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Convergence properties of recurrence sequences

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INTRODUCTION

In this monograph we study the asymptotic behaviour of certain linear recurrence sequences. We recall that a linear recurrence sequence is a sequence $\{x_n\}_{n\geq N}$ satisfying a recurrence relation of the form

$$(0.1) \qquad p_k(n)x_{n+k} + p_{k-1}(n)x_{n+k-1} + \dots + p_0(n)x_n = 0 \qquad (n \geq N)$$

where $\{p_k(n)\}_{n\geq N},\dots,\{p_0(n)\}_{n\geq N}$ are sequences with terms belonging to some number field K. We call (0.1) a linear recurrence. If $p_0(n)p_k(n) \neq 0$ for all $n \geq N$, then (0.1) has k solutions $\{x_n^{(i)}\}_{n\geq N}$ $(i = 1,\dots,k)$ which are linearly independent over the field K, and are each uniquely determined by any k subsequent values $x_r^{(i)},\dots,x_{r+k-1}^{(i)}$ where $r \in \mathbb{Z}$, $r \geq N$. We call k the order of (0.1). In this work, we only consider recurrences for which $p_0(n)p_k(n) \neq 0$ and we take for K either of the fields $\mathbb{Q}$,$\mathbb{R}$ or $\mathbb{C}$ supplied with the usual absolute value as metric.

If $\pi_i = \lim_{n\to\infty} p_i(n)$ exists for $i \in \{0,\dots,k\}$ with $\pi_i \in \mathbb{C}$, the characteristic polynomial P of (0.1) is defined as $P(X) = \pi_k X^k + \dots + \pi_1 X + \pi_0$. The zeros of P give an indication about the asymptotic behaviour of the solutions of the linear recurrence. For example, if $\{x_n\}_{n\geq N}$ is a solution of (0.1) and if $\alpha = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists, then clearly $P(\alpha) = 0$. On the other hand, one might wonder whether for every recurrence of type (0.1) having a characteristic polynomial $P \in \mathbb{C}[X]$, it is true that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists for every non-trivial solution $\{x_n\}_{n\geq N}$ of the recurrence. This problem was first stated and partly solved by H.Poincaré, who proved that if all zeros of P have distinct absolute values, then $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists for all non-trivial solutions $\{x_n\}_{n\geq N}$ of the recurrence. As an extension of this result, it was proved by O.Perron [Pe1] that in this case for every zero α of P the recurrence has a solution $\{y_n\}_{n\geq N}$ such that $\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \alpha$. This result is known as the Theorem of Poincaré and Perron. We state it below in its complete form:

Theorem 1. Suppose we have a linear recurrence of the form (0.1) with $p_0(n),\dots,p_k(n) \in \mathbb{C}$ and $p_k(n)p_0(n) \neq 0$ for all $n \geq N$. If the characteristic polynomial of (0.1) exists and has zeros $\alpha_1,\dots\alpha_k$ with $|\alpha_1| < \dots < |\alpha_k|$, then the recurrence has solutions $\{x_n\}_{n\geq N}$ such that $\lim_{n\to\infty} \frac{x_{n+1}^{(i)}}{x_n^{(i)}} = \alpha_i$ $(i = 1,\dots,k)$.

The next problem was to describe the behaviour of the solutions in case the characteristic polynomial has zeros with equal moduli. At first it was conjectured that similar results as Theorem 1 would hold in this case. However, Perron was able to give a few counterexamples for some second-order recurrences whose characteristic polynomial has two zeros with equal moduli, thus showing that the result of Theorem 1 is not generally valid if we omit the condition on the absolute values of the zeros of P ([Pe2]). Nevertheless, if we impose some additional conditions on the behaviour of the coefficients of the recurrence, we can obtain results similar to Theorem 1. As an example we state the following result by O.Perron [Pe2]:

Theorem 2. Consider the second-order linear recurrence

(0.2) $\quad u_{n+2} - (2 + \eta_1(n))\cdot u_{n+1} + (1 + \eta_0(n))\cdot u_n = 0 \qquad (n \geq N)$

where $\eta_0(n), \eta_1(n)$ are $\mathbb{Z}_{\geq N}$-valued functions such that $\lim_{n\to\infty} \eta_0(n) = \lim_{n\to\infty} \eta_1(n) = 0$ and such that $\eta_1(n) \geq 0$ and $\eta_1(n) - \eta_0(n) \geq 0$ for sufficiently large values of n. Then $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ for all non-trivial solutions $\{x_n\}_{n\geq N}$ of (0.2).

On the other hand, Perron formulated and proved a result of a slightly different type which does not suffer from the restriction on the moduli of the zeros of the characteristic polynomial. We state it here:

Theorem 3. Suppose we have a linear recurrence of the type (0.1) with $p_0(n), \ldots, p_k(n) \in \mathbb{C}$ and $p_0(n)p_k(n) \neq 0$. If the characteristic polynomial of (0.1) exists and has zeros $\alpha_1, \ldots \alpha_k$ (counted according to their multiplicities), then the recurrence has linearly independent solutions $\{x_n^{(1)}\}_{n\geq N}, \ldots, \{x_n^{(k)}\}_{n\geq N}$ such that $\limsup_{n\to\infty} \sqrt{|u_n^{(i)}|} = |\alpha_i| \quad (i = 1, \ldots, k)$.

After that, the attention was restricted to special types of linear recurrences, which have rational functions as coefficients or where the coefficients can be developed in factorial series. (a factorial series is a series of the form $\sum_{i=0}^{\infty} \frac{a_i \cdot i!}{n(n+1)\ldots(n+i)}$ with $a_0, a_1, \ldots$ complex numbers.) If the coefficients of the recurrence satisfy certain conditions (for details, see [N]), the solutions can be developed into convergent factorial series. In this way, extensions of the Poincaré-Perron Theorem may be obtained for this special type of recurrences. We state one important result, in order to give an impression of the kind of results occurring in this context:

Theorem 4. Suppose we have a linear recurrence of type (0.1) with $p_i(X) = \sum_{s=0}^{\infty} c_{is}(X + i)(X + i + 1)\cdot\ldots\cdot(X + i + s - 1)$ and $c_{0p}c_{kp} \neq 0$ $(i = 0,\ldots,k)$. Put $f_j(X) = \sum_{i=0}^{k} c_{ij}\cdot X^i$ $(j = 0,\ldots,p)$. Suppose that α is a zero of multipicity ℓ-j of $f_{p-j}(X)$ for $j = 0,\ldots,\ell-1$. Then the recurrence has ℓ linearly independent solutions $\{x_n^{(i)}\}_{n\geq N}$ such that

$$\lim_{n\to\infty} \frac{u_n^{(i)}}{\alpha^n\cdot n^{\beta(i)}\cdot(\log n)^{r(i)}} = 1$$

for $i = 1,\ldots,\ell$ and certain explicitly calculable numbers $\beta(i) \in \mathbb{C}$, $r(i) \in \mathbb{Z}$, $0 \leq r(i) < \ell$.

The proof of this theorem can be found in [N], page 324-6. In the same work a more extensive treatment of this type of recurrences can be found as well.

For several decades there was no activity in this area, but new interest arose when it appeared that linear recurrence sequences play an important role in irrationality proofs (compare Apéry's proof of the irrationality of $\zeta(3)$). Moreover, linear recurrences of order two occur in the theory of orthogonal polynomials (see e.g. [M1],[M2]).

In this work, we resume the investigation of linear recurrences of more general type, with coefficients in some subfield of $\mathbb{C}$ and having a characteristic polynomial, and we derive some generalizations of Theorem 1. Hereafter we outline the contents of this study.

Chapter 1. Here the concept of a shift (or recurrence) operator is introduced and some algebraic properties are derived. We also introduce matrix recurrences, i.e. recurrences of the type $M_n x_n = x_{n+1}$ $(n \in \mathbb{N})$, where the M_n are non-singular $k \times k$-matrices with entries in some number field K and the x_n are k-dimensional vectors with entries in the same field K. This appears to be a somewhat more general concept than linear recurrences and some of the results can be formulated more elegantly in terms of matrix recurrences.

Chapter 2. This chapter stands somewhat apart from the rest of the work. It is dedicated to linear recurrences with coefficients in $\mathbb{Q}[X]$ and whose solutions are sequences of rational numbers. To every so-called *rational recurrence* we can adjoin the set of real numbers α such that $\alpha = \lim_{n\to\infty} \frac{u_n}{v_n}$ for two solutions $\{u_n\}_{n\geq N}$ and $\{v_n\}_{n\geq N}$ of the recurrence with u_n and v_n rational numbers for all n. We shall prove the following two results:

(i) The union of such sets taken over all rational recurrences is a countable subfield of $\mathbb{R}$, containing e.g. the numbers $e, \pi, \log k$ (for $k \in \mathbb{Q}$), $\zeta(k)$ (for $k \in \mathbb{Z}$, $k \geq 2$).

(ii) The union of such sets taken over all rational recurrences with constant coefficients is equal to the set of real algebraic numbers.

Chapter 3. The aim of this chapter is to provide a decomposition of matrix recurrences into smaller-sized matrix recurrences whose limit matrices have only eigenvalues with equal moduli. Indeed, the following result follows immediately from Theorem 3.2: Suppose that M is a matrix in $\mathbb{C}^{k,k}$ of the form

$$M = \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_l \end{bmatrix}$$

where $R_1, \ldots, R_l$ are square matrices such that all eigenvalues of R_i have smaller moduli than those of R_{i+1} $(i = 1, \ldots, l-1)$. Let $\{M_n\}$ be a sequences of $k \times k$-matrices converging (entrywise) to M. Then there exist matrices $S_1, S_2, \ldots$ in $\mathbb{C}^{k,k}$ such that S_n converges (entrywise) to the identity matrix and a matrix

$$M_n^* = \begin{bmatrix} R_{1n} & & & \\ & R_{2n} & & \\ & & \ddots & \\ & & & R_{ln} \end{bmatrix}$$

such that R_{in} converges to R_i (for $i = 1, \ldots, l$) and $M_n = S_{n+1} \cdot M_n^* \cdot S_n^{-1}$.
From this result it is easy to prove the following generalization of Theorem 1 (which is an easy consequence of Theorem 3.3):

Theorem 5. Let

(0.3) $\qquad p_k(n)x_{n+k} + \ldots + p_0(n)x_n = 0$

be a linear recurrence with complex coefficients such that $p_0(n)p_k(n) \neq 0$ $(n \in \mathbb{N})$ and let P be its characteristic polynomial. Suppose that P has zeros $\alpha_1, \ldots, \alpha_k$ (counted according to multiplicities) and that $|\alpha_1| = \ldots = |\alpha_l|$ and $|\alpha_j| \neq |\alpha_1|$ for $j = l+1, \ldots, k$. Then there exist l linearly independent solutions $\{x_n^{(1)}\}_{n \geq N}, \ldots, \{x_n^{(l)}\}_{n \geq N}$ of (0.3) and a linear recurrence of order l

(0.4) $\qquad q_l(n)x_{n+l} + \ldots + q_0(n)x_n = 0$

such that $\{x_n^{(1)}\}_{n \geq N}, \ldots, \{x_n^{(l)}\}_{n \geq N}$ constitute a basis of solutions of (0.4) and such that (0.4) has characteristic polynomial $Q(X) = (X - \alpha_1) \cdot \ldots \cdot (X - \alpha_l)$.

Note that the case $l = 1$ immediately yields the Poincaré-Perron Theorem. The last part of the chapter consists of a quantitative refinement of this result and implies that the order of convergence of $M_n - M$ and the order of convergence of $M_n^* - M$ are the same.
(In fact, Theorem 3 now also follows immediately from Theorem 5, as can be easily seen.)

Chapter 4. This chapter is dedicated to linear recurrences with fast converging coefficients. Since Theorem 1 is valid for recurrences with constant coefficients, even without the restriction on the absolute values of the zeros of the characteristic polynomial, one would expect the same result to hold if the coefficients are not constants, but converge fast enough. In fact, the next result is a direct consequence of Theorem 3.15 and Corollary 4.2:

Theorem 6. Consider the linear recurrence

(0.5) $\quad p_k(n)x_{n+k} + \dots + p_0(n)x_n = 0$

with $\lim\limits_{n\to\infty} p_i(n) = \pi_i$ $(i = 0,\dots,k)$, $\pi_k\cdot\pi_0 \neq 0$ and $p_0(n)p_k(n) \neq 0$ for $n \geq N$, where, in addition, $\sum\limits_{n=N}^{\infty} n^{L-1}\cdot|\ p_i(n) - \pi_i\ |$ converges for all i, where L is the maximum of the multiplicities of the zeros of the characteristic polynomial of (0.5). Let α be a zero of P with multiplicity l. Then (0.5) has l linearly independent solutions $\{x_n^{(1)}\}_{n\geq N},\dots,\{x_n^{(l)}\}_{n\geq N}$ such that

$$\lim_{n\to\infty} \frac{x_n^{(i)}}{\alpha^n\cdot n^{i-1}} = 1 \quad \text{for } i = 1,\dots,l.$$

Chapter 4 gives, in addition, a quantitative result, where the rate of convergence of the series $\sum\limits_{n=N}^{\infty} n^{L-1}\cdot|\ p_i(n) - \pi_i\ |$ is related to the rates of convergence of the differences $\dfrac{x_n^{(i)}}{\alpha^n\cdot n^{i-1}} - 1$ for $i = 1,\dots,l$.

Chapters 5 and 6. These chapters deal with linear second-order recurrences where the characteristic polynomial has two zeros with equal moduli. Results similar to Theorems 2 and 4 of this introduction are derived for case where the coefficients behave neatly. For such recurrences, we meet largely two types of behaviour of the solutions:

(i) For each zero α_i of the characteristic polynomial there exists a solution $\{x_n^{(i)}\}_{n\geq N}$ of the recurrence such that $\lim\limits_{n\to\infty} \dfrac{x_{n+1}^{(i)}}{x_n^{(i)}} = \alpha_i$ $(i = 1,2)$. Moreover, $\lim_{n\to\infty} \frac{x_n^{(2)}}{x_n^{(1)}} = 0$, so that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists for all non-trivial solutions $\{x_n\}_{n\geq N}$ of the recurrence. Recurrences of this type can be called 'hyperbolic', in accordance with the terminology for sequences of fractional linear maps, where hyperbolicity implies the existence of two limit points, one of which is stable, whereas the other is unstable.

(ii) For each zero α_i there is a solution $\{x_n^{(i)}\}_{n\geq N}$ of the recurrence such that $\lim_{n\to\infty} \frac{x_{n+1}^{(i)}}{x_n^{(i)}} = \alpha_i$ $(i = 1,2)$, but now $\lim_{n\to\infty} \frac{x_n^{(2)}}{x_n^{(1)}}$ and $\lim_{n\to\infty} \frac{x_n^{(1)}}{x_n^{(2)}}$ do not exist, whereas $\lim_{n\to\infty} \left|\frac{x_n^{(2)}}{x_n^{(1)}}\right|$ does exist. In particular, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ does not exist for any solution $\{x_n\}_{n\geq N}$ of the recurrence that is not linearly dependent of either $\{x_n^{(1)}\}_{n\geq N}$ or $\{x_n^{(2)}\}_{n\geq N}$. Recurrences of this type can be called 'elliptic'.

For example, the linear recurrence

(0.6) $\qquad x_{n+2} - 2\cdot x_{n+1} + (1 - \eta(n))x_n = 0$

with $\eta(n) \in \mathbb{R}$, $\lim_{n\to\infty} \eta(n) = 0$, has two linearly independent solutions $\{x_n^{(1)}\}_{n\geq N}$ and $\{x_n^{(2)}\}_{n\geq N}$ such that $x_n^{(1)}, x_n^{(2)} \in \mathbb{R}$ $(n \in \mathbb{N})$, $\lim_{n\to\infty} \frac{x_n^{(2)}}{x_n^{(1)}} = 0$ and $\lim_{n\to\infty} \frac{x_{n+1}^{(i)}}{x_n^{(i)}} = 1$ for $i = 1,2$ if $\eta(n) \geq 0$ for n large enough. On the other hand, if $\eta(n) < 0$ and $n^2\cdot|\eta(n)| > 1/4 + \varepsilon$ for some $\varepsilon > 0$ and n large enough, then $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ does not exist for any solution $\{x_n\}_{n\geq N}$ of (0.6) with $x_n \in \mathbb{R}$ $(n \in \mathbb{N})$. Further, if $\{\eta(n)\}$ satisfies suitable regularity conditions, then there exist linearly independent solutions $\{y_n^{(1)}\}_{n\geq N}$ and $\{y_n^{(2)}\}_{n\geq N}$ of (0.6) with $y_n^{(i)} \in \mathbb{C}$ $(i = 1,2)$ such that $\lim_{n\to\infty} \frac{y_{n+1}^{(i)}}{y_n^{(i)}} = 1$ for $i = 1,2$ and with $|y_n^{(1)}| = |y_n^{(2)}|$ for all $n \geq N$.

If the coefficients behave more irregularly, however, then it may occur that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ does not exist for any solution $\{x_n\}_{n\geq N}$ of the recurrence or that it exists for only one solution $\{x_n\}_{n\geq N}$ of the recurrence (up to multiplication by a scalar in $\mathbb{C}$). Some counterexamples are given in Chapters 5 and 6. In Chapter 5, the case that the zeros of the characteristic polynomial are equal is treated, in Chapter 6 the zeros are not equal, but have equal moduli.

Chapter 7. This chapter contains the solution of a problem posed by O.Perron ([Pe3]), about the convergence of a certain type of continued fractions. A simple application of the results of the preceding chapters provides necessary and sufficient convergence conditions. It will be seen that the continued fractions which converge, are exactly those which are related (in the manner described in Chapter 7) to linear recurrences of hyperbolic type.

A more extensive survey of this study with a special emphasis to application of the results to recurrences with coefficients in $\mathbb{R}[X]$, can be found in [Kl].

CHAPTER ONE

PRELIMINARY CONCEPTS AND RESULTS

§1. Recurrence operators.

Let K be some field with characteristic zero. For $m \in \mathbb{Z}$, we consider sequences $\{u_n\}_{n\geq m}$ with $u_n \in K$ $(n \geq m)$ and with the following addition and multiplication: $\{u_n\}_{n\geq m} + \{v_n\}_{n\geq m} = \{u_n + v_n\}_{n\geq m}$, $\{u_n\}_{n\geq m} \cdot \{v_n\}_{n\geq m} = \{u_n v_n\}_{n\geq m}$. Multiplication of a sequence by a number in K is defined by $\lambda \cdot \{u_n\}_{n\geq m} = \{\lambda \cdot u_n\}_{n\geq m}$. We define an equivalence relation on this set by $\{u_n\}_{n\geq m} \sim \{v_n\}_{n\geq m'}$ if and only if there exists some number $M \geq m,m'$ such that $u_n = v_n$ for $n \geq M$. Let $\mathscr{S}(K)$ be the set of equivalence classes with respect to this equivalence relation. The addition and multiplication defined above can be extended into $\mathscr{S}(K)$ in the obvious manner. In this way, $\mathscr{S}(K)$ becomes a ring. An element of $\mathscr{S}(K)$ will be denoted by $\{u_n\},\{v_n\}$, etc. and we shall refer to them simply by the word *sequence*, instead of equivalence class of sequences. (In order to indicate that a certain fact is true for all members of a sequence $\{u_n\}$ we shall simply write "for $n \in \mathbb{N}$" or something alike).

In $\mathscr{S}(K)$ we can consider certain subsets of sequences. By K[X] and K(X) we refer to the sets of sequences $\{u(n)\}$, where $u \in K[X]$ and K(X), respectively. Clearly, K(X) is a field with the above addition and multiplication. More in general, we shall denote by $\mathcal{O} = \mathcal{O}(K)$ any field of sequences in $\mathscr{S}(K)$ with the addition and multiplication defined above. If $\{u_n\} \in \mathcal{O}(K)$ for some field $\mathcal{O}(K)$, then the inverse of $\{u_n\}$ is clearly $\{u_n^{-1}\}$.

We define *shift operators* onto sequences in $\mathscr{S}(K)$ as follows:

(i) The *elementary shift operator* T is defined by $T(\{u_n\}) = \{u_{n+1}\}$.

(ii) For $p_k, p_{k-1}, \ldots, p_0 \in \mathscr{S}(K)$ the shift operator $R = p_kT^k + p_{k-1}T^{k-1} + \ldots + p_0$ maps $\{u_n\} \in \mathscr{S}(K)$ into $\{p_k(n)u_{n+k} + p_{k-1}(n)u_{n+k-1} + \ldots + p_0(n)u_n\} \in \mathscr{S}(K)$.

For T^0, the *identity operator*, we shall also write I. If R is some shift operator and $\{u_n\}$ a sequence in $\mathscr{S}(K)$, we shall often write $R(u_n)$ instead of $R(\{u_n\})$.

In the sequel we shall restrict our attention to the set of shift operators of the form $R = p_kT^k + \ldots + p_0$, with $p_0,\ldots,p_k = \mathscr{S}(K)$ $(k \geq 0)$ such that either $R = 0$ or $p_k(n)p_0(n) \neq 0$ for all n. We denote this set by $\mathfrak{R}(K)$. We call k the *order* of R and denote it by ord(R). The order of the zero-operator is not defined.

If $\mathcal{O}$ is a field of sequences such that $T(\mathcal{O}) \subseteq \mathcal{O}$, we consider the set $\mathcal{O}[T]$ of shift operators with coefficients in $\mathcal{O}$. We define an addition and multiplication of operators as follows: If $R_1, R_2 \in \mathcal{O}[T]$, then $R_1 + R_2$ and $R_1R_2 = R_1 \cdot R_2$ are defined by $(R_1 + R_2)(u_n) = R_1(u_n) + R_2(u_n)$ and $R_1 \cdot R_2(u_n) = R_1(R_2(u_n))$ for any sequence $\{u_n\}$ in $\mathcal{S}(K)$. (Note that this definition determines their form uniquely). It is obvious that $\mathcal{O}[T]$ becomes a ring in this way. We shall denote this ring of operators by $\mathfrak{R}(\mathcal{O},K)$. Note that, if $R_1, R_2 \in \mathfrak{R}(\mathcal{O},K)$, and $R_1, R_2 \neq 0$, then $\operatorname{ord}(R_1 \cdot R_2) = \operatorname{ord}(R_1) + \operatorname{ord}(R_2)$.

For $R \in \mathfrak{R}(K)$ we consider the set $Z(R)$ of sequences $\{u_n\}$ in $\mathcal{S}(K)$ such that $R(u_n) = 0$. In this case, we call $\{u_n\}$ a *zero* of R. Clearly, $\{0\} \in Z(R)$ for all $R \in \mathfrak{R}(K)$, and if $\{u_n\}, \{v_n\} \in Z(R)$, then $\lambda \cdot \{u_n\} \in Z(R)$ for any $\lambda \in K$ and $\{u_n\} + \{v_n\} \in Z(R)$. Hence, $Z(R)$ is a vector space over K.

Remark 1.1.1. If $R \in \mathfrak{R}(K)$ and $\{u_n\} \in Z(R)$, then

(1.1) $$p_k(n)u_{n+k} + p_{k-1}(n)u_{n+k-1} + \dots + p_0(n)u_n = 0 \qquad (n \in \mathbb{N}).$$

Let r be a positive integer with $r \geq k$. By applying (1.1) repeatedly, we obtain that there exist sequences $q_0, \dots, q_{k-1}$ in $\mathcal{S}(K)$ such that $q_0(n) \neq 0$ for all n and

$$u_{n+r} = q_{k-1}(n)u_{n+k-1} + \dots + q_0(n)u_n \qquad (n \in \mathbb{N}).$$

Hence we see that the values of u_n are uniquely determined by k subsequent values $u_m, \dots, u_{m+k-1}$. Moreover, if we define $\{u_n^{(j)}\}$ $(j = 1, \dots, k)$ by $u_{m+i-1}^{(j)} = \delta_{ij}$ $(i = 1, \dots k)$, $\{u_n^{(1)}\}, \dots, \{u_n^{(k)}\}$ are linearly independent over K. So we find that $\dim_K Z(R) = \operatorname{ord}(R)$, and $\{u_n^{(1)}\}, \dots, \{u_n^{(k)}\}$ constitutes a basis of $Z(R)$.

Remark 1.1.2. If $\{u_n^{(1)}\}, \dots, \{u_n^{(k)}\}$ is a basis of $Z(R)$, then we can write (1.1) in the form of a sequence of determinants

(1.2) $$\begin{vmatrix} u_{n+k}^{(1)} & \dots & u_{n+k}^{(k)} & u_{n+k} \\ \vdots & & \vdots & \vdots \\ u_n^{(1)} & \dots & u_n^{(k)} & u_n \end{vmatrix} = 0 \qquad (n \in \mathbb{N}).$$

Indeed if $\{u_n\}$ is a solution of (1.1), then $\{u_n\}$ is a linear combination of the basis sequences $\{u_n^{(1)}\}, \dots, \{u_n^{(k)}\}$ with coefficients in K, hence $\{u_n\}$ satisfies (1.2). Conversely, if the $\{u_n^{(i)}\}$ $(i=1,\dots,k)$ form a basis of solutions of a linear recurrence, then

$$u^{(i)}_{n+k} = r_{k-1}(n)u^{(i)}_{n+k-1} +..+ r_0(n)u^{(i)}_n$$

for i=1,2,...,k, and sequences $\{r_j(n)\} \in \mathcal{S}(K)$ (j = 0,...,k-1). Since the $\{u^{(i)}_n\}^{\infty}_{n=m}$ are linearly independent (i = 1,2,...,k), the coefficients $r_{k-1}(n),\ldots,r_0(n)$ can be determined by Cramer's rule from expression (1.2). The coefficients are uniquely determined, since otherwise $\{u_n\}$ would satisfy a recurrence of lower order, which would contradict the fact that there are k linearly independent solutions.

By (1.1) we see that the zeros of R satisfy a linear recurrence of order k. Because of this fact, we shall often refer to shift operators in $\mathfrak{R}(K)$ as *recurrence operators*. From now on, we shall denote a recurrence operator by a capital letter (I,T,P,Q,R,S,V,W,...). Note that it follows immediately from Remark 1.1.2 that a recurrence operator in $\mathfrak{R}(K)$ is, up to (left) multiplication, uniquely defined by its set of zeros.

It is evident that the set $\mathfrak{R}(K)$ is not closed under addition of operators, as defined above. On the other hand, it is closed under multiplication, and, if $R_1,R_2 \in \mathfrak{R}(K)$ and $R_1,R_2 \neq 0$, then $\mathrm{ord}(R_1\cdot R_2) = \mathrm{ord}(R_1) + \mathrm{ord}(R_2)$.

Let $R \in \mathfrak{R}(K)$. Write $R = p_kT^k +...+ p_0I$. Suppose that $p_i(n)$ converges to some number π_i in the metrical completion $\bar{K}$ of K (with respect to some metric on K). for i = 1,...,k. We define the *characteristic polynomial* χ_R of R as follows:

$$\chi_R(X) = \pi_kX^k +...+ \pi_0.$$

§2. The algebra $\mathfrak{R}(\mathcal{O},K)$.

It is clear from §1 that for K and $\mathcal{O} = \mathcal{O}(K)$ given, the set $\mathfrak{R}(\mathcal{O},K)$ is an algebra over $\mathcal{O}$ with the addition and multiplication of operators as defined above. We recall that we consider only fields $\mathcal{O}$ such that $T(\mathcal{O}) \subset \mathcal{O}$. Note that if $R \in \mathfrak{R}(\mathcal{O},K)$, then certainly $R \in \mathfrak{R}(K)$, so that the concepts defined for recurrence operators in $\mathfrak{R}(K)$ are also valid for operators in $\mathfrak{R}(\mathcal{O},K)$. It is not difficult to see that multiplication in $\mathfrak{R}(\mathcal{O},K)$ is not commutative in general. However, multiplication and addition are both associative and the distributive law holds between them. Note that multiplication on the left side by a function $p \in \mathcal{O}$ is the same as multiplication on the left side by the operator pI and that for $p \neq 0$ the sets Z(R) and Z(pR) are equal. We now define a divisor of an operator as follows:

If $R,S,V \in \mathfrak{R}(\mathcal{O},K)$ and $R = S\cdot V$ we call V a *(right) divisor* of R and write $V|R$.

S is then called a *left divisor* of R. Note that $V|R$ implies $Z(V) \subset Z(R)$. Conversely, if $R,V \in \mathfrak{R}(\mathcal{O},K)$, and $Z(V) \subset Z(R)$, then $V|R$. For we can find $P,Q \in \mathfrak{R}(\mathcal{O},K)$ such that $R = P\cdot V + Q$, where $\mathrm{ord}(Q) < \mathrm{ord}(V)$ if $Q \neq 0$. Then $Z(V) \subset Z(Q)$. However, if $Q \neq 0$, then $\dim Z(V) \leq \dim Z(Q)$, which is impossible. Hence $Q = 0$ and $V|R$.

For $R,S \in \mathfrak{R}(\mathcal{O},K)$ we define the *greatest common divisor* (R,S) of R and S as the monic operator $V \in \mathfrak{R}(\mathcal{O},K)$ of largest order such that $V|R$ and $V|S$.

Proposition 1.1. *Let* $R,S \in \mathfrak{R}(\mathcal{O},K)$. *The following statements are valid*:

1. (R,S) *exists and is uniquely determined.*
   Moreover, the Euclidean algorithm can be applied in $\mathfrak{R}(\mathcal{O},K)$ *to find* (R,S).
2. *There exist* $P,Q \in \mathfrak{R}(\mathcal{O},K)$ *such that* $P\cdot R + Q\cdot S = (R,S)$.
3. $Z((R,S)) = Z(R) \cap Z(S)$. *Conversely, if* $V \in \mathfrak{R}(\mathcal{O},K)$, *and* $Z(V) = Z(R) \cap Z(S)$, *then there exists a* $p \in \mathbb{Q}$, $p \neq 0$, *such that* $pV = (R,S)$.

Proof: (1). The Euclidian algorithm can be applied to linear operators in the same way as with polynomials in some domain K[X] (with K some field). We obtain that for two operators R and S in $\mathfrak{R}(\mathcal{O},K)$ there exist operators P and Q in $\mathfrak{R}(\mathcal{O},K)$ such that $R = Q\cdot S + P$ and either $\mathrm{ord}(P) < \mathrm{ord}(S)$ or $P = 0$. The existence of the greatest common divisor (R,S) follows from the Euclidian algorithm. For uniqueness, see (3).
(2). This follows immediately from the Euclidian algorithm. We leave the details of (1) and (2) to the reader.
(3) Put $V = (R,S)$. Then there exist R_1 and R_2 in $\mathfrak{R}(\mathcal{O},K)$ such that $R = R_1\cdot V$ and $S = S_1\cdot V$, and $(R_1,S_1) = I$. So $Z(V) \subset Z(R) \cap Z(S)$.
Let $\{u_n\} \in Z(R) \cap Z(S)$. Then $\{V(u_n)\} \in Z(R_1) \cap Z(S_1) = \{\{0\}\}$, which implies $\{u_n\} \in Z(V)$. For the converse, we use that a monic operator is uniquely determined by its set of zeros. □

§3. Reducible operators.

Suppose $R \in \mathfrak{R}(\mathcal{O},K)$. R is called *reducible* if $R = R_1\cdot R_2$ where $R_1,R_2 \in \mathfrak{R}(\mathcal{O},K)$ and $\mathrm{ord}(R_1) \geq 1$, $\mathrm{ord}(R_2) \geq 1$. Otherwise, R is called *irreducible*.

If R is of first order, we can solve the equation

$$R(u_n) = 0 \tag{1.3}$$

Put $R = pT - q$, where $p,q \in \mathcal{O}(K)$. Then, for $\{u_n\} \in Z(R)$, and m large enough,

$\frac{u_{n+1}}{u_n} = \frac{q(n)}{p(n)}$ $(n \geq m)$, so $u_n = \lambda \prod_{k=m}^{n-1} r(k)$ for $\lambda \in K$ and $r = \frac{q}{p}$.

If R is the product of first order operators, (1.3) can be solved by subsequently solving first order operator equations. For example, if $R = S \cdot V$, where S and V are of first order, we can first find $\{v_n\}$ by solving $S(v_n) = 0$, as described above, and then solve the inhomogeneous recurrence equation

(1.4) $$V(u_n) = v_n.$$

Put $V = p(T - r)$, where $p,r \in \mathcal{O}(K)$. Then we have

$$u_{n+1} - r(n)u_n = \frac{v_n}{p(n)}.$$

Hence,

$$u_n = \lambda \cdot t(n) + t(n) \cdot \sum_{\ell=m}^{n-1} \frac{v_\ell}{p(\ell)t(\ell+1)}$$

where $\lambda \in K$ and $t(n) = \prod_{\ell=m}^{n-1} r(\ell)$.

§4. Derived operators; The lowest common multiple of two operators.

In this section, we fix K and $\mathcal{O} = \mathcal{O}(K)$ and write $\mathfrak{R}$ for $\mathfrak{R}(\mathcal{O},K)$. We define the concept of a derived operator:

Let $R,S \in \mathfrak{R}$, $S \neq 0$. The *S-derived* of R is the monic operator W such that $Z(W) = \{\{S(u_n)\} | \{u_n\} \in Z(R)\}$. We denote W by R/S.

Proposition 1.2. *Let* $R,S \in \mathfrak{R}$, $S \neq 0$. *Then* $R/S \in \mathfrak{R}$ *and*

$$\mathrm{ord}((R,S)) + \mathrm{ord}(R/S) = \mathrm{ord}(R).$$

Proof: S induces a homomorphism σ from Z(R) onto S(Z(R)). Clearly, $\mathrm{Ker}\ \sigma = Z((R,S))$. Hence, $\mathrm{ord}(R) = \dim Z(R) = \dim Z((R,S)) + \dim S(Z(R))$. Let $V \in \mathfrak{R}(K)$ be the monic operator such that $Z(V) = S(Z(R))$. By Remark 1.1.2, such an operator exists. We prove that $V \in \mathfrak{R}$. Put $\ell = \mathrm{ord}(V) = \mathrm{ord}(R) - \mathrm{ord}((R,S))$. Then $V = T^\ell + q_{\ell-1} \cdot T^{\ell-1} + \ldots + q_0$ with $q_{\ell-1},\ldots,q_0 \in \mathcal{F}(K)$. Put $R = R_1 \cdot (R,S)$, $S = S_1 \cdot (R,S)$. There exist $W_0,\ldots,W_\ell \in \mathfrak{R}(\mathcal{O},K)$ such that $R_1 | (T^i S_1 - W_i)$, $\mathrm{ord}(W_i) < \mathrm{ord}(R_1)$ $(i = 0,\ldots,\ell)$. Since $Z(R_1) \subset Z(V \cdot S_1)$, the operator $W_\ell + q_{\ell-1} \cdot W_{\ell-1} + \ldots + q_0 \cdot W_0$ is identically zero on $Z(R_1)$. But then it must be identically zero on $\mathcal{F}(K)$. We have $W_j = \sum_{j=0}^{\ell-1} w_{jh} \cdot T^h$ with $w_{jh} \in \mathcal{O}(K)$ for all j,h. Hence, $-w_{\ell h} = \sum_{j=0}^{\ell-1} q_j w_{jh}$ for all h. From this we obtain that

$q_{\ell-1},\ldots,q_0 \in \mathcal{O}(K)$. So, $V \in \mathfrak{R}(\mathcal{O},K)$ and, by definition, $V = R/S$. The second assertion now follows immediately from $Z(R/S) = S(Z(R))$. □

Remark 1.4.1. It follows from the proof above that if $R = R_1 \cdot V$, $S = S_1 \cdot V$ for $R,S,V,R_1,S_1 \in \mathfrak{R}$, then $R/S = R_1/S_1$.
Remark 1.4.2. Clearly, $R/I = R$ for all $R \in \mathfrak{R}$. Hence, by Remark 1.4.1, if $R = R_1 \cdot V$, then $R/V = R_1$.
Remark 1.4.3. Since $Z(I) = \{\{0\}\}$, we have $I/R = I$ for all $R \in \mathfrak{R}$, $R \neq 0$.
Remark 1.4.4. From Remark 1.4.1 and 1.4.2 it follows that, if $R,S,V \in \mathfrak{R}$, $V \neq 0$ and $R \cdot V = S \cdot V$, then $R = S$.

Suppose $R,S \in \mathfrak{R}$. The monic operator $V \in \mathfrak{R}$ of smallest order such that $Z(V) \supset Z(R) \cup Z(S)$ is the operator that has as zeros the linear combinations of zeros of R and zeros of S. (Notation: $Z(V) = Z(R) + Z(S)$.). It is evident that an operator V with $Z(V) \supset Z(R) + Z(S)$ exists. On the other hand, that there exists a $V \in \mathfrak{R}$ with $Z(V) = Z(R) + Z(S)$ is made clear by the following proposition.

Proposition 1.3. *Let* $R,S \in \mathfrak{R}$, $R,S \neq 0$, *and* R,S *monic*. *Then* $(R/S) \cdot S = (S/R) \cdot R$ *and* $Z((R/S) \cdot S) = Z(S) + Z(R)$.

Proof: $Z((R/S) \cdot S) = Z(S) \cup \{\{u_n\} \mid \{S(u_n)\} \in Z(R/S)\} =$
$\{\{u_n\} \mid \{S(u_n)\} = \{S(v_n)\}$ for some $\{v_n\} \in Z(R)\} =$
$\{\{u_n\} \mid \{u_n\} = \{v_n\} + \{t_n\}$ for $\{v_n\} \in Z(R), \{t_n\} \in Z(S)\} = Z(R) + Z(S)$. The alleged identity follows since the expression on the right-hand side is symmetrical in R and S. □

We define the ***lowest common multiple*** $[R,S]$ of $R,S \in \mathfrak{R}$ as the monic operator V such that $Z(V) = Z(R) + Z(S)$.
By Proposition 1.3, $[R,S] = (R/S) \cdot S$ if $S \neq 0$, and $[R,0] = 0$. Clearly, $[R,S] = [S,R]$, and, if $R,S \in \mathfrak{R}$, then also $[R,S] \in \mathfrak{R}$.

Remark 1.4.5. We have the following identity:

$$\mathrm{ord}(R) + \mathrm{ord}(S) = \mathrm{ord}((R,S)) + \mathrm{ord}([R,S]).$$

Remark 1.4.6. $[R,S]$ is the unique monic operator of smallest order such that both $R \mid [R,S]$ and $S \mid [R,S]$.

A further property is the following:

Proposition 1.4. *If* $A,S,R \in \mathfrak{R}$, *and* $S,R \neq 0$, *then the following identity holds*: $A/SR = (A/R)/S$.

Proof: $Z(A/SR) = \{\{(S\cdot R)(u_n)\} \mid \{u_n\} \in Z(A)\} = \{\{S(v_n)\} \mid \{v_n\} \in Z(A/R)\} = Z((A/R)/S)$.

Remark 1.4.7. If $R,S \in \mathfrak{R}$, $S \neq 0$, and P and Q in $\mathfrak{R}$ are such that $S = Q\cdot R + P$, $P \neq 0$, then $R/S = R/P$. In particular, it is no restriction of generality if we assume $ord(S) < ord(R)$ when dealing with R/S.

Remark 1.4.8. If $R,S \in \mathfrak{R}$, $(R,S) = I$ and $S \neq 0$, so that $R/S \in \mathfrak{R}$ is well-defined and $ord(R/S) = ord(R)$, we can find a $V \in \mathfrak{R}$ such that $(R/S)/V = R$.

Proof: By Proposition 1.1(c), there exist $P,Q \in \mathfrak{R}$ such that $P\cdot R + Q\cdot S = I$. We show that we can take Q for the operator V. If $\{u_n\} \in Z(R)$, then $(Q\cdot S)(u_n) = u_n$, hence $Z((R/S)/Q) = Z(R/QS) = \{\{(Q\cdot S)(v_n)\} \mid \{v_n\} \in Z(R)\} = \{\{v_n\} \mid \{v_n\} \in Z(R)\} = Z(R)$. But then it follows that $(R/S)/Q = R$. □

Finally, we show that if $R,S \in \mathfrak{R}$, $S \neq 0$, then a factorization of R in irreducible factors induces a factorization of R/S in irreducible factors.

