13.6633	13.0035
0.0500	14.9880	15.0445	15.9684	14.9659
0.0250	16.8411	16.9164	18.1542	16.8118
0.0100	19.1685	19.2683	20.9187	19.1296
0.0050	20.8615	20.9797	22.9409	20.8155
0.0025	22.5099	22.6465	24.9177	22.4568
0.0010	24.6345	24.7952	27.4753	24.5721

All the critical values, calculated from the a.d. of $\underline{v}_i$, from an approximation to the a.d. or from the exact moments of the $\underline{v}_i$ may be used as *approximate* critical values for the performance of the test.

The *exact* critical values would have to be based on the exact distribution of $\underline{v}_i$ which is unavailable to us. See also section 9.6.

The outcomes of $\underline{v}_i$, i=1,...,4, for the data of table 9.1.1 are as given in the following table.

Table 9.2.7. Outcomes of the four test-statistics for the data of table 9.1.1.

statistic	outcome
$\underline{v}_1$	17.24
$\underline{v}_2$	16.46
$\underline{v}_3$	22.49
$\underline{v}_4$	17.03

The outcomes are significant at the 5% level for all four tests. Actually, the data of table 9.1.1 were obtained from a simulation of the experiment under the alternative $a_{(1)}$. This explains the fact that the outcome of $\underline{v}_3$ is the highest of the four, because this statistic was designed especially to work against $a_{(1)}$.

9.3. SIMULATION RESULTS (UNDER H_0)

For each of the four statistics considered, we obtained 1000 pseudo-observations, under H_0, by generating for each experiment E_i a pseudo-random word. The words of each experiment were combined and an outcome of $\underline{v}_i$ was calculated. In this way we were able to make (pseudo-) estimates of the right-tail probabilities of the critical values of the preceding sections. These results thus also give an impression of the actual level of significance of the tests as compared to the nominal level α.

Table 9.3.1. Estimates of the right-tail probabilities (e.r.t.p.) of the critical values $k^{(i)}_{A,1-\alpha}$ of method A, under H_0, obtained by simulation. (See remark 9.3.1.).

α	$k^{(1)}_{A,1-\alpha}$	e.r.t.p	$k^{(2)}_{A,1-\alpha}$	e.r.t.p	$k^{(3)}_{A,1-\alpha}$	e.r.t.p	$k^{(4)}_{A,1-\alpha}$	e.r.t.p
0.1000	13.3730	0.073	13.4016	0.107	14.0336	0.100	13.3616	0.104
0.0500	15.5293	0.038	15.5824	0.058	16.6753	0.038	15.5073	0.035
0.0250	17.5685	0.015	17.6505	0.017	19.2285	0.020	17.5345	0.015
0.0100	20.1432	0.004	20.2682	0.009	22.5142	0.005	20.0902	0.007
0.0050	22.0240	0.003	22.1850	0.002	24.9520	0.003	21.9550	0.002
0.0025	23.8545	0.000	24.0545	0.002	27.3545	0.001	23.7745	0.000
0.0010	26.2245	0.000	26.4745	0.001	30.4745	0.000	26.1245	0.000

Table 9.3.2. Estimates of the right-tail probabilities (e.r.t.p.) of the critical values $k_{B,1-\alpha}^{(i)}$ of method B, under H_0, obtained by simulation.

α	$k_{B,1-\alpha}^{(1)}$	e.r.t.p	$k_{B,1-\alpha}^{(2)}$	e.r.t.p	$k_{B,1-\alpha}^{(3)}$	e.r.t.p
0.1000	13.3754	0.103	13.4090	0.084	14.0786	0.086
0.0500	15.5283	0.055	15.5795	0.043	16.6105	0.031
0.0250	17.5627	0.019	17.6313	0.017	19.0232	0.014
0.0100	20.1278	0.011	20.2193	0.006	22.0879	0.004
0.0050	21.9996	0.006	22.1084	0.003	24.3374	0.001
0.0025	23.8262	0.001	23.9522	0.001	26.5416	0.001
0.0010	26.1855	0.001	26.3342	0.001	29.4001	0.000

Table 9.3.3. Estimates of the right-tail probabilities (e.r.t.p.) of the critical values $k_{C,1-\alpha}^{(i)}$ of method C, under H_0, obtained by simulation.

α	$k_{C,1-\alpha}^{(1)}$	e.r.t.p	$k_{C,1-\alpha}^{(2)}$	e.r.t.p	$k_{C,1-\alpha}^{(3)}$	e.r.t.p	$k_{C,1-\alpha}^{(4)}$	e.r.t.p
0.1000	13.0182	0.115	13.0558	0.103	13.6633	0.101	13.0035	0.095
0.0500	14.9880	0.060	15.0445	0.061	15.9684	0.046	14.9659	0.044
0.0250	16.8411	0.034	16.9164	0.030	18.1542	0.024	16.8118	0.017
0.0100	19.1685	0.012	19.2683	0.011	20.9187	0.009	19.1296	0.005
0.0050	20.8615	0.005	20.9797	0.006	22.9409	0.003	20.8155	0.003
0.0025	22.5099	0.003	22.6465	0.002	24.9177	0.000	22.4568	0.002
0.0010	24.6345	0.002	24.7952	0.000	27.4753	0.000	24.5721	0.000

REMARK 9.3.1. Due to high costs of computer time, the simulations have not been made for each α separately. Therefore, the estimates of the right-tail probabilities in tables 9.3.1., 9.3.2. and 9.3.3. are dependent columnwise.

