CWI Tracts

CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands
Telephone 31 - 20 592 9333, telex 12571 (mactr nl),
telefax 31 - 20 592 4199

CWI is the nationally funded Dutch institute for research in Mathematics and Computer Science.

The solution of a one-dimensional Stefan problem

C. Vuik

1991 Mathematics Subject Classification: 35J25.
ISBN 90 6196 419 9
NUGI-code: 811

Printed in the Netherlands

Preface

This monograph contains the first three chapters of my thesis: *The solution of a class of Stefan problems* . The existence and uniqueness of the solution of a one-dimensional moving boundary problem (Stefan problem) are proved. Since all the existence results are proved in a constructive way, numerical approximation schemes are easily obtained from the theoretical results.

Contents

Chapter 0
Introduction and summary

1. The origin of the Stefan problem

In this monograph we study some classes of Stefan problems.

An example of a Stefan problem is the melting of ice in water. In a mathematical model the temperature of the water satisfies the heat equation. Remark that initially the configuration of the water is known. However after some time part of the ice is melted and so the domain whereon the heat equation must be satisfied, is transformed. The domain can be found using the assumption that the heat disappeared from the water has been used to melt the ice.

More generally a Stefan problem is a problem where the domain, on which a parabolic partial differential equation should be satisfied, is unknown and must be found as part of the solution.

This class of problems is named after J. Stefan (who is also known from the Stefan Boltzmann constant). In 1891 Stefan has written a paper on the formation of ice in the polar seas [Stefan, 1891]. He gives a mathematical model which describes the formation of ice and compares the solution of this model with measurements obtained from polar expeditions.

Many more phenomena can be described by a Stefan problem. Some examples are: the freezing of wet soil, the freezing of food, the production of iron in a blast furnace, the decrease of oxygen in a muscle in the vicinity of a clotted bloodvessel and the etching techniques used for the production of microelectronic devices.

After 1940 there is a considerable increase of publications on the Stefan problem. This is illustrated by Table 1, which is obtained from the list of publications given in [Cannon, 1984].

period	1931–1940	1941–1950	1951–1960	1961–1970	1971–1980	1981–1982
papers	0.1	1.8	4	7.8	23.3	54.4

Table 1. Average number of papers published in one year.

2. Historical survey

We first discuss some classes of Stefan problems for which existence and uniqueness results are known from the literature. Subsequently we mention the class of problems we are going to study and compare these with the earlier ones.

We remark that an elaborate historical survey until 1960 can be found in [Rubinstein, 1971] and that a list of publications on the Stefan problem is given in [Cannon, 1984].

An existence theorem related to the following Stefan problem was published in [Evans, 1951]:

Problem 1

Given the constants T,a,B > 0, find functions S and C such that

$$\frac{\partial C(x,t)}{\partial t} - \frac{\partial^2 C(x,t)}{\partial x^2} = 0 \quad , x \in (0,S(t)),\ t \in (0,T), \tag{2.1}$$

$$S(0) = 0 \quad , \tag{2.2}$$

$$\frac{\partial C}{\partial x}(0,t) = -a \quad , t \in (0,T), \tag{2.3}$$

$$C(S(t),t) = 0 \quad , t \in (0,T), \tag{2.4}$$

$$-\frac{\partial C}{\partial x}(S(t),t) = B\,\frac{dS(t)}{dt}, \quad t \in (0,T). \tag{2.5}$$

Boundary condition (2.5) is also known as the Stefan condition. In [Douglas, 1957], uniqueness of the solution of Problem 1 is shown.

Existence and uniqueness for a more general class of Stefan problems was shown in [Kyner, 1959]. In this paper, the Stefan condition is:

$$-f(S(t),t,\frac{\partial C}{\partial x}(S(t),t))\,\frac{\partial C}{\partial x}(S(t),t) + b = \frac{dS(t)}{dt}\ , t \in (0,T), \tag{2.6}$$

where f is a given positive function with continuous second order derivatives and b is a non–negative constant. In the sequel Problem 1 with (2.5) replaced by (2.6) will be referred to as Problem 2.

The most general existence and uniqueness results given in the literature are proved in [Rubinstein, 1971] and [Fasano & Primicerio, 1977]. The Stefan problem investigated in [Fasano & Primicerio, 1977] is:

Problem 3

Given the functions C_0,g,B and μ and the constants T,b > 0 find S and C such that

$$\frac{\partial C(x,t)}{\partial t} - \frac{\partial^2 C(x,t)}{\partial x^2} = 0 \quad , x \in (0,S(t)),\ t \in (0,T), \tag{2.7}$$

$$C(x,0) = C_0(x),\ x \in [0,b],\ S(0) = b, \tag{2.8}$$

(2.9) $$\frac{\partial C}{\partial x}(0,t) = g(C(0,t),t) \quad , t \in (0,T),$$

(2.10) $$C(S(t),t) = 0 \quad , t \in (0,T),$$

(2.11) $$\frac{\partial C}{\partial x}(S(t),t) = B(S(t),t)\,\frac{dS(t)}{dt} + \mu(S(t),t),\ t \in (0,T).$$

They assume that $|B(x,t)| \geq B_0 > 0$, B has a continuous first order derivatives and μ is a Lipschitz continuous function. Since we are mainly interested in a comparison of the different Stefan conditions we omit the assumptions on C_0 and g.

Comparing Problem 3 with Problem 2 we note the following differences:
- since $S(0) > 0$ in Problem 3 the function C satisfies initial condition (2.8),
- the constant a in (2.3) is replaced by the function g,
- there is an extra term $\mu(S(t),t)$ in the Stefan condition (2.11),
- the functions in the Stefan condition satisfy weaker smoothness conditions,
- the absence of sign conditions imposed on the constants and functions.

In [Fasano & Primicerio, 1977] it was shown that if T is small enough then Problem 3 has a unique solution. Subsequently they specify a subclass of these Stefan problems such that the unique solution exists for every $T > 0$. Finally they give an example such that the assumptions of Problem 3 hold but there is no solution for T large enough.

In this monograph we consider the following Stefan problem:
Problem 4

Given $T > 0$, the function C_0, the multifunction $\bar{B}$ and the functional $\bar{G}$, find functions S and C such that:

(2.12) $$\frac{\partial C(x,t)}{\partial t} - \frac{\partial^2 C(x,t)}{\partial x^2} = 0 \quad , x \in (-\infty, S(t)),\ t \in (0,T],$$

(2.13) $$C(x,0) = C_0(x)\ ,\ x \in (-\infty, 0],\ S(0) = 0,$$

(2.14) $$C(S(t),t) = 0 \quad , t \in [0,T],$$

(2.15) $$\bar{G}(S,f,t) \in \bar{B}(S(t)),\ t \in [0,T],$$

$$\text{where}\quad f(t) = \int_{-\infty}^{\infty} [C_0(x) - C(x,t)]\,dx,\ t \in [0,T],$$

(for easy notation, we define $C(x,t) = 0$, $x \in (S(t),\infty)$, $t \in [0,T]$ and $C_0(x) = 0$, $x \in (0,\infty)$). We prove existence and uniqueness of the solution of Problem 4 under certain conditions imposed on C_0, $\bar{B}$ and $\bar{G}$.

The main differences between Problem 4 (and given results) and Problem 3 (and given results) are:

Domain

In Problem 4 the diffusion equation is posed on an unbounded domain. In the literature (see [Fasano & Primicerio, 1977; p. 697]) there are conjectures that results shown for a bounded domain can also be shown for an unbounded domain.

Stefan condition

In Problem 1, 2 and 3 the Stefan condition is a differential equation. In Problem 4 the Stefan condition is a functional integral equation. So Problem 4 can be used to describe a more general class of Stefan conditions than the class described with Problem 1, 2 or 3 (compare [Va; p. 32, Example 3]). Furthermore it follows from a comparison between Problems 3 and 4 given in [Vc; p. 28, Remark 6.2 ii)] that the smoothness conditions imposed on the functions in the Stefan condition (2.15) are weaker than the smoothness conditions in [Fasano & Primicerio, 1977].

Existence

In contrast with the existence results given in the literature we prove existence in a constructive way. So our existence proof suggests a numerical solution method. In a numerical simulation properties of the numerical iterates are in agreement with those of the analytical iterates as given in the theory. Another difference is that our conditions are such that the solution exists for every $T > 0$.

Uniqueness

It is known that on an unbounded domain the solution of the diffusion equation is unique only if it is an element of a certain function class. In order to prove uniqueness we will assume that the function C is bounded. In the literature uniqueness of the solution is shown under the assumption that S is a differentiable function. We prove uniqueness under the assumption that S is a continuous function.

3. Summary of this monograph

§ 3.1 Introduction

In this monograph we prove existence and uniqueness of the solution of the Stefan problem as described by Problem 4 ((2.12),...,(2.15)). In the Chapters 1, 2 and 3 some of the conditions on $\bar{G}$ are different.

Our work was motivated by an etching technique which is used for the production of microelectronic devices [Notten, 1989; Chapter 8]. We use this application to illustrate our theory. In this case, the function C describes the concentration of the etching agent whereas S(t) denotes the time-dependent position of the interface between solid and liquid. In most etching problems the vessel containing the liquid is very large with respect to the area wherein the etching agent shows a noticeable decrease. This is one of our reasons to pose the Stefan problem on an unbounded domain.

We start with a discussion of our Stefan condition (2.15). Integrating Condition (2.5):

$$(3.1) \qquad -\frac{\partial C}{\partial x}(S(t),t) = B\,\frac{dS(t)}{dt}, \quad t \in [0,T],$$

in the time direction and using the diffusion equation (2.12) and the equation $\frac{\partial C}{\partial x}(-\infty,t) = 0$ yields

$$(3.2) \qquad -\int_{-\infty}^{\infty} [C(x,t) - C_0(x)]\,dx = B\,S(t), \; t \in [0,T],$$

which is of the form (2.15) with $\bar{G}(S,f,t) = f(t)$ and $\bar{B}(x) = Bx$.

The Stefan condition (3.2) has the following features:
- Condition (3.2) is the mass balance in integral form. This means that the loss of etching agent is proportional to the loss of solid. The proportionality constant B is given by the chemical properties of the etching agent and the solid.
- The function S in (3.1) should be differentiable whereas in (3.2) it is sufficient that S be continuous. This is attractive from a physical point of view. Furthermore it suggests that the smoothness conditions in (2.15) are weaker than in (2.11). This suggestion is proved in [Vc; p. 28, Remark 6.2 ii)].

Obviously Condition (2.15) is a generalization of Condition (3.2). Since (2.15) does not look very natural we note that it was obtained by first proving some existence results for a Stefan problem using Condition (3.2) and then trying to find the most general condition under which such kinds of proof could still be given.

In this monograph we impose the following conditions upon the function C_0, the multifunction $\bar{B}$ and the functional $\bar{G}$. The function C_0 should be a monotone decreasing Lipschitz continuous function with $C_0(0) = 0$ and $\lim_{x\to -\infty} C_0(x) = 1$. The multifunction $\bar{B}$ should be the inverse of a nondecreasing Lipschitz continuous function. The Lipschitz constant is denoted by $\frac{1}{B_1}$. The functional $\bar{G}$ should be

Lipschitz continuous in its first and second argument with respect to the sup–norm. The Lipschitz constants are denoted by G_1 and G_2.

Now we shall discuss the chapters seperately. In every chapter we impose some extra conditions upon $\bar{G}$.

§ 3.2 Chapter 1

In this chapter we assume in addition that $\bar{G}$ is an increasing Lipschitz continuous function in its third argument (time).

The main result of this chapter is the following: introduce the operator $\mathcal{T}$, operating on a function S as follows

$$\mathcal{T}(S)(t) = \bar{B}^{-1}(\bar{G}(S,f_s,t)),\ t \in [0,T].$$

In this formula the function f_s is defined by $f_s(t) = \int_{-\infty}^{\infty} [C_0(x) - C_s(x,t)]\, dx,\ t \in [0,T]$ where C_s satisfies the equations (2.12), (2.13) and (2.14) for the given function S. Then it is proved that

$$\sup_{t\in[0,T]} |\mathcal{T}(S_1)(t) - \mathcal{T}(S_2)(t)| \le \frac{G_1+G_2}{B_1} \sup_{t\in[0,T]} |S_1(t) - S_2(t)|.$$

Hence, if $\frac{G_1+G_2}{B_1} < 1$ then Banach's fixed point theorem implies that there is a unique function $\bar{S}$ such that $\bar{S} = \mathcal{T}(\bar{S})$. We show that $(\bar{S},C_{\bar{S}})$ is the unique solution of Problem 4. Furthermore Banach's theorem implies that for every initial function S_0 the iterates given by $S_i = \mathcal{T}(S_{i-1})$, i = 1,2,.. converge to the function $\bar{S}$. The given numerical solution method is based on this property.

Applications

1. Define $\bar{B}(x) = B_1x$ and $\bar{G}(S,f,t) = f(t),\ t \in [0,1]$ (see [Va; p. 30, Remark 6.3 i)]). Remark that in this case (2.15) is equivalent to (3.2). The inverse $\bar{B}^{-1}$ is an increasing Lipschitz continuous function with Lipschitz constant $\frac{1}{B_1}$ and $\bar{G}$ is Lipschitz continuous in its first and second argument with $G_1 = 0$ and $G_2 = 1$. For every $B_1 > 1$, the inequality $\frac{G_1+G_2}{B_1} < 1$ holds, which implies that there is a unique solution of this Stefan problem. This Stefan problem describes an etching technique where the etching properties are constant. For $B_1 = 2$ the numerically calculated iterates are given in [Va; p. 42, Figure 1]. Note that the iterates converge and alternate. It can be proved

for this example that the iterates form an alternating sequence (see [Va; p. 30, Theorem 4.11 i)]).

2. Define $\bar{B}(x) = \begin{cases} 2x & , x \in [0,0.02) \\ [0.04,0.07] & , x = 0.02 \\ 0.07 + 3(x-0.02), & x \in (0.02,\infty) \end{cases}$, (see Figure 1 for the graph of $\bar{B}$) and $\bar{G}(S,f,t)=\begin{cases} f(t) & , t \in [0,\frac{1}{2}] \\ \frac{3}{2} f(t) - \frac{1}{2} f(\frac{1}{2}), & t \in (\frac{1}{2},1] \end{cases}$ ([Va; p.31, Example 1]).

Using an etching problem the following interpretation of $\bar{B}$ and $\bar{G}$ may be given (where, in fact, $\bar{B}$ describes the properties of the solid whereas $\bar{G}$ describes the properties of the etching agent).

Interpretation of $\bar{B}$ From the definition of $\bar{B}$ it follows that the solid in $[0,\infty)$ can be divided into three parts. In the first part $[0,0.02)$ the proportionality constant B_1 is 2 whereas in the third part $(0.02,\infty)$ B_1 equals 3. This means that the solid in $[0,0.02)$ has a larger etching rate than the solid in $(0.02,\infty)$. These parts are separated by the second part in 0.02. The solid in 0.02 can be seen as a limit situation: there is a finite amount of solid in an infinitely thin layer (in a heat problem this phenomenon is known as a "heat capacity").

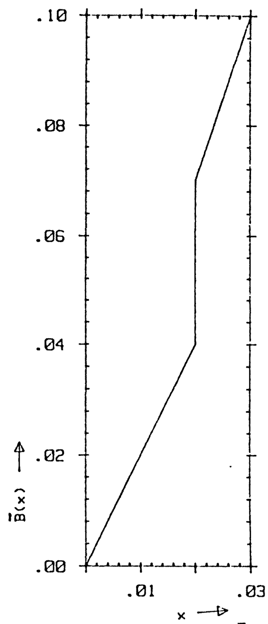

Figure 1. The multifunction $\bar{B}$.

Interpretation of $\bar{G}$ At $t = \frac{1}{2}$ the proportionality constant jumps from 1 to $\frac{3}{2}$. A reason can be that before $t = \frac{1}{2}$ there is a different chemical reaction at the interface S then after $t = \frac{1}{2}$.

The numerically calculated iterates are given in [Va; p. 43, Figure 2]. The iterates converge and alternate. Assuming that the third iterate is a good approximation of the solution we compare its properties with the given interpretation of $\bar{B}$ and $\bar{G}$. In $[0,0.02)$ the interface moves with a certain speed. In 0.02 the boundary halts until the infinitely thin layer is etched away. Thereafter the boundary moves again but with a lower speed than in $[0,0.02)$. At $t = \frac{1}{2}$ we observe a sudden change in speed

$(\frac{dS}{dt}(\frac{1}{2}+h) \simeq \frac{3}{2}\frac{dS}{dt}(t-h))$. Note that these properties show a close correspondence with the given interpretations. Further we note that the function $t \mapsto S(t)$ is not differentiable. Thus the existence results shown in the literature (eq. [Fasano & Primicerio, 1977]) can not be applied.

3. As a final application of Chapter 1 we consider the following "optimal control" problem. Suppose a layer of ice of 0° is immersed in hot water. After a given time the ice should be melted. If the heat of the water is insufficient to achieve this result, then one uses a heat source with a given strength. To minimize energy costs the heat source is turned on as late as possible.

A mathematical model for this application is given by Problem 4 if $\bar{B}$ and $\bar{G}$ are defined as follows: $\bar{B}(x) = 2x$ and

$$\bar{G}(S,f,t) = f(t) + \int_0^t g(f,\tau)d\tau,\ t \in [0,1] \text{ where}$$

$$g(f,t) = \begin{cases} 0 & ,\ t \in [0, 1 + (f(1) - 0.48)/0.75\,] \\ 0.75 & ,\ t \in (1 + (f(1) - 0.48)/0.75,\ \infty) \end{cases}$$ ([Va; p. 32, Example 3]).

In this model, C is the temperature of the water and S(t) is the interface between ice and water.

<u>Interpretation of $\bar{B}$</u> $\bar{B}$ describes the amount of heat needed to melt a certain amount of ice.

<u>Interpretation of $\bar{G}$</u> The first term in $\bar{G}$ models the heat released from the water up to time t, whereas the second term models the heat obtained from the heat source. The quantities in the functional g can be interpreted as follows: the thickness of the ice layer is 0.24, the amount of heat needed to melt this layer is $0.48 = \bar{B}(0.24)$, the ice should have melted at $t = 1$ and the strength of the heat source equals 0.75. We note that the heat source is not turned on ($g(f,t) = 0,\ t \in [0,1]$) if the amount of heat ($f(1)$) released from the water at $t = 1$ is greater than the heat ($\bar{B}(0.24)$) which is needed to melt the ice layer. On the other hand if $f(1) \leq \bar{B}(0.24)$ then the heat source is turned on at $t_1 = 1 + (f(1) - 0.48)/0.75$. In this case suppose $\{\bar{S}, C_{\bar{S}}\}$ is the solution of Problem 4. It now follows from $\bar{G}(\bar{S},f,t) \in \bar{B}(\bar{S}(t)),\ t \in [0,1]$ and

$\bar{G}(\bar{S},f,1) = f(1) + \int_{t_1}^{1} 0.75\, d\tau = f(1) - (f(1)\text{-}0.48) = 0.48$ that $\bar{S}(1) = \frac{1}{2}\bar{G}(\bar{S},f,1) = 0.24$.

So the ice layer will have melted just in time.

Remark that for this choice of $\bar{G}$, $\bar{G}(S,f,t)$ depends on $f(1)$ for every $t \in [0,1]$. Hence this application can not be described with Problem 3.

The numerically calculated iterates are given in [Va; p. 45, Figure 4]. The iterates converge but do not alternate. Assuming that the third iterate is a good approximation of the solution $\bar{S}$ we note that $\bar{S}(1) = 0.24$. Furthermore the calculation yields that the heat source should be turned on at $t_1 = 0.84$. As a result of this the slope of the graph of S shows a sudden increase at $t_1 = 0.84$.

§ 3.3 Chapter 2

In this chapter we assume in addition that $\bar{G}$ is an increasing Lipschitz continuous function in its third argument and that $\bar{G}(S,f,t)$ only depends on $S|_{[0,t]}$ and $f|_{[0,t]}$. An interpretation of this condition is that $\bar{G}(S,f,t)$ depends on the history and the present. This suggests that in many physical applications the condition is satisfied.

In Chapter 1 we prove existence and uniqueness if the inequality $\frac{G_1+G_2}{B_1} < 1$ holds. In many numerical experiments however we observe that the iterates converge even if $\frac{G_1+G_2}{B_1} > 1$. So we expect that existence and uniqueness can be proved under a weaker condition on B_1, G_1 and G_2. We first note that weakened conditions on B_1, G_1 and G_2 may not permit that $\frac{G_1}{B_1}$ could be greater than or equal to one since, as appears from the following example in that case uniqueness can not, in general, be shown. Define $\bar{B}(x) = x$ and $\bar{G}(S,f,t) = S(t)$, $t \in [0,1]$. Since $B_1 = 1$, $G_1 = 1$ and $G_2 = 0$ it follows that $\frac{G_1}{B_1} = 1$. However every function S satisfies $\bar{G}(S,f,t) \in \bar{B}(S(t))$, so uniqueness is lost. Secondly we note that in most numerical experiments iterates converge faster for small t than for large t (compare [Va; p. 42, Figure 1]). This observation leads to the following strategy: prove existence for small t and repeat this argument until existence is proved on [0,T]. This explains the extra condition on $\bar{G}$ because restricting the problem to $[0,t_1]$ with $t_1 < T$ is only possible if the problem does not depend on $S|_{(t_1,T]}$ or $f_{(t_1,T]}$.

In this chapter we prove existence and uniqueness assuming that the extra conditions on $\bar{G}$ and the inequality $\frac{G_1}{B_1} < 1$ hold.

In the existence proof the interval [0,T] is divided into subintervals with length h. It is proved that

$$\sup_{t\in[0,h]} |\mathcal{T}(S_1)(t) - \mathcal{T}(S_2)(t)| \leq \frac{G_1+2G_2\ L\sqrt{\frac{h}{\pi}}}{B_1} \sup_{t\in[0,h]} |S_1(t) - S_2(t)|,$$

where L is a Lipschitz constant of C_0. This estimate depends on h. For small h this estimate is better than the estimate given in Chapter 1. In [Vb; p. 25, Corollary 5.6] it is shown that this estimate is optimal for small h. Since $\frac{G_1}{B_1} < 1$ we can take h sufficiently small such that Banach's theorem implies that there is a unique function $\bar{S}$ which satisfies the equation $\bar{S}(t) = \mathcal{T}(\bar{S}(t))$, $t \in [0,h]$. Repeating this procedure on every subinterval we prove existence and uniqueness of the solution of Problem 4. The given numerical solution method is equivalent to the method given in Chapter 1.

Applications

1. Define $\bar{B}(x) = B_1x$ and $\bar{G}(S,f,t) = f(t)$, $t \in [0,1]$ (compare § 3.2, Application 1 and [Vb; p. 20, Remark 5.2i)]). Note that $\bar{G}(S,f,t)$ only depends on f(t) so the extra conditions on $\bar{G}$ are also satisfied. Since $G_1 = 0$ and $G_2 = 1$ it follows that for every $B_1 > 0$ there is a unique solution. For $B_1 = 0.25$, $L = 0.25$ and $L = 2$ the numerically calculated iterates are given in [Vb; p. 34 and 35, Figures 2 and 3]. The figures suggest that the rate of convergence depends on L. This corresponds to the fact that the estimate also depends on L, but it is in contrast with Chapter 1 where the rate of convergence only depends on B_1. Another illustration of the different convergence behaviour for $B_1 > 1$ and $B_1 \leq 1$ is the example given in [Vb; § 5.4, p. 27,..,31]. In this example we take $C_0(x) = 1$, $x \in (-\infty,0]$. This function C_0 does not satisfy our conditions, however C_0 can be seen as the limit function of the sequence $\{C_0^n\}_{n \geq 1}$ where the functions C_0^n are given by $C_0^n(x) = \min\{1,-nx\}$, $x \in (-\infty,0]$. For these functions our conditions are satisfied. For the initial function C_0 we observe a fast convergence of the iterates for $B_1 = 10$, whereas for $B_1 = 0.28$ the iterates are divergent. To explain this we note that for $B_1 = 0.28$ the rate of convergence for every C_0^n depends on $L_n = n$. Since L_n goes to infinity for $n \to \infty$ we expect that the rate of convergence goes to zero. This corresponds to the fact that for $B_1 = 0.28$ the iterates are divergent. On the other hand for $B_1 = 10$ the rate of convergence does not depend on L_n. So we are not surprised that for this C_0 and $B_1 = 10$ the iterates converge.

2. In [Vb; p. 21, Example 3] we show that the results of Chapter 1 are not contained in Chapter 2.

3. In [Vb; p. 20, Remark 5.2 ii)] we compare our results with the results given in [Fasano & Primicerio, 1977]. It appears that the class of Stefan conditions considered in [Fasano & Primicerio, 1977] is a subclass of the Stefan conditions considered in our work. Furthermore it appears that in our results the smoothness conditions are weaker.

§ 3.4 Chapter 3

In this chapter we prove existence and uniqueness of the solution of a Stefan problem without the condition that $\bar{G}$ is an increasing function of its third argument. This has consequences for both the existence proof and the applications. A typical application is the freezing of supercooled water. It is known that this physical phenomenon can be instable. There are examples where the speed of the interface (S(t)) between ice and water goes to infinity. This is known as "blow up" of the speed of S. We impose an additional condition on $\bar{G}$ (compare [Vc; p. 5, Condition 2.3]) such that "blow up" does not occur. An interpretation of this condition is: the function $t \mapsto \bar{G}(S,f,t)$ should be Lipschitz continuous for every S and f which are element of a certain function class.

The main part of Chapter 3 consists of the proof that if the conditions are satisfied then there is no "blow up". Furthermore the estimates given in the Chapters 1 and 2 are adapted. Using these estimates we prove existence and uniqueness of a solution of Problem 4 in a constructive way. The existence proof and the numerical solution method show a strong resemblance with the proofs and solution methods given in the Chapters 1 and 2.

Applications

1. Characteristic phenomena that can be described with a Stefan problem are: the solidification of a liquid with a temperature below its melting temperature (as we already noticed), and the formation of a crystal from a supersaturated solution. Problem 4 with $\bar{B}(x) = B_1x$ and $\bar{G}(S,f,t) = -f(t)$, $t \in [0,1]$ is a mathematical model for crystal growth [Vc; p. 29, Remark 6.2 i)]. In this application, C is the concentration of the solute, S is the interface between the liquid and the crystal and B_1 is the ratio between the loss of solute and the growth of the crystal. Assuming that $B_1 > 1$ our conditions are satisfied and existence and uniqueness follow from our existence theorem. For $B_1 = 2$ the numerically calculated iterates are given in [Vc; p. 37, Figure 2]. Note that the iterates converge and are monotone decreasing. It can be proved for this example that the iterates form a monotone decreasing sequence (see [Va; p. 30, Theorem 4.11 i)]). Furthermore we show that if $B_1 < 1$ then after some time "blow up" occurs [Vc; p. 29, Example 2].

Note that in the first applications in Sections 3.2, 3.3 and 3.4 we consider the same Stefan condition $\int_{-\infty}^{\infty} [C_0(x) - C(x,t)]dx = B\, S(t)$, $t \in [0,T]$

for different values of B. In Chapter 1 we prove existence assuming that $B > 1$. In Chapter 2 this condition is weakened and we show existence for $B > 0$. Finally in Chapter 3 we prove existence for $B < -1$ and show with an example that if $-1 < B < 0$ then for T large enough the Stefan problem has no solution (i.e. the speed of S blows up).

2. Define $\bar{B}(x) = 2x$ and $\bar{G}(S,f,t) = \begin{cases} -f(t) & , t \in [0,0.25] \\ -2f(0.25) + f(t), & t \in (0.25,1] \end{cases}$.

A possible interpretation is: a solute precipitates until $t = 0.25$. Thereafter the solute behaves itself as an etching agent. The numerically calculated iterates are given in [Vc; p. 38, Figure 3]. The iterates are convergent. Note that the iterates do not form an alternating nor a monotone sequence. Assuming that the third iterate is a good approximation of the solution it follows that the crystal increases for $t \in [0,0.25]$ and decreases for $t \in (0.25,1]$. In $t = 0.25$ the function S is not differentiable.

3. In [Vc; p. 28, Remark 6.2 ii)] we compare our results with the results given in [Fasano & Primicerio, 1977]. They prove existence and uniqueness for T small enough without a condition which implies that "blow up" does not occur. An advantage is that they can also prove existence for problems where after some time "blow up" does occur. A disadvantage is, however, that even if the solution exists for every $T > 0$ the existence proof holds for T small enough only.

4. In [Vc; p. 30, Example 3] we show that the results of Chapters 1 and 2 are not contained in Chapter 3.

Literature

1. Cannon, J.R.
The one–dimensional heat equation.
Addison Wesley, Reading. (1984)

2. Douglas, J.
A uniqueness theorem for for the solution of a Stefan problem.
Proc. Amer. Math. Soc. 8 402–408. (1957)

3. Evans, G.W.
A note on the existence of a solution to a problem of Stefan.
Q. Appl. Math. 9 185–193. (1951)

4. Fasano, A., Primicerio, M.
General free boundary problems for the heat equation I.
J. Math. Anal. Appl. 57 694–723. (1977)

5. Kyner, W.T.
An existence and uniqueness theorem for a nonlinear Stefan problem.
J. Math. Mech. 8 483–498 (1959)

6. Notten, P.H.L.
Electrochemical study of the etching of III–V semiconductors.
Thesis, Eindhoven University of Technology. (1989)

7. Rubinstein, L.I.
The Stefan Problem.
American Mathematical Society, Providence. (1971)

8. Stefan, J.
Über die Theorie der Eisbildung, ins besondere über die Eisbildung im Polarmeere.
Ann. Physik. Chemie 42 269–286 (1891)

Va Vuik, C. (Chapter 1 of this monograph)
The solution of a one–dimensional Stefan problem I
Preprint 491 State University Utrecht (1987)

Vb Vuik, C. (Chapter 2 of this monograph)
The solution of a one–dimensional Stefan problem II
Preprint 493 State University Utrecht (1988)

Vc Vuik, C. (Chapter 3 of this monograph)
The solution of a one–dimensional Stefan problem III
Preprint 505 State University Utrecht (1988)

Chapter 1

The solution of a one-dimensional Stefan problem I

1. Introduction

This work was motivated by an etching technique which is used for the production of microelectronic devices. A mathematical model of this etching technique is equivalent to a Stefan problem ([Kuiken,1984], [Vuik & Cuvelier, 1985]). This model is the subject of this paper.

Let $x \in \mathbb{R}$ be the space co-ordinate and t the time co-ordinate. For the time $t=0$ we suppose the region $x \geq 0$ to consist of a solid and the region $x < 0$ to be filled by an etching liquid. We denote by $S(t)$ the position of the time-dependent boundary between solid and liquid. The function C describes the concentration of the etching agent.

In dimensionless form the problem is: given $T > 0$, $B > 0$ and $C_0:(-\infty,0]\to[0,1]$, find sufficiently smooth functions $S:[0,T]\to\mathbb{R}$ and $C:\{(x,t)\,|\,t\in[0,T],\ x\in(-\infty,S(t)]\}\to\mathbb{R}$ such that

$$(1.1)\quad \begin{cases} \dfrac{\partial C(x,t)}{\partial t} - \dfrac{\partial^2 C(x,t)}{\partial x^2} = 0 & \text{for } x\in(-\infty,S(t)),t\in(0,T], \\ C(x,0) = C_0(x) & \text{for } x\in(-\infty,0], \\ C(S(t),t) = 0 & \text{for } t\in[0,T], \end{cases}$$

$$(1.2)\quad -\frac{1}{B}\int_{-\infty}^{\infty} [C(x,t)-C_0(x)]dx = S(t) \text{ for } t\in[0,T].$$

Remark 1.3. For easy notation, we define $C(x,t)=0$ for $x\in(S(t),\infty),t\in[0,T]$ and $C_0(x)=0$ for $x\in(0,\infty)$.

Physical interpretation

The transport of the etching agent is described by the diffusion equation. The concentration for $t=0$ is a given function C_0 with $0\leq C_0(x)\leq 1$ for $x\in(-\infty,0]$. The condition $C(S(t),t)=0$ for $t\in[0,T]$ corresponds to a fast chemical reaction on the moving boundary (see[Kuiken, 1984],[Vuik & Cuvelier, 1985]). The boundary condition (1.2) is the Stefan condition in integral form. This condition reflects the assumption that the loss of etching agent is proportional to the loss of solid. The proportionality constant B is given by the chemical properties of the etching agent and the solid. In practice, $B>>1$ for an etching problem, see [Kuiken,Kelly,Notten, 1986; p.1220, Table 1], where this quantity is called β, which in all their examples is at least 100. We use the property $B>1$ in our existence theorem.

Existence and uniqueness results of the solution for a one-dimensional Stefan problem are reported in [Evans, 1951],[Douglas, 1957], [Kyner, 1959], [Friedman, 1964], [Rubinstein, 1971], [Cannon & Primicerio, 1971], [Hill & Kotlow,1972] and [Fasano & Primicerio, 1977].

This paper is organized as follows. In Section 2 we give some definitions and basic lemmas. The section also includes known results about the solution of the diffusion equation.

In Section 3 we specify the Stefan problem. The problem has the following features. First of all, the problem is defined on an unbounded one-dimensional space-domain. Secondly, the Stefan condition (1.2) on the time-dependent boundary is given in a generalized form.

Several existence theorems for the solution of the diffusion equation are given in Section 4. Also in Section 4, we compare the losses of etching agent for two different time-dependent boundaries (Lemma 4.2, Theorem 4.11). Finally we prove some properties of the loss of etching agent seen as a function of time.

In Section 5, Theorem 5.11 we prove by a contraction argument the existence and uniqueness of a solution for the Stefan problem where $S\in C[0,T]$ and the function $C\in C^{2,1}(Q_S)\cap C(\bar{Q}_S)$ is bounded.

In section 6 we give some examples of the Stefan problem specified in Section 3. Furthermore, we compute numerical solutions for these examples.

2. Preliminaries

In this section we summarize known results about the solution of the diffusion equation. After that we give some basic lemmas.

2.1 Definitions

For a given $T>0$ we define the following function spaces:

$$O=\{S\in C[0,T]\,|\,S(0)=0\},$$

$$P=\{S\in O\,|\,S \text{ is monotone non-decreasing}\},$$

and for $K>0$, $\bar{M}_K=\{S\in P\,|\,S(t+h)-S(t)\le Kh,\, h\ge 0;\, t+h,t\in[0,T]\}$,

$$M_K=\bar{M}_K\cap C^2[0,T].$$

We use the norm $\|S\|_\infty=\sup_{t\in[0,T]}|S(t)|$ on these spaces. At the end of this section, Corollary 2.8 states that $\bar{M}_K$ is the closure of M_K in $C[0,T]$ with respect to the ∞-norm.

For a given function $S\in C[0,T]$ the set $Q_S\subset R^2$ is defined by $Q_S=\{(x,t)\,|\,x\in(-\infty,S(t)),\ t\in(0,T)\}$. For a given constant R such that $-R<\min_{t\in[0,T]}S(t)$ we define the set $Q_S^R\subset R^2$ by $Q_S^R=\{(x,t)\,|\,x\in(-R,S(t)),t\in(0,T)\}$. The closure of a set $Q\subset R^2$ is denoted by $\bar{Q}$.

2.2 The function spaces $C^{\ell,\ell/2}(\bar{Q}_S)$, $C^{2,1}(Q_S)$ and $C^\ell[a,b]$

We use the following function spaces defined in [Ladyženskaja, Solonnikov, Ural'ceva, 1968; p.7]. For a given $\ell\in R^+\backslash N$, $C^{\ell,\ell/2}(\bar{Q}_S)$ is the Banach space of continuous functions f on $\bar{Q}_S$, having continuous derivatives $\frac{\partial^{r+p}f}{\partial t^r\partial x^p}$ for $2r+p<\ell$ and a finite norm $\|f\|^{\ell,\ell/2}$. Here the norm $\|f\|^{\ell,\ell/2}$ is defined by: let $[\ell]$ be the largest integer less than ℓ,

$$\|f\|^{\ell,\ell/2}=\sum_{j=0}^{[\ell]}\left(\sum_{2r+p=j}\max_{\bar{Q}_S}\left|\frac{\partial^{r+p}f}{\partial t^r\partial x^p}\right|\right)+\sum_{2r+p=[\ell]}\left\langle\frac{\partial^{r+p}f}{\partial t^r\partial x^p}\right\rangle_x^{(\ell-[\ell])}+$$

$$+\sum_{0<\ell-2r-p<2}\left\langle\frac{\partial^{r+p}f}{\partial t^r\partial x^p}\right\rangle_t^{\frac{\ell-2r-p}{2}},$$

and $\langle f\rangle_x^\alpha=\sup\left\{\frac{|f(x',t)-f(x'',t)|}{|x'-x''|^\alpha}\ \middle|\ (x',t),(x'',t)\in\bar{Q}_S;\,|x'-x''|\le 1\right\}$,

$$<f>_t^\alpha = \sup\left\{ \frac{|f(x,t')-f(x,t'')|}{|t'-t''|^\alpha} \;\middle|\; (x,t'),(x,t'')\in\bar{Q}_S; |t'-t''|\le 1 \right\} \text{ for } \alpha\in(0,1).$$

$C^{2,1}(Q_S)$ is the set of continuous functions f on Q_S, having continuous derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial f}{\partial t}$.

For a given $\ell\in\mathbb{R}^+\backslash\mathbb{N}$ and $[a,b]\subset\mathbb{R}$, $C^\ell[a,b]$ is the Banach space of continuous functions f on [a,b], having continuous derivatives $\frac{d^p f}{dx^p}$ for $p<\ell$, and a finite norm $\|f\|^\ell$. Here the norm $\|f\|^\ell$ is defined by:

$$\|f\|^\ell = \sum_{j=0}^{[\ell]} \max_{[a,b]} \left|\frac{d^j f}{dx^j}\right| + <\frac{d^{[\ell]}f}{dx^{[\ell]}}>^{(\ell-[\ell])} \text{ and}$$

$$<f>^\alpha = \sup\left\{ \frac{|f(x')-f(x'')|}{|x'-x''|^\alpha} \;\middle|\; x',x''\in[a,b]; |x'-x''|\le 1 \right\} \text{ for } \alpha\in(0,1).$$

2.3 Existence theorems and a maximum principle

The following definitions are given in [Ladyženskaja e.a., 1968;p.317-320]. For $S_0(t)=0$, $t\in[0,T]$ we suppose that the functions a_2,a_1,a_0: $\bar{Q}_{S_0}\to\mathbb{R}$ are given and that there is a constant $a>0$ such that $a_2(x,t)\ge a$ for $(x,t)\in\bar{Q}_{S_0}$. The opera:or L is defined by

$$L(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}) = \frac{\partial u}{\partial t} - a_2(x,t)\frac{\partial^2 u}{\partial x^2} + a_1(x,t)\frac{\partial u}{\partial x} + a_0(x,t)u.$$

The following equations are considered:

$$(2.1)\quad \begin{cases} L(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t})u(x,t)=f(x,t), x\in(-\infty,0) t\in(0,T] , \\ u(x,0)=\varphi(x), x\in(-\infty,0],\; u(0,t)=\Phi(t),\; t\in[0,T]. \end{cases}$$

To obtain a smooth solution u of these equations, it is necessary that L,f,φ and Φ satisfy certain compatibility conditions in (0,0). To this end we introduce the operator

$$A(x,t,\frac{\partial}{\partial x})u = a_2(x,t)\frac{\partial^2 u}{\partial x^2} - a_1(x,t)\frac{\partial u}{\partial x} - a_0(x,t)u$$

and define the function $u^{(0)}(x)=\varphi(x), x\in(-\infty,0]$. Furthermore, we define the functions $u^{(k)}:(-\infty,0]\to\mathbb{R}, k=1,2,..$ by the following recursion:

$$u^{(k+1)}(x) = \sum_{j=0}^{k}\binom{k}{j}A^{(j)}(x,0,\frac{\partial}{\partial x})u^{(k-j)}(x) + \frac{\partial^k f(x,t)}{\partial t^k}\Big|_{t=0} ,$$

where $A^{(j)}(x,t,\frac{\partial}{\partial x})u = \frac{\partial^j a_2(x,t)}{\partial t^j}\frac{\partial^2 u}{\partial x^2} - \frac{\partial^j a_1(x,t)}{\partial t^j}\frac{\partial u}{\partial x} - \frac{\partial^j a_0(x,t)}{\partial t^j}u.$

We say that compatibility conditions of order $m\ge 0$ are fulfilled if

$u^{(k)}(0) = \frac{d^k\Phi}{dt^k}(0)$ for $k=0,1,..,m$.

Theorem 2.2 [Ladyženskaja e.a., 1968; p.320, Theorem 5.2].
Suppose $\ell \in \mathbb{R}^+\backslash\mathbb{N}$ is given and the coefficients of the operator L belong to $C^{\ell,\ell/2}(\bar{Q}_{S_0})$. Then for any $f\in C^{\ell,\ell/2}(\bar{Q}_{S_0})$, $\varphi\in C^{\ell+2}(-\infty,0]$, $\Phi\in C^{\frac{\ell+2}{2}}[0,T]$ satisfying the compatibility conditions of order $[\ell/2]+1$ there is a unique bounded function $u\in C^{\ell+2,\frac{\ell+2}{2}}(\bar{Q}_{S_0})$ that satisfies the equations given in (2.1).

Lemma 2.3 [Friedman, 1964; p.80].
For $S\in C[0,T]$ and $-R < \min_{t\in[0,T]} S(t)$, suppose that u_m is a sequence of functions satisfying: $\frac{\partial u_m(x,t)}{\partial t} - \frac{\partial^2 u_m(x,t)}{\partial x^2} = 0$, $x\in(-R,S(t)), t\in(0,T]$, $u_m(x,0)=\varphi_m(x)$, $x\in[-R,S(0)]$, $u_m(-R,t)= \psi_m(t)$, $u_m(S(t),t)=\Phi_m(t)$, $t\in[0,T]$. If $\varphi_m\to\varphi$ uniformly on $[-R,S(0)]$ and $\psi_m\to\psi$, $\Phi_m\to\Phi$ uniformly on $[0,T]$, then the sequence u_m is uniformly convergent on $\bar{Q}_S^R$ to a function u, the derivatives $\frac{\partial u_m}{\partial x}$, $\frac{\partial^2 u_m}{\partial x^2}$ and $\frac{\partial u_m}{\partial t}$ converge uniformly on closed subsets of $\{(x,t)|x\in(-R,S(t)),t\in(0,T]\}$ to the corresponding derivatives of u and

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ x\in(-R,S(t)),\ t\in(0,T],$$

$u(x,0)=\varphi(x)$, $x\in[-R,S(0)]$, $u(-R,t)=\psi(t)$, $u(S(t),t)=\Phi(t)$, $t\in[0,T]$.

Lemma 2.4 (maximum principle).
Suppose $S\in C[0,T]$. If the bounded function $u\in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ satisfies:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ x\in(-\infty,S(t)),\ t\in(0,T],$$

then $\min\left\{ \inf_{x\in(-\infty,S(0)]} u(x,0),\ \min_{t\in[0,T]} u(S(t),t) \right\} \le u(\tilde{x},\tilde{t}) \le$

$\max\left\{ \sup_{x\in(-\infty,S(0)]} u(x,0),\ \max_{t\in[0,T]} u(S(t),t) \right\}$ for $(\tilde{x},\tilde{t})\in\bar{Q}_S$.

Proof. First of all we consider the right-hand inequality. Since u is bounded there is a constant β such that $|u(x,t)|\le\beta$, $(x,t)\in\bar{Q}_S$.

Define $\mu = \max\left\{ \sup_{x\in(-\infty,S(0)]} u(x,0),\ \max_{t\in[0,T]} u(S(t),t) \right\}$, $\sigma_1= \min_{t\in[0,T]} S(t)$

and $\sigma_2 = \max_{t\in[0,T]} S(t)$.

Suppose there is a point $(\xi,\tau)\in Q_S$ such that $u(\xi,\tau)>\mu$. Then we define an auxiliary function $\varphi:\mathbb{R}\to\mathbb{R}$ by

$\varphi(x)=\mu+\left(\frac{u(\xi,\tau)-\mu}{2(\xi-\sigma_2)}\right)(x-\sigma_2)$ and a constant $R=\min\left\{\sigma_1-1,\ \xi-1,\ \sigma_2+\frac{2(\xi-\sigma_2)(\beta-\mu)}{(u(\xi,\tau)-\mu)}\right\}$.

The function $h:\bar{Q}_S^R\to\mathbb{R}$ given by $h(x,t)=\varphi(x) - u(x,t)$ satisfies

$$\frac{\partial h(x,t)}{\partial t} - \frac{\partial^2 h(x,t)}{\partial x^2} = 0,\ x\in(R,S(t)),\ t\in(0,T],$$

$h(x,0)\geq 0,\ x\in[R,0],\ h(R,t)\geq 0,\ h(S(t),t)\geq 0,\ t\in[0,T]$.

The inequalities follow from the inequality $S(t)\leq\sigma_2$, $t\in[0,T]$ and the definition of φ and R.

According to the minimum principle for bounded domains (see [Protter & Weinberger, 1967; p.168, Theorem 2]) it follows that $h(x,t)\geq 0$, $(x,t)\in\bar{Q}_S^R$.

Since $(\xi,\tau)\in Q_S^R$ we obtain

$$0\leq h(\xi,\tau)=\mu+\frac{u(\xi,\tau)-\mu}{2(\xi-\sigma_2)}(\xi-\sigma_2)-u(\xi,\tau),$$

which implies $u(\xi,\tau)\leq\mu$. This is a contradiction.

The left-hand inequality can be derived from the right-hand inequality by using the function $-u$. ∎

<u>Lemma 2.5</u>

For $S\in C[0,T]$ suppose u_m is a sequence of bounded functions such that $u_m\in C^{2,1}(Q_S)\cap C(\bar{Q}_S)$ and

$$\frac{\partial u_m(x,t)}{\partial t} - \frac{\partial^2 u_m(x,t)}{\partial x^2} = 0,\ x\in(-\infty,S(t)),\ t\in(0,T],$$

$u_m(x,0) = \varphi_m(x),\ x\in(-\infty,S(0)],\ u_m(S(t),t)=\Phi_m(t),\ t\in[0,T]$.

If $\varphi_m\to\varphi$ uniformly on $(-\infty,S(0)]$ and $\Phi_m\to\Phi$ uniformly on $[0,T]$

then the function $u:\bar{Q}_S\to\mathbb{R}$ defined by $u(x,t)=\lim_{m\to\infty} u_m(x,t)$, $(x,t)\in\bar{Q}_S$

is bounded, $u\in C^{2,1}(Q_S)\cap C(\bar{Q}_S)$ and satisfies

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ x\in(-\infty,S(t)),\ t\in(0,T],$$

$u(x,0) = \varphi(x),\ x\in(-\infty,S(0)],\ u(S(t),t) = \Phi(t),\ t\in[0,T]$.