Proposition 1.5. *Let* $R,S \in \mathfrak{R}$, $S \neq 0$, *and suppose* $R = R_1\cdot \ldots \cdot R_k$ *where* $R_1,\ldots,R_k$ *are irreducible over* $\mathfrak{R}$. *Then*

$$R/S = (R_1/S_1)\cdot \ldots \cdot (R_k/S_k)$$

where $S_k = S$, $S_{j-1} = S_j/R_j$ $(j = 2,\ldots,k)$ *and* $R_1/S_1,\ldots,R_k/S_k$ *lie in* $\mathfrak{R}$ *and are irreducible over* $\mathfrak{R}$.

Moreover, if $(R,S) = I$, *then* $ord(R_j/S_j) = ord(R_j)$ *for* $j = 1,\ldots,k$.

We prove a lemma before proving the proposition.

Lemma 1.6. *Suppose* $R,S \in \mathfrak{R}$, $S \neq 0$ *and* R *irreducible over* $\mathfrak{R}$. *Then* R/S *is irreducible over* $\mathfrak{R}$.

Proof: Suppose $R/S = V_1\cdot V_2$, $V_1,V_2 \in \mathfrak{R}$ and $r := ord(V_2) > 0$. Then $(V_2S) \mid [R,S]$ and $ord(V_2\cdot S) > r$. If $R|S$, then $[R,S] = q\cdot S$ for some $q \in \mathcal{O}$, hence $r = 0$, which yields a contradiction. If $R \nmid S$, then $(R,S) = I$. In that case, $Z(V_2S) = Z(S) + M$, where $M \subset Z(R)$ and $r = \dim(M) > 0$. Then, $Z((V_2S,R)) = M$. Since $(V_2S,R)|R$ and R is irreducible, we obtain $r = ord(R)$. □

Proof of Proposition 1.5.: Put $R = R_1\cdot R^*$, where $R_1,R^* \in \mathfrak{R}$ and R_1 is irreducible. We proceed by induction on k. For $k = 1$ the assertion follows immediate-

ly from Lemma 1.5. Suppose the assertion is true for $\ell \le k-1$. Then $R^*/S = (R_2/S_2)\cdot\ldots\cdot(R_k/S_k)$, where $S_k = S$, $S_{j-1} = S_j/R_j$ $(j = 3,\ldots,k)$. We shall prove that $R/S = (R_1/S_1)\cdot R^*/S$. Firstly, it is clear that R^*/S divides R/S. Put $R/S = R_1^*\cdot(R^*/S)$. We calculate R_1^*. Put $S_1 = S/R^*$. Using Propositions 1.3 and 1.4 we obtain: $R_1^*\cdot(R^*/S)\cdot S = (R/S)\cdot S = (S/R)\cdot R = ((S/R^*)/R_1)\cdot R_1\cdot R^* =$
$= (S_1/R_1)\cdot R_1\cdot R^* = (R_1/S_1)\cdot S_1\cdot R^* = (R_1/S_1)\cdot(S/R^*)\cdot R^* = (R_1/S_1)\cdot(R^*/S)\cdot S.$
Hence, by Remark 1.4.4, $R_1^* = R_1/S_1$. So we see $S_1 \in \mathfrak{R}$ and $S_1 = S/R^* =$
$= S_k/R_2\ldots R_k = (S_k/R_k)/R_2\ldots R_{k-1} = S_{k-1}/R_2\ldots R_{k-1} = S_{k-2}/R_2\ldots R_{k-2} = \ldots =$
$= S_2/R_2$.
Moreover, since $S_1 \in \mathfrak{R}$ and R_1 irreducible, R_1/S_1 is irreducible by Lemma 1.5. Furthermore, if $(R,S) = I$, then $\mathrm{ord}(R/S) = \mathrm{ord}(R)$. Since for $j = 1,\ldots,k$,

$$\mathrm{ord}(R_j/S_j) \le \mathrm{ord}(R_j)$$

we have that $\mathrm{ord}(R_j/S_j) = \mathrm{ord}(R_j)$ for all j. □

We can determine the lowest common multiple $[R,S]$ of two operators $R,S \in \mathfrak{R}$ in the following way:

Suppose that application of the Euclidian algorithm gives the following chain of equalities:

$R = Q_1\cdot S + R_1$, $S = Q_2\cdot R_1 + R_2,\ldots,$ $R_{n-2} = Q_n\cdot R_{n-1} + R_n$, $R_{n-1} = Q\cdot R_n$,

where $R_n = (R,S)$, $\mathrm{ord}(R_n) < \mathrm{ord}(R_{n-1}) < \ldots < \mathrm{ord}(R_1) < \mathrm{ord}(S)$ and $Q,Q_j,R_j \in \mathfrak{R}$ for $j = 1,\ldots,n$. Put $R = R_{-1}$, $S = R_0$.

Clearly, $[R_n,R_{n-1}] = R_{n-1} = Q\cdot R_n$. If we have that $[R_j,R_{j+1}] = V_j\cdot R_j =$ $= W_j\cdot R_{j+1}$ for $V_j,W_j \in \mathfrak{R}$ and some $j \in \{0,\ldots,n-1\}$, we can find $[R_j,R_{j-1}]$ as follows: Since $R_{j-1} = Q_{j+1}\cdot R_j + R_{j+1}$, we have that $V_j\cdot R_j = W_j(R_{j-1} - Q_{j+1}\cdot R_j)$, hence $(V_j + W_j\cdot Q_{j+1})\cdot R_j + W_j\cdot R_{j-1}$. We claim that $W_j\cdot R_{j-1} = q\cdot[R_j,R_{j-1}]$ for some $q \in \mathcal{O}$. Suppose this this not so. It is evident that both W_j and $V_j + W_j\cdot Q_{j+1}$ lie in $\mathfrak{R}$, so that both R_{j-1} and R_j divide $W_j\cdot R_{j-1}$. Hence $[R_j,R_{j-1}]|W_j\cdot R_{j-1}$. So there must be some $W \in \mathfrak{R}$ of order ≥ 1 such that W is a left divisor of both W_j and $V_j + W_j\cdot Q_{j+1}$. Then W is a left divisor of V_j. Hence there exist operators V_j^* and W_j^* such that $V_j = W\cdot V_j^*$, $W_j = W\cdot W_j^*$. From $V_j\cdot R_j = W_j\cdot R_{j+1}$ we derive $V_j^*\cdot R_j = W_j^*\cdot R_{j+1}$, so $[R_j,R_{j+1}]$ divides $V_j^*\cdot R_j$, in contradiction with $\mathrm{ord}(V_j^*\cdot R_j) < \mathrm{ord}(V_j\cdot R_j) = \mathrm{ord}([R_j,R_{j+1}])$. So, by subsequently lowering the value of j, we finally obtain [R,S] in this way.

The derived operator R/S can now be obtained by simply dividing [R,S] by S and by left multiplication with a suitable factor in $\mathcal{O}$. Note that the assertions of Proposition 1.2 also follow from the above construction.

§5. Some properties of operators in $\mathfrak{R}(K)$.

In this section, we study a few properties of the set $\mathfrak{R}(K)$, which we will need in later chapters.

(i) Let $q = \{q_n\} \in \mathcal{S}(K)$, $q_n \neq 0$ for all n. Put $S = q\cdot I$. Then $S \in \mathcal{S}(K)$, $\mathrm{ord}(S) = 0$. Let $R \in \mathfrak{R}(K)$, $\mathrm{ord}(R) = k$. As in §4, we define R/S as the monic operator such that $S(Z(R)) = Z(R/S)$. Then $R/S \in \mathfrak{R}(K)$, and $\mathrm{ord}(R/S) = k$. More explicitly, let $R = p_kT^k+\ldots+p_0$. Then $R/S = r_k^{-1}(r_kT^k+\ldots+r_0)$, where $r_j(X) = \frac{p_j(X)}{q(X+j)}$ $(j = 0,\ldots,k)$.

In later chapters we shall apply this procedure quite often and refer to it as a *zeroth-order transformation* of the operator R. (R/S is called a *zeroth-order transform* of R). Note that if $\lim_{x\to\infty} q(x) = q$ and χ_R exists, $\chi_R \in \overline{K}[X]$, then $\chi_{R/S} = c\chi_R$ for some $c \in \overline{K}$. On the other hand, if $\lim_{x\to\infty} \frac{q(X+1)}{q(X)} = \ell$, then $\chi_{R/S}(X) = \chi_R(X/\ell)$. If $\{u_n\} \in Z(R)$ and $u_n \neq 0$ for all n, we may take $q = \{u_n^{-1}\}$. In that case, $\{1\} \in Z(R/S)$.

(ii) If $S,V,R \in \mathfrak{R}(K)$, $\mathrm{ord}(S) > 0$, $\mathrm{ord}(V) > 0$ and $S\cdot V = R$, then we call V a *(formal) divisor of* R, and S·V a *(formal) factorization of* R. As in §4, we write R/V for the monic operator q·S ($q \in \mathcal{S}(K)$). For instance, if $R \in \mathfrak{R}(K)$, $\{u_n\} \in Z(R)$ and $u_n \neq 0$ for all n, R admits of a formal factorization of the form $R = S\cdot(T - \frac{u_{n+1}}{u_n})$, for some $S \in \mathfrak{R}(K)$.

Remark 1.5.1. With the extension of the definition of a derived operator to the set $\mathfrak{R}(K)$, Proposition 1.4 remains valid for $A,S,R \in \mathfrak{R}(K)$.

§6. Matrix Recurrences.

It often appears convenient to study recurrences not in the form (1.1), but as *matrix recurrences*, that is, recurrences of the type

(1.5) $$M_nx_n = x_{n+1} \qquad (n\geq m)$$

where $M_m, M_{m+1},\ldots$ is a sequence of non-singular matrices in $K^{k,k}$ where K is some number field and $x_m, x_{m+1}, \dots$ is a sequence of vectors in K^k.

We shall further identify two matrix recurrences defined by sequences $\{M_n^{(1)}\}$ and $\{M_n^{(2)}\}$, respectively, if $M_n := M_n^{(1)} = M_n^{(2)}$ for all n larger than some number N, and we shall indicate them by $[M_n]$. Similarly, we identify two sequences of matrices, or vectors, if their members are equal from a certain index N on, and we shall write $\{M_n\}, \{x_n\}$, etc. (Compare §1.1, where we did something similar for sequences of numbers). In practice, we shall often assume $n \geq 0$ or 1, if this does not affect our conclusions.

By $\mathcal{M}(K)$ we denote the set of matrix recurrences where the matrices have coefficients in the field K, and the solutions are sequences of numbers in K as well.

From now on, we suppose that K is a subfield of the field of complex numbers. A recurrence operator $R \in \mathcal{R}(K)$ corresponds to a matrix recurrence in $\mathcal{M}(K)$ in the following way: Let $R = p(T^k - q_{k-1}T^{k-1} - \dots - q_0)$, with $p, q_0, \dots, q_{k-1}$ sequences in $\mathcal{S}(K)$. We define a sequence of matrices $\{M_n^R\}$, where

$$(1.6) \quad M_n^R = \begin{bmatrix} q_{k-1}(n) & q_{k-2}(n) & \dots & q_1(n) & q_0(n) \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Clearly, M_n^R is non-singular for all n. We call $[M_n^R]$ the matrix recurrence *associate to* R. The sequences $\{x_n\}$ that satisfy (1.5) for $M_n = M_n^R$ are precisely those for which $x_n^T = (u_{n+k-1}, \dots, u_n)$ where $\{u_n\} \in Z(R)$. (By x^T we denote the transpose of the vector x).

If $R \in \mathcal{R}$ and χ_R exists, then the sequence $\{M_n^R\}$ converges (entrywise) to a matrix M^R, where $[M^R]$ is the (constant) matrix recurrence associate to the (constant) operator $\chi_R(T) = T^k + \pi_{k-1}T^{k-1} + \dots + \pi_0$, which can be obtained by replacing X in the expression for $\chi_R(X)$ by the shift operator T. It is a well-known fact from linear algebra that the eigenvalues of the matrix M^R are precisely the zeros of $\chi_R(X)$, whereas each eigenvalue has geometric multiplicity one and the algebraic multiplicity of each eigenvalue is equal to the multiplicity of the corresponding zero in χ_R. (Thus, the characteristic polynomial of the matrix M^R is $c \cdot \chi_R$, where c is some non-zero complex number.)

In the sequel we shall denote the limit matrix of a sequence of matrices $\{M_n\}$ by $\lim M_n$.

It sometimes appears useful not to consider the matrix recurrence (1.5), but a matrix recurrence

$$(1.7) \qquad (U^{-1}M_nU)y_n = y_{n+1}$$

where U is an invertible matrix in $K^{k,k}$. Note that (1.7) is essentially the same matrix recurrence as (1.5), with $x_n = Uy_n$ for all n. We call (1.7) a *conjugate matrix recurrence* of (1.5). Note that

$$\lim U^{-1}M_nU = U^{-1}(\lim M_n)U$$

if $\lim M_n$ exists.

A procedure we shall often apply is to consider instead of (1.5) a conjugate matrix recurrence such that the limit matrix is in so-called Jordan normal form. We shall shortly recall the definition of a (complex or real) Jordan normal form.(See any text on linear algebra for a more extensive exposition.)

Let $M \in \mathbb{C}^{k,k}$. Then there exists an invertible matrix $U \in \mathbb{C}^{k,k}$ such that $U^{-1}MU$ is of the form

$$(1.8) \qquad \begin{bmatrix} B(\alpha_1,m_1) & & & 0 \\ & B(\alpha_2,m_2) & & \\ & & \ddots & \\ 0 & & & B(\alpha_\ell,m_\ell) \end{bmatrix}$$

where $\alpha_1,\ldots,\alpha_\ell$ are the eigenvalues of M, repeated according to geometric multiplicity, and $B(\alpha,\ell) = \alpha\cdot I + J$, where I is the identity matrix in $\mathbb{C}^{\ell,\ell}$ and J is the matrix in $\mathbb{C}^{\ell,\ell}$ such that

$$(1.9) \qquad J = \begin{bmatrix} 0 & 1 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 1 & \ldots\ldots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & 1 \\ 0 & 0 & 0 & \ldots\ldots & 0 \end{bmatrix}$$

(In the sequel, we shall denote all matrices of this form by J, if it is clear what the dimension is.)

The form (1.8) is uniquely determined up to permutation of the matrices $B(\alpha,\ell)$ and is called the *(complex) Jordan normal form* of the matrix M.

In the same way, if $K = \mathbb{R}$, to every matrix M can be found a real-valued matrix U, such that $U^{-1}MU$ is of the following form:

$$(1.10) \qquad \begin{bmatrix} C(\alpha_1,m_1) & & & 0 \\ & C(\alpha_2,m_2) & & \\ & & \ddots & \\ 0 & & & C(\alpha_\ell,m_\ell) \end{bmatrix}.$$

Here $\alpha_1,\ldots,\alpha_q$ are the real eigenvalues of M $(q\le\ell)$, and $\alpha_{q+1},\ldots,\alpha_\ell$, $\bar\alpha_{q+1},\ldots,\bar\alpha_\ell$ are the non-real eigenvalues of M, counted according to their geometric multiplicities, and $C(\alpha,\ell) = B(\alpha,\ell)$ if $\alpha \in \mathbb{R}$. If $\alpha \notin \mathbb{R}$, and $\alpha = \beta + i\gamma$ (where $\beta,\gamma \in \mathbb{R}$), then $C(\alpha,\ell) \in \mathbb{R}^{\ell,\ell}$ and has the form

(1.11) $$C(\alpha,\ell) = \begin{bmatrix} A(\alpha) & I & 0 & \cdots & 0 \\ 0 & A(\alpha) & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & I \\ 0 & 0 & 0 & \cdots & A(\alpha) \end{bmatrix}$$

where I is the identity matrix in $\mathbb{R}^{2,2}$, and $A(\alpha)$ is the matrix

(1.12) $$A(\alpha) = \begin{bmatrix} \beta & -\gamma \\ \gamma & \beta \end{bmatrix}.$$

The form (1.10) is called the *real Jordan normal form* of the matrix M and is uniquely determined up to permutation of the matrices $C(\alpha_j,m_j)$.

Lemma 1.7. *If* $R \in \mathfrak{R}(K)$ *for* $K = \mathbb{C}$ *or* $K = \mathbb{R}$, *and* χ_R *exists, then the eigenvalues of the limit matrix* $\lim M_n^R$ *are the zeros of* χ_R *and each eigenvalue has geometric multiplicity one.*

Proof: This follows from a simple calculation. □

It follows immediately from the above considerations that not each matrix recurrence is the conjugate of a matrix recurrence $[M_m^R]$ corresponding to a linear recurrence operator R. In particular, if $\lim M_n$ has eigenvalues with geometric multiplicity greater than one, there is no such R.

Suppose $R,S \in \mathfrak{R}(K)$ for some field K. We derive the matrix corresponding to R/S. In the first place, note that we can assume without loss of generality that $\mathrm{ord}(S) < \mathrm{ord}(R)$, and $(R,S) = I$ (by Remark 1.4.7). Let $\{M_n^R\}$ be the sequence of matrices, corresponding to R. Since $r = \mathrm{ord}(R) > \mathrm{ord}(S)$, there exist invertible matrices $S_n \in K^{r,r}$ $(n \in \mathbb{N})$ such that

(1.13) $$S_n \begin{bmatrix} u_{n+r-1} \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} S(u_{n+r-1}) \\ \vdots \\ S(u_n) \end{bmatrix}$$

Then, $M_n^{R/S} = S_{n+1} M_n^R S_n^{-1}$ $(n \in \mathbb{N})$.

CHAPTER TWO

RATIONAL OPERATORS

§1. Introduction.

In this chapter, we take $K = \mathbb{Q}$ and $\mathcal{O}(K)$ the field of sequences of the form $\{r(n)\}$ with $r \in \mathbb{Q}(X)$. Put $\mathfrak{Rat} = \mathfrak{R}(\mathcal{O}(X),\mathbb{Q})$. If $R \in \mathfrak{Rat}$, we call R a *rational operator*. If R is a rational operator of order k, and a zero $\{u_n\}$ of R has k initial values $u_\ell,\ldots,u_{\ell+k-1}$ in $\mathbb{Q}$, then $u_n \in \mathbb{Q}$ for all $n \geq \ell$.

Let $R \in \mathfrak{Rat}$. For $\{u_n\},\{v_n\} \in Z(R)$, we consider the sequence of quotients $\frac{u_n}{v_n}$ for $n \geq \ell$. If its limit exists, it is a real number .We define the set L(R) by

$$L(R) = \{\alpha \in \mathbb{R} |\ \alpha = \lim_{n\to\infty} \frac{u_n}{v_n} \text{ for } \{u_n\},\{v_n\} \in Z(R)\}.$$

Since Z(R) is a vector space over $\mathbb{Q}$, it is clear that $\mathbb{Q} \subset L(R) \subset \mathbb{R}$ if $\text{ord}(R) > 0$. We define $\mathcal{L}$ as the union of all sets L(R), where $R \in \mathfrak{Rat}$.

The aim of this chapter is to prove the following two facts:
(1) $\mathcal{L}$ is a field.
(2) The union of the sets L(R) where R runs through the set of rational operators with constant coefficients is equal to the set of real algebraic numbers $\overline{\mathbb{Q}} \cap \mathbb{R}$.
Hence, in particular, we have that $\overline{\mathbb{Q}} \cap \mathbb{R} \subset \mathcal{L} \subset \mathbb{R}$. That $\mathcal{L} \neq \mathbb{R}$ follows immediately from the following lemma.

Lemma 2.1. *$\mathcal{L}$ is a countable set.*

Proof: For $R \in \mathfrak{Rat}$, the set Z(R) is a k-dimensional vector space over $\mathbb{Q}$, so that L(R) is countable. Since $\mathfrak{Rat} \subset \bigcup_{k=0}^{\infty} \mathbb{Q}(X)^k$, the set $\mathfrak{Rat}$ is a countable set, hence the union $\mathcal{L}$ of the sets L(R) for $R \in \mathfrak{Rat}$ is also countable.

On the other hand, $\mathcal{L}$ contains real transcendental numbers. It can be shown without any effort that the numbers of the form

$$\sum_{N=0}^{\infty} \prod_{n=0}^{N-1} q(n) \qquad \text{for } q \in \mathbb{Q}(X) \text{ and } q(n) \neq 0,\ q(n)^{-1} \neq 0 \text{ for } n \geq 0,$$

if the sum converges, lie in some L(R) where R is some reducible rational operator of order 2. Namely, $R = (T-q)(T-1)$. So we obtain for instance the

numbers e^k ($k \in \mathbb{Q}$), $\log k$ ($k \in \mathbb{Q}$, $k>1$), $\arctan k$ ($k \in \mathbb{Q}$, $|k| \leq 1$) if we take $q(n) = \frac{k}{n+1}$, $q(n) = \frac{k-1}{k}\cdot\frac{n+1}{n+2}$, $q(n) = -\frac{2n+1}{2n+3}\cdot k^2$, respectively ($n \in \mathbb{N}$).
As announced above, the following result is valid:

Theorem 2.2. *$\mathcal{L}$ is a field.*

We shall use the following lemma.

Lemma 2.3. *Let* $R,S \in \mathcal{R}at$. *There exists an operator* $V \in \mathcal{R}at$ *such that* $Z(V) \supset \{\{u_n v_n\} \mid \{u_n\} \in Z(R)$ *and* $\{v_n\} \in Z(S)\}$ *and* $\mathrm{ord}(V) \leq \mathrm{ord}(R)\cdot\mathrm{ord}(S)$.

Proof: Put $r = \mathrm{ord}(R)$ and $s = \mathrm{ord}(S)$. For $k,\ell \in \mathbb{Z}_{\geq 0}$ there exist $p_{r-1,k},\dots,p_{0,k},q_{s-1,\ell},\dots,q_{0,\ell} \in \mathbb{Q}(X)$ such that

$$\{u_{n+k}\} = p_{r-1,k}\{u_{n+r-1}\} + \dots + p_{0,k}\{u_n\}$$

and

$$\{v_{n+\ell}\} = q_{s-1,\ell}\{v_{n+s-1}\} + \dots + q_{0,\ell}\{v_n\}$$

for all $\{u_n\} \in Z(R)$ and $\{v_n\} \in Z(S)$ respectively. Hence, each of the rs+1 sequences $\{u_n v_n\},\dots,\{u_{n+rs} v_{n+rs}\}$ can be written as a linear combination of the rs sequences $\{u_{n+i} v_{n+j}\}$ ($0 \leq i \leq r-1$, $0 \leq j \leq s-1$) with coefficients in $\mathbb{Q}(X)$ depending only on R and S. Thus, the rs+1 sequences $\{u_{n+j} v_{n+j}\}$ ($0 \leq j \leq rs$) are linearly dependent over $\mathbb{Q}(X)$. So we can find a number $t \leq rs$ and rational functions $r_0,\dots,r_{t-1}$ such that

$$(T^t + r_{t-1}T^{t-1} + \dots + r_0)(u_n v_n) = 0$$

for all $\{u_n\} \in Z(R)$ and $\{v_n\} \in Z(S)$. Put $V = T^t + r_{t-1}T^{t-1} + \dots + r_0$. Then $\mathrm{ord}(V) = t \leq rs$ and V is the desired operator. □

Proof of Theorem 2.2.: Suppose $\alpha,\beta \in \mathcal{L}$. Then there exist $R,S \in \mathcal{R}at$ and $\{u_n^{(1)}\},\{u_n^{(2)}\} \in Z(R)$, $\{v_n^{(1)}\},\{v_n^{(2)}\} \in Z(S)$ such that

$$\lim_{n\to\infty} \frac{u_n^{(1)}}{v_n^{(1)}} = \alpha, \quad \lim_{n\to\infty} \frac{u_n^{(2)}}{v_n^{(2)}} = \beta.$$

Also, $\{-u_n^{(1)}\} \in Z(R)$, hence $-\alpha = \lim_{n\to\infty} \frac{-u_n^{(1)}}{v_n^{(1)}} \in \mathcal{L}$, and for $\alpha \neq 0$,

$$1/\alpha = \lim_{n\to\infty} \frac{v_n^{(1)}}{u_n^{(1)}} \in \mathcal{L}.$$

Finally, we show that $\alpha\beta$ and $\alpha+\beta$ lie in $\mathcal{L}$. By Lemma 2.3, there exists a

$V \in \mathcal{R}at$ such that $\{u_n^{(i)} v_n^{(j)}\} \in Z(V)$ for $i,j \in \{1,2\}$. Hence

$$\alpha\beta = \lim_{n\to\infty} \frac{u_n^{(1)} u_n^{(2)}}{v_n^{(1)} v_n^{(2)}} \in L(V) \subset \mathcal{L},$$

and

$$\alpha+\beta = \lim_{n\to\infty} \frac{u_n^{(1)} v_n^{(2)} + u_n^{(2)} v_n^{(1)}}{v_n^{(1)} v_n^{(2)}} \in L(V) \subset \mathcal{L}.$$

§2. Rational operators with constant coefficients.

In this section we shall consider the set of rational operators with constant coefficients. Thus, the field $\mathcal{O}(K)$ is the field of constant sequences with terms in $\mathbb{Q}$, which we shall, by abuse of notation, denote by $\mathbb{Q}$. We shall prove that the union of the sets $L(R)$ where R runs through the set $\mathfrak{R}(\mathbb{Q},\mathbb{Q})$ is the set of real algebraic numbers. Note that for $R \in \mathfrak{R}(\mathbb{Q},\mathbb{Q})$, the characteristic polynomial χ_R of R exists and is irreducible if and only if R is irreducible in $\mathbb{Q}[X]$. (In fact, $R = \chi_R(T)$.)

We first prove a lemma about the form of a (rational) root.

Proposition 2.4. *Let* $R \in \mathfrak{R}(\mathbb{Q},\mathbb{Q})$ *be of order* k. *Write* $\chi_R(X) = \prod_{j=1}^{\ell} P_j(X)^{e_j}$, *where* $P_j(X) = \prod_{i=1}^{d_j} (X-\alpha_{ji}) \in \mathbb{Q}[X]$ *are distinct irreducible polynomials in* $\mathbb{Q}[X]$. *Then, for* $\{u_n\} \in Z(R)$,

$$u_n = \sum_{j=1}^{\ell} \sum_{m=1}^{e_j} \sum_{i=1}^{d_j} Q_{mj}(\alpha_{ji}) \cdot \alpha_{ji}^n \cdot n^{m-1} \tag{2.1}$$

where $Q_{mj} \in \mathbb{Q}[X]$ *and* $\deg(Q_{mj}) \le d_j - 1$ $(m = 1,\dots,e_j;\ j = 1,\dots,\ell)$.

Proof: A basis of the zeros of R over $\mathbb{C}$ is

$\{\ \{\alpha_{ji}^n \cdot n^{m-1}\} \mid m = 1,\dots,e_j;\ i = 1,\dots,d_j;\ j = 1,\dots,\ell\ \}$.

Put $v_{jn}^{(s)} = \sum_{i=1}^{d_j} \alpha_{ji}^{n+s-1}$ $(s = 1,\dots,d_j;\ j = 1,\dots,\ell)$. We claim that $\{\ \{v_{jn}^{(s)} \cdot n^{m-1}\} \mid m = 1,\dots,e_j;\ s = 1,\dots,d_j;\ j = 1,\dots,\ell\ \}$ is a basis of $Z(R)$. Firstly, $v_{jn}^{(s)} \in \mathbb{Q}$, since it is an elementary symmetrical form in the zeros of $P_j(X)$, so that $\{v_{jn}^{(s)} \cdot n^{m-1}\} \in Z(R)$. Since there are exactly k different zeros of this form, it remains to be shown that they are linearly independent.

Suppose

$$\{0\} = \sum_m \sum_s \sum_j \lambda_{msj}\{n^{m-1}v_{jn}^{(s)}\} = \sum_m \sum_s \sum_j \sum_{i=1}^{d_j} \lambda_{msj}\{n^{m-1}\cdot\alpha_{ji}^{n+s-1}\} =$$

$$= \sum_m \sum_j \sum_i \sum_{s=1}^{d_j} \lambda_{msj}\cdot\alpha_{ji}^{s-1}\{\alpha_{ji}^n\cdot n^{m-1}\}.$$

Since the sequences $\{\alpha_{ji}^n\cdot n^{m-1}\}$ ($m = 1,\dots,e_j$; $i = 1,\dots,d_j$; $j = 1,\dots,\ell$) form a basis of zeros over $\mathbb{C}$, we obtain that $\sum_{s=1}^{d_j}\lambda_{msj}\alpha_{ji}^{s-1} = 0$ for all m,j,i. Thus, $\sum_{s=1}^{d_j} \lambda_{msj}X^{s-1}$ is a polynomial of degree smaller than d_j with roots $\alpha_{j1},\dots,\alpha_{jd_j}$, so it must be identically zero, which implies $\lambda_{msj} = 0$ for all m,s,j. Hence, for $\{u_n\} \in Z(R)$,

$$\{u_n\} = \sum_m \sum_s \sum_j C_{msj}\{n^{m-1}\cdot v_{jn}^{(s)}\}$$

where $C_{msj} \in \mathbb{Q}$ for all m,s,j. So we obtain

$$\{u_n\} = \sum_m \sum_s \sum_j \sum_i C_{msj}\{n^{m-1}\cdot\alpha_{ji}^{n+s-1}\} = \sum_m \sum_i \sum_j \sum_{s=1}^{d_j} C_{msj}\cdot\alpha_{ji}^{s-1}\{n^{m-1}\cdot\alpha_{ji}^n\} =$$

$$= \sum_m \sum_j \sum_{i=1}^{d_j} Q_{mj}(\alpha_{ji})\{\alpha_{ji}^n\cdot n^{m-1}\},$$

where $Q_{mj}(X) = \sum_{s=1}^{d_j} C_{msj}X^{s-1} \in \mathbb{Q}[X]$ and $\deg(Q_{mj}) \le d_j-1$. □

We use this result to investigate the set L(R). First we treat the case that R is irreducible over $\mathbb{Q}$.

Proposition 2.5. *Suppose* $R \in \mathfrak{R}(\mathbb{Q},\mathbb{Q})$ *is irreducible. Put* $\chi_R(X) = c\cdot\prod_{j=1}^{k}(X-\alpha_j)$, *where* $c \in \mathbb{Q}$, $\alpha_1,\dots,\alpha_k \in \mathbb{C}$.
(a). *If* $|\alpha_1|=\dots=|\alpha_\ell| > |\alpha_j|$ *for* $j > \ell$, *then* $L(R) \subset \mathbb{Q}(\alpha_1) \cap \dots \cap \mathbb{Q}(\alpha_\ell) \cap \mathbb{R}$.
(b). *If* $|\alpha_1| > |\alpha_j|$ *for* $j > 1$, *then* $L(R) \subset \mathbb{Q}(\alpha_1) \subset \mathbb{R}$.

Proof: Suppose $|\alpha_1| = \dots = |\alpha_\ell| > |\alpha_j|$ for $j \ge \ell$. Let $\{u_n\},\{v_n\} \in Z(R)$. By Proposition 2.4, there exist $\pi_1,\pi_2 \in \mathbb{Q}[X]$ of degree $\le$ k-1 such that

$$u_n = \sum_{i=1}^{k} \pi_1(\alpha_i)\cdot\alpha_i^n, \quad v_n = \sum_{i=1}^{k} \pi_2(\alpha_i)\cdot\alpha_i^n.$$

Suppose $\{v_n\} \ne \{0\}$. Note that $\pi_i(\alpha_j) = 0$ implies $\pi_i = 0$ $(i \in \{1,2\})$. Then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{k} \pi_1(\alpha_i)\cdot\alpha_i^n}{\sum_{i=1}^{k} \pi_2(\alpha_i)\cdot\alpha_i^n} = \lim_{n\to\infty} \frac{\sum_{i=1}^{\ell} \pi_1(\alpha_i)\cdot\gamma_i^n}{\sum_{i=1}^{\ell} \pi_2(\alpha_i)\cdot\gamma_i^n}$$

where $\gamma_i = \alpha_i/\alpha_1$, hence $|\gamma_i| = 1$ for $i = 1,\ldots,\ell$. Since the denominator in the rightmost quotient is bounded from above, we have that, if the limit exists and is L, then

$$\lim_{n\to\infty} \sum_{i=1}^{\ell} \rho(\alpha_i)\cdot\gamma_i^n = 0,$$

where $\rho = \pi_1 - L\cdot\pi_2$. We show that this implies $\rho(\alpha_j) = 0$ for $j = 1,\ldots,\ell$. Since all γ_j are distinct, there exists a $\delta > 0$ such that $|\gamma_j - \gamma_i| > \delta$ for $i \neq j$. Let $\varepsilon > 0$. Choose N so large that for $n > N$

$$\left| \sum_{i=1}^{\ell} \rho(\alpha_i)\cdot\gamma_i^n \right| < \varepsilon\cdot\frac{\delta^{\ell-1}}{2^{\ell-1}}.$$

Then

$$\left| \sum_{i=2}^{\ell} \rho(\alpha_i)\cdot\gamma_i^n\cdot(\gamma_i - \gamma_1) \right| \leq \left| \sum_{i=1}^{\ell} \rho(\alpha_i)\cdot\gamma_i^{n+1} \right| + \left| \sum_{i=1}^{\ell} \rho(\alpha_i)\cdot\gamma_i^n\cdot\gamma_1 \right| < 2\cdot\varepsilon\cdot\frac{\delta^{\ell-1}}{2^{\ell-1}}.$$

Proceeding in this way, we obtain

$$\left| \rho(\alpha_\ell)\cdot\gamma_\ell^n\cdot(\gamma_\ell - \gamma_1)\cdot\ldots\cdot(\gamma_\ell - \gamma_{\ell-1}) \right| < 2^{\ell-1}\cdot\varepsilon\cdot\frac{\delta^{\ell-1}}{2^{\ell-1}} < \varepsilon\cdot\delta^{\ell-1}$$

for $n > N$. Since $|\gamma_\ell - \gamma_i| > \delta$ and $|\gamma_i| = 1$ for $i = 1,\ldots\ell-1$, this yields $|\rho(\alpha_\ell)| < \varepsilon$. Since ε can be chosen arbitrarily small, we obtain that $\rho(\alpha_\ell) = 0$. In the same way we prove that $\rho(\alpha_i) = 0$ for $i = 1,\ldots,\ell$. But then, by the definition of ρ,

$$L = \frac{\pi_1(\alpha_i)}{\pi_2(\alpha_i)} \quad \text{for } i = 1,\ldots,\ell.$$

Hence, if L exists, it lies in $\mathbb{Q}(\alpha_1) \cap \ldots \cap \mathbb{Q}(\alpha_\ell)$. Also, $L \in \mathbb{R}$, for if none of the α_i $(i = 1,\ldots,\ell)$ is real, then there is an $m \in \{2,\ldots,\ell\}$ such that $\alpha_m = \overline{\alpha_1}$. Then,

$$L = \frac{\pi_1(\alpha_1)}{\pi_2(\alpha_1)} = \frac{\pi_1(\overline{\alpha_1})}{\pi_2(\overline{\alpha_1})} = \frac{\overline{\pi_1(\alpha_1)}}{\overline{\pi_2(\alpha_1)}} = \overline{L}$$

which implies $L \in \mathbb{R}$.

If $\ell = 1$, then clearly $\alpha_1 \in \mathbb{R}$, hence $\mathbb{Q}(\alpha_1) \subset \mathbb{R}$. Moreover, let $L \in \mathbb{Q}(\alpha_1)$.

Then $L = \pi(\alpha_1)$ for some $\pi \in \mathbb{Q}[X]$ with $\deg \pi \leq k-1$. Take $u_n = \sum_{i=1}^{k} \pi(\alpha_i)\cdot\alpha_i^n$ and $v_n = \sum_{i=1}^{k} \alpha_i^n$ for $n = 0,1,\ldots$. Then $\lim_{n\to\infty} \frac{u_n}{v_n} = \pi(\alpha_1) = L$, so that indeed

$L(R) = \mathbb{Q}(\alpha_1)$. □

Finally, we consider the case that R is reducible.

Proposition 2.6. *If* $R \in \mathfrak{R}(\mathbb{Q},\mathbb{Q})$, *then* $L(R) \subset \bar{\mathbb{Q}} \cap \mathbb{R}$.

Proof: Put $R = \prod_{j=1}^{\ell} R_j(T)^{e_j}$, where $R_j(X) = \prod_{i=1}^{d_j} (X-\alpha_{ji})$ is irreducible over $\mathbb{Q}[X]$ and the R_j $(j = 1,\dots,\ell)$ are distinct. By Proposition 2.4, for any $\{u_n\},\{v_n\} \in Z(R)$, there exist P_{mj},Q_{mj} in $\mathbb{Q}[X]$ $(j = 1,\dots,\ell;\ m = 1,\dots,e_j)$ such that

$$u_n = \sum_{j=1}^{\ell} \sum_{m=1}^{e_j} \sum_{i=1}^{d_j} P_{mj}(\alpha_{ji}) \cdot \alpha_{ji}^n \cdot n^{m-1},$$

$$v_n = \sum_{j=1}^{\ell} \sum_{m=1}^{e_j} \sum_{i=1}^{d_j} Q_{mj}(\alpha_{ji}) \cdot \alpha_{ji}^n \cdot n^{m-1}.$$

Let μ be the smallest integer such that $P_{mj} = Q_{mj} = 0$ for $m > \mu$ and $j = 1,\dots,\ell$. Then, if $\lim_{n\to\infty} \frac{u_n}{v_n}$ exists and is equal to L, say, then

$$\lim_{n\to\infty} \sum_{j=1}^{\ell} \sum_{i=1}^{d_j} \rho_{\mu j}(\alpha_{ji}) \cdot \gamma_{ji}^n = 0$$

where $\rho_{\mu j} = P_{\mu j} - L \cdot Q_{\mu j}$ $(j = 1,\dots,l)$ and $\gamma_{ij} = \frac{\alpha_{ji}}{\alpha}$, where α is $\max_{i,j} |\alpha_{ji}|$ taken over all j such that not both $P_{\mu j}$ and $Q_{\mu j}$ are identically zero. Hence $|\gamma_{ji}| \le 1$ for all i and j such that not $P_{\mu j} = Q_{\mu j} = 0$, and for at least one pair i,j the number γ_{ji} has absolute value one. We then proceed as in the proof of Proposition 2.5. and obtain

$$L = \lim_{n\to\infty} \frac{P_{\mu J}(\alpha_{JI})}{Q_{\mu J}(\alpha_{JI})}$$

for some $J \in \{1,\dots,\ell\}$, $I \in \{1,\dots,d_j\}$ such that $|\alpha_{JI}| = \alpha$.

In particular, $L \in \bar{\mathbb{Q}}$. The fact that $L \in \mathbb{R}$ follows by the same argument as in the proof of Proposition 2.5. □

Now we come to the final result.

Theorem 2.7. $L = \overline{\mathbb{Q}} \cap \mathbb{R}$

Proof: By Proposition 2.6, for every $R \in \mathfrak{R}(\mathbb{Q},\mathbb{Q})$, the set $L(R)$ is a subset of $\overline{\mathbb{Q}} \cap \mathbb{R}$. Conversely, take $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}$. We prove that $\alpha \in L(R)$ for some $R \in \mathfrak{R}(\mathbb{Q},\mathbb{Q})$. In §2.1 we saw that for any $R \in \mathfrak{R}(\mathbb{Q},\mathbb{Q})$ with $\operatorname{ord}(R) > 0$, the set of rational numbers is a subset of $L(R)$. So we can suppose $\alpha \notin \mathbb{Q}$. Choose $q \in \mathbb{Q}$ such that $\alpha + q$ is smaller in absolute value than all of its conjugates. Since $\alpha + q \neq 0$, the number $\frac{1}{\alpha + q}$ is larger in absolute value than all its conjugates. Let P be the minimal polynomial of $\frac{1}{\alpha + q}$ over $\mathbb{Q}[X]$ and choose $R \in \mathfrak{R}(\mathbb{Q},\mathbb{Q})$ such that $P = \chi_R$. By Proposition 2.5(b), $L(R) = \mathbb{Q}(\frac{1}{\alpha + q})$. Since $\alpha \in \mathbb{Q}(\frac{1}{\alpha + q})$, we obtain that $\alpha \in L(R)$. □

CHAPTER THREE

A FACTORIZATION THEOREM

§1. Introduction.

