CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands
Telephone 31 - 20 592 9333, telex 12571 (mactr nl),
telefax 31 - 20 592 4199

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Multimedians in metric and normed spaces

E.R. Verheul

1980 Mathematics Subject Classification: 06C99, 06D99, 46B20, 46B30, 46B25, 52A01, 51F20, 52A43, 54E50.
ISBN 90 6196 420 2
NUGI-code: 811

Printed in the Netherlands

FOREWORD

For me one of the highlights of highschool-mathematics was the introduction of the concept inner product. Problems that in earlier schoolyears could only be handled with geometric methods, were now be poured in an algebraic form and were then easily solved with a combination of algebraic manipulation and geometric insight. The strength of this combination has made a tremendous impression on me.

During my later study of mathematics this feeling became even stronger, when it turned out that this combination was also fruitful in the more general context of inner product spaces. Apart from their beautiful structure, inner product spaces have been studied extensively for the many applications they produced, e.g. in Fourier-analysis.

In the winter of 1987 I was introduced to the subject of median algebras by Dr. Marcel van de Vel — they play an important role in his monograph "Theory of Convex Structures". When it appeared to me that the combination of algebraic manipulation and geometric insight was vital in this area too I became very interested. Median algebras have — sometimes implicitly— appeared in rather different disciplines of mathematics such as discrete mathematics, lattice theory and topology, each of which use their own language. This makes it rather difficult to get an overall view of the subject. When, with the help of Marcel, I obtained that view, it seemed to me that one of the reasons median algebras were not that popular was the seeming lack of applications. Furthermore it surprised me that the highly natural class of median algebras arising from metric spaces and normed spaces, was hardly studied.

With this monograph, based on my dissertation, I therefore hope to achieve the following things:

- Giving an introduction to the subject of median algebras.
- Presenting a theory of median algebras arising from metric and normed spaces.
- Connecting the theory of median algebras to the real world by means of applications.

Acknowledgements

I would like to express my indebtedness to Dr. Marcel van de Vel. Professionally, he has been responsible for the creation of a research environment that was extremely stimulating to work in. His valuable suggestions on median convexities and other subjects were a constant source of inspiration. Socially, I have come to know him as a very nice person to be with, and to sport with.

I am grateful to Prof. Dr. H.-J. Bandelt from Hamburg with whom I came in contact shortly before completing my doctorate research. The combinatorical viewpoints that were developed in the research which I did with him and Dr. M. van de Vel were particularly helpful in solving some remaining problems I had.

I would like to thank Prof. Dr. J. van Mill, for his constant interest in my research, and his willingness to help me whenever topological problems were concerned.

I am grateful to my former high school teachers in mathematics, Drs. R. Birkhoff, Dr. F. Kuiper, and Drs. C. Reedijk for guiding and stimulating me on my first steps in mathematics.

When doing mathematical research it is rather unfortunate that many ideas come to mind outside office hours. I therefore have a profound admiration for my wife Jacqueline Meltzer, who never got disturbed when I suddenly had to work out an idea, and who comforted me when the idea was not that successful.

Finally, I want to thank my parents for their support and encouragement to pursue my own interests.

E. V.

Amsterdam, the Netherlands
October, 1992

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SYMBOLS

$\mathbb{N} = \{1,2,..\}$,	set of natural numbers
$\mathbb{Z},\mathbb{Q},\mathbb{R}$,	set of integers, rationals and reals respectively
$A \Delta B$,	symmetric difference of A and B
A',	complement of A
2^X,	power set of X
$\times_{i\in I} X_i$,	Cartesian product of sets $(X_i)_{i\in I}$
$\lvert A \rvert$, $\#(A)$,	cardinality of A
$\bar{A}$, $cl(A)$,	(topological) closure of A
A^o, $int(A)$,	interior of A
$a \sqcap b$,	collection of maximal lowerbounds of points a,b in a partial order
$a \wedge b$,	infimum (meet) of points a,b in a lattice
$a \sqcup b$,	collection of minimal upperbounds of points a,b in a partial order
$a \vee b$,	supremum (join) of points a,b in a lattice
$inf(A)$,	infimum of A
$min(A)$,	minimum of A
$sup(A)$,	supremum of A
$max(A)$,	maximum of A
$D(b,r)$,	closed ball at a point b of radius r in metric space
$diam(A)$,	diameter of A, i.e. $sup\{\rho(a,b) \mid a,b \in A\}$
$\rho(A,B)$,	distance between A and B, i.e. $inf\{\rho(a,b) \mid a \in A \; ; b \in B\}$
ρ_H,	Hausdorff metric
$co(A)$,	convex hull of A
$\overline{co}(A)$,	convex closure of A, i.e. the closure of $co(A)$
$sp(A)$,	linear span of A
$\overline{sp}(A)$,	linear closure of A, i.e. the closure of $sp(A)$
$ker(f)$,	kernel of a linear function f
$[f]$,	equivalence class of integrable functions
iff,	if and only if
■,	end of proof

INTRODUCTION

-das mittelding- das wahre in allen sachen,
kennt und schätzt man izt nimmer.
(W.A. Mozart)

A point p in a metric space (X, ρ) is (metrically) *between* a, $b \in X$ provided that $\rho(a,b) = \rho(a,p) + \rho(p,b)$. This definition was first formulated by Menger in 1928 (see [56]). The (metric) *interval* joining two points a, b of X is the set $I_\rho(a,b)$ consisting of all points between a and b.

In this thesis we consider *modular* metric spaces (X, ρ), which have the property that all sets of type

$$M_X(a,b,c) = I_\rho(a,b) \cap I_\rho(b,c) \cap I_\rho(c,a)$$

are nonempty. For example, if X is the real line $\mathbb{R}$ with its natural metric, then $M_{\mathbb{R}}(a,b,c)$ consists of one point only, namely the middle one of a, b, c. More generally, the operator M_X is single-valued if X equals $\mathbb{R}^n$ with the "sum-norm". It is properly multivalued if X equals $\mathbb{R}^n$ with the "max-norm". These states of the values of M_X (singlevalued/multivalued) correspond with rather opposite situations.

In 1947, Birkhoff and Kiss [15] considered a ternary (so-called *median*) operator on a distributive lattice, and discussed its properties. One year later, Avann [5] formulated a general concept of a *median algebra*. This is a set with a ternary operation possessing a few natural properties, which are fulfilled, for instance, by the single-valued operator of $\mathbb{R}^n$ with the sum-norm. The Birkhoff-Kiss operator is a metric-free example. The subsequent papers [72], [73], [74], of Sholander provide some characterizations of median algebras appealing to the intuitive meaning of the word "median". Some of his results were used later by Avann [6] to conclude that if the operator M_X of a modular metric space X is single-valued, then it gives a median operator. To emphasize that the set $M_X(a,b,c)$ can have more than one point, the operator M_X is called a *multimedian* operator. In the fortuitous situation that M_X is single-valued, the space (X, ρ) is a *median metric space*.

More recently, median algebras have been studied from the viewpoint of convexity. Sholander's concept of median betweenness gives rise to an interval operator which, in turn, leads to a natural description of *convex sets*. A convexity arising from a median algebra has a number of properties reminding of traditional convexity in vector spaces, specifically in inner product spaces. For instance, disjoint convex sets can be separated with complementary half-spaces, and certain convex sets (such as polytopes) allow for a natural projection similar to metric nearest point projections. In fact, an important part of this thesis deals with a combination of traditional convexity and median convexity: a study of modular normed spaces.

As can be seen from the 1983 survey paper of Bandelt and Hedlíková [8] median algebras were not widely known outside of lattice theory or graph theory, until recently. Nonetheless, median spaces have occurred in a somewhat disguised form in topology (normally supercompact spaces and superextensions —see van Mill's dissertation [58]). Here, the idea is to use a closed subbase with the following intersection property: any collection of subbase members that meet two by two meet altogether. Formally replacing the closed subbase by the collection of closed balls in a metric space leads to a description of hyperconvex metrics, the study of which goes back to Nachbin [63] and to Aronszaijn and Panitchpakdi [4]. Hyperconvex metric spaces, and more general spaces with the "(3,2) Intersection Property of balls" yield another type of example of modular spaces. In these spaces the values of the operator M_X are usually genuinely multivalued.

In this dissertation we present a study of modular metric spaces —in which the ***median*** metric spaces play a prominent role— thereby combining viewpoints from median convexity and that of intersection properties of balls. This study, which is mainly set in the disciplines of metric geometry and analysis, is applicable in rather various situations, such as modular lattices, graphs and Banach spaces. In this fashion, we obtain new examples of spaces with the "(3,2) Intersection Property of balls" and we are able to solve a problem of Aronszaijn and Panitchpakdi on the completion of these spaces in the affirmative. We also give an application of the developed theory in chapter VII, where results on shortest network of line segments interconnecting an arbitrary set (Steiner trees) are presented.

Organization

Some basic information on partial orders, (multi-)lattices, and convexity is presented in Chapter I, culminating in a general (metric-free) theory of modular spaces. Some standard results have been provided with a proof in order to make our treatment somewhat self-contained. Among other things, it is shown that median spaces are modular spaces with an abundance of convex sets.

Chapter II specializes to modular ***metric*** spaces (as described above). Here it is shown that the modularity condition corresponds with the "(3,2) Intersection Property of balls". Apart from the early work of Nachbin and of Aronszaijn and Panitchpakdi, this topic received attention from Isbell [40], [41], Lima [48], en Lindenstrauss [50], [51]. Prominent examples are: $L_1(\mu)$-spaces, $K_1(\mu)$-spaces, and metric ("valuated") lattices. Particular attention is given to extension properties of contractive maps, completeness, and weak topology (that is, the topology generated by the convex closed sets). Also an explicit description of the completion of a modular metric space is given. This description, which consists of adding convergence points of "decreasing" or "increasing" sequences, resembles the classical Carathéodory extension theorem for measures to σ-algebras. This is the starting point for a "Heine-Borel" type theorem, characterizing weak compactness in modular metric spaces.

Chapter III further specializes to modular ***normed*** spaces. Among the important results are: a characterization of $L_1(\mu)$-spaces as median Banach spaces, or, as Banach spaces with an additive orthogonality. The latter justifies the above claim of analogy with inner product spaces. As a consequence, each median normed space embeds (linearly) isometrically into an $L_1(\mu)$ space.

Chapter IV provides characterizations of spaces of type $l_1(I)$ in terms of the Hausdorff property of the weak topology (in the sense of Chapter II). These results are used in a decomposition of modular Banach spaces into two modular ones. One factor has no linear functionals compatible with the metric (sub)convexity (equivalently, it has no non-trivial metrically convex bodies), whereas the other has a point-separating collection of such functionals. ***Median*** Banach spaces of the first type correspond with atom-free $L_1(\mu)$ spaces, whereas spaces of the second type correspond with $l_1(I)$ spaces.

Some questions on isometric embedding of median metric spaces into $L_1(\mu)$ spaces are considered in Chapter V. We use results of Assouad and Deza [7] to show that $L_1(\mu)$ spaces are not only universal median normed spaces but in fact are universal median ***metric*** spaces: each median metric space embeds isometrically into an $L_1(\mu)$ space. Completely different techniques have been used to show that a median metric space embeds isometrically into $l_1(n)$ iff it can be embedded as a median subalgebra.

Chapter VI deals with the amalgamation of modular (metric) spaces. This construction results into a unique modular space extending the original ones, and which is median if the original spaces both are. Particular attention goes to the extension of metrics which are "adapted" to an interval operator in the sense that balls around convex sets are convex. The results are applied to construct median or hyperconvex metrics on collapsible polyhedra in an elegant and natural way. This construction generalizes and simplifies that of Mai and Tang [53].

Finally, an application of median geometry is given Chapter VII. Here the topic is the theory of Steiner trees, which deals with the following type of problem. How can you design a network connecting all consumers that minimizes the quantity of material used? This type of problem arises in the design of telephone networks, oil pipelines, and electrical circuitry. The main result of this chapter is that in general median metric space such trees exist and can be found in a finite number of steps. This generalizes and strengthens a result of M. Hanan [35] in the plane. The method employed by Hanan is rather technical and ad hoc as it involves highly specific constructions in the plane. In contrast, we have based our methods on a fairly well developed geometry of median metric spaces. In particular, there is no need to restrict to two dimensions.

Somewhat more detailed summaries can be found at the beginning of each chapter, with the exception of chapter I.

Note on notation: when referring to a result within this dissertation, we only specify its chapter number if it differs from the current one.

CHAPTER I

PRELIMINARIES

§ 1 Partial orders

1.1 Lattices. Let $(P, \leq)$ be a *poset* (partially ordered set). If $x,y \in P$ have an infimum (resp. supremum) then it will be denoted by $x \wedge y$ (resp. $x \vee y$). A *semi-lattice* is a poset S in which each pair of points x,y has an infimum. A *lattice* is a partially ordered set in which each pair of points has an infimum and a supremum. If K,L are lattices, then $f{:}K \to L$ is called a *(lattice) homomorphism* provided

$$f(x \wedge y) = f(x) \wedge f(y) \quad ; \quad f(x \vee y) = f(x) \vee f(y),$$

for all $x,y \in K$. Note that a homomorphism is order preserving. A bijective homomorphism is called an *isomorphism*. A least (resp. greatest) element of a lattice —if any— is called a *unit,* and will be denoted by 0 (resp. 1).

A lattice L is called *distributive* provided that

$$\forall x,y,z \in L : x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

or equivalently,

$$\forall x,y,z \in L : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A lattice L is called *modular* provided that

$$a \leq c \text{ implies } a \vee (b \wedge c) = (a \vee b) \wedge c \quad (a,b,c \in L).$$

Modularity is considerably more general than distributivity. For instance, nearly all lattices arising from algebraic considerations are modular, but usually not distributive.

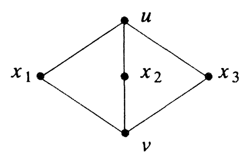

Fig. 1.1A: The lattice $K_{2,3}$.

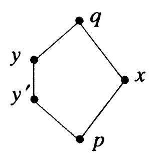

Fig. 1.1B: The lattice N_5

Figure 1.1A depicts the non-distributive lattice $K_{2,3}$, and Figure 1.1B depicts the non-modular lattice N_5. In fact the following hold. A lattice is modular if and only if it does not contain the lattice N_5 as a sublattice, and a modular lattice is distributive if and only if it does not contain the lattice $K_{2,3}$. See [13] or [28].

1.2 Lattice groups. A partial order $\leq$ on an Abelian group G with zero element 0 is called a *group order,* provided that $0 \leq x$ implies $a \leq x+a$ for all $a,x \in G$. In this situation, the pair $(G,\leq)$ is called an Abelian *ordered group.* Observe that a (non-trivial) ordered group can not have units. If $\leq$ constitutes a lattice, then $(G,\leq)$ is called an Abelian *lattice (ordered) group.* See [13], where it is also shown that a lattice group is a distributive lattice, and that any Abelian torsion free group can be made into an Abelian lattice group.

The most prominent example arises from vector spaces. A group order $\leq$ on a real vector space V with the property that $0 \leq x$ implies $0 \leq \lambda x$ for all $\lambda \in \mathbb{R}_0^+$, is called a *vector order.* The pair $(V,\leq)$ is called an *ordered vector space* in this situation. If $\leq$ constitutes a lattice, then $(V,\leq)$ is called a *vector lattice,* or a *Riesz space.* See the book of Luxemburg and Zaanen [52].

For a point x in an Abelian lattice group G, we define the *positive part* by $x^+ = x \vee 0$, the *negative part* by $x^- = -(x \wedge 0)$, and the *modulus* by $|x| = x^+ + x^-$. We remark that Birkhoff [13] defines the negative part of x differently by taking $x^- = x \wedge 0$. Our notation is most common in the theory of Riesz spaces.

1.3 Boolean algebras. Let L be distributive lattice with units $0,1$. A point $a \in L$ is called an *atom* if for $b \in L$ with $0 \leq b \leq a$ either $b=0$ or $b=a$. Two points $x,x' \in L$ are *complementary* provided that

$$x \wedge x' = 0 \quad ; \quad x \vee x' = 1.$$

In these circumstances x' is a *complement* of x. A point $x \in L$ can have at most one complement. The lattice L is called *complemented* if every $x \in L$ has a complement.

A *Boolean algebra* $(A, \vee, \wedge)$ is a complemented, distributive lattice. As a (classical) example of Boolean algebras we have the following. A collection of subsets $\mathcal{A}$ of a set X is called an *algebra of sets* if

(1) $\varnothing, X \in \mathcal{A}$,

(2) $A,B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$,

(3) $A,B \in \mathcal{A}$ then $A \Delta B \in \mathcal{A}$.

One can easily verify that the triple $(\mathcal{A}, \cup, \cap)$ yields a Boolean algebra. ([1]) In fact by the "Stone representation theorem" (Theorem 1.5) each Boolean algebra can be seen as an algebra of sets. An algebra of sets closed under taking countable unions is called a *σ-algebra.*

A homomorphism between Boolean algebras, is a lattice homomorphism with the additional property that $p(0)=0$ and $p(1)=1$. Hence, there is no such thing as a trivial Boolean homomorphism.

The following is a well-known connection between ultra-filters and (Boolean) homomorphisms with values in $\{0,1\}$. Let X be a set. Then $\mathcal{F}$ is an ultra-filter on X iff $\mathcal{F} = p^{-1}(1)$ for some homomorphism of the power set 2^X into $\{0,1\}$. In a natural fashion one can define a notion of

[1] The use of a script character refers to an algebra of sets, as opposed to an abstract Boolean algebra.

filter on a Boolean algebra. By using maximal filters one obtains:

1.4 Lemma. (e.g. [1, p. 197-199]) *Let* $(A, \vee, \wedge)$ *be a Boolean algebra and* $0 \neq a \in X$, *then there exists a homomorphism* $h : A \to \{0,1\}$ *with* $p(a)=1$. ■

We remark that there is a correspondence between the power set 2^X and the function space $\{0,1\}^X$: a set $A \subseteq X$ corresponds with the *characteristic function* χ_A on A, defined by $\chi_A(x)=1$ iff $x \in A$. It is well-known that $\{0,1\}^X$ endowed with the topology of pointwise convergence, yields a compact Hausdorff space.

Let $(A, \vee, \wedge)$ be a Boolean algebra. Then $B(A)$ denotes the subset of $\{0,1\}^A$ consisting of all homomorphisms between A and $\{0,1\}$. Endow $B(A)$ with the relative topology of $\{0,1\}^A$. The following is a rather straightforward consequence of Lemma 1.4.

1.5 Theorem. (Stone representation theorem) *Let* $(A, \vee, \wedge)$ *be a Boolean algebra. Then,*

(1) The function space $B(A)$ *is a compact Hausdorff topological space.*

(2) The function $j : (A, \vee, \wedge) \to 2^{B(A)} : j(a) = \{p \in B(A) \mid p(a)=1\}$, *is a homomorphism. The values of j are closed and open in* $B(A)$. ■

1.6 Measures on Boolean algebras; Measure spaces. Let $(A, \vee, \wedge)$ be a Boolean algebra. A function $\mu : A \to [0,\infty]$ is called a *(finitely-additive) measure* on A provided

(1) $\mu(0)=0$,

(2) μ is finitely additive, i.e. if $a_1, \cdots, a_n$ are pairwise disjoint elements of A, i.e. $a_i \wedge a_j = 0$ for distinct i,j, then

$$\mu(\textstyle\sum_{i=1}^n a_i) = \sum_{i=1}^n \mu(a_i).$$

The binary function $\rho_\mu(a,b) = \mu(a \vee b) - \mu(a \wedge b)$ yields a pseudo-metric on $A_{fin} = \{a \in A \mid \mu(a) < \infty\}$. This pseudo-metric space will be denoted by $K(A,\mu)$. The quotient of A_{fin} obtained by dividing out zero sets will be denoted by $K_1(A,\mu)$.

If $(\mathcal{A}, \vee, \wedge)$ is an algebra of sets with unit X, then a measure μ on $\mathcal{A}$ is called *σ-additive* if for each sequence $(A_i)_{i=1}^\infty$ of pairwise disjoint sets in $\mathcal{A}$ whose union is also in $\mathcal{A}$ the following equality holds

$$\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i).$$

If $\mathcal{A}$ is a σ-algebra, then the triple $(X, \mathcal{A}, \mu)$ is called a *measure space*. For general reference see the book of Royden [69]. A prominent type of example is given by the *counting* measure μ on the σ-algebra 2^I for some index set I, which is given by:

$$\mu(F) = \begin{cases} \#(F) & \textit{if } F \textit{ is finite} \\ \infty & \textit{otherwise.} \end{cases} \qquad (F \subseteq I)$$

The following result is classical.

1.7 Theorem. (Carathéodory extension theorem) *A σ-additive measure* μ *on an algebra of sets* $(\mathcal{A}, \vee, \wedge)$ *with unit X can be extended to a σ-additive measure on the smallest σ-algebra* $\mathcal{A}'$ *in X containing* $\mathcal{A}$. *If X is the union of countable many elements of* $\mathcal{A}$ *of finite measure, then this extension is unique.* ■

In a measure space one can define the "integral" $\int_X f\,d\mu$ for a *measurable* function $f: X \to [-\infty, \infty]$. If $\int_X |f|\,d\mu$ is finite, then f is called *absolutely integrable*. The collection of integrable functions on X is a pseudo-normed space with pseudo-norm $||f|| = \int_X |f|\,d\mu$. If f is an integrable function, then $[f]$ denotes the class of measurable functions g that equal f "almost everywhere" (i.e. $\mu(\{x \mid f(x) \neq g(x)\}) = 0$). One then defines

$$L_1(X, \mathcal{A}, \mu) = \{[f] \mid f \textit{ is absolutely integrable}\}.$$

In literature these Banach spaces are commonly called $L_1(\mu)$ spaces. If μ is the counting measure on an index set I then the associated Banach space is usually denoted by $l_1(I)$. For I finite, say $|I| = n$, the space $l_1(I)$ is just $(\mathbb{R}^n, ||.||_s)$ where $||.||_s$ denotes the *sum norm*, which is defined by

$$||(x_1, \cdots, x_n)||_s = |x_1| + \cdots + |x_n|.$$

The space $K_1(\mathcal{A}, \mu)$ (which is usually denoted by $K_1(X, \mathcal{A}, \mu)$) corresponds with the closed subset $\{[\chi_U] \mid U \in \mathcal{A}\ ;\ \mu(U) < \infty\}$ of $L_1(X, \mathcal{A}, \mu)$. Hence, $K_1(X, \mathcal{A}, \mu)$ is a complete metric space.

We obtain a nice application of the theorems of Carathéodory and Stone. See [64].

1.8 Theorem. *Let $(A, \vee, \wedge)$ be a Boolean algebra, and let μ be a measure on A. Then there is a σ-algebra $\mathcal{A}'$ with a σ-additive measure μ', such that A is (isomorphic with) a subalgebra of $\mathcal{A}$, and μ' extends μ.*

Proof: By the Stone representation theorem we can look upon A as an algebra $\mathcal{A}$ consisting of clopen sets in $B(A)$, and we can regard μ as being defined on $\mathcal{A}$. Suppose that $A_1, A_2, \cdots$ is a sequence of pairwise disjoint members of $\mathcal{A}$ such that their union is in $\mathcal{A}$. As elements of $\mathcal{A}$ are clopen sets of $B(A)$, we see that $A_n = \varnothing$ for large enough n. Hence μ is trivially σ-additive. Applying the Carathéodory extension theorem to μ and $\mathcal{A}$ concludes the proof of the theorem. ■

§ 2 Convex structures

2.1 Convexities. The following notions are taken from the monograph of van de Vel [79]. A family $\mathcal{C}$ of subsets of a set X is called a *convexity* on X if

(C-1) $\varnothing, X$ are in $\mathcal{C}$.

(C-2) $\mathcal{C}$ is stable for intersections.

(C-3) $\mathcal{C}$ is stable for updirected unions, i.e. if $\mathcal{D} \subseteq \mathcal{C}$ is non-empty and updirected then $\cup\mathcal{D}$ is in $\mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a *convex structure*, and members of $\mathcal{C}$ are called *($\mathcal{C}$-)convex*. Each $A \subseteq X$ is contained in a smallest convex set, the *convex hull* of A or $co(A)$ for short. The convex hull of a finite set is called a *polytope*. A polytope spanned by two points is called a *segment*. A subset H of X is called a *($\mathcal{C}$-)halfspace* provided both $H, X \setminus H$ are members of $\mathcal{C}$. Let us say that two disjoint subsets A, B in X are separated by a $\mathcal{C}$-halfspace H provided $A \subseteq H$ and $B \cap H = \varnothing$. This notion gives rise to the following separation axioms S_1, S_2, S_3, S_3 on $\mathcal{C}$ as follows:

S_1: Points in X are $\mathcal{C}$-convex.
S_2: distinct points in X can be separated by $\mathcal{C}$-halfspaces.
S_3: Each $\mathcal{C}$-convex subset C in X can be separated by $\mathcal{C}$-halfspaces from points $q \notin C$.
S_4: Each pair of disjoint convex subsets in X can be separated by $\mathcal{C}$-halfspaces.

Axiom S_4 is usually called the *Kakutani separating property*. Clearly, S_2 implies S_1, and under assumption of S_1, $S_4 \to S_3 \to S_2$.

2.2 Interval operators. An operator $I : X^2 \to 2^X$ is called an *interval operator* if it has the following properties for all $a,b \in X$:
(I-1) *Convexity of points:* $I(a,a) = \{a\}$.
(I-2) *Extensiveness:* $a,b \in I(a,b)$.
(I-3) *Symmetry:* $I(a,b) = I(b,a)$.
The pair (X,I) is called an *interval space*. A subset C of X is called *star-shaped* at a point $c_0 \in C$ provided that

$$\forall c \in C \ : \ I(c_0,c) \subseteq C.$$

The subset C is called *I-convex*, if C is star-shaped at all of its points. One can easily verify that the collection of all I-convex subsets yields a convexity (the *I-convexity*). The hull operator of this convexity will be denoted by co_I. We give some examples.

2.3 Segment operator of convexity. Let $(X,\mathcal{C})$ be an S_1-convexity. Then the segment operator, i.e. the operator assigning to each pair of points the segment between those points, is an interval operator.

2.4 Standard interval operator. Let V be a real vector space. The standard interval operator co is given by

$$co(a,b) = \{t \cdot a + (1-t) \cdot b \mid 0 \le t \le 1\},$$

for all $a,b \in V$. The induced convexity is called *standard*.

2.5 Metric interval operator. Let (X,ρ) be a metric space. The metric interval operator I_ρ is defined by

$$I_\rho(a,b) = \{x \in X \mid \rho(a,x) + \rho(x,b) = \rho(a,b)\},$$

for all $a,b \in X$. The I_ρ-convex subsets are called *ρ-convex*, *geodesically convex*, or simply *metric-convex*. The concept of metric intervals originates from Menger [56].

As an illustration we mention that the metric intervals of an inner product space coincide with the standard intervals.

2.6 Lattice interval operator. Let $(L,\vee,\wedge)$ be a lattice. The lattice interval operator I_l is defined by

$$I_l(a,b) = \{x \in L \mid (a \wedge x) \vee (b \wedge x) = x = (a \vee x) \wedge (b \vee x)\},$$

for all $a,b \in L$. This lattice interval operator was introduced by Glivenko [30]. See also the book of Blumenthal and Menger, [16], where other interval operators on lattices are considered as well. The I_l-convex subsets are also called *l-convex*.

The following result summarizes some of the properties of lattice interval operators. In the sequel, $[a,b]$ will denote the order interval between two points $a \leq b$ in a lattice.

2.7 Proposition. *Let L be a lattice. Then the following are true for all $a,b,x \in L$:*

(1) $x \in I_l(a,b)$ *implies* $a \wedge b \leq x \leq a \vee b$.

(2) $a \wedge b \in I_l(a,b)$; $a \vee b \in I_l(a,b)$.

(3) *If $a \leq b$ then $I_l(a,b)$ is exactly the order interval between a,b.*

(4) *The convexity induced by I_l consists of all order-convex sublattices of L.*

Proof: For a proof of (1), consider the following (in)equalities:

$x \leq x \vee (a \wedge b) \leq (a \vee x) \wedge (x \vee b) = x,$

in which the equality is by assumption. Hence equality holds throughout, showing that $a \wedge b \leq x$. The other inequality is derived similarly. For a proof of (2), for $x = a \wedge b$, we find that $(a \wedge x) \vee (x \wedge b) = x \vee x = x$ and $(a \vee x) \wedge (x \vee b) = a \wedge b = x$. Hence $x \in I_l(a,b)$. Similar computations yield $a \vee b \in I_l(a,b)$. For a proof of (3), suppose that $a \leq b$. If $a \leq x \leq b$, then $(a \vee x) \wedge (x \vee b) = x$. Considering the dual formula as well, we find that $x \in I_l(a,b)$. The opposite implication follows from the first part of the lemma. For a proof of (4), to find the convex hull $co(a,b)$ of two points $a,b \in L$, one should start with $I_l(a,b)$. Then take all pairs $a',b' \in I_l(a,b)$ and add $I_l(a',b')$. Then repeat the process until the set stabilizes. By combining parts (2) and (3) we see that the convex hull of a,b, must include the entire order-interval $[a \wedge b, a \vee b]$. By using part (1), it can be seen that no other points are obtained during the stabilization process. ■

Let us introduce some further terminology. A point c of an interval space (X,I) is said to be *in between* a and b if and only if $c \in I(a,b)$. These so-called "betweenness" relations are studied by several authors, among which are Pitcher and Smiley [66], and Sholander [72], [73], [72].

2.8 An alternative way of describing interval spaces. Let X be a set. An operator $M: X^3 \to 2^X$ is called a *mixing-operator* provided the following conditions are fulfilled:

(M-1) *Absorption* or *majority rule* : $M(a,a,b) = \{a\}$.

(M-2) *Symmetry* : if σ is any permutation of a, b, c then $M(\sigma(a),\sigma(b),\sigma(c)) = M(a,b,c)$.

By the *standard mixing-operator* of an interval space (X,I) is meant the operator M defined by the formula $M(a,b,c) = I(a,b) \cap I(a,c) \cap I(b,c)$. Sometimes we shall use the notation M_I or M_X for this operator.

Starting with a mixing-operator one can construct three interval operators $I_M^1 \subseteq I_M^2 \subseteq I_M^3$ on X by taking:

(i) $I_M^1(a,b) = \{x \in X \mid \{x\} = M(a,b,x)\}$,

(ii) $I_M^2(a,b) = \{x \in X \mid x \in M(a,b,x)\}$,

(iii) $I_M^3(a,b) = \cup\{M(a,b,x) \mid x \in X\}$, for $a,b \in X$.

In general the mixing-operators induced by I_M^i $(i = 1,2,3)$ do are not equal to M. The following result is straightforward:

2.9 Proposition. *The following formulae hold for a mixing operator M of an interval operator I on X:*

$$I_M^2(a,b)=I_M^3(a,b)=I(a,b) \quad (a,b\in X). \qquad \blacksquare$$

2.10 Morphisms, subspaces and products of interval spaces. Let (X_i,I_i) $(i=1,2)$ be interval spaces, and let $f:X_1 \to X_2$. A function f is called *interval preserving*, or *IP* provided

$$f(I_1(a,b)) \subseteq I_2(f(a),f(b)),$$

for all $a,b\in X_1$. One can verify that f also "preserves" the induced convexities on X_1 and X_2 in the sense that preimages under f of I_2-convex sets are I_1-convex sets. As an example of IP-functionals we offer the following. Consider the convexity on $\{0,1\}$ consisting of all subsets of $\{0,1\}$. Let H be a $\mathcal{C}$-halfspace, then the functional $f:X \to \{0,1\}$ that is 1 on H and 0 elsewhere is an IP-functional.

Let (X,I) be an interval space, and let $Y\subseteq X$. Then the operator $I_Y: Y^2 \to 2^Y$ defined by $I_Y(y_1,y_2)=I(y_1,y_2)\cap Y$ is an interval operator on Y. The operator I_Y is called the *relative interval operator*. The mixing operator M_Y of Y induced by this operator, i.e. $M_Y=M_I\cap Y$, equals the *relative mixing operator*.

Let (X_i,I_i) for $i\in I$ be a collection of interval spaces. On the Cartesian product $\prod_{i\in I}X_i$ the *product interval operator* I_π is defined by

$$I_\pi((a_i)_{i\in I},(b_i)_{i\in I}) = \prod_{i\in I} I_i(a_i,b_i),$$

for all $(a_i)_{i\in I},(b_i)_{i\in I}\in \prod_{i\in I}X_i$.

A straightforward verification shows that I_π is indeed an interval operator on the product. The mixing operator M_π of $\prod_{i\in I}X_i$ takes the form $\prod_{i\in I}M_i$, where M_i denotes the mixing operator of X_i.

2.11 Median operators. A *median* (operator) m on a set X is a function $m: X^3 \to X$ with the following properties:

(M-1) *Absorption* or *majority rule* : $m(a,a,b)=a$.

(M-2) *Symmetry* : if σ is any permutation of a, b, c then $m(\sigma(a),\sigma(b),\sigma(c))=m(a,b,c)$.

(M-3) *Transitive rule* : $m(m(a,b,c),d,c)=m(a,m(b,c,d),c)$.

The pair (X,m) is called a *median algebra*. By virtue of the symmetry one can look upon the transitive rule of medians as a "swapping" rule (which might be easier to remember): in the expression $m(m(a,b,c),d,c)$ —in which the point c occurs at two levels— one may exchange the point d with either of a,b.

A median can be seen as a (singlevalued) mixing operator. Sholander [72], [73], [74], presented several axiom systems for median algebras. The most prominent type of example is that of a distributive lattice (e.g. a Boolean algebra) with

$$m(a,b,c)=(a\wedge b)\vee(a\wedge c)\vee(b\wedge c).$$

Note that by distributivity the right-hand side is not changed if we permute the roles of $\wedge$ and $\vee$. This example first appeared in a 1947 paper [15] of Birkhoff and Kiss.

The median m of a totally ordered lattice, e.g. $\mathbb{R}$, is given by

$m(a,b,c) =$ the middle one of a,b,c.

2.12 Median interval operator. One can readily verify that for a median operator the interval operations I_m^1, I_m^2 and I_m^3 coincide. This yields a canonical *median interval operator*, which is denoted by I_m. The I_m-convex subsets are called *median-convex*, or simply *m-convex*. We remark that median intervals are m-convex. See [73], or Theorem 4.24.

We mention the following characterization of median interval space, which shall be particularly motivating. We refer to [73] or [79].

2.13 Theorem. *The following are equivalent for an interval space* (X,I).

(1) (X,I) *is derived from a median.*

(2) (X,I) *has the following properties for all* $x,y,z \in X$

(i) *If* $z \in I(x,y)$ *then* $I(x,z) \subseteq I(x,y)$.

(ii) $|\, I(x,y) \cap I(x,z) \cap I(y,z)\, | = 1$.

Moreover, the expression between the bars in (ii) defines the ambient median.

Proof: For a self-contained proof ([2]): implication $(1) \to (2)$ is Example (iv) of Paragraph 3.1, and implication $(2) \to (1)$ is Proposition 4.1 combined with Corollary 4.17. ■

We now introduce a particular intersection property of sets. Consider any cardinal number $\aleph \geq 2$. A collection $\mathcal{C}$ of subsets of a set X has the *($\aleph$,2) Intersection Property* (briefly *($\aleph$,2)-IP*) provided every subcollection of cardinality $\leq \aleph$ consisting of sets meeting two by two has a non-empty intersection. We shall be mainly interested in the case that $\aleph$ is a finite cardinal. The collection $\mathcal{C}$ has the *finite intersecting property* (briefly *(F,2)-IP)* if it has the (n,2)-IP for all finite cardinal numbers n. The collection $\mathcal{C}$ has the *arbitrary intersecting property* (briefly *(A,2)-IP)* if it has the ($\aleph$,2)-IP for all cardinal numbers $\aleph$.

The (F,2)-IP can be used to characterize median convexity as follows. See also [79].

2.14 Theorem. *The following are equivalent for an* S_1 *convexity* $\mathcal{C}$.

(1) *The convexity* $\mathcal{C}$ *is derived from a median.*

(2) $\mathcal{C}$ *is* S_2 *and has the (F,2)-IP.*

(3) $\mathcal{C}$ *is* S_4 *and has the (F,2)-IP.*

Proof: For a self-contained proof ([2]): Implication $(3) \to (2)$ is evident. As to implication $(2) \to (1)$, according to Theorem 2.13 it suffices to show that the mixing operator M induced by the segment operator $co(-,-)$ of $\mathcal{C}$ is single-valued (it is non-empty by the (F,2)-IP). To this end, let x,y be two distinct points in a value $M(a,b,c)$ of the mixing operator. Separate x,y by a halfspace H. We may assume that $a,b \in H$. This implies that the whole set $M(a,b,c) \subseteq co(a,b)$ is contained in H, a contradiction. Implication $(1) \to (3)$, is Theorem 4.19. ■

From (2) we conclude in particular that distinct points in a median algebra can be separated by halfspaces. Applying this result to the lattice convexity of a Boolean algebra we obtain Lemma 1.4.

[2] One should better skip this proof at first reading.

2.15 Halfspace reasoning. In median convexity there is a particular form of calculus involving convex sets and separating halfspaces that can be used in several situations. The following example might be illustrative: A median algebra (X,m) satisfies (we write xyz for $m(x,y,z)$):

(M-3′) *5-point transitive rule* : $((abu)(abv)w) = (ab(uvw))$. ([3])

For a proof, let $a,b,u,v,w \in X$ and suppose that (M-3′) does not hold. Then by the Kakutani separation property there exists an m-halfspace H that contains the point $ab(uvw)$ and misses the point $(abu)(abv)w$ (see Figure 2.15).

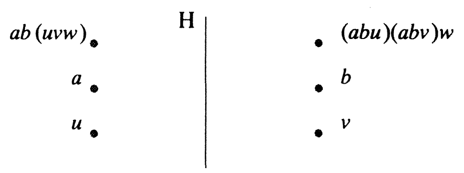

Fig. 2.15: halfspace reasoning

We repeatedly use the following consequence of Theorem 2.13: the point (xyz) is the only point in the intersection $I_m(x,y) \cap I_m(x,z) \cap I_m(y,z)$. If both points a, b are members of H, then $(abu),(abv) \in I_m(a,b) \subseteq H$ and hence $(abu)(abv)w \in H$, a contradiction. Assuming $a,b \in X \setminus H$ yields a similar contradiction. We may assume that $a \in H$ and $b \in X \setminus H$. With this configuration of the points a,b it is impossible that either $u,v \in H$ or $u,v \in X \setminus H$ holds. Hence without loss of generality we may assume that $u \in H$ and $v \in X \setminus H$. Now all points except w have been placed. If $w \in H$, then $(abu)(abv)w \in H$, a contradiction. If $w \in X \setminus H$, then $ab(uvw) \in X \setminus H$, another contradiction.

Under assumptiom of (M-1) en (M-2) one can deduce (M-3) from (M-3′) by simple algebraic manupilation. To prove the reverse implicaton in an ***algebraic*** fashion seems not not easy — see [47].

2.16 Morphisms, subspaces and products of median algebras. Let (X_i,m_i) $(i=1,2)$ be median algebras, and let $f: X_1 \to X_2$. Then the function f is called *median preserving* (*MP*), or a *homomorphism* provided

$$f(m_1(a,b,c)) = m_2(f(a),f(b),f(c)),$$

for all $a,b,c \in X_1$. One can verify that a MP function also preserves median intervals. Let A be a Boolean algebra, and let f be a function $A \to \{0,1\}$ with $f(0)=0$ and $f(1)=1$. Then f is a median homomorphism iff f is a Boolean homomorphism.

Let (X_i,m_i) for $i \in I$ be a collection of median algebras. On the Cartesian product $\prod_{i \in I} X_i$ the *product median* m_π is defined by

[3] Many authors —e.g. Birkhoff [13], and Sholander [72], [73], [74]— define median operators by (M-1), (M-2) and (M-3′).

$$m_\pi((a_i)_{i\in I},(b_i)_{i\in I}),(c_i)_{i\in I})=\prod_{i\in I}m_i(a_i,b_i,c_i),$$

for all $(a_i)_{i\in I},(b_i)_{i\in I},(c_i)_{i\in I}\in\prod X_{i\in I}$. A straightforward verification shows that m_π is a median operator. The interval operation on the product induced by m_π is precisely the product of the interval operators I_{m_i}. This leads to the standard median operator of $\mathbb{R}^n$.

If all (X_i,I_i) are equal to $\mathbb{R}$ with the standard interval operator, then the (standard) median operator of the Riesz space $\mathbb{R}^I$ equals the product median.

The following is a generalization of the "Stone representation theorem".

2.17 Theorem. *Let (X,m) be a median algebra and let $b\in X$. Then*

(1) The function space $B_b(X)=\{f:X\to\{0,1\}\mid f$ is a homomorphism with $f(b)=0\}$ is a compact Hausdorff topological space.

(2) The function $j:(X,m)\to 2^{B_b(X)}:j(a)=\{p\in B_b(X)\mid p(a)=1\}$ is a homomorphism. The values of j are closed and open in $B(X)$. ■

2.18 Corollary. (cf. [8, Theorem 1.5]) *Each median algebra is isomorphic with a median stable subset of a power set. Moreover, each finite median algebra embeds in a finite Boolean algebra, and hence in a Euclidean space equipped with the product median.* ■

A subset Y of a median algebra (X,m) is *median stable* provided $m(Y^3)\subseteq Y$. In particular, the restricted map $m_{\mid Y^3}$ yields a median algebra. The *median stabilization* of a subset Z, $med(Z)$ for short, is the smallest median stable subset including Z.

One of the remarkable properties of median stabilization, though rather difficult to find in literature, is that the stabilization of a finite set is finite. A proof of this appears in Chapter V.

2.19 Theorem. *Let (X,m) be a median algebra, and let $Z\subseteq X$. A point $p\in X$ is not contained in the median stabilization of Z, if and only if then there exist halfspaces G,H such that*

$$p\in G\cap H\ ;\ G\cap H\cap Z=\varnothing.$$

Proof: The "if" part of the theorem is evident. First we shall prove the "only if" part of the theorem for finite Z. To this end, as $med(Z)$ is finite and $p\notin med(Z)$ the S_3 property of the (median) convexity of X gives rise to a finite number of halfspaces $H_1,\cdots,H_n$ such that:

$$p\in\cap_{i=1}^n H_1\ ;\ med(Z)\cap\cap_{i=1}^n H_i=\varnothing.$$

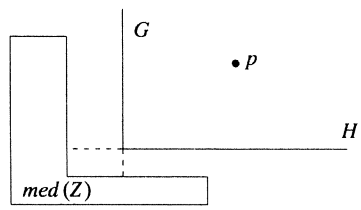

Fig. 2.19: separating a point from a median stable set

Observe that the sets $med(Z) \cap H_i$ $(i=1,2 \cdots n)$ are relatively convex in the median algebra $(med(Z),m)$. Hence by the finite intersection property of median convexity (Theorem 2.14) there must be halfspaces G,H among $H_1, \cdots, H_n$ such that $G \cap H \cap med(Z) = \emptyset$.

The proof that the "only if" part of the theorem holds for arbitrary Z uses the following compactness argument. For $V \subseteq Z$ finite let $B(V)$ be the collection of pairs (g,h), where g,h are median preserving functions $X \to \{0,1\}$ with $g(p)=h(p)=0$ and $(g(v),h(v)) \neq (0,0)$ for all $v \in V$.

One can easily verify that the sets $B(V)$ are compact subsets of $B_p(X) \times B_p(X)$ — see Theorem 2.17. Moreover, the sets $B(V)$ have the finite intersection property by the first part of the proof. Hence by compactness there exists a pair (g,h) that is contained in $B(V)$ for every finite set $V \subseteq Z$. One can easily see that the associated halfspaces $G=g^{-1}(0)$, $H=h^{-1}(0)$ are as desired. ■

We end this section with some remarks on computing the median stabilization of a finite subset Z in a median algebra (X,m). An iterative process is as follows; define $Z_1 = Z$ and

$$Z_{n+1} = m(Z_n,Z_n,Z_n) \quad (n \in \mathbb{N}) \quad ; \; Z_\infty = \cup_{n \in \mathbb{N}} Z_n.$$

Then $med(Z)=Z_\infty$. As mentioned earlier the median stabilization of a finite subset is finite, hence from certain n on the Z_n equal $med(Z)$. Information on this n is useful for computational purpose. To this end, define the *median stabilization degree (msd)* of a median algebra X as the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that $med(Z)=Z_n$ for all $Z \subseteq X$, see [10]. For instance, the msd of $\mathbb{R}$ is zero (evident) and the msd of $\mathbb{R} \times \mathbb{R}$ is one. We outline a proof of the latter statement. Let $Z \subseteq \mathbb{R} \times \mathbb{R}$ be finite and $p \in med(Z)$. By the use of a translation we may assume that p equals the origin. As the lattice of points generated by Z is median stable, there exist points $a=(0,a_2)$, $b=(b_1,0)$ in Z. If either a_2,b_1 is zero then we are done. Otherwise, by Theorem 2.19 there exists a point $c=(c_1,c_2)$ such that (c_1,c_2) and (b_1,a_2) are in opposite quadrants. Hence, the median of a,b,c equals p, i.e. $p \in Z_1$.

In a similar fashion one can show that $msd(\mathbb{R}^n) \leq n-1$. See [10], where it is actually shown that $msd(\mathbb{R}^n)$ grows like $\log_{1.5}(n)$.

In many situations — for instance in $\mathbb{R}^n$ — it is possible to obtain a finite median stable subset X containing Z at forehand. The following is then an alternative method to compute $med(Z)$. Find all halfspaces in X and intersect them, then throw away all points in X that are seperated — in the sense of the previous theorem — from Z by these sets.

For example, let Z be a subset of $\mathbb{R}^2$ with the product median, and let X be the lattice of points generated by Z. Then the halfspaces are the intersections of X of halfspaces parallel to x-axis or y-axis. Hence the number of intersections involved is in the order of $\#Z \times \#Z$.

§ 3 Geometric interval operators

In the study of interval spaces induced by metrics and medians, many proofs and definitions involve only a few general properties. These properties are used as an axiom system for geometric interval operators. The resulting theory can be applied in other situations as well, e.g. (modular) lattices, and it provides us with convenient terminology.

3.1 Basepoint orders. Let (X,I) be an interval space and let $b \in X$. The following defines a reflexive relation on X:

$$x \leq_b y \Leftrightarrow x \in I(b,y).$$

The relation $\leq_b$ is called the *basepoint relation* at b. The following questions are natural.

(i) Are basepoint relations transitive? That is, are intervals star-shaped at their endpoints, viz. does $c \in I(a,b)$ imply $I(a,c) \subseteq I(a,b)$?

(ii) Are basepoint relations anti-symmetric? Viz. if $I(a,b) = I(a,b')$ then $b = b'$

(iii) Are the relations $(I(a,b), \leq_a)$ and $(I(b,a), \leq_b)$ mutually inverse for all $a,b \in X$? Viz. if $a,b,x,y \in X$ satisfy $x,y \in I(a,b)$ and $x \in I(a,y)$, then $y \in I(x,b)$. See Figure 3.1.

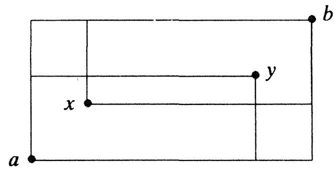

Fig. 3.1: the inversion law

For a counterexample to (i), consider the set $X = \{1,2,3,4\}$, and define an interval operator I on X by $I(i,j) = \{i,j\}$ for $\{i,j\} \neq \{1,3\}, \{1,4\}$ and $I(1,3) = \{1,2,3\}$; $I(1,4) = \{1,3,4\}$.
For a counterexample to (ii), (iii), take the indiscrete interval operator of a set with at least three points. We now come to the following definition. An interval operator I on a set X is called *geometric* if it satisfies:

(G-1) *star-shapedness*: If $c \in I(a,b)$, then $I(a,c) \subseteq I(a,b)$.

(G-2) *inversion law*: If $a,b,x,y \in X$ satisfy $x,y \in I(a,b)$ and $x \in I(a,y)$, then $y \in I(x,b)$.

The above examples show that the axioms (G-1) and (G-2) are independent. Axioms (G-1) and (G-2) can also be formulated in the following way:

(3.2) For all $a,b,x,y \in X$ the statements $y \in I(a,b)$, $x \in I(a,y)$ imply $x \in I(a,b)$, $y \in I(x,b)$.

Axioms (G-1) and (G-2) are considered by several authors, e.g. Blumenthal & Menger, Hedlíková, and Sholander. Hedlíková [37] uses the term "ternary space" for geometric interval space.

Note that the product of geometric interval spaces is geometric. As further examples of geometric interval operators we have the following:

(i) the (standard) segment operator of a linear vector space (evident),

(ii) the metric interval operator of a metric space (straightforward),
(iii) the lattice interval operator of a modular lattice (see Theorem 4.2 below),
(iv) the median interval operator of a median algebra (X,m) ([37], [73]).

For a proof of the last statement, let $y \in I_m(a,b)$ and $x \in I_m(a,y)$. Then,

$$m(a,x,b) = m(a,m(a,x,y),b) = m(a,m(a,b,y),x) = m(a,y,x) = x,$$

showing that $x \in I_m(a,b)$. On the other hand,

$$m(x,y,b) = m(m(a,x,y),y,b) = m(m(a,b,y),y,x) = m(y,y,x) = y,$$

showing that $y \in I_m(x,b)$. Whence I_m is geometric by (3.2). Let us show how the formula

$$m(a,b,c) = I_m(a,b) \cap I_m(a,c) \cap I_m(b,c) \qquad (a,b,c \in X), \tag{*}$$

which is mentioned in Theorem 2.13, can be deduced from the geometric properties. First, it is clear that the point $m(a,b,c)$ is at least contained in the right-hand set of (*). Next, let z be another point in this set. Then

$$z = m(z,a,b) = m(m(z,a,c),a,b) = m(m(a,b,c),a,z).$$

Whence, $z \in I_m(a,m(a,b,c))$. Similarly we obtain $z \in I_m(b,m(a,b,c))$. Hence $z = m(a,b,c)$.

The following result summarizes some of the properties of geometric interval operators.

3.3 Proposition. *Let (X,I) be a geometric interval space and let $a,b,c \in X$. Then,*

(1) Every basepoint relation of X is a partial order.
In particular the interval operators I_M^1 and I_M^2 coincide.

(2) The mixing operator M has the following property:

$$\forall a,b,c \in X \quad c \in M(a,b,c) \Leftrightarrow M(a,b,c) = \{c\}. \tag{3.3.3}$$

Proof: For a proof of the non-trivial part of statement (2): let $x \in M(a,b,c)$. That is, $x \in I(b,c)$. By assumption we have $c \in I(b,a)$ so (G-1) yields $I(b,c) \subseteq I(b,a)$. We can now apply (G-2) to the points b,a,x,c which yields $c \in I(x,a)$. As the last set is contained in $I(a,c)$ we can apply (G-2) to the points a,c,c,x which yields $x \in I(c,c) = \{c\}$. Statement (1) follows from (2). ■

3.4 Gates. Let (X,I) be a geometric interval space and let C be a subset. A *gate* of x in C is a point $c_x \in C$ such that $c_x \in I(x,c)$ for all $c \in C$. See [21], [37], [42].

Clearly, c_x is the smallest element of C in the basepoint order with basepoint x. As $\leq_x$ is a partial order by Proposition 3.3, the point x can have at most one gate in C. The subset C is called *gated*, provided every $x \in X$ admits a gate in C. Hedlíková [37] and Isbell [42], use the name "Chebyshev sets" for gated sets. The induced *gate function* $x \to c_x$ is denoted by p_C. For example, if (X,m) is a median algebra, then any interval $I_m(a,b)$ is gated. The induced gate function is given by $x \to m(a,b,x)$. (See [79], or Corollary 4.15).

We mention some properties of gates in a geometric interval space (X,I).

(3.4.1) Let x have a gate p in $C \subseteq X$. If D is a subset of X which is star-shaped at x and meets C, then $p \in C \cap D$.

(3.4.2) Any gated subset of X is convex. See [42].

For a proof of statement (3.4.2), let C be a gated subset of (X,I). Consider $a,b\in C$, and $c\in I(a,b)$. Let c' be the gate of c in C. Then by definition we have $c'\in I(c,a)$, and $c'\in I(c,b)$. By axiom (G-1) the last formula yields $c'\in I(c,b)\subseteq I(a,b)$. Hence we can apply (G-2) to the points a,b,c',c which yields $c\in I(c',b)$. By axiom (G-2) that is $I(c,b)\subseteq I(c',b)$, i.e. $I(c,b)=I(c',b)$. By Proposition 3.3(1) we conclude $c=c'\in C$, as desired.

Let C be a gated subset of X.

(3.4.3) If $D\subseteq C$ is gated in (C,I_C), then D is gated in (X,I). See [42].

(3.4.4) (transitive rule) If D is a gated subset of X not disjoint from C, then $C\cap D$ is also gated in X and $p_{C\cap D}=p_C\circ p_D$ (composition product). See [42].

(3.4.5) If D is any gated subset of X, then the composition $P=p_C\circ p_D$ is idempotent, i.e. $P^2=P$. In particular, for any element $c_1\in C$ the points $n_1=p_D(c_1)$, $n_2=p_C(n_1)$ are *mutual* gates, i.e. n_2 is the gate of n_1 in D and vice versa.

3.5 Proposition. *The collection of gated subsets of a geometric interval space has the (F,2)-IP.*

Proof: Let (X,I) be a geometric interval space. First, the proposition is trivially true for collections consisting of two members. Next, assume that the proposition is true for collections of cardinality less than $n>2$ and suppose that $D_1,\cdots,D_n$ is a collection of gated sets in X that meet two by two. Then by the inductive hypotheses there is a point

$x\in\cap_{i=2}^{n}D_i$.