<u>Proof.</u> With the maximum principle we deduce

$$|u_m(x,t) - u_k(x,t)| \to 0 \text{ for } m,k\to\infty \text{ uniformly for } (x,t)\in\bar{Q}_S.$$

Thus, for an arbitrary constant R such that $-R < \min_{t\in[0,T]} S(t)$

$$\sup_{t\in[0,T]} |u_m(-R,t)-u_k(-R,t)| \to 0 \text{ for } m,k\to\infty .$$

Define $\psi(t)=\lim_{m\to\infty} u_m(-R,t)$, $t\in[0,T]$. Since $u_m\in C^{2,1}(Q_S^R)\cap C(\bar{Q}_S^R)$ we find from Lemma 2.3 that the function given by $u(x,t)=\lim_{m\to\infty} u_m(x,t)$, $(x,t)\in \bar{Q}_S^R$ is an element of $C^{2,1}(Q_S^R)\cap C(\bar{Q}_S^R)$ and satisfies

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ x\in(-R,S(t)),\ t\in(0,T],$$

$u(x,0) = \varphi(x)$, $x\in[-R,S(0)]$, $u(-R,t) = \psi(t)$, $u(S(t),t) = \Phi(t)$, $t\in[0,T]$.

Furthermore it follows that

$$|u(\tilde{x},\tilde{t})| \le \sup_m \left\{ \sup_{x\in(-\infty,0]} |\varphi_m(x)|,\ \max_{t\in[0,T]} |\Phi_m(t)| \right\},\ (\tilde{x},\tilde{t})\in\bar{Q}_S^R ,$$

thus u is bounded on $\bar{Q}^R_S$. This proves the lemma because R is arbitrary. ■

2.4 Approximation results

Lemma 2.6.

Suppose the function $\varphi:(-\infty,0]\to[0,1]$ is such that $\varphi(0)=0$, $\lim_{x\to-\infty}\varphi(x)=1$ and there is a constant $K > 0$ such that $-Kh \le \varphi(x+h) - \varphi(x)\le 0$, $h\ge 0$; $x+h \in(-\infty,0]$. Then there is a sequence of functions $\{\varphi_n\}_{n\ge 1}$ such that

i) $\varphi_n\in C^\infty(-\infty,0]$, $0 \le -\dfrac{d\varphi_n(x)}{dx} \le K$, $x\in(-\infty,0]$,

ii) $\varphi_{n-1}(x) \le \varphi_n(x) \le \varphi(x)$, $x\in(-\infty,0]$, $n=2,3,\dots$,

iii) $\lim_{n\to\infty} \int_{-\infty}^{0} [\varphi(x) - \varphi_n(x)]dx=0$, $\lim_{n\to\infty} \sup_{x\in(-\infty,0]} [\varphi(x)-\varphi_n(x)]=0$,

iv) $\varphi_n(0) = \dfrac{d\varphi_n}{dx}(0) = \dfrac{d^2\varphi_n}{dx^2}(0) = 0$ and $\lim_{x\to-\infty} \varphi_n(x) = 1$.

Furthermore there is a function $\varphi^+:(-\infty,0]\to[0,1]$ such that $\varphi^+\in C^\infty(-\infty,0]$, $\varphi(x)\le\varphi^+(x)$, $x\in(-\infty,0]$ and $\int_{-\infty}^{0} [\varphi^+(x)-\varphi(x)]dx < \infty$.

Proof. Suppose $g\in C^\infty(\mathbb{R})$ is a function with the following properties $g(x) = g(-x)\ge 0$, $x\in[-\frac{1}{2},\frac{1}{2}]$, $g(x)=0$, $x\in\mathbb{R}\setminus[-\frac{1}{2},\frac{1}{2}]$ and $\int_{-\infty}^{\infty} g(x)dx = 1$. Define $g_n:\mathbb{R}\to[0,\infty)$ by $g_n(x)=ng(nx)$, $x\in\mathbb{R}$, $\varphi(x)=0$, $x\in(0,\infty)$,

$\bar{\varphi}_n(x) = \varphi(x+\frac{1}{n})$, $x\in\mathbb{R}$ and $\varphi_n(x) = \int_{-\infty}^{\infty} \bar{\varphi}_n(\tau)g_n(x-\tau)d\tau$, $x\in(-\infty,0]$.

Since $\varphi:(-\infty,0]\to[0,1]$ the definition of φ_n yields $\varphi_n:(-\infty,0]\to[0,1]$.

i) It is easily seen that $\varphi_n \in C^\infty(-\infty,0]$. This together with

$$\varphi_n(x+h) - \varphi_n(x) = \int_{-\infty}^{\infty} [\bar{\varphi}_n(\tau+h) - \bar{\varphi}_n(\tau)] g_n(x-\tau)d\tau, \quad x, x+h \in (-\infty,0]$$

implies $-K \le \frac{d\varphi_n(x)}{dx} \le 0, \; x\in(-\infty,0]$.

ii) From the definition it follows that

$$\varphi_{n-1}(x) = \int_{-\infty}^{\infty} \varphi(\tau+\frac{1}{n-1})(n-1)g((n-1)(x-\tau))d\tau = \int_{-\infty}^{\infty} \varphi(\frac{1}{n-1}(y+1))g((n-1)x-y)dy,$$

$$\varphi_n(x) = \int_{-\infty}^{\infty} \varphi(\tau+\frac{1}{n})ng(n(x-\tau))d\tau = \int_{-\infty}^{\infty} \varphi(\frac{1}{n}(y+x+1))g((n-1)x-y)dy.$$

Since $g(x) \ge 0, x\in\mathbb{R}$, $g((n-1)x-y)=0$, $y \le (n-1)x-\frac{1}{2}$ and

$\varphi(\frac{1}{n-1}(y+1)) \le \varphi(\frac{1}{n}(y+x+1))$, $y \ge (n-1)x-1$ it follows

that $\varphi_{n-1}(x) \le \varphi_n(x)$, $x\in(-\infty,0]$. The inequality $\varphi_n(x) \le \varphi(x)$, $x\in(-\infty,0]$

holds because $g(n(x-\tau))=0$, $\tau \le x - \frac{1}{2n}$ and $\varphi(\tau+\frac{1}{n}) \le \varphi(x), \tau \ge x-\frac{1}{n}$.

iii) From the inequality $\varphi(x+\frac{2}{n}) \le \varphi_n(x) \le \varphi(x)$, $x\in(-\infty,0]$ we deduce

that

$$0 \le \int_{-\infty}^{0} [\varphi(x)-\varphi_n(x)]dx \le \lim_{N\to\infty} \int_{-N}^{0} [\varphi(x) - \varphi(x+\frac{2}{n})]dx$$

$$= \lim_{N\to\infty} \int_{-N-\frac{n}{2}}^{-N} \varphi(x+\frac{2}{n})dx = \frac{2}{n}, \quad \text{thus } \lim_{n\to\infty} \int_{-\infty}^{0} [\varphi(x)-\varphi_n(x)]dx = 0.$$

Since φ is Lipschitz-continuous it follows that

$\varphi(x) - \frac{2K}{n} \le \varphi(x+\frac{2}{n}) \le \varphi_n(x) \le \varphi(x)$, $x\in(-\infty,0]$ thus

$\lim_{n\to\infty} \sup_{x\in(-\infty,0]} [\varphi(x)-\varphi_n(x)]=0$.

iv) It is easily seen that $\varphi_n(x)=0$, $x\in[-\frac{1}{2n},0]$. This combined with

$\varphi_n \in C^\infty(-\infty,0]$ yields $\varphi_n(0) = \frac{d\varphi_n}{dx}(0) = \frac{d^2\varphi_n}{dx^2}(0)=0$. The inequality

$\varphi(x+\frac{2}{n}) \le \varphi_n(x) \le \varphi(x), x\in(-\infty,0]$ implies $\lim_{x\to-\infty} \varphi_n(x)=1$.

The function $\varphi^+:(-\infty,0]\to[0,1]$ defined by $\varphi^+(x) = \int_{-\infty}^{\infty} \varphi(\tau-1)g(x-\tau)d\tau$

has the required properties. ∎

Lemma 2.7.

i) For $\epsilon>0$ and $S\in P$ there is a function $S^- \in P\cap C^2[0,T]$ such that $S^-(t) \le S(t)$, $t\in[0,T]$ and $\|S-S^-\|_\infty < \epsilon$.

ii) For $\epsilon>0$, $K>0$ and $S\in\bar{M}_K$, there are functions $S^+, S^- \in M_K$ such that $S^-(t) \le S(t) \le S^+(t)$, $t\in[0,T]$ and $\|S^+-S^-\|_\infty < 2\epsilon$.

i) Proof. From $S\in C[0,T]$ we know S is uniformly continuous. Hence, for $\epsilon>0$ there is an $N\in\mathbb{N}$ such that if $|t-\tau|<\frac{T}{N}$ and $t,\tau\in[0,T]$ then $|S(t)-S(\tau)|<\frac{\epsilon}{4}$. Take $h=T/N$ and $t_i=ih$ for $i=0,\dots,N$. Define the function $\hat{S}:[0,T]\to\mathbb{R}$ as follows: $\hat{S}\in C[0,T]$, $\hat{S}(t_i)=S(t_i)$, $i=0,\dots,N$ and $\hat{S}$ is a linear function on $[t_i,t_{i+1}]$, $i=0,\dots,N-1$. It is easily seen that $\|\hat{S}-S\|_\infty < \frac{\epsilon}{4}$.

The function $\bar{S}:[0,T]\to\mathbb{R}$ is given by $\bar{S}(t)=\max\{0,\hat{S}(t)-\frac{\epsilon}{2}\}$, $t\in[0,T]$. There exists a strictly increasing sequence $\tau_i\in[0,T]$, $i=0,\dots,m+1$ with $m\le N$ such that, $\tau_0=0$, $\tau_{m+1}=T$ and $\bar{S}\in C^2([0,T]\setminus\bigcup_{i=1}^{m}\{\tau_i\})$.

From the definition of $\bar{S}$ we obtain: for $i=0,\dots,m$ there are $a_i,b_i\in\mathbb{R}$ such that $\bar{S}(t)=a_it+b_i$, $t\in[\tau_i,\tau_{i+1}]$ and $0\le a_i\le\frac{\epsilon}{4h}$.

For $\mu\in(0,\frac{1}{4}\min_{i=0,\dots,m}(\tau_{i+1}-\tau_i)]$ the function $S^-:[0,T]\to\mathbb{R}$ is defined by:

$$S^-(t)=\begin{cases}\bar{S}(\tau_i-\mu)+(t-(\tau_i-\mu))\left(\frac{a_{i-1}+a_i}{2}\right)-\frac{2\mu}{\pi}\left(\frac{a_i-a_{i-1}}{2}\right)\cos\left((t-\tau_i)\frac{\pi}{2\mu}\right), & t\in[\tau_i-\mu,\tau_i+\mu],\ i=1,\dots,m\\ \bar{S}(t)\ , & t\in[0,T]\setminus\bigcup_{i=1}^{m}[\tau_i-\mu,\tau_i+\mu].\end{cases}$$

For μ small enough the function S^- has the required properties.

ii) The function S^- is constructed in the same way as in part i). Since S is Lipschitz continuous the inequalities $0\le a_i\le K$ hold for $i=0,\dots,m$. This together with part i) implies $S^-\in M_K$.

We use the auxiliary function $\varphi:[0,T]\to\mathbb{R}$ given by $\varphi(t)=Kt-S(t)$ for $t\in[0,T]$. From the above it is clear that there is a function $\varphi^-\in M_K$ such that $\varphi^-(t)\le\varphi(t)\le\varphi^-(t)+\epsilon$, $t\in[0,T]$. Define $S^+\in M_K$ by $S^+(t)=Kt-\varphi^-(t)$, $t\in[0,T]$. It is easily seen that $S^+(t)\ge S(t)\ge S^+(t)-\epsilon$, $t\in[0,T]$. ⊠

Corollary 2.8. $\bar{M}_K$ is the closure of M_K in $C[0,T]$ with respect to the ∞-norm.

3. Statement of the problem

In this section we specify the Stefan problem. This will appear to be a more general problem than the problem mentioned in Section 1.

We shall always impose Condition 3.1:

Condition 3.1.

The function C_0 in (1.1) should be an element of the set Cond.3.1:= $\{\varphi:(-\infty,0]\to\mathbb{R}\,|\,\varphi$ is a monotone decreasing Lipschitz continuous function with $\varphi(0)=0$ and $\lim_{x\to-\infty}\varphi(x)=1\}$.

Let L be a Lipschitz constant of the function C_0.

Occasionally we shall impose the following stronger condition:

Condition 3.2.

The function C_0 in (1.1) should be an element of the set Cond 3.2:=

$\{\varphi\in C^{3+\alpha}(-\infty,0]$ for an $\alpha\in(0,1)\,|\,\varphi\in$Cond 3.1 and $\frac{d\varphi}{dx}(0) = \frac{d^2\varphi}{dx^2}(0)=0\}$.

In the sequel, $\bar{B}:[0,\infty)\to[0,\infty)$ will denote a multifunction (see [Smithson,1972]) and, $\bar{G}:P\times\bar{M}_L\times[0,T]\to[0,\infty)$ will denote a functional, both subject to the following condition, which will always be imposed:

Condition 3.3.

The multifunction $\bar{B}$ should be surjective and there should be a $B\in(0,\infty)$ so that for $x\in[0,\infty)$, $h>0$, $y_1\in\bar{B}(x)$ and $y_2\in\bar{B}(x+h)$ the inequality $y_2-y_1\geq Bh$ holds.

The functional $\bar{G}$ should be such that:

i) $\bar{G}(S,f,0)=0$ for $S\in P$, $f\in\bar{M}_L$.

ii) There is a $\gamma\in\mathbb{R}$ such that $0\leq\bar{G}(S,f,t+h)-\bar{G}(S,f,t)\leq\gamma h$ for $h\geq0$; $t+h$, $t\in[0,T]$, $S\in P$ and $f\in\bar{M}_L$.

iii) There are constants $G_1,G_2\in\mathbb{R}$ so that

$$\sup_{t\in[0,T]}|\bar{G}(S_1,f,t)-\bar{G}(S_2,f,t)| \leq G_1\,\|S_1 - S_2\|_\infty\ ,S_1,S_2\in\bar{M}_{\gamma/B},\ f\in\bar{M}_L,$$

$$\sup_{t\in[0,T]}|\bar{G}(S,f_1,t) - \bar{G}(S,f_2,t)| \leq G_2\,\|f_1-f_2\|_\infty,\ S\in\bar{M}_{\gamma/B},\ f_1,f_2\in\bar{M}_L.$$

In order to state a Stefan problem, we first state a reduced problem: for a given function $S\in O$, find a bounded function $C_S \in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ that satisfies the equations in (1.1) for this function S.

For any $S\in O$, for which the reduced problem has a solution C_S, we define $C_S(x,t)=0$, $x>S(t)$, $t\in[0,T]$. Let the function $f_S:[0,T]\to\mathbb{R}$ be defined by $f_S(t)= -\int_{-\infty}^{\infty} [C_S(x,t) - C_0(x)]dx$.

The Stefan problem can be stated as follows: find a function $S\in P$ and a solution C_S of the reduced problem such that

(3.4) $\quad \tilde{G}(S,f_S,t) \in \tilde{B}(S(t))$, $t\in[0,T]$.

In Section 6 we give some examples of the Stefan problem defined above.

Definition 3.5.

We define the function spaces $M=M\gamma/B$ and $\tilde{M}=\tilde{M}\gamma/B$.

4. Properties of the reduced problem

In this section we study the properties of solutions of the reduced problem. These properties are used in Section 5 to prove an existence theorem for the solution of the Stefan problem. Some properties are stated for their own sake.

4.1 Existence theorem for C_S if $S\in C^2[0,T]$ and Condition 3.2 holds

Theorem 4.1.

Suppose Condition 3.2 holds. Then for any $S\in C^2[0,T]$ with $S(0)=0$, the reduced problem has a unique solution C_S and it satisfies

$C_S\in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q}_S)$ with the same α as in Condition 3.2.

Proof.

Our proof is based on Theorem 2.2. First of all, the reduced problem is transformed into a stationary domain problem. Define the following transformation:

$$\binom{y}{\tau} = F(x,t)=\binom{x-S(t)}{t} \text{ for } x \in \mathbb{R} \text{ , } t\in[0,T].$$

Since the Jacobian $\det(DF(x,t)) = \begin{vmatrix} 1 & \frac{-dS(t)}{dt} \\ 0 & 1 \end{vmatrix} = 1 \neq 0$, F is a C^2-diffeomorphism.

Define $S_0(t)=0$, $t\in[0,T]$. It is easily seen that C_S is a solution of (1.1) if and only if the function $\tilde{C}$, defined by $\tilde{C}(y,\tau)= C_S \circ F^{-1}(y,\tau)=C_S(x,t)$, $(y,\tau)\in\bar{Q}_{S_0}$ solves the problem:

$$\frac{\partial\tilde{C}(y,\tau)}{\partial\tau} - \frac{\partial^2\tilde{C}(y,\tau)}{\partial y^2} - \frac{dS(\tau)}{d\tau}\frac{\partial\tilde{C}(y,\tau)}{\partial y} = 0, \; y\in(-\infty,0), \; \tau\in(0,T],$$

$$\tilde{C}(y,0) = C_0(y) \text{ , } y\in(-\infty,0] \text{ , } \tilde{C}(0,\tau)=0 \text{ , } \tau\in[0,T].$$

For this problem we check the conditions of Theorem 2.2 for the same α as in Condition 3.2.

i) Since $S\in C^2[0,T]$, the coefficients of the parabolic equation belong to $C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{Q}_{S_0})$.

ii) The parabolic equation is homogeneous and the boundary function is $C^\infty[0,T]$; Condition 3.2. says that $C_0\in C^{3+\alpha}(-\infty,0]$.

iii) Since Condition 3.2 states that $C_0(0) = \frac{dC_0}{dx}(0) = \frac{d^2C_0}{dx^2}(0) = 0$, the compatibility conditions of order 1 are fulfilled. So there is a unique bounded solution $\tilde{C}$ of this problem and $\tilde{C} \in C^{3+\alpha, \frac{3+\alpha}{2}}(\bar{Q}_{S_0})$.

Theorem 4.1 now follows, since the components of F are in $C^{3+\alpha, \frac{3+\alpha}{2}}(\mathbb{R}\times[0,T])$.■

4.2 Estimation of $\frac{\partial C_S}{\partial x}$

Lemma 4.2.

Suppose Condition 3.2 holds. Let $S_1, S_2 \in C^2[0,T]$, $S_1(0) = S_2(0) = 0$ and $\frac{dS_1(t)}{dt} \le \frac{dS_2(t)}{dt}$ for $t\in[0,T]$, then the inequality

$$-\frac{\partial C_{S_2}}{\partial x}(S_2(t),t) \le -\frac{\partial C_{S_1}}{\partial x}(S_1(t),t) \text{ holds for } t\in[0,T].$$

Proof.

Choose $\hat{t}\in[0,T]$ and define $\delta = S_2(\hat{t}) - S_1(\hat{t})$ and $C(x,t) = C_{S_2}(x+\delta,t) - C_{S_1}(x,t)$, $x\in(-\infty, S_2(t)-\delta]$, $t\in[0,\hat{t}]$. It is easily seen that C is a bounded solution of:

$$\frac{\partial C(x,t)}{\partial t} - \frac{\partial^2 C(x,t)}{\partial x^2} = 0\ ,\quad x\in(-\infty,S_2(t)-\delta),\ t\in(0,\hat{t}],$$

$C(x,0) = C_0(x+\delta) - C_0(x)$, $x\in(-\infty,-\delta]$, $C(S_2(t)-\delta,t) = -C_{S_1}(S_2(t)-\delta,t)$, $t\in[0,\hat{t}]$.

The maximum principle applied to C_{S_1} gives $C_{S_1}(x,t)\ge 0, (x,t)\in\bar{Q}_{S_1}$.

The condition $\frac{dS_1(t)}{dt} \le \frac{dS_2(t)}{dt}$ implies $S_2(t)-\delta \le S_1(t)$ and hence $-C_{S_1}(S_2(t)-\delta,t) \le 0$ for $t\in[0,\hat{t}]$. From Condition 3.2 we know $\frac{dC_0(x)}{dx} \le 0$, $x\in(-\infty,0]$ and so $C_0(x+\delta)-C_o(x) \le 0$. Combination of these inequalities with the maximum principle yields $C(x,t) \le 0$ for $x\in(-\infty,S_2(t)-\delta]$, $t\in[0,\hat{t}]$.

This together with $C(S_2(\hat{t})-\delta,\hat{t}) = C_{S_2}(S_2(\hat{t}),\hat{t}) - C_{S_1}(S_1(\hat{t}),\hat{t}) = 0$ implies

$$\frac{\partial C}{\partial x}(S_2(\hat{t})-\delta,\hat{t}) \ge 0.$$

Substitution of $C(x,t) = C_{S_2}(x+\delta,t) - C_{S_1}(x,t)$ gives the result

$$-\frac{\partial C_{S_1}}{\partial x}(S_1(\hat{t}),\hat{t}) \ge -\frac{\partial C_{S_2}}{\partial x}(S_2(\hat{t}),\hat{t}).$$ ■

Lemma 4.3.

Suppose Condition 3.2 holds and $S\in P \cap C^2[0,T]$. Then the solution C_S of the reduced problem has the property

$$0 \le -\frac{\partial C_S(x,t)}{\partial x} \le \sup_{\bar{x}\in(-\infty,0]} -\frac{dC_0}{dx}(\bar{x}) \quad , \quad (x,t)\in\bar{Q}_S.$$

Proof.

Define $S_0(t)=0$, $t\in[0,T]$ and the functions

$$\hat{C}_0(x)= \begin{cases} C_0(x), & x\in(-\infty,0] \\ -C_0(-x), & x\in(0,\infty) \end{cases} \quad , \quad \hat{C}(x,t)= \begin{cases} C_{S_0}(x,t), & x\in(-\infty,0], t\in[0,T] \\ -C_{S_0}(-x,t), & x\in(0,\infty), \ t\in[0,T] \end{cases} ,$$

where C_{S_0} is the solution of the reduced problem.

The boundary condition $C_{S_0}(0,t)=0$, $t\in[0,T]$ yields $\frac{\partial C_{S_0}}{\partial t}(0,t)=0$.

Together with $\frac{\partial C_{S_0}}{\partial t}(0,t)-\frac{\partial^2 C_{S_0}}{\partial x^2}(0,t)=0$ we obtain $\frac{\partial^2 C_{S_0}}{\partial x^2}(0,t)=0$,

so $\frac{\partial^2\hat{C}(x,t)}{\partial x^2}$ is a continuous function.

Since $\hat{C}$ is an odd function of x

$$\lim_{x\uparrow 0} \frac{\partial\hat{C}(x,t)}{\partial x} = \lim_{x\downarrow 0} \frac{\partial\hat{C}(x,t)}{\partial x} \text{ and } \lim_{x\uparrow 0} \frac{\partial^3\hat{C}(x,t)}{\partial x^3} = \lim_{x\downarrow 0} \frac{\partial^3\hat{C}(x,t)}{\partial x^3}.$$

Hence $\hat{C}\in C^{3+\alpha,\frac{3+\alpha}{2}}(\mathbb{R}\times[0,T])$ and $\hat{C}$ is a solution of

$$\frac{\partial\hat{C}(x,t)}{\partial t} - \frac{\partial^2\hat{C}(x,t)}{\partial x^2} = 0 \quad , \quad x\in \mathbb{R}, \ t\in(0,T],$$

$$\hat{C}(x,0) = \hat{C}_0(x) \quad , \quad x\in \mathbb{R}.$$

The maximum principle for this Cauchy problem [Ladyženskaja e.a., 1968; p.18, Theorem 2.5] yields:

$$\inf_{\bar{x}\in \mathbb{R}} \frac{d\hat{C}_0}{dx}(\bar{x}) \le \frac{\partial\hat{C}(x,t)}{\partial x} \le \sup_{\bar{x}\in \mathbb{R}} \frac{d\hat{C}_0}{dx}(\bar{x}), \quad x\in \mathbb{R}, \ t\in[0,T].$$

It is sufficient to consider the infimum of $\frac{d\hat{C}_0}{dx}$ on $(-\infty,0]$, because $\frac{d\hat{C}_0}{dx}$ is an even function. From Condition 3.2 we know

$$\frac{d\hat{C}_0(x)}{dx} \le 0, \ x\in \mathbb{R} \text{ so } 0\le -\frac{\partial C_{S_0}(x,t)}{\partial x} \le \sup_{\bar{x}\in(-\infty,0]} -\frac{dC_0}{dx}(\bar{x}), \quad (x,t)\in\bar{Q}_{S_0}.$$

For an arbitrary $S\in P\cap C^2[0,T]$, the bounded function $\frac{\partial C_S}{\partial x}$ satisfies:

$$\frac{\partial}{\partial t}\frac{\partial C_s(x,t)}{\partial x} - \frac{\partial^2}{\partial x^2}\frac{\partial C_S(x,t)}{\partial x} = 0 \quad , \quad x\in(-\infty,S(t)), \ t\in(0,T],$$

$$\frac{\partial C_S}{\partial x}(x,0) = \frac{dC_0(x)}{dx} \quad , \quad x\in(-\infty,0].$$

Since $\frac{dS(t)}{dt} \geq \frac{dS_0(t)}{dt} = 0$, $t\in[0,T]$, we obtain from Lemma 4.2

$$-\frac{\partial C_S}{\partial x}(S(t),t) \leq -\frac{\partial C_{S_0}}{\partial x}(0,t) \ , \ t\in[0,T].$$

Furthermore $0\leq -\frac{\partial C_S}{\partial x}(S(t),t)$, $t\in[0,T]$ because $C_S(x,t)\geq 0$, $(x,t)\in\bar{Q}_S$ and $C_S(S(t),t)=0$, $t\in[0,T]$. These inequalities together with the maximum principle yield

$$0 \leq -\frac{\partial C_S(x,t)}{\partial x} \leq \sup_{\bar{x}\in(-\infty,0]} -\frac{dC_0}{dx}(\bar{x}) \ , \ (x,t)\in\bar{Q}_S. \qquad ■$$

4.3 <u>The relation $f_S(t) = -\int_0^t \frac{\partial C_S}{\partial x}(S(\tau),\tau)d\tau$, $t\in[0,T]$</u>

The function f_S is defined in Section 3 after the formulation of the reduced problem.

<u>Lemma 4.4.</u>

For $S\in\mathcal{O}$ suppose there is a solution C_S of the reduced problem, then $\lim_{x\to-\infty} C_S(x,t)=1$ for $t\in[0,T]$.

<u>Proof.</u>

The maximum principle together with Condition 3.1 yields $0\leq C_S(x,t)\leq 1$, $(x,t)\in\bar{Q}_S$. Since $\lim_{x\to-\infty} C_0(x)=1$, there is for $\epsilon>0$ an $N_1 < \min_{t\in[0,T]} S(t)$ such that $1 - C_0(x)<\frac{\epsilon}{2}$, $x\in(-\infty,N_1]$.

The function $\varphi:(-\infty,N_1]\times[0,T]\to\mathbb{R}$ given by

$$\varphi(x,t) = -(1-\tfrac{\epsilon}{2})\mathrm{erf}\left(\frac{x-N_1}{2\sqrt{t+1}}\right)$$ is a bounded solution of

$$\frac{\partial\varphi(x,t)}{\partial t} - \frac{\partial^2\varphi(x,t)}{\partial x^2} = 0 \ , \ x\in(-\infty,N_1), \ t\in(0,T],$$

$$\varphi(x,0) = -(1-\tfrac{\epsilon}{2})\mathrm{erf}(\tfrac{x-N_1}{2}), \ x\in(-\infty,N_1], \ \varphi(N_1,t) = 0, \ t\in[0,T].$$

Since $-1\leq \mathrm{erf}(\frac{x-N_1}{2}) \leq 0$ for $x\in(-\infty,N_1]$ the maximum principle yields $C_S(x,t) \geq \varphi(x,t)$, $x\in(-\infty,N_1]$, $t\in[0,T]$.

Furthermore there is an $N_2<0$ such that $\mathrm{erf}(y)\leq -1+\frac{\epsilon}{2}$ for $y\in(-\infty,N_2]$, because $\lim_{y\to-\infty} \mathrm{erf}(y)=-1$. This implies

$1\geq C_S(x,t) \geq \varphi(x,t)\geq(1-\frac{\epsilon}{2})(1-\frac{\epsilon}{2})>1-\epsilon$ for $x\in(-\infty,N_1+2N_2\sqrt{T+1})$, $t\in[0,T]$.

Thus $\lim_{x\to-\infty} C_S(x,t)=1$ for $t\in[0,T]$. ■

<u>Remark 4.5.</u> If $C_S\in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q}_S)$ and $\lim_{x\to-\infty} C_s(x,t)=1$, $t\in[0,T]$ then it is easily seen that $\lim_{x\to-\infty} \frac{\partial C_s(x,t)}{\partial x}=0$, $t\in[0,T]$.

Lemma 4.6.

Suppose Condition 3.2 holds. Then for any $S\in C^2[0,T]$ with $S(0)=0$, the solution of the reduced problem satisfies:

$$\int_{-\infty}^{\infty}[C_S(x,\hat{t})-C_0(x)]dx = \int_0^{\hat{t}} \frac{\partial C_S}{\partial x}(S(t),t)dt \;,\hat{t}\in[0,T].$$

Thus $f_S(t)=-\int_0^t \frac{\partial C_S}{\partial x}(S(\tau),\tau)d\tau,\ t\in[0,T]$.

Proof.

Choose $\hat{t}\in[0,T]$ and R such that $-R<\min_{t\in[0,T]} S(t)$, We define

$\Sigma =\{(x,t)|x=S(t),\ t\in(0,\hat{t})\}$ and $Q=\{(x,t)|x\in(-R,S(t)),t\in(0,\hat{t})\}$.

According to Theorem 4.1 $C_S\in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q}_S)$.

With the divergence theorem of Gauss [Hayman & Kennedy, 1976; p.22, Theorem 1.9] we obtain:

$$\iint_Q \operatorname{div}\begin{pmatrix} \frac{\partial C_S(x,t)}{\partial x} \\ -C_S(x,t)\end{pmatrix}dxdt = \int_{\partial Q}\begin{pmatrix} \frac{\partial C_S(x,t)}{\partial x} \\ -C_S(x,t)\end{pmatrix}\begin{pmatrix} n_x \\ n_t\end{pmatrix}ds,$$

where $\begin{pmatrix} n_x \\ n_t\end{pmatrix}$ is the outward pointing unit normal and

$\operatorname{div}(\cdot) = (\frac{\partial\cdot}{\partial x}\ ,\ \frac{\partial\cdot}{\partial t})^T$.

This is equivalent to

$$\iint_Q[\frac{\partial^2 C_S(x,t)}{\partial x^2} - \frac{\partial C_S(x,t)}{\partial t}]dxdt = \int_{-R}^{0} C_0(x)dx - \int_{-R}^{S(\hat{t})} C_S(x,\hat{t})dx$$

$$+ \int_\Sigma[\frac{\partial C_S(x,t)}{\partial x}n_x - C_S(x,t).n_t]ds - \int_0^{\hat{t}} \frac{\partial C_S}{\partial x}(-R,t)dt.$$

The left-hand side is zero because C_S is a solution of (1.1). Since $C_S(x,t)=0$, $(x,t)\in\Sigma$ the integral over Σ reduces to

$\int_\Sigma \frac{\partial C_S(x,t)}{\partial x}.n_x ds$. Substitution of the parametrization

$x=S(\tau)$, $t=\tau$ of Σ and using $\begin{pmatrix} n_x \\ n_t\end{pmatrix}\begin{pmatrix} \frac{dS}{d\tau} \\ 1\end{pmatrix}=0$, $n_x \geq 0$, $n_x^2+n_t^2=1$ yields

$$\int_{\Sigma} \frac{\partial C_S(x,t)}{\partial x} n_x ds = \int_0^{\hat{t}} \frac{\partial C_S}{\partial x}(S(\tau),\tau) n_x \sqrt{\left(\frac{n_t}{n_x}\right)^2+1} d\tau = \int_0^{\hat{t}} \frac{\partial C_S}{\partial x}(S(\tau),\tau) d\tau.$$

Thus,

$$\int_{-R}^{\infty} [C_S(x,\hat{t})-C_0(x)]dx - \int_0^{\hat{t}} \frac{\partial C_S}{\partial x}(S(t),t)dt + \int_0^{\hat{t}} \frac{\partial C_S}{\partial x}(-R,t)dt=0.$$

Since $C_S \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q}_S)$, the inequality $|\frac{\partial C_S}{\partial x}(-R,t)| \le \|C_S\|^{3+\alpha,\frac{3+\alpha}{2}}$ holds for $t\in[0,T]$. From Remark 4.5 and the Lebesgue dominated convergence theorem it now follows that

$$\lim_{R\to\infty} \int_0^{\hat{t}} \frac{\partial C_S}{\partial x}(-R,t)dt = \int_0^{\hat{t}} \lim_{R\to\infty} \frac{\partial C_S}{\partial x}(-R,t)dt=0.$$

This yields $\lim_{R\to\infty} \int_{-R}^{\infty} [C_S(x,\hat{t}) - C_0(x)]dx = \int_0^{\hat{t}} \frac{\partial C_S}{\partial x}(S(t),t)dt.$ ■

4.4 Existence theorem for C_S if $S\in C^2[0,T]$

Definition 4.7. We apply Lemma 2.6 to C_0 and define a sequence $\{C_n\}_{n\ge1}$ and a function C^+ as given in this lemma.

The function $C_S^n:\bar{Q}_S\to\mathbb{R}$ for $n\ge1$, if it exists, is defined as a solution of the reduced problem, with initial condition $C_S^n(x,0)=C_n(x)$, $x\in(-\infty,0]$.

Lemma 4.8.

For every $A > 0$, $\delta > 0$ there is an $N > 0$ such that if $S\in\mathcal{O}$, $S(t)\in[-A,A]$, $t\in[0,T]$, and the functions C_S^n and C_S exist, then $C_S^n(x,t)\le C_S(x,t)$, $(x,t)\in\bar{Q}_S$, $n=1,2,\ldots$ and $\int_{-\infty}^{-N} [C_S(x,t) - C_S^n(x,t)]dx < \delta$, $n=1,2,\ldots$.

Proof. Take $S^-(t)=-A$ and $S^+(t)=A$, $t\in[0,T]$. Define the functions $C_0^-\in C^{3\frac{1}{2}}(-\infty,-A]$ and $C_0^+\in C^{3\frac{1}{2}}(-\infty,A]$ with the following properties: the functions $x\to C_0^{\pm}(x \pm A)$ are elements of Cond 3.2, $C_0^-(x)\le C_1(x)$, $x\in(-\infty,-A]$, $C_0(x)\le C_0^+(x)$, $x\in(-\infty,0]$ and $\int_{-\infty}^{\infty} [C_0^+(x)-C_0^-(x)]dx<\infty$. It is easy to see that C_0^+ can be such that $C_0^+(x)=C^+(x)$, $x\in(-\infty,0]$.

An application of Theorem 4.1 proves that there are unique solutions $C_{S^\pm} \in C^{3\frac{1}{2},\,1\frac{3}{4}}(\bar{Q}_{S^\pm})$ of the reduced problem, with initial conditions $C_S\pm(x,0) = C_0^{\pm}(x)$, $x\in(-\infty,\pm A]$.

According to the definition of C_0^-, C_0^+ and Lemma 4.6 there is an $N>0$ large enough such that

$$\left|\int_{-\infty}^{-N}[C_0^{\pm}(x)-C_0(x)]dx\right| < \frac{\delta}{4} \text{ and } \left|\int_{-\infty}^{-N}[C_S\pm(x,T)-C_0^{\pm}(x)]dx\right| < \frac{\delta}{4} .$$

Since $\int_{-\infty}^{-N}[C_S\pm(x,t)-C_0^{\pm}(x)]dx = \int_0^t \frac{\partial C_S\pm}{\partial x}(-N,\tau)d\tau$ and Lemma 4.3 yields $\frac{\partial C_S\pm(x,t)}{\partial x} \le 0$, $(x,t)\in\bar{Q}_S\pm$ we obtain:

$$\left|\int_{-\infty}^{-N}[C_S\pm(x,t)-C_0(x)]dx\right| \le \left|\int_{-\infty}^{-N}[C_S\pm(x,T) - C_0^{\pm}(x)]dx\right| + \left|\int_{-\infty}^{-N}[C_0^{\pm}(x)-C_0(x)]dx\right| < \frac{\delta}{2},\ t\in[0,T]$$

From the maximum principle we obtain

$$C_S-(x,t) \le C_S^n(x,t) \le C_S(x,t) \le C_S+(x,t), (x,t)\in\bar{Q}_S+,\ n=1,2,\dots .$$

This implies for $R\in \mathbb{R}$, $R>N$

$$\int_{-R}^{-N}[C_S(x,t)-C_S^n(x,t)]dx \le \int_{-\infty}^{-N}[C_S+(x,t)-C_S-(x,t)]dx \le \delta,\ t\in[0,T],\ n=1,2,\dots .$$

Thus $\int_{-\infty}^{-N}[C_S(x,t)-C_S^n(x,t)]dx$ exists and is less than δ. ⊠

<u>Theorem 4.9.</u>

For a given function $S\in O\cap C^2[0,T]$ there is a unique solution C_S of the reduced problem, and $f_S\in P$ (for f_S see Section 3).

If $S\in P \cap C^2[0,T]$, then $f_S\in\tilde{M}_L$, and $0 \le -\frac{\partial C_S(x,t)}{\partial x} \le L$ for $(x,t)\in Q_S$.

<u>Proof.</u>

The functions C_n of Definition 4.7 are elements of Cond.3.2 with $\alpha=\frac{1}{2}$

According to Theorem 4.1 the unique functions C_S^n exist and $C_S^n \in C^{3\frac{1}{2},1\frac{3}{4}}(\bar{Q}_S)$. Since $C_S^n:\bar{Q}_S\to[0,1]$, $C_n\to C_0$ uniformly on $(-\infty,0]$ and $C_S^n(S(t),t)=0$, $t\in[0,T]$, $n=1,2,\dots$ we deduce from Lemma 2.5 and the maximum principle that the function $C_S(x,t) = \lim_{n\to\infty} C_S^n(x,t), (x,t)\in\bar{Q}_S$ is the unique solution of the reduced problem.

Define $f_S^n(t)=-\int_{-\infty}^{\infty}[C_S^n(x,t)-C_n(x)]dx$, $t\in[0,T]$, $n=1,2,\dots$.

An application of the maximum principle yields:

$$0 \le C_S^k(x,t) \le C_S^n(x,t),\ (x,t)\in\bar{Q}_S,\ 1 \le k \le n.$$

Since $C_S^k(S(t),t) = C_S^n(S(t),t)=0$, $t\in[0,T]$ the inequalities

$0\le -\frac{\partial C_S^k}{\partial x}(S(t),t) \le -\frac{\partial C_S^n}{\partial x}(S(t),t)$ hold for $t\in[0,T]$.

According to Lemma 4.6 this implies $f_S^k(t) \le f_S^n(t)$, $t\in[0,T]$, $1 \le k \le n$.

On the other hand we know that

$$f_S^n(t) - f_S^k(t) \le \int_{-\infty}^{\infty} [C_n(x)-C_k(x)]dx \le \int_{-\infty}^{\infty} [C_0(x)-C_k(x)]dx.$$

Thus, (4.10) $\|f_S^n - f_S^k\|_\infty \le \int_{-\infty}^{\infty} [C_0(x) - C_k(x)]dx.$

The inequalities $0\le -\frac{\partial C_S^n}{\partial x}(S(t),t) \le \|C_S^n\|^{3\frac{1}{2},1\frac{3}{4}}$ combined with Lemma 4.6 yield $f_S^n\in P$, $n=1,2,\ldots$. This together with inequality (4.10) and Lemma 2.6 iii) demonstrates that $\{f_S^n\}_{n\ge1}$ is a Cauchy sequence in the complete space $(P,\|\cdot\|_\infty)$. Thus, $\lim_{n\to\infty} f_S^n(t)=\hat{f}_S(t)$, $t\in[0,T]$ exists and is an element of P.

Given $\delta>0$, Lemma 4.8 demonstrates that there is an $N>0$ such that

$|\int_{-\infty}^{-N} [C_S(x,t)-C_S^n(x,t)]dx|<\delta$. It is easy to see that

$$f_S(t)=f_S^n(t) + \int_{-N}^{\infty} [C_S^n(x,t) - C_S(x,t)]dx$$

$$+ \int_{-\infty}^{\infty} [C_0(x) - C_n(x)]dx + \int_{-\infty}^{-N} [C_S^n(x,t)-C_S(x,t]dx.$$

This implies that f_S exists and

$$|f_S(t)-f_S^n(t)| \le |\int_{-N}^{\infty} [C_S^n(x,t) - C_S(x,t)]dx| + |\int_{-\infty}^{\infty} [C_0(x) - C_n(x)]dx| + \delta,$$

$t\in[0,T]$, $n=1,2,\ldots$. Since C_S^n converges uniformly to C_S for $n\to\infty$, we obtain with Definition 4.7

$$|f_S(t) - \hat{f}_S(t)| = \lim_{n\to\infty} |f_S(t)-f_S^n(t)| \le \delta,\ t\in[0,T].$$

Because δ can be chosen arbitrarily small, we conclude $f_S(t)=\hat{f}_S(t)$, $t\in[0,T]$ and thus $f_S\in P$.

For $S\in P\cap C^2[0,T]$, the Lemmas 4.3 and 2.6 i) yield $0\le -\frac{\partial C_S^n}{\partial x}(S(t),t)\le L$, $t\in[0,T]$, $n=1,2,\ldots$. Thus, $f_S^n\in\tilde{M}_L$, $n=1,2,\ldots$. Since $(\tilde{M}_L,\|\cdot\|_\infty)$ is a complete space, we conclude that for $S\in P$ the function f_S exists and is an element of $\tilde{M}_L$.

Furthermore for $S\in P$, the Lemmas 4.3 and 2.6 i) yield $0\le -\frac{\partial C_S^n(x,t)}{\partial x} \le L, (x,t)\in\bar{Q}_S$. According to Lemma 2.5 it follows that $0\le -\frac{\partial C_S(x,t)}{\partial x} \le L$, $(x,t)\in Q_S$. ∎

4.5 f_S depends continuously on S

Theorem 4.11.

i) If S_1, $S_2 \in 0$, $S_1(t) \le S_2(t)$, $t\in[0,T]$ and there are solutions C_{S_1}, C_{S_2} of the reduced problem, then

$$C_{S_1}(x,t) \le C_{S_2}(x,t),\ (x,t)\in\bar{Q}_{S_2} \text{ and } \int_{-\infty}^{\infty}[C_{S_2}(x,t)-C_{S_1}(x,t)]dx \le \|S_1-S_2\|_\infty,\ t\in[0,T].$$

ii) If $S_1, S_2 \in \tilde{M}_K$ and there are solutions C_{S_1}, C_{S_2} of the reduced problem, then $\int_{-\infty}^{\infty}|C_{S_2}(x,t)-C_{S_1}(x,t)|dx \le \|S_1-S_2\|_\infty$, $t\in[0,T]$.

If f_{S_1}, f_{S_2} exist, this implies $\|f_{S_1}-f_{S_2}\|_\infty \le \|S_1-S_2\|_\infty$.

Proof.

i) Define $\delta = \|S_1-S_2\|_\infty$ and the function $\tilde{C}$ by $\tilde{C}(x,t)=C_{S_2}(x+\delta,t)$ for $x\in(-\infty,S_2(t)-\delta]$, $t\in[0,T]$. This function $\tilde{C}$ satisfies

$$\frac{\partial\tilde{C}(x,t)}{\partial t} - \frac{\partial^2\tilde{C}(x,t)}{\partial x^2} = 0\ ,\ x\in(-\infty,S_2(t)-\delta),\ t\in(0,T],$$

$\tilde{C}(x,0)=C_0(x+\delta)$, $x\in(-\infty,-\delta]$, $\tilde{C}(S_2(t)-\delta,t)=0$, $t\in[0,T]$.

Since C_0 is a monotone decreasing fun:tion and $S_2(t)-\delta \le S_1(t)$, $t\in[0,T]$, application of the maximum principle yields:

$\tilde{C}(x,t) \le C_{S_1}(x,t) \le C_{S_2}(x,t)$, $(x,t)\in\bar{Q}_{S_2}$.

Thus, for $N\in\mathbb{R}$, we have the following estimation

$$\int_N^{\infty}[C_{S_2}(x,t)-C_{S_1}(x,t)]dx \le \int_N^{\infty}[C_{S_2}(x,t)-C_{S_2}(x+\delta,t)]dx$$

$$= \int_N^{S_2(t)} C_{S_2}(x,t)dx - \int_{N-\delta}^{S_2(t)-\delta} C_{S_2}(x+\delta,t)dx + \int_{N-\delta}^{N} C_{S_2}(x+\delta,t)dx$$

$$= \int_{N-\delta}^{N} C_{S_2}(x+\delta,t)dx \le \delta = \|S_1-S_2\|_\infty\ ,\ \text{because}$$

$C_{S_2}(x+\delta,t)\le 1$, $x\in\mathbb{R}$, $t\in[0,T]$. Thus, $\int_{-\infty}^{\infty}[C_{S_2}(x,t)-C_{S_1}(x,t)]dx$ exists and is less than $\|S_1-S_2\|_\infty$ for $t\in[0,T]$.

ii) Suppose there is an $\epsilon>0$ such that

$$\max_{t\in[0,T]} \int_{-\infty}^{\infty}\left|C_{S_2}(x,t) - C_{S_1}(x,t)\right|dx \ge \|S_1-S_2\|_\infty + \epsilon.$$

Define the functions $\bar{S}$, $\underline{S}\in\tilde{M}_K$ as follows $\bar{S}(t)=\max\{S_1(t),S_2(t)\}$

and $\underline{S}(t)=\min\{S_1(t),S_2(t)\}$, $t\in[0,T]$. According to Lemma 2.7 ii) there are functions $S^+,S^-\in M_K$ such that $\bar{S}(t)\leq S^+(t)\leq\bar{S}(t)+\frac{\epsilon}{4}$ and $\underline{S}(t)-\frac{\epsilon}{4}\leq S^-(t)\leq\underline{S}(t)$ for $t\in[0,T]$. Since $S^+,S^-\in M_K$ Theorem 4.9 says that there are solutions C_{S^+} and C_{S^-} of the reduced problem. Application of the maximum principle yields:

$$C_{S^-}(x,t) \leq C_{S_1}(x,t),\ C_{S_2}(x,t) \leq C_{S^+}(x,t),\ (x,t)\in\bar{Q}_{S^+}$$

and thus

$$\text{for } N\in\mathbb{R}: \int_N^\infty |C_{S_2}(x,t) - C_{S_1}(x,t)|dx \leq \int_N^\infty [C_{S^+}(x,t) - C_{S^-}(x,t)]dx.$$

With part i) of this theorem it follows that $\int_{-\infty}^\infty |C_{S_2}(x,t)-C_{S_1}(x,t)|dx$ exists and $\int_{-\infty}^\infty |C_{S_2}(x,t) - C_{S_1}(x,t)|dx \leq \|S^+-S^-\|_\infty \leq \|S_1-S_2\|_\infty + \frac{\epsilon}{2}$ for $t\in[0,T]$. This is a contradiction. ∎

4.6 Properties of the function f_S

Theorem 4.12.

i) If $S\in P$ and there is a solution C_S of the reduced problem then $f_S\in\tilde{M}_L$.

ii) If $S\in O$ and there is a solution C_S of the reduced problem then $f_S\in P$.