Suppose $[M_n] \in \mathcal{M}(\mathbb{C})$ is a matrix recurrence and $\lim M_n = M$, where $M \in \mathbb{C}^{k,k}$ and M has eigenvalues $\alpha_1,\ldots,\alpha_k$ with $|\alpha_1| < |\alpha_2| < \ldots < |\alpha_k|$. Then for each $j \in \{1,2,\ldots,k\}$ there is a solution $\{x_n^{(j)}\}$ of $[M_n]$ such that $\frac{x_n^{(j)}}{|x_n^{(j)}|}$ converges to an eigenvector of M, corresponding to the eigenvalue α_j. Conversely, for each non-trivial solution $\{x_n\}$ of the matrix recurrence the quotient $\frac{x_n}{|x_n|}$ converges to an eigenvector of M. The above facts were proved by O.Perron [Pe1] and H.Poincaré [Po], respectively. (In fact, Poincaré stated his result not for matrix recurrences, but only for ordinary linear recurrences.)

If we apply the above result to recurrence operators, we obtain a result that is known as 'Poincaré's theorem for difference equations'. It reads as follows:

Suppose $R \in \mathcal{R}(\mathbb{C})$ and $\chi_R(X) = c\cdot\prod_{j=1}^{k}(X-\alpha_j)$, where $c,\alpha_1,\ldots,\alpha_k \in \mathbb{C}$, $c \neq 0$, and $|\alpha_1| < |\alpha_2| < \ldots < |\alpha_k|$. Then R has divisors $S_1,\ldots,S_k \in \mathcal{R}(\mathbb{C})$ such that $\chi_{S_j}(X) = X-\alpha_j$ for $j = 1,\ldots,k$. (Or, which is equivalent, R has zeros $\{u_n^{(j)}\}$ such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha_j$ $(j = 1,\ldots,k)$).

(Note that, if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ exists for $\{u_n\} \in Z(R)$, it must be equal to a root of χ_R).

If the limit matrix M has several eigenvalues with the same absolute value, or, which amounts to the same, the characteristic polynomial has several roots with the same absolute value, it is in general not true that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ exists for $\{u_n\} \in Z(R)$ or that for a solution $\{x_n\}$ of $[M_n]$ the quotient $\frac{x_n}{|x_n|}$ converges to an eigenvector of M. For counterexamples, see for instance [Pe2], Remark 3.1.1, Proposition 5.3 and §6.1. However, if a recurrence operator R can be factorized in such a way that each factor of R has a characteristic polynomial where all roots have distinct absolute values, the behaviour of the zeros of R can be derived from the behaviour of the zeros of the factors. In particular, for second-order operators we have the following result, of which we shall make use in a later chapter.

Proposition 3.1. *Suppose* $R \in \mathfrak{R}(\mathbb{R})$ *with* $\chi_R(X) = (X-a)^2$ *for* $a \in \mathbb{R}$, $a \neq 0$, *has a (real) zero* $\{u_n\} \in Z(R)$ *such that* $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = a$.
Then $\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = a$ *for all zeros* $\{w_n\} \neq \{0\}$ *in* $Z(R)$.

Proof: R can be factorized as $R = c_n \cdot (T - p) \cdot (T - \frac{u_{n+1}}{u_n})$, where $\lim_{n\to\infty} p(n) = a$ and $c_n, p(n) \in \mathbb{R}$ $(n \in \mathbb{N})$. Without loss of generality we may assume $c_n = 1$ for all n. Put $S = u_n^{-1} \cdot I$. Then $R/S = (T - q) \cdot (T - 1)$ where $q_n = p(n) \cdot \frac{u_{n+1}}{u_{n+2}}$, hence $\lim_{n\to\infty} q_n = 1$ and $q_n \in \mathbb{R}$ $(n \in \mathbb{N})$. For $\{v_n\} \in Z(R/S)$, we have
$v_{n+1} - v_n = \lambda \cdot q_\ell \cdot \ldots \cdot q_{n-1}$ for ℓ so large that $q_n > 0$ for $n \geq \ell$ and $\lambda \in \mathbb{C}$.
Then $v_n = \mu + \lambda \cdot \sum_{m=\ell}^{n-1} q_\ell \cdot \ldots \cdot q_{m-1}$ $(n \geq \ell)$. If $\lambda = 0$, then $\frac{v_{n+1}}{v_n} = 1$ for all $n \geq \ell$. If $\lambda \neq 0$ and $\sum_{m=\ell}^{\infty} q_\ell \cdot \ldots \cdot q_{m-1}$ diverges, then

$$\lim_{n\to\infty} \frac{v_{n+1}}{v_n} = 1 + \lambda \cdot \lim_{n\to\infty} \frac{q_\ell \cdot \ldots \cdot q_{n-1}}{\mu + \lambda \cdot \sum_{m=\ell}^{n-1} q_\ell \cdot \ldots \cdot q_{m-1}} = 1 + \lim_{n\to\infty} \frac{q_\ell \cdot \ldots \cdot q_{n-1}}{\sum_{m=\ell}^{n-1} q_\ell \cdot \ldots \cdot q_{m-1}}$$

If $\lambda \neq 0$ and $\sum_{m=\ell}^{\infty} q_\ell \cdot \ldots \cdot q_{m-1}$ converges, then $v_n = \mu' - \lambda \cdot \sum_{m=n}^{\infty} q_\ell \cdot \ldots \cdot q_{m-1}$, so that

$$\lim_{n\to\infty} \frac{v_{n+1}}{v_n} = 1 + \lambda \cdot \lim_{n\to\infty} \frac{q_\ell \cdot \ldots \cdot q_{n-1}}{\mu' + \lambda \cdot \sum_{m=n}^{\infty} q_\ell \cdot \ldots \cdot q_{m-1}} = 1 \quad \text{if } \mu' \neq 0$$

and

$$\lim_{n\to\infty} \frac{v_{n+1}}{v_n} = 1 - \lim_{n\to\infty} \frac{q_\ell \cdot \ldots \cdot q_{n-1}}{\sum_{m=n}^{\infty} q_\ell \cdot \ldots \cdot q_{m-1}} \quad \text{if } \mu' = 0.$$

We show that,if $\{p_n\}$ is a sequence of positive numbers, for which

$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1$, then

$$\lim_{n\to\infty} \frac{p_n}{\sum_{k=n}^{\infty} p_k} = 0 \quad \text{if } \sum_{k=0}^{\infty} p_k \text{ converges}$$

and

$$\lim_{n\to\infty} \frac{p_n}{\sum_{k=0}^{n-1} p_k} = 0 \quad \text{if } \sum_{k=0}^{\infty} p_k \text{ diverges.}$$

First suppose that the sum diverges. Choose $\varepsilon > 0$. Take N so large that $\frac{p_{n+1}}{p_n} < 1 + \varepsilon$ for $n > N$. Then, for $n > N$,

$$\frac{\sum_{k=0}^{n-1} p_k}{p_n} = \frac{\sum_{k=0}^{N-1} p_k + \sum_{k=N}^{n-1} p_k}{p_n} > \sum_{k=N}^{n-1} (p_k/p_n) > \sum_{j=1}^{n-N} (\frac{1}{1+\varepsilon})^j.$$

Hence,

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} (p_k/p_n) \geq \sum_{j=1}^{\infty} (1+\varepsilon)^{-j} = \frac{1}{\varepsilon}.$$

If $\sum_{k=0}^{\infty} p_k$ converges, we choose N so large that $\frac{p_{n+1}}{p_n} > 1 - \varepsilon$ for $n > N$. Then, for $n > N$,

$$\sum_{k=n}^{\infty} (p_k/p_n) > \sum_{j=0}^{\infty} (1-\varepsilon)^j = \frac{1}{\varepsilon}.$$

Thus, $\lim_{n\to\infty} \frac{v_{n+1}}{v_n} = 1$ for $\{v_n\} \neq \{0\}$, $\{v_n\} \in Z(R/S)$.

For $\{w_n\} \in Z(R)$, we have $\left\{ \frac{w_n}{u_n} \right\} \in Z(R/S)$, so that, if $\{w_n\} \neq \{0\}$

$$\lim_{n\to\infty} \frac{w_{n+1}}{w_n} \cdot \frac{u_n}{u_{n+1}} = 1, \text{ which implies } \lim_{n\to\infty} \frac{w_{n+1}}{w_n} = a. \qquad \square$$

Remark 3.1.1. If $R \in \mathfrak{R}(\mathbb{C})$, the assertion of Proposition 3.1 is not generally true. In order to see this, consider the following example:

Let $N_1, N_2, \ldots$ be a monotonically increasing sequence of positive integers such that $N_n \to \infty$ as $n \to \infty$. Let $R = (T - \exp(i\phi_n)) \cdot (T - 1)$ where $\phi_n = \frac{2\pi}{N_j}$ for $n \geq 0$ and $N_1 + N_2 + \ldots + N_{j-1} \leq n < N_1 + N_2 + \ldots + N_j$ $(j \in \mathbb{N})$.
Clearly, $\chi_R(X) = X^2 - 2X + 1$ and R has a zero {1}. Put $v_n = \sum_{k=0}^{n-1} \exp(i\Phi_k)$

$(n \geq 0)$, where $\Phi_n \equiv \frac{2\pi k}{N}$ mod 2π for $n = N_1 + \dots + N_{j-1} + k$ $(0 \leq k \leq N_j)$. Then $\{v_n\} \in Z(R)$, $\{v_n\} \neq \{0\}$. Further, for all j, $v_{N_1+\dots+N_j} = 0$. So $\lim_{n\to\infty} \frac{v_{n+1}}{v_n}$ does not exist. For a zero $\{u_n\} \in Z(R)$, we have $\{u_n\} = \lambda\cdot\{v_n\} + \mu\cdot\{1\}$. If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ exists, it must be equal to 1. On the other hand, if we take $n = N_1 + \dots + N_j$, then $\lim_{j\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\lambda\cdot\exp(i\Phi_n) + \mu}{\mu} = \frac{\lambda + \mu}{\mu}$. Hence, λ must be zero. So only for $\{u_n\} = \{\mu\} \neq \{0\}$ does $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ exist.

In this chapter we shall derive a factorization theorem for matrix recurrences. This result will enable us to derive a generalization of Poincaré's theorem for the solutions of a matrix recurrence and, consequently, for the zeros of a recurrence operator.

Theorem 3.2. *Put*

$$M = \begin{bmatrix} R_1 & & & 0 \\ & R_2 & & \\ & & \ddots & \\ 0 & & & R_\ell \end{bmatrix}$$

where $R_j \in K^{k_j,k_j}$ $(j = 1,\dots,\ell)$, $\sum_{j=1}^{\ell} k_j = k$ *,and all eigenvalues in* $\mathbb{C}$ *of* R_j *have smaller absolute values than all eigenvalues in* $\mathbb{C}$ *of* R_{j+1} $(j = 1,\dots,\ell-1)$. *Further, let* $[M_n] \in \mathcal{M}(K)$, *where* $M_n \in K^{k,k}$, M_n *invertible and* $\lim M_n = M$. *Then there exists a sequence of matrices* $\{B_n\}$ *with* $B_n \in K^{k,k}$, B_n *invertible* $(n \in \mathbb{N})$, *such that*

(3.1) $\lim B_n = I$

(3.2) $$B_{n+1}\cdot M_n\cdot B_n^{-1} = \begin{bmatrix} R_{1n} & & & 0 \\ & R_{2n} & & \\ & & \ddots & \\ 0 & & & R_{\ell n} \end{bmatrix}$$

where $R_{jn} \in K^{k_j,k_j}$ *and* $\lim R_{jn} = R_j$ $(j = 1,\dots,\ell)$.

Applying Theorem 3.2 yields the following result for the zeros of recurrence operators:

Theorem 3.3. *Suppose* $R \in \mathfrak{R}(K)$, *where* $K = \mathbb{R}$ *or* $K = \mathbb{C}$, *and* $\chi_R(X) = c \cdot \prod_{j=1}^{\ell} P_j(X)$, *with* $c \in \mathbb{C}$, $c \neq 0$, *and* $P_j(X) = \prod_{i=1}^{d_j} (X - \alpha_{ji})$ *such that* $\alpha_{ji} \in \mathbb{C}$ $(j = 1,\dots,\ell;$ $i = 1,\dots,d_j)$, $\alpha_j = |\alpha_{j1}| = \dots = |\alpha_{jd_j}|$ *and* $\alpha_1,\dots,\alpha_\ell$ *are distinct non-negative real numbers. Then* $R = c \cdot S_1 \cdot S_2 \cdot \dots \cdot S_\ell$ *where* $S_1, S_2, \dots, S_\ell \in \mathfrak{R}(K)$ *and* $\chi_{S_i} = P_i$ $(i = 1,2,\dots,\ell)$.

Corollary 3.4. *If* $R \in \mathfrak{R}(K)$ *for* $K = \mathbb{R}$ *or* $\mathbb{C}$, *and* $\chi_R = c \cdot \prod_{j=1}^{k} (X - \alpha_j)$ *and* $|\alpha_1| \neq |\alpha_j|$ *for* $j = 2,\dots,k$, *then* R *has a zero* $\{u_n\}$ *such that* $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha_1$. *(Note that* $u_n \in K$ *by definition).*

Proof: By Theorem 3.3, $R = V \cdot S$, where $\chi_S(X) = X - \alpha_1$. Hence, $S = q \cdot (T - r)$, with $\lim_{n\to\infty} r(n) = \alpha_1$. For $\{v_n\} \in Z(S)$, $\{v_n\} \neq \{0\}$, we have $\frac{v_{n+1}}{v_n} = r(n)$ $(n \in \mathbb{Z}_{\geq m})$, so that $\lim \frac{v_{n+1}}{v_n} = \alpha_1$. Further, since $Z(S) \subset Z(R)$, $\{v_n\}$ is also a zero of R. □

Note that taking $d_1 = d_2 = \dots = d_\ell = 1$ in Theorem 3.3 yields Poincaré's Theorem.

Before we prove Theorem 3.2 and Theorem 3.3, we need some more definitions and facts. This will be the subject of the next section.

§2. Some more facts about matrix recurrences.

Unless stated otherwise, we take for the field K either $\mathbb{R}$ or $\mathbb{C}$. Let $[M_n] \in \mathcal{M}(K)$. ($M_0, M_1, \dots$ are non-singular $k \times k$ matrices by definition). If $\{x_n^{(1)}\},\dots,\{x_n^{(k)}\}$ are solutions of the matrix recurrence, we can write

(3.3) $$M_n \cdot X_n = X_{n+1}$$

where $x_n^{(i)} = \begin{pmatrix} x_{n1}^{(i)} \\ \vdots \\ x_{nk}^{(i)} \end{pmatrix} \in K^k$ and $X_n = \left(x_n^{(1)}\ x_n^{(2)} \dots x_n^{(k)} \right) \in K^{k,k}$

$(i = 1,\dots,k)$.

If we choose the k solutions such that $x_0^{(1)},\dots,x_0^{(k)}$ are linearly independent over K, we see by (3.3) that then $\{x_n^{(1)}\},\dots,\{x_n^{(k)}\}$ are linearly independent.

In the sequel we shall need to speak about minors of the matrices M_n, X_n, and $M = \lim M_n$ and we shall introduce a simple notation for them.

Let $I = \{i_1,\dots,i_m\}$ and $J = \{j_1,\dots,j_m\}$ be subsets of $\{1,\dots,k\}$ with m elements, such that $i_1 < i_2 < \dots < i_m$ and $j_1 < j_2 < \dots < j_m$. We denote the minor determinant

$$\begin{vmatrix} a_{i_1 j_1} & \cdots\cdots & a_{i_1 j_m} \\ \vdots & & \vdots \\ a_{i_m j_1} & \cdots\cdots & a_{i_m j_m} \end{vmatrix}$$

of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

by $D_{I,J}^{(m)}(A)$.

Let $I_1, I_2, \dots I_\mu$ ($\mu = \binom{k}{m}$) be the μ subsets with m elements of $\{1,\dots,k\}$, ordered in such a way that, if $i_1 < \dots < i_m$ and $j_1 < \dots < j_m$, then $\{i_1,\dots,i_m\} < \{j_1,\dots,j_m\}$ if $i_\ell = j_\ell$ for $1 \le \ell \le L-1$ and $i_L < j_L$ for some $L \in \{1,\dots,m\}$. The $\mu \times \mu$-matrix $(b_{ij})_{i,j=1,\dots,m}$ with $b_{ij} = D_{I_i,I_j}^{(m)}(A)$ is denoted by $A^{(m)}$. Note that $A^{(k)} = \det A$.

Lemma 3.5. *Let* $A,B,C \in \mathbb{C}^{k,k}$ $(k \in \mathbb{N})$ *such that* $A \cdot B = C$ *and let* $\ell \in \mathbb{N}$, $\ell \le k$. *Then* $A^{(\ell)} \cdot B^{(\ell)} = C^{(\ell)}$ *and* $\det A^{(\ell)} = (\det A)^\nu$ *where* $\nu = \binom{m-1}{\ell-1}$.

Proof: See for instance [K], page 321. □

The following lemma applies to the matrix recurrences $[M_n^{(m)}]$.

Lemma 3.6. *Let* $[M_n] \in \mathcal{M}(K)$ *and* $\{x_n^{(1)}\},\dots,\{x_n^{(k)}\}$ *a basis of solutions. A basis of solutions of* $[M_n^{(m)}]$ *is given by the* $\binom{k}{m}$ *column vectors of* $X_n^{(m)}$, *where* $X_n = (x_n^{(1)},\dots,x_n^{(k)})$. $(1 \le m \le k)$.

Proof: $M_n \cdot X_n = X_{n+1}$ and $\det X_n \neq 0$ $(n \geq 0)$. By Lemma 3.5, $M_n^{(m)} \cdot X_n^{(m)} = X_{n+1}^{(m)}$ and $\det X_n^{(m)} \neq 0$ $(n \geq 0)$. Moreover, the matrix $X_n^{(m)}$ has the required dimension. □

Remark 3.2.1. If A has eigenvalues $\alpha_1,\dots,\alpha_k$ (written according to multiplicities), then $A^{(m)}$ has as its eigenvalues all numbers of the form $\alpha_{i_1} \cdot \ldots \cdot \alpha_{i_m}$, where $1 \leq i_1 < \ldots < i_m \leq k$.

Finally, we introduce the norm of a matrix. Let $A \in K^{k,\ell}$ ($K = \mathbb{R}$ or $K = \mathbb{C}$), $k,\ell \in \mathbb{N}$. The *norm* $\| A \|$ of the matrix A is defined as

$$(3.4) \qquad \| A \| = \max_{x \neq 0} \frac{|Ax|}{|x|}.$$

The norm has the following properties. Let $A \in K^{k,\ell}, B \in K^{m,n}$.

1. $\| A \| = 0$ if and only if $A = 0$.
2. For $\lambda \in K$, $\| \lambda A \| = |\lambda| \cdot \| A \|$.
3. $\| A + B \| \leq \| A \| + \| B \|$ if $k = m$, $\ell = n$.
4. $\| A \cdot B \| \leq \| A \| \cdot \| B \|$ if $\ell = m$.
5. If $k = \ell$ and α is an eigenvalue of A, then $\| A \| \geq |\alpha|$.

It is in general not true that $\| A \| = \max |\alpha|$, where the maximum is taken over the eigenvalues of A. Nevertheless, the following fact is true:

Lemma 3.7. *Let $\varepsilon > 0$ and $A \in K^{k,k}$. Let α be one of the complex eigenvalues of A with the greatest absolute value. Then there exists a matrix $U \in K^{k,k}$ such that $\| U^{-1}AU \| \leq |\alpha| + \varepsilon$.*

Proof: By §1.6, we can find a matrix $V \in K^{k,k}$ such that $V^{-1}AV = D + \mathcal{J}$, where D is a diagonal matrix of the form

$$D = \begin{pmatrix} D_1 & & & & \\ & D_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & D_\ell \end{pmatrix}$$

where $D_j \in \mathbb{C}$ if $K = \mathbb{C}$ and $D_j \in \mathbb{R}$ or $D_j \in \mathbb{R}^{2,2}$ if $K = \mathbb{R}$ (see §1.6 for details) and

$$\mathcal{J} = \begin{bmatrix} 0 & J_1 & 0 & \cdot & 0 \\ 0 & 0 & J_2 & & \cdot \\ \vdots & & & & \cdot \\ & & & & J_{\ell-1} \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

where $J_j = 0$ or 1 if $D_j \in \mathbb{R}$ and $J_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if $D_j \in \mathbb{R}^{2,2}$.

Further, we can find a diagonal matrix W such that $W^{-1}JW = \varepsilon\cdot\mathcal{J}$. In fact, take W as follows:

$$W = \begin{bmatrix} E_1 & & & \\ & E_2 & & \\ & & \cdot & \\ & & & E_\ell \end{bmatrix}$$

where $E_j = \varepsilon^{j-1}$ or $E_j = \varepsilon^{j-1}\cdot I$, where I is the 2×2 identity matrix $(1 \le j \le \ell)$, the choice between $E_j \in \mathbb{R}$ and $E_j \in \mathbb{R}^{2,2}$ depending on the fact whether $D_j \in \mathbb{R}$ or $D_j \in \mathbb{R}^{2,2}$. (If $K = \mathbb{C}$, then obviously $E_j \in \mathbb{R}$ for all j.)

Take $U = V\cdot W$. Then $U^{-1}AU = D + \varepsilon\cdot J$. For $j = 1,\dots,\ell$,

$$|(U^{-1}AUx)_j| \le |\alpha|\cdot|x_j| + \varepsilon\cdot|x_{j+1}|$$

where x is a vector such that $x^T = (\, x_1\ x_2\ \cdots\ x_\ell\,)$ with $x_i \in \mathbb{C}$ if $K = \mathbb{C}$ and, if $K = \mathbb{R}$, then $x_i \in \mathbb{R}$ if $D_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^2$ if $D_i \in \mathbb{R}^{2,2}$.

Thus,

$$|U^{-1}AUx|^2 \le |\alpha|^2\cdot|x|^2 + \varepsilon^2\cdot|x|^2 + 2|\alpha|\cdot\varepsilon\cdot\sum_{j=1}^{\ell-1}|x_j|\cdot|x_{j+1}| \le (|\alpha| + \varepsilon)^2\cdot|x|^2. \qquad \square$$

Lemma 3.8. *Let* $A,B \in K^{k,k}$ *such that* A *is non-singular and* $\| B \| < \| A^{-1}\|^{-1}$. *Then* $A + B$ *is non-singular and*

$$\|(A + B)^{-1}\| \le \frac{1}{\| A^{-1}\|^{-1} - \| B \|}.$$

Proof: Take $x \in K^k$, $x \ne 0$. Then

$$\frac{|Ax|}{|x|} = \frac{|Ax|}{|A^{-1}(Ax)|} \ge \frac{1}{\| A^{-1}\|} \quad \text{and} \quad \frac{|Bx|}{|x|} \le \| B \|.$$

Hence,

$$\min_{x\ne 0} \frac{|Ax| - |Bx|}{x} \ge \| A^{-1}\|^{-1} - \| B \| > 0$$

and

$$|(A + B)x| \ge ||Ax| - |Bx|| > 0,$$

so that $A + B$ is non-singular. Moreover,

$$\|(A + B)^{-1}\| = \max_{y \neq 0} \frac{|(A + B)^{-1}y|}{|y|} = \max_{x \neq 0} \frac{|x|}{|(A + B)x|}$$

$$\leq \frac{1}{\min\limits_{x \neq 0} \frac{|Ax| - |Bx|}{|x|}} \leq \frac{1}{\| A^{-1}\|^{-1} - \| B \|}.$$ □

Remark 3.2.2. If $R \in \mathfrak{R}(K)$ and $R = p_k T^k + \ldots + p_1 T + p_0$, we define the *norm* $N_n(R)$ of R as $N_n(R) = \max\{|p_i(n)| \mid 0 \leq i \leq k \}$.

§3. The main theorem.

Now we are ready to prove the main result of this chapter. Again, let K be either $\mathbb{R}$ or $\mathbb{C}$. For a matrix $A \in K^{k,k}$ we denote the entry in the i-th row and the j-th column by A_{ij} $(i,j \in \{1,\ldots,k\})$. We shall prove the theorem in several steps.

Lemma 3.9. *Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of invertible matrices in $K^{k,k}$ such that $\lim A_n^{-1} \cdot A_{n+1} = A$ and A has only eigenvalues in $\mathbb{C}$ with absolute values smaller than one.*
Then the series $\sum_{\ell=0}^{\infty} A_\ell$ converges (entrywise) and

$$\lim A_n^{-1} \cdot \sum_{\ell=0}^{\infty} A_{n+\ell} = (I - A)^{-1}.$$

Proof: First suppose that A is in complex Jordan normal form. Put $E_n = A_n^{-1} \cdot A_{n+1} - A$. Let B be the matrix that is obtained from A by replacing the elements on the diagonal by their absolute values. For N large enough there exists a matrix E such that, for all $i,j \in \{1,\ldots,k\}$ and $n \geq N$, $|(E_n)_{ij}| \leq E_{ij}$ and such that $B + E$ has still eigenvalues in $\mathbb{C}$ with absolute values smaller than one. Then, for $p \in \mathbb{N}$, $n \geq N$, we define

$$G_{np} = A_n^{-1} \cdot \sum_{\ell=0}^{p} A_{n+\ell} - (I - A)^{-1} = \sum_{\ell=0}^{p} \prod_{m=0}^{\ell-1} (A + E_{n+m}) - (I - A)^{-1}$$

and

$$H_p = \sum_{\ell=0}^{p} (B + E)^{\ell} - (I - B)^{-1}.$$

Hence H_p converges to a matrix H as $p \to \infty$.
Now choose $\varepsilon > 0$. Put $\varepsilon' = \frac{\varepsilon}{\max\limits_{i,j} |H_{ij}|}$. Since $\lim E_n = 0$,

$$|(E_n)_{ij}| < \varepsilon' \cdot E_{ij} \text{ for n large enough, } i,j \in \{1,\ldots,k\}.$$

Hence,

$$|(G_{np})_{ij}| < \varepsilon' \cdot H_{ij} < \varepsilon \quad \text{for all p,i,j and n large enough.}$$

If A is not in complex Jordan normal form, then there is a matrix $U \in \mathbb{C}^{k,k}$ such that $U^{-1}AU$ is in complex Jordan normal form. By what has been proved above,

$$\lim U^{-1}A_n^{-1} \cdot \sum_{\ell=0}^{\infty} A_{n+\ell} \cdot U = (I - U^{-1}AU)^{-1} = (U^{-1}(I - A)U)^{-1}$$

so that

$$U^{-1} \cdot (\lim A_n^{-1} \cdot \sum_{\ell=0}^{\infty} A_{n+\ell}) \cdot U = U^{-1}(I - A)^{-1}U$$

and the result follows. □

Lemma 3.10. *Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of matrices in $K^{k,k}$, converging to some matrix A. Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of vectors in K^k with $\lim_{n\to\infty} \varepsilon_n = 0$. Then the following assertion holds:*

If A has only eigenvalues in $\mathbb{C}$ with absolute values smaller than one, every sequence $\{x_n\}$ satisfying the following inhomogeneous recurrence relation

$$x_{n+1} = A_n x_n + \varepsilon_n \qquad (n \in \mathbb{N})$$

converges to zero.

Proof: Let β be an eigenvalue of A with maximal absolute value. Let $\varepsilon > 0$ be such that $|\beta| + 4\varepsilon < 1$. We can find a matrix $U \in K^{k,k}$ such that $U^{-1}AU = D + \varepsilon \cdot J$, where $\| D \| = |\beta|$ and $\| J \| \le 1$. Put $U^{-1}A_nU = D_n + \varepsilon \cdot J$. Then $\{D_n\}_{n=0}^{\infty}$ is a sequence of matrices converging to D. Further, let for $n \ge 0$,

$$y_n = U^{-1}x_n.$$

Then $\{y_n\}$ satisfies the equation

$$y_{n+1} = (D_n + \varepsilon \cdot J)y_n + U^{-1}\varepsilon_n \qquad (n \ge 0).$$

Let N be so large that for $n \ge N$

$$\| D_n \| \le |\beta| + \varepsilon.$$

Then,

$$|y_{n+1}| \le (|\beta| + 2\varepsilon) \cdot |y_n| + \delta_n \qquad (n \ge N),$$

where $\delta_n = |U^{-1}\varepsilon_n|$. Hence, $\delta_n \to 0$ as $n \to \infty$. Consider the recurrence relation

$$z_{n+1} = (|\beta| + 2\varepsilon)z_n + \delta_n \qquad (n \ge N)$$

and take $z_N = |y_N|$ as the initial value. Then $|y_n| \le z_n$ for $n \ge N$. Moreover,

$$z_{n+1} \le \max \{(|\beta| + 2\varepsilon)(1 + \varepsilon)z_n,\ \delta_n(1 + \tfrac{1}{\varepsilon})\}$$

for $n \ge N$. Since $(|\beta| + 2\varepsilon)(1 + \varepsilon) < 1$, we obtain that $z_n \to 0$, from which it follows that $x_n \to 0$ as $n \to \infty$, irrespective of the initial value. □

Lemma 3.11. *Let* A *be a matrix of the form* $\begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$, *where* $R \in K^{\ell,\ell}$ *and* $S \in K^{m,m}$ $(m = k-\ell)$ *such that all eigenvalues of* R *in* $\mathbb{C}$ *have smaller moduli than all eigenvalues of* S *in* $\mathbb{C}$. *Further, let* $\{D_n\}_{n=0}^{\infty}$ *be a sequence of matrices in* $K^{k,k}$, *such that* $\lim D_n = 0$ *and* $A + D_n$ *invertible for all* n. *Then there exists a sequence* $\{B_n\}_{n=0}^{\infty}$ *of matrices in* $K^{k,k}$ *with*

$$B_n = \begin{bmatrix} I & C_n \\ 0 & I \end{bmatrix}$$

where $C_n \in K^{\ell,m}$ $(n \geq 0)$ *and* $\lim C_n = 0$, *such that*

$$(B_{n+1} \cdot (A + D_n) \cdot B_n^{-1})_{ij} = 0$$

for all n *large enough and for* $i \in \{1,\ldots,\ell\}$ *and* $j \in \{\ell+1,\ldots,k\}$, *and*

$$\| B_{n+1} - I \| \leq \delta \cdot \| B_n - I \| + c \cdot \| D_n \|$$

for some $\delta < 1$, $c \in \mathbb{R}$, $c > 0$ *and for all* n *large enough.*

Proof: Note that it is sufficient to prove the lemma for any conjugate matrix of A. So we may suppose that

$$A + D_n = \begin{bmatrix} R_n^* & Q_n \\ P_n & S_n^* \end{bmatrix}$$

with $R_n^* \in K^{\ell,\ell}$ $(n \geq 0)$ and $\| R \| \leq |\beta| + \varepsilon$, $\| S^{-1} \| \leq \frac{1}{|\gamma| - \varepsilon}$ where β is an eigenvalue of R with greatest absolute value and γ an eigenvalue of S with smallest absolute value, and ε is such that $0 < \varepsilon < (|\gamma| - |\beta|)/6$. Then, for $n \geq N$,

$$\| R_n^* \| < |\beta| + 2\varepsilon, \ \|(S_n^*)^{-1}\| < \frac{1}{|\gamma| - 2\varepsilon}, \ \| P_n \| < \varepsilon, \ \| Q_n \| < \varepsilon.$$

Now choose $C_N = 0$ and define $\{C_n\}_{n>N}$ in the following way:

$$C_{n+1} = (R_n^* \cdot C_n - Q_n) \cdot (S_n^* - P_n \cdot C_n)^{-1}. \tag{3.5}$$

We show that $S_n^* - P_n \cdot C_n$ is indeed invertible for $n \geq N$. Suppose that $C_N, \ldots, C_n$ are well-defined and that $\| C_m \| < 1$ for $N \leq m \leq n$. Then

$$\| P_n \cdot C_n \| < \varepsilon < |\gamma| - 2\varepsilon < \|(S_n^*)^{-1}\|^{-1}.$$

Hence, by Lemma 3.8, $S_n^* - P_n \cdot C_n$ is non-singular, and

$$\|(S_n^* - P_n \cdot C_n)^{-1}\| \leq \frac{1}{\|(S_n^*)^{-1}\|^{-1} - \| P_n \cdot C_n \|} < \frac{1}{|\gamma| - 3\varepsilon}.$$

Thus, C_{n+1} is well-defined, and

$$\| C_{n+1} \| \leq (\| C_n \| \cdot \| R_n^* \| + \| Q_n \|) \cdot \|(S_n^* - P_n \cdot C_n)^{-1}\| < \frac{|\beta| + 3\varepsilon}{|\gamma| - 3\varepsilon} < 1.$$

Moreover, we have the following inequality :

$$(3.6)\qquad \| C_{n+1} \| \le \| C_n \| \cdot \frac{|\beta| + 3\varepsilon}{|\gamma| - 3\varepsilon} + \| Q_n \| \cdot \frac{1}{|\gamma| - 3\varepsilon}.$$

Let $\{y_n\}_{n\ge N}$ be the sequence of positive real numbers such that

$$(3.7)\qquad y_{n+1} = y_n \cdot \frac{|\beta| + 3\varepsilon}{|\gamma| - 3\varepsilon} + \| Q_n \| \cdot \frac{1}{|\gamma| - 3\varepsilon}$$

and $y_N = 0$. Then $\| C_n\| \le y_n$ for $n \ge N$. Further, we can apply Lemma 3.10 to (3.7), with $K = \mathbb{R}$ and $k = 1$, since

$$\frac{|\beta| + 3\varepsilon}{|\gamma| - 3\varepsilon} < 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\| Q_n \|}{|\gamma| - 3\varepsilon} = 0$$

and find that $\lim_{n\to\infty} y_n = 0$, so that $\lim_{n\to\infty} C_n = 0$ as well.
Put

$$B_n = \begin{bmatrix} I & C_n \\ 0 & I \end{bmatrix} \qquad (n \ge N).$$

Then

$$B_n^{-1} = \begin{bmatrix} I & -C_n \\ 0 & I \end{bmatrix} \quad \text{and} \quad \lim B_n = I.$$

Also,

$$B_{n+1} \cdot (A + D_n) \cdot B_n^{-1} = \begin{bmatrix} R_n + C_{n+1} \cdot P_n & 0 \\ P_n & S_n - P_n \cdot C_n \end{bmatrix}.$$

The last assertion of the theorem follows from (3.6) and the fact that $\| Q_n \| \ll \| D_n \|$. □

Lemma 3.12. *Let* $\{A_n\}$, $\{B_n\}$ *be sequences of non-singular matrices in* $K^{k,k}$ *and* $K^{\ell,\ell}$, *respectively, and* $\lim A_n = A$, $\lim B_n = B$, *while all eigenvalues in* $\mathbb{C}$ *of* A *have smaller absolute values than all eigenvalues in* $\mathbb{C}$ *of* B. *Further, let* $\{D_n\}$ *be a sequence of matrices in* $K^{\ell,k}$ *converging to the zero matrix. Then the recurrence relation*

$$(3.8)\qquad X_{n+1} \cdot A_n = B_n \cdot X_n + D_n \qquad (n \in \mathbb{Z}_{\ge 0})$$

has a solution $\{C_n\}$, $C_n \in K^{\ell,k}$, *such that* $\lim C_n = 0$ *and*

$$\| C_n \| \le c' \cdot \sum_{k=n}^{\infty} \| D_k \| \cdot \delta^{k-n}$$

for some number $0 < \delta < 1$ *and some constant* c' *independent of* n.

Proof: Solving (3.8), we find

$$(3.9)\qquad (B_{n-1} \cdot \ldots \cdot B_0)^{-1} \cdot C_n \cdot (A_{n-1} \cdot \ldots \cdot A_0) =$$

$$= C_0 + \sum_{k=0}^{n-1} (B_k\cdot\ldots\cdot B_0)^{-1}\cdot D_k\cdot(A_{k-1}\cdot\ldots\cdot A_0).$$

By multiplying A, A_n, B, B_n, D_n $(n \geq 0)$ by a suitable constant $c \in K^*$, we can, by Lemma 3.7, find numbers $\varepsilon > 0$, $m \in \mathbb{N}$ and matrices $U \in K^{k,k}$, $V \in K^{\ell,\ell}$, such that

(3.10) $\quad \| c\cdot U^{-1}A_jU \| < 1 - \varepsilon \quad$ and $\quad \| c^{-1}\cdot V^{-1}B_j^{-1}V \| < 1 - \varepsilon \quad$ for $j \geq m$.

Using the properties of the matrix norm, we obtain that the sum

$$\sum_{k=m}^{\infty} (B_k\cdot\ldots\cdot B_m)^{-1}\cdot D_k\cdot(A_{k-1}\cdot\ldots\cdot A_m)$$

converges to some matrix in $K^{\ell,k}$ for any $m \in \mathbb{N}$. Now choose

$$C_0 = -\sum_{k=0}^{\infty} (B_k\cdot\ldots\cdot B_0)^{-1}\cdot D_k\cdot(A_{k-1}\cdot\ldots\cdot A_0)$$

as the initial value for the recurrence sequence defined by (3.8). Then, since all A_n $(n \in \mathbb{N})$ are invertible,

$$C_n = \sum_{k=n}^{\infty} (B_k\cdot\ldots\cdot B_n)^{-1}\cdot D_k\cdot(A_{k-1}\cdot\ldots\cdot A_n). \qquad (3.11)$$

Since $\{D_n\}$ converges to zero, $\{C_n\}$ converges to zero as well. The last inequality now follows easily from (3.10) and (3.11). □

We now come to the proof of Theorem 3.2.

Proof: We proceed by induction to ℓ. For $\ell = 1$, take $B_n = I$ for all n. Suppose the assertion is true for $\ell = 1,\ldots,L-1$. Put

$$S = \begin{bmatrix} R_2 & & & 0 \\ & R_3 & & \\ & & \ddots & \\ 0 & & & R_L \end{bmatrix}.$$

Then

$$M = \begin{bmatrix} R_1 & 0 \\ 0 & S \end{bmatrix}$$

and all eigenvalues in $\mathbb{C}$ of R have smaller absolute values than all eigenvalues in $\mathbb{C}$ of S. By Lemma 3.11, there exists a sequence $\{B_n'\}$, $B_n' \in K^{k,k}$, such that

$$\lim B_n' = I \qquad (3.12)$$

$$B_{n+1}'\cdot M_n\cdot B_n'^{-1} = \begin{bmatrix} R_n^* & 0 \\ Q_n^* & S_n^* \end{bmatrix} \qquad (3.13)$$

where $Q_n^* \in K^{k-k_1,k_1}$. Since R_n^* and S_n^* are non-singular and $\lim R_n^* = R_1$, $\lim S_n^* = S$, $\lim Q_n^* = 0$, Lemma 3.12 yields that the recurrence equation

$$X_{n+1} \cdot R_n^* = S_n^* \cdot X_n - Q_n^* \qquad (n \geq 0)$$

has a solution $\{C_n\}$ such that $\lim C_n = 0$.

Put $B_n^* = \begin{bmatrix} I & 0 \\ C_n & I \end{bmatrix}$ $(n \in \mathbb{N})$. Then

$$(3.14) \qquad B_{n+1}^* \cdot B'_{n+1} \cdot M_n \cdot (B_n^* \cdot B'_n)^{-1} = \begin{bmatrix} R_n^* & 0 \\ 0 & S_n^* \end{bmatrix},$$

$$(3.15) \qquad \lim B_n^* \cdot B'_n = I.$$

By the induction hypothesis, there exist matrices $F_n \in K^{k-k_1,k-k_1}$ $(n \in \mathbb{N})$ such that

$$\lim F_n = I,$$

$$F_{n+1} \cdot S_n^* \cdot F_n^{-1} = \begin{bmatrix} R_{2n} & & & 0 \\ & R_{3n} & & \\ & & \ddots & \\ 0 & & & R_{Ln} \end{bmatrix}$$

where $\lim R_{jn}^* = R_j$ $(j = 2,\ldots,n)$.

