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Monotonicity properties of infinitely divisible distributions

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Monotonicity Properties of Infinitely Divisible Distributions

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Chapter 1

PRELIMINARIES

1.1 Introduction and summary

The field of infinitely divisible random variables has grown during the last few decades to take a permanent position in the theory of probability. Almost all standard text books on probability now include at least one chapter devoted to the field of infinite divisibility. This is mainly due to its importance in solving the general central limit problem and to its applications to stochastic processes with stationary independent increments.

The notion of infinite divisibility originated in connection with the central limit problem. The classical central limit problem is concerned with characterizing the random variables X which can be obtained as

$$\sum_{k=1}^{n} a_n \mathrm{X}_k + b_n \xrightarrow{w} \mathrm{X} \text{ as } n \to \infty,$$

where (X_k) is a sequence of identically distributed random variables with finite variance a and expectation b, and $a_n = (na)^{-1/2}$ and $b_n = -nb\,(a_n)^{-1/2}$. It turns out (cf. Loève (1977)) that X must be a normally distributed random variable with mean zero and variance one. Natural generalizations of the classical central limit problem (and thus of the normal distribution) is to drop the condition that the X_k's have finite mean and variance (these limit random variables are called stable) and to consider limits where (X_k) does not necessarily have finite mean and variance and are not necessarily identically distributed (these limit random variables are called self-decomposable). Infinitely divisible random variables generalize the stable and self-decomposable ones as they are defined as the solution of the following (general) central limit problem:

$$\sum_{k=1}^{n} \mathrm{X}_{k,n} \xrightarrow{w} \mathrm{X} \text{ as } n \to \infty,$$

with some conditions on (a_n), (b_n) and $(X_{k,n})$.

Though practical applications have started to appear (cf. for example Ahmad and Abouammoh (1977), Thorin (1977), Carasso (1987), Keilson and Servi (1987) and Hansen and Willekens (1989)), infinite divisibility is (still) mainly of theoretical interest. In this monograph the emphasis is on the theory. Examples and practical applications are, as a rule, not sought.

Our starting point is the Lévy canonical representation of infinitely divisible distributions (cf. Theorem 1.3.2 of this monograph), where the characteristic function of an infinitely divisible distribution function F is related with a function M, called the Lévy spectral function. We are interested in characterizing the distributions F which have a Lévy spectral function M satisfying some monotonicity requirement.

In Chapter 2 we give a review of known monotonicity results in infinite divisibility and we present a curious connection with analogous results in renewal theory. In Chapters 3 and 4 we consider non-negative infinitely divisible random variables whose Lévy spectral functions are either absolutely continuous or supported by the non-negative integers. Chapter 3, which is based on the article *On Moment Sequences and Infinitely Divisible Sequences*, Hansen and Steutel (1988), studies these Lévy spectral functions in the context of moment sequences and moment functions. The results of *On Logconcave and Logconvex Infinitely Divisible Sequences and Densities*, Hansen (1988), are given in Chapter 4, where log-concave and log-convex Lévy spectral functions are considered. The set of infinitely divisible distributions with α-unimodal Lévy spectral functions is characterized in Chapter 5. Chapter 6 deviates from the theme of this monograph, as it studies infinitely divisible random variables with α-unimodal Lévy spectral functions as limits of sums of triangular arrays of random variables and as limits of sums of shrunken random variables.

1.2 Notations and conventions

In this section we list notations and conventions, which will be used throughout this monograph, often without further reference.

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_- := (-\infty, 0]$, $\mathbb{N}_0 := \{0, 1, ...\}$ and $\mathbb{N}_+ := \{1, 2, ...\}$. Random variables will always be one-dimensional and real-valued. They will be denoted by the capitals X, Y, Z, ... The distribution functions of X and Y will be denoted by F and G; their densities, if they exist, by f and g; their characteristic functions by ϕ and γ, with ϕ defined by

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x), \quad t \in \mathbb{R}.$$

If the random variables X and Y are non-negative then we denote their Laplace-

Stieltjes transforms by $\hat{f}$ and $\hat{g}$, where $\hat{f}$ is given by

$$\hat{f}(\tau)=\int_{\mathbb{R}_+} e^{-\tau x}\, dF(x), \quad \tau\in\mathbb{R}_+.$$

If the random variables are non-negative and concentrated on the non-negative integers $\mathbb{N}_0$, then we call their distributions discrete and denote them by $(p_n)_{n=0}^{\infty}$, $(q_n)_{n=0}^{\infty}$, $(r_n)_{n=0}^{\infty}$, ... Their probability generating functions will be denoted by P, Q, R, ..., with P defined by

$$P(z)=\sum_{n=0}^{\infty} p_n z^n \ , \quad |z|\le 1 .$$

In general, we shall denote the generating function of a sequence $(a_n)_{n=0}^{\infty}$ by the corresponding capital letter A, where

$$A(z)=\sum_{n=0}^{\infty} a_n z^n \ , \quad |z|\le r \ ,$$

for some $r\in\mathbb{R}_+$. All sequences considered in this monograph will be real-valued and indexed by $\mathbb{N}_0$, and henceforth denoted by (a_n), (b_n) etc..

A classification of a set C is a class $\{\, C_t \mid t\in I\}$ with $I\subseteq\mathbb{R}$ an index set, such that the sets C_t are non-decreasing, i.e., for $t(1)<t(2)<\ldots<t(n)$ with $t(i)\in I$, we have that $C_{t(1)}\subseteq C_{t(2)}\subseteq\ldots\subseteq C_{t(n)}\subseteq C$.

A set C of characteristic functions is said to be closed under limits if any characteristic function which is the limit of a sequence of characteristic functions in C is itself a member of C.

The letters M and N will always denote Lévy spectral functions (cf. Theorem 1.3.2 of Section 1.3) and H will always be a canonical measure (cf. Theorem 1.3.3). The sequences (p_n) and (r_n) will always be assumed to be related through equation (1.2) of Theorem 1.3.4.

1.3 Infinitely divisible distributions

We begin this section with a definition of an infinitely divisible random variable (cf. Lukacs (1970), p. 107). This definition is equivalent to the one given in the introduction (cf. Theorem 1.3.11).

DEFINITION 1.3.1. A random variable X is said to be infinitely divisible if for every positive integer n, there exists independent and identically distributed random variables $X_{1n}, X_{2n}, \ldots, X_{nn}$ such that

$$X\overset{d}{=}X_{1n}+X_{2n}+\ldots+X_{nn} \ ,$$

with $\stackrel{d}{=}$ denoting equality in distribution. □

We say that ϕ, $\hat{f}$, P or F is infinitely divisible if it stems from an infinitely divisible random variable. We state three representation theorems for infinitely divisible distributions on $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}_0$. The proof of the first theorem may be found as Theorem 5.5.2 in Lukacs (1970), of the second partly in Feller (1971), p. 450, Theorem 2 and partly in Steutel (1970), p. 86, Theorem 4.2.4 and the proof of the last theorem in Feller (1968), p. 290 and Steutel (1970), p. 83, Corollary 4.2.1.

THEOREM 1.3.2. *A function ϕ is an infinitely divisible characteristic function if and only if it can be written in the form*

$$\ln \phi(t) = ita_\phi - \tfrac{1}{2}\sigma_\phi^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} k(t,x)\,dM(x) \ ,$$

where $a_\phi \in \mathbb{R}$, $\sigma_\phi^2 \in \mathbb{R}_+$, $k(t,x) = e^{itx} - 1 - itx(1+x^2)^{-1}$, and such that the function M (called the Lévy spectral function) satisfies

(i) *$M(x)$ is non-decreasing on $(-\infty,0)$ and $(0,\infty)$;*

(ii) *$M(-\infty) = M(\infty) = 0$;*

(iii) *The integrals $\int_{-1}^{0} x^2\,dM(x)$ and $\int_{0}^{1} x^2\,dM(x)$ are finite.*

The representation is unique.

THEOREM 1.3.3. *A function $\hat{f}$ is an infinitely divisible Laplace-Stieltjes transform if and only if it can be written in the form*

$$\ln \hat{f}(\tau) = \int_0^\infty (e^{-\tau x} - 1)x^{-1}\,dH(x) \, ,$$

where the function H (called the canonical measure) is non-decreasing. Equivalently, $\hat{f}$ is infinitely divisible if and only if its distribution function F satisfies

$$\int_0^x u\,dF(u) = \int_0^x F(x-u)\,dH(u) \ , \ x \in \mathbb{R}_+ \, . \tag{1.1}$$

Necessarily $\int_1^\infty x^{-1}\,dH(x) < \infty$. The representations are unique.

THEOREM 1.3.4. *A function P with $P(0) > 0$ is an infinitely divisible probability generating function if and only if it can be written in the form*

$$\ln P(z) = -\theta\,(1 - G(z)) \ ,$$

where $\theta > 0$ *and* G *is a probability generating function with distribution* (g_n) *such that* $G(0)=0$. *Equivalently,* P *is infinitely divisible with* $P(0)>0$ *if and only if its distribution* (p_n) *satisfies* $p_0 > 0$ *and*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_{n-k}\, r_k \ , \ n \in \mathbb{N}_0 \ , \tag{1.2}$$

with (r_n) *(the canonical measure of* (p_n)*) non-negative; necessarily* $r_n = \theta(n+1)g_{n+1}$ *and* $\sum_{k=0}^{\infty} r_k/(k+1) = \theta < \infty$. *The representations are unique.*

Relations between the function M in Theorem 1.3.2, the function H in Theorem 1.3.3 and the sequence (r_n) in Theorem 1.3.4 are derived in van Harn (1978), Section 1.7. Equations (1.1) and (1.2) are derived from the canonical representation of the integral transform of the corresponding infinitely divisible distribution by differentiation (and then inverting the integral transforms). Since the derivative of a characteristic function does not necessarily exist, we do not have a representation similar to (1.1) and (1.2) in Theorem 1.3.2. We will use the following notations throughout this monograph.

NOTATION 1.3.5. Let $ID(\mathbb{R})$, $ID(\mathbb{R}_+)$ and $ID(\mathbb{N}_0)$ denote the set of infinitely divisible characteristic functions, Laplace-Stieltjes transforms and probability generating functions, respectively. □

NOTATION 1.3.6. An infinitely divisible characteristic function ϕ is uniquely determined by the triple $[a_\phi, \sigma_\phi^2, M]$ in Theorem 1.3.2. An infinitely divisible characteristic function with Lévy spectral function M and constants a_ϕ and σ_ϕ^2 (cf. Theorem 1.3.2) will therefore be denoted by $\phi = [a_\phi, \sigma_\phi^2, M]$. □

The next two theorems give some useful properties and another characterization of $ID(I)$, $I \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{N}_0\}$. For a proof of the first theorem we refer to Theorems 5.3.2 and 5.3.3 in Lukacs (1970). The second theorem is a basic consequence of Definition 1.3.1.

THEOREM 1.3.7. *For* $I = \mathbb{R}, \mathbb{R}_+$ *or* $\mathbb{N}_0$ *the sets* $ID(I)$ *(cf. Notation 1.3.5) are multiplication semigroups, closed under limits.*

THEOREM 1.3.8. *The following equivalences hold (cf. Notation 1.3.5)*

(i) $\phi \in ID(\mathbb{R})$ *if and only if* $\phi^{1/n}$ *is a characteristic function for all* $n \in \mathbb{N}_+$;

(ii) $\hat{f} \in ID(\mathbb{R}_+)$ *if and only if* $\hat{f}^{1/n}$ *is a Laplace-Stieltjes transform for all* $n \in \mathbb{N}_+$;

(iii) $P \in ID(\mathbb{N}_0)$ *if and only if* $P^{1/n}$ *is a probability generating function for all* $n \in \mathbb{N}_+$.

Before continuing we need two definitions.

DEFINITION 1.3.9. A sequence (X_k) of random variables is said to be bounded if there exists a constant $c \geq 0$ such that

$$\mathbb{P}(|X_k| \leq c) = 1, \quad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

DEFINITION 1.3.10. By a triangular array of random variables is meant a double sequence of random variables $(X_{k,n})$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N}_+$ (hence forth denoted $(X_{k,n})$), such that the random variables $X_{1,n}, \ldots, X_{n,n}$ of the n-th row are mutually independent.

The triangular array $(X_{k,n})$, is said to be uniformly asymptotically negligible (uan) if $X_{k,n} \to 0$ in probability, uniformly in k as $n \to \infty$, i.e., if for every $\varepsilon > 0$

$$\lim_{n \to \infty} \sup_{1 \leq k \leq n} \mathbb{P}(|X_{k,n}| \geq \varepsilon) = 0. \qquad \square$$

The triangular array of characteristic functions $(\phi_{k,n})$ and the triangular array of distribution functions $(F_{k,n})$ will be called uan if they stem from a uan triangular array of random variables. As mentioned in the introduction, the notion of infinitely divisibility originated in the context of the central limit theorem. We now state a theorem which shows that the set of infinitely divisible random variables is the solution of the so called general central limit theorem. For a proof we refer to Theorem A, p. 321 in Loève (1977).

THEOREM 1.3.11. *A random variable* X *is infinitely divisible if and only if there exists a uan triangular array* $(X_{k,n})$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N}_+$, *of random variables such that*

$$\sum_{k=1}^{n} X_{k,n} \xrightarrow{w} X \text{ as } n \to \infty.$$

Here $\xrightarrow{w}$ denotes weak convergence. The next theorem shows how the Lévy spectral function M and the constant σ_ϕ in Theorem 1.3.2 are determined by the sums of the distribution functions of $(X_{k,n})$. This theorem is proved in Loève (1977) as Criterion B, p. 326.

THEOREM 1.3.12. *Let* X *be a random variable with characteristic function* ϕ *and let* $(X_{k,n})$, $k=1,2,\dots,n$, $n \in \mathbb{N}_+$ *be a uan triangular array of random variables with distribution functions* $(F_{k,n})$, $k=1,2,\dots,n$, $n \in \mathbb{N}_+$. *There exists a sequence* (b_n) *such that*

$$\sum_{k=1}^{n} X_{k,n} + b_n \xrightarrow{w} X \text{ as } n \to \infty,$$

if and only if

(i) *there exists a function* M *satisfying (i) and (iii) of Theorem 1.3.2 such that*

$$M_n := \sum_{k=1}^{n} F_{k,n} \xrightarrow{w} M \text{ as } n \to \infty$$

outside every neighbourhood of the origin.

(ii) $$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \sum_{k=1}^{n} \Big[\int_{|x| \le \varepsilon} x^2 \, dF_{k,n}(x) - \Big(\int_{|x| \le \varepsilon} x \, dF_{k,n}(x) \Big)^2 \Big] = \sigma_\phi^2.$$

Necessarily ϕ *is infinitely divisible with* $\phi = [a_\phi, \sigma_\phi^2, M]$ *for some* $a_\phi \in \mathbb{R}$.

We finish this section with two lemmas, which will be used in Chapter 6 and the Appendix. For a proof we refer to Loève (1977), p. 314, Theorem A.

LEMMA 1.3.13. *The triangular array of characteristic functions* $(\phi_{k,n})$, $k=1,2,\dots,n$, $n \in \mathbb{N}_+$, *is uan if and only if*

$$\lim_{n \to \infty} \sup_{1 \le k \le n} |\phi_{k,n}(t) - 1| = 0,$$

uniformly on every finite interval.

LEMMA 1.3.14. *If* $(X_{k,n}^{(1)})$ *and* $(X_{k,n}^{(2)})$ *are uan triangular arrays, then* $(X_{k,n}^{(1)} + X_{k,n}^{(2)})$ *is a uan triangular array.*

1.4 Self-decomposable and stable distributions

The sets of self-decomposable and stable distributions are two important subsets of the set of infinitely divisible distributions. Stable distributions are widely studied (cf. Lukacs (1970)) as they provide a natural generalization of the normal distribution; self-decomposable distributions are, in turn, a generalization of the stable distributions. In Chapters 5 and 6 we generalize both concepts.

A random variable X is called stable if there exists sequences (a_n) and (b_n) of real numbers with $a_n \geq 0$ and a sequence (X_k) of independent and identically distributed random variables such that $(X_{k,n})$ with $X_{k,n} = a_n X_k$ is uan and

$$\sum_{k=1}^{n} a_n X_k + b_n \xrightarrow{w} X \quad \text{as } n \to \infty , \tag{1.3}$$

If (X_k) are not necessarily identically distributed then X is called self-decomposable. The uan condition on $(a_n X_k)$ implies that $a_n \to 0$ and $a_{n+1} / a_n \to 1$ as $n \to \infty$. Let the linear operator T_t be defined by $T_t x = tx$. Then (1.3) can be rewritten as

$$\sum_{k=1}^{n} T_{a_n} X_k + b_n \xrightarrow{w} X \quad \text{as } n \to \infty . \tag{1.4}$$

In Chapter 6 and in the Appendix, we consider limits of the form (1.4), with T_{a_n} replaced by a more general operator U_{a_n}. Self-decomposability of X is equivalent to

$$X \overset{d}{=} cX' + X_c , \tag{1.5}$$

for all $c \in (0,1)$, where X' and X_c are independent and X' is distributed as X. For the corresponding characteristic functions this means that for every $c \in (0,1)$ there exists a characteristic function ϕ_c such that

$$\phi(t) = \phi(ct)\,\phi_c(t) \ , \ t \in \mathbb{R} . \tag{1.6}$$

The above results can be derived from Loève (1977), Section 24. If $\phi_c(t) = \phi((1-c^{\delta})^{1/\delta}\, t)\, e^{iat}$ for some $a \in \mathbb{R}$ and $\delta \in (0,2]$, then ϕ is stable (cf. the proof of Theorem 5.7.2, Lukacs (1970)). If X is non-negative, then we call $\hat{f}$ self-decomposable (on $\mathbb{R}_+$) if

$$\hat{f}(\tau) = \hat{f}(c\tau)\,\hat{f}_c(\tau) \, , \ \tau \in \mathbb{R}_+ ,$$

and stable (on $\mathbb{R}_+$) if $\hat{f}_c(\tau) = \hat{f}((1-c^{\delta})^{1/\delta}\,\tau)$. Steutel and van Harn (1979) proposed a discrete analogue of self-decomposability and stability. A random variable X is said to be discrete self-decomposable if for every $c \in (0,1)$ there exists a random variable X_c, independent of X' such that

$$X \overset{d}{=} c \otimes X' + X_c , \tag{1.7}$$

with X' and X identically distributed. The random variable $c \otimes X$ is defined in distribution by

$$P_{c\otimes X}(z)=P_X(1-c(1-z))\,,\ |z|\le 1\,,$$

with P_X denoting the probability generating function of X. For a probabilistic interpretation of $\otimes$ see Steutel and van Harn (1979). In terms of probability generating functions (1.7) reads

$$P(z)=P(1-c(1-z))P_c(z)\,,\ |z|\le 1\,,\ c\in(0,1)\,,$$

for some probability generating functions P_c. If $P_c(z)=P(1-(1-c^\delta)^{1/\delta}(1-z))$ for some $\delta>0$, then P is said to be (discrete) stable.

We now give another notation and list a series of representation theorems for self-decomposable and stable distributions. The proof of Theorem 1.4.1 can be found in Lukacs (1970), Section 5.7. This proof contains a minor error, which is corrected in Hall (1981). The proof of Theorem 1.4.5 is given in Lukacs (1970), Section 5.11. For the proofs for distributions on $\mathbb{R}_+$ we refer to Feller (1971), Section XIII.6 and those for distributions on $\mathbb{N}_0$ to Steutel and van Harn (1979).

THEOREM 1.4.1. *A function ϕ is the characteristic function of a stable distribution if and only if ϕ is either normal or ϕ can be written in the form*

$$\ln\phi(t)=ita_\phi-c\,|t|^\delta(1+i\beta\,\mathrm{sgn}(t)\,w(|t|,\delta))\,,$$

where $c\ge 0$, $|\beta|\le 1$, $\delta\in(0,2)$ and $a_\phi\in\mathbb{R}$. The function $w(|t|,\delta)$ is given by

$$w(|t|,\delta)=\begin{cases}\tan(\pi\delta/2) & \text{if } \delta\ne 1\\ -(2/\pi)\ln|t| & \text{if } \delta=1\end{cases}.$$

Equivalently, ϕ is the characteristic function of a stable distribution if and only if ϕ is infinitely divisible and either (cf. Theorem 1.3.2) $\sigma_\phi^2>0$ and $M(x)\equiv 0$ or $\sigma_\phi^2=0$ and $M(x)=C_1\,|x|^{-\delta}$ for $x<0$ and $M(x)=-C_2x^{-\delta}$ for $x>0$. The parameters satisfy $\delta\in(0,2)$, $C_1\ge 0$, $C_2\ge 0$ and $C_1+C_2\ge 0$. The parameter δ is called the exponent of stability of ϕ.

THEOREM 1.4.2. *A function $\hat{f}$ is the Laplace-Stieltjes transform of a stable distribution on $\mathbb{R}_+$ if and only if it can be written in the form*

$$\ln\hat{f}(\tau)=-\lambda\tau^\delta\,,\ \tau\in\mathbb{R}_+\,,$$

with $\lambda\ge 0$ and $\delta\in(0,1]$. The parameter δ is called the exponent of stability of $\hat{f}$.

THEOREM 1.4.3. *A function P is the probability generating function of a stable distribution on $\mathbb{N}_0$ if and only if it can be written in the form*

$$\ln P(z)=-\lambda(1-z)^\delta\,,\ |z|\le 1,$$

with $\lambda\ge 0$ and $\delta\in(0,1]$. The parameter δ is called the exponent of stability of P.

The degenerate distribution is trivially self-decomposable and stable. It is therefore usual in the literature not to call the degenerate distribution a stable distribution. For our purposes it is however desirable to include the degenerate distributions in the set of stable distributions. We therefore introduce the following notation.

NOTATION 1.4.4. The characteristic function of a stable, possibly degenerate, distribution with exponent δ, will be denoted by $\phi_{\mathrm{STABLE}(\delta)}$. Similarly, we denote by $P_{\mathrm{STABLE}(\delta)}$ and $\hat{f}_{\mathrm{STABLE}(\delta)}$ the probability generating function and the Laplace-Stieltjes transform of a stable, possibly degenerate, distribution with exponent δ. □

THEOREM 1.4.5. *A function ϕ is the characteristic function of a self-decomposable distribution if and only if ϕ is infinitely divisible with Lévy spectral function M (cf. Theorem 1.3.2) having left and right derivatives and such that $|x|M'(x)$ is non-decreasing on $(-\infty,0)$ and non-increasing on $(0,\infty)$.*

THEOREM 1.4.6. *A function P is the probability generating function of a self-decomposable distribution if and only if it can be written in the form*

$$\ln P(z) = \int_0^1 \ln Q\,(1-v\,(1-z))\,v^{-1}\,dv\;,$$

with Q a unique infinitely divisible probability generating function. Equivalently, P is a self-decomposable probability generating function if and only if it is infinitely divisible and its canonical measure (r_n) (cf. Theorem 1.3.4) is non-increasing.

The analogue of Theorem 1.4.6 for distributions on $\mathbb{R}_+$ is proved in van Harn et. al. (1982) and mentioned for distributions on $\mathbb{R}$ in Steutel and van Harn (1979). In Chapter 5 we prove two theorems which include these analogues. We finish this section with a definition, which we use in Chapters 5 and 6.

DEFINITION 1.4.7. The characteristic function ϕ is said to be in the domain of normal attraction of a stable characteristic function with exponent δ if and only if for suitable (b_n), $\lim_{n\to\infty} e^{ib_n t}\phi^n((1/n)^{1/\delta}t) = \phi_{\mathrm{STABLE}(\delta)}(t)$. □

1.5 Motivation and methodology

In this monograph we study the relationship between an infinitely divisible distribution and its Lévy spectral function (or canonical measure). We prove that an infinitely divisible distribution whose Lévy spectral function (or canonical measure) possesses some (well-known) monotonicity property, for example complete monotonicity, log-concavity, log-convexity or α-unimodality, belongs to a (well-known) set of distributions, for example the set of mixtures of geometric distributions, strongly unimodal distributions or α-self-decomposable distributions.

It turns out that in studying monotonicity properties in this context, it is easiest to first consider the discrete case (cf. Theorem 1.3.4). Here the canonical measure can be explicitly expressed in terms of the probabilities, which makes developing and proving hypotheses and constructing counterexamples easier than in the case of distributions on $\mathbb{R}$ or $\mathbb{R}_+$. In many cases we can even prove the equivalent result for distributions on $\mathbb{R}_+$ from those on $\mathbb{N}_0$ by applying some simple argument (cf. Sections 3.5, 4.3 and 5.5). Also, by studying the discrete case we develop insight which can be helpful in proving the result for $\mathbb{R}$ (cf. Section 5.4).

As is seen in Chapter 2, many of the monotonicity results obtained in renewal theory and infinite divisibility are quite similar. This observation led us to consider log-convexity and moment sequences in the context of infinite divisibility (cf. Theorems 2.4.2 and 2.4.3 of the next chapter).

Chapter 2

INFINITELY DIVISIBLE SEQUENCES AND RENEWAL SEQUENCES

2.1 Introduction

Much of renewal theory is concerned with determining properties of the renewal function. This includes study of the relationship between the renewal function and its underlying distribution. Similarly, the interplay between an infinitely divisible distribution and its Lévy spectral function plays an important role in the field of infinite divisibility. Many of the results obtained in these two, very different, fields are quite similar. This correspondence, between results in infinite divisibility and in renewal theory, proves to be very useful (cf. Chapters 3 and 4). In this chapter we give a brief review of the results concerning monotonicity properties in these two fields. Section two of this chapter gives a short introduction to renewal theory. The following two sections state the results, with little mention of possible applications. For a more complete description we refer to the references. The last section discusses the interplay between renewal sequences and infinitely divisible sequences.

2.2 A taste of renewal theory

In this section we give a brief introduction to renewal theory. For simplicity we restrict ourselves to renewal sequences on $\mathbb{N}_0$. For a more rigorous introduction to renewal theory we refer to Feller (1968). Let E be an event (for example a renewal) and define the probability distribution (f_n) by

$$f_n := \mathbf{P}(\text{E occurs for the first time at time } n+1)\,,\ n \in \mathbb{N}_0\,.$$

Define the sequence of probabilities (u_n) (called the renewal sequence with underlying distribution (f_n)) by

$$u_n := \mathbb{P}(\text{E occurs at time } n)\ ,\ u_0 = 1,\ n \in \mathbb{N}_+ .$$

It then follows that (f_n) and (u_n) are related by

$$u_{n+1} = \sum_{k=0}^{n} u_{n-k} f_k\ ,\ u_0=1,\ n \in \mathbb{N}_0\ . \tag{2.1}$$

It is usual in the literature (cf. for example Feller (1968)) to define f_n as the probability that E occurs for the first time at time n, with $f_0 = 0$. We prefer our approach because it results in a recurrence relation which, in appearance, resembles equation (1.2). The probability f_n can be interpreted as the probability that a machine first breaks down at time $n+1$ given that it broke down at time zero. If a broken down machine gets fixed instantaneously, and after reparation is 'as good as new', then u_n gives the probability that the machine breaks down at time n. In many practical situations the distribution (f_n) is known, or at least some property (for example a monotonicity property) is known, and the sequence (u_n) or its behaviour in some sense, is sought. One of the most important results from renewal theory is:

$$\text{if } \mu := \sum_{n=0}^{\infty} (n+1) f_n < \infty \text{ and } (f_n) \text{ is aperiodic, then } \lim_{n\to\infty} u_n = \mu^{-1}.$$

In the following section we give a review of monotonicity results in renewal theory.

2.3 Monotonicity results in renewal theory

To avoid repetition, we only review the results in discrete renewal theory, i.e., on sequences related through (2.1). The analogous results for general renewal functions, related through the so-called renewal equation (cf. Ross (1983)), are also true. In fact in many cases the result for renewal functions can be obtained by applying a limiting argument to the result on renewal sequences (cf. Hansen and Frenk (1989)). The list is by no means complete and will be given with very few comments to applications.

Kaluza (1928) was the first (to the author's knowledge) to study sequences related through (2.1). Among other results he proved

THEOREM 2.3.1. *Let the sequences* (u_n) *and* (f_n) *be related by (2.1). The following implications hold.*

(i) *If* $f_n \geq 0,\ n \in \mathbb{N}_0$ *then* $u_n \geq 0,\ n \in \mathbb{N}_0$;

(ii) *If* (u_n) *is log-convex then* $f_n \geq 0,\ n \in \mathbb{N}_0$.

The definitions of log-convexity and log-concavity are given in Section 4.2. In de Bruijn and Erdős (1953), for part (i), and in Hansen and Frenk (1989), for parts (ii) and (iii), we find the following theorem.

THEOREM 2.3.2. *Let the sequences* (u_n) *and* (f_n) *be related by (2.1). Suppose* $\sum f_n \le 1$ *and* $\sum (n+1) f_n = \mu < \infty$. *Let*

$$F(n) := \sum_{k=0}^{n} f_k \, , \; F_1(n) := \mu^{-1} \sum_{k=0}^{n} (1 - F(k)) \, .$$

The following implications hold.

(i) *If* (f_n) *is log-convex then* (u_n) *is log-convex;*

(ii) *If* $(1 - F(n))$ *is log-convex then* (u_n) *is non-increasing;*

(iii) *If* $(1 - F_1(n))$ *is log-convex then* $u_n \ge \mu^{-1}$, $n \in \mathbb{N}_0$.

For condition (i) in Theorem 2.3.2 it is only necessary to assume that $f_n \ge 0$ for $n \in \mathbb{N}_0$. The analogue of Theorem 2.3.2, parts (ii) and (iii), for distributions on $\mathbb{R}_+$ was first proved in Brown (1980), by coupling methods. Hansen and Frenk (1989) provides a simpler proof by applying a limiting argument to Theorem 2.3.2. The conditions on (f_n) given in (ii) and (iii) are related to distributions with decreasing failure rates and increasing mean residual life-times. Note that

(f_n) log-convex $\Rightarrow$ $(1 - F(n))$ log-convex $\Rightarrow$ $(1 - \mu^{-1} F_1(n))$ log-convex.

(u_n) log-convex $\Rightarrow$ (u_n) non-increasing $\Rightarrow$ $u_n \ge \mu^{-1}$, $n \in \mathbb{N}_0$.

Horn (1970) studied (2.1) in the context of moment sequences. For the definition of moment sequences we refer to Notation 3.2.1. Horn (1970) dropped the condition that the sequences are non-negative and convergent.

THEOREM 2.3.3. *Let the sequences* (u_n) *and* (f_n) *be related by (2.1). Then*

(i) (u_{n+1}) *is a Hamburger moment sequence on* $\mathbb{R}$ *if and only if* (f_n) *is a Hamburger moment sequence on* $\mathbb{R}$;

(ii) (u_n) *is a Stieltjes moment sequence on* $\mathbb{R}_+$ *if and only if* (f_n) *is a Stieltjes moment sequence on* $\mathbb{R}_+$;

(iii) (u_n) *is a Hausdorff moment sequence on* $[0, 1]$ *if and only if* (f_n) *is a Hausdorff moment sequence on* $[0, 1]$ *with* $\sum f_n \le 1$.

Kaluza (1928) proved part (ii) and the 'if' part of (iii), with a proof different from Horn's (1970). Theorem 2.3.3 is stronger then Theorem 2.3.2 in the sense that

> (a_n) Hausdorff moment sequence $\Rightarrow$ (a_n) Stieltjes moment sequence $\Rightarrow$ (a_n) log-convex.