Proof.

i) According to Lemma 2.7 i) there is a sequence of functions $S_n\in P\cap C^2[0,T]$, $n=1,2,\ldots$ such that $\lim_{n\to\infty}\|S-S_n\|_\infty=0$ and $S_n(t)\leq S(t)$, $t\in[0,T]$. Theorem 4.9 says that there are unique solutions C_{S_n}, $n=1,2,\ldots$ and $f_{S_n}\in\tilde{M}_L$.
With Theorem 4.11 ii) we obtain $\|f_{S_n}-f_{S_k}\|_\infty \leq \|S_n-S_k\|_\infty$.
This implies that $\{f_{S_n}\}_{n\geq1}$ is a Cauchy sequence in the complete space $(\tilde{M}_L,\|\cdot\|_\infty)$. Hence, $\lim_{n\to\infty} f_{S_n}(t)=\hat{f}_S(t), t\in[0,T]$ exists and $\hat{f}_S\in\tilde{M}_L$.
Application of Theorem 4.11 i) yields

$$0=\lim_{n\to\infty} \|S-S_n\|_\infty \geq \lim_{n\to\infty} \sup_{t\in[0,T]} \left|\int_{-\infty}^\infty [C_{S_n}(x,t)-C_S(x,t)]dx\right| =$$

$$= \lim_{n\to\infty} \sup_{t\in[0,T]} \left|-\int_{-\infty}^\infty [C_S(x,t)-C_0(x)]dx + \int_{-\infty}^\infty [C_{S_n}(x,t) - C_0(x)]dx\right| = \|f_S-\hat{f}_S\|_\infty.$$

Thus, f_S exists and is an element of $\tilde{M}_L$.

ii) Take $\bar{S}_n \in O \cap C^2[0,T]$ such that $\|S-\bar{S}_n\|_\infty \le \frac{1}{2n}$, $n=1,2,\ldots$.

Define $S_n(t)=\bar{S}_n(t)-\frac{1}{n}$, $S_n^+(t)=\bar{S}_n(t)+\frac{1}{n}$, $t\in[0,T]$ and $C_n:(-\infty,-\frac{1}{n}]\to[0,1]$ by $C_n(x)=\min\{C_o(x), -(x+\frac{1}{n})L\}$, $x\in(-\infty,-\frac{1}{n}]$, $n=1,2,\ldots$.

According to Theorem 4.9 the equations

$$\frac{\partial C(x,t)}{\partial t} - \frac{\partial^2 C(x,t)}{\partial x^2} = 0,\ x\in(-\infty,S_n(t)),\ t\in(0,T],$$

$C(x,0)=C_n(x)$, $x\in(-\infty,-\frac{1}{n}]$, $C(S_n(t),t)=0$, $t\in[0,T]$,

have unique bounded solutions $C_{S_n}\in C^{2,1}(Q_{S_n})\cap C(\bar{Q}_{S_n})$, $n=1,2,\ldots$.

Furthermore the functions $f_{S_n}(t)=-\int_{-\infty}^{\infty}[C_{S_n}(x,t)-C_n(x)]dx$

exist and $f_{S_n}\in P$.

An application of the maximum principle yields

$C_{S_n}(x,t) \le C_S(x,t) \le C_{S_n}(x-\frac{2}{n},t)$, $(x,t)\in\bar{Q}_{S_n^+}$. This implies that

$|\int_{-\infty}^{\infty}[C_{S_n}(x,t)-C_S(x,t)]dx|$ exists and is less than $2/n$ for $t\in[0,T]$

and $n=1,2,\ldots$ see the proof of Theorem 4.11 i). It is easy to see that

$f_S(t)=f_{S_n}(t) + \int_{-\infty}^{\infty}[C_{S_n}(x,t)-C_S(x,t)]dx + \int_{-\infty}^{\infty}[C_0(x)-C_n(x)]dx$, $t\in[0,T]$ and

$n=1,2,\ldots$. This implies that f_S exists and $\|f_S-f_{S_n}\|_\infty \le \frac{3}{n}$, $n=1,2,\ldots$.

Since $f_{S_n}\in P$ and the space $(P,\|\cdot\|_\infty)$ is complete if follows that $f_S\in P$. ■

4.7 Existence theorem for C_S if $S\in\tilde{M}$

Theorem 4.13.

For a given function $S\in\tilde{M}$, there is a unique solution C_S of the reduced problem, $0\le\frac{-\partial C_S(x,t)}{\partial x}\le L$, $(x,t)\in Q_S$ and $f_S\in\tilde{M}_L$.

Proof.

According to Lemma 2.7 ii) there is a sequence of functions $S_n\in M$, $n=1,2,\ldots$ such that $\lim_{n\to\infty} \|S-S_n\|_\infty=0$ and $S(t)\le S_n(t)$, $t\in[0,T]$.

Applying Theorem 4.9 yields that there are unique solutions C_{S_n}, $n=1,2,\ldots$ of the reduced problem.

Define $\Phi_n(t)=C_{S_n}(S(t),t)$ and $\Phi(t)=0$, $t\in[0,T]$. We know $C_{S_n}\in C^{2,1}(Q_{S_n})\cap C(\bar{Q}_{S_n})$ hence,

$$C_{S_n}(S(t),t)-C_{S_n}(S_n(t),t) + \int_{S_n(t)}^{S(t)} \frac{\partial C_{S_n}}{\partial x}(\zeta,t)d\zeta = \int_{S_n(t)}^{S(t)} \frac{\partial C_{S_n}}{\partial x}(\zeta,t)d\zeta.$$

From Theorem 4.9 it follows that $0 \le -\frac{\partial C_{S_n}(x,t)}{\partial x} \le L$, $(x,t) \in Q_{S_n}$,

thus $\|\Phi_n - \Phi\|_\infty \le L\ \|S-S_n\|_\infty$.

Since $C_{S_n}:\bar{Q}_S \to [0,1]$, $C_{S_n}(x,0)=C_0(x)$, $x\in(-\infty,0]$, $n=1,2,\ldots$ and $\Phi_n \to \Phi$ uniformly on $[0,T]$ we obtain from Lemma 2.5 and the maximum principle that the function $C_S(x,t)=\lim_{n\to\infty} C_{S_n}(x,t)$, $(x,t)\in\bar{Q}_S$ is the unique solution of the reduced problem. Since $0 \le \frac{-\partial C_{S_n}(x,t)}{\partial x} \le L$, $(x,t)\in Q_S$, Lemma 2.5 yields $0 \le \frac{-\partial C_S(x,t)}{\partial x} \le L$, $(x,t)\in Q_S$.

It follows from Theorem 4.12 i) that f_S exists and $f_S \in \bar{M}_L$. ∎

5. The solution of the Stefan problem

In this section we prove existence and uniqueness of a solution for the Stefan problem. To prove the existence theorem we use the results of Section 4, particularly Theorem 4.11. Since the existence theorem is proved in a constructive way, the proof suggests an approximation scheme by which the Stefan problem can be solved numerically.

5.1 Definition of the operator $\mathcal{T}$

Definition 5.1.

For $\bar{B}$ as in Condition 3.3, we define $\bar{B}^{-1}(y)=\{x\in[0,\infty)\,|\,y\in\bar{B}(x)\}$.

Remark 5.2.

Using condition 3.3 it is easy to see that $\bar{B}^{-1}:[0,\infty)\to[0,\infty)$ is a function with the properties $\bar{B}^{-1}(0)=0$ and $0\le\bar{B}^{-1}(y+h)-\bar{B}^{-1}(y)\le\frac{h}{\underline{B}}$, $h\ge0; y\in[0,\infty)$.

Lemma 5.3.

The expression $\bar{B}^{-1}(\bar{G}(S,f_S,t))$ is properly defined for $S\in\bar{M}, t\in[0,T]$. Furthermore it defines a function of t which is an element of $\bar{M}$.

Proof.

For $S\in\bar{M}$, Theorem 4.13 states that f_S exists and $f_S\in\bar{M}_L$.

This together with Condition 3.3 implies that $\bar{G}(S,f_S,t)$ exists for $t\in[0,T]$ and $\bar{G}(S,f_S,t)\in[0,\infty)$. Using Remark 5.2 it follows that $\bar{B}^{-1}(\bar{G}(S,f_S,t))$ is properly defined for $S\in\bar{M}$ and $t\in[0,T]$.

Condition 3.3 combined with Remark 5.2 yields $\bar{B}^{-1}(\bar{G}(S,f_S,0))=0$.

Since $S\in\bar{M}$ and $f_S\in\bar{M}_L$ we know from Condition 3.3 that

$$0\le\bar{G}(S,f_S,t+h)-\bar{G}(S,f_S,t)\le\gamma h,\ h\ge0;\ t+h,\ t\in[0,T].$$

According to Remark 5.2 we obtain

$$0 \le \bar{B}^{-1}(\bar{G}(S,f_S,t+h)) - \bar{B}^{-1}(\bar{G}(S,f_S,t)) \le \frac{1}{\underline{B}}(\bar{G}(S,f_S,t+h) - \bar{G}(S,f_S,t)) \le \frac{\gamma}{\underline{B}}h,$$

$h \ge 0$; $t+h, t\in[0,T]$. Thus, with Definition 3.5 it is proved that $\bar{B}^{-1}(\bar{G}(S,f_S,t)), S\in\bar{M}$ is a function of t, which is an element of $\bar{M}$. ∎

Definition 5.4.

The operator $\mathcal{T}:\tilde{M}\to\tilde{M}$ is defined as follows:

$$\mathcal{T}(S)(t)=\bar{B}^{-1}(\bar{G}(S,f_S,t)) \ , \ t\in[0,T].$$

5.2 Existence and uniqueness of the solution for the Stefan problem

Lemma 5.5.

The inequality $\|\mathcal{T}(S_1)-\mathcal{T}(S_2)\|_\infty \le \frac{G_1+G_2}{B}\|S_1-S_2\|_\infty$ holds for $S_1,S_2\in\tilde{M}$.

Proof.

From Condition 3.3, Definition 5.4 and Remark 5.2 we obtain

$$\|\mathcal{T}(S_1)-\mathcal{T}(S_2)\|_\infty = \sup_{t\in[0,T]}\left|\bar{B}^{-1}(\bar{G}(S_1,f_{S_1},t)) - \bar{B}^{-1}(\bar{G}(S_2,f_{S_2},t))\right|$$

$$\le \sup_{t\in[0,T]} \frac{1}{B}\left|\bar{G}(S_1,f_{S_1},t) - \bar{G}(S_2,f_{S_2},t)\right|$$

$$\le \sup_{t\in[0,T]} \frac{1}{B}\left|\bar{G}(S_1,f_{S_1},t) - \bar{G}(S_2,f_{S_1},t)\right| + \sup_{t\in[0,T]} \frac{1}{B}\left|\bar{G}(S_2,f_{S_1},t) - \bar{G}(S_2,f_{S_2},t)\right|$$

$$\le \frac{G_1}{B}\|S_1-S_2\|_\infty + \frac{G_2}{B}\|f_{S_1}-f_{S_2}\|_\infty.$$

This together with Theorem 4.11 ii) yields

$$\|\mathcal{T}(S_1) - \mathcal{T}(S_2)\|_\infty \le \frac{G_1+G_2}{B}\|S_1-S_2\|_\infty.$$ ∎

Theorem 5.6.

If $G_1+G_2 < B$, then there is a unique function $\bar{S}\in\tilde{M}$, such that $\mathcal{T}(\bar{S})=\bar{S}$.

Proof.

We know that $(\tilde{M},\|\cdot\|_\infty)$ is a complete metric space and $\tilde{M}\neq\phi$.

From Lemma 5.5 it follows that $\mathcal{T}$ is a contraction on $\tilde{M}$ for $G_1+G_2 < B$.

Thus, we can apply the Banach fixed point theorem to prove the existence and uniqueness of the function $\bar{S}\in\tilde{M}$ such that $\mathcal{T}(\bar{S})=\bar{S}$. ∎

Now we are able to state one of our main theorems:

Main Theorem 5.7.

The Stefan problem: "to find a function $S\in P$ and a solution C_S of the reduced problem such that $\bar{G}(S,f_S,t)\in\bar{B}(S(t))$, $t\in[0,T]$", has a unique solution if $G_1+G_2 < B$. [1]

If we denote the solution of the Stefan problem by the pair $(\bar{S},C_{\bar{S}})$ then $\bar{S}\in\bar{M}$.

Proof.

Existence. If conditions 3.1 and 3.3 hold and $G_1+G_2 < B$, then Theorem 5.6 states that there is a fixed point $\bar{S}\in\bar{M}$ of the operator $\mathcal{T}:\bar{M}\to\bar{M}$. This implies that there is a unique solution $C_{\bar{S}}$ of the reduced problem and $\bar{S}(t)=\bar{B}^{-1}(\bar{G}(\bar{S},f_{\bar{S}},t))$, $t\in[0,T]$. With Definition 5.1, the last equation is equivalent to $\bar{G}(\bar{S},f_{\bar{S}},t)\in\bar{B}(\bar{S}(t))$, $t\in[0,T]$. Thus, the pair $(\bar{S},C_{\bar{S}})$ is a solution of the Stefan problem, and $\bar{S}$ is an element of $\bar{M}$.

Uniqueness. Suppose the pair (S,C_S) satisfies the Stefan problem. Since $S\in P$, Lemma 4.12 i) implies that $\bar{G}(S,f_S,t)$ exists for $t\in[0,T]$. From $\bar{G}(S,f_S,t)\in\bar{B}(S(t))$, $t\in[0,T]$, we obtain $S(t)=\bar{B}^{-1}(\bar{G}(S,f_S,t))$, $t\in[0,T]$. Condition 3.3 combined with Remark 5.2 yields $S\in\bar{M}$. This implies that $\mathcal{T}(S)$ is defined and $\mathcal{T}(S)=S$. Since the fixed point $\bar{S}$ of $\mathcal{T}$ is unique the theorem is proved. ∎

[1] Remember that our convention is to impose Conditions 3.1 and 3.3.

5.3 Further uniqueness results

In the sequel, $\bar{B}:\mathbb{R}\to\mathbb{R}$ will denote a multifunction and $\bar{G}:O\times P\times[0,T]\to[0,\infty)$ will denote a functional, both subject to the following condition:

Condition 5.8.

The restriction of $\bar{B}$ to $[0,\infty)$ and the restriction of $\bar{G}$ to $P\times\bar{M}_L\times[0,T]$ satisfy the requirements of Section 3. Furthermore, the multifunction $\bar{B}$ should be such that for $x\in\mathbb{R}, h>0$, $y_1\in\bar{B}(x)$, $y_2\in\bar{B}(x+h)$ the inequality $y_2-y_1>0$ holds.

The functional $\bar{G}$ should be such that $\bar{G}(S,f,0)=0$, $S\in O, f\in P$, and $0\le\bar{G}(S,f,t+h) - \bar{G}(S,f,t)$, $h\ge0$; $t+h$, $t\in[0,T]$, $S\in O, f\in P$.

Definition 5.9.

Define $\bar{B}^{-1}(y)=\{x\in\mathbb{R} \mid y\in\bar{B}(x)\}$, and $I=\bar{B}(\mathbb{R})$.

Remark 5.10.

Using Conditions 3.3 and 5.8 it is easy to see that $\bar{B}^{-1}:I\to\mathbb{R}$ is a function with the properties $\bar{B}^{-1}(0)=0$ and

$0 \le \bar{B}^{-1}(y+h) - \bar{B}^{-1}(y)$, $h\ge0$; $y\in I$.

Our second main theorem can be stated as follows:

Main Theorem 5.11.

The Stefan problem: "to find a function $S\in O$ and a solution C_S of the reduced problem such that $\bar{G}(S,f_S,t)\in\bar{B}(S(t))$, $t\in[0,T]$", has a unique solution if Condition 5.8 holds and $G_1+G_2 < B$. [2]

If we denote the solution of the Stefan problem by the pair $(\bar{S},\bar{C}_S)$ then $\bar{S}\in\bar{M}$.

[2] Remember that our convention is to impose Condition 3.1.

Proof.

Existence. Using Conditions 3.1, 5.8, $G_1+G_2 < B$ and $\tilde{M}\subset O$ it follows from Theorem 5.6 that there is a fixed point $\bar{S}\in\tilde{M}$ of the operator $\mathcal{T}:\tilde{M}\to\tilde{M}$. This implies that there is a unique solution $C_{\bar{S}}$ of the reduced problem and $\bar{S}(t)=\bar{B}^{-1}(\bar{G}(\bar{S},f_{\bar{S}},t))$, $t\in[0,T]$. With definition 5.9, the last equation is equivalent to $\bar{G}(\bar{S},f_{\bar{S}},t)\in\bar{B}(\bar{S}(t))$, $t\in[0,T]$. Thus, the pair $(\bar{S},C_{\bar{S}})$ is a solution of the Stefan problem, and $\bar{S}$ is an element of $\tilde{M}$.

Uniqueness. Suppose the pair (S,C_S) satisfies the Stefan problem. Since $S\in O$, Lemma 4.12 ii) implies that $\bar{G}(S,f_S,t)$ exists for $t\in[0,T]$. From $\bar{G}(S,f_S,t)\in\bar{B}(S(t))$, $t\in[0,T]$ and $[0,\infty)\subset I$ we obtain $S(t)=\bar{B}^{-1}(\bar{G}(S,f_S,t))$, $t\in[0,T]$. Condition 5.8 combined with Remark 5.10 yields $S\in P$. Applying Lemma 4.12 i) yields $f_S\in\tilde{M}_L$. With Condition 3.3 and Remark 5.2, this implies that $S\in\tilde{M}$. Thus, $\mathcal{T}(S)$ is defined and $\mathcal{T}(S)=S$. Since the fixed point $\bar{S}$ of $\mathcal{T}$ is unique the theorem is proved. ■

Remark 5.12.

In order to compare Theorems 5.7 and 5.11 we note the following. Both theorems state the existence of a solution of the Stefan problem in $\tilde{M}$ (a set of Lipschitz continuous monotone increasing functions). However they differ with respect to uniqueness: Theorem 5.7 imposes weaker conditions on $\bar{B}$ and $\bar{G}$ than Theorem 5.11, but proves only uniqueness of the solution in the set P, which contains monotone increasing functions only, whereas Theorem 5.11 even proves uniqueness in the set O, which also contains non-monotone functions.

6. Applications of our main theorems

In this section we first consider the Stefan problem mentioned in Section 1. After that we look at some other examples. Finally we give a scheme by which the Stefan problem is solved numerically.

6.1 The classical Stefan problem

In the sequel we suppose that $T>0$ and $C_0(-\infty,0]\to[0,1]$ are given. Furthermore, Condition 3.1 holds for this function C_0.

For a given bounded integrable function $b:\mathbb{R}\to[B_1,\infty)$ with $B_1>0$ we define $\bar{B}:\mathbb{R}\to\mathbb{R}$ by $\bar{B}(x)=\int_0^x b(\zeta)d\zeta$, $x\in\mathbb{R}$.

This function $\bar{B}$ satisfies Condition 5.6.

Suppose that the bounded continuous functions g,h: $\mathbb{R}\times[0,T]\to[0,\infty)$ are given and that there are $g_1,g_2,h_1\in\mathbb{R}$ such that:

$$|g(x_1,t)-g(x_2,t)|\le g_1|x_1-x_2| \;;\; x_1,x_2\in[0,\infty),\ t\in[0,T],$$
$$|g(x,t_1)-g(x,t_2)|\le g_2|t_1-t_2| \;;\; x\in[0,\infty),\quad t_1,t_2\in[0,T],$$
$$|h(x_1,t)-h(x_2,t)|\le h_1|x_1-x_2| \;;\; x_1,x_2\in[0,\infty),\ t\in[0,T].$$

For given $\delta\ge0$ and $S\in O$ we define $S_\delta:[-\delta,T]\to\mathbb{R}$ by

$$S_\delta(t)=\begin{cases}0, & t\in[-\delta,0)\\ S(t), & t\in[0,T]\end{cases} \text{ and}$$

$$\bar{G}(S,f,t)=\int_0^t g(S(\tau),\tau)df(\tau)+\int_0^t h(S_\delta(\tau-\delta),\tau)d\tau,\ S\in O,\ f\in P, t\in[0,T].$$

The first integral is a Lebesgue-Stieltjes integral.

Remark 6.1.

For $S\in P$, $f\in\bar{M}_L$, $t\in[0,T]$ we know that f is absolutely continuous and thus: $\bar{G}(S,f,t)=\int_0^t g(S(\tau),\tau)\frac{df(\tau)}{d\tau}d\tau+\int_0^t h(S_\delta(\tau-\delta),\tau)d\tau$.

Lemma 6.2.

The functional $\bar{G}$ satisfies Condition 5.8.

Proof.

It is easily seen that $\bar{G}(S,f,0)=0$, $S\in O, f\in P$. Since $f\in P$ and $g(S(t),t)\ge0$, $h(S_\delta(t-\delta),t)\ge0$, $t\in[0,T]$ we have

$0 \le \bar{G}(S,f,t+\epsilon) - \bar{G}(S,f,t)$, $\epsilon \ge 0$; $t+\epsilon$, $t\in[0,T]$, $S\in O$, $f\in P$.

It remains to be shown that $\bar{G}$ satisfies Condition 3.3 ii),iii).

Since $g(x,t), h(x,t)\ge 0$, $x\in\mathbb{R}$, $t\in[0,T]$ we obtain

$$0\le\bar{G}(S,f,t+\epsilon)-\bar{G}(S,f,t) \le \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t) \int_t^{t+\epsilon} \frac{df(\tau)}{d\tau}d\tau + \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} h(x,t)\epsilon$$

$$\le \left\{ \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} Lg(x,t) + \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} h(x,t) \right\} \epsilon, \ \epsilon\ge 0;\ t+\epsilon,\ t\in[0,T],\ S\in P,\ f\in\bar{M}_L.$$

Thus Condition 3.3 ii) holds.

For $G_1=(g_1L+h_1)T$ it is easy to verify that the inequality

$\sup_{t\in[0,T]} |\bar{G}(S_1,f,t) - \bar{G}(S_2,f,t)| \le G_1\|S_1-S_2\|_\infty$ holds for S_1, $S_2\in\bar{M}$, $f\in\bar{M}_L$.

If $S\in\bar{M}$ then $g(S(t),t)$ is a Lipschitz continuous function of t and the Lipschitz constant equals $g_1 \frac{\gamma}{B_1} + g_2$. With integration by parts we obtain

$$\int_0^t g(S(\tau),\tau)\frac{df(\tau)}{d\tau}d\tau = g(S(t),t)f(t) - \int_0^t f(\tau) \frac{dg(S(\tau),\tau)}{d\tau} d\tau,\ t\in[0,T],\ S\in\bar{M},\ f\in\bar{M}_L.$$

From this we derive the following inequalities:

$$\sup_{t\in[0,T]} |\bar{G}(S,f_1,t) - \bar{G}(S,f_2,t)| \le \left\{ \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t) + \int_0^T \left|\frac{dg(S(\tau),\tau)}{d\tau}\right| d\tau \right\} \|f_1-f_2\|_\infty$$

$$\le \left\{ \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t) + T(\frac{\gamma}{B_1}g_1+g_2) \right\} \|f_1-f_2\|_\infty,\ S\in\bar{M},\ f_1,\ f_2\in\bar{M}_L.$$

Thus Condition 3.3 iii) holds for $G_2=\left\{ \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t) + T(\frac{\gamma}{B_1}g_1+g_2) \right\}$. ∎

Remark 6.3.

i) The problem mentioned in Section 1 equivalent to the Stefan problem specified in Section 3 if we choose $b(x)=B_1$, $x\in\mathbb{R}$ and $g(x,t)=1$, $h(x,t)=0$, $x\in\mathbb{R}$, $t\in[0,T]$.

ii) Suppose the problem defined above is a mathematical model of an etching technique. Then the functions b and g reflect the ratio between the loss of etching agent and the loss of solid. This ratio depends on the time and the position of the moving boundary. Furthermore, if the function h is not identically zero, then there is also a loss of solid independent of the loss of etching agent.

Theorem 5.11 yields that there is a unique solution of the Stefan problem if $G_1+G_2< B$. This inequality holds if

$$(g_1(L+\frac{\gamma}{B_1})+g_2+h_1)T + \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t)<B_1 \text{ with } \gamma = \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} Lg(x,t) + \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} h(x,t).$$

<u>Lemma 6.4.</u>

If $\sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t)<B_1$ then there is a unique solution of this Stefan problem for $t\in[0,T]$.

<u>Proof.</u>

Take $T_1=\min\left\{ T,(B_1 - \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t))/(g_1(L+\frac{\gamma}{B_1})+g_2+h_1) \right\}.$

Existence and uniqueness on $[0,T_1]$ follow from Theorem 5.11.

If $T_1=T$ the lemma is proved.

In the other case denote the unique solution for $t\in[0,T_1]$ by $(\bar{S},C_{\bar{S}})$.

Define $\bar{M}^1=\{S\in\bar{M}\,|\,S(t)=\bar{S}(t),\ t\in[0,T_1]\}$ and $\bar{M}_L^1=\{f\in\bar{M}_L\,|\,f(t)=f_{\bar{S}}(t),t\in[0,T_1]\}$.

The following inequalities are easily verified,

$$\sup_{t\in[0,T]}|\bar{G}(S_1,f,t) - \bar{G}(S_2,f,t)| \le \{(g_1L+h_1)(T-T_1)\}\|S_1-S_2\|_\infty,\ S_1,\ S_2\in\bar{M}^1,\ f\in\bar{M}_L^1$$

and $$\sup_{t\in[0,T]}|\bar{G}(S,f_1,t) - \bar{G}(S,f_2,t)| \le \left\{ \sup_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t) + (\frac{\gamma g_1}{B_1}+g_2)(T-T_1) \right\} \|f_1-f_2\|_\infty,$$

$S\in\bar{M}^1$, f_1, $f_2\in\bar{M}_L^1$.

For $T_2 = \min\{T,2T_1\}$, Theorem 5.11 yields that there is a unique solution for the Stefan problem on $[0,T_2]$. Repetition of this procedure using $T_n = \min\{T,nT_1\}$ until $T_n=T$ proves the lemma. ∎

6.2 <u>Other examples</u>

<u>Example 1.</u>

Given $B_1,B_3>0$, $B_2\ge0$ and $\ell\ge0$, define $\bar{B}:\mathbb{R}\to\mathbb{R}$ by

$$\bar{B}(x)=\begin{cases} B_1x & ,\ x\in(-\infty,\ell) \\ [B_1\ell,B_1\ell+B_2] & ,\ x = \ell \\ B_1\ell+B_2+B_3(x-\ell) & ,\ x\in(\ell,\infty). \end{cases}$$

Furthermore, for given constants $g_1, g_2>0$ and $t_1\in(0,T)$ define

$$\bar{G}(S,f,t)=\begin{cases} g_1 f(t) & , t\in[0,t_1] \\ g_1 f(t_1)+g_2(f(t)-f(t_1)) & , t\in(t_1,T] \end{cases}, \quad S\in O,\ f\in P .$$

Condition 5.8 is satisfied and $B=\min\{B_1,B_3\}$, $\gamma=L\max\{g_1,g_2\}$, $G_1=0$ and $G_2=g_2+|g_1-g_2|$. If $g_2+|g_1-g_2|<\min\{B_1,B_3\}$ then Theorem 5.11 yields that there is a unique solution of the Stefan problem.

Remark 6.5.

This example can be a mathematical model of an etching technique, where the etching properties discontinuously depend on the time and the position of the time-dependent boundary.

Example 2.

Given $B_1>0$ define $\bar{B}:R\to R$ by $\bar{B}(x)=B_1x$, $x\in R$.

Given $\delta>0$ and $g_\delta:R\to[0,\infty)$, which is an integrable function such that $\mathrm{supp}(g_\delta)\subset[0,\delta)$ and $\int_{-\infty}^{\infty} g_\delta(\zeta)d\zeta=1$. Define $f_\delta(t)=\begin{cases} 0 & , t\in[-\delta,0) \\ f(t) & , t\in[0,T] \end{cases}$,

$$\bar{G}(S,f,t) = \int_{-\infty}^{\infty} f_\delta(\tau)g_\delta(t-\tau)d\tau,\quad t\in[0,T],\ S\in O,\ f\in P.$$

Condition 5.8 is satisfied and $B=B_1$, $\gamma=L$, $G_1=0$ and $G_2=1$.

Theorem 5.11 yields that there is a unique solution of the Stefan problem if $B_1>1$.

Remark 6.6.

A physical interpretation of this example is: it lasts some time before a loss of etching agent results in a loss of solid.

Example 3.

Given $B_1>0$ define $\bar{B}:R\to R$ by $\bar{B}(x)=B_1x$, $x\in R$. Given $g_1>0$ and $s_1\in R$ such that $0\le s_1\le Tg_1/B_1$ we define the functional

$$g:P\times R\to[0,\infty) \text{ by } g(f,t) = \begin{cases} 0 & , t\in[0,T+(f(T)-B_1s_1)/g_1] \\ g_1 & , t\in(T+(f(T)-B_1s_1)/g_1,\infty). \end{cases}$$

Define $\bar{G}(S,f,t)=f(t)+\int_0^t g(f,\tau)d\tau$, $t\in[0,T]$, $S\in O$, $f\in P$.

It is easy to see that

$$0\le\bar{G}(S,f,t+h)-\bar{G}(S,f,t)\le f(t+h)-f(t)+\int_t^{t+h} g(f,\tau)d\tau\le(L+g_1)h,\quad h\ge0;\ t+h,\ t\in[0,T],$$

$S \in P$, $f \in \bar{M}_L$ and

$$\sup_{t \in [0,T]} |\bar{G}(S,f_1,t) - \bar{G}(S,f_2,t)| \le \|f_1 - f_2\|_\infty + \left| \int_{T+(f_1(T)-B_1 s_1)/g_1}^{T+(f_2(T)-B_1 s_1)/g_1} g_1 d\tau \right| \le 2\|f_1 - f_2\|_\infty ,$$

$S \in \bar{M}$, $f \in \bar{M}_L$.

Thus Condition 5.8 holds with $B=B_1$, $\gamma=L+g_1$, $G_1=0$ and $G_2=2$.

Theorem 5.11 yields that there is a unique solution of the Stefan problem if $B_1>2$.

Remark 6.7.

For the solution $(\bar{S}, C_{\bar{S}})$ we know that $B_1\bar{S}(t)=f_{\bar{S}}(t) + \int_0^t g(f_{\bar{S}},\tau)d\tau$, $t \in [0,T]$.

If $f_{\bar{S}}(T) \ge B_1 s_1$ then $g(f_{\bar{S}},t)=0$, $t \in [0,T]$, thus $\bar{S}(T) = \frac{1}{B_1} f_{\bar{S}}(T) \ge s_1$.

In the other case $\bar{S}(T) = \frac{1}{B_1} f_{\bar{S}}(T) + \frac{1}{B_1} \int_{T+(f_{\bar{S}}(T)-B_1 s_1)/g_1}^{T} g_1 d\tau = s_1$.

Thus, $T+(f_{\bar{S}}(T) - B_1 s_1)/g_1$ can be seen as the point in time at which a source with strength g_1 is turned on so that $\bar{S}(T) \ge s_1$.

Remark 6.8.

In these examples the multifunctions $\bar{B}$ and the functionals $\bar{G}$ satisfy Condition 5.8. Thus, we can always apply Theorem 5.11. Now, we shall consider a mathematical model of a physical problem, such that Condition 5.8 does not hold.

For a given constant $\ell>0$ we suppose that for the time $t=0$, the region $x<0$ is filled with an etching liquid, the region $0 \le x < \ell$ is filled with a solid and the region $x \ge \ell$ is filled with another solid. Furthermore, we suppose that the two solids have different etching properties.

If we assume beforehand that the time-dependent boundary S is a monotone increasing function of t, then a possible mathematical model could be a Stefan problem as in Section 3 with $\bar{B}$ and $\bar{G}$ given by:

$$\bar{B}(x) = \begin{cases} B_1 x & , \ x \in [0,\ell) \\ B_1\ell + B_2(x-\ell) & , \ x \in [\ell,\infty) \end{cases} \text{ with } B_1, B_2 > 0 \text{ and}$$

$\bar{G}(S,f,t)=f(t)$, $t \in [0,T]$, $S \in P$, $f \in \bar{M}_L$ (see Example 1). The proportionality constants B_1 and B_2 are determined by the chemical properties of the etching agent and the solids.

However, if for some reason or other it cannot be ruled out beforehand that the boundary moves backwards, then it is not at all certain that the same model will still apply since in that case the two solids should precipitate in the reverse of the order in which they were etched away. In this model Condition 3.3 holds. If $B_1>1$ and $B_2>1$, existence and uniqueness of the solution in P follow from Theorem 5.7.

<u>Example 4.</u>

Suppose the positive constants a, T and B_1 are given. The given function $u_0:[a,\infty)\to[0,\infty)$ is such that the function $R\to(-R+a)u_0(-R+a)$ is an element of Cond 3.1. With $R=\sqrt{x_1^2+x_2^2+x_3^2}$ we define a 3-dimensional rotation-symmetrical Stefan problem as follows (see also [Friedman, 1964; p.234-236]):

$$\frac{\partial u(R,t)}{\partial t}-\frac{\partial^2 u(R,t)}{\partial R^2}-\frac{2}{R}\frac{\partial u(R,t)}{\partial R}=0,\ R\in(\tilde{S}(t),\infty),\ t\in(0,T],$$

$$u(R,0)=u_0(R),\ R\in[a,\infty),\ u(\tilde{S}(t),t)=0,\ t\in[0,T],$$

$$-\frac{1}{B_1}\frac{\partial u}{\partial R}(\tilde{S}(t),t)=\frac{d\tilde{S}(t)}{dt},\ t\in[0,T],\ \tilde{S}(0)=a.$$

Define $x=-R+a$, $C(x,t)=Ru(R,t)$, $R\in[\tilde{S}(t),\infty)$, $t\in[0,T]$, $C_0(x)=Ru_0(R)$, $R\in[a,\infty)$ and $S(t)=-\tilde{S}(t)+a$, $t\in[0,T]$.

It can be shown that S,C satisfy the following equations:

$$\frac{\partial C(x,t)}{\partial t}-\frac{\partial^2 C(x,t)}{\partial x^2}=0,\ x\in(-\infty,S(t)),\ t\in(0,T],$$

$$C(x,0)=C_0(x),\ x\in(-\infty,0],\ C(S(t),t)=0,\ t\in[0,T],$$

$$\frac{1}{B_1}\frac{\partial C}{\partial x}(S(t),t)=(-S(t)+a)\left(-\frac{dS(t)}{dt}\right),\ t\in[0,T],\ S(0)=0.$$

Integration to t of the last equation gives

$$\int_0^t\frac{1}{B_1}\frac{\partial C}{\partial x}(S(\tau),\tau)d\tau=\frac{1}{2}(-S(t)+a)^2-\frac{1}{2}a^2,\ t\in[0,T].$$

This is equivalent to $S(t)=a-\sqrt{a^2+\frac{2}{B_1}\int_0^t\frac{\partial C}{\partial x}(S(\tau),\tau)d\tau}$, $t\in[0,T]$.

Since $f_S(t)=-\int_0^t\frac{\partial C_s}{\partial x}(S(\tau),\tau)d\tau$ the

Stefan problem is equivalent to the Stefan problem specified in Section 3 if the inequality $2TL<B_1a^2$ holds and we choose $\bar{B}(x)=x$, $x\in[0,\infty)$,

$\bar{G}(S,f,t) = a-\sqrt{a^2 - \frac{2f(t)}{B_1}}$, $S\in P$, $f\in\bar{M}_L$, $t\in[0,T]$.

With the definition of C_0 it follows that C_0 satisfies Condition 3.1.

It is easy to prove that Condition 3.3 holds with $B=1$,

$\gamma = \dfrac{L}{B_1\sqrt{a^2-\frac{2TL}{B_1}}}$, $G_1 = 0$ and $G_2 = \dfrac{1}{B_1\sqrt{a^2-\frac{2TL}{B_1}}}$.

Thus Theorem 5.7 yields that there is a unique solution of the Stefan problem if $B_1\sqrt{a^2-\frac{2TL}{B_1}} > 1$.

Remark 6.9.

i) This example can be a mathematical model of the following etching problem. An etching liquid contains a solid ball with radius a, and the initial concentration of the etching agent is rotation-symmetric.

ii) In this example the functional $\bar{G}$ satisfies Condition 3.3 so we can apply Theorem 5.7 to obtain existence and uniqueness of solution. However the conditions of Theorem 5.11 are not fulfilled because for every $T>0$ there is a function $f\in P$ such that $a^2-\frac{2f(T)}{B_1} < 0$, which implies that $\bar{G}(S,f,T) = a - \sqrt{a^2-\frac{2f(T)}{B_1}}$ is not defined in $\mathbb{R}$.

6.3 Numerical experiments

In this subsection we present some numerical results to illustrate the existence theorems of Section 5. We first consider a Stefan problem which can be solved analytically. Then we compare a numerical solution of this problem with the analytical one. After that we give some numerical solutions of the examples considered in Section 6.2.

It is easy to see that for $B\in[1,\infty)$ there is a unique constant k such that $k = \frac{2}{B\sqrt{\pi}}\left(\dfrac{1}{1+\text{erf}(\frac{k}{2})}\right)e^{-(\frac{k}{2})^2}$. We find an approximation of k by the successive substitution process

$$k_{i+1} = \frac{2}{B\sqrt{\pi}}\left(\frac{1}{1+\text{erf}(\frac{k_i}{2})}\right)e^{-(\frac{k_i}{2})^2},\ i=0,1,\dots \text{ and } k_0=0.$$

Take $C_0(x)=1 - (1+\text{erf}(\frac{x+k}{2}))/(1+\text{erf}(\frac{k}{2}))$, $x\in(-\infty,0]$.

Then the functions $S(t)=k(\sqrt{t+1}-1)$, $t\in[0,1]$ and

$$C(x,t) = 1-\left(1+\mathrm{erf}\left(\frac{x+k}{2\sqrt{t+1}}\right)\right)/(1+\mathrm{erf}(\tfrac{k}{2})) \ , \ x\in(-\infty,S(t)], \ t\in[0,1]$$

satisfy

$$\frac{\partial C(x,t)}{\partial t} - \frac{\partial^2 C(x,t)}{\partial x^2} = 0 \ , \ x\in(-\infty,S(t)), \ t\in(0,1],$$

$C(x,0) = C_0(x)$, $x\in(-\infty,0]$, $C(S(t),t)=0$, $t\in[0,1]$ and

$$S(t) = -\frac{1}{B}\int_{-\infty}^{S(t)} [C(x,t) - C_0(x)]dx \ , \ t\in[0,1].$$

Choose $B=2$. Since for this choice of B and C_0 the conditions of Theorem 5.11 are fulfilled (compare Section 6.1), there is a unique solution $(\bar{S},C_{\bar{S}})$ of the Stefan problem. Furthermore, the operator $\mathcal{T}:\bar{M}\to\bar{M}$ is a contraction and it is known that for a given function $S_0\in\bar{M}$ the sequence of functions $S_{\ell+1}=\mathcal{T}(S_\ell)$, $\ell=0,1,\ldots$ is convergent and

$\lim_{\ell\to\infty} S_\ell=\bar{S}$.

With this in mind we compute a numerical solution as follows.

Take $M=100$, $N=200$, $\Delta x=0.1$, $\Delta t=0.005$, $eps=10^{-5}$ and k as given above.

a) Set $\ell=0$ and $S_j^{(0)}=0$, $j=0,1,\ldots,N$.

b) Compute $C_{i,j}^{(\ell)}$, which is an approximation of $C(-10+i\Delta x+S_j^{(\ell)}, j\Delta t)$ for $i=0,\ldots,M$ and $j=0,\ldots,N$ as follows:

$$C_{i,0}^{(\ell)} = 1-(1+\mathrm{erf}((-10+i\Delta x+k)/2))/(1+\mathrm{erf}(\tfrac{k}{2})), \ i=0,\ldots,M,$$

$$C_{0,j}^{(\ell)} = 1 \ , \ C_{M,j}^{(\ell)} = 0 \ , \ j=1,\ldots,N \text{ and}$$

$$C_{i,j}^{(\ell)}= C_{i,j-1}^{(\ell)} + \frac{\Delta t}{(\Delta x)^2}\left(C_{i-1,j-1}^{(\ell)} - 2C_{i,j-1}^{(\ell)} + C_{i+1,j-1}^{(\ell)}\right) +$$

$$\frac{1}{2\Delta x}\left(S_j^{(\ell)} - S_{j-1}^{(\ell)}\right)\left(C_{i+1,j-1}^{(\ell)} - C_{i-1,j-1}^{(\ell)}\right) \text{ for } i=2,\ldots,M-1, \ j=1,\ldots,N.$$

c) With $f_0^{(\ell)} = 0$, $f_j^{(\ell)} = -\sum_{n=1}^{j}\left[\frac{1}{2}\left(C_{M,n}^{(\ell)} - C_{M-1,n}^{(\ell)}\right) + \frac{1}{2}\left(C_{M,n-1}^{(\ell)} - C_{M-1,n-1}^{(\ell)}\right)\right]\cdot\frac{\Delta t}{\Delta x}$, $j=1,\ldots,N$, we obtain $S_j^{(\ell+1)} = \frac{1}{B} f_j^{(\ell)}$, $j=0,\ldots,N$.

If $\max_{j\in\{0,\dots,N\}} |S_j^{(\ell+1)} - S_j^{(\ell)}| < eps$ go to d)

otherwise $\ell := \ell + 1$ go to b).

d) $\hat{\ell} := \ell + 1$.

Define $S^{(\ell)} \in C[0,1]$, $\ell = 0,\dots,\hat{\ell}$ as follows $S^{(\ell)}(j\Delta t) = S_j^{(\ell)}$, $j=0,\dots,N$

and $S^{(\ell)}$ is a linear function on $[j\Delta t, (j+1)\Delta t]$, $j=0,\dots,N-1$.

Remark 6.10.

In b) we compute a numerical solution of the diffusion equation with initial and boundary conditions. In the computation of $C_{i,j}^{(\ell)}$ the convective term reflects the fact that at every time-step the grid is translated to the right over a distance $S_j^{(\ell)} - S_{j-1}^{(\ell)}$.

Since $1-C(-10+k,1) < 10^{-4}$, we consider the numerical solution $C_{i,j}^{(\ell)}$ of the diffusion equation on the domain

$\{(x,t)|x\in[-10 + S^{(\ell)}(t), S^{(\ell)}(t)], t\in[0,1]\}$.

The numerical solution turns out to satisfy:

$\max_{j\in\{0,\dots,N\}} | S(j\Delta t) - S_j^{(\ell)} | \leq 0.0014$ for $\hat{\ell}=6$ and

$\max_{i\in\{0,\dots,M\}} | C(-10+i\Delta t+S_N^{(\hat{\ell})},1) - C_{i,N}^{(\hat{\ell})} | \leq 0.0005$. Thus the numerical approximation shows good agreement with the exact solution of the Stefan problem.

In Figure 1 we have plotted $S^{(\ell)}$, $\ell=1,2,3$. In this case the iterates, form an alternating sequence. It follows from Lemma 5.5 that $\|\mathcal{T}(S_1) - \mathcal{T}(S_2)\|_\infty \leq 0.5 \|S_1 - S_2\|_\infty$, $S_1,S_2\in \bar{M}$.

Table 1 shows the distances $\|S^{(\ell)} - S^{(\ell-1)}\|_\infty$ for $\ell=1,\dots,6$.

It appears that

$$\max_{\ell\in\{1,\dots,5\}} \frac{\|S^{(\ell+1)} - S^{(\ell)}\|_\infty}{\|S^{(\ell)} - S^{(\ell-1)}\|_\infty} \leq 0.13.$$

ℓ	$\|S^{(\ell)}-S^{(\ell-1)}\|_\infty$
1	0.2051
2	0.268×10^{-1}
3	0.2932×10^{-2}
4	0.2914×10^{-3}
5	0.2644×10^{-4}
6	0.2217×10^{-5}

Table 1.

Remark 6.11.

We also report numerical results for the examples of Section 6.2. Obviously, in these examples we use another relation to obtain $S_j^{(\ell+1)}$ in step c). This relation is a numerical analogue of the relation $S_{\ell+1} = \mathcal{T}(S_\ell)$.

ℓ	$\|S^{(\ell)}-S^{(\ell-1)}\|_\infty$			
1	0.1657	0.2037	0.24	0.232
2	0.1804×10^{-1}	0.2625×10^{-1}	0.211×10^{-1}	0.3717×10^{-1}
3	0.1655×10^{-2}	0.2809×10^{-2}	0.2148×10^{-2}	0.4673×10^{-2}
4	0.1377×10^{-3}	0.2706×10^{-3}	0.1980×10^{-3}	0.5288×10^{-3}
5	0.1047×10^{-4}	0.2359×10^{-4}	0.1688×10^{-4}	0.5405×10^{-4}
6	0.7349×10^{-6}	0.1884×10^{-5}	0.1313×10^{-5}	0.5074×10^{-5}

Table 2.

Figure 2 and the first column of Table 2 show the results of the numerical solution for Example 1. We choose $B_1=2$, $B_2=0.03$, $B_3=3$, $\ell=0.02$, $g_1=1$, $g_2=1.5$ and $t_1=0.5$. Table 2 shows that $\|S^{(6)}-S^{(5)}\|_\infty \le 10^{-5}$

and $\max_{\ell\in\{1,\dots,5\}} \dfrac{\|S^{(\ell+1)}-S^{(\ell)}\|_\infty}{\|S^{(\ell)}-S^{(\ell-1)}\|_\infty} \le 0.11$.

The estimation in Lemma 5.5 yields $\dfrac{\|\mathcal{T}(S_2)-\mathcal{T}(S_1)\|_\infty}{\|S_2-S_1\|_\infty} \le 0.75$.

Figure 3 and the second column of Table 2 show the results of the numerical solution for Example 2. We choose

$B_1=2$, $\delta=\frac{1}{50}$ and $g_\delta:\mathbb{R}\to[0,\infty)$ as $g_\delta(x)=\begin{cases} 50 & x\in[0,\frac{1}{50}) \\ 0 & x\in\mathbb{R}\setminus[0,\frac{1}{50}) \end{cases}$.

