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Towards a symmetrical theory of generalised functions

J.J. Lodder

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'But much yet remains to be said.'

PREFACE

This tract reports the result of my efforts to find a new mathematical basis for generalised functions, in such a way that multiplication of generalised functions is always possible.

This search was motivated by a dissatisfaction on my part with the standard formulation of analysis, and by problems in mathematical physics for which existing theories of generalised functions are not adequate. The application to quantum field theory is indicated only briefly. The tract develops the theory as a mathematical subject on its own merits.

The characteristic properties of the symmetrical theory are the existence of a symmetrical inner product, in which every generalised function also serves as a test function, and the possibility to multiply all generalised functions. The ability to assign finite values to generalised functions at every point is a natural consequence.

In many respects the symmetrical theory resembles a Hilbert space, but it also contains Dirac's δ-function. The generalised functions also supports the usual operator algebra, without the domain problems of Hilbert space. The symmetrical theory combines the good properties of distributions and Hilbert spaces, while avoiding the difficulties in these theories.

It will be seen that the symmetrical theory implies a great research program, which is at present far too difficult to tackle in its full generality. As a first step this tract presents the construction of a simple model for the symmetrical theory of generalised functions, in order to show that the program is feasible.

The simple model contains many new generalised functions in addition to most of the usual special functions. The core of the book is the definition of the usual operator algebra and the constuction of a product algebra.

The presentation has been kept informal, since I felt that the presentation of new concepts, together with some heuristic considerations for the choices made in the construction, is at this stage more important than the deployment of formalism.

The material in this book should be accessible to graduate students. On the technical level some familiarity with standard analytical function theory is necessary. Only some standard results on analytic continuation are assumed. Use of special functions has been kept to a minimum. Some familiarity with distribution theory and functional analysis is helpful, but not necessary for following the main line of the construction. The subject matter is in principle elementary, but it is necessary to develop a new way of looking at familiar concepts. This may take some getting used to.

The tract is intended for a wide readership, ranging from working mathematicians to physicists who only need some of the results. To increase accessability many examples are presented. Remarks in the text give additional explanations, which may be skipped. Each chapter begins with an introduction outlining the contents. Long chapters end with a summary. In addition to the usual index an index to formulae has been provided for reference purposes. The tract may also serve as a reference work, in which results needed for applications can be looked up.

The results of this book are applicable in areas of mathematical physics where multiplication or convolution of generalised functions is unavoidable, such as non-linear partial differential equations and the perturbation expansions in quantum field theory.

It will be seen that replacement of distribution theory (as the mathematical explanation of Dirac's δ-function) by a symmetrical theory of generalised functions has far-reaching implications for mathematical analysis as a whole. These are outlined in a closing chapter.

It is my hope that the challenge posed by the program for a symmetrical theory will be taken up by others. Further applications will be published elsewhere.

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Correspondence address:
J. J. Lodder
Oudegracht 331b
3511 PC Utrecht
The Netherlands
Tel. 31/(0)30 314153

TABLE OF CONTENTS

continued

continued

continued

CHAPTER 1

INTRODUCTION

This tract presents a program and a simple model for a new theory, the symmetrical theory of generalised functions.

The characteristic difference with distribution theory is the existence of a symmetrical inner product, in which generalised functions and test functions are treated on an equal footing. Every generalised function also serves as a test function. In particular, the Dirac δ–function is treated on equal footing with ordinary functions and not just as a linear functional.

The aim of this symmetrical theory is the mathematical treatment of singular objects such as the Dirac δ–function, divergent integrals or limits, and products of generalised functions. The symmetrical theory makes it possible to define these in cases where they are undefined or infinite in the sense of standard analysis or distribution theory.

The standard method to deal with subjects of this kind is distribution theory, and it is the best possible method in the context of standard analysis. Nevertheless distribution theory does not allow multiplication of distributions, and it does not uniquely define the needed divergent integrals and convolutions. Ad hoc assumptions, for instance regularization methods, are necessary to obtain meaningful results of computations.

The construction of the symmetrical theory of generalised functions is a first step towards an alternative 'explanation' of divergent integrals and δ–functions. It avoids the limitations of distribution theory and Hilbert space theory, and it combines many of the good properties of these theories.

At present the aim of constructing a definitive, or even in some sense complete, symmetrical theory of generalised functions is far too ambitious. The much simpler aim of this book is the construction of a demonstration model, in order to show that an alternative to distribution theory is possible. The model will be kept as small and simple as possible, yet it will be sufficiently strong to give an unambiguous meaning to the undefined expressions which occur in applications such as quantum field theory.

Despite the present immature state of the program it has implications for mathematical analysis. It suggests the need for a reconsideration of the foundations. A reconstruction of analysis, using concepts of function, limit, derivative, and integral which are appropriate for a symmetrical theory of generalised functions is called for. This will entail a large amount of work.

1.1 A historical perspective

In the eighteenth century the subject of analysis consisted mainly of the investigation of the properties of analytic, or piecewise analytic functions, and indeed a function was understood in this way as an (in principle explicit) relation between quantities. The notion of infinitely large and infinitesimal quantities was used freely, often in ways no longer recognized as correct.

This changed early in the nineteenth century. Two of the driving forces behind this change were the logical problems of the infinite and the infinitesimal in the foundations of the calculus, and the development of the theory of Fourier series and integrals.

A drastic generalization of the function concept became necessary as a consequence of Fourier's insistence that the sum of a Fourier series should be accepted as a function when the series converges, and that only integrability is needed for the existence of Fourier coefficients. The practical need for a sufficiently general and powerful theory of the Fourier transform has remained an important source of new developments in analysis ever since.

Dirichlet's proof of Fourier's theorem, for functions with a finite number of maxima and minima, made it clear that not only the concept of a function, but also the theory of integration had to be generalised. The function concept was generalised to what is now known as the Dirichlet function concept, in which functions are understood as arbitrary mappings. The needed generalization of the concept of the integral was given by Riemann. When the concept of bounded variation was added by Jordan the first major Fourier theory was conceptually complete.

In the same period Cauchy and Abel initiated a program for the rigorization of analysis by avoiding infinite quantities and eliminating infinitesimals. Mathematicians and philosophers have made a distinction between the actual and the potential infinite since classical antiquity. The first is 'really' infinite, the second can merely be made as large as one wishes. The rigorization program required the complete elimination of the actual infinite (and the infinitesimal) from analysis, and its replacement by a potential infinite. This program was carried to completion by the efforts of Weierstrass and his school. The infinitesimal and the infinite were replaced by the arbitrarily small and the arbitrarily large. The program supplied a rigorous foundation for the Riemann integral, and consequently for the Fourier transform.

Classical analysis and Fourier theory reached its modern form with Parseval and Lebesgue, with the emergence of measure theory and the $\mathcal{L}_2$ theory of the Fourier transform. With the introduction of the concept of a Hilbert space, and the recognition that the Fourier operator is a unitary operator in Hilbert space, the second major Fourier theory was complete.

The applications of the theory of integral transforms, and Fourier theory in particular, were never free from difficulties caused by the limitations of the classical framework. The development of the operational calculus by Heaviside, and the introduction of the δ–function by Dirac, created formalisms which are not easily absorbed in the classical framework.

The successes of quantum field theory, in particular quantum electrodynamics, in the years around 1950, were obtained by means of complicated manipulations with generalised functions and divergent integrals. For lack of an adequate mathematical foundation, these great advances were made in a heuristic and experimental manner. It is possible to reinterpret these computations in the context of renormalization theory in terms of distributions, but this has the character of an explanation after the fact. It takes much effort to accomplish, and it does not help much with further developments.

At about the same time, and independently of these developments, Dirac's δ–function was 'explained' in a mathematically rigorous way by Schwartz by means of the theory of distributions. Distribution theory was a major accomplishment. It is sufficiently strong to give a rigorous meaning to many (but not all) applications of the Dirac δ–function. The special case of the tempered distributions is the third major Fourier theory.

Distribution theory explains the δ–function as a linear functional instead of a function. Likewise all other distributions are defined as linear functionals only. It is in general not possible to assign a value at a point to a distribution. The multiplication of distributions is thereby excluded.

This limits the straightforward application of distribution theory to linear problems. Solutions of the multiplication problem are possible in special cases, but a general method has not emerged. Yet the divergence problems of the perturbation expansion of quantum field theory arise as products of generalised functions. The limitations of distribution theory are also manifest in the theory of non-linear partial differential equations, in particular when shock waves arise.

None of the Fourier theories referred to above is adequate to deal with all problems which arise in attempts to apply the Fourier transform to the solution of problems. Moreover, the different Fourier theories are incompatible in their formal properties, so that it is often impossible to combine the useful properties of the various theories.

In particular, none of the available Fourier theories is adequate for the applications to quantum mechanics and quantum field theory. On the one hand in quantum mechanics it seems to be necessary to work in a Hilbert space, on the other hand in quantum field theory singular objects such as δ–functions cannot be missed. This often makes uneasy compromises in the mathematical apparatus unavoidable.

The symmetrical theory of generalised functions will (when it has been sufficiently developed) constitute a fourth Fourier theory. It combines the symmetry and the possibility of multiplication of Hilbert space theory with the presence of δ–functions and the absence of domain considerations of distribution theory.

When the symmetrical theory of generalised functions is applied to quantum field theory the result is the automatic disappearance of divergence problems. The divergence problem never arises in the context of symmetrical generalised functions, so it does not have to be eliminated.

The program outlined in this book will make it unavoidable to reconsider the foundations of analysis, in particular the function concept and the treatment of the infinite. This will involve a reconsideration of some aspects of the historical context outlined briefly above.

1.2 Introduction to this work

The starting point of this work is a rejection of the premise underlying distribution theory, which is that objects such as the Dirac δ–function should be considered as linear functionals, and no more than that.

An alternative explanation of the δ–function as a symmetrical generalised function is constructed instead. 'Symmetrical' refers to the existence of a symmetrical inner or scalar product, and to the absence of a distinction between generalised functions and test functions.

The symmetrical theory of generalised functions is not only a Fourier theory. It provides an integration theory which can give a well defined meaning to some divergent integrals, and it allows the definition of limits and derivatives which are undefined or divergent in a standard sense.

It is well known that the multiplication problem of generalised functions does not have a solution in the context of distribution theory. In this work the existence of a solution to the multiplication problem is imposed as a requirement on the mathematical theory which explains the δ–function. It will be shown to be possible to construct a theory of generalised functions which satisfies this requirement.

The resulting symmetrical theory of generalised functions is logically independent of distribution theory. It is in some respects less general, but on the other hand it also contains many objects which do not have a non-zero counterpart in distribution theory.

Compared to the vast amount of work which has been done in the context of functional analysis and distribution theory, the aims and results of this book are modest. As indicated by its title it is only a first step towards a new theory of symmetrical generalised functions.

The lack of an adequate function concept has forced me to fall back on the eighteenth century concept of a function as an explicitly described object. For this reason it might be more appropriate to call the present first attempt a theory of generalised special functions. It is to be hoped that a synthesis between the different function concepts can be effected in the future.

A consequence of building a better Fourier theory, or a better theory of generalised functions, is the need for a rebuilding of analysis as a whole. It seems that $\mathcal{L}_2$–theory and distribution theory are the best that is possible within the confines of standard analysis. If these theories are considered to be not good enough for all purposes, the need for a reconstruction of analysis gradually unfolds from attempts to construct a better theory. It turns out to be not very important which aims one uses as a starting point. Any attempt to add substantially new properties to distribution theory leads to the need for drastic revisions.

Starting from simple wishes for a somewhat better Fourier theory or distribution theory, the need for reconstructing analysis gradually follows from the consequences. In this book the course of the evolution of ideas as it actually occurred is more or less followed.

Once the simple model has been constructed, it can be used as a guide to the needed new conceptual framework. This conceptual framework is in turn needed to develop stronger models.

It is not clear at present where this development may lead, and whether it will end with a unique largest model, which will be complete on its own terms. It is even less clear what kind of foundations will be needed to support the resulting structure.

It will be necessary to reconsider the function concept, and consequently the limit, derivative and integral concepts. Perhaps more controversially, it seems also necessary to reconsider the role played by the infinite in analysis. It may be useful to restore the actual infinite to carefully delimited place in analysis.

Nonstandard analysis in its present form is not directly suitable for this purpose, since it is equivalent to standard analysis, but it may be possible to adapt it in such a way that it will be more suitable.

The present treatment relies on the analysis of asymptotic behaviour, which replaces standard concepts of limit and convergence. The standard concept of asymptotics will also need generalization in future developments.

1.3 Outline of the contents

The organization of the rest of the book is as follows.

In the next chapter, Ch. 2, the requirements imposed on a symmetrical theory of generalised functions are outlined. An outline of the construction of the simple model is given at the end of Ch. 2. The third chapter presents a trivial model in order to show that it is possible to satisfy all requirements of Ch. 2.

In the Chs. 4–11 the construction of a simple model for a symmetrical theory of generalised functions is given. It is verified in Ch. 12 that the model satisfies the requirements of Ch. 2.

In keeping with the aims outlined in the previous section the model is kept as small as possible. Generality is not an objective at present. Only closure under the operators is imposed. Convergence of sequences of generalised functions is defined in Ch. 19, but this generalised convergence concept has not been used as yet to enlarge the model.

The next two chapters, Chs. 13–14, deal with the values, the limits, and the support of the generalised functions. The integration over the support of the generalised functions is defined, and it is shown that the fundamental theorem of the calculus holds for the generalised functions in an appropriate sense.

In the next three chapters, Chs. 15–18, the transformation properties of the generalised functions under translations and scale transformations are derived, with the applications to physical problems in mind.

The convergence of sequences, and the equivalence of the weak topology for generalised functions, are defined in Ch. 19. Periodic functions and Fourier sums are added to the model in Ch. 20. In Ch. 21 the Hilbert transform is added, and the analytic boundary properties of the generalised functions are derived. These are useful for applications.

Only the local infinities, such as divergencies near a point, are treated in this book. The set theoretic aspects are left for the future, with the exception of appendix F, where an indication of the application to Cantor sets is given.

The application of the symmetrical generalised functions to the computations of quantum field theory will be worked out elsewhere. The results of methods for the regularization of integrals are compared with the generalised function results in Ch. 22.

Finally the implications of the program, as elucidated by the model, for analysis and its foundations, are discussed in the closing Chs. 23–24.

It should be kept in mind that the model is only a first step towards a symmetrical theory of generalised functions. This may be a suitable argument to justify the informal presentation. Theorems and proofs are avoided, only the verification of the most important properties is given. It seems to me that the present model is too small, it has too much the character of a tentative first step, to justify the deployment of formal apparatus. It seems better to continue the development to see where it may lead.

CHAPTER 2

REQUIREMENTS AND PROPERTIES

In this chapter a list of properties required of a symmetrical theory of generalised functions is given. It is pointed out that the construction of a complete model satisfying these requirements, starting from scratch is far too difficult a task. Instead the feasibility and usefulness of a symmetrical theory of generalised functions is demonstrated by constructing a simple model which is still large enough for many applications where products or convolutions of generalised functions are unavoidable.

The final section of this chapter gives an outline of the construction of the simple model in Chs. 4–11. The restrictions imposed to obtain the simple model are indicated.

2.1 Contents of the model

The space of generalised functions (**GF**) should contain at least an element such as Dirac's δ–function. The δ–function should have properties such as its defining property

$$\int_{-\infty}^{\infty} dx\ \delta(x)\, f(x) = f(0), \tag{2.1}$$

it should vanish when multiplied by x,

$$x \cdot \delta(x) = 0, \tag{2.2}$$

it is the derivative of a jump and the Fourier transform of a constant

$$\frac{d}{dx} H(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} dy\ e^{ixy} = \delta(x), \tag{2.3}$$

and it should appear as the limit of a sequence which sharpens up

$$\lim_{\epsilon \downarrow 0} \epsilon^{-1} f(\epsilon^{-1} x) = \delta(x) \cdot \int_{-\infty}^{\infty} dx\ f(x), \tag{2.4}$$

as postulated for the first time by Dirac [Dir]. The properties listed above hold when $f(x)$ is suitably restricted. The last of Dirac's requirements (2.4) is difficult to implement in distribution theory. It will be derived in Ch. 19.

The space of generalised functions should be a linear vector space. It is convenient to use a complex linear space. The symbol $\mathit{0}$ or $\mathit{0}(x)$ is used for the zero element.

The space of generalised functions should contain many ordinary functions as generalised functions. In particular the usual spaces of test functions used in distribution theory, such as the Schwartz space $\mathcal{S}$ of $\mathbf{C}^{\infty}$–functions of rapid decrease, should be included in the space of generalised functions.

2.2 The scalar product

The distinctive property of a symmetrical theory of generalised functions is the existence of a symmetrical scalar (or inner) product $\mathbf{GF} \times \mathbf{GF} \to \mathbb{C}$ satisfying

$$\langle f, g \rangle = \langle g, f \rangle^*, \tag{2.5}$$

$\forall f, g \in \mathbf{GF}$, in addition to the linearity conditions

$$\langle f, \alpha g \rangle = \alpha \langle f, g \rangle = \langle \alpha^* f, g \rangle, \tag{2.6}$$

and

$$\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle. \tag{2.7}$$

Following physics conventions the product is referred to as a scalar product instead of an inner product, the * denotes complex conjugation, and it is applied to the part to the left of the comma of the scalar product.

Remark 2.1 The term 'scalar product' is used instead of 'inner product'. The term inner product usually means a positive definite inner product. The term scalar product is used for all bilinear maps from the Cartesian product space to the scalars of the vector space, without implying the positive definite property. The scalar product will also be scalar in the sense that it is invariant under a large class of linear transformations of the space of generalised functions. In intermediate stages of the construction the term scalar product will also be used for non-symmetrical mappings to the complex numbers.

The symmetrical theory has the symmetry property (2.5) in common with the $\mathcal{L}_2$ theory of square integrable functions and Hilbert space in general. It contrasts with distribution theory where one side of the bracket is a 'good' function and the other side is not a function at all but a linear functional.

The requirement of symmetry means that there is no distinction between test functions and generalised functions. Every generalised function also serves as a test function. Equivalently every generalised function is always defined as a linear functional on the whole space of generalised functions.

Complex conjugation is defined for generalised functions in such a way that for every generalised function $f(x) \in \mathbf{GF}$ there is a complex conjugate element $f(x)^* \in \mathbf{GF}$. It is defined in such a way that complex conjugation of generalised functions is an extension of the complex conjugation of ordinary functions. For the scalar product the standard property

$$\langle f, g \rangle^* = \langle f^*, g^* \rangle, \tag{2.8}$$

should hold.

It seems to be impossible to construct a symmetrical theory of generalised functions with a positive definite scalar product. The possibilities

$$\langle f, f \rangle < 0, \tag{2.9}$$

and also

$$\langle f, f \rangle = 0, \qquad f \neq 0, \tag{2.10}$$

are left open, and the weaker requirement of non-degeneracy,

$$\langle f, g \rangle = 0, \quad \forall g \in \mathbf{GF} \Rightarrow f = 0, \tag{2.11}$$

is adopted instead. The notation $\|f\|^2$ for $\langle f, f \rangle$ is not used since $\|f\|$ does not have the standard properties of a norm.

Remark 2.2 For the construction of models the requirement of non-degeneracy is in the first place a heuristic principle. If it is not satisfied the model is either too large or too small. This can be cured by eliminating the degenerate elements or by introducing additional elements to lift the degeneracy. Lacking a norm, the property of non-degeneracy is indispensable for defining convergence in the scalar product. (Compare Ch. 19.)

2.3 Operators

The space of generalised functions **GF** should serve as domain for the usual operators of mathematical physics. In particular the differential operator $\mathcal{D}$, the multiplication by x operator $\mathcal{X}$, and all polynomials in these operators, should be defined for all generalised functions $\in$ **GF**. The standard commutation relation for these operators

$$[\mathcal{D}, \mathcal{X}] := \mathcal{D}\mathcal{X} - \mathcal{X}\mathcal{D} = \mathcal{I}, \tag{2.12}$$

should be satisfied. Here $\mathcal{I}$ is the identity operator.

Remark 2.3 The domain of all operators is the whole space of generalised functions. This eliminates domain considerations, which are always a nuisance in Hilbert space theory.

Remark 2.4 The standard proof [Neu] of uniqueness of representations of the commutation relation (2.12) does not apply. It refers to representations in Hilbert space. The space of generalised functions is not a Hilbert space since the scalar product is not positive definite.

The operators should be extensions of the operators acting on standard functions in the sense that

$$\mathcal{D} f(x) = \frac{d}{dx} f(x), \tag{2.13}$$

and

$$\mathcal{X} f(x) = x \cdot f(x), \tag{2.14}$$

in those cases that $f(x)$ is a function for which the action of the operators is defined in a standard sense. An example is provided by the space $\mathcal{S}$ of rapidly decreasing $\mathbf{C}^\infty$–functions, introduced by Schwartz [Sch2].

Remark 2.5 The required operator properties of the theory of generalised functions are the same as the properties these operators have on $\mathcal{S} \times \mathcal{S}$, considered as a subspace of $\mathcal{L}_2 \times \mathcal{L}_2$. In many respects the space of symmetrical generalised functions resembles the usual Schwartz space $\mathcal{S}$ and the $\mathcal{L}_2$ Hilbert space.

The Fourier operator $\mathcal{F}$ should be defined on the whole space of generalised functions and it should be unitary. The operators $\mathcal{D}$ and $\mathcal{X}$ should be unitarily equivalent under the action of the Fourier operator.
Following Bateman [Erd2], the Fourier operator will be normalized as

$$\mathcal{F}\, f(x) = \int_{-\infty}^{\infty} dy\; e^{-ixy} f(y), \tag{2.15}$$

and

$$\mathcal{F}^{\dagger}\, f(x) = \int_{-\infty}^{\infty} dy\; e^{ixy} f(y), \tag{2.16}$$

when the integral has a standard meaning. The superscript dagger † indicates the adjoint operator, which is required to exist also. With this normalization the algebraic properties of the $\mathcal{F}$–operator are

$$\mathcal{F}^2 = 2\pi\mathcal{P}, \tag{2.17}$$

and

$$\mathcal{F}^{\dagger}\mathcal{F} = 2\pi\mathcal{I}, \tag{2.18}$$

so the inverse is

$$\mathcal{F}^{-1} = (2\pi)^{-1}\mathcal{F}^{\dagger} = (2\pi)^{-1}\mathcal{F}\mathcal{P}. \tag{2.19}$$

The operator $\mathcal{P}$ is the parity operator defined by

$$\mathcal{P}\, f(x) = f(-x), \tag{2.20}$$

for ordinary functions. It has the property

$$\mathcal{P}^2 = \mathcal{I}, \qquad \text{so} \qquad \mathcal{P}^{-1} = \mathcal{P}, \tag{2.21}$$

so it can only have eigenvalues ± 1.

Remark 2.6 It will be clear that the operators $\mathcal{X}$ and $\mathcal{P}$ imply a fixed origin of the coordinate system. At this stage the functions are considered as functions on $\mathbb{R}$, so the special point is the point $x = 0$. The translation of generalised functions will be taken up in Ch. 15.

With the adopted normalization of $\mathcal{F}$ the unitary equivalence of the operators $\mathcal{D}$ and $\mathcal{X}$ takes the forms

$$\mathcal{D} = i\mathcal{F}^{-1}\mathcal{X}\mathcal{F} = -i\mathcal{F}\mathcal{X}\mathcal{F}^{-1}, \tag{2.22}$$

and

$$\boldsymbol{\mathcal{X}} = i\boldsymbol{\mathcal{F}}^{-1}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{F}} = -i\boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{F}}^{-1}. \tag{2.23}$$

The inversion of the order in (2.22) and (2.23) follows from the relations

$$\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{P}}^{-1} = -\boldsymbol{\mathcal{X}}, \quad \text{and} \quad \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{P}}^{-1} = -\boldsymbol{\mathcal{D}}, \tag{2.24}$$

between the operators $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{X}}$ and the parity operator. The unit function $I(x)$ will be taken as a generalised function with the defining property

$$\boldsymbol{\mathcal{D}}\, I(x) = \mathit{0}(x). \tag{2.25}$$

This implies the existence of a generalised function $\boldsymbol{\delta}(x)$,

$$\boldsymbol{\delta}(x) := (2\pi)^{-1}\boldsymbol{\mathcal{F}}\, I(x), \tag{2.26}$$

with the property

$$\boldsymbol{\mathcal{X}}\, \boldsymbol{\delta}(x) = \mathit{0}(x). \tag{2.27}$$

The normalization of $I(x)$ and $\boldsymbol{\delta}(x)$ is fixed up to a sign by requiring

$$\langle\, I(x)\,,\boldsymbol{\delta}(x)\,\rangle = 1. \tag{2.28}$$

The sign follows later when $\boldsymbol{\delta}$ will be identified with the Dirac δ–function.

Remark 2.7 For the time being read δ for $\boldsymbol{\delta}$ until $\boldsymbol{\delta}$ has been defined in Ch. 7. In this book 'Dirac's δ–function' denotes the intuitively understood object proposed by Dirac, without implying a rigorization. The notation δ is reserved for the distribution in the sense of Schwartz, and $\boldsymbol{\delta}$ always indicates the symmetrical generalised function. The generalised function $I(x)$ is the generalised function equivalent of the ordinary unit function, which has the value one everywhere. It also coincides with the multiplicative unit element to be introduced below.

With the extra factor 2π resulting from the normalization (2.15) the unitarity of the Fourier operator takes the form

$$\langle\, \boldsymbol{\mathcal{F}}f\,,\boldsymbol{\mathcal{F}}g\,\rangle = 2\pi\,\langle\, f\,,g\,\rangle. \tag{2.29}$$

Formula (2.29) will be referred to as Parseval's equality, since it is the extension of Parseval's equality on $\mathcal{L}_2$ to generalised functions.

Remark 2.8 Sometimes it will be useful to introduce the normalized Fourier operator $\bar{\boldsymbol{\mathcal{F}}}$ defined by $\bar{\boldsymbol{\mathcal{F}}} := (2\pi)^{-1/2}\boldsymbol{\mathcal{F}}$, which is a unitary operator.

Remark 2.9 The operators $\boldsymbol{\mathcal{X}}$ and $i\boldsymbol{\mathcal{D}}$ cannot be made selfadjoint. It will even be impossible to provide an adjoint operator. The operators are almost selfadjoint in the sense that the adjoint exists when a small subset of exceptional cases is eliminated. When the adjoint exists the operators are selfadjoint.

2.4 Products

It is a well-known property of distribution theory that multiplication and convolution of distributions is in general not possible. [Sch3].

For the symmetrical theory of generalised functions the existence of a product and a convolution is required. So for all f, $g \in \mathbf{GF}$ there is a unique element $f \bullet g \in \mathbf{GF}$.
The product $\bullet : \mathbf{GF} \times \mathbf{GF} \to \mathbf{GF}$ must have the properties:

Linear
$$(\alpha f) \bullet g = \alpha (f \bullet g), \tag{2.30}$$

Commutative
$$f \bullet g = g \bullet f, \tag{2.31}$$

Distributive
$$f \bullet (g + h) = f \bullet g + f \bullet h, \tag{2.32}$$

Non-degenerate
$$f \bullet g = 0, \quad \forall g \in \mathbf{GF} \Rightarrow f = 0, \tag{2.33}$$

Existence of a unit element I

$$\exists I \in \mathbf{GF} : I \bullet f = f, \quad \forall f \in \mathbf{GF}, \tag{2.34}$$

Agreement with the scalar product

$$\langle f , g \rangle = \langle I , f^* \bullet g \rangle, \tag{2.35}$$

Agreement with the pointwise product

$$(f \bullet g)(x) \Big|_{x=x_0} = \left(f(x) \Big|_{x=x_0} \right) \cdot \left(g(x) \Big|_{x=x_0} \right), \tag{2.36}$$

whenever the generalised functions are locally equal to standard functions.

Remark 2.10 The requirement of the existence of a unit element (2.34) implies the non-degeneracy of the product (2.33). It has been given as a separate entry since there are large subspaces of the model which do not contain a unit element for multiplication, but in which the weaker property (2.33) holds.

Remark 2.11 In Ch. 13 the values at a point of generalised functions will be defined. The pointwise property holds in the standard sense, and it holds more generally for generalised functions.

Warning: It is not possible to define values of generalised functions in such a way that (2.36) holds generally in the sense of generalised functions. Nevertheless the term 'pointwise' will be applied to the generalised function product in order to distinguish it from the convolution product.

Remark 2.12 It will be necessary to identify the zero element for differentiation I defined by (2.25) and (2.28) with the unit element I of the product defined by (2.34). The notation anticipates this.

The product cannot be associative, in general

$$f \bullet (g \bullet h) \neq (f \bullet g) \bullet h. \tag{2.37}$$

The non-associativity of the product cannot be avoided, not even by making the product non-commutative. It will be seen to be an unavoidable consequence of the requirement (2.34) of the existence of a unit element for the product.

Remark 2.13 This non-associativity also appears in distribution theory for convolution products, when an attempt is made to define products such as $x \bullet x^{-1} \bullet \delta$, or the equivalent convolutions.

Remark 2.14 It is also possible to define a non-commutative asymmetrical generalised function product, indicated with the symbols $\circ\bullet$ and $\bullet\circ$, which is more but not completely associative. The definitions are given in Sec. 8.2 and Sec. 9.2. Only one of these is really needed since the asymmetrical products are related by

$$f \circ\bullet g = g \bullet\circ f, \tag{2.38}$$

for all f and g. It is convenient to define these products first. The commutative symmetrical product $\bullet$ is obtained from the asymmetrical products by symmetrization. The generalised function products are defined in Ch. 8 and Ch. 9.

More interesting product properties arise in combination with the operators introduced in the previous section. In particular Leibniz's rule must hold for the differentiation of a generalised function product

$$\mathcal{D}(f \bullet g) = \mathcal{D}f \bullet g + f \bullet \mathcal{D}g. \tag{2.39}$$

The operator $i\mathcal{D}$ does not have to be selfadjoint, since we have from Leibniz's rule

$$\langle \mathcal{D}f\,, g \rangle = -\langle f\,, \mathcal{D}g \rangle + \langle I\,, \mathcal{D}(f^* \bullet g) \rangle. \tag{2.40}$$

and the surface term on the right-hand side is not necessarily zero. Correspondingly we cannot have the left-multiplicative rule

$$\mathcal{X}(f \bullet g) = \mathcal{X}f \bullet g, \tag{2.41}$$
WRONG!

and the right-multiplicative rule

$$\mathcal{X}(f \bullet g) = f \bullet \mathcal{X}g, \tag{2.42}$$
WRONG!

for the operator $\mathcal{X}$, since in combination with (2.35) this would imply selfadjointness of the operator $\mathcal{X}$. This is impossible, since $\mathcal{X}$ and $\mathcal{D}$ should be unitarily equivalent. The symmetrized product rule, which will be called the semi-Leibniz rule,

$$\mathcal{X}(f \bullet g) = \tfrac{1}{2}\mathcal{X}f \bullet g + \tfrac{1}{2}f \bullet \mathcal{X}g, \tag{2.43}$$

WRONG!

is also incorrect in general. The conditions on f and g under which the rules for $\mathcal{X}$ reduce to the semi-Leibniz rule (2.43) or even to the multiplicative rules (2.42) and (2.41) will be derived in Ch. 10. Only a small subspace has to be excluded. It is also possible to derive the necessary correction terms.

The product should be constructed in such a way that parity is conserved. Therefore the parity operator should act on the product as

$$\mathcal{P}\big(f(x) \bullet g(x)\big) = \mathcal{P}\,f(x) \bullet \mathcal{P}\,g(x). \tag{2.44}$$

If f and g are parity eigenfunctions,

$$\mathcal{P}f = p_f \cdot f \quad \text{and} \quad \mathcal{P}g = p_g \cdot g, \tag{2.45}$$

with $p_f = \pm 1$ and $p_g = \pm 1$, then

$$\mathcal{P}(f \bullet g) = p_f p_g \cdot (f \bullet g). \tag{2.46}$$

Remark 2.15 The product will be constructed in such a way that it also conserves the scaling properties of the generalised functions. If f and g are homogeneous,

$$f(ax) = a^{\lambda} f(x) \quad \text{and} \quad g(ax) = a^{\mu}\, g(x), \tag{2.47}$$

$\forall a \in \mathbb{R}_+$ with $\lambda, \mu \in \mathbb{C}$, then

$$(f \bullet g)(ax) = a^{\lambda+\mu}(f \bullet g)(x), \tag{2.48}$$

in agreement with the corresponding properties of the pointwise product of ordinary functions. The scale transformations will be defined for generalised functions in Ch. 16.

2.5 Convolution products

Corresponding to the pointwise product there is the convolution product $(*)$. defined as the Fourier image of the pointwise product by

$$f * g := \mathcal{F}^{-1}(\mathcal{F}f \bullet \mathcal{F}g) = 2\pi\mathcal{F}(\mathcal{F}^{-1}f \bullet \mathcal{F}^{-1}g). \tag{2.49}$$

The properties of the convolution product follow at once from the corresponding properties (2.30–36) of the pointwise product and the properties of the Fourier transform.
The convolution is:

Linear

$$(\alpha f) * g = \alpha(f * g), \tag{2.50}$$

Commutative

$$f * g = g * f, \tag{2.51}$$

Distributive

$$f * (g + h) = f * g + f * h, \tag{2.52}$$

Non-degenerate

$$f * g = 0, \quad \forall g \in \mathbf{GF} \Rightarrow f = 0, \tag{2.53}$$

With unit element δ

$$\delta * f = f, \quad \forall f \in \mathbf{GF}, \tag{2.54}$$

In agreement with the scalar product

$$\langle f(x) , g(x) \rangle = \langle \delta(x) , f(x)^* * \mathcal{P} g(x) \rangle. \tag{2.55}$$

In agreement with the integral form of the convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} dy\, f(y)\, g(x - y), \tag{2.56}$$

when f and g are standard functions, and when the integral is defined in a standard sense.

The convolution cannot be associative. If it were associative, the pointwise product would be associative by unitary equivalence, and this is known to be impossible.

The operator relations for the convolution follow by unitary equivalence from the corresponding rules for the pointwise product. There is the Leibniz rule for the $\mathcal{X}$ operator

$$\mathcal{X}(f * g) = \mathcal{X}f * g + f * \mathcal{X}g, \tag{2.57}$$

acting on the convolution product. The complicated rule for the $\mathcal{D}$ operator will sometimes, but not always, simplify to the semi-Leibniz rule

$$\mathcal{D}(f * g) = \tfrac{1}{2}\mathcal{D}f * g + \tfrac{1}{2}f * \mathcal{D}g, \tag{2.58}$$

WRONG!

which may simplify to the multiplicative rule

$$\mathcal{D}(f * g) = \mathcal{D}f * g = f * \mathcal{D}g. \tag{2.59}$$

WRONG!

As with the dot product the simple rule holds when a small subspace is excluded.

Since the convolution and the pointwise product are unitarily equivalent, it is only a matter of convenience which is defined first.

Remark 2.16 It is convenient to define also the left- and right-sided convolution products $\circ *$ and $* \circ$ as the Fourier images of $\circ \bullet$ and $\bullet \circ$, which are the corresponding pointwise products.

2.6 Summary of the properties

The requirements and properties listed above are not given a priori. They were found by constructing a model with properties that are as 'good' as possible. The guiding principle is that it is attempted to conserve as much as possible of the 'good' properties of $\mathcal{S} \times \mathcal{S}$, considered as subspace of the symmetrical structure $\mathcal{L}_2 \times \mathcal{L}_2$, when singular objects such as Dirac delta functions are added, while keeping the symmetry.

The main differences between a space of symmetrical generalised functions and the Hilbert space $\mathcal{L}_2$ are:

1) The scalar product is not positive definite, there is no norm.
2) The pointwise product and the convolution are not associative.
3) The differential and multiplication operators are not selfadjoint.
4) The operators are defined without any domain restriction.
5) The product operation does not take us out of the space.
6) Evaluation of divergent integrals is possible.
7) Sets of measure zero may contribute to integrals.

It is not obvious that all requirements listed in the previous section can be satisfied at the same time. A trivial model, satisfying all requirements, is constructed in the next chapter to demonstrate the possibility. The chapters 4 to 11 are devoted to the construction of a small model which is strong enough for many applications. An outline of the construction is gives in the last section of this chapter.

The construction of a maximal, or in some sense complete model, for the requirements listed above is an open question, which requires much further work. The simple model given here will first be constructed in an informal manner. In Ch. 12 it will be shown to possess the required properties.

2.7 Other properties

Apart from the properties which were required in the previous sections the simple model has many other useful and beautiful properties. Some of these will be derived in the second half of the book. From their fundamental character it is to be expected that these properties are not particular to the simple model.

There seems to be some freedom of choice in deciding which properties should be required, and which should be derived. The choice made in this work seems fairly natural.

A first group of properties, Chs. 13–14, concerns the value and the support of generalised functions. The real number system is not adequate for supporting the generalised functions, but it seems plausible that an extension is possible in such a way that the support property of the product holds: the support of the product is the intersection of the supports of the factors.

Also it is always possible to assign finite values to generalised functions at every point. In consequence generalised functions are integrable over every finite or infinite interval. It will be shown that a generalised function always possesses a primitive, and that the fundamental theorem of the calculus holds for generalised functions.

A second group of properties, Chs. 15–18, derives from groups of operators defined on the generalised functions. For instance the translation group is defined on all generalised functions. An important result is the demonstration that it is possible to define the scale transformation on all generalised functions in such a way that the product is scale invariant. Consequently the scale operator is unitary. This makes meaningful applications to physics possible.

Thirdly it is shown in Ch. 19 that generalised functions can also be obtained as generalised limits of sequences. This may well take a more prominent place in a future exposition of the subject.

In the fourth place it is shown in Ch. 21 that the model possesses further analytic properties of another kind. Some generalised functions (not all) are limits (in a sense to be defined) of functions which are analytic in an imaginary half-plane [Tich].

It will be shown that the generalised function product has the property that if $f(x)$ and $g(x)$ are generalised functions of argument $x = \operatorname{Re}\lambda$ satisfying

$$f(x) = \lim_{\operatorname{Im}\lambda\downarrow 0} \tilde{f}(\lambda), \qquad \text{and} \qquad g(x) = \lim_{\operatorname{Im}\lambda\downarrow 0} \tilde{g}(\lambda), \tag{2.60}$$

with $\tilde{f}(\lambda)$ and $\tilde{g}(\lambda)$ analytic functions in the half-plane $\operatorname{Im}\lambda > 0$, then the generalised function product satisfies

$$f(x) \bullet g(x) = \lim_{\operatorname{Im}\lambda\downarrow 0} \tilde{f}(\lambda) \cdot \tilde{g}(\lambda). \tag{2.61}$$

This is referred to as the analytic boundary property. It is a derived property of the model that was not used for its construction.

Finally it is shown in Ch. 20 that periodic functions can easily be added as a special case.

Taken together the required and derived properties are sufficiently general to allow us to use the symmetrical generalised functions as a tool for mathematical analysis and its applications.

2.8 Outline of the construction

This section gives an outline of the organization of the following chapters. The actual construction of the model is based on the method of constructing functionals, which depend analytically on a complex parameter, which can be continued analytically. The starting point is a preliminary class of functions (**PC**) with the property that x^λ is a linear functional on **PC**, analytic in λ, which can be continued analytically as far as one wishes.

It is not known what conditions should be imposed on **PC** to guarantee the existence of the analytic continuation. This leads to the

Fundamental unsolved problem: Give necessary (and sufficient) conditions on $f(x) \in \mathbf{PC}$ such that the analytic continuation (possibly multiple valued) of the functional

$$\tilde{f}(\lambda) := \int_0^a dx\, x^\lambda\, f(x), \tag{2.62}$$

to almost the entire complex λ–plane exists. In other words the analytic continuation of $\tilde{f}(\lambda)$ should give rise to at most isolated singularities in the complex λ–plane. (Or more generally a Riemann surface.) The occurrence of natural boundaries is not allowed.

The solution to this fundamental unsolved problem seems to be unknown. Therefore it is not possible to consider the space **PC** in general. Instead the construction will be based on small subspaces of **PC** with simple properties. These are obtained by imposing strong sufficient conditions, which are not at all necessary. This is equivalent to prescribing simple singular behaviour of the (generalised) functions. Solving the unsolved problem amounts to characterizing the worst singularities which could be incorporated in a future symmetrical theory of generalised functions.

Remark 2.17 Even the requirement of isolated singularities may be stronger than necessary. All that is actually needed is an appropriate generalization of Laurent's theorem. When limits and Fourier integrals in the sense of generalised functions are available it may well be possible to generalize to more complicated singularities, such as isolated density points of singularities. This lies far beyond the scope of this book. It will be necessary to proceed stepwise.

In this work the subspace $\mathbf{PC}_\lambda \subset \mathbf{PC}$ is used, which is restricted in such a way that the analytic continuations will be meromorphic in the entire λ–plane.

The subspace $\mathbf{PC}_\lambda$ consists of functions $\mathbb{R} \to \mathbb{C}$ which are $\mathbf{C}^\infty$ with the exception of a finite number of singularities. At the singularities a simple asymptotic behaviour in terms of powers of x, logarithms, and oscillating exponentials is required. It will be seen in Ch. 4 that this restriction does indeed give rise to meromorphic analytic continuations.

A preliminary integration is then defined using Hadamard's partie finie, or equivalently by means of analytic continuation. A preliminary scalar product is defined by combining the pointwise multiplication with the preliminary integral. The preliminary versions of the operators are derived from their standard definitions.

The resulting symmetrical structure $\mathbf{PC}_\lambda \times \mathbf{PC}_\lambda$, with the corresponding preliminary operators, has many defects. Closure under the preliminary operators holds, but standard properties such as Leibniz's rule for differentiating products, and Parseval's equality do not hold.

These shortcomings are more easily remedied when the elements of the preliminary class are considered as linear functionals on $\mathbf{PC}$, that is as elements of $\mathbf{PC}'$. Again the space $\mathbf{PC}'$ is hard to characterize. As in the case of $\mathbf{PC}$ only small subspaces of $\mathbf{PC}'$ are used. These subspaces have well-defined properties fixed by construction. The first subspace which occurs naturally is the subspace $\mathbf{PC}'_\lambda$ which contains the elements of $\mathbf{PC}_\lambda$, now considered as linear functionals on $\mathbf{PC}_\lambda$. Closure of $\mathbf{PC}'_\lambda$ under the operators is achieved by noting that the elements of $\mathbf{PC}_\lambda$ are restricted in such a way that they have a local Mellin transform, which is a meromorphic function of its complex argument. The residues of this meromorphic function are easily found. This leads naturally to the introduction of many singular generalised functions, which are localized at or near a point. These local generalised functions measure the coefficients in the asymptotic expansion of the function on which they act.

The subclass $\mathbf{PC}'_\lambda$ combined with these localized generalised functions is easily closed under the usual operators to $\overline{\mathbf{PC}}'_\lambda$ in Ch. 6. The class $\mathbf{PC}_\lambda$ is then completed to $\overline{\mathbf{PC}}_\lambda$ by introducing an injective mapping $\mathcal{M} : \mathbf{PC}_\lambda \to \overline{\mathbf{PC}}'_\lambda$, and by requiring the inverse mapping $\mathcal{M}^{-1} : \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}_\lambda$ to exist. The action of the operators on $\overline{\mathbf{PC}}_\lambda$ is then found by transferring them from $\overline{\mathbf{PC}}'_\lambda$ in Ch. 7.

The product and convolution on $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$ are found in Ch. 8 by analytic means. This involves an explicit symmetrization to obtain a commutative product and consequently a symmetrical scalar product. In order to transfer the product and convolution from $\overline{\mathbf{PC}}'_\lambda$ to $\overline{\mathbf{PC}}_\lambda$ special mappings $\mathcal{M}_\mathcal{X}$ and $\mathcal{M}_\mathcal{D}$ are constructed in Ch. 9, which have the additional property of commuting with either the $\mathcal{X}$ or the $\mathcal{D}$ operator. In this way the good product properties in $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$ are conserved in $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda$.

The generalised function product may differ from the preliminary pointwise product by additional δ–functions at singular points.

Although the model contains objects called $\tilde{\delta}$–functions which have many properties (2.1–4) in common with the δ–distribution in the sense of Schwartz [Sch1], it is necessary to keep a distinction in mind between symmetrical generalised functions and distributions. The symmetrical theory of generalised functions and distribution theory are logically independent explanations of Dirac's δ–function although they have of course many properties in common. The generalised function product on $\overline{\mathbf{PC}}_\lambda$ then gives rise to a symmetrical scalar product $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \mathbb{C}$, with better properties than the preliminary scalar product.

Finally the symmetrical structures $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$ are joined to give the full symmetrical structure $\mathbf{GF}_s \times \mathbf{GF}_s$ of simple generalised functions. This completes the construction of the model. Again the space $\mathbf{GF}_s$ can be thought of as no more than a small subspace of an as yet unknown and much larger space $\mathbf{GF}$ satisfying the requirements of this chapter. The construction of such a complete symmetrical theory of generalised functions is far beyond the scope of this work.

The model has been restricted to one independent variable. This is not a trivial restriction which is easily lifted. Unless the problem factorizes into one-dimensional problems, it entails consideration of the more complicated singularities in more dimensions. In this respect the symmetrical theory is not really less general than distribution theory, where comparable difficulties arise in actual computations, since the spaces of test functions must be adapted to the singularities which one allows to occur.

In the final chapters the symmetrical theory of generalised functions will be compared with distribution theory and with different methods of defining products of distributions. A program for further work is outlined, and the possible implications of the program for the foundations of analysis are discussed briefly.

CHAPTER 3

A TRIVIAL MODEL

In order to show that all requirements listed in the previous chapter can be met a trivial model $\mathbf{GF}_t$ is constructed. It is far to small to serve a useful purpose. It only serves as an illustration that the list of requirements of Ch. 2 can be satisfied. The trivial model is even somewhat better than required. The trivial model $\mathbf{GF}_t$ contains the basic elements

$$x^p, \qquad \text{and} \qquad \delta^{(q)}, \tag{3.1}$$

$\forall p, q \in \mathbb{N}$. It also contains all finite linear combinations of these elements with the standard linearity properties of a linear vector space. Therefore it also contains the element $\mathit{0} := 0 \cdot x^p = 0 \cdot \delta^{(q)}$ as the zero element.

The basic elements in (3.1) are defined to be real, so complex conjugation does not yield more elements.

More neutral names for the elements such as a_p and b_p could be used but the present choice is more in agreement with standard usage and heuristic interpretation.

The symmetrical scalar products of the basic elements are defined by

$$\langle x^p \, , x^q \rangle := 0, \qquad \langle x^p \, , \delta^{(q)} \rangle := \delta_{p,q} \, , \qquad \langle \delta^{(p)} \, , \delta^{(q)} \rangle := 0, \tag{3.2}$$

where $\delta_{p,q}$ is the Kronecker δ–symbol.

The model contains elements that have a negative scalar product with themselves. An example is provided by the element $x^0 - \delta^{(0)}$, which has a negative squared norm.

The operators $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{D}}$ are defined $\forall p \in \mathbb{N}$ by

$$\boldsymbol{\mathcal{X}}\, x^p := x^{p+1}, \qquad \boldsymbol{\mathcal{D}}\, x^{p+1} := (p+1)\, x^p, \tag{3.3}$$

$$\boldsymbol{\mathcal{X}}\, \delta^{(p+1)} := \delta^{(p)}, \qquad \boldsymbol{\mathcal{D}}\, \delta^{(p)} := -(p+1)\, \delta^{(p+1)}, \tag{3.4}$$

$$\boldsymbol{\mathcal{X}}\, \delta^{(0)} := \mathit{0}, \qquad \boldsymbol{\mathcal{D}}\, x^0 := \mathit{0}\,. \tag{3.5}$$

The Fourier operator $\boldsymbol{\mathcal{F}}$ is defined by

$$\boldsymbol{\mathcal{F}}\, x^p := 2\pi p!\, (-i)^p\, \delta^{(p)}, \qquad \boldsymbol{\mathcal{F}}\, \delta^{(p)} := \tfrac{(-i)^p}{p!}\, x^p, \tag{3.6}$$

with the corresponding inverse

$$\boldsymbol{\mathcal{F}}^{-1} x^p = i^p p!\; \delta^{(p)}, \qquad \boldsymbol{\mathcal{F}}^{-1}\, \delta^{(p)} = \tfrac{i^p}{2\pi p!}\, x^p. \tag{3.7}$$

The parity of the elements x^p and $\delta^{(p)}$ then follows from (3.7) and (2.20),

$$\mathcal{P}\, x^p = (-)^p x^p, \qquad \mathcal{P}\, \delta^{(p)} = (-)^p \delta^{(p)} . \tag{3.8}$$

The products are defined by

$$x^p \bullet x^q := x^{p+q}, \tag{3.9}$$

$$\delta^{(q)} \bullet x^p = x^p \bullet \delta^{(q)} := \begin{cases} 0 & p > q, \\ \delta^{(q-p)} & q \geq p, \end{cases} \tag{3.10}$$

$$\delta^{(p)} \bullet \delta^{(q)} := 0, \tag{3.11}$$

so $I := x^0$ is the unit element of the product.
The corresponding convolutions are

$$x^p * x^q := 0, \tag{3.12}$$

$$\delta^{(q)} * x^p = x^p * \delta^{(q)} := \begin{cases} 0 & q > p, \\ \frac{p!}{q!\,(p-q)!} x^{p-q} & p \geq q, \end{cases} \tag{3.13}$$

$$\delta^{(p)} * \delta^{(q)} := \tfrac{(p+q)!}{p!\,q!}\, \delta^{(p+q)}, \tag{3.14}$$

so $\delta := \delta^{(0)}$ is the unit element for convolution.

One readily verifies by direct computation that the trivial model is even better than required in Ch. 2. The operators $\mathcal{X}$ and $\mathcal{D}$ are selfadjoint and the product and convolution are associative.

Remark 3.1 One may of course interpret x^p as x to the power p, and $\delta^{(p)}$ as the p^{th} derivative of Dirac's δ–function divided by $(-)^p p!$.

The trivial model can be constructed in the context of distribution theory if it is considered as a subspace of the tempered distributions, by imposing associativity, commutativity, and Parseval's equality.

Remark 3.2 Instead of verifying the properties of $\mathbf{GF}_t$ by direct computation one can also wait for the completion of a larger model such as $\mathbf{GF}_s$ and recover the trivial model as a subspace.

Remark 3.3 The trivial model can easily be enlarged by adding translated elements and/or the Schwartz space $\mathcal{S}$ of rapidly disappearing $\mathbf{C}^\infty$ functions, but this does not yield a more useful model.

The construction of the trivial model may serve as an example of the method for the construction of a larger but still fairly simple model in the next chapters. The approach is constructive, without regard for completeness or topological considerations. When the model is completed it will be shown that it has the desired properties by construction.

CHAPTER 4

PRELIMINARIES

This chapter introduces a preliminary class of generalised functions, and preliminary versions of the integral, scalar product, operators, product, and convolution, by classical means. The preliminary properties are unsatisfactory, but they can serve as a starting point for the construction of simple model for a symmetrical theory of generalised functions.

4.1 A preliminary class of (generalised) functions

Until a general method has been found the construction of a model for a symmetrical theory of generalised functions has to be a somewhat experimental procedure. Considerations of utility lead to the inclusion of some elements. Closedness under the usual operators and under multiplication then forces the inclusion of many other elements. On the other hand it has been attempted to keep the model as small as possible, insisting only on closedness under the operators.

Example 4.1 Inclusion of a function with a jump discontinuity in the finite, such as $e^{-|x|}\mathrm{sgn}(x)$, leads by Fourier transformation to functions behaving as x^{-1} at infinity. Repeated application of operators $\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{D}}$, and $\boldsymbol{\mathcal{F}}$, then leads to functions diverging as arbitrary powers of x at infinity and as arbitrary powers of x^{-1} in the finite, multiplied by arbitrary positive powers of $\log|x|$.

The preliminary class will contain the ordinary functions allowed as generalised functions in the model. A natural possibility would be to start with a suitable subclass of the distributions, which is closed under the usual operators. This course is not taken. Instead the elements of the preliminary class (abbreviated in the following as **PC**) will be considered to be ordinary functions $\mathbb{R} \to \mathbb{C}$. In the following attention will be restricted to the subset $\mathbf{PC}_\lambda \subset \mathbf{PC}$ to be defined below. It will be too bothersome to mention the restriction to the subspace $\mathbf{PC}_\lambda$ explicitly, so 'preliminary class' will usually refer to $\mathbf{PC}_\lambda$.

Remark 4.1 The preliminary class $\mathbf{PC}_\lambda$ does not contain δ–functions. Its completion in the sense of generalised functions, $\overline{\mathbf{PC}}_\lambda$, defined in Ch. 7, does contain objects called $\bar{\delta}$–functions. In the restricted context of distribution theory the $\bar{\delta}$–functions have of the properties of δ–functions. As generalised functions they do not satisfy all of Dirac's requirements (2.1–4).

The heuristics outlined in Ex. 4.1 lead naturally to the choice of functions which are infinitely continuously differentiable ($\mathbf{C}^\infty$), with the exception of a finite number of exceptional points (piecewise $\mathbf{C}^\infty$), as the minimal generalization with respect to the spaces of test functions of distribution theory. It is then necessary to restrict the singular behaviour at the exceptional points. This is done by requiring suitable asymptotic behaviour near the singularity.

It is required that functions $f(x) \in \mathbf{PC}_\lambda$ are asymptotic to a formal power series of the form

$$f(x) \sim f_a(x; x_0+) := \sum_{j=0}^{\infty} \sum_{k=0}^{K_j} c_{jk}(x - x_0)^{\lambda_j} \log^k(x - x_0), \tag{4.1}$$

at the positive side of every point $x_0 \in \mathbb{R}$, with $K_j \in \mathbb{N}$, and $\lambda_j, c_{jk} \in \mathbb{C}$. The powers and logarithms are taken on the principal branch. The coefficients c_{jk} depend on the point x_0. They are written as $c_{jk}(x_0+)$ where necessary for clarity.
The $\{\lambda_j\}$ are assumed to be ordered by their real parts

$$-\infty < \operatorname{Re} \lambda_0 \leq \operatorname{Re} \lambda_1 \leq \cdots \leq \operatorname{Re} \lambda_j \cdots < \infty, \tag{4.2}$$

and they are restricted by requiring the pointset

$$\{\lambda_j : \operatorname{Re} \lambda_j < a\}, \tag{4.3}$$

to be finite $\forall a \in \mathbb{R}$. This means that every negative real half-plane $\operatorname{Re} \lambda < a$ contains only finitely many $\{\lambda_j\}$ values.

A sequence $\{\lambda_j\} \subset \mathbb{C}$ with the properties (4.3) and (4.2) will be called an ascending $\{\lambda_j\}$ sequence. Reversing the inequality signs in (4.2), or equivalently replacing $\{\lambda_j\}$ by $\{-\lambda_j\}$ results in a descending $\{\lambda_j\}$ sequence. A finite union of ascending sequences is ascending, an intersection of ascending and descending sequences is finite.
In the following, a notation such as

$$\sum_{j=0}^{\operatorname{Re} \lambda_j < a} < \infty, \tag{4.4}$$

will be used to indicate the sum over j, until the superscript condition is false. By restriction (4.3) this is a finite sum. The sum over powers of logarithms appearing in (4.1) is also required to be finite. The restrictions on the allowed λ–values imply that $f \in \mathbf{PC}_\lambda$ cannot grow faster than a power of x anywhere.

The formal series appearing in (4.1) is asymptotic to $f(x)$ in the sense of Poincaré, that is $\forall \alpha \in \mathbb{C}$

$$\lim_{x \downarrow x_0} (x - x_0)^\alpha \Big(f(x) - \sum_{j=0}^{\operatorname{Re} \lambda_j < \operatorname{Re} \alpha} \sum_{k=0}^{K_j} c_{jk}(x - x_0)^{\lambda_j} \log^k(x - x_0) \Big) = 0. \tag{4.5}$$

By restriction (4.3) the number of subtractions in (4.5) is finite.

A proof that the powers x^{λ_j} can serve as basis for unique asymptotic expansions may be found in [B&H]. The logarithms are easily added, since after removing more strongly growing terms and multiplying by $x^{-\lambda_j}$, the dominant behaviour is by requirement (4.1) a finite degree polynomial in $\log|x|$. The term 'non-asymptotic' $\not\sim$ is defined by

$$f(x) \not\sim x^{\alpha} \log^{q}(x) H(x) \Longleftrightarrow (\alpha, q) \notin \{\lambda_j, q\}, \tag{4.6}$$

where $\{\lambda_j, q\}$ is the set of powers appearing in the asymptotic expansion. It is usually clear from the context which point is meant, so the point $x = 0$ does not have to be indicated in the notation. Non-asymptotic is in relation to a given asymptotic set. It means that the specified behaviour is absent from the subset describing the given function.

More generally the notation $\perp$ will be defined as proper generalization of $\not\sim$. It will also indicate that the the behaviour on both sides is different.

The same kind of asymptotic requirements hold at the negative side of every point with $x - x_0$ replaced by $x_0 - x$. The coefficients $c_{jk}(x_0-)$ in the left, and $c_{jk}(x_0+)$ in the right asymptotic expansion are not necessarily equal. It is sometimes convenient to use $c_{jk}(x_0e)$ and $c_{jk}(x_0o)$ for the coefficients in the asymptotic expansions in terms of $|x-x_0|^{\lambda}$ and $|x-x_0|^{\lambda}\,\mathrm{sgn}(x)$ respectively.

Remark 4.2 The asymptotic property is required at every point. It is automatically satisfied at points where $f \in \mathbf{PC}_{\lambda}$ is $\mathbf{C}^{\infty}$. A $\mathbf{C}^{\infty}$–function possesses a Taylor series at every point, which is of the required form. The Taylor series is asymptotic to the function, even if it does not converge to the function anywhere.

Example 4.2 The $\mathbf{C}^{\infty}$–function

$$f(x) := e^{-|x|^{-1}}, \tag{4.7}$$

is asymptotic to its Taylor series at $x = 0$, but the sum of the Taylor series, which has all coefficients equal to zero, differs from $f(x)$ except at $x = 0$.

Example 4.3 For every given formal Taylor series there is an analytic function, analytic in a sector containing the positive real axis, which is asymptotic to the given Taylor series, even if the radius of convergence is zero. (Ritt's theorem) [Hen]. So for every given formal Taylor series there is a $\mathbf{C}^{\infty}$–function asymptotic to it.

Remark 4.3 The restriction to a finite number of logarithmic terms is necessary, since we have formally

$$x^{\lambda} = e^{\lambda \log(x)} = \sum_{j=0}^{\infty} \tfrac{1}{j!} \lambda^{j} \log^{j} x, \tag{4.8}$$

which would lead to non-uniqueness. Moreover the powers of the logarithm do not qualify as an asymptotic sequence. At $x = 0$ and $x = \infty$ the function $|\log^{k+1}|x||$ dominates $|\log^{k}|x||$, instead of being dominated by it.

Since there are singularities in the finite it is to be expected that the Fourier operator will transform these into singularities at infinity. The asymptotic behaviour of Fourier transforms of functions $f(x) \in \mathbf{PC}_\lambda$ with asymptotic expansions of the form (4.1) is well known. A derivation up to the first power of the logarithm may be found in Lighthill [Lig]. This is easily extended to arbitrary powers of the logarithm. The asymptotic behaviour of $f \in \mathbf{PC}_\lambda$ at infinity, both at $+\infty$ and $-\infty$, is required to be the Fourier transform of the allowed singular behaviour in the finite,

$$f(x) \sim f_a(x;+\infty) := \sum_{j=0}^{J} e^{iy_j x} \sum_{k=0}^{\infty} \sum_{l=0}^{L_k} c_{jkl}\, x^{-\lambda_k - 1} \log^l x, \qquad (4.9)$$

with $J, L_k \in \mathbb{N}$, and $y_j \in \mathbb{R}$, and c_{jkl}, $\lambda_k \in \mathbb{C}$. At $x = -\infty$ we have the same requirement with x replaced where necessary by $-x$.

Reflecting the finite number of allowed singular points in the finite only a finite sum over exponentials is allowed at infinity. The $\{\lambda_k\}$ are restricted in the same way (4.3) as before. At infinity only descending sequences of $\{\lambda_j\}$ values are allowed in the asymptotic expansions.

Since every $f(x) \in \mathbf{PC}_\lambda$ is piecewise $\mathbf{C}^\infty$, all its derivatives are by definition piecewise $\mathbf{C}^\infty$. All derivatives $f^{(n)}(x) \in \mathbf{PC}_\lambda$ are required to admit an asymptotic expansion of the same type as $f(x) \in \mathbf{PC}_\lambda$ itself.

Example 4.4 The condition is necessary. The function

$$f(x) := e^{-x^{-2}} \sin(e^{x^{-4}}), \qquad (4.10)$$

is piecewise $\mathbf{C}^\infty$, and asymptotic to $f_a(x;0+) = 0$ at $x = 0+$, but for $x \downarrow 0$ its derivative cannot be bounded by a power of x. The derivatives of $f(x)$ do not admit an asymptotic expansion of the required type.

Example 4.5 The functions sgn(x), tanh(x), sin(x), cos(x), all finite Fourier sums, cos(log$|x|$), the unit function $I(x)$, all polynomials in x, the Bessel functions $\mathbf{J}_\nu(x)$, and many other special functions of limited growth are elements of the preliminary class.

Remark 4.4 The usual spaces of test functions of distribution theory, such as the Schwartz spaces $\mathcal{S}$ or $\mathcal{D}$, are subspaces of $\mathbf{PC}_\lambda$.

Remark 4.5 The functions cosh(x), tan(x), and most periodic functions do not belong to $\mathbf{PC}_\lambda$. The periodic continuation of functions of bounded support $\in \mathbf{PC}_\lambda$, and the resulting infinite Fourier series and sums will be added as a special case in Ch. 20.

Remark 4.6 At this stage the elements of $\mathbf{PC}_\lambda$ are still ordinary functions defined on a union of open intervals, obtained by deleting a finite number of points from $\mathbb{R}$. They will be defined as generalised functions in Ch. 7 by adding the behaviour at the singular points.

Remark 4.7 In this book the standard concept of an asymptotic expansion due to Poincaré (4.5) has been used. This is not the natural concept from the standpoint of generalised function theory. It has been used nevertheless to keep contact with a well-known starting point. The generalization to an asymptotic concept which fits better with the symmetrical generalised functions lies outside the scope of this work.

4.2 A preliminary product

The class $\mathbf{PC}_\lambda$ is closed under pointwise multiplication. The product of piecewise $\mathbf{C}^\infty$–functions is piecewise $\mathbf{C}^\infty$, the product of two functions is asymptotic to the formal product of the asymptotic series, the product of the asymptotic series is of the required type, and the finiteness conditions are obviously met. This preliminary product satisfies the requirement (2.30–36). It is commutative, distributive and even associative. The unit function $I(x)$ is the unit element of the product, and the product is non-degenerate. It will be made to agree with the preliminary scalar product in Sec. 4.4.

The preliminary product of $f \in \mathbf{PC}_\lambda$ and $g \in \mathbf{PC}_\lambda$ will be written as fg or $f(x)\,g(x)$, in contrast with the generalised function product, (to be defined in Ch. 8 and Ch. 9), which will be indicated by the fat centred dot $f \bullet g$ or $f(x) \bullet g(x)$.

The preliminary product is defined on a union of open intervals. This is sufficient since single points do not contribute to preliminary integrals. The generalised function product will agree with the pointwise product (2.36) on the open intervals, but in addition it will also give the behaviour at the singular points.

Remark 4.8 In Sec. 4.6 a preliminary convolution product is introduced. It is less convenient than the pointwise product and it will not be used for further developments. Its relation to the generalised function convolution will appear in Ch. 22, where the regularization of convolution integrals is derived.

4.3 Preliminary integration on the preliminary class

The elements of the preliminary class are ordinary functions. In general these functions will not be integrable in the classical sense, since there can be divergences as arbitrary powers of x. The method of the partie finie, which was originally introduced by Hadamard [Had], can be used to define an integral on the preliminary class. This is done by defining the integral for the functions which are allowed in the asymptotic expansions and by removing as many terms as necessary to make the integrals converge in the classical sense.

The basic integrals we need are

$$\int_0^a dx\, x^\alpha = \frac{a^{\alpha+1}}{\alpha+1}, \tag{4.11}$$

and

$$\int_a^\infty dx\, x^\alpha = -\frac{a^{\alpha+1}}{\alpha+1}, \tag{4.12}$$

which are valid (convergent) for $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\alpha < -1$ respectively.

The integrals are now defined for other values of α by analytic continuation with respect to α. More generally we define $\forall a \in \mathbb{R}_+$, $\forall \alpha \in \mathbb{C}$ the preliminary value of the integral as

$$\mathrm{Pre}\int_0^a dx\, x^\alpha \log^q(x) := \operatorname*{Res}_{\lambda=\alpha} q!\,(\lambda-\alpha)^{-q-1}\int_0^a dx\, x^\lambda, \tag{4.13}$$

which leads for $\alpha \neq -1$ to the result

$$\mathrm{Pre}\int_0^a dx\, x^\alpha \log^q(x) = \frac{(-)^q q!}{(\alpha+1)^{q+1}} + \left(\frac{\partial}{\partial\alpha}\right)^q \left(\frac{a^{\alpha+1}-1}{\alpha+1}\right), \tag{4.14}$$

and for $\alpha = -1$ to

$$\mathrm{Pre}\int_0^a dx\, x^{-1} \log^q(x) = \frac{\log^{q+1}(a)}{(q+1)!}. \tag{4.15}$$

The preliminary character of the integral is indicated by the 'Pre' appearing in front of it. The generalised function version of the integral will be defined in Ch. 14. The preliminary value of the integral is equal to the generalised function value of the integral over the 'open' intervals between singularities. The meaning of this statement will be defined in Ch. 14.

In (4.13) the analytic continuation with respect to λ is understood, and 'Res' stands for the operation of taking the residue. This is understood in the sense of standard analytic function theory. The standard definition (A.3) has been written out in appendix A.

At infinity the asymptotic expansions may also contain oscillating exponentials. For $k \neq 0$ we have to evaluate integrals of the form

$$\int_a^\infty dx\; e^{ikx} x^\alpha \log^q(x) = \left(\frac{\partial}{\partial\alpha}\right)^q \int_a^\infty dx\; e^{ikx} x^\alpha. \tag{4.16}$$

For $q = 0$, substituting $ikx := -y$, and rotating the integration contour, converts the integral into

$$\int_a^\infty dx\; e^{ikx} x^\lambda = (ik)^{-\alpha-1}\int_{ika}^\infty dy\; e^{-y} y^\alpha = (ik)^{-\alpha-1}\,\Gamma(\alpha+1, ika), \tag{4.17}$$

which is the definition [Erd1] of the incomplete Γ–function. Since the incomplete Γ–function can be continued analytically to an entire function of its first argument (4.17) can be used to define the preliminary integral

$$\mathrm{Pre}\int_a^\infty dx\, e^{ikx}x^\alpha \log^q(x) := \left(\frac{\partial}{\partial\alpha}\right)^q \Big((ik)^{-\alpha-1}\,\Gamma(\alpha+1, ika)\Big), \tag{4.18}$$

for all values of α when $k \neq 0$. Logarithms are again introduced by formal differentiation with respect to α. For $k = 0$ the integrals are obtained by direct integration as in (4.13)

$$\mathrm{Pre}\int_a^\infty dx\, x^\alpha \log^q(x) := -\mathrm{Pre}\int_0^a dx\, x^\alpha \log^q(x), \tag{4.19}$$

or by taking a residue.

The integrals from $-\infty$ to $-a$ are found by replacing x by $-x$. Integrals near other points are obtained by replacing x by $x - x_0$ or $x_0 - x$.

The preliminary integration of arbitrary functions $f \in \mathbf{PC}_\lambda$ is now defined by choosing a finite partition of $\mathbb{R}$ not coinciding with any of the singular points of $f(x)$, subtracting a sufficient number of terms of the asymptotic expansions to the left and right of every singular point to make the integral absolutely convergent in the classical sense, and integrating the pieces and the remainder separately. Since this gives a finite sum of finite contributions the result is finite. Let $f \in \mathbf{PC}_\lambda$ have singularities at

$$-\infty < x_0 < x_1 < \cdots < x_n < \infty. \tag{4.20}$$

Choose a partition $\{a_j\}$ of $\mathbb{R}$ such that

$$-\infty < a_0, < x_0 < a_1 < x_1 < \cdots\cdots < x_n < a_{n+1} < \infty. \tag{4.21}$$

In agreement with Hadamard's definition of the partie finie the preliminary integral is then defined as

$$\mathrm{Pre}\int_{-\infty}^\infty dx\; f(x) := \mathrm{Pre}\int_{-\infty}^{a_0} dx\; f(x) + \mathrm{Pre}\int_{a_{n+1}}^\infty dx\; f(x) + \\ + \sum_{j=0}^n \Big(\mathrm{Pre}\int_{a_j}^{x_j} dx\; f(x) + \mathrm{Pre}\int_{x_j}^{a_{j+1}} dx\; f(x)\Big). \tag{4.22}$$

A typical term in (4.22) is defined by

$$\mathrm{Pre}\int_{x_j}^{a_{j+1}} dx\; f(x) := \\ := \int_{x_j}^{a_{j+1}} dx\,\Big(f(x) - \sum_{k=0}^{\mathrm{Re}\,\lambda_k>-1}\sum_{l=0}^{L_k} c_{kl}\,(x_j+)(x-x_j)^{\lambda_k}\log^l(x-x_j)\Big) + \\ + \sum_{k=0}^{\mathrm{Re}\,\lambda_k>-1}\sum_{l=0}^{L_k} c_{kl}(x_j+)\,\mathrm{Pre}\int_{x_j}^{a_{j+1}} dx\,(x-x_j)^{\lambda_k}\log^l(x-x_j). \tag{4.23}$$

The terms at infinity are defined in terms of asymptotic expansions in x^λ. This makes the integral dependent on the choice of the origin. We will return to this point in Ch. 15, where the translation of generalised functions is defined.

The first integral above converges absolutely in a standard sense, so it does not need a 'Pre'. The integrals appearing in the sums have been defined by (4.14) and (4.15) above. The value of integral defined by (4.22) is finite, since it is a finite sum of finite contributions. It is clear from the definition that the preliminary integral coincides with the standard integral when this is defined in a standard sense. It also coincides with Hadamard's definition of the partie finie, so 'Pre' may also be read as valeur principale or principal value. The difference is that the preliminary integral is used in this book as a starting point for constructing a better definition of the integral.

The value of the preliminary integral does not depend on the choice of the partition, since $f(x) \in \mathbf{PC}_\lambda$ is $\mathbf{C}^\infty$ between singularities. The loss in one term resulting from a shift of a point of the partition is balanced by the gain of the next. It is also possible to subtract more terms of the asymptotic expansion than necessary for obtaining convergence, without changing the value of the integral. Integrals between finite limits are obviously included as a special case of the integral over the infinite interval.

Remark 4.9 It will be shown in Ch. 12 and Ch. 14 that it is not necessary to insert partition points explicitly. Integrals can also be evaluated by other methods.

Remark 4.10 The preliminary integral of a (strictly) positive function can be negative. By (4.11) the preliminary integral

$$\mathrm{Pre}\int_0^1 dx\, x^{-2} = -1, \tag{4.24}$$

is an example a positive function with a negative integral. This will remain so in the definitive version. It also provides an example of a function $x^{-1}H(x)H(1-x)$ which would have a negative squared norm, if there was a norm.

Remark 4.11 In Ch. 13 the concepts of limit and value will be defined for generalised functions in such a way that many integrals can be evaluated by the substitution of the limits of integration into a primitive function, in accordance with the fundamental theorem Prop. 14.2 of the calculus. This is usually a more convenient method for the actual computation of integrals.

Remark 4.12 The behaviour of the integral under scale transformations will be defined in Ch. 16. The more general question of the possibility of transformations of the dummy variable in integrals will be left open for the future. The behaviour of integrals, and more generally of generalised functions, under translations of the integration variable will be defined in Ch. 15. The so called surface terms at infinity, which arise by translations, are relevant in quantum field theory.

The integral defined in this section is a preliminary integral. It corresponds with integral in the sense of generalised functions over a union of open intervals. The integral in the sense which is appropriate for the generalised functions will be defined in Ch. 14.

4.4 A preliminary scalar product

Since the preliminary class is closed under the pointwise product introduced in Sec. 4.1 the integral introduced in the previous section gives rise to a symmetrical scalar product, defined by

$$\langle f , g \rangle_{\text{pre}} := \text{Pre}\int_{-\infty}^{\infty} dx \, \big(f(x)^* \cdot g(x)\big). \tag{4.25}$$

Following physical convention the * denotes complex conjugation, and it is placed on the left in the otherwise symmetrical scalar product.

Like the integral the preliminary scalar product contains only the contributions from the open intervals. The good properties of the scalar product can be realised only when the operators have been defined in the generalised function sense on the completed space of generalised functions.

The scalar product can be used to induce a preliminary topology on the preliminary class. This preliminary weak topology is not suitable as a basis for further development of the model. It lacks the analyticity properties needed for the construction of the model.

Example 4.6 The integrals of the functions x^λ are not analytic in the parameter λ, since by (4.11), and (4.15), we have

$$\lim_{\lambda \to -1} \text{Pre}\int_0^a dx \, x^\lambda \neq \text{Pre}\int_0^a dx \, x^{-1}. \tag{4.26}$$

if the limit is understood in the standard sense.

When the model has been completed the concepts of limit and convergence will be redefined in Ch. 13 and Ch. 19 in a way which is more appropriate to a theory of generalised functions.

4.5 Preliminary operators

The standard operators, $\mathcal{X}$, $\mathcal{D}$, $\mathcal{P}$, and $\mathcal{F}$, are defined in a standard sense, or almost a standard sense, on the preliminary class.

The operator $\mathcal{X}_{\text{pre}}$ is simply defined as $x\cdot$. It is defined as multiplication by the (generalised) function $f(x) := x$. Since the function $f(x) = x$ belongs to $\mathbf{PC}_\lambda$ this defines

$$\mathcal{X}_{\text{pre}} \, f(x) := x \cdot f(x) \,, \tag{4.27}$$

as an element of $\mathbf{PC}_\lambda$.

The operator $\mathcal{D}_{\text{pre}}$ is defined on the open intervals between singularities by

$$\mathcal{D}_{\text{pre}}\, f(x) := \frac{d}{dx}\, f(x)\,. \tag{4.28}$$

This defines $\mathcal{D}_{\text{pre}} f \in \mathbf{PC}_\lambda$, since $f \in \mathbf{PC}_\lambda$ is $\mathbf{C}^\infty$ between singularities, and the derivative is asymptotic to the formal derivative of the asymptotic expansion. This formal derivative is again of the required type. The preliminary derivative at the singular points is left undefined. It will be defined in the sense of generalised functions in Ch. 7 when the preliminary class has been completed. With this preliminary definition the preliminary class is closed under differentiation but information is lost. Any piecewise constant function has a zero preliminary derivative.

The Fourier operator presents somewhat more difficulties. It is not defined in a standard sense, since the integral

$$\int_{-\infty}^{\infty} dy\; e^{-ixy} f(y), \tag{4.29}$$

need not converge in a classical sense. Therefore the integral has to be defined in the sense of Sec. 4.4 by

$$\mathcal{F}_{\text{pre}}\, f(x) := \left\langle\, e^{ixy}\,,\, f(y)\,\right\rangle_{\text{pre}}. \tag{4.30}$$

The preliminary class is closed under the pre-Fourier operator. The pre-Fourier transform is $\mathbf{C}^\infty$ up to a finite number of exceptional points given by the exponentials in the asymptotic expansion at infinity, since by subtraction the original function can be made to vanish faster than any given power at infinity, and the pre-Fourier transform of the subtracted terms is an entire function. It can be shown, using standard methods for the asymptotic estimation of Fourier transforms [Lig], that the asymptotic behaviour at the singular points and at infinity is again of the required type.

Remark 4.13 The pre-Fourier operator also has a large zero space

$$\mathcal{F}_{\text{pre}}\big(e^{ikx}\big) = \mathit{0}(x). \tag{4.31}$$

It destroys information since it gives zero $\forall k \in \mathbb{R}$.

The parity operator does not need a pre. It is defined by

$$\mathcal{P}\, f(x) = f(-x), \tag{4.32}$$

$\forall f \in \mathbf{PC}_\lambda$. It is clear that (4.32) defines again an element $\in \mathbf{PC}_\lambda$.

The preliminary operators are now defined on the whole of $\mathbf{PC}_\lambda$, and the subspace $\mathbf{PC}_\lambda$ is closed under the action of the preliminary operators. As a result of the classical definition of the operators, in principle as limits using the $\forall\,\delta > 0\,\exists\epsilon > 0$ formalism, the properties of the preliminary operators are unsatisfactory.

The operators $\mathcal{X}_{\text{pre}}$ and $\mathcal{D}_{\text{pre}}$ are not (unitarily) equivalent. The operator $\mathcal{D}_{\text{pre}}$ has a large zero-space by (4.28), but the operator $\mathcal{X}_{\text{pre}}$ has only the trivial zero-space spanned by the element $0(x)$. The pre-Fourier operator is not unitary by (4.31), and the algebraic properties (2.22–23) do not hold.

4.6 Preliminary convolutions

It might also be also possible to use a preliminary convolution product

$$\big(f * g\big)(x) := \big\langle\, f(y)^* \,, g(x-y)\,\big\rangle_{\text{pre}}, \tag{4.33}$$

defined by partie finie of the usual convolution integral, but it not clear to me how this should be done. The approach based on the pointwise product is more transparent and direct, so this line will be followed instead.

The desired unitary equivalence of the convolution and the pointwise product, (2.49), will be implemented in Ch. 10 by defining the convolution as the Fourier image of the pointwise product.

Basing the construction on the pointwise product, instead of using the convolution, avoids the problems associated with the lack of uniqueness and obviousness of the regularization methods which must be used to impose a meaning on the divergent convolution integrals.

In Ch. 22 the converse will be discussed. Regularizations can be derived from the known generalised function products. A regularization is not necessary, but it may be a convenient method for evaluating convolution integrals.

It will also be made plausible that it is impossible to construct a satisfactory generalised function product on basis of ad-hoc assumptions about the regularization of convolution integrals.

4.7 Summary of the preliminaries

The ordinary functions which will become generalised functions have been taken as piecewise $\mathbf{C}^\infty$–functions, with a restricted asymptotic behaviour near their singular points. The usual operators are then defined in a more or less standard sense. These preliminary operators do not have satisfactory properties as a consequence of their standard definitions. The preliminary class is closed under the preliminary operators and under the pointwise product.

In the next chapters the pre-operators are lifted to the linear functionals on the preliminary class, in such a way that they have suitable analyticity properties. This makes the closure of the space of linear functionals under the operators possible. The analytic properties of the linear functionals make it possible to define the operators on the linear functionals with better properties than those of the preliminary operators of this chapter. The operators are then (in Ch. 7) pulled back from $\mathbf{PC}'_\lambda$ to a completed preliminary class $\overline{\mathbf{PC}}_\lambda$, with greatly improved properties.

CHAPTER 5

LINEAR FUNCTIONALS ON THE PRELIMINARY CLASS

The usual procedure for defining linear functionals on the preliminary class would be to introduce a topology on the preliminary class, followed by the introduction of the dual space $\mathbf{PC}'$ of all continuous linear functionals on $\mathbf{PC}$.

This course will not be followed here for two reasons. The first is that the preliminary weak topology (defined in Sec. 4.4) is not suitable for this purpose. The second is that any dual space constructed in this way will be much too large. This will make it impossible to effect the closure of $\mathbf{PC}$ in such a way that a symmetrical structure satisfying (2.5) results.

Instead in the following the undefined concept of the space of 'all' linear functionals on $\mathbf{PC}$ is used loosely. This is indicated as the space $\mathbf{PC}'$ of 'the' linear functionals on $\mathbf{PC}$. Actually only the subspace $\mathbf{PC}_\lambda \subset \mathbf{PC}$ is used as a basis for the construction. Likewise only small subspaces of $\mathbf{PC}'$, with properties fixed by construction, will occur in the model.

A situation as in $\mathcal{L}_2$–theory, where the function space $\mathcal{L}_2$ can be identified with the space of linear functionals on $\mathcal{L}_2$, would be preferable. I do not known if this can be achieved for the symmetrical generalised functions.

5.1 The preliminary class as linear functionals

An obvious class of linear functionals on the preliminary class $\mathbf{PC}_\lambda$ can be generated from $\mathbf{PC}_\lambda$ itself by means of the preliminary scalar product introduced in Sec. 4.4. An element $f \in \mathbf{PC}_\lambda$ generates an element $f' \in \mathbf{PC}'_\lambda$, which is defined as a linear functional on $\mathbf{PC}_\lambda$ by

$$\langle g , f' \rangle := \langle g , f \rangle_{\mathrm{pre}}. \tag{5.1}$$

This definition defines both a class of linear functionals on $\mathbf{PC}_\lambda$ and a natural embedding $\mathbf{PC}_\lambda \to \mathbf{PC}'$. In the following the symbol $\mathcal{M} : \mathbf{PC}_\lambda \to \mathbf{PC}'_\lambda$

$$\mathcal{M} : f' = \mathcal{M} f, \tag{5.2}$$

will be used. The subspace $\mathcal{M}\mathbf{PC}_\lambda \subset \mathbf{PC}'$ will be referred to as $\mathbf{PC}'_\lambda$.

For the time being there is no symmetry in the scalar product defined by (5.1). By convention the elements of $\mathbf{PC}_\lambda$ will be placed on the left side of the scalar product, $\langle \; , \; \rangle : \mathbf{PC}_\lambda \times \mathbf{PC}'_\lambda \to \mathbb{C}$. Symmetry will be restored in Ch. 8 and Ch. 11 when the model has been completed.

When it is necessary to distinguish between functions $f \in \mathbf{PC}_\lambda$ and linear functionals on $\mathbf{PC}_\lambda$ a 'prime' $'$ is added to the function symbol in a suitable way, for instance as in

$$x'^\lambda H(x), \quad \text{or} \quad x^\lambda H'(x) \quad \text{or} \quad |x'|^\lambda \operatorname{sgn}(x). \tag{5.3}$$

The prime should not be interpreted as differentiation. It will be seen in the following that one prime is sufficient, so only one prime will be used.

In the next section subclasses of analytic functionals are introduced. These are linear functionals $\in \mathbf{PC}'_\lambda$ which depend analytically on a complex parameter. The primed powers will be defined there (5.19) as residues.

5.2 Analytic functionals

Each term in the asymptotic expansion (4.1) gives rise to an analytic functional in a complex parameter. For example the function $f(x)^* \in \mathbf{PC}_\lambda$, with $a \in \mathbb{R}_+$ positive,

$$f(x)^* := x^\lambda \big(H(x) - H(x-a)\big), \tag{5.4}$$

defines for every $g(x) \in \mathbf{PC}_\lambda$ an analytic function $\tilde{g}(\lambda) : \mathbb{C} \to \mathbb{C}$ by

$$\tilde{g}(\lambda) := \big\langle f(x)\, , g(x) \big\rangle_{\text{pre}}, \tag{5.5}$$

or equivalently in integral notation

$$\tilde{g}(\lambda) := \mathrm{Pre}\!\int_0^a dx\, x^\lambda\, g(x). \tag{5.6}$$

Consider $\tilde{g}(\lambda)$ as a function of λ defined on the complex λ-plane. First restrict $g(x) \in \mathbf{PC}_\lambda$ to $g(x) \in \mathbf{PC}_\lambda \cap \mathbf{C}^\infty(0,a]$.
Then from the asymptotic expansion (4.1)

$$g(x) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{K_j} c_{jk}\, x^{\lambda_j} \log^k(x), \tag{5.7}$$

and the ordering of the $\{\lambda_j\}$ imposed in (4.2), it is clear that the integral (5.6) converges absolutely for $\mathrm{Re}\,\lambda > -1 - \mathrm{Re}\,\lambda_0$

By standard results on the analyticity of the Mellin transform [Tich], it is known that $\tilde{g}(\lambda)$ is analytic in the right half-plane $\mathrm{Re}\,\lambda > -1 - \mathrm{Re}\,\lambda_0$. This analytic function of λ possesses a meromorphic analytic continuation to the whole λ-plane, with the exception of finitely many singular points. The analytic continuation to the half-plane $\mathrm{Re}\,\lambda > \alpha$ is given explicitly by

$$\begin{aligned}\tilde{g}(\lambda) := \int_0^a dx\, x^\lambda \Big(g(x) - \sum_{j=0}^{\mathrm{Re}\,\lambda_j > -\alpha-1} \sum_{k=0}^{K_j} c_{jk}\, x^{\lambda_j} \log^k x \Big) + \\ + \sum_{j=0}^{\mathrm{Re}\,\lambda_j > -\alpha-1} \sum_{k=0}^{K_j} c_{jk}\, \mathrm{Pre}\!\int_0^a dx\, x^{\lambda+\lambda_j} \log^k x.\end{aligned} \tag{5.8}$$

The first integral converges absolutely in the half-plane $\mathrm{Re}\,\lambda > -\alpha - 1$. The integrals occurring in the finite sums in the second line of (5.8) have been defined as a Hadamard partie finie in Sec. 4.3. As a function of λ these terms are analytic in the entire λ-plane, with the exception of the singular point at $\lambda = -\lambda_j - 1$, where there is a pole. One sees from the explicit form of the analytic continuation (5.8) that the analytic continuation is indeed meromorphic, since it is meromorphic in any half-plane.
Finally the function $\tilde{g}(\lambda)$ is defined at the exceptional points $\{\lambda_j\}$ by

$$\tilde{g}(\lambda_j) := \operatorname*{Res}_{\lambda=\lambda_j} \frac{\tilde{g}(\lambda)}{\lambda - \lambda_j}\,, \tag{5.9}$$

in agreement the preliminary definition of the integral, and therefore in agreement with Hadamard's definition of the partie finie.

In what follows the analytic continuation and the completion by taking a residue are always understood. Standard results on the uniqueness of the analytic continuation are used throughout. By Cauchy's theorem and definition (5.9) the function $\tilde{g}(\lambda)$ satisfies

$$\tilde{g}(\lambda) = \operatorname*{Res}_{\mu=\lambda} \frac{\tilde{g}(\mu)}{\mu - \lambda}\,, \tag{5.10}$$

in the entire complex λ-plane, including the possible singular points.

The analytic functional $x^\lambda\big(H(x) - H(x-a)\big)$ is now extended from the restricted space $\mathbf{PC}_\lambda \cap \mathbf{C}^\infty(0,a]$ to $\mathbf{PC}_\lambda$ as a whole by noting that $\forall\, g(x) \in \mathbf{PC}_\lambda$ the integral

$$\tilde{g}(\lambda) := \mathrm{Pre}\!\int_a^b dx\, x^\lambda\, g(x), \tag{5.11}$$

with $a, b \in \mathbb{R}$, $b > a > 0$, is an entire function of λ. This follows from standard arguments, (differentiability with respect to λ), when the integral $\int_a^b dx\; g(x)$ converges absolutely. It can always be made convergent by choosing a suitable partition between singularities, subtracting a sufficient number of terms of the asymptotic expansions of $g(x)$ at the singular points, and integrating these separately. A typical subtracted term is of the form

$$\mathrm{Pre}\!\int_a^{x_j} dx\, x^\lambda (x_j - x)^{\lambda_j} \log^k(x_j - x). \tag{5.12}$$

By a suitable change of variable this can be reduced to a standard representation [Erd1] of the Eulerian incomplete B–function, or a k–times repeated derivative of the incomplete B–function. The incomplete B–function is an entire function of its first argument, [Erd1], so the singularities away from $x = 0$ gives a contribution which is a finite sum of entire functions.

Therefore, $\forall\, g(x) \in \mathbf{PC}_\lambda$, the function $\tilde{g}(\lambda)$ is meromorphic in the entire λ–plane. In much the same way one sees that $\forall\, g \in \mathbf{PC}_\lambda$ the functions

$$\text{Pre}\int_{x_0}^{x_0+a} dx\,(x-x_0)^\lambda\, g(x), \quad \text{and} \quad \text{Pre}\int_{x_0-a}^{x_0} dx\,(x_0-x)^\lambda\, g(x), \tag{5.13}$$

are meromorphic linear functionals $\in \mathbf{PC}'_\lambda$. Likewise one sees that in an environment of $+\infty$ or $-\infty$ the functionals

$$\text{Pre}\int_a^\infty dx\, x^\lambda\, e^{ikx} g(x), \quad \text{and} \quad \text{Pre}\int_{-\infty}^{-a} dx\,(-x)^\lambda\, e^{ikx} g(x), \tag{5.14}$$

are meromorphic analytic functionals. They are by (4.9) entire functions of λ for almost all values of the parameter k, with the possible exception of finitely many k–values where the asymptotic expansion of $g(x)$ contains a factor oscillating as e^{-ikx}, resulting in behaviour of $e^{ikx}g(x)$ as a power.

Remark 5.1 It is convenient to remember that behaviour as $|x|^\alpha \log^q |x|$ in the finite or at infinity gives rise to a pole in the λ–plane at $\lambda = -\alpha - 1$ of order $q + 1$.

The local behaviour of a function $f \in \mathbf{PC}_\lambda$, as given by its asymptotic expansion of the form (4.1) near a point, is transformed into the location and order of the poles of its local Mellin transform. This is exploited in Sec. 5.4 to define localized generalised functions, which selectively measure the separate coefficients in the asymptotic expansions.

Remark 5.2 For almost all points $x_0 \in \mathbb{R}$ the elements $f(x) \in \mathbf{PC}_\lambda$ are $\mathbf{C}^\infty(x_0)$, and consequently asymptotic to a Taylor series. For these values of x_0 the poles of $\tilde{f}(x)$ are first order, located at the negative integers $-p-1 \in \mathbb{C}$ in the complex λ–plane.

Example 5.1 The function

$$f(x) := \begin{cases} e^{-x} & x \ge 0, \\ 0 & x < 0, \end{cases}, \tag{5.15}$$

is mapped into

$$\langle\, f(x)\,,x^\lambda\,\rangle = \text{Pre}\int_0^\infty dx\, x^\lambda\, e^{-x} = \Gamma(\lambda+1), \tag{5.16}$$

which is [Erd1] the Eulerian Γ–function. The Γ–function has poles at $\lambda = -p-1$, with residues $(-)^p/p!$ equal to the corresponding coefficients in the power series of e^{-x} at the point $x = 0+$.

The generalised function $x^\lambda H'(x) = x'^\lambda H(x)$ is defined as an element of $\mathbf{PC}'_\lambda$ by

$$\langle\, f(x)\,,x^\lambda H'(x)\,\rangle := \langle\, f(x)\,,x^\lambda H(x)\,\rangle_{\text{pre}}, \tag{5.17}$$

$\forall\, f(x) \in \mathbf{PC}_\lambda$. The more general functions and the linear combinations

$$x'^\lambda \log^q(x) H(x), \quad \text{and} \quad |x'|^\lambda \log^q|x|, \quad \text{and} \quad |x'|^\lambda \log^q|x| \operatorname{sgn}^m(x), \tag{5.18}$$

are defined in the same way. It is possible to leave out the scalar products in the definition (5.17). Then one writes

$$x'^\alpha H(x) = \operatorname*{Res}_{\lambda=\alpha} (\lambda - \alpha)^{-1} x^\lambda H(x), \tag{5.19}$$

with the understanding that the appropriate scalar product has to be added to both sides. Similar notation is used for the powers of x on other intervals. More generally we define the logarithms $\in \mathbf{PC}'_\lambda$ by

$$x'^\alpha \log^q(x) H(x) = \operatorname*{Res}_{\lambda=\alpha} q!\, (\lambda - \alpha)^{-q-1} x^\lambda H(x), \tag{5.20}$$

in agreement with the standard properties of the complex powers. It would also be possible to define preliminary logarithms $\in \mathbf{PC}_\lambda$ and to take residues to obtain logarithms in $\mathbf{PC}'_\lambda$.

Ordinary function $\in \mathbf{PC}'_\lambda$ will be indicated by $f'(x)$, or by explicitly adding primed Heaviside functions to the function symbol.

It follows at once that formal differentiation with respect to α yields the result

$$\frac{\partial}{\partial \alpha}\, x'^\alpha \log^q(x) H(x) = x'^\alpha \log^{q+1}(x) H(x), \tag{5.21}$$

in agreement with standard expectations.

5.3 Analytic properties of the partie finie

In Sec. 4.3 the partie finie was defined by analytic continuation for the powers and logarithms, and by a subtraction procedure for arbitrary $f(x) \in \mathbf{PC}_\lambda$. Now that the analytic properties of the powers are available it is possible to unify these cases.

From (5.8) it is seen that

$$\operatorname{Pre}\!\int_{-\infty}^{\infty} dx\, |x|^\alpha f(x) = \operatorname*{Res}_{\lambda=\alpha} (\lambda - \alpha)^{-1} \int_{-\infty}^{\infty} dx\, |x|^\lambda f(x), \tag{5.22}$$

with the special case

$$\operatorname{Pre}\!\int_{-\infty}^{\infty} dx\; f(x) = \operatorname*{Res}_{\lambda=0} \lambda^{-1} \int_{-\infty}^{\infty} dx\, |x|^\lambda f(x). \tag{5.23}$$

In particular for the finite interval we have

$$\operatorname{Pre}\!\int_{0}^{a} dx\; f(x) = \operatorname*{Res}_{\lambda=0} \lambda^{-1} \int_{0}^{a} dx\, x^\lambda f(x). \tag{5.24}$$

This is sometimes a convenient method to evaluate integrals from the analytic functionals when these are known explicitly.

Example 5.2 The integral

$$\text{Pre}\int_0^\infty dx\, x^{-1} e^{-x} = \operatorname*{Res}_{\lambda=-1} (\lambda+1)^{-1} \Gamma(\lambda+1) = \psi(1), \tag{5.25}$$

can be evaluated as a residue. The ψ–function is the logarithmic derivative [Erd1] of the Γ–function.

Remark 5.3 In the following the preliminary integral will be identified with the generalised function integral over an 'open' interval $(0+, a-)$. The generalised function integrals over 'closed' intervals are defined in Ch. 14, together with a preliminary definition of 'open' and 'closed' in the sense of generalised functions.

The scalar product was defined by means of the integral by (4.25). Therefore the scalar product inherits the analytic properties of the integral.
This gives the result

$$\langle\, f(x)\,,\, g'(x)\,\rangle = \operatorname*{Res}_{\lambda=0} \lambda^{-1} \,\langle\, f(x)\,,\, |x|^\lambda g(x)\,\rangle. \tag{5.26}$$

The scalar product appearing above can be interpreted both as the preliminary scalar product on $\mathbf{PC}_\lambda$, or as the generalised function scalar product $\mathbf{PC}_\lambda \times \mathbf{PC}'_\lambda \to \mathbb{C}$.

At first sight nothing has been gained by going from $\mathbf{PC}_\lambda$ to $\mathbf{PC}'_\lambda$, but the gain is having good analytic properties such as (5.26), in contrast to the undefined result obtained by replacing the residue in (5.26) by a limit.

5.4 Localized functionals

In the previous section it was shown that the linear functionals on $\mathbf{PC}_\lambda$, generated by the powers of the variable x are analytic functions in a half-plane, with a meromorphic analytic continuation. Therefore corresponding to every $f \in \mathbf{PC}_\lambda$ we have four families of meromorphic functions generated by the scalar products

$$\langle\, f(x)\,,\, (x-x_0)^\lambda \big(H(x-x_0) - H(x-a-x_0)\big)\,\rangle, \tag{5.27}$$

$$\langle\, f(x)\,,\, (x_0-x)^\lambda \big(H(x_0-x) - H(x_0-x+a)\big)\,\rangle, \tag{5.28}$$

$$\langle\, f(x)\,,\, x^\lambda\, e^{ik_0x} H(a-x)\,\rangle, \tag{5.29}$$

$$\langle\, f(x)\,,\, (-x)^\lambda\, e^{ik_0x} H(-x-a)\,\rangle, \tag{5.30}$$

parametrized by $x_0 \in \mathbb{R}$ and $k_0 \in \mathbb{R}$. At each point $x_0 \in \mathbb{R}$ there are the left and right local Mellin transforms. At each $k_0 \in \mathbb{R}$ there is a left and a right Mellin transform, localized at $x = -\infty$ and $x = +\infty$ respectively, of the function $e^{ik_0x} f(x)$.

These local transforms also depend on the value of a, which determines the length of the integration interval. This dependence on a is irrelevant for the pole parts of the meromorphic function. As shown in the previous section a different choice of a changes the local Mellin transform by adding an entire function of the variable λ.

Example 5.3 The integral

$$\mathrm{Pre}\int_0^a dx\, x^\lambda = \frac{a^{\lambda+1}}{\lambda+1}, \tag{5.31}$$

defined in Sec. 4.3 can be written as

$$\mathrm{Pre}\int_0^a dx\, x^\lambda = \frac{1}{\lambda+1} + \frac{a^{\lambda+1}-1}{\lambda+1}. \tag{5.32}$$

The first term is a pole part independent of a. The second term does depend on a, but it is an entire function of the variable λ.

Remark 5.4 In contrast with this the value assigned to the meromorphic function at the pole location by (5.9) does depend on the choice of a. Some of the consequences of this dependence will be demonstrated in Ch. 16, where the invariance under scale transformations is discussed.

Conversely the coefficients of the powers of x occurring in the asymptotic expansion can be found from the local Mellin transform.

Following the method introduced by Gelfand and Shilov [G&S], the 'eta-down' generalised function, $\eta_\downarrow(x)$, is defined by

$$\langle\, f(x)\,,\eta_\downarrow(x)\,\rangle := \operatorname*{Res}_{\lambda=-1}\ \langle\, f(x)\,, x^\lambda\big(H(x)-H(x-a)\big)\,\rangle. \tag{5.33}$$

This expression will be non-zero if and only if the scalar product on the right-hand side of (5.33) has a first order pole at $\lambda=-1$. From the foregoing it is known that this will be the case when the asymptotic expansion at $x=0+$ contains a constant term proportional to x^0. By (5.32) the result of taking the residue in (5.33) does not depend the choice of a. It is not necessary to take a smaller than the distance to the next singular point, since the integration near the next singular point will produce by (4.17) only entire terms, which do not contribute to the residue in (5.33).

Example 5.4 In the special case that $f(x)\in \mathbf{PC}_\lambda$ is continuous at $x=0$, we obtain

$$\langle\, f(x)\,,\eta_\downarrow(x)\,\rangle = f(0)^*, \tag{5.34}$$

so apart from the conventional * the generalised function $\eta_\downarrow(x)$ is in this special case equivalent to the δ–distribution.

Example 5.5 In the special case that $f \in \mathbf{PC}_\lambda$ is discontinuous at $x = 0$, but such that

$$f(0+) := \lim_{x \downarrow 0} f(x) \tag{5.35}$$

exists, we obtain

$$\langle\, f(x)\,,\eta_\downarrow(x)\,\rangle = f(0+)^*. \tag{5.36}$$

In the sense of distribution theory the expression (f, δ) is in this case undefined. It can be defined only when the space of test functions has been suitably adapted.

Example 5.6 For the function $f(x) \in \mathbf{PC}_\lambda$, defined by

$$f(x) := x^{-n}\, e^{-x}, \tag{5.37}$$

we obtain

$$\langle\, f(x)\,,\eta_\downarrow(x)\,\rangle = (-)^n/n!\,. \tag{5.38}$$

This shows that the generalised function $\eta_\downarrow(x)$ can measure the constant part of a function, even when it is hidden under divergent terms. This possibility is absent in distribution theory, unless it is specially defined by a suitable subtraction procedure.

Remark 5.5 Dirac's generalised function $\delta(x)$ and Schwartz's distribution δ have a traditional normalization, due to Dirac [Dir] which does not carry over easily to symmetrical generalised functions. Therefore it was necessary to choose a different letter, η instead of δ. The δ will be reintroduced in Ch. 7 as $\check{\delta}$, with a different normalization, and a different meaning.

A larger class of localized generalised functions $\eta_\downarrow^{(\alpha,q)}(x)$ is defined by

$$\langle\, f(x)\,,\eta_\downarrow^{(\alpha,q)}(x)\,\rangle := \operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{1}{q!}(\lambda+\alpha+1)^q\,\langle\, f(x)\,, x^\lambda\big(H(x)-H(x-a)\big)\,\rangle, \tag{5.39}$$

with $a \in \mathbb{R}_+$, $\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$.

The generalised function $\eta_\downarrow^{(\alpha,q)}(x)$ measures the coefficient of $x^\alpha(-\log(x))^q$, in the asymptotic expansion (4.1) of $f(x) \in \mathbf{PC}_\lambda$, at the positive side of the point $x = 0$. In particular we obtain the result

$$\langle\, x^\alpha \log^q(x)\, H(x)\,,\eta_\downarrow^{(\beta,r)}(x)\,\rangle = \begin{cases} (-)^q & \alpha^* = \beta, \quad q = r, \\ 0 & \text{otherwise}, \end{cases} \tag{5.40}$$

when the η–function acts on a power.

Example 5.7 The generalised functions such as $\eta_\downarrow(x)$ can be used to measure the behaviour of functions such as $\sqrt{x}$,

$$\langle\, \eta_\downarrow^{(0.5)}(x)\,,\sqrt{x}\,\rangle = 1. \tag{5.41}$$

The η–functions are particularly suitable to detect scaling behaviour. An example is given in appendix F, where the action of the $\eta_\downarrow$–functions on the Cantor's staircase function is computed. This is an example where η–functions with a transcendental number in the index occur naturally.

By the uniqueness of the different powers in the asymptotic expansions allowed by (4.1) different powers are not asymptotic to each other

$$|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \not\sim |x|^\beta \log^r|x| \operatorname{sgn}^n(x), \tag{5.42}$$

when $\alpha \neq \beta$, or $q \neq r$, or $m \neq n$. Correspondingly the $\eta_\downarrow$–functions are independent for different coefficients. The symbol $\perp$ is used to indicate this

$$\eta_\downarrow^{(\alpha,q)}(x) \perp \eta_\downarrow^{(\beta,r)}(x), \tag{5.43}$$

when the indices are different. The relation $\perp$ is not related to the vanishing of the scalar product.

Remark 5.6 In the superscript of the η–functions indices which have the value zero are usually omitted,

$$\eta_\downarrow^{(\alpha)}(x) \equiv \eta_\downarrow^{(\alpha,0)}(x), \quad \text{and} \quad \eta(x) \equiv \eta^{(0)}(x) \equiv \eta^{(0,0)}(x), \tag{5.44}$$

as in (5.41) and in the definition (5.33) of $\eta_\downarrow(x)$ given above.

Remark 5.7 The normalization of the η–functions chosen here differs by a factor $(-)^q/q!$ from that of earlier work, ([Lod1]: equations (I.23) and (I.25)). Instead of $q!\,\log^q|x|$ measured by the 'old' $\eta_\downarrow$–functions, the 'new' $\eta_\downarrow$–functions measure $|\log|x||^q$. This is done to facilitate the generalization to non-integral powers of $\log|x|$ in the future. From

$$\int_0^1 dx\, x^\lambda \big(-\log(x)\big)^\nu = \Gamma(\nu+1)\,(\lambda+1)^{-\nu-1}, \tag{5.45}$$

one sees that arbitrary complex powers of the logarithm will give rise to branchpoints in the λ–plane. For $\nu = q \in \mathbb{N}$ these reduce to poles. (The poles in (5.45) at $(-\nu-1) \in \mathbb{N}$ come from the upper limit).

In the same way as for the right asymptotic behaviour one finds the left asymptotic behaviour by defining the 'eta-up' functions $\eta_\uparrow(x)$ by

$$\begin{aligned}\big\langle f(x)\,, \eta_\uparrow^{(\alpha,q)}(x) \big\rangle &:= \\ &:= \operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{1}{q!}(\lambda+\alpha+1)^q \big\langle f(x)\,, (-x)^\lambda \big(H(-x) - H(-x-a)\big) \big\rangle,\end{aligned} \tag{5.46}$$

so $\eta_\uparrow^{(\alpha,q)}(x)$ is related to $\eta_\downarrow^{(\alpha,q)}(x)$ by

$$\eta_\downarrow^{(\alpha,q)}(x) = \eta_\uparrow^{(\alpha,q)}(-x). \tag{5.47}$$

For many computations it is easier to introduce the symmetrical and the antisymmetrical linear combinations by

$$\eta_s^{(\alpha,q)}(x) := \tfrac{1}{2}\,\eta_\downarrow^{(\alpha,q)}(x) + \tfrac{1}{2}\,\eta_\uparrow^{(\alpha,q)}(x), \tag{5.48}$$

and

$$\eta_a^{(\alpha,q)}(x) := \tfrac{1}{2}\,\eta_\downarrow^{(\alpha,q)}(x) - \tfrac{1}{2}\,\eta_\uparrow^{(\alpha,q)}(x), \tag{5.49}$$

for all values of the indices. The symmetrical one, η_s, measures the average of the left and right values, the antisymmetrical one, η_a, measures *half* the jump across a singular point.