This is easily verified by using Schwarz' inequality. Part (i) of the following, and last, theorem of this section is proved in Kingman (1972), p. 7, Theorem 1.4. The other two parts are easily proved.

THEOREM 2.3.4. *Let the sequences* (u_n) *and* (f_n) *be related by (2.1). Then.*

(i) $u_{n+m} \geq u_n u_m$ *for* $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$;

(ii) *If* $u_0 u_1 > 0$ *then* $u_n > 0$, $n \in \mathbb{N}_0$;

(iii) *If* $f_0 \in [0,1]$ *and* $f_{n+1} / f_n \leq 1 - f_0$ *then* (u_n) *is non-increasing.*

2.4 Monotonicity results of infinitely divisible distributions

As in Section 2.3 we only review the results for discrete distributions. Most results are here also true for distributions on $\mathbb{R}_+$ and can usually be obtained by some limiting argument (cf. Chapter 4) or some other argument (cf. Chapters 3 and 5) from those on $\mathbb{N}_0$. The analogues of Theorems 2.3.2 and 2.3.3 are proved in Chapters 4 and 3, respectively. For completeness we also state them here.

THEOREM 2.4.1. *Let the sequences* (p_n) *and* (r_n) *be related by (1.2), i.e., by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_{n-k} r_k \,, \; p_0 \geq 0, \; n \in \mathbb{N}_0 \,. \tag{2.2}$$

The following implications hold.

(i) *If* $r_n \geq 0$, $n \in \mathbb{N}_0$ *then* $p_n \geq 0$, $n \in \mathbb{N}_0$;

(ii) *If* (p_n) *is log-convex then* $r_n \geq 0$, $n \in \mathbb{N}_0$.

Part (ii) of Theorem 2.4.1 was first proved in Steutel (1970), Theorem 4.2.2. For a different proof see Section 4.2. Part (ii) states that all log-convex distributions are infinitely divisible. The absolutely continuous analogue of Theorem 2.4.1 (ii) is proved as Theorem 4.2.6, p. 89 in Steutel (1970). The following theorem and its absolutely continuous analogue is proved in Chapter 4.

THEOREM 2.4.2. *Let the sequences* (p_n) *and* (r_n) *be related by (2.2). The following implications hold.*

(i) *If* (r_n) *is log-convex and* $r_0^2 \le r_1$ *then* (p_n) *is log-convex;*

(ii) *If* (r_n) *is log-concave and* $r_0^2 \ge r_1$ *then* (p_n) *is log-concave.*

In Chapter 3 we prove the analogue of Theorem 2.3.3 for infinitely divisible sequences, i.e., sequences (p_n) and (r_n) related by (2.2). This analogue also provides an alternative proof of Theorem 2.3.3 (cf. Section 3.4).

THEOREM 2.4.3. *Let the sequences* (p_n) *and* (r_n) *be related by (2.2). Then*

(i) (p_{n+1}) *is a Hamburger moment sequence on* $\mathbb{R}$ *if and only if* $(r_n/(n+1))$ *is a Hamburger moment sequence on* $\mathbb{R}$ *with* $\mu \le \lambda$ *(cf. (3.6));*

(ii) (p_n) *is a Stieltjes moment sequence on* $\mathbb{R}_+$ *if and only if* $(r_n/(n+1))$ *is a Stieltjes moment sequence on* $\mathbb{R}_+$ *with* $\mu \le \lambda$ *(cf. (3.6));*

(iii) (p_n) *is a Hausdorff moment sequence on* $[0, 1]$ *if and only if* $(r_n/(n+1))$ *is a Hausdorff moment sequence on* $[0, 1]$ *with* $\mu \le \lambda$ *(cf. (3.6)).*

The absolutely continuous analogue of part (iii) is proved in Steutel (1970), Theorem 2.12.1. Parts (iii) and (iv) of the following theorem are proved in Steutel and van Harn (1979). The first part is due to F.W. Steutel (private communication) and can be proved as Theorem 1.4, p.7 in Kingman (1972) is proved. The second part is proved in Steutel (1970), Theorem 4.2.3.

THEOREM 2.4.4. *Let the sequences* (p_n) *and* (r_n) *be related by (2.2). Then*

(i) $\binom{m+n}{m} p_{n+m} \ge p_n p_m$ *for* $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$;

(ii) *If* $p_0 p_1 > 0$ *then* $p_n > 0$, $n \in \mathbb{N}_0$;

(iii) *If* $r_0 \in [0,1]$ *and* (r_n) *is non-increasing then* (p_n) *is non-increasing;*

(iv) *If* (r_n) *is non-increasing then* (p_n) *is unimodal.*

Condition (iv) implies that all discrete self-decomposable distributions are unimodal (cf. Theorem 1.4.6).

2.5 Renewal sequences and infinitely divisible sequences

As seen in the two previous sections there is a connection between the behaviour of sequences related through (2.1) and sequences related through (2.2). This section is intended to give some understanding to why this is true.

Let (p_n) be a sequence of real numbers and define the two sequences (r_n) and (f_n) through

$$p_{n+1} = \sum_{k=0}^{n} p_{n-k} f_k \,, \; n \in \mathbb{N}_0 \,, \tag{2.3}$$

$$(n+1) p_{n+1} = \sum_{k=0}^{n} p_{n-k} r_k \,, \; n \in \mathbb{N}_0 \,. \tag{2.4}$$

Equation (2.4) is more general than equation (2.3) in the sense that if the sequence (f_n) is non-negative, then (r_n) is also non-negative (cf. Steutel (1970), p.83). In fact, if (p_n) is a probability distribution and (f_n) is non-negative, then (p_n) is a compound geometric distribution and hence infinitely divisible.

Taking generating functions on both sides of (2.3) and (2.4) and eliminating $P(z)$ yields

$$R(z) = -\frac{d}{dz} \ln(1 - zF(z)) \,. \tag{2.5}$$

Let $f_{-1} := -1$. Equation (2.5) is equivalent to

$$(n+1)[(-1)^n f_n] = \sum_{k=0}^{n} [(-1)^{n-k-1} f_{n-k-1}]\, [(-1)^k r_k] \,, \; n \in \mathbb{N}_0 . \tag{2.6}$$

One could therefore expect that the relationship between (p_n) and (r_n) is similar to that between (f_n) and (r_n). Hence, if (r_n) has some property which (p_n) inherits and $((-1)^n r_n)$ also has this property, then $((-1)^{n-1} f_{n-1})$ will also inherit this property. In a similar way one can start with (p_n). Equation (2.6) does not only provide an idea as to why the theorems in Sections 2.3 and 2.4 are similar, but also provides alternative proofs of some of the results in Section 2.3 (see Chapter 3).

Chapter 3

MOMENT SEQUENCES AND MOMENT FUNCTIONS

3.1 Introduction

In this chapter we consider sequences (p_n) and (r_n) related through equation (1.2), i.e., through

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k}\,, \quad n \in \mathbb{N}_0\,, \tag{3.1}$$

where either (p_n) or $(r_n/(n+1))$ is a moment sequence (cf. Notation 3.2.1), and we consider functions f and h related through (cf. (1.1))

$$x\, f(x) = \int_0^x f(x-u)\, h(u)\, du\,, \quad x \in \mathbb{R}_+\,, \tag{3.2}$$

where either f or $h(u)/u$ is a moment function (cf. Definition 3.5.1). The discrete case is considered in the same vein as renewal sequences were by Horn (1970) (cf. Theorems 2.3.3 and 2.4.3). The proofs in the discrete case are based on a theorem in monotone matrix function theory by Bendat and Sherman (1955). At first glance it may seem as though this approach uses too powerful tools, but its great advantage is that it results in some very elegant proofs. The somewhat more straightforward, but tedious, approach used to prove Theorems 3.3.2 and 3.3.7 can also be adapted to prove Theorems 3.3.4 and 3.3.8, but fails to prove Theorem 3.3.1 (cf. Remark 3.3.3).

In all but the last section of this chapter we will consider equation (3.1), and its absolutely continuous analogue (3.2), outside its probabalistic context; for (3.1) we drop the conditions that $r_n \geq 0$ and $\sum r_n/(n+1) < \infty$ (thus (p_n) need not be non-negative and $\sum p_n$ need not be convergent) and take, for convenience, $p_0 = 1$; for (3.2) we drop the condition that $\int_1^\infty h(x)/x\, dx < \infty$ (thus $\int f\, dx$ need not be finite). In Section 3.2 we give some preliminaries. Statements (i), (ii) and (iii) of Theorem 2.4.3 will be proved as separate theorems in Section 3.3. The absolutely continuous analogue is proved in Section 3.5 using the results on infinitely divisible moment sequences. Using equation (2.6) and Theorem 2.4.3 we prove Theorem 2.3.3 in Section 3.4. Its absolutely continuous analogue is mentioned in Section 3.6. Our result on Hausdorff moment sequences has applications in probability theory, as sequences of this type are mixtures of geometric distributions. This case and its absolutely continuous analogue will be investigated in more detail in Section 3.7.

3.2 Preliminaries

For ease in notation we introduce the following sets.

NOTATION 3.2.1. Let $MS(\mathbb{R})$, $MS(\mathbb{R}_+)$ and $MS([0,1])$ denote the set of Hamburger, Stieltjes and Hausdorff moment sequences, respectively, i.e., for $I \in \{\mathbb{R}, \mathbb{R}_+, [0,1]\}$ let

$$(a_n) \in MS(I) \text{ if and only if } a_n = \int_I x^n d\mu(x)\,,\ n \in \mathbb{N}_0\,, \tag{3.3}$$

where μ is a nonnegative measure. Also let the sets $MS^*(I)$ (cf. (3.3)) be given by

$$(a_n) \in MS^*(I) \text{ if and only if } (a_n) \in MS(I) \text{ with } \mu \le \lambda\ ; \tag{3.4}$$

here λ denotes Lebesgue measure and $\mu_1 \le \mu_2$ means that $\mu_1(B) \le \mu_2(B)$ for all Borel sets B, i.e., that μ_1 is absolutely continuous with respect to μ_2 and the Radon-Nikodym derivative $d\mu_1 / d\mu_2 \le 1$. We also define (compare (3.3) and (3.4)) the set $MS_T(I)$ by

$$(a_n) \in MS_T(I) \text{ if and only if } (a_n) \in MS(I) \text{ and } \mu \text{ is supported by } [-T, T]\,. \tag{3.5}$$

and let $MS_T^*(I) = MS_T(I) \cap MS^*(I)$. In a similar way we define $MS(\mathbb{R}_-)$ and $MS^*(\mathbb{R}_-)$. The measure μ in (3.3) is called the representing measure of the sequence (a_n). □

According to equation (2.1) the functions U and F are related by

$$z\,F(z) = 1 - \frac{1}{U(z)}\,, \tag{3.6}$$

and P and R, according to (3.1), by

$$\tilde{R}(z) := z \sum_{n=0}^{\infty} \frac{r_n}{n+1} z^n = \log P(z)\ . \tag{3.7}$$

The symbols U, F, P and $\tilde{R}$ in (3.6) and (3.7) can be regarded as formal power series. We only use them as functions when μ in (3.3) is supported by $[-T,T]$. Then U, F, P and $\tilde{R}$ are well-defined functions of the form

$$f(z) = \int_{[-T,T]} (1-xz)^{-1}\, d\mu(x)\,,\ z \in (-T^{-1}, T^{-1})\,.$$

Hence they are Stieltjes transforms and so the measure μ is unique.

As will become apparent, the essential difference between (3.6) and (3.7) in the context of moment sequences stems from the fact that

$1 - \dfrac{1}{w}$ maps the upper half-plane onto the upper half-plane,

$\log w$ maps the upper half-plane onto the strip $0 < \operatorname{Im} w < \pi$.

This difference explains the difference between Theorems 2.3.3 and 2.4.3 and somewhat complicates the proof of Theorem 2.4.3. We will use the following lemmas describing some properties of moment sequences. The proofs of the first two lemmas can be found in Shohat and Tamarkin (1943). The third lemma follows from Helly's first theorem and the corollary to Theorem 25.12, p. 292 in Billingsley (1979).

LEMMA 3.2.2. $(a_n) \in MS(\mathbb{R}_+)$ *if and only if* $(a_n) \in MS(\mathbb{R})$ *and* $(a_{n+1}) \in MS(\mathbb{R})$.

LEMMA 3.2.3. $(a_n) \in MS([0,1])$ *if and only if* $(a_n) \in MS(\mathbb{R}_+)$ *and* a_n *is bounded.*

LEMMA 3.2.4. *Let* $I \in \{\mathbb{R}, \mathbb{R}_+, [0,1]\}$. *If* $(a_n(t)) \in MS(I)$ *(or* $MS^*(I)$*) for all* $t \geq t_0$, *and if* $\lim_{t\to\infty} a_n(t) = a_n$, $n \in \mathbb{N}_0$, *then* $(a_n) \in MS(I)$ *(or* $MS^*(I)$*).*

REMARK 3.2.5. In proving Theorem 2.4.3 we will use Lemma 3.2.4 together with truncation of integrals. If (p_n) is given by

$$p_n = \int_{\mathbb{R}} x^n\, d\mu(x)\,, \qquad n \in \mathbb{N}_0\,,$$

and (r_n) is defined by (3.1), then we define $(p_n(T))$ by

$$p_n(T) = \int_{|x| \leq T} x^n\, d\mu(x)\,, \qquad n \in \mathbb{N}_0.$$

If we now define $(r_n(T))$ from $(p_n(T))$ by means of (3.1), then $p_n(T) \to p_n$ and $r_n(T) \to r_n$ as $T \to \infty$. In a similar way one can start from (r_n). □

Before we proceed we need a definition.

DEFINITION 3.2.6. For $T > 0$, let A_T denote the set of real analytic functions on $(-T^{-1}, T^{-1})$ that have an analytic continuation to the upper half-plane, and that are either constant or map the upper half-plane into itself; A_T^* denotes the subset of functions in A_T that map the upper half-plane into the strip $0 < \operatorname{Im} z < \pi$. □

The following lemma, proved in Bendat and Sherman (1955), is as basic here as it was in Horn (1970).

LEMMA 3.2.7. *Let $T > 0$ and let C be a real-valued function defined on $(-T^{-1}, T^{-1})$. Then the following statements are equivalent.*

(i) $C \in A_T$;

(ii) $C(x) = \sum_0^\infty c_n x^n$ *for* $x \in (-T^{-1}, T^{-1})$ *with* $(c_{n+1}) \in MS_T(\mathbb{R})$;

(iii) $C(z) = C(0) + \int_{[-T,T]} \frac{z}{1-tz} d\mu(t) \quad (\operatorname{Im} z > 0)$.

Lemma 3.2.7 shows that (c_{n+1}) is a (truncated) Hamburger moment sequence if and only if its generating function (seen as a mapping) maps the upper half plane into itself. Hence U has this mapping property if and only if $zF(z)$ does (cf. (3.6)). The next lemma enables us to use Lemma 3.2.7 in the situation of equation (3.7).

LEMMA 3.2.8. *Let $T > 0$ and let μ be a finite measure on $[-T, T]$. Let f be defined by*

$$f(z) = \int_{[-T,T]} \frac{z}{1-zt} d\mu(t) \quad (\operatorname{Im} z > 0) . \tag{3.8}$$

Then $\mu \le \lambda$ (cf. (3.4)) if and only if $0 < \operatorname{Im} f(z) < \pi$ for $\operatorname{Im} z > 0$.

PROOF. Clearly, $\operatorname{Im} f(z) > 0$ if $\operatorname{Im} z > 0$. If $\mu \le \lambda$ then for $\operatorname{Im} z > 0$ we have

$$\operatorname{Im} f(z) \le \operatorname{Im} \int_{[-T,T]} \frac{z}{1-zt} \, dt = \arg(1 + zT) - \arg(1 - zT) < \pi \ .$$

Now suppose that $0 < \operatorname{Im} f(z) < \pi$. The function

$$g(s) := -f(-(s+T)^{-1}) = \int_{[0,2T]} \frac{1}{s+t} \, d\mu(t - T)$$

is a Stieltjes transform. From the inversion formula for such transforms (cf. Widder (1946), p. 340) we have for any ξ_1, ξ_2 with $0 < \xi_1 < \xi_2$, and writing $\tilde{\mu}(\xi) = (\mu(\xi + 0) + \mu(\xi - 0))/2$,

$$\tilde{\mu}(\xi_2) - \tilde{\mu}(\xi_1) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{\xi_1}^{\xi_2} \operatorname{Im} f\left(\frac{1}{s - i\eta}\right) ds \le \xi_2 - \xi_1 \ ,$$

since $\operatorname{Im}(1/\bar{z}) > 0$ if and only if $\operatorname{Im} z > 0$. It follows that $\mu \le \lambda$. □

COROLLARY 1. *Let f be given by (3.8) and let $c > 0$. Then $0 < Im f(z) < c\pi$ for $Im\, z > 0$ if and only if $\mu \le c\lambda$.*

Now from Lemmas 3.2.7 and 3.2.8 we obtain

LEMMA 3.2.9. *Let $T > 0$ and let C be a real-valued function on $(-T^{-1}, T^{-1})$. Then the following three statements are equivalent.*

(i) $C \in A_T^*$;

(ii) $C(x) = \sum_{n=0}^{\infty} c_n x^n$ *for* $x \in (-T^{-1}, T^{-1})$ *with* $(c_{n+1}) \in MS_T^*(\mathbb{R})$;

(iii) $C(x) = C(0) + \int_{[-T,T]} \frac{x}{1-xt}\, d\mu(t)$,

for a measure $\mu \le \lambda$.

The next lemma is an analogue for our situation of Lemma B in Horn (1970). It reduces to this lemma if all $*$'s are deleted. We will need both versions of the lemma, and we will refer to it as Lemma 3.2.10* if we need it with the $*$'s and as Lemma 3.2.10 otherwise. These lemmas are the key lemmas for the results of the next section, as they enable us to study moment sequences by considering the mapping properties of their generating functions. Here we use $A_T^* \subset A_T$ and $MS_T^*(I) \subset MS_T(I)$, which follows from Notation 3.2.1 and Definition 3.2.6.

LEMMA 3.2.10.* (3.2.10.) *Let $T > 0$ and let (c_n) be a sequence of real numbers such that $C(x) = \sum_0^{\infty} c_n x^n$ converges for all $x \in (-T^{-1}, T^{-1})$. Then (cf. Definition 3.2.6 and equation (3.5))*

(a) $(c_n) \in MS_T^*(\mathbb{R})$ *if and only if* $x\,C(x) \in A_T^*$;

(b) $(c_{n+1}) \in MS_T^*(\mathbb{R})$ *if and only if* $C(x) \in A_T^*$;

(c) $(c_n) \in MS_T^*(\mathbb{R}_+)$ *if and only if* $x\,C(x) \in A_T^*$ *and* $C(x) \in A_T$;

(d) $(c_{n+1}) \in MS_T^*(\mathbb{R}_+)$ *if and only if* $C(x) \in A_T^*$ *and* $x^{-1}(C(x)-C(0)) \in A_T$.

PROOF. (a) and (b) follow directly from Lemma 3.2.9; the proofs of (c) and (d) are quite similar. We prove (c). First, let $(c_n) \in MS_T^*(\mathbb{R}_+)$. Then by (a) we have $x\,C(x) \in A_T^*$ and by (c) of Lemma 3.2.10 that $C(x) \in A_T$. Conversely, if $xC(x) \in A_T^*$, then $(c_n) \in MS_T^*(\mathbb{R})$ by (a), whereas $C(x) \in A_T$ by (b) of Lemma 3.2.10 implies that $(c_{n+1}) \in MS_T(\mathbb{R})$. From Lemma 3.2.2 and the fact that $MS_T^*(\mathbb{R}) \subset MS_T(\mathbb{R})$ we conclude that $(c_n) \in MS_T(\mathbb{R}_+)$. Finally, since moment sequences on $MS_T(I)$ have unique representing measures we have $(c_n) \in MS_T^*(\mathbb{R}) \cap MS_T(\mathbb{R}_+) = MS_T^*(\mathbb{R}_+)$. □

The following lemma is immediate from Lemma 3.2.2.

LEMMA 3.2.11. *$(c_n) \in MS_T^*(\mathbb{R}_+)$ if and only if $(c_n) \in MS_T^*(\mathbb{R})$ and $(c_{n+1}) \in MS(\mathbb{R})$.*

3.3 Moment sequences and infinitely divisible sequences

We are now ready to prove Theorem 2.4.3. We present the statements (i), (ii) and (iii) as separate theorems.

3.3.1. Hamburger moment sequences

THEOREM 3.3.1. *Let (p_n) and (r_n) be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k\, r_{n-k}\,, \quad n \in \mathbb{N}_0 \,.$$

Then (cf. (3.3) and (3.4))

$$(p_{n+1}) \in MS(\mathbb{R}) \ \textit{if and only if} \ \left(\frac{r_n}{n+1}\right) \in MS^*(\mathbb{R})\,.$$

PROOF. In view of Lemma 3.2.4 and Remark 3.2.5 we only have to prove the equivalence for $MS_T(\mathbb{R})$ and $MS_T^*(\mathbb{R})$. Since (cf. (3.7)) the generating functions of (p_n) and $(r_n/(n+1))$ are related by $\tilde{R}(z) = \log P(z))$, in view of Lemma 3.2.10* (3.2.10) it is sufficient to prove that $P \in A_T$ if and only if $\log P \in A_{T'}^*$ for some $T' > 0$. This last statement is true since (cf. Definition 3.2.6) P (is constant or) maps the upper half-plane into itself if and only if $\log P$ (is constant or) maps the upper half-plane into the strip $0 < \operatorname{Im} z < \pi$. Finally, $P(z)$ is convergent for $|z| < T$ with $T > 0$ if and only if $\log P(z)$ converges for $|z| < T'$ for some $T' > 0$. □

The next theorem, of some interest in its own right, is a preparation for the proof of (ii) in Theorem 2.4.3. Its proof is similar to the proof of Theorem 2.12.1 in Steutel (1970) concerning mixtures of exponential distributions.

THEOREM 3.3.2. *Let (p_n) and (r_n) be related as in Theorem 3.3.1. Then*

$$(p_n) \in MS(\mathbb{R}) \text{ if and only if } \frac{r_n}{n+1} = b_n - (-1)^n c_n \text{ with } (b_n), (c_n) \in MS^*(\mathbb{R}_+) .$$

PROOF. Any sequence in $MS(\mathbb{R})$ can be approximated by sequences in $MS_T(\mathbb{R})$ as follows. If $(p_n) \in MS_T(\mathbb{R})$ has a discrete representing measure with positive atoms q_i at t_i $(i = 1, 2, \dots, N)$ where $-T < t_N < \dots < t_{l+1} < 0 < t_l < \dots < t_1 < T$, then $P(x)$ takes the form

$$P(x) = \sum_{i=1}^{N} q_i/(1 - xt_i) = Q(x) / \prod_{i=1}^{N} (1 - xt_i),$$

with Q a polynomial of degree at most $N-1$. Observe that

$$P(x) < 0 \text{ for } x \downarrow t_k^{-1}, \text{ for } k = 1, 2, \dots, l ,$$
$$P(x) > 0 \text{ for } x \uparrow t_k^{-1}, \text{ for } k = 1, 2, \dots, l ,$$
$$P(x) > 0 \text{ for } x \downarrow t_k^{-1}, \text{ for } k = l+1, 2, \dots, N ,$$
$$P(x) < 0 \text{ for } x \uparrow t_k^{-1}, \text{ for } k = l+1, 2, \dots, N .$$

Since Q is continuous on $\mathbb{R}$ except maybe at t_k^{-1}, $k = 1, 2, \dots, N$, we see that Q has at least $N-2$ zeros, denoted by s_k^{-1}, $k = 1, 2, \dots, l-1, l+2, \dots, N$, which satisfy

$$-T < t_N < s_N < \dots < s_{l+2} < t_{l+1} < 0 < t_l < s_{l-1} < \dots < s_1 < t_1 < T.$$

The graph of $P(x)$ is sketched in figure 3.1 (see the next page). If a zero of Q, s^{-1} say $(s \neq 0)$, is the site of a local extremum of Q we must have

$$0 = P'(x)\,|_{x=s^{-1}} = \int_{[-T,T]} \frac{t}{(1-s^{-1}t)^2}\, d\mu(t),$$

$$0 = P(x)\,|_{x=s^{-1}} = \int_{[-T,T]} \frac{1-s^{-1}t}{(1-s^{-1}t)^2}\, d\mu(t),$$

which implies that

$$0 = \int_{[-T,T]} \frac{1}{(1-s^{-1}t)^2}\, d\mu(t),$$

i.e., that $\mu = 0$, *a.e.*. Thus no zero of Q is the site of a local extremum and so the reciprocal of the $(N-1)$st zero, s say, must lie in (t_{l+1}, t_l). Let $s_{l+1} = \min(0,s)$ and $s_l = \max(0,s)$. If Q only has $N-2$ zeros (this corresponds to $s = 0$ above and is the case when $\sum q_i t_i = 0$), then let $s_l = s_{l+1} = 0$. In either case

$$P(x)=\prod_{k=1}^{N}(1-xs_k)\,/\,\prod_{i=1}^{N}(1-xt_i). \tag{3.9}$$

It follows that $\tilde{R}$ can be written as (cf. (3.7))

$$\tilde{R}(x)=\log P(x)=\int_0^T \frac{x}{1-xt}\,d\mu_{N,1}(t)-\int_0^T \frac{x}{1-xt}\,d\mu_{N,2}(t)\ ,$$

with $\mu'_{N,1}(t)=\sum_{k=1}^{l}1_{(s_k,t_k)}(t)$, and $\mu'_{N,2}(t)=\sum_{k=l+1}^{N}1_{(-s_k,-t_k)}(t)$, so $\mu_{N,1}$ and $\mu_{N,2}$ are bounded by Lebesgue measure, i.e., $r_n\,/\,(n+1)$ has the desired property. A limiting argument completes the proof.

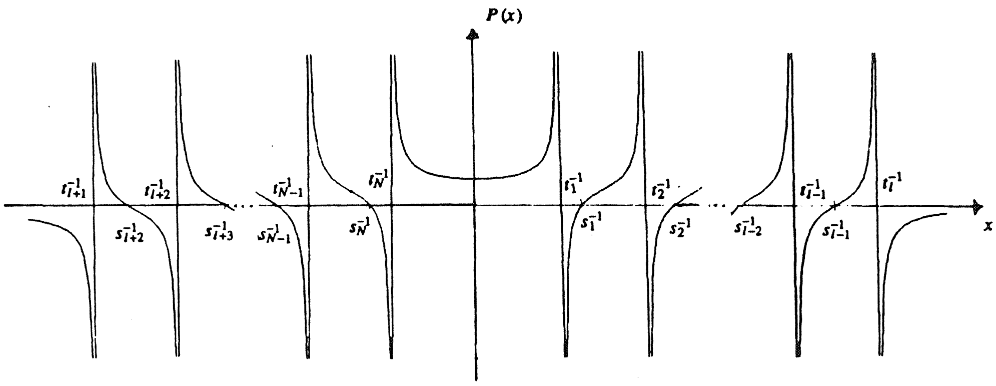

figure 3.1

Conversely, we approximate the representing measures μ_1 and μ_2 of (b_n) and (c_n), respectively. For $k=1,2,\ldots,N-1$, let

$$t_{k,N}=T(N-k)/N\ ,$$

$$s_{k,N}=t_{k,N}-\mu_1(0,t_{k,N}]+\mu_1(0,t_{k+1,N}]\ ,$$

$$\mu'_{N,1}(t)=\sum_{k=1}^{N-1}1_{(s_{k,N},t_{k,N})}(t)\ ,$$

$$\mu_{N,1}(0,t_{k,N}]=\mu_1(0,t_{k,N}]\ .$$

For $k=N,N+1,\ldots,2N-1$, let

$$s_{k,N}=T(N-k)/N\ ,$$

$$t_{k,N}=s_{k,N}-\mu_2(0,-s_{k+1,N}]+\mu_2(0,-s_{k,N}]\ ,$$

$$\mu'_{N,2}(t)=\sum_{k=N}^{2N-1}1_{(-s_{k,N},-t_{k,N})}(t)\ ,$$

$$\mu_{N,2}(0,-s_{k,N}] = \mu_2(0,-s_{k,N}] \,.$$

Since $\mu_{N,1} \xrightarrow{w} \mu_1$ and $\mu_{N,2} \xrightarrow{w} \mu_2$ we have by Helly's first and second theorems (cf. (3.7))

$$P(x) = \lim_{N\to\infty} \exp\left\{ \int_0^T \frac{x}{1-xt}\, d\mu_{N,1}(t) - \int_0^T \frac{x}{1+xt}\, d\mu_{N,2}(t) \right\}$$

$$= \lim_{N\to\infty} \prod_{k=1}^{2N-1} \frac{1-xs_k}{1-xt_k} = \lim_{N\to\infty} \sum_{k=1}^{2N-1} \frac{q_k}{1-xt_k} = \int_{[-T,T]} \frac{1}{1-xt}\, d\mu(t) \,.$$

□

COROLLARY 1. *$(p_n) \in MS(\mathbb{R})$ if and only if $(r_{n+1}/(n+2)) \in MS(\mathbb{R})$ with representing measure μ satisfying*

$$\int_{-\infty}^{\infty} |t|^{-1} d\mu(t) < \infty \text{ and } \mu(B) \le \int_B |t|\, dt \text{ for all Borel sets } B \,.$$

PROOF. Let μ_b and μ_c be the representing measures of (b_n) and (c_n) in Theorem 3.3.2, and let $d\mu_1 = t d\mu_b$, $d\mu_2 = t d\mu_c$. Then the measure μ defined by

$$\mu(-\infty, t] = \begin{cases} \mu_2(-t,\infty) & t \le 0 \\ \mu_2(0,\infty) + \mu_1(0,t) & t > 0 \end{cases} ,$$

satisfies the requirements. □

REMARK 3.3.3. Suppose that $(p_{n+1}) \in MS_T(\mathbb{R})$ with a discrete representing measure and $p_0 = 1$. Proceeding as in the proof of Theorem 3.3.2 we see that P (the generating function of (p_n)) takes the form

$$P(x) = 1 + \sum_{i=1}^{N} q_i x/(1-xt_i) = Q(x) \Big/ \prod_{i=1}^{N} (1-xt_i) ,$$

with Q a polynomial of degree at most N. As in the proof of Theorem 3.3.2 we can identify $N-2$ zeros of Q by observing the sign of $P(x)$ as $x \to t_k^{-1}$ from the left and from the right (cf. figure 3.1). The method of proof used to prove Theorem 3.3.2 now fails to prove Theorem 3.3.1 since the last two zeros cannot be confined to the interval (t_{l+1}, t_l). □

3.3.2. Stieltjes moment sequences

THEOREM 3.3.4. *Let (p_n) and (r_n) be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}\,, \quad n \in \mathbb{N}_0\,.$$

Then (cf. (3.3) and (3.4))

$$(p_n) \in MS(\mathbb{R}_+) \text{ if and only if } \left(\frac{r_n}{n+1}\right) \in MS^*(\mathbb{R}_+)\,.$$

PROOF. For the first part we restrict attention to $MS_T(\mathbb{R}_+)$ and $MS_T^*(\mathbb{R}_+)$ (cf. Remark 3.2.5). If $(p_n) \in MS_T(\mathbb{R}_+)$, then by Lemma 3.2.2 we have $(p_n) \in MS_T(\mathbb{R})$ and $(p_{n+1}) \in MS_T(\mathbb{R}) \subset MS(\mathbb{R})$. Hence, by the proof of Theorem 3.3.1 and the corollary to Theorem 3.3.2 one has $(r_n/(n+1)) \in MS_T^*(\mathbb{R})$ and $(r_{n+1}/(n+2)) \in MS(\mathbb{R})$. By Lemma 3.2.11, then $(r_n/(n+1)) \in MS_T^*(\mathbb{R}_+)$.
Conversely, by Theorem 3.3.2 with $c_n = 0$ $(n = 0, 1, \ldots)$ we have $(p_n) \in MS(\mathbb{R})$. Since $MS^*(\mathbb{R}_+) \subset MS^*(\mathbb{R})$, by Theorem 3.3.1 we also have $(p_{n+1}) \in MS(\mathbb{R})$, and therefore, by Lemma 3.2.2, $(p_n) \in MS(\mathbb{R}_+)$. □

COROLLARY 1. *Let (u_n) and (v_n) be in $MS(\mathbb{R}_+)$, with generating functions U and V and let $W := U^a V^b$ be the generating function of (w_n). Then $(w_n) \in MS(\mathbb{R}_+)$ if $a \geq 0$, $b \geq 0$ and $a + b \leq 1$.*

PROOF. See (3.7) and (3.4). □

REMARK 3.3.5. Theorem 3.3.4 can also be proved without use of Theorem 3.3.2 in a similar way as its analogue in Horn (1970) is proved. This proof, however, uses Theorem 2.3.3 (ii). As mentioned in Remark 3.3.3, the proof of Theorem 3.3.2 can be adapted to give yet another proof of Theorem 3.3.4. □

REMARK 3.3.6. If (p_n) and $(r_n/(n+1))$ are in $MS(\mathbb{R})$ then by Theorem 3.3.1 and Lemma 3.2.2, $(p_n) \in MS(\mathbb{R}_+)$ and by the corollary to Theorem 3.3.2 and Lemma 3.2.2, $(r_n/(n+1)) \in MS(\mathbb{R}_+)$. Hence, as in the renewal case (cf. Horn (1970)), (p_n) and $(r_n/(n+1))$ cannot both be in $MS(\mathbb{R})$ without being in $MS(\mathbb{R}_+)$. □

It is interesting to note the difference between Theorems 3.3.1 and 3.3.4. Theorem 3.3.4 considers (p_n) as a moment sequence, whereas Theorem 3.3.1 considers (p_{n+1}). This shift in indices is necessary to ensure that the sequence (p_n) has the right sign. For example, if $(r_n/(n+1)) \in MS^*(\mathbb{R})$, then $r_0 > 0$ and since $p_0 > 0$

we must have $p_1 > 0$, which $(p_{n+1}) \in MS(\mathbb{R})$ ensures. This observation led us to consider the set $MS(\mathbb{R}_-)$, where the odd terms of the sequence are non-positive.