Table 2 shows that $\|S^{(6)}-S^{(5)}\|_\infty \le 10^{-5}$ and

$\max_{\ell\in\{1,\dots,5\}} \dfrac{\|S^{(\ell+1)}-S^{(\ell)}\|_\infty}{\|S^{(\ell)}-S^{(\ell-1)}\|_\infty} \le 0.13$.

The estimation in Lemma 5.5 yields $\dfrac{\|\mathcal{T}(S_2)-\mathcal{T}(S_1)\|_\infty}{\|S_2-S_1\|_\infty} \le 0.5$.

Figure 4 and the third column of Table 2 show the results of the numerical solution for Example 3. We choose $B_1=2$, $g_1=0.75$ and $s_1=0.24$.

Table 2 shows that $\|S^{(6)}-S^{(5)}\|_\infty \le 10^{-5}$ and

$$\max_{\ell\in\{1,\dots,5\}} \frac{\|S^{(\ell+1)}-S^{(\ell)}\|_\infty}{\|S^{(\ell)}-S^{(\ell-1)}\|_\infty} \le 0.1.$$

The estimation in Lemma 5.5 yields $\frac{\|\mathcal{T}(S_2)-\mathcal{T}(S_1)\|_\infty}{\|S_2-S_1\|_\infty} \le 1$.

It follows from Figure 4 that in this example the iterates do not form an alternating sequence. The numerical approximation of the quantity $T + (f_{\bar{s}}(T) - B_1 s_1)/g_1$ mentioned in Remark 6.7 is 0.8363.

Figure 5 and the fourth column of Table 2 show the results of the numerical solution for Example 4. We choose $B_1=2$ and $a=1$. Table 2 shows that $\|S^{(6)}-S^{(5)}\|_\infty \le 10^{-5}$ and $\max_{\ell\in\{1,\dots,5\}} \frac{\|S^{(\ell+1)}-S^{(\ell)}\|_\infty}{\|S^{(\ell)}-S^{(\ell-1)}\|_\infty} \le 0.16$.

Since $L=0.455$ the estimation in Lemma 5.5 yields $\frac{\|\mathcal{T}(S_2)-\mathcal{T}(S_1)\|_\infty}{\|S_2-S_1\|_\infty} \le 0.68$.

References.

1. J.R.Cannon, M.Primicerio
Remarks on the one-phase Stefan problem for the heat equation with the flux prescribed on the fixed boundary.
J.Math.An.Appl. 35 361-373 (1971)

2. J.Douglas
A uniqueness theorem for the solution of a Stefan problem.
Proc.Amer.Math.Soc. 8 402-408 (1957)

3. G.W.Evans
A note on the existence of a solution to a problem of Stefan.
Q.Appl.Math. 9 185-193 (1951)

4. A.Fasano, M.Primicerio
General free boundary problems for the heat equation I.
J.Math.Anal.Appl. 57 694-723 (1977)

5. A.Friedman
Partial Differential Equations of Parabolic Type.
Prentice-Hall, Englewood Cliffs (1964)

6. W.K.Hayman, P.B.Kennedy
Subharmonic Functions I.
Academic Press, London (1976)

7. C.D.Hill, D.B.Kotlow
Classical solutions in the large of a two-phase free boundary problem I,II.
Arch.Rat.Mech.Anal. 45 63-78
47 369-379 (1972)

8. H.K.Kuiken
Etching: a two dimensional mathematical approach.
Proc.R.Soc. London A392 199-225 (1984)

9. H.K.Kuiken, J.J.Kelly, P.H.L.Notten
Etching profiles at resist edges I.
J.Electrochem. Soc. 133 1217-1226 (1986)

10. W.T.Kyner
An existence and uniqueness theorem for a nonlinear Stefan problem.
J.Math.Mech. 8 483-498 (1959)

11. O.A.Ladyženskaja, V.A.Solonnikov, N.N.Ural'ceva
Linear and quasilinear equations of parabolic type.
American Mathematical Society, Providence (1968)

12. M.H.Protter, H.F.Weinberger
Maximum principles in differential equations.
Prentice-Hall, Englewood Cliffs (1967)

13. L.I.Rubinstein
The Stefan Problem.
American Mathematical Society, Providence. (1971)

14. R.E.Smithson
Multifunctions
Nieuw Arch.Wisk.(3) 20 31-53 (1972)

15. C.Vuik, C.Cuvelier
Numerical solution of an etching problem.
J.Comp.Physics 59 247-263 (1985)

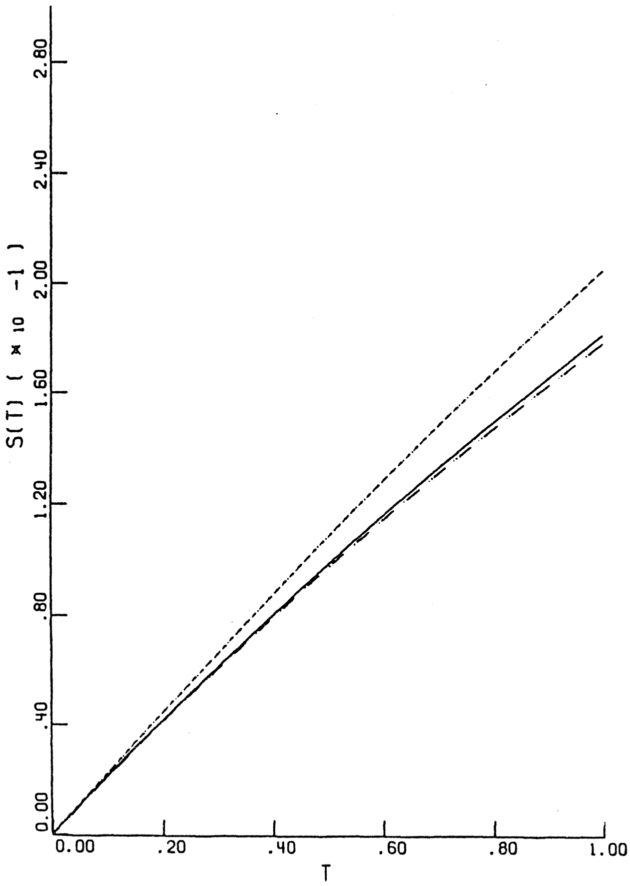

Figure 1

--- iterate 1

— — iterate 2

____ iterate 3

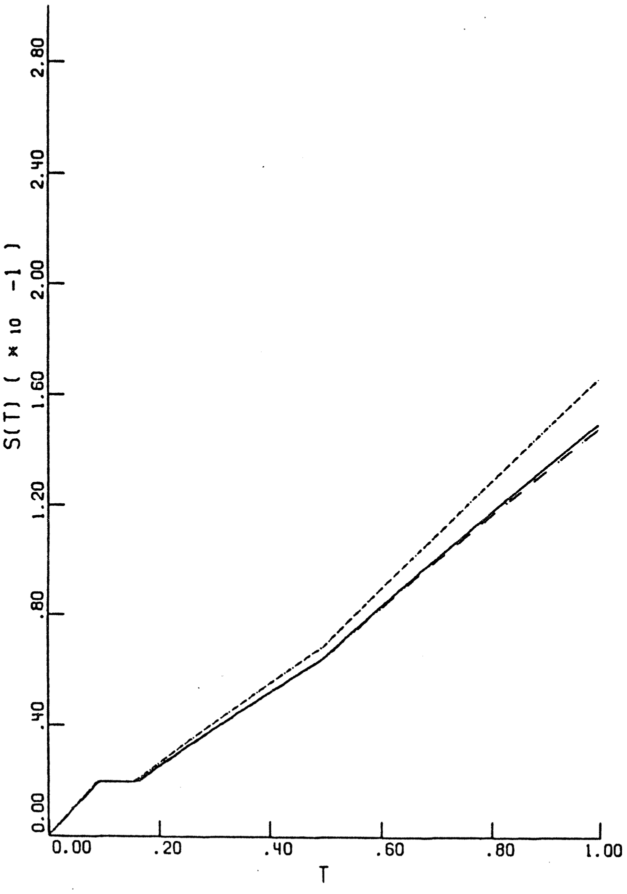

Figure 2

--- iterate 1

— — iterate 2

_____ iterate 3

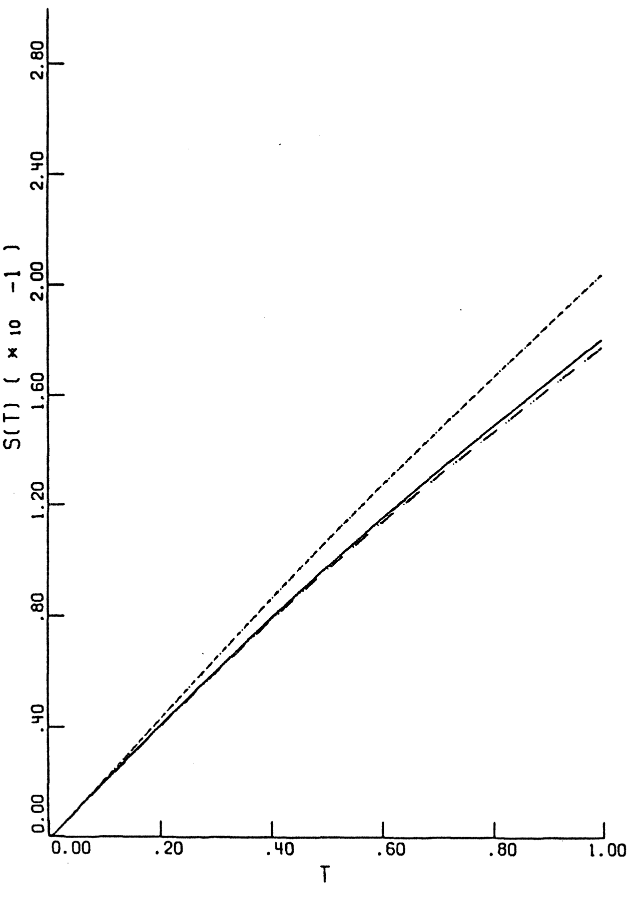

Figure 3

--- iterate 1
— — iterate 2
____ iterate 3

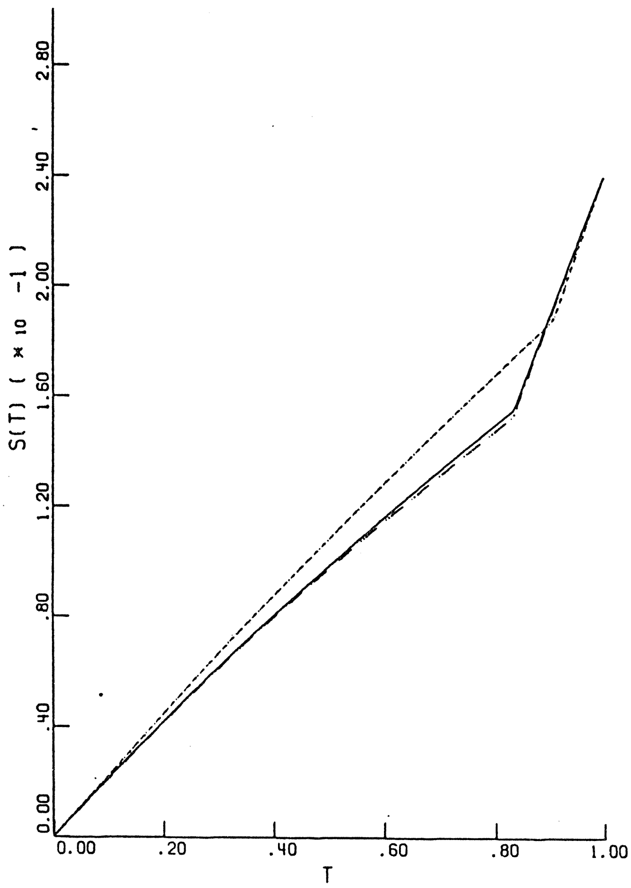

Figure 4

--- iterate 1
— — iterate 2
——— iterate 3

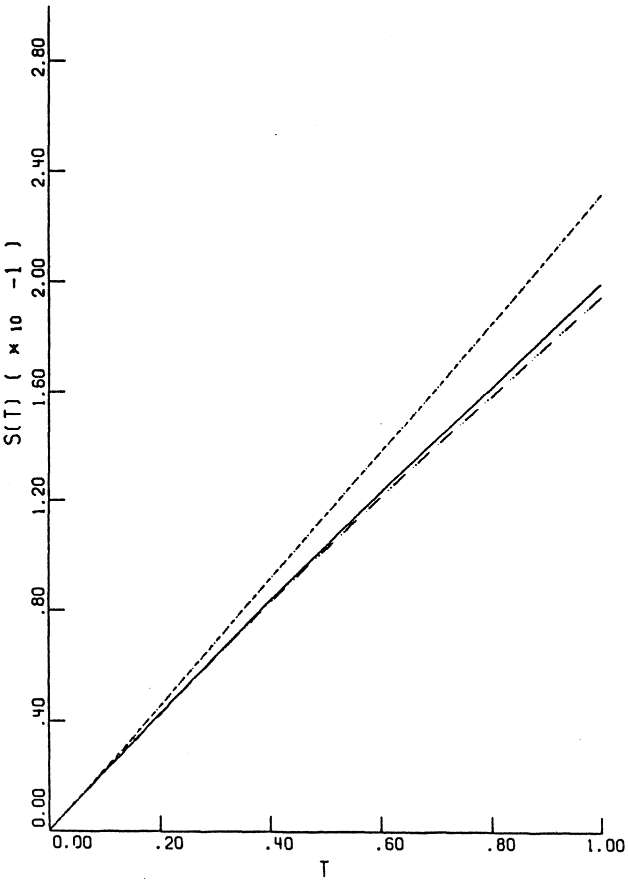

Figure 5

--- iterate 1
- - iterate 2
____ iterate 3

Chapter 2

The solution of a one-dimensional Stefan problem II

1. Introduction

In this paper we study a one-dimensional Stefan problem. An example of such a problem is: given $T > 0$, $B > 0$ and $C_0:(-\infty,0]\to[0,1]$, find sufficiently smooth functions $S:[0,T]\to\mathbb{R}$ and $C:\{(x,t)|x\in(-\infty,S(t)],\ t\in[0,T]\}\to\mathbb{R}$ such that:

$$(1.1)\qquad \begin{cases} \dfrac{\partial C(x,t)}{\partial t} - \dfrac{\partial^2 C(x,t)}{\partial x^2} = 0, x\in(-\infty,S(t)),\ t\in(0,T], \\ C(x,0)=C_0(x), x\in(-\infty,0],\ C(S(t),t)=0,\ t\in[0,T], \end{cases}$$

$$(1.2)\qquad -\frac{1}{B}\int_{-\infty}^{\infty}[C(x,t)-C_0(x)]dx=S(t), t\in[0,T].$$

For easy notation, we define $C(x,t)=0$ for $x\in(S(t),\infty)$, $t\in[0,T]$ and $C_0(x)=0$ for $x\in(0,\infty)$.

An existence and uniqueness theorem for this Stefan problem, if $B > 1$ and certain conditions on C_0 are imposed, is given in [Vuik, 1987]. Furthermore it follows from [Vuik, 1987] that this Stefan problem can be seen as a mathematical model for an etching technique. The function C describes the concentration of the etching agent and S(t) denotes the position of the time dependent boundary between the solid and the etching liquid. The proportionality constant B which is determined by the chemical properties of the etching agent and the solid appears to be greater than one. In the existence and uniqueness theorem of [Vuik, 1987] we have used the fact that B is greater than one.

In this paper we prove an existence and uniqueness theorem which is also valid for $0 < B \le 1$. In [Schulze, Beckett, Howarth and Poots, 1983] the solidification of steel is described. They use a mathematical model which is equivalent to the given Stefan problem. In this case for $t=0$ the region $x \ge 0$ consists of liquid steel and the region $x<0$ consists of solid steel. The normalized melting temperature of steel is equal to zero. The temperature of the liquid steel equals zero, and the temperature of the solid steel is described by the function -C. The position of the time dependent boundary between solid and liquid is denoted by S(t). The proportionality constant B in this example is 0.28 (see [Schulze e.a., 1983;p.336, Table 1], where this quantity is called β (Stefan number)). Thus a Stefan problem with $0 < B \leq 1$ is also interesting from a physical and technical point of view.

This paper is organized as follows. In Section 2 we give some definitions and specify the Stefan problem. We need some results from [Vuik, 1987], these are summarized in Section 3.

In Section 4 we prove an estimate for the solution of the diffusion equation (Theorem 4.1) and show that this estimate is sharp. Thereafter we define an operator $\mathcal{T}$. From the estimate given in Theorem 4.1 it follows that $\mathcal{T}^m$ is a contraction for m large enough. Using this property of the operator $\mathcal{T}$ we prove in our Main Theorem 4.17 the existence and the uniqueness of a solution for the Stefan problem for every $B > 0$, where $S \in C[0,T]$ and the function $C \in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ is bounded.

In Section 5 we give some examples of the Stefan problem specified in Section 2, and compute numerical solutions for these examples. Furthermore we give some invariance properties of the estimate given in Theorem 4.1 and the Stefan problem. From this it follows that the estimate given in Theorem 4.1 is optimal in a certain sense. Finally we give an example where the function C_0 is such that $\lim_{x \uparrow 0} C_0(x) > 0$. It appears that our results (e.g. $\mathcal{T}^m$ is a contraction for m large enough) do not hold for this example.

2. Statement of the problem

In this section we give some definitions and specify a Stefan problem.

2.1 Definitions

For a given $T > 0$ we define the following function spaces:

$$O = \{S \in C[0,T] \mid S(0)=0\},$$

$$P = \{S \in O \mid S \text{ is monotone non-decreasing}\},$$

and for $K > 0$, $\bar{M}_K = \{S \in P \mid S(t+h)-S(t) \le Kh,\ h \ge 0;\ t+h, t \in [0,T]\}$,

$$M_K = \bar{M}_K \cap C^2[0,T].$$

We define $\|f\|_{\hat{t},\infty} = \sup_{t\in[0,\hat{t}]} |f(t)|$, $\hat{t}\in[0,T]$ and $\|f\|_\infty = \|f\|_{T,\infty}$.

It follows from [Vuik, 1987; Corollary 2.8] that $\bar{M}_K$ is the closure of M_K in $C[0,T]$ with respect to the ∞-norm.

For a given function $S \in C[0,T]$ the set $Q_S \subset \mathbb{R}^2$ is defined by $Q_S = \{(x,t) \mid x \in (-\infty, S(t)),\ t \in (0,T)\}$. The closure of a set $Q \subset \mathbb{R}^2$ is denoted by $\bar{Q}$.

2.2 The function spaces $C^{\ell,\ell/2}(\bar{Q}_S)$, $C^{2,1}(Q_S)$ and $C^{\ell}[a,b]$

We use the following function spaces defined in [Ladyženskaja, Solonnikov, Ural'ceva, 1968; p.7]. For a given $\ell \in \mathbb{R}^+ \backslash \mathbb{N}$, $C^{\ell,\ell/2}(\bar{Q}_S)$ is the Banach space of continuous functions f on $\bar{Q}_S$, having continuous derivatives $\dfrac{\partial^{r+p} f}{\partial t^r \partial x^p}$ for $2r+p<\ell$ and a finite norm $\|f\|^{\ell,\ell/2}$. Here the norm $\|f\|^{\ell,\ell/2}$ is defined by : let $[\ell]$ be the largest integer less than ℓ,

$$\|f\|^{\ell,\ell/2} = \sum_{j=0}^{[\ell]} \left(\sum_{2r+p=j} \max_{\bar{Q}_S} \left| \frac{\partial^{r+p} f}{\partial t^r \partial x^p} \right| \right) + \sum_{2r+p=[\ell]} \left\langle \frac{\partial^{r+p} f}{\partial t^r \partial x^p} \right\rangle_x^{\ell-[\ell]} + \sum_{0<\ell-2r-p<2} \left\langle \frac{\partial^{r+p} f}{\partial t^r \partial x^p} \right\rangle_t^{\frac{\ell-2r-p}{2}},$$

and

$$\langle f \rangle_x^\alpha = \sup \left\{ \frac{|f(x',t)-f(x'',t)|}{|x'-x''|^\alpha} \,\middle|\, (x',t),(x'',t) \in \bar{Q}_S;\ |x'-x''| \le 1 \right\},$$

$$\langle f \rangle_t^\alpha = \sup \left\{ \frac{|f(x,t')-f(x,t'')|}{|t'-t''|^\alpha} \,\middle|\, (x,t'),(x,t'') \in \bar{Q}_S;\ |t'-t''| \le 1 \right\} \text{ for } \alpha \in (0,1).$$

$C^{2,1}(Q_S)$ is the set of continuous functions f on Q_S, having continuous derivatives $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial f}{\partial t}$.

For a given $\ell \in R^+ \setminus N$ and $[a,b] \subset R$, $C^\ell[a,b]$ is the Banach space of continuous functions f on $[a,b]$, having continuous derivatives $\frac{d^p f}{dx^p}$ for $p<\ell$, and a finite norm $\|f\|^\ell$. Here the norm $\|f\|^\ell$ is defined by $\|f\|^\ell = \sum_{j=0}^{[\ell]} \max_{[a,b]} |\frac{d^j f}{dx^j}| + \langle \frac{d^{[\ell]} f}{dx^{[\ell]}} \rangle^{\ell-[\ell]}$

and $\langle f \rangle^\alpha = \sup\left\{ \frac{|f(x')-f(x")|}{|x'-x"|^\alpha} \;\middle|\; x',x" \epsilon [a,b]; |x'-x"| \leq 1 \right\}$ and $\alpha \epsilon (0,1)$.

2.3 Statement of the problem

Suppose $C_0:(-\infty,0] \to [0,1]$ is a given function. We shall always impose Condition 2.1:

Condition 2.1.

The function C_0 should be an element of the set Cond 2.1:= $\{\phi:(-\infty,0] \to R \mid \phi$ is a monotone decreasing Lipschitz continuous function with $\phi(0)=0$ and $\lim_{x \to -\infty} \phi(x)=1\}$. Let L be a Lipschitz constant of the function C_0.

Occasionally we shall impose the following stronger condition:

Condition 2.2.

The function C_0 should be an element of the set Cond 2.2:= $\{\phi \epsilon C^{3+\alpha}(-\infty,0]$ for an $\alpha \epsilon (0,1) \mid \phi \epsilon$Cond 2.1 and $\frac{d\phi}{dx}(0) = \frac{d^2\phi}{dx^2}(0)=0\}$.

In the sequel, $\bar{B}:R \to R$ will denote a multifunction (see[Smithson, 1972]) and $\bar{G}:O \times P \times [0,T] \to [0,\infty)$ will denote a functional, both subject to the following condition, which will always be imposed:

Condition 2.3.

The multifunction $\bar{B}$ should be such that:

i) $0 \epsilon \bar{B}(0)$.

ii) For $x \epsilon R$, $h>0, y_1 \epsilon \bar{B}(x)$ and $y_2 \epsilon \bar{B}(x+h)$ the inequality

$y_2 - y_1 > 0$ holds. There is a $B \in (0,\infty)$ such that for $x \in [0,\infty)$, $h>0$, $y_1 \in \bar{B}(x)$ and $y_2 \in \bar{B}(x+h)$ the inequality $y_2 - y_1 \geq Bh$ holds.

iii) The restriction of $\bar{B}$ to $[0,\infty)$ should be surjective on $[0,\infty)$.

The functional $\bar{G}$ should be such that:

iv) $\bar{G}(S,f,0)=0$, $S \in O$, $f \in P$.

v) $0 \leq \bar{G}(S,f,t+h) - \bar{G}(S,f,t)$, $h \geq 0$; $t+h, t \in [0,T]$, $S \in O, f \in P$. There is a $\gamma \in [0,\infty)$ such that $\bar{G}(S,f,t+h) - \bar{G}(S,f,t) \leq \gamma h$ for $h \geq 0$; $t+h$, $t \in [0,T]$, $S \in P$ and $f \in \tilde{M}_L$.

vi) There are constants G_1, $G_2 \in R$ so that

$$\sup_{t \in [0,T]} |\bar{G}(S_1,f,t) - \bar{G}(S_2,f,t)| \leq G_1 \|S_1 - S_2\|_\infty, \; S_1, S_2 \in \tilde{M}_{\gamma/B}, \; f \in \tilde{M}_L,$$

$$\sup_{t \in [0,T]} |\bar{G}(S,f_1,t) - \bar{G}(S,f_2,t)| \leq G_2 \|f_1 - f_2\|_\infty, S \in \tilde{M}_{\gamma/B}, \; f_1, f_2 \in \tilde{M}_L.$$

vii) For every $t \in [0,T]$, $S \in \tilde{M}_{\gamma/B}$ and $f \in \tilde{M}_L$, $\bar{G}(S,f,t)$ should only depend on $S|_{[0,t]}$ and $f|_{[0,t]}$.

In order to state our Stefan problem, we first state a reduced problem: for a given function $S \in O$, find a bounded solution $C_S \in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ of the equations in (1.1) for this function S.

For any $S \in O$, for which the reduced problem has a solution C_S, we define $C_S(x,t)=0$, $x > S(t)$, $t \in [0,T]$. Let the function $f_S:[0,T] \to R$ be defined by $f_S(t) = -\int_{-\infty}^{\infty} [C_S(x,t) - C_0(x)]dx$.

A Stefan problem can be stated as follows: find a function $S \in O$ and a solution C_S of the reduced problem such that

$$\bar{G}(S,f_S,t) \in \bar{B}(S(t)), \; t \in [0,T].$$

3. Preliminaries

In this section we summarize known results about the solution of the diffusion equation and quote some basic lemmas.

Lemma 3.1 (maximum principle) [Vuik, 1987; Lemma 2.4].

Suppose $S \in C[0,T]$. If the bounded function $u \in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ satisfies:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, x \in (-\infty, S(t)), t \in (0,T],$$

then
$$\min \{ \inf_{x \in (-\infty, S(0)]} u(x,0), \min_{t \in [0,T]} u(S(t),t) \} \le u(\tilde{x},\tilde{t}) \le$$

$$\max \{ \sup_{x \in (-\infty, S(0)]} u(x,0), \max_{t \in [0,T]} u(S(t),t) \} \text{ for } (\tilde{x},\tilde{t}) \in \bar{Q}_S.$$

Lemma 3.2 [Vuik, 1987; Lemma 2.6].

Suppose the function $\phi:(-\infty,0] \to [0,1]$ is an element of Cond 2.1. For every $\epsilon>0$ there is a function $\phi^-:(-\infty,0] \to [0,1]$ which is an element of Cond 2.2, $\phi^-(x) \le \phi(x)$, $x \in (-\infty,0]$ and $\int_{-\infty}^{0} [\phi(x)-\phi^-(x)]dx < \epsilon$, and a Lipschitz constant of ϕ is also a Lipschitz constant of ϕ^-.

In the same way it can be shown that there is a function $\phi^+:(-\infty,\epsilon] \to [0,1]$ such that $x \to \phi^+(x+\epsilon)$ is an element of Cond 2.2, $\phi(x) \le \phi^+(x)$, $x \in (-\infty,0]$ and $\int_{-\infty}^{0} [\phi^+(x)-\phi(x)]dx + \int_{0}^{\epsilon} \phi^+(x)dx < \epsilon$, and a Lipschitz constant of ϕ is also a Lipschitz constant of ϕ^+.

Lemma 3.3 [Vuik, 1987; Lemma 2.7].

For $\epsilon>0$, $K>0$ and $S \in \bar{M}_K$, there are functions S^+, $S^- \in M_K$ such that $S^-(t) \le S(t) \le S^+(t)$, $t \in [0,T]$ and $\|S^+ - S^-\|_\infty < \epsilon$.

Theorem 3.4 [Vuik, 1987; Theorems 4.1, 4.13].

i) Suppose Condition 2.2 holds. Then for any $S \in C^2[0,T]$ with $S(0)=0$, the reduced problem has a unique solution C_S and it satisfies $C_S \in C^{3+\alpha, \frac{3+\alpha}{2}}(\bar{Q}_S)$ with the same α as in Condition 2.2.

ii) For a given function $S \epsilon \tilde{M}_{7/B}$, there is a unique solution C_S of the reduced problem, $0 \leq \frac{-\partial C_S(x,t)}{\partial x} \leq L, (x,t) \epsilon Q_S$ and $f_S \epsilon \tilde{M}_L$.

Lemma 3.5 [Vuik, 1987; Lemma 4.6].

Suppose Condition 2.2 holds. Then for any $S \epsilon C^2[0,T]$ with $S(0)=0$, the solution C_S of the reduced problem satisfies:

$$\int_{-\infty}^{\infty} [C_S(x,t) - C_0(x)] dx = \int_0^t \frac{\partial C_S}{\partial x}(S(\tau), \tau) d\tau .$$

Lemma 3.6 [Vuik, 1987; Theorem 4.11, 4.12].

i) If S_1, $S_2 \epsilon O$, $S_1(t) \leq S_2(t)$, $t \epsilon [0,T]$ and there are solutions C_{S_1}, C_{S_2} of the reduced problem, then $C_{S_1}(x,t) \leq C_{S_2}(x,t), (x,t) \epsilon \bar{Q}_{S_2}$ and

$$\int_{-\infty}^{\infty} [C_{S_2}(x,t) - C_{S_1}(x,t)] dx \leq \|S_1 - S_2\|_{\bar{t},\infty}, \quad t \epsilon [0,\bar{t}], \ \bar{t} \epsilon [0,T].$$

ii) If $S \epsilon P$ and there is a solution C_S of the reduced problem then $f_S \epsilon \tilde{M}_L$.

iii) If $S \epsilon O$ and there is a solution C_S of the reduced problem then $f_S \epsilon P$.

4 The solution of the Stefan problem

In this section we prove an important property of solutions of the reduced problem. After that we use this property to prove the existence and uniqueness of a solution of the Stefan problem.

4.1 An estimation of the quantity $\|f_{S_1}-f_{S_2}\|_{t,\infty}$

In this subsection we give in Theorem 4.1 an overestimate of $\|f_{S_1}-f_{S_2}\|_{t,\infty}$. After that we illustrate this estimate with a numerical example. Finally we prove an underestimate for the quantity $\|f_{S_1}-f_{S_2}\|_{t,\infty}$.

Theorem 4.1.

Suppose $S_1,S_2\epsilon\bar{M}_{\gamma/B}$, $t_1,t_2\epsilon[0,T]$ and $t_1\leq t_2$ then the following inequality holds:

$$\|f_{S_1}-f_{S_2}\|_{t_2,\infty}\leq\|S_1-S_2\|_{t_1,\infty}+2L\sqrt{\frac{t_2-t_1}{\pi}}\|S_1-S_2\|_{t_2,\infty}.$$

Proof.

Define $\bar{S}(t)=\max(S_1(t),S_2(t))$ and $\underline{S}(t)=\min(S_1(t),S_2(t))$, $t\epsilon[0,T]$.

It is easily seen that $\bar{S},\underline{S}\epsilon\bar{M}_{\gamma/B}$ and $\|S_1-S_2\|_{t,\infty}=\|\bar{S}-\underline{S}\|_{t,\infty}$, $t\epsilon[0,T]$.

From Theorem 3.4 ii) follows that $C_{\bar{S}},C_{\underline{S}}$ exist and $f_{\bar{S}},f_{\underline{S}}\epsilon\bar{M}_L$.

Application of the maximum principle yields

$$(4.2)\qquad \|f_{S_1}-f_{S_2}\|_{t_2,\infty}\leq\max_{t\epsilon[0,t_2]}|f_{\bar{S}}(t)-f_{\underline{S}}(t)|=\int_{-\infty}^{\infty}[C_{\bar{S}}(x,\hat{t})-C_{\underline{S}}(x,\hat{t})]dx \text{ for some } \hat{t}\epsilon[0,t_2].$$

If $\hat{t}\epsilon[0,t_1]$ then Lemma 3.6 i) states that

$$\int_{-\infty}^{\infty}[C_{\bar{S}}(x,\hat{t})-C_{\underline{S}}(x,\hat{t})]dx\leq\|\bar{S}-\underline{S}\|_{t_1,\infty}=\|S_1-S_2\|_{t_1,\infty}.$$

In this case the theorem is proved.

Now we suppose that $\hat{t}\epsilon(t_1,t_2]$. According to Lemma 3.3 for every $\epsilon>0$ there is an $S^-\epsilon C^2[t_1,\hat{t}]$ such that $S^-(t_1)=\underline{S}(t_1)$, $\underline{S}(t)-\epsilon\leq S^-(t)\leq\underline{S}(t)$, $t\epsilon[t_1,\hat{t}]$ and $0\leq\frac{dS^-(t)}{dt}\leq\frac{\gamma}{B}, t\epsilon[t_1,\hat{t}]$.

Define $S^-(t)=\underline{S}(t)$, $t\epsilon[0,t_1)$, $S^+(t)=S^-(t)+\|\bar{S}-\underline{S}\|_{\hat{t},\infty}+\epsilon$, $t\epsilon[0,\hat{t}]$ and

$$Q^{\pm}=\{(x,t)|x\epsilon(-\infty,S^{\pm}(t)),\ t\epsilon(t_1,\hat{t})\}.$$

With Theorem 3.4 ii) it is easily seen that the functions $x \to C_{\bar{S}}(x+\bar{S}(t_1),t_1)$ and $x \to C_{\underline{S}}(x+\underline{S}(t_1),t_1)$ are elements of Cond 2.1. Thus it follows from Lemma 3.2 that there are functions $C_0^{\pm}:(-\infty, S^{\pm}(t_1)] \to [0,1]$ such that the functions $x \to C_0^{\pm}(x+S^{\pm}(t_1))$ are elements of Cond 2.2,

$C_0^-(x) \le C_{\underline{S}}(x,t_1)$, $x \in (-\infty, S^-(t_1)]$,

$C_{\bar{S}}(x,t_1) \le C_0^+(x)$, $x \in (-\infty, S^+(t_1)]$, $\int_{-\infty}^{S^-(t_1)} [C_{\underline{S}}(x,t_1) - C_0^-(x)]dx < \epsilon$

and $\int_{-\infty}^{S^+(t_1)} [C_0^+(x) - C_{\bar{S}}(x,t_1)]dx < \epsilon$.

It follows from Theorem 3.4 i) that there are functions $C^{\pm} \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q}^{\pm})$ such that

$$\frac{\partial C^{\pm}(x,t)}{\partial t} - \frac{\partial^2 C^{\pm}(x,t)}{\partial x^2} = 0, x \in (-\infty, S^{\pm}(t)), \ t \in (t_1, \hat{t}],$$

$$C^{\pm}(x,t_1) = C_0^{\pm}(x), x \in (-\infty, S^{\pm}(t_1)], \ C^{\pm}(S^{\pm}(t),t) = 0, \ t \in [t_1, \hat{t}].$$

According to the maximum principle we have

$$\int_{-\infty}^{\infty} [C_{\bar{S}}(x,\hat{t}) - C_{\underline{S}}(x,\hat{t})]dx \le \int_{-\infty}^{\infty} [C^+(x,\hat{t}) - C^-(x,\hat{t})]dx.$$

Application of Lemma 3.5 yields

$$\int_{-\infty}^{\infty} [C^+(x,\hat{t}) - C^-(x,\hat{t})]dx = \int_{-\infty}^{\infty} [C_0^+(x) - C_0^-(x)]dx + \int_{t_1}^{\hat{t}} [\frac{\partial C^+}{\partial x}(S^+(\tau),\tau) - \frac{\partial C^-}{\partial x}(S^-(\tau),\tau)]d\tau.$$

Furthermore we deduce from the definition of C_0^+, C_0^- and Lemma 3.6 i):

$$\int_{-\infty}^{\infty} [C_0^+(x) - C_0^-(x)]dx \le \int_{-\infty}^{\infty} [C_{\bar{S}}(x,t_1) - C_{\underline{S}}(x,t_1)]dx + 2\epsilon$$

$$\le \|\bar{S} - \underline{S}\|_{t_1,\infty} + 2\epsilon = \|S_1 - S_2\|_{t_1,\infty} + 2\epsilon.$$

This together with inequality (4.2) yields

$$\|f_{S_1} - f_{S_2}\|_{t_2,\infty} \le \|S_1 - S_2\|_{t_1,\infty} + \int_{t_1}^{\hat{t}} [\frac{\partial C^+}{\partial x}(S^+(\tau),\tau) - \frac{\partial C^-}{\partial x}(S^-(\tau),\tau)]d\tau + 2\epsilon. \tag{4.3}$$

It remains to estimate the quantity: $\int_{t_1}^{\hat{t}} [\frac{\partial C^+}{\partial x}(S^+(\tau),\tau) - \frac{\partial C^-}{\partial x}(S^-(\tau),\tau)]d\tau$.

For this purpose we introduce the function $\hat{C}$ given by

$\hat{C}(x,t) = C^+(x + \|S^+ - S^-\|_{\hat{t},\infty}, t) - C^-(x,t)$, $(x,t) \in \mathbb{R} \times [t_1, \hat{t}]$ which satisfies:

$$\frac{\partial \hat{C}(x,t)}{\partial t} - \frac{\partial^2 \hat{C}(x,t)}{\partial x^2} = 0, \ x \in (-\infty, S^-(t)), \ t \in (t_1, \hat{t}],$$

$$\hat{C}(S^-(t),t) = 0, \ t \in [t_1, \hat{t}] \text{ and } -L \le \frac{\partial \hat{C}(x,t)}{\partial x} \le L, \ (x,t) \in \bar{Q}^-.$$

We deduce from the maximum principle:

$$0 \le C^+(x,t) - C^-(x,t),\ (x,t)\epsilon\bar{Q}^-.$$

Since Theorem 3.4 ii) states that the function $x\to C^+(x,t)$ is Lipschitz continuous the inequality

$$-L\|S^+-S^-\|_{\hat{t},\infty} \le C^+(x+\|S^+-S^-\|_{\hat{t},\infty},t)-C^+(x,t)$$

holds for $(x,t)\epsilon\bar{Q}^-$. Addition of these inequalities yields

$$(4.4)\qquad -L\|S^+-S^-\|_{\hat{t},\infty} \le \hat{C}(x,t),\ (x,t)\epsilon\bar{Q}^-.$$

Choose $\delta_1 > 0$. Since $\lim_{x\to-\infty} \mathrm{erf}(x) = -1$, there is a $\delta_2 > 0$ such that

$$(1+\delta_1)\mathrm{erf}(-\|S^+-S^-\|_{\hat{t},\infty}/2\sqrt{\delta_2}) < -1.$$

Furthermore we choose $\tau\ \epsilon(t_1,\hat{t}]$ and define

$$\bar{C}(x,t)=(1+\delta_1)L\|S^+-S^-\|_{\hat{t},\infty}\mathrm{erf}\left(\frac{x-S^-(\tau)}{2\sqrt{t-t_1+\delta_2}}\right),(x,t)\epsilon(-\infty,S^-(\tau)]\times[t_1,\tau].$$

The function $\bar{C}$ satisfies:

$$\frac{\partial\bar{C}(x,t)}{\partial t} - \frac{\partial^2\bar{C}(x,t)}{\partial x^2} = 0,(x,t)\epsilon(-\infty,S^-(\tau))\times(t_1,\tau].$$

Since $S^-(t)-S^-(\tau) \le 0$, $t\epsilon[t_1,\tau]$ it follows that $\bar{C}(S^-(t),t) \le 0 = \hat{C}(S^-(t),t),t\epsilon[t_1,\tau]$.

From the definition of $\bar{C}$ and inequality (4.4) we obtain

$$\bar{C}(x,t_1) \le -L\|S^+-S^-\|_{\hat{t},\infty}\le \hat{C}(x,t_1),\ x\epsilon(-\infty,S^-(t_1)-\|S^+-S^-\|_{\hat{t},\infty}].$$

Since the function $x\to\bar{C}(x,t_1)$ is convex and the function $x\to\hat{C}(x,t_1)$ is Lipschitz continuous, these inequalities imply

$$\bar{C}(x,t_1) \le L(x-S^-(t_1)) \le \hat{C}(x,t_1)-\hat{C}(S^-(t_1),t_1)=\hat{C}(x,t_1),x-S^-(t_1) \in [-\|S^+-S^-\|_{\hat{t},\infty},0].$$

Application of the maximum principle yields:

$$\bar{C}(x,t) \le \hat{C}(x,t),\ x\epsilon(-\infty,S^-(t)],\ t\epsilon[t_1,\tau].$$

With $\bar{C}(S^-(\tau),\tau) = \hat{C}(S^-(\tau),\tau)=0$ we deduce the following:

$$\frac{\partial\hat{C}}{\partial x}(S^-(\tau),\tau) \le \frac{\partial\bar{C}}{\partial x}(S^-(\tau),\tau)= \frac{(1+\delta_1)L\|S^+-S^-\|_{\hat{t},\infty}}{\sqrt{\pi}\sqrt{\tau-t_1+\delta_2}}.$$

This inequality holds for every $\delta_1 > 0$ and $\tau\epsilon(t_1,\hat{t}]$. Since the function $t\to \frac{\partial\hat{C}}{\partial x}(S^-(t),t)$ is continuous, the inequality

$$\frac{\partial\hat{C}}{\partial x}(S^-(\tau),\tau) \le \frac{L\|S^+-S^-\|_{\hat{t},\infty}}{\sqrt{\pi(\tau-t_1)}} \text{ holds for } \tau \in [t_1,\hat{t}].$$

Using this estimate in (4.3) yields

$$\|f_{S_1}-f_{S_2}\|_{t_2,\infty} \le \|S_1-S_2\|_{t_1,\infty}+2L\sqrt{\frac{\hat{t}-t_1}{\pi}}\|S^+-S^-\|_{\hat{t},\infty}+2\epsilon$$

$$\le \|S_1-S_2\|_{t_1,\infty}+ 2L\sqrt{\frac{\hat{t}-t_1}{\pi}}\|S_1-S_2\|_{\hat{t},\infty}+ (2+2L\sqrt{\tfrac{T}{\pi}})\epsilon.$$

Since ϵ is arbitrary it follows that

$$\|f_{S_1}-f_{S_2}\|_{t_2,\infty}\le \|S_1-S_2\|_{t_1,\infty}+ 2L\sqrt{\frac{t_2-t_1}{\pi}}\|S_1-S_2\|_{t_2,\infty}. \qquad \boxtimes$$

To obtain a better insight into the results of Theorem 4.1 we perform the following numerical experiment. Take $C_0(x) = \min\left\{-\frac{1}{4}x,1\right\}$, $x \epsilon(-\infty,0]$, $S_1(t) = 0$, $t \epsilon[0,1]$, $S_2(t) = \min\{t,0.01\}$, $t \epsilon[0,1]$ and compute a numerical approximation for C_{S_1} and C_{S_2}. The inequality

$$\|f_{S_1}-f_{S_2}\|_{t,\infty} \le 2L\sqrt{\tfrac{t}{\pi}}\ \|S_1-S_2\|_{t,\infty}, t \ \epsilon\ [0,1]$$

follows from Theorem 4.1 with $t_1=0$ and $t_2=t$. Figure 1 shows the function $t \to 2L\sqrt{\frac{t}{\pi}}\ \|S_1-S_2\|_{t,\infty}$ and the numerical approximation for $\|f_{S_1}-f_{S_2}\|_{t,\infty}$ as a function of t. The figure suggests that with this choice of C_0, S_1 and S_2 the estimate given in Theorem 4.1 is sharp.

This numerical experiment suggests a proof of the sharpness of the estimate of Theorem 4.1. To this end we formulate Lemma 4.5. The remainder of this paper can be read without studying this lemma and its proof.

Lemma 4.5.

Given $\epsilon > 0$ we can find t_1 and functions C_0, S_1 and S_2 such that

$$\|f_{S_1}-f_{S_2}\|_{t_1,\infty} \ge (1-\epsilon)2L\sqrt{\tfrac{t_1}{\pi}}\ \|S_1-S_2\|_{t_1,\infty}.$$

Proof.

Define $C_0(x) = \min\{-x,1\}$, $x \ \epsilon\ (-\infty,0]$, $L = 1$, $T = \frac{1}{16}$. Choose $t_1\epsilon[0,T]$ and $\delta\ \epsilon\ (0,t_1)$, and define $S_1(t)=0$, $t\ \epsilon\ [0,T]$ and $S_2(t)=\min\{t,\delta\}$, $t\ \epsilon\ [0,T]$. From Theorem 3.4 i) it follows that there are functions C_{S_1} and C_{S_2} which are solutions of the reduced problem.

In the first part of the proof we define two auxiliary functions ϕ_1 and ϕ_2, which are solutions of the diffusion equation. Using the maximum principle and some known inequalities for the erf. function we prove that the differences between the functions $x \to C_{S_2}(x,t)$ and $x \to \frac{\delta-x}{\delta+1}$ are small for $x \in [0,\delta]$.

Define $\phi_1(x,t) = \frac{-x}{\delta+1} - \operatorname{erfc}\left(\frac{x+1}{2\sqrt{t}}\right), x \in [-1,S_2(t)],\ t \in [0,T]$.

The function ϕ_1 satisfies:

$$\frac{\partial\phi_1(x,t)}{\partial t} - \frac{\partial^2\phi_1(x,t)}{\partial x^2} = 0, x \in (-1,S_2(t)), t \in (0,T],$$

$$\phi_1(x,0) = \frac{-x}{\delta+1},\ x \in (-1,0],\ \phi_1(-1,t) = \frac{-\delta}{\delta+1},\ \phi_1(S_2(t),t) \le 0,\ t \in [0,T].$$

From [Abramowitz & Stegun, 1972;p.298, inequality 7.1.13] we have

$$\operatorname{erfc}(y) \le \frac{2}{\sqrt{\pi}}\frac{e^{-y^2}}{y+\sqrt{y^2+\frac{4}{\pi}}} \le e^{-y^2}, y \ge 0.$$

This inequality and the maximum principle yield

$$\frac{-x}{\delta+1} - e^{\frac{-1}{16t_1}} \le \phi_1(x,t) \le C_{S_2}(x,t),\ x \in [-\tfrac{1}{2},\ S_2(t)],\ t \in [0,t_1]. \tag{4.6}$$

Define $\delta(t_1) = e^{\frac{-1}{16t_1}}$ and the function $\phi_2: [-\frac{1}{2},\delta]\times[\delta,t_1] \to \mathbb{R}$ by

$$\phi_2(x,t) = \frac{\delta-x}{\delta+1} - 2(\delta(t_1)+\delta)\operatorname{erf}\left(\frac{-x+\delta}{2\sqrt{t-\delta}}\right), x \in [-\tfrac{1}{2},\delta],\ t \in [\delta,t_1].$$

The function ϕ_2 satisfies the diffusion equation for $x \in (-\frac{1}{2},\delta)$, $t \in (\delta,t_1]$.