Inspection of the tables 9.3.1., 9.3.2. and 9.3.3. shows that in almost all cases the approximate critical values are slightly too high. This means that the actual level of the test is lower than the nominal level α. We shall call such tests "timid" (such tests are usually called "conservative"), in contrast with the tests where the actual level is higher than the nominal level α, which we shall call "bold". When critical values are used which are obtained by method C for $\underline{v}_1$, the test that we recommend, "bold" tests are obtained. Of course we have to take the inaccuracy into account resulting from the fact that we have only estimates of the right-tail probabilities at our disposal. The general tendency is however clear enough.

Furthermore it seems that the estimates in table 9.3.3 (method C) are generally closer to the nominal values of α than in the other two cases. Therefore we recommend method C for the approximation of the critical values in all cases.

Another impression of the goodness of the approximation using the exact moments of $\underline{v}$ (method C) may be obtained from the following figures.

Figure 9.3.1. Pseudo-empirical distribution function of 1000 simulated observations of $\underline{v}_1$ (dashed line) and distribution function of an adapted χ^2-distribution. (The same simulation results as for table 9.3.3).

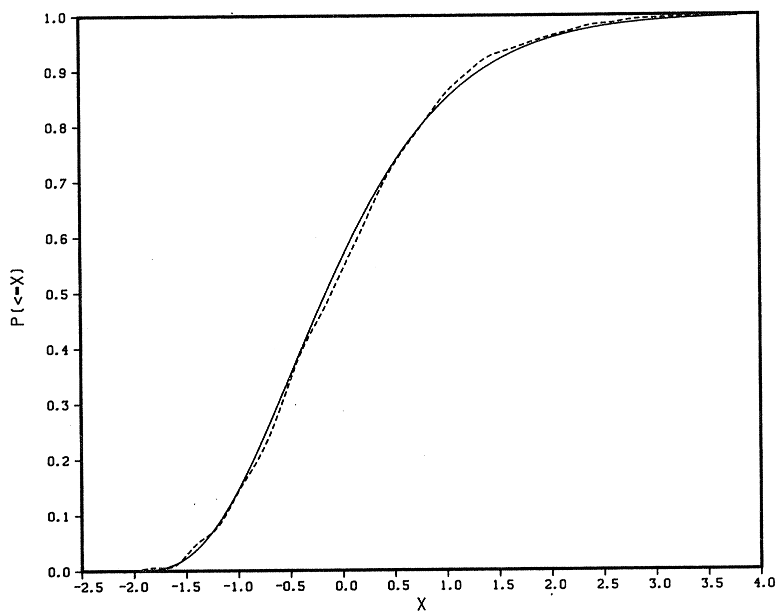


Figure 9.3.2. Pseudo-empirical distribution function of 1000 simulated observations of $\underline{v}_2$ (dashed line) and distribution function of an adapted χ^2-distribution. (The same simulation results as for table 9.3.3).

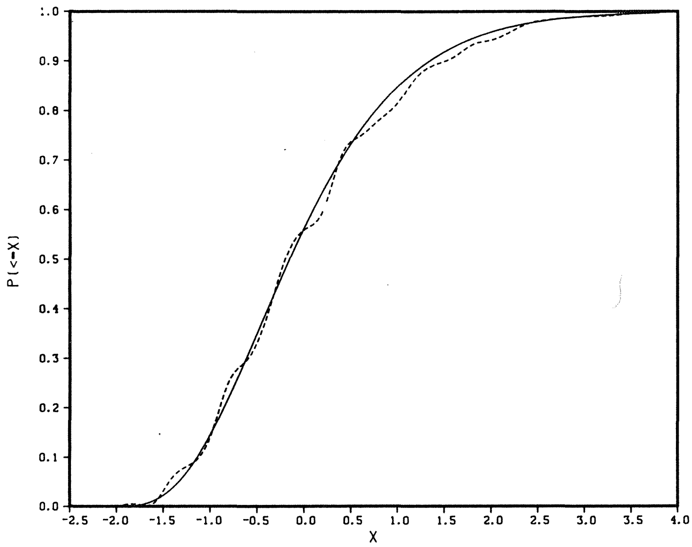


Figure 9.3.3. Pseudo-empirical distribution function of 1000 simulated observations of $\underline{v}_3$ (dashed line) and distribution function of an adapted χ^2-distribution. (The same simulation results as for table 9.3.3).

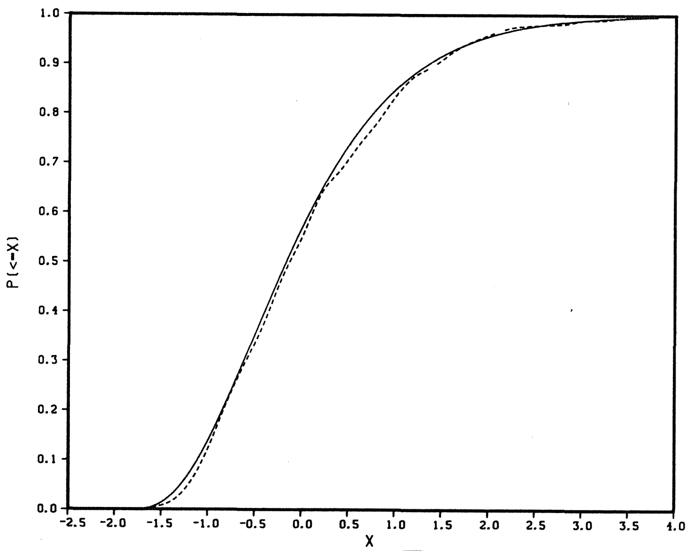
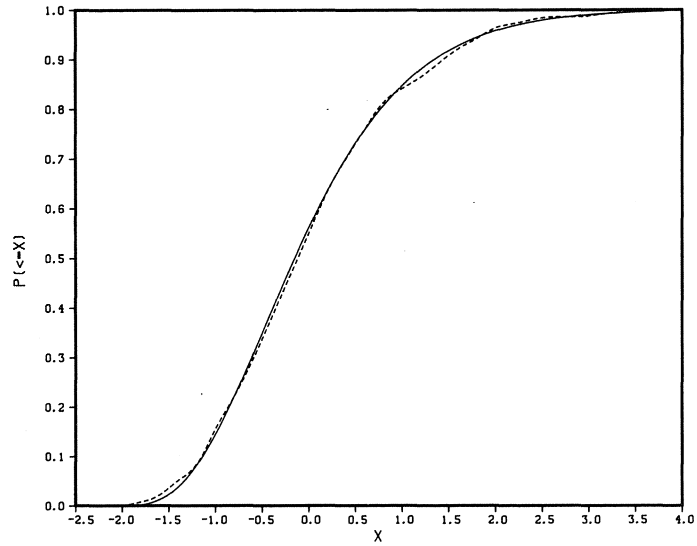