Put $B_n = \begin{bmatrix} I & 0 \\ 0 & F_n \end{bmatrix} \cdot B_n^* \cdot B'_n$ $(n \in \mathbb{N})$. Then $B_n \in K^{k,k}$ and

$$(3.16) \qquad \lim B_n = I,$$

$$(3.17) \qquad B_{n+1} \cdot M_n \cdot B_n^{-1} = \begin{bmatrix} I & 0 \\ 0 & F_{n+1} \end{bmatrix} \cdot \begin{bmatrix} R_n^* & 0 \\ 0 & S_n^* \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & F_n^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} R_n^* & & & 0 \\ & R_{2n} & & \\ & & \ddots & \\ 0 & & & R_{Ln} \end{bmatrix}. \qquad \square$$

The following theorem prepares the proof of Theorem 3.3.

Theorem 3.13. *Let* $1 \leq \ell \leq k$. *Let* $[M_n] \in \mathcal{M}(K)$ *and* $\lim M_n = M$. *Suppose* M *has the form*

(3.18) $$M = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$$

where $R \in K^{\ell,\ell}$ *and* $S \in K^{k-\ell,k-\ell}$, R *and* S *have eigenvalues* $\alpha_1,\ldots\alpha_\ell$ *and* $\alpha_{\ell+1},\ldots\alpha_k$ *respectively (counted according to their multiplicities) and* $|\alpha_i| \neq |\alpha_j|$ *if* $i \in \{1,\ldots,\ell\}$ *and* $j \in \{\ell+1,\ldots,k\}$. *Then there are* ℓ *linearly independent solutions* $\{x_n^{(1)}\},\ldots,\{x_n^{(\ell)}\}$ *of* $[M_n]$ *such that, for* $X_n = (x_n^{(1)},\ldots,x_n^{(\ell)})$,

(3.19) $$\lim_{x\to\infty} \frac{D_{I,I}^{(\ell)}(X_{n+1})}{D_{I,I}^{(\ell)}(X_n)} = \alpha_1\cdot\ldots\cdot\alpha_\ell,$$

(3.20) $$\lim_{x\to\infty} \frac{D_{J,I}^{(\ell)}(X_n)}{D_{I,I}^{(\ell)}(X_n)} = 0,$$

where $I = \{1,\ldots,\ell\}$ *and* J *is any subset of* $\{1,\ldots,k\}$ *with* ℓ *elements, different from* I.

Proof: First suppose that

$$M = \begin{bmatrix} R_1 & & & 0 \\ & R_2 & & \\ & & \ddots & \\ 0 & & & R_m \end{bmatrix}$$

where all eigenvalues of R_j have smaller absolute values than all eigenvalues of R_{j+1} $(j = 1,\ldots,m-1)$, and $R_j \in K^{k_j,k_j}$, $\sum_{j=1}^{m} k_j = k$. By Theorem 3.2, there exists a sequence $\{B_n\}$, $B_n \in K^{k,k}$, such that

$$\lim B_n = I$$

and

$$B_{n+1}M_nB_n^{-1} = \begin{bmatrix} R_{1n} & & & 0 \\ & R_{2n} & & \\ & & \ddots & \\ 0 & & & R_{mn} \end{bmatrix} \qquad (n \in \mathbb{N})$$

where $R_{jn} \in K^{k_j,k_j}$ and $\lim R_{jn} = R_j$ $(j = 1,\ldots,m)$. Suppose that each of the R_j takes either all of its eigenvalues from the set $\{\alpha_1,\ldots,\alpha_m\}$ or from the set $\{\alpha_{m+1},\ldots,\alpha_k\}$. For $j = 1,\ldots,\ell$, the matrix recurrence $[R_{jn}] \in \mathcal{M}(K)$ has k_j linearly independent solutions $\{y_n^{(p)}\}$, with

$p - \ell_j = p - \sum_{i=1}^{j-1} k_i \in \{1,\dots,k_j\}$, and, for $p-\ell_j \in \{1,\dots,k_j\}$, $x_n^{(p)} = \begin{bmatrix} x_{n1}^{(p)} \\ \vdots \\ x_{nk}^{(p)} \end{bmatrix}$,

where $\begin{bmatrix} x_{n,\ell_j+1}^{(p)} \\ \vdots \\ x_{n,\ell_{j+1}}^{(p)} \end{bmatrix} = y_n^{(p)}$ and $x_{ni}^{(p)} = 0$ if $i-\ell_j \notin \{1,\dots,k_j\}$.

Put $X_n = (x_n^{(1)},\dots,x_n^{(k)})$ $(n \in \mathbb{N})$. Then,

$$B_{n+1}\cdot M_n \cdot B_n^{-1}\cdot X_n = X_{n+1} \quad (n \in \mathbb{N})$$

and, for $I_j = \{\ell_j+1,\dots,\ell_{j+1}\}$ and $J \in \{1,\dots,k\}$, with $|J| = k_j$, $J \neq I_j$ we have

(3.21) $$\frac{D_{I_j,I_j}^{(k_j)}(X_{n+1})}{D_{I_j,I_j}^{(k_j)}(X_n)} = \det R_{jn},$$

(3.22) $$D_{J,I_j}^{(k_j)}(X_n) = 0.$$

Note that $0 \neq \det R_{jn} \longrightarrow \det R_j$ as $n \longrightarrow \infty$, and $\det R_j = \alpha_{\ell_j+1}\cdot\ldots\cdot\alpha_{\ell_{j+1}}$, where $\alpha_{\ell_j+1},\dots,\alpha_{\ell_{j+1}}$ are the eigenvalues of R_j in $\mathbb{C}$, counted according to their multiplicities. A basis of solutions for $[M_n]$ is given by the columns of $B_n^{-1}\cdot X_n$. Then, by Lemma 3.5, with J some subset of $\{1,\dots,k\}$ with ℓ elements and $j \in \{1,\dots,m\}$,

$$D_{J,I_j}^{(k_j)}(B_n^{-1}\cdot X_n) = \sum_K D_{J,K}^{(k_j)}(B_n^{-1})\cdot D_{K,I_j}^{(k_j)}(X_n) = D_{J,I_j}^{(k_j)}(B_n^{-1})\cdot D_{I_j,I_j}^{(k_j)}(X_n).$$

So, taking into account that $B_n \longrightarrow I$ and hence that $B_n^{[m]} \longrightarrow I$ (where I is the identity matrix in $K^{k,k}$ and $K^{\mu,\mu}$ with $\mu = \binom{k}{m}$, respectively),

$$\lim_{n\to\infty} \frac{D_{I_j,I_j}^{(k_j)}(B_{n+1}^{-1}\cdot X_{n+1})}{D_{I_j,I_j}^{(k_j)}(B_n^{-1}\cdot X_n)} = \lim_{n\to\infty} \frac{D_{I_j,I_j}^{(k_j)}(B_{n+1}^{-1})}{D_{I_j,I_j}^{(k_j)}(B_n^{-1})}\cdot \lim_{n\to\infty} \frac{D_{I_j,I_j}^{(k_j)}(X_{n+1})}{D_{I_j,I_j}^{(k_j)}(X_n)}$$

$$= \alpha_{\ell_j+1}\cdot\ldots\cdot\alpha_{\ell_{j+1}}$$

and, for $J \neq I_j$,

$$\lim_{n\to\infty} \frac{D^{(k_j)}_{J,I_j}(B_n^{-1}\cdot X_n)}{D^{(k_j)}_{I_j,I_j}(B_n^{-1}\cdot X_n)} = \lim_{n\to\infty} \frac{D^{(k_j)}_{J,I_j}(B_n^{-1})}{D^{(k_j)}_{I_j,I_j}(B_n^{-1})} = 0.$$

In the general case

$$M = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix} \qquad (R \in K^{\ell,\ell} \text{ and } S \in K^{k-\ell,k-\ell})$$

there exist $U_1 \in K^{\ell,\ell}$ and $U_2 \in K^{k-\ell,k-\ell}$ such that, for $U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$, we have

$$U^{-1}MU = \begin{bmatrix} R_{\sigma(1)} & & & 0 \\ & R_{\sigma(2)} & & \\ & & \ddots & \\ 0 & & & R_{\sigma(m)} \end{bmatrix}$$

where $\sigma(1),\ldots,\sigma(m)$ is some permutation of the numbers $1,\ldots,m$ and $R_1,\ldots,R_m$ are as above. Further, there exists a matrix $P \in K^{k,k}$ which permutes the matrices $R_{\sigma(1)},\ldots,R_{\sigma(m)}$ in such a way that

$$P^{-1}U^{-1}MUP = \begin{bmatrix} R_1 & & & 0 \\ & R_2 & & \\ & & \ddots & \\ 0 & & & R_m \end{bmatrix}.$$

By Theorem 3.2, there exists a sequence $\{B_n\}$, $B_n \in K^{k,k}$ such that, for $n \in \mathbb{N}$, we have $\lim B_n = I$ and

$$B_{n+1}P^{-1}U^{-1}M_nUPB_n^{-1} = \begin{bmatrix} R_{1n} & & & 0 \\ & R_{2n} & & \\ & & \ddots & \\ 0 & & & R_{mn} \end{bmatrix} \qquad (n \in \mathbb{N}).$$

Hence, for $F_n = UPB_n(UP)^{-1}$,

$$F_{n+1}M_nF_n^{-1} = \begin{bmatrix} R_n & 0 \\ 0 & S_n \end{bmatrix}$$

where $\lim R_n = R$, $\lim S_n = S$, and $\lim F_n = \lim UPB_n(UP)^{-1} = I$. Applying the result obtained in the first part of the proof, we find that there exist linearly independent solutions $\{x_n^{(1)}\},\ldots,\{x_n^{(\ell)}\}$ such that the assertions of the theorem hold. □

Corollary 3.14. *Let* $[M_n] \in \mathcal{M}(K)$ *and* $\lim M_n = M$. *Suppose that* M *has eigenvalues* $\alpha_1,\ldots\alpha_k$ *(counted according to their multiplicities) where*

$|\alpha_1| \le |\alpha_2| \le \dots \le |\alpha_k|$. *If* h *and* m *are such that* $0 \le h < m \le k$ *and* $|\alpha_h| < |\alpha_{h+1}|$ *or* $h = 0$, $|\alpha_m| < |\alpha_{m+1}|$ *or* $m = k$, *then* $[M_n]$ *has* m-h *linearly independent solutions* $\{x_n^{(h+1)}\},\dots,\{x_n^{(m)}\}$ *such that for each sequence* $\{x_n\}$, *where* $\{x_n\} = \lambda_1\cdot\{x_n^{(h+1)}\} + \dots + \lambda_{m-h}\cdot\{x_n^{(m)}\}$, $\{x_n\} \ne \{0\}$, $\lambda_1,\dots\lambda_{m-h} \in K$, *we have*

$$(M - \alpha_{h+1}I)\cdot\ldots\cdot(M - \alpha_m I)\frac{x_n}{|x_n|} \longrightarrow 0 \qquad (n \to \infty).$$

Proof: A transformation matrix U can be found such that $U^{-1}MU$ is in (real or complex) Jordan normal form. U is determined up to permutation of the block matrices $C_{\alpha,j}$ and $B_{\alpha,j}$ respectively (see §1.6 for the notation). Then we can choose U such that

$$U^{-1}MU = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$$

where R has eigenvalues $\alpha_{h+1},\dots\alpha_m$ and S has eigenvalues $\alpha_1,\dots\alpha_h$ and $\alpha_{m+1},\dots,\alpha_k$. Applying Theorem 3.13, we find that the matrix recurrence $[U^{-1}M_nU]$ has $\ell = m\text{-}h$ linearly independent solutions $\{y_n^{(1)}\},\dots,\{y_n^{(\ell)}\}$ such that, for $Y_n = (y_n^{(1)},\dots,y_n^{(k)})$, $I = \{1,\dots,\ell\}$, and J any subset of $\{1,\dots,k\}$ with ℓ elements, $J \ne I$,

$$\lim_{n\to\infty} \frac{D_{J,I}^{(\ell)}(X_n)}{D_{I,I}^{(\ell)}(X_n)} = 0 \tag{3.23}$$

Hence it follows that, for $y_n^{(i)} = \begin{bmatrix} y_{n1}^{(i)} \\ \vdots \\ y_{nk}^{(i)} \end{bmatrix}$ $(i = 1,\dots,\ell)$,

$$\lim_{n\to\infty} \frac{y_{nj}^{(i)}}{|y_n^{(i)}|} = 0 \qquad (j = \ell+1,\dots,k).$$

To show this, take $J_{qj} = I \cup \{j\}\backslash\{q\}$ for $j = \ell+1,\dots k$; $q = 1,\dots,\ell$. Then

$$\frac{D_{J_{qj},I}^{(\ell)}(Y_n)}{D_{I,I}^{(\ell)}(Y_n)} = \pm z_{qjn},$$

where $z_{1jn},\dots,z_{\ell jn}$ are the solutions of the set of linear equations

$$y_{n1}^{(i)}\cdot z_{1jn} + \dots + y_{n\ell}^{(i)}\cdot z_{\ell jn} = y_{nj}^{(i)} \qquad (i = 1,\dots,\ell). \tag{3.24}$$

By (3.23), $z_{qjn} \in K$ and $\lim_{n\to\infty} z_{qjn} = 0$ for all q,j. Hence, by (3.24),

$$\lim_{n\to\infty} \frac{y^{(i)}_{nj}}{|y^{(i)}_n|} = 0 \quad \text{for } i = 1,\ldots,\ell,\ j = \ell+1,\ldots,k.$$

Since $(R - \alpha_{h+1}I)\cdot\ldots\cdot(R - \alpha_m I)$ is the zero matrix,

$$U^{-1}(M - \alpha_{h+1})\cdot\ldots\cdot(M - \alpha_m I)\cdot U \frac{y_n}{|y_n|} \longrightarrow 0 \quad (n \to \infty),$$

with $\{y_n\} = \lambda_1\cdot\{y_n^{(1)}\} + \ldots + \lambda_\ell\cdot\{y_n^{(\ell)}\}$ and $\{y_n\} \neq 0$, $\lambda_1,\ldots\lambda_\ell \in K$. Put $Uy_n = x_n$. Then $\{x_n\}$ is a root of $[M_n]$. By the properties of the matrix norm,

$$|x_n| = |Uy_n| \geq \frac{|y_n|}{\| U^{-1} \|},$$

where $\| U^{-1}\| \neq 0$. Thus,

$$(M - \alpha_{h+1}I)\ldots(M - \alpha_m I) \frac{x_n}{|x_n|} \longrightarrow 0 \quad (n \to \infty),$$

as asserted. □

We apply Theorem 3.13 to linear recurrence operators in order to obtain Theorem 3.3.

Proof of Theorem 3.3: We prove the following statement, from which we can easily prove the theorem by induction.

Let $R \in \mathfrak{R}(K)$, $K = \mathbb{R}$ or $\mathbb{C}$. Let $\chi_R(X) = P(X)\cdot Q(X)$, with $P,Q \in K[X]$ monic polynomials and all zeros in $\mathbb{C}$ of P have larger absolute values than all zeros of Q. Then $R = S_1\cdot S_2 = R_2\cdot R_1$, where $S_1,S_2,R_1,R_2 \in \mathfrak{R}(K)$ and $\chi_{R_1} = \chi_{S_1} = P$, $\chi_{R_2} = \chi_{S_2} = Q$.

We shall only prove that R has a divisor S with $\chi_S = Q$. The other result goes similarly. Put $m = \deg Q$. Let $\beta_1,\ldots,\beta_m$ be the zeros of Q and $\beta_{m+1},\ldots,\beta_k$ those of P. Let $[M_n^R]$ be the matrix recurrence associated with R. Finally, let $M^R = \lim M_n^R$. Consider the constant recurrence operator $\chi_R(T) \in \mathfrak{R}(K)$ which is formed by replacing all instances of X in the expression for $\chi_R(X)$ by the shift operator T. In the same way we define the operators $P(T)$ and $Q(T)$. Note that $P(T),Q(T) \in \mathfrak{R}(K)$ and $\chi_R(T) = P(T)\cdot Q(T) = Q(T)\cdot P(T)$. Let $\{u_n^{(1)}\},\ldots,\{u_n^{(m)}\}$ be a basis of $Z(Q(T))$ and $\{u_n^{(m+1)}\},\ldots,\{u_n^{(k)}\}$ be a basis of $Z(P(T))$. It is easy to see that such bases exist: If $K = \mathbb{C}$, the matter is quite trivial. If $K = \mathbb{R}$, we first choose a basis of complex roots. This can be chosen in such a way that for each basis sequence $\{x_n\}$, also $\{\bar{x}_n\}$ is a basis sequence. If $\{x_n\}$ is not a sequence of real numbers, then we choose $\{x_n + \bar{x}_n\}$ and $\{x_n - \bar{x}_n\}$ instead of $\{x_n\}$ and $\{\bar{x}_n\}$. Clearly, $\{u_n^{(1)}\},\ldots,\{u_n^{(k)}\}$ is a basis of $Z(\chi_R(T))$. Further, let

$$U = (\upsilon^{(1)},\dots,\upsilon^{(k)}), \text{ where } \upsilon^{(i)} = \begin{bmatrix} u^{(i)}_{k-1} \\ \vdots \\ u^{(i)}_0 \end{bmatrix} \quad (i = 1,\dots,k).$$

From the construction of U it follows that $U \in K^{k,k}$ and that U is non-singular. Moreover,

$$U^{-1}\cdot M^R\cdot U = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

with $N_1 \in K^{m,m}$ and $N_2 \in K^{k-m,k-m}$ and N_1,N_2 have characteristic polynomials Q and P, respectively. By Theorem 3.13, the matrix recurrence $[U^{-1}\cdot M^R_n\cdot U]$ has m linearly independent solutions $\{x^{(1)}_n\},\dots,\{x^{(m)}_n\}$ such that, for $X_n = (x^{(1)}_n,\dots,x^{(m)}_n)$, $I = \{1,\dots,m\}$ and J some other subset of $\{1,\dots,k\}$ with m elements,

$$\lim_{n\to\infty} \frac{D^{(m)}_{I,I}(X_{n+1})}{D^{(m)}_{I,I}(X_n)} = \beta_1\cdot\ldots\cdot\beta_m, \tag{3.25}$$

$$\lim_{n\to\infty} \frac{D^{(m)}_{J,I}(X_n)}{D^{(m)}_{I,I}(X_n)} = 0. \tag{3.26}$$

Put, for $i = 1,\dots,m$,

$$z^{(i)}_n = Ux^{(i)}_n \qquad (n \in \mathbb{N}).$$

Then, $\{z^{(1)}_n\},\dots,\{z^{(m)}_n\}$ are linearly independent solutions of $[M^R_n]$. Hence, for

$$S = \begin{vmatrix} y^{(1)}_{n+m} & \cdots & y^{(m)}_{n+m} & T^m \\ \vdots & & \vdots & \vdots \\ y^{(1)}_{n+1} & \cdots & y^{(m)}_{n+1} & T \\ y^{(1)}_n & \cdots & y^{(m)}_n & I \end{vmatrix}$$

where $z^{(i)}_n = \begin{bmatrix} y^{(i)}_{n+k-1} \\ \vdots \\ y^{(i)}_n \end{bmatrix} \quad (i = 1,\dots,k),$

we have $S \in \mathfrak{R}(K)$, $\mathrm{ord}(S) = m$ and $S|R$. It remains to prove that $\chi_S = Q$. Let $I = \{1,\dots,m\}$ and, for $q = 0,1,\dots,m$, define $J_q := \{k-m,\dots,k-1,k\}\backslash\{k-q\}$.

Put $Y_n = (z^{(1)}_n,\dots,z^{(m)}_n)$. Then $Y_n \in K^{k,m}$ $(n \in \mathbb{N})$. It follows from the definition of S that

$$\chi_S(X) = \lim_{n\to\infty} \sum_{j=0}^{m} (-1)^{m-j} \cdot \frac{D^{(m)}_{J_j,I}(Y_n)}{D^{(m)}_{J_m,I}(Y_n)} \cdot X^j .$$

Note that $D^{(m)}_{J_m,I}(Y_n) \neq 0$ for n large enough. We calculate χ_S.

Since $U \cdot X_n = Y_n$, we have

$$\sum_K D^{(m)}_{J_j,K}(U) \cdot D^{(m)}_{K,I}(X_n) = D^{(m)}_{J_j,I}(Y_n) \qquad \text{for } j = 0,\dots,m$$

where the sum is taken over all subsets K of $\{1,\dots,k\}$ with m elements. Since $D^{(m)}_{J_m,I}(U) \neq 0$, which follows from the definition of U, we have, by (3.26),

$$\lim_{n\to\infty} \frac{D^{(m)}_{J_j,I}(Y_n)}{D^{(m)}_{J_m,I}(Y_n)} = \frac{D^{(m)}_{J_j,I}(U)}{D^{(m)}_{J_m,I}(U)} .$$

Hence,

$$\chi_S(X) = \sum_{j=0}^{m} (-1)^{m-j} \cdot \frac{D^{(m)}_{J_j,I}(U)}{D^{(m)}_{J_m,I}(U)} \cdot X^j = \begin{vmatrix} u^{(1)}_m & \cdots & u^{(m)}_m & X^m \\ \vdots & & \vdots & \vdots \\ u^{(1)}_1 & \cdots & u^{(m)}_1 & X \\ u^{(1)}_0 & \cdots & u^{(m)}_0 & 1 \end{vmatrix} \cdot \left[D^{(m)}_{J_m,I}(U)\right]^{-1} = Q(X)$$

by the definition of $\{u^{(j)}_n\}$ $(j = 1,\dots,m)$. □

§4. Order of convergence of the solutions.

In the last section we derived that for a solution $\{x_n\}$ of a matrix recurrence, the quotient $\frac{x_n}{|x_n|}$ tends to the union of generalized eigenspaces corresponding to the eigenvalues of the limit matrix having some common absolute value. In this section we shall derive a result about the order of convergence of the solutions, which appears to follow fairly easily from the results of §3.3.

Remark 3.4.1. If $[M_n] \in \mathcal{M}(K)$ and $M = \lim M_n$ exists, we call an eigenvalue α of M *simple*, if it has (algebraic) multiplicity one and if M has no other eigenvalues with the same absolute value as α. Similarly, if $R \in \mathcal{R}(K)$, we call a zero α of χ_R simple, if it has multiplicity one and if χ_R has no other zeros with the same absolute value as α. If M (or χ_R) has only simple eigenvalues (zeros), we call $[M_n]$ (or R) simple. On the other hand, if $M = \lim M_n$ (or χ_R) exists and has not only simple eigenvalues, we call $[M_n]$ (or R) *non-simple*.

Theorem 3.15. *Let* $f:\mathbb{N} \longrightarrow \mathbb{R}^+$ *be a monotonically non-increasing function such that* $\lim_{n\to\infty} f(n) = 0$ *and* $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$.
Let $[M_n] \in \mathcal{M}(K)$ $(K = \mathbb{R}$ or $\mathbb{C})$ *with* $\lim M_n = M$. *Suppose that* M *is of the form*

$$M = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$$

where all eigenvalues of R *have distinct absolute values from all eigenvalues of* S *and suppose that* $\| M_n - M \| = \mathcal{O}(f(n))$.
Then there exists a sequence $\{B_n\}$, $B_n \in K^{k,k}$, *such that*

(i) $\lim B_n = I$ *and* $\| I - B_n\| = \mathcal{O}(f(n))$

(ii) $B_{n+1}M_nB_n^{-1} = \begin{bmatrix} R_n & 0 \\ 0 & S_n \end{bmatrix}$

where $\lim R_n = R$, $\lim S_n = S$.

(iii) $\| R_n - R \| = \mathcal{O}(f(n))$ *and* $\| S_n - S \| = \mathcal{O}(f(n))$, *and if* $\sum n^j \cdot \| M_n - M \|$ *converges for some* $j \in \mathbb{R}$, *then both* $\sum n^j \cdot \| R_n - R \|$ *and* $\sum n^j \cdot \| S_n - S \|$ *converge.*

Proof: (i) and (ii) follow from Theorem 3.2. We only have to prove (iii).

Let $M_n = \begin{bmatrix} R_n^* & Q_n \\ P_n & S_n^* \end{bmatrix}$

By Lemma 3.11 there exists a sequence $\{B'_n\}$, $B'_n \in K^{k,k}$, such that

(i) $\lim B'_n = I$

(ii) $B'_{n+1} M_n (B'_n)^{-1} = \begin{bmatrix} R_n & 0 \\ P_n & S_n \end{bmatrix}$

where $\lim R_n = R$, $\lim S_n = S$ and $\lim P_n = 0$.

(iii) $\| B'_{n+1} - I \| \le \| B'_n - I \| \cdot \delta + c \cdot \| M_n - M \| \le \| B'_n - I \| \cdot \delta + c_1 f(n)$

where $0 < \delta < 1$ and $c, c_1 \in \mathbb{R}_{>0}$.

By (iii),

$$\frac{\| B'_{n+1} - I \|}{\delta^{n+1}} - \frac{\| B'_n - I \|}{\delta^n} \le \delta^{-n+1} c_1 f(n).$$

Hence,

$$\| B'_n - I \| \le c_2 \cdot \sum_{k=0}^{n} f(k) \cdot \delta^{n-k}.$$

Let N be so large that, for $n \ge N$,

$$\left| \frac{f(n+1)}{f(n)} - 1 \right| < \frac{1 - \delta}{2}$$

Then, for $n \ge N$,

$$\frac{1}{f(n)} \cdot \sum_{k=0}^{n} f(k) \cdot \delta^{n-k} \le \frac{1}{f(n)} \cdot \sum_{k=0}^{N} f(k) \cdot \delta^{n-k} + \sum_{k=N+1}^{n} \left[\frac{1+\delta}{2} \right]^{-n+k} \cdot \delta^{n-k}$$

$$\le \frac{f(0)}{f(n)} \cdot \frac{\delta^{n-N}}{1-\delta} + \frac{1+\delta}{1-\delta}.$$

Using the fact that $\delta^n \cdot (f(n))^{-1} \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain

$$\| B'_n - I \| = \mathcal{O}(f(n)).$$

Hence

$$(3.27) \qquad \| M - B'_{n+1} M_n (B'_n)^{-1} \| \le \| (I - B'_{n+1}) \cdot M \| + \| B'_{n+1} M \cdot (B'_n)^{-1} \cdot (B'_n - I) \| + \| B'_{n+1} \cdot (M_n - M) \cdot (B'_n)^{-1} \| = \mathcal{O}(f(n))$$

and, by (iii),

$$\sum_{n=0}^{N} n^j \cdot \| (B'_n - I) \| \ll \sum_{n=0}^{N} \sum_{k=0}^{n} n^j \cdot \| M_k - M \| \cdot \delta^{n-k}$$
$$= \sum_{k=0}^{N} \sum_{n=k}^{\infty} n^j \cdot \| M_k - M \| \cdot \delta^{n-k} \ll \sum_{k=0}^{\infty} k^j \cdot \| M_k - M \|,$$

since $\sum_{n=0}^{\infty} n^j \cdot \delta^n$ converges, and

$$\sum_{n=k}^{\infty} n^j \cdot \delta^{n-k} = k^j \cdot \sum_{n=k}^{\infty} (n/k)^j \cdot \delta^{n-k} = k^j \cdot \sum_{n=0}^{\infty} (1 + n/k)^j \cdot \delta^n \ll k^j \cdot \sum_{n=0}^{\infty} n^j \cdot \delta^n$$

so that, by the first inequality of (3.27), $\sum n^j \cdot \| M - B'_{n+1} M_n (B'_n)^{-1} \|$ converges. By Lemma 3.12, we can choose a sequence of matrices $\{C_n\}$ in such a way that

$$C_n \to 0 \quad \text{and} \quad C_{n+1} \cdot R_n = S_n \cdot C_n - P_n, \ \| C_n \| << \cdot \sum_{k=n}^{\infty} \| P_k \| \cdot \delta^{k-n}$$

for some number $0 < \delta < 1$. Since $\| P_n \| = \mathcal{O}(f(n))$ we have, by the properties of f, that $\| C_n \| = \mathcal{O}(f(n))$. Put

$$B^*_n = \begin{pmatrix} I & 0 \\ C_n & I \end{pmatrix} \quad \text{and} \quad B_n = B^*_n \cdot B'_n.$$

Then

$$B_{n+1} M_n B_n^{-1} = \begin{pmatrix} R_n & 0 \\ 0 & S_n \end{pmatrix}$$

and

$$\| B_n - I \| = \mathcal{O}(f(n)) \quad (\text{for} \quad n \to \infty).$$

Furthermore, it follows from (3.27) and (ii) that

$$\| M - B_{n+1} M_n (B_n)^{-1} \| = \mathcal{O}(f(n)) \quad (n \to \infty).$$

Moreover, if $\sum n^j \cdot \| M_n - M \|$ converges, then $\sum n^j \cdot \| M - B_{n+1} M_n (B_n)^{-1} \|$ converges as well. □

Corollary 3.16. *Let* $[M_n] \in \mathcal{M}(K)$, *with* $\lim M_n = M$ *and let* $\alpha \neq 0$ *be a simple eigenvalue of* M. *Further, suppose that* $\sum \| M_n - M \|$ *converges. Then there exists a solution* $\{x_n\}$ *of* $[M_n]$ *such that* $\frac{x_n}{\alpha^n}$ *converges to an eigenvector of* M *that corresponds to the eigenvalue* α.

Proof: From Theorem 3.15, it follows that there exists a permutation matrix U and a sequence of matrices $\{B_n\}$ such that

(a) $\lim B_n = I$

(b) $B_{n+1} U M_n U^{-1} B_n^{-1} = \begin{pmatrix} \alpha + \delta_n & 0 \\ 0 & S_n \end{pmatrix}$

where $\sum |\delta_n| < \infty$.

Put $N_n = B_{n+1} U M_n U^{-1} B_n^{-1}$. The matrix recurrence $[N_n]$ has a solution $\{x_n\}$ with $x_n^T = (x_{n1}, 0, 0, \ldots, 0)$, such that

$$\frac{x_{n+1,1}}{x_{n,1}} = \alpha + \delta_n.$$

Then

$$x_{n1} = \alpha^n \cdot x_{01} \cdot \prod_{k=0}^{n-1} (1 + \delta_k/\alpha)$$

so that $\lim_{n\to\infty} \frac{x_n}{\alpha^n} = \lambda\cdot e_1$ for some $\lambda \in K^*$ (with $e_1^T = (1,0,\ldots,0)$). Then $\{U^{-1}B_n^{-1}x_n\}$ is a solution of $[M_n]$ and $(B_nU)^{-1}x_n = U^{-1}(x_n + \xi_n)$ where $\lim_{n\to\infty} \frac{\xi_n}{|x_n|} = 0$, so

$$\lim_{n\to\infty} \frac{(B_nU)^{-1}x_n}{\alpha^n} = \lambda\cdot U^{-1}e_1$$

and $U^{-1}e_1$ is the eigenvector of M corresponding to the eigenvalue α. □

Corollary 3.17. *Let* $R \in \mathfrak{R}(K)$ *be simple and such that* $\sum N_n(R - \chi_R(T))$ *converges. Then for all zeros* $\alpha \neq 0$ *of* χ_R, R *has a zero* $\{v_n\}$ *such that*

$$\lim_{n\to\infty} \frac{v_n}{\alpha^n} = 1.$$

Proof: Apply Corollary 3.16 to $[M_n^R]$. Each solution $\{x_n\}$ of this matrix recurrence is of the form

$$x_n^T = (u_{n+k-1},\ldots,u_{n+1},u_n)$$

with $\{u_n\} \in Z(R)$ and $\| M^R - M_n^R \| << N_n(\chi_R(T) - R)$.
Since $\frac{x_n}{\alpha^n}$ converges to the eigenvector of $\lim M_n^R$ corresponding to the eigenvalue α (see Ch.1,§6), we have that $\lambda = \lim \frac{u_n}{\alpha^n}$ exists and $\lambda \neq 0$. Dividing u_n by λ yields the desired result. □

CHAPTER FOUR

FAST CONVERGENCE

§1. Introduction.

From Corollary 3.16 it follows that for a simple matrix recurrence $[M_n]$ the solutions behave very much like the solutions of the constant matrix recurrence $[\lim M_n]$ if $\sum \| M_n - \lim M_n \|$ converges. In this chapter we investigate the case that $[M_n]$ is non-simple. We shall derive a condition on the convergence rate of the sequence $\{M_0, M_1, \ldots\}$ in order that the solutions of $[M_n]$ 'behave like' the solutions of $[\lim M_n]$. First of all, however, we must define more precisely what we mean by similar behaviour of solutions.

We can interpret Corollary 3.16 in the following way: For each solution $\{x_n\}$ of a simple matrix recurrence $[M_n]$ there is a solution $\{y_n\}$ of the constant matrix recurrence $[\lim M_n]$ such that

$$(4.1) \qquad \lim_{n\to\infty} \frac{x_n - y_n}{|y_n|} = \lim_{n\to\infty} \frac{x_n - y_n}{|x_n|} = 0.$$

Condition (4.1) seems to be a good definition of similar behaviour of two solutions $\{x_n\}$ and $\{y_n\}$. We write $\{x_n\} \sim \{y_n\}$ if $\{x_n\}$ and $\{y_n\}$ satisfy (4.1). Note that $\lim_{n\to\infty} \frac{x_n - y_n}{|y_n|} = 0$ implies $\lim_{n\to\infty} \frac{x_n - y_n}{|x_n|} = 0$ and conversely.

It should be clear that, if we want to generalize the results of Corollaries 3.16 and 3.17, we have to exclude the case that the limit matrix M of the sequence $\{M_n\}$ has eigenvalues zero. For if M has an eigenvalue zero with multiplicity ℓ, then $[M]$ has ℓ solutions $\{x_n^{(1)}\}, \ldots, \{x_n^{(\ell)}\}$ with $x_0^{(1)}, \ldots, x_0^{(\ell)}$ linearly independent and $x_n^{(1)} = \ldots = x_n^{(\ell)} = 0$ for $n \geq \ell$, so that definition (4.1) does not make sense. (With the aid of Theorem 3.8 one can show that in this case $[M_n]$ has ℓ linearly independent solutions $\{y_n^{(1)}\}, \ldots, \{y_n^{(\ell)}\}$ such that $y_n^{(i)} \to 0$ as $n \to \infty$ $(i = 1, \ldots, \ell)$).

We now state the results of this chapter first for matrix recurrences and after that for recurrence operators. We define the ***minimal polynomial*** of a matrix $M \in \mathbb{C}^{k,k}$ as the monic polynomial of smallest degree > 0 in $\mathbb{C}[X]$ such that $P(M) = 0$. Further, we denote by $\mathcal{M}$ the set of equivalence classes of bounded monotonic functions $f: \mathbb{N} \to \mathbb{R}_{>0}$ under the same equivalence relation as the one defined in Chapter 1, §1, i.e. $f \sim g \Leftrightarrow f(n) = g(n)$ for n large enough.

Theorem 4.1. *Let* $[M_n] \in \mathcal{M}(\mathbb{C})$ *and* $M = \lim M_n$. *Let* $P \in \mathbb{C}[X]$ *be the minimal polynomial of* M *and let* L *be the maximum of the multiplicities of the zeros of* P. *Suppose that* M *has no eigenvalue zero and that* $\sum n^{L-1}\cdot\frac{1}{f(n)}\cdot\| M_n - M \|$ *converges for some* $f \in \mathcal{M}$.
Then there is a bijection between the solutions $\{x_n\}$ *of* $[M_n]$ *and* $\{y_n\}$ *of* [M] *such that* $\{x_n\} \sim \{y_n\}$. *Moreover, we have*

$$\frac{x_n - y_n}{|y_n|} = o(f(n)) \qquad (n \to \infty).$$

Applying Theorem 4.1 to recurrence operators yields the following result:

Corollary 4.2. *Let* $R \in \mathcal{R}(\mathbb{C})$, $\mathrm{ord}(R) = k$, *such that* χ_R *exists and* $\chi_R(0) \neq 0$, *and let* L *be the maximum of the multiplicities of* χ_R. *Suppose that* $\sum n^{L-1}\cdot\frac{1}{f(n)}\cdot N_n(\chi_R(T) - R)$ *converges for some* $f \in \mathcal{M}$. *Then for each basis of zeros* $\{v_n^{(1)}\},\ldots,\{v_n^{(k)}\}$ *of* $\chi_R(T)$ *such that* $\lim_{n\to\infty} \frac{v_{n+1}^{(i)}}{v_n^{(i)}}$ *exists* $(i = 1,\ldots,k)$ *there exists a basis of zeros* $\{u_n^{(1)}\},\ldots,\{u_n^{(k)}\}$ *of* R *such that*

$$\frac{u_n^{(i)}}{v_n^{(i)}} - 1 = o(f(n)) \qquad (n \to \infty;\ i = 1,\ldots,k).$$

Before proving Theorem 4.1 and Corollary 4.2 it will be useful to recall some properties of the solutions of the constant matrix recurrence [M]. This will be the subject of the next section. On account of the second assertion of Theorem 3.15(iii) it will be sufficient to assume that M has only eigenvalues with the same absolute value. Since M has no eigenvalue zero, we can normalize such that all eigenvalues have absolute value one.

§2. The constant matrix recurrence.

We assume that $M \in \mathbb{C}^{k,k}$ has only eigenvalues with absolute value one. There exists a conjugate matrix $\tilde{M}$ such that

$$\tilde{M} = \begin{bmatrix} \alpha_1\cdot B(g_1) & & 0 \\ & \ddots & \\ 0 & & \alpha_\ell\cdot B(g_\ell) \end{bmatrix}$$

where B(g) is a $g \times g$-matrix of the form $I + J$, with J as in (1.9).

It is clear that, if $\{(x_n^{(i,j)}\}; 1 \le j \le g_i\}$ is a basis of solutions for $[\alpha_i \cdot B(g_i)]$, then $\{(z_n^{(i,j)}\}; 1 \le i \le \ell, 1 \le j \le g_i\}$ constitutes a basis of solutions of $[\bar{M}]$, where

$$z_n^{(i,j)} = \begin{bmatrix} 0 \\ x_n^{(i,j)} \\ 0 \end{bmatrix} \quad \text{and} \quad \bar{M} \cdot z_n^{(i,j)} = \begin{bmatrix} 0 \\ \alpha_i \cdot B(g_i) \cdot x_n^{(i,j)} \\ 0 \end{bmatrix}.$$

We determine a basis $\{(x_n^{(i,j)}\}\}$. For $\{x_n\}$ a solution of $[\alpha \cdot B(g)]$,

$$x_{n+1} = \alpha \cdot (I + J) x_n \qquad (n \in \mathbb{N}).$$

Put $x_0^{(i,j)} = e_j$, where e_j is the j-th unit vector. For $m \in \mathbb{Z}$

$$B(g)^m = (I + J)^m = I + \binom{m}{1} \cdot J + \binom{m}{2} \cdot J^2 + \dots + \binom{m}{g-1} \cdot J^{g-1}.$$

Hence,

$$(4.2) \qquad x_n^{(i,j)} = \alpha_i^n \cdot B(g_i)^n \cdot e_j = \alpha_i^n \cdot (e_j + \binom{n}{1} \cdot e_{j-1} + \dots + \binom{n}{g-1} \cdot e_{j-g+1})$$

where $e_i = 0$ for $i \le 0$.

So, $(x_n^{(i,j)}, e_\ell) \ne 0$ if and only if $\ell \in \{j-g+1, \dots, j\}$, and it becomes clear that $\{(x_n^{(i,j)}\}; 1 \le j \le g_i\}$ is a basis of solutions of $[\alpha_i \cdot B(g_i)]$. Then, $\{(z_n^{(i,j)}\}; 1 \le i \le \ell, 1 \le j \le g_i\}$ is a basis of solutions of $[\bar{M}]$. Moreover, for $j = 1, \dots, g_i - 1$,

$$\frac{|x_n^{(i,j)}|}{|x_n^{(i,j+1)}|} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

and

$$\frac{|z_n^{(i,j)}|}{|z_n^{(i',j')}|} \longrightarrow 0 \quad \text{if} \quad j < j'$$

and the latter quotient is bounded for all i, i' if $j = j'$.

If $\{z_n\}$ is an arbitrary non-trivial solution of $[\bar{M}]$, then

$$\{z_n\} = \sum_{i,j} \lambda_{ij} \cdot \{z_n^{(i,j)}\}$$

with $\lambda_{ij} \in \mathbb{C}$ $(1 \le i \le \ell; 1 \le j \le g_i)$, not all λ_{ij} being zero. Then

$$(4.3) \qquad 0 < |z_n^{(i,j)}| < c \cdot |z_n|$$

with $c \in \mathbb{R}_{>0}$ depending only on the λ_{ij} for (i,j) such that not $\lambda_{ij'} = 0$ for $i \in \{1, \dots, g_{j'}\}$ and $j' \ge j$.