If x' is the gate of x in D_1, then by (3.4.1), $x'\in D_1\cap D_i$ (for $i=2,3,\cdots,n$), and x' is a point as desired. ∎

§ 4 Modular spaces

As stated earlier, the interval operator of a median algebra is geometric. ([4]) Whence in view of Theorem 2.13 median algebras correspond with geometric interval spaces with a single-valued mixing operator (see also Corollary 4.17). So it is natural to call such interval spaces *median*. A *modular space* is defined to be a geometric interval space such that the mixing operator $M_I:(a,b,c)\to I(a,b)\cap I(a,c)\cap I(b,c)$ takes non-empty values. In this situation the mixing operator is called the *multimedian*.

From Theorem 2.13 one deduces that an interval space satisfying the star-shapedness axiom (G-1) with a singlevalued mixing operator is median —hence the remaining axiom (G-2) follows automatically. This phenomenon can be explained by the following —alternative— axiom system for modular spaces.

[4] The results of this section were obtained by H.-J. Bandelt, M. van de Vel and the author. See [11].

4.1 Proposition. *Let (X,I) be an interval space that satisfies axiom (G-1) and such that the mixing operator only takes non-empty values. Then (X,I) is modular iff the mixing operator M has property (3.3.3).*

Proof: The "only if part" is Proposition 3.3. Conversely, let $a,b,x,y \in X$ such that $x,y \in I(a,b)$ and $x \in I(a,y)$. By star-shapedness we obtain that $M(x,y,b) \subseteq M(a,y,b) = \{y\}$. As by assumption the value $M(x,y,b)$ is non-empty, we conclude that this set equals $\{y\}$, that is $y \in I(y,b)$ as desired. ■

One can find simple examples of interval spaces, which do not satisfy the star-shapedness axiom (G-1), but do have a non-empty mixing operator. Whence axiom (G-1) can not be omitted in Proposition 4.1. The following example, which was first mentioned by Sholander [73], is rather interesting.

We first introduce some concepts. A *Steiner point* of three points a,b,c in a metric space (X,ρ) is a point in X that minimalizes the expression

$$\rho(a,s)+\rho(b,s)+\rho(c,s) \quad (s \in X).$$

We mention that in modular ***metric*** space (see below) a point in the multimedian is a Steiner point of the ambient three points (Corollary II: 1.12). In Chapter VII will investigate the relation between Steiner points and medians in greater depth. For an inner product space $(X, <.,.>)$ we define the *Steiner* intervals by

$$I_S(a,b) = \{z \in X \mid <a-z, b-z> \leq -\tfrac{1}{2} \cdot ||a-z|| \cdot ||b-z||\},$$

for all $a,b \in X$. Geometrically this means that z is in between a and b iff the angle between the vectors $a-z, b-z$ is more than 120 degrees. Then the value $M(a,b,c)$ of the mixing operator consists of one point, namely the (unique!) Steiner point of the triple a,b,c. See for instance [18]. That the Steiner intervals do not satisfy axiom (G-1) is a direct verification. In particular, these intervals are not Steiner-convex either —actually, it follows from the results in chapter IV that the Steiner convexity only consists of the singletons, the empty set and the whole X.

Many examples of modular space arise from lattice theory.

4.2 Theorem. *Let L be a lattice, and let M_l be the mixing operator of L (induced by the lattice intervals). Then the following are equivalent:*

(1) L is modular.

(2) The interval operator I_l satisfies axiom (G-1).

(3) M_l only takes non-empty values.

(4) (L,I_l) is a modular space.

In any of the above situations the value $M_l(a,b,c)$ contains the point $(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$. Moreover, M_l is single-valued iff L is distributive.

Proof: For a proof of implications (2),(3) $\rightarrow$ (1), suppose that L is not modular. Then it contains the lattice N_5, (see Fig. 1.1B for notation), as a sublattice.

A straightforward verification shows that $p \in {}_l(x,y)$ and $y' \in I(p,y)$. Hence, if I_l satisfies the star-shapedness axiom (G-1) then $y' \in I(x,y)$. That is $y' = (x \vee y') \wedge (y' \vee y) = y$. This settles implication (2) $\to$ (1).

For a proof of implication (3) $\to$ (1), let $z \in M(x,y',y)$ then (cf. Proposition 2.7) $y' \le z \le y$, hence $y = y \wedge q = (y \vee z) \wedge (z \vee x) = z$. Similarly we have $y' = y' \vee p = (y' \wedge z) \vee (z \wedge x) = z$. Whence $y = y'$, a contradiction. Implication (4) $\to$ (1) now trivially follows.

For (1) $\to$ (2): Let $b \in I_l(a,c)$ and $d \in I_l(a,b)$. Hence (Proposition 2.7), $a \wedge c \le b$ and $a \wedge b \le d$. These inequalities are used in the following calculations:

$$d = (a \wedge d) \vee (d \wedge b) = (a \wedge d) \vee \Big\{ (d \wedge ((a \wedge b) \vee (b \wedge c)) \Big\} = (a \wedge d) \vee \Big\{ ((a \wedge b) \vee (d \wedge b \wedge c) \Big\}$$
$$= (a \wedge d) \vee (a \wedge b) \vee (d \wedge b \wedge c) \le (a \wedge d) \vee (d \wedge c) \le d.$$

The third equality is modularity. Hence $d = (a \wedge d) \vee (d \wedge c)$. Dually $d = (a \vee d) \wedge (d \vee c)$. That is, $d \in I_l(a,c)$ as desired.

For (1) $\to$ (3): Let $a,b,c \in L$, and consider the point $m = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$. By using the modular law we find

$$a \vee m = a \vee \Big\{ (b \vee c) \wedge \{ (c \vee a) \wedge (a \vee b) \} \Big\} = \{ a \vee (b \vee c) \} \wedge \{ (c \vee a) \wedge (a \vee b) \} = (c \vee a) \wedge (a \vee b).$$

Similar equalities hold for $b \vee m$ and $c \vee m$. If follows that $(a \vee m) \wedge (m \vee b) = m$, with similar formulas for all other combinations. For a dual formula consider the following computation:

$$(a \wedge m) \vee (b \wedge m) = \{(a \wedge m) \vee b\} \wedge m = \{(a \wedge (a \vee m) \wedge (m \vee b)) \vee b\} \wedge m$$
$$= \{(a \wedge (m \vee b)) \vee b\} \wedge m = \{(a \vee b) \wedge (m \vee b)\} \wedge m = m.$$

The first and fourth equality are applications of modularity, whereas the second equality follows from substituting $m = (a \vee m) \wedge (m \vee b)$. Similar formulas hold for other combinations. Therefore, $m \in I_l(a,b) \cap I_l(b,c) \cap I_l(c,a)$. We remark that by duality the point $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ is also contained in the value $M(a,b,c)$ of the mixing operator.

In view of implications (1) $\to$ (2),(3), all that needs to be verified for implication (1) $\to$ (4) is that I_l satisfies the inversion axiom (G-2). To this end, let $a,b \in L$ and $y \in I_l(a,b)$, $x \in I_l(a,y)$. By Proposition 2.7(2), the latter condition implies $a \wedge y \le x$, hence $a \wedge y \le x \wedge y$. We conclude,

$$(a \wedge y) \vee (y \wedge b) \le (x \wedge y) \vee (y \wedge b).$$

Now the left hand side equals y by assumption, whereas the right hand side is less or equal to y. Whence equality holds troughout, that is $(x \wedge y) \vee (y \wedge b) = y$. Similarly we obtain $(x \vee y) \wedge (y \vee b) = y$. We conclude that $y \in I_l(x,b)$. Observe that modularity is not required in this part of the proof. Whence the lattice interval operator of ***any*** lattice satisfies the inversion axiom (G-2). This was earlier observed by Blumenthal and Menger [16, ex. 1 p. 67].

For a proof of the last statement, if the mixing operator of L is single-valued, then in view of an earlier remark $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ for all $a,b,c \in L$. It is well-known that the latter property is equivalent with distributivity of L. See [13]. ■

The equivalence of statements (1) and (2) are taken from E. Pitcher and F. Smiley in [66]. The equivalence of statements (1) and (3) and the concluding statements are taken from H. Draškovičová [20]. From the proof of implication (1) $\to$ (4) one can also deduce

4.3 Metric interval spaces. In this thesis we are mainly interested in ***metric*** interval spaces; by abuse of language a metric space is called modular (resp. median) if the underlying metric interval space is modular (resp. median).

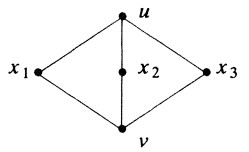

Fig. 4.3A: $K_{2,3}$

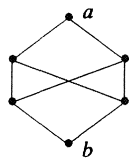

Fig. 4.3B: $K_{3,3}$ minus edge

For example, the standard median of $\mathbb{R}$, induced by the order, is also the metric median induced by the standard metric on $\mathbb{R}$. Similarly, the standard median of $\mathbb{R}^n$ is derived from the sum-norm (see II: §1).

Modular (metric) space was first studied in the context of graphs. These so-called *modular graphs* (compare II: 1.1), where introduced by Howorka [38]. See also the paper of Bandelt and Mulder [9]. The modular graphs depicted in Figures 4.3A, 4.3B, shall be of particular interest.

As further examples of modular metric spaces we mention the metric spaces of type $K_1(\mu)$, $L_1(\mu)$. (see Theorems II: 1.8 and II: 1.9).

4.4 Some general results.

(4.4.1) The convexity of a modular space has the (F,2)-IP.

In view of Theorem 2.14 the previous result states that we can look upon median spaces as modular spaces with an abundance of convex subsets.

(4.4.2) Let C be a convex subset of a modular space and let $b \in C$. Then any minimal element in $(C, \leq_b)$ is a minimum, i.e. a gate.

From the following result we obtain a method of verifying whether a multimedian is a median.

4.5 Theorem. *Let I_1, I_2 be two modular interval operators on a set X with respective multimedians M_1, M_2. If $M_1(a,b,c) \cap M_2(a,b,c) \neq \varnothing$ for all $a,b,c \in X$, then $I_1 = I_2$ and $m_1 = M_2$. In particular, if $I_1(a,b) \subseteq I_2(a,b)$ for all $a,b \in X$ then $I_1 = I_2$.*

Proof: Let $a,b \in X$, and let $x \in I_1(a,b)$, i.e. $M_1(a,b,x) = \{x\}$ by Proposition 3.3. By assumption the sets $M_1(a,b,x)$, $M_2(a,b,x)$ meet, whence $x \in M_2(a,b,x)$, i.e. $x \in I_2(a,b)$ by using Proposition 3.3 once more. We have shown that $I_1(a,b)$ is contained in $I_2(a,b)$. Similarly we obtain the other inclusion, hence $I_1(a,b) = I_2(a,b)$. ■

This result has several interesting consequences.

4.6 Corollary. (cf. Th. 4.24) *The segment operator of a modular space is geometric iff all intervals are convex.*

Proof: Segments being convex, the segment operator clearly satisfies axiom (G-1). Also, as the multimedian of (X,I) is contained in the mixing operator of the segment operator the mixing operator of (X, co_I) only takes non-empty values. From the previous theorem we conclude that the segment operator of a modular space satisfies axiom (G-2) if and only if the interval operator equals the segment operator, i.e. if the intervals I are I-convex. ■

4.7 Corollary. *Let (X,I) be the a modular space. If there exists a median m on X such that $m(a,b,c)\in M(a,b,c)$ for all $a,b,c\in X$, then m and M coincide.* ■

A function $f:X_1\to X_2$ between modular spaces (X_1,I_1) and (X_2,I_2) is called *multimedian preserving*, if the sets $f(M_1(a,b,c))$, $M_2(f(a),f(b),f(c))$ meet for all $a,b,c\in X$ (in fact, it follows that the first set is ***always*** included in the second). Two modular spaces are called *isomorphic* if there exists a bijective multimedian preserving map.

A subset Y of a modular space is called *multimedian stable* provided that the value $M(y_1,y_2,y_3)$ of the multimedian meets Y for all $y_1,y_2,y_3\in Y$. We remark that for median spaces these notions coincide with the earlier introduced notions "median preserving" and "median stable".

Observe that the image of a multimedian preserving function is multimedian stable. With "halfspace reasoning" (see 2.15) one can verify that gate functions in median space preserve the median. A similar property for gate functions in ***multi***median space does not hold.

It is not difficult to show that a multimedian preserving function is interval preserving (and vice versa). The following result is somewhat stronger.

4.8 Theorem. *Let (X_1,I_1) and (X_2,I_2) be modular spaces, and let $f:X_1\to X_2$ be a surjective multimedian preserving map. Then,*

$$f(I_1(x_1,x_2))=I_2(f(x_1),f(x_2)) \quad\text{and}\quad f(M_1(x_1,x_2,x_3))=M_2(f(x_1),f(x_2),f(x_3)).$$

for all $x_1,x_2,x_3\in X_1$. In particular, f maps I_1-convex subsets onto I_2-convex subsets.

Proof: Let $a,b\in X_2$ and $a'\in f^{-1}(a)$, $b'\in f^{-1}(b)$. We take $\bar{I}(a,b)=f(I_1(a',b'))$.

First, we shall verify that the definition of $\bar{I}(a,b)$ does not depend on the choice of a',b'. To this end, suppose $a''\in f^{-1}(a)$, $b''\in f^{-1}(b)$. Let $z\in f(I_1(a',b'))$, say $z=f(z')$ with $z'\in I_1(a',b')$. On the one hand,

$$\{z\}=\{f(z')\}=f(M_1(a',b',z')), \tag{1}$$

and on the other hand,

$$M_2(a,b,z)=M_2(f(a'),f(b'),f(z')). \tag{2}$$

As the sets on the right-hand side of (1) and (2) meet, we conclude from Proposition 3.3 that $\{z\}=M_2(f(a'),f(b'),f(z'))$. Now as the sets $f(M_1(a'',b'',z'))$ and $M_2(f(a''),f(b''),f(z'))$ $(=\{z\})$ meet we obtain that the set $f(M_1(a'',b'',z'))$ contains the point z. Whence there exists a $z''\in M_1(a'',b'',z')\subseteq I_1(a'',b'')$ with $f(z'')=z$. We have shown that $f(I_1(a',b'))\subseteq f(I_1(a'',b''))$. The other inclusion is similar. From the above we also conclude that

$$\bar{I}(a,b)\subseteq I_2(a,b) \tag{3}$$

for all $a,b\in X_2$. Secondly, we show that $\bar{I}$ is geometric. To this end, let $y\in\bar{I}(a,b)$ and $x\in\bar{I}(a,y)$. Hence there are preimages y',a',b' of y,a,b respectively such that $y'\in I_1(a',b')$. As $\bar{I}(a,y)=f(I_1(a',y'))$ there exists a preimage x' of x such that $x'\in I_1(a',y')$. By the geometric properties of I_1 we have $x'\in I_1(a',b')$ and $y'\in I_1(x',b')$. By definition $x\in\bar{I}(a,b)$ and $y\in\bar{I}(x,b)$. Therefore $\bar{I}$ is geometric by (3.2). Finally, a simple set-theoretic argument shows that $(M$

denotes the mixing operator of $\tilde{I}$)

$$f(M_1(a',b',c')) \subseteq \tilde{M}(a,b,c) \tag{4}$$

for all $a,b,c \in X_2$ and $a=f(a')$, $b=f(b')$, $c=f(c')$.

From formulae (3), (4) it follows that $(X_2,\tilde{I})$ is a modular space comparable with (X_2,I_2). So from Theorem 4.5 we deduce that $\tilde{M}=M_2$, $\tilde{I}=I_2$, and the theorem follows. ■

We remark that the surjectivity in the previous result is not essential. If $f:X_1 \to X_2$ is multimedian preserving, then $Im(f)$ equipped with the relative interval operator is also modular (its multimedian equals the relative multimedian).

We obtain two corollaries about a (not necessarily surjective) function $f:X_1 \to X_2$ between modular spaces (X_1,I_1) and (X_2,I_2).

4.9 Corollary. *If f is a multimedian preserving function, then*

$$f(M_1(a,b,c)) \subseteq M_2(f(a),f(b),f(c)).$$ ■

4.10 Corollary. *The following are equivalent.*

(1) f is multimedian preserving.

(2) f is interval preserving.

If (X_2,I_2) is median then properties (1),(2) are also equivalent with

(3) f inverts I_2-convex subsets into I_1-convex subsets.

Proof: The equivalence of properties (1) and (2) follows from Theorem 4.8 and the remarks after this lemma. Implication $(2) \to (3)$ holds for all geometric interval spaces. For a proof of implication $(3) \to (2)$, let $a,b \in X_1$. As median intervals are m-convex we have the following formula

$$f(co_1(a,b)) \subseteq co_2(f(a),f(b)) = I_2(f(a),f(b)). \tag{4}$$

Hence we have

$$f(I_1(a,b)) \subseteq f(co_1(a,b)) \subseteq I_2(f(a),f(b)),$$

as desired. ■

4.11 Corollary. *Let (X_1,I_1) and (X_2,I_2) be modular spaces, and let $f:X_1 \to X_2$ be a surjective multimedian preserving map. If $G \subseteq X_1$ is gated then so is $f(G)$, and the following equality holds.*

$$p_{f(G)} \circ f = f \circ p_G.$$ ■

4.12 Modular space and (multi)lattices. The following lemma summarizes some of the properties of basepoint orders in modular space.

4.13 Lemma. *Let (X,I) be a modular space, and let $z,a,b \in X$.*

(1) If $m \le_z a,b$, then $M(m,a,b) \subseteq M(z,a,b)$, and for any $m' \in M(m,a,b)$ we have $m \le_z m'$.

(2) The value $M(z,a,b)$ of the multimedian equals the set of maximal lower bounds of the points a,b in $\le_z$.

(3) If $n \in I(a,b)$ then there exists an $n' \in M(z,a,b)$ with $n' \le_z n$.

Proof: The proof of statement (1) is straightforward.

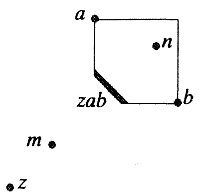

Fig. 4.13: basepoint orders in modular space

For a proof of (2), let $m \leq_z a,b$. By (1) we obtain an element $m' \in M(z,a,b)$ with $m \leq_z m' \leq_z a,b$. Next, suppose that $n \in X$ satisfies $m' \leq_z n \leq_z a,b$. By the above reasoning we may assume that $n \in M(z,a,b)$. Using the geometric properties of the intervals one can verify that

$$n \in I(m',a) \cap I(m',b) \cap I(a,b) = M(m',a,b).$$

The latter right-hand side equals $\{m'\}$ by Proposition 3.3. For a proof of (3), let $m_1 \in M(z,a,w)$ and $m_2 \in M(z,b,w)$. By the inclusion $I(m_1,m_2) \subseteq I(m_1,b) \subseteq I(a,b)$, we obtain $M(z,m_1,m_2) \subseteq M(z,a,b)$. Hence any point $n' \in M(z,m_1,m_2)$ is as desired. ■

As a consequence of Lemma 4.13(3) we obtain the following.

4.14 Corollary. *Let (X,I) be a multimedian space. Then the following are equivalent for* $a,b,c \in X$:

(1) $M(a,b,c)$ *is a singleton.*

(2) $M(a,b,c)$ *is I-convex.*

Proof: Implication $(1) \rightarrow (2)$ is trivial. Conversely, let the set $M(a,b,c)$ be I-convex, and let $m_1, m_2 \in M(a,b,c)$. Then the subset $M(c,m_1,m_2)$ is contained in $M(a,b,c)$. Hence by Lemma 4.13(2) any $m \in M(c,m_1,m_2)$ is a maximal lower bound of the points b,c in $\leq_c$. However $m \leq_c m_1, m_2 \leq_c a,b$, whence $m = m_1 = m_2$. ■

4.15 Corollary. *Let* $I(a,b)$ *be an interval in a modular space (X,I). Then a point* $z \in X$ *has a gate in* $I(a,b)$ *iff* $M(z,a,b)$ *is a singleton. Moreover, in this situation the unique member of* $M(z,a,b)$ *is the gate of z in* $I(a,b)$. ■

Let X, Y be sets and let $F : X \rightarrow 2^Y$ be a multivalued function. For $A \subseteq X$ there is a natural definition of $F(A)$ as

$$F(A) = \bigcup_{a \in A} F(a).$$

This convention enables us to use notation of type $M(A,b,c)$, $M(a,B,c)$ in modular space. Sometimes —when no ambiguity arises— we shall write *xyz* instead of $M(x,y,z)$ and *(xyz)bc* instead of $M(M(x,y,z),b,c)$ etc..

The following is a "four-point transitive rule" for modular space.

4.16 Theorem. *Let (X,I) be a modular space, and let $a,b,c,d \in X$. Then the following hold:*

(1) $\forall x \in (abc)dc\ \exists y \in a(bcd)c : x \in I(c,y)$.

(2) *If abc is a singleton then $(abc)dc \subseteq a(bcd)c$.*

Proof: For a proof of statement (1), let $m \in abc$, and $x \in mdc$. Hence we have $x \leq_c d$ and $x \leq_c m \leq_c a,b$. By Lemma 4.13 we obtain an $n \in bcd$ with $x \leq_c n$. Using this result once more yields a $y \in anc \subseteq a(bcd)c$ with $n \leq_c y$. The point y is clearly as desired. For a proof of (2), let $x \in (abc)dc$. By using statement (1) twice we obtain $y \in a(bcd)c$ and $x' \in (abc)dc$ with $x \leq_z y \leq_c x'$. Hence $x = y = x'$ by Lemma 4.13(2). ■

Repeated application of (1) leads to a chain in $\leq_c$ of points which alternate between the sets $M(M(a,b,c),d,c)$, $M(M(d,b,c),a,c)$ and $M(M(a,d,c),b,c)$. In modular graphs, all bounded chains of all basepoint orders are finite, and it follows that the three composed multimedians must have a point in common.

We do not know whether sets of type $M(M(a,b,c),d,c)$ are always closed —they probably are not. (see also the remarks prior to Example II: 2.20). Hence the above result can not be directly extended to general (complete) modular metric spaces.

4.17 Corollary. (= Th. 2.13) *A singlevalued multimedian is a median operator.* ■

We now derive two classical results on median convexity. See also [79].

4.18 Theorem. *Let (X,I) be a median interval space.*

(1) *If $a,b,c \in X$ then the set $I(a,I(b,c))$ ([5]) is gated. Moreover, $I(a,I(b,c)) = I(I(a,b),c)$.*

(2) *If $C \subseteq X$ is convex, and $x \in X$ then the convex hull of $\{x\} \cup C$ is given by $I(x,C)$.*

Proof: Let $x \in X$ and let $p = m(a,m(b,c,x),x)$. First, the point p is contained in the set $I(a,I(b,c))$. Next, let $r \in I(a,I(b,c))$, that is $r \in I(a,s)$ for some $s \in I(b,c)$. As $p \leq_x b,c$ we conclude that $p \leq_x s$, by virtue of Lemma 4.13. Also $p \leq_x a$, hence by a similar use of Lemma 4.13 we obtain that $p \leq_x r$. Whence p is the gate of x in $I(a,I(b,c))$.

In particular, we obtain that the set $I(a,I(b,c))$ is convex. Evidently, the convex hull of the points $a.b,c$ contains the set $I(a,I(b,c))$. Whence $co(a,b,c) = I(a,I(b,c))$, and the last statement of (1) easily follows.

Statement (2) simply follows from (1). ■

Property (2) of Theorem 4.18 is known as *Join-hull commutativity,* see [79].

4.19 Theorem. (=Th. 2.14) *The convexity of a median interval space has the Kakutani separation property.*

Proof: Let A, B be disjoint convex subsets of a median interval space (X,I). By using Zorn's Lemma we first find a convex set $G \supseteq A$ maximal with the property that it avoids B, and then a convex set $H \supseteq B$ maximal with the property that it misses G. Suppose that $p \in X \setminus (G \cup H)$.

---

[5] See the convention of p. 20.

By Join-hull commutativity the sets $I(p,G)$, $I(p,H)$ are convex. Hence by maximality of G and H there exist $g \in G$, and $h \in H$ such that $I(p,g)$ meets H (in say h'), and $I(p,h)$ meets G (in say g'). As G,H are down-directed in $\leq_p$ we may assume that $g' \leq_p g$ and $h' \leq_p h$. See Figure 4.19 below.

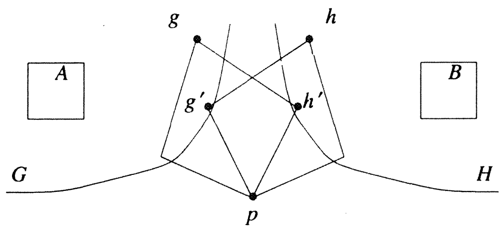

Fig. 4.19: Separating disjoint convex subsets

Hence, $g' \leq_p g,h$ and $h' \leq_p g,h$. In view of Lemma 4.13 the point $M(p,g,h)$ lies in both intervals $I(g',g)$ and $I(h',h)$. That is, $M(p,g,h) \in H \cap G = \varnothing$, a contradiction. We conclude that G and H are complementary halfspaces. ■

4.20 Multilattices. The following notion provides us with a different viewpoint of basepoints orders in modular space. Let $(P,\leq)$ be a partial order and $a,b \in P$. The collection of maximal lowerbounds (resp. minimal upperbounds) will be denoted by $a \sqcap b$ (resp. $a \sqcup b$). P is called a *multilattice* if for each $a,b \in P$ the following hold:

(1) the set $a \sqcap b$ is non-empty, and if $u \leq a,b$ then there exists a $u' \in a_1 \sqcap b_1$ with $u \leq u'$,

(2) the set $a \sqcup b$ is non-empty, and if $a,b \leq u$ then there exists a $u' \in a_1 \sqcup b_1$ with $u' \leq u$.

The collection of all such u as described in (1) (resp. (2)) will be denoted by $(a \sqcap b)_u$ (resp. $(a \sqcap b)_u$). See the paper of Benado, [12]. If P (only) satisfies condition (1), then P is called a *semi-multilattice*. From parts (1) and (2) of Lemma 4.13 we conclude the following.

(4.21) Partial orders of type $(\leq_a, I(a,b))$ in modular space are multilattices. Moreover, $(x \sqcap y)_u = M(u,x,y)$ and $(x \sqcup y)_u = M(b,x,y)$ (for $u \leq_a x,y$).

In [12] an extended version of modularity and distributivity for multilattices is introduced. A multilattice L is *modular* provided the following holds. If $p \leq q \in L$ and $p \leq x$, $y \leq q$ are such that (cf. Figure 1.1B)

$$p \in x \sqcap y;\ q \in x \sqcup y,$$

and if y' is such that $p \leq y' \leq y$ (in particular, $p \in x \sqcap y'$) and $q \in x \sqcup y'$, then $y = y'$. A multilattice L is *distributive* provided the above conclusion holds with the hypothesis "$p \leq y' \leq y$" replaced by "$p \in x \sqcap y'$". See [12].

4.22 Proposition. *Let (X,I) be a modular space, then all multilattices of type $(\leq_a, I(a,b))$ are modular.*

Proof: Let $x,y,p,q \in I(a,b)$ satisfy

$$p \leq_a x, y \leq_a q\ ;\ M(x,y,p) = p\ ;\ M(x,y,q) = q,$$

and let y' be such that $p \leq_a y' \leq_a y$ and $M(x,y',q) = q$. We have $y' \in I(p,y)$ and hence $y \in I(y',q)$.

As $q \in I(x,y')$ we see that $y \in I(y',q) \subseteq I(y',x)$ But $y' \in I(p,y) \subseteq I(x,y)$, whence $y'=y$. ■

Somewhat confusingly, modular MULTIlattices need not be modular interval spaces (cf. Th. 4.25), even when they occur as an interval of a modular space.

4.23 Example. *A modular graph with an interval that is not a modular graph.*

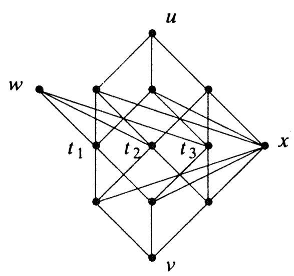

Fig. 4.23: non-modular interval in a modular graph.

Figure 4.23 indicates a modular graph G, in which the interval $uv = G \setminus \{w\}$ is not a modular graph (use the points t_i for i = 1, 2, 3).

It is a natural question whether the intervals of a modular space are distributive multilattices. The following result gives a characterization.

4.24 Theorem. *The following are equivalent for a modular space X:*

(1) Partial orders of type $(\leq_a, I(a,b))$ are distributive lattices.
(2) Partial orders of type $(\leq_a, I(a,b))$ are distributive multilattices.
(3) The "join" operation is associative, i.e. $I(a,I(b,c)) = I(I(a,b),c)$
(4) X does not contain $K_{2,3}$ as a multimedian stable subspace.
(5) All intervals of X are convex.
(6) X is median.

Proof: Let $\{u,v,x_1,x_2,x_3\}$ be a multimedian stable subset of X isomorphic with $K_{2,3}$ such that u, v correspond with the elements of $K_{2,3}$ of degree three (see Figure 4.3A). As $M(x_i,x_j,u)=u$ and $M(x_i,x_j,v)=v$ for distinct $1 \leq i,j \leq 3$, the multilattice $(uv, \leq_u)$ cannot be distributive. This proves implications $(1) \to (2) \to (4)$.

Let $x_1, x_2, x_3 \in X$ and $u \neq v \in M(x_1,x_2,x_2)$. If $x_1' \in M(x_1,u,v)$, then $x_1' \in uv$ and $u, v \in M(x_1',x_2,x_3)$. So, without loss of generality, $x_1,x_2,x_3 \in uv$. The points x_1,x_2,x_3 are evidently distinct and satisfy $M(x_i,x_j,u)=\{u\}$ and $M(x_i,x_j,v)=\{v\}$ for distinct $1 \leq i,j \leq 3$. It appears that $\{u,v,x_1,x_2,x_3\}$ is a multimedian-stable subspace of X isomorphic with $K_{2,3}$, establishing the implication $(4) \to (6)$.

[6] See the convention of p. 20.

Implication (6)$\to$(1) is well-known, for a direct proof: By (4.21) and Proposition 4.22 each partial order of type $(\leq_a, I(a,b))$ is a modular lattice. If such lattice is not distributive then it contains a sublattice of type $K_{2,3}$ —see Paragraph 1.1. Consult Figure 1.1A for notation. Now elementary basepoint order considerations yield that the multimedian $M(x_1,x_2,x_3)$ contains both u and v, a contradiction.

For a proof of implication (3)$\to$(4), suppose X includes a multimedian stable subspace isomorphic with $K_{2,3}$. Then (see Figure 4.3A) $x_1 \in I(u,v) \subseteq I(I(x_2,x_3),v)$. Now $M(x_1,x_2,x_3)$ does not contain x_1 (for otherwise $M(x_1,x_2,x_3) = \{x_1\}$), and as $I(x_2,I(x_3,v)) \subseteq I(x_2,I(x_3,x_2)) = I(x_3,x_2)$, we conclude that $x_1 \notin I(x_2,I(x_3,v))$.

Finally, implications (6)$\to$(3) and (6)$\to$(5) are well-known, whereas (5)$\to$(6) follows from Corollary 4.14. ■

Implication (5)$\to$(6) is a strengthing of [16, Theorem 2.23] and [62, Theorem 3.16] where this result is shown for metric —see II: §1— lattices and modular graphs (with their geodesic metric) respectively.

In regard of Example 4.23, one may wonder how the situation is with modular (proper) lattices. Here is a characterization.

4.25 Theorem. *The following are equivalent for a modular space.*

(1) All intervals are modular lattices.

(2) The space does not include a multimedian-stable subspace isomorphic with the modular graph $K_{3,3}$ minus an edge.

Proof: First, note that if a modular space X includes a $K_{3,3}$ minus an edge, then (see Figure 4.3B) the "top" and "bottom" points a, b span an interval of X which is properly a multilattice. Next, suppose that X has an interval ab which is not a lattice. Then there exist $u, v \in ab$ with distinct maximal lower bounds x, y in the basepoint order $\leq_a$. Note that none of u, v, x, y can be equal to either a or b and that $x, y \in M(u,v,a)$. If $u' \in M(x,y,b)$, then $u' \in ab$ and x, y are still maximal lower bounds of u', v. So, without loss of generality, we also have $u, v \in M(x,y,b)$. All conditions obtained so far remain valid if b is replaced with a point of $M(u,v,b)$ and if a is replaced with a point of $M(x,y,a)$. We now arrive at a multimedian-stable configuration

$$\{ a, b, u, v, x, y \}$$

in X, which apparently is a copy of $K_{3,3}$ minus an edge. ■

Benado [12] shows that modular multilattices satisfy a "Jordan-Hölder" type theorem. The same result then holds for all modular spaces by Proposition 4.22. We present a simple direct proof.

4.26 Theorem. *Let X be a modular space. If there exists a finite maximal chain joining two points a, $b \in X$, then all maximal chains joining a,b are of the same length* ([7]).

[7] The length is understood to be the number of ***steps*** in the chain, in other words: the number of elements minus one.

Proof: We verify by induction on $n \geq 1$ that if there is a maximal chain of length n between two points a, b of X, then all chains $a \mapsto b$ are finite and of length $\leq n$. For $n = 1$, the conclusion is evident. Assume the result to be valid for maximal chains of length $n \geq 1$. Let $\alpha : a \mapsto b$ be a maximal chain of length $n+1$, and consider a finite chain $\beta : a \mapsto b$. Let u, resp. v, be the last element $\neq b$ in α, resp. β. Note that u, b are neighbors and that the part of α from a to u is a maximal chain of length n. If $v \in au$ then by inductive assumption, β cannot be longer than $n+1$. We assume $v \notin au$, or, equivalently, $u \notin vb$. As $M(u,v,b)$ must contain (at least) one of u, b, we see that $b \in M(u,v,b)$. Take $x \in M(u,v,a)$. Application of Proposition 4.22 shows that x, v are neighbors. By induction, there is a maximal chain from a to u via x of length n. Composing its part up to x with the edge to v we obtain a maximal chain from a to v via x of length at most n. By induction, the part of β up to v is of length $\leq n$, and β itself is of length $\leq n+1$. ∎

CHAPTER II

MODULAR METRIC SPACES

Until now, we considered modular spaces from the viewpoint of general interval spaces. We now specialize to metric intervals of type

$$I_\rho(a,b)=\{x \mid \rho(a,x)+\rho(x,b)=\rho(a,b)\}$$

in a metric space (X,ρ). Modular metric spaces are related with spaces having the (3,2) Intersection Property of balls. It was an open problem of Aronszaijn and Panitchpakdi in [4], whether completions of such spaces still have the (3,2) Intersection Property of balls.

We show that metric multimedians are non-expansive multifunctions, and we use this result to settle the previous problem in the affirmative. It turns out that such completions arise by a procedure similar to that occurring in the proof of the Carathéodory extension theorem of measures to σ-algebras. See sections 1,2, and 3.

In sections 4 and 5 we show that metric completeness of modular metric space can be expressed in terms of "weak" compactness of order-bounded subsets.

The main result of section 6 is that ***all*** median operators are "metric" if we allow metrics with values in Riesz spaces, and that many of the results and techniques on metric medians extend to the general situation. ([1])

§ 1 Examples; connections with (3,2)-IP of balls

We first present some examples of modular metric spaces: finite modular spaces, $L_1(\mu)$ spaces, $K_1(\mu)$ spaces, and more generally: metric lattices.

1.1 Finite modular spaces. Let (X,I) be a geometric interval space. Then two points $a,b\in X$ are *neighbors* if $ab=\{a,b\}$. We say that (X,I) is *discrete* provided bounded chains in basepoint orders are finite. In this situation the neighbor relation induces a graphical structure with the set X as vertices. Moreover, by the geometric properties of the interval operator, each step in a maximal chain $a \mapsto b$ in the basepoint order of a represents an edge. Whence the induced graph is connected. If the interval operator, resulting from the geodesic metric, coincides with the given operator of X, then X is called *graphic.*

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[1] Sections 1, 2 and 4 as well as the first half of section 3 (until paragraph 3.8) are taken from [85]. The results of the second half of section 3 and the whole of section 5 were obtained by M. van de Vel and the author. See [83].

1.2 Lemma. *If (X,I) is a discrete modular space, then each geodesic from $a \in X$ to $b \in X$ is increasing in $(I(a,b), \leq_a)$.*

Proof: The result is evident for geodesics of length one. Let $n \geq 1$, and assume that the result holds for all geodesics of length at most n. Let $a = a_0, \cdots, a_{n+1} = b$ be a geodesic of length $n+1$. By the induction hypotheses the geodesic $a = a_0, \cdots, a_n$ is increasing in $(I(a,a_n), \leq_a)$. As the points a_n, b are neighbors, the value $M(a,a_n,b)$ of the multimedian must be equal to either a_n or b. If the first situation occurs then $a_n \leq_a b$ and we are done. So assume the last situation occurs, i.e. $b \in I(a,a_n)$. By virtue of Theorem I: 4.26 any maximal chain in $(I(a,b), \leq_a)$ is of length n. But this contradicts the assumption that $a_0, \cdots, a_n$ is a geodesic. ∎

1.3 Theorem. (cf. [46]) *Any discrete modular space is graphic.*

Proof: From Lemma 1.2 we conclude that all points of X can be connected with finite paths and that the I-intervals are contained in the geodesic intervals I_g. By Theorem I: 4.5 we conclude that $I = I_g$. ∎

1.4 Metric lattices. A *valuation* on a lattice L is a function $v : L \to \mathbb{R}$ which satisfies:

$$v(x \vee y) + v(x \wedge y) = v(x) + v(y) \ (x, y \in L).$$

If v satisfies, $x < y$ implies $v(x) < v(y)$ $(x, y \in L)$, then v is called *positive* and the pair (L, v) is called a *metric lattice.* If v satisfies, $x \leq y$ implies $v(x) \leq v(y)$ $(x, y \in L)$, then v is called *isotone* and the pair (L, v) is called a *pseudo-metric lattice.* Valuations on lattices were first introduced by Glivenko [30], [31]. See also Birkhoff [13, chapter 10], or Blumenthal & Menger [16, chapter 2], where valuations are assumed to take non-negative values only. However, this restriction plays no major role in their work. In section 6 we allow valuations to have values in any Abelian lattice group.

We remark that valuations on Riesz spaces in the sense of Schaefer in [71] are not valuations in our sense —however the restrictions of such valuations to the positive cone are.

The name "metric" lattice is explained by the following formula which determines a (pseudo-)metric on a (pseudo-)metric lattice L:

$$\rho(x,y) = v(x \vee y) - v(x \wedge y) \ (x, y \in L),$$

see [13]. Here are some straightforward examples of metric lattices, taken from [16].

(1.4.2) Let $L = \mathbb{N}$, ordered as follows $a \leq b$ iff a divides b. Note that $a \wedge b$ and $a \vee b$ are the greatest common divisor, respectively the smallest common multiple of a, b. Take

$$v(a) = log(a).$$

(1.4.3) Let S be the collection of finite dimensional subspaces of some vector space V, and let

$$v(M) = dim(M),$$

for $M \in S$.

(1.4.4) Let (A, μ) be a Boolean algebra with a (finitely-additive) measure. Note that $K_1(A, \mu)$ inherits the (distributive) lattice structure of A. The following yields a positive valuation.

$v(\bar{a}) = \mu(a)$,

where $\bar{a}$ denotes the class of $a \in A$.

(1.4.1) Let C^+ be the positive cone of an $L_1(\mu)$ space and let v be (the restriction of) the norm,

$v(x) = ||x||$.

A norm $||.||$ on a Riesz space X is called a *Riesz norm* provided it is compatible with the Riesz modulus; that is, $|x| \le |y|$ implies $||x|| \le ||y||$ for all $x,y \in X$. In this situation the triple $(X, \le, ||.||)$ is called a *normed Riesz space,* if — in addition— $||.||$ is complete, then this triple is called a *Banach lattice.* Observe that $||\,x\,|| = ||\,|x|\,||$ for all points x in a normed Riesz space. An *L-space* is a normed Riesz space, such that $||x+y|| = ||x|| + ||y||$ for all positive x,y in X. See [71]. The most prominent example of L-spaces are the Banach lattices of type $L_1(\mu)$. In fact, the famous Kakutani representation theorem ([44], [49]) states that each complete L-space is linearly isometric with an $L_1(\mu)$ space. We mention that the subspaces of $L_1([0,1])$ consisting of *Riemann integrable functions* and of the *essentially bounded functions* also yield L-spaces. See [69].

The following example of metric lattices does not seem to be widely known.

1.5 Proposition. *Any L-space X equipped with*

$v(x) = ||x^+|| - ||x^-||$

yields a metric lattice. Moreover, the metric induced by v equals that induced by $||.||$.

Proof: We first show that v is *additive,* i.e. $v(x+y) = v(x) + v(y)$ for all $x,y \in X$. First, for positive points in X additivity holds by assumption. Next, for general points x,y in X use the equality

$(x+y)^+ + x^- + y^- = (x+y)^- + x^+ + y^+,$

and the first result. We now obtain the following equalities:

$v(x) + v(y) = v(x+y) = v((x \wedge y) + (x \vee y)) = v(x \wedge y) + v(x \vee y).$

That is, v is a valuation. To show that v is positive, consider points $x < y$. Then $y = x + (y-x)$ and as $y - x > 0$ we conclude $v(y) = v(x) + v(y-x) > v(x)$

The proof of the last statement follows from the additivity of v, and the formula $x \vee y - x \wedge y = |x-y|$, which is valid for points x,y in any Riesz space. ∎

Observe that if X is a space of type $L_1(X, \mathcal{A}, \mu)$, then the valuation v, as described in the previous proposition, is given by $v([f]) = \int_X f d\mu$ $([f] \in L_1(X, \mathcal{A}, \mu))$.

1.6 Lemma. *A metric lattice L is modular, and metric betweenness in L is equivalent with geometric betweenness.*

Proof: The first statement appears in [13, p. 232], and [16, p. 58], and the second statement is taken from [30]. ∎

In section 6 (Lemma 6.1) we shall prove a generalization of Lemma 1.6. From Lemma 1.6 together with Theorem I: 4.2 we obtain:

1.7 Theorem. *A metric lattice L is a modular metric space. The lattice L is a median metric space if and only if L is distributive.* ∎

We now obtain two "concrete" examples of median metric spaces, which shall turn out to be universal in the sense that all median metric spaces can be isometrically embedded in them (see chapter V).

We recall from paragraph I: 2.16, that a product space of type $\mathbb{R}^X$ can be equipped with the product median m_π derived from $\mathbb{R}$.

1.8 Theorem. *Any L-space is a median normed space. More specifically, if $(N,\mathcal{A},\mu)$ is a measure space then the metric median of the Banach L-space $L_1(N,\mathcal{A},\mu)$ is given by*

$m'([f],[g],[h]) = [m_\pi(f,g,h)]$ *for all* $[f],[g],[h] \in L_1(N,\mathcal{A},\mu)$. ∎

1.9 Theorem. *Let $(N,\mathcal{A},\mu)$ be a measure space. Then the space $K_1(N,\mathcal{A},\mu)$ is a complete median stable subset of $L_1(N,\mathcal{A},\mu)$.* ∎

1.10 Sharp radii. Let (X,ρ) be a metric space, and let $(x_1,x_2,x_3) \in X^3$. Using elementary linear algebra one can see there is a unique triple (r_1^s,r_2^s,r_3^s) in $\mathbb{R}^3$, such that

$$\rho(x_i,x_j) = r_i^s + r_j^s \text{ for } i \neq j \in \{1,2,3\}. \tag{1.10.1}$$

In fact we have

$$\begin{aligned} r_1^s &= \tfrac{1}{2}(\rho(x_1,x_2)+\rho(x_1,x_3)-\rho(x_2,x_3)) \\ r_2^s &= \tfrac{1}{2}(\rho(x_1,x_2)-\rho(x_1,x_3)+\rho(x_2,x_3)) \\ r_3^s &= \tfrac{1}{2}(-\rho(x_1,x_2)+\rho(x_1,x_3)+\rho(x_2,x_3)). \end{aligned} \tag{1.10.2}$$

We call them the *sharp radii* corresponding to (x_1,x_2,x_3). This enables us to define a function, $(r_1^s,r_2^s,r_3^s): X^3 \to \mathbb{R}^3$. By the triangle inequality of ρ the numbers r_i^s $(i=1,2,3)$ are non-negative.

1.11 Proposition. *Let (X,ρ) be a metric space with the metric mixing operator M. Then the following are equivalent for $x_1,x_2,x_3,m \in X$.*

(1) $m \in M(x_1,x_2,x_3)$.

(2) $\rho(x_i,m) = r_i^s$ *for* $i=1,2,3$.

(3) $\rho(x_i,m) \leq r_i^s$ *for* $i=1,2,3$.

(4) $\rho(x_1,m)+\rho(x_2,m)+\rho(x_3,m) = \tfrac{1}{2}(\rho(x_1,x_2)+\rho(x_1,x_3)+\rho(x_2,x_3))$.

In particular, the distance of x_i $(i=1,2,3)$ to a member m of $M(x_1,x_2,x_3)$ is independent of the choice of m.

Proof: First, let $m \in M(x_1,x_2,x_3)$. Then taking $r_i = \rho(x_i,m)$ yields a solution of (1.10.1). Hence implication (1) $\to$ (2) follows. Implication (2) $\to$ (3) is evident. Next, by the triangle inequality of ρ any m as described in (3) actually satisfies $\rho(x_i,m) = r_i^s$ $(i=1,2,3)$. By invoking the definition of sharp radii we obtain implication (3) $\to$ (4). Finally, as the left-hand side of (4) equals

$$\tfrac{1}{2}\big(\rho(x_1,m)+\rho(x_2,m)+\rho(x_1,m)+\rho(x_3,m)+\rho(x_2,m)+\rho(x_3,m)\big),$$

implication (4) $\to$ (1) can be deduced by use of the triangle inequality of ρ. ∎

A simple application of the triangle inequality shows that for arbitrary m the left-hand side of equality 1.11(4) is minorized by the right-hand side. From this observation we deduce the following:

1.12 Corollary. *With the notation of Proposition 1.11: If $M(x_1,x_2,x_3)$ is non-empty, then this set consists of the collection of Steiner points of the triple x_1,x_2,x_3* ■

It follows from Proposition 1.11(4) that for points a,b,c in a modular graph G with geodesic metric ρ the number $\rho(a,b)+\rho(a,c)+\rho(b,c)$ must be even, that is G is bipartite.

In contrast with Corollary 1.12, not each Steiner point of the points x_1,x_2,x_3 needs to be a member of the mixing operator $M(x_1,x_2,x_3)$. Indeed, consider any inner product space X of dimension ≥ 2 and let x_1,x_2,x_3 be some affinely independent points in X. Then these points have a Steiner point (see Chapter I), but the mixing operator of these points is empty (recall that the metric intervals of X coincide with the standard intervals). In this light, the following result due to Avann [6] is somewhat surprising:

If for all points a,b,c in a graph G with its geodesic metric ρ, there is a unique point that minimalizes the expression $\rho(a,x)+\rho(b,x)+\rho(c,x)$ $(x \in G)$, then G is a median graph.

In the proof of Proposition 1.11 we encountered the following principle, derived from the triangle inequality of a metric. If $x \in D(x_1,r_1) \cap D(x_2,r_2)$ and $r_1+r_2=\rho(a,b)$ then $\rho(x,x_i)=r_i$ $(i=1,2)$. We shall use this principle without further reference.

We now introduce a well-known intersection property of balls. By abuse of language we say that a metric space (X,ρ) has the (*,2)-IP (where * denotes a cardinal number, "F" or "A") if and only if ρ is convex and if the collection of closed balls has the (*,2)-IP. We recall that a metric ρ is *convex* if for all $x,y \in X$ and $0 \leq t \leq 1$ there exists a $z \in X$ with $\rho(x,z)=t \cdot \rho(x,y)$ and $\rho(y,z)=(1-t)\cdot\rho(x,y)$.

Metric spaces with the $(\aleph,2)$-IP property were introduced by Aronszaijn and Panitchpakdi in [4], who use the name "hyperconvex metrics". See also the work of Isbell [40], [41], who uses the name "injective metrics". The term "(n,2)-IP" was introduced by Lindenstrauss for normed spaces in [50].

The following is a different description of the $(\aleph,2)$-IP.

(1.13) Let $\aleph$ be a cardinal number. A metric space (X,ρ) has the $(\aleph,2)$-IP iff for every collection of closed balls $\{D(x_i,r_i)\}_{i\leq\aleph}$ in X with $\rho(x_i,x_j)\leq r_i+r_j$ $(i,j\leq\aleph)$ we have

$$\bigcap_{i=1}^{n} D(x_i,r_i)\neq\varnothing.$$

1.14 Examples. Let (X,τ) be any metric space. A continuous function $f:X\to\mathbb{R}$ is called *bounded* provided $sup_{x\in X} f(x)<\infty$. The set consisting of all bounded functions $X\to\mathbb{R}$, denoted by $B_\tau(X)$, is a normed space with norm $||f||=sup_{x\in X} f(x)$. If we endow the set I with the discrete topology then the corresponding space of bounded functions shall be denoted by simply $B(I)$. One can easily verify that spaces of type $B_\tau(X)$ have the (F,2)-IP, compare the proof of [4, Theorem 1, p. 431] where this is shown for compact X.

Let X be a metric space with the (F,2)-IP, and let $IF(X)$ be the collection of all non-empty intersections of finitely many balls in X. It follows from the argument used by Sine in [75, Theorem 15], that $IF(X)$ endowed with the Hausdorff metric (see section 2) also has the (F,2)-IP.

From Proposition 1.11 we conclude that the value $M(x_1,x_2,x_3)$ of a metric mixing operator is the intersection of the closed balls $D(x_i,r_i^s)$. Hence, metric spaces with the (3,2)-IP, e.g. spaces of type $B_\tau(X)$, are modular. We show a converse of this result.

1.15 Lemma. *Let* (X,ρ) *be a metric space. Let* $x_1,x_2,x_3 \in X$, *and* $r_1,r_2,r_3 \geq 0$. *If for any* $i \neq j$ *in* $\{1,2,3\}$

$$\rho(x_i,x_j) \leq r_i + r_j,$$

then each r_i *can be replaced by a number* $\bar{r}_i$ *such that* $0 \leq \bar{r}_i \leq r_i$ *and at least two of the three inequalities become equalities.*

Proof: Take

$$\bar{r}_1 = max(\rho(x_1,x_2) - r_2, \rho(x_1,x_3) - r_3, 0)$$
$$\bar{r}_2 = max(\rho(x_1,x_2) - \bar{r}_1, \rho(x_2,x_3) - r_3, 0)$$
$$\bar{r}_3 = max(\rho(x_1,x_3) - \bar{r}_1, \rho(x_2,x_3) - \bar{r}_2, 0).$$

It is easy to see that the $\bar{r}_i$ are non-negative, and a case study shows that the $\bar{r}_i$ are as desired. ∎

There is simpler proof of the previous lemma using a "continuity" argument. However the appearing formulae shall be of later use.

The next lemma roughly states that in modular metric spaces the intersection of three balls can be replaced by an intersection of two balls. This lemma was partially inspired by the proof of [48, Theorem 3.2].

1.16 Lemma. *Let* (X,ρ) *be a modular space. If* $x_1,x_2,x_3 \in X$ *and* $r_1,r_2,r_3 \geq 0$ *are such that*

$$\rho(x_1,x_2) = r_1 + r_2$$
$$\rho(x_1,x_3) = r_1 + r_3$$
$$\rho(x_2,x_3) \leq r_2 + r_3$$

and if $r_i^s = r_i^s(x_1,x_2,x_3)$ $(i = 1,2,3)$, *then* $r_1^s \geq r_1$ *and*

$$D(x_1,r_1) \cap D(x_2,r_2) \cap D(x_3,r_3) = \bigcup_{m \in M(x_1,x_2,x_3)} D(x_1,r_1) \cap D(m, r_1^s - r_1).$$

Proof: By invoking the definition of sharp radii we obtain the following (in)equalities

$$r_i^s = \tfrac{1}{2}(\rho(x_1,x_2) + \rho(x_1,x_3) - \rho(x_2,x_3)) \geq \tfrac{1}{2}(r_1 + r_2 + r_1 + r_3 - (r_2 + r_3)) = r_1.$$

First we shall prove the inclusion from right to left. To this end, take $m \in M_X(x_1,x_2,x_3)$. By Proposition 1.11 we have $\rho(x_1,m) = r_1^s$. Now let $w \in D(x_1,r_1) \cap D(m, r_1^s - r_1)$. Then $\rho(x_1,w) = r_1$ and $\rho(w,m) = r_1^s - r_1 = \rho(x_1,m) - r_1$. Hence we obtain

$$\rho(x_2,w) \leq \rho(x_2,m) + \rho(m,w) = \rho(x_2,m) + \rho(x_1,m) - r_1 = \rho(x_1,x_2) - r_1 = r_2.$$

One similarly shows that $\rho(x_3,w) \leq r_3$, establishing the inclusion from right to left of the theorem.