Figure 9.3.4. Pseudo-empirical distribution function of 1000 simulated observations of $\underline{v}_4$ (dashed line) and distribution function of an adapted χ^2-distribution. (The same simulation results as for table 9.3.3).



9.4. SIMULATION RESULTS (UNDER ALTERNATIVES) & POWER

To make an estimate of the power of the four tests considered, we also generated 1000 pseudo-observations of $\underline{v}_1,\ldots,\underline{v}_4$, under each of the two alternatives $a_{(1)}$ and $a_{(2)}$ that were defined in section 9.1. Estimates of the right-tail probabilities of the (approximate) critical values of section 9.2 are given in the tables 9.4.3 - 9.4.8.

In each case the estimate is compared with the approximation to the power of the test as calculated from formula (4.4.6),

$$(9.4.1) \qquad P\left(\sum_{\tau=1}^{r} \lambda_\tau (\underline{u}_\tau + \omega_\tau)^2 \geq k\right),$$

with $\underline{\vec{u}} \sim N(\vec{0},I_r)$, $\Sigma_{1\cdot} = BB'$, $r = \text{rank } B'QB$, $\lambda_1,\ldots,\lambda_r$ the (positive) eigenvalues of $B'QB$, k a critical value and

$$(9.4.2) \qquad \vec{\omega} = \Lambda_+^{-1} P_+' B' Q \vec{\delta}_* .$$

Table 9.4.1. Elements of the dispersion matrix of $\vec{\underline{t}}_*$, under $a_{(1)}$, i.e. the elements of $\Sigma_{1.}(a_{(1)})$.

0.1636	-0.0685	-0.0951	-0.0392	0.0432	-0.0041	-0.0412	-0.0087	0.0498	-0.0417	0.0170	0.0247	-0.0417	0.0170	0.0247
-0.0685	0.1142	-0.0456	0.0189	-0.0386	0.0197	0.0146	-0.0128	-0.0018	0.0175	-0.0314	0.0139	0.0175	-0.0314	0.0139
-0.0951	-0.0456	0.1407	0.0202	-0.0046	-0.0157	0.0265	0.0214	-0.0480	0.0242	0.0144	-0.0385	0.0242	0.0144	-0.0385
-0.0392	0.0189	0.0202	0.1185	-0.0751	-0.0434	-0.0088	0.0149	-0.0061	-0.0352	0.0206	0.0146	-0.0352	0.0206	0.0146
0.0432	-0.0386	-0.0046	-0.0751	0.1780	-0.1029	-0.0094	-0.0471	0.0565	0.0206	-0.0461	0.0255	0.0206	-0.0461	0.0255
-0.0041	0.0197	-0.0157	-0.0434	-0.1029	0.1462	0.0182	0.0322	-0.0504	0.0146	0.0255	-0.0401	0.0146	0.0255	-0.0401
-0.0412	0.0146	0.0265	-0.0088	-0.0094	0.0182	0.1053	-0.0162	-0.0891	-0.0276	0.0055	0.0222	-0.0276	0.0055	0.0222
-0.0087	-0.0128	0.0214	0.0149	-0.0471	0.0322	-0.0162	0.1185	-0.1023	0.0050	-0.0293	0.0243	0.0050	-0.0293	0.0243
0.0498	-0.0018	-0.0480	-0.0061	0.0565	-0.0504	-0.0891	-0.1023	0.1914	0.0227	0.0238	-0.0465	0.0227	0.0238	-0.0465
-0.0417	0.0175	0.0242	-0.0352	0.0206	0.0146	-0.0276	0.0050	0.0227	0.1528	-0.0591	-0.0937	-0.0482	0.0159	0.0323
0.0170	-0.0314	0.0144	0.0206	-0.0461	0.0255	0.0055	-0.0293	0.0238	-0.0591	0.1476	-0.0885	0.0159	-0.0408	0.0248
0.0247	0.0139	-0.0385	0.0146	0.0255	-0.0401	0.0222	0.0243	-0.0465	-0.0937	-0.0885	0.1822	0.0323	0.0248	-0.0571
-0.0417	0.0175	0.0242	-0.0352	0.0206	0.0146	-0.0276	0.0050	0.0227	-0.0482	0.0159	0.0323	0.1528	-0.0591	-0.0937
0.0170	-0.0314	0.0144	0.0206	-0.0461	0.0255	0.0055	-0.0293	0.0238	0.0159	-0.0408	0.0248	-0.0591	0.1476	-0.0885
0.0247	0.0139	-0.0385	0.0146	0.0255	-0.0401	0.0222	0.0243	-0.0465	0.0323	0.0248	-0.0571	-0.0937	-0.0885	0.1822

Table 9.4.2. Elements of the dispersion matrix of $\underline{\vec{t}}_*$, under $a_{(2)}$, i.e. the elements of $\Sigma_{1.}(a_{(2)})$.