Lemma 4.3. *Let $\mathfrak{M}$ be as above, and let $x \in \mathbb{C}^k$, $x^T = (x_1,\dots,x_k)$, and $n \in \mathbb{Z}$, $n \neq 0$. Put $\ell_m = \sum_{i=1}^{m-1} g_i$ for $m = 0,\dots,\ell$. Then, for $\ell_m < j \le \ell_{m+1}$,*

$$|(\mathfrak{M}^n x)_j| \le c_0 \cdot |n|^{\ell_{m+1}-j} \cdot |x|$$

where c_0 is some constant, depending only on $\mathfrak{M}$.

Proof: Put $q = j - \ell_m$. Note that $\ell_{m+1} - j = g_m - q$. By (4.2),

$$(4.4) \qquad (\mathfrak{M}^n x)_j = \sum_{i=1}^{k} x_i \cdot (\mathfrak{M}^n e_i)_j = \sum_{i=\ell_m+1}^{\ell_{m+1}} x_i \cdot (\alpha_m^n \cdot B(g_m)^n \cdot e_{i-\ell_m})_q$$

$$= \alpha_m^n \cdot \sum_{i=1}^{g_m} x_{i+\ell_m} \cdot \Big(\sum_{p=0}^{i-1} \binom{n}{p} \cdot e_{i-p} \Big)_q = \alpha_m^n \cdot \sum_{i=1}^{g_m} x_{i+\ell_m} \cdot \binom{n}{i-q}.$$

Hence, for n large enough,

$$|(\mathfrak{M}^n x)_j| \le g_m \cdot \left| \binom{n}{g_m - q} \right| \cdot |x| \le c_1 \cdot |x| \cdot |n|^{g_m - q}$$

where c_1 depends only on $\mathfrak{M}$. For small n, we use the inequality

$$|(\mathfrak{M}^n x)_j| \le \| \mathfrak{M}^n \| \cdot |x|.$$

Hence,

$$|(\mathfrak{M}^n x)_j| \le c_0 \cdot |x| \cdot |n|^{g_m - q}$$

for $n \neq 0$ and c_0 depending only on $\mathfrak{M}$. □

§3. The proof of Theorem 4.1.

We introduce the notation $\sum_{(n)}$: Let $\{x_n\}_{n=N}^{\infty}$ be a sequence of numbers, vectors or matrices, $x_n \in \mathbb{C}$ (or $\mathbb{C}^k$, $\mathbb{C}^{k,m}$ respectively, for some numbers k and m).

If $\sum x_k$ converges, then $\sum_{(n)} x_k := \sum_{k=n}^{\infty} x_k$.

If $\sum x_k$ diverges, then $\sum_{(n)} x_k := \sum_{k=N}^{n-1} x_k$.

The proof of Theorem 4.1 goes in two steps.

Proposition 4.4.: *Let $M \in \mathbb{C}^{k,k}$ be such that M has only eigenvalues with absolute value one. Let L be the maximum of the multiplicities of the zeros of the minimal polynomial of M. Further, let $\{D_n\}$ be a sequence of matrices in*

$\mathbb{C}^{k,k}$. *Finally, let* $\{x_n\}$ *be a sequence of vectors in* $\mathbb{C}^k$. *Then the inhomogeneous matrix recurrence*

(4.5) $$y_{n+1} = M\cdot y_n + D_n\cdot x_n$$

has a solution $\{y_n^{(0)}\}$ *such that*

$$|y_n^{(0)}| \le c\cdot \sum_{i=1}^{L} n^{i-1}\cdot\Big(\sum_{(n)} k^{L-i}\cdot|x_k|\cdot\| D_k\|\Big)$$

with c *depending only on* M.

Proof: If $\{y_n\}$ is a solution of (4.5), then

$$y_n = M^n\cdot\Big(y_0 + \sum_{k=0}^{n-1} M^{-k-1}\cdot D_k x_k\Big).$$

Put

$$M = \begin{bmatrix} \alpha_1\cdot B(g_1) & & 0 \\ & \ddots & \\ 0 & & \alpha_\ell\cdot B(g_\ell) \end{bmatrix}$$

as in §4.2, and put $\ell_m = \sum_{i=1}^{m-1} g_i$ for $m = 1,2,\dots,\ell+1$. Then, by Lemma 4.3, for $j = 1,\dots,k$,

$$|(M^{-k-1}\cdot D_k x_k)_j| \le c_0\cdot k^{\ell_{m+1}-j}\cdot\| D_k\|\cdot|x_k| \qquad (k \in \mathbb{Z},\ k \ne 0).$$

If $\sum_{k=0}^{\infty} k^{\ell_{m+1}-j}\cdot\| D_k\|\cdot|x_k|$ converges, we choose

$$y_{0j}^{(0)} = -\sum_{k=0}^{\infty} (M^{-k-1}\cdot D_k x_k)_j.$$

If $\sum_{k=0}^{\infty} k^{\ell_{m+1}-j}\cdot\| D_k\|\cdot|x_k|$ diverges, we choose $y_{0j}^{(0)} = 0$.

Put $z_n^{(0)} = y_0^{(0)} + \sum_{k=0}^{n-1} M^{-k-1}\cdot D_k x_k$ $(n \in \mathbb{N})$ where $y_0^{(0)} = \begin{pmatrix} y_{01}^{(0)} \\ \vdots \\ y_{0k}^{(0)} \end{pmatrix}$. Then

$$|z_{nj}^{(0)}| \le c_2\cdot\sum_{(n)} k^{\ell_{m+1}-j}\cdot|x_k|\cdot\| D_k\| \qquad (n \ge 0)$$

with c_2 depending only on M. Finally, put $y_n^{(0)} = M^n\cdot z_n^{(0)}$ $(n \ge 0)$. Then $\{y_n^{(0)}\}$ is a solution of (4.5), and by (4.4) with $\ell_m < j \le \ell_{m+1}$ and $q = j - \ell_m$, we have that, for some constant c_3 only depending on M,

$$|y_{nj}^{(0)}| \le |(M^n\cdot z_n^{(0)})_j| \le \sum_{i=1}^{g_m} |z_{n,i+\ell_m}^{(0)}|\cdot\binom{n}{i-q}$$

$$\le c_3 \cdot \sum_{i=1}^{g_m} n^{i-q} \cdot \sum_{(n)} k^{g_m - i} \cdot |x_k| \cdot \| D_k \| \le c_3 \cdot \sum_{i=1}^{L} n^{i-q} \cdot \sum_{(n)} k^{L-i} \cdot |x_k| \cdot \| D_k \|$$

since L is the maximum of the numbers g_m $(m = 1, \dots, \ell)$.

Hence,

$$|y_n^{(0)}| \le c_4 \cdot \sum_{i=1}^{L} n^{i-1} \sum_{(n)} k^{L-i} \cdot |x_k| \cdot \| D_k \| .$$

with c_4 some constant depending only on M. □

Lemma 4.5. *Let* $\{d_k\}$ *be a sequence of non-negative real numbers and* $m \in \mathbb{R}$ *such that* $\sum_{k=0}^{\infty} d_k \cdot k^m$ *converges. Then, for* $f : \mathbb{N} \to \mathbb{R}_{>0}$ *monotonic,*

$$\frac{1}{f(n)} \cdot \sum_{(n)} d_k \cdot k^m \cdot f(k) \to 0 \quad as \quad n \to \infty.$$

Proof: If f is bounded, then $\sum_{k=0}^{\infty} d_k \cdot k^m \cdot f(k)$ converges and

$$\sum_{(n)} d_k \cdot k^m \cdot f(k) << f(n) \cdot \sum_{(n)} d_k \cdot k^m .$$

If $\lim_{n \to \infty} f(n) = \infty$, then if $\sum_{k=0}^{\infty} d_k \cdot k^m \cdot f(k)$ converges, clearly $\sum_{(n)} d_k \cdot k^m \cdot f(k) \to 0$ as $n \to \infty$, hence $\frac{1}{f(n)} \cdot \sum_{(n)} d_k \cdot k^m \cdot f(k) \to 0$ *as* $n \to \infty$.

Suppose that $\sum_{k=0}^{\infty} d_k \cdot k^m \cdot f(k)$ diverges. Choose $\varepsilon > 0$. Let $R = \left| \sum_{k=0}^{\infty} d_k \cdot k^m \right|$ and let $N \in \mathbb{N}$ be such that $\left| \sum_{(n)} d_k \cdot k^m \right| < \varepsilon$ for $n \ge N$. Then, for $n \ge N$,

$$\sum_{(n)} d_k \cdot k^m \cdot f(k) \le f(N) \cdot \sum_{k=0}^{N-1} d_k \cdot k^m + f(n) \cdot \sum_{k=N}^{n-1} d_k \cdot k^m \le R \cdot f(N) + \varepsilon \cdot f(n).$$

Hence,

$$\frac{1}{f(n)} \cdot \sum_{(n)} d_k \cdot k^m \cdot f(k) \le \frac{R \cdot f(N)}{f(n)} + \varepsilon < 2\varepsilon$$

for n large enough. Since $\varepsilon > 0$ can be chosen arbitrarily small, the assertion follows. □

Proof of Theorem 4.1. We may assume, without loss of generality, that M has only eigenvalues with absolute value one. Let $\{x_n\}$ be a non-trivial solution of [M]. Put $D_n = M_n - M$. Then $\sum_{n=0}^{\infty} n^{L-1} \cdot \frac{1}{f(n)} \cdot \| D_n \|$ converges. Consider the inhomogeneous matrix recurrence

(4.6) $$y_{n+1} = M\cdot y_n + D_n\cdot x_n \qquad (n \in \mathbb{N}).$$

According to §4.2 (and (4.2) in particular), $|x_n| \sim c\cdot n^q$ for some $c \in \mathbb{R}$, $c \neq 0$ and $0 \leq q \leq L-1$. By Lemma 4.5,

$$\sum_{(n)} k^m\cdot|x_k|\cdot\| D_k\| = |x_n|\cdot n^m\cdot o(f(n)) \quad \text{for } m \in \mathbb{R}.$$

By Proposition 4.4, the recurrence (4.6) has a solution $\{y_n^{(0)}\}$ such that

$$|y_n^{(0)}| \leq c_0\cdot \sum_{i=1}^{L} n^{i-1}\cdot\sum_{(n)} k^{L-i}\cdot|x_k|\cdot\| D_k\| = |x_n|\cdot o(f(n))$$

with c_0 depending only on M. Define t_n such that for $n \in \mathbb{N}$

$$t_n\cdot f(n)\cdot|x_n| = c_0\cdot \sum_{i=1}^{L} n^{i-1}\cdot\sum_{(n)} k^{L-i}\cdot|x_k|\cdot\| D_k\|.$$

Then $\lim_{n\to\infty} t_n = 0$ and $|y_n^{(0)}| \leq t_n\cdot f(n)\cdot|x_n|$. We may assume that $t_n \leq \frac{1}{2}$ for $n \geq 0$. We show that a sequence $\{y_n^{(1)}\},\{y_n^{(2)}\},\ldots$ can be found such that

(i) $$y_{n+1}^{(i)} = M\cdot y_n^{(i)} + D_n\cdot y_n^{(i-1)} \qquad (i \geq 1)$$

(ii) $$|y_n^{(i)}| \leq 2^{-i}\cdot t_n\cdot f(n)\cdot|x_n| \qquad (i \geq 0).$$

We proceed by induction.

Suppose that $\{y_n^{(1)}\},\ldots,\{y_n^{(j-1)}\}$ exist such that (i) and (ii) hold for $i \leq j-1$. Consider the inhomogeneous matrix recurrence

(4.7) $$y_{n+1} = M\cdot y_n + D_n\cdot y_n^{(j-1)} \qquad (n \in \mathbb{N}).$$

Since $|y_n^{(j-1)}| \leq 2^{1-j}\cdot t_n\cdot f(n)\cdot|x_n|$ for $n \in \mathbb{N}$, we can rewrite (4.7) as

(4.8) $$y_{n+1} = M\cdot y_n + D_n^{(j)}\cdot x_n$$

where $\| D_n^{(j)}\| \leq 2^{-j}\cdot\| D_n\|$.

Applying Proposition 4.4, we find that (4.8) has a solution $\{y_n^{(j)}\}$ such that

$$|y_n^{(j)}| \leq c_0\cdot \sum_{i=1}^{L} n^{i-1}\cdot\sum_{(n)} k^{L-i}\cdot|x_k|\cdot\| D_k^{(j)}\| \leq 2^{-j}\cdot t_n\cdot f(n)\cdot|x_n| \quad (n \in \mathbb{N}).$$

Since $\{y_n^{(j)}\}$ is also a solution of (4.7), it satisfies conditions (i) and (ii) for $j = i$. Put

$$w_n = \sum_{i=o}^{\infty} y_n^{(i)} \qquad (n \in \mathbb{N}).$$

Clearly, the sum converges for $n \geq 0$, and

$$|w_n| \leq \sum_{i=0}^{\infty} |y_n^{(i)}| \leq 2\cdot t_n\cdot f(n)\cdot|x_n|.$$

Hence,

$$\lim_{n\to\infty} \frac{|w_n|}{|x_n|} = 0 \quad \text{and} \quad \frac{|w_n|}{|x_n|} = o(f(n)).$$

Moreover, since

$$y_{n+1}^{(i+1)} = M\cdot y_n^{(i+1)} + D_n\cdot y_n^{(i)} \qquad (i \geq 0)$$

and

$$y_{n+1}^{(0)} = M\cdot y_n^{(0)} + D_n\cdot x_n \qquad (n \in \mathbb{N}),$$

$\{w_n\}$ satisfies

$$w_{n+1} = M\cdot w_n + D_n\cdot w_n + D_n\cdot x_n .$$

Further,

$$x_{n+1} = M\cdot x_n$$

so that, if we define $z_n = w_n + x_n \quad (n \in \mathbb{N})$,

$$z_{n+1} = M\cdot z_n + D_n\cdot z_n = M_n\cdot z_n$$

and

$$\frac{z_n - x_n}{|x_n|} = o(f(n)).$$

In particular, for any non-trivial solution $\{x_n\}$ of [M] there exists a solution $\{z_n\}$ of $[M_n]$ such that $\{x_n\} \sim \{z_n\}$.

Now let $\{x_n^{(1)}\},\dots,\{x_n^{(k)}\}$ be a basis of solutions of [M] such that for $\{x_n\} = \lambda_1\{x_n^{(1)}\} + \dots + \lambda_k\{x_n^{(k)}\}$ the quotient $\left|\frac{x_n^{(i)}}{x_n}\right| \leq C$ if $\lambda_i \neq 0$. (C depending only on M and the coefficients $\lambda_1,\dots\lambda_k$ but independent of i and n). That such a basis exists, has been shown in §4.2 (cf. (4.3)). (Although it has been shown only for M in Jordan normal form, the result generalizes easily for general M). Let $\{z_n^{(1)}\},\dots,\{z_n^{(k)}\}$ be solutions of $[M_n]$ such that

$$\frac{z_n^{(i)} - x_n^{(i)}}{|x_n^{(i)}|} = o(f(n)) \qquad (1 \leq i \leq k).$$

We show that $\{z_n^{(1)}\},\dots,\{z_n^{(k)}\}$ form a basis of solutions of $[M_n]$. Suppose this is not so. Then there exist $\lambda_1,\dots,\lambda_k \in \mathbb{C}$, not all zero, such that

$$\{\ 0\ \} = \lambda_1\cdot\{z_n^{(1)}\} + \dots + \lambda_k\cdot\{z_n^{(k)}\}.$$

Then, for $\{x_n\} = \lambda_1\cdot\{x_n^{(1)}\} + \dots + \lambda_k\cdot\{x_n^{(k)}\}$,

$$1 = \left|\frac{x_n}{x_n}\right| \leq \sum_{i=1}^{k} |\lambda_i|\cdot\left|\frac{z_n^{(i)} - x_n^{(i)}}{x_n^{(i)}}\right|\cdot\left|\frac{x_n^{(i)}}{x_n}\right| \longrightarrow 0 \quad (n \to \infty),$$

which yields a contradiction.

Now let $\{z_n\}$ be an arbitrary solution of $[M_n]$. Then

$$\{z_n\} = \mu_1\cdot\{z_n^{(1)}\} + \dots + \mu_k\cdot\{z_n^{(k)}\}.$$

Put $\{x_n\} = \mu_1\cdot\{x_n^{(1)}\} + \ldots + \mu_k\cdot\{x_n^{(k)}\}$. Then

$$\left|\frac{z_n - x_n}{x_n}\right| \le \sum_{i=1}^{k} |\mu_i|\cdot\left|\frac{z_n^{(i)} - x_n^{(i)}}{x_n^{(i)}}\right|\cdot\left|\frac{x_n^{(i)}}{x_n}\right| = o(f(n)).\square$$

Proof of Corollary 4.2. Let $[M_n^R]$ be the matrix recurrence associate to R. Put $M^R = \lim M_n^R$. Since all eigenvalues of M^R have geometric multiplicity one, the minimal polynomial of M^R is χ_R (up to a non-zero factor). Hence, $\sum n^{L-1}\cdot\frac{1}{f(n)}\cdot\| M^R - M_n^R \|$ converges. Let $\{u_n^{(1)}\},\ldots,\{u_n^{(k)}\}$ be a basis of $Z(\chi_R(T))$ such that

$$\lim_{n\to\infty} \frac{u_{n+1}^{(i)}}{u_n^{(i)}} = \alpha_i \qquad \text{(where } \alpha_1,\ldots,\alpha_k \text{ are the zeros of } \chi_R\text{)}.$$

It is clear from §4.2 that such a basis exists. Put

$$x_n^{(i)} = \begin{bmatrix} u_{n+k-1}^{(i)} \\ \vdots \\ u_n^{(i)} \end{bmatrix} \qquad (i = 1,\ldots,k;\ n \in \mathbb{N}).$$

$\{u_n^{(1)}\},\ldots,\{u_n^{(k)}\}$ is a basis of solutions of [M]. By Theorem 4.1 there exist solutions $\{y_n^{(1)}\},\ldots,\{y_n^{(k)}\}$ of $[M_n^R]$ such that

$$\left|\frac{y_n^{(i)} - x_n^{(i)}}{x_n^{(i)}}\right| = o(f(n)), \tag{4.9}$$

where $y_n^{(i)}$ is of the form

$$y_n^{(i)} = \begin{bmatrix} v_{n+k-1}^{(i)} \\ \vdots \\ v_n^{(i)} \end{bmatrix} \qquad (i = 1,\ldots,k;\ n \in \mathbb{Z}_{\ge m})$$

with $\{v_n^{(i)}\} \in Z(R)$ $(i = 1,\ldots,k)$. Since $\{y_n^{(1)}\},\ldots,\{y_n^{(k)}\}$ is a basis of solutions of $[M_n^R]$, $\{v_n^{(1)}\},\ldots,\{v_n^{(k)}\}$ is a basis of Z(R). Moreover, by (4.9) we have that

$$\left|\frac{u_n^{(i)} - v_n^{(i)}}{x_n^{(i)}}\right| = o(f(n)) \qquad (i = 1,\ldots k).$$

Since $\lim_{n\to\infty} \frac{u_{n+1}^{(i)}}{u_n^{(i)}} = \alpha_i$ and $\alpha_i \ne 0$, we find that

$$\frac{u_n^{(i)}}{v_n^{(i)}} - 1 = o(f(n)) \qquad (i = 1,\dots,k).\square$$

If $\{M_n\}$ converges very fast, we can easily derive a factorization theorem for $\{M_n\}$, like in Chapter 3.

Theorem 4.6: *Let $[M_n] \in \mathcal{M}(\mathbb{C})$, $M = \lim M_n$. Let $P \in \mathbb{C}[X]$ be the minimal polynomial of M, and let L be the maximum of the multiplicities of the zeros of P. Suppose that M has no eigenvalue zero and that $\sum n^{2L-2}\cdot\frac{1}{f(n)}\cdot\| M - M_n\|$ converges for some $f \in \mathcal{M}$. Then there exists a sequence of $k \times k$-matrices $\{S_n\}$ such that*

(i) $\qquad \| S_n - I \| = o(f(n)) \quad (n \to \infty)$

(ii) $\qquad M_n = S_{n+1}\cdot M\cdot S_n^{-1} \quad (n \in \mathbb{N})$.

Proof: Without loss of generality we may suppose that M has only eigenvalues with absolute value one and is in Jordan normal form. Then, using the notation introduced in §4.2, we may assume that $M = \bar{M}$, whence $\| M^n\| \ll \max_{i,j} |x_n^{(i,j)}| \le$ $\le c\cdot|n|^{L-1}$, where c is independent of n ($n \in \mathbb{Z}$). Put $D_n = M_n - M$. Consider the inhomogeneous recurrence equation

$$(4.10) \qquad Y_{n+1} = M\cdot Y_n + D_n\cdot M^n$$

where $Y_n \in \mathbb{C}^{k,k}$ ($n \in \mathbb{N}$). The solution of (4.10) is

$$Y_n = M^n\cdot(Y_0 + \sum_{k=0}^{n-1} M^{-k-1}\cdot D_k\cdot M^k) \qquad (n \in \mathbb{N}).$$

Since

$$\left\| \sum_{k=0}^{n-1} M^{-k-1}\cdot D_k\cdot M^k \right\| \ll \sum_{k=0}^{n-1} \| D_k\|\cdot\| M^k \|^2 \ll \sum_{k=0}^{n-1} \| D_k\|\cdot k^{2L-2}$$

and since the latter sum converges, we have that $\sum_{k=0}^{\infty} M^{-k-1}\cdot D_k\cdot M^k$ converges. Choose $Y_0^{(0)} = -\sum_{k=0}^{\infty} M^{-k-1}\cdot D_k\cdot M^k$ and let $\{Y_n^{(0)}\}$ be a solution of (4.10). Then $Y_n^{(0)} = -M^n\cdot \sum_{k=n}^{\infty} M^{-k-1}\cdot D_k\cdot M^k$ ($n \in \mathbb{N}$). Put $t_n\cdot f(n) = \sum_{k=n}^{\infty} \| M^{-k-1}\cdot D_k\cdot M^k \|$. Then $\lim_{n\to\infty} t_n = 0$ and $\| Y_n^{(0)}\cdot M^{-n} \| \le t_n\cdot f(n)$ ($n \in \mathbb{N}$). Without loss of generality we may assume that $t_n \le 1/2$ for $n \in \mathbb{N}$. We show that a sequence $\{Y_n^{(1)}\},\{Y_n^{(2)}\},\dots$

can be found such that

(i) $$Y^{(i)}_{n+1} = M\cdot Y^{(i)}_n + D_n\cdot Y^{(i-1)}_n \qquad (i \geq 1,\ n \in \mathbb{N}),$$

(ii) $$\| Y^{(i)}_n \cdot M^{-n} \| \leq 2^{-i}\cdot t_n\cdot f(n) \qquad (i \geq 0,\ n \in \mathbb{N}).$$

We proceed by induction. Suppose that $\{Y^{(1)}_n\},\dots,\{Y^{(j-1)}_n\}$ exist such that (i) and (ii) hold for $i \leq j-1$. Consider the recurrence equation

(4.11) $$Y_{n+1} = M\cdot Y_n + D_n\cdot Y^{(j-1)}_n.$$

Since $\| Y^{(j-1)}_n\cdot M^{-n} \| \leq 2^{-1+j}\cdot t_n\cdot f(n)$ for $n \in \mathbb{N}$, we can rewrite (4.11) as

(4.12) $$Y_{n+1} = M\cdot Y_n + D^{(j)}_n\cdot M^n,$$

where $\| D^{(j)}_n \| \leq 2^{-j}\cdot \| D_n \|$. As above, and by the definition of t_n, we find that (4.12) has a solution $\{Y^{(j)}_n\}$ with $\| Y^{(j)}_n\cdot M^{-n} \| \leq 2^{-j}\cdot t_n\cdot f(n)$ $(n \in \mathbb{N})$. As $\{Y^{(j)}_n\}$ is also a solution of (4.11) it satisfies conditions (i) and (ii) for $i = j$. Put $W_n = \sum_{i=0}^{\infty} Y^{(i)}_n$ $(n \in \mathbb{N})$. Clearly, the sum converges for all n and

$$\| W_n\cdot M^{-n} \| \leq \sum_{i=0}^{\infty} \| Y^{(i)}_n M^{-n} \| \leq 2\cdot t_n\cdot f(n).$$

Hence, $\| W_n\cdot M^{-n} \| = o(f(n))$. Moreover, since

$$Y^{(i)}_{n+1} = M\cdot Y^{(i)}_n + D_n\cdot Y^{(i-1)}_n \qquad (i \geq 1)$$

and

$$Y^{(0)}_{n+1} = M\cdot Y^{(0)}_n + D_n\cdot M^n \qquad \text{for } n \in \mathbb{N},$$

we obtain that

$$W_{n+1} = M\cdot W_n + D_n\cdot W_n + D_n\cdot M^n = M_n\cdot W_n + D_n\cdot M^n,$$

so that

$$Z_{n+1} := M^{n+1} + W_{n+1} = M_n\cdot(M^n + W_n) = M_n\cdot Z_n \qquad (n \in \mathbb{N})$$

and $\| Z_n\cdot M^{-n} - I \| = o(f(n))$ as $n \to \infty$. Put $S_n = Z_n\cdot M^{-n}$ $(n \in \mathbb{N})$. Then

$$S_{n+1}\cdot M\cdot S^{-1}_n = Z_{n+1}\cdot M^{-n}\cdot(Z_n\cdot M^{-n})^{-1} = Z_{n+1}\cdot Z^{-1}_n) = M_n$$

for all n, and

$$\| S_n - I \| = o(f(n)) \qquad (n \to \infty).\square$$

CHAPTER FIVE

SECOND-ORDER RECURRENCES (1)

§1. Introduction.

In both this and the following chapter we shall study the behaviour of the solutions of operators in $\mathfrak{R}(\mathbb{C})$ of order two which have a monic characteristic polynomial. It will be useful to introduce the concept of an eigenvalue of an operator.

Let $R \in \mathfrak{R}(\mathbb{C})$, $\chi_R \in \mathbb{C}[X]$. If $\chi_R(\alpha) = 0$, we call α an *eigenvalue* of R.
We distinguish three cases:
(1). The eigenvalues have distinct absolute values.
(2). The eigenvalues are equal.
(3). The eigenvalues have the same absolute value, but are not equal.
The last case will be treated in chapter six.

Let $R \in \mathfrak{R}(\mathbb{C})$, $\chi_R(X) = (X-\alpha)(X-\beta)$. The associated matrix recurrence $[M_n^R]$ has limit matrix $M^R = \begin{bmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{bmatrix}$, which has eigenvalues α and β. The geometric multiplicity of α is one. Conversely, let $[M_n]$ be some matrix recurrence of order two where $M = \lim M_n$ exists and has no eigenvalues with geometric multiplicity two (which amounts to saying that the minimal polynomial of M has degree two). By linear algebra, there exists a conjugate matrix recurrence $[N_n]$ with $N = \lim N_n = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}$, where α and β are the eigenvalues of M. For a solution $\{x_n\}$ of $[N_n]$ we have:

$$(5.1) \qquad \begin{aligned} (x_{n+1})_1 &= (\alpha + \delta_{11}(n))\cdot(x_n)_1 + (1 + \delta_{12}(n))\cdot(x_n)_2 \\ (x_{n+1})_2 &= \delta_{21}(n)\cdot(x_n)_1 + (\beta + \delta_{22}(n))\cdot(x_n)_2 \end{aligned}$$

where $(\delta_{ij}(n)) = N_n - N \quad (n \in \mathbb{N})$. Hence,

$$(x_{n+2})_1 = (\alpha + \delta_{11}(n+1))\cdot(x_{n+1})_1 + (1 + \delta_{12}(n+1))\cdot\delta_{21}(n)\cdot(x_n)_1$$
$$+ (1 + \delta_{12}(n+1))\cdot(\beta + \delta_{22}(n))\cdot\frac{(x_{n+1})_1 - (\alpha + \delta_{11}(n))\cdot(x_n)_1}{1 + \delta_{12}(n)},$$

so that $\{(x_n)_1\}$ is a root of a recurrence operator $R \in \mathfrak{R}(\mathbb{C})$ with characteristic polynomial $\chi_R(X) = (X-\alpha)(X-\beta)$. Since $\{x_n\}$ is completely determined by $\{(x_n)_1\}$ (by (5.1)) and dim Z(R) is equal to the number of linearly independent solutions of $[N_n]$, it follows that for each zero $\{y_n\} \in Z(R)$ there exists a corresponding solution $\begin{pmatrix} y_n \\ z_n \end{pmatrix}$ of $[N_n]$, where

$$z_n \cdot (1 + \delta_{12}(n)) = y_{n+1} - (\alpha + \delta_{11}(n)) \cdot y_n.$$

It thus follows that it is no restriction of generality to study second-order operators instead of second order matrix recurrences, if the limit matrix has only eigenvalues with geometric multiplicity one. (So we exclude the case that the limit matrix is $\alpha \cdot I$ for $\alpha \in \mathbb{C}$ and I the identity matrix).

Let R be as above. If we want to investigate the behaviour of the zeros of R, it is sufficient to study the behaviour of the zeros of one of the zeroth-order transforms of R. We shall normalize the operators in the following way: Put $R = T^2 - p_n \cdot T - q_n$. Suppose that $p_n, q_n \neq 0$ for $n \geq N$. (If $p_n = 0$ for all $n \in \mathbb{N}$, then it is easy to calculate the zeros of R). Put

$$S = \prod_{k=N}^{n-2} (2/p_k) \cdot I = s_n \cdot I.$$

Then

$$R/S = s_{n+2} \cdot (T^2 - p_n \cdot T - q_n) \cdot \frac{1}{s_n} = T^2 - p_n \cdot \frac{s_{n+2}}{s_{n+1}} \cdot T - q_n \cdot \frac{s_{n+2}}{s_n} = T^2 - 2 \cdot T - \frac{4 \cdot q_n}{p_n p_{n-1}}$$

for $n \geq N + 1$.

Remark 5.1.1. Note that the normalized operators do not always have a characteristic polynomial. If $R \in \mathfrak{R}(\mathbb{C})$, $\chi_R(X) = X^2 - \alpha^2$ for $\alpha \in \mathbb{C}^*$, and $R/S = T^2 - 2 \cdot T + Q_n$ is a zeroth-order transform of R, then $\lim |Q_n| = \infty$.

§2. Simple operators of order two.

This case has been treated in Chapter 3. We shall state the result of Theorem 3.15 for recurrence operators.

Theorem 5.1. *Let $R \in \mathfrak{R}(K)$, $K = \mathbb{R}$ or $\mathbb{C}$, $\mathrm{ord}(R) = 2$ and $\chi_R(X) = (X-\alpha)(X-\beta)$, where $\alpha, \beta \in K$ and $|\alpha| \neq |\beta|$. Suppose that $f: \mathbb{N} \longrightarrow \mathbb{R}_{>0}$ is a monotonically non-increasing function such that $\lim_{n\to\infty} f(n) = 0$ and $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$ and such that $N_n(R - \chi_R(T)) = \mathcal{O}(f(n))$. Then $R = (T - \beta_n)(T - \alpha_n)$ with $\alpha_n, \beta_n \in K$ for all n and $\alpha_n - \alpha = \mathcal{O}(f(n))$, $\beta_n - \beta = \mathcal{O}(f(n))$. Moreover, if $\sum N_n(R - \chi_R(T)) <$ $< \infty$, then R has zeros $\{u_n^{(1)}\}, \{u_n^{(2)}\}$ such that*

$$\lim_{n\to\infty} \frac{u_n^{(1)}}{\alpha^n} = \lim_{n\to\infty} \frac{u_n^{(2)}}{\beta^n} = 1$$

(unless α or β is zero, in which case one of the limits is not defined).

Proof: Let $U \in K^{2,2}$ such that $U^{-1}M^RU = \begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}$. Put $M'_n = U^{-1}M^R_nU$. By Theorem 3.15, there exists a sequence of matrices $\{B_n\}$, $B_n \in K^{2,2}$, with $\| B_n - I\| = \mathcal{O}(f(n))$ such that

$$B_{n+1}M'_nB_n^{-1} = \begin{bmatrix}\alpha'_n & 0\\ 0 & \beta'_n\end{bmatrix}$$

where $\alpha'_n,\beta'_n \in K$ and $\alpha - \alpha'_n = \mathcal{O}(f(n))$, $\beta - \beta'_n = \mathcal{O}(f(n))$.

The solutions of $[B_{n+1}M'_nB_n^{-1}]$ are of the form $\{B_nU^{-1}\begin{bmatrix}u_{n+1}\\ u_n\end{bmatrix}\}$ for $\{u_n\} \in Z(R)$.

Let $V \in \mathfrak{R}(K)$ be such that $\mathrm{ord}(V) = 1$ and $V(u_n) = (B_nU^{-1}\begin{bmatrix}u_{n+1}\\ u_n\end{bmatrix})_1$ for $\{u_n\} \in Z(R)$. Then $\{V(u_n)\} \in Z(T - \alpha'_n)$. Hence, $R = r_n\cdot(T - \alpha'_n)\cdot V$ for some $\{r_n\}$, $r_n \in K$. The operator V is of the form $V = c\cdot b_n(T - \beta_n)$, where $c \in K^*$, and $b_n,\beta_n \in K$ for all n, and $b_n - 1 = \mathcal{O}(f(n))$. Moreover, since $\alpha - \alpha'_n = \mathcal{O}(f(n))$ and $\beta_{n+1} - \beta + \alpha'_n\cdot\frac{b_n}{b_{n+1}} - \alpha = \mathcal{O}(f(n))$, we have that $\beta_n - \beta = \mathcal{O}(f(n))$ and $\alpha'_n\cdot\frac{b_n}{b_{n+1}} - \alpha = \mathcal{O}(f(n))$. Put $\alpha_n := \alpha'_n\cdot\frac{b_n}{b_{n+1}}$. This yields the desired result.
The second assertion follows immediately from Corollary 3.17. □

§3. Non-simple operators with two equal eigenvalues: Fast convergence.

Let $R \in \mathfrak{R}(K)$, $\mathrm{ord}(R) = 2$, $\chi_R(X) = (X - \alpha)^2$. We suppose that $\alpha \neq 0$. If $\alpha = 0$ and the coefficients of R behave neatly, there is in many cases a zeroth-order transform of R with eigenvalues that are not both zero. The following result follows from Corollary 4.2:

Corollary 5.2. *Let* $R \in \mathfrak{R}(K)$, $\mathrm{ord}(R) = 2$, *and* $\chi_R(X) = (X-\alpha)^2$, $\alpha \neq 0$. *Suppose that* $\sum n\cdot N_n(R - \chi_R(T)) < \infty$. *Then* R *has zeros* $\{u_n^{(1)}\}$ *and* $\{u_n^{(2)}\}$ *such that*

$$\lim_{n\to\infty} \frac{u_n^{(1)}}{n\cdot\alpha^n} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{u_n^{(2)}}{\alpha^n} = 1.$$

Proof: For $K = \mathbb{C}$ the result follows immediately from Corollary 4.2. For $K = \mathbb{R}$ the result follows from the complex case by replacing $\{u_n^{(j)}\}$ by $\{(u_n^{(j)} + \bar{u}_n^{(j)})/2\}$ for $j = 1,2$. □

If the coefficients of R converge more slowly, the result of Corollary 5.2

is no longer valid. There are even cases in which there are no zeros $\{u_n\} \in Z(R)$ such that $\frac{u_{n+1}}{u_n}$ converges to an eigenvalue of R. The following case is an example of such a result.

Proposition 5.3. *Let* $R \in \mathfrak{R}(\mathbb{R})$, $R = T^2 - 2\cdot T + 1 - C_n$, *where* $n(n+1)\cdot C_n = - 1/4 - d_n$ $(n \in \mathbb{N})$ *with* $\lim_{n\to\infty} d_n = 0$ *and* $\sum_{n=1}^{\infty} d_n/n = +\infty$. *Then* R *has no (real) zeros* $\{u_n\}$ *such that* $\frac{u_{n+1}}{u_n}$ *converges.*

Proof: Let $\{u_n\} \in Z(R)$. Note that if the limit exists, then it is equal to 1. Put $g_n = \frac{u_{n+1}}{u_n} - 1$. Then $\{g_n\}$ satisfies

$$g_{n+1} = \frac{g_n + C_n}{g_n + 1}. \tag{5.2}$$

Without loss of generality we may assume that $-d_n < 1/4$ for all n. Then it is clear that $g_{n+1} < g_n$ as long as $g_n > -1$ and if $g_n < -1$, then $g_{n+1} > 0$. Hence, the sequence $\{g_n\}$ decreases monotonically in the neighbourhood of 0. So, if $\lim g_n = 0$, we have that $0 < g_n < 1$ for $n > N$. Then $0 < ng_n < n$ for $n > N$ and

$$ng_n - (n+1)g_{n+1} = \frac{(ng_n)^2 - ng_n - n(n+1)C_n}{n + ng_n} > \frac{(ng_n - 1/2)^2 + d_n}{2n} \geq \frac{d_n}{2n}.$$

Then, by $\sum d_n/n = +\infty$, we see that $ng_n < 0$ for some $n > N$, which yields a contradiction. Hence, $\{g_n\}$ does not converge and therefore $\frac{u_{n+1}}{u_n}$ does not converge either. □

The aim of the rest of this chapter is to investigate some other cases for which the behaviour of the coefficients is regular. We shall see that in many of these cases the operator $R \in \mathfrak{R}(\mathbb{C})$ has a zero $\{u_n\}$ such that $\frac{u_{n+1}}{u_n}$ converges. First we make some preparations.

For the rest of this chapter we suppose that the recurrence operator is normalized in the way described in §5.1, unless stated otherwise. Hence we put $R = T^2 - 2\cdot T + Q(n)$, where $Q(n) = 1 - C_n$, $\lim_{n\to\infty} C_n = 0$. If $\{u_n\}$ is a non-trivial zero of R, we put $g_n = \frac{u_{n+1}}{u_n} - 1$. Then $\{g_n\}$ satisfies (5.2). Further, if $S = \frac{1}{u_n}\cdot I$, then $\{1\} \in Z(R/S)$ and

(5.3) $$R/S = T^2 - 2\cdot\frac{u_{n+1}}{u_{n+2}}\cdot T + \frac{u_n}{u_{n+2}}\cdot Q(n) = (T - \frac{1-g_{n+1}}{1+g_{n+1}})\cdot(T - 1).$$

We first investigate the case that $n^2\cdot C_n$ converges to some non-zero complex number.

Theorem 5.4. *Let* $R \in \mathfrak{R}(\mathbb{C})$, $R = T^2 - 2\cdot T + 1-C_n$, *with* $\lim_{n\to\infty} n^2\cdot C_n = \gamma$ *for* $\gamma \in \mathbb{C}$, $\gamma \notin \{r \in \mathbb{R} \mid r \leq -1/4\}$. *Then* R *has zeros* $\{u_n^{(1)}\}$ *and* $\{u_n^{(2)}\}$ *such that*

(i) $$\lim_{n\to\infty} n\cdot(\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1) = \alpha \quad \textit{and} \quad \lim_{n\to\infty} n\cdot(\frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1) = 1-\alpha,$$

where α *is the root of* $X^2 - X - \gamma$ *with* $\mathrm{Re}\ \alpha > 1/2$.

(ii) *If* $\sum |n\cdot C_n - \gamma/n|$ *converges, then* R *has zeros* $\{v_n^{(1)}\}$ *and* $\{v_n^{(2)}\}$ *such that*

$$\lim_{n\to\infty} \frac{v_n^{(1)}}{n^\alpha} = \lim_{n\to\infty} \frac{v_n^{(2)}}{n^{1-\alpha}} = 1.$$

Corollary 5.4. *Under the conditions of the first part of Theorem* 5.4, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ *for every non-trivial zero* $\{u_n\}$ *of* R. *If* $C_n \in \mathbb{R}$ $(n \in \mathbb{N})$, *then we can find* $\{u_n^{(1)}\},\{u_n^{(2)}\},\{v_n^{(1)}\},\{v_n^{(2)}\}$ *such that* $u_n^{(1)},u_n^{(2)},v_n^{(1)},v_n^{(2)} \in \mathbb{R}$ $(n \in \mathbb{N})$.