From the definitions it follows that

$$C_{S_2}(\delta,t) - \phi_2(\delta,t) = 0,\ t \in [\delta,t_1].$$

Since $\operatorname{erf}\left(\frac{\frac{1}{2}+\delta}{2\sqrt{t-\delta}}\right) \ge \operatorname{erf}(1) \ge \frac{1}{2},\ t \in [\delta,\frac{1}{16}]$

we obtain, using (4.6) the following inequalities:

$$C_{S_2}(-\tfrac{1}{2},t) - \phi_2(-\tfrac{1}{2},t) \ge \frac{1}{2(\delta+1)} - \delta(t_1) - \left(\frac{\delta+\frac{1}{2}}{\delta+1} - (\delta(t_1)+\delta)\right) = \frac{\delta^2}{\delta+1} \ge 0,\ t \in [\delta,t_1$$

and

$$C_{S_2}(x,\delta) - \phi_2(x,\delta) \ge \frac{-x}{\delta+1} - \delta(t_1) - \left(\frac{\delta-x}{\delta+1} - 2(\delta(t_1)+\delta)\right)$$

$$= \delta(t_1) + 2\delta - \frac{\delta}{\delta+1} \ge 0, x \in [-\tfrac{1}{2},\delta).$$

Application of the maximum principle yields

$$C_{S_2}(x,t) - \phi_2(x,t) \ge 0,\ x \in [-\tfrac{1}{2},\delta],\ t \in [\delta,t_1]. \tag{4.7}$$

Choose $t_0 \in (\delta,t_1)$. Since $\frac{d\,\operatorname{erf}(y)}{dy} \le \frac{2}{\sqrt{\pi}}, y \in \mathbb{R}$

it follows with inequality (4.7) that

$$(4.8) \qquad C_{S_2}(0,t) \geq \phi_2(0,t) = \frac{\delta}{\delta+1} - 2(\delta(t_1)+\delta)\operatorname{erf}\left(\frac{\delta}{2\sqrt{t-\delta}}\right)$$

$$\geq \frac{\delta}{\delta+1} - 2(\delta(t_1)+\delta)\frac{\delta}{\sqrt{\pi}\sqrt{t_0-\delta}}, \quad t \in [t_0,t_1].$$

Inequality (4.8) combined with $C_{S_1}(0,t) = 0$, $t \in [0,t_1]$ yields an underestimate of the difference $C_{S_2}(0,t), -C_{S_1}(0,t)$, $t \in [t_0,t_1]$. This enables us to define an auxiliary function ϕ_3 such that $\|f_{S_1}-f_{S_2}\|_{t_1,\infty} \geq \int_{-\infty}^{0} \phi_3(x,t_1)dx$.

After choosing δ, t_0 and t_1 in an appropriate way the lemma is proved. The mentioned function ϕ_3 is given by:

$$\phi_3(x,t) = \left(\frac{\delta}{\delta+1} - 2(\delta(t_1)+\delta)\frac{\delta}{\sqrt{\pi}\sqrt{t_0-\delta}}\right)\operatorname{erfc}\left(\frac{-x}{2\sqrt{t-t_0}}\right), \quad x \in (-\infty,0], \quad t \in [t_0,t_1]$$

and satisfies the following equations:

$$\frac{\partial\phi_3(x,t)}{\partial t} - \frac{\partial^2\phi_3(x,t)}{\partial x^2} = 0, \quad x \in (-\infty,0), \quad t \in (t_0,t_1],$$

$$\phi_3(x,t_0) = 0, \quad x \in (-\infty,0) \text{ and } \phi_3(0,t) = \frac{\delta}{\delta+1} - 2(\delta(t_1)+\delta)\frac{\delta}{\sqrt{\pi}\sqrt{t_0-\delta}}, \quad t \in [t_0,t_1].$$

Furthermore for every $\mu > 0$ it follows from the maximum principle and inequality (4.8) that

$$C_{S_2}(x,t) - C_{S_1}(x,t) \geq \phi_3(x-\mu,t), \quad x \in (-\infty,0], \quad t \in [t_0,t_1],$$

thus

$$C_{S_2}(x,t) - C_{S_1}(x,t) \geq \phi_3(x,t), \quad x \in (-\infty,0], \quad t \in [t_0,t_1].$$

This implies

$$(4.9) \qquad \|f_{S_1}-f_{S_2}\|_{t_1,\infty} \geq \int_{-\infty}^{\infty}[C_{S_2}(x,t_1)-C_{S_1}(x,t_1)]dx \geq \int_{-\infty}^{0}\phi_3(x,t_1)dx.$$

From [Abramowitz & Stegun, 1972;p.299, 7.2.1 and 7.2.5] it follows that

$$\int_0^\infty \operatorname{erfc}(y)dy = \frac{1}{\sqrt{\pi}}.$$

This implies

$$\int_{-\infty}^{0}\phi_3(x,t_1)dx = \left(\frac{\delta}{\delta+1} - 2(\delta(t_1)+\delta)\frac{\delta}{\sqrt{\pi}\sqrt{t_0-\delta}}\right)\frac{2\sqrt{t_1-t_0}}{\sqrt{\pi}}.$$

Combination with inequality (4.9) yields

$$\|f_{S_1}-f_{S_2}\|_{t_1,\infty} \geq \left(\frac{\delta}{\delta+1} - 2(\delta(t_1)+\delta)\frac{\delta}{\sqrt{\pi}\sqrt{t_0-\delta}}\right)\frac{2\sqrt{t_1-t_0}}{\sqrt{\pi}}.$$

Since $2L\sqrt{\frac{t_1}{\pi}}\|S_1-S_2\|_{t_1,\infty} = 2\sqrt{\frac{t_1}{\pi}}\,\delta$ we obtain

$$\frac{\|f_{S_1}-f_{S_2}\|_{t_1,\infty}}{2L\sqrt{\frac{t_1}{\pi}}\ \|S_1-S_2\|_{t_1,\infty}} \geq \left(\frac{1}{\delta+1} - \frac{2(\delta(t_1)+\delta)}{\sqrt{\pi}\sqrt{t_0+\delta}} \right)\sqrt{\frac{t_1-t_0}{t_1}}.$$

With the choice $t_0 = t_1^2$ and $\delta = \frac{3}{4}t_1^2$ the inequality becomes

$$\frac{\|f_{S_1}-f_{S_2}\|_{t_1,\infty}}{2L\sqrt{\frac{t_1}{\pi}}\|S_1-S_2\|_{t_1,\infty}} \geq \frac{\sqrt{1-t_1}}{1+\frac{3}{4}t_1^2} - \frac{4\sqrt{1-t_1}e^{\frac{-1}{16t_1}}}{\sqrt{\pi}t_1} - \frac{3t_1\sqrt{1-t_1}}{\sqrt{\pi}}.$$

Since
$$\lim_{t_1\downarrow 0} \left\{ \frac{\sqrt{1-t_1}}{1+\frac{3}{4}t_1^2} - \frac{16\sqrt{1-t_1}e^{\frac{-1}{16t_1}}}{\pi t_1} - \frac{3t_1\sqrt{1-t_1}}{\sqrt{\pi}} \right\} = 1$$

there is an $t_1 \in (0,T]$ such that

$$\|f_{S_1}-f_{S_2}\|_{t_1,\infty} \geq (1-\epsilon)2L\ \sqrt{\frac{t_1}{\pi}}\|S_1-S_2\|_{t_1,\infty}.$$

This proves the lemma. ⊠

4.2 Definition of the operator $\mathcal{T}$

Definition 4.10.

We define $\bar{B}^{-1}(y) = \{x \in R | y \in \bar{B}(x)\}$ and $I = \bar{B}(R)$.

Remark 4.11.

Using Condition 2.3 it is easy to see that $\bar{B}^{-1}: I\rightarrow R$ is a function with the properties $\bar{B}^{-1}(0) = 0$, $0 \leq \bar{B}^{-1}(y+h)-\bar{B}^{-1}(y), h \geq 0$, $y,y+h \in I$ and $\bar{B}^{-1}(y+h) - \bar{B}^{-1}(y) \leq \frac{h}{B}$, $h \geq 0$, $y \in [0,\infty)$.

Lemma 4.12.

For $S \in \bar{M}_{\gamma/B}$ the function $t \rightarrow \bar{B}^{-1}(\bar{G}(S,f_S,t))$ is an element of $\bar{M}_{\gamma/B}$.

Proof.

For $S \in \bar{M}_{\gamma/B}$, Theorem 3.4 ii) states that f_S exists and $f_S \in \bar{M}_L$. This together with Condition 2.3 implies that $\bar{G}(S,f_S,t)$ exists for $t \in [0,T]$ and $\bar{G}(S,f_S,t) \in [0,\infty)$. Since $[0,\infty) \subset I$ it follows from Remark 4.11 that $\bar{B}^{-1}(\bar{G}(S,f_S,t))$ is properly defined for $S \in \bar{M}_{\gamma/B}$ and $t \in [0,T]$. Condition 2.3 combined with Remark 4.11 yields $\bar{B}^{-1}(\bar{G}(S,f_S,0))=0$. Since $S \in \bar{M}_{\gamma/B}$ and $f_S \in \bar{M}_L$ we know from Condition 2.3 that

$$0 \leq \bar{G}(S,f_S,t+h) - \bar{G}(S,f_S,t) \leq \gamma h,\ h \geq 0; t+h,\ t \in [0,T].$$

According to Remark 4.11 we obtain that $\bar{B}^{-1}(\bar{G}(S,f_S,t))$ is a function of t which is an element of $\bar{M}_{7/B}$. ⊠

Definition 4.13.

The operator $\mathcal{T}:\bar{M}_{7/B}\to\bar{M}_{7/B}$ is defined as follows:

$$\mathcal{T}(S)(t) = \bar{B}^{-1}(\bar{G}(S,f_S,t)),\ t \in [0,T].$$

4.3 Existence and uniqueness of the solution of the Stefan problem

Lemma 4.14.

Conditions 2.3 vi), vii) hold if and only if

$$\sup_{t\in[0,\hat{t}]} |\bar{G}(S_1,f,t)-\bar{G}(S_2,f,t)| \le G_1\|S_1-S_2\|_{\hat{t},\infty}, \hat{t} \in[0,T],\ S_1,S_2 \in \bar{M}_{7/B}, f \in \bar{M}_L,$$

$$\sup_{t\in[0,\hat{t}]} |\bar{G}(S,f_1,t)-\bar{G}(S,f_2,t)| \le G_2\|f_1-f_2\|_{\hat{t},\infty},\ \hat{t} \in[0,T],\ S \in \bar{M}_{7/B}, f_1,f_2 \in \bar{M}_L.$$

Proof.

⇒ : For $\hat{t} \in[0,T]$ and $S_1,S_2 \in \bar{M}_{7/B}$ we define $\bar{S}_1,\ \bar{S}_2 \in \bar{M}_{7/B}$ as follows: $\bar{S}_i(t)=S_i(t),\ t \in[0,\hat{t}],\ \bar{S}_i(t)=S_i(\hat{t}),\ t \in(\hat{t},T],\ i = 1,2.$
From Condition 2.3 vi), vii) it follows that

$$\sup_{t\in[0,\hat{t}]} |\bar{G}(S_1,f,t)-\bar{G}(S_2,f,t)| \le \sup_{t\in[0,T]} |\bar{G}(\bar{S}_1,f,t)-\bar{G}(\bar{S}_2,f,t)|$$
$$\le G_1\ \|\bar{S}_1-\bar{S}_2\|_\infty = G_1\|S_1-S_2\|_{\hat{t},\infty}.$$

The inequality

$$\sup_{t\in[0,\hat{t}]} |\bar{G}(S,f_1,t) - \bar{G}(S,f_2,t)| \le G_2\|f_1-f_2\|_{\hat{t},\infty}$$

can be shown in the same way.

⇐ : For $\hat{t}=T$ the inequalities are equivalent with the inequalities mentioned in Condition 2.3 vi).

Suppose $S \in \bar{M}_{7/B}$, $f \in \bar{M}_L$ and $t \in [0,T]$ are given. For all $\hat{S} \in \bar{M}_{7/B}$ and $\hat{f} \in \bar{M}_L$ with $\hat{S}(\tau)=S(\tau),\ \hat{f}(\tau)=f(\tau),\ \tau \in [0,t]$ the inequalities imply $\bar{G}(S,f,\tau)=\bar{G}(\hat{S},\hat{f},\tau),\ \tau \in [0,t]$. Thus $\bar{G}(S,f,t),\ t \in [0,T]$ depends only on $S|_{[0,t]}$ and $f|_{[0,t]}$. ⊠

Theorem 4.15.

If $G_1 < B$, then there is a unique function $\bar{S} \in \bar{M}_{\gamma/B}$ such that $\mathcal{T}(\bar{S})=\bar{S}$.

Proof.

Define $h = \frac{T}{N}$, $N \in \mathbb{N}$ and choose N large enough such that $\rho_1 := \frac{G_1}{B} + \frac{2LG_2}{B}\sqrt{\frac{h}{\pi}} < 1$.

Define $\rho_2 = \frac{G_2}{B}$ and $t_j=jh$, $j=0,\dots,N$.

For given $S_1,S_2 \in \bar{M}_{\gamma/B}$ we define $\sigma_j^{(m)}=\|\mathcal{T}^m(S_1)-\mathcal{T}^m(S_2)\|_{t_j,\infty}$, $j=0,1,\dots,N$, $m \ge 0$.

From the definitions it follows that $\sigma_i^{(m)} \le \sigma_j^{(m)}$, $i \le j$, $m \ge 0$.

Application of Lemma 4.14 and Theorem 4.1 yields

$$\|\mathcal{T}^m(S_1)-\mathcal{T}^m(S_2)\|_{t_j,\infty} \le \frac{G_1}{B}\|\mathcal{T}^{m-1}(S_1)-\mathcal{T}^{m-1}(S_2)\|_{t_j,\infty} + \frac{G_2}{B}\|f_{\mathcal{T}^{m-1}(S_1)}-f_{\mathcal{T}^{m-1}(S_2)}\|_{t_j,\infty}$$

$$\le \frac{G_2}{B}\|\mathcal{T}^{m-1}(S_1)-\mathcal{T}^{m-1}(S_2)\|_{t_{j-1},\infty} + \frac{G_1+2G_2L\sqrt{\frac{h}{\pi}}}{B}\|\mathcal{T}^{m-1}(S_1) - \mathcal{T}^{m-1}(S_2)\|_{t_j,\infty}.$$

From the definitions we conclude

$$\sigma_j^{(m)} \le \rho_2\sigma_{j-1}^{(m-1)} + \rho_1\sigma_j^{(m-1)}, j=1,\dots,N, \ m \ge 1.$$

With induction to n we shall show that the inequality

$$(4.16) \qquad \sigma_n^{(m)} \le \rho_1^m \sum_{j=0}^{n-1}\left(\frac{m\rho_2}{\rho_1}\right)^j \sigma_n^{(0)} \text{ holds for } n \ge 1 \text{ and } m \ge 0.$$

Since $\sigma_0^{(m)}=0$, $m \ge 0$ it is easily seen that $\sigma_1^{(m)} \le \rho_1^{m-i}\,\sigma_1^{(i)}$, $m \ge 0$, $i=0,\dots,m$ and thus $\sigma_1^{(m)} \le \rho_1^m\sigma_1^{(0)}$, $m \ge 0$.

Suppose that for an $n \ge 1$ inequality (4.16) holds for every $m \ge 0$.

Using the inequality $\sigma_{n+1}^{(m)} \le \rho_2\sigma_n^{(m-1)} + \rho_1\sigma_{n+1}^{(m-1)}$ yields

$$\sigma_{n+1}^{(m)} \le \rho_2 \sum_{i=0}^{m-1} \rho_1^{m-1-i}\,\sigma_n^{(i)} + \rho_1^m\,\sigma_{n+1}^{(0)} \le$$

$$\le \rho_2 \sum_{i=0}^{m-1} \rho_1^{m-1-i}\,\rho_1^i \sum_{j=0}^{n-1}\left(\frac{i\rho_2}{\rho_1}\right)^j \sigma_n^{(0)} + \rho_1^m\,\sigma_{n+1}^{(0)} \le$$

$$\le \rho_2\rho_1^{m-1}\, m \sum_{j=0}^{n-1}\left(\frac{m\rho_2}{\rho_1}\right)^j \sigma_n^{(0)} + \rho_1^m\,\sigma_{n+1}^{(0)} \le \rho_1^m \sum_{j=1}^{n}\left(\frac{m\rho_2}{\rho_1}\right)^j \sigma_n^{(0)} + \rho_1^m\,\sigma_{n+1}^{(0)}, \ m \ge 1.$$

Since $\sigma_n^{(0)} \le \sigma_{n+1}^{(0)}$ we deduce $\sigma_{n+1}^{(m)} \le \rho_1^m \sum_{j=0}^{n}\left(\frac{m\rho_2}{\rho_1}\right)^j \sigma_{n+1}^{(0)}$. $m \ge 0$.

This shows that inequality (4.16) holds for $n \ge 1$ and $m \ge 0$.

Since $\rho_1 < 1$ we know that

$$\lim_{m\to\infty} \rho_1^m \sum_{j=0}^{N-1} \left(\frac{m\rho_2}{\rho_1}\right)^j = \lim_{m\to\infty} e^{m\ln(\rho_1)} \cdot \sum_{j=0}^{N-1} \left(\frac{m\rho_2}{\rho_1}\right)^j = 0.$$

Choose $\hat{m}$ large enough such that $\rho_3 := \rho_1^{\hat{m}} \sum_{j=0}^{N-1} \left(\frac{\hat{m}\rho_2}{\rho_1}\right)^j < 1.$

With inequality (4.16) it follows that

$$\|\mathcal{T}^{\hat{m}}(S_1) - \mathcal{T}^{\hat{m}}(S_2)\|_\infty \le \rho_3 \|S_1 - S_2\|_\infty .$$

We know that $(\bar{M}_{\gamma/B}, \|.\|_\infty)$ is a complete metric space and $\bar{M}_{\gamma/B} \neq \phi$.

Furthermore we have just shown that $\mathcal{T}^{\hat{m}}$ is a conctraction on $\bar{M}_{\gamma/B}$ and $\mathcal{T}$ is a continuous operator on $\bar{M}_{\gamma/B}$. According to the Banach fixed point theorem there is a unique function $\bar{S} \in \bar{M}_{\gamma/B}$ such that $\mathcal{T}(\bar{S}) = \bar{S}$. ⊠

Now we are able to state our main theorem.

Main Theorem 4.17.

The Stefan problem :"to find a function $S \in O$ and a solution C_S of the reduced problem such that $\bar{G}(S,f_S,t) \in \bar{B}(S(t))$, $t \in [0,T]$", has a unique solution if $G_1 < B$.*

If we denote the solution of the Stefan problem by the pair $(\bar{S},C_{\bar{S}})$ then $\bar{S} \in \bar{M}_{\gamma/B}$.

Proof.

Existence. Using Conditions 2.1 and 2.3, $G_1 < B$ and $\bar{M}_{\gamma/B} \subset O$ it follows from Theorem 4.15 that there is a fixed point $\bar{S} \in \bar{M}_{\gamma/B}$ of the operator $\mathcal{T}$. This implies that there is a unique solution $C_{\bar{S}}$ of the reduced problem and $\bar{S}(t) = \bar{B}^{-1}(\bar{G}(\bar{S},f_{\bar{S}},t))$, $t \in [0,T]$.

With definition 4.10, the last equation is equivalent to $\bar{G}(\bar{S},f_{\bar{S}},t) \in \bar{B}(\bar{S}(t))$, $t \in [0,T]$.

Thus the pair $(\bar{S},C_{\bar{S}})$ is a solution of the Stefan problem, and $\bar{S}$ is an element of $\bar{M}_{\gamma/B}$.

Uniqueness. Suppose the pair (S,C_S) satisfies the Stefan problem. Since $S \in O$, Lemma 3.6 iii) implies that $\bar{G}(S,f_S,t)$ exists for $t \in [0,T]$.

From $\bar{G}(S,f_S,t) \in \bar{B}(S(t))$, $t \in [0,T]$ and $[0,\infty) \subset I$ we obtain $S(t) = \bar{B}^{-1}(\bar{G}(S,f_S,t))$, $t \in [0,T]$. Condition 2.3 combined with

* Remember that our convention is to impose Conditions 2.1 and 2.3 .

Remark 4.11 yields S ϵ P. Application of Lemma 3.6 ii) yields $f_S \epsilon \tilde{M}_L$.

With Condition 2.3 and Remark 4.11 this implies that S ϵ $\tilde{M}_{7/B}$.

Thus $\mathcal{T}(S)$ is defined and $\mathcal{T}(S) = S$. Since the fixed point $\tilde{S}$ of $\mathcal{T}$ is unique the theorem is proved. ⊠

Remark 4.18.

i) We compare Theorem 4.17 of the present paper with Theorem 5.11 of [Vuik,1987] and note the following. Both theorems state the existence of a solution for the Stefan problem in $\tilde{M}_{7/B}$ and the uniqueness of the solution in the set O. However they impose different conditions. Theorem 5.11 from [Vuik,1987] is proved for $G_1 + G_2 < B$, whereas we prove Theorem 4.17 if the inequality $G_1 < B$ holds and the additional Condition 2.3 vii) is satisfied. An interpretation of Condition 2.3 vii) is that the expression $\tilde{G}(S,f,t)$ only depends on the history and the present and does not depend on the future. With this interpretation it is easy to see that in many physical applications Condition 2.3 vii) is satisfied.

ii) One may prove an analogue of Theorem 5.7 of [Vuik, 1987] where the condition $G_1 + G_2 < B$ is replaced by Condition 2.3 vii) and $G_1 < B$.

5 Applications of the main theorem

In this section we give multifunctions $\bar{B}$ and functionals $\bar{G}$ which satisfy Condition 2.3 and for which the corresponding Stefan problem can be seen as a mathematical model of a physical problem. After that we give invariance properties of the reduced and Stefan problem, and present some numerical results. Finally we state an example wherein the function C_0 is not an element of Cond 2.1. It appears that in some cases the sequence given by $S_{i+1} = \mathcal{T}(S_i)$, $i \geq 0$ does not converge.

5.1 Examples

In this subsection we suppose that $T > 0$ and $C_0:(-\infty,0]\to[0,1]$ are given. Furthermore C_0 is an element of Cond 2.1.

Example 1.

For a given bounded integrable function $b:\mathbb{R}\to[B_1,\infty)$ with $B_1 > 0$ we define $\bar{B}:\mathbb{R}\to\mathbb{R}$ by $\bar{B}(x) = \int_0^x b(\xi)d\xi$, $x \in \mathbb{R}$. It is easy to see that this function $\bar{B}$ satisfies Conditions 2.3 i),ii),iii) and $B = B_1$.

Suppose that the bounded continuous functions $g,h:\mathbb{R}\times[0,T]\to[0,\infty)$ are given and that for every $k > 0$ there are $g_1,g_2,h_1 \in \mathbb{R}$ such that

$$|g(x_1,t)-g(x_2,t)| \leq g_1|x_1-x_2|;x_1,x_2 \in [0,k],\ t \in [0,T],$$

$$|g(x,t_1)-g(x,t_2)| \leq g_2|t_1-t_2|;x \in [0,k],\ t_1,t_2 \in [0,T],$$

$$|h(x_1,t)-h(x_2,t)| \leq h_1|x_1-x_2|;x_1,x_2 \in [0,k],\ t \in [0,T].$$

For given $\delta \geq 0$ and $S \in O$ we define $S_\delta:[-\delta,T]\to\mathbb{R}$ by

$$S_\delta(t) = \begin{cases} 0 & ,t \in [-\delta,0) \\ S(t) & ,t \in [0,T] \end{cases} \text{ and}$$

$$\bar{G}(S,f,t) = \int_0^t g(S(\tau),\tau)df(\tau) + \int_0^t h(S_\delta(\tau-\delta),\tau)d\tau,\ S \in O,\ f \in P,\ t \in [0,T].$$

The first integral is a Lebesgue-Stieltjes integral.

It can be shown that Conditions 2.3 iv), v), vi) hold for this functional $\bar{G}$

with $\gamma = \sup\limits_{\substack{x\in[0,\infty)\\ t\in[0,T]}} Lg(x,t) + \sup\limits_{\substack{x\in[0,\infty)\\ t\in[0,T]}} h(x,t)$, $k = \frac{\gamma T}{B_1}$

$G_1 = (g_1L+h_1)T$ and $G_2 = \left\{ \sup\limits_{\substack{x\in[0,\infty)\\ t\in[0,T]}} g(x,t) + T\left(\frac{\gamma}{B_1}g_1 + g_2\right) \right\}$

(compare [Vuik,1987; Lemma 6.2]). Furthermore it is easy to see that for

$t \in [0,T]$, $S \in \tilde{M}_{7/B}$ and $f \in \tilde{M}_L$, $\tilde{G}(S,f,t)$ only depends on $S|_{[0,t]}$ and $f|_{[0,t]}$. Thus Condition 2.3 vii) holds.

Theorem 5.1.

If the functions b,g and h satisfy the conditions mentioned in Example 1 then the corresponding Stefan problem has a unique solution.

Proof.

For $S_1, S_2 \in \tilde{M}_{7/B_1}$, $f \in \tilde{M}_L$, $t_1, t_2 \in [0,T]$ and $t_1 \le t_2$ it is easily seen that

$$\sup_{t \in [0,t_2]} |\tilde{G}(S_1,f,t) - \tilde{G}(S_2,f,t)| \le G_1 \|S_1 - S_2\|_{t_1,\infty} + (g_1 L + h_1)(t_2 - t_1)\|S_1 - S_2\|_{t_2,\infty}.$$

Application of Theorem 4.1 and Lemma 4.14 yields

$$\|\mathcal{T}(S_1) - \mathcal{T}(S_2)\|_{t_2,\infty} \le \frac{G_1+G_2}{B} \|S_1 - S_2\|_{t_1,\infty} + \frac{1}{B}\left\{ (g_1 L + h_1)(t_2 - t_1) + 2LG_2\sqrt{\frac{t_2 - t_1}{\pi}} \right\} \|S_1 - S_2\|_{t_2,\infty}.$$

With this inequality it can be shown that $\mathcal{T}^m$ is a contraction for m large enough. This implies that the Stefan problem has a unique solution (compare the proofs of Theorem 4.15, 4.17). ⊠

Remark 5.2.

i) The problem given in Section 1 is equivalent to the Stefan problem specified in Section 2 if we choose $b(x) = B_1$, $x \in \mathbb{R}$ where $B_1 > 0$, $g(x,t) = 1$ and $h(x,t) = 0$, $x \in \mathbb{R}$, $t \in [0,T]$. It follows from Theorem 5.1 that there is a unique solution of the Stefan problem for every $B_1 > 0$.

ii) Most Stefan problems considered in the literature are of the following form: to find a function $S \in \dot{O} \cap C^1[0,T]$ and a solution C_S of the reduced problem such that

$$\frac{dS(t)}{dt} = -g(S(t),t)\frac{\partial C_S}{\partial x}(S(t),t) + h(S(t),t), \quad t \in [0,T],$$

where the bounded continuous function $(x,t) \to g(x,t)$ has continuous derivatives $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial t}$ and the bounded continuous function $(x,t) \to h(x,t)$ is Lipschitz continuous in x (see [Fasano & Primicerio, 1977]). We will refer to this problem as Problem 1.

Since $\quad f_S(t) = -\int_0^t \frac{\partial C_S}{\partial x}(S(\tau),\tau)d\tau$ if $C_0 \in$ Cond 2.2, $S \in C^2[0,T]$ and $S(0) = 0$, a more general formulation of the problem is the following: to find $S \in O$ and a solution C_S of the reduced problem such that

$$S(t) = \int_0^t g(S(\tau),\tau)\, df_S(\tau) + \int_0^t h(S(\tau),\tau)d\tau,\ t \in [0,T].$$

We will refer to this problem as Problem 2. Remark that Problem 2 is equivalent to the Stefan problem considered in Example 1 with $\delta = 0$. If we make the additional assumption that $g,h:\mathbb{R}\times[0,T]\to[0,\infty)$ then it follows from Theorem 5.1 that Problem 2 has a unique solution.

Remark that the differential equation in Problem 1 is transformed to an integral equation in Problem 2. This enables us to impose weaker conditions on the smoothness of the functions g and h. To prove existence and uniqueness for a solution of Problem 1 it is supposed that the function g has continuous derivatives $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial t}$ whereas to prove existence and uniqueness for a solution of Problem 2 it is sufficient that the function g is Lipschitz continuous. See also Example 1 of [Vuik,1987] where we analyse a discontinuous function g.

Example 2.

Given $B_1 > 0$ define $\bar{B}:\mathbb{R}\to\mathbb{R}$ by $\bar{B}(x) = B_1 x,\ x \in \mathbb{R}$. Given $\delta > 0$ and an integrable function $g_\delta:\mathbb{R}\to[0,\infty)$ such that $\mathrm{supp}(g_\delta) \subset [0,\delta)$ and $\int_{-\infty}^{\infty} g_\delta(\xi)d\xi = 1$.

Define $f_\delta(t) = \begin{cases} 0 & ,t\in[-\delta,0) \\ f(t) & ,t\in[0,T] \end{cases}$,

$$\bar{G}(S,f,t) = \int_{-\infty}^{\infty} f_\delta(\tau)g_\delta(t-\tau)d\tau,\ t \in [0,T],\ S \in O,\ f \in P.$$

Conditions 2.3 i),...,vi) are satisfied for $B = B_1$, $\gamma = L$, $G_1 = 0$ and $G_2 = 1$. Since $\mathrm{supp}(g_\delta) \subset [0,\delta)$, $\bar{G}(S,f,t)$ only depends on $S|_{[0,t]}$ and $f|_{[0,t]}$. Thus Condition 2.3 vii) is also satisfied. Theorem 4.17 states that there is a unique solution of the Stefan problem for every $B_1 > 0$.

Example 3.

Given $B_1 > 0$ define $\bar{B}:\mathbb{R}\to\mathbb{R}$ by $\bar{B}(x) = B_1 x,\ x \in \mathbb{R}$. Given $g_1 > 0$ and $s_1 \in \mathbb{R}$ such that $0 \le s_1 \le Tg_1/B_1$ we define the functional $g:P\times\mathbb{R}\to[0,\infty)$ by

$$g(f,t) = \begin{cases} 0 & t \in [0,T+(f(T)-B_1 s_1)/g_1], \\ g_1 & t \in (T+(f(T)-B_1 s_1)/g_1,\infty). \end{cases}$$

Define $\bar{G}(S,f,t) = f(t) + \int_0^t g(f,\tau)d\tau$, $t \in [0,T]$, $S \in O$ and $f \in P$.

Conditions 2.3 i),...,vi) are satisfied for $B = B_1$, $\gamma = L+g_1$, $G_1 = 0$ and $G_2 = 2$. However it is easily seen from the definition of g and $\bar{G}$ that Condition 2.3 vii) does not hold. Thus we can not apply Theorem 4.17. If $B_1 > 2$ the existence and the uniqueness of the solution of this Stefan problem are given by Theorem 5.11 of [Vuik, 1987] (compare Remark 4.18i)).

Remark 5.3.

i) Since Theorem 4.17 can not be applied to the Stefan problem of Example 3, we use Theorem 4.1 to obtain the following. Application of Theorem 4.1 with $t_1 = 0$ and $t_2 = T$ gives $\|f_{S_1}-f_{S_2}\|_\infty \le 2L\sqrt{\frac{T}{\pi}}\|S_1-S_2\|_\infty$.

With this inequality and $G_1 = 0$ we derive

$$\|\mathcal{T}(S_1)-\mathcal{T}(S_2)\|_\infty \le \frac{2G_2L}{B_1}\sqrt{\frac{T}{\pi}}\|S_1-S_2\|_\infty .$$

If $B_1 > 2G_2L\sqrt{\frac{T}{\pi}}$ then the operator $\mathcal{T}$ is a contraction on $\bar{M}_{\gamma/B}$.

This implies that the Stefan problem has a unique solution for $B_1 > 2G_2L\sqrt{\frac{T}{\pi}}$ (compare the proof of Theorem 4.17).

ii) From Theorem 4.17 it follows that there is a unique solution of a Stefan problem if the Conditions 2.1, 2.3 and $G_1 = 0$ are satisfied. For a Stefan problem where the Conditions 2.1, 2.3 i),...,vi) and $G_1 = 0$ hold we can prove that there is a unique solution only if T is sufficiently small (see Remark 5.3 i)). With the interpretation of Condition 2.3 vii) given in Remark 4.18 i) we note an analogy between these results and the existence and uniqueness results for a Volterra respectively Fredholm integral equation given, e.g., in [Mikhlin, 1957;p.1-19].

5.2 Invariance properties of the reduced and Stefan problem

From [Effros & Kazdan, 1971;§3] it follows that the diffusion equation is only invariant under transformations of the form

$$(x,t) \to (ax+y_0, a^2t+\tau_0), \quad a \neq 0.$$

Without loss of generality we can restrict ourselves to the case $y_0 = 0$,

$\tau_0 = 0$ and $a > 0$. First of all we analyse the estimate given in Theorem 4.1 and show that the ratio between the left- and right-hand side is invariant under these transformations. From this it follows that the estimate of Theorem 4.1 is optimal in a certain class of estimates. After that we formulate a condition such that the Stefan problem has an invariance property and illustrate this with an example.

Suppose $C_0 \in$ Cond 2.1, T, $S \in O$ are given and $C_S:\bar{Q}_S \to [0,1]$ is a solution of the reduced problem. Define for $a > 0$ the following transformation $(x,t) \to (y,\tau)$ where $y = ax$ and $\tau = a^2t$, and the functions

$$C_0^a(y) = C_0\left(\frac{y}{a}\right),\ y \in (-\infty,0],\ S^a(\tau) = a\ S\left(\frac{\tau}{a^2}\right),\ \tau \in [0,a^2T]$$

and

$$C_S^a(y,\tau) = C_S\left(\frac{y}{a},\frac{\tau}{a^2}\right),\ y \in (-\infty,S^a(\tau)],\ \tau \in [0,a^2T].$$

It is easy to see that if L is a Lipschitz constant of C_0 then $L^a = \frac{L}{a}$ is a Lipschitz constant of C_0^a. Furthermore C_S^a satisfies the transformed reduced problem:

$$\frac{\partial C_S^a(y,\tau)}{\partial \tau} - \frac{\partial^2 C_S^a(y,\tau)}{\partial y^2} = 0,\ y \in (-\infty,S^a(\tau)),\ \tau \in (0,a^2T],$$

$$C_S^a(y,0) = C_0^a(y),\ y \in (-\infty,0],\ C_S^a(S^a(\tau),\tau) = 0,\ \tau \in [0,a^2T].$$

Define $f_S^a(\tau) = -\int_{-\infty}^{\infty} [C_S^a(y,\tau) - C_0^a(y)]dy,\ \tau \in [0,a^2T]$. It follows that $f_S^a(\tau) = a\ f_S\left(\frac{\tau}{a^2}\right),\ \tau \in [0,a^2T]$. Using Theorem 4.1 the following lemma is obvious:

<u>Lemma 5.4.</u>

Suppose $S_1,S_2 \in \bar{M}_{7/8}$, $\tau_1,\tau_2 \in [0,a^2T]$ and $\tau_1 \le \tau_2$ then the following inequality holds:

$$\|f_{S_1}^a - f_{S_2}^a\|_{\tau_2,\infty} \le \|S_1^a - S_2^a\|_{\tau_1,\infty} + 2L^a\sqrt{\frac{\tau_2-\tau_1}{\pi}}\ \|S_1^a - S_2^a\|_{\tau_2,\infty}$$

and the ratio of the left- and right-hand side is invariant.

In the following lemma we define a class of estimates for $\|f_{S_1} - f_{S_2}\|_{t_1,\infty}$, and prove some properties of these estimates. From this lemma it follows that the estimate given in Theorem 4.1 is optimal in a certain sense.

Lemma 5.5.

Suppose Λ: $[0,\infty) \times [0,\infty) \to [0,\infty)$ is such that for every $K,T,L > 0$, $t_1 \in [0,T]$ and S_1, $S_2 \in \bar{M}_K$ the following inequality holds:

$$\|f_{S_1} - f_{S_2}\|_{t_1,\infty} \le \Lambda(L,t_1)\|S_1 - S_2\|_{t_1,\infty}.$$

Then there is a function $\lambda:[0,\infty) \to [0,\infty)$ with the properties:

i) $\|f_{S_1} - f_{S_2}\|_{t_1,\infty} \le \lambda(L\sqrt{t_1})\|S_1 - S_2\|_{t_1,\infty}$,

ii) $\lambda(L\sqrt{t_1}) \le \Lambda(L,t_1)$,

iii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $\lambda(x) \ge (1-\epsilon)\frac{2x}{\sqrt{\pi}}$ for $x < \delta$.

Proof.

Define the function $\Lambda_{inf}:[0,\infty) \times [0,\infty) \to [0,\infty)$ by

$$\Lambda_{inf}(L,t_1) = \inf_{a>0} \Lambda(\tfrac{L}{a}, a^2t_1).$$

Since $\dfrac{\|f_{S_1} - f_{S_2}\|_{t_1,\infty}}{\|S_1 - S_2\|_{t_1,\infty}} = \dfrac{\|f^a_{S_1} - f^a_{S_2}\|_{\tau_1,\infty}}{\|S^a_1 - S^a_2\|_{\tau_1,\infty}} \le \Lambda(L^a,\tau_1)$ with $\tau_1 = a^2t_1$

it follows that $\|f_{S_1} - f_{S_2}\|_{t_1,\infty} \le \Lambda_{inf}(L,t_1)\|S_1 - S_2\|_{t_1,\infty}$.

From the definition we know that $\Lambda_{inf}(L,t_1) = \Lambda_{inf}(\frac{L}{a}, a^2t_1)$, $a > 0$ thus $\Lambda_{inf}(L,t_1) = \Lambda_{inf}(1,L^2t_1)$.

Hence the function $\lambda:[0,\infty) \to [0,\infty)$ given by $\lambda(x) = \Lambda_{inf}(1,x^2)$, $x \in [0,\infty)$ satisfies i) and ii). This together with Lemma 4.5 yields that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\lambda(x) \ge (1-\epsilon)\,\frac{2x}{\sqrt{\pi}} \text{ for } x < \delta. \qquad \boxtimes$$

Since the estimate with $\lambda(x) = \frac{2}{\sqrt{\pi}}x$, $x \in [0,\infty)$ is shown in Theorem 4.1 we obtain from Lemma 5.5 the following corollary:

Corollary 5.6.

In the class of estimates given in Lemma 5.5 there is no estimate which is asymptotically better for $L\sqrt{t_1} \to 0$ then the estimate given in Theorem 4.1.

Suppose the multifunction $\bar{B}$: $\mathbb{R} \to \mathbb{R}$ and the functional $\bar{G}$: $O\times P\times[0,T] \to \mathbb{R}$ are given for every $T > 0$. We define a class $\mathscr{S}(\bar{B},\bar{G})$ which consists of Stefan problems of the form:"given $T > 0$ and a function $x \to C_0(x)$, $x \in (-\infty,0]$, find a function $S \in O$ and a solution C_S of the reduced problem such that $\bar{G}(S,f_S,t) \in \bar{B}(S(t))$, $t \in [0,T]$".

We now formulate a sufficient condition on $\bar{B}$ and $\bar{G}$ such that the class $\mathscr{S}(\bar{B},\bar{G})$ is invariant under transformations of the form $(x,t) \to (ax,a^2t)$, $a > 0$.

Condition 5.7.

$\bar{B}$ and $\bar{G}$ should be such that $\bar{B}$ satisfies Condition 2.3 i),ii),iii) and

$$\bar{B}^{-1}(\bar{G}(S^a,f^a,\tau)) = a\bar{B}^{-1}(\bar{G}(S,f,t)),$$

for every $a > 0$, $S \in O$, $f \in P$ and $t \in [0,T]$.

By straightforward computation it follows that if a Stefan problem of $\mathscr{S}(\bar{B},\bar{G})$ has a solution (S,C_S) and Condition 5.7 holds then the pair (S^a,C_S^a) is also a solution of a Stefan problem of the class $\mathscr{S}(\bar{B},\bar{G})$.

From this we conclude:

Theorem 5.8.

If Condition 5.7 holds then the class $\mathscr{S}(\bar{B},\bar{G})$ is invariant under transformations of the form $(x,t) \to (ax,a^2t)$, $a > 0$.

Example 5.9.

Suppose that $B_1 > 0$ and the bounded functions g,h: $\mathbb{R}\times[0,\infty) \to \mathbb{R}$ which are continuous on $\mathbb{R} \times (0,\infty)$, are given. Define $\bar{B}(x) = B_1x$, $x \in \mathbb{R}$ and

$$\bar{G}(S,f,t) = \int_0^t g(S(\mu),\mu)df(\mu) + \int_0^t h(s(\mu),\mu)d\mu,\ S \in O,\ f \in P\ ,\ t \in [0,T],$$

(compare Example 5.1). If there are functions $\bar{g},\bar{h}$: $\mathbb{R} \to \mathbb{R}$ such that

$g(S(t),t) = \bar{g}\left(\frac{S(t)}{\sqrt{t}}\right)$ and $h(S(t),t) = \frac{1}{\sqrt{t}}\,\bar{h}\left(\frac{S(t)}{\sqrt{t}}\right)$

then $\bar{B}$ and $\bar{G}$ satisfies Condition 5.7.

5.3 Numerical experiments

The proof of Theorem 4.15 suggests that if $\frac{1}{B}$ or L increase then the convergence slows down. We have no proof for that. However, the following numerical example suggests that this is really the case.

Define for a given $B_1 > 0$ $\bar{B}: \mathbb{R} \to \mathbb{R}$ by $\bar{B}(x) = B_1 x,\ x \in \mathbb{R}$,

$T = 1,\ \bar{G}(S,f,t) = f(t),\ t \in [0,1],\ S \in O,\ f \in P$ and for $L > 0$ given we define $C_0(x) = \min\{-Lx, 1\},\ x \in (-\infty, 0]$.

Conditions 2.1 and 2.3 are satisfied for $B = B_1$, $\gamma = L$, $G_1 = 0$ and $G_2 = 1$. Since $B > G_1$, Theorem 4.17 yields that there is a unique solution (S,C) of the Stefan problem.

We compute a numerical solution for this problem as follows.

Take $M = 100$, $N = 200$, $\Delta x = 0.1$, $\Delta t = 0.005$ and $eps = 10^{-2}$.

a) Set $\ell = 0$ and $S_j^{(0)} = 0,\ j = 0,\dots,N$.

b) Compute $C_{ij}^{(\ell)}$, which is an approximation of $C(-10+i\Delta x+S_j^{(\ell)}, j\Delta t)$ for $i = 0,\dots,M$ and $j = 0,\dots,N$ as follows:

$$C_{i,0}^{(\ell)} = C_0(-10+i\Delta x),\quad i=0,\dots,M,$$

$$C_{0,j}^{(\ell)} = 1,\quad C_{M,j}^{(\ell)} = 0,\quad j = 1,\dots,N \text{ and}$$

$$C_{ij}^{(\ell)} = C_{i,j-1}^{(\ell)} + \frac{\Delta t}{(\Delta x)^2}\left(C_{i-1,j-1}^{(\ell)} - 2C_{i,j-1}^{(\ell)} + C_{i+1,j-1}^{(\ell)}\right) + \frac{1}{2\Delta x}\left(S_j^{(\ell)} - S_{j-1}^{(\ell)}\right)\left(C_{i+1,j-1}^{(\ell)} - C_{i-1,j-1}^{(\ell)}\right)$$

for $i = 2,\dots,M-1,\ j = 1,\dots,N$.

c) With $f_0^{(\ell)} = 0$, $f_j^{(\ell)} = -\sum_{n=1}^{j}\left(\frac{1}{2}\left(C_{M,n}^{(\ell)} - C_{M-1,n}^{(\ell)}\right) + \frac{1}{2}\left(C_{M,n-1}^{(\ell)} - C_{M-1,n-1}^{(\ell)}\right)\right)\frac{\Delta t}{\Delta x}$, $j = 1,\dots,N$, we obtain $S_j^{(\ell+1)} = \frac{1}{B_1} f_j^{(\ell)},\ j = 0,\dots,N$.

If $\max_{j\in\{0,\dots,N\}} |S_j^{(\ell+1)} - S_j^{(\ell)}| < eps$ go to d) or else $\ell := \ell + 1$ go to b).

d) $\hat{\ell} := \ell + 1$.

For several choices of B_1 and L we compute the number of iterates $\hat{\ell}$ which is needed to obtain $\max_{j\in\{0,1,\ldots,N\}} |S_j^{(\hat{\ell})} - S_j^{(\hat{\ell}-1)}| < eps$.

The results of these computations are given in the Tables 1 and 2.

B_1	$\hat{\ell}$
1	5
0.5	8
0.25	13
0.125	21
0.0625	33

Table 1. The number of iterates for L = 1.

L	$\hat{\ell}$
0.25	6
0.5	9
1	13
2	17
4	21

Table 2. The number of iterates for B_1 = 0.25.

From the Tables 1 and 2 it appears indeed that the convergence of the sequence of numerical approximations S_i slows down if $\frac{1}{B_1}$ or L increases.

In the Figures 2 and 3 we have plotted some iterates for B_1 = 0.25, L = 0.25 and L = 2. In the figures we observe that at first the iterates converge fast on a small time interval but slow on the whole time interval. In the end the iterates converge fast on the whole time interval (compare the proof of Theorem 4.15). Remark that in this example the iterates form an alternating sequence. Furthermore the odd numbered iterates form a monotone decreasing sequence whereas the even numbered iterates form a monotone increasing sequence.

5.4 Example where C_0 is such that $\lim_{x\uparrow 0} C_0(x) > 0$

In this subsection we consider an example with an initial function C_0 which is not an element of Cond 2.1. It appears that for a given function $S_0 \in P$ the sequence S_i given by $S_{i+1} = \mathcal{T}(S_i)$, $i \geq 0$ does not converge.

Define $C_0(x) = 1$, $x \in (-\infty,0]$, $\bar{B}(x) = B_1 x$, $B_1 > 0$, $x \in \mathbb{R}$ and $\bar{G}(S,f,t) = f(t)$, $t \in [0,T]$. It follows from Subsection 5.1 that Condition 2.3 is satisfied for this choice of $\bar{B}$ and $\bar{G}$ with $B = B_1$, $G_1 = 0$ and $G_2 = 1$.

For $k \geq 0$ we define the function $S \in \mathcal{O}$ by $S(t) = k\sqrt{t}$, $t \in [0,T]$ and

$$C_S(x,t) = 1 - \left(1+\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)\Big/\left(1+\mathrm{erf}\left(\frac{k}{2}\right)\right), (x,t) \in \bar{Q}_S.$$

It is easily seen that the function C_S satisfies:

$$\frac{\partial C_S(x,t)}{\partial t} - \frac{\partial^2 C_S(x,t)}{\partial x^2} = 0 \quad , \ x \in (-\infty, S(t)), \ t \in (0,T],$$

$C_S(x,0) = C_0(x)$, $x \in (-\infty,0)$ and $C_S(S(t),t) = 0$, $t \in (0,T]$.