0.1360	-0.0573	-0.0787	-0.0340	0.0143	0.0197	-0.0340	0.0143	0.0197	-0.0340	0.0143	0.0197	-0.0340	0.0143	0.0197
-0.0573	0.1225	-0.0652	0.0143	-0.0306	0.0163	0.0143	-0.0306	0.0163	0.0143	-0.0306	0.0163	0.0143	-0.0306	0.0163
-0.0787	-0.0652	0.1438	0.0197	0.0163	-0.0360	0.0197	0.0163	-0.0360	0.0197	0.0163	-0.0360	0.0197	0.0163	-0.0360
-0.0340	0.0143	0.0197	0.1473	-0.0596	-0.0876	-0.0377	0.0151	0.0227	-0.0377	0.0151	0.0227	-0.0377	0.0151	0.0227
0.0143	-0.0306	0.0163	-0.0596	0.1837	-0.1241	0.0151	-0.0510	0.0359	0.0151	-0.0510	0.0359	0.0151	-0.0510	0.0359
0.0197	0.0163	-0.0360	-0.0876	-0.1241	0.2117	0.0227	0.0359	-0.0586	0.0227	0.0359	-0.0586	0.0227	0.0359	-0.0586
-0.0340	0.0143	0.0197	-0.0377	0.0151	0.0227	0.1473	-0.0596	-0.0876	-0.0377	0.0151	0.0227	-0.0377	0.0151	0.0227
0.0143	-0.0306	0.0163	0.0151	-0.0510	0.0359	-0.0596	0.1837	-0.1241	0.0151	-0.0510	0.0359	0.0151	-0.0510	0.0359
0.0197	0.0163	-0.0360	0.0227	0.0359	-0.0586	-0.0876	-0.1241	0.2117	0.0227	0.0359	-0.0586	0.0227	0.0359	-0.0586
-0.0340	0.0143	0.0197	-0.0377	0.0151	0.0227	-0.0377	0.0151	0.0227	0.1473	-0.0596	-0.0876	-0.0377	0.0151	0.0227
0.0143	-0.0306	0.0163	0.0151	-0.0510	0.0359	0.0151	-0.0510	0.0359	-0.0596	0.1837	-0.1241	0.0151	-0.0510	0.0359
0.0197	0.0163	-0.0360	0.0227	0.0359	-0.0586	0.0227	0.0359	-0.0586	-0.0876	-0.1241	0.2117	0.0227	0.0359	-0.0586
-0.0340	0.0143	0.0197	-0.0377	0.0151	0.0227	-0.0377	0.0151	0.0227	-0.0377	0.0151	0.0227	0.1473	-0.0596	-0.0876
0.0143	-0.0306	0.0163	0.0151	-0.0510	0.0359	0.0151	-0.0510	0.0359	0.0151	-0.0510	0.0359	-0.0596	0.1837	-0.1241
0.0197	0.0163	-0.0360	0.0227	0.0359	-0.0586	0.0227	0.0359	-0.0586	0.0227	-0.0586	0.0359	-0.0876	-0.1241	0.2117

For each of the two alternatives considered, the quantities $\Sigma_{1.}$ and B have to be calculated. The matrices $\Sigma_{1.}(a_{(1)})$ and $\Sigma_{1.}(a_{(2)})$ are given in the tables 9.4.1 and 9.4.2 on pages 136 & 137. Furthermore we need the eigenvalues $\lambda_1,\ldots,\lambda_r$ and the vectors $\vec{\omega}$ for each of the four choices of Q and each of the two choices of a. For those readers who might wish to check the calculations, we give the eigenvalues and the components of $\vec{\omega}$ in each of the eight cases in the following tables.

Table 9.4.3. Eigenvalues $\lambda_1,\ldots,\lambda_8$ and components of $\vec{\omega}$, for $Q_1,\ldots,Q_4$ and $a_{(1)}$.

τ	$\lambda_\tau^{(1)}$	$\omega_\tau^{(1)}$	$\lambda_\tau^{(2)}$	$\omega_\tau^{(2)}$
1	1.1173	0.0000	1.2150	0.0000
2	1.1173	0.0000	1.2150	0.0000
3	1.1173	0.0000	1.2150	0.0000
4	0.8645	0.0000	0.7966	-1.7953
5	0.8645	0.0000	0.7850	0.0000
6	0.8645	0.0000	0.7850	0.0000
7	0.8220	2.1830	0.7850	0.0000
8	0.6488	0.4574	0.6611	1.3235

τ	$\lambda_\tau^{(3)}$	$\omega_\tau^{(3)}$	$\lambda_\tau^{(4)}$	$\omega_\tau^{(4)}$
1	1.3511	0.0000	1.1021	0.0000
2	1.3511	0.0000	1.1021	0.0000
3	1.3511	0.0000	1.1021	0.0000
4	1.2425	2.2260	0.8809	0.0000
5	0.5142	0.0000	0.8809	0.0000
6	0.5142	0.0000	0.8809	0.0000
7	0.5142	0.0000	0.8109	2.2153
8	0.3088	-0.1401	0.6611	-0.2586

Table 9.4.4. Eigenvalues $\lambda_1,\ldots,\lambda_8$ and components of $\vec{\omega}$, for $Q_1,\ldots,Q_4$ and $a_{(2)}$.