THEOREM 3.3.7. *Let (p_n) and (r_n) be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}\,, \quad n \in \mathbb{N}_0\,.$$

Then (cf. (3.3) and (3.4))

$$(p_{n+1}) \in MS(\mathbb{R}_-) \text{ if and only if } \left(\frac{r_n}{n+1}\right) \in MS^*(\mathbb{R}_-)\,.$$

PROOF. Any sequence in $MS(\mathbb{R}_-)$ can be approximated by sequences in $MS_T(\mathbb{R}_-)$ as follows. If $p_0 = 1$ and $(p_{n+1}) \in MS_T(\mathbb{R}_-)$ has a discrete representing measure with positive atoms q_i at t_i where $(i = 1, 2, \ldots, N)$ and $-T < t_N < \ldots < t_{N-1} < \ldots < t_1 < 0$, then $P(x)$ takes the form

$$P(x) = 1 + \sum_{i=1}^{N} q_i x/(1 - xt_i) = Q(x) \Big/ \prod_{i=1}^{N} (1 - xt_i)\,,$$

with Q a polynomial of degree at most N. Observe that

$$P(x) < 0 \text{ for } x \downarrow t_k^{-1}\,, \text{ for } k = 1, 2, \ldots, N\,,$$

$$P(x) > 0 \text{ for } x \uparrow t_k^{-1}\,, \text{ for } k = 1, 2, \ldots, N\,,$$

thus, since Q is continuous on $\mathbb{R}$ except maybe at t_k^{-1}, $k = 1, 2, \ldots, N$, we see that Q has at least $N - 1$ zeros, denoted by s_k^{-1}, $k = 1, 2, \ldots, N-1$, and satisfying

$$-T < t_N < s_{N-1} < t_{N-1} < s_{N-2} < \ldots < s_2 < t_2 < s_1 < t_1 < 0.$$

The graph of $P(x)$ is sketched in figure 3.2 (see the next page). Since $P(x) \to -\infty$ as $x \downarrow t_N^{-1}$ and $P(0) = 1$, then the last zero, s_N^{-1} say, must satisfy $-T' < s_N < t_N$ for some $T' > 0$. Hence

$$P(x) = \prod_{k=1}^{N} (1 - xs_k) \Big/ \prod_{i=1}^{N} (1 - xt_i),$$

and so

$$\tilde{R}(x) = \log P(x) = \int_{-T'}^{0} \frac{x}{1-xt}\, d\mu_N(t)\,,$$

with $\mu'_N(t) = \sum_{k=1}^{N} 1_{(s_k, t_k)}(t)$, so μ_N is bounded by Lebesgue measure, i.e., $r_n / (n+1)$ has the desired property. A limiting argument completes the proof.

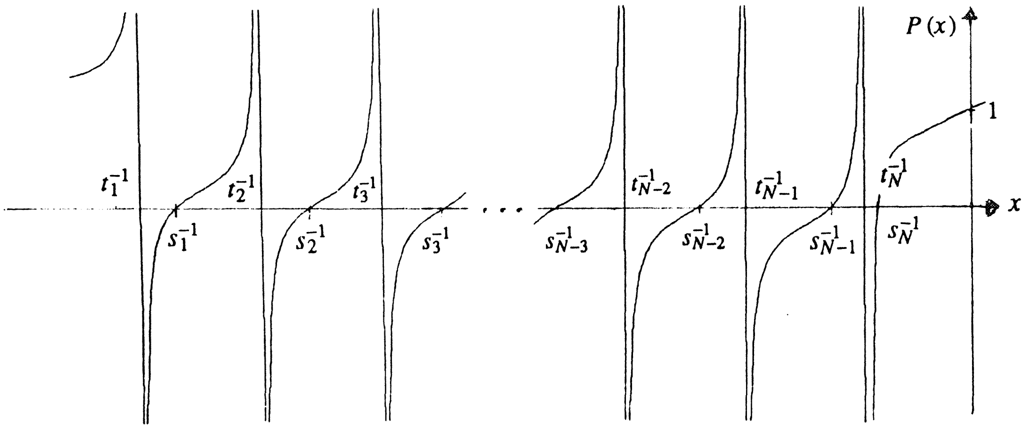

figure 3.2

The converse is proved as in Theorem 3.3.2. □

3.3.3. Hausdorff moment sequences

THEOREM 3.3.8. *Let* (p_n) *and* (r_n) *be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}\,, \quad n \in \mathbb{N}_0\,.$$

Then (cf. (3.3) and (3.4)).

$$(p_n) \in MS([0,1]) \text{ if and only if } \left(\frac{r_n}{n+1}\right) \in MS^*([0,1])\,.$$

PROOF. If $(p_n) \in MS([0,1])$, then (p_n) is non-increasing and since $p_0 = 1$, we have $p_n \leq 1$. By Lemma 3.2.3, $(p_n) \in MS(\mathbb{R}_+)$. By Theorem 3.3.4, $(r_n/(n+1)) \in MS^*(\mathbb{R}_+)$ and by (3.1), $p_n \leq 1$ implies that $r_n \leq n+1$. Hence $(r_n/(n+1)) \in MS^*([0,1])$ by Lemma 3.2.3.

Conversely, if $(r_n/(n+1)) \in MS^*([0,1])$, then by Lemma 3.2.3, $(r_n/(n+1)) \in MS^*(\mathbb{R}_+)$ and $r_n \leq 1$. By Theorem 3.3.4, $(p_n) \in MS(\mathbb{R}_+)$ and by (3.1), $r_n \leq 1$ implies that $p_n \leq 1$ and hence, by Lemma 3.2.3, $(p_n) \in MS([0,1])$. □

COROLLARY 1. *Let* (u_n) *and* (v_n) *be in* $MS([0,1])$ *with generating functions* U *and* V *and let* $W = U^a V^b$ *be the generating function of* (w_n). *Then* $(w_n) \in MS([0,1])$ *if* $a \geq 0$, $b \geq 0$ *and* $a + b \leq 1$.

COROLLARY 2. *Let* (p_n) *and* (r_n) *be related by (3.1). Then (cf. (3.3) and (3.4))*

$$(p_n) \in MS([0,T]) \text{ if and only if } \left(\frac{r_n}{n+1}\right) \in MS^*([0,T]),$$

for any $T>0$.

PROOF. If $(p_n) \in MS([0,T])$ then $(T^{-n}p_n) \in MS([0,1])$. From (3.1) it follows that

$$(n+1)[T^{-n-1}p_{n+1}] = \sum_{k=0}^{n} [T^{-k}p_k]\,[T^{-n+k-1}r_{n-k}], \; n \in \mathbb{N}_0 .$$

Hence, by Theorem 3.3.8, $(T^{-n-1}\frac{r_n}{n+1}) \in MS^*([0,1])$. Observe that

$$r_n/(n+1) = T^{n+1} \int_0^1 x^n \, d\mu(x) = \int_0^T y^n \, d\,T\mu(y/T) .$$

Since $T\mu(y/T)$ is bounded by Lebesgue measure, then $(r_n/(n+1)) \in MS^*([0,T])$. The converse is proved similarly. □

3.4 Moment sequences and renewal sequences

Let the two sequences (p_n) and (r_n) be related through (3.1). Define the sequence (f_n) by

$$p_{n+1} = \sum_{k=0}^{n} p_k f_{n-k}, \quad n \in \mathbb{N}_0 . \tag{3.10}$$

In Section 2.5 we proved that for $f_{-1} := -1$

$$(n+1)[(-1)^n f_n] = \sum_{k=0}^{n} [(-1)^{k-1} f_{k-1}]\,[(-1)^{n-k} r_{n-k}], \; n \in \mathbb{N}_0 . \tag{3.11}$$

We now give an alternative proof of Theorem 2.3.3.

PROOF (of Theorem 2.3.3). From the definitions of $MS(\mathbb{R})$ and $MS^*(\mathbb{R})$ it is clear that

(a) $(a_n) \in MS(\mathbb{R})$ (or $MS^*(\mathbb{R})$) iff $((-1)^n a_n) \in MS(\mathbb{R})$ (or $MS^*(\mathbb{R})$);

(b) $(a_n) \in MS(\mathbb{R}_+)$ (or $MS^*(\mathbb{R}_+)$) iff $((-1)^n a_n) \in MS(\mathbb{R}_-)$ (or $MS^*(\mathbb{R}_-)$).

Theorem 2.3.3 (i) is proved by noting the equivalence of the following statements.

$$\begin{aligned}
(p_{n+1}) \in MS(\mathbb{R}) &\text{ iff } (r_n/(n+1)) \in MS^*(\mathbb{R}) && \text{(cf. (3.1) and Theorem 3.3.1)}\\
&\text{ iff } ((-1)^n r_n/(n+1)) \in MS^*(\mathbb{R}) && \text{(cf. statement (a))}\\
&\text{ iff } ((-1)^n f_n) \in MS(\mathbb{R}) && \text{(cf. (3.11) and Theorem 3.3.1)}\\
&\text{ iff } (f_n) \in MS(\mathbb{R}). && \text{(cf. statement (a))}
\end{aligned}$$

For the result on Stieltjes moment sequences we note the following.

$$\begin{aligned}
(p_n) \in MS(\mathbb{R}_+) &\text{ iff } (r_n/(n+1)) \in MS^*(\mathbb{R}_+) && \text{(cf. (3.1) and Theorem 3.3.4)}\\
&\text{ iff } ((-1)^n r_n/(n+1)) \in MS^*(\mathbb{R}_-) && \text{(cf. statement (b))}\\
&\text{ iff } ((-1)^n f_n) \in MS(\mathbb{R}_-) && \text{(cf. (3.11) and Theorem 3.3.7)}\\
&\text{ iff } (f_n) \in MS(\mathbb{R}_+). && \text{(cf. statement (b))}
\end{aligned}$$

The result on $MS([0,1])$ is easiest proved as in Horn (1970). □

The same reasoning for $MS(\mathbb{R}_-)$ yields

THEOREM 3.4.1. *Let (u_n) and (f_n) be related by equation (2.1), i.e., by*

$$u_{n+1} = \sum_{k=0}^{n} u_k\, f_{n-k}\,, \quad n \in \mathbb{N}_0\,.$$

Then (cf. (3.3) and (3.4))

$$(u_{n+1}) \in MS(\mathbb{R}_-) \text{ if and only if } (f_n) \in MS(\mathbb{R}_-)\,.$$

3.5 Moment functions and infinitely divisible functions

Let F and H be two non-decreasing, right continuous, not necessarily bounded functions on $\mathbb{R}_+$ related through equation (1.1) of Theorem 1.3.3, i.e., through

$$\int_0^x u\, dF(u) = \int_0^x F(x-u)\, dH(u)\,, \quad x \in \mathbb{R}_+\,. \tag{3.12}$$

Let F and H be absolutely continuous with Radon-Nikodym derivatives f and h, respectively. Then

$$x f(x) = \int_0^x f(x-u)\, h(u) du\,, \quad x \in \mathbb{R}_+\,. \tag{3.13}$$

In this section we consider (3.13) where either f or $h(u)/u$ are moment functions (cf.

Definition 3.5.1). In the proof of Theorem 3.5.3 of this section it is essential whether or not the distribution function of f or of $h(u)/u$ is bounded. We therefore state the theorem in terms of the densities of the distribution functions in (3.12), instead of in terms of the functions in (3.13). We consider (3.13) in the context of moment functions instead of (3.12), partly because (3.13) is the analogue of (3.1) and partly because $c-F$ is a bounded moment function for some $c>0$ if and only if f is a moment function with $\int f = c$.

The set $MS([0,1])$ coincides with the set of completely monotone sequences (cf. Feller (1971), Section VII,3). The set of completely monotone functions is equal to the set of Laplace-Stieltjes transforms (cf. Feller (1971), Section XIII,4). In order to keep the analogy between completely monotone sequences and completely monotone functions we introduce the following definition.

DEFINITION 3.5.1. Let $MF((0,\infty))$ and $MF((0,1])$ denote the sets of Stieltjes and Hausdorff moment functions, respectively, i.e., for $I \in \{(0,\infty), (0,1]\}$ let

$$f \in MF(I) \text{ if and only if } f(\tau) = \int_{-\ln(I)} e^{-x\tau}\, d\mu(x), \quad \tau \in \mathbb{R}_+, \tag{3.14}$$

where μ is a nonnegative measure and $-\ln((0,1]) = [0,\infty)$ and $-\ln((0,\infty)) = \mathbb{R}$. Also let the sets $MS^*(I)$ (cf. 3.14)) be given by

$$f \in MF^*(I) \text{ if and only if } f \in MF(I) \text{ with } \mu \le \lambda .$$

The measure μ in (3.14) is called the representing measure of f. □

We will need the following lemma. The proof is almost identical with that of Lemma 3.2.4 and therefore omitted.

LEMMA 3.5.2. *Let $I \in \{(0,\infty), (0,1]\}$. If $f_t \in MF(I)$ (or $MF^*(I)$) for all $t \ge t_0$, and if $\lim_{t\to\infty} f_t(\tau) = f(\tau)$, $\tau \in \mathbb{R}_+$, then $f \in MF(I)$ (or $MF^*(I)$).*

Part (ii) of the following theorem was proved by Steutel (1970), p. 44, Theorem 2.12.1, for distribution functions. We slightly generalize this result in

THEOREM 3.5.3. *Let F and H be related by (3.12) and suppose they have densities f and h, respectively. Then*

(i) $f \in MF((0,\infty))$ *if and only if* $\tau^{-1} h(\tau) \in MF^*((0,\infty))$;

(ii) $f \in MF((0,1])$ *if and only if* $\tau^{-1} h(\tau) \in MF^*((0,1])$.

PROOF. Theorem 2.12.1 of Steutel (1970) proves part (ii) for bounded F. Any function F with density $f \in MF((0,1])$ can be written as a limit of bounded functions F_t with densities $f_t \in MF((0,1])$. For each F_t a function H_t can be found satisfying (3.12). By Theorem 3.5.3 (ii) for bounded F, $\tau^{-1} h_t(\tau) \in MF^*((0,1])$. Since $F_t \to F$ and $H_t \to H$, then by Lemma 3.5.2, $\tau^{-1} h(\tau) \in MF^*((0,1])$. Similarly when starting with $\tau^{-1} h(\tau)$.

Part (i) can be proved from part (ii) as follows; if $f \in MF((0,\infty))$, then by truncation of integrals we have a sequence $f_t \in MF((0,t])$ such that $f_t \to f$. Since $f_t \in MF((0,t])$ if and only if $t^{-\tau} f_t(\tau) \in MF((0,1])$, then by part (ii) $t^{-\tau}\tau^{-1} h_t(\tau) \in MF^*((0,1])$ and hence $\tau^{-1} h_t(\tau) \in MF^*((0,t])$. By Lemma 3.5.2, on letting $t \to \infty$ it follows that $\tau^{-1} h(\tau) \in MF^*((0,\infty))$. Similarly when starting with $\tau^{-1} h(\tau)$. □

REMARK 3.5.4. Part (i) of Theorem 3.5.3 can also be proved by applying a limiting argument to Theorem 3.3.8 as in the proof of Theorem 4.3.2 (see Chapter 4). Yet another way to prove Theorem 3.5.3 (i) (or another way to prove Theorem 3.3.8 using Theorem 3.5.3 (i)) is as follows:

Suppose $f \in MF((0,1])$. By Steutel (1969) f is infinitely divisible and therefore satisfies (3.13). For any $\tau \le c$, $c \ge 1$, $\hat{f}$ can be put in the form (cf. (3.14))

$$\begin{aligned}\hat{f}(\tau) &= \int_0^\infty x(x+\tau/c)^{-1}\, d\mu(cx) \\ &= \int_0^1 (1-x(1-\tau/c))^{-1}\, d\mu_{1,c}(x) := P_c(1-\tau/c)\,, \end{aligned} \tag{3.15}$$

where $d\mu_{1,c}(x) = -(1-x)d\mu(c(x^{-1}-1))$. Let $(p_n(c))$ have generating function P_c. Then $(p_n(c)) \in MS([0,1])$. Let $(r_n(c))$ be defined by using $(p_n(c))$ in (3.1) and let R_c denote its generating function. By Theorem 3.3.8, $(r_n(c)/(n+1)) \in MS^*([0,1])$, and hence

$$R_c(z) = \int_0^1 (1-xz)^{-2}\, d\mu_{1,c}(x) = \int_0^\infty c\,(y+c(1-z))^{-2} d\mu_{2,c}(y)\,, \tag{3.16}$$

with $d\mu_{2,c}(y) = -c\,((y/c)+1)^2\, d\mu_{1,c}(((y/c)+1)^{-1})$. Hence $\mu_{2,c} \le \lambda$. Observe that (cf. (3.13))

$$\hat{h}(\tau) = -\frac{d}{d\tau}\ln \hat{f}(\tau) = -\frac{d}{d\tau}\ln P_c(1-\tau/c) = c^{-1} R_c(1-\tau/c)\,. \tag{3.17}$$

From (3.16) and (3.17) it follows that $h(\tau)/\tau \in MF^*((0,1])$. The converse is shown similarly. □

3.6 Moment functions and renewal functions

Let F be a distribution function on $\mathbb{R}_+$ with density f and Laplace-Stieltjes transform $\hat{f}$. Suppose F is a compound geometric distribution, i.e.,

$$F(x) = (1-p) \sum_{n=0}^{\infty} p^n G^{*n}(x), \quad x \in \mathbb{R}_+ ,$$

with G a distribution function, having density g, and G^{*n} being the n-th convolution power of G. Applying the same type of argumentation as used in Remark 3.5.4 to Theorem 2.3.3 or applying a limit argument to Theorem 2.3.3 as in Hansen and Frenk (1988), it can be shown that (this is partly shown in Sumita and Masuda (1987), Theorem 3.3.2, p. 644 and is proved in Frenk (1988), both with proofs similar to that of Theorem 3.3.2))

$$f \in MF((0,1]) \text{ if and only if } g \in MF((0,1]) .$$

Let $U_p := (1-p)^{-1} F$ and $U := \lim_{p \to 1} U_p$. If u is the density of U, then by Lemma 3.5.2,

$$u \in MF((0,1]) \text{ if and only if } g \in MF((0,1]) . \tag{3.18}$$

The function U (called the renewal function (cf. Ross (1983))) is the unique solution of the renewal equation,

$$U(x) = 1 + \int_0^x G(x-y)\, dU(y), \quad x \in \mathbb{R}_+ . \tag{3.19}$$

Using the method of proof of Theorem 3.5.3 on statement (3.18) we obtain

THEOREM 3.6.1. *Let U and G be related by (3.19) and suppose they have densities u and g respectively. Then*

(i) $u \in MF((0,\infty))$ *if and only if* $g \in MF((0,\infty))$;

(ii) $u \in MF((0,1])$ *if and only if* $g \in MF((0,1])$ *and* $\int_0^\infty g(x)\, dx \le 1$.

3.7 Applications and special cases

The main occurrence of (3.1) is in infinite divisibility. In this context (p_n) is a Hausdorff moment sequence if and only if (p_n) is a mixture of geometric distributions, i.e.,

$$p_n = \int_0^1 x^n (1-x)\, dF(x), \quad n \in \mathbb{N}_0 , \tag{3.20}$$

where F is a distribution function on [0, 1]. Theorem 3.3.8 can be restated as

THEOREM 3.7.1. *(p_n) in (3.1) is a mixture of geometric distributions if and only if $(g_n) = (r_n/\{\theta(n+1)\})$ (cf. Theorem 1.3.4) is a mixture of geometric distributions with F in (3.20) satisfying $(1-t)dF \le d\lambda$ (cf. (3.4)). Equivalently, the probability generating function P is of the form*

$$P(z) = \int_0^1 (1-x)/(1-xz)\, dF(x),$$

with F a distribution function on [0, 1], *if and only if P can be represented as*

$$\ln P(z) = \int_z^1 \int_0^1 (1-uv)^{-2}\, d\mu(v)\, du,$$

and μ bounded by Lebesgue measure. The representation is unique.

The continuous analogue of Theorem 3.7.1 is implicit in Theorem 2.12.1 of Steutel (1970).

As a curiosity we prove that it is possible to have $(p_n) = (g_{n+1})$ in Theorem 3.7.1, i.e., $zP(z) = G(z)$ in Theorem 1.3.4. Solving for (p_n) in $P(z) = P(0) \exp[\theta z P(z)]$ one finds, e.g. by Lagrange expansion,

$$p_n = \frac{(n+1)^n}{(n+1)!} (\theta e^{-\theta})^n e^{-\theta}, \quad n \in \mathbb{N}_0 . \tag{3.21}$$

Since

$$\frac{n^n}{n!} = \frac{1}{\pi} \int_0^\pi \left(\frac{\sin x}{x} \exp[x \cot x]\right)^n dx, \quad n \in \mathbb{N}_0,$$

as has been proved by Bouwkamp (1986), (p_n) is a Hausdorff moment sequence. The distribution in (3.21) and its continuous analogue are busy period distributions. This application is discussed in Steutel and Hansen (1988).

We finish this chapter by stating a conjecture, the continuous analogue of which is discussed in Steutel (1970) (p.28, 94). In probabilistic terms the conjecture is that mixtures of negative binomial distributions of order 2, i.e., of probabilities of the form $(n+1)p^n(1-p)^2$ $(n = 0, 1, \ldots)$, are infinitely divisible. We state the conjecture more formally as follows.

CONJECTURE. If (a_n) is a Hausdorff moment sequence, then $(p_n) := ((n+1)a_n)$ satisfies (3.1) with $r_k \ge 0$.

Note that the conjecture is true if and only if the analogous conjecture in Steutel (1970) concerning mixtures of gamma two distributions is true (this can be seen by applying the method used to couple distributions on $\mathbb{N}_0$ and $\mathbb{R}_+$ in Remark 3.5.4). It is however numerically easier to look for counter examples in the discrete case than in the gamma two case. We computed(*) the first 200 r_k's for 20 different mixtures of negative binomial distributions of order two and did not find any negative r_k's.

(*) My thanks go to A.A Stoorvogel for his programming help.

Chapter 4

LOGCONCAVE AND LOGCONVEX SEQUENCES AND DENSITIES

4.1 Introduction

Log-concavity and log-convexity of functions and sequences in probability has been of interest to several authors, e.g. Karlin (1968). Ibragimov (1956) calls a distribution strongly unimodal if its convolution with any unimodal distribution is unimodal. He proves that the set of strongly unimodal probability densities is equal to the set of log-concave densities. An equivalent result for log-concave discrete probability distributions has been proved by Keilson and Gerber (1971). Much work has been done on the unimodality of infinitely divisible distributions (cf. Wolfe (1971), Yamazato (1978) and Sato and Yamazato (1978)), but little on strong unimodality. The study of log-concave functions and sequences is thus a relatively unknown field in probability, with important applications in the fields of statistics and optimization. Log-convexity is of interest in the study of reliability and of infinitely divisible random variables. Steutel (1970) proves that all log-convex discrete probability distributions are infinitely divisible (Theorem 2.4.1 (ii)). The absolutely continuous analogue is also proved in Steutel (1970).

In this chapter we consider distributions of non-negative infinitely divisible random variables whose canonical measures are either absolutely continuous or supported by the integers. We prove that for such distributions to be log-concave (log-convex), it is sufficient that their canonical measures be log-concave (log-convex). Our results in the discrete case contain an analogue of Yamazato's (1982) concavity result (it also provides an alternative proof of this result), and an analogue to the convexity result for renewal sequences in de Bruijn and Erdös (1953) (cf. Theorem 2.3.2 (i)).

4.2 Discrete distributions

In this section we consider infinitely divisible discrete probability distributions (p_n) on $\mathbb{N}_0$. A sequence (a_n) is log-concave if (a_n) is non-negative and $(\log(a_n))$ is concave (here $\ln 0 := -\infty$), or equivalently if $a_n \geq 0$ and

$$a_n^2 \geq a_{n+1}\, a_{n-1}\ , \qquad n \in \mathbb{N}_+\ . \tag{4.1}$$

If the sequence satisfies (4.1) with strict inequality, then the sequence is said to be strictly log-concave. Similarly, (a_n) is log-convex if $a_n \geq 0$ and the sequence satisfies

$$a_n^2 \leq a_{n+1}\, a_{n-1}\ , \qquad n \in \mathbb{N}_+\ . \tag{4.2}$$

(a_n) is said to be strictly log-convex if (4.2) is satisfied with strict inequality. Log-convex sequences are sometimes called Kaluza sequences, since Kaluza (1928) was the first to study sequences satisfying (4.2). Karlin (1968) calls a sequence satisfying (4.1) a Pólya frequency sequence of order 2.

A probability distribution (p_n) on $\mathbb{N}_0$ with $p_0 > 0$ is infinitely divisible if and only if it satisfies

$$(n+1)\, p_{n+1} = \sum_{k=0}^{n} r_k\, p_{n-k}\ , \qquad n \in \mathbb{N}_0\ , \tag{4.3}$$

with non-negative r_k and, necessarily, $\sum_{k=0}^{\infty} r_k/(k+1) < \infty$ (cf. Theorem 1.3.4). Theorem 2.4.1 part (ii) states that all log-convex distributions are infinitely divisible. This can be proved by induction since

$$r_n\, p_n\, p_0 = p_n\, p_{n+1} + \sum_{k=0}^{n-1} r_k (p_{n+1}\, p_{n-k-1} - p_{n-k}\, p_n)$$

is positive if (p_n) is strictly log-convex and noting that any log-convex sequence can be written as a limit of strictly log-convex sequences. Not all log-concave distributions are infinitely divisible since (cf. (4.3))

$$r_1 = p_0^{-2}\,(2 p_2 p_0 - p_1^2)$$

is not necessarily non-negative when (p_n) is log-concave. We also note that log-concave and log-convex distributions do in fact exist, for example the generalized logarithmic series distribution is log-convex (cf. Hansen and Willekens (1989)) and the binomial and Poisson distributions are log-concave (cf. Keilson and Gerber (1971)). Also the geometric distribution is both log-convex and log-concave.

The proofs of the two theorems in this section rely on two equations derived from (4.3). Though easily verified by (4.3), the equations were rather hard to find. Because of their importance we state them in a lemma.

LEMMA 4.2.1. *Let* (p_n) *and* (r_n) *be related by (4.3) and let* $p_{-1} := 0$. *Then*

$$m\,(m+2)\,(p_{m+1}^2 - p_m\, p_{m+2}) = p_{m+1}\,(r_0\, p_m - p_{m+1}) \tag{4.4}$$
$$+\sum_{l=0}^{m}\sum_{k=0}^{l}(p_{m-l}\,p_{m-k-1} - p_{m-k}\,p_{m-l-1})\,(r_{k+1}\,r_l - r_{l+1}\,r_k)\ ;$$

$$r_{m+1}\,(m+2)\,(p_{m+1}p_{m+3} - p_{m+2}^2) = p_{m+1}\,(r_{m+2}\,p_{m+2} - r_{m+1}\,p_{m+3}) \tag{4.5}$$
$$+\sum_{k=0}^{m}(p_{m-k}\,p_{m+2} - p_{m+1}\,p_{m-k+1})\,(r_{m+2}\,r_k - r_{k+1}\,r_{m+1})\ .$$

Relation (4.4) is a discrete analogue of equation (10) in Yamazato (1982), whereas (4.5) is an analogue of formula (7) in de Bruijn and Erdös (1953). We will need the following lemma.

LEMMA 4.2.2. *Let* (p_n) *and* (r_n) *be related by (4.3) with* $p_0 > 0$. *Then*

(i) *if* $p_n^2 > p_{n-1}\,p_{n+1}$ *for* $n = 1, 2, \ldots, m$ *then* $r_0\,p_m - p_{m+1} > 0$;

(ii) *if* (r_n) *is strictly log-convex and* $r_0^2 - r_1 < 0$ *then* $r_{m+2}\,p_{m+2} - r_{m+1}\,p_{m+3} > 0$.

PROOF. If $p_n^2 > p_{n-1}\,p_{n+1}$ for $n = 1, 2, \ldots, m$, then p_{n+1}/p_n is decreasing for $n = 1, 2, \ldots, m$, so $r_0 = p_1/p_0 > p_{m+1}/p_m$.

If (r_n) is strictly log-convex, then (r_{n+1}/r_n) is increasing. Hence,

$$\begin{aligned}(m+3)\,p_{m+3} &= p_{m+2}\,r_0 + \sum_{k=1}^{m+2} p_{m+2-k}\,r_{k-1}\,\frac{r_k}{r_{k-1}}\\ &< p_{m+2}\,r_0 + (m+2)\,p_{m+2}\max_{1\le k\le m+2}\left\{\frac{r_k}{r_{k-1}}\right\}\\ &< p_{m+2}\,\frac{r_{m+2}}{r_{m+1}} + (m+2)\,p_{m+2}\,\frac{r_{m+2}}{r_{m+1}}\ .\end{aligned}$$

□

We are now ready to prove the two main results of this section. We begin with the result on log-concave sequences.