Remark 5.8 It is possible to obtain $\eta_s^{(\alpha,q)}(x)$ as a residue from

$$\langle\, f(x)\,,\eta_s^{(\alpha,q)}(x)\,\rangle = \tfrac{1}{2}\operatorname*{Res}_{\lambda=-\alpha-1}\tfrac{1}{q!}(\lambda+\alpha+1)^q\,\langle\, f(x)\,,|x|^\lambda H(a-|x|)\,\rangle, \tag{5.50}$$

by combining (5.39) and (5.46) with (5.48).

Example 5.8 For the signum function, $\operatorname{sgn}(x)$, which is fully defined (as an element $\in \mathbf{PC}_\lambda$) by

$$\operatorname{sgn}(x) := \begin{cases} 1 & x > 0, \\ \text{undefined} & x = 0, \\ -1 & x < 0, \end{cases} \tag{5.51}$$

one obtains

$$\langle\,\operatorname{sgn}(x)\,,\eta_a^{(\alpha,q)}(x)\,\rangle = \begin{cases} 1 & \alpha = 0,\ q = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.52}$$

It should be remembered that by definition (5.49) the η_a–function measures half the jump of the sgn–function.

For integer values of the parameter α on the η–function it is often more convenient to define the linear combinations

$$\eta^{(p,q)}(x) := \tfrac{1}{2}\,\eta_\downarrow^{(p,q)}(x) + \tfrac{1}{2}(-)^p\eta_\uparrow^{(p,q)}(x),$$

and

$$\sigma^{(p,q)}(x) := \tfrac{1}{2}\,\eta_\downarrow^{(p,q)}(x) - \tfrac{1}{2}(-)^p\eta_\uparrow^{(p,q)}(x), \tag{5.53}$$

which measure $x^p|\log|x||^q$, and $x^p|\log|x||^q\operatorname{sgn}(x)$ respectively.

It is sometimes convenient to use the notation

$$\eta_s^{(\alpha,q)}(x)\operatorname{sgn}^m(x) := \begin{cases} \eta_s^{(\alpha,q)}(x), & m = 0 \pmod 2, \\ \eta_a^{(\alpha,q)}(x), & m = 1 \pmod 2, \end{cases} \tag{5.54}$$

for keeping the computations general. This notation will be shown to agree with the product of generalised functions in Ch. 9. The special notations for the η–functions are summarized in Table 5.1.

The notation of the special cases can be simplified by introducing the generalization to complex subscripts of the Kronecker's δ–symbol defined by

$$\delta_{\alpha,\beta} := \begin{cases} 1 & \alpha^* = \beta, \\ 0 & \alpha^* \neq \beta. \end{cases} \tag{5.55}$$

The previous formula (5.40) then takes the form

$$\left\langle x^\alpha \log^q(x)\,,\eta_1^{(\beta,r)}(x)\right\rangle = (-)^q\delta_{\alpha,\beta}\,\delta_{q,r}\,. \tag{5.56}$$

A further simplification of the notation can be obtained by introducing another generalization of the Kronecker symbol defined by

$$\delta_{m,n}^{\text{mod}2} := \tfrac{1}{2}\left(1+(-)^{m+n}\right) = \begin{cases} 0 & m \neq n \pmod 2, \\ 1 & m = n \pmod 2. \end{cases} \tag{5.57}$$

The 'Kronecker delta mod two' tests if the parity of m and n is equal. Equation (5.56) can be rewritten with Kronecker δ's as

$$\left\langle |x|^\alpha \log^q|x|\,\text{sgn}^m(x)\,,\eta_s^{(\beta,r)}(x)\,\text{sgn}^n(x)\right\rangle = (-)^q\delta_{\alpha,\beta}\,\delta_{q,r}\,\delta_{m,n}^{\text{mod}2}\,. \tag{5.58}$$

Despite its first appearance this notation will be found to simplify many computations. It avoids the necessity of considering odd and even special cases separately.

Considered as a function of the variable α or α^*, the function $\delta_{\alpha,\beta}$ is zero for almost all values of α with the exception of a finite set of points. Nevertheless it is convenient to consider it as an analytic function, with the property that it is zero in the punctured complex plane $\alpha \in \mathbb{C}, \alpha \neq \beta^*$.

In the following the term 'zeromorphic' is used to indicate an analytic function with this property in the complex plane, which is punctured by omitting a discrete set of points. The simplest example of a zeromorphic function is of course the generalization of the Kronecker's δ–symbol. Conversely every zeromorphic function can be written as a formal sum over Kronecker's δ–symbols.

Example 5.9 The analytic function $\tilde{g}(\alpha)$ defined by

$$\tilde{g}(\alpha) := \operatorname*{Res}_{\lambda=\alpha} \Gamma(\lambda), \tag{5.59}$$

where $\Gamma(\lambda)$ is the Eulerian Γ–function, is zeromorphic. The exceptional points are at $\alpha = -p,\ p \in \mathbb{N}$. Using the Kronecker's δ–symbol the function $\tilde{g}(\alpha)$ can be written explicitly as

$$\tilde{g}(\alpha) = \sum_{p=0}^{\infty} \tfrac{(-)^p}{p!}\,\delta_{\alpha,-p}\,, \tag{5.60}$$

in the form of a formal sum of Kronecker's δ–symbols. This formal sum notation will often be used to indicate exceptional cases explicitly.

Remark 5.9 An obvious property of zeromorphic functions is that

$$\operatorname*{Res}_{\lambda=\alpha}(\lambda-\alpha)^k f(\lambda) = 0, \tag{5.61}$$

$\forall\alpha \in \mathbb{C}$, $\forall k \in \mathbb{Z}$, and for every zeromorphic function.

Following the same method one can also define generalised functions at infinity. The same definition is used, up to a minus sign arising from the fact that the relevant term now comes from the lower limit of the integration interval, and a factor $(-)^q$ since the function $\log^q|x|$ is positive at infinity. The generalised function 'eta-slash-up', $\not\eta_\uparrow(x)$, is defined by

$$\langle\, f(x), \not\eta_\uparrow^{(\alpha,q)}(x)\,\rangle := -\operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{(-)^q}{q!}(\lambda+\alpha+1)^q \,\langle\, f(x), x^\lambda H(x-a)\,\rangle. \quad (5.62)$$

As usual we take $a \in \mathbb{R}_+$ positive. The resulting scalar product does not depend on the value of a.

The slash through the greek letter such as η serves to distinguish between an $\eta(x)$–function in the finite, and the similar $\not\eta(x)$–function at infinity. The equivalent of the measurement formula (5.58) for the $\not\eta_\uparrow$–functions is

$$\langle\, x^\alpha \log^q(x) H(x)\,, \not\eta_\uparrow^{(\beta,r)}(x)\,\rangle = \delta_{\alpha,\beta}\,\delta_{q,r}\,, \quad (5.63)$$

without the minus signs $(-)^q$. This was absorbed in the definition, in keeping with the positive sign of the logarithm at infinity.
In the special case $\alpha = q = 0$ the function $\not\eta_\uparrow(x) = \not\eta_\uparrow^{(0,0)}(x)$ measures the constant part at $x = +\infty$ of $f(x) \in \mathbf{PC}_\lambda$.

Example 5.10 When the limit

$$\lim_{x\uparrow\infty} f(x) =: f(+\infty) \quad (5.64)$$

exists, one obtains

$$\langle\, f(x)\,, \not\eta_\uparrow(x)\,\rangle = f(+\infty)^*. \quad (5.65)$$

The result (5.65) is again independent of the choice made for a. Equation (5.65) also gives the general result when $f(+\infty)$ is interpreted as the asymptotic coefficient of x^0 in the asymptotic expansion of $f(x)$ at $x = +\infty$. In Ch. 13 a generalised limit will be defined in terms of the scalar product.

Example 5.11 The $\not\eta$–function averages out oscillations at infinity, as in

$$\langle\, e^{ikx}x^\alpha\,, \not\eta_\uparrow(x)\,\rangle = \begin{cases} 0 & k \neq 0, \\ \delta_{\alpha,0} & k = 0, \end{cases} \quad (5.66)$$

so it measures an average behaviour at infinity.
As before the generalised function 'eta-slash-down' $\not\eta_\downarrow^{(\alpha,q)}(x)$ at minus infinity is defined by

$$\not\eta_\downarrow^{(\alpha,q)}(x) := \not\eta_\uparrow^{(\alpha,q)}(-x), \quad (5.67)$$

so it can be found as a residue from

$$\not\eta_\downarrow^{(\alpha,q)}(x) := -\operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{(-)^q}{q!}(\lambda+\alpha+1)^q(-x)^\lambda H(-x-a). \quad (5.68)$$

Many equations in this work are invariant under the substitutions

$$\eta_{\downarrow}^{(\alpha,q)}(x) \leftrightarrow (-)^q \not\eta_{\uparrow}^{(\alpha,q)}(x) \quad \text{and} \quad \not\eta_{\downarrow}^{(\alpha,q)}(x) \leftrightarrow (-)^q \eta_{\uparrow}^{(\alpha,q)}(x), \tag{5/69}$$

since the definitions (5.39) and (5.62) have been constructed in this way. In the following a slash will be added to the equation number to indicate this. This convention reduces the number of equations which have to be written out in full.

As in the finite, one can define the even and odd linear combinations

$$\not\eta_{s}^{(\alpha,q)}(x) := \tfrac{1}{2}\, \not\eta_{\uparrow}^{(\alpha,q)}(x) + \tfrac{1}{2}\, \not\eta_{\downarrow}^{(\alpha,q)}(x), \tag{5/70}$$

and

$$\not\eta_{a}^{(\alpha,q)}(x) := \tfrac{1}{2}\, \not\eta_{\uparrow}^{(\alpha,q)}(x) - \tfrac{1}{2}\, \not\eta_{\downarrow}^{(\alpha,q)}(x), \tag{5/71}$$

with the arrows reversed in accordance with (5.69). These generalised functions detect functions behaving at infinity as

$$\left\langle\, |x|^{\alpha} \log^{q}|x| \,,\, \not\eta_{s}^{(\beta,r)}(x) \,\right\rangle = \delta_{\alpha,\beta}\, \delta_{q,r}\,, \tag{5/72}$$

and

$$\left\langle\, |x|^{\alpha} \log^{q}|x| \operatorname{sgn}(x) \,,\, \not\eta_{a}^{(\beta,r)}(x) \,\right\rangle = \delta_{\alpha,\beta}\, \delta_{q,r}\,. \tag{5/73}$$

The linear combinations $\not\eta$ and $\not\phi$, are defined for $p \in \mathbb{Z}$, $q \in \mathbb{N}$, by

$$\not\eta^{(p,q)}(x) := \tfrac{1}{2}\, \not\eta_{\uparrow}^{(p,q)}(x) + \tfrac{1}{2}(-)^p \not\eta_{\downarrow}^{(p,q)}(x), \tag{5/74}$$

and

$$\not\phi^{(p,q)}(x) := \tfrac{1}{2}\, \not\eta_{\uparrow}^{(p,q)}(x) - \tfrac{1}{2}(-)^p \not\eta_{\downarrow}^{(p,q)}(x). \tag{5/75}$$

These linear combinations measure the coefficients of the powers at infinity

$$\left\langle\, x^{p} \log^{q}|x| \,,\, \not\eta^{(r,s)}(x) \,\right\rangle = \delta_{r,p}\, \delta_{q,s}\,, \tag{5/76}$$

and

$$\left\langle\, x^{p} \log^{q}|x| \operatorname{sgn}(x) \,,\, \not\phi^{(r,s)}(x) \,\right\rangle = \delta_{r,p}\, \delta_{q,s}\,. \tag{5/77}$$

The general case is again written as $\not\eta_{s}^{(\alpha,q)}(x) \operatorname{sgn}^{m}(x)$, with the same special cases as for $\eta_{s}^{(\alpha,q)}(x) \operatorname{sgn}^{m}(x)$. The special cases are tabulated on the next page.

For convenience the special notations for the η and $\not\eta$–functions are collected in the following table, with $\alpha \in \mathbb{C}$, $p \in \mathbb{Z}$, and $q \in \mathbb{N}$.

Table 5.1

Special notations for $\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$ and $\not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$

α	m (mod 2)	q	$\eta_{\dots}^{\dots}$	$\eta_{\dots}^{\dots}$ measures
α	m	0	$\eta_s^{(\alpha)}(x)\,\mathrm{sgn}^m(x)$	$\lvert x\rvert^\alpha\,\mathrm{sgn}^m(x)$
α	0	q	$\eta_s^{(\alpha,q)}(x)$	$\lvert x\rvert^\alpha\lvert\log\lvert x\rvert\rvert^q$
α	1	q	$\eta_a^{(\alpha,q)}(x)$	$\lvert x\rvert^\alpha\lvert\log\lvert x\rvert\rvert^q\,\mathrm{sgn}(x)$
p	p	q	$\eta^{(p,q)}(x)$	$x^p\lvert\log\lvert x\rvert\rvert^q$
p	$p+1$	q	$\sigma^{(p,q)}(x)$	$x^p\lvert\log\lvert x\rvert\rvert^q\,\mathrm{sgn}(x)$
p	p	0	$\eta^{(p)}(x)$	x^p
p	$p+1$	0	$\sigma^{(p)}(x)$	$x^p\,\mathrm{sgn}(x)$
0	0	0	$\eta(x)$	$I(x)$
0	1	0	$\sigma(x)$	$\mathrm{sgn}(x)$

and idem for $\not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$ with an added slash

The same specialization applies to the $\eta_\downarrow$–functions and the other η–functions with arrows.

Combination of the definitions (5.39) and (5.62) of the η–functions as residues gives

$$\operatorname*{Res}_{\lambda=-\alpha-1}(\lambda+\alpha+1)^q\,|x|^\lambda\,\mathrm{sgn}^m(x) = 2(-)^q q!\,\big((-)^q\eta_s^{(\alpha,q)}(x)-\not\eta_s^{(\alpha,q)}(x)\big)\,\mathrm{sgn}^m(x), \tag{5.78}$$

which is the basis for computing the Fourier transforms of the η–functions.

Remark 5.10 The linear combination of η and $\not\eta$ as in (5.78) occurs often. For convenience the $(-)^q$ will always be combined with the η–function, and the minus sign with the $\not\eta$–function. Both terms then measure $\log^q|x|$, both in the finite and at infinity.

Example 5.12 It is often obvious how the split between the finite and infinity should be effected without introducing explicit cutoffs. The Mellin transform of the function arctan(x) is given by [Erd2]

$$\int_0^\infty dx\,x^\lambda\arctan(x) = \frac{\pi}{2(\lambda+1)\sin\frac{\pi}{2}\lambda}, \tag{5.79}$$

which has poles at $\lambda = 2m$, and an additional pole at $\lambda = -1$. These correspond to the powers x^{2m+1}, and an additional power x^0. Computing the residues at the poles gives the scalar products with the η–functions. Splitting between $x = 0+$ and $x = +\infty-$ by hand, we find

$$\langle\,\eta_\downarrow^{(\alpha,q)}(x)\,,\arctan(x)\,\rangle = \delta_{q,0}\sum_{p=0}^{\infty}\delta_{\alpha,2p+1}\,\tfrac{(-)^p}{2p+1}, \tag{5.80}$$

and consequently the asymptotic expansion

$$\arctan(x) \sim x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots = \sum_{j=0}^{\infty} \tfrac{(-)^j}{2j+1} x^{2j+1}, \tag{5.81}$$

for $x \downarrow 0$, and likewise,

$$\arctan(x) \sim \tfrac{\pi}{2} - x^{-1} + \tfrac{1}{3}x^{-3} + \cdots = \tfrac{\pi}{2} - \sum_{j=0}^{\infty} \tfrac{(-)^j}{2j+1} x^{-2j-1}, \tag{5.82}$$

for $x \uparrow \infty$, in agreement with the elementary standard result. The asymptotic expansion (5.81) actually converges for small x, but this is irrelevant.

Formal differentiation with respect to α applied to the definitions of the η–functions yields

$$\frac{\partial}{\partial\alpha}\, \eta_s^{(\alpha,q+1)}(x)\,\mathrm{sgn}^m(x) = \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \tag{5/83}$$

and

$$\frac{\partial}{\partial\alpha}\, \not\eta_s^{(\alpha,q+1)}(x)\,\mathrm{sgn}^m(x) = -\,\not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \tag{5/84}$$

for $q \geq 0$, and to

$$\frac{\partial}{\partial\alpha}\, \eta_s^{(\alpha,0)}(x)\,\mathrm{sgn}^m(x) = \frac{\partial}{\partial\alpha}\, \not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = 0(x), \tag{5/85}$$

in agreement with the corresponding behaviour of the powers.

The complex conjugate of the η–functions is defined in the obvious way by taking the complex conjugate of defining equations such as (5.85). The result is obviously

$$\left(\eta^{(\alpha,q)}(x)\right)^* = \eta^{(\alpha^*,q)}(x), \tag{5.86}$$

since $(x^\lambda)^* = x^{\lambda^*}$ for $x \in \mathbb{R}_+$.

Remark 5.11 The generalised functions at infinity do not have a support in $\mathbb{R}$. Nevertheless they are treated as localized generalised functions. The concept of the support of the generalised functions will be defined in Ch. 13 in a heuristic way.

5.5 Translated arguments and exponentials

Now that the singular functions of argument x are defined, the generalised functions of argument $x - x_0$ are defined by

$$\eta_{\downarrow}^{(\alpha,q)}(x-x_0) := \operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{1}{q!}(\lambda+\alpha+1)^q (x-x_0)^\lambda \big(H(x-x_0) - H(x-x_0-a)\big), \tag{5.87}$$

and idem at infinity

$$\not{\eta}_{\uparrow}^{(\alpha,q)}(x-x_0) := -\operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{(-)^q}{q!}(\lambda+\alpha+1)^q (x-x_0)^\lambda H(x-x_0-a). \tag{5.88}$$

The definitions of the functions $\eta_{\uparrow}(x - x_0)$ and $\not{\eta}_{\downarrow}(x - x_0)$ are the same with $x - x_0$ replaced by $x_0 - x$, and with adapted Heaviside functions.

Here some preliminary remarks on the subject of translations are necessary. A more elaborate treatment will be given in Ch. 15.

In the finite the η–functions at different locations, $\eta(x - x_1)$, and $\eta(x - x_2)$, with $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, are linearly independent in the sense that an η–function at one point cannot be written as an effectively finite linear combination of η–functions at other points. The relation $\perp$ will also be used to indicate this.

Example 5.13 An attempt to define (anticipating Ch. 15)

$$\eta_{\downarrow}^{(\alpha,q)}(x-x_0) := e^{-x_0\mathcal{D}}\eta_{\downarrow}^{(\alpha,q)}(x) = \sum_{j=0}^{\infty} \tfrac{1}{j!} {x_0}^j \mathcal{D}^j\, \eta_{\downarrow}^{(\alpha,q)}(x), \tag{5.89}$$

WRONG!

fails. The resulting formal linear combination is not an allowed expression in the model.

At infinity the function $f(x) := (x - x_0)^\lambda H(x - x_0)$ is asymptotic to its Taylor expansion, which is found from the binomial theorem,

$$f(x) \sim f_a(x;+\infty) := \sum_{j=0}^{\infty} \tbinom{\lambda}{j}(-x_0)^j x^{\lambda-j} H(x), \tag{5.90}$$

so the function $\not{\eta}_{\uparrow}^{(\alpha,q)}(x - x_0)$ of argument $(x - x_0)$ can be expressed as

$$\begin{aligned}\not{\eta}_{\uparrow}^{(\alpha,q)}(x-x_0) &= -\operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{(-)^q}{q!}(\lambda+\alpha+1)^q (x-x_0)^\lambda H(x-x_0+a)\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-)^{j+k} {x_0}^j \tbinom{q+k}{k}\tbinom{-\alpha-1}{j}^{(k)} \not{\eta}_{\uparrow}^{(\alpha+j,q+k)}(x), \end{aligned}\tag{5.91}$$

in terms of of the functions $\not{\eta}_{\uparrow}(x)$ of argument (x). The derivatives of the binomial coefficients occurring in (5.91) are defined by (B.5) in appendix B.

Remark 5.12 The infinite sum over the indices appearing in (5.91) is only formally infinite. The functions $f(x) \in \mathbf{PC}_\lambda$ are restricted to a finite number of effective powers and a finite number of logarithmic terms. The infinite sum has only a finite number non-zero terms when a scalar product with an element $f(x) \in \mathbf{PC}_\lambda$ is evaluated.
The function $f(x) \in \mathbf{PC}_\lambda$ defined by

$$f(x) := x^\lambda \, e^{ikx} H(x-a), \tag{5.92}$$

with $a \in \mathbb{R}_+$, has the residue

$$e^{ikx} \eta\!\!\!/_{\uparrow}^{(\alpha,q)}(x) := -\operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{(-)^q}{q!} (\lambda+\alpha+1)^q \, e^{ikx} x^\lambda H(x-a). \tag{5.93}$$

The usual linear combinations are defined in the usual way. The corresponding measurement formula is

$$\left\langle \, e^{ik_1x} |x|^\alpha \log^q |x| \operatorname{sgn}^m(x) \, , e^{ik_2x} \eta\!\!\!/_{s}^{(\beta,r)}(x) \operatorname{sgn}^n(x) \, \right\rangle = \delta_{k_1,k_2} \, \delta_{\alpha,\beta} \, \delta_{q,r} \, \delta_{m,n}^{\mathrm{mod}2} \, . \tag{5.94}$$

The functions $e^{ik_1x} \eta\!\!\!/_{\uparrow}^{(\alpha,q)}(x) \perp e^{ik_2x} \eta\!\!\!/_{\uparrow}^{(\alpha,q)}(x)$ are again linearly independent in their k–dependence.
In the finite at $x = 0+$ the function

$$f(x) := e^{ikx} x^\lambda H(x), \tag{5.95}$$

is asymptotic to the formal power series

$$f_a(x; 0+) := \sum_{j=0}^{\infty} \tfrac{1}{j!} (ik)^j x^{\lambda+j} H(x), \tag{5.96}$$

as one sees by expanding the exponential.
Therefore the function $e^{ikx} \eta_{\downarrow}^{(\alpha,q)}(x)$ can be expressed as

$$e^{ikx} \eta_{\downarrow}^{(\alpha,q)}(x) = \sum_{j=0}^{\infty} \tfrac{1}{j!} (ik)^j \, \eta_{\downarrow}^{(\alpha-j,q)}(x), \tag{5.97}$$

in terms of $\eta_{\downarrow}$–functions. The sum is again only formally an infinite sum.

Remark 5.13 The asymptotic (binomial and exponential) series used above are actually convergent,. This is completely irrelevant in the context of generalised functions.

The general case is now clear. The function $e^{ikx}(x-x_0)^\lambda H(x-x_0)$ has the residue

$$\operatorname*{Res}_{\lambda=-\alpha-1}(\lambda+\alpha+1)^q\,(x-x_0)^\lambda\,e^{ikx}H(x-x_0) =$$
$$= e^{ikx_0}\sum_{j=0}^{\infty}\tfrac{1}{j!}(ikx)^j\,\eta_\downarrow^{(\alpha,q+j)}(x-x_0) +$$
$$+\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-)^{j+k}(x_0)^j\tbinom{q+k}{k}\tbinom{-\alpha-1}{j}^{(k)}\,e^{ikx_0}\eta_\uparrow^{(\alpha+j,q+k)}(x). \quad (5.98)$$

The corresponding formulæ for the function $e^{ikx}(x_0-x)^\lambda H(x_0-x)$ and the usual linear combinations are easily found, so it is not necessary to write them out. The translation operators acting on all generalised functions will be defined in Ch. 15.

5.6 Linear combinations

It is important to characterize the allowed linear combination of η–functions. Some examples of allowed and forbidden linear combinations occurred in the previous section.

Remark 5.14 The four families of η–functions depend on a complex parameter, which can take arbitrary complex values. The η–functions at different parameter values have to be considered as linearly independent elements of the space of generalised functions. The space of generalised functions will be very dense in an (as yet undefined) topological sense. This is not really a problem in the limited context of this book, since only finite linear combinations of these elements are allowed as generalised functions. Expressions such as

$$\int d\alpha\, f(\alpha)\,\eta^{(\alpha)}, \quad \text{or} \quad \int dy\; f(y)\,\eta(y), \qquad (5.99)$$

are in general undefined.

Following the examples given above, the space $\mathbf{PC}'_\eta$ of allowed linear combinations of η-functions is defined to contain all linear combinations of the form

$$\sum_{\{x_j\}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{1} c_{jklm}\,\eta_s^{(\alpha_k,l)}(x-x_j)\,\operatorname{sgn}^m(x-x_j), \qquad (5.100)$$

with $\{x_j\}\subset\mathbb{R}$ a finite subset of the reals, $\{\alpha_k\}\subset\mathbb{C}$ a descending sequence of complex numbers in agreement with restriction (4.2), and with arbitrary coefficients $c_{jklm}\in\mathbb{C}$.

Likewise the allowed linear combinations at infinity are of the form

$$\sum_{\{x_j\}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{1} c_{jklm}\, e^{ix_jx}\,\not\eta_s^{(\alpha_k,l)}(x)\,\mathrm{sgn}^m(x), \tag{5.101}$$

with the same requirements as in (5.100) except that $\{\alpha_k\} \subset \mathbb{C}$ now has to be an ascending sequence instead of a descending one. The notation $\mathbf{PC}'_{\not\eta}$ is used for the subspace of $\mathbf{PC}'$ spanned by the elements of (5.101).

Instead of using the definition (5.39) it would be possible to define new η–functions by expressions of the form

$$\operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{1}{q!}(\lambda+\alpha+1)^q\,\tilde{g}(\lambda)|x|^{\lambda}, \tag{5.102}$$

with an additional arbitrary analytic (preferably entire) function $\tilde{g}(\lambda)$. This yields nothing new, the resulting new η–functions are allowed linear combinations of the old η–functions, provided $\tilde{g}(\lambda)$ is meromorphic with poles of order less that q.

Example 5.14 A useful example occurs in Ch. 16, where the scaled η–functions appear by taking $\tilde{g}(\lambda) := a^{\lambda}$. It is also convenient to introduce equivalence classes of η–functions by identifying linear combinations obtained by modifying the analytic function in (5.102), for instance by identifying η–functions with different values of the parameter a. This will be done in Ch. 18.

One easily sees that the scalar products of the allowed linear combinations with an element $\in \mathbf{PC}_{\lambda}$ contains only finitely many non-zero terms, since by (4.2) the intersection between an ascending sequence and a descending sequence of $\{\alpha_k\}$ values is finite. With the restriction (4.1) the allowed linear combinations (5.100) and (5.101) are effectively finite as linear functionals. It will be seen when products have been defined that all expressions of the form

$$\sum \cdots \eta^{\cdots}_{\cdots} := g(x) \bullet \sum_{\text{allowed}} \cdots \eta^{\cdots}_{\cdots}, \tag{5.103}$$

are again allowed linear combinations.

The availability of so many localized generalised functions gives the symmetrical theory of generalised functions a far greater analysing power than distribution theory. This is a general property of symmetrical theories of generalised functions. Enlarged models can be expected to contain many more localized generalised functions than the simple model developed in this tract.

In the following chapter the usual operators will be defined on the part of $\mathbf{PC}'_{\lambda}$ which has now been constructed, and $\mathbf{PC}'_{\lambda}$ will be completed to $\overline{\mathbf{PC}}'_{\lambda}$ to close it under the operators. It will be seen that $\mathbf{PC}'_{\eta}$ and $\mathbf{PC}'_{\not\eta}$ can also be obtained from $\mathbf{PC}'_\lambda$ by imposing closure under differentiation, but the direct definitions of this chapter seem preferable.

A table summarizing the contents of the various $\mathbf{PC}^{\cdots}_{\cdots}$ classes is presented at the beginning of Ch. 9.

CHAPTER 6

OPERATORS ON THE LINEAR FUNCTIONALS

In this chapter the usual operators are defined on the space $\mathbf{PC}'$ of linear functionals on $\mathbf{PC}$, at least on the subspaces which were explicitly constructed in the previous chapter. Since there is an embedding $\mathcal{M} : \mathbf{PC}_\lambda \to \mathbf{PC}'_\lambda$, the preliminary operators can be transferred from $\mathbf{PC}_\lambda$ to $\mathbf{PC}'_\lambda$. The natural definition of the element $\mathcal{O}g' \in \mathbf{PC}'_\lambda$ is derived from the requirement

$$\langle f , \mathcal{O}g' \rangle := \langle f , \mathcal{O}_{\text{pre}}\, g \rangle_{\text{pre}}, \tag{6.1}$$

$\forall f \in \mathbf{PC}_\lambda$. The scalar product is required to be non-degenerate, so (6.1) defines a unique element $\in \mathbf{PC}'_\lambda$.

This does not settle the definition fully. In Sec. 4.3, and Sec. 5.3, it was seen that two equivalent definitions of the preliminary scalar product are available, either by repeated subtractions, or as a residue. It was shown that both methods give the same value to the scalar product. This changes when the operator $\mathcal{O}$ introduces an additional function of the complex variable in the residue.

The definition (6.1) is completed by choosing the analytic method to calculate the residue. This conserves the good analytic properties of the analytic functionals introduced in the previous chapter. An operator $\mathcal{O}$ is therefore defined on the analytic functionals by

$$\langle f , \mathcal{O} \operatorname*{Res}_{\lambda=\cdots} \cdots x^\lambda \rangle := \operatorname*{Res}_{\lambda=\cdots} \cdots \langle f , \mathcal{O}_{\text{pre}}\, x^\lambda \rangle, \tag{6.2}$$

in terms of its preliminary version, by interchanging the operator and the taking of the residue. It is understood that all functions of λ which are added by the action of the operator are expanded in a Laurent series when the residue is computed. Only operators which give meromorphic additional functions of λ are considered in the following. This condition is automatically satisfied for the usual operators, so it is not an actual restriction. It should be clear that analytic functions of λ cannot be taken out of residues. In general

$$\operatorname*{Res}_{\lambda=\alpha} \cdots \tilde{f}(\lambda) \cdots x^\lambda \neq \tilde{f}(\alpha) \operatorname*{Res}_{\lambda=\alpha} \cdots x^\lambda. \tag{6.3}$$

Even when the function $\tilde{f}(\lambda)$ is entire the non-zero terms in its Taylor expansion will give rise to additional η–functions.

When operators are defined by taking residues the unsatisfactory preliminary properties of the preliminary class are converted into the good analytic properties of $\mathbf{PC}'_\lambda$. The subtraction scheme for computing the scalar product would result in properties of $\mathbf{PC}'_\lambda$ which are no better than the preliminary properties of $\mathbf{PC}_\lambda$.

For 'ordinary' functions $f'(x) \in \mathbf{PC}'_\lambda$ the operators in the sense of generalised functions are defined by adding the appropriate localized generalised functions. These are obtained from the known action of the operator on the asymptotic expansions of $f(x) \in \mathbf{PC}_\lambda$.

$$\mathcal{O}\, f'(x) = \mathcal{OM}\, f(x) := \mathcal{MO}_{\mathrm{pre}}\, f(x) + \\ + \mathcal{OM} \sum_{x_0} f_a(x; x_0) - \mathcal{MO}_{\mathrm{pre}} \sum_{x_0} f_a(x; x_0). \quad (6.4)$$

The map of the asymptotic expansion is understood as formally the same asymptotic expansion, with primes added to make the separate terms elements of $\mathbf{PC}'_\lambda$. The sum over x_0 including $x_0 = \pm\infty$ is by definition finite. The operators acting on the asymptotic expansions yield allowed linear combinations of localized generalised functions.

This somewhat roundabout definition is made necessary by the constructive approach followed in this book. The elements of $\mathbf{PC}'_\lambda$ are defined by construction rather than in terms of other objects with known properties. It is for instance unknown a priori if it is possible (or even desirable) to make the operators selfadjoint. (selfadjointness properties will be derived in Ch. 12). It is not possible to define the operators in the same way as in distribution theory. For instance simply postulating

$$\langle\, \mathcal{X} f(x)\,, g(x)\,\rangle = \langle\, f(x)\,, \mathcal{X} g(x)\,\rangle, \quad (6.5)$$

may not yield a suitable theory of generalised functions. This makes it impossible to define operators on $\mathbf{PC}'_\lambda$ by transfer of preliminary operators on $\mathbf{PC}_\lambda$. Conversely the operators (in the sense of generalised function theory) on $\mathbf{PC}_\lambda$ will be found in Ch. 7 from the corresponding operators on $\mathbf{PC}'_\lambda$ by inverting the mapping.

6.1 Multiplication

The simplest case is the $\mathcal{X}$ operator. Its preliminary version is defined on $\mathbf{PC}_\lambda$ by

$$\mathcal{X}_{\mathrm{pre}}\, f(x) := x \cdot f(x) = (x - x_0 + x_0) \cdot f(x), \quad (6.6)$$

so its action on the powers and logarithms is

$$\mathcal{X}_{\mathrm{pre}}|x - x_0|^\alpha \log^q|x - x_0| \operatorname{sgn}(x - x_0) = |x - x_0|^{\alpha+1} \log^q|x - x_0| + \\ + x_0|x - x_0|^\alpha \log^q|x - x_0| \operatorname{sgn}(x - x_0), \quad (6.7)$$

and

$$\mathcal{X}_{\mathrm{pre}}|x - x_0|^\alpha \log^q|x - x_0| = |x - x_0|^{\alpha+1} \log^q|x - x_0| \operatorname{sgn}(x - x_0) + \\ + x_0|x - x_0|^\alpha \log^q|x - x_0|. \quad (6.8)$$

The translation by x_0 merely adds $x_0\mathcal{I}$ to the $\boldsymbol{\mathcal{X}}$ operator. It is not necessary to write the x_0 in all cases, since it can be easily added.

Transferred to the λ-plane the $\boldsymbol{\mathcal{X}}$ operator produces only a translation $\lambda := \lambda + 1$. Since no additional function of λ is introduced by the $\boldsymbol{\mathcal{X}}$ operator, the action of $\boldsymbol{\mathcal{X}}$ can be transferred directly to the powers and logarithms in $\mathbf{PC}'_\lambda$, with the result

$$\boldsymbol{\mathcal{X}}|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) = |x'|^{\alpha+1} \log^q|x| \operatorname{sgn}^{m+1}(x), \tag{6.9}$$

where the notation $\operatorname{sgn}^m(x)$, with $m \in \mathbb{Z}$, has been introduced to combine the odd and even cases.

The action of $\boldsymbol{\mathcal{X}}$ on the η-functions follows from the definitions (6.2) and (6.6) and the analyticity, with the results

$$\boldsymbol{\mathcal{X}}\, \eta_\downarrow^{(\alpha,q)}(x - x_0) = \eta_\downarrow^{(\alpha-1,q)}(x - x_0) + x_0\, \eta_\downarrow^{(\alpha,q)}(x - x_0), \tag{6/10}$$

and

$$\boldsymbol{\mathcal{X}}\, e^{ikx} \eta\!\!\!/_\uparrow^{(\alpha,q)}(x) = e^{ikx} \eta\!\!\!/_\uparrow^{(\alpha-1,q)}(x). \tag{6/11}$$

For the linear combinations $\eta_s^{(\alpha,q)}(x)$ and $\eta_a^{(\alpha,q)}(x)$, the operator $\boldsymbol{\mathcal{X}}$ behaves as

$$\boldsymbol{\mathcal{X}}\, \eta_s^{(\alpha,q)}(x) = \eta_a^{(\alpha-1,q)}(x), \tag{6/12}$$

and

$$\boldsymbol{\mathcal{X}}\, \eta_a^{(\alpha,q)}(x) = \eta_s^{(\alpha-1,q)}(x). \tag{6/13}$$

For the linear combinations $\eta^{(p,q)}$ and $\sigma^{(p,q)}$ the operator $\boldsymbol{\mathcal{X}}$ gives

$$\boldsymbol{\mathcal{X}}\, \eta^{(p,q)}(x) = \eta^{(p-1,q)}(x), \tag{6/14}$$

and

$$\boldsymbol{\mathcal{X}}\, \sigma^{(p,q)}(x) = \sigma^{(p-1,q)}(x). \tag{6/15}$$

The operator $\boldsymbol{\mathcal{X}}$ has odd parity, so it changes the parity of the functions it acts on.

Remark 6.1 The behaviour of the function $\eta(x) = \eta^{(0)}(x)$ under the $\boldsymbol{\mathcal{X}}$ operator

$$\boldsymbol{\mathcal{X}}\, \eta(x) = \eta^{(-1)}(x), \tag{6/16}$$

contrasts with property

$$\boldsymbol{\mathcal{X}}\, \delta(x) = \mathit{0}(x), \tag{6.17}$$

of the Dirac δ-function in the sense of distribution theory, which will be derived for the generalised function $\boldsymbol{\delta}(x)$ in the next chapter. The non-zero result is made possible by the enlarged space of test functions, which contains elements behaving as x^{-1} near $x = 0$.

6.2 Differentiation

The pre-differential operator $\mathcal{D}_{\text{pre}}$ is simply differentiation in the standard sense.

$$\mathcal{D}_{\text{pre}}\, f(x) := \frac{df}{dx}, \tag{6.18}$$

which is well defined between singularities.
Acting on the powers $\in \mathbf{PC}_\lambda$ gives

$$\mathcal{D}_{\text{pre}}(x - x_0)^\lambda H(x - x_0) = \lambda(x - x_0)^{\lambda-1} H(x - x_0). \tag{6.19}$$

From this we find for the derivatives of the powers $\in \mathbf{PC}'_\lambda$, (putting $\lambda = \alpha + \lambda - \alpha$, and invoking the analyticity)

$$\begin{aligned}\mathcal{D}\,(x - x_0)^\alpha \log^q(x - x_0) H'(x - x_0) &= \operatorname*{Res}_{\lambda=\alpha} q!\,(\lambda - \alpha)^{-q-1} \lambda\, x^{\lambda-1} H(x) = \\ &= \alpha(x - x_0)^{\alpha-1} \log^q(x - x_0) H'(x - x_0) + \\ &\quad + q(x - x_0)^{\alpha-1} \log^{q-1}(x - x_0) H'(x - x_0),\end{aligned} \tag{6.20}$$

in agreement with the standard result for $q > 0$.
For the special case $q = 0$, there is no logarithmic term and we obtain

$$\begin{aligned}\mathcal{D}\,(x - x_0)^\alpha H'(x - x_0) &= \alpha(x - x_0)^{\alpha-1} H'(x - x_0) + \\ &\quad + \eta_\downarrow^{(-\alpha)}(x - x_0) - \not\eta_\uparrow^{(-\alpha)}(x - x_0),\end{aligned} \tag{6.21}$$

and in particular

$$\mathcal{D}\, H'(x) = \eta_\downarrow(x) - \not\eta_\uparrow(x). \tag{6.22}$$

Using the Kronecker's δ–symbol (5.55), and taking $x_0 = 0$ for convenience of notation, these can be combined into

$$\begin{aligned}\mathcal{D}\, x^\alpha \log^q(x) H'(x) &= \alpha x^{\alpha-1} \log^q(x) H'(x) + \\ &\quad + q(1 - \delta_{q,0}) x^{\alpha-1} \log^{q-1}(x) H'(x) + \\ &\quad + \delta_{q,0}\big(\eta_\downarrow^{(-\alpha)}(x) - \not\eta_\uparrow^{(-\alpha)}(x)\big),\end{aligned} \tag{6.23}$$

which is now valid for all special values of the parameters.
For the odd and even functions this result can be written as

$$\begin{aligned}\mathcal{D}\,|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) &= \alpha|x'|^{\alpha-1} \log^q|x| \operatorname{sgn}^{m+1}(x) + \\ &\quad + q(1 - \delta_{q,0})|x'|^{\alpha-1} \log^{q-1}|x| \operatorname{sgn}^{m+1}(x) + \\ &\quad + 2\,\delta_{q,0}\big(\eta_s^{(-\alpha)}(x) - \not\eta_s^{(-\alpha)}(x)\big) \operatorname{sgn}^{m+1}(x).\end{aligned} \tag{6.24}$$

It is seen that in contrast with distribution theory, there is (6.21) always a singular term when a power of x is differentiated in the space $\mathbf{PC}'_\lambda$.
In the special case $\alpha = m = p \in \mathbb{N}$, $q = 0$, of the function x'^p we find

$$\mathcal{D}\, x'^p = p \cdot x'^{p-1} + 2\,\sigma^{(-p)}(x) - 2\,\not\sigma^{(-p)}(x), \tag{6.25}$$

which reduces for $p = 0$ to

$$\mathcal{D}\, I'(x) = 2\,\sigma(x) - 2\,\not\sigma(x). \tag{6.26}$$

This shows that the function $I'(x)$ is not the zero element (7.49) for the differential operator. In fact the space $\mathbf{PC}'_\lambda$ does not contain an identity element I which satisfies $\mathcal{D}\, I = 0$. The existence of a preferred point is inherent in functions belonging to $\mathbf{PC}'_\lambda$.

Remark 6.2 It is necessary to give the special point when specifying elements of $\mathbf{PC}'_\lambda$. The function $I'(x - x_0)$ has the derivative

$$\mathcal{D}\, I'(x - x_0) = 2\,\sigma(x - x_0) - 2\,\not\sigma(x - x_0), \tag{6.27}$$

in agreement with (6.25) after specialization to $\alpha = q = m = 0$, so the generalised function $I'(x - x_0)$ is not equal to $I'(x)$.

Example 6.1 In the special case $\alpha = -p$, $m = p + 1$ we obtain

$$\mathcal{D}\, x'^{-p}\,\mathrm{sgn}(x) = -p \cdot x'^{-p-1}\,\mathrm{sgn}(x) + 2\,\eta^{(p)}(x) - 2\,\not\eta^{(p)}(x), \tag{6.28}$$

which reduces in the special case $p = 0$ to

$$\mathcal{D}\,\mathrm{sgn}'(x) = 2\,\eta(x) - 2\,\not\eta(x), \tag{6.29}$$

so in contrast with the situation in distribution theory the function $\mathrm{sgn}'(x)$ has a derivative both in the finite and at infinity.

In terms of the mapping $\mathcal{M} : \mathbf{PC}_\lambda \to \mathbf{PC}'_\lambda$, the difference between the direct transfer and the definition (6.1) appears as

$$\mathcal{M}\mathcal{D}_{\mathrm{pre}}|x|^\alpha = \alpha|x'|^{\alpha-1}\,\mathrm{sgn}(x), \tag{6.30}$$

in contrast with

$$\mathcal{D}\mathcal{M}|x|^\alpha = \alpha|x'|^{\alpha-1}\,\mathrm{sgn}(x) + \eta_a^{(-\alpha)}(x) - \not\eta_a^{(-\alpha)}(x). \tag{6.31}$$

This also corresponds with the difference between the subtraction method and the analytic method for defining the operators.

The derivatives of the η–functions also follow from the analyticity. Substitution of the definition of $\mathcal{D}$ into (5.39) or (5.62) gives

$$\begin{aligned}\mathcal{D}\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = {}& -(\alpha+1)\,\eta_s^{(\alpha+1,q)}(x)\,\mathrm{sgn}^{m+1}(x) + \\ & + (q+1)\,\eta_s^{(\alpha+1,q+1)}(x)\,\mathrm{sgn}^{m+1}(x),\end{aligned} \tag{6/32}$$

both in the finite and at infinity, and

$$\begin{aligned}\mathcal{D}\, e^{ikx}\not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = {}& -(\alpha+1)\,e^{ikx}\not\eta_s^{(\alpha+1,q)}(x)\,\mathrm{sgn}^{m+1}(x) + \\ & - (q+1)\,e^{ikx}\not\eta_s^{(\alpha+1,q+1)}(x))\,\mathrm{sgn}^{m+1}(x) + \\ & + ik\,e^{ikx}\not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^{m+1}(x),\end{aligned} \tag{6/33}$$

at infinity and also in the finite. The difference of the signs in the second term between (6.32) and (6.33) comes from the different normalization (5.39) versus (5.62) of η and $\not\eta$.

Remark 6.3 As indicated by the slashes at the formula numbers (6.32) and (6.33) one of the two formulæ is superfluous.

Example 6.2 For the linear combinations $\eta_\uparrow$ and $\not\eta_\uparrow$ we find that the sense of the arrows is unchanged by differentiation. The derivatives of $\eta_\downarrow$ and $\eta_\uparrow$ are found from (6.32) as

$$\mathcal{D}\,\eta_\downarrow(x) = -\,\eta_\downarrow^{(1)}(x) + \eta_\downarrow^{(1,1)}(x), \tag{6/34}$$

and

$$\mathcal{D}\,\eta_\uparrow(x) = +\,\eta_\uparrow^{(1)}(x) - \eta_\uparrow^{(1,1)}(x), \tag{6/35}$$

in agreement with expectation.

For future reference the repeated derivatives are also computed as residues. For the η–functions the result is

$$\mathcal{D}^p\,\eta_s^{(\alpha,q)}\,\mathrm{sgn}^m(x) = p!\sum_{k=0}^{p}\binom{q+k}{k}\binom{-\alpha-1}{p}^{(k)}\eta_s^{(\alpha+p,q+k)}(x)\,\mathrm{sgn}^{m+p}(x),$$

and (6/36)

$$\mathcal{D}^p\,\not\eta_s^{(\alpha,q)}\,\mathrm{sgn}^m(x) = p!\sum_{k=0}^{p}(-)^k\binom{q+k}{k}\binom{-\alpha-1}{p}^{(k)}\not\eta_s^{(\alpha+p,q+k)}(x)\,\mathrm{sgn}^{m+p}(x), \tag{6/37}$$

with the derived binomial coefficients defined in appendix B. In particular the Stirling numbers of the first kind $(1)_p^{[k]}$ appear in the repeated derivative of the η–function

$$\mathcal{D}^p\,\eta(x) = \sum_{k=0}^{p}(-)^k(1)_p^{[k]}\,\eta^{(p,k)}(x). \tag{6.38}$$

For the powers the same computation gives

$$\begin{aligned}\mathcal{D}^p|x'|^\alpha\log^q|x|\,\mathrm{sgn}^m(x) &= p!\sum_{k=0}^{\min(p,q)}\binom{q}{k}\binom{\alpha}{p}^{(k)}|x'|^{\alpha-p}\log^{q-k}|x|\,\mathrm{sgn}^{m+p}(x) + \\ &+ 2p!\sum_{k=0}^{p-q-1}\frac{(-)^k q!\,k!}{(q+k+1)!}\binom{\alpha}{p}^{(k+q+1)}\times \\ &\times\left((-)^k\eta_s^{(-\alpha+p-1,k)}(x) - \not\eta_s^{(-\alpha+p-1,k)}(x)\right)\mathrm{sgn}^{m+p}(x).\end{aligned} \tag{6.39}$$

The second summation in (6.39) may be empty. In this case it should be omitted. The differentiation yields η–functions when the condition $p > q$ is satisfied.

The subspace $\mathbf{PC}'_\lambda$ is not closed under differentiation, but it is seen from the preceding results that $\mathbf{PC}'_\lambda \oplus \mathbf{PC}'_\eta \oplus \mathbf{PC}'_{\not\eta}$ is closed under differentiation.

It remains to define the differential operator for arbitrary $f'(x) \in \mathbf{PC}'_\lambda$. This is left for Sec. 6.4.

6.3 The Fourier operator

The Fourier operator is defined for good functions $f(x) \in \mathcal{S}$ by

$$\mathcal{F}\, f(x) = \int_{-\infty}^{\infty} dy\; e^{-ixy} f(y), \tag{6.40}$$

following the convention (2.15) used in [Erd2]. The corresponding adjoint operator is then normalized to

$$\mathcal{F}^{\dagger}\, f(x) = \int_{-\infty}^{\infty} dy\; e^{ixy} f(y). \tag{6.41}$$

It will be shown in Ch. 12 that the complex conjugate which appears above is actually also the adjoint operator. On $\mathcal{S} \times \mathcal{S}$ the Fourier operator has all the properties listed in Sec. 2.3.

For the powers of x the preliminary Fourier transform is well-known. The Fourier integral

$$\int_{0}^{\infty} dy\; e^{-ixy} y^{\lambda}, \tag{6.42}$$

can be found [G&S] by substituting $ixy := y$, and rotating the integration contour, which results in

$$\begin{aligned} &\mathcal{F}_{\text{pre}}\, x^{\lambda} H(x) = \\ &\quad = \Gamma(\lambda+1)\big(e^{-i\frac{\pi}{2}(\lambda+1)} x^{-\lambda-1} H(x) + e^{+i\frac{\pi}{2}(\lambda+1)} (-x)^{-\lambda-1} H(-x)\big), \end{aligned} \tag{6.43}$$

where $\Gamma(\lambda+1)$ is the Eulerian Γ–function. In the same manner one finds

$$\begin{aligned} &\mathcal{F}_{\text{pre}}(-x)^{\lambda} H(-x) = \\ &\quad = \Gamma(\lambda+1)\big(e^{+i\frac{\pi}{2}(\lambda+1)} x^{-\lambda-1} H(x) + e^{-i\frac{\pi}{2}(\lambda+1)} (-x)^{-\lambda-1} H(-x)\big), \end{aligned} \tag{6.44}$$

in agreement with the result obtained by substituting $x \to -x$.

Remark 6.4 In the symmetrical theory of generalised functions there is complete symmetry under the Fourier transform. There is no need to distinguish between k–space and x–space, as in some other Fourier theories. The letter x will be used also for the Fourier transformed variable.

It is often more convenient to use the linear combinations

$$\mathcal{F}_{\text{pre}}|x|^{\lambda} = 2\,\Gamma(\lambda+1)\cos\tfrac{\pi}{2}(\lambda+1)\,|x|^{-\lambda-1}, \tag{6.45}$$

and

$$\mathcal{F}_{\text{pre}}|x|^{\lambda}\operatorname{sgn}(x) = 2i\,\Gamma(\lambda+1)\sin\tfrac{\pi}{2}(\lambda+1)\,|x|^{-\lambda-1}\operatorname{sgn}(x), \tag{6.46}$$

which are parity eigenfunctions. The formulæ (6.45) and (6.46) can be combined into

$$\mathcal{F}_{\text{pre}}|x|^{\lambda}\operatorname{sgn}^{m}(x) = -2i^{m}\,\Gamma(\lambda+1)\sin\tfrac{\pi}{2}(\lambda+m)\,|x|^{-\lambda-1}\operatorname{sgn}^{m}(x). \tag{6.47}$$

In this way the separate consideration of the odd and even case is avoided.

Remark 6.5 Even though only the values $m = 0$ and $m = 1$ are needed it turns out to be convenient to keep $m \in \mathbb{Z}$ arbitrary. The function $i^m \sin \frac{\pi}{2}(\lambda + m)$ is periodic in m with period two, as it should be.

The Fourier transforms found above hold in a classical sense for $-1 < \operatorname{Re}\lambda < 0$. Again by analytic continuation with respect to λ, these formulæ can be extended to the entire complex λ-plane, with the exception of the integers, $\lambda = p$, $p \in \mathbb{Z}$, where poles and/or zeroes may be found.

Formal differentiation with respect to λ gives the Fourier transforms with log-functions. It is convenient for this purpose to define the Laurent coefficients

$$c_j(\alpha, m) = \operatorname*{Res}_{\lambda=\alpha} (\lambda - \alpha)^{-j-1}\, \Gamma(\lambda + 1) \sin \tfrac{\pi}{2}(\lambda + m), \tag{6.48}$$

of the function $f(\lambda) := \Gamma(\lambda + 1) \sin \frac{\pi}{2}(\lambda + m)$. The explicit value of these coefficients and some of their properties are derived in appendix C. It is simpler to find the Fourier transform of the logarithms $\in \mathbf{PC}_\lambda$, including the δ-functions which may arise, by computing the Fourier transform first in $\mathbf{PC}'_\lambda$.

For future reference the Fourier transform of a power on a finite interval is also given. It is found in the same way,

$$\mathcal{F}_{\text{pre}}|x|^\lambda H\big(|x| - a\big) = -2i^m\, \gamma(\lambda + 1, ia|x|) \sin \tfrac{\pi}{2}(\lambda + m)\, |x|^{-\lambda-1}, \tag{6.49}$$

where $\gamma(-\lambda - 1, ia|x|)$ is now the incomplete γ-function. [Erd1]. By subtraction of (6.49) from (6.47), or directly by evaluating the integral one also finds

$$\mathcal{F}_{\text{pre}}|x|^\lambda H\big(a - |x|\big) = -2i^m\, \Gamma(\lambda + 1, ia|x|) \sin \tfrac{\pi}{2}(\lambda + m)\, |x|^{-\lambda-1}, \tag{6.50}$$

where $\Gamma(\lambda + 1, ia|x|)$ is the other [Erd1] incomplete Γ-function.

Now that the pre-Fourier operator is known for the powers $x^\lambda \in \mathbf{PC}_\lambda$ the Fourier operator acting on the powers and logarithms $\in \mathbf{PC}'_\lambda$ is found in accordance with (6.1) by computing the appropriate residues

$$\begin{aligned} &\mathcal{F}|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) := && (6.51)\\ &\quad - 2i^m \operatorname*{Res}_{\lambda=\alpha} q!\, (\lambda - \alpha)^{-q-1}\, \Gamma(\lambda + 1) \sin \tfrac{\pi}{2}(\lambda + m)\, |x|^{-\lambda-1} \operatorname{sgn}^m(x). \end{aligned}$$

Expansion of the special functions in a Laurent series at $\lambda = \alpha$, and calculation of the residue gives

$$\begin{aligned} &\mathcal{F}|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) = \\ &\quad - 2i^m q! \sum_{j=0}^{q+1} \tfrac{(-)^j}{j!}\, c_{q-j}(\alpha, m) |x'|^{-\alpha-1} \log^j|x| \operatorname{sgn}^m(x) + \\ &\quad + 4i^m q! \sum_{j=0}^{\infty} j!\, c_{q+j+1}(\alpha, m) \big((-)^j \eta_s^{(\alpha,j)}(x) - \not\eta_s^{(\alpha,j)}(x)\big) \operatorname{sgn}^m(x), \end{aligned} \tag{6.52}$$

with coefficients $c_{...}$ defined in appendix C.

Frequently occurring special cases are the broken powers

$$\mathcal{F}x'^p\,\mathrm{sgn}(x) = 2(-i)^{p+1}p!\,x'^{-p-1} + \\ + 4i^{p+1}\sum_{j=0}^{\infty} j!\,c_{j+1}(p,p+1)\big((-)^j\sigma^{(p,j)}\,x - \not\sigma^{(p,j)}(x)\big), \quad (6.53)$$

with the special case

$$\mathcal{F}\,\mathrm{sgn}'(x) = -2i\,x'^{-1} + 4i\sum_{j=0}^{\infty} j!\,c_{j+1}(0,1)\big((-)^j\sigma^{(0,j)}(x) - \not\sigma^{(0,j)}(x)\big). \quad (6.54)$$

For the integral powers (6.52) specializes to

$$\mathcal{F}\,x'^p = 4i^p\sum_{j=0}^{\infty} j!\; c_{j+1}(p,p)\big((-)^j\eta^{(p,j)}(x) - \not\eta^{(p,j)}(x)\big), \quad (6.55)$$

with the special case

$$\mathcal{F}\,I'(x) = 2\pi\big(\eta(x) - \not\eta(x)\big) + \\ + 4\sum_{j=1}^{\infty} j!\; c_{j+1}(0,0)\big((-)^j\eta^{(0,j)}(x) - \not\eta^{(0,j)}(x)\big), \quad (6.56)$$

where the value of the coefficient $c_1(0,0) = \frac{\pi}{2}$ given in table C.1 has been substituted.

The Fourier transform of the η–functions is found in the same way by computing the appropriate residue

$$\mathcal{F}\big((-)^q\eta_s^{(\alpha,q)}(x) - \not\eta_s^{(\alpha,q)}\big)\,\mathrm{sgn}^m(x) = \quad (6.57) \\ -i^m \operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{(-)^q}{2q!}(\lambda+\alpha+1)^q\,\Gamma(\lambda+1)\sin\tfrac{\pi}{2}(\lambda+m)\,|x|^{-\lambda-1}\,\mathrm{sgn}^m(x).$$

The residue generates again η–functions, unless $\alpha = r = p$, and $q = 0$, where a power reappears. When the Kronecker δ is used to combine the exceptional cases, the evaluation of the residue gives

$$\mathcal{F}\big((-)^q\,\eta_s^{(\alpha,q)}(x) - \not\eta_s^{(\alpha,q)}(x)\big)\,\mathrm{sgn}^m(x) = \\ + 2i^m\sum_{j=-1}^{\infty} \tfrac{(-)^q(q+j)!}{q!}(1-\delta_{q,0}\,\delta_{j,-1})\,c_j(-\alpha-1,m)\times \\ \times\big((-)^{q+j}\eta_s^{(-\alpha-1,q+j)}(x) - \not\eta_s^{(-\alpha-1,q+j)}(x)\big)\,\mathrm{sgn}^m(x) + \\ -i^m\,\delta_{q,0}\,c_{-1}(-\alpha-1,m)|x'|^\alpha\,\mathrm{sgn}^m(x). \quad (6.58)$$

In these expressions the generalised functions in the finite and at infinity occur together. It is possible to separate these by introducing suitable smooth cutoff functions, but it is done more easily by definition. From the asymptotic behaviour of the Fourier transform [Lig] it is known that the Fourier transform of a singularity in the finite will be a singularity at infinity, so the splitting is effected by defining

$$\mathcal{F}(\text{any } \eta) := \text{only } \not\eta\text{–functions,} \tag{6.59}$$

and vice versa

$$\mathcal{F}(\text{any } \not\eta) := \text{only } \eta\text{–functions.} \tag{6.60}$$

This is not possible in the special case $\alpha = p \in \mathbb{N}$, $m = p \pmod 2$, $q = 0$, where we obtain after substituting the explicit form (C.18) of the coefficient $c_{-1}(-p-1, p)$

$$\mathcal{F}\big(\eta^{(p)}(x) - \not\eta^{(p)}(x)\big) = \tfrac{(-i)^p}{p!}\, x'^p + \tag{6/61}$$
$$+\, 2i^p \sum_{j=0}^{\infty} j!\, c_j(-p-1, p)\big((-)^j \eta_s^{(-p-1,j)}(x) - \not\eta_s^{(-p-1,j)}(x)\big)\, \mathrm{sgn}^p(x).$$

No suitable generalised function which can serve as $\mathcal{F}\eta$ or $\mathcal{F}\not\eta$ exists as yet in the model,so it is necessary to create one by definition,

$$\mathcal{F}\eta^{(p)}(x) := \tfrac{(-i)^p}{p!}\big(x'^p + \theta^{(p)}(x)\big) - 2i^p \sum_{j=0}^{\infty} j!\, c_j(-p-1, p)\, \not\phi^{(-p-1,j)}(x),$$

and (6.62)

$$\mathcal{F}\not\eta^{(p)}(x) := \tfrac{(-i)^p}{p!}\,\theta^{(p)}(x) - 2i^p \sum_{j=0}^{\infty} (-)^j j!\, c_j(-p-1, p)\, \sigma^{(-p-1,j)}(x), \tag{6.63}$$

and consequently, after applying $\mathcal{F}$ again and using (C.27) to re-arrange the summations, we find

$$\mathcal{F}\theta^{(p)}(x) = 4i^p \sum_{j=0}^{\infty} j!\, c_{j+1}(p, p)\, \not\eta^{(p,j)}(x). \tag{6.64}$$

Inverting the Fourier transform yields

$$\theta^{(p)}(x) = \tfrac{2}{\pi}(-i)^p \mathcal{F} \sum_{j=0}^{\infty} j!\, c_{j+1}(p, p)\, \not\eta^{(p,j)}(x), \tag{6.65}$$

which gives $\theta^{(p)}(x)$ as an (inverse) Fourier transform.

The translated functions $\theta^{(p)}(x-x_0)$ are defined similarly by

$$\theta^{(p)}(x-x_0) = -4i^p \mathcal{F}^{-1} \sum_{j=0}^{\infty} (-)^j j!\, c_{j+1}(p,p)\, e^{-ixx_0} \not{\eta}^{(p,j)}(x). \tag{6.66}$$

The space $\mathbf{PC}'_{\theta^p}$ of all allowed linear combinations of $\theta^{(p)}$–functions contains all elements of the form

$$\sum_{x_j \in \mathbb{R}} \sum_{k=0}^{\infty} c_{jk}\, \theta^{(k)}(x-x_j), \tag{6.67}$$

with $\{x_j\} \subset \mathbb{R}$ a finite subset, and $c_{jk} \in \mathbb{C}$ arbitrary. This agrees with the restrictions (4.1) imposed on $\mathbf{PC}_\lambda$.
The closure of $\mathbf{PC}'_\lambda$ under the operators can now be defined as

$$\overline{\mathbf{PC}}'_\lambda := \mathbf{PC}'_\lambda \oplus \mathbf{PC}'_\eta \oplus \mathbf{PC}'_{\not\eta} \oplus \mathbf{PC}'_{\theta^p}. \tag{6.68}$$

The space $\overline{\mathbf{PC}}'_\lambda$ defined above contains all elements from the undefined space $\mathbf{PC}'$ that will be included in the simple model for a symmetrical theory of generalised functions. The definition of the functions $\theta^{(p)}(x-x_0)$ as linear functionals on $\mathbf{PC}_\lambda$ will be postponed until the next chapter, when $\mathbf{PC}_\lambda$ has been completed to $\overline{\mathbf{PC}}_\lambda$.

Remark 6.6 This is an example where the non-degeneracy of the scalar product (2.11) is used as a heuristic principle. The scalar product of the generalised functions $\theta^{(p)}(x-x_0)$ with all elements $f(x) \in \mathbf{PC}_\lambda$ would be zero, but the $\theta^{(p)}(x-x_0)$ are necessary, so $\mathbf{PC}_\lambda$ must be enlarged.

The general form of the action of the Fourier operator acting on the η–functions can now be written as

$$\begin{aligned} \mathcal{F}\, \eta_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x) = -2i^m \sum_{j=-1}^{\infty} & \tfrac{(q+j)!}{q!} (1-\delta_{q,0}\,\delta_{j,-1}) \times \\ & \times c_j(-\alpha-1,m)\, \not{\eta}_s^{(-\alpha-1,q+j)}(x)\, \mathrm{sgn}^m(x) + \\ & + \delta_{q,0} \sum_{p=0}^{\infty} \delta_{\alpha,p}\, \delta_{p,m}^{\mathrm{mod}2}\, \tfrac{(-i)^p}{p!} \left(x' + \theta^{(p)}(x)\right), \end{aligned} \tag{6.69}$$

and

$$\begin{aligned} \mathcal{F}\, \not{\eta}_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x) = -2i^m \sum_{j=-1}^{\infty} & \tfrac{(-)^j (q+j)!}{q!} (1-\delta_{q,0}\,\delta_{j,-1}) \times \\ & \times c_j(-\alpha-1,m)\, \eta_s^{(-\alpha-1,q+j)}(x)\, \mathrm{sgn}^m(x) + \\ & + \delta_{q,0} \sum_{p=0}^{\infty} \delta_{\alpha,p}\, \delta_{p,m}^{\mathrm{mod}2}\, \tfrac{(-i)^p}{p!}\, \theta^{(p)}(x), \end{aligned} \tag{6.70}$$

valid for arbitrary values of the parameters. The formal summations over p come from the substitution of the explicit form (C.18) of the $c_{-1}(-p-1,p)$ coefficient.

In all equations up to this point, such as (6.70) or (6.69) one may verify that it is possible to differentiate formally with respect to complex parameters such as α. This is of course not accidental. Since differentiation with respect to α and the operators act on different pieces of equation (6.1), we have in general

$$\frac{\partial}{\partial\alpha}\,\mathcal{O}\ldots^{\cdots\alpha\cdots} = \mathcal{O}\,\frac{\partial}{\partial\alpha}\ldots^{\cdots\alpha\cdots}, \tag{6.71}$$

for all elements in $\overline{\mathbf{PC}}'_\lambda$. The validity of this property is confined to the subspace $\overline{\mathbf{PC}}'_\lambda$. It will not carry over to the whole model.

The action of the operators $\mathcal{X}$ and $\mathcal{D}$ on the function $\theta^{(p)}(x)$ follows from the known action of these operators on the function $\not{\eta}^{(p)}(x)$, and from the requirements (2.22–23) of unitary equivalence,

$$\mathcal{X}\,\theta^{(p)}(x) := i\mathcal{F}^{-1}\mathcal{D}\mathcal{F}\,\theta^{(p)}(x), \tag{6.72}$$

and

$$\mathcal{D}\,\theta^{(p)}(x) := i\mathcal{F}^{-1}\mathcal{X}\mathcal{F}\,\theta^{(p)}(x). \tag{6.73}$$

Evaluation of the operators gives

$$\mathcal{X}\,\theta^{(p)}(x) = \theta^{(p+1)}(x), \tag{6.74}$$

and

$$\mathcal{D}\,\theta^{(p+1)}(x) = (p+1)\,\theta^{(p)}(x) - 2\,\sigma^{(-p-1)}(x), \tag{6.75}$$

and especially

$$\mathcal{D}\,\theta^{(0)}(x) = \mathcal{D}\,\theta(x) = -2\,\sigma(x). \tag{6.76}$$

Heuristically the function $\theta^{(p)}(x)$ can be considered to be the function

$$f(x) := x^p, \tag{6.77}$$

restricted to an infinitesimal surrounding of the point $x = 0$. This is made plausible by noting that

$$\mathcal{D}\big(x'^{p+1} + \theta^{(p+1)}(x)\big) = (p+1)\big(x'^p + \theta^{(p)}(x)\big) - 2\,\not{\sigma}^{(-p-1)}(x), \tag{6.78}$$

which shows that the 'gap' in the function x'^p in the finite at $x = 0$, which caused a σ–function upon differentiation, (6.25), has been 'bridged' by the addition of the $\theta^{(p)}(x)$–function. The definition of the support of elements such as $\theta(x)$ will be postponed until Ch. 13.

The normalization of the θ–functions has been chosen with this interpretation in mind. In particular one can think of the function $\theta(x) = \theta^{(0)}(x)$ as the generalised function version of the ordinary function

$$\theta_{ord}(x) := \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases} \tag{6.79}$$

This function cannot be non-zero in distribution theory, since it is equivalent to zero in the Lebesgue measure. As a distribution it is therefore equivalent to the zero distribution.

Anticipating the results of Ch. 19, the limit properties of $\theta(x)$ are also in agreement with this interpretation. The sequence $\exp(-nx^2)$ will be shown to go to $\theta(x)$ for $n \to \infty$, in the sense defined in Ch. 19.

Remark 6.7 Even after completing $\mathbf{PC}'_\lambda$ to $\overline{\mathbf{PC}}'_\lambda$ it still does not contain by (6.78) an zero element satisfying (2.25), since the derivative at infinity of $I'(x) + \theta(x)$ remains non-zero.

Remark 6.8 The operators $\mathcal{X}$ and $\mathcal{D}$ are invertible on $\mathbf{PC}'_\lambda$. This is to be expected since there is no zero element. The space $\overline{\mathbf{PC}}'_\lambda$ is not closed under $\mathcal{X}^{-1}$ and $\mathcal{D}^{-1}$ however. The exceptions are the expressions

$$\mathcal{X}^{-1}\,\theta(x) \quad \text{and} \quad \mathcal{D}^{-1}\,\not\eta(x), \tag{6.80}$$

which will have a meaning in the full model $\mathbf{GF}_s$, but not in the restricted subspace $\mathbf{PC}'_\lambda$. The inverse operators will be defined for all generalised functions in Ch. 14.

6.4 Operators on ordinary functions

The operators are now extended by means of the definition (6.4) to the 'ordinary' functions $f'(x) \in \mathbf{PC}'_\lambda$.

The extension of the multiplication operator $\mathcal{X}$ to ordinary functions is obvious. No additional η–functions arise in this case, since none are produced when the operator $\mathcal{X}$ acts on the asymptotic expansions

$$\mathcal{X}\,f'(x) = \mathcal{X}\mathcal{M}\,f(x) = \mathcal{M}\mathcal{X}\,f(x). \tag{6.81}$$

By (6.32) and (6.32) the operator $\mathcal{D}$ adds η–functions only when $f(x)$ behave as $|x|^\lambda \operatorname{sgn}^m(x)$ in the finite, or as $e^{ikx}|x|^\lambda \operatorname{sgn}^m(x)$ at infinity. When the asymptotic coefficients are measured with the appropriate η and $\not\eta$–functions the derivative can be written as

$$\begin{aligned}
\mathcal{D}\,f'(x) = \mathcal{D}\mathcal{M}\,f(x) = \mathcal{M}\mathcal{D}\,f(x) &+ \\
+ \sum_{x_0\in\mathbb{R}}\sum_{\lambda\in\mathbb{C}}\sum_{m} \langle\, f(x)\,, \eta_s^{(\lambda^*,0)}(x-x_0)\operatorname{sgn}^m(x)\,\rangle\, &\eta_s^{(-\lambda,0)}(x-x_0)\operatorname{sgn}^{m+1}(x) \\
+ \sum_{k\in\mathbb{R}}\sum_{\lambda\in\mathbb{C}}\sum_{m} \langle\, f(x)\,, e^{ikx}\not\eta_s^{(\lambda^*,0)}(x)\operatorname{sgn}^m(x)\,\rangle\, &\times \\
\times\, e^{ikx}\big(\eta_s^{(-\lambda,0)}(x) + ik\,\not\eta_s^{(-\lambda-1,0)}(x)\big)&\operatorname{sgn}^{m+1}(x).
\end{aligned} \tag{6.82}$$

This is again a formally infinite sum over η–functions. Since $f \in \mathbf{PC}_\lambda$ has been restricted (4.1) to a finite number of poles in a negative real half-plane in the finite, the contribution from a finite point is a descending sequence of η–functions. Likewise the contributions from infinity are ascending sequences of $\not\eta$–functions. In both cases only a finite number of terms in the formally infinite sums will contribute.

Remark 6.9 The form of the asymptotic expansion at infinity, and consequently the form of the resulting sum of $\not\eta$–functions, will depend on the choice of origin. The value of a scalar product involving such a sum does not depend on this choice.

To summarize: The derivative of an ordinary function, $f(x)$, considered as an element $f'(x) \in \mathbf{PC}'_\lambda$, is the ordinary derivative plus the η–functions resulting from the differentiation of its asymptotic expansion near the singularities.

The embedding $\mathcal{M} : \mathbf{PC}_\lambda \to \mathbf{PC}'_\lambda$ can be modified at points $x_0 \in \mathbb{R}$ where $f(x) \in \mathbf{PC}_\lambda$ is $\mathbf{C}^\infty(x_0)$. There $f(x) \in \mathbf{PC}_\lambda$ is asymptotic to a Taylor series

$$f(x) \sim \sum_{j=0}^{\infty} c_j (x - x_0)^j. \tag{6.83}$$

The map $\mathcal{M}$ can be changed by adding the appropriate $\theta^{(p)}$–functions

$$f'(x) := \mathcal{M}_{new} f(x) := \mathcal{M}_{old} f(x) + \sum_{j=0}^{\infty} c_j\, \theta^{(j)}(x - x_0), \tag{6.84}$$

to the image $\mathcal{M}_{old}\, f(x)$. This makes $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ also $\mathbf{C}^\infty$, without having additional σ^{-p-1}–functions upon differentiation. In this way $\mathbf{C}^\infty$–points and singular points are treated on the same footing. The map $\mathcal{M}_{new}$ is a map from $\mathbf{PC}_\lambda$ to $\overline{\mathbf{PC}}'_\lambda$. Under the map $\mathcal{M}_{new}$ the singular points remain the same. No new singular points in $\overline{\mathbf{PC}}'_\lambda$ as in Ex. 6.1 are introduced.

Remark 6.10 This modification does not alter the mapping $\mathcal{M}$ essentially. In the following the notation $\mathcal{M}_\mathcal{X}$ will be used for the completed mapping $\mathcal{M}_{new}$ for reasons which will become clear in Ch. 9. Conversely one can also think of (6.84) as the natural extension of the $\mathbf{C}^\infty$–concept to $\mathbf{PC}'_\lambda$. By the usual restrictions only a finite number of points $x_0 \in \mathbb{R}$ with singular powers $(x' - x_0)^p$ are allowed.

Adding the derivatives of the θ–functions in (6.84) to (6.82) defines the differential operator for arbitrary elements $\in \overline{\mathbf{PC}}'_\lambda$.

The Fourier operator can be extended to all $f'(x) \in \overline{\mathbf{PC}}'_\lambda$, in accordance with (6.4). The result is

$$\begin{aligned}
\mathcal{F} f'(x) &= \mathcal{F}\mathcal{M}\, f(x) = \mathcal{M}\mathcal{F}_{\text{pre}}\, f(x) + \\
&+ \sum_{\{x_0 \in \mathbb{R}\}} \sum_{\{\lambda_j \in \mathbb{C}\}} \sum_{q=0}^{\infty} \sum_{m} \langle\, f(x)\,, \not{\eta}_s^{(\lambda^*,q)}(x)\,\mathrm{sgn}^m(x)\,\rangle \times \\
&\quad \times +4i^m q! \sum_{j=0}^{\infty} (-)^j j!\, c_{q+j+1}(\alpha, m)\, \eta_s^{(\alpha,j)}(x)\,\mathrm{sgn}^m(x) + \\
&+ \sum_{\{x_0 \in \mathbb{R}\}} \sum_{\{\lambda_j \in \mathbb{C}\}} \sum_{q=0}^{\infty} \sum_{m} \langle\, f(x)\,, e^{ix_0x} \not{\eta}_s^{(\lambda^*,q)}(x)\,\mathrm{sgn}^m(x)\,\rangle \times \\
&\quad \times -4i^m q! \sum_{j=0}^{\infty} j!\, c_{q+j+1}(\alpha, m)\, \eta_s^{(\alpha,j)}(x - x_0)\,\mathrm{sgn}^m(x - x_0),
\end{aligned} \tag{6.85}$$

which again displays the relevant asymptotic behaviour. Despite its awful appearance when written out (6.85) still represents an allowed linear combination.

Example 6.3 A useful example, which will be used for the verification of Parseval's equality;; in Ch. 12, is the Fourier transform of the damped power

$$\mathcal{F}_{\text{pre}}(\pm x)^\lambda\, e^{\mp ax} H(\pm x) = \Gamma(\lambda + 1)(\pm ix + a)^{-\lambda-1}, \tag{6.86}$$

which can be found in the same way as (6.43). The corresponding Fourier transform in $\overline{\mathbf{PC}}'_\lambda$ is found either from (6.86) or by taking residues. The result is (again for parity eigenfunctions)

$$\begin{aligned}
&\mathcal{F}|x'|^\alpha\, e^{-a|x|}\mathrm{sgn}^m(x) = \\
&\quad = -\Gamma^{[-1]}(\alpha + 1)\big((ix + a)^{-\alpha-1)} \log(ix + a) + (-)^m(i \to -i)\big)\, I'(x) + \\
&\quad + \Gamma^{[0]}(\alpha + 1)\big((ix + a)^{-\alpha-1} + (-)^m(i \to -i)\big)\, I'(x) + \\
&\quad - 4i^m \sum_{j=0}^{\infty} \tfrac{1}{j!}(-a)^j \sum_{k=0}^{\infty} k!\; c_{k+1}(\alpha + j, m)\, \not{\eta}_s^{(-a+j,k)}(x)\,\mathrm{sgn}^m(x),
\end{aligned} \tag{6.87}$$

where the binomial theorem can be used to find the asymptotic expansion of the powers of $(ix + a)$ at infinity. The discontinuities at $x = 0$ transform into $\not{\eta}$-functions under the Fourier transform. The Fourier transform does not contain η-functions in agreement with the smooth behaviour of $f(x)$ at infinity.

This lengthy exposition has served its purpose. The usual operators, and the operator algebra generated by them, is defined on $\overline{\mathbf{PC}}'_\lambda$, and $\overline{\mathbf{PC}}'_\lambda$ is closed under the operators.

Now the operators can be transferred to $\mathbf{PC}_\lambda$, and $\mathbf{PC}_\lambda$ can be closed under the operators to $\overline{\mathbf{PC}}_\lambda$, while keeping most of the good properties of the operators on $\overline{\mathbf{PC}}'_\lambda$. This will be done in the next chapter.

CHAPTER 7

OPERATORS ON THE COMPLETED PRELIMINARY CLASS

It became clear on several occasions in the preceding chapters Chs. 4–6 that the preliminary class $\mathbf{PC}_\lambda$ of ordinary functions is too small. The function $\theta(x) \in \overline{\mathbf{PC}}'_\lambda$ lacks a suitable test function $\in \mathbf{PC}_\lambda$ to distinguish it from zero. The preliminary differentiation of a step function gives zero, and the preliminary Fourier transform of the function $I(x)$ is also zero.

The space $\overline{\mathbf{PC}}'_\lambda$, with the operators defined in Ch. 6 does not have these shortcomings. It can therefore be used to define the operators in the sense of generalised functions on $\mathbf{PC}_\lambda$. The simplest idea would be to transfer the operators from $\overline{\mathbf{PC}}'_\lambda$ to $\mathbf{PC}_\lambda$ by means of the natural map $\mathcal{M}$ defined by (5.2) and its inverse $\mathcal{M}^{-1}$,

$$\mathcal{O}\, f(x) := \mathcal{M}^{-1}\mathcal{O}\mathcal{M}\, f(x), \tag{7.1}$$

for all $f(x) \in \mathbf{PC}_\lambda$. In this simple form this does not work, since $\mathbf{PC}_\lambda$ is too small. In Sec. 7.1 the space $\mathbf{PC}_\lambda$ is first completed to $\overline{\mathbf{PC}}_\lambda$, and the map $\mathcal{M}^{-1} : \mathbf{PC}'_\lambda \to \mathbf{PC}_\lambda$ is completed to $\overline{\mathcal{M}}^{-1} : \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}_\lambda$ in a minimal way, adding as little to $\mathbf{PC}_\lambda$ as possible. The map $\mathcal{M}$ is also completed to $\overline{\mathcal{M}}$.
The operators are then defined on $\overline{\mathbf{PC}}_\lambda$ by analogy with (7.1) by

$$\mathcal{O}\, f(x) := \overline{\mathcal{M}}^{-1}\mathcal{O}\overline{\mathcal{M}}\;\, f(x), \tag{7.2}$$

for all $f(x) \in \overline{\mathbf{PC}}_\lambda$. Since $\overline{\mathcal{M}}\; f(x)$ is an element $\in \overline{\mathbf{PC}}'_\lambda$, the expression $\mathcal{O}\overline{\mathcal{M}}\; f(x)$ has been defined as an element of $\overline{\mathbf{PC}}'_\lambda$ in the previous chapter. When $\mathbf{PC}_\lambda$ has been completed in such a way that the expression $\overline{\mathcal{M}}^{-1}\overline{\mathbf{PC}}'_\lambda$ is always defined as an element of $\overline{\mathbf{PC}}_\lambda$, the previous equation (7.2) defines the action of the operators on $\overline{\mathbf{PC}}_\lambda$.

7.1 Completion of the preliminary class

Some freedom exists in the choice of the completion $\overline{\mathbf{PC}}_\lambda$ of $\mathbf{PC}_\lambda$, and consequently in the construction of the inverse map $\overline{\mathcal{M}}^{-1}$. This freedom is restricted by taking the minimal extension of $\mathbf{PC}_\lambda$ which completes $\mathbf{PC}_\lambda$ to $\overline{\mathbf{PC}}_\lambda$ under the usual operators.

This minimal extension is obtained by giving a zero inverse image $\in \overline{\mathbf{PC}}_\lambda$ to as many elements of $\overline{\mathbf{PC}}'_\lambda$ with point support as possible. This is not always possible. The operators can convert an element $\in \mathbf{PC}'_\lambda$ with support which is larger than a point into an element with point support. In the preceding chapter it was found that this happens in the cases

$$\mathcal{D}\, I'(x) = 2\,\sigma(x) - 2\,\not\sigma(x), \tag{7.3}$$

and

$$\mathcal{D}\,\mathrm{sgn}'(x) = 2\,\eta(x) - 2\,\not\eta(x), \tag{7.4}$$

and

$$\mathcal{F}\big(I'(x) + \theta(x)\big) = 2\pi\,\eta(x) + \sum_{j=1} \cdots \eta^{(0,j)}(x) \cdots, \tag{7.5}$$

by equations (6.26), (6.29), and (6.56). The converse happens in the case

$$\mathcal{F}\,\eta^{(p)}(x) = x'^{p} + \theta^{(p)}(x) + \sum_{j=0}^{\infty} \cdots \not\phi^{(-p-1,j)}(x), \tag{7.6}$$

which is the only case in which an element with point support acquires a support greater than a point.

The collection of elements on the right-hand side of (7.3)–(7.6) is still too large, since it will be further enlarged by the action of the operators. Two considerations can be used to restrict the completion of $\mathbf{PC}_\lambda$.

The first is the requirement (7.49) that there should be a zero element for differentiation, the unit function $I(x) \in \mathbf{GF}$, with zero derivative. Since such an element is not present in $\overline{\mathbf{PC}}'_\lambda$, it must be in $\overline{\mathbf{PC}}_\lambda$. This leads to the choice

$$\mathcal{M}^{-1}\, I'(x) = I(x), \tag{7.7}$$

and consequently one has to make the choice

$$\mathcal{M}^{-1}\,\sigma(x) = \mathcal{M}^{-1}\,\not\phi(x) = 0(x), \tag{7.8}$$

in order to obtain a unit element with a zero derivative.

A second consideration is that the element $\theta(x) \in \overline{\mathbf{PC}}'_\lambda$ lacks a suitable test function $f(x) \in \mathbf{PC}_\lambda$ to distinguish it from zero. By the requirement of non-degeneracy of the scalar product (2.11), an element $\in \mathbf{GF}$ is non-zero if and only if it has a non-zero scalar product with at least one other generalised function. Since $\theta(x)$ is localized at $x = 0$, its test function should also be localized at the origin. This leads to the definition of the element $\not\delta(x) \in \overline{\mathbf{PC}}_\lambda$ by

$$\not\delta(x) := \bar{\mathcal{M}}^{-1}\,\eta(x), \tag{7.9}$$

with a corresponding extension of the map $\mathcal{M}^{-1}$ to $\bar{\mathcal{M}}^{-1}$.

Remark 7.1 The preceding definition is a purely formal one. All properties of the object $\not\delta(x)$ follow from definition (7.9), and the subsequent use of $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}^{-1}$ to define operators and products. The same was true for the definition (7.7), but there it was not immediately clear, since the object $I(x)$ appears to be well understood.

Remark 7.2 In contrast with the usage in distribution theory 'test function' is used to indicate a generalised function which has non-zero scalar product with the given function, and zero scalar product with all other generalised functions. Of course, by the requirement of symmetry, all generalised functions are test functions if the term 'test function' is used in the sense of distribution theory. The generalised function $\delta(x)$ tests for the presence of the function $\theta(x)$ and visa-versa.

More generally $\mathbf{PC}_\lambda$ is completed to $\overline{\mathbf{PC}}_\lambda$ by adding the (finite linear combinations of the) elements $\delta^{(p)}(x - x_0) \in \overline{\mathbf{PC}}_\lambda$, defined by

$$\delta^{(p)}(x - x_0) := \bar{\mathcal{M}}^{-1}\, \eta^{(p)}(x - x_0). \tag{7.10}$$

It will be demonstrated that these additional elements are sufficient for achieving operator completion with the good properties required in Ch. 2. One sees that the minimal completion is obtained by giving non-zero inverse image only to those elements with point support $\in \overline{\mathbf{PC}}'_\lambda$, which can be converted by an operator into an element with support larger than a point. Only the $\eta^{(p)}$–functions appearing in (7.6) meet this criterion. It will be seen that the definition (7.10) leads to the results

$$\mathcal{D}\, I(x) = 0(x) \quad \text{and} \quad \mathcal{X}\, \delta(x) = 0(x), \tag{7.11}$$

in agreement with the unitary equivalence of the $\mathcal{X}$ and $\mathcal{D}$ operators.

Remark 7.3 The product will be defined in accordance with the requirement (2.36) in such a way that it is pointwise. Therefore it does not make further extension of $\overline{\mathbf{PC}}_\lambda$ necessary. The convolution product likewise has the property that the convolution of elements with point support has again a point support.

The general form of the inverse map $\bar{\mathcal{M}}^{-1}$ acting on the η–functions is defined in agreement with (7.10) and (7.8) as

$$\bar{\mathcal{M}}^{-1}\, \eta_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x) = \delta_{q,0} \sum_{p=0}^{\infty} \delta_{\alpha,p}\, \delta_{p,m}^{\mathrm{mod}2}\, \delta^{(p)}(x), \tag{7.12}$$

and

$$\bar{\mathcal{M}}^{-1}\, \not{\eta}_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x) = 0(x), \tag{7.13}$$

$\forall \alpha \in \mathbb{C}$, $\forall m \in \mathbb{Z}$, $\forall q \in \mathbb{N}$.

Remark 7.4 The definitions (7.13) and (7.12) have the special cases

$$\bar{\mathcal{M}}^{-1}\, \eta^{(p)}(x) = \delta^{(p)}(x), \tag{7.14}$$

and

$$\bar{\mathcal{M}}^{-1}\, \sigma^{(p)} = 0(x), \tag{7.15}$$

Combining these gives for the one-sided η–functions

$$\bar{\mathcal{M}}^{-1}\,\eta_{\downarrow}^{(p)}(x) = \delta^{(p)}(x), \tag{7.16}$$

but

$$\bar{\mathcal{M}}^{-1}\,\eta_{\uparrow}^{(p)}(x) = (-)^p\delta^{(p)}(x), \tag{7.17}$$

as it should, since the $\eta_{\downarrow}$ and $\eta_{\uparrow}$–functions are normalized to detect $|x|^p$, while $\delta^{(p)}(x)$ detects x^p. This explains the minus signs.

The complex conjugates δ^* of the δ-functions are defined to be equal to themselves. All δ–functions are real, as they are in distribution theory.

One may notice at this point that formal differentiation with respect to parameters such as α does not commute with the inverse mapping.

Example 7.1 By (5.83–85) we have

$$\frac{\partial}{\partial\alpha}\,\bar{\mathcal{M}}^{-1}\,\eta^{(\alpha,1)}(x) = \bar{\mathcal{M}}^{-1}\,0(x) = 0(x), \tag{7.18}$$

but inverting the order gives

$$\bar{\mathcal{M}}^{-1}\,\frac{\partial}{\partial\alpha}\,\eta^{(\alpha,1)}(x) = \bar{\mathcal{M}}^{-1}\,\eta^{(\alpha)}(x) = \sum_{p=0}^{\infty}\delta_{\alpha,p}\,\delta^{(p)}(x) \neq 0(x), \tag{7.19}$$

for $\alpha = p \in \mathbb{N}$.

Therefore one expects that in general differentiation with respect to α will not commute with the operators. In special cases however $\frac{\partial}{\partial\alpha}$ may commute with an operator. This will be seen when the action of the operators has been worked out.

The space $\mathbf{PC}_\delta$ of allowed linear combinations of δ–functions is defined to be the image under $\bar{\mathcal{M}}^{-1}$ of the allowed linear combinations of η–functions,

$$\mathbf{PC}_\delta := \bar{\mathcal{M}}^{-1}\big(\mathbf{PC}'_\eta \oplus \mathbf{PC}'_{\not\eta}\big) = \mathcal{M}^{-1}\mathbf{PC}'_\eta, \tag{7.20}$$

so the space $\mathbf{PC}_\delta$ contains the linear combinations of the form

$$\sum_{\{x_j\}}\sum_{k=0}^{K_j} c_{jk}\,\delta^{(k)}(x - x_j), \tag{7.21}$$

with $\{x_j\} \subset \mathbb{R}$ a finite subset of the real numbers, and with coefficients $c_{jk} \in \mathbb{C}$ arbitrary.

In contrast with the linear combinations of the η–functions the allowed linear combinations of δ–functions are explicitly finite. This follows immediately since a descending $\{\alpha_k\}$ sequence can contain only a finite number of positive integers.

The completion $\overline{\mathbf{PC}}_\lambda$ of $\mathbf{PC}_\lambda$ is now defined as

$$\overline{\mathbf{PC}}_\lambda := \mathbf{PC}_\lambda \oplus \mathbf{PC}_{\bar{\delta}}. \tag{7.22}$$

It will be seen in the following sections that $\overline{\mathbf{PC}}_\lambda$ is indeed closed under the operators.

The 'delta-bar' $:= \bar{\delta}$ symbol has been introduced to avoid the inconvenient normalization of the customary δ–function. It also helps to avoid confusion between the distribution δ and the generalised function $\bar{\delta}(x)$. The properties of the generalised function $\bar{\delta}(x)$ will be derived from its definition (7.10) in the following sections. Apart from the normalization it has many properties in common with Dirac's δ–function, and with the distribution δ. Some properties of the δ–function are taken over by the η–function however.

Remark 7.5 In the following a distinction between distribution theory and the symmetrical theory of generalised functions will be made by using 'distribution' exclusively for a distribution in the sense of the theory of distributions [Schl] of Schwartz. 'Generalised function' or simply 'function' (as in $\bar{\delta}$–function) will be reserved for an object in the sense of this book. It is unfortunate that 'generalised function' is such an over-used term, denoting with some authors not only distributions, but also all kinds of other generalizations of the standard function concept. 'Symmetrical generalised functions' will be used when it is necessary to emphasize the distinction.

At this point it might be asked if it would have been easier to introduce $\overline{\mathbf{PC}}_\lambda$ from the start as a subclass of the tempered distributions. This would make it closed under the usual operators from the start. There are two reasons for not doing this.

In the first place it will be necessary to keep in mind a clear distinction between the generalised functions $\bar{\delta}(x)$ and $\eta(x)$ on one hand, and the distribution δ on the other hand. Generalised functions in the sense of this book and distributions are logically independent explanations of Dirac's δ–function. Moreover the properties (2.1–4) postulated by Dirac for the δ–function are shared in generalised function theory by the $\bar{\delta}$ and η–function in ways which will become clear in the following chapters. A comparison will be made in Ch. 23 where the symmetrical theory of generalised functions is compared with distribution theory.

A second reason for not starting with distributions is that the inverse mapping and the completion of $\mathbf{PC}_\lambda$ to $\overline{\mathbf{PC}}_\lambda$ which has been chosen is the minimal completion. It does not seem to be necessary to make this minimal choice. Other choices for the completion would have less resemblance to distribution theory.

Remark 7.6 Since the Schwartz space $\mathcal{S}$ of test functions is contained as a subspace both in $\overline{\mathbf{PC}}_\lambda$ and in $\overline{\mathbf{PC}}'_\lambda$, all symmetrical generalised functions are also (tempered) distributions. Many non-zero generalised functions correspond to the zero distribution for lack of a suitable test function $\in \mathcal{S}$ to distinguish them from zero.

Example 7.2 The generalised function $\eta^{(\alpha,q)}(x-x_0)$ is for $\alpha \neq p \in \mathbb{N}$, or $q > 0$, equivalent to the zero distribution. It has a zero scalar product with the good functions $f(x) \in \mathcal{S}$. The same holds $\forall \alpha \in \mathbb{C}$ for the generalised functions $\not\eta^{(\alpha,q)}(x)$, since good functions disappear at infinity faster than any power.

For the ordinary functions the inverse map was already defined. In particular for the powers we have

$$\overline{\mathcal{M}}^{-1}|x'|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x) = |x|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x). \tag{7.23}$$

This has the special case

$$\overline{\mathcal{M}}^{-1}x'^{p} = x^{p}, \tag{7.24}$$

so we can take

$$\overline{\mathcal{M}}^{-1}\,\theta^{(p)}(x) = \mathit{0}(x), \tag{7.25}$$

since the element $x^p \in \mathbf{PC}_\lambda$, which is $\mathbf{C}^\infty(-\infty,\infty)$, already has local properties at the point $x=0$ corresponding to the inclusion of the function $\theta^{(p)}(x)$ at the origin.

The completion chosen in this section is minimal in the sense that the only cases where a localized generalised function from $\overline{\mathbf{PC}}'_\lambda$ has a non-zero counterpart in $\overline{\mathbf{PC}}_\lambda$ is the unavoidable case (7.6) where an operator converts an element with point support into an element with support greater than a point.

Remark 7.7 It would be possible to take a maximal completion of $\mathbf{PC}_\lambda$ by giving a non-zero counterpart $\in \overline{\mathbf{PC}}_\lambda$ to every element $\in \overline{\mathbf{PC}}'_\lambda$, but this would leave us without elements δ and I satisfying $\mathcal{D}\,I = \mathit{0} = \mathcal{X}\,\delta$, so no improvement with respect to staying in $\overline{\mathbf{PC}}'_\lambda$ would result. Since the completion should not be maximal we may as well take the minimal completion.

7.2 The remaining scalar products

In this section the remaining undefined scalar products are computed by imposing Parseval's equality (2.29). The scalar product of the $\theta(x)$–function

$$\langle\, \delta(x)\,, \theta(x)\,\rangle = \langle\, I(x)\,, \not\eta(x)\,\rangle = 1, \tag{7.26}$$

is in agreement with the naïve interpretation of the function $\theta(x)$.
More generally we find for translated arguments

$$\langle\, \delta^{(p)}(x-x_1)\,, \theta^{(q)}(x-x_2)\,\rangle = \delta_{x_1,x_2}\,\delta_{p,q}\,. \tag{7.27}$$

It also follows from Parseval's equality that

$$\left\langle \delta^{(p)}(x - x_1)\,,\, \eta^{(\alpha,q)}(x - x_2) \right\rangle = 0, \tag{7.28}$$

and

$$\left\langle \delta^{(p)}(x - x_1)\,,\, \not\eta^{(\alpha,q)}(x - x_2) \right\rangle = 0. \tag{7.29}$$

$\forall x_1, x_2 \in \mathbb{R}$, $\forall \alpha \in \mathbb{C}$, and $\forall p, q \in \mathbb{N}$.
The scalar product of the δ–function with the powers and logarithms $\in \overline{\mathbf{PC}}'_\lambda$ is zero,

$$\left\langle \delta^{(p)}(x)\,,\, |x'|^{\alpha} \log^{q}|x| \operatorname{sgn}^{m}(x) \right\rangle = 0. \tag{7.30}$$

In particular

$$\left\langle \delta(x)\,,\, I'(x) \right\rangle = 0, \tag{7.31}$$

but

$$\left\langle \delta^{(p)}(x)\,,\, x'^{q} + \theta^{(q)}(x) \right\rangle = \delta_{p,q}\,. \tag{7.32}$$

The product of a $\delta^{(p)}$–function with a $\mathbf{C}^{\infty}$–function $\in \overline{\mathbf{PC}}'_\lambda$ is, in accordance with (7.30) and (7.32), entirely due to the $\theta^{(p)}$–function which may or may not be present at the location of its support.

7.3 The operators $\mathcal{X}$ and $\mathcal{D}$ on the preliminary class

The operators are now defined on $\overline{\mathbf{PC}}_\lambda$ by pulling them back from $\overline{\mathbf{PC}}'_\lambda$, in accordance with the definition (7.2). The preliminary operators were already defined between singular points. All that is needed is the addition of the appropriate δ–functions at the singular points.

The multiplication operator does not change with respect to its preliminary version. It does not introduce additional δ–functions.
The action of the operator $\mathcal{X}$ on the δ–functions is

$$\begin{aligned} \mathcal{X}\, \delta^{(p+1)}(x) &:= \overline{\mathcal{M}}^{-1} \mathcal{X} \overline{\mathcal{M}}\ \delta^{(p+1)}(x) = \overline{\mathcal{M}}^{-1} \mathcal{X}\, \eta^{(p+1)}(x) = \\ &= \overline{\mathcal{M}}^{-1}\, \eta^{(p)}(x) = \delta^{(p)}(x), \end{aligned} \tag{7.33}$$

and in particular for $p = 0$

$$\mathcal{X}\, \delta(x) = \overline{\mathcal{M}}^{-1} \mathcal{X} \overline{\mathcal{M}}\ \delta(x) = \overline{\mathcal{M}}^{-1}\, \eta^{(-1)}(x) = 0(x). \tag{7.34}$$

One sees that under the operator $\mathcal{X}$ the generalised functions $\delta^{(p)}(x)$ transform in the same way as the distributions $\delta^{(p)}$ (apart from the different normalization). The action of $\mathcal{X}$ on the translated functions is

$$\mathcal{X}\, \delta^{(p+1)}(x - x_0) = \delta^{(p)}(x - x_0) + x_0 \cdot \delta^{(p+1)}(x - x_0), \tag{7.35}$$

and

$$\boldsymbol{\mathcal{X}}\,\bar{\delta}(x - x_0) = x_0\,\bar{\delta}(x - x_0), \tag{7.36}$$

which again shows the localization of the $\bar{\delta}$–functions. According to (7.36) the functions $\bar{\delta}(x - x_0)$ are eigenfunctions of the $\boldsymbol{\mathcal{X}}$ operator

$$\boldsymbol{\mathcal{X}}\,\bar{\delta}(x - x_0) = x_0\,\bar{\delta}(x - x_0), \tag{7.37}$$

in agreement with the corresponding property of the δ–distribution.
For the powers of x one obtains the unremarkable result

$$\boldsymbol{\mathcal{X}}|x|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x) = |x|^{\alpha+1}\log^{q}|x|\,\mathrm{sgn}^{m+1}(x), \tag{7.38}$$

as expected from elementary algebra.
The differential operator $\boldsymbol{\mathcal{D}}$ acting on the $\bar{\delta}$–functions gives

$$\boldsymbol{\mathcal{D}}\,\bar{\delta}^{(p)}(x - x_0) = \bar{\boldsymbol{\mathcal{M}}}^{-1}\boldsymbol{\mathcal{D}}\bar{\boldsymbol{\mathcal{M}}}\;\bar{\delta}^{(p)}(x - x_0) = -(p+1)\,\bar{\delta}^{(p+1)}(x - x_0), \tag{7.39}$$

as expected from distribution theory.

Remark 7.8 In contrast to the standard normalization the superscript in parenthesis on the $\bar{\delta}$–functions does not indicate a repeated derivative. Instead we have

$$\boldsymbol{\mathcal{D}}^{p}\;\bar{\delta}(x) = (-)^{p}p!\;\bar{\delta}^{(p)}(x), \tag{7.40}$$

and conversely, when $\bar{\delta}^{(p)}(x)$ is temporarily interpreted as a distribution

$$\bar{\delta}^{(p)}(x) = (-)^{p}\delta^{(p)}\,/p!\,, \tag{7.41}$$

in contrast with the usual normalization of the $\delta^{(p)}$ distribution. For $p = 0$ the bar in $\bar{\delta}^{(0)}$ could be omitted, since $\delta^{(0)}$ and $\bar{\delta}^{(0)}$ have the same normalization. This is never done to avoid confusion between generalised functions and distributions.

Acting on the powers and logarithms, now considered as elements of $\overline{\mathbf{PC}}_{\lambda}$ instead of $\mathbf{PC}'_{\lambda}$, the $\boldsymbol{\mathcal{D}}$ operator gives additional $\bar{\delta}$–functions in the special case that the function $f(x) \in \overline{\mathbf{PC}}_{\lambda}$ behaves as $(x - x_0)^{-p}\log^{q}|x - x_0|\,\mathrm{sgn}(x - x_0)$ at a singular point.

In this and many following formulæ the notation can be simplified, and all exceptional cases can be combined in one formula by using the generalization to complex argument of Kronecker's δ–symbol (5.55) introduced previously. The result of the computation of the derivative then takes the form

$$\begin{aligned}\boldsymbol{\mathcal{D}}|x|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x) = {} & +\alpha|x|^{\alpha-1}\log^{q}|x|\,\mathrm{sgn}^{m+1}(x) + \\ & + q(1 - \delta_{q,0})|x|^{\alpha-1}\log^{q-1}|x|\,\mathrm{sgn}^{m+1}(x) + \\ & + 2\,\delta_{q,0}\sum_{p=0}^{\infty}\delta_{\alpha,-p}\,\delta_{p,m+1}^{\mathrm{mod}2}\,\bar{\delta}^{(p)}(x).\end{aligned} \tag{7.42}$$

In particular for $q = 0$ this reduces to

$$\mathcal{D}\, x^{-p}\,\mathrm{sgn}(x) = -p\, x^{-p-1}\,\mathrm{sgn}(x) + 2\,\delta^{(p)}(x), \tag{7.43}$$

and

$$\mathcal{D}\, x^{-p} = -p\, x^{-p-1}, \tag{7.44}$$

in agreement with the corresponding result in distribution theory.
The corresponding formulæ in terms of Heaviside's step functions are

$$\mathcal{D}\, x^{-p} H(x) = -p\, x^{-p-1} H(x) + \delta^{(p)}(x), \tag{7.45}$$

and

$$\mathcal{D}\, x^{-p} H(-x) = -p\, x^{-p-1} H(-x) - \delta^{(p)}(x). \tag{7.46}$$

The different behaviour of $H(x) \in \overline{\mathbf{PC}}_\lambda$ and $H'(x) \in \overline{\mathbf{PC}}'_\lambda$ should be noted. The primed Heaviside functions $H'(x)$ and $H'(-x) \in \mathbf{PC}'_\lambda$ have *different* derivatives at $x = 0$,

$$\mathcal{D}H'(x) = \eta_\downarrow(x) - \not\eta_\uparrow(x), \quad \text{and} \quad \mathcal{D}H'(-x) = -\,\eta_\uparrow(x) + \not\eta_\downarrow(x), \tag{7.47}$$

but the unprimed Heaviside functions $H(x)$ and $H(-x) \in \mathbf{PC}_\lambda$ have the same derivative,

$$\mathcal{D}H(x) = -\mathcal{D}H(-x) = \delta(x), \tag{7.48}$$

(up to a sign). This is a consequence of the choice of the minimal completion of $\mathbf{PC}_\lambda$.

The equality of the derivatives in (7.48) can be interpreted as meaning that the support of the δ function is the point $x = 0$ itself, while the $\eta_{\uparrow\downarrow}$-functions reside on the positive and negative infinitesimal surroundings of the point, at $x = 0+$ and $x = 0-$ respectively. This illustrates the inadequacy of the real number system as a support for the generalised functions. This matter will be the subject of Ch. 13.
For the unit function $I(x) \in \mathbf{PC}_\lambda$ the previous equation is equivalent to

$$\mathcal{D}\, I(x) = \mathcal{D}\big(H(x) + H(-x)\big) = \delta(x) - \delta(x) = \mathit{0}(x), \tag{7.49}$$

in contrast with equation (6.26)

$$\mathcal{D}\, I'(x) = \mathcal{D}\big(H'(x) + H'(-x)\big) = 2\,\sigma(x) - 2\,\not\sigma(x) \neq \mathit{0}(x), \tag{7.50}$$

found in the previous chapter.

The derivatives of the ordinary functions are found from the corresponding derivatives in $\overline{\mathbf{PC}}'_\lambda$ by

$$\mathcal{D}\, f(x) := \bar{\mathcal{M}}^{-1}\mathcal{D}\bar{\mathcal{M}}\;\; f(x), \tag{7.51}$$

in the same way. Additional δ–functions appear in the derivative when there is a jump in the coefficient of the power x^{-p} in the asymptotic expansions. Measuring the jump with the appropriate σ–function, the result can be written as

$$\mathcal{D}\, f(x) = \mathcal{D}_{\text{pre}}\, f(x) + 2 \sum_{\{x_j\}} \sum_{p=0}^{\infty} \left\langle \sigma^{(-p)}(x - x_j)\, ,\, f(x) \right\rangle \delta^{(p)}(x - x_j). \quad (7.52)$$

As a consequence of the restrictions imposed on $\mathbf{PC}_\lambda$ both the sum over singular points, and the sum over powers are actually finite. The sum is a finite linear combination of δ–functions.

Remark 7.9 Anticipating the results of Ch. 16, where the scaling and homogeneity properties of generalised functions are defined, we expect $\delta^{(p)}$–functions to appear when the result of a computation is locally of parity $(-)^p$, and homogeneous of degree $-p-1$.

7.4 The Fourier transform on the preliminary class

The Fourier operator on $\overline{\mathbf{PC}}_\lambda$ is also found directly from the Fourier operator on $\overline{\mathbf{PC}}'_\lambda$.
The Fourier transform of the δ–functions is

$$\delta^{(p)}(x - x_0) := \mathcal{M}^{-1}\mathcal{F}\mathcal{M}\ \delta^{(p)}(x - x_0) = e^{ixx_0}(-ix)^p/p!\,, \quad (7.53)$$

which has the special cases

$$\mathcal{F}\,\delta^{(p)}(x) = \tfrac{(-i)^p}{p!}\, x^p, \quad (7.54)$$

and

$$\mathcal{F}\,\delta(x) = I(x). \quad (7.55)$$

The δ–function and the unit function $I(x)$ are indeed a Fourier pair.
The parity of the δ–functions is found by computing the square of the $\mathcal{F}$ operator,

$$\mathcal{P}\,\delta^{(p)}(x) = (2\pi)^{-1}\mathcal{F}^2\,\delta^{(p)}(x) = (-)^p\delta^{(p)}(x), \quad (7.56)$$

in agreement with the corresponding behaviour of the standard δ–function.
The Fourier transform of the powers and logarithms is computed by application of the definition (7.2) as

$$\begin{aligned}\mathcal{F}|x|^\alpha \log^q|x|\,\text{sgn}^m(x) = &-2i^m q! \sum_{j=0}^{q+1} \tfrac{(-)^j}{j!}\, c_{q-j}(\alpha, m)|x|^{-\alpha-1}\log^j|x|\,\text{sgn}^m(x)\\ &+ 4i^m q! \sum_{p=0}^{\infty} \delta_{\alpha,p}\, \delta_{p,m}^{\text{mod2}}\, c_{q+1}(p, m)\, \delta^{(p)}(x). \quad (7.57)\end{aligned}$$

Specialization to $\alpha = m = p \in \mathbb{N}$ gives

$$\mathcal{F}\, x^p \log^q|x| = -2i^p q! \sum_{j=0}^{q} \tfrac{(-)^j}{j!}\, c_{q-j}(p,p) x^{-p-1} \log^j|x| \operatorname{sgn}(x) + 4i^p q!\; c_{q+1}(p,p)\, \delta^{(p)}(x), \tag{7.58}$$

which is the case where additional δ–functions occur.
For $q = 0$ this reduces to the well-known result

$$\mathcal{F}\, x^p = 4i^p\, c_1(p,p)\, \delta^{(p)}(x) = 2\pi(-i)^p p!\; \delta^{(p)}(x), \tag{7.59}$$

where the explicit values of $c_0(p,p) = 0$ and $c_1(p,p)$, found in appendix C, table C.1, have been substituted.
For the computation of Hilbert transforms one needs the special case

$$\mathcal{F} x^{-1} = -i\pi \operatorname{sgn}(x), \tag{7.60}$$

in agreement with the standard result.