As for a proof of the reverse inclusion, let $w \in D(x_1,r_1) \cap D(x_2,r_2) \cap D(x_3,r_3)$. We have the following sequence of equalities.

$$\begin{aligned} M(x_1,x_2,x_3) \cap D(w,r_1^s - r_1) &= D(x_1,r_1^s) \cap D(x_2,r_2^s) \cap D(x_3,r_3^s) \cap D(w,r_1^s - r_1) \\ &= D(x_1,r_1^s) \cap M(x_2,x_3,w) \\ &= M(x_2,x_3,w) \neq \varnothing. \end{aligned} \tag{1}$$

The first equality in formula (1) only invokes Proposition 1.11. As for the second equality in (1), we shall first verify the following formulae:

$$\begin{aligned} \rho(x_2,x_3) &= r_2^s + r_3^s \\ \rho(x_2,w) &= r_2^s + r_1^s - r_1 \\ \rho(x_3,w) &= r_3^s + r_1^s - r_1. \end{aligned} \tag{2}$$

The first equation of (2) follows by definition of sharp radii. As for the second equation of (2), it is clear that $r_2^s + r_1^s = \rho(x_1,x_2) = r_1 + r_2$ and hence $r_2 = r_2^s + r_1^s - r_1$. Now as $\rho(x_2,w) = r_2$ the second equality of (2) is clear. One similarly verifies the third equality of (2). By applying Proposition 1.11 we obtain:

$$D(x_2,r_2^s) \cap D(x_3,r_3^s) \cap D(w,r_1^s - r_1) = M(x_2,x_3,w).$$

For a proof of the third equality in (1); it is easy to see that $w \in I_\rho(x_1,x_2) \cap I_\rho(x_1,x_3)$. By the geometric property of metric intervals we obtain:

$$\begin{aligned} M(x_2,x_3,w) &= I_\rho(x_2,w) \cap I_\rho(x_3,w) \cap I_\rho(x_2,x_3) \\ &\subseteq I_\rho(x_2,x_1) \cap I_\rho(x_3,x_1) \cap I_\rho(x_2,x_3) = M(x_1,x_2,x_3). \end{aligned}$$

Now as $M(x_1,x_2,x_3) \subseteq D(x_1,r_1^s)$, the third equality of (1) is proven.

After taking $m \in M(x_1,x_2,x_3) \cap D(w,r_1^s - r_1)$ we find that $w \in D(x_1,r_1) \cap D(m,r_1^s - r_1)$, establishing the inclusion from left to right of the lemma. ■

The following theorem shows that with respect to the (3,2)-IP, the multimedian is obtained by a crucial intersection of three balls.

1.17 Theorem. *The following are equivalent for a metric space (X, ρ).*

(1) *(X, ρ) is a modular metric space and ρ is a convex metric.*

(2) *(X, ρ) has the (3,2)-IP.*

Proof: Implication $(2) \to (1)$ has been observed earlier. For a proof of implication $(1) \to (2)$, take any $x_1,x_2,x_3 \in X$ and $r_1,r_2,r_3 \geq 0$ satisfying

$$\rho(x_i,x_j) \leq r_i + r_j \quad \forall \ 1 \leq i,j \leq 3.$$

By Lemma 1.15 we may assume

$$\begin{aligned} \rho(x_1,x_2) &= r_1 + r_2 \\ \rho(x_1,x_3) &= r_1 + r_3 \\ \rho(x_2,x_3) &\leq r_2 + r_3. \end{aligned}$$

Let r_i^s be the sharp radii. By Lemma 1.16 $r_1^s \geq r_1$. By assumption $M(x_1,x_2,x_3)$ is non-empty.

Let $m \in M(x_1,x_2,x_3)$. As (X,ρ) is metrically convex we have

$$D(x_1,r_1) \cap D(m,r_1^s - r_1) \neq \emptyset$$

and by the previous lemma

$$D(x_1,r_1) \cap D(x_2,r_2) \cap D(x_3,r_3) \neq \emptyset. \qquad \blacksquare$$

See Theorem 2.14, for a description of modular metric spaces with the (3,2)-IP in terms of connectedness.

1.18 Remarks. It is essential in (1) above that the metric be convex. However, it is possible to weaken the definition of a "convex" metric, and the "(n,2)-IP" by replacing the non-negative real numbers by the non-negative part of any (additive) subgroup of $\mathbb{R}$, e.g. $\mathbb{N} \cup \{0\}$ or $\mathbb{Q}_0^+$. In this way we also obtain a more general version of "adapted" metric. Certain parts of the theory (in particular Lemma 1.15 and Theorem 1.17) are still valid in the adapted setting. With such modifications, the theory applies to connected graphs (with its geodesic metric) as well.

As a corollary to Theorem 1.17 we obtain that modular normed spaces have the (3,2)-IP. This result was first proved by Å. Lima in [48]. In the works of Hanner, [36], it is shown that, modulo linear isometrics, there are but finitely many norms on $\mathbb{R}^n$ ($n \in \mathbb{N}$) with the (3,2)-IP. Whence, the same holds for modular norms on $\mathbb{R}^n$. As spaces of type $L_1(\mu)$ are median (Theorem 1.8) they have the (3,2)-IP. The following result was first proven by Lindenstrauss in [51, p. 491]. We give a more direct proof.

1.19 Corollary. *Let X be an $L_1(\mu)$ space. Then,*

X has the (4,2)-IP iff $\dim(X) \leq 2$.

Proof: For a proof of the implication from left to right, we will show that an $L_1(\mu)$ space X of dimension greater than 2 cannot have the (4,2)-IP. We assume that $X = L_1(Y,\mathcal{A},\mu)$. It is left to the reader to ascertain that three linearly independent measurable functions in Y give rise to three pairwise disjoint measurable sets U_1,U_2,U_3 of positive measure in the basic measure space. Let X_{U_i} be the characteristic function of U_i $(i=1,2,3)$, and consider the following radii $r_i \in \mathbb{R}$ and points a_i of X:

$$a_i = \frac{1}{\mu(U_i)} \cdot [X_{U_i}] \quad (i=1,2,3) \ ; \ a_4 = a_1 + a_2 + a_3 \ \textit{and} \ r_i = 1 \quad (i=1,2,3,4)$$

Whence, $\| a_i - a_j \| = 2$ for all $i \neq j$. Now consider the following equalities

$$0 = M(a_1,a_2,a_2) = D(a_1,r_1) \cap D(a_2,r_2) \cap D(a_3,r_3),$$

$$a_1 + a_2 = M(a_2,a_3,a_4) = D(a_2,r_2) \cap D(a_3,r_3) \cap D(a_4,r_4).$$

The first and third equality follow from Theorem 1.8, whereas the second and fourth equality follow from Proposition 1.11. Therefore we must have

$$D(a_1,r_1) \cap D(a_2,r_2) \cap D(a_3,r_3) \cap D(a_4,r_4) = \emptyset.$$

The implication from right to left is evident. $\blacksquare$

1.20 Pointed products. If $\{(X_i,\rho_i)\}_{i=1}^n$ is a finite collection of metric spaces then there are several metrics on the Cartesian product $\prod_{i=1}^n X_i$. We are particularly interested in the "sum" metric on this product, given by

$$\rho((x_i)_{i=1}^n,(y_i)_{i=1}^n)=\sum_{k=1}^n \rho_i(x_i,y_i).$$

A generalization to products of arbitrarily many metric spaces goes as follows. Let $(X_i,\rho_i)_{i\in I}$ be a collection of metric spaces and let $b=(b_i)_{i\in I}\in\prod_{i\in I}X_i$. The *pointed product* $l_1^b(X_i \mid i\in I)$ at b, is the set of $(x_i)_{i\in I}$ with $x_i\in X_i$ such that

(1) $x_i\neq b_i$ for at most countably many $i\in I$ (say $i_1,i_2,\cdots$).

(2) $\sum_{k=1}^{\infty}\rho_{i_k}(x_{i_k},b_{i_k})<\infty$.

On a pointed product we take the following ("sum") metric ρ. If $(x_i)_{i\in I},(y_i)_{i\in I}\in l_1^b(X_i \mid i\in I)$, then there are only countable many elements of I such that $x_i\neq b_i$ or $y_i\neq b_i$. Enumerate them as $i_1,i_2,\cdots$, and define

$$\rho((x_i)_{i\in I},(y_i)_{i\in I})=\sum_{k=1}^{\infty}\rho_{i_k}(x_{i_k},y_{i_k}).$$

A straightforward calculation shows that ρ indeed is a metric. If all X_i are normed spaces then one usually takes $b_i=0$ for all $i\in I$. If all X_i equal $\mathbb{R}$ then the pointed product at 0 equals the well-known space $l_1(I)$.

A pointed product $l_1^b(X_i \mid i\in I)$ is a convex subset of the product space $(\prod_{i\in I}X_i,I_\pi)$, and the metric interval operator I_{ρ_s} equals the relative interval operator. In particular, the metric mixing operator of a pointed product $l_1^b(X_i \mid i\in I)$ equals the relative mixing operator M_π. Whence, a pointed product is a modular metric space provided all factor spaces are.

1.21 Lemma. *Let $l_1^b(X_i \mid i\in I)$ be a pointed product with metric ρ. Then ρ is a convex metric if and only if every ρ_i is.*

Proof: We shall only show the "if" part as the other part is obvious. Let $x=(x_i)_{i\in I}$, $y=(y_i)_{i\in I}\in l_1^b(X_i \mid i\in I)$, and let $0\leq t\leq 1$. For each $i\in I$ we can find a point $z=z_i\in X_i$, such that

$$\rho_i(x_i,z_i)=t\cdot\rho(x_i,y_i) \ : \ \rho_i(y_i,z_i)=(1-t)\cdot\rho(x_i,y_i).$$

This yields a point $z=(z_i)_{i\in I}$ in the pointed product with $\rho(x,z)=t\cdot\rho(x,y)$ and $\rho(y,z)=(1-t)\cdot\rho(x,y)$, as desired. ■

As a consequence of Theorem 1.17 and Lemma 1.21 we arrive at the following result, which was obtained by Lindenstrauss [50] in case all factors are Banach spaces.

1.22 Theorem. *Let (X,ρ) be the pointed product of $(X_i,\rho_i)_{i\in I}$. Then, (X,ρ) has the (3,2)-IP if and only if every (X,ρ_i) has the (3,2)-IP.* ■

§ 2 Calculus in modular metric spaces

2.1 The (n,2)-IP versus the Hausdorff metric. We recall some well-known notions. Let (X,ρ) be a metric space and let

$$2^X_{bc}=\{\varnothing\neq A\subseteq X \mid A \textit{ closed and bounded }\}.$$

Clearly, if (X,ρ) is a modular space then the multimedian takes its values in 2^X_{bc}. The "distance" between two sets $A,B\in 2^X_b$, is given by $inf\{\rho(a,b)\mid a\in A\ \ b\in B\}$. Obviously this does not yield a genuine (pseudo)metric on 2^X_{bc}. The *Hausdorff distance* between $A,B\in 2^X_{bc}$ is given by

$$\rho_H(A,B)=max\,(\sup_{a\in A}\rho(a,B),\sup_{b\in B}\rho(b,A)).$$

We also mention the *Pompéiu distance* between A,B — cf. [67]— which is given by

$$\rho_P(A,B)=½(\sup_{a\in A}\rho(a,B)+\sup_{b\in B}\rho(b,A)).$$

One can easily verify that ρ_H and ρ_P yield metrics on 2^X_{bc}, the *Hausdorff metric,* and the *Pompéiu metric* respectively. Observe that for $A\in 2^X_{bc}$ and $x\in X$ we have

$$\rho_H(\{x\},A)=\sup_{a\in A}\rho(x,A).$$

We usually drop the singleton's braces. Hence, if $M(a,b,c)$ is a value of a metric multimedian then (Proposition 1.11):

$$\rho_H(a,M(a,b,c))=\rho(a,M(a,b,c))=½(\rho(a,b)+\rho(a,c)-\rho(b,c)).$$

We shall use this without further reference.

Unless stated otherwise we endow 2^X_{bc} with the Hausdorff metric. We have introduced the Pompéiu metric —which is not commonly used— as it turns out to be a useful tool. From the following inequalities it follows that ρ_H and ρ_P are equivalent.

(2.2) If $A,B\in 2^X_{bc}$ then $\rho(A,B)\leq\rho_P(A,B)\leq\rho_H(A,B)\leq 2\cdot\rho_P(A,B)$.

We mention the following problem which plays an important role in the paper [4] of Aronszaijn and Panitchpakdi. Consider a non-empty intersection of a finite collection of closed balls in a metric space, and suppose that we vary the involved radii and points a "little" —such that the new intersection remains non-empty. Then what can we say about the Hausdorff metric between these intersections? The following result —which is shown by a straightforward verification— gives a partial answer.

2.3 Proposition. *Let $\cap_{i=1}^{n-1}D(x_i,r_i)$ and $\cap_{i=1}^{n-1}D(x'_i,r'_i)$ be non-empty intersections in a metric space with the $(n,2)$-IP. Then,*

$$\rho_H(\cap_{i=1}^{n-1}D(x_i,r_i)\,,\,\cap_{i=1}^{n-1}D(x'_i,r'_i))\leq\max_{1\leq i\leq n-1}(\,|r_i-r'_i|+\rho(x_i,x'_i)\,).$$ ■

No similar result is known when intersecting n balls in a metric space with the $(n, 2)$-IP. From the previous result we infer that the multimedian of a metric space with the (4,2)-IP is a Lipschitz map of factor 2 with respect to the sum-metric on X^3. We shall show below that ***all*** multimedians are such Lipschitz mappings, but the proof is more elaborate.

2.4 Contractivity of multimedians.

2.5 Lemma. *Let (X,ρ) be a modular space and let $a,b,c,d \in X$. Then,*

(1) $\rho_H(c, M(abc,d,c)) = \rho_H(c, M(a,bcd,c))$.

(2) $\rho_H(c, M(abc,d,c)) = \frac{1}{2}(\rho(c,d) + \rho(c,abc) - \rho(d,abc))$.

(3) $\rho(c,d) + \rho(c,abc) - \rho(d,abc) = \rho(c,a) + \rho(c,bcd) - \rho(a,bcd)$.

(4) $\frac{1}{2}(\rho(c,a) + \rho(c,b) + \rho(b,a)) + \rho(d,abc) = \frac{1}{2}(\rho(c,b) + \rho(c,d) + \rho(d,b)) + \rho(a,bcd)$.

Proof: For the proof of formula (1), take any $x \in M(abc,d,c)$. By Theorem I: 4.16, there exists a $y \in M(a,bcd,c)$ such that $x \in I(c,y)$. Hence $\rho(c,x) \le \rho(c,y)$. As we took $x \in M(abc,d,c)$ arbitrarily we obtain that the left-hand side of (1) is less than or equal to the right-hand side. By permuting the roles of a and d we obtain the other inequality.
For a proof of the formulae (2),(3) and (4), take any $x \in M(abc,d,c)$. There is a $z \in abc$ such that $x \in zdc$, hence by Proposition 1.11

$$\rho(c,x) = \tfrac{1}{2}(\rho(c,z) + \rho(c,d) - \rho(d,z)) = \tfrac{1}{2}(\rho(c,abc) + \rho(c,d) - \rho(d,z)).$$

(2) easily follows from this. Combining formulae (1) and (2) yields formula (3). For a proof of (4), by Proposition 1.11(2) we have $\rho(c,abc) = \frac{1}{2}(\rho(c,a) + \rho(c,b) - \rho(a,b))$, and a similar equality holds for $\rho(c,bcd)$. Substituting this in equality (3) yields

$$\rho(c,d) + \tfrac{1}{2}(\rho(c,a) + \rho(c,b) - \rho(a,b)) - \rho(d,abc) = \rho(c,a) + \tfrac{1}{2}(\rho(c,b) + \rho(c,d) - \rho(b,d)) - \rho(a,bcd).$$

By subtracting $\rho(c,d) + \rho(c,a) + \rho(c,b)$ on both sides and dropping all minus signs we obtain formula (4). ■

2.6 Theorem. *If a,b,c,d,x are points in a modular metric space (X,ρ), then*

(1) $\rho(a,abc) + \rho(b,abc) + \rho(c,abc) + \rho(d,abc) = \rho(a,bcd) + \rho(b,bcd) + \rho(c,bcd) + \rho(d,bcd)$.

(2) $\rho(a,abc) + \rho(b,abc) + \rho(c,abc) + \rho(d,abc) \le \rho(a,x) + \rho(b,x) + \rho(c,x) + \rho(d,x)$.

Proof: Part (1) follows from Proposition 1.11(4) combined with Lemma 2.5(4).
For a proof of the second part of the corollary, let $x \in X$. Without loss of generality we may assume that $x \in I_\rho(a,b)$. Indeed, take $x' \in abx$ then

$$\begin{aligned}\rho(a,x') + \rho(b,x') + \rho(c,x') + \rho(d,x') &= \rho(a,x) - \rho(x,x') + \rho(b,x) - \rho(x,x') + \rho(c,x') + \rho(d,x') \\ &= \rho(a,x) + \rho(b,x) + \rho(c,x') - \rho(x,x') + \rho(d,x') - \rho(x,x') \\ &\le \rho(a,x) + \rho(b,x) + \rho(c,x) + \rho(d,x).\end{aligned}$$

The inequality is the triangle inequality of ρ. Hence we may now conclude that

$$\begin{aligned}\rho(a,x) + \rho(b,x) + \rho(c,x) + \rho(d,x) &= \rho(a,abx) + \rho(b,abx) + \rho(c,abx) + \rho(x,abx) + \rho(d,x) \\ &= \rho(a,abc) + \rho(b,abc) + \rho(c,abc) + \rho(x,abc) + \rho(d,x) \\ &\ge \rho(a,abc) + \rho(b,abc) + \rho(c,abc) + \rho(d,abc)\end{aligned}$$

The second equality is the first part of the corollary and the inequality is the triangle inequality of ρ. ■

Observe that the second part of the previous corollary implies that connecting four points in median metric space with one of the (four) medians gives a minimal connecting network using one extra point.

Let (X,ρ) be a metric space and $A \subseteq X$. A point $x \in X$ is said to have a *metric nearest point* in A if there is a point $a \in A$ with $\rho(x,a)=\rho(x,A)$. The set A is said to admit metric nearest points if every $x \in X$ has a metric nearest point in A. The following states that values of multimedians consist of metric nearest points.

2.7 Proposition. *Let (X,ρ) be a modular space, and let $a,b \in X$. Then the interval $I(a,b)$ admits metric nearest points. Furthermore, if $x \in X$ then any member of $M(x,a,b)$ is a metric nearest point of x in $I(a,b)$.*

Proof: Let $x \in X$ and take any $z \in M(x,a,b)$. By construction of M, we have

$$\rho(x,z) = ½(\rho(x,a)+\rho(x,b)-\rho(a,b)).$$

Now let $c \in I(a,b)$, that is $\rho(a,b)=\rho(a,c)+\rho(c,b)$. Substituting this in the previous equation yields $\rho(x,z) = ½(\rho(x,a)-\rho(a,c)+\rho(x,b)-\rho(c,b))$. Using the triangle inequality twice gives $\rho(x,z) \le ½(\rho(x,c)+\rho(x,c)) = \rho(x,c)$. So z is a metric nearest point of x in $I(a,b)$. ■

The following result which states that values of the multimedian admit metric nearest points from certain "directions" shall be quite fruitful. See also Example 2.20.

2.8 Proposition. *Let (X,ρ) be a modular space and let $a,a',b,c \in X$. Then, compare Figure 2.8, any point d in $a'bc$ has a metric nearest point p in abc. Moreover, $p \in I(a,d)$.*

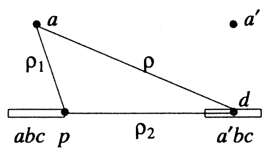

Fig. 2.8: $\rho_1+\rho_2=\rho$

Proof: The point d is contained in the interval $I(b,c)$ so by Lemma I: 4.13(3) there exists a $p \in abc$ such that $p \in I(a,d)$. To verify that p is a metric nearest point of d in abc let $x \in abc$. Then,

$$\rho(d,x) \ge \rho(d,a)-\rho(a,x) = \rho(d,a)-\rho(a,p) = \rho(d,p).$$

In which the inequality is triangle inequality, the first equality is Proposition 1.11 and the second equality expresses $p \in I(a,d)$. ■

As a consequence of Proposition 2.8 we have the following.

2.9 Proposition. *Let (X, ρ) be a modular space let $a, a', b, c \in X$ and $d \in a'bc$. Then the following hold (see Figure 2.8).*

(1) $\rho(d, abc) = \rho(a, d) - \rho(a, abc)$.

(2) $\rho(a'bc, abc) = \rho(a, a'bc) - \rho(a, abc)$

(3) $\sup_{x \in a'bc} \rho(x, abc) = \rho_H(a, a'bc) - \rho_H(a, abc)$. ∎

A map $F : (X, \rho) \to (Y, \rho')$ between (pseudo)metric spaces that satisfies

$$\rho'(F(x), F(y)) \leq M \cdot \rho(x, y)$$

for some fixed constant M and all $x, y \in X$ is called a Lipschitz map with Lipschitz factor M. If M can be taken 1 then F is called non-expansive.

The next theorem states that a metric multimedian is non-expansive with regard to the sum-metric on X^3 and the Pompéiu metric on 2^X_{bc}.

2.10 Theorem. *Let (X, ρ) be a modular space and let $a, a', b, b', c, c' \in X$, then*

$$\rho_P(abc, a'b'c') \leq \rho(a, a') + \rho(b, b') + \rho(c, c').$$

Proof: Consider the following (in)equalities:

$$\begin{aligned}\rho_P(abc, a'bc) &= \tfrac{1}{2}\Big(\sup_{d \in a'bc} \rho(d, abc) + \sup_{d \in abc} \rho(d, a'bc)\Big)\\ &= \tfrac{1}{2}(\rho_H(a, a'bc) - \rho_H(a, abc) + \rho_H(a', abc) - \rho_H(a', a'bc))\\ &\leq \tfrac{1}{2}(\rho_H(\{a\}, \{a'\}) + \rho_H(\{a\}, \{a'\})) = \rho(a, a').\end{aligned}$$

The second equality is Proposition 2.9(3) twice, and the inequality is the triangle inequality of ρ_H. The theorem now follows from the triangle inequality of ρ_P. ∎

The following two results are obtained from (2.2).

2.11 Theorem. *If abc and $a'b'c'$ are values of a metric multimedian, then*

$$\rho(abc, a'b'c') \leq \rho(a, a') + \rho(b, b') + \rho(c, c').$$

In particular, a metric median is contractive with respect to the sum metric on X^3. ∎

2.12 Theorem. *The multimedian of a modular metric space is a Lipschitz map with factor 2.* ∎

From the previous result we conclude that a multimedian is Lower Semi Continuous (LSC) (see for instance [22]). By the famous Michael selection theorem [57] we obtain that the multimedian of a modular Banach space admits a continuous selection— observe that in these circumstances the values of the multimedian are (standard) convex and complete.

Simple examples in the modular graph $K_{2,3}$ show that the Lipschitz factor 2 appearing in the previous corollary is sharp. We present some consequences of the Lipschitz property in two, rather different, circumstances: discrete and connected modular spaces.

2.13 Theorem. *Let G be a modular graph and let $a,b,c\in G$. Then each pair of points in $M(a,b,c)$ can be joined by a geodesic of which the vertices alternate between $M(a,b,c)$ and its complement.*

Proof: Let $m_1\neq m_2$ be points in $M(a,b,c)$, and let $a'\in M(a,m_1,m_2)$. As $\rho(a,m_1)=\rho(a,m_2)$ (Proposition 1.11) we also have $\rho(a',m_1)=\rho(a',m_2)$. Whence $\rho(m_1,m_2)=2\cdot\rho(a,m_1)$. In particular, the distance between m_1 and m_2 is even. So if $\rho(m_1,m_2)=2$ there is nothing left to be proved. Assume that $\rho(m_1,m_2)>2$. By virtue of Proposition 1.11 we have

$$\rho(a',m_1)=\rho(a',m_2)=\tfrac{1}{2}\rho(m_1,m_2).$$

So we can take a point $x\in I(a',m_1)$ distinct from m_1 and a'. See Figure 2.13.

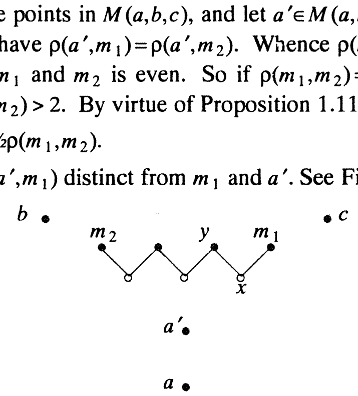

Fig. 2.13: A value of the multimedian in a graph

As $m_2\in M(a',b,c)$ the Lipschitz factor two of M (Theorem 2.12) enables us to take a point $y\in M(x,b,c)$ with $\rho(m_2,y)\leq 2\cdot\rho(a',x)$. By using the Lipschitz property once more we conclude that $\rho(m_1,y)\leq 2\cdot\rho(m_1,x)$ —observe that $m_1=M(m_1,b,c)$. By the calculations

$$\rho(m_1,y)+\rho(y,m_2)\leq 2\cdot(\rho(m_1,x)+\rho(x,a'))=\rho(m_1,m_2),$$

we derive that the last three inequalities are equalities, which implies $u\in I(m_1,m_2)\setminus\{m_1,m_2\}$. Also, from Lemma I: 4.13(1) we obtain that $M(x,b,c)\subseteq M(a,b,c)$, that is $y\in M(a,b,c)$. The result now easily follows with induction. ■

2.14 Theorem. *Let (X,ρ) be a modular metric space with (metrically) complete intervals. Then the following are equivalent:*

(1) All intervals of X are connected

(2) X is arc-wise-connected.

(3) ρ is metrically convex.

(4) X is connected.

Proof: Implication (1)$\to$(3) is easy, and in fact holds for all metric spaces. Indeed, let $a,b\in X$ and $0\leq s\leq\rho(a,b)$. If the closed balls $D(a,s)$, and $D(b,\rho(a,b)-s)$ are disjoint, then the open balls $B(a,s)$, $B(b,\rho(a,b))$ yield a separation of $I(a,b)$. Blumenthal and Menger, [16, Theorem 6.2], show that in a complete metric space with a convex metric, distinct points can be connected with an isometric arc. So implications (3)$\to$(2)$\to$(1) are valid for all metric spaces with complete intervals.

Implication (1)$\to$(4) is evident. For a proof of implication (4)$\to$(3), let a,b be distinct points in X. In view of [16, Theorem 6.2] we only have to show that the set $I(a,b)\setminus\{a,b\}$ is non-empty. To this end, assume to the contrary that $I(a,b)=\{a,b\}$. Let $\varepsilon=\tfrac{1}{2}\rho(a,b)$. By connectivity of X, we can find a finite sequence $x_1, x_2, \cdots, x_n$ of points in X such that $x_1 = a$, $x_n = b$ and $\rho(x_i, x_{i+1}) < \varepsilon$. Consider the sets $M_i = M(a,b,x_i) \subseteq I(a,b)$ $(1 \leq i \leq n)$. For each i we have

$$\rho_H(M_i, M_{i+1}) \leq 2\cdot\rho(x_i, x_{i+1}) < \rho(a,b). \qquad (*)$$

Note that each set M_i must be one of $\{a\},\{b\}$. As $M_1 = \{a\}$ and $M_n = \{b\}$, there exists a $j : 1 \leq j \leq n$ such that $M_j = \{a\}$ and $M_{j+1} = \{b\}$. This contradicts formula (*). ■

Consider a connected median metric space X. The metric interval between points $a,b \in X$ is the image of the function $X \to X : x \to m(a,b,x)$. As the median is continuous (it is even non-expansive), the interval $I(a,b)$ is connected. Hence conditions (1),(4) of the above corollary are equivalent for all median metric spaces regardless of completeness of intervals. It seems that the above argument can not be adapted to general modular metric spaces.

2.15 Corollary. *Let $(N, \mathcal{A}, \mu)$ be a measure space. Then the following are equivalent (see Theorem 1.9):*

(1) $K_1(N, \mathcal{A}, \mu)$ is connected.

(2) $(N, \mathcal{A}, \mu)$ is atomless.

Proof: By Theorem 1.9, $K_1(N, \mathcal{A}, \mu)$ is a complete median metric space. It is well-known that (2) is equivalent with metrical convexity of $K_1(N, \mathcal{A}, \mu)$. Hence, property (1) is equivalent with metrical convexity of $K_1(N, \mathcal{A}, \mu)$ by Theorem 2.14. ■

We conclude that in the situation of Corollary 2.15 the range of μ, i.e. the set $\{\mu(A) < \infty \mid A \in \mathcal{A}\}$, is a closed interval in $\mathbb{R}$. This is a special case of a theorem of Liapounoff.

2.16 A transitive rule for metric multimedians. Recall the four-point transitive rule for medians: $((abc)dc) = (a(bcd)c)$. We shall extend the domain of a ***multi***median such that this operator is also defined on triples (A,b,c) in which A is a (closed) subset and b,c are points, and we shall show that the transitive rule —which can then be formulated— holds.

The earlier encountered extension of the multimedian $M(A,b,c)$ (see p. 20) does not obey the four-point transitive rule. Actually, by the aid of Theorem I: 4.24(4) one can verify that the four-point transitive rule is satisfied for this extension if and only if M is a median.

We need some notions. Let (X,d) be a metric space. If $A \subseteq X$ is non-empty and closed, and $x \in X$, then we put

$$I[x,A] = I[A,x] = \{y \in X \mid \rho(x,y) + \rho(y,A) = \rho(x,A)\}. \qquad (2.16.1)$$

The set $I[x,A]$ is closed and usually different from $I(x,A)$. We now define

$$[Abc] = I(A,d) \cap I(A,c) \cap I(d,c), \qquad (2.16.2)$$

and similarly we define $[bAc]$ and $[bcA]$ (which equals $[Abc]$). ([2]) If $A = \{a\}$ is a singleton then $[Abc]$ is just $[abc]$, i.e. $[-,-,-]$ is an extension of M.

We make two assertions. Let $A \subseteq X$ be closed and $x \in X$. Then we have the following (triangle) inequality:

[2] We use rectangular brackets [,] to avoid ambiguity with the earlier introduced notation $I(x,A)$, $M(A,b,c)$ etc.

$$\rho(x,A) \le \rho(x,y) + \rho(y,A). \tag{T1}$$

If all points of A have the same distance to x, then

$$\rho(x,y) \le \rho(x,A) + \rho(A,y). \tag{T2}$$

Let A be a closed set. In analogy with earlier notation we let

$$\begin{aligned} r_1 &= ½(\rho(A,b) + \rho(A,c) - \rho(b,c)) \\ r_2 &= ½(\rho(A,b) - \rho(A,c) + \rho(b,c)) \\ r_3 &= ½(-\rho(A,b) + \rho(A,c) + \rho(b,c)). \end{aligned} \tag{2.16.3}$$

denote the *sharp radii* corresponding with the triple A,b,c. By triangle inequality (T1) the numbers r_2, r_3 are non-negative. For r_1, the type of assumption prior to (T2) is required.

The following is a simple extension of Proposition 1.11.

2.17 Proposition. *Let (X,ρ) be a metric space with the metric mixing operator M. Then the following are equivalent for a non-empty, closed set A and points $x_2,x_3,m \in X$.*

(1) $m \in M[A,x_2,x_3]$.

(2) $\rho(A,m) = r_1^s$ *and* $\rho(x_i,m) = r_i^s$ *for* $i = 2,3$.

(3) $\rho(A,m) \le r_1^s$ *and* $\rho(x_i,m) \le r_i^s$ *for* $i = 2,3$.

(4) $\rho(A,m) + \rho(x_2,m) + \rho(x_3,m) = ½(\rho(A,x_2) + \rho(A,x_3) + \rho(x_2,x_3))$. ■

We emphasize on the fact that $[(abc)dc]$ can be empty. In Example 2.20 we shall give conditions to avoid this. We are now able to show the announced transitive rule of the multimedian.

2.18 Theorem. *Let (X,ρ) be a modular space. Then,*

$$\forall a,b,c,d \quad [(abc)dc] = [a(bcd)c]. \tag{2.18.1}$$

Proof: Let r_{abc}, r_d, r_c be the sharp radii corresponding with abc,d,c, and let r_a, r_{bcd}, r'_c be the sharp radii for a,bcd,c. With the aid of Proposition 2.17, we first show that a point $z \in [(abc)dc]$ is in $[a(bcd)c]$. The opposite inclusion then directly follows from symmetry considerations.

Step 1: $\rho(c,z) = r'_c$. As $\rho(c,z) = r_c$ we must prove $r_c = r'_c$, or, explicitly,

$$½(\rho(abc,c) - \rho(abc,d) + \rho(c,d)) = ½(-\rho(a,bcd) + \rho(a,c) + \rho(bcd,c)),$$

but this is just Lemma 2.5(3).

Step 2: $\rho(a,z) = r_a$. For this consider the following inclusion

$$I[c,abc] \subseteq I(c,a). \tag{1}$$

To show this take $x \in I(c,abc)$, so $\rho(c,x) + \rho(x,abc) = \rho(abc,c)$. Now

$$\rho(c,a) = \rho(c,abc) + \rho(abc,a) \ge \rho(c,x) + \rho(x,a) \ge \rho(c,a).$$

The first equality follows by definition of abc, whereas the second equality is by assumption. The first inequality is (T2). We conclude that all inequalities above are equalities. This gives $\rho(c,x) + \rho(x,a) = \rho(c,a)$, that is: $x \in I(a,c)$.

Now by construction of $[(abc)cd]$, we have $z \in I(c,abc)$. So by (1), $\rho(c,z) + \rho(z,a) = \rho(c,a)$. Thus we obtain

$$\rho(z,a)=\rho(c,a)-\rho(c,z)=\rho(c,a)-r'_c=r_a$$

as desired.

Step 3: $\rho(bcd,z)\leq r_{bcd}$. By Theorem 2.6(1)

$$\rho(z,bcd)+\rho(b,bcd)+\rho(c,bcd)+\rho(d,bcd)=\rho(z,cdz)+\rho(b,cdz)+\rho(c,cdz)+\rho(d,cdz).$$

Now notice that $dcz=z$, which gives

$$\rho(z,bcd)+\rho(b,bcd)+\rho(c,bcd)+\rho(d,bcd)=\rho(b,z)+\rho(c,z)+\rho(d,z),$$

or

$$\rho(z,bcd)=\rho(b,z)+\rho(c,z)+\rho(d,z)-(\rho(b,bcd)+\rho(c,bcd)+\rho(d,bcd)).$$

Now consider the following (in)equalities,

$$\begin{aligned}&\rho(c,z)+\rho(z,d)=\rho(c,d),\\&\rho(b,bcd)+\rho(c,bcd)+\rho(d,bcd)=\tfrac{1}{2}(\rho(b,c)+\rho(b,d)+\rho(c,d)),\\&\rho(b,z)\leq\rho(b,abc)+\rho(abc,z).\end{aligned}\tag{2}$$

The first equality is implied by $z\in[(abc)dc\,]$, the second is Proposition 1.11(4), whereas the last inequality is just the triangle inequality (T1). From the (in)equalities in (2) we obtain that

$$\begin{aligned}\rho(z,bcd)&\leq\rho(b,abc)+\rho(abc,z)+\rho(c,d)-\tfrac{1}{2}(\rho(b,c)+\rho(b,d)+\rho(c,d)),\\&=\rho(b,abc)+\rho(abc,z)+\tfrac{1}{2}(-\rho(b,c)-\rho(b,d)+\rho(c,d).\end{aligned}$$

As $\rho(abc,z)=\rho(abc,c)-\rho(c,z)$ we have

$$\begin{aligned}\rho(z,bcd)&\leq\rho(b,abc)+\rho(abc,c)-\rho(c,z)+\tfrac{1}{2}(-\rho(b,c)-\rho(b,d)+\rho(c,d)),\\&=\rho(b,c)-\rho(c,z)+\tfrac{1}{2}(-\rho(b,c)-\rho(b,d)+\rho(c,d)),\\&=\tfrac{1}{2}(\rho(b,c)-\rho(b,d)+\rho(c,d))-\rho(c,z).\end{aligned}$$

As $\rho(c,z)=r'_c$ and $\rho(c,bcd)=\tfrac{1}{2}(\rho(b,c)-\rho(b,d)+\rho(c,d))$, we now come to the desired inequality

$$\rho(z,bcd)\leq\rho(c,bcd)-r'_c=r_{bcd},$$

where the last equality is by definition of the sharp radii. ■

In the following theorem we present a sufficient condition for the set $[(abc)dc\,]$ being non-empty.

2.19 Theorem. *Let (X,ρ) be a modular space and let $a,b,c,d\in X$. If $M(a,b,c)$ admits metric nearest points and if $N\subseteq M(a,b,c)$ is the resulting set of metric nearest points of d, then*

$$[(abc)dc\,]=(N,d,c)\subseteq(abc,d,c).\tag{1}$$

In particular, $[(abc)dc\,]$ is non-empty.

Proof: As the statement on inclusion in (1) is obvious, we concentrate on the equality. We first show that the left-hand side is contained in the right-hand side. Let y be an element of the left-hand side in (1). That is (cf. Proposition 2.17),

$$\rho(y,abc)+\rho(y,c)+\rho(y,d)=\tfrac{1}{2}(\rho(abc,c)+\rho(abc,d)+\rho(c,d)).$$

Let x be a metric nearest point of y in abc. Then using the previous equation,

$$\rho(y,x)+\rho(y,c)+\rho(y,d)=\tfrac{1}{2}(\rho(abc,c)+\rho(abc,d)+\rho(c,d)),$$

$$\leq \tfrac{1}{2}(\rho(x,c)+\rho(x,d)+\rho(c,d)).$$

In which the last inequality is by definition of distance to set. Using the triangle inequality three times we obtain

$$\rho(y,x)+\rho(y,c)+\rho(y,d) \leq \tfrac{1}{2}(\rho(x,y)+\rho(y,c)+\rho(x,y)+\rho(y,d)+\rho(c,y)+\rho(y,d))$$
$$=\rho(y,x)+\rho(y,c)+\rho(y,d).$$

So all inequalities are in fact equalities. In particular $\rho(abc,d)=\rho(x,d)$, that is: x is a metric nearest point of d in abc, and

$$\rho(y,x)+\rho(y,c)+\rho(y,d)=\tfrac{1}{2}(\rho(x,c)+\rho(x,d)+\rho(c,d)),$$

that is $y \in M(x,d,c)$ (Proposition 1.11(4)). We conclude that y is contained in the right-hand side.

For a proof that the right-hand side of (1) is contained in the left-hand side, let x be a metric nearest point of d in abc and let $y \in xdc$. Then by Proposition 1.11(4)

$$\rho(y,x)+\rho(y,d)+\rho(y,c)=\tfrac{1}{2}(\rho(x,d)+\rho(x,c)+\rho(d,c)).$$

Now $\rho(x,d)=\rho(abc,d)$ by assumption and $\rho(x,c)=\rho(abc,c)$ (as $x \in abc$). We obtain that

$$\rho(y,x)+\rho(y,d)+\rho(y,c)=\tfrac{1}{2}(\rho(abc,d)+\rho(abc,c)+\rho(d,c)).$$

We arrive at

$$\rho(y,abc)+\rho(y,d)+\rho(y,c) \leq \rho(y,x)+\rho(y,d)+\rho(y,c)=\tfrac{1}{2}(\rho(abc,d)+\rho(abc,c)+\rho(d,c)).$$

By using (T1) we conclude that the inequality is an equality, that is: $y \in [abc,d,c]$ by Proposition 2.17. ∎

We remark that —under the conditions of the previous theorem— there is another description of $[(abc)dc]$, namely as the points of $M(abc,c,d)$ that realize the Hausdorff distance of c to $M(abc,c,d)$. Compare Lemma 2.5(1).

2.20 Example. *Let (X,ρ) be a modular space with multimedian M and $a,b,c \in X$. In each of the following two cases, $M(a,b,c)$ admits metric nearest points.*

(1) $M(a,b,c)$ is compact.

(2) (X,ρ) has the (4,2)-IP.

We do not know "more reasonable" conditions, such as completeness of the metric, under which values of multimedians in modular metric space admit metric nearest points.

§ 3 The completion of modular metric spaces

Let (X,ρ) be any metric space. We shall implicitly use the following property of the hyperspace 2^X_{bc}. If $(B_n)_{n=1}^{\infty}$ is converging to B in 2^X_{bc}, then for every $b \in B$ there exists a sequence $(b_n)_{n=1}^{\infty}$, with $b_n \in B_n$ for all $n \in \mathbb{N}$, converging to b. In particular, if $(A_n)_{n=1}^{\infty}$ is converging to A in 2_{bc} and $A_n \subseteq B_n$ for all $n \in \mathbb{N}$, then $A \subseteq B$.

3.1 Theorem. *If (X,ρ) is a modular space then so is its completion $(\tilde{X},\tilde{\rho})$. Moreover, the multimedian of $\tilde{X}$ is the unique continuous extension of $\overline{M}:X^3 \to 2^{\tilde{X}}_{bc}$, given by the description*

$(a,b,c) \to cl_{\tilde{X}}(M_X(a,b,c)),$

to the whole of $\tilde{X}^3$.

Proof: We consider X as a dense subspace of $\tilde{X}$. We denote the Hausdorff metric on $2^{\tilde{X}}_{bc}$ by $\tilde{\rho}_H$. Let $M_{\tilde{X}}(\tilde{a},\tilde{b},\tilde{c})$ denote the mixing-operator of $(\tilde{X},\tilde{\rho})$. By Theorem 2.12 the multimedian $M_X:X^3 \to 2^X_{bc}$ is a Lipschitz map with Lipschitz factor 2, with respect to the Hausdorff metric on 2^X_{bc}. Hence the same holds for $\overline{M}$, with respect to the Hausdorff metric on $2^{\tilde{X}}_{bc}$. By [22, p. 298] $(2^{\tilde{X}}_{bc},\rho_H)$ is a complete metric space. As X^3 is a dense subset of $\tilde{X}^3$, we can (uniquely) extend $\overline{M}$ to a Lipschitz map (with Lipschitz factor 2) $\tilde{M}$ to the whole of $\tilde{X}^3$. Proposition 1.11 together with a continuity argument show that $\tilde{M}(a,b,c) \subseteq M_{\tilde{X}}(a,b,c)$, for $a,b,c \in \tilde{X}$. In particular, $\tilde{X}$ is modular. For the other inclusion, Lemma 2.5(4) together with a continuity argument show the following equality for $a,b,c,d \in \tilde{X}$:

$$½(\tilde{\rho}(c,a)+\tilde{\rho}(c,b)+\tilde{\rho}(b,a))+\tilde{\rho}(d,\tilde{M}(a,b,c)) = ½(\tilde{\rho}(c,b)+\tilde{\rho}(c,d)+\tilde{\rho}(d,b))+\tilde{\rho}(a,\tilde{M}(b,c,d)). \tag{1}$$

Now let $m \in M_{\tilde{X}}(a,b,c)$, that is $M_{\tilde{X}}(m,b,c) = M_{\tilde{X}}(b,c,m) = \{m\}$. Now as $\tilde{M}(b,c,m)$ is contained in $M_{\tilde{X}}(b,c,m)$ we must have $\tilde{M}(b,c,m) = \{m\}$. Hence from equality (1) with $d = m$ one deduces:

$$½(\tilde{\rho}(c,a)+\tilde{\rho}(c,b)+\tilde{\rho}(b,a))+\tilde{\rho}(m,\tilde{M}(a,b,c)) = ½(\tilde{\rho}(c,b)+\tilde{\rho}(c,m)+\tilde{\rho}(m,b))+\tilde{\rho}(a,m), \text{ or}$$

$$\tilde{\rho}(m,\tilde{M}(a,b,c)) = ½(\tilde{\rho}(c,b)+\tilde{\rho}(c,m)+\tilde{\rho}(m,b)+2\cdot\tilde{\rho}(a,m)) - ½(\tilde{\rho}(c,a)+\tilde{\rho}(c,b)+\tilde{\rho}(b,a)).$$

Now as m is a member of both $I_{\tilde{\rho}}(a,c)$ and $I_{\tilde{\rho}}(a,b)$, we obtain the equality:

$$\tilde{\rho}(m,\tilde{M}(a,b,c)) = ½(\tilde{\rho}(c,b)+\tilde{\rho}(c,a)+\tilde{\rho}(b,a)) - ½(\tilde{\rho}(c,a)+\tilde{\rho}(c,b)+\tilde{\rho}(b,a)) = 0.$$

By the closedness of $\tilde{M}(a,b,c)$, we conclude that $m \in \tilde{M}(a,b,c)$. ■

For any metric space (X,ρ) the set $\{\{x\} \mid x \in X\}$ is a closed subset of 2^X_{bc}. We obtain the following corollary from Theorem 3.1:

3.2 Corollary. *The completion of a median metric space is a median metric space.* ■

A more straightforward proof of the previous corollary goes as follows. As a metric median is uniformly continuous it extends to a uniformly continuous ternary operation m' of the completion X'. By continuity m' is a median which selects from the multimedian of X'. Hence by Corollary I: 4.7 m' equals the multimedian, i.e. X' is a median metric space.

Aronszaijn and Panitchpakdi [4, p. 419] have shown that if (X,ρ) has the (n+1,2)-IP then its completion $(\tilde{X},\tilde{\rho})$ has the (n,2)-IP. Whether $(\tilde{X},\tilde{\rho})$ also has the (n+1,2)-IP is an open problem. For normed spaces the problem was settled in the affirmative by Lindenstrauss [50]. We are now able to solve this problem for $n = 2$:

3.3 Corollary. *If (X,ρ) has the (3,2)-IP then so has its completion $(\tilde X,\tilde\rho)$.*

Proof: By Theorem 3.1 $(\tilde X,\tilde\rho)$ is a modular space. By the quoted result in [4] $(\tilde X,\tilde\rho)$ has the (2,2)-IP, which means that $(\tilde X,\tilde\rho)$ has a convex metric. The theorem now follows from Theorem 1.17. ■

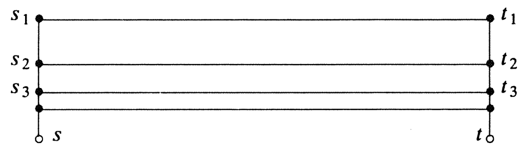

Fig. 3.3: the (2,2)-IP is not inherited by the completion

As an illustration we show that the (2,2)-IP is not inherited by the completion. Consider the points

$$s=(0,0),\ t=(1,0),\ s_n=(0,2^{1-n}),\ t_n=(1,2^{1-n})\ (n\in\mathbb{N}).$$

Let $X=[s_1,(0,0))\cup[t_1,(1,0))\cup_{n\in\mathbb{N}}[s_n,t_n]$, see Figure 3.3. Then the geodesic metric ρ on X is convex, and the completion of (X,ρ) equals $X\cup\{s,t\}$. One can easily verify that $I_{\tilde\rho}(s,t)=\{s,t\}$.

The following result describes the interval operator of the completion of a modular metric space.

3.4 Theorem. *Let (X,ρ) be a modular space with completion $(\tilde X,\tilde\rho)$. Then,*

(1) The (metric) interval function $I: X^2\to 2^X_{bc}$ of X is contractive with regard to the sum metric on X^2.

(2) The interval function $I_{\tilde\rho}$ of the (modular) space $(\tilde X,\tilde\rho)$ is the unique extension of $\bar I: X^2\to 2^{\tilde X}_{bc}$, given by the description

$$(a,b)\to cl_{\tilde X}(I(a,b)).$$

Proof: Let $M,\tilde M$ be the multimedians of X and $\tilde X$ respectively. See Theorem 3.1. For a proof of part (1), take $a,a',b'\in X$. By the triangle inequality of ρ_H it suffices to show that

$$\rho_H(I(a,b),I(a',b))\le\rho(a,a'). \tag{3}$$

To this end, take $x\in I(a,b)$, then

$$\begin{aligned}\rho(x,M(a',b,x)) &= \tfrac{1}{2}(\rho(x,a')+\rho(x,b)-\rho(a',b))\\ &= \tfrac{1}{2}(\rho(x,a')+\rho(a,b)-\rho(x,a)-\rho(a',b))\le\rho(a,a').\end{aligned}$$

As the set $M(a',b,x)$ is contained in $I(a',b)$ we have established that $\rho(x,I(a',b))\le\rho(a,a')$. By permuting the roles of a and a' one arrives at (3). For a proof of part (2), we shall first show the following equality:

$$\bar I(a,b)=I_{\tilde\rho}(a,b), \tag{4}$$

for all $a,b\in X$. The inclusion from left to right follows for simple topological reasons. For a proof of the inclusion from right to left, take $z \in I_{\tilde\rho}(a,b)$. Let $(z_n)_{n=1}^{\infty}$ be a sequence in X converging to z. Construct a sequence $(c_n)_{n=1}^{\infty}$ in $I(a,b)$ by choosing $c_n \in M(a,b,z_n)$ $(n \in \mathbb{N})$. Then,

$$\rho(c_n,z) \leq \rho_H(\tilde M(a,b,z_n),z) = \rho_H(\tilde M(a,b,z_n),\tilde M(a,b,z)) \leq 2\cdot\rho(z_n,z).$$

The first inequality follows by definition of Hausdorff metric, the equality is evident, whereas the last inequality applies that $\tilde M$ is a Lipschitz map (with factor 2). Hence $c_n \in M(a,b,z_n)$ converges to z, and we have shown (4). To conclude the proof of part (2), as the space $(\tilde X,\tilde\rho)$ is modular its interval function $I_{\tilde\rho}$ is continuous by part (1). In view of equality (4), $I_{\tilde\rho}$ is the (unique) continuous extension of $\bar I$. ■

3.5 Corollary. *Let (X,ρ) be a modular space, with completion $\tilde X$. Suppose that C is a subset of X and $a \in C$. Then,*

(1) If C is ρ-convex (resp. star-shaped at a) in X then so is its closure in X.

(2) If C is ρ-convex (resp. star-shaped at a) in X then so is the completion $Cl_{\tilde X}(C)$ of C in $\tilde X$. ■

3.6 Corollary. *Let (X,ρ) be a modular space. Then the collection of ρ-convex subsets in 2^X_{bc} is closed.* ■

We remark that the last three results do not hold in general metric spaces. To this end, let T be the (closed) triangle in the plane spanned by the origin, $e_1=(1,0)$ and $e_2=(0,1)$, and let X be $T \cup \{(1,1)\}$ minus the open convex segment $(0,e_1)$. We endow X with the restriction of the sum-metric ρ. Then the following are easily verified. The interval $I_\rho(0,e_1)$ only consists of $0,e_1$, while the interval between those points in the completion of X equals the segment $[0,e_1]$ (cf. Th. 3.4). X minus the points e_1,e_2, and $(1,1)$ is ρ-convex, but its closure is not (cf. Cor. 3.5). The set X minus the point $(1,1)$ can be obtained as a limit of ρ-convex subsets in the hyperspace metric, but is not ρ-convex itself (cf. Cor. 3.6).

3.7 Decreasing and increasing sequences. We recall that a partially ordered set D is *downdirected* provided for each $d_1,d_2 \in D$ there exists $d \in D$ with $d \leq d_1$ and $d \leq d_2$. The concept of an *updirected* set is defined dually. A function of an updirected set D to a set X is usually called a *net*. We often use notation of type $(p_i)_{i\in D}$ for nets. If X is also ordered, then a net that respects this order, i.e. $i \leq j$ implies $p_i \leq p_j$, is called increasing. The concept of a *decreasing* net is defined dually. If the partial order is a basepoint order with basepoint b, then a sequence or a net in X which is decreasing (increasing) in $\leq_b$ will be called *b-decreasing* (*b-increasing*).

The following proposition is a simple generalization of [2, Lemma 2.8].

3.8 Proposition. *Let (X,ρ) be a metric space, and let $P=(p_i)_{i\in D}$ be a net in X. Then the following hold:*

(1) In either of the following situations a net $P=(p_i)_{i\in D}$ is Cauchy:

(i) The net P is decreasing in $\leq_b$.

(ii) The net P is bounded and increasing in $\leq_b$.

(2) If P as described in (i) (resp. (ii)) converges to $p \in X$, then $p = \inf(D)$ (resp. $p = \sup(D)$).

Proof: For a proof of (i), let $\varepsilon > 0$. Define $R = \inf\{\rho(b,p_i) \mid i \in D\}$. There exists an $i \in D$ such that $\rho(b,p_i) < R + \frac{1}{2}\cdot\varepsilon$. Then for all $j \geq i$ we have $R \leq \rho(b,p_j) \leq \rho(b,p_i) < R + \frac{1}{2}\cdot\varepsilon$. Consequently, $\rho(p_i,p_j)=\rho(b,p_i)-\rho(b,p_j)<\frac{1}{2}\cdot\varepsilon$. Whence, for all $j,k\geq i$

$$\rho(p_j,p_k)\leq\rho(p_j,p_i)+\rho(p_i,p_k)<\tfrac{1}{2}\cdot\varepsilon+\tfrac{1}{2}\cdot\varepsilon=\varepsilon.$$

A similar argument works under the assumptions of (ii).

For a proof of (2), let P be as described in (i). Clearly $p\in I(b,p_i)$ for all $i\in D$, i.e. p is a lower bound of the p_i in $\leq_b$. Suppose that $q\leq_b p_i$, i.e. $\rho(b,p_i)=\rho(b,q)+\rho(q,p_i)$, for all $i\in D$. Then by continuity $\rho(b,p)=\rho(b,p)+\rho(q,p)$. That is $q\leq_b p$. The proof of the remaining part of (2) is similar. ■

The converse of statement (2) in the previous proposition is in general not true: the subset $(1,2)$ of the metric space $\{0\}\cup(1,2)\cup\{3\}$ has the origin as infimum and 3 as supremum, however these points are not adherent to $(1,2)$. There is an affirmative result for normed spaces, see Proposition III: 2.5.

By the previous lemma we come to the following notion. A metric space (X,ρ) is *downconverging* relative to b (briefly, *b-downconverging*) provided each b-decreasing sequence in X converges to a point of X. Similarly, X is *upconverging* relative to b (or, *b-upconverging*) provided each bounded, b-increasing sequence in X converges to a point of X. If both conditions hold relative to the basepoint b, then we say that X is *converging* relative to b (or, *b-converging*).