THEOREM 4.2.3. *Let* (p_n) *and* (r_n) *be related by*

$$(n+1)\,p_{n+1} = \sum_{k=0}^{n} r_k\, p_{n-k}\,, \qquad n \in \mathbb{N}_0\,, \tag{4.3}$$

with $r_k \geq 0$, $p_0 > 0$ *and let* (r_n) *be log-concave. Then*

(p_n) *is log-concave if and only if* $r_0^2 - r_1 \geq 0$.

PROOF. Suppose that (r_n) is strictly log-concave and $r_0^2 - r_1 > 0$, then (r_n) is positive and hence (p_n) is positive. Observe that (cf. (4.3))

$$2\,(p_1^2 - p_0\,p_2) = p_0^2\,(r_0^2 - r_1)\,. \tag{4.6}$$

By using (4.6), Lemma 4.2.2 (i) and applying induction to (4.4) we see that (p_n) is strictly log-concave. The proof is completed by noting that any log-concave sequence can be written as a limit of strictly log-concave sequences. □

The following theorem states the result on log-convex sequences. It is the infinitely divisible analogue of Theorem 2.3.2 (i).

THEOREM 4.2.4. *Let* (p_n) *and* (r_n) *be related by*

$$(n+1)\,p_{n+1} = \sum_{k=0}^{n} r_k\, p_{n-k}\,,\ n \in \mathbb{N}_0\,,$$

with $r_k \geq 0$, $p_0 > 0$ *and let* (r_n) *be log-convex. Then*

(p_n) *is log-convex if and only if* $r_0^2 - r_1 \leq 0$.

PROOF. As in Theorem 4.2.3 except that Lemma 4.2.2 (ii) is used and induction is applied to (4.5). □

It is curious to note the difference in equations (4.4) and (4.5). We were not able to find an equation of the form (4.4) to prove Theorem 4.2.4 or one of the form (4.5) to prove Theorem 4.2.3.

REMARC 4.2.5. The assumption that (p_n) is a probability distribution is not used in the proofs of Theorems 4.2.3 and 4.2.4. These theorems are thus true for arbitrary non-negative sequences related by (4.3). □

4.3 Absolutely continuous distributions

In this section infinitely divisible probability distributions F on $\mathbb{R}_+$ with absolutely continuous canonical measures are considered. We obtain two results on the log-concavity and log-convexity of the densities of F, which are analogues to those obtained in Section 4.2. The result on log-concave densities is proved in Yamazato (1982). We here propose a proof based on applying a limiting argument to Theorem 4.2.3. This proof can easily be adapted to log-convex densities, thus giving the absolutely continuous analogue of Theorem 4.2.4.

A function f on $\mathbb{R}$ is log-concave (log-convex) on an interval I if f is positive on I and $\log(f)$ is concave (convex) on I. The function f is said to be log-concave (log-convex) if $I = \{x \mid f > 0\}$ is an interval and f is log-concave (log-convex) on I. As in the discrete case, f is strictly log-concave (strictly log-convex) if $\ln(f)$ is strictly concave (strictly convex).

A distribution function F on $(0, \infty)$ is infinitely divisible if and only if there exists a non-decreasing measure H such that

$$\int_0^x u \, dF(u) = \int_0^x F(x-u) \, dH(u), \tag{4.7}$$

$$\int_1^\infty u^{-1} \, dH(u) < \infty , \tag{4.8}$$

where H and F determine each other uniquely (cf. Theorem 1.3.3). If F and H have densities f and h, then

$$x f(x) = \int_0^x h(x-u) f(u) \, du . \tag{4.9}$$

Without loss of generality we assume that $\inf\{ x \mid f(x) > 0\} = 0$. It is shown in Steutel (1970) that all absolutely continuous distributions with log-convex densities are infinitely divisible. As in the discrete case, not all distributions having log-concave densities are infinitely divisible, e.g. $f(x) = c \exp(-x^2)$ for $x \in (0, \infty)$.

We begin with a lemma.

LEMMA 4.3.1. *Let f and h be related by (4.9). Suppose h is monotone on* $(0, \varepsilon)$ *for some* $\varepsilon > 0$ *and* $0 < f(0+) < \infty$. *Then* $h(0+) = 1$.

PROOF. Suppose h is non-increasing on $(0, \varepsilon)$, then $h(0+) > 0$. By (4.9) the function f is continuous. From (4.9) it follows that for $0 < x < \varepsilon$

$$h(0+) \geq x f(x) / \int_0^x f(u) \, du ,$$

$$h(x) \le x\, f(x) / \int_0^x f(u)\, du\ .$$

As $x \to 0$ the right hand sides tend to one, so $h(0+) = 1$. Similarly if h is non-decreasing. □

THEOREM 4.3.2. (Yamazato) *Let F be an infinitely divisible distribution function on $(0,\infty)$ with an absolutely continuous canonical measure H. Let f and h be the densities of F and H respectively, and assume that h is log-concave. Then*

f is log-concave if and only if $h(0+) \ge 1$.

PROOF. Suppose h is log-concave and $h(0+) > 1$, then h must be continuous on I. Define $(r_n(k))$ by

$$r_n(k) = h\left(\frac{n+1}{k}\right), \qquad n \in \mathbb{N}_0\ ,$$

and any $k \in \mathbb{N}_0$. Then $(r_n(k))$ is log-concave, and since $h(0+) > 1$ we have $(r_0(k))^2 > r_1(k)$, for all sufficiently large k. By (4.8) and the continuity of h we see that $\sum r_n(k)/(n+1) < \infty$. For fixed k define $(p_n(k))$ by

$$(n+1)\, p_{n+1}(k) = \sum_{l=0}^{n} p_{n-l}(k)\, r_l(k)\ , \qquad n \in \mathbb{N}_0\ , \tag{4.10}$$

$$p_0(k) = k \exp\left(-\sum_{n=0}^{\infty} r_n(k)/(n+1)\right) > 0\ , \tag{4.11}$$

with $\sum p_n(k) = k$. By Theorem 4.2.3 and Remark 4.2.5 the sequence $(p_n(k))$ is log-concave. Let

$$F_k(x) = \sum_{\substack{n \ge 0 \\ n \le kx}} k^{-1} p_n(k)\ , \tag{4.12}$$

$$H_k(x) = \sum_{\substack{n \ge 0 \\ n \le kx}} k^{-1} r_n(k)\ . \tag{4.13}$$

From (4.10) and (4.11) it follows that

$$\int_{[0,x+k^{-1}]} u\, dF_k(u) = \int_{[0,x]} F_k(x-u)\, dH_k(u)\ , \tag{4.14}$$

$$\frac{n+1}{k} p_{n+1}(k) = \int_{[0,\frac{n}{k}]} h\left(\frac{n+1}{k} - u\right) dF_k(u)\ . \tag{4.15}$$

By Helly's first theorem (cf. Feller (1971)) there is a subsequence $(F_{k(s)})$ converging weakly to some distribution function, F_{limit} say, as $s \to \infty$. Hence, since $H_k \to H$, by Helly's second theorem

$$\int_{[0,x]} u\, dF_{limit}(u) = \int_{[0,x]} F_{limit}(x-u)\, dH(u) \ .$$

Since H uniquely determines F in (4.7) we must have $F = F_{limit}$. Let

$$f_k(x) = (p_{n+1}(k))^{kx-n} \cdot (p_n(k))^{n+1-xk} \ , \qquad x \in [\frac{n}{k}, \frac{n+1}{k}) \ , \tag{4.16}$$

then f_k is a log-concave function of x. Let $n \to \infty$ and $k \to \infty$ in such a way that $k^{-1}(n+1) \to x$, then it follows from (4.9), (4.15) and (4.16) that

$$x\, f_{limit}(x) := \lim_{\substack{k\to\infty \\ n\to\infty \\ k^{-1}(n+1)\to x}} \frac{n+1}{k} f_k(\frac{n+1}{k}) = \int_{[0,x)} h(x-u)\, dF(u) \overset{a.e.}{=} x\, f(x) \ .$$

Since log-concavity is preserved under convergence, F has a log-concave density f_{limit}. As any log-concave function with $h(0+) \geq 1$ can be written as a limit of log-concave functions (h_k) with $h_k(0+) > 1$, this completes the first part of the proof.

Conversely, if f and h are log-concave then h is monotone on $(0,\varepsilon)$ for some $\varepsilon > 0$. If $0 < f(0+) < \infty$ then $h(0+) = 1$ by Lemma 4.3.1. If f is log-concave then $f(0+)$ cannot be infinite. If $f(0+) = 0$, then f is non-decreasing on $(0,\varepsilon)$ and

$$x\, f(x) \leq f(x) \int_0^x h(u)\, du \ .$$

Letting $x \to 0$ yields $h(0+) \geq 1$. □

The proof of Theorem 4.3.2 can easily be adapted to log-convex densities by using Theorem 4.2.4 instead of Theorem 4.2.3. We then obtain

THEOREM 4.3.3. *Let F be an infinitely divisible distribution function with an absolutely continuous canonical measure H. Let f and h be the densities of F and H respectively, and assume that h is log-convex. Then*

f is log-convex if and only if $h(0+) \leq 1$.

4.4 Applications and counterexamples

In this section we define a set of infinitely divisible distributions in terms of their canonical measures and determine under what conditions a distribution in this set is log-concave or log-convex. An application of this result shows that if (p_n) is log-concave (log-convex) then (r_n) is not necessarily log-concave (log-convex). Finally, we characterize the log-convex discrete stable distributions.

Let D denote the set of distributions having canonical measures (r_n) of the form

$$r_n = (n+1)\left\{ \int_0^b y^n \, dm(y) + \int_c^a y^n \, dy \right\}, \qquad n \in \mathbb{N}_0, \tag{4.17}$$

for fixed a, b and c with $0 \le b \le 1$, $0 \le c \le a \le 1$, m bounded by Lebesgue measure and

$$\int_a^b dm(y) < b - a, \qquad \text{if } b > a,$$

$$\int_0^c dm(y) < c, \qquad \text{if } c > 0.$$

The proof of Theorem 4.2 in Yamazato (1982) can be adapted to prove the following theorem if Theorem 3.3.8 of Chapter 3 and Corollary 2 to Theorem 3.3.8 is used in the same fashion as Lemma 4.2 in Yamazato (1982).

THEOREM 4.4.1. *Let (p_n) and (r_n) be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \qquad n \in \mathbb{N}_0,$$

with non-negative r_k and $p_0 > 0$. Let $(p_n) \in D$, then

(i) *if $c = 0$ and $a \ge b$ then (p_n) is log-concave;*

if $c \ge 0$ and $a < b$ then (p_n) is not log-concave;

(ii) *if $c \ge 0$ and $a \ge c \ge b$ then (p_n) is log-convex;*

if $c \ge 0$ and $a \ge b > c$ then (p_n) is not log-convex;

if $c \ge 0$ and $b > a > c$ then (p_n) is not log-convex;

if $c \ge 0$ and $b < a = c$ then (p_n) is log-convex.

REMARK 4.4.2. The absolutely continuous analogue of Theorem 4.4.1 can be obtained by applying the same type of limiting argument as in the proof of Theorem 4.3.2. □

REMARK 4.4.3. Let m in (4.11) be Lebesgue measure on (d,b), and zero otherwise. Then $r_n = b^n - d^n + a^n$ and $r_n^2 - r_{n+1} r_{n-1} < 0$ for large n if $a > b > d > 0$, whereas (p_n) is log-concave by Theorem 4.4.1 (i). Similarly (r_n) is asymptotically log-concave if $0 = d < b < c < a$, whereas (p_n) is log-convex by Theorem 4.4.1 (ii). Hence, if (p_n) is log-concave (log-convex) then (r_n) is not necessarily log-concave (log-convex) (compare with Theorems 4.2.3 and 4.2.4). □

REMARK 4.4.4. Theorem 4.4.1 characterizes D in terms of log-convexity and almost completely in terms of log-concavity. We were not able to prove that if $a \geq b > c > 0$, then (p_n) is log-concave (cf. part (i) of Theorem 4.4.1). □

A discrete analogue of an absolutely continuous stable distribution was introduced by Steutel and van Harn (1979) and discussed in Section 1.4. Steutel and van Harn (1979) proved that a distribution (p_n) is discrete stable with exponent δ if and only if its generating function is of the form (cf. Theorem 1.4.3)

$$P(z) = \exp(-\lambda(1-z)^{\delta}) , \quad \delta \in (0,1] , \ \lambda \geq 0 .$$

Taking generating functions on both sides of (4.3) and comparing with the Taylor series expansion of $-\lambda(1-z)^{\delta}$ one sees that (r_n) is strictly log-convex and that $r_0^2 - r_1 < 0$ if and only if $\delta < 1 - r_0$. Applying Theorem 4.2.4 to these observations gives

THEOREM 4.4.5. *Let (p_n) be discrete stable with exponent δ. Then*

(p_n) is strictly log-convex if and only if $\lambda < \delta^{-1} - 1$.

The canonical density h of an absolutely continuous stable distribution on $(0,\infty)$ is of the form $cx^{-\delta}$, $\delta \in (0,1]$, hence h is log-convex and $h(0+) = \infty$. Applying Theorem 4.3.3 we have, rather unexpectedly, that there are no log-convex stable densities on $(0,\infty)$.

Chapter 5

A GENERALIZED SELF-DECOMPOSABILITY

5.1 Introduction

Functional equations have been a helpful tool in representing subsets of the set of infinitely divisible distributions. The definitions of self-decomposable and stable distributions in terms of a functional equation for their characteristic function or probability generating function (cf. Section 1.4) are well-known examples; for other examples see van Harn (1978). O'Connor (1979a) shows that membership in the set of infinitely divisible distributions with unimodal Lévy spectral functions is related with the solutions of the functional equation (cf. equation (1.6))

$$\phi(t)=\phi^{\beta}(ct)\,\phi_c(t) \qquad , c\in(0,1),\ t\in\mathbb{R}, \tag{5.1}$$

with $\beta=c$ and where ϕ and ϕ_c are characteristic functions. Jurek (1985) calls such characteristic functions shrinking-self-decomposable, or s-self-decomposable for short. All self-decomposable characteristic functions are s-self-decomposable, as follows easily from the fact that self-decomposable characteristic functions are infinitely divisible. Interpolating between (5.1) with $\beta=c$ and (1.6), O'Connor (1979b) studies equation (5.1) with $\beta=c^{1-\alpha}$, $\alpha\in(0,1)$. In O'Connor (1981) the case $\alpha\in(1,3)$ is considered. In this chapter we study the case $\beta=c^{\alpha}$, $\alpha\in\mathbb{R}$. This case has also been studied in Jurek (1988) and (1989), where some of the results of this chapter are also proved. For a detailed comparison we refer to Remarks 5.4.11 and 5.4.12.

We consider random variables X on $\mathbb{R}$, on $\mathbb{R}_+$ and on $\mathbb{N}_0$. We introduce a one parameter family of functional equations, of the form (5.1), satisfied by the characteristic function, Laplace-Stieltjes transform, or probability generating function of X, depending on whether the random variable has support on $\mathbb{R}$, on $\mathbb{R}_+$ or on $\mathbb{N}_0$. Our equations for random variables on $\mathbb{R}$ include O'Connor's and have as special cases the functional equation defining self-decomposable distributions and s-self-decomposable distributions. We establish a canonical form for the integral transform satisfying these functional equations, show that these integral transforms are infinitely divisible, and have Lévy spectral functions that are, in an extended sense, α-unimodal. These results include those of O'Connor and Jurek. It is also shown that this one parameter family of functional equations provides a classification of the set of infinitely divisible random variables.

5.2 α-Unimodality

A random variable X with distribution function F and density f is said to be unimodal, with mode at x_0 (not necessarily unique), if $f(x)$ is non-decreasing for $x < x_0$ and non-increasing for $x > x_0$. Throughout this chapter we assume that $x_0 = 0$, i.e., if a function is said to be unimodal (or α-unimodal) it is understood that its mode is at the origin. Khintchine (1938) showed that X is unimodal (at zero) if X = UY, with U and Y independent and U uniform on $(0,1)$. Olshen and Savage (1970) generalized this concept; a random variable is said to be α-unimodal (at zero) if it is of the form $U^{1/\alpha}Y$, with U and Y independent and U uniformly distributed on $(0,1)$ and $\alpha \neq 0$. If Y has distribution function G, then

$$f(x) = \alpha x^{\alpha-1} \int_x^{\infty} v^{-\alpha} dG(v), \ x \in \mathbb{R}_+ ,$$

$$f(x) = \alpha |x|^{\alpha-1} \int_{-\infty}^{x} |v|^{-\alpha} dG(v), \ x \in \mathbb{R}_- .$$

This result corresponds to Corollary 2 p. 28 in Olshen and Savage (1970). Hence, f is α-unimodal if and only if $|x|^{1-\alpha} f(x)$ is non-decreasing on $(-\infty,0)$ and non-increasing on $(0,\infty)$. We will use α-unimodality in connection with Lévy spectral functions, so a more general definition is needed.

DEFINITION 5.2.1. A function C (not necessarily non-negative) is said to be α-unimodal for some $\alpha \in \mathbb{R}$, if $|x|^{1-\alpha} C(x)$ is bounded from below and non-decreasing on $(-\infty, 0)$ and non-increasing on $(0, \infty)$ or equivalently if there exists constants $\lambda_1, \lambda_2 \in \mathbb{R}$ and a function N, non-decreasing on $(-\infty, 0)$ and $(0, \infty)$ such that

$$C(x) = \begin{cases} x^{\alpha-1} \left(\int_x^{\infty} v^{-\alpha} dN(v) + \lambda_1 \right), & x > 0 \\ |x|^{\alpha-1} \left(\int_{-\infty}^{x} |v|^{-\alpha} dN(v) + \lambda_2 \right), & x < 0 \end{cases}, \tag{5.2}$$

and such that the integrals converge for every $x \in \mathbb{R}\backslash\{0\}$. □

An analogue of α-unimodality for discrete distributions on $\mathbb{N}_0$ is introduced in Abouammoh (1987). An equivalent, but different, definition was given by Steutel (1988). Steutel (1988) calls a random variable discrete α-unimodal (at zero) if X $= \mathrm{U}^{1/\alpha} \otimes \mathrm{Y}$ with U and Y independent and U uniform on $(0,1)$. The multiplication operator $\otimes$ is defined in Section 1.4. Since we only consider discrete random variables on $\mathbb{N}_0$, α-unimodality at zero is equivalent to α-monotonicity. If X and Y are $\mathbb{N}_0$ valued random variables with probability generating functions G and S, respectively, then

$$G(z) = \alpha \int_0^1 S(1 - v(1-z))\, v^{\alpha-1}\, dv\ .$$

Expanding the integral on the right hand side Steutel (1988) shows that (g_n) is α-monotone if and only if $(n!\,\Gamma(n+\alpha)^{-1} g_n)$ is non-increasing or equivalently if $(n+\alpha)\, g_n \geq (n+1)\, g_{n+1}$. We generalize this definition to

DEFINITION 5.2.2. A sequence (r_n) is said to be α-monotone, for $\alpha > 0$, if $(n!\,\Gamma(n+\alpha)^{-1} r_n)$ is bounded from below and is non-increasing, or equivalently if there exists $\lambda_1 \in \mathbb{R}$ and a non-negative sequence (h_n) such that

$$r_n = \alpha \frac{\Gamma(n+\alpha)}{n!} \left[\sum_{k=n}^{\infty} \frac{k!}{\Gamma(k+1+\alpha)} h_k + \lambda_1 \right] \quad , n \in \mathbb{N}_0\ . \tag{5.3}$$

A sequence (r_n) is said to be zero-monotone if $r_0 = \lambda$ and $r_n = 0$ for $n \geq 1$ (cf. (5.3)), i.e., (r_n) is the canonical measure of the Poisson distribution. □

REMARK 5.2.3. It is immediate from Definitions 5.2.1 and 5.2.2 that if a function f or a sequence (a_n) is α_0-unimodal then it is α-unimodal for every $\alpha \geq \alpha_0$. □

5.3 Distributions on $\mathbb{N}_0$

The starting point of this chapter was the work on random variables supported by $\mathbb{N}_0$. In this special class of random variables the probabilities themselves can be found explicitly and the corresponding Lévy spectral functions are easily computed. This provided a good source of intuition and insight which was very helpful in establishing the results of Section 5.4. In this section this discrete analogue is discussed. We begin with a definition.

DEFINITION 5.3.1 Let the random variable X on $\mathbb{N}_0$ have distribution (p_n) and probability generating function P. The function P is said to be α-self-decomposable for some $\alpha \in \mathbb{R}$ and to belong to the set $S_\alpha(\mathbb{N}_0)$, if for every $c \in (0,1)$ there exists a probability generating function P_c such that

$$P(z) = P^{c^\alpha}(1 - c(1-z))\, P_c(z) \qquad , \ |z| \le 1 . \tag{5.4}$$

If $\alpha = 0$ in Definition 5.3.1, then (5.4) provides the functional equation defining discrete self-decomposable probability generating functions as given in Section 1.4.

Let (r_n) be $(\alpha+1)$-monotone, $\alpha > -1$, $\alpha \neq 0$ and let $\tilde{H}(z) = \sum_{n=0}^{\infty} h_n / (n+1)\, z^{n+1}$. By using

$$\int_0^1 v^{\beta-1} (1-v)^{\gamma-1}\, dv = \Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\beta+\gamma)^{-1} ,$$

we obtain

$$\begin{aligned} \tilde{R}(z) &:= \sum_{n=0}^{\infty} r^n / (n+1)\, z^{n+1} \\ &= \int_0^1 (\tilde{H}(1 - v(1-z)) - \tilde{H}(1-v))\, v^{\alpha-1}\, dv + \lambda_2 - \lambda_2 (1-z)^{-\alpha} , \end{aligned} \tag{5.5}$$

where $\lambda_2 = \lambda_1 (1+\alpha)\,\Gamma(1+\alpha)\,(-\alpha)^{-1}$. Hence (r_n) in (5.3) is a canonical measure, i.e., $r_n \ge 0$ and $\sum r_n / (n+1) < \infty$ (cf. Theorem 1.3.4), if and only if (h_n) is, with $\lambda_1 \ge 0$, $\lambda_1 = 0$ for $\alpha \ge 0$. Similarly if $\alpha = 0$.

Suppose (p_n) has canonical measure (r_n) with (r_n) $(\alpha+1)$-monotone, $\alpha > -1$ and suppose (q_n) is an infinitely divisible distribution with canonical measure (h_n). By (5.5) and Theorem 1.3.4,

$$\begin{aligned} \ln P(z) &= \tilde{R}(z) + \ln P(0) = \tilde{R}(z) - \tilde{R}(1) \\ &= \int_0^1 (\tilde{H}(1 - v(1-z)) - \tilde{H}(1))\, v^{\alpha-1}\, dv - \lambda_2 (1-z)^{-\alpha} \\ &= \int_0^1 \ln Q(1 - v(1-z))\, v^{\alpha-1}\, dv - \lambda_2 (1-z)^{-\alpha} . \end{aligned}$$

We now prove

THEOREM 5.3.2. *Let $\alpha \in (-1,\infty)$ and let P be a probability generating function. The following statements are equivalent.*

(i) *P is α-self-decomposable;*

(ii) $\ln P(z) = \int_0^1 \ln Q(1-v(1-z))\, v^{\alpha-1}\, dv - \lambda (1-z)^{-\alpha}$, $\lambda \geq 0$ *and* Q *is a unique infinitely divisible probability generating function*;

(iii) (p_n) *is infinitely divisible and its canonical measure* (r_n) *is* $(\alpha+1)$*-monotone.*

Furthermore, λ *in (ii) is zero if* $\alpha \geq 0$ *and* P_c *is infinitely divisible for every* $c \in (0,1)$ *if* P *is* α*-self-decomposable.*

PROOF. (ii) $\Leftrightarrow$ (iii) has been proved above. We now prove (i) $\Leftrightarrow$ (ii). Let P satisfy (5.4), let $r>0$ and $c_n \in (0,1)$ for $n \in \mathbb{N}_+$ such that $r\,(1-c_n)^{-1} \in \mathbb{N}_0$, then

$$\ln Q_{r,n}(z) := \ln (P_{c_n}(z))^{r\,(1-c_n)^{-1}}$$
$$= r \left\{ \frac{\ln P(z) - \ln P(1-c_n(1-z))}{1-c_n} + \frac{1-c_n^\alpha}{1-c_n} \ln P(1-c_n(1-z)) \right\}$$

is the logarithm of a probability generating function. Let c_n be such that $c_n \uparrow 1$ as $n \to \infty$, then

$$\ln Q_r(z) := \lim_{n\to\infty} \ln Q_{r,n}(z) = r\,(-(1-z)P'(z)/P(z) + \alpha \ln P(z)). \tag{5.6}$$

Since $Q_r(z) \to 1$ as $z \to 1$ (cf. Lemma 1, Steutel and van Harn (1979)), by Theorem 1.3.7, Q_r is a probability generating function for every $r > 0$, and thus $Q := Q_1$ is an infinitely divisible probability generating function. Equation (5.6) gives rise to the following differential equation,

$$\ln Q(z) = -(1-z) \frac{d}{dz} \ln P(z) + \alpha \ln P(z),$$

which has (ii) as unique solution.

Conversely, if P is as in (ii), then P satisfies (5.4) with

$$\ln P_c(z) = \int_c^1 \ln Q(1-v(1-z))\, v^{\alpha-1}\, dv\ . \tag{5.7}$$

Let $(p_n(c))$ be the probability distribution corresponding to P_c and define $(r_n(c))$ using $(p_n(c))$ in (1.2). Then

$$\sum_{n=0}^{\infty} r_n(c)\, z^n := R_c(z) = \frac{d}{dz} \ln P_c(z)$$
$$= \int_c^1 H(1-v(1-z))\, v^\alpha\, dv\ ,$$

which has positive coefficients. Hence P_c is infinitely divisible. □

COROLLARY 1. *P is discrete stable with exponent* δ *(cf. Theorem 1.4.3), then* $P \in S_\alpha(\mathbb{N}_0)$ *for every* $\alpha \geq -\delta$.

COROLLARY 2. *The Poisson distribution is the only distribution in* $S_{-1}(\mathbb{N}_0)$*, and its canonical measure is* α*-monotone with* $\alpha = 0$.

Note that for $\alpha > 0$ the sequence (r_n) is $(\alpha+1)$-monotone if and only if $(r_n/(n+1))$ is α-monotone. Since all log-convex sequences and all Hausdorff moment sequences are non-increasing we have (cf. Theorem 3.3.8 and Theorem 4.2.4)

COROLLARY 3. *If* (r_n) *is log-convex, then* (p_n) *is 0-self-decomposable.*

COROLLARY 4. *If* (p_n) *is a mixture of geometric distributions, then* (p_n) *is 1-self-decomposable.*

REMARK 5.3.3. It follows from the proof of Theorem 5.3.2 that a necessary and sufficient condition for the convergence of the integral in Theorem 5.3.2 (ii) is that the canonical measure (h_n) of the infinitely divisible probability generating function Q satisfies

$$\sum_{k=0}^{\infty} \frac{(k+1)!}{\Gamma(k+2+\alpha)} \frac{h_k}{k+1} < \infty, \tag{5.8}$$

for $\alpha \in (-1,0)$ (recall that (r_n) is $(\alpha+1)$-monotone). By Stirling's formula (5.8) is equivalent to

$$\sum_{k=0}^{\infty} (k+1)^{-\alpha} \frac{h_k}{k+1} < \infty.$$

If $\alpha = 0$ then it is necessary and sufficient that

$$\sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{1}{n+1} \frac{h_k}{k+1} < \infty,$$

or equivalently

$$\sum_{k=0}^{\infty} \ln(k+1) \frac{h_k}{k+1} < \infty.$$

□

Berg and Forst (1983) introduced the set of n-times self-decomposable probability generating functions $L_n(\mathbb{N}_0)$ inductively by letting $L_0(\mathbb{N}_0) = S_0(\mathbb{N}_0)$ and (cf. (5.4))

$$L_{n+1}(\mathbb{N}_0) = \{P \in L_n(\mathbb{N}_0) \mid P_c \in L_n(\mathbb{N}_0),\ c \in (0,1)\}.$$

Hence $L_1(\mathbb{N}_0)$ is the subset of $S_0(\mathbb{N}_0)$, where P_c is 0-self-decomposable for every $c \in (0,1)$. Similarly, let $S_\alpha^0(\mathbb{N}_0) := S_\alpha(\mathbb{N}_0)$ and let

$$S_\alpha^{n+1}(\mathbb{N}_0) = \{P \in S_\alpha^n(\mathbb{N}_0) \mid P_c \in S_\alpha^n(\mathbb{N}_0),\ c \in (0,1)\}.$$

We conclude this section with a characterization of the probability generating functions in $S_\alpha^n(\mathbb{N}_0)$.

THEOREM 5.3.4. *Let $P \in S_\alpha(\mathbb{N}_0)$ with P_c given by (5.4) and Q by Theorem 5.3.2. Then $Q \in S_\alpha^n(\mathbb{N}_0)$ if and only if $P \in S_\alpha^{n+1}(\mathbb{N}_0)$, $n \in \mathbb{N}_0$.*

PROOF. Suppose $Q \in S_\alpha^n(\mathbb{N}_0)$. Observe that

$$\begin{aligned}\ln P_c(z) &= \ln P(z) - c^\alpha \ln P(1-c(1-z)) \\ &= \int_0^1 [\ln Q(1-v(1-z)) - c^\alpha \ln Q(1-cv(1-z))]\, v^{\alpha-1}\, dv \\ &= \int_0^1 \ln Q_c(1-v(1-z))\, v^{\alpha-1}\, dv ,\end{aligned}$$

for some probability generating function Q_c. Obviously, if Q can be decomposed $n+1$-times in this way, then so can P. Hence $P \in S_\alpha^{n+1}(\mathbb{N}_0)$.

Conversely, if $P \in S_\alpha^{n+1}(\mathbb{N}_0)$, then $P_c \in S_\alpha^n(\mathbb{N}_0)$ for every $c \in (0,1)$. By the proof of Theorem 5.3.2

$$\ln Q(z) = \lim_{c \downarrow 1} (1-c)^{-1} \ln P_c(z).$$

Obviously $S_\alpha^n(\mathbb{N}_0)$ is closed under limits and so $Q \in S_\alpha^n(\mathbb{N}_0)$. □

5.4 Distributions on $\mathbb{R}$

Throughout this section we will understand by the derivative of a Lévy spectral function M a right continuous function M' defined by

$$M'(x+) = M_+'(x),$$
$$M'(x-) = M_-'(x),$$

where M_+' and M_-' are respectively the right and left derivatives, $f(x+) = \lim_{y\downarrow x} f(y)$ and $f(x-) = \lim_{y\uparrow x} f(y)$. The assumption of right continuity of M' is non-essential and is assumed for uniqueness only. The results of this section are also true if we were to make M' left continuous or set M' equal to any (linear) combination of M_+' and M_-' at points where $M_+' \neq M_-'$.

We begin with a definition

DEFINITION 5.4.1. Let the random variable X on $\mathbb{R}$ have distribution function F and characteristic function ϕ. The function ϕ is said to be α-self-decomposable for some $\alpha \in \mathbb{R}$, if for every $c \in (0,1)$ there exists a characteristic function ϕ_c such that

$$\phi(t) = \phi^{c^\alpha}(ct)\, \phi_c(t) \qquad , t \in \mathbb{R}. \tag{5.8}$$

For $\alpha = 0$ we have the functional equation defining self-decomposable characteristic functions and for $-2 < \alpha \leq 1$ the equations corresponding to O'Connors class $L_{1-\alpha}$. Our results include those of O'Connor, but our proofs differ. For a detailed comparison we refer to Remark 5.4.11.