From the definitions it follows that

$$\mathcal{T}(S)(t) = \frac{-1}{B_1} \int_{-\infty}^{\infty} [C_S(x,t) - C_0(x)]dx = \frac{2\ \exp(-(\frac{k}{2})^2)}{B_1\sqrt{\pi}(1+\mathrm{erf}(\frac{k}{2}))} \sqrt{t}, \ t \in [0,T].$$

For a given $k_0 \geq 0$ we define $S_0(t) = k_0\sqrt{t}$, $t \in [0,T]$ and $S_{i+1} = \mathcal{T}(S_i)$, $i \geq 0$.

It follows that $S_{i+1}(t) = k_{i+1}\sqrt{t}$, $t \in [0,T]$ with

$$k_{i+1} = \frac{2}{B_1\sqrt{\pi}} \frac{\exp(-(\frac{k_i}{2})^2)}{(1+\mathrm{erf}(\frac{k_i}{2}))} \ , \ i \geq 0.$$

The first column of Table 3 shows the computed iterates for $k_0 = 0$ and $B_1 = 10$. The table suggests that the sequence $\{k_i\}$ converges very fast. The second column of Table 3 shows the results for $B_1 = 0.28$ and $k_0 = 0$ (the value of B_1 is motivated by the example of the solidification of steel mentioned in the introduction), whereas the third column shows the results for $B_1 = 0.28$ and $k_0 = 2$. In both cases the sequence $\{k_i\}$ seems to be divergent.

B_1 \ i	10	0.28	0.28
0	.00000000000000E+00	.00000000000000E+00	.20000000000000E+01
1	.11283791670955E+00	.40299255967697E+01	.80454015224521E+00
2	.10575394486570E+00	.34829757729564E-01	.23961218545097E+01
3	.10619274629613E+00	.39510705880979E+01	.50227994826367E+00
4	.10616554230449E+00	.40781319185908E-01	.29616443267558E+01
5	.10616722875706E+00	.39376636196297E+01	.22901993883374E+00
6	.10616712420874E+00	.41877248584397E-01	.35240638226608E+01
7	.10616713069001E+00	.39351969976127E+01	.90919372991648E-01
8	.10616713028822E+00	.42081660394389E-01	.38255087236915E+01
9	.10616713031312E+00	.39347369996965E+01	.52099084807601E-01
10	.10616713031158E+00	.42119877347319E-01	.39122229620512E+01
11	.10616713031167E+00	.39346510008064E+01	.44027842204842E-01
12	.10616713031167E+00	.42127025562986E-01	.39303585818360E+01
13	.10616713031167E+00	.39346349154019E+01	.42485161526590E-01
14	.10616713031167E+00	.42128362697328E-01	.39338290499032E+01
15	.10616713031167E+00	.39346319064937E+01	.42195399553968E-01
16	.10616713031167E+00	.42128612823503E-01	.39344810570076E+01
17	.10616713031167E+00	.39346313436434E+01	.42141154387612E-01
18	.10616713031167E+00	.42128659612578E-01	.39346031218370E+01
19	.10616713031167E+00	.39346312383556E+01	.42131005715959E-01
20	.10616713031167E+00	.42128668365037E-01	.39346259590014E+01
21	.10616713031167E+00	.39346312186602E+01	.42129107233743E-01
22	.10616713031167E+00	.42128670002290E-01	.39346302310891E+01
23	.10616713031167E+00	.39346312149760E+01	.42128752098024E-01
24	.10616713031167E+00	.42128670308557E-01	.39346310302388E+01
25	.10616713031167E+00	.39346312142868E+01	.42128685665552E-01

Table 3 The computed values of k_i for $i = 0,\dots,25$.

We make a further investigation for the case $k_0 = 2$ and $B_1 = 0.28$.

Since the function $f_{B_1}: k \to \frac{2}{B_1\sqrt{\pi}} \frac{\exp(-(\frac{k}{2})^2)}{(1+\mathrm{erf}(\frac{k}{2}))}$

is monotone decreasing, the inequalities

$$k_{2i+3} \le k_{2i+1} \le k_{2i} \le k_{2i+2}, \quad i \ge 0$$

hold for the sequence $\{k_i\}$ in the third column of Table 3. Thus the sequence $\{k_i\}$ with $k_0 = 2$ and $B_1 = 0.28$ is divergent.

In Section 4 we have proven the existence and the uniqueness of the solution of a Stefan problem using the convergent sequence $\{S_i\}$ given by $S_{i+1} = \mathcal{T}(S_i)$. In this example we have shown that there are values of B_1 such that the sequence $\{S_i\}$ is divergent. Nevertheless, a further analysis yields that for every $B_1 > 0$ there is a solution of this Stefan problem in the set

$$\{S \in \dot{O} | S(t) = k\sqrt{t},\ t \in [0,T],\ k \geq 0\}$$

which is unique in this set. Denote this solution by $\bar{S}(t) = k_{B_1}\sqrt{t},\ t \in [0,T]$. Since f_{B_1} is monotone decreasing it can be shown that if $k_0 = 2$ and $B_1 = 0.28$ then $k^+, k^- \in \mathbb{R}$ given by $\lim_{i\to\infty} k_{2i} = k^+$ and $\lim_{i\to\infty} k_{2i+1} = k^-$ satisfy the inequalities $k^- < k_{B_1} < k^+$. Since there is a unique solution of the Stefan problem of the form $S(t) = k\sqrt{t},\ t \in [0,T]$, the functions S^+, S^- given by $S^+(t) = k^+\sqrt{t},\ \bar{S}(t) = k^-\sqrt{t},\ t \in [0,T]$ are no solutions of the Stefan problem.

Remark 5.10.

In this remark we compare the results on the convergence of the iterates in this example with the results obtained in the proof of Theorem 4.15 of the present paper and in Lemma 5.5 of [Vuik, 1987].

For this comparison we approximate the initial function C_0 given by $C_0(x) = 1,\ x \in (-\infty,0]$ with the sequence of functions $\{C_0^n\}_{n\geq 1}$ where C_0^n, $n \geq 1$ is defined by $C_0^n(x) = \min\{1,-nx\},\ x \in (-\infty,0]$.

Since C_0^n satisfies Condition 2.1 it follows from [Vuik, 1987; Lemma 5.5] that if $B > G_1 + G_2$ then $\lim_{i\to\infty} S_i^n = S^n$ exists and $\|S_{i+1}^n - S^n\|_\infty \leq \frac{G_1+G_2}{B}\|S_i^n - S^n\|_\infty$.

This yields an overestimate of the contraction factor which does not depend on n.

Thus if $B > G_1 + G_2$ we would not be surprised if the sequence $\{S_i\}_{i\geq 0}$ belonging to the initial function C_0, converges. The first column of Table 3 suggests indeed that for $B_1 = 10$, where the inequality $B > G_1 + G_2$ holds, the sequence $\{S_i\}_{i\geq 0}$ is convergent.

On the other hand if $B \in (G_1,G_2+G_2)$ then the proof of Theorem 4.15 and the numerical experiments suggest that the speed of convergence of the sequence $\{S_i^n\}_{i\geq 0}$ depends on the Lipschitz constant $L_n = n$ of C_0^n.

Since L_n goes to infinity for $n \to \infty$ we expect that the speed of convergence goes to zero. Thus if $B \in (G_1,G_1+G_2)$ it seems possible that the sequence $\{S_i\}_{i\geq 0}$ is divergent. For $B_1 = 0.28$ the constant B is an element of (G_1, G_1+G_2) and it follows indeed from Table 3 that the sequence $\{S_i\}_{i\geq 0}$ is divergent.

References.

1. M. Abramowitz, I.A. Stegun
Handbook of Mathematical Functions
Dover Publications New York (1972)

2. E.G. Effros, J.L. Kazdan
On the Dirichlet problem for the heat equation
Indiana Univ. Math. J. 20 683-693 (1971)

3. A. Fasano, M. Primicerio
General free boundary problems for the heat equation I
J. Math. Anal. Appl. 57 694-723 (1977)

4. O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva
Linear and quasilinear equations of parabolic type
American Mathematical Society, Providence (1968)

5. S.G. Mikhlin
Integral equations and their applications to certain problems in mechanics, mathematical physics and technology
Pergamon Press London (1957)

6. H.J. Schulze, P.M. Beckett, J.A. Howarth, G. Poots
Analytical and numerical solutions to two dimensional moving interface problems with applications to the solidification of killed steel ingots
Proc. R. Soc. Lond. A 385 313-343 (1983)

7. R.E. Smithson
Multifunctions
Nieuw Arch. Wisk. (3) 20 31-53 (1972)

8. C. Vuik (Chapter 1 of this monograph)
The solution of a one-dimensional Stefan problem I (1987)

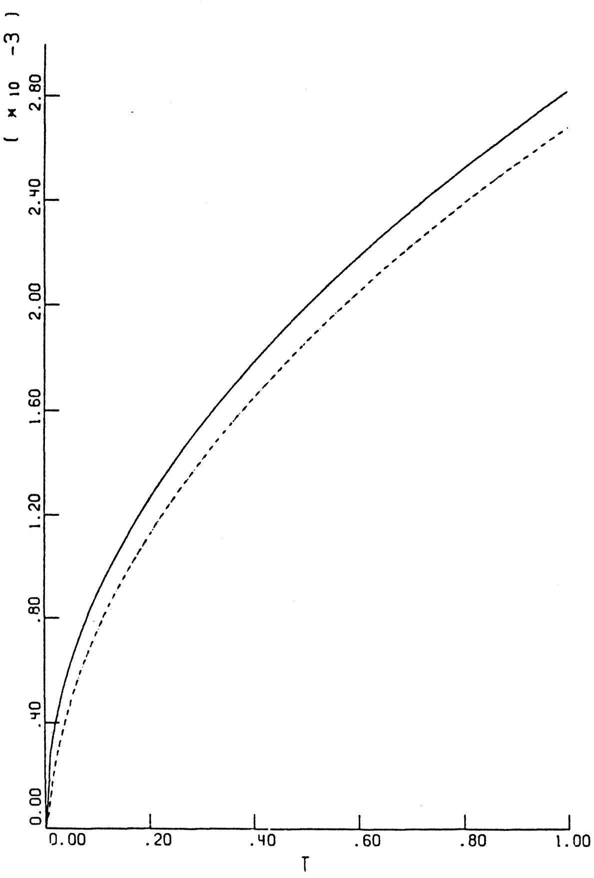

Figure 1 —— $2L\sqrt{\frac{t}{\pi}}\ \|S_1 - S_2\|_{t,\infty}$

-.-.- $\|f_{S_1} - f_{S_2}\|_{t,\infty}$

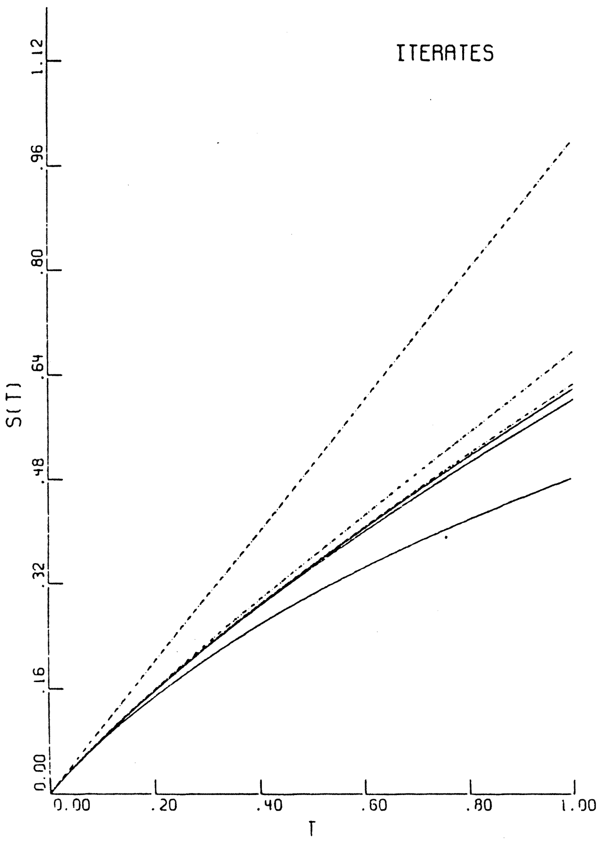

Figure 2 -.-.- odd numbered iterates

——— even numbered iterates

B = 0.25 , L = 0.25

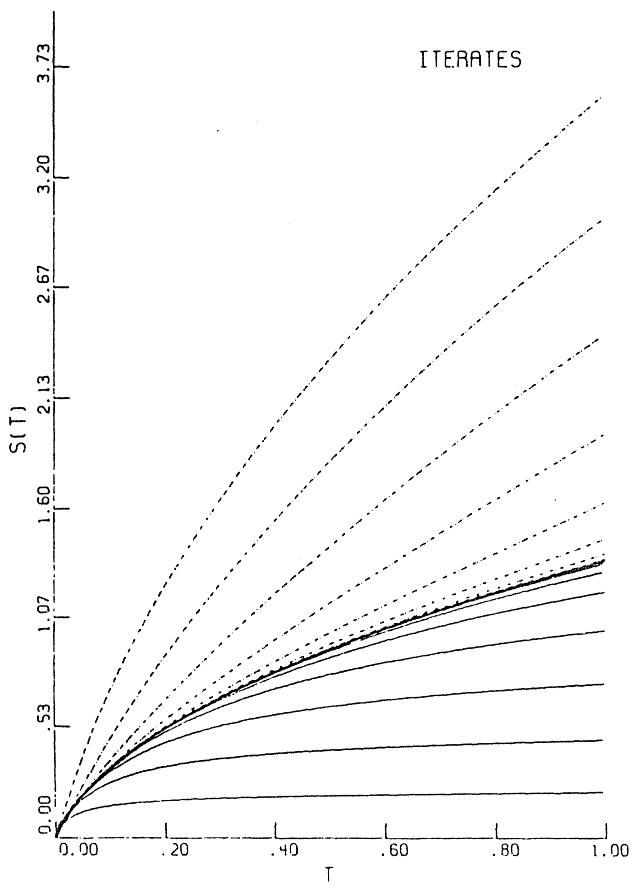

Figure 3 -.-.- odd numbered iterates

——— even numbered iterates

B = 0.25 , L = 2

The solution of a one-dimensional Stefan problem III

1. Introduction

The subject of this paper is a more general one-dimensional Stefan problem then the one considered in [Va] and [Vb]. In Section 2 we specify the Stefan problem. In [Va] and [Vb] we have shown existence and uniqueness of the solution of a Stefan problem under certain conditions. From these conditions it follows that a solution of the Stefan problem is a monotone increasing function. Some of these conditions are weakened in this paper. A consequence of this is that a solution of the Stefan problem can be a non-monotone function. Another consequence is that there are Stefan problems such that the existence of the solution follows from theorems shown in this paper, whereas the existence theorems of [Va] or [Vb] are not applicable. Examples of such problems are:

i) a mathematical model which describes the freezing of a supercooled liquid (reported in [Carslaw & Jaeger, 1959; p.287], [Parker, 1970; p.175] and [Moerbecke, 1974; Section 4.1]),

ii) a mathematical model which describes the growth of a crystal in a supersaturated solution (reported in [Parker, 1970; p.178]).

We now describe this model ii).

Let $x \in \mathbb{R}$ be the space co-ordinate and t the time co-ordinate. For the time $t = 0$ we suppose the region $x \geq 0$ to consist of a crystal and the region $x < 0$ to consist of a supersaturated solution. The position of the time dependent boundary between the liquid and the crystal is denoted by S(t). The function C describes the concentration of the solute. In dimensionless form the problem is: given $T > 0$, $B > 0$ and C_o: $(-\infty,0] \to [0,1]$, find sufficiently smooth functions

S: $[0,T] \to \mathbb{R}$ and C: $\{(x,t) | x \in (-\infty, S(t)],\ t \in [0,T]\} \to \mathbb{R}$
such that

$$(1.1) \qquad \begin{cases} \dfrac{\partial C(x,t)}{\partial t} - \dfrac{\partial^2 C(x,t)}{\partial x^2} = 0 \ , \ x \in (-\infty, S(t)),\ t \in (0,T], \\ C(x,0) = C_o(x),\ x \in (-\infty, 0] \ ,\ C(S(t),t) = 0,\ t \in [0,T], \end{cases}$$

$$(1.2) \qquad \frac{1}{B} \int_{-\infty}^{\infty} [C(x,t) - C_o(x)]dx = S(t) \ ,\ t \in [0,T].$$

For easy notation, we define $C(x,t) = 0,\ x \in (S(t),\infty),\ t \in [0,T]$ and $C_o(x) = 0,\ x \in (0,\infty)$.

Existence and uniqueness results for the solution of a Stefan problem which can be applied to the given example are reported in [Rubinstein, 1971], [Sherman, 1971], [Moerbecke, 1974] and [Fasano & Primicerio, 1977]. In Section 6 we compare the existence and uniqueness theorem of the present paper with the results given in the literature. In [Va] and [Vb] we have shown existence and uniqueness for the solution of a Stefan problem with the following condition

$$-\frac{1}{B} \int_{-\infty}^{\infty} [C(x,t) - C_o(x)]dx = S(t),\ t \in [0,T]$$

instead of (1.2).

We now give a short description of the contents of the present paper. In Section 2 we state several definitions and conditions and we specify a Stefan problem which is more general then (1.1) and (1.2). Some results from [Va] and [Vb] are summarized in Section 3 which also includes an approximation result. In Section 4 we estimate the function $\frac{\partial C}{\partial x}$. After that we give existence theorems for the solution of the diffusion equation. Finally given two time dependent boundaries we compare the solutions of the equations given in (1.1). In Section 5 we define an operator $\mathcal{T}$ and state two fixed point theorems for this operator. Using these theorems we prove existence and uniqueness for a solution of the Stefan problem and give a comparison with the results given in [Va], [Vb].

Finally in Section 6 we give some examples of the Stefan problem specified in Section 2. In one of these examples Condition 2.3v) which is used in the proof of our Main Theorem, is not satisfied and there is no solution of the given Stefan problem. For some examples we compute a numerical solution.

2. Statement of the problem

In this section we give some definitions and specify a Stefan problem.

2.1 Definitions

For a given $T > 0$ we define the following function spaces:

$O = \{S \in C[0,T] \mid S(0) = 0\}$, $P = \{S \in O \mid S$ is monotone non-decreasing$\}$,

and for $K_1, K_2 \in \bar{\mathbb{R}}$; $K_1, K_2 \geq 0$, $t_1 > 0$

$$\tilde{M}_{K_1,K_2,t_1} = \{S \in O \mid -\frac{K_1 h}{2\sqrt{t+t_1}} \leq S(t+h) - S(t) \leq \frac{K_2 h}{2\sqrt{t+t_1}},\ h \geq 0;\ t+h,\ t \in [0,T]\}$$

and $M_{K_1,K_2,t_1} = \tilde{M}_{K_1,K_2,t_1} \cap C^2[0,T]$, where we use the conventions:

$-\infty < a$, $a < \infty$ for every $a \in \mathbb{R}$, $-\infty < \infty$, $0.\infty = \infty.0 = 0$,

$\infty a = a\infty = \infty$ and $\infty(-a) = (-a)\infty = (-\infty)a = a(-\infty) = -\infty$ for $a \in \mathbb{R}$, $a > 0$.

$\tilde{M}$ denotes the set of lower Lipschitz continuous functions: $S \in O$ and there is an $L \geq 0$ such that $-Lh \leq S(t+h) - S(t)$, $h \geq 0$; $t+h$, $t \in [0,T]$.

Note that for every $t_1 > 0$ $\tilde{M} = \bigcup_{K\geq 0} \tilde{M}_{K,\infty,t_1}$.

We define $\|f\|_{\hat{t},\infty} = \sup_{t\in[0,\hat{t}]} |f(t)|, \hat{t} \in [0,T]$ and $\|f\|_\infty = \|f\|_{T,\infty}$.

In Section 3 we prove that $\tilde{M}_{K_1,K_2,t_1}$ is the closure of M_{K_1,K_2,t_1} in $C[0,T]$ with respect to the ∞-norm.

For a given function $S \in C[0,T]$ the set $Q_S \subset \mathbb{R}^2$ is defined by $Q_S = \{(x,t) \mid x \in (-\infty, S(t)),\ t \in (0,T)\}$. The closure of a set $Q \subset \mathbb{R}^2$ is denoted by $\bar{Q}$.

2.2 The function spaces $C^{\ell,\ell/2}(\bar{Q}_S)$, $C^{2,1}(Q_S)$ and $C^{\ell}[a,b]$

We use the following function spaces defined in [Ladyženskaja, Solonnikov, Ural'ceva, 1968; p.7]. For a given $\ell \in \mathbb{R}^+\backslash\mathbb{N}$, $C^{\ell,\ell/2}(\bar{Q}_S)$ is the Banach space of continuous functions f on $\bar{Q}_S$, having continuous derivatives $\frac{\partial^{r+p}f}{\partial t^r \partial x^p}$ for $2r+p < \ell$ and a finite norm $\|f\|^{\ell,\ell/2}$.

Here the norm $\|f\|^{\ell,\ell/2}$ is defined by: let $[\ell]$ be the largest integer less than ℓ,

$$\|f\|^{\ell,\ell/2} = \sum_{j=0}^{[\ell]} \left(\sum_{2r+p=j} \max_{\bar{Q}_S} \left|\frac{\partial^{r+p}f}{\partial t^r \partial x^p}\right| \right) + \sum_{2r+p=[\ell]} \left\langle\frac{\partial^{r+p}f}{\partial t^r \partial x^p}\right\rangle_x^{\ell-[\ell]} + \sum_{0<\ell-2r-p<2} \left\langle\frac{\partial^{r+p}f}{\partial t^r \partial x^p}\right\rangle_t^{\frac{\ell-2r-p}{2}},$$

and $\langle f\rangle_x^\alpha = \sup \left\{ \frac{|f(x',t)-f(x'',t)|}{|x'-x''|^\alpha} \;\middle|\; (x',t),\ (x'',t) \in \bar{Q}_S;\ |x'-x''| \le 1 \right\}$,

$\langle f\rangle_t^\alpha = \sup \left\{ \frac{|f(x,t')-f(x,t'')|}{|t'-t''|^\alpha} \;\middle|\; (x,t'),\ (x,t'') \in \bar{Q}_S;\ |t'-t''| \le 1 \right\}$ for $\alpha \in (0,1)$.

$C^{2,1}(Q_S)$ is the set of continuous functions f on Q_S, having continuous derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial f}{\partial t}$.

For a given $\ell \in \mathbb{R}^+\backslash\mathbb{N}$ and $[a,b] \subset \mathbb{R}$, $C^\ell[a,b]$ is the Banach space of continuous functions f on $[a,b]$, having continuous derivatives $\frac{d^p f}{dx^p}$ for $p < \ell$, and a finite norm $\|f\|^\ell$. Here the norm $\|f\|^\ell$ is defined by $\|f\|^\ell = \sum_{j=0}^{[\ell]} \max_{[a,b]} \left|\frac{d^j f}{dx^j}\right| + \left\langle\frac{d^{[\ell]}f}{dx^{[\ell]}}\right\rangle^{\ell-[\ell]}$ and

$\langle f\rangle^\alpha = \sup \left\{ \frac{|f(x')-f(x'')|}{|x'-x''|^\alpha} \;\middle|\; x',\ x'' \in [a,b];\ |x'-x''| \le 1 \right\}$ and $\alpha \in (0,1)$.

2.3 Conditions

Suppose C_o: $(-\infty,0] \to [0,1]$ is a given function. We shall always impose Condition 2.1:

Condition 2.1.

The function C_o should be an element of the set Cond 2.1:= $\{\varphi: (-\infty,0] \to \mathbb{R} \mid \varphi$ is a monotone decreasing Lipschitz continuous function with $\varphi(0) = 0$ and $\lim_{x\to-\infty} \varphi(x) = 1\}$. Let L be a Lipschitz constant of the function C_o.

Occasionally we shall impose the following stronger condition:

Condition 2.2.

The function C_o should be an element of the set Cond 2.2:= $\{\varphi \in C^{3+\alpha}(-\infty,0]$ for an $\alpha \in (0,1) \mid \varphi \in$ Cond 2.1 and $\frac{d\varphi}{dx}(0) = \frac{d^2\varphi}{dx^2}(0) = 0\}$.

In the sequel, $\tilde{B}$: $\mathbb{R} \to \mathbb{R}$ will denote a multifunction (see

[Smithson, 1972]) and $\bar{G}$: $O \times P \times [0,T] \to \mathbb{R}$ will denote a functional, both subject to the following condition, which will always be imposed:

Condition 2.3.

The multifunction $\bar{B}$ should be such that:

i) $0 \in \bar{B}(0)$.

ii) There is a $B \in (0,\infty)$ such that for $x \in \mathbb{R}$, $h > 0$, $y_1 \in \bar{B}(x)$ and $y_2 \in \bar{B}(x+h)$ the inequality $y_2 - y_1 \geq Bh$ holds.

iii) $\bar{B}$ is surjective.

The functional $\bar{G}$ should be such that:

iv) $\bar{G}(S,f,0) = 0$, $S \in O$, $f \in P$.

v) There are constants γ_1, γ_2, $\gamma_3 \geq 0$ and $\hat{t} > 0$ such that for every $t_1 \in (0,\hat{t}]$, K_1, $K_2 \geq 0$ the inequality

$|\bar{G}(S,f,t+h) - \bar{G}(S,f,t)| \leq \dfrac{\gamma_1 K_1 + \gamma_2 K_2 + \gamma_3}{2\sqrt{t+t_1}}\, h$ holds for

$h \geq 0$; $t+h$, $t \in [0,T]$, $S \in \tilde{M}_{K_1,\infty,t_1}$, $f \in \tilde{M}_{0,K_2,t_1}$

with $\gamma_1 + \gamma_2 < B$ and $\gamma_2 > 0$.

To motivate the formulation of Lemmas 2.4, 2.7 and Definitions 2.6 and 2.9 we remark the following: using $\bar{B}$ and $\bar{G}$ we shall define in Section 5 an operator $\mathcal{T}$ and a function space as in [Vb] such that $\mathcal{T}$ maps the function space into itself. In [Vb] it follows from [Vb; Condition 2.3] that the function $t \mapsto \bar{G}(S,f,t)$ is an element of P which implies that $\mathcal{T}$: $P \to P$. However an important difference between the Condition 2.3 of [Vb] and the Condition 2.3 of the present paper is that in the latter the function $t \mapsto \bar{G}(S,f,t)$ is not necessarily an element of P. Thus it is possible that $S \in P$ but $\mathcal{T}(S) \notin P$. From this it follows that we look for a function space which includes non-monotone functions and is mapped into itself by the operator $\mathcal{T}$. With an appropriate choice of K_1, K_2 and t_1 (Definitions 2.6 and 2.9) we shall show in Section 5 that $\mathcal{T}: \tilde{M}_{K_1,K_2,t_1} \to \tilde{M}_{K_1,K_2,t_1}$.

Lemma 2.4.

For B, γ_1, γ_2 and γ_3 as in Condition 2.3 there is a unique constant $k_1 \in (0,\infty)$ such that for $\epsilon_o = \frac{B-\gamma_1-\gamma_2}{2\gamma_2}$ the equation

(2.5) $$Bk_1 = \gamma_1 k_1 + \frac{2\gamma_2(1+\epsilon_o)\exp(-\left(\frac{k_1}{2}\right)^2)}{\sqrt{\pi}(1+\mathrm{erf}\left(\frac{-k_1}{2}\right))} + \gamma_3 \text{ holds.}$$

Proof.

We define the function $\varphi: [0,\infty) \to \mathbb{R}$ by

$$\varphi: k \mapsto \frac{1}{B}\left\{ \gamma_1 k + \frac{2\gamma_2(1+\epsilon_o)\exp(-\left(\frac{k}{2}\right)^2)}{\sqrt{\pi}\ \mathrm{erfc}\left(\frac{k}{2}\right)} + \gamma_3 \right\}.$$

It is obvious that equation (2.5) is equivalent to $k_1 = \varphi(k_1)$. First of all we show that there is a constant $k_1 \in (0,\infty)$ such that $k_1 = \varphi(k_1)$. After that we show that this constant is unique.

From the inequalities (see [Abramowitz & Stegun, 1972; p.298, inequality 7.1.13])

$$\frac{1}{k+\sqrt{k^2+2}} < e^{k^2}\int_k^\infty e^{-y^2}\,dy \le \frac{1}{k+\sqrt{k^2+\frac{4}{\pi}}},\ k \ge 0,$$

it follows that

$$\frac{\sqrt{\pi}}{2k}\left(\frac{k}{2} + \sqrt{(\tfrac{k}{2})^2 + \tfrac{4}{\pi}}\right) \le \frac{\exp(-\left(\frac{k}{2}\right)^2)}{k\ \mathrm{erfc}\ (\frac{k}{2})} < \frac{\sqrt{\pi}}{2k}\left(\frac{k}{2} + \sqrt{(\tfrac{k}{2})^2+2}\right), k > 0$$

and thus $\lim_{k\to\infty} \frac{\exp(-(\frac{k}{2})^2)}{k\ \mathrm{erfc}\ (\frac{k}{2})} = \frac{\sqrt{\pi}}{2}$. This implies $\lim_{k\to\infty} \frac{\varphi(k)}{k} = \frac{\gamma_1+\gamma_2(1+\epsilon_o)}{B}$.

Using the definition of ϵ_o and the inequality $\gamma_1+\gamma_2 < B$ we deduce $\lim_{k\to\infty} \frac{\varphi(k)}{k} < 1$. On the other hand we have $\varphi(0) > 0$ because $\gamma_2 > 0$. Since φ is a continuous function it follows that there is a constant $k_1 \in (0,\infty)$ such that $k_1 = \varphi(k_1)$.

Using again inequality 7.1.13 of [Abramowitz & Stegun, 1972] it follows that $\frac{d\varphi(k)}{dk} > 0$ and $\frac{d}{dk}\left(\frac{\varphi(k)}{k}\right) < 0$, $k \in (0,\infty)$.

The last inequality combined with the fact that $\frac{\varphi(k_1)}{k_1} = 1$ yields $k < \varphi(k)$, $k \in [0,k_1)$ and $k > \varphi(k)$, $k \in (k_1,\infty)$. Thus there is a unique constant $k_1 \in (0,\infty)$ such that equation (2.5) holds. ⊠

Definition 2.6.

Define $\epsilon_o = \frac{B-\gamma_1-\gamma_2}{2\gamma_2}$, k_1 such that (2.5) holds and

$k_2 = \frac{2(1+\epsilon_o)\exp(-\left(\frac{k_1}{2}\right)^2)}{\sqrt{\pi}(1+\text{erf}\left(-\frac{k_1}{2}\right))}$. Furthermore we define the functions

$\tilde{S}:[0,T] \to (-\infty,0]$ and $\tilde{C}$: $\bar{Q}_{\tilde{S}} \to [0,\infty)$ by $\tilde{S}(t) = -k_1\sqrt{t}$, $t \in [0,T]$

and $\tilde{C}(x,t) = (1+\epsilon_o) - \frac{(1+\epsilon_o)(1+\text{erf}(\frac{x}{2\sqrt{t}}))}{(1+\text{erf}(\frac{-k_1}{2}))}$, $(x,t) \in \bar{Q}_{\tilde{S}}$.

Lemma 2.7.

There is a $\tilde{t} > 0$ such that for $t \in (0,\tilde{t}]$ the inequality

(2.8) $\quad C_o(x) \le \tilde{C}(x+\tilde{S}(t),t)$ holds for $x \in (-\infty,0]$.

Proof.

Since $\lim_{t\downarrow 0} \tilde{C}(-\frac{1}{L} + \tilde{S}(t),t) = 1 + \epsilon_o$ there is a $\tilde{t} > 0$ such that for $t \in (0,\tilde{t}]$, $\tilde{C}(-\frac{1}{L} + \tilde{S}(t),t) \ge 1$. The function $x \mapsto \tilde{C}(x+\tilde{S}(t),t)$, $x \in (-\infty,0]$ is monotone decreasing thus $C_o(x) \le 1 \le \tilde{C}(x+\tilde{S}(t),t)$, $x \in (-\infty,-\frac{1}{L}]$, $t \in (0,\tilde{t}]$.
Using the fact that the function $x \mapsto \tilde{C}(x+\tilde{S}(t),t)$, $x \in (-\infty,0]$ is concave, together with $\tilde{C}(-\frac{1}{L}+\tilde{S}(t),t) \ge 1$ and $\tilde{C}(\tilde{S}(t),t) = 0$ we deduce

$C_o(x) \le -Lx \le \tilde{C}(x+\tilde{S}(t),t)$, $x \in [-\frac{1}{L},0]$, $t \in (0,\tilde{t}]$.

This proves the lemma. ⊠

Definition 2.9.

For $\hat{t}$ as in Condition 2.3v) and $\tilde{t}$ as in Lemma 2.7 we define $t_o = \min(\hat{t},\tilde{t})$ and for $K_1,K_2 \in \mathbb{R}$; $K_1,K_2 \ge 0$ the following function spaces $\tilde{M}_{K_1,K_2} = \tilde{M}_{K_1,K_2,t_o}$ and $M_{K_1,K_2} = M_{K_1,K_2,t_o}$.

We shall always impose the following condition:

Condition 2.10.

The functional $\bar{G}$ should be such that there are constants $G_1, G_2 \in \mathbb{R}$ such that

$$\sup_{t\in[0,T]} |\bar{G}(S_1,f,t)-\bar{G}(S_2,f,t)| \le G_1\|S_1-S_2\|_\infty,\ S_1,S_2 \in \bar{M}_{k_1,k_1},\ f \in \bar{M}_{0,k_2},$$

$$\sup_{t\in[0,T]} |\bar{G}(S,f_1,t) - \bar{G}(S,f_2,t)| \le G_2\|f_1-f_2\|_\infty,\ S \in \bar{M}_{k_1,k_1},\ f_1,f_2 \in \bar{M}_{0,k_2}.$$

Occasionally we shall impose the following condition:

Condition 2.11.

For every $t \in [0,T]$, $S \in \bar{M}_{k_1,k_1}$ and $f \in \bar{M}_{0,k_2}$, $\bar{G}(S,f,t)$ should only depend on $S|_{[0,t]}$ and $f|_{[0,t]}$.

2.4 Statement of a Stefan problem

In order to state our Stefan problem, we first state a reduced problem: for a given function $S \in O$, find a bounded solution $C_S \in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ of the equations in (1.1) for this function S. For any $S \in O$, for which the reduced problem has a solution C_S, we define $C_S(x,t) = 0$, $x > S(t)$, $t \in [0,T]$. Let the function $f_S:[0,T] \to \mathbb{R}$ be defined by $f_S(t) = -\int_{-\infty}^{\infty} [C_S(x,t) - C_o(x)]dx$.

A Stefan problem can be stated as follows: to find a lower Lipschitz continuous function S and a solution C_S of the reduced problem such that $\bar{G}(S,f_S,t) \in \bar{B}(S(t))$, $t \in [0,T]$.

3 Preliminaries

In this section we summarize known results about the solution of the diffusion equation and show that every function $\bar{S} \in \bar{M}_{K_1,K_2,t_1}$ can be approximated by a function $S \in M_{K_1,K_2,t_1}$.

3.1 Known results

Lemma 3.1 (maximum principle) [Va; Lemma 2.4].

Suppose $S \in C[0,T]$. If the bounded function $u \in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ satisfies:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ x \in (-\infty, S(t)),\ t \in (0,T],$$

then $$\min\left(\inf_{x \in (-\infty, S(0)]} u(x,0),\ \min_{t \in [0,T]} u(S(t),t) \right) \le u(\tilde{x},\tilde{t}) \le$$

$$\max\left(\sup_{x \in (-\infty, S(0)]} u(x,0),\ \max_{t \in [0,T]} u(S(t),t) \right) \text{ for } (\tilde{x},\tilde{t}) \in \bar{Q}_S.$$

Lemma 3.2 [Va; Lemma 2.6].

Suppose the function φ: $(-\infty,0] \to [0,1]$ is an element of Cond 2.1.

For every $\epsilon > 0$ there is a function φ^-: $(-\infty,0] \to [0,1]$ which is an element of Cond 2.2, $\varphi^-(x) \le \varphi(x)$, $x \in (-\infty,0]$, $\int_{-\infty}^{\infty} [\varphi(x)-\varphi^-(x)]dx < \epsilon$ and a Lipschitz constant of φ is also a Lipschitz constant of φ^-.

In the same way it can be shown that there is a function φ^+: $(-\infty,\epsilon] \to [0,1]$, such that $x \mapsto \varphi^+(x+\epsilon)$ is an element of Cond 2.2, $\varphi(x) \le \varphi^+(x)$, $x \in (-\infty,0]$, $\int_{-\infty}^{0} [\varphi^+(x)-\varphi(x)]dx + \int_{0}^{\epsilon} \varphi^+(x)dx < \epsilon$ and a Lipschitz constant of φ is also a Lipschitz constant of φ^+.

Lemma 3.3 [Va; Lemma 2.5].

For $S \in C[0,T]$ suppose u_m is a sequence of bounded functions such that $u_m \in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ and $\frac{\partial u_m(x,t)}{\partial t} - \frac{\partial^2 u_m(x,t)}{\partial x^2} = 0$, $x \in (-\infty,S(t))$, $t \in (0,T]$.

Define the functions φ_m and Φ_m by $\varphi_m(x) = u_m(x,0)$, $x \in (-\infty,S(0)]$ and $\Phi_m(t) = u_m(S(t),t)$, $t \in [0,T]$.

If $\varphi_m \to \varphi$ uniformly on $(-\infty,S(0)]$ and $\Phi_m \to \Phi$ uniformly on $[0,T]$ then the function u: $\bar{Q}_S \to \mathbb{R}$ defined by $u(x,t) = \lim_{m\to\infty} u_m(x,t)$, $(x,t) \in \bar{Q}_S$ is bounded, $u \in C^{2,1}(Q_S) \cap C(\bar{Q}_S)$ and satisfies

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ x \in (-\infty,S(t)),\ t \in (0,T],$$

$u(x,0) = \varphi(x)$, $x \in (-\infty,S(0)]$, $u(S(t),t) = \Phi(t)$, $t \in [0,T]$.

Theorem 3.4 [Va; Theorem 4.1].

Suppose Condition 2.2 holds. Then for any $S \in O \cap C^2[0,T]$, the reduced problem has a unique solution C_S and it satisfies $C_S \in C^{3+\alpha, \frac{3+\alpha}{2}}(\bar{Q}_S)$ with the same α as in Condition 2.2.

Lemma 3.5 [Va; Lemma 4.2].

Suppose Condition 2.2 holds. Let $S_1, S_2 \in O \cap C^2[0,T]$ and $\frac{dS_1(t)}{dt} \le \frac{dS_2(t)}{dt}$ for $t \in [0,T]$, then the inequality $-\frac{\partial C_{S_2}}{\partial x}(S_2(t),t) \le -\frac{\partial C_{S_1}}{\partial x}(S_1(t),t)$ holds for $t \in [0,T]$.

Lemma 3.6 [Va; Lemma 4.6].

Suppose Condition 2.2 holds. Then for any $S \in O \cap C^2[0,T]$, the solution C_S of the reduced problem satisfies:

$$\int_{-\infty}^{\infty} [C_S(x,t) - C_o(x)]dx = \int_0^t \frac{\partial C_S}{\partial x}(S(\tau),\tau)d\tau.$$

Lemma 3.7 [Va; Theorems 4.11, 4.12 ii)].

i) If $S_1, S_2 \in O$, $S_1(t) \le S_2(t)$, $t \in [0,T]$ and there are solutions C_{S_1}, C_{S_2} of the reduced problem, then $C_{S_1}(x,t) \le C_{S_2}(x,t)$, $(x,t) \in \bar{Q}_{S_2}$ and $\int_{-\infty}^{\infty} [C_{S_2}(x,t) - C_{S_1}(x,t)]dx \le \|S_1 - S_2\|_{\bar{t},\infty}$, $t \in [0,\bar{t}]$, $\bar{t} \in [0,T]$.

ii) If $S \in O$ and there is a solution of the reduced problem then $f_S \in P$.

Lemma 3.8 [Vb; Lemma 4.14].

Conditions 2.10, 2.11 hold if and only if

$$\sup_{t \in [0,\bar{t}]} |\bar{G}(S_1,f,t) - \bar{G}(S_2,f,t)| \le G_1 \|S_1 - S_2\|_{\bar{t},\infty}, \quad \bar{t} \in [0,T],\ S_1,S_2 \in \bar{M}_{k_1,k_1},\ f \in \bar{M}_{0,k_2},$$

$$\sup_{t \in [0,\bar{t}]} |\bar{G}(S,f_1,t) - \bar{G}(S,f_2,t)| \le G_2 \|f_1 - f_2\|_{\bar{t},\infty}, \quad \bar{t} \in [0,T],\ S \in \bar{M}_{k_1,k_1},\ f_1,f_2 \in \bar{M}_{0,k_2}.$$

3.2 Approximation results

Lemma 3.9.

For $\epsilon > 0$, $K_1, K_2 \ge 0$, $t_1 > 0$ and $S \in \bar{M}_{K_1,K_2,t_1}$ there are functions $S^+, S^- \in M_{K_1,K_2,t_1}$ such that $S^-(t) \le S(t) \le S^+(t)$, $t \in [0,T]$

and $\|S^+-S^-\|_\infty < 2\epsilon$.

Proof.

The function $\hat{S} \in \tilde{M}_{0,K_1+K_2,t_1}$ is defined by $\hat{S}(t) = S(t) + K_1\sqrt{t+t_1}$, $t \in [0,T]$.

Take $N \in \mathbb{N}$, $N > \frac{4T}{\epsilon\sqrt{t_1}}(K_1+K_2)$. For $h = \frac{T}{N}$ we define $S_N \in P$

as follows:

$$S_N(ih) = \begin{cases} \hat{S}((i+2)h)-\hat{S}(2h), & i=0,1,\dots,N-2 \\ \hat{S}(T) - \hat{S}(2h), & i=N-1,N \end{cases},$$

and S_N is a linear function on $[ih, (i+1)h]$, $i=0,\dots,N-1$. It is easily seen that $\|\hat{S}-S_N\|_\infty < \frac{3\epsilon}{8}$.

If the function $\bar{S} \in P$ is given by $\bar{S}(t) = \max(0, S_N(t)-\frac{\epsilon}{2})$, $t \in [0,T]$ then there is a strictly increasing sequence $\tau_i \in [0,T]$, $i=0,\dots,m+1$, with $m \le N$ such that $\tau_o = 0$, $\tau_{m+1} = T$ and $\bar{S} \in C^2([0,T]\setminus\bigcup_{i=1}^{m}\{\tau_i\})$.

From the definition of $\bar{S}$ it follows that there are $a_i, b_i \in \mathbb{R}$ such that $\bar{S}(t) = a_i t + b_i$, $t \in [\tau_i, \tau_{i+1}]$ and $0 \le a_i \le \frac{K_1+K_2}{2\sqrt{\tau_{i+2}+t_1}}$, $i=0,\dots,m-1$ and $a_m=0$.

For $\mu \in (0, \frac{1}{4}\min_{i=0,\dots,m}(\tau_{i+1}-\tau_i)]$ the function S_μ is defined by:

$$S_\mu(t)= \begin{cases} \bar{S}(\tau_i-\mu)+(t-(\tau_i-\mu))(\frac{a_{i-1}+a_i}{2})-\frac{2\mu}{\pi}(\frac{a_i-a_{i-1}}{2})\cos((t-\tau_i)\frac{\pi}{2\mu}), t\in[\tau_i-\mu,\tau_i+\mu], i=1,\dots,m, \\ \bar{S}(t), t\in[0,T]\setminus\bigcup_{i=1}^{m}[\tau_i-\mu,\tau_i+\mu]. \end{cases}$$

For μ small enough S_μ has the following properties:

$$S_\mu \in M_{0,K_1+K_2,t_1},\ S_\mu(t) \le \hat{S}(t) \le S_\mu(t) + \epsilon,\ t \in [0,T].$$

Define for this function S_μ the function S^- by $S^-(t) = S_\mu(t) - K_1\sqrt{t+t_1}$, $t \in [0,T]$.

It follows that $S^- \in M_{K_1,K_2,t_1}$ and $S^-(t) \le S(t) \le S^-(t) + \epsilon$, $t \in [0,T]$.

To find S^+ we note that the function $t \mapsto -S(t)$ is an element of $\tilde{M}_{K_2,K_1,t_1}$.

From the construction of S^- it follows that there is a function $\tilde{S} \in M_{K_2,K_1,t}$ such that $\tilde{S}(t) \le -S(t) \le \tilde{S}(t) + \epsilon$, $t \in [0,T]$, thus the function S^+ given by $S^+(t) = -\tilde{S}(t)$, $t \in [0,T]$ has the required properties. ⊠

Corollary 3.10.

$\tilde{M}_{K_1,K_2,t_1}$ is the closure of M_{K_1,K_2,t_1} in $C[0,T]$ with respect to the ∞-norm.

4. Properties of the reduced problem

In this section we give some properties of solutions of the reduced problem. These properties are used in Section 5 to prove an existence theorem for the solution of the Stefan problem.

4.1 Estimation of $\frac{\partial C_S}{\partial x}$

Theorem 4.1.

Suppose Condition 2.2 holds and $S \in M_{k_1,\infty}$. Then the solution C_S of the reduced problem has the properties:

i) $0 \le -\frac{\partial C_S}{\partial x}(S(t),t) \le \frac{k_2}{2\sqrt{t+t_o}}$, $t \in [0,T]$,

ii) $0 \le -\frac{\partial C_S(x,t)}{\partial x} \le \frac{k_2}{2\sqrt{t_o}}$, $(x,t) \in \bar{Q}_S$.