It turns out that "monotone" sequences provide as much information as the more general decreasing or increasing nets.

3.9 Theorem. *Let (X,ρ) be a metric space and let $b\in X$. Then the following hold.*

(1) If X is b-downconverging, then any decreasing net in $\leq_b$ converges.

(2) If X is b-upconverging, then any increasing bounded net in $\leq_b$ converges.

Proof: For a proof of (1), let $P=(p_i)_{i\in D}$ be a decreasing net in $\leq_b$, and let $R=\inf\{\rho(b,p_i)\mid i\in D\}$. Take any point $j(1)\in D$ such that $\rho(b,p_{j(1)})<R+2^{-1}$. By induction, having constructed $j(1)\leq\cdots\leq j(n)$ in D such that $p_{j(p)}\leq_b p_{j(q)}$ and $\rho(b,p_{j(p)})<R+2^{-p}$ for $1\leq p\leq q\leq n$, we take $j(n+1)\geq j(n)$ in D, such that $\rho(b,p_{j(n+1)})<R+2^{-(n+1)}$. This gives a b-decreasing sequence $(p_{j(n)})_{n=1}^{\infty}$, which by assumption converges to some point p. Evidently, $\rho(b,p)=R$.

Let $\varepsilon>0$ and fix $N\in\mathbb{N}$ such that $2^{-N}<\varepsilon$. Then for any $i\geq j(N)$ in D, we have that (cf. the proof of Proposition 3.8) $\rho(p,p_i)<2\cdot\varepsilon$. Hence the net P converges to p.

The proof of statement (2) is largely the same. ■

In a geometric interval space (X,I), the following multivalued "cone" function can be defined on the collection of finite sequences of X.

(I1) $\text{cone}(a)=\{x\}$,

(I2) $\text{cone}(a_1,a_2,\cdots,a_{n+1})=\cup\{I(m,a_{n+1})\mid m\in\text{cone}(a_1,a_2,\cdots,a_n)\}$ $(n\geq 1)$.

Observe that $\text{cone}(a,b)=I(a,b)$ $(a,b\in X)$, and that $\text{cone}(a,b,c)=I(I(a,b),c)$ —compare Theorem I: 4.18.

In view of Theorem I: 4.24 the cone function is symmetric iff X is median. Moreover, in a median algebra (X,m), the set $\text{cone}(a_1,a_2,\cdots,a_n)$ is gated, and the gate function f_n takes the following form:

(3.10) $f_2(x)=m(a_1,a_2,x)$; $f_{k+1}(x)=m(a_{k+1},f_k(x),x)$ $(2\leq k<n)$.

See [79]. For a direct proof proceed as in Theorem I: 4.18, where (3.10) is shown for $n=3$.

In particular, cone$(a_1,a_2,\cdots,a_n)$ is precisely the convex hull of the points $a_1,a_2,\cdots,a_n$ (i.e. a polytope). Although the cone function is not symmetric in a general interval spaces (see above), the following can be shown by means of induction.

(3.11) Let (X,I) be a geometric interval space, and let $b\in X$. If $a_1,\cdots,a_n$ is a finite collection in X that is totally ordered in $\leq_b$, then

$$cone(a_1,\cdots,a_n)=I(\min_{i=1}^n a_i,\ \max_{i=1}^n a_i).$$

The following result describes the behavior of the cone operator with respect to the Hausdorff metric ρ_H on (bounded) sets. This result is a generalization of Theorem 3.4(1).

3.12 Lemma. *The following inequality holds for all finite sequences $(a_i)_{i=1}^n,(b_i)_{i=1}^n$ in a modular metric space (X,ρ):*

$$\rho_H(\mathrm{cone}(a_1,\cdots,a_n),\mathrm{cone}(b_1,\cdots,b_n))\leq\sum_{i=1}^n\rho(a_i,b_i).$$

Proof: By the triangle inequality of ρ_H it suffices to prove that

$$\rho_H(\mathrm{cone}(a_1,\cdots,a_n),\mathrm{cone}(a_1,\cdots,a_{n-1},b_n))\leq\rho(a_n,b_n).$$

We must verify that for each $x\in\mathrm{cone}(a_1,\cdots,a_n)$,

$$\rho(x,\mathrm{cone}(a_1,\cdots,a_{n-1},b_n))\leq\rho(a_n,b_n).$$

By invoking the definition of cone, there is a $z\in\mathrm{cone}(a_1,\cdots,a_{n-1})$ such that $x\in I(z,a_n)$. Consider $y=M(b_n,z,x)$. As

$$M(b_n,z,x)\subseteq I(b_n,z)\subseteq\mathrm{cone}(a_1,\cdots,b_n),$$

we find that $y\in\mathrm{cone}(a_1,\cdots,b_n)$. Consider the following (in)equalities:

$$\begin{aligned}\rho(x,\mathrm{cone}(a_1,\cdots,b_n))&\leq\rho(x,y)\\ &=\tfrac12(-\rho(b_n,z)+\rho(b_n,x)+\rho(z,x))\\ &=\tfrac12(-\rho(b_n,z)+\rho(b_n,x)+\rho(a_n,z)-\rho(a_n,x)\\ &=\tfrac12(\rho(a_n,z)-\rho(b_n,z)+\rho(b_n,x)-\rho(a_n,x))\\ &\leq\rho(a_n,b_n).\end{aligned}$$

The first equality only uses Proposition 1.11, and the last inequality involves $x\in I(a_n,z)$. ■

From this lemma, combined with the remarks prior to it, we obtain:

3.13 Corollary. *Let (X,ρ) be a median metric space, and let $(\tilde X,\tilde\rho)$ be its completion. Then for each $\tilde x\in\tilde X$ and for each $\varepsilon>0$ there is a ρ-convex subset C of the original space X such that*

$$\tilde x\in Cl_{\tilde X}(C);\ diam(C)<\varepsilon.$$

■

A subset C of a modular space X is called *multimedian at* a point $0 \in X$ provided that $M(0,c_1,c_2) \subseteq C$ for all $c_1,c_2 \in C$.

The following subsets of a modular metric space X are multimedian at 0: the whole X, any ρ-convex subset of X and any star-shaped subset at 0. In particular, any closed ball around 0 and any interval starting at 0 is multimedian at 0.

The following simple fact is stated here for later reference (cf. Corollary 3.5).

(3.14) Let (X,ρ) be a modular metric space, and let $(\tilde{X},\tilde{\rho})$ denote its completion. If C is multimedian at $b \in X$ then so are the closure in X and in $\tilde{X}$.

Consider a fixed basepoint 0 of metric space (X,ρ), and let $(\tilde{X},\tilde{\rho})$ denote the completion of X. If C is a non-empty subset of X, then C_0 (resp. C^0) will denote the subspace of $\tilde{X}$, consisting of all limits of 0–decreasing (0–increasing) Cauchy sequences in C. We now come to one of the main results of this section.

3.15 Theorem. *Let (X,ρ) be a modular metric space, and let $C \subseteq X$ be multimedian at* 0. *Then the completion of C is given by $(C_0)^0$.*

The proof of Theorem 3.14 requires a construction to transform sequences in a modular space into increasing and decreasing sequences. As this construction shall be of later use also, we formulate it as a lemma.

3.16 Lemma. *Let (X,I) be a modular space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Then there exist sequences $P^k = (p_n^k)_{n=k}^{\infty}$ in X satisfying the following conditions.*

(1:k) $p_k^k = x_k$.

(2:k) $p_n^k \in M(0,p_{n-1}^k,x_n)$ *for* $n > k$.

(3:k) $p_{n+1}^{k-1} \leq_0 p_n^k$ *for* $n \geq k$.

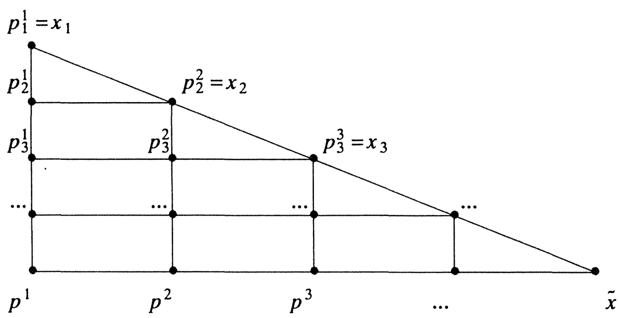

Fig. 3.16: construction of the sequences p_n^k

Proof of Lemma 3.16: For $k = 1$, we take $p_1^1 = x_1$, and, recursively,

$p_n^1 \in M(p_{n-1}^1, 0, x_n)$.

See Figure 3.16. Next, let $l>1$ and suppose that sequences P^k satisfying (1:k), (2:k), and (3:k) have been defined for $1 \le k < l$. Take $p_l^l = x_l$, and, recursively,

$p_{n+1}^l \in M(p_{n+1}^{l-1}, p_n^l, x_{n+1})$.

By induction on n, we verify the following formula:

$$p_n^l \in M(0, p_{n-1}^l, x_n) \text{ (for } n>l) \text{ and } p_{n+1}^{l-1} \le_0 p_n^l \text{ (for } n \ge l). \qquad (*)$$

First, $p_{l+1}^{l-1} \le_0 p_l^{l-1} \le_0 x_l = p_l^l$, which settles the above formula in case $n=l$. Next, assume that formula (*) holds for some $n \ge l$. Then $p_{n+1}^{l-1} \le_0 x_{n+1}$ by (2:l-1), and $p_{n+1}^{l-1} \le_0 p_n^l$ by (*). Whence by the construction of p_{n+1}^l and by Lemma I: 4.13, we obtain:

$p_{n+1}^l \in M(p_{n+1}^{l-1}, p_n^l, x_{n+1}) \subseteq M(0, p_n^l, x_{n+1})$; $p_{n+1}^{l-1} \le_0 p_{n+1}^l$.

In particular, $p_{n+2}^{l-1} \le_0 p_{n+1}^{l-1} \le_0 p_{n+1}^l$, as required in (*) for $n+1$. ■

Proof of Theorem 3.14: Let $\tilde{C}$ be the closure of C in the completion $\tilde{X}$ of X. Let $x \in \tilde{C}$ and fix a sequence $(x_n)_{n=1}^\infty$ in C converging to x. Without loss of generality, $\rho(x,x_n) \le 2^{-n}$. Let the sequences $P^k = (p_n^k)_{n=k}^\infty$ be as constructed in Lemma 3.16. As C is multimedian at 0, it follows from formulae (1:k), (2:k), (as stated in Lemma 3.16) that $p_n^k \in C$ $(n \ge k)$. From formula (2:k) we conclude that each sequence $P^k = (p_n^k)_{n=k}^\infty$ is 0-decreasing. Hence it converges to a point p^k in C_0. By (1:k), $p_k^k = x_k \in \mathrm{cone}(x_k)$. If $n \ge k$ and $p_n^k \in \mathrm{cone}(x_k, \cdots, x_n)$, then, since $p_{n+1}^k \in I(p_n^k, x_{n+1})$ by (2:k), we conclude by induction that

$p_{n+1}^k \in \mathrm{cone}_\rho(x_k, x_{k+1}, \cdots, x_{n+1}) \subseteq \mathrm{cone}_{\tilde{\rho}}(x_k, x_{k+1}, \cdots, x_{n+1})$.

The completion of a modular metric space being modular (Theorem 3.1) we can apply Lemma 3.12 (with one sequence constant and equal to x) to the effect that

$$\tilde{\rho}_H(x, \mathrm{cone}_{\tilde{\rho}}(x_k, x_{k+1} \cdots, x_n)) \le \sum_{j=k}^{n} \rho(x, x_j) \le \sum_{j=k}^{n} 2^{-j} \le 2^{-(k-1)}.$$

In particular, $\rho(x,p_n^k) \le 2^{-(k-1)}$, for $n \ge k$. Hence, $\rho(x,p^k) \le 2^{-(k-1)}$ and $(p^k)_{k=1}^\infty$ converges to x. From formula (3:k) we deduce that $p^{k-1} \le_0 p^k$. So, $(p^k)_{k=1}^\infty$ is a 0–increasing sequence in C_0, which is bounded by the above result. Hence this sequence converges in $(C_0)^0$, showing that $x \in (C_0)^0$. ■

3.17 Corollary. *Let (X,ρ) be a modular space. Then a subset multimedian at b is complete if and only if it is b-converging.* ■

3.18 Corollary. *The completion of a modular space (X,ρ) is given by $(X_0)^0$.* ■

From this corollary we obtain the following characterization of completeness of modular metric spaces.

3.19 Theorem. *The following are equivalent for a modular metric space X.*
(1) *X is converging in some basepoint order.*
(3) *X is upconverging in all basepoint orders.*
(2) *X is converging in all basepoint orders.*
(4) *X is complete.* ■

3.20 Corollary. *Let (X,ρ) be a modular space with completion $(\tilde{X},\tilde{\rho})$. Then the following formula holds for all $a,b\in X$:*

$$(I_\rho(a,b))_a)_b = (I_\rho(a,b))_a)^a = Cl_{\tilde{X}}(I_\rho(a,b)) = I_{\tilde{\rho}}(a,b).$$

Proof: For a proof of the first equality: by the inversion axiom (G-2) any increasing sequence in $(\leq_a, I_{\tilde{\rho}}(a,b))$ is decreasing in $(\leq_b, I_{\tilde{\rho}}(a,b))$. The second equality follows from Theorem 3.15. Finally, the last equality is shown in Theorem 3.4(2). ■

3.21 Theorem. *Let (X,ρ) be a modular space, and let $(\tilde{X},\tilde{\rho})$ denote its completion. Then the following are equivalent:*
(1) *X is downconverging in each of its base-point orders.*
(2) *All intervals of X are complete.*
(3) *X is a metric-convex subset of $\tilde{X}$.*

Proof: Implications $(1)\to(2)\to(3)$ follow from Corollary 3.20 and implications $(3)\to(2)\to(1)$ are evident. ■

In contrast with Theorem 3.19, down(up-)convergence in one base-point order need not imply down(up-)convergence for other base-points. For a simple example, consider the following median subalgebra X of the plane (coordinate-wise median)

$$X = \{ (x_1,x_2) \mid 0 < x_i < 1 \textit{ for } i = 1, 2; x_1 + x_2 \leq 1 \} \cup \{ (0,0) \}.$$

This space is down-converging in the order of $(½,½)$, but not in the order of $(0,0)$. On the other hand, the space is up-converging from the viewpoint of $(0,0)$, but not from the viewpoint of $(½,½)$. Moreover, $(X^0)_0$ is not complete as it does not contain the points $(1,0)$ and $(0,1)$. Compare with Corollary 3.18.

We mention an affirmative result: a normed space upconverging in one point is upconverging in all of its points. See Proposition III: 2.5 and Corollary III: 2.6.

§ 4 Gated sets in (modular) metric spaces

The following result summarizes some properties of gated sets in general metric spaces.

4.1 Proposition. *Let (X,ρ) be a metric space and let $(\tilde{X},\tilde{\rho})$ denote its completion. Then the following hold for a gated set $C\subseteq X$.*
(1) *The gate function $p_C : X \to C$ is contractive.*
(2) *C is closed and ρ-convex.*
(3) *The set $Cl_{\tilde{X}}(C)$ is gated in $\tilde{X}$; its gate function is the unique continuous extension of*

$p_C : X \to \bar{C}$ to the whole of $\bar{X}$.

Proof: For a proof of (1), take $x,y \in X$. By definition of gate, $p(x) \in I(x,p(y))$, that is $\rho(x,p(x)) + \rho(p(x),p(y)) = \rho(x,p(y))$. Similarly we obtain, $\rho(y,p(y)) + \rho(p(y),p(x)) = \rho(y,p(x))$. Adding these equalities we obtain

$$2\cdot\rho(p(x),p(y)) = \rho(x,p(y)) - \rho(p(y),y) + \rho(y,p(x)) - \rho(p(x),x) \le 2\cdot\rho(x,y),$$

by the triangle inequality. That C is ρ-convex is just (I: 3.4.2). As $C = \{x \in X \mid x = p(x)\}$, the rest of statement (2) follows from (1).

For a proof of (3), as $p_C : X \to \bar{C}$ is contractive we can uniquely extend this function to a contractive map $\bar{p} : \bar{X} \to \bar{C}$. A routine argument shows that $\bar{p}$ is the gate function of $\bar{C}$. ■

Part (1) of the previous proposition was first shown by A. Dress and R. Scharlau in [21]. From Proposition 4.1(3) we conclude that gated sets in arbitrary metric spaces behave well with respect to taking completions. This is contrary to the behavior of geodesically convex subsets (cf. Corollary 3.5, and the remarks prior to Paragraph 3.7).

In a general metric space a closed (or even complete) geodesically convex subset need not be gated. We work towards such a result for modular metric spaces.

Let (X,ρ) be a metric space, and let $b \in X$. Any subset C of X downdirected in $\le_b$ can be seen as a Cauchy net by Proposition 3.8. Hence under some form of completeness this net converges to the gate of b in C. For instance if C is b-downconverging, or if there exists a $c \in C$ such that $I(b,c) \cap C$ is complete. The above applies in particular if X is a modular metric space, and if C is multimedian at b. We conclude to the following.

4.2 Proposition. *Let (X,ρ) be a downconverging modular metric space. If a closed subset $C \subseteq X$ is multimedian at $b \in X$, then b has a gate in C. In particular, a subset of X is gated if and only if it is ρ-convex and closed.* ■

We remark that by Theorem 3.21 a modular metric space is downconverging iff all intervals of X are (metrically) complete.

The following result compares ***metric*** nearest points with ***order*** nearest points (i.e., with gates) in a modular metric space.

4.3 Proposition. *Let (X,ρ) be a modular space and let C be a subset of X which is multimedian at $b \in X$. Then the following are equivalent for $p \in X$.*

(1) p is the gate of b in C.

(2) p is a metric nearest point of b in C.

Proof: Implication (1)$\to$(2) is evident. For a proof of the other implication let $p \in C$ be as described in (2). Let $c \in C$ and suppose $m \in M(b,p,c) \setminus \{p\}$. Then evidently $\rho(b,m) < \rho(b,p)$. As $m \in C$, this contradicts (2). We conclude $M(b,p,c) = \{p\}$ for all $c \in C$, i.e. p is the gate of b in C. ■

In Proposition I: 3.5 it is shown that gated sets in a general geometric interval space have the (F,2)-IP. We now come to stronger intersection properties of gated sets in a metric space. The following notion is convenient. Let B be a subset of a metric space (X,ρ) and let $b \in X$. The subset B is called *b-upbounded*, or *upbounded at b* if any sequence $(b_i)_{i=1}^{\infty}$ in B increasing in the basepoint order $\leq_b$ is bounded.

One can easily verify that replacing "sequence" by "net" in the last definition yields an equivalent notion.

4.4 Remark. Each bounded subset of X is evidently b-upbounded. The converse does not hold, not even for geodesically convex subsets. For $i \in \mathbb{N}$ we let $e_i \in l_1(\mathbb{N})$ denote the unit vector with the i-th coordinate one and all other coordinates zero. Then the (median stable) subset C of $l_1(\mathbb{N})$, defined by

$$C = \{\lambda \cdot e_i \mid i \in \mathbb{N}, -i \leq \lambda \leq i\},$$

is 0-upbounded but not bounded. Note that C is star-shaped at 0 and complete. There is an affirmative result for geodesically convex subsets in a modular ***normed*** space. For instance: in a modular Banach space a geodesically convex subset upbounded at ***one*** of its points is bounded. See Theorem III: 5.10.

The following lemma shall be used in different situations.

4.5 Lemma. *Let (X,ρ) be a metric space and $b \in X$. Let $\mathcal{G}$ be a downdirected collection of closed subsets gated at b. If $C \subseteq X$ is a b-upbounded b-upconverging subset, star-shaped at b such that $C \cap G \neq \emptyset$ for all $G \in \mathcal{G}$ then*

$$C \cap \bigcap \mathcal{G} \neq \emptyset.$$

Proof: For $G \in \mathcal{G}$ we let n_G denote the gate of b in G. By formula (I: 3.4.1) $n_G \in C$. One easily verifies that the net $\{n_G \mid G \in \mathcal{G}\}$ directed by $\mathcal{G}$, is increasing in $\leq_b$. As $\{n_G \mid G \in \mathcal{G}\}$ is contained in C it is bounded. Hence this net converges, say to a point $p \in C$ by assumption. As all $G \in \mathcal{G}$ are closed, p is a member of every such G, i.e.

$$p \in C \cap \bigcap \mathcal{G}. \qquad \blacksquare$$

We now derive two results: one concerning general metric space, and one for modular metric space.

4.6 Theorem. *Let (X,ρ) be a metric space and $b \in X$. Let $\mathcal{G}$ be a collection of gated subsets. If $C \subseteq X$ is a b-upbounded b-upconverging subset, star-shaped at b, and if C meets $\cap \mathcal{F}$ for all finite subcollections of $\mathcal{G}$, then*

$$C \cap \bigcap \mathcal{G} \neq \emptyset.$$

Proof: By formula (I: 3.4.4) the intersection of finite members of $\mathcal{G}$ is gated, and by Proposition 4.1(2) gated sets are closed. Whence the collection

$$\{\cap \mathcal{F} \mid \mathcal{F} \subseteq \mathcal{G} \text{ finite }\},$$

satisfies the assumption of Lemma 4.5. $\blacksquare$

For a downconverging modular metric space, gated sets are precisely the non-empty convex closed subsets by Proposition 4.2. For such spaces, Theorem 4.6 is a particular case of the following result.

4.7 Theorem. *Let (X,ρ) be a modular metric space and $b \in X$. Let $\mathfrak{D}$ be a collection of closed subsets multimedian at b. If $C \subseteq X$ is a complete b-upbounded subset, star-shaped at b such that C meets $\cap\mathcal{F}$ for each finite subcollection of $\mathfrak{D}$, then*

$$C \cap \bigcap \mathfrak{D} \neq \emptyset.$$

Proof: As in the proof of the previous theorem we consider the collection

$$\mathfrak{D}_{fin} = \{\cap\mathcal{F} \mid \mathcal{F} \subseteq \mathfrak{D} \ \ finite \ \}.$$

This collection consists of closed subsets multimedian at b meeting C. As C is complete, any member of $\mathfrak{D}_{fin}$ has a gate from b. Then we apply Lemma 4.5. ■

The proof of Theorem 4.7 only uses that C is b-converging. However this property is equivalent with completeness of C by Corollary 3.17. Theorem 4.7 applies in particular for a collection of closed ρ-convex subsets. In the next section we shall interpret Theorem 4.7 in terms of a "weak" topology.

In view of Proposition 4.2 a non-empty intersection of gated sets is gated. The question now arises whether this holds for arbitrary metric spaces, and what the gate function of the intersection looks like. For convenience we introduce the following. Let X be a Hausdorff topological space, and let $(p_i)_{i\in I}$ be a collection of functions of X into itself, such that their composition products commute two by two. Let $\mathcal{F}$ be the collection of finite subsets of I, direct $\mathcal{F}$ by inclusion. The collection $(p_i)_{i\in I}$ is called *composable* if for every $x \in X$ the net

$$(\textstyle\prod_{i\in F} p_i(x))_{F\in\mathcal{F}},$$

directed by $\mathcal{F}$, converges in X. The limit function is called the *composition* of the family $(p_i)_{i\in I}$.

We can now prove the following result.

4.8 Theorem. *Let (X,ρ) be a downconverging metric space, and let $(C_i)_{i\in I}$ be a collection of gated subsets of X with a non-empty intersection. Then the associated gate functions p_i $(i \in I)$ are composable, and their composition yields the gate function of $\cap_{i\in I} C_i$.*

Proof: Let $x \in X$ and let $F \subseteq I$ be finite. By formula (I: 3.4.4) the functions p_i commute pairwise, and the function $\prod_{i\in F} p_i$ is the gate function of the set $\cap_{i\in F} C_i$. Hence if $c \in \cap_{i\in I} C_i$ then the net

$$(\textstyle\prod_{i\in F} p_i(x))_{F\in\mathcal{F}},$$

is decreasing in $(I(x,c), \leq_c)$. By assumption it converges to say $q \in I(x,c)$. As $c \in \cap_{i\in I} C_i$ was arbitrary, we conclude that q is the gate of x in $\bigcap_{i\in I} C_i$. ■

The condition that X be downconverging, cannot be removed from Theorem 4.8.

We end this section with some results concerning upbounded subsets. It is natural to ask whether a subset that is upbounded at ***one*** of its points is upbounded at ***all*** of it points.

4.9 Proposition. *Let (X,ρ) be a modular metric space and let $C \subseteq X$ be multimedian at both points $0,c$. If C is 0-upbounded and 0-downconverging, then C is c-upbounded.*

Proof: Assume to the contrary that $(x_n)_{n=1}^{\infty}$ is a c-increasing sequence in C that is unbounded. Let the sequences $P^k=(p_n^k)_{n=k}^{\infty}$ in X be as constructed in Lemma 3.16. See Figure 3.16. We argue as in the proof of Theorem 3.15. As C is multimedian at 0, it follows from formulae (1:k), (2:k) (as stated in Lemma 3.16) that $p_n^k \in C$ $(n \geq k)$. From formula (2:k) we conclude that each sequence $P^k=(p_n^k)_{n=k}^{\infty}$ is 0-decreasing. Hence it converges to a point p^k in C. By (3-k) we deduce that the sequence $(p^k)_{k=1}^{\infty}$ is 0-increasing. By (1:k), $p_k^k = x_k \in \text{cone}(x_k)$. If $n \geq k$ and $p_n^k \in \text{cone}(x_k, \cdots, x_n)$, then, since $p_{n+1}^k \in I(p_n^k, x_{n+1})$ by (2:k), we conclude by induction that

$$p_{n+1}^k \in \text{cone}_\rho(x_k, x_{k+1}, \cdots, x_{n+1}). \tag{1}$$

The right-hand side of formula (1) equals $I_\rho(x_k,x_{n+1})$ by (3.11). By Proposition 2.7 we obtain $\rho(c,p_{n+1}^k) \geq \rho(c,x_k)$, hence $\rho(c,p^k) \geq \rho(c,x_k)$. We conclude that the sequence $(p_k)_{k=1}^{\infty}$ is 0-increasing and unbounded. This contradicts the assumption that C is 0-upbounded. ■

From the previous result we conclude that in a downconverging modular metric space a geodesically convex subset that is upbounded at one point is upbounded at all points.

We next consider the question whether the closure of an upbounded subset is upbounded.

4.10 Proposition. *Let (X,ρ) be a downconverging modular metric space and let C be a subset of X, star-shaped at the point $0 \in C$. If C is 0-upbounded, then the completion of C is 0-upbounded. In particular the closure of C in X is 0-upbounded.*

Proof: Let $\tilde{C} \subseteq \tilde{X}$ denote the respective completions of C and X. Assume to the contrary that $(x_n)_{n=1}^{\infty}$ is a 0-increasing sequence in $\tilde{C}$ that is unbounded. We construct a sequence $(y_n)_{n=1}^{\infty}$ in C with the following properties:

(i) $\rho(x_n,y_n) \leq 2^{-n}$,

(ii) $y_n \leq_0 x_n$

(iii) $y_1 \leq_0 y_2 \leq_0 \cdots \leq_0 y_n$

for all $n \in \mathbb{N}$. To this end, let $n \in \mathbb{N}$. By Theorem 3.15 there is a 0-increasing sequence $(z_k^n)_{k=1}^{\infty}$ in C converging to x_n. Let $k \in \mathbb{N}$ be such that $\rho(z_k^1,x_1) \leq 2^{-1}$ and let $y_1 = z_k^1$. Suppose that $y_1, y_2, \cdots, y_n$ have been constructed such as in (i), (ii) and (iii) for some $n \geq 1$. As metric multimedians are Lipschitz (Theorem 2.12), there exists a $k \in \mathbb{N}$ such that

$$\rho_H(x_{n+1}, M_{\tilde{\rho}}(y_n, x_{n+1}, z_k^{n+1})) \leq 2^{-(n+1)}.$$

Choose $y_{n+1} \in M_{\tilde{\rho}}(y_n,x_{n+1},z_k^{n+1})$. Then y_{n+1} satisfies (i), (ii) and (iii) by construction. As $\tilde{C}$ is star-shaped at 0 (Corollary 3.5) we have that $y_{n+1} \in \tilde{C}$. Also, $y_{n+1} \in I_{\tilde{\rho}}(y_n,z_k^{n+1})$. In view of Theorem 3.21, the latter interval is contained in X. Whence, $y_{n+1} \in \tilde{C} \cap X = C$. This concludes the induction.

By properties (i) and (iii) we obtain that $(y_n)_{n=1}^{\infty}$ is an unbounded 0-increasing sequence in C, contradicting the assumption that C is 0-upbounded. ■

§ 5 Weak topologies in modular metric spaces

5.1 Motivation. Let (X,ρ) be a (modular) metric space. The *weak(ρ) topology* or, *weak(metric) topology*, is generated by the collection of closed ρ-convex subsets in X. We usually denote this topology by τ_w.

The construction of the weak(metric) topology is similar to the construction of the standard weak topology in a locally convex topological vector space V. Indeed, the collection of closed (standard) convex subsets of V yields a closed subbase for the standard weak topology.

In connection with the Alexander subbase lemma, see [22], Theorem 4.7 can be used to derive that a subset of a modular metric space that is complete and both star-shaped and upbounded at some point, e.g. a complete ball, is weakly(metric) compact. This result makes it interesting to work in modular metric spaces with a Hausdorff weak(metric) topology. The Hausdorff property of this topology requires the existence of many ρ-convex subsets. As, roughly speaking, modular (metric) spaces with an abundance of geodesically convex subsets correspond with ***median*** (metric) spaces (cf. Theorem I: 2.14, and I: 4.4(1)), the weak(metric) topology of properly ***multi***median spaces is usually not Hausdorff. Compare the situation in ***normed*** spaces: a modular normed space with a Hausdorff weak(norm) topology is median. See chapter IV.

We introduce a topology on a modular metric space that is less attached to geodesically convex subsets. Let (X,ρ) be a modular metric space and let $b\in X$. The *basepoint topology* at b, briefly the *b-topology*, $\tau(b)$ is the topology generated by the collection of closed subsets in X that are multimedian at b. Clearly, $\tau_w \subseteq \tau(b)$. Note that closed balls around b and intervals which have b as endpoint are subbase members of $\tau(b)$.

5.2 Completeness and weak(metric) compactness. The next two theorems (which are proved simultaneously) show the similarity between the basepoint topology of ***modular*** metric space and the weak(metric) topology of ***median*** metric space.

5.3 Theorem. *Let (X,ρ) be a modular metric space, and let $C\subseteq X$ be a subset star-shaped at $b\in C$. Then the following are equivalent:*

(1) C is b-upbounded and complete.

(2) C is b-compact.

5.4 Theorem. *Let (X,ρ) be a median metric space, and let $C\subseteq X$ be a subset star-shaped at $b\in C$. Then the following are equivalent:*

(1) C is b-upbounded and complete.

(2) C is weakly(ρ) compact.

Proof: We refer to the above two theorems as (A) and (B). Let X and C be as stated in either of the theorems. Implications $(1)\to(2)$ of (A) and (B) follow from the Alexander subbase lemma combined with Theorem 4.7. For a proof of the converse implications, let $(x_i)_{i=1}^{\infty}$ be a b-increasing sequence in C. In the situation (A) we let $A_i=\cup\{x_j \mid j\geq i\}$ and in the situation (B) we let $A_i=\cup\{I_\rho(x_i,x_j) \mid j\geq i\}$ for $i\in\mathbb{N}$. Compare Figure 5.4.

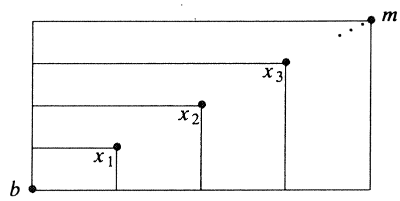

Fig. 5.4: a b-increasing sequence

Situation (A): As a subset A_i is evidently multimedian at b —compare (3.11)— so is its metric closure $cl_X(A_i)$ by (3.14). Whence, the latter set is b-closed in X, and consequently the intersection $C \cap A_i$ is b-compact in X.

Situation (B): Each subset A_i is ρ-convex, see Theorem I: 4.24(5), therefore so is its metric closure $cl_X(A_i)$ by Corollary 3.5. Consequently, the latter set is weakly(ρ)-closed, and whence the intersection $C \cap cl_X(A_i)$ is weakly(ρ) compact.

Now consider any of the situations (A), (B). As the sets $C \cap cl_X(A_i)$ have the finite intersection property, there exists a point $a \in \cap_{i=1}^{\infty} C \cap A_i$, by compactness. Then,

$$\rho(b,x_i) = \rho(b,A_i) = \rho(b, cl_X(A_i) \cap C)) \leq \rho(b,a).$$

In which the first equality holds by virtue of (3.11). Hence the sequence $(x_i)_{i=1}^{\infty}$ is bounded. Whence C is 0-upbounded.

For a proof of completeness of C, let $(x_i)_{i=1}^{\infty}$ be a bounded b-increasing sequence in C. As $(x_i)_{i=1}^{\infty}$ is a Cauchy sequence, it converges to a point m of the completion $(\tilde{X},\tilde{\rho})$ of (X,ρ). Compare Figure 5.4. In the situation (A) we let $A_i = Cl_{\tilde{X}}(\{x_j \mid j \geq i\})$ and in the situation (B) we let $A_i = I_{\tilde{\rho}}(x_i,m)$.

More or less similar as above, we obtain that each set $X \cap A_i$ is b-closed in X in situation (A), and weakly(ρ) closed in situation (B). Whence in situation (A) the sets $C \cap A_i$ are b-compact subsets in X, and in situation (B) the sets $C \cap A_i$ are weakly(ρ) compact.

Now consider any of the situations (A), (B). On one hand as the $C \cap A_i$ evidently have the finite intersection property we conclude that $\cap\{K_i \mid i \in \mathbb{N}\} \neq \emptyset$. On the other hand, we have that $\cap_{i \in \mathbb{N}} A_i = \{m\}$. Hence $\{m\} = \cap\{K_i \mid i \in \mathbb{N}\}$, and in particular $m \in C$.

We obtain that C is b-upconverging. Similarly we obtain that C is b-downconverging. Hence C is b-converging, thus C is complete by Corollary 3.17. ■

We remark that there exist geodesically convex, weakly(metric) compact subsets that are not bounded. See Remark 4.4. From Theorems 5.3, 5.4 we obtain the following corollaries.

5.5 Corollary. *Let (X,ρ) be a modular space. Then the following are equivalent:*

(1) X is complete.

(2) There exists a point $b \in X$ such that all closed balls at b are b-compact.

(3) For all $b \in X$ closed balls with b as a center are b-compact. ■

5.6 Corollary. *Let (X,ρ) be a median space. Then the following are equivalent:*

(1) *X is complete.*

(2) *All closed balls of X are weakly(ρ) compact.* ■

We now obtain two extensions of Theorem 3.21.

5.7 Theorem. *Let (X,ρ) be a modular space. Then the following are equivalent:*

(1) *X is downconverging.*

(2) *For each $b\in X$ any closed subset of X multimedian at a point b has a gate from b.*

(3) *For each $b\in X$ intervals having b as an endpoint are b-compact.*

5.8 Theorem. *The following are equivalent for a median metric space (X,ρ).*

(1) *X is downconverging.*

(2) *Each closed geodesically convex subset of X is gated.*

(3) *All intervals are weakly(ρ)-compact.*

Proof: We refer to the above two Theorems as (A) and (B). Implications $(1)\rightarrow(3)$ of (A) and (B) follow from Theorems 5.3 and 5.4. The converse implications follow by Theorem 3.21 combined with Theorems 5.3 and 5.4.

Implications $(1)\rightarrow(2)$ of Theorem (A) and (B) follow from Proposition 4.2. For a proof of the implications $(2)\rightarrow(1)$, let $b\in X$ and let $(x_i)_{i=1}^{\infty}$ be a b-decreasing sequence. In the situation (A) we let $K=\{x_1,x_2,\cdots\}$ and in the situation (B) we let A be $K=\cup_{i=1}^{\infty}I(x_1,x_i)$.

Situation (A): The set K is multimedian at b —compare (3.11)— hence so is its closure $cl_X(K)$ by (3.14). So by assumption b has a gate p in $cl_X(K)$.

Situation (B): As the set K arises from an increasing union of intervals it is ρ-convex, hence so is its closure $cl_X(K)$ by (3.14). So by assumption b has a gate p in $cl_X(K)$.

Now consider any of the situations (A), (B). Let $D(p,r)$ be a closed ball around p. As $p\in cl_X(K)$ there exists a $k\in D(p,r)\cap K$, that is $k\in I(x_n,x_1)$ for some $n\in\mathbb{N}$. By the geometric properties of the metric intervals we obtain that $x_m\in I(p,k)$ for all $m\geq n$. As balls are star-shaped at their center, we have $x_m\in D(b,r)$. Whence $(x_i)_{i=1}^{\infty}$ converges to p. ■

5.9 Relative weak topologies. Let (X,ρ) be a modular space. If $Y\subseteq X$ is a subset with $M_I(y_1,y_2,y_3)\subseteq Y$ for all $y_1,y_2,y_3\in Y$, then the two natural "b-topologies" on Y —the relative topology w.r.t. the b-topology of X, and (the coarser) intrinsic b-topology of Y— coincide. Indeed, let C be a subbase member of the intrinsic b-topology of Y, i.e. C is closed and multimedian at b in Y. As C is multimedian at b in Y, and as $M_I(Y^3)\subseteq Y$ one readily verifies that C is multimedian at b in X. By 3.14 the closure $Cl_X(C)$ is multimedian at b in X. Hence $C=Cl_X(C)\cap Y$ is a relatively closed subset of Y with respect to the b-topology of X.

A similar problem —and more relevant, see chapter IV— is whether the relative and the intrinsic weak(metric) topology of a median stable subset of a median metric space coincide. The answer is positive but requires some effort. It turns out the answer is positive even in a broader setting.

By a *topological median algebra* is meant a set X with a Hausdorff topology and a median operator on X which is continuous in this topology. See [79]. In these circumstances, the *weak topology* of X is generated by the subbase consisting of all closed convex sets. This definition is in accordance with the metric situation. Analogous one defines the notion of a *basepoint topology* in X.

Finally, X is *locally star-shaped* provided for each $p \in X$ and each neighborhood U of p there is a neighborhood V of p such that $I(p,x) \subseteq U$ whenever $x \in V$. For compact median algebras, this condition can easily be derived from the (assumed) continuity of the median operator. See van Mill and van de Vel [60]. If X is a median metric space, then the median operator is continuous and each metric ball is star-shaped from its center. Observe that the proofs of Theorems 5.7 and 5.8 implicitly use that modular metric space is locally star-shaped.

5.10 Proposition. *Let X be a locally star-shaped median algebra and let $Y \subseteq X$ be a median stable subset. Then the weak topology of Y equals the relative weak topology, derived from X.*

Proof: By definition, a convex closed subset of X has a relatively convex, relatively closed trace on Y. Conversely, let $C \subseteq Y$ be a relatively convex, relatively closed set. We verify that $Cl_X(co(C)) \cap Y = C$. Note that the closure of a convex set is convex. Next, let $y \in Cl_X(co(C)) \cap Y$ and fix a net $(d_j)_{j \in J}$ in $co(C)$ converging to y. For each j we fix a finite set $F_j \subseteq C$ with $d_j \in co_Y(F_j)$. The polytope $co_X(F_j)$ is gated; for a description of its gate function, see (3.10). In view of this description the gate c_j of $y \in Y$ is in the relative polytope $co_Y(F_j)$. In particular, $c_j \in C$.

Let $U \subseteq X$ be a neighborhood of y and let $V \subseteq U$ be as in the definition of local star-shapedness. For some $j_0 \in J$ and all $j \geq j_0$ we find $d_j \in V$, whence $c_j \in I(y,d_j) \subseteq U$. So the net $(c_j)_{j \in J}$ converges to y, showing that $y \in Cl_Y(C) = C$. ■

5.11 Separation properties of the weak(metric) topology.

5.12 Proposition. *Let (X,ρ) be a modular space such that the weak(ρ) topology is Hausdorff. Then the following hold.*
(1) If X is downconverging, then the weak(ρ) topology is regular.
(2) If X has complete balls, then the weak(ρ) topology is normal.

Proof: For a proof of statement (1), let $x \in X$ and let B be a closed interval-convex subset of X. We conclude from Theorem 5.7 that B is gated. Let p be the gate of x in B. By applying the Hausdorff property to the points p,x, we find closed interval-convex subsets $C_1, \cdots, C_n$, $D_1, \cdots, D_m$ of X with

$$p \notin \bigcup_{i=1}^{n} C_i \ , \ x \notin \bigcup_{i=1}^{m} D_i \ \text{ and } \ X = \bigcup_{i=1}^{n} C_i \cup \bigcup_{i=1}^{m} D_i. \tag{3}$$

Now let Q be the union of all sets C_i or D_i meeting B, and take $U = X \setminus Q$. Then $x \notin U$. Indeed, suppose that $x \in C_i$ and $C_i \cap B \neq \varnothing$. Then clearly the gate p of x in B is in C_i, contradicting formula (3). The same formula states that x is not a member of any D_i. As U is a (base) open element of the weak(ρ) topology, the proof is complete.

For a proof of statement (2), as observed at the beginning of this section, complete balls in modular metric space are weakly(metric) compact. We conclude that the weak(ρ) topology of X is σ-compact Hausdorff, and hence that the weak(ρ) topology is normal. ■

5.13 Proposition. *Let (X, ρ) be a median metric space with separable complete intervals. If for each countable set A the set $\overline{co}(A)$ is weakly(ρ) normal, then X is weakly(ρ) normal.*

Proof: The continuity of the interval function (Theorem 3.4), together with join-hull commutativity of a median algebra (Theorem I: 4.18), imply that each polytope of X is also separable. Now if $A \subseteq X$ is countable, then $co(A)$ obtains as the union of countably many separable spaces of type $co(F)$, with $F \subseteq A$ finite. It follows that $\overline{co}(A)$ is also separable for countable sets A. We note that by the completeness of intervals, each non-empty convex closed set of X has a gate function. See Proposition 4.2.

Let $A,\ B \subseteq X$ be two disjoint and weakly(ρ) closed sets, let $A_0 \subseteq A$, respectively $B_0 \subseteq B$ be singletons, let $K_0 = \overline{co}(A_0 \cup B_0)$, and let $p_0 \colon X \to K_0$ be the gate function.

Assume that we have constructed sequences of countable sets $A_0 \subseteq \cdots \subseteq A_n \subseteq A$, $B_0 \subseteq \cdots \subseteq B_n \subseteq B$, together with convex closed sets $K_i = \overline{co}(A_i \cup B_i)$ (where $0 \le i \le n$), such that if $p_i \colon X \to K_i$ denotes the gate mapping, then $p_i(A) \subseteq \overline{p_i(A_{i+1})}$ and $p_i(B) \subseteq \overline{p_i(B_{i+1})}$ for $i = 0, \cdots, n-1$. Now $p_n(A)$ is a subset of the separable metric space K_n, and hence there is a countable set $A_{n+1} \subseteq A$ with $p_n(A) \subseteq \overline{p_n(A_{n+1})}$. Similarly, there is a countable set $B_{n+1} \subseteq B$ with $p_n(B) \subseteq \overline{p_n(B_{n+1})}$. We may assume that $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$. Then put $K_{n+1} = \overline{co}(A_{n+1} \cup B_{n+1})$.

Having completed the inductive construction, we put $A_\infty = \cup_{n \in \mathbb{N}} A_n$, $B_\infty = \cup_{n \in \mathbb{N}} B_n$, and we let p_∞ denote the gate map onto $K_\infty = \overline{co}(A_\infty \cup B_\infty)$. Observe that $K_\infty = Cl(\cup_{n=1}^{\infty} K_n)$, and that p_∞ is the pointwise limit of the maps p_n.

Let $a \in A$ and consider a weak(ρ) neighborhood U of $p_\infty(a)$. By passing to a smaller neighborhood if necessary, we may assume that U is of type $X \setminus (\cup_{i=1}^{m} C_i)$, where the sets C_i are convex closed in X. Suppose $A_\infty \subseteq \cup_{i=1}^{m} C_i$. As $p_n(a)$ converges to $p_\infty(a)$ we have $p_n(a) \in U$ for large enough n. We may assume that each set C_i which is met by A_∞ is also met by each of the sets $A_n, A_{n+1}, \cdots$. We consider a point $a_{n+1} \in A_{n+1}$ such that $p_n(a_{n+1}) \in U$, say: $a_{n+1} \in C_i$. However, $A_n \subseteq K_n$ meets C_i as well and hence $p_n(a_{n+1})$ should be in C_i.

We have shown that $p_\infty(a)$ is weakly(ρ) adherent to A_∞. Note that $p_\infty(A_\infty) = A_\infty$, and hence that $p_\infty(A) \subseteq Cl_w(p_\infty(A_\infty))$. Here Cl_w stands for weak(ρ) closure. We conclude that $p_\infty(A) \subseteq A$. In the same way, one can show that $p_\infty(B) \subseteq B$. It follows that $p_\infty(A)$ and $p_\infty(B)$ have a disjoint weak(ρ) closure. By assumption, these sets can be separated with weakly(ρ) open subsets of K_∞.

The gate function p_∞ is continuous (Proposition 4.1) and convexity preserving (see the remarks prior to Theorem I: 4.8). Hence taking inverse images under p_∞ yields the desired separation of A and B. ■

5.14 Corollary. *If (X,ρ) is a median-stable subset of $l_1(I)$ with complete relative intervals, then X is weakly(ρ) normal.*

Proof: We first note that by the definition of weak(metric) topology, the original and weak(metric) closure of a convex set in a modular metric space are always the same. Let $A \subseteq X$ be a countable set. Since the weak(metric) convex closure of a countable subset of $l_1(I)$ is essentially a subset of $l_1(\mathbb{N})$, it follows that $\overline{co}(A)$ (convex closure in $l_1(I)$) is a weakly(metric) metrizable set. See Theorem IV: 4.2. Now, the convex closure of A relative to X is a subset of the above one, and the choice between intrinsic weak(metric) or relative weak(metric) topology is indifferent by Proposition 5.10. Application of Proposition 5.13 gives the desired result. ■

Compare Proposition 5.12 with the above corollary. It is not known whether median metric spaces are isometrically embeddable in some $l_1(I)$ space under the assumption of being weakly(metric) Hausdorff. See Theorem IV: 2.9, for an affirmative result on median ***normed*** spaces. The complete median metric spaces that can be embedded in an $l_1(I)$ space evidently correspond with ***closed*** median stable subsets. The assumption in Corollary 5.14, on completeness of relative intervals, leads one to relatively convex subsets of closed median stable sets in $l_1(I)$, in regard of Theorem 3.21.

5.15 Comparing b-topologies and weak(metric) topologies. The following two results indicate that a b-topology only "looks" in one direction, whereas the weak(metric) topology "looks" in all directions.

5.16 Theorem. *Let X be a locally star-shaped median algebra. Then, the weak topology of X is the largest topology on X coarser than each b-topology, viz.,*

$$\tau_w = \bigcap_{x\in X} \tau(x),$$

Proof: The inclusion from left to right is evident. As for the reverse inclusion, let C be a member of the right-hand side. Let $x \notin C$. By assumption C is closed in τ_x. Hence there exist closed subsets $D_1, D_2, \cdots, D_n$, multimedian at x with

$$x \notin \cup_{i=1}^n D_i \quad ; \quad C \subseteq \cup_{i=1}^n D_i.$$

Let U be a star-shaped neighborhood of x avoiding $D_1, \cdots, D_n$. Suppose that $y \in U \cap co(D_i)$. Then there exists a finite subset F of D_i with $y \in co(F)$. As D_i is downdirected in $\leq_x$, there exists a $d \in D_i$ with $d \leq_x f$ for all $f \in F$. As $y \in co(F)$, we also have that $d \leq_x y$. But then $d \in I(x,y) \subseteq U$, a contradiction. We conclude that $x \notin \cup_{i=1}^n cl_X(co(D_i))$. In other words, x is not weakly(metric) adherent to C. As $x \notin C$ was arbitrary we obtain $C \in \tau_w$. ■

One can also show:

5.17 Theorem. *Let (X,ρ) be a downconverging modular metric space. If the weak(ρ) topology is Hausdorff, then*

$$\tau_w = \bigcap_{x\in X} \tau(x).$$

■

It is well-known that two topologies $\tau_1 \subseteq \tau_2$ on a set X with (X,τ_1) Hausdorff and (X,τ_2) compact, coincide. This leads to the following result.

5.18 Theorem. *If (X,ρ) is a complete bounded modular space with a Hausdorff weak(ρ) topology, then $\tau_w = \tau(b)$ for all $b \in X$. If, in addition, (X,ρ) is compact, then $\tau_\rho = \tau_w$ as well.* ■

§ 6 All median operators are G-metric

Let G be an Abelian lattice group G. One may think of a Riesz space. A *G-metric* on a set X is a map $\rho : X^2 \to G$ satisfying the following conditions:

(i) $\rho(a,b) > 0$ if $a \neq b$; $\rho(a,a) = 0$,

(ii) $\rho(a,b) = \rho(b,a)$,

(iii) $\rho(a,b) \leq \rho(a,c) + \rho(c,b)$,

for all $a,b,c \in X$. See Hung [39]. The pair (X,ρ) is called a *G-metric space*. In analogy to the situation in metric intervals one verifies that a G-metric interval operator—with its obvious meaning— is geometric. This finally leads us to the class of modular/median G-metric spaces.

By a similar replacement of the real numbers by G in the definition of a valuation one obtains the concept of a *G-valuation*, and a *G-metric lattice*. In analogy to the situation in metric lattices one verifies that the formula

$$\rho(x,y) = |x \vee y| - |x \wedge y| \quad (x,\ y \in L),$$

yields a G-metric if v is a positive G-valuation —see [13, ex. 4 p. 234]).

Observe that the mapping $v : G \to G$ given by $v(x) = x$ yields a positive G-valuation. The induced G-metric is simply given by $\rho(x,y) = |x-y|$, where $|.|$ denotes the modulus operator of G (see I: 1.2).

The following is a modification of Lemma 1.6.

6.1 Lemma. *A G-metric lattice L is modular, and G-metric betweenness in L is equivalent with lattice-betweenness.*

Proof: We reason as in [13, p. 232], [16, p. 58]. If L is non-modular, then it contains the lattice N_5 as a sublattice. In the notation of Figure I: 1.1B we then have

$$v(x) + v(y) = v(x \wedge y) + v(x \vee y) = v(p) + v(q) = v(x \wedge y') + v(x \vee y') = v(x) + v(y').$$

That is, $v(y') = v(y)$. Whence $y = y'$ as v is positive, a contradiction.

Denote the G-metric intervals of L by $I_\rho(.,.)$. Next, let $a,b \in L$ and $x \in I_l(a,b)$, that is $(a \wedge x) \vee (b \wedge x) = x = (a \vee x) \wedge (b \vee x)$.

$$\begin{aligned}\rho(a,x) + \rho(x,b) &= v(a \vee x) - v(a \wedge x) + v(b \vee x) - v(b \wedge x)\\ &= v(a \vee x) + v(b \vee x) - (v(a \wedge x) + v(b \wedge x))\\ &= v(a \vee b) + v(x) - (v(a \wedge b) + v(x))\end{aligned}$$

$$= \nu(a \vee b) - \nu(a \wedge b) = \rho(a,b).$$

The third equality uses $(a \vee x) \vee (b \vee x) = a \vee b$ and $(a \wedge x) \wedge (b \wedge x) = a \wedge b$, see Proposition I: 2.7(1).
We find that $I_l(a,b) \subseteq I_\rho(a,b)$ and by Theorem I: 4.5 this gives $I_l = I_\rho$. ∎

From the previous result together with Theorem I: 4.2 we obtain:

6.2 Theorem. *A G-normed lattice L is a modular G-metric space, and the metric multimedian and the lattice multimedian coincide. The lattice L is a median G-metric space if and only if L is distributive.* ∎

As an (Abelian) lattice group is distributive [13], we conclude from Theorem 6.2 that the (standard) median operator of an Abelian lattice group G is induced by the G-metric intervals. Whence, we can look upon Riesz spaces as (linear) G-normed median spaces. We shall show in chapter III, that (genuine) normed median spaces correspond with (subspaces of) $L_1(\mu)$ spaces. The following is a generalization of this.

6.3 Theorem. *Each median space corresponds with a median stable subset of a Riesz space. In particular, all median operators are G-metric.*

Proof: Let (X,m) be a median algebra. Then X can be seen as a median stable subset of a Boolean algebra $\mathcal{A}$, see Corollary I: 2.18. Now the collection of step-functions on $\mathcal{A}$, yields a Riesz space L (see [52, p. 178]), in which $\mathcal{A}$, and hence X, occurs as a median stable subset. In view of Theorem 6.2 the map $\rho: L^2 \to L$ given by $\rho(r,s) = |r-s|$, yields a G-metric that generates the median of L. By taking the restriction of ρ to X we obtain a G-metric as desired. ∎

In chapter V we shall show that median ***metric*** spaces correspond with median subsets of normed median spaces, the $L_1(\mu)$ spaces.

Let m be the median of a median G-metric space (X,ρ). Many of the results on metric medians derived in the present chapter extend to the G-metric situation. For instance, a G-median m is contractive with respect to the sum-metric on X^3 — which evidently yields an Abelian lattice group. We also mention an extended version of Theorem 2.6(1) (writing xyz for $m(x,y,z)$):

$$\rho(a,abc) + \rho(b,abc) + \rho(c,abc) + \rho(d,abc) = \rho(a,bcd) + \rho(b,bcd) + \rho(c,bcd) + \rho(d,bcd).$$

for all $a,b,c,d \in X$.