We will use the following two lemmas; the proof of the first lemma is similar to the first part of the proof of Theorem 5.11.1 in Lukacs (1970). The discrete counterpart of the second lemma is discussed in the second paragraph of Section 5.3.

LEMMA 5.4.2. *If ϕ satisfies (5.8), then ϕ has no real zeros.*

PROOF. If ϕ satisfies (5.8), then $\psi = |\phi|^2$ is a characteristic function which also satisfies (5.8). Suppose ψ has zeros. Since ψ is continuous and $\psi(0) = 1$, there exists a t_0 such that $\psi(t_0) = 0$ while $\psi(t) \neq 0$ for $|t| < t_0$. It follows from (5.8) that $\psi_c(t_0) = 0$ while $\psi(t) \neq 0$ for $|t| < t_0$. By Theorem 4.1.2 on p. 69 of Lukacs (1970), with $n = 1$ and $t = t_0/2$, we have

$$4(1 - \psi_c(t_0/2)) \geq 1 - \psi_c(t_0) = 1 .$$

Since $\psi_c(t_0/2) = \psi(t_0/2) / \psi^{c^\alpha}(ct_0/2)$ is continuous in c, we obtain a contradiction by choosing c sufficiently close to 1. So ψ and ψ_c, and hence also ϕ and ϕ_c, have no real zeros. $\square$

LEMMA 5.4.3. *Let $\alpha > -2$ and let M be such that $M'(x)$ is α-unimodal, i.e.,*

$$M'(x) = \begin{cases} x^{\alpha-1}\left(\int_x^\infty v^{-\alpha}\, dN(v) + \lambda_1 \right), & x > 0 \\ |x|^{\alpha-1}\left(\int_{-\infty}^x |v|^{-\alpha}\, dN(v) + \lambda_2 \right), & x < 0 \end{cases} , \tag{5.9}$$

with the integrals converging for every $x \in \mathbb{R}\setminus\{0\}$ and $\int_1^\infty \ln v\, dN(v) < \infty$ if $\alpha = 0$. Then M is a Lévy spectral function if and only if N can be chosen to be a Lévy spectral function and $\lambda_1, \lambda_2 \in [0,\infty)$ and $\lambda_1 = \lambda_2 = 0$ for $\alpha \geq 0$.

PROOF. First let M be a Lévy spectral function satisfying (5.9). We have to show that N satisfies the requirements (ii) and (iii) of Theorem 1.3.2 and that λ_1 and λ_2 satisfy the requirements of the lemma.

By Theorem 1.3.2 (i) and (ii), $M'(x) \geq 0$ and $M(\infty)=0$. Since $\lim_{x\to\infty} x^{1-\alpha} M'(x)=\lambda_1$ we have $\lambda_1 \in [0,\infty)$. Similarly $\lambda_2 \in [0,-\infty)$. From (5.9) it follows that for $y>0$

$$-M(y)=\int_y^\infty M'(x)\,dx=\int_y^\infty x^{\alpha-1}\left(\int_x^\infty v^{-\alpha}\,dN(v)+\lambda_1\right)dx\,. \tag{5.10}$$

Since $0 \leq -M(y)<\infty$, we must have that $\int_y^\infty \lambda_1 x^{\alpha-1}\,dx<\infty$ and hence that $\lambda_1=0$ for $\alpha \geq 0$. A similar reasoning for $y<0$ yields $\lambda_2=0$ for $\alpha \geq 0$.

Requirement (ii) of Theorem 1.3.2. Let $\alpha \leq 0$. By (5.9)

$$\infty > |M'(x)| \geq |x^{-1}\int_x^\infty dN(v)+\lambda_1 x^{\alpha-1}|\,,$$

hence $|N(\infty)|<\infty$. Similarly $|N(-\infty)|<\infty$. Let $\alpha>0$. Observe that for $y>0$ (cf. (5.10))

$$-M(y)=\alpha^{-1}\int_y^\infty (1-(y/v)^\alpha)\,dN(v)\,.$$

By (5.9) $\int_y^\infty (y/v)^\alpha\,dN(v)<\infty$, so $\int_y^\infty dN(v)$ converges and hence $|N(\infty)|<\infty$. Similarly $|N(-\infty)|<\infty$.

Requirement (iii) of Theorem 1.3.2. Let $0<\delta<\varepsilon$. From (5.9) it follows for $\alpha>-2$, that

$$\int_\delta^\varepsilon u^2\,dM(u)=(\alpha+2)^{-1}\Big[(\varepsilon^{\alpha+2}-\delta^{\alpha+2})\Big(\int_\varepsilon^\infty v^{-\alpha}\,dN(v)+\lambda_1\Big)$$
$$-\delta^{\alpha+2}\int_\delta^\varepsilon v^{-\alpha}\,dN(v)+\int_\delta^\varepsilon v^2\,dN(v)\Big]. \tag{5.11}$$

Since $\delta^{\alpha+2} \leq v^{\alpha+2}$ for $v \in (\delta,\varepsilon)$, the right hand side of (5.11) is bounded from below by

$$(\alpha+2)^{-1}\Big[(\varepsilon^{\alpha+2}-\delta^{\alpha+2})\Big(\int_\varepsilon^\infty v^{-\alpha}\,dN(v)+\lambda_1\Big)\Big]. \tag{5.12}$$

Let $\delta=0$, then the left hand side of (5.11) tends to zero as $\varepsilon \to 0$ and so (5.12) tends to zero as $\varepsilon \to 0$. By letting $\delta \to 0$ in (5.11) it follows that N satisfies condition (iii) of Theorem 1.3.2. Similarly for $0>\delta>\varepsilon$.

Conversely, suppose N is a Lévy spectral function such that the integrals in (5.9) converge. We have to show that M satisfies requirements (ii) and (iii) of Theorem

1.3.2. Assume with out loss of generality that $\lambda_1 = \lambda_2 = 0$.

Requirement (ii) of Theorem 1.3.2 . Let $z > y > 0$ and let $\alpha \neq 0$. By (5.9) we have that

$$0 \le M(z) - M(y) = \int_y^z M'(x)\,dx$$

$$= \alpha^{-1} [z^{\alpha} \int_z^{\infty} v^{-\alpha}\,dN(v) - y^{\alpha} \int_y^{\infty} v^{-\alpha}\,dN(v) + N(z) - N(y)]$$

$$\le \alpha^{-1} [z^{\alpha} \int_z^{\infty} v^{-\alpha}\,dN(v) - N(y)] .$$

For $\alpha < 0$, $\lim_{z\to\infty} z^{\alpha} \int_z^{\infty} v^{-\alpha}\,dN(v) = 0$, and so $|M(\infty)| < \infty$. If $\alpha > 0$, then

$$\lim_{z\to\infty} z^{\alpha} \int_z^{\infty} v^{-\alpha}\,dN(v) = \lim_{z\to\infty} z^{\alpha} \int_z^{\infty} (z/v)^{\alpha}\,dN(v) \le \lim_{z\to\infty} N(z) = 0 ,$$

and hence $|M(\infty)| < \infty$. For $\alpha = 0$, observe that

$$0 \le M(z) - M(y) = \int_y^z \ln v\,dN(v) + N(y)\ln y - N(z)\ln z .$$

By the condition on N in the lemma for $\alpha = 0$, $\lim_{z\to\infty} N(z)\ln z = 0$ and hence $|N(\infty)| < \infty$. A similar argument for $z < y < 0$ shows that $|M(-\infty)| < \infty$.

Requirement (iii) of Theorem 1.3.2 . Let $0 < \delta < \varepsilon$. The right hand side of (5.11) is bounded from above by

$$(\alpha+2)^{-1} [\varepsilon^{\alpha+2} \int_{\varepsilon}^{\infty} v^{-\alpha}\,dN(v) + \int_{\delta}^{\varepsilon} v^2\,dN(v)] . \tag{5.13}$$

Letting $\delta \to 0$ in (5.13) and (5.11) we see that $\int_0^{\varepsilon} u^2\,dM(v) < \infty$. A similar argument holds for $0 > \delta > \varepsilon$. □

COROLLARY 1. *Let N be a Lévy spectral function such that the integrals in (5.9) converge for $\alpha \neq 0$ and $\int_1^{\infty} \ln v\,dN(v) < \infty$ if $\alpha = 0$. Then,*

(i) $$\lim_{x\to\infty} x^{\alpha} \int_x^{\infty} v^{-\alpha}\,dN(v) = 0 = \lim_{x\to-\infty} |x|^{\alpha} \int_{-\infty}^{x} |v|^{-\alpha}\,dN(v) , \ \alpha \in (-\infty,\infty) ;$$

(ii) $$\lim_{x\to 0^+} x^{\alpha+2} \int_x^{\infty} v^{-\alpha}\,dN(v) = 0 = \lim_{x\to 0^-} |x|^{\alpha+2} \int_{-\infty}^{x} |v|^{-\alpha}\,dN(v) , \ \alpha \in (-2,\infty).$$

PROOF. This corollary is proved for $\alpha \in (0,1]$ in O'Connor (1979b). Part (ii) follows from the proof of Lemma 5.4.3. Part (i) is trivial for $\alpha < 0$ since the integrals always converge. For $\alpha \geq 0$ part (i) is evident from

$$| x^{\alpha} \int_{x}^{\infty} v^{-\alpha} dN(v) | \leq | N(x) | .$$

□

REMARK 5.4.4. If $M'(x)$ is α-unimodal then for $0 < x \leq \varepsilon$,

$$x^2 M'(x) \geq x^{\alpha+1} \int_{\varepsilon}^{\infty} v^{-\alpha} dN(v) .$$

Integrating over $(0, \varepsilon)$ it follows, by Theorem 1.3.2 (iii), that there are no α-unimodal Lévy spectral functions with $\alpha \leq -2$. □

In Section 1.4 we defined the linear operator T_r on $\mathbb{R}$ by $T_r(x) = rx$ and showed that it was closely related with the random variables having 0-self-decomposable characteristic functions. Let T_r operate on set functions by

$$T_r M(B) := M(T_r^{-1} B) = M(\{x \in \mathbb{R} \mid T_c(x) \in B\}) ,$$

for any Borel set B. The following lemma gives a connection between T_r and the notion of α-unimodality.

LEMMA 5.4.5. *Let $\alpha \in \mathbb{R}$, β_ε be the set of Borel sets on $\mathbb{R} \backslash (-\varepsilon, \varepsilon)$, for any $\varepsilon > 0$, and M a Lévy spectral function. The following statements are equivalent.*

(i) $M(\mathrm{B}) \geq c^{\alpha} T_c M(\mathrm{B})$, $c \in (0,1)$, $\mathrm{B} \in \beta_\varepsilon$;

(ii) *M has left and right derivatives on $\mathbb{R} \backslash \{0\}$ and $M'(x)$ is α-unimodal.*

PROOF. First we prove (i) $\Rightarrow$ (ii). Let $\alpha \in \mathbb{R}$, fix $\varepsilon > 0$ and let $\mathrm{B} = (a, b)$, $0 < \varepsilon \leq a < b$. Then (i) is equivalent to

$$c^{\alpha} (M(b/c) - M(a/c)) \leq M(b) - M(a). \tag{5.14}$$

If $\alpha \leq 0$, then M is convex and hence by Theorem A, p. 4, Roberts and Varberg (1973), M is absolutely continuous on (ε, ∞). Suppose $\alpha > 0$ and let $w(x) := M(x^{1/\alpha})$. From (5.14) with $x = (b/c)^{\alpha}$, $y = (a/c)^{\alpha}$, $x' = b^{\alpha}$ and $y' = a^{\alpha}$ it follows that

$$\frac{w(x) - w(y)}{x - y} \leq \frac{w(x') - w(y')}{x' - y'} .$$

Hence w is convex. By Theorem A, p. 4, Roberts and Varberg (1973), w and hence also M is absolutely continuous on (ε, ∞). Observe that

$$\int_B M'(x)\, dx = M(B) \geq c^{\alpha} T_c M(B) = c^{\alpha-1} \int_B T_c M'(x)\, dx , \tag{5.15}$$

with $B=(a\,,b)$. Differentiating (5.15) with respect to a and multiplying both sides by $a^{1-\alpha}$, we see that $a^{1-\alpha}M'(a)$ is non-increasing on $\mathbb{R}_+$. Similarly for $b<a\le\varepsilon<0$ we obtain that $a^{1-\alpha}M'(a)$ is non-decreasing on $\mathbb{R}_-$

The converse is proved for $B=(a\,,b)$, $0<\varepsilon\le a<b$, by observing that (cf. (5.9))

$$\begin{aligned} M(B) &= \int_B M'(x)\,dx = \int_B x^{\alpha-1}\Big(\int_x^\infty v^{-\alpha}\,dN(v)\Big)\,dx \\ &= c^\alpha \int_{T_c^{-1}B} x^{\alpha-1}\Big(\int_{cx}^\infty v^{-\alpha}\,dN(v)\Big)\,dx \\ &\ge c^\alpha \int_{T_c^{-1}B} M'(x)\,dx = c^\alpha\, T_cM(B)\,. \end{aligned}$$

□

If $M(\mathbb{R})<\infty$, then by Lemma 5.4.5, $M(\mathbb{R})\ge c^\alpha M(\mathbb{R})$, $c\in(0\,,1)$. Hence

COROLLARY 1. *If M is a Lévy spectral function having left and right derivatives on $\mathbb{R}\backslash\{0\}$ and where $M'(x)$ is α-unimodal, $\alpha<0$, then M must be unbounded.*

Before proving a representation theorem for α-self-decomposable characteristic functions we will prove a preparatory lemma, whose counterpart for discrete distributions is discussed in the third paragraph of Section 5.3.

LEMMA 5.4.6. *Let ϕ be an infinitely divisible characteristic function with Lévy spectral function M having left and right derivatives on $\mathbb{R}\backslash\{0\}$ and such that $M'(x)$ is α-unimodal. Then,*

(i) *if $\alpha\in[0\,,\infty)$ there exists an infinitely divisible characteristic function γ such that*

$$\ln\phi(t)=\int_0^1 \ln\gamma(vt)\,v^{\alpha-1}\,dv\,;$$

(ii) *if $\alpha\in(-2\,,0)$ there exists an infinitely divisible characteristic function γ and a stable characteristic function $\phi_{\mathrm{STABLE}(-\alpha)}(t)$, possibly degenerate, such that*

$$\ln\phi(t)=\int_0^1 \ln\gamma(vt)\,v^{\alpha-1}\,dv+\ln\phi_{\mathrm{STABLE}(-\alpha)}(t)\,.$$

PROOF. We partly follow the proof of Theorem 2 in Alf and O'Connor (1977). The proof of (i) is very similar to that of (ii) for $\alpha \in (-1,0)$, so we only prove (ii). For ϕ as given in the lemma, there exists a Lévy spectral function N and $\lambda_1, \lambda_2 \in [0,\infty)$ such that N and M are related by (5.9). Define the Lévy spectral function M_1 by

$$M_1(x)=\begin{cases} M(x)-\alpha^{-1}\lambda_1 x^{\alpha} & , x>0 \\ M(x)+\alpha^{-1}\lambda_2 |x|^{\alpha} & , x<0 \end{cases} \quad . \tag{5.16}$$

Hence M_1 is an α-unimodal Lévy spectral function with the same N in (5.9) as M, but with $\lambda_1=\lambda_2=0$. Define an infinitely divisible characteristic function γ using N for M in Theorem 1.3.2. We now wish to evaluate the integral in (ii) and show, by choosing a_γ and σ_γ appropriately, that it is equal to $\ln\phi(t)$. The manner in which a_γ must be selected is closely related to the proof of Theorem 5.7.3 in Lukacs (1970). We consider two cases.

CASE I, $\alpha \in (-1,0)$. For ease of notation we define for $x>0$

$$L(t,x)=x^{\alpha}\int_0^1 v^{\alpha-1}k(vt,x)\,dv=\int_0^x (e^{itv}-1)v^{\alpha-1}dv-\frac{ix^{\alpha+1}t}{(\alpha+1)(1+x^2)} . \tag{5.17}$$

From the first equality it follows that $L(t,x)/x^{\alpha+2}$ is bounded as $x\to 0$. Note that

$$\frac{\partial L(t,x)}{\partial x}=x^{\alpha-1}[k(t,x)+2ix^3t/(\alpha+1)(1+x^2)^2] .$$

Observe that

$$\int_0^1 \ln\gamma(vt)v^{\alpha-1}dv-i(\alpha+1)^{-1}a_\gamma t+\tfrac{1}{2}(\alpha+2)^{-1}\sigma_\gamma^2 t^2$$

$$=\int_{-\infty}^{\infty}\int_0^1 v^{\alpha-1}k(vt,x)\,dv\,dN(x)$$

$$=\int_0^{\infty}L(t,x)x^{-\alpha}\,dN(x)+\int_0^{\infty}L(-t,x)x^{-\alpha}\,d{-N(-x)}$$

$$=\lim_{\varepsilon\to 0}\Big[\frac{-L(t,x)}{x^{\alpha+2}}x^{\alpha+2}\int_x^{\infty}v^{-\alpha}\,dN(v)\Big|_{x=\varepsilon}^{\infty}+\int_{\varepsilon}^{\infty}\frac{\partial L(t,x)}{\partial x}\int_x^{\infty}v^{-\alpha}\,dN(v)dx\Big]$$

$$+\lim_{\varepsilon\to 0}\Big[\frac{-L(-t,x)}{x^{\alpha+2}}x^{\alpha+2}\int_x^{\infty}v^{-\alpha}\,d{-N(-v)}\Big|_{x=\varepsilon}^{\infty}+\int_{\varepsilon}^{\infty}\frac{\partial L(-t,x)}{\partial x}\int_x^{\infty}v^{-\alpha}\,d{-N(-v)}dx\Big]$$

$$=\int_{\mathbb{R}\setminus\{0\}}k(t,x)dM_1(x)+2(\alpha+1)^{-1}\int_{\mathbb{R}\setminus\{0\}}itx^3/(1+x^2)^2\,dM_1(x) , \tag{5.18}$$

where the second equality is obtained by integrating by parts, and the final expression by observing that $L(t,x)/x^{\alpha+2}$ is bounded as $x\to 0$ and using Corollary 1 to Lemma 5.4.3. Let a_γ and σ_γ be defined by

$$(\alpha+1)^{-1} a_\gamma + 2(\alpha+1)^{-1} \int_{\mathbb{R}\setminus\{0\}} x^3/(1+x^2)^2\, dM_1(x) = a_\phi , \tag{5.19a}$$

$$\tfrac{1}{2}(\alpha+2)^{-2}\sigma_\gamma^2 = \tfrac{1}{2}\sigma_\phi^2 . \tag{5.19b}$$

From (5.18) we obtain (cf. Theorem 1.4.1 and (5.16))

$$\int_0^1 \ln\gamma(vt)\, v^{\alpha-1}\, dv = i(\alpha+1)^{-1} a_\gamma t + \tfrac{1}{2}(\alpha+2)^{-2}\sigma_\gamma^2 t^2$$

$$+2(\alpha+1)^{-1} \int_{\mathbb{R}\setminus\{0\}} itx^3/(1+x^2)^2\, dM_1(x) + \int_{\mathbb{R}\setminus\{0\}} k(t,x)\, dM_1(x)$$

$$= ia_\phi t + \tfrac{1}{2}\sigma_\phi^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} k(t,x)\, dM(x) - \ln\phi_{\mathrm{STABLE}(-\alpha)}(t) .$$

Hence ϕ is of the desired form.

CASE II, $\alpha \in (-2,-1]$. For ease of notation we define

$$L^*(t,x) = \int_0^x (e^{itv} - 1 - itv) v^{\alpha-1}\, dv \ . \tag{5.20}$$

Note that $L^*(t,x)/x^{\alpha+2}$ is bounded as $x \to \infty$ and that

$$\frac{\partial L^*(t,x)}{\partial x} = x^{\alpha-1}\,[\,k(t,x) - itx^3/(1+x^2)\,] .$$

Since the integrals in (5.9) converge we can let

$$a_\gamma = -\int_{\mathbb{R}\setminus\{0\}} v^3/(1+v^2)\, dN(v) \ . \tag{5.21}$$

Also define σ_γ^2 by (5.19b). Analogous to Case I, we have that

$$\int_0^1 \ln\gamma(vt) v^{\alpha-1}\, dv + \tfrac{1}{2}\sigma_\phi^2 t^2$$

$$= \int_{\mathbb{R}\setminus\{0\}} \int_0^1 (e^{itvx} - 1 - itvx)\, v^{\alpha-1}\, dv\, dN(x)$$

$$= \int_0^\infty L^*(t,x) x^{-\alpha}\, dN(x) + \int_0^\infty L^*(-t,x) x^{-\alpha}\, d{-N}(-x)$$

$$= \lim_{\varepsilon\to 0}\Big[\frac{-L^*(t,x)}{x^{\alpha+2}} x^{\alpha+2} \int_x^\infty v^{-\alpha}\, dN(v) \Big|_{x=\varepsilon}^{\infty} + \int_\varepsilon^\infty \frac{\partial L^*(t,x)}{\partial x} \int_x^\infty v^{-\alpha}\, dN(v) dx\Big]$$

$$+ \lim_{\varepsilon\to 0}\Big[\frac{-L^*(-t,x)}{x^{\alpha+2}} x^{\alpha+2} \int_x^\infty v^{-\alpha}\, d{-N}(-v) \Big|_{x=\varepsilon}^{\infty} + \int_\varepsilon^\infty \frac{\partial L^*(-t,x)}{\partial x} \int_x^\infty v^{-\alpha}\, d{-N}(-v) dx\Big]$$

$$= \int_{\mathbb{R}\setminus\{0\}} k(t,x) dM_1(x) - \int_{\mathbb{R}\setminus\{0\}} itx^3/(1+x^2)\, dM_1(x) \ .$$

Rewriting as in Case I, we see that ϕ has the desired form. $\square$

We are now ready to prove the two main theorems of this section (and in the author's opinion also two of the nicest results in this monograph).

THEOREM 5.4.7. *Let* $\alpha \geq 0$ *and let* ϕ *be a characteristic function. The following statements are equivalent.*

(i) ϕ *is* α*-self-decomposable, and* ϕ' *exists on* $\mathbb{R}\setminus\{0\}$ *with* $\lim_{t\to 0} t\,\phi'(t) = 0$;

(ii) *There exists a unique infinitely divisible characteristic function* γ *such that*

$$\ln \phi(t) = \int_0^1 \ln \gamma(vt)\, v^{\alpha-1}\, dv\ ;$$

(iii) ϕ *is infinitely divisible with Lévy spectral function M having left and right derivatives and such that* $M'(x)$ *is* α*-unimodal;*

(iv) ϕ *is* α*-self-decomposable and* ϕ_c *is an infinitely divisible characteristic function for every* $c \in (0,1)$.

PROOF. First we prove (i) $\Leftrightarrow$ (ii). Let $r > 0$ and $c_n \in (0,1)$ for $n \in \mathbb{N}_+$ such that $r(1-c_n)^{-1} \in \mathbb{N}_0$. By (5.8)

$$\ln \gamma_{r,n}(t) := \ln\left(\phi_{c_n}(t)\right)^{r(1-c_n)^{-1}} = r\left\{\frac{\ln\phi(t) - \ln\phi(c_n t)}{1-c_n} + \frac{1-c_n^{\alpha}}{1-c_n}\ln\phi(c_n t)\right\}, \quad (5.22)$$

is the logarithm of a characteristic function. Let c_n be such that $c_n \uparrow 1$ as $n \to \infty$, then

$$\ln \gamma_r(t) := \lim_{n\to\infty} \ln \gamma_{r,n}(t) = r\left(t\,\phi'(t)/\phi(t) + \alpha \ln \phi(t)\right). \quad (5.23)$$

Since $\gamma_r(t) \to 1$ as $t \to 0$, by Theorem 1.3.7, γ_r is a characteristic function for every $r > 0$, and thus $\gamma := \gamma_1$ is an infinitely divisible characteristic function. Equation (5.23) gives rise to the following differential equation:

$$t^{\alpha-1} \ln \gamma(t) = t^{\alpha} \frac{d}{dt} \ln \phi(t) + \alpha\, t^{\alpha-1} \ln \phi(t) = \frac{d}{dt}\, t^{\alpha} \ln \phi(t). \quad (5.24)$$

Hence ϕ is given by (ii). Conversely, if ϕ is as in (ii), then ϕ' exists on $\mathbb{R}\setminus\{0\}$ with $t\phi'(t) \to 0$ as $t \to 0$ and ϕ satisfies (5.8) with

$$\ln \phi_c(t) = \int_c^1 \ln \gamma(vt)\, v^{\alpha-1}\, dv\ . \quad (5.25)$$

The characteristic function ϕ_c is infinitely divisible and so (ii) $\Rightarrow$ (iv) is also proved.

Suppose (iv) is satisfied. Since $\phi = \lim_{c\to 0} \phi_c$, by Theorem 1.3.7, ϕ is infinitely divisible with some Lévy spectral function M. Since ϕ and ϕ_c are related by (5.8), the Lévy spectral function M_c of ϕ_c is given by

$$M_c(x) = M(x) - c^{\alpha} M(x/c)$$

(cf. Theorem 1.3.2 and Lukacs (1970), p. 163). M_c is non-decreasing by Theorem 1.3.2 (i), hence from Lemma 5.4.5 we see that (iii) is satisfied.

The proof is completed by applying Lemma 5.4.6 to get (iii) $\Rightarrow$ (ii). □

THEOREM 5.4.8. *Let $\alpha \in (-2, 0)$ and let ϕ be a characteristic function. The following statements are equivalent.*

(i) *ϕ is α-self-decomposable, ϕ' exists on $\mathbb{R} \setminus \{0\}$ with $\lim_{t \to 0} t\,\phi'(t) = 0$;*

(ii) *There exists a unique infinitely divisible characteristic function γ and a stable, possibly degenerate, characteristic function $\phi_{\mathrm{STABLE}(-\alpha)}(t)$ such that*

$$\ln \phi(t) = \int_0^1 \ln \gamma(vt)\, v^{\alpha-1}\, dv + \ln \phi_{\mathrm{STABLE}(-\alpha)}(t)\,;$$

(iii) *ϕ is infinitely divisible with Lévy spectral function M having left and right derivatives and such that $M'(x)$ is α-unimodal;*

(iv) *ϕ is α-self-decomposable and ϕ_c is an infinitely divisible characteristic function for every $c \in (0, 1)$.*

PROOF. Lemma 5.4.6 proves that (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) is shown as in Theorem 5.4.7. It remains to prove (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). As in the proof of Theorem 5.4.7, equation (5.8) gives rise to the differential equation (5.24). Integrating on both sides of (5.24) over (t_0, t), $t_0 > 0$ yields

$$\int_{t_0}^{t} u^{\alpha-1} \ln \gamma(u)\, du = t^{\alpha} \ln \phi(t) - t_0^{\alpha} \ln \phi(t_0)\,,$$

and hence

$$\ln \phi(t) = \int_{t_0}^{t} \ln \gamma^{1/\alpha}(u)\, d(u/t)^{\alpha} + t^{-\alpha} t_0^{\alpha} \ln \phi(t_0)\,,$$

for all $t \geq t_0$. By (5.8) we have

$$\ln \phi_c(t) = \int_c^1 \ln \gamma(vt)\, v^{\alpha-1} dv\,, \tag{5.26}$$

for all $t \geq 0$. Similarly, we obtain that ϕ_c is of the form (5.26) for $t \leq 0$. So ϕ_c is infinitely divisible and condition (iv) is satisfied.

By Theorem 1.3.8 (i), $\phi_c^{c^{-\alpha}}$ is a characteristic function for every c such that $c^{\alpha} \in \mathbb{N}_+$. From (5.8) it follows that

$$\phi^{c^{-\alpha}}(t) = \phi(ct)\,\phi_c^{c^{-\alpha}}(t)\,.$$

So $\phi^{c^{-\alpha}}(t)$ is a characteristic function for every c such that $c^{\alpha} \in \mathbb{N}_+$ and therefore, by Theorem 1.3.8 (i), ϕ is infinitely divisible. Since ϕ and ϕ_c are related by (5.8) the Lévy spectral function M_c of ϕ_c is given by

$$M_c(x) = M(x) - c^{\alpha} M(x/c)$$

(cf. Theorem 1.3.2 and Lukacs (1970), p. 163). M_c is non-increasing by Theorem 1.3.2 (i), hence from Lemma 5.4.5 we see that (iii) is satisfied. □

It is interresting to note (as was done to me by L. de Haan) the exceptional role that the stable characteristic function (with exponent α) plays in Theorem 5.4.8 (ii). This resembles the exceptional role that the normal characteristic function plays in Theorem 1.3.2. The stable charaecteristic function with exponent α has an 'extreme' Lévy spectral function M in the sense that $x^{1-\alpha}M'(x)$ is constant and that it is the only Lévy spectral function that satisfies Lemma 5.4.5 (i) with equality. Contrary to the case of Theorem 1.3.2, the tail behaviour of the distribution function of $\phi_{\mathrm{STABLE}(-\alpha)}$ is not much different than that of the integral in Theorem 5.4.8 (ii) (cf. Steutel (1974) and his reference list).

NOTATION 5.4.9. A characteristic function ϕ belongs to the set $S_{\alpha}(\mathbb{R})$ if ϕ satisfies any of the conditions (i) - (iii) of Theorems 5.4.7 and 5.4.8. □

If ϕ is stable with exponent δ, then, by Theorem 1.4.1, ϕ has a Lévy spectral function M such that $M'(x)$ is $(-\delta)$-unimodal and hence (cf. Remark 5.2.3) α-unimodal for all $\alpha \geq -\delta$. We thus have the following corollary to Theorem 5.4.8 (iii).

