CWI Tract 68

Matrix and operator extensions

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CONTENTS

INTRODUCTION

0.1. Extension problems. This book concerns extension problems for linear operators. They can be described roughly as follows. Given a part of an operator, find, if possible, the complementary part such that the full operator has certain prescribed properties. To pose the problems in a more precise way, consider vector spaces X and Y and let Q be a projection on $\mathscr{L} = \mathscr{L}(X,Y)$, the space of linear operators acting from X to Y. Assume that A is in the image of Q, and let $\mathscr{E}$ be the set of operators in $\mathscr{L}$ satisfying a certain property, (P) say. The general problem is to describe the set

$$\mathscr{S}_{\mathscr{E},Q}(A) := \mathscr{E} \cap \{ B \in \mathscr{L} \mid Q(B) = A \}.$$

This set of solutions, $\mathscr{S}_{\mathscr{E},Q}(A)$, consists of all operators B that have the property (P) and that are equal to A on the part which is left invariant by Q. An element B in $\mathscr{S}_{\mathscr{E},Q}(A)$ is called an *extension* of A with property (P). The phrase "describe the set" concerns questions like the following. For which A is the set of solutions non-empty? When is $\mathscr{S}_{\mathscr{E},Q}(A)$ a singleton? Give a description of $\mathscr{S}_{\mathscr{E},Q}(A)$ in terms of the given part A, etc.

0.2. Three classes. The setting described in the previous subsection is very general, and a large variety of problems can be put in this context. In this book we treat three classes of problems of this type, namely:

- positive extension problems;
- strictly contractive extension problems;
- minimal rank extension problems.

In the positive extension problems that we shall treat, the operators act on $X = Y = \mathbb{C}^n$ or on $X = Y = l_2$, the Hilbert space of square summable sequences of complex numbers, and hence they may be represented as finite $n \times n$ or semi-infinite matrices $B = \left[b_{ij} \right]$. The given part is a symmetric band consisting of $2p + 1$ diagonals, say, centered around the main diagonal and the extensions are required to be positive definite. In other words, $\mathscr{E}$ is the set of all positive definite operators and the projection Q is given by

$$Q(B) = C = \left[c_{ij} \right] , c_{ij} = \begin{cases} 0 & \text{if } |j-i| > p, \\ b_{ij} & \text{if } |j-i| \leq p. \end{cases}$$

In our contractive extension problem the operators again act on $X = Y = \mathbb{C}^n$ or on $X = Y = l_2$. But now the given part is of triangular form and consists of all diagonals below a given diagonal, the q-th say. The extensions are required to have operator norm

less than one. Thus Q is the projection defined by

$$Q(B) = C = \left[c_{ij} \right] \ , c_{ij} = \begin{cases} 0 & \text{if } j - i > q, \\ b_{ij} & \text{if } j - i \leqq q, \end{cases} \tag{0.1}$$

and $\mathscr{E}$ is the set of all strictly contractive operators.

In the minimal rank extension problems we consider sets $\mathscr{E}_n \subset \mathscr{L}$ defined by

$$\mathscr{E}_n = \{ B \in \mathscr{L} \mid \operatorname{rank} B \leqq n \ \}.$$

The given parts are of triangular form, and thus the projection Q may be of the form (0.1). Here we shall also consider integral operators acting on $X = Y = L_2[0,1]$, the usual Lesbesgue space of square integrable functions defined on $[0,1]$, and in that case the projection Q is defined by $Q(B) = C$, where

$$B : L_2[0,1] \rightarrow L_2[0,1] \ \ ; (B\phi)(t) = \int_0^1 k(t,s)\phi(s)ds \ , 0 \leqq t \leqq 1 \ ,$$

$$C : L_2[0,1] \rightarrow L_2[0,1] \ \ ; (C\phi)(t) = \int_0^t k(t,s)\phi(s)ds \ , 0 \leqq t \leqq 1 \ .$$

The minimal rank extension problems we shall deal with ask to determine for a given A in the image of Q the smallest number $\ell = \ell(A)$ for which $\mathscr{S}_{\mathscr{E}_\ell,Q}(A) \neq \varnothing$ and to describe the set $\mathscr{S}_{\mathscr{E}_\ell,Q}(A)$. In other words, given a lower triangular operator A we want to determine the smallest possible rank of an extension of A and all minimal rank extensions of A.

We shall not restrict the attention to scalar matrices only, but we allow the entries to be matrices or, more generally, operators acting on Hilbert spaces. Positive and strictly contractive extension problems also appear in the context of integral operators (see H. Dym and I. Gohberg [24]). The general method for dealing with positive and strictly contractive extension problems developed in this thesis may also be applied to such operators. Our problems also concern special subclasses of operators. E.g., as a variant of the positive extension problem we shall consider the case when the operators in $\mathscr{E}$ are required to be (block) Toeplitz operators, i.e., $b_{ij} = b_{i+1,j+1}$ for all i and j. In the latter form the positive extension problem may be rephrased as an extension problem concerning (matrix- or operator-valued) functions on the unit circle.

In what follows we treat the above problems in more detail, and we describe some of our main results.

0.3. Positive extension problems. Consider the following problem. Let B_{ij} be given

matrices for $|j-i| \leqq q$. Find the remaining matrices B_{ij}, $|j-i| > q$, such that the block matrix $B = \left[B_{ij} \right]_{i,j=1}^{n}$ is positive definite (shortly: $B > 0$). This problem was introduced by H. Dym and I. Gohberg [23] who proved that a solution exists if and only if

$$\begin{bmatrix} B_{ii} & \cdots & B_{i,i+q} \\ \vdots & & \vdots \\ B_{i+q,i} & \cdots & B_{i+q,i+q} \end{bmatrix} > 0 , i = 1,...,n-q. \tag{0.2}$$

In a paper by J.A. Ball and I. Gohberg [5] a finite dimensional version of the shift invariant subspace approach of J.A. Ball and J.W. Helton [6] was used to derive a full parametrization of the set of all solutions via a linear fractional representation. The coefficients in this linear fractional map are determined using a theorem of Beurling-Lax type. In this book we shall present two other methods to obtain such a linear fractional representation and, moreover, we shall give explicit formulas for the coefficients in this linear fractional map in terms of the given data. The following two theorems are among our main results.

THEOREM 0.1. *Let* $B_{ij} = B_{ji}^{*}$ *be given matrices for* $|j-i| \leqq q$ ($\leqq n-1$), *and assume that condition (0.2) is satisfied. For* $0 < j-i \leqq q$ *define the matrix* Z_{ij} *by*

$$Z_{ij} = -\beta_{ij}\gamma_{ij}^{-1}\varsigma_{ij} ,$$

where β_{ij}, γ_{ij} *and* ς_{ij} *are given via the partitioning*

$$\left[B_{rs} \right]_{r,s=i}^{j} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} & B_{ij} \\ \beta_{ij}^{*} & \gamma_{ij} & \varsigma_{ij} \\ B_{ij}^{*} & \varsigma_{ij}^{*} & \eta_{ij} \end{bmatrix} .$$

Put

$$\Lambda_p = \left[\delta_{i+p,j}(B_{ij}+Z_{ij}) \right]_{i,j=1}^{n} , p = 1,...,q ,$$

and let $\Delta_0, \ldots, \Delta_q$ *be defined recursively by*

$$\Delta_0 = \text{diag} \left[\tfrac{1}{2}B_{ii} \right]_{i=1}^{n} , \Delta_1 = \tfrac{1}{2}\Delta_0^{-\frac{1}{2}}\Lambda_1\Delta_0^{-\frac{1}{2}} ,$$

$$\Delta_p = \tfrac{1}{2}\overset{\leftarrow p-1}{\underset{k=1}{\Pi}}(I-\Delta_k\Delta_k^{*})^{-\frac{1}{2}}\Delta_0^{-\frac{1}{2}}\Lambda_p\Delta_0^{-\frac{1}{2}}\overset{\rightarrow p-1}{\underset{k=1}{\Pi}}(I-\Delta_k^{*}\Delta_k)^{-\frac{1}{2}} , p = 2,...,q.$$

Put

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \Delta_0^{1\!/\!2} & \Delta_0^{1\!/\!2} \\ -\Delta_0^{-1\!/\!2} & \Delta_0^{-1\!/\!2} \end{bmatrix} \overset{\to}{\underset{p=1}{\Pi}}^q \begin{bmatrix} (I-\Delta_p\Delta_p^*)^{-1\!/\!2} & (I-\Delta_p\Delta_p^*)^{-1\!/\!2}\Delta_p \\ (I-\Delta_p^*\Delta_p)^{-1\!/\!2}\Delta_p^* & (I-\Delta_p^*\Delta_p)^{-1\!/\!2} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} .$$

Then the set of all $n \times n$ *block matrices* $F = \begin{bmatrix} F_{ij} \end{bmatrix}_{i,j=1}^n$ *with* $F_{ij} = B_{ij}$, $-q \leq j-i \leq q$, *and* $F > 0$ *is the set of all matrices of the form*

$$F = (\alpha G + \beta)^{*-1}(I - G^*G)(\alpha G + \beta)^{-1},$$

where $G = \begin{bmatrix} G_{ij} \end{bmatrix}_{i,j=1}^n$ *is any strictly contractive block matrix with* $G_{ij} = 0$, $j-i \leq q$. *The correspondence is 1-1. Moreover,*

$$\det F = (\overset{n}{\underset{i=1}{\Pi}} \det B_{ii}) (\underset{0<p\leq q}{\Pi} \det(I-\Delta_p^*\Delta_p)) \det(I-G^*G).$$

Here $\overset{\to}{\underset{p=i}{\Pi}}^j H_p = H_i \cdot\cdot\cdot H_j$ *and* $\overset{\leftarrow}{\underset{p=i}{\Pi}}^j H_p = H_j \cdot\cdot\cdot H_i$ *for* $i \leq j$, *and these* matrix products are defined to be the identity matrix for $i > j$. *When a matrix has a zero* number of rows or columns the formulas have to be interpreted in the usual way.

THEOREM 0.2. *Let* $B_{ij} = B_{ji}^*$ *be given matrices for* $|j-i| \leq q$ $(\leq n-1)$, *and* *assume that condition* (0.2) *is satisfied. For* $p = 1,...,n$ *let*

$$\begin{bmatrix} Y_{pp} \\ \vdots \\ Y_{\beta(p),p} \end{bmatrix} = \begin{bmatrix} B_{pp} & \cdots & B_{p,\beta(p)} \\ \vdots & & \vdots \\ B_{\beta(p),p} & \cdots & B_{\beta(p),\beta(p)} \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} ,$$

and

$$\begin{bmatrix} X_{\gamma(p),p} \\ \vdots \\ X_{p,p} \end{bmatrix} = \begin{bmatrix} B_{\gamma(p),\gamma(p)} & \cdots & B_{\gamma(p),p} \\ \vdots & & \vdots \\ B_{p,\gamma(p)} & \cdots & B_{pp} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} ,$$

where $\beta(p) = \min\{n,p+q\}$ *and* $\gamma(p) = \max\{1,p-q\}$. *Define* $n \times n$ *triangular block* *matrices* $U = \begin{bmatrix} U_{ij} \end{bmatrix}_{i,j=1}^n$ *and* $V = \begin{bmatrix} V_{ij} \end{bmatrix}_{i,j=1}^n$ *by*

$$V_{ij} = \begin{cases} Y_{ij}Y_{jj}^{-1\!/\!2} & ,j \leq i \leq \beta(j); \\ 0 & , \text{elsewhere}; \end{cases}$$

$$U_{ij} = \begin{cases} X_{ij}X_{jj}^{-\frac{1}{2}} & , \gamma(j) \le i \le j; \\ 0 & , \text{elsewhere.} \end{cases}$$

Then the set of all $n \times n$ block matrices $F = \left[F_{ij} \right]_{i,j=1}^{n}$ *with* $F_{ij} = B_{ij}$,

$-q \le j-i \le q$, *and* $F > 0$ *is the set of all matrices of the form*

$$F = (G^*V^*+U^*)^{-1}(I-G^*G)(VG+U)^{-1} ,$$

where $G = \left[G_{ij} \right]_{i,j=1}^{n}$ *is any strictly contractive block matrix with* $G_{ij} = 0$, $j-i \le q$.
The correspondence is 1-1. Moreover,

$$\det F = |\det U|^{-2}\det(I-G^*G).$$

The two theorems are obtained in quite different ways. Theorem 0.1 is derived via a sequential approach developed in Chapter I, which may be viewed as an adapted version of the classical Schur algorithm used in complex function theory (see I. Schur [61, 62]). It consists of recursively applying an elementary linear fractional map, which reduces the problem in each step to a simpler one, and finally to a trivial one. Theorem 0.2 appears as a corollary of the so-called band method, a general scheme for dealing with extension problems introduced by H. Dym and I. Gohberg [24, 27] and developed further by I. Gohberg, M.A. Kaashoek and H.J. Woerdeman [38, 39, 40]. This method is described in the next subsection.

0.4. The band method. Here a positive extension problem is considered in an algebra $\mathcal{M}$ with a unit e and an involution $*$. The algebra admits a direct sum decomposition of the form

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2^0 + \mathcal{M}_d + \mathcal{M}_3^0 + \mathcal{M}_4,$$

where $\mathcal{M}_1, \mathcal{M}_2^0, \mathcal{M}_d, \mathcal{M}_3^0$ and $\mathcal{M}_4$ are linear subspaces of $\mathcal{M}$ satisfying the following two conditions:

(i) $e \in \mathcal{M}_d$, $\mathcal{M}_1 = \mathcal{M}_4^*$, $\mathcal{M}_2^0 = (\mathcal{M}_3^0)^*$, $\mathcal{M}_d = \mathcal{M}_d^*$;

(ii) the following multiplication table describes some additional rules on the multiplication in $\mathcal{M}$:

	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_-^0$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}_c$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_4$

where

$$\mathcal{M}_+^0 := \mathcal{M}_1 + \mathcal{M}_2^0 \,,\, \mathcal{M}_-^0 := \mathcal{M}_3^0 + \mathcal{M}_4 \,,$$

$$\mathcal{M}_c := \mathcal{M}_2^0 + \mathcal{M}_d + \mathcal{M}_3^0 \,.$$

An element $a \in \mathcal{M}$ is called *positive definite* in $\mathcal{M}$ if there exists an invertible element $c \in \mathcal{M}$ such that $a = c^*c$. In this context the positive extension problem reads as follows. Let $k = k^* \in \mathcal{M}_c$ be given. Find all *positive extensions* $b \in \mathcal{M}$ of k, i.e., find all b such that $b = m_1 + k + m_1^*$ with $m_1 \in \mathcal{M}_1$ and b positive definite in $\mathcal{M}$.

To put Theorem 0.2 in this context, take $\mathcal{M}$ to be the algebra of $n \times n$ block matrices with involution the usual adjoint of a matrix and with the identity matrix as unit, and let

$$\mathcal{M}_1 = \mathcal{M}_4^* = \{\, \left[a_{ij} \right]_{i,j=1}^{n} \mid a_{ij} = 0 \,, j-i \leqq q \,\},$$

$$\mathcal{M}_2^0 = \mathcal{M}_3^{0*} = \{\, \left[a_{ij} \right]_{i,j=1}^{n} \mid a_{ij} = 0 \,, j-i > q \text{ and } j-i \leqq 0 \,\},$$

$$\mathcal{M}_d = \{\, \left[a_{ij} \right]_{i,j=1}^{n} \mid a_{ij} = 0 \,, i \neq j \,\}.$$

In the band method an important role is played by a positive extension of a special type, called the *band extension*, which, by definition, is a positive extension b of $k \in \mathcal{M}_c$ with the additional property that $b^{-1} \in \mathcal{M}_c$. Thus the band extension of the positive band extension problem in Theorem 0.2 is the unique extension F such that $(F^{-1})_{ij} = 0$ for $j - i \geqq q$. In H. Dym and I. Gohberg [23] it was shown (assuming (0.2) is satisfied) that the unique band extension is given by $U^{*-1}U^{-1} = V^{*-1}V^{-1}$, where U and V are as in Theorem 0.2. Thus Theorem 0.2 shows that the coefficients of the linear fractional map which describes all solutions can be read off from the Cholesky factors of the band extension. It turns out − and this is the second main feature of the band method − that this principle holds in general. The first main principle in the context of the band method is

that the construction of the band extension may be reduced to solving a linear equation.

As a further illustration of the band method we consider an extension problem in the Wiener algebra W on the unit circle. Recall that W consists of all complex valued functions f on the unit circle $\mathbf{T}$ of the form

$$f(\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j f_j \, , \lambda \in \mathbf{T} \, ,$$

$$\sum_{j=-\infty}^{\infty} |f_j| < \infty.$$

The involution * on W is defined by

$$f^*(\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j f^*_{-j} = f(\lambda)^* \, , |\lambda| = 1,$$

and the unit is the function $e(\lambda) = 1$.

Given complex numbers $a_j = \bar{a}_{-j}$, $|j| \leq m$, a function $f \in W$ is called a *positive extension* of the given band $\{ a_j \mid |j| \leq m \}$ whenever $f(\lambda) > 0$ for $|\lambda| = 1$ and $f_j = a_j$ for $|j| \leq m$. The problem to find all positive extensions of a given band can be put in the context of the band method by choosing the subspaces in the following manner:

$$\mathcal{M}_1 = \mathcal{M}_4^* = \{ f \in W \mid f_j = 0, j \leq m \},$$

$$\mathcal{M}_2^0 = \mathcal{M}_3^{0*} = \{ f \in W \mid f_j = 0, j \leq 0 \text{ and } j > m \},$$

$$\mathcal{M}_d = \{ f \in W \mid f_j = 0, j \neq 0 \}.$$

In a similar way the positive extension problem in the operator Wiener algebra may be put in the context of the band method, and we shall use this (in Section III.3) to derive a linear fractional description of all positive extensions of a given operator band. In other words, given the operator band $\{ A_j \mid |j| \leq m \}$ we shall determine all positive definite block Toeplitz matrices

$$B = \begin{bmatrix} B_0 & B_{-1} & B_{-2} & \cdots \\ B_1 & B_0 & B_{-1} & \cdots \\ B_2 & B_1 & B_0 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

such that $B_j = A_j$, $|j| \leq m$, and the symbol of B is in the Wiener algebra.

In this book we illustrate the band method only on the above mentioned examples.

Other examples may be found in I. Gohberg, M.A. Kaashoek and H.J. Woerdeman [38, 39, 40].

0.5. Strictly contractive extension problems. Consider the following problem. Let A_{ij} be given matrices for $j \leq i$. Find the remaining matrices A_{ij}, $j > i$, such that the block matrix $A = \left(A_{ij} \right)_{i,j=1}^{n}$ has operator norm less than 1. From the distance formula of W. Arveson [3] it follows that the necessary and sufficient conditions for the existence of a solution are

$$\left\| \begin{bmatrix} A_{i1} & \cdots & A_{i,i} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{n,i} \end{bmatrix} \right\| < 1, i = 1,\ldots,m.$$

In [5] J.A. Ball and I. Gohberg showed that the set of all solutions may be represented via a linear fractional map, for which they used again their finite dimensional version of the shift invariant subspace approach. For the 2×2 case the contractive completion problem was treated earlier by Gr. Arsene and A. Gheondea [2], and by C. Davis, W.M. Kahan and W.F. Weinberger [17].

Our aim is to present an explicit description of the set of all solutions via a linear fractional map of which the coefficients are directly given in terms of the original data. As in the positive extension problem we do this in two ways: a sequential way and via the band method which applies to strictly contractive extensions as well. The sequential approach yields the following result for the scalar case.

THEOREM 0.3. *Let* a_{ij} *be given complex numbers, where* $1 \leq j \leq i \leq n$, *and suppose that*

$$\left\| \left(a_{ij} \right)_{i=p,j=1}^{n \quad p} \right\| < 1 , p = 1,\ldots,n.$$

For $j \leq i$ *define the number* h_{ij} *by*

$$h_{ij} = \frac{(a_{ij} + \beta_{ij}(I - \alpha_{ij}^{*}\alpha_{ij})^{-1}\alpha_{ij}^{*}\gamma_{ij})}{(1 - \beta_{ij}(I - \alpha_{ij}^{*}\alpha_{ij})^{-1}\beta_{ij}^{*})^{\frac{1}{2}}.(1 - \gamma_{ij}^{*}(I - \alpha_{ij}\alpha_{ij}^{*})^{-1}\gamma_{ij})^{\frac{1}{2}}} ,$$

where α_{ij}, β_{ij} *and* γ_{ij} *are given via the partitioning*

$$\left(a_{rs} \right)_{r=i,s=1}^{n \quad j} = \begin{bmatrix} \beta_{ij} & a_{ij} \\ \alpha_{ij} & \gamma_{ij} \end{bmatrix} .$$

Furthermore, let

$$\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \overset{\rightarrow}{\underset{p=-n+1}{\Pi}} {}^{0} \begin{bmatrix} (I-\Delta_p\Delta_p^*)^{-\frac{1}{2}} & (I-\Delta_p\Delta_p^*)^{-\frac{1}{2}}\Delta_p \\ (I-\Delta_p^*\Delta_p)^{-\frac{1}{2}}\Delta_p^* & (I-\Delta_p^*\Delta_p)^{-\frac{1}{2}} \end{bmatrix},$$

where

$$\Delta_p = \left[\delta_{i+p,j} h_{ij} \right]_{i,j=1}^{n}, p = -n+1,...,0.$$

Then the set of all matrices $F = \left[f_{ij} \right]_{i,j=1,}^{n} \in \mathbb{C}^{n \times n}$ *with* $f_{ij} = a_{ij}$, $j \leq i$, *and* $\|F\| < 1$ *is the set of all matrices of the form*

$$F = F(G) = (\theta_{11}G + \theta_{12})(\theta_{21}G + \theta_{22})^{-1},$$

where $G = \left[g_{ij} \right]_{i,j=1}^{n}$ *is any matrix with* $\|G\| < 1$ *and* $g_{ij} = 0$ *for* $j \leq i$. *The correspondence is 1-1. Moreover,*

$$\det(I - F(G)^* F(G)) = \{ \underset{j \leq i}{\Pi} (1 - |h_{ij}|^2) \} \det(I - G^* G).$$

The band method yields the following result.

THEOREM 0.4. *Let* a_{ij} *be given complex numbers, where* $1 \leq j \leq i \leq n$, *and suppose that*

$$S_p := \left[a_{ij} \right]_{i=p,j=1}^{n \quad p}$$

has norm strictly less than one for $p = 1,...,n$. *Put*

$$\begin{bmatrix} \hat{\alpha}_{ii} \\ \hat{\alpha}_{i+1,i} \\ \vdots \\ \hat{\alpha}_{ni} \end{bmatrix} = (I - S_i S_i^*)^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, i = 1,...,n,$$

$$\begin{bmatrix} \hat{\beta}_{i,i} \\ \vdots \\ \hat{\beta}_{n-1,i} \\ \hat{\beta}_{ni} \end{bmatrix} = S_i(I - S_i^* S_i)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, i = 1,...,n,$$

$$
\begin{bmatrix} \hat{\gamma}_{1i} \\ \hat{\gamma}_{2i} \\ \vdots \\ \hat{\gamma}_{i,i} \end{bmatrix} = S_i^*(I - S_i S_i^*)^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} , \ i = 1,\dots,n ,
$$

$$
\begin{bmatrix} \hat{\delta}_{1i} \\ \vdots \\ \hat{\delta}_{i-1,i} \\ \hat{\delta}_{ii} \end{bmatrix} = (I - S_i^* S_i)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} , \ i = 1,\dots,n ,
$$

and let

$$
\alpha := \begin{bmatrix} \alpha_{ij} \end{bmatrix}_{i,j=1}^{n} , \ \alpha_{ij} = \begin{cases} \hat{\alpha}_{ij}\hat{\alpha}_{jj}^{-\frac{1}{2}} , i \geq j; \\ 0 \qquad , i < j; \end{cases}
$$

$$
\beta := \begin{bmatrix} \beta_{ij} \end{bmatrix}_{i,j=1}^{n} , \ \beta_{ij} = \begin{cases} \hat{\beta}_{ij}\hat{\delta}_{jj}^{-\frac{1}{2}} , i \geq j; \\ 0 \qquad , i < j; \end{cases}
$$

$$
\gamma := \begin{bmatrix} \gamma_{ij} \end{bmatrix}_{i,j=1}^{n} , \ \gamma_{ij} = \begin{cases} \hat{\gamma}_{ij}\hat{\alpha}_{jj}^{-\frac{1}{2}} , i \leq j; \\ 0 \qquad , i > j; \end{cases}
$$

$$
\delta := \begin{bmatrix} \delta_{ij} \end{bmatrix}_{i,j=1}^{n} , \ \delta_{ij} = \begin{cases} \hat{\delta}_{ij}\hat{\delta}_{jj}^{-\frac{1}{2}} , i \leq j; \\ 0 \qquad , i > j. \end{cases}
$$

Then the set of all matrices $F = \begin{bmatrix} f_{ij} \end{bmatrix}_{i,j=1}^{n} \in \mathbb{C}^{n \times n}$ *with* $f_{ij} = a_{ij}$, $j \leq i$, *and* $\|F\| < 1$ *is the set of all matrices of the form*

$$
F = T(G) = (\alpha G + \beta)(\gamma G + \delta)^{-1},
$$

where $G = \begin{bmatrix} g_{ij} \end{bmatrix}_{i,j=1}^{n}$ *is any matrix with* $\|G\| < 1$ *and* $g_{ij} = 0$ *for* $j \leq i$. *The correspondence is 1-1. Moreover,*

$$
\det(I - T(G)^* T(G)) = |\det \delta|^{-2} \det(I - G^* G).
$$

Theorem 0.3 and its block matrix analogue are the first main results in Chapter I. Theorem 0.1 is proved using the block matrix version of Theorem 0.3 and the fact that the positive extension problem may be reduced to a strictly contractive extension problem by using an inverse scattering principle. Theorem 0.4, in contrast, is proved as a corollary of the results on positive extensions. Here we use the observation that

$$\begin{bmatrix} I & A \\ A^* & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ A^* & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I-A^*A \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

is positive definite if and only if $\|A\| \leq 1$. Note that whenever a lower triangular part of A is given, the given part of $\begin{bmatrix} I & A \\ A^* & I \end{bmatrix}$ is of band type.

As another application of the band method we shall also consider the following strictly contractive extension problem. Given complex numbers ϕ_j, $j \leq 0$, determine all f in the Wiener algebra W such that $|f(\lambda)| < 1$ for $|\lambda| = 1$ and $f_j = \phi_j$ for $j \leq 0$. This problem is connected with the well-known theorem of Nehari (see, e.g., V.M. Adamjan, D.Z. Arov and M.G. Krein [1] and H. Dym and I. Gohberg [26]), and its matrix version plays an important role in H_∞-control theory. In this book we solve the problem in the more general setting of the operator Wiener algebra.

0.6. Maximum entropy principles. Consider again the positive extension problem for block matrices: let B_{ij} be given matrices for $|j-i| \leq q$, and find the remaining matrices B_{ij}, $|j-i| > q$, such that the block matrix $B = \begin{bmatrix} B_{ij} \end{bmatrix}_{i,j=1}^{n}$ is positive definite. Let D_B denote the middle (diagonal) factor in the U^*DU decomposition (where U is upper triangular with identities as its diagonal entries) of a positive extension B of the given band, and write B_0 for the band extension. In [23] H. Dym and I. Gohberg showed that for any positive extension B of the given band

$$D_B \leq D_{B_0}, \tag{0.3}$$

and equality holds in (0.3) if and only if $B = B_0$. As a corollary (see [23]) it follows that the band extension is the unique positive extension for which the determinant is as large as possible. (From the determinant formulas in Theorem 0.1 and 0.2 it is clear that one obtains the extension with largest possible determinant only by choosing $G = 0$.) This corollary is usually referred to as the maximum entropy principle for the matrix case. For the positive extension problem in the Wiener algebra, considered in Subsection 0.3, there is also a maximum entropy principle. It identifies the band extension as the unique positive

extension f for which the entropy integral

$$\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \log f(e^{i\theta}) d\theta$$

is as large as possible (see J.P. Burg [13], and also H. Dym and I. Gohberg [22]). It turns out that both maximum entropy principles may be viewed as special cases of a general maximum entropy principle that we shall derive in the abstract setting of the band method. This general maximum entropy principle identifies the band extension as the unique positive extension for which the multiplicative diagonal is maximal.

We shall derive an abstract maximum entropy principle also for strictly contractive extension problems. The concrete maximum entropy principle for block matrices mentioned above will play a role in the sequential approach.

0.7. Minimal rank extension problems. As indicated before this class of problems concerns mainly three kinds of operators: finite matrices, semi-infinite matrices and integral operators. Let us start by describing the minimal rank extension problem for the latter case.

Let $K : L_2^m[0,1] \to L_2^n[0,1]$ be an integral operator with an $n \times m$ matrix kernel k defined on the square $[a,b] \times [a,b]$. So

$$(Kf)(t) = \int\limits_0^1 k(t,s)f(s)ds \ , 0 \leq t \leq 1 \ .$$

An integral operator H with kernel h is called a *finite rank extension* of the lower triangular part of K if rank $H(=: \text{rank } h) < \infty$ and

$$h(t,s) = k(t,s) \ , 0 \leq s < t \leq 1.$$

For instance, the Volterra operator

$$(Vf)(t) = \int\limits_0^t f(s)ds \ , 0 \leq t \leq 1,$$

on $L_2[0,1]$ has a finite rank extension of rank 1. The minimal rank extension problem we are interested in asks to determine all *minimal rank extensions* of the lower triangular part of a given integral operator K, i.e., to find all finite rank extensions of the lower triangular part of K with smallest possible rank. Problems of this type originated in I. Gohberg and M.A. Kaashoek [35], where they appear in connection with minimal realizations of boundary value systems.

The analysis of the minimal rank extension problem for integral operators is based on

its matrix analogue, which is of interest in its own right. Let matrices A_{ij}, $1 \leqq j \leqq i \leqq n$, be given. Find matrices A_{ij}, $1 \leqq i < j \leqq n$, such that the block matrix

$$A = \left[A_{ij} \right]_{i,j=1}^{n}$$

has lowest possible rank. Such an A is called a *minimal rank extension* of the given lower triangular part $\mathcal{A} := \{ A_{ij} \mid 1 \leqq j \leqq i \leqq n \}$, and its rank is called the *minimal lower rank* of $\mathcal{A}$. In this book we shall prove, among others, the following results.

THEOREM 0.5. *Let* $\mathcal{A} = \{ A_{ij} \mid 1 \leqq j \leqq i \leqq n \}$ *be a given lower triangular part. The minimal lower rank* $\ell(\mathcal{A})$ *of* $\mathcal{A}$ *is given by*

$$\ell(\mathcal{A}) = \sum_{p=1}^{n} \operatorname{rank} A^{(p,p)} - \sum_{p=1}^{n-1} \operatorname{rank} A^{(p+1,p)} ,$$

where

$$A^{(p,q)} = \begin{bmatrix} A_{p1} & \cdots & A_{pq} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nq} \end{bmatrix} .$$

Furthermore, $\mathcal{A}$ *has only one minimal rank extension if and only if*

$$\operatorname{rank} A^{(p,p)} = \operatorname{rank} A^{(p+1,p)} = \operatorname{rank} A^{(p+1,p+1)} , p = 1,...,n-1.$$

THEOREM 0.6. *Assume that the lower triangular part of the integral operator* $K : L_2^m[0,1] \rightarrow L_2^n[0,1]$ *with kernel* k *has a finite rank extension. Then the rank* ℓ *of a minimal rank extension of the lower triangular part of* K *is equal to*

$$\ell = \max_{\pi, \tau_\pi} \lambda(\pi, \tau_\pi, k),$$

where the maximum is taken over all partitions π *of* $[0,1]$ *and all corresponding sets of intermediate points* τ_π. *Here, for* $\pi = \{\alpha_0, \cdots, \alpha_n\}$ *and* $\tau_\pi = \{\tau_1, \cdots, \tau_n\}$, *the number* $\lambda(\pi, \tau_\pi, k)$ *is defined by*

$$\lambda(\pi, \tau_\pi, k) := \sum_{i=1}^{n} \operatorname{rank} k^{\tau_i} - \sum_{i=1}^{n-1} \operatorname{rank} k^{\alpha_i}, \tag{0.4}$$

where k^β *denotes the restriction of* k *to the rectangle* $[\beta,1] \times [0,\beta]$. *Furthermore, the lower triangular part has only one minimal rank extension if and only if* $\operatorname{rank} k^\beta$ *is independent of* $\beta \in (0,1)$.

We identify a large class of partitions π and corresponding sets τ_π for which the maximum in (0.4) is attained.

The minimal rank extension problems will be treated in the context of operators that are triangular relative to chains of orthogonal projections. The general results will be specified for various types of operators.

0.8. General patterns. The matrix versions of the three classes of problems can also be considered for the case when the given entries in the matrix do not form a band (positive extensions) or a triangle (strictly contractive/minimal rank extensions). For more general patterns of given entries there are many open problems. One of them, connected with the minimal rank extension problem, will be discussed in some detail. For positive extension problems relative to more general patterns we refer to R. Grone, C.R. Johnson, E.M. de Sá and H. Wolkowitz [45], where the existence of a solution, in particular one with maximal determinant, is the main topic, and to H. Nelis, P. Dewilde and E. Deprettere [57] which concerns the case of multi-band patterns and its relation with questions appearing in electrical engineering. For the contractive extension problem results on existence of a solution relative to more general patterns appeared in C.R. Johnson and L. Rodman [46].