If a lattice group G is (conditionally) complete and totally ordered, then some of the results for metric multimedians extend to the G-metric situation. For instance, a G-metric multimedian is a Lipschitz map of factor 2 with respect to the "Hausdorff G-metric" and the sum-metric on X^3.

CHAPTER III

MODULAR NORMED SPACES

The present chapter is devoted to modular ***normed*** spaces. In chapter II (Theorem II: 1.8) an example of a median normed spaces appeared: an (abstract) L-space e.g. an $L_1(\mu)$ space. One of the aims of this chapter is to prove that ***all*** median Banach spaces are of the latter type. This is not only an interesting result in its own right but it also opens new perspectives in the study of $L_1(\mu)$ spaces: this result describes $L_1(\mu)$ spaces as Banach spaces with a special median convexity. Descriptions of $L_1(\mu)$ spaces in terms of metric betweenness have appeared earlier in literature. See [76, Theorem 3], where it shown that a Banach lattice is an $L_1(\mu)$ iff the "lattice betweenness" coincides with the metric betweenness.

In the process of showing this characterization, it turned out that there are many similarities between inner product (i. p.) spaces and median normed spaces. In fact, it is possible to use techniques from i. p. spaces to get elegant proofs and results in $L_1(\mu)$ spaces. An example of such a similarity is that both types of spaces are characterized by a "tri-spherical intersection property". See the paper of Comfort and Gordon [17], for such a result in i. p. space. The most striking similarity lies in the notion of "orthogonality".

In sections 3 and 4 we characterize modular normed spaces with additive orthogonality. In section 3 we do this for a distinct class of normed spaces, among which are the finite dimensional ones. Here the characterization is self-contained and uses techniques coming from Hilbert space. The general characterization appears in section 4, and involves the Kakutani representation theorem.

In section 5 we describe some of the rather peculiar properties of geodesically convex subsets in modular normed space. From these properties new characterizations of $L_1(\mu)$ spaces are obtained.

§ 1 Introduction and motivation

1.1 Orthogonality. By a *modular normed space*, we mean a normed vector space such that the induced metric space is modular. In other words, the mixing-operator of a modular normed space only takes non-empty values. By a *median normed space*, we mean a normed space such that the induced metric space is median. In this situation, the values of the mixing-operator are singletons, and the mixing operator is called a *normed median*. The term modular Banach space should speak for itself.

Metric intervals and geodesically convex subsets of a normed space are usually called *norm intervals* and *norm-convex* respectively. Note that a norm interval I is invariant under translation and under multiplication with scalars, i.e.

$$I(x+p,x+q)=x+I(p,q) \ ; \ \lambda \cdot I(p,q)=I(\lambda \cdot p,\lambda \cdot q),$$

for all points p,q,x and $\lambda \in \mathbb{R}$. The term "star-shaped" in normed space is always used with respect to the norm intervals, and ***not*** with respect to the standard convex intervals.

We first recall the following. Two points x,y of a normed space X are called *orthogonal in the Pythagorean sense*, briefly $x \perp_P y$, provided $||x+y||^2=||x||^2+||y||^2$. See [3], or [43]. Let us call a binary relation R on X *additive* if $x\,R\,z$ and $y\,R\,z$ implies $(x+y)\,R\,z$. The following result of James is well-known, see [43].

A normed space is linearly isometric with an i. p. space iff $\perp_P$ *is additive.*

Let us next consider two points x,y in a normed X that satisfy $||x+y||=||x||+||y||$. In view of the above, it might be natural to call such points "orthogonal", however it turns out that the name *codirectional*, as introduced in the paper of Alfsen and Effros [2], is more appropriate. In [2] the points x,y are called *antidirectional* if $||x-y||=||x||+||y||$. Directionality gives rise to a notion of orthogonality as follows; the points x,y are called *median orthogonal*, briefly $x \perp y$, provided x,y are both codirectional and antidirectional. Further motivation of this definition shall arise later.

One can easily verify that two points x,y in a normed space are orthogonal iff both values $M(0,x,y)$, $M(0,x,-y)$ of the mixing operator equal zero (observe that the conditions $0 \in M(0,x,y)$ and $M(0,x,y)=\{0\}$ are equivalent). In an i. p. space the value $M(0,x,y)$ of the mixing operator is empty iff the points $x,y \in X$ are independent —in particular, i. p. spaces of dimension at least two are not modular. So in these circumstances the points x,y are median orthogonal iff one of the points is the origin. Hence, in particular situations (e.g. in i. p. spaces) additivity of the median orthogonality does not provide any information, simply because all orthogonal points are trivial.

A modular normed space has an abundance of orthogonal points (cf. §4), and it turns out that in such spaces the additivity of $\perp$ characterizes ***median*** normed spaces. In particular it follows that $L_1(\mu)$ spaces correspond with modular Banach spaces with an additive orthogonality.

1.2 Riesz spaces. We compare our notion of "orthogonality" with a similar notion from the theory of Riesz spaces. As usual we let $|x|$ denote $x \vee 0 - x \wedge 0$ $(x \in L)$. Two points $x,y \in L$ are called *Riesz-orthogonal* provided $|x| \wedge |y|=0$. See [52], where it is shown that Riesz-orthogonality of the points x,y is equivalent with the property $|x+y|=|x-y|=|x|+|y|$. That is, the (L-metric) medians $m(0,x,y)$, $m(0,x,-y)$ both equal zero (Theorem II: 6.2). Whence Riesz-orthogonality coincides with "median orthogonality". We remark that Riesz-orthogonality is additive.

§ 2 Preliminaries

Let X be a normed space with unit ball B. A convex subset F of B is called a *face* of B whenever $p,q \in F$ if $\lambda \cdot p+(1-\lambda) \cdot q \in F$ for $p,q \in B$ and $\lambda \in (0,1)$. The face F is called proper if $F \neq B$. A one-point set $\{p\}$ is a face iff p is an extreme point of B. By Zorn's lemma there exist *maximal* proper faces of the unit ball. See [2], where it is also shown that these faces are closed. If $||p||=1$ then we define $face(p)=\cap\{F \mid F \textit{ is a face containing } p\}$. We shall call these faces *minimal*.

A subset C of X is called a *cone* if $\lambda \cdot C \subseteq C$ for all non-negative λ. A cone C is convex iff $C+C \subseteq C$. One calls a cone C *proper* if $C \cap -C=(0)$. If $C-C=X$, then C is called a *generating cone*. The linear span of a convex cone C (i.e. the smallest linear subspace containing C) is given by the set $C-C$. If A is any set in X then the cone generated by A, $cone(A)$, is the smallest cone in X containing A.

A cone C in X is called *facial* if C is generated by a proper face F of B (i.e. $C=cone(F)$). If $0 \neq p \in X$ then the cone generated by $face(p/||p||)$ is denoted by $C(p)$. If $p=0$ then we let $C(p)=\{0\}$. The cone $C(p)$ is the smallest facial cone containing p. Every facial cone is convex and proper. It shall be convenient to relate facial cones with norm-intervals. To this end, let us mention the following results of Alfsen and Effros [2].

2.1 Lemma. ([2, Lemma 2.7]). *Let C be a (standard) convex cone in a normed space X. The following are equivalent:*

(1) C is a facial cone.

(2) C is star-shaped (w. r. t. the norm, see §1) at the origin and every pair $x,y \in C$ is codirectional. ■

2.2 Lemma. ([2, Lemma 2.6]). *If p is a point of a normed space X, then:*

$$C(p)=\bigcup_{\lambda \geq 0} I(0,\lambda \cdot p)=\bigcup_{\lambda \geq 0} \lambda \cdot I(0,p).$$ ■

2.3 Lemma. ([2, Lemma 2.3]). *The following are equivalent for points x,y in a normed space X.*

(1) There is a facial cone containing x,y.

(2) $x,y \in C(x+y)$.

(3) $||x+y||=||x||+||y||$. ■

For any proper, convex cone C in a vector space, we let $\leq_C$ denote the vector order induced by C, i.e. $x \leq_C y$ iff $y-x \in C$. From Lemma 2.1 we conclude the following result, which shall be of later use.

(2.4) Let C be a facial cone. Then the vector order $\leq_C$ on C coincides with the basepoint order at zero.

From Lemma 2.3 we deduce also that if two points are codirectional then so are non-negative multiples of these points. There is a similar result for antidirectional points. In particular, if two points are orthogonal then so are all multiples of these points. We shall use these properties without further reference.

The following result is a simple characterization of ***normed*** (linear) spaces that are upconverging. Its proof is a modification of [71, Ch. II, Proposition 8.2], which concerns completeness of L-spaces.

2.5 Proposition. *The following are equivalent for a normed space* $(X, \|.\|)$.

(1) *There is a* $b \in X$ *such that every b-increasing sequence bounded in norm has a supremum in* $(X, \leq_b)$.

(2) *There is a* $b \in X$ *such that the basepoint order* $\leq_b$ *is upconverging.*

(3) *All basepoint orders are upconverging.*

Proof: From the invariance of norm intervals under translation it simply follows that if (1) or (2) hold for some $b \in X$, then they hold for all $b \in X$. This implies in particular that statements (2) and (3) are equivalent. Implication (3)$\rightarrow$(1) is Proposition II: 3.8. For a proof of implication (1)$\rightarrow$(2), by the above remark we may assume that $b = 0$. We first prove the following intermediate result.

(4) Let $(x_n)_{n=1}^{\infty}$ be a sequence of codirectional points in X with $\|x_n\| < 2^{-2n}$, and let $s_i = \sum_{n=1}^{i} x_n$ for $i \in \mathbb{N}$. Then the sequence $(s_i)_{i=1}^{\infty}$ converges in X.

To this end, let $y_n = 2^n \cdot x_n$ and $z_i = \sum_{n=1}^{i} y_n$ for $i \in \mathbb{N}$. Then the sequences $(s_i)_{i=1}^{\infty}$, $(z_i)_{i=1}^{\infty}$ are clearly bounded, and increasing in $\leq_0$ by Lemma 2.3. We denote the suprema of these sequences by s,z respectively. Then for all $k,n \in \mathbb{N}$ with $k > n$ we have $s_k - s_n \leq_0 2^{-(n+1)} y$ —compare (2.4). Hence $s - s_n \leq_0 2^{-(n+1)} y$, and consequently $\|s - s_n\| \leq 2^{-(n+1)} \|y\|$. Whence $s_n \rightarrow s$, which concludes the proof of statement (4).

To obtain implication (4)$\rightarrow$(2); given any bounded 0-increasing sequence, it suffices to show that some subsequence converges. Let $(p_n)_{n=1}^{\infty}$ be a subsequence that satisfies $\|p_{n+1} - p_n\| < 2^{-2n}$. Let $x_n = p_{n+1} - p_n$ for $n \in \mathbb{N}$, then all x_n are codirectional —compare Lemmas 2.1 and 2.1. Hence the sequence $(x_n)_{n=1}^{\infty}$ satisfies the assumptions of (4). From this statement it now follows that $s_n = p_n - p_1$ constitutes a converging sequence in X. ■

We do not know whether the dual version of Proposition 2.5 —concerning decreasing sequences and infima— holds.

A normed space that is upconverging at all of its points need not be complete. Indeed, as the norm intervals of an inner product space coincide with the (complete) convex segments, any non-complete inner product space yields a counterexample. By combining Proposition 2.5 and Theorem II: 3.19, we obtain the following affirmative result for ***modular*** normed spaces, which generalizes [71, Proposition 8.2].

2.6 Corollary. *The following are equivalent for a modular normed space X.*

(1) *There is a* $b \in X$ *such that every b-increasing sequence bounded in norm has a supremum in* $(X, \leq_b)$.

(2) *There is a* $b \in X$ *such that the basepoint order* $\leq_b$ *is upconverging.*

(3) *X is complete.* ■

In contrast with Corollaries II: 3.18, and 2.6 the completion of a modular normed space X is not given by X^0. See Remark III: 3.7.

The two notions of directionality can be formulated in terms of an "inner product" as follows: for $x,y \in X$ we define the *median inner product* by

$<x,y> = \frac{1}{2}(||x|| + ||y|| - ||x-y||)$.

Then the points x,y are codirectional iff $<x, -y> = 0$.

2.7 Proposition. (continuity of the inner product) *Let $(X, ||.||)$ be a normed space, then the following equality holds for all $x,y \in X$:*

$<x,y> \leq Max(||x||, ||y||)$.

In particular, the inner product is a uniformly continuous function of X^2 into $\mathbb{R}$.

Proof: Let $x,y \in X$. By the triangle inequality we have $||y|| \leq ||x|| + ||x-y||$. This implies

$\frac{1}{2}(||x|| + ||y|| - ||x-y||) \leq ||x||$.

The left-hand side is precisely $<x,y>$. Permuting the role of x and y concludes the proof of the proposition. ■

If m is a point in the value $M(x,y,0)$ of the mixing operator, then the $<x,y> = ||m||$ (cf. Proposition II: 1.11). For convenience we write $||M(x,y,0)||$ for $||m||$.

The following is a well-known description of when a convex subset in an i. p. space admits metric nearest points.

Let C be a convex subset of an i. p. space $(X, <.,.>_I)$, and let $x \in X$, $p \in C$. T.f.a.e.:

(1) $<x-p, c-p>_I \leq 0 \quad \forall c \in C$.

(2) p is a (unique) metric nearest point of x in C.

We work towards such a description in modular normed space.

2.8 Proposition. *Let C be a subset of a normed space $(X, ||.||)$, and let $x \in X$, $p \in C$. Then the following are equivalent.*

(1) $<x-p, c-p> = 0 \quad \forall c \in C$.

(2) p is the gate of x in C.

If X is modular and C is norm-convex, then conditions (1), (2) are also equivalent with:

(3) p is a metric nearest point of x in C.

Proof: For a proof of the equivalence of statements (1) and (2), by the definition of (metric) mixing operator M, a point p is the gate of x in C iff $M(x,c,p) = p$ for all $c \in C$. As the mixing operator is translation invariant, the last is equivalent with $M(x-p, c-p, 0) = 0$ for all $c \in C$, that is $<x-p, c-p> = 0$ for all $c \in C$. The last statement is shown in Proposition II: 4.2. ■

Let X be a normed space. For a subspace N of X we define the *orthogonal complement* by $N^{\perp} = \{x \in X \mid x \perp z \ \forall z \in N\}$. By the multiplicative stability of orthogonality, the line through a member of $N^{\perp}$ is contained in $N^{\perp}$. That is, $N^{\perp}$ is a (non-proper) cone. This cone is not necessarily a subspace of X. See Theorem 2.11 for a partial characterization.

We remark that the set $N^{\perp}$ is generally not comparable with the *complementary cone* N' of N as introduced in [2]. However if X is modular then $N^{\perp} \subseteq N'$ and equality holds if N is norm-convex. In fact, for ***complete*** subspaces N equality holds iff N is norm-convex (compare Theorem 2.9 below with [2, Theorem 2.9]).

We recall the following concepts. Let V be any vector space, and let W_1 and W_2 be cones in V. Then V is called *decomposable* into W_1 and W_2 if every $x \in V$ can be written as $x = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$. In this case w_i is called the W_i *component* $(i = 1,2)$. If the points w_1, w_2 are uniquely determined by x, then we use the term *unique decomposition*. We now come to a decomposition theorem for norm-convex subspaces.

2.9 Theorem. *The following are equivalent for a subspace N of a normed space X:*

(1) N has a gate function p.

(2) Every point $x \in X$ can be decomposed into a point of N and of $N^{\perp}$.

(3) Every point $x \in X$ can be uniquely decomposed into a point of N and of $N^{\perp}$.

For such a subspace N and a point $x \in X$, the unique decomposition into $N, N^{\perp}$ is given by

$$x = p(x) + (x - p(x)).$$

If X is a modular space with complete intervals, then (1), (2), (3) are also equivalent with:

(4) N is closed and norm-convex.

Proof: We first show the following assertion:

(*) If X can be decomposed in N and $N^{\perp}$ then this decomposition is unique. Moreover, the function $p : X \to N$, assigning to $x \in X$ its N-component, is the gate function of N.

To this end, let $x = \tilde{x} + x^{\perp}$ be a decomposition of $x \in X$ in N and $N^{\perp}$. Hence, $x - \tilde{x} = x^{\perp} \in N^{\perp}$, and thus by Proposition 2.8 the point $\tilde{x}$ is the gate of x in N. As gates are unique, so is the decomposition of X in N and $N^{\perp}$. The rest of assertion (*) easily follows.

For a proof of implication $(1) \to (2)$, take $x \in X$. By Proposition 2.8, $x - p(x) \in N^{\perp}$. Therefore $x = p(x) + (x - p(x))$ is a decomposition of x in N and $N^{\perp}$. Implications $(2) \to (3)$ and $(3) \to (1)$ directly follow from assertion (*).

The last part of the lemma is shown in Proposition II: 4.2 for general modular metric spaces ■

As $N^{\perp}$ is a cone, the unicity part of Theorem 2.9 yields:

2.10 Corollary. *A gate function p on a subspace of a normed spaces satisfies: $p(\lambda \cdot x) = \lambda \cdot p(x)$ for every $x \in X$ and $\lambda \in \mathbb{R}$.* ■

The following theorem characterizes when a subspace N has a ***linear*** gate function.

2.11 Theorem. *The following are equivalent for a subspace N of a normed space X.*

(1) N has a gate function and $N^{\perp}$ is a subspace.

(2) N has a linear gate function.

(3) There is a linear projection $p : X \to N$ satisfying $||x|| = ||p(x) - x|| + ||p(x)||$ for all $x \in X$.

Moreover, in any of the above situations $N^{\perp}$ has a gate function given by $x \to x - p(x)$.

Proof: The proof of implication $(1) \to (2)$ immediately follows from the uniqueness of the decomposition of X in N and $N^{\perp}$. For a proof of implication $(2) \to (3)$, the norm condition occurring in (3) simply states that $p(x) \in I(x, 0)$, which is valid for every gate function on a subspace of X.

For a proof of implication (3)$\to$(1), we first show that p is the gate function of N. To this end, take $x\in X$ and $z\in N$. By assumption and by linearity of p, we have

$$||x-z|| = ||p(x-z)-(x-z)|| + ||p(x-z)|| = ||p(x)-p(z)-(x-z)|| + ||p(x)-p(z)||,$$

and as p is a projection: $||x-z|| = ||p(x)-x|| + ||p(x)-z||$. Whence $p(x)\in I(x,z)$. As $z\in N$ was arbitrary we conclude that p is the gate function on N. Next, we shall show that $N^\perp$ has also a gate function. Consider the map $p':X\to N^\perp : x\to x-p(x)$. By the assumption of (3) $||x|| = ||p'(x)-x|| + ||p'(x)||$ for all $x\in X$. Clearly, p' is a linear projection on $N^\perp$. The foregoing argument (applied to p' instead of p) yields that p' is the gate function of $N^\perp$. In particular, $N^\perp$ is norm-convex. Finally, as metric intervals are (standard) convex, $N^\perp$ is (standard) convex. As $N^\perp$ is a cone, $N^\perp$ is a subspace. ■

We remark that subspaces N satisfying (3) were introduced by Cunningham [19], who calls these spaces *L-summands*; the associated projections are called *L-projections*. See also [2]. Let $Ext(C)$ denote the set of extreme points of a convex subset C in a linear space.

2.12 Corollary. *Let X be a normed space with unit ball B. If $N\subseteq X$ is a subspace, then:*

(1) If N has a gate function and N is of codimension 1, then N is an L-summand.

(2) If N is a non-trivial L-summand in X, then the following equality holds:

$Ext(B) = Ext(B\cap N)\cup Ext(B\cap N^\perp)$.

Proof: In view of Theorem 2.9, there exists a $0\neq p\in N^\perp$. We show that $N^\perp$ equals the line through p. Assume to the contrary that $q\in N^\perp$, is not a member of this line, i.e. p,q are independent points. As the codimension of N is one, there exists a non-zero element $l\in N$, and a scalar λ such that $q=\lambda\cdot p+l$. As $q\in N^\perp$ we have $M(0,\lambda\cdot p+l,l)=0$, that is $M(-l,\lambda\cdot p,0)=-l$. However the last left-hand side is zero as $\lambda\cdot p\in N^\perp$, a contradiction. In particular we conclude that $N^\perp$ is linear. Whence N is an L-summand by Theorem 2.11.

For a proof of (2), let p be the L-projection of N. First, let e be an extreme point of B, and let $s=||p(e)||$, $t=||e-p(e)||$. By Theorem 2.11(3) we have $s+t=1$. If both s,t are non-zero, then the equality $e=s\cdot[p(e)s^{-1}]+t\cdot[(e-p(e))t^{-1}]$ would contradict the assumption that e is extreme. So one of s,t is zero, whence e is in the right-hand side of the formula in (2). For a proof of the other inclusion, let e be an extreme point of $B\cap N$. Assume to the contrary that $e=t\cdot a+(1-t)\cdot b$ for some $0<t<1$ and $a,b\neq e$ in the unit sphere. After taking images under the linear projection p, we obtain that e is a convex combination of the points $p(a),p(b)$. As an L-projection p is non-expansive (in fact all gate functions are non-expansive) the points $p(a),p(b)$ are contained in the unit ball of X. Hence one of these points, say $p(a)$, equals e. So in particular $||p(a)||=1$. So after evaluating the formula $||a|| = ||p(a)-a|| + ||p(a)||$, we conclude that $a=e$, a contradiction. We similarly obtain the other part of the inclusion. ■

The following result shows that many normed spaces have norm-convex subspaces.

2.13 Proposition. *Let $(X, ||.||)$ be a normed space with unit ball B.*

(1) The following are equivalent for a unit vector $e \in X$:

(i) e is an extreme point of B.

(ii) $C(e) = \{\lambda \cdot e \mid \lambda \in \mathbb{R}\}$.

(iii) $I(0,e) = \{\lambda \cdot e \mid \lambda \in [0,1]\}$.

In particular, a line through an extreme point of the unit ball is a norm-convex subspace.

(2) If X is modular, then non-antipodal extreme points are mutually orthogonal.

Proof: The proof of part (1) follows from Lemma 2.2.

For a proof of part (2), take different, non-antipodal extreme points e_1, e_2 of B. Clearly e_1 and e_2 are independent. To show the orthogonality of e_1 and e_2, it suffices to show that $M(0,e_1,e_2) = 0$. To this end, take $m \in M(0,e_1,e_2) \neq \varnothing$. By the minimality of the intervals $I(0,e_1)$ and $I(0,e_2)$ and the definition of $M(0,e_1,e_2)$, we obtain that m is a multiple of both e_1 and e_2. By the independence of e_1 and e_2 we conclude that m equals zero. ■

We obtain a simple description of facial cones in modular normed space.

2.14 Corollary. *Let X be a modular normed space with completion $\tilde{X}$. Then,*

(1) A convex cone C is facial iff it is star-shaped at the origin and proper.

(2) The completion of a facial cone in X is a facial cone in $\tilde{X}$.

In particular, a point which is extreme in the unit ball of X is extreme in the unit ball of $\tilde{X}$.

Proof: The "only if" part of (1) is Lemma 2.1. Conversely, let C be a proper, convex cone, star-shaped at the origin. In view of Lemma 2.1 we only have to verify that all points in C are codirectional. To this end, let $x,y \in C$. Consider the following inclusions:

$$M(0,x,-y) \subseteq I(0,x) \cap I(0,-y) \subseteq C \cap -C = \{0\}. \tag{3}$$

The first inclusion only invokes the definition of the mixing operator, the second inclusion holds by star-shapedness at 0 of C, whereas the equality holds by properness of C. As the set $M(0,x,-y)$ is non-empty we conclude that $M(0,x,-y) = \{0\}$, that is x,y are codirectional.

For a proof of the statement (2), let $C(F)$ be a facial cone in X. Let $Cl_{\tilde{X}}(C(F))$ be the completion of $C(F)$. We shall apply Lemma 2.1. It is shown in Corollary II: 3.5, that if C is a subset of a modular metric space X star-shaped at a point c, then the completion of C is star-shaped at c in the completion of X. The cone $C(F)$ is star-shaped at the origin, whence so is the completion $Cl_{\tilde{X}}(C(F))$ of $C(F)$. As points in $Cl_{\tilde{X}}(C(F))$ are clearly codirectional, we can apply Lemma 2.1 to conclude that $Cl_{\tilde{X}}(C(F))$ is a facial cone.

The last statement follows directly from statement (2) and Proposition 2.13(1). ■

2.15 Problem. *Is the completion of a maximal face of the unit ball in modular normed space maximal in the completion?*

It is well-known that $L_1([0,1])$ has no extreme points in its unit ball. From the previous corollary we conclude that the L-space R consisting of the Riemann integrable functions on $[0,1]$ can not have extreme points in its unit ball either.

§ 3 Characterizing a class of modular normed spaces

Note that if the ***co***directionality relation of a normed space X is additive, then so is the ***an-ti***directionality relation (and vice versa). Let us say that X has an *additive directionality* if either of the above conditions is satisfied.

Clearly, an additive directionality implies an additive orthogonality. In section 4 we shall show that for modular normed space the reverse implication also holds.

We start with a fundamental lemma.

3.1 Lemma. *A median normed space has an additive directionality.*

Proof: Let $x,y,z\in X$ be such that both pairs x,z and y,z are antidirectional, that is: $M(0,x,z)=M(0,y,z)=0$. Let $w=(x+y)/2$, then by convexity $w\in I(x,y)$. Using the five-point transitive rule of medians (see I: 2.15) we obtain the following equalities:

$$M(0,z,w)=M(0,z,M(x,y,w))=M(M(0,z,x),M(0,z,y),w)=M(0,0,z)=0.$$

So the points w and z are antidirectional, hence so are $2\cdot w=x+y$ and z. ■

Let $(X,||.||)$ be a modular normed space. A collection $\{a_i \mid i\in I\}$ in a normed space is called *orthonormal* if each a_i is a unit vector and distinct a_i,a_j are median orthogonal. The following is easily verified.

3.2 Proposition. *Let $(X,||.||)$ be a modular normed space with an additive orthogonality. Let A be a subset of X, then*

(1) $x\in(sp(A))^\perp$ *iff* $x\perp a$ $\forall a\in A$.

(2) If A is an orthonormal set then A is independent. ■

If $f_i:X\to X$ for $i=1,2,\cdots,n$ are functions, then the composition $f_1\circ\cdots\circ f_n$ will be denoted by $\prod_{i=1}^n f_i$.

3.3 Lemma. *Let $(X,||.||)$ be a modular normed space with an additive orthogonality, and let $a_1,\cdots,a_n$ be non-antipodal extreme points of the unit ball. If p_i, $p_i^\perp$ are the gate functions of $sp(a_i)$ and $sp(a_i)^\perp$ respectively ($i=1,2,\cdots,n$), then*

(1) The gate function of the subspace $sp(a_1,\cdots,a_n)^\perp$ is given by the composition $\prod_{i=1}^n p_i^\perp$.

(2) For every $x\in X$ the following equality holds: $x=\sum_{i=1}^n p_i(x)+(\prod_{i=1}^n p_i^\perp)(x)$.

(3) The linear span $sp(a_1,\cdots,a_n)$ is norm-convex and the gate function of this span is given by $\sum_{i=1}^n p_i$.

Proof: For a proof of part (1) of the lemma, by Proposition 3.2 we obtain the equality

$$sp(a_1,\cdots,a_n)^\perp=\cap_{i=1}^n sp(a_i)^\perp. \tag{4}$$

By the transitive rule of gate functions —(I: 3.4.4)— we obtain that the gate function of the subspace $sp(a_1,\cdots,a_n)^\perp$ equals $\prod_{i=1}^n p_i^\perp$, showing part (1).

For a proof of part (2) we shall show by induction on k that the following equality holds for all $1\le k\le n$

$$x=\sum_{i=1}^k p_i(x)+(\prod_{i=1}^k p_i^\perp)(x). \tag{Q(k)}$$

To this end, by Proposition 2.13(1), the subspace $sp(a_1)$ is norm-convex so (Q(1)) obtains from Theorem 2.11. Assume $Q(k)$ for $1 \le k \le n-1$. Applying Theorem 2.11 with respect to the norm-convex set $sp(a_{k+1})$ and the point $(\prod_{i=1}^{k} p_i^{\perp})(x)$ yields:

$$(\textstyle\prod_{i=1}^{k} p_i^{\perp})(x)) = p_{k+1}((\prod_{i=1}^{k} p_i^{\perp})(x)) + p_{k+1}^{\perp}((\prod_{i=1}^{k} p_i^{\perp})(x)). \tag{5}$$

By Proposition 3.2, the norm-convex line through a_{k+1} is contained in the set $sp(a_1, \cdots, a_k)^{\perp}$. We have already shown that $\prod_{i=1}^{k} p_i^{\perp}$ is the gate function on this set. Now by the transitive rule of gate functions ((I: 3.4.4)) we conclude that

$$p_{k+1}(\textstyle\prod_{i=1}^{k} p_i^{\perp}(x)) = p_{k+1}(x).$$

Substituting this in (5) one can deduce Q(k+1) from Q(k), completing the proof of part (2).

Part (3) directly follows from Theorem 2.9. ∎

As the span of a set is determined by its finite subcollections, we conclude:

3.4 Corollary. *Let $(X, ||.||)$ be a modular normed space with an additive orthogonality. Let E be a collection of extreme points of the unit ball. Then the linear span $sp(E)$ of E is a norm-convex subspace of X.* ∎

We need the following well-known lemma.

3.5 Lemma. *Let $F: I \to [0,\infty)$ be a function such that $\sum_{i=1}^{\infty} F(i_i) < \infty$ for every countable subset $\{i_1, i_2, \cdots\}$ of I. Then there are at most countable $i \in I$ with $F(i) > 0$.* ∎

A normed space X (say, with unit ball B) is said to have the *Krein-Milman property*, briefly (K-M) property, if $\overline{sp}(Ext(B)) = X$. If we denote the antipodal relation by A, i.e. $x A y$ iff $x = \pm y$, then the cardinality of $Ext(B)/A$ is called the *extremity* of X. Observe that the extremity of $l_1(I)$ is simply the cardinality of I.

3.6 Theorem. *Let X be a modular normed space with the (K-M) property. Let $\tilde{X}$ denote the completion of X. If $\aleph$ is the extremity of X, then the following statements are equivalent:*

(1) $\tilde{X}$ is linearly isometric with $l_1(\aleph)$.

(2) X has a normed median.

(3) X has an additive directionality.

(4) X has an additive orthogonality.

Proof: Implications $(1) \to (2)$, $(3) \to (4)$ are directly verified, whereas implication $(2) \to (3)$ is Lemma 3.1. For a proof of implication $(4) \to (1)$, let B denote the unit ball of X. Choose a complete representation set $R \subseteq Ext(B)$ of $Ext(B)/A$. For each $e \in R$ we let p_e denote its gate function. As the image of p_e is one dimensional, we can find for every $x \in X$ a scalar $\lambda_e(x)$ such that the equality $p_e(x) = \lambda_e(x) \cdot e$ holds. Let $e_1, e_2, \cdots, e_n$ be a (finite) subset of R. By Lemma 3.3(2) the map $\sum_{i=1}^{n} p_{e_i}$ is the gate function of the subspace $sp(e_1, e_2, \cdots, e_n)$. Now consider the following (in)equalities:

$$||x|| = ||\textstyle\sum_{i=1}^{n} p_{e_i}(x)|| + ||x - \sum_{i=1}^{n} p_{e_i}(x)|| \ge ||\sum_{i=1}^{n} p_{e_i}(x)|| = \sum_{i=1}^{n} |\lambda_{e_i}(x)|.$$

In which the first equality is Theorem 2.9 and the last equality follows by orthonormality of the

$e_1, \cdots e_n$. Whence, for a countable subset $\{e_1, e_2, \cdots\}$ of R we have the inequality

$$\sum_{i=1}^{\infty} |\lambda_{e_i}(x)| \le ||x|| < \infty. \tag{5}$$

Now define the function $J : X \to l_1(R)$ by

$$J(x)_e = \lambda_e(x) \quad (e \in R).$$

Lemma 3.5 insures that J takes it values in $l_1(R)$. Moreover, J restricted to the linear span of $Ext(B)$ yields a linear isometry. As the image of this map is dense in X, we obtain a linear isometry of $\tilde{X}$ onto $l_1(R)$. ∎

3.7 Remark. Not every (dense) median stable subspace of an $l_1(I)$ space has the (K-M) property. To this end, let ν be the measure on the power set of $\mathbb{Z}$ which is induced by defining $\mu(\{x\}) = 2^{-|x|}$ $(x \in \mathbb{Z})$ on the atoms. Also consider the following algebra of sets in $\mathbb{Z}$:

$$\mathcal{A} = \{D \subseteq \mathbb{Z} \mid \exists m \in \mathbb{N}_{>1} \ : \ D + m = D\}.$$

Compare [34, p. 10]. Then the Riesz space Y consisting of step-functions on members of $\mathcal{A}$ (see [52, p. 178]) can be considered as a dense median stable subspace of the Banach space $X = L_1(\mathbb{Z}, 2^{\mathbb{Z}}, \mu)$ (which is linearly isometric with $l_1(\mathbb{N})$). Then none of the extreme points in the unit ball of X —which correspond with the singletons of $\mathbb{Z}$— lie in Y. Whence in view of Corollary 2.14 the unit ball of Y possesses no extreme points.

Actually, by virtue of Lemma 2.13 none of the extreme points in the unit ball of X can be approximated by an increasing sequence in the basepoint order $(Y, \le_0)$. It even follows that these extreme points can not be approximated by an increasing sequence in the Riesz order of Y. Compare the remarks at the end of II: §3.

3.8 Corollary.

(1) *A median Banach space is reflexive iff it is finite dimensional.*

(2) *Modulo linear isometries the only norm on $\mathbb{R}^n$ with a Banach median is the sum-norm (i.e. $||(x_1, \cdots, x_n)|| = \sum_{i=1}^{n} |x_i|$).*

(3) *Let X be a median Banach space. Then there is an index set I and a median Banach space Y such that X is linearly isometric with the product $Y \times l_1(I)$ equipped with the sum-norm, and where the unit ball of Y has no extreme points (unless Y is trivial). Modulo linear isometries this decomposition is unique.*

Proof: For a proof of (1), if a Banach space X is reflexive then it has property (K-M). By Theorem 3.6 we conclude that a reflexive median normed space is an $l_1(\aleph)$ space. It is well-known that $l_1(\aleph)$ is non-reflexive, if $\aleph$ is infinite. Part (2) follows as any finite dimensional Banach space has property (K-M).

For a proof of part (3), let N be the linear span of $Ext(B)$. By Corollary 3.4, N is a norm-convex subspace. By Corollary II: 3.5 the closure $\bar{N}$ is also norm-convex. Lemma 3.1 states that $N^{\perp}$ is a subspace. Hence taking $Y = N^{\perp}$ yields a decomposition of X in Y and N as described in Theorem 2.11. By Corollary 2.12(2) we obtain that the unit ball of Y has no extreme points. Clearly N has the (K-M) property. Let I be an index set of cardinality equal to the extremity of N. Then N is linearly isometric with $l_1(I)$ by Theorem 3.6. For a proof of the unicity of such decompositions, let $Y' \times l_1(J)$ be another decomposition as described in (3). The extremity of Y' is 0, hence by Corollary 2.12(2) we conclude that the cardinality of J equals the extremity of X. So the cardinalities of J and I are equal, showing that $l_1(I)$ and $l_1(J)$ are linearly isometric. This implies that Y' and Y are also linearly isometric. ∎

Corollary 3.8(3) is well-known for $L_1(\mu)$ spaces (which turn out to be precisely the median Banach spaces). In chapter IV we will show a generalization of Corollary 3.8(3): each modular Banach space can be (uniquely) decomposed into a "rigid" part and an $l_1(I)$ part.

§ 4 Characterizing modular spaces with additive orthogonality

4.1 A-spaces. Let X be a normed space (say, with unit ball B). Then X is called a *CL-space* if there is a maximal face F of B with $X = C(F) - C(F)$. The latter is implied by $B = co(F \cup -F)$. The concept of a CL-space was introduced by Fullerton in [25] for Banach spaces. We shall say that X is *CL-generated* by $C(F)$. The following (stronger) notion was introduced by Alfsen and Effros [2]. The space X is called an *A-space* if there exists a proper, convex cone C in X, such that

(1) $\forall x,y \in C : ||x+y|| = ||x|| + ||y||$.

(2) For every $x \in X$ there are $c_1, c_2 \in C$ with $x = c_1 - c_2$ and $||x|| = ||c_1|| + ||c_2||$.

Observe that c_1, c_2 are orthogonal. If for all x the corresponding c_1, c_2 are unique, then X is called a *uniquely generated A-space*. We shall say that X is (uniquely) *A-generated* by C. One can easily verify that if X is an A-space then the corresponding (A-generating) cone must be (maximal) facial. See [2, Corollary 5.2].

It is shown by Lima that modular Banach spaces are A-spaces, see [48, Theorem 2.5]. We also mention that a modular normed space is CL-generated if and only if it is A-generated (Lemma 4.7). The next proposition —a modification of the result that positive operators between Banach lattices are continuous— gives a relation between norm-intervals and Banach A-spaces. This result shall be of particular use for modular Banach spaces. See IV: §2.

4.2 Proposition. *Let $(X_1, ||.||_1)$ be a Banach A-space and let $(X_2, ||.||_2)$ be a normed space. Then each linear function $f : X_1 \to X_2$ that preserves the norm-intervals is continuous.*

Proof: Let D_1 be the unit ball of X_1 and let X be A-generated by the cone C. One can easily verify that the norm of f equals $\sup\{ ||f(x)||_2 \mid x \in C \cap D_1 \}$. Hence, if f is not bounded, then there exists a sequence $x_n \in C$ with $||x_n||_1 \to 0$, and $||f(x_n)||_2 \to \infty$. By completeness the Cauchy sequence $\sum_{n=1}^{k} x_n$ $(k \in \mathbb{N})$ converges to a point $p \in C$. For each $n \in \mathbb{N}$ we have $p - x_n \in C$, hence

$$||p - x_n||_1 + ||x_n||_1 = ||p||_1.$$

That is $x_n \in I_1(0,p)$. So by assumption on f we have $f(x_n) \in I_2(0, f(p))$. In particular we conclude $||f(x_n)||_2 \le ||f(p)||_2$. We obtain a contradiction. ∎

The completeness in Proposition 4.2 can not be omitted. See the remarks after Corollary IV: 2.7.

4.3 Corollary. *Two complete A-spaces on a vector space that induce the same norm-intervals are equivalent.* ∎

The following relates decompositions in A-spaces with maximal and minimal points. We use the notation introduced in I: 4.20.

4.4 Proposition. *Let X be a normed space, A-generated by a convex cone C and let $x=x^+-x^-$ be an orthogonal decomposition with $x^+,x^-\in C$. Then with respect to the vector order $\leq_C$ we have $x^+\in x\sqcup 0$, $x^-\in x\sqcap 0$.*

Proof: Suppose that $c\in C$ satisfies that $x\leq_C c\leq_C x^+$ (see Figure 4.4), hence, $x^+-c\geq_C 0$ and $c\in I(x,x^+)$.

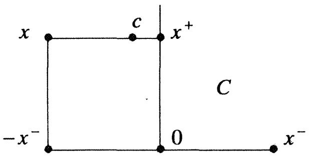

Fig. 4.4: an orthogonal decomposition

As $x^+\in I(0,x)$ we obtain from the geometric properties of metric intervals that $x^+\in I(0,c)$. Now we obtain

$$\begin{aligned}||c-x^+||+||x^-+x^+|| &= ||c||-||x^+||+||x^-||+||x^+|| \\ &= ||x^-||+||c||=||x^-+c||,\end{aligned}$$

in which the last equality follows as the points x^-,c are codirectional. We conclude that $c-x^+\in I(0,x^-+c)$. As the cone C is star-shaped at the origin, $c-x^+\in C$, whence $x^+-c\leq_C 0$. We conclude that $x^+=c$, and therefore $x^+\in x\sqcup 0$. The second part of the proposition follows from the formula $x-(x\sqcup 0)=x\sqcap 0$, or by symmetric reasoning. ∎

We remark that the order $\leq_C$ in the previous proposition need not induce a multilattice structure on X.

4.5 Vector multilattices. A *vector multilattice* is a multilattice derived from a vector order. We observe that the many well-known formulae from the theory of Riesz space (such as $x+y-(x\wedge y)=x\vee y$) are in fact valid for general vector orders with the obvious changes. The following is easily verified

4.6 Proposition. *Let $\leq$ be a vector order on a vector space X such that the positive cone C generates X. Then, $\leq$ is a vector multilattice (resp. lattice) iff $(C,\leq)$ is a semi-multilattice (resp. semi-lattice).* ∎

4.7 Lemma. *Let $C(F)$ be a facial cone in a modular normed space, and let $Y=C(F)-C(F)$. Then the following hold:*
(1) The order $\leq_{C(F)}$ yields a vector multilattice on Y.
(2) Y is A-generated by $C(F)$.
In particular, a modular normed space is CL-generated if and only if it is A-generated.

Proof: By Lemma 2.1, $C(F)$ is an updirected union of intervals of type $I(0,c)$ $(c\in C)$. As the partial orders $(I(0,c),\leq_0)$ are semi-multilattices by (I: 4.21), so is the order $(C(F),\leq_0)$. The latter order coincides with $(C(F),\leq_C)$, by (2.4). Hence we can apply Proposition 4.6. For a proof of statement (2), let $x\in X$ and $m\in x\sqcup 0$. Hence, $x-m\in x\sqcap 0$. In particular the points $m,m-x$ are codirectional. We also have $0\in x\sqcap(m-x)$. As the basepoint order and the vector order coincide on the positive cone (2.4), we obtain that $M(m,m-x,0)=0$ —see (I: 4.21). So the points $m,m-x$ are also antidirectional. ■

4.8 Problem. *Is the subspace Y of Lemma 4.7 modular?* Compare Proposition 4.10.

The following lemma shows that under certain restrictions orthogonal decompositions are unique.

4.9 Lemma. *Let X be a modular normed space with an additive orthogonality. Let F be a face of the unit ball, and let $c_1,c_2,d_1,d_2\in C(F)$ such that $c_1\perp c_2$, $d_1\perp d_2$. If $c_1-c_2=d_1-d_2$, then $c_i=d_i$ $(i=1,2)$.*

Proof: We derive the following formula, which takes the greatest part of the lemma's proof:

$$c_1\perp c_2+d_2. \tag{1}$$

To this end, as $c_1,c_2+d_2\in C(F)$ we conclude from Lemma 2.1 that the points c_1 and c_2+d_2 are codirectional. We shall show that these points are also antidirectional, i.e. that

$$M(0,c_1,c_2+d_2)=0. \tag{2}$$

Let x denote d_1-d_2 $(=c_1-c_2)$. By the assumption on orthogonality we obtain that $||x||=||d_1||+||d_2||=||c_1||+||c_2||$. First, consider the following sequence of triangle inequalities:

$$\begin{aligned}||c_1+d_1||+||c_2+d_2|| &\leq ||c_1||+||d_1||+||c_2||+||d_2||,\\ &=||x||+||x||=||2\cdot x||=||c_1+d_1-(c_2+d_2)||,\\ &\leq ||c_1+d_1||+||c_2+d_2||.\end{aligned}$$

We conclude that all inequalities are in fact equalities. Hence $0\in I(c_1+d_1,c_2+d_2)$, that is:

$$0=M(0,c_1+d_1,c_2+d_2). \tag{3}$$

Next consider the following equalities:

$$\begin{aligned}||d_1||+||c_1-(c_2+d_2)|| &= ||d_1||+||2\cdot d_2-d_1||\\ &=||d_1||+2\cdot||d_2||+||d_1||=2\cdot(||d_1||+||d_2||)\\ &=||2\cdot x||=||c_1+d_1-(c_2+d_2)||.\end{aligned}$$

The second equality uses $d_1 \perp d_2$. We conclude that $c_1 \in I(c_1+d_1, c_2+d_2)$. As the interval $I(c_1+d_1, c_2+d_2)$ is star-shaped at its endpoint c_2+d_2, we obtain the following inclusion.

$$I(c_1, c_2+d_2) \subseteq I(c_1+d_1, c_2+d_2). \tag{4}$$

Finally, we assert the following:

$$\begin{aligned} M(0,c_1,c_2+d_2) &= I(0,c_1) \cap I(0,c_2+d_2) \cap I(c_1,c_2+d_2) \\ &\subseteq I(0,c_1+d_1) \cap I(0,c_2+d_2) \cap I(c_1+d_1,c_2+d_2) = M(0,c_1+d_1,c_2+d_2) = 0. \end{aligned} \tag{5}$$

The first and second equality of (5) only invoke the definition of M. The third equality of (5) is just assertion (3). We shall prove the inclusion in (5). As c_1 and d_1 are in a common facial cone (namely, $C(F)$), Lemma 2.3 states that $||c_1|| + ||d_1|| = ||c_1+d_1||$, i.e. $c_1 \in I(0,c_1+d_1)$. By the star-shapedness of the interval $I(0,c_1+d_1)$ at the end point c_1+d_1, we obtain that $I(0,c_1) \subseteq I(0,c_1+d_1)$. This formula together with (4) shows the inclusion in (5). Assertion (5) yields equality (2), establishing formula (1).

By assumption, $c_1 \perp c_2$, and by the multiplicative stability of orthogonality we also have $c_1 \perp -2 \cdot c_2$. Now by additivity of orthogonality , together with formula (1), we conclude that

$$c_1 \perp c_2 - d_2. \tag{6}$$

By permuting the role of the c and d, we obtain

$$d_1 \perp c_2 - d_2. \tag{7}$$

By using the additivity once more on the formulae (6) and (7), we obtain that $c_1 - d_1 \perp c_2 - d_2$. By assumption, the points $c_1 - d_1$ and $c_2 - d_2$ are equal. As the origin is the only point orthogonal to itself we conclude that $c_1 - d_1 = c_2 - d_2 = 0$, which concludes the proof of this lemma. ■

4.10 Proposition. *Let $C(F)$ be a facial cone in a modular normed space, and let $Y = C(F) - C(F)$. Then the following are equivalent.*

(1) Y has an additive orthogonality.

(2) Y is uniquely A-generated by $C(F)$.

(3) For all $x, y \in C(F)$ the value $M(x,y,0)$ of the multimedian is a singleton.

(4) Y is an L-space.

Proof: Implication $(4) \to (1)$ follows from Lemma 3.1. Implication $(1) \to (2)$, follows from Lemmas 4.7 and 4.9. For a proof of implication $(2) \to (3)$, suppose that $m_1, m_2 \in M(x,y,0)$. Then the points $x - m_i, m_i - y$ are contained in opposite facial cones, hence they are antidirectional. Also as $||x - m_i|| + ||m_i - y|| = ||x - y||$ these points are codirectional. So the points $x - m_i, m_i - y$ are orthogonal ($i = 1,2$). We conclude that

$$x - m_1 = x - m_2 \ ; \ m_1 - y = m_2 - y,$$

from the definition of uniquely A-generated space. Whence, $m_1 = m_2$. If $C(F)$ satisfies property (3), then $(C(F), \leq_0)$ is a semi-lattice. Hence, from Proposition 4.6 we conclude that Y is a vector lattice with an additive positive cone, i.e. X is an L-space. ■

The equivalence of statements (2), (4) was first shown by Lima in [48, Corollary 3.8].

4.11 Theorem. *The following are equivalent for a normed space X.*

(1) X is a modular CL-space with an additive orthogonality.

(2) X is a modular A-space with an additive orthogonality.

(3) X is a uniquely generated modular A-space space.

(4) X is an L-space. ■

Before stating the main result of this section, let us discuss a description of additive orthogonality in terms of an additive property of the median inner product $<.,.>$. As a straightforward calculation shows that $<x,x+x> = <x,x> = ||x||$ for all x in a normed space X, we can not expect this inner product to obey the formula $<x+y,z> = <x,z> + <y,z>$ for all $x,y,z \in X$. However, by considering a few cases one verifies that the (median) inner product of $\mathbb{R}$ satisfies

$$<x+y,z> \leq <x,z> + <y,z>$$

for all $x,y,z \in \mathbb{R}$. Motivated by this we call the median inner product *subadditive* if the latter inequality holds. By using the subadditivity of $\mathbb{R}$ point-wisely we deduce the following:

4.12 Lemma. *The median inner product of an $L_1(\mu)$ space is subadditive.* ■

4.13 Theorem. *The following are equivalent for a modular normed space X.*

(1) The completion $\tilde{X}$ of X is linearly isometric with an $L_1(\mu)$ space.

(2) X is a median-stable subspace of an $L_1(\mu)$ space.

(3) X has a normed median.

(4) The intervals of X are norm-convex.

(5) X has a subadditive inner product.

(6) The directionality of X is additive.

(7) The orthogonality of X is additive.

(8) Every subspace of X of dimension at most three, has an additive orthogonality.

Proof: We derive the following sequences of implications:

$(1) \to (2) \to (5) \to (6) \to (7) \to (8)$; $(3) \leftrightarrow (4) \to (7) \leftrightarrow (8)$: $(7) \to (3) \to (1)$.

As spaces of type $L_1(\mu)$ are median (Theorem II: 1.8) implication $(1) \to (2)$ is clear. Implication $(2) \to (5)$ is Lemma 4.12, and implications $(5) \to (6) \to (7) \to (8)$ are evident. The equivalence of statements (3) and (4) is shown in Theorem I: 4.24, for general (metric) modular space. Implication $(3) \to (7)$ is Lemma 3.1, and the equivalence of statements (7) and (8) is a straightforward verification. For a proof of implication $(7) \to (3)$, let $x \in X$. By Proposition 4.10 the facial space $C(x) - C(x)$ is an L-space. As the metric interval $I(0,x)$ is contained in $C(x)$ by Lemma 2.2, the ordered set $(I(0,x), \leq_0)$ is a distributive lattice. By translation we conclude that all ordered sets of type $(I(a,b), \leq_a)$ with $a,b \in X$ are distributive lattices. Whence X is a median normed space, see Theorem I: 4.24. For a proof of implication $(3) \to (1)$, as the completion of a median metric space is median (Corollary II: 3.2) $\tilde{X}$ is a median Banach space. As a modular median Banach space is a CL-space by [48, Theorem 2.5], we conclude that X is a complete L-space. By the classical Kakutani representation theorem, see [49], X is linearly isometric with an $L_1(\mu)$ space. ■

The previous result states that the $L_1(\mu)$ spaces are ***universal*** median normed spaces: each median normed space corresponds with a median-stable subspace of an $L_1(\mu)$ space. In chapter V we will show that $L_1(\mu)$ spaces are universal median ***metric*** spaces: each median metric space corresponds with a median-stable subspace of an $L_1(\mu)$ space.

4.14 Problem. *Does there exist a median normed space that is not an L-space?*

4.15 Example. The L-space B consisting of all essentially bounded Lebesgue integrable functions on the unit interval yields an example of a norm-convex, whence median-stable, subspace of $L_1([0,1])$. The space $R \subseteq B$ consisting of all Riemann integrable functions on the unit interval yields an example of a median-stable subspace of $L_1([0,1])$, which is not norm-convex. Indeed, take a positive Lebesgue measurable function f on the unit interval, that is (point-wise) below the function c identically one, i.e. $[f] \in I(0,[c])$, and not Riemann integrable. Actually, the norm-convex hull of R, i.e. the smallest norm-convex set containing R, is B.

§ 5 Norm-convex subsets

The following is one of the main results of this section.

5.1 Theorem. *Let X be a modular normed space. Suppose that C is a norm-convex proper cone in X. Let Y be the subspace $C-C$. Then,*

(1) The subspace Y is an L-space that is norm-convex in X.

(2) C is a gated subset of Y. Moreover, if $p' : Y \to C$ is the gate function of C, then the unique orthogonal decomposition of a point $y \in Y$ is given by $y = p'(y) - p'(-y)$.

(3) If C is gated in X, and if $p : X \to C$ is the corresponding gate function, then Y is gated in X and $X \to Y : x \to p(x) - p(-x)$ is the gate function of Y.

Proof: From Corollary 2.14 we conclude that C is a facial cone. Evidently Y is a norm-convex subset of X. By Lemma 4.7, Y is A-generated by C. Let $y \in Y$, and let $y = c^+ - c^-$ be an orthogonal decomposition with members of C. Proposition 4.4 states that the point c^+ is maximal in the ordered set $(C \cap I(0,y), \leq_0)$. Whence by (I: 4.4.2) we conclude that c^+ is the gate of y in C. Similarly we obtain that $-c^-$ is the gate of y is $-C$. As gates are unique, we conclude from Proposition 4.10 that Y is an L-space. At the same time, we have shown that C is gated in Y and that the gate map behaves as described in (2).

For a proof of statement (3), we shall first show that if Y has a gate function, say q, then it has the form described in (3). Indeed, let $x \in X$. As the point $q(x)$ is a member of Y we can use (2) to obtain $q(x) = p(q(x)) - p(-q(x))$. Now consider the following equalities:

$$q(x) = p(q(x)) - p(q(-x)) = q(p(x)) - q(p(-x)) = p(x) - p(-x).$$

The first equality follows from Corollary 2.10, where the equality $-q(x) = q(-x)$ is shown. The second equality follows the commutativity of gate functions on non-disjoint subsets, see (I: 3.4.4). The last equality is evident as q is a projection. To prove the existence of gates in Y we consider two cases:

(i) C is complete. We shall show that Y is complete, which immediately implies that Y is gated. To this end, by Proposition II: 4.1, the gate function of C, p, is contractive. Hence the image under p of a Cauchy sequence yields a Cauchy sequence. Using the decomposition as described in (2), we obtain that Y is complete.
(ii) general case: As usual let $\bar{X}$, $\bar{C}$ denote the completions of X and C respectively, and let $\bar{p}$ be the gate function of $\bar{C}$. By the previous case we conclude that $\bar{C}-\bar{C}$ has a gate function $\bar{q}$ of the right form, i.e. $\bar{q}(x)=\bar{p}(x)-\bar{p}(-x)$ for all $x\in\bar{X}$. So we are done if we show that $\bar{p}(x)=p(x)$ for all $x\in X$. To this end, $\bar{p}(x)$ is the unique metric point of x in $\bar{C}$. On the other hand $p(x)$ realizes the distance from x to C. Hence, as $\rho(x,C)=\rho(x,\bar{C})$ we are done. ■

The following corollary is geometrically obvious but a straightforward proof seems quite difficult.