COROLLARY 1. *ϕ is stable with exponent δ, then $\phi \in S_{\alpha}(\mathbb{R})$ for every $\alpha \geq -\delta$.*

The proof of (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) is also valid for $\alpha \leq -2$. By Remark 5.4.4 and observing that the normal characteristic function is α-self-decomposable if and only if $\alpha \geq -2$, we obtain

COROLLARY 2. *The normal characteristic function is the only characteristic function in $S_{-2}(\mathbb{R})$ and in the case $\alpha < -2$, $S_{\alpha}(\mathbb{R})$ contains only degenerate characteristic functions.*

By (5.26) and Theorem 5.4.8 (iii) we see that

$$c^{\alpha} \ln \phi(ct) = \ln \phi(t) - \ln \phi_c(t) = \int_0^c \ln \gamma(vt)\, v^{\alpha-1}\, dv + \ln \phi_{\mathrm{STABLE}(-\alpha)}(t)\,.$$

Let $c = (n+1)^{\alpha^{-1}}$. Letting $n \to \infty$ we have (cf. Definition 1.4.7)

COROLLARY 3. *Let $-2 \le \alpha < 0$. If $\phi \in S_\alpha(\mathbb{R})$, then ϕ is in the domain of normal attraction of a stable characteristic function of order $-\alpha$.*

REMARK 5.4.10. It follows from the proof of Theorem 5.4.8 that the the integral in Theorem 5.4.8 (ii) converges if and only if the Lévy spectral function N of the (unique) infinitely divisible characteristic function γ satisfies

$$\int_{|v|\ge 1} v^{-\alpha}\, dN(v) < \infty\ , \ \alpha \in (-2,0)\,,$$

and γ has no degenerate component if $\alpha \in (-2,-1]$, i.e., a_γ is of the form (5.21). For $\alpha = 0$ it is necessary and sufficient that the Lévy spectral function N satisfies

$$\int_{|v|\ge 1} |v|^{-1} N(v)\, dv = \int_{|x|\ge 1} \ln |x|\, dN(x) < \infty\,,$$

and hence

$$\int_{|x|\ge 1} \ln |x|\, dN(x) < \infty.$$

Finally, the integral in Theorem 5.4.7 (ii) converges for any characteristic function γ, provided $\alpha > 0$. □

REMARK 5.4.11. O'Connor (1979b) proves Theorem 5.4.7 for $\alpha \in (0,1)$. In his proof he uses the results of O'Connor (1979a), which proves Theorem 5.4.7 for $\alpha = 1$. We prove Theorem 5.4.7 for any fixed $\alpha \ge 0$ and we do not use results for other known fixed α. O'Connor (1981) does not consider condition (i) of Theorem 5.4.8, but the condition

(i′) ϕ is α-self-decomposable and infinitely divisible,

and proceeds to prove (i′) $\Leftrightarrow$ (iii) for $\alpha \in (-2,0)$ and (ii) $\Leftrightarrow$ (iii) for $\alpha \in (-1,0)$. We prefer (i) to (i′) as it is usually easier to verify that ϕ' exists and $t\,\phi'(t) \to 0$ as $t \to 0$ rather than ϕ is infinitely divisible. The proofs are also simpler if condition (i′) is used instead of (i). O'Connor tried but was unable to prove (ii) $\Leftrightarrow$ (iii) for $\alpha \in (-2,-1]$. We were able to do so by making the appropriate choices of a_γ and σ_γ in the proof of Lemma 5.4.6. □

REMARK 5.4.12. Jurek (1988) and (1989) obtains some of the results of this section. Our work, however, was done independently and almost simultaneously of Jurek's. Jurek (1988) and (1989) defines $S_\alpha(\mathbb{R})$ as the set of characteristic functions ϕ which can be obtained as limits of the following kind (cf. Theorem 6.2.3 (ii));

$$\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} (1/n)^\alpha \ln \phi_j(t/n) ,$$

where (ϕ_j) is a sequence of infinitely divisible characteristic functions. He then proves that $\phi \in S_\alpha(\mathbb{R}) \Leftrightarrow$ (i′) and that (i′) $\Leftrightarrow$ (ii) of Theorems 5.4.7 and 5.4.8 for $\alpha > -2$. Jurek however assumes that ϕ stems from a symmetric random variable in the case that $\alpha \in (-2, -1]$. The key to understanding why this assumption is not necessary lays in equation (5.21) in the proof of Lemma 5.4.6. By choosing a_γ as in (5.21) we ensure that the integral in Theorem 5.4.8 (ii) converges. If ϕ stems from a symmetric random variable, then so does γ and hence N (the Lévy spectral function of γ) is symmetric. If N is symmetric, then $a_\gamma = 0$ in (5.21) and the integral in Theorem 5.4.8 (ii) will always converge. If $N \equiv 0$, then by (5.21) $a_\gamma = 0$ if the integral in Theorem 5.4.8 (ii) is to converge. Furthermore, if we assume that $a_\gamma = 0$ and require that the integral in Theorem 5.4.8 (ii) converges, then necessarily

$$\int_{\mathbb{R} \setminus \{0\}} v^3/(1+v^2) dN(v) = 0 ,$$

which (for example) symmetry of N ensures. □

REMARK 5.4.13. The proof of Theorem 5.3.4 can easily be adapted to prove the analogue of Theorem 5.3.4 for distributions on $\mathbb{R}$ (and $\mathbb{R}_+$). □

5.5 Distributions on $\mathbb{R}_+$

In this section we use the results of Section 5.3, to prove the analogue of Theorems 5.3.2, 5.4.7 and 5.4.8 for random variables supported by $\mathbb{R}_+$. The theorem is easily proved by Theorems 5.4.7 and 5.4.8. It is stated separately because it takes a simpler form (as in Theorem 5.3.2), and, as will be shown, the α-self-decomposable distributions on $\mathbb{R}_+$ and on $\mathbb{N}_0$ are closely related.

DEFINITION 5.5.1. Let the random variable X on $\mathbb{R}_+$ have distribution function F and Laplace-Stieltjes transform $\hat{f}$. The function $\hat{f}$ is said to be α-self-decomposable and belong to $S_\alpha(\mathbb{R}_+)$, for some $\alpha \in \mathbb{R}$, if for every $c \in (0,1)$ there exists a Laplace-Stieltjes transform $\hat{f}_c$ such that

$$\hat{f}(\tau) = \hat{f}^{c^\alpha}(c\tau) \hat{f}_c(\tau) \qquad , \tau \in \mathbb{R}_+. \tag{5.27}$$

It can be shown (cf. Theorem 2.7.1 of van Harn (1978)) that X is infinitely divisible and supported by $\mathbb{R}_+$ (cf. Theorem 1.3.3) if and only if the characteristic function ϕ of X is given by Theorem 1.3.2 with $M \equiv 0$ for $x < 0$, $a_\phi = \sigma_\phi^2 = 0$ and $x M'(x) = H'(x)$ for $x > 0$. Furthermore $\hat{f}$ is analytic and $\tau \hat{f}'(\tau) \to 0$ as $\tau \to 0$ (cf. Steutel and van Harn (1979)). We have thus proved

THEOREM 5.5.2. *Let* $\alpha \in (-1, \infty)$ *and let* $\hat{f}$ *be a Laplace-Stieltjes transform. The following statements are equivalent.*

(i) $\hat{f}$ *is* α*-self-decomposable;*

(ii) $\ln \hat{f}(\tau) = \int_0^1 \ln \hat{q}(v\tau) v^{\alpha-1}\, dv - \lambda \tau^{-\alpha}$, $\lambda \geq 0$ *and* $\hat{q}$ *a unique infinitely divisible Laplace-Stieltjes transform;*

(iii) $\hat{f}$ *is infinitely divisible with canonical measure H having left and right derivatives and such that* $H'(x)/x$ *is* α*-unimodal on* $\mathbb{R}_+$.

Furthermore λ *in (ii) is zero if* $\alpha \geq 0$ *and* $\hat{f}_c$ *is infinitely divisible for every* $c \in (0,1)$ *if* $\hat{f}$ *is* α*-self-decomposable.*

Since all log-convex functions and all completely monotone functions are non-increasing we have (cf. Theorems 4.3.3 and 3.5.3)

COROLLARY 1. *If* $H'(x)/x$ *is log-convex, then* $\hat{f} \in S_0(\mathbb{R}_+)$.

COROLLARY 2. *If F is a mixture of exponential distributions, then* $\hat{f} \in S_1(\mathbb{R}_+)$.

Goldie (1967) proved that there is a correspondence between probability generating functions in $ID(\mathbb{N}_0)$ and Laplace-Stieltjes transforms in $ID(\mathbb{R}_+)$. A similar result was proved in Forst (1979) between $S_0(\mathbb{N}_0)$ and $S_0(\mathbb{R}_+)$. We now prove a lemma which generalizes these results by giving a correspondence between probability generating functions in $S_\alpha(\mathbb{N}_0)$ and Laplace-Stieltjes transforms in $S_\alpha(\mathbb{R}_+)$.

LEMMA 5.5.3. *Let F be a distribution function on* $\mathbb{R}_+$ *and define the probability distribution* $(p_n(s))$ *on* $\mathbb{N}_0$ *by*

$$p_n(s) = (n!)^{-1} \int_0^\infty e^{-st} (st)^n dF(t) \ .$$

The following statements are equivalent

(i) *the probability generating function of* $(p_n(s))$ *is in* $S_\alpha(\mathbb{N}_0)$ *for all* $s \in \mathbb{R}_+$;

(ii) *the Laplace-Stieltjes transform of* F *is in* $S_\alpha(\mathbb{R}_+)$.

PROOF. The probability generating function P_s of $(p_n(s))$ and the Laplace-Stieltjes transform $\hat{f}$ of F are related by

$$\hat{f}(s(1-z)) = P_s(z).$$

The lemma now follows from (5.4) and (5.27). □

REMARK 5.5.4. Theorem 5.5.2 can also be proved by using Theorem 5.3.2, Lemma 5.5.3, Lemma 4.2.1 of Steutel (1970) and Theorem 1, p. 439 of Feller (1971). □

REMARK 5.5.5. Multiple α-self-decomposable characteristic functions are studied in Hansen (1989b) and (a generalization and a unification of) multiple α-self-decomposable distributions on $\mathbb{R}_+$ and $\mathbb{N}_0$ is considered in Hansen (1989a). In Hansen (1989c) characteristic functions satisfying

$$\phi(t) = \phi^{c^\alpha}(c^{-1}t)\,\phi_c(t) \qquad , t \in \mathbb{R},$$

are characterized. □

5.6 A classification of ID(I)

From Theorems 5.4.7 and 5.4.8 it is clear that $S_\alpha(\mathbb{R}) \subseteq ID(\mathbb{R})$, and that $S_\alpha(\mathbb{R})$ is closed under multiplication and limits. If $\alpha(1) > \alpha(2)$ and $\phi \in S_{\alpha(2)}(\mathbb{R})$, then (cf. (5.8))

$$\phi(t) = \phi^{c^{\alpha(1)}}(ct)\,\phi^{c^{\alpha(2)} - c^{\alpha(1)}}(ct)\,\phi_c(t).$$

Since ϕ is infinitely divisible, $\phi^{c^{\alpha(2)} - c^{\alpha(1)}}$ is a characteristic function and hence $\phi \in S_{\alpha(1)}(\mathbb{R})$. From Corollary 2 to Theorem 5.4.8 it follows that $S_\alpha(\mathbb{R})$, $\alpha < -2$, contains only degenerate characteristic functions. Theorem 5.4.7 (ii) implies that for any $\phi \in ID(\mathbb{R})$,

$$\phi_\alpha(t) := \exp\Big(\int_0^1 \ln \phi^\alpha(vt)\, v^{\alpha-1}\, dv\Big) = \exp\Big(\int_0^1 \ln \phi(vt)\, dv^\alpha\Big) ,$$

is in $S_\alpha(\mathbb{R})$ for all $\alpha > 0$, with the integral converging by Remark 5.4.10. By Helly's second theorem $\phi_\alpha \to \phi$ as $\alpha \to \infty$. Hence

$$\bigcup_{\alpha \in \mathbb{R}} S_\alpha(\mathbb{R}) = ID(\mathbb{R}) .$$

Similarly for $S_\alpha(\mathbb{R}_+)$ and $S_\alpha(\mathbb{N}_0)$. We collect these results in the following theorem.

THEOREM 5.6.1. *For $I = \mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{N}_0$, the sets $S_\alpha(I)$ are multiplication semigroups, closed under limits and provide a classification of $ID(I)$, i.e.,*

(i) *If $\alpha_2 < \alpha_1$ then $S_{\alpha_2}(I) \subset S_{\alpha_1}(I)$;*

(ii) $ID(I) = \bigcup_{\alpha \in \mathbb{R}} S_\alpha(I)$;

(iii) *$S_\alpha(I)$ is closed under limits and multiplication for every α.*

Chapter 6

α-SELF-DECOMPOSABILITY AND LIMIT LAWS

6.1 Introduction

Limit distributions of sums of independent random variables has been a central topic in probability theory and statistics for many years. The classical central limit problem and its successive generalizations to stable and self-decomposable random variables are well-known examples. Also, the set of infinitely divisible random variables can be described as the solution of the so-called general central limit problem (cf. Loève (1977), compare Theorem 1.3.11). Recently, Jurek (1981) introduced the set of s-self-decomposable random variables, which are defined as limits of sums of 'shrunken' random variables. Jurek (1985) showed that this set coincides with the set of random variables having characteristic functions in $S_1(\mathbb{R})$ (see Notation 5.4.9). The set of 1-self-decomposable random variables satisfying (i) of Theorem 5.4.8 is included in the set of infinitely divisible random variables and in turn includes the sets of self-decomposable and stable random variables.

In contrast to the previous chapters we only study random variables on $\mathbb{R}$ in this chapter. We consider random variables with characteristic functions in $S_\alpha(\mathbb{R})$ as limits of sums of independent normed or 'shrunken' random variables. Using the representations of $S_\alpha(\mathbb{R})$ obtained in the previous chapter, we present in Section 6.2 four limit forms of α-self-decomposable random variables. One of these limit forms is the one used by Jurek (1988) and (1989) to define $S_\alpha(\mathbb{R})$. The method used is easily adapted to $S_\alpha(\mathbb{N}_0)$. In Section 6.3 we introduce two shrinking operators. Both operators are closely connected with α-unimodality. The first operator is a generalization of Jurek's (1981) shrinking operator and the second operator, a stochastic shrinking operator, is a generalization of the well-known linear operator T_t

(cf. Section 1.4). In the same vein as Jurek (1981) we consider limit distributions of sums of 'generalized shrunken' independent random variables and show that the set of these limit distributions is equal to $S_\alpha(\mathbb{R})$, $\alpha \geq 0$. In Section 6.3 we also characterize the set of distributions obtained as limits of sums of 'generalized shrunken' independent identically distributed random variables. In Section 6.4 we discuss $S_\alpha(\mathbb{R})$ for $\alpha \in (-2,0]$ in the context of limit distributions in detail. Among other results we show that $S_\alpha(\mathbb{R})$, $\alpha \in (-2,0]$, contains limit distributions of T_c-normed sums of block wise identically distributed random variables. We conclude in Section 6.5 with a few comments and remarks.

6.2 Sums of uan triangular arrays

In this section we use equation (5.8) and the results of Section 5.4 to describe random variables X with characteristic functions ϕ in $S_\alpha(\mathbb{R})$ as limits of sums of uan random variables. Our approach is similar to that in Lukacs (1970), Theorem 5.11.1 and its corollary.

Let $\phi_{\mathrm{STABLE}(-\alpha)}(t) = 1$ for $\alpha \geq 0$. Writing the integrals in Theorems 5.4.7 and 5.4.8 as limits of Riemann sums we have (cf. Notation 5.4.9)

THEOREM 6.2.1. *Let $\alpha > 0$ and let ϕ be a characteristic function. The following statements are equivalent.*

(i) $\phi \in S_\alpha(\mathbb{R})$;

(ii) *There exists an infinitely divisible characteristic function γ such that*

$$\ln \phi(t) = \alpha^{-1} \lim_{n\to\infty} \sum_{j=1}^{n} (j/n) \ln \gamma^{1/j}((j/n)^{1/\alpha}\, t);$$

(iii) *There exists an infinitely divisible characteristic function γ and a stable characteristic function $\phi_{\mathrm{STABLE}(-\alpha)}(t)$ such that*

$$\ln \phi(t) = \ln \phi_{\mathrm{STABLE}(-\alpha)}(t) + \lim_{n\to\infty} \sum_{j=1}^{n} (j/n)^{\alpha} \ln \gamma^{1/j}((j/n)\, t).$$

Conditions (i) and (iii) are also equivalent for $\alpha \leq 0$.

In Theorem 6.2.1 we see that the logarithm of any characteristic function in $S_\alpha(\mathbb{R})$ can be written as a limit of a weighted sum of logarithms of identical characteristic functions. In the next two theorems we obtain a less restrictive limiting form of ϕ than in Theorem 6.2.1. The proof of the first theorem is simpler and almost identical with the proof of the second and is therefore omitted.

THEOREM 6.2.2. *Let $\alpha > 0$ and let ϕ be a characteristic function. The following statements are equivalent.*

(i) $\phi \in S_\alpha(\mathbb{R})$;

(ii) *There exist infinitely divisible characteristic functions $(\phi_j)_1^\infty$ such that*

$$\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} (1/n) \ln \phi_j((1/n)^{1/\alpha}\, t);$$

(iii) *There exist infinitely divisible characteristic function $(\phi_j)_1^\infty$ such that*

$$\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} (1/n)^\alpha \ln \phi_j(t/n).$$

Conditions (i) and (iii) are also equivalent for $\alpha = 0$.

THEOREM 6.2.3. *Let $\alpha \in (-2, 0)$ and let ϕ be a characteristic function. The following statements are equivalent.*

(i) $\phi \in S_\alpha(\mathbb{R})$;

(ii) *There exist infinitely divisible characteristic functions $(\phi_j)_1^\infty$ such that*

$$\ln \phi(t) = \lim_{n \to \infty} \sum_{j=1}^{n} (1/n)^\alpha \ln \phi_j(t/n).$$

PROOF. Suppose $\phi \in S_\alpha(\mathbb{R})$ and let $\alpha \neq -1$. Letting $j^{\alpha-1} \ln \gamma(jt) = \ln \phi_j(t)$, $j \geq 2$ and $\phi_1 = \gamma \cdot \phi_{\mathrm{STABLE}(-\alpha)}$ in Theorem 6.2.1 (iii), we see that (i) implies (ii). A similar argument holds for $\alpha = -1$. We now prove (ii) $\Rightarrow$ (i). Let (k_n) be a sequence of real numbers such that $k_n < n$ and $(k_n/n) \to c \in (0,1)$ as $n \to \infty$. Observe that

$$\sum_{j=1}^{n} (1/n)^\alpha \ln \phi_j(t/n) = \sum_{j=1}^{k_n} (1/k_n)^\alpha (k_n/n)^\alpha \ln \phi_j((t/k_n)(k_n/n)) + \sum_{j=k_n+1}^{n} (1/n)^\alpha \ln \phi_j(t/n). \tag{6.1}$$

Letting $n \to \infty$ in (6.1), the left hand side tends, by definition, to $\ln \phi(t)$, the first term on the right hand side tends, by definition, to $c^\alpha \ln \phi(ct)$ and so the second term tends to a limit, which by Theorems 1.3.8 and 1.3.7, is the logarithm of some infinitely divisible characteristic function, $\ln \phi_c(t)$ say. By Theorem 5.4.8 (iv) $\phi \in S_\alpha(\mathbb{R})$. □

Jurek (1988) uses condition (ii) of Theorem 6.2.3 to define $S_\alpha(\mathbb{R})$.

REMARK 6.2.4. The implications (i) $\Rightarrow$ (ii) for $\alpha>0$ and (i) $\Rightarrow$ (iii) for $\alpha>-2$ of Theorem 6.2.2 can also be proved directly by noting that (cf. (5.8))

$$\begin{aligned}\sum_{j=1}^{n}(j/n)\ln\phi_{c_j}((j/n)^{1/\alpha}t) &= \sum_{j=1}^{n}(j/n)[\ln\phi((j/n)^{1/\alpha}t)-((j-1)/j)\ln\phi(((j-1)/n)^{1/\alpha}t)]\\ &= \sum_{j=1}^{n}(j/n)\ln\phi((j/n)^{1/\alpha}t)-\sum_{j=1}^{n}((j-1)/n)\ln\phi(((j-1)/n)^{1/\alpha}t)\\ &= \ln\phi(t),\end{aligned}$$

with $\alpha>0$ and $c_j=((j-1)/j)^{1/\alpha}$ and observing that

$$\sum_{j=1}^{n}(j/n)^{\alpha}\ln\phi_{c_j}((j/n)t)=\ln\phi(t)\quad,\ \alpha>0\,,$$

$$(1/n)^{\alpha}\ln\phi(t/n)+\sum_{j=2}^{n}(j/n)^{\alpha}\ln\phi_{c_j}((j/n)t)=\ln\phi(t)\quad,\ \alpha\in(-2,0]\,,$$

where $c_j=(j-1)/j$. □

REMARK 6.2.5. For $\alpha>0$ we can replace the existence of a sequence of infinitely divisible characteristic functions (ϕ_j) in Theorem 6.2.2 (ii) by the existence of a uan triangular array $(\phi_{k,n})$, where

$$\ln\phi_{k,n}(t)=1/n\sum_{j=k}^{n}\ln\phi_j((1/n)^{\alpha}t)\quad,k=1,2,\dots,n\,,\ n\in\mathbb{N}_+\,,$$

for some sequence of characteristic functions (ϕ_j). The implication (ii) $\Rightarrow$ (i) is then proved, as in Theorem 6.2.2, by observing that

$$\begin{aligned}\sum_{j=1}^{n}(1/n)\ln\phi_j((1/n)^{\alpha}t) &= \sum_{j=1}^{k_n}(1/k_n)(k_n/n)\ln\phi_j((1/k_n)^{1/\alpha}(k_n/n)^{1/\alpha}t)\\ &\quad+\sum_{j=k_n+1}^{n}\ln\phi_{j,n}(t)+k_n\ln\phi_{k_n+1,n}(t)\,,\end{aligned}\tag{6.2}$$

and letting $n\to\infty$ such that $(k_n/n)\to c\in(0,1)$. Observe that the sum of the last two terms in (6.2) tends to the logarithm of an infinitely divisible characteristic function by the uan property of $(\phi_{k,n})$, Theorem 1.3.12, and Lemma 1.3.14. Conversely, by Remark 6.2.4 we need only show that $(\phi_{k,n})$ is uan, where

$$\ln\phi_{k,n}(t)=1/n\sum_{j=k}^{n}\ln\phi_{c_j}((j/n)^{1/\alpha}t)\quad,k=1,2,\dots,n\,,\ n\in\mathbb{N}_+\,,$$

and $c_j=((j-1)/j)^{1/\alpha}$. An infinitely divisible characteristic function γ has no real zeros, hence for any $T>0$ there exists a $C>0$ such that $|\ln\gamma(t)|\le C$, $t\in[-T,T]$. From (5.25), letting $a(j)=(j/n)^{1/\alpha}$, we obtain

$$|\ln \phi_{k,n}(t)| \le \frac{C}{n} \sum_{j=k}^{n} \frac{n}{j} \int_{a(j-1)}^{a(j)} u^{\alpha-1} du \le \frac{C}{\alpha} \frac{\ln n}{n} \to 0 \text{ as } n \to \infty ,$$

uniformly on $[-T, T]$. By Lemma 1.3.13, $(\phi_{k,n})$ is uan. Similarly we can replace the condition that (ϕ_j) be infinitely divisible in part (iii) by the condition that the triangular array $(\phi_{k,n})$ be uan (this is done in O'Connor (1979b) for $\alpha \in (0,1)$) where

$$\ln \phi_{k,n}(t) = (1/n)^{\alpha} \sum_{j=k}^{n} \ln \phi_j((j/n)t) \quad , k = 1, 2, \dots, n \, , \, n \in \mathbb{N}_+ \, .$$

□

REMARK 6.2.6. We would like to find a suitable definition of the generalized stable laws in $S_\alpha(\mathbb{R})$ analogous to the definition of the stable laws corresponding to the classical self-decomposable laws, i.e., to find the characteristic functions in $S_\alpha(\mathbb{R})$ which can be written as a limit of weighted sums of identical characteristic functions. All the theorems in this section are of the same form as Theorem 6.2.1, where ϕ is equal to the limit of weighted sums of identical characteristic functions. In this setting, the stable laws in $S_\alpha(\mathbb{R})$ are just the limit laws in $S_\alpha(\mathbb{R})$. Therefore the approach of this section does not suggest a reasonable definition generalizing classical stability. □

6.3 Sums of shrunken random variables

Throughout this section we will study $S_\alpha(\mathbb{R})$ for $\alpha \ge 0$ only. Define the one-parameter family of non-linear shrinking operators $(U_{\alpha,t})$, $\alpha > 0$, by

$$U_{\alpha,t}x = \begin{cases} 0 & \text{if } |x| \le t \\ (1 - (\frac{t}{|x|})^{\alpha})^{1/\alpha} x & \text{if } |x| > t \end{cases} , \tag{6.3}$$

for $t \ge 0$. When $\alpha = 1$, we regain the shrinking operator U_t defined in Jurek (1981). In figure 6.1 on the next page we sketch $U_{\alpha,t}$ as a function in x and α. Let Y_t be a random variable defined by $\mathbb{P}(Y_t = 1) = t^{\alpha} = 1 - \mathbb{P}(Y_t = 0)$. We then define the one-parameter family of linear stochastic operators $(T_{\alpha,t})$, $\alpha \ge 0$, by

$$T_{\alpha,t}x = Y_t \, T_t x \quad , \, t \in [0, 1] \, . \tag{6.4}$$

If $\alpha = 0$ then $T_{\alpha,t}$ reduces to the linear operator T_t (cf. Section 1.4). In this section we show, using the results of the Appendix, that the set of random variables X given by

$$\sum_{k=1}^{n} U_{\alpha,t_n} X_k + b_n \xrightarrow{w} X \quad \text{as } n \to \infty , \tag{6.5}$$

$$\sum_{k=1}^{n} T_{\alpha,t_n}^{(k)} X_k + b_n \xrightarrow{w} X \quad \text{as } n \to \infty , \tag{6.6}$$

for suitable (t_n), (b_n), (X_k) and $(T_{\alpha,t_n}^{(k)})$, is equal to the set of random variables with characteristic functions in $S_\alpha(\mathbb{R})$. We also characterize the random variables X obtained as (6.5) and (6.6) where X_k are identically distributed.

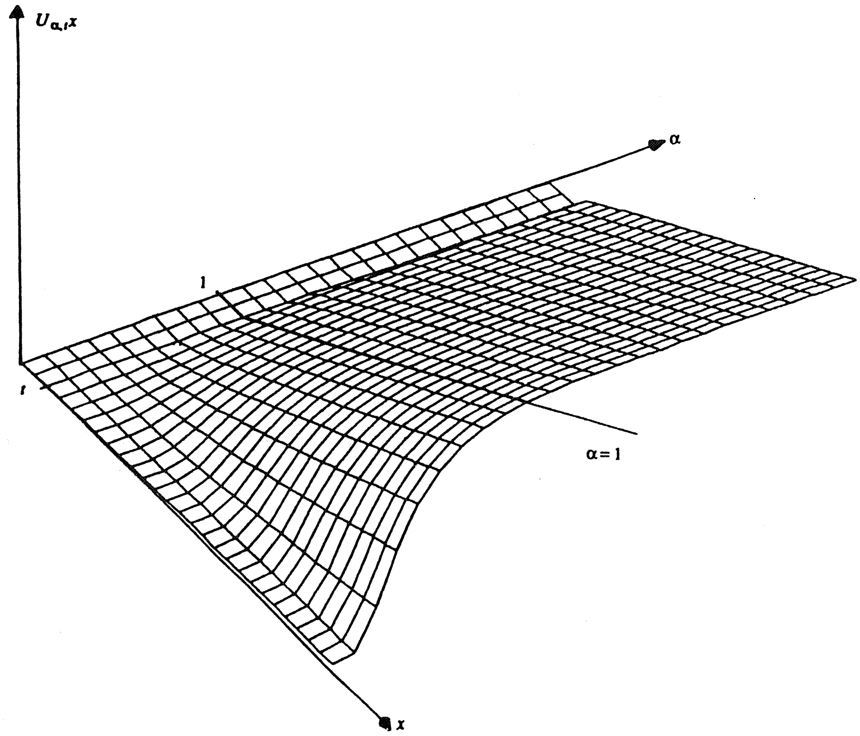

figure 6.1

6.3.1 $U_{\alpha,t}$–shrunken random variables.

Let the random variable X have distribution function F and denote by F_U the distribution function of $U_{\alpha,t}(\mathrm{X})$. Let $U_{\alpha,t}$ and $U_{\alpha,t}^{-1}$ act on Borel sets B by

$$U_{\alpha,t}B = \{\, y \mid U_{\alpha,t}x = y, \text{ for } x \in B \,\} \text{ and } U_{\alpha,t}^{-1}B = \{\, x \mid U_{\alpha,t}x \in B \,\} .$$

Let $I = (a,b)$, $0 < a < b$, then

$$F_U(I) := \mathbb{P}(U_{\alpha,t}\mathrm{X} \in I) = \mathbb{P}(\mathrm{X} \in U_{\alpha,t}^{-1}I) = F(U_{\alpha,t}^{-1}I).$$

We therefore let the operator $U_{\alpha,t}$ operate on set functions by $U_{\alpha,t}F(B) = F(U_{\alpha,t}^{-1}B)$, for any Borel set B. The following lemma gives a connection between $U_{\alpha,t}$ and the notion of α-unimodality.

LEMMA 6.3.1. *Let $\alpha > 0$, $\varepsilon > 0$ and let β_ε be the set of Borel sets on $\mathbb{R}\backslash(-\varepsilon, \varepsilon)$. Further let M be a Lévy spectral function. The following statements are equivalent.*

(i) *M has left and right derivatives on $\mathbb{R}\backslash\{0\}$ and $M'(x)$ is α-unimodal;*

(ii) $M(\mathrm{B}) \geq U_{\alpha,t}M(\mathrm{B})$, $t \in (0, \infty)$, $\mathrm{B} \in \beta_\varepsilon$.

PROOF. First we prove (ii)$\Rightarrow$(i). Let $\alpha \in \mathbb{R}$, fix $\varepsilon > 0$ and let $\mathrm{B} = (a, b)$, $0 < \varepsilon \leq a < b$. (ii) is equivalent to

$$M((b^\alpha + t^\alpha)^{1/\alpha}) - M((a^\alpha + t^\alpha)^{1/\alpha}) \leq M(b) - M(a). \tag{6.7}$$

Let $w(x) := M(x^{1/\alpha})$. From (6.7) with $x = b^\alpha + t^\alpha$, $y = a^\alpha + t^\alpha$, $x' = b^\alpha$ and $y' = a^\alpha$ it follows that

$$\frac{w(x) - w(y)}{x - y} \leq \frac{w(x') - w(y')}{x' - y'} . \tag{6.8}$$

Hence w is convex. By Proposition 16, p. 109, Royden (1968), w and hence also M has left and right derivatives on (ε, ∞). Letting $a \to b$ and hence $y \to x$ and $y' \to x'$ in (6.8) we obtain

$$(b^\alpha + t^\alpha)^{-1+1/\alpha} M'((b^\alpha + t^\alpha)^{1/\alpha}) \leq b^{1-\alpha} M(b) .$$

Let $c^\alpha = b^\alpha + t^\alpha$, then $c \geq b$ and

$$c^{1-\alpha} M'(c) \leq b^{1-\alpha} M'(b).$$

For $B = (a,b)$, $0 > \varepsilon \geq b > a$ the proof is similar. So M' is α-unimodal.

The implication (i)$\Rightarrow$(ii) is proved for $B = (a,b)$, $0 < \varepsilon \leq a < b$, by observing that (cf. (5.9))

$$\begin{aligned} M(B) &= \int_B M'(x)\,dx = \int_B \Big(\int_x^\infty v^{-\alpha}\, dN(v)\Big)\, d\alpha^{-1} x^\alpha \\ &= \int_{U_{\alpha,t}^{-1}B} \Big(\int_{(x^\alpha - t^\alpha)^{1/\alpha}}^\infty v^{-\alpha}\, dN(v)\Big)\, d\alpha^{-1}(x^\alpha - t^\alpha) \\ &\geq \int_{U_{\alpha,t}^{-1}B} M'(x)\,dx = U_{\alpha,t} M(B) . \end{aligned}$$

□

We are now ready to prove

THEOREM 6.3.2. *Let X have characteristic function ϕ. Then $\phi \in S_\alpha(\mathbb{R})$, $\alpha > 0$ if and only if there exists a sequence (b_n) such that*

$$\sum_{k=1}^n U_{\alpha,t_n} \mathrm{X}_k + b_n \xrightarrow{w} \mathrm{X} \quad \textit{as } n \to \infty , \tag{6.5}$$

where (t_n) is a non-negative, non-decreasing sequence and (X_k) is an independent unbounded sequence of random variables such that the triangular array $(U_{\alpha,t_n} X_k)$ is uan.