The rules for estimating the leading term of the asymptotic behaviour of the Fourier transform are apparent from (7.58). The results found above compare directly with the results given in [Lod1]. When a correction for the different normalization is made the results agree. The main difference is that in [Lod1] only the leading term of the Fourier transform appears.

Remark 7.10 The Fourier transforms of the powers cannot be differentiated formally with respect to α. Even when the undefined derivative of the Kronecker δ in (7.58) is ignored, the coefficient of the δ–function which would be obtained by formal differentiation is incorrect. This is to be expected, since the inverse mapping and formal differentiation with respect to α do not commute.

The Fourier transform of the ordinary functions $f(x) \in \mathbf{PC}_\lambda$ can be found in the same way by application of (7.2)

$$\mathcal{F}\, f(x) = \overline{\mathcal{M}}^{-1} \mathcal{F} \overline{\mathcal{M}}\; f(x). \tag{7.61}$$

The Fourier transform differs from the preliminary Fourier transform by δ–functions, when behaviour as $x^p\, e^{ikx}$ at infinity is present.
The previous result on $\overline{\mathbf{PC}}'_\lambda$ takes the form

$$\mathcal{F}\, f(x) = \mathcal{F}_{\text{pre}}\, f(x) + + \sum_{\{x_j\}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \langle\, f(x)\,,\, e^{ix_j x} \not{\eta}^{(p,q)}(x)\,\rangle \times \times +4i^p q!\; c_{q+1}(p,p)\, \delta^{(p)}(x - x_j), \tag{7.62}$$

when the $\not{\eta}$–functions are used to measure the relevant terms in the asymptotic expansion of $f(x)$ at infinity. The sums over δ–functions which occur in (7.62) are again actually finite.

Example 7.3 Functions which vanish at infinity, such as (6.86), cannot have δ–functions in their Fourier transform

$$\begin{aligned}\mathcal{F}|x|^{\alpha}\, e^{-a|x|}\mathrm{sgn}^m(x) &= \\ &= -\,\Gamma^{[-1]}(\alpha+1)\big((ix+a)^{-\alpha-1)}\log(ix+a) + (-)^m(i \to -i)\big) + \\ &\quad + \Gamma^{[0]}(\alpha+1)\big((ix+a)^{-\alpha-1} + (-)^m(i \to -i)\big). \qquad (7.63)\end{aligned}$$

Remark 7.11 The method of calculating the action of operators acting on $\overline{\mathbf{PC}}_\lambda$ first in $\overline{\mathbf{PC}}'_\lambda$, keeping only relevant terms, and transferring back to $\overline{\mathbf{PC}}_\lambda$, is an efficient tool to calculate the action of operators on $\overline{\mathbf{PC}}_\lambda$.

Since $\overline{\mathbf{PC}}_\lambda$ can be interpreted as a subspace of the (tempered) distributions the method introduced in this chapter is also an efficient calculational tool in the context of distribution theory.

It has now been shown by explicitly defining all operators on $\overline{\mathbf{PC}}_\lambda$ that the space $\overline{\mathbf{PC}}_\lambda$ is closed under the action of the operators.

CHAPTER 8

PRODUCTS OF LINEAR FUNCTIONALS

In this chapter the scalar product of the class $\overline{\mathbf{PC}}'_\lambda$ with itself is defined. In agreement with this scalar product, a pointwise product $\bullet : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}'_\lambda$, and also a convolution product $* : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}'_\lambda$ is defined with the properties required in Ch. 2 for a product and convolution of generalised functions.

It is not necessary to follow the constructive approach used in this chapter. In earlier work [K&L] it was shown that there exists a product on $\mathbf{PC}'$, derived from the pointwise product on $\mathcal{S}$ by considering $\mathbf{PC}'$ as the bi-dual of $\mathcal{S}$.

As indicated in Ch. 2 it is convenient to do more. It will be seen that a generalised function product (on $\overline{\mathbf{PC}}'_\lambda$, not on $\mathbf{PC}_\lambda$ or $\mathbf{GF}$) can be either associative or commutative, but not both. The commutative non-associative product is the natural choice since it agrees with the requirement of symmetry of the scalar product. The associative non-commutative product has a richer algebraic structure, and the commutative product can be easily derived from it by symmetrization. Therefore both products are defined in this chapter. Conversion between the two possibilities is not difficult. In other theories of multiplication of distributions often only one of these options is worked out, so a comparison is also easier when both a commutative and a non-commutative version of the product is available.

It should be kept in mind that the choice between a commutative and an associative product algebra exists only in $\overline{\mathbf{PC}}'_\lambda$, not in $\overline{\mathbf{PC}}_\lambda$ or $\mathbf{GF}$ as a whole. A separate section is devoted to the algebraic properties of the commutators and associators.

8.1 Linear functionals on the linear functionals

In the previous chapters the spaces $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ were equipped with suitably defined operators. The scalar product $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbb{C}$ is still asymmetrical since the spaces $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ have different properties. First the scalar product is symmetrized on $\overline{\mathbf{PC}}'_\lambda$ to a symmetrical scalar product $\langle\ ,\ \rangle : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbb{C}$. As an intermediate step asymmetrical scalar products are introduced. These are then symmetrized.

The product $\bullet : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}'_\lambda$ is defined in such a way that it agrees with the symmetrized scalar product.

In the foregoing $\overline{\mathbf{PC}}'_\lambda$ was understood as the space of linear functionals on the preliminary class $\mathbf{PC}_\lambda$, given by the scalar product $\mathbf{PC}_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbb{C}$. This relation can be read the other way. The preliminary class $\mathbf{PC}_\lambda$ can also be considered as a class of linear functionals on $\overline{\mathbf{PC}}'_\lambda$. The elements of $\overline{\mathbf{PC}}'_\lambda$ can now be defined as linear functionals on $\overline{\mathbf{PC}}'_\lambda$ by taking suitable residues.

Care is needed here, since the elements of $\overline{\mathbf{PC}}'_\lambda$ were already defined as residues. A double residue involves an analytic function in two variables, and the taking of two residues of an analytic function in two variables may depend on the order in which the residues are evaluated.
The ordinary functions present no difficulty. We can define

$$\langle f'(x) \,, g'(x) \rangle := \langle f(x) \,, g'(x) \rangle, \tag{8.1}$$

$\forall g'(x) \in \overline{\mathbf{PC}}'_\lambda$, which says that as far as the scalar product with $\overline{\mathbf{PC}}'_\lambda$ is concerned the element $f(x) \in \mathbf{PC}_\lambda$ can be identified with $f'(x) \in \overline{\mathbf{PC}}'_\lambda$.
In particular we obtain

$$\langle |x'|^\lambda \operatorname{sgn}^m(x) \,, g'(x) \rangle = \langle |x|^\lambda \operatorname{sgn}^m(x) \,, g'(x) \rangle, \tag{8.2}$$

with $g'(x) \in \mathbf{PC}'_\lambda$ an ordinary function. This could be used to define

$$\langle \eta_s^{(\alpha,q)}(x) \,, g'(x) \rangle := \operatorname*{Res}_{\lambda+\alpha=1} \tfrac{1}{2q!} (\lambda+\alpha+1)^q \langle |x|^\lambda \,, g'(x) \rangle, \tag{8.3}$$
WRONG!

This definition could give rise to difficulties, since elements $g'(x) \in \overline{\mathbf{PC}}'_\lambda$ are also defined as residues. In particular one can take (5.20)

$$g'(x) := |x'|^\beta \log^r|x| \operatorname{sgn}^n(x) = \operatorname*{Res}_{\mu=\beta} r!\, (\mu-\beta)^{-r-1} |x|^\mu \operatorname{sgn}^n(x). \tag{8.4}$$

Then we have to compute a double residue

$$\operatorname*{Res}_{\lambda=-\alpha-1} (\lambda+\alpha+1)^q \operatorname*{Res}_{\mu=\beta} (\mu-\beta)^{-r-1} \cdots\cdots (\lambda,\mu). \tag{8.5}$$

To obtain the simplest case, putting $\alpha = \beta = q = r = 0$, and going over to one-sided η–functions, reduces (8.5) to

$$\langle \eta_\downarrow(x) \,, H'(x) \rangle = \operatorname*{Res}_{\lambda=-1} \operatorname*{Res}_{\mu=0} \mu^{-1} \frac{a^{\lambda+\mu+1}}{\lambda+\mu+1}, \tag{8.6}$$

where the definitions (5.39) and (5.19) of $\eta_\downarrow(x)$ and $H'(x)$ have been substituted.

The anticipated difficulty does indeed occur. The evaluation of the residues (taking $a = 1$ for simplicity since the result does not depend on a), depends on the order in which the evaluation is carried out. If the λ residue is computed first the answer is zero.

$$\operatorname*{Res}_{\mu=0} \mu^{-1} \operatorname*{Res}_{\lambda=-1} (\lambda+\mu+1)^{-1} = \operatorname*{Res}_{\mu=0} \mu^{-1} \delta_{\mu,0} = 0. \tag{8.7}$$

Taking the λ–residue first results in a zeromorphic function of μ, so the second μ–residue is zero. If the μ–residue is evaluated first the result is merely the substitution $\mu := 0$, and the result of taking the second λ–residue is

$$\operatorname*{Res}_{\lambda=-1}\operatorname*{Res}_{\mu=0}\mu^{-1}(\lambda+\mu+1)^{-1} = \operatorname*{Res}_{\lambda=-1}(1-\delta_{\lambda,-1})(\lambda+1)^{-1} = 1. \tag{8.8}$$

The 'left-first' scalar product is defined by the prescription that the residue connected with the term of the left-hand side of the scalar product has to be evaluated first.
From the preceding computation of residues (8.7) and (8.8) we find

$$\langle\!\!\circ\, \eta_{\downarrow}(x)\,,H'(x)\,\rangle = 0, \tag{8.9}$$

and

$$\langle\!\!\circ\, H'(x)\,,\eta_{\downarrow}(x)\,\rangle = 1, \tag{8.10}$$

where the notation $\langle\!\!\circ\,,\,\rangle$ has been introduced to indicate the left-first scalar product. The circle through the angle bracket in the scalar product symbol has been chosen to remind us that after all the residue is obtained by evaluating a contour integral.

In the same way the 'right-first' scalar product is defined by taking the residue on the right-hand side of the scalar product first. This results in

$$\langle\,\eta_{\downarrow}(x)\,,H'(x)\,\rangle\!\!\!\circ = 1, \tag{8.11}$$

and

$$\langle\,H'(x)\,,\eta_{\downarrow}(x)\,\rangle\!\!\!\circ = 0, \tag{8.12}$$

where in similar notation $\langle\,,\,\rangle\!\!\!\circ$ indicates the right-first scalar product.

Remark 8.1 The lack of interchangeability is not particular to the computation of residues. The same phenomenon occurs when one attempts to compute the analogous double limit

$$\lim_{\substack{a\downarrow 0\\ b\downarrow 0}}\int_{-\infty}^{\infty}dx\,b^{-1}H(x)H(a-x)H(b-x) = \lim_{\substack{a\downarrow 0\\ b\downarrow 0}} b^{-1}\min(a,b) = \begin{cases}0 & a \text{ first},\\ 1 & b \text{ first},\end{cases} \tag{8.13}$$

which also depends on the order in which the limits are evaluated.

The left- and right-first scalar products defined above do not qualify as a symmetrical scalar product, as required by (2.5) for a symmetrical theory of generalised functions.

It follows from the requirement of symmetry of the scalar product (2.5) that we have to define the symmetrical scalar product $\langle\,,\,\rangle$ by symmetrization

$$\langle\,,\,\rangle := \tfrac{1}{2}\langle\!\!\circ\,,\,\rangle + \tfrac{1}{2}\langle\,,\,\rangle\!\!\!\circ. \tag{8.14}$$

The symmetry property (2.5) of the scalar product is obviously satisfied,

$$\langle f , g \rangle = \langle g , f \rangle^*, \tag{8.15}$$

since the left- and right-first scalar products satisfy

$$\langle\!\langle f(x) , g(x) \rangle^* = \langle g(x) , f(x) \rangle\!\rangle, \tag{8.16}$$

by construction.
For the special cases (8. 9–12) computed above we obtain

$$\langle H'(x) , \eta_\downarrow(x) \rangle = \langle \eta_\downarrow(x) , H'(x) \rangle = \tfrac{1}{2}, \tag{8.17}$$

in contrast with the earlier result

$$\langle \eta_\downarrow(x) , H(x) \rangle = 1. \tag{8.18}$$

This shows that $H'(x)$ and $H(x)$ are *different* generalised functions, which will remain different when $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ will be combined (in Ch. 11) to the space $\mathbf{GF}_s$ of symmetrical generalised functions.
It is also possible to effect a trivial generalization by defining

$$2 \langle f(x) , g(x) \rangle_\rho := (1+\rho) \langle\!\langle f(x) , g(x) \rangle + (1-\rho) \langle f(x) , g(x) \rangle\!\rangle, \tag{8.19}$$

which reduces to one of the previous cases for $\rho = -1, 0,$ or 1. Another choice of the parameter ρ will lead to a product which is neither commutative nor associative.

In the following it will be often sufficient to give only the symmetrical form of the products in cases where the left-sided form equals the right-sided form. The general case of a scalar product with one η–function becomes

$$\langle\!\langle \eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x) , |x'|^\beta \log^r|x| \operatorname{sgn}^n(x) \rangle = 0, \tag{8/20}$$

and

$$\langle \eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x) , |x'|^\beta \log^r|x| \operatorname{sgn}^n(x) \rangle\!\rangle = (-)^q \delta_{\alpha,\beta}\, \delta_{q,r}\, \delta_{m,p}^{\mathrm{mod}2}, \tag{8/21}$$

and idem with left and right interchanged.
Averaging (8.21) and (8.20) gives the symmetrical scalar product

$$\langle \eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x) , |x'|^\beta \log^r|x| \operatorname{sgn}^n(x) \rangle = \tfrac{1}{2} (-)^q \delta_{\alpha,\beta}\, \delta_{q,r}\, \delta_{m,p}^{\mathrm{mod}2}. \tag{8/22}$$

The corresponding formulæ for the usual linear combinations are similar. The same factor $\frac{1}{2}$ occurs in the symmetrical version in all cases. Likewise there is the same $\frac{1}{2}$ in all symmetrical scalar products at infinity.

The scalar product of an η–function at $x = x_0$ or at $x = \pm\infty$ and an ordinary function $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ is again defined as the scalar product with the corresponding asymptotic series

$$\langle\, \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,,\, f'(x)\,\rangle := \langle\, \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,,\, f'_{asy}(x;0)\,\rangle, \qquad (8.23)$$

and idem at infinity. The sum implied in (8.23) is always finite in consequence of the restrictions imposed on the allowed asymptotic form (4.1) of $f(x)$.

The result of a scalar product with $\eta^{(\alpha,q)}(x)$ or $\not\eta^{(\alpha,q)}(x)$ is again a zeromorphic analytic function of the argument α. Therefore the scalar product of two η–functions can be obtained by computing another residue with the result

$$\langle\, \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,,\, \eta_s^{(\beta,r)}(x)\,\mathrm{sgn}^n(x)\,\rangle = 0, \qquad (8/24)$$

$\forall \alpha, \beta \in \mathbb{C}$, $q, r \in \mathbb{N}$, $m, n \in \mathbb{Z}$, since any zeromorphic function has residue zero everywhere. In the same way we obtain

$$\langle\, \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,,\, \not\eta_s^{(\beta,r)}(x)\,\mathrm{sgn}^n(x)\,\rangle = 0, \qquad (8/25)$$

and

$$\langle\, \not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,,\, \not\eta_s^{(\beta,r)}(x)\,\mathrm{sgn}^n(x)\,\rangle = 0, \qquad (8/26)$$

for all values of the parameters. It is not necessary to distinguish the left-first and the right-first case, since both are zero.

Finally the symmetrical scalar products involving the $\theta^{(p)}$–functions are found by imposing Parseval's equality (2.29) for the scalar product.

The results are

$$\langle\, \theta^{(p)}(x)\,,\, \theta^{q}(x)\,\rangle = 0, \qquad (8.27)$$

by (8.26) and

$$\langle\, \theta^{(p)}(x)\,,\, \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,\rangle = \tfrac{1}{2}\,\delta_{q,0}\,\delta_{\alpha,p}\,\delta_{p,m}^{\mathrm{mod}2}\,, \qquad (8.28)$$

by (8.22) and the Fourier transforms found in Ch. 6. The non-zero case is

$$\langle\, \theta^{(p)}(x)\,,\, \eta^{(q)}(x)\,\rangle = \tfrac{1}{2}\,\delta_{p,q}\,, \qquad (8.29)$$

while the scalar product

$$\langle\, \theta^{(p)}(x)\,,\, \not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,\rangle = 0, \qquad (8.30)$$

of $\theta^{(p)}(x)$ with anything at infinity is zero. The scalar product of a $\theta^{(p)}$–function and an ordinary function $f'(x) \in \mathbf{PC}'_\lambda$ is also defined to be zero. This exhausts the elements of $\overline{\mathbf{PC}}'_\lambda$.

We now have a symmetrical scalar product, $\langle\ ,\ \rangle : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbb{C}$, defined on the whole space, with the required properties.

The next step is the definition of a $\bullet : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}_\lambda$ product, which generates this scalar product by

$$\langle f'(x) , g'(x) \rangle = \langle I(x) , f'(x)^* \bullet g'(x) \rangle, \tag{8.31}$$

$\forall f'(x), g'(x) \in \overline{\mathbf{PC}}'_\lambda$, with product properties which are as simple as possible. Corresponding to the left-first scalar product there will be a left-first generalised function product, while the symmetrical product corresponds to the symmetrical scalar product. Afterwards the scalar products can be redefined from the products.

8.2 The pointwise product on $\overline{\mathbf{PC}}'_\lambda$

The 'pointwise' product of generalised functions, indicated by the fat centered dot $\bullet$, will be defined in such a way that it is local in the sense of this work. This means that the support (in the sense of generalised functions, to be introduced in Ch. 13) of the product is contained in the supports of the factors. The pointwise product is an extension of the pointwise product of ordinary functions $f(x), g(x) \in \mathcal{S}$. This justifies the name 'pointwise' given to the $\bullet$ product to distinguish it from the convolution product, which is as usual indicated by $*$ the centered asterisk.

Remark 8.2 The method followed in this book is the opposite of the method used previously in [K&L]. There the simple product on $\overline{\mathbf{PC}}'_\lambda$ was derived from the more complicated product on $\mathbf{PC}_\lambda$. Here the product on $\overline{\mathbf{PC}}'_\lambda$ is defined first. This product is then transferred by a suitable mapping to $\mathbf{PC}_\lambda$. In [K&L] an automatic method was outlined to obtain a canonical product on $\overline{\mathbf{PC}}_\lambda$ (there called simply $\mathbf{PC}$), by considering $\overline{\mathbf{PC}}_\lambda$ as a bi-dual of the Schwartz space $\mathcal{S}$ of test functions. The results of this procedure agree with the definition adopted here, which is based on choosing the simplest product which agrees with the scalar product.

The product indicated by $\bullet$ is the product associated with the symmetrical scalar product. It is commutative but not associative. It is derived from the products associated with the left-first and the right-first scalar products. These are indicated by $\circ\!\bullet$ and $\bullet\!\circ$ respectively. These products are not commutative, but they can be shown to be associative.

The left-sided product $\circ\!\bullet$ is defined in such a way that it agrees with the left-sided scalar product in the sense that

$$\langle\!\!\langle f(x) , g(x) \rangle = \langle I(x) , f(x)^* \circ\!\bullet\, g(x) \rangle. \tag{8.32}$$

Likewise the right-sided product $\bullet\!\circ$ satisfies

$$\langle f(x) , g(x) \rangle\!\!\rangle = \langle I(x) , f(x)^* \bullet\!\circ\, g(x) \rangle. \tag{8.33}$$

Comparing (8.33) and (8.32) with (8.16) one sees that that the left- and right-sided products can be related by

$$f(x) \circ\!\bullet\, g(x) = g(x) \bullet\!\circ\, f(x), \tag{8.34}$$

so it is sufficient to calculate only one (usually the left-first) case. When only the symmetrical product is given it agrees with both the left and the right-sided product unless stated otherwise.

Remark 8.3 Anticipating the results of the next chapter it is mentioned that there will be only one unit element $I \in \overline{\mathbf{PC}}_\lambda$ for all products. Scalar products with the unit function do not depend on the choice of left or right.

The symmetrical product defined by

$$f(x) \bullet g(x) := \tfrac{1}{2}\, f(x) \circ\!\bullet\, g(x) + \tfrac{1}{2}\, f(x) \bullet\!\circ\, g(x), \tag{8.35}$$

can also be written in the forms

$$\begin{aligned} f(x) \bullet g(x) &= \tfrac{1}{2}\, f(x) \circ\!\bullet\, g(x) + \tfrac{1}{2}\, g(x) \circ\!\bullet\, f(x) = \\ &= \tfrac{1}{2}\, f(x) \bullet\!\circ\, g(x) + \tfrac{1}{2}\, g(x) \bullet\!\circ\, f(x). \end{aligned} \tag{8.36}$$

The symmetrical product is obviously commutative by construction. The product commutator is defined by

$$\left[f(x) \,{}_{\circ\bullet}\, g(x)\right] := f(x) \circ\!\bullet\, g(x) - f(x) \bullet\!\circ\, g(x), \tag{8.37}$$

where the lowered product symbol takes the place of the comma in the usual form of the commutator. It can by (8.34) also be written as the difference of the left and right-sided products

$$\left[f(x) \,{}_{\circ\bullet}\, g(x)\right] = f(x) \circ\!\bullet\, g(x) - f(x) \bullet\!\circ\, g(x). \tag{8.38}$$

The asymmetrical products can be recovered from to symmetrical product and the product commutator by

$$f(x) \circ\!\bullet\, g(x) = f(x) \bullet g(x) + \tfrac{1}{2}\left[f(x) \,{}_{\circ\bullet}\, g(x)\right], \tag{8.39}$$

and

$$f(x) \bullet\!\circ\, g(x) = f(x) \bullet g(x) - \tfrac{1}{2}\left[f(x) \,{}_{\circ\bullet}\, g(x)\right], \tag{8.40}$$

The algebraic properties of the commutators are the subject of the next section.

To get started the products of the basic functions are defined in the simplest possible way by inspection of the corresponding scalar product.

From the scalar products (8.24) and (8.26) we are led by (8.32) to define the products

$$\eta^{(\alpha,q)}(x) \bullet \eta^{(\beta,r)}(x) := 0(x), \tag{8/41}$$

and

$$\not\eta^{(\alpha,q)}(x) \bullet \not\eta^{(\beta,r)}(x) := 0(x), \tag{8/42}$$

for all values of the indices. Of course we have to define

$$\eta^{(\alpha,q)}(x) \bullet \not\eta^{(\beta,r)}(x) = 0(x), \tag{8/43}$$

since the supports do not overlap. The left- and right-first versions of these products are also zero.

The products of the η–functions and the powers can be read off directly from the corresponding scalar products. From the left-sided scalar products one is led to

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \circ\bullet |x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) := 0(x), \tag{8/44}$$

and

$$|x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) \circ\bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) := \\ := \begin{cases} 0(x) & r > q, \\ (-)^r \eta_s^{(\alpha-\beta,q-r)}(x)\,\mathrm{sgn}^{m+n}(x) & q \geq r, \end{cases} \tag{8/45}$$

in agreement with (8.20) and (8.21). The corresponding right-sided products are

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet\circ |x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) := \\ := \begin{cases} 0(x) & r > q, \\ (-)^r \eta_s^{(\alpha-\beta,q-r)}(x)\,\mathrm{sgn}^{m+n}(x) & q \geq r, \end{cases} \tag{8/46}$$

and

$$|x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) \bullet\circ \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) := 0(x), \tag{8/47}$$

as one also sees by inverting left and right in (8.45) and (8.44).
The symmetrized product is the same up to a factor $\frac{1}{2}$

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet |x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) = \\ = |x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) \bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \\ = \frac{1}{2} \begin{cases} 0(x) & r > q, \\ (-)^r \eta_s^{(\alpha-\beta,q-r)}(x)\,\mathrm{sgn}^{m+n}(x) & q \geq r. \end{cases} \tag{8/48}$$

Frequently occurring special cases are

$$I'(x) \bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \tfrac{1}{2}\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \tag{8/49}$$

and

$$\mathrm{sgn}'(x) \bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \tfrac{1}{2}\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^{m+1}(x), \tag{8/50}$$

and the first powers

$$x' \bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \tfrac{1}{2}\,\eta_s^{(\alpha-1,q)}(x)\,\mathrm{sgn}^{m+1}(x), \tag{8/51}$$

and

$$x'^{-1} \bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \tfrac{1}{2}\,\eta_s^{(\alpha+1,q)}(x)\,\mathrm{sgn}^{m+1}(x), \tag{8/52}$$

which occur in connection with the operators $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{X}}^{-1}$. The corresponding left-first products lack the factor $\frac{1}{2}$, the corresponding right-first products are zero.

The products (left, right, and symmetrical) of the η–functions and the ordinary functions are again obtained by taking the product of the η–function with the asymptotic expansion

$$\eta^{(\alpha,q)}(x-x_0) \bullet f'(x) := \eta^{(\alpha,q)}(x-x_0) \bullet f'_a(x;x_0), \tag{8.53}$$

where the separate terms have been defined above. As before this leads to formally infinite sums with only a finite number of effectively non-zero terms.

The $\bullet$ product of ordinary functions is put equal to the pointwise product, in agreement with the scalar product of the ordinary functions,

$$f'(x) \bullet g'(x) := \boldsymbol{\mathcal{M}}\big(f(x)\cdot g(x)\big). \tag{8.54}$$

In particular for the powers and logarithms

$$|x'|^\alpha \log^q|x|\,\mathrm{sgn}^m(x) \bullet |x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) = |x'|^{\alpha+\beta} \log^{q+r}|x|\,\mathrm{sgn}^{m+n}(x), \tag{8.55}$$

the standard result is recovered. This is also valid for the left- and right-sided versions of this product.

The products of the $\theta^{(p)}$–functions are fixed by the requirement that

$$f(x) := x'^p + \theta^{(p)}(x), \tag{8.56}$$

behaves in the finite at $x=0$ as the function x^p. In particular the function $I'(x)+\theta(x)$ should behave at $x=0$ as the unit element for all products. This will be the case if the left/right character of the θ–functions is defined to be the opposite of the corresponding powers.

This consideration leads to the definitions

$$\theta^{(p)}(x) \circ\!\bullet\, \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) := O(x), \tag{8.57}$$

and

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \circ\!\bullet\, \theta^{(p)}(x) := \eta_s^{(\alpha-p,q)}(x)\,\mathrm{sgn}^{m+p}(x), \tag{8.58}$$

The corresponding symmetrical product is

$$\theta^{(p)}(x) \bullet \eta_s^{(\alpha,q)}\,\mathrm{sgn}^m(x) = \tfrac{1}{2}\,\eta_s^{(\alpha-p,q)}(x)\,\mathrm{sgn}^{m+p}(x). \tag{8.59}$$

For $p = 0$ this reduces to the special case

$$\theta(x) \bullet \eta^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \tfrac{1}{2}\,\eta^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x). \tag{8.60}$$

The functions $\theta^{(p)}(x)$ regularize the positive integral powers $x'^p \in \overline{\mathbf{PC}}'_\lambda$ in the sense that the linear combination $x'^p + \theta^{(p)}(x)$ is insensitive to the choice of the left- and right-sided products

$$\begin{aligned}\big(x'^p + \theta^{(p)}(x)\big) \circ\!\bullet\, \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) &= \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \circ\!\bullet\, \big(x'^p + \theta^{(p)}(x)\big)\\ &= \eta_s^{(\alpha-p,q)}(x)\,\mathrm{sgn}^{m-p}(x),\end{aligned} \tag{8.61}$$

for all the η–functions. The same holds for the right-sided products.
In particular for $p = 1$ and $p = 0$ one has

$$\big(I'(x) + \theta(x)\big) \bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \tag{8.62}$$

and

$$\big(x' + \theta^{(1)}(x)\big) \bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \eta_s^{(\alpha-1,q)}(x)\,\mathrm{sgn}^{m+1}(x), \tag{8.63}$$

in which the 'hole' at $x = 0$ has been filled by adding the correct amount of θ–function. The linear combination (8.62) is the best approximation to a unit element for multiplication in $\overline{\mathbf{PC}}'_\lambda$. The exceptional elements are the generalised functions at infinity, which receive a factor $\frac{1}{2}$.

Since the class $\overline{\mathbf{PC}}'_\lambda$ contains only the functions $\theta^{(p)}(x - x_0)$ as localized powers, only the positive integral powers in the finite, without additional logarithms, can be made regular in this way. The terminology 'regular power' and 'singular power' is used in the following to distinguish between the two cases.
The product of $\theta(x)$ with itself is also found in this way from

$$\big(I'(x) + \theta(x)\big) \bullet \theta(x) := \theta(x), \tag{8.64}$$

so the simplest definition is

$$\theta(x) \bullet \theta(x) := \theta(x), \tag{8.65}$$

and consequently

$$I'(x) \bullet \theta(x) = O(x). \tag{8.66}$$

More generally these considerations lead to

$$\theta^{(p)}(x) \bullet \theta^{q}(x) := \theta^{(p+q)}(x), \tag{8.67}$$

in agreement with the naïve interpretation of the θ–functions. Analogously to the case of the positive integral powers (8.61), the left- and right-sided products of the $\theta^{(p)}$–functions are the same as the symmetrical product.

The product of θ–functions and η–functions located at different points is of course defined to be zero. Otherwise the product would not be pointwise. Likewise, the product of the $\theta^{(p)}$–functions with the ordinary functions $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ is defined to be zero at the singular points,

$$\theta^{(p)}(x) \bullet f'(x) := O(x). \tag{8.68}$$

At the regular points $f'(x) \in \mathbf{PC}'_\lambda$ is by definition asymptotic to a Taylor series, so the product can be defined by

$$\begin{aligned} \theta^{(p)}(x-x_0) \bullet f'(x) &:= \theta^{(p)}(x-x_0) \bullet f_a(x;x_0) = \\ &= \sum_{j=0}^{\infty} \tfrac{1}{j!}\, f^{(j)}(x_0)\, \theta^{(p+j)}(x-x_0), \end{aligned} \tag{8.69}$$

at the points where $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ is $\mathbf{C}^\infty$. This assumes of course that $f(x)$ is regular at $x = x_0$, in the sense that it contains the appropriate $\theta^{(p)}(x - x_0)$ functions. If these are omitted by taking a residue in $|x - x_0|^\lambda$ the result is again zero. The preceding equations (8.68) and (8.69) can be combined to

$$\begin{aligned} \theta^{(p)}(x-x_0) \bullet f'(x) &= \theta^{(p)}(x-x_0) \bullet f_a(x;x_0) = \\ &= \sum_{\{j:\lambda_j = k \in \mathbb{N}\}} \tfrac{1}{k!}\, f^{(k)}(x_0)\, \theta^{(p+k)}(x-x_0), \end{aligned} \tag{8.70}$$

which leaves only the contribution from the regular terms of the asymptotic expansion.
The three different products $\circ\bullet, \bullet, \bullet\circ : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}'_\lambda$ are now defined for all pairs $f'(x), g'(x) \in \overline{\mathbf{PC}}'_\lambda$.

The scalar product defined previously can be recovered from the commutative product by

$$\langle\, f(x)\,, g(x)\,\rangle := \langle\, I(x)\,, f(x)^* \bullet g(x)\,\rangle, \tag{8.71}$$

where it should be noted that $I(x)$ is the element $I(x) \in \mathbf{PC}_\lambda$, not $I'(x) \in \overline{\mathbf{PC}}'_\lambda$.

8.3 Associativity and commutativity

Multiple products can be defined as repeated products of pairs. It is clear that the symmetrical product involving more than two factors is not necessarily associative.

Example 8.1 The triple product

$$\eta_\downarrow(x) \bullet \big(H'(x) \bullet H'(x)\big) = \eta_\downarrow(x) \bullet H'(x) = \tfrac{1}{2}\,\eta_\downarrow(x), \tag{8.72}$$

differs from the rearranged form

$$\big(\eta_\downarrow(x) \bullet H'(x)\big) \bullet H'(x) = \tfrac{1}{2}\,\eta_\downarrow(x) \bullet H'(x) = \tfrac{1}{4}\,\eta_\downarrow(x), \tag{8.73}$$

by a factor two.

The non-commutative left- and right-sided products are associative as long as the order of the factors is respected.

For all $f'(x)$, $g'(x)$, and $h'(x) \in \overline{\mathbf{PC}}'_\lambda$ one easily verifies that

$$\big(f'(x) \circ\!\bullet\, g'(x)\big) \circ\!\bullet\, h'(x) = f'(x) \circ\!\bullet \big(g'(x) \circ\!\bullet\, h'(x)\big), \tag{8.74}$$

and also for the right-sided products

$$\big(f'(x) \bullet\!\circ\, g'(x)\big) \bullet\!\circ\, h'(x) = f'(x) \bullet\!\circ \big(g'(x) \bullet\!\circ\, h'(x)\big). \tag{8.75}$$

There are no factors $\frac{1}{2}$ to spoil the associativity. One can also go back to the scalar product to verify that the residues corresponding to (8.75) can be taken in arbitrary order. It is of course impossible to change the order of the terms in (8.75) cyclically as this interchanges non-commuting products.

For an arbitrary product $\diamond$ the product commutator is defined by analogy with the corresponding definition (8.37) in the previous section by

$$[f \,{\scriptstyle\diamond}\, g] := f \diamond g - g \diamond f. \tag{8.76}$$

The kind of product is indicated by the lowered product symbol, which replaces the comma in the usual commutator. It is not convenient to use the standard commutator notation $[f, g]$ since there are too many different products.

Likewise the associator of an arbitrary product $\diamond$ is defined by

$$[f \,{\scriptstyle\diamond}\, g \,{\scriptstyle\diamond}\, h] := (f \diamond g) \diamond h - f \diamond (g \diamond h). \tag{8.77}$$

The associator expresses the lack of associativity of a product. Like the more familiar commutator it can be used to convert a product into a form with parenthesis placed differently.

The general commutator properties are

$$\big[f(x) \circ\!\bullet\, g(x)\big] = -\big[f(x) \bullet\!\circ\, g(x)\big], \tag{8.78}$$

and consequently

$$\big[f(x) \bullet g(x)\big] = 0(x). \tag{8.79}$$

In addition in $\overline{\mathbf{PC}}'_\lambda$ there are the associator properties (8.75) of the non-commutative products

$$\big[f(x) \circ\!\bullet\, g(x) \circ\!\bullet\, h(x)\big] = \big[f(x) \bullet\!\circ\, g(x) \bullet\!\circ\, h(x)\big] = 0(x), \tag{8.80}$$

$\forall f, g, h \in \overline{\mathbf{PC}}'_\lambda$, which will *not* carry over to model as a whole.

Using the associativity of the asymmetrical products the associator of the symmetrical product can be expressed in the commutators of the symmetrical products. The result is

$$\begin{aligned}\big[f(x) \bullet g(x) \bullet h(x)\big] &= \tfrac{1}{4}\big[g(x) \circ\!\bullet\, \big[f(x) \circ\!\bullet\, h(x)\big]\big] = \\ &= \tfrac{1}{4}\big[g(x) \bullet\!\circ\, \big[f(x) \bullet\!\circ\, h(x)\big]\big].\end{aligned} \tag{8.81}$$

The general case of a product which is neither associative nor commutative occurs in the next chapter.

Remark 8.4 Symmetrization can always be used to convert any associative non-commutative algebra into a commutative non-associative algebra.

The basic non-vanishing product commutators are

$$\begin{aligned}&\big[|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) \circ\!\bullet\, \eta_s^{(\beta,r)}(x) \operatorname{sgn}^n(x)\big] = \\ &\qquad = \begin{cases} 0(x) & q > r, \\ (-)^q \eta_s^{(\beta-\alpha, r-q)}(x) \operatorname{sgn}^{m+n}(x) & q \le r, \end{cases}\end{aligned} \tag{8/82}$$

and

$$\big[\theta^{(p)}(x) \circ\!\bullet\, \eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x)\big] = -\,\eta_s^{(\alpha-p,q)}(x) \operatorname{sgn}^{m+p}(x), \tag{8.83}$$

so for the regular powers we have

$$\big[\big(x'^p + \theta^{(p)}(x)\big) \circ\!\bullet\, \eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x)\big] = 0(x). \tag{8.84}$$

All other product commutators also vanish.

The asymmetrical products of the basic functions are commutative unless one of the factors is an η–function, and the other factor is a non-integral power. For positive integral powers the product is commutative when the correct amount of θ–function is present. In the terminology introduced with (8.61) a power of x product commutes with an η–function if and only if it is regular. The result of any commutator is always an η– or an $\not\eta$–function, so the commutator space is spanned by the allowed linear combinations of η– and $\not\eta$–functions.

From the relation (8.81) between the associator of the symmetrical product and the commutators of the asymmetrical products one sees that the symmetrical product of basic functions is non-associative if and only if both of the repeated commutators in (8.81) are non-zero. This can happen only in the case that one, and no more than one, of the outer factors is an η–function, and if both the other factors are singular powers. The only nonvanishing associator of the symmetrical product is thus

$$\left[|x'|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x)\bullet|x'|^{\beta}\log^{r}|x|\,\mathrm{sgn}^{n}(x)\bullet\eta_{s}^{(\gamma,s)}(x)\,\mathrm{sgn}^{p}(x)\right]= \\ =\begin{cases} 0(x) & q+r>s, \\ \frac{1}{4}(-)^{q+r}\eta_{s}^{(\gamma-\alpha-\beta,s-q-r)}(x)\,\mathrm{sgn}^{p-m-n}(x) & q+r\le s, \end{cases} \qquad (8/85)$$

and likewise with $\circ\bullet$ replaced by $\bullet\circ$, or with the η–function on the left. Associators with the η–function in the middle such as

$$\left[H'(x)\bullet\eta_{1}(x)\bullet H'(x)\right]=0(x), \qquad (8.86)$$

are always zero.

Products of ordinary functions with η–functions are associative or commutative if and only if the products of the η–functions with all components of the asymptotic expansions are associative or commutative. The associator space of the symmetrical product is the same as the commutator space of the asymmetrical product.

In $\overline{\mathbf{PC}}'_{\lambda}$ the commutators of the asymmetrical products satisfy the cyclic commutator property (the Jacobi identity)

$$\left[f\circ\!\bullet\left[g\circ\!\bullet h\right]\right]+\left[g\circ\!\bullet\left[h\circ\!\bullet f\right]\right]+\left[h\circ\!\bullet\left[f\circ\!\bullet g\right]\right]=0\,. \qquad (8.87)$$

Likewise the associator of the symmetrical product satisfies the cyclic associator property

$$\left[f\bullet g\bullet h\right]+\left[g\bullet h\bullet f\right]+\left[h\bullet f\bullet g\right]=0, \qquad (8.88)$$

as one verifies by elementary algebra. These properties do not carry over to the model as a whole. The general case occurs in the next section.

8.4 The convolution product on $\overline{\mathbf{PC}}'_\lambda$

The convolution products $*$, and also $\circ *$ and $* \circ$, are defined as the Fourier image of the corresponding pointwise products by

$$f' \circ * \, g' := \mathcal{F}^{-1}(\mathcal{F}f' \circ \bullet \, \mathcal{F}g'), \tag{8.89}$$

and

$$f' * g' := \mathcal{F}^{-1}(\mathcal{F}f' \bullet \mathcal{F}g'). \tag{8.90}$$

Conversely one has

$$f' \bullet g' = \mathcal{F}(\mathcal{F}^{-1}f' * \mathcal{F}^{-1}g'), \tag{8.91}$$

as one sees by replacing $\mathcal{F}f'$ by f'.

It is convenient to define the left-sided convolution products with the same handedness as the ordinary product,

$$f \circ * \, g := \mathcal{F}^{-1}\big(\mathcal{F}f \circ \bullet \, \mathcal{F}g\big), \tag{8.92}$$

since this makes $\mathcal{D}$ on $\circ *$ obey the same rules as $\mathcal{X}$ on $\circ \bullet$.

Remark 8.5 It follows from the parity properties of the product and the Fourier operator that equation (8.90) can also be written in the form

$$f' * g' = 2\pi \mathcal{F}\big(\mathcal{F}^{-1}f' \bullet \mathcal{F}g'\big). \tag{8.93}$$

These two forms are equal.

The properties of the convolution product follow at once from the corresponding properties of the pointwise product. The two products are unitarily equivalent, and the product algebras are isomorphic. Therefore it is unnecessary to rewrite the previous sections for the convolution products.

Computation of convolutions is a straightforward application of the definition. The actual computation of convolutions is a tedious exercise however. The formulæ are much bigger than the corresponding pointwise product formulæTherefore only some illustrative examples are given.

The convolution product does not have a unit element in $\overline{\mathbf{PC}}'_\lambda$. The best approximation to a unit element is the generalised function $\mathcal{F}^{-1}\big(I'(x)+\theta(x)\big)$

$$\mathcal{F}^{-1}\big(I'(x) + \theta(x)\big) = \eta(x) + \tfrac{2}{\pi} \sum_{j=1}^{\infty} (-)^j j! \; c_{j+1}(0,0) \, \eta^{0,j)}(x). \tag{8.94}$$

The exceptional cases are the localized generalised functions $\eta^{(\alpha,q)}(x - x_0)$,

$$\eta(x) * \eta_s^{(\alpha,q)}(x - x_0) \operatorname{sgn}^m(x - x_0) = \tfrac{1}{2}\, \eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x - x_0), \tag{8.95}$$

with $(\alpha, q) \neq (p, 0)$, and $\theta^{(p)}(x - x_0)$

$$\eta(x) * \theta^{(p)}(x - x_0) = \tfrac{1}{2}\, \theta^{(p)}(x - x_0), \tag{8.96}$$

in the finite, which receive an additional factor $\frac{1}{2}$. This contrasts with the generalised functions at infinity

$$\eta(x) * \not\eta_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x) = \not\eta_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x), \tag{8.97}$$

for which $\eta(x)$ is the unit element.

More generally, convolution with the generalised function $(-)^p p!\, \eta^{(p)}(x)$ corresponds to repeated differentiation, again with an additional factor $\frac{1}{2}$ for the exceptional elements in the finite.
The convolution of the η–functions with each other is

$$\begin{aligned}\eta_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x) * \eta_s^{(\beta,r)}(x)\, \mathrm{sgn}^n(x) &= \\ = + \tfrac{1}{2}\, \delta_{q,0} \sum_{p=0}^{\infty} \tfrac{(-)^p}{p!}\, \delta_{\alpha,p}\, \delta_{m,p}^{\mathrm{mod}2}\, \mathcal{D}^p\, \eta_s^{(\beta,r)}(x)\, \mathrm{sgn}^n(x) &+ \\ + \tfrac{1}{2}\, \delta_{r,0} \sum_{p=0}^{\infty} \tfrac{(-)^p}{p!}\, \delta_{\beta,p}\, \delta_{n,p}^{\mathrm{mod}2}\, \mathcal{D}^q\, \eta_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x), &\end{aligned} \tag{8.98}$$

with the repeated derivative given explicitly by (6.36).

Convolution of an arbitrary generalised function $f'(x)$ with an $\eta^{(\alpha,q)}$ function with $(\alpha, q) \neq (p, 0)$ yields an effectively finite sum of $\theta^{(p)}(x - x_j)$ and $\eta(x - x_j)$ functions at the discrete set of singular points $\{x_j\} \in \mathbb{R}$ of $f'(x)$.

For generalised functions which are ordinary functions $\in \overline{\mathbf{PC}}'_\lambda \cap \mathcal{L}_2$ the convolution product takes the classical form

$$f'(x) * g'(x) = \int_{-\infty}^{\infty} dy\; f'(x - y)\, g'(y), \tag{8.99}$$

by standard theorems. [Tich].

The convolution product does not have the support property of the pointwise product. The convolution of a generalised function at infinity with a function of bounded support is usually non-zero.

Example 8.2 The convolution product

$$\not\eta(x) * \eta(x) = \eta(x). \tag{8.100}$$

is non-zero. Likewise the convolutions $\not\eta(x) * H(1 - |x|)$ and $\not\eta(x) * e^{-x^2}$ are non-zero. For integrable ordinary functions the leading term of the convolution equals

$$\not\eta(x) * f'(x) = (2\pi)^{-1} \not\eta(x) \int_{-\infty}^{\infty} dx\; f(x) + \cdots, \tag{8.101}$$

as one might guess from the classical form of the convolution integral.

The possibility of obtaining the convolution product by regularization of convolution integrals will be discussed in Ch. 22. This will be shown to be impossible. There seems to be no natural way to obtain a non-associative product by regularization.

The causality aspects of the convolution will be left for Ch. 21, when the convolution on $\mathbf{GF}_s$ is available, and the Hilbert transform has been introduced.

8.5 Operator and product properties

The product and convolution $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}'_\lambda$ have all the properties required in Ch. 2. It is sufficient to consider only the pointwise product. By unitary equivalence, using (2.22–23) and the definition of the convolution (2.49) all properties can be transferred to the convolution product when $\mathcal{X}$ and $i\mathcal{D}$ are interchanged.

Actually the operator properties in $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$ are better than required. The operator $\mathcal{X}$ is multiplicative on $\overline{\mathbf{PC}}'_\lambda$

$$\mathcal{X}(f' \bullet g') = \mathcal{X}f' \bullet g' = f' \bullet \mathcal{X}g', \tag{8.102}$$

$\forall f', g \in \overline{\mathbf{PC}}'_\lambda$, and for all choices left/right/symmetrical of the product. Verification is done by inspection of the products of the basic functions.

The product satisfies the Leibniz rule (2.39) for $\mathcal{D}$ on $\circ\bullet$ and $\bullet\circ$, and therefore on $\bullet$

$$\mathcal{D}(f' \bullet g') = (\mathcal{D}f') \bullet g' + f' \bullet (\mathcal{D}g'), \tag{8.103}$$

as one verifies for $\circ\bullet$ and $\bullet\circ$ by direct computation. The Leibniz rule for $\mathcal{X}$ on the convolutions

$$\mathcal{X}(f' * g') = \mathcal{X}f' * g' + f' * \mathcal{X}g'. \tag{8.104}$$

then follows from (2.22–23) and (2.49) by Fourier transformation.

The Leibniz rule also holds for multiple products, since these can only be defined as repeated pair-wise products. Therefore the Leibniz rule has to be applied pair-wise.

In the commutative version the parentheses have to be respected in keeping with the non-associativity

$$\begin{aligned}\mathcal{D}\big(f \bullet (g \bullet h)\big) &= (\mathcal{D}f) \bullet (g \bullet h) + f \bullet \mathcal{D}(g \bullet h) \\ &= (\mathcal{D}f) \bullet (g \bullet h) + f \bullet \big((\mathcal{D}g) \bullet h\big) + f \bullet \big(g \bullet (\mathcal{D}h)\big)\end{aligned} \tag{8.105}$$

with some superfluous additional parenthesis inserted for clarity.

In the associative version of the product the parentheses are irrelevant, but the order of the terms has to be respected in keeping with the non-commutativity.

Example 8.3 It is incorrect to differentiate the as yet undefined expression $\left(H'\right)^m(x)$, assuming for lack of a better definition that $\left(H'\right)^m(x) = H'(x)$, to obtain by misuse of Leibniz's rule

$$\mathcal{D}\left(H'\right)^m(x) = m \cdot \left(H'\right)^{m-1}(x) \bullet \mathcal{D}H'(x) = \tfrac{m}{2} \cdot \left(\eta_\downarrow(x) - \eta\!\!\!/_\uparrow(x)\right),$$

and

$$\mathcal{D}\left(H'\right)^m(x) = \mathcal{D}H'(x) = \left(\eta_\downarrow(x) - \eta\!\!\!/_\uparrow(x)\right), \tag{8.106}$$

$\forall m \in \mathbb{N}$, which would be a contradiction for $m \neq 2$. It is correct to differentiate the repeated product of three terms pair-wise,

$$\mathcal{D}\left(H'(x) \bullet \left(H'(x) \bullet H'(x)\right)\right) = \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4}\right)\left(\eta_\downarrow(x) - \eta\!\!\!/_\uparrow(x)\right) = \mathcal{D}H'(x), \tag{8.107}$$

with the (not unexpected) correct result. By induction this can be extended to all $m \in \mathbb{N}$ if the as yet undefined expression $\left(H'\right)^m(x)$ is defined by

$$\left(H'\right)^{m+2}(x) := H'(x) \bullet \left(H'\right)^{m+1}(x), \qquad \left(H'\right)^1(x) := H'(x), \tag{8.108}$$

or even by an arbitrary insertion of properly nested parenthesis, which are respected by the differential operator.

In the non-commutative product version the multiple products can also be differentiated correctly but in this case the order of the terms has to be respected.

Example 8.4 The previous product (8.107) of Heaviside functions is in the non-commutative left-sided version

$$\mathcal{D}\left(H'(x) \circ\!\bullet\, H'(x) \circ\!\bullet\, H'(x)\right) = (0 + 0 + 1)\left(\eta_\downarrow(x) - \eta\!\!\!/_\uparrow(x)\right) = \mathcal{D}H'(x), \tag{8.109}$$

which is also correct. The same holds in the right-sided version. The term which gives the non-zero contribution depends on the choice of the right- or left-sided version of the product but the result is correct in both cases.

Remark 8.6 It is best to indicate multiplication dots explicitly and to avoid undefined expressions such as $H'^m(x)$ or $H'(x)^m$, unless the meaning is clear from the context.

The parity operator can be taken into the product in agreement with its standard behaviour (2.44)

$$\mathcal{P}\left(f'(x) \bullet g'(x)\right) = \mathcal{P}\, f'(x) \bullet \mathcal{P}\, g'(x). \tag{8.110}$$

The operator $\mathcal{X}$ is selfadjoint in the scalar product on $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$, since it is multiplicative (8.102) in $\overline{\mathbf{PC}}'_\lambda$.

$$\left\langle \mathcal{X} f', g' \right\rangle = \left\langle f', \mathcal{X} g' \right\rangle = \left\langle I, \mathcal{X}(f \bullet g) \right\rangle. \tag{8.111}$$

Therefore by unitary equivalence of $\mathcal{X}$ and $i\mathcal{D}$ the operator $i\mathcal{D}$ is also selfadjoint in $\overline{\mathbf{PC}}'_\lambda$

$$\langle \mathcal{D}f' , g' \rangle = - \langle f' , \mathcal{D}g' \rangle . \tag{8.112}$$

$\forall f', g' \in \overline{\mathbf{PC}}'_\lambda$. In combination with the Leibniz rule (2.39) for differentiation of a product, this implies that

$$\langle I , \mathcal{D} f'(x) \rangle = 0, \tag{8.113}$$

$\forall f'(x) \in \overline{\mathbf{PC}}'_\lambda$. In integral notation (to be introduced in Ch. 14) this appears in the form

$$\int_{-\infty}^{\infty} dx \, \mathcal{D} f'(x) = 0, \tag{8.114}$$

$\forall f'(x) \in \overline{\mathbf{PC}}'_\lambda$. The good functions $f(x) \in \mathcal{S}$ also satisfy (8.114).

It should be noted that the selfadjoint character of these operators holds only on $\overline{\mathbf{PC}}'_\lambda$. It will not be possible to carry over this property to the whole model. The selfadjointness properties are collected in Sec. 12.6.

The Fourier operator is (up to the normalization) unitary in the scalar product. (Parseval's equality (2.29)),

$$\langle f' , g' \rangle = (2\pi)^{-1} \langle \mathcal{F}f' , \mathcal{F}g' \rangle , \tag{8.115}$$

$\forall f', g' \in \overline{\mathbf{PC}}'_\lambda$. This will be verified in Ch. 12.

It is clear that the subspace $\overline{\mathbf{PC}}'_\lambda \subset \mathbf{GF}$ has much better operator properties than required for $\mathbf{GF}$ as a whole.

8.6 Summary of the product properties

The product properties are in agreement with the requirements of Ch. 2.

The product satisfies Leibniz's rule for differentiation, and the operator $\mathcal{X}$ is both left- and right-multiplicative.

The asymmetrical products are completely associative, and the symmetrical product is almost associative. The only source of non-associativity is the factor $\frac{1}{2}$ resulting from the symmetrization.

The subspace $\overline{\mathbf{PC}}'_\lambda$ is closed under the product, and conversely every element $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ can be written (in many ways) as the product of two other elements.

The only shortcoming of the product on $\overline{\mathbf{PC}}'_\lambda$ is the absence of a unit element for the product as required in Ch. 2.

In the next chapter the product and convolution are transferred to $\overline{\mathbf{PC}}_\lambda$ in such a way that the product acquires a unit element, while keeping as many of the good properties of $\overline{\mathbf{PC}}'_\lambda$ as possible.

It will be seen there that the gain of a unit element for the product must be paid for by a loss of associativity of the products and the loss of complete selfadjointness of the operators $\mathcal{D}$ and $\mathcal{X}$.

It may well be preferable in many cases to avoid the need for a unit element for the product (and a zero element for differentiation) so that it remains possible to work entirely in the space $\overline{\mathbf{PC}}'_{\lambda}$.

CHAPTER 9

PRODUCTS ON THE COMPLETED PRELIMINARY CLASS

In this chapter the pointwise product and the convolution product are transferred from the space of linear functionals $\overline{\mathbf{PC}}'_\lambda$ to the completed preliminary class $\overline{\mathbf{PC}}_\lambda$. This is accomplished, while keeping most of the good properties of $\mathbf{PC}'_\lambda$, by choosing suitable transfer mappings. The product $\bullet : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}_\lambda$ is then defined $\forall f, g \in \overline{\mathbf{PC}}_\lambda$ by

$$f \bullet g := \mathcal{M}^{-1}(\mathcal{M}f \bullet \mathcal{M}g), \tag{9.1}$$

with $\mathcal{M}f,\ \mathcal{M}g \in \overline{\mathbf{PC}}'_\lambda$ and $\mathcal{M}^{-1} : \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}_\lambda$ to be defined below. The left-sided and the right-sided products are also defined by (9.1), with $\bullet$ replaced by $\circ\bullet$ and $\bullet\circ$ respectively. The properties of the product are defined in the next chapter.

9.1 Mappings

This section begins with a summary of the mappings defined previously. The linear functionals on $\mathbf{PC}_\lambda$ and a mapping $\mathcal{M} : \mathbf{PC}_\lambda \to \mathbf{PC}'_\lambda$ were defined in Sec. 5.1. This mapping was extended in Sec. 7.1 to $\overline{\mathcal{M}} : \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}'_\lambda$, defined on the completion of $\mathbf{PC}_\lambda$ in such a way that its inverse $\overline{\mathcal{M}}^{-1} : \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}_\lambda$ is as simple as possible. The natural mapping $\mathcal{M}$ is generated by

$$\overline{\mathcal{M}}\ \delta^{(p)}(x - x_0) = \eta^{(p)}(x - x_0), \tag{9.2}$$

and

$$\begin{aligned}\overline{\mathcal{M}}\ |x - x_0|^\alpha \log^q|x - x_0| \operatorname{sgn}^m(x - x_0) = \\ = |x' - x_0|^\alpha \log^q|x - x_0| \operatorname{sgn}^m(x - x_0),\end{aligned} \tag{9.3}$$

for the δ–functions and the powers. For ordinary functions at their $\mathbf{C}^\infty$–points the mapping identifies the function and the linear functional.
The inverse mapping $\overline{\mathcal{M}}^{-1} : \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}_\lambda$ can be read off directly from the formulæ above,

$$\overline{\mathcal{M}}^{-1}\, \eta^{(p)}(x - x_0) = \delta^{(p)}(x - x_0), \tag{9.4}$$

for the non-zero case, and more generally for all η–functions

$$\overline{\mathcal{M}}^{-1}\, \eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x)(x) = \delta_{q,0} \sum_{p=0}^{\infty} \delta_{\alpha,p}\, \delta_{m,0}^{\mathrm{mod}2}\, \delta^{(p)}(x), \tag{9.5}$$

with the non-zero cases indicated explicitly. For the powers we obtain the simple result

$$\overline{\mathcal{M}}^{-1}|x' - x_0|^\alpha \log^q|x - x_0| \operatorname{sgn}^m(x - x_0) =$$
$$= |x - x_0|^\alpha \log^q|x - x_0| \operatorname{sgn}^m(x - x_0). \tag{9.6}$$

There are many elements $\in \overline{\mathbf{PC}}'_\lambda$ which are not in the range $\overline{\mathbf{PC}}'_{\mathcal{M}} \subset \overline{\mathbf{PC}}'_\lambda$ of the map $\overline{\mathcal{M}}$. For the complement of $\overline{\mathbf{PC}}'_{\mathcal{M}}$ the symbol $\overline{\mathbf{PC}}'_{\not\mathcal{M}} = \overline{\mathbf{PC}}'_\lambda \ominus \overline{\mathbf{PC}}'_{\mathcal{M}}$ is used. The symbols $\boldsymbol{P}_{\mathcal{M}}$ and $\boldsymbol{P}_{\not\mathcal{M}}$ are used for the respective projection operators. The projection operator $\boldsymbol{P}_{\mathcal{M}}$ eliminates all θ–functions and all η–functions with the exception of $\eta^{(p)}(x - x_0)$. It leaves only those elements $f'(x) \in \overline{\mathbf{PC}}'_{\mathcal{M}} \subset \overline{\mathbf{PC}}'_\lambda$ which have a non-zero counterpart in distribution theory.
This is equivalent to

$$f'(x) \in \overline{\mathbf{PC}}'_{\mathcal{M}} \iff \exists\, g(x) \in \mathcal{S} : \langle f'(x) , g(x) \rangle \neq 0, \tag{9.7}$$

and conversely for the complement $\overline{\mathbf{PC}}'_{\not\mathcal{M}}$

$$f'(x) \in \overline{\mathbf{PC}}'_{\not\mathcal{M}} \iff \langle f'(x) , g(x) \rangle = 0 \;\; \forall\, g(x) \in \mathcal{S}, \tag{9.8}$$

where $\mathcal{S}$ is the Schwartz space of $\mathbf{C}^\infty$–functions of rapid decrease.
For the elements of $\overline{\mathbf{PC}}'_{\not\mathcal{M}}$ the inverse mapping is defined to be zero,

$$\overline{\mathcal{M}}^{-1}\, \theta^{(p)}(x) = \overline{\mathcal{M}}^{-1}\, \not\eta^{(\alpha,q)}(x) = \mathit{0}(x), \tag{9.9}$$

for all elements which have no counterpart as distributions.

The notation $\mathcal{M}^{-1}$ may be somewhat misleading. It is accurate on the restriction $\overline{\mathbf{PC}}'_{\mathcal{M}} = \boldsymbol{P}_{\mathcal{M}}\overline{\mathbf{PC}}'_\lambda \subset \overline{\mathbf{PC}}'_\lambda$, but not on $\overline{\mathbf{PC}}'_\lambda$ as a whole. The operator relations between $\mathcal{M}$ and $\mathcal{M}^{-1}$ are

$$\overline{\mathcal{M}}\;\overline{\mathcal{M}}^{-1} = \mathcal{I} - \boldsymbol{P}_{\not\mathcal{M}} = \boldsymbol{P}_{\mathcal{M}}, \tag{9.10}$$

and

$$\overline{\mathcal{M}}^{-1}\overline{\mathcal{M}} = \mathcal{I}, \tag{9.11}$$

$\forall\, f(x) \in \overline{\mathbf{PC}}_\lambda$. The excepted subspace can be seen explicitly in this form.

For convenience the contents of the various **PC**–classes and the relations between them are collected in the following table.

Table 9.1

PC...	Contents of PC...
$\mathbf{PC}$	An unspecified class of 'ordinary' functions used as a starting point
$\mathbf{PC}_\lambda$	Piecewise $\mathbf{C}^\infty$–functions with power type singularities
$\mathbf{PC}_\delta$	All allowed linear combinations of δ–functions
$\overline{\mathbf{PC}}_\lambda$	The direct sum of the two previous classes
$\mathbf{PC}'$	The undefined class of 'all' linear functionals on $\mathbf{PC}$
$\mathbf{PC}'_\lambda$	The space $\mathbf{PC}_\lambda$, considered as linear functionals on $\mathbf{PC}_\lambda$
$\mathbf{PC}'_\eta$	All allowed linear combinations of η–functions
$\mathbf{PC}'_{\not\eta}$	All allowed linear combinations of $\not\eta$–functions
$\mathbf{PC}'_{\theta^p}$	All allowed linear combinations of $\theta^{(p)}$–functions
$\overline{\mathbf{PC}}'_\lambda$	The direct sum of the previous four classes
$\overline{\mathbf{PC}}'_{\mathcal{M}}$	The range of the map $\mathcal{M}$ and also the elements $\in \overline{\mathbf{PC}}'_\lambda$ which are non-zero as distributions
$\overline{\mathbf{PC}}'_{\not\mathcal{M}}$	All elements of $\overline{\mathbf{PC}}'_\lambda$ which are zero as distributions
$\mathbf{PC}^\oplus_\lambda$	The direct sum $\overline{\mathbf{PC}}_\lambda \oplus \overline{\mathbf{PC}}'_\lambda$
$\mathbf{GF}$	An unspecified model for the requirements of Ch. 2
$\mathbf{GF}_t$	The trivial model of Ch. 3
$\mathbf{GF}_s$	The direct sum $\mathbf{PC}^\oplus_\lambda$, reduced by identifying the common elements of $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$, (to be defined in Ch. 11)

The next table gives the properties of the various maps.

Table 9.2

Map	Domain	Range	···jective
$\mathcal{M}$	$\mathbf{PC}_\lambda$	$\mathbf{PC}'_\lambda$	bijective
$\mathcal{M}^{-1}$	$\mathbf{PC}'_\lambda$	$\mathbf{PC}_\lambda$	bijective
$\overline{\mathcal{M}}$	$\overline{\mathbf{PC}}_\lambda$	$\overline{\mathbf{PC}}'_\lambda$	injective
$\overline{\mathcal{M}}^{-1}$	$\overline{\mathbf{PC}}'_\lambda$	$\overline{\mathbf{PC}}_\lambda$	surjective
$\overline{\mathcal{M}}$	$\overline{\mathbf{PC}}_\lambda$	$\overline{\mathbf{PC}}'_{\mathcal{M}}$	bijective
$\overline{\mathcal{M}}^{-1}$	$\overline{\mathbf{PC}}'_{\mathcal{M}}$	$\overline{\mathbf{PC}}_\lambda$	bijective

The relations between the various $\mathbf{PC}$–spaces are collected in the following diagram

$$
\begin{array}{ccc}
\mathbf{PC}_\lambda & \overset{\mathcal{M}^{-1}}{\longleftarrow} \; \overset{\mathcal{M}}{\longrightarrow} & \mathbf{PC}'_\lambda \\
\Big\downarrow \oplus \mathbf{PC}_\delta & & \Big\downarrow \oplus \mathbf{PC}'_\eta \oplus \mathbf{PC}'_{\not\eta} \oplus \mathbf{PC}'_{\theta P} \\
\overline{\mathbf{PC}}_\lambda & \overset{\overline{\mathcal{M}}^{-1}}{\longleftarrow} \; \overset{\overline{\mathcal{M}}}{\longrightarrow} & \overline{\mathbf{PC}}'_\lambda = \overline{\mathbf{PC}}'_{\mathcal{M}} \oplus \overline{\mathbf{PC}}'_{\not\mathcal{M}} \\
\Big\updownarrow & & \Big\downarrow P_{\mathcal{M}} \\
\overline{\mathbf{PC}}_\lambda & \overset{\overline{\mathcal{M}}^{-1}}{\longleftarrow} \; \overset{\overline{\mathcal{M}}}{\longrightarrow} & \overline{\mathbf{PC}}'_{\mathcal{M}} = P_{\mathcal{M}}\left(\overline{\mathbf{PC}}'_\lambda\right)
\end{array}
\tag{9.12}
$$

which summarizes the steps taken in the construction.
The mapping in the top line of the preceding diagram $\mathcal{M} : \mathbf{PC}_\lambda \to \mathbf{PC}'_\lambda$ is bijective. This is no longer the case in the second line, since $\overline{\mathbf{PC}}'_\lambda$ has been enlarged more than $\overline{\mathbf{PC}}_\lambda$. The completed map $\overline{\mathcal{M}} : \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}'_\lambda$ has a large zero space $\overline{\mathbf{PC}}'_{\not\mathcal{M}} \subset \overline{\mathbf{PC}}'_\lambda$. After removing this zero space with the projection operator $P_{\mathcal{M}} = \overline{\mathcal{M}}^{-1}\overline{\mathcal{M}}$ the mapping in the third line $\overline{\mathcal{M}} : \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}'_{\mathcal{M}}$ is again bijective. The term δ-equivalent will be used to denote elements $\in \overline{\mathbf{PC}}'_\lambda$ which have the same image under $P_{\mathcal{M}}$. The properties of the maps are summarized in table 9.2 on the preceding page.

In Ch. 11 the spaces $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ are joined to form the space $\mathbf{GF}_s$ of symmetrical generalised functions. The mappings then reduce to $\overline{\mathcal{M}} : \mathbf{GF}_s \to \mathbf{GF}_s$.

The mapping $\overline{\mathcal{M}} : \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}'_\lambda$ and its inverse $\overline{\mathcal{M}}^{-1} : \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}_\lambda$ given above are now renamed to $\mathcal{M}_{\mathcal{X}}$ and $\mathcal{M}_{\mathcal{X}}^{-1}$ for reasons which will become clear.
It is possible to use the mapping $\mathcal{M}_{\mathcal{X}}$ to transfer products from $\overline{\mathbf{PC}}'_\lambda$ to $\overline{\mathbf{PC}}_\lambda$ by putting

$$f \bullet g := \mathcal{M}_{\mathcal{X}}^{-1}(\mathcal{M}_{\mathcal{X}} f \bullet \mathcal{M}_{\mathcal{X}} g). \tag{9.13}$$

WRONG!

This will not lead to a product with good properties. In particular Leibniz's rule (2.39) for the differentiation of a product will not be satisfied in $\overline{\mathbf{PC}}_\lambda$, even though it is satisfied in $\overline{\mathbf{PC}}'_\lambda$. Leibniz's rule in $\overline{\mathbf{PC}}_\lambda$ could be derived in $\overline{\mathbf{PC}}_\lambda$ if the mapping $\mathcal{M}_{\mathcal{X}}$ (and its inverse) commuted with differentiation, but

$$\mathcal{M}_{\mathcal{X}} \mathcal{D} f(x) \neq \mathcal{D} \mathcal{M}_{\mathcal{X}} f(x), \tag{9.14}$$

for many elements $f(x) \in \mathbf{PC}_\lambda$.

Example 9.1 The obvious counterexample by (7.49) and (6.26) is the unit function

$$0(x) = \mathcal{M}_{\mathcal{X}}\mathcal{D}\, I(x) \neq \mathcal{D}\mathcal{M}_{\mathcal{X}}\, I(x) = \mathcal{D}\, I'(x) = 2\,\sigma(x) - 2\,\not{\sigma}(x). \tag{9.15}$$

This phenomenon occurs more generally for arbitrary powers

$$\mathcal{M}_{\mathcal{X}}\mathcal{D}|x|^{\alpha}\operatorname{sgn}(x) = \mathcal{M}_{\mathcal{X}}\,\alpha|x|^{\alpha-1} = \alpha|x'|^{\alpha-1}, \tag{9.16}$$

but

$$\begin{aligned}\mathcal{D}\mathcal{M}_{\mathcal{X}}|x|^{\alpha}\operatorname{sgn}(x) = \mathcal{D}\,|x'|^{\alpha}\operatorname{sgn}(x) &= \\ = \alpha|x'|^{\alpha-1} &+ 2\big(\eta_s^{(-\alpha)}(x) - \not{\eta}_s^{(-\alpha)}(x)\big),\end{aligned} \tag{9.17}$$

valid when $\alpha \neq 2p$, $p \in \mathbb{N}$.

By inspection it is seen that the mapping $\mathcal{M}_{\mathcal{X}}$ almost commutes with the $\mathcal{X}$ operator. It satisfies

$$\mathcal{M}_{\mathcal{X}}\mathcal{X}\, f(x) = \mathcal{X}\mathcal{M}_{\mathcal{X}}\, f(x), \tag{9.18}$$

on a large subspace of $\overline{\mathbf{PC}}_{\lambda}$.

It is not possible to have a mapping which commutes completely with the operator $\mathcal{X}$, since the operator $\mathcal{X}$ has by (7.33) a zero element $\delta(x) \in \overline{\mathbf{PC}}_{\lambda}$, but it does not have a zero element in $\overline{\mathbf{PC}}'_{\lambda}$.

On the restricted space $(\mathcal{I} - \boldsymbol{P}_{\delta^{(0)}})\overline{\mathbf{PC}}_{\lambda}$ the mapping $\mathcal{M}_{\mathcal{X}}$ does commute with the operator $\mathcal{X}$. In obvious notation $\boldsymbol{P}_{\delta^{(0)}}$ and $\boldsymbol{P}_I$ denote the one-dimensional projection operators on the indicated one-dimensional subspaces. The commutation rule of $\mathcal{X}$ and $\mathcal{M}_{\mathcal{X}}$ is

$$\big[\mathcal{X}\,, \mathcal{M}_{\mathcal{X}}\big] := \mathcal{X}\mathcal{M}_{\mathcal{X}} - \mathcal{M}_{\mathcal{X}}\mathcal{X} = \mathcal{X}\mathcal{M}_{\mathcal{X}}\boldsymbol{P}_{\delta^{(0)}}, \tag{9.19}$$

with the one dimensional excepted subspace span $\delta^{(0)}$ indicated explicitly.

It is now easy to construct a mapping $\mathcal{M}_{\mathcal{D}}$ which almost commutes with differentiation. The operators $\mathcal{X}$ and $\mathcal{D}$ are unitarily equivalent under the Fourier transform by equations (2.22–23). Therefore the mapping $\mathcal{M}_{\mathcal{D}}$, defined by

$$\mathcal{M}_{\mathcal{D}} := \mathcal{F}^{-1}\mathcal{M}_{\mathcal{X}}\mathcal{F}, \tag{9.20}$$

almost commutes with the differential operator $\mathcal{D}$. The commutation relation is the Fourier transform of (9.19),

$$\big[\mathcal{D}\,, \mathcal{M}_{\mathcal{D}}\big] := \mathcal{D}\mathcal{M}_{\mathcal{D}} - \mathcal{M}_{\mathcal{D}}\mathcal{D} = \mathcal{D}\mathcal{M}_{\mathcal{D}}\boldsymbol{P}_I. \tag{9.21}$$

The exceptional element is of course the unit function $I(x)$, since it is the Fourier transform of the element $\delta(x)$

It should be noted that in (9.20) the Fourier operator acts on the space $\overline{\mathbf{PC}}_\lambda$, while the inverse Fourier operator acts on $\overline{\mathbf{PC}}'_\lambda$.
Since the unitary equivalence (2.22–23) can be written in the forms

$$\mathcal{D} = i\mathcal{F}^{-1}\mathcal{X}\mathcal{F} = -i\mathcal{F}\mathcal{X}\mathcal{F}^{-1}, \tag{9.22}$$

we are at liberty to define either

$$\mathcal{M}_\mathcal{D} := \mathcal{F}^{-1}\mathcal{M}_\mathcal{X}\mathcal{F}, \qquad \text{or} \qquad \mathcal{M}_\mathcal{D} := \mathcal{F}\mathcal{M}_\mathcal{X}\mathcal{F}^{-1}. \tag{9.23}$$

Both expressions are equal since the parity operator $\mathcal{P}$ commutes with the maps $\mathcal{M}_\mathcal{X}$ and $\mathcal{M}_\mathcal{D}$,

$$\mathcal{P}\mathcal{M}_\mathcal{X} = \mathcal{M}_\mathcal{X}\mathcal{P}, \qquad \mathcal{M}_\mathcal{D}\mathcal{P} = \mathcal{P}\mathcal{M}_\mathcal{D}, \tag{9.24}$$

as one verifies by inspection.

When the definition of the Fourier operator on $\overline{\mathbf{PC}}_\lambda$ is substituted into (9.20) the map $\mathcal{M}_\mathcal{D}$ can also be written in the form

$$\mathcal{M}_\mathcal{D} = \mathcal{F}^{-1}P_\mathcal{M}\mathcal{F}\mathcal{M}_\mathcal{X}. \tag{9.25}$$

For the actual computation of the products and convolutions several possibilities are now open. A choice has to be made on the basis of convenience. The pointwise product has to be computed with the $\mathcal{M}_\mathcal{D}$ mapping which is more complicated than the $\mathcal{M}_\mathcal{X}$ mapping. On the other hand the $\mathcal{M}_\mathcal{X}$ mapping is simple, but the convolution products on $\overline{\mathbf{PC}}'_\lambda$ are more complicated expressions than the pointwise products. Moreover the pointwise product and the convolution product can always be found from each other by Fourier transformation.

The following choice is made. First the $\mathcal{M}_\mathcal{D}$–mapping is calculated explicitly for the δ–functions and the powers. It can be extended easily to the 'ordinary' functions. Then the product is calculated explicitly for the same special cases. The extension to 'ordinary' functions is made by means of the asymptotic expansions. Finally the convolution products are calculated in the simplest way, in some cases by Fourier transformation of the pointwise product, in some cases by direct computation of convolutions in $\overline{\mathbf{PC}}'_\lambda$ and the $\mathcal{M}_\mathcal{X}$ mapping.
Computing the $\mathcal{M}_\mathcal{D}$ mapping by (9.20) gives for the $\delta^{(p)}$–functions

$$\begin{aligned}\mathcal{M}_\mathcal{D}\,\delta^{(p)}(x) &:= \mathcal{F}^{-1}\mathcal{M}_\mathcal{X}\mathcal{F}\,\delta^{(p)}(x) = \\ &= \tfrac{2(-)^p}{\pi p!}\sum_{j=0}^{\infty}(-)^j j!\; c_{j+1}(p,p)\,\eta^{(p,j)}(x),\end{aligned} \tag{9.26}$$

with the c_j–coefficients defined as usual by (6.48). For the powers we obtain from (9.20) by straightforward computation

$$\begin{aligned}\mathcal{M}_{\mathcal{D}}|x|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x) &= |x'|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x)+\\ &+2\sum_{j=0}^{\infty}d_{jq}(\alpha,m)\big((-)^{j}\eta_{s}^{(-\alpha-1,j)}(x)-\not\eta_{s}^{(-\alpha-1,j)}(x)\big)\,\mathrm{sgn}^{m}(x)+\\ &+\tfrac{2}{\pi}q!\sum_{j=0}^{\infty}\tfrac{(-)^{j}}{j!}\,c_{q+1}(j,j)\,\delta_{\alpha,j}\,\delta_{j,m}^{\mathrm{mod}2}\,\theta^{(j)}(x).\end{aligned} \tag{9.27}$$

The d–coefficient is a combination of c–coefficients given explicitly by

$$\begin{aligned}d_{jq}(\alpha,m) &:= -\tfrac{2}{\pi}q!\,j!\sum_{l=-1}^{j}(-)^{l+j}c_{l}(-\alpha-1,m)\,c_{q+j+1-l}(\alpha,m)=\\ &= -\tfrac{2}{\pi}q!\,j!\sum_{l=-1}^{q}(-)^{l+q}c_{l}(\alpha,m)\,c_{q+j+1-l}(-\alpha-1,m).\end{aligned} \tag{9.28}$$

The conversion between the two forms is derived in appendix C. Comparison of the two forms gives the symmetry property of the d-coefficients

$$d_{jq}(\alpha,m)=d_{qj}(-\alpha-1,m). \tag{9.29}$$

This is sometimes useful to diminish the number of terms in summations.

Remark 9.1 The map $\mathcal{M}_{\mathcal{D}}$ obviously respects the parity of the generalised functions, but it does not respect the one-sidedness of the Heaviside functions. The expression $\mathcal{M}_{\mathcal{D}}\,x^{\lambda}H(x)$ contains not only $\eta_{\downarrow}^{(-\lambda-1,j)}(x)$–functions, but also $\eta_{\uparrow}^{(-\lambda-1,j)}(x)$–functions. It will be seen in the next section that the product of a Heaviside function on the positive side and a Heaviside function on the negative side is not necessarily zero.

Remark 9.2 The $\not\eta_s$–functions do not contribute to any product, so they can be omitted to save superfluous work.

For ordinary functions $f\in\mathbf{PC}_{\lambda}$ we find $\mathcal{M}_{\mathcal{D}}f=f'\in\mathbf{PC}'_{\lambda}$, plus a sum of η–functions at the singular points. This is defined to be the sum of η–functions generated by the mapping by $\mathcal{M}_{\mathcal{D}}$ of the corresponding asymptotic series. In formula this can be written as

$$\mathcal{M}_{\mathcal{D}}\,f(x)=\mathcal{M}_{\mathcal{X}}\,f(x)+\sum_{\{x_j\}}\big(\mathcal{M}_{\mathcal{D}}\,f_a(x;x_j)-\mathcal{M}_{\mathcal{X}}\,f_a(x;x_j)\big), \tag{9.30}$$

The sum over singular points is finite, and the resulting linear combination of η–functions is again an allowed generalised function.

The inverse mapping $\mathcal{M}_{\mathcal{D}}^{-1}$ is also found by direct computation, which gives

$$\mathcal{M}_{\mathcal{D}}^{-1}\,\eta_{s}^{(\alpha,q)}(x)\,\mathrm{sgn}^{m}(x)=\delta_{q,0}\sum_{p=0}^{\infty}\delta_{\alpha,p}\,\delta_{p,m}^{\mathrm{mod}2}\,\delta^{(p)}(x), \tag{9.31}$$

and

$$\mathcal{M}_{\mathcal{D}}^{-1}\,\theta^{(p)}(x) = \mathit{0}(x), \tag{9.32}$$

and

$$\mathcal{M}_{\mathcal{D}}^{-1}|x'|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x) = |x|^{\alpha}\log^{q}|x|\,\mathrm{sgn}^{m}(x). \tag{9.33}$$

The only case which needs special attention is

$$\mathcal{M}_{\mathcal{D}}^{-1}x'^{-p-1}\log^{q}|x|\,\mathrm{sgn}^{m}(x) = x^{-p-1}\log^{q}|x|\,\mathrm{sgn}^{m}(x), \tag{9.34}$$

where it must be verified by explicit computation that no additional $\delta^{(p)}(x)$–function is introduced.

It is not immediately clear that the operators $\mathcal{M}_{\mathcal{D}}$ and $\mathcal{M}_{\mathcal{D}}^{-1}$ have the same relation as $\mathcal{M}_{\mathcal{X}}$ and $\mathcal{M}_{\mathcal{X}}^{-1}$, since $\mathcal{F}$ and $P_{\mathcal{M}}$ do not commute. One must verify that the operators $\mathcal{M}_{\mathcal{D}}$ and $\mathcal{M}_{\mathcal{D}}^{-1}$ are related in the same way (9.11) and (9.33)5 as the operators $\mathcal{M}_{\mathcal{X}}$ and $\mathcal{M}_{\mathcal{X}}^{-1}$,

$$\mathcal{M}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}^{-1} = \mathcal{I} - P_{\not\mathcal{M}} = P_{\mathcal{M}}, \tag{9.35}$$

$\forall\, f'(x) \in \overline{\mathbf{PC}}'_{\lambda}$, and

$$\mathcal{M}_{\mathcal{D}}^{-1}\mathcal{M}_{\mathcal{D}} = \mathcal{I}, \tag{9.36}$$

$\forall\, f(x) \in \overline{\mathbf{PC}}_{\lambda}$. The only case which needs attention is that of the powers. Computation gives

$$\mathcal{M}_{\mathcal{D}}^{-1}\mathcal{M}_{\mathcal{D}}\,x^{-p-1}\log^{q}|x|\,\mathrm{sgn}(x) = x^{-p-1}\log^{q}|x|\,\mathrm{sgn}(x), \tag{9.37}$$

without additional $\delta^{(p)}$–functions. The coefficient of the $\delta^{(p)}$–functions which do appear in (9.37) equals $d_{0q}(-p-1, p)$, which is zero by (C.38).

Comparing the mappings $\mathcal{M}_{\mathcal{D}}$ and $\mathcal{M}_{\mathcal{X}}$ one sees (9.30) that the leading terms agree. The differences appear in the higher terms. The inverse mappings are identical

$$\mathcal{M}_{\mathcal{D}}^{-1} = \mathcal{M}_{\mathcal{X}}^{-1}, \tag{9.38}$$

since there are no elements in $\overline{\mathbf{PC}}_{\lambda}$ which can be used to distinguish between them.

Now that the mappings $\mathcal{M}_{\mathcal{D}}$ and $\mathcal{M}_{\mathcal{D}}^{-1}$ are known explicitly, it is possible to define a pointwise product on $\overline{\mathbf{PC}}_{\lambda}$ which satisfies Leibniz's rule for differentiation.

9.2 The pointwise product on $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda$

The products $\circ\bullet$, $\bullet\circ$, and $\bullet$ are defined by transfer from $\overline{\mathbf{PC}}'_\lambda$ using the mapping $\mathcal{M}_\mathcal{D}$ instead of $\mathcal{M}_\mathcal{X}$ as in (9.13). This results in a product which obeys Leibniz's rule for differentiation. This will be shown in the next chapter. The 'left-sided' product $\circ\bullet : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}_\lambda$ is defined as

$$f \circ\bullet\, g := \mathcal{M}_\mathcal{D}^{-1}\big(\mathcal{M}_\mathcal{D} f \circ\bullet\, \mathcal{M}_\mathcal{D}\, g\big), \tag{9.39}$$

and idem for the 'right-sided' product. The symmetrical product can be obtained either by symmetrization, or by transfer from $\overline{\mathbf{PC}}'_\lambda$.

$$f \bullet g := \mathcal{M}_\mathcal{D}^{-1}\big(\mathcal{M}_\mathcal{D} f \bullet \mathcal{M}_\mathcal{D}\, g\big). \tag{9.40}$$

The result is the same for both methods.

Given the maps $\mathcal{M}_\mathcal{D}$ and $\mathcal{M}_\mathcal{D}^{-1}$ and the $\bullet$ product on $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$ the computation of the product $\bullet : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}_\lambda$ (and also $\circ\bullet$ and $\bullet\circ$) is only a matter of carrying out the necessary substitutions. In order to save effort it may be convenient to insert the projection operator $P_\mathcal{M}$ in (9.40)

$$f \bullet g = \mathcal{M}_\mathcal{D}^{-1} P_\mathcal{M}\big(\mathcal{M}_\mathcal{D} f \bullet \mathcal{M}_\mathcal{D}\, g\big), \tag{9.41}$$

since it is useless to carry along terms which will vanish in the final result. The simplest product is that of the δ–functions

$$\delta^{(p)}(x - x_1) \bullet \delta^{(q)}(x - x_2) = 0(x), \tag{9.42}$$

$\forall p, q \in \mathbb{N}$ and $\forall x_1, x_2 \in \mathbb{R}$. The product of δ–functions located at different points is obviously zero since the supports do not overlap.

Remark 9.3 In (9.42) x_1 and x_2 have to be considered as parameters. Considered as a generalised function of more variables expressions such as

$$\delta(x - x_1)\,\delta(x - x_2) = \delta(x_1 - x_2)\,\delta(x - x_2), \tag{9.43}$$

could be written. This lies outside the scope of this book.

As a special case of (9.42) we have the result

$$\delta(x) \bullet \delta(x) = 0(x), \tag{9.44}$$

for the product of two δ–functions located at the same point.

Remark 9.4 The result $\delta \bullet \delta = 0$ is often found surprising. This is the result of mental pictures of the δ–function as a limit of a sharply peaked function. The relation of this picture to the generalised functions η and δ is given in Ch. 19 when convergence of sequences of generalised functions has been defined.

In distribution theory the result is not infinite, it is undefined, and any answer (including $\delta(0) = 0$) can be obtained for $\delta(0)$ by suitably adapting the undefined double limit. The result (9.44) arises in a natural way in all attempts to define products of distributions, since the zero distribution is the only distribution which has even parity, is homogeneous of degree -2, and has its support contained in a point. (The support of the zero distribution is empty, so it is contained in any point.)

The left- and right-sided products of δ–functions are also zero in all cases. For the left-sided product of a power and a δ–function we find

$$|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \circ\bullet\, \delta^{(r)}(x) = \tfrac{2(-)^r q!}{\pi r!}\, c_{q+1}(r,r) \sum_{j=0}^{\infty} \delta_{r-\alpha,j}\, \delta^{\text{mod}2}_{m+r,j}\, \delta^{(j)}(x), \tag{9.45}$$

since the $\theta^{(p)}$ term in (9.27) does not contribute to a left-sided product. For the special case $\alpha = m = p \in \mathbb{N}$, $q = 0$, we find

$$x^p \circ\bullet\, \delta^{(r)}(x) = \begin{cases} 0(x) & p < r, \\ \delta^{(r-p)}(x) & p \geq r, \end{cases} \tag{9.46}$$

by substituting the value $c_1(r,r) = \frac{\pi}{2}(-)^r r!$ from table C.2.

For the right-sided product only the θ–functions in (9.27) contribute and we obtain

$$|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \bullet\circ\, \delta^{(r)}(x) = \tfrac{2q!}{\pi} \sum_{j=0}^{r} \tfrac{(-)^j}{j!}\, c_{q+1}(j,j)\, \delta_{\alpha,j}\, \delta^{\text{mod}2}_{m,j}\, \delta^{(r-j)}(x), \tag{9.47}$$

which reduces for $\alpha = m = p \in \mathbb{N}$, $q = 0$ to

$$x^p \bullet\circ\, \delta^{(r)}(x) = x^p \circ\bullet\, \delta^{(r)}(x) = \begin{cases} 0(x) & p < r, \\ \delta^{(r-p)}(x) & p \geq r, \end{cases} \tag{9.48}$$

in agreement with the left-sided version (9.46). In particular for $q = 0$ we find

$$I(x) \circ\bullet\, \delta^{(p)}(x) = I \bullet\circ\, \delta^{(p)}(x) = \delta^{(p)}(x), \tag{9.49}$$

in agreement with the identification of the unit function $I(x)$ as the unit element for multiplication.

For the sgn–function one finds

$$\operatorname{sgn}(x) \circ\bullet\, \delta^{(p)}(x) = \operatorname{sgn}(x) \bullet\circ\, \delta^{(p)}(x) = 0(x), \tag{9.50}$$

as expected.

For the negative powers specialization to $\alpha = m = -p-1$ gives

$$x^{-p-1}\log^q|x| \circ\!\bullet\, \delta^{(r)}(x) = \tfrac{2(-)^r q!}{\pi r!}\, c_{q+1}(r,r)\, \delta^{(r+p+1)}(x), \tag{9.51}$$

and correspondingly

$$x^{-p-1}\log^q|x| \circ\!\bullet\, \delta^{(r)}(x) = 0(x). \tag{9.52}$$

Therefore for $q = 0$ we have the special cases

$$x^{-p-1} \circ\!\bullet\, \delta^{(r)}(x) = \delta^{(r+p+1)}(x), \tag{9.53}$$

$$x^{-p-1} \bullet\!\circ\, \delta^{(r)}(x) = 0(x), \tag{9.54}$$

Symmetrization gives the symmetrical product of a δ–function and a power is in its general form

$$\begin{aligned} &|x|^\alpha \log^q|x| \,\mathrm{sgn}^m(x) \bullet \delta^{(r)}(x) = \\ &\quad + \tfrac{q!}{\pi} \sum_{j=0}^{r} \delta_{\alpha,j}\, \delta_{j,m}^{\mathrm{mod2}} \big(\tfrac{(-)^j}{j!}\, c_{q+1}(j,j) + \tfrac{(-)^r}{r!}\, c_{q+1}(r,r) \big)\, \delta^{(r-j)}(x) + \\ &\quad + \tfrac{q!}{\pi} \sum_{j=0}^{\infty} \delta_{\alpha,-j-1}\, \delta_{j+1,m}^{\mathrm{mod2}}\, \tfrac{(-)^r}{r!}\, c_{q+1}(r,r)\, \delta^{(r+j+1)}(x), \end{aligned} \tag{9.55}$$

where the summation in (9.45) has been split and rearranged.
For positive integral powers specialization to $\alpha = m = p \in \mathbb{N}$ gives

$$\begin{aligned} &x^p \log^q|x| \bullet \delta^{(r)}(x) = \\ &= \begin{cases} 0(x) & p > r, \\ \tfrac{q!}{\pi} \big(\tfrac{(-)^r}{r!}\, c_{q+1}(r,r) + \tfrac{(-)^p}{p!}\, c_{q+1}(p,p) \big)\, \delta^{(r-p)}(x) & r \ge p. \end{cases} \end{aligned} \tag{9.56}$$

For the case $q = 0$ this reduces to

$$x^p \bullet \delta^{(r)}(x) = \begin{cases} 0(x) & p > r, \\ \delta^{(r-p)}(x) & r \ge p. \end{cases} \tag{9.57}$$

The result for positive powers is in agreement with the operator $\mathcal{X} \cdots = x \bullet \cdots$ and with the corresponding formula in distribution theory.
For the special case $q = 1$ one obtains

$$x^p \log|x| \bullet \delta^{(r)}(x) = \begin{cases} 0(x) & p > r, \\ \tfrac{1}{2} \big(\psi(p+1) + \psi(r+1) \big)\, \delta^{(r-p)}(x) & r \ge p. \end{cases} \tag{9.58}$$

For $p = r = 0$ we obtain the result

$$\log^q|x| \bullet \delta(x) = \tfrac{q!}{\pi}\, c_{q+1}(0,0)\, \delta(x), \tag{9.59}$$

with the often occurring special case

$$\log|x| \bullet \delta(x) = \psi(1)\, \delta(x). \tag{9.60}$$

The function ψ is the logarithmic derivative of the Γ–function, and $\psi(1)$ is the Euler-Mascheroni constant. This result (9.60) has to be modified when scale transformations are considered. This will be done in Ch. 18 when the scale transformation has been defined.

Remark 9.5 The result (9.60) contrasts with the result (8.48)

$$\log|x| \bullet \eta(x) = 0(x), \tag{9.61}$$

which was found in the previous chapter.

Specializing to $\alpha = m = -p - 1$, $p \in \mathbb{N}$ gives

$$x^{-p-1} \log^q|x| \bullet \delta^{(r)}(x) = \tfrac{(-)^r q!}{\pi r!}\, c_{q+1}(r,r)\, \delta^{(r+p+1)}(x). \tag{9.62}$$

and by taking $q = 0$ and $q = 1$ one obtains

$$x^{-p-1} \bullet \delta^{(r)}(x) = \tfrac{1}{2}\, \delta^{(r+p+1)}(x), \tag{9.63}$$

and

$$x^{-p-1} \log|x| \bullet \delta^{(r)}(x) = \tfrac{1}{2}\, \psi(r+1)\, \delta^{(r+p+1)}(x), \tag{9.64}$$

The factor $\frac{1}{2}$ in (9.64) and (9.63) should be noted.

Remark 9.6 In the product formulæ given above various factors $\frac{1}{2}$ occur. It is often surprising to see how these factors $\frac{1}{2}$ are necessary to give correct answers to complicated computations. As a result of the presence of the θ–function in $\overline{\mathbf{PC}}'_\lambda$ these factors $\frac{1}{2}$ automatically come out correctly. In other approaches it seems to be necessary to put them in by hand.

Example 9.2 By combining (9.57) and (9.63) one obtains the standard example of a non-associative product

$$(x^{-1} \bullet x) \bullet \delta(x) = \delta(x), \quad \text{but} \quad x^{-1} \bullet (x \bullet \delta(x)) = 0(x), \tag{9.65}$$

and

$$(x \bullet x^{-1}) \bullet \delta(x) = \delta(x), \quad \text{but} \quad x \bullet (x^{-1} \bullet \delta(x)) = \tfrac{1}{2}\, \delta(x), \tag{9.66}$$

and

$$(x \bullet \delta(x)) \bullet x^{-1} = 0(x), \quad \text{but} \quad x \bullet (\delta(x) \bullet x^{-1}) = \tfrac{1}{2}\, \delta(x). \tag{9.67}$$

The non-associativity of the product on $\overline{\mathbf{PC}}_\lambda$ cannot be avoided, not even by using a non-commutative product. It is an inevitable consequence of introducing a δ–function.

The left-sided product of arbitrary powers and logarithms is found to be

$$|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \circ\!\bullet\, |x|^\beta \log^r|x| \operatorname{sgn}^n(x) = |x|^{\alpha+\beta} \log^{q+r}|x| \operatorname{sgn}^{m+n}(x) + \\ + 2\, d_{qr}(\beta, n) \sum_{p=0}^{\infty} \delta_{\alpha+\beta,-p-1}\, \delta^{\mathrm{mod2}}_{m+n,p}\, \delta^{(p)}(x). \tag{9.68}$$

The right-sided product has the same form

$$|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \circ\!\bullet\, |x|^\beta \log^r|x| \operatorname{sgn}^n(x) = |x|^{\alpha+\beta} \log^{q+r}|x| \operatorname{sgn}^{m+n}(x) + \\ + 2\, d_{rq}(\alpha, m) \sum_{p=0}^{\infty} \delta_{\alpha+\beta,-p-1}\, \delta^{\mathrm{mod2}}_{m+n,p}\, \delta^{(p)}(x), \tag{9.69}$$

but with a different d–coefficient. The symmetrical product is found by symmetrization,

$$|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \bullet |x|^\beta \log^r|x| \operatorname{sgn}^n(x) = |x|^{\alpha+\beta} \log^{q+r}|x| \operatorname{sgn}^{m+n}(x) + \\ + \big(d_{rq}(\alpha, m) + d_{qr}(\beta, n)\big) \sum_{p=0}^{\infty} \delta_{\alpha+\beta,-p-1}\, \delta^{\mathrm{mod2}}_{m+n,p}\, \delta^{(p)}(x). \tag{9.70}$$

The form of the result (9.70) is explicitly symmetrical in the factors of the product. It is seen by inspecting the Kronecker d's that $\delta^{(p)}(x)$–functions arise if and only if the result of a product is proportional to a negative integral power of the form $x^{-p-1} \log^q|x| \operatorname{sgn}(x)$.
Specializing (9.69) to $q = r = 0$ gives

$$|x|^\alpha \operatorname{sgn}^m(x) \bullet |x|^\beta \operatorname{sgn}^n(x) = |x|^{\alpha+\beta} \operatorname{sgn}^{m+n}(x) + \\ - \big(d_{00}(\alpha, m) + d_{00}(\beta, n)\big) \sum_{p=0}^{\infty} \delta_{\alpha+\beta,-p-1}\, \delta^{\mathrm{mod2}}_{p,m+n}\, \delta^{(p)}(x). \tag{9.71}$$

The coefficient of the δ–function in (9.71) is for $\alpha, \beta \notin \mathbb{Z}$ given explicitly by

$$\big(d_{00}(\alpha, m) + d_{00}(\beta, n)\big) = \\ = \tfrac{1}{2}\big(\psi(\alpha+1) + \psi(-\alpha) + \psi(\beta+1) + \psi(-\beta) + \frac{(-)^m \pi}{\sin \pi\alpha} + \frac{(-)^n \pi}{\sin \pi\beta}\big). \tag{9.72}$$

Rewriting (9.71) in terms of Heaviside step functions gives

$$x^\alpha H(x) \bullet x^\beta H(x) = x^{\alpha+\beta} H(x) + \\ - \tfrac{1}{4}\big(\psi(\alpha+1) + \psi(-\alpha) + \psi(\beta+1) + \psi(-\beta)\big) \sum_{p=0}^{\infty} \delta_{\alpha+\beta,-p-1}\, \delta^{(p)}(x), \tag{9.73}$$

and

$$x^\alpha H(x) \bullet (-x)^\beta H(-x) = -\tfrac{\pi}{2}(\sin \pi\beta)^{-1} \sum_{p=0}^{\infty} \delta_{\alpha+\beta,-p-1}\, \delta^{(p)}(x),$$

$$= -\tfrac{\pi}{2}(\sin \pi\alpha)^{-1} \sum_{p=0}^{\infty} (-)^p \delta_{\alpha+\beta,-p-1}\, \delta^{(p)}(x). \quad (9.74)$$

This shows that products involving $H(x)$ and $H(-x)$ are not necessarily zero. It will be seen in Ch. 13 that this is in agreement with the supports of these functions which overlap at $x = 0$. When generalised functions of argument $(x{\pm}io)$ are introduced in Ch. 21 this overlap will make a satisfactory product definition for functions of argument $(x{\pm}io)$ possible.
For integral powers only the cases in which additional δ–functions may occur are interesting. For the special case of the function x^p one obtains

$$x^p \circ\!\bullet\, |x|^\alpha \log^q|x| \operatorname{sgn}^m(x) = x^p \bullet\!\circ\, |x|^\alpha \log^q|x| \operatorname{sgn}^m(x)$$
$$= |x|^{\alpha+p} \log^q|x| \operatorname{sgn}^{m+p}(x), \quad (9.75)$$

so products with regular powers x^p never add δ–functions. The same result for the δ–functions was obtained in (9.46) and (9.48). Therefore the functions $x^p \in \overline{\mathbf{PC}}_\lambda$ are special in the sense that they have the property

$$x^p \circ\!\bullet\, f(x) = f(x) \circ\!\bullet\, x^p = x^p \bullet\!\circ\, f(x) = f(x) \bullet\!\circ\, x^p, \quad (9.76)$$

for all $f(x) \in \overline{\mathbf{PC}}_\lambda$. It will be seen in Ch. 11 that (9.76) also holds for all $f(x) \in \mathbf{GF}_s$. This makes it possible to identify the operator $\boldsymbol{\mathcal{X}}$ with the generalised function multiplication by $x\,\bullet$

$$\boldsymbol{\mathcal{X}}\, f(x) = x \bullet f(x), \quad (9.77)$$

for all generalised functions.
For the case with the sgn–function on the other side one finds

$$x^p \operatorname{sgn}(x) \bullet x^{-p-1-q} = x^{-q-1} \operatorname{sgn}(x) - 2\big(\psi(p+1) + \psi(p+q+1)\big)\, \delta^{(q)}(x), \quad (9.78)$$

In particular one has the special cases

$$\operatorname{sgn}(x) \bullet x^{-1} = |x|^{-1} - 2\,\psi(1)\, \delta(x), \quad (9.79)$$

and

$$\operatorname{sgn}(x) \bullet |x|^{-1} = x^{-1}. \quad (9.80)$$

This shows again that care is needed with the notation and that the generalised function multiplication dot $\bullet$ has to appear explicitly. Undefined notations such as $1/x$ or $1/|x|$ should be avoided.

Example 9.3 It follows from the preceding formulæ that

$$H(x) \bullet |x|^{-1} = x^{-1}H(x) \neq H(x) \bullet x^{-1} = x^{-1}H(x) - \psi(1)\,\delta(x). \tag{9.81}$$

The notation is consistent and logical, but I made the mistake of equating both sides of (9.81) by omitting the δ–function several times. This point will recur in Ch. 14 in connection with the definition of the integral.

There remains the uninteresting case

$$x^{-p-1}\,\mathrm{sgn}(x) \bullet x^{-q-1} = x^{-p-q-2}\,\mathrm{sgn}(x) - \psi(q+1)\,\delta^{(p+q+1)}(x), \tag{9.82}$$

which completes the products of powers where additional $\delta^{(p)}$–functions occur.

The products of δ–functions and ordinary functions is again defined as the product of the δ–function with the asymptotic expansion of the ordinary function. The result is non-zero only when the function $f(x)$ is considered to belong in $\overline{\mathbf{PC}}_\lambda$.

$$\delta^{(p)}(x - x_0) \bullet f(x) := \delta^{(p)}(x - x_0) \bullet f_a(x; x_0), \tag{9.83}$$

which is by (4.1) always a finite linear combination of δ–functions. The result of the computation becomes (with the substitution of an appropriate measurement formula for the asymptotic coefficients, and taking $x_0 = 0$ for convenience)

$$\begin{aligned} f(x) \circ\!\bullet\, \delta^{(p)}(x) = \sum_{j=0}^{\infty} \langle\, \boldsymbol{\mathcal{X}}^{j+1} f(x)\,, \delta(x) \,\rangle\, \delta^{(p+j+1)}(x) \,+ \\ + \sum_{j=0}^{p} \langle\, f(x)\,, \delta^{(j)}(x) \,\rangle\, \delta^{(p-j)}(x), \end{aligned} \tag{9.84}$$

for the left-sided products, and

$$f(x) \bullet\!\circ\, \delta^{(p)}(x) = \sum_{j=0}^{p} \langle\, f(x)\,, \delta^{(j)}(x) \,\rangle\, \delta^{(p-j)}, \tag{9.85}$$

for the right-sided product.

Only the asymptotic terms proportional to $x^p \log^q|x|$ and $x^{-p-1}\log^q|x|$ contribute. For $p = 0$ one obtains the symmetrized result

$$f(x) \bullet \delta(x - x_0) = \tfrac{1}{2} \sum_{j=0}^{\infty} (1 + \delta_{j,0})\, \langle\, \boldsymbol{\mathcal{X}}^{j} f(x)\,, \delta(x) \,\rangle\, \delta^{(j)}(x), \tag{9.86}$$

which often occurs in applications. When negative powers are absent this reduces to

$$f(x) \bullet \delta(x) = \big\langle\, \delta(x)\,, f(x) \,\big\rangle\, \delta(x), \tag{9.87}$$

in agreement with distribution theory when $f(x)$ is regular at $x = 0$, without logarithmic terms.

The generalised function product of ordinary functions $f(x) \bullet g(x) \in \mathbf{PC}_\lambda$ is defined as the product of the asymptotic expansions added to the ordinary product, which results in

$$f(x) \bullet g(x) := f(x) \cdot g(x) + \sum_{x_j \in \mathbb{R}} \boldsymbol{P}_\delta\big(f_a(x;x_j) \bullet g_a(x;x_j)\big). \tag{9.88}$$

The generalised function product differs from the preliminary product by at most a finite linear combination of δ–functions, located at finitely many points.

9.3 The convolution product

Computation of the convolution product $* : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}_\lambda$ is now straightforward. Either we Fourier transform the pointwise product

$$f * g := \mathcal{F}^{-1}\big(\mathcal{F}f \bullet \mathcal{F}g\big), \tag{9.89}$$

or we calculate directly

$$f * g = \mathcal{M}_{\mathcal{X}}^{-1}\big(\mathcal{M}_{\mathcal{X}} f * \mathcal{M}_{\mathcal{X}} g\big), \tag{9.90}$$

from the convolution on $\overline{\mathbf{PC}}'_\lambda$ and the transfer mappings $\mathcal{M}_{\mathcal{X}}$ and $\mathcal{M}_{\mathcal{X}}^{-1}$. The results are the same for both methods, since the map $\mathcal{M}_{\mathcal{X}}$ is the Fourier transform of the map $\mathcal{M}_{\mathcal{D}}$ by (9.20). The first method is preferable in the simple cases.