τ	$\lambda_\tau^{(1)}$	$\omega_\tau^{(1)}$	$\lambda_\tau^{(2)}$	$\omega_\tau^{(2)}$
1	1.2275	-3.0368	1.2119	-3.0466
2	1.0444	0.0000	1.0237	0.0000
3	0.9321	0.0766	0.9020	-0.1258
4	0.8187	-0.3097	0.8386	-0.1907
5	0.7778	0.0000	0.8037	0.0000
6	0.6420	-0.1065	0.6580	0.1293
7	0.6023	-0.1952	0.5799	-0.0350
8	0.4519	0.0766	0.4860	0.1656

τ	$\lambda_\tau^{(3)}$	$\omega_\tau^{(3)}$	$\lambda_\tau^{(4)}$	$\omega_\tau^{(4)}$
1	1.5245	2.3184	1.2158	3.0036
2	1.4650	0.0000	0.9673	0.0000
3	1.2669	-1.0061	0.9017	-0.2752
4	0.8740	-0.4125	0.8549	0.0000
5	0.4039	0.0000	0.8329	-0.4308
6	0.3942	1.2943	0.6329	-0.2373
7	0.3595	-1.0356	0.5859	-0.1811
8	0.2645	-0.2725	0.5098	0.0850

We now have

Table 9.4.5. Estimates of the right-tail probabilities (e.r.t.p) of the critical values $k^{(i)}_{A,1-\alpha}$ of method A, under $a_{(1)}$, obtained by simulation. The values in brackets give the approximate power (a.p) calculated from (4.4.6).

α	$k^{(1)}_{A,1-\alpha}$	e.r.t.p (a.p)	$k^{(2)}_{A,1-\alpha}$	e.r.t.p (a.p)	$k^{(3)}_{A,1-\alpha}$	e.r.t.p (a.p)	$k^{(4)}_{A,1-\alpha}$	e.r.t.p (a.p)
0.1000	13.3730	0.304 (0.314)	13.4016	0.281 (0.291)	14.0336	0.435 (0.395)	13.3616	0.298 (0.313)
0.0500	15.5293	0.184 (0.201)	15.5824	0.153 (0.181)	16.6753	0.290 (0.269)	15.5073	0.167 (0.200)
0.0250	17.5685	0.113 (0.125)	17.6505	0.071 (0.110	19.2285	0.185 (0.178)	17.5345	0.088 (0.125)
0.0100	20.1432	0.048 (0.065)	20.2682	0.029 (0.055)	22.5142	0.077 (0.099)	20.0902	0.043 (0.065)
0.0050	22.0240	0.030 (0.039)	22.1850	0.017 (0.032)	24.9520	0.038 (0.063)	21.9550	0.024 (0.039)
0.0025	23.8545	0.015 (0.023)	24.0545	0.007 (0.018)	27.3545	0.023 (0.039)	23.7745	0.013 (0.023)
0.0010	26.2245	0.006 (0.012)	26.4745	0.000 (0.009)	30.4745	0.010 (0.020)	26.1245	0.005 (0.012)

Table 9.4.6. Estimates of the right-tail probabilities (e.r.t.p) of the critical values $k^{(i)}_{B,1-\alpha}$ of method B, under $a_{(1)}$, obtained by simulation. The values in brackets give the approximate power (a.p) calculated from (4.4.6).

α	$k^{(1)}_{B,1-\alpha}$	e.r.t.p (a.p)	$k^{(2)}_{B,1-\alpha}$	e.r.t.p (a.p)	$k^{(3)}_{B,1-\alpha}$	e.r.t.p (a.P)	$k^{(4)}_{B,1-\alpha}$	e.r.t.p (a.p)
0.1000	13.3754	0.335 (0.314)	13.4090	0.283 (0.291)	14.0786	0.398 (0.392)	13.3616	0.322 (0.313)
0.0500	15.5283	0.192 (0.201)	15.5795	0.172 (0.181)	16.6105	0.276 (0.271)	15.5073	0.191 (0.200)
0.0250	17.5627	0.109 (0.126)	17.6313	0.083 (0.110)	19.0232	0.180 (0.184)	17.5345	0.117 (0.125)
0.0100	20.1278	0.051 (0.066)	20.2193	0.037 (0.056)	22.0879	0.092 (0.108)	20.0902	0.053 (0.065)
0.0050	21.9996	0.022 (0.040)	22.1084	0.026 (0.033)	24.3374	0.059 (0.071)	21.9550	0.026 (0.039)
0.0025	23.8262	0.013 (0.024)	23.9522	0.010 (0.019)	26.5416	0.038 (0.046)	23.7745	0.012 (0.023)
0.0010	26.1855	0.003 (0.012)	26.3342	0.004 (0.009)	29.4001	0.016 (0.025)	26.1245	0.003 (0.012)

Table 9.4.7. Estimates of the right-tail probabilities (e.r.t.p) of the critical values $k^{(i)}_{C,1-\alpha}$ of method C, under $a_{(1)}$, obtained by simulation. The values in brackets give the approximate power (a.p) calculated from (4.4.6).