It turns out that, in general, in the minimal rank extension problem the minimal possible rank is not only determined by the ranks of fully specified submatrices as is the case for the triangular patterns (cf. Theorem 0.5). This focusses the attention upon those patterns of specified entries for which the minimum is so determined. It is shown that it is necessary that the bipartite graph of the pattern be chordal, and some evidence is given for the conjecture that this is also sufficient. In this conjecture the triangular patterns once again play an important role.

0.9. Description of contents. This book consists of two parts with a total of five chapters. In Part A positive and strictly contractive extension problems are treated. Chapter I concerns the sequential approach. The remaining two chapters in this part concern the band method (Chapter II) and its applications (Chapter III). In Part B minimal rank extension problems are treated. Chapter IV concerns matrices and Chapter V operators. This book is based upon results that already found a place in the literature, in papers written by or co-authored by the present author. At the end of each part, in a brief section of comments, we list the papers involved.

PART A
POSITIVE AND STRICTLY CONTRACTIVE EXTENSIONS

This part, which consists of three chapters, treats positive and strictly contractive extension problems. In Chapter I a sequential approach is used to derive linear fractional forms describing all strictly contractive and positive extensions for block matrices. Chapter II concerns the band method. In Chapter III the results obtained using the band method are specified for the algebra of operator matrices and the operator Wiener algebra.

CHAPTER I. BLOCK MATRICES: A SEQUENTIAL APPROACH

In this chapter we treat the positive and strictly contractive extension problem for block matrices using a sequential approach. The main aim is to derive explicit linear fractional descriptions for all solutions. In Section 1 some elementary facts concerning linear fractional maps are recalled. Section 2, which deals with the 2×2 strictly contractive extension problem, provides a first step in the proof and illustrates the methods used to solve the general problem. Section 3 describes the elimination procedure which is used in Section 4 to solve the general $(n \times m)$ strictly contractive extension problem. In Section 5 the positive extension problem is reduced to a strictly contractive one. Section 6 contains the solution of the positive extension problem. Section 7 deals with the Toeplitz case.

I.1. Linear fractional maps with matrix coefficients

In this section we collect together some elementary facts on linear fractional maps with matrix coefficients. As general references for this topic we mention B. Schwarz and A. Zaks [63], [64] and the references given there. Also some special types of linear fractional maps, which will be used for solving the problem of strictly contractive extensions, are introduced in this section.

Let A, B, C and D be matrices of size $p \times p, p \times q, q \times p$ and $q \times q$, respectively. Using them as blocks we define the following $(p+q) \times (p+q)$ matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{1.1}$$

We consider here only nonsingular $(p+q) \times (p+q)$ matrices M, i.e., we assume throughout that

$$\det M = \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq 0.$$

Under this condition, define the linear fractional maps $\mathcal{M}_M$ and $\hat{\mathcal{M}}_M$ by

$$\mathcal{M}_M(G) = (AG+B)(CG+D)^{-1},$$

$$\hat{\mathcal{M}}_M(G) = (A-GC)^{-1}(-B+GD),$$

where the variable G is a $p \times q$ matrix. We call the matrix M the matrix defining the map $\mathcal{M}_M$. The matrix $\mathcal{M}_M(G)$ is defined only on the set of matrices G for which $CG+D$ is invertible. Analogous remarks hold for $\hat{\mathcal{M}}_M$.

PROPOSITION 1.1. *Let M, N be nonsingular $(p+q) \times (p+q)$ matrices. Then (on the appropiate domains)*

(i) $\mathcal{M}_M \circ \mathcal{M}_N = \mathcal{M}_{MN}$;

(ii) $\hat{\mathcal{M}}_M \circ \hat{\mathcal{M}}_N = \hat{\mathcal{M}}_{NM}$;

(iii) $\mathcal{M}_M = \hat{\mathcal{M}}_{M^{-1}}$;

Proof. Let M be given by (1.1) and $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. First note that (on the appropiate domain) $\mathcal{M}_M(G) = K$ if and only if

$$\begin{bmatrix} I & -K \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} G \\ I \end{bmatrix} = 0,$$

and the latter identity holds if and only if $\hat{\mathcal{M}}_M(K) = G$.

For (i) note that $\mathcal{M}_M(\mathcal{M}_N(G)) = K$ if and only if

$$0 = \begin{bmatrix} I & -K \end{bmatrix} M \begin{bmatrix} \mathcal{M}_N(G) \\ I \end{bmatrix} = \begin{bmatrix} I & -K \end{bmatrix} MN \begin{bmatrix} G \\ I \end{bmatrix} (cG+d)^{-1}.$$

But this is equivalent to $\mathcal{M}_{MN}(G) = K$.

One proves (ii) analogously. So let us finish with (iii). Since $\mathcal{M}_M(G) = K$ if and only if $\hat{\mathcal{M}}_M(K) = G$, we obtain $\mathcal{M}_M = (\hat{\mathcal{M}}_M)^{-1}$. On the other hand, by (ii), $\hat{\mathcal{M}}_M \circ \hat{\mathcal{M}}_{M^{-1}} = \hat{\mathcal{M}}_{M^{-1}} \circ \hat{\mathcal{M}}_M = \hat{\mathcal{M}}_I$, which is the identity map. So $(\hat{\mathcal{M}}_M)^{-1} = \hat{\mathcal{M}}_{M^{-1}}$. Now (iii) follows. $\square$

In this paper we shall deal with linear fractional maps of which the defining matrix has additional symmetry properties. Let J be the following $(p+q) \times (p+q)$ matrix:

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \tag{1.2}$$

where I_r denotes the identity matrix of size $r \times r$. The matrix M is called J *–unitary* if $M^* J M = J$ and $MJM^* = J$. For $p = q$ the matrix M is called $(\tilde{J}, J)$ *–unitary* if $M^* \tilde{J} M = J$ and $MJM^* = \tilde{J}$, where J is as in (1.2) and $\tilde{J}$ is the $2p \times 2p$ matrix

$$\tilde{J} = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}. \tag{1.3}$$

The symbol $\|G\|$ denotes the largest singular value (the operator norm) of the matrix G.

PROPOSITION 1.2. *Let M be as in* (1.1).

(i) If M is J-*unitary, then* $\mathcal{M}_M$ *and* $\hat{\mathcal{M}}_M$ *map* $\{G \mid \|G\| < 1\}$ *into* $\{G \mid \|G\| < 1\}$. *Furthermore, if* $\det(CG+D) \neq 0$, *then*

$$\det(I - \mathcal{M}_M(G)^* \mathcal{M}_M(G)) = |\det(CG+D)|^{-2} \det(I - G^* G). \qquad (1.4)$$

(ii) *If M is* $(\tilde{J}, J)$-*unitary, then* $\mathcal{M}_M$ *maps* $\{G \mid \|G\| < 1\}$ *into* $\{G \mid G+G^* > 0\}$ *and* $\hat{\mathcal{M}}_M$ *maps* $\{G \mid G+G^* > 0\}$ *into* $\{G \mid \|G\| < 1\}$. *Furthermore, if* $\det(CG+D) \neq 0$, *then*

$$\det(\mathcal{M}_M(G) + \mathcal{M}_M(G)^*) = |\det(CG+D)|^{-2} \det(I - G^* G). \qquad (1.5)$$

Proof. Let us prove (ii). One proves (i) analogously. Suppose that M is $(\tilde{J}, J)$-unitary and let G be a $p \times q$ matrix with $\|G\| < 1$. Then

$$I - G^* G = -\begin{bmatrix} G^* & I \end{bmatrix} M^* \tilde{J} M \begin{bmatrix} G \\ I \end{bmatrix} = \qquad (1.6)$$

$$= (AG+B)^*(CG+D) - (CG+D)^*(AG+B)$$

is strictly positive. Suppose that $(CG+D)x = 0$ for some $x \neq 0$. Using (1.6), it follows that $<(I-G^* G)x, x> = 0$, giving a contradiction. Since $CG+D$ is square, we obtain that $\det(CG+D) \neq 0$. Hence $\mathcal{M}_M(G)$ is well-defined. Rewriting (1.6) gives

$$I - G^* G = (CG+D)^*(\mathcal{M}_M(G)^* + \mathcal{M}_M(G))(CG+D), \qquad (1.7)$$

so $\mathcal{M}_M(G)^* + \mathcal{M}_M(G) > 0$. Analogously, one proves that $\hat{\mathcal{M}}_M$ maps $\{G \mid G+G^* > 0\}$ into $\{G \mid \|G\| < 1\}$. The identity (1.5) is a direct consequence of (1.7). $\square$

The following types of linear fractional maps will be used to solve the strictly contractive extension problem. Let Δ be a $p \times q$ matrix with norm less than 1. Define

$$T_\Delta(G) := W_{\Delta^*}(G+\Delta)(\Delta^* G+I)^{-1} W_\Delta^{-1},$$

$$\hat{T}_\Delta(G) := W_\Delta^{-1}(I - G\Delta^*)^{-1}(G-\Delta)W_{\Delta^*},$$

where G is a $p \times q$ matrix and $W_B := (I - B^* B)^{-\frac{1}{2}}$ for $\|B\| < 1$. For a positive definite matrix N the symbol $N^{-\frac{1}{2}}$ denotes the inverse of the usual positive definite square root $N^{\frac{1}{2}}$ of N. Note that $T_\Delta = \mathcal{M}_{M(\Delta)}$ and $\hat{T}_\Delta = \hat{\mathcal{M}}_{M(\Delta)}$, where $M(\Delta)$ is the J-unitary (with J as

in (1.2)) matrix

$$M(\Delta) = \begin{bmatrix} W_{\Delta^.} & W_{\Delta^.\Delta} \\ W_{\Delta\Delta^*} & W_\Delta \end{bmatrix}.$$ (1.8)

The following corollary is now a direct consequence of Propositions 1.1 and 1.2.

COROLLARY 1.3. *Let* $\|\Delta\| < 1$. *Then*

$$T_\Delta : \{ G \mid \|G\| < 1 \} \rightarrow \{ G \mid \|G\| < 1 \}$$

is bijective, and its inverse is $\hat{T}_\Delta$. *Furthermore, if* $\det(\Delta^* G + I) \neq 0$, *then*

$$\det(I - T_\Delta(G)^* T_\Delta(G)) = \det(I - \Delta^*\Delta) \, |\det(\Delta^* G + I)|^{-2} \det(I - G^* G).$$

I.2. The strictly contractive extension problem: the 2×2 case

In this section we shall explain the method used in this chapter on the 2×2 case. Part of the results are used later for the general case. Let α, β and γ be given matrices. We want to find all matrices F of the form

$$F = \begin{bmatrix} \beta & X \\ \alpha & \gamma \end{bmatrix}$$ (2.1)

with norm (strictly) less than one. Suppose that such an extension F has been found. Then $\|\alpha\| < 1$, and by Corollary 1.3

$$\hat{T}_{\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}} \begin{bmatrix} \beta & X \\ \alpha & \gamma \end{bmatrix} = \begin{bmatrix} \beta W_\alpha & X + \beta W_\alpha \alpha^* W_{\alpha^.}\gamma \\ 0 & W_{\alpha^.}\gamma \end{bmatrix}$$ (2.2)

is well-defined and has also norm less than one. In particular, $B := \beta W_\alpha$ and $C := W_{\alpha^.}\gamma$ are strict contractions. Using again Corollary 1.3 we obtain that

$$\hat{T}_{\begin{bmatrix} \beta W_\alpha & 0 \\ 0 & W_{\alpha^.}\gamma \end{bmatrix}} \circ \hat{T}_{\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}} \begin{bmatrix} \beta & X \\ \alpha & \gamma \end{bmatrix} = \begin{bmatrix} 0 & W_{B^.}(X + B\alpha^* C)W_C \\ 0 & 0 \end{bmatrix}$$ (2.3)

is well-defined and has norm less than one. In other words, if $\|F\| < 1$, then (2.3) is of the form $\begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix}$ with G norm less than one.

Conversely, let us start with $\begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix}$ with $\|G\| < 1$, and assume that $\left\| \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \right\| < 1$

and $\| \begin{bmatrix} \alpha & \gamma \end{bmatrix} \| < 1$. The latter conditions imply that $B := \beta W_\alpha$ and $C := W_\alpha \cdot \gamma$ are strict contractions, and hence the 2×2 block matrix

$$T_{\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}} \circ T_{\begin{bmatrix} \beta W_\alpha & 0 \\ 0 & W_\alpha \cdot \gamma \end{bmatrix}} \begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \beta & W_B^{-1} G W_C^{-1} - B\alpha^* C \\ \alpha & \gamma \end{bmatrix} \tag{2.4}$$

is well-defined and has norm less than one. In particular, the right hand side of (2.4) solves the extension problem. The above calculations describe in a nutshell the elimination procedure which we shall use later. We summarize the results in the following theorem.

THEOREM 2.1. *Let* α,β *and* γ *be matrices of sizes* $p_2 \times q_1$, $p_1 \times q_1$ *and* $p_2 \times q_2$, *respectively, satisfying* $\| \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \| < 1$ *and* $\| \begin{bmatrix} \alpha & \gamma \end{bmatrix} \| < 1$. *Put*

$$\Delta_{-1} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}, \Delta_0 = \begin{bmatrix} \beta W_\alpha & 0 \\ 0 & W_\alpha \cdot \gamma \end{bmatrix}.$$

Then all the matrices of the form

$$\begin{bmatrix} \beta & * \\ \alpha & \gamma \end{bmatrix} \tag{2.5}$$

with norm less than one are given by

$$F(G) = T_{\Delta_{-1}} \circ T_{\Delta_0} \begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix}, \tag{2.6}$$

where G *is any* $p_1 \times q_2$ *matrix with norm less than one. The correspondence is 1-1. Furthermore, for such a matrix* G,

$$\det(I - F(G)^* F(G)) = \det(I - \alpha^* \alpha).\det(I - \beta(I - \alpha^* \alpha)^{-1} \beta^*) \tag{2.7}$$

$$\det(I - \gamma^*(I - \alpha\alpha^*)^{-1} \gamma).\det(I - G^* G).$$

Proof. It remains to prove (2.7). Let $\tilde{G} = \begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix}$. Since $\det(\Delta_0^* \tilde{G} + I) = 1$, we obtain by Corollary 1.3

$$\det(I - T_{\Delta_0}(\tilde{G})^* T_{\Delta_0}(\tilde{G})) = \det(I - \Delta_0^* \Delta_0).\det(I - \tilde{G}^* \tilde{G}). \tag{2.8}$$

Put $\hat{G} = T_{\Delta_0}(\tilde{G})$. An analogous reasoning yields

$$\det(I - T_{\Delta_{-1}}(\hat{G})^* T_{\Delta_{-1}}(\hat{G})) = \det(I - \Delta_{-1}^* \Delta_{-1}).\det(I - \hat{G}^* \hat{G}). \tag{2.9}$$

Inserting (2.8) in (2.9) and using the identity $\det(I - \tilde{G}^* \tilde{G}) = \det(I - G^* G)$, we obtain

$$\det(I - F(G)^* F(G)) = \det(I - \Delta_0^* \Delta_0) \det(I - \Delta_{-1}^* \Delta_{-1}) \det(I - G^* G),$$

and (2.7) follows immediately. $\square$

The (1,2) entry of the right hand side in (2.4) is equal to

$$X = -\beta(I - \alpha^* \alpha)^{-1} \alpha^* \gamma + (I - \beta(I - \alpha^* \alpha)^{-1} \beta^*)^{1/2} G(I - \gamma^*(I - \alpha\alpha^*)^{-1} \gamma)^{1/2}. \quad (2.10)$$

Thus all strictly contractive extensions $\begin{pmatrix} \beta & * \\ \alpha & \gamma \end{pmatrix}$ are given by (2.1) with X as in (2.10) and $\|G\| < 1$. In this form the solution of the extension problem for 2×2 matrices appears in [17] and [2].

From (2.7) it is clear that

$$\det(I - F(0)^* F(0)) > \det(I - F(G)^* F(G))$$

for $G \neq 0$. So $F(0)$ is the unique strictly contractive extension F of (2.5) for which $\det(I - F^* F)$ is maximal. According to (2.10) the corresponding X is given by

$$X_0 = -\beta(I - \alpha^* \alpha)^{-1} \alpha^* \gamma. \quad (2.11)$$

I.3. The elimination procedure

In this section the principle of reducing the strictly contractive extension problem to one of simpler form by creating zero diagonals, illustrated in the previous section on the 2×2 case, will be described for the $n \times m$ block case. As suggested by the approach followed in Section 2, to solve the strictly contractive extension problem for arbitrary block matrices one has to understand the behaviour of the linear fractional maps T_Δ and $\hat{T}_\Delta$ on upper triangular (relative to some diagonal) block matrices.

Let $\Omega^{n \times m}$ denote the set of all $n \times m$ block matrices $A = \begin{bmatrix} A_{ij} \end{bmatrix}_{i=1, j=1}^{n \quad m}$ with a fixed block structure. Thus for each (i, j) the matrix A_{ij} has some fixed size independent of A. In the set $\Omega^{n \times m}$ addition is a well-defined operation. When multiplying $A \in \Omega^{n \times m}$ and $B \in \Omega^{m \times p}$, we shall assume that the product $A_{ij} B_{jk}$ makes sense. When inverting or taking the square root of $A \in \Omega^{n \times n}$, the diagonal elements A_{ii} are assumed to be square. Define for $p \in \mathbb{Z}$ the p-*th diagonal map* $\mathscr{D}_p : \Omega^{n \times m} \to \Omega^{n \times m}$ by

$$\mathscr{D}_p(A) = \begin{bmatrix} \delta_{i+p, j} A_{ij} \end{bmatrix}_{i=1, j=1}^{n \quad m},$$

where δ_{ij} denotes the Kronecker delta, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. Note that for $p \leq -n$ and $p \geq m$ the matrix $\mathscr{D}_p(A)$ is equal to 0. Of course, we are not interested in those values of p, but for notational convenience they are not excluded. We refer to $\mathscr{D}_p(A)$ as the p-th diagonal of A. Let $\mathscr{L}_p : \Omega^{n \times m} \to \Omega^{n \times m}$ be defined by

$$\mathscr{L}_p(A) = \sum_{q \leq p} \mathscr{D}_q(A) .$$

We call $\mathscr{L}_p(A)$ the lower triangular part of A relative to the p-th diagonal. Furthermore, let $\mathbf{U}_p^{n \times m}$ denote the set of upper triangular matrices relative to the p-th diagonal, i.e.,

$$\mathbf{U}_p^{n \times m} = \{ A \in \Omega^{n \times m} \mid \mathscr{L}_{p-1}(A) = 0 \} = \{ A \in \Omega^{n \times m} \mid A = \sum_{q \geq p} \mathscr{D}_q(A) \}.$$

For $A \in \Omega^{n \times m}$ and $q \in \mathbf{Z}$ put

$$m_q(A) = \max_{i \in \mathbf{Z}} \left\| \begin{bmatrix} A_{i1} & \cdots & A_{i,i+q} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{n,i+q} \end{bmatrix} \right\| .$$

It is easy to see that $m_q(A) < 1$ implies that $\|A_{ij}\| < 1$ for $j - i \leq q$. Since

$$\| \mathscr{D}_p(A) \| = \max_i \| A_{i,i+p} \| ,$$

we conclude that $\| \mathscr{D}_p(A) \| < 1$ whenever $m_q(A) < 1$ and $p \leq q$.

The elimination procedure may now be described by the following two propositions.

PROPOSITION 3.1. Let $A \in \mathbf{U}_p^{n \times m}$, $\| \mathscr{D}_p(A) \| < 1$ and $q \geq p$. Then

(i) $\hat{T}_{\mathscr{D}_p(A)}(A) \in \mathbf{U}_{p+1}^{n \times m}$;

(ii) If $\mathscr{L}_q(A) = \mathscr{L}_q(K)$, then $\mathscr{L}_q(\hat{T}_{\mathscr{D}_p(A)}(A)) = \mathscr{L}_q(\hat{T}_{\mathscr{D}_p(A)}(K))$;

(iii) If $m_q(A) < 1$, then $m_q(\hat{T}_{\mathscr{D}_p(A)}(A)) < 1$.

PROPSITION 3.2. Let $p \leq q$, and let $G, \check{G} \in \mathbf{U}_{p+1}^{n \times m}$ be such that $\mathscr{L}_q(G) = \mathscr{L}_q(\check{G})$. Furthermore, let $\Delta = \mathscr{D}_p(\Delta)$ have norm less than 1. Then

$$\mathscr{L}_q(T_\Delta(G)) = \mathscr{L}_q(T_\Delta(\check{G})) .$$

Moreover,

$$\det(I - T_\Delta(G)^* T_\Delta(G)) = \det(I - \Delta^* \Delta) \det(I - G^* G). \tag{3.1}$$

We shall prove these two propositions at the end of this section. First we deduce the

following theorem.

THEOREM 3.3. *Let $A \in \Omega^{n \times m}$ be such that $m_q(A) < 1$. Make the following sequence of matrices:*

$$\Sigma_{-n+1} := A \ , \ \Delta_{-n+1} = \mathcal{D}_{-n+1}(A)$$

$$\Sigma_i := \hat{T}_{\Delta_{i-1}}(\Sigma_{i-1}) \ , \ \Delta_i = \mathcal{D}_i(\Sigma_i), i = -n+2,...,q+1. \tag{3.2}$$

Then the set $\{ F(G) \mid G \in U_{q+1}^{n \times m}, \ \|G\| < 1 \}$, where

$$F(G) := T_{\Delta_{-n+1}}(\ \cdots \ (T_{\Delta_q}(G)) \ \cdots \ ) \ ,$$

is the set of all $F \in \Omega^{n \times m}$ such that $\mathcal{L}_q(F) = \mathcal{L}_q(A)$ and $\|F\| < 1$. The correspondence between these sets is 1-1. Furthermore,

$$\det(I - F(G)^* F(G)) = \prod_{p=-n+1}^{q} \det(I - \Delta_p \Delta_p^*) \det(I - G^* G) \ . \tag{3.3}$$

Proof. Using Proposition 3.1 one sees, by induction, that the matrices in (3.2) are well-defined, $m_q(\Sigma_p) < 1$ and $\Sigma_p \in U_p^{n \times m}$ for $p = -n+1,...,q+1$. Let $\|F\| < 1$ be such that $\mathcal{L}_q(A) = \mathcal{L}_q(F)$. Put $G_{-n+1} = F$, $G_j = \hat{T}_{\Delta_j}(G_{j-1})$, $j = -n+2,...,q+1$. By repeatedly applying Corollary 1.3 and Proposition 3.1(ii) one obtains that $\|G_j\| < 1$ and $\mathcal{L}_q(G_j) = \mathcal{L}_q(\Sigma_j)$, $j = -n+1,...,q+1$. In particular, $\|G_{q+1}\| < 1$ and $\mathcal{L}_q(G_{q+1}) = \mathcal{L}_q(\Sigma_{q+1}) = 0$. Hence for $G = G_{q+1} \in U_{q+1}^{n \times m}$ we have that $F = F(G)$ and $\|G\| < 1$, and thus F is of the desired form.

Conversely, let $G \in U_{q+1}^{n \times m}$ with $\|G\| < 1$. Put $F_{q+1} = G$ and $F_j = T_{\Delta_j}(F_{j+1})$ for $j = -n+1,...,q$. Since $\|F_{q+1}\| < 1$, it follows from Corollary 1.3, by induction, that $\|F_j\| < 1$ for $j = -n+1,...,q$. Furthermore, $\mathcal{L}_q(F_{q+1}) = 0 = \mathcal{L}_q(\Sigma_{q+1})$. By repeatedly applying Proposition 3.2 we obtain that $\mathcal{L}_q(F_j) = \mathcal{L}_q(\Sigma_j)$ for $j = -n+1,...,q$. So $\|F_{-n+1}\| < 1$ and $\mathcal{L}_q(F_{-n+1}) = \mathcal{L}_q(\Sigma_{-n+1}) = \mathcal{L}_q(A)$. This proves that $F_{-n+1} = F(G)$ is a contraction such that $\mathcal{L}_q(F(G)) = \mathcal{L}_q(A)$.

The 1-1 correspondence follows immediately from the fact that T_{Δ_i} is invertible for $i = -n+1,...,q$.

Finally, a repeated application of the determinant formula in Proposition 3.2 yields (3.3). $\square$

The construction of the linear fractional map $F(G)$ in Theorem 3.3 may be viewed as a variation of the Schur algorithm which is used to solve the Nevanlinna-Pick interpolation

problem (see, e.g., [61] and [62]).

Let A be such that $m_q(A) < 1$. Recall from [5] the following definition. The *maximum entropy solution* of the strictly contractive extension problem for A is the unique strictly contractive extension of A which maximizes the number $\det(I - F^* F)$, i.e., the maximum entropy solution F_0 has the following properties: $\mathscr{L}_q(F_0) = \mathscr{L}_q(A)$, $\|F_0\| < 1$ and

$$\det(I - F_0^* F_0) > \det(I - F^* F),$$

where $F \neq F_0$ varies over the set $\{ F \mid \|F\| \leq 1 \text{ and } \mathscr{L}_q(F) = \mathscr{L}_q(A) \}$. Note that in Theorem 3.3 the maximium entropy solution is obtained by taking $G = 0$. The latter statement follows immediately from (3.3).

For the proofs of Propositions 3.1 and 3.2 we need to analyze further the behaviour of the linear fractional maps T_Δ and $\hat{T}_\Delta$. In what follows $\Delta \in \Omega^{n \times m}$ is a *diagonal*, i.e., $\Delta = \mathscr{D}_p(\Delta)$ for some p. Furthermore, we assume that $\|\Delta\| < 1$. Then $W_\Delta \in \Omega^{m \times m}$ and $W_{\Delta^*} \in \Omega^{n \times n}$ are both invertible square block diagonal matrices. To be more precise, if $\Delta = \left[\Delta_{ij} \right]_{i=1, j=1}^{n \quad m} = \mathscr{D}_p(\Delta)$, then

$$W_\Delta = \text{diag} \left[W_{\Delta_{j-p,j}} \right]_{j=1}^{m}, \quad W_{\Delta^*} = \text{diag} \left[W_{\Delta^*_{i,p+i}} \right]_{i=1}^{n},$$

where we use the convention that $\Delta_{ij} = 0$ for $(i,j) \notin \{1,...,n\} \times \{1,...,m\}$. Here $\text{diag} \left[Z_i \right]_{i=1}^{k} = \text{diag} \left[Z_1,...,Z_k \right]$ denotes the $k \times k$ block diagonal matrix with Z_i as the (i,i)th entry $(i = 1,...,k)$. If $\Delta_{ij} = 0$, then $W_{\Delta_{ij}} = I$. So, for instance, if $n + p \leq m$, then

$$W_\Delta = \begin{cases} \text{diag} \left[W_{\Delta_{1-p,1}}, \cdots, W_{\Delta_{n,n+p}}, I, \cdots, I \right] & , p \leq 0, \\[2mm] \text{diag} \left[I, \ldots, I, W_{\Delta_{1,1+p}}, \ldots, W_{\Delta_{n,n+p}}, I, ..., I \right] & , p > 0. \end{cases}$$

Let E_i denote the square block matrix which has on the (i,i)-th place an identity matrix and zeros elsewhere. The number of blocks in E_i and their sizes should be clear from the context.

LEMMA 3.4. *Let* $A = \left[A_{ij} \right]_{i=1, j=1}^{n \quad m} \in U_p^{n \times m}$, *and fix* $r \in \{1,...,n\}$ *and* $s \in \{1,...,m\}$ *such that* $s - r = p$. *Put* $\Delta = E_r A E_s$, *and assume that* $\|\Delta\| < 1$. *Then*

$$(\hat{T}_\Delta(A))_{ij} = \begin{cases} A_{ij} & , i > r \text{ or } j < s; \\ 0 & , i = r, j = s; \\ A_{is}(I - A_{rs}^* A_{rs})^{-\frac{1}{2}} & , i < r, j = s; \\ (I - A_{rs} A_{rs}^*)^{-\frac{1}{2}} A_{rj} & , i = r, j > s; \\ A_{ij} + A_{is}(I - A_{rs}^* A_{rs})^{-1} A_{rs}^* A_{rj} & , i < r, j > s. \end{cases}$$

In particular, if $j - i > p$, *then*

$$(\hat{T}_\Delta(A))_{ij} = (W_{\Delta^*})_{ii}(A_{ij} + Z_{ij})(W_\Delta)_{jj} ,$$

where

$$Z_{ij} = \begin{cases} A_{is} W_{A_{r^*}} A_{rs}^* W_{A_{r^*}} A_{rj} & , i < r, j > s, \\ 0 & , \text{otherwise.} \end{cases}$$

Proof. Straightforward calculations. $\square$

Let P_i ($i = 1,\dots,n$) and Q_j ($j = 1,\dots,m$) be the block matrices of size $(m - i + 1) \times n$ and $m \times j$, respectively, given by

Here and in the sequel the blank entries in the matrices denote zeroes. For convenience we also introduce $P_i = I$ ($i \le 0$), $P_i = 0$ ($i \ge n + 1$), $Q_i = 0$ ($i \le 0$) and $Q_i = I$ ($i \ge m + 1$). Note that

$$P_i A Q_j = \begin{bmatrix} A_{i1} & \cdots & A_{ij} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nj} \end{bmatrix}.$$

LEMMA 3.5. *Let* $A \in U_p^{n \times m}$ *and* $\Delta = \mathcal{D}_p(A)$ *have norm less than one. Then* $\hat{T}_\Delta(A)$ *exists and*

$$P_k \hat{T}_\Delta(A)Q_l = \hat{T}_{P_k \Delta Q_l}(P_k A Q_l) , \ k = 1,...,n , \ l = 1,...,m .$$

Proof. Since $A\Delta^* \in U_0^{n \times n}$, and its main diagonal has norm less than one, we get that $I - A\Delta^*$ is invertible. Hence $\hat{T}_\Delta(A)$ is well defined. Let $\Delta_i = E_i \Delta$, $i = 1,...,n-p$, then $\Delta = \sum_{i=1}^{n-p} \Delta_i$. Now $M(\Delta_{n-p}) \cdots M(\Delta_1) = M(\Delta)$, which is easy to check because of the abundance of zeroes in the matrices $M(\Delta_i)$. Hence

$$\hat{T}_\Delta = \hat{T}_{\Delta_1} \circ \cdots \circ \hat{T}_{\Delta_{n-p}}. \tag{3.5}$$

·Let $(k,l) \in \{1,...,n\} \times \{1,...,m\}$. Note that by Lemma 3.4 the block matrix $\hat{T}_{\Delta_{k+1}} \circ \cdots \circ \hat{T}_{\Delta_{n-p}}(A)$ belongs to $U_p^{n \times m}$ and its (i,j)th entry is the matrix A_{ij} for $i < k+1$. So, in particular,

$$\Delta_k = E_k A E_{p+k} = E_k(\hat{T}_{\Delta_{k+1}} \circ \cdots \circ \hat{T}_{\Delta_{n-p}}(A)) E_{p+k}.$$

Thus it suffices to prove (for $i \in \{1,...,n-p\}$) that

$$P_k \hat{T}_{\Delta_i}(B)Q_l = \hat{T}_{P_k \Delta_i Q_l}(P_k B Q_l),$$

where $B \in U_p^{n \times m}$ and $E_i B E_{p+i} = \Delta_i$. The latter follows directly from Lemma 3.4. $\Box$

Proof of Proposition 3.1. Denote $\Delta = \mathcal{D}_p(A)$. In order to prove (ii), note that $\mathcal{L}_q(A) = \mathcal{L}_q(K)$ if and only if $P_i A Q_{i+q} = P_i K Q_{i+q}$, $i = 1,...,n$. Furthermore, if $\mathcal{L}_q(A) = \mathcal{L}_q(K)$, then $\Delta = \mathcal{D}_p(A) = \mathcal{D}_p(K)$. Using Lemma 3.5 we obtain

$$P_i \hat{T}_\Delta(A)Q_{i+q} = \hat{T}_{P_i \Delta Q_{i+q}}(P_i A Q_{i+q}) = \hat{T}_{P_i \Delta Q_{i+q}}(P_i K Q_{i+q}) = P_i \hat{T}_\Delta(K)Q_{i+q}$$

for $i = 1,...,n$. But then (ii) follows immediately.

For (i), use (ii) with $q = p$ and $K = \Delta$. Since $\mathcal{L}_p(\hat{T}_\Delta(\Delta)) = 0$, this gives the desired result.

Finally, we prove (iii). If $\|P_i A Q_{i+q}\| < 1$, then Corollary 1.3 yields that $\hat{T}_{P_i \Delta Q_{i+q}}(P_i A Q_{i+q}) = P_i \hat{T}_\Delta(A)Q_{i+q}$ has norm less than one. Using this observation for all admissible i, we obtain (iii). $\Box$

Proof of Proposition 3.2. First note that $\Delta^* G \in U_1^{m \times m}$. Hence $I + \Delta^* G$ is invertible, and $T_\Delta(G)$ is well defined. Denote $H = T_\Delta(G)$ and $\tilde{H} = T_\Delta(\tilde{G})$. Since $P_i G Q_{i+q} = P_i \tilde{G} Q_{i+q}$, we get that

$$\hat{T}_{P_i \Delta Q_{i+q}}(P_i H Q_{i+q}) = P_i G Q_{i+q} = P_i \tilde{G} Q_{i+q} = \hat{T}_{P_i \Delta Q_{i+q}}(P_i \tilde{H} Q_{i+q}).$$

The injectivity of $\hat{T}_{P_i \Delta Q_{i+q}}$ yields $P_i H Q_{i+q} = P_i \bar{H} Q_{i+q}$.