5.2 Corollary. *Let X be a modular normed space. Then the following are equivalent for a facial cone $C(F)$.*
(1) The intervals of type $I(0,p)$ with $p\in C(F)$ are norm-convex.
(2) The cone $C(F)$ is norm-convex.
Hence, for $p\in X$: $I(0,p)$ is norm-convex iff $C(p)$ is norm-convex.

Proof: For a proof of implication $(1)\to(2)$, let $x,y\in C(F)$. As $C(F)$ is a facial cone, $x,y\in I(0,x+y)\subseteq C(F)$. By assumption the interval $I(x,y)$ is contained in $I(0,x+y)$ and hence in $C(F)$. As the points x,y were arbitrary we conclude that $C(F)$ is norm-convex.
For a proof of the other implication, assume that $C(F)$ is norm-convex. Let $p\in C(F)$. By the previous theorem the subspace $Y=C(F)-C(F)$ is an L-space. Hence, the (relative) interval $I(0,p)\cap Y$ is norm-convex in Y. As Y is a norm-convex subspace of X, the interval $I(0,p)$ is norm-convex in X. ■

The following result is a nice characterization of $L_1(\mu)$ spaces.

5.3 Theorem. *Let X be a modular normed space. Then the following are equivalent:*
(1) X is linearly isometric with an $L_1(\mu)$ space.
(2) X is complete and there is a maximal facial cone C in X that is norm-convex.
(3) There is a complete facial cone C in X that is norm-convex and satisfies $C-C=X$.

Proof: Implication $(1)\to(2)$ is obvious, whereas implication $(2)\to(3)$ follows as every complete modular normed space is A-generated by [48, Theorem 2.5]. For a proof of implication $(3)\to(1)$, as C admits gates, we conclude from Theorem 5.1 that $C-C$ is an L-space. As remarked in the proof of Theorem 5.1 (case (i)) the space $C-C$ is complete. ■

We now arrive at a characterization of median normed spaces.

5.4 Theorem. *Let X be a modular normed space. Then the following are equivalent:*
(1) X is a median normed space.
(2) Every minimal facial cone of X is norm-convex.
(3) Every facial cone of X is norm-convex.
(4) Every point of X is contained in a norm-convex facial cone.

Proof: For a proof of implication $(1)\rightarrow(2)$, by Theorem 4.13 the intervals are norm-convex. Hence by Corollary 5.2 we conclude that every (minimal) facial cone is norm-convex. Implication $(2)\rightarrow(3)$ also follows from Corollary 5.2. Implication $(3)\rightarrow(4)$ is evident. Hence we are left with the proof of implication $(4)\rightarrow(1)$. In view of Theorem 4.13 we have show that metric intervals are norm-convex. To this end, it suffices to show this for intervals of type $I(0,p)$ with $p\in X$. By assumption there is a proper, norm-convex cone C containing p. By Corollary 5.2 we deduce that the interval $I(0,p)$ is norm-convex. ■

5.5 Constructing proper, norm-convex cones. Suppose that D is a norm-convex subset of a modular normed space which has the origin as an extreme point. Then one readily verifies that the cone $C(D)$ of D is proper and norm-convex. So from Theorem 5.1(1) and Theorem I: 4.24(5) we obtain the following:

5.6 Proposition. *Let D be a norm-convex subset in a modular normed space X with at least one extreme point. Then*

(1) *The affine span of D is a norm-convex affine space which, up to translation, is an L-space.*

(2) *Any interval of type $I(a,b)$ $(a,b\in D)$, is norm-convex.*

In particular, D is a median metric space. ■

Clearly, we can not replace the existence of an extreme point by the condition $D\neq X$ in the above proposition. Indeed, consider the cone $X\times\mathbb{R}_0^+$ in the space $X\times\mathbb{R}$. Where X is a (non-median) modular normed space, and the product is equipped with the sum norm.

The following proposition gives a way to find extreme points in certain norm-convex subsets.

5.7 Proposition. *Let X be a modular normed space with completion $\tilde{X}$. Let D be a norm-convex subset of X and $a,b\in D$. Then,*

(1) *If b is maximal in the order $(C,\leq_a)$ then b is an extreme point of D.*

(2) *Any extreme point of D is an extreme point of $Cl_{\tilde{X}}(D)$.*

Proof: For a proof of the first statement, without loss of generality we may assume that $a=0$. Assume to the contrary b is not extreme, that is $b=\frac{1}{2}(x+y)$ for some $x,y\in D\setminus\{b\}$. We shall first verify the following formula:

$$M(y,b,x+y)=b. \tag{3}$$

To this end, let $m\in M(y,b,x+y)$. By norm-convexity of D we obtain that $m\in D$. Next, consider the following formula:

$$m\in I(b,x+y)\subseteq I(0,x+y). \tag{4}$$

The inclusion follows as b is in the standard interval between 0, and $x+y$, and as norm intervals are star-shaped at the endpoints. The geometric properties of metric intervals together with formula (4) yield $b\in I(0,m)$, hence $b=m$ by maximality of b. This shows formula (3) From (3) we deduce that $b\in I(y,x+y)$, i.e.

$$||\tfrac{1}{2}(x-y)|| + ||\tfrac{1}{2}(x+y)|| = ||x||.$$

This formula also states that $b \in I(0,x)$, which implies $b = x$ by maximality of b. We arrive at a contradiction, hence b is an extreme point of D.

For a proof of the second statement, let e be an extreme point of D. By the use of a translation we may assume that e equals the origin. In this case the cone $C(D)$ of D is a proper, norm-convex cone. Now by Corollary 2.14(1) we conclude that this cone is facial. By Corollary 2.14(2) we conclude that the cone $Cl_{\tilde{X}}(C(D))$ is a facial cone of $\tilde{X}$, and is hence proper. From this it follows that the origin is extreme in $Cl_{\tilde{X}}(D)$. ■

5.8 Remark. Suppose that C is a norm-convex, complete subset of a modular normed space X, and that C is upbounded at a point $x_1 \in C$. Using a maximal chain one can find a point x_2 which is maximal in the order $(C, \leq_{x_1})$ (cf. the proof of [2, Theorem 2.9]). In the same fashion one can change x_1 into a point maximal in the order $(C, \leq_{x_2})$. Such points are called *mutually maximal* in C.

We now arrive at a nice characterization of norm-convex intervals in terms of ***bounded***, norm-convex subsets.

5.9 Theorem. *The following are equivalent for points x,y in a modular normed space X.*
(1) The interval $I(x,y)$ is norm-convex.
(2) There exists a bounded, norm-convex subset B of X containing x,y.
(3) There exists a norm-convex subset B of X containing x,y, together with an extreme point.
Hence, X is a median normed space iff every pair of points in X is contained in some norm-convex set as described in (2) or (3).

Proof: Implication (1)→(2) is evident. For a proof of implication (2)→(1), let B be as described in (2). As usual let $\tilde{X}$ denote the completion of X. We shall show that the interval $I_{\tilde{X}}(x,y)$ is norm-convex in $\tilde{X}$ This immediately implies that the interval $I(x,y)$ is norm-convex in X. To this end, by Corollary II: 3.5, $cl_{\tilde{X}}(B)$ is a (bounded, complete) norm-convex subset. Hence, by Proposition 5.7(1) B has an extreme point, say e. By Proposition 5.6 we conclude that the interval $I_{\tilde{X}}(x-e, y-e)$ is norm-convex in $\tilde{X}$. Hence, so is the interval $I_{\tilde{X}}(x,y)$, as desired.

Implication (1)→(3) follows from the fact that either endpoint x,y is extreme in the interval $I(x,y)$. Finally, implication (3)→(1) follows from Proposition 5.6.

The last statement follows directly from Theorem 4.13. ■

The following result shows an even stronger relation between bounded, norm-convex subsets and intervals.

5.10 Theorem. *The following are equivalent for a subset D in a modular normed space X.*
(1) D is gated and there are mutually maximal points in D.
(2) D is a norm-convex interval.
In particular, a complete, norm-convex subset of X that is upbounded at one of its points is an interval.

Proof: Implication (2)$\rightarrow$(1) is evident. For a proof of implication (1)$\rightarrow$(2), let x_1,x_2 be mutually maximal points in D. Denote the gate function of D by p. We may assume that x_1 equals the origin. The cone $C(D)$ is norm-convex and proper in X. Let $z \in D \subseteq C(D)$, then the points x_2,z are codirectional, that is $x_2,z \in I(0,x_2+z)$. We verify that

$$x_2,z \in I(0,p(x_2+z)). \tag{3}$$

To this end, let n denote either of x_2,z. By definition of a gate we have $p(x_2+z) \in I(n,x_2+z)$. By the geometric properties of metric intervals we obtain that $n \in I(0,p(x_2+z))$, i.e. formula (3). Hence by maximality of x_2 in $(D, \leq_0)$, $x_2 = p(x_2+z)$. Now formula (4) states that $z \in I(0,x_2)$, as desired.

The last statement of the theorem follows from Remark 5.8 and the equivalence of statements (1) and (2). ■

5.11 Corollary. *Let $(X, ||.||)$ be a downconverging modular normed space. Then any norm-convex subset C upbounded at one of its points is bounded.*

Proof: By Proposition II: 4.10 the completion $\tilde{C}$ of C is a complete, norm-convex subset of the completion $\tilde{X}$ of X that is upbounded at one of its points (actually, C and $\tilde{C}$ are upbounded at ***all*** points by Proposition II: 4.9). Hence $\tilde{C}$ is a (bounded) interval by Theorem 5.10. ■

As a consequence of Theorem 5.10 we conclude that if C is a bounded, closed, norm-convex subset of an $L_1(\mu)$ space, then there exist integrable functions f,g such that

$$C = \{[h] \mid h(x) \textit{ is in between } f(x) \textit{ and } g(x) \textit{ almost everywhere } \}.$$

5.12 Remarks. The equivalence of statements (1) and (2) in Theorem 5.9, as well as Theorem 5.10, are not true in a general modular metric space. As a counterexample to the first assertion just consider the (bounded) unit ball of $(\mathbb{R}^3, ||.||_{\max})$. For a counterexample to the second assertion, consider the following (median-stable) subset T of $(\mathbb{R}^2, ||.||_s)$:

$$T = \{x \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1 \wedge x_2 = 0\} \cup \{x \in \mathbb{R}^2 \mid x_1 = 0 \wedge -1 \leq x_2 \leq 1\}.$$

Let C be an arbitrary subset of a modular normed space. The following is an iterative process to obtain the norm-convex hull of C. Define

$$C_1 = C \quad ; \quad C_{n+1} = \cup_{x,y \in C_n} I(x,y) \quad (n \in \mathbb{N}).$$

Then the norm-convex hull of C is given by $\cup_{n \in \mathbb{N}} C_n$. One can easily verify that $diam(C_n) \leq 3^n diam(C)$, hence the C_n are bounded iff C is bounded.

Starting with a two point set $C = \{x,y\}$, Theorem 5.9 gives the following equivalences:

- The interval $I(x,y)$ is not norm-convex.
- $diam(C_n)$ tends to infinity,
- the C_n do not stabilize after finite steps.

Soltan showed in [77] that if an interval $I(x,y)$ in the space $(\mathbb{R}^3, ||.||_{\max})$ is not norm-convex, then its convex hull is the whole of $\mathbb{R}^3$. See also chapter IV.

CHAPTER IV

DECOMPOSING MODULAR BANACH SPACES

The result of Soltan [77], that $\mathbb{R}^3$ with the "max" norm has no norm-convex bodies except for $\mathbb{R}^3$ itself, is particularly motivating for our approach below. Let us say that a normed space is ***rigid*** if it has no norm-convex bodies except for itself. This condition is shown to be equivalent with the non-existence of functionals which preserve the metric convexity. This may also motivate the use of the term "rigid". See section 1, where we develop some general results concerning (non-)rigidity for so-called "vector convexities" in linear spaces. These results are applied in the following sections to the norm-convexity of a modular normed space.

Spaces of type $l_1(I)$ are opposed to rigid ones, in the sense that they are characterized among modular Banach spaces by the existence of a separating collection of functionals that "preserve" the metric convexity. The abundance of such functionals in a normed space X corresponds with the Hausdorff property of the weak(norm) topology of X. See section 2.

In section 3, we characterize $l_1(I)$ spaces in terms of the existence of ***metrics*** that "preserve" the norm convexity.

The main result of this chapter appears in section 4. This states that each modular Banach space can be decomposed into a rigid space and an $l_1(I)$ space. The decomposition involves the "sum" norm on a product of two factors, and is a generalization of the decomposition of an $L_1(\mu)$ space into an atom free part and an $l_1(I)$ part — compare Corollary III: 3.8 (3). (')

§ 1 Vector convexities

For non-defined terms see [45]. Let X be a vector space. A convexity $\mathcal{C}$ on X is called a *vector convexity* if

(V-1) $\mathcal{C}$ consists of (standard) convex subsets.

(V-2) $\mathcal{C}$ is stable under translations.

(V-3) $\mathcal{C}$ is stable under homotheties with positive coefficients, i.e. if $C \in \mathcal{C}$ and $x \in X$, $\lambda > 0$ then $x + \lambda \cdot (C - x) \in \mathcal{C}$.

' The results of this chapter, except the results appearing in §3, were obtained by van de Vel and the author. See [81].

Note that by axiom (V-2) it suffices to check axiom (V-3) for points $x \in C$. The convexity $\mathcal{C}$ is called *symmetric* if $U \in \mathcal{C}$ implies $-U \in \mathcal{C}$. Evidently, a halfspace in $\mathcal{C}$ is a (standard) halfspace in X. A (linear) functional on X is called a *$\mathcal{C}$-functional*, if the preimage of each halfspace in $\mathbb{R}$ is a halfspace of $\mathcal{C}$. If the convexity $\mathcal{C}$ is induced by an interval operator then $\mathcal{C}$-functionals correspond with the interval-preserving functionals (cf. Corollary I: 4.10). Two sets A,B in X are *$\mathcal{C}$-separated* (resp. *strongly* $\mathcal{C}$-separated) if there exists a continuous $\mathcal{C}$-functional that separates (resp. strongly separates) the subsets A and B.

As an example of a convexity (resp. vector convexity) we mention the geodesic convexity of a metric (resp. normed) space. Recall that geodesically convex subsets in normed space are usually called *norm-convex*. In any vector space, the collection of all affine subsets is a vector convexity.

1.1 Topological vector spaces. We are interested in vector convexities on ***topological*** vector spaces. We remark that any vector space X can be endowed with a Hausdorff, locally convex vector topology. To this end, a point x in a subset $A \subseteq X$ is called a *core point*, if each line through x meets A in a (line-) open set. See the book of Kelley and Namioka, [45], who use the term "radial at x". The collection of all symmetric, convex subsets in X in which the origin occurs as a core point, forms a local base for a Hausdorff, locally convex topology, known as the *core topology*. See [45, exercise 6I].

Let X be a topological vector space, and let C be a *convex body* in X, i.e., a convex subset with non-empty interior. If $c \in int(C)$, then the following hold (see [45, ch. 4]):

$$int(C) = c + \bigcup_{0 \le \lambda < 1} \lambda \cdot (C - c). \tag{1.1.1}$$

$$cl(C) = c + \bigcap_{\lambda > 1} \lambda \cdot (C - c). \tag{1.1.2}$$

In particular we conclude that $int(C) = int(\bar{C})$. One can apply the above equalities to obtain the well-known fact that a half-line starting in an interior point of a convex set intersects the boundary of C in at most one point. Throughout, the origin of a vector space is denoted by 0.

1.2 Lemma. *Let X be a topological vector space, and let C be a convex, open subset of X. Then C is a halfspace, iff the boundary of C is convex, iff the boundary of C is a closed hyperplane.* ■

This well-known result follows from the fact that two disjoint convex subsets A,B with A open can be separated by a (continuous) functional. See [70, Theorem 3.4].

1.3 Proposition. *Let X be a topological vector space, and let $\mathcal{C}$ be a vector convexity on X. Then the following hold for each $U \in \mathcal{C}$ with non-empty interior.*

(1) *$int(U)$ and $cl(U)$ are members of $\mathcal{C}$.*

(2) *If U is open, and if $u \in cl(U)$, $\lambda > 0$, then $\lambda(U - u) \in \mathcal{C}$*

(3) *If $\mathcal{C}$ is symmetric, and if U is an open halfspace then the hyperplane $cl(U) \cap X \setminus U$ and the halfspaces $int(X \setminus U)$, $cl(U)$, $X \setminus U$ are also members of $\mathcal{C}$.*

Proof: Statement (1) directly follows from formulae (1.1.1) and (1.1.2) (observe that the union in (1.1.1) involves a totally ordered collection of sets). For a proof of statement (2) consider the following calculation:

$$\lambda(U-u) = int\,(cl\,(\lambda(U-u))) = int\,(\lambda\cdot(cl\,(U-u))) = int\,(\lambda\cdot(cl\,(U)-u)).$$

By using axioms (V-2) and (V-3) of a vector convexity and statement (1), we obtain that $int\,(\lambda\cdot(cl\,(U)-u))$ is a member of $\mathcal{C}$, hence so is $\lambda(U-u)$. Statement (3) is an immediate consequence of (1). ■

The following result is of crucial importance. Throughout, $\partial(U)$ denotes the boundary of a set U.

1.4 Lemma. *Let X be a topological vector space, and let $\mathcal{C}$ be a vector convexity on X. If $U\in\mathcal{C}$ is an open set and if $C\subseteq\partial(U)$ is convex, then there exists a (standard) open halfspace H of X including U, disjoint with C, and such that $H\in\mathcal{C}$.*

Proof: The collection $\mathcal{O}$ of all $O\in\mathcal{C}$, which are open and disjoint with C, is partially ordered by inclusion. As every chain in $\mathcal{O}$ evidently has an upper bound, we can apply Zorn's lemma to obtain a maximal element H in $\mathcal{O}$. We claim the following equality for any $d\in\partial(H)$:

$$H = d + \bigcup_{\lambda>0}\lambda\cdot(H-d). \tag{1}$$

To this end, denote the right-hand side of (1) by $H(d)$. Clearly, $H(d)$ is an open set including H, and $\lambda\cdot(H-d)\in\mathcal{C}$ for all $\lambda>0$ by Proposition 1.3(2). As the sets appearing in the union (1) constitute a chain, this shows that $H(d)\in\mathcal{C}$. For a point $c\in C(IN\partial(H))$ it is evident that $C\cap H(c)=\varnothing$. Hence $H(c)\in\mathcal{O}$, and by the maximality of H we obtain that $H(c)=H$. Suppose that $a\in H(b)$ for two points $a\neq b$ of ∂H. Then there exist $\lambda>0$ and $x\in H$ such that $a=b+\lambda(x-b)$. If $\lambda\leq 1$ then $a\in(b,x]$, which implies that $a\in H$, a contradiction. We conclude that $\lambda>1$. Hence

$$b = a+\frac{\lambda}{\lambda-1}(x-a)\in H(a).$$

In other words: $a\in H(b)$ implies $b\in H(a)$. This allows to conclude that $C\cap H(d)=\varnothing$ for any $d\in\partial(H)$ and in particular, that $H(d)=H$ as required in (1). For, if c is in the intersection, then $d\in H(c)=H$, a contradiction.

We use formula (1) to show that the boundary of H is convex. Suppose that there exist $d_1,d_2\in\partial H$ such that $(d_1,d_2)\cap H\neq\varnothing$. Fix $e\in H$ and $t\in(0,1)$ such that $e=t\cdot d_1+(1-t)\cdot d_2$. Then $d_1=d_2+t^{-1}(e-d_2)$. In view of (1) this means that $d_1\in H$, a contradiction. Now apply Lemma 1.2 to conclude that H is an open halfspace. Clearly this set is as desired. ■

1.5 Theorem. *Let X be a topological vector space, let $\mathcal{C}$ be a symmetric vector convexity on X, and let $U\in\mathcal{C}$ be a body. Then the following hold.*

(1) If $C\subseteq\partial U$ is non-empty and convex, then C and U can be $\mathcal{C}$-separated. In particular, each $u\in U$ can be strongly separated from C.

(2) If $c\notin cl\,(U)$, then c and U can be strongly $\mathcal{C}$-separated.

Proof: For a proof of statement (1), by applying Lemma 1.4 and Proposition 1.3(3) we obtain an open halfspace H which contains the interior of U and includes C in its boundary. Clearly H corresponds with a continuous $\mathcal{C}$-functional separating between C and U. As f cannot be constant on U, the second part of (1) follows at once.

For a proof of statement (2), let u be any point in $int(U)$. The ray through c starting at u intersects the boundary of U in precisely one point, and we may assume that this point is the origin. Now we can use (1) to separate the origin and U. Suppose that c lies in the closed halfspace H containing U. Then the segment $[u,c)$ lies in the interior of H. Hence the origin lies in the interior of H, a contradiction. ■

The above result does not hold if the point $x \notin \partial(U)$ is replaced by a $\mathcal{C}$-convex set disjoint with $\bar{U}$. A simple counter-example can be constructed as follows. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are (symmetric) vector convexities on X, then so is the collection of all sets of type

$$C_1 \cap C_2, \ (C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2).$$

Consider the convexity which results in this way from the "rectangular" convexity of $\mathbb{R}^2$ and the affine convexity of $\mathbb{R}^2$ consisting of all lines parallel to the line $x+y=0$. Then no rectangular body can be separated from a "special" line.

A vector convexity on a topological vector space is said to be *rigid* provided it contains no non-trivial convex body. The following characterization of rigidity is a direct consequence of Theorem 1.5

1.6 Corollary. *Let X be a topological vector space with a symmetric vector convexity $\mathcal{C}$. Then the following are equivalent:*

(1) X does not admit non-trivial continuous $\mathcal{C}$-functionals.

(2) X is rigid. ■

In a topological vector space with a symmetric vector convexity $\mathcal{C}$, the *weak*($\mathcal{C}$) topology is defined to be generated by the closed subbase consisting of all convex sets which are closed in the original topology. If X is a normed space and if $\mathcal{C}$ is the collection of all norm-convex sets in X, then this topology is precisely the "weak(norm) topology", introduced in II: §5. If $\mathcal{C}$ consists of all (standard) convex subsets of X, and if X is a locally convex topological vector space, then the above topology is a *standard* weak topology, i.e., a topology on X generated by a collection of continuous functionals (see [70, § 3.8]). We note that, in general, a weak($\mathcal{C}$) topology need not be a standard weak topology. See Theorem 1.8.

The weak($\mathcal{C}$) topology of a vector space X is T_1 precisely if $\mathcal{C}$ contains all singletons of X. The next proposition describes when this topology is Hausdorff.

We introduce two notions of boundedness, the first of which is well-known. A (convex) subset C of a ***topological*** vector space is called *bounded* if for every neighborhood U of the origin there exists a real number $s>0$ such that $C \subseteq s \cdot U$. See [45, p. 44]. A (convex) set C of a (***general***) vector space is called *line-bounded* if the intersection of C with any line in X is a bounded subset of the line.

1.7 Proposition. *Let X be a topological vector space, and let $\mathcal{C}$ be a symmetric vector convexity. Then the following are equivalent:*

(1) The weak($\mathcal{C}$) topology of X is Hausdorff.

(2) Every pair of distinct points in X can be strongly $\mathcal{C}$-separated.

If, in addition, X has a bounded (standard) convex body, then statements (1) and (2) are also equivalent with:

(3) The $\mathcal{C}$-convex hull of a bounded set is line-bounded.

Proof: Implication (2)$\rightarrow$(1) is evident. For a proof of implication (1)$\rightarrow$(2), let x,y be two distinct points in X. Then there are two disjoint neighborhoods of x and y in the weak($\mathcal{C}$) topology of X. This implies that there exist closed sets $C_1, \cdots, C_n, D_1, \cdots, D_m \in \mathcal{C}$ with

$$y \notin \bigcup_{i=1}^{n} C_i,\ x \notin \bigcup_{i=1}^{m} D_i \text{ and } X = \bigcup_{i=1}^{n} C_i \cup \bigcup_{i=1}^{m} D_i.$$

We may assume that this covering of X is irreducible. Then each of the sets $C_1, \cdots, C_n, D_1, \cdots, D_m$ must have a non-empty interior. We may assume that $x \in C_i$. Then the point y and the set C_i satisfy the conditions of Theorem 1.5. Consequently, we can strongly separate C_i (a fortiori x) and y.

Next we derive the implication (3)$\rightarrow$(2). To this end, let $x,y \in X$. By (3) we obtain a $\mathcal{C}$-convex, line-bounded convex body B in X. We may assume that $0 \in B$. As B is line-bounded, there exists a λ such that $x + \lambda \cdot B$ avoids y. Hence by Theorem 1.5 (applied to y and $x + \lambda \cdot B$) we can strongly $\mathcal{C}$-separate x and y.

Finally, we establish the implication (2)$\rightarrow$(3). Let B be a bounded set and consider the collection $(f_i)_{i \in I}$ of all continuous $\mathcal{C}$-functionals. As the f_i are bounded on B there exists an $i \in I$ such that $f(B) \subseteq [-\gamma_i, \gamma_i]$ ($\gamma_i \in \mathbb{R}$). Now the subset

$$\cap_{i \in I} f_i^{-1}([-\gamma_i, \gamma_i]),$$

is a $\mathcal{C}$-convex line-bounded set containing B, which gives (3). ∎

For a normed space $(X, ||.||)$, let $\beta(X, ||.||)$ be the diameter of the smallest norm-convex set including the unit ball B of X. Soltan [77], showed that $\beta(\mathbb{R}^n, ||.||) < \infty$ (where $||.||$ is any norm on $\mathbb{R}^n$) if and only if there exists a point-separating collection of norm-functionals $f_1, \cdots, f_n$ on $(\mathbb{R}^n, ||.||)$. See [77]. Proposition 1.7 leads to the following generalization of this result: the norm-convex hull of B is line-bounded if and only if there exists a point-separating collection of continuous norm-functionals on X.

From the previous proposition we deduce that if a weak($\mathcal{C}$) topology is Hausdorff, then the collection of continuous $\mathcal{C}$-functionals $\mathcal{F}$ is point separating. Hence this collection of functionals determines a standard weak topology, coarser than the original weak($\mathcal{C}$) topology. The question arises whether these topologies coincide. The next result gives a simple characterization. If X is a vector space, and if $\mathcal{C}$ is a vector convexity on X, then a topology on X (not necessarily a vector topology) is *locally $\mathcal{C}$-convex* provided each point has a neighborhood base of $\mathcal{C}$-convex sets. See [79].

1.8 Theorem. *Let X be a topological vector space, and let $\mathcal{C}$ be a symmetric vector convexity that contains all singletons of X. Then the following are equivalent:*

(1) The weak($\mathcal{C}$) topology is a vector topology.

(2) The weak($\mathcal{C}$) topology is regular.

(3) The weak($\mathcal{C}$) topology is locally $\mathcal{C}$-convex.

(4) The weak($\mathcal{C}$) topology is generated by the collection of continuous $\mathcal{C}$-functionals of X.

Proof: We first establish the following intermediate statement.

(5) Let N be a weakly($\mathcal{C}$) closed neighborhood of $q \in X$. Then there exists a finite number of continuous $\mathcal{C}$-functionals $(f_i)_{i=1}^n$, together with real numbers $(\lambda_i)_{i=1}^n$, such that
$q \in \cap_{i=1}^n f_i^{-1}((-\infty,\lambda_i]) \subseteq N$.

We may assume that $N \neq X$. By the definition of weak($\mathcal{C}$) topology, there exist $\mathcal{C}$-convex closed subsets $C_1, \cdots, C_n$ of X such that

$$q \notin \bigcup_{i=1}^{n} C_i \,;\, N \cup (\bigcup_{i=1}^{n} C_i) = X. \tag{6}$$

As all sets appearing in the last equation are closed, we may assume that all C_i have a non-empty interior. By Theorem 1.5, there is a continuous $\mathcal{C}$-functional f_i such that

$$f_i(q) < \inf \{f_i(x) \mid x \in C_i\}. \tag{7}$$

Denote the right-hand side of (7) by λ_i $(i = 1,2, \cdots, n)$. Then by (6) we obtain that the set $\cap\{f_i^{-1}((-\infty,\lambda_i)) \mid i = 1,2, \cdots, n\}$ is as desired.

Having established (5), we proceed as follows. A T_1 vector topology is regular, showing implication $(1) \to (2)$. The intermediate statement directly gives the implications $(2) \to (3)$ and $(2) \to (4)$, whereas the implication $(4) \to (1)$ is trivial.

We are left with a proof of $(3) \to (2)$. Let C be a weakly($\mathcal{C}$) closed set in X and let $q \notin C$. Then C can be covered with a finite number of $\mathcal{C}$-convex closed sets not containing q. To produce a $\mathcal{C}$-closed neighborhood of q disjoint with C, it is therefore sufficient to consider C to be a $\mathcal{C}$-convex closed set itself. Let N be a convex neighborhood of q disjoint with C. If $\bar{N} \cap C = \varnothing$, then we are done. If, on the other hand, the convex set $C_0 = \bar{N} \cap C$ is non-empty, then by Theorem 1.5(1), there is a continuous $\mathcal{C}$-functional f such that $f(q) < \inf(f(C_0))$. For $\varepsilon > 0$ sufficiently small, the set $U = N \cap f^{-1}((\infty, f(q)+\varepsilon))$ is a convex weak($\mathcal{C}$) neighborhood of q and $\bar{U} \cap C_0 = \varnothing$. Evidently, $\bar{U} \cap C = \varnothing$. ∎

The argument in (5) above shows that in a vector convexity $\mathcal{C}$ with singletons and with a regular weak($\mathcal{C}$)-topology, each convex closed set of $\mathcal{C}$ can be separated from a point outside by a ***finite collection*** of continuous $\mathcal{C}$-functionals. The example mentioned after Theorem 1.5 also shows that, for general vector convexities, this cannot be improved to a situation where only ***one*** ([2]) $\mathcal{C}$-functional is needed. For modular normed spaces, however, it is possible to derive such a result. See Theorem 2.9.

[2] It is not difficult to verify that if $f_1, \cdots, f_n$ are (continuous) linear functionals separating a point p from a convex closed set C, then there is a linear combination $\sum_{i=1}^n c_i f_i$ which separates p from C. In circumstances as above, such combinations need not be $\mathcal{C}$-functionals.

§ 2 Weak(norm) topology in modular normed spaces

The norm-convexity of a modular normed space has somewhat stronger properties than a ***general*** vector convexity. See Lemma 2.2 below, and the remarks after Theorem 2.9.

From Theorem I: 2.14 and Proposition 1.7 we obtain the following.

2.1 Theorem. *If the weak(norm) topology of a modular normed space X is Hausdorff, then X is median.* ∎

In the next results we study the weak(norm) topology of a modular normed space more extensively. It turns out that dense median stable subspaces of $l_1(I)$ spaces are precisely the modular normed spaces with a regular weak(norm) topology.

We start with three lemmas. The second lemma concerns ***continuous*** norm-functionals, and the first lemma states that in ***Banach*** spaces continuity of such functionals is self-provided.

2.2 Lemma. *Any norm-functional on a modular Banach space is continuous.*

Proof: As each norm-functional is norm-interval preserving (cf. Corollary I: 4.10(3)), the theorem follows from Proposition III: 4.2. ∎

2.3 Corollary. *The following are equivalent for a modular Banach space.*

(1) *X is rigid.*

(2) *No norm-convex, proper subset of X has a core point.*

Proof: Implication $(2) \rightarrow (1)$ is evident. For a proof of the converse implication, assume to the contrary that $C \neq X$ is a norm-convex subset in X with a core point. If we endow X with the core topology, then C becomes a norm-convex body. Whence by Theorem 1.5 there exists a non-trivial norm-functional f on X. By Lemma 2.2 the functional f is continuous in the norm-topology. We conclude that X is not rigid, a contradiction. ∎

2.4 Lemma. *Let X be a modular normed space with completion $\tilde{X}$, and let H be a norm-convex closed hyperplane of X through the origin. Then,*

(1) *The completion $\tilde{H}$ of H in $\tilde{X}$ is the kernel of a continuous norm-functional of $\tilde{X}$.*

(2) *The median orthogonal complement $\tilde{H}^{\perp}$ of $\tilde{H}$ in $\tilde{X}$ is the span of an extreme point in the unit ball of $\tilde{X}$.*

Proof: For a proof of statement (1), it is well-known that H is the kernel of a continuous functional f on X. Let $\tilde{f}:\tilde{X} \rightarrow \mathbb{R}$ be the (unique) continuous extension of f. We first show that $\tilde{H}$ is the kernel of $\tilde{f}$.

Obviously, $\tilde{H} \subseteq \tilde{f}^{-1}(0)$. As for the reverse inclusion, let $x \in \tilde{f}^{-1}(0)$, and pick a point $e \in X$ with $f(e)=1$. As X is dense in $\tilde{X}$, there is a sequence $(x_n)_{n=1}^{\infty}$ in X converging to x. Let $k_n = x_n - f(x_n) \cdot e$ $(n \in \mathbb{N})$. Then $k_n \in H$ and

$$||f(x_n) \cdot e|| = ||(k_n + f(x_n) \cdot e) - k_n|| \geq \rho(x_n, H).$$

Hence, as $f(x_n) \rightarrow \tilde{f}(x) = 0$ we conclude that the $\rho(x, H) = 0$, i.e. $x \in \tilde{H}$.

Our proof of statement (1) is complete if we show that $\tilde{f}^{-1}([0,\infty))$ is norm-convex in $\tilde{X}$. For convenience let $C_+ = \{x \in \tilde{X} \mid \tilde{f}(x) > 0\}$, and $C_- = -C_+$. First, in Theorem II: 3.1 it is shown that the completion of a modular metric space Y is modular, and in Corollary II: 3.5 it is shown that the completion of a norm-convex subset of Y is norm-convex in the completion. Therefore, $\tilde{X}$ is a modular Banach space, and $\tilde{H}$ is norm-convex in $\tilde{X}$. Next, let $a,b \in \tilde{f}^{-1}([0,\infty))$ (see Figure 2.4).

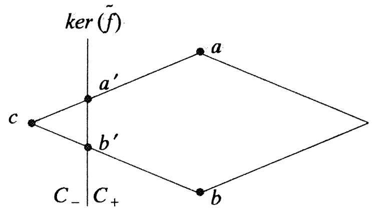

Fig. 2.4: the kernel of a norm-functional

Striving for a contradiction, suppose that $c \in I(a,b)$ with $c \in C_-$. As $\tilde{H} = \tilde{f}^{-1}(0)$ is norm-convex, we may assume that $a \in C_+$. Then the interval $I(a,c)$ meets the open sets C_+, C_-. As intervals are connected (they are even convex), the interval $I(a,c)$ meets $\tilde{H}$ in say a'. Hence, $a' \in I(a,c) \subseteq I(a,b)$. By star-shapedness of intervals at either endpoint, $c \in I(a',b)$. By applying the same method to b if $b \notin \tilde{H}$ we conclude that c is metrically between two points a',b' in $\tilde{H}$. However as $\tilde{H}$ is norm-convex we conclude that $c \in \tilde{H}$, contradicting the assumption that $c \in C_-$.

For a proof of statement (2), as $\tilde{H}$ is norm-convex and complete this hyperplane is gated. By virtue of Corollary III: 2.12(1), $\tilde{X}$ is linearly isometric with $\tilde{H} \times \tilde{H}^{\perp}$. Clearly $\tilde{H}^{\perp}$ is of dimension 1. Hence any point e in $\tilde{H}^{\perp}$ of norm 1, is extreme in the unit ball of $\tilde{H}^{\perp}$ and spans the whole of $\tilde{H}^{\perp}$. By Corollary III: 2.12(2) any such point is also extreme in the unit ball of $\tilde{X}$. ∎

2.5 Remarks. It is not true that a hyperplane, which occurs as a convex set of a symmetric vector convexity $\mathcal{C}$, corresponds with a $\mathcal{C}$-functional: just consider the "affine" convexity of a vector space. For general normed spaces, each hyperplane of the metric convexity induces a norm-functional (cf. the proof of Lemma 2.4). The following is not known:

2.6 Problem. *Does the canonical extension of a continuous norm-functional to the metric completion preserve the metric convexity?*

Lemma 2.4 states that the answer to the above is affirmative for *modular* normed spaces.

It is well-known, and easy to show, that the extreme points of the unit ball in an $L_1(\mu)$ space correspond with characteristic functions on atoms of the ambient σ-algebra. Hence from Lemma 2.4(2) one can deduce that the (continuous) norm-functionals on X are precisely the evaluation mapping on atoms. We obtain the following corollary.

2.7 Corollary. *Let $X = L_1(N, \mathcal{A}, \mu)$ be an $L_1(\mu)$ space. Then the following are equivalent:*

(1) *The unit ball of X has no extreme points.*

(2) *There is no norm-convex subset $C \neq X$ with a core point.*

(3) *The measure space $(N, \mathcal{A}, \mu)$ is atom free.* ■

From the previous result we conclude that the space $L_1([0,1])$ has no (non-trivial) norm-functionals. We are now able to give an example of a discontinuous norm-functional on a (non-complete) median normed space. Indeed, let R, B be the subspaces of $L_1([0,1])$ consisting of the collections of Riemann integrable functions and essentially bounded integrable functions respectively. Then R is median stable and B is norm-convex in $L_1([0,1])$ (see Example III: 4.15). If we endow these spaces with the core topology, then the characteristic function of the unit interval is a core point of the positive cones of R and B. Hence by Theorem 1.5 there exist non-trivial norm-functionals on R and B. By Lemma 2.4 these functionals cannot be norm-continuous.

2.8 Lemma. *Let X be a normed space. If $x \in X$ has a gate c in a subset C of X, and if a norm-functional f strongly separates x and c, then f also strongly separates x and C.*

Proof: Straightforward. ■

The following result is a strengthening of Theorem 1.8.

2.9 Theorem. *For a modular normed space X the following are equivalent.*

(1) *Relative to the weak(norm) topology, X is a topological vector space.*

(2) *Each norm-convex closed subset A of X can be strongly norm-separated from points $q \notin A$.*

(3) *The weak(norm) topology of X is generated by a collection of continuous norm-functionals that separates any norm-convex closed set A from points $q \notin A$.*

(4) *The completion of X is linearly isometric with $l_1(I)$ for some index set I.*

Proof: We shall show the following sequence of implications:

$(1) \to (2) \to (4) \to (3) \to (1)$.

For a proof of implication $(1) \to (2)$, let A be a norm-convex closed subset of X, and let $q \notin A$. By Theorem 1.8(4), there exist continuous norm-functionals $f_1, \cdots, f_n$ and scalars $\lambda_1, \cdots, \lambda_n$ such that

$$f_i(q) < \lambda_i \text{ for } i = 1, 2, \cdots, n \; ; \; A \subseteq \bigcup_{i=1}^{n} f_i^{-1}([\lambda_i, \infty)). \tag{5}$$

If none of the functionals $f_1, \cdots, f_n$ strongly separate A from q, then the convex sets

A, and $f^{-1}(-\infty, \lambda_i)$ for $i = 1, 2, \cdots, n$,

meet two by two (each halfspace meets A by assumption, whereas the halfspaces meet in q). The convexity of a modular (metric) space has the (F,2)-IP, see (I: 4.4.1). In our situation, this conflicts with (5).

For a proof of implication $(2) \to (4)$, let $\tilde{X}$ denote the completion of X. From Theorem 2.1 we conclude that X is at least a median space, hence by Corollary II: 3.2 so is $\tilde{X}$. Let $Ext(\tilde{B})$ be the set of extreme points in the unit ball $\tilde{B}$ of $\tilde{X}$. We shall now show that $\tilde{X}$ has the (K-M) property, i.e., the convex closure of $Ext(\tilde{B})$ in $\tilde{X}$ equals $\tilde{B}$. To this end, if $\tilde{X}$ fails to have the (K-M) property then the linear closure $N = \overline{sp}(Ext(\tilde{B}))$ is not the whole of $\tilde{X}$. As X is dense in $\tilde{X}$ there exists an $x \in X \setminus N$. In Corollary III: 3.4 it is shown that N is a norm-convex subspace of $\tilde{X}$. Hence as N is complete, it is gated. Let n be the gate of x in N.

We claim that n and x can be strongly norm-separated in $\tilde{X}$. ([3]) By Corollary II: 3.13, we can find a (small) norm-convex closed convex set U in X that avoids x, and such that $n \in Cl_{\tilde{X}}(U)$. By assumption on X there exists a continuous norm-functional f strongly separating U and x. The extension $\tilde{f}$ of f to $\tilde{X}$ is a norm-functional by Lemma 2.4. Clearly $\tilde{f}$ strongly separates $Cl_{\tilde{X}}(U)$ and x, and a fortiori, $\tilde{f}$ strongly separates n and x.

Invoking the definition of the gate n we conclude that $\tilde{f}$ strongly separates x and N as well, i.e. $N \subseteq ker(\tilde{f})$. By Lemma 2.4(2) there exists a point $e \in Ext(\tilde{B})$ spanning $\tilde{f}^{-1}(0)^{\perp}$. Clearly $e \notin N$, contradicting the definition of N.

We have shown that $\tilde{X}$ is a median Banach space with the (K-M) property. In Theorem III: 3.6 it is shown that such a space is linearly isometric with $l_1(I)$ for some index set I.

For a proof of implication $(4) \to (3)$, let X be a dense median stable subspace of $l_1(I)$ for some index set I. We shall show that the collection of coordinate functions of $l_1(I)$, restricted to X, is as described in (3). To this end, as these functionals are clearly continuous norm-functionals the topology τ generated by them is coarser than the weak(norm) topology of X. We shall show that τ contains a subbase of the weak(norm) topology, namely the closed norm-convex subsets of $l_1(I)$. Suppose that C is a closed and norm-convex subset of X. Take $x \notin C$. By Corollary II: 3.5 the completion $\tilde{C}$ of C is a gated set of the completion $l_1(I)$ of X. Let y be the gate of x in $\tilde{C}$. As $x \neq y$ we can find a coordinate function p strongly separating x and y. Hence by Lemma 2.8, p strongly separates $\tilde{C}$ and y. In particular the restriction of p to X strongly separates C and x. This shows that C is closed in τ, i.e. $C \in \tau$. Finally, implication $(3) \to (1)$ is standard. ■

The previous result asserts that $l_1(I)$ as well as its dense median stable subsets have a regular weak(norm) topology. By Corollary II: 5.14 this topology appears to be even normal on norm-convex subsets.

A difference between Theorems 1.8 and 2.9 is that the latter requires only one functional for separating a point from a convex set. This type of extension cannot be obtained in the general setting of Theorem 1.8.

The next two corollaries (which are proved simultaneously) provide some additional information on isometric embedding in $l_1(I)$ spaces.

2.10 Corollary. *If a modular normed space X can be isometrically embedded in an $l_1(I)$ space, then X is median and its weak(norm) topology is regular. Conversely, if X is a median normed space with a Hausdorff weak(norm) topology, and if all metric intervals of X are complete, then X embeds isometrically in an $l_1(I)$ space.*

[3] At this stage of the proof we could not decide whether $\tilde{X}$ has a regular or even a Hausdorff weak(norm) topology. Otherwise, Proposition 1.7 would have done the job.

2.11 Corollary. *The only $L_1(\mu)$ spaces allowing for an isometric embedding in an $l_1(I)$ space are the $l_1(J)$ spaces with $|J| \leq |I|$.*

Proof: First, let X be a modular normed space which is isometrically embedded in an $l_1(I)$ space. In particular, X is a median stable set in $l_1(I)$. In Proposition II: 5.10 it is shown that the weak(metric) topology of a median stable subset of a median metric space is the relative topology derived from the weak(metric) topology of the ambient space. Now $l_1(I)$ has a regular weak(norm) topology by Theorem 2.9 and hence the weak(norm) topology of X is regular as well. Similarly, the median subspace $\bar{X}$ has a regular weak(norm) topology. Hence by Theorems 1.8 2.9 the Banach space $\bar{X}$ is an isometric $l_1(J)$ space where, evidently, $|J| \leq |I|$. By Theorem III: 4.13 the class of median Banach spaces consists exactly of the $L_1(\mu)$ spaces, and we obtain Corollary 2.11 in this way.

On the other hand, it follows from Proposition II: 5.12 that if a median metric space with complete intervals has a Hausdorff weak(metric) topology, then this topology is even regular. Another application of Theorems 1.8 and 2.9 gives the result. ■

2.12 Problem. *Is each modular normed space with a Hausdorff weak(norm) topology isometrically embeddable in an $l_1(I)$ space? Or, equivalently, is the Hausdorff property of the weak(norm) topology inherited by the completion?*

Observe that above mentioned spaces are at least median (Theorem 2.1). We can also deduce the following.

2.13 Proposition. *Let X be a normed space with a Hausdorff weak(norm) topology. Then either the completion $\tilde{X}$ of X has infinitely many extreme points in its unit ball, or X is finite dimensional.*

Proof: Let $\tilde{B}$ denote the unit ball of $\tilde{X}$. By Proposition 1.7 there at least exists a closed norm-hyperplane H_1. Without loss of generality $0 \in H_1$. In view of Lemma 2.4(2) there is an extreme point of $\tilde{B}$ not in the closure $\tilde{H}_1$ of H_1. As H_1 is norm-convex it is also a modular normed space. By the earlier cited Proposition II: 5.12 the weak(norm) topology of H_1 is the relative topology derived from the weak(norm) topology of the whole X. In particular the weak(norm) topology of H_1 is Hausdorff. Hence, we can apply the above to H_1, to conclude that there exists a closed norm-hyperplane H_2 in H_1 and an extreme point e_1 in the unit ball of the completion $\tilde{H}_1$ that is not in $\tilde{H}_2$. By Corollary III: 2.12(2), e_2 is also an extreme point of $\tilde{B}$. If X is not finite dimensional, then we can continue this argument to obtain infinitely many extreme points in $\tilde{B}$. ■

§ 3 Adapted metrics

A metric d on X is called *compatible* with a geometric interval space (X,I) provided the following holds:

If C is I-convex then so is the closed ball $D_d(C,r)$ around C.

It is possible to give an equivalent definition in terms of open balls. Note that the discrete metric on X is compatible with (X,I). As every convex set is an updirected union of polytopes we obtain:

(3.1) A metric d is compatible with a geometric interval space (X,I) provided closed balls around polytopes are convex.

The following is easily verified.

(3.2) A compatible metric respects basepoint orders in the following sense: if $x \leq_b y$ then $d(b,x) \leq d(b,y)$.

Let X be a topological space. Equip $\mathbb{R}^X$ with the product interval operator I_π. Then the subspace $B_\tau(X)$ is I_π-convex, hence I_m restricted to $B_\tau(X)$ yields a median interval space.

3.3 Example. *The supremum-norm of $B_\tau(X)$ is compatible with the median convexity of this space.*

Proof: Let $\leq$ be the product order of $\mathbb{R}^X$, i.e. $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. Let $F = \{f_1, \cdots, f_n\}$ be a finite subset of $\mathbb{R}^X$, and let p, q respectively denote the point-wise minimum and maximum of these functions. It is well-known and easy to prove that the median convex hull of F consists of the order segment $[p,q]$. Hence polytopes of $\mathbb{R}^X$ correspond with order intervals of type $[p,q]$. As $B_\tau(X)$ is an I_π-convex subset of $\mathbb{R}^X$, the same holds for $B_\tau(X)$ with the additional property that the functions p and q are bounded. One can easily verify that a closed (norm-)ball of radius $r \geq 0$, around such a segment $[p,q]$ equals the order segment $[p-r, q+r]$ —note that the median of the points $p,q,f \in \%B_\tau(X)$ yields a metric nearest point of f in $[p,q]$. Whence closed ball around polytopes are convex. By (3.1) we conclude that the supremum norm is as desired. ■

In view of (I: 4.4.1) a convex metric compatible with a convexity with the (F,2)-IP has the (F,2)-IP of balls. This partially motivates our interest in compatible metrics. The following result summarizes some of the properties of such metrics.

3.4 Lemma. *Let (X,I) be a geometric interval space and let d be compatible with I. Then the following hold for $B \subseteq X$ gated and $C \subseteq X$ convex:*

(1) If $c \in C$, then $d(p_B(c), C) = d(B,C)$. In particular, p_B is a metric nearest point function and B is closed in (X,d).

(2) If C is gated, then any pair of mutual gates b,c of B,C realize the distance between B and C, i.e. $d(B,C) = d(b,c)$.

(3) If (X,I) is median, then p_B is contractive w.r.t. (X,d).

Proof: For a proof of (1), denote $r=d(B,C)$. If $\varepsilon>0$, then the ball $D(C,r+\varepsilon)$ meets B. As C is convex so is this ball, and consequently $p_B(c)$ is a member of it ((I: 3.4.1)). Whence $d(p_B(c),C)\le r+\varepsilon$. As $\varepsilon>0$ was arbitrary, we obtain statement (1). Statement (2) follows from statement (1).

For a proof of statement (3), let $x,y\in X$. If $a\in I(x,p_B(x))$ then by definition of gate $a\le_x p_B(x)\le_x p_B(y)$. Hence by (3.2) $d(p_B(x),p_B(y))\le d(a,p_B(y))$. Whence

$$d(I(x,p_B(x)),p_B(y))=d(p_B(x),p_B(y)). \tag{*}$$

Now the ball $D(I(x,p_B(x)),d(x,y))$ contains y and meets B (in $p_B(x)$). From (I: 3.4.1) we conclude that $p_B(y)\in D(I(x,p_B(x)),d(x,y))$, i.e. $d(I(x,p_B(x)),p_B(y))\le d(x,y)$. By formula (*) this amounts to $d(p_B(x),p_B(y))\le d(x,y)$, as desired. ■

3.5 Proposition. *Let (X,m) be a median algebra, and let d be a compatible metric. Then the restriction of d to a median stable subset Y of X yields a compatible metric on Y. If moreover d is a convex metric and Y is m-convex then the restriction of d to Y is also a convex metric.*

Proof: Let d' denote the restriction of d to Y. We aim at the use of (3.1). Let F be a finite subset of Y, and let $r>0$. Consider the following equality:

$$D_{d'}(co_Y(F),r)=D_d(co_X(F),r)\cap Y. \tag{1}$$

To this end, the inclusion from left to right in (1) is evident. Let y be a member of the right-hand side in (1). The polytope $co_X(F)$ is gated, see (II: 3.10). By Lemma 3.4 the gate p of y in $co_X(F)$ is a metric nearest point in $co_X(F)$ with respect to d, i.e. $d(y,p)\le r$. By the description of the gate p in (II: 3.10), if follows that p is also the gate of y in $co_Y(F)$. Whence, y is a member of the right-hand side of (1).

For a proof of the last statement of the proposition, let $a,b\in Y$ and $0\le s\le d(a,b)$. Then the closed balls in $D(a,s)$, $D(b,d(a,b)-s)$ and the convex set Y meet two by two. Hence these sets meet altogether, which implies that the restriction of d to Y is convex. ■

We now come to a topological version of compatibility. Let (X,I) be a geometric interval space, and let τ be a topology on X. Let τ_w be the weak topology of τ, i.e. the topology generated by the collection of closed I-convex subsets of X. The metric d is said to be *adapted* to the triple (X,I,τ) if:

(i) d is compatible with (X,I),

(ii) The topology τ_d on X induced by d satisfies: $\tau_w\subseteq\tau_d\subseteq\tau$.

The notion of adapted metric was introduced by van Mill and van de Vel in [61] for compact topological median algebras.

Metrics adapted to the triple (X,I_ρ,τ_ρ) derived from a metric space (X,ρ), shall be of special interest. The next result follows from Propositions II: 5.10 and 3.5.

3.6 Proposition. *Let X be a locally star-shaped median algebra, and let d be an adapted metric. Then the restriction of d to a median stable subset Y of X yields an adapted metric on Y.* ■

3.7 Corollary. *Let Y be a median stable subset of a space of type $l_1(I)$. Then the supremum norm of $B(I)$ restricted to Y yields a metric adapted to the sum-norm of $l_1(I)$.*

Proof: Observe that for each topological space (X,τ), the topological median algebra $(B_\tau(X), I_\pi, \tau_{\|.\|})$ is locally star-shaped (even "locally convex"). The space $l_1(I)$ is an I_π-convex subspace of $B(I)$. Hence by Example 3.3 and Proposition 3.6, the sup-norm restricted to $l_1(I)$ yields an adapted norm onto $l_1(I)$ endowed with the relative sup-norm topology. The latter topology is evidently coarser than the topology induced by the sum-norm of $l_1(I)$. Also, by virtue of Theorem 2.9 the weak(sum-norm) of $l_1(I)$ is generated by the coordinate projections $p_i : l_1(I) \to \mathbb{R}$ $(i \in I)$. Hence the weak(sum-norm) topology of $l_1(I)$ is coarser than the relative sup-norm topology on $l_1(I)$. This shows that the sup-norm restricted to $l_1(I)$ is adapted to the sum-norm. Applying Proposition 3.6 concludes the proof. ■

It is possible to weaken the notions of a compatible or an adapted metric for a geometric interval space (X,I) by requiring that closed balls around ***singletons*** are I-convex. In this situation we say that a metric ρ is *point-compatible* with (X,I) (or: *point-adapted* to a triple (X,I,τ)).