α	$k^{(1)}_{C,1-\alpha}$	e.r.t.p (a.p)	$k^{(2)}_{C,1-\alpha}$	e.r.t.p (a.p)	$k^{(3)}_{C,1-\alpha}$	e.r.t.p (a.p)	$k^{(4)}_{C,1-\alpha}$	e.r.t.p (a.p)
0.1000	13.0182	0.365 (0.336)	13.0558	0.309 (0.312)	13.6633	0.405 (0.415)	13.0035	0.339 (0.335)
0.0500	14.9880	0.216 (0.226)	15.0445	0.209 (0.204)	15.9684	0.272 (0.299)	14.9659	0.220 (0.225)
0.0250	16.8411	0.128 (0.149)	16.9164	0.314 (0.131)	18.1542	0.175 (0.213)	16.8118	0.130 (0.148)
0.0100	19.1685	0.071 (0.084)	19.2683	0.079 (0.072)	20.9187	0.095 (0.133)	19.1296	0.073 (0.084)
0.0050	20.8615	0.039 (0.054)	20.9797	0.044 (0.045)	22.9409	0.052 (0.092)	20.8155	0.041 (0.054)
0.0025	22.5099	0.020 (0.034)	22.6465	0.021 (0.028)	24.9177	0.031 (0.063)	22.4568	0.018 (0.034)
0.0010	24.6345	0.007 (0.019)	24.7952	0.011 (0.015)	27.4753	0.020 (0.038)	24.5721	0.007 (0.019)

Table 9.4.8. Estimates of the right-tail probabilities (e.r.t.p) of the critical values $k^{(i)}_{A,1-\alpha}$ of method A, under $a_{(2)}$, obtained by simulation. The values in brackets give the approximate power (a.p) calculated from (4.4.6).

α	$k^{(1)}_{A,1-\alpha}$	e.r.t.p (a.p)	$k^{(2)}_{A,1-\alpha}$	e.r.t.p (a.p)	$k^{(3)}_{A,1-\alpha}$	e.r.t.p (a.p)	$k^{(4)}_{A,1-\alpha}$	e.r.t.p (a.p)
0.1000	13.3730	0.694 (0.677)	13.4016	0.724 (0.672)	14.0336	0.619 (0.593)	13.3616	0.701 (0.671)
0.0500	15.5293	0.590 (0.565)	15.5824	0.607 (0.557)	16.6753	0.482 (0.463)	15.5073	0.575 (0.558)
0.0250	17.5685	0.478 (0.462)	17.6505	0.478 (0.453)	19.2285	0.370 (0.352)	17.5345	0.464 (0.455)
0.0100	20.1432	0.349 (0.346)	20.2682	0.360 (0.336)	22.5142	0.239 (0.237)	20.0902	0.332 (0.340)
0.0050	22.0240	0.281 (0.274)	22.1850	0.293 (0.263)	24.9520	0.166 (0.172)	21.9550	0.261 (0.268)
0.0025	23.8545	0.208 (0.214)	24.0545	0.208 (0.204)	27.3545	0.121 (0.123)	23.7745	0.201 (0.209)
0.0010	26.2245	0.122 (0.153)	26.4745	0.150 (0.143)	30.4745	0.067 (0.077)	26.1245	0.147 (0.148)

Table 9.4.9. Estimates of the right-tail probabilities (e.r.t.p) of the critical values $k^{(i)}_{B,1-\alpha}$ of method B, under $a_{(2)}$, obtained by simulation. The values in brackets give the approximate power (a.p) calculated from (4.4.6).

α	$k^{(1)}_{B,1-\alpha}$	e.r.t.p (a.p)	$k^{(2)}_{B,1-\alpha}$	e.r.t.p (a.p)	$k^{(3)}_{B,1-\alpha}$	e.r.t.p (a.p)	$k^{(4)}_{B,1-\alpha}$	e.r.t.p (a.p)
0.1000	13.3754	0.680 (0.676)	13.4090	0.716 (0.671)	14.0786	0.637 (0.590)	13.3616	0.681 (0.671)
0.0500	15.5283	0.588 (0.565)	15.5795	0.588 (0.557)	16.6105	0.491 (0.466)	15.5073	0.571 (0.558)
0.0250	17.5627	0.470 (0.463)	17.6313	0.469 (0.454)	19.0232	0.378 (0.360)	17.5345	0.464 (0.455)
0.0100	20.1278	0.327 (0.347)	20.2193	0.347 (0.338)	22.0879	0.265 (0.250)	20.0902	0.363 (0.340)
0.0050	21.9996	0.262 (0.275)	22.1084	0.275 (0.266)	24.3374	0.171 (0.187)	21.9550	0.271 (0.268)
0.0025	23.8262	0.187 (0.215)	23.9522	0.208 (0.207)	26.5416	0.117 (0.138)	23.7745	0.205 (0.209)
0.0010	26.1855	0.130 (0.153)	26.3342	0.121 (0.146)	29.4001	0.070 (0.091)	26.1245	0.138 (0.148)

Table 9.4.10. Estimates of the right-tail probabilities (e.r.t.p) of the critical values $k^{(i)}_{C,1-\alpha}$ of method C, under $a_{(2)}$, obtained by simulation. The values in brackets give the approximate power (a.p) calculated from (4.4.6).

α	$k^{(1)}_{C,1-\alpha}$	e.r.t.p (a.p)	$k^{(2)}_{C,1-\alpha}$	e.r.t.p (a.p)	$k^{(3)}_{C,1-\alpha}$	e.r.t.p (a.p)	$k^{(4)}_{C,1-\alpha}$	e.r.t.p (a.p)
0.1000	13.0182	0.735 (0.695)	13.0558	0.729 (0.690)	13.6633	0.663 (0.612)	13.0035	0.705 (0.690)
0.0500	14.9880	0.623 (0.593)	15.0445	0.629 (0.586)	15.9684	0.528 (0.496)	14.9659	0.601 (0.586)
0.0250	16.8411	0.518 (0.498)	16.9164	0.527 (0.489)	18.1542	0.417 (0.397)	16.8118	0.500 (0.491)
0.0100	19.1685	0.403 (0.388)	19.2683	0.433 (0.378)	20.9187	0.305 (0.289)	19.1296	0.398 (0.381)
0.0050	20.8615	0.319 (0.317)	20.9797	0.326 (0.307)	22.9409	0.222 (0.224)	20.8155	0.317 (0.310)
0.0025	22.5099	0.260 (0.257)	22.6465	0.251 (0.247)	24.9177	0.169 (0.173)	22.4568	0.252 (0.251)
0.0010	24.6345	0.183 (0.192)	24.7952	0.169 (0.183)	27.4753	0.114 (0.121)	24.5721	0.180 (0.186)

REMARK 9.4.1. The same remark applies as in remark 9.3.1. Notice furthermore that the last column of table 9.4.4 contains the same critical values as the last column of table 9.4.3. The estimates, however, are independent of the estimates of table 9.4.3. They are given to make a better comparison possible.