To prove (3.1) note that $\Delta^* G \in U_1^{m \times m}$. Hence $\det(I + \Delta^* G) = 1$. Use now Corollary 1.3. $\square$

I.4. The strictly contractive extension problem: the general case

In this section we prove the first main theorem for the strictly contractive extension problem. Recall that for $A \in \Omega^{n \times m}$ the block matrix $\mathcal{L}_q(A)$ stands for the lower triangular part of A relative to the qth diagonal. We use the following notations: $\overset{\rightarrow}{\underset{p=i}{\Pi}}{}^{j} H_p =$
$H_i \cdots H_j$ and $\overset{\leftarrow}{\underset{p=i}{\Pi}}{}^{j} H_p = H_j \cdots H_i$, where $i \leq j$. In the case when $i > j$ these matrix products are defined to be the identity matrix.

THEOREM 4.1. *Let* $A = \left[A_{ij} \right]_{i=1,j=1}^{n \quad m} \in \Omega^{n \times m}$ *and* $q \in \{-n+1,...,m-1\}$ *be given. Suppose that*

$$\left\| \begin{bmatrix} A_{i1} & \cdots & A_{i,i+q} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{n,i+q} \end{bmatrix} \right\| < 1, i = 1,...,m-q.$$

For $j - i \leq q$ *define*

$$Z_{ij} = \beta_{ij}(I - \alpha_{ij}^* \alpha_{ij})^{-1} \alpha_{ij}^* \gamma_{ij},$$

where α_{ij}, β_{ij} *and* γ_{ij} *are given via the partitioning*

$$\left[A_{rs} \right]_{r=i,s=1}^{n \quad j} = \begin{bmatrix} \beta_{ij} & A_{ij} \\ \alpha_{ij} & \gamma_{ij} \end{bmatrix}. \tag{4.1}$$

Furthermore, let

$$\Lambda_p = \left[\delta_{i+p,j}(A_{ij} + Z_{ij}) \right]_{i=1,j=1}^{n \quad m}, p = -n+1,...,q,$$

and $\Delta_p, p = -n+1,...,q$, *be given by*

$$\Delta_{-n+1} = \Lambda_{-n+1},$$

$$\Delta_p = \overset{\leftarrow}{\underset{k=-n+1}{\Pi}}{}^{p-1} (I - \Delta_k \Delta_k^*)^{-\frac{1}{2}} \Lambda_p \overset{\rightarrow}{\underset{k=-n+1}{\Pi}}{}^{p-1} (I - \Delta_k^* \Delta_k)^{-\frac{1}{2}}. \tag{4.2}$$

Put

$$
\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \overset{\rightarrow}{\underset{p=-n+1}{\Pi}}{}^{q} \begin{bmatrix} (I-\Delta_p\Delta_p^*)^{-\frac{1}{2}} & (I-\Delta_p\Delta_p^*)^{-\frac{1}{2}}\Delta_p \\ (I-\Delta_p^*\Delta_p)^{-\frac{1}{2}}\Delta_p^* & (I-\Delta_p^*\Delta_p)^{-\frac{1}{2}} \end{bmatrix} ,
$$

Then the set of all $F \in \Omega^{n \times m}$ with $\mathscr{L}_q(A) = \mathscr{L}_q(F)$ and $\|F\| < 1$ is the set of all matrices of the form

$$
F = F(G) = (\theta_{11}G + \theta_{12})(\theta_{21}G + \theta_{22})^{-1},
$$

where G is any matrix with $\mathscr{L}_q(G) = 0$ and $\|G\| < 1$. The correspondence is 1-1. Moreover,

$$
\det(I - F(G)^*F(G)) = \underset{p \leq q}{\Pi} \det(I - \Delta_p^*\Delta_p)\det(I - G^*G), \tag{4.3}
$$

and

$$
\det(I - F(G)^*F(G)) = \underset{j-i \leq q}{\Pi} \det(I - H_{ij}^*H_{ij})\det(I - G^*G), \tag{4.4}
$$

where

$$
H_{ij} = (I - \beta_{ij}(I - \alpha_{ij}^*\alpha_{ij})^{-1}\beta_{ij}^*)^{-\frac{1}{2}}
$$

$$
(A_{ij} + \beta_{ij}(I - \alpha_{ij}^*\alpha_{ij})^{-1}\alpha_{ij}^*\gamma_{ij})(I - \gamma_{ij}^*(I - \alpha_{ij}\alpha_{ij}^*)^{-1}\gamma_{ij})^{-\frac{1}{2}} .
$$

It may happen that the matrix α_{ij} in (4.1) has a zero number of columns or a zero number of rows. In the first case α_{ij} should be understood as the linear map from the zero space to $\mathbb{C}^p$, where p stands for the number of rows of α_{ij} ($=$ number of rows of γ_{ij}). In the second case α_{ij} should be understood as the linear map from $\mathbb{C}^p$ to the zero space, and here p stands for the number of columns of α_{ij} ($=$ the number of columns of β_{ij}). A similar interpretation applies to the other matrices in (4.1). In all such cases Z_{ij} is a matrix with zero entries of the same size as A_{ij}.

The scalar version of the above theorem is the following.

THEOREM 4.2. *Let $q \in \{-n+1,\ldots,m-1\}$ and a_{ij} be given complex numbers for $1 \leq i \leq n$, $1 \leq j \leq m$ and $j-i \leq q$. Suppose that*

$$
\| \begin{bmatrix} a_{ij} \end{bmatrix}_{i=p,j=1}^{n \quad p+q} \| < 1 , \quad p = 1,\ldots,m-q.
$$

For $j-i \leq q$ define the number h_{ij} by

$$h_{ij} = \frac{(a_{ij}+\beta_{ij}(I-\alpha_{ij}^{*}\alpha_{ij})^{-1}\alpha_{ij}^{*}\gamma_{ij})}{(1-\beta_{ij}(I-\alpha_{ij}^{*}\alpha_{ij})^{-1}\beta_{ij}^{*})^{\frac{1}{2}}.(1-\gamma_{ij}^{*}(I-\alpha_{ij}\alpha_{ij}^{*})^{-1}\gamma_{ij})^{\frac{1}{2}}} , \tag{4.5}$$

where α_{ij}, β_{ij} and γ_{ij} are given via the partitioning

$$\left[a_{rs} \right]_{r=i,s=1}^{n \quad j} = \begin{bmatrix} \beta_{ij} & a_{ij} \\ \alpha_{ij} & \gamma_{ij} \end{bmatrix} .$$

Furthermore, let

$$\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \overset{q}{\underset{p=-n+1}{\Pi}} \begin{bmatrix} (I-\Delta_p\Delta_p^{*})^{-\frac{1}{2}} & (I-\Delta_p\Delta_p^{*})^{-\frac{1}{2}}\Delta_p \\ (I-\Delta_p^{*}\Delta_p)^{-\frac{1}{2}}\Delta_p^{*} & (I-\Delta_p^{*}\Delta_p)^{-\frac{1}{2}} \end{bmatrix} ,$$

where

$$\Delta_p = \left[\delta_{i+p,j}h_{ij} \right]_{i=1,j=1}^{n \quad m} , p = -n+1,...,q.$$

Then the set of all matrices $F = \left[f_{ij} \right]_{i=1,j=1}^{n \quad m} \in \mathbb{C}^{n \times m}$ with $f_{ij} = a_{ij}$, $j-i \leqq q$, and $\|F\| < 1$ is the set of all matrices of the form

$$F = F(G) = (\theta_{11}G+\theta_{12})(\theta_{21}G+\theta_{22})^{-1},$$

where $G = \left[g_{ij} \right]_{i=1,j=1}^{n \quad m}$ is any matrix with $\|G\| < 1$ and $g_{ij} = 0$ for $j-i \leqq q$. The correspondence is 1-1. Moreover,

$$\det(I-F(G)^{*}F(G)) = \{ \underset{j-i\leqq q}{\Pi} (1- |h_{ij}|^{2}) \}\det(I-G^{*}G).$$

We shall prove Theorem 4.1 using Theorem 3.3. The following proposition yields explicit formulas for the matrices Δ_i appearing in Theorem 3.3.

PROPOSTION 4.3. Let $A = \left[A_{ij} \right]_{i=1,j=1}^{n \quad m} \in \Omega^{n \times m}$ be given such that $m_{m-2}(A) < 1$. Consider

$$A_X = \begin{bmatrix} A_{11} & \cdots & A_{1,m-1} & X \\ A_{21} & \cdots & A_{2,m-1} & A_{2m} \\ \vdots & & \vdots & \vdots \\ A_{n1} & \cdots & A_{n,m-1} & A_{nm} \end{bmatrix} .$$

Then $\|A_X\| < 1$ if and only if for some strictly contractive G

$$X = Z + LGR, \qquad (4.6)$$

where

$$L = (\overset{\rightarrow}{\underset{p=-n+1}{\Pi}} {}^{m-2} (I - \Delta_p \Delta_p^*)^{\frac{1}{2}})_{11} \ , \ R = (\overset{\leftarrow}{\underset{p=-n+1}{\Pi}} {}^{m-2} (I - \Delta_p^* \Delta_p)^{\frac{1}{2}})_{mm} \ ,$$

$$Z = -\beta W_\alpha \alpha^* W_\alpha \cdot \gamma,$$

with Δ_p, $p = -n+1,...,m-1$, *defined by* (3.2) *and*

$$\alpha = \begin{bmatrix} A_{21} & \cdots & A_{2,m-1} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{n,m-1} \end{bmatrix} , \gamma = \begin{bmatrix} A_{2m} \\ \vdots \\ A_{nm} \end{bmatrix} ,$$

$$\beta = \begin{bmatrix} A_{11} & \cdots & A_{1,m-1} \end{bmatrix}.$$

Furthermore, if $\|G\| < 1$ *and X is given by* (4.6), *then*

$$\det(I - G^* G) = \det(I - H^* H) \ , \qquad (4.7)$$

where

$$H = (I - \beta(I - \alpha^* \alpha)^{-1} \beta^*)^{-\frac{1}{2}} (X - Z)(I - \gamma^*(I - \alpha \alpha^*)^{-1} \gamma)^{-\frac{1}{2}} \ . \qquad (4.8)$$

If, in addition, the element in the right upper corner is scalar, then $G = H$.

Proof. Let $\Sigma_{-n+1}(X) = A_X$ and $\Sigma_p(X) = \hat{T}_{\Delta_{p-1}}(\Sigma_{p-1}(X))$, $p = -n+2,...,m-1$. Since $\mathcal{L}_{m-2}(A_X) = \mathcal{L}_{m-2}(A)$ we get that $\mathcal{L}_{m-2}(\Sigma_p(X)) = \mathcal{L}_{m-2}(\Sigma_p)$ (where Σ_p is defined by (3.2)), $p = -n+1,...,m-1$. Suppose that $\|A_X\| < 1$. Then also $\|\Sigma_{m-1}(X)\| < 1$. Furthermore, $\mathcal{L}_{m-2}(\Sigma_{m-1}(X)) = \mathcal{L}_{m-2}(\Sigma_{m-1}) = 0$, so $(\Sigma_{m-1}(X))_{ij} = 0$ for $(i,j) \neq (1,m)$. Let $G = (\Sigma_{m-1}(X))_{1m}$. Then $\|G\| < 1$. Let us start with proving that

$$G = L^{-1}(X - \hat{Z})R^{-1}, \qquad (4.9)$$

where $\hat{Z}$ is some matrix. Later we will show that $\hat{Z} = Z$. Put $G_p = (\Sigma_p(X))_{1m}$, $p = -n+1,...,m-1$. By induction we prove that

$$G_p = \hat{L}_p(X - \hat{Z}_p)\hat{R}_p, \ p = -n+1,...,m-1, \qquad (4.10)$$

where $\hat{L}_p = \overset{\leftarrow}{\underset{k=-n+1}{\Pi}} {}^{p-1} (W_{\Delta_k})_{11}$, $\hat{R}_p = \overset{\rightarrow}{\underset{k=-n+1}{\Pi}} {}^{p-1} (W_{\Delta_k})_{mm}$ and $\hat{Z}_p$ is some matrix independent of X. Here $\hat{L}_{-n+1}$ and $\hat{R}_{-n+1}$ should be understood as being the identity matrix.

Equation (4.10) clearly holds for $p = -n+1$. Suppose that (4.10) holds for $p-1$. Consider now $\Sigma_p(X) = \hat{T}_{\Delta_{p-1}}(\Sigma_{p-1}(X))$. Let $\Delta_{p-1}^{(\nu)} = E_\nu \Delta_{p-1} E_{\nu+p-1}$, $\nu = 1,\ldots,n$. Then $\Delta_{p-1} = \sum_{\nu=1}^{n} \Delta_{p-1}^{(\nu)}$. As in the proof of Lemma 3.5 we obtain

$$\hat{T}_{\Delta_{p-1}} = \hat{T}_{\Delta_{p-1}^{(1)}} \circ \cdots \circ \hat{T}_{\Delta_{p-1}^{(n)}}.$$

Since $\Delta_{p-1} = \mathcal{D}_{p-1}(\Sigma_{p-1}) = \mathcal{D}_{p-1}(\Sigma_{p-1}(X))$, we have that $\Delta_{p-1}^{(n)} = E_n \Sigma_{p-1}(X) E_{n+p-1}$. Furthermore, $\|\Delta_{p-1}^{(n)}\| < 1$, so we can use Lemma 3.4 and obtain

$$(\hat{T}_{\Delta_{p-1}^{(n)}}(\Sigma_{p-1}(X)))_{1m} = (W_{\Delta_{p-1}^{(n)*}})_{11}(G_{p-1}-Z_{p-1}^{(n)})(W_{\Delta_{p-1}^{(n)}})_{mm},$$

where $Z_{p-1}^{(n)}$ is some matrix independent of X. Applying $\hat{T}_{\Delta_{p-1}^{(n-1)}}$ on $\hat{T}_{\Delta_{p-1}^{(n)}}(\Sigma_{p-1}(X))$ we get, using the same kind of arguments, on the $(1,m)$th entry of the result the matrix

$$(W_{\Delta_{p-1}^{(n-1)*}})_{11}\{(\hat{T}_{\Delta_{p-1}^{(n)}}(\Sigma_{p-1}(X)))_{1m} - Z_{p-1}^{(n-1)}\}(W_{\Delta_{p-1}^{(n-1)}})_{mm} =$$

$$(W_{\Delta_{p-1}^{(n-1)*}})_{11}(W_{\Delta_{p-1}^{(n)*}})_{11}(G_{p-1}-\tilde{Z})(W_{\Delta_{p-1}^{(n)}})_{mm}(W_{\Delta_{p-1}^{(n-1)}})_{mm},$$

where

$$\tilde{Z} = (W_{\Delta_{p-1}^{(n)*}})_{11}^{-1}Z_{p-1}^{(n-1)}(W_{\Delta_{p-1}^{(n)}})_{mm}^{-1}+Z_{p-1}^{(n)}.$$

Proceeding in this way we obtain

$$G_p = (\Sigma_p(X))_{1m} = \{\hat{T}_{\Delta_{p-1}^{(1)}}(\cdots(\hat{T}_{\Delta_{p-1}^{(n)}}(\Sigma_{p-1}(X)))\cdots)\}_{1m} =$$

$$= \{\overset{\rightarrow}{\underset{\nu=1}{\Pi}}^{n}(W_{\Delta_{p-1}^{(\nu)*}})_{11}\}(G_{p-1}-Z')\{\overset{\leftarrow}{\underset{\nu=1}{\Pi}}^{n}(W_{\Delta_{p-1}^{(\nu)}})_{mm}\} =$$

$$= (W_{\Delta_{p-1}^*})_{11}(G_{p-1}-Z')(W_{\Delta_{p-1}})_{mm},$$

where Z' is some matrix independent of X. The induction hypothesis yields that

$$G_p = (W_{\Delta_{p-1}^*})_{11}\{(\overset{\leftarrow}{\underset{k=-n+1}{\Pi}}^{p-2}W_{\Delta_k^*})_{11}(X-\hat{Z}_{p-1})(\overset{\rightarrow}{\underset{k=-n+1}{\Pi}}^{p-2}W_{\Delta_k})_{mm}-Z'\}(W_{\Delta_{p-1}})_{mm} =$$

$$= (\overset{\leftarrow}{\underset{k=-n+1}{\Pi}}^{p-1}W_{\Delta_k^*})_{11}(X-\hat{Z}_p)(\overset{\rightarrow}{\underset{k=-n+1}{\Pi}}^{p-1}W_{\Delta_k})_{mm},$$

where

$$\hat{Z}_p = \hat{Z}_{p-1}+(\overset{\leftarrow}{\underset{k=-n+1}{\Pi}}^{p-2}W_{\Delta_k^*})_{11}^{-1}Z'(\overset{\rightarrow}{\underset{k=-n+1}{\Pi}}^{p-2}W_{\Delta_k})_{mm}^{-1}.$$

Hence we proved (4.10). Since $G = G_{m-1}$, equation (4.9) holds.

Summarizing, we proved that

$$A_X = T_{\Delta_{-n+1}} \circ \cdots \circ T_{\Delta_{m-2}}(\tilde{G}), \tag{4.11}$$

where $\tilde{G} = \Sigma_{m-1}(X) \in \Omega_{m-2}^{n \times m}$ has norm less than one and $G = (\tilde{G})_{1m}$ is given by (4.9). Conversely, by Theorem 3.3 (with $q = m-1$), if we let $\tilde{G}$ vary over the set $\{\tilde{G} \mid \tilde{G} \in \Omega_{m-2}^{n \times m}, \|\tilde{G}\| < 1\}$, we obtain via formula (4.11) all A_X with $\|A_X\| < 1$. The corresponding X is given via (4.9) where $G = (\tilde{G})_{1m}$. Furthermore, the determinant formula of Theorem 3.3 gives that

$$\det(I - A_X^* A_X) = c_1 \det(I - \tilde{G}^* \tilde{G}) = c_1 \det(I - G^* G), \tag{4.12}$$

where $c_1 > 0$. So the maximum entropy solution A_{X_0} is obtained for $G = 0$, giving $X_0 = \hat{Z}$. On the other hand, viewing A_X as the 2×2 block matrix

$$\begin{bmatrix} \beta & X \\ \alpha & \gamma \end{bmatrix}$$

one concludes from the results in Section 2 that (4.8) gives a 1-1 correspondence between the set $\{H \mid \|H\| < 1\}$ and all matrices X such that $\|A_X\| < 1$ (use the description given in (2.10)). Furthermore,

$$\det(I - A_X^* A_X) = c_2 \det(I - H^* H), \tag{4.13}$$

where $c_2 > 0$ (use formula (2.7)). Hence we can conclude that the maximum entropy solution A_{X_0} is obtained for $H = 0$, which according to (4.8) corresponds to $X_0 = Z$. So $Z = X_0 = \hat{Z}$. But then (4.9) implies (4.6). Further, using (4.12) and (4.13) with $X = X_0$, we get that

$$c_1 = \det(I - A_{X_0}^* A_{X_0}) = c_2.$$

Now (4.7) follows from (4.12) and (4.13).

Finally, if the X is scalar, so are G and H. Equation (4.7) then implies $|G| = |H|$. Since both G and H are a product of $X - Z$ and some positive number, we obtain equality. □

Since L and R are invertible, equation (4.6) gives a 1-1 correspondence between X and G. The proof of Proposition 4.3 shows that G as a function of X is given by

$$G = (\Sigma_{m-1}(X))_{1m} = (\hat{T}_{\Delta_{m-2}}(\cdots (\hat{T}_{\Delta_{-n+1}}(A_X)) \cdots))_{1m}. \tag{4.14}$$

Proof of Theorem 4.1. In order to prove that the map $F(G)$ in Theorem 4.1 is the same as in Theorem 3.3., we have to prove that the Δ_p's in (4.2) are the same as in (3.2). When this is done one uses Proposition 1.1(i) to conclude that both maps $F(G)$ are the same. Fix $p \in \{-n+1,...,q\}$. According to (3.2) the (r,s)th coefficient in Δ_p, where $r-s = p$, is equal to the (r,s)th coefficient in Σ_p. Since

$$P_r \Sigma_p Q_s = \hat{T}_{P_r \Delta_{p-1} Q_s}(\cdot \cdot \cdot (\hat{T}_{P_r \Delta_{-n+1} Q_s}(P_r A Q_s)) \cdot \cdot \cdot)$$

(use Lemma 3.5), we can focus only on $P_r A Q_s$ and find ourselves in the situation of Proposition 4.3 with $X = A_{rs}$. The matrix $(\Sigma_p)_{rs}$ plays according to (4.14) the role of the G in Proposition 4.3. So using (4.6) we can conclude that

$$(\Delta_p)_{rs} = (\Sigma_p)_{rs} = \overset{\leftarrow}{\underset{\nu=-n+1}{\Pi}}^{p-1} (W_{(P_r \Delta_s Q_s)}{}^{\cdot})_{rr} (A_{rs} + Z_{rs}) \overset{\rightarrow}{\underset{\nu=-n+1}{\Pi}}^{p-1} (W_{P_r \Delta_s Q_s})_{ss}$$

$$= (\overset{\leftarrow}{\underset{\nu=-n+1}{\Pi}}^{p-1} W_{\Delta_{\cdot}})_{rr} (A_{rs} + Z_{rs})(\overset{\rightarrow}{\underset{\nu=-n+1}{\Pi}}^{p-1} W_{\Delta_{\cdot}})_{ss}.$$

Calculating the (r,s)th entry of the right hand side of (4.2) one obtains the same matrix. Hence the definitions of Δ_p in (3.2) and (4.2) coincide.

Formula (4.3) follows directly from Theorem 3.3. Rewriting (4.3) we get

$$\det(I - F(G)^* F(G)) = (\underset{p \leq q}{\Pi} \{ \underset{r-s=p}{\Pi} \det(I - (\Delta_p)_{rs}^*(\Delta_p)_{rs}) \}) \det(I - G^* G).$$

By Proposition 4.3 we have that $\det(I - (\Delta_p)_{rs}^*(\Delta_p)_{rs}) = \det(I - H_{rs}^* H_{rs})$. But then (4.4) follows. $\square$

Proof of Theorem 4.2. Let $r-s = p$. Note that $(\Delta_p)_{rs}$ is scalar. Using the last sentence of Proposition 4.3 with G replaced by $(\Delta_p)_{rs}$ we get that $(\Delta_p)_{rs} = H_{rs}$. Theorem 4.1 now implies Theorem 4.2. $\square$

Let $\theta := \left[\theta_{ij} \right]_{i,j=1}^{2}$ be as in Theorem 4.1. Since $\theta = \overset{\rightarrow}{\underset{p=-n+1}{\Pi}}^{q} M(\Delta_p)$, where $M(\Delta)$ is defined in (1.8), the matrix θ is J-unitary. Using this, one calculates that

$$F(0)(I - F(0)^* F(0))^{-1} = \theta_{12} \theta_{22}^{-1} \{\theta_{22}^{*-1}(\theta_{22}^* \theta_{22} - \theta_{12}^* \theta_{12})\theta_{22}^{-1}\}^{-1} = \theta_{12} \theta_{22}^*.$$

It is not hard to see that $\mathscr{D}_p(\theta_{12}) = 0$ for $p > q$ and $\theta_{22} \in U_0^{m \times m}$. Hence for the maximum entropy solution $F(0)$ we have

$$\mathscr{D}_p(F(0)(I - F(0)^* F(0))^{-1}) = 0 , p > q.$$

In other words, $F(0)(I-F(0)^*F(0))^{-1}$ is lower triangular with respect to the q-th diagonal. This result also appears in Theorem III.9 in [5]. We shall return to this connection in Section III.2.

I.5. Reduction of the positive extension problem to a strictly contractive one

To solve the problem of positive extensions we need the following linear fractional maps. Let Δ be a positive definite matrix. Define

$$R_\Delta(G) := \Delta^{1/2}(G+I)(-G+I)^{-1}\Delta^{1/2},$$

$$\hat{R}_\Delta(G) := \Delta^{1/2}(\Delta+G)^{-1}(G-\Delta)\Delta^{-1/2},$$

where G is a matrix of the same size as Δ. Let $N(\Delta)$ be the $(\bar{J},J)$-unitary matrix

$$N(\Delta) = \tfrac{1}{2}\sqrt{2} \begin{bmatrix} \Delta^{1/2} & \Delta^{1/2} \\ -\Delta^{-1/2} & \Delta^{-1/2} \end{bmatrix}. \tag{5.1}$$

Note that R_Δ and $\hat{R}_\Delta$ are the linear fractional maps $\mathcal{M}_{N(\Delta)}$ and $\hat{\mathcal{M}}_{N(\Delta)}$, respectively. The following corollary is now a direct consequence of Propositions 1.1 and 1.2.

COROLLARY 5.1. *Let* $\Delta > 0$. *Then*

$$R_\Delta : \{ G \mid \|G\| < 1\} \to \{ G \mid G+G^* > 0\}$$

is bijective, and its inverse is $\hat{R}_\Delta$. *Furthermore, if* $\det(I-G) \neq 0$, *then*

$$\det(R_\Delta(G)+R_\Delta(G)^*) = \det 2\Delta \ |\det(I-G)|^{-2} \det(I-G^*G). \tag{5.2}$$

The next theorem shows that the positive extension problem may be solved by reduction to a strictly contractive extension problem. First we introduce some notations. Let

$$B = \begin{bmatrix} B_{ij} \end{bmatrix}_{i,j=1}^{n} \in \Omega^{n \times n}$$ and fix a number $q \geq 0$. We write $\mathrm{diag}_q(B) > 0$ if

$$\begin{bmatrix} B_{ii} & \cdots & B_{i,i+q} \\ \vdots & & \vdots \\ B_{i+q,i} & \cdots & B_{i+q,i+q} \end{bmatrix} > 0 \ , \ i = 1,...,n-q.$$

With a $B \in \Omega^{n \times n}$ satisfying $\mathrm{diag}_q(B) > 0$ we associate the following strict upper triangular matrix $\bar{B}$

$$\bar{B} = \hat{R}_{\frac{1}{2}\mathcal{D}_0(B)}(\frac{1}{2}\mathcal{D}_0(B) + \sum_{i>0} \mathcal{D}_i(B)). \tag{5.3}$$

We shall later see that $m_q(\bar{B}) < 1$. The next theorem shows that the positive extension problem for B reduces to a strictly contractive extension problem for $\bar{B}$. The main idea here can also be interpreted as a reduction principle which employs inverse scattering (see, e.g., [19]).

THEOREM 5.2. *Let* $q \geq 0$, *and let* $B = \left[B_{ij} \right]_{i,j=0}^{n} \in \Omega^{n \times n}$ *be such that*

diag $_q(B) > 0$. *Put* $\Delta = \frac{1}{2}\mathcal{D}_0(B)$. *Then the set*

$$\{ R_\Delta(F) + R_\Delta(F)^* \mid \|F\| < 1, \mathcal{L}_q(F) = \mathcal{L}_q(\bar{B}) \}$$

is the set of all $C = \left[C_{ij} \right]_{i,j=1}^{n}$ *with* $C > 0$ *and* $C_{ij} = B_{ij}$, $|j-i| \leq q$. *The correspondence is 1-1. Furthermore,*

$$\det(R_\Delta(F) + R_\Delta(F)^*) = \det 2\Delta \det(I - F^*F). \tag{5.4}$$

Let P_{ij}, $1 \leq i \leq j \leq n$, be the block matrix of size $(j - i + 1) \times n$ given by

$$P_{ij} = \begin{array}{c} \quad\;\; i \quad\;\; j \\ \quad\;\; \downarrow \quad\; \downarrow \\ \left[\begin{array}{cccc} & I & & \\ & & \cdot & \\ & & & \cdot \\ & & & & I \end{array} \right] \end{array}.$$

So, if $B = \left[B_{ij} \right]_{i,j=1}^{n}$, then

$$P_{ij} B P_{ij}^* = \begin{bmatrix} B_{ii} & \cdots & B_{ij} \\ \vdots & & \vdots \\ B_{ji} & \cdots & B_{jj} \end{bmatrix}.$$

Note that if $q \geq 0$ and $A, K \in U_0^{n \times n}$, then $\mathcal{L}_q(A) = \mathcal{L}_q(K)$ if and only if $P_{i,i+q} A P_{i,i+q}^* = P_{i,i+q} K P_{i,i+q}^*$, $i = 1,\ldots,n-q$.

LEMMA 5.3. *Let* $A \in U_0^{n \times n}$ *with* $\Delta = \mathcal{D}_0(A) > 0$. *Then* $\hat{R}_\Delta(A)$ *is well-defined and*

$$P_{ij} \hat{R}_\Delta(A) P_{ij}^* = \hat{R}_{P_{ij}\Delta P_{ij}^*}(P_{ij} A P_{ij}^*), \quad 1 \leq i \leq j \leq n.$$

Proof. Since $A + \Delta \in U_0^{n \times n}$ and its main diagonal 2Δ is positive definite, $\Delta + A$ is invertible. So $\hat{R}_\Delta(A)$ is well-defined. Write

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \Delta = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix},$$

such that $A_{22} = P_{ij}AP_{ij}^*$ and $\Delta_2 = P_{ij}\Delta P_{ij}^*$. Straightforward calculations yield the lemma. $\square$

PROPOSITION 5.4. *Let $q \geq 0$ and $A \in U_0^{n \times n}$ with $\mathscr{D}_0(A) > 0$. Then*

(i) $\hat{R}_{\mathscr{D}_0(A)}(A) \in U_1^{n \times n}$;

(ii) *If $\mathscr{L}_q(A) = \mathscr{L}_q(K)$, then $\mathscr{L}_q(\hat{R}_{\mathscr{D}_0(A)}(A)) = \mathscr{L}_q(\hat{R}_{\mathscr{D}_0(A)}(K))$;*

(iii) *If $\mathrm{diag}_q(A + A^*) > 0$, then $m_q(\hat{R}_{\mathscr{D}_0(A)}(A)) < 1$;*

Proof. Put $\Delta = \mathscr{D}_0(A)$. Since $(A + \Delta)^{-1} \in U_0^{n \times n}$ and $A - \Delta \in U_1^{n \times n}$, (i) follows directly.

For (ii), note the following. If $P_{i,i+q}AP_{i,i+q}^* = P_{i,i+q}KP_{i,i+q}$, then by Lemma 5.3

$$P_{i,i+q}\hat{R}_\Delta(A)P_{i,i+q}^* = P_{i,i+q}\hat{R}_\Delta(K)P_{i,i+q}^*.$$

Using this for $i = 1,...,n - q$ yields (ii).

If $P_{i,i+q}(A + A^*)P_{i,i+q}^* > 0$, $i = 1,...,n - q$, then, by Corollary 5.1, the matrices

$$P_{i,i+q}\hat{R}_\Delta(A)P_{i,i+q}^* = \hat{R}_{P_{i,i+q}\Delta P_{i,i+q}^*}(P_{i,i+q}AP_{i,i+q}^*), \quad i = 1,...,n - q,$$

have norm less than one. Since also $\hat{R}_\Delta(A) \in U_1^{n \times n}$, it follows that $m_q(\hat{R}_\Delta(A)) < 1$. $\square$

PROPOSITION 5.5. *Let $q \geq 0$ and $K, \tilde{K} \in U_1^{n \times n}$ be such that $\mathscr{L}_q(K) = \mathscr{L}_q(\tilde{K})$. Furthermore, let $\Delta = \mathscr{D}_0(\Delta)$ be positive definite. Then*

$$\mathscr{L}_q(R_\Delta(K)) = \mathscr{L}_q(R_\Delta(\tilde{K})).$$

Furthermore,

$$\det(R_\Delta(K) + R_\Delta(K)^*) = \det 2\Delta \, \det(I - K^*K).$$

Proof. Put $H = R_\Delta(K)$ and $\tilde{H} = R_\Delta(\tilde{K})$. Since $P_{i,i+q}KP_{i,i+q}^* = P_{i,i+q}\tilde{K}P_{i,i+q}^*$, we get by Lemma 5.3 that

$$\hat{R}_{P_{i,i+q}\Delta P_{i,i+q}^*}(P_{i,i+q}HP_{i,i+q}^*) = \hat{R}_{P_{i,i+q}\Delta P_{i,i+q}^*}(P_{i,i+q}\tilde{H}P_{i,i+q}^*).$$

But then the injectivity of the map $\hat{R}_{P_{i,i+q} \Delta P^*_{i,i+q}}$ yields $P_{i,i+q} H P^*_{i,i+q} = P_{i,i+q} \bar{H} P^*_{i,i+q}$.

The determinant formula follows directly from the determinant formula in Corollary 5.1. $\square$

Proof of Theorem 5.2. Let $C = \left[C_{ij} \right]_{i,j=1}^{n}$ be such that $C > 0$ and $C_{ij} = B_{ij}$, $|j-i| \leq q$. Let $K \in U_0^{n \times m}$ be such that $K + K^* = C$ and $\mathcal{D}_0(K) = \Delta$. Put $F = \hat{R}_\Delta(K)$. By Proposition 5.4(ii), $\mathcal{L}_q(F) = \mathcal{L}_q(\bar{B})$. Furthermore, Corollary 5.1 yields $\|F\| < 1$.

Conversely, let $K = R_\Delta(F)$ where $\|F\| < 1$ and $\mathcal{L}_q(F) = \mathcal{L}_q(\bar{B})$. Corollary 5.1 gives that $K + K^* > 0$. With Proposition 5.5 one concludes that $\mathcal{L}_q(K) = \mathcal{L}_q(R_\Delta(\bar{B}))$. Since $B = R_\Delta(\bar{B}) + R_\Delta(\bar{B})^*$, the matrix $C := K + K^*$ satisfies $\mathcal{D}_p(C) = \mathcal{D}_p(B)$ for $-q \leq p \leq q$.

Proposition 5.4(i) yields $F \in U_1^{n \times n}$. Thus $\det(I - F) = 1$. Now use (5.2) to obtain (5.4). $\square$

To finish this section let us illustrate the reduction procedure on the 3×3 case.