Proof.

i) We use the functions $\tilde{S}$, $\tilde{C}$ which are defined in Definition 2.6.
For t_o as in Definition 2.9 the function $\hat{S}:[0,T] \to \mathbb{R}$ defined by
$\hat{S}(t) = \tilde{S}(t+t_o) - \tilde{S}(t_o)$, $t \in [0,T]$ is an element of $O \cap C^2[0,T]$.
Application of Theorem 3.4 yields:

there is a unique solution $C_{\hat{S}}$ of the reduced problem and $C_{\hat{S}} \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q}_{\hat{S}})$.
It is easy to verify that the function $(x,t) \mapsto \tilde{C}(x+\tilde{S}(t_o), t+t_o)$ is a solution of the reduced problem for the time-dependent boundary $\hat{S}$ and the initial function $x \mapsto \tilde{C}(x+\tilde{S}(t_o),t_o)$.
Inequality (2.8) implies $C_o(x) \le \tilde{C}(x+\tilde{S}(t_o),t_o)$ for $x \in (-\infty,0]$.
This combined with the maximum principle yields

$$C_{\hat{S}}(x,t) \le \tilde{C}(x+\tilde{S}(t_o),\ t+t_o),\ (x,t) \in \bar{Q}_{\hat{S}}.$$

Since $C_{\hat{S}}(\hat{S}(t),t) = 0$ and $\tilde{C}(\hat{S}(t)+\tilde{S}(t_o),t+t_o) = \tilde{C}(\tilde{S}(t+t_o),t+t_o) = 0$, $t \in [0,T]$

it follows that

(4.2) $$-\frac{\partial C_{\hat S}}{\partial x}(\hat S(t),t) \le -\frac{\partial \tilde C}{\partial x}(\hat S(t)+\tilde S(t_o),\ t+t_o),\ t \in [0,T].$$

From Definition 2.6 we know

$$\frac{\partial \tilde C(x,t)}{\partial x} = -\frac{(1+\epsilon_o)\exp(-\frac{x^2}{4t})}{(1+\mathrm{erf}(\frac{-k_1}{2}))\sqrt{\pi t}},\ x \in (-\infty,\tilde S(t)],\ t \in (0,\infty),$$

thus $$-\frac{\partial \tilde C}{\partial x}(\hat S(t_o)+\tilde S(t_o),\ t+t_o) = \frac{2(1+\epsilon_o)\exp(-(\frac{k_1}{2})^2)}{\sqrt{\pi}(1+\mathrm{erf}(\frac{-k_1}{2}))2\sqrt{t+t_o}} = \frac{k_2}{2\sqrt{t+t_o}},\ t \in [0,T].$$

This together with inequality (4.2) yields

(4.3) $$-\frac{\partial C_{\hat S}}{\partial x}(\hat S(t),t) \le \frac{k_2}{2\sqrt{t+t_o}},\ t \in [0,T].$$

From the definition of $\hat S$ and $M_{k_1,\infty}$ we know that for every $S \in M_{k_1,\infty}$ the inequality $\frac{d\hat S(t)}{dt} \le \frac{dS(t)}{dt}$ holds for $t \in [0,T]$.

Application of Lemma 3.5 yields

$$-\frac{\partial C_S}{\partial x}(S(t),t) \le -\frac{\partial C_{\hat S}}{\partial x}(\hat S(t),t),\ t \in [0,T].$$

From this and inequality (4.3) we deduce that

$$-\frac{\partial C_S}{\partial x}(S(t),t) \le \frac{k_2}{2\sqrt{t+t_o}},\ t \in [0,T].$$

On the other hand application of the maximum principle gives $C_S(x,t) \ge 0$, $(x,t) \in \bar Q_S$. Since $C_S(S(t),t) = 0$, $t \in [0,T]$ this implies

$$0 \le -\frac{\partial C_S}{\partial x}(S(t),t),\ t \in [0,T].$$

ii) The bounded function $\frac{\partial C_S}{\partial x}$ satisfies

$$\frac{\partial}{\partial t}\left(\frac{\partial C_S(x,t)}{\partial x}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial C_S(x,t)}{\partial x}\right) = 0,\ x \in (-\infty,S(t)),\ t \in (0,T],$$

$$\frac{\partial C_S}{\partial x}(x,0) = \frac{dC_o(x)}{dx},\ x \in (-\infty,0].$$

Using the inequalities $0 \le -\frac{\partial C_S}{\partial x}(S(t),t) \le \frac{k_2}{2\sqrt{t+t_o}}$, $t \in [0,T]$ and the maximum principle we conclude:

(4.4) $$0 \le -\frac{\partial C_S(x,t)}{\partial x} \le \max\left\{L,\ \frac{k_2}{2\sqrt{t_o}}\right\},\ (x,t) \in \bar Q_S.$$

From the proof of Lemma 2.7 it follows that $\tilde C(-\frac{1}{L}+\tilde S(t_o),t_o) \ge 1$.

This together with $\tilde C(\tilde S(t_o),t_o) = 0$ implies that there is a $\xi \in (-\frac{1}{L},0)$ such that $-\frac{\partial \tilde C}{\partial x}(\xi+\tilde S(t_o),t_o) \ge L$.

Since the function $x \mapsto \tilde C(x+\tilde S(t_o),t_o)$ for $x \in (\ \infty,0]$ is concave and $-\frac{\partial\tilde{C}}{\partial x}(\tilde{S}(t_o),t_o) = \frac{k_2}{2\sqrt{t_o}}$ the last inequality implies $\frac{k_2}{2\sqrt{t_o}} \geq L$. This combined with inequality (4.4) yields $0 \leq -\frac{\partial C_S(x,t)}{\partial x} \leq \frac{k_2}{2\sqrt{t_o}}$, $(x,t) \in \bar{Q}_S$. ⊠

4.2 Existence results for C_S

The proof of the following lemma is analogous to the proof of [Va; Theorem 4.9] if we use Theorem 4.1 of the present paper instead of [Va; Lemma 4.3].

Lemma 4.5.

For a given function $S \in O \cap C^2\,[0,T]$ there is a unique solution C_S of the reduced problem and $f_S \in P$. If $S \in M_{k_1,\infty}$, then

$f_S \in \bar{M}_{0,k_2}$ and $0 \leq -\frac{\partial C_S(x,t)}{\partial x} \leq \frac{k_2}{2\sqrt{t_o}}$ for $(x,t) \in Q_S$.

Theorem 4.6.

For a given function $S \in \bar{M}_{k_1,k_1}$ there is a unique solution C_S of the reduced problem and $0 \leq -\frac{\partial C_S(x,t)}{\partial x} \leq \frac{k_2}{2\sqrt{t_o}}$, $(x,t) \in Q_S$.

Proof.

According to Lemma 3.9 there is a sequence of functions $S_n \in M_{k_1,k_1}$ $n = 1,2,\ldots$ such that $\lim_{n\to\infty} \|S_n-S\|_\infty = 0$ and $S(t) \leq S_n(t)$, $t \in [0,T]$.

Application of Lemma 4.5 yields that there are unique solutions C_{S_n}, $n = 1,2,\ldots$ of the reduced problem.

Define $\Phi_n(t) = C_{S_n}(S(t),t)$ and $\Phi(t) = 0$, $t \in [0,T]$. We know that $C_{S_n} \in C^{2,1}(Q_{S_n}) \cap C(\bar{Q}_{S_n})$ hence

$$C_{S_n}(S(t),t) = C_{S_n}(S_n(t),t) + \int_{S_n(t)}^{S(t)} \frac{\partial C_{S_n}}{\partial x}(\xi,t)d\xi = \int_{S_n(t)}^{S(t)} \frac{\partial C_{S_n}}{\partial x}(\xi,t)d\xi,\ t \in [0,T].$$

From Lemma 4.5 it follows that $0 \leq -\frac{\partial C_{S_n}(x,t)}{\partial x} \leq \frac{k_2}{2\sqrt{t_o}}$, $(x,t) \in Q_{S_n}$,

thus $\quad \|\Phi_n-\Phi\|_\infty \leq \frac{k_2}{2\sqrt{t_o}}\|S_n-S\|_\infty$.

Since $C_{S_n}: \bar{Q}_S \to [0,1]$, $C_{S_n}(x,0) = C_o(x)$, $x \in (-\infty,0]$, $n = 1,2,\ldots$ and $\Phi_n \to \Phi$ uniformly on $[0,T]$ we obtain from Lemma 3.3 and the maximum principle that the function $C_S(x,t) = \lim_{n\to\infty} C_{S_n}(x,t)$, $(x,t) \in \bar{Q}_S$ is the unique solution of the reduced problem. The inequalities $C \leq -\frac{\partial C_{S_n}(x,t)}{\partial x} \leq \frac{k_2}{2\sqrt{t_o}}$,

$(x,t) \in Q_S$, together with Lemma 3.3 yield

$$0 \leq -\frac{\partial C_S(x,t)}{\partial x} \leq \frac{k_2}{2\sqrt{t_o}}, \quad (x,t) \in Q_S. \qquad \boxtimes$$

4.3 Properties of the function f_S

Theorem 4.7.

If $S_1, S_2 \in \tilde{M}_{k_1,k_1}$ and f_{S_1}, f_{S_2} exist then the inequality

$$\|f_{S_1} - f_{S_2}\|_{\bar{t},\infty} \leq \|S_1 - S_2\|_{\bar{t},\infty} \text{ holds for } \bar{t} \in [0,T].$$

Proof.

Define the functions S^+, $S^- \in \tilde{M}_{k_1,k_1}$ as follows $S^+(t) = \max\{S_1(t), S_2(t)\}$ and $S^-(t) = \min\{S_1(t), S_2(t)\}$, $t \in [0,T]$. Theorem 4.6 says that there are solutions C_{S^+} and C_{S^-} of the reduced problem. Application of the maximum principle yields:

$$C_{S^-}(x,t) \leq C_{S_1}(x,t),\ C_{S_2}(x,t) \leq C_{S^+}(x,t),\ (x,t) \in \bar{Q}_{S^+}, \text{ and}$$

thus for $N \in \mathbb{R}$

$$\int_N^\infty |C_{S_2}(x,t) - C_{S_1}(x,t)|dx \leq \int_N^\infty [C_{S^+}(x,t) - C_{S^-}(x,t)]dx.$$

From Lemma 3.7i) it follows that $\int_{-\infty}^{\infty} [C_{S^+}(x,t) - C_{S^-}(x,t)]dx$ exist and

$$\int_{-\infty}^{\infty} |C_{S_2}(x,t) - C_{S_1}(x,t)|dx \leq \|S_1 - S_2\|_{\bar{t},\infty},\ t \in [0,\bar{t}],\ \bar{t} \in [0,T].$$

This inequality combined with the definition of f_S given in Subsection 2.4 proves the theorem. $\boxtimes$

Lemma 4.8.

If $S \in \tilde{M}_{k_1,\infty}$ and there is a solution C_S of the reduced problem then $f_S \in \tilde{M}_{0,k_2}$.

Proof.

Take $S_n \in M_{k_1,\infty}$ such that $\|S - S_n\|_\infty \leq \frac{1}{2n}$, $n = 1,2,\ldots$.

Define $S_n^-(t) = S_n(t) - \frac{1}{n}$, $S_n^+(t) = S_n(t) + \frac{1}{n}$, $t \in [0,T]$ and $C_n: (-\infty, -\frac{1}{n}] \to [0,1]$ by $C_n(x) = \min\{C_o(x), -(x+\frac{1}{n})L\}$, $x \in (-\infty, -\frac{1}{n}]$, $n = 1,2,\ldots$.

According to Lemma 4.5 the equations

$$\frac{\partial C(x,t)}{\partial t} - \frac{\partial^2 C(x,t)}{\partial x^2} = 0,\ x \in (-\infty, S_n^-(t)),\ t \in (0,T],$$

$C(x,0) = C_n(x),\ x \in (-\infty, -\frac{1}{n}],\ C(S_n^-(t),t) = 0,\ t \in [0,T],$

have a unique bounded solution $C_{S_n^-} \in C^{2,1}(Q_{S_n^-}) \cap C(\bar{Q}_{S_n^-})$, for

$n = 1,2,\ldots$ Furthermore the functions $f_{S_n^-}(t) = -\int_{-\infty}^{\infty} [C_{S_n^-}(x,t) - C_n(x)]dx$

exist and $f_{S_n^-} \in \tilde{M}_{0,k_2}$.

Application of the maximum principle yields

$$C_{S_n^-}(x,t) \le C_S(x,t) \le C_{S_n^-}(x-\tfrac{2}{n},t),\ (x,t) \in \bar{Q}_{S_n^+}.$$

This implies that

$$0 \le \int_{-\infty}^{\infty} [C_S(x,t) - C_{S_n^-}(x,t)]dx \le \tfrac{2}{n} \text{ for } t \in [0,T],\ n = 1,2,\ldots\ .$$

Since the quantity $f_{S_n^-}(t) + \int_{-\infty}^{\infty} [C_{S_n^-}(x,t) - C_S(x,t)]dx + \int_{-\infty}^{\infty} [C_o(x) - C_n(x)]dx$ is finite and equals $f_S(t)$ for $t \in [0,T]$, $n = 1,2,\ldots$, it follows that f_S exists and $\|f_S - f_{S_n^-}\|_\infty \le \frac{3}{n}$, $n = 1,2,\ldots$. From $f_{S_n^-} \in \tilde{M}_{0,k_2}$ and the completeness of the space $(\tilde{M}_{0,k_2}, \|\cdot\|_\infty)$ we conclude that $f_S \in \tilde{M}_{0,k_2}$. ⊠

Theorem 4.9.

Suppose $S_1, S_2 \in \tilde{M}_{k_1,k_1}$, $t^-, t^+ \in [0,T]$ and $t^- \le t^+$ then the following inequality holds:

$$\|f_{S_1} - f_{S_2}\|_{t^+,\infty} \le \|S_1 - S_2\|_{t^-,\infty} + \frac{k_2}{1+\operatorname{erf}(\frac{-k_1}{2})} \sqrt{\frac{t^+ - t^-}{\pi t_o}}\ \|S_1 - S_2\|_{t^+,\infty}.$$

Proof.

Define $\bar{S}(t) = \max\{S_1(t), S_2(t)\}$ and $\underline{S}(t) = \min\{S_1(t), S_2(t)\}$, $t \in [0,T]$.

It is easily seen that $\bar{S}$, $\underline{S} \in \tilde{M}_{k_1,k_1}$ and $\|S_1 - S_2\|_{t,\infty} = \|\bar{S} - \underline{S}\|_{t,\infty}$, $t \in [0,T]$.

From Theorems 4.6, 4.8 follows that $C_{\bar{S}}$, $C_{\underline{S}}$ exist and $f_{\bar{S}}$, $f_{\underline{S}} \in \tilde{M}_{0,k_2}$.

Application of the maximum principle yields

$$(4.10)\quad \|f_{S_1} - f_{S_2}\|_{t^+,\infty} \le \max_{t \in [0,t^+]} |f_{\bar{S}}(t) - f_{\underline{S}}(t)| = \int_{-\infty}^{\infty} [C_{\bar{S}}(x,\hat{t}) - C_{\underline{S}}(x,\hat{t})]dx$$

for some $\hat{t} \in [0,t^+]$.

If $\hat{t} \in [0,t^-]$ then Theorem 4.7 states that

$$\int_{-\infty}^{\infty}[C_{\bar{S}}(x,\hat{t})-C_{\underline{S}}(x,\hat{t})]dx \le \|\bar{S}-\underline{S}\|_{t^-,\infty} = \|S_1-S_2\|_{t^-,\infty}.$$

In this case the theorem is proved.

Now we suppose that $\hat{t} \in (t^-,t^+]$. According to Lemma 3.9 for every $\epsilon > 0$ there is an $S^- \in C^2[t^-,\hat{t}]$ such that $S^-(t^-) = \underline{S}(t^-)$, $\underline{S}(t) - \epsilon \le S^-(t) \le \underline{S}(t)$, $t \in [t^-,\hat{t}]$ and $|\frac{dS^-(t)}{dt}| \le \frac{k_1}{2\sqrt{t+t_o}}$, $t \in [t^-,\hat{t}]$.

Define $S^-(t) = \underline{S}(t)$, $t \in [0,t^-)$, $S^+(t) = S^-(t) + \|\bar{S}-\underline{S}\|_{\hat{t},\infty} + \epsilon$, $t \in [0,\hat{t}]$ and $Q^{\pm} = \{(x,t)|x \in (-\infty,S^{\pm}(t)),\ t \in (t^-,\hat{t})\}$.

From Theorem 4.6 we know that the functions $x \mapsto C_{\bar{S}}(x+\bar{S}(t^-),t^-)$ and $x \mapsto C_{\underline{S}}(x+\underline{S}(t^-),t^-)$ are elements of Cond 2.1.

A Lipschitz constant of these functions is $\frac{k_2}{2\sqrt{t_o}}$. Thus it follows from Lemma 3.2 that there are functions $C_o^{\pm}: (-\infty,S^{\pm}(t^-)] \to [0,1]$ such that the functions $x \mapsto C_o^{\pm}(x+S^{\pm}(t^-))$ are elements of Cond 2.2, $C_o^-(x) \le C_{\underline{S}}(x,t^-)$, $x \in (-\infty,S^-(t^-)]$, $C_{\bar{S}}(x,t^-) \le C_o^+(x)$, $x \in (-\infty,S^+(t^-)]$,

$$\int_{-\infty}^{S^-(t^-)} [C_{\underline{S}}(x,t^-)-C_o^-(x)]dx < \epsilon \text{ and } \int_{-\infty}^{S^+(t^-)} [C_o^+(x)-C_{\bar{S}}(x,t^-)dx < \epsilon.$$

It follows from Theorem 3.4 that there are functions $C^{\pm} \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{Q}^{\pm})$ such that

$$\frac{\partial C^{\pm}(x,t)}{\partial t} - \frac{\partial^2 C^{\pm}(x,t)}{\partial x^2} = 0,\ x \in (-\infty,S^{\pm}(t)),\ t \in (t^-,\hat{t}],$$

$$C^{\pm}(x,t^-) = C_o^{\pm}(x),\ x \in (-\infty,S^{\pm}(t^-)],\ C^{\pm}(S^{\pm}(t),t) = 0,\ t \in [t^-,\hat{t}].$$

According to the maximum principle we have

$$\int_{-\infty}^{\infty}[C_{\bar{S}}(x,\hat{t})-C_{\underline{S}}(x,\hat{t})]dx \le \int_{-\infty}^{\infty}[C^+(x,\hat{t})-C^-(x,\hat{t})]dx.$$

Application of Lemma 3.6 yields

$$\int_{-\infty}^{\infty}[C^+(x,\hat{t})-C^-(x,\hat{t})]dx = \int_{-\infty}^{\infty}[C_o^+(x)-C_o^-(x)]dx + \int_{t^-}^{\hat{t}}[\frac{\partial C^+}{\partial x}(S^+(\tau),\tau) - \frac{\partial C^-}{\partial x}(S^-(\tau),\tau)]d\tau.$$

Furthermore we deduce from the definition of C_o^+, C_o^- and Lemma 3.7i)

$$\int_{-\infty}^{\infty}[C_o^+(x)-C_o^-(x)]dx \le \int_{-\infty}^{\infty}[C_{\bar{S}}(x,t^-)-C_{\underline{S}}(x,t^-)]dx + 2\epsilon$$

$$\le \|\bar{S}-\underline{S}\|_{t^-,\infty} + 2\epsilon = \|S_1-S_2\|_{t^-,\infty} + 2\epsilon.$$

This together with inequality (4.10) yields

$$(4.11)\qquad \|f_{S_1}-f_{S_2}\|_{t^+,\infty} \le \|S_1-S_2\|_{t^-,\infty} + \int_{t^-}^{\hat{t}} [\frac{\partial C^+}{\partial x}(S^+(\tau),\tau) - \frac{\partial C^-}{\partial x}(S^-(\tau),\tau)]d\tau + 2\epsilon.$$

It remains to estimate the quantity:

$$(4.12)\qquad \int_{t^-}^{\hat{t}} [\frac{\partial C^+}{\partial x}(S^+(\tau),\tau) - \frac{\partial C^-}{\partial x}(S^-(\tau),\tau)]d\tau = \int_{t^-}^{\hat{t}} \frac{\partial \hat{C}}{\partial x}(S^-(\tau),\tau)d\tau,$$

with $\hat{C}(x,t) = C^+(x+\|S^+-S^-\|_{\hat{t},\infty}, t) - C^-(x,t)$, $(x,t) \in \mathbb{R} \times [t^-,\hat{t}]$.

The function $\hat{C}$ satisfies:

$$\frac{\partial \hat{C}(x,t)}{\partial t} - \frac{\partial^2 \hat{C}(x,t)}{\partial x^2} = 0 \quad , x \in (-\infty, S^-(t)),\ t \in (t^-,\hat{t}],$$

$\hat{C}(S^-(t),t) = 0$, $t \in [t^-,\hat{t}]$ and $\frac{-k_2}{2\sqrt{t_o}} \le \frac{\partial \hat{C}(x,t)}{\partial x} \le \frac{k_2}{2\sqrt{t_o}}$, $(x,t) \in \bar{Q}^-$.

We deduce from the maximum principle $0 \le C^+(x,t) - C^-(x,t)$, $(x,t) \in \bar{Q}^-$.

Since Theorem 4.6 states that the function $x \mapsto C^+(x,t)$ is Lipschitz continuous, the inequality

$$\frac{-k_2}{2\sqrt{t_o}} \|S^+-S^-\|_{\hat{t},\infty} \le C^+(x+\|S^+-S^-\|_{\hat{t},\infty},t) - C^+(x,t) \text{ holds for } (x,t) \in \bar{Q}^-.$$

Adding of these inequalities yields

$$(4.13)\qquad \frac{-k_2}{2\sqrt{t_o}} \|S^+-S^-\|_{\hat{t},\infty} \le \hat{C}(x,t),\ (x,t) \in \bar{Q}^-.$$

We introduce an auxiliary function $\bar{C}$ to estimate $\frac{\partial \hat{C}}{\partial x}(S^-(\tau),\tau)$.

To this end we choose $\delta_1 > 0$ and note that since $\lim_{x\to-\infty} \text{erf}(x) = -1$, there is a $\delta_2 \in (0,t_o]$ such that

$$(1+\delta_1)\ \text{erf}(-\|S^+-S^-\|_{\hat{t},\infty} / 2\sqrt{\delta_2}) < -1.$$

Furthermore we choose $\tau \in (t^-,\hat{t}]$ and with $\tilde{S}$ as in Definition 2.6 we define

$$\bar{C}(x,t) = (1+\delta_1)\ \frac{k_2}{2\sqrt{t_o}}\ \|S^+-S^-\|_{\hat{t},\infty}\ \frac{\text{erf}\left(\dfrac{x-S^-(\tau)+\tilde{S}(\tau-t^-+\delta_2)}{2\sqrt{t-t^-+\delta_2}}\right)-\text{erf}(\frac{-k_1}{2})}{1 + \text{erf}\ (\frac{-k_1}{2})}, x \in \mathbb{R}, t \in [t^-,\tau].$$

Since $\hat{C}$ and $\bar{C}$ are solutions of the diffusion equation it follows from the maximum principle that if

$\bar{C}(S^-(t),t) \le \hat{C}(S^-(t),t)$, $t \in [t^-,\tau]$ and $\bar{C}(x,t^-) \le \hat{C}(x,t^-)$, $x \in (-\infty, S^-(t^-))]$

then $\quad \bar{C}(x,t) \le \hat{C}(x,t)$, $x \in (-\infty, S^-(t)]$, $t \in [t^-,\tau]$.

From the inequalities $\frac{dS^-(t)}{dt} \ge \frac{-k_1}{2\sqrt{t+t_o}} \ge \frac{-k_1}{2\sqrt{t+\delta_2-\hat{t}}} = \frac{d\tilde{S}}{dt}(t-t^-+\delta_2)$

we know that $S^-(\tau)-S^-(t) \ge \tilde{S}(\tau-t^-+\delta_2)-\tilde{S}(t-t^-+\delta_2)$, $t \in [t^-,\tau]$.

This implies that $S^-(t) \le \tilde{S}(t-t^-+\delta_2) + S^-(\tau) - \tilde{S}(\tau-t^-+\delta_2)$, $t \in [t^-,\tau]$

and thus $\check{C}(S^-(t),t) \le 0 = \hat{C}(S^-(t),t)$, $t \in [t^-,\tau]$.

From the definition of $\check{C}$ and inequality (4.13) we obtain

$$\check{C}(x,t^-) \le \frac{-k_2}{2\sqrt{t_o}} \|S^+-S^-\|_{\hat{t},\infty} \le \hat{C}(x,t^-),\ x \in (-\infty, S^-(t^-) - \|S^+-S^-\|_{\hat{t},\infty}].$$

Since the function $x \mapsto \check{C}(x,t^-)$ is convex and the function $x \mapsto \hat{C}(x,t^-)$ is Lipschitz continuous, these inequalities imply

$$\check{C}(x,t^-) \le \frac{k_2}{2\sqrt{t_o}} (x-S^-(t^-)) \le \hat{C}(x,t^-) - \hat{C}(S^-(t^-),t^-) = \hat{C}(x,t^-),$$

$$x \in [S^-(t^-)-\|S^+-S^-\|_{\hat{t},\infty},\ S^-(t^-)].$$

Thus the inequality $\check{C}(x,t) \le \hat{C}(x,t)$ holds for $x \in (-\infty, S^-(t)]$, $t \in [t^-,\tau]$.

This inequality combined with $\check{C}(S^-(\tau),\tau) = \hat{C}(S^-(\tau),\tau) = 0$ yields

$$\frac{\partial\hat{C}}{\partial x}(S^-(\tau),\tau) \le \frac{\partial\check{C}}{\partial x}(S^-(\tau),\tau) = \frac{(1+\delta_1)k_2\|S^+-S^-\|_{\hat{t},\infty} \exp(-\left(\frac{k_1}{2}\right)^2)}{2\sqrt{\pi t_o}(1+\mathrm{erf}(\frac{-k_1}{2}))\sqrt{\tau-t^-+\delta_2}}.$$

This inequality holds for every $\delta_1 > 0$ and $\tau \in (t^-,\hat{t}]$. Since the function $t \mapsto \frac{\partial\hat{C}}{\partial x}(S^-(t),t)$ is continuous the inequality

$$\frac{\partial\hat{C}}{\partial x}(S^-(\tau),\tau) \le \frac{k_2\|S^+-S^-\|_{\hat{t},\infty}}{2\sqrt{\pi t_o}(1+\mathrm{erf}(\frac{-k_1}{2}))\sqrt{\tau-t^-}} \text{ holds for } \tau \in [t^-,\hat{t}].$$

Using this estimate in equation (4.12) combined with inequality (4.11) yields

$$\|f_{S_1}-f_{S_2}\|_{t^+,\infty} \le \|S_1-S_2\|_{t^-,\infty} + \frac{k_2}{1+\mathrm{erf}(\frac{-k_1}{2})} \sqrt{\frac{\hat{t}-t^-}{\pi t_0}}\ \|S^+-S^-\|_{\hat{t},\infty} + 2\epsilon$$

$$\le \|S_1-S_2\|_{t^-,\infty} + \frac{k_2}{1+\mathrm{erf}(\frac{-k_1}{2})} \sqrt{\frac{\hat{t}-t^-}{\pi t_o}}\ \|S_1-S_2\|_{\hat{t},\infty} + (2+ \frac{k_2}{1+\mathrm{erf}(\frac{-k_1}{2})} \sqrt{\frac{T}{\pi t_o}})\ \epsilon.$$

Since ϵ is arbitrary it follows that

$$\|f_{S_1}-f_{S_2}\|_{t^+,\infty} \le \|S_1-S_2\|_{t^-,\infty} + \frac{k_2}{1+\mathrm{erf}(\frac{-k_1}{2})} \sqrt{\frac{t^+-t^-}{\pi t_o}}\ \|S_1-S_2\|_{t^+,\infty}. \quad \boxtimes$$

5. The solution of the Stefan problem

In this section we define an operator $\mathcal{T}$: $\tilde{M}_{k_1,k_1} \to \tilde{M}_{k_1,k_1}$. Under certain conditions we can prove that there is a unique fixed point of the operator $\mathcal{T}$. Using this property of $\mathcal{T}$ we prove in our main theorem that there is a unique solution of the Stefan problem. Finally we compare this theorem with the existence and uniqueness theorems given in [Va] and [Vb].

5.1 Definition of the operator $\mathcal{T}$

Definition 5.1.

We define $\bar{B}^{-1}(y) = \{x \in \mathbb{R} | y \in \bar{B}(x)\}$.

Remark 5.2.

Using Condition 2.3i), ii), iii) it is easy to see that $\bar{B}^{-1}: \mathbb{R} \to \mathbb{R}$ is a function with the properties $\bar{B}^{-1}(0) = 0$, $0 \le \bar{B}^{-1}(y+h) - \bar{B}^{-1}(y) \le \frac{h}{B}$, $h \ge 0$, $y \in \mathbb{R}$.

Lemma 5.3.

For $S \in \tilde{M}_{k_1,k_1}$ the function $t \mapsto \bar{B}^{-1}(\bar{G}(S,f_S,t))$ is an element of $\tilde{M}_{k_1,k_1}$.

Proof.

For $S \in \tilde{M}_{k_1,k_1}$, the Theorems 4.6, 4.8 yield that f_S exists and $f_S \in \tilde{M}_{0,k_2}$. From Condition 2.3 and Remark 5.2 it follows that $\bar{B}^{-1}(\bar{G}(S,f_S,t))$ is properly defined for $S \in \tilde{M}_{k_1,k_1}$ and $t \in [0,T]$. Condition 2.3iv) combined with Remark 5.2 yields $\bar{B}^{-1}(\bar{G}(S,f_S,0)) = 0$.

Since $S \in \tilde{M}_{k_1,k_1}$ and $f_S \in \tilde{M}_{0,k_2}$ we deduce from Remark 5.2 and Condition 2.3v) the following inequality:

$$|\bar{B}^{-1}(\bar{G}(S,f_S,t+h)) - \bar{B}^{-1}(\bar{G}(S,f,t))| \le \frac{\gamma_1 k_1+\gamma_2 k_2+\gamma_3}{2B\sqrt{t+t_o}}\, h,\ h \ge 0;\ t+h,\ t \in [0,T].$$

From equation (2.5) and Definition 2.6 it follows that $\frac{\gamma_1 k_1+\gamma_2 k_2+\gamma_3}{B} = k_1$, hence

$$|\bar{B}^{-1}(\bar{G}(S,f_S,t+h)) - \bar{B}^{-1}(\bar{G}(S,f_S,t))| \le \frac{k_1 h}{2\sqrt{t+t_o}},\ h \ge 0;\ t+h,\ t \in [0,T].$$

Thus the function $t \mapsto \bar{B}^{-1}(\bar{G}(S,f_S,t))$ is an element of $\tilde{M}_{k_1,k_1}$. ⊠

Definition 5.4.

The operator $\mathcal{T}: \tilde{M}_{k_1,k_1} \to \tilde{M}_{k_1,k_1}$ is defined as follows:

$$\mathcal{T}(S)(t) = \bar{B}^{-1}(\bar{G}(S,f_S,t)),\ t \in [0,T].$$

5.2 Fixed point theorems for the operator $\mathcal{T}$

Theorem 5.5.

If $G_1 + G_2 < B$ then there is a unique function $\bar{S} \in \tilde{M}_{k_1,k_1}$ such that $\mathcal{T}(\bar{S}) = \bar{S}$.

Proof.

From Definition 5.4, Remark 5.2 and Condition 2.10 we obtain

$\|\mathcal{T}(S_1) - \mathcal{T}(S_2)\|_\infty \le \frac{G_1}{B} \|S_1-S_2\|_\infty + \frac{G_2}{B} \|S_1-S_2\|_\infty$ for $S_1,S_2 \in \tilde{M}_{k_1,k_1}$.

Application of Theorem 4.7 yields $\|f_{S_1}-f_{S_2}\|_\infty \le \|S_1-S_2\|_\infty$ thus

$$\|\mathcal{T}(S_1) - \mathcal{T}(S_2)\|_\infty \le \frac{G_1+G_2}{B} \|S_1-S_2\|_\infty.$$

Since $G_1 + G_2 < B$ this implies that the operator $\mathcal{T}$ is a contraction on the complete metric space $(\tilde{M}_{k_1,k_1}, \|.\|_\infty)$. According to the Banach fixed point theorem there is a unique function $\tilde{S} \in \tilde{M}_{k_1,k_1}$ such that $\mathcal{T}(\tilde{S}) = \tilde{S}$. ⊠

Theorem 5.6.

If $G_1 < B$ and Condition 2.11 holds then there is a unique function $\tilde{S} \in \tilde{M}_{k_1,k_1}$ such that $\mathcal{T}(\tilde{S}) = \tilde{S}$.

Proof.

Define $h = \frac{T}{N}$, $N \in \mathbb{N}$ and choose N large enough such that

$$\rho_1 := \frac{G_1}{B} + \frac{k_2G_2\sqrt{\frac{h}{\pi t_o}}}{B(1+\mathrm{erf}(\frac{-k_1}{2}))} < 1. \text{ Define } \rho_2 = \frac{G_2}{B} \text{ and } \tau_j = jh,\ j = 0,\dots,N.$$

For given $S_1,S_2 \in \tilde{M}_{k_1,k_1}$ we define

$$\sigma_j^{(m)} = \|\mathcal{T}^m(S_1) - \mathcal{T}^m(S_2)\|_{\tau_j,\infty},\ j = 0,1,\dots,N,\ m\ge0.$$

From the definitions it follows that $\sigma_i^{(m)} \le \sigma_j^{(m)}$ $i\le j$, $m\ge0$.

Application of Lemma 3.8 and Theorem 4.9 combined with Remark 5.2 yields

$$\|\mathcal{T}^m(S_1)-\mathcal{T}^m(S_2)\|_{\tau_j,\infty} \le \frac{G_1}{B} \|\mathcal{T}^{m-1}(S_1)-\mathcal{T}^{m-1}(S_2)\|_{\tau_j,\infty} + \frac{G_2}{B} \|f_{\mathcal{T}^{m-1}(S_1)}-f_{\mathcal{T}^{m-1}(S_2)}\|_{\tau_j,\infty}$$

$$\le \frac{G_2}{B} \|\mathcal{T}^{m-1}(S_1)-\mathcal{T}^{m-1}(S_2)\|_{\tau_{j-1},\infty} + \left(\frac{G_1}{B} + \frac{k_2G_2\sqrt{\frac{h}{\pi t_o}}}{B(1+\mathrm{erf}(\frac{-k_1}{2}))} \right) \|\mathcal{T}^{m-1}(S_1)-\mathcal{T}^{m-1}(S_2)\|_{\tau_j,\infty}.$$

From the definitions we conclude

$$\sigma_j^{(m)} \le \rho_2\, \sigma_{j-1}^{(m-1)} + \rho_1\, \sigma_j^{(m-1)},\ j=1,\dots,N,\ m\ge1.$$

Using this it can be proved with induction to n that the inequality

(5.7) $\quad \sigma_n^{(m)} \le \rho_1^m \sum_{j=0}^{n-1} \left(\frac{m\rho_2}{\rho_1}\right)^j \sigma_n^{(0)}$ holds for $n\ge1$ and $m\ge0$

(compare the proof of [Vb; Theorem 4.15]).

Since $\rho_1 < 1$ we know that

$$\lim_{m\to\infty} \rho_1^m \sum_{j=0}^{N-1} \left(\frac{m\rho_2}{\rho_1}\right)^j = \lim_{m\to\infty} e^{m\ln(\rho_1)} \sum_{j=0}^{N-1} \left(\frac{m\rho_2}{\rho_1}\right)^j = 0.$$

Choose $\hat{m}$ large enough such that $\rho_3 := \rho_1^{\hat{m}} \sum_{j=0}^{N-1} \left(\frac{\hat{m}\rho_2}{\rho_1}\right)^j < 1$.

With inequality (5.7) it follows that $\|\mathcal{T}^{\hat{m}}(S_1)-\mathcal{T}^{\hat{m}}(S_2)\|_\infty \le \rho_3\|S_1-S_2\|_\infty$.

We know that $(\bar{M}_{k_1,k_1}, \|\cdot\|_\infty)$ is a complete metric space and $\bar{M}_{k_1,k_1} \ne \emptyset$. Furthermore we have just shown that $\mathcal{T}^{\hat{m}}$ is a contraction on $\bar{M}_{k_1,k_1}$ and $\mathcal{T}$ is a continuous operator on $\bar{M}_{k_1,k_1}$. According to the Banach fixed point theorem there is a unique function $\bar{S} \in \bar{M}_{k_1,k_1}$ such that $\mathcal{T}(\bar{S}) = \bar{S}$. ⊠

5.3 Existence and uniqueness of the solution of the Stefan problem

In this subsection we prove our main theorem, and compare this theorem with the existence and uniqueness theorems given in [Va] and [Vb].

Main Theorem 5.8.

If one of the following conditions holds:

i) $G_1 + G_2 < B$, or

ii) $G_1 < B$ and $\bar{G}$ is such that Condition 2.11 holds, or

iii) $\bar{G}$ is such that Condition 2.11 holds and there is a function $\lambda \in O$ such that the inequality

(5.9) $\quad \sup_{t\in[0,t^+]} |\bar{G}(S_1,f,t)-\bar{G}(S_2,f,t)| \le G_1\|S_1-S_2\|_{t^-,\infty} + \lambda(t^+-t^-)\|S_1-S_2\|_{t^+,\infty}$

holds for $S_1,S_2 \in \bar{M}_{k_1,k_1}$, $f \in \bar{M}_{0,k_2}$; $t^-,t^+ \in [0,T]$ and $t^- \le t^+$, then the Stefan problem: "to find a lower Lipschitz continuous function S in the sense of Subsection 2.1 and a solution C_S of the reduced problem such that $\bar{G}(S,f_S,t) \in \bar{B}(S(t))$, $t \in [0,T]$", has a unique solution.

Remember that our convention is to impose Conditions 2.1, 2.3 and 2.10. If we denote the solution of the Stefan problem by the pair $(\bar{S}, C_{\bar{S}})$ then $\bar{S} \in \tilde{M}_{k_1,k_1}$.

<u>Proof.</u>

i) Existence. Since $G_1 + G_2 < B$ it follows from Theorem 5.5 that there is a fixed point $\bar{S} \in \tilde{M}_{k_1,k_1}$ of the operator $\mathcal{T}$. This implies that there is a unique solution $C_{\bar{S}}$ of the reduced problem and $\bar{S}(t) = \bar{B}^{-1}(\bar{G}(\bar{S}, f_{\bar{S}}, t))$, $t \in [0,T]$. With Definition 5.1 the last equation is equivalent to $\bar{G}(\bar{S}, f_{\bar{S}}, t) \in \bar{B}(\bar{S}(t))$, $t \in [0,T]$. Thus the pair $(\bar{S}, C_{\bar{S}})$ is a solution of the Stefan problem and $\bar{S}$ is an element of $\tilde{M}_{k_1,k_1}$.

Uniqueness. Suppose the pair (S, C_S) satisfies the Stefan problem. Since $S \in \tilde{M}$ and $\tilde{M} = \bigcup_{K\geq 0} \tilde{M}_{K,\infty}$ there is a $K \in [0,\infty)$ such that $S \in \tilde{M}_{K,\infty}$. This together with Lemma 3.7ii) yields $f_S \in P$. Thus $\bar{G}(S, f_S, t)$ exists for $t \in [0,T]$ and $\bar{G}(S, f_S, t) \in \bar{B}(S(t))$, $t \in [0,T]$. Using Remark 5.2 it follows that

(5.10) $S(t) = \bar{B}^{-1}(\bar{G}(S, f_S, t))$, $t \in [0,T]$.

To show that $S \in \tilde{M}_{k_1,k_1}$ we distinguish the cases $K \in [0,k_1]$ and $K \in (k_1,\infty)$. In the first case it follows from Lemma 4.8 that $f_S \in \tilde{M}_{0,k_2}$. Equation (5.10) combined with Condition 2.3iv), v) and Remark 5.2 yields $S(0) = 0$ and

$$|S(t+h) - S(t)| \leq \frac{\gamma_1 k_1 + \gamma_2 k_2 + \gamma_3}{2B\sqrt{t+t_o}}\, h, \ h\geq 0; \ t+h, \ t \in [0,T].$$

Using (2.5) and Definition 2.6 it follows that $S \in \tilde{M}_{k_1,k_1}$.

In the second case $K \in (k_1,\infty)$ we define $N_1 = K$, $\hat{S}(t) = -K\sqrt{t}$, $t \in [0,T]$

and $$\hat{C}(x,t) = 1+\epsilon_o - \frac{(1+\epsilon_o)\left(1+\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)}{\operatorname{erfc}\left(\frac{K}{2}\right)}, \ (x,t) \in Q_{\hat{S}}.$$

From the maximum principle it follows with $\tilde{S}$, $\tilde{C}$ as in Definition 2.6 and $\tilde{t}$ as in Lemma 2.7 that

$$\tilde{C}(x+\tilde{S}(\tilde{t}),\tilde{t}) \le \hat{C}(x+\hat{S}(\tilde{t}),\tilde{t}),\; x \in (-\infty,0].$$

Using this inequality it can be shown in the same way as in Lemma 4.8 that $f_S \in \tilde{M}_{0,\hat{K}}$ with $\hat{K} = \dfrac{2(1+\epsilon_o)\exp(-(\frac{K}{2})^2)}{\sqrt{\pi}\,\mathrm{erfc}(\frac{K}{2})}$.

Define the function φ as in the proof of Lemma 2.4: $\varphi:[0,\infty) \to \mathbb{R}$

$$\varphi:k \mapsto \frac{1}{B}\left\{ \gamma_1 k + \frac{2\gamma_2(1+\epsilon_o)\exp(-(\frac{k}{2})^2)}{\sqrt{\pi}\,\mathrm{erfc}(\frac{k}{2})} + \gamma_3 \right\}.$$

From $f_S \in \tilde{M}_{0,\hat{K}}$, (5.10), Remark 5.2 and Condition 2.3iv), v) it follows that $S \in \tilde{M}_{N_2,N_2}$ with $N_2 = \varphi(N_1)$. Repetition of this argument yields $S \in \tilde{M}_{N_i,N_i}$ with $N_i = \max\{k_1,\varphi(N_{i-1})\}$, $i\ge 2$.

Using the properties of φ derived in the proof of Lemma 2.4 we obtain $k_1 = \varphi(k_1) < \varphi(k) < k$, $k \in (k_1,\infty)$. Thus the sequence $(N_i)_{i\ge 1}$ is monotone decreasing and bounded from below which implies that there is an $\hat{N} \in [k_1,\infty)$ such that $\lim_{i\to\infty} N_i = \hat{N}$. Since φ is continuous we have $\hat{N} = \varphi(\hat{N})$. However, since there is a unique constant $k_1 \in (0,\infty)$ such that $k_1 = \varphi(k_1)$ it follows that $\hat{N} = k_1$ and $S \in \tilde{M}_{k_1,k_1}$.

With (5.10) and the fact that $S \in \tilde{M}_{k_1,k_1}$ it follows that $\mathcal{T}(S)$ is defined and $\mathcal{T}(S) = S$. Since the fixed point $\bar{S}$ of $\mathcal{T}$ is unique we conclude that the Stefan problem has a unique solution.

ii) Using Theorem 5.6 instead of Theorem 5.5 the proof is analogous to the proof of part i).

iii) Application of Lemma 3.8 and Theorem 4.9 combined with Remark 5.2 and inequality (5.9) yields

$$\|\mathcal{T}(S_1)-\mathcal{T}(S_2)\|_{t^+,\infty} \le \frac{G_1+G_2}{B}\|S_1-S_2\|_{t^-,\infty} + \left\{ \frac{\lambda(t^+-t^-)}{B} + \frac{k_2G_2\sqrt{\frac{t^+-t^-}{\pi t_o}}}{B(1+\mathrm{erf}(\frac{-k_1}{2}))} \right\} \|S_1-S_2\|_{t^+,\infty},$$

$S_1,S_2 \in \tilde{M}_{k_1,k_1}$; $t^-,t^+ \in [0,T]$ and $t^- \le t^+$.

From this inequality and the fact that $\lim_{t\to 0} \lambda(t) = 0$ it can be shown that $\mathcal{T}^m$ is a contraction for m large enough (compare the proof of Theorem 5.6). Using this property instead of Theorem 5.5 the proof is analogous to the proof of part i). ⊠

Remark 5.11.

i) We compare the existence and uniqueness results of the present paper with the results given in [Va; Theorem 5.11] and [Vb; Theorem 4.17]. We note the following differences: in the present paper we show existence and uniqueness of the solution of the Stefan problem without the condition that the function $t \mapsto \tilde{G}(S,f,t)$ is monotone increasing which was imposed in [Va; Condition 3.3ii)] and [Vb; Condition 2.3v)]. However in the present paper we impose the extra condition $\gamma_1 + \gamma_2 < B$ (Condition 2.3v)).
Furthermore in [Va], [Vb] we have shown uniqueness of the solution in the set O whereas in the present paper we show uniqueness of the solution in the set $\tilde{M}$ which consists of lower Lipschitz continuous functions.

ii) In Section 6 we shall give an example where Condition 2.3v) does not hold whereas the conditions of [Va; Theorem 5.11] and [Vb; Theorem 4.17] are satisfied. This implies that the results of [Va], [Vb] are not contained in Theorem 5.8 of the present paper.

6. Examples and numerical experiments

In this section we give an example of the Stefan problem specified in Section 2. Using this example we compare our Main Theorem 5.8 with the results given in the literature. After that we give an example wherein the conditions of Theorem 5.8 are satisfied except for Condition 2.3v) and show that this example has no solution. Furthermore we show with an example that the results of [Va], [Vb] are not contained in the present paper. Finally we give some numerical results.

6.1 Examples

In this subsection we suppose that $T > 0$ and C_o: $(-\infty,0] \to [0,1]$ are given. Furthermore C_o is an element of Cond 2.1.

Example 1.

For a given bounded integrable function b: $\mathbb{R} \to [B_1,\infty)$ with $B_1 > 0$ we define $\bar{B}: \mathbb{R} \to \mathbb{R}$ by $\bar{B}(x) = \int_0^x b(\xi)d\xi$, $x \in \mathbb{R}$. It is easy to see that this function $\bar{B}$ satisfies Conditions 2.3i), ii), iii).

Suppose that the bounded continuous functions g, h: $\mathbb{R} \times [0,T] \to \mathbb{R}$ are given and that for every $K > 0$ there are $g_1,g_2,h_1 \in \mathbb{R}$ such that

$$|g(x_1,t)-g(x_2,t)| \le g_1|x_1-x_2| \quad ; x_1,x_2 \in [-K,K],\ t \in [0,T],$$

$$|g(x,t_1)-g(x,t_2)| \le g_2|t_1-t_2| \quad ; x \in [-K,K],\ t_1,t_2 \in [0,T],$$

$$|h(x_1,t)-h(x_2,t)| \le h_1|x_1-x_2| \quad ; x_1,x_2 \in [-K,K],\ t \in [0,T].$$

We define $\bar{G}(S,f,t) = \int_0^t g(S(\tau),\tau)df(\tau) + \int_0^t h(S(\tau),\tau)d\tau$, $S \in O$, $f \in P$, $t \in [0,T]$.

The first integral is a Lebesgue Stieltjes integral.

First of all we check Conditions 2.3iv), v). From the definition it follows that $\bar{G}(S,f,0) = 0$, $S \in O$, $f \in P$. Since f is a monotone increasing function we deduce

$$|\bar{G}(S,f,t+\epsilon) - \bar{G}(S,f,t)| = |\int_t^{t+\epsilon} g(S(\tau),\tau)df(\tau) + \int_t^{t+\epsilon} h(S(\tau),\tau)d\tau|$$

$$\le \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |g(x,t)|(f(t+\epsilon)-f(t)) + \epsilon \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |h(x,t)|.$$

This implies that if $B_1 > \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |g(x,t)|$ then Condition 2.3v)

holds with $\gamma_1 = 0$, $\gamma_2 = \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |g(x,t)|$, $\gamma_3 = 2\sqrt{T+1} \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |h(x,t)|$

and $\hat{t} = 1$ (if $g(x,t) = 0$, $x \in \mathbb{R}$, $t \in [0,T]$ then we take $\gamma_2 = \frac{1}{2} B_1$).