Proof of Corollary: Let $\{u_n\} \in Z(R)$, $\{u_n\} \neq \{0\}$. Then $\{u_n\} = \lambda\cdot\{u_n^{(1)}\} + \mu\cdot\{u_n^{(2)}\}$ with $\lambda,\mu \in \mathbb{C}$, not both zero. By Theorem 5.4(i), $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$. Hence, if $\lambda \neq 0$, then

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\mu\cdot(u_{n+1}^{(2)}/u_{n+1}^{(1)}) + \lambda}{\mu\cdot(u_n^{(2)}/u_n^{(1)}) + \lambda}\cdot\frac{u_{n+1}^{(1)}}{u_n^{(1)}} = 1,$$

and if $\lambda = 0$, the result follows immediately from Theorem 5.4(i). If $C_n \in \mathbb{R}$, $\{\bar{u}_n^{(2)}\} \in Z(R)$. Since $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n} = 0$ for all $\{u_n\} \in Z(R)$ linearly independent with $\{u_n^{(2)}\}$, we must have that $\{u_n\}$ and $\{\bar{u}_n\}$ are linearly dependent. So, by multiplication with a suitable constant, we can take $u_n \in \mathbb{R}$ $(n \in \mathbb{N})$. Since for $\{u_n^{(1)}\}$ we can take any zero linearly independent with $\{u_n^{(2)}\}$, we can choose $u_n^{(1)} \in \mathbb{R}$ $(n \in \mathbb{N})$ as well. The same argument applies to $\{v_n^{(1)}\},\{v_n^{(2)}\}$. □

For the proof of Theorem 5.4 we need some lemmas.

Lemma 5.5. *Let $\{a_n\}$ be a sequence of real numbers satisfying*

(5.4) $$a_{n+1} = (1 - e_n)\cdot a_n + e'_n$$

where $e_n, e'_n \in \mathbb{R}$, $e_n > 0$, $\lim_{n\to\infty} e_n = 0$, $\sum_{n=1}^{\infty} e_n$ *diverges and* $\frac{e'_n}{e_n} \longrightarrow 0$ *as* $n \longrightarrow \infty$. *Then* $\{a_n\}$ *converges to zero.*

Proof: Note that

(5.5) $$a_{n+1} < a_n \iff \frac{e'_n}{e_n} < a_n.$$

Choose $0 < \varepsilon < 1/2$. Let N be so large that $e_n < \varepsilon$ and $|e'_n| < e_n\cdot\varepsilon$ for $n \ge N$.

If $|a_{n+1}| > \varepsilon$ for some $n \ge N$, then $|a_n| > \frac{\varepsilon - |e'_n|}{1 - e_n} \ge \varepsilon$. Hence, either $|a_n| > \varepsilon$ for all $n \ge N$, or $|a_n| \le \varepsilon$ for $n \ge N' \ge N$. In the former case, we have by (5.3) and (5.4) that $|a_{n+1}| < |a_n|$ for $n \ge N$. Then $\{|a_n|\}$ converges to some number $a \ge \varepsilon$. On the other hand, by $\lim_{n\to\infty} e'_n/e_n = 0$,

$$2\cdot e_n\cdot a > |\ a_{n+1} - a_n\ | = |\ e_n\cdot a_n - e'_n\ | > e_n\cdot a/2$$

for n large enough. Since $\lim_{n\to\infty} e_n = 0$, $\sum e_n$ diverges and $\{a_n\}$ is monotonic for $n \ge N$, this yields a contradiction. Thus, $|a_n| \le \varepsilon$ for $n \ge N'$. As ε is arbitrary, we have that $\lim_{n\to\infty} a_n = 0$. □

Lemma 5.6. *Let $\{\lambda_n\}$ be a sequence of non-zero complex numbers such that* $\frac{\lambda_{n+1}}{\lambda_n} = 1 + o(n^{-1})$. *Let* $\alpha \in \mathbb{C}$, Re $\alpha > 1$. *Then*

$$\lim_{n\to\infty} \frac{\sum_{k=n}^{\infty} \lambda_k\cdot k^{-\alpha}}{\lambda_n\cdot n^{1-\alpha}} = \frac{1}{\alpha - 1}.$$

Proof: Put $\frac{\lambda_{n+1}}{\lambda_n} = 1 + \frac{\gamma_n}{n}$, where $\gamma_n \in \mathbb{C}$, $\gamma_n \longrightarrow 0$ $(n \longrightarrow \infty)$. Choose $\varepsilon > 0$. Let $N' \ge N$ be so large that $|\gamma_n| < \varepsilon$ for $n \ge N'$. Then

$$\left| \frac{\lambda_{n+k}}{\lambda_n} - 1 \right| = \left| \prod_{\ell=n}^{n+k-1} (1 + \frac{\gamma_\ell}{\ell}) - 1 \right| \le \frac{\varepsilon\cdot k}{n},$$

since the evaluation of the expression in the second term gives a sum of terms such that each of their moduli are smaller than ε times the corresponding term that appears on evaluating the expression $\prod_{\ell=n}^{n+k-1} (1 + \frac{1}{\ell}) - 1$, which is equal to k/n. Hence,

$$\sum_{k=n}^{\infty} \frac{\lambda_k}{\lambda_n} \cdot k^{-\alpha} - \frac{1}{\alpha - 1} \cdot n^{1-\alpha} = \sum_{k=n}^{\infty} k^{-\alpha} - \frac{1}{\alpha - 1} \cdot n^{1-\alpha} + \sum_{k=n}^{\infty} \frac{\varepsilon_{nk} \cdot k}{n} \cdot k^{-\alpha},$$

where $|\varepsilon_{nk}| < \varepsilon$ $(n \geq N', k \geq n)$. By the formula of Euler-MacLaurin,

$$\sum_{k=n}^{\infty} k^{-\alpha} - \frac{1}{\alpha - 1} \cdot n^{1-\alpha} = O(|n^{-\alpha}|).$$

Moreover,

$$\left| \sum_{k=n}^{\infty} \frac{\varepsilon_{nk} \cdot k}{n} \cdot k^{-\alpha} \right| \leq \frac{\varepsilon \cdot c(\alpha)}{|n^{\alpha - 1}|},$$

with $c(\alpha)$ some positive number depending only on α. So we have

$$\left| \frac{\sum_{k=n}^{\infty} \frac{\lambda_k}{\lambda_n} \cdot k^{-\alpha} - \frac{1}{\alpha - 1} \cdot n^{1-\alpha}}{n^{1-\alpha}} \right| \leq \varepsilon \cdot c_1(\alpha),$$

where $c_1(\alpha)$ is some positive number depending only on α. Since ε can be chosen arbitrarily small, we obtain the desired result. □

Lemma 5.7. *If* Re $\alpha - 1/2 = r$ *for* $\alpha \in \mathbb{C}$, *then*

$$\lim_{n \to \infty} (|n + \alpha| - |n + 1 - \alpha|) = 2r.$$

Proof: Put $\alpha = r + is + 1/2$. Then

$$\lim_{n \to \infty} (|n + r + is + 1/2| - |n - r + is + 1/2|)$$

$$= \lim_{n \to \infty} \frac{(n + r + 1/2)^2 + s^2 - (n - r + 1/2)^2 - s^2}{|n + r + is + 1/2| + |n - r + is + 1/2|}$$

$$= \lim_{n \to \infty} \frac{4rn + o(n)}{2n + o(n)} = 2r.$$ □

Proof of Theorem 5.4: Put $h_n = n(\frac{u_{n+1}}{u_n} - 1) - \alpha$ for $\{u_n\} \in Z(R)$. Then

$$h_{n+1} = \frac{(1 + \frac{1-\alpha}{n}) \cdot h_n + d_n}{h_n/n + 1 + \alpha/n} \quad (n \in \mathbb{N}) \tag{5.6}$$

with $d_n = (n + 1) \cdot C_n - \gamma/n = o(n^{-1})$. So, if $|h_n| < |n + \alpha|$,

$$|h_{n+1}| \leq \frac{|1 + \frac{1-\alpha}{n}| \cdot |h_n| + |d_n|}{|1 + \alpha/n| - |h_n|/n}.$$

We show that $\lim_{n \to \infty} h_n = 0$ for some solution $\{h_n\}$ of (5.6). (Note that this implies $\lim_{n\to\infty} n\cdot(\frac{u_{n+1}}{u_n} - 1) = \alpha$ for some zero $\{u_n\}$ of R). Choose $0 < \varepsilon < r/4$, $\varepsilon < 1$ for $r = \mathrm{Re}\,\alpha - 1/2 > 0$. Choose N so large that for $n \geq N$

$$n\cdot|d_n| < r\varepsilon/2, \quad |n + \alpha| - |n + 1 - \alpha| > r, \text{ and } N > 2|\alpha|.$$

Take some sequence $\{u_n\} \in Z(R)$ such that

$$3\varepsilon/4 \leq \left| N\cdot(\frac{u_{N+1}}{u_N} - 1) - \alpha \right| \leq \varepsilon.$$

Then, with $\{h_n\}$ as defined above, $|h_n| \leq \varepsilon < |n + \alpha|/2$ $(n \geq N)$ implies

$$|h_{n+1}| \leq \frac{|n + 1 - \alpha|\cdot\varepsilon + r\varepsilon/2}{|n + \alpha| - \varepsilon} < \varepsilon.$$

So $|h_n| < \varepsilon$ for all $n \geq N$. Then

$$(- |n + \alpha| + |n + 1 - \alpha|)\cdot|h_n| + n\cdot|d_n| + |h_n|^2$$
$$< |h_n|\cdot(|h_n| - r) + r\varepsilon/2 < 3\varepsilon\cdot(\varepsilon - r)/4 + r\varepsilon/2 < -\varepsilon^2/4,$$

so that

$$|h_{n+1}| - |h_n| < \frac{-\varepsilon^2}{4|n + \alpha|} < 0$$

as long as $|h_n| > 3\varepsilon/4$. By subsequently changing the value of ε properly, we find that $\lim_{n\to\infty} h_n = 0$. Further, if $\sum |d_n|$ converges, then $\sum \frac{|h_n|}{n}$ converges as well. For by (5.6) we have

$$h_{n+1} = h_n\cdot(1 + \frac{1 - 2\alpha - h_n}{n}) + d'_n$$

where $\sum |d'_n| < \infty$. Putting

$$\Gamma_n = \prod_{k=N}^{n-1} (1 + \frac{1 - 2\alpha - h_k}{k})$$

for $n \geq N$ we obtain

$$h_n = \Gamma_n\cdot h_N + \Gamma_n\cdot \sum_{k=N}^{n-1} \frac{d'_k}{\Gamma_{k+1}}.$$

Since $|1 + \frac{1 - 2\alpha - h_n}{n}| < 1 - \delta'/n$ for some $\delta' > 0$ and $n \geq N' \geq N$, we have that $\Gamma_n \longrightarrow 0$ as $n \longrightarrow \infty$ and that $\sum_{k=N}^{\infty} \frac{|\Gamma_n|}{n} < \infty$, so that

$$\sum_{k=N'}^{\infty} \frac{|h_n|}{n} \leq \sum_{k=N'}^{\infty} \frac{|\Gamma_n|}{n}\cdot|h_N| + \sum_{n=N'}^{\infty} \frac{|\Gamma_n|}{n}\cdot \sum_{k=N}^{N'-1} \frac{|d'_k|}{|\Gamma_{k+1}|} + \sum_{n=N'}^{\infty} \frac{|\Gamma_n|}{n}\cdot \sum_{k=N'}^{n-1} \frac{|d'_k|}{|\Gamma_{k+1}|}$$
$$< c_1 + c_2\cdot \sum_{k=N'}^{\infty} |d'_k|\cdot k^{\varepsilon}\cdot \sum_{n=k}^{\infty} n^{-1-\varepsilon} < c_1 + c_3\cdot \sum_{k=N'}^{\infty} |d'_k| < \infty,$$

where c_2, c_3 are constants depending only on $\varepsilon, \varepsilon'$ and α, and c_1 is a constant depending on $\{h_n\}$ and $\{d'_n\}$. Then $\sum |g_n - \alpha/n|$ converges, so that

$$\frac{u_{n+1}}{u_n} = (1 + \alpha/n)(1 + \varepsilon_n) \qquad (n \geq N)$$

with $\sum |\varepsilon_n| < \infty$, which implies $\lim_{n\to\infty} \frac{u_n}{n^\alpha} = \lambda \in \mathbb{C}^*$. Now choose $\{v_n^{(1)}\} = \{u_n/\lambda\}$. Then $\{v_n^{(1)}\} \in Z(R)$ and $\lim_{n\to\infty} \frac{v_n^{(1)}}{n^\alpha} = 1$.

For the second part of the proof, put $S = \frac{1}{u_n^{(1)}}\cdot I$. Then, as in (5.3), $R/S = (T - \frac{1-g_{n+1}}{1+g_{n+1}})(T - 1)$ for some $\{g_n\}$. Let $\{w_n\} \in Z(T - \frac{1-g_{n+1}}{1+g_{n+1}})$, $\{w_n\} \neq \{0\}$. Since $g_n \sim \alpha/n$ and $\mathrm{Re}\,\alpha > 1/2$ we obtain

$$w_n = \lambda_n \cdot \prod_{k=N}^{n-1} (1 - 2\alpha/k) \qquad (n \geq N),$$

where $\frac{\lambda_{n+1}}{\lambda_n} = 1 + o(n^{-1})$ and if $\sum |d_k|$ converges, then $\sum |g_k - \alpha/k|$ converges, as we saw above, so that $\lim_{n\to\infty} \lambda_n \in \mathbb{C}^*$. Since $n^{2\alpha}\cdot \prod_{k=N}^{n-1} (1 - 2\alpha/k) = c(\alpha)\cdot\Gamma(2\alpha)$ for some $c(\alpha) \in \mathbb{C}^*$ depending only on α and N (see e.g. [W],page 237), we have that $w_n = \lambda'_n \cdot n^{-2\alpha}$, where $\frac{\lambda'_{n+1}}{\lambda'_n} = 1 + o(n^{-1})$ and $\lim_{n\to\infty} \lambda'_n \in \mathbb{C}^*$ if $\sum |d_k| < \infty$. Hence, $\sum_{n=N}^{\infty} w_n$ converges absolutely. Put $v_n = u_n \cdot \sum_{k=n}^{\infty} w_k$ $(n \geq N)$. Then

$\{v_n\} \in Z(R)$ and

$$\lim_{n\to\infty} n\cdot(\frac{v_{n+1}}{v_n} - 1) = \lim_{n\to\infty} n\cdot(\frac{u_{n+1}}{u_n} - 1) - \lim_{n\to\infty} n\cdot\frac{u_{n+1}}{u_n}\cdot w_n\cdot(\sum_{k=n}^{\infty} w_k)^{-1}$$

provided that both limits exist. Using Lemma 5.6 and the fact that $\lim_{n\to\infty} n\cdot(\frac{u_{n+1}}{u_n} - 1) = \alpha$ we find

$$\lim_{n\to\infty} n\cdot(\frac{v_{n+1}}{v_n} - 1) = \alpha - \lim_{n\to\infty} (1 + \alpha/n + o(n^{-1}))\cdot(-1 + 2\alpha + o(1)) = 1-\alpha.$$

Note that $\lim_{n\to\infty} \frac{v_n}{u_n} = 0$. Further, if $\sum |d_n|$ converges,

$$w_n = \lambda'_n \cdot n^{-2\alpha} \text{ and } u_n = \mu_n \cdot n^\alpha,$$

where $\lim_{n\to\infty} \lambda'_n$ and $\lim_{n\to\infty} \mu_n$ exist and are unequal to zero, so that

$$v_n = \mu_n \cdot n^{\alpha} \cdot \left(\sum_{k=n}^{\infty} \lambda' \cdot k^{-2\alpha} + \sum_{k=n}^{\infty} (\lambda'_k - \lambda') \cdot k^{-2\alpha} \right) = \mu'_n \cdot n^{1-\alpha}.$$

Now put $v_n^{(2)} = \frac{v_n}{\mu'}$ where $\mu' := \lim_{n\to\infty} \mu'_n \in \mathbb{C}^*$. Then $\lim_{n\to\infty} \frac{v_n^{(2)}}{n^{1-\alpha}} = 1.$ □

We now investigate the case that $\lim_{n\to\infty} n^2 \cdot C_n$ is real and $\leq -1/4$. We have already seen that, for $R \in \mathfrak{R}(\mathbb{R})$, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ does not exist for any zero $\{u_n\}$ of R if $\lim_{n\to\infty} n^2 \cdot C_n < -1/4$. For $R \in \mathfrak{R}(\mathbb{C})$ the situation is different, however. The following result is true:

Theorem 5.8. *Let* $R \in \mathfrak{R}(\mathbb{C})$, $R = T^2 - 2 \cdot T + 1 - C_n$, *where* $\lim_{n\to\infty} n^2 \cdot C_n = \gamma$ *and* $\gamma \in \mathbb{R}$, $\gamma \leq -1/4$. *Put* $d_n = (n + 1)C_n - \gamma/n$.
The following assertions hold:

(i) *Suppose there exists a sequence* $\{d'_n\}$ *such that* $|d_n| \leq d'_n$, $\sum_{n=N}^{\infty} d'_n$ *converges and* $\left(\sum_{k=n}^{\infty} d'_k\right)^2 \leq n d'_n/4$. *If* $\alpha^2 - \alpha - \gamma = 0$, R *has a zero* $\{u_n\}$ *such that*

$$\lim_{n\to\infty} n \cdot \left(\frac{u_{n+1}}{u_n} - 1\right) = \alpha.$$

(ii) *If moreover* $\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} d'_k\right) \cdot \frac{1}{n}$ *converges, then* R *has zeros* $\{u_n^{(1)}\}$ *and* $\{u_n^{(2)}\}$ *such that*

$$\lim_{n\to\infty} \frac{u_n^{(1)}}{n^{\alpha}} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{u_n^{(2)}}{n^{\alpha'}} = 1$$

if $\gamma < -1/4$, *where* α, α' *are the roots of* $X^2 - X - \gamma$, *and*

$$\lim_{n\to\infty} \frac{u_n^{(1)}}{n^{1/2}} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{u_n^{(2)}}{n^{1/2} \log n} = 1$$

if $\gamma = -1/4$.

We prove the following lemma:

Lemma 5.9. *Let* $\{d_n\}$ *be a sequence of non-negative real numbers such that* $\sum_{n=1}^{\infty} d_n$ *converges and* $\left(\sum_{k=n}^{\infty} d_k\right)^2 \leq n d_n/4$ *for* $n \geq N$. *Then the following assertions are valid:*

(i) *The recurrence*

$$(5.7) \qquad x_{n+1} = \frac{x_n - d_n}{1 + x_n/n}$$

has a solution $\{x_n^{(0)}\}$ *such that* $\lim_{n\to\infty} x_n^{(0)} = 0$ *and* $\{x_n^{(0)}\}$ *is a monotonically decreasing sequence for* $n \geq N$.

(ii) *There exists a sequence* $\{d_n'\}$, *where* $d_n \leq d_n'$ *for all* n, *such that* $\lim_{n\to\infty} d_n' = 0$ *and* $\{2\cdot\sum_{k=n}^{\infty} d_k\}$ *is a solution of*

$$(5.8) \qquad x_{n+1} = \frac{x_n - d_n'}{1 + x_n/n}.$$

Proof: (i). Put $D_n = \sum_{k=n}^{\infty} d_k$ $(n \geq N)$. If $n \geq N$ and $x_n \geq 2D_n$, then

$$x_{n+1} \geq \frac{2D_n - d_n}{1 + 2D_n/n} \geq 2D_{n+1}.$$

Hence, if $x_N^{(0)} \geq 2D_N$, then $x_n^{(0)} \geq 2D_n$ for $n \geq N$, where $\{x_n^{(0)}\}$ is defined by (5.7). On the other hand,

$$x_{n+1}^{(0)} - x_n^{(0)} = \frac{-d_n - (x_n^{(0)})^2/n}{1 + x_n^{(0)}/n} < 0,$$

so that $\{x_n^{(0)}\}$ converges to some limit $x \geq 0$. If $x > 0$, then $x_{n+1} - x_n << -x^2/n$, so that $\{x_n\}$ cannot converge. So $x = 0$.

(ii). Note that $d_n' := 2D_n - 2D_{n+1}(1 + 2D_n/n) \geq d_n$ for all n. □

Proof of Theorem 5.8: Let $\{0\} \neq \{u_n\} \in Z(R)$ and put

$$h_n = n\cdot\left(\frac{u_{n+1}}{u_n} - 1\right) - 1/2 - i\beta$$

where $\beta \in \mathbb{R}$, $\gamma = -1/4 - \beta^2$. Then $\{h_n\}$ satisfies (5.6) with $\alpha = 1/2 + i\beta$. Let $\{k_n\}$ be a sequence of positive numbers satisfying

$$(5.9) \qquad k_{n+1} = \frac{k_n - d_n'}{1 + k_n/n}$$

and k_n tends monotonically to zero. The existence of such a sequence is guaranteed by Lemma 5.9. Let $N' \geq N$ be such that $k_n \leq N'$ and $d_n' \leq 1$ for $n \geq N'$. Define $U_n = \{z \in \mathbb{C} |\ |z| \leq k_n\}$ for $n \geq N'$. Then U_n is a compact set and $\bigcap_{n=N'}^{\infty} U_n = \{0\}$. We show that for each $m \geq N'$ a sequence $\{h_n^{(m)}\}$ exists such that $h_n^{(m)} \in U_n$ for $N' \leq n \leq m$ and such that $\{h_n^{(m)}\}$ satisfies (5.6). Indeed, by (5.6),

$$h_n = \frac{h_{n+1}(n + 1/2 + i\beta)/n - d_n}{1 + (1/2 - i\beta - h_{n+1})/n}.$$

Take $h_m^{(m)} \in U_m$. Then $|h_m^{(m)}| \le k_m < k_{N'} \le N'$ and, for $N' \le n < m$, if $|h_n^{(m)}| < N'$, then

$$|h_n^{(m)}| = \frac{|h_{n+1}^{(m)}||(n + 1/2 + i\beta)/n| + |d_n|}{|1 + (1/2 - i\beta)/n| - |h_{n+1}^{(m)}|/n} \le \frac{|h_{n+1}^{(m)}| + d'_n}{1 - |h_{n+1}^{(m)}|/n} \tag{5.10}$$

so that from $h_{n+1}^{(m)} \in U_{n+1}$ it follows that $h_n^{(m)} \in U_n$. Note that this implies $|h_n^{(m)}| < N'$ for $N' \le n < m$. Now consider the sequences $H_n = \{h_n^{(n+j)}\}_{j\ge0}$. All elements of $H_{N'}$ lie in $U_{N'}$, which is a compact set, so that $H_{N'}$ has at least one limit point, $l_{N'}$, say. Let $\{l_n\}_{n\ge N'}$ be a solution of (5.9). By continuity, l_n is a limit point of U_n for $n \ge N'$, so that, in particular, $l_n \in U_n$ $(n \ge N')$. Hence, $\lim_{n\to\infty} l_n = 0$. Let $\{u_n^{(0)}\}$ be such that, for $n \ge N$,

$$l_n = n\cdot(\frac{u_{n+1}^{(0)}}{u_n^{(0)}} - 1) - 1/2 - i\beta.$$

Then $\{u_n^{(0)}\} \in Z(R)$ and

$$\lim_{n\to\infty} n\cdot(\frac{u_{n+1}^{(0)}}{u_n^{(0)}} - 1) = 1/2 + i\beta.$$

If $\beta \ne 0$, we can in the same way find a zero $\{v_n^{(0)}\}$ of R such that

$$\lim_{n\to\infty} n\cdot(\frac{v_{n+1}^{(0)}}{v_n^{(0)}} - 1) = 1/2 - i\beta.$$

Put $D'_n = \sum_{k=n}^{\infty} d'_k$. Note that if we substitute $f_n := 2d'_n - 4D'_nD'_{n+1}/n$ for d'_n in (5.9), we have $f_n \ge d'_n$, $\lim_{n\to\infty} f_n = 0$ and $k_n := 2\cdot\sum_{k=n}^{\infty} d'_k$ is a solution of (5.9). Further, suppose that $\sum_n D'_n/n$ converges. Let $\{v_n^{(1)}\} \in Z(R)$ such that

$$|n\cdot(\frac{v_{n+1}^{(1)}}{v_n^{(1)}} - 1) - 1/2 - i\beta| \le 2D'_n \quad (n \ge N).$$

By the first part of the proof and the above remark, such a $\{v_n^{(1)}\}$ exists. We have

$$\frac{v_{n+1}^{(1)}}{v_n^{(1)}} = (1 + (1/2 + i\beta)/n)\cdot(1 + \delta_n/n)$$

where $|\delta_n| << D'_n$. Hence, $v_n^{(1)}\cdot n^{-1/2-i\beta} \to \lambda_1 \in \mathbb{C}^*$ as $n \to \infty$. Choose $u_n^{(1)} = v_n^{(1)}/\lambda_1$. Similarly, if $\beta \ne 0$, we can find $\{u_n^{(2)}\}$ such that

$$\lim_{n\to\infty} n^{-1/2+i\beta}\cdot u_n^{(2)} = 1.$$

Now suppose $\beta = 0$. Put $S = (u_n^{(1)})^{-1}\cdot I$. Then, by (5.3),

$$R/S = (T - \frac{1-g_{n+1}}{1+g_{n+1}})\cdot(T - 1), \text{ where } g_n = \frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1.$$

Let $\{0\} \neq \{w_n\} \in Z(R/(S(T-1)))$. Then $\frac{w_{n+1}}{w_n} = (1 - 1/n)(1 + \delta_n/n)$, where $|\delta_n| << D'_n$. So $w_n = \lambda_n\cdot n^{-1}$, where $\lambda_n \to \lambda \in \mathbb{C}^*$ as $n \to \infty$. Without loss of generality we may assume that $\lambda = 1$. Put $u_n^{(2)} = u_n^{(1)}\cdot\sum_{k=1}^{n-1} w_k$. Then $\{u_n^{(2)}\} \in Z(R)$ since $\{w_n\} \in Z(R/(S(T-1)))$. Moreover, $n\cdot w_n \to 1$ $(n \to \infty)$. We prove that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} w_k}{\log n} = 1.$$

Choose $\varepsilon > 0$. Let N_0 be so large that $|n\cdot w_n - 1| < \varepsilon$ for $n \geq N_0$. Then

$$(\log n)^{-1}\cdot|\sum_{k=1}^{n-1}(w_k - 1/k)| \leq \frac{c\cdot N_0}{\log n} + \frac{1}{\log n}\cdot\sum_{k=N_0}^{n-1}\varepsilon/k < \frac{c\cdot N_0}{\log n} + 2\varepsilon < 3\varepsilon$$

for n large enough. Hence, $\frac{u_n^{(2)}}{n^{1/2}\log n} \to 1$ as $n \to \infty$. □

Remark 5.3.1. That the condition $4D_n^2 \leq nd_n$ is not far from best possible can be seen from the following example:

Take $d_n = (n\cdot\log^2 n)^{-1}$. Then $\frac{D_n^2}{n\cdot d_n} \to 1$ as $n \to \infty$. Consider the recurrence

(*) $$x_{n+1} = \frac{x_n - d_n}{1 + x_n/n}$$

If $\lim_{n\to\infty} x_n^{(0)} = 0$ for some solution $\{x_n^{(0)}\}$, then $x_n^{(0)} > 0$ for $n \geq N$.
Hence, $x_{n+1} < x_n - d_n$, so that $x_n^{(0)} > \sum_{k=n}^{\infty} d_k$ for $n \geq N$. Then,

$$x_{n+1}^{(0)} \leq \frac{x_n^{(0)} - d_n}{1 + D_n/n}.$$

Then $0 < x_n^{(0)}\cdot\Gamma_n \leq x_N^{(0)} - \sum_{k=N}^{n-1} d_k\cdot\Gamma_k$, where $\Gamma_n = \prod_{k=N}^{n-1}(1 + D_k/k)$ for $n \geq N$.
Hence, $\sum_{n=N}^{\infty} d_n\cdot\Gamma_n$ must converge. On the other hand,

$$\sum_{n=N}^{\infty} d_n \cdot \Gamma_n > \sum_{n=N}^{\infty} \frac{1}{n\cdot\log^2 n} \cdot \prod_{k=N}^{n-1} (1 + \frac{1}{k\cdot\log k})$$
$$> \sum_{n=N}^{\infty} \frac{1}{n\cdot\log^2 n} \cdot \prod_{k=N}^{n-1} \exp(\frac{1}{k\cdot\log k} - 1/k^2)$$
$$>> \sum_{n=N}^{\infty} \frac{1}{n\cdot\log^2 n} \cdot e^{\log\log n - 2/n} >> \sum_{n=N}^{\infty} \frac{1}{n\cdot\log n} = \infty.$$

Remark 5.3.2. The number 4 in the inequality $4D_n^2 \le nd_n$ cannot be improved, as we shall show below: Let $\{d_n\}$ be some sequence of non-negative real numbers such that $\sum d_n$ converges and such that $\varepsilon\cdot nd_n < D_nD_{n+1}$ for some number $\varepsilon > 0$, where $D_n = \sum_{k=n}^{\infty} d_k$ $(n \in \mathbb{N})$. Consider the recurrence (*) of Remark 5.3.1. If (*) has some real solution $\{x_n^{(0)}\}$ such that $\lim_{n\to\infty} x_n^{(0)} = 0$, then $x_n^{(0)} > 0$ for $n \ge N$. By

(**) $\quad x_n^{(0)} - x_{n+1}^{(0)} = x_n^{(0)}\cdot x_{n+1}^{(0)}/n + d_n$

we infer that $x_n^{(0)} - x_{n+1}^{(0)} > d_n$, so that $x_n^{(0)} > D_n$ for $n \ge N$. Using that $D_nD_{n+1}/n > \varepsilon\cdot d_n$ we obtain by (**) that $x_n^{(0)} - x_{n+1}^{(0)} > (1 + \varepsilon)\cdot d_n$, so that $x_n^{(0)} > (1 + \varepsilon)\cdot D_n$ for $n \ge N$. Continuing in the same way, by repeatedly applying (**) and the inequality $D_nD_{n+1}/n > \varepsilon\cdot d_n$, we obtain a sequence $\{\varepsilon_h\}_{h=0}^{\infty}$ of positive real numbers ε_h $(h \ge 0)$ defined by $\varepsilon_0 = \varepsilon$ and $\varepsilon_h = \varepsilon\cdot(1 + \varepsilon_{h-1})^2$ for $h \ge 1$, and such that $x_n^{(0)} > (1 + \varepsilon_h)\cdot D_n$ for $n \ge N$ and all h. Since obviously $\varepsilon_1 > \varepsilon_0$, we have that $\varepsilon_h > \varepsilon_{h-1}$ for all $h > 0$. Now suppose that $\varepsilon > 1/4$. Put $E = \lim_{h\to\infty} \varepsilon_h$. If $E \in \mathbb{R}$, it satisfies the equation $E = \varepsilon\cdot(1 + E)^2$. However, since $\varepsilon > 1/4$, the equation $X = \varepsilon\cdot(1 + X)^2$ has no real solutions. Hence $E = \infty$ and, consequently, the recurrence (*) cannot have a real solution that converges.

§4. Non-simple operators with two equal eigenvalues: Slow convergence (hyperbolic case).

In [Pe2] Perron showed the following fact:

If $R = T^2 - (2 - \eta_1(n))\cdot T + (1 - \eta_0(n))$, where $\eta_0(n),\eta_1(n) \in \mathbb{R}$ for all n and $\lim_{n\to\infty} \eta_0(n) = \lim_{n\to\infty} \eta_1(n) = 0$, then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ for all $\{u_n\} \in Z(R)$, $\{u_n\} \neq \{0\}$ if $\eta_1(n) \geq 0$ and $\eta_0(n) - \eta_1(n) \geq 0$ for $n \geq N$. In fact, it can be shown that the condition $\eta_1(n) \geq 0$ can be omitted. Let $\{v_n\} \in Z(R)$, $\{v_n\} \neq \{0\}$. Put $g_n = \frac{v_{n+1}}{v_n} - 1$. Then

$$g_{n+1} = \frac{(1 - \eta_1(n))\cdot g_n + \eta_0(n) - \eta_1(n)}{1 + g_n}.$$

Let $N' \geq N$ be so large that $|\eta_1(n)| < 1/2$ for $n \geq N$. Let $g_{N'} \geq 0$. Then, since $\eta_0(n) - \eta_1(n) \geq 0$, we have $g_n \geq 0$ for $n \geq N'$. Let $\zeta(n)$ be the largest root of $X^2 - |\eta_1(n)|\cdot X + \eta_1(n)-\eta_0(n)$. Then $\zeta(n) \geq 0$. Put $\xi(n) = \max(g_n, \max_{m\geq n-1} \zeta(n))$ $(n \in \mathbb{N})$. Clearly, $\xi(n) \geq 0$ for $n \geq N'$. We show that $\{\xi(n)\}_{n\geq N'}$ is a monotonically non-increasing sequence. For if $g_n \geq \zeta(n)$, then $g_{n+1} \leq g_n$, so that $\xi(n+1) \leq \xi(n)$. If $g_n < \zeta(n)$, then $g_{n+1} < \zeta(n)$ as well, so that again $\xi(n+1) \leq \xi(n)$. Since $\zeta(n)$ tends to zero as $n \to \infty$, we have that either $\lim_{n\to\infty} g_n = 0$ or $g_n > \max_{m\geq n-1} \zeta(m)$ for n large enough. In the latter case, $\{g_n\}$ decreases monotonically for $n \geq N_0$, so that $\{g_n\}$ converges to some number $g \geq 0$. Then $\lim_{n\to\infty} \frac{v_{n+1}}{v_n} = 1+g$, so that $g \neq 0$ is impossible. Hence, by Proposition 3.1, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ for all zeros $\{u_n\} \neq \{0\}$ of R. □

Remark: It can moreover be shown that there is a unique zero $\{v_n\}$ (up to a multiplicative constant) such that $\frac{v_{n+1}}{v_n} - 1 \leq 0$ for all n. By symmetry, $\{v_n\}$ can be taken real-valued. Furthermore, we have $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ for all $\{u_n\} \in Z(R)$ linearly independent with $\{v_n\}$. (See [K2]).

In the sequel, we shall generalize Perron's result in several directions.

For instance, if η_0 and η_1 converge fast to zero, or if their behaviour is in some other way regular (whatever we may mean by this rather vague term does not concern us here yet), it will appear this similar statements about the behaviour of the zeros can be made as in the case above. As in the preceding sections, we consider the normalized operator $R = T^2 - 2\cdot T + 1 - C_n$, $R \in \mathfrak{R}(\mathbb{C})$. If $\{C_n\}$ does not converge so fast that the conditions of Corollary 5.2 are satisfied, it will be necessary to impose additional conditions on the behaviour of $\{C_n\}$. For example, Theorem 5.10 holds if $|\arg C_n| < \pi - \varepsilon$ for some positive real number ε and n large enough, and $\lim_{n\to\infty} (\sqrt{C_n^{-1}} - \sqrt{C_{n+1}^{-1}}) = 0$, where we define $\sqrt{z}$ for $z \in \mathbb{C}$ such that $-\pi/2 < \arg \sqrt{z} < \pi/2$ if $z \neq 0$ and z is not a negative real number. Note that this condition implies that $\lim_{n\to\infty} n^2|C_n| = \infty$. Indeed we have for any $\varepsilon > 0$ that $|\sqrt{C_n^{-1}} - \sqrt{C_{n+1}^{-1}}| < \varepsilon$ for $n \geq N(\varepsilon)$, which implies $|\sqrt{C_n^{-1}}| < 2\varepsilon n$ for n large enough, so that $n^2\cdot|C_n| > (4\varepsilon^2)^{-1}$ for n large enough. Since ε can be chosen arbitrarily small, we have $\lim_{n\to\infty} n^2|C_n| = \infty$. In particular, $\sum |\sqrt{C_n}|$ diverges (see Remark 5.4.1). With a view to later applications, we shall impose even weaker conditions on $\{C_n\}$:

Theorem 5.10. *Let* $R \in \mathfrak{R}(\mathbb{C})$, $R = T^2 - 2\cdot T + 1 - C_n$, *where* $\lim_{n\to\infty} C_n = -d$ *for some non-negative real number* d, *and moreover* $\sum \operatorname{Re} \sqrt{C_n} = +\infty$ *and* $C_{n+1}/C_n - 1 = o(\operatorname{Re} \sqrt{C_n})$. *Then* R *has zeros* $\{u_n^{(1)}\}$ *and* $\{u_n^{(2)}\}$ *such that*

$$\lim_{n\to\infty} \sqrt{C_n^{-1}}\left(\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1\right) = 1 \quad \textit{and} \quad \lim_{n\to\infty} \sqrt{C_n^{-1}}\left(\frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1\right) = -1.$$

and, in addition,

$$\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0.$$

Corollary 5.10. *Let* R *be as in Theorem* 5.10. *Then* $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ *exists for all non-trivial zeros* $\{u_n\}$ *of* R. *Moreover, if* $C_n \in \mathbb{R}$ $(n \in \mathbb{R})$, *then we can find linearly independent zeros* $\{u_n^{(1)}\},\{u_n^{(2)}\} \in Z(R)$ *such that* $u_n^{(1)}, u_n^{(2)} \in \mathbb{R}$ *for* $n \in \mathbb{N}$.

Proof of Corollary: Let $\{u_n\} \in Z(R)$, $\{u_n\} \neq \{0\}$. Apply Theorem 5.10. Since

$\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ are linearly independent, there exist $\lambda,\mu \in \mathbb{C}$, not both zero, such that $\{u_n\} = \lambda\cdot\{u_n^{(1)}\} + \mu\cdot\{u_n^{(2)}\}$. Further, since $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$, we have

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}} = 1 - \lim_{n\to\infty} \sqrt{C_n} \quad \text{if } \lambda = 0,$$

and

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}}\cdot\frac{\lambda + \mu\cdot u_{n+1}^{(2)}/u_{n+1}^{(1)}}{\lambda + \mu\cdot u_n^{(2)}/u_n^{(1)}} = 1 + \lim_{n\to\infty} \sqrt{C_n} \quad \text{if } \lambda \neq 0.$$

For the proof of the second assertion, compare Corollary 5.4. □

Lemma 5.11: *Let* $\{\gamma_n\},\{\delta_n\},\{\varepsilon_n\}$ *be sequences of complex numbers such that* $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \gamma_n = 0$, $|1 - \delta_n| \le 1$ *for all* n *large enough,* $\lim_{n\to\infty} |1 - \delta_n| = 1$ *and* $\sum_{n=1}^{\infty} (1 - |1 - \delta_n|) = +\infty$. *Moreover,* suppose that $\lim_{n\to\infty} \frac{\varepsilon_n}{1 - |1 - \delta_n|} = 0$ *and* that $\frac{|\gamma_n|}{1 - |1 - \delta_n|}$ *is bounded. Then the recurrence*

$$f_{n+1} = \frac{f_n\cdot(1 - \delta_n) + \varepsilon_n}{1 + \gamma_n\cdot f_n} \tag{5.11}$$

has a solution $\{f_n^{(0)}\}$ *such that* $\lim_{n\to\infty} f_n^{(0)} = 0$. *Moreover, if* $\lim_{n\to\infty} \frac{\gamma_n}{1 - |1 - \delta_n|} = 0$, *then* $\lim_{n\to\infty} f_n = 0$ *for all solutions* $\{f_n\}$ *of (5.11) but one. For the remaining solution* $\{f_n^{(\infty)}\}$ *we have* $\lim_{n\to\infty} f_n^{(\infty)} = \infty$.

Proof: Let M be such that $\frac{|\gamma_n|}{1 - |1 - \delta_n|} < M$ for all n. Let N be so large that for $n \ge N$ both $4\cdot|\varepsilon_n| < 1 - |1 - \delta_n|$ and $|1 - \delta_n| < 1$. If $\{f_n\}$ satisfies (5.11) and $|f_m| \le f = \frac{1}{2M}$ for some $m \ge N$, then

$$|f_{m+1}| \le \frac{|f_m|(1 - (1 - |1 - \delta_m|)) + |\varepsilon_m|}{1 - |\gamma_m|\cdot f} \le$$

$$|f_m|\cdot(1 - (1 - |1 - \delta_m|)/2) + 2\cdot|\varepsilon_m| \le f\cdot(1 - (1 - |1 - \delta_m|)/2) + 2\cdot|\varepsilon_m| < f,$$

since $|\gamma_m|\cdot f < 1/2$ for all $m \ge N$.