We remark —we shall not use this— that in Propositions 3.5 and 3.6, "compatible", resp. "adapted" can be replaced by "point-compatible", resp. "point-adapted". We now come to the main theorem.

3.8 Theorem. *For a modular normed space the following are equivalent:*

(1) X has an adapted metric.

(2) X has a point-adapted metric.

(3) For each norm-convex closed subset A of X and each point $q \notin A$ there is a norm-convex norm-neighborhood of q disjoint with A.

(4) The completion of X is linearly isometric with $l_1(I)$ for some index set I.

Proof: Implication $(1) \to (2)$ is trivial. For a proof of implication $(2) \to (3)$, let d be a point-adapted metric on X. Let A and q be as prescribed in (3). As the weak topology is coarser than the topology of (X,d) there exists a closed ball $D(q,r)$ with $r > 0$ avoiding A. As the topology of d is coarser than the norm topology of $l_1(I)$ we conclude that $D(q,r)$ is a norm-convex norm-neighborhood of q. For a proof of implication $(3) \to (4)$, let A be a norm-convex closed subset of X, and let $q \notin A$. By assumption there exists a norm-convex (norm-)neighborhood U of q avoiding A. Let $\tilde{X}$ denote the completion of X and let $\tilde{U}$ and $\tilde{A}$ be the closure in $\tilde{X}$ of U and A, respectively. By a simple topological argument we obtain that $int(\tilde{U}) \cap X$ equals the X-interior of the X-closure of U, which in turn equals the X-interior of U (U is a convex body of X). Whence, $int(\tilde{U})$ avoids A and so

$$int(\tilde{U}) \cap \tilde{A} = \emptyset. \tag{5}$$

By Corollary II: 3.5 the sets $\tilde{A}$, $\tilde{U}$ are complete, norm-convex subset of the completion of X. In particular, the set $\tilde{A}$ is gated. Let $p \in \tilde{A}$ be the gate of q in $\tilde{A}$. As $int(\tilde{U})$ is norm-convex by Proposition 1.3, we deduce from formula (5) that $p \notin int(\tilde{U})$. Hence by Theorem 1.5 we can find a continuous norm-functional f separating U and p. Hence f strongly separates q and p and in view of Lemma 2.8, f also strongly separates a and $\tilde{A}$. In particular the restriction of f to X strongly separates q and A. By Theorem 2.9 we obtain that the completion of X is a space of

type $l_1(I)$.
The proof of implication (4)$\rightarrow$(1) is Corollary 3.7. ∎

§ 4 The decomposition theorem

4.1 Theorem. *Let X be a modular normed space with complete intervals. Then there exist an index set I, a subspace Z of $l_1(I)$, and a rigid modular normed space Y such that X is isometric with the product $Y \times Z$, equipped with the sum norm. In particular, the spaces Y and Z have complete intervals. The decomposition is unique up to isometry.*

Proof: Let $(H_i)_{i\in S}$ be the collection of all norm-convex closed hyperplanes through the origin, and let $Y = \cap\{H_i \mid i\in S\}$. By Corollary III: 2.12 any H_i $(i\in S)$ is an L-summand. Let p_i denote the L-projection of H_i.

It follows from Theorem II: 4.8 that Y is gated, and that the gate function p_Y of Y is the pointwise limit of linear functions —namely, finite compositions of the p_i— and is hence linear. We conclude that Y is an L-summand. Hence, X decomposes into linear orthogonal complements Y and $Y^\perp$: each $x\in X$ decomposes as

$$x = y + y^\perp, \quad (y\in Y,\ y^\perp\in Y^\perp).$$

By definition of (median) orthogonality, $||x|| = ||y|| + ||y^\perp||$ holds, and hence X is linearly isometric with $Y{\times}Y^\perp$, equipped with the sum-norm. Observe that both Y and $Y^\perp$ are closed subspaces of X, hence they have complete intervals.

First, we show that Y is rigid. Suppose that f is a non-trivial continuous norm-functional on Y. Then $H = f^{-1}(0)$ is a non-trivial closed norm-hyperplane of Y. One can easily verify that the subspace $H' = H + Y^\perp$ is a norm-hyperplane in X containing the origin, conflicting with the construction of Y.

Next we shall show that each pair of distinct points $x,y\in Y^\perp$ can be strongly separated by a continuous norm-functional on X. Indeed, as $Y\cap Y^\perp = \{0\}$ the point $x-y$ can not be a member of Y. Hence there exists an $i\in S$ such that the norm-hyperplane H_i misses $x-y$. Clearly this implies that x and y can be strongly separated by the corresponding norm-functional on X.

From Theorem 2.9 we conclude that the subspace $Z = Y^\perp$ is linearly isometric with a dense subspace of $l_1(I)$ for some index set I. Moreover, as Z has complete intervals, it follows from Theorem II: 3.21 that Z is norm-convex in $l_1(I)$. Hence, decompositions as stated in the theorem at least exist.

For a proof of unicity of such a decomposition under linear isometries, we shall show the following, even stronger, result. Let V be an L-summand of X. If V is rigid and if distinct points of $V^\perp$ can be strongly separated by norm-functionals of $V^\perp$, then $V = Y$. To this end, as continuous norm-functionals on X must be trivial on V, we have $V \subseteq Y$. For a proof of the other inclusion, suppose to the contrary that $y\in Y\setminus V$. As X decomposes into V and $V^\perp$, we can write $y = d + d^\perp$ for some $d\in V$ and $d^\perp\in V^\perp$. Clearly, $d^\perp \neq 0$. By assumption we can find a continuous norm-functional f on $V^\perp$, that strongly separates $d^\perp$ from the origin. By taking $\bar{f}(v+v^\perp) = f(v^\perp)$ for all $v\in V$, $v^\perp\in V^\perp$, we extend f to a continuous norm-functional $\bar{f}$ on the whole of X. However, $\bar{f}(y)=f(d^{\perp})\neq 0$, which is impossible by the construction of Y. ■

As a simple application of Theorem 4.1, one can verify that

(i) A modular space of type $(\mathbb{R}^2, ||.||)$ is linearly isometric with $l_1(2)$;

(ii) A modular space of type $(\mathbb{R}^3, ||.||)$ is either rigid, or linearly isometric with $l_1(3)$.

These results also follow from the work of Hanner [36], where it is shown that, modulo linear isometries, there are but finitely many modular norms on $\mathbb{R}^n$ $(n\in\mathbb{N})$. If $n=2$ then there is only one such norm, and for $n=3$ there are exactly two of them: the sum- and maximum-norm.

From Corollary 2.7 we conclude that for $L_1(\mu)$ spaces Theorem 4.1 yields the well-known result that an $L_1(\mu)$ space is decomposable in a product of an atom free $L_1(\mu)$ space and an $l_1(I)$ space (cf. Corollary III: 3.8(3)).

Application: characterization of $l_1(I)$ spaces among modular Banach spaces in terms of properties of the weak(norm) topology.

Let $X=l_1(I)$ for some index set. By virtue of Theorem 2.9, the weak(norm) topology is generated by the coordinate functions $p_i: l_1(I)\to\mathbb{R}$ $(i\in I)$. That is, the weak(norm) topology is a standard weak topology. Depending on the cardinality of I, one of the following three situations occurs.

CASE I: The index set I is uncountable. Evidently, no point in $l_1(I)$ has a countable weak(norm) neighborhood base. Hence, the weak(norm) topology of $l_1(I)$ is not metrizable. Also, by Proposition II: 5.12 the completeness of X, together with the Hausdorff property of its weak(norm) topology, imply the regularity of this topology.

CASE II: The cardinality of the index set I is countably infinite. Clearly $l_1(\mathbb{N})$ with the weak(norm) topology is a topological subspace of $\mathbb{R}^{\mathbb{N}}$ with the product topology. In particular the weak(norm) topology is metrizable. On the other hand, it is well-known that a weak topology is normable if and only if the ambient vector space is finitely dimensional (cf. ([70, § 3.8]). Therefore, the weak(norm) topology of $l_1(\mathbb{N})$ is not normable (for a direct proof of this see e.g. [59, Lemma 1.2.1]).

CASE III: The cardinality of the index set I is finite, say $|I|=n$. The weak(norm) topology of $l_1(n)$ is normable.

Combining these cases with Theorem 2.9 we have shown the following result:

4.2 Theorem. *Let X be a modular Banach space.*

(1) The weak(norm) topology of X is Hausdorff if and only if X is linearly isometric with $l_1(I)$ for some index set I.

(2) The weak(norm) topology of X is metrizable if and only if X is linearly isometric with $l_1(I)$ for some countable set I.

(3) The weak(norm) topology of X is normable if and only if X is linearly isometric with $l_1(I)$ for some finite set I. ■

By Proposition 1.7 and Corollary 2.10 the modular Banach spaces in which the unit ball has a line-bounded norm-hull are precisely the $l_1(I)$ spaces. The following is a characterization of modular normed spaces in which the latter hull is genuinely bounded.

4.3 Theorem. *Let X be a modular normed space. Then the following conditions are equivalent:*
(1) X is linearly isometric with an $l_1(n)$ space for some $n \in \mathbb{N}$.
(2) X contains a bounded norm-convex body.
(3) X contains a norm-convex body with an extreme point.

Proof: Let $\tilde{X}$ be the completion of X. Then $\tilde{X}$ is also a modular normed space by Theorem II: 3.1. Consider the following additional statement:
(4) $\tilde{X}$ contains a norm-convex body with an extreme point.
We shall show the following implications:

$$\begin{tikzcd} & (2) \arrow[dr] & & \\ (1) \arrow[ur] \arrow[dr] & & (4) \arrow[r] & (1) \\ & (3) \arrow[ur] & & \end{tikzcd}$$

Note that the implications $(1) \to (2),(3)$ are evident. For a proof of implication $(2) \to (4)$, let C be a bounded norm-convex body in X. By Corollary II: 3.5(2) the completion $\tilde{C}$ of C in $\tilde{X}$ is a norm-convex subset of $\tilde{X}$. Obviously, $\tilde{C}$ has a non-empty interior and is bounded. By virtue of Remark III: 5.8 and Proposition III: 5.7(1), a complete bounded norm-convex subset has an extreme point. Therefore $\tilde{C}$ has an extreme point.

Next, for a proof of implication $(3) \to (4)$, let C be a norm-convex subset of X with an extreme point e. In Proposition III: 5.7(2) it is shown that under these conditions the point e remains extreme in the completion $\tilde{C}$ of C in $\tilde{X}$. By Corollary II: 3.5(2) once again, $\tilde{C}$ is a norm-convex subset of $\tilde{X}$.

Finally, for a proof of implication $(4) \to (1)$, let C be a norm-convex body with an extreme point e of $\tilde{X}$. We may assume that e equals the origin. By Theorem 4.1 we can regard $\tilde{X}$ as a product $Y \times l_1(I)$ (equipped with the sum-norm) of modular Banach spaces, in which the space Y is rigid. Let us assume that Y is non-trivial. As the origin is an extreme point of C, the subset $C \cap Y$ cannot contain non-trivial antipodal points, hence $C \cap Y$ is a proper norm-convex body of Y. This is impossible as Y is rigid. We conclude that $Y = \{0\}$, that is, $\tilde{X}$ is linearly isometric with $l_1(I)$. Assume that the cardinality of I is infinite. Let $x \neq 0$ be a member of C. Observe that if $x_i > 0$ then, as C is norm-convex, all members of C have a non-negative i-th coordinate. As I is infinite, every (non-trivial) ball contains two points x,y such that $0 \neq x_i = -y_i$ for some $i \in I$. But then C cannot have interior points. We conclude that $\tilde{X}$ is linearly isometric with $l_1(n)$ for some $n \in \mathbb{N}$. In particular $\tilde{X}$ is finite dimensional, whence $X = \tilde{X}$. ■

We remark that condition (3) is equivalent with the existence of a bounded, norm-convex subset of X containing the unit ball. Another equivalent condition is, that the norm-convex hull of a bounded set B (i.e. the smallest norm-convex set containing B) is bounded.

CHAPTER V

ISOMETRIC EMBEDDINGS OF MEDIAN SPACES

In the first section of this chapter we discuss two techniques which are of particular interest to us.

The paper [7] written by P. Assuoad and M. Deza contains two chapters of a projected (but not achieved) book on metric subspaces of $L_1(\mu)$ spaces. Among other things, they show that a metric space is isometrically embeddable in an $L_1(\mu)$ space, whenever its ***finite*** subsets are. As [7] is rather obscure we give full proofs of these results here.

Superextensions were introduced by de Groot in [32] in a topological setting. A fundamental property of the superextension of a space X is that it behaves as a free algebra with members of X as generators: any function of the original set to a median algebra extends to a median preserving function, defined over the superextension.

In section 2 we apply these techniques to show that median metric spaces correspond with metric subspaces of $L_1(\mu)$ and $K_1(\mu)$ spaces.

In section 3 we use completely different techniques to show that a median metric space embeds isometrically in an $l_1(n)$ iff it embeds as a median subalgebra. (^{1})

§ 1 Preliminaries

1.1 Finite subspaces of $L_1(\mu)$. We shall prove the following theorem.

1.2 Theorem. *A metric space (X,ρ) is isometrically embeddable in an $L_1(\mu)$ space iff the metric subspace $(F,\rho_{|F})$ is isometrically embeddable in an $L_1(\mu)$ space for all finite subsets F of X.*

For a proof of this theorem we descend to the (larger) class of pseudo-metric spaces. A pseudo-metric space (X,ρ) is *L-embeddable* if it is isometrically embeddable in an $L_1(\mu)$ space. First we introduce a special kind of L-embeddable spaces. A pseudo-metric space is called *K-embeddable* if it isometrically embeddable in the pseudo-metric space $K(X,\mathcal{A},\mu)$ for some measure space $(X,\mathcal{A},\mu)$.

1.3 Lemma. ([7, Proposition 1.11]) *A pseudo-metric space (X,ρ) is L-embeddable iff it is K-embeddable.*

^{1} The results of this chapter were obtained by van de Vel and the author. See [82].

Proof: All we need to verify is that $L_1(\mu)$ spaces themselves are K-embeddable. To this end, let $(N,\mathcal{A},\mu)$ be a measure space. Consider the product measure space:

$$(N\times\mathbb{R},\mathcal{A}\otimes\mathcal{A}_L,\mu\otimes\mu_L).\ (^2)$$

We denote $\mu\otimes\mu_L$ by ν. For $\bar{f}\in L_1(N,\mathcal{A},\mu)$ we define

$$E(\bar{f})=\{(x,t)\in N\times\mathbb{R}\mid t>f(x)\}.$$

Now we verify the following formulae:

(1) $E(\bar{f})\Delta E(\bar{0})\in\mathcal{A}\otimes\mathcal{A}_L$

(2) $\int|\bar{f}|\,d\mu=\nu(E(\bar{f})\Delta E(\bar{0}))$.

To this end, formulae (1) and (2) are evident for step functions of $(N,\mathcal{A},\mu)$; the general case easily follows from this.
Hence, the function $\bar{f}\to E(\bar{f})\Delta E(\bar{0})$ is an isometry of $L_1(N,\mathcal{A},\mu)$ into $K(N\times\mathbb{R},\mathcal{A}\otimes\mathcal{A}_L,\mu\otimes\mu_L)$. ■

Let X be a set. Define $J_X:X\to 2^{2^X}$ by

$$J_X(x)=\{S\mid x\in S\subseteq X\}.$$

Let $\sigma(X)$ denote the σ-algebra on 2^X generated by the sets $(J_X(x))_{x\in X}$.

1.4 Lemma. *If a pseudo-metric space (X,ρ) is K-embeddable, then there is a measure μ on $\sigma(X)$ such that (X,ρ) is isometrically embeddable in $K(2^X,\sigma(X),\mu)$.*

Proof: Let $f:X\to K(N,\mathcal{A},\mu)$ be an isometric embedding. The function $f^*:(N,\mathcal{A})\to(2^X,\sigma(X))$ defined by $f^*(n)=\{x\in X\mid n\in f(x)\}$, is measurable as (we write J for J_X)

$$f^{*^{-1}}(J(x))=f(x). \tag{1}$$

This allows us to define a measure $\mu_{f.}$ by taking

$$\mu_{f.}(U)=\mu(f^{*-1}(U)),$$

for $V\in\sigma(X)$. By virtue of equality (1) we find that J takes values in $K(2^X,\sigma(X),\mu_{f.})$. We show that J is in fact an isometrical embedding of X into $K(2^X,\sigma(X),\mu_{f.})$. To this end, let $x,y\in X$. Consider the following equalities:

$$f^{*-1}(J(x)\Delta J(y))=f^{*-1}(J(x))\Delta f^{*-1}(J(y))=f(x)\Delta f(y).$$

By assumption, the μ-measure of this set equals $\rho(x,y)$. ■

Surprisingly, the following corollary is not stated in [7].

1.5 Corollary. *If a finite metric space (X,ρ) is isometrically embeddable in an $L_1(\mu)$ space, then it is isometrically embeddable in $(\mathbb{R}^{2^{|X|}},||.||_s)$.*

Proof: If (X,ρ) is isometrically embeddable in an $L_1(\mu)$ space, then it is isometrically embeddable in $K(2^X,\sigma(X),\mu)$ for some measure μ. Let N be the number of atoms in $\sigma(X)$, then clearly $N\le 2^{|X|}$. One can easily verify that $K(2^{2^X},\sigma(X),\mu)$ isometrically embeds in $\mathbb{R}^N$. ■

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2 Some textbooks on measure theory, e.g. [33], define the notion "product measure space" only for σ-finite measure spaces. However this restriction is not essential; there is a canonic way to define the product measure space for arbitrary measure spaces. See for instance [69, p. 304].

In section 2 we shall show that a finite metric median space X can be embedded in $\mathbb{R}^{|X|-1}$.

1.6 Lemma. *Let $(d_i)_{i\in D}$ be a net of pseudo-metrics on a (fixed) set X converging pointwisely to a pseudo-metric d. Then, if (X,ρ_i) is K-embeddable for each $i\in D$ then so is (X,d).*

Proof: By Lemma 1.4, for each $i\in D$ there exists a measure μ_i on $\sigma(X)$ such that J_X is an isometric embedding of (X,d_i) in $K(2^{2^X},\sigma(X),\mu_{d_i})$. As $[0,\infty]^{\sigma(X)}$, endowed with the topology of pointwise convergence, is compact, the net $(\mu_{d_i})_{i\in D}$ has a converging subnet. Let μ be the limit of this subnet. Then μ is a ***finitely***-additive measure, and J_X is an isometric embedding of (X,ρ) into $K(2^{2^X},\sigma(X),\mu)$. Observe that the measure μ need not be ***countable*** additive. However, by Theorem I: 1.8 we can isometrically embed $K(2^{2^X},\sigma(X),\mu)$, in $K_1(X',\mathcal{A},\mu')$ for some measure space $(X',\mathcal{A},\mu')$. Whence, (X,ρ) is K-embeddable. ■

Proof of Theorem 1.2: The "only if" part of the theorem is evident. For a proof of the "if" part: fix $b\in X$. Let F be a finite subset of X. By $r_F: X\to F\cup\{b\}$ we denote the following retraction:

$$r_F(x)=\begin{cases} x & \text{if } x\in F \\ b & \textit{elsewhere.}\end{cases}$$

Now define the pseudo-metric d_F on X by $d_F(x,y)=\rho(r_F(x),r_F(y))$.

By assumption, the space (X,d_F) is L-embeddable, and whence (by Lemma 1.3) it is K-embeddable. Let $\mathcal{F}$ be the set of all finite subsets of X. Clearly ρ is the pointwise limit of the net $(d_Y)_{Y\in\mathcal{F}}$ where $\mathcal{F}$ is directed by inclusion. Hence by Lemma 1.4 ρ is K-embeddable, whence L-embeddable. ■

1.7 Remark. By Corollary 1.5 we can reformulate Theorem 1.2 as:

A metric space (X,ρ) is isometrically embeddable in an $L_1(\mu)$ space iff the metric subspace $(F,\rho_{|F})$ is isometrically embeddable in $(\mathbb{R}^{2^{|F|}},||.||_s)$ for all finite subsets F of X.

1.8 Superextensions. Let X be a set. A linked system in 2^X is a (non-empty) subcollection of 2^X consisting of pairwise intersecting sets. See [84]. The set of all maximal linked systems (*mls's*) in 2^X, denoted by $\lambda(X)$, is called the *superextension* of X. ([3]) The superextension $\lambda(\{1,\cdots,n\})$ is simply denoted by $\lambda(n)$. For instance one can easily verify that for each $x\in X$ the collection of type $\{S \mid x\in S\subseteq X\}$, is a maximal linked system. Whence, the mapping $J_X: X\to\lambda(X)$ is a (set-theoretic) embedding. If $X=\{1,\cdots,n\}$ for some $n\in\mathbb{N}$, then this embedding is simply denoted by J_n.

Let $l\in\lambda(X)$. The following observations are easily verified.

(1) If $A\subseteq X$, then either $A\in l$ or $X\setminus A\in l$, but not both.

(2) If $A\in l$ and $B\supseteq A$, then $B\in l$.

---

[3] Observe that this definition uses Zorn's lemma.

Denote the median of the (distributive) lattice 2^{2^X} by m_*. The following is implicitly shown in [84, p. 4].

1.9 Lemma. *$\lambda(X)$ is a median stable subset of 2^{2^X}.*

Proof: We have to verify that $m_*(l_1,l_2,l_3)$ is an mls for each $l_1,l_2,l_3 \in \lambda(X)$. By definition we have $m_*(l_1,l_2,l_3)=(l_1 \cap l_2)\cup(l_1 \cap l_3)\cup(l_2 \cap l_3)$. Hence the collection $m_*(l_1,l_2,l_3)$ is at least a linked system. Suppose that $A \subseteq X$ meets every element of $m_*(l_1,l_2,l_3)$, then at least two of the following formulae hold:

$A \in l_1$; $A \in l_2$; $A \in l_3$.

Indeed, if for instance $A \notin l_1$ and $A \notin l_2$, then $X \setminus A \in l_1$ and $X \setminus A \in l_2$. Hence, $X \setminus A \in m_*(l_1,l_2,l_3)$, contradicting the assumption on A. From this observation we deduce that $A \in m_*(l_1,l_2,l_3)$. ■

In general the median subspaces $\lambda(X)$, $\sigma(X)$ of 2^{2^X} are not comparable. It follows from [84], that the superextension of a ***finite*** set X, is the median stabilization of $\{J_X(x) \mid x \in X\}$. Whence in this situation $\lambda(X) \subseteq \sigma(X)$.

The following result is due to Verbeek [84]. We give an alternative proof.

1.10 Theorem. *Let X be a median space. Each function $f:\{1,2,..,n\} \to F$ can be extended (with respect to the standard embedding J_n) to a unique median preserving mapping $f':\lambda(n) \to X$.*

Proof: The unicity of such an extension follows as $\lambda(n)$ is the median stabilization of $l_n(1), \cdots, l_n(n)$.

For $m \in \lambda(n)$ we take:

$$f'(m)=\bigcap_{M \in m} co(f(M)).$$

The proof that f' is as desired goes in three steps.

First for $m \in \lambda(n)$, and for $M_1,M_2 \in m$ we have that $f(M_1)\cap f(M_2) \neq \varnothing$. Hence, the convex sets appearing in the right-hand side of formula (1) meet two by two. As the cardinality of m is finite, the right-hand side of (1) is at least non-empty.

Next, suppose that $x,y \in f'(m)$. There exists a halfspace H with $x \in H$ and $y \notin H$. Precisely one of the formulae $f^{-1}(H) \in m$, $f^{-1}(X \setminus H) \in m$ holds, say the first one. But then $co(f(f^{-1}(H)))$ misses y, contradicting the assumption that $y \in f'(m)$. This shows that f' determines a function $\lambda(n) \to X$, as desired. That f' extends the standard embedding J_n is evident.

Finally, we verify that f' is median preserving. To this end, let $l_1,l_2,l_2 \in \lambda(X)$. Striving for a contradiction we suppose that

$a=m(f'(l_1),f'(l_2),f'(l_3))$; $b=f'(m_*(l_1,l_2,l_3))$; $a \neq b$.

Let H be a halfspace in X with $a \in H$ and $b \notin H$. Clearly at least two of $f'(l_1),f'(l_2),f'(l_3)$ must lie in H, say $f'(l_1),f'(l_2)$. Hence there are $M_i \in l_i$ $(i=1,2)$, with $co(f(M_i)) \subseteq H$. But now we have $M_1 \cup M_2 \in m_*(l_1,l_2,l_3)$, and therefore $b=f'(m_*(l_1,l_2,l_3)) \in co(f(M_1 \cup M_2)) \subseteq H$, a contradiction. ■

The following result shall be of crucial importance.

1.11 Corollary. *Let (X,m) be a median space, and let $F \subseteq X$ be a finite set. Then the median stabilization $med(F)$ consists of at most $2^{2^{|F|}}$ points.*

Proof: By Theorem 1.10 there exists a median preserving function $f' : \lambda(\{1,2,..,n\}) \to X$, that contains F in its image. Clearly $Im(f')$ is a median stable subset of F, i.e. $med(F) \subseteq Im(f')$. Also as $2^{2^{|F|}}$ consists of $2^{2^{|F|}}$ points, the cardinality of the subset $\lambda(\{1,2,..,n\})$ is majorized by that number. Hence so is the set $Im(f)$. ■

§ 2 Isometric embedding in $L_1(\mu)$ spaces

2.1 Lemma. *Let (X,m) be a median algebra, and let C be a subset with a gate function p_C. Then the set*

$$cone(C,a) = \{ b \in X \mid b \in I(a, p_C(b)) \}$$

is the smallest convex subset of X including C and a.

Proof: The only problem is to see that the right-hand side of the above formula is m-convex. This can be obtained by "halfspace reasoning", see I: 2.15. ■

We will refer to $cone(C,a)$ as the *cone* with basis C with apex a. Note that $cone(C,a) = co(C \cup \{ a \})$.

2.2 Lemma. *Let (X,ρ) be a median metric space, and let C be a subset with gate function p_C. If $C \times \mathbb{R}$ is equipped with the sum metric ρ_s then the function*

$$f_C : cone(C,a) \to C \times \mathbb{R}, \quad x \to (p(x), \rho(x, p(x))),$$

satisfies the following equality for all $x \in cone(C,a)$:

$$\rho_s(f_C(x), f_C(a)) = \rho(a,x).$$

Proof: Let $x \in cone(C,a)$. By Lemma 2.1, we have $x \in I(a, p_C(x))$ and hence that

$$\rho(a,x) + \rho(x, p(x)) = \rho(a, p(x)). \qquad (*)$$

Now consider the following equalities:

$$\rho(p(x), p(a)) + \rho(p(a), a) - \rho(p(x), x) = \rho(a, p(x)) - \rho(p(x), x) = \rho(a,x). \qquad (**)$$

The first equality follows by definition of the gate $p(a)$. The second equality is just property (*). From (**) we conclude that

$$\rho(p(a), a) - \rho(p(x), x) = \rho(a,x) - \rho(p(a), p(x)).$$

As p is contractive (Proposition II: 4.1), the right-hand side of the equality is non-negative. Therefore, (**) can be rewritten as

$$\rho(p(x), p(a)) + | \rho(p(a), a) - \rho(p(x), x) | = \rho(a,x),$$

which gives the result. ■

2.3 Theorem. *A median metric space (X,ρ) with n points is isometrically embeddable in $l_1(n-1)$.*

Proof: We establish the theorem by induction on n. For n=1, the situation is clear. Suppose the result holds for median metric spaces with at most n points, and suppose that X has n+1 points. Let $(C_i)_{i=1}^m$ be a maximal chain of non-empty convex sets in X. We consider $C=C_{m-1}$. By virtue of the maximality of the chain, there is a point $a \in X$ such that $X=co(C \cup \{ a \})$. We shall show that the map $f_C : X \to C \times \mathbb{R}$, considered in the previous lemma, is an isometry.

To this end take $x,y \in cone(C,a)$. If both x,y are members of C then, evidently, $\rho_s(f_C(x),f_C(y))=\rho(x,y)$. If $x \in cone(C,a) \setminus C$, then $cone(C,x)=cone(C,a)$ (otherwise we could insert an extra term $cone(C,x)$ in the above chain). By Lemma 2.2 we conclude that $\rho_s(f_C(x),f_C(y))=\rho(x,y)$. Therefore f_C is an isometry. Now C has at most n points and by inductive assumption, it embeds isometrically in $l_1(n-1)$. This yields an isometric embedding of X in $l_1(n-1)\times\mathbb{R}$ (sum norm), which in turn is isometric with $l_1(n)$. ■

2.4 Theorem. *For a metric space (X,ρ), the following conditions are equivalent.*

(1) *(X,ρ) is a median metric space.*
(2) *(X,ρ) is a median stable subspace of an $L_1(\mu)$-space.*
(3) *(X,ρ) is a median stable subspace of a $K_1(\mu)$-space.*

Proof: The implications (3),(2)$\to$(1) are obvious. For a proof of implication (1)$\to$(2), take any finite set F in X. By Corollary 1.11 the median stabilization $med(F)$ of F is finite. Hence the metric space $(med(F),\rho_{|med(F)})$ is median. By Theorem 2.3, we conclude that $(med(F),\rho_{|med(F)})$ is isometrically embeddable in a (finite dimensional) $L_1(\mu)$ space. In particular, its metric subspace $(F,\rho_F)\subset(med(F),\rho_{|med(F)})$ is isometrically embeddable in an $L_1(\mu)$ space. Now Theorem 1.2 and Lemma 1.3 finish the proof of the theorem. ■

§ 3 Congruences and optimality

We recall the following results. Let (X,m) be a median algebra. If $f : X \to Y$ is a surjective, median preserving function, then f maps convex sets onto convex sets (Theorem I: 4.8). Moreover, if the convex set $C \subseteq X$ is gated, then so is $f(C)$, and the respective gate functions commute with f—see Corollary I: 4.11. We finally remark that a gate function preserves the median, as one can verify by "halfspace reasoning" (see I: 2.15).

For each pair of disjoint convex sets C,D in a median algebra X there is a homomorphism $X \to \{0,1\}$ mapping C to 0 and D to 1 —see Theorem I: 4.19. This result is frequently applied in case $C=\{ a \}$, $D=\{ b \}$. In a situation where $I(a,b)=\{ a,b \}$, the commuting of homomorphisms with gate maps implies that, up to interchanging 0,1, there can be only one homomorphism separating such a pair of points.

By a *congruence relation* on X is meant an equivalence relation $\equiv$ such that $m(a_1,b,c) \equiv m(a_2,b,c)$ whenever $a_1 \equiv a_2$. For a congruence "$\equiv$" on X, consider the quotient $q: X \to X/\equiv$. An induced median $\tilde{m}$ can be defined on $X/\equiv$ as follows:

$$\tilde{m}(q(a),q(b),q(c)) = qm(a,b,c).$$

By definition, q is a median-preserving function. Conversely, every median-preserving function of median algebras induces a congruence relation on its domain in the obvious way.

3.1 Lemma. ([8]) *Each congruence relation on a product $X_1 \times X_2$ of two median algebras X_1, X_2 is the product of a congruence on X_1 with a congruence on X_2.*

Proof: Let $X = X_1 \times X_2$ and fix a point of reference $c = (c_1, c_2) \in X$. The following two sets are gated:

$$C_1 = \{c_1\} \times X_2; \, C_2 = X_1 \times \{c_2\}.$$

Let $q: X \to \tilde{X}$ be the quotient function, associated with the given congruence on X. We obtain convex sets $q(C_i) = \tilde{C}_i \subseteq \tilde{X}$ which are gated. Let $p_i: X \to C_i$ and $\tilde{p}_i: \tilde{X} \to \tilde{C}_i$ denote the gate functions ($i = 1, 2$). This gives a median preserving function

$$\tilde{p} = (\tilde{p}_1, \tilde{p}_2): \tilde{X} \to \tilde{C}_1 \times \tilde{C}_2,$$

which we show to be an isomorphism.

As homomorphisms commute with gate functions, we have equalities $\tilde{p}_i \circ q = q \circ p_i$ for $i = 1, 2$. It follows that $\tilde{p}$ is surjective. Let $b = (b_1, b_2) \in X$. By using the idempotent law of the constituting factor medians, we see that $m((c_1,b_2),c,(b_1,c_2)) = c$. Hence c is the infimum in $(X, \leq_c)$ of (c_1,b_2) and (b_1,c_2). Similarly, b is the infimum in $(X, \leq_b)$ of (b_1,c_2) and (c_1,b_2). As median preserving functions also preserve basepoint orders, we find that the point $q(b)$ is the infimum in $(\tilde{X}, \leq_{q(b)})$ of $u = q(c_1,b_2)$ and $v = q(b_1,c_2)$. Hence $q(b)$ is the supremum of u, v in the basepoint order of $q(c)$. Now $u = \tilde{p}_1 q(b)$ and $v = \tilde{p}_2 q(b)$, and injectivity easily follows. ∎

If each non-trivial congruence relation on a median algebra X identifies some points of the subset Y, then Y is said to be *optimal* in X.

3.2 Proposition. *Let Y be a subset of a product $X_1 \times X_2$ of two median algebras. Then there exist median-preserving quotients*

$$q_i: X_i \to \tilde{X}_i$$

of X_i ($i = 1, 2$) such that $q_1 \times q_2$ is injective on Y and $q(Y)$ is optimal in $\tilde{X}_1 \times \tilde{X}_2$.

Proof: Evidently, if a nest of congruence relations is given on a median algebra X, then its union is again a congruence relation. A simple application of Zorn's lemma yields a congruence relation $\equiv$ on $X_1 \times X_2$, maximal with the property that no two distinct points of Y are related. By Lemma 3.1, the relation $\equiv$ splits over the factors, yielding the desired quotients. ∎

In the next results we operate on product sets $X_h \times X_v$. The labels "h" and "v" refer to the viewpoint of a "horizontal" resp. "vertical" factor. The coordinate projections are denoted by

$$\pi_h : X_h \times X_v \to X_h;\ \pi_v : X_h \times X_v \to X_v.$$

A pair of points $a \neq b$ in $X_h \times X_v$ is called *horizontal* (*vertical*) provided the vertical (horizontal) projection π_v (π_h) identifies the pair. A pair which is neither horizontal nor vertical is called *skew*. A few lemmas are required.

3.3 Lemma. *Let a,b,c,d be four points in a product of two median algebras, and let*

$$c = m(c,d,a);\ d = m(c,d,b).$$

If a, b is a horizontal (vertical) pair, then so is c, d.

Proof: Just use that the product's median commutes with both factor projections. ∎

The diagonal of $[0,1]^2$ is a typical example of a non-optimally embedded algebra. The (subalgebra) interval between $(0,0)$ and $(1,1)$ allows no horizontal pair, and yet the pair of end-points is not a vertical one. In contrast we have:

3.4 Lemma. *Let X be a median algebra which is optimally embedded in $X_h \times X_v$, and let a, $b \in X$. If $I_X(a,b)$ includes no horizontal pair, then a, b is a vertical pair.*

Proof: Assume to the contrary that

$$a_h = \pi_h(a) \neq b_h = \pi_h(b),$$

and consider the smallest congruence relation $\equiv$ on the product space which identifies the convex set $C = I_h(a_h,b_h) \times \{ b_v \}$, viz.,

$$u \equiv v \Leftrightarrow \exists\, u',\ v' \in C : u = m(u',u,v);\ v = m(v',u,v) \qquad (*)$$

(cf. Bandelt and Hedlikova [8]). Suppose $u \neq v$ in X are congruent under $\equiv$, and let $u',v' \in C$ be corresponding points as in (*). Being in C, these points constitute a horizontal pair. By Lemma 3.3, u,v are horizontal, and so are the projections

$$\bar{u} = m(a,b,u);\ \bar{v} = m(a,b,v)$$

onto the interval $I(a,b)$. These points are in $I_X(a,b)$ since X is median stable, so by assumption, $\bar{u} = \bar{v}$. The definition of gate gives that

$$\bar{u} \in I_X(u,u'),$$

whereas the intersection of the intervals $I_X(u,v)$ and $co_X(u,u')$ consists of u only. Hence, u is the gate of $\bar{u}$ in $I_X(u,v)$. Similarly, v is the gate of $\bar{v}$ in $I_X(u,v)$. As $\bar{u} = \bar{v}$, this is a contradiction. We conclude that some further identification can be performed on $X_h \times X_v$ without touching at X, and the embedding is not optimal. ∎

3.5 Construction. Let (X,ρ) be a median metric space which is embedded as a subalgebra of $X_h \times X_v$, and let a, $b \in X$. We construct three real numbers

$$\rho_h(a,b),\ \rho_v(a,b),\ \rho_s(a,b) \geq 0$$

as follows. Let $F \subseteq I_X(a,b)$ be a finite median-stable set including a, b, and let $K \subseteq F$ be a maximal totally ordered set (a maximal chain) in the basepoint order $\leq_a$ joining a, b. We let $\rho_{h,K}$ (resp. $\rho_{v,K}$, $\rho_{s,K}$) be the sum of all distances between successive points of K which constitute a horizontal (resp., vertical, skew) pair. Note that

$$\rho_{h,K} + \rho_{v,K} + \rho_{s,K} = \rho(a,b),$$

since each point other than a is taken from the metric interval between its predecessor and b.

We first verify that these numbers do not depend on the particular choice of K. Suppose $K' \subseteq F$ is another maximal chain joining a, b. In the finite median algebra F, each "atomic pair" (i.e., a pair of type x, y such that $I_F(x,y) = \{x,y\}$) can be separated by exactly one homomorphism $X \to \{0,1\}$. If we pick one atomic pair of successors in K, then the corresponding map separates between a, b, and hence it somewhere cuts an atomic pair of K'. As this chain is increasing, only one such pair is cut. This establishes a bijective correspondence ([4]) between successor pairs of K and of K'. Two corresponding pairs — say: $u < v$ in K and $u' < v'$ in K' — yield mutual gates u, u' and v, v', and hence they are of the same type and at the same distance.

So we arrive at three numbers $\rho_h(F)$, $\rho_v(F)$, $\rho_s(F) \geq 0$, the sum of which equals $\rho(a,b)$. If $G \supseteq F$ is another finite median-stable subset of $I_X(a,b)$, then each maximal chain K in F extends to a maximal chain L in G. Evidently, a horizontal (resp. vertical) atomic pair of K subdivides into atomic pairs of L which are exclusively horizontal (resp. vertical). A skew atomic pair of K may subdivide into a mixture of all three types. In each case, the distances sum up to the distance of the original atomic pair of K. We conclude that

$$\rho_h(F) \leq \rho_h(G),\ \rho_v(F) \leq \rho_v(G),\ \rho_s(F) \geq \rho_s(G).$$

Now the collection of all finite median-stable subsets of $I_X(a,b)$ is updirected under inclusion. The previous observations yield three numbers

$$\rho_h(a,b) = \sup_F \rho_h(F);\ \rho_v(a,b) = \sup_F \rho_v(F);\ \rho_s(a,b) = \inf_F \rho_s(F),$$

such that

$$\rho(a,b) = \rho_h(a,b) + \rho_v(a,b) + \rho_s(a,b).$$

3.6 Proposition. *Let (X, ρ) be a metric median space which is optimally (algebraically) embedded into a product $X_h \times X_v$ of median algebras. Then:*

(1) Each of the functions ρ_h, ρ_v, ρ_s is a pseudo-metric on X.

(2) A pair of points $a, b \in X$ is horizontal (vertical) iff $\rho_v(a,b) = 0$ ($\rho_h(a,b) = 0$). In either case, $\rho_s(a,b) = 0$.

(3) If a, a' and b, b' are horizontal pairs, then $\rho_s(a,b) = \rho_s(a',b')$ and $\rho_v(a,b) = \rho_v(a',b')$. A similar statement holds for vertical pairs.

(4) If $c \in I_X(a,b)$ then $\rho_*(a,b) = \rho_*(a,c) + \rho_*(c,b)$, where the subscript "$*$" denotes any of h, v, s.

(5) If $c \in X$ and $\pi_h(c) \notin \pi_h I_X(a,b)$, then $\rho_h(a,b) < \rho_h(a,c) + \rho_h(c,b)$. A similar formula works for the projection π_v.

Proof: (1): The properties

$\rho_*(a,b) = 0$ if $a = b$, and

[4] This argument also shows that maximal chains between two points are of equal length. Compare Theorem I: 4.26.

$\rho_*(a,b) = \rho_*(b,a),$

are straightforward for all three functions. As for the triangle inequality, consider $a, b, c \in X$. It suffices to verify the property on a collection of finite median stable subsets of X including a, b, c and inducing a cofinal collection of subsets on each of the intervals $I_X(a,b)$, $I_X(a,c)$, $I_X(c,b)$.

To this end, consider three finite median stable subsets F_{ab}, F_{ac}, F_{cb} of the respective intervals and let F be the (finite) median stabilization of $F_{ab} \cup F_{ac} \cup F_{cb}$. Consider maximal chains $K_{ac}, K_{cb} \subseteq F$ joining the points referred to by the label. We project the points of $K_{ac} \cup K_{cb}$ into $I_X(a,b)$ by the map $m(a,b,\cdots)$. The images are in $F \cap I_X(a,b)$ by median stability. An atomic pair is either identified, or it maps to another atomic pair of F, in which case the corresponding endpoints form mutual gates and the image pair is of the same type. Finally, pairs which correspond under mutual gate mappings are isometric (since gate projections are non-expansive).

(2): If $a, b \in X$ is a vertical pair of points then evidently each discrete chain in $I_X(a,b)$ is built with vertical pairs. Hence there is no contribution to ρ_h or to ρ_s. The converse follows from Lemma 3.4

(3): Think of two horizontal pairs. By the triangle inequality we have

$$\rho_s(a,b) \leq \rho_s(a,a') + \rho_s(a',b') + \rho_s(b',b).$$

The first and third term are zero by (2). The opposite equality obtains similarly. The same kind of argument works with ρ_s replaced by ρ_v.

(4): The argument is a simplification of the one given in (1): consider all finite median stable subsets of $I_X(a,b)$ which contain a, b, c. As the choice of a maximal chain in a given median stable subset is irrelevant, we need only consider chains through c. The result follows easily.

(5): Let $d = m(a,b,c)$. Then d is in each of the intervals $I_X(a,b)$, $I_X(a,c)$, $I_X(c,b)$, and three applications of (4) give

$$\rho_h(a,c) + \rho_h(c,b) - 2\cdot\rho_h(c,d) = \rho_h(a,b).$$

Now the pair of points c, d is not vertical, since $\pi_h(d)$ is in $I_X(\pi_h(a),\pi_h(b))$ and $\pi_h(c)$ is not. Hence, by (2), $\rho_h(c,d) > 0$. ■

3.7 Corollary. *Let the median metric space (X,ρ) be optimally embedded as a subalgebra of $X_h \times X_v$, and suppose that the coordinate projections map X surjectively onto each of the factors. Then there exists a metric ρ_h on X_h and a metric ρ_v on X_v with the following properties.*

(1) ρ_h generates the median of X_h and ρ_v generates the median of X_v.

(2) $\rho = \rho_h + \rho_v$.

Proof: Given two points $a_h, b_h \in X_h$, choose pre-images $a, b \in X$. With the above notation, we put

$$\rho_h(a_h,b_h) = \rho_h(a,b) + \tfrac{1}{2}\rho_s(a,b). \qquad (*)$$

Note that if different representatives a', b' are taken in X, then a, a' and b, b' are vertical pairs and by Proposition 3.6(3), the terms at the right-hand side of (*) remain unchanged. This defines a metric by Proposition 3.6(1) and (2), which is compatible with the median of X_h by Proposition 3.6(4) and (5). After defining a metric ρ_v on X_v in the same way, we obtain that the metric ρ of X satisfies $\rho = \rho_h + \rho_v$, as required. ■

3.8 Proposition. *Let (X,ρ) be a median metric space which embeds algebraically into a totally ordered set. Then there is an isometric embedding of (X,ρ) into the real line.*

Proof: X is assumed to be a median stable subset of a totally ordered set L. This yields that the median convexity of X is the relative convexity, induced from the (standard) order convexity of L. In particular, the interval function of X is derived from the relative total order induced on X. In the sequel we consider $X=L$.

Define a function $f{:}X\to\mathbb{R}$ as follows. Fix $0\in X$ and put

$$f(x)=\begin{cases}\rho(0,x) & \text{if } x>0\\ -\rho(0,x) & \text{if } x\leq 0.\end{cases}$$

To see that f is an isometric embedding into $\mathbb{R}$, let $a,b\in X$. For reasons of symmetry we need only consider the following two possibilities.

CASE I: $0<a<b$. Following the order-theoretic definition of $I(a,b)$, we have $a\in I(0,b)$. Following the metric definition of $I(a,b)$, we find $\rho(0,a)+\rho(a,b)=\rho(0,b)$. Hence $f(b)-f(a)=\rho(a,b)$.

CASE II: $a<0<b$. This time, we have $\rho(a,0)+\rho(0,b)=\rho(a,b)$, and hence $-f(a)+f(b)=\rho(a,b)$. ■

Combining the previous results leads us to the following.

3.9 Theorem. *Let (X,ρ) be a median metric space which embeds algebraically into a product of n totally ordered sets. Then there is an isometric embedding of (X,ρ) into $l_1(n)$.*

Proof: We proceed by induction on the number n of factors. For n=1 this is the previous result. Let X be algebraically embedded into $\prod_{i=1}^{n+1}L_i$. By Proposition 3.2, we obtain an optimal embedding of X into a product of a quotient of $\prod_{i=1}^{n}L_i$ with a quotient of L_{n+1}. As median-preserving functions also preserve intervals and basepoint orders, it is evident that the quotient of a totally ordered median algebra is totally ordered. By (inductive) application of Lemma 3.1, we see that the former quotient is again a product of n totally ordered sets. Let X_h be the projection image of X into this product, and let X_v denote the other projection image of X. By Corollary 3.7, we obtain median metrics on each of the spaces X_h,X_v, such that the corresponding "sum metric" on $X_h \times X_v$ agrees with the given metric of X. By the induction hypothesis, these metric factor spaces can be isometrically re-embedded in, respectively, $l_1(n)$, and $\mathbb{R}$. The sum-metric on the product of these two spaces corresponds exactly with the sum-norm of $l_1(n+1)$. ■

We remark that the restriction to finitely many factors is essential: there exists a median metric space which is algebraically embeddable in $\mathbb{R}^{\mathbb{N}}$ but can not be isometrically embedded in $l_1(\mathbb{N})$. Indeed, consider the following dense subspace of $L_1([0,1])$

$$W = \{f;\mathbb{Q} \to \mathbb{R} \mid f \textit{ is continuous}\}.$$

In view of Corollary IV: 2.11, W cannot be isometrically embedded in any $l_1(I)$ space.

The application of Theorem 3.9 requires a method of verifying whether a median (metric) space is algebraically in a product of totally ordered sets. The following result was found by E. Evans [23].

3.10 Theorem. *The following are equivalent for a median algebra* (X,m).

(1) $m(a,b,c) \in \{a,b,c\} \quad \forall a,b,c \in X.$

(2) *Either, X is embeddable in a totally ordered set (as a subalgebra), or it is a graphic square.* ■

Combining the previous theorem with Proposition 3.8 we obtain:

3.11 Corollary. *A median metric space X satisfying formula 3.10(1) is either a metric subspace of* $\mathbb{R}$ *or a graphic square.* ■

AMALGAMATING SPACES

Suppose we have two geometric interval spaces (X_1,I_1) and (X_2,I_2) such that $X_1 \cap X_2$ is gated in both spaces. In section 1 we shall show that the interval operators I_1, I_2 can be extended in a canonical way to the whole of $X = X_1 \cup X_2$. This yields an interval operator I on X, which is the unique geometric extension to X of I_1, I_2, such that the sets X_1, X_2 are gated. Uniqueness of such extensions is relevant as this enables us to recover a geometric interval space from a cover of (two) convex sets.

In section 2 we obtain a construction to extend compatible metrics on (X_1,I_1) and (X_2,I_2) to the whole of (X,I). As compatible metrics on median spaces have the (F,2)-IP, we this yields a method to construct such metrics on certain median interval spaces. This construction is applied in section 3 to create median and hyperconvex metrics on collapsible polyhedra.

§ 1 The amalgamation of geometric interval spaces

For convenience we introduce the following convention. Let $X_1, X_2, \cdots, X_n, Y$ be sets. A set of functions $f_i : X_i^2 \to Y$ $(i = 1,2, \cdots, n)$ is said to be *matching* if for all $1 \le i,j \le n$ the mappings f_i, f_j coincide on $(X_i \cap X_j)^2$ (this set may be empty). Subsets of type $X_i \cap X_j$ with $i \ne j$ are called *connectors*.

We now come to an extension theorem for geometric interval operators. This result was inspired by a result of van de Vel [80, Theorem 3.1], where an extension theorem for (topological) median convexities is shown (cf. Theorem 1.4 below).

1.1 Theorem. *Suppose that (X_1,I_1) and (X_2,I_2) are matching geometric interval spaces, with a commonly gated connector. Let $p_i : X_i \to X_1 \cap X_2$ be the gate function $(i = 1,2)$. Then there is one and only one geometric interval operator I on $X_1 \cup X_2$ that extends I_1 and I_2 with the property that an I-interval that meets X_1 and X_2 also meets $X_1 \cap X_2$. If $a \in X_1$ and $b \in X_2$, then I is given by*

$$I(a,b) = I_1(a, p_2(b)) \cup I_2(p_1(a), b). \tag{1.1.1}$$

Proof: Let I be the extension of I_1, I_2 satisfying formula (1.1.1). Then I is at least a (well-defined) interval operator on $X_1 \cup X_2$. Let $x_i \in X_i$ and let $\le^i_{x_i}$ denote the (original) basepoint order of (X_i,I_i) $(i = 1,2)$. The basepoint relation of I shall be denoted by $\le_x$

$(x \in X_1 \cup X_2)$. The following formula describes the compatibility of $\leq$ and $\leq^i$. If $a \in X_1$, $b \in X_2$ and $x \in X_1$, then

$$x \leq_a b \iff x \leq_a^1 p_2(b). \tag{2}$$

A similar formula holds for $x \in X_2$. The implication from right to left in (2) is evident. Let $x \leq_a b$. In view of the definition of I we only have to consider the situation that $x \in I_2(p_1(a),b)$ (see Figure 1.1).

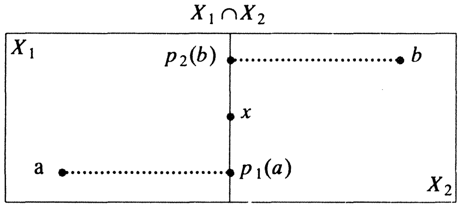


Fig. 1.1: extending matching interval spaces

As x is a member of the connector we have $p_2(b) \leq_b^2 x$ by definition of a gate. By the geometric property of I_2 we conclude $x \in I_2(p_1(a),p_2(b))$, i.e. $x \in I_1(p_1(a),p_2(b))$. By the geometric property of I_1 the last set is contained in $I_1(a,p_2(b))$.

For a proof that I is geometric we use (I: 3.2). To this end, let $a,b,x,y \in X_1 \cup X_2$ and $y \leq_a b$ and $x \leq_a y$. We have to show that

$$x \leq_b a,\ y \leq_b x. \tag{4}$$

We may assume that $a,x \in X_1$ and $b,y \in X_2$, since by formula (2) other situations reduce to the original interval spaces. With the aid of formula (2) and the definition of gate we obtain the following implications:

$$x \leq_a y \Rightarrow \begin{cases} x \leq_a^1 p_1(x) \leq_a^1 p_2(y) & (*) \\ p_1(x) \in I_1(p_1(a),p_2(y)) & (**) \end{cases} ;\ y \leq_a b \Rightarrow \begin{cases} y \leq_b^2 p_2(y) \leq_b^2 p_1(a) & (+) \\ p_2(y) \in I_2(p_2(b),p_1(a)) & (++) \end{cases}$$

Combining (**) and (+) yields $y \leq_b^2 p_1(x)$, i.e. $y \leq_b x$ by formula (2). We obtain $p_2(y) \leq_a^1 p_2(b)$ by using the definition of the gate $p_1(a)$ on (++). Combining this with formula (*) yields $x \leq_a^1 p_2(b)$, i.e. $x \leq_a b$ by formula (2). This completes the proof of (4), showing that I is geometric.

One can readily verify that an interval $I(a,b)$ meeting X_1 and X_2 will meet $X_1 \cap X_2$ in the gate of a (or b) in $X_1 \cap X_2$.

For a proof of unicity, suppose that $\bar{I}$ is another geometric extension of I_1,I_2 onto $X_1 \cup X_2$ as described. Let $a \in X_1$ and $b \in X_2$, and let $z \in \bar{I}(a,b) \cap X_1 \cap X_2$. Then

$$p_1(a) \in I_1(a,z) = \bar{I}(a,z) \subseteq \bar{I}(a,b).$$

Hence the point $p_1(a)$ is the gate of a in X_2. Similarly the point $p_2(b)$ is the gate of b in X_1. By the assumed geometrical property of $\bar{I}$ we obtain that the sets $I(a,p_2(b))$ and $I_2(p_1(a),b)$ are subsets of $\bar{I}(a,b)$, i.e. $I(a,b)$ is contained in $\bar{I}(a,b)$. Let $x \in \bar{I}(a,b)$, say $x \in X_1$. We established earlier that $p_2(b)$ is the gate of b in X_1, so in particular $p_2(b) \in \bar{I}(b,x)$. By the assumed geometric property of $\bar{I}$ we deduce that $x \in \bar{I}(p_2(b),a)$, that is $x \in I(a,b)$. As $a \in X_1$ and $b \in X_2$ were arbitrary we conclude that $\bar{I} = I$. ∎

The geometric interval space $(X_1 \cup X_2, I)$ described in the previous theorem, is called the *(geometric) amalgamation* of (X_1,I_1) and (X_2,I_2). We remark that if we do not require the intersection-property of intervals appearing in the previous theorem, then there may be more than one extension on $X_1 \cup X_2$. See Theorem 1.4 below for an affirmative result on modular spaces.