Likewise one may define the the left- and right-sided convolution products $\circ *$, and $* \circ$ either by Fourier transformation of the corresponding pointwise product in accordance with (8.89),

$$f \circ\!* \, g := \mathcal{F}^{-1}\big(\mathcal{F}f \circ\!\bullet \mathcal{F}g\big), \tag{9.91}$$

or from the similar convolutions on $\overline{\mathbf{PC}}'_\lambda$ by using the $\mathcal{M}_{\mathcal{X}}$ mapping.
The convolution product has a unit element $\delta(x)$

$$\delta(x) * f(x) = f(x), \tag{9.92}$$

$\forall\, f(x) \in \overline{\mathbf{PC}}_\lambda$, since $\delta(x)$ is the Fourier transform of $I(x)$, which is the unit element of the pointwise product. In the same way one finds that convolution with the generalised function $(-)^p p!\ \delta^{(p)}(x)$ is equivalent to repeated differentiation

$$\delta^{(p)}(x) * f(x) = \tfrac{(-)^p}{p!}\,\mathcal{D}^p f(x). \tag{9.93}$$

This holds generally for arbitrary $f(x) \in \overline{\mathbf{PC}}_\lambda$, since it is the Fourier image of the property $\boldsymbol{\mathcal{X}}^p\, f(x) = x^p \bullet f(x)$ of the $\boldsymbol{\mathcal{X}}$ operator.
In particular one obtains for the δ–functions

$$\delta^{(p)}(x) * \delta^{(q)}(x) = \tfrac{(p+q)!}{p!\,q!}\; \delta^{(p+q)}(x) = \tfrac{(-)^p}{p!} \boldsymbol{\mathcal{D}}^p\, \delta^{(q)}(x), \tag{9.94}$$

and for δ–functions convoluted with powers

$$\begin{aligned} \delta^{(p)}(x) * |x|^\alpha \log^q|x| \operatorname{sgn}^m(x) &= \tfrac{(-)^p}{p!} \boldsymbol{\mathcal{D}}^p \big(|x|^\alpha \log^q|x| \operatorname{sgn}^m(x)\big) = \\ &= p! \sum_{k=0}^{\min(p,q)} \tbinom{q}{k} \tbinom{\alpha}{p}^{(k)} |x'|^{\alpha-p} \log^{q-k}|x| \operatorname{sgn}^{m+p}(x) + \\ &\quad + 2p! \sum_{k=0}^{\infty} \tfrac{1}{q+1} \tbinom{\alpha}{p}^{(q+1)} \delta_{-\alpha+p-1,k}\, \delta^{\mathrm{mod}2}_{m+p,k}\, \delta^{(k)}(x), \end{aligned} \tag{9.95}$$

and where the derivatives of the binomial coefficients are zero when the sum over k is empty.
The convolutions of the powers and logarithms are again rather cumbersome to compute. Additional polynomials arise in combination with the logarithms at infinity when the convolution behaves as $x^p \log^q|x|$ at infinity. This was expected from the appearance of additional $\delta^{(p)}$–functions in the pointwise products behaving as $x^{-p-1} \log^q|x| \operatorname{sgn}(x)$ in the finite.
The only special case which is needed is

$$x^{-1} * x^{-1} = -\pi^2\, \delta(x), \tag{9.96}$$

which appears in Ch. 21 when the inverse of the Hilbert operator has to be found.

The convolution of ordinary functions $\in \mathbf{PC}_\lambda$ is best computed by means of the Fourier transform method. The appearance of polynomials at infinity can be read off from the asymptotic expansion of the factors at infinity. Additional terms may appear when the convolution product is homogeneous of degree $p \in \mathbb{N}$ and of parity $(-)^p$ at infinity.
The relation of the convolution product to the standard convolution integral

$$f(x) * g(x) = \int_{-\infty}^{\infty} dy\; f(y)\, g(x-y), \tag{9.97}$$

will be discussed in Ch. 22 when the limits of generalised functions have been defined. Since the convolution product of the generalised functions is known, it is possible to assess the correctness of the various 'regularization methods' which can be used to define the convolution as a 'regularization' of the undefined divergent integrals which are obtained by attempting to take a regularization of (9.97) as a definition of the convolution product.

9.4 Uniqueness of the products

It is clear that the map $\mathcal{M}_{\mathcal{X}}$ is not fully determined by the commutation requirement (9.19)

$$\mathcal{X}\mathcal{M}_{\mathcal{X}} - \mathcal{M}_{\mathcal{X}}\mathcal{X} = \mathcal{X}\mathcal{M}_{\mathcal{X}} P_{\delta(0)}. \tag{9.98}$$

For $\delta^{(p)}(x)$ one could define instead of (9.2)

$$\mathcal{M}'_{\mathcal{X}}\, \delta^{(p)}(x) := \sum_{j=0}^{\infty} c_j\, \eta^{(p,j)}(x). \tag{9.99}$$

It follows from commutativity (9.99) that the coefficients in (9.99) can depend only on j, not on p. Likewise for the powers one can make the more general choice

$$\mathcal{M}_{\mathcal{X}}|x|^{\alpha} \log^q|x| \operatorname{sgn}^m(x) := |x'|^{\alpha} \log^q|x| \operatorname{sgn}^m(x) + \\ + p(\alpha) \sum_{j=0}^{\infty} c_{q,m;j}\, \eta_s^{(-\alpha-1,j)} \operatorname{sgn}^m(x), \tag{9.100}$$

where $p(\alpha) = p(\alpha+1)$ is an arbitrary periodic function of α with period one. The periodicity and the factoring out of the α dependence again follow from the commutation requirement (9.19).

The different possibilities for the standardization can be parametrized by a countable number of of parameters, since the periodic function $p(\alpha)$ is fully characterized by its Fourier coefficients.

In what follows the choice standardization defined in the previous section is kept, which corresponds to the choice

$$c_j = \delta_{j,0} \quad \text{and} \quad p(\alpha) = 0. \tag{9.101}$$

The standardization of the map $\mathcal{M}_{\mathcal{D}}^{-1}$ follows immediately by Fourier transformation of (9.101).

The freedom to choose a different standardization of the mappings implies a freedom the products of δ–functions and logarithms one sees that the coefficient in

$$\delta(x) \bullet \log^q|x| = c_q\, \delta(x), \tag{9.102}$$

can be chosen arbitrarily be changing he standardization. It will be shown in Chs. 16–18 that this freedom of standardization does not affect the results of computations with generalised functions when invariance under scale transformations is imposed. In Ch. 17 a slightly more general standardization will be introduced to accommodate indeterminate generalised functions.

When the freedom to choose a different standardization is kept it follows that the method of this chapter defines a (countably) infinite family of products on the generalised functions, and therefore also on the corresponding distributions. It seems that the product definitions for this subspace of the distributions, which have been proposed in the literature, can be obtained by choosing the standardization appropriately. This chapter provides a covering theory for these approaches to the multiplication problem. This remark will be further discussed in Ch. 23.

There are good reasons for preferring the standardization (9.101) or its indeterminate generalization (17.26). These will become clear in the following chapters when the consequences of the choice (9.101) are seen. The standardization will be discussed in Chs. 14, 16 and Ch. 18, 19, and Ch. 22, where it appears again in explicit form.

The algebraic properties and the operator properties of the product on $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda$ are derived in the next chapter.

CHAPTER 10

PRODUCT PROPERTIES

In this chapter some of the properties of the pointwise product are derived. The associativity and commutativity of the product are investigated. Rules are found for the validity of the multiplicative property of the operator $\boldsymbol{\mathcal{X}}$. The exceptional subspaces in which the operators $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{D}}$ are not selfadjoint are characterized. The chapter closes with a summary of the product properties.

10.1 Associativity and commutativity of the products

The left- and right-sided products on $\overline{\mathbf{PC}}'_\lambda$ were found to be associative (but not commutative). One may ask if this property carries over to $\overline{\mathbf{PC}}_\lambda$. This is best seen by attempting to derive associativity on $\overline{\mathbf{PC}}_\lambda$ from associativity on $\overline{\mathbf{PC}}'_\lambda$. Transfer of a left-sided triple product results in

$$(f \circ\!\bullet\, g) \circ\!\bullet\, h = \mathcal{M}_{\mathcal{D}}^{-1}\big(\mathcal{M}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}^{-1}(\mathcal{M}_{\mathcal{D}} f \circ\!\bullet\, \mathcal{M}_{\mathcal{D}} g) \circ\!\bullet\, \mathcal{M}_{\mathcal{D}} h\big), \tag{10.1}$$

and idem for the other term of the associator. If it were allowed to replace the expression $\mathcal{M}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}^{-1}$ by $\mathcal{I}$, then associativity would follow. However it is known from (9.35) that we have instead

$$\mathcal{M}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}^{-1} = \mathcal{I} - P_{\mathcal{M}}, \tag{10.2}$$

Using associativity in $\overline{\mathbf{PC}}'_\lambda$ the associator in $\overline{\mathbf{PC}}_\lambda$ is found as

$$\begin{aligned}[f \circ\!\bullet\, g \circ\!\bullet\, h] = {}& \mathcal{M}_{\mathcal{D}}^{-1}\big(P_{\mathcal{M}}(\mathcal{M}_{\mathcal{D}} f \circ\!\bullet\, \mathcal{M}_{\mathcal{D}} g) \circ\!\bullet\, \mathcal{M}_{\mathcal{D}} h\big) + \\ & - \mathcal{M}_{\mathcal{D}}^{-1}\big(\mathcal{M}_{\mathcal{D}} f \circ\!\bullet\, P_{\mathcal{M}}(\mathcal{M}_{\mathcal{D}} g \circ\!\bullet\, \mathcal{M}_{\mathcal{D}} h)\big).\end{aligned} \tag{10.3}$$

Now we have $\mathcal{M}_{\mathcal{D}}^{-1} P_{\mathcal{M}} f = 0$ from the definition of the projection operator, but this does not imply that $\mathcal{M}_{\mathcal{D}}^{-1}\big(P_{\mathcal{M}}(f \circ\!\bullet\, g) \circ\!\bullet\, h\big) = 0$.

Example 10.1 Substituting $f := \operatorname{sgn}(x)$, $g := \operatorname{sgn}(x)$, $h := \delta(x)$ in (10.3) gives

$$\delta(x) = \big(\operatorname{sgn}(x) \circ\!\bullet \operatorname{sgn}(x)\big) \circ\!\bullet\, \delta(x) \neq \operatorname{sgn}(x) \circ\!\bullet \big(\operatorname{sgn}(x) \circ\!\bullet\, \delta(x)\big) = 0(x), \tag{10.4}$$

which is clearly not associative.

Product commutators the asymmetrical products are found by straightforward computation. This results in

$$\begin{aligned}\big[|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \circ\!\bullet\, \delta^{(p)}(x)\big] &= \\ = \tfrac{2}{\pi} \tfrac{(-)^p}{p!} q!\, c_{q+1}(p,p) &\sum_{j=0}^{\infty} \delta_{p-\alpha,j}\, \delta^{\mathrm{mod}2}_{m+p,j}\, \delta^{(j)}(x) + \\ - \tfrac{2}{\pi} q! \sum_{j=0}^{p} \tfrac{(-)^j}{j!}\, c_{q+1}(j,j)\, &\delta_{\alpha,j}\, \delta^{\mathrm{mod}2}_{m,j}\, \delta^{p-j}(x),\end{aligned} \tag{10.5}$$

for the δ–function, and

$$[|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \circ_\bullet |x|^\beta \log^r|x| \operatorname{sgn}^n(x)] =$$

$$= 2\big(d_{qr}(\beta, n) - d_{rq}(\alpha, m)\big) \sum_{p=0}^{\infty} \delta_{-\alpha-\beta-1,p}\, \delta^{\mathrm{mod}2}_{m+n,p}\, \delta^{(p)}(x), \qquad (10.6)$$

for the powers. Inspection of (10.5) and (10.6) reveals that the coefficient of $\delta^{(0)}(x)$ is zero in both cases, by cancellation and by the symmetry property (9.29) of the d–coefficients. That is, $\forall\, f(x), g(x) \in \overline{\mathbf{PC}}_\lambda$,

$$\delta(x) \neq [f(x) \circ_\bullet g(x)]. \qquad (10.7)$$

Again the notation $\perp$ will be used to indicate this

$$\delta^{(0)}(x) \perp [f(x) \circ_\bullet g(x)], \qquad (10.8)$$

where $\perp$ with respect to a set now means $\perp$ with respect to all elements in the set.

The commutator space is spanned by the allowed linear combinations of δ–functions without the one-dimensional subspace spanned by $\delta(x)$. This makes the scalar product in $\overline{\mathbf{PC}}_\lambda$ unique.

Associators involve somewhat more work. Associators containing three or two δ–functions are obviously zero. The associator involving one δ–function on the right equals

$$[|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \circ_\bullet |x|^\beta \log^r|x| \operatorname{sgn}^n(x) \circ_\bullet \delta^{(p)}(x)] =$$

$$= \sum_{j=0}^{\infty} e_j\, \delta_{p-\alpha-\beta,j}\, \delta^{\mathrm{mod}2}_{p+m+n,j}\, \delta^{(j)}(x), \qquad (10.9)$$

with the coefficient e_j given by

$$e_j = \frac{c_{q+r+1}(p,p)}{c_1(p,p)} - \frac{c_{r+1}(p,p)}{c_1(p,p)} \sum_{k=0}^{\infty} \delta_{p-\beta,k}\, \delta^{\mathrm{mod}2}_{p+m,k}\, \frac{c_{q+1}(k,k)}{c_1(k,k)}. \qquad (10.10)$$

The associator in (10.9) is zero in the special case $q = r = 0$. The other associators with one δ–function are similar in form.
The left-sided associator of the powers equals

$$[|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \circ_\bullet |x|^\beta \log^r|x| \operatorname{sgn}^n(x) \circ_\bullet |x|^\gamma \log^s|x| \operatorname{sgn}^o(x)] =$$

$$= 2 \sum_{p=0}^{\infty} \delta_{\alpha+\beta+\gamma,-p-1}\, \delta^{\mathrm{mod}2}_{m+n+o,p}\, \delta^{(p)}(x) \times$$

$$\times \Big(d_{q+r,s}(\gamma, o) - d_{q,r+s}(\gamma, o) + \tfrac{2s!}{\pi}\, d_{qr}(\beta, n) \sum_{j=0}^{p} \tfrac{(-)^j}{j!}\, \delta_{\gamma,j}\, \delta^{\mathrm{mod}2}_{o,j}\, c_{s+1}(j,j) +$$

$$- \tfrac{2q!}{\pi}\, d_{rs}(\gamma, o) \sum_{j=0}^{\infty} \tfrac{(-)^j}{j!}\, \delta_{\alpha-p,j}\, \delta^{\mathrm{mod}2}_{m+p,j}\, c_{q+1}(j,j)\Big). \qquad (10.11)$$

The associator space of the left-sided product is spanned by the allowed linear combinations of δ–functions, including $\delta^{(0)}(x)$.

The associator of the symmetrical product can be expressed in terms of associators and commutators of the left-sided product as

$$\begin{aligned} 4[f \bullet g \bullet h] = &+[f \circ\!\bullet\, g \circ\!\bullet\, h] + [f \circ\!\bullet\, h \circ\!\bullet\, g] + [g \circ\!\bullet\, f \circ\!\bullet\, h] + \\ &- [g \circ\!\bullet\, h \circ\!\bullet\, f] - [h \circ\!\bullet\, g \circ\!\bullet\, f] - [h \circ\!\bullet\, f \circ\!\bullet\, g] + \\ &+ [g \circ\!\bullet\, [f \circ\!\bullet\, h]]. \end{aligned} \tag{10.12}$$

The commutator in (10.12) implies non-associative parenthesis. In the special case (8.80) of the product on $\overline{\mathbf{PC}}'_\lambda$ the product is associative and only the double commutator in the last line survives. The associators of the symmetrical product are easily evaluated by carrying out the appropriate substitutions. The formulæ become rather large so they are not written out here.

There are two reasons for the lack of associativity of the symmetrical product on $\overline{\mathbf{PC}}_\lambda$.

The first is the lack of associativity of the left- and right-sided products represented by the associator terms in (10.12). This cannot be avoided in $\overline{\mathbf{PC}}_\lambda$. An example was given above.

The second is the occurrence of factor $\frac{1}{2}$ resulting from the symmetrization, as in the product on $\overline{\mathbf{PC}}'_\lambda$. This is given by the double commutator term.

Example 10.2 The product

$$\tfrac{1}{2}\,\delta^{(2)}(x) = \left(x^{-1} \bullet x^{-1}\right) \bullet \delta(x) \neq x^{-1} \bullet \left(x^{-1} \bullet \delta(x)\right) = \tfrac{1}{4}\,\delta^{(2)}(x), \tag{10.13}$$

is not associative even though none of the sub-products are zero, and even though the corresponding left- and right-sided products

$$x^{-1} \circ\!\bullet\, x^{-1} \circ\!\bullet\, \delta(x) = \delta(x) \bullet\!\circ\, x^{-1} \bullet\!\circ\, x^{-1} = \delta^{(2)}(x), \tag{10.14}$$

are associative.

The second type of non-associativity can be avoided by going to the left- and right-sided products. This leaves the first source of non-associativity.

It is therefore seen that on $\overline{\mathbf{PC}}_\lambda$ it is possible to have a commutative product, but it is not possible to have an associative product. This is the reason for preferring the commutative product, the inconvenience of non-associativity being unavoidable.

For an arbitrary product $\diamond$, which is neither associative, nor commutative, elementary algebra yields the cyclic commutator property

$$\begin{aligned} &[a \diamond [b \diamond c]] + [b \diamond [c \diamond a]] + [c \diamond [a \diamond b]] = \\ &\quad = - \sum_{\text{perm}} (-)^{\text{perm}} [a \diamond b \diamond c], \end{aligned} \tag{10.15}$$

where the sum with the signature is over the six permutations. It follows that an associative product satisfies the cyclic commutator property.
Conversely the cyclic associator property of an arbitrary product is

$$[a \diamond b \diamond c] + [b \diamond c \diamond a] + [c \diamond a \diamond b] = \\ = [a \diamond b \diamond c] + [b \diamond c \diamond a] + [c \diamond a \diamond b], \qquad (10.16)$$

so a non-associative product satisfies the cyclic associator property if it is commutative.

Remark 10.1 As usual the commutator separation symbol takes precedence, so no parenthesis are inserted around the products in (10.16).

On $\overline{\mathbf{PC}}_\lambda$ the left- and right-sided products satisfy neither the cyclic associator property, nor the cyclic commutator property. The symmetrical product is commutative, so it satisfies the cyclic associator property. Being commutative it satisfies the cyclic commutator property trivially.

10.2 Operator properties of the product

The operator properties on $\overline{\mathbf{PC}}_\lambda$ are more complicated than the corresponding properties on $\overline{\mathbf{PC}}'_\lambda$. The multiplication operator $\mathcal{X}$ satisfies the simple property

$$\mathcal{X}(f \bullet g) = (\mathcal{X}f) \bullet g = f \bullet (\mathcal{X}g), \qquad (10.17)$$

on $\overline{\mathbf{PC}}'_\lambda$. This property does *not* carry over to $\overline{\mathbf{PC}}_\lambda$.

Example 10.3 The standard example in which (10.17) is violated is

$$\delta(x) = (\mathcal{X}\, x^{-1}) \bullet \delta(x) \neq \mathcal{X}(x^{-1} \bullet \delta(x)) = \tfrac{1}{2}\, \delta(x) \neq x^{-1} \bullet (\mathcal{X}\, \delta(x)) = 0(x), \qquad (10.18)$$

in agreement with the non-associative products (9.65–67).

This was to be expected since the map $\mathcal{M}_\mathcal{D}$ does not commute with the operator $\mathcal{X}$, even though it does commute with $\mathcal{M}_\mathcal{D}^{-1}$. If one attempts to derive the property (10.18) for the left-sided product on $\overline{\mathbf{PC}}'_\lambda$ one obtains

$$\begin{aligned} \mathcal{X}(f \circ\!\bullet\, g) &= \mathcal{M}_\mathcal{D}^{-1}(\mathcal{X}\mathcal{M}_\mathcal{D} f \circ\!\bullet\, \mathcal{M}_\mathcal{D} g) = \\ &= \mathcal{X}f \circ\!\bullet\, g + \mathcal{M}_\mathcal{D}^{-1}([\mathcal{X}, \mathcal{M}_\mathcal{D}] f \circ\!\bullet\, \mathcal{M}_\mathcal{D} g) = \\ &= f \circ\!\bullet\, \mathcal{X}g + \mathcal{M}_\mathcal{D}^{-1}(\mathcal{M}_\mathcal{D} f \circ\!\bullet\, [\mathcal{X}, \mathcal{M}_\mathcal{D}] g), \end{aligned} \qquad (10.19)$$

and one has to verify by explicit computation whether the second terms in (10.19) and (10.18) vanish.

For the left-sided product one finds the left-left rule

$$\mathcal{X}(f \circ\!\bullet\, g) = (\mathcal{X}f) \circ\!\bullet\, g \neq f \circ\!\bullet\, (\mathcal{X}g), \qquad (10.20)$$

which is that the operator $\mathcal{X}$ is multiplicative on the left-sided factor of the left-sided product. Correspondingly in the opposite order there is the right-right rule for the right-sided product

$$\mathcal{X}(f \bullet\circ g) = f \bullet\circ (\mathcal{X}g) \neq (\mathcal{X}f) \bullet\circ g. \tag{10.21}$$

The validity of the product rules (10.20) and (10.21) must be verified by inspection. Instead of working out the operator commutators in (10.18) and (10.19) it is more convenient to work out the basic products. In the product equation (9.45) it is clear that the operator $\mathcal{X}$ can be taken into the left factor, since the coefficient of $\delta(x)$ does not depend on α. The same holds for the right-sided product by symmetry. The same argument does not hold when one attempts to do the same for the $\mathcal{X}$ operator acting on the second factor of the left-sided product.

For the symmetrical product the multiplicative properties of $\mathcal{X}$ are more complicated. Expressing the symmetrical product in left-sided products an rearranging terms by means of product commutators yields the forms

$$\begin{aligned}\mathcal{X}(f \bullet g) &= \mathcal{X}f \bullet g - \tfrac{1}{2}\mathcal{X}[f \circ\bullet g] + \tfrac{1}{2}[\mathcal{X}f \circ\bullet g] = \\ &= f \bullet \mathcal{X}g + \tfrac{1}{2}\mathcal{X}[f \circ\bullet g] - \tfrac{1}{2}[f \circ\bullet \mathcal{X}g] = \\ &= \tfrac{1}{2}\mathcal{X}f \bullet g + \tfrac{1}{2}f \bullet \mathcal{X}g + \tfrac{1}{4}[\mathcal{X}f \circ\bullet g] - \tfrac{1}{4}[f \circ\bullet \mathcal{X}g].\end{aligned} \tag{10.22}$$

The validity of the multiplicative rules (2.41–43) depends on the vanishing and/or cancellation of the product commutators in (10.22). This is not difficult since the product commutators are known explicitly. In the example given above the multiplication rule is the semi-Leibniz rule.

Care is needed only when additional $\delta^{(p)}$–functions appear in products. This happens when the product is homogeneous of degree $-p-1$ with parity $p+1$.
Subtracting both forms yields the error term for the multiplicative rule

$$\mathcal{X}f \bullet g - f \bullet \mathcal{X}g = -\mathcal{X}[f \circ\bullet g] + \tfrac{1}{2}[\mathcal{X}f \circ\bullet g] + \tfrac{1}{2}[f \circ\bullet \mathcal{X}g]. \tag{10.23}$$

This can also be used to shift the operator $\mathcal{X}$ in a product. It is also possible to replace the operator $\mathcal{X}$ by $x\bullet$. Using the special property (9.76) the error terms in the multiplicative rule can also be expressed in terms of associators as

$$\begin{aligned}\mathcal{X}(f \bullet g) &= \mathcal{X}f \bullet g + \tfrac{1}{2}[x \circ\bullet f \circ\bullet g] + \tfrac{1}{2}[g \circ\bullet f \circ\bullet x], \\ &= f \bullet \mathcal{X}g + \tfrac{1}{2}[x \circ\bullet g \circ\bullet f] + \tfrac{1}{2}[f \circ\bullet g \circ\bullet x].\end{aligned} \tag{10.24}$$

The commutator form seems more convenient however.

The pointwise product defined in Ch. 9 satisfies Leibniz's rule for the differentiation of a product

$$\mathcal{D}\big(f(x) \bullet g(x)\big) = \big(\mathcal{D} f(x)\big) \bullet g(x) + f(x) \bullet \big(\mathcal{D} g(x)\big). \tag{10.25}$$

This follows for products not involving the identity element from the commutation of $\mathcal{M}_{\mathcal{D}}$ and $\mathcal{D}$,

$$\mathcal{M}_{\mathcal{D}}\mathcal{D} = \mathcal{D}\mathcal{M}_{\mathcal{D}}, \qquad \text{and} \qquad \mathcal{M}_{\mathcal{D}}^{-1}\mathcal{D} = \mathcal{D}\mathcal{M}_{\mathcal{D}}^{-1}, \tag{10.26}$$

and from Leibniz's rule (8.103) in $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$. For products with the identity element $I(x)$ it follows by direct computation.

$$\mathcal{D}\big(I(x) \bullet f(x)\big) = 0(x) \bullet f(x) + I(x) \bullet \mathcal{D} f(x) = \mathcal{D} f(x), \tag{10.27}$$

since $I(x)$ is both the unit element of the multiplication and the zero element for differentiation. It is seen that it is necessary that the identity element of the product coincides with the zero element of the differential operator.

10.3 The scalar products on $\overline{\mathbf{PC}}_\lambda$

Now that the products have been defined it is possible to define the scalar product $\langle\, , \rangle : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \mathbb{C}$ in terms of the generalised function product. This is done by putting for the left-sided scalar product

$$\langle\!\!\langle f(x)\, , g(x) \rangle := \langle\, I(x)\, , f^*(x) \circ\!\bullet\, g(x) \,\rangle, \tag{10.28}$$

for all f and g in $\overline{\mathbf{PC}}_\lambda$. It is sufficient to define the scalar product of $\mathbf{PC}_\lambda$ with the unit function $I(x)$. Correspondingly one has the right-sided product

$$\langle f(x)\, , g(x) \rangle\!\!\rangle := \langle\, I(x)\, , f^*(x) \bullet\!\circ\, g(x) \,\rangle, \tag{10.29}$$

and the symmetrical scalar product

$$\langle f(x)\, , g(x) \rangle := \langle\, I(x)\, , f(x)^* \bullet g(x) \,\rangle, \tag{10.30}$$

which may be obtained by symmetrizing the scalar product, or equivalently by using the symmetrical product.

In the following it is sufficient to consider only the symmetrical scalar product. In the previous section it was shown (10.7) that product commutators do not contain the element $\delta(x)$. Therefore in $\overline{\mathbf{PC}}_\lambda$ product commutators do not contribute to scalar products. The left-first scalar product equals the symmetrical scalar product in all cases. There is only one scalar product $\langle\; , \;\rangle : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda$

$$\langle\!\!\langle f\, , g \rangle = \langle f\, , g \rangle = \langle f\, , g \rangle\!\!\rangle, \tag{10.31}$$

$\forall f, g \in \overline{\mathbf{PC}}_\lambda$. The scalar product on $\overline{\mathbf{PC}}_\lambda$ agrees with the preliminary scalar product (4.25) between singularities.

Scalar products with the unit function can be defined as

$$\langle I(x)\, , f(x) \rangle := \langle \delta(x)\, , \mathcal{F} f(x) \rangle = \langle I(x)\, , \delta(x) \bullet \mathcal{F} f(x) \rangle, \tag{10.32}$$

in accordance with Parseval's equality (2.29) and definition (10.30). It is known from the previous section that any product with a δ–function is of the form

$$\delta(x) \bullet f(x) = \sum_{p=0}^{\infty} c_p \, \delta^{(p)}(x), \tag{10.33}$$

$\forall f(x) \in \overline{\mathbf{PC}}_\lambda$, Only finitely many c_p coefficients can be non-zero. Therefore all scalar products can be reduced to the special case

$$\langle I(x) , \delta^{(p)}(x) \rangle = \delta_{p,0} , \tag{10.34}$$

in agreement with the normalization (2.28) of the δ–function. Compared to the preliminary scalar product defined in Sec. 4.4 only the $\delta^{(0)}(x)$–functions at the singular points give additional contributions.
The product formula for the δ–function has the consequence

$$\langle \delta(x) , f(x) \rangle = f(0), \tag{10.35}$$

when $f(0)$ is defined in a standard sense. For symmetrical generalised functions the scalar product (10.35) is always defined, and conversely in Ch. 13 equation (10.35) will be taken as the definition of the value of a generalised function.

Remark 10.2 It should be kept in mind that $\eta(x)$ and $\delta(x)$ behave similarly only for continuous functions. For the logarithm one obtains

$$\langle \delta(x) , \log^q|x| \rangle = \tfrac{(-)^q}{\pi} q! \; c_{q+1}(0,0), \tag{10.36}$$

but

$$\langle \eta(x) , \log^q|x| \rangle = 0. \tag{10.37}$$

The η–functions are more selective than the δ–function.

For the scalar product of the powers one obtains

$$\begin{aligned} \langle |x|^\alpha \log^q|x| \operatorname{sgn}^m(x) , |x|^\beta \log^r|x| \operatorname{sgn}^n(x) \rangle &= \\ &= \delta_{\alpha,-\beta-1} \, \delta_{m,n}^{\mathrm{mod}2} \, d_{rq}(\alpha^*, m) + d_{qr}(\beta, n)\big). \end{aligned} \tag{10.38}$$

Numerically the result of (10.38) is usually zero. The only exception occurs when the products are proportional to $|x|^{-1} \log^{q+r}|x|$.
One obtains as a special case of (10.38) for $\alpha \notin \mathrm{N}$

$$\langle |x|^\alpha , |x|^{-\alpha-1} \rangle = \tfrac{1}{2} \, \psi(\alpha+1) + \tfrac{1}{2} \, \psi(-\alpha) + \frac{\pi}{\sin \pi\alpha} . \tag{10.39}$$

The value of the scalar product does depend on α, even though the naïve (but incorrect) answer for the scalar product is

$$\langle I(x) , |x|^{-1} \rangle = 0, \tag{10.40}$$

independently of α.

Specializing (10.38) to scalar products with the unit function gives

$$\langle\, I(x)\,,\, |x|^\alpha \log^q|x|\, \mathrm{sgn}^m(x)\,\rangle = 0, \tag{10.41}$$

$\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$. In particular for $\alpha = -1$ rewriting (10.41) in integral notation results in

$$\int_{-\infty}^{\infty} dx\, |x|^{-1} \log^q|x| = 0. \tag{10.42}$$

This result can also be obtained from (10.36) by Parseval's equality and the completion formula (C.27) of the c_j–coefficients. The simple appearance of (10.42) is a consequence of the choice of the standardization (9.101) of the products.
In Ch. 14 the integral in the sense of generalised functions will be defined as a special case of the scalar product. When written in integral notation the scalar product (10.42) takes the form

$$\int_{-\infty}^{\infty} dx\, |x|^\alpha \log^q|x|\, \mathrm{sgn}^m(x) := \langle\, I(x)\,,\, |x|^\alpha \log^q|x|\, \mathrm{sgn}^m(x)\,\rangle = 0, \tag{10.43}$$

$\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$, in agreement with Hadamard's definition of the partie finie used in Sec. 4.3 as a starting point.

In contrast with the situation in $\overline{\mathbf{PC}}'_\lambda$ the operators $\boldsymbol{\mathcal{X}}$ and $i\boldsymbol{\mathcal{D}}$ are not selfadjoint in the scalar product $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \mathbb{C}$ defined above.

Example 10.4 In the special case

$$\begin{aligned} i = \langle\, I(x)\,,\, i\,\delta(x)\,\rangle = \langle\, I(x)\,,\, i\boldsymbol{\mathcal{D}}H(x)\,\rangle \neq \\ \neq \langle\, i\boldsymbol{\mathcal{D}}\, I(x)\,,\, H(x)\,\rangle = \langle\, 0(x)\,,\, H(x)\,\rangle = 0, \end{aligned} \tag{10.44}$$

one sees this immediately. It can be obtained as the Fourier transform of the more familiar example

$$1 = \langle\, \delta(x)\,,\, \boldsymbol{\mathcal{X}}\, x^{-1}\,\rangle \neq \langle\, \boldsymbol{\mathcal{X}}\, \delta(x)\,,\, x^{-1}\,\rangle = 0, \tag{10.45}$$

which corresponds with the example of a non-associative product (9.65–67) given above. In general one has

$$\langle\, i\boldsymbol{\mathcal{D}}\, f(x)\,,\, g(x)\,\rangle = \langle\, f(x)\,,\, i\boldsymbol{\mathcal{D}}\, g(x)\,\rangle - \langle\, I(x)\,,\, i\boldsymbol{\mathcal{D}}\big(f(x)^* \bullet g(x)\big)\,\rangle, \tag{10.46}$$

from Leibniz's rule for differentiation of products.

One sees from the failure of the multiplicative product property (10.22) for the $\boldsymbol{\mathcal{X}}$ operator that the selfadjointness of the operator $\boldsymbol{\mathcal{X}}$ on $\overline{\mathbf{PC}}'_\lambda$ does not carry over to $\overline{\mathbf{PC}}_\lambda$. It is not difficult to characterize the failure of the selfadjointness for the operator $\boldsymbol{\mathcal{X}}$. It is selfadjoint when the multiplicative rule holds, so it is not selfadjoint in scalar products in which the error term (10.23) for the multiplicative rule contributes. Since the commutator space of the products does not contain the element $\delta^{(0)}(x)$, only the second term in (10.23) with $\boldsymbol{\mathcal{X}}$ outside the commutator can contribute. This is the case if and only if the product is equal to $|x|^{-2}\log^q|x|\,\mathrm{sgn}(x)$, or to $\delta^{(1)}(x)$. For arbitrary generalised functions selfadjointness depends on the vanishing and/or cancellation of the cross-terms of degree -2 in the product of the asymptotic expansions at $x = 0$. By unitary equivalence the operator $\boldsymbol{\mathcal{D}}$ is not selfadjoint in scalar products where the asymptotic expansion of the convolution product of the factors contains a term of the form $x\log^q|x|$ at $x = \pm\infty$.

Remark 10.3 Anticipating the results of Ch. 14 it is easy to characterize the failure of the selfadjointness of the operator $i\boldsymbol{\mathcal{D}}$. It fails when the product $f(x) \bullet g(x)$ behaves as $\mathrm{sgn}(x)\log^q|x|$ at infinity. Consequently the operator $\boldsymbol{\mathcal{X}}$ fails to be selfadjoint in the scalar product when the convolution $f(x) * g(x)$ behaves as $x^{-1}\log^q|x|$ at $x = 0$. This apparent lack of symmetry between the operators $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{X}}$ is a consequence of the choice of the standardization (9.101) of the product.

Remark 10.4 It follows from the existence of a zero element that the operators $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{D}}$ do not possess an adjoint in $\overline{\mathbf{PC}}_\lambda$. There does not exist an operator $i\boldsymbol{\mathcal{D}}^\dagger$ such that

$$\langle\, f(x)\,, i\boldsymbol{\mathcal{D}}\, g(x)\,\rangle = \langle\, i\boldsymbol{\mathcal{D}}^\dagger f(x)\,, g(x)\,\rangle, \tag{10.47}$$

$\forall f, g \in \mathbf{GF}_s$. The same conclusion is also implied by the existence of a zero element.

The exceptions to the selfadjointness of the operators $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{D}}$ will be characterized in Sec. 12.6. The operators $\boldsymbol{\mathcal{X}}$ and $i\boldsymbol{\mathcal{D}}$ are almost selfadjoint in the sense that for an element $f(x) \in \overline{\mathbf{PC}}_\lambda$ it is sufficient to exclude at most a 'small' subspace for the allowed function $g(x)$ in order to obtain a selfadjoint operator in the remaining space. It is necessary to exclude a subspace of $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda$. It is not sufficient to exclude a subspace of $\overline{\mathbf{PC}}_\lambda$ only.

The lack of selfadjointness of the operator $i\boldsymbol{\mathcal{D}}$ is an inevitable consequence of the introduction of a non-zero unit element with zero derivative everywhere, including infinity. Likewise its Fourier transform $\delta(x)$ causes the lack of selfadjointness of the operator $\boldsymbol{\mathcal{X}}$.

A choice is inevitable, either one has a unit element for the product and a zero element for differentiation, or one has selfadjoint operators and the possibility of an associative product. The model constructed in this book has both possibilities realised, each in its own subspace $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ respectively. The model as a whole cannot have all possibilities realized at the same time.

The Fourier operator is unitary in the scalar product defined in this section, (Parseval's equality),

$$\langle \mathcal{F} f(x) , \mathcal{F} g(x) \rangle = 2\pi \langle f(x) , g(x) \rangle, \tag{10.48}$$

apart from the normalization. This is easily verified for the special cases given above. The verification in the general case will be given in Ch. 12.

10.4 Summary of the product properties

In the two previous chapters the product has been extended from a product on the linear functionals to a product on the original space $\overline{\mathbf{PC}}_\lambda$. The algebraic structure of the product on $\overline{\mathbf{PC}}_\lambda$ is richer and more complicated than the corresponding product on $\overline{\mathbf{PC}}'_\lambda$.

The product algebra is neither commutative, nor associative. A commutative product can be obtained by symmetrization. The non-associativity cannot be removed.

The product satisfies Leibniz's rule for differentiation, but the operator $\mathcal{X}$ is not multiplicative in the product.

The product algebra generates a symmetrical scalar product, but the operators $\mathcal{X}$ and $\mathcal{D}$ are not completely selfadjoint in the scalar product. This is an unavoidable consequence of the introduction of a unit element for the product, which is a zero element for the operators.

Parseval's equality, which is equivalent to unitarity of the Fourier operator holds without exception.

The product and convolution on $\overline{\mathbf{PC}}_\lambda$ can be transfered directly to an important subspace of the (tempered) distributions. Some remarks on the connection with other definitions of products and/or convolutions on the distributions will be made in Ch. 23.

CHAPTER 11

THE SIMPLE MODEL

In the previous five chapters the operator and product properties of the preliminary class were found by a large detour. The generalised function properties were found from the preliminary properties of Ch. 4, by first transferring everything to the space $\overline{\mathbf{PC}}'_\lambda$ of linear functionals on the preliminary class, and then back to the preliminary class.

The preliminary class $\overline{\mathbf{PC}}_\lambda$ by itself is already a model for a symmetrical theory of generalised functions. It satisfies all requirements listed in Ch. 2. It lacks the analysing power and the simplicity of computation present in the linear functionals $\overline{\mathbf{PC}}'_\lambda$. The space $\overline{\mathbf{PC}}'_\lambda$ is not a model by itself, since it lacks the required unit elements I and δ.

The simple model referred to in the chapter title is obtained by combining $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ into the space $\mathbf{GF}_s$, which is the simple model for a symmetrical generalised functions.

As an intermediate step the space $\mathbf{PC}^\oplus_\lambda$ is defined as the direct sum of $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ by

$$\mathbf{PC}^\oplus_\lambda := \overline{\mathbf{PC}}_\lambda \oplus \overline{\mathbf{PC}}'_\lambda. \tag{11.1}$$

This space is too large however, since the spaces $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ have many elements, such as the Schwartz space $\mathcal{S}$, in common. The space of simple symmetrical generalised functions $\mathbf{GF}_s$ is obtained from $\mathbf{PC}^\oplus_\lambda$ by identifying the common parts of $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$.

Remark 11.1 The properties of $\overline{\mathbf{PC}}'_\lambda$ were obtained from the preliminary properties of $\overline{\mathbf{PC}}_\lambda$ by taking suitable residues. The generalised function properties of $\overline{\mathbf{PC}}_\lambda$ differ from the preliminary properties only for exceptional isolated values of the complex parameters. Instead of combining $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ it is also possible, now that we have the space $\overline{\mathbf{PC}}_\lambda$ as a space of generalised functions, to rederive the properties of $\overline{\mathbf{PC}}'_\lambda$, and consequently of $\mathbf{GF}_s$ from the known generalised function properties of the space $\overline{\mathbf{PC}}_\lambda$. Since the results are the same the choice is a matter of convenience.

Two sections are devoted to the discussion of the relations between symmetrical generalised functions on one hand, and distributions and ordinary functions on the other hand.

11.1 Local power functions

The functions $H(x) \in \overline{\mathbf{PC}}_\lambda$ and $H'(x) \in \overline{\mathbf{PC}}'_\lambda$ differ only at $x = 0$ and $x = \infty$. In between they are indistinguishable. The difference is defined as an element $\in \mathbf{PC}^\oplus_\lambda$ by

$$\theta_\downarrow(x) + \not\theta_\uparrow(x) := H(x) - H'(x). \tag{11.2}$$

More generally the functions 'theta-down' and 'theta-slash-up' are defined by

$$(-)^q\theta_\downarrow^{(\alpha,q)}(x) + \not\theta_\uparrow^{(\alpha,q)}(x) := x^\alpha \log^q(x)H(x) - x^\alpha \log^q(x)H'(x). \qquad (11.3)$$

The notation anticipates a splitting of this difference into a part in the finite and a part at infinity. As before the slash indicates a location at infinity. The sign convention in (11.3) with a + sign at infinity is different from the corresponding formula (5.78) for the η–functions.

The generalised functions $(-)^q\theta_\downarrow^{(\alpha,q)}(x)$ can be interpreted heuristically as the restriction of the function $x^\alpha \log^q(x)$ to a positive infinitesimal environment of the point $x = 0$. Anticipating the section on products it may be noted that this heuristic interpretation is supported by product formulæ such as

$$\theta_\downarrow^{(\alpha,q)}(x) = \theta(x) \bullet (-)^q x^\alpha \log^q(x)H(x). \qquad (11.4)$$

The 'theta-up' $\theta_\uparrow$ and the 'theta-slash-down' $\not\theta_\downarrow$ functions are defined similarly by

$$\theta_\uparrow^{(\alpha,q)}(x) := \theta_\downarrow^{(\alpha,q)}(-x). \qquad (11.5)$$

The convention (5.69) for adding slashes to formula numbers can de extended with the definitions

$$\theta_\downarrow^{(\alpha,q)}(x) \leftrightarrow (-)^q\not\theta_\uparrow^{(\alpha,q)}(x), \quad \text{and} \quad \not\theta_\downarrow^{(\alpha,q)}(x) \leftrightarrow (-)^q\theta_\uparrow^{(\alpha,q)}(x), \qquad (11.6)$$

in accordance with the transformations of the η–functions. A slash at a formula number now indicates that the formula remains valid when slashes are added where possible, with cancellation of double slashes.

As for the η–functions it is convenient to define the even and odd θ–functions by

$$\theta_s^{(\alpha,q)}(x) := \theta_\downarrow^{(\alpha,q)}(x) + \theta_\uparrow^{(\alpha,q)}(x), \qquad (11/7)$$

and

$$\theta_a^{(\alpha,q)}(x) := \theta_\downarrow^{(\alpha,q)}(x) - \theta_\uparrow^{(\alpha,q)}(x). \qquad (11/8)$$

The functions $\theta_s^{(\alpha,q)}(x)$ behave as $(-)^q|x|^\alpha \log^q|x|$ near $x = 0$, while the functions $\theta_a^{(\alpha,q)}(x)$ behave as $(-)^q|x|^\alpha \log^q|x| \operatorname{sgn}(x)$. In contrast with the complementary formulæ (5.48–49) for the η–functions no factor $\frac{1}{2}$ is included in the definitions (11.8) and (11.7).

As usual the notation

$$\theta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x) := \begin{cases} \theta_s^{(\alpha,q)}(x) & m = 0 \pmod 2, \\ \theta_a^{(\alpha,q)}(x) & m = 1 \pmod 2, \end{cases} \qquad (11/9)$$

combines the two cases. For integral values of the index it is convenient to define

$$\theta^{(p,q)}(x) := \theta_{\downarrow}^{(p,q)}(x) + (-)^p \theta_{\uparrow}^{(p,q)}(x), \tag{11/10}$$

and

$$\tau^{(p,q)}(x) := \theta_{\downarrow}^{(p,q)}(x) - (-)^p \theta_{\uparrow}^{(p,q)}(x), \tag{11/11}$$

which behave as $(-)^q x^p \log^q|x|$, and $(-)^q x^p \log^q|x| \operatorname{sgn}(x)$ at $x = 0$ respectively. At infinity there are the same linear combinations with slashes added. The special notations for the θ–functions are collected in the following table.

Table 11.1

Special notations for $\theta_s^{(\alpha,q)}(x)\operatorname{sgn}^m(x)$ and $\not\theta_s^{(\alpha,q)}(x)\operatorname{sgn}^m(x)$

α	m (mod 2)	q	$\theta^{:::}_{...}(x)$	corresponding $\eta^{:::}_{...}(x)$
α	m	q	$\theta_s^{(\alpha)}(x)\operatorname{sgn}^m(x)$	$\eta_s^{(\alpha)}(x)\operatorname{sgn}^m(x)$
α	0	q	$\theta_s^{(\alpha,q)}(x)$	$\eta_s^{(\alpha,q)}(x)$
α	1	q	$\theta_a^{(\alpha,q)}(x)$	$\eta_a^{(\alpha,q)}(x)$
p	p	q	$\theta^{(p,q)}(x)$	$\eta^{(p,q)}(x)$
p	$p+1$	q	$\tau^{(p,q)}(x)$	$\sigma^{(p,q)}(x)$
p	p	0	$\theta^{(p)}(x)$	$\eta^{(p)}(x)$
p	$p+1$	0	$\tau^{(p)}(x)$	$\sigma^{(p)}(x)$
0	0	0	$\theta(x)$	$\eta(x)$
0	1	0	$\tau(x)$	$\sigma(x)$

and idem for $\not\theta_s^{(\alpha,q)}(x)\operatorname{sgn}^m(x)$ with slashes added

The symbol $\theta^{(p)}(x)$ has been defined twice in two different ways. In this section the function $\theta^{(p)}(x)$ was defined as the difference in the finite between the functions x^p and x'^p, while the function $\theta^{(p)}(x)$ was defined previously in (6.62) as the Fourier transform of $\not\eta^{(p)}(x) + \cdots$. It will be seen in the next section that the two definitions agree.

The properties of the functions $\theta_s^{(\alpha,q)}(x)\operatorname{sgn}^m(x)$ follow from the known properties of the subspaces $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ by linearity.

11.2 Scalar products

The scalar products

$$\langle\ ,\ \rangle : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbb{C} \quad \text{and} \quad \langle\ ,\ \rangle : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \mathbb{C}, \tag{11.12}$$

were defined in Sec. 8.1 and Sec. 10.3. The scalar product

$$\langle\ ,\ \rangle : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}_\lambda \to \mathbb{C}$$

is defined in agreement with (8.14) to by imposing symmetry

$$\langle\ ,\ \rangle : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}_\lambda \to \mathbb{C} := \langle\ ,\ \rangle : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbb{C}. \tag{11.13}$$

This gives the scalar product

$$\langle\ ,\ \rangle : \mathbf{PC}^\oplus_\lambda \times \mathbf{PC}^\oplus_\lambda \to \mathbb{C}, \tag{11.14}$$

as

$$\mathbf{PC}^\oplus_\lambda \times \mathbf{PC}^\oplus_\lambda \to \mathbb{C} := (\overline{\mathbf{PC}}_\lambda \oplus \overline{\mathbf{PC}}'_\lambda) \times (\overline{\mathbf{PC}}_\lambda \oplus \overline{\mathbf{PC}}'_\lambda) \to \mathbb{C}, \tag{11.15}$$

where all terms are now defined. The same definition applies to the left-first and the right-first scalar products.

The scalar product defines the elements of $\mathbf{PC}^\oplus_\lambda$ as linear functionals on $\mathbf{PC}^\oplus_\lambda$. This is only a small part of their properties as generalised functions.

11.3 Operators on the generalised functions

The action of the operators on $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ was completely defined in Ch. 6 and Ch. 7, and both $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ are closed under the operators. Therefore the operators are defined on $\mathbf{PC}^\oplus_\lambda$ by their action on $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$. Only for the $\theta_s^{(\alpha,q)}$–functions introduced in the previous section some care is needed. Subtraction always yields a sum of a θ–function in the finite, and a $\not\theta$–function at infinity, which has to be split. As before it is convenient to resolve this splitting problem by definition.

Remark 11.2 When limits of sequences of generalised functions are defined in Ch. 19 it can be shown that the splitting by definition which is used in this chapter agrees with the behaviour of suitably chosen limit sequences.

The operators $\mathcal{X}$ and $\mathcal{D}$ are defined to be local, in the sense that they do not enlarge the support of the generalised functions on which they act. They convert functions in the finite into functions in the finite. Splitting into parts in the finite and parts at infinity gives in the case of the $\mathcal{X}$ operator

$$\mathcal{X}\,\theta_s^{(\alpha,q)}(x) = \theta_a^{(\alpha+1,q)}(x), \quad \text{and} \quad \mathcal{X}\,\not\theta_s^{(\alpha,q)}(x) = \not\theta_a^{(\alpha+1,q)}(x), \tag{11/16}$$

for all possible subscripts.

For the differential operator $\mathcal{D}$ we obtain

$$\begin{aligned}\mathcal{D}\,\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^{m+1}(x) = {} & +\alpha\,\theta_s^{(\alpha-1,q)}(x)\,\mathrm{sgn}^m(x) + \\ & + q(1-\delta_{q,0})\,\theta_s^{(\alpha-1,q-1)}(x)\,\mathrm{sgn}^m(x) + \\ & - 2\,\delta_{q,0}\,\eta_s^{(-\alpha)}(x)\,\mathrm{sgn}^m(x), \end{aligned} \tag{11.17}$$

and

$$\begin{aligned}\mathcal{D}\,\not\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^{m+1}(x) = {} & +\alpha\,\not\theta_s^{(\alpha-1,q)}(x)\,\mathrm{sgn}^m(x) + \\ & + q(1-\delta_{q,0})\,\not\theta_s^{(\alpha-1,q-1)}(x)\,\mathrm{sgn}^m(x) + \\ & + 2\,\delta_{q,0}\,\not\eta_s^{(-\alpha)}(x)\,\mathrm{sgn}^m(x), \end{aligned} \tag{11.18}$$

which again shows that the θ–functions behave as the powers.

The exceptional cases for differentiation are θ, $\not\theta$, τ, and $\not\tau$. In these cases θ–functions are converted into η and σ–functions. The results are

$$\mathcal{D}\,\theta(x) = -2\,\sigma(x), \quad \text{and} \quad \mathcal{D}\,\tau(x) = 2\,\delta(x) - 2\,\eta(x), \tag{11.19}$$

and at infinity

$$\mathcal{D}\,\not\theta(x) = 2\,\not\sigma(x), \quad \text{and} \quad \mathcal{D}\not\tau(x) = 2\,\not\eta(x), \tag{11.20}$$

The equivalent formulæ for the one-sided linear combinations are

$$\mathcal{D}\,\theta_{\downarrow}(x) = \delta(x) - \eta_{\downarrow}(x), \tag{11.21}$$

and

$$\mathcal{D}\,\not\theta_{\uparrow}(x) = \not\eta_{\uparrow}(x). \tag{11.22}$$

This illustrates the intermediate position of the $\theta_{\downarrow}$–function between the spaces $\overline{\mathbf{PC}}_{\lambda}$ and $\overline{\mathbf{PC}}'_{\lambda}$. It also serves as a reminder that the choice $\delta(x) \neq \eta(x)$ is necessary to obtain an interesting theory.

The Fourier transforms of the θ–functions are also found by subtraction. From the asymptotic behaviour of the Fourier transform [Lig] it is known that the Fourier transform of a singularity in the finite leads to a singularity at infinity.

Therefore the splitting is defined as

$$\begin{aligned}\mathcal{F}\,\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = {} & \\ = -2i^m(-)^q q! \sum_{j=0}^{q+1} & \tfrac{(-)^j}{j!}\, c_{q-j}(\alpha,m)\,\not\theta_s^{(-\alpha-1,j)}(x)\,\mathrm{sgn}^m(x) + \\ + 4i^m(-)^q q! & \sum_{j=0}^{\infty} j!\, c_{q+j+1}(\alpha,m)\,\not\eta_s^{(\alpha,j)}(x)\,\mathrm{sgn}^m(x), \end{aligned} \tag{11.23}$$

and

$$\mathcal{F}\,\not\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) =$$
$$- 2i^m q! \sum_{j=0}^{q+1} \tfrac{1}{j!}\, c_{q-j}(\alpha,m)\,\theta_s^{(-\alpha-1,j)}(x)\,\mathrm{sgn}^m(x) +$$
$$- 4i^m q! \sum_{j=0}^{\infty} (-)^j j!\, c_{q+j+1)}(\alpha,m)\,\eta_s^{(\alpha,j)}(x)\,\mathrm{sgn}^m(x) +$$
$$+ 4i^m q! \sum_{p=0}^{\infty} \delta_{\alpha,p}\,\delta_{p,m}^{\mathrm{mod}2}\, c_{q+1}(p,m)\,\not\delta^{(p)}(x). \tag{11.24}$$

For integer parameter values this can be written as

$$\mathcal{F}\,\theta^{(p,q)}(x) = -2i^p(-)^q q! \sum_{j=0}^{q+1} \tfrac{(-)^j}{j!}\, c_{q-j}(p,p)\,\not\tau^{(-p-1,j)}(x) +$$
$$+ 4i^m(-)^q q! \sum_{j=0}^{\infty} c_{q+j+1}(p,p)\,\not\eta^{(p,j)}(x), \tag{11/25}$$

and

$$\mathcal{F}\,\tau^{(p,q)}(x) = -2i^p(-)^q q! \sum_{j=0}^{q+1} \tfrac{(-)^j}{j!}\, c_{q-j}(p,p)\,\not\theta^{(-p-1,j)}(x)$$
$$+ 4i^m(-)^q q! \sum_{j=0}^{\infty} c_{q+j+1}(p,p)\,\not\phi^{(p,j)}(x). \tag{11/26}$$

In particular the previous special case (6.64)

$$\mathcal{F}\,\theta^{(p)}(x) = 4i^p \sum_{j=0}^{\infty} j!\, c_{j+1}(p,p)\,\not\eta^{(p,j)}(x), \tag{11.27}$$

is recovered as a special case by putting $q = 0$. This shows that the different definitions of $\theta^{(p)}(x)$ do indeed agree. The Fourier transform of the difference of the functions x^p and x'^p at $x = 0$ equals the Fourier transform of $\theta^{(p)}(x)$ as defined previously in (6.64).
This completes the definition of the operators on the θ–functions.

Remark 11.3 The transformation of the θ–functions under the usual operators is dual to the corresponding formulæ for the η–functions. The θ–functions transform as the powers, the η–functions measure the powers. The measurement formula in the finite

$$\langle\, \theta_s^{(\alpha,q)}(x)\,,\, \eta_s^{(\beta,r)}(x) \,\rangle = \tfrac{1}{2}\,\delta_{\alpha,\beta}\,\delta_{q,r}\,, \tag{11/28}$$

is by Parseval's equality in agreement with the corresponding formula

$$\langle\, \not\theta_s^{(\alpha,q)}(x)\,,\, \not\eta_s^{(\beta,r)}(x) \,\rangle = \tfrac{1}{2}\,\delta_{\alpha,\beta}\,\delta_{q,r}\,, \tag{11/29}$$

at infinity.

It is convenient to extend the definition of the maps to $\mathbf{GF}_s$ as a whole. The map $\mathcal{M} : \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}'_\lambda$ is completed by the definition

$$\mathcal{M}\, f'(x) := \mathcal{I}\, f'(x) = f'(x), \tag{11.30}$$

$\forall\, f'(x) \in \overline{\mathbf{PC}}'_\lambda$. This makes $\mathcal{M}$ into a projection operator on the subspace $\overline{\mathbf{PC}}'_\lambda$,

$$\mathcal{M}\mathcal{M} = \mathcal{M}^2 = \mathcal{M}. \tag{11.31}$$

The zero space of $\mathcal{M}$ is spanned by the elements

$$\left(\delta^{(p)}(x - x_0) - \eta^{(p)}(x - x_0)\right) \in \mathbf{GF}_s, \tag{11.32}$$

$\forall p \in \mathbb{N}$, $\forall x_0 \in \mathbb{R}$, and the elements

$$\theta_s^{(\alpha,q)}(x - x_0)\,\mathrm{sgn}^m(x - x_0) \in \mathbf{GF}_s, \tag{11.33}$$

$\alpha \in \mathbb{C}$, $(\alpha, q) \neq (p \in \mathbb{N}, 0)$, $x_0 \in \mathbb{R}$, and the elements

$$e^{ikx}\not\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \in \overline{\mathbf{PC}}'_\lambda \subset \mathbf{GF}_s, \tag{11.34}$$

$\forall\alpha \in \mathbb{C}$, $\forall q, m \in \mathbb{N}$, $\forall k \in \mathbb{R}$.
The projection operators $P_{\mathcal{M}}$ and $P_{\not\mathcal{M}}$ are completed by defining

$$P_{\mathcal{M}}\overline{\mathbf{PC}}_\lambda := \mathcal{I}\overline{\mathbf{PC}}_\lambda, \tag{11.35}$$

and

$$P_{\not\mathcal{M}}\overline{\mathbf{PC}}_\lambda := \{0\}, \tag{11.36}$$

which makes $P_{\mathcal{M}}$ a projection on the elements of $\mathbf{GF}_s$ which are non-zero as distributions.
Likewise $\mathcal{M}^{-1}$ is completed by the definition

$$\mathcal{M}^{-1}\, f(x) := \mathcal{I}\, f(x) = f(x), \tag{11.37}$$

$\forall\, f(x) \in \overline{\mathbf{PC}}_\lambda$. This definition makes $\mathcal{M}^{-1}$ into a projection operator on the subspace $\overline{\mathbf{PC}}_\lambda$. In addition to its zero space in $\overline{\mathbf{PC}}'_\lambda$ it has the same zero space as $\mathcal{M}$. In particular the elements $\theta^{(p)}(x)$ are in the zero space of $\mathcal{M}^{-1}$, but not in the zero space of $\mathcal{M}$.

It is obvious that $\mathcal{M}^{-1}$ is not the inverse of $\mathcal{M}$, since non-trivial projections do not have an inverse. The relation between the operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ remains (9.10)

$$\mathcal{M}^{-1}\mathcal{M} = \mathcal{M}\mathcal{M}^{-1} = P_{\mathcal{M}} = \mathcal{I} - P_{\not\mathcal{M}}, \tag{11.38}$$

so that these operators are each others inverse on the subspace $P_{\mathcal{M}}\mathbf{GF}_s$. The restricted mappings $\mathcal{M} : \overline{\mathbf{PC}}_\lambda \to P_{\mathcal{M}}\overline{\mathbf{PC}}'_\lambda$ and $\mathcal{M}^{-1} : P_{\mathcal{M}}\overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}_\lambda$ are bijective.

The same holds for the map $\mathcal{M}_{\mathcal{X}} = \mathcal{M}$, and for $\mathcal{M}_{\mathcal{D}} = \mathcal{F}^{-1}\mathcal{M}_{\mathcal{X}}\mathcal{F}$. The zero-space of $\mathcal{M}_{\mathcal{D}}$ is found by applying $\mathcal{F}^{-1}$ to the zero-space of $\mathcal{M}_{\mathcal{X}}$.

Remark 11.4 Despite this extension of the maps there remains a difference between $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$. Operators are defined first on $\overline{\mathbf{PC}}'_\lambda$, and then transferred by $\mathcal{M}^{-1}\mathcal{O}\mathcal{M}$ to $\overline{\mathbf{PC}}_\lambda$.

11.4 The product of generalised functions

The products $\circ\bullet$, $\bullet\circ$, $\bullet$ have to be defined on $\in \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda$. Most of the formulæ are given only for the symmetrical product. In most cases the three products are equal, and it is not necessary to write them all out.
The products

$$\bullet : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}_\lambda, \tag{11.39}$$

and

$$\bullet \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \overline{\mathbf{PC}}'_\lambda, \tag{11.40}$$

have already been defined. It remains to find the products

$$\bullet : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbf{PC}^\oplus_\lambda, \tag{11.41}$$

and

$$\bullet : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}_\lambda \to \mathbf{PC}^\oplus_\lambda, \tag{11.42}$$

to complete the product

$$\bullet : \mathbf{PC}^\oplus_\lambda \times \mathbf{PC}^\oplus_\lambda \to \mathbf{PC}^\oplus_\lambda, \tag{11.43}$$

and consequently the product

$$\bullet : \mathbf{GF}_s \times \mathbf{GF}_s \to \mathbf{GF}_s. \tag{11.44}$$

It would be possible to abstract the product $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda$ from the scalar product as in Ch. 8 for the products on $\overline{\mathbf{PC}}'_\lambda$. Now that the product on $\overline{\mathbf{PC}}_\lambda$ is known, it is possible to work back from the product on $\overline{\mathbf{PC}}_\lambda$ by taking appropriate residues. In this way the product $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbf{PC}^\oplus_\lambda$ is found first. By taking a second residue, with the usual symmetrization (8.14), the product $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$ can be recovered. It agrees with the product defined previously.
As a starting point the product (9.70),

$$|x|^\lambda \operatorname{sgn}^m(x) \bullet |x|^\mu \operatorname{sgn}^n(x) = \\ = |x|^{\lambda+\mu} \operatorname{sgn}^{m+n}(x) + \sum_{p=0}^{\infty} \delta_{\lambda+\mu,-p-1} \cdots \delta^{(p)}(x), \tag{11.45}$$

can be taken. The explicit dependence of $\cdots$ on λ and μ is irrelevant, since the Kronecker δ is zeromorphic. Taking a residue on λ gives

$$|x'|^\alpha \log^q |x| \operatorname{sgn}^m(x) \bullet |x|^\mu \operatorname{sgn}^n(x) = |x'|^{\alpha+\mu} \operatorname{sgn}^{m+n}(x). \tag{11.46}$$

The δ–function part drops out, since it is a zeromorphic function of the variable λ. Multiplying by a power of $\lambda+\alpha+1$ and taking the residue gives after splitting

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet |x|^\mu \log^r|x|\,\mathrm{sgn}^n(x) = \\ = \begin{cases} 0(x) & r>q, \\ (-)^r \eta_s^{(\alpha-\mu,q-r)}(x)\,\mathrm{sgn}^{m+n}(x) & q \geq r, \end{cases} \quad (11/47)$$

and

$$\not\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet |x|^\mu \log^r|x|\,\mathrm{sgn}^n(x) = \\ = \begin{cases} 0(x) & r>q, \\ \not\eta_s^{(\alpha-\mu,q-r)}(x)\,\mathrm{sgn}^{m+n}(x) & q \geq r, \end{cases} \quad (11/48)$$

in agreement with the scalar product.
Products in $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda$ involving δ–functions follow from (7.33)

$$\delta^{(p)}(x) \bullet |x|^\alpha \log^q|x|\,\mathrm{sgn}^m(x) = \sum_{j=-p}^{\infty} \delta_{-\alpha,j} \cdots \delta^{(p+j)}(x). \quad (11.49)$$

The right-hand side is again zeromorphic in α, so by taking a residue, we obtain

$$\delta^{(p)}(x) \bullet |x'|^\alpha \log^q|x|\,\mathrm{sgn}^m(x) = 0(x), \quad (11.50)$$

and

$$\delta^{(p)}(x) \bullet \eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = 0(x), \quad (11.51)$$

$\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$.

The product involving ordinary functions is now defined as the pointwise product between singularities. At the singularities it is defined as the product of the asymptotic series. This gives

$$f(x) \bullet g'(x) = \big(f(x) \cdot g(x)\big)'(x).$$

If one of the factors has a prime the product has a prime.

The product $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$ is now recovered in the usual way (compare Ch. 8). If another residue is taken starting from (11.48) the result is zero, if we start with (11.46) we obtain one. The left-sided products are obtained by taking the residue corresponding to the left-sided factor in the residue first, the right-sided product by taking the opposite order. By symmetrization as in Ch. 8, the product $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}'_\lambda$ is recovered.
A typical result is

$$|x'|^a \log^q|x|\,\mathrm{sgn}^m(x) \bullet |x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) = |x'|^{\alpha+\beta} \log^{q+r}|x|\,\mathrm{sgn}^{m+n}(x), \quad (11/52)$$

which also holds for the other versions of the product of the powers. For the products of the η–functions and the powers one recovers the result (8.44–45)

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet\!\circ\, |x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) = \\ = \begin{cases} \mathit{0}(x) & q > r, \\ (-)^q \eta_s^{(\beta-\alpha,r-q)}(x)\,\mathrm{sgn}^{m+n}(x) & q \le r, \end{cases} \quad (11/53)$$

and

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \circ\!\bullet\, |x'|^\beta \log^r|x|\,\mathrm{sgn}^n(x) = \mathit{0}(x), \quad (11/54)$$

in agreement with the results of Ch. 8. The product of η–functions is always zero,

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet \eta_s^{(\beta,r)}(x)\,\mathrm{sgn}^n(x) = \mathit{0}(x), \quad (11/55)$$

again independently of the kind of product.

Finally, repeated subtractions of (11.52) and (11.46) from (9.70), yields the products of the θ–functions with themselves

$$\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet \theta_s^{(\beta,r)}(x)\,\mathrm{sgn}^n(x) = \theta_s^{(\alpha+\beta,q+r)}(x)\,\mathrm{sgn}^{m+n}(x) + \\ + \big(d_{rq}(\alpha,m) + d_{qr}(\beta,n)\big)(-)^{q+r}\sum_{p=0}^{\infty} \delta_{-\alpha-\beta-1,p}\,\delta_{m+n,p}^{\mathrm{mod}2}\,\delta^{(p)}(x), \quad (11.56)$$

and idem at infinity

$$\not\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet \not\theta_s^{(\beta,r)}(x)\,\mathrm{sgn}^n(x) = \not\theta_s^{(\alpha+\beta,q+r)}(x)\,\mathrm{sgn}^{m+n}(x), \quad (11.57)$$

without δ–functions. The formulæ (11.56) and (11.57) will be needed for verifying Parseval's equality in Ch. 12.
It is seen from (11.56) that we have in the special case $(\beta, r) = (0,0)$

$$|x|^\alpha \log^q|x|\,\mathrm{sgn}^m(x) \bullet \theta(x) = \theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \quad (11/58)$$

without additional δ–functions. This is in agreement with the naïve interpretation of the θ–functions. In particular by further specialization one recovers the result (8.65)

$$\theta(x) \bullet \theta(x) = \theta(x). \quad (11.59)$$

By subtraction of (11.54) and (11.47) one sees that the left- and right-sided products of the θ–functions behave oppositely to the corresponding products of primed powers,

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \circ\!\bullet\, \theta_s^{(\beta,r)}(x)\,\mathrm{sgn}^n(x) = \\ = \begin{cases} \mathit{0}(x) & q > r, \\ \eta_s^{(\beta-\alpha,r-q)}(x)\,\mathrm{sgn}^{m+n}(x) & q \le r, \end{cases} \quad (11/60)$$

and

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet\!\circ\, \theta_s^{(\beta,r)}(x)\,\mathrm{sgn}^n(x) =^{\cdot} \mathit{0}(x), \qquad (11/61)$$

without the factor $(-)^q$, since $\theta_s^{(\alpha,q)}(x)$ is defined as a positive function. The products of θ–functions and δ–functions are found as

$$\begin{aligned}\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet \delta^{(r)}(x) &= \\ &= \tfrac{q!}{\pi}\sum_{j=0}^{r}\delta_{\alpha,j}\,\delta_{j,m}^{\mathrm{mod2}}\left(\tfrac{(-)^j}{j!}\,c_{q+1}(j,j) + \tfrac{(-)^r}{r!}\,c_{q+1}(r,r)\right)\delta^{(r-j)}(x) + \\ &\quad + \tfrac{q!}{\pi}\sum_{j=0}^{\infty}\delta_{\alpha,-j-1}\,\delta_{j+1,m}^{\mathrm{mod2}}\,\tfrac{(-)^r}{r!}\,c_{q+1}(r,r)\,\delta^{(r+j+1)}(x), \end{aligned} \qquad (11.62)$$

in agreement with the corresponding product of the unprimed powers. The primed powers do not contribute. Therefore one finds the property

$$\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) \bullet \delta^{(r)}(x) = (-)^q|x|^\alpha \log^q|x|\,\mathrm{sgn}^m(x) \bullet \delta^{(r)}(x), \qquad (11.63)$$

in agreement with the interpretation of the θ–functions.
The previous equation (11.62) has the important special case, (substituting the value $c_1(0,0) = \frac{\pi}{2}$ from table C.1),

$$\theta(x) \bullet \delta(x) = \delta(x). \qquad (11.64)$$

This does not include a factor $\frac{1}{2}$, since both terms in (11.62) contribute a factor $\frac{1}{2}$ to (11.64).

Remark 11.5 The result of a product from $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}_\lambda$, as in (11.64), is not necessarily $\in \overline{\mathbf{PC}}'_\lambda$.

The product $\bullet : \overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}'_\lambda \to \mathbf{PC}^\oplus_\lambda$ is now completely defined, and by commutativity also the product $\bullet : \overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}_\lambda \to \mathbf{PC}^\oplus_\lambda$. Therefore the product $\bullet : \mathbf{PC}^\oplus_\lambda \times \mathbf{PC}^\oplus_\lambda \to \mathbf{PC}^\oplus_\lambda$ can now be defined by

$$\mathbf{PC}^\oplus_\lambda \bullet \mathbf{PC}^\oplus_\lambda := (\overline{\mathbf{PC}}_\lambda \oplus \overline{\mathbf{PC}}'_\lambda) \bullet (\overline{\mathbf{PC}}_\lambda \oplus \overline{\mathbf{PC}}'_\lambda), \qquad (11.65)$$

in agreement with the definition of $\mathbf{PC}^\oplus_\lambda$.

The product $\mathbf{PC}^\oplus_\lambda \times \mathbf{PC}^\oplus_\lambda \to \mathbf{PC}^\oplus_\lambda$ has all the properties required in Ch. 2. In particular one verifies by direct computation that the generalised function $I(x)$ is indeed the unit element of all products

$$I(x) \circ\!\bullet\, f(x) = I(x) \bullet f(x) = I(x) \bullet\!\circ\, f(x) = f(x), \qquad (11.66)$$

for all $f \in \mathbf{PC}^\oplus_\lambda$.

The product of all pairs of elements has been explicitly defined as an element of $\mathbf{PC}^\oplus_\lambda$, so the model is closed under the generalised function multiplication.

11.5 Convolution of generalised functions

The convolution product $\mathbf{PC}_\lambda^\oplus \times \mathbf{PC}_\lambda^\oplus \to \mathbf{PC}_\lambda^\oplus$ is best defined as the Fourier image of the pointwise product

$$f(x) * g(x) := \mathcal{F}^{-1}\big(\mathcal{F} f(x) \cdot \mathcal{F} g(x)\big). \qquad (11.67)$$

The convolution can also be defined by defining a 'regularization' of the convolution integrals but this method is more arbitrary. The regularization of divergent integrals will be defined in Ch. 22. It is easier and less arbitrary to derive the correct regularization from the known convolution.
The identity element of the convolution is $\delta(x) = \delta$, since

$$\delta(x) * f(x) = f(x), \qquad (11.68)$$

$\forall f(x) \in \mathbf{PC}_\lambda^\oplus$. The function $\delta(x)$ is the Fourier transform of the unit function. Calculation of the remaining convolutions $\forall f(x) \in \mathbf{PC}_\lambda^\oplus$ is a tedious exercise which yields no new insights.

11.6 The simple model

It remains to complete the construction by removing the superfluous parts of $\mathbf{PC}_\lambda^\oplus$. The spaces $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ have much in common.

Example 11.1 The Schwartz space $\mathcal{S}$ of $\mathbf{C}^\infty$–functions of rapid decrease at infinity is a subset of both $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$.

This extends immediately to all $\mathbf{C}^\infty$ pieces between singularities. It is only at the singular points that elements of $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ differ.

Now the space of simple generalised functions is obtained by identifying the $\mathbf{C}^\infty$ parts. This leaves only a summing up of the possible singular behaviour at the singular points. The simple model $\mathbf{GF}_s$ can be written as a direct sum of pieces as

$$\mathbf{GF}_s = \mathbf{PC}_\lambda \oplus \mathbf{PC}_\delta \oplus \mathbf{PC}'_\theta \oplus \mathbf{PC}'_\eta \oplus \mathbf{PC}'_{\eta'}, \qquad (11.69)$$

or equivalently as

$$\mathbf{GF}_s = \mathbf{PC}'_\lambda \oplus \mathbf{PC}'_\eta \oplus \mathbf{PC}'_{\eta'} \oplus \mathbf{PC}'_\theta \oplus \mathbf{PC}_\delta, \qquad (11.70)$$

since $\mathbf{PC}'_\theta$ contains the localized difference between $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$.

Conversely the splitting is no longer unique, any $\mathbf{C}^\infty$–function can be assigned either to $\overline{\mathbf{PC}}_\lambda$ or to $\overline{\mathbf{PC}}'_\lambda$. It is only at the singular points that the characterization of the singular behaviour is unique.

It follows that the space $\mathbf{GF}_s$ has the required properties, since by construction the space $\mathbf{PC}_\lambda^\oplus$ has them. This will be verified in the next chapter.

11.7 Generalised functions as ordinary functions

In the previous chapters the generalised functions were constructed starting with a class of ordinary functions. Conversely one can associate an ordinary function $f(x_0)$ with a generalised function by

$$f(x_0) := \langle \delta(x - x_0) \, , f(x) \rangle, \tag{11.71}$$

which can be used to identify generalised functions with ordinary functions. The function $f(x_0)$ will be defined as the value of the generalised function in Ch. 13.

The converse is not uniquely possible. Corresponding to an 'ordinary' function $f(x) : \mathbb{R} \to \mathbb{R}$, which satisfies the restrictions imposed in Ch. 4, there are in general many different generalised functions. These functions differ in the interpretation of their behaviour at the singular points as either of the type of $\mathbf{PC}'_\lambda$, or of type $\mathbf{PC}_\lambda$. In fact the number of different choices is much larger, since the choice of adding a prime can be made for each separate term in the asymptotic expansion at each singular point. Arbitrary linear combinations of the two possibilities are also allowed. A generalised function is not determined by its values.

The local behaviour of an ordinary function $f'(x)$, which is piecewise constant, considered as a generalised function $\in \overline{\mathbf{PC}}'_\lambda$, can be demonstrated by writing out the function in terms of $\theta^{(p)}$ and Heaviside stepfunctions. The local behaviour of a piecewise constant ordinary generalised function $\in \overline{\mathbf{PC}}'_\lambda$ can be characterized by the three numbers

$$f'(x) = c_{--} \, H'(-x) + c_0 \, \theta(x) + c_{++} \, H'(x), \tag{11.72}$$

which can be measured by

$$\begin{aligned} c_0 &= \langle \delta(x) \, , f'(x) \rangle, \\ c_{--} &= \langle 2 \, \eta_\uparrow(x) - \delta(x) \, , f'(x) \rangle, \\ c_{++} &= \langle 2 \, \eta_\downarrow(x) - \delta(x) \, , f'(x) \rangle. \end{aligned} \tag{11.73}$$

By contrast an ordinary function $f(x) \in \mathbf{PC}_\lambda$, with no more than a jump discontinuity, can be written as

$$f(x) = c_{--} \, H(-x) + c_{++} \, H(x), \tag{11.74}$$

with its coefficients measured by

$$c_- = \langle \eta_\uparrow(x) \, , f(x) \rangle, \qquad c_+ = \langle \eta_\downarrow(x) \, , f(x) \rangle. \tag{11.75}$$

It is characterized by only two numbers.
In the general case of a function $\in \mathbf{GF}_s$ with a jump discontinuity

$$f(x) = c_{--} \, H'(-x) + c_{0-} \, \theta_\uparrow(x) + c_{0+} \, \theta_\downarrow(x) + c_{++} \, H'(x), \tag{11.76}$$

the coefficients can be measured using the asymmetrical scalar products

$$c_{--} = \langle \eta_{\uparrow}(x) \, , f(x) \rangle\!\!\circ, \tag{11.77}$$

$$c_{0-} = \circ\!\!\langle \eta_{\uparrow}(x) \, , f(x) \rangle, \tag{11.78}$$

$$c_{0+} = \circ\!\!\langle \eta_{\downarrow}(x) \, , f(x) \rangle, \tag{11.79}$$

$$c_{--} = \langle \eta_{\downarrow}(x) \, , f(x) \rangle\!\!\circ. \tag{11.80}$$

The $\not\delta(x)$–function is not needed, since it measures $\frac{1}{2}(c_{0-} + c_{0+})$ at the jump.

Corresponding to an ordinary function with a singularity such as a jump, there are two different generalised functions. Both have a jump discontinuity, but of a different kind. The difference shows up for instance in their different derivatives. Which one is appropriate in a given problem depends on the problem to which analysis is applied. Roughly speaking it will be seen that the 'smooth' Heaviside functions $H'(x) \in \overline{\mathbf{PC}}'_{\lambda}$ are appropriate when the functions are seen as the limit of a sequence of smooth function. The 'sharp' Heaviside functions $H(x) \in \overline{\mathbf{PC}}_{\lambda}$ are appropriate when the functions involved are 'really' discontinuous rather than very (perhaps infinitely) steep. Of course linear combinations of sharp and smooth discontinuities are also allowed.

notation The following conventions will be used. Ordinary function without further mention will be assumed to have sharp singularities, that is as $f(x) \in \overline{\mathbf{PC}}_{\lambda}$. If all singularities are weak a prime is added to the function symbol, as in $f'(x) \in \overline{\mathbf{PC}}'_{\lambda}$. Primes in a function symbol as in $(x' \pm io)^{\alpha}$, (compare Ch. 21), indicate that a residue should be taken, so the appearance of η–functions is to be expected. The type of singularity can also be indicated be the explicit appearance of Heaviside functions, and/or θ–functions, as in $e^{-|x|}I'(x)$ versus $e^{-|x|}$.

Mixed behaviour at different singularities is also possible. It is allowed to choose any type of singularity for any term in the asymptotic expansion at any point, including regular points. The functions $I'(x)$, $I(x)$, and $I'(x) + \theta(x)$ are different.

Remark 11.6 The distinction between the functions $I(x)$ and $I'(x) + \theta(x)$ ties in with the traditional distinction between the actual and the potential infinite in an aesthetically pleasing way. This remark will be taken up again in Ch. 24.

Example 11.2 In Ch. 19 it will be found that the smooth functions have limit properties such as

$$\lim_{a \downarrow 0} e^{-ax^2} = I'(x) + \theta(x) + \sum_{j=0}^{\infty} \cdots \not\phi^{(-1,j)}(x) \neq I(x), \tag{11.81}$$

where the limit has to be interpreted as convergence in the sense of the symmetrical theory of generalised functions, to be defined in Ch. 19.

It will be seen that the sharp and the smooth functions correspond with different idealizations. Both are useful in an appropriate context.

Example 11.3 If in quantum electrodynamics electrons are considered to be point-particles, the sharp functions are appropriate. If the electron is only very small (finite but of a size which is too small to be measured) the smooth singularity is an appropriate idealization for its description.

11.8 Generalised functions as distributions

The Schwartz spaces of test functions $\mathcal{S}$ and $\mathcal{D}$ are contained in the preliminary class, and therefore also in the classes $\overline{\mathbf{PC}}'_\lambda$ and $\overline{\mathbf{PC}}_\lambda$, and in $\mathbf{GF}_s$. This means that all symmetrical generalised functions are always defined as linear functionals on a space of test functions, that is as distributions.

Many generalised functions are zero as distributions for lack of a suitable test function in $\mathcal{S}$ to distinguish them from zero. The contents of $\overline{\mathbf{PC}}_\lambda$ are non-zero as distributions. The elements of $\overline{\mathbf{PC}}'_\lambda$ are non-zero when they are unchanged by the projection operator $\boldsymbol{P}_{\mathcal{M}}$ defined in Ch. 9. The difference between an element $f(x) \in \overline{\mathbf{PC}}_\lambda$ and the corresponding element $f'(x) := \mathcal{M}\, f(x) \in \overline{\mathbf{PC}}'_\lambda$ is also zero in the sense of distribution theory.

Example 11.4 The difference $\delta(x) - \eta(x)$ is zero as a distribution, since

$$\langle\, f(x)\,, \delta(x)\,\rangle = \langle\, f(x)\,, \eta(x)\,\rangle = f(0), \tag{11.82}$$

$\forall\, f(x) \in \mathcal{S}$. The generalised function $\theta(x)$ is needed to distinguish between $\eta(x)$ and $\delta(x)$.

Zero as distributions are all θ–functions, all generalised functions at infinity, and all η–functions except $\eta^{(p)}(x - x_0)$, with $p \in \mathbb{N}$, and $x_0 \in \mathbb{R}$. The reduction of the symmetrical generalised functions to the elements which have a non-zero equivalent among the distributions is a projection. One is free to choose the either a projection on $\overline{\mathbf{PC}}_\lambda$ or $\overline{\mathbf{PC}}'_\lambda$. The transfer map $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$ can be used to convert between the two possibilities.

The distributional aspect of the symmetrical generalised functions is only a part of their properties as generalised functions. Generalised functions have more properties than distributions, since they are not exclusively defined as linear functionals.

11.9 Summary of the contents of the model

The model contains several independent families of elements localized at a point. At the point $x = 0\pm$ there are the families

$$\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \tag{11.83}$$

and

$$\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \tag{11.84}$$

$\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$, $\forall m \in \mathbb{Z}$. Moreover at the point $x = 0$ there is the family

$$\delta^{(p)}(x), \tag{11.85}$$

$\forall p \in \mathbb{N}$. Many linear combinations of these elements are also generalised functions $\in \mathbf{GF}_s$. In particular one verifies that $\forall\, g(x) \in \mathbf{GF}_s$ the elements

$$g(x) \bullet \eta_s^{(\alpha,q)}(x), \quad \text{and} \quad g(x) \bullet \theta_s^{(\alpha,q)}(x), \quad \text{and} \quad g(x) \bullet \delta^{(p)}(x), \tag{11.86}$$

are elements $\in \mathbf{GF}_s$. These are admissible linear combinations of generalised functions. This does not exhaust the linear combinations. As noted before there are many formally infinite linear combinations which are effectively finite, since only finitely many terms in the linear combination are non-zero in any scalar product with an element $\in \mathbf{GF}_s$.
At infinity there are the families of elements

$$\not{\theta}^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \quad \text{and} \quad \not{\eta}^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \tag{11.87}$$

$\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$, $\forall m \in \mathbb{Z}$. There are no δ-slash functions in the model. For the linear combinations at infinity the same restrictions apply as in the finite. Effectively only finite linear combinations occur in both cases. This implies that the allowed α–values in a linear combination of $\not{\eta}$–functions are an ascending sequence, while $\not{\theta}$–functions contain only descending sequences.

Remark 11.7 The model $\mathbf{GF}_s$ contains two subspaces which are also closed under the operators. These are obtained by restricting the complex parameters such as α and the allowed powers in the asymptotic expansions to $\alpha \in \mathbb{Z}$ or $\alpha \in \mathbb{R}$. No significant simplification is obtained in this way, so we keep the general case $\alpha \in \mathbb{C}$.

The simple model is now complete. In the following chapters some of its properties will be derived.

CHAPTER 12

PROPERTIES AND VERIFICATION

In this chapter the list of requirements given in Ch. 2 is followed and it is verified that the requirements are satisfied. Only an outline of the verification is given in most cases. This chapter also serves to collect partial results spread over several previous chapters.

Given the exploratory character of the work presented in this book it does not seem appropriate to develop formalism and to give formal proofs of theorems. It seems better to develop the formalism further, with the aim of developing more general methods.

12.1 Contents of the model

The model contains not only the Dirac δ–function, it even contains two different objects $\delta(x)$, and $\eta(x)$, which both have δ–like properties.
The first and the third of the properties (2.1–4) postulated by Dirac are shared by both δ and η. Both are the derivative of a step-function, $H(x)$ and $H'(x)$ respectively. Both are the Fourier transform of a 'constant' function. Only small changes in the interpretation are necessary. Both functions measure a (different) aspect of the value at $x = 0$ of another generalised function.
The second property of the δ–function (2.2)

$$\boldsymbol{\mathcal{X}}\,\delta(x) = x \cdot \delta(x) = \mathit{0}(x), \tag{12.1}$$

is satisfied by $\delta(x)$, but not by $\eta(x)$.
The fourth property (2.4) has not been dealt with yet. It will be shown in Ch. 19 that it is satisfied by $\eta(x)$, but not by $\delta(x)$. It will be necessary to redefine the limit concept for this purpose.

The model does not contain an element which satisfies all of Dirac's requirements. Conversely for all of Dirac's requirements there is an element which satisfies it. These elements are different for different requirements. In a wider sense the model is in agreement with the spirit in which Dirac proposed the δ–function. It satisfies an appropriate generalization of Dirac's requirements.

12.2 The scalar product

The scalar product $\langle\, ,\, \rangle : \mathbf{GF}_s \times \mathbf{GF}_s \to \mathbf{GF}_s$ is obviously linear by construction. The symmetry must be verified.

Property 12.1 The scalar product satisfies

$$\langle f(x), g(x) \rangle = \langle g(x), f(x) \rangle^*. \tag{12.2}$$

Verification: For real functions this is obviously true since the product is commutative by construction, so we have for real $f(x), g(x)$

$$\begin{aligned}\langle f(x), g(x) \rangle &= \langle I(x), f(x) \bullet g(x) \rangle = \\ &= \langle I(x), g(x) \bullet f(x) \rangle = \langle g(x), f(x) \rangle.\end{aligned} \tag{12.3}$$

For arbitrary $f(x), g(x)$ we obtain

$$\langle f, g \rangle = \langle I, f^* \bullet g \rangle = \langle I, g \bullet f^* \rangle = \langle g^*, f^* \rangle, \tag{12.4}$$

so it remain to verify that the * can be taken out of the scalar product,

$$\langle f(x), g(x) \rangle = \langle f(x)^*, g(x)^* \rangle^*, \tag{12.5}$$

It is sufficient to show this in the special case

$$\langle I, f^* \rangle = \langle I, f \rangle^*. \tag{12.6}$$

Since all scalar products have been reduced by the definition of the scalar product to the scalar product $\langle I, \delta \rangle = 1$, it is sufficient to have $\delta^* = \delta$.

From the complex conjugation of the powers $(x^\lambda)^* = x^{\lambda^*}$ one is led to $(x'^\lambda)^* = x'^{\lambda^*}$ and $\left(\eta^{(\alpha,q)}(x)\right)^* = \eta^{(\alpha^*,q)}(x)$, so $\eta = (\eta)^*$ is real for real α, in particular for $\alpha = 0$. By the postulate of minimal completion $\delta(x)$ is also taken to be real. (It would be possible to redefine $\delta := e^{i\varphi}\delta$, but this leads only to avoidable complications.) The complex conjugates can be taken out of the scalar products by definition of the complex conjugation. □

12.3 Integration on the preliminary class

Remark 12.1 This section is superfluous since the needed results have been verified already in Ch. 4. It is included to show that the conditions imposed there can easily be weakened and that it is not necessary to insert an explicit partition in order to define the integral.

The preliminary integral, defined in Sec. 4.3, on a restricted preliminary class $\mathbf{PC}_\lambda$ agrees with Hadamard's definition [Had] of the partie finie. It is always well-defined and finite.

The restrictions imposed upon the preliminary class are much stronger than necessary to allow the definition of the integral. The finiteness conditions are already implied by the analyticity requirements and the required asymptotic form are far to restrictive for this purpose of integration. It is sufficient to impose on $f(x) \in \mathbf{PC}$ the condition

$$\forall x_0 \in \mathbb{R}\, \exists \lambda_{x_0} \in \mathbb{R}\, \exists a_{x_0+} \in \mathbb{R}_+ : \tilde{g}(\lambda; x_0+) := \int_{x_0}^{x_0+b} dx\, (x-x_0)^\lambda f(x) < \infty, \tag{12.7}$$

for $x_0 < b < a_{x_0+}$ and $\operatorname{Re}\lambda > \lambda_{x_0}$. A similar condition is imposed to the left of every point. By standard arguments [Tich] it is known that the function $\tilde{g}(\lambda; x_0+)$ is an analytic function of λ in the half-plane $\operatorname{Re}\lambda > \lambda_{x_0}$. Again the existence of a (for simplicity single-valued) analytic continuation to the whole λ–plane is required for all points $x_0 \in \mathbb{R}$.

Remark 12.2 For the definition of the integral continuation of $\tilde{g}(\lambda; x_0)$ to an environment of $\lambda = 0$ is already sufficient, but the continuation to the whole λ–plane is needed for a satisfactory theory of generalised functions.

Associated with every point $x_0 \in \mathbb{R}$ there is now an open interval $x_0 - a_{x_0-} < x < x_0 + a_{x_0+}$ with the property that $f(x) \in \mathbf{PC}$ has an integral on all subintervals $(x_0 - b_, x_0 + b_+)$ defined by

$$\int_{x_0-b_-}^{x_0+b+} dx\, f(x) := \tag{12.8}$$

$$:= \operatorname*{Res}_{\lambda=0} \lambda^{-1} \left(\int_{x_0-b_-}^{x_0} dx\, (x_0-x)^\lambda f(x) + \int_{x_0}^{x_0+b_+} dx\, (x-x_0)^\lambda f(x) \right),$$

$\forall\, b_- < a_{x_0}$, and $\forall b_+ < a_{x_0+}$. These open intervals around every point $x_0 \in \mathbb{R}$ obviously cover any compact subset $[a, b] \subset \mathbb{R}$. By the Heine-Borel theorem every open cover of a compact set $\subset \mathbb{R}$ has a finite subcover. Associated with each open set in the subcover is a point $x_j \in \mathbb{R}$. These points can be arranged in order as

$$a \leq x_0 < x_1 < \cdots\cdots x_j \leq b, \tag{12.9}$$

The integral over a compact subset $[a, b] \in \mathbb{R}$ is now defined by choosing a finite partition

$$a \leq x_0 < \alpha_0 < x_1 < a_1 < \cdots\cdots < a_{j-1} < x_j \leq b, \tag{12.10}$$

with each partition point a_j in the intersection of the open environments of x_j and x_{j+1}. The integral is now defined as the finite sum of the subintegrals. It is obviously finite and it does not depend on the choice of the partition. If we also require analytic integrability on the intervals $(a+, \infty)$, and $(-\infty, a-)$ with $a_-, a_+ \in \mathbb{R}_+$, the integral is defined from $-\infty$ to ∞ by

$$\int_{-\infty}^{\infty} dx\, f(x) := \int_{-\infty}^{a_-} + \int_{a_-}^{a_+} + \int_{a_+}^{\infty} . \tag{12.11}$$

It has been shown that only requirements on analyticity and analytic continuation are needed. The finiteness and the asymptotic expansions then follow. The simple model is obtained by imposing the restriction to meromorphic functions.

The integral over finite intervals (in the sense of generalised functions) is redefined in Ch. 14 in terms of scalar products. It will be shown that this definition is a generalisation of the partie finie integral which served as starting point. The contributions from isolated singular points appear in addition to the partie finie integral in a way which generalises Dirac's requirement.

12.4 Operator algebra

The general pattern of the verification of an algebraic property of the operators is similar in all cases. First the validity of the property in a preliminary sense on the preliminary class is established. Then the unrestricted validity of the property on $\mathbf{PC}'_\lambda$ is deduced using analytic continuation and evaluation of residues. This must be extended to the completion $\overline{\mathbf{PC}}'_\lambda$ by explicit computation. Finally the property is established on $\overline{\mathbf{PC}}_\lambda$ by transfer, using mappings such as $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}^{-1}$. It then holds for $\mathbf{GF}_s$ as a whole by linearity. Only some typical cases are written out fully.

It will be shown first that the canonical commutation relation (2.12) for the operators $\mathcal{X}$ and $\mathcal{D}$ holds.

Property 12.2

$$\big[\mathcal{D}\,,\mathcal{X}\big]\, f(x) := \big(\mathcal{D}\mathcal{X} - \mathcal{X}\mathcal{D}\big)\, f(x) = \mathcal{I}\, f(x) = f(x). \tag{12.12}$$

$\forall\, f(x) \in \mathbf{GF}_s$

Verification: The commutation relation (12.12) holds for the preliminary operators acting on the powers

$$\big[\mathcal{D}_{\text{pre}}\,, \mathcal{X}_{\text{pre}}\big] x^\lambda H(x) = x^\lambda H(x), \tag{12.13}$$

for all values of λ. By taking residues in accordance with the definition one sees that it holds for the elements $|x'|^\lambda \log^q|x| \operatorname{sgn}^m(x) \in \mathbf{PC}'_\lambda$ This implies the validity for the $\theta^{(p)}$–functions since these functions behave as the corresponding powers under $\mathcal{X}$ and $\mathcal{D}$. The property also holds for the η–functions $\eta^{(\alpha,q)}(x) \operatorname{sgn}^m(x)$ and $\not{\eta}_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x)$. The commutation relation is now verified for all of $\overline{\mathbf{PC}}'_\lambda$, since $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ is $\mathbf{C}^\infty$ between singularities. The property holds on $\overline{\mathbf{PC}}_\lambda$ by transfer,

$$\begin{aligned}\mathcal{D}\mathcal{X}\, f(x) &= \mathcal{M}^{-1}\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\mathcal{X}\mathcal{M}\, f(x) = \mathcal{M}^{-1}\mathcal{D}P_{\mathcal{M}}\mathcal{X} = \\ &= \mathcal{M}^{-1}\mathcal{D}\mathcal{X}\mathcal{M}\, f(x) - \mathcal{M}^{-1}\mathcal{D}P_{\not\mathcal{M}}\mathcal{X}\mathcal{M}\, f(x),\end{aligned} \tag{12.14}$$

and the second term is zero since $\mathcal{M}^{-1}$ commutes with $\mathcal{D}$ an $\mathcal{X}$,

$$[\mathcal{M}^{-1},\mathcal{D}] = \mathcal{M}^{-1}\mathcal{D} - \mathcal{D}\mathcal{M}^{-1} = \mathcal{M}^{-1}\mathcal{D} - \mathcal{M}^{-1}\mathcal{D}\mathcal{M}\mathcal{M}^{-1} = 0, \tag{12.15}$$

and by definition

$$\mathcal{M}^{-1} = \mathcal{M}^{-1}P_{\mathcal{M}}, \qquad \text{so} \qquad \mathcal{M}^{-1}P_{\not\mathcal{M}} = 0. \tag{12.16}$$

Therefore

$$\mathcal{D}\mathcal{X} - \mathcal{X}\mathcal{D} = \mathcal{M}^{-1}(\mathcal{D}\mathcal{X} - \mathcal{X}\mathcal{D})\mathcal{M} = \mathcal{M}^{-1}\mathcal{I}\mathcal{M} = \mathcal{I}, \tag{12.17}$$

By linearity the commutation relation holds on $\mathbf{GF}_s$ as a whole. □

The demonstration of the validity of the properties of the Fourier operator is slightly more complicated. We have the algebraic properties

Property 12.3

$$\mathcal{F}\mathcal{F} = 2\pi\mathcal{P}, \tag{12.18}$$

and in the same way

$$\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = 2\pi\mathcal{I}, \tag{12.19}$$

and consequently

$$\mathcal{F}^{-1} = (2\pi)^{-1}\mathcal{F}^*. \tag{12.20}$$

Verification: The property holds as a theorem in $\mathcal{L}_2$. Therefore it holds for the function $x^\lambda H(x)H(a-x)$ for $\operatorname{Re}\lambda > -\frac{1}{2}$, and by analytic continuation for all values of λ. It also holds at infinity for the functions $x^\lambda\, e^{ikx}H(x-a)$, with $\operatorname{Re}\lambda < -\frac{1}{2}$. The property holds for the powers and η–functions in $\mathbf{PC}'_\lambda$ by taking suitable residues, for example

$$\begin{aligned}\mathcal{F}\mathcal{F}\,\eta_s^{(\alpha,q)}\operatorname{sgn}^m(x) &= \mathcal{F}\Big(\operatorname*{Res}_{\lambda=-\alpha-1}\tfrac{1}{q!}(\lambda+\alpha+1)^q\,\mathcal{F}|x|^\lambda\operatorname{sgn}^m(x)\Big) = \\ &= \operatorname*{Res}_{\lambda=-\alpha-1}\tfrac{1}{q!}(\lambda+\alpha+1)^q\,2\pi\mathcal{P}|x|^\lambda\operatorname{sgn}^m(x) = \\ &= 2\pi\mathcal{P}\,\eta_s^{(\alpha,q)}\operatorname{sgn}^m(x).\end{aligned} \tag{12.21}$$

The property is extended to $\overline{\mathbf{PC}}'_\lambda$ by computation of the action of $\mathcal{F}\mathcal{F}$ on the θ–functions.

$$\mathcal{F}\mathcal{F}\,\theta^{(p)}(x) = \mathcal{F}4i^p\sum_{j=0}^{\infty} j!\,c_{j+1}(p,p)\,\not\eta^{(p,j)}(x), = (-)^p\theta^{(p)}(x), \tag{12.22}$$

as one sees by explicit computation using the completion formula (C.27) of the c_j–coefficients. The property holds for the ordinary functions in $\overline{\mathbf{PC}}'_\lambda$ by linearity, since a function $\in \overline{\mathbf{PC}}'_\lambda$ can be converted into an $\mathcal{L}_2$ function by a finite number of subtractions of terms for which the property has been shown to hold. Finally the property can be transferred to $\overline{\mathbf{PC}}_\lambda$ by transfer with the mapping $\mathcal{M}$.

$$\mathcal{F}\mathcal{F} = \mathcal{M}^{-1}\mathcal{F}\mathcal{M}\mathcal{M}^{-1}\mathcal{F}\mathcal{M} = 2\pi\mathcal{P} + \mathcal{M}^{-1}\mathcal{F}P_{\mathcal{M}}\mathcal{F}\mathcal{M}, \qquad (12.23)$$

and the last term in (12.23) equals zero since $\mathcal{F}$ commutes with $P_{\mathcal{M}}$.
The related property with $\mathcal{F}$ replaced by $\mathcal{F}^*$ follows in the same way. In the next section Parseval's equality will be verified. Then it will be possible to replace the complex conjugate $\mathcal{F}^*$ by the adjoint $\mathcal{F}^\dagger$. □

Example 12.1 By explicit computation we find

$$\begin{aligned}\mathcal{F}\mathcal{F}|x|^\lambda \operatorname{sgn}^m(x) &= \mathcal{F}\big(-2i^m\,\Gamma(\lambda+1)\sin\tfrac{\pi}{2}(\lambda+m)\,|x|^{-\lambda-1}\operatorname{sgn}^m(x)\big) = \\ &= 4(-)^m\Gamma(\lambda+1)\sin\tfrac{\pi}{2}(\lambda+m)\,\Gamma(-\lambda)\sin\tfrac{\pi}{2}(-\lambda-1+m)\,|x|^\lambda\operatorname{sgn}^m(x) = \\ &= 2\pi(-)^m|x|^\lambda\operatorname{sgn}^m(x) = 2\pi\mathcal{P}|x|^\lambda\operatorname{sgn}^m(x), \qquad (12.24)\end{aligned}$$

by the completion formula (C.7)' of the Γ–function.
Next there is the unitary equivalence (2.22–23) of the operators $\mathcal{X}$ and $i\mathcal{D}$.

Property 12.4 Unitary equivalence of $\mathcal{X}$ and $i\mathcal{D}$

$$\mathcal{X} = i\mathcal{F}^{-1}\mathcal{D}\mathcal{F} = -i\mathcal{F}\mathcal{D}\mathcal{F}^{-1}, \qquad (12.25)$$

and

$$\mathcal{D} = i\mathcal{F}^{-1}\mathcal{X}\mathcal{F} = -i\mathcal{F}\mathcal{X}\mathcal{F}^{-1}, \qquad (12.26)$$

Verification: The verification proceeds along the same lines as the previous case. Again the property holds as a theorem in $\mathcal{L}_2 \cap \mathbf{PC}_\lambda$, it holds for the powers by analytic continuation, it holds for the primed powers and η–functions by taking residues. It is not necessary to verify the property for the θ–functions, since the operators $\mathcal{X}$ and $\mathcal{D}$ were defined on the θ–functions by (12.26) and (12.25). It again holds for the ordinary functions $\in \overline{\mathbf{PC}}'_\lambda$ by subtraction. The transfer to $\overline{\mathbf{PC}}_\lambda$ is also possible, since

$$\mathcal{M}^{-1}\mathcal{F}^{-1}\mathcal{M}\mathcal{M}^{-1}\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\mathcal{F} = \mathcal{M}^{-1}\mathcal{F}\mathcal{D}\mathcal{F}\mathcal{M}, \qquad (12.27)$$

and both $\mathcal{D}$ and $\mathcal{F}^{-1}$ commute with $P_{\mathcal{M}}$. □

As an immediate consequence we have

Property 12.5 The transfer map $\mathcal{M}_{\mathcal{D}}$ is equivalent to $\mathcal{M}_{\mathcal{X}} = \mathcal{M}$ for the definition of the operators.

Verification: It has to be shown that

$$\mathcal{M}_{\mathcal{D}}^{-1}\mathcal{D}\mathcal{M}_{\mathcal{D}} = \mathcal{M}_{\mathcal{X}}^{-1}\mathcal{D}\mathcal{M}_{\mathcal{X}}, \tag{12.28}$$

$\forall f(x) \in \overline{\mathbf{PC}}_\lambda$. Direct computation gives

$$\begin{aligned}\mathcal{M}_{\mathcal{D}}^{-1}\mathcal{D}\mathcal{M}_{\mathcal{D}} = \mathcal{F}^{-1}\mathcal{M}_{\mathcal{X}}^{-1}\mathcal{F}\mathcal{D}\mathcal{F}^{-1}\mathcal{M}_{\mathcal{X}}\mathcal{F} =\\ -i\mathcal{F}^{-1}\mathcal{X}\mathcal{F} = \mathcal{D} = \mathcal{M}_{\mathcal{X}}^{-1}\mathcal{D}\mathcal{M}_{\mathcal{X}},\end{aligned} \tag{12.29}$$

by application of the previous result. The property also holds for all polynomials in the operators $\mathcal{X}$ and $\mathcal{D}$. □

12.5 Operators and products

The properties of the operator $\mathcal{X}$ have been found already in Chs. 8 and 10, so they are only summarized.

Property 12.6 The operator $\mathcal{X}$ is multiplicative in $\overline{\mathbf{PC}}'_\lambda$

$$\mathcal{X}\big(f'(x)\bullet g'(x)\big) = \big(\mathcal{X}\, f'(x)\big)\bullet g'(x) = f'(x)\bullet\big(\mathcal{X}\, g'(x)\big), \tag{12.30}$$

□

Property 12.7 The operator $\mathcal{X}$ is not multiplicative in $\overline{\mathbf{PC}}_\lambda$. A sufficient condition for the validity of the multiplicative rule is the absence of terms in the product which are homogeneous of degree $-p-1$ with parity $(-)^p$. □

Property 12.8 The operator $\mathcal{X}$ is also multiplicative in mixed products $\overline{\mathbf{PC}}'_\lambda \times \overline{\mathbf{PC}}_\lambda$. □

The next property to be verified is Leibniz's rule for the differentiation of the product.

Property 12.9 Leibniz's rule

$$\mathcal{D}\big(f(x)\bullet g(x)\big) = \big(\mathcal{D}\, f(x)\big)\bullet g(x) + f(x)\bullet\big(\mathcal{D}\, g(x)\big). \tag{12.31}$$

holds for the differentiation of a product.

Verification: Between singular points the elements of $\mathbf{GF}_s$ are $\mathbf{C}^\infty$, so Leibniz's rule holds by standard arguments. At the singular points in $\overline{\mathbf{PC}}'_\lambda$ one verifies Leibniz's rule by direct computation for the products of the powers and the η–functions. By subtraction a function remains for which

Leibniz's rule holds in a standard sense. Also one verifies Leibniz's for the θ–functions, since these behave as the integral powers. Then Leibniz's rule also holds on $\overline{\mathbf{PC}}_\lambda$ by transfer, since the transfer map $\mathcal{M}_\mathcal{D}$ has been constructed in such a way that it commutes with differentiation.

$$\begin{aligned}\mathcal{D}(f \bullet g) &= \mathcal{D}\mathcal{M}_\mathcal{D}^{-1}(\mathcal{M}_\mathcal{D} f \bullet \mathcal{M}_\mathcal{D} g) = \mathcal{M}_\mathcal{D}^{-1}(\mathcal{D}(\mathcal{M}_\mathcal{D} f \bullet \mathcal{M}_\mathcal{D} g)) = \\ &= \mathcal{M}_\mathcal{D}^{-1}(\mathcal{D}\mathcal{M}_\mathcal{D} f \bullet g + f \bullet \mathcal{D}\mathcal{M}_\mathcal{D} g) = \\ &= \mathcal{D} f \bullet g + f \bullet \mathcal{D} g + \mathcal{M}_\mathcal{D}^{-1}([\mathcal{D}, \mathcal{M}_\mathcal{D}] f \bullet g + f \bullet [\mathcal{D}, \mathcal{M}_\mathcal{D}] g) \\ &= \mathcal{D} f \bullet g + f \bullet \mathcal{D} g. \end{aligned} \tag{12.32}$$

The last line follows from (9.21)so the commutator equals $\mathcal{D}\mathcal{M}_\mathcal{D} P_I$. Its action on any generalised function is a generalised function at infinity, which is annihilated by the inverse map. □

12.6 Selfadjoint properties

In this section the existence of adjoint operators is investigated. The conditions for lack of selfadjointness of the operators $\mathcal{X}$ and $i\mathcal{D}$ is established. For the operators $i\mathcal{D}$ and $\mathcal{X}$ it was already shown in Ch. 6 that these operators are selfadjoint in this subspace.

Property 12.10 The operator $\mathcal{X}$ is selfadjoint in $\overline{\mathbf{PC}}'_\lambda$.