EXAMPLE: the 3×3 block case. Let $\alpha, \beta, \gamma, \zeta$ and η be given matrices such that

$$\begin{bmatrix} \alpha & \beta \\ \beta^* & \gamma \end{bmatrix} > 0 \, , \quad \begin{bmatrix} \gamma & \zeta \\ \zeta^* & \eta \end{bmatrix} > 0 \, .$$

We want to find all matrices X such that

$$B_X = \begin{bmatrix} \alpha & \beta & X \\ \beta^* & \gamma & \zeta \\ X^* & \zeta^* & \eta \end{bmatrix} > 0.$$

Note that

$$\tfrac{1}{2}\mathcal{D}_0(B_X) + \mathcal{D}_1(B_X) + \mathcal{D}_2(B_X) = \begin{bmatrix} \tfrac{1}{2}\alpha & \beta & X \\ 0 & \tfrac{1}{2}\gamma & \zeta \\ 0 & 0 & \tfrac{1}{2}\eta \end{bmatrix} \, , \quad \Delta = \begin{bmatrix} \tfrac{1}{2}\alpha & 0 & 0 \\ 0 & \tfrac{1}{2}\gamma & 0 \\ 0 & 0 & \tfrac{1}{2}\eta \end{bmatrix} \, ,$$

and thus

$$\bar{B}_X = \begin{bmatrix} 0 & \alpha^{-1/2}\beta\gamma^{-1/2} & \alpha^{-1/2}(X - \beta\gamma^{-1}\zeta)\eta^{-1/2} \\ 0 & 0 & \gamma^{-1/2}\zeta\eta^{-1/2} \\ 0 & 0 & 0 \end{bmatrix} \, .$$

All Y such that

$$C_Y = \begin{bmatrix} 0 & \alpha^{-\frac12}\beta\gamma^{-\frac12} & Y \\ 0 & 0 & \gamma^{-\frac12}\zeta\eta^{-\frac12} \\ 0 & 0 & 0 \end{bmatrix}$$

has norm less than one, are given by (cf. (2.10))

$$Y = (I-\alpha^{-\frac12}\beta\gamma^{-1}\beta^*\alpha^{-\frac12})^{\frac12} \, G \, (I-\eta^{-\frac12}\zeta^*\gamma^{-1}\zeta\eta^{-\frac12})^{\frac12} \,,$$

where $\|G\| < 1$. So

$$X = \beta\gamma^{-1}\zeta + \alpha^{\frac12} Y \eta^{\frac12} =$$

$$\beta\gamma^{-1}\zeta + \alpha^{\frac12}(I-\alpha^{-\frac12}\beta\gamma^{-1}\beta^*\alpha^{-\frac12})^{\frac12} \, G \, (I-\eta^{-\frac12}\zeta^*\gamma^{-1}\zeta\eta^{-\frac12})^{\frac12}\eta^{\frac12} \,.$$

The unique solution C_Y with maximal determinant is obtained for $Y = 0$. The matrix $B_X \,(> 0)$ for which the determinant is as large as possible (the so-called 'maximum entropy solution') is therefore B_{X_0}, where $X_0 = \beta\gamma^{-1}\zeta$.

I.6. The positive extension problem

THEOREM 6.1. *Let* $B = \begin{bmatrix} B_{ij} \end{bmatrix}_{i,j=1}^{n} \in \Omega^{n \times n}$ *and* $q \in \{0,...,n-1\}$ *be given. Suppose that*

$$\begin{bmatrix} B_{ii} & \cdots & B_{i,i+q} \\ \vdots & & \vdots \\ B_{i+q,i} & \cdots & B_{i+q,i+q} \end{bmatrix} > 0 \, , \, i = 1,...,n-q.$$

For $0 < j-i \leq q$ *define the matrix* Z_{ij} *by*

$$Z_{ij} = -\beta_{ij}\gamma_{ij}^{-1}\zeta_{ij} \,,$$

where β_{ij}, γ_{ij} *and* ζ_{ij} *are given via the partitioning*

$$\begin{bmatrix} B_{rs} \end{bmatrix}_{r,s=i}^{j} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} & B_{ij} \\ \beta_{ij}^* & \gamma_{ij} & \zeta_{ij} \\ B_{ij}^* & \zeta_{ij}^* & \eta_{ij} \end{bmatrix} \,. \tag{6.1}$$

Furthermore, let

$$\Lambda_p = \begin{bmatrix} \delta_{i+p,j}(B_{ij}+Z_{ij}) \end{bmatrix}_{i,j=1}^{n} \,, p = 1,...,q \,,$$

and Δ_p, $p = 0,...,q$, *be given by*

$$\Delta_0 = \text{diag} \left[\tfrac{1}{2} B_{ii} \right]_{i=1}^n , \ \Delta_1 = \tfrac{1}{2}\Delta_0^{-\frac{1}{2}}\Lambda_1\Delta_0^{-\frac{1}{2}} ,$$

$$\Delta_p = \tfrac{1}{2}\overset{\leftarrow p-1}{\underset{k=1}{\Pi}}(I-\Delta_k\Delta_k^*)^{-\frac{1}{2}}\Delta_0^{-\frac{1}{2}}\Lambda_p\Delta_0^{-\frac{1}{2}}\overset{\rightarrow p-1}{\underset{k=1}{\Pi}}(I-\Delta_k^*\Delta_k)^{-\frac{1}{2}} , p = 2,...,q.$$

Put

$$\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \begin{bmatrix} \Delta_0^{\frac{1}{2}} & \Delta_0^{\frac{1}{2}} \\ -\Delta_0^{-\frac{1}{2}} & \Delta_0^{-\frac{1}{2}} \end{bmatrix} \overset{\rightarrow}{\underset{p=1}{\Pi}} q \begin{bmatrix} (I-\Delta_p\Delta_p^*)^{-\frac{1}{2}} & (I-\Delta_p\Delta_p^*)^{-\frac{1}{2}}\Delta_p \\ (I-\Delta_p^*\Delta_p)^{-\frac{1}{2}}\Delta_p^* & (I-\Delta_p^*\Delta_p)^{-\frac{1}{2}} \end{bmatrix} .$$

Then the set of all $F = \left[F_{ij} \right]_{i,j=1}^n \in \Omega^{n \times n}$ *with* $F_{ij} = B_{ij}$, $-q \leq j-i \leq q$ *and* $F > 0$ *is the set of all matrices of the form* $F = T(G)+T(G)^*$, *where*

$$T(G) = (\theta_{11}G+\theta_{12})(\theta_{21}G+\theta_{22})^{-1}$$

and G *is any matrix with* $L_q(G) = 0$ *and* $\|G\| < 1$. *The correspondence is 1-1. More-over,*

$$\det(T(G)+T(G)^*) = \overset{n}{\underset{i=1}{\Pi}}\det B_{ii} \underset{0<p\leq q}{\Pi}\det(I-\Delta_p^*\Delta_p)\det(I-G^*G), \qquad (6.2)$$

and

$$\det(T(G)+T(G)^*) = \overset{n}{\underset{i=1}{\Pi}}\det B_{ii} \underset{0<j-i\leq q}{\Pi}\det(I-H_{ij}^*H_{ij})\det(I-G^*G), \qquad (6.3)$$

where

$$H_{ij} = (\alpha_{ij}-\beta_{ij}\gamma_{ij}^{-1}\beta_{ij}^*)^{-\frac{1}{2}}(B_{ij}-\beta_{ij}\gamma_{ij}^{-1}\varsigma_{ij})(\eta_{ij}-\varsigma_{ij}^*\gamma_{ij}^{-1}\varsigma_{ij})^{-\frac{1}{2}}$$

and $\alpha_{ij},\beta_{ij},\gamma_{ij},\varsigma_{ij}$ *and* η_{ij} *are as in* (6.1).

For the scalar problem we have the following result.

THEOREM 6.2. *Let* $q \in \{0,...,n-1\}$, *and* $b_{ij} = \bar{b}_{ji}$ *be given complex numbers for* $1 \leq i,j \leq n$ *and* $0 \leq j-i \leq q$. *Suppose that*

$$\left[b_{ij} \right]_{i,j=p}^{p+q} > 0 , \ p = 1,...,n-q.$$

For $0 < j-i \leq q$ *define the number* h_{ij} *by*

$$h_{ij} = \frac{b_{ij}-\beta_{ij}\gamma_{ij}^{-1}\varsigma_{ij}}{(\alpha_{ij}-\beta_{ij}\gamma_{ij}^{-1}\beta_{ij}^*)^{\frac{1}{2}}(\eta_{ij}-\varsigma_{ij}^*\gamma_{ij}^{-1}\varsigma_{ij})^{\frac{1}{2}}}, \qquad (6.4)$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \varsigma_{ij}$ *and* η_{ij} *are given via the partitioning*

$$\left[b_{rs} \right]_{r,s=i}^{j} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} & b_{ij} \\ \beta_{ij}^* & \gamma_{ij} & \varsigma_{ij} \\ b_{ij}^* & \varsigma_{ij}^* & \eta_{ij} \end{bmatrix}.$$

Furthermore, let

$$\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \begin{bmatrix} \Delta_0^{\frac{1}{2}} & \Delta_0^{\frac{1}{2}} \\ -\Delta_0^{-\frac{1}{2}} & \Delta_0^{-\frac{1}{2}} \end{bmatrix} \overset{\rightarrow}{\underset{p=1}{\Pi}} q \begin{bmatrix} (I-\Delta_p\Delta_p^*)^{-\frac{1}{2}} & (I-\Delta_p\Delta_p^*)^{-\frac{1}{2}}\Delta_p \\ (I-\Delta_p^*\Delta_p)^{-\frac{1}{2}}\Delta_p^* & (I-\Delta_p^*\Delta_p)^{-\frac{1}{2}} \end{bmatrix},$$

where

$$\Delta_0 = \text{diag} \left[\tfrac{1}{2}b_{ii} \right]_{i=1}^{n}, \Delta_p = \left[\delta_{i+p,j}h_{ij} \right]_{i,j=1}^{n}, p = 1,\dots,q.$$

Then the set of all matrices $F = \left[f_{ij} \right]_{i,j=1}^{n} \in \mathbb{C}^{n \times n}$ *with* $f_{ij} = b_{ij}$, $-q \leqq j-i \leqq q$, *and* $F > 0$ *is the set of all matrices of the form* $F = T(G)+T(G)^*$, *where*

$$T(G) = (\theta_{11}G + \theta_{12})(\theta_{21}G + \theta_{22})^{-1}$$

and $G = \left[g_{ij} \right]_{i,j=1}^{n}$ *is any matrix with* $\|G\| < 1$ *and* $g_{ij} = 0$ *for* $j-i \leqq q$. *The correspondence is 1-1. Moreover,*

$$\det(T(G)+T(G)^*) = \overset{n}{\underset{i=1}{\Pi}} b_{ii} \cdot \underset{0<j-i\leqq q}{\Pi} (1-|h_{ij}|^2) \cdot \det(I-G^*G).$$

We shall prove Theorem 6.1 using Theorem 5.2.

LEMMA 6.3. *Let* $q \geqq 0$ *and* $B = \left[B_{ij} \right]_{i,j=1}^{n}$ *such that* $\text{diag}_q(B) > 0$. *Denote* $\Delta = \tfrac{1}{2}\mathscr{D}_0(B)$ *and let* $\bar{B} = \left[\bar{B}_{ij} \right]_{i,j=1}^{n}$ *be defined as in (5.3). Fix* (r,s) *such that* $0 < s-r \leqq q$. *Then*

$$\bar{B}_{rs} + b(I-a^*a)^{-1}a^*c = \tfrac{1}{2}\Delta_{rr}^{-\frac{1}{2}}(B_{rs}-\beta\gamma^{-1}\varsigma)\Delta_{ss}^{-\frac{1}{2}}, \tag{6.5}$$

where a, b, c, β, γ *and* ς *are given via the partitionings*

$$\left[\bar{B}_{ij} \right]_{i=r,j=1}^{n,\ s} = \begin{bmatrix} b & \bar{B}_{rs} \\ a & c \end{bmatrix}, \left[B_{ij} \right]_{i,j=r}^{s} = \begin{bmatrix} \alpha & \beta & B_{rs} \\ \beta^* & \gamma & \varsigma \\ B_{rs}^* & \varsigma^* & \eta \end{bmatrix}.$$

Furthermore, if

$$\hat{H} = (I - b(I - a^*a)^{-1}b^*)^{-\frac{1}{2}}(\bar{B}_{rs} + b(I - a^*a)^{-1}a^*c)(I - c^*(I - aa^*)^{-1}c)^{-\frac{1}{2}}$$

and

$$H = (\alpha - \beta\gamma^{-1}\beta^*)^{-\frac{1}{2}}(B_{rs} - \beta\gamma^{-1}\zeta)(\eta - \zeta^*\gamma^{-1}\zeta)^{-\frac{1}{2}},$$

then

$$\det(I - \hat{H}^*\hat{H}) = \det(I - H^*H). \tag{6.6}$$

When B_{rs} is scalar, then $\hat{H} = H$.

Proof. Let us start by proving that we may assume that $q = n - 1$, $r = 1$ and $s = n$. Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be given by

$$\left[\bar{B}_{ij}\right]_{i,j=r}^{s} = \begin{pmatrix} \hat{b} & \bar{B}_{rs} \\ \hat{a} & \hat{c} \end{pmatrix}.$$

Since $\bar{B} \in U_1^{n \times n}$, we have that

$$b = \begin{bmatrix} 0 & \hat{b} \end{bmatrix}, a = \begin{bmatrix} 0 & \hat{a} \\ 0 & 0 \end{bmatrix}, c = \begin{bmatrix} \hat{c} \\ 0 \end{bmatrix}.$$

It is easy to see that if in all the expressions in the lemma we replace a, b and c by $\hat{a}$, $\hat{b}$ and $\hat{c}$, respectively, the results do not change. So only the entries (i,j), $i,j \in \{r,...,s\}$, of the block matrices B and $\bar{B}$ are of importance. So without loss of generality we can assume that the entry (r,s) is in the upper right corner.

Consider the self-adjoint block matrix $B_X \in \Omega^{n \times n}$ which has an X on the $(1,n)$th place and B_{ij} on the (i,j)th place where $0 \leq |j-i| < q = n-1$. Note that B and B_X have the same main diagonal. Further, we denote by $\hat{B}_Y$ the block matrix $\bar{B}$ where $\bar{B}_{1n}$ is replaced by Y. Suppose that X is chosen so that $B_X > 0$. (For instance, $X = B_{1n}$.) Let $\bar{B}_X$ be defined as in (5.3) with B replaced by B_X. From Proposition 5.2(ii) it follows that $\bar{B}_X = \hat{B}_Y$ for some Y. Moreover,

$$Y = (\hat{B}_Y)_{1n} = (\bar{B}_X)_{1n} = \frac{1}{2}\Delta_{11}^{-\frac{1}{2}}(X + \Phi)\Delta_{nn}^{-\frac{1}{2}},$$

for some matrix Φ independent of X. Adding $b(I - a^*a)^{-1}a^*c$ on both sides gives

$$Y + b(I - a^*a)^{-1}a^*c = \frac{1}{2}\Delta_{11}^{-\frac{1}{2}}(X + \tilde{\Phi})\Delta_{nn}^{-\frac{1}{2}}, \tag{6.7}$$

where $\tilde{\Phi}$ is some matrix independent of X. Since

$$\hat{B}_Y = \begin{bmatrix} b & Y \\ a & c \end{bmatrix},$$

we get that for this 2×2 strictly contractive extension problem, where Y is the unknown, the maximum entropy solution is $\hat{B}_{Y_0}$ with $Y_0 = -b(I-a^*a)^{-1}a^*c$. Following (6.7) the corresponding X is $-\tilde{\Phi}$. On the other hand, viewing B_X as the 3×3 matrix

$$B_X = \begin{bmatrix} \alpha & \beta & X \\ \beta^* & \gamma & \zeta \\ X^* & \zeta^* & \eta \end{bmatrix},$$

we see that the maximum entropy solution B_{X_0} is obtained for $X_0 = \beta\gamma^{-1}\zeta$. Hence $-\tilde{\Phi} = \beta\gamma^{-1}\zeta$, and thus

$$Y+b(I-a^*a)^{-1}a^*c = \tfrac{1}{2}\Delta_{11}^{-\frac{1}{2}}(X-\beta\gamma^{-1}\zeta)\Delta_{nn}^{-\frac{1}{2}}. \tag{6.8}$$

Substituting $X = B_{1n}$ and $Y = \overline{B}_{1n}$ in (6.8) yields (6.5).

Let

$$\hat{H}_Y = (I-b(I-a^*a)^{-1}b^*)^{-\frac{1}{2}}(Y+b(I-a^*a)^{-1}a^*c)(I-c^*(I-aa^*)^{-1}c)^{-\frac{1}{2}},$$

$$H_X = (\alpha-\beta\gamma^{-1}\beta^*)^{-\frac{1}{2}}(X-\beta\gamma^{-1}\zeta)(\eta-\zeta^*\gamma^{-1}\zeta)^{-\frac{1}{2}}.$$

We know that (cf. Section 2)

$$\det(I-\hat{B}_Y^*\hat{B}_Y) = c_0\det(I-\hat{H}_Y^*\hat{H}_Y),$$

where $c_0 > 0$. So with (5.4) we may conclude that

$$\det B_X = \det(\overline{B}_X+\overline{B}_X^*) = \tilde{c}\det(I-\hat{H}_Y^*\hat{H}_Y), \tag{6.9}$$

where $\tilde{c} > 0$. On the other hand, one calculates that $B_X = \Phi_X^*\Phi_X$, where

$$\Phi_X = \begin{bmatrix} (\alpha-\beta\gamma^{-1}\beta^*)^{\frac{1}{2}} & 0 & (\alpha-\beta\gamma^{-1}\beta^*)^{-\frac{1}{2}}(X-\beta\gamma^{-1}\zeta) \\ 0 & \gamma^{\frac{1}{2}} & 0 \\ 0 & 0 & \{(\eta-\zeta^*\gamma^{-1}\zeta)^{\frac{1}{2}}(I-H_X^*H_X)(\eta-\zeta^*\gamma^{-1}\zeta)^{\frac{1}{2}}\}^{\frac{1}{2}} \end{bmatrix}$$

$$\begin{bmatrix} I & 0 & 0 \\ \gamma^{-1}\beta^* & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & \gamma^{-1}\zeta \\ 0 & 0 & I \end{bmatrix}.$$

So,

$$\det B_X = |\det \Phi_X|^2 = c' \det(I - H_X^* H_X),\qquad (6.10)$$

where $c' > 0$. If $X = X_0$ and $Y = Y_0$ we obtain by (6.9) and (6.10) that

$$\bar{c} = \det B_{X_0} = c'.$$

Equations (6.9) and (6.10) now give that

$$\det(I - \hat{H}_Y^* \hat{H}_Y) = \det(I - H_X^* H_X).$$

Filling in $X = B_{1n}$ and $Y = \bar{B}_{1n}$ we obtain (6.6).

If B_{1n} is scalar, then so are $\bar{B}_{1n}$, H and $\hat{H}$. Identity (6.6) then yields $|H| = |\hat{H}|$. Using (6.5) we see that $\hat{H}$ is a product of a positive number and $B_{1n} - \beta \gamma^{-1} \zeta$. For H the same is true. Hence we obtain equality. $\square$

Proof of Theorem 6.1. Let $B = \left[B_{ij} \right]_{i,j=1}^{n}$ be such that diag $_q(B) > 0$. By Theorem 5.2 the set of all matrices C with $P_{i,i+q} C P_{i,i+q}^* = P_{i,i+q} B P_{i,i+q}^*$, $i = 1,...,n-q$, and $C > 0$ is the set

$$\{ K + K^* \mid K = R_{\Delta_0}(F) , \|F\| < 1 , \mathscr{L}_q(F) = \mathscr{L}_q(\bar{B}) \}.$$

Denote $\bar{B} = \left[\bar{B}_{ij} \right]_{i,j=1}^{n}$. Theorem 4.1 gives a linear fractional description of all F with $\|F\| < 1$ and $\mathscr{L}_q(F) = \mathscr{L}_q(\bar{B})$. Let us denote the sequence of Δ's we obtain from Theorem 4.1 by $\hat{\Delta}_{-n+1}, \ldots, \hat{\Delta}_q$. Note that since $\bar{B} \in U_1^{n \times n}$, we get that $\hat{\Delta}_j = 0$ for $j \leq 0$. Further, using (6.5) we see that $\Delta_j = \hat{\Delta}_j$, $j = 1,...,q$. With Theorem 4.1 we conclude that the set

$$\{ T_{\Delta_1} \circ \cdots \circ T_{\Delta_q}(G) \mid \|G\| < 1, \mathscr{L}_q(G) = 0 \}$$

is the set of all F with $\|F\| < 1$ and $\mathscr{L}_q(F) = \mathscr{L}_q(\bar{B})$. Note that $T(G) = R_{\Delta_0} \circ T_{\Delta_1} \circ \cdots \circ T_{\Delta_q}(G)$ and the first part of the theorem is proved.

For formula (6.2) one uses the formulas (5.4) and (4.3). Let $\hat{H}_{ij}$ denote the matrices H_{ij} we obtain by applying Theorem 4.1 on the matrix $\bar{B}$. Use identities (5.4) and (4.4) to obtain

$$\det(T(G)^* + T(G)) = \det(2\Delta_0) \prod_{j-i \leq q} \det(I - \hat{H}_{ij}^* \hat{H}_{ij}) \det(I - G^* G).$$

With Lemma 6.3 one may conclude that

$$\det(I - \hat{H}_{ij}^* \hat{H}_{ij}) = \det(I - H_{ij}^* H_{ij}).$$

Now (6.3) follows. $\square$

Proof of Theorem 6.2. We use the notations introduced in the proof of Theorem 6.1. Let $r - s = p > 0$. As remarked in the proof of Theorem 4.2 we have that $(\hat{\Delta}_p)_{rs} = \hat{H}_{rs}$. Using the last sentence of Lemma 6.3 we obtain

$$(\Delta_p)_{rs} = (\hat{\Delta}_p)_{rs} = \hat{H}_{rs} = H_{rs}.$$

Now Theorem 6.2 follows directly from Theorem 6.1. $\square$

Let $\theta := \left[\theta_{ij} \right]_{i,j=1}^{2}$ be as in Theorem 6.1. Note that $\theta = \sqrt{2}N(\Delta_0)\overset{\rightarrow}{\underset{p=1}{\overset{q}{\Pi}}} M(\Delta_p)$, where $N(\Delta)$ and $M(\Delta)$ are given by (5.1) and (1.8), respectively. Since $N(\Delta_0)$ is $(\tilde{J},J)$-unitary and $M(\Delta_p)$ is J-unitary, the matrix $\frac{1}{2}\sqrt{2}\theta$ is $(\tilde{J},J)$-unitary. Using this, one calculates that $T(0) + T(0)^* = 2\theta_{22}^{*-1}\theta_{22}^{-1}$. It is not hard to see that $\theta_{22} = \sum_{p=0}^{q-1} \mathcal{D}_p(\theta_{22})$, so that the self-adjoint matrix $(T(0) + T(0)^*)^{-1}$ belongs to $U_{-q}^{n \times n}$. In other words, the entries of $(T(0) + T(0)^*)^{-1}$ are zero outside the given band. Also $T(0) + T(0)^*$ is the unique extension with maximal determinant (cf. formula (6.2)). These connections will become more transparent in Section III.1.

The set of all solutions of the positive extension problem may also be parametrized via so-called choice sequences (see [16]), or triangular choice schemes (see [31], where also the contractive extension problem appears). These methods yield determinant formulas of the type appearing in Theorems 6.1 and 6.2. These choice-sequences and triangular choice schemes are related to the $\Delta_0, \cdots, \Delta_q$ defined in Theorem 6.1. The scalar versions in (6.4) may be recognized as partial correlation coefficients (PARCOR's) (see [54]). The papers [16] and [31] do not contain linear fractional descriptions for the sets of all solutions.

Independently, H.Dym [20] and P. Dewilde and E.F.A. Deprettere [18] obtained also a linear fractional description for the set of all positive extensions of a given band. Their methods are similar to the one used here, however, their formulas are less explicit than the ones given here.

I.7. The Toeplitz case

In this section we consider the problems of strictly contractive and positive extensions for the class of Toeplitz block matrices. A block matrix $A = \left[A_{ij} \right]_{i=1,j=1}^{n \quad m}$ is called *Toeplitz* if $A_{ij} = A_{i+1,j+1}$ for all admissible i and j. We denote the class of $n \times m$ Toeplitz block matrices by $T\Omega^{n \times m}$. By $TU_p^{n \times m}$ we denote all block Toeplitz block matrices A which are upper triangular relative to the p-th diagonal, i.e., $\mathcal{L}_p(A) = 0$.

Let us first look at the strictly contractive case. The following example shows that a lower triangular part of a Toeplitz block matrix A which satisfies the condition $m_q(A) < 1$ need not have a strictly contractive Toeplitz extension F (i.e., an $F \in T\Omega^{n \times m}$ with $\|F\| < 1$ and $\mathcal{L}_q(F) = \mathcal{L}_q(A)$). This conclusion may also be drawn from the results in [47].

EXAMPLE 7.1. Consider the following lower triangular part of a Toeplitz matrix.

$$\begin{bmatrix} 0 & & & & \\ 7/10 & 0 & & & \\ 0 & 7/10 & 0 & & \\ 7/10 & 0 & 7/10 & 0 & \\ 0 & 7/10 & 0 & 7/10 & 0 \end{bmatrix}. \tag{7.1}$$

In the right upper corner of the first two columns there is an unknown entry. To find this entry means to solve a 2×2 strictly contractive extension problem. By using (2.10) we get that the (1,2)th entry should be a complex number in the disk $\{ z \in \mathbb{C} \mid |z|^2 < 0.02 \}$. Next we determine the (2,3)th entry. For this we only consider the submatrix obtained by leaving out the first row and the last two columns. Then again a 2×2 extension problem appears. With (2.10) we may conclude that the (2,3)th entry should be in the disk $\{ z \in \mathbb{C} \mid |z + \frac{343}{510}| < \frac{2}{51} \}$. Since the two disks have an empty intersection, the entries on the positions (1,2) and (2,3) cannot be the same complex number. Therefore the lower part of (7.1) has no strictly contractive Toeplitz extension. Note that all given submatrices have norm strictly less than one.

Thus in general the condition that all given submatrices have norm strictly less than one does not imply the existence of a strictly contractive Toeplitz extension. We shall see that the situation is different if we work in the class $TU_0^{n \times n}$ of all upper triangular Toeplitz block matrices. First a few preliminary results.

LEMMA 7.2. (i) *If* $A \in TU_p^{n \times n}$, $(p \geq 0)$, *such that* $\Delta = \mathcal{D}_p(A)$ *has norm less than one, then* $\hat{T}_\Delta(A) \in TU_{p+1}^{n \times n}$;

(ii) *If* $G \in TU_{p+1}^{n \times n}$, $(p \geq 0)$, *and* $\Delta = \mathcal{D}_p(\Delta)$ *has norm less than one, then* $T_\Delta(G) \in TU_p^{n \times n}$;

Proof. In the calculations there appear only upper triangular Toeplitz block matrices, except for W_Δ and W_{Δ^*}. In the latter matrices the non-Toeplitz part disappears because of multiplication with zeroes. So to prove the lemma one only has to use the fact that the set $TU_0^{n \times n}$ of upper triangular Toeplitz block matrices is closed under addition, multiplication and inversion (provided the inverse exists). $\square$

PROPOSITION 7.3. *Let* $q \geq 0$ *and let* $A \in TU_0^{n \times n}$ *be such that* $m_q(A) < 1$. *Define* $\Sigma_p, \Delta_p, p = 0,...,q$, *by*

$$\Sigma_0 = A, \Delta_0 = \mathcal{D}_0(A),$$
$$\Sigma_p = \hat{T}_{\Delta_{p-1}}(\Sigma_{p-1}), \Delta_p = \mathcal{D}_p(\Sigma_p), p = 1,...,q. \tag{7.2}$$

Then $\Delta_0, ..., \Delta_q$ *are block Toeplitz matrices and the set* $\{ F(G) \mid G \in T\Omega_q^{n \times n}, \|G\| < 1 \}$, *where*

$$F(G) = T_{\Delta_0} \circ \cdots \circ T_{\Delta_q}(G),$$

is the set of all $F \in T\Omega^{n \times n}$ *with* $\mathcal{L}_q(F) = \mathcal{L}_q(A)$ *and* $\|F\| < 1$. *The correspondence is 1-1. Furthermore,*

$$\det(I - F(G)^* F(G)) = \prod_{p=0}^{q} \det(I - \Delta_p^* \Delta_p) \det(I - G^* G). \tag{7.3}$$

Proof. The proof of this proposition is similar to the proof of Theorem 3.3. (Note that in this case the matrices $\Delta_{-n+1}, ..., \Delta_{-1}$ appearing in Theorem 3.3 are zero.) One has to realize, however, that in each step one stays in the set $T\Omega^{n \times n}$. This is ensured by Lemma 7.2. $\square$

The above proposition leads to the following conclusion.

If in Theorem 4.1 one starts with a square upper triangular block Toeplitz matrix A, *i.e.,* $A = \left[A_{j-i} \right]_{i,j=0}^{n-1}$ *with* $A_p = 0$ *for* $p < 0$, *and if* q *is a nonnegative integer, then the set of all block Toeplitz matrices* $F = \left[F_{j-i} \right]_{i,j=0}^{n-1}$ *such that* $\|F\| < 1$ *and* $F_p = A_p$ *for* $p \leq q$ *is the set of all matrices* $F(G)$, *where* $F(G)$ *is constructed as in Theorem 4.1 and* $G = \left[G_{j-i} \right]_{i,j=0}^{n-1}$ *is any block Toeplitz matrix with* $G_p = 0$, $p \leq q$, *and*

$\|G\| < 1$. *The correspondence is 1-1.*

For the scalar Toeplitz case the construction of $F(G)$ can be simplified considerably. The final result is the following.

THEOREM 7.4. *Let* $q \in \{0,...,n-1\}$ *and* a_p *be given complex numbers for* $0 \leq p \leq q$, *and let* $a_p = 0$ *for* $p < 0$. *Suppose that the Toeplitz matrix* $\left[a_{j-i} \right]_{i,j=0}^{q}$ *has norm less than one. Define the numbers* $h_0, \ldots, h_q$ *by*

$$h_0 = a_0 \; ; \; h_1 = \frac{a_1}{1-|a_0|^2} \; ;$$

$$h_p = (a_p - z_p) \frac{\det S_{p-1}}{\det S_p}, p = 2,...,q,$$

where $S_r = I - A_r^* A_r$, $A_r = \left[a_{j-i} \right]_{i,j=0}^{r-1}$, *and*

$$z_p = \frac{(-1)^{p-1}}{\det S_{p-1}} \cdot \det \begin{bmatrix} a_1 & a_2 & \cdots & a_{p-1} & 0 \\ s_{00} & s_{01} & \cdots & s_{0,p-2} & s_{0,p-1} \\ \vdots & \vdots & & \vdots & \vdots \\ s_{p-2,0} & s_{p-2,1} & \cdots & s_{p-2,p-2} & s_{p-2,p-1} \end{bmatrix},$$

with s_{ij} *given by* $S_p = \left[s_{ij} \right]_{i,j=0}^{p-1}$. *Put* $\Delta_p = \left[\delta_{i+p,j} h_p \right]_{i,j=0}^{n-1}$, $p = 0,...,q$. *Furthermore, let*

$$\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \overset{\rightarrow}{\Pi}_{p=0}^{q} \begin{bmatrix} I & \Delta_p \\ \Delta_p^* & I \end{bmatrix}.$$

Then the set of all Toeplitz matrices $F = \left[f_{j-i} \right]_{i,j=0}^{n-1} \in \mathbb{C}^{n \times n}$ *with* $f_p = a_p, p \leq q$, *and* $\|F\| < 1$ *is the set of all matrices of the form*

$$F = F(G) = (\theta_{11} G + \theta_{12})(\theta_{21} G + \theta_{22})^{-1},$$

where $G = \left[g_{j-i} \right]_{i,j=0}^{n-1}$ *is any Toeplitz matrix with* $\|G\| < 1$ *and* $g_p = 0$ *for* $p \leq q$. *The correspondence is 1-1. Moreover,*

$$\det(I - F(G)^* F(G)) = \overset{q}{\underset{j=0}{\Pi}} (1 - |h_j|^2)^{n-j} \det(I - G^* G).$$

Proof. We can apply Theorem 4.2 to obtain a linear fractional description for the set of *all* strictly contractive extensions. From the results in Ya. L. Geronimus [32] it follows that, for $j - i = p$ (≥ 0), the number h_{ij} in Theorem 4.2 is equal to h_p. Furthermore, note that if $\Delta = \left[\delta_{i+p,j} h \right]_{i,j=0}^{n-1}$ ($p \geq 0$), where $|h| < 1$, and $G \in U_{p+1}^{n \times n}$, then

$$T_\Delta(G) = (G + \Delta)(\Delta^* G + I)^{-1}. \tag{7.4}$$

This follows from the fact that premultiplying the right hand side of (7.4) with W_Δ· comes down to dividing all entries by $\sqrt{1 - |h|^2}$ and postmultiplying it with W_Δ^{-1} comes down to multiplying all entries with $\sqrt{1 - |h|^2}$. Formula (7.4) shows that the linear fractional map in Theorem 4.2 for this special case is equal to $F(G)$. Apply the conclusion preceding this theorem and the proof is complete. $\square$

We now come to the positive extension problem. We want to use the reduction described in Section I.5. The fact that using this reduction we stay in the class of Toeplitz matrices is the content of the following lemma.