We now choose $\{f_n^{(0)}\}$ such that it satisfies (5.11) and such that $|f_N^{(0)}| \le f$. Then $|f_n^{(0)}| \le f$ for all $n \ge N$ and

$$|f_{n+1}^{(0)}| \le |f_n^{(0)}|\cdot(1 - (1 - |1 - \delta_n|)/2) + 2\cdot|\varepsilon_n|.$$

Application of Lemma 5.5 now yields that $\lim_{n\to\infty} f_n^{(0)} = 0$.

Now suppose that in addition $\lim_{n\to\infty} \dfrac{\gamma_n}{1 - |1 - \delta_n|} = 0$. Put $h_n = f_n - f_n^{(0)}$ for all $n \in \mathbb{N}$. Then $\{h_n\}$ satisfies the recurrent relation

$$(5.12) \qquad h_{n+1} = \frac{h_n\cdot(1 - \delta_n - f_{n+1}^{(0)}\cdot\gamma_n)}{1 + \gamma_n f_n^{(0)} + \gamma_n h_n} \qquad (n \in \mathbb{N}).$$

Put $\dfrac{1 - \delta_n - f_{n+1}^{(0)}\cdot\gamma_n}{1 + \gamma_n f_n^{(0)}} = 1 - \delta_n + \tau_n$ $(n \in \mathbb{N})$. Then

$$\tau_n = \frac{(1 - \delta_n)\cdot\gamma_n\cdot f_n^{(0)} + f_{n+1}\cdot\gamma_n^{(0)}}{1 + \gamma_n f_n^{(0)}},$$

so that $\lim_{n\to\infty} \dfrac{\tau_n}{1 - |1 - \delta_n|} = 0$. Consequently, (5.12) can be written as

$$(5.13) \qquad h_{n+1} = \frac{(1 - \delta_n^*)\cdot h_{n+1}}{1 + \gamma_n^*\cdot h_n} \quad (n \in \mathbb{N}),$$

where $|1 - \delta_n^*| \le 1$ for almost every $n \in \mathbb{N}$ and moreover, $\sum_{n=1}^{\infty} (1 - |1 - \delta_n^*|) = +\infty$, $\lim_{n\to\infty} \dfrac{\gamma_n^*}{1 - |1 - \delta_n^*|} = 0$. Solving (5.13) explicitly yields

$$(5.14) \qquad h_n = \prod_{k=1}^{\infty}(1 - \delta_k^*)\cdot\Big[\, h_1 - \sum_{l=n}^{\infty} \gamma_l^*\cdot \prod_{k=1}^{l-1}(1 - \delta_k^*)\Big]^{-1}$$

since the sum $\sum_{l=1}^{\infty} \gamma_l^*\cdot \prod_{k=1}^{l-1}(1 - \delta_k^*)$ converges absolutely, by

$$\sum_{l=1}^{\infty} |\gamma_l^*|\cdot \prod_{k=1}^{l-1}|1 - \delta_k^*| \ll \sum_{l=1}^{\infty} (1 - |1 - \delta_l^*|)\cdot \prod_{k=1}^{l-1}(1 - \delta_k^*) = 1.$$

Thus, if we take $h_1 \in \mathbb{C}$, $h_1 \ne 0$ or $h_1 = \infty$, we find that $h_n \to 0$ as $n \to \infty$. On the other hand, if we take $h_1 = 0$, then

$h_n^{-1} = -\sum_{l=n}^{\infty} \gamma_l^*\cdot \prod_{k=1}^{n-1}(1 - \delta_k^*)$, so that $|h_n^{-1}| \le \max_{l\ge n} \dfrac{\gamma_l^*}{1 - |1 - \delta_l^*|}$, while the latter expression tends to zero as $n \to \infty$. □

Remark 5.4.1. Since $|\delta_n| \ge 1 - |1 - \delta_n|$ in Lemma 5.11 the lemma is valid in particular if $\lim_{n\to\infty} \varepsilon_n/\delta_n = 0$, $\lim_{n\to\infty} \delta_n = 0$, $|\arg \delta_n| < \pi/2 - \varepsilon$ for some $\varepsilon > 0$ and n large enough, where $\sum \delta_n$ diverges and $\left|\frac{\gamma_n}{\delta_n}\right|$ is bounded.

Lemma 5.12. *Let* $R \in \mathfrak{R}(\mathbb{C})$, ord $R = 2$, *such that* R *has non-trivial zeros* $\{u_n^{(1)}\},\{u_n^{(2)}\},\{v_n\}$ *which are pairwise linearly independent and*

$$\lim_{n\to\infty} a_n\cdot\left[\frac{u_{n+1}^{(i)}}{u_n^{(i)}} - 1\right] = \alpha \quad (i = 1,2) \quad \textit{and} \quad \lim_{n\to\infty} a_n\cdot\left[\frac{v_{n+1}}{v_n} - 1\right] = \beta$$

for some sequence of non-zero complex numbers $\{a_n\}$ *and complex numbers* α *and* β $(\alpha \neq \beta)$. *Then*

$$\lim_{n\to\infty} \frac{v_n}{u_n^{(i)}} = 0 \quad (i = 1,2).$$

Proof: Put $\zeta_n = \frac{v_n}{u_n^{(1)}}$. Since $\{u_n^{(1)}\}$ and $\{v_n\}$ are linearly independent zeros of R, we may put $\{u_n^{(2)}\} = \lambda\cdot\{u_n^{(1)}\} + \mu\cdot\{v_n\}$, where $\lambda,\mu \neq 0$. Without loss of generality we can assume that $\lambda = 1$. Then

$$\alpha = \lim_{n\to\infty} a_n\cdot\left[\mu\cdot\frac{v_{n+1} - v_n}{u_n^{(1)}(1 + \mu\zeta_n)} + \frac{u_{n+1}^{(1)} - u_n^{(1)}}{u_n^{(1)}(1 + \mu\zeta_n)}\right] =$$

$$\lim_{n\to\infty} \frac{a_n}{1 + \mu\zeta_n}\cdot\left[\mu\zeta_n\cdot\frac{v_{n+1} - v_n}{v_n} + \frac{u_{n+1}^{(1)} - u_n^{(1)}}{u_n^{(1)}}\right].$$

Subtracting $\beta = \lim_{n\to\infty} a_n\cdot\left[\frac{v_{n+1}}{v_n} - 1\right]$ yields

$$\lim_{n\to\infty} \frac{a_n}{1 + \mu\zeta_n}\cdot\left[-\frac{v_{n+1} - v_n}{v_n} + \frac{u_{n+1}^{(1)} - u_n^{(1)}}{u_n^{(1)}}\right] = \alpha - \beta \neq 0.$$

Moreover, by

$$\lim_{n\to\infty} a_n\cdot\left[-\frac{v_{n+1} - v_n}{v_n} + \frac{u_{n+1}^{(1)} - u_n^{(1)}}{u_n^{(1)}}\right] = \alpha - \beta$$

we obtain, using that the numbers a_n are non-zero,

$\lim_{n\to\infty} \frac{1}{1 + \mu\zeta_n} = 1$, whence (by $\mu \neq 0$) $\lim_{n\to\infty} \zeta_n = 0$. □

Proof of Theorem 5.10: By the conditions on the behaviour of $\{C_n\}$, we have that $\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 1$. Let $\{u_n\} \in Z(R)$, $\{u_n\} \neq \{0\}$ and $g_n = \frac{u_{n+1}}{u_n} - 1$. Put

$$f_n = \frac{g_n - \sqrt{C_n}}{g_n + \sqrt{C_n}} \quad (n \in \mathbb{N}).$$

Then $\{f_n\}$ satisfies

$$(5.15) \qquad f_{n+1} = \frac{f_n \cdot (1 - \delta_n) + \varepsilon_n}{1 + \gamma_n \cdot f_n}$$

where $1 - \delta_n = \frac{1 - \sqrt{C_n}}{1 + \sqrt{C_n}}$, $\varepsilon_n = \frac{\sqrt{C_n} - \sqrt{C_{n+1}}}{\sqrt{C_n} + \sqrt{C_{n+1}}}$ and $\gamma_n = \varepsilon_n \cdot (1 - \delta_n)$. Since $1 - |1 - \delta_n| \sim c \cdot \mathrm{Re}\,\sqrt{C_n}$ (for some $c \in \mathbb{R}_{>0}$ depending only on d) we have, by the conditions on $\{C_n\}$ that γ_n, δ_n and ε_n satisfy the conditions of Lemma 5.11, including the condition that $\lim_{n\to\infty} \frac{\gamma_n}{1 - |1 - \delta_n|} = 0$. Hence, we have that (5.15) has a solution $\{f_n^{(\infty)}\}$ such that $\lim_{n\to\infty} f_n^{(\infty)} = \infty$, whereas for the other solutions $\{f_n\}$ of (5.15) $\lim_{n\to\infty} f_n = 0$. Let $\{u_n^{(1)}\},\{u_n^{(2)}\}$ be such that

$$\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1 = \sqrt{C_n} \cdot \frac{1 + f_n^{(0)}}{1 - f_n^{(0)}}, \qquad \frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1 = \sqrt{C_n} \cdot \frac{1 + f_n^{(\infty)}}{1 - f_n^{(\infty)}}$$

where $\{f_n^{(0)}\}$ is some solution of (5.15) for which $\lim_{n\to\infty} f_n^{(0)} = 0$.

Then $\{u_n^{(i)}\} \in Z(R)$ $(i = 1,2)$ and

$$\lim_{n\to\infty} \frac{1}{\sqrt{C_n}} \cdot \left[\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1 \right] = 1, \qquad \lim_{n\to\infty} \frac{1}{\sqrt{C_n}} \cdot \left[\frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1 \right] = -1.$$

Moreover, since $\lim \frac{1}{\sqrt{C_n}} \cdot \left[\frac{u_{n+1}}{u_n} - 1 \right] = 1$ for all zeros $\{u_n\}$ of R that are linearly independent with $\{u_n^{(2)}\}$, Lemma 5.12 ensures that $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$. (Note that for all non-trivial zeros $\{u_n\}$ of R there is some solution $\{f_n\}$ of (5.15) such that $\frac{u_{n+1}}{u_n} - 1 = \sqrt{C_n} \cdot \frac{1 + f_n}{1 - f_n}$.) □

§5. Non-simple operators with two equal eigenvalues: Slow convergence (elliptic case).

Let $R = T^2 - 2\cdot T + (1 - C_n)$. If the numbers C_n lie on the negative real axis, or sufficiently close to it, the behaviour of the zeros of the recurrence is rather different from the behaviour in the cases treated above. For one thing, there is generally not a subdominant zero, i.e. a zero $\{v_n\}$ such that $\lim_{n\to\infty} \frac{v_n}{u_n} = 0$ for all zeros $\{u_n\}$ linearly independent with $\{v_n\}$. We shall show that (provided that the $\{C_n\}$ behave not too irregularly) the behaviour of the zeros is rather similar to the behaviour we encounter in the case that $C_n = C < 0$ $(n \in \mathbb{N})$, where there are two linearly independent zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that $\lim_{n\to\infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}}$ and $\lim_{n\to\infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}}$ exist and $\lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 1$. We define for $z \in \mathbb{C}$, $z \neq 0$, the principal value of the argument Arg z such that $-\pi < \operatorname{Arg} z \leq \pi$.

Theorem 5.13. *Let* R *and* $\{C_n\}$ *be as above. Suppose that* $\lim_{n\to\infty} C_n = -d$ *for some* $d \in \mathbb{R}$, $d \geq 0$, *and* $\sqrt{-C_n^{-1}} - \sqrt{-C_{n+1}^{-1}}$ *converges to* 0 *monotonically as* $n \to \infty$. *Moreover, suppose that the series* $\sum \left| \sqrt{-C_{n-1}^{-1}} - 2\cdot\sqrt{-C_n^{-1}} + \sqrt{-C_{n+1}^{-1}} \right|$, $\sum \left| \sqrt{C_{n+1}/C_n} - \sqrt{C_n/C_{n-1}} \right|$, $\sum \left| \sqrt{C_{n+1}} - \sqrt{C_n} \right|$, $\sum \left| \operatorname{Im} \sqrt{-C_n} \right|$ *and* $\sum \left| \operatorname{Im} \sqrt{C_{n+1}/C_n} \right|$ *converge. Then* R *has zeros* $\{u_n^{(1)}\}$ *and* $\{u_n^{(2)}\}$ *such that*

$$\lim_{n\to\infty} \frac{1}{\sqrt{-C_n}}\left[\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1 \right] = i \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{\sqrt{-C_n}}\left[\frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1 \right] = -i$$

and $\lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 1$, *whereas* $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}}$ *does not exist. Further, if* $d = 0$, *then for all zeros* $\{u_n\}$ *of* R *which are not of the form* $\{u_n\} = \lambda\cdot\{u_n^{(1)}\} + \mu\cdot\{u_n^{(2)}\}$ *with* $|\lambda| = |\mu|$, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$. *On the other hand, if* $\{u_n\} = \lambda\cdot\{u_n^{(1)}\} + \mu\cdot\{u_n^{(2)}\}$ *with* $|\lambda| = |\mu|$, *then* $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ *does not exist (for all d).*

We use the following lemmas:

Lemma 5.14. *The recurrence relation*

$$k_{n+1} = \frac{\zeta_n \cdot k_n + \varepsilon_n}{\gamma_n \cdot k_n + 1} \qquad (n \in \mathbb{N}) \tag{5.16}$$

where $\{\gamma_n\},\{\varepsilon_n\},\{\zeta_n\}$ *are complex-valued sequences such that both* $\sum_{n=1}^{\infty} (|\varepsilon_n| + |\gamma_n|)$ *and* $\sum_{n=1}^{\infty} \Big| |\zeta_n| - 1 \Big|$ *converge, has solutions* $\{k_n^{(0)}\}$ *and* $\{k_n^{(\infty)}\}$ *such that* $\lim_{n\to\infty} k_n^{(0)} = 0$ *and* $\lim_{n\to\infty} k_n^{(\infty)} = \infty$. *Moreover,* $\lim_{n\to\infty} |k_n|$ *exists in* $\mathbb{P}^1(\mathbb{C})$ *for all solutions* $\{k_n\}$ *of (5.16).*

Proof: First we assume that $\sum_{n=1}^{\infty} |\zeta_n - 1|$ converges. Consider the matrix recurrence $[I + D_n]$ where $D_n = \begin{bmatrix} \zeta_n - 1 & \varepsilon_n \\ \gamma_n & 1 \end{bmatrix}$. A sequence $\{k_n\}$ is a solution of (5.16) if and only if k_n is of the form $k_n = \frac{x_{n1}}{x_{n2}}$ $(n \in \mathbb{N})$ for some non-trivial solution $\{x_n\}$ (with $x_n = \begin{pmatrix} x_{n1} \\ x_{n2} \end{pmatrix}$) of $[I + D_n]$. Without loss of generality we may assume that $\| D_n \| < 1$ for all n. Since $\sum_{n=1}^{\infty} \| D_n \|$ converges, it follows that the sequence $\{(I + D_n)\cdot(I + D_{n-1})\cdot \ldots \cdot(I + D_1)\}_{n=1}^{\infty}$ converges to some non-singular limit matrix $F \in \mathbb{C}^{2,2}$. Obviously $\{X_n\} = \{(I + D_n)\cdot(I + D_{n-1})\cdot \ldots \cdot(I + D_1)\cdot F^{-1}\}$ is a complete solution of $[I + D_n]$ and $\lim X_n = I$. Put $X_n = (x_n^{(1)}\ x_n^{(2)})$ $(n \in \mathbb{N})$. Then $x_n^{(1)} \to e_1$, $x_n^{(2)} \to e_2$ $(n \to \infty)$ where e_i is the i-th unit vector in $\mathbb{C}^2$ $(i = 1,2)$. It now suffices to define $k_n^{(0)} = \frac{x_{n1}^{(1)}}{x_{n2}^{(1)}}$ and $k_n^{(\infty)} = \frac{x_{n1}^{(2)}}{x_{n2}^{(2)}}$ $(n \in \mathbb{N})$. Moreover, it is clear that $\lim_{n\to\infty} x_n$ exists in $\mathbb{C}^2$ for all solutions $\{x_n\}$ of $[I + D_n]$.

Now for the general case. We may assume that $\Big| |\zeta_n| - 1 \Big| < 1$ for all n. Put $\zeta_n/|\zeta_n| = e_n$ $(N \in \mathbb{N})$. Then $|e_n| = 1$ and $\{h_n\} := \{k_n(e_{n-1}\cdot \ldots \cdot e_1)^{-1}\}$ satisfies the recurrence relation

$$h_{n+1} = \frac{|\zeta_n|\cdot h_n + \varepsilon_n^*}{\gamma_n^* \cdot h_n + 1} \qquad (n \in \mathbb{N}) \tag{5.17}$$

where $\varepsilon_n^* = \varepsilon_n(e_n\cdot \ldots \cdot e_1)^{-1}$ and $\gamma_n^* = \gamma_n(e_{n-1}\cdot \ldots \cdot e_1)$ $(n \in \mathbb{N})$. Application of the lemma for the case that $\sum_{n=1}^{\infty} |\zeta_n - 1| < \infty$ to (5.17) yields the result.
(Remark: Note that the proof even yields that $\lim_{n\to\infty} k_n(e_{n-1}\cdot \ldots \cdot e_1)^{-1}$ exists for all solutions $\{k_n\}$ of (5.16) and, conversely, that for every $\alpha \in \mathbb{C} \cup \{\infty\}$ there exists a solution $\{k_n\}$ of (5.16) such that $\lim_{n\to\infty} k_n(e_{n-1}\cdot \ldots \cdot e_1)^{-1} = \alpha$.)

Lemma 5.15. *Consider the recurrence relation*

$$k_{n+1} = \frac{k_n + e_n \cdot r_n}{k_n \cdot r_n + e_n} \tag{5.18}$$

where $r_n, e_n \in \mathbb{C}$ $(n \in \mathbb{N})$, $\lim_{n\to\infty} \frac{r_n}{e_{n+1} - 1} = 0$, $\sum_{n=1}^{\infty} \Big| |e_n| - 1 \Big|$ *converges and also*

$$\sum_{n=1}^{\infty} \left| \frac{r_n}{1 - e_{n+1}} - \frac{r_{n+1}}{1 - e_{n+2}} \right|, \ \sum_{n=1}^{\infty} \left| \frac{r_n}{e_{n+1} - 1} \right| \cdot | \operatorname{Im} r_n |, \ \sum_{n=1}^{\infty} |e_n - e_{n+1}|$$

converge. Then (5.15) has solutions $\{k_n^{(0)}\}$ *and* $\{k_n^{(\infty)}\}$ *with*

$$\lim_{n\to\infty} k_n^{(0)} = 0, \quad \lim_{n\to\infty} k_n^{(\infty)} = \infty.$$

Moreover, for all solutions $\{k_n\}$ *of (5.15) the limit* $\lim_{n\to\infty} |k_n|$ *exists in* $\mathbb{P}^1(\mathbb{C})$.

Proof: We define complex-valued sequences $\{h_n\}, \{\rho_n\}, \{\hat{\rho}_n\}, \{\varepsilon_n\}, \{s_n\}, \{\hat{s}_n\}$, $\{a_n\}, \{\hat{a}_n\}$ by

$h_n = k_n(e_{n-1} \cdot \ldots \cdot e_1)$, $\rho_n = r_n(e_n \cdot \ldots \cdot e_1)$, $\hat{\rho}_n = r_n(e_n \cdot \ldots \cdot e_1)^{-1}$, $\varepsilon_n = e_n/|e_n|$,

$$s_n = \frac{r_n}{e_{n+1} - 1}, \quad \hat{s}_n = \frac{r_n}{e_{n+1}^{-1} - 1}, \quad a_n = F(s_n^2 \cdot \varepsilon_{n+1}), \quad \hat{a}_n = F(\hat{s}_n^2 \cdot \varepsilon_{n+1}) \qquad (n \in \mathbb{N})$$

where $F(z) = (-1 + \sqrt{1 + 4z})/2z$ (and $F(0) = 1$, in accordance with our convention in the choice of the branch of the square root). $\{h_n\}$ satisfies the recurrent relation

$$h_{n+1} = \frac{h_n + \rho_n}{h_n \cdot \hat{\rho}_n + 1} \qquad (n \in \mathbb{N}). \tag{5.19}$$

Note that the conditions of the theorem imply that $\sum_{n=1}^{\infty} |\varepsilon_n - \varepsilon_{n+1}|$, $\sum_{n=1}^{\infty} |s_n - s_{n+1}|$, and $\sum_{n=1}^{\infty} |\hat{s}_n - \hat{s}_{n+1}|$ converge and that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \hat{s}_n = 0$.

Since $\sum_{n=1}^{N} \rho_n = \sum_{n=1}^{N} s_n \cdot (e_n - 1) \cdot e_{n-1} \cdot \ldots \cdot e_1 = \sum_{n=1}^{N} (s_n - s_{n+1}) \cdot e_n \cdot \ldots \cdot e_1 - s_1$ $+ s_{N+1} \cdot e_N \cdot \ldots \cdot e_1$, it follows that $\sum_{n=1}^{\infty} \rho_n$ converges and that for $N \in \mathbb{N}$

$$\sum_{n=N}^{\infty} \rho_n = \sum_{n=N}^{\infty} (s_n - s_{n+1}) \cdot e_n \cdot \ldots \cdot e_1 - s_N \cdot e_{N-1} \cdot \ldots \cdot e_1 =: \sum_{n=N}^{\infty} \sigma_n - s_N \cdot e_{N-1} \cdot \ldots \cdot s_1$$

with $\sum_{n=1}^{\infty} |\sigma_n|$ a converging series. A similar formula holds for $\sum_{n=1}^{\infty} \hat{\rho}_n$, with s_n, e_n replaced by $\hat{s}_n, e_n^{-1}$, respectively $(n \in \mathbb{N})$. Further, since $F'(z)$ is bounded in the neighbourhood of $z = 0$ and

$$|a_n - a_{n+1}| = |\, F(s_n^2 \cdot \varepsilon_{n+1}) - F(s_{n+1}^2 \cdot \varepsilon_{n+2}) \,| = \left| \int_{s_n^2 \varepsilon_{n+1}}^{s_{n+1}^2 \varepsilon_{n+2}} F'(\zeta) d\zeta \right| <<$$

$$<< |s_{n+1}^2 \varepsilon_{n+2} - s_n^2 \varepsilon_{n+1}|,$$

so that $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges. In a similar manner we can show that $\sum_{n=1}^{\infty} |\hat{a}_n - \hat{a}_{n+1}|$ converges. If we define sequences $\{y_n\}, \{\hat{y}_n\}$ by

$$y_n = a_n s_n (e_n \cdot \ldots \cdot e_1) \text{ and } \hat{y}_n = \hat{a}_n \hat{s}_n (e_n \cdot \ldots \cdot e_1)^{-1} \quad (n \in \mathbb{N}),$$

we have that

$$\beta_n := y_{n+1} - y_n - \rho_n + y_n y_{n+1} \hat{\rho}_n = (a_{n+1} - a_n) \cdot s_{n+1} \cdot e_{n+1} \cdot \ldots \cdot e_1 + (a_n - 1)\rho_n$$
$$- a_n \sigma_n + a_n a_{n+1} s_n s_{n+1} r_n (e_{n+1} \cdot \ldots \cdot e_1).$$

Since

$$a_n a_{n+1} s_n s_{n+1} r_n (e_{n+1} \cdot \ldots \cdot e_1) - a_n \cdot s_n \cdot r_n \varepsilon_{n+1} (e_n \cdot \ldots \cdot e_1) = a_n (e_n \cdot \ldots \cdot e_1) \cdot$$
$$\left[(a_{n+1} - a_n) s_n s_{n+1} e_{n+1} + a_n (s_{n+1} - s_n) s_n e_{n+1} + a_n s_n^2 \cdot (e_{n+1} - \varepsilon_{n+1}) \right]$$

and

$$\lim_{n\to\infty} a_n = 1, \quad \sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty, \quad \sum_{n=1}^{\infty} |a_n \cdot \sigma_n| < \infty, \quad \sum_{n=1}^{\infty} |s_n - s_{n+1}| < \infty,$$
$$\sum_{n=1}^{\infty} |e_n - \varepsilon_n| = \sum_{n=1}^{\infty} \Big| |e_n| - 1 \Big| < \infty,$$

we have, by

$$(5.20) \qquad a_n - 1 + a_n^2 \cdot s_n^2 \cdot \varepsilon_{n+1} = 0 \qquad (n \in \mathbb{N})$$

that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} |\beta_n|$ converges. A corresponding result holds for $\{\hat{\beta}_n\}$ $:= \{\hat{y}_{n+1} - \hat{y}_n - \hat{\rho}_n + \hat{y}_n \hat{y}_{n+1} \rho_n\}$. Put

$$\alpha_n = 1 - y_{n+1}\hat{\rho}_n + \hat{y}_n \rho_n - y_n \hat{y}_{n+1}, \quad \hat{\alpha}_n = 1 - \hat{y}_{n+1}\rho_n + y_n \hat{\rho}_n - \hat{y}_n y_{n+1} \quad (n \in \mathbb{N}).$$

Then, by $\sum_{n=1}^{\infty} |\beta_n| < \infty$, we have for all n

$$(5.21) \qquad a_n = \frac{(1 + \rho_n \hat{y}_n)(1 + \hat{\rho}_n y_n) - (\rho_n + y_n)(\hat{\rho}_n + \hat{y}_n)}{1 + \hat{\rho}_n y_n} + \gamma_n,$$

$$\hat{a}_n = \frac{(1 + \rho_n \hat{y}_n)(1 + \hat{\rho}_n y_n) - (\rho_n + y_n)(\hat{\rho}_n + \hat{y}_n)}{1 + \rho_n \hat{y}_n} + \hat{\gamma}_n$$

where $\{\gamma_n\}, \{\hat{\gamma}_n\}$ are sequences such that $\sum_{n=1}^{\infty} |\gamma_n| < \infty$ and $\sum_{n=1}^{\infty} |\hat{\gamma}_n| < \infty$. With the aid of (5.20) we derive

$$\hat{\rho}_n y_n = r_n (e_n \cdot \ldots \cdot e_1)^{-1} \cdot a_n s_n (e_n \cdot \ldots \cdot e_1) = r_n a_n s_n = r_n (1 - a_n)^{1/2} \cdot (\bar{\varepsilon}_{n+1})^{1/2}$$

and similarly

$$\rho_n\hat{y}_n = r_n(1 - \hat{a}_n)^{1/2}\cdot(\varepsilon_{n+1})^{1/2}.$$

Using (5.20) and the estimates

$$|r_n(\bar{s}_n - \hat{s}_n)| \le |r_n|\cdot\left|\frac{\bar{r}_n}{\bar{e}_{n+1}-1} - \frac{r_n}{\bar{e}_{n+1}-1}\right| + |r_n|\cdot\left|\frac{r_n}{\bar{e}_{n+1}-1} - \frac{r_n}{e_{n+1}^{-1}-1}\right|$$

$$\ll |s_n|\cdot|\operatorname{Im} r_n| + |s_n\hat{s}_n|\cdot\Big||e_n| - 1\Big|,$$

$$|\bar{a}_n - \hat{a}_n| = |F(\bar{s}_n^2\cdot\bar{\varepsilon}_{n+1}) - F(\hat{s}_n\cdot\bar{\varepsilon}_{n+1})| \ll |\bar{s}_n - \hat{s}_n|,$$

we obtain that

$$\sum_{n=1}^{\infty}\left|\rho_n\hat{y}_n - \overline{\hat{\rho}_n y_n}\right| = \sum_{n=1}^{\infty}|r_n(1-\hat{a}_n)^{1/2} - \bar{r}_n(1-\bar{a}_n)^{1/2}| \ll$$

$$\ll \sum_{n=1}^{\infty}|s_n|\cdot|\operatorname{Im} r_n| + \sum_{n=1}^{\infty}|r_n|\cdot|\hat{a}_n\hat{s}_n - \bar{a}_n\bar{s}_n|$$

$$\ll \sum_{n=1}^{\infty}|s_n|\cdot|\operatorname{Im} r_n| + \sum_{n=1}^{\infty}|r_n|\cdot|\hat{s}_n - \bar{s}_n|$$

$$\ll \sum_{n=1}^{\infty}|s_n|\cdot|\operatorname{Im} r_n| + \sum_{n=1}^{\infty}\Big||e_n| - 1\Big| < \infty,$$

so that, by (5.21),

$$\sum_{n=1}^{\infty}\left|\left|\frac{\alpha_n}{\hat{\alpha}_n}\right| - 1\right| = \sum_{n=1}^{\infty}\left|\left|\frac{1+\rho_n\hat{y}_n}{1+\hat{\rho}_n y_n}\right| - 1\right|\cdot|1+\tilde{\gamma}_n| < \infty,$$

$\{\tilde{\gamma}_n\}$ being some sequence such that $\sum_{n=1}^{\infty}|\tilde{\gamma}_n|$ converges. Now define for all solutions $\{h_n\}$ of (5.19) a corresponding sequence $\{g_n\}$ by $g_n = \dfrac{h_n - y_n}{1 + h_n\hat{y}_n}$ $(n \in \mathbb{N})$. Then the sequences $\{g_n\}$ are the solutions of the recurrence

$$(5.22)\qquad g_{n+1} = \frac{\alpha_n g_n + \beta_n}{\hat{\beta}_n g_n + \hat{\alpha}_n}\qquad (n \in \mathbb{N}).$$

The recurrence (5.22) satisfies the conditions of Lemma 5.14 (note that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\hat{\alpha}_n = 1$), so that (5.22) has solutions $\{g_n^{(0)}\},\{g_n^{(\infty)}\}$ such that

$$\lim_{n\to\infty} g_n^{(0)} = 0,\quad \lim_{n\to\infty} g_n^{(\infty)} = \infty,$$

whereas $\lim_{n\to\infty}|g_n|$ exists for all solutions $\{g_n\}$ of (5.22)

Now define sequences $\{k_n^{(0)}\}$ and $\{k_n^{(\infty)}\}$ by

$$k_n^{(0)} = \frac{g_n^{(0)} + y_n}{1 + g_n^{(0)}\hat{y}_n}\cdot(e_{n-1}\cdot\ldots\cdot e_1)^{-1} \quad\text{and}\quad k_n^{(\infty)} = \frac{g_n^{(\infty)} + y_n}{1 + g_n^{(\infty)}\hat{y}_n}\cdot(e_{n-1}\cdot\ldots\cdot e_1)^{-1}$$

for $n \in \mathbb{N}$. Then $\{k_n^{(0)}\}$ and $\{k_n^{(\infty)}\}$ are solutions of (5.18) and, since $\prod_{n=1}^{\infty}|e_n|$ converges, we have that $\lim_{n\to\infty} k_n^{(0)} = 0$ and $\lim_{n\to\infty} k_n^{(\infty)} = \infty$.

Finally, if $\{k_n\}$ is an arbitrary solution of (5.18), then it has the form

$$k_n = \frac{g_n + y_n}{1 + g_n \cdot \hat{y}_n} \cdot (e_{n-1} \cdot \ldots \cdot e_1)^{-1} \qquad (n \in \mathbb{N})$$

so that indeed $\lim_{n\to\infty} |k_n|$ exists. □

If $\{k_n\}$ is a solution of (5.18) other than $\{k_n^{(0)}\}$ and $\{k_n^{(\infty)}\}$, then $k = \lim_{n\to\infty} |k_n|$ exists and $k \in \mathbb{C}\backslash\{0\}$. Hence,

$$(5.23) \qquad \arg k_{n+1} - \arg k_n = \arg \frac{k_n + e_n r_n}{k_n(e_n + k_n r_n)} = \arg \frac{1}{e_n} + \arg \frac{1 + e_n r_n k_n^{-1}}{1 + \bar{e}_n r_n k_n}.$$

$$= \arg \frac{1}{e_n} + \mathcal{O}(r_n).$$

Since $|e_n - 1| = |\arg e_n| \cdot (1 + o(1)) + \mathcal{O}(|e_n| - 1)$ for $e_n \longrightarrow 1$ and $\frac{r_n}{e_{n+1} - 1} \longrightarrow 0$ $(n \longrightarrow \infty)$, it follows that

$$(5.24) \qquad \arg k_{n+1} - \arg k_n = \arg \frac{1}{e_n} \cdot (1 + o(1)) + \mathcal{O}(|r_n - r_{n+1}|) + \mathcal{O}(|e_n| - 1).$$

We have moreover that

$$(-1 + e_{n+1}) \cdot \left[\frac{r_n}{e_{n+1} - 1} - \frac{r_{n-1}}{e_n - 1} \right] = (r_n - r_{n-1}) + \frac{r_{n-1}}{e_n - 1} \cdot (e_n - e_{n+1}),$$

so that $\sum_{n=1}^{\infty} |r_n - r_{n+1}|$ converges. Now let $\arg \frac{1}{e_n} = \operatorname{Arg} \frac{1}{e_n}$ and assume that the sign of $\operatorname{Arg} \frac{1}{e_n}$ is constant (i.e. independent of $n \in \mathbb{N}$). It then follows by (5.24) and $\sum_{n=1}^{\infty} |r_n - r_{n+1}| < \infty$, $\sum_{n=1}^{\infty} \left| |e_n| - 1 \right| < \infty$, that $\{\arg k_n\}$ is a converging sequence if and only if $\sum_{n=1}^{\infty} \arg \frac{1}{e_n}$ converges, i.e. if and only if $\prod_{n=1}^{\infty} e_n$ converges. So we have the following lemma:

Lemma 5.16. *Consider the recurrence relation (5.18) of Lemma 5.15 and suppose in addition that either* $\operatorname{Arg} e_n > 0$ *for all* $n \in \mathbb{N}$ *or* $\operatorname{Arg} e_n < 0$ *for all* $n \in \mathbb{N}$. *If* $\prod_{n=1}^{\infty} e_n$ *diverges, then the only converging solutions of (5.18) are* $\{k_n^{(0)}\}$ *and* $\{k_n^{(\infty)}\}$.

Proof of Theorem 5.13: Let $\{u_n\}$ be a non-trivial zero of R. Put

$$(5.25) \qquad k_n = \frac{u_{n+1} - (i \cdot \sqrt{-C_n} + 1) \cdot u_n}{u_{n+1} + (i \cdot \sqrt{-C_n} - 1) \cdot u_n}.$$

Then $\{k_n\}$ satisfies the recurrence relation (5.18) where

$$r_n = \frac{\sqrt{-C_n} - \sqrt{-C_{n+1}}}{\sqrt{-C_n} + \sqrt{-C_{n+1}}} \quad \text{and} \quad e_n = \frac{i - \sqrt{-C_n}}{i + \sqrt{-C_n}}.$$

Note that the conditions of Lemma 5.15 are satisfied, because $|\ \mathrm{Im}\ r_n\ | \sim |\ \mathrm{Im}\ \sqrt{C_{n+1}/C_n}\ |$, $|e_n| - 1 \sim c\cdot\mathrm{Im}\ \sqrt{-C_n}$ for some $c \in \mathbb{R}$, $c \neq 0$, $|e_n - e_{n+1}| \sim c'\cdot|\ \sqrt{C_{n+1}} - \sqrt{C_n}\ |$ for some $c' \in \mathbb{R}_{>0}$, and $r_n \sim (1 - \sqrt{C_{n+1}/C_n})/2$, $(1 - e_n)^{-1} \sim \frac{i}{2}\cdot\frac{1}{\sqrt{-C_n}}$. Note that the condition that $\lim_{n\to\infty} \sqrt{-C_n^{-1}} - \sqrt{-C_{n+1}^{-1}} = 0$ implies that $|\ \sqrt{C_n}\ | \gg n$. Since $\mathrm{Re}\ \sqrt{-C_n} > 0$ by definition, it follows that $\sum \mathrm{Re}\ \sqrt{-C_n} = +\infty$. Since $\tan \arg e_n = \dfrac{2\cdot\mathrm{Re}\ \sqrt{-C_n}}{1 + |C_n|} > 0$ $(n \in \mathbb{N})$, it thus follows by Lemma 5.16 that all solutions of (5.18) except for $\{k_n^{(0)}\}$ and $\{k_n^{(\infty)}\}$ diverge. Now define $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ by

$$k_n^{(0)} = \frac{u_{n+1}^{(1)} - (i\cdot\sqrt{-C_n} + 1)\cdot u_n^{(1)}}{u_{n+1}^{(1)} + (i\cdot\sqrt{-C_n} - 1)\cdot u_n^{(1)}}, \quad k_n^{(\infty)} = \frac{u_{n+1}^{(2)} - (i\cdot\sqrt{-C_n} + 1)\cdot u_n^{(2)}}{u_{n+1}^{(2)} + (i\cdot\sqrt{-C_n} - 1)\cdot u_n^{(2)}}.$$

Put $\tau_n = \dfrac{u_n^{(2)}}{u_n^{(1)}}$ and $\zeta_n^{(i)} = \dfrac{u_{n+1}^{(i)}/u_n^{(i)} - 1}{i\cdot\sqrt{-C_n}}$ $(i = 1,2;\ n \in \mathbb{N})$. Let $\{u_n\} \in Z(R)$, $\{u_n\} = \lambda\cdot\{u_n^{(1)}\} + \mu\cdot\{u_n^{(2)}\}$, with $\lambda\cdot\mu \neq 0$, and let $\{k_n\}$ be the corresponding solution of (5.18) (by (5.25)). Then

$$(5.26) \qquad -1 + 2\cdot(1 - k_n)^{-1} = \frac{u_{n+1}/u_n - 1}{i\cdot\sqrt{-C_n}} = \zeta_n^{(1)} + (\zeta_n^{(2)} - \zeta_n^{(1)})\cdot\frac{\tau_n}{\lambda/\mu + \tau_n}$$

for all n, so that

$$\tau_n = -\lambda/\mu\cdot\frac{(k_n - 1)(\zeta_n^{(1)} - 1) + 2\cdot k_n}{(k_n - 1)(\zeta_n^{(2)} + 1) + 2} \qquad (n \in \mathbb{N}).$$

Since $\{|k_n|\}$ converges to some positive number k and $\zeta_n^{(1)} \to 1$, $\zeta_n^{(2)} \to -1$ as $n \to \infty$, we conclude that $\lim_{n\to\infty} |\tau_n|/|k_n| = |\lambda/\mu|$. Moreover, since $\{k_n\}$ does not converge, $\{\tau_n\}$ does not converge either. Clearly, $\lim_{n\to\infty} |\tau_n| = |\lambda/\mu|\cdot k \neq 0$, so that, by taking $|\lambda/\mu|\cdot k\cdot\{u_n^{(1)}\}$ instead of $\{u_n^{(1)}\}$, effect that $\lim_{n\to\infty} \left|\dfrac{u_n^{(2)}}{u_n^{(1)}}\right| = 1$. Furthermore, if $d = 0$ and we choose $\{u_n\} = \lambda\cdot\{u_n^{(1)}\} + \mu\cdot\{u_n^{(2)}\}$ with $|\lambda| \neq |\mu|$, then $k = \lim_{n\to\infty} k_n \neq 1$, so that $\{(1 - k_n)^{-1}\}$ is a bounded sequence and, by (5.26) we infer that $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 1$. Finally, let $\{u_n\} = \lambda\cdot\{u_n^{(1)}\} + \mu\cdot\{u_n^{(2)}\}$ for some $\lambda,\mu \in \mathbb{C}\backslash\{0\}$ with $|\lambda| = |\mu|$, and let $d = 0$. Let the curve $\mathcal{C}_n$ be defined by

$\mathcal{C}_n = \{\ z_n(\lambda,\mu) = \dfrac{\lambda\cdot u^{(1)}_{n+1} + \mu\cdot u^{(2)}_{n+1}}{\lambda\cdot u^{(1)}_n + \mu\cdot u^{(2)}_n}\ ,\ |\lambda| = |\mu| = 1\ \}$. $\mathcal{C}_n$ is a closed Jordan-curve in $\mathbb{P}^1(\mathbb{C})$ (it is the image of the unit circle under a fractional linear map) and $\lim_{n\to\infty} \mathcal{C}_n = \mathbb{R} \cup \{\infty\}$. Hence, for every $M \in \mathbb{N}$, $\mathcal{C}_n$ has points z with $|z| \geq M$ for all $n \geq N_M$. Put $z_n = \dfrac{u_{n+1}}{u_n} = 1$ for $\{u_n\} \in Z(R)$. Then

$$(5.27)\qquad z_{n+1} = \frac{z_n + C_n}{z_n + 1} \quad (n \in \mathbb{N}).$$

So, if $|z_{n+1}| \leq m < 1/2$, and $|C_n| \leq m$, we have that $|z_n| \leq \dfrac{m + |C_n|}{1 - m} \leq 4m$. Hence, if $|z_n| \geq M > 2$, then $|z_{n+1}| \geq M/4$ for all $n \geq N_M$. Consequently, for every solution $\{z_n(\lambda,\mu)\}$ of (5.27) with $|\lambda| = |\mu| \neq 0$, $|z_n(\lambda,\mu)|$ becomes larger than 1/2 infinitely often. But then $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n}$ can not exist for $\{u_n\} = \lambda\cdot\{u^{(1)}_n\} + \mu\cdot\{u^{(2)}_n\}$ with $|\lambda| = |\mu|$. If $d \neq 0$, this is trivial. □

Remark 5.5.1. If $\{C_n\}$ is a real sequence, then we have that $\{\bar{u}^{(1)}_n\} = \{u^{(2)}_n\}$. If $\{u_n\} = \lambda\cdot\{u^{(1)}_n\} + \mu\cdot\{u^{(2)}_n\}$ with $|\lambda| = |\mu|$, then clearly $u_n \in \mathbb{R}$ (up to some multiplicative factor). Since $\lim_{n\to\infty} \sqrt{-C^{-1}_n} - \sqrt{-C^{-1}_{n+1}} = 0$, we see that $C_n < -1/n^2$ for n large enough, and indeed, by Proposition 5.3 we have that $\lim_{n\to\infty} \dfrac{u_{n+1}}{u_n}$ does not exist. This yields once more the last statement of Theorem 5.13.