According to these results it appears that $\underline{v}_3$ has the highest power against $a_{(1)}$, which is in agreement with the fact that Q_3 was especially chosen so that $\underline{v}_3$ would react on this kind of alternative. Furthermore, $\underline{v}_1$ and $\underline{v}_4$ are equally good, though $\underline{v}_1$ seems to perform slightly better than $\underline{v}_4$.

Against $a_{(2)}$ it is $\underline{v}_2$ that appears to work best, which is again not surprising because there are preferences for three positions in $a_{(2)}$. Again $\underline{v}_1$ and $\underline{v}_4$ are competetive while now $\underline{v}_3$ seems to work worst.

It seems that the highest power is obtained when the critical values have been determined by method C. The results may be misleading, however, because we have not made a correction for the fact that the actual levels of the tests are not equal to the nominal levels. Therefore we compare "bold" and "timid" tests which possibly gives a distorted picture of the situation. The results are nevertheless supported by the results of Pitman-efficiencies (section 9.5).

Finally we observe that the approximate power of the tests as calculated from (4.4.6) agrees in a satisfactory way with the simulation results. The agreement improves, as can be expected, as the number of experiments, m, increases. This follows from some more simulations which we made and which are not reproduced here because of the limited space.

9.5. PITMAN & BAHADUR EFFICIENCIES

The asymptotic relative Pitman efficiency (ARPE) of $\underline{v}(Q_i)$ with respect to $\underline{v}(Q_j)$ is equal to (cf. (5.2.2))

$$(9.5.1) \qquad e_{ij}(\beta) = \frac{H_j^{-1}(\beta)}{H_i^{-1}(\beta)}, \qquad \beta \in (\alpha,1),$$

where $H_i(\eta)$ is given by (5.2.3). Using an iterative zero-searching procedure the inverse values of $H(\eta)$ were calculated. Because $e_{ij}(\beta)$ depends on α,β

and a, we cannot give a complete survey of the results. Therefore, we shall only give the results for $\alpha = 0.05$ and $\beta = 0.25$, 0.50 and 0.75 and for $a_{(1)}$ and $a_{(2)}$. For other values of α and β the ARPE's show generally the same pattern.

Table 9.5.1. ARPE's for the 4 tests considered, with $\alpha = 0.05$ and $\beta = 0.25$ for $a_{(1)}$.

$H^{-1}(0.25)$		10.43	11.55	7.85	10.47
	test no.	1	2	3	4
10.43	1	1.000	1.107	0.753	1.004
11.55	2	0.903	1.000	0.680	0.906
7.85	3	1.329	1.471	1.000	1.334
10.47	4	0.996	1.103	0.750	1.000

Table 9.5.2. ARPE's for the 4 tests considered, with $\alpha = 0.05$ and $\beta = 0.50$ for $a_{(1)}$.

$H^{-1}(0.50)$		20.82	22.68	16.35	20.89
	test no.	1	2	3	4
20.82	1	1.000	1.089	0.785	1.003
22.68	2	0.918	1.000	0.721	0.921
16.35	3	1.273	1.387	1.000	1.278
20.89	4	0.997	1.086	0.783	1.000

Table 9.5.3. ARPE's for the four tests considered, with $\alpha = 0.05$ and $\beta = 0.75$ for $a_{(1)}$.

$H^{-1}(0.75)$		33.69	36.25	27.28	36.33
	test no.	1	2	3	4
33.69	1	1.000	1.076	0.810	1.078
36.25	2	0.929	1.000	0.753	1.002
27.28	3	1.235	1.329	1.000	1.332
36.33	4	0.927	0.998	0.751	1.000

Table 9.5.4. ARPE's for the four tests considered, with $\alpha = 0.05$ and $\beta = 0.25$ for $a_{(2)}$.

$H^{-1}(0.25)$		3.74	3.73	4.70	3.75
	test no.	1	2	3	4
3.74	1	1.000	0.997	1.257	1.003
3.73	2	1.003	1.000	1.260	1.005
4.70	3	0.796	0.794	1.000	0.798
3.75	4	0.997	0.995	1.253	1.000

Table 9.5.5. ARPE's for the four tests considered, with $\alpha = 0.05$ and $\beta = 0.50$ for $a_{(2)}$.

$H^{-1}(0.50)$		7.46	7.45	9.32	7.47
	test no.	1	2	3	4
7.46	1	1.000	0.999	1.249	1.001
7.45	2	1.001	1.000	1.251	1.003
9.32	3	0.800	0.799	1.000	0.802
7.47	4	0.999	0.997	1.248	1.000

Table 9.5.6. ARPE's for the four tests considered, with $\alpha = 0.05$ and $\beta = 0.75$ for $a_{(2)}$.