LEMMA 7.5. (i) *If* $A \in TU_0^{n \times n}$ *with* $\Delta = \mathcal{D}_0(A) > 0$, *then* $\hat{R}_\Delta(A) \in TU_1^{n \times n}$;

(ii) *If* $G \in TU_1^{n \times n}$, *and* $\Delta = \mathcal{D}_0(\Delta) \in T\Omega^{n \times n}$ *is positive definite, then* $R_\Delta(G) \in TU_0^{n \times n}$.

Proof. Use the fact that the set $TU_0^{n \times n}$ is closed under addition, multiplication and inversion (provided the inverse exists). $\square$

The above lemma, Theorem 5.2 and Proposition 7.3 lead to the following conclusion.

If in Theorem 6.1 one starts with a block matrix B which, in addition, is Toeplitz, i.e.,

$B = \left[B_{j-i} \right]_{i,j=0}^{n-1}$, *then the set of all block Toeplitz matrices* $F = \left[F_{j-i} \right]_{i,j=0}^{n-1}$ *such that* $F > 0$ *and* $F_p = B_p$ *for* $0 \leq p \leq q$ *is the set of all matrices of the form* $F = T(G) + T(G)^*$, *where* $T(G)$ *is constructed as in Theorem 6.1 and* $G = \left[G_{j-i} \right]_{i,j=0}^{n-1}$ *is any block Toeplitz matrix with* $G_p = 0$, $p \leq q$, *and* $\|G\| < 1$. *The correspondence is 1-1.*

Here also are some simplifications in the scalar case. We have the following result.

THEOREM 7.6. *Let* $q \in \{0,...,n-1\}$ *and* $b_p = \bar{b}_{-p}$ *be given complex numbers for* $0 \leq p \leq q$. *Suppose that the Toeplitz matrix* $\left[b_{j-i} \right]_{i,j=0}^{q}$ *is positive definite. Define the numbers* $h_0, \ldots, h_q$ *by*

$$h_0 = \frac{1}{2} b_0 \,; h_1 = \frac{b_1}{b_0} \,;$$

$$h_p = (b_p - z_p) \frac{\det T_{p-1}}{\det T_p} , p = 2,\ldots,q ,$$

where $T_p = \left[b_{j-i} \right]_{i,j=0}^{p-1}$ *and*

$$z_p = \frac{(-1)^p}{\det T_{p-1}} \cdot \det \begin{bmatrix} b_1 & . & . & b_{p-2} & b_{p-1} & 0 \\ b_0 & . & & & b_{p-2} & b_{p-1} \\ \bar{b}_1 & & & & & b_{p-2} \\ \vdots & & & & & \vdots \\ \bar{b}_{p-2} & . & . & \bar{b}_1 & b_0 & b_1 \end{bmatrix} .$$

Put $\Delta_p = \left[\delta_{i+p,j} h_p \right]_{i,j=0}^{n-1}, p = 0,\ldots,q.$ *Furthermore, let*

$$\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \begin{bmatrix} \Delta_0 & \Delta_0 \\ -I & I \end{bmatrix} \cdot \overset{\rightarrow}{\underset{p=1}{\Pi}}{}^q \begin{bmatrix} I & \Delta_p \\ \Delta_p^* & I \end{bmatrix} .$$

Then the set of all Toeplitz matrices $F = \left[f_{j-i} \right]_{i,j=0}^{n-1} \in \mathbb{C}^{n \times n}$ *with* $f_p = b_p$,

$-q \leqq p \leqq q$, *and* $F > 0$ *is the set of all matrices of the form* $F = T(G) + T(G)^*$, *where*

$$T(G) = (\theta_{11}G + \theta_{12})(\theta_{21}G + \theta_{22})^{-1}$$

and $G = \left[g_{j-i} \right]_{i,j=0}^{n-1}$ *is any Toeplitz matrix with* $\|G\| < 1$ *and* $g_p = 0$ *for* $p \leqq q$. *The correspondence is 1-1. Moreover,*

$$\det(T(G)^* + T(G)) = b_0^n \overset{q}{\underset{j=1}{\Pi}} (1 - |h_j|^2)^{n-j} \det(I - G^* G).$$

Proof. We can apply Theorem 6.2 to obtain a linear fractional description for the set of *all* strictly contractive extensions. From the results in [53] it follows that for $j - i = p (\geqq 0)$, the number h_{ij} in Theorem 6.2 is equal to h_p. Furthermore, note that if $G \in \Omega_0^{n \times n}$, then

$$R_{\Delta_0}(G) = (\Delta_0 G + \Delta_0)(-G + I)^{-1}.$$

This and the conclusion concerning T_Δ in the proof of Theorem 7.4 prove that for this special case the linear fractional map from Theorem 6.2 is equal to $T(G)$. Apply the conclusion preceding this theorem and the proof is complete. $\square$

An analysis of the Toeplitz positive and contractive extension problem also appears in

[15], [28], [29] and [30], where choice sequence approaches is used. These papers do not contain linear fractional descriptions for the sets of all solutions.

CHAPTER II: THE BAND METHOD

In this chapter the band method, which concerns a general scheme for dealing with positive and strictly contractive extension problem, is introduced. Again the main aim is to describe all solutions explicitely via linear fractional maps. Section 1 concerns the positive extension problem and Section 2 the strictly contractive extension problem. Section 3 deals with maximum entropy principles.

II.1. Positive extensions

Let $\mathcal{M}$ be an algebra with a unit e and an involution $*$. We suppose that $\mathcal{M}$ admits a direct sum decomposition of the form

$$\mathcal{M} = \mathcal{M}_1 \dotplus \mathcal{M}_2^0 \dotplus \mathcal{M}_d \dotplus \mathcal{M}_3^0 \dotplus \mathcal{M}_4, \tag{1.1}$$

where $\mathcal{M}_1, \mathcal{M}_2^0, \mathcal{M}_d, \mathcal{M}_3^0$ and $\mathcal{M}_4$ are linear subspaces of $\mathcal{M}$ and the following conditions are satisfied:

(i) $e \in \mathcal{M}_d$, $\mathcal{M}_1 = \mathcal{M}_4^*$, $\mathcal{M}_2^0 = (\mathcal{M}_3^0)^*$, $\mathcal{M}_d = \mathcal{M}_d^*$;

(ii) the following multiplication table describes some additional rules on the multiplication in $\mathcal{M}$:

	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_-^0$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}_c$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_4$

$$\tag{1.2}$$

where

$$\mathcal{M}_+^0 := \mathcal{M}_1 \dotplus \mathcal{M}_2^0 \ , \ \mathcal{M}_-^0 := \mathcal{M}_3^0 \dotplus \mathcal{M}_4 \ , \tag{1.3}$$

$$\mathcal{M}_c := \mathcal{M}_2^0 \dotplus \mathcal{M}_d \dotplus \mathcal{M}_3^0 \ .$$

Some additional notations are

$$\mathcal{M}_+ := \mathcal{M}_+^0 \dotplus \mathcal{M}_d \ , \ \mathcal{M}_- := \mathcal{M}_-^0 \dotplus \mathcal{M}_d \ , \tag{1.4}$$

$$\mathcal{M}_2 := \mathcal{M}_2^0 \dotplus \mathcal{M}_d \ , \ \mathcal{M}_3 := \mathcal{M}_3^0 \dotplus \mathcal{M}_d \ .$$

Note that $\mathcal{M}_1$ (resp. $\mathcal{M}_4$) is a two-sided ideal of the subalgebra $\mathcal{M}_+$ (resp. $\mathcal{M}_-$). Furthermore, if $d \in \mathcal{M}_d$ is invertible, then $d^{-1} \in \mathcal{M}_d$.

If $\mathcal{A}$ is an algebra with a unit and an involution *, we say that an element $a \in \mathcal{A}$ is *nonnegative definite* in $\mathcal{A}$ (notation: $a \geq_{\mathcal{A}} 0$) if there exists an element $c \in \mathcal{A}$ such that $a = c^* c$. The element $a \in \mathcal{A}$ is called *positive definite* in $\mathcal{A}$ (notation: $a >_{\mathcal{A}} 0$) if there exists an invertible element $c \in \mathcal{A}$ such that $a = c^* c$. We shall write $b \geq_{\mathcal{A}} a$ instead of $b - a \geq_{\mathcal{A}} 0$, and $b >_{\mathcal{A}} a$ instead of $b - a >_{\mathcal{A}} 0$. When $\mathcal{A} = \mathcal{M}$ we shall omit the subscript $\mathcal{M}$.

Let us introduce the following two types of factorizations for positive elements in $\mathcal{M}$. Let $b \in \mathcal{M}$ be positive definite in $\mathcal{M}$. We shall say that b admits a *left spectral factorization* (relative to the decomposition (1.1)) if $b = b_+ b_+^*$ for some $b_+ \in \mathcal{M}_+$ with $b_+^{-1} \in \mathcal{M}_+$. We shall say that b admits a *right spectral factorization* (relative to the decomposition (1.1)) if $b = b_- b_-^*$ for some $b_- \in \mathcal{M}_-$ with $b_-^{-1} \in \mathcal{M}_-$. Note that b admits a left spectral factorization if and only if b^{-1} admits a right spectral factorization.

We shall use the symbols P_i ($i = 1,...,4$), P_i^0 ($i = 2,3$), $P_\pm$, $P_\pm^0$, P_c and P_d to denote the natural projections of $\mathcal{M}$ onto the subspaces of the same index along their natural complement in $\mathcal{M}$. Thus, for instance,

$$P_+ = P_1 + P_2 , P_- = P_3 + P_4 , P_c = P_2 + P_3^0 = P_2^0 + P_d + P_3^0 = P_2^0 + P_3 .$$

Let $k = k^* \in \mathcal{M}_c$. An element $b \in \mathcal{M}$ is called a *positive extension* of k if $P_c b = k$ and b is positive definite in $\mathcal{M}$. A positive extension b of k is called a *(positive) band extension* of k if in addition $b^{-1} \in \mathcal{M}_c$. In what follows we will just speak about a band extension and omit the adjective positive.

THEOREM 1.1. *Let* $k = k^* \in \mathcal{M}_c$. *The element* k *has a band extension* b *which admits a left and a right spectral factorization if and only if the equations*

$$P_2(kx) = e , P_3(ky) = e , \tag{1.5}$$

have solutions x *and* y *with the following properties:*

(i) $x \in \mathcal{M}_2$, $y \in \mathcal{M}_3$,

(ii) x *and* y *are invertible*, $x^{-1} \in \mathcal{M}_+$, $y^{-1} \in \mathcal{M}_-$,

(iii) $P_d x$ *and* $P_d y$ *are positive definite in* $\mathcal{M}_d$.

Moreover, if such an element b *exists, then* b *is unique and given by*

$$b = x^{*-1}(P_d x)x^{-1} = y^{*-1}(P_d y)y^{-1} . \qquad (1.6)$$

In the proof of Theorem 1.1 we will need the following two lemmas.

LEMMA 1.2. *If* $b_{\pm} \in \mathcal{M}_{\pm}$ *is invertible with inverse in* $\mathcal{M}_{\pm}$ *and* $b = b_{\pm} b_{\pm}^{*}$ *belongs to* $\mathcal{M}_c$, *then* $b_{\pm} \in \mathcal{M}_c \cap \mathcal{M}_{\pm}$.

Proof. Since $b_{+} = bb_{+}^{*-1}$, $b \in \mathcal{M}_c$ and $b_{+}^{*-1} \in \mathcal{M}_{-}$, we get that $b_{+} \in \mathcal{M}_2 \dotplus \mathcal{M}_3^0 \dotplus \mathcal{M}_4$. But then, since $b_{+} \in \mathcal{M}_{+}$, we obtain that $b_{+} \in \mathcal{M}_2 = \mathcal{M}_c \cap \mathcal{M}_{+}$. The proof of the minus version is analogous. $\square$

LEMMA 1.3. *Let* $x_{\pm} \in \mathcal{M}_{\pm}$ *be invertible with* $x_{\pm}^{-1} \in \mathcal{M}_{\pm}$. *Then* $P_d x_{\pm}$ *is invertible and* $(P_d x_{\pm})^{-1} = P_d x_{\pm}^{-1}$.

Proof. Write $x_{\pm} = P_d x_{\pm} + P_{\pm}^0 x_{\pm}$ and $x_{\pm}^{-1} =: y_{\pm} = P_d y_{\pm} + P_{\pm}^0 y_{\pm}$. Writing out the products $x_{\pm} y_{\pm}$ and $y_{\pm} x_{\pm}$, which are equal to e, and by applying the projection P_d one obtains that $P_d x_{\pm} P_d y_{\pm} = P_d y_{\pm} P_d x_{\pm} = P_d e = e$, and the lemma is proved. $\square$

Proof of Theorem 1.1. Let b be a band extension of k, and let $b^{-1} = uu^* = vv^*$, where $u^{\pm 1} \in \mathcal{M}_{+}$ and $v^{\pm 1} \in \mathcal{M}_{-}$. Since $b^{-1} \in \mathcal{M}_c$, Lemma 1.2 yields that $u \in \mathcal{M}_2$ and $v \in \mathcal{M}_3$. Put $x = u(P_d u^*)$ and $y = v(P_d v^*)$. Then $x \in \mathcal{M}_2$, $y \in \mathcal{M}_3$, and $x^{-1} \in \mathcal{M}_{+}$, $y^{-1} \in \mathcal{M}_{-}$. Furthermore, $P_d x = (P_d u)(P_d u)^*$ and $P_d y = (P_d v)(P_d v)^*$ are positive definite in $\mathcal{M}_d$, and

$$(x^*)^{-1}(P_d x)x^{-1} = (u^*)^{-1}u^{-1} = b = (v^*)^{-1}v^{-1} = (y^*)^{-1}(P_d y)y^{-1}.$$

Since $P_c b = k$ we have that $b = P_1 b + k + P_4 b$. So using multiplication table (1.2)

$$P_2(kx) = P_2(bx - (P_1 b)x - (P_4 b)x) = P_2(bx) = P_2((x^*)^{-1}P_d x) = e,$$

where for the last equality one uses Lemma 1.3. In much the same way one proves that $P_3(ky) = e$.

Conversely, suppose that x and y exist such that all the conditions in the theorem are fullfilled. Let $\hat{b}$ be defined by $\hat{b} = b_1 + k + b_1^*$, where $b_1 = -P_1(kx)x^{-1} \in \mathcal{M}_1$. Then $\hat{b}x = -P_1(kx) + kx + b_1^* x$, and using the multiplication table (1.2) we get that $P_1(\hat{b}x) = 0$ and $P_2(\hat{b}x) = P_2(kx) = e$. So $\hat{b}x \in e + \mathcal{M}_-^0$. Since $P_d x$ is positive definite (in $\mathcal{M}_d$), $P_d x = P_d x^*$, and hence $x^* \hat{b}x \in P_d x + \mathcal{M}_-^0$. From $k = k^*$ it follows that $(x^* \hat{b}x)^* = x^* \hat{b}x$, and hence $x^* \hat{b}x \in P_d x + \mathcal{M}_+^0$. This can only happen when $x^* \hat{b}x = P_d x$. So we get that

$$\hat{b} = x^{*-1}(P_d x)x^{-1}. \qquad (1.7)$$

Now $\hat{b}^{-1} = x(P_d x)^{-1}x^* \in \mathcal{M}_c$. Further, using that $P_d x$ is positive definite in $\mathcal{M}_d$, we see from (1.7) that $\hat{b}$ admits a right spectral factorization.

Analogously, one proves that

$$\bar{b} := -(y^{-1})^*(P_4(ky))^* + k - P_4(ky)y^{-1}$$

is equal to $(y^*)^{-1}(P_d y)y^{-1}$. But then $\bar{b}$ is a band extension of k which admits a left spectral factorization.

We finish the proof by proving that if k has a band extension f which admits a right spectral factorization and a band extension g which admits a left spectral factorization, then $f = g$. This then yields $\hat{b} = \bar{b} =: b$ and also the uniqueness of b. So let f and g be as above and write $f^{-1} = uu^*$, $u \in \mathcal{M}_2$, $u^{-1} \in \mathcal{M}_+$, and $g^{-1} = vv^*$, $v \in \mathcal{M}_3$, $v^{-1} \in \mathcal{M}_-$ (use Lemma 1.2). Put $h := f^{-1} - g^{-1}$. Then h belongs to $\mathcal{M}_c$. Since $P_c f = P_c g = k$, we have that $g - f = z_1 + z_1^*$ for some $z_1 \in \mathcal{M}_1$. Using that $h = f^{-1}(z_1 + z_1^*)g^{-1}$, we obtain $u^{-1}hv^{*-1} = u^*(z_1 + z_1^*)v$. Because of the multiplication table (1.2) the left hand side belongs to $\mathcal{M}_+ + \mathcal{M}_3^0$ and hence $0 = P_4(u^*(z_1 + z_1^*)v) = u^* z_1^* v$. Thus $z_1^* = 0$. But then $f = g$ follows. $\square$

Note that in the last paragraph of the proof of Theorem 1.1 we actually proved the following result.

THEOREM 1.4. *Let $k = k^* \in \mathcal{M}_c$, and suppose that k has a band extension f which admits a right spectral factorization and a band extension g which admits a left spectral factorization. Then $f = g$.*

To describe the set of all positive extension of a given $k \in \mathcal{M}_c$, we need extra requirements on the algebra $\mathcal{M}$. We shall assume that $\mathcal{M}$ is a *-subalgebra of a B*-algebra $\mathfrak{R}$ with norm $\|.\|_{\mathfrak{R}}$, and $\mathfrak{R}$ has a unit e which belongs to $\mathcal{M}$. Further, we assume that the following two axioms hold:

AXIOM (A1). If $f \in \mathcal{M}$ is invertible in $\mathfrak{R}$, then $f^{-1} \in \mathcal{M}$;

AXIOM (A2). If $f_n \in \mathcal{M}_+$, $f \in \mathcal{M}$ and $\lim_{n \to \infty} \|f_n - f\|_{\mathfrak{R}} = 0$, then $f \in \mathcal{M}_+$.

Note that if $e - f^* f$ is positive definite in $\mathcal{M}$, then $e - f^* f$ is positive definite in $\mathfrak{R}$, and hence $\|f\|_{\mathfrak{R}} < 1$.

THEOREM 1.5. *Let $\mathcal{M}$ be a *-subalgebra of a B*-algebra $\mathfrak{R}$ such that the unit e of $\mathfrak{R}$ belongs to $\mathcal{M}$, and assume that Axioms (A1) and (A2) hold. Let $k = k^* \in \mathcal{M}_c$ and suppose that k has a band extension b which admits a left and a right spectral factorization:*

$$b = u^{*-1}u^{-1} = v^{*-1}v^{-1} \ , \ u^{\pm 1} \in \mathcal{M}_+ \ , \ v^{\pm 1} \in \mathcal{M}_-. \tag{1.8}$$

Then each positive extension of k is of the form

$$T(g) = (g^* v^* + u^*)^{-1}(e - g^* g)(vg + u)^{-1}, \tag{1.9}$$

where g is an element of $\mathcal{M}_1$ such that $e - g^* g$ is positive definite in $\mathcal{M}$. Furthermore, formula (1.9) gives a 1-1 correspondence between all such g and all positive extensions of k. Alternatively, each positive extension of k is of the form

$$S(f) = (f^* u^* + v^*)^{-1}(e - f^* f)(uf + v)^{-1}, \tag{1.10}$$

where f is an element of $\mathcal{M}_4$ such that $e - f^* f$ is positive definite in $\mathcal{M}$. Furthermore, formula (1.10) gives a 1-1 correspondence between all such f and all positive extensions of k.

In the proof of Theorem 1.5 we need the following lemmas.

LEMMA 1.6. *Let $g \in \mathcal{M}_\pm$ be such that $e - g^* g$ is positive definite in $\mathcal{R}$. Then $e - g$ is invertible and $(e - g)^{-1} \in \mathcal{M}_\pm$.*

Proof. Let $g \in \mathcal{M}_+$ be such that $e - g^* g$ is positive definite in $\mathcal{R}$. Since $\mathcal{R}$ is a B*-algebra $\| g \|_{\mathcal{R}} < 1$. But then $(e - g)^{-1}$ exists in $\mathcal{R}$, and because of Axiom (A1) the element $(e - g)^{-1}$ belongs to $\mathcal{M}$. Using Axiom (A2) and the fact that

$$\| (e - g)^{-1} - \sum_{j=0}^{n} g^j \|_{\mathcal{R}} \to 0 \ , \ n \to \infty,$$

we get that $(e - g)^{-1} \in \mathcal{M}_+$. The minus version one obtains by applying the involution.
□

LEMMA 1.7. *Let $z \in \mathcal{M}_\pm$ be such that $z + z^*$ is positive definite in $\mathcal{R}$. Then z is invertible and $z^{-1} \in \mathcal{M}_\pm$.*

Proof. Write $z + z^* = aa^*$ with $a \in \mathcal{R}$ invertible in $\mathcal{R}$. For $\epsilon > 0$ we have

$$(e - \epsilon z^*)(e - \epsilon z) = e - \epsilon[(z + z^*) - \epsilon z^* z] = e - \epsilon a[e - \epsilon a^{-1} z^* z(a^{-1})^*]a^*.$$

Choose $\epsilon > 0$ such that $\| \epsilon^{1/2} z(a^{-1})^* \| < 1$. Then, since $\mathcal{R}$ is a B*-algebra, we obtain that $e - \epsilon a^{-1} z^* z(a^{-1})^* = g_\epsilon g_\epsilon^*$ for some invertible $g_\epsilon \in \mathcal{R}$. Now

$$e - (e - \epsilon z^*)(e - \epsilon z) = \epsilon a g_\epsilon g_\epsilon^* a^*$$

is positive definite in $\mathcal{R}$. So by Lemma 1.6 the element $z = \epsilon^{-1}(e - (e - \epsilon z))$ is invertible in $\mathcal{R}$, and $z^{-1} \in \mathcal{M}_+$. □

Proof of Theorem 1.5. Write $b = c + c^*$ with $c \in \mathcal{M}_+$ and $P_d c = \frac{1}{2} P_d k$, and define

$$L(g) = (-c^* vg + cu)(vg + u)^{-1}$$

for all g for which $u + vg$ is invertible. Then

$$L(g) + L(g)^* = (g^* v^* + u^*)^{-1}\{(g^* v^* + u^*)(-c^* vg + cu) +$$

$$+ (-g^* v^* c + u^* c^*)(vg + u)\}(vg + u)^{-1}$$

$$= (g^* v^* + u^*)^{-1}\{u^*(c^* + c)u - g^* v^*(c^* + c)vg\}(vg + u)^{-1} = T(g),$$

whenever $u + vg$ is invertible. If $u + vg$ is invertible, then the same is true for $u^{-1}vg + c$ and one checks that

$$L(g) = c - v^{*-1}(e + gu^{-1}v)^{-1} gu^{-1}. \qquad (1.11)$$

Suppose that $g \in \mathcal{M}_1$ and $e - g^* g$ is positive definite in $\mathcal{M}$. Since $v \in \mathcal{M}_3$, we have that $vg \in \mathcal{M}_+$, so $u^{-1}vg \in \mathcal{M}_+$. Further, since $e - (u^{-1}vg)^*(u^{-1}vg) = e - g^* g$ is positive definite in $\mathfrak{R}$, Lemma 1.6 yields that $(e + u^{-1}vg)^{-1} \in \mathcal{M}_+$. In particular, $T(g)$ and $L(g)$ are well defined. Note that $T(g)$ clearly is positive definite in $\mathcal{M}$. Use now the multiplication table to show that $(e + gu^{-1}v)^{-1} = e - g(e + u^{-1}vg)^{-1}u^{-1}v \in \mathcal{M}_+$, and consequently, that $L(g) \in c + \mathcal{M}_1$. But then $T(g) = L(g) + L(g)^* \in b + \mathcal{M}_1 + \mathcal{M}_3$. Hence $T(g)$ is a positive extension of k.

Conversely, suppose that a is a positive extension of k. Write $a = z + z^*$ with $z \in \mathcal{M}_+$ and $P_d z = \frac{1}{2}P_d k$, and put $w := z - c \in \mathcal{M}_1$. Since $b + a$ is positive definite in $\mathfrak{R}$, we get that $v^*(b + a)v = v^*(b + b + w + w^*)v = 2e + v^* wv + v^* w^* v$ is positive definite in $\mathfrak{R}$. From Lemma 1.2 it follows that $v \in \mathcal{M}_- \cap \mathcal{M}_c$, and thus $v^* wv \in \mathcal{M}_+$. Lemma 1.7 yields that $e + v^* wv$ is invertible and its inverse belongs to $\mathcal{M}_+$. Put now $g := -(e + v^* wv)^{-1}v^* wu$. By Lemma 1.2 the elements v^* and u are in $\mathcal{M}_2$. Since $w \in \mathcal{M}_1$ we get that $g \in \mathcal{M}_1$. Furthermore, $vg + u = -v(e + v^* wv)^{-1}v^* wu + u = (e + vv^* w)^{-1}u$ is invertible, and

$$L(g) = (c^* v(e + v^* wv)^{-1}v^* wu + cu)u^{-1}(e + vv^* w)$$

$$= (c^* - c^*(e + vv^* w)^{-1} + c)(e + vv^* w) = b(e + b^{-1}w) - c^* = z.$$

Hence $a = L(g) + L(g)^* = T(g)$. Since a is positive definite in $\mathcal{M}$, it follows that $e - g^* g$ is positive definite in $\mathcal{M}$. Since the map $g \to T(g)$ is one-one we have established

the desired 1-1 correspondence.

In order to prove the alternative representation (1.10) one proceeds in an analogous way. Let

$$K(f) = (-cuf + c^* v)(v + uf)^{-1},$$

and use the axioms to show that for $f \in \mathcal{M}_4$ with $e - f^* f$ positive definite in $\mathcal{M}$ the element $K(f)$ is well defined. Then calculations show that $S(f) = K(f) + K(f)^* \in b + \mathcal{M}_1 + \mathcal{M}_4$ is a positive extension of k. Conversely, let a be a positive extension of k and write $a = z + z^*$ with $z \in \mathcal{M}_-$ and $P_d z = \frac{1}{2} P_d k$. One uses Lemma 1.7 to show that $e + u^*(z - c^*) u$ is invertible, and one introduces $f := -(e + u^*(z - c^*) u)^{-1} u^*(z - c^*) v$. Then $f \in \mathcal{M}_4$, $e - f^* f$ is positive definite in $\mathcal{M}$ and $K(f) = z$. But then the desired one-one correspondence (1.10) is established. $\square$

Theorems 1.1 and 1.4 are similar to some results in [24] but now concern positive extensions in the setting of an algebra with an involution. In [24] the algebra has no involution and the extensions are required to be invertible. Theorem 1.5 is a new result inspired by earlier concrete versions (see [20]).

II.2. Strictly contractive extensions

Let $\mathcal{B}$ be a vector space, and suppose that $\mathcal{B}$ admits a direct sum decomposition

$$\mathcal{B} = \mathcal{B}_- \dotplus \mathcal{B}_+ ,$$

where $\mathcal{B}_-$ and $\mathcal{B}_+$ are subspaces of $\mathcal{B}$. We are interested in the following problem: given $\phi \in \mathcal{B}_-$, when does there exist an element $\psi \in \mathcal{B}$ such that $\|\psi\| < 1$ (for some specified norm) and $\psi - \phi \in \mathcal{B}_+$? Such an element ψ is called a strictly contractive extension of ϕ. Furthermore, if a strictly contractive extension of ϕ exists, we want to describe all strictly contractive extensions of ϕ. We shall solve the problem using the results in the previous section. In order to be able to do this we need some more structure on $\mathcal{B}$. In what follows we shall assume that $\mathcal{B}$ can be embedded in an algebra of 2×2 matrices with a unit and an involution.

We shall assume that the space $\mathcal{B}$ appears as the space of $(1,2)$-elements of the following algebra of 2×2 block matrices:

$$\mathcal{M} = \left\{ f = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}, d \in \mathcal{D} \right\}.$$

Here $\mathscr{A}$ and $\mathscr{D}$ are algebras with identities $e_{\mathscr{A}}$ and $e_{\mathscr{D}}$, respectively, and involutions *, and $\mathscr{C}$ is a vector space which is isomorphic to $\mathscr{B}$ via an operator * whose inverse is also denoted by *, such that for every choice of $a \in \mathscr{A}$, $b \in \mathscr{B}$, $c \in \mathscr{C}$ and $d \in \mathscr{D}$:

$$bc \in \mathscr{A}, (bc)^* = c^* b^* ; ab \in \mathscr{B}, (ab)^* = b^* a^* ;$$

$$bd \in \mathscr{B}, (bd)^* = d^* b^* ; ca \in \mathscr{C}, (ca)^* = a^* c^* ; \qquad (2.1)$$

$$dc \in \mathscr{C}, (dc)^* = c^* d^* ; cb \in \mathscr{D}, (cb)^* = b^* c^* ;$$

It is easy to see that $\mathscr{M}$ is an algebra (with respect to the natural rules for matrix multiplication and addition) with unit

$$e := \begin{bmatrix} e_{\mathscr{A}} & 0 \\ 0 & e_{\mathscr{D}} \end{bmatrix}.$$

We define an involution * on $\mathscr{M}$ by setting

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* := \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}.$$

We will assume some additional structure within each of the four spaces $\mathscr{A} - \mathscr{D}$. The algebras $\mathscr{A}$ and $\mathscr{D}$ are assumed to admit direct sum decompositions

$$\mathscr{A} = \mathscr{A}_-^0 \,\dot{+}\, \mathscr{A}_d \,\dot{+}\, \mathscr{A}_+^0, \quad \mathscr{D} = \mathscr{D}_-^0 \,\dot{+}\, \mathscr{D}_d \,\dot{+}\, \mathscr{D}_+^0 \qquad (2.2)$$

in which all six of the newly indicated spaces are subalgebras and are such that

$$e_{\mathscr{A}} \in \mathscr{A}_d, (\mathscr{A}_-^0)^* = (\mathscr{A}_+^0)^*, (\mathscr{A}_d)^* = \mathscr{A}_d, \qquad (2.3)$$

$$e_{\mathscr{D}} \in \mathscr{D}_d, (\mathscr{D}_-^0)^* = (\mathscr{D}_+^0)^*, (\mathscr{D}_d)^* = \mathscr{D}_d,$$

and the inclusions

$$\mathscr{A}_d \mathscr{A}_\pm^0 \subset \mathscr{A}_\pm^0, \mathscr{A}_\pm^0 \mathscr{A}_d \subset \mathscr{A}_\pm^0, \mathscr{D}_d \mathscr{D}_\pm^0 \subset \mathscr{D}_\pm^0, \mathscr{D}_\pm^0 \mathscr{D}_d \subset \mathscr{D}_\pm^0 \qquad (2.4)$$

are in force. It is then readily checked that

$$\mathscr{A}_\pm := \mathscr{A}_\pm^0 \,\dot{+}\, \mathscr{A}_d, \mathscr{D}_\pm := \mathscr{D}_\pm^0 \,\dot{+}\, \mathscr{D}_d$$

are algebras. Moreover, if $a \in \mathscr{A}_d$ (resp. $d \in \mathscr{D}_d$) and is invertible, then $a^{-1} \in \mathscr{A}_d$ (resp. $d^{-1} \in \mathscr{D}_d$). Finally, we suppose that $\mathscr{B}$ and $\mathscr{C}$ admit decompositions

$$\mathscr{B} = \mathscr{B}_- \,\dot{+}\, \mathscr{B}_+, \mathscr{C} = \mathscr{C}_- \,\dot{+}\, \mathscr{C}_+, \qquad (2.5)$$

where $\mathscr{B}_{\pm} \subset \mathscr{B}$ and $\mathscr{C}_{\pm} \subset \mathscr{C}$ are subspaces satisfying

$$\mathscr{C}_- = \mathscr{B}_+^* , \mathscr{C}_+ = \mathscr{B}_-^* ,$$

$$\mathscr{B}_{\pm} \mathscr{D}_{\pm} \subset \mathscr{B}_{\pm} , \mathscr{A}_{\pm} \mathscr{B}_{\pm} \subset \mathscr{B}_{\pm} ,$$

$$\mathscr{C}_{\pm} \mathscr{B}_{\pm} \subset \mathscr{D}_{\pm}^0 , \mathscr{B}_{\pm} \mathscr{C}_{\pm} \subset \mathscr{A}_{\pm}^0 , \tag{2.6}$$

$$\mathscr{C}_{\pm} \mathscr{A}_{\pm} \subset \mathscr{C}_{\pm} , \mathscr{D}_{\pm} \mathscr{C}_{\pm} \subset \mathscr{C}_{\pm} .$$

Now let us introduce the following subspaces of $\mathscr{M}$:

$$\mathscr{M}_1 = \begin{bmatrix} 0 & \mathscr{B}_+ \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathscr{B}_+ \right\} ,$$

$$\mathscr{M}_2^0 = \begin{bmatrix} \mathscr{A}_+^0 & \mathscr{B}_- \\ 0 & \mathscr{D}_+^0 \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a \in \mathscr{A}_+^0 , b \in \mathscr{B}_- , d \in \mathscr{D}_+^0 \right\} ,$$

$$\mathscr{M}_d = \begin{bmatrix} \mathscr{A}_d & 0 \\ 0 & \mathscr{D}_d \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a \in \mathscr{A}_d , d \in \mathscr{D}_d \right\} ,$$

$$\mathscr{M}_3^0 = \begin{bmatrix} \mathscr{A}_-^0 & 0 \\ \mathscr{C}_+ & \mathscr{D}_-^0 \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \mid a \in \mathscr{A}_-^0 , c \in \mathscr{C}_+ , d \in \mathscr{D}_-^0 \right\} ,$$

$$\mathscr{M}_4 = \begin{bmatrix} 0 & 0 \\ \mathscr{C}_- & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \mid c \in \mathscr{C}_- \right\} .$$

Note that (1.1) holds and that this decomposition satisfies the conditions (i) and (ii) in Section II.1. With respect to positive elements we assume that the algebra $\mathscr{M}$ satisfies the following axiom

AXIOM (A0a). If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is positive definite in $\mathscr{M}$, then a is positive definite in $\mathscr{A}$ and d is positive definite in $\mathscr{D}$.