Secondly we check Condition 2.10. With $G_1 = g_1k_2\sqrt{T} + h_1T$ it follows that the inequality

$$\sup_{t\in[0,T]} |\bar{G}(S_1,f,t)-\bar{G}(S_2,f,t)| \le G_1\|S_1-S_2\|_\infty \text{ holds for } S_1,S_2 \in \bar{M}_{k_1,k_1}$$

and $f \in \bar{M}_{0,k_2}$. If $S \in \bar{M}_{k_1,k_1}$ then $t \mapsto g(S(t),t)$ is a Lipschitz continuous function and a Lipschitz constant is $\frac{k_1}{2\sqrt{t_o}} g_1+g_2$.

With integration by parts we obtain

$$\int_0^t g(S(\tau),\tau)df(\tau) = g(S(t),t)f(t) - \int_0^t f(\tau)dg(S(\tau),\tau),\ t \in [0,T],\ S \in \bar{M}_{k_1,k_1},\ f \in \bar{M}_{0,k_2}.$$

Using this equation we derive with $G_2 = \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |g(x,t)| + T(\frac{k_1}{2\sqrt{t_o}} g_1+g_2)$

the following inequalities:

$$\sup_{t\in[0,T]} |\bar{G}(S,f_1,t)-\bar{G}(S,f_2,t)| \le \left\{ \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |g(x,t)| + \int_0^T d|g(S(\tau),\tau)| \right\} \|f_1-f_2\|_\infty$$

$$\le G_2\|f_1-f_2\|_\infty,\ S \in \bar{M}_{k_1,k_1},\ f_1,f_2 \in \bar{M}_{0,k_2}.$$

Thirdly we remark that for every $t \in [0,T]$, $S \in \bar{M}_{k_1,k_1}$ and $f \in \bar{M}_{0,k_2}$, $\bar{G}(S,f,t)$ only depends on $S|_{[0,t]}$ and $f|_{[0,t]}$.
Thus Condition 2.11 is also satisfied.

Finally we show that inequality (5.9) holds. From the inequalities

$$\sup_{t\in[0,t^+]} |\bar{G}(S_1,f,t) - \bar{G}(S_2,f,t)| \le$$

$$\sup_{t\in[0,t^-]} \left\{ \int_0^t |g(S_1(\tau),\tau)-g(S_2(\tau),\tau)|df(\tau) + \int_0^t |h(S_1(\tau),\tau)-h(S_2(\tau),\tau)|d\tau \right\}$$

$$+ \sup_{t\in[t^-,t^+]} \left\{ \int_{t^-}^t |g(S_1(\tau),\tau)-g(S_2(\tau),\tau)|df(\tau) + \int_{t^-}^t |h(S_1(\tau),\tau)-h(S_2(\tau),\tau)|d\tau \right\}$$

$$\le G_1\|S_1-S_2\|_{t^-,\infty} + \left(\frac{g_1k_2}{2\sqrt{t_o}} + h_1 \right)(t^+-t^-)\|S_1-S_2\|_{t^+,\infty} \text{ with}$$

$S_1,S_2 \in \bar{M}_{k_1,k_1}$, $f \in \bar{M}_{0,k_2}$; $t^+,t^- \in [0,T]$ and $t^- \le t^+$,

it follows that inequality (5.9) holds with $\lambda(t) = \left(\frac{g_1k_2}{2\sqrt{t_o}} + h_1 \right)t,\ t \in [0,T]$.

Application of Theorem 5.8iii) yields:

<u>Theorem 6.1.</u>

A Stefan problem which is described by a $\bar{B}$ and $\bar{G}$ as given above such that $\sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |g(x,t)| < B_1$ has a unique solution.

If we denote the solution by the pair (S,C_S) then $S \in \bar{M}_{k_1,k_1}$.

Remark 6.2.

i) The Stefan problem given in Section 1 is equivalent to the Stefan problem specified in Section 2 if we choose $b(x) = B_1$, $x \in \mathbb{R}$, where $B_1 > 0$, $g(x,t) = -1$ and $h(x,t) = 0$, $x \in \mathbb{R}$, $t \in [0,T]$. It follows from Theorem 6.1 that there is a unique solution of the Stefan problem for every $B_1 > 1$.

ii) We now compare Theorem 6.1 with the results obtained in the literature (see also [Vb; Remark 5.2ii)]). Most Stefan problems considered in the literature are of the following form: to find a function $S \in O \cap C^1(0,T)$ and a solution C_S of the reduced problem such that

$$\frac{dS(t)}{dt} = -g(S(t),t)\,\frac{\partial C_S}{\partial x}\,(S(t),t) + h(S(t),t),\ t \in (0,T),$$

where the bounded continuous function $(x,t) \mapsto g(x,t)$ has continuous derivatives $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial t}$ and the bounded continuous function $(x,t) \mapsto h(x,t)$ is Lipschitz continuous in x (see [Fasano & Primicerio, 1977]). We will refer to this problem as Problem 1. A more general formulation of the problem is: to find a lower Lipschitz continuous function S and a solution C_S of the reduced problem such that

$$S(t) = \int_0^t g(S(\tau),\tau)\,df_S(\tau) + \int_0^t h(S(\tau),\tau)\,d\tau,\ t \in [0,T].$$

We will refer to this problem as Problem 2.

Remark that our smoothness conditions on the functions g and h (Theorem 6.1) are weaker than the conditions imposed in [Fasano & Primicerio, 1977]. With the additional condition: $|g(x,t)| < 1$, $(x,t) \in \mathbb{R} \times [0,\infty)$ it follows from Theorem 6.1 that Problem 2 has a unique solution for every $T \in (0,\infty)$. If this condition is not imposed then there is an example which has no solution for every $T > 0$ (see the following example). Since the condition $|g(x,t)| < 1$, $(x,t) \in \mathbb{R} \times [0,\infty)$ is not imposed in

[Fasano & Primicerio, 1977; Theorem 3] they show that if C_o, g and h are given then there is a $T^* > 0$ such that Problem 1 has a solution for $T \in (0,T^*)$. In [Fasano & Primicerio, 1977; Theorem 5] it is shown that for $T \in (0,T^*)$ the solution of Problem 1 is unique in the set $\{S \in O | S$ is Lipschitz continuous in $[0,T]$ and continuously differentiable in $(0,T)\}$. In [Fasano & Primicerio, 1977; Theorem 7] and [Fasano & Primicerio, 1981] sufficient conditions are given to ensure that $T^* = \infty$.

Example 2.

In this example we choose a function g as in Problem 2 such that the condition $|g(x,t)| < 1$, $(x,t) \in \mathbb{R} \times [0,T]$ does not hold and show that for a given function C_o there is no solution of Problem 2 for T arbitrary large (compare [Fasano & Primicerio, 1981; Theorem 2.2]).

Choose $T \in (0,\infty)$ and define $C_o(x) = \text{erf}\,(-\frac{x}{2})$, $x \in (-\infty,0]$, $\tilde{B}(x) = x$, $x \in \mathbb{R}$ and $\tilde{G}(S,f,t) = -2f(t)$, $S \in O$, $f \in P$, $t \in [0,T]$. With $B = 1$, $\gamma_1 = 0$, $\gamma_2 = 2$, $\gamma_3 = 0$ and $\hat{t} = 1$ it follows that $\tilde{B}$ and $\tilde{G}$ satisfy Condition 2.3 except for the inequality $\gamma_1+\gamma_2 < B$. Suppose that the pair (S,C_S) is a solution of the Stefan problem. We shall prove the following inequalities:

(6.3) $$\frac{-4}{\sqrt{\pi}} \le S(t) \le \frac{-4\sqrt{t+1}+4}{\sqrt{\pi}}, \quad t \in [0,T].$$

These inequalities yield a contradiction for $t > 3$. This shows that the Stefan problem has no solution for $T > 3$.

To prove the right-hand inequality of (6.3) we define the function $S_o(t) = 0$, $t \in [0,T]$ and the solution of the reduced problem $C_{S_o}(x,t) = \text{erf}\left(\frac{-x}{2\sqrt{t+1}}\right)$, $x \in (-\infty,0]$, $t \in [0,T]$. From [Abramowitz & Stegun, 1972; p.229, 7.2.1 and 7.2.5] it follows that

$\int_0^\infty \operatorname{erfc}(y)dy = \frac{1}{\sqrt{\pi}}$. This implies:

$$f_{S_o}(t) = -\int_{-\infty}^{\infty}[C_{S_o}(x,t)-C_o(x)]dx = \int_0^\infty \operatorname{erfc}\left(\frac{x}{2\sqrt{t+1}}\right)dx - \int_0^\infty \operatorname{erfc}\left(\frac{x}{2}\right)dx = \frac{2\sqrt{t+1}-2}{\sqrt{\pi}}.$$

Since $S \in O$ we know from Lemma 3.7ii) that $f_S \in P$. This combined with $S(t) = -2f_S(t)$, $t \in [0,T]$ yields $S(t) \le 0 = S_o(t)$, $t \in [0,T]$.

From Lemma 3.7i) it follows that $C_S(x,t) \le C_{S_o}(x,t)$, $(x,t) \in \bar{Q}_{S_o}$ and thus

$$S(t) = -2f_S(t) \le -2f_{S_o}(t) = \frac{-4\sqrt{t+1}+4}{\sqrt{\pi}},\ t \in [0,T].$$

To prove the left-hand inequality of (6.3) we note that

$C_S(x,t) \le C_{S_o}(x,t) \le C_o(x)$, $x \in (-\infty,0]$, $t \in [0,T]$ and thus

$$S(t) = -2f_S(t) = 2\int_{-\infty}^{\infty}[C_S(x,t)-C_o(x)]dx \le -2\int_{S(t)}^{0} C_o(x)dx =$$

$$-4\int_0^{\frac{-S(t)}{2}} [1-\operatorname{erfc}(y)]dy \le 2S(t) + 4\int_0^\infty \operatorname{erfc}(y)dy = 2S(t) + \frac{4}{\sqrt{\pi}},\ t \in [0,T].$$

From this we conclude $S(t) \ge -\frac{4}{\sqrt{\pi}}$, $t \in [0,T]$.

Example 3.

We shall give a $\bar{B}$ and $\bar{G}$ such that existence and uniqueness of the solution of the Stefan problem follows from [Va; Theorem 5.11] or [Vb; Theorem 4.17] but not from Main Theorem 5.8 of the present paper (see also Remark 5.11 ii)).

Define $\bar{B}(x) = x$, $x \in \mathbb{R}$ and $\bar{G}(S,f,t) = (f(t))^2$, $S \in O$, $f \in P$, $t \in [0,T]$.

For $f(t) = K_2\sqrt{t+2t_1}$ and $t = h = \frac{1}{2}T$ it follows that

$$|\bar{G}(S,f,t+h)-\bar{G}(S,f,t)| = (f(t+h)-f(t))(f(t+h)+f(t)) \ge$$

$$\frac{K_2}{2\sqrt{t+h+2t_1}}\, hK_2\ \sqrt{t+h+2t_1} = \frac{1}{\sqrt{2}}K_2^2\ \sqrt{T+2t_1}\ \frac{h}{2\sqrt{t+t_1}}.$$

This implies that there is no constant $\gamma_2 \ge 0$ such that for every $K_2 \ge 0$ the inequality

$$|\bar{G}(S,f,t+h)-\bar{G}(S,f,t)| \le \frac{\gamma_1K_1+\gamma_2K_2+\gamma_3}{2\sqrt{t+t_1}}\ h \text{ holds for } h \ge 0;\ t+h,\ t \in [0,T],$$

$S \in \bar{M}_{K_1,\infty,t_1}$ and $f \in \bar{M}_{0,K_2,t_1}$.

Thus Condition 2.3 of the present paper is not satisfied.

On the other hand, it is easy to check that Condition 5.8 of [Va] and Condition 2.3 of [Vb] hold for $B = 1$, $\gamma = 2L^2T$, $G_1 = 0$ and $G_2 = 2LT$. From [Vb; Theorem 4.17] we conclude that the Stefan problem has a unique solution. Moreover if $2LT < 1$ then it follows from [Va; Theorem 5.11] too, that the Stefan problem has a unique solution.

6.2 Numerical experiments

In this subsection we present some numerical results to illustrate our Main Theorem 5.8. In the introduction we have noted that if the pair (S,C_S) is a solution of a Stefan problem as given in Section 2, then S can be a non-monotone function. In the following experiments we have chosen $\bar{B}$ and $\bar{G}$ such that in the first experiment S is a monotone increasing function, in the second experiment S is a monotone decreasing function and in the third experiment S is a non-monotone function of t. If $\bar{B}$ and $\bar{G}$ satisfy the conditions of Theorem 5.8 then the corresponding Stefan problem has a unique solution which we denote by the pair $(\bar{S},C_{\bar{S}})$. Furthermore for a given function $S_o \in \bar{M}_{k_1,k_1}$ it follows that the sequence of functions $\{S_i\}_{i\geq 0}$ defined by $S_{i+1} = \mathcal{T}(S_i)$, $i\geq 0$ converges and $\lim_{i\to\infty} S_i = \bar{S}$.

With this in mind we compute a sequence of functions $\hat{S}_i \in O$, $i = 0,1,..$ using a numerical analogue of the relation $S_{i+1} = \mathcal{T}(S_i)$ (the numerical scheme is the same as in [Va; Section 6.3]). In the following experiments we take $C_o(x) = \min\{1,-x\}$, $x \in (-\infty,0]$, $T = 1$ and $\hat{S}_o(t) = 0$, $t \in [0,T]$.

For some choices of $\bar{B}$ and $\bar{G}$ we can use the following remark to show that the sequence $\{S_i\}_{i\geq 0}$ is monotone or alternating.

Remark 6.4.

If $S_1, S_2 \in O$, $S_1(t) \le S_2(t)$, $t \in [0,T]$ and f_{S_1}, f_{S_2} exist then it follows from Lemma 3.7i) that $f_{S_1}(t) \ge f_{S_2}(t)$, $t \in [0,T]$.

Experiment 1. We define $\bar{B}(x) = 2x$, $x \in \mathbb{R}$ and $\bar{G}(S,f,t) = f(t)$, $S \in O$, $f \in P$ and $t \in [0,T]$. This can be a mathematical model of an etching problem. Condition 2.3 and 2.10 hold with $B = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = 0$, $\hat{t} = 1$, $G_1 = 0$ and $G_2 = 1$. Since $G_1 + G_2 < B$ the conditions of Theorem 5.8i) are fulfilled. Using Remark 6.4 it follows that if we choose $S_o(t) = 0$, $t \in [0,T]$ then the sequence $\{S_i\}_{i\ge0}$ is alternating. The numerical results are given in the first column of Table 1 and Figure 1. Remark that the iterates in Figure 1 form an alternating sequence.

i	$\lVert\hat{S}_i - \hat{S}_{i-1}\rVert_\infty$		
1	0.3596	0.3596	0.1238
2	0.735×10^{-1}	0.8431×10^{-1}	0.4208×10^{-1}
3	0.1281×10^{-1}	0.1967×10^{-1}	0.2713×10^{-2}
4	0.2082×10^{-2}	0.4344×10^{-2}	0.7184×10^{-3}
5	0.3117×10^{-3}	0.9×10^{-3}	0.4264×10^{-4}
6	0.4343×10^{-4}	0.1751×10^{-3}	0.8903×10^{-5}
7	0.5667×10^{-5}	0.3212×10^{-4}	
8		0.5579×10^{-5}	

Table 1.

Experiment 2. We define $\bar{B}(x) = 2x$, $x \in \mathbb{R}$ and $\bar{G}(S,f,t) = -f(t)$, $S \in O$, $f \in P$, $t \in [0,T]$. This can be a mathematical model which describes the growth of a crystal in a supersaturated solution. Conditions 2.3 and 2.10 hold with $B = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = 0$, $\hat{t} = 1$, $G_1 = 0$ and $G_2 = 1$. Since $G_1 + G_2 < B$ the conditions of Theorem 5.8i) are fulfilled. Using Remark 6.4 it follows that if we choose $S_o(t) = 0$, $t \in [0,T]$ then the sequence $\{S_i\}_{i\geq 0}$ is monotone decreasing. The numerical results are given in the second column of Table 1 and Figure 2. The iterates in Figure 2 form a monotone decreasing sequence.

Experiment 3. We define $\tilde{B}(x) = 2x$, $x \in R$ and

$$\tilde{G}(S,f,t) = \begin{cases} -f(t) & , t \in [0,0.25] \\ -2f(0.25) + f(t), & t \in (0.25,T] \end{cases}, \quad S \in \mathcal{O},\ f \in P.$$

The corresponding Stefan problem is a mixture of the two problems given above. For $t \in [0, 0.25]$ this problem describes the growth of a solid in a superaturated solution and for $t \in (0.25,T]$ it describes the etching of a solid. Conditions 2.3, 2.10 and 2.11 hold with $B = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = 0$, $\hat{t} = 1$, $G_1 = 0$ and $G_2 = 3$. Since $G_1 < B$ the conditions of Theorem 5.8ii) are fulfilled. The numerical results are given in the third column of Table 1 and Figure 3. The iterates in Figure 3 do not form an alternating nor a monotone sequence.

References

1. M. Abramowitz, I.A. Stegun
Handbook of mathematical functions
Dover Publications New York (1972)

2. H.S. Carslaw, J.C. Jaeger
Conduction of heat in solids
Oxford University Press London (1959)

3. A. Fasano, M. Primicerio
General free boundary problems for the heat equation I
J. Math. Anal. Appl. 57 694-723 (1977)

4. A. Fasano, M. Primicerio
New results on some classical parabolic free-boundary problems
Quart. Appl. Math. 38 439-460 (1981)

5. O. A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva
Linear and quasilinear equations of parabolic type
American Mathematical Society Providence (1968)

6. P. van Moerbecke
An optimal stopping problem for linear reward
Acta Math. 132 111-151 (1974)

7. R.L. Parker
Crystal growth mechanisms: energetics, kinetics and transport
Solid State Physics 25 152-298 (1970)

8. L.I. Rubinstein
The Stefan problem
American Mathematical Society Providence (1971)

9. B. Sherman

General one-phase Stefan problems and free boundary problems for the heat equation with Cauchy data prescribed on the free boundary

SIAM J. Appl. Math. 20 557-570 (1971)

10. R. E. Smithson

Multifunctions

Nieuw Arch. Wisk. (3) 20 31-53 (1972)

Va. C. Vuik (Chapter 1 of this monograph)

The solution of a one-dimensional Stefan problem I (1987)

Vb. C. Vuik (Chapter 2 of this monograph)

The solution of a one-dimensional Stefan problem II (1988)

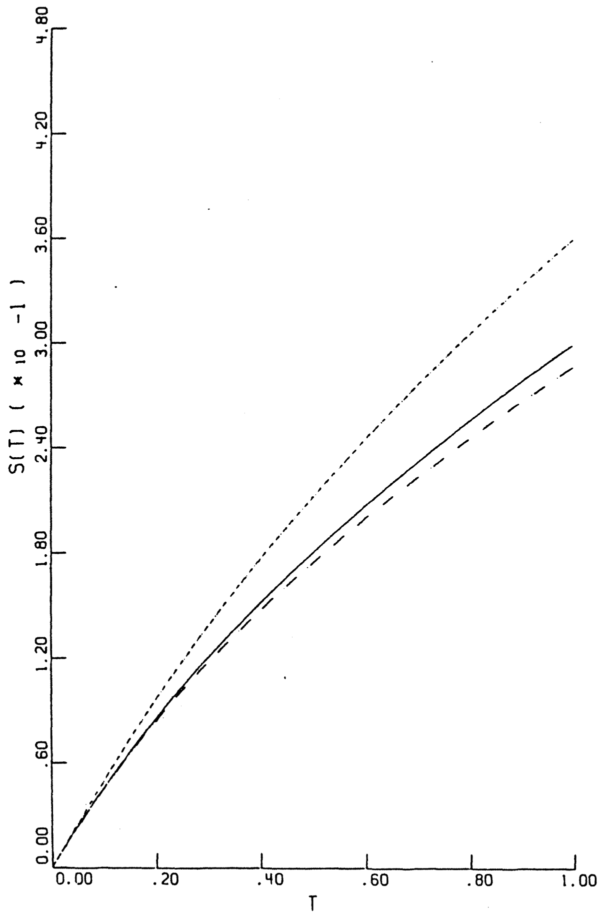

Figure 1 --- iterate 1

— — iterate 2

—— iterate 3

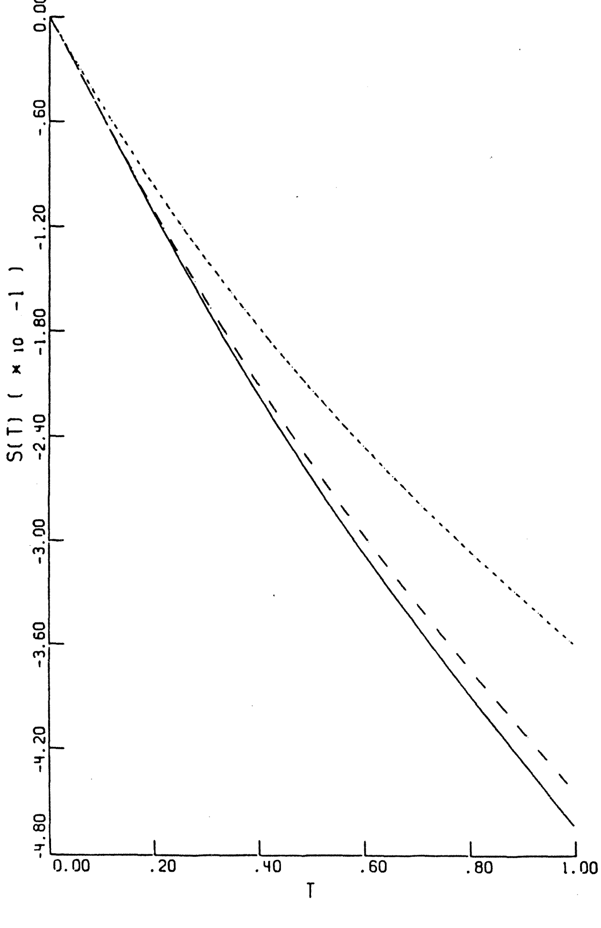

Figure 2

--- iterate 1

— —iterate 2

——iterate 3

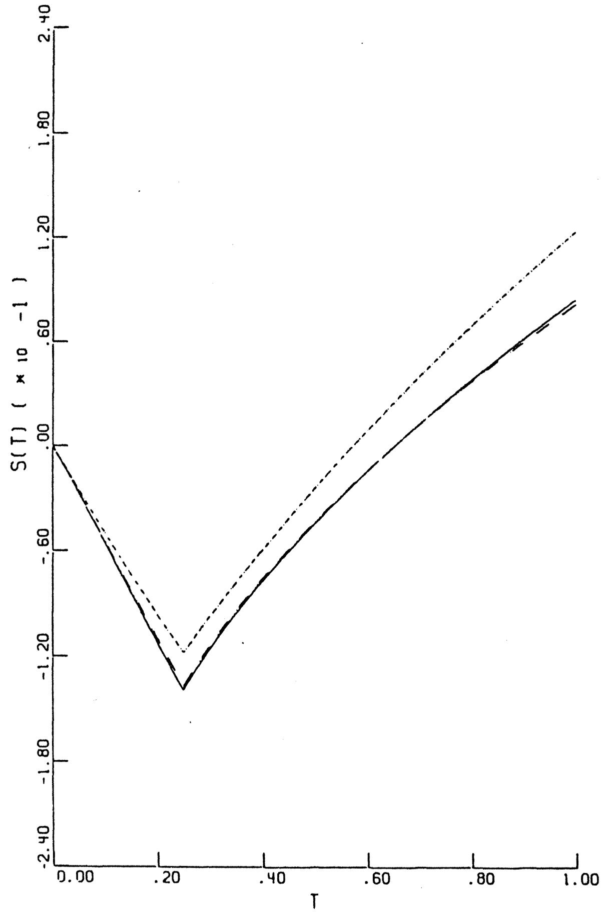

Figure 3 --- iterate 1

— — iterate 2

—— iterate 3

INDEX

CWI TRACTS

1 D.H.J. Epema. *Surfaces with canonical hyperplane sections.* 1984.

2 J.J. Dijkstra. *Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility.* 1984.

3 A.J. van der Schaft. *System theoretic descriptions of physical systems.* 1984.

4 J. Koene. *Minimal cost flow in processing networks, a primal approach.* 1984.

5 B. Hoogenboom. *Intertwining functions on compact Lie groups.* 1984.

6 A.P.W. Böhm. *Dataflow computation.* 1984.

7 A. Blokhuis. *Few-distance sets.* 1984.

8 M.H. van Hoorn. *Algorithms and approximations for queueing systems.* 1984.

9 C.P.J. Koymans. *Models of the lambda calculus.* 1984.

10 C.G. van der Laan, N.M. Temme. *Calculation of special functions: the gamma function, the exponential integrals and error-like functions.* 1984.

11 N.M. van Dijk. *Controlled Markov processes; time-discretization.* 1984.

12 W.H. Hundsdorfer. *The numerical solution of nonlinear stiff initial value problems: an analysis of one step methods.* 1985.

13 D. Grune. *On the design of ALEPH.* 1985.

14 J.G.F. Thiemann. *Analytic spaces and dynamic programming: a measure theoretic approach.* 1985.

15 F.J. van der Linden. *Euclidean rings with two infinite primes.* 1985.

16 R.J.P. Groothuizen. *Mixed elliptic-hyperbolic partial differential operators: a case-study in Fourier integral operators.* 1985.

17 H.M.M. ten Eikelder. *Symmetries for dynamical and Hamiltonian systems.* 1985.

18 A.D.M. Kester. *Some large deviation results in statistics.* 1985.

19 T.M.V. Janssen. *Foundations and applications of Montague grammar, part 1: Philosophy, framework, computer science.* 1986.

20 B.F. Schriever. *Order dependence.* 1986.

21 D.P. van der Vecht. *Inequalities for stopped Brownian motion.* 1986.

22 J.C.S.P. van der Woude. *Topological dynamix.* 1986.

23 A.F. Monna. *Methods, concepts and ideas in mathematics: aspects of an evolution.* 1986.

24 J.C.M. Baeten. *Filters and ultrafilters over definable subsets of admissible ordinals.* 1986.

25 A.W.J. Kolen. *Tree network and planar rectilinear location theory.* 1986.

26 A.H. Veen. *The misconstrued semicolon: Reconciling imperative languages and dataflow machines.* 1986.

27 A.J.M. van Engelen. *Homogeneous zero-dimensional absolute Borel sets.* 1986.

28 T.M.V. Janssen. *Foundations and applications of Montague grammar, part 2: Applications to natural language.* 1986.

29 H.L. Trentelman. *Almost invariant subspaces and high gain feedback.* 1986.

30 A.G. de Kok. *Production-inventory control models: approximations and algorithms.* 1987.

31 E.E.M. van Berkum. *Optimal paired comparison designs for factorial experiments.* 1987.

32 J.H.J. Einmahl. *Multivariate empirical processes.* 1987.

33 O.J. Vrieze. *Stochastic games with finite state and action spaces.* 1987.

34 P.H.M. Kersten. *Infinitesimal symmetries: a computational approach.* 1987.

35 M.L. Eaton. *Lectures on topics in probability inequalities.* 1987.

36 A.H.P. van der Burgh, R.M.M. Mattheij (eds.). *Proceedings of the first international conference on industrial and applied mathematics (ICIAM 87).* 1987.

37 L. Stougie. *Design and analysis of algorithms for stochastic integer programming.* 1987.

38 J.B.G. Frenk. *On Banach algebras, renewal measures and regenerative processes.* 1987.

39 H.J.M. Peters, O.J. Vrieze (eds.). *Surveys in game theory and related topics.* 1987.

40 J.L. Geluk, L. de Haan. *Regular variation, extensions and Tauberian theorems.* 1987.

41 Sape J. Mullender (ed.). *The Amoeba distributed operating system: Selected papers 1984-1987.* 1987.

42 P.R.J. Asveld, A. Nijholt (eds.). *Essays on concepts, formalisms, and tools.* 1987.

43 H.L. Bodlaender. *Distributed computing: structure and complexity.* 1987.

44 A.W. van der Vaart. *Statistical estimation in large parameter spaces.* 1988.

45 S.A. van de Geer. *Regression analysis and empirical processes.* 1988.

46 S.P. Spekreijse. *Multigrid solution of the steady Euler equations.* 1988.

47 J.B. Dijkstra. *Analysis of means in some non-standard situations.* 1988.

48 F.C. Drost. *Asymptotics for generalized chi-square goodness-of-fit tests.* 1988.

49 F.W. Wubs. *Numerical solution of the shallow-water equations.* 1988.

50 F. de Kerf. *Asymptotic analysis of a class of perturbed Korteweg-de Vries initial value problems.* 1988.

51 P.J.M. van Laarhoven. *Theoretical and computational aspects of simulated annealing.* 1988.

52 P.M. van Loon. *Continuous decoupling transformations for linear boundary value problems.* 1988.

53 K.C.P. Machielsen. *Numerical solution of optimal control problems with state constraints by sequential quadratic programming in function space.* 1988.

54 L.C.R.J. Willenborg. *Computational aspects of survey data processing.* 1988.

55 G.J. van der Steen. *A program generator for recognition, parsing and transduction with syntactic patterns.* 1988.

56 J.C. Ebergen. *Translating programs into delay-insensitive circuits.* 1989.

57 S.M. Verduyn Lunel. *Exponential type calculus for linear delay equations.* 1989.

58 M.C.M. de Gunst. *A random model for plant cell population growth.* 1989.

59 D. van Dulst. *Characterizations of Banach spaces not containing l^1.* 1989.

60 H.E. de Swart. *Vacillation and predictability properties of low-order atmospheric spectral models.* 1989.

61 P. de Jong. *Central limit theorems for generalized multilinear forms.* 1989.

62 V.J. de Jong. *A specification system for statistical software.* 1989.

63 B. Hanzon. *Identifiability, recursive identification and spaces of linear dynamical systems, part I.* 1989.

64 B. Hanzon. *Identifiability, recursive identification and spaces of linear dynamical systems, part II.* 1989.

65 B.M.M. de Weger. *Algorithms for diophantine equations.* 1989.

66 A. Jung. *Cartesian closed categories of domains.* 1989.

67 J.W. Polderman. *Adaptive control & identification: Conflict or conflux?.* 1989.

68 H.J. Woerdeman. *Matrix and operator extensions.* 1989.

69 B.G. Hansen. *Monotonicity properties of infinitely divisible distributions.* 1989.

70 J.K. Lenstra, H.C. Tijms, A. Volgenant (eds.). *Twenty-five years of operations research in the Netherlands: Papers dedicated to Gijs de Leve.* 1990.

71 P.J.C. Spreij. *Counting process systems. Identification and stochastic realization.* 1990.

72 J.F. Kaashoek. *Modeling one dimensional pattern formation by anti-diffusion.* 1990.

73 A.M.H. Gerards. *Graphs and polyhedra. Binary spaces and cutting planes.* 1990.

74 B. Koren. *Multigrid and defect correction for the steady Navier-Stokes equations. Application to aerodynamics.* 1991.

75 M.W.P. Savelsbergh. *Computer aided routing.* 1992.

76 O.E. Flippo. *Stability, duality and decomposition in general mathematical programming.* 1991.

77 A.J. van Es. *Aspects of nonparametric density estimation.* 1991.

78 G.A.P. Kindervater. *Exercises in parallel combinatorial computing.* 1992.

79 J.J. Lodder. *Towards a symmetrical theory of generalized functions.* 1991.

80 S.A. Smulders. *Control of freeway traffic flow.* 1993.

81 P.H.M. America, J.J.M.M. Rutten. *A parallel object-oriented language: design and semantic foundations.* 1992.

82 F. Thuijsman. *Optimality and equilibria in stochastic games.* 1992.

83 R.J. Kooman. *Convergence properties of recurrence sequences.* 1992.

84 A.M. Cohen (ed.). *Computational aspects of Lie group representations and related topics. Proceedings of the 1990 Computational Algebra Seminar at CWI, Amsterdam.* 1991.

85 V. de Valk. *One-dependent processes.* 1993.

86 J.A. Baars, J.A.M. de Groot. *On topological and linear equivalence of certain function spaces.* 1992.

87 A.F. Monna. *The way of mathematics and mathematicians.* 1992.

88 E.D. de Goede. *Numerical methods for the three-dimensional shallow water equations.* 1993.

89 M. Zwaan. *Moment problems in Hilbert space with applications to magnetic resonance imaging.* 1993.

90 C. Vuik. *The solution of a one-dimensional Stefan problem.* 1993.

91. E.R. Verheul. *Multimedians in metric and normed spaces.* 1993.

MATHEMATICAL CENTRE TRACTS

1 T. van der Walt. *Fixed and almost fixed points.* 1963.

2 A.R. Bloemena. *Sampling from a graph.* 1964.

3 G. de Leve. *Generalized Markovian decision processes, part I: model and method.* 1964.

4 G. de Leve. *Generalized Markovian decision processes, part II: probabilistic background.* 1964.

5 G. de Leve, H.C. Tijms, P.J. Weeda. *Generalized Markovian decision processes, applications.* 1970.

6 M.A. Maurice. *Compact ordered spaces.* 1964.

7 W.R. van Zwet. *Convex transformations of random variables.* 1964.

8 J.A. Zonneveld. *Automatic numerical integration.* 1964.

9 P.C. Baayen. *Universal morphisms.* 1964.

10 E.M. de Jager. *Applications of distributions in mathematical physics.* 1964.

11 A.B. Paalman-de Miranda. *Topological semigroups.* 1964.

12 J.A.Th.M. van Berckel, H. Brandt Corstius, R.J. Mokken, A. van Wijngaarden. *Formal properties of newspaper Dutch.* 1965.

13 H.A. Lauwerier. *Asymptotic expansions.* 1966, out of print; replaced by MCT 54.

14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

15 R. Doornbos. *Slippage tests.* 1966.

16 J.W. de Bakker. *Formal definition of programming languages with an application to the definition of ALGOL 60.* 1967.

17 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 1.* 1968.

18 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 2.* 1968.

19 J. van der Slot. *Some properties related to compactness.* 1968.

20 P.J. van der Houwen. *Finite difference methods for solving partial differential equations.* 1968.

21 E. Wattel. *The compactness operator in set theory and topology.* 1968.

22 T.J. Dekker. *ALGOL 60 procedures in numerical algebra, part 1.* 1968.

23 T.J. Dekker, W. Hoffmann. *ALGOL 60 procedures in numerical algebra, part 2.* 1968.

24 J.W. de Bakker. *Recursive procedures.* 1971.

25 E.R. Paërl. *Representations of the Lorentz group and projective geometry.* 1969.

26 European Meeting 1968. *Selected statistical papers, part I.* 1968.

27 European Meeting 1968. *Selected statistical papers, part II.* 1968.

28 J. Oosterhoff. *Combination of one-sided statistical tests.* 1969.

29 J. Verhoeff. *Error detecting decimal codes.* 1969.

30 H. Brandt Corstius. *Exercises in computational linguistics.* 1970.

31 W. Molenaar. *Approximations to the Poisson, binomial and hypergeometric distribution functions.* 1970.

32 L. de Haan. *On regular variation and its application to the weak convergence of sample extremes.* 1970.

33 F.W. Steutel. *Preservation of infinite divisibility under mixing and related topics.* 1970.

34 I. Juhász, A. Verbeek, N.S. Kroonenberg. *Cardinal functions in topology.* 1971.

35 M.H. van Emden. *An analysis of complexity.* 1971.

36 J. Grasman. *On the birth of boundary layers.* 1971.

37 J.W. de Bakker, G.A. Blaauw, A.J.W. Duijvestijn, E.W. Dijkstra, P.J. van der Houwen, G.A.M. Kamsteeg-Kemper, F.E.J. Kruseman Aretz, W.L. van der Poel, J.P. Schaap-Kruseman, M.V. Wilkes, G. Zoutendijk. *MC-25 Informatica Symposium.* 1971.

38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

39 H. Bavinck. *Jacobi series and approximation.* 1972.

40 H.C. Tijms. *Analysis of (s,S) inventory models.* 1972.

41 A. Verbeek. *Superextensions of topological spaces.* 1972.

42 W. Vervaat. *Success epochs in Bernoulli trials (with applications in number theory).* 1972.

43 F.H. Ruymgaart. *Asymptotic theory of rank tests for independence.* 1973.

44 H. Bart. *Meromorphic operator valued functions.* 1973.

45 A.A. Balkema. *Monotone transformations and limit laws.* 1973.

46 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 1: the language.* 1973.

47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

52 P.C. Baayen (ed.). *Topological structures.* 1974.

53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

55 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 1: theory of designs, finite geometry and coding theory.* 1974.

56 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 2: graph theory, foundations, partitions and combinatorial geometry.* 1974.

57 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 3: combinatorial group theory.* 1974.

58 W. Albers. *Asymptotic expansions and the deficiency concept in statistics.* 1975.

59 J.L. Mijnheer. *Sample path properties of stable processes.* 1975.

60 F. Göbel. *Queueing models involving buffers.* 1975.

63 J.W. de Bakker (ed.). *Foundations of computer science.* 1975.

64 W.J. de Schipper. *Symmetric closed categories.* 1975.

65 J. de Vries. *Topological transformation groups, 1: a categorical approach.* 1975.

66 H.G.J. Pijls. *Logically convex algebras in spectral theory and eigenfunction expansions.* 1976.

68 P.P.N. de Groen. *Singularly perturbed differential operators of second order.* 1976.

69 J.K. Lenstra. *Sequencing by enumerative methods.* 1977.

70 W.P. de Roever, Jr. *Recursive program schemes: semantics and proof theory.* 1976.

71 J.A.E.E. van Nunen. *Contracting Markov decision processes.* 1976.

72 J.K.M. Jansen. *Simple periodic and non-periodic Lamé functions and their applications in the theory of conical waveguides.* 1977.

73 D.M.R. Leivant. *Absoluteness of intuitionistic logic.* 1979.

74 H.J.J. te Riele. *A theoretical and computational study of generalized aliquot sequences.* 1976.

75 A.E. Brouwer. *Treelike spaces and related connected topological spaces.* 1977.

76 M. Rem. *Associons and the closure statement.* 1976.

77 W.C.M. Kallenberg. *Asymptotic optimality of likelihood ratio tests in exponential families.* 1978.

78 E. de Jonge, A.C.M. van Rooij. *Introduction to Riesz spaces.* 1977.

79 M.C.A. van Zuijlen. *Emperical distributions and rank statistics.* 1977.

80 P.W. Hemker. *A numerical study of stiff two-point boundary problems.* 1977.

81 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 1.* 1976.

82 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 2.* 1976.

83 L.S. van Benthem Jutting. *Checking Landau's "Grundlagen" in the AUTOMATH system.* 1979.

84 H.L.L. Busard. *The translation of the elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?), books vii-xii.* 1977.

85 J. van Mill. *Supercompactness and Wallman spaces.* 1977.

86 S.G. van der Meulen, M. Veldhorst. *Torrix I, a programming system for operations on vectors and matrices over arbitrary fields and of variable size.* 1978.

88 A. Schrijver. *Matroids and linking systems.* 1977.

89 J.W. de Roever. *Complex Fourier transformation and analytic functionals with unbounded carriers.* 1978.

90 L.P.J. Groenewegen. *Characterization of optimal strategies in dynamic games.* 1981.

91 J.M. Geysel. *Transcendence in fields of positive characteristic.* 1979.

92 P.J. Weeda. *Finite generalized Markov programming.* 1979.

93 H.C. Tijms, J. Wessels (eds.). *Markov decision theory.* 1977.

94 A. Bijlsma. *Simultaneous approximations in transcendental number theory.* 1978.

95 K.M. van Hee. *Bayesian control of Markov chains.* 1978.

96 P.M.B. Vitányi. *Lindenmayer systems: structure, languages, and growth functions.* 1980.

97 A. Federgruen. *Markovian control problems; functional equations and algorithms.* 1984.

98 R. Geel. *Singular perturbations of hyperbolic type.* 1978.

99 J.K. Lenstra, A.H.G. Rinnooy Kan, P. van Emde Boas (eds.). *Interfaces between computer science and operations research.* 1978.

100 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 1.* 1979.

101 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 2.* 1979.

102 D. van Dulst. *Reflexive and superreflexive Banach spaces.* 1978.

103 K. van Harn. *Classifying infinitely divisible distributions by functional equations.* 1978.

104 J.M. van Wouwe. *Go-spaces and generalizations of metrizability.* 1979.

105 R. Helmers. *Edgeworth expansions for linear combinations of order statistics.* 1982.

106 A. Schrijver (ed.). *Packing and covering in combinatorics.* 1979.

107 C. den Heijer. *The numerical solution of nonlinear operator equations by imbedding methods.* 1979.

108 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science III, part 1.* 1979.

109 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science III, part 2.* 1979.

110 J.C. van Vliet. *ALGOL 68 transput, part I: historical review and discussion of the implementation model.* 1979.

111 J.C. van Vliet. *ALGOL 68 transput, part II: an implementation model.* 1979.

112 H.C.P. Berbee. *Random walks with stationary increments and renewal theory.* 1979.

113 T.A.B. Snijders. *Asymptotic optimality theory for testing problems with restricted alternatives.* 1979.

114 A.J.E.M. Janssen. *Application of the Wigner distribution to harmonic analysis of generalized stochastic processes.* 1979.

115 P.C. Baayen, J. van Mill (eds.). *Topological structures II, part 1.* 1979.

116 P.C. Baayen, J. van Mill (eds.). *Topological structures II, part 2.* 1979.

117 P.J.M. Kallenberg. *Branching processes with continuous state space.* 1979.

118 P. Groeneboom. *Large deviations and asymptotic efficiencies.* 1980.

119 F.J. Peters. *Sparse matrices and substructures, with a novel implementation of finite element algorithms.* 1980.

120 W.P.M. de Ruyter. *On the asymptotic analysis of large-scale ocean circulation.* 1980.

121 W.H. Haemers. *Eigenvalue techniques in design and graph theory.* 1980.

122 J.C.P. Bus. *Numerical solution of systems of nonlinear equations.* 1980.

123 I. Yuhász. *Cardinal functions in topology - ten years later.* 1980.

124 R.D. Gill. *Censoring and stochastic integrals.* 1980.

125 R. Eising. *2-D systems, an algebraic approach.* 1980.

126 G. van der Hoek. *Reduction methods in nonlinear programming.* 1980.

127 J.W. Klop. *Combinatory reduction systems.* 1980.

128 A.J.J. Talman. *Variable dimension fixed point algorithms and triangulations.* 1980.

129 G. van der Laan. *Simplicial fixed point algorithms.* 1980.

130 P.J.W. ten Hagen, T. Hagen, P. Klint, H. Noot, H.J. Sint, A.H. Veen. *ILP: intermediate language for pictures.* 1980.

131 R.J.R. Back. *Correctness preserving program refinements: proof theory and applications.* 1980.

132 H.M. Mulder. *The interval function of a graph.* 1980.

133 C.A.J. Klaassen. *Statistical performance of location estimators.* 1981.

134 J.C. van Vliet, H. Wupper (eds.). *Proceedings international conference on ALGOL 68.* 1981.

135 J.A.G. Groenendijk, T.M.V. Janssen, M.J.B. Stokhof (eds.). *Formal methods in the study of language, part I.* 1981.

136 J.A.G. Groenendijk, T.M.V. Janssen, M.J.B. Stokhof (eds.). *Formal methods in the study of language, part II.* 1981.

137 J. Telgen. *Redundancy and linear programs.* 1981.

138 H.A. Lauwerier. *Mathematical models of epidemics.* 1981.

139 J. van der Wal. *Stochastic dynamic programming, successive approximations and nearly optimal strategies for Markov decision processes and Markov games.* 1981.

140 J.H. van Geldrop. *A mathematical theory of pure exchange economies without the no-critical-point hypothesis.* 1981.

141 G.E. Welters. *Abel-Jacobi isogenies for certain types of Fano threefolds.* 1981.

142 H.R. Bennett, D.J. Lutzer (eds.). *Topology and order structures, part 1.* 1981.

143 J.M. Schumacher. *Dynamic feedback in finite- and infinite-dimensional linear systems.* 1981.

144 P. Eijgenraam. *The solution of initial value problems using interval arithmetic; formulation and analysis of an algorithm.* 1981.

145 A.J. Brentjes. *Multi-dimensional continued fraction algorithms.* 1981.

146 C.V.M. van der Mee. *Semigroup and factorization methods in transport theory.* 1981.

147 H.H. Tigelaar. *Identification and informative sample size.* 1982.

148 L.C.M. Kallenberg. *Linear programming and finite Markovian control problems.* 1983.

149 C.B. Huijsmans, M.A. Kaashoek, W.A.J. Luxemburg, W.K. Vietsch (eds.). *From A to Z, proceedings of a symposium in honour of A.C. Zaanen.* 1982.

150 M. Veldhorst. *An analysis of sparse matrix storage schemes.* 1982.

151 R.J.M.M. Does. *Higher order asymptotics for simple linear rank statistics.* 1982.

152 G.F. van der Hoeven. *Projections of lawless sequences.* 1982.

153 J.P.C. Blanc. *Application of the theory of boundary value problems in the analysis of a queueing model with paired services.* 1982.

154 H.W. Lenstra, Jr., R. Tijdeman (eds.). *Computational methods in number theory, part I.* 1982.

155 H.W. Lenstra, Jr., R. Tijdeman (eds.). *Computational methods in number theory, part II.* 1982.

156 P.M.G. Apers. *Query processing and data allocation in distributed database systems.* 1983.

157 H.A.W.M. Kneppers. *The covariant classification of two-dimensional smooth commutative formal groups over an algebraically closed field of positive characteristic.* 1983.

158 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science IV, distributed systems, part 1.* 1983.

159 J.W. de Bakker, J. van Leeuwen (eds.). *Foundations of computer science IV, distributed systems, part 2.* 1983.

160 A. Rezus. *Abstract AUTOMATH.* 1983.

161 G.F. Helminck. *Eisenstein series on the metaplectic group, an algebraic approach.* 1983.

162 J.J. Dik. *Tests for preference.* 1983.

163 H. Schippers. *Multiple grid methods for equations of the second kind with applications in fluid mechanics.* 1983.

164 F.A. van der Duyn Schouten. *Markov decision processes with continuous time parameter.* 1983.

165 P.C.T. van der Hoeven. *On point processes.* 1983.

166 H.B.M. Jonkers. *Abstraction, specification and implementation techniques, with an application to garbage collection.* 1983.

167 W.H.M. Zijm. *Nonnegative matrices in dynamic programming.* 1983.

168 J.H. Evertse. *Upper bounds for the numbers of solutions of diophantine equations.* 1983.

169 H.R. Bennett, D.J. Lutzer (eds.). *Topology and order structures, part 2.* 1983.