The following result, which is easily verified, provides us with a different description of the geometric amalgamation.

1.2 Proposition. *Let (X,I) be a geometric interval space and let X_1, X_2 be subsets of X such that $X_1 \cup X_2 = X$ and $X_1 \cap X_2$ is gated in X. Then following are equivalent.*
(1) Every I-interval meeting X_1 and X_2 also meets $X_1 \cap X_2$.
(2) The subsets X_1, X_2 are gated in X.
In the above situation, the gate maps $X \to X_1$ and $X_2 \to X_1 \cap X_2$ coincide on X_2. ∎

1.3 Matching modular spaces. For modular interval spaces there is a simpler description of the geometric amalgamation.

1.4 Theorem. *Suppose that (X_1,I_1) and (X_2,I_2) are matching modular interval spaces with a commonly gated connector. Then,*
(1) The amalgamation interval operator I of X_1,X_2 is the unique modular interval operator on $X_1 \cup X_2$ that extends I_1 and I_2.
(2) If $a \in X_1$ and $b,c \in X_2$ then,

$$M(a,b,c) = M_2(p_1(a),b,c).$$

In particular, if (X_1,I_1) and (X_2,I_2) are median then so is the amalgamation.

Proof: The validity of statement (2) follows from the description of the amalgamation interval operator (Formula 1.1.1) and Formula (2) appearing in the proof of Theorem 1.1.
It immediately follows from (2) that the geometric amalgamation of (X_1,I_1) and (X_2,I_2) is modular. We are left with the unicity part of (2). To this end, let $\bar{I}$ be another interval operator as described in the theorem. Aiming at the use of Theorem 1.1, Let $a \in X_1$ and $b \in X_2$. Let $p_1(a)$ be the gate of a in $X_1 \cap X_2$. By assumption there is a point

$$x \in \bar{I}(a,p_1(a)) \cap \bar{I}(a,b) \cap \bar{I}(b,p_1(a)).$$

As $\bar{I}$ extends I_1,I_2 we conclude that $x \in I_1(a,p_1(a)) \cap I_2(b,p_1(a))$, i.e. $x \in X_1 \cap X_2$. Hence, $\bar{I}(a,b)$ meets $X_1 \cap X_2$. We can now apply Theorem 1.1 to obtain $\bar{I} = I$. ∎

1.5 Matching metric interval spaces. A collection $\{(X_i,\rho_i) \mid i = 1,2,\cdots,n\}$ of metric spaces is said to be *matching* provided the metric functions are matching, and all connectors $X_i \cap X_j$ are closed in X_i $(1 \le i \ne j \le n)$.

Let (X_1,ρ_1) and (X_2,ρ_2) be two matching metric spaces with a non-empty connector. We construct an extension ρ of ρ_1, ρ_2 on $X_1 \cup X_2$ as follows.

$$\rho(x_1,x_2) = \inf_{c \in X_1 \cap X_2} \rho_1(x_1,c)+\rho_2(c,x_2) \qquad (x_1 \in X_1, x_2 \in X_2).$$

It is easy to see that ρ is a metric on $X \cup X_2$. This metric is called the *path-metric* with respect to ρ_1, ρ_2. In general the path-metric is not the only extension of ρ_1 and ρ_2, but it is the ***largest*** metric extension on $X_1 \cup X_2$. The induced topology on $X \cup X_2$ is independent of the extension and equals the "Whitehead topology", see below.

1.6 Theorem. *Let (X_1,I_1) and (X_2,I_2) be matching interval spaces with a commonly gated connector. Suppose that I_1,I_2 are derived from matching metrics ρ_1 and ρ_2 on X_1 and X_2 respectively. Then the following hold.*

(1) The path-metric ρ w.r.t. ρ_1,ρ_2 is the unique metric on $X_1 \cup X_2$ extending ρ_1 and ρ_2 and inducing the amalgamation interval operator I.

(2) For $a \in X_1$ and $b \in X_2$ we have:

$$\rho(a,b) = \rho_1(a,p_1(a)) + \rho_2(p_1(a),b).$$

Moreover, if the metrics ρ_1, ρ_2 are convex then so is ρ.

Proof: First, we will show equality (2). To this end, let $a \in X_1$ and $b \in X_2$. For each $c \in X_1 \cap X_2$,

$$\begin{aligned}\rho_1(a,p_1(a))+\rho_2(p_1(a),b) &\leq \rho_1(a,p_1(a))+\rho_2(p_1(a),c)+\rho_2(c,b)\\ &= \rho_1(a,p_1(a))+\rho_1(p_1(a),c)+\rho_2(c,b) = \rho_1(a,c)+\rho_2(c,b).\end{aligned}$$

The inequality is the triangle inequality of ρ_2. The first equality is due to compatibility of the metrics ρ_1,ρ_2 on $X_1 \cap X_2$, and the second equality only invokes the definition of the gate $p_1(a)$. By virtue of the definition of a path-metric we conclude to equality (2).

Secondly, from the equality in (2) it follows that the metric interval operator of ρ equals I. Finally, for a proof of the unicity part of statement (1), note that a metric extending ρ_1 and ρ_2 and inducing I must have the form as described in (2). ■

1.7 Repeated matchings; topological properties. The following result describes the convex and gated sets of the geometric amalgamation.

1.8 Proposition. *A non-empty subset C of a geometric amalgamation $(X_1 \cup X_2, I)$ is convex (resp. gated) if and only if*

(i) $C \cap X_i$ is convex (resp. gated or empty) in (X_i,I_i).

(ii) If C meets X_1 and X_2 then it meets $X_1 \cap X_2$.

Proof: By Proposition 1.2 X_i is a gated subset of the amalgamation $(i=1,2)$. If C is a convex subset of the amalgamation meeting X_1 (say in x_1) and X_2 (say in x_2), then the gate of x_1 onto X_2 is a member of $I(x_1,x_2) \subseteq C$. The subset $C \cap X_i$ is evidently convex in (X_i,I_i) $(i=1,2)$. If C is gated in the amalgamation, then the intersection of gated sets $C \cap X_i$ is either empty or gated by (I: 3.4.4).

Conversely, let C be a subset as described in (i), (ii) with respect to "convex". If C is contained in either X_1 or X_2, then C is evidently convex in the amalgamation. Otherwise C meets $X_1 \cap X_2$, say in z. For any $x_1 \in X_1$ and $x_2 \in X_2$ we have

$$I_1(x_1,p_2(x_2)) \subseteq I_1(x_1,z) \subseteq C \ ; \ I_2(x_2,p_1(x_1)) \subseteq I_2(x_2,z) \subseteq C.$$

Hence in view of Theorem 1.1 $I(x_1,x_2) \subseteq C$. Whence C is a convex subset of the amalgamation.

Suppose C is as described in (i), (ii) with respect to "gated". If C meets $X_1 \cap X_2$, then the gate of $x_i \in X_i$ in $C \cap X_i$ is the gate of x_i in the whole of C $(i=1,2)$. Indeed, assume $i=1$, and let $z \in C \cap X_2$. Then $p_2(z) \in C \cap X_2$. Hence the gate of x_1 in $C \cap X_1$ is contained in $I(x_1,p_2(z))$ which is a subset of $I(x_1,z)$. If C is contained in X_1 or X_2, then we can apply (I: 3.4.3) to obtain that C is gated in the amalgamation. ■

We are interested in topologies on geometric interval space such that convex sets are connected and gated sets are closed. Such topologies arise frequently, cf. Theorem II: 2.14 and Proposition II: 4.1. These topologies enable us to formulate condition (ii) of Proposition 1.8 in terms of (topological) closure and connectedness. We work towards such a description for convex and gated subsets for an arbitrary finite sequence of matchings.

First we introduce some notions. A collection of topological spaces $(X_i,\tau_i)_{i=1}^n$ is said to be ***matching*** provided for all $1 \le i,j \le n$ the relative topologies induced on the connector $X_i \cap X_j$ by X_i and X_j coincide and the subset $X_i \cap X_j$ is closed in X_i. The *Whitehead topology* on $\cup\{X_i \mid i=1,2,\cdots,n\}$ is defined as follows. A subset F of the union is closed if and only if $A \cap X_i$ is closed in (X_i,τ_i) for all i. There is a similar description for open sets.

(1.9) The Whitehead topology is the unique topology on $\cup\{X_i \mid i=1,2,\cdots,n\}$ such that the spaces (X_i,τ_i) occur as closed subspaces $(i=1,2,\cdots,n)$.

From (1.9) it follows that any extension of two matching metric spaces with a non-empty closed connector induces the Whitehead topology. Unless stated to the contrary we take the Whitehead topology on the union of matching topological spaces.

1.10 Lemma. *Let (X_1,I_1) and (X_2,I_2) be matching geometric interval spaces. Let $\mathcal{B}_i$ $(i=1,2)$ be a cover of X_i consisting of gated sets endowed with a topology such that $\mathcal{B}_i$ is matching and*

(i) A subset $C \subseteq X_i$ is convex iff C is connected and for each $G \in \mathcal{B}_i$ the set $C \cap G$ is convex.

(ii) A non-empty subset $C \subseteq X_i$ is gated iff C is connected and for each $G \in \mathcal{B}_i$ the set $C \cap G$ is gated or empty.

Then with respect to the Whitehead topology on the geometric amalgamation,

(iii) A subset $C \subseteq X_1 \cup X_2$ is convex iff C is connected and for each $G \in \mathcal{B}_1 \cup \mathcal{B}_2$ the set $C \cap G$ is convex.

(iv) A non-empty subset $C \subseteq X_1 \cup X_2$ is gated iff C is connected and for each $G \in \mathcal{B}_1 \cup \mathcal{B}_2$ the set $C \cap G$ is gated or empty.

Proof: Note that the cover $\mathcal{B}_1 \cup \mathcal{B}_2$ is matching. Let $C \subseteq X_1 \cup X_2$ be a convex subset of the amalgamation, meeting both X_1 and X_2. Then C meets $X_1 \cap X_2$ by Proposition 1.8. Hence $C = (C \cap X_1) \cup (C \cap X_2)$ is a connected subset of $X_1 \cup X_2$. Next let $C \subseteq X_1 \cup X_2$ be connected and for all $G \in \mathcal{B}_1 \cup \mathcal{B}_2$ the set $C \cap G$ is convex. Let $i = 1,2$. As all intersections of type $C \cap G$ with $G \in \mathcal{B}_i$ are connected by assumption, the subset $C \cap X_i$ splits into a finite number of components, which by virtue of (i) are convex in X_i. Let $\mathcal{C}_i$ denote the collection of such components.

Assume that K,L are distinct components in $\mathcal{C}_1$. As C is connected in $X_1 \cup X_2$ there exists a sequence of successively intersecting members $\mathcal{C}_1 \cup \mathcal{C}_2$, starting with K and ending with L. Note that the members of this chain must alternate between $\mathcal{C}_1$ and $\mathcal{C}_2$. In particular, there exists a member $L' \in \mathcal{C}_1$ different from K, and a member $M \in \mathcal{C}_2$, that meets both K and L'. But then $M \cap X_1 \cap X_2$ is a convex set, hence connected, subset in $C \cap X_1$ meeting K and L'. That is $K = L'$, a contradiction.

We conclude that the set $C \cap X_1$ $(i = 1,2)$ is connected and hence convex in (X_1, I_1) by assumption (i). Similarly we obtain that the set $C \cap X_2$ is convex in (X_2, I_2). As C meets the connector by connectivity, we obtain that C is convex in the amalgamation by Proposition 1.8. We have shown statement (iii). The proof of statement (iv) is similar. ■

Let $\{(X_i, I_i, \tau_i) \mid i = 1,2, \cdots, n\}$ be a finite collection of geometric interval spaces and topologies such that for all $i = 1,2, \cdots, n$, convex subsets in X_i are connected, and gated subsets in X_i are closed. From the previous lemma we conclude that ***if*** there is a matching procedure yielding a geometric interval operator on the union $\cup \{(X_i, I_i, \tau_i) \mid i = 1,2, \cdots, n\}$, then:

a non-empty subset of the union is convex (gated) iff it is connected and intersects each X_i in a convex (gated or empty) part. (*)

In particular, the convexity on the union is independent of the matching procedure. If all (X_i, I_i) are modular, then so is the interval operator on the union obtained by matching (use induction and Theorem 1.4). As a median interval between two points equals the convex hull of these points, we deduce that the median interval operator is independent of the matching procedure. We don't know whether this holds for ***general*** modular spaces.

There is a special type of space where the matching techniques are applicable, the connected cubical polyhedra. See section 3. Van de Vel has shown that if there is a median convexity on a cubical polyhedron (which need not come from a matching procedure!) that extends the interval operators on the cubes, then the convexity must be of type (*).

If (X_i, I_i, τ_i) are as above, and if the I_i are derived from matching (convex) metrics ρ_i that induce the topology τ_i $(i = 1,2, \cdots, n)$, then an interval operator on $\cup \{(X_i, I_i, \tau_i) \mid i = 1,2, \cdots, n\}$ obtained by a matching procedure is also induced by a (convex) metric. This follows by induction and Theorem 1.6. Somewhat surprisingly it turns out that this metric is independent of the matching procedure. In fact something stronger holds.

The following notions are well-known. Let $A \subseteq X$ be sets and let Ω be a cover of X. A *chain* of Ω on A is a finite sequence $U_1, \cdots, U_n \in \Omega$ such that $U_i \cap U_{i+1} \cap A \neq \varnothing$ for all $i = 1,2, \cdots, n-1$. Two points $a,b \in A$ are *connected* on A by a chain of Ω if there is a chain of Ω on A such that $a \in U_1$ and $b \in U_n$.

1.11 Lemma. *Let $n \in \mathbb{N}$ and let $\Omega = \{(X_i, \rho_i) \mid i = 1, \cdots, n\}$ be a collection of matching metric spaces. There is at most one metric ρ on $X = \cup_{i=1}^{n} X_i$ such that*
(i) ρ extends ρ_i for each $i = 1, 2, \cdots, n$.
(ii) Each $a, b \in X$ can be connected by a chain of Ω on $I_\rho(a,b)$.

Proof: Suppose that d is another metric on $\cup_{i=1}^{n} X_i$ with properties (i) and (ii). We verify the following statement by induction on $k \geq 1$:

If $a, b \in X$ are connected by a chain $(X_{i_j})_{j=1}^{k}$ on $I_\rho(a,b)$, then $\rho(a,b) \geq d(a,b)$. (*)

Formula (*) trivially holds for $k = 1$. Assume that (*) holds for some $k \in \mathbb{N}$. Let $a, b \in X$ and let $(X_{i_j})_{j=1}^{k+1}$ be a chain of Ω on $I_\rho(a,b)$. Fix $x \in I_\rho(a,b) \cap X_{i_k} \cap X_{i_{k+1}}$. Then $\rho(a,x) = d(a,x)$ and $\rho(x,b) \geq d(x,b)$ by the induction hypotheses. As $x \in I_\rho(a,b)$ we obtain

$$\rho(a,b) = \rho(a,x) + \rho(x,b) \geq d(a,x) + d(x,b) \geq d(a,b).$$

This concludes the induction. By permuting the roles of ρ and d we obtain $d(a,b) \geq \rho(a,b)$. Whence $d(a,b) = \rho(a,b)$. ■

1.12 Corollary. *If $(X_i, \rho_i)_{i=1}^{n}$ are matching metric spaces ($n \in \mathbb{N}$), then there is at most one metric ρ on $\cup\{X_i \mid i = 1, 2, \cdots, n\}$ with connected intervals and extending all ρ_i.* ■

1.13 Corollary. *Let $(X_i, \rho_i)_{i=1}^{n}$ be a finite collection of matching modular metric spaces.*
(1) If for all $i \leq n$ the metric ρ_i is convex, then there is at most one convex modular metric ρ on $\cup\{X_i \mid i = 1, 2, \cdots, n\}$ extending all ρ_i.
(2) If the set $\cup\{X_i \mid i = 1, 2, \cdots, n\}$ is connected, then there is at most one modular metric ρ on this union extending all ρ_i.

Proof: If all (X_i, ρ_i) are complete, then so is the metric ρ on the union. By Theorem II: 2.14 ρ has connected intervals in both circumstances (1) and (2). Hence we can apply Corollary 1.12. The proof of the general case follows by taking completions (by Theorem II: 3.1 the completion of a modular metric space is modular). ■

1.14 Problem. *Does there exist a metric as in 1.11 or 1.12 or 1.13?*

§ 2 Matching adapted metrics

The following theorem, which is formulated in a general setting, states that two compatible metrics on the summands of a geometric amalgation can be extended to the amalgamation.

2.1 Theorem. *Let (X_1, I_1), and (X_2, I_2) be matching geometric interval spaces with a commonly gated connector, and let d_1, d_2 be convex compatible metrics on (X_1, I_1), (X_2, I_2) respectively, that match. If all balls of type*

$$D_{d_i}(x_i, d_i(x_i, X_1 \cap X_2))$$

for $i = 1, 2$ and $x_i \in X_i$ are gated, then there is one and only one convex metric d on $X_1 \cup X_2$ extending d_1, d_2 that is compatible with $(X_1 \cup X_2, I)$. The metric d equals the path-metric w.r.t.

d_1,d_2.

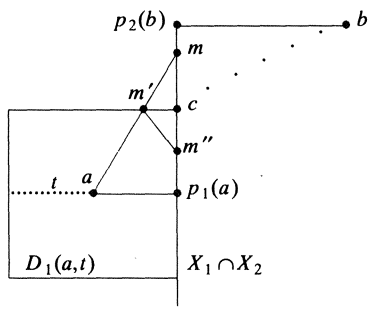
Proof: Let d denote the path-metric w.r.t. d_1,d_2. The closed balls of d_1,d_2,d are denoted by, respectively D_1,D_2,D. By p_i $(i=1,2)$ we denote the gate function $X_i \to X_1 \cap X_2$.

Assertion: Let $a \in X_1$, $b \in X_2$ and $t = d_1(a, X_1 \cap X_2)$. Then there exists a $c \in X_1 \cap X_2$ with

$$d(a,b)=d_1(a,c)+d_2(c,b) \;;\; d_1(a,c)=t. \tag{1}$$

In particular, c realizes the infimum appearing in the definition of path-metric.

To this end, by Lemma IV: 3.4(1) the convex set $D_1(a,t)$ meets $X_1 \cap X_2$ (e.g. in $p_1(a)$). Hence, the subset $D_1(a,t) \cap X_1 \cap X_2$ is gated in $X_1 \cap X_2$. Let c be the gate of b in $D_1(a,t) \cap X_1 \cap X_2$. We claim that c satisfies (1). See the figure below.



By Lemma IV: 3.4(1), c realizes the distance of b to $D_1(a,t) \cap X_1 \cap X_2$. Now let $m \in X_1 \cap X_2$. As d_1 is convex we can find $m' \in X_1$ with

$$d_1(a,m)=d_1(a,m')+d_1(m',m), \text{ and } d_1(a,m')=t.$$

That is,

$$m' \in D_1(a,t),\ m' \in D_1(m, d(a,m)-t). \tag{2}$$

Let m'' be the gate of m' in $X_1 \cap X_2$. Then by virtue of formula (2) and (I: 3.4.1):

$$m'' \in D_1(a,t) \cap X_1 \cap X_2,\ m'' \in D_1(m, d(a,m)-t). \tag{3}$$

Note that m' and m'' lie in the boundaries of the balls that occur in (2), (3). In particular, we have $d_1(a,m'')=t=d_1(a,c)$. Hence we have the following inequality:

$$d_1(a,c)+d_2(c,b) \le d_1(a,m'')+d_2(m'',b). \tag{4}$$

From (3) we also obtain the following equalities:

$$d_1(a,m)=d_1(a,m')+d_1(m',m)=d_1(a,m'')+d_1(m'',m).$$

That is

$$d(a,m'')=d(a,m)-d(m'',m). \tag{5}$$

Now, the triangle inequality of d and formula (5) give

$$d_1(a,m'')+d_2(m'',b) \le d_1(a,m'')+d_2(m'',m)+d_2(m,b) \le d_1(a,m)+d_2(m,b).$$

Whence, $d_1(a,c)+d_2(c,b) \le d_1(a,m)+d_2(m,b)$, by formula (4). As $m \in X_1 \cap X_2$ was arbitrary we conclude to formula (1) by invoking the definition of d.

From formula (1) one can easily deduce that d is convex. Next, let C be a convex subset of the amalgamation and $r > 0$. We consider two cases.

Case (i): $C \cap X_1 \cap X_2 \neq \emptyset$. In this situation we claim that the following formula holds for $i = 1,2$:

$$D(C,r) \cap X_i = D_i(C \cap X_i, r). \tag{6}$$

We may assume that $i = 1$. The inclusion from right to left is evident. For a proof of the other inclusion, let y be a member of the left-hand side of (6). We shall show the following:

For all $\varepsilon > 0$ there exists a $c \in C \cap X_1$ such that $d_1(y,c) < r + \varepsilon$. (7)

Take $\varepsilon > 0$. By definition of distance to the subset C, there exists a $c \in C$ such that $d(y,c) < r + \varepsilon$. If $c \in X_1$ we are done. Otherwise, by the Assertion there exists an $m \in X_1 \cap X_2$ such that

$$d_1(y,m) + d_2(m,c) = d(y,c) < r + \varepsilon.$$

Now let $s = d_2(m,c)$. Then the ball $D_2(m,s)$ contains c and, being convex, it also contains the gate $p_2(c)$ of c. Hence we have $d_1(y,m) + d_2(m,p_2(c)) < r + \varepsilon$, so $d(y,p_2(c)) < r + \varepsilon$. This finishes the proof of (7). From (7) we conclude that $d_1(y, C \cap X_1) \leq r$, i.e. y is a member of the right-hand side in (6).

By applying Proposition 1.8 to formula (1) we conclude that $D(C,r)$ is convex in the amalgamation.

Case (ii): $C \cap X_1 \cap X_2 = \emptyset$. By Proposition 1.8 this means that the subset C is exclusively contained in X_1 or X_2. We may assume that $C \subseteq X_1$. Let $t = d_1(C, X_2 \cap X_1)$. Then,

$$D(C,s) = D_1(C,s) \quad (0 \leq s \leq t). \tag{8}$$

To this end, as the inclusion from right to left in (8) is evident, let x be a member of the left-hand side of (8). Suppose that $x \notin X_1$. As X_1 is d-closed there exists an $\varepsilon > 0$ such that $D(x,\varepsilon)$ avoids X_1. Also there is a $c \in C$ with $d(c,x) < s + \varepsilon$. Now by the Assertion there is a point $y \in X_1 \cap X_2$, with $d(c,x) = d(c,y) + d(y,x)$. Whence $d(c,y) < s$, a contradiction.

From formula (8) we conclude that $D(C,s)$ is convex for each $s \leq t$. As d is a convex metric, we have the following equality for $s > t$,

$$D(C,s) = D(D(C,t), s-t). \tag{9}$$

By (8) we have $D(C,t) = D_1(C,t) \subseteq X_1$, whereas Lemma IV: 3.4(1) implies that $D_1(C,t) \cap X_1 \cap X_2 \neq \emptyset$. Hence, we can apply case (i) to the right-hand side of (9), which yields that $D(C,r)$ is convex in the amalgamation.

Finally, let d' be another metric as described, and let the closed balls of d' be denoted by D'. As d' extends d_1, d_2 we have $d' \leq d$. Let $a \in X_1$, $b \in X_2$ and let $t = d_1(a, X_1 \cap X_2)$. Then,

$$D_1(a,t) = D'(a,t). \tag{10}$$

As the inclusion from left to right in (10) is evident, let y be a member of the right-hand side of (10). By Proposition 1.2 the gate $p_1(a)$ of a in $X_2 \cap X_2$ is also the gate of a in the whole of X_2. So we obtain from Lemma IV: 3.4(1) that

$$d'(a,X_2) = d'(a, X_1 \cap X_2) = d'(a, p_1(a)) = t. \tag{11}$$

Now observe that each point different from y in the d'-interval $I_{d'}(a,y)$ must lie in X_1, and that y

is the limit of such points by the metrical convexity of d'. As the gated set X_1 is d'-closed (Lemma IV: 3.4(1)), we obtain that $y \in X_1$. Whence y is contained in the left-hand side of (10).

We also conclude from formula (11) that $d'(a,b) \geq t$. As the metric d' is convex we can find $m \in X_1 \cup X_2$ with

$$d'(a,m)+d'(m,b)=d'(a,b) \ ; \ d'(a,m)=t.$$

In view of formula (10), $m \in D_1(a,t) \subseteq X_1$, hence the gate m' of m in $X_1 \cap X_2$ is also contained in $D_1(a,t)$. Similarly, $m' \in D'(b,d'(a,b)-t)$ as $m \in D'(b,d'(a,b)-t)$. Whence,

$$d_1(a,m')+d_2(m',b) \leq t+d'(a,b)-t=d'(a,b).$$

By invoking the definition of the path-metric d, we conclude $d(a,b) \leq d'(a,b)$. As $a \in X_1$ and $b \in X_2$ were arbitrary we have shown $d=d'$. ■

2.2 Theorem. *Let (X_i,ρ_i) and (X_2,ρ_2) be complete modular metric spaces such that the connector is gated in both spaces and let d_i $(i=1,2)$ be convex adapted metrics on X_i that match. Then the path-metric of d_1,d_2 is the unique convex metric on the union of X_1, X_2 that is adapted to the path-metric of ρ_1,ρ_2.*

Proof: Closed balls around points in (X_i,d_i) are ρ_i-convex and closed, hence they are gated by virtue of Theorem II: 5.7. Hence by Theorem 2.1 the path-metric d of d_1,d_2 is the unique convex compatible metric on the geometric amalgamation of the X_1 and X_2. Using Theorem 1.6 we obtain that d is adapted to the path-metric of ρ_1,ρ_2. ■

§ 3 Application: special metrics on collapsible polyhedra

We recall the following definitions of Mai and Tang [53]. Let K be a cubical complex. A subset Y of $|K|$ is called a *generalized cuboid,* abbreviated *GC*, if Y is connected and for every cube I^k the subset $Y \cap I^k$ is either empty or takes the form

(3.1) $$\{(y_1, \cdots ,y_k) \mid s_i \leq y_i \leq t_i \, , \, i=1,2, \cdots ,k\} \, ,$$

for certain $s_i \leq t_i$ $(i=1,2, \cdots ,k)$ in I. Alternatively, if we equip all cubes of K with the sum-metric then Y is connected and meets every cube in a gated subset.

K is called *collapsible* if there is a sequence of subcomplexes $K_0,K_1, \cdots ,K_n$ of K, and non-empty subcomplexes L_i of K_i $(i=1,2, \cdots ,n)$, such that K_0 is a point, $K_n=K$, and $K_{i+1}=K_i \cup L_i \times I$, where

$$L_i \times I = \{c \times \{0\}, c \times I, c \times \{1\} \mid c \in L_i\} \ \ (i=0,1, \cdots ,n-1).$$

K is called *regular* if each L_i is a GC of K_i

The following is easily verified.

3.2 Lemma. *Let (X_1,I_1), (X_2,I_2) be geometric interval spaces with compatible metrics d_1, d_2 respectively. Then the max-metric d_m* on $X_1 \times X_2$ *given by*

$$d_m((x_1,x_2),(y_1,y_2)) = max(d_1(x_1,y_1), d_2(y_1,y_2)),$$

for $x_1,y_1 \in X_1$ and $x_2,y_2 \in X_2$ is compatible with $(X_1 \times X_2, I_1 \times I_2)$. Moreover, d_m is convex if d_1 and d_2 are. ■

3.3 Theorem. *Let K be a regular collapsible cubical polyhedron. Then the following hold.*

(1) There exists a unique convex median metric ρ on K such that the restriction of ρ to each cube of K is the sum-metric. Moreover, a subset C of K is gated precisely if C is a GC.

(2) There exists a unique convex metric d adapted to ρ such that d restricted to each cube of K yields the max-metric.

Proof: The unicity of statements (1) and (2) follow from Corollary 1.13. For a proof of existence of these metrics, let K_i,L_i be as in the definition of collapsible cubical polyhedron. We shall show that (1) and (2) hold for K_i with induction on i. For $i=0$ this is obvious. Let $i>0$, and let d_i,ρ_i be metrics satisfying (1) and (2) for K_i. By the induction hypotheses, L_i is a gated subset of K_i. Hence the restriction of ρ_i to L_i yields a convex, median metric. So the sum-metric ρ_s on $L_i \times I$ is also a convex median metric. Moreover the subset L_i occurs as a gated subset of $L_i \times I$. Hence we can apply Theorem 1.6 to obtain a convex median metric ρ_{i+1} on $K_i \cup L_i \times I = K_{i+1}$. First, the restriction of ρ_{i+1} to cubes of K_{i+1} is the sum-metric. Next, the description of gated sets of K_{i+1} is as desired by Lemma 1.10. This concludes the induction step for statement (1).

By the induction hypotheses (K_i,d_i) is a convex metric adapted to ρ_i. Using Proposition IV: 3.6 we obtain that the restriction of d_i to L_i is a convex metric, adapted to the restriction of ρ_i to L_i. Hence the max-metric d_m on $L_i \times I$ yields a convex metric adapted to d_s —see Lemma 3.2. We can now apply Theorem 2.2 to obtain an adapted metric of ρ_{i+1} that extends d_i and d_m. This adapted metric evidently satisfies (2). This concludes the induction step for statements (1), (2). ■

In [53, Lemma 1] it is shown that a collapsible polyhedron S (see [78]) can be subdivided to a regular collapsible cubical complex K such that the polyhedron of any subcomplex of S is exactly the polyhedron of the corresponding subcomplex of K. Hence, from Theorem 3.3 we obtain the following.

3.4 Theorem. *If S is a collapsible polyhedron, then there are metrically-convex metrics ρ,d on S, such that (S,ρ) is a median metric space, and d is adapted to ρ.* ■

3.5 Corollary. ([53, p. 336]) *A collapsible polyhedron admits a hyperconvex metric.* ■

It is an open problem whether the converses of Theorem 3.4 or Corollary 3.5, hold.

CHAPTER VII

MEDIANS VERSUS STEINER TREES

The theory of Steiner trees deals with the following type of problem. How can you design a network connecting all consumers and minimizing the quantity of material used? This type of problem arises in the design of telephone networks, oil pipelines, and electrical circuitry. The main result of this chapter is that in general median metric space such trees exist and can be found in a finite number of steps. This generalizes and strengthens a result of M. Hanan [35] in the plane. The method employed by Hanan is rather technical and ad-hoc as it involves highly specific constructions in the plane. In contrast, we have based our methods on a fairly well developed geometry of median metric spaces. In particular, there is no need to restrict to two dimensions. ($^{\prime}$)

§ 1 Introduction

Let (X,ρ) be a metric space. Let $G=(V,E)$ be a graph in X, i.e. $V \subseteq X$. The *length* of an edge is the distance between its vertices and the *length* of G is the sum of all edge lengths.

Let $C=\{c_1, \cdots, c_n\}$ be a subset of X. The graph G is *connecting* C if G is connected and $C \subseteq V \subseteq X$. In this circumstance vertices in $V \setminus C$ are called *additional*. Suppose that we have a graph of minimal length connecting C. By minimality, removing an edge must result in a disconnected graph. Hence the connecting graph must be a tree (see for instance [24, Theorem 2.1]). From now on, a graph connecting C of minimal length will be called a *Steiner tree* of C, after J. Steiner who considered the case $n=3$. In this case at most one additional point is required (see below), i.e. the Steiner point as encountered in section 4 of Chapter I.

In general a Steiner tree will contain several additional points, however the number of such points can be restricted. To show this we recall the following (well-known) formula for a tree (T,E) (cf. [24, pp. 22-24]):

$$\#End(T) - 2 = \sum_{deg(p) \geq 3} deg(p) - 2, \qquad (*)$$

in which the *degree*, $deg(p)$, of a point $p \in T$ is the number of neighbors of p, and $End(T)$ denotes the endpoints of T, i.e. the points of degree 1.

Now consider a tree (T,E) connecting C. We can discard any additional point of degree < 3 to obtain a Steiner tree connecting C where all additional vertices have degree ≥ 3. In particu-

$^{\prime}$ The results of this chapter were obtained by van de Vel and the author.

lar, endpoints of the tree must be members of C. It then follows from formula (*) that the number of additional vertices does not exceed $\#C-2$. Hence without loss of generality we may assume that any tree connecting C is a tree with at most $\#C-2$ additional points. This observation implies the existence of a Steiner tree in situations where the set of (potential) additional points can be taken compact, e.g. when closed balls are compact.

Most research on Steiner trees is done in the context of the Euclidean plane. For a survey see [29], a less technical survey can be found in the January 1989 edition of Scientific American [14].

In a 1961 paper Z.A. Melzak [54] gave a finite algorithm for finding Steiner trees in the Euclidean plane — see also the book of Melzak [55]. Fifteen years later it was shown by Garey, Graham and Johnson [27] that the computation of Steiner trees in the Euclidean plane is "NP-hard". The NP-hard problems — NP stands for Non-deterministic Polynomial — are a wide class of problems with the following important property:
- No NP-hard problem is known to be solvable by a polynomial time-bounded algorithm. If ***any*** NP-hard problem can be solved in such a fashion then ***all*** NP-hard problems can be solved in such a fashion.

In contrast, finding a tree of minimal length ***without*** introducing extra points can be solved in polynomial time, see [24, p. 26]. The class of NP-hard problems includes many problems notorious for their computational difficulty, such as the traveling salesman problem, the graph chromatic number problem, tautology testing, and clique finding. It is widely believed (though not yet proved) that ***no*** NP-hard problem can be solved in polynomial time. Hence, NP-hardness is a very strong indication for inherent intractability.

As an example of this, calculating a Steiner tree in the Euclidean plane on 29 points was close to the limit of computing capabilities in 1989 ([14]). The following is quoted from [68].
When the Long Lines Department of the Telephone Company establishes a communications hookup for a customer, federal tariffs require that the billing rate is proportional to the length of a Steiner tree connecting the customers termini. In light of the previous this kind of billing is not attainable.

Sometimes other metrics than the Euclidean are considered. Most notably is the Steiner tree problem for points in the plane endowed with sum metric, which has an important application in printed circuitry. Here n points on an insulated plate are to be electrically connected. For important technical reasons the nozzle that sprays the thin metal lines onto the plate can only move vertically or horizontally.

It was shown by M. Hanan in 1966 [35] that in these circumstances a Steiner tree can be found in the lattice generated by the original points. Hence, such trees can be found in a finite number of steps. Unfortunately, just as in the Euclidean case the complexity of the problem is NP-hard, as is shown by Garey and Johnson [26].

As medians are Steiner points (Corollary II: 1.12), and as the sum metric on the plane is one of the most prominent examples of a median metric, the question arises whether there are general results on Steiner trees for median metric spaces. The answer, which is affirmative, shall be the topic of the next section.

§ 2 The main result

In this section we will prove that in general median metric space a Steiner tree exists and can be found in the (finite!) median stabilization of the original points. Hence, also these trees can be found in a finite number of steps. We mention that two methods to compute the median stabilization were described at the end of section 2 in Chapter I.

From now on we let (X, ρ) be a median metric space and $C \subseteq X$.

2.1 Lemma. *Let $G \subseteq X$ be gated with gate function p, and let $x \in X$. If $y \in co(G \cup \{x\})$ then $\rho(G,y) \le \rho(G,x)$.*

Proof: By JHC there exists a $c \in G$ with $y \in I(x,c)$. On the one hand $\rho(x,c)$ equals

$$\rho(x,y)+\rho(y,c)=\rho(x,y)+\rho(y,p(y))+\rho(p(y),c).$$

On the other hand $\rho(x,c)$ equals

$$\rho(x,p(x))+\rho(p(x),c)=\rho(x,p(x))+\rho(p(x),p(y))+\rho(p(y),c).$$

Comparing these expressions yields

$$\rho(y,p(y))=\rho(x,p(x))+\rho(p(x),p(y))-\rho(x,y)\le\rho(x,p(x)).$$

The latter inequality holds by contractivity of p. ■

The following result is fundamental.

2.2 Theorem. *There exist Steiner trees connecting C. In fact, there is a Steiner tree with vertices in the median stabilization of C.*

Proof: It suffices to prove the result for finite median metric spaces X. Indeed, start with any tree connecting C, say with vertices in V. Then the median stabilization of V is a finite median metric space, hence by assumption there exists a tree in $med(C)$ whose length does not exceed the original one. This then shows the theorem for general X.

If X is finite then the first statement is evident. For a proof of the second; let $S=(T,E)$ be a Steiner tree connecting C such that the number points in $med(T)$ is minimal. Suppose that $med(T)\neq med(C)$. From now on, we consider $med(T)$ as the ambient median metric space. By Theorem I: 2.19 there exist halfspaces H_1, H_2 in $med(T)$ such that

$$H_1 \cap H_2 \neq \emptyset \quad ; \quad H_1 \cap H_2 \cap C=\emptyset.$$

We may assume that H_1 and H_2 are minimal with this property. The convex hull of any point in H_i $(i=1,2)$ and the whole of $(H_i)'$ equals $med(T)$, as otherwise we could contradict the minimality of H_i. So by Lemma 2.1 the distance $\rho(s,(H_i)')$ for $s \in H_1 \cap H_2$ and $i=1,2$ only depends on i. We will denote these distances by ρ_i. For $i=1,2$ we let p_i denote the gate function $X \to (H_i)'$.

By adding extra points if necessary we may assume that if a point $s \in H_1 \cap H_2$ has a neighbor in $(H_i)'$ then this neighbor equals the gate of s in $(H_i)'$. Let n_1 be the number of points in $H_1 \cap H_2$ which have a neighbor in $(H_1)'$, but not in $(H_2)'$. Define n_2 analogously.

For $i=1,2$, let $F_i : T \to (H_1 \cap H_2)'$ be the identity on points of $(H_1 \cap H_2)'$ and $F_i(s)=p_i(s)$ for points of $H_1 \cap H_2$. Consider the graph G_i $(i=1,2)$ with vertex set equal to the image of F_i; two vertices $F_i(u)$, $F_i(v)$ are neighbors iff u and v are.

One can easily show the following estimations:

$$length(G_1) \le length(S)+(n_1-n_2)\rho_1 \quad ; \quad length(G_2) \le length(S)+(n_2-n_1)\rho_2$$

Hence n_1 must equal n_2 and both G_1, G_2 must be Steiner trees connecting C in $(H_1 \cap H_2)'$.

Being the union of two convex sets, the set $(H_1 \cap H_2)'$ is median stable. Moreover it is a genuine subset of med(T), thus contradicting the minimality assumption on T. Hence $T \subseteq med(C)$. ■

The lattice generated by a set of points in the plane yields a median stable set, which is generally larger than the median stabilization. Hence Theorem 2.2 is a strengthing of the result of Hanan in the plane. Our result is moreover applicable to other spaces such as $\mathbb{R}^n$ with the sum metric ($n \in \mathbb{N}$), or more general spaces of type $L_1(\mu)$ or $K_1(\mu)$ (e.g. probabilistic spaces).

With the use of the amalgamation technique we developed in Chapter VI, we can construct "tailor-made" median metric spaces. As an illustration of this, one could say that Hanan was designing a telephone network for Manhattan without taking the heights of the buildings in account — the consumers of the network are all supposed to be located on the ground floor. There is a natural concept of distance between points of Manhattan. Inside a building the distance involved is the sum metric; the distance between points in different buildings is the distance between the projections on the ground floor plus the respective heights. This metric space can be constructed by "repeated amalgamation of buildings with the plane". In view of Theorem VI: 1.6 this metric is median, hence Theorem 2.2 is applicable.

We observe that not ***every*** Steiner tree connecting C lies in the median stabilization as simple examples in the plane show. However each Steiner system lies in the (median) convex hull of C. This result is similar to the situation in the Euclidean plane. Each Steiner tree in the Euclidean plane lies in the convex hull of the original points ([29, 3.5]). We need the following result.

2.3 Lemma. *Let $G \subseteq X$ be gated with gate function p, and let $x_1, \cdots, x_n \in X$ such that $x_1 \in G$. Then*

$$\sum_{i=1}^{n-1} \rho(x_i, x_{i+1}) \ge \rho(p(x_n), x_n) + \sum_{i=1}^{n-1} \rho(p(x_i), p(x_{i+1})).$$

Proof: By induction on n. For n=2 the result follows directly from definition of gate. As to the step $n \to n+1$, use that

$$\rho(x_n, p(x_n)) + \rho(x_n, x_{n+1}) \ge \rho(x_{n+1}, p(x_n)) = \rho(x_{n+1}, p(x_{n+1})) + \rho(p(x_n), p(x_{n+1})).$$ ■

2.4 Proposition. *Let $G=(V,E)$ be a connecting graph of $C \subseteq X$ let $c_0 \in C$ and let $p: X \to co(C \setminus \{c_0\})$ be the gate map. Then the graph $G'=p(G) \cup \{p(c_0)\}, E')$ with*

$$E'=\{p(u)p(v) \mid uv \in E\} \cup \{c_0 p(c_0)\}$$

connects C and its length does not exceed that of G.

Proof: Consider a path γ from c_0 into $co(C \setminus \{c_0\})$. Then, by Lemma 2.3 and the contractivity of p we obtain:

$$\begin{aligned}\textstyle\sum_{uv \in E} \rho(u,v) &= \textstyle\sum_{uv \in \gamma} \rho(u,v) + \sum_{uv \notin \gamma} \rho(u,v) \\ &\leq \rho(p(c_0), c_0) + \textstyle\sum_{uv \in \gamma} \rho(p(u), p(v)) + \sum_{uv \notin \gamma} \rho(p(u), p(v)).\end{aligned}$$

■

2.5 Theorem. *The vertices of a Steiner tree connecting C are contained in the convex hull of C.*

Proof: Suppose that $T = (V,E)$ is a Steiner tree connecting C and that m is an additional vertex. T is evidently also a Steiner tree of $C \cup \{m\}$. Consider the graph as constructed in Proposition 2.4 with respect to m. Suppose that $m \notin co(C)$. Let p be the gate of m in $co(C)$. By removing the vertex m and vertex $mp(m)$ we obtain a graph connecting C of smaller length, a contradiction. Hence $m \in co(C)$. ■

BIBLIOGRAPHY

[1] J.C. Abbot, *Sets, lattices and boolean algebras*, Allyn and Bacon Inc., Boston, (1969) 282pp.

[2] E. Alfsen, E. Effros, *Structure in real Banach spaces*, Ann. of Math. 96, (1972) pp. 98-173.

[3] D. Amir, *Characterizations of inner product spaces*, Birkhäuser Verlag, (1986) pp. 200.

[4] N. Aronszajn, P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. 6, (1956) pp. 405-439.

[5] S.P. Avann, *Ternary distributive sem-lattices*, Bull. Amer. Soc. 54, (1948) p. 79.

[6] S.P. Avann, *Metric ternary distributive semi-lattices*, Proc. Amer. Math. Soc. 12, (1961) pp. 407-414.

[7] P. Assouad, M. Deza, *Metric subspaces of* L^1, Publications Mathematiques D' Orsay, 82.03, Université de Paris-Sud, Bat. 425, (1982) 47pp.

[8] H.-J. Bandelt, J. Hedlíková, *Median algebras*, Discrete Math., 45, (1983) pp. 1-30.

[9] H.-J. Bandelt, H. M. Mulder, *Pseudo-modular graphs*, Discrete Math., (1986) pp. 245-260.

[10] H.-J. Bandelt, M. van de Vel, *The median stabilization degree of a median algebra*, in preparation.

[11] H.-J. Bandelt, M. van de Vel, E. Verheul, *Modular interval spaces*, submitted.

[12] M. Benado, *Les ensembles partiellement ordonnés et le théorème de raffinemente de Schreier, II (Théorie de multistructures)*, Čech. J. Math., 5(80), (1955) pp. 308-344.

[13] G. Birkhoff, *Lattice theory*, AMS Coll. Publ. XXV, Providence, R.I, (1967) 418pp.

[14] M.W. Bern, R.L. Graham, *The shortest-network problem*, Scientific American, Januari 1989, pp. 66-71.

[15] G. Birkhoff, S. Kiss, *A ternary operator in distributive lattices*, Bull. Amer. Math. Soc. 53, (1947) pp. 749-752.

[16] L.M. Blumenthal, K. Menger, *Studies in geometry*, Freeman, San Fransisco, (1970) 512pp.

[17] W. Comfort, H. Gordon, *Inner product spaces and the tri-spherical intersection property*, Proc. Amer. Math. Soc. 12, (1961) pp. 327-329.

[18] R. Courant, H. Robbins, *What is mathematics?*, Oxford University Press, New York, (1941).

[19] F. Cunningham Jr., *L-structure in L-spaces*, Trans. Amer. Math. Soc. 95, (1960) pp. 274-299.

[20] H. Draškovičová , *Über die Relation "zwischen" in V erbänden*, Matematicko-Fyzikálny Časopis SAV., 16, 1, (1966) pp. 13-20.

[21] A. Dress, R. Scharlau, *Gated sets in metric spaces,* Aequationes Math. 34, (1987) pp. 112-120.

[22] R. Engelking, *General topology*, PWN - Polish Scientific Publishers, Warszawa, (1977) 529pp.

[23] E. Evans, *Median lattices and convex subalgebras*, Colloquia Mathematica Societatis János Bolyai Vol. 29 (Universal algebra), Esztergon, Hungary, (1977) pp. 225-240.

[24] S. Even, *Graph algorithms*, Computer Science Press, Potomac, MD (1979) pp. 249.

[25] R. Fullerton, *Geometrical characterizations of certain function spaces,* Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem; Pergamon Oxford, 1961, pp. 227-236.

[26] M.R. Garey, D.S. Johnson, *The rectilinear Steiner tree problem is NP-complete*, Siam J. Appl. Math. 32, (1977) pp. 826-834.

[27] M.R. Garey, R.L. Graham, D.S. Johnson, *The complexity of computing Steiner minimal trees*, Siam J. Appl. Math. 32, (1977) pp. 835-859.

[28] H. Gericke, *Lattice theory*, George G. Harrap & Co, (1963) 185pp.

[29] E.N. Gilbert, H.O. Pollak, *Steiner minimal trees*, SIAM J. Appl. Math. (1968), pp. 1-29.

[30] V. Glivenko, *Géometrie des systèmes de choses normées*, Amer. J. Math. 58, (1936) pp. 799–828.

[31] V. Glivenko, *Contributions à l' etude des systèmes de choses normées*, Amer. J. Math. 59, (1937), pp. 941-956.

[32] J. de Groot, *Supercompactness and superextension*, Contributions to extension theory of topological structures, Symp. Berlin 1967, Deutsche Verlag der Wissenschaften, Berlin, (1969) pp. 89-90.

[33] P.R. Halmos, *Measure theory*, Van Nostrand, (1950) 304pp. (republished by Springer in 1974).

[34] P.R. Halmos, *Lectures on boolean algebras*, Van Nostrand, (1967) pp. 147.

[35] M. Hanan, *On Steiner's problem with rectilinear distance*, Siam J. Appl. Math. 14, (1966) pp. 255-265.

[36] O. Hanner, *Intersection of translates of convex bodies*, Math. Scand. 4, (1956) pp. 65-87.

[37] J. Hedlíková, *Ternary spaces, media and Chebyshev sets*, Czech. Math. J. 33, (1983) pp. 212-228.

[38] E. Howorka, *Betweenness in graphs*, Abstracts Amer. Math. Soc. 2, (1981) *783-06-5.

[39] H. H. Hung, *The algamation property for G-metric spaces*, Proc. Amer. Math. Soc. 37, (1973) pp. 53-58.

[40] J.R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. 39, (1964) pp. 65-76.

[40] J.R. Isbell, *s admits an injective metric*, Proc. Amer. Math. Soc 28, (1971) pp. 259-261.

[42] J.R. Isbell, *Median algebra*, Trans. Amer. Math. Soc. 260, (1980) pp. 319-362.

[43] R.C. James, *Orthogonality in normed linear spaces*, Duke Math. J. 12, (1945) pp. 291-301.

[44] S. Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. 42 (2), (1941) pp. 523-537.

[45] J.L. Kelley, I. Namioka, *Linear topological spaces*, Van Nostrand, (1950) 256pp. (republished by Springer in 1974).

[46] M. Kolibiar, *Über metrische Vielverbände*, Acta Fac. Rer. Nat. Univ. Commenianae Math. 4, (1959) pp. 187-203.

[47] M. Kolibiar, T. Marcisová, *On a question of J. Hashimoto*, Mat. Časopis 24, (1974) pp. 179-185.

[48] Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc., 227, (1977) pp. 1-62.

[49] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, (1979) pp. 243.

[50] J. Lindenstrauss, *Extension of compact operators,* Memoirs of the Amer. Math. Soc. 48, (1964) 112pp.

[51] J. Lindenstrauss, *On the extension of operators with a finite dimensional range,* Illinois J. Math., (1964) pp. 488-499.

[52] W. Luxemburg, A. Zaanen, *Riesz spaces,* North-Holland Publishing Company, Amsterdam, (1971) pp. 514.

[53] J.-H. Mai, Y. Tang, *An injective metrization for collapsible polyhedra*, Proc. Amer. Math. Soc. 88, (1983) pp. 333-337.

[54] Z.A. Melzak, *On the problem of Steiner*, Canad. Math. Bull. 4, (1983) pp. 143-148.

[55] Z.A. Melzak, *Companion to concrete mathematics*, Wiley-Interscience, New York, (1973), pp. 270.

[56] K. Menger, *Untersuchungen über allgemeine Metrik,* Math. Ann. 100, (1928) pp. 75-163.

[57] E. Michael, *Continuous selections I*, Ann. of Math. 63 (1956) pp. 361-382.

[58] J. van Mill, *Supercompactness and Wallman spaces*, Math. Centre Tracts 85, Amsterdam, (1977).

[59] J. van Mill, *Infinite-dimensional topology: prerequisites and introduction*, North-Holland, Amsterdam, (1988) pp. 401.

[60] J. van Mill, M. van de Vel, *On an internal property of absolute retracts*, Topology Proceedings 4, (1979) pp. 193-200.

[61] J. van Mill, M. van de Vel, *Convexity preserving mappings in subbase convexity theory*, Proc. Kon. Ned. Akad. Wet 81, (1978) pp. 76-90.

[62] H.M. Mulder, *The interval function of a graph*, Math. Centre Tracts 132, Mathematisch Centrum, Amsterdam, Dissertation (1980) 191pp.

[63] L. Nachbin, *A theorem of the Hahn-Banach type,* Trans. Amer. Math. Soc. 68, (1950) pp. 28-46.

[64] J. Neveu, *Mathematische grundlagen der wahrscheinlichkeitstheorie*, Oldenbourg verlag, (1969) 264pp.

[65] J. Nieminen, *The ideal structure of simple ternary algebras*, Colloq. Math. 40, (1978) pp. 23-29.

[66] E. Pitcher, M. Smiley, *Transitivities of betweenness*, Trans. Amer. Soc. 52, (1942) pp. 95-114.

[67] D. Pompéiu, *Sur la continuiteé des fonctions de variables complexes*, Ann. Fac. Sci. Toulouse 7, (1905) pp. 265-315.

[68] F.P. Prepata, M.I. Shamos, *Computational geometry*, Springer, New York, (1985), pp. 398.

[69] H.L. Royden, *Real analysis*, Macmillan, New-York, (1988) 444pp.

[70] W. Rudin, *Functional analysis*, Mc Graw-Hill, (1973) 397pp.

[71] H.H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, (1974) 376pp.

[72] M. Sholander, *Trees, lattices, order, and betweenness*, Proc. Amer. Math. Soc. 3, (1952) pp. 369-381.

[73] M. Sholander, *Medians and betweenness*, Proc. Amer. Math. Soc. 5, (1954) pp. 801-807.

[74] M. Sholander, *Medians, lattices and trees*, Proc. Amer. Math. Soc. 5, (1954) pp. 808-812.

[75] R. Sine, *Hyperconvexity and nonexpansive multifunctions*, Trans. Amer. Math. Soc. 315, (1989) pp. 755-767.

[76] M.F. Smiley, *A comparison of algebraic, metric, and lattice betweenness*, Bull. Amer. Math. Soc. 49, (1943) pp. 246-252.

[77] V.P. Soltan, *On the diameter of d-convex hull* (Russian), Mat. Issled. 9 No 1, (1974) pp. 94-109.

[78] J.R. Stallings, *Lectures on polyhedral topology*, Tata Institute of Fundamental research, Bombay, (1968) 260pp.

[79] M. van de Vel, *Abstract, topological and uniform convex structures*, report WS-353, Free University Amsterdam, (1989).

[80] M. van de Vel, *Matching binary convexities*, Top. Appl. 16, (1983) pp. 207-235.

[81] M. van de Vel, E. Verheul, *Decomposing multimedian Banach spaces into a rigid and an $l_1(I)$-part*, report WS-373, Amsterdam, (1990), submitted.

[82] M. van de Vel, E. Verheul, *Isometric embedding of median algebras into $L_1(\mu)$ spaces*, submitted.

[83] M. van de Vel, E. Verheul, *Topology of median spaces*, submitted.

[84] A. Verbeek, *Superextensions of topological spaces*, Math. Centre Tracts 41, Mathematisch Centrum, Amsterdam, (1972) 155pp.

[85] E. Verheul, *Modular metric spaces*, submitted. A preliminary version appeared as report WS-358, Amsterdam, (1989).

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48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

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50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

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