Note that Axiom (A0a) implies that for $b \in \mathscr{B}$ the element $e_{\mathscr{D}} + b^* b$ is positive definite in $\mathscr{D}$ and the element $e_{\mathscr{A}} + bb^*$ is positive definite in $\mathscr{A}$. This follows immediately from the observation that

$$\begin{bmatrix} e_{\mathscr{A}} & b \\ 0 & e_{\mathscr{D}} \end{bmatrix} \begin{bmatrix} e_{\mathscr{A}} & 0 \\ b^* & e_{\mathscr{D}} \end{bmatrix} , \begin{bmatrix} e_{\mathscr{A}} & 0 \\ b^* & e_{\mathscr{D}} \end{bmatrix} \begin{bmatrix} e_{\mathscr{A}} & b \\ 0 & e_{\mathscr{D}} \end{bmatrix}$$

are positive definite elements in $\mathcal{M}$. Further, from the equations

$$k_h := \begin{bmatrix} e_{\mathcal{A}} & h \\ h^* & e_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ h^* & e_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} e_{\mathcal{A}} & 0 \\ 0 & e_{\mathcal{D}}-h^*h \end{bmatrix} \begin{bmatrix} e_{\mathcal{A}} & h \\ 0 & e_{\mathcal{D}} \end{bmatrix}$$

$$= \begin{bmatrix} e_{\mathcal{A}} & h \\ 0 & e_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} e_{\mathcal{A}}-hh^* & 0 \\ 0 & e_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} e_{\mathcal{A}} & 0 \\ h^* & e_{\mathcal{D}} \end{bmatrix}$$

one obtains that for $h \in \mathcal{B}$ the element k_h is positive definite in $\mathcal{M}$ if and only if $e_{\mathcal{D}}-h^*h$ is positive definite in $\mathcal{D}$, or equivalently, $e_{\mathcal{A}}-hh^*$ is positive definite in $\mathcal{A}$. From now on we shall write e for both $e_{\mathcal{A}}$ and $e_{\mathcal{D}}$.

Let $\phi \in \mathcal{B}_-$ be given. An element $\psi \in \mathcal{B}$ is called a *strictly contractive extension* of ϕ if $\psi-\phi \in \mathcal{B}_+$ and $e-\psi^*\psi$ is positive definite in $\mathcal{D}$. Recall that an element $d \in \mathcal{D}$ is called positive definite in $\mathcal{D}$ if there exists an invertible element $c \in \mathcal{D}$ such that $d = cc^*$. The term "strictly contractive extension" is justified by the fact that in a B*-algebra an element b has norm less than one if and only if $e-b^*b$ is positive definite. We are interested to find all strictly contractive extensions of a given $\phi \in \mathcal{B}_-$. We call $g \in \mathcal{B}$ a *(strictly contractive) triangular extension* of ϕ if g is a strictly contractive extension of ϕ and $g(e-g^*g)^{-1}$ belongs to $\mathcal{B}_-$. In what follows we will omit the words strictly contractive and just talk about a triangular extension. As in Section II.1 we say that a positive element $d \in \mathcal{D}$ admits a *left (right) spectral factorization* (with respect to the decomposition of $\mathcal{D}$ in (2.2)) if there is an invertible $c \in \mathcal{D}_+$ ($\mathcal{D}_-$) such that $d = cc^*$ and $c^{-1} \in \mathcal{D}_+$ ($\mathcal{D}_-$). In $\mathcal{A}$ we have similar definitions. Note that k_ϕ admits a right spectral factorization in $\mathcal{M}$ if and only if $e-\phi^*\phi$ admits a right spectral factorization in $\mathcal{D}$ and that k_ϕ admits a left spectral factorization in $\mathcal{M}$ if and only if $e-\phi\phi^*$ admits a left spectral factorization in $\mathcal{A}$.

The following theorem is an application of Theorem II.1.1. We need some additional notation. If $\mathcal{E}_0$ is a subspace of the space $\mathcal{E}$, we let $P_{\mathcal{E}_0}$ denote the projection in $\mathcal{E}$ on $\mathcal{E}_0$ along a natural complement. So, for instance, $P_{\mathcal{A}_+}$ is the projection on $\mathcal{A}_+$ along $\mathcal{A}_-^0$.

THEOREM 2.1. *Let $\mathcal{M}$ be an algebra as above, and assume that Axiom (A0a) is satisfied. Let $\phi \in \mathcal{B}_-$. The element ϕ has a triangular extension g such that $e-gg^*$ admits a left and $e-g^*g$ admits a right spectral factorization if and only if the equations*

$$e = a -P_{\mathcal{A}_-}(\phi(P_{\mathcal{E}_+}(\phi^*a))) \, , \, e = d -P_{\mathcal{D}_+}(\phi^*P_{\mathcal{B}_-}(\phi d)) \, . \tag{2.7}$$

have solutions a and d with the following properties:

(i) $a \in \mathcal{A}_-$, $d \in \mathcal{D}_+$,

(ii) a and d are invertible, $a^{-1} \in \mathcal{A}_-$, $d^{-1} \in \mathcal{D}_+$,

(iii) $P_{\mathcal{A}_d} a$ and $P_{\mathcal{D}_d} d$ are positive definite in $\mathcal{A}_d$ and $\mathcal{D}_d$, respectively.

In that case ϕ has a unique triangular extension g for which $e - g g^*$ admits a left and $e - g^* g$ admits a right spectral factorization, and this g is given by

$$ g := b d^{-1} = a^{*-1} c^* , $$

where

$$ b = P_{\mathcal{B}_-}(\phi d) , c = P_{\mathcal{C}_+}(\phi^* a) . \tag{2.8} $$

The spectral factorizations of $e - g g^*$ and $e - g^* g$ are given by

$$ e - g g^* = a^{*-1}(P_{\mathcal{A}_d} a) a^{-1} , e - g^* g = d^{*-1}(P_{\mathcal{D}_d} d) d^{-1} . $$

In many applications the algebra $\mathcal{M}$ has the additional property that every positive definite element admits a left and right spectral factorization. For such an algebra $\mathcal{M}$ the hypothesis of Theorem 2.1 imply that there exists a unique triangular extension of ϕ.

Proof. We will transform the strictly contractive extension problem into a positive extension problem in $\mathcal{M}$.

Let $\phi \in \mathcal{B}_-$ be given and put

$$ k_\phi := \begin{bmatrix} e & \phi \\ \phi^* & e \end{bmatrix} . $$

Clearly, k_ϕ is an element of $\mathcal{M}_c$. Suppose that k is a positive extension of k_ϕ. (The definition is in Section 1). Since $k - k_\phi \in \mathcal{M}_1 \dot{+} \mathcal{M}_4$ and k is selfadjoint, k has to be equal to k_ψ for some $\psi \in \mathcal{B}$. Also $\psi - \phi \in \mathcal{B}_+$. Further, since

$$ \begin{bmatrix} e & 0 \\ 0 & e - \psi^* \psi \end{bmatrix} = \begin{bmatrix} e & 0 \\ -\psi^* & e \end{bmatrix} \begin{bmatrix} e & \psi \\ \psi^* & e \end{bmatrix} \begin{bmatrix} e & -\psi \\ 0 & e \end{bmatrix} , \tag{2.9} $$

Axiom (A0a) gives that $e - \psi^* \psi$ is positive definite in $\mathcal{D}$. So a positive extension k_ψ of k_ϕ gives a strictly contractive extension ψ of ϕ. The converse is also true. Indeed, suppose that ψ is a strictly contractive extension of ϕ. Since $\psi - \phi \in \mathcal{B}_+$, the element $k_\psi - k_\phi$ belongs to $\mathcal{M}_1 \dot{+} \mathcal{M}_4$. Since $e - \psi^* \psi$ is positive definite in $\mathcal{D}$, equation (2.9) yields that k_ψ is positive definite in $\mathcal{M}$. So the strictly contractive extension problem in $\mathcal{B}$ is equivalent to a positive

extension problem in $\mathcal{M}$. Further, note that

$$
\begin{bmatrix} e & \psi \\ \psi^* & e \end{bmatrix}^{-1} = \begin{bmatrix} * & \psi(e-\psi^*\psi)^{-1} \\ * & * \end{bmatrix}.
$$

So $k_\psi^{-1} \in \mathcal{M}_c$ if and only if $\psi(e-\psi^*\psi)^{-1} \in \mathcal{B}_-$. Thus k_ψ is a band extension of k_ϕ if and only if ψ is a triangular extension of ϕ.

We are now ready to prove the theorem. Suppose that g is a triangular extension of ϕ such that

$$
e - g^*g = r^*r \; , \; e - gg^* = s^*s \; ,
$$

where $r^{\pm 1} \in \mathcal{D}_+$ and $s^{\pm 1} \in \mathcal{A}_-$. Since g is a triangular extension of ϕ we have that k_g is a band extension of k_ϕ, and since

$$
k_g = \begin{bmatrix} e & 0 \\ g^* & e \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & r^* \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} e & g \\ 0 & e \end{bmatrix},
$$

the matrix k_g admits a left spectral factorization (with respect to the decomposition (1.1)). Analogously, k_g admits a right spectral factorization. By Theorem 1.1 there are

$$
x = \begin{bmatrix} \hat{d} & -b \\ 0 & d \end{bmatrix} \in \mathcal{M}_2, y = \begin{bmatrix} a & 0 \\ -c & \hat{a} \end{bmatrix} \in \mathcal{M}_3,
$$

such that $x^{-1} \in \mathcal{M}_+$, $y^{-1} \in \mathcal{M}_-$, $P_d x$ and $P_d y$ are positive definite in $\mathcal{M}_d$, and equations (1.5) hold with $k = k_\phi$. Writing out (1.5) one obtains that $\hat{a} = \hat{d} = e$, and equations (2.7) and (2.8) hold. Furthermore,

$$
x^{-1} = \begin{bmatrix} e & bd^{-1} \\ 0 & d^{-1} \end{bmatrix} \in \mathcal{M}_+,
$$

yields that $d^{-1} \in \mathcal{D}_+$. Analogously, $a^{-1} \in \mathcal{A}_-$. Since $P_d x$ is positive definite in $\mathcal{M}_d$, we have that

$$
P_d x = \begin{bmatrix} e & 0 \\ 0 & P_{\mathcal{D}_d} d \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & q^* \end{bmatrix} \begin{bmatrix} * & 0 \\ 0 & q \end{bmatrix},
$$

where $\begin{bmatrix} * & 0 \\ 0 & q \end{bmatrix}$ is invertible in $\mathcal{M}_d$. Thus, in particular, $P_{\mathcal{D}_d} d = q^*q$ is positive definite in $\mathcal{D}_d$. Analogously, $P_{\mathcal{A}_d} a$ is positive definite in $\mathcal{A}_d$. Furthermore, from (1.6) with band

extension k_g one obtains that $g = a^{*-1}c^* = bd^{-1}$.

Conversely, suppose that a and d exist as in the theorem. Put $k = k_\phi$,

$$x := \begin{bmatrix} e & -b \\ 0 & d \end{bmatrix} \in \mathcal{M}_2, y := \begin{bmatrix} a & 0 \\ -c & e \end{bmatrix} \in \mathcal{M}_3,$$

where b and c are given by (2.8). One easily checks that with this choice of x and y the conditions in Theorem 1.1 are satisfied. Applying Theorem 1.1 gives that

$$k_g = \begin{bmatrix} e & g \\ g^* & e \end{bmatrix} := \begin{bmatrix} a^* & -c^* \\ 0 & e \end{bmatrix}^{-1} \begin{bmatrix} P_{\mathcal{D}_d}a & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} a & 0 \\ -c & e \end{bmatrix}^{-1} \qquad (2.10)$$

$$= \begin{bmatrix} e & 0 \\ -b^* & d^* \end{bmatrix}^{-1} \begin{bmatrix} e & 0 \\ 0 & P_{\mathcal{M}_d}d \end{bmatrix} \begin{bmatrix} e & -b \\ 0 & d \end{bmatrix}^{-1},$$

is the unique band extension of k_ϕ. But then $g = a^{*-1}c^* = bd^{-1}$ is the unique triangular extension. Using equation (2.9) with g instead of ψ and equation (2.10) one obtains that $e - g^*g = d^{*-1}(P_{\mathcal{D}_d}d)d^{-1}$, so that $e - g^*g$ admits a right spectral factorization. Analogously one shows that $e - gg^* = a^{*-1}(P_{\mathcal{M}_d}a)a^{-1}$ admits a left spectral factorization. □

If $\phi \in \mathcal{B}$, we let $\Xi := P_{\mathcal{A}_-}\phi : \mathcal{C}_+ \rightarrow \mathcal{A}_-$ denote the operator defined by the following action:

$$\Xi(c) = (P_{\mathcal{A}_-}\phi)(c) := P_{\mathcal{A}_-}(\phi c) , c \in \mathcal{C}_+.$$

We shall employ this notation also for other subspaces.

THEOREM 2.2. *Let $\mathcal{M}$ be a Banach algebra satisfying Axiom (A0a). Let $\phi \in \mathcal{B}_-$ be given. Introduce the following operators*

$$\Xi := P_{\mathcal{A}_-}\phi : \mathcal{C}_+ \rightarrow \mathcal{A}_- ; \Xi_* := P_{\mathcal{C}_+}\phi^* : \mathcal{A}_- \rightarrow \mathcal{C}_+ ; \qquad (2.11)$$

$$\tilde{\Xi} := P_{\mathcal{B}_-}\phi : \mathcal{D}_+ \rightarrow \mathcal{B}_- ; \tilde{\Xi}_* := P_{\mathcal{D}_+}\phi^* : \mathcal{B}_- \rightarrow \mathcal{D}_+.$$

Suppose that for each $0 \leq \epsilon \leq 1$ the operators $I - \epsilon^2\Xi\Xi_*$ and $I - \epsilon^2\tilde{\Xi}_*\tilde{\Xi}$ are invertible, and that the elements

$$P_{\mathcal{A}_d}[(I - \epsilon^2\Xi\Xi_*)^{-1}e], P_{\mathcal{D}_d}[(I - \epsilon^2\tilde{\Xi}_*\tilde{\Xi})^{-1}e] \qquad (2.12)$$

are positive definite in $\mathcal{A}_d$ and $\mathcal{D}_d$, respectively. Let $r \in \mathcal{A}_d$ and $s \in \mathcal{D}_d$ be such that

$$P_{\mathcal{A}_d}[(I - \Xi\Xi_*)^{-1}e] = r^*r , P_{\mathcal{D}_d}[(I - \tilde{\Xi}_*\tilde{\Xi})^{-1}e] = s^*s ,$$

and put

$$\alpha := ((I - \Xi\Xi_*)^{-1}e)r^{-1}, \gamma := P_{\mathscr{C}_+}(\phi^*\alpha),$$ (2.13)

$$\delta := ((I - \tilde{\Xi}_*\tilde{\Xi})^{-1}e)s^{-1}, \beta := P_{\mathscr{B}_-}(\phi\delta).$$

Then ϕ has a unique triangular extension g for which $e - gg^*$ admits a left and $e - g^*g$ admits a right spectral factorization, and this g is given by

$$g := \beta\delta^{-1} = \alpha^{*-1}\gamma^*.$$

Moreover, $\alpha^{\pm 1} \in \mathscr{A}_-$, $\delta^{\pm 1} \in \mathscr{D}_+$, and the spectral factorizations of $e - gg^*$ and $e - g^*g$ are given by

$$e - gg^* = \alpha^{*-1}\alpha^{-1}, \ e - g^*g = \delta^{*-1}\delta^{-1}.$$

Proof. For $0 \leq \epsilon \leq 1$, put

$$a_\epsilon := (I - \epsilon^2\Xi\Xi_*)^{-1}e, c_\epsilon := P_{\mathscr{C}_+}(\epsilon\phi^*a_\epsilon),$$

$$d_\epsilon := (I - \epsilon^2\tilde{\Xi}_*\tilde{\Xi})^{-1}e, b_\epsilon := P_{\mathscr{B}_-}(\epsilon\phi d_\epsilon),$$

$$x_\epsilon := \begin{bmatrix} e_{\mathscr{A}} & -b_\epsilon \\ 0 & d_\epsilon \end{bmatrix}, y_\epsilon := \begin{bmatrix} a_\epsilon & 0 \\ -c_\epsilon & e_{\mathscr{D}} \end{bmatrix}.$$

Note that $x_\epsilon \in \mathscr{M}_+$ and $y_\epsilon \in \mathscr{M}_-$ for $0 \leq \epsilon \leq 1$. Clearly, the elements introduced above are analytic in the real variable ϵ. For $k_\epsilon = \begin{bmatrix} e_{\mathscr{A}} & \epsilon\phi \\ \epsilon\phi^* & e_{\mathscr{D}} \end{bmatrix}$ we have that

$$P_3(k_\epsilon y_\epsilon) = P_3 \begin{bmatrix} a_\epsilon - \epsilon\phi c_\epsilon & \epsilon\phi \\ \epsilon\phi^*a_\epsilon - c_\epsilon & e_{\mathscr{D}} \end{bmatrix} = e,$$

since $P_{\mathscr{C}_+}(\epsilon\phi^*a_\epsilon - c_\epsilon) = 0$ and

$$P_{\mathscr{A}_-}(a - \epsilon\phi c_\epsilon) = P_{\mathscr{A}_-}(a_\epsilon - \epsilon\phi P_{\mathscr{C}_+}(\epsilon\phi^*a_\epsilon)) = (I - \epsilon^2\Xi\Xi_*)a_\epsilon = e.$$

Analogously, one calculates that $P_2(k_\epsilon x_\epsilon) = e$. If ϵ is small enough, $0 \leq \epsilon < \sigma (\leq 1)$ say, then the elements a_ϵ and d_ϵ are invertible in $\mathscr{A}_-$ and $\mathscr{D}_+$, respectively. Further, by assumption, for $0 \leq \epsilon \leq 1$ the elements $P_{\mathscr{A}_d}(a_\epsilon)$ and $P_{\mathscr{D}_d}(d_\epsilon)$ are positive definite in $\mathscr{A}_d$ and $\mathscr{D}_d$, respectively. Hence, for $0 \leq \epsilon \leq 1$, the elements $P_d(x_\epsilon)$ and $P_d(y_\epsilon)$ are positive definite in $\mathscr{M}_d$. Now Theorem II.1.1 yields that for $0 \leq \epsilon < \sigma$

$$x_\epsilon^{*-1}(P_d x_\epsilon)x_\epsilon^{-1} = y_\epsilon^{*-1}(P_d y_\epsilon)y_\epsilon^{-1}$$

is the unique band extension of k_ϵ. It follows that

$$x_\epsilon(P_d x_\epsilon)^{-1}x_\epsilon^* = y_\epsilon(P_d y_\epsilon)^{-1}y_\epsilon^* \qquad (2.14)$$

holds for $0 \leq \epsilon < \sigma$, and by analyticity (2.14) also holds for $0 \leq \epsilon \leq 1$. Calculating the
(1,1) element of (2.14) we obtain

$$a_\epsilon(P_{\mathscr{A}_d} a_\epsilon)^{-1}a_\epsilon^* = e + b_\epsilon(P_{\mathscr{D}_d} d_\epsilon)^{-1}b_\epsilon^*.$$

Axiom (A0a) implies that the right hand side is positive definite, which gives that a_ϵ is
invertible for $0 \leq \epsilon \leq 1$. Indeed,

$$a_\epsilon^+ := (P_{\mathscr{A}_d} a_\epsilon)^{-1}a_\epsilon^*(e + b_\epsilon(P_{\mathscr{D}_d} d_\epsilon)^{-1}b_\epsilon^*)^{-1}$$

is a right inverse of a_ϵ. Further, $\epsilon \mapsto a_\epsilon^+$ is analytic. Consider

$$a_\epsilon^+ a_\epsilon - e.$$

For $0 \leq \epsilon < \sigma$ this equals zero. But then $a_\epsilon^+ a_\epsilon - e$ is zero for $\epsilon \in [0,1]$, proving the inver-
tibility of a_ϵ. Furthermore, since $P_{\mathscr{A}_+} a_\epsilon^{-1} = 0$ for $0 \leq \epsilon < \sigma$, we get by analyticity that
this also holds for $\epsilon = 1$. So a_1 is, in fact, invertible in $\mathscr{A}_-$. Analogously, d_1 is invertible
in $\mathscr{D}_+$. But then the theorem follows directly from Theorem 2.1. $\square$

THEOREM 2.3. *Assume that the algebra $\mathscr{M}$ is a *-subalgebra of the B^*-algebra $\mathfrak{R}$
where the unit e of $\mathfrak{R}$ belongs to $\mathscr{M}$, and assume that Axioms (A0a), (A1) and (A2) hold.
Let $\phi \in \mathscr{B}_-$, and suppose that ϕ has a triangular extension g such that $e - gg^*$ admits a
left and $e - g^*g$ admits a right spectral factorization. Let $\alpha \in \mathscr{A}_-$ and $\delta \in \mathscr{D}_-$ be inver-
tible elements such that $\alpha^{-1} \in \mathscr{A}_-$, $\delta^{-1} \in \mathscr{D}_+$ and*

$$(e - gg^*)^{-1} = \alpha\alpha^* , (e - g^*g)^{-1} = \delta\delta^* ,$$

and put

$$\beta = P_{\mathscr{B}_-}(\phi\delta) , \gamma = P_{\mathscr{C}_+}(\phi^*\alpha).$$

Then each strictly contractive extension $\psi \in \mathscr{B}$ of ϕ is of the form

$$\psi = (\alpha h + \beta)(\gamma h + \delta)^{-1}, \qquad (2.15)$$

*where h is an element in $\mathscr{B}_+$ such that $e - h^*h$ is positive definite in $\mathscr{D}$. Furthermore,
equation (2.15) gives a one-one correspondence between all such h and all strictly*

contractive extensions ψ of ϕ. Alternatively, each strictly contractive extension $\psi \in \mathcal{B}$ of ϕ is of the form

$$\psi = (\alpha^* + f^* \beta^*)^{-1} (\gamma^* + f^* \delta^*), \tag{2.16}$$

where f is an element in $\mathcal{B}_+$ such that $e - ff^*$ is positive definite in $\mathcal{A}$. Furthermore, equation (2.16) gives a one-one correspondence between all such f and all strictly contractive extensions ψ of ϕ.

Proof. Put

$$u = \begin{pmatrix} -e & \beta \\ 0 & -\delta \end{pmatrix} , v = \begin{pmatrix} \alpha & 0 \\ -\gamma & e \end{pmatrix}.$$

Now apply Theorem 1.5 (in particular (1.9)) to obtain that all positive extensions k_ψ of $k_\phi \in \mathcal{M}_c$ are of the form

$$k_\psi = \begin{bmatrix} e & (\alpha h + \beta)(\gamma h + \delta)^{-1} \\ (\gamma h + \delta)^{*-1}(\alpha h + \beta)^* & e \end{bmatrix},$$

where $\begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_1$ is such that $\begin{bmatrix} e & 0 \\ 0 & e - h^* h \end{bmatrix}$ is positive definite in $\mathcal{M}$. Moreover,

there is a one-one correspondence between all such h and all positive extensions of k_ϕ. But then the one-one correspondence (2.15) follows.

The alternative form (2.16) is obtained by using formula (1.10) in Theorem 1.5. $\square$

The approach in this section is the same as the abstract approach in [27], except that here the algebra is enriched with an involution. Both Theorems 2.2 and 2.3 are new, but they are inspired by earlier concrete versions in [26], [27] (see also [20]).

II.3. Maximum entropy principles

This section concerns maximum entropy principles in the general setting of the band method. These principles provide alternative ways to identify the band extension (in the positive extension problem) and the triangular extension (in the strictly contractive extension problem).

3.1. Positive extensions. Let $\mathcal{M}$ be an algebra with unit e and involution * and with the structure described in the first paragraph of Section II.1. We introduce the following

notions. Let b be a positive definite element of $\mathcal{M}$ which admits a right spectral factorization $b = b_-b_-^*$, $b_-^{\pm 1} \in \mathcal{M}_-$. We define the *right multiplicative diagonal* $\Delta_r(b)$ of b to be the element

$$\Delta_r(b) := P_d(b_-)P_d(b_-)^*.$$

The right multiplicative diagonal of b is well-defined. Indeed, suppose that $b = c_-c_-^*$, $c_-^{\pm 1} \in \mathcal{M}_-$, is another right spectral factorization of b. Then

$$d := c_-^{-1}b_- = c_-^* b_-^{*-1} \in \mathcal{M}_- \cap \mathcal{M}_+ = \mathcal{M}_d.$$

Hence $b_- = c_-d$, thus $c_-c_-^* = b_-b_-^* = c_-dd^*c_-^*$, giving that $dd^* = e$. But then, since $d \in \mathcal{M}_d$,

$$P_d(b_-)P_d(b_-)^* = P_d(c_-d)P_d(c_-d)^* =$$

$$= P_d(c_-)dd^*P_d(c_-^*) = P_d(c_-)P_d(c_-)^*.$$

Note that $\Delta_r(b)$ is positive definite in $\mathcal{M}_d$. This follows from $P_d(b_-)^{-1} = P_d(b_-^{-1})$ (see Lemma 1.3). When b admits a left spectral factorization $b = b_+b_+^*$, $b_+^{\pm 1} \in \mathcal{M}_+$, we define its *left multiplicative diagonal* $\Delta_l(b)$ to be the element

$$\Delta_l(b) := P_d(b_+)P_d(b_+)^*.$$

Again $\Delta_l(b)$ is well-defined and $\Delta_l(b)$ is positive definite in $\mathcal{M}_d$.

Recall that an element $a \in \mathcal{M}$ is called nonnegative definite in $\mathcal{M}$ if there is an element $c \in \mathcal{M}$ such that $a = c^*c$. With respect to nonnegative definite elements we introduce the following two axioms.

AXIOM (A3). The element $P_d(c^*c)$ is nonnegative definite in $\mathcal{M}$ for all $c \in \mathcal{M}$.

AXIOM (A4). If $P_d(c^*c) = 0$, then $c = 0$.

When $\mathcal{M}$ satisfies these two axioms we have the following general maximum entropy principle.

THEOREM 3.1. *Let $\mathcal{M}$ be as above, and assume that $\mathcal{M}$ satisfies Axioms (A3) and (A4). Let $k \in \mathcal{M}_c$ have a band extension b which admits a right (left) spectral factorization. Then for any positive extension a of k which admits a right (left) spectral factorization*

$$\Delta_r(b) \geqq \Delta_r(a) \quad (\Delta_l(b) \geqq \Delta_l(a)).\tag{3.1}$$

Furthermore, equality holds in (3.1) if and only if $a = b$.

We need the following lemma.

LEMMA 3.2. (i) *Suppose that* b *admits a right spectral factorization. Then there are unique* $m_- \in \mathcal{M}_-^0$ *and* $d \in \mathcal{M}_d$ *such that* $(e + m_-)^{-1} \in \mathcal{M}_-$ *and*

$$b = (e + m_-)d(e + m_-)^*. \tag{3.2}$$

Moreover, $d = \Delta_r(b)$.

(ii) *Suppose that* b *admits a left spectral factorization. Then there are unique* $m_+ \in \mathcal{M}_+^0$ *and* $d \in \mathcal{M}_d$ *such that* $(e + m_+)^{-1} \in \mathcal{M}_+$ *and*

$$b = (e + m_+)d(e + m_+)^*. \tag{3.3}$$

Moreover, $d = \Delta_l(b)$.

Proof. Let us prove (i). Let $b = b_- b_-^*$ with $b_-^{\pm 1} \in \mathcal{M}_-$, be a right spectral factorization. Put $m_- := b_- P_d(b_-)^{-1} - e$ and $d = \Delta_r(b)$. It is easy to check that $m_- \in \mathcal{M}_-^0$, $(e + m_-)^{-1} \in \mathcal{M}_-$ and that (3.2) holds. Suppose now that

$$b = (e + \hat{m}_-)\hat{d}(e + \hat{m}_-^*),$$

where $\hat{m}_- \in \mathcal{M}_-^0$, $(e + \hat{m}_-)^{-1} \in \mathcal{M}_-$ and $d \in \mathcal{M}_d$. Then

$$(e + \hat{m}_-)^{-1}(e + m_-) = \hat{d}(e + \hat{m}_-)^*(e + m_-)^{*-1}d^{-1}. \tag{3.4}$$

Applying P_d to both sides of (3.4) gives $e = \hat{d}d^{-1}$, which implies that $d = \hat{d}$. Apply now P_- to both sides of (3.4). Then

$$(e + \hat{m}_-)^{-1}(e + m_-) = \hat{d}d^{-1} = e.$$

Thus $\hat{m}_- = m_-$.

The proof of (ii) is similar. $\square$

Proof of Theorem 3.1. Let k have a band extension b, let a be a positive extension of k, and suppose that both admit a right spectral factorization

$$a = (e + a_-)\Delta_r(a)(e + a_-)^*, \quad b = (e + b_-)\Delta_r(b)(e + b_-)^*,$$

with $a_-, b_- \in \mathcal{M}_-^0$ and $(e + a_-)^{-1}, (e + b_-)^{-1} \in \mathcal{M}_-$. Since $b^{-1} \in \mathcal{M}_c$, Lemma 1.2 implies $(e + b_-)^{-1} \in \mathcal{M}_3$. Write $a = b + a - b$, and observe that

$$(e + b_-)^{-1}(e + a_-)\Delta_r(a)(e + a_-)^*(e + b_-)^{*-1} = \tag{3.5}$$

$$\Delta_r(b) + (e + b_-)^{-1}(a - b)(e + b_-)^{*-1}.$$

Since a and b are both positive extensions of k,

$$a - b = m_1 + m_1^*$$

for some $m_1 \in \mathcal{M}_1$. Then $(e + b_-)^{-1} m_1 \in \mathcal{M}_+^0$. From this we obtain that

$$P_d((e + b_-)^{-1}(a - b)(e + b_-)^{*-1}) = 0.$$

Write $(e + b_-)^{-1}(e + a_-) = e + w$ with $w \in \mathcal{M}_-^0$. Applying P_d on equation (3.5) gives

$$\Delta_r(b) = P_d((e + w)\Delta_r(a)(e + w)^*) =$$

$$P_d(\Delta_r(a) + w\Delta_r(a) + \Delta_r(a)w^* + w\Delta_r(a)w^*) = \Delta_r(a) + P_d(w\Delta_r(a)w^*) \geqq \Delta_r(a),$$

where in the proof of the last inequality we use Axiom (A3). Furthermore, $\Delta_r(a) = \Delta_r(b)$ if and only if $P_d(w\Delta_r(a)w^*) = 0$. Since $\Delta_r(a) > 0$, we obtain from Axiom (A4) that the latter equality holds if and only if $w = 0$. But then $a = b$.

The proof of the left version is similar. $\square$

Theorem 3.1 is inspired by earlier concrete versions (see [22], [23]).

Let $\mathcal{M}_d^>$ denote the set of all elements in $\mathcal{M}_d$ that are positive definite:

$$\mathcal{M}_d^> := \{ a \in \mathcal{M}_d \mid a > 0 \}.$$

We call a function $F : \mathcal{M}_d^> \to \mathbb{R}$ strictly monotone if $d_1 \geqq d_2$ and $d_1 \neq d_2$ implies $F(d_1) > F(d_2)$.

COROLLARY 3.3 *Assume that $\mathcal{M}$ satisfies Axioms (A3) and (A4), and let $F : \mathcal{M}_d^> \to \mathbb{R}$ be strictly monotone. Let $k \in \mathcal{M}_c$. Suppose that k has a band extension b which admits a right (left) spectral factorization. Then for any positive extension a of k which admits a right (left) spectral factorization*

$$F(\Delta_r(b)) \geqq F(\Delta_r(a)) \quad (F(\Delta_l(b)) \geqq F(\Delta_l(a))). \tag{3.6}$$

Furthermore, equality holds in (3.6) if and only if $a = b$.

Proof. Follows directly from Theorem 3.1. $\square$

3.2. Strictly contractive extensions.
In this subsection $\mathcal{M}$ is the algebra

$$\mathcal{M} = \left\{ f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}, d \in \mathcal{D} \right\},$$

with the properties described in the second and third paragraph of Section II.2. We shall use the following axiom.

AXIOM (A0b). If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonnegative definite in $\mathcal{M}$, then a is nonnegative definite in $\mathcal{A}$ and d is nonnegative definite in $\mathcal{D}$.

The notions of right and left multiplicative diagonals of elements in $\mathcal{A}$ and $\mathcal{D}$ we introduce in the same way as is done in Subsection 3.1 for elements in $\mathcal{M}$. We now have the following theorems.

THEOREM 3.4. *Let* $\mathcal{M}$ *satisfy Axioms (A0a), (A0b), (A3) and (A4). Let* $\phi \in \mathcal{B}_-$, *and suppose that* ϕ *has a triangular extension* g *such that* $e - g^* g$ *admits a right spectral factorization. Then for any strictly contractive extension* ψ *of* ϕ *such that* $e - \psi^* \psi$ *admits a right spectral factorization*

$$\Delta_r(e - g^* g) \geqq \Delta_r(e - \psi^* \psi) \ ,$$

and equality holds if and only if $\psi = g$.