Verification: This follows immediately from the validity of the multiplicative rule (12.30)

$$\mathcal{X}(f'(x) \bullet g'(x)) = \mathcal{X} f'(x) \bullet g'(x) = f'(x) \bullet \mathcal{X} g'(x), \tag{12.33}$$

valid $\forall f'(x), g'(x) \in \overline{\mathbf{PC}}'_\lambda$. □

Property 12.11 The operator $i\mathcal{D}$ is selfadjoint in $\overline{\mathbf{PC}}'_\lambda$: $\forall f'(x)$, $g'(x) \in \overline{\mathbf{PC}}'_\lambda$

$$\langle i\mathcal{D} f'(x), g'(x) \rangle = \langle f'(x), i\mathcal{D} g'(x) \rangle, \tag{12.34}$$

Verification: In $\overline{\mathbf{PC}}'_\lambda$ there are no stock-terms at infinity,

$$\langle I(x), \mathcal{D} f'(x) \rangle = \langle \delta(x), \mathcal{F}\mathcal{D} f'(x) \rangle = i \langle \delta(x), \mathcal{X}\mathcal{F} f'(x) \rangle = 0, \tag{12.35}$$

$\forall f(x)$ in $\overline{\mathbf{PC}}'_\lambda$, since $\overline{\mathbf{PC}}'_\lambda$ does not contain an element such that $\mathcal{X} \bullet f'(x) = \theta(x)$. It follows from Leibniz's rule that the adjoint $\mathcal{D}^\dagger = -\mathcal{D}$ exists, and that the operator $i\mathcal{D}$ is selfadjoint in $\overline{\mathbf{PC}}'_\lambda$. Equivalently one can Fourier transform the previous property. □

On the other hand it is known that the operators $\boldsymbol{\mathcal{X}}$ and $i\boldsymbol{\mathcal{D}}$ do not possess an adjoint in $\overline{\mathbf{PC}}_\lambda$. It remains to characterize the exceptions.

Property 12.12 The operator $\boldsymbol{\mathcal{X}}$ is selfadjoint in $\overline{\mathbf{PC}}_\lambda$ iff

$$\langle \boldsymbol{\mathcal{X}} f(x), g(x) \rangle = \langle f(x), \boldsymbol{\mathcal{X}} g(x) \rangle \Longleftrightarrow \boldsymbol{\mathcal{X}}[f(x) \circ\!\bullet\, g(x)] = \mathit{0}(x), \quad (12.36)$$

at $x = 0$. It is sufficient (but not necessary) that

$$f(x) \bullet g(x) \perp \delta^{(1)}(x) \quad \text{and} \quad f(x) \bullet g(x) \perp |x|^{-2} \log^q|x| \operatorname{sgn}(x), \quad (12.37)$$

so the product should not be homogeneous of degree -2 with negative parity. Equivalently the product of their Fourier transforms should be $\not\sim x \log^q|x|$ at infinity.

Verification: See Sec. 10.3. □

Property 12.13 The necessary and sufficient condition for the validity of

$$\langle \boldsymbol{\mathcal{D}} f(x), g(x) \rangle = -\langle f(x), \boldsymbol{\mathcal{D}} g(x) \rangle, \quad (12.38)$$

in $\overline{\mathbf{PC}}_\lambda$ is

$$\langle \not{\phi}(x), f(x) \bullet g(x) \rangle = 0, \quad (12.39)$$

Verification: The stock-term equals

$$\begin{aligned} \langle I(x), \boldsymbol{\mathcal{D}}\big(f(x)\, f \bullet g(x)\big) \rangle &= \langle \delta(x), \boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{D}}\big(f(x) \bullet g(x)\big) \rangle = \\ &= i\, \langle \delta(x), \boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{F}}\big(f(x) \bullet g(x)\big) \rangle. \end{aligned} \quad (12.40)$$

It is non-zero if and only if the expansion of $f(x) \bullet g(x)$ at infinity contains terms of the form $\operatorname{sgn}(x)$. Since $\overline{\mathbf{PC}}'_\lambda$ does not contribute it is sufficient to exclude $\not{f}(x)$. It is not necessary to exclude $\not{f}^{(0,q)}(x)$ with $q > 0$. One easily verifies from the Fourier and product properties that

$$\langle \delta(x), i\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{F}} \not{f}^{(0,q)}(x) \rangle = 2\,\delta_{q,0} = \langle I(x), \boldsymbol{\mathcal{D}} \operatorname{sgn}(x) \rangle. \quad (12.41)$$

This special behaviour of $\boldsymbol{\mathcal{D}}$ at infinity is a fortunate result of the standardization (9.101). It is indeed a good reason to prefer this standardization. (See also Rem. 12.4 in the next section however). □

It is *not* possible to exclude a small subspace for $f(x)$ and $g(x)$ separately. Instead it is necessary to exclude a subspace of $\overline{\mathbf{PC}}_\lambda \times \overline{\mathbf{PC}}_\lambda$. Given $f(x)$, there is given a descending sequence of λ values which occur in its asymptotic expansion. Correspondingly there is an ascending sequence of forbidden $-\lambda-1$ values for the asymptotic expansion of $g(x)$. Since the asymptotic expansion of $g(x)$ is also required to be descending, the number of non-zero terms which actually appears is finite.

Remark 12.3 Translations, which will be defined for generalised functions in Ch. 15 change the asymptotic behaviour at infinity. If it is desirable that $i\mathcal{D}$ should also be selfadjoint for the translated functions as well it is necessary to exclude behaviour as $x^p\,\mathrm{sgn}(x)$ at infinity.
Remark 12.4 Likewise scale transformations, to be defined in Ch. 16 transform $\log^q|x|\,\mathrm{sgn}(x)$ into $\big(\log(a) + \log|x|\big)^q\,\mathrm{sgn}(x)$, so if these transformations are allowed behaviour as $x^p \log^q|x|\,\mathrm{sgn}(x)$ at infinity should be excluded.

The fundamental theorem of the calculus Prop. 14.2 is closely related to the selfadjointness of the operator $\mathcal{D}$. It will be shown in Sec. 14.3 that the fundamental theorem holds for generalised functions in an appropriate sense, despite the lack of complete selfadjointness of the differential operator.

12.7 Parseval's equality

Next the existence of an adjoint of the Fourier operator must be demonstrated. This amounts to the verification of Parseval's equality.

Property 12.14 Parseval's equality

$$\big\langle \mathcal{F} f(x)\,,\mathcal{F} g(x)\big\rangle = 2\pi\,\big\langle f(x)\,,g(x)\big\rangle, \tag{12.42}$$

$\forall\, f(x), g(x) \in \mathbf{GF}_s$.

Verification: For zero scalar products the result is trivial, it is only necessary to verify the non-zero cases. For the powers $|x'|^\alpha \log^q|x|\,\mathrm{sgn}^m(x) \in \overline{\mathbf{PC}}'_\lambda$ the property holds trivially. Only $(-)^j\eta_s^{(\alpha,q)}(x) - \not\eta_s^{(\alpha,q)}(x)$ occurs as a linear combination of η–functions in the Fourier transforms. All scalar products are zero. For the $\theta^{(p)}(x)$–functions the scalar products have been defined by invoking Parseval, so Parseval holds by construction.

In order to extend this to $\overline{\mathbf{PC}}_\lambda$ the elements $\theta^{(\alpha,q)}(x)$ must be added. Adding one θ–function still gives a zero result. For two θ–functions one obtains a non-zero scalar product only for $\beta = -\alpha - 1$, $m = n$, as one sees from the product property (8.65). (For convenience $\alpha^* = \alpha$ is taken. The stars can easily be restored if desired)

$$\big\langle\, \theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,,\theta_s^{(-\alpha-1,r)}(x)\,\mathrm{sgn}^m(x)\,\big\rangle = d_{rq}(\alpha,m) + d_{qr}(-\alpha-1,n), \tag{12.43}$$

while comparison with the Fourier transform of $\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$

$$\mathcal{F}\,\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = -2i^m q! \sum_{j=0}^{q+1} \frac{(-)^{q+j}}{j!}\, c_{q-j}(\alpha,m)\,\not\theta_s^{(-\alpha-1,j)}(x)\,\mathrm{sgn}^m(x) +$$

$$+ 4i^m(-)^q q! \sum_{j=0}^{\infty} j!\, c_{q+j+1}(\alpha,m)\,\not\eta_s^{(\alpha,j)}(x)\,\mathrm{sgn}^m(x), \tag{12.44}$$

yields the same answer. The c–coefficients in the cross-terms combine to the correct d–coefficient

$$\begin{aligned}
&\langle \mathcal{F}\,\theta^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\,,\mathcal{F}\,\theta^{(-\alpha-1,r)}(x)\,\mathrm{sgn}^m(x)\,\rangle = \\
&= 4q!\,r!\sum_{j=-1}^{q+1}(-)^j c_{q-j}(\alpha,m)\,c_{q+r-j+1}(-\alpha-1.m)+(q\to r,\alpha\to-\alpha-1) \\
&= 2\pi\big(d_{rq}(\alpha,m)+d_{qr}(-\alpha-1,n)\big).
\end{aligned} \tag{12.45}$$

The contribution from the δ–function at the origin in (12.43) equals the cross-products at infinity.

The same result is obtained for the functions $\not\theta^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$ at infinity. Adding the θ and $\not\theta$–functions to the powers in $\overline{\mathbf{PC}}'_\lambda$ gives the validity of Parseval's equality for the powers in $\overline{\mathbf{PC}}_\lambda$. This is not yet sufficient since products of powers may diverge both in the finite and at infinity. Therefore the validity is also established for the functions $e^{-a|x|}|x'|^\lambda\,\mathrm{sgn}^m(x)$. The logarithms can easily be added, but this yields no new insight, while it makes the formulæ more cumbersome. The scalar product follows from the definition of the Γ–function

$$\begin{aligned}
&\langle\, |x|^{\alpha^*}\,\mathrm{sgn}^m(x)\,,|x|^\beta\,e^{-a|x|}\mathrm{sgn}^n(x)\,\rangle = \\
&\quad = 2\,\delta^{\mathrm{mod}2}_{m,n}\,a^{-\alpha-\beta-1}\big(\Gamma^{[0]}(\alpha+\beta+1)-\Gamma^{[-1]}(\alpha+\beta+1)\log(a)\big)+ \\
&\quad -\tfrac{4}{\pi}\,\delta^{\mathrm{mod}2}_{m,n}\,a^{-\alpha-\beta-1}\,c_0(\alpha,m)\,c_1(-\alpha-1,n)\sum_{j=0}^{\infty}\tfrac{(-)^j}{j!}\,\delta_{-\alpha-\beta-1,j}\,,
\end{aligned} \tag{12.46}$$

The Fourier transform of the damped power has been calculated in Ch. 7. For the verification of Parseval's equality we need the integral

$$\int_0^\infty dx\,x^\lambda(ix+a)^\mu=(-i)^{\lambda+1}a^{\lambda+\mu+1}\frac{\Gamma(\lambda+1)\,\Gamma(-\lambda-\mu-1)}{\Gamma(-\mu)}, \tag{12.47}$$

since the integral is a standard representation [Erd1] of the Eulerian B–function. Carrying out the necessary substitutions one finds that Parseval's equality holds for the scalar product (12.46). Since Parseval's equality has already been verified for the θ–function (12.46) holds both in $\overline{\mathbf{PC}}'_\lambda$ and in $\overline{\mathbf{PC}}_\lambda$.

Finally Parseval's equality is now seen to hold for arbitrary generalised functions. by subtracting a finite number of terms of the asymptotic expansions and damped powers, such that an $\mathcal{L}_2$ function remains. By the restrictions imposed on $\mathbf{PC}_\lambda$ this is always possible. Parseval's equality has been shown to hold for the subtracted terms. For the remainder Parseval's equality holds in a classical sense. Combining these gives Parseval's equality for the generalised functions. □

The validity of Parseval's equality for the symmetrical scalar product is the key result of the symmetrical theory of generalised functions.

It is possible to convince oneself that the brief outlines of the verifications can be worked out to complete proofs of theorems. Fully writing this all out will result in a book by itself however. In the present state of the program it seems more appropriate to me to continue the development to see where it may lead. An outline is presented in Ch. 24.

CHAPTER 13

VALUES, LIMITS, AND THE SUPPORT

A definition of the value of a generalised function at a point follows naturally from the model. An attempt is made to define the concept of the support of generalised functions. The real number system is not suitable as a support for the generalised functions. In its place an ad hoc modification is made which lacks a proper foundation at present. Infinitesimal environments of points are used as if they were ordinary points. The values of generalised functions on the other hand are found as scalar products, which have values in the standard complex numbers. Limits of generalised functions are also defined by means of scalar products.

13.1 Values of generalised functions

It is possible to assign point values to generalised functions, but conversely the point values do not determine the generalised function fully.

Corresponding to the generalised function $f(x) \in \mathbf{GF}_s$ one defines the complex number valued functions $\mathbb{R} \to \mathbb{C}$ on (for the time being) the real axis $\forall x_0 \in \mathbb{R}$ by

$$f(x_0) := \langle \delta(x - x_0) \, , f(x) \rangle, \tag{13.1}$$

and also

$$f(x_0+) := \langle \eta_{\downarrow}(x - x_0) \, , f(x) \rangle, \tag{13.2}$$

$$f(x_0-) := \langle \eta_{\uparrow}(x - x_0) \, , f(x) \rangle, \tag{13.3}$$

which are called the value of the generalised function at the point $x = x_0$, and the limiting value on the positive and negative side of the same point. Similarly one can define the value of the p'th derivative at and around the point x_0 by

$$f^{(p)}(x_0) := p! \, \langle \delta^{(p)}(x - x_0) \, , f(x) \rangle, \tag{13.4}$$

$$f^{(p)}(x_0+) := p! \, \langle \eta_{\downarrow}^{(p)}(x - x_0) \, , f(x) \rangle, \tag{13.5}$$

$$f^{(p)}(x_0-) := p! \, \langle (-)^p \eta_{\uparrow}^{(p)}(x - x_0) \, , f(x) \rangle. \tag{13.6}$$

The terminology is justified, since at $\mathbf{C}^\infty$–points all three values coincide, and are equal to the standard value of the ordinary function corresponding to the generalised function. It is on occasion convenient to use the notation

$$f(x_0) = f(x)\Big|_{x=x_0} = f(x)\Big|_{x_0}, \tag{13.7}$$

in particular in connection with the evaluation of definite integrals.

According to these definitions the unprimed Heaviside step function has the values

$$H(x_0) = \begin{cases} 0 & x_0 \leq 0-\,, \\ \frac{1}{2} & x_0 = 0\,, \\ 1 & x_0 \geq 0+\,, \end{cases} \tag{13.8}$$

and correspondingly for $H(-x)$. The primed Heaviside function has the values

$$H'(x_0) = \begin{cases} 0 & x_0 \leq 0-\,, \\ 0 & x_0 = 0\,, \\ \frac{1}{2} & x_0 = 0+\,, \\ 1 & x_0 > 0+\,, \end{cases} \tag{13.9}$$

which are different from the values (13.8) of the unprimed step functions. Likewise the value assigned to an 'ordinary' function $f(x)$ at a point of discontinuity depends on its interpretation, either as a generalised function $f(x) \in \mathbf{PC}$, or as $f'(x) \in \mathbf{PC}'$.

The generalised function $\theta(x)$ cannot be described as the function which is one at $x = 0$ and zero everywhere else, since this fixes the generalised function only modulo the class of generalised functions of value zero everywhere. It has the values

$$\theta(x_0) = \begin{cases} 0 & x_0 < 0-\,, \text{ and } x_0 > 0+\,, \\ \frac{1}{2} & x_0 = 0-\,, \text{ and } x_0 = 0+\,, \\ 1 & x_0 = 0, \end{cases} \tag{13.10}$$

in the sense of generalised functions.
Correspondingly the generalised function $\theta_\downarrow(x)$ has the values

$$\theta_\downarrow(x_0) = \begin{cases} 0 & x_0 < 0, \\ \frac{1}{2} & x_0 = 0\,, \text{ and } x_0 = 0+\,, \\ 0 & x_0 > 0+\,, \end{cases} \tag{13.11}$$

since the value at $x = 0$ is divided between $\theta_\downarrow$ and $\theta_\uparrow$.

Remark 13.1 It is also possible to define the value by means of the left-sided or the right-sided product, or by means of an arbitrary linear combination. This leaves the values of the ordinary functions $\in \overline{\mathbf{PC}}_\lambda$ unchanged by (10.7), but it changes the values of $\theta_\downarrow(x)$ and $H'(x)$ at $x = 0+$.

Similarly the values near $x = \pm\infty$ are defined by

$$f(+\infty-) := \big\langle \not{\eta}_\uparrow(x)\,, f(x)\,\big\rangle, \tag{13.12}$$

for the value below $+\infty$ and

$$f(-\infty+) := \big\langle \not{\eta}_\downarrow(x)\,, f(x)\,\big\rangle, \tag{13.13}$$

for the value above $-\infty$. The model does not contain a $\not{\delta}$ ('delta-slash') function, so the values at the points $\pm\infty$ cannot be defined by means of

a scalar product. Instead the values at $\pm\infty$ will be defined in such a way that the fundamental theorem of the calculus Prop. 14.2 holds. This will be verified in the next chapter.
For the unprimed powers $\in \overline{\mathbf{PC}}_\lambda$ the values at $\pm\infty$ are

$$|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \Big|_{\pm\infty} := \pm^m \delta_{\alpha,0}\, \delta_{q,0}\,, \tag{13.14}$$

in agreement with the standardization (9.101) of the maps and products. If another standardization is chosen one has to modify the values at infinity or give up the fundamental theorem Prop. 14.2 of the calculus. The former seems preferable, since the values at infinity are conventional anyway.
For the primed powers one has to assign the values

$$\big(|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x)\big) \Big|_{\pm\infty} := 0, \tag{13.15}$$

in agreement with the corresponding behaviour of these functions at $x = 0$.
Oscillatory functions at $x = \pm\infty$ have the value zero by definition

$$e^{ikx}|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \Big|_{\pm\infty} = 0, \tag{13.16}$$

for $k \neq 0$, in agreement with their response to $\not{\eta}_\uparrow(x)$.

For the $\not{\theta}_\uparrow$ and $\not{\theta}_\downarrow$–functions there is a minor difference. The points $\pm\infty$ are considered as distinct points, in contrast with the point $x = 0$. Therefore these functions have the values

$$\not{\theta}_\uparrow(x_0) = \begin{cases} 0 & x_0 \leq +\infty-\,, \\ \frac{1}{2} & x_0 = +\infty-\,, \\ 1 & x_0 = +\infty, \end{cases} \tag{13.17}$$

As a consequence the unit function $I(x_0) = 1$ has the value one everywhere, but the function $I'(x) + \theta(x)$ has the values

$$I'(x) + \theta(x) = \begin{cases} 0 & x_0 = \pm\infty, \\ \frac{1}{2} & \pm\infty\mp\,, \\ 1 & -\infty+ < x < +\infty-\,, \end{cases} \tag{13.18}$$

in agreement with a naïve interpretation of these functions.

The same symbol is used for generalised function and its value. The distinction is usually clear from the context. It can also be made clear by distinguishing in the notation between 'fixed' and 'variable' variables, using for instance $f(x)$ for the generalised function and $f(x_0)$ for its value, in accordance with standard usage.

For convenience and for reference purposes the values of all commonly occurring piecewise constant functions are collected in the following table.

Table 13.1

Values of piecewise constant generalised functions

$f(x) = \cdots$	$-\infty$	$-\infty+$	-1	$0-$	0	$0+$	1	$+\infty-$	$+\infty$
$H(x)$	0	0	0	0	$\frac{1}{2}$	1	1	1	1
$H'(x)$	0	0	0	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0
$H(-x)$	1	1	1	1	$\frac{1}{2}$	0	0	0	0
$H'(-x)$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	0	0
$I(x)$	1	1	1	1	1	1	1	1	1
$I'(x)$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0
$\mathrm{sgn}(x)$	-1	-1	-1	-1	0	1	1	1	1
$\mathrm{sgn}'(x)$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0
$\theta_{\downarrow}(x)$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\theta_{\uparrow}(x)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
$\theta(x)$	0	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0
$\not\theta_{\uparrow}(x)$	0	0	0	0	0	0	0	$\frac{1}{2}$	1
$\not\theta_{\downarrow}(x)$	1	$\frac{1}{2}$	0	0	0	0	0	0	0

The values assigned to generalised functions are usually not very surprising. An exception may be the case of the logarithms, where application of the definition gives

$$\left(\log|x|\right)\Big|_{x=0} := \left\langle \log|x| \,, \delta(x) \right\rangle = \psi(1), \tag{13.19}$$

by (9.60) and more generally

$$\left(\log^q|x|\right)\Big|_{x=0} = \tfrac{2}{\pi} q!\; c_{q+1}(0,0), \tag{13.20}$$

by (9.59), with the c_j–coefficients defined by (6.48) as Laurent coefficients.

For the ordinary functions the values at any point is defined as the evaluation of its asymptotic expansion at that point.

The value at a point is a linear functional on the generalised functions. For values defined as scalar products this follows from the linearity of the scalar product, for the values at infinity it is clear by inspection.

When it is necessary to display the value explicitly it is convenient to introduce an evaluation operator $\mathcal{E} : \mathbf{GF}_s \to \mathbf{PC}_\lambda$, which converts a generalised function into its value. The function $\mathcal{E}\, f(x_0) \in \mathbf{PC}_\lambda : \mathbb{R} \to \mathbb{C}$ is defined $\forall x_0 \in \mathbb{R}$ by

$$\mathcal{E}\, f(x_0) := \left\langle\, \delta(x - x_0)\, ,\, f(x)\, \right\rangle, \tag{13.21}$$

which is easily seen to define an element of the preliminary class. It can therefore be considered as a generalised function $\in \overline{\mathbf{PC}}_\lambda$, since the ordinary functions $\in \overline{\mathbf{PC}}_\lambda$ have been identified with the corresponding generalised functions. For all $f(x) \in \mathbf{GF}_s$ the operator $\mathcal{E}' : \mathbf{GF}_s \to \mathbf{PC}'_\lambda$ is defined as

$$\mathcal{E}'\, f(x) := \mathcal{M} \left\langle\, \delta(y - x)\, ,\, f(y)\, \right\rangle, \tag{13.22}$$

which assigns the corresponding generalised function $\in \mathbf{PC}'_\lambda$ to $f(x)$. The difference appears only when other operators are applied.

Remark 13.2 The distinctions made above may appear overly subtle, but the framework lends itself naturally to make these distinctions.

Remark 13.3 In earlier work [K&L] a more general evaluation functional was used. This was arbitrary except for its agreement with the standard value of a continuous function. Nevertheless it can serve as basis for a product. The evaluation operator of this section follows from the product, which is fixed by the requirements of analyticity.

The evaluation operator $\mathcal{E}$ is a projection operator

$$\mathcal{E}\mathcal{E} = \mathcal{E}, \tag{13.23}$$

so it can be expected to destroy information. It is clear that information is indeed lost by passing from the generalised function to its value(s).

Example 13.1 The generalised function $\delta(x)$ has the value zero everywhere, since

$$\delta(x_0) := \left\langle\, \delta(x - x_0)\, ,\, \delta(x)\, \right\rangle = 0, \tag{13.24}$$

$\forall x_0 \in \mathbb{R}$. Also its limiting values from above and below are zero.

$$\delta(x_0+) := \left\langle\, \eta_{\downarrow}(x - x_0)\, ,\, \delta(x)\, \right\rangle = 0, \tag{13.25}$$

and

$$\delta(x_0-) := \left\langle\, \eta_{\uparrow}(x - x_0)\, ,\, \delta(x)\, \right\rangle = 0. \tag{13.26}$$

Likewise the generalised functions η and $\not\eta$ are everywhere zero. (Read this as: Have their value equal to zero everywhere).

This shows that a non-zero generalised function can have its value zero everywhere. This is not surprising and no disadvantage of the theory. Generalised functions are constructed for the purpose of overcoming the limitations of the standard function concept. It is therefore not surprising that the concept of a function as a mapping of the reals into itself fails to describe all the properties of the generalised functions.

A generalised function with value zero necessarily has a discrete support. Corresponding to a given set of values there is a simplest generalised function, which is obtained by requiring a zero scalar product with other generalised functions with point support. By the requirement of non-degeneracy of the scalar product this defines it uniquely.

The pointwise property cannot be carried over completely to the generalised functions. It is *not* true that the value of the product of two generalised functions equals the ordinary product of their values. The equation

$$\langle \delta(x) , f(x) \bullet g(x) \rangle = \langle \delta(x) , f(x) \rangle \cdot \langle \delta(x) , g(x) \rangle, \qquad (13.27)$$

WRONG!

does *not* hold in general, not even when the generalised value of the generalised functions is taken.

Example 13.2 Two obvious counter examples in which the product of the values at $x = 0$ is incorrect are obtained by taking $f(x) = g(x) = \mathrm{sgn}(x)$, or by taking $f(x) = x^{-1}$, $g(x) = x$. In both cases the left-hand side of (13.27) equals one, but the right-hand side is zero.

It is an ancient folklore that zero times infinity may be finite. In the theory of generalised functions one could say that even zero times zero may be finite. This is probably no help to the understanding, and it is safer to keep in mind that the value product property (13.27) does not hold for generalised functions.

13.2 Limiting values of generalised functions

At points where a generalised function has no worse than a jump discontinuity the value from above

$$f(x_0+) = \lim_{x \downarrow x_0} f(x), \qquad (13.28)$$

is equal to the limit in the standard sense. Therefore the limit in the sense of generalised functions is defined by

$$\operatorname*{Lim}_{x \downarrow 0} f(x) := \langle \eta_{\downarrow}(x) , f(x) \rangle = f(0+), \qquad (13.29)$$

where the notation 'Lim' is used to indicate the limit in the sense of generalised functions. A limit in the classical sense is always indicated by using 'lim'. Both limits agree when both exist. The generalised limit exists in many cases where the standard limit does not exist.

Example 13.3 The generalised limit

$$\operatorname*{Lim}_{x \downarrow 0} x^{-n} e^{-x} = (-)^n / n!, \qquad (13.30)$$

exists $\forall n \in \mathbb{N}$, the standard limit exists only for $n = 0$.

Conversely it is not obvious that the generalised limit is really a generalization of the standard limit concept. A proof that the generalised limit always exists when the standard limit exists depends on a resolution of the difficult problems of the existence of analytic continuation which cannot be resolved at present. In cases it is not necessary to distinguish between the the two limit concepts.
More generally the limits at other points and at infinity are defined by

$$\operatorname*{Lim}_{x\downarrow x_0} f(x) := \langle \eta_\downarrow(x-x_0)\,,f(x)\rangle = f(x_0+), \tag{13.31}$$

$$\operatorname*{Lim}_{x\uparrow x_0} f(x) := \langle \eta_\uparrow(x-x_0)\,,f(x)\rangle = f(x_0-), \tag{13.32}$$

$$\operatorname*{Lim}_{x\uparrow \infty} f(x) := \langle \not\eta_\uparrow\,,f(x)\rangle = f(+\infty-), \tag{13.33}$$

$$\operatorname*{Lim}_{x\downarrow -\infty} f(x) := \langle \not\eta_\downarrow(x)\,,f(x)\rangle = f(-\infty+). \tag{13.34}$$

By construction the generalised limit always exists as a well defined finite complex number.

Remark 13.4 One may notice that the value of a generalised function at $x = \infty$ defined in the previous section cannot always be obtained as a limit or a generalised limit.

Example 13.4 One has for the generalised function $I'(x)$

$$0 = I'(+\infty) \neq \operatorname*{Lim}_{x\uparrow\infty} I'(x) = I'(+\infty-) = \tfrac{1}{2}, \tag{13.35}$$

as defined in the previous section.
It is convenient to remember the (limiting) values of the powers

$$\operatorname*{Lim}_{x\downarrow 0} x^\alpha \log^q(x) = \operatorname*{Lim}_{x\uparrow\infty} x^\alpha \log^q(x) = \delta_{\alpha,0}\,\delta_{q,0}\,, \tag{13.36}$$

and in particular

$$\operatorname*{Lim}_{x\downarrow 0} x^\alpha = \operatorname*{Lim}_{x\uparrow\infty} x^\alpha = \begin{cases} 0 & \alpha \neq 0, \\ 1 & \alpha = 0, \end{cases} \tag{13.37}$$

$\forall \alpha \in \mathbb{C}$. This contrasts with the more complicated standard result

$$\lim_{x\downarrow 0} x^\alpha = \begin{cases} 0 & \operatorname{Re}\alpha > 0, \\ 1 & \alpha = 0, \\ ? & \operatorname{Re}\alpha \leq 0,\ \operatorname{Im}\alpha \neq 0, \\ \infty & \operatorname{Re}\alpha < 0,\ \operatorname{Im}\alpha = 0. \end{cases} \tag{13.38}$$

One easily gets used to the generalised limits, which greatly simplify the calculation of definite integrals. This will be demonstrated in the next chapter when the fundamental theorem of the calculus, Prop. 14.2, is available.

13.3 The support of the generalised functions

The theory of generalised functions leads naturally to the concept of infinitesimal surroundings of the points in the finite and to infinite environments at plus and minus infinity. The real number system as formalized in the nineteenth century does not have the room to accommodate this. Nonstandard analysis is at first sight more suitable for this purpose. It is not clear however if nonstandard analysis is the most suitable way of providing the generalised functions with a support in an intuitively attractive way. Therefore this matter is left open and the support of the generalised functions is defined in a heuristic way without proper foundation.

Disclaimer 13.1 The author does not accept any responsibility whatsoever for any interpretation that may placed on the contents of this section. It is not based on any foundations and has been included only for its possible heuristic value. It is hoped that the heuristic value will not be offset by causing too much discussion or confusion.

The support of the generalised functions consists of the real numbers $x_0 \in \mathbb{R}$, and their positive and negative infinitesimal environments. The positive infinitesimal environment of the point x_0 is written as x_0+ and treated as if it were an ordinary point $x_0 \in \mathbb{R}$. The 'points' $+\infty-$ and $-\infty+$ are introduced as an environment of plus and minus infinity.

The supports of the singular generalised functions are now fixed by definition, starting from the well known properties of the 'ordinary functions' $\in \mathbf{PC}_\lambda$. The definition is a generalization of the concept that the support should be the closure of the set of points where the function is nonzero. This notion is well defined for $\mathbf{C}^\infty$–functions. It can be generalised to $\mathbf{PC}_\lambda$ by putting

$$\operatorname{support} H(x) = \{0, +\infty\}, \tag{13.39}$$

and likewise for the Heaviside function on the negative side

$$\operatorname{support} H(-x) := \{-\infty, 0\}, \tag{13.40}$$

and likewise for the same functions including powers or logarithms.

The notion of the support can now be transferred to $\mathbf{PC}'_\lambda$. Since the $\eta_\downarrow$–function has the scalar products

$$\langle\, \eta_\downarrow(x)\, , H(x)\, \rangle = 1, \quad \text{and} \quad \langle\, \eta_\downarrow(x)\, , H(-x)\, \rangle = 0, \tag{13.41}$$

the support of $\eta_\downarrow$ cannot include the point $x = 0$. On the other hand the support does not include the interval $\{\epsilon, \infty\}$ for any finite $\epsilon > 0$. Therefore the 'point' $x = 0+$ is introduced in order to provide a support for $\eta_\downarrow(x)$,

$$\operatorname{support} \eta_\downarrow(x) := \{0+\}, \tag{13.42}$$

and by analogy

$$\text{support}\, H'(x) := \{0+, +\infty-\}. \tag{13.43}$$

The support space is obviously not the space of the real numbers. It is not even properly defined at present. Therefore it also lacks a topology. The standard distinction between open and closed intervals is therefore also undefined.

The question of topology of the support is also left open for the time being. It is not necessary to define the concepts open or closed for the supports of the generalised functions. The neutral symbols $\{\cdots\}$ have been used instead. In the next chapter open and closed intervals are defined in connection with the integral.
The points $+\infty-$ and $-\infty+$ are introduced as support for the $\not\delta$–functions,

$$\text{support}\, \not\delta_{\uparrow}(x) := \{+\infty-\}, \tag{13.44}$$

and correspondingly at $-\infty+$

$$\text{support}\, \not\delta_{\downarrow}(x) := \{-\infty+\}. \tag{13.45}$$

The definition of the support is now carried back to the completion $\overline{\mathbf{PC}}_\lambda$ of $\mathbf{PC}_\lambda$ by including the δ–functions. From the supports of the Heaviside functions and the scalar products

$$\langle\, H(x)\, ,\, \delta(x)\, \rangle = \tfrac{1}{2}, \quad \text{and} \quad \langle\, H'(x)\, ,\, \delta(x)\, \rangle = 0, \tag{13.46}$$

it is seen that the support in $\mathbb{R}$ of the $\delta(x)$–function is

$$\text{support}\, \delta(x) = \{0\}, \tag{13.47}$$

and more generally

$$\text{support}\, \delta^{(p)}(x - x_0) = \{x_0\}, \tag{13.48}$$

$\forall x_0 \in \mathbb{R}, \forall p \in \mathbb{N}$, in agreement with the corresponding support of the δ–distributions in distribution theory.
Finally the support of the θ–fuctions is found by imposing linearity,

$$\text{support}\, \theta_{\downarrow}(x) := \{0, 0+\}, \quad \text{and} \quad \text{support}\, \theta_{\uparrow}(x) := \{0-, 0\}, \tag{13.49}$$

as the intersection of the supports of the different Heaviside functions. Since there is only one point $x = 0$ the support of $\theta(x)$ becomes

$$\text{support}\, \theta(x) = \{0-, 0+\}, \tag{13.50}$$

and likewise at infinity

$$\text{support}\, \not\theta_{\downarrow}(x) = \{-\infty, -\infty+\}, \quad \text{and} \quad \text{support}\, \not\theta_{\uparrow}(x) = \{+\infty-, +\infty\}, \tag{13.51}$$

but the points at $\pm\infty$ are not identified with each other, so the support of $\not\theta(x)$ consists of two disjunct pieces.

Remark 13.5 It is advisable in this context not to omit the plus sign in $+\infty$ for clarity.

For the ordinary generalised functions the point $x_0 \in \mathbb{R}$ belongs by definition to the support of $f(x) \in \mathbf{GF}_s$ if a δ–function at the point gives a non-zero result, or if the point x_0 is a density point of the support in order to include trivial zeroes. The support at the singular points follows from the interpretation of the corresponding asymptotic expansions as belonging either to $\overline{\mathbf{PC}}_\lambda$ or $\overline{\mathbf{PC}}'_\lambda$.

The standard difficulties inherent in the definition of subsets of the reals do not arise here as yet. The limited content of the model considered in this book makes unnecessary to consider more than a finite union of intervals.

Remark 13.6 The elements of the support have been taken to be ordered in the natural way, with the point x_0+ greater than x_0 but smaller than any standard real number greater than x_0.

The definitions of the support of the generalised functions fit in with the definitions of the integration over finite intervals, which will be given in the next chapter.

It will be obvious that the acceptance of the concept of a support for the generalised functions requires a restructuring of the real number system. Nonstandard analysis may be suitable for this purpose, but its suitability in this respect has not yet been explored. Some further remarks on the use of nonstandard analysis may be found in Ch. 23.

The provisional solution adopted here has the standard continuum three times, in addition to four different 'points' at infinity. The standardization in this tract allows the identification of the points $+\infty$ and $-\infty$, since all values in these points agree.

CHAPTER 14

INTEGRATION

In this chapter integrals of generalised functions are defined as appropriate scalar products with piecewise constant functions. The inverse derivative, and therefore a primitive function, is defined for all generalised functions. The fundamental theorem of the calculus holds for generalised functions. The (generalised) value of an integral can be found by substituting the limits of integration into the primitive function, if this is done in accordance with the definitions of the values of generalised functions given in the previous chapter.

14.1 Integration between arbitrary limits

The integral in the sense of generalised functions is defined as a special case of the scalar product. In Sec. 4.4 the preliminary scalar product was defined in terms of the preliminary integral. Now that the definitive scalar product is available, it can be used for the definition of the integral. In particular the integral from $-\infty$ to ∞ is defined in the sense of generalised functions by

$$\int_{-\infty}^{\infty} dx\, f(x) := \langle\, I(x)\,,\, f(x)\,\rangle, \tag{14.1}$$

as the scalar product with the unit function.

In Ch. 13 the limiting values of generalised functions were defined as scalar products with an η–function, and the point values were defined as a scalar product with a δ–function. A generalised function is said to be piecewise constant when the ordinary function of $x_0 \in \mathbb{R}$

$$f(x_0) := \langle\, \delta(x - x_0)\,,\, f(x)\,\rangle$$

is a piecewise constant ordinary function of its argument $x_0 \in \mathbb{R}$.

Decompositions of the unit function into piecewise constant functions with the values one or zero, and possibly the value $\frac{1}{2}$ in boundary points, lead automatically to restricted integrals. In $\overline{\mathbf{PC}}_\lambda$ there is the decomposition

$$I(x) = H(-x) + H(x), \tag{14.2}$$

while in $\overline{\mathbf{PC}}'_\lambda$ there is the decomposition

$$I'(x) = H'(-x) + H'(x), \tag{14.3}$$

and consequently

$$I'(x) + \theta(x) = H'(-x) + \theta(x) + H'(x). \tag{14.4}$$

Combining these gives in $\mathbf{PC}^{\oplus}_{\lambda}$ the decomposition

$$I(x) = \not\theta_{\downarrow}(x) + H'(-x) + \theta_{\uparrow}(x) + \theta_{\downarrow}(x) + H'(x) + \not\theta_{\uparrow}(x), \tag{14.5}$$

which again illustrates the inadequacy of the real number system as a support for the generalised functions.

Remark 14.1 For the Lebesgue integral a decomposition involving $\theta(x)$ is pointless, since single points do not contribute to a Lebesgue integral.

Restricted integrals are now defined by suitable scalar products such a

$$\int_{-\infty+}^{0-} dx\ f(x) := \langle\, H'(-x)\,,\, f(x)\,\rangle, \tag{14.6}$$

and

$$\int_{0}^{0+} dx\ f(x) := \langle\, \theta_{\downarrow}(x)\,,\, f(x)\,\rangle, \tag{14.7}$$

and idem for other intervals, in self-evident notation. For the $\theta_{\downarrow}$–functions the left and right-sided scalar products are not necessarily equal. This makes it possible to define a slightly more general integral by

$$\int_{0}^{0+)} dx\ f(x) := \lessdot\, \theta_{\downarrow}(x)\,,\, f(x)\,\rangle, \tag{14.8}$$

and

$$\int_{0}^{0+]} dx\ f(x) := \langle\, \theta_{\downarrow}(x)\,,\, f(x)\,\gtrdot. \tag{14.9}$$

Of course it is convenient to choose the corresponding definitions for the primed Heaviside functions,

$$\int_{[0+}^{a} dx\ f(x) := \lessdot\, H'(x)\,,\, f(x)\,\rangle, \tag{14.10}$$

and

$$\int_{(0+}^{a} dx\ f(x) := \langle\, H'(x)\,,\, f(x)\,\gtrdot, \tag{14.11}$$

since this makes the integration intervals complementary if the parenthesis and square brackets are used according to the same rules as the standard notations for open and closed intervals. In order to show the consistency of this notation one must show that

$$\int_{0}^{a} dx = \int_{0}^{0+)} dx + \int_{[0+}^{a} dx = \int_{0}^{0+]} dx + \int_{(0+}^{a} dx\,. \tag{14.12}$$

This is the case since it follows from the the explicit product formula (9.68) that

$$\lessdot\, \mathrm{sgn}(x)\,,\, f(x)\, \rangle = \langle\, \mathrm{sgn}(x)\,,\, f(x)\, \gtrdot, \tag{14.13}$$

$\forall\, f(x) \in \mathbf{PC}_\lambda$, and therefore $\forall\, f(x) \in \mathbf{GF}_s$.

The same notation is applied at $= \pm\infty$, but by splitting the integration interval it is also possible to combine an open interval near $x = 0$ with a closed interval at infinity and vice versa.

Remark 14.2 Although the notation and terminology of open and closed intervals has been taken over from the corresponding standard usage no topology is implied. All integrals should be interpreted as convenient alternative notation for the corresponding scalar products, which have been defined in the previous chapters. Readers who do not think this notation convenient can restrict themselves to scalar product notation.

The integral with a 0+ at the upper limit is by the preceding definitions the symmetrized version corresponding with the symmetrized scalar product,

$$\int_0^{0+} dx = \tfrac{1}{2} \int_0^{0+)} dx + \tfrac{1}{2} \int_0^{0+]} dx\,, \tag{14.14}$$

with a $\theta_\downarrow(x)$–function, in agreement with the definitions of Ch. 8. This holds generally, no delimiter equals the average of a parenthesis and a square bracket.

Remark 14.3 The notations 0+) and 0+] indicate limit processes, in particular this notation indicates orders in which limits are to be taken. It does not seem convenient to interpret these as points in the support.

The notation for the limits of integration is collected in the following table.

Table 14.1

Scalar product with	of type	lower limit	upper limit
$\not\theta_{\downarrow}(x)$	left	$-\infty$	$-\infty+)$
$\not\theta_{\downarrow}(x)$	right	$-\infty$	$-\infty+]$
$H'(-x)$	left	$[-\infty+$	$0-]$
$H'(-x)$	right	$(-\infty+$	$0-)$
$H'(-x)$	sym	$-\infty+$	$0-$
$H(-x)$	all types	$-\infty$	0
$\theta_{\uparrow}(x)$	sym	$0-$	0
$\theta_{\downarrow}(x)$	left	0	$0+)$
$\theta_{\downarrow}(x)$	right	0	$0+]$
$\theta_{\downarrow}(x)$	sym	0	$0+$
$H'(x)$	left	$[0+$	$+\infty-]$
$H'(x)$	right	$(0+$	$+\infty-)$
$H(x)$	all types	0	$+\infty$
$\not\theta_{\uparrow}(x)$	sym	$+\infty-$	$+\infty$
$I'(x) + \theta(x)$	sym	$-\infty+$	$+\infty-$
$I(x)$	all types	$-\infty$	$+\infty$

In the table 'sym' stands for symmetrical, and 'all' means that it does not matter which scalar product is taken. Not all cases have been listed in the table, but one easily adds the other cases by analogy with the cases listed. Usually only the integrals corresponding to the symmetrical scalar products are needed.

The generalization to a finite number of partition points in the finite is immediate. In keeping with the self-imposed restrictions of this book only finite decompositions of the unit function and finite sums of partial integrals are considered.

The preliminary integral of Sec. 4.3 can be identified with the integral in the sense of generalised functions over the 'open' intervals obtained by omitting all singular points of the integrand.

This set of definitions has some consequences which one needs to get used to. For integrals over the whole range there is no problem with the notation, since

$$\int_{-\infty}^{\infty} dx\, x^{-1} = \langle\, I(x)\,, x^{-1}\,\rangle = \langle\, \delta(x)\,, -i\pi\,\mathrm{sgn}(x)\,\rangle = 0, \tag{14.15}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, |x|^{-1} &= \langle\, I(x)\,, |x|^{-1}\,\rangle = \langle\, \delta(x)\,, 2\big(\log|x| - \psi(1)\, I(x)\big)\,\rangle = \\ &= 2\,\langle\, I(x)\,, \delta(x) \bullet \big(\log|x| - \psi(1)\, I(x)\big)\,\rangle = \\ &= 2\big(\psi(1) - \psi(1)\big) = 0,\end{aligned} \tag{14.16}$$

as one finds by repeated application of Parseval's equality.

Care with the notation is needed for integrals over partial intervals. For the special case of integration starting at $x = 0$ one finds

$$\begin{aligned}\int_{0}^{\infty} dx\, x^{-1} &= \langle\, H(x)\,, x^{-1}\,\rangle = \tfrac{1}{2}\,\langle\, \mathrm{sgn}(x)\,, x^{-1}\,\rangle = \\ &= \tfrac{1}{2}\,\langle\, I(x)\,, \mathrm{sgn}(x) \bullet x^{-1}\,\rangle = -\,\psi(1) = -\int_{-\infty}^{0} dx\, x^{-1},\end{aligned} \tag{14.17}$$

by (9.79), but

$$\begin{aligned}\int_{0}^{\infty} dx\, |x|^{-1} &= \langle\, H(x)\,, |x|^{-1}\,\rangle = \tfrac{1}{2}\,\langle\, \mathrm{sgn}(x)\,, |x|^{-1}\,\rangle = \\ &= \tfrac{1}{2}\,\langle\, I(x)\,, \mathrm{sgn}(x) \bullet |x|^{-1}\,\rangle = 0 = \int_{-\infty}^{0} dx\, |x|^{-1},\end{aligned} \tag{14.18}$$

by (9.80), so the behaviour at $x = 0$ is really relevant when the integration interval includes the point $x = 0$.

Remark 14.4 Another way to make a mistake is to put

$$\int_{-\infty}^{\infty} dx\, |x|^{-1} = 2\int_{0}^{\infty} dx\, x^{-1}. \tag{14.19}$$

WRONG!

Written in this form the reason is obvious, on the right-hand side the contribution from the point $x = 0$ is counted twice, on the left-hand side it is only counted once.

It is difficult to get used to the fact that single points may contribute to integrals.

The same care is needed when integrating Heaviside functions, since we have by construction

$$\int_{0}^{\infty} dx\, x^{-1} H(x) = \langle\, H(x)\,, x^{-1} H(x)\,\rangle = -\,\tfrac{1}{2}\,\psi(1), \tag{14.20}$$

which differs by a factor $\frac{1}{2}$ from the corresponding formula (14.17) without the additional Heaviside function.

The naïve expressions

$$\int_0^\infty \frac{dx}{x}, \quad \text{and} \quad \int_0^a \frac{dx}{x}, \tag{14.21}$$

do not have a unique meaning in the theory of generalised functions. It is necessary to specify the integrand as a generalised function before a well defined generalised function integral exists.

Remark 14.5 In the next section it will be seen that the value of the integrals is in agreement with the fundamental theorem of the calculus,

$$\int_0^\infty dx\, x^{-1} = \log|x| \Big|_0^{+\infty} = \psi(1), \tag{14.22}$$

and

$$\int_0^\infty dx\, |x|^{-1} = \log|x| \operatorname{sgn}(x) \Big|_0^{+\infty} = 0, \tag{14.23}$$

which again shows the relevance of the behaviour of the primitive function at $x = 0$. Increasing the lower limit to $0+$ makes both integrals equal to zero.

All integrals over all intervals are perfectly well defined. Mistakes can be caused only by improper use of the notation.

Remark 14.6 This precision of notation will become less necessary when the indeterminacy of the integral has been introduced in Ch. 18. When attention is restricted to determinate integrals, or when the indeterminacy is introduced explicitly, the difference between (14.23) and (14.22) is often irrelevant.

Remark 14.7 When ordinary functions are considered as primed functions $\in \overline{\mathbf{PC}}'_\lambda$ boundary points such as $x = 0$ do not contribute to the integrals.

By definition scalar products can always be converted into integrals over the whole range from $-\infty$ to ∞. The possibility of reducing the scalar product to an integral over a smaller interval depends on the support of the generalised function which appears as the integrand.

In distribution theory the support is by definition closed, and all integrals can be reduced to an integral over any open interval containing the support. Taking the interval of integration equal to the support may give erroneous contributions from the endpoints. This is clear from the examples given above.

This also holds in the sense of generalised functions. For generalised functions there is the additional possibility that the support may be open.

Then a complementary rule holds. A generalised function with an open support must be integrated over at least a closed interval containing the support.

Example 14.1 To measure the amount of δ–function present in an element of $\overline{\mathbf{PC}}_\lambda$ one must integrate over $(0-, 0+)$, to observe the amount of an $\eta_\downarrow$–function one has to integrate over $[0, a]$, with $a \in \mathbb{R}_+$. It does not matter in this case if the upper limit is open or closed. The lower limit has to be closed in order to obtain correct answers.

The behaviour of integrals, when limits are taken by varying the endpoints of the region of integration, will be left until the convergence of sequences of generalised functions has been defined in Ch. 19. This will allow the definition of the closed interval at for instance $[0, 0+]$ as a limit $[0, 0+] := \lim_{a\downarrow 0}[0, a]$. Then the rules for integration over the support take the intuitively appealing form that integrals have to be taken over a region which is infinitesimally greater than the support of the integrand.

Remark 14.8 Integrals with a logarithmic divergence are not invariant under scale transformations. This will be discussed in Ch. 16 when the scale transformation properties of the generalised functions are defined.

Remark 14.9 The integral in the sense of generalised functions is a different concept, defined for a different class of integrands, than the standard Riemann or Lebesgue integral. This makes it somewhat pointless to argue about their relative generality. These different integral concepts should not be confused even though the same integral symbol is used for both.

14.2 Inverse operators

The inverse of the $\mathcal{D}$ and $\mathcal{X}$ operators has not yet been defined. Strictly speaking these operators do not have an inverse since they have a zero element, but they almost have an inverse. The definition follows the by now familiar pattern. The preliminary operator $\mathcal{X}_{\text{pre}}$ was defined in Sec. 4.5 as $x\cdot$. The preliminary operator $\mathcal{X}_{\text{pre}}^{-1}$ is defined on $\mathbf{PC}_\lambda$ by

$$\mathcal{X}_{\text{pre}}^{-1} f(x) := x^{-1} \cdot f(x), \tag{14.24}$$

which defines again an element of the preliminary class $\mathbf{PC}_\lambda$. Unless $f(x)$ happens to be zero and regular at $x = 0$ the element $x^{-1} \cdot f(x)$ has an additional singular point at $x = 0$, which is obviously of the required type.

The preliminary operator $\mathcal{X}_{\text{pre}}^{-1}$ is transferred to the linear functionals by taking residues. The result is

$$\begin{aligned} \mathcal{X}^{-1}|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) &= |x'|^{\alpha-1} \log^q|x| \operatorname{sgn}^{m+1}(x) = \\ &= x^{-1}\bullet |x'|^\alpha \log^q|x| \operatorname{sgn}^{m+1}(x) = \\ &= x'^{-1}\bullet |x'|^\alpha \log^q|x| \operatorname{sgn}^{m+1}(x), \end{aligned} \tag{14.25}$$

for the powers. For the η–functions we obtain after splitting into parts in the finite and at infinity,

$$\begin{aligned}\mathcal{X}^{-1}\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) &= \eta_s^{(\alpha+1,q)}\,\mathrm{sgn}^{m+1}(x) = \\ &= x^{-1}\bullet\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^{m+1}(x) = \\ &= x'^{-1}\circ\bullet\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^{m+1}(x) = \\ &= 2x'^{-1}\bullet\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^{m+1}(x),\end{aligned} \qquad (14/26)$$

where the factor 2 in the formula for the commutative product with x'^{-1} should be noted. At infinity the same formula holds with a slash added.
The operator $\mathcal{X}^{-1}$ can be extended to $(\mathcal{I}-\boldsymbol{P}_{\theta(0)})\overline{\mathbf{PC}}'_\lambda$ by defining

$$\mathcal{X}^{-1}\,\theta^{(p+1)}(x) := \theta^{(p)}(x), \qquad (14.27)$$

in agreement with the naïve interpretation of the $\theta^{(p)}$–functions, but this will come later.
The operator $\mathcal{X}^{-1}$ is related to the operator $x'^{-1}\bullet$ by

$$\mathcal{X}^{-1}\,f'(x) = x^{-1}\bullet\,f'(x) = x'^{-1}\circ\bullet\,f'(x) \neq x'^{-1}\bullet\circ\,f'(x), \qquad (14.28)$$

$\forall\,f'(x)\in\overline{\mathbf{PC}}'_\lambda$. In terms of the symmetrical generalised function product and $x'^{-1}\bullet$ one obtains the more complicated relation

$$\begin{aligned}\mathcal{X}^{-1}\,f(x) &= 2x'^{-1}\bullet\,\boldsymbol{P}_{\eta\not\eta}\,f(x) + x'^{-1}\bullet\,(\mathcal{I}-\boldsymbol{P}_{\eta\not\eta})\,f(x) = \\ &= x'^{-1}\bullet\,(\mathcal{I}+\boldsymbol{P}_{\eta\not\eta})\,f(x),\end{aligned} \qquad (14.29)$$

where $\boldsymbol{P}_{\eta\not\eta}$ is the projection operator on the allowed linear combinations of η and $\not\eta$–functions.
The operator $\mathcal{X}^{-1}$ is pulled back to $\overline{\mathbf{PC}}_\lambda$ in accordance with (7.2) by putting

$$\mathcal{X}^{-1}\,f(x) := \overline{\mathcal{M}}^{-1}\mathcal{X}^{-1}\overline{\mathcal{M}}\;\,f(x), \qquad (14.30)$$

$\forall\,f(x)\in\mathbf{PC}$. The results for $\mathcal{X}^{-1}$ acting on $\overline{\mathbf{PC}}_\lambda$ are

$$\mathcal{X}^{-1}|x|^\alpha\log^q|x|\,\mathrm{sgn}^m(x) = |x|^{\alpha-1}\log^q|x|\,\mathrm{sgn}^{m+1}(x), \qquad (14.31)$$

and

$$\mathcal{X}^{-1}\,\delta^{(p)}(x) = \delta^{(p+1)}(x), \qquad (14.32)$$

in agreement with expectations. The relationship between $\mathcal{X}^{-1}$ and $x^{-1}\bullet$ remains the same as in $\overline{\mathbf{PC}}'_\lambda$,

$$\mathcal{X}^{-1}\,f(x) = x^{-1}\circ\bullet\,f(x), \qquad (14.33)$$

but again

$$\mathcal{X}^{-1} f(x) = 2x^{-1} \bullet \boldsymbol{P}_\delta f(x) + x^{-1} \bullet \left(\mathcal{I} - \boldsymbol{P}_\delta\right) f(x) = \\ = x^{-1} \bullet \left(\mathcal{I} + \boldsymbol{P}_\delta\right) f(x), \tag{14.34}$$

$\forall f(x) \in \overline{\mathbf{PC}}_\lambda$, since $x^{-1} \in \overline{\mathbf{PC}}_\lambda$ maps into $x'^{-1} \in \overline{\mathbf{PC}}'_\lambda$. The operator $\boldsymbol{P}_{\delta^{(p)}}$ is the projection operator on the space of (finite linear combinations of) the $\delta^{(p)}(x)$-functions at $x = 0$. The extension to singular functions at $x \neq 0$ is found from the asymptotic expansion

$$x^{-1} \sim \sum_{j=0}^{\infty} \binom{-1}{j} (x - x_0)^j x_0{}^{-j-1}, \tag{14.35}$$

of the function x^{-1} at the point $x = x_0$.

Example 14.2 For the η-functions this gives

$$\mathcal{X}^{-1} \eta_s^{(\alpha,q)}(x - x_0) = \sum_{j=0}^{\infty} \binom{-1}{j} x_0{}^{-j-1} \eta_s^{(\alpha-j,q)}(x - x_0), \tag{14.36}$$

which is an allowed linear combination.

By linearity the operator is now defined on $\mathbf{GF}_s$. This results in the formula

$$\mathcal{X}^{-1} \theta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x) = \theta_s^{(\alpha-1,q)}(x) \operatorname{sgn}^{m+1}(x), \tag{14.37}$$

in agreement with the earlier definition (14.27). It is seen that $\overline{\mathbf{PC}}'_\lambda$ is not closed under the operator $\mathcal{X}^{-1}$ even though it is closed under $\mathcal{X}$, since

$$\mathcal{X}^{-1} \theta(x) = \theta^{(-1)}(x), \tag{14.38}$$

and $\theta^{(-1)}(x) \notin \overline{\mathbf{PC}}'_\lambda$. The element $\theta(x)$ is the only exceptional element for the operator $\mathcal{X}^{-1}$ in $\overline{\mathbf{PC}}'_\lambda$.

For the ordinary functions $\mathbf{GF}_s$ the operator $\mathcal{X}^{-1}$ is again equal to its preliminary version. No additional singular functions at the singular points arise.

As the next step it must be seen in how far the name $\mathcal{X}^{-1}$ is justified. On $\overline{\mathbf{PC}}'_\lambda$ one sees by direct computation that

$$\mathcal{X}\mathcal{X}^{-1} = \mathcal{X}^{-1}\mathcal{X} = \mathcal{I}. \tag{14.39}$$

Transfer to $\overline{\mathbf{PC}}_\lambda$ yields

$$\mathcal{X}\mathcal{X}^{-1} = \mathcal{I}, \tag{14.40}$$

but

$$\mathcal{X}^{-1}\mathcal{X} = \mathcal{I} - \boldsymbol{P}_{\delta^{(0)}}, \tag{14.41}$$

in agreement with the well known zero space of the operator $\boldsymbol{\mathcal{X}}$.

The inverse differential operator $\mathcal{D}^{-1}$ is defined in the same way. Its preliminary version is defined between singularities by

$$\mathcal{D}^{-1} f(x) := \int^x dx\, f(x), \tag{14.42}$$

with an arbitrary lower limit of integration.
In particular we obtain for $\lambda \neq -1$

$$\mathcal{D}^{-1}_{\text{pre}}|x|^\lambda \operatorname{sgn}^m(x) = (\lambda+1)^{-1}|x|^{\lambda+1}\operatorname{sgn}^{m+1}(x), \tag{14.43}$$

where the lower limit can be taken as either 0 or $\pm\infty$. For the purpose of taking residues it is irrelevant what happens at $\lambda = -1$.

The action of $\mathcal{D}^{-1}$ on $\overline{\mathbf{PC}}'_\lambda$ is found by taking residues, using the binomial theorem to expand $(\lambda+1)^{-1}$ at $\lambda = \alpha$ in powers of $(\lambda - \alpha)$.

$$\begin{aligned}\mathcal{D}^{-1}|x'|^\alpha \log^q|x|\operatorname{sgn}^m(x) &= \operatorname*{Res}_{\lambda=\alpha} q!\,(\lambda-\alpha)^{-q-1}(\lambda+1)^{-1}|x|^{\lambda+1}\operatorname{sgn}^{m+1}(x) = \\ &= (-)^q q! \sum_{j=0}^{q} \tfrac{(-)^j}{j!}(\alpha+1)^{j-q-1}|x'|^{\alpha+1}\log^j|x|\operatorname{sgn}^{m+1}(x) + \\ &\quad - 2(-)^q q! \sum_{j=0}^{\infty} j!\,(\alpha+1)^{-j-q-2} \times \\ &\quad \times \left((-)^j \eta_s^{(-\alpha-2,j)}(x) - \not{\eta}_s^{(-\alpha-2,j)}(x)\right)\operatorname{sgn}^{m+1}(x),\end{aligned} \tag{14.44}$$

for $\alpha \neq -1$ and

$$\mathcal{D}^{-1}|x'|^{-1}\log^q|x|\operatorname{sgn}^m(x) = \log^{q+1}|x'|\operatorname{sgn}^{m+1}(x), \tag{14.45}$$

for $\alpha = -1$. No η–functions arise in this case since it is not necessary to expand a function of λ.

For the η–functions the general case becomes, (after taking residues and splitting between the finite and infinity)

$$\mathcal{D}^{-1}\,\eta_s^{(\alpha,q)}(x)\operatorname{sgn}^m(x) = -\sum_{j=0}^{\infty} \tfrac{(q+j)!}{q!}\,\alpha^{-j-1}\,\eta_s^{(\alpha-1,q+j)}(x)\operatorname{sgn}^{m+1}(x), \tag{14/46}$$

for $\alpha \neq 0$. The same holds for $\not{\eta}$ with slashes added.
For $\alpha = 0$ and $q > 0$ taking residues gives

$$\mathcal{D}^{-1}\,\eta_s^{(0,q)}(x)\operatorname{sgn}^m(x) = -q^{-1}\,\eta_s^{(-1,q-1)}(x)\operatorname{sgn}^{m+1}(x), \tag{14.47}$$

but for the exceptional case $\alpha = 0, q = 0$ one obtains

$$\mathcal{D}^{-1}\left(\eta_s(x) - \not{\eta}_s(x)\right)\operatorname{sgn}^m(x) = \tfrac{1}{2}|x'|^0\operatorname{sgn}^{m+1}(x), \tag{14.48}$$

which cannot be split into a part at $x = 0$ and a part at infinity as we did in the general case. This problem cannot be resolved in the restricted space $\overline{\mathbf{PC}}'_\lambda$. A solution will be possible in $\mathbf{GF}_s$ as a whole.
The operator $\mathcal{D}^{-1}$ is now transferred to $\overline{\mathbf{PC}}_\lambda$ by

$$\mathcal{D}^{-1} f(x) := \mathcal{M}^{-1}\mathcal{D}^{-1}\mathcal{M} f(x), \tag{14.49}$$

$\forall f(x) \in \overline{\mathbf{PC}}_\lambda$.
For the powers this results in

$$\begin{aligned} &\mathcal{D}^{-1}|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) = \\ &\quad = (-)^q q! \sum_{j=0}^{q} \tfrac{(-)^j}{j!} (\alpha+1)^{j-q-1}|x|^{\alpha+1} \log^j|x| \operatorname{sgn}^{m+1}(x) + \\ &\quad + 2q!\, (-)^{q+1}(\alpha+1)^{-q-2} \sum_{p=0}^{\infty} \delta_{-\alpha-2,p}\, \delta^{\mathrm{mod}2}_{m+1,p}\, \delta^{(p)}(x), \end{aligned} \tag{14.50}$$

with the special case

$$\mathcal{D}^{-1}x^{-p-2} \operatorname{sgn}(x) = -(p+1)^{-1}x^{-p-1} \operatorname{sgn}(x) - 2(p+1)^{-2}\, \delta^{(p)}(x), \tag{14.51}$$

For the δ–functions one obtains the obvious result

$$\mathcal{D}^{-1}\, \delta^{(p+1)}(x) = -(p+1)^{-1}\, \delta^{(p)}(x), \tag{14.52}$$

with the exceptional case

$$\mathcal{D}^{-1}\, \delta(x) = \tfrac{1}{2} \operatorname{sgn}(x), \tag{14.53}$$

in agreement with the corresponding result in distribution theory.

Remark 14.10 It should be noted again that the function $\operatorname{sgn}(x) \in \overline{\mathbf{PC}}_\lambda$ does not have a non-zero component of its derivative at infinity, simply because by definition the minimal completion of $\mathbf{PC}_\lambda$ chosen in Ch. 7 does not contain anything at infinity which could serve as derivative for the function $\operatorname{sgn}(x)$. Introducing such an element does not serve a useful purpose, since the reason for having $\overline{\mathbf{PC}}_\lambda$ next to $\overline{\mathbf{PC}}'_\lambda$ is the desire to include a unit element for the multiplication.

The operator $\mathcal{D}^{-1}$ is now defined on $\mathbf{GF}_s$ by linearity. The primitives of the θ–functions are found by subtraction. It is not useful to write out the results in the general case.
The splitting problem for the η–function can now be resolved. For the exceptional case $\alpha = q = m = 0$ in (14.48) one obtains from (14.53)

$$\mathcal{D}^{-1}\big(\delta(x) - \eta(x) + \not\eta(x)\big) = \tfrac{1}{2}\big(\operatorname{sgn}(x) - \operatorname{sgn}'(x)\big) = \tfrac{1}{2}\big(\tau(x) + \not\tau(x)\big), \tag{14.54}$$

This can be split between the finite and infinity with the results

$$\mathcal{D}^{-1}\big(\delta(x) - \eta(x)\big) = \tfrac{1}{2}\,\tau(x), \tag{14.55}$$

and

$$\mathcal{D}^{-1}\,\not\eta(x) = \tfrac{1}{2}\not\tau(x), \tag{14.56}$$

which gives $\mathcal{D}^{-1}\not\eta(x)$ a support at infinity. This allows us to find the primitives of $\eta(x)$ and $\not\eta(x)$ separately with the additional result

$$\mathcal{D}^{-1}\,\eta(x) = \tfrac{1}{2}\big(\operatorname{sgn}(x) - \tau(x)\big) = \tfrac{1}{2}\big(\operatorname{sgn}'(x) + \not\tau(x)\big). \tag{14.57}$$

The function $\eta(x)$ has a non-zero primitive in the finite, and its primitive does not have a derivative at infinity.
Likewise for the σ-functions one can write

$$\mathcal{D}^{-1}\big(\sigma(x) - \not\sigma(x)\big) = \tfrac{1}{2}\,I'(x) = \tfrac{1}{2}\big(I(x) - \theta(x) - \not\theta(x)\big), \tag{14.58}$$

which can be split by defining

$$\mathcal{D}^{-1}\,\sigma(x) = -\tfrac{1}{2}\,\theta(x), \qquad \mathcal{D}^{-1}\not\sigma(x) = \tfrac{1}{2}\not\theta(x), \tag{14.59}$$

since the unit function has a zero derivative.

For ordinary functions the primitives can be found by subtraction of a sufficient number of terms of the asymptotic expansions at the singular points. For ordinary functions located in the finite the primitive function contains the sharp signum function at infinity.

Example 14.3 For the function $f(x) := \cosh^{-2}(x)$ one finds

$$\mathcal{D}^{-1}\cosh^{-2}(x) = \tanh(x), \tag{14.60}$$

and not

$$\mathcal{D}^{-1}\cosh^{-2}(x) = \tanh'(x), \tag{14.61}$$

WRONG!

since differentiation yields

$$\mathcal{D}\tanh'(x) = \cosh^{-2}(x) - 2\not\eta(x) \neq \cosh^{-2}(x), \tag{14.62}$$

with an additional $\not\eta$-function.

The sharp signum function gives the correct result for the validity of the fundamental theorem of the calculus, (compare the next section.) The operators $\mathcal{D}^{-1}$ and $\mathcal{X}^{-1}$ have the expected operator properties, such as unitary equivalence under Fourier transformation.

Property 14.1 The operators $\mathcal{D}^{-1}$ and $\mathcal{X}^{-1}$ are unitarily equivalent.

$$\mathcal{X}^{-1} = -i\mathcal{F}^{-1}\mathcal{D}^{-1}\mathcal{F} = i\mathcal{F}\mathcal{X}^{-1}\mathcal{F}^{-1}, \tag{14.63}$$

and also

$$\mathcal{D}^{-1} = -i\mathcal{F}^{-1}\mathcal{X}\mathcal{F} = i\mathcal{F}\mathcal{X}^{-1}\mathcal{F}^{-1}, \tag{14.64}$$

Verification: Direct computation gives

$$\mathcal{X}\big(-i\mathcal{F}^{-1}\mathcal{D}^{-1}\mathcal{F}\big) = \mathcal{F}^{-1}\mathcal{D}\mathcal{F}\mathcal{F}^{-1}\mathcal{D}^{-1}\mathcal{F} = \mathcal{F}^{-1}\mathcal{I}\mathcal{F} = \mathcal{I}, \tag{14.65}$$

and

$$\big(-i\mathcal{F}^{-1}\mathcal{D}^{-1}\mathcal{F}\big)\mathcal{X}^{-1} = \mathcal{F}^{-1}\big(\mathcal{I} - P_I\big)\mathcal{F} = \mathcal{I} - P_{\delta(0)}, \tag{14.66}$$

by using the unitary equivalence of $\mathcal{X}$ and $\mathcal{D}$. □

The relation between the operators $\mathcal{D}$ and $\mathcal{D}^{-1}$ follows by Fourier transformation of the corresponding result for $\mathcal{X}$.

$$\mathcal{D}\mathcal{D}^{-1} = \mathcal{I}, \tag{14.67}$$

$$\mathcal{D}^{-1}\mathcal{D} = \mathcal{I} - P_I, \tag{14.68}$$

since the one-dimensional excepted subspace is spanned by I, which is the Fourier transform of the exceptional element $\delta(x)$ for the $\mathcal{X}$ operator. The operators $\mathcal{X}^{-1}$ and $\mathcal{D}^{-1}$ have a negative parity. We have

$$\mathcal{P}\mathcal{X}^{-1}\mathcal{P} = -\mathcal{X}^{-1}, \qquad \mathcal{P}\mathcal{X}^{-1} = -\mathcal{X}^{-1}\mathcal{P}, \tag{14.69}$$

and

$$\mathcal{P}\mathcal{D}^{-1}\mathcal{P} = -\mathcal{D}^{-1}, \qquad \mathcal{P}\mathcal{D}^{-1} = -\mathcal{D}^{-1}\mathcal{P}, \tag{14.70}$$

so these operators convert parity eigenfunctions into parity eigenfunctions with the opposite parity.

Remark 14.11 The operators $\mathcal{X}^{-1}$ and $\mathcal{D}^{-1}$ are defined uniquely. It is of course possible to include an arbitrary multiple of the zero element of $\mathcal{X}$ or $\mathcal{D}$ in the definition of the inverse operators but this is not done here. In this way confusion with the indeterminate constant introduced by the scale transformations is avoided. The inverse operators preserve the parity and scaling properties (Ch. 16) of the generalised functions.

Remark 14.12 It is also possible to use the unitary property (14.64) for the definition of $\mathcal{D}^{-1}$ in terms of $\mathcal{X}^{-1}$, but the direct derivation of the properties is more transparent.

From the unitary equivalence of $\mathcal{X}^{-1}$ and $\mathcal{D}^{-1}$ one finds the inverse operator as

$$\mathcal{D}^{-1} f(x) = -i\mathcal{F}^{-1}\big(x^{-1}\bullet \mathcal{F} f(x)\big) = \tfrac{1}{2}\operatorname{sgn}(x) * f(x), \tag{14.71}$$

which can be written in the form of an integral as

$$\mathcal{D}^{-1} f(x) = \tfrac{1}{2}\int_{-\infty}^{x} dy\, f(y) - \tfrac{1}{2}\int_{x}^{\infty} dy\, f(y). \tag{14.72}$$

This result is also valid in a classical sense when the integrals converge. The generalised interpretation of (14.72) will be discussed in Ch. 22.

14.3 The fundamental theorem of the calculus

In Ch. 4 the integral was defined as a residue of an auxiliary analytic function. This definition is often cumbersome to apply to the actual computation of integrals. The value of a generalised function on the other hand can often be found by inspection, when its asymptotic expansion is known. Therefore it is convenient to define a primitive generalised function for each generalised function in such a way that the fundamental theorem of the calculus holds. This has been accomplished in principle in the previous section by the definition of the operator $\mathcal{D}^{-1}$, since the primitive function $F(x)$ defined by

$$F(x) := \mathcal{D}^{-1} f(x), \tag{14.73}$$

obviously satisfies

$$\mathcal{D}\, F(x) = f(x), \tag{14.74}$$

by (14.67). It is often easy to find the primitive function by inspection, good luck, or by consulting a table of primitives, and to verify by explicit differentiation that it satisfies (14.74) in the sense of generalised functions.

With the primitive defined above one verifies the property referred to in calculus books as the fundamental theorem of the calculus,

Property 14.2 $\forall f(x) \in \mathbf{GF}_s$, and for all integration intervals

$$\int_a^b dx\; f(x) = F(x)\Big|_a^b = F(b) - F(a), \tag{14.75}$$

if $F(x)$ satisfies

$$\mathcal{D}\, F(x) = f(x). \tag{14.76}$$

Verification: Let $h(x)$ be a piecewise constant function of the kind used in the previous section to define integrals over intervals. The derivative of $h(x)$ contains in general two η– or δ–functions. In case of a Heaviside function with a $H(x)$ singularity at infinity there is only one η–like function, in the special case $h(x) = I(x)$ there is none. From Leibniz's rule we have

$$\mathcal{D}\big(h(x) \bullet F(x)\big) = \mathcal{D}\, h(x) \bullet F(x) + h(x) \bullet f(x). \tag{14.77}$$

Taking the scalar product with the unit function gives

$$\big\langle\, I(x)\,, \mathcal{D}\, h(x) \bullet F(x) \,\big\rangle + \big\langle\, I(x)\,, h(x) \bullet f(x) \,\big\rangle = \big\langle\, I(x)\,, \mathcal{D}\big(h(x) \bullet F(x)\big) \,\big\rangle = 0, \tag{14.78}$$

This can be rewritten as

$$\big\langle\, h(x)\,,\, f(x) \,\big\rangle = \big\langle\, \mathcal{D}\, h(x)\,,\, F(x) \,\big\rangle + \big\langle\, I(x)\,, \mathcal{D}\big(h(x) \bullet F(x)\big) \,\big\rangle, \tag{14.79}$$

From Parseval's equality we have

$$\langle I(x)\,, \mathcal{D}\big(h(x) \bullet F(x)\big) \rangle = \langle \delta(x)\,, i\mathcal{X}\mathcal{F}\big(h(x) \bullet F(x)\big) \rangle, \tag{14.80}$$

so the result is non-zero iff for some q

$$\mathcal{F}\big(h(x) \bullet F(x)\big) \sim x^{-1} \log^q |x|. \tag{14.81}$$

Transforming back it follows that the stock-term is non-zero iff for some q

$$h(x) \bullet F(x) \sim \log^q |x| \operatorname{sgn}(x). \tag{14.82}$$

Since there is complete freedom to choose the values of functions at infinity arbitrarily the values at $\pm\infty$ have been chosen in the previous chapter in such a way that the stock-term at infinity equals the result of the missing derivative of the sharp Heaviside functions at $x = \pm\infty$. The fundamental theorem of the calculus holds by derivation in the finite, and by convention at infinity. □

It is of course possible to choose a standardization of the maps and products differing from (9.101), with correspondingly different values for

$$\int_{-\infty}^{\infty} |x|^{-1} \log^q |x| = c_q \neq 0, \tag{14.83}$$

and to choose an incompatible set of values at infinity. This would destroy the validity of the fundamental theorem, but there seems to be no good reason for doing this.

Remark 14.13 It is also possible to obtain an indeterminate form of the fundamental theorem by replacing the determinate values of the primitive by the indeterminate values. These will be defined in Ch. 16. The indeterminate form of the fundamental theorem is often very convenient for physical computations, in particular in quantum field theory.

Finally one may remark that the inelegant need to insert partition points, which was necessary for the definition of the preliminary integral in Ch. 4, can be avoided by invoking the fundamental theorem. The singularities which contribute to the integral give rise to a $\operatorname{sgn}(x)$–type behaviour, and only the total $\operatorname{sgn}(x)$ behaviour from all singularities and regular behaviour combined contributes to the value of the integral. The location of the singularities is irrelevant.

CHAPTER 15

TRANSLATIONS OF GENERALISED FUNCTIONS

Up to this point many aspects of the theory of generalised functions supposed the existence of a preferred point, the point zero, or the origin of the coordinate system, when the functions are used to describe a physical quantity. The integrals of generalised functions are not always invariant under a change of the origin, or equivalently a shift of the function. In this chapter the origin is considered to be fixed and the functions are shifted.

15.1 Translations

The translations are defined following the same pattern as for the other operators. First the preliminary translation operators $\mathcal{T}_{\text{pre}}(x_0, 0)$ are defined on $\mathbf{PC}_\lambda$ by

$$\mathcal{T}_{\text{pre}}(x_0, 0)\, f(x) := f(x - x_0), \tag{15.1}$$

in accordance with the standard definition of the translations.

Remark 15.1 The translations with the second parameter equal to zero are referred to as coordinate translations, in order to distinguish them from the wave number translations introduced in the next section, and the phase plane translations,

One readily sees that this preliminary definition does defines an element of $\mathbf{PC}_\lambda$. In the finite this is obvious. The asymptotic expansion at infinity has the correct form. For $x \gg x_0 > 0$ and likewise for $x \ll x_0 < 0$ one expands

$$(x - x_0)^\lambda = \sum_{j=0}^{\infty} \tbinom{\lambda}{j} (-x_0)^j x^{(\lambda - j)}, \tag{15.2}$$

by the binomial theorem. For the logarithm near $x = +\infty$ one has

$$\log(x - x_0) = \log(x) \sum_{j=0}^{\infty} \tfrac{1}{j+1} {x_0}^{j+1} x^{-j-1}, \tag{15.3}$$

so the asymptotic expansion of shifted powers of the logarithm is also of the required form. Substituting (15.3) and (15.2) into the asymptotic expansion of an arbitrary $f(x) \in \mathbf{PC}_\lambda$ is rather cumbersome, but the result is obviously an asymptotic expansion of the required form.
The translations are now transferred to $\overline{\mathbf{PC}}'_\lambda$ by taking residues.
For the translated powers $\in \overline{\mathbf{PC}}'_\lambda$ one finds

$$\begin{aligned}\mathcal{T}(x_0, 0)|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) &:= \operatorname*{Res}_{\lambda=\alpha} q!\,(\lambda - \alpha)^{-q-1} |x - x_0|^\lambda \operatorname{sgn}^m(x - x_0) \\ &= |x' - x_0|^\alpha \log^q|x - x_0| \operatorname{sgn}^m(x - x_0). \end{aligned}\tag{15.4}$$

Considered as a function of (x) the asymptotic expansion at infinity and the corresponding η–functions may be found by expanding the shifted power using the binomial theorem. The result is

$$
\begin{aligned}
&|x' - x_0|^\alpha \log^q|x - x_0| \operatorname{sgn}^m(x - x_0) \sim \\
&\quad \sim \sum_{j=0}^{\infty} |x'|^{\alpha-j} \operatorname{sgn}^{m+j}(x) \sum_{k=0}^{j} \tbinom{q}{k} \tbinom{\alpha+j}{j}^{(k)} \log^{q-k}(x) + \qquad (15.5)\\
&\quad + 2q! \sum_{j=0}^{\infty} (-x_0)^j \sum_{k=0}^{j} \tfrac{(-)^k k!\, q!}{(q+k+1)!} \tbinom{\alpha}{j+q+1}^{(q+k+1)} \not\eta_s^{(-\alpha+q+j,k)}(x) \operatorname{sgn}^{m+j}(x),
\end{aligned}
$$

which again includes a formally infinite sum of $\not\eta$–functions. The derivatives of the binomial coefficients which occur in the expansion are defined in appendix B. In the finite the shifted power cannot be expanded, so no additional η–functions arise.
For the the translated η–functions one finds in the same way

$$
\mathcal{T}(x_0, 0)\, \eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x) = \eta^{(\alpha,q)}(x - x_0) \operatorname{sgn}^m(x - x_0), \qquad (15.6)
$$

and

$$
\mathcal{T}(x_0, 0)\, \not\eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x) = \not\eta_s^{(\alpha,q)}(x - x_0) \operatorname{sgn}^m(x - x_0). \qquad (15.7)
$$

The translated $\not\eta$–functions can be expressed in terms of the $\not\eta(x)$–functions by expanding the power of x near infinity by the binomial theorem, followed by a Taylor expansion of the binomial coefficients. The result is

$$
\begin{aligned}
&\not\eta_s^{(\alpha,q)}(x - x_0) \operatorname{sgn}^m(x - x_0) = \qquad (15.8)\\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-)^{j+k} x_0{}^j \tbinom{q+k}{k} \tbinom{-\alpha-1}{j}^{(k)} \not\eta_s^{(\alpha+j,q+k)}(x) \operatorname{sgn}^{m+j}(x),
\end{aligned}
$$

in agreement with the computation (5.91) of the translated $\not\eta$–functions at infinity.
For ordinary functions $f(x) \in \overline{\mathbf{PC}}'_\lambda$ one defines

$$
\begin{aligned}
\mathcal{T}(x_0, 0) f'(x) :=\ & \bar{\mathcal{M}}\ \mathcal{T}_{\text{pre}}(x_0, 0) \bar{\mathcal{M}}^{-1} f'(x) + \qquad (15.9)\\
& + \mathcal{T}(x_0, 0) f'_a(x; \pm\infty) - \bar{\mathcal{M}}\ \mathcal{T}_{\text{pre}}(x_0, 0) \bar{\mathcal{M}}^{-1} f'_a(x; \pm\infty).
\end{aligned}
$$

Additional $\not\eta$–functions at infinity arise from the translation of the asymptotic series. Even though (15.9) looks formidable when fully written out it is obviously an allowed generalised function at infinity.

Remark 15.2 The results do not depend on the choice of $\mathcal{M}_{\mathcal{X}}$ or $\mathcal{M}_{\mathcal{D}}$, so the $\bar{\mathcal{M}}$ without a subscript is used.

The translations are now defined on $\overline{\mathbf{PC}}_\lambda$ by

$$\mathcal{T}(x_0,0)\, f(x) := \bar{\mathcal{M}}^{-1}\mathcal{T}(x_0,0)\bar{\mathcal{M}}\ \ f(x), \qquad (15.10)$$

$\forall\, f(x) \in \overline{\mathbf{PC}}_\lambda$. This yields only the translations of the δ–functions

$$\mathcal{T}(x_0,0)\,\delta^{(p)}(x) = \delta^{(p)}(x - x_0), \qquad (15.11)$$

in agreement with expectations. For the powers $\in \overline{\mathbf{PC}}_\lambda$ the inverse map destroys the generalised functions at infinity, so in this case the translation operator equals its preliminary version.
The translation operator has one eigenfunction

$$\mathcal{T}(x_0,0)\, I(x) = I(x - x_0) = I(x), \qquad (15.12)$$

the unit function $I(x) \in \overline{\mathbf{PC}}_\lambda$. It does not have an eigenfunction $\in \overline{\mathbf{PC}}'_\lambda$.

The infinitesimal generator of the translations is the operator $i\mathcal{D} = \mathcal{K}$. This follows for entire analytic functions from the Taylor series

$$e^{ia\mathcal{K}} f(x) = e^{-a\mathcal{D}} f(x) = \sum_{j=0}^{\infty} \tfrac{1}{j!}(-a)^j\mathcal{D}^j\, f(x) = f(x-a), \qquad (15.13)$$

which converges in this case for all values of a.

For generalised functions the meaning of the operator $e^{-a\mathcal{D}}$ is defined as the corresponding translation operator,

$$e^{-a\mathcal{D}} f(x) := \mathcal{T}(a,0)\, f(x) = f(x-a), \qquad (15.14)$$

$\forall\, f(x) \in \mathbf{GF}_s$. It remains to find out in which cases the computation of the translated function by expansion of the exponential is possible.

It will be seen that this is the case only at infinity. For the $\eta\!\!\!/$-functions one finds after rearranging the summations

$$e^{-a\mathcal{D}}\big(\eta\!\!\!/_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)\big) =$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j} (-)^{j+k} a^j \binom{q+k}{k}\binom{-\alpha-1}{j}^{(k)} \eta\!\!\!/_s^{(\alpha+j,q+k)}(x)\,\mathrm{sgn}^{m+j}(x), \quad (15.15)$$

in agreement with the result (15.8) derived by evaluating the relevant residue directly.
For the operators $\mathcal{X}$ and $\mathcal{D}$ one obtains the transformation properties under translations

$$e^{a\mathcal{D}}\mathcal{X}\, e^{-a\mathcal{D}} = e^{a[\mathcal{D},\cdots]}\mathcal{X} = \mathcal{X} + a\mathcal{I}, \qquad (15.16)$$

and

$$e^{a\mathcal{D}}\mathcal{D}\, e^{-a\mathcal{D}} = e^{a[\mathcal{D},\cdots]}\mathcal{D} = \mathcal{D}, \qquad (15.17)$$

in agreement with the natural interpretation of the translation operator.

15.2 Wave number translations

The wave number translations are defined $\forall\, f(x) \in \mathbf{GF}_s$ by

$$\mathcal{T}(0,k_0)\, f(x) = e^{ik_0x}\bullet f(x), \tag{15.18}$$

$\forall\, f(x) \in \mathbf{GF}_s$. Since e^{ik_0x} is an element of $\mathbf{GF}_s$ this defines the wave number translation as an operator $\mathbf{GF}_s \to \mathbf{GF}_s$. This makes the usual detour by way of the mappings superfluous.
The infinitesimal generator of the wave number translations is the operator $\mathcal{X}$. This follows at once from the result (9.77) that the multiplication operator $\mathcal{X}$ can be identified with the multiplication with the generalised function $x\,\bullet$,

$$\mathcal{X}\, f(x) = x \bullet f(x), \tag{15.19}$$

$\forall\, f(x) \in \mathbf{GF}_s$.
As in the case of the coordinate translation operator it must be verified if the exponential can be expanded. This is the case in the finite, but not at infinity, since application of the expanded form of the exponential form of the operator yields descending sequences of η–functions,

$$e^{ia\mathcal{X}}\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \sum_{j=0}^{\infty} \tfrac{1}{j!}\,(ia)^j\,\eta_s^{(\alpha-j,q)}(x)\,\mathrm{sgn}^{m+j}(x). \tag{15.20}$$

Only descending sequences of η–functions in the finite, and ascending sequences of $\not\eta$–functions at infinity are allowed linear combinations. Descending sequences of $\not\eta$–functions and ascending sequences of η–functions are not allowed.

The wave number translations of the derivatives of the functions $\delta^{(p)}(x)$ are given by

$$e^{ia\mathcal{X}}\delta^{(p)}(x) = \sum_{j=0}^{p} \tfrac{1}{j!}\,(ia)^j\,\mathcal{X}^j\,\delta^{(p)}(x) = \sum_{j=0}^{p} \tfrac{1}{j!}\,(ia)^j\,\delta^{(p-j)}(x), \tag{15.21}$$

which is an explicitly finite linear combination. This result can also be obtained by application of the inverse mapping to (15.20)
At infinity the operator $e^{ia\mathcal{X}}$ is defined as

$$e^{ia\mathcal{X}} f(x) := \mathcal{T}(0,a)\, f(x) = e^{iax}\bullet f(x), \tag{15.22}$$

without the possibility of expanding the exponential.

The wave number translations have only one eigenfunction, the element $\delta(x) \in \mathbf{PC}_\lambda$, which is the Fourier transform of the eigenfunction of the coordinate translations,

$$\mathcal{T}(0,k_0)\,\delta(x) = e^{ik_0x}\bullet\delta(x) = \delta(x), \tag{15.23}$$

in agreement with the corresponding result in distribution theory.

The wave number translation is unitarily equivalent to the translation operator

$$\mathcal{T}(0,a) = \mathcal{F}^{-1}\mathcal{T}(a,0)\mathcal{F}, \tag{15.24}$$

as one sees from the unitary equivalence (2.22–23) of their infinitesimal generators. A proof can be constructed in the same way as given in the corresponding proof of the unitary equivalence of the operators $\mathcal{X}$ and $\mathcal{D}$. Correspondingly there is a symmetry between behaviour in the finite and at infinity.

The combined translations $\mathcal{T}(x_0, k_0)$ are introduced in the last section of this chapter.

15.3 Surface terms

Despite the fact that the unit function is invariant under translations and despite the suggestion going out from the notation

$$\int_{-\infty}^{\infty} dx\ f(x) = \langle\, I(x)\,,\, f(x)\,\rangle, \tag{15.25}$$

the integral of the generalised functions is *not* invariant under translations. The definition of the preliminary integral, at least the part near $x = \pm\infty$, depends on the existence of a preferred point which serves as the origin. The reason for this is the necessity to find asymptotic expansions near $x = \pm\infty$, and the form of the asymptotic expansion depends on the variable, x or $x - x_0$, in which one expands.

Example 15.1 The integral

$$\int_{-\infty}^{\infty} dx\ \mathrm{sgn}(x) = 0, \tag{15.26}$$

equals zero by symmetry, but the integral

$$\int_{-\infty}^{\infty} dx\ \mathrm{sgn}(x - x_0) = |x - x_0|\,\Big|_{-\infty}^{+\infty} = -2x_0, \tag{15.27}$$

is non-zero. Similarly we find

$$\int_{-\infty}^{\infty} dx\ \tanh(x) = \log\cosh(x)\,\Big|_{-\infty}^{+\infty} = 0, \tag{15.28}$$

but

$$\int_{-\infty}^{\infty} dx\ \tanh(x - x_0) = \log\cosh(x - x_0)\,\Big|_{-\infty}^{+\infty} = -2x_0, \tag{15.29}$$

showing that the same occurs for $\mathbf{C}^{\infty}(-\infty, +\infty)$ functions.

The reason for the occurrence of surface terms is obvious. The difference of two bounded functions, which behave asymptotically as $\mathrm{sgn}(x)$, is a function which is (absolutely) integrable in a standard sense. By linearity its integral in the sense of generalised functions has to agree with the standard value of the integral,

$$\int_{-\infty}^{\infty} dx\,\big(\mathrm{sgn}(x) - \mathrm{sgn}(x - x_0)\big) = 2\int_{0}^{x_0} dx\ = 2x_0, \tag{15.30}$$

in agreement with (15.27) and (15.26).
In other words the translation operators are in general not completely unitary.

$$\langle\, \mathcal{T}(x_0, 0)\, f(x)\, ,g(x)\,\rangle \neq \langle\, f(x)\, , \mathcal{T}(x_0, 0)^{\dagger}\, g(x)\,\rangle, \tag{15.31}$$

in the scalar product. By unitary equivalence the same holds for the wave number translations.

The failure of complete unitarity of the translation operators is also expected from the lack of complete selfadjointness of the infinitesimal generators $\mathcal{X}$ and $i\mathcal{D}$. The situation is different in the subspaces $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$. In $\overline{\mathbf{PC}}'_\lambda$ the generators are selfadjoint, the scalar product is translation invariant and the translation operators are unitary. In $\overline{\mathbf{PC}}_\lambda$ there are no generalised functions at infinity. Therefore the scalar product is not translation invariant and the translation operator is not unitary in $\overline{\mathbf{PC}}_\lambda$, and therefore in $\mathbf{GF}_s$.

Example 15.2 The scalar product

$$0 = \langle\, \mathrm{sgn}(x)\, , I(x)\,\rangle \neq \langle\, \mathrm{sgn}(x - x_0)\, , I(x - x_0)\,\rangle = -2x_0, \tag{15.32}$$

is not translation invariant.
Example 15.3 The previous example takes in $\overline{\mathbf{PC}}'_\lambda$ the form

$$\int_{-\infty}^{\infty} dx\, \mathcal{T}(x_0, 0)\, \mathrm{sgn}'(x) = -2x_0 + 2x_0 \int_{-\infty}^{\infty} dx\ \not\eta(x) = 0, \tag{15.33}$$

where the additional contribution of the $\not\eta$–function cancels with the contribution of the finite.

The wave number translation operator is unitary in $\overline{\mathbf{PC}}'_\lambda$, but it is not unitary in $\overline{\mathbf{PC}}_\lambda$, in agreement with the lack of unitarity of translation operators.
Example 15.4 The scalar product

$$\begin{aligned} 0 = \langle\, \delta(x)\, , x^{-1}\,\rangle \neq \langle\, e^{ik_0x}\bullet \delta(x)\, , e^{ik_0x}\bullet x^{-1}\,\rangle = \\ = \langle\, \delta(x)\, , ikx \bullet x^{-1}\,\rangle = ik_0, \end{aligned} \tag{15.34}$$

is not invariant under wave number translations. In $\overline{\mathbf{PC}}'_\lambda$ on the other hand the corresponding scalar product is invariant.

$$0 = \left\langle e^{ik_0x}\eta(x)\, ,\, e^{ik_0x}x^{-1}\right\rangle = \left\langle \cdot\sum_{j=0}^{\infty} \tfrac{1}{j!}(ik_0)^j\,\eta^{(-j.0)}\, ,\, \sum_{k=0}^{\infty}\tfrac{1}{k!}(ik_0)^k x^{k-1}\right\rangle = 0. \tag{15.35}$$

This example is of course the Fourier transform of the previous example.

For the application to physical problems this means that there is a choice. If the ordinary functions which describe the situation are considered as elements of $\overline{\mathbf{PC}}_\lambda$, then surface terms may appear when the origin is shifted. If they are considered as elements of $\overline{\mathbf{PC}}'_\lambda$ these terms are canceled by the contributions of the $\eta\!\!/$-functions at infinity.

Nevertheless the value assigned to the integral does depend on the choice of the origin in $\overline{\mathbf{PC}}_\lambda$, or equivalently on the amount of $\eta\!\!/$ assumed to be present at infinity in $\overline{\mathbf{PC}}'_\lambda$. A unique value can be assigned to the integral of a function behaving as $x^p\,\mathrm{sgn}(x)$ at infinity only when there is a preferred choice for the origin of the coordinate system. The grounds for preferring an origin cannot be supplied by a mathematical analysis. Only the nature of the problem itself can supply a preferred point.

The additional terms which arise when the origin is shifted are called surface terms since they reside on a 'surface' at infinity.

Example 15.5 Surface terms arise frequently when the theory of generalised functions or distribution theory is applied to computations in quantum field theory.

Remark 15.3 In special relativity coordinate space does not have a preferred origin, wave number space on the other hand has one, since the origin in wave number space is not changed by Lorentz transformations. A homogeneous field in coordinate space is homogeneous for all Lorentz observers. The relevant symmetry is the Poincaré group. This remark will recur in Ch. 22 when the regularization of integrals is discussed.

15.4 Phase plane translations

This section is not directly relevant to the main line of the book. It may well be skipped. It outlines an extension which is useful in some applications.

The phase plane translations are defined as the transformations generated by arbitrary linear combinations of the operators $\mathcal{X}$ and $i\mathcal{D} = \mathcal{K}$,

$$\mathcal{T}(a,b) := e^{-a\mathcal{D}+ib\mathcal{X}} = e^{i(a\mathcal{K}+b\mathcal{X})}. \tag{15.36}$$

The Baker-Hausdorff lemma is frequently used In working out exponential operators. In its simplest form it reads

$$e^{\mathcal{A}+\mathcal{B}} = e^{\mathcal{A}}e^{\mathcal{B}}e^{-\frac{1}{2}\left[\mathcal{A},\mathcal{B}\right]} = e^{\mathcal{B}}e^{\mathcal{A}}e^{\frac{1}{2}\left[\mathcal{A},\mathcal{B}\right]}, \quad \text{and} \quad e^{\mathcal{A}}e^{\mathcal{B}} = e^{\mathcal{B}}e^{\mathcal{A}}e^{\left[\mathcal{A},\mathcal{B}\right]}, \tag{15.37}$$

as one sees by straightforward algebra or by consulting a textbook. It holds in this form on condition that

$$[\mathcal{A}, [\mathcal{A}, \mathcal{B}]] = [\mathcal{B}, [\mathcal{A}, \mathcal{B}]] = 0. \tag{15.38}$$

Using the Baker-Hausdorff lemma the general translation (15.36) can also be written in the forms

$$\mathcal{T}(a,b) = e^{iab/2} e^{ia\mathcal{K}} e^{ib\mathcal{X}} = e^{-iab/2} e^{ib\mathcal{X}} e^{ia\mathcal{K}}, \tag{15.39}$$

which can serve to define the general translation operator in terms of the special translations defined above.

Remark 15.4 The general translations defined above generate a projective representation of the translation group of the two dimensional Euclidean plane. Using the Baker-Hausdorff lemma one sees that the group property becomes

$$\mathcal{T}(a_1,b_1)\mathcal{T}(a_2,b_2) = e^{i(a_1b_2-a_2b_1)/2}\mathcal{T}(a_1+a_2, b_1+b_2). \tag{15.40}$$

The representation is only a ray representation, but this is not bothersome in applications such as quantum mechanics where phase factors are irrelevant.

It is possible to introduce more general phase plane transformations by allowing quadratic infinitesimal generators. The operator $\mathcal{X}^2 - \mathcal{D}^2$ generates rotations of the phase plane. This is not worked out. Only the special case of the Fourier operator, which corresponds to a rotation of the phase plane by $\frac{\pi}{2}$ occurs in this book. The unitary equivalence (2.22–23) of $\mathcal{X}$ and $\mathcal{D}$ may serve as an example. Likewise the translations transform under rotations into

$$\mathcal{F}^{-1}\mathcal{T}(a,b)\mathcal{F} = \mathcal{T}(-b,a). \tag{15.41}$$

The parity operator $\mathcal{P} = \mathcal{F}^2$ corresponds to a rotation over π, which is a reflection of the phase plane.

It is in principle simple to work out the general case, but the special functions which appear take up to much space.

The scale transformations generated by $\frac{1}{2}\mathcal{X}\mathcal{D} + \frac{1}{2}\mathcal{D}\mathcal{X}$ are the subject of the next chapter. Together these transformations generate the affine group of transformations of the Euclidean plane which leave the area element unchanged. The group is is represented as a group of transformations of the generalised functions. This remark is not further worked out in this book.

CHAPTER 16

SCALE TRANSFORMATIONS AND HOMOGENEITY

In the previous chapter attention was focussed on the point $x = 0$ as the origin of the coordinate system. In this chapter the point one receives similar attention. Mathematically the number one is a well defined concept.

When analysis is applied to the description of the world as in physical theory, a measurement procedure, involving a system of units, is needed to reduce quantity to number. Different choices of a system of units lead to different numbers and functions. A further discussion is given in Ch. 18. The relation between these numbers is usually a scale transformation.

In this chapter the transformation of generalised functions under scale transformations is investigated, and homogeneity is defined for generalised functions. Indeterminate functions are introduced to simplify the scale transformation and homogeneity properties of the generalised functions. This is useful in many applications of the theory of generalised functions and it simplifies the mathematical structure.

16.1 Definition of the scale transformations

The scale transformation on the generalised functions corresponds in a classical sense to the change in the independent variable $x := ax$, with $a \in \mathbb{R}_+$ a non-negative real number. The scale transformation is indicated by introducing the scale transformation operator $\boldsymbol{\mathcal{S}}(a)$ with the property

$$\boldsymbol{\mathcal{S}}(a)\, x := ax, \tag{16.1}$$

and more generally for ordinary functions,

$$\boldsymbol{\mathcal{S}}(a)\, f(x) := f(ax). \tag{16.2}$$

It is sometimes convenient to introduce the normalized scale operator defined by

$$\bar{\boldsymbol{\mathcal{S}}}(a) := \sqrt{a}\, \boldsymbol{\mathcal{S}}(a), \tag{16.3}$$

which is almost unitary. It will be shown to be a unitary operator on a large subspace of $\mathbf{GF}_s$. If desired the scale transform can be defined for negative non-zero values of the parameter (a) by

$$\boldsymbol{\mathcal{S}}(-a)\, f(x) := \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{S}}(a)\, f(x) = f(-ax), \tag{16.4}$$

where $\boldsymbol{\mathcal{P}}$ is the parity operator. The general case can always be recovered by replacing a by $-a$ or $|a|$ where necessary.

For generalised functions the behaviour under scale transformations is somewhat more complicated than the corresponding transformations of ordinary functions. To get started the definition (16.2) is taken as a preliminary definition of the scale transformations on $\mathbf{PC}_\lambda$. Following the pattern established in previous chapters we begin with a preliminary scale transformation on $\mathbf{PC}_\lambda$, not including $\not\delta$–functions. This is used to find the scale transformations on $\overline{\mathbf{PC}}'_\lambda$. The results are then pulled back to $\overline{\mathbf{PC}}_\lambda$, now including the $\not\delta$–functions.
Application of the definition (16.2) to the powers of x gives

$$\boldsymbol{\mathcal{S}}_{\mathrm{pre}}(a)x^\lambda H(x) := (ax)^\lambda H(ax) := a^\lambda \cdot x^\lambda H(x), \tag{16.5}$$

and

$$\boldsymbol{\mathcal{S}}_{\mathrm{pre}}(a)(-x)^\lambda H(-x) := (-ax)^\lambda H(-ax) := a^\lambda \cdot (-x)^\lambda H(-x). \tag{16.6}$$

where $\boldsymbol{\mathcal{S}}_{\mathrm{pre}}(a)$ is used to indicate the preliminary scale transformation. The scale transformation is now defined on $\overline{\mathbf{PC}}'_\lambda$ by taking residues. For the primed powers and logarithms we find

$$\begin{aligned}\boldsymbol{\mathcal{S}}(a)\big(|x'|^\alpha\big) &:= \operatorname*{Res}_{\lambda=\alpha} q!\,(\lambda-\alpha)^{-q-1}\, a^\lambda |x|^\lambda \operatorname{sgn}^m(x) = \\ &= a^\alpha |x'|^\alpha \sum_{j=0}^{q} \tbinom{q}{j} \log^j|x'| \log^{q-j}(a) \operatorname{sgn}^m(x) + \\ &\quad + 2a^\alpha \sum_{j=0}^{\infty} \tfrac{(-)^j j!\,q!}{(q+j+1)!} \log^{q+j+1}(a) \times \\ &\qquad \times \big((-)^j \eta_s^{(-\alpha-1,j)}(x) - \not\eta_s^{(-\alpha-1,j)}\big) \operatorname{sgn}^m(x).\end{aligned} \tag{16.7}$$

To obtain (16.7) the Taylor series

$$a^\lambda = e^{\lambda \log(a)} = a^\alpha \sum_{j=0}^{\infty} \tfrac{1}{j!} (\lambda-\alpha)^j \log^j(a), \tag{16.8}$$

for the exponential has been substituted. The sum over η–functions appearing in (16.7) is as usual only formally infinite. The expression $\log(a) \in \mathbb{R}$ is simply a real number, the logarithm $\log(a)$ does not have to be interpreted as a generalised function of a.
Specialization of (16.7) to the unit function $I'(x) \in \overline{\mathbf{PC}}'_\lambda$ gives

$$I'(ax) = I'(x) + 2\sum_{j=0}^{\infty} \tfrac{(-)^j}{j+1} \log^{j+1}(a)\big((-)^j \sigma^{(-1,j)} - \not\sigma^{(-1,j)}(x)\big), \tag{16.9}$$

so the function $I'(x)$ is not invariant under scale transformations.

The η–functions of the scaled argument are also defined as a residue by

$$\begin{aligned}
&\boldsymbol{S}(a)\big((-)^q \eta_s^{(\alpha,q)}(x) - \not\eta_s^{(\alpha,q)}(x)\big)\,\mathrm{sgn}^m(x) := \\
&:= \operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{1}{q!}(\lambda+\alpha+1)^q\, a^\lambda |x|^\lambda\, \mathrm{sgn}^m(x) = \qquad (16.10)\\
&= a^{-\alpha-1} \sum_{j=0}^{\infty} \tfrac{(-)^j (q+j)!}{q!\,j!} \log^j(a) \big((-)^{q+j} \eta_s^{(\alpha,q+j)}(x) - \not\eta_s^{(\alpha,q+j)}(x)\big)\,\mathrm{sgn}^m(x).
\end{aligned}$$

This expression is again split into parts in the finite and parts at infinity in the obvious way, by keeping the scale transform of an η–function in the finite in the finite. This gives

$$\eta_s^{(\alpha,q)}(ax)\,\mathrm{sgn}^m(ax) := a^{-\alpha-1} \sum_{j=0}^{\infty} \tfrac{(q+j)!}{q!\,j!} \log^j(a)\, \eta_s^{(\alpha,q+j)}(x)\,\mathrm{sgn}^m(x), \qquad (16/11)$$

in the finite, and

$$\not\eta_s^{(\alpha,q)}(ax)\,\mathrm{sgn}^m(ax) := a^{-\alpha-1} \sum_{j=0}^{\infty} \tfrac{(-)^j (q+j)!}{q!\,j!} \log^j(a)\, \not\eta_s^{(\alpha,q+j)}(x)\,\mathrm{sgn}^m(x), \qquad (16/12)$$

at infinity.

Anticipating the result of transfer to $\overline{\mathbf{PC}}_\lambda$ one expects that the unit function $I(x) \in \overline{\mathbf{PC}}_\lambda$ cannot change under scale transformations as there is only one unit element

$$\boldsymbol{S}(a)\, I(x) := I(x). \qquad (16.13)$$

The scale transformation of the $\theta(x)$ function has to be defined as the opposite of the scaling (16.9) of the generalised function $I'(x) \in \overline{\mathbf{PC}}'_\lambda$ at the point $x = 0$.

$$\theta(ax) := \boldsymbol{S}(a)\,\theta(x) := \theta(x) - 2\sum_{j=0}^{\infty} \tfrac{1}{j+1} \log^j(a)\, \sigma^{(-1,j)}(x). \qquad (16.14)$$

More generally we obtain by extending the smoothness to x^p

$$\theta^{(p)}(ax) := \boldsymbol{S}(a)\,\theta^{(p)}(x) := a^p\,\theta^{(p)}(x) - 2a^p \sum_{j=0}^{\infty} \tfrac{1}{j+1} \log^j(a)\, \sigma^{(-p-1,j)}(x). \qquad (16.15)$$

Further on this will be seen to agree with the result obtained by transferring the scale transformation to the powers in $\overline{\mathbf{PC}}_\lambda$ and taking differences.

The scale transform of the ordinary functions $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ is now defined by combining the preliminary scale transformations with the η–functions generated by applying the scale transformation to the asymptotic expansion at the points $x = 0$, and $\pm\infty$,

$$\boldsymbol{S}(a)\, f'(x) = \boldsymbol{\mathcal{M}S}_{\mathrm{pre}}(a)\, f(x) + \boldsymbol{S}(a)\boldsymbol{\mathcal{M}} f_a(x;0) - \boldsymbol{\mathcal{M}S}_{\mathrm{pre}}(a) f_a(x;0), \qquad (16.16)$$

and idem at $x = \pm\infty$. This completes the definition of the scale transformation on $\overline{\mathbf{PC}}'_\lambda$.

Example 16.1 The scale transform of a $\mathbf{C}^\infty$–function such as

$$\mathcal{S}(a)\, e^{-x^2} = e^{-a^2x^2}, \tag{16.17}$$

does not contain additional generalised functions. A slightly more complicated scale transform is

$$\begin{aligned}\mathcal{S}(a)\, e^{-|x'|} &= \mathcal{S}(a) \sum_{j=0}^{\infty} \tfrac{(-)^j}{j!} |x'|^j = \\ &= e^{-a|x'|} + \sum_{j=0}^{\infty} a^j \sum_{k=0}^{\infty} \tfrac{k!}{(k+j+1)!} \log^k(a)\, \eta_s^{(-j-1,k)}(x),\end{aligned} \tag{16.18}$$

which is again an allowed linear combination of η–functions, with finitely many effectively non-zero terms.