$H^{-1}(0.75)$		12.06	12.06	14.94	12.08
	test no.	1	2	3	4
12.06	1	1.000	1.000	1.239	1.002
12.06	2	1.000	1.000	1.239	1.002
14.94	3	0.807	0.807	1.000	0.809
12.08	4	0.998	0.998	1.237	1.000

These results confirm clearly the simulation results of the preceding section. I.e. $\underline{v}_3$ (the "directed" test) performs best against $a_{(1)}$. Second best is $\underline{v}_1$ (the "χ^2-type" test), though only slightly better than $\underline{v}_4$ (the "asymptotic χ^2-type" test). The "equal weights" test, $\underline{v}_2$ performs definitely worse against $a_{(1)}$.

The situation under $a_{(2)}$ is different, fully in accordance with our expectations. As in the simulation results, the "equal weights" test performs best against $a_{(2)}$. Again $\underline{v}_1$ is second best and is slightly better than $\underline{v}_4$. This time the directed test, $\underline{v}_3$ performs worst.

The approximation to $e_{ij}(\beta)$, (cf. (2.5.11)),

$$e^*_{ij} = \frac{\vec{\zeta}\,'Q_i\vec{\zeta}}{\vec{\zeta}\,'Q_j\vec{\zeta}} \tag{9.5.2}$$

is much easier to use, because no eigenvalues etc. have to be calculated. We have

Table 9.5.7. Components of the vectors $\vec{\zeta}$ for $a_{(1)}$ and $a_{(2)}$.

ν		$\vec{\zeta}^{(1)}$	$\vec{\zeta}^{(2)}$
	1	0.2600	0.2683
1	j 2	-0.1033	-0.1020
	3	-0.1567	-0.1663
	1	-0.0650	-0.1200
2	j 2	0.0258	0.2783
	3	0.0392	-0.1583
	1	-0.0650	-0.1463
3	j 2	0.0258	-0.1870
	3	0.0392	0.3333
	1	-0.0650	-0.0010
4	j 2	0.0258	0.0053
	3	0.0392	-0.0043
	1	-0.0650	-0.0010
5	j 2	0.0258	0.0053
	3	0.0392	-0.0043

Table 9.5.8. Approximate ARPE's for the four tests considered, calculated according to (5.2.11), for $a_{(1)}$.

$\vec{\zeta}'Q_i\vec{\zeta}$		0.405	0.373	0.616	0.402
	test no.	1	2	3	4
0.405	1	1.000	1.086	0.657	1.007
0.373	2	0.921	1.000	0.606	0.928
0.616	3	1.521	1.561	1.000	1.532
0.402	4	0.993	1.078	0.653	1.000

Table 9.5.9. Approximate ARPE's for the four tests considered, calculated according to (5.2.11), for $a_{(2)}$.

$\vec{\zeta}'Q_i\vec{\zeta}$		1.132	1.144	1.069	1.125
	test no.	1	2	3	4
1.132	1	1.000	0.990	1.059	1.006
1.144	2	1.011	1.000	1.070	1.017
1.069	3	0.944	0.934	1.000	0.950
1.125	4	0.994	0.983	1.052	1.000

The reader may judge for himself whether he thinks these approximations good enough for his purposes. In any case, the general tendency is the same as in the 'exact' ARPE cases.

Finally, the asymptotic relative Bahadur efficiency (ARBE) is equal to (cf. (5.4.1))

$$(9.5.3) \qquad E_{ij}(a) = \frac{\frac{1}{\lambda_1^{(i)}} \vec{\zeta}' Q_i \vec{\zeta}}{\frac{1}{\lambda_1^{(j)}} \vec{\zeta}' Q_j \vec{\zeta}}$$

Using the data of tables 9.1.2, 9.2.1, 9.5.8 and 9.5.9 we find

Table 9.5.10. ARBE's for the four tests considered, calculated according to (5.4.1), for $a_{(1)}$.

$\frac{1}{\lambda_1^{(i)}} \vec{\zeta}' Q_i \vec{\zeta}$		0.378	0.329	0.402	0.402
	test no.	1	2	3	4
0.378	1	1.000	1.149	0.940	0.940
0.329	2	0.870	1.000	0.818	0.818
0.402	3	1.063	1.222	1.000	1.000
0.402	4	1.063	1.222	1.000	1.000

Table 9.5.11. ARBE's for the four tests considered, calculated according to (5.4.1), for $a_{(2)}$.

$\frac{1}{\lambda_1^{(i)}} \vec{\zeta}' Q_i \vec{\zeta}$		1.056	1.010	0.697	1.125
	test no.	1	2	3	4
1.056	1	1.000	1.046	1.515	0.939
1.010	2	0.956	1.000	1.449	0.898
0.697	3	0.660	0.690	1.000	0.620
1.125	4	1.065	1.114	1.614	1.000

It is clear that the simulation results of section 9.4 are more in accordance with the ARPE's than with the ARBE's. The ARPE therefore seems to be the better measure of asymptotic relative efficiencies in our case.

9.6. CONCLUDING REMARKS

By computer generation of all possible $\Pi_{i=1}^{m} N_i$ different combinations of words and calculation of $\underline{v}(Q)$ for each combination, it is in principle possible to obtain the exact distribution of $\underline{v}(Q)$. However, the number of possibilities becomes soon prohibitive. For instance in our example we have

$$\prod_{i=1}^{m} N_i = 3.24 \times 10^{12}. \qquad (9.6.1)$$

So only for relatively small m, it can be done in practice. The interested reader is referred to DIK (1979), which shows that, under H_0, the number of possibilities can be reduced a little by symmetry arguments, and which gives some results, under H_0, for the case that $Q = I_n \times G_g$.

He may also find there results of simulations of $\underline{w}(Q)$, i.e. in the unconditional situation. Some remarks are made there also on the effects of the deletion of "useless" observations.

We are aware that the numerical examples that we have given do not, in any way, cover all the possible situations that can occur. But the examples given support clearly the recommendations that we have given in this thesis.

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