THEOREM 3.5. *Let* $\mathcal{M}$ *satisfy Axioms (A0a), (A0b), (A3) and (A4). Let* $\phi \in \mathcal{B}_-$, *and suppose that* ϕ *has a triangular extension* g *such that* $e - gg^*$ *admits a left spectral factorization. Then for any strictly contractive extension* ψ *of* ϕ *such that* $e - \psi\psi^*$ *admits a left spectral factorization*

$$\Delta_l(e - gg^*) \geqq \Delta_l(e - \psi\psi^*) \ ,$$

and equality holds if and only if $\psi = g$.

We prove Theorem 3.4. The proof of Theorem 3.5 is similar.

Proof of Theorem 3.4. The assumptions in the theorem imply together with Axiom (A0a) that k_g is a band extension of k_ϕ which admits a right spectral factorization. Furthermore, k_ψ is a positive extension of k_ϕ which admits a left spectral factorization. Using (2.9) one finds that

$$\Delta_r(k_\psi) = \begin{bmatrix} e & 0 \\ 0 & \Delta_r(e - \psi^* \psi) \end{bmatrix} ,$$

and we have an analogous formula for $\Delta_r(k_g)$. Since (A3) and (A4) hold in $\mathcal{M}$ we obtain from Theorem 3.1 that

$$\Delta_r(k_g) - \Delta_r(k_\psi) = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_r(e - g^* g) - \Delta_r(e - \psi^* \psi) \end{bmatrix}$$

is nonnegative definite in $\mathcal{M}$. Axiom (A0b) implies

$$\Delta_r(e - g^* g) \geqq \Delta_r(e - \psi^* \psi).$$

Further, if equality holds, then $\Delta_r(k_g) = \Delta_r(k_\psi)$. From Theorem 3.1 we obtain that $k_g = k_\psi$, and hence $g = \psi$. $\square$

As in Subsection 3.1 any strictly monotone function on $\mathcal{D}_d^{\,\vec{}}$ (or on $\mathcal{A}_d^{\,\vec{}}$) can be used to pick out the triangular extension.

CHAPTER III. THE BAND METHOD: APPLICATIONS

In this chapter the band method is applied to some concrete positive and strictly con-
tractive extension problems. Sections 1 and 2 concern extension problems for finite opera-
tor matrices, and Sections 3 and 4 treat extension problems for operator valued functions
from the Wiener algebra.

III.1. Operator matrices: positive extensions

In this section we specify the results of Section II.1 for the algebra of $n \times n$ operator
matrices. An element of this algebra has the following form:

$$ T = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} . $$

Here A_{ij}, $1 \leqq i,j \leqq n$, is a bounded linear operator from a Hilbert space H_j into a Hil-
bert space H_i; shortly $A_{ij} \in \mathscr{L}(H_j, H_i)$. Note that T is an operator on the Hilbert space
$H_1 \oplus \cdots \oplus H_n$. The notation $T > 0$ means that T is positive definite. We write I_j for
the identity operator on H_j.

THEOREM 1.1. *For* $1 \leqq i,j \leqq n$, $|j-i| \leqq p$, *let* $A_{ij} = A_{ji}^{*}$ *be a given operator
acting from a Hilbert space* H_j *into a Hilbert space* H_i, *and suppose that*

$$ \begin{bmatrix} A_{ii} & \cdots & A_{i,i+p} \\ \vdots & & \vdots \\ A_{i+p,i} & \cdots & A_{i+p,i+p} \end{bmatrix} > 0 , i = 1,...,n-p. \tag{1.1} $$

For $q = 1,...,n$, *let*

$$ \begin{bmatrix} Y_{qq} \\ \vdots \\ Y_{\beta(q),q} \end{bmatrix} = \begin{bmatrix} A_{qq} & \cdots & A_{q,\beta(q)} \\ \vdots & & \vdots \\ A_{\beta(q),q} & \cdots & A_{\beta(q),\beta(q)} \end{bmatrix}^{-1} \begin{bmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{bmatrix} , \tag{1.2} $$

and

$$ \begin{bmatrix} X_{\gamma(q),q} \\ \vdots \\ X_{q,q} \end{bmatrix} = \begin{bmatrix} A_{\gamma(q),\gamma(q)} & \cdots & A_{\gamma(q),q} \\ \vdots & & \vdots \\ A_{q,\gamma(q)} & \cdots & A_{qq} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_q \end{bmatrix} , \tag{1.3} $$

The band method: applications

where $\beta(q) = \min\{n, p+q\}$ *and* $\gamma(q) = \max\{1, q-p\}$. *Let the* $n \times n$ *triangular operator matrices* U *and* V *be defined by*

$$V_{ij} = \begin{cases} Y_{ij} Y_{jj}^{-\frac{1}{2}} & , j \leq i \leq \beta(j); \\ 0 & , \text{elsewhere;} \end{cases} \tag{1.4}$$

$$U_{ij} = \begin{cases} X_{ij} X_{jj}^{-\frac{1}{2}} & , \gamma(j) \leq i \leq j; \\ 0 & , \text{elsewhere.} \end{cases} \tag{1.5}$$

Then the $n \times n$ *operator matrix* F *given by the following factorizations of its inverse*

$$F := U^{*-1} U^{-1} = V^{*-1} V^{-1} \tag{1.6}$$

is the unique positive definite operator matrix with $F_{ij} = A_{ij}$, $|j-i| \leq p$, *and* $(F^{-1})_{ij} = 0$, $|j-i| > p$.

Proof. We will obtain this theorem as a special case of Theorem II.1.1. Let $\mathcal{M}$ be the algebra of $n \times n$ operator matrices considered in this section. The unit in $\mathcal{M}$ is the identity operator on $H_1 \oplus \cdots \oplus H_n$ and the involution * on $\mathcal{M}$ is the usual adjoint of an operator between Hilbert spaces. Put

$$\mathcal{M}_1 = \{ \left[F_{ij} \right]_{i,j=1}^n \mid F_{ij} = 0 , j-i \leq p \},$$

$$\mathcal{M}_2^0 = \{ \left[F_{ij} \right]_{i,j=1}^n \mid F_{ij} = 0 , j-i > p \text{ and } j-i \leq 0 \},$$

$$\mathcal{M}_d = \{ \left[F_{ij} \right]_{i,j=1}^n \mid F_{ij} = 0 , i \neq j \},$$

$$\mathcal{M}_3^0 = \{ \left[F_{ij} \right]_{i,j=1}^n \mid F_{ij} = 0 , j-i \geq 0 \text{ and } j-i < -p \},$$

$$\mathcal{M}_4 = \{ \left[F_{ij} \right]_{i,j=1}^n \mid F_{ij} = 0 , j-i \geq -p \}.$$

It is easy to see that

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2^0 + \mathcal{M}_d + \mathcal{M}_3^0 + \mathcal{M}_4$$

and that the above subspaces satisfy the conditions (i) and (ii) in Section II.1.

Let $K = \left[K_{ij} \right]_{i,j=1}^{n}$, where $K_{ij} = A_{ij}$ for $|j-i| \leq p$ and $K_{ij} = 0$ otherwise. A direct computation shows that

$$P_2(KX) = I \ , \ P_3(KY) = I \ ,$$

where $X = \left[X_{ij} \right]_{i,j=1}^{n}$ and $Y = \left[Y_{ij} \right]_{i,j=1}^{n}$ are the upper and lower block band matrices of which the entries in the band $|j-i| \leq p$ are given by (1.2) and (1.3), respectively, and which have zero entries outside this band. Since Y_{qq} is the $(1,1)$-block element in the left upper corner of the inverse of a positive definite operator matrix, the element Y_{qq} is positive definite. Similarly X_{qq} is positive definite, and hence the main diagonals of X and Y are positive definite. But then X and Y are invertible and $X^{-1} \in \mathcal{M}_+$ and $Y^{-1} \in \mathcal{M}_-$. In this way it follows from Theorem II.1.1 that the operator matrix F defined in Theorem 1.1 is precisely the unique band extension of K. $\square$

We say that an $n \times n$ operator matrix F is a *positive extension* of the band $\{ A_{ij} \mid |j-i| \leq p \}$ if F is positive definite and $F_{ij} = A_{ij}$ for $|j-i| \leq p$. The extension F in Theorem 1.1 is called the *band extension* of $\{ A_{ij} \mid j-i \leq p \}$, i.e., a positive extension F of $\{ A_{ij} \mid j-i \leq p \}$ is called the band extension of $\{ A_{ij} \mid j-i \leq p \}$ if $(F^{-1})_{ij} = 0$ for $j-i > p$. Note that condition (1.1) is clearly a necessary condition for the existence of a positive extension of the band $\{ A_{ij} \mid |j-i| \leq p \}$. By applying Theorem II.1.5 in the setting described in the proof of Theorem 1.1 we obtain the following description for the set of all positive extensions of a given band.

THEOREM 1.2. *Let* $A_{ij} = A_{ji}^*$, $1 \leq i,j \leq n$, $|j-i| \leq p$, *be given operators acting from a Hilbert space* H_j *into a Hilbert space* H_i. *In order that there exists a positive extension of the band* $\{ A_{ij} \mid |j-i| \leq p \}$ *it is necessary and sufficient that*

$$\begin{bmatrix} A_{ii} & \cdots & A_{i,i+p} \\ \vdots & & \vdots \\ A_{i+p,i} & \cdots & A_{i+p,i+p} \end{bmatrix} > 0, \ i = 1,...,n-p. \tag{1.7}$$

Assume that the latter conditions hold. Let U *and* V *be the* $n \times n$ *operator matrices defined in* (1.2)-(1.5). *Then each positive extension* F *of the given band is of the form*

$$F = (G^*V^* + U^*)^{-1}(I - G^*G)(VG + U)^{-1} \ , \tag{1.8}$$

where G *is a strictly contractive (in operator norm)* $n \times n$ *operator matrix with* $G_{ij} = 0$, $j-i \leq p$. *Furthermore, formula* (1.8) *gives a 1-1 correspondence between all such* G *and all positive extensions* F.

Proof. Let $\mathcal{M}$, $\mathcal{M}_1 - \mathcal{M}_4$ be as in the proof of Theorem 1.1. Since $\mathcal{M}$ endowed with the operator norm is a B*-algebra and $\mathcal{M}_+$ is a closed subalgebra of $\mathcal{M}$, Axioms (A1) and (A2) are fulfilled automatically (with $\mathfrak{R} = \mathcal{M}$). Now apply Theorem II.1.5 to obtain the theorem. $\square$

Let us remark that there is an alternative description for the set of all positive extensions of a given band, which one obtains from (1.10) in Chapter II.

Next we specify the general maximum entropy principle for the case considered in this section. It is known (see [7]) that a positive definite operator A on $H_1 \oplus \cdots \oplus H_n$ admits a $U^* D U$ factorization as follows.

$$
A = \begin{bmatrix} I_1 & & & \\ U_{12}^* & \cdot & & 0 \\ \vdots & & \ddots & \\ U_{1n}^* & \cdots & U_{n-1,n}^* & I_n \end{bmatrix} \text{diag} \left[\Delta_i^r(A) \right]_{i=1}^n \begin{bmatrix} I_1 & U_{12} & \cdots & U_{1n} \\ & \cdot & & \vdots \\ 0 & & \ddots & U_{n-1,n} \\ & & & I_n \end{bmatrix} , \quad (1.9)
$$

where $\Delta_i^r(A)$ is a positive definite operator on H_i $(i = 1,...,n)$. We call

$\text{diag} \left[\Delta_i^r(A) \right]_{i=1}^n$ *the right multiplicative diagonal* of A. The right multiplicative diagonal of A is given by the following identities:

$$
\Delta_1^r(A) = A_{11} , \quad (1.10)
$$

$$
\Delta_i^r(A) := A_{ii} - \begin{bmatrix} A_{i1} & \cdots & A_{i,i-1} \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{1,i-1} \\ \vdots & & \vdots \\ A_{i-1,1} & \cdots & A_{i-1,i-1} \end{bmatrix}^{-1} \begin{bmatrix} A_{1i} \\ \vdots \\ A_{i-1,i} \end{bmatrix} , i = 2,...,n.
$$

Also A admits a $L^* D L$ factorization:

$$
A = \begin{bmatrix} I_1 & V_{12}^* & \cdots & V_{1n}^* \\ & \cdot & & \vdots \\ 0 & & \ddots & V_{n-1,n}^* \\ & & & I_n \end{bmatrix} \text{diag} \left[\Delta_i^l(A) \right]_{i=1}^n \begin{bmatrix} I_1 & & & \\ V_{12} & \cdot & & 0 \\ \vdots & & \ddots & \\ V_{1n} & \cdots & V_{n-1,n} & I_n \end{bmatrix} . \quad (1.11)
$$

We call $\text{diag} \left[\Delta_i^l(A) \right]_{i=1}^n$ *the left multiplicative diagonal of* A. The left multiplicative diagonal of A is given by

$$\Delta_i^l(A) := A_{ii} - \begin{bmatrix} A_{i,i+1} & \cdots & A_{in} \end{bmatrix} \begin{bmatrix} A_{i+1,i+1} & \cdots & A_{n,i+1} \\ \vdots & & \vdots \\ A_{i+1,n} & \cdots & A_{nn} \end{bmatrix}^{-1} \begin{bmatrix} A_{i+1,i} \\ \vdots \\ A_{ni} \end{bmatrix}, \ i = 1,...,n-1,$$

$$\Delta_n^l(A) = A_{nn} \qquad\qquad\qquad (1.12)$$

THEOREM 1.3. *For* $1 \leq i,j \leq n$, $|j-i| \leq p$, *let* $A_{ij} = A_{ji}^*$ *be a given operator acting from a Hilbert space* H_j *into a Hilbert space* H_i, *and suppose that*

$$\begin{bmatrix} A_{ii} & \cdots & A_{i,i+p} \\ \vdots & & \vdots \\ A_{i+p,i} & \cdots & A_{i+p,i+p} \end{bmatrix} > 0, \ i = 1,...,n-p.$$

Put

$$M_i := \begin{bmatrix} 0 & \cdots & 0 & I_i \end{bmatrix} \begin{bmatrix} A_{\gamma(i),\gamma(i)} & \cdots & A_{\gamma(i),i} \\ \vdots & & \vdots \\ A_{i,\gamma(i)} & \cdots & A_{ii} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_i \end{bmatrix}, \ i = 1,...,n, \quad (1.13)$$

where $\gamma(i) = \max\{1,i-p\}$. *Then for the right multiplicative diagonal*

$$\text{diag} \ \left[\Delta_i^r(A) \right]_{i=1}^n \ \textit{of any positive extension } A \textit{ of the given band, the following inequalities}$$

hold:

$$\Delta_i^r(A) \leq M_i^{-1} , \ i = 1,...,n. \qquad\qquad (1.14)$$

Moreover, equality holds for all i *in* (1.14) *if and only if* A *is the band extension of the given band.*

Proof. Let $\mathcal{M}$, $\mathcal{M}_1 - \mathcal{M}_4$ be as in the proof of Theorem 1.1. Let us check the Axioms (A3) and (A4). Clearly, Axiom (A3) is fulfilled since the diagonal entries of a nonnegative definite operator matrix are themselves nonnegative definite. To check Axiom (A4), let $A = \begin{bmatrix} A_{ij} \end{bmatrix}_{i,j=1}^n \geq 0$, and suppose that $A_{ii} = 0$, $i = 1,...,n$. Consider $\Phi = \begin{bmatrix} 0 & A_{ij} \\ A_{ij}^* & 0 \end{bmatrix}$ for some $j > i$. Since $A \geq 0$, we have that $\Phi \geq 0$. Thus for all $v \in H_j$

$$< \Phi \begin{bmatrix} -A_{ij}v \\ v \end{bmatrix}, \begin{bmatrix} -A_{ij}v \\ v \end{bmatrix} > = -2\|A_{ij}v\|^2 \geq 0.$$

Hence we may conclude that $A_{ij} = 0$, $j \neq i$. Thus $\mathcal{M}$ satisfies Axiom (A4). Now we may apply Theorem II.3.1.

Let F be the band extension of the given band. From the factorization $U^{*-1}U^{-1}$ of F in (1.6) it follows that diag $\left[M_i^{-1} \right]$ is precisely the right multiplicative diagonal of F. Indeed, the operator M_i is precisely the operator X_{ii}. Since any positive definite operator matrix admits a right spectral factorization, the theorem follows directly from Theorem II.3.1. $\square$

An analogous result holds for the left multiplicative diagonal of a positive extension. In that case the M_i have to be replaced by $\bar{M}_i$, where

$$
\bar{M}_i := \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} A_{ii} & \cdots & A_{i,\beta(i)} \\ \vdots & & \vdots \\ A_{\beta(i),i} & \cdots & A_{\beta(i),\beta(i)} \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} , i = 1,\ldots,n ,
$$

with $\beta(i) = \min\{n, p + i\}$.

It turns out that in the finite dimensional case the determinant provides a suitable strictly monotone function on the set of positive definite diagonal block matrices. This observation yields the following corollary, which also appeared in [23].

COROLLARY 1.4. *For* $1 \leq i,j \leq n$, $|j - i| \leq p$, *let* $A_{ij} = A_{ji}^*$ *be a given operator acting from a finite dimensional Hilbert space* H_j *into a finite dimensional Hilbert space* H_i, *and suppose that*

$$
\begin{bmatrix} A_{ii} & \cdots & A_{i,i+p} \\ \vdots & & \vdots \\ A_{i+p,i} & \cdots & A_{i+p,i+p} \end{bmatrix} > 0, i = 1,\ldots,n-p.
$$

Then for any positive extension A of the given band

$$
\det A \leq \prod_{i=1}^{n} \det M_i^{-1} , \tag{1.16}
$$

where M_i *is defined in* (1.13). *Moreover, equality holds in* (1.16) *if and only if A is the unique band extension of the given band.*

Proof. Let $f : \mathcal{M}_d^> \rightarrow \mathbb{R}$ be defined by $f(A) = \det A$. Here $\mathcal{M}_d^>$ denotes the set of positive definite diagonal operator matrices. Then f is strictly monotone (for the definition see Section II.3) and the corollary follows immediately from Corollary 3.3 in Chapter II. $\square$

Note that this corollary identifies the band extension as the unique positive extension for which the determinant is maximal. Thus we recover the remark made at the end of Section I.6.

Theorem 1.1, 1.2 and 1.3 also have stationary versions which concern block Toeplitz matrices. To prove these one has to specify further the band method (incuding the proofs) for block Toeplitz matrices. It turns out that in this case the matrices U and V are not Toeplitz, but nevertheless the final parametrization theorem is an analogue of Theorem 1.2.

For the block matrix case Theorems 1.1 and 1.3, and Corollary 1.4 appeared in [23].

III.2. Operator matrices: strictly contractive extensions

In this section we specify the results of Section II.2 for the space of $n \times m$ operator matrices. An element of this space has the following form:

$$T = \begin{pmatrix} \phi_{11} & \cdots & \phi_{1m} \\ \vdots & & \vdots \\ \phi_{n1} & \cdots & \phi_{nm} \end{pmatrix}.$$

Here ϕ_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$, is a bounded linear operator from a Hilbert space H_j into a Hilbert space $\tilde{H}_i$; shortly $\phi_{ij} \in \mathcal{L}(H_j, \tilde{H}_i)$. We let $\|T\|$ denote the usual operator norm of $T : H_1 \oplus \cdots \oplus H_m \to \tilde{H}_1 \oplus \cdots \oplus \tilde{H}_n$.

Fix $-n < p < m$. For $1 \leq i \leq n$, $1 \leq j \leq m$, $j - i \leq p$, let $\phi_{ij} \in \mathcal{L}(H_j, \tilde{H}_i)$. An operator matrix T is called a *strictly contractive extension* of the given lower triangular part $\{ \phi_{ij}, j - i \leq p \}$ if $\|T\| < 1$ and for $j - i \leq p$ the (i,j)th element of T is equal to ϕ_{ij}. Let $r(j) = \max\{1, j - p\}$ and $s(j) = \min\{m, j + p\}$. Clearly a necessary condition for the existence of a strictly contractive extension is the following:

$$\left\| \left(\phi_{ij} \right)_{i=k, j=1}^{n, \; s(k)} \right\| < 1 \;, k = r(1), \ldots, n. \tag{2.1}$$

For $j = 1, \ldots, n$ let

$$S_j := \begin{pmatrix} \phi_{j1} & \cdots & \phi_{j, s(j)} \\ \vdots & & \vdots \\ \phi_{n1} & \cdots & \phi_{n, s(j)} \end{pmatrix} : \overset{s(j)}{\underset{k=1}{\oplus}} H_k \to \overset{n}{\underset{k=j}{\oplus}} \tilde{H}_k,$$

if $j + p \geq 1$, and

$$S_j = 0 : (0) \to \overset{n}{\underset{k=j}{\oplus}} \tilde{H}_k,$$

if $j + p < 1$. For $j = 1, \ldots, m$ let

$$R_j := \begin{bmatrix} \phi_{r(j),1} & \cdots & \phi_{r(j),j} \\ \vdots & & \vdots \\ \phi_{n1} & \cdots & \phi_{nj} \end{bmatrix} : \overset{j}{\underset{k=1}{\oplus}} H_k \rightarrow \overset{n}{\underset{k=r(j)}{\oplus}} \tilde{H}_k,$$

if $j - p \leq n$, and

$$R_j = 0 : \overset{j}{\underset{k=1}{\oplus}} H_k \rightarrow (0),$$

if $j - p > n$. Obviously, (2.1) implies that $\|S_j\| < 1$, $1 \leq j \leq n$, and also $\|R_j\| < 1$, $1 \leq j \leq m$. The converse statement holds trivially.

THEOREM 2.1. *For* $1 \leq i \leq n$, $1 \leq j \leq m$, $j - i \leq p$, *let* ϕ_{ij} *be a given operator acting from a Hilbert space* H_j *into a Hilbert space* $\tilde{H}_i$, *and suppose that* (2.1) *holds. Put*

$$\begin{bmatrix} \hat{\alpha}_{ii} \\ \hat{\alpha}_{i+1,i} \\ \vdots \\ \hat{\alpha}_{ni} \end{bmatrix} = (I - S_i S_i^*)^{-1} \begin{bmatrix} I_{\tilde{H}_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} , i = 1,\dots,n, \tag{2.2}$$

$$\begin{bmatrix} \hat{\beta}_{r(j),j} \\ \vdots \\ \hat{\beta}_{n-1,j} \\ \hat{\beta}_{nj} \end{bmatrix} = R_j (I - R_j^* R_j)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{H_j} \end{bmatrix} , j = 1,\dots,m, \tag{2.3}$$

$$\begin{bmatrix} \hat{\gamma}_{1i} \\ \hat{\gamma}_{2i} \\ \vdots \\ \hat{\gamma}_{s(i),i} \end{bmatrix} = S_i^* (I - S_i S_i^*)^{-1} \begin{bmatrix} I_{\tilde{H}_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} , i = 1,\dots,n, \tag{2.4}$$

$$\begin{bmatrix} \hat{\delta}_{1j} \\ \vdots \\ \hat{\delta}_{j-1,j} \\ \hat{\delta}_{jj} \end{bmatrix} = (I - R_j^* R_j)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{H_j} \end{bmatrix} , j = 1,\dots,m, \tag{2.5}$$

and let

$$\alpha := \begin{bmatrix} \alpha_{ij} \end{bmatrix}_{i,j=1}^{n} , \alpha_{ij} = \begin{cases} \hat{\alpha}_{ij} \hat{\alpha}_{jj}^{-\frac{1}{2}} , i \geq j; \\ 0 \quad , i < j; \end{cases} \tag{2.6}$$

$$\beta := \left[\beta_{ij} \right]_{i=1,j=1}^{n\ \ m}, \beta_{ij} = \left\{ \begin{array}{ll} \hat{\beta}_{ij}\hat{\delta}_{jj}^{-\frac{1}{2}} & , i \geqq r(j); \\ 0 & , i < r(j); \end{array} \right.$$ (2.7)

$$\gamma := \left[\gamma_{ij} \right]_{i=1,j=1}^{m\ \ n}, \gamma_{ij} = \left\{ \begin{array}{ll} \hat{\gamma}_{ij}\hat{\alpha}_{jj}^{-\frac{1}{2}} & , i \leqq s(j); \\ 0 & , i > s(j); \end{array} \right.$$ (2.8)

$$\delta := \left[\delta_{ij} \right]_{i,j=1}^{m}, \delta_{ij} = \left\{ \begin{array}{ll} \hat{\delta}_{ij}\hat{\delta}_{jj}^{-\frac{1}{2}} & , i \leqq j; \\ 0 & , i > j; \end{array} \right.$$ (2.9)

Then the operator matrix G defined by

$$G := \beta\delta^{-1} = \alpha^{*-1}\gamma^{*}$$

is the unique strictly contractive extension of the given lower triangular part $\{ \phi_{ij} \mid j-i \leqq p \}$ *with* $(G(I-G^{*}G)^{-1})_{ij} = 0$ *for* $j-i > p$.

Proof. We will obtain this theorem as a special case of Theorem II.2.1. Let

$$\mathcal{A}_{+}^{0} = \mathcal{A}_{-}^{0*} = \{ \left[F_{ij} \right]_{i,j=1}^{n} \mid F_{ij}:\tilde{H}_{j} \rightarrow \tilde{H}_{i} , F_{ij} = 0, j-i \leqq 0 \},$$

$$\mathcal{A}_{d} = \{ \left[F_{ij} \right]_{i,j=1}^{n} \mid F_{ij}:\tilde{H}_{j} \rightarrow \tilde{H}_{i} , F_{ij} = 0, j-i \neq 0 \},$$

and define $\mathcal{D}_{\pm}^{0}$ and $\mathcal{D}_{d}$ in the same way with $\tilde{H}_{k}$ replaced by H_{k} and n replaced by m. Furthermore, let

$$\mathcal{B}_{+} = \mathcal{C}_{-}^{*} = \{ \left[F_{ij} \right]_{i=1,j=1}^{n\ \ m} \mid F_{ij}:H_{j} \rightarrow \tilde{H}_{i} , F_{ij} = 0, j-i \leqq p \},$$

$$\mathcal{B}_{-} = \mathcal{C}_{+}^{*} = \{ \left[F_{ij} \right]_{i=1,j=1}^{p\ \ q} \mid F_{ij}:H_{j} \rightarrow \tilde{H}_{i} , F_{ij} = 0, j-i > p \}.$$

Let $\mathcal{A} - \mathcal{D}$ be given via (2.2) and (2.5) in Section II.2. On these spaces we define the operations * as the usual adjoint of an operator between Hilbert spaces. We endow the space $\mathcal{M} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ of $(n+m) \times (n+m)$ operator matrices with the usual operator norm. It is easy to see that the conditions (2.1)-(2.6) in Section II.2 are satisfied.

Let $\phi = \left[\phi_{ij} \right]_{i=1,j=1}^{n\ \ m}$, where $\phi_{ij} = 0$ for $j-i > p$. Consider the operator

$I - \epsilon^2 \Xi \Xi_*$, where Ξ and Ξ_* are defined in (2.11) in Section II.2. Applying this operator to an element $A \in \mathcal{A}_-$ gives an element in $\mathcal{A}_-$ whose columns are described by the following equations:

$$\left[((I - \epsilon^2 \Xi \Xi_*)(A))_{ij} \right]_{i=j}^{n} = (I - \epsilon^2 S_j S_j^*) \left[A_{ij} \right]_{i=j}^{n} , \quad j = 1,\ldots,n.$$

It is not hard to see that $I - \epsilon^2 \Xi \Xi_*$ is invertible for all $0 \leq \epsilon \leq 1$ if and only if $\|S_j\| < 1$, $j = 1,\ldots,n$. Analogously, one shows that the operators $I - \epsilon^2 \tilde{\Xi}_* \tilde{\Xi}$, $0 \leq \epsilon \leq 1$, are all invertible if and only if condition (2.1) holds. Assume that (2.1) holds. Then for $0 \leq \epsilon \leq 1$ the first element appearing in (2.12) in Section II.2 is the diagonal operator matrix with as (i,i)th element the $(1,1)$ element of the positive operator matrix $(I - \epsilon^2 S_i S_i^*)^{-1}$. So clearly this diagonal operator matrix is positive definite in $\mathcal{A}_d$. Analogously, one proves that the second element in (2.12) in Section II.2 is positive definite in $\mathcal{D}_d$. Applying now Theorem II.2.2 one obtains (using the above calculations) the operator matrices α, β, γ, and δ given in this theorem. $\square$

We shall call the extension G in Theorem 2.1 the *triangular extension* of $\{ \phi_{ij} \mid j - i \leq p \}$. For the description for the set of all strictly contractive extensions of a given lower triangular part one now simply applies Theorem II.2.3. Since condition (2.1) is necessary for the existence of a strictly contractive extension, we obtain the following result.

THEOREM 2.2. *For* $1 \leq i \leq n$, $1 \leq j \leq m$, $j - i \leq p$, *let* ϕ_{ij} *be a given operator acting from a Hilbert space* H_j *into a Hilbert space* $\tilde{H}_i$. *Then the lower triangular part* $\{ \phi_{ij} \mid j - i \leq p \}$ *has a strictly contractive extension if and only if* (2.1) *holds. Suppose that* (2.1) *holds, and let* α, β, γ, *and* δ *be defined by* (2.2)-(2.9). *Then each strictly contractive extension* F *of the given lower triangular part is of the form*

$$F = (\alpha E + \beta)(\gamma E + \delta)^{-1} , \tag{2.10}$$

where $E = \left[E_{ij} \right]_{i=1, j=1}^{n \quad m}$ *is a strictly contractive operator matrix with* $E_{ij} = 0$, $j - i \leq p$. *Furthermore,* (2.10) *gives a one-one correspondence between all such* E *and all strictly contractive extensions* F *of the given lower triangular part.*

Proof. Introduce the spaces $\mathcal{A} - \mathcal{D}$ with their decompositions as in the proof of Theorem 2.1. The space $\mathcal{M} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ clearly satisfies Axiom (A0a). Since the algebra of $(n + m) \times (n + m)$ operator matrices endowed with the usual adjoint and the usual

operator norm is a B*-algebra and $\mathcal{M}_+ \subset \mathcal{M}$ is a closed subalgebra, Axioms (A1) and (A2) are fulfilled automatically (with $\mathfrak{R} = \mathcal{M}$). Now apply Theorem II.2.3 to obtain the theorem. $\square$

Let us remark that there is an alternative description for the set of all positive extensions, which one obtains from (2.16) in Section II.2.

The general maximum entropy result in Theorem II.3.4 specified for the case considered here yields the following theorem.

THEOREM 2.3. *For* $1 \leq i \leq n$, $1 \leq j \leq m$, $j - i \leq p$, *let* ϕ_{ij} *be a given operator acting from a Hilbert space* H_j *into a Hilbert space* $\tilde{H}_i$, *and suppose that* (2.1) *holds. Put*

$$L_i = \begin{bmatrix} 0 & \cdots & 0 & I_{H_i} \end{bmatrix} (I - R_i^* R_i)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{H_i} \end{bmatrix}, \quad i = 1,...,m. \tag{2.11}$$

Then for the right multiplicative diagonal diag $\left[\Delta_i^r (I - \psi^* \psi) \right]$ *of* $I - \psi^* \psi$, *where* ψ *is a strictly contractive extension of the given lower triangular part, the following inequalities hold:*

$$\Delta_i^r (I - \psi^* \psi) \leq L_i^{-1}, \quad i = 1,...,m. \tag{2.12}$$

Moreover, equality holds for all i *in* (2.12) *if and only if* ψ *is the unique triangular extension of the given lower triangular part.*

Proof. Introduce the spaces $\mathcal{A} - \mathcal{D}$ and $\mathcal{M}$ as in the proof of Theorem 2.1 along with their decompositions. Clearly Axioms (A0a) and (A0b) are satisfied. As in the proof of Theorem III.1.3 one checks that Axioms (A3) and (A4) are satisfied. Note that from Theorem 2.1 it follows that diag $\left[L_i^{-1} \right]_{i=1}^{n}$ is the right multiplicative diagonal of $I - G^* G$, where G is the triangular extension of the given band. Apply now Theorem II.3.4 to obtain the theorem. $\square$

Theorem II.3.5 yields a left analogue.

For the block matrix case, as in Section III.1, the determinant can be used to identify the triangular extension of a given lower triangular part.

COROLLARY 2.4. *For* $1 \leq i \leq n$, $1 \leq j \leq m$, $j - i \leq p$, *let* ϕ_{ij} *be a given operator acting from a finite dimensional Hilbert space* H_j *into a finite dimensional Hilbert space* $\tilde{H}_i$, *and suppose that* (2.1) *holds. If* ψ *is a strictly contractive extension of the given*

lower triangular part, then

$$\det(I - \psi^* \psi) \leq \prod_{i=1}^{m} \det L_i^{-1} \,, \tag{2.13}$$

where $L_1, \ldots, L_m$ are defined in (2.11). Moreover, equality holds for all i in (2.13) if and only if ψ is the unique triangular extension of the given lower triangular part.

In a somewhat different form the maximum entropy principle in Corollary 2.4 may also be found in [10] (in the setting of a maximum distance problem) and in [5].