The scale transformation on $\overline{\mathbf{PC}}_\lambda$ is now defined $\forall\, f(x) \in \overline{\mathbf{PC}}_\lambda$ as

$$\mathcal{S}(a)\, f(x) := \mathcal{M}^{-1}\mathcal{S}(a)\mathcal{M}\, f(x). \tag{16.19}$$

following the general pattern established in Ch. 7.

It can be shown, compare Prop. 12.5, that the result does not depend on the choice of $\mathcal{M}_{\mathcal{X}}$ or $\mathcal{M}_{\mathcal{D}}$ to effect the transfer, so the subscript is omitted. The computation is done most conveniently using $\mathcal{M}_{\mathcal{X}}$ however.

The δ–functions transform under scale transformations in the same way as the corresponding distributions

$$\mathcal{S}(a)\, \delta^{(p)}(x) = \delta^{(p)}(ax) = a^{-p-1}\, \delta^{(p)}(x). \tag{16.20}$$

For the powers and logarithms $\in \overline{\mathbf{PC}}_\lambda$ we obtain

$$\begin{aligned}\mathcal{S}(a)\big(|x|^\alpha \log^q|x| \operatorname{sgn}^m(x)\big) &= a^\alpha |x|^\alpha \sum_{j=0}^{q} \tbinom{q}{j} \log^j|x| \log^{q-j}(a) \operatorname{sgn}^m(x) + \\ &+ 2(q+1)^{-1} \log^{q+1}(a) \sum_{p=0}^{\infty} \delta_{-\alpha-1,p}\, \delta_{m,p}^{\mathrm{mod}2}\, \delta^{(p)}(x).\end{aligned} \tag{16.21}$$

The sum in the first part is the binomial expansion of $\big(\log(a)+\log|x|\big)^q$. The additional δ–functions in the second term appear for $\alpha = -p-1$, $m = p$. In particular one finds

$$\mathcal{S}(a)\, |x|^{-1} = a^{-1}|x|^{-1} + 2\log(a)\, \delta(x). \tag{16.22}$$

In the special case $\alpha = 0$, $q = 1$, one recovers the standard result

$$\boldsymbol{\mathcal{S}}(a)\log|x| = \log|ax| = \log|x| + \log(a), \tag{16.23}$$

in agreement with the classical defining property of the logarithm.

For ordinary functions $f(x) \in \overline{\mathbf{PC}}_\lambda$ additional δ-functions appear under scale transformations when the asymptotic expansion of $f(x)$ at $x = 0$ contains terms behaving as

$$x^{-p-1}\log^q|x|\,\mathrm{sgn}(x), \tag{16.24}$$

in agreement with the result found above.

The scale transformation has a fixed point at $x = 0$. By combining the scale transformations with the translations introduced in the previous chapter this fixed point can be shifted. The scale transformation around the point $x = x_0$ is defined as

$$\boldsymbol{\mathcal{S}}(a; x_0, 0) := \boldsymbol{\mathcal{T}}^{-1}(x_0, 0)\boldsymbol{\mathcal{S}}(a)\boldsymbol{\mathcal{T}}(x_0, 0). \tag{16.25}$$

The action of the translated scale transformations is the same as above with all arguments x replaced by $x - x_0$.

At infinity it is convenient to introduce the momentum translated scale transformations by

$$\boldsymbol{\mathcal{S}}(a; 0, p_0) := \boldsymbol{\mathcal{T}}^{-1}(0, p_0)\boldsymbol{\mathcal{S}}(a)\boldsymbol{\mathcal{T}}(0, p_0), \tag{16.26}$$

to simplify the transformation properties of oscillating exponentials at infinity. It is also possible to introduce one-sided scale transformations acting on one side of the point $x = 0$ only.

The scale transformations are now defined on $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ and therefore on $\mathbf{GF}_s$. In the next sections some properties of the scale transform are found.

16.2 Scaling of the scalar product and unitarity

The scalar product in the standard sense should transform under scale transformations as

$$\langle\, \boldsymbol{\mathcal{S}}(a)\, f(x)\, ,\, \boldsymbol{\mathcal{S}}(a)\, g(x)\,\rangle = a^{-1}\,\langle\, f(x)\, ,\, g(x)\,\rangle. \tag{16.27}$$

This is seen for ordinary functions $\in \mathcal{L}_2$ by writing the scalar product in the form of an integral and by applying the change of variable $x := ax$

$$\langle\, f(x)\, ,\, g(x)\,\rangle = \int_{-\infty}^{\infty} dx\; f(x)^*\, g(x) = a\int_{-\infty}^{\infty} dx\; f(ax)^* g(ax). \tag{16.28}$$

In terms of the scale transformation operator this means that it should have the property

$$\boldsymbol{\mathcal{S}}^{\dagger}(a) = a^{-1}\boldsymbol{\mathcal{S}}^{-1}(a) = a^{-1}\boldsymbol{\mathcal{S}}(a^{-1}). \tag{16.29}$$

In terms of the normalized version (16.3) this means that the normalized scale operator $\bar{\boldsymbol{\mathcal{S}}}(a)$ should be unitary,

$$\bar{\boldsymbol{\mathcal{S}}}^{\dagger}(a) \stackrel{?}{=} \bar{\boldsymbol{\mathcal{S}}}^{-1}(a) = \bar{\boldsymbol{\mathcal{S}}}(a^{-1}). \tag{16.30}$$

WRONG!

One verifies by inspection that this is the case in the subspace $\overline{\mathbf{PC}}'_{\lambda}$. As one may anticipate from the lack of selfadjointness of the operators $\boldsymbol{\mathcal{X}}$ and $i\boldsymbol{\mathcal{D}}$ the unitarity does not carry over to $\overline{\mathbf{PC}}_{\lambda}$.

Example 16.2 The scalar product

$$\left\langle\, |x|^{-1} \,,\, I(x) \,\right\rangle = 0$$

transforms under scale transformation into

$$\left\langle\, a^{-1}|x|^{-1} + 2\log(a)\,\delta(x) \,,\, I(x) \,\right\rangle = 2\log(a), \tag{16.31}$$

in disagreement with (16.27).

As with selfadjointness the number of exceptional cases is small. These can be characterized more easily when the scaling of the product has been defined in the next section.

16.3 Operator properties of the scale operator

The scale transformations are a group. For the scale transformations at a point this is the multiplication group of the positive real numbers. By taking $\log(a)$ as the parameter rather than a itself this becomes the addition group of the reals. The translated scale transformations also form a group. The group property takes the form

$$\boldsymbol{\mathcal{S}}(a_1)\boldsymbol{\mathcal{S}}(a_2) = \boldsymbol{\mathcal{S}}(a_1 a_2). \tag{16.32}$$

The infinitesimal generator of the scale transformations is the operator $\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{D}}$. For entire analytic functions one verifies this by expansion in a Taylor series

$$\begin{aligned} a^{\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{D}}} &= e^{\log(a)\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{D}}} f(x) = e^{\log(a)\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{D}}} \sum_{p=0}^{\infty} c_p\, x^p = \\ &= \sum_{p=0}^{\infty} c_p\, e^{p\log(a)} x^p = \sum_{p=0}^{\infty} c_p (ax)^p = f(ax), \end{aligned} \tag{16.33}$$

so formally any Taylor series is transformed by the scale factor.
The infinitesimal generator for the normalized scale operator is the almost selfadjoint operator $\frac{1}{2}(\mathcal{X}\mathcal{D}+\mathcal{D}\mathcal{X})$,

$$\bar{\mathcal{S}}(a) = a^{\frac{1}{2}(\mathcal{X}\mathcal{D}+\mathcal{D}\mathcal{X})} = a^{\frac{1}{2}} a^{\mathcal{X}\mathcal{D}} = \sqrt{a}\,\mathcal{S}(a), \tag{16.34}$$

obtained from $\mathcal{X}\mathcal{D}$ by using the commutation relation (2.12) in agreement with the definitions (16.1) and (16.3).

The next step is to investigate the validity of the expansion of the exponential form of the scale operator when it acts on the generalised functions. For the powers and logarithms one verifies by direct computation that

$$a^{\mathcal{X}\mathcal{D}}\big(|x|^{\lambda}\log^{q}|x|\,\mathrm{sgn}^{m}(x)\big) = |ax|^{\lambda}\big(\log|x| + \log(a)\big)^{q}\,\mathrm{sgn}^{m}(ax), \tag{16.35}$$

with the scaled power given by (16.21). The same result is found for the primed powers.
For the η–functions one finds also that the exponential form of the scale transform acting on an η–function results in an allowed linear combination of η–functions,

$$a^{\mathcal{X}\mathcal{D}}\,\eta_{s}^{(\alpha,q)}(x)\,\mathrm{sgn}^{m}(x) = \mathcal{S}(a)\big(\eta_{s}^{(\alpha,q)}(x)\,\mathrm{sgn}^{m}(x)\big), \tag{16/36}$$

with the scale transform given by (16.10). The same holds for the δ–functions.

For ordinary functions the scaled asymptotic expansions can therefore be found by application of the exponential form of the scale operator. The convergence (if any) is not changed by application of a scale transform, only the radius of convergence is scaled. Between singular points the scaled function is simply found from the definition (16.1) of the scale transformation.
The operators $\mathcal{X}$ and $\mathcal{D}$ transform under scale transformations into the scaled forms

$$\mathcal{S}(-a)\mathcal{X}\mathcal{S}(a) = e^{-\log(a)[\mathcal{X}\mathcal{D},\quad]}\mathcal{X} = a^{-1}\mathcal{X}, \tag{16.37}$$

and

$$\mathcal{S}(-a)\mathcal{D}\mathcal{S}(a) = e^{-\log(a)[\mathcal{X}\mathcal{D},\quad]}\mathcal{D} = a\mathcal{D}, \tag{16.38}$$

in agreement with the expected behaviour.

In the subspace $\overline{\mathbf{PC}}'_{\lambda}$ the operators $\mathcal{X}$ and $\mathcal{D}$, and therefore also the operator $\frac{1}{2}(\mathcal{X}\mathcal{D}+\mathcal{D}\mathcal{X})$, are selfadjoint. Correspondingly the normalized scale operator $\bar{\mathcal{S}}(a)$ is unitary in $\overline{\mathbf{PC}}'_{\lambda}$. In the subspace $\overline{\mathbf{PC}}_{\lambda}$ these operators are not selfadjoint. and there is no reason to expect $\mathcal{S}(a)$ to be unitary in $\mathbf{GF}_s$. By Ex. 16.2 given in the previous section it is seen that unitarity of the scale operator does not hold in $\overline{\mathbf{PC}}_{\lambda}$.

The Fourier operator does not commute with the scale transformations. Instead the Fourier operator transforms the scale operator into its inverse

$$\mathcal{F}^{-1}\mathcal{S}(a)\mathcal{F} = a\mathcal{S}(a^{-1}). \tag{16.39}$$

The additional factor a is due to the normalization of the scale operator. It can be avoided by taking the unitary scale operator.

16.4 Scaling of the product

The product of generalised functions does not commute with scale transformations. It is in general not true that

$$\boldsymbol{S}(a)\,f(x)\bullet\boldsymbol{S}(a)\,g(x)=\boldsymbol{S}(a)\big(f(x)\bullet g(x)\big). \qquad (16.40) \ \textbf{WRONG!}$$

The product of the scale transforms does not in general equal the scale transform of the product.

Example 16.3 One has the scale transform of the product

$$\begin{aligned}\boldsymbol{S}(a)\big(x^{-1}\bullet\operatorname{sgn}(x)\big)&=\boldsymbol{S}(a)\big(|x|^{-1}-2\,\psi(1)\,\delta(x)\big)=\\&=a^{-1}\big(|x|^{-1}+2\big(\log(a)-\psi(1)\big)\,\delta(x)\big),\end{aligned} \qquad (16.41)$$

but the product of the scaled factors is

$$\begin{aligned}\boldsymbol{S}(a)x^{-1}\bullet\boldsymbol{S}(a)\operatorname{sgn}(x)&=a^{-1}x^{-1}\bullet\operatorname{sgn}(x)=\\&=a^{-1}\big(|x|^{-1}-2\,\psi(1)\,\delta(x)\big),\end{aligned} \qquad (16.42)$$

so the results are different and (16.40) does not hold in general.

This can also be seen from the infinitesimal generator acting on a product

$$\begin{aligned}\boldsymbol{X}\boldsymbol{D}\big(f(x)\bullet g(x)\big)&=\boldsymbol{X}\big(\boldsymbol{D}f(x)\bullet g(x)+f(x)\bullet\boldsymbol{D}g(x)\big)\neq\\&\neq\boldsymbol{X}\boldsymbol{D}f(x)\bullet g(x)+f(x)\bullet\boldsymbol{X}\boldsymbol{D}g(x).\end{aligned} \qquad (16.43)$$

The failure of the multiplicative rule (2.41–43) for the $\boldsymbol{X}$ operator makes it impossible to take the generator into the product.

The scale transformation does commute with the product when the factors are suitably restricted, for instance when the factors of the product are restricted to $\overline{\mathbf{PC}}'_\lambda$.

Example 16.4 The previous example in $\overline{\mathbf{PC}}'_\lambda$ takes the form

$$\begin{aligned}\boldsymbol{S}(a)\big(x'^{-1}\bullet\operatorname{sgn}(x)\big)&=\boldsymbol{S}(a)|x'|^{-1}=a^{-1}|x'|^{-1}+\\&+2a^{-1}\sum_{j=0}^{\infty}\tfrac{(-)^j}{j+1}\log^j(a)\big((-)^j\eta^{(0,j)}(x)-\not\eta^{(0,j)}(x)\big),\end{aligned} \qquad (16.44)$$

and by direct computation one verifies that

$$\begin{aligned}\boldsymbol{S}(a)x'^{-1}\bullet\boldsymbol{S}(a)\operatorname{sgn}(x)&=a^{-1}|x'|^{-1}+\\&+2a^{-1}\sum_{j=0}^{\infty}\tfrac{(-)^j}{j+1}\log^j(a)\big((-)^j\eta^{(0,j)}(x)-\not\eta^{(0,j)}(x)\big),\end{aligned} \qquad (16.45)$$

so in this case the scale transformation commutes with the product. One easily verifies this for $\overline{\mathbf{PC}}'_\lambda$ as a whole by computing the scale transform of the basic products of the powers and the η–functions.

Remark 16.1 As noted previously some care is needed with the notation. Multiplication dots should be indicated explicitly since

$$\boldsymbol{\mathcal{S}}(a)\big(x^{-1}\bullet \operatorname{sgn}(x)\big) \neq \boldsymbol{\mathcal{S}}(a)|x|^{-1}, \tag{16.46}$$

in accordance with the previous exceptional case given in Ex. 16.3.

The lack of a product on the distributions which transforms correctly under scale transformations has been an obstacle for the application of products of generalised functions to physical problems. The availability of the product on $\overline{\mathbf{PC}}'_\lambda$ which is scale invariant removes this obstacle.

16.5 Homogeneity of generalised functions

A function is called homogeneous of degree λ when it has the property

$$f(ax) := \boldsymbol{\mathcal{S}}(a)\, f(x) = a^\lambda\, f(x), \tag{16.47}$$

where $a^\lambda = e^{\lambda \log|a|}$ is the principal value of the power. It is called associated homogeneous of degree λ and order n when

$$f(ax) = a^\lambda\, f(x) + a^\lambda \sum_{j=0}^{n} \log^j(a) f_j(x), \tag{16.48}$$

where the functions $f_j(x)$ are also associated homogeneous of degree λ and order n or less. Often the number of terms appearing in the sum is infinite. The degree is then infinite. The term associated homogeneous is often used without mentioning the degree, associated homogeneous is often called simply homogeneous.

Remark 16.2 It may happen that the scaling property (16.48) holds only for $x > 0$ or $x < 0$, or the value of λ may be different on the left and the right side. This may be called left- or right-sided homogeneity, but the distinction will usually not be made. For example, the generalised function

$$x^\lambda H(x), \tag{16.49}$$

is left homogeneous of degree 0, and it is right homogeneous of degree λ.

Homogeneous functions are by definition (16.47) eigenfunctions of the scale transformation operator. Associated homogeneous functions are eigenfunctions of the scale transformation to leading order. In the next section the concept of indeterminacy is introduced to convert associated homogeneity into homogeneity. This makes the distinction irrelevant in many cases.

From the transformation property (16.10) one sees that the η–functions at the origin are associated homogeneous

$$\eta^{(\alpha,q)}(ax) = a^{-\alpha-1}\,\eta^{(\alpha,q)}(x) + a^{-\alpha-1} \sum_{j=1}^{\infty} \tfrac{(q+j)!}{q!\,j!} \log^j(a)\,\eta^{(\alpha,q+j)}(x), \quad (16/50)$$

of degree $-\alpha-1$ and infinite order. The same holds for the $\eta\!\!\!/$-functions at infinity.
For the primed powers x'^{λ} we find from (16.7) that these are

$$(ax')^{\alpha}H'(ax) = a^{\alpha}x'^{\alpha}H'(x) + \\ + 2a^{\alpha}\sum_{j=0}^{\infty}\tfrac{(-)^j}{j+1}\log^j(a)\big((-)^j\eta^{(-\alpha-1,j)}(x) - \eta\!\!\!/_{\uparrow}^{(-\alpha-1,j)}(x)\big), \quad (16.51)$$

also associated homogeneous of degree α of infinite order.
For primed powers and logarithms $\in \overline{\mathbf{PC}}'_{\lambda}$ we find

$$|ax'|^{\alpha}\log^q|ax|\,\mathrm{sgn}^m(ax) = a^{\alpha}|x'|^{\alpha}\sum_{j=0}^{q}\tbinom{q}{j}\log^j(a)\log^{q-j}|x|\,\mathrm{sgn}^m(x) + \\ + 2a^{\alpha}\sum_{j=0}^{\infty}\tfrac{(-)^j q!\,j!}{(q+j+1)!}\log^{q+j+1}(a)\times \\ \times\big((-)^j\eta^{(-\alpha-1,q+j)}(x) - \eta\!\!\!/_{s}^{(-\alpha-1,j)}(x)\big)\,\mathrm{sgn}^m(x), \quad (16.52)$$

so the finite order homogeneity of the standard logarithm is transformed into infinite order.

In accordance with their interpretation the functions $\theta^{(p)}(x) \in \overline{\mathbf{PC}}'_{\lambda}$ are associated homogeneous of infinite order since

$$\theta^{(p)}(ax) = a^p\,\theta^{(p)}(x) - 2a^p\sum_{j=0}^{\infty}\tfrac{1}{j+1}\log^{j+1}(a)\,\sigma^{(-p-1,j)}(x), \quad (16.53)$$

Going back to $\overline{\mathbf{PC}}_{\lambda}$ it is seen from (16.20)

$$\delta\!\!\!/^{(p)}(ax) = a^{-p-1}\,\delta\!\!\!/^{(p)}(x), \quad (16.54)$$

that the $\delta\!\!\!/^{(p)}$-functions are homogeneous of degree $-p-1$, in agreement with the comparable properties of the distributions $\delta^{(p)}$.
For the powers and logarithms $\in \overline{\mathbf{PC}}_{\lambda}$ we have by (16.21)

$$|ax|^{\alpha}\log^q|ax|\,\mathrm{sgn}^m(ax) = a^{\alpha}|x|^{\alpha}\big(\log|x| + \log(a)\big)^q\,\mathrm{sgn}^m(x) + \\ + 2\sum_{p=0}^{\infty}\delta_{\alpha,-p-1}\,\delta_{p,m}^{\mathrm{mod}2}\,a^{-p-1}\log^{q+1}(a)\,\delta\!\!\!/^{(p)}(x), \quad (16.55)$$

so these functions are associated homogeneous of finite order q.

This is in agreement with classical analysis, with the exception of the case $x^{-p-1}\log^q|x|\,\mathrm{sgn}(x)$, where an additional $\delta\!\!\!/^{(p)}$-function appears. This increases the order by one.

Going now to the whole space $\mathbf{GF}_s$, the homogeneity properties are found by subtraction. The $\theta^{(\alpha,q)}(x)$ functions are associated homogeneous of degree α of order ∞, since by subtraction of (16.52) and (16.55) one finds

$$\begin{aligned} \theta_s^{(\alpha,q)}(ax)\,\mathrm{sgn}^m(x) = {} & +a^\alpha \sum_{j=0}^{q} \tbinom{q}{j} \log^j(a)\,\theta_s^{(\alpha,q-j)}(x)\,\mathrm{sgn}^m(x) + \\ & + 2a^\alpha \sum_{p=0}^{\infty} \delta_{\alpha,-p-1}\,\delta_{p,m}^{\mathrm{mod2}} \log^{q+1}(a)\,\delta^{(p)}(x) + \\ & - 2a^\alpha \sum_{j=0}^{\infty} \tfrac{q!\,j!}{(q+j+1)!} \log^{q+j+1}(a)\,\eta_s^{(-\alpha-1,j)}(x)\,\mathrm{sgn}^m(x), \end{aligned} \tag{16.56}$$

in agreement with the special case (16.53) found above. At infinity this simplifies to

$$\begin{aligned} \not{\theta}_s^{(\alpha,q)}(ax)\,\mathrm{sgn}^m(x) = {} & a^\alpha \sum_{j=0}^{q} \tbinom{q}{j} \log^j(a)\,\not{\theta}_s^{(\alpha,q-j)}(x)\,\mathrm{sgn}^m(x) + \\ & + 2a^\alpha \sum_{j=0}^{\infty} \tfrac{(-)^j\,q!\,j!}{(q+j+1)!} \log^{q+j+1}(a)\,\not{\eta}_s^{(-\alpha-1,j)}(x)\,\mathrm{sgn}^m(x), \end{aligned} \tag{16.57}$$

since there are no δ–slash functions in the model.

Remark 16.3 The basic generalised functions

$$\eta^{(\alpha,q)}(x), \quad \text{and} \quad \theta^{(\alpha,q)}(x), \quad \text{and} \quad \delta^{(p)}(x),$$

with point support in the finite, and their counterparts

$$\not{\eta}^{(\alpha,q)}(x), \quad \text{and} \quad \not{\theta}^{(\alpha,q)}(x), \tag{16.58}$$

at infinity are (associated) homogeneous.

The usual operators $\mathcal{X}$, $\mathcal{D}$, $\mathcal{F}$, and $\mathcal{P}$ are homogeneous in the following sense. They convert a homogeneous function of degree λ into a homogeneous function of degree $\lambda+1$, $\lambda-1$, $-\lambda-1$, and λ respectively. This is often convenient to predict beforehand where additional δ–functions will appear in a formula. This can occur only when the parity and degree of homogeneity are correct.

Homogeneity also involves a preferred point $x = 0$. Homogeneous functions with respect to a different point x are defined by replacing the scale transform by the translated scale transform operator.

At infinity the generalised functions $\not{\eta}^{(\alpha,q)}(x)$ are (associated) homogeneous.

The generalised functions $e^{ikx}\not{y}^{(\alpha,q)}(x)$ are eigenfunctions of $\boldsymbol{S}(a;0,p_0)$, the momentum shifted scaling operator.

The terms which are allowed in the asymptotic expansions of generalised functions by the requirement (4.1) are all (associated) homogeneous. The generalised function is homogeneous only when all terms in its asymptotic expansion have the same order. This can happen only when the number of terms in the asymptotic expansion is finite.

For homogeneous generalised functions the power counting of the degree of divergence is used with the standard meaning although divergence is an irrelevant concept for generalised functions. Thus a limit is said to be logarithmically, linearly, quadratically $\cdots$ divergent when the it concerns a function, whose leading term is associated homogeneous of degree 0, -1, -2, $\cdots$ in the finite and/or 0, 1, 2, $\cdots$ at infinity. Likewise for an integral with an integrand of degree -1, -2, -3, $\cdots$ in the finite, or of degree -1, 0, 1, $\cdots$ at infinity.

In the next chapter the distinction between homogeneous and associated homogeneous is changed by introducing indeterminate generalised functions, for which the distinction vanishes.

CHAPTER 17

INDETERMINACY CALCULUS

The generalised functions, as defined in the previous chapter, do not possess satisfactory transformation properties under scale transformations. The scale transform of a product does not equal the product of the scale transformed factors, and the scale operator is not unitary in the scalar product. This becomes important when it is necessary to evaluate expressions which are divergent in a standard sense.

The associated terms generated by the scale transformation are often a nuisance, in particular in cases where it can be seen beforehand that these terms will not contribute to a final result. Carrying them along in the meantime can involve much useless work.

The scale transform can be made unitary, the product can be made to commute with scale transformations, and the superfluous associated terms can be avoided by introducing indeterminate generalised functions.

17.1 Indeterminate generalised functions

In $\overline{\mathbf{PC}}_\lambda$ the indeterminate logarithm is defined as an equivalence class of generalised functions by

$$ {}_i\log|x| := \log|x| - \boldsymbol{C}\, I(x), \tag{17.1}$$

for any value of the indeterminate constant $\boldsymbol{C}$. The indeterminate logarithm is the orbit of the generalised function $\log|x|$ under the group of scale transformations. Usually the generalised function $I(x)$ will be omitted to shorten the notation.

Remark 17.1 The indeterminate logarithm can be looked at in a different way. In standard analysis the logarithm may be defined by

$$\log|x| := \int_1^{|x|} dy\, y^{-1}. \tag{17.2}$$

The indeterminate logarithm is obtained by replacing the one at the lower limit of integration by an unspecified different point. For generalised functions the indeterminate logarithm can also be found as

$$ {}_i\log|x| := \int_{e^{\boldsymbol{C}}}^{|x|} dy\, y^{-1}, \tag{17.3}$$

as an integral with an unspecified lower limit. This point will return in the next section, where the indeterminate form of the fundamental theorem of the calculus is given in its indeterminate form.

Remark 17.2 For physical quantities the indeterminate logarithm can also written as

$$ {}_i\log|x| = \log\big(|x|/\mathrm{u}_x\big) - \boldsymbol{C}, \tag{17.4} $$

where u_x is an arbitrary unit for measuring the the physical quantity x. This has the advantage that the argument of the logarithm is explicitly dimensionless. Suppose it is useful to compare different lengths on a logarithmic scale. The expression $\log(\mathit{length})$ is undefined. One may take

$$ \log\left(\frac{\mathit{length}}{1\,\mathrm{meter}}\right) \quad \text{or} \quad \log\left(\frac{\mathit{length}}{1\,\mathrm{inch}}\right), \tag{17.5} $$

as the numerical value of $\log(\mathit{length})$. In cases where the units are irrelevant one does not have to specify them. The indeterminate logarithm can be used for this purpose. This remark will be worked out in the next chapter.

The indeterminacy of the logarithm can be extended to other generalised functions. This is done by noting that the scale transformation as defined in Sec. 16.1 is an expansion in powers of $\log(a)$. The indeterminate generalised functions are now defined by replacing $\log(a)$ by $\log(a) - \boldsymbol{C}$ in their scale transforms, followed by putting a equal to one, or more formally

$$ {}_if(x) := \boldsymbol{S}(e^{-\boldsymbol{C}})\, f(x). \tag{17.6} $$

In $\overline{\mathbf{PC}}'_\lambda$ this leads to (bringing the irrelevant exponential factor to the other side for clarity)

$$ \begin{aligned} e^{\alpha \boldsymbol{C}}{}_i|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) &= |x'|^\alpha \big(\log|x| - \boldsymbol{C}\big)^q + \\ &+ 2\sum_{j=0}^{\infty} \tfrac{q!\,j!}{(q+j+1)!} \boldsymbol{C}^{q+j+1} \big((-)^j \eta_s^{(-\alpha-1,j)}(x) - \eta_s^{(-\alpha-1,j)}(x)\big), \end{aligned} \tag{17.7} $$

for the indeterminate powers and to

$$ e^{-(\alpha+1)\boldsymbol{C}}{}_i\eta^{(\alpha,q)}(x) := \eta^{(\alpha,q)}(x) + \sum_{j=1}^{\infty} \tfrac{(-)^j (q+j)!}{q!\,j!} \boldsymbol{C}^j\, \eta^{(\alpha,q+j)}(x), \tag{17/8} $$

for the indeterminate η functions.
Correspondingly in $\overline{\mathbf{PC}}_\lambda$ there are the indeterminate powers

$$ \begin{aligned} e^{\alpha \boldsymbol{C}}{}_i|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) &= |x|^\alpha \sum_{j=0}^{q} (-)^j \tbinom{q}{j} \boldsymbol{C}^j \log^{q-j}|x| \operatorname{sgn}^m(x) + \\ &+ 2(-)^{q+1}\boldsymbol{C}^{q+1} \sum_{p=0}^{\infty} \delta_{\alpha,-p-1}\, \delta_{p,m}^{\mathrm{mod}2}\, \delta^{(p)}(x), \end{aligned} \tag{17.9} $$

either by definition or by the usual inverse mapping from $\overline{\mathbf{PC}}'_\lambda$. It may be noted that the sum in (17.9) has at most one non-zero term, the sum in (17.8) is infinite, so the indeterminacy is less in $\overline{\mathbf{PC}}_\lambda$.
The $\delta^{(p)}$–functions $\in \overline{\mathbf{PC}}_\lambda$ are determinate.

$$e^{-(p+1)\boldsymbol{C}}{}_i\delta^{(p)}(x) = \delta^{(p)}(x), \tag{17.10}$$

so the notation ${}_i\delta^{(p)}$ will not be used.

The $\boldsymbol{C}$'s in the exponentials never contribute to scalar products, so the exponentials may be omitted without changing any numerical result. These exponentials are omitted in the following to save superfluous writing. This is equivalent to the preceding prescription.

Remark 17.3 Strictly speaking one should introduce notation such as

$${}_i\eta^{(\alpha,q)}(x;\boldsymbol{C}) \quad \text{instead of} \quad {}_i\eta^{(\alpha,q)}(x), \tag{17.11}$$

with the indeterminate constant indicated explicitly. The same should be done for the powers, products (compare Rem. 17.8), and also for indeterminate integrals. This will not be done in this book. In actual computations it is quite easy to keep track of the scaling of the various factors without using this cumbersome notation.

An advantage of having indeterminate functions is that associated homogeneous functions can be made homogeneous by defining the scale transformations as

$$\boldsymbol{S}(a)_i f(x;\boldsymbol{C}) = \boldsymbol{S}(a)\boldsymbol{S}(e^{-\boldsymbol{C}})\, f(x) = \boldsymbol{S}(a\cdot e^{-\boldsymbol{C}})\, f(x) = {}_i f(x;\boldsymbol{C}+\log(a)), \tag{17.12}$$

with the indeterminate constant changed to

$$\boldsymbol{C} := \boldsymbol{C} + \log(a). \tag{17.13}$$

This is the basic rule of the indeterminacy. In shortened notation one may write

$$\boldsymbol{S}(a)\,{}_i\eta^{(\alpha,q)}(x) = a^{-\alpha-1}\,{}_i\eta^{(\alpha,q)}(x), \tag{17.14}$$

with the understanding that the substitution (17.13) has been carried out.

The order of the associated homogeneity of the determinate function is equal to the degree of the highest power of $\boldsymbol{C}$ which appears in the corresponding indeterminate function. In the following the explicit distinction between determinate and indeterminate will be omitted. No confusion should result. The indeterminacy is signalled by the explicit appearance of a power of the indeterminate constant.

Remark 17.4 At first sight this leads to confusing notation such as

$$|x|^{-1} = |x|^{-1} - 2\boldsymbol{C}\,\delta(x), \tag{17.15}$$

which should be interpreted as

$$_i|x|^{-1} = |x|^{-1} - 2\boldsymbol{C}\,\delta(x), \tag{17.16}$$

This notation is actually convenient when one is used to it.

It is possible to use the indeterminacy in this informal way since indeterminate final results are usually meaningless.

Adding the indeterminate constant to formulæ adds information which keeps track of the scale transformation properties. Information is lost by replacing $\boldsymbol{C} + 37$ by $\boldsymbol{C}$, or by $\boldsymbol{C} + 1$. but this is irrelevant when the final result is indeterminate. If the final result yields $\boldsymbol{C} - \boldsymbol{C} = 0$ it is necessary to keep all numbers occurring together with the $\boldsymbol{C}$ consistently. This is also advisable in the calculation of intermediate results, unless the final result is known beforehand to be indeterminate.

The indeterminate form of an arbitrary generalised function at a point is obtained as the indeterminate form of its asymptotic expansion. A generalised function is defined to be determinate when the indeterminate constant $\boldsymbol{C}$ (and therefore its powers) is absent from its indeterminate version. The terms in its asymptotic expansion are therefore homogeneous but not associated homogeneous.

Example 17.1 The generalised function

$$f(x) := e^{-x^2}, \tag{17.17}$$

$\in \overline{\mathbf{PC}}_\lambda$ is determinate, the function

$$f(x) := |x|^{-1}\,e^{-|x|}, \tag{17.18}$$

either $\in \overline{\mathbf{PC}}_\lambda$, or $\in \overline{\mathbf{PC}}'_\lambda$ is indeterminate. The elements of $\overline{\mathbf{PC}}'_\lambda$ are indeterminate, unless they are $\mathbf{C}^\infty$ functions of rapid decrease at infinity. (The space $\mathcal{S}$ of Schwartz).

Example 17.2 The function

$$f(x) := x^{-1} = {}_if(x)$$

considered as an element $\in \overline{\mathbf{PC}}_\lambda$ is determinate, the function

$$f(x) := |x|^{-1}$$

is indeterminate. Its indeterminate version is

$$_if(x) := |x|^{-1} - 2\boldsymbol{C}\,\delta(x), \tag{17.19}$$

Both functions are indeterminate as elements $\in \overline{\mathbf{PC}}'_\lambda$, with the corresponding indeterminate versions

$$ {}_i x'^{-1} = |x'|^{-1} - 2\sum_{j=0}^{\infty} \boldsymbol{C}^{j+1}\big((-)^j \sigma^{(0,j)}(x) - \not\sigma^{(0,j)}(x)\big), \qquad (17.20) $$

and

$$ {}_i|x'|^{-1} = x'^{-1} - 2\sum_{j=0}^{\infty} \boldsymbol{C}^{j+1}\big((-)^j \eta^{(0,j)} - \not\eta^{(0,j)}(x)\big), \qquad (17.21) $$

involving η–functions.

17.2 Operators on indeterminate functions

The action of the operators on the indeterminate functions follows at once from the known action on the determinate functions. Many formulæ can be simplified in cases where terms occurring together with the indeterminate constant can be ignored

Example 17.3 In the Fourier transform (6.69) of the (indeterminate) η–function, with $\alpha \neq p \in \mathbb{N}$,

$$ \mathcal{F}\,\eta_s^{(\alpha,q)}(x) = -2\sum_{j=0}^{\infty} \tfrac{(q+j)!}{q!}\, c_j(-\alpha-1,0)\,\not\eta_s^{(-\alpha-1,q+j)}(x), \qquad (17.22) $$

infinitely terms can usually be omitted leaving

$$ \begin{aligned} \mathcal{F}\,\eta_s^{(\alpha,q)}(x) &= 2\,\Gamma(-\alpha)\sin\tfrac{\pi}{2}(\alpha+1)\,\not\eta_s^{(-\alpha-1,q)}(x) + \\ &+ \cdots \boldsymbol{C}\,\not\eta_s^{(-\alpha-1,q+1)}(x) + \cdots \boldsymbol{C}^2 \cdots, \end{aligned} \qquad (17.23) $$

since the $\boldsymbol{C}$'s have to cancel in the final result. When the leading term happens to vanish it is necessary to take an additional term. With some experience it is usually possible to see beforehand which terms are relevant.

The indeterminate functions can also be shifted, again by replacing x by $x - x_0$. The shifted indeterminate functions are eigenfunctions of the translated scaling operator. It is now possible to define an indeterminate equivalent for the ordinary functions both in $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$. This is done by replacing everywhere the asymptotic expansion by its indeterminate equivalent. This is possible by virtue of the fact that all terms in the allowed asymptotic expansions are (associated) homogeneous. Since no indeterminacy occurs at the $\mathbf{C}^\infty$ points this replacement has to be made only at the finitely many singular points. It is allowed to choose a different $\boldsymbol{C}_j$ for each side of each singular point. All the $\boldsymbol{C}_j$'s can be made equal since a scale transformation at x_j changes C to $\boldsymbol{C} + \log(a_j)$ with a_j arbitrary. Therefore the $\boldsymbol{C}_j$ will be dropped and the same $\boldsymbol{C}$ is used at every singular point. This includes ∞ since $\boldsymbol{C}$ can be changed by by a scale transformation at an arbitrary finite point.

Remark 17.5 In this book only the simplest case, in which the scaling is the same at every point, is considered. More complicated cases are easily added by introducing a gauge field to relate the scaling at different points.

17.3 Indeterminate products

The main shortcoming of the product defined in Ch. 9 is the lack of proper transformation of the product under scale transformations. This is a consequence of the choice of standardization (9.101) of the map $\mathcal{M}_{\mathcal{X}}$, and thereby of the product. The map $\mathcal{M}_{\mathcal{X}}$ does not commute with scale transformations.

Example 17.4 For the simplest case $\delta^{(p)}(x)$ one obtains

$$\mathcal{S}(a)\mathcal{M}_{\mathcal{X}}\,\delta(x) = \mathcal{S}(a)\,\eta(x) = a^{-1}\,\eta(x) + a^{-1}\sum_{j=1}^{\infty}\log^j(a)\,\eta^{(0,j)}(x), \quad (17.24)$$

but

$$\mathcal{M}_{\mathcal{X}}\mathcal{S}(a)\,\delta(x) = a^{-1}\,\eta(x), \quad (17.25)$$

so the results are different.

The remedy is now obvious. It is sufficient to change the standardization of the product by defining a new ${}_i\mathcal{M}_{\mathcal{X}}$ by

$${}_i\mathcal{M}_{\mathcal{X}} := \mathcal{S}(e^{C})\mathcal{M}_{\mathcal{X}}\mathcal{S}(e^{-C}), \quad (17.26)$$

and correspondingly for the inverse.

For the generalised functions $\delta^{(p)}$ and $|x|^{\alpha}$ the new standardization is

$${}_i\mathcal{M}_{\mathcal{X}}\,\delta^{(p)}(x) = \sum_{j=0}^{\infty}(-\boldsymbol{C})^j\,\eta^{(p,j)}, \quad (17.27)$$

and

$$\begin{aligned}{}_i\mathcal{M}_{\mathcal{X}}|x|^{\alpha}\log^q|x|\,\mathrm{sgn}^m(x) = |x'|^{\alpha}\log^q|x|\,\mathrm{sgn}^m(x) + & \quad (17.28)\\ +\sum_{j=0}^{\infty}(-\boldsymbol{C})^{q+j+1}\,\eta_s^{(-\alpha-1,q)}(x)\,\mathrm{sgn}^m(x).&\end{aligned}$$

This will be referred to as the indeterminate standardization when it it necessary to make a distinction with respect to the old determinate standardization.

From the transformation of the scale operator under Fourier (16.39) and the relation (9.20) between $\mathcal{M}_{\mathcal{D}}$ and $\mathcal{M}_{\mathcal{X}}$ one sees that the corresponding ${}_i\mathcal{M}_{\mathcal{D}}$ is given by

$${}_i\mathcal{M}_{\mathcal{D}} = \mathcal{S}(e^{-C})\mathcal{M}_{\mathcal{D}}\mathcal{S}(e^{C}). \quad (17.29)$$

The new mapping is not completely invariant under scale transformations

$$\mathcal{S}(a^{-1})_i\mathcal{M}_{\mathcal{D}}\mathcal{S}(a) = \mathcal{S}(e^{-(C+\log(a))})_i\mathcal{M}_{\mathcal{D}}\mathcal{S}(e^{C+\log(a)}), \tag{17.30}$$

but the scale transformation merely replaces C by $C + \log(a)$, in agreement with (17.13). Therefore in the indeterminate sense defined in the previous section the map ${}_i\mathcal{M}_{\mathcal{D}}$ is invariant under scale transformations.

The product is now defined by complete analogy with the corresponding definitions in Ch. 9. From the scaling (16.37) and (16.38) of the operators $\mathcal{X}$ and $\mathcal{D}$ it is clear that the new map ${}_i\mathcal{M}_{\mathcal{X}}$ again almost commutes with the operator $\mathcal{X}$

$${}_i\mathcal{M}_{\mathcal{X}}\mathcal{X} - \mathcal{X}_i\mathcal{M}_{\mathcal{X}} = -\mathcal{X}_i\mathcal{M}_{\mathcal{X}}P_{\delta(0)}. \tag{17.31}$$

Therefore the operator properties of the product are not changed by the indeterminate standardization. As in the previous section it is not necessary to introduce a special notation such as ${}_i\bullet$ for the new product definition. The presence of explicit C's signals the necessity of using the new product.

In $\overline{\mathbf{PC}}'_\lambda$ the scale transformation commutes with the product. The product of the indeterminate functions in $\overline{\mathbf{PC}}'_\lambda$ equals the indeterminate version of the product.

Remark 17.6 Even though the indeterminacy is easily handled informally (and correctly!) by just adding C's where required, this procedure seems to sloppy and difficult to understand to some readers. For once the C's will be written out in full splendour by defining

$${}_i\mathcal{M}_{\mathcal{D}}(C) := \mathcal{S}(e^{C})\mathcal{M}_{\mathcal{D}}\mathcal{S}(e^{-C}), \tag{17.32}$$

$${}_i\mathcal{M}_{\mathcal{D}}^{-1}(C) := \mathcal{S}(e^{C})\mathcal{M}_{\mathcal{D}}^{-1}\mathcal{S}(e^{-C}), \tag{17.33}$$

$$f(x)\,{}_i\bullet_C\,g(x) := {}_i\mathcal{M}_{\mathcal{D}}^{-1}(C)\big({}_i\mathcal{M}_{\mathcal{D}}(C)\,f(x)\bullet{}_i\mathcal{M}_{\mathcal{D}}(C)\,g(x)\big), \tag{17.34}$$

with the scale of the product indicated explicitly. Application of the scale operator to the product of indeterminate functions, and use of the interchangeability of product and scale in $\overline{\mathbf{PC}}'_\lambda$ gives the result

$$\begin{aligned}\mathcal{S}(a)\big({}_if(x;C)\,{}_i\bullet_C\,{}_ig(x;C) &= \mathcal{S}(a)_if(x;C)\,{}_i\bullet_{C+\log(a)}\,\mathcal{S}(a)_ig(x;C) = \\ &= {}_if(x;C+\log(a))\,{}_i\bullet_{C+\log(a)}\,{}_ig(x;C+\log(a)).\end{aligned} \tag{17.35}$$

In (17.35) he scale of the functions and the product is indicated explicitly.

Formally the product has been made scale invariant by defining a product which depends explicitly on the scale. In the following explicit references to a scale will usually be omitted. Of course one can effect a trivial generalization in (17.35) by replacing C, C, and C by C_1, C_2, and C_3. This does not affect the determinacy or indeterminacy of any result, it only introduces additional arbitrariness.

Usually (17.35) will simply be written as

$$\mathcal{S}(a)\big(f(x) \bullet g(x)\big) = \mathcal{S}(a)\, f(x) \bullet \mathcal{S}(a)\, g(x), \tag{17.36}$$

and consequently

$$\big\langle\, \mathcal{S}(a)\, f(x)\,,\, \mathcal{S}(a)\, g(x)\,\big\rangle = a^{-1}\,\big\langle\, f(x)\,,\, g(x)\,\big\rangle, \tag{17.37}$$

with the indeterminacy and the scaling (17.13) understood.

Since the determinate product of generalised functions and the scale transformation do not commute, it can happen that the indeterminate product of determinate functions is indeterminate and visa versa.

Example 17.5 The following examples show that all possibilities actually occur for the indeterminate product in $\overline{\mathbf{PC}}_\lambda$.

$(d \bullet d = d)$:

$$x^{-1} \bullet x^{-1} = x^{-2}, \tag{17.38}$$

$(d \bullet d = i)$:

$$x^{-1} \bullet \operatorname{sgn}(x) = |x|^{-1} - 2(\psi(1) + \boldsymbol{C})\, \delta(x), \tag{17.39}$$

$(i \bullet d = d)$:

$$\big(|x|^{-1} - 2\boldsymbol{C}\, \delta(x)\big) \bullet \operatorname{sgn}(x) = x^{-1}, \tag{17.40}$$

$(i \bullet d = i)$:

$$\big(|x|^{-1} - 2\boldsymbol{C}\, \delta(x)\big) \bullet I(x) = \big(|x|^{-1} - 2\boldsymbol{C}\, \delta(x)\big), \tag{17.41}$$

$(i \bullet i = d)$:

$$\big(|x|^{-1} - 2\boldsymbol{C}\, \delta(x)\big) \bullet \big(|x|^{-1} - 2\boldsymbol{C}\, \delta(x)\big) = x^{-2}, \tag{17.42}$$

$(i \bullet i = i)$:

$$\big(\log|x| + \boldsymbol{C}\, I(x)\big) \bullet \big(\log|x| + \boldsymbol{C}\, I(x)\big) = \log^2|x| + 2\boldsymbol{C}\log|x| + \boldsymbol{C}^2 I(x). \tag{17.43}$$

The introduction of the indeterminate product serves its purpose. We now have the property

Property 17.1 The indeterminate product commutes with scale transformations

$$\mathcal{S}(a)\big(f(x) \bullet g(x)\big) = \mathcal{S}(a)\, f(x) \bullet \mathcal{S}(a)\, g(x), \tag{17.44}$$

provided that the indeterminate constant is adjusted appropriately by

$$\boldsymbol{C} := \boldsymbol{C} + \log(a), \tag{17.45}$$

as in (17.13). □

As an immediate consequence we have

Property 17.2 The scale transformation operator is unitary (up to the normalization) in the indeterminate scalar product

$$\langle \boldsymbol{S}(a)\, f(x)\, , \boldsymbol{S}(a)\, g(x) \rangle = a^{-1} \langle f(x)\, , g(x) \rangle, \tag{17.46}$$

again with the substitution (17.45). □

Example 17.6 The previous example Ex. 16.2 of a non-unitary scalar product now becomes

$$\langle |x|^{-1} - 2\boldsymbol{C}\, \delta(x)\, , I(x) \rangle = -2\boldsymbol{C}, \tag{17.47}$$

and idem scaled

$$\langle |x|^{-1} - 2\boldsymbol{C}\, \delta(x) - 2\log(a)\, \delta(x)\, , I(x) \rangle = -2\boldsymbol{C} - 2\log(a), \tag{17.48}$$

in agreement with (17.45).

17.4 More indeterminacy

The whole content of this book, and in fact the whole of analysis, can be written either in determinate or indeterminate form. Which form is preferable depends on the relevance of scale transformations.

The concept of the value of a generalised function can be extended to the indeterminate generalised functions introduced in Sec. 17.1. Obviously the values of indeterminate generalised functions may become indeterminate. The indeterminate logarithm $\in \mathbf{PC}_\lambda$ has the values

$$
\begin{aligned}
{}_i\log(0+) &= \langle \eta_\downarrow(x)\, , {}_i\log(x) \rangle = -\boldsymbol{C}, & (17.49)\\
{}_i\log(0) &= \langle \delta(x)\, , {}_i\log|x| \rangle = \psi(1) - \boldsymbol{C}, & (17.50)
\end{aligned}
$$

The indeterminate logarithm $\in \mathbf{PC}'$ has the values

$$
\begin{aligned}
{}_i\log'(0+) &= \langle \eta_\downarrow(x)\, , {}_i\log|x'| \rangle = -\tfrac{1}{2}\boldsymbol{C}, & (17.51)\\
{}_i\log'(0) &= \langle \delta(x)\, , {}_i\log|x'| \rangle = 0, & (17.52)
\end{aligned}
$$

as one sees by substituting the explicit form of the indeterminate logarithm. The values for the primed functions again have the additional factor $\frac{1}{2}$. The other half of the indeterminacy is located in the corresponding θ–functions. More generally we obtain for the value of the unprimed powers of the logarithm at $x = 0$

$$
\begin{aligned}
\left({}_i|x|^\alpha \log^q|x| \operatorname{sgn}^m(x)\right)\Big|_0 &= \langle \delta(x)\, , {}_i|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \rangle = \\
&= \tfrac{2q!}{\pi}\, \delta_{\alpha,0}\, \delta_{m,0}^{\mathrm{mod}2} \sum_{j=0}^{q} (-)^{q+j} \tbinom{q}{j}\, c_{j+1}(0,0) \boldsymbol{C}^{q-j}. & (17.53)
\end{aligned}
$$

The limiting values in the unprimed case are

$$
\begin{aligned}
{}_i|x|^\alpha \log^q|x| \operatorname{sgn}^m(x)\Big|_{0+} &= \big\langle \eta_\downarrow(x)\,,\, {}_i|x|^\alpha \log^q|x| \operatorname{sgn}^m(x) \big\rangle = \\
&= (-\boldsymbol{C})^q\, \delta_{\alpha,0}^{\mathrm{mod}2}\,. \qquad (17.54)
\end{aligned}
$$

There is again an additional factor $\frac{1}{2}$ in the primed case. The corresponding determinate values of the logarithm are recovered by omitting the $\boldsymbol{C}$'s.

In Ch. 13 the limit was defined at a fixed value (equal to one) of the scale parameter. It is of course also possible to define an indeterminate limit by

$$
{}_i\!\operatorname*{Lim}_{x\downarrow 0} f(x) := a \big\langle \eta_\downarrow(ax)\,,\, f(x) \big\rangle. \qquad (17.55)
$$

Rewriting this in integral notation

$$
{}_i\!\operatorname*{Lim}_{x\downarrow 0} f(x) := \int_0^b d(ax)\ \eta_\downarrow(ax)\, f(x), \qquad (17.56)
$$

makes the reason for the factor a in (17.55) clear.
Substitution of the explicit form of the scaled $\eta_\downarrow$–function (16.10),

$$
\eta_\downarrow(ax) = a^{-1} \sum_{j=0}^{\infty} \log^j(a)\, \eta_\downarrow^{(0,j)}(x), \qquad (17.57)
$$

shows that the difference between the determinate and the indeterminate version appears only for the logarithm and its powers. For the special case of the first power one obtains

$$
{}_i\!\operatorname*{Lim}_{x\downarrow 0} \log(x) = {}_i\!\log(0+) = \log(a) := -\boldsymbol{C}, \qquad (17.58)
$$

with $\log(a) := -\boldsymbol{C}$ the arbitrary scale factor. It does not matter if the logarithm is made indeterminate or the limit is made indeterminate. The result is the same in both cases,

$$
{}_i\!\operatorname*{Lim}_{x\downarrow 0} \log^q(x) = \operatorname*{Lim}_{x\downarrow 0} {}_i\!\log^q(x) = \log^q(a) := (-)^q \boldsymbol{C}^q, \qquad (17.59)
$$

as one sees from the explicit form (17.57) of the scaled $\eta_\downarrow$–function.

The fundamental theorem of the calculus can also be applied in its indeterminate form by replacing the values of the primitive by the indeterminate values. This results in

$$ {}_i\log(b) = {}_i\int_{0+}^{b} dx\, x^{-1} = \log(b) - \boldsymbol{C}, \tag{17.60} $$

This corresponds with replacing the factor x^λ in Hadamard's prescription for the partie finie by $(e^{-\boldsymbol{C}}x)^\lambda$. The indeterminate form of the fundamental theorem is especially suitable for physical computations, which may have indeterminate results. The indeterminacy of the integral is entirely due to the indeterminacy of the integrand since

$$ \int_{-\infty}^{\infty} dx\; f(x) := \langle\, I(x)\,,\, f(x)\,\rangle, \tag{17.61} $$

and products with the unit function, (or the δ–function), never add indeterminacy.

In the next chapter the consequences of the introduction of indeterminate generalised functions for the applications of the theory to physical problems are elucidated.

CHAPTER 18

INDETERMINACY AND MEASUREMENT

This chapter is not directly related to the main line of this tract. It deals with the application of indeterminate generalised functions to physics.

In the previous chapter indeterminate generalised functions were introduced to simplify the scaling behaviour of the generalised functions. The indeterminate versions of the generalised functions are often more suitable to represent physical quantities than the corresponding determinate expressions.

The first section summarizes some measurement conventions. The determinacy and indeterminacy of the numbers, representing the results of computations of physical quantities, is the subject of the next section.

In the theory of generalised functions the results of physical computations are found as scalar products and integrals. At present it seems that indeterminacy in the results of physical computations cannot be avoided in the case of quantum field theory.

The last section gives some properties which are useful when determinacy or indeterminacy has to be established.

18.1 Measurement and unit systems

This section summarizes the conventional relations between physical quantities and mathematical numbers. It is best skipped by readers interested mainly in the mathematical aspects of the generalised functions.

In order to make it possible to apply (generalised) functions, whose argument and value are (usually real) numbers, to the description of the physical world it is necessary to convert physical quantities to numbers. The need for generalised functions derives from the desire to idealize certain situations, for instance point particles instead of very small particles, in order to simplify physical theories.

The reduction of quantity to number is accomplished by a measurement procedure, using a suitable system of units. This consists of a few fundamental units and many derived units. It is a matter of convention which units are considered to be fundamental. The relations between fundamental and derived units are formalized by a system of dimensions of physical quantities.

In the following the variable x will denote both a real number and a physical quantity converted into a real number by measuring it with respect to the unit u_x. The unit will be written explicitly only when this is necessary for clarity.

A system of units is an artificial construct, which allows much arbitrariness. The simplest change in a unit system is to change the magnitude of an unit. It is also possible to change the quantities considered as fundamental and/or the relations between derived and fundamental quantities. Both changes multiply the real number representing the physical quantity by a constant. This constant may be different for different physical quantities.

Example 18.1 If length is measured in meters instead of in inches the measured value changes by a factor 0.0254. Therefore the numerical value of the argument of a function of the measured value changes by the same factor.

Remark 18.1 So called dimensionless quantities, which are already a number in a suitable unit system, are not immune to changes in the unit system. Whether or not a given quantity is dimensionless and also its value depends on the system of units which is adopted.

Example 18.2 In some cases feet/mile may seem to be a more convenient unit for measuring angles than a degree or a radian. Whether this unit is dimensionless or not is a matter of convention.

Example 18.3 The ratio of an electric field strength and a magnetic field strength is dimensionless in natural unit systems. It is also dimensionless in the Gaussian unit system. In other systems of units (such as the MKSA system) it has the dimension of a velocity.

More complicated changes are also possible. but these are not considered here.

The results of (physical) computations are (usually real) numbers, which are interpreted as physical quantities by inverting the measurement conventions. These real numbers arise in the theory of generalised functions as suitable scalar products.

Example 18.4 The procedure of 'substituting values in the functions' takes the form of a scalar product with a δ–function,

$$f(x_0) := \left\langle \delta(x - x_0)\, , f(x) \right\rangle, \tag{18.1}$$

in agreement with the definitions (13.1) of the values of generalised functions introduced in Ch. 13. It is this step which converts generalised functions into ordinary functions.

Example 18.5 The predictions of quantum mechanics are expectation values, derived from 'amplitudes' $f(x)$ by computing scalar products of the form

$$\langle f(x), \mathcal{O}\, f(x)\rangle := \left\langle f(x)\mathcal{O}\, , f(x) \right\rangle, \tag{18.2}$$

where $f(x)$ is a (generalised) function and $\mathcal{O}$ a (selfadjoint) operator which represents a physical quantity.

Remark 18.2 In physical terminology a quantity is invariant under a change of units if it changes in the correct way. For example if the result of a computation is a length it should change by a factor .0254 under the change of units of Ex. 18.1. If it is a volume it should change by a factor $(.0254)^3$.

Likewise the ratio of field strengths of Ex. 18.3 should not change under the change of units of Ex. 18.1, if a natural type of unit system is employed. It should change by a factor .0254 in the second type of unit system. It might be better to call this covariant.

Deciding what the transformation of a physical quantity under changes in the unit system should be is therefore somewhat complicated.

In physical terminology 'invariant' is used for 'transforms correctly under scale transformations'. In mathematical terminology this might be called covariant.

Natural question: Is the result of computations in physics (involving generalised functions) invariant under changes in the measurement conventions?

In terms of generalised functions this reduces to the question: Is the scalar product, and consequently the assignment of values to generalised functions, invariant under scale transformations?

It would be nice if the assignment of values to generalised functions could be made invariant under scale transformations, but it is known from the previous chapter that this is not possible. The desirable behaviour of the scalar product is lacking in the case

$$\log(0) \mathrel{:\overset{?}{=}} \langle\, \delta(x)\,, \log|x|\,\rangle = \psi(1), \tag{18.3}$$

as one sees from the product (9.60). From the scale transformation property (16.21) it is known that (18.3) transforms into

$$\langle\, \delta(x)\,, \log|ax|\,\rangle = \psi(1) + \log(a), \tag{18.4}$$

under scale transformations. This is a consequence of the lack of complete unitarity of the scale operator.

From a physical standpoint this is unavoidable. An expression such as

$$\int_0^a dx\, x^{-1} \overset{?}{=} \log(a), \tag{18.5}$$

cannot be invariant under a change of units. Mathematically the logarithm can be fixed by defining log(1) to be zero, physically one may put log(1 meter) or log(1 inch) equal to zero, but not both at the same time.

It is therefore necessary to distinguish quantities which transform correctly under scale transformations, which have a physical interpretation, from quantities which do not transform correctly, which are therefore (physically) meaningless.

Even when the result of all computations are fully determinate, the indeterminate functions are still a useful tool for performing physical computations when the choice of a unit system is irrelevant, or when it is convenient to have indeterminate intermediate results.

Example 18.6 The equation for an adiabatic curve

$$p^{\gamma-1}T^{\gamma} = \text{constant}, \tag{18.6}$$

in thermodynamics can be written as

$$(\gamma-1)\,{}_i\!\log(p) + \gamma\,{}_i\!\log(T) = \boldsymbol{C}, \tag{18.7}$$

or

$$(\gamma-1)\,{}_i\!\log(p/\mathrm{u}_p) + \gamma\,{}_i\!\log(T/\mathrm{u}_T) = \boldsymbol{C}, \tag{18.8}$$

where u_p and u_T are arbitrary units of pressure and temperature. It is of course possible to write this in determinate form in terms of the determinate logarithm as

$$(\gamma-1)\log(p/p_0) + \gamma\log(T/T_0) = 0, \tag{18.9}$$

where p_0 and T_0 are the coordinates of an arbitrary point on the adiabatic curve. Of course a point on the curve has to be known for this to be possible.

Remark 18.3 The indeterminate form (18.7) is useful even when no point (p_0, T_0) on the curve is known, since it gives the functional dependence of the physical quantities.

Remark 18.4 It often happens that ratios of physical quantities can be measured to much greater precision than the values of these quantities expressed in a unit system. If desired an arbitrary unit can be introduced in these cases.

Remark 18.5 The habit of introducing units is so strong that it is customary to use the term 'arbitrary units', when a quantity has not been measured in relation to a standardized unit system. This is in practice equivalent to working with indeterminate quantities.

Mathematically inclined readers may wonder how it is possible that the result of a well defined computation of a physical quantity may be indeterminate. Indeterminacy might be acceptable for intermediate results but one might think that physical results should always be determinate. This was indeed the case before the invention of quantum field theory. Even though classical field theory produced divergent results in some cases, these results were always determinate. Quantum field theory by contrast [Lod2] produces divergent results which can be indeterminate. An example occurs in the computation of the electromagnetic mass correction of the electron.

When the result of a computation is indeterminate it is necessary to appeal to experiment to supply a definite value. This may not always be possible. It is not known if this reliance on experimental values for fundamental parameters is a permanent feature of quantum field theory. One might hope for a fully determinate physics, but this hope may well be in vain.

Answer to natural question: The answer is no, but conversely physical meaning is attached only to determinate results. Indeterminate results function as free parameters of the physical theory.

Remark 18.6 It has been thought for a long time that ordinary functions $f(x) : \mathbb{R} \to \mathbb{R}$ should be adequate as a basis for theoretical physics. With the invention of quantum field theory the use of generalised functions has become unavoidable, even though the results of the theory appear in the form of real numbers, which can be compared with the outcome of experiments.

Remark 18.7 Although physical terminology has been used throughout the same considerations would apply to any attempt to generalize the interpretation of the integral as 'the area under the curve' to include the integral (18.9).

In the following determinate and indeterminate are used synonymously with invariant and not invariant under scale transformations.

18.2 Indeterminate computations

Indeterminate expressions occur naturally when scalar products and integrals of the indeterminate generalised functions are computed, or when logarithmically divergent integrals are evaluated. Some examples were given in Sec. 14.3. Some further examples will be given in Ch. 22, where the evaluation of divergent convolution integrals is discussed.

Determinacy of integrals is not the same as convergence. A convergent integral is always determinate, a divergent integral can be both determinate or indeterminate.

Example 18.7 The integral

$$\int_{-\infty}^{a} dx\, x^{\lambda} H(x) = \begin{cases} (\lambda+1)^{-1} a^{\lambda+1} & \lambda \neq -1, \\ \log(a) & \lambda = -1, \end{cases} \tag{18.10}$$

with $\lambda \in \mathbb{C}$ and $a \in \mathbb{R}_{+}$, is determinate for $\lambda \neq -1$, it is indeterminate for $\lambda = -1$, it is divergent, but determinate for $\operatorname{Re}\lambda < -1$.

Example 18.8 When one computes the (logarithmically divergent) integral

$$\int_{0}^{\infty} dx\, x^{-1}\, e^{-x} = \psi(1) - \boldsymbol{C}, \tag{18.11}$$

one obtains an indeterminate result. The same holds for the (linearly divergent) integral

$$\int_0^\infty dx\, x^{-2}\, e^{-x} = -\psi(2) + \boldsymbol{C}, \tag{18.12}$$

but the integral of the sum of the previous integrands gives

$$\int_0^\infty dx\,(1+x)x^{-2}\, e^{-x} = \psi(1) - \boldsymbol{C} - \psi(2) + \boldsymbol{C} = -1, \tag{18.13}$$

which is a determinate result, even though the integral in (18.13) is still linearly divergent.

An advantage of having indeterminate integrals is that the cumbersome difference between the integrals

$$\int_{-\infty}^a dx\, x^{-1} \bullet H(x), \tag{18.14}$$

and

$$\int_0^a dx\, x^{-1}, \tag{18.15}$$

becomes irrelevant. The contributions from the additional δ–functions cancel or do not cancel together with the $\boldsymbol{C}$'s. If they do not cancel the integral is indeterminate and therefore (physically) meaningless.

Indeterminate functions are very suitable for performing physical computations when the results of the computations may or may not be determinate.

Example 18.9 The potential caused by an electrical line charge (The Green function of the Laplace equation in two dimensions), of charge density λ is found as

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{\lambda}{4\pi} \int_{-\infty}^\infty dx\,(x^2+r^2)^{-1/2} = \\ &= \frac{\lambda}{4\pi} \log(\sqrt{x^2+r^2} + x)\Big|_{-\infty}^{+\infty} = \\ &= \frac{-\lambda}{2\pi}\left(\log(r/r_0) + 2\log 2 + \boldsymbol{C}\right). \end{aligned} \tag{18.16}$$

This is given in convenient form by the indeterminate logarithm. The value of the potential at a point is meaningless, and it cannot be fixed in a natural way. It is of course possible to fix the value of the potential by any suitable convention, but this is physically irrelevant.

The electrical field produced by a line charge,

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\lambda}{2\pi}\frac{\boldsymbol{r}}{r^2}, \tag{18.17}$$

is given by the gradient of the potential, which is of course determinate.

Example 18.10 The second order correction to the mass of the electron in the perturbation expansion of quantum electrodynamics is (compare [Lod2]) linearly divergent and indeterminate, the mass correction of the photon is quadratically divergent and determinate. One does not know this before the integrals representing these corrections have been found, for example by explicit computation. The determinacy can then be established, for instance by evaluating the integrals.

The use of an indeterminate constant takes care of the bookkeeping of the scale transformation properties automatically. In complicated computations it is a useful check on the correctness of the computation, since all powers of $\boldsymbol{C}$ have to vanish in the final result, when this is known to be determinate. It is even possible to split a determinate expression into indeterminate parts, and to recombine these later into the determinate final result. (if no errors have been committed)! The introduction of the indeterminacy does not spoil the linearity of the theory of generalised functions.

The manipulation of the indeterminate constant is analogous to the corresponding rule for the indefinite constant which appears in the indefinite integral. By the fundamental theorem of the calculus Ch. 14, the definite integral of a function $f(x)$ is the definite difference

$$\int_a^b dx\ f(x) = \big(F(b) + C\big) - \big(F(a) + C\big) = F(b) - F(a), \tag{18.18}$$

of two values of the indefinite primitive $F(x) + \boldsymbol{C}$. Of course the same value of C has to be chosen in both to obtain the correct result.

Example 18.11 In the definite integral one may take $C + 37$ at the upper limit and C at the lower limit. The definite integral is then equal to the area under the curve plus 37. There is no limit to the complications which can be created arbitrarily.

Likewise there is the possibility of adding a different number to C at each singular point. This corresponds to choosing a different definition of the logarithm at each singular point. It is convenient to define the logarithm in the same way everywhere. Other conventions are possible but not useful in the context of this book.

Remark 18.8 The possibility of obtaining a determinate result by addition of the contributions from different singularities obviously implies the possibility to relate the scaling at different points. Here only the simplest case is considered. It corresponds to a homogeneous space with the same properties everywhere. A more complicated possibility is obtained by introducing a gauge field to relate the scaling at different points.

Remark 18.9 Actually the indeterminate constant is closely related to the indefinite constant which appears in the indefinite integral. Instead of the choice made in this book, one can define the operator $\mathcal{D}^{-1}$ only up to a multiple of the unit function. In the Fourier transformed picture this means that the function x^{-1} has been made indeterminate. The function x^{-1} should then be replaced by $x^{-1} + \boldsymbol{C}\,\delta(x)$. This choice was not made in this work. It would destroys the definite parity of the operator $\mathcal{D}^{-1}$. Only the function $|x|^{-1} - 2\boldsymbol{C}\,\delta(x)$ is allowed to be indeterminate by a $\delta(x)$ function.

If one does not like the indeterminacy can always work with determinate expressions. It is then necessary to find out by other means if the results of computations are scale invariant and (physically) meaningful.

18.3 Determinacy

It is often useful to have criteria to decide beforehand whether generalised functions, scalar products, or integrals are determinate. The contents of the preceding are summarized by listing some properties.

Property 18.1 A generalised function $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ is determinate iff $f'(x) \in \overline{\mathbf{PC}}'_\lambda \cap \mathcal{S} = \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space of $\mathbf{C}^\infty$–functions of rapid decrease. □

An equivalent statement is

Property 18.2 A generalised function $f'(x) \in \overline{\mathbf{PC}}'_\lambda$ is determinate iff

$$\langle\, \eta^{(\alpha,q)}(x - x_0)\,,\, f'(x)\,\rangle = \sum_{p=0}^{\infty} \delta_{\alpha,p}\,\delta_{q,0}\,\delta_{p,m}^{\mathrm{mod}2}\,\langle\, \eta^{(p)}(x - x_0)\,,\, f'(x)\,\rangle$$

$\forall \alpha \in \mathbb{C}$, $\forall x_0 \in \mathbb{R}$, $\forall q \in \mathbb{N}$, and

$$\langle\, e^{ikx} \not\eta^{(\alpha,q)}(x)\,,\, f(x)\,\rangle = 0, \tag{18.19}$$

$\forall k \in \mathbb{R}$, $\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$. □

Likewise in $\overline{\mathbf{PC}}_\lambda$ one finds

Property 18.3 A generalised function $f(x) \in \overline{\mathbf{PC}}_\lambda$ is determinate iff

$$\langle\, \eta_s^{(\alpha,q+1)}(x - x_0)\,,\, f(x)\,\rangle = 0, \tag{18.20}$$

$\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$, $\forall x_0 \in \mathbb{R}$, and

$$\langle\, \sigma^{(-p-1,q)}(x - x_0)\,,\, f(x)\,\rangle = 0, \tag{18.21}$$

$\forall p, q \in \mathbb{N}$, $\forall x_0 \in \mathbb{R}$, and

$$\langle\, e^{ikx} \not\eta_s^{(\alpha,q+1)}(x)\,,\, f(x)\,\rangle = 0, \tag{18.22}$$

$\forall k \in \mathbb{R}$, $\forall \alpha \in \mathbb{C}$, $\forall q \in \mathbb{N}$. □

The various indeterminacies in $\overline{\mathbf{PC}}_\lambda$ and $\overline{\mathbf{PC}}'_\lambda$ do not cancel, so we have

Property 18.4 A generalised function $f(x) \in \mathbf{GF}_s$ is determinate iff $f(x) = f_1(x) + f_2(x)$, with $f_1 \in \overline{\mathbf{PC}}_\lambda$ and $f_2(x) \in \overline{\mathbf{PC}}'_\lambda$ and with $f_1(x)$ and $f_2(x)$ separately determinate.

Verification: From left to right by inspection, from right to left is trivial □

In the scalar product the indeterminacy of different terms may cancel. In view of the definition of the scalar product it is sufficient to consider the determinacy of the integral. An integral can be determinate only if all powers of the indeterminate constant cancel, so we have

Property 18.5 A generalised function $f(x) \in \overline{\mathbf{PC}}'_\lambda$ has a determinate integral iff

$$\sum_{x_j \in \mathbb{R}} \left\langle (-)^k \sigma^{(-1,k)}(x - x_j)\, ,\, f(x) \right\rangle = \left\langle \not\sigma^{(-1,k)}(x)\, ,\, f(x) \right\rangle, \qquad (18.23)$$

$\forall k \in \mathbb{N}$. As usual the sum over singular points is by definition finite and the number of equalities which have to be checked is also effectively finite.

Verification: By inspection it is seen that the generalised functions which occur in the indeterminacy all have a zero integral with the exception of $\eta(x - x_0)$, and $\not\eta(x - x_0)$, so these terms have to add to zero. □

In $\overline{\mathbf{PC}}_\lambda$ the situation is less satisfactory, since there is an asymmetry between the finite and infinity. This leads to

Property 18.6 A generalised function $f(x) \in \overline{\mathbf{PC}}_\lambda$ has a determinate integral iff

$$\sum_{x_j \in \mathbb{R}} \left\langle \sigma^{(-1,k)}(x - x_0)\, ,\, f(x) \right\rangle = 0, \qquad (18.24)$$

$\forall k \in \mathbb{N}$.

Verification: In $\overline{\mathbf{PC}}_\lambda$ only the $\delta^{(0)}(x - x_j)$ function contributes. As for the property 12.4 the equality is the necessary and sufficient condition for the cancellation of the contributions of the $\delta(x - x_0)$ functions □

Since adding the generalised function $\tau^{(-1)}(x)$ does not change the determinacy of the integral

$$\int_{-\infty}^{\infty} dx\, |x'|^{-1}, \qquad (18.25)$$

while it does change the response to a $\sigma^{(-1)}$–function, it is not possible to combine the previous properties into a statement valid for $\mathbf{GF}_s$ as a whole. Examples Ex. 18.8 and Ex. 18.10 provide some cases where cancellation of the indeterminacy occurs.

It is obvious from the properties listed above that only logarithmically divergent integrals contribute to the indeterminacy. More precisely one sees from the properties of the $\sigma^{(-1)}(x)$ functions that it is only the logarithmically divergent terms in the asymptotic expansions that matter. The occurrence of more strongly divergent terms is irrelevant.

Finally one may verify that the different choices for the standardization will affect the outcome of a computation only when the result of the computation is indeterminate.

There is often some arbitrariness in deciding which generalised functions should be used to represent an 'ordinary' function. The determinacy may depend on this choice.

Example 18.12 We have

$$ {}_{i}|x'|^{-1} = |x'|^{-1} - 2\boldsymbol{C}\big(\eta(x) - \not{\eta}(x)\big), \tag{18.26} $$

which does not contribute to the integral, but

$$ |x|^{-1} = |x|^{-1} - 2\boldsymbol{C}\,\not{\delta}(x), \tag{18.27} $$

which does contribute.

Mathematical analysis is no help here. It is necessary to formulate the whole physical theory in terms of generalised functions from the beginning, instead of attempting to translate 'ordinary' functions into generalised functions when mathematical difficulties occur.

CHAPTER 19

CONVERGENCE OF SEQUENCES

In this chapter the scalar product of the generalised functions is used to define a convergence concept on the generalised functions. Formally this definition of convergence is the generalised function equivalent of the weak convergence in Hilbert space. Since the space of generalised functions is enlarged with respect to Hilbert space by the addition of many singular generalised functions with point support, the 'weak' convergence also becomes much stronger. Weak convergence of generalised functions implies for instance pointwise convergence.

It is therefore useful to consider also partial convergence. This is defined as weak convergence with respect to the scalar product with a subset instead of the whole space.

The standard limit concept is not the most suitable limit concept for defining convergence of generalised functions. The generalised limit concept developed in Ch. 13 is more suitable for this purpose.

It will be seen that many standard results are recovered with an adapted interpretation. For instance, a sequence of increasingly peaked functions with integral one will converge to an η–function in a sense to be developed below. There are many sequences which converge to a non-zero limit as generalised functions, even though they converge to zero in the sense of distribution theory. It is also possible to interchange operators and limits, as one is used to in distribution theory.

There is no generalised function equivalent of the strong convergence in Hilbert space induced by convergence in the norm, since there is no norm for the generalised functions.

19.1 Sequences of generalised functions

We consider one parameter families of generalised functions $f(x;a)$, with $a \in \mathbb{R}$, and investigate the limiting behaviour as a tends to a limiting value, for instance for $a\downarrow 0$. In keeping with the standard definition of weak convergence the sequence $f(x;a)$ is defined to converge (weakly) to $f(x)$ for $a\downarrow 0$ when

$$\lim_{a\downarrow 0} \langle f(x;a)\,,g(x)\rangle = \langle f(x)\,,g(x)\rangle, \tag{19.1}$$

$\forall\, g(x) \in \mathbf{GF}_s$.

Weak limits are defined uniquely. This follows immediately from the requirement of non-degeneracy (2.11) of the scalar product. Any generalised function that has zero scalar product with all other generalised functions is by definition zero.

Limits for other values of a are defined analogously. It is usually sufficient to consider the limits $a{\downarrow}0$ and $a{\uparrow}\infty$. For convenience of notation $a \in \mathbb{R}_+$ will be taken as a positive. The case where a runs through the positive integers is easily included as a special case when summation of sequences of numbers has been defined in Ch. 20.

It should be kept in mind that in (19.1) we have a parametrized family of generalised functions of one independent variable. At this stage of the development it is not possible to consider it as a generalised function in two variables. This makes the exposition somewhat unsatisfactory at times, but this cannot be remedied in the context of this book.

Despite the name, 'weak' convergence is actually a strong property for generalised functions. The space $\mathbf{GF}_s$ contains many elements such as $\delta(x)$ which are not contained in the Hilbert space $\mathcal{L}_2$. This makes 'weak' convergence for generalised functions a much stronger property. Since the terms 'weak' and 'strong' are not really appropriate for generalised functions the distinction will be dropped, and only the term convergence will be used.

The convergence defined above is such a strong property that it is useful to consider also partial convergence. A sequence $f(x;a)$ is defined to converge partially to $f(x)$ with respect to a subset $\mathbf{S} \subset \mathbf{GF}_s$ when

$$\lim_{a} \langle f(x;a), g(x) \rangle = \langle f(x), g(x) \rangle, \tag{19.2}$$

$\forall\, g(x) \in \mathbf{S} \subset \mathbf{GF}_s$. The subset $\mathbf{S} \subset \mathbf{GF}_s$ does not have to be a subspace. It is of course possible that the sequence actually converges partially on a larger subset than the subset on which convergence has been proved. A completely convergent sequence converges partially on every subset. In the following the subset $\mathbf{S}$ will be called the test set or the convergence set of the partial convergence.

Example 19.1 By taking for the convergence set $\mathbf{S}$ the subset containing the elements $\delta(x-x_0)$, for all points $x_0 \in \mathbb{R}$, one obtains the generalised function equivalent of pointwise convergence everywhere. Pointwise convergence is obviously not included in weak convergence in Hilbert space.

Convergence in the sense of distribution theory is included as a special case. It corresponds to partial convergence on the usual Schwartz spaces of test functions, such as $\mathcal{S}$ or $\mathcal{D}$, which are subspaces of the generalised functions.

The drawback to having only partial convergence is that the limit of a partially convergent sequence may not be defined uniquely. Any generalised function that has zero scalar product with the whole test set can be added to the limit. For example if the convergence is no stronger than the pointwise convergence of Ex. 19.1, any allowed linear combination of δ–functions can be added to the limit.

The advantage of the concept of partial convergence is that it is frequently unnecessary to establish complete convergence. For computing scalar products it is sufficient to know that the functions which actually occur in the scalar products belong to the convergence set.

A proof of complete convergence will usually proceed through successive stages in which convergence on successively larger subsets is established.

The convergence in the sense of the generalised functions is therefore a very flexible concept. There is no loss of power or convenience with respect to convergence in the sense of distribution theory. If convergence in the sense of distribution theory is adequate one can stop once partial convergence with respect to $\mathcal{S}$ has been proved, if necessary it is possible to do more.

Instead of considering convergence in the symmetrized scalar product it is also possible to consider convergence in the left- or right-first scalar products defined in Ch. 8. This will be seen to hold advantages in some cases.

It has tacitly been assumed so far that the limit needed in the definition of the concept of convergence is the standard limit. This is indeed a possible definition, but it is not the most suitable choice for the generalised functions.

Example 19.2 Consider the 'Gaussian' sequence $f(x;a)$ defined by

$$f(x;a) := a\,e^{-a^2x^2}, \tag{19.3}$$

which is the standard example of a sequence converging (when $a\uparrow\infty$) to $\sqrt{\pi}$ times the $\delta(x)$–function. Therefore it converges partially, with respect to the subset $\mathcal{S} \subset \mathbf{GF}_s$, to both $\delta(x)$ and $\eta(x)$. It is not possible to distinguish these on $\mathcal{S}$. Actually using only $\mathcal{S}$ the limit is as yet undefined by any element in the subspace $\boldsymbol{P}_{\mathcal{M}}\mathbf{GF}_s$, but it will be seen that the convergence can be strengthened.

In order to test for convergence on larger subspaces we compute the scalar product

$$\left\langle a\,e^{-a^2x^2}, |x|^\lambda \right\rangle = a^{-\lambda}\,\Gamma(\tfrac{1}{2}\lambda + \tfrac{1}{2}), \tag{19.4}$$

which follows (substituting $a^2x^2 \to y$) from the definition of the Γ–function.

The result is valid in a classical sense for $\operatorname{Re}\lambda > -1$, and in the sense of generalised functions for $\lambda \neq -2p-1$. One sees that the standard limit of the sequence exists as a finite number only for $\operatorname{Re}\lambda > 0$ or $\lambda = 0$. This restriction is clearly unnecessary when the generalised limit properties (13.36) found in Ch. 13, such as $\operatorname{Lim} a^\lambda = \delta_{\lambda,0}$, are used to define the convergence.

The example given above suggests that a more useful convergence concept is obtained by replacing convergence in the standard sense,

$$f(x;a) \to f(x) \iff \lim_a \left\langle f(x;a)\,, g(x) \right\rangle = \left\langle f(x)\,, g(x) \right\rangle, \tag{19.5}$$

which was tacitly assumed in the definition (19.1), by

$$f(x;a) \to f(x) \iff \operatorname*{Lim}_a \left\langle f(x;a)\,, g(x) \right\rangle = \left\langle f(x)\,, g(x) \right\rangle. \tag{19.6}$$

The distinction will be kept by writing 'Lim' or 'lim' also for limits of sequences.

Strictly speaking this is not a satisfactory way to proceed, since it would be much better to define $f(x;a)$ from the beginning as a generalised function in two variables, but this is not possible in the context of this book. Actually the situation is often better than it seems at first sight. In the non-zero contributions the factors a and x can in many cases be combined in such a way that only functions of argument ax remain.

19.2 Increasingly peaked sequences

In this section special attention is paid to sequences which (according to naïve expectation) converge to localized generalised functions such as the η–function. To get started we return to the sequence (19.3) of Gaussians. The relevant scalar products with $\overline{\mathbf{PC}}_\lambda$ are those which have a singularity at $x = 0$

$$\langle\, a\, e^{-a^2x^2}, \delta^{(2p)}(x)\,\rangle = \tfrac{(-)^p}{p!}\, a^{2p+1}, \tag{19.7}$$

and

$$\langle\, a\, e^{-a^2x^2}, |x|^\lambda\,\rangle = a^{-\lambda}\,\Gamma\,(\tfrac{1}{2}\lambda + \tfrac{1}{2}). \tag{19.8}$$

Taking the analytical limit in (19.7) gives zero, in (19.8) it gives

$$\begin{aligned}\operatorname*{Lim}_{a\uparrow\infty}\ \langle\, a\, e^{-a^2x^2}, |x|^\lambda\,\rangle &= \operatorname*{Lim}_{a\uparrow\infty} a^{-\lambda}\,\Gamma(\tfrac{1}{2}\lambda + \tfrac{1}{2}) = \\ &= \sqrt{\pi}\,\delta_{\lambda,0} = \sqrt{\pi}\ \langle\, \eta(x)\,, |x|^\lambda\,\rangle,\end{aligned} \tag{19.9}$$

which is valid for $\lambda \neq -2p-1$. The value for the exceptional cases is obtained by taking a residue.

Therefore we have established partial convergence $f(x;a) \to \eta(x)$, with respect to the powers $\in \overline{\mathbf{PC}}_\lambda$, and consequently to all functions $\in \overline{\mathbf{PC}}_\lambda$ which have an asymptotic expansion in terms of the simple powers.

This result is easily extended to include logarithmic terms. Taking a residue on λ, which is (at a regular point) equivalent to differentiating equation (19.6) repeatedly with respect to λ, gives

$$\langle\, a\, e^{-a^2x^2}, |x|^\lambda \log^q|x|\,\rangle = a^{-\lambda} \sum_{k=0}^{q} (-)^k \tbinom{q}{k}\, 2^{k-q}\, \Gamma^{(q-k)}\,(\tfrac{1}{2}\lambda + \tfrac{1}{2}) \log^k(a). \tag{19.10}$$

At the singular points $\Gamma^{(j)}$ is replaced by $j!\ \Gamma^{[j]}$, which is the residue defined in appendix **A**.
In the generalised limit only the $k = 0$ term of the sum contributes,

$$\operatorname*{Lim}_{a\uparrow\infty}\ \langle\, f(x;a)\,, |x|^\lambda \log^q|x|\,\rangle = 2^{-q}\,\Gamma^{(q)}(\tfrac{1}{2})\,\delta_{\lambda,0}\,. \tag{19.11}$$

In particular for $q = 1$ and $\lambda = 0$ this gives the result

$$\operatorname*{Lim}_{a\uparrow\infty} \langle a\, e^{-a^2x^2}, \log|x| \rangle = \tfrac{1}{2}\,\Gamma^{(1)}(\tfrac{1}{2}) \neq \langle \delta(x), \log|x| \rangle \neq \langle \eta(x), \log|x| \rangle, \tag{19.12}$$

Comparing the result (19.12) with the corresponding scalar product of the η–functions

$$\langle |x|^{\lambda} \log^q|x|, \eta_s^{(\beta,r)}(x) \rangle = (-)^q \delta_{\lambda,\beta}\, \delta_{q,r}, \tag{19.13}$$

one finds as a result of the limit computation

$$\begin{aligned} \operatorname*{Lim}_{a\uparrow\infty} a\, e^{-a^2x^2} &= \sum_{j=0}^{\infty} (-2)^j\, \Gamma^{(j)}(\tfrac{1}{2})\, \eta^{(0,j)}(x) \\ &= \sqrt{\pi}\, \eta(x) + \sum_{j=1}^{\infty} (-2)^j\, \Gamma^{(j)}(\tfrac{1}{2})\, \eta^{(0,j)}(x). \end{aligned} \tag{19.14}$$

This result for the limit reproduces the scalar products with $\overline{\mathbf{PC}}_{\lambda}$ correctly. The first term in (19.14) is in accordance with expectations, but the occurrence of the other terms might be an unexpected result. The interpretation of these additional terms will become clear in the following.

The coefficients in the linear combination of $\eta^{(0,j)}(x)$ functions depend on the sequence of generalised functions used to approximate the η–function.

Example 19.3 Taking instead of $f(x;a) = a\, e^{-a^2x^2}$ the sequence

$$f(x;a) := a\, e^{-ax} H(x), \tag{19.15}$$

and repeating the computation yields

$$\operatorname*{Lim}_{a\uparrow\infty} a\, e^{-ax} H(x) = \sum_{j=0}^{\infty} (-)^j \Gamma^{(j)}(1)\, \eta_{\downarrow}^{(0,j)}(x), \tag{19.16}$$

and consequently

$$\operatorname*{Lim}_{a\uparrow\infty} a\, e^{-a|x|} = 2 \sum_{j=0}^{\infty} (-)^j \Gamma^{(j)}(1)\, \eta^{(0,j)}(x), \tag{19.17}$$

with coefficients differing from those appearing in (19.14). This is a generally the case.

The situation in the symmetrical theory of generalised functions differs fundamentally from the corresponding situation [Lig] in distribution theory.

In distribution theory the δ–function can be defined as an equivalence class of sequences converging to it, such as (19.17) or (19.15). The distinction between these sequences cannot be made by means of the test functions of distribution theory.

For generalised functions the linear combination of η–functions which appears does depend on the particular approximating sequence. The class of all sequences which converge partially on $\mathcal{S}$ to the η–function is an inherently richer object than the δ–distribution.

The difference between non-equivalent sequences can be observed by taking a scalar product with a function having a $|x|^0 \log^q|x|$ type behaviour at $x = 0$. In the context of distribution theory such test functions are not allowed, since the resulting limit does not exist classically.

The situation outlined above is satisfactory. Equivalent sequences in the sense of distribution theory are not equivalent in the sense of symmetrical generalised functions. Yet there is only one unique η–function. The difference between different sequences can be characterized completely by the different linear combinations of $\eta^{(0,j)}(x)$–functions which appear in the limit.

Remark 19.1 In Fourier transformed language the approximation of δ–functions corresponds to the regularization of integrals s which diverge at infinity. This subject will be dealt with in Ch. 22.

Remark 19.2 The result of taking the limit is not invariant under scale transformations of the parameter a. The generalised limit $\operatorname{Lim}_{a\uparrow\infty} \log(a)$ is not invariant under scale transformations. One can always replace the parameter a by $2a$, which changes the result of the computation by replacing $\log(a)$ by $\log(a) + \log 2$, which gives an additional term in the limit. By straightforward computation one obtains for $f(x;a) := 2a\, e^{-4a^2x^2}$

$$\operatorname*{Lim}_{a\uparrow\infty} 2a\, e^{-4a^2x^2} = \sum_{j=0}^{\infty} \eta^{(0,j)}(x) \sum_{k=0}^{j} \binom{j}{k} (-\log 2)^k\, \Gamma^{(j-k)}(\tfrac{1}{2}), \tag{19.18}$$

which has the same leading term as (19.14). The higher terms are different however. The sequence (19.18) is different from the sequence (19.3).

The examples given above make it again clear why it is not possible to obtain a satisfactory theory of multiplication of generalised functions on the basis of taking suitable limits of sequences. It is not even possible to obtain a satisfactory regularization in his way. (Compare Ch. 22).

The scalar products of δ–like functions with logarithms

$$\langle\, \delta(x)\, , \log|x|\, \rangle = \psi(1), \quad \text{and} \quad \langle\, \eta(x)\, , \log|x|\, \rangle = 0, \tag{19.19}$$

are perfectly well defined, but the values of these scalar products cannot be obtained by the limiting procedure outlined above. Any value can be obtained be choosing a suitable approximating sequence.

The improved approach using the methods developed in this book is possible by relating the definitions of the generalised functions $\log|x|$, $\eta(x)$,

and $\delta(x)$ to the same scale. Everything is derived from the simple powers x^λ. The η–functions with other indices are also obtained easily as limits.
As a special case we compute

$$\operatorname*{Lim}_{a\uparrow\infty} a^\alpha\, e^{-a^2x^2} = \sum_{j=0}^{\infty}(-2)^j j!\; \Gamma^{[j]}(\tfrac{1}{2}\alpha)\, \eta_s^{(\alpha-1,j)}(x) + \\ + \sum_{p=0}^{\infty}\delta_{\alpha,-2p}\, \tfrac{(-)^p}{p!}\, \theta^{(2p)}(x), \tag{19.20}$$

with $\Gamma^{[j]}$ the residue of the Γ–function defined in appendix A. The additional $\theta^{(p)}$–functions arise from the fact that for these special values of α testing with $\delta^{(p)}(x)$ yields a non-zero result in the limit.
In particular for $\alpha = 0$ we obtain

$$\operatorname*{Lim}_{a\uparrow\infty} e^{-a^2x^2} = \theta(x) + \sum_{j=0}^{\infty}(-2)^j j!\; \Gamma^{[j]}(0)\, \sigma^{(-1,j)}(x), \tag{19.21}$$

in agreement with the naïve interpretation of $\theta(x)$ as the function which equals one at $x = 0$ and zero otherwise. The result differs from

$$a^\alpha \operatorname*{Lim}_{a\uparrow\infty} e^{-ax} H(x) = \sum_{j=0}^{\infty}(-)^j j!\; \Gamma^{[j]}(\alpha)\, \eta_\downarrow^{(\alpha-1,j)}(x) + \\ + \sum_{p=0}^{\infty}\tfrac{(-)^p}{p!}\, \delta_{-\alpha,p}\, \theta_\downarrow^{(p)}(x). \tag{19.22}$$

Only the leading terms agree (up to the normalization).

It is a simple matter to obtain a sequence having $\eta^{(0,1)}(x)$ as its leading term in the limit. Any difference of two normalized sequences with leading term $\eta(x)$ will do.

Remark 19.3 In distribution theory it is necessary to approximate derivatives of the δ–distribution by sequences which are designed to have their lower moments equal to zero, otherwise the limits of the scalar products will diverge in a classical sense.

Example 19.4 In distribution theory one has the limits

$$\lim_{a\uparrow\infty} a\, e^{-a^2x^2} = \sqrt{\pi}\;\delta(x), \tag{19.23}$$

and consequently

$$\lim_{a\uparrow\infty} 2a^3x\, e^{-a^2x^2} = \sqrt{\pi}\;\delta^{(1)}(x), \tag{19.24}$$

and differentiating once more

$$\lim_{a\uparrow\infty} (2a^5x^2 - a^3)\, e^{-a^2x^2} = \sqrt{\pi}\; \delta^{(2)}(x). \tag{19.25}$$

It is necessary to use these more complicated forms to avoid divergence of the limits. In the sense of generalised functions (19.25), with 'lim' replaced by Lim, also converges to $\sqrt{\pi}\; \eta^{(2)}(x) + \cdots$, but it is possible to use the simpler form (19.20) with $\alpha = 3$ instead.

19.3 Convergence on $\overline{\mathbf{PC}}'_\lambda$

As the next step one may attempt to extend the convergence to $\overline{\mathbf{PC}}'_\lambda$ by taking residues with respect to λ to obtain $|x'|^\lambda$. As found before the result depends on the order in which the residues are computed.

Using again the notation of the left-first and right-first scalar products to indicate which residue is to be computed first we define

$$\operatorname*{Lim}_a \,\langle\, f(x;a)\,, |x'|^\alpha \,\rangle\!\!\rangle := \operatorname*{Lim}_a \operatorname*{Res}_{\lambda=\alpha} (\lambda-\alpha)^{-1} \langle\, f(x;a)\,, |x|^\lambda \,\rangle, \tag{19.26}$$

and

$$\operatorname*{Lim}_a \,\langle\!\!\langle f(x;a)\,, |x'|^\alpha \,\rangle := \operatorname*{Res}_{\lambda=\alpha} \operatorname*{Lim}_a (\lambda-\alpha)^{-1} \langle\, f(x;a)\,, |x|^\lambda \,\rangle. \tag{19.27}$$

Symmetrization is defined by

$$\operatorname*{Lim}_a \,\langle\, f(x;a)\,, |x'|^\alpha \,\rangle := \tfrac{1}{2} \operatorname*{Lim}_a \operatorname{Res} \cdots + \tfrac{1}{2} \operatorname{Res} \operatorname*{Lim}_a \cdots,. \tag{19.28}$$

As before in Ch. 8 only one term contributes, so the net result is simply the introduction of the usual half.

Taking the residue first substitutes α for λ, and then the Lim is non-zero, taking the Lim first leaves a residue of a zeromorphic function, which is zero.

Scalar products with θ–functions are again found as differences

$$\operatorname*{Lim}_{a\uparrow\infty} \,\langle\, f(x;a)\,, \theta_s^{(\alpha)}(x) \,\rangle = \operatorname*{Lim}_{a\uparrow\infty} \,\langle\, f(x;a)\,, |x|^\alpha \,\rangle - \operatorname*{Lim}_{a\uparrow\infty} \,\langle\, f(x;a)\,, |x'|^\alpha \,\rangle. \tag{19.29}$$

Application of these definitions to the standard example

$$f(x;a) := \pi^{-1/2} a\, e^{-a^2x^2}, \tag{19.30}$$

yields

$$\operatorname*{Lim}_{a\uparrow\infty} \,\langle\!\!\langle f(x;a)\,, |x'|^\lambda \,\rangle = 0, \tag{19.31}$$

and

$$\operatorname*{Lim}_{a\uparrow\infty} \,\langle\, f(x;a)\,, |x'|^\lambda \,\rangle\!\!\rangle = \delta_{\lambda,0}\,. \tag{19.32}$$

Symmetrization gives

$$\operatorname*{Lim}_{a\uparrow\infty} \left\langle f(x;a)\,,|x'|^\lambda \right\rangle = \tfrac{1}{2}\,\delta_{\lambda,0} = \left\langle \eta(x)\,,|x'|^\lambda \right\rangle. \tag{19.33}$$

This extends the convergence of (19.30) to $\overline{\mathbf{PC}}'_\lambda$.
Taking the difference with the result for the unprimed powers yields

$$\operatorname*{Lim}_{a\uparrow\infty} \left\langle f(x;a)\,,\theta_s^{(\lambda)}(x) \right\rangle = \tfrac{1}{2}\,\delta_{\lambda,0} = \left\langle \eta(x)\,,\theta_s^{(\lambda)}(x) \right\rangle, \tag{19.34}$$

in accordance with the scalar product with the η–function.
Putting $\lambda = 0$ gives

$$\operatorname*{Lim}_{a\uparrow\infty} \left\langle f(x;a)\,,\theta(x) \right\rangle = \tfrac{1}{2} = \left\langle \eta(x)\,,\theta(x) \right\rangle \neq \left\langle \boldsymbol{\delta}(x)\,,\theta(x) \right\rangle = 1. \tag{19.35}$$

The leading term of the limit of the sequence is the η–function, not the $\boldsymbol{\delta}$–function. The same is the case for the higher terms in the series since there are no $\boldsymbol{\delta}^{(\alpha,j)}$–functions.
With this extension of the definition of convergence we have

Property 19.1

$$\operatorname*{Lim}_{a\uparrow\infty} a\,e^{-a^2x^2} = \sum_{j=0}^{\infty}(-2)^j\,\Gamma^{(j)}(\tfrac{1}{2})\,\eta^{(0,j)}(x), \tag{19.36}$$

converges completely on $\mathbf{GF}_s$ to its limit. □

It will be seen that most sequences which occur naturally allow for extension of the partial convergence to complete convergence.

19.4 Dirac's limit property

After the rather special examples considered in the previous section it is time to introduce a more general limit property due to Dirac.

In his book [Dir] Dirac mentions several properties (2.1–4) that the δ–function should have. The first three of these are obviously satisfied by the distribution δ, but the fourth, (2.4)

$$\operatorname*{Lim}_{\epsilon\downarrow 0} \epsilon^{-1} f(\epsilon^{-1}x) = \delta(x)\cdot\int_{-\infty}^{\infty} dx\ f(x), \tag{19.37}$$

is usually omitted, since it is somewhat problematical in distribution theory.

Dirac's notation is used in this section, even though it would be more consistent with the notation of the rest of this chapter to write

$$\operatorname*{Lim}_{a\uparrow\infty} a\,f(ax) = \boldsymbol{\delta}(x)\cdot\int_{-\infty}^{\infty} dx\ f(x). \tag{19.38}$$

The first question is how (19.37) should be interpreted. Dirac is not clear on this point, but from the general context it is clear that only conditions which are necessary to give meaning to the right-hand side should be imposed. It is therefore sufficient that $f(x)$ should be (Lebesgue) integrable. An often used example is the rectangular function $f(x;\epsilon) := \epsilon^{-1}H(x)H(\epsilon - x)$ of width ϵ and height ϵ^{-1}, which is worked out in the next example.

It is clear that (19.37) is not valid in the sense of distribution theory with this interpretation. In distribution theory the distribution δ can be defined as an equivalence class of sequences of $\mathbf{C}^{\infty}$–functions, for instance from the Schwartz spaces $\mathcal{S}$ or $\mathcal{D}$. The rectangular function is not an element of these spaces, so Dirac's sequence cannot be used as a definition of the δ–function in this case.

It is possible in the context of distribution theory to give a more useful interpretation to (19.37), by considering the function $f(x)$ as a distribution, which happens to be equal (in the sense of distribution theory) to an ordinary function. With this interpretation the sequence (19.37) becomes a sequence of distributions, which converges to the δ–distribution in the sense of distribution theory. This is a rather weak result, since the convergence is only with respect to the test functions $\in \mathcal{D}$ or $\in \mathcal{S}$.

It is of course possible to work harder and to show that the result can be extended to functions which are continuous at the origin, but this requires more effort, and it does not produce a general result.

The situation is different when the symmetrical theory is used, with the convergence concept defined above. Considering only the positive real axis for convenience, and taking $f(x)$ real valued, one finds for the scalar product with a power

$$\langle\, f(x;\epsilon)\,,\, x^{\lambda}H(x)\,\rangle = \int_0^{\infty} dx\, x^{\lambda}\epsilon^{-1}\, f(\epsilon^{-1}x). \tag{19.39}$$

By substituting $y := \epsilon^{-1}x$ this takes the simple form

$$\langle\, f(x;\epsilon)\,,\, x^{\lambda}H(x)\,\rangle = \epsilon^{\lambda}\int_0^{\infty} dy\; f(y)\, y^{\lambda} := \epsilon^{\lambda}\,\tilde{f}(\lambda), \tag{19.40}$$

From the limit properties (13.36) one sees that the limit $\epsilon{\downarrow}0$ exists when $\tilde{f}(\lambda)$ is analytic at the origin. It is known from the results of Ch. 5 that this is the case when the asymptotic expansion of $f(x)$ at $x = 0+$ does not contain terms of the form $x^{-1}\log^{q}(x)$.
The result of taking the limit (partially on the powers) is

$$\operatorname*{Lim}_{\epsilon\downarrow 0}\epsilon^{-1}\, f(x;\epsilon) = \eta_{\scriptscriptstyle\downarrow}(x)\int_0^{\infty} dy\; f(y), \tag{19.41}$$

in agreement with Dirac's requirement. The result (19.41) holds not only for integrable functions. It holds more generally, the only requirement is the absence of logarithmically divergent terms.

The general result (converging completely) is

$$\operatorname*{Lim}_{\epsilon\downarrow 0} f(x;\epsilon) = \sum_{j=0}^{\infty} (-)^j \tilde{f}^{(j)}(0)\ \eta_{\downarrow}^{(0,j)}(x), \tag{19.42}$$

valid when $\tilde{f}(\lambda)$ is analytic at $\lambda = 0$.

Example 19.5 The rectangle limit. Taking the natural example $f(x) = H(x)H(1-x)$ one obtains

$$\operatorname*{Lim}_{\epsilon\downarrow 0} \epsilon^{-1} H(x)H(\epsilon - x) = \sum_{j=0}^{\infty} j!\, \eta_{\downarrow}^{(0,j)}(x), \tag{19.43}$$

with the first term in agreement with naïve expectations.

Remark 19.4 Dirac's property cannot be extended simply to all generalised functions by considering $f(x;\epsilon) := \epsilon^{\lambda} \boldsymbol{S}(\epsilon^{-1})\, f(x)$ The naïve change of variable $y := \epsilon^{-1}x$ is not justified in general. It was shown in Ch. 16 that the scale operator is not completely unitary. The exception is again given by the functions behaving as $x^{-p-1} \log^q|x| \operatorname{sgn}(x)$ at the origin.

Example 19.6 As a special case one may take the function $f(x) = \delta(x-1)$, or equivalently $\eta(x-1)$. This yields the the intuitively appealing result

$$\operatorname*{Lim}_{\epsilon\downarrow 0} \delta(x - \epsilon) = \eta_{\downarrow}(x), \tag{19.44}$$

without additional higher terms. By comparison with (19.40) one sees that this is (essentially) the only sequence without higher terms. Only the function $\delta(x-1)$ has the function $\tilde{f}(\lambda) = 1$ as its Mellin transform. One may of course add functions with a zero Mellin transform such as $\delta(x-1) - \eta(x-1)$. By computing the required scalar products one verifies that the convergence in (19.44) is complete.

Example 19.7 When we replace $\delta(x-1)$ by $\delta(x-2)$ we obtain

$$\operatorname*{Lim}_{\epsilon\downarrow 0} \delta(x - 2\epsilon) = \sum_{j=0}^{\infty} (-\log 2)^j j!\, \eta_{\downarrow}^{(0,j)}(x), \tag{19.45}$$

so by taking the difference we obtain a sequence with $\eta_{\downarrow}^{(0,1)}(x)$ as leading term,

$$\operatorname*{Lim}_{\epsilon\downarrow 0} \big(\delta(x-\epsilon) - \delta(x - 2\epsilon)\big) = \log 2\ \eta_{\downarrow}^{(0,1)}(x) + \cdots . \tag{19.46}$$

One can also construct sequences converging to logarithmic η-functions

$$\operatorname*{Lim}_{\epsilon\downarrow 0} \epsilon\, \delta^{(1)}(x - \epsilon) = \eta_{\downarrow}^{(0,1)}(x), \tag{19.47}$$

without additional terms. Similarly one can construct sequences converging to the higher logarithmic η-functions with $j > 1$.

Dirac's limit property is easily generalised to scaled sequences of the general form $a^{\lambda} \boldsymbol{S}(a)\, f(x)$. Many of the limits which occur in this chapter, such as (19.3), are special cases of Dirac's limit property.

19.5 Limits at infinity

Instead of increasingly peaked sequences one can also consider sequences which become progressively wider. The method of computation remains the same. Computation of the analog of the examples given in the previous section results in

$$\operatorname*{Lim}_{a\downarrow 0} e^{-a^2x^2} = I'(x) + \theta(x) + \sum_{j=0}^{\infty} 2^{-j} j!\, \Gamma^{[j]}(0)\, \not\phi^{(-1,j)}(x), \tag{19.48}$$

in agreement with the limit calculated in (19.20).

Remark 19.5 In the limit calculation the scalar product with a power has been computed, and the η–functions have been put either at $x = 0$ or $x = \infty$ in such a way that the correct result is obtained. This can also be obtained by splitting the integration interval. The result is the splitting of the Γ–function into incomplete Γ– and γ–functions with the correct limiting behaviour. This has not been written out fully since it leads only to technical complications without additional insight.

An example which will be needed in the next chapter in connection with the analytic properties of the Hilbert transform is

$$\operatorname*{Lim}_{a\downarrow 0} e^{-ax} H(x) = \theta_{\downarrow}(x) + H'(x) + \sum_{j=0}^{\infty} j!\, \Gamma^{[j]}(0)\, \not\eta_{\uparrow}^{(-1,j)}(x). \tag{19.49}$$

A particularly instructive special case is given by

$$\operatorname*{Lim}_{a\uparrow\infty} H(a+x)H(a-x) = I'(x) + \theta(x), \tag{19.50}$$

without additional $\not\phi$–functions. From the lack of $\not\phi^{(-1,j)}(x)$ one concludes that the integral in the sense of generalised functions satisfies

$$\operatorname*{Lim}_{a\uparrow\infty} \int_0^a dx\ f(x) = \int_0^{\infty-} dx\ f(x), \tag{19.51}$$

$\forall f(x) \in \mathbf{GF}_s$, including the logarithmically divergent integrals behaving as $|x|^{-1}\log^q|x|$ at infinity. For ordinary integrands (not containing generalised functions at infinity) the upper limit can be taken as ∞. This example will recur in worked out form in Ch. 22.

Fourier transformation or direct computation yields the corresponding example

$$\operatorname*{Lim}_{a\uparrow\infty} \pi^{-1}x^{-1}\sin(ax) = \sum_{j=0}^{\infty} (-)^j j!\, c_{j+1}(0,0)\, \eta^{(0,j)}(x) = \boldsymbol{\mathcal{M}}_{\mathcal{D}}\, \delta(x). \tag{19.52}$$

There are indeed good reasons for preferring the Dirichlet kernel. Consequently the generalised limit

$$\operatorname*{Lim}_{a\uparrow\infty} \left\langle \pi^{-1}x^{-1}\sin(ax)\,,\log^q|x|\right\rangle = c_{j+1}(0,0) = \left\langle \delta(x)\,,\log^q|x|\right\rangle, \qquad (19.53)$$

gives the correct products of the δ-function with the powers of the logarithm. The sequences (19.52) and (19.50) are preferred sequences. A different choice of the values of generalised functions at infinity, and consequently a different choice for the values of the powers of the logarithm at $x = 0$ in Ch. 13, and another choice for the standardization of the product in Ch. 9, would result in another preferred sequence. The present choice is a highly convenient one. This remark will be taken up again in Ch. 22.