PROOF. In this proof we will frequently use the results of the Appendix. First we will show that the operator $U_{\alpha,t}$ satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5. Fix $\alpha > 0$ and let $f_t := U_{\alpha,t}$, $t \in [0,\infty)$. By (6.2) we have

$$f_s \circ f_t(x) = f_s(\operatorname{sgn}(x)(|x|^\alpha - t^\alpha)_+^{1/\alpha}) = \operatorname{sgn}(x)(|x|^\alpha - t^\alpha - s^\alpha)_+^{1/\alpha} = f_{s \oplus t}(x),$$

where $(x)_+$ is equal to x if $x > 0$ and zero otherwise and $s \oplus t$ is defined by

$$s \oplus t = (s^\alpha + t^\alpha)^{1/\alpha}.$$

Obviously $f_0(x) = x$ and so $f_0 \circ f_t = f_t$ Hence $S = (\{f_t\}_{t \in \mathbb{R}_+}, \circ)$ is a composition semigroup with respect to the semigroup $(\mathbb{R}_+, \oplus)$. If $t \geq s$ then

$$|f_t(x)| = (|x|^\alpha - t^\alpha)_+^{1/\alpha} \leq (|x|^\alpha - s^\alpha)_+^{1/\alpha} = |f_s(x)| \leq |x|,$$

and so f_t in fact does shrink its arguments. From (6.2) it follows that $f_t(x)$ is continuous in both t and x ($f_t(x)$ is differentiable in both t and x on (t,∞)), that f_t is unbounded ($f_t(x) \to \infty$ as $x \to \infty$) and that f_t is one-to-one on (t,∞).

Suppose X is obtained by (6.5). Since $(U_{\alpha,t_n} X_k)$ is uan, X is infinitely divisible and hence $\phi = [a_\phi, \sigma_\phi^2, M]$. By Corollary 1 to Theorem A.3.2 in the Appendix, M satisfies the inequality of Lemma 6.3.1 (ii). By Lemma 6.3.1 and Theorem 5.4.7, $\phi \in S_\alpha(\mathbb{R})$.

The converse is proved in three steps. First it is shown that the symmetric normal distribution is a limit as described in the theorem; secondly it is shown that a random variable in $S_\alpha(\mathbb{R})$ without normal component is of the desired type and finally these two results are combined to prove the theorem.

Let X be a symmetric normal random variable, i.e., $\phi = [0, \sigma_\phi^2, 0]$. Let $F^{(1)}$ and $F^{(2)}$ be Weibull distributions, defined by

$$F^{(1)}(x) = 1 - \exp(-\tfrac{1}{2}\theta x^{2\alpha}),\ x \geq 0,$$

$$F^{(2)}(x) = \exp(-\tfrac{1}{2}\theta |x|^{2\alpha}),\ x < 0.$$

and let G be given by

$$G(B) = \tfrac{1}{2} F^{(1)}(B \cap \mathbb{R}_+) + \tfrac{1}{2} F^{(2)}(B \cap (-\infty, 0)).$$

Choose θ such that

$$\sigma_\phi^2 = \int_0^\infty x^2 \, d(1 - e^{-\theta x^\alpha}).$$

Define (t_n) by

$$n = t_n^2 \exp(\tfrac{1}{2}\theta\, t_n^{2\alpha}).$$

Hence $t_n \to \infty$ as $n \to \infty$. We will prove that the triangular array $(F_{k,n})$ with $F_{k,n} = U_{\alpha,t_n} G$, $k = 1, 2, \ldots, n$ is uan and satisfies conditions (i) and (ii) of Theorem 1.3.12 with $M \equiv 0$ and hence that X is of the form (6.5).

The uan property. Let $\varepsilon > 0$ be fixed. Observe that

$$F_{k,n}(\{\, |x| \geq \varepsilon \,\}) = U_{\alpha,t_n} G(\{\, |x| \geq \varepsilon \,\}) = e^{-\frac{1}{2}\theta(\varepsilon^\alpha + t_n^\alpha)^2}.$$

Hence $(F_{k,n})$ tends to zero outside every neighbourhood of the origin, uniformly in k, as $n \to \infty$ and so $(F_{k,n})$ is uan.

Condition (i) of Theorem 1.3.12. Let $\varepsilon > 0$ be fixed. Observe that

$$\sum_{k=1}^{n} F_{k,n}(\{\, |x| \geq \varepsilon \,\}) = nU_{\alpha,t_n} G(\{\, |x| \geq \varepsilon \,\}) = t_n^2 e^{-\frac{1}{2}\theta(\varepsilon^{2\alpha} + 2t_n^\alpha \varepsilon^\alpha)},$$

which tends to zero as $n \to \infty$. Hence X is infinitely divisible with $M \equiv 0$.

Condition (ii) of Theorem 1.3.12. Let $\varepsilon > 0$ be fixed. Observe that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x| \leq \varepsilon} x^2 \, dF_{k,n}(x) = \int_0^{\varepsilon t_n} x^2 \, d(1 - e^{-\theta x^\alpha} e^{-\frac{1}{2}\theta(x/t_n)^{2\alpha}})$$

$$\to \int_0^{\infty} x^2 \, d(1 - e^{-\theta x^\alpha}) = \sigma_\phi^2 \text{ as } n \to \infty.$$

Hence condition (ii) is also satisfied.

Next, let X be a random variable in $S_\alpha(\mathbb{R})$ without normal component, i.e., $\phi = [a_\phi, 0, M]$. By Theorem A.3.2 of the Appendix, X can be obtained as a limit of the form (6.5).

We now have proved that for any $\phi \in S_\alpha(\mathbb{R})$ with $\phi = [a_\phi, \sigma_\phi^2, M]$ we can find distribution functions $F_{k,n}^{(1)} = U_{\alpha,t_n} G$ and $F_{k,n}^{(2)} = U_{\alpha,t_n} F_k^{(2)}$ such that

$$\prod_{k=1}^{n} \phi_{k,n}^{(1)} \to \phi^{(1)} := [0, 2\sigma_\phi^2, 0],$$

$$\prod_{k=1}^{n} \phi_{k,n}^{(2)} \to \phi^{(2)} := [2a_\phi, 0, 2M].$$

Let

$$F_k := \tfrac{1}{2}(G + F_k^{(2)}).$$

It can be verified that the sequence (X_k) is unbounded and that $(U_{\alpha,t_n} F_k)$ is uan. Let $\phi_{k,n}$ be the characteristic functions of $U_{\alpha,t_n} F_k$. In view of Theorem 1.3.12

$$\prod_{k=1}^{n} \phi_{k,n} \to \phi := [a_\phi, \sigma_\phi^2, M]. \qquad \square$$

THEOREM 6.3.3. *Let* X *have characteristic function* ϕ. *Then* ϕ *is infinitely divisible, i.e.,* $\phi=[a_\phi, \sigma_\phi^2, M]$ *with either* $M\equiv 0$ *and* $\sigma_\phi^2\geq 0$, *or* $\sigma_\phi^2=0$ *and*

$$M(x)=\begin{cases} -C_1 e^{-p_1 x^\alpha} & x>0 \\ C_2 e^{-p_2 |x|^\alpha} & x<0 \end{cases},$$

with $\alpha>0$, $p_1>0$, $p_2>0$, $C_1\geq 0$, $C_2\geq 0$ *and* $C_1+C_2>0$ *if and only if there exists a sequence* (b_n) *such that*

$$\sum_{k=1}^{n} U_{\alpha,t_n} \mathrm{X}_k + b_n \xrightarrow{w} \mathrm{X} \qquad \text{as } n\to\infty,$$

where (t_n) *is a non-negative, non-decreasing sequence and* (X_k) *is an independent not bounded sequence of identically distributed random variables such that the triangular array* $(U_{\alpha,t_n}\mathrm{X}_k)$ *is uan.*

PROOF. In this proof we will frequently use the results of the Appendix. From the proof of Theorem 6.3.2, we know that $U_{\alpha,t}$ satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5 of the Appendix.

Let $\phi=[a_\phi, \sigma_\phi^2, M]$ be as in the theorem. If $\phi=[a_\phi, \sigma_\phi^2, 0]$, then by the proof of Theorem 6.3.2, X is of the form (6.5), with (X_k) identically distributed. Theorem A.4.1 of the Appendix proves the "only if" part for $\phi=[0,0,M]$.

Suppose X can be obtained as a limit as described in the theorem. By Theorem 1.3.12, X is infinitely divisible with $\phi=[a_\phi, \sigma_\phi^2, M]$. By Theorem A.4.1, there exists a semigroup homomorphism q from $(\mathbb{R}_+, \oplus)$ to $([1,\infty), \cdot)$, with $\oplus$ defined in the proof of Theorem 6.3.2, such that

$$M(B)=q(t)\,U_{\alpha,t}M(B)\,. \tag{6.9}$$

Let $g(x) := q(x^{1/\alpha})$. Then since q is a homomorphism

$$g(x)g(y)=q(x^{1/\alpha})\,q(y^{1/\alpha})=q((x+y)^{1/\alpha})=g(x+y).$$

Since $g(x)\geq 1$ and $g(0)=1$ then $g(x)=\exp(px)$ and so

$$q(x)=\exp(px^\alpha)\,,$$

for some $p\geq 0$. If $p=0$, then by repeated use of (6.9) we have

$$M(B)=U_{\alpha,tn^{1/\alpha}}M(B)\,.$$

Letting $n\to\infty$ we see that $M\equiv 0$ and hence ϕ is normal. Now let $p>0$ and $B=(a^\alpha, b^\alpha)$ with $0<a\leq b$, $t^\alpha=s$ and let $w(x) := M(x^{1/\alpha})$. By (6.9)

$$\frac{w(a)-w(b)}{a-b}=e^{ps}\,\frac{w(a+s)-w(b+s)}{(a+s)-(b+s)}.$$

We know from Theorems 6.3.2 and 5.4.7 that M' exists. Letting $b\to a$ we obtain that

$$e^{pa}\, w'(a) = e^{p(a+s)}\, w'(a+s) = q\ ,$$

with q a non-negative constant. Hence $w(x) = ce^{-px}$ and therefore

$$M(x) = w(x^\alpha) = ce^{-px^\alpha}\ ,$$

with $c \in \mathbb{R}_-$. Likewise for $0 > b \geq a$. Hence M is of the desired form.

It remains to be shown that if M is not identically zero then $\sigma_\phi^2 = 0$. Let F be the distribution function of (X_k) and let $M_n := nU_{\alpha,t_n}F$. Also define

$$R_n(s) := M_n(\{\ |x| > s\ \}) := nU_{\alpha,t_n}F(\{\ |x| > s\ \})\,.$$

Let t be such that $R_n(s)$ is non-zero in a neighbourhood of t and let $(k(n))$ be a sequence such that $(t_{k(n)}^\alpha - t_n^\alpha)^{1/\alpha} \to t$ as $n \to \infty$ (this is possible by Lemma A.3.1 of the Appendix). Observe that

$$\begin{aligned} R_{k(n)}(s) &= n\,(k(n)/n)\, U_{\alpha,t_n}F(\{\ |x| > s\ \}) \\ &= (k(n)/n)\, M_n(\{\ |x| > (s^\alpha + t_{k(n)}^\alpha - t_n^\alpha)^{1/\alpha}\ \}) = (k(n)/n)\, R_n(s^\alpha + t_{k(n)}^\alpha - t_n^\alpha)^{1/\alpha})\,. \end{aligned}$$

Since $M_n \to M$, $R_n(s)$ converges for all $s > 0$ to a not identically zero limit, $R(s)$ say, and we must have $\lim_{n\to\infty} k(n)/n = c < \infty$. Integrating by parts we have

$$\int_{|x|\leq\varepsilon} x^2\, dM_{k(n)}(x) = -\varepsilon^2 R_{k(n)}(\varepsilon) + 2\int_0^\varepsilon s\, R_n((s^\alpha + t_{k(n)}^\alpha - t_n^\alpha)^{1/\alpha})\, ds\ .$$

Letting $n \to \infty$ we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|\leq\varepsilon} x^2\, dF_{k,n}(x) \leq 2c \int_0^\varepsilon s\, R((s^\alpha + t^\alpha)^{1/\alpha})\, ds \leq 2c\, R(t)\,\varepsilon\ ,$$

which tends to zero as $\varepsilon \to 0$. Hence, by Theorem 1.3.12, $\sigma_\phi^2 = 0$. □

6.3.2 $\mathrm{T}_{\alpha,t}$*–shrunken random variables.*

Let the random variable X have distribution function F and denote by F_{T} the distribution function of $\mathrm{T}_{\alpha,t}\mathrm{X}$. Let $I = (a,b)$, $0 < a < b$. We then define F_{T} by

$$F_{\mathrm{T}}(I) := \mathbb{P}(\mathrm{T}_{\alpha,t}\mathrm{X} \in I) = t^\alpha\, \mathbb{P}(\mathrm{X} \in T_t^{-1}I) = t^\alpha F(T_t^{-1}I).$$

We therefore let the operator $\mathrm{T}_{\alpha,t}$ operate on set functions by $\mathrm{T}_{\alpha,t}F(B) = t^\alpha F(T_t^{-1}B)$, for any Borel set B and with T_t defined in Section 1.4. Hence for any Borel set B of $\mathbb{R}\backslash\{-\varepsilon, \varepsilon\}$, $\varepsilon > 0$,

$$M(\mathrm{B}) \geq c^\alpha\, T_c M(\mathrm{B}) \text{ if and only if } M(\mathrm{B}) \geq \mathrm{T}_{\alpha,c}M(\mathrm{B})\,. \tag{6.10}$$

We are now ready to prove

THEOREM 6.3.4. *Let* X *have characteristic function* ϕ. *Then* $\phi \in S_\alpha(\mathbb{R})$ $(\alpha \geq 0)$ *if and only if there exists a sequence* (b_n) *such that*

$$\sum_{k=1}^{n} \mathrm{T}_{\alpha,t_n}^{(k)} \mathrm{X}_k + b_n \xrightarrow{w} \mathrm{X} \qquad \text{as } n \to \infty\,, \tag{6.6}$$

where (t_n) *is a non-negative, non-increasing sequence with* $t_1 \leq 1$, (X_k) *is an independent unbounded sequence of random variables such that* $(\mathrm{T}_{\alpha,t_n}^{(k)} \mathrm{X}_k)$ *is uan and the random operators* $\mathrm{T}_{\alpha,t_n}^{(k)}$ *are independent and identically distributed.*

PROOF. In this proof we will frequently use the results of the Appendix. First we will show that $\mathrm{T}_{\alpha,t}$ satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5 of the Appendix. Fix $\alpha > 0$. Let $U_t := \mathrm{T}_{\alpha,1/t}, f_t := T_{1/t}, t \in [1,\infty)$ and $p(t) := t^{-\alpha}$. We have

$$f_s \circ f_t(x) = f_s(t^{-1}x) = s^{-1}\,t^{-1}\,x = f_{s \oplus t}(x),$$

where $s \oplus t$ is defined by

$$s \oplus t = t \cdot s.$$

Obviously $f_1(x) = x$ and so $f_1 \circ f_t = f_t$. Hence $S = (\{f_t\}_{t \in [1,\infty)}, \circ)$ is a composition semigroup with respect to the semigroup $([1,\infty), \oplus)$. If $t \geq s$ then

$$|f_t(x)| = |t^{-1}x| \leq |s^{-1}x| = |f_s(x)| \leq |x|,$$

and so f_t in fact does shrink its arguments. From the definition of T_t (cf. Section 1.4) it follows that $f_t(x)$ is continuous in both t and x ($f_t(x)$ is differentiable in both t and x on $(0,\infty)$), that f_t is unbounded ($f_t(x) \to \infty$ as $x \to \infty$) and that f_t is one-to-one on $(0,\infty)$. The operator $\mathrm{T}_{\alpha,t}$ thus satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5 of the Appendix.

Suppose X is obtained by (6.6). Since the sequence $(\mathrm{T}_{\alpha,t_n} \mathrm{X}_k)$ is uan, X is infinitely divisible and so $\phi = [a_\phi, \sigma_\phi^2, M]$. By (6.10) and Corollary 1 to Theorem A.3.2 in the Appendix M satisfies the inequality of Lemma 5.4.5 (ii). By Lemma 5.4.5 and Theorem 5.4.7, $\phi \in S_\alpha(\mathbb{R})$.

The converse is proved in three steps. First it is shown that the symmetric normal distribution is a limit as described in the theorem; secondly it is shown that a random variable without normal component is of the desired type and finally these two results are combined to prove the theorem. The last step is identical with the last step of the proof of Theorem 6.3.2 and therefore omitted.

Let X be a symmetric normal random variable, i.e., $\phi = [0, \sigma_\phi^2, 0]$. Let Y be a unbounded random variable on $\mathbb{R}$ with finite second moment and let it have distribution function G. Let

$$\sigma_\phi^2 = \int_{-\infty}^{\infty} x^2\, dG(x)\,.$$

Let (t_n) be defined by

$$t_n = n^{-1/(\alpha+2)} .$$

We will prove that the triangular array $(F_{k,n})$ with $F_{k,n} = \mathrm{T}_{\alpha,t_n} G$, $k = 1, 2, \ldots, n$ is uan and satisfies conditions (i) and (ii) of Theorem 1.3.12 with $M \equiv 0$ and hence that X is of the form (6.6).

The uan property. Let $\varepsilon > 0$ be fixed. Observe that

$$F_{k,n}(\{ |x| \geq \varepsilon \}) = \mathrm{T}_{\alpha,t_n} G(\{ |x| \geq \varepsilon \}) \leq t_n^{\alpha} = n^{-\alpha/(\alpha+2)} .$$

Hence $(F_{k,n})$ tends to zero outside every neighbourhood of the origin, uniformly in k, as $n \to \infty$ and so $(F_{k,n})$ is uan.

Condition (i) of Theorem 1.3.12. Let $\varepsilon > 0$ be fixed. Observe that

$$\sum_{k=1}^{n} F_{k,n}(\{ |x| \geq \varepsilon \}) = n \mathrm{T}_{\alpha,t_n} G(\{ |x| \geq \varepsilon \})$$

$$= n^{2/(\alpha+2)} G(\{ |x| \geq n^{1/(\alpha+2)} \varepsilon \}). \tag{6.11}$$

Since Y has finite second moment, $\lim_{x \to \infty} x^2 (1 - G(\varepsilon x)) = 0$ for $\varepsilon > 0$. Hence (6.11) tends to zero as $n \to \infty$. So X is infinitely divisible with $M \equiv 0$.

Condition (ii) of Theorem 1.3.12. Let $\varepsilon > 0$ be fixed. Observe that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \int_{|x| \leq \varepsilon} x^2 \, dF_{k,n}(x) = n t_n^{\alpha+2} \int_0^{\varepsilon/t_n} y^2 \, dG(y)$$

$$\to \int_0^{\infty} y^2 \, dG(y) = \sigma_\phi^2 \quad \text{as } n \to \infty .$$

Hence condition (ii) is also satisfied.

Next, let X be a random variable without normal component, i.e., $\phi = [a_\phi, 0, M]$. By Theorem A.3.2 of the Appendix, X can be obtained as a limit of the form (6.6). □

THEOREM 6.3.5. *Let* X *have characteristic function* ϕ. *Then* ϕ *is stable if and only if there exists a sequence* (b_n) *such that*

$$\sum_{k=1}^{n} \mathrm{T}_{\alpha,t_n}^{(k)} \mathrm{X}_k + b_n \xrightarrow{w} \mathrm{X} \qquad \text{as } n \to \infty ,$$

where (t_n) *is a non-increasing sequence in* $(0,1]$, (X_k) *are independent not bounded identically distributed random variables and the random operators* $\mathrm{T}_{\alpha,t_n}^{(k)}$ *are independent and distributed as* T_{α,t_n}.

PROOF. In this proof we will use the results of the Appendix. From the proof of Theorem 6.3.4, we know that $T_{\alpha,t}$, $t \in (0,1]$ satisfies Assumptions A.2.1, A.2.2, A.2.3 and A.2.5 of the Appendix.

Let $\phi = [a_\phi, \sigma_\phi^2, M]$ be stable. If $\phi = [a_\phi, \sigma_\phi^2, 0]$, then by the proof of Theorem 6.3.4, X is of the form (6.6), with (X_k) identically distributed. Theorem A.4.1 of the Appendix proves the 'only if' part for $\phi = [0, 0, M]$.

Suppose X can be obtained as a limit as described in the theorem. Observe that

$$\mathbb{P}(|X_{k,n}| \geq \varepsilon) := \mathbb{P}(|T_{\alpha,t_n} X_k| \geq \varepsilon) = t_n^\alpha \, \mathbb{P}(|T_{t_n} X_k| \geq \varepsilon) \leq \mathbb{P}(|T_{1/t_n} X_k| \geq \varepsilon).$$

Since the sequence $(T_{1/t_n} X_k)$ is uan, $(T_{\alpha,t_n} X_k)$ is uan and hence X is infinitely divisible with characteristic function ϕ of the form $\phi = [a_\phi, \sigma_\phi^2, M]$. From Theorem 1.3.12 we have that

$$\lim_{n\to\infty} \sum_{k=1}^{n} T_{\alpha,t_n} F_k = \lim_{n\to\infty} n\, t_n^\alpha T_{t_n} F = M ,$$

outside every neighbourhood of the origin. Also

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|\leq\varepsilon} x^2 \, dT_{\alpha,t_n} F_k(x) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|\leq\varepsilon} x^2 \, dnt_n^\alpha T_{t_n} F(x) = \sigma_\phi^2 .$$

Obviously, if $n\, t_n^\alpha$ is bounded then $M \equiv 0$ and so ϕ is normal. Suppose M is not identically zero, then $n\, t_n^\alpha \to \infty$ as $n \to \infty$. Let $N(n) = n\, t_n^\alpha$. Then

$$M = \lim_{n\to\infty} N(n) T_{t_n} F , \tag{6.12}$$

outside every neighbourhood of the origin and

$$\sigma_\phi^2 = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|\leq\varepsilon} x^2 \, dN(n) T_{t_n} F(x) .$$

By Lemma A.3.1 of the Appendix, $t_{n+1}/t_n \to 1$ as $n \to \infty$. Observe that (cf. (6.12))

$$(N(n+1) - N(n))\, T_{t_n} F = (((n+1) t_{n+1}^\alpha / n t_n^\alpha) - 1) N(n)\, T_{t_n} F \to 0 , \tag{6.13}$$

as $n \to \infty$ outside every neighbourhood of the origin. Similarly

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|\leq\varepsilon} x^2 \, d(N(n+1) - N(n))\, T_{t_n} F(x) = 0.$$

For every $l \in \mathbb{N}_+$ there exists an $n \in \mathbb{N}_+$ such that $N(n) \leq l < N(n+1)$. Let $(t_l')_{l=1}^\infty$ be defined by

$$t_l' = t_n \quad , N(n) \leq l < N(n+1) \, , \, l = 1,2,3,\dots .$$

For any $l \in \mathbb{N}_+$ let n be such that $N(n) \leq l < N(n+1)$. Then (cf. (6.12) and (6.13))

$$\lim_{l\to\infty} \sum_{k=1}^{l} T_{t_l'} F = \lim_{l\to\infty} l\, T_{t_l'} F = \lim_{l\to\infty} (N(n) T_{t_n} F + (l - N(n)) T_{t_n} F) = M ,$$

outside every neighbourhood of the origin. Similarly

$$\sigma_\phi^2 = \lim_{\varepsilon\to 0}\lim_{l\to\infty}\sum_{k=1}^{l}\int_{|x|\le\varepsilon} x^2\, d\, l\, T_{t_l'}F(x)\,.$$

By Theorem 1.3.12 we see that

$$\sum_{k=1}^{l} T_{t_l'}\mathrm{X}_k \to \mathrm{X} \quad \text{as } l\to\infty,$$

where (X_k) are independent and identically distributed with distribution function F. Hence X is stable (cf. Section 1.4). □

REMARK 6.3.6. We can also use Theorem A.4.1 to prove the "if" part of Theorem 6.3.5. This proof however must be split into two steps. The first (which we give below) shows that M is of the desired form. In the second step it is shown that if M is not identically zero then $\sigma_\phi^2 = 0$. This second step is quite involved (cf. proof of Theorem 6.3.3) and therefore omitted.

Suppose X can be obtained as a limit as described in the theorem. From the proof of Theorem 6.3.4 we know that X can be normal. By Theorem A.4.1 there exists a semigroup homomorphism q from $([1,\infty),\oplus)$ to $([1,\infty),\cdot)$ with $\oplus$ defined in the proof of Theorem 6.3.4, such that

$$M(B) = q(t)\, t^{-\alpha}\, T_{1/t} M(B)\,. \tag{6.14}$$

Since $q(x)\ge 1$ and $q(1)=1$ we have that $q(x)=x^\gamma$ for some $\gamma\ge 0$. If $\gamma=0$ then by repeated use of (6.11) we see that $M\equiv 0$. Let $\gamma>0$ and $B=(a,b)$ with $0<a\le b$. By (6.14)

$$\frac{M(a)-M(b)}{a-b} = t^\gamma\, t^{-\alpha}\, \frac{M(at)-M(bt)}{a-b}.$$

We know from Theorems 6.3.4 and 5.4.7 that M' exists. Letting $b\to a$ we obtain that

$$a^{\gamma-\alpha+1}\, M'(a) = (at)^{\gamma-\alpha+1}\, M'(at) = c_1\,,$$

with c_1 a non-negative constant. Hence

$$M(x) = c\, x^{\alpha-\gamma}\,,$$

with $c \in \mathbb{R}_-$. Since M is a Lévy spectral function if and only if $(\alpha-\gamma)\in(0,2)$ we must have $\gamma\in(\alpha,\alpha+2)$. Likewise for $0>b\ge a$. Hence M is the Lévy spectral function of a stable distribution. □

REMARK 6.3.7. The unbounded condition on X_k in the theorems of this section is assumed, since if (X_k) were bounded and $t_n\to 0$ ($t_n\to\infty$) then the limit in (6.6) ((6.5)) would be degenerate. □

6.4 Subsets of the set of self-decomposable limit laws

In this section we consider the subsets $S_\alpha(\mathbb{R})$, $\alpha \in (-2,0]$ of the self-decomposable limit laws. We show that random variables X whose characteristic functions are in $S_\alpha(\mathbb{R})$ can be written as (1.4), where (X_k) are independent and block wise identically distributed random variables. This result is in some sense, the counterpart of Theorem 6.3.2 for $\alpha \in (-2,0]$.

Let $\mathrm{Y}(t)$ be a stochastic process with independent stationary increments with $\mathrm{Y}(0)=0$ with probability one (cf. Feller (1971)) and let (X_k) be a sequence of independent random variables, all distributed as $\mathrm{Y}(1)$. Then

$$\mathrm{Y}(s) \overset{d}{=} \sum_{k=1}^{s} \mathrm{X}_k \,, \quad s \in \mathbb{R}_+ \,,$$

where a non-integer sum of the X_k is defined by the non-integer power of its characteristic function. Let the stochastic operator $\mathrm{T}^*_{\alpha,t}$ act on infinitely divisible random variables $\mathrm{Y}(1)$ by

$$\mathrm{T}^*_{\alpha,t}\mathrm{Y}(1) \overset{d}{=} t\mathrm{Y}(t^\alpha) \,.$$

Hence, for any infinitely divisible random variable X we have that

$$\mathrm{T}^*_{\alpha,t}\mathrm{X} \overset{d}{=} \sum_{k=1}^{t^\alpha} t\mathrm{X}_k \,,$$

where (X_k) are independent random variables distributed as X. If $\alpha \in \mathbb{R}_-$ and $t \in (0,1]$, then $t^\alpha \geq 1$. As $\mathrm{T}^*_{\alpha,t}\mathrm{X}$ produces t^α copies of X we can interpret $\mathrm{T}^*_{\alpha,t}$ with $\alpha \in \mathbb{R}_-$ and $t \in (0,1]$, as a stochastic 'breeding' operator. Let (t_n) be a sequence of non-increasing real numbers with $t_1 = 1$ and $t_n \to 0$ as $n \to \infty$. Theorem 6.2.3 (iii) states that a random variable X has a characteristic function in $S_\alpha(\mathbb{R})$, $\alpha > -2$, if and only if there exists a sequence (X_j) of independent infinitely divisible random variables such that

$$\sum_{j=1}^{n} \mathrm{T}^*_{\alpha,t_n}\mathrm{X}_j \overset{w}{\to} \mathrm{X} \,. \tag{6.15}$$

Thus (6.15) is a limit of a sum of a triangular array as in Theorem 1.3.12.

To eliminate the condition that (X_k) be infinitely divisible we introduce a new stochastic 'breeding' operator $\mathrm{T}_{\alpha,t}$ as follows.

$$\mathrm{T}_{\alpha,t}\mathrm{X} \overset{d}{=} \sum_{k=1}^{[t^\alpha]} t\mathrm{X}_k \,, \quad t \in (0,1] \,, \; \alpha \in (-2,0] \,,$$

where $[x]$ denotes the integer part of x and (X_k) are independent random variables distributed as X. The following theorem gives us another classification of the $S_\alpha(\mathbb{R})$ limit laws. For a proof we refer to Theorem A.5.1 of the Appendix.

THEOREM 6.4.1. *Let* $\alpha \in (-2,0]$ *and let* X *have characteristic function* ϕ. *Then* $\phi \in S_\alpha(\mathbb{R})$ *if and only if there exists a sequence* (b_n) *such that*

$$\sum_{k=1}^{n} T_{\alpha,t_n} X_k + b_n \xrightarrow{w} X \qquad as\ n \to \infty,$$

where (t_n) *is a non-negative, non-increasing sequence and* (X_k) *are independent not bounded random variables such that the sequence* $(T_{\alpha,t_n} X_k)$ *is independent and uan.*

The set $S_\alpha(\mathbb{R})$ thus contains characteristic functions ϕ whose random variables X are weak limits of normed sums of block-wise identically distributed random variables. The size of these blocks is $[t_n^\alpha]$. For $\alpha = 0$, the block size is one and the limit in Theorem 6.4.1 reduces to the limit defining classical self-decomposable random variables (cf. Section 1.4).

6.5 Remarks and comments

Jurek (1988) and (1989) gives the following interpretation of Theorem 6.2.3: Let $(Y_k(t))$ be a sequence of stochastic processes with independent stationary increments with $Y_k(0) = 0$ with probability one (cf. Feller (1971)) and let X_k be distributed as $Y_k(1)$. Rewritting Theorem 6.2.3 (ii) in terms of stochastic processes with independent stationary increments and letting $t_n = 1/n$ we get

$$\frac{Y_1(n^{-\alpha}) + Y_2(n^{-\alpha}) + ... + Y_n(n^{-\alpha})}{n} \xrightarrow{w} X \quad \text{as } n \to \infty.$$

Note that $n^{-1}\, Y_j(n^{-\alpha}) = T^*_{\alpha,t} Y(1)$, with $T^*_{\alpha,t}$ defined in Section 6.4. The other theorems of Section 6.2 can also be reformulated in terms of limit distributions of sums of T_c-shrunken (or $T^*_{\alpha,t}$-shrunken) stochastic processes with independent stationary increments.

The shrinking operators introduced in Section 6.3 do have some practical justification. One could for example imagine a situation where a signal X must be measured. $U_{\alpha,t}$ can then be interpreted as follows; if the signal is too small then our instruments can not register the signal and if the signal is registered, then we only measure a fraction of the strength of the true signal. Likewise, we can interpret $T_{\alpha,t}$ as: with a certain probability we do not receive the signal, and if the signal is received then we only measure a fraction of the strength of the true signal.