§5. Applications.

We denote by $\mathcal{M}er(K)$ the set of convergent Laurent series in 1/n with a finite principal part and whose coefficients lie in the field K, i.e. $f \in \mathcal{M}er(K) \Rightarrow f(n) = F(1/n)$ where F(z) is meromorphic in z = 0 ($n \in \mathbb{N}$, large enough). We let K be either of the fields $\mathbb{R}$ or $\mathbb{C}$.

Let $R = T^2 - (2 + p(n))\cdot T + (1 + q(n))$, where $p,q \in \mathcal{M}er(K)$. We define the *order* of $r \in \mathcal{M}er(\mathbb{R})$ or $\mathcal{M}er(\mathbb{C})$ by ord r = d if $\lim_{x\to\infty} r(x)\cdot x^d \in \mathbb{C}^*$. If $r \equiv 0$, we define ord $r = \infty$. (So, the order is just the multiplicity of $x = \infty$ as a zero of r, counted negative if $x = \infty$ is a pole of r.) We suppose ord p > 0, ord q > 0. Put

$$r(X) = 1 - \frac{4(1 + q(X))}{(2 + p(X))(2 + p(X-1))}.$$

Then $r \in \mathcal{M}er(K)$, ord $r > 0$ and $R^* = T^2 - 2\cdot T + (1 - r(n))$ is a zeroth-order transform of R. By Theorem 5.4(ii) and Corollary 5.4, Theorem 5.8(ii), Theorem 5.10 and Theorem 5.13 we obtain the following facts:

1. If ord $r \geq 2$ and $\gamma = \lim_{x\to\infty} r(x)\cdot x^2 \neq -1/4$, then R^* has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{u_n^{(1)}}{n^\alpha} = 1, \quad \lim_{n\to\infty} \frac{u_n^{(2)}}{n^\beta} = 1,$$

where α and β are the roots of the polynomial $X^2 - X - \gamma$, and moreover, $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$ if $\gamma > -1/4$ whereas $\lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 1$ and $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}}$ does not exist if $\gamma < -1/4$.

2. If ord $r = 2$ and $\lim_{x\to\infty} r(x)\cdot x^2 = -1/4$, then R^* has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{u_n^{(1)}}{\sqrt{n}\cdot\log n} = 1, \quad \lim_{n\to\infty} \frac{u_n^{(2)}}{\sqrt{n}} = 1.$$

In particular, $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$.

3. If ord $r = 1$ and $\lim_{x\to\infty} r(x)\cdot x$ is not a negative real number, then R^* has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that $u_n^{(1)}, u_n^{(2)} \in K$ $(n \in \mathbb{N})$, $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$ and

$$\lim_{n\to\infty} \frac{1}{\sqrt{r(n)}}\left(\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1\right) = 1, \quad \lim_{n\to\infty} \frac{1}{\sqrt{r(n)}}\left(\frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1\right) = -1.$$

4. If ord $r = 1$ and $r = \lim_{x\to\infty} r(x)\cdot x < 0$, then R^* has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{1}{\sqrt{-r(n)}}\left(\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1\right) = i \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\sqrt{-r(n)}}\left(\frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1\right) = -i.$$

Moreover, if $r_1 = \lim_{x\to\infty} x\cdot(r(x) - r) \in \mathbb{R}$, then we can choose $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that $\lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 1$ whereas $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}}$ does not exist, whereas, if r_1 is not a real number, then $\lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 0$ or infinity, as in 3.

(In fact, $\{u_n^{(2)}\} = \{\bar{u}_n^{(1)}\}$ if $K = \mathbb{R}$.) Note that corresponding results for R follow immediately from those of R^*.

CHAPTER SIX

SECOND-ORDER RECURRENCES (2)

§1. Introduction.

In this chapter we shall treat non-simple operators with two distinct eigenvalues α and β such that $|\alpha| = |\beta|$. As in the previous chapter, we shall have to impose additional conditions on the behaviour of the operator $R - \chi_R(T)$ in order to ensure convergence of $\frac{u_{n+1}}{u_n}$ for $\{u_n\} \in Z(R)$.

Indeed there exist operators of the above type such that for none of their zeros $\{u_n\}$ the quotient $\frac{u_{n+1}}{u_n}$ converges. For instance, take $R = T^2 - (1 + \frac{(-1)^n}{n})$. Let $\{u_n\} \in Z(R)$, $\{u_n\} \neq \{0\}$. Then $\frac{u_{2n+1}}{u_{2n}} \longrightarrow 0$ and $\left|\frac{u_{2n}}{u_{2n-1}}\right| \longrightarrow \infty$ as $n \longrightarrow \infty$, unless $u_0 = 0$.

There also exist operators R such that $\frac{u_{n+1}}{u_n}$ converges to only one of the roots of χ_R for all non-trivial zeros $\{u_n\}$ of R. For instance, let $R = (p_n \cdot T + p_{n+1})(T - 1)$, where $p_n = 1 + \frac{(-1)^n}{n}$, hence $\chi_R(X) = X^2 - 1$. A zero $\{u_n\}$ of R has the form $u_n = \lambda \cdot \sum_{k=0}^{n-1} (-1)^k \cdot p_k + \mu$. Hence,

$$\frac{u_{n+1}}{u_n} = 1 + \frac{\lambda \cdot (-1)^n \cdot p_n}{\lambda \cdot \sum_{k=0}^{n-1} (-1)^k \cdot p_k + \mu} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

We first treat the case that $R - \chi_R(T)$ converges fast. The result follows immediately from Theorem 4.1.

Corollary 6.1. *Let $R \in \mathfrak{R}(\mathbb{C})$, $\chi_R(T) = (X - \alpha)(X - \beta)$, where $|\alpha| = |\beta|$, $\alpha \neq \beta$. Suppose that $\sum N_n(R - \chi_R(T)) < \infty$. Then R has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that*

$$\lim_{n\to\infty} \frac{u_n^{(1)}}{\alpha^n} = \lim_{n\to\infty} \frac{u_n^{(2)}}{\beta^n} = 1.$$

If $R - \chi_R(T)$ converges more slowly, we shall have to investigate the matter in more detail. The case that $\alpha \neq -\beta$ is covered by Theorems 5.10 and 5.13, Corollary 6.1 and Theorem 6.2 below. Some of the results overlap.

§2. R has two opposite non-zero eigenvalues.

We present two results. One of them is a decomposition theorem for matrices as in Chapter 3. The second results uses the fact that the operator R′ which has zeros $\{u_{2n}\}$, where $\{u_n\} \in Z(R)$, has a characteristic polynomial with two equal non-zero eigenvalues, so that the results of Chapter 5 can be applied. In principle, a similar method can be applied whenever the ratios of the eigenvalues are roots of unity.

Theorem 6.2. *Let* $R = T^2 + P(n)\cdot T + Q(n)$, *where* $\lim_{n\to\infty} P(n) = p$, $\lim_{n\to\infty} Q(n) = q$, *and* $X^2 + pX + q = (X - \alpha_1)(X - \alpha_2)$ *with* $|\alpha_1| = |\alpha_2|$, $\alpha_1 \neq \alpha_2$. *Suppose that* $\sum_{n=1}^{\infty} |P(n) - P(n+1)| < \infty$, $\sum_{n=1}^{\infty} |Q(n) - Q(n+1)| < \infty$, *and that there exists some sequence of non-negative real numbers* $\{d_n\}$, $\sum_{n=1}^{\infty} d_n < \infty$, *such that* Re $\overline{P(n)}\cdot\sqrt{P(n)^2 - 4Q(n)}$ *is semi-definite for a fixed branch of the square root* ($0 \leq \arg \sqrt{z} < \pi$, *say*). *Then* R *has zeros* $\{u_n^{(1)}\},\{u_n^{(2)}\}$ *such that* $u_n^{(i)} = \alpha_i(n-1)\cdot\ldots\cdot\alpha_i(1)\cdot(1 + o(1))$ *where* $\alpha_1(n),\alpha_2(n)$ *are the zeros of* $\mathcal{P}_n(X) = X^2 + P(n)\cdot X + Q(n)$ *such that* $\alpha_i(n) \to \alpha_i$ ($i = 1,2$).

We first give a 'matrix decomposition lemma':

Lemma 6.3. *Let* $\{M_n\}$ *be a sequence of matrices in* $\mathbb{C}^{2,2}$ *with* $\sum_{n=1}^{\infty} \| M_n - M_{n+1} \| < \infty$, *and with* M_n *having eigenvalues* α_n *and* β_n *such that* $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} \beta_n = \beta$ *where* $|\alpha| = |\beta|$, $\alpha \neq \beta$, *and such that there exists a sequence* $\{d_n\}$ *of non-negative real integers with* $|\alpha_n| \leq |\beta_n| + d_n$ *for all* n *and* $\sum_{n=1}^{\infty} d_n < \infty$. *Then there exist matrices* $F_n \in GL(2,\mathbb{C})$ *which converge to some matrix* $F \in GL(2,\mathbb{C})$ *such that*

$$F_{n+1}M_nF_n^{-1} = \begin{bmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{bmatrix}.$$

Proof: There exists a sequence $\{U_n\}$, $U_n \in GL(2,\mathbb{C})$ such that

$$U_nM_nU_n^{-1} = \begin{bmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{bmatrix}$$

and $\lim U_n = U \in GL(2,\mathbb{C})$. Furthermore, $\sum_{n=1}^{\infty} |U(n) - U(n+1)| < \infty$, so that

$$U_{n+1}M_nU_n^{-1} = \begin{bmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{bmatrix} + D_n,$$

where $\sum_{n=1}^{\infty} \| D_n \|$ converges. By the assumptions of the lemma we can, by Lemma 5.14, find a sequence $\{V_n\}$, $V_n \in GL(2,\mathbb{C})$, such that $\lim V_n = I$ and

$$V_{n+1}U_{n+1}M_nU_n^{-1}V_n^{-1} = \begin{bmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{bmatrix}.$$

Now let $F_n = V_nU_n$ ($n \in \mathbb{N}$). □

For the proof of Theorem 6.2 we simply apply Lemma 6.3 with $M_n = M_n^R$ ($n \in \mathbb{N}$). Note that $|\alpha_1(n)|^2 - |\alpha_2(n)|^2 = Re\ \overline{P(n)}\cdot\sqrt{P(n)^2 - 4Q(n)}$ so that $|\alpha_1(n)/\alpha_2(n)| - 1$ has constant sign for all n if and only if $Re\ \overline{P(n)}\cdot\sqrt{P(n)^2 - 4Q(n)}$ has. The matrix recurrence $[F_{n+1}M_nF_n^{-1}]$ (with $\{F_n\}$ as in Lemma 6.3) has solutions $\{F^{-1}F_n\cdot\begin{bmatrix} u_{n+1}-\alpha_2u_n \\ u_{n+1}-\alpha_1u_n \end{bmatrix}\}$ where $F^{-1}F_n \longrightarrow I$ as $n \longrightarrow \infty$ and where $\{u_n\}$ is a zero of R. Hence R has a zero $\{u_n^{(1)}\}$ such that $\lim_{n\to\infty} \frac{u_{n+1}^{(1)} - \alpha_1\cdot u_n^{(1)}}{u_{n+1}^{(1)} - \alpha_2\cdot u_n^{(1)}} = 0$, and $u_{n+1}^{(1)} - \alpha_2\cdot u_n^{(1)} = \alpha_1(n-1)\cdot\ldots\cdot\alpha_1(1)\cdot(1 + o(1))$, from which it can easily be deduced that $u_n^{(1)} = \lambda_1\cdot\alpha_1(n-1)\cdot\ldots\cdot\alpha_1(1)\cdot(1 + o(1))$ for $\lambda_1 = (\alpha_1 - \alpha_2)^{-1} \neq 0$, and for all n. The corresponding fact for $\{u_n^{(2)}\}$ goes, of course, similarly. □

Corollary 6.4. *Let R be as in Theorem 6.2. If* $\left|\sum_{n=1}^{\infty} Re\ \overline{P(n)}\cdot\sqrt{P(n)^2 - 4Q(n)}\right|$ *converges, then* $\lim_{n\to\infty}\left|\frac{u_n^{(2)}}{u_n^{(1)}}\right|$ *exists, but* $\lim_{n\to\infty}\frac{u_n^{(2)}}{u_n^{(1)}}$ *does not exist, where* $\{u_n^{(1)}\}$ *and* $\{u_n^{(2)}\}$ *are as in Theorem 6.2. Moreover,* $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}$ *exists if and only if* $\{u_n\}$ *is linearly dependent of either of the* $\{u_n^{(i)}\}$ *(i = 1,2). On the other hand, if* $\left|\sum_{n=1}^{\infty} Re\ \overline{P(n)}\cdot\sqrt{P(n)^2 - 4Q(n)}\right|$ *diverges, then* $\lim_{n\to\infty}\left|\frac{u_n^{(2)}}{u_n^{(1)}}\right|$ *is either zero or infinity. In this case,* $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}$ *exists for all non-zero* $\{u_n\}$ *in Z(R).*

Proof: It suffices to note that
$Re\ \overline{P(n)}\cdot\sqrt{P(n)^2 - 4Q(n)} = |\alpha_1(n)|^2 - |\alpha_2(n)|^2$. □

Let $R = T^2 - P(n)\cdot T - Q(n)$. If $\chi_R(X) = X^2 - \alpha^2$ for some $\alpha \neq 0$, and $P(n) \neq 0$ for all $n \in \mathbb{N}$, we can normalize R as in Chapter 5, §1, thus obtaining $R/S = T^2 - 2\cdot T + 1 - C_n$ for some S, where $\lim_{n\to\infty} |C_n| = \infty$ (see Remark 5.1.1). If $\{u_n\} \in Z(R/S)$, then $\{z_n\}$, with $z_n = \dfrac{u_{n+1} - \sqrt{C_n}\cdot u_n}{u_{n+1} + \sqrt{C_n}\cdot u_n}$ $(n \in \mathbb{N})$, satisfies

$$(6.1) \qquad z_{n+1} = \frac{(1 - \delta_n)\cdot z_n + \varepsilon_n}{\gamma_n \cdot z_n + 1} \quad (n \in \mathbb{N}),$$

where

$$(6.2) \qquad 1 - \delta_n = \frac{1 - \sqrt{C_n}}{1 + \sqrt{C_n}}, \; \varepsilon_n = \frac{\sqrt{C_n} - \sqrt{C_{n+1}}}{\sqrt{C_n} + \sqrt{C_{n+1}}}, \text{ and } \gamma_n = \varepsilon_n(1 - \delta_n) \quad (n \in \mathbb{N}).$$

We use the following lemma to investigate (6.1).

Lemma 6.5. *Let $\{\delta_n\},\{\varepsilon_n\}$ be sequences of complex numbers with* $\lim_{n\to\infty} \delta_n = 2$, $|1 - \delta_n| \le 1$ (for all $n \ge N$), $\sum_{n=1}^{\infty} (1 - |1 - \delta_n|) = \infty$, and $|\varepsilon_n\varepsilon_{n-1}| + |\varepsilon_n(1 - \delta_n) + \varepsilon_{n-1}| = o(1 - |1 - \delta_n|)$ $(n \to \infty)$. *Then the recurrence*

$$(6.3) \qquad z_{n+1} = \frac{(1 - \delta_n)\cdot z_n + \varepsilon_n}{(1 - \delta_n)\varepsilon_n \cdot z_n + 1} \quad (n \in \mathbb{N})$$

has solutions $\{z_n^{(0)}\}$ and $\{z_n^{(\infty)}\}$ such that $\lim_{n\to\infty} z_n^{(0)} = 0$, $\lim_{n\to\infty} z_n^{(\infty)} = \infty$. *Further, if $\{z_n\} \neq \{z_n^{(\infty)}\}$ is a solution of (6.3), then* $\lim_{n\to\infty} z_n = 0$.

Proof: For all $n \in \mathbb{N}$, we have

$$(6.4) \qquad z_{n+2} = \frac{(1 - \delta_n)(1 - \delta_{n+1} + \varepsilon_n\varepsilon_{n+1})\cdot z_n + (\varepsilon_{n+1} + \varepsilon_n(1 - \delta_{n+1}))}{(1 - \delta_n)(\varepsilon_n + \varepsilon_{n+1}(1 - \delta_{n+1}))\cdot z_n + (1 + \varepsilon_n\varepsilon_{n+1}(1 - \delta_{n+1}))}.$$

Application of Lemma 5.11 immediately yields the result for the sequences $\{z_{2n}\}$. Defining $\{z_n\}$ for n odd by (6.3), we obtain the result for all n. □

The result of Lemma 6.5 allows us to conclude that

Corollary 6.6. *Let* $R \in \mathfrak{R}(\mathbb{C})$, $R = T^2 - 2\cdot T + 1 - C_n$, *where* $C_n \in \mathbb{C}$, $\lim_{n\to\infty} |C_n| = \infty$, $\sum_{n=1}^{\infty} \operatorname{Re} \frac{1}{\sqrt{C_n}} = \infty$, $(\sqrt{C_{n+1}/C_n} - 1)\cdot(\sqrt{C_n/C_{n-1}} - 1) = o(\operatorname{Re} \frac{1}{\sqrt{C_n}})$, $\frac{1}{\sqrt{C_{n+1}}} - \frac{1}{\sqrt{C_{n-1}}} = o(\operatorname{Re} \frac{1}{\sqrt{C_n}})$, $\sqrt{C_{n+1}/C_n} - \sqrt{C_n/C_{n-1}} = o(\operatorname{Re} \frac{1}{\sqrt{C_n}})$. *Then R has zeros* $\{u_n^{(1)}\}$ *and* $\{u_n^{(2)}\}$ *such that*

$$\lim_{n\to\infty} \frac{1}{\sqrt{C_n}}\cdot\frac{u^{(1)}_{n+1}}{u^{(1)}_n} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{\sqrt{C_n}}\cdot\frac{u^{(2)}_{n+1}}{u^{(2)}_n} = -1, \quad \textit{and} \quad \lim_{n\to\infty} \frac{u^{(2)}_n}{u^{(1)}_n} = 0.$$

Proof: Define sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ by (6.2) and apply Lemma 6.5. Then $\{u^{(1)}_n\}$ and $\{u^{(2)}_n\}$ can be defined by

$$z_n^{(0)} = \frac{u^{(1)}_{n+1} - \sqrt{C_n}\cdot u^{(1)}_n}{u^{(1)}_{n+1} + \sqrt{C_n}\cdot u^{(1)}_n} \quad \text{and} \quad z_n^{(\infty)} = \frac{u^{(2)}_{n+1} - \sqrt{C_n}\cdot u^{(2)}_n}{u^{(2)}_{n+1} + \sqrt{C_n}\cdot u^{(2)}_n}.$$

Since for any $\{u_n\} \in Z(R)$ which is linearly independent with $\{u_n^{(2)}\}$, we have $\lim_{n\to\infty} \frac{1}{\sqrt{C_n}}\cdot\left(\frac{u_{n+1}}{u_n} - 1\right) = 1$, Lemma 5.12 allows us to conclude that $\lim_{n\to\infty} \frac{u^{(2)}_n}{u^{(1)}_n} = 0$. □

Remark 6.2.1. Direct application of Lemma 5.11 to R (where R is as in Corollary 6.6) would give as conditions on $\{C_n\}$ (so that the statement of Cor.6.6 holds):

$$\lim_{n\to\infty} |C_n| = \infty, \quad \sum_{n=1}^{\infty} \mathrm{Re}\,\frac{1}{\sqrt{C_n}} = \infty, \quad \sqrt{C_{n+1}/C_n} - 1 = o(\mathrm{Re}\,\frac{1}{\sqrt{C_n}}).$$

The conditions of Corollary 6.6 are obviously weaker. For example, if $C_n = C(1/n)$, where $C(x)$ is a Laurent series at $x = 0$, $C(x) = \alpha\cdot x^{-2}(1 + \mathcal{O}(x))$ for $\alpha \in \mathbb{C}$, α not a non-positive real number, then Corollary 6.6 may be applied, whereas $\sqrt{C_{n+1}/C_n} - 1 \neq o(\mathrm{Re}\,\frac{1}{\sqrt{C_n}})$.

Remark 6.2.2. A similar theorem for the elliptic case (where $\lim_{n\to\infty} |C_n| = \infty$, $\sum_{n=1}^{\infty} \left|\mathrm{Re}\,\frac{1}{\sqrt{C_n}}\right| < \infty$) can be derived from Lemma 5.15. Since, however, no new ideas are involved, and since for most interesting cases Theorem 6.2 suffices, we will not pursue this matter any further.

§3. Applications.

1. Let $R = T^2 - p(n)\cdot T - (1 + q(n))$, where $p,q \in \mathcal{Mer}(\mathbb{C})$. Suppose that ord $p > 0$, ord $q > 0$, so that $\chi_R(X) = X^2 - 1$. We can apply a zeroth-order transformation onto R such that the resulting operator R' is of the form $R' = T^2 - p^*(n)T - (1 + q^*(n))$, where ord p^* = ord p and ord $q^* \geq 2$. In particular, we can take

$$(6.12)\qquad R' = T^2 - \frac{p(n)}{1 + q(n)/2}\cdot T - \frac{1 + q(n)}{(1 + q(n)/2)(1 + q(n-1)/2)}.$$

We distinguish two cases:

(i) ord $p \geq 2$. We can apply Corollary 6.1 to R' and find that R has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that $u_n^{(1)}, u_n^{(2)} \in \mathbb{R}$ $(n \in \mathbb{N})$ if $p,q \in \mathcal{Mer}(\mathbb{R})$ and

$$\lim_{n\to\infty} u_n^{(1)}\cdot\prod_{k=1}^{n-2}(1 + q(k)/2)^{-1} = \lim_{n\to\infty} (-1)^n\cdot u_n^{(2)}\cdot\prod_{k=1}^{n-2}(1 + q(k)/2)^{-1} = 1.$$

(ii) ord $p = 1$. Let $R^* = T^2 - 2\cdot T - \dfrac{4(1 + q(n))}{p(n)p(n-1)}$.

If $p(x) = \frac{a}{x} + \mathcal{O}(x^{-2})$ with $ai \notin \mathbb{R}$, we may apply Corollary 6.6 to R^* and find that it has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ with

$$\lim_{n\to\infty} \frac{\sqrt{a^2}}{2n}\cdot\frac{u_{n+1}^{(1)}}{u_n^{(1)}} = 1, \quad \lim_{n\to\infty} \frac{\sqrt{a^2}}{2n}\cdot\frac{u_{n+1}^{(2)}}{u_n^{(2)}} = -1, \quad \lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0,$$

where $\sqrt{a^2}$ is the square root of a with positive real part.

Hence, R has zeros $\{v_n^{(1)}\},\{v_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{\sqrt{a^2}}{a}\cdot\frac{v_{n+1}^{(1)}}{v_n^{(1)}} = 1, \quad \lim_{n\to\infty} \frac{\sqrt{a^2}}{a}\cdot\frac{v_{n+1}^{(2)}}{v_n^{(2)}} = -1 \quad\text{and}\quad \lim_{n\to\infty} \frac{v_n^{(2)}}{v_n^{(1)}} = 0.$$

The same conclusion can be reached if we apply Theorem 6.2 and Corollary 6.4.

If $p(x) = \frac{a}{x} + \mathcal{O}(x^{-2})$ with $ai \in \mathbb{R}$, $a \neq 0$, we apply Theorem 6.2 and Corollary 6.4 and obtain that R has zeros $\{v_n^{(1)}\},\{v_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{v_{n+1}^{(1)}}{v_n^{(1)}} = 1, \quad \lim_{n\to\infty} \frac{v_{n+1}^{(2)}}{v_n^{(2)}} = -1 \quad\text{and}\quad \lim_{n\to\infty} \left|\frac{v_n^{(2)}}{v_n^{(1)}}\right| = 1,$$

whereas $\lim_{n\to\infty} \dfrac{v_n^{(2)}}{v_n^{(1)}}$ does not exist.

2. Let $R = T^2 - (-1)^n\cdot p(n)\cdot T - (1 + q(n))$, where $p,q \in \mathcal{Mer}(\mathbb{R})$, ord $p > 0$, ord $q > 0$, so that $\chi\,(X) = X^2 - 1$.

(i). If ord $p \geq 2$, we can apply Corollary 6.1 and find that R has zeros $\{u_n^{(1)}\},\{u_n^{(2)}\}$ with $u_n^{(1)}, u_n^{(2)} \in \mathbb{R}$ $(n \in \mathbb{N})$ and

$$\lim_{n\to\infty} u_n^{(1)}\cdot\prod_{k=1}^{n-2}(1 + q(k)/2)^{-1} = \lim_{n\to\infty} (-1)^n\cdot u_n^{(2)}\cdot\prod_{k=1}^{n-2}(1 + q(k)/2)^{-1} = 1.$$

(ii). If ord $p = 1$, we put $R^* = T^2 - 2\cdot T + \dfrac{4(1 + q(n))}{p(n)p(n-1)}$.

As in 1(ii), we find that R^* has zeros $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{p(n)}{2}\cdot\frac{u_{n+1}^{(1)}}{u_n^{(1)}} = i, \quad \lim_{n\to\infty} \frac{p(n)}{2}\cdot\frac{u_{n+1}^{(2)}}{u_n^{(2)}} = -i.$$

Hence, R has zeros $\{v_n^{(1)}\}$, $\{v_n^{(2)}\}$ such that $\lim_{n\to\infty} (-1)^n\cdot\frac{v_{n+1}^{(j)}}{v_n^{(j)}} = (-1)^{j-1}\cdot i$ $(j = 1,2)$.

3. Let $R = T^2 - p(n)\cdot T - q(n)$, where $p,q \in \mathcal{Mer}(\mathbb{C})$ and $\chi_R(X) = (X-\alpha)(X-\beta)$, with $\alpha,\beta \in \mathbb{C}$, and $|\alpha| = |\beta|$, $\alpha \neq \beta$, $\alpha \neq -\beta$. Applying Theorem 6.2 and Corollary 6.4 (or, alternatively, Theorems 5.10 and 5.13) to

$R^* = T^2 - 2\cdot T - \frac{4\cdot q(n)}{p(n)p(n-1)}$, we find that R has zeros $\{u_n^{(1)}\},\{u_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}} = \alpha \quad \text{and} \quad \lim_{n\to\infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}} = \beta,$$

and, moreover, $\lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 1$ if $\frac{4\cdot q(n)}{p(n)p(n-1)} = a + \frac{b}{n} + \mathcal{O}(n^{-2})$ with $b \in \mathbb{R}$

(note that $a \in \mathbb{R}$ in any case), whereas $\lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 0$ or infinity if $b \notin \mathbb{R}$.

In fact, $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$ if and only if $b\cdot\frac{\text{Re }\alpha}{\text{Im }\alpha} < 0$.

CHAPTER SEVEN.

APPLICATION TO CONTINUED FRACTIONS.

We shall conclude with an application of the above results, which constitutes an answer to the following problem, posed by Perron [Pe3]:

Consider the continued fraction

$$\frac{q(1)|}{|p(1)} + \frac{q(2)|}{|p(2)} + \ldots + \frac{q(n)|}{|p(n)} + \ldots \tag{7.1}$$

where $p,q \in \mathcal{M}er(\mathbb{C})$, $p,q \neq 0$. If $\lim_{n\to\infty} \frac{q(1)|}{|p(1)} + \frac{q(2)|}{|p(2)} + \ldots + \frac{q(n)|}{|p(n)}$ exists or if $\lim_{n\to\infty} p(1) + \frac{q(2)|}{|p(2)} + \ldots + \frac{q(n)|}{|p(n)} = 0$, we say that the continued fraction (7.1) *converges in a broad sense*. The problem is to determine for which $p,q \neq 0$ the expression (7.1) converges in a broad sense.

Consider the recurrence operator $R = T^2 - p(n)\cdot T - q(n)$ $(n \in \mathbb{N})$. Without loss of generality we may suppose $p(n),q(n) \neq 0$ for $n \in \mathbb{N}$. Let $\{u_{n-2}\}_{n\geq 1}$, $\{v_{n-2}\}_{n\geq 1}$ be the zeros of R for which $u_{-1} = 1$, $u_0 = 0$, $v_{-1} = 0$, $v_0 = 1$. It is then clear that $\{u_{n-2}\}$ and $\{v_{n-2}\}$ are linearly independent. Moreover,

$$\frac{u_n}{v_n} = \frac{q(1)|}{|p(1)} + \frac{q(2)|}{|p(2)} + \ldots + \frac{q(n)|}{|p(n)} \quad (n \in \mathbb{N}). \tag{7.2}$$

Therefore the continued fraction (7.1) converges in a broad sense if and only if either $\lim_{n\to\infty} \frac{u_n}{v_n}$ exists or $\lim_{n\to\infty} \frac{v_n}{u_n} = 0$. On the other hand, if $\lim_{n\to\infty} \frac{u_n}{v_n} = \zeta$, then $\lim_{n\to\infty} \frac{u_n - \zeta\cdot v_n}{v_n} = 0$ and $\{u_{n-2} - \zeta\cdot v_{n-2}\} \in Z(R)$. Thus, we have that (7.1) converges in a broad sense if and only if the corresponding recurrence operator R has linearly independent real zeros $\{u_n\}$ and $\{w_n\}$ such that $\lim_{n\to\infty} \frac{w_n}{u_n} = 0$.

We consider the 'normalized' zeroth-order transform R^* of R:

$$R^* = T^2 - 2\cdot T - \frac{4\cdot q(n)}{p(n)p(n-1)}.$$

(Note that $p(n) \neq 0$ for $n \in \mathbb{N}$, so that R^* is well defined). Since R^* is a zeroth-order transform of R, its zeros are of the form $\{\rho(n)x_n\}$, where

$\{x_n\} \in Z(R)$, $\rho(n) \in \mathbb{C}^*$ for $n \geq 1$ and $\rho(n)$ depends only on $\{p(n)\}$. So the answer to our problem boils down to the answer of the problem for which p,q the operator R^* has two linearly independent real zeros $\{u_n\},\{w_n\}$, such that $\lim_{n\to\infty} \frac{w_n}{u_n} = 0$. Put $r(n) = 1 + \frac{4\cdot q(n)}{p(n)p(n-1)}$. Then $r \in \mathcal{Mer}(\mathbb{C})$, $r(n) \neq 1$ or ∞ for $n \geq N$. Put $\imath = \lim_{x\to\infty} r(x)$.

(i) By Poincaré's theorem (or Chapter 3) we have that for $\imath \in \mathbb{C}$, $\imath$ not a non-positive real number R^* has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}} = 1 + \sqrt{\imath} \quad \text{and} \quad \lim_{n\to\infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}} = 1 - \sqrt{\imath}. \quad \text{Hence,} \quad \lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0.$$

(ii) If $\imath \in \mathbb{R}$, $\imath < 0$, we can apply Application 3 of Chapter 6, §3 and obtain that R^* has zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that

$\lim_{n\to\infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}} = 1 + \sqrt{\imath}$ and $\lim_{n\to\infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}} = 1 - \sqrt{\imath}$. Moreover, if $r(x) = \imath + \frac{\mathfrak{s}}{x} + \mathcal{O}(x^{-2})$ and $\mathfrak{s} \notin \mathbb{R}$, then $\lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 0$ or infinity. On the other hand, if $\mathfrak{s} \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}}$ does not exist. In the latter case, we can not find two linearly independent zeros $\{u_n\}$ and $\{w_n\}$ of R^* (so, neither of R) such that $\lim_{n\to\infty} \frac{w_n}{u_n} = 0$. Indeed, this would imply that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{v_{n+1}}{v_n}$ for all zeros $\{v_n\}$ of R^* that are linearly independent with $\{w_n\}$, which is not possible.

(iii) If $\imath = 0$, we can apply the results of Chapter 5 to obtain that

(a) If ord $r \geq 2$ and $\gamma = \lim_{n\to\infty} r(n)\cdot n^2 \geq -1/4$, then R^* has two zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$.

(b) If ord $r \geq 2$ and $\gamma < -1/4$ (γ as in (b)), then R^* has two zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that $\lim_{n\to\infty} \frac{u_n^{(1)}}{n^\alpha} = 0$ and $\lim_{n\to\infty} \frac{u_n^{(2)}}{n^\beta} = 0$, where α and β are the two zeros of the polynomial $X^2 - X - \gamma$. It is clear that $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}}$ can not exist. As in (ii), the conclusion is that there cannot be zeros $\{w_n\}$ and $\{u_n\}$ of R such that the limit of their quotients is zero.

(iv) If ord r = 1 and $r(x) = \frac{a}{x} + \mathcal{O}(x^{-2})$, then R^* has non-trivial zeros $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that $\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0$ if and only if a is not a negative real number.

(v) If $\mathfrak{r} = +\infty$ or $-\infty$, we can apply the results of Chapter 6:

(a) If ord $r \le -3$, we put $R' = T^2 - 2\cdot T + (-1)^s n^d(1 + t(n))$ with $t \in \mathcal{M}er(\mathbb{C})$, and $(-1)^s = -1$ or 1 if $\mathfrak{r} = -\infty$ or $+\infty$, respectively. We consider the zeroth-order transform R'' of R':

$$R'' = T^2 - \frac{2\cdot(n + 1/2)^{-d/2}}{1 + t(n)/2}\cdot T + (-1)^s\cdot\frac{(1 - 1/4n^2)^{-d/2}\cdot(1 + t(n))}{(1 + t(n-1)/2)(1 + t(n)/2)}.$$

Since $\frac{(1 + t(n))}{(1 + t(n-1)/2)(1 + t(n)/2)} = 1 + \mathcal{O}(n^{-2})$ and $d \ge 3$, we can apply Corollary 6.1 to R'' and find, as in (ii), that R'' can not have linearly independent zeros $\{u_n\}$ and $\{w_n\}$ for which $\lim_{n\to\infty} \frac{w_n}{u_n} = 0$, so neither can R.

(b) If d = - ord r = 1 or 2, we can reason as in §3 of Chapter 6 and obtain:

If $r(x) = ax^d + \mathcal{O}(x^{d-1})$, $a \ne 0$, then R^* has two linearly independent zeros $\{u_n\}$ and $\{w_n\}$ for which $\lim_{n\to\infty} \frac{w_n}{u_n} = 0$, if and only if a is not a negative real number. (One can apply Cor.6.6 or Th.6.2 and Cor.6.4 (Cor.6.6 only for a not negative real) to a suitable zeroth-order transform in the manner described for ord $r \le -3$.)

If we apply these results to the continued fraction (7.1), we obtain the following result:

Theorem 7.1: *Consider the continued fraction*

$$(7.1) \qquad \frac{q(1)|}{|p(1)} + \frac{q(2)|}{|p(2)} + \ldots + \frac{q(n)|}{|p(n)} + \ldots,$$

where $p,q \in \mathcal{M}er(\mathbb{C})$, $p,q \ne 0$. *Put* $r(x) = 1 + \frac{4\cdot q(x)}{p(x)p(x-1)}$. *The expression (7.1) converges in a broad sense if and only if one of the following conditions is satisfied:*

(1) ord $r \ge 2$ *and* $\lim_{x\to\infty} r(x)\cdot x^2 \ge -1/4$.

(2) ord r = 1 *and* $\lim_{x\to\infty} r(x)\cdot x$ *is not a negative real number.*

(3) ord r = 0 *unless both* $\mathfrak{r} = \lim_{x\to\infty} r(x) < 0$ *and* $\lim_{x\to\infty} (r(x) - \mathfrak{r})\cdot x \in \mathbb{R}$.

(4) ord r = -1 *and* $\lim_{x\to\infty} r(x)\cdot x^{-1}$ *is not a negative real number.*

(5) ord r = -2 *and* $\lim_{x\to\infty} r(x)\cdot x^{-2}$ *is not a negative real number.*

A final remark. Suppose that (7.1) converges in a broad sense. Put $y_n = \frac{q(n)|}{|p(n)} + \frac{q(n+1)|}{|p(n+1)} + \dots$ $(n \in \mathbb{N})$. Then $y_n = \dfrac{q(n)}{p(n) + y_{n+1}}$, which yields $y_n \cdot y_{n+1} + p(n) \cdot y_n - q(n) = 0$. So we find, if $y_1^{-1} \neq 0$, that $\{w_{n-2}\} = \{(-1)^{n-1} \cdot y_{n-1} \cdot y_{n-2} \cdot \ldots \cdot y_1\}_{n \geq 1}$ is a zero of the recurrence operator $R = T^2 - p(n) \cdot T - q(n)$. We show that $\{w_{n-2}\}$ is a subdominant zero of R, in other words: If $\{x_n\} \in Z(R)$ linearly independent with $\{w_n\}$, then $\lim_{n\to\infty} \frac{w_n}{x_n} = 0$. Indeed, let $\{u_n\}$ and $\{v_n\}$ be as above. So, $\{u_{n-2}\}, \{v_{n-2}\} \in Z(R)$ and $u_{-1} = v_0 = 1$, $u_0 = v_{-1} = 0$. Let $\zeta = \lim_{n\to\infty} \frac{u_n}{v_n}$. Then $y_1 = \zeta$, $\zeta \in \mathbb{C}$. Hence $w_{-1} = 1$, $w_0 = -\zeta$, so that $\{w_n\} = \{u_n\} - \zeta \cdot \{v_n\}$, so $\lim_{n\to\infty} \frac{w_n}{v_n} = \lim_{n\to\infty} \frac{u_n - \zeta \cdot v_n}{v_n} = 0$. Finally, if $y_1^{-1} \neq 0$, then $\lim_{n\to\infty} \frac{v_n}{u_n} = 0$, and we define $\{w_{n-2}\} = \{(-1)^n \cdot y_{n-1} \cdot y_{n-2} \cdot \ldots \cdot y_2\}_{n \geq 2}$. Then $w_0 = 1$, $w_1 = p(1)$. Hence, $\{w_n\} = \{v_n\}$, so that $\lim_{n\to\infty} \frac{w_n}{u_n} = 0$. Thus, for $\{w_n\}$ as defined above, we have that $\lim_{n\to\infty} \frac{w_n}{x_n} = 0$ for all $\{x_n\} \in Z(R)$ linearly independent with $\{w_n\}$.

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REGISTER OF TERMS

Only the page where the terms are defined is indicated.

INDEX OF SYMBOLS

Only symbols that are not conventional or have a specific meaning in this work are listed below; the number indicates the page where they are defined.

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