19.6 Operators and limits

One of the great virtues of distribution theory is the ability to interchange limits and operators freely. It remains to investigate in how far this will hold for sequences of generalised functions.

For operators which have an adjoint the situation is simple.

Property 19.2 Operators which have an adjoint can be interchanged with completely convergent limits.

Verification: We have

$$\operatorname*{Lim}_{a} \left\langle \mathcal{O}\, f(x;a)\,, g(x)\right\rangle = \operatorname*{Lim}_{a} \left\langle f(x;a)\,, \mathcal{O}^\dagger\, g(x)\right\rangle, \qquad (19.54)$$

from the definition of the adjoint. By complete convergence this equals

$$\operatorname*{Lim}_{a} \left\langle f(x;a)\,, \mathcal{O}^\dagger\, g(x)\right\rangle = \left\langle f(x)\,, \mathcal{O}^\dagger\, g(x)\right\rangle, \qquad (19.55)$$

so by shifting the operator back we have

$$\operatorname*{Lim}_{a} f(x;a) = f(x) \Rightarrow \operatorname*{Lim}_{a} \mathcal{O}\, f(x;a) = \mathcal{O}\, f(x), \qquad (19.56)$$

valid when the operator $\mathcal{O}$ has an adjoint. In particular limits can be interchanged with selfadjoint operators. □

The simplest case is obtained by considering the Fourier operator. It has an adjoint $\mathcal{F}^\dagger = 2\pi\mathcal{F}^{-1}$, so the limit of the Fourier transforms is the Fourier transform of the limit. By computing the Fourier transform of a sequence such as (19.3) one obtains an identity involving Γ–functions. Quite complicated identities for analytic functions can be produced in this way.

For the differential operator the situation is more complicated. The possibility of interchanging limit and differentiation would follow from selfadjointness ness of the operator $i\mathcal{D}$ by

$$\operatorname*{Lim}_{a} \left\langle \mathcal{D}\, f(x;a)\,, g(x)\right\rangle = -\left\langle f(x;a)\,, \mathcal{D}\, g(x)\right\rangle, \qquad (19.57)$$

WRONG!

Assuming complete convergence the right-hand side converges, and using selfadjointness again we obtain

$$\operatorname*{Lim}_{a} \left\langle \mathcal{D} f(x;a), g(x) \right\rangle = \left\langle \mathcal{D} f(x), g(x) \right\rangle. \tag{19.58}$$

WRONG!

Therefore the interchange of differentiation with the limit is allowed whenever the operator $\mathcal{D}$ is selfadjoint in the relevant scalar products. In particular this is the case in the subspace $\overline{\mathbf{PC}}'_\lambda$. However it was seen in Ch. 10 that the differential operator does not have an adjoint when $\overline{\mathbf{PC}}'_\lambda$ is enlarged to $\mathbf{GF}_s$ by the addition of $\overline{\mathbf{PC}}_\lambda$. It is necessary to investigate the exceptional cases separately. The lack of selfadjointness in $\overline{\mathbf{PC}}_\lambda$ is caused by the existence of a stock-term at infinity

$$\left\langle \mathcal{D} f(x), g(x) \right\rangle = -\left\langle f(x), \mathcal{D} g(x) \right\rangle + \left\langle I(x), \mathcal{D}\big(f(x) \bullet g(x)\big) \right\rangle, \tag{19.59}$$

which equals

$$\left\langle I(x), \mathcal{D}\big(f(x) \bullet g(x)\big) \right\rangle = f(x) \bullet g(x) \Big|_{-\infty}^{+\infty} = \left\langle \not{\delta}(x), f(x) \bullet g(x) \right\rangle, \tag{19.60}$$

by the fundamental theorem Prop. 14.2 of the calculus.

Only the term proportional to $\dot{\operatorname{sgn}}(x)$ in the asymptotic expansion at infinity of $f(x) \bullet g(x)$ may spoil the possibility of interchanging limit and derivative. It can be shown that products of the form $f(x) \bullet g(x) \bullet \not{\delta}(x)$ are associative $\forall f, g \in \overline{\mathbf{PC}}_\lambda$, so the stock-term can be rewritten as

$$\operatorname*{Lim}_{a} \left\langle f(x;a), \not{\delta}(x) \bullet g(x) \right\rangle = \left\langle f(x), \not{\delta}(x)\, g(x) \right\rangle, \tag{19.61}$$

which holds by the assumption of complete convergence. Therefore we have the result:

Property 19.3 The derivative of the limit of a completely convergent sequence of generalised functions equals the limit of the derivative of the sequence. □

For the operator $\mathcal{X}$ we have the same situation, since $\mathcal{X}$ and $\mathcal{D}$ are unitarily equivalent under Fourier. The multiplication by x operator can be interchanged with the limit when the operator $\mathcal{X}$ is selfadjoint in the relevant scalar products. This is again the case in the subspace $\overline{\mathbf{PC}}'_\lambda$. In $\overline{\mathbf{PC}}_\lambda$ he exceptional case occurs when $f(x) * g(x) \sim x^{-1}$ at $x = 0$. As in the case of the operator $\mathcal{D}$ one finds that the operator $\mathcal{X}$ can be interchanged with the limit.

19.7 Completed limits

It was seen in the previous section that peaked sequences such as (19.3) converge completely to $\eta + \cdots$ on $\mathbf{GF}_s$. They also converge partially to δ on a subset $\subset \mathbf{GF}_s$. Sequences of increasingly peaked functions do not converge completely to the δ–function.

Example 19.8 It is possible to construct sequences converging completely to $\delta(x)$ of the form

$$f(x;a) := f_1(a)\,\delta(x) + f_2(x;a), \tag{19.62}$$

with $\operatorname{Lim} f_1(a) = 1$, and $\operatorname{Lim} f_2(x;a) = 0(x)$. The result of taking the limit is obviously the δ–function, but it has been put in explicitly.

It is sometimes useful to possess limiting procedures which yield δ–functions. This can be achieved by defining completed limits by

$$\overline{\operatorname{Lim}}\, f(x;a) := \overline{\mathcal{M}}^{-1} \operatorname{Lim} \mathcal{M}\, f(x;a). \tag{19.63}$$

This obviously yields a limit which is an element of $\overline{\mathbf{PC}}_\lambda$. In particular the standard examples take the form

$$\overline{\operatorname{Lim}}_{a\uparrow\infty}\; a\, e^{-a^2x^2} = \sqrt{\pi}\, \dot{\overline{\mathcal{M}}}^{-1}\big(\eta(x) + \cdots\big) = \sqrt{\pi}\; \delta(x), \tag{19.64}$$

and

$$\overline{\operatorname{Lim}}_{a\downarrow 0}\; e^{-ax^2} = \overline{\mathcal{M}}^{-1}\big(I'(x) + \theta(x) + \cdots\big) = I(x), \tag{19.65}$$

and

$$\overline{\operatorname{Lim}}_{a\downarrow 0}\; e^{-ax} H(x) = \overline{\mathcal{M}}^{-1}\big(H'(x) + \cdots\big) = H(x). \tag{19.66}$$

An application of this definition will be given in the next chapter.

19.8 On topology

When the standard limit (19.5) is used to define convergence it is possible to apply standard methods to derive a topology from convergence in the scalar product. The topology is based on properties of the numerical value of scalar products, for instance on the requirement that a given scalar product be smaller than a given $\epsilon > 0$.

With the generalised limit (19.6) this is no longer possible. The definition of the generalised limit is not based on the numerical values of scalar products, but instead on the asymptotic behaviour of the scalar product as a function of the limiting parameter.

Example 19.9 The sequence $f(x;a) := a^{-1}\, e^{-\pi x^2/a^2}$ was shown to converge completely. Its scalar product with the δ–function equals a^{-1}, with a going to zero. The generalised limit is nevertheless zero, since asymptotic behaviour for large a does not contain a constant part.

What is needed is a measure of the closeness of an element of a limiting sequence to the limit. Suppose that $\mathrm{Lim}_{a\downarrow 0}\big(f(x;a) - f(x)\big) = O(x)$. Then we may consider the scalar product

$$d(a;g) := \big\langle\, f(x;a) - f(x)\,,\, g(x)\,\big\rangle, \tag{19.67}$$

for a suitable test function $g(x) \in \mathbf{GF}_s$. In distribution theory test functions $g(x) \in \mathcal{S}$ are required to be asymptotic to a Taylor series, so the distance function $d(a;g)$ is also asymptotic to a Taylor series.

When generalised limits are considered the closeness function is in general asymptotic to a more general expression, within the limits imposed by (4.1). As a measure of the distance $\epsilon(a)$ between $f(x;a)$, and $f(x)$, with respect to the generalised function $g(x)$ one can take any finite number of the leading terms of the asymptotic expansion

$$\epsilon(a;g) := \sum_{|\lambda_j|<\rho} \sum_{k} c_{jk}\, a^{|\lambda_j|} \log^k |x|, \tag{19.68}$$

with the exponent of the power replaced by its absolute value. The c_{jk} are the coefficients in the asymptotic expansion of $d(a;g)$ for $a\downarrow 0$. The potentially bothersome divergence of the logarithms is irrelevant, since a distance function with $\lambda_j = 0$ cannot occur in the case of convergence.

In particular one can take only the leading term, corresponding to the highest power of the logarithm of the power of a closest to $\lambda = 0$. In the previous example the newly defined distance is a instead of a^{-1}. It is of course impossible to use the asymptotic expansion as a whole, since no assumptions about convergence have been made. For $\mathrm{Re}\,\lambda_j > 0$ the modulus sign in (19.68) is superfluous and we have convergence in a standard sense. This is also the case when the asymptotic expansion vanishes except for a remainder term.

This asymptotic analysis is an essential component of a symmetrical theory of generalised functions. It will remain even if it were possible to avoid the use of analytic methods to investigate the asymptotic behaviour.

Remark 19.6 It is often thought that convergence is indispensable in order to give a computational content to analysis. This is not the case. The numbers which appear as the result of generalised function computations can be obtained by determining coefficients in asymptotic expansions.

It is possible to consider limit processes, it does not seem to be possible to define the concept of a neighbourhood in the space of generalised functions in such a way that it agrees with the generalised convergence concept.

Topological vector spaces are not suitable at present to serve as basis for a symmetrical theory of generalised functions. It is not possible to obtain adequate symmetry properties in this way. Further thought on this subject is necessary.

19.9 Conclusion

It has become clear that many limit processes can be handled on basis of the definitions given above. The limit concept is stronger, yet the computational convenience of distribution theory has not been lost. It is still possible to interchange limits and operators in the relevant cases.

It is also clear that the limit properties are less suitable for the definition of singular functions. In cases where divergence occurs the limit approach lacks the necessary power.

Limits in the sense of smaller than ϵ have to be replaced by an asymptotic analysis for small ϵ. In this respect the symmetrical theory of generalised functions differs fundamentally from both distribution theory and standard analysis.

The subject of closure under convergence in some sense is left open. Regaining an unified theory, which should combine the good properties of standard analysis with the additional power of the symmetrical generalised function, will require a large amount of work.

CHAPTER 20

SUMMATION AND PERIODIC FUNCTIONS

The summation of infinite series is to a large extent analogous to the integration of functions. A summation theory for numerical sequences, which is suitable for the generalised functions is developed along the lines of Ch. 4.

Periodic generalised functions, and consequently Fourier series are not yet allowed as generalised functions. This is easily remedied. Only the comb of equally spaced δ–functions has to be added as a generalised function. Then the multiplication of the δ–comb with an arbitrary generalised function $\in \mathbf{GF}_s$ gives the Fourier transforms of the periodic functions which are compatible with the rest of the model. Every generalised function with a bounded support can be continued to a periodic generalised function. Its Fourier coefficients are found in the form of the standard integral over the period, which is now interpreted in the sense of generalised function theory.

Since the product of a δ–function and a Fourier sum is then well defined, the theory of Fourier sums also gives the possibility of evaluating many sums as a product of a periodic generalised function and a δ–function. A sequences of numbers $\{a_n\}$ can be represented by a generalised function, which is a sum of of δ–functions at $x = n$ with weight a_n. The summation of infinite series which results in this way is compatible with the evaluation of integrals as defined in Ch. 14.

20.1 Preliminary summation

In the standard sense sums of sequences of numbers are defined by convergence of the series of partial sums. This concept is on one hand more general than needed for the purpose of the model developed in this book, on the other hand it does not provide a sum for many (divergent) sequences which arise naturally when periodic generalised functions are defined in a way which is compatible with the symmetrical theory.

As in the case of functions we begin by defining a preliminary class of sequences with suitable asymptotic properties. The sequences $\{a_n\}$ are considered as functions of the variable n. These functions of n are required to satisfy the same asymptotic conditions (4.9) as the functions which belong to the preliminary class,

$$a_n \sim \sum_{j=0}^{J} \sum_{k=0}^{\infty} \sum_{l=0}^{L_k} c_{jkl}\, e^{ib_j n} n^{\lambda_k} \log^l(n), \tag{20.1}$$

with $b_n \in \mathbb{R}$, and $c_{jkl}, \lambda_k \in \mathbb{C}$. The sums over j and l are finite, the upper limit L_k of the l–summation may depend on k. The λ_k are again required to be a descending sequence in the complex λ–plane, in the sense defined in Sec. 4.1. As before asymptotic means asymptotic in the standard sense due to Poincaré. The exponential oscillations are usually absent, which greatly simplifies the computations. The asymptotic behaviour is then of the form

$$a_n \sim \sum_{k=0}^{\infty} \sum_{l=0}^{L_k} c_{kl}\, n^{\lambda_k} \log^l(n), \tag{20.2}$$

with the same restrictions as above. In the simplest case the elements a_n are asymptotic to a power of n for $n\uparrow\infty$. The logarithms are easily obtained by formal differentiation with respect to λ.

The basic sum we need is therefore

$$\sum_{n=1}^{\infty} n^{\lambda} = \zeta(-\lambda), \tag{20.3}$$

valid in a standard sense for $\operatorname{Re}\lambda < -1$. The function $\zeta(\lambda)$ is the Riemann zeta function. Some of the properties of the ζ–function and its generalizations are listed in appendix D.

The ζ–function is analytic in the entire λ–plane, with the exception of a simple pole of first order at $\lambda = 1$. The residue at the pole is equal to 1. The ζ–function is therefore a meromorphic function of λ. For sums the function $\zeta(-\lambda)$ is the discrete equivalent of the function $(\lambda + 1)^{-1}$, which occurs in the case of integrals. The discrete case is technically more difficult, but there is no fundamental difference.

The needed preliminary sums are defined as in Ch. 4 by

$$\operatorname{Pre}\sum_{n=1}^{\infty} e^{ibn} n^{\alpha} \log^l(n) := \operatorname*{Res}_{\lambda=\alpha} (\lambda - \alpha)^{-1} \sum_{n=1}^{\infty} e^{ibn} n^{\lambda} \log^l(n), \tag{20.4}$$

where as usual the analytical continuation needed to define the residue has been assumed.

Remark 20.1 The existence of the analytic continuation should not be taken for granted. Some counterexamples with a natural boundary along the imaginary axis may be found in [Tich]. The conditions on the asymptotic expansion imposed above are sufficient to assure the existence of the analytic continuation, but these conditions are much stronger than necessary. As before the aim is to keep the model as small as possible.

In particular one finds for the sum of the powers, after consulting appendix D or [Erdl] for the Laurent expansion of the ζ–function

$$\operatorname{Pre}\sum_{n=1}^{\infty} n^{\lambda} = \begin{cases} \zeta(-\lambda) & \lambda \neq -1, \\ \psi(1) & \lambda = -1, \end{cases} \tag{20.5}$$

in agreement with the standard result for $\operatorname{Re}\lambda < -1$.

The preliminary sum of a sequence is defined as

$$\mathrm{Pre}\sum_{n=1}^{\infty} a_n := \sum_{n=1}^{\infty}\Big(a_n - \sum_{j=0}^{J}\sum_{k=0}^{\mathrm{Re}\,\lambda_k\geq -1}\sum_{l=0}^{L_k} c_{jkl}\, e^{ib_j n} n^{\lambda_k}\log^l(n)\Big) + \\ + \sum_{j=0}^{J}\sum_{k=0}^{\mathrm{Re}\,\lambda_k<-1} \mathrm{Pre}\sum_{l=0}^{L_k} c_{jkl}\, e^{ib_j n} n^{\lambda_k}\log^l(n), \qquad (20.6)$$

by complete analogy with the corresponding formula (4.22) for the integral. The first sum is convergent in a standard sense, and the number of subtracted terms is finite since the $\{\lambda_k\}$ are required to be a descending sequence. The preliminary sums in the second line have been defined above in the sense of generalised functions.

Remark 20.2 In expressions such as (20.6) there are two different types of sum, the (effectively finite) formal summations over terms of the asymptotic expansion , and generalised summations over a_n. This should not lead to confusion even though the same summation-symbol is used in both cases.

It follows from the definition of the sum that it has the analytical property

$$\mathrm{Pre}\sum_{n=1}^{\infty} a_n = \operatorname*{Res}_{\lambda=0} \lambda^{-1}\sum_{n=1}^{\infty} a_n n^{\lambda}. \qquad (20.7)$$

This could also be used as a definition of the sum.

The special functions which arise in the summation of sequences are special cases of the function (Bateman's notation),

$$\Phi(z,s,\nu) := \sum_{n=0}^{\infty}(\nu+n)^{-s} z^n. \qquad (20.8)$$

The definition of the function $\Phi(z,s,\nu)$ and many derived properties may be found in [Erd1].

The Riemann ζ–function and the generalised ζ–function are special cases of (20.8). The generalised ζ–function is obtained by taking $z = 1$,

$$\zeta(s,\nu) = \Phi(1,s,\nu) = \sum_{n=0}^{\infty}(\nu+n)^{-s}, \qquad (20.9)$$

and the Riemann ζ–function is obtained by specializing to $\nu = 1$,

$$\zeta(s) = \zeta(s,1) = \Phi(1,s,1) = \sum_{n=1}^{\infty} n^{-s}. \qquad (20.10)$$

The properties of these functions mentioned in [Erd1] will be used in the following without proof. Some formulæ are collected for reference purposes in appendix D.

All sums defined so far are sums from $n = 1$ to infinity. Sums starting at another n–value are defined by

$$\mathrm{Pre}\sum_{n=p}^{\infty} a_n := \mathrm{Pre}\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{p-1} a_n. \tag{20.11}$$

Sums over negative n–values are defined analogously. In the allowed asymptotic expansions n is replaced by $-n$ or $|n|$, and the sums are defined by replacing n by $-n$ in the definitions.

In Fourier theory one encounters sums from $-\infty$ to ∞. These are defined in the obvious way by

$$\mathrm{Pre}\sum_{n=-\infty}^{\infty} a_n := \mathrm{Pre}\sum_{n=-\infty}^{-1} a_n + a_0 + \mathrm{Pre}\sum_{n=1}^{\infty} a_n. \tag{20.12}$$

If the sequences are defined on part of the integers the sum is interpreted as a sum over all integers by taking undefined a_n–values to be zero.

It will be clear that the phenomenon of surface terms in the evaluation of integrals recurs in the definition of sums. The term at $n = 0$ has a preferred position, and the substitution $m := n + p$ may yield an additional contribution at infinity.

Example 20.1 Taking $a_n = n^0 = 1$, $\forall n \in \mathbb{Z}$, one obtains from the definitions

$$\mathrm{Pre}\sum_{n=1}^{\infty} n^0 = \zeta(0) = -\tfrac{1}{2}, \tag{20.13}$$

$$\mathrm{Pre}\sum_{n=p}^{\infty} n^0 = \mathrm{Pre}\sum_{n=1}^{\infty} (n+p-1)^0 = \tfrac{1}{2} - p, \tag{20.14}$$

which illustrates the dependence on the choice of an origin.
Summing over all positive and negative integers n gives

$$\mathrm{Pre}\sum_{n=-\infty}^{\infty} (n+p)^0 = \mathrm{Pre}\sum_{n=-\infty}^{\infty} n^0 = \sum_{-\infty}^{-1} + \sum_{0}^{0} + \sum_{1}^{\infty} = -\tfrac{1}{2} + 1 - \tfrac{1}{2} = 0. \tag{20.15}$$

The result is independent of the choice of origin since $a_n = n^0$ does not have a jump at infinity.

The dependence of sums on the choice of an origin is unavoidable when the summand has jumps at infinity. Expressions such as

$$\sum_{n=p}^{\infty} 1 = ??? \qquad \text{or} \qquad 1+1+1+\cdots = ???\,, \tag{20.16}$$

WRONG!

are and remain meaningless. It is of course possible to give these expressions an arbitrary meaning, but it is better to avoid this in order to prevent unnecessary confusion.

20.2 Scalar products of sequences

A preliminary symmetrical scalar product can be defined on the sequences by

$$\langle\, a_n\,,\, b_n \,\rangle_{\text{pre}} := \text{Pre}\sum_{n=-\infty}^{\infty} a_n^* b_n\,, \tag{20.17}$$

by analogy with the scalar product of functions.
As before the sequences can be considered as linear functionals on the sequences by

$$\langle\, a_{n'}\,,\, b_n \,\rangle^* := \langle\, b_n\,, a_{n'} \,\rangle := \langle\, b_n\,,\, a_n \,\rangle_{\text{pre}}\,. \tag{20.18}$$

The primed powers are obtained as

$$\langle\, b_n\,,\, n'^{\alpha} \log^q(n) \,\rangle = \operatorname*{Res}_{\lambda=\alpha} q!\,(\lambda-\alpha)^{-q-1} \,\langle\, b_n\,,\, n^{\lambda} \,\rangle_{\text{pre}}. \tag{20.19}$$

In particular the primed Heaviside sequence $H'(n)$ is defined for positive n by

$$H'(n) := \operatorname*{Res}_{\lambda=0} n^{\lambda} H(n), \tag{20.20}$$

and likewise for negative n. The primed unit sequence is now defined by

$$I'(n) := H'(-n) + \delta_{n,0} + H'(n). \tag{20.21}$$

The primed unit sequence equals one in the finite, the residue determines its limiting behaviour at infinity.

The generalised functions at infinity can also be defined for sequences. For example the generalised function $\not\eta_{\uparrow}^{(\alpha,q)}(n)$ may be defined for sequences by

$$\langle\, a_n\,,\, \not\eta_{\uparrow}^{(\alpha,q)}(n) \,\rangle := \operatorname*{Res}_{\lambda=-\alpha-1} \tfrac{(-)^q}{q!}\,(\lambda+\alpha+1)^q\ \text{Pre}\sum_{n=1}^{\infty} a_n n^{\lambda}. \tag{20.22}$$

It measures the coefficient of $n^{\alpha^*}\log^q(n)$ in the asymptotic expansion of a_n at $n = +\infty$

$$\left\langle \not\delta_{\uparrow}^{(\alpha,q)}(n)\,,\, n^{\beta}\log^{r}(n)\right\rangle = \delta_{\alpha,\beta}\,\delta_{q,r}\,, \tag{20.23}$$

in agreement with the corresponding measurement formula for functions.

Limits of sequences are defined in an analogous manner. In particular, the limit for $n\uparrow\infty$ of a sequence is defined by

$$\operatorname*{Lim}_{n\uparrow\infty} a_n := \left\langle \not\delta_{\uparrow}(n)\,,\, a_n \right\rangle. \tag{20.24}$$

Application of the definition (20.24) to the discrete powers gives

$$\operatorname*{Lim}_{n\uparrow\infty} n^{\alpha}\log^{q}(n) = \delta_{q,0}\,\delta_{\alpha,0}\,, \tag{20.25}$$

in agreement with the corresponding result (13.36) for the continuous powers of x.
The standard limit is in this case

$$\lim_{n\uparrow\infty} n^{\alpha}\log^{q}(n) = \begin{cases} 0 & \operatorname{Re}\alpha < 0, \\ 1 & \operatorname{Re}\alpha = 0,\ q = 0, \\ \infty & \operatorname{Re}\alpha = 0,\ q > 0, \\ \infty & \operatorname{Re}\alpha > 0,\ \operatorname{Im}\alpha = 0, \\ \text{undefined} & \operatorname{Re}\alpha > 0,\ \operatorname{Im}\alpha \neq 0. \end{cases} \tag{20.26}$$

The sum in the sense of generalised functions is defined by

$$\sum_{n=1}^{+\infty-} a_n := \left\langle H'(n)\,,\, a_n \right\rangle, \tag{20.27}$$

by analogy with the corresponding integral.

Sums and limits can be defined for sequences in complete analogy with the corresponding definitions for generalised functions. Perhaps it would have been better to start with summation instead of integration, but the integrals are simpler. In order to proceed it is necessary to define operators on the sequences. Instead of doing this directly it is more convenient to consider the sequences as generalised functions, and to define the operators on the corresponding generalised functions first.

20.3 The comb of δ–functions

The material of this section provides the necessary tools to include periodic functions, and consequently Fourier series, in the symmetrical theory of generalised functions. When the Fourier theory of the periodic functions is available it can be used to complete the theory of sequences.

In order to add periodic functions it is sufficient to add a single one, since the others arise from the operator and product properties. The best choice for this periodic function is the comb of δ–functions defined as a formal expression by

$$\text{III}(x) := \sum_{n=-\infty}^{\infty} \delta(x-n)\,. \tag{20.28}$$

The sum is as yet a formal sum. This generalised function is also called Dirac's comb by some authors. The function $\text{III}(ax+b)$ is defined in the same way. Since the factor a can be taken out of the δ–function it is sufficient to consider the argument $(x-b)$ for the translated combs.
One frequently encounters the special case

$$\underline{\text{III}}(x) := \sum_{n=-\infty}^{\infty} \delta(x-2\pi n) = (2\pi)^{-1}\,\text{III}(x/2\pi), \tag{20.29}$$

which occurs as the Fourier transform of (20.28).
The scalar product of a generalised function and a comb of δ–functions gives rise to the formal expression

$$\langle\, \text{III}(x)\,,\, f(x)\,\rangle = \sum_{n=-\infty}^{\infty} a_n. \tag{20.30}$$

The coefficients a_n are given by

$$a_n = \langle\, \delta(x-n)\,,\, f(x)\,\rangle, \tag{20.31}$$

The scalar product takes the form of an infinite sum of the kind which has been defined in the previous section. The correct asymptotic behaviour of the terms in the sums follows at once from the asymptotic behaviour (4.9) imposed on the generalised functions. Therefore the sum is well defined.

The product of the comb with an arbitrary generalised function $g(x) \in \overline{\mathbf{PC}}_\lambda$ is a rough comb of the form

$$f(x) \bullet \text{III}(x) = \sum_{n=-\infty}^{\infty} \sum_{p=0}^{P_n} a_{n,m}\, \delta^{(p)}(x-n), \tag{20.32}$$

with the coefficients $a_{n,m}$ given by (9.86)

$$a_{n,m} = \sum_{q=0}^{\infty} \tfrac{1}{\pi} q!\, c_{q+1}(0,0)\, \langle\, \delta(x-n)\,,\, \boldsymbol{\mathcal{X}}^m\, g(x)\,\rangle, \tag{20.33}$$

since the product of any generalised function with a δ–function is by (9.86) again a linear combination of δ–functions at the same point. The coefficient in (20.33) follows from (9.53–54) for the product of a negative power with a δ–function.

This expression should also be defined as a generalised function. From the requirements imposed on the preliminary class it follows that the number of points where $P_n > 0$ must be finite. For every $f(x) \in \mathbf{PC}_\lambda$ there is an N_f such that $f(x)$ is $\mathbf{C}^\infty$ for $|x| > N_f$. Therefore we have $P_n = 0$ for all $n > N_f$. The infinite q-summations are also effectively finite.

It remains to define the action of the operators on the combs. The operators $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{D}}$ do not present any difficulties. The derivatives of the comb are given by

$$\boldsymbol{\mathcal{D}}^p\, ш(x) := (-)^p p!\, ш^{(p)}(x) = (-)^p p! \sum_{n=-\infty}^{\infty} \delta^{(p)}(x-n), \tag{20.34}$$

which is again easily accommodated as a generalised function.

Somewhat more effort is required to define the Fourier transform of the comb. Formally one obtains

$$\boldsymbol{\mathcal{F}}\, ш(x) = \sum_{n=-\infty}^{\infty} e^{inx}. \tag{20.35}$$

In the sense of distribution theory it is known that

$$\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi \sum_{n=-\infty}^{\infty} \delta(x-2\pi n) = 2\pi\, ш(x). \tag{20.36}$$

Apart from a scale transformation the comb is its own Fourier transform,

$$\boldsymbol{\mathcal{F}}\, ш(x) = 2\pi\, ш(x) = ш(x/2\pi). \tag{20.37}$$

This will be taken as the definition of the Fourier transform of the comb.

Remark 20.3 Application of (20.37) to a test function $\in \mathcal{S}$ yields a special case of Poisson's summation formula [Lig]. For example, taking a Gaussian gives

$$\sum_{n=-\infty}^{\infty} e^{-a^2 n^2} = a^{-1}\sqrt{\pi} \sum_{n=-\infty}^{\infty} e^{-\pi^2 \alpha^{-2} n^2}. \tag{20.38}$$

Poisson's summation formula is easily extended to larger classes of functions by approximating them with test functions.

It must be shown that (20.37) is a good definition in the sense of generalised functions. This is equivalent to demonstrating that Parseval's equality holds with definition (20.38).

The result holds for the contribution from the finite by Parseval's equality, so it is sufficient to consider the asymptotic regime. For convenience this can be taken from $n = 1$ since the results holds for the contributions from the finite.
Taking $f(x) = x^\lambda H(x)$ gives

$$\langle x^\lambda H(x)\,,\text{Ш}(x)\,\rangle = \sum_{n=1}^{\infty} n^\lambda = \zeta(-\lambda), \tag{20.39}$$

while the Fourier transform yields

$$\begin{aligned}\langle\,\Gamma(\lambda+1)\big(e^{-\frac{\pi}{2}(\lambda+1)}x^{-\lambda-1}H(x) + e^{i\frac{\pi}{2}(\lambda+1)}(-x)^{-\lambda-1}H(-x)\big)\,,\text{Ш}(x)\,\rangle = \\ = -2(2\pi)^{-\lambda-1}\sin\tfrac{\pi}{2}\lambda\,\Gamma(\lambda+1)\,\zeta(\lambda+1).\end{aligned} \tag{20.40}$$

Equating the right-hand sides of (20.40) and (20.39) yields Riemann's functional equation (D.6) of the ζ–function,

$$\zeta(-\lambda) = -2(2\pi)^{-\lambda-1}\sin\tfrac{\pi}{2}\lambda\,\Gamma(\lambda+1)\,\zeta(\lambda+1), \tag{20.41}$$

valid for $\lambda \neq -1$. For $\lambda = -1$ both sides have the same residue. Parseval's equality has therefore been verified for this special case.

Shifting the comb $\text{Ш}(x)$ to $\text{Ш}(x-a)$ and making the same calculation yields Hurwitz's equation, (D.5) for the generalised ζ–function. Adding oscillations at infinity yields Lerch's formula (D.4) (after some transformation). Considering $f(x) := x^\lambda\, e^{-bx}$ also leads to Lerch's formula. Finally powers of logarithms can be added by differentiation with respect to λ. These formulæ and their proof may be found in [Erd1].

The convergence is established for arbitrary generalised functions by subtracting a sufficient number of terms of the asymptotic expansion until a remainder is obtained for which Poisson's summation formula holds in a classical sense.

Now that the Fourier transform of the comb has been established it is possible to define the infinite formal sum (20.35) as

$$\sum_{n=-\infty}^{\infty} e^{inx} := 2\pi\sum_{n=-\infty}^{\infty}\delta(x-2\pi n) = \text{Ш}(x). \tag{20.42}$$

This result can also be obtained by approximating a periodic function with suitable sequences.

20.4 Periodic functions

The definition of periodic generalised functions must be constructed in such a way that it is an extension of the classical concept of periodicity. Classically a function $f_p(x)$ is periodic with period one when it satisfies

$$f_p(x+1) = f_p(x), \tag{20.43}$$

$\forall x \in \mathbb{R}$. This definition cannot be taken over for generalised functions, since generalised functions are not determined by their values. It was seen in Ch. 13 that there is a large class of generalised functions with value zero everywhere. Therefore it is necessary to return to the concept of infinite repetition. Consider any generalised function $f_o(x)$ with support contained in the interval $\{-\frac{1}{2}, \frac{1}{2}\}$. The corresponding periodic generalised function $f_p(x)$ is defined by the formal expression

$$f_p(x) := \sum_{n=-\infty}^{\infty} \mathcal{T}(n,0)\, f_o(x), \tag{20.44}$$

which formally represents an infinite repetition of the same function. The coordinate translation operator $\mathcal{T}(n,0)$ has been defined in Ch. 15. With this definition a periodic function is invariant under translation over a period, so instead of (20.43) we now have

$$\mathcal{T}(1,0)\, f_p(x) = f_p(x). \tag{20.45}$$

A convention is necessary to decide if the point $-\frac{1}{2}$ or the point $\frac{1}{2}$ belongs to the interval $\{-\frac{1}{2}, \frac{1}{2}\}$. The choice is irrelevant for the resulting periodic function, as long as double counting is avoided.

Example 20.2 Taking $f_o(x) := \delta(x)$ results in $f_p(x) = ш(x)$.

The Fourier transform of the periodic function defined above can be read of immediately from the Fourier properties (15.24) of the translation operator. The result is

$$\mathcal{F}\big(\mathcal{T}(n,0)\, f_o(x)\big) = \mathcal{T}(0,n)\big(\mathcal{F}\, f_o(x)\big). \tag{20.46}$$

The action of the momentum translation operator $\mathcal{T}(0,n)$ on the generalised functions corresponds by (15.18) to generalised function multiplication by the generalised function e^{inx}, so the Fourier transform of a periodic function takes the form

$$\mathcal{F}\, f_p(x) = \big(\mathcal{F}\, f_o(x)\big) \bullet \sum_{n=-\infty}^{\infty} e^{inx}. \tag{20.47}$$

The sum of exponentials which occurs here has been defined as a generalised function in the previous section. It equals the comb of δ-functions. The Fourier transform of a periodic function can also be written in the form

$$\mathcal{F}\, f_p(x) = 2\pi\big(\mathcal{F}\, f_o(x)\big) \bullet \sum_{n=-\infty}^{\infty} \delta(x-2\pi n) = 2\pi \mathcal{F}\, f_o(x) \bullet ш(x). \tag{20.48}$$

This can be simplified by using the product property (9.86) of the δ–function, which states that the product of any generalised function and a δ–function is again a finite linear combination of $\delta^{(p)}$–functions. Ignoring the contributions at infinity for the time being, and assuming derivatives to be absent we obtain for the finite part of the Fourier transform

$$\mathcal{F} f_p(x) = 2\pi \sum_{n=-\infty}^{\infty} a_n \, \delta(x - 2\pi n), \tag{20.49}$$

with the Fourier coefficients a_n given by

$$a_n = \langle \mathcal{F} f_o(x) \, , \delta(x - n) \rangle = \mathcal{F} f_o(n), \tag{20.50}$$

This can be converted by means of Parseval's equality to

$$a_n = \langle e^{2\pi i n x} \, , f_o(x) \rangle = \int_{-\infty}^{\infty} dx \; e^{-2\pi i n x} f_o(x) = \int_{-1/2}^{1/2} dx \; e^{-2\pi i n x} f_o(x), \tag{20.51}$$

which coincides with the standard form of the Fourier coefficient.

Example 20.3 Testing the Fourier transform with a $\theta(x - 2\pi n)$ function yields

$$a_n = (2\pi)^{-1} \langle \theta(x - 2\pi n) \, , \mathcal{F} f_p(x) \rangle = \langle e^{2\pi i n x} \not{\eta}(x) \, , f_p(x) \rangle, \tag{20.52}$$

which shows that the Fourier coefficient a_0 is equal to the constant part of the periodic function at infinity.

More generally the coefficient a_n can be found by multiplication with any function having the value 1 at $x = 2\pi n$ and the value zero at all other integral multiples of 2π.

In distribution theory it is possible to consider instead of periodic distributions the distributions of a space of test functions of bounded support. If it is desirable to consider periodic distributions on the support $(-\infty, \infty)$, it is necessary [Lig] to obtain the Fourier coefficients by means of so called 'unitary' or 'smudge' functions, since the classical Fourier integral (20.51) is undefined in the sense of distribution theory. A unitary function is an element $U(x) \in \mathcal{S}$ such that $U(x)$ has support contained in $|x| \leq 1$, and such that

$$\sum_{n=-\infty}^{\infty} U(x - n) = 1, \tag{20.53}$$

$\forall x \in \mathbb{R}$. Its Fourier transform is (the restriction to the real axis of) an entire analytic function, which equals one at $x = 0$ and zero at every integer multiple of 2π. The Fourier coefficients can be computed as

$$a_n = \int_{-\infty}^{\infty} dx \; e^{-2\pi i n x} f_p(x) \, U(x), \tag{20.54}$$

with $U(x)$ an arbitrary smudge function.

In practice the requirements imposed on unitary functions are so restrictive that the actual computation of Fourier coefficients by means of (20.54) is impracticable.

In the symmetrical theory of generalised functions these problems do not arise. Equation (20.54) is also a correct (but unnecessarily awkward) way to compute Fourier coefficients in the sense of generalised functions. The use of smudge functions is not necessary. In the symmetrical theory of generalised functions there is more freedom to compute Fourier coefficients in any convenient way.

The standard formula for the Fourier coefficients corresponds to choosing $U(x) = H(1 - |x|)$. The corresponding observing function is the Fourier transform $(x-n)^{-1}\sin(x-n)$. This choice is allowed for periodic generalised functions, it is forbidden for periodic distributions.

For integrable function the Fourier transform of a function of bounded support is known to be the restriction to the real axis of an entire analytic function of exponential type, [Tich]. This carries over to ordinary generalised functions of bounded support.

Other localized generalised functions, such as $\theta^{(\alpha,q)}$ or $\eta^{(\alpha,q)}$ can also generate periodic generalised functions, which do not have a counterpart in distribution theory. These new periodic functions do not have Fourier coefficients in the finite. Their Fourier transform is located at infinity.

Example 20.4 Consider the periodic function

$$f_p(x) := \sum_{n=-\infty}^{\infty} \theta(x-n) = \sum_{n=-\infty}^{\infty} \mathcal{T}(n,0)\,\theta(x), \tag{20.55}$$

which which might be called a θ–comb by analogy with the δ–comb. Its Fourier transform is given by

$$\begin{aligned} \mathcal{F} f_p(x) &= \mathcal{F}\theta(x) \sum_{n=-\infty}^{\infty} e^{2\pi i n x} = \mathcal{F}\theta(x) \bullet \text{Ш}(x)) = \\ &= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} j!\, c_{j+1}(0,0)\, e^{2\pi i n x} \eta^{(0,j)}(x). \end{aligned} \tag{20.56}$$

The scalar product of this expression with all generalised functions $\in \mathbf{GF}_s$ is well defined. By (4.9) its scalar product with all generalised functions contains only finitely many non-zero terms.

Generally the Fourier transform of a periodic function takes the form

$$\begin{aligned} \mathcal{F} f_p(x) &= \big(f_1(x) + f_2(x)\big) \bullet \sum_{n=-\infty}^{\infty} e^{inx} = \\ &= \sum_{n=-\infty}^{\infty} a_n\, \delta(x - 2\pi n) + f_2(x) \bullet \sum_{n=-\infty}^{\infty} e^{inx}, \end{aligned} \tag{20.57}$$

with $f_2(x)$ an allowed generalised function at infinity.

20.5 Completion of the sequences

Application of the limit properties of sequences leads to the question in how far periodic functions can be considered as an infinite repetition of the period. This is conveniently discussed in terms of the comb of δ–functions. The treatment parallels the corresponding treatment for sequences, so it is not necessary to write out the details. The result is

$$\operatorname*{Lim}_{N\uparrow\infty}\sum_{n=1}^{N}\delta(x-n) := \sum_{n=1}^{+\infty-}\delta(x-n) = H'(x)\bullet\sum_{n=1}^{\infty}\delta(x-n). \tag{20.58}$$

Likewise in the symmetrical case one obtains

$$\operatorname*{Lim}_{N\uparrow\infty}\sum_{n=-N}^{N}\delta(x-n) = \sum_{-\infty+}^{+\infty-}\delta(x-n). \tag{20.59}$$

The formal sum which served as a starting point can be recovered as the completed limit

$$\operatorname*{\overline{Lim}}_{N\uparrow\infty}\sum_{n=-N}^{N}\delta(x-n) = \sum_{n=-\infty}^{\infty}\delta(x-n). \tag{20.60}$$

The same holds for the sums of sequences

$$\operatorname*{Lim}_{N\uparrow\infty}\sum_{n=1}^{N}a_n = \sum_{n=1}^{+\infty-}a_n. \tag{20.61}$$

and the completion

$$\operatorname*{\overline{Lim}}_{N\uparrow\infty}\sum_{n=1}^{N}a_n = \sum_{n=1}^{+\infty}a_n. \tag{20.62}$$

The two expressions do not differ

$$\sum_{n=1}^{+\infty-}a_n = \sum_{n=1}^{+\infty}a_n, \tag{20.63}$$

since there is a contribution from infinity only when this is explicitly indicated by the appearance of a generalised function at infinity. The completion is simpler for sequences than for functions.

The periodic generalised functions are only partially recovered as limits. In particular the periodic functions at infinity are missing.

20.6 Conclusion

This chapter presents an outline of the introduction of periodic symmetrical generalised functions. It is not yet a complete definition in the sense of generalised functions. The products of periodic functions have been defined only for periodic functions with the same period. In order to lift this restriction it is necessary to enlarge the allowed asymptotic behaviour at infinity. Instead of the finite sum of exponentials allowed by (4.9), it is now necessary to allow more complicated expressions of the form

$$\sum_{j_1=-\infty}^{+\infty} \sum_{j_2=-\infty}^{+\infty} \cdots\cdots \sum_{j_n=-\infty}^{+\infty} e^{i \sum_{k=1}^{n} b_{j_k} x} \cdots\cdots , \qquad (20.64)$$

An extension of the model in this direction does not give rise to new difficulties, since only finitely many periodic functions have to be multiplied. This extension is not included in this work since it is not immediately useful for applications, and it does not lead to more insight in the structure of the model.

CHAPTER 21

HILBERT TRANSFORMS AND CAUSALITY

The Hilbert transform of a generalised functions is defined as the convolution of that function with the generalised function $(-\pi x)^{-1}$. It can be found by straightforward computation from the Fourier transform and the multiplication by the signum function, which are already defined for all generalised functions. As an application the Kramers-Kronig relations for causal functions are derived.

21.1 The Hilbert transform

The Hilbert transform of a generalised function is defined as the convolution of that function with the generalised function $-\pi^{-1}x^{-1}$,

$$\mathcal{H} f(x) := -\pi^{-1}x^{-1} * f(x). \tag{21.1}$$

The normalization used here is the same as the normalization used in the tables of the Bateman manuscript project. [Erd1]. With this choice of normalization the Hilbert transform of a real function is real, but its square is negative.
In cases where the integral is defined classically the Hilbert transform takes the form

$$\mathcal{H} f(x) = -\pi^{-1}\int_{-\infty}^{\infty} dy\, \frac{f(y)}{x-y}, \tag{21.2}$$

where the integral is to be understood classically as a principal value.

In accordance with the definition of the convolution of generalised functions, the Hilbert transform of a generalised function can be computed by Fourier transformation as

$$\mathcal{H} f(x) = i\mathcal{F}^{-1}\big((\mathcal{F} f(x)) \cdot \operatorname{sgn}(x)\big), \tag{21.3}$$

where the factor i comes from $\mathcal{F}x^{-1} = -i\pi \operatorname{sgn}(x)$. Since the Fourier transform and the multiplication is defined for all generalised functions the computation of Hilbert transforms is straightforward.
The Hilbert transform would be its own inverse (up to the sign) if the relation

$$\mathcal{H}\mathcal{H} = -\mathcal{I}, \qquad (21.4)\ \textbf{WRONG!}$$

would hold. In cases where the convolution product is associative the identity (21.4) follows from

$$\mathcal{H}\mathcal{H} f(x) = \pi^{-2}\, x^{-1} * \big(x^{-1} * f(x)\big), \tag{21.5}$$

and the convolution product (9.96).

$$x^{-1} * x^{-1} = -\pi^2\,\delta(x). \tag{21.6}$$

Unfortunately the convolution product (21.5) is in general not associative. This can be seen by Fourier transformation to the equivalent non-associative pointwise product

$$\operatorname{sgn}(x) \bullet \big(\operatorname{sgn}(x) \bullet f(x)\big) \neq \big(\operatorname{sgn}(x) \bullet \operatorname{sgn}(x)\big) \bullet f(x) = I(x) \bullet f(x) = f(x). \tag{21.7}$$

The standard example in which inequality holds in (21.7) is

$$O(x) = \operatorname{sgn}(x) \bullet \big(\operatorname{sgn}(x) \bullet \delta^{(p)}(x)\big) \neq \big(\operatorname{sgn}(x) \bullet \operatorname{sgn}(x)\big) \bullet \delta^{(p)}(x) = \delta^{(p)}(x), \tag{21.8}$$

as found before. Correspondingly the convolution product is not associative when $f(x)$ is a polynomial in x.

The Hilbert transform has a zero space containing all polynomials $\in \overline{\mathbf{PC}}_\lambda$. Instead of (21.4) only the weaker equality

$$\mathcal{H}\mathcal{H} = \mathcal{I} - \boldsymbol{P}_{x^p}, \tag{21.9}$$

holds. On the subspace $(\mathcal{I} - \boldsymbol{P}_{x^p})\mathbf{G}\hat{\mathbf{F}}_s$ the Hilbert transform has a unique inverse. In general the inverse of the Hilbert transform is defined only up to an arbitrary polynomial.

The Hilbert transforms of the generalised functions are computed by straightforward application of the definition. As usual it is convenient to begin with the subspace $\overline{\mathbf{PC}}'_\lambda$. Computation of the Hilbert transform of the powers gives

$$\begin{aligned}
\mathcal{H}|x'|^\alpha \log^q|x| \operatorname{sgn}^m(x) &= \\
&= (-)^q q! \sum_{j=0}^{q+1} \tfrac{(-)^j}{j!}\, h_{q-j}(\alpha,m)|x'|^\alpha \log^j|x| \operatorname{sgn}^{m+1}(x) + \\
&\quad - 2(-)^q q! \sum_{j=0}^{\infty} j!\, h_{q+j+1}(\alpha,m) \times \\
&\quad \times \big((-)^j \eta_s^{(-\alpha-1,j)}(x) - \not\eta^{(-a-1,j)}(x)\big) \operatorname{sgn}^{m+1}(x),
\end{aligned} \tag{21.10}$$

valid for $(\alpha, q) \neq (p, 0)$. The coefficients $h_{\dots}$

$$h_l(\alpha,m) := \tfrac{2}{\pi} \sum_{j=-1}^{l+1} (-)^j c_j(\alpha,m)\, c_{l-j}(-\alpha-1, m+1), \tag{21.11}$$

are defined by (C.42) in appendix C. The corresponding computation gives in the exceptional case

$$\mathcal{H}\, x' = -2\sum_{j=0}^{\infty} j!\, h_{j+1}(p,p)\big((-)^j \eta^{(-p-1,j)}(x) - \not\eta^{(-p-1,j)}(x)\big). \tag{21.12}$$

For the corresponding $\theta^{(p)}$-functions one obtains

$$\mathcal{H}\, \theta^{(p)}(x) = 2\sum_{j=0}^{\infty} (-)^j j!\, h_{j+1}(p,p)\, \eta^{(-p-1,j)}(x), \tag{21.13}$$

so combining (21.13) and (21.12) gives

$$\mathcal{H}\big(x'^p + \theta^{(p)}(x)\big) = 2\sum_{j=0}^{\infty} j!\, h_{j+1}(p,p)\, \not\eta^{(-p-1,j)}(x). \tag{21.14}$$

One sees that the result of taking the Hilbert transform of a power $\in \overline{\mathbf{PC}}'_\lambda$ is always non-zero. The $\not\eta$-functions at infinity cannot be removed, so the Hilbert transform of a polynomial in $\overline{\mathbf{PC}}'_\lambda$ is a non-zero generalised function at infinity.
For the η-functions computation gives

$$\begin{aligned} &\mathcal{H}\, \eta_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x) = \\ &= \sum_{j=-1}^{\infty} \tfrac{(-)^j (q+j)!}{q!} \big(1-\delta_{q,0}\,\delta_{j,-1}\big)\, h_j(-\alpha-1,m)\, \eta_s^{(\alpha,q+j)}(x)\, \mathrm{sgn}^{m+1}(x) + \\ &+ \tfrac{1}{\pi}\, \delta_{q,0} \sum_{p=0}^{\infty}\Big(\delta_{\alpha,-p-1}\, \theta^{(p)}(x) - p!\, \delta_{\alpha,p}\big(x'^{-p-1} + \not\theta^{(-p-1)}(x)\big)\Big). \end{aligned} \tag{21.15}$$

As expected one sees that the Hilbert transform of a localized function in the finite is always located in the finite at the same place. The exceptions arise either on the way out or coming back.
At infinity one finds almost the same result,

$$\begin{aligned} &\mathcal{H}\, \not\eta_s^{(\alpha,q)}(x)\, \mathrm{sgn}^m(x) = \\ &= \sum_{j=-1}^{\infty} \tfrac{(-)^j (q+j)!}{q!} \big(1-\delta_{q,0}\big)\,\delta_{j,-1}\, h_j(-\alpha-1,m)\, \eta_s^{(\alpha,q+j)}(x)\, \mathrm{sgn}^{m+1}(x) + \\ &+ \tfrac{1}{\pi}\, \delta_{q,0} \sum_{p=0}^{\infty}\Big(\delta_{\alpha,-p-1}\big(x'^p + \theta^{(p)}(x)\big) - p!\, \delta_{\alpha,p}\, \not\theta^{(-p-1)}(x)\Big), \end{aligned} \tag{21.16}$$

The wave number shifted $\not{\eta}$–functions (with $x_0 \neq 0$) are unchanged up to a sign by the Hilbert transform

$$\mathcal{H}\, e^{ixx_0} \not{\eta}_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x) = \mathrm{sgn}(x_0)\, e^{ix_0x} \not{\eta}_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x), \qquad (21.17)$$

since in this cases multiplication with the signum function changes only a sign.

The Hilbert transform does not have a zero element in $\overline{\mathbf{PC}}'_\lambda$. On the other hand $\overline{\mathbf{PC}}'_\lambda$ is not closed under the Hilbert transform, since we have

$$\mathrm{sgn}(x) \bullet \theta^{(p)}(x) = \tau^{(p)}(x) \notin \overline{\mathbf{PC}}'_\lambda .$$

and consequently

$$\mathcal{H} \not{\eta}^{(p)}(x) = -\tfrac{p!}{\pi}\, \not{\theta}^{(-p-1)}(x) - \sum_{j=0}^{\infty} (-)^j j!\, h_l(p, p+1)\, \not{\phi}^{(p,j)}(x), \qquad (21.18)$$

so the subspace $\overline{\mathbf{PC}}'_\lambda$ is not closed under the Hilbert operator.

In $\overline{\mathbf{PC}}_\lambda$ one computes the Hilbert transform either directly or by transfer, with the results for the powers in

$$\mathcal{H}|x|^\alpha \log^q|x|\,\mathrm{sgn}^m(x) = (-)^q q! \sum_{j=0}^{q+1} \tfrac{(-)^j}{j!}\, h_{q-j}(\alpha, m)|x|^\alpha \log^j|x|\,\mathrm{sgn}^{m+1}(x) +$$

$$- 2(-)^q q! \sum_{p=0}^{\infty} \delta_{-\alpha-1,p}\, \delta_{m+1,p}\, h_{l+1}(\alpha, m)\, \delta^{(p)}(x), \qquad (21.19)$$

valid for $(\alpha, q) \neq (p, 0)$. The exceptional case is trivial

$$\mathcal{H}\, x^p = 0(x), \qquad (21.20)$$

$\forall p \in \mathbb{N}$. For the $\delta^{(p)}(x)$–functions one finds

$$\mathcal{H}\, \delta^{(p)}(x) = -\tfrac{1}{\pi} x^{-p-1}\,, \qquad (21.21)$$

in agreement with the corresponding result in distribution theory.

Remark 21.1 Alternatively one can transfer the results which were found in the subspace $\overline{\mathbf{PC}}'_\lambda$ by application of the mappings

$$\mathcal{H}\, f(x) = \bar{\mathcal{M}}^{-1} \mathcal{H} \bar{\mathcal{M}}\; f(x), \qquad (21.22)$$

as one sees by substituting the definitions.

In $\overline{\mathbf{PC}}_\lambda$ the Hilbert transform has a zero space containing all polynomials, and the repeated Hilbert transform is not the identity. One sees again that the introduction of the unit element $I(x)$ makes it impossible to have a fully invertible Hilbert transform.

21.2 Functions of argument $(x\pm io)$

An important special case which occurs often is related to functions with support on the positive real axis. In linear response theory the requirement that the response should follow its cause leads naturally to Green's functions with support on the positive real axis. It follows classically that the real and imaginary parts of the Fourier transform of the response function are related by a Hilbert transform. These connections are traditionally known as dispersion relations. It may happen that the convolution integrals appearing in dispersion relations diverge in a classical sense. It is then necessary to subtract the divergent part, which leads to subtracted dispersion relations.

In the sense of generalised functions the dispersion relations are always well defined and there is no need for subtractions.

The definitions follow the by now familiar path. First the preliminary functions are defined by

$$(x\pm io)^{\alpha}_{\mathrm{pre}} = x^{\alpha}H(x) + e^{\pm i\pi\alpha}(-x)^{\alpha}H(-x). \tag{21.23}$$

Somewhat more generally one can define

$$\begin{aligned}\left(e^{i\varphi}(x\pm io)\right)^{\alpha}_{\mathrm{pre}} &= e^{i\varphi\alpha}(x\pm io)^{\alpha}_{\mathrm{pre}} = \\ &= e^{i\varphi\alpha}x^{\alpha}H(x) + e^{i(\varphi\pm\pi)\alpha}(-x)^{\alpha}H(-x),\end{aligned} \tag{21.24}$$

which often occurs in formulæ, especially with $\varphi = \pm i\pi/2$.
The corresponding functions in $\overline{\mathbf{PC}}'_{\lambda}$ are again defined by taking appropriate residues. In self evident notation the powers are defined by

$$\left(e^{i\varphi}(x'\pm io)\right)^{\alpha}\log^{q}\left(e^{i\varphi}(x\pm io)\right) := \operatorname*{Res}_{\lambda=\alpha} q!\,(\lambda-\alpha)^{-q-1}\,e^{i\varphi\alpha}(x\pm io)^{\alpha}_{\mathrm{pre}}, \tag{21.25}$$

and the corresponding η–functions of argument $(x\pm io)$ are defined by

$$(-)^{q}\eta^{(\alpha,q)}_{\varphi}(x\pm io) - \not\eta^{(\alpha,q)}_{\varphi}(x\pm io) := \operatorname*{Res}_{\lambda=\alpha}\ \tfrac{(-)^{q}}{2q!}\,(\lambda+\alpha+1)^{q}\,e^{i\varphi\lambda}(x\pm io)^{\lambda}_{\mathrm{pre}}, \tag{21.26}$$

which is as usual split in η_{φ}–functions at $x=0$ and $\not\eta_{\varphi}$–functions at infinity. To simplify the notation the subscript φ is omitted if $\varphi = 0$.

Straightforward computation of the residue, expanding the exponentials in a Taylor series, and recombining terms by the binomial theorem gives for the η–functions

$$\begin{aligned}\eta^{(\alpha,q)}_{\varphi}(x\pm io) &= \tfrac{1}{2}\,e^{-i\varphi(\alpha+1)}\sum_{j=0}^{\infty}\tbinom{q+j}{j}(i\varphi)^{j}\,\eta^{(\alpha,q+j)}_{\downarrow}(x) + \\ &\quad + \tfrac{1}{2}\,e^{-i(\varphi\pm\pi)(\alpha+1)}\sum_{j=0}^{\infty}\tbinom{q+j}{j}\,i^{j}(\varphi\pm\pi)^{j}\,\eta^{(\alpha,q+j)}_{\uparrow}(x) = \\ &= \sum_{j=0}^{\infty}\tfrac{1}{j!}(i\varphi)^{j}\,\eta^{(\alpha,j)}(x\pm io),\end{aligned} \tag{21/27}$$

and idem at infinity. In particular for $\varphi = 0$ one obtain the special case

$$\eta^{(\alpha,q)}(x \pm io) = \tfrac{1}{2}\,\eta_{\downarrow}^{(\alpha,q)}(x) + \tfrac{1}{2}\,e^{\mp i\pi(\alpha+1)}\sum_{j=0}^{\infty}\tbinom{q+j}{j}(\pm i\pi)^j\,\eta_{\uparrow}^{(\alpha,q+j)}(x). \tag{21/28}$$

Remark 21.2 It is tempting to introduce notation such as $\eta\big(e^{i\varphi}(x \pm io)\big)$, but this might cause confusion by suggesting that η–functions of complex argument have been defined, which is not the case. Functions of argument $\big(e^{i\varphi}(x \pm io)\big)$ are functions defined on the real axis.

Evaluation of the residue gives in case of the powers gives

$$\begin{aligned}
&\big(e^{i\varphi}(x' \pm io)\big)^{\alpha}\log^{q}\big(e^{i\varphi}(x' \pm io)\big) = \\
&= e^{i\varphi\alpha}x'^{\alpha}\big(\log(x) + i\varphi\big)^{q}H(x) + \\
&\quad + e^{i(\varphi\pm\pi)\alpha}(-x)^{\alpha}\big(\log(-x) + i(\varphi \pm \pi)\big)^{q}H(-x) + \\
&\quad + e^{i\varphi\alpha}\sum_{j=0}^{\infty}\tfrac{(-)^j q!\,j!}{(q+j+1)!}(i\varphi)^{q+j+1}\big((-)^j\eta_{\downarrow}^{(-\alpha-1,j)}(x) - \eta_{\uparrow}^{(-\alpha-1,j)}(x)\big) + \\
&\quad + e^{i(\varphi\pm\pi)\alpha}\sum_{j=0}^{\infty}\tfrac{(-)^j q!\,j!}{(q+j+1)!}\big(i(\varphi \pm \pi)\big)^{q+j+1}\big((-)^j\eta_{\uparrow}^{(-\alpha-1,j)}(x) - \eta_{\downarrow}^{(-\alpha-1,j)}(x)\big).
\end{aligned} \tag{21.29}$$

In particular for $\varphi = 0$ one obtains

$$\begin{aligned}
&(x' \pm io)^{\alpha}\log^{q}(x' \pm io) = \\
&\quad = x^{\alpha}\log^{q}(x)H'(x) + e^{\pm i\pi\alpha}(-x)^{\alpha}\big(\log(-x) \pm i\pi\big)^{q}H'(-x) + \\
&\quad + e^{\pm i\pi\alpha}\sum_{j=0}^{\infty}\tfrac{(-)^j q!\,j!}{(q+j+1)!}(\pm i\pi)^{q+j+1}\big((-)^j\eta_{\uparrow}^{(-\alpha-1,j)}(x) - \eta_{\downarrow}^{(-\alpha-1,j)}(x)\big).
\end{aligned} \tag{21.30}$$

One sees that the side on which the η–functions appear depends on the choice of phase on the positive real axis. Consequently it is not true that exponentials can be taken out of the primed powers,

$$e^{i\varphi\alpha}(x' \pm io)^{\alpha} \neq \big(e^{i\varphi}(x' \pm io)\big)^{\alpha}, \tag{21.31}$$

as the η–functions are different. This even holds if φ is a multiple of 2π.

The corresponding functions in $\overline{\mathbf{PC}}_{\lambda}$ are again defined by application of the inverse mapping,

$$\big(e^{i\varphi}(x \pm io)\big)^{a}\log^{q}\big(e^{i\varphi}(x \pm io)\big) := \overline{\mathcal{M}}^{-1}\big(e^{i\varphi}(x' \pm io)\big)^{\alpha}\log^{q}\big(e^{i\varphi}(x' \pm io)\big), \tag{21.32}$$

These functions differ from the corresponding preliminary functions by at most $\delta^{(p)}$–functions at $x = 0$. In particular one recovers the standard result from distribution theory

$$\left(e^{i\varphi}(x\pm io)\right)^{\alpha} \log^{q}(x\pm io) = \left(e^{i\varphi}(x\pm io)\right)^{\alpha} \log^{q}\left(e^{i\varphi}(x\pm io)\right)_{\text{pre}} + \sum_{p=0}^{\infty} \delta_{\alpha,-p-1}\left((i\varphi)^{q+1} e^{i\varphi\alpha} + (i(\varphi\pm\pi))^{q+1} e^{i(\varphi\pm\pi)\alpha}\right) \delta^{(p)}(x), \tag{21.33}$$

which reduces for $\varphi = 0$ to the more familiar form

$$(x\pm io)^{\alpha} \log^{q}(x\pm io) = x^{\alpha} \log^{q}(x) H(x) + e^{\pm i\pi}(-x)^{\alpha}\left(\log^{q}(-x) \pm i\pi\right) H(-x) + \sum_{p=0}^{\infty} \delta_{\alpha,-p-1}\, (-)^{p+1} (\pm i\pi)^{q+1} \delta^{(p)}(x). \tag{21.34}$$

In particular for $q = 0$ one obtains

$$(x\pm io)^{-p-1} = x^{-p-1} \mp i\pi\, \delta^{(p)}(x). \tag{21.35}$$

This agrees with the standard result when the different normalization of the δ–function is taken into account.
It is also possible to define $\delta^{(p)}$–functions of argument $(x\pm io)$ by

$$\delta_{\varphi}^{(p)}(x\pm io) := \bar{\mathcal{M}}^{-1}\, \eta_{\varphi}^{(p)}(x\pm io), \tag{21.36}$$

but fortunately this is not necessary, since it does not produce any new generalised functions. Computation of the inverse mapping simply gives

$$\delta_{\varphi}^{(p)}(x\pm io) = 0(x), \tag{21.37}$$

so the definition (21.36) is superfluous, and it will not be used in this book.

This completes the definition of the powers and η–functions of argument $(x\pm io)$. Ordinary functions (such as Bessel functions) can be defined as functions of argument $(x\pm io)$ at singular points if the asymptotic expansion at these points is of the required form.

The operators are defined in the usual way taking residues and using the mappings. Some typical results are

$$\mathcal{X}\left(e^{i\varphi}(x'\pm io)\right)^{\alpha} = e^{-i\varphi}\left(e^{i\varphi}(x'\pm io)\right)^{\alpha+1} \tag{21.38}$$

and

$$\mathcal{X}\, \eta_{\varphi}^{(\alpha,q)}(x\pm io) = e^{-i\varphi}\eta_{\varphi}^{(\alpha-1,q)}(x\pm io), \tag{21.39}$$

for the operator $\mathcal{X}$, and

$$\mathcal{D}\big(e^{i\varphi}(x'\pm io)\big)^{\alpha} = \alpha\, e^{i\varphi}\big(e^{i\varphi}(x'\pm io)\big)^{\alpha-1} + 2\, e^{i\varphi}\big(\eta_{\varphi}^{(-\alpha)}(x\pm io) - \not\eta_{\varphi}^{(-\alpha)}(x\pm io)\big), \tag{21.40}$$

which corresponds in $\overline{\mathbf{PC}}_{\lambda}$ with

$$\mathcal{D}\big(e^{i\varphi}(x\pm io)\big)^{\alpha} = \alpha\, e^{i\varphi}\big(e^{i\varphi}(x\pm io)\big)^{\alpha-1}, \tag{21.41}$$

which holds both for integral and non-integral values of α.
For the η_{φ}–functions the result is formally the same as for the ordinary η–functions

$$\mathcal{D}\,\eta_{\varphi}^{(\alpha,q)}(x\pm io) = -(\alpha+1)\, e^{i\varphi}\eta_{\varphi}^{(\alpha+1,q)}(x\pm io) + (q+1)\, e^{i\varphi}\eta_{\varphi}^{(\alpha+1,q+1)}(x\pm io). \tag{21.42}$$

Finally the Fourier transforms take the form

$$\mathcal{F}(x'\pm io)^{\alpha} = -2\,\operatorname*{Res}_{\lambda=\alpha}(\lambda-\alpha)^{-1}\,\Gamma(\lambda+1)\, e^{\pm i\frac{\pi}{2}\lambda}\sin\pi\lambda\,(\pm x)^{-\lambda-1}H'(\pm x), \tag{21.43}$$

It is not useful to compute this residue explicitly. It does not reduce to a sum over the usual c_j–coefficients.
In $\overline{\mathbf{PC}}_{\lambda}$ (21.43) reduces to

$$\mathcal{F}(x\pm io)^{\alpha} = \begin{cases} -2\, e^{\pm i\frac{\pi}{2}\lambda}\sin\pi\lambda\;\Gamma(\alpha+1)\, x^{-\alpha-1}H(x) & \alpha\neq p, \\ 2\pi(-i)^{p}\,\delta^{(p)}(x)/p! & \alpha = p, \end{cases} \tag{21.44}$$

in agreement with the corresponding formula in distribution theory.
Products behave as expected. By direct computation one finds in $\overline{\mathbf{PC}}'_{\lambda}$

$$\big(e^{i\varphi}(x'\pm io)\big)^{\alpha}\log^{q}(x\pm io)\bullet\big(e^{i\varphi}(x'\pm io)\big)^{\beta}\log^{r}(x\pm io) = = \big(e^{i\varphi}(x'\pm io)\big)^{\alpha+\beta}\log^{q+r}(x\pm io), \tag{21.45}$$

where the sum over factorials

$$\sum_{j=0}^{r}\frac{(-)^{j}(l+j)!}{j!\,(r-j)!\,(q+l+j+1)!} = \frac{l!\,(q+r)!}{q!\,r!\,(q+r+l+1)!}, \tag{21.46}$$

can be evaluated using the toolkit supplied in [G,K&P].
For the η–functions one finds

$$\big(e^{i\varphi}(x'\pm io)\big)^{\alpha}\log^{q}(x'\pm io)\bullet\eta_{\varphi}^{(\beta,r)}(x'\pm io) = \begin{cases}\eta_{\varphi}^{(\beta-\alpha,r-q)}(x'\pm io) & r\geq q, \\ O(x) & r<q. \end{cases} \tag{21/47}$$

As always the product of the η–functions remains zero. It is seen that the product conserves the causal character of the functions. These properties carry over immediately to $\overline{\mathbf{PC}}_\lambda$. This results in

$$\left(e^{i\varphi}(x \pm io)\right)^{\alpha} \log^{q}(x \pm io) \bullet \left(e^{i\varphi}(x \pm io)\right)^{\beta} \log^{r}(x \pm io) = \\ = \left(e^{i\varphi}(x \pm io)\right)^{\alpha+\beta} \log^{q+r}(x \pm io), \qquad (21.48)$$

while the product with $\delta^{(p)}(x)$–functions becomes trivial. There is only one kind of $\delta^{(p)}$–function.

This simplicity of the product is lost when functions of argument $(x+io)$ are multiplied with functions of argument $(x-io)$. It is also lost when functions with a different choice of phase on the positive real axis are multiplied together. Finally the products of elements $\in \overline{\mathbf{PC}}'_\lambda$ with elements from $\overline{\mathbf{PC}}_\lambda$ is not simple.

The simple product properties for functions of argument $(x \pm io)$ are easily obtained from those of the corresponding powers by defining functions of argument $(x \pm io)$ in terms of asymptotic expansions in powers and logarithms of argument $(x \pm io)$.

Example 21.1 It is instructive to compare the the different ways in which the product properties are realized in $\overline{\mathbf{PC}}'_\lambda$ and $\overline{\mathbf{PC}}_\lambda$. In $\overline{\mathbf{PC}}'_\lambda$ we compute

$$(x' \pm io)^{\alpha} \bullet (x' \pm io)^{-\alpha-1} = (x' \pm io)^{-1}, \qquad (21.49)$$

and the η–functions come from products of $H'(x)$ and $\eta(x)$–functions on the same side of the origin. In $\overline{\mathbf{PC}}_\lambda$ the computation

$$(x \pm io)^{\alpha} \bullet (x \pm io)^{-\alpha-1} = (x \pm io)^{-1}, \qquad (21.50)$$

also gives the correct amount of δ–function, but the result now comes from cross-terms of the form $(x)^{\alpha} H(x) \bullet (-x)^{-\alpha-1} H(-x)$.

The simple product properties for the functions of argument $(x \pm io)$ correspond under Fourier transformation to simple convolution properties of the causal functions with support on the positive real axis. The convolution of two functions of positive support is again a function of positive support.

21.3 Boundary values of analytic functions

In distribution theory Titchmarsh's theorem is valid. Every distribution on the real axis is the sum of the boundary values of an analytic function in the imaginary upper half-plane and one in the lower half-plane,

$$f(x) = f_{+}(x) + f_{-}(x), \qquad (21.51)$$

with f_+ and $f_-(x)$ defined by

$$f_+(x) := \mathcal{F}^{-1}\big(H(x) \bullet (\mathcal{F} f(x))\big), \tag{21.52}$$
$$f_-(x) := \mathcal{F}^{-1}\big(H(-x) \bullet (\mathcal{F} f(x))\big). \tag{21.53}$$

These functions are known as the positive and negative frequency parts, or as analytic signals. In the following the term analytic parts will be used.
It follows immediately from the definition that

$$f_+(x) + f_-(x) = f(x), \tag{21.54}$$
$$f_+(x) - f_-(x) = -i\mathcal{H} f(x). \tag{21.55}$$

and conversely

$$f_+(x) = \tfrac{1}{2}\big(f(x) - i\mathcal{H}(x)\big), \tag{21.56}$$
$$f_-(x) = \tfrac{1}{2}\big(f(x) + i\mathcal{H}(x)\big), \tag{21.57}$$

(with the 'wrong' sign). These definitions can be taken over for the symmetrical generalised functions.

Instead of considering a new function space of analytic functions by allowing the Fourier transformed variable to be complex, it is more convenient to keep the Fourier operator as a mapping of $\mathbf{GF}_s$ into itself, and to consider instead the limiting behaviour of the generalised functions

$$f_+(x; a) := \mathcal{F}^{-1}\big(e^{-ax} H(x) \bullet (\mathcal{F} f(x))\big), \tag{21.58}$$
$$f_-(x; a) := \mathcal{F}^{-1}\big(e^{ax} H(-x) \bullet (\mathcal{F} f(x))\big). \tag{21.59}$$

as $a \downarrow 0$. The interesting case occurs when these functions have a support which is larger than a point. This is the case in $\overline{\mathbf{PC}}_\lambda$ and in the corresponding part $\overline{\mathbf{PC}}'_{\mathcal{M}}$ of $\overline{\mathbf{PC}}'_\lambda$.

As in Ch. 14 in the definition of the generalised function integral, the increased analysing power allows us to split the frequency domain into more parts than is possible in distribution theory.
The 'zero frequency part' may be defined by

$$f_0(x) := \mathcal{F}^{-1}\big(\theta(x) \circ\!\bullet \mathcal{F} f(x)\big), \tag{21.60}$$

which has the advantage that the functions x^p do not have to be split. Likewise one may define the 'positive infinitesimal frequency part' by

$$f_{0+}(x) := \mathcal{F}^{-1}\big(\theta_\downarrow(x) \bullet\!\circ \mathcal{F} f(x)\big), \tag{21.61}$$

and the 'strictly positive frequency' part by

$$f_{++}(x) := \mathcal{F}^{-1}\big(H'(x) \bullet\!\circ \mathcal{F} f(x)\big). \tag{21.62}$$

Finally the 'positive infinite frequency' part may be defined by

$$f_{+\infty-}(x) := \mathcal{F}^{-1}\big(\emptyset_{\uparrow}(x) \bullet\!\circ\, \mathcal{F} f(x)\big). \tag{21.63}$$

The standard positive frequency part may be recovered as

$$f_{+} := \tfrac{1}{2} f_0(x) + f_{0+}(x) + f_{++}(x) + f_{+\infty-}(x). \tag{21.64}$$

Analogous definitions apply on the negative frequency side.

If the analytic parts are considered as functions of the complex variable $z = x + ia$, the strictly positive frequency part is analytic in the half-plane $\operatorname{Im} z < 0$. The negative frequency part is analytic in the upper half-plane.

Fourier transforming the limit properties (19.49) of the Heaviside functions or by direct computation one finds the corresponding limit properties for the analytic parts.
In the special case of the powers one finds

$$\operatorname*{Lim}_{a\downarrow 0}(x \pm ia)^{\alpha} = (x{\pm}io)^{\alpha}, \tag{21.65}$$

The limiting behaviour is as expected, one finds

Property 21.1 Analytic boundary property
The limit of an analytic part equals the corresponding function of argument $(x'{\pm}io)$.

Verification: The property holds for the powers by direct computation. Therefore it holds for the asymptotic expansions, and therefore also for the ordinary functions. □

The product properties are also as expected

Property 21.2 The product of the limits equals the limit of the product.
If

$$\operatorname*{Lim}_{a\downarrow 0} f(x \pm ia) = f(x'{\pm}io), \quad \text{and} \quad \operatorname*{Lim}_{a\downarrow 0} g(x \pm ia) = g(x'{\pm}io),$$

then

$$\operatorname*{Lim}_{a\downarrow 0} f \cdot g(x \pm ia) = f \bullet g\,(x'{\pm}io), \tag{21.66}$$

Verification: As above it is sufficient to verify the property for the powers. □

The usual formulæ from distribution theory can be recovered by taking the completed limit, for example

$$\operatorname*{\overline{Lim}}_{a\downarrow 0}(x \pm ia)^{-1} = (x{\pm}io)^{-1} = x^{-1} \mp i\pi\,\delta(x), \tag{21.67}$$

The formula simplify, since there is only one δ–function. The difference between left and right of a singularity is no longer observable.

21.4 Causality

Many systems in which things vary in time, for instance electrical networks with varying applied voltages, can be described by linear (differential) equations. Physical systems satisfy the condition that effects follow their causes. In mathematical terms this means that the Green function of a causal system should be zero for negative values of the time. Therefore a generalised function is defined to be causal when it satisfies the condition

$$H(-x) \bullet f(x) = 0(x), \tag{21.68}$$

or equivalently

$$H(x) \bullet f(x) = f(x), \tag{21.69}$$

since the two Heaviside functions sum to the unit function. Combining (21.69) and (21.68) gives

$$\operatorname{sgn}(x) \bullet f(x) = f(x). \tag{21.70}$$

By Fourier transformation and the definition of the Hilbert transform this can be rewritten as

$$\mathcal{F}^{-1} f(x) = -i\,\mathcal{H}\mathcal{F}^{-1} f(x). \tag{21.71}$$

The Hilbert transform anti-commutes with the parity operator

$$\mathcal{H}\mathcal{P} = -\mathcal{P}\mathcal{H}, \tag{21.72}$$

so (21.71) can be written more conveniently in terms of the Fourier operator

$$\mathcal{F} f(x) = i\,\mathcal{H}\mathcal{F} f(x). \tag{21.73}$$

Since it is easily seen that the Hilbert transform commutes with taking real parts, the expression (21.73) can be reduced to the pair

$$\operatorname{Re}\mathcal{F} f(x) = -\mathcal{H}\operatorname{Im}\mathcal{F} f(x), \tag{21.74}$$

and

$$\operatorname{Im}\mathcal{F} f(x) = \mathcal{H}\operatorname{Re}\mathcal{F} f(x), \tag{21.75}$$

which is valid when $f(x)$ is causal. When $f(x)$ is a causal function the real and imaginary parts of its Fourier transform are a Hilbert pair. This is a reason for adopting the definition (21.69).

In the terminology of linear response theory the response to an applied $\delta(x)$ input is called the impulse response function. Its Fourier transform is called the spectral response function.

The relations (21.75) and (21.74) are called Kramers-Kronig relations or dispersion relations. When a system is causal its spectral response satisfies a dispersion relation.

The standard definition of causality (21.68) has some properties which one has to get used to. Application of the definition shows that the generalised functions

$$H(x), \quad \text{and} \quad \eta_{\downarrow}(x), \tag{21.76}$$

are causal. More generally, $\forall\, f'(x) \in \overline{\mathbf{PC}}'_{\lambda}$ the generalised function $H(x) \bullet f'(x)$ is causal. The $\delta(x)$–function is not causal, since

$$H(x) \bullet \delta(x) = \tfrac{1}{2}\, \delta(x) \neq \delta(x). \tag{21.77}$$

It may be preferable to define 'strict causality' by

$$H'(x) \circ\!\bullet\, f(x) = f(x). \tag{21.78}$$

The generalised functions $\eta_{\downarrow}(x)$, and $H'(x)$ are strictly causal, the generalised function $H(x)$ is not.

An idealized resistor, described by Ohm's law is not strictly causal or even causal in the standard sense. Ohm's law

$$\mathrm{I} = \mathrm{V}/\mathrm{R}, \tag{21.79}$$

leads to an impulse response given by a $\delta(x)$–function, the spectral response is given by the unit function. The response is not causal, the spectral response does not satisfy a dispersion relation of the form (21.71). Subtraction of the constant part at infinity leaves $0(x)$, so there is no subtracted dispersion relation either. In the sense of distribution theory this is all that can be done.

In the sense of this book this problem disappears if the impulse response is assumed to be given by $\eta_{\downarrow}(x)$ instead of $\delta(x)$. This is a causal function satisfying (21.68–69). The corresponding spectral response is found by Fourier transformation

$$\begin{aligned} \mathcal{F}\,\eta_{\downarrow}(x) = I'(x) + \theta(x) + & \\ - 2\sum_{j=0}^{\infty} j!\big(c_j(-1,p)\, \not{\phi}^{(-1,j)} &+ i\, c_j(-1,p+1)\, \not{\eta}^{(-1,j)}(x)\big). \end{aligned} \tag{21.80}$$

The real and imaginary parts of (21.80) are indeed by (21.16) and (21.14) a Hilbert pair. The spectral response is causal, and it satisfies a dispersion relation (21.74–75). An $\eta_{\downarrow}(x)$ response is physically indistinguishable from a $\delta(x)$ response, since the response

$$\mathrm{I}(x) = \eta_{\downarrow}(x) * \mathrm{V}(x)/\mathrm{R}, \tag{21.81}$$

is different from the non-causal response

$$I(x) = \delta(x) * V(x)/R, \tag{21.82}$$

only when the applied voltage contains infinitely high frequencies, which are absent in realistic applied voltages.

One can interpret the causal resistor and the non-causal resistor as different idealizations of realizable resistors. The causal resistor is in a sense more realistic, since physically realizable resistors are equivalent to a network with a small inductance in series and a small capacitor in parallel. These non-Ohmic contributions to its impedance cause the spectral response to go to zero for high frequencies.

The non-causal resistor has an actually infinite frequency response, so it cannot be realized. The causal resistor corresponds to the limit of a actually realizable resistor, when the frequency response is (by idealization) assumed to extend to arbitrarily high frequencies.

The completed limits defined in Ch. 19 can be used to obtain a unit frequency response function. In agreement with the naïve interpretation the actual infinite cannot be realized by a limit process.

The example given above illustrates that the symmetrical theory of generalised functions makes it both possible and necessary to consider the idealizations one wants to allow.

CHAPTER 22

ON REGULARIZATION

Obtaining the results of mathematical computations often takes the form of the evaluation of definite integrals. When the integrands of these integrals are (Lebesgue) integrable the evaluation is well defined, at least in principle.

In many cases of interest, in particular in quantum field theory, the integrals turn out to be divergent, and therefore meaningless. It is necessary to inject meaning into the integrals afterwards. Regularization and renormalization are the terms used for this process. It seems to be impossible to give a general method beforehand, instead the regularization method is defined afterwards on basis of physical ideas as to the desirability of the results to be obtained.

Opinions of the author This is a curious and unacceptable situation. It seems to have been accepted mainly on basis of its empirical success and the lack of an alternative.

In a good physical theory, resting on a sound mathematical basis, it must be clear beforehand how computations are to be carried out. The results should always be well defined, leaving no room for different interpretations. There should be no room nor necessity for ad-hoc regularizations devised afterwards on basis of the desirability of their results.

Clearly this state of affairs may have two possible causes. The physical theory might be incorrect, or the mathematical apparatus brought to bear on it might be inadequate.

The correctness of the physical theory as a description of nature is not really relevant in this context. Whether electrons are point particles or very small finite-sized particles, with a size which is (forever?) too small to be observed is irrelevant.

An adequate mathematical apparatus should have the possibility to handle both cases. Even if a charged point particle does not exist in nature it is a plausible idealization. Simplification by idealization and exploration of the consequences is what mathematical analysis should accomplish.

The root of the problem is the occurrence of actual infinities in the case of a theory of point particles. The interaction energy of an extended particle tend to infinity when the size of the particle tends to zero. The interaction energies of point particles are infinite to begin with.

By banishing the actual infinite from analysis (compare Sec. 1.1) the possibility of supplying an adequate foundation for a quantum field theory of point particles has been lost.

The difficulty of the incorporation of Dirac's δ–function in analysis, and the impossibility of solving the multiplication problem for distributions can be seen as mathematical consequences of the same lack of adequate treatment of the infinite.

Of course the problem is not that classical analysis as we know it is in any way incorrect. It is merely that the rigorization of analysis (as it has been generally accepted) is not adequate as a basis for the mathematical treatment of actual infinities. It only deals with limit processes which yield finite results.

In this view the persistence of divergence problems in quantum field theory does not indicate a lack of adequate understanding of the physical world. Instead it indicates the need for a better treatment of the infinite in analysis. The efforts reported in this tract may be a first step in this direction. **end of opinions.**

Example 22.1 The question: Is the perturbation expansion in quantum electrodynamics gauge invariant? can be answered with:

1) No,
2) That depends on the regularization,
3) Of course! The regularization *must* be chosen in such a way that the result is gauge invariant.

Textbooks on quantum electrodynamics generally favour the third answer but disagree on the proper method of achieving this result, [QED]. There is no universal agreement that the possibility of the third answer has been proved, [F,H,R&W] although this is generally assumed.

22.1 Perturbations in quantum field theory

This section provides only the briefest possible outline. It may be skipped.

A quantum field theory is defined by specifying a suitable Lagrangian density. In many cases this consists of a free part, leading to equations for free fields which can be solved trivially, and an interaction term which can only be handled by means of a perturbation expansion.

The perturbation expansion results in products of propagators (Green functions) of the free fields. These propagators involve repeated products of (modified) Bessel functions of argument $(x^2 + a^2 - io)$, and therefore of the generalised functions x^{-2}, $\delta(x^2)$, $x^{2n}H(x^2)$, and $x^{2n}\log|x^2|$. These products are undefined in the sense of distribution theory.

The undefined products are then Fourier transformed into undefined (because divergent) convolution integrals, assuming (without justification) that the standard integral formula for the convolution, (2.56) or (22.13), is valid also when it yields divergent integrals. (This turns out to be incorrect in Sec. 22.6).

The resulting undefined convolution integrals are then subjected to a regularization process which defines their meaning. The quantum field theory is said to be renormalizable when this is possible by introducing no more than a finite number of empirical parameters. The results are in often spectacular agreement with experiment.

There are several difficulties with this procedure.

1) The validity of the classical convolution formula must be taken on faith.
2) The equivalence of coordinate and momentum representations is lost, since the renormalization can only be carried out in the momentum representation.
3) The unitarity of the Fourier transformation from coordinate to momentum representation is also a matter of faith, the results cannot be evaluated in the original coordinate representation.
4) The presence of δ–functions is not compatible with the use of a Hilbert space for the state vectors.
5) It is only a prescription, which does not rest on a sound foundation.

Despite these problems the situation is not as bad as it might seem from the above discussion. The results of the renormalization can be shown to be correct up to finite renormalizations.

22.2 Standard regularizations

In a standard treatment it is necessary to define a regularization of the divergent integral in order to obtain finite predictions of the theory. There are many ad-hoc regularization schemes available in the literature.

It can be seen that the products, integrals, and convolutions which occur in quantum field theory are well defined in the sense of the symmetrical theory of generalised functions. Also the limit processes involved in the regularization are well defined (Ch. 19) in the sense of this tract. Therefore the result of a regularization method can be compared with the result in the sense of generalised function theory.

When the results do not agree it is possible to obtain the necessary corrections. Regularization methods are thereby reduced to more or less convenient computation methods, which can be used to evaluate integrals in the sense of this tract.

In the following the complications inherent in the Minkowski geometry of space-time are ignored. Instead the equivalent problems are discussed for functions of one independent variable. This exhibits the most important points.

A first class of regularization methods is obtained by introducing a cutoff function depending on a parameter, in such a way hat the cutoff function approaches the unit function in a suitable limit. This approach fits in with the limiting properties of sequences of generalised functions found in Ch. 19. These limiting properties can be used to evaluate the cutoff regularization methods, and to evaluate the corrections which have to be made to a given cutoff procedure.

A second class of methods uses analytic techniques resembling those used in this work. These methods allow for even more arbitrariness than the cutoff methods. It is possible to obtain the generalised function results in this way.

It will be seen in the last section that it is incorrect to assume that the convolution product can be obtained by regularization of the classical form of the convolution integral. Consequently it is also impossible to find the generalised function products from assumed regularizations of the convolution integral.

The theory of generalised functions allows less arbitrariness than regularization methods. Moreover it is possible to show explicitly where the remaining arbitrariness resides, and how it is fixed.

22.3 Cutoff regularization

A cutoff regularization of an integral is defined by choosing a cutoff function $f(x;a)$, such that it tends to the unit function in the limit. Only results with correct scaling properties are of interest, so we can restrict attention to scaling limits of the type considered first by Dirac. This means that we choose cutoff functions of the type $\mathrm{Lim}_{a\uparrow\infty}\, \boldsymbol{S}(a^{-1})\, f(x) = f(a^{-1}x)$, normalized to $f(0) = 1$. Calculating the scalar product with a power yields

$$\int_{-\infty}^{\infty} dx\, |x|^{\lambda}\, f(a^{-1}x) = a^{-\lambda-1} \int_{-\infty}^{\infty} dx\, |x|^{\lambda}\, f(x) := a^{-\lambda-1}\, \tilde{f}(\lambda), \tag{22.1}$$

which gives for the limit in the sense of Ch. 19

$$\mathrm{Lim}_{a\uparrow\infty} f(a^{-1}x) = \big(I'(x) + \theta(x)\big) + \sum_{j=0}^{\infty} \tilde{f}^{(j)}(-1)\, \not{\phi}^{(-1,j)}(x). \tag{22.2}$$

It is immediately clear from (22.2) that the difficulties with a regularization occur in the case of integrands which have logarithmically divergent terms of the form $|x|^{-1}\log^{q}|x|$ in their asymptotic expansion at infinity.

Remark 22.1 When 'ordinary' functions considered as elements $\in \overline{\mathbf{PC}}_{\lambda}$ without a generalised function at infinity are considered as integrands the difference between the upper limit $+\infty-$ and $+\infty$ vanishes.

Comparing the limit property (22.2) with the definition of the integral in the sense of generalised functions in Ch. 14

$$\int_{-\infty}^{\infty} dx\; f(x) := \big\langle\, I(x)\, ,\, f(x)\, \big\rangle, \tag{22.3}$$

one sees that the regularization will reproduce the values of the integral if and only if all coefficients $\tilde{f}^{(j)}(-1) = 0$ are zero.

This is the case when we choose $f(x) = H(1+x)H(1-x)$, or $f(a^{-1}x) = H(a+x)H(a-x)$. The limit property now takes the form

$$\operatorname*{Lim}_{a\uparrow\infty} \int_{-a}^{a} dx\ f(x) = \int_{-\infty+}^{+\infty-} dx\ f(x) = \int_{-\infty}^{\infty} dx\ f(x), \tag{22.4}$$

in agreement with expectations. For other cutoff functions the expected result does not hold. For example from (19.48) we have

$$\operatorname*{Lim}_{a\downarrow 0} \int_{0}^{\infty} dx\ e^{-a^2x^2} f(x) = \int_{0}^{\infty} dx\ f(x) + \sum_{j=0}^{\infty} \Gamma^{(j)}(\tfrac{1}{2}) \langle \phi^{(-1,j)}(x)\, , f(x) \rangle \neq$$
$$\neq \int_{0}^{\infty} dx\ f(x). \tag{22.5}$$

The difference appears when the integral diverges logarithmically.

If one wants to use an arbitrary cutoff function it is not difficult to recover the generalised function results by computing the appropriate Mellin coefficients, and by correcting (22.5) accordingly.

22.4 Analytic regularizations

A second method to define divergent integrals is based on the observation that it is often possible to generalise one of the parameters of the integrand to a complex number, or to introduce an additional function depending on a complex parameter, in such a way that the integral becomes a meromorphic function of the parameter. Divergences are signalled by the appearance of poles. The value of the integral is then defined by taking a suitable residue.

Example 22.2 The power of x satisfies

$$\operatorname*{Res}_{\lambda=\alpha} (\lambda-\alpha)^{-1}\, x^{\lambda} H(x) = x^{\alpha} H'(x). \tag{22.6}$$

Correspondingly the integral in the sense of generalised functions satisfies

$$\int_{-\infty+}^{+\infty-} dx\ f(x) = \operatorname*{Res}_{\lambda=0}\ \lambda^{-1} \int_{-\infty}^{\infty} dx\, |x|^{\lambda} f(x), \tag{22.7}$$

in agreement with the result (5.22) obtained in Ch. 4.

Remark 22.2 It does not matter if one writes

$$\text{Res} \cdots x^{\lambda} \bullet f(x), \quad \text{or} \quad \text{Res} \cdots x^{\lambda} f(x),$$

since the difference is zeromorphic.

Analytic regularizations are not less arbitrary than cutoff regularizations. One may add an arbitrary (entire) function $\tilde{g}(\lambda)$ of λ to (22.6) to obtain

$$\operatorname*{Res}_{\lambda=0} \tilde{g}(\lambda)\, |x|^{\lambda} = \tilde{g}(0)\, I'(x) + \sum_{j=0}^{\infty} \tfrac{1}{j!}\, \tilde{g}^{(j+1)}(0)\, \not\delta^{(-1,j)}(x), \tag{22.8}$$

with corresponding additional terms in the regularization of integrands containing a logarithmically divergent term in their asymptotic expansion at infinity.

There is more arbitrariness in analytic methods, since it is largely a matter of taste which parameter one wants to complexify.

Example 22.3 In the method of dimensional regularization one defines

$$\int_{-\infty}^{\infty} dx\; f(x) := \operatorname*{Res}_{n=1} (n-1)^{-1} \int d^n r\; g(n)\, f(\mathbf{r}), \tag{22.9}$$

with $f(\mathbf{r})$ a suitable generalisation of $f(x)$ to n-dimensional space. This would give the same result as the generalised function computation, except that it is customary to introduce additional analytic functions of $g(n)$ in the residue.

Straightforward dimensional regularization with $g(n) = 1$ reproduces the generalised function results. Choice of a different analytic function in (22.8) corresponds to a different standardization of the generalised function product.

Remark 22.3 In the method of dimensional regularization, as used in quantum field theory, the auxiliary analytic function depends on the integral to be regularized. This destroys the linearity of the integral ($\int(f+g) \neq \int f + \int g$), and introduces an arbitrariness which cannot be accommodated in a mathematical theory based on first principles.

22.5 Arbitrariness and standardization

It is clear that all regularization methods allow some arbitrariness. The same possibility for allowing arbitrariness exists in the symmetrical theory of generalised functions. It was seen in Ch. 10 and Ch. 14 that the values of the integrals

$$\int_{-\infty}^{\infty} dx\, |x|^{-1} \log^q |x| := c_q, \tag{22.10}$$

can be chosen arbitrarily. This arbitrariness also appears in integrals diverging s more strongly than logarithmically when the asymptotic expansion at infinity contains a logarithmically divergent term.

When this arbitrariness is allowed no meaning can be attached to the differences of divergent integrals of this type, since the scaling of the different logarithms is not related. The scale operator is not unitary in this approach. The lack of unitarity can be compensated by the freedom to choose the regularization prescription.

For the generalised function product the arbitrariness can be fixed, up to the single indeterminate constant C, by the requirement that the generalised function product should transform properly under scale transformations. By (17.36) the scale transform of the (properly standardized) product equals the product of the scaled factors

$$\mathcal{S}(a)\big(f(x) \bullet g(x)\big) = \mathcal{S}(a)\, f(x) \bullet \mathcal{S}(a)\, g(x), \tag{22.11}$$

and consequently by (17.37) the normalized scale transformation is unitary

$$\langle\, \bar{\mathcal{S}}(a)\, f(x)\,, \bar{\mathcal{S}}(a)\, g(x)\,\rangle = \langle\, f(x)\,, g(x)\,\rangle, \tag{22.12}$$

in the corresponding scalar product.

This fixes the standardization, and hence the arbitrary constants in the evaluation of divergent integrals. The mathematical development in Ch. 13 and Ch. 14 used this standardization on grounds of simplicity and mathematical elegance.

The availability of a scale invariant product opens the possibility that the now well defined finite renormalizations can have a physical interpretation, [Lod2]. This possibility remains to be explored.

22.6 Convolutions and surface terms

The discussion in the previous section focussed on the logarithmically divergent integrals, since the arbitrariness resides there. The phenomenon of surface terms must also be considered. It was shown in Ch. 15 that the value of a linearly divergent integral will change by a finite quantity when the integrand is translated.

This will introduce a lack of determinacy when there is no reason to prefer an origin for the coordinate system. This is usually assumed to be the case in quantum field theory.

The problem can be converted in mathematical terms as an attempt to define products of generalised functions by regularization of convolutions. This is based on the convolution formula

$$\begin{aligned}\mathcal{F}\big(\mathcal{F}^{-1} f(x) \bullet \mathcal{F}^{-1} g(x)\big) &= f(x) * g(x) = \\ &= \int_{-\infty}^{\infty} dy\ f(y)\, g(x-y) = \int_{-\infty}^{\infty} dy\ f(x-y)\, g(y),\end{aligned} \tag{22.13}$$

WRONG!

which is valid in a standard sense in $\mathcal{L}_2$. One may (in $\mathcal{L}_2$) even take the same integral with y replaced by $y + a$, for arbitrary $a \in \mathbb{R}$, In the standard treatment one assumes (without justification) that the integral formula (22.13) for the convolution also holds for generalised functions, even when the integral diverges. This makes the convolution product undefined by a surface term.

Example 22.4 Consider the product $x^{-1} \bullet \delta(x)$. This product is well defined for generalised functions, and given by $x^{-1} \bullet \delta(x) = \frac{1}{2}\,\delta^{(1)}(x)$. This result does not depend on the formalism of this book, any other definition would violate the analytic boundary property.
Fourier transformation of the product (using $\mathcal{F}x^{-1} = 2i\,\mathrm{sgn}(x)$) gives

$$I(x) * \mathrm{sgn}(x) = x^1 = x. \tag{22.14}$$

On the other hand the convolution integrals equal

$$\int_{-\infty}^{\infty} dy\ \mathrm{sgn}(y)\, I(x-y) = \int_{-\infty}^{\infty} dy\ \mathrm{sgn}(y) = 0, \tag{22.15}$$

and

$$\int_{-\infty}^{\infty} dy\ \mathrm{sgn}(x-y-a)\, I(a+y) = \int_{-\infty}^{\infty} \mathrm{sgn}(y-x+a) = 2x-2a, \tag{22.16}$$

By symmetrization we obtain after inverting the Fourier transform

$$x^{-1} \bullet \delta(x) = \tfrac{1}{2}\,\delta^{(1)}(x) + \tfrac{1}{2} ia\,\delta(x), \tag{22.17}$$

WRONG!

The result is easy to understand. Instead of $x^{-1} \bullet \delta(x)$ we have computed

$$\begin{aligned}\left(e^{iax}x^{-1}\right) \bullet \left(e^{-iax}\delta(x)\right) &= \left(x^{-1} + ia + O(x) + \right) \bullet \delta(x) = \\ &= \tfrac{1}{2}\,\delta^{(1)} + \tfrac{1}{2} ia\,\delta(x) \neq x^{-1} \bullet \delta(x).\end{aligned} \tag{22.18}$$

The factor $\frac{1}{2}$ is also incorrect since the lack of associativity of the product does not have a counterpart in the regularization of the integrals.

One sees that it is not possible to find the products of generalised functions from the regularization of convolution integrals.

The formal convolution theorem (22.13) cannot be made to hold for generalised functions, and the product cannot be obtained from the regularization of the convolution.

Whether the arbitrariness inherent in the surface terms will actually appear in the results of computations depends on the physical system in question.

Example 22.5 Quantum field theory is invariant under the Poincaré group, which contains Lorentz transformations and translations of the coordinates. In momentum space there is a preferred origin, marked by $\delta(x)$, which corresponds to the Fourier transform of a constant function.

Remark 22.4 The arbitrariness in the scaling behaviour has been fixed by using a more appropriate mathematical formalism, the arbitrariness in the surface terms cannot be fixed on mathematical grounds. Fortunately it can be fixed on physical grounds, at least in a homogeneous space-time.

Arguments in quantum field theory, which depends for their validity on the assumption of freely chosen surface terms, are incorrect.

22.7 Conclusion

It seems plausible that the symmetrical theory of generalised functions is sufficiently powerful to give rigorous meaning to the products, convolutions, and integrals needed in the perturbation expansion of quantum field theory, [Lod2]. The divergent integrals in quantum field theory appear as well defined convolution products in the sense of generalised function theory. Instead of evaluating convolutions it is simpler in many cases to evaluate the pointwise products directly, without performing the Fourier transformation.

In contrast to the standard approach there is complete symmetry under Fourier transformation, and all computations can be done either in the coordinate representation or the momentum representation. The choice is a matter of convenience only.

The characteristic difference between the generalised function method and the regularization methods is the absence of arbitrary finite renormalizations. Only one single arbitrary mass scale remains. It appears explicitly in the form of the indeterminate constant C and its powers.

Physical results do not contain C's, conversely any result that does contain C's has no physical significance. It will not transform correctly under scale transformations.

The new possibility [Lod2] is that determinate finite results may be hidden under divergences. These results are arbitrary by a finite renormalization in the standard sense.

The physical consequences of the possibility that finite renormalizations may be determinate remains to be explored. Much work remains to be done along these lines.

CHAPTER 23

MULTIPLICATION AND THE INFINITE

The simple model provides a covering theory from which many product algebras on subspaces of the distributions can be obtained by suitable specialization.

There are many theories of more general objects than distributions. (Usually these are also called generalized functions). In some of these theories multiplication is also possible. A review will not be attempted.

Instead a criterion will be discussed by which the different possible theories can be compared. The question is what one wants to accomplish by constructing a theory of generalized functions.

The actual infinite and its role in mathematics, in particular for theories of generalized functions is discussed. The need for infinitesimals and the relevance of nonstandard analysis is also discussed briefly.

23.1 A practical impossibility argument

Let us assume that a multiplication of all (tempered) distributions has been achieved. Equivalently one may assume that multiplication has been achieved in a larger theory of generalized functions, in such a way that a reduction to the distributions is possible. Consider products of the form $\delta \cdot f$, which are by hypothesis well defined distributions. From the support requirement it follows that the result of the product must be a finite linear combination of the $\delta^{(p)}$–distributions, since there are no other objects with point support in the distributions.

Consider in particular the function $f(x)$ defined by

$$f(x) := \sum_{n=1}^{\infty} n^{-2} e^{in!\,n^2 x}. \tag{23.1}$$

The periodic function $f(x)$ is bounded, $|f(x)| \leq \sum n^{-2} = \pi^2/6$, and $f(x)$ is continuous. Continuity is established by a simple estimate or by remarking that (23.1) is a Fourier series with quadratically vanishing coefficients. Therefore (23.1) also defines a tempered distribution. It is not a generalised function in the sense of this book however. The Fourier coefficients do not satisfy the required asymptotics.

Its product with the δ–function is by assumption well defined, so it must equal

$$\delta(x) \bullet f(x) = \delta(x) \sum_{n=1}^{\infty} n^{-2} = \tfrac{\pi^2}{6}\, \delta(x), \tag{23.2}$$

assuming that product and summation can be interchanged. Since $f(x)$ is bounded the derivatives of δ are absent.

Remark 23.1 The ability to interchange sums, differentiation, etc. freely is one of the justifications for having distribution theory.

Being a distribution, $f(x)$ is differentiable and its derivative is again a distribution. Taking the first derivative, interchanging summation and differentiation, and multiplying with the δ–function gives

$$\delta(x) \bullet \sum_{n=1}^{\infty} n!\, e^{in!\, n^2 x} = \delta(x) \sum_{n=1}^{\infty} n!\,. \qquad (23.3)$$

By hypothesis the product is a well defined distribution, so we should know the meaning of the expression $\sum n!$ as a finite real number. The distributional derivative of a (standard) non-differentiable continuous function is an example of a distribution which cannot be given a value at a point. Only integrals with test functions $\in \mathcal{S}$ are well defined.

The choice of the sequence $a_n := n!$ in (23.3) is of course arbitrary. Any sequence of numbers $\{a_n\} \subset \mathbb{R}$ can be substituted for $n!$. Therefore the hypothesis that distributions can be multiplied implies that for arbitrary sequences $\{a_n\} \subset \mathbb{R}$,

$$\sum_{n=1}^{\infty} a_n = ??? = \text{a well defined real number???} \qquad (23.4)$$

As a special case we may choose the sequence $a_n := \int_n^{n+1} dx\, f(x)$, for an arbitrary function $f(x) \in \mathbf{C}^{\infty}(0, \infty)$ to reach the conclusion that

$$\int_1^{\infty} dx\, f(x) = ??? = \text{a well defined real number???}, \qquad (23.5)$$

for arbitrary functions $\in \mathbf{C}^{\infty}(1, \infty)$, no matter how strongly divergent at infinity. (Of course integrability in the finite is already sufficient). The conditions (23.4) and (23.5) are necessary, there is no assurance that they are also sufficient.

Remark 23.2 The special case $\sum n!$ may be evaluated in a future generalization of the theory, but it seems impossible to achieve this in the general case.

So, the hypothesis that multiplication of distributions is possible without losing the useful properties of distribution theory implies the possibility of assigning a finite value to every divergent expression whatsoever.

One cannot exclude the possibility that the definition of a finite value for every divergence will be possible in a mathematically satisfactory way in some future theory, but for the time being it is necessary to conclude that a product on the distributions satisfying the minimal requirements used here is not possible.

The success of other approaches to the multiplication problem can be judged by their success in coming to grips with the problem of defining finite values for divergent quantities.

The simple model developed in this book is again the minimal model in this respect. It defines finite values for divergent quantities behaving as powers and logarithms, and as δ–functions.

It is partially possible to avoid the problem by defining values of generalized functions in a larger class of objects. Again the yardstick by which these efforts can be gauged is the possibility to relate these more general objects to standard real numbers.

23.2 On the nature of the infinite

(Continued from Sec. 1.1). The nature of the infinite has been a subject for speculative thought for philosophers, theologians, and of course mathematicians from the earliest times. More recently the infinite has also made its appearance in physics, in particular in (quantum) field theory. The correct handling of the infinities which arise there has been the subject of a very large research effort.

A philosophical distinction, which was explicitly made already by Aristotle, is the distinction between the actual and the potential infinite. The actual infinite is defined to be really infinite, without possibility of approaching it in any way. By contrast a potentially infinite quantity is defined to be a quantity which can be made larger than any given quantity.

Example 23.1 In terms of an actual infinite parallel lines in Euclidean geometry may be said to intersect at infinity. Euclid did not accept this. A possible restatement in terms of a potential infinite is that parallel lines will not intersect, no matter how far they are continued.

In the eighteenth century the calculus was developed in an intuitive way, without the availability of a sound logical basis. Infinite quantities and infinitesimals were freely used and manipulated, and in the right hands this yielded spectacular results.

In the nineteenth century the desire for a sound logical foundation of these operations with infinities led to the formulation of a program for the rigorization of analysis [Kli]. The content of this program was the elimination of the actual infinite and the infinitesimals from analysis, and therefore (it was then thought) from mathematics in general. The program was brought to completion by the efforts of the Weierstrass school. Many manipulations with infinities were given a sound basis by the development of the theory of analytic continuation. Divergent series were brought under control by

Poincaré's theory of asymptotics. It is these ingredients which also lie at the basis of this book.

As far as analysis is concerned the elimination of the actual infinite has been achieved, provided that the real number system is taken for granted.

It has been thought that this demonstrated the practical necessity of avoiding the actual infinite. The development of nonstandard analysis by Robinson [Robi] made this position very difficult to maintain.

23.3 The impossibility result of Schwartz

There is a well known impossibility result due to Schwartz, [Sch3], which states that an associative algebra containing the δ–fuction and the continuous functions is not possible. This result is not directly relevant to the efforts in this book, since the product of the generalised functions, when reduced to the intersection with the distributions, is inherently non-associative. It has been demonstrated in Ch. 10 that this lack of associativity is actually an advantage. It drastically simplifies the theory by allowing Leibniz' rule to hold for differentiation of multiple products (8.105).

Remark 23.3 In other theories, with an associative product and Leibniz's rule, it seems unavoidable to introduce infinitely many different Heaviside functions, with different δ-like derivatives. This leads to very large theories, with contents that are difficult to characterize.

Although Schwartz's result is not directly relevant, it offers heuristic support for the choices made in this work.