APPENDIX

LIMIT DISTRIBUTIONS OF SUMS OF SHRUNKEN RANDOM VARIABLES

A.1 Introduction

In this appendix we consider random variables X obtained as limits of sums of random variables as follows;

$$\sum_{k=1}^{n} U_{t_n}^{(k)} \mathrm{X}_k \xrightarrow{w} \mathrm{X} + b_n \quad \text{as } n \to \infty\,, \tag{A.1}$$

where (b_n) is a sequence of real numbers, (X_k) are independent, unbounded random variables, the triangular array $(U_{t_n}^{(k)}\mathrm{X}_k)$ is uan, and $(U_{t_n}^{(k)})_{k=1}^{n}$ are mutually independent and independent of (X_k) and all distributed as U_t, with U_t a 'generalized stochastic shrinking' operator defined by

$$U_t\mathrm{X} = \mathrm{Y}_t\, f_t(\mathrm{X})\,, \tag{A.2}$$

where X and Y_t are independent and $\mathbb{P}(\mathrm{Y}_t=1)=p(t)=1-\mathbb{P}(\mathrm{Y}_t=0)$. Let δ_0 be a distribution function with total mass at zero and let B be a Borel set. The distribution function of the random variable $U_t\mathrm{X}$ is given by

$$\begin{aligned} F_{U_t\mathrm{X}}(B) &:= p(t)\,\mathbb{P}(f_t(\mathrm{X})\in B) + (1-p(t))\,\delta_0(B) \\ &= p(t)\,\mathbb{P}(\mathrm{X}\in \{\, x \mid f_t(x)\in B \,\}) + (1-p(t))\,\delta_0(B)\,. \end{aligned} \tag{A.3}$$

The notion of a shrinking operator originated from Jurek (1981). Many of the proofs in this appendix are based on the proofs of the corresponding theorem in Jurek (1981), where (A.3) is considered with $U_t = U_{1,t}$ where $U_{\alpha,t}$ is defined in (6.3). As special cases of U_t we have the shrinking operators $U_{\alpha,t}$ and $\mathrm{T}_{\alpha,t}$ introduced in Section 6.4. The assumptions on the 'shrinking functions' f_t, the norming sequence (t_n) and the probabilities $p(t)$ are stated in Section A.2. In Section A.3 we characterize the random variables X in (A.1) in terms of an inequality on the Lévy spectral function of X. The special case where (X_k) are identically distributed is studied in Section A.4. In Section A.5 we give a classification of the set of random variables of the form (A.1) and in Section A.6 we list a few examples.

A.2 Preliminaries

Let f_t and f_t^{-1} act on Borel sets B by

$$f_t B = \{ y \mid f_t(x) = y \text{ for } x \in B \} \text{ and } f_t^{-1} B = \{ x \mid f_t(x) \in B \}. \tag{A.4}$$

We begin by listing the assumptions which we make on f_t, (t_n) and $p(t)$ in (A.1) and (A.2).

ASSUMPTION A.2.1. We make the following assumptions on $U_{t_n} X_k$;

(i) (X_k) are independent, unbounded random variables;

(ii) $(Y_{t_n}^{(k)})_{k=1}^{n}$ are mutually independent and independent of (X_k);

(iii) The triangular array $(U_{t_n}^{(k)} X_k)$ is uan;

(iv) The norming sequence (t_n) is non-decreasing. □

ASSUMPTION A.2.2. We make the following assumptions on f_t;

(i) $S = (\{f_t\}_{t \in [0,\infty)}, \circ)$, is a commutative composition semigroup with respect to the semigroup $([0,\infty), \oplus)$, i.e.,

for all $t,s \in [0,\infty)$; $f_t \circ f_s = f_{t \oplus s}$;

for all $t \in [0,\infty)$; $f_0 \circ f_t = f_t = f_t \circ f_0$;

(ii) $\{f_t\}_{t \in [0,\infty)}$ are shrinking operators, i.e.,

for all $t,s \in [0,\infty)$ with $t > s$, $|f_t(x)| < |f_s(x)|$, $x \in \mathbb{R}$;

for all $x \in \mathbb{R}$, $\lim_{t \to \infty} f_t(x) = 0$;

(iii) $f_t(x)$ is continuous in both t and x, unbounded in x and for any interval I, not containing zero, $f_t(f_t^{-1}(I)) = I$. □

ASSUMPTION A.2.3. We make the following assumptions on $p(t)$;

(i) p is a semigroup homomorphism from $([0,\infty),\oplus)$ to $((0,1],\cdot)$, i.e.,

$$p(s\oplus t)=p(s)p(t) \text{ for all } t,s\in[0,\infty). \qquad \Box$$

Assumption A.2.1 (iii) ensures that the limit in (A.1) exists, namely an infinitely divisible random variable. Part (i) is assumed, since if (X_k) was bounded and $t_n\to\infty$ with $f_\infty\equiv 0$, then (A.1) would have a degenerate limit. Requirement (iv) is equivalent with the assumption that (t_n) be monotone ((t_n) non-increasing implies that (s_n) with $s_n=t_n^{-1}$ is non-decreasing) which is a normal assumption in central limit problems.

Assumption A.2.2 (ii) states that f_t in fact does shrink its arguments and provides an ordering of $\{f_t\}$. Conditions (i) and (iii) of Assumption A.2.2 as well as Assumption A.2.3 are essential for solving the limit problem on hand.

As a consequence of our assumptions we have the following lemma.

LEMMA A.2.4. *Let $\{f_t\}_{t\in[0,\infty)}$ and $p(t)$ satisfy Assumptions A.2.2 and A.2.3, respectively. Then*

(i) *For all $s,t\in[0,\infty)$ with $t\ge s$ and any interval I not containing zero:*

$$f_t\circ f_s^{-1}(I)=f_{t\ominus s}(I) \text{ with } t\ominus s\in[0,\infty);$$

(ii) *$p(t)$ is continuous, non-increasing and non-zero on $[0,\infty)$ with $p(0)=1$ and for all $t,s\in[0,\infty)$ with $t\ge s$*

$$p(t\ominus s)=p(t)/p(s).$$

PROOF. We first prove part (i). Let $s,t\in[0,\infty)$ be arbitrary with $t\ge s$. We can always select a $y\in[0,\infty)$ such that

$$|f_{s\oplus y}(x)|\ge|f_t(x)|,$$

for all $x\in\mathbb{R}$ and $s\oplus y\le t$. By Assumption A.2.2 (ii) it is impossible to have $w\le t$ and $|f_w(x_1)|>|f_t(x_1)|$ and $|f_w(x_2)|<|f_t(x_2)|$ for some $x_1,x_2\in\mathbb{R}$. Hence, by the continuity of f_t in t, there exists a unique $y\in[0,\infty)$ such that $f_{y\oplus s}=f_t$. We therefore have

$$f_t\circ f_s^{-1}(x)=f_y(x) := f_{t\ominus s}(x),$$

for any $x\neq 0$.

For part (ii) let $y=t\ominus s$ and hence $t=y\oplus s$. Observe that (cf. Assumption A.2.3)

$$p(t \ominus s) p(s) = p(y \oplus s) = p(t). \qquad \square$$

If $p(t_n) \to c > 0$ as $n \to \infty$, then upon replacing X_k in (A.1) by $Y_k = ZX_k$ with $\mathbb{P}(Z=0) = 1 - \mathbb{P}(Z=1) = 1-c$ and $p(t)$ by $p^*(t) \equiv 1$, we obtain the same limit problem. We therefore make the following and last assumption.

ASSUMPTION A.2.5. $p(t)$ is either constantly one or $p(t)$ is non-increasing with $\lim_{t \to \infty} p(t) = 0$. $\square$

We conclude this preliminary section with a lemma and a notation.

LEMMA A.2.6. *If* $F_n \xrightarrow{w} F$ *and* $t_n \to t$ *as* $n \to \infty$, *then* $U_{t_n} F_n \xrightarrow{w} U_t F$ *as* $n \to \infty$.

PROOF. Let $x_n \to x$, then by the continuity of $f_t(x)$ in both t and x, $U_{t_n} x_n \to U_t x$. The lemma now follows from Theorem 5.5 in Billingsley (1968). $\square$

NOTATION A.2.7. Let $U(S,p)$ denote the set of characteristic functions whose random variables can be described as limits of the form (A.1), under Assumptions A.2.1, A.2.2, A.2.3 and A.2.5. $\square$

A.3 A characterization of U(S,p)

Let the operators U_t and U_t^{-1} act on set functions by

$U_t F(B) = p(t) F(f_t^{-1} B)$ and $U_t^{-1} F(B) = p(t)^{-1} F(f_t B)$.

Let $t,s \in [0,\infty)$ with $t \geq s$. From Assumptions A.2.2 (i) and Lemma A.2.4 we have

$$\begin{aligned} U_t U_s^{-1} F(B) &= p(t)/p(s) F(f_s \circ f_t^{-1} B) = p(t \ominus s) F((f_t \circ f_s^{-1})^{-1} B) \\ &= p(t \ominus s) F(f_{t \ominus s}^{-1} B) = U_{t \ominus s} F(B), \end{aligned} \tag{A.5}$$

and that (cf. (A.5))

$$\begin{aligned} U_t F(B) &= p(t) F(f_s \circ f_s^{-1} \circ f_t^{-1} B) = p(t) p(s)/p(s) F(f_s \circ f_t^{-1} \circ f_s^{-1} B) \\ &= p(s) U_t U_s^{-1} F(f_s^{-1} B) = p(s) U_{t \ominus s} F(f_s^{-1} B) \\ &= U_s U_{t \ominus s} F(B) = U_{t \ominus s} U_s F(B). \end{aligned} \tag{A.6}$$

Before proving the main theorem of this section, we prove a preparatory lemma.

LEMMA A.3.1. *Let ϕ be infinitely divisible with $\phi=[a_\phi, \sigma_\phi^2, M]$. If $\phi \in U(S,p)$ and if M is not identically zero, then*

(i) $t_n \to \infty$ *as* $n \to \infty$;

(ii) $\hat{t}_n := t_{n+1} \ominus t_n \to 0$ *as* $n \to \infty$.

PROOF. We first prove part (i). Suppose (i) is not true, then there exists $t_0 \in [0,\infty)$ such that $t_n \le t_0 < \infty$ for all $n \in \mathbb{N}_+$. By Assumptions A.2.2 (ii) and A.2.5 we have for each $\varepsilon > 0$,

$$\mathbb{P}(|U_{t_n}^{(k)} X_k| \ge \varepsilon) = p(t_n)\mathbb{P}(|f_{t_n}(X_k)| \ge \varepsilon) \ge p(t_0)\mathbb{P}(|f_{t_0}(X_k)| \ge \varepsilon).$$

Since $(U_{t_n}^{(k)} X_k)$ is uan, then

$$\sup_k \mathbb{P}(|f_{t_0}(X_k)| \ge \varepsilon) = 0.$$

From Assumption A.2.2 (iii), f_t is unbounded and hence (X_k) is bounded. This contradicts Assumption A.2.1 (i). Hence (i) holds.

To prove part (ii) suppose $(\hat{t}_n)$ has a limit point $t_0 \in (0,\infty)$. Then there exists a subsequence $(k(n))$ such that $\hat{t}_{k(n)} \to t_0$. By Theorem 1.3.12 it follows that outside every neighbourhood of the origin,

$$M_n := \sum_{k=1}^{n} U_{t_n} F_k \xrightarrow{w} M \text{ as } n \to \infty. \tag{A.7}$$

Let I be a continuity set of M, with I and interval bounded away from the origin. From (A.7) and (A.6) it follows that

$$\begin{aligned} M_{k(n)+1}(I) &= U_{t_{k(n)+1}} F_{k(n)+1}(I) + \sum_{k=1}^{k(n)} U_{t_{k(n)+1} \ominus t_{k(n)}} U_{t_{k(n)}} F_k(I) \\ &= U_{t_{k(n)+1}} F_{k(n)+1}(I) + U_{\hat{t}_{k(n)}} M_{k(n)}(I). \end{aligned} \tag{A.8}$$

Letting $n \to \infty$ in (A.8) we obtain, by Lemma A.2.6, (A.7) and the uan property of $(U_{t_n}^{(k)} X_k)$ that

$$M(I) = U_{t_0} M(I). \tag{A.9}$$

By repeated use of (A.9) we see that

$$M(I) = U_{t_0^{\oplus k}} M(I) \le M(f_{t_0^{\oplus k}}^{-1} I),$$

for $k \in \mathbb{N}_+$. Applying Assumption A.2.2 (ii) yields $M(I)=0$, which contradicts the assumption of the lemma. □

THEOREM A.3.2. *Let ϕ be infinitely divisible with $\phi=[a_\phi, 0, M]$. Then $\phi \in U(S,p)$ if and only if for every $t \in (0,\infty)$ and for every Borel set B of $\mathbb{R}\backslash(-\varepsilon, \varepsilon)$, for any $\varepsilon > 0$, the*

Lévy spectral function M satisfies

$$M(B) \geq U_t M(B).$$

PROOF. We first prove the 'only if' part. If M vanishes identically then the 'only if' part is trivial. Suppose M is not identically zero. From Lemma A.3.1 it follows that for any $t \in (0,\infty)$ there exists a subsequence $(k(n))$ such that $k(n) \geq n$ and

$$\tilde{t}_n := t_{k(n)} \ominus t_n \to t \text{ as } n \to \infty . \tag{A.10}$$

For any continuity set I of M, with I an interval bounded away from the origin, we have (cf. (A.7) and (A.6))

$$\begin{aligned} M_{k(n)}(I) &= \sum_{k=1}^{k(n)} U_{t_{k(n)}} F_k(I) \geq \sum_{k=1}^{n} U_{t_{k(n)} \ominus t_n} U_{t_n} F_k(I) \\ &= \sum_{k=1}^{n} U_{\tilde{t}_n} U_{t_n} F_k(I) = U_{\tilde{t}_n} M_n(I) . \end{aligned}$$

Letting $n \to \infty$ we have by (A.7), Lemma A.2.6 and (A.10) that M satisfies the inequality of the theorem for $B = I$. Since the Borel sets on $\mathbb{R} \backslash (-\varepsilon, \varepsilon)$ are generated by the intervals bounded away from zero, the 'only if' part is proved.

Conversely, suppose M satisfies the inequality of the theorem. If M vanishes identically then the 'if' part is trivial. Suppose M does not vanish identically. Suppose (r_k) is a non-decreasing sequence satisfying conditions (i) and (ii) of Lemma A.3.1 and such that $p(r_k)^{-1} \in \mathbb{N}_+$ (this is possible by Assumption A.2.5). Define L_s by

$$L_s = M - U_s M .$$

Let $s_k = r_k \ominus r_{k-1}$, $k = 2, 3, \ldots$ and $s_1 = \infty$. Then $L_{s_1} = M$ and (cf. (A.7) and (A.5))

$$\sum_{k=1}^{n} U_{r_n} U_{r_k}^{-1} L_{s_k} = M . \tag{A.11}$$

We now approximate $(f_{r_k}^{-1} L_{s_k})$ by distribution functions (F_k) and proceed to show that $(U_{r_n} F_k)$ satisfies the conditions of Theorem 1.3.12.

Define (ε_k) by

$$\varepsilon_k = \inf \{ \varepsilon > 0 \mid L_{s_k}(\{ |x| > \varepsilon \}) \leq 1 \} \ , k \in \mathbb{N}_+ ,$$

and (η_k) by $\eta_k = L_{s_k}(|x| > \varepsilon_k)$, $k \in \mathbb{N}_+$. Also let

$$F_k^*(B) = f_{r_k}^{-1} L_{s_k}(B \cap \{ |x| > \varepsilon_k \}) + (1 - \eta_k)\delta_0(B) \ , k \in \mathbb{N}_+ ,$$

for any Borel set B of $\mathbb{R}$, with δ_0 a distribution function with total mass at zero. Finally, let $k(0) = 0$, $k(n) = \sum_{k=1}^{n} p(r_k)^{-1}$ and

$$t_l = r_n \text{ for } k(n-1) < l \leq k(n) , n \in \mathbb{N}_+ ;$$

$$F_l = F_n^* \text{ for } k(n-1) < l \leq k(n) , n \in \mathbb{N}_+ .$$

Observe that $\varepsilon_k \to 0$ as $n \to \infty$, since, if $\varepsilon_k \geq \varepsilon_0 > 0$, then $L_{s_k}(\{ |x| > \varepsilon_0 \}) \geq 1$ and so (cf. Lemma A.3.1 (ii))

$$1 \leq \lim_{k\to\infty} L_{s_k}(\{ |x| > \varepsilon_0 \}) = M(\{ |x| > \varepsilon_0 \}) - \lim_{k\to\infty} U_{s_k} M(\{ |x| > \varepsilon_0 \}) = 0 .$$

For every $n \in \mathbb{N}_+$ we can choose an $N \in \mathbb{N}_+$ such that $k(N-1) < n \leq k(N)$. Thus

$$\begin{aligned} M_n &:= \sum_{l=1}^{n} U_{t_n} F_l = \sum_{l=1}^{k(N)} U_{t_n} F_l - \sum_{l=n+1}^{k(N)} U_{t_n} F_l \\ &= \sum_{k=1}^{N} p(r_k)^{-1} U_{r_N} F_k^* - (k(N) - n) U_{r_N} F_N^* . \end{aligned} \tag{A.12}$$

The uan property. For every $\varepsilon > 0$, define the sequence $(j(n))$ by

$$\sup_{1\leq k\leq n} U_{t_n} F_k(\{ |x| > \varepsilon \}) = U_{t_n} F_{j(n)}(\{ |x| > \varepsilon \}).$$

If $(j(n))$ is bounded, then by Lemmas A.2.6 and A.3.1

$$\lim_{n\to\infty} \sup_{1\leq k\leq n} U_{t_n} F_k(\{ |x| > \varepsilon \}) = 0 . \tag{A.13}$$

Suppose $(j(n))$ is unbounded. Observe that $F_{j(n)} = F^*_{i(n)}$ for some $i(n) \leq N$. Hence by the definition of F_k and Lemma A.2.4 (i),

$$U_{t_n} F_{j(n)}(\{ |x| > \varepsilon \}) \leq U_{r_N} f^{-1}_{r_{i(n)}} L_{i(n)}(\{ |x| > \varepsilon \}) = p(r_N) f_{r_N \ominus r_{k(n)}} L_{i(n)}(\{ |x| > \varepsilon \}) .$$

From Assumption A.2.2 (ii) and the unboundedness of f_t we have that

$$f^{-1}_{r_N \ominus r_{k(n)}}(\{ |x| > \varepsilon \}) \subseteq (\{ |x| > \varepsilon \}) .$$

Hence

$$U_{t_n} F_{j(n)}(\{ |x| > \varepsilon \}) \leq p(r_N) L_{i(n)}(\{ |x| > \varepsilon \}) .$$

Since $i(n) \to \infty$ as $n \to \infty$ and so $s_{i(n)} \to \infty$ as $n \to \infty$, $L_{s_{i(n)}} \to 0$. Formula (A.13) is thereby satisfied. The triangular array $(U_{t_n} F_k)$ is therefore uan.

Condition (i) of Theorem 1.3.12. Let I be a continuity set of M with I an interval bounded away from the origin. Since $\varepsilon_k \to 0$ as $k \to \infty$ there exists a k_0 such that $\varepsilon_k \notin I$ for all $k > k_0$. Observe that (cf. (A.12), the definition of F_k^*, (A.5) and (A.6))

$$\begin{aligned} M_{k(N)}(I) &= \sum_{k=1}^{k_0} p(r_k)^{-1} U_{r_N} F_k^*(I) + \sum_{k=k_0}^{N} p(r_k)^{-1} U_{r_N} f^{-1}_{r_k}(M - L_{s_k} M)(I) \\ &= \sum_{k=1}^{k_0} p(r_k)^{-1} U_{r_N} F_k^*(I) + \sum_{k=k_0}^{N} U_{r_N} U^{-1}_{r_k} M(I) - \sum_{k=k_0}^{N} U_{r_N \ominus r_k} U_{r_k} U^{-1}_{r_{k-1}} M(I) \\ &= \sum_{k=1}^{k_0} p(r_k)^{-1} U_{r_N} F_k^*(I) + M(I) - U_{t_N \ominus t_{k_0 - 1}} M(I) . \end{aligned}$$

Letting $N \to \infty$ we obtain

$$M_{k(N)} \xrightarrow{w} M \text{ as } N \to \infty \tag{A.14}$$

outside every neighbourhood of the origin. Observe that

$$(k(N)-n)U_{r_N}F_N^*(I) \le p(r_n)^{-1}U_{r_N}F_N^*(I) \le L_{s_N}(I)\,. \tag{A.15}$$

Since $L_{s_N}(I) \to 0$ as $N \to \infty$ we have by (A.12), (A.14) and (A.15), that $M_n \xrightarrow{w} M$ outside every neighbourhood of the origin. Hence condition (ii) of Theorem 1.3.12 is satisfied.

Condition (ii) of Theorem 1.3.12. By the definition of F_k we have (cf. (A.12) and (A.11))

$$\begin{aligned}
\sum_{l=1}^{n} \int_{|x|\le\varepsilon} x^2\, dU_{t_n}F_l(x) &\le \sum_{k=1}^{N} \int_{|x|\le\varepsilon} x^2\, dp(r_k)^{-1}U_{r_N}F_k^*(x) \\
&\le \sum_{k=1}^{N} \int_{|x|\le\varepsilon} x^2\, dU_{r_N}U_{r_k}^{-1}L_{s_k}(x) \\
&= \int_{|x|\le\varepsilon} x^2\, dM(x)\,.
\end{aligned}$$

Letting $\varepsilon \to \infty$, we see that condition (ii) of Theorem 1.3.12 is met. □

In the 'only if' part of the proof of Theorem A.3.2 it was not necessary to assume that $\sigma_\phi^2 = 0$. We therefore have

COROLLARY 1. *If $\phi = [a_\phi, \sigma_\phi^2, M]$ and $\phi \in U(S, p)$, then M satisfies the inequality of Theorem A.3.2.*

A.4 Stability in U(S,p)

In this section we consider (A.1) with the added assumption that the sequence (X_k) is identically distributed. We call such limit distributions $U(S, p)$-stable.

THEOREM A.4.1. *Let ϕ be infinitely divisible with $\phi = [a_\phi, 0, M]$. Then ϕ is $U(S, p)$-stable if and only if there exists a semigroup homomorphism g from $([0,\infty), \oplus)$ to $([1,\infty), \cdot)$, such that for every $t \in (0,\infty)$ and for every Borel set B of $\mathbb{R}\backslash(-\varepsilon,\varepsilon)$, for any $\varepsilon > 0$, the Lévy spectral function M satisfies*

$$M(B) = g(t)\,U_t M(B)\,. \tag{A.16}$$

PROOF. We first prove the 'only if' part. If M vanishes identically then the 'only if' part is trivial. Suppose M is not identically zero. From Lemma A.3.1 it follows that for any $t \in (0,\infty)$ there exists a subsequence $(k(n))$ such that $k(n) \geq n$ and (A.10) is satisfied. For any continuity set I of M, with I an interval bounded away from the origin, we have (cf. (A.6))

$$M_{k(n)}(I) := k(n) U_{t_{k(n)}} F(I) = k(n)/n \cdot U_{\tilde{t}_n} M_n(I) .$$

Letting $n \to \infty$ we have by (A.7), Lemma A.2.6 and (A.10) that both $M_{k(n)}$ and $U_{\tilde{t}_n} M_n$ converge and hence $k(n)/n$ must converge. Let $g(t) = \lim_{n \to \infty} k(n)/n$. Then M satisfies (A.16) with $B = I$. Since the Borel sets on $\mathbb{R} \backslash (-\varepsilon, \varepsilon)$ are generated by the intervals bounded away from zero, (A.16) holds for all Borel sets B.

Suppose there is a $t_0 > 0$ such that $g(t_0) \leq 1$. By repeated use of (A.16) we see that

$$M(B) \leq U_{t^{\oplus k}} M(B) = p(t^{\oplus k}) f_{t^{\oplus k}} M(B),$$

for $k \in \mathbb{N}_+$. Applying Assumption A.2.2 (ii) we have $M \equiv 0$. Hence $g(t) > 1$ on $(0, \infty)$. Observe that $M(B) = g(t \oplus s)\, U_{t \oplus s}(B)$ and

$$M(B) = g(t) M(B) = g(t) g(s)\, U_s U_t M(B) = g(s) g(t)\, U_{t \oplus s} M(B) .$$

Hence g is a semigroup homomorphism from $([0,\infty),\oplus)$ to $([1,\infty),\cdot)$.

Conversely, suppose M satisfies (A.16). If M vanishes identically then the 'if' part is trivial. Suppose M does not vanish identically. Let (t_n) be a non-decreasing sequence satisfying conditions (i) and (ii) of Lemma A.3.1 and such that $g(t_n) = n$ (this is possible since g is unbounded and continuous). Define $\bar{\varepsilon}$ by

$$\bar{\varepsilon} = \inf \{ \varepsilon > 0 \mid M(\{ |x| > \varepsilon \}) \leq 1 \},$$

and set $\eta = M(\{ |x| > \bar{\varepsilon}\})$. Define the probability distribution F by

$$F(B) = M(B \cap \{ |x| > \bar{\varepsilon} \}) + (1-\eta)\delta_0(B),$$

for any Borel set B of $\mathbb{R}$. We now show that $(U_{t_n} F)$ satisfies the conditions of Theorem 1.3.12.

Condition (i) of Theorem 1.3.12. Let I be an interval bounded away from the origin. For any a there exists an N such that for all $n \geq N$

$$f_{t_n}^{-1}(I) \cap \{ |x| > a \} = f_{t_n}^{-1}(I) .$$

If this were not true then $f_{t_n}^{-1}(I) \supset B$ for all n large and for some non-empty Borel set B contained in $\{ |x| \leq a\}$. Hence $I = f_{t_n} f_{t_n}^{-1}(I) \supset f_{t_n}(B)$. By Assumption A.2.2 (ii) for any $x \in \mathbb{R}$, $f_{t_n}(x) \to 0$ as $n \to \infty$, contradicting the assumption that I is bounded away from the origin. Hence for n large

$$\begin{aligned} n U_{t_n} F(I) &= g(t_n) p(t_n) (M(\{ |x| > \bar{\varepsilon} \} \cap f_{t_n}^{-1}(I)) + (1-\eta)\delta_0(f_{t_n}^{-1}(I))) \\ &= g(t_n) U_{t_n} M(I) = M(I) . \end{aligned} \tag{A.17}$$

Condition (i) of Theorem 1.3.12 is thus satisfied.

The uan property. For any $n \geq N$ and any $\varepsilon > 0$ we have by (A.17)

$$\sup_{1 \leq k \leq n} U_{t_n} F(\{ |x| \geq \varepsilon \}) = n^{-1} M(\{ |x| \geq \varepsilon \}).$$

Letting $n \to \infty$ we see that $(U_{t_n} F)$ is uan.

Condition (ii) of Theorem 1.3.12. Finally, observe that for $n \geq N$, (cf. (A.17))

$$\sum_{k=1}^{n} \int_{|x| \leq \varepsilon} x^2 \, dU_{t_n} F(x) = \int_{|x| \leq \varepsilon} x^2 \, dM(x).$$

Condition (ii) of Theorem 1.3.12 is now met upon letting $n \to \infty$ and $\varepsilon \to 0$. □

A.5 Some subsets of U(S,p)

Let the stochastic 'breeding operator' B_t be defined by

$$B_t X \overset{d}{=} \sum_{k=1}^{[q(t)]} U_t^{(k)} X_k,$$

where $X_1, \ldots, X_{[q(t)]}$ are independent random variables, all distributed as X and $q(t)$ is a semigroup homomorphism from $([0,\infty), \oplus)$ to $([1,\infty), \cdot)$. We are interested in the random variables X which have the form

$$\sum_{k=1}^{n} B_{t_n}^{(k)} X_k + b_n \overset{w}{\to} X \text{ as } n \to \infty, \tag{A.18}$$

where (b_n) is a sequence of real numbers, $(B_{t_n}^{(k)} X_k)$ are uan, (X_k) are independent and unbounded, $B_{t_n}^{(k)}$ are independent and identically distributed, and all independent of (X_k) and (t_n) is non-decreasing. We therefore introduce the notation

NOTATION A.5.1. Let $B(S, p, q)$ denote the set of characteristic functions whose random variables can be described as limits of the form (A.18). □

Let F_k be the distribution function of X_k. Then

$$M_n^* := \sum_{k=1}^{n} \sum_{l=1}^{[q(t)]} U_{t_n} F_k = [q(t)] \sum_{k=1}^{n} U_{t_n} F_k. \tag{A.19}$$

As in the proof of Lemma A.3.1 (i) we see that the uan property of $(B_{t_n} X_k)$ implies that $t_n \to \infty$ as $n \to \infty$. If $q(t) \to c < \infty$ as $t \to \infty$, then $c = \lim_{n\to\infty} q(t^{\oplus n}) = cq(t)$ and hence $q(t) = 1$ for all t. Hence assume without loss of generality that

$$q(t) \to \infty \text{ as } t \to \infty.$$

It follows from (A.19) and Theorem 1.3.12, that M_n^* tends to a Lévy spectral function

M and so

$$\sum_{k=1}^{n} U_{t_n} F_k \to 0 \text{ as } n \to \infty,$$

outside every neighbourhood of the origin. Therefore M_n^* and M_n with

$$M_n := q(t_n) \sum_{k=1}^{n} U_{t_n} F_k ,$$

have the same limits. The proof of Theorem A.3.2 can now easily be adapted to prove

THEOREM A.5.2. *Let ϕ be infinitely divisible with $\phi = [a_\phi, 0, M]$. Then $\phi \in B(S, p, q)$ if and only if there exists a semigroup homomorphism q from $([0,\infty), \oplus)$ to $([1,\infty), \cdot)$, such that for every $t \in (0,\infty)$ and for every Borel set B of $\mathbb{R}\backslash(-\varepsilon,\varepsilon)$, for any $\varepsilon > 0$, the Lévy spectral function M satisfies*

$$M(B) \geq q(t)\, U_t M(B) .$$

A.6 Examples

We conclude this appendix with a few examples. A special case of Example A.6.1 yields the shrinking operator $T_{\alpha,t}$ of Section 6.4.1 and of Example A.6.2, the operator $U_{\alpha,t}$ considered in Section 6.4.2.

EXAMPLE A.6.1.

$$f_t(x) = t^\gamma x , \quad t \in (0,1] , \ \gamma > 0,$$

$$p(t) = t^\alpha , \quad t \in (0,1] , \ \alpha \geq 0,$$

$$s \oplus t = s\, t .$$

Hence $U_t = T_{\alpha/\gamma, t^\gamma}$. □

EXAMPLE A.6.2.

$$f_t(x) = \operatorname{sgn}(x)\,(|x|^\alpha - t^\alpha)_+^{1/\alpha} , \quad t \in [0,\infty) , \ \alpha > 0,$$

$$p(t) = e^{-pt^\alpha} , \quad t \in [0,\infty),$$

$$s \oplus t = (s^\alpha + t^\alpha)^{1/\alpha} .$$

Hence $U_t = U_{\alpha,t}$ if $p = 0$. □

EXAMPLE A.6.3.

$f_t(x) = \operatorname{sgn}(x)\,[(|x|^{\alpha}+1)^t - 1]^{1/\alpha}\,, \quad t \in (0,1]\,,\ \alpha > 0\,,$

$p(t) = t^{\gamma}\,, \quad t \in (0,1]\,,\ \gamma \geq 0\,,$

$s \oplus t = s\,t.$ □

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