23.4 On more general theories

There exists a vast literature on generalizations of the distribution concept, that is theories which admit more singular objects than δ–functions. No attempt will be made to review these efforts.

Typically there may be a non-zero element $\delta \bullet \delta$, or there may be infinitely many different Heaviside functions, with different δ-like derivatives.

A guiding idea of the symmetrical theory of generalised functions is that the multiplication problem for distributions cannot be solved because there are to many distributions. The δ–function has been explained mathematically by imbedding it in a larger class of objects than necessary. This may be convenient for many purposes, but it makes the multiplication problem intractable.

Remark 23.4 The natural method in mathematics is to generalise as much as possible. For instance in solving differential equations it is customary to attempt an existence proof for solutions in some very general class of objects. As the next step regularity results are proved to define properties of solutions. This approach may be convenient, but it becomes self-defeating if the generality makes the multiplications needed to obtain solutions impossible, [Lod3].

It goes against the spirit of this work to attempt a solution of the multiplication problem by construction of larger classes of objects. since distribution theory is already held to be too large.

Nevertheless it may be possible to obtain a symmetrical theory of generalised functions from a larger theory, such as [Col], by a suitable reduction. More work is needed to investigate this possibility.

The test for the usefulness of larger theories is in accordance with the heuristic of the previous sections the possibility of obtaining finite results in cases where standard computations diverge.

23.5 Nonstandard analysis

The construction of nonstandard analysis by Robinson [Robi] was a very important result for the interpretation of analysis. It showed clearly that the results of analysis are independent of the standard rigorization. Indeed it opens up the possibility that there may be many ways to create a logically consistent foundation for analysis.

The emphasis on the avoidance of the actual infinite, which was the characteristic property of the standard formulation, is absent from the nonstandard formulation. In nonstandard analysis it is possible to operate freely with infinitesimals and with infinite numbers within the rules of internal set theory [Robe].

A simple model for nonstandard analysis can be constructed by considering nonstandard real numbers as equivalence classes of sequences of standard real numbers. In this model infinitesimals are identified with (equivalence classes of) sequences which converge to zero, infinite numbers with diverging sequences. The correspondence $\mathbb{R}^* \to \mathbb{R}$ takes the form of identifying convergent sequences with their limit.

It is well known that nonstandard analysis is equivalent to standard analysis in the sense that standard proofs exist for the standard part of all nonstandard results. Nonstandard analysis does not help us with with the interpretation of divergent quantities. Standard divergent expressions become nonstandard infinite numbers.

Distribution theory can be reformulated in terms of nonstandard analysis [Robi].

Example 23.2 The δ–function in nonstandard analysis becomes an equivalence class of functions $\mathbb{R}^* \to \mathbb{R}$. These functions take on infinite values in the infinitesimal environment of the point $x = 0$. and are infinitesimal everywhere else.

Unfortunately the reformulation of distribution theory in terms of nonstandard analysis is no help with the multiplication problem. The nonstandard version of the theory is equivalent to the standard approach. This is best seen when distributions are considered as equivalence classes of sequences of test functions [Lig].

For instance an object such as $\delta \cdot \delta$ may be defined as a nonstandard function by choosing particular elements of the equivalence class representing the δ–function, but this does not help us with the interpretation of the expression $\delta \cdot \delta$. The nonstandard approach supplies only a conceptual improvement with respect to the standard approach.

As pointed out above, the problem of the multiplication of generalised functions should not be separated from the problem of giving a meaning to infinite expressions in the form of standard real numbers.

Example 23.3 The value of the δ–function at $x = 0$ is undefined in a standard sense, It is a well defined infinite number for a given representant of the δ–function in nonstandard analysis. This infinite number is different for different representants of the δ–function. Therefore the value of the δ–function at $x = 0$ cannot be given a unique nonstandard value. The reinterpretation of $\mathfrak{d}(x)$ as a symmetrical generalised function supplies the value zero.

Remark 23.5 The conceptual simplifications offered by nonstandard analysis are best seen in situations where one has to deal with one limit process. In cases with two non-commuting limits rewriting the example given in Rem. 8.1 as

$$\epsilon_1^{-1} \min(\epsilon_1, \epsilon_2), \tag{23.6}$$

with ϵ_1 and ϵ_2 infinitesimal is not really helpful.

This may be a reason why nonstandard analysis has not proved useful in the construction of a solution for the multiplication problem.

The additional property of the model presented in this book can be interpreted in terms of nonstandard analysis as the construction of a new mapping $\mathbb{R}^* \nrightarrow \mathbb{R}$ with the property that it maps some, but not all, infinite nonstandard numbers into the finite standard real numbers. The mapping of finite nonstandard real numbers is not changed.

The usual mapping $\mathbb{R}^* \rightarrow \mathbb{R}$ can be based on the standard convergence concept. Likewise the new map $\mathbb{R}^* \nrightarrow \mathbb{R}$ can be based on the generalised convergence concept.

Example 23.4 When the equivalent of the restrictions imposed in this work is imposed, the nonstandard infinite values of the δ–function are all mapped into the standard real number zero.

$$\mathfrak{d}(0) := \left\langle I(x) \, , \mathfrak{d}(x) \bullet \mathfrak{d}(x) \right\rangle = \left\langle I(x) \, , O(x) \right\rangle = 0, \tag{23.7}$$

as found in Ch. 9.

Of course the limited character of the model allows only for the interpretation of a small subset of the infinite numbers. Clearly much stronger theories are possible.

Nevertheless, the systematic mapping of infinite numbers into finite numbers is an essential step on the way to progress with the multiplication problem. It is also what separates the work in this book from other approaches to the multiplication problem.

This mapping of infinite numbers into finite numbers is a new contribution, and it is the aspect which makes a break with many standard concepts unavoidable.

The results obtained so far have a rather limited character. It is to be feared that this will be unavoidable for a long time to come. Somewhere between standard (and nonstandard) analysis, where no infinities are interpreted, and complete multiplication of the distributions, which requires the interpretation of all infinities, the further development of symmetrical theories of generalised functions will remain the art of the possible.

23.6 Other distribution products.

There exists a large amount of literature in which products and/or convolutions are defined on restricted subspaces of the distributions. Some examples are [Fsh], [Kel], [LB-G], [A,M&S] [Obe], and [Ita]. The subspace of the distributions usually coincides, or is contained in, the subspace $\overline{\mathbf{PC}}_\lambda$ of the simple model.

Actually the simple model provides more. There is some freedom in the choice of the standardization of the product, and there is also the choice between symmetrical and asymmetrical products. By suitable specialization many special multiplication theories can be derived.

The special cases found in the literature obtain a product by postulating product properties, for instance the Leibniz rule for differentiation (2.39) or the analytic boundary property. The products which result in this way are special cases of the more general product obtained in this work, which can be obtained by choosing the appropriate standardization.

The symmetrical theory provides a covering theory for multiplication theories on the subspace of the distributions which it contains.

23.7 Conclusion

The evaluation in finite terms of infinite quantities is a good criterion for comparing theories of generalized functions with the possibility of multiplication.

The symmetrical theory of generalised functions differs fundamentally from other approaches to the multiplication problem. It does not fit in with distribution theory and other approaches to generalized functions.

It is a new theory of generalised functions, which is logically independent of other theories of generalized functions. In particular it is independent of distribution theory.

CHAPTER 24

PROGRAM, OUTLOOK, AND CONCLUSION

This chapter outlines the possibility of larger models for the symmetrical theory of generalised functions. All restrictions imposed of the simple model are sufficient, none are necessary. This leaves much room for generalization.

The development of new concepts, which is necessitated by the acceptance of symmetry as a starting point, is by no means complete. In particular the definition of asymptotic expansions needs generalization. The standard (Poincaré's) concept used in this book is not really adequate for the purpose. The analysis of asymptotic behaviour should serve as a basis for theories of generalised functions, instead of on the approach to a limit.

The construction of a model for a symmetrical theory of generalised functions makes it clear that an alternative to distribution theory is possible. It is necessary to work out the consequences further in order to demonstrate the usefulness of the new approach.

What remains to be done is the reconstruction of parts of analysis, in such a way that generalised functions are incorporated from the beginning.

24.1 Larger models

More work is needed on the analysis of singular behaviour of functions. The method used in this book can be taken much further. Analysis of singular behaviour in the real was effected by Mellin transforming functions on the real axis into analytic functions on the (punctured) complex plane. (It is also possible to employ other integral transforms). The starting point was the analysing power which is available (without recourse to generalised methods) for meromorphic functions.

The analysis of the behaviour in the complex plane near a singular point λ_0 is effected by analysing the behaviour of the analytic function of $\lambda = \lambda_0 + \rho\, e^{i\varphi}$ as a function of the argument φ on a sufficiently small circle $\rho =$ constant, around the singular point. (The circle may be deformed arbitrarily as long as no other singular points are included). For single-valued analytic functions this function is simply a periodic function of the argument φ. The analysis involves no more than computing its Fourier coefficients. (It is actually sufficient to find the constant part.)

More complicated singular behaviour in the real leads to multiple-valued analytic functions. (Compare (5.45), which shows that non-integral powers of the logarithm give rise to branchpoints). The argument φ now runs from $-\infty$ to $+\infty$, and considered as a function of φ the the analytic function is no longer periodic.

As a function of φ, it can be considered as a generalised function of the argument φ, which makes it amenable to analysis with the tools supplied by the simple model. This in turn allows the analysis of more complicated singular behaviour in the finite. It is not clear to me at present if this bootstrap process [Mün] will lead to an infinite regression.

It would be very much preferable if general methods could be found, but it seems likely that further development of models will be necessary before adequate general insight can be obtained.

Generalization will result in the enlargement of the two parameter index set (α, q) of the η–functions to an infinite dimensional index space. The number of singular generalised functions with point support will equal the number of allowed types of singular behaviour. (In distribution theory only non-singular local behaviour as x^p can be detected).

It seems plausible that a symmetrical theory of generalised functions can be built on the basis of the exponential-logarithmic functions [Har] introduced by P. du Bois-Raymond.

The difficult question (which will be discussed further in the next section) is how many different types of asymptotic behaviour one may have in a function, if it is required that the different asymptotic components can be distinguished uniquely. This is the inverse of the standard question in asymptotics: given a function, which is defined in some way, find its asymptotic behaviour.

The questions posed above could be answered on basis of a solution to the fundamental unsolved problem (2.62) referred to in Ch. 2.

The extension to more variables is not trivial. It is necessary to consider more the complicated singularities which are possible in more dimensions. The simple arrow on the η–functions will have to be replaced by a description of the approach to the singularity. For the time being only factorizable problems can be considered.

The greater generality of distribution theory in this respect is only apparent. In actually handling complicated singularities it is necessary to design appropriately adapted spaces of test functions, and coordinate transformations immediately cause great difficulties.

A definitive model is not in sight, nor do I know if there can be such a thing. There is room and need for much further work.

24.2 On asymptotics

Throughout this work this work the standard definition, due to Poincaré, based on the standard limit concept, has been employed. In the abbreviated notation for (4.5) the asymptotic condition is

$$f(x) \sim f_{asy}(x;0+) \iff \lim_{x\downarrow 0} x^{\alpha}\big(f(x) - f_{asy}(x;0+)\big) = 0, \tag{24.1}$$

$\forall\alpha \in \mathbb{C}$.

It is natural in the context of generalised function theory to replace this by the generalised limit concept,

$$f(x) \approx f_{asy}(x;0+) \Longleftrightarrow \operatorname*{Lim}_{x\downarrow 0} x^{\alpha}\big(f(x) - f_{asy}(x;0+)\big) = 0, \tag{24.2}$$

$\forall\alpha \in \mathbb{C}$, or equivalently

$$\big\langle\, \eta_{\downarrow}(x)\,,\, x^{\alpha}\big(f(x) - f_{asy}(x;0)\big)\,\big\rangle = 0. \tag{24.3}$$

An example at infinity is provided by the oscillating functions

$$f(x) := e^{ikx} \approx \begin{cases} 0(x) & k \neq 0, \\ I(x) & k = 0, \end{cases} \tag{24.4}$$

which are according to this definition asymptotic to zero for $x{\uparrow}\infty$, even though the standard limit does not exist.

This weaker form of the asymptotic condition might therefore be referred to as asymptotic. In the finite this concept allows us to handle singularities with increasingly rapid oscillations, for instance of the form

$$\sin(x^{-1}) \approx 0(x), \tag{24.5}$$

which are in the mean asymptotic to zero.

Standard asymptotic theory might be called a posteriori in the sense that it attempts to solve the problem: Given a function defined in some (possibly implicit) way, find its asymptotic expansion up to some given accuracy.

As a consequence of this work the opposite question has to be posed: How many different types of asymptotic behaviour can be distinguished uniquely in presence of each other.

It is not sufficient to find a finite number of terms in an asymptotic expansion. An order symbol O(something) is not sufficient. (Compare Ex. 4.4, where differentiation converts terms of order zero into dominant terms).

It is necessary to characterize all terms which may occur, in order to know if the required analytic continuations exist. It is also necessary to be sure that the class of allowed asymptotic expansions is closed under the operator algebra which should be defined on all generalised functions.

It is unclear if this problem of a priori asymptotics does have a unique solution. There might be different, incompatible solutions, or a sequence of increasingly large classes without a closure in the form of a largest class. This subject must be left for the future. In this tract the problem is avoided by the restriction (4.1) to a well known asymptotic set. [B&H].

It is often thought that convergence is necessary in order to define precise numerical results, and that asymptotic expansions cannot serve in this respect, since numerically an asymptotic expansion yield only a finite, non-zero precision. This is correct, but the results of generalised function computations appear as the values of coefficients in asymptotic expansions, which can (in principle) be found with arbitrary precision. There is no need to attempt to sum asymptotic expansions, although this remains as a useful approximation method.

24.3 On foundations

As indicated in Sec. 1.1, the theory of the Fourier transform has been a source new developments in mathematics right from its inception by Fourier.
The apparently simple questions:

Which 'functions' allow a Fourier expansion?

and conversely:

What is the meaning of Fourier sums and integrals?

have led to the basic questions:

What is a function, and what are limits, derivatives, and integrals?

The foundations of analysis, as constructed before the introduction of the Dirac δ–function, do not leave room for the existence of a δ–function. The situation was saved by considering δ as a linear functional only. Consequently, even the simplest possible Fourier integral (2.3) and sum (20.35) do not represent a function in the classical sense.

The requirement that multiplication of (generalised) functions must always be defined for all generalised functions makes this compromise solution untenable.

The symmetrical theory of generalised functions is the natural next step. Once this has been accepted it is unavoidable to consider generalised functions and ordinary functions on the same footing.

This poses unsurmountable problems on basis of the standard concept of functions as maps $\mathbb{R} \to \mathbb{R}$. In this work I have of necessity fallen back on the eighteenth century concept (enunciated by Euler) [Eul], of a function as an analytic expression.

In this sense it is perhaps inappropriate to talk about generalised functions. In a sense all generalised functions introduced in this work, including the δ–function, are special functions.

If the program outlined above were fully completed, the result would be a general theory of special functions. This may well be enough for the applications of analysis.

The really difficult problem is building a synthesis, which combines the availability of δ–functions with a general function concept. This should of course be done without losing the general existence of derivatives, limits, and integrals. As pointed out in the previous chapter this involves a reconsideration of the handling of infinite quantities in mathematics.

As discussed in Sec. 19.8 the standard tools from topology are not suitable in the presently available form. Only in the special case of the $\mathcal{L}_2$ Hilbert space does a symmetrical structure result when linear functionals are defined.

A solution is not in sight, and I have to leave the reader at this point with open problems which will take much effort to settle satisfactorily.

24.4 Outlook

The reader who has followed the outline of the preceding chapters will have acquired some feeling for the inherent problems of the construction of symmetrical theories of generalised functions. It is not that the construction involves difficult concepts or methods. Indeed, given some familiarity with the theory of analytic continuation, the material lends itself to an elementary presentation.

The difficulty is rather that it is necessary to unlearn concepts which have been part of an education in mathematics for a long time.

The most drastic break with tradition is the need to reinterpret infinite quantities as finite quantities. Strictly speaking this is not a reinterpretation, infinite quantities do not exist in standard analysis, they are undefined. Even though we are at liberty to define undefined quantities in any (consistent) way we choose, it takes time and effort to regard the new interpretations as natural.

The first step is the replacement of 'smaller than any given ϵ' by 'does not contain a term ϵ^0 asymptotically' in the asymptotic expansion for small ϵ. This forces us to forget standard visualizations of open environments as small surroundings, and consequently standard notions of topology.

The second step must be a reconsideration of the standard function concept. The standard (Dirichlet) function concept arose from the realization of the insufficiency of the concept of functions as relations between quantities. A new function concept must be developed in such a way that generalised functions can be accommodated.

The name 'generalised functions' is a misnomer in this respect. Generalised functions are really special functions in the sense that these functions have been constructed explicitly. It is to be expected that also in future theories the families of generalised functions will be parametrized by finite, or at most countable index sets. Generalised functions are special in the same sense that analytic functions are special.

Further progress will entail a better understanding of asymptotic behaviour. The method of mapping to analytic functions will probably remain indispensable for progress in this respect. It allows for the application of the well developed classification of singularities of analytic functions.

It is difficult to imagine at present what a definitive theory of generalised function would be like. Instead one can think of a series of increasingly comprehensive models, which are defined constructively.

The situation is analogous to the theory of computable real numbers, ever large classes can be defined and constructed, but the class of all computable real numbers remains an intractable concept.

Perhaps progress with the fundamental unsolved problem (2.62) of the existence of analytic continuations will make a less constructive approach possible.

The present work is only a first start towards a symmetrical theory of generalised functions. It may be seen as no more than an indication that it is necessary and useful to overstep the boundaries which have traditionally been imposed by classical analysis and distribution theory.

Nevertheless the symmetrical theory of generalised functions is strong enough to deal with the divergence problems which arise in quantum field theory. The common denominator of both quantum field theory and the theory of multiplication of generalised functions is the need to come to grips with problems which necessitate the handling of (actual) infinities.

Assigning finite values to infinite quantities in a sensible and meaningful way will remain the challenge for future enlarged theories of generalised functions. Only the simplest special case has been dealt with as yet.

24.5 Propositions

Let me now summarize some of my conclusions, which arise from the work reported in this tract.

1) The requirement of the possibility of multiplication should be added to the requirements which are traditionally imposed on Dirac's δ–function.
2) (Occam's razor) The number of generalised functions is not to be multiplied without good reason.
3) As a consequence one has the requirement that it must be possible to evaluate generalised functions at a point.
4) Given 1) and 2) the existence of a symmetrical scalar product should be required. Mathematically in order to remain as close to a Hilbert space as possible, physically in order to have applicability to quantum mechanics.
5) It is possible to construct a model for a symmetrical theory of generalised functions. It allows the definition of operators and products.
6) This model is applicable to computations in quantum field theory. Applications of the symmetrical theory are possible without worrying about the foundations.
7) More work is needed to explore more general theories of symmetrical generalised functions.
8) The theory is, and should be, based on asymptotics instead of on convergence.
9) It is necessary to reconsider the foundations of analysis.
10) The constructive approach should be extended to analysis as a whole.

There is no need for the reader to agree with the propositions enumerated above. These propositions represent my interpretation of the work and its meaning.

It is to be hoped that more work will be carried out to enlarge our understanding of analysis and its applications. The nature of the infinite in mathematics and physics needs much further thought. The standard foundations of analysis must be reconsidered.

APPENDIX A. Laurent coefficients

For every single valued analytic function $f(\lambda)$, with only isolated singularities, there is for every point α in the complex λ–plane a formal Laurent series

$$f(\lambda) = \sum_{j=-\infty}^{\infty} (\lambda - \alpha)^j \, f^{[j]}(\alpha). \tag{A.1}$$

The Laurent coefficients $f^{[j]}(\alpha)$ are defined by

$$f^{[j]}(\alpha) := \operatorname*{Res}_{\lambda=\alpha} (\lambda - \alpha)^{-j-1} \, f(\lambda). \tag{A.2}$$

The Res operation stands for the taking of the residue. This is defined in the standard way as

$$\operatorname*{Res}_{\lambda=\alpha} f(\lambda) := \oint_{|\lambda-\alpha|=0+} d\lambda \, f(\lambda) := \tfrac{1}{2\pi} \lim_{\rho \downarrow 0} \int_0^{2\pi} d\varphi \, \rho \, e^{i\varphi} f(\alpha + \rho \, e^{i\varphi}). \tag{A.3}$$

For analytic functions with isolated singularities the limit is trivial, since its argument does not depend on ρ when ρ is sufficiently small. In the special case of meromorphic functions the Laurent series has at most a finite number of singular, ($j < 0$), terms. The Laurent series actually converges to $f(\lambda)$ in a punctured disk centered at $\lambda = \alpha$, but this is irrelevant for the purpose of generalised function theory. It is sufficient that the residues exist.

If the function $f(\lambda)$ is analytic at $\lambda = \alpha$, the Laurent series reduces to a Taylor series

$$f(\lambda) = \sum_{j=0}^{\infty} (\lambda - \alpha)^j \, f^{[j]}(\alpha), \quad \text{with} \quad f^{[j]}(\alpha) = \frac{1}{j!} \, f^{(j)}(\alpha)\,. \tag{A.4}$$

The $f^{(j)}$ is the j-th derivative. In the following the square bracket superscript is always used for a residue, the superscript in parenthesis is used for a derivative. A raised prime never means a derivative. It serves to distinguish between objects $\in$ **PC** defined as functions, and objects $\in$ **PC**$'$ defined as linear functionals.

The Laurent series for a product of analytic functions is the product of the corresponding series. Collecting the same powers of $(\lambda - \alpha)$ we find

$$(f \cdot g)^{[j]}(\alpha) = \sum_{k=-\infty}^{\infty} f^{[k]}(\alpha) \, g^{[j-k]}(\alpha), \tag{A.5}$$

which is convenient for the actual computation of Laurent coefficients. For the case that both f and g are meromorphic, the summation in (A.5) is finite.

APPENDIX B. Binomial coefficients, Pochhammer symbols

The binomial coefficients $\binom{\lambda}{n}$ are polynomials in the complex variable $\lambda \in \mathbb{C}$, of degree $n \in \mathbb{N}$ defined by

$$\binom{\lambda}{0} := 1, \qquad \binom{\lambda}{n+1} := \frac{\lambda - n}{n+1} \binom{\lambda}{n}. \tag{B.1}$$

The explicit form of the polynomial is

$$\binom{\lambda}{n} = \frac{1}{n!}\, \lambda(\lambda-1)\cdots(\lambda-n+1). \tag{B.2}$$

They are related to the Eulerian Γ–function by

$$\binom{\lambda}{n} = \frac{1}{n!}\,\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} = \frac{(-)^n}{n!}\,\frac{\Gamma(n-\lambda)}{\Gamma(-\lambda)}, \tag{B.3}$$

for $\lambda \neq p \in \mathbb{Z}$, and by a limit when a Γ–function is singular. From this one immediately finds the well known special values

$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!}, \qquad \text{and} \qquad \binom{-m-1}{n} = \frac{(-)^n(m+n)!}{m!\,n!}, \tag{B.4}$$

which are valid for integer values $\lambda = m \in \mathbb{N}$.
The derivatives of the binomial coefficients with respect to λ are now written as

$$\binom{\lambda}{n}^{(k)} := \left(\frac{\partial}{\partial\lambda}\right)^k \binom{\lambda}{n}. \tag{B.5}$$

By repeated application of the Leibniz rule for differentiating a product one finds the explicit form

$$\binom{\lambda}{n}^{(k)} = \sum\sum\cdots\cdots\sum_{-1<j_1<j_2<\cdots<j_{n-k}<n} \frac{k!}{n!}(\lambda-j_1)(\lambda-j_2)\cdots\cdots(\lambda-j_{n-k}). \tag{B.6}$$

It is a polynomial of degree $n-k$ in λ and it is zero for $k > n$.
The Taylor series of the binomial coefficient at an arbitrary $\alpha \in \mathbb{C}$ is given by

$$\binom{\lambda}{n} = \sum_{k=0}^{n} \frac{1}{k!} \binom{\alpha}{n}^{(k)} (\lambda-\alpha)^k, \tag{B.7}$$

which is obviously a polynomial of degree n in λ.
The Taylor series of the derivative of the binomial coefficients at $\lambda = \alpha$ is

$$\binom{\lambda}{n}^{(k)} = \sum_{j=0}^{n-k} \frac{1}{j!} \binom{\alpha}{n}^{(k+j)} (\lambda-\alpha)^j, \tag{B.8}$$

and in particular for $\alpha = 0$

$$\binom{\lambda}{n}^{(k)} = \sum_{j=0}^{n-k} \frac{1}{j!} \binom{0}{n}^{(k+j)} \lambda^j, \tag{B.9}$$

which gives the derivatives of the binomial coefficient as a polynomial of degree $n - k$ in the variable λ. The derivative of the binomial coefficient does not have the property that it is integer for integer values of λ. The values also alternate in sign.

For computational purposes it is therefore preferable to introduce the Pochhammer symbols by

$$(\lambda)_0 = 1, \qquad (\lambda)_{n+1} = (\lambda + n)\,(\lambda)_n\,. \tag{B.10}$$

This gives the explicit form

$$(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = (-)^n \frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda-n)}. \tag{B.11}$$

The derived Pochhammer symbols are now defined $\forall k \in \mathbb{Z}$ by

$$(\lambda)_n^{[k]} := \operatorname*{Res}_{\mu=\lambda} (\mu - \lambda)^{-k-1} (\mu)_n\,, \tag{B.12}$$

by taking residues. This gives the explicit values

$$(\lambda)_n^{[k]} = \begin{cases} \frac{1}{k!} \left(\frac{\partial}{\partial\lambda}\right)^k (\lambda)_n & 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{B.13}$$

The derived Pochhammer symbols are zero unless $0 \le k \le n$, since the Pochhammer symbol is a polynomial of degree n in λ. It should be noted that in contrast with the derivative of the binomial coefficient a factorial $k!$ is included in the definition. This simplifies some formulæ. For $n = 0$, and $\forall\lambda \in \mathbb{C}$, one has the special value

$$(\lambda)_0^{[k]} = \delta_{0,k}\,. \tag{B.14}$$

The properties of the Pochhammer symbols are easily found from the definition. The corresponding formulæ for the binomial coefficients are then found from the connections

$$(\lambda)_n^{[k]} = \frac{n!}{k!}(-)^{n+k}\binom{-\lambda}{n}^{(k)}, \qquad \binom{\lambda}{n}^{(k)} = \frac{k!}{n!}(-)^{n+k}(-\lambda)_n^{[k]}. \tag{B.15}$$

Some of the properties of the derived Pochhammer symbols are:
Explicit form as a factored polynomial with integer coefficients

$$(\lambda)_n^{[k]} = \sum_{-1<j_1<j_2<\cdots<j_{n-k}<n} \sum \cdots\cdots \sum (\lambda + j_1)(\lambda + j_2)\cdots\cdots(\lambda + j_{n-k}). \tag{B.16}$$

Inversion formula

$$(\lambda)_n^{[k]} = (-)^{n+k}(1 - \lambda - n)_n^{[k]}. \tag{B.17}$$

Recursion by n

$$(\lambda)_{n+1}^{[k]} = (\lambda + n)\,(\lambda)_n^{[k]} + (\lambda)_n^{[k-1]}. \tag{B.18}$$

Recursion with respect to λ

$$(\lambda)_n^{[k]} = \lambda\,(\lambda + 1)_{n-1}^{[k]} + (\lambda + 1)_{n-1}^{[k-1]}. \tag{B.19}$$

Taylor series

$$(\lambda)_n^{[k]} = \sum_{j=0}^{n-k} \binom{j+k}{j} (\alpha)_n^{[j+k]}(\lambda - \alpha)^j. \tag{B.20}$$

Explicit polynomial form in powers of λ

$$(\lambda)_n^{[k]} = \sum_{j=0}^{n-k} \binom{j+k}{j} (0)_n^{[j+k]}\,\lambda^j. \tag{B.21}$$

The explicit values of the integer coefficients $(0)_n^{[k]}$ are best calculated recursively by using the recursion formula (B.18) with the initial conditions

$$(0)_n^{[0]} = \delta_{0,n} \qquad (0)_n^{[n+1]} = 0. \tag{B.22}$$

The results are

$$(0)_n^{[k]} = \begin{array}{ccccccc} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 2 & 3 & 1 & & & \\ 0 & 6 & 11 & 6 & 1 & & \\ 0 & 24 & 50 & 35 & 10 & 1 & \\ 0 & 120 & 274 & 225 & 85 & 15 & 1 \end{array} = (1)_{n-1}^{[k-1]}. \tag{B.23}$$

The values of these special derived Pochhammer symbols are known [Erd1] as the Stirling numbers of the first kind. A large table can be found in [A&S], with a trivial difference in the sign convention.

APPENDIX C. Laurent coefficients for Fourier transforms

The different forms of the derivatives of the trigonometric functions

$$\frac{d}{d\lambda}\cos\lambda = -\sin\lambda, \qquad \frac{d}{d\lambda}\sin\lambda = \cos\lambda, \tag{C.1}$$

can be written by means of the shift relations,

$$\sin(\lambda \pm \tfrac{\pi}{2}) = \pm\cos\lambda, \qquad \cos(\lambda \pm \tfrac{\pi}{2}) = \mp\sin\lambda, \tag{C.2}$$

in the same form,

$$\frac{d}{d\lambda}\cos\lambda = \pm\cos(\lambda \pm \tfrac{\pi}{2}), \qquad \frac{d}{d\lambda}\sin\lambda = \pm\sin(\lambda \pm \tfrac{\pi}{2}). \tag{C.3}$$

This can be used to put the Taylor series of these functions in the forms

$$\sin\lambda = \sum_{j=0}^{\infty} \tfrac{1}{j!}\sin(\alpha + \tfrac{\pi}{2}j)\,(\lambda-\alpha)^j, \tag{C.4}$$

and

$$\sin\tfrac{\pi}{2}\lambda = \sum_{j=0}^{\infty} \tfrac{1}{j!}(\tfrac{\pi}{2})^j \sin\tfrac{\pi}{2}(\alpha+j)\,(\lambda-\alpha)^j, \tag{C.5}$$

and idem for the cosine.
This avoids the cumbersome period 4 of the usual form of the Taylor series.
The Eulerian Γ–function satisfies the functional equation

$$\Gamma(\lambda+1) = \lambda\cdot\Gamma(\lambda), \tag{C.6}$$

and the completion formula,

$$\Gamma(\lambda)\,\Gamma(1-\lambda)\sin\pi\lambda = \pi. \tag{C.7}$$

Repeated application of the functional equation (C.6) gives

$$\Gamma(\lambda+n) = (\lambda)_n\,\Gamma(\lambda), \quad \text{and} \quad \Gamma(\lambda) = (-)^n(1-\lambda)_n\,\Gamma(\lambda-n), \tag{C.8}$$

which connects the Γ–functions and the Pochhammer symbols.

The Laurent series of the Γ–function at the singular points $\lambda = -p$, $p \in \mathbb{N}$ is easily found by division from the completion equation. The result is

$$\Gamma(\lambda) = \tfrac{(-)^p}{p!}\big((\lambda+p)^{-1} + \psi(p+1) + {} \\ - \tfrac{1}{2}(\lambda+p)\big(\psi^{(1)}(p+1) - \psi^2(p+1) - \tfrac{\pi^2}{3}\big) + O(\lambda+p)^2\big), \tag{C.9}$$

where $\psi(\lambda)$ is the logarithmic derivative

$$\psi(\lambda) := \frac{1}{\Gamma(\lambda)} \frac{\partial}{\partial \lambda} \Gamma(\lambda), \tag{C.10}$$

of the Γ–function. At other points the Laurent series of the Γ–function is the Taylor series.
Going over to half angles and using the shift formulæ (C.2) the completion formula (C.7) can be put in the forms

$$\Gamma(\lambda) \sin(\tfrac{\pi}{2}\lambda)\, \Gamma(1-\lambda) \sin \tfrac{\pi}{2}(1-\lambda) = \tfrac{\pi}{2}, \tag{C.11}$$

and

$$\Gamma(\lambda) \cos(\tfrac{\pi}{2}\lambda)\, \Gamma(1-\lambda) \cos \tfrac{\pi}{2}(1-\lambda) = \tfrac{\pi}{2}. \tag{C.12}$$

These can be combined into the form (putting $\lambda := \lambda + 1$),

$$\Gamma(\lambda+1) \sin \tfrac{\pi}{2}(\lambda+m)\, \Gamma(-\lambda) \sin \tfrac{\pi}{2}(-\lambda-1+m) = \tfrac{\pi}{2}, \tag{C.13}$$

for arbitrary $m \in \mathbb{Z}$.

The Laurent coefficients $c_j(\alpha, m)$ needed for the computation of the Fourier transforms of generalised functions can all be expressed in the Laurent coefficients of the function $f(\lambda) := \Gamma(\lambda+1) \sin \frac{\pi}{2}(\lambda+m)$,

$$c_j(\alpha, m) := \operatorname*{Res}_{\lambda=\alpha} (\lambda-\alpha)^{-j-1}\, \Gamma(\lambda+1) \sin \tfrac{\pi}{2}(\lambda+m). \tag{C.14}$$

It follows immediately from the definition (C.14) that it is possible to differentiate formally with respect to α, with the result

$$\frac{\partial}{\partial \alpha} c_j(\alpha, m) = (j+1)\, c_{j+1}(\alpha, m). \tag{C.15}$$

This is useful only for $\alpha \neq p \in \mathbb{Z}$.
The coefficients c_j clearly have the property

$$c_j(\alpha, m+2n) = (-)^n c_j(\alpha, m), \tag{C.16}$$

$\forall n \in \mathbb{Z}$, since the sine has that property.

Remark C.1 Despite the fact that these Laurent coefficients are periodic in m with period 4 it is convenient to keep $m \in \mathbb{Z}$ arbitrary.

Since the sine function is entire, and the Γ–function has only first order poles, we have immediately

$$c_j(\alpha, m) = 0, \qquad \text{for} \qquad j < -1. \tag{C.17}$$

For $j = -1$ the coefficient $c_{-1}(\alpha, m)$ is non-zero only for $\alpha = -p-1$, $p \in \mathbb{N}$, where a pole of the Γ–function is found. The explicit form can be written as

$$c_{-1}(\alpha, m) = \begin{cases} 0 & \alpha \neq -p-1, \\ \frac{(-)^p}{p!} \sin \frac{\pi}{2}(\alpha + m) & \alpha = -p-1, \end{cases} \tag{C.18}$$

which can be written in one line by means of Kronecker's δ–symbol as

$$c_{-1}(\alpha, m) = -\sum_{p=0}^{\infty} \delta_{\alpha,-p-1}\, \delta_{p,m}^{\mathrm{mod}2}\, \tfrac{1}{p!} (-)^{(m+p)/2}\,, \tag{C.19}$$

which is an explicitly zeromorphic function.

For $\alpha \neq -p-1$, $p \in \mathbb{N}$, the function $f(\lambda) = \Gamma(\lambda+1) \sin \frac{\pi}{2}(\lambda + m)$ is analytic, and its Taylor coefficients can be found by repeated differentiation. This results in the explicit form

$$c_j(\alpha, m) = \sum_{k=0}^{j} \tfrac{1}{k!\,(j-k)!}\, \Gamma^{(k)}(\alpha+1) (\tfrac{\pi}{2})^{j-k} \sin \tfrac{\pi}{2}(\alpha + m + j - k), \tag{C.20}$$

so

$$c_0(\alpha, m) = \Gamma(\alpha+1) \sin \tfrac{\pi}{2}(\alpha + m), \tag{C.21}$$

and

$$c_1(\alpha, m) = c_0(\alpha, m)\big(\psi(\alpha+1) + \tfrac{\pi}{2} \cot \tfrac{\pi}{2}(\alpha + m)\big), \tag{C.22}$$

valid for $\alpha \neq -p-1$, $p \in \mathbb{N}$. Since the singular points are isolated c_j can be expressed in terms of c_0,

$$c_j(\alpha, m) = \operatorname*{Res}_{\lambda=\alpha} (\lambda - \alpha)^{-j-1}\, c_0(\lambda, m), \tag{C.23}$$

by substituting (C.20) with $j = 0$ into (C.14).

For the special case that $\alpha = p \in \mathbb{N}$, $m = p$ or $m = p+1$, the explicit values of the coefficients, up to terms involving the second derivatives of the Γ–function, are collected in the following table.

Table C.1

$c_0(p,p)$	$= 0$
$c_1(p,p)$	$= \frac{\pi}{2}(-)^p p!$
$c_2(p,p)$	$= \frac{\pi}{2}(-)^p p!\, \psi(p+1)$
$c_3(p,p)$	$= \frac{\pi}{4}(-)^p p!\, (\psi^{(1)}(p+1) + \psi^2(p+1) - \frac{\pi^2}{12})$
$c_0(p,p+1)$	$= (-)^p p!$
$c_1(p,p+1)$	$= (-)^p p!\, \psi(p+1)$
$c_2(p,p+1)$	$= \frac{1}{2}(-)^p p!\, (\psi^{(1)}(p+1) + \psi^2(p+1) - \frac{\pi^2}{4})$

The c–coefficients at other values of the second parameter are found from

$$c_j(\alpha, m) = \begin{cases} (-)^{(m-p)/2} c_j(\alpha, p) & m - p = 0 \pmod 2, \\ (-)^{(m-p-1)/2} c_j(\alpha, p+1) & m - p = 1 \pmod 2. \end{cases} \tag{C.24}$$

Only the values at $m = 0$,

$$c_j(p, 0) = \begin{cases} (-)^{p/2} c_j(p, p) & p = 0 \pmod 2, \\ (-)^{(p-1)/2} c_j(p, p+1) & p = 1 \pmod 2, \end{cases} \tag{C.25}$$

and $m = 1$,

$$c_j(p, 1) = \begin{cases} (-)^{p/2} c_j(p, p+1) & p = 0 \pmod 2, \\ (-)^{(p-1)/2} c_j(p, p) & p = 1 \pmod 2, \end{cases} \tag{C.26}$$

are usually needed for actual computations.
The c–coefficients satisfy the completion formula

$$\sum_{j=-1}^{l+1} (-)^j c_j(\alpha, m)\, c_{l-j}(-\alpha - 1, m) = \tfrac{\pi}{2}\, \delta_{l,0}, \tag{C.27}$$

which is obtained by substituting the Laurent series into the completion formula (C.13), using the Laurent product formula (A.5). A slightly more general form is obtained by using the functional equation of the Γ–function in the form (C.8)

$$\sum_{j=-1}^{l+1} (-)^j c_j(\alpha + n, m - n)\, c_{l-j}(-\alpha - 1, m) = \tfrac{\pi}{2} (-)^l (\alpha)_n^{[l]}, \tag{C.28}$$

which reduces to the previous form (C.27) for $n = 0$. In particular for $\alpha \notin \mathbb{Z}$ we have the special cases

$$c_0(\alpha, m)\, c_0(-\alpha - 1, m) = \tfrac{\pi}{2}, \tag{C.29}$$

and

$$c_0(\alpha, m)\, c_1(-\alpha - 1, m) = c_1(\alpha, m)\, c_0(-\alpha - 1, m). \tag{C.30}$$

The completion formula (C.28) can be used to shorten summations,

$$(-)^q \sum_{j=-1}^{q} (-)^j c_j(\alpha, j)\, c_{q+r+1-j}(-\alpha - 1, m) =$$

$$= (-)^r \sum_{j=-1}^{r} (-)^j c_j(-\alpha - 1, m)\, c_{q+r+1-j}(\alpha, m), \tag{C.31}$$

as one finds from (C.28) by putting $l = q+r+1$ and rearranging summations. The completion formula (C.28) can be used conveniently to calculate the c–coefficients at the singular points $\alpha = -p-1, \quad p \in \mathbb{N}$. Substituting $l = -1, l = 0$ gives the starting conditions. By isolating the highest term, the completion equation (C.28) can be rewritten for $l > 0$ and $\forall p \in \mathbb{N}$ as

$$c_l(-p-1,p) = \frac{2(-)^{p+l+1}}{\pi p!} \sum_{j=-1}^{l-1} (-)^j c_j(-p-1,p)\, c_{l-j+1}(p,p), \tag{C.32}$$

$$c_l(-p-1,p+1) = \frac{(-)^{p+l+1}}{p!} \sum_{j=0}^{l-1} (-)^j c_j(-p-1,p+1)\, c_{l-j}(p,p+1), \tag{C.33}$$

where the explicit values of $c_0(p,p+1)$ or $c_1(p,p)$ have been substituted. These forms of the completion relation can be used to compute the c_j–coefficients at the singular points recursively.
The results are collected in the following table.

Table C.2

$c_{-1}(-p-1,p+1) = 0$
$c_0(-p-1,p+1) = \frac{\pi}{2}(-)^p/p!$
$c_1(-p-1,p+1) = \frac{\pi}{2}(-)^p\psi(p+1)/p!$
$c_2(-p-1,p+1) = \frac{\pi}{4}(-)^{p+1}\left(\psi^{(1)}(p+1) - \psi^2(p+1) - \frac{\pi^2}{4}\right)/p!$
$c_{-1}(-p-1,p) = (-)^{p+1}/p!$
$c_0(-p-1,p) = (-)^{p+1}\psi(p+1)/p!$
$c_1(-p-1,p) = \frac{1}{2}(-)^p\left(\psi^{(1)}(p+1) - \psi^2(p+1) - \frac{\pi^2}{12}\right)/p!$
$c_2(-p-1,p) = \cdots\ \psi^{(2)}\ \cdots$ calculate for yourself

The table has been completed again up to second derivatives of the Γ–function. It is obviously possible to compute all coefficients recursively in this way but the amount of work involved rapidly becomes excessive.

It is often useful to relate different c–coefficients. The functional equation for the Γ–function (C.6) can be converted into

$$c_{j+1}(\alpha+1, m+1) = -(\alpha+1)\, c_{j+1}(\alpha, m) - c_j(\alpha, m), \tag{C.34}$$

by computing the residue. Unfortunately this relation cannot be used to enlarge the tables given above.

Dividing the functional equation (C.6) by λ and computing residues gives a formula for increasing α in the form

$$c_j(\alpha, m) = \sum_{k=-1}^{j} (-\alpha-1)^{k-j-1} c_k(\alpha+1, m+1), \tag{C.35}$$

valid for $\alpha \neq -1$. For $\alpha = -1$ one obtains the relation (C.34) at $\alpha = -1$.

For the computation of products it is convenient to introduce a coefficient $d_{jq}(\alpha, m)$, with $\alpha \in \mathbb{C}$, $r, q \in \mathbb{N}$, $m \in \mathbb{Z}$ by

$$d_{qr}(\alpha, m) := -\tfrac{2}{\pi} q!\, r! \sum_{j=-1}^{r} (-)^{j+r} c_j(\alpha, m)\, c_{q+r+1-j}(-\alpha-1, m) =$$

$$= -\tfrac{2}{\pi} q!\, r! \sum_{j=-1}^{q} (-)^{j+q} c_j(-\alpha-1, m)\, c_{q+r+1-j}(\alpha, m), \qquad \text{(C.36)}$$

where the different forms are related by (C.31). It follows from the two different forms of (C.36) that the d–coefficients are related by

$$d_{qr}(\alpha, m) = d_{rq}(-\alpha-1, m). \qquad \text{(C.37)}$$

The d–coefficients are again periodic in m with period 2. In the special case $q = 0$ one obtains from the second form

$$d_{0r}(\alpha, m) = -\tfrac{2}{\pi} r! \left(c_0(-\alpha-1, m)\, c_{r+1}(\alpha, m) - c_{-1}(-\alpha-1, m)\, c_{r+2}(\alpha, m)\right), \qquad \text{(C.38)}$$

of which only the first term survives, unless $\alpha = p \in \mathbb{N}$. Further specialization gives

$$d_{00}(\alpha, m) = -\psi(\alpha+1) - \tfrac{\pi}{2} \cot \tfrac{\pi}{2}(\alpha + m) = d_{00}(-\alpha-1, m). \qquad \text{(C.39)}$$

For integral values of the argument substitution of the c–coefficients gives the special values

$$d_{q0}(p, p) = d_{0r}(-p-1, p) = 0, \qquad \text{(C.40)}$$

and

$$d_{00}(p, p+1) = d_{00}(-p-1, p+1) = -\psi(p+1), \qquad \text{(C.41)}$$

which occur in products of integral powers.

For computing Hilbert transforms it is convenient to introduce coefficients $h_l(\alpha, m)$ defined by

$$h_l(\alpha, m) := \tfrac{2}{\pi} \sum_{j=-1}^{l+1} (-)^j c_j(\alpha, m)\, c_{l-j}(-\alpha-1, m+1), \qquad \text{(C.42)}$$

which differs from the completion formula (C.27) only by the change of m to $m+1$. The trivial zeros of the h–coefficients are located at

$$h_0(p, p) = h_0(-p-1, p+1) = 0. \qquad \text{(C.43)}$$

The h–coefficients satisfy the reflection property

$$(-)^{l+1} h_l(-\alpha-1, m+1) = h_l(\alpha, m) = (-)^l h_l(-\alpha-1, m-1), \qquad \text{(C.44)}$$

since the summation can be changed from j to $l - j$.

APPENDIX D. Generalised zeta functions

In this appendix some properties of the Riemann ζ-function and its generalizations are listed. Proofs and more complicated formulæ may be found in [Erd1].

The special functions which arise in the summation of sequences are special cases of the function (Bateman's notation) defined by

$$\Phi(z, s, \nu) := \sum_{n=0}^{\infty} (\nu + n)^{-s} z^n, \tag{D.1}$$

for $\nu \neq -p - 1$, $|z| < 1$, $\operatorname{Re} s > -1$, and by analytic continuation elsewhere. The generalised ζ–function is obtained by putting $z = 1$,

$$\zeta(s, \nu) := \Phi(1, s, \nu) = \sum_{n=0}^{\infty} (\nu + n)^{-s}, \tag{D.2}$$

and the Riemann ζ–function is obtained by specializing to $\nu = 1$,

$$\zeta(s) := \zeta(s, 1) = \Phi(1, s, 1) = \sum_{n=1}^{\infty} n^{-s}. \tag{D.3}$$

The properties of these functions and their derivation as given in [Erd1] will be used in the following without proof.
The function $\Phi(e^\alpha, -\lambda, \nu)$ satisfies the functional equation

$$\begin{aligned} \Phi(e^\alpha, -\lambda, \nu) &= i\, e^{-\alpha\nu} (2\pi)^{-\lambda-1}\, \Gamma(\lambda + 1) \times \\ &\times \left(e^{i\frac{\pi}{2}\lambda} \Phi\big(e^{-2\pi i\nu}, 1 + \lambda, \tfrac{\alpha}{2\pi i}\big) - e^{i\frac{\pi}{2}(\nu-\lambda)} \Phi\big(e^{2\pi i\nu}, 1 + \lambda, \tfrac{1-\alpha)}{2\pi i}\big)\right). \end{aligned} \tag{D.4}$$

This functional equation for the function $\Phi(z, -\lambda, \nu)$ is known as Lerch' equation.

Specializing to $z = 1$, or $\alpha = 0$ and writing out the Φ–function gives Hurwitz' formula for the generalised ζ–function

$$\zeta(-\lambda, , \nu) = 2(2\pi)^{-\lambda-1}\, \Gamma(\lambda + 1) \sum_{n=1}^{\infty} n^{-\lambda-1} \sin(2\pi n\nu - \tfrac{\pi}{2}\lambda). \tag{D.5}$$

while further specialization to $\nu = 1$ gives Riemann's functional equation

$$\zeta(-\lambda) = -2(2\pi)^{-\lambda-1}\, \Gamma(\lambda + 1) \sin \tfrac{\pi}{2}\lambda\; \zeta(\lambda + 1). \tag{D.6}$$

for the ζ–function.

The ζ–function is analytic as a function of λ, with the exception of a simple pole at $\lambda = -1$ with residue 1 and constant term equal to $\psi(1)$. The generalised ζ–function has the same pole and residue, but its constant part at the pole equals $\psi(\nu)$.

APPENDIX E. Operator algebra

This appendix lists some well known operator properties for ease of reference and to fix the notation. The infinitesimal generators are tabulated.
The commutator of two operators is defined by

$$[\mathcal{A}, \mathcal{B}] := \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}, \tag{E.1}$$

and the repeated commutator is defined by

$$[\mathcal{A}, \mathcal{B}]_{n+1} := [\mathcal{A}, [\mathcal{A}, \mathcal{B}]_n], \qquad \text{with} \qquad [\mathcal{A}, \mathcal{B}]_0 := \mathcal{B}, \tag{E.2}$$

so

$$[\mathcal{A}, \mathcal{B}]_n = [\mathcal{A}, \cdots]^n \mathcal{B}, \tag{E.3}$$

By straightforward expansion of the exponentials and reordering terms one sees that

$$e^{\mathcal{A}} \mathcal{B} e^{-\mathcal{A}} = \sum_{j=0}^{\infty} \tfrac{1}{j!} [\mathcal{A}, \cdots^{j}] \mathcal{B} = \sum_{j=0}^{\infty} \tfrac{1}{j!} [\mathcal{A}, \mathcal{B}]_j = e^{[\mathcal{A}, \cdots]} \mathcal{B}. \tag{E.4}$$

For the reordering of operators in exponentials the Baker-Hausdorff lemma is needed in its simplest form. If

$$[\mathcal{A}, [\mathcal{A}, \mathcal{B}]] = [\mathcal{B}, [\mathcal{A}, \mathcal{B}]] = 0$$

then

$$e^{\mathcal{A}+\mathcal{B}} = e^{\mathcal{A}} e^{\mathcal{B}} e^{-\frac{1}{2}[\mathcal{A},\mathcal{B}]} = e^{\mathcal{B}} e^{\mathcal{A}} e^{\frac{1}{2}[\mathcal{A},\mathcal{B}]},$$

and

$$e^{\mathcal{A}} e^{\mathcal{B}} = e^{\mathcal{B}} e^{\mathcal{A}} e^{[\mathcal{A},\mathcal{B}]}, \tag{E.5}$$

as one sees by straightforward algebra or by consulting a textbook.

For one parameter families of unitary operators $\mathcal{U}(a)$ depending additively on a parameter a, it is convenient to look for an infinitesimal generator $\mathcal{A}$ in order to obtain the operator family in the form

$$\mathcal{U}(a) = e^{ia\mathcal{A}}, \tag{E.6}$$

This presents some difficulties in the case of the operators acting on the generalised functions since the formal expression resulting from the application of the exponentiated operator may be undefined as a generalised function. Moreover it is very useful to have the exponential form also for operators that are not completely unitary, but almost unitary.

Such families are easily generated by operator differential equations of the form

$$\frac{d}{da}\,\mathcal{U}(a) = i\big[\mathcal{A}\,,\mathcal{U}(a)\big], \tag{E.7}$$

with the formal solution

$$\mathcal{U}(a) = e^{ia\left[\mathcal{A},\cdots\right]}\mathcal{U}(0) = e^{-ia\mathcal{A}}\mathcal{U}(0)\,e^{ia\mathcal{A}}, \tag{E.8}$$

by (E.4). If the infinitesimal generator $\mathcal{A}$ is selfadjoint the operators $\mathcal{U}(a)$ are unitary.

When the generalised functions are transformed by a unitary operator the scalar products transform according to

$$\big\langle\, f(x)\,, \mathcal{O}\,g(x)\,\big\rangle \to \big\langle \mathcal{U}\,f(x)\,, \mathcal{O}\mathcal{U}\,g(x)\,\big\rangle = \big\langle\, f(x)\,, \mathcal{U}^{\dagger}\mathcal{O}\mathcal{U}\,g(x)\,\big\rangle, \tag{E.9}$$

Therefore one can also consider transformations of the operators instead of transformations of the functions. The transformed operators are given by

$$\mathcal{O} := \mathcal{U}^{\dagger}\mathcal{O}\mathcal{U} = \mathcal{U}^{-1}\mathcal{O}\mathcal{U}, \tag{E.10}$$

if the operator $\mathcal{U}$ is unitary.

It is often more convenient to consider the operator $\mathcal{U}(a)$ as a (super)operator transforming the operators, since it avoids the question if the exponentials can be expanded.

The operator family then satisfies an operator differential equation

$$\frac{d}{da}\,\mathcal{O}(a) = -i\big[\mathcal{A}\,, \mathcal{O}(a)\big], \tag{E.11}$$

with the corresponding formal solution

$$\mathcal{O}(a) = e^{-ia\left[\mathcal{A},\cdots\right]}\,\mathcal{O}(0) = e^{ia\mathcal{A}}\mathcal{O}(0)\,e^{-ia\mathcal{A}}, \tag{E.12}$$

Therefore instead of the operators acting on the generalised functions we consider the same operators as (super)operators acting on the operators. This transformation is known in quantum mechanics as the transformation between the Schrödinger and the Heisenberg picture. In the following the difference between the operator $\mathcal{A}$ acting on the generalised functions and the corresponding superoperator acting on the operators on the generalised functions will be ignored.

Families of unitary superoperators are best characterized by their action on the operators $\mathcal{X}$ and $\mathcal{D}$.

It will be convenient to introduce the operator $\mathcal{K}$ as

$$\mathcal{K} := -i\mathcal{D}, \tag{E.13}$$

in order to replace the antihermitian character of the operator $\mathcal{D}$ by an operator with a hermitian character. The minus sign is traditional in quantum mechanics, together with the commutation relation

$$[\mathcal{X}, \mathcal{K}] = i\mathcal{I}. \tag{E.14}$$

The operator $\mathcal{K}$ is known as the wave number operator. The choice made above makes $f(x) = e^{ikx}$ a wave function with positive wave number.

The operator phase plane is defined to be the two dimensional space spanned by the operators $\mathcal{X}$ and $\mathcal{K}$, so all operators $\mathcal{O}$ of the form

$$\mathcal{O} := a\mathcal{X} + b\mathcal{K}, \tag{E.15}$$

with a, $b \in \mathbb{R}$. It is convenient to characterize the operators by their transformation of the phase plane. Only the simple cases where the infinitesimal generator is a linear or quadratic polynomial in the operators $\mathcal{X}$ and $\mathcal{K}$ (or $\mathcal{D}$) are considered.

The families of transformations of the generalised functions are all subgroups of the (affine) group of linear transformations of the phase plane which invariant area element. The infinitesimal generators are given in the following table

Table E.1

Infinitesimal generator	Phase plane transformations
$\mathcal{X}$	Wavenumber translations
$\mathcal{K} = -i\mathcal{D}$	Translations
$a\mathcal{X} + b\mathcal{K}$	General phase plane translations
$\mathcal{X}\mathcal{D}$	Scale transformations
$\frac{1}{2}(\mathcal{X}\mathcal{D} + \mathcal{D}\mathcal{X})$	Normalized scale transformations
$\mathcal{X}^2 - \mathcal{D}^2$	Rotations
$\mathcal{X}^2 + \mathcal{D}^2$	Lorentz transformations

The phase rotations occur in this tract only for the special cases of rotations over multiples of $\frac{\pi}{2}$ given by the powers of the Fourier operator. The general rotation can also be written out for the generalised functions, but this involves too many special functions properties (parabolic cylinder functions) for the purpose of this tract.

The Lorentz transformations are scale transformations rotated over $\frac{\pi}{4}$. When the scale and Lorentz transformations are omitted the Euclidean group of the plane remains.

One can also consider many other one parameter subgroups, such as shifted scale transformations. Many operator properties such as (16.39) are easy to remember by thinking of the corresponding phase plane transformations. The whole group can be represented as a transformation group of the generalised functions.

APPENDIX F. Cantor's staircase function

The staircase function introduced by Cantor is well known as an example of a monotonously increasing function with zero derivative almost everywhere, which is not constant. It does not equal the Lebesgue integral of its derivative. It is the standard textbook example of a function which is continuous, but not absolutely continuous.

The Cantor staircase (it would be unkind to associate it with the devil) is defined for $0 \leq x \leq 1$ by

$$\mathcal{C}a_0(x) := x, \qquad \mathcal{C}a_{n+1}(x) := \begin{cases} \frac{1}{2}\,\mathcal{C}a_n(3x) & 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{2} & \frac{1}{3} \leq x \leq \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2}\,\mathcal{C}a_n(3x-2) & \frac{2}{3} \leq x \leq 1, \end{cases} \tag{F.1}$$

and

$$\mathcal{C}a(x) := \lim_{n\to\infty} \mathcal{C}a_n(x). \tag{F.2}$$

It follows immediately from the definition that the Cantor function satisfies the scaling property

$$\mathcal{C}a(3x) = 2\,\mathcal{C}a(x), \quad \mathcal{C}a(x) = \tfrac{1}{2}\,\mathcal{C}a(3x), \tag{F.3}$$

for $0 \leq x \leq \frac{1}{3}$, and conversely

$$2\,\mathcal{C}a(x/3) = \mathcal{C}a(x), \quad \mathcal{C}a(x/3) = \tfrac{1}{2}\,\mathcal{C}a(x), \tag{F.4}$$

which can be used to define a continuation of the Cantor function to the positive real axis $0 \leq x < \infty$. The Cantor function does not satisfy the requirements imposed in Ch. 4 on generalised functions. Nevertheless it will be treated as such for the time being. In order to find its properties as a generalised function we compute the integral

$$\tilde{\mathcal{C}a}(\lambda) := \int_0^1 dx\, x^\lambda\, \mathcal{C}a(x), \tag{F.5}$$

which converges for $\operatorname{Re}\lambda$ sufficiently large. Splitting the interval $(0,1)$ into the subintervals $(0, \frac{1}{3})$, and $(\frac{1}{3}, 1)$, and using the scaling property (F.3) gives

$$\tilde{\mathcal{C}a}(\lambda) = 2^{-1}3^{-\lambda-1}\,\tilde{\mathcal{C}a}(\lambda) + \int_{1/3}^1 dx\, x^\lambda\, \mathcal{C}a(x), \tag{F.6}$$

which can be rearranged to

$$\tilde{\mathcal{C}a}(\lambda) = 2(2 - 3^{-\lambda-1})^{-1} \int_{1/3}^1 dx\, x^\lambda\, \mathcal{C}a(x). \tag{F.7}$$

The integral occurring in (F.7) is an entire function of λ, since it is defined and differentiable for all $\lambda \in \mathbb{C}$. The pre-factor has simple poles at the zeroes of the denominator. These occur at the points $\lambda_k \in \mathbb{C}$ satisfying

$$3^{-\lambda_k - 1} = 2, \qquad (-\lambda_k - 1)\log(3) = \log(2) + 2k\pi i, \tag{F.8}$$

$\forall k \in \mathbb{Z}$, so at

$$\lambda_k := -1 - {}^3\!\log(2) - 2k\pi i\, {}^3\!\log(e), \tag{F.9}$$

$\forall k \in \mathbb{Z}$. The notation is simplified by introducing ${}^3\!\log(x) = \log(x)/\log(3)$, which is the logarithm to base 3.

The Mellin transform is analytic for $\operatorname{Re}\lambda > -1 - {}^3\!\log(2)$, with a meromorphic analytic continuation to the entire λ-plane. The residue c_k, at the pole located at λ_k, is equal to

$$c_k := \operatorname*{Res}_{\lambda=\lambda_k} \tilde{\mathcal{C}}(\lambda) = {}^3\!\log(e) \int_{\frac{1}{3}}^{1} dx\, x^{\lambda_k}\, \mathcal{C}(x), \tag{F.10}$$

which is $\forall k \in \mathbb{Z}$ a well defined complex number, since the integrand is continuous and bounded.
The scalar product of the Cantor function with the $\eta_\downarrow$-functions is therefore well defined,

$$\left\langle \mathcal{C}(x)\, , \eta_\downarrow^{(\alpha,q)}(x) \right\rangle = \delta_{q,0} \sum_{k=-\infty}^{\infty} c_k\, \delta_{\alpha,-1-\lambda_k}\,. \tag{F.11}$$

This leads us to suspect that Cantor's staircase function is asymptotic to the formal power series

$$\mathcal{C}(x) \sim \sum_{k=0}^{\infty} c_k\, x^{-1-\lambda_k}. \tag{F.12}$$

The staircase function does not satisfy the restrictions imposed on the preliminary class in Ch. 4, since the number of poles on the line $\operatorname{Re}\lambda = -1 - {}^3\!\log(2)$ is not finite. It can be easily included in future generalizations of the model.

The expansion obtained in (F.12) is not only asymptotic to the Cantor function, it actually converges to it on the whole real axis. To see this the expansion (F.12) can be written in the form

$$\mathcal{C}(x) \sim x^{{}^3\!\log(2)} \sum_{k=-\infty}^{\infty} c_k\, e^{2k\pi i\, {}^3\!\log(x)}, \tag{F.13}$$

with coefficients rewritten as

$$c_k = \int_{-1}^{0} d\big({}^3\!\log(x)\big)\; e^{-2k\pi i\, {}^3\!\log(x)} x^{-{}^3\!\log(2)}\, \mathcal{C}(x), \tag{F.14}$$

in the form of an integral over the variable ${}^3\!\log(x)$.

It is clear from the scaling formula (F.3) that the function

$$c(x) := 2^{-{}^3\log(x)}\, \mathcal{C}a(x) = x^{-{}^3\log(2)}\, \mathcal{C}a(x), \tag{F.15}$$

has the property

$$c(x) = c(3x) = c(x/3), \tag{F.16}$$

$\forall x \in \mathbb{R}_+$, so the function $c(x)$ is a periodic function of the argument $\log(x)$ with period $\log(3)$. The expansion (F.13) can therefore be interpreted as a Fourier series. Moreover it is a Fourier series of a function of bounded variation, so it is known to converge to the function. As a consequence it follows that Cantor's staircase function possesses a convergent expansion in terms of powers of its argument x.

Remark F.1 To verify the bounded variation one remarks that the Cantor staircase function is bounded by

$$(x/2)^{{}^3\log(2)} \leq \mathcal{C}a(x) \leq x^{{}^3\log(2)}, \tag{F.17}$$

so the function $c(x)$ is bounded by

$$2^{-1} < 2^{-{}^3\log(2)} \leq c(x) \leq 1. \tag{F.18}$$

The logarithm of $c(x)$ is the difference of two monotonic functions, so it is of bounded variation. Therefore in combination with the bound (F.18) it follows that $c(x)$ is itself of bounded variation.

The derivative at $x = 0$ of the staircase function can be expected to be given by a sum of $\eta_{\downarrow}$-functions

$$\mathcal{D}\, \mathcal{C}a(x)\Big|_{x=0} = \sum_{k=-\infty}^{\infty} c_k\, \eta_{\downarrow}^{({}^3\log(2)+2k\pi i\,{}^3\log(e),0)}(x), \tag{F.19}$$

with the same coefficients as before.

At this stage it is not possible to demonstrate that the Cantor function equals the integral of its derivative in the sense of generalised functions.

It is clear however that the standard treatment by means of Lebesgue's measure theory is inadequate. It is of course correct that standard derivative

$$\left(\frac{d}{dx}\right)_{Lebesgue} \mathcal{C}a(x) = 0, \qquad \text{a.e.}, \tag{F.20}$$

equals zero almost everywhere, but this statement ignores the nature of the singularities completely.

APPENDIX Y. Formula index

This appendix lists definitions, results of operations on, and indeterminacies of generalised functions. The tables give both a reference to a formula number and a pagenumber. The formulæ are listed in order of appearance in the text. Results are indicated by the = sign. Missing pieces of large formulæ or arguments are indicated by $\cdots$.

Y.1 Definitions

This table gives references to places where symbols are defined, to key formulæ, or to the place of first occurrence of a symbol.

Definition	Remark or result	Form nr.	Page nr.
δ	The distribution delta	2.1	7
$\langle \cdots , \cdots \rangle$	Symmetrical scalar product	2.5	8
$0 = 0(x)$	Zero generalised function	2.10	9
$[\cdots , \cdots]$	Operator commutator	2.12	9
$\mathcal{D}$	Differential operator	2.12	9
$\mathcal{X}$	Multiplication by x operator	2.12	9
$\mathcal{I}$	Identity operator	2.12	9
$\mathcal{D}$	Extension of $\frac{d}{dx}$	2.13	9
$\mathcal{F}$	Fourier operator	2.15	10
$\mathcal{F}^{-1}$	Inverse Fourier operator	2.19	10
$\mathcal{P}$	Parity operator	2.20	10
$I(x)$	Unit function	2.25	11
$\delta(x)$	Generalised function delta-bar	2.26	11
$\cdot$	Standard pointwise product	2.36	12
$\circ\bullet$	Left-first product	2.38	13
$\bullet\circ$	Right-first product	2.38	13
$*$	Convolution product	2.49	15
x^p, $\delta^{(q)}$	Trivial elements	3.1	21
$f_a(x; x_0+)$	Asymptotic expansion of $f(x)$ at $x = x_0+$	4.1	24
$\not\sim$	Not in asymptotic expansion	4.6	25
$\Gamma(\cdots , \cdots)$	Second incomplete Γ–function	4.17	28
$\mathrm{Pre}\int$	Preliminary integral	4.22	29
$\langle \cdots , \cdots \rangle_{\mathrm{pre}}$	Preliminary scalar product	4.25	31
$\mathcal{X}_{\mathrm{pre}}$	Preliminary multiplication by x	4.27	31
$\mathcal{D}_{\mathrm{pre}}$	Preliminary differential operator	4.28	32
$\mathcal{F}_{\mathrm{pre}}$	Preliminary Fourier operator	4.30	32

continued

continued

$\overline{\mathbf{PC}}'_\lambda$	Closure of $\mathbf{PC}'_\lambda$ under operators	6.68	65
θ_{ord}	Naïve theta function	6.79	67
$\mathcal{X}^{-1}$	Inverse of $\mathcal{X}$	6.80	67
$\mathcal{O}$	Any operator on $\overline{\mathbf{PC}}_\lambda$	7.2	71
$\bar{\mathcal{M}}$	Completed map $\overline{\mathbf{PC}}_\lambda \to \overline{\mathbf{PC}}'_\lambda$	7.2	71
$\bar{\mathcal{M}}^{-1}$	Inverse of the map $\mathcal{M}^{-1}$	7.2	71
$\bar{\delta}(x)$, $\bar{\delta}^{(p)}(x)$	Generalised functions 'delta-bar'	7.10	73
$\mathbf{PC}_{\bar{\delta}}$	Space of allowed linear combinations of $\bar{\delta}^{(p)}$ functions	7.20	74
$\overline{\mathbf{PC}}_\lambda$	The completed preliminary class	7.22	75
$\lessdot \cdots , \cdots \rangle$	'Left-first' scalar product	8.9	85
$\langle \cdots , \cdots \gtrdot$	'Right-first' scalar product	8.11	85
$\langle \cdots , \cdots \rangle$	Symmetrical scalar product	8.14	85
$\langle \cdots , \cdots \rangle_\rho$	Trivial generalisation	8.19	86
$\circ\bullet$	Left-first product on $\overline{\mathbf{PC}}'_\lambda$	8.32	88
$\bullet\circ$	Right-first product on $\overline{\mathbf{PC}}'_\lambda$	8.33	88
$\bullet$	Symmetrical product on $\overline{\mathbf{PC}}'_\lambda$	8.35	89
$[\cdots \circ\bullet \cdots]$	Product commutator	8.37	89
$\langle \cdots , \cdots \rangle$	Scalar product recovered	8.71	93
$[f \diamond g]$	Arbitrary product commutator	8.76	94
$[f \diamond g \diamond h]$	Arbitrary product associator	8.77	94
$[\cdots \circ\bullet \cdots \circ\bullet \cdots]$	Product associator	8.80	95
$\circ *$	Left-first convolution	8.89	97
$*$	Symmetrical convolution	8.93	97
$\bullet$	Symmetrical product on $\overline{\mathbf{PC}}_\lambda$	9.1	103
$\bar{\mathcal{M}}$	Completed map	9.2	103
$\bar{\mathcal{M}}^{-1}$	Inverse of completed map	9.4	103
$\overline{\mathbf{PC}}'_{\mathcal{M}}$	Subspace of $\overline{\mathbf{PC}}'_\lambda$. See also table 9.1	9.7	104
$\overline{\mathbf{PC}}'_{\not\mathcal{M}}$	Complement of $\overline{\mathbf{PC}}'_{\mathcal{M}}$	9.8	104
$P_{\mathcal{M}}$, $P_{\not\mathcal{M}}$	Corresponding projections	9.8	104
$\mathcal{M}_{\mathcal{X}}$, $\mathcal{M}_{\mathcal{X}}^{-1}$	$\bar{\mathcal{M}}$, $\bar{\mathcal{M}}^{-1}$ renamed	9.13	106
$P_{\bar{\delta}^{(0)}}$	Projection on $\bar{\delta}^{(0)}(x)$	9.19	107
$\mathcal{M}_{\mathcal{D}}$	Fourier image of $\mathcal{M}_{\mathcal{X}}$	9.20	107
P_I	Projection on $I(x)$	9.21	107
$d_{qr}(\cdots,\cdots)$	d-coefficient	9.28	109
$\mathcal{M}_{\mathcal{D}}^{-1}$	Inverse of $\mathcal{M}_{\mathcal{D}}$	9.31	110
$\circ\bullet$	'Left-first' product, on $\overline{\mathbf{PC}}_\lambda$	9.39	111

continued

continued

$\bar{\mathcal{S}}(a)$	Unitary scale transformation	16.3	195
$\mathcal{S}(a; x_0, 0)$	Translated scale transformation	16.25	199
$\mathcal{S}(a; 0, p_0)$	Wave translated scale transformation	16.26	199
$a^{\mathcal{X}\mathcal{D}}$	Infinitesimal generator of scale	16.33	200
${}_i\log\lvert x\rvert$	Indeterminate logarithm	17.1	207
$\boldsymbol{C}$	Indeterminate constant	17.7	208
${}_i\lvert x'\rvert^{\alpha}$	Indeterminate power	17.7	208
${}_i\eta^{(\alpha,q)}(x)$	Indeterminate η function	17.8	208
${}_if(x)$	Indeterminate version of $f(x)$	17.19	210
${}_i\mathcal{M}_{\mathcal{X}}$	Indeterminate mapping	17.26	212
${}_i\mathcal{C}$	Explictly scale dependent product	17.32	213
$\bullet$	Indeterminate product	17.38	214
${}_i\log(0+)$	Indeterminate value	17.49	215
${}_i\mathrm{Lim}$	Indeterminate limit	17.55	216
$\langle f(x), \mathcal{O}\, f(x)\rangle$	Expectation value	18.2	220
C	Indefinite constant in integral	18.18	225
lim	Standard weak convergence	19.1	229
Lim	Generalised 'weak' convergence	19.6	231
$H(a+x)H(a-x)$	Preferred sequence to $I'(x)$	19.50	240
$\pi^{-1}x^{-1}\sin(ax)$	Preferred sequence to $\delta(x)$	19.52	240
$\mathrm{L\overline{im}}$	Completed limit	19.63	243
$\zeta(s)$	Riemann's zeta function	20.3	248
${}_{\text{pre}}\sum$	Preliminary sum	20.6	249
$\Phi(z, s, \nu)$	Lerch's transcendent	20.8	249
$\zeta(s, \nu)$	Generalised zeta function	20.9	249
$\langle\, a_n\, , b_n\, \rangle$	Scalar product of sequences	20.17	251
$\text{Ш}(x)$	The comb of δ's, Dirac's comb	20.28	253
$\text{Ш}(x)$	Dirac's comb, rescaled	20.29	253
f_p	Any periodic function, period one	20.43	256
f_o	Idem, the restriction to one period	20.44	256
a_n	Fourier coefficient	20.50	257
$U(x)$	Smudge function	20.53	257
$\mathcal{H}$	The Hilbert operator	21.2	261
$(x \pm io)^{\alpha}$	Analytic boundary functions	21.23	265
$\left(e^{i\varphi}(x \pm io)\right)^{\alpha}$	Analytic boundary functions	21.24	265
$\eta_{\varphi}^{(\alpha,q)}(x \pm io)$	Corresponding η–functions	21.26	265
$\bar{\mathcal{M}}^{-1}\mathcal{H}\bar{\mathcal{M}}$	Definition	21.27	265
$\left(e^{i\varphi}(x' \pm io)\right)^{\alpha}$	Computation	21.29	266
$(x \pm io)^{\alpha}\log^{q}(x \pm io)$	Standard result	21.34	267

continued

$f_+,\ f_-$	Positive, negative frequency parts	21.52	270
$d^n r$	N-dimensional volume element	22.9	280
$f^{[j]}$	Laurent coefficient	A.2	301
Res	Residue at pole	A.3	301
$\binom{\lambda}{n}^{(k)}$	Derivative of binomial coefficient	B.5	302
$(\lambda)_n^{[k]}$	Residue of Pochhammer symbol	B.12	303
$(0)_n^{[k]}$	Stirling numbers of the first kind	B.23	304
ψ	Logarithmic derivative of Γ	C.10	306
$c_j(\,,\,)$	Laurent coefficient for Fourier	C.14	306
$d_{qr}(\,,\,)$	Combination of c_j coefficients for product	C.36	310
$h_j(\,,\,)$	Combination of c_j coefficients for Hilbert transform	C.42	310
$\Phi(z,s,\nu)$	Lerch's transcendent function	D.1	311
$\zeta(s,\nu)$	Generalised ζ–function	D.2	311
$\zeta(s)$	Riemann's zeta function	D.3	311
$\mathcal{C}a(x)$	Cantor's staircase function	F.1	315

end of table Y.1.

Y.2 The operator $\boldsymbol{\mathcal{X}}$

Formulæ for the operator $\boldsymbol{\mathcal{X}}$.

$\boldsymbol{\mathcal{X}}$-formula	Remark or result	Form nr.	Page nr.
$[\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{X}}]$	Commutator	2.12	9
$\boldsymbol{\mathcal{X}}\,f(x)$	Extension requirement	2.14	9
$\boldsymbol{\mathcal{F}}^{-1}\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{F}}$	$= -i\boldsymbol{\mathcal{D}}$, Unitary equivalence	2.22	10
$\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{P}}$	Parity $= -\boldsymbol{\mathcal{X}}$	2.24	11
$\boldsymbol{\mathcal{X}}\,\delta(x)$	Annihilation requirement	2.27	11
$\boldsymbol{\mathcal{X}}(f \bullet g)$	Multiplicative rules	2.41	13
$\boldsymbol{\mathcal{X}}(f * g)$	Leibniz's rule (convolution)	2.57	15
$\boldsymbol{\mathcal{X}}_{\mathrm{pre}}\,f(x)$	$\boldsymbol{\mathcal{X}}_{\mathrm{pre}} = x \cdot f(x)$	6.6	56
$\boldsymbol{\mathcal{X}}\lvert x'\rvert^{\alpha}\log^q\lvert x\rvert\,\mathrm{sgn}^m(x)$	$= \lvert x'\rvert^{\alpha+1}\log^q\lvert x\rvert\,\mathrm{sgn}^{m+1}(x)$	6.9	57
$\boldsymbol{\mathcal{X}}\,\eta_{\downarrow}^{(\alpha,q)}(x-x_0)$	$= \eta_{\downarrow}^{(\alpha-1,q)}(x-x_0) + \cdots$	6.10	57
$\boldsymbol{\mathcal{X}}\,e^{ikx}\not\eta_{\uparrow}^{(\alpha,q)}(x)$	$= e^{ikx}\not\eta_{\uparrow}^{(\alpha-1,q)}(x)$	6.11	57
$\boldsymbol{\mathcal{X}}\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$	$= \eta_a^{(\alpha-1,q)}(x)$	6.12	57
$\boldsymbol{\mathcal{X}}\,\eta^{(p,q)}(x)$	$= \eta^{(p-1,q)}(x)$	6.14	57
$\boldsymbol{\mathcal{X}}\,\sigma^{(p,q)}(x)$	$= \sigma^{(p-1,q)}(x)$	6.15	57

continued

$\mathcal{X}\,\eta(x)$	$=\eta^{(-1)}(x)$	6.16	57
$\mathcal{X}\,\theta^{(p)}(x)$	$=\theta^{(p+1)}(x)$	6.72	66
$\mathcal{X}\,f'(x)$	General element	6.81	67
$\mathcal{X}\,\delta^{(p+1)}(x)$	$=\delta^{(p)}(x)$	7.33	77
$\mathcal{X}\,\delta(x)$	$=0(x)$	7.34	77
$\mathcal{X}\,\delta(x-x_0)$	$=x_0\,\delta(x-x_0)$	7.36	78
$\mathcal{X}\lvert x\rvert^\alpha\log^q\lvert x\rvert$	$=\lvert x\rvert^{\alpha+1}\log^q\lvert x\rvert\,\mathrm{sgn}(x)$	7.38	78
$\mathcal{X}(f'\bullet g')$	Multiplicative rule on $\overline{\mathbf{PC}}'_\lambda$	8.102	99
$\langle\,\mathcal{X}f'\,,g'\,\rangle$	Selfadjointness in $\overline{\mathbf{PC}}'_\lambda$	8.111	100
$\mathcal{X}\,f(x)$	$=x\bullet f(x)$	9.77	116
$\mathcal{X}\big(f\circ\!\bullet\,g\big)$	Left-left multiplicative rule	10.20	126
$\mathcal{X}\big(f\bullet g\big)$	Multiplicative rules on $\overline{\mathbf{PC}}_\lambda$	10.22	127
$\mathcal{X}f\bullet g-f\bullet\mathcal{X}g$	Error term	10.23	127
$\mathcal{X}\,\theta_s^{(\alpha,q)}(x)$	$=\theta_a^{(\alpha+1,q)}(x)$	11.16	136
$[\mathcal{D}\,,\mathcal{X}]$	Commutation rule	12.12	152
$\mathcal{F}^{-1}\mathcal{D}\mathcal{F}$	$=-i\mathcal{X}$, Unitary equivalence	12.25	154
$\mathcal{X}\big(f'(x)\bullet g'(x)\big)$	Multiplicative rules	12.30	155
$\langle\,\mathcal{X}f'\,,g'\,\rangle$	Selfadjointness in $\overline{\mathbf{PC}}_\lambda$	12.33	156
$\langle\,\mathcal{X}f\,,g\,\rangle$	Selfadjointness in $\overline{\mathbf{PC}}_\lambda$	12.36	157
$\mathcal{X}_{\text{pre}}^{-1}$	Pre inverse operator	14.24	177
$\mathcal{X}^{-1}\lvert x'\rvert^\alpha\log^q\lvert x\rvert\,\mathrm{sgn}^m(x)$	$=\lvert x'\rvert^{\alpha-1}\log^q\lvert x\rvert\,\mathrm{sgn}^{m+1}(x)$	14.25	177
$\mathcal{X}^{-1}\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$	$=\eta_s^{(\alpha+1,q)}(x)\,\mathrm{sgn}^{m+1}(x)$	14.26	178
$\mathcal{X}^{-1}\,f'(x)$	$=x^{-1}\bullet f'(x)$ on $\overline{\mathbf{PC}}'_\lambda$	14.28	178
$\mathcal{X}^{-1}\,f(x)$	$\neq x^{-1}\bullet f(x)$ on $\overline{\mathbf{PC}}_\lambda$	14.29	178
$\mathcal{X}^{-1}\lvert x\rvert^\alpha\log^q\lvert x\rvert\,\mathrm{sgn}^m(x)$	$=\lvert x\rvert^{\alpha-1}\log^q\lvert x\rvert\,\mathrm{sgn}(x)$	14.31	178
$\mathcal{X}^{-1}\,\delta^{(p)}(x)$	$=\delta^{(p+1)}(x)$	14.32	178
$\mathcal{X}^{-1}\,\theta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$	$=\theta_s^{(\alpha-1,q)}(x)$	14.37	179
$\mathcal{X}^{-1}\,\theta(x)$	$=\theta^{(-1)}(x)$	14.38	179
$\mathcal{X}\mathcal{X}^{-1}$	Almost inverse	14.39	179
$\mathcal{F}^{-1}\mathcal{X}^{-1}\mathcal{F}$	$=i\mathcal{D}^{-1}$, Unitary equivalence	14.64	183
$e^{a\mathcal{D}}\mathcal{X}\,e^{-a\mathcal{D}}$	Translation of $\mathcal{X}$ operator	15.16	189
$\mathcal{S}(-a)\mathcal{X}\mathcal{S}(a)$	Scaling of $\mathcal{X}$	16.37	201
$\mathrm{Lim}\,\mathcal{X}=\mathcal{X}\,\mathrm{Lim}$	Interchangeable	19.62	242
$\mathcal{X}\big(e^{i\varphi}(x'\pm io)\big)^\alpha$	$=e^{-i\varphi}\big(e^{i\varphi}(x'\pm io)\big)^{\alpha+1}$	21.38	267
$\mathcal{X}\,\eta_\varphi^{(\alpha,q)}(x\pm io)$	$=\cdots\eta_\varphi^{(\alpha-1,q)}\cdots$	21.39	267

end of table Y.2.

Y.3 The operator $\mathcal{D}$

Formulæ for the operator $\mathcal{D}$.

$\mathcal{D}$-formula	Remark or result	Form nr.	Page nr.
$[\mathcal{D}, \mathcal{X}]$	Canonical commutator	2.12	9
$\mathcal{D}\, f(x)$	Extension requirement	2.13	9
$\mathcal{F}^{-1}\mathcal{D}\mathcal{F}$	$= -i\mathcal{X}$, Unitary equivalence	2.23	11
$\mathcal{P}\mathcal{D}\mathcal{P}$	Parity $= -\mathcal{D}$	2.24	11
$\mathcal{D}(f \bullet g)$	Leibniz rule	2.39	13
$\mathcal{D}(f \bullet g)$	Multiplicative rules	2.58	16
$\mathcal{D}_{\text{pre}}(x - x_0)^{\lambda} H(x - x_0)$	$= \lambda(x - x_0)^{\lambda-1} H(x - x_0)$	6.19	58
$\mathcal{D}\,(x - x_0)^{\alpha} \log^q(x - x_0)$	$= \alpha(x - x_0)^{\alpha} \cdots + \cdots$	6.23	58
$\mathcal{D}\,\lvert x'\rvert^{\alpha} \log^q\lvert x\rvert \operatorname{sgn}^m(x)$	$= \alpha\lvert x'\rvert^{\alpha-1} \log^q\lvert x\rvert \operatorname{sgn}^{m+1}(x) + \ldots$	6.24	58
$\mathcal{D}\, I'(x)$	$= 2\,\sigma(x) - 2\,\not\phi(x)$	6.26	59
$\mathcal{D}\,\eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x)$	$= -(\alpha + 1)\,\eta_s^{(\alpha+1,q)} \ldots$	6.32	59
$\mathcal{D}\, e^{ikx} \not\eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x)$	$= -(\alpha + 1)\, e^{ikx} \not\eta_s^{(\alpha+1,q)} \ldots$	6.33	59
$\mathcal{D}^p\, \eta_s^{(\alpha,q)} \operatorname{sgn}^m(x)$	$= \cdots \eta_s^{(\alpha+p,q)}(x) \cdots$	6.36	60
$\mathcal{D}\,\theta^{(p)}(x)$	$= p\,\theta^{p-1}(x) + \cdots$	6.73	66
$\mathcal{D}\, f'(x)$	General element $\in \overline{\mathbf{PC}}'_{\lambda}$	6.82	67
$\mathcal{D}\,\delta^{(p)}(x)$	$= -(p + 1)\,\delta^{(p+1)}(x)$	7.39	78
$\mathcal{D}^p\ \delta(x)$	$= (-)^p p!\ \delta^{(p)}(x)$	7.40	78
$\mathcal{D}\lvert x\rvert^{\alpha} \log^q\lvert x\rvert \operatorname{sgn}^m(x)$	$= \alpha\lvert x\rvert^{\alpha-1} \log^q\lvert x\rvert \operatorname{sgn}^{m+1}(x) + \cdots$	7.42	78
$\mathcal{D}H(x)$	$= \delta(x)$	7.48	79
$\mathcal{D}\, I(x)$	$= \mathit{0}(x)$	7.49	79
$\mathcal{D}\, f(x)$	General element $\in \overline{\mathbf{PC}}_{\lambda}$	7.52	80
$\mathcal{D}\big(H'\big)^m$	$\mathcal{D}$ on multiple products	8.106	100
$\langle \mathcal{D}f', g' \rangle$	Selfadjointness in $\overline{\mathbf{PC}}'_{\lambda}$	8.112	101
$\mathcal{D}\big(f(x) \bullet g(x)\big)$	Leibniz rule	10.25	127
$\mathcal{D}\,\theta_s^{(\alpha,q)}(x)$	$= \alpha\,\theta_a^{(\alpha-1,q)}(x) + \cdots$	11.17	137
$\mathcal{D}\,\theta(x)$	$= -2\,\sigma(x)$	11.19	137
$\mathcal{D}\,\tau(x)$	$= 2\,\delta(x) - 2\,\eta(x)$	11.19	137
$\mathcal{D}\,\not\theta(x)$	$= 2\,\not\phi(x)$	11.20	137
$\mathcal{D}\not\tau(x)$	$= 2\not\eta(x)$	11.20	137
$[\mathcal{D}, \mathcal{X}]$	Commutation rule	12.12	152
$\mathcal{F}^{-1}\mathcal{X}\mathcal{F}$	$= -i\mathcal{D}$, Unitary equivalence	12.25	154
$\mathcal{D}\big(f(x) \bullet g(x)\big)$	Leibniz rule	12.31	155

continued

$\langle \mathcal{D} f', g' \rangle$	Selfadjointness in $\overline{\mathbf{PC}}'_\lambda$	12.34	156
$\langle \mathcal{D} f, g \rangle$	Selfadjointness in $\overline{\mathbf{PC}}_\lambda$	12.39	157
$\mathcal{D}^{-1}_{\text{pre}}$	Preliminary inverse operator	14.42	180
$\mathcal{D}^{-1}\lvert x'\rvert^\alpha \log^q\lvert x\rvert \operatorname{sgn}^m(x)$	$= (\alpha+1)^{-1}\lvert x\rvert^{\alpha+1}\cdots$	14.44	180
$\mathcal{D}^{-1}\lvert x'\rvert^{-1} \log^q\lvert x\rvert \operatorname{sgn}^m(x)$	$= (q+1)^{-1}\log^{q+1}\lvert x'\rvert\cdots$	14.45	180
$\mathcal{D}^{-1}\,\eta_s^{(\alpha,q)}(x)\operatorname{sgn}^m(x)$	$= -\alpha^{-1}\,\eta_s^{(\alpha-1,q)}(x)\cdots$	14.46	180
$\mathcal{D}^{-1}\,\eta_s^{(0,q)}(x)$	$= -q^{-1}\,\eta_s^{(-1,q-1)}(x)$	14.47	180
$\mathcal{D}^{-1}\lvert x\rvert^\alpha \log^q\lvert x\rvert \operatorname{sgn}^m(x)$	$= (\alpha+1)^{-1}\lvert x\rvert^{\alpha+1}\cdots$	14.50	181
$\mathcal{D}^{-1}\,\delta^{(p+1)}(x)$	$= -(p+1)^{-1}\,\delta^{(p)}(x)$	14.52	181
$\mathcal{D}^{-1}\,\delta(x)$	$= \frac{1}{2}\operatorname{sgn}(x)$	14.53	181
$\mathcal{D}^{-1}\,\not\eta(x)$	$= \frac{1}{2}\not\tau(x)$	14.56	182
$\mathcal{D}^{-1}\,\eta(x)$	$= \frac{1}{2}\big(\operatorname{sgn}(x) - \tau(x)\big)$	14.57	182
$\mathcal{D}^{-1}\cosh^{-2}(x)$	$= \tanh(x)$	14.60	182
$\mathcal{F}^{-1}\mathcal{D}^{-1}\mathcal{F}$	$= i\mathcal{X}^{-1}$, Unitary equivalence	14.63	183
$\mathcal{D}^{-1}\mathcal{D}$	Almost inverse	14.67	183
$\mathcal{D}^{-1}\,f(x)$	Indefinite integral	14.72	183
$F(x) := \mathcal{D}^{-1}\,f(x)$	Primitive function	14.73	184
$e^{a\mathcal{D}}\mathcal{D}\,e^{-a\mathcal{D}}$	Translation of $\mathcal{D}$ operator	15.17	189
$\mathcal{S}(-a)\mathcal{D}\mathcal{S}(a)$	Scaling of $\mathcal{D}$	16.38	201
$\operatorname{Lim}\mathcal{D} = \mathcal{D}\operatorname{Lim}$	Interchangeable	19.58	242
$\mathcal{D}^p\,Ш(x)$	Derivative of comb	20.34	254
$\mathcal{D}\big(e^{i\varphi}(x'\pm io)\big)^\alpha$	$= \alpha\,e^{i\varphi}\big(e^{i\varphi}(x'\pm io)\big)^{\alpha-1}$	21.40	268
$\mathcal{D}\,\eta_\varphi^{(\alpha,q)}(x\pm io)$	$= -(\alpha+1)\,e^{i\varphi}\eta_\varphi^{(\alpha+1,q)}(x)\cdots$	21.42	268

end of table Y.3.

Y.4 The operator $\mathcal{F}$

Formulæ for the operator $\mathcal{F}$.

$\mathcal{F}$-formula	Remark or result	Form nr.	Page nr.
$\mathcal{F}\,f(x)$	Normalization	2.15	10
$\mathcal{F}^\dagger\,f(x)$	Normalization	2.16	10
$\mathcal{F}^{-1}\,f(x)$	Normalization	2.19	10
$\mathcal{F}^{-1}\mathcal{X}\mathcal{F}$	$= -i\mathcal{D}$, Unitary equivalence	2.22	10
$\mathcal{F}^{-1}\mathcal{D}\mathcal{F}$	$= -i\mathcal{X}$, Unitary equivalence	2.23	11
$\mathcal{F}\,I = \delta$	Requirement	2.26	11
$\langle \mathcal{F}f, \mathcal{F}g \rangle$	Parseval's equality	2.29	11

continued

end of table Y.4.

Y.5 Mappings

This table lists formulæ for the various maps used in the construction of the products.

continued

$\overline{\mathcal{M}}^{-1}\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)(x)$	$= \delta_{q,0}\,\delta_{\alpha,p}\,\delta_{m,0}^{\mathrm{mod}2}\,\delta^{(p)}(x)$	9.5	103
$\overline{\mathcal{M}}^{-1}\|x\|^\alpha \log^q\|x\|\,\mathrm{sgn}^m(x)$	$= \|x\|^\alpha \log^q\|x\|$	9.6	104
$\overline{\mathcal{M}}^{-1}\,\theta^{(p)}(x)$	$= 0(x)$	9.9	104
$\overline{\mathcal{M}}\ \overline{\mathcal{M}}^{-1}$	Exceptional subspace	9.10	104
$\overline{\mathcal{M}}\ ,\overline{\mathcal{M}}^{-1}$	Table of properties	9.12	105
$\mathcal{M},\overline{\mathcal{M}}^{-1}$	Commutative diagram	9.12	106
$\mathcal{M}_{\mathcal{X}}$	Standardization: $=\overline{\mathcal{M}}$	9.13	106
$\mathcal{M}_{\mathcal{X}}^{-1}$	Standardization: $=\overline{\mathcal{M}}^{-1}$	9.13	106
$[\mathcal{X}\,,\mathcal{M}_{\mathcal{X}}]$	Commutator	9.19	107
$\mathcal{M}_{\mathcal{D}} = \mathcal{F}^{-1}\mathcal{M}_{\mathcal{X}}\mathcal{F}$	Definition	9.20	107
$[\mathcal{D}\,,\mathcal{M}_{\mathcal{D}}]$	Commutator	9.21	107
$\mathcal{M}_{\mathcal{D}}\,\delta^{(p)}(x)$	$=\eta^{(p)}(x)+\cdots$	9.26	108
$\mathcal{M}_{\mathcal{D}}\|x\|^\alpha\log^q\|x\|\,\mathrm{sgn}^m(x)$	$=\|x'\|^\alpha(x)\cdots+\cdots$	9.27	109
$\mathcal{M}_{\mathcal{D}}\, f = \mathcal{M}_{\mathcal{X}}\, f+\cdots$	General element	9.30	109
$\mathcal{M}_{\mathcal{D}}^{-1}\,\eta_s^{(\alpha,q)}(x)\,\mathrm{sgn}^m(x)$	$=\delta_{q,0}\,\delta_{\alpha,p}\,\delta_{p,m}^{\mathrm{mod}2}\,\delta^{(p)}(x)$	9.31	110
$\mathcal{M}_{\mathcal{D}}^{-1}\,\theta^{(p)}(x)$	$=0(x)$	9.32	110
$\mathcal{M}_{\mathcal{D}}^{-1}\|x'\|^\alpha\log^q\|x\|\,\mathrm{sgn}^m(x)$	$=x^{-p-1}\log^q\|x\|\,\mathrm{sgn}^m(x)$	9.33	110
$\mathcal{M}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}^{-1}$	Exceptional subspace	9.35	110
$\mathcal{M}'_{\mathcal{X}}\,\delta^{(p)}(x)$	Alternative standardization	9.99	120
$\mathcal{M}'_{\mathcal{X}}\|x\|^\alpha\log^q\|x\|\,\mathrm{sgn}^m(x)$	Alternative standardization	9.100	120
$\mathcal{M}\,f'(x)$	Extension of $\mathcal{M}$ to $\overline{\mathbf{PC}}'_\lambda$	11.30	139
$\mathcal{M}^{-1}\,f(x)$	Extension of $\mathcal{M}^{-1}$ to $\overline{\mathbf{PC}}_\lambda$	11.37	139
$\mathcal{M}^{-1}\mathcal{M}$	Exceptional subspace	11.38	139
$\mathcal{M}_{\mathcal{D}},\ \mathcal{M}_{\mathcal{X}}$	Equivalence for operator definition	12.28	155
${}_i\mathcal{M}_{\mathcal{X}}$	Indeterminate standardization	17.26	212
${}_i\mathcal{M}_{\mathcal{X}}\,\delta^{(p)}(x)$	Indeterminate standardization	17.27	212
${}_i\mathcal{M}_{\mathcal{X}}\|x\|^\alpha\log^q\|x\|\,\mathrm{sgn}^m(x)$	Indeterminate standardization	17.28	212
${}_i\mathcal{M}_{\mathcal{D}}$	Indeterminate standardization	17.29	212
${}_i\mathcal{M}_{\mathcal{D}}(C)$	Explictly scale dependent map	17.32	213
$\overline{\mathcal{M}}^{-1}\mathcal{H}\overline{\mathcal{M}}$	Transfer of Hilbert	21.22	264

end of table Y.5.

Y.6 Translation operators

This table lists various formulæ involving the translation operators.

Translation	Remark or result	Form nr.	Page nr.
$\mathcal{T}_{\text{pre}}(x_0, 0)$	Preliminary translation	15.1	187
$\mathcal{T}(x_0, 0)\lvert x'\rvert^\alpha \log^q\lvert x\rvert \operatorname{sgn}^m(x)$	$= \lvert x' - x_0\rvert^\alpha \cdots$	15.5	188
$\mathcal{T}(x_0, 0)\,\eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x)$	$= \eta^{(\alpha,q)}(x - x_0)\cdots$	15.6	188
$\mathcal{T}(x_0, 0)\,\delta^{(p)}(x)$	$= \delta^{(p)}(x - x_0)$	15.11	189
$\mathcal{T}(x_0, 0)\, I(x)$	$I(x)$ is translation invariant	15.12	189
$\mathcal{T}_{\text{pre}}(0, k_0) = e^{ik_0x}\bullet$	Preliminary wave translation	15.18	190
$e^{ia\mathcal{X}}\eta_s^{(\alpha,q)}(x) \operatorname{sgn}^m(x)$	Wave translation of η	15.20	190
$e^{ia\mathcal{X}}\delta^{(p)}(x)$	Wave translation of $\delta^{(p)}$	15.21	190
$\mathcal{T}(0, k_0)\,\delta(x)$	Eigenfunction	15.23	190
$\mathcal{F}^{-1}\mathcal{T}(a, 0)\mathcal{F}$	Unitary equivalence	15.24	191
$\langle\, \mathcal{T}(x_0, 0)\, f(x)\,, \mathcal{T}(x_0, 0)\, g(x)\,\rangle$	Almost unitary	15.31	192
$\mathcal{T}(a, b)$	General phase plane translations	15.36	193
$\mathcal{T}(a_1, b_1)\mathcal{T}(a_2, b_2)$	Group property, projective representation	15.40	194
$\mathcal{F}^{-1}\mathcal{T}(a, b)\mathcal{F}$	Unitary equivalence, $= \mathcal{T}(-b, a)$	15.41	194
$\sum_{n=-\infty}^{\infty} \mathcal{T}(n, 0)$	Repeated period	20.44	256

end of table Y.6.

Y.7 Scale transformation operators

This table lists various formulæ involving the scale operators

Scale formula	Remark or result	Form nr.	Page nr.
$\mathcal{S}(a)$	Preliminary definition	16.2	195
$\mathcal{S}(a)\big(\lvert x'\rvert^\alpha \log^q\lvert x\rvert \operatorname{sgn}^m(x)\big)$	$= a^\alpha\lvert x'\rvert^\alpha \log^q\lvert x\rvert \operatorname{sgn}^m(x) + \cdots$	16.7	196
$\mathcal{S}(a)\, I'(x)$	$\neq I'(x)$ not scale eigenfunction	16.10	197
$\mathcal{S}(a)\,\eta^{(\alpha,q)}(x) \operatorname{sgn}^m(x)$	$= a^{-\alpha-1}\,\eta_s^{(\alpha,q)}(x)\cdots$	16.11	197
$\mathcal{S}(a)\, I(x)$	$= I(x)$: is scale eigenfunction	16.13	197
$\mathcal{S}(a)\,\theta^{(p)}(x)$	$= a^p\,\theta^{(p)}(x) - \cdots$	16.15	197

continued

$\boldsymbol{S}(a)\, f'(x)$	Scale transformation in $\overline{\mathbf{PC}}'_\lambda$	16.16	197
$\boldsymbol{S}(a)\, \delta^{(p)}(x)$	Scaling of δ–functions	16.20	198
$\boldsymbol{S}(a)\|x\|^\alpha \log^q\|x\| \operatorname{sgn}^m(x)$	$= a^\alpha \|x\|^\alpha \cdots$	16.21	198
$\boldsymbol{S}(a)\, \|x\|^{-1}$	$= \|x\|^{-1} + 2\log(a)\, \delta(x)$	16.22	198
$\boldsymbol{S}(a) \log\|x\|$	Scaling of the logarithm	16.23	199
$\langle\, \boldsymbol{S}(a)\, f(x)\, , \boldsymbol{S}(a)\, g(x)\, \rangle$	$\boldsymbol{S}(a)$ unitary in $\overline{\mathbf{PC}}'_\lambda$	16.27	199
$\langle\, \boldsymbol{S}(a)\, f(x)\, , \boldsymbol{S}(a)\, g(x)\, \rangle$	$\boldsymbol{S}(a)$ not unitary in $\overline{\mathbf{PC}}_\lambda$	16.27	199
$\boldsymbol{S}(a_1)\boldsymbol{S}(a_2)$	$= \boldsymbol{S}(a_1 a_2)$, Group property	16.32	200
$a^{\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{D}}}$	Exponential form of scale	16.33	200
$\frac{1}{2}(\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{X}})$	Selfadjoint generator	16.34	201
$\boldsymbol{S}(-a)\boldsymbol{\mathcal{X}}\boldsymbol{S}(a)$	Scaling of the operator $\boldsymbol{\mathcal{X}}$	16.37	201
$\boldsymbol{S}(-a)\boldsymbol{\mathcal{D}}\boldsymbol{S}(a)$	Scaling of the operator $\boldsymbol{\mathcal{D}}$	16.38	201
$\boldsymbol{\mathcal{F}}^{-1}\boldsymbol{S}(a)\boldsymbol{\mathcal{F}}$	Fourier transform of scale	16.39	201
$\boldsymbol{S}(a)\, f(x) \bullet \boldsymbol{S}(a)\, g(x)$	Scaling of product	16.40	202
$\boldsymbol{S}(a)\big(f(x) \bullet g(x)\big)$	Scaling of indeterminate product	17.44	214
$\langle\, \boldsymbol{S}(a)\, f(x)\, , \boldsymbol{S}(a)\, g(x)\, \rangle$	Unitarity, indeterminate	17.46	215

end of table Y.7.

Y.8 Indeterminacy

This section lists the indeterminacy of the generalised functions. Only the leading component is given when $\cdots$ appears.

Indeterminacy	Remark or result	Form nr.	Page nr.
${}_i\log\|x\|$	$= -\boldsymbol{C}\, I(x)$	17.1	207
${}_i\|x'\|^\alpha$	$= \frac{1}{q+1}\boldsymbol{C} + \cdots$	17.7	208
$\eta_s^{(\alpha,q)}(x)$	$= -(q+1)\boldsymbol{C}\, \eta_s^{(\alpha,q+1)}(x) + \cdots$	17.8	208
$\|x\|^\alpha$	$= -2\boldsymbol{C} \sum \delta_{\alpha,-p-1}\, \delta_{p,m}^{\mathrm{mod}2}\, \delta^{(p)}(x)$	17.9	208
$\delta^{(p)}(x)$	$= 0(x)$: Determinate	17.10	209
$\boldsymbol{S}(a)\, {}_i\eta^{(\alpha,q)}(x)$	Scaling of indeterminate functions	17.12	209
e^{-x^2}	$= 0(x)$: Determinate	17.17	210
x^{-1}	$= 0(x)$: Determinate	17.19	210
$\|x\|^{-1}$	$= -2\boldsymbol{C}\, \delta(x)$	17.19	210
x'^{-1}	$= -2C\, \sigma(x) + 2\boldsymbol{C}\, \phi(x) + \cdots$	17.20	211
$\|x'\|^{-1}$	$= -2\boldsymbol{C}\, \eta(x) + \cdots$	17.21	211
$\boldsymbol{S}(a)\big({}_i f(x;\boldsymbol{C})\, {}_i\bullet_{\boldsymbol{C}}\, {}_i g(x;\boldsymbol{C})$	Scale invariant product	17.35	213

continued

Y.9 Hilbert transforms

Formulæ for the operator $\mathcal{H}$

Y.10 Tables

Tables for various conventions and properties are listed

APPENDIX Z. Product tables

The tables in this appendix give the formula numbers of products of the basic functions. The first table lists the asymmetrical products, left-sided when reading southwest to northeast, or right-sided when reading the opposite way. It may be necessary to specialize or rename parameters to obtain the desired product. The factors in the product are only partially indicated for lack of space. The first block gives elements in $\overline{\mathbf{PC}}'_\lambda$, the second in $\overline{\mathbf{PC}}_\lambda$, while the last block contains the mixed elements.

$\circ\bullet/\nearrow\!\swarrow/\bullet\circ$	$\eta^{(\beta,r)}$	$\not\eta^{(\beta,r)}$	$x'\log$	$\theta^{(q)}$	$\delta^{(p)}$	$x\log$	$\theta^{(\beta,r)}$	$\not\theta^{(\beta,r)}$
$\eta^{(\alpha,q)}$	*0*	*0*	*0*	8.58	*0*	11.46	11.59	*0*
$\not\eta^{(\alpha,q)}$	*0*	*0*	*0*	*0*	*0*	11.47	*0*	11.59
$x'\log$	8.45	8.45	8.55	*0*	*0*	11.45	*0*	*0*
$\theta^{(p)}$	*0*	*0*	*0*	8.65	11.62	11.57	11.55	*0*
$\delta^{(p)}$	*0*	*0*	*0*	11.62	*0*	9.47	11.62	*0*
$x\log$	11.46	11.47	11.45	11.57	9.45	9.68	11.55	11.56
$\theta^{(\alpha,q)}$	*0*	*0*	*0*	11.55	11.61	11.55	11.55	*0*
$\not\theta^{(\alpha,q)}$	*0*	*0*	*0*	*0*	*0*	11.56	*0*	11.56

Entries marked *0* indicate that the product in question equals zero. The second table lists the symmetrical products

$\bullet/\nearrow\!\swarrow/\bullet$	$\eta^{(\beta,r)}$	$\not\eta^{(\beta,r)}$	$x'\log$	$\theta^{(q)}$	$\delta^{(p)}$	$x\log$	$\theta^{(\beta,r)}$	$\not\theta^{(\beta,r)}$
$\eta^{(\alpha,q)}$	*0*	*0*	8.48	8.59	*0*	11.46	11.59	*0*
$\not\eta^{(\alpha,q)}$	*0*	*0*	8.48	*0*	*0*	11.47	*0*	11.59
$x'\log$	8.48	8.48	11.45	*0*	*0*	11.45	*0*	*0*
$\theta^{(p)}$	8.59	*0*	*0*	8.67	11.61	11.55	11.55	*0*
$\delta^{(p)}$	*0*	*0*	*0*	11.61	*0*	9.55	11.61	*0*
$x\log$	11.46	11.47	11.45	11.55	9.55	9.70	11.55	11.56
$\theta^{(\alpha,q)}$	11.59	*0*	*0*	11.55	11.61	11.55	11.55	*0*
$\not\theta^{(\alpha,q)}$	*0*	11.59	*0*	*0*	*0*	11.56	*0*	11.56

It is symmetrical, since it lists the symmetrical products. Some entries agree with the previous table. In these cases the left- and right-sided products are equal.

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INDEX

This index lists keywords only. A large index to formulæ, symbols, and notations is given in appendix Y. For frequently occuring keywords only a selection of the pagenumbers is given.

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