III.3. The operator Wiener algebra: positive extensions

Let H be a complex Hilbert space. We write $\mathcal{L}(H)$ for the Banach algebra of all bounded linear operators on H. By $W_H(\mathbf{T})$ we denote the *operator Wiener algebra* on the unit circle $\mathbf{T}$, which consists of all operator valued functions F on $\mathbf{T}$ of the form

$$F(\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j F_j \,, \lambda \in \mathbf{T} \,, \tag{3.1}$$

where $F_j \in \mathcal{L}(H)$ for each j and

$$\sum_{j=-\infty}^{\infty} \|F_j\| < \infty.$$

As usual we refer to F_j as the j-th Fourier coefficient of the function F. On $W_H(\mathbf{T})$ there is a natural involution, namely for F as in (3.1) we have

$$F^*(\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j F_{-j}^* = F(\lambda)^* \,, |\lambda| = 1.$$

Also $W_H(\mathbf{T})$ has a unit, namely the function $e(\lambda) = I$. Here I stands for the identity operator on H.

LEMMA 3.1. *Let $F \in W_H(\mathbf{T})$. Then F is positive definite in $W_H(\mathbf{T})$ if and only if $F(\lambda)$ is a positive definite operator on H for each $\lambda \in \mathbf{T}$. In that case we may write*

$$F(\lambda) = (I + U(\lambda))^* D (I + U(\lambda)) \,, \lambda \in \mathbf{T} \,, \tag{3.2}$$

where D is a positive definite operator on H and U is an element of $W_H(\mathbf{T})$ such that the j-th Fourier coefficients of U and $(e+U)^{-1}-e$ are zero for each $j \leq 0$ (or for each $j \geq 0$). Such factorizations are unique.

Proof. Assume that $F(\lambda) > 0$ for each $|\lambda| = 1$ and $F \in W_H(\mathbf{T})$. By Theorem 0.1 in [42] the function F admits a canonical factorization

$$F(\lambda) = G_-(\lambda)G_+(\lambda) , \quad |\lambda| = 1 ,$$

of which the factors G_-, G_+ are in $W_H(\mathbf{T})$. Since $F(\lambda)$ is positive definite for each $\lambda \in \mathbf{T}$,

$$F(\lambda) = G_+^*(\lambda)G_-^*(\lambda) , \lambda \in \mathbf{T} ,$$

is again a canonical factorization of F, and hence there exists an invertible operator Δ in $\mathscr{L}(H)$ such that $G_+ = \Delta G_-^*$. It follows that Δ is positive definite, and

$$F(\lambda) = (G_-(\lambda)\Delta^{\frac{1}{2}})(G_-(\lambda)\Delta^{\frac{1}{2}})^* , \lambda \in \mathbf{T} ,$$

which implies that F is positive definite as an element of $W_H(\mathbf{T})$. This proves the sufficiency. The proof of the necessity is trivial. Putting $D := G_-(\infty)\Delta G_-(\infty)^*$ and $U = G_-(\infty)^{*-1}G_-^* -e$ we get the desired factorization. The uniqueness of the factorization follows in the same way as the uniqueness of the factorization in Lemma II.3.2. $\square$

In this section we solve the following problem. Given $A_j = A_{-j}^*$, $|j| \leq m$, in $\mathscr{L}(H)$, determine all $F \in W_H(\mathbf{T})$ such that

(i) $F(\lambda)$ is a positive definite operator for $|\lambda| = 1$,

(ii) $F_j = A_j$ for $|j| \leq m$.

Such a function F will be called a *positive extension* of the band $\{ A_j \mid |j| \leq m \}$. In what follows H^m is the Hilbert space equal to the direct sum of m copies of H.

THEOREM 3.2. *Let* $A_j = A_{-j}^*$, $|j| \leq m$, *be given operators in* $\mathscr{L}(H)$, *and assume that*

$$\Gamma := \begin{bmatrix} A_0 & A_{-1} & \cdots & A_{-m} \\ A_1 & A_0 & \cdots & A_{-m+1} \\ \cdot & \cdot & \cdot & \cdot \\ A_m & A_{m-1} & \cdots & A_0 \end{bmatrix} \tag{3.3}$$

is a positive definite operator on H^{m+1}. *Put*

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_m \end{bmatrix} := \Gamma^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} , \quad \begin{bmatrix} Y_{-m} \\ \vdots \\ Y_{-1} \\ Y_0 \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} , \tag{3.4}$$

$$U(\lambda) = \sum_{j=0}^{m} \lambda^j X_j X_0^{-\frac{1}{2}} , \quad V(\lambda) = \sum_{j=-m}^{0} \lambda^j Y_j Y_0^{-\frac{1}{2}} .$$ (3.5)

Then

$$F(\lambda) := U(\lambda)^{*-1} U(\lambda)^{-1} = V(\lambda)^{*-1} V(\lambda)^{-1} , \quad \lambda \in \mathbf{T} ,$$ (3.6)

is in $W_H(\mathbf{T})$ *and* F *is the unique function in* $W_H(\mathbf{T})$ *such that* $F(\lambda) > 0$, $|\lambda| = 1$, *the j-th coefficient* $F_j = A_j$ *for* $|j| \leqq m$, *and for* $|j| > m$ *the j-th coeffcient of* F^{-1} *is equal to zero.*

Proof. First we show that X_0 and Y_0 are positive definite operators in $\mathscr{L}(H)$. Put $\Gamma_n = \left[A_{i-j} \right]_{i,j=0}^{n}$. Since $\Gamma_n = \Gamma$ is positive definite, the operators $\Gamma_0, \Gamma_1, \cdots, \Gamma_m$ are invertible. Hence (see [7]) the operator matrix Γ_m factors as $\Gamma_m = UDL$, where U (resp. L) is an upper (resp. lower) triangular $(m+1) \times (m+1)$ operator matrix with I's on the diagonal and D is a $(m+1) \times (m+1)$ diagonal matrix with positive definite operators $D_0, \ldots, D_m$ on the diagonal. Now let $X_0, \cdots, X_m$ be defined by (3.4). Then

$$\begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \Gamma \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_m \end{bmatrix} = UD \begin{bmatrix} X_0 \\ * \\ \vdots \\ * \end{bmatrix} = U \begin{bmatrix} D_0 X_0 \\ * \\ \vdots \\ * \end{bmatrix} ,$$

and hence $X_0 = D_0^{-1} > 0$. In a similar way one proves that $Y_0 > 0$.

Since X_0 and Y_0 are positive definite, the functions U and V in (3.5) are well-defined. Next, we show that $U(\lambda)$ is invertible for all $|\lambda| \leqq 1$. To do this put $U_j := X_j X_0^{-1}$ for $j = 0, \ldots, m$, and let

$$K := \begin{bmatrix} -U_1 & I & 0 & \ldots & 0 \\ -U_2 & 0 & I & \ldots & 0 \\ . & & . & . & . & . \\ -U_{m-1} & 0 & 0 & \ldots & I \\ -U_m & 0 & 0 & \ldots & 0 \end{bmatrix} .$$

Note that

$$\begin{bmatrix} X_0^* & X_1^* & \ldots & X_m^* \end{bmatrix} \Gamma = \begin{bmatrix} I & 0 & \ldots & 0 \end{bmatrix}$$

and $X_0^* = X_0$. Hence we can use the same arguments as in steps 1 and 2 of the proof of

Theorem 2.1 in [43] to show that

$$\Gamma_{m-1} - K^* \Gamma_{m-1} K = \text{diag}\,(X_0^{-1}, 0, \cdots, 0)\,, \tag{3.7}$$

$$\Gamma_{m-1} - (K^*)^m \Gamma_{m-1} K^m =$$

$$\begin{bmatrix} I & U_1^* & \cdots & U_{m-1}^* \\ 0 & I & \cdots & U_{m-2}^* \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & U_1^* \\ 0 & 0 & \cdots & I \end{bmatrix} \begin{bmatrix} X_0^{-1} & 0 & \cdots & 0 \\ 0 & X_0^{-1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & X_0^{-1} \end{bmatrix} \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ U_1 & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ U_{m-2} & U_{m-3} & \cdots & I & 0 \\ U_{m-1} & U_{m-2} & \cdots & U_1 & I \end{bmatrix}. \tag{3.8}$$

Recall that $\Gamma_m = \Gamma$ is positive definite. So the same is true for Γ_{m-1}. Hence $\Gamma_{m-1} = E^* E$ for some invertible operator E on H^m. But then (3.8) implies that

$$I - [E^{*-1}(K^*)^m E^*][E K^m E^{-1}]$$

is a positive operator on H^m, and thus $\|E K^m E^{-1}\| < 1$. It follows that $E K^m E^{-1}$ has its spectrum in the open unit disc $\mathbb{D}$, and using similarity and the spectral mapping theorem, we may conclude that the spectrum of K is in $\mathbb{D}$. Note that

$$\begin{bmatrix} 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & I & \cdots & 0 \\ I & 0 & \cdots & 0 \end{bmatrix} K \begin{bmatrix} 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & I & \cdots & 0 \\ I & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 0 & -U_m \\ I & \cdots & 0 & 0 & -U_{m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & I & 0 & -U_2 \\ 0 & \cdots & 0 & I & -U_1 \end{bmatrix} \quad (=: C_L). \tag{3.9}$$

The right hand side of (3.9) is the so-called second companion matrix of the operator polynomial

$$L(\lambda) := U_m + \lambda U_{m-1} + \cdots + \lambda^{m-1} U_1 + \lambda^m I.$$

Since K and C_L are similar we know that $\lambda - C_L$ is invertible for all $|\lambda| \geq 1$. Now use that $\lambda - C_L$ is a linearization (cf., [36], Section 2.1) of $L(\lambda)$. It follows that $L(\lambda)$ is invertible for all $|\lambda| \geq 1$. But $U(\lambda) = \lambda^m L(\lambda^{-1}) X_0^{-\frac{1}{2}}$. So $U(\lambda)$ is invertible for all $0 < |\lambda| \leq 1$. Also $U(0) = X_0^{\frac{1}{2}}$ is invertible, and we have proved that $U(\lambda)$ has the desired invertibility properties. In a similar way one proves that $V(\lambda)$ is invertible for all $|\lambda| \geq 1$.

We are now ready to use the general scheme of Section II.1 and to apply Theorem

II.1.1. Let

$$\mathcal{M}_1 = \{ F \in W_H(\mathsf{T}) \mid F_j = 0, j \le m \},$$

$$\mathcal{M}_2^0 = \{ F \in W_H(\mathsf{T}) \mid F_j = 0, j \le 0 \text{ and } j > m \},$$

$$\mathcal{M}_d = \{ F \in W_H(\mathsf{T}) \mid F_j = 0, j \ne 0 \},$$

$$\mathcal{M}_3^0 = \{ F \in W_H(\mathsf{T}) \mid F_j = 0, j < -m \text{ and } j \ge 0 \},$$

$$\mathcal{M}_4 = \{ F \in W_H(\mathsf{T}) \mid F_j = 0, j \ge -m \}.$$

Put $k(\lambda) = \sum\limits_{j=-m}^{m} \lambda^j A_j$. Then $k = k^* \in \mathcal{M}_c$ and (by Lemma 3.1) a function $F \in W_H(\mathsf{T})$ satisfies (i) and (ii) above if and only if F is a positive extension of k (in the sense of Section II.1). Define

$$x(\lambda) := \sum_{j=0}^{m} \lambda^j X_j \,,\, y(\lambda) := \sum_{j=-m}^{0} \lambda^j Y_j \,,\, \lambda \in \mathsf{T}.$$

It is easy to see that equations (II.1.5) are satisfied. Since $x = UX_0^{\frac{1}{2}}$ has no spectrum in $\mathbb{D}$ we obtain that $x^{-1} \in \mathcal{M}_+$. Also $P_d x = X_0$ is positive definite in $\mathcal{M}_d = \mathcal{L}(H)$. Analogously, $y^{-1} \in \mathcal{M}_-$ and $P_d y$ is positive definite in $\mathcal{M}_d$. Theorem II.1.1 now yields that k has a unique band extension and that this band extension is given by (II.1.6). It follows that F defined in (3.6) is the unique band extension of k. $\square$

The positive extension F of $\{A_j \mid |j| \le m \}$ obtained in Theorem 3.2 is called the *band extension* of $\{A_j \mid |j| \le m \}$.

To get all positive extensions we shall apply Theorem II.1.5. This requires to prove that $W_H(\mathsf{T})$ satisfies the Axioms (A1) and (A2) for a suitable B*-algebra $\mathcal{R}$. For $\mathcal{R}$ we take the algebra $C_H(\mathsf{T})$ of all $\mathcal{L}(H)$-valued continuous functions on T endowed with the usual supremum norm. With this norm and the involution * defined by $F^*(\lambda) = F(\lambda)^*$, $|\lambda| = 1$, the algebra $C_H(\mathsf{T})$ is a B*-algebra. The function $e(\lambda) = I$, $\lambda \in \mathsf{T}$, is the unit of $C_H(\mathsf{T})$. Note that $W_H(\mathsf{T})$ is a *-subalgebra of $C_H(\mathsf{T})$, and the unit e belongs to $W_H(\mathsf{T})$. Axiom 1 is fulfilled because of the Bochner-Phillips theorem [11], which states that $F \in W_H(\mathsf{T})$ is invertible in $W_H(\mathsf{T})$ if and only if $F(\lambda)$ is an invertible operator on H for each $\lambda \in \mathsf{T}$. The latter statement may be rephrased as F is invertible in $C_H(\mathsf{T})$. To prove that Axiom (A2) is satisfied, take $F \in W_H(\mathsf{T})$, and assume that there exists a sequence $F^{(1)}, F^{(2)}, \cdots$ in $\mathcal{M}_+$ such that $F^{(n)} \to F$ in the norm of $C_H(\mathsf{T})$. Since

$F^{(n)} \in \mathscr{M}_+$, we know that $(F^{(n)})_j = 0$ for $j < 0$. It follows that

$$F_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) e^{-ij\theta} d\theta =$$

$$= \lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} F^{(n)}(e^{i\theta}) e^{-ij\theta} d\theta = \lim_{n \to \infty} (F^{(n)})_j = 0 , j < 0.$$

Thus $F \in \mathscr{M}_+$, and Axiom (A2) is fulfilled.

THEOREM 3.3. *Let* $A_j = A^*_{-j}$, $|j| \le m$, *be a given set of operators on* H. *In order that there exists a positive extension of the given band* $\{ A_j \mid |j| \le m \}$ *it is necessary and sufficient that the operator* Γ, *defined in* (3.3), *is positive definite on* H^{m+1}. *Suppose that this condition is satisfied, and define* $U(\lambda)$ *and* $V(\lambda)$ *by* (3.5). *Then each positive extension* E *of the given band is of the form*

$$E(\lambda) = (V(\lambda)G(\lambda) + U(\lambda))^{*-1}(I - G(\lambda)^* G(\lambda))(V(\lambda)G(\lambda) + U(\lambda))^{-1}, \lambda \in \mathbf{T}, (3.10)$$

where G *is an element in* $W_H(\mathbf{T})$ *such that* $\|G(\lambda)\| < 1$, $|\lambda| = 1$, *and* $G_j = 0, j \le m$. *Furthermore, formula* (3.10) *gives a one-one correspondence between all such* G *and all positive extensions* E *of the given band.*

Proof. In order to prove that the positive definiteness of Γ is a necessary condition, let E be a positive extension of the given band. Then the operator on $l_2(H)$ (the space of all square summable sequences with elements in H) defined via the matrix

$$\left[E_{i-j} \right]_{i,j=1}^{\infty}$$

is positive definite. Since Γ is a principle submatrix of this infinite matrix, Γ is a positive definite operator. The rest of the theorem is a direct application of Theorem II.1.5, where it should be noted that $e - G^* G$ is positive definite in $\mathscr{M} = W_H(\mathbf{T})$ if and only if $\|G(\lambda)\| = \|G(\lambda)^*\| < 1, \lambda \in \mathbf{T}. \quad \square$

There is an alternative description for the set of all positive extensions of a given band which one obtains from (1.10) in Section II.1.

Let $E \in W_H(\mathbf{T})$ and assume that $E(\lambda) = \sum_{j=-\infty}^{\infty} E_j \lambda^j$ is positive definite for all $\lambda \in \mathbf{T}$. By Lemma 3.1 there exists a unique function W in $W_H(\mathbf{T})$ and a unique positive definite operator $\Delta_r(E)$ on H such that W and $(e + W)^{-1} - e$ has zero nonpositive Fourier coefficients and

$$E(\lambda) = (I + W(\lambda))^* \Delta_r(E)(I + W(\lambda)) \;,\; \lambda \in \mathbf{T}. \tag{3.11}$$

We refer to $\Delta_r(E)$ as the *right multiplicative diagonal* of E. It is known (cf. [7]) that $\Delta_r(E)$ is given by

$$\Delta_r(E)^{-1} = \begin{bmatrix} I & 0 & \cdots \end{bmatrix} \begin{bmatrix} F_0 & F_{-1} & \cdots \\ F_1 & F_0 & \cdots \\ \vdots & \vdots & \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \end{bmatrix}. \tag{3.12}$$

THEOREM 3.4. *Let* $A_j = A^*_{-j}$, $|j| \leq m$, *be given operators in* $\mathscr{L}(H)$, *and assume that*

$$\Gamma := \begin{bmatrix} A_0 & A_{-1} & \cdots & A_{-m} \\ A_1 & A_0 & \cdots & A_{-m+1} \\ \cdot & \cdot & \cdot & \cdot \\ A_m & A_{m-1} & \cdots & A_0 \end{bmatrix}$$

is a positive definite operator on H^{m+1}. *Put*

$$M_r = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \Gamma^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.13}$$

Then for the right multiplicative diagonal $\Delta_r(E)$ *of a positive extension* E *of the given band the following inequality holds:*

$$\Delta_r(E) \leq M_r^{-1}. \tag{3.14}$$

Moreover, equality holds in (3.14) *if and only if* E *is the unique band extension of the given band.*

Proof. Let $\mathscr{M}$, $\mathscr{M}_1 - \mathscr{M}_4$ be as in the proof of Theorem 3.2. If a function E,
$E(\lambda) = \sum\limits_{i=-\infty}^{\infty} E_i \lambda^i$, is nonnegative semi-definite in $W_H(\mathbf{T})$, then the operator

$$\left[E_{j-i} \right]_{i,j=0}^{\infty} : l_2(H) \to l_2(H) , \tag{3.15}$$

where $l_2(H)$ is the Hilbert space of square summable sequences of elements in H, is non-negative semi-definite. But then E_0 is a nonnegative definite operator on H. This proves that $\mathscr{M}$ satisfies Axiom (A3). When $E_0 = 0$, the nonnegative definiteness of the operator

in (3.15) implies that all E_j are zero. This follows from the arguments used in the first paragraph of the proof of Theorem III.1.3. Hence $\mathcal{M}$ satisfies Axiom (A4).

Using the formula for the band extension in (3.6) one easily sees that M_r^{-1} is the right multiplicative diagonal of this band extension. Applying Theorem II.3.1 in the setting described here one obtains the theorem. $\square$

For the case when H is finite dimensional it is known (see [13], and also [22]) that the band extension is characterized as the unique extension which maximizes the entropy integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det E(e^{i\theta}) d\theta. \tag{3.16}$$

Note that when E is factorized as in (3.11) this integral equals the number $\log \det \Delta_r(E)$. Indeed, inserting in (3.11) in (3.16) gives a sum of three integrals. Furthermore, since $e + W$ and its inverse are analytic in a neigbourhood of the unit disc, the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det (I + W(e^{i\theta})) d\theta$$

equals zero, and hence the assertion follows. Since "log det" is a strictly monotone function on the set of positive definite matrices, Corollary II.3.3 yields the result stated in th beginning of this paragraph.

For the matrix Wiener algebra case Theorem 3.2 was obtained before in [22] (see also [55], [56]), and a linear fractional description of all positive extensions of a given band appears in [20] without proof, and with proof in [21].

III.4. The operator Wiener algebra: strictly contractive extensions

In this section we solve the following problem. Given ϕ_j, $j \leq 0$, in $\mathcal{L}(H)$ such that $\sum_{j=-\infty}^{0} \|\phi_j\| < \infty$, determine all $F \in W_H(\mathbf{T})$ such that

(j) $F(\lambda)$ is a strictly contractive operator for $|\lambda| = 1$ (i.e., $\|F(\lambda)\| < 1, \lambda \in \mathbf{T}$),

(jj) $F_j = \phi_j$ for $j \leq 0$.

Such a F is called a *strictly contractive extension* of ϕ, where $\phi(\lambda) = \sum_{j=-\infty}^{0} \lambda^j \phi_j, \lambda \in \mathbf{T}$.

In what follows $l_2(H)$ is the Hilbert space of sequences $(\eta_j)_{j=0}^{\infty}$, $\eta_j \in H$, such that

$\sum_{j=0}^{\infty} \| \eta_j \|^2 < \infty$. The space $l_1(H)$ is defined analogously.

THEOREM 4.1. *Let* ϕ_j, $j \leq 0$, *be given operators on H with* $\sum_{j=-\infty}^{0} \| \phi_j \| < \infty$, *and*

assume that the Hankel operator matrix

$$
\Lambda := \begin{bmatrix} \phi_0 & \phi_{-1} & \phi_{-2} & \cdots \\ \phi_{-1} & \phi_{-2} & \phi_{-3} & \cdots \\ \phi_{-2} & \phi_{-3} & \phi_{-4} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} ,
$$

(4.1)

seen as an operator on $l_2(H)$, *has norm less than one. Put*

$$
\begin{bmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_{-1} \\ \vdots \end{bmatrix} = (I - \Lambda\Lambda^*)^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \end{bmatrix} , \quad \begin{bmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \\ \vdots \end{bmatrix} = \Lambda^* \begin{bmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_{-1} \\ \vdots \end{bmatrix} ,
$$

(4.2)

$$
\begin{bmatrix} \hat{\delta}_0 \\ \hat{\delta}_1 \\ \vdots \end{bmatrix} = (I - \Lambda^*\Lambda)^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \end{bmatrix} , \quad \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_{-1} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} \hat{\delta}_0 \\ \hat{\delta}_1 \\ \vdots \end{bmatrix} ,
$$

(4.3)

and let

$$
\alpha(\lambda) = \sum_{j=-\infty}^{0} \hat{\alpha}_j \hat{\alpha}_0^{-\frac{1}{2}} \lambda^j , \quad \gamma(\lambda) = \sum_{j=0}^{\infty} \hat{\gamma}_j \hat{\alpha}_0^{-\frac{1}{2}} \lambda^j ,
$$

(4.4)

$$
\beta(\lambda) = \sum_{j=-\infty}^{0} \hat{\beta}_j \hat{\delta}_0^{-\frac{1}{2}} \lambda^j , \quad \delta(\lambda) = \sum_{j=0}^{\infty} \hat{\delta}_j \hat{\delta}_0^{-\frac{1}{2}} \lambda^j .
$$

Then the function g *given by*

$$
g(\lambda) := \beta(\lambda)\delta(\lambda)^{-1} = \alpha(\lambda)^{*-1}\gamma(\lambda)^* , \quad \lambda \in \mathbf{T} ,
$$

is the unique function $g \in W_H(\mathbf{T})$ *such that* $g_j = \phi_j$ *for* $j \leq 0$, $\| g(\lambda) \| < 1$ *for* $|\lambda| = 1$, *and* $(g(I - g^*g)^{-1})_j = 0$ *for* $j > 0$.

Proof. Take $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ to be the Banach algebra $W_H(\mathbf{T})$. Put

$$
\mathscr{A}_+^0 = \mathscr{B}_+ = \mathscr{C}_-^* = \mathscr{D}_+^0 = \{ f \in W_H(\mathbf{T}) \mid f_j = 0 , j \leq 0 \};
$$

$$
\mathscr{A}_-^0 = \mathscr{D}_-^0 = \{ f \in W_H(\mathbf{T}) \mid f_j = 0 , j \geq 0 \};
$$

$$\mathscr{B}_- = \mathscr{C}_+^* = \{ f \in W_H(\mathbf{T}) \mid f_j = 0, j > 0 \};$$

$$\mathscr{A}_d = \mathscr{D}_d = \mathscr{L}(H).$$

One easily checks that the induced decompositions satisfy the desired algebraic structure as described in the second paragraph of Section II.2. Our aim is to apply Theorem II.2.2.

Let $\epsilon \in [0,1]$. First we show that $I - \epsilon^2 \Lambda \Lambda^*$ acts as an invertible operator on $l_1(H)$. Since $\|\Lambda\| < 1$ the operator

$$\begin{bmatrix} I & \epsilon \Lambda \\ \epsilon \Lambda^* & I \end{bmatrix}$$

on $l_2(H) \oplus l_2(H)$ is invertible. Let $\bar{\Lambda}$ denote the matrix Λ with the columns in the reversed order, i.e., we identify sequences $(\eta_j)_{j=0}^{\infty}$ with sequences $(\xi_j)_{j=-\infty}^{0}$. Put $D_\lambda = \text{diag}$ $(I, \lambda I, \lambda^2 I, \cdots)$ and $\bar{D}_\lambda = \text{diag} (\cdots, \lambda^2 I, \lambda I, I)$. Consider the operator valued function

$$F(\lambda) := \begin{bmatrix} D_\lambda^* & 0 \\ 0 & \bar{D}_\lambda \end{bmatrix} \begin{bmatrix} I & \epsilon \bar{\Lambda} \\ \epsilon \bar{\Lambda}^* & I \end{bmatrix} \begin{bmatrix} D_\lambda & 0 \\ 0 & \bar{D}_\lambda^* \end{bmatrix}, \ |\lambda| = 1.$$

Note that $F(\lambda) = \sum\limits_{i=-\infty}^{\infty} \lambda^i Z_i$, where

$$Z_i = Z_{-i}^* = \begin{bmatrix} 0 & M_i \\ 0 & 0 \end{bmatrix}, i < 0, Z_0 = \begin{bmatrix} I & M_0 \\ M_0^* & I \end{bmatrix},$$

and

$$M_i = \begin{bmatrix} \cdots & 0 & \epsilon\phi_i & 0 & 0 & \cdots & 0 & 0 \\ \cdots & 0 & 0 & \epsilon\phi_i & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \epsilon\phi_i \\ \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

So the norm of the operator Z_i, $i \neq 0$, on the Hilbert space $l_2(H) \oplus l_2(H)$, equals $\epsilon \|\phi_i\|$. Since $\sum\limits_{j=-\infty}^{0} \|\phi_j\| < \infty$ the function F belongs to the Wiener algebra $W_{l_2(H) \oplus l_2(H)}(\mathbf{T})$. Further, $F(\lambda)$ is invertible for each $|\lambda| = 1$. Using Theorem 1 in [11] we get that F^{-1} belongs also to the operator Wiener algebra of the Hilbert space $l_2(H) \oplus l_2(H)$. Since

$$\begin{bmatrix} I & \epsilon\tilde{\Lambda} \\ \epsilon\tilde{\Lambda}^* & I \end{bmatrix}^{-1} = \begin{bmatrix} D_\lambda & 0 \\ 0 & \tilde{D}_\lambda^* \end{bmatrix} F^{-1}(\lambda) \begin{bmatrix} D_\lambda^* & 0 \\ 0 & \tilde{D}_\lambda \end{bmatrix}$$

we conclude that

$$\begin{bmatrix} I & \epsilon\tilde{\Lambda} \\ \epsilon\tilde{\Lambda}^* & I \end{bmatrix}^{-1} = \sum_{j=-\infty}^{\infty} \Delta_j ,$$

where $\sum_{j=-\infty}^{\infty} \|\Delta_j\| < \infty$. In fact,

$$\Delta_j = \begin{bmatrix} P_j & Q_j \\ R_j & S_j \end{bmatrix} : l_2(H) \oplus l_2(H) \rightarrow l_2(H) \oplus l_2(H),$$

where P_j, R_j, Q_j and S_j each have at most one nonzero diagonal. Because of this, Δ_j is also a well-defined operator on $l_1(H) \oplus l_1(H)$ and its norm as an operator on $l_1(H) \oplus l_1(H)$ is at most $\|P_j\| + \|Q_j\| + \|R_j\| + \|S_j\| \leq 4\|\Delta_j\|$. Hence

$$\begin{bmatrix} I & \epsilon\Lambda \\ \epsilon\Lambda^* & I \end{bmatrix}^{-1}$$

is bounded as an operator acting in $l_1(H) \oplus l_1(H)$ since its norm is bounded by $4 \sum_{j=-\infty}^{\infty} \|\Delta_j\|$. It follows that $I - \epsilon^2\Lambda\Lambda^*$ is invertible on $l_1(H)$.

Note that in the present case the operator $I - \epsilon^2 \Xi \Xi_*$, where Ξ and Ξ_* are defined as in (2.11) in Section II.2, acts in the following way. If $A(\lambda) = \sum_{j=0}^{\infty} A_j \lambda^j$ belongs to $\mathscr{B}_+$, then $B(\lambda) = \sum_{j=-\infty}^{0} B_j \lambda^j := (I - \epsilon^2 \Xi \Xi_*)A$ is given by

$$B_j x = [(I - \epsilon^2\Lambda\Lambda^*)(A_k x)_{k=0}^{\infty}]_j , j = 0, -1, \dots , x \in H. \tag{4.5}$$

Since $I - \epsilon^2\Lambda\Lambda^*$ is invertible on $l_1(H)$ we get that $I - \epsilon^2 \Xi \Xi_*$ is invertible. Further, $\pi_d[(I - \epsilon^2 \Xi \Xi_*)^{-1}e] = \{(I - \epsilon^2\Lambda\Lambda^*)^{-1}\}_{00}$ is a positive definite operator on H. Analogously, one proves that $I - \epsilon\tilde{\Xi}_*\tilde{\Xi}$ is invertible and that $\pi_d[(I - \epsilon^2\tilde{\Xi}_*\tilde{\Xi})^{-1}e]$ is positive definite. Now we may apply Theorem II.2.2. By using the description of the operator $I - \Xi\Xi_*$ given above (via equation (4.5)) and an analogous description for $I - \tilde{\Xi}_*\tilde{\Xi}$ one obtains the theorem. □

We call the strictly contractive extension g of ϕ in Theorem 4.1 the *triangular*

extension of ϕ. Thus a stictly contractive extension g of ϕ is the triangular extension of ϕ if the j-th Fourier coefficient of $g(I-g^*g)^{-1}$ is zero for $j > 0$.

THEOREM 4.2. *Let* ϕ_j, $j \leqq 0$, *be given operators in* $\mathscr{L}(H)$ *with* $\displaystyle\sum_{j=-\infty}^{0} \|\phi_j\| < \infty$.

In order that the function $\phi(\lambda) = \displaystyle\sum_{j=-\infty}^{0} \lambda^j \phi_j$ *has a strictly contractive extension it is necessary and sufficient that* Λ, *defined in* (4.1), *seen as an operator on* $l_2(H)$, *has norm less than one. Assume that the latter condition is satisfied, and let* $\alpha(\lambda),\beta(\lambda),\gamma(\lambda)$ *and* $\delta(\lambda)$ *be defined by* (4.2)-(4.4). *Then each strictly contractive extension* ψ *of* ϕ *is of the form*

$$\psi(\lambda) = (\alpha(\lambda)h(\lambda)+\beta(\lambda))(\gamma(\lambda)h(\lambda)+\delta(\lambda))^{-1}, \lambda \in \mathbf{T} , \tag{4.6}$$

where h is an element in $W_H(\mathbf{T})$ *such that* $\|h(\lambda)\| < 1$, $|\lambda| = 1$, *and* $h_j = 0$, $j \leqq 0$. *Furthermore, formula* (4.6) *gives a one-one correspondence between all such h and all strictly contractive extensions ψ of f.*

Proof. If $\psi(\lambda) = \displaystyle\sum_{i=-\infty}^{\infty} \lambda^i \psi_i$ is a strictly contractive extension of ϕ, then the matrix

$\left[\psi_{j-i}\right]_{i,j=-\infty}^{\infty}$ (viewed as an operator on the Hilbert space of square summable doubly-infinite sequences of elements of H) has norm strictly less than one. But then Λ, being part of this matrix, also has norm strictly less than one. Hence $\|\Lambda\| < 1$ is a necessary condition for the existence of a strictly contractive extension.

It remains to check that the conditions of Theorem II.2.3 are satisfied. Let $\mathscr{A}-\mathscr{D}$ be as in the proof of Theorem 4.1. Then the matrix algebra $\mathscr{M} = \begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix}$ may be identified with $W_{H\oplus H}(\mathbf{T})$. The latter is a *-subalgebra of the B*-algebra $C_{H\oplus H}(\mathbf{T})$, and hence all the axioms are fulfilled (compare the paragraph preceding Theorem III.3.3). Now we may apply Theorem II.2.3 and obtain the desired result. $\square$

Theorem II.3.4 yields the following maximum entropy principle.

THEOREM 4.3. *Let* ϕ_j, $j \leqq 0$, *be given operators on H with* $\displaystyle\sum_{j=-\infty}^{0} \|\phi_j\| < \infty$, *and assume that the Hankel operator matrix*

$$\Lambda := \begin{bmatrix} \phi_0 & \phi_{-1} & \phi_{-2} & \cdots \\ \phi_{-1} & \phi_{-2} & \phi_{-3} & \cdots \\ \phi_{-2} & \phi_{-3} & \phi_{-4} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

seen as an operator on $l_2(H)$, has norm less than one. Put

$$M_r = \begin{bmatrix} I & 0 & \ldots \end{bmatrix} (I - \Lambda^* \Lambda)^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \end{bmatrix}. \tag{4.7}$$

Then for any strictly contractive extension ψ of the function $\phi(\lambda) = \sum\limits_{j=-\infty}^{0} \lambda^j \phi_j$ the follow-ing inequality holds

$$\Delta_r(I - \psi^* \psi) \leq M_r^{-1}, \tag{4.8}$$

where $\Delta_r(I - \psi^* \psi)$ denotes the right multiplicative diagonal of $I - \psi^* \psi$. Moreover, equality holds in (4.8) if and only if ψ is the unique triangular extension of ϕ.

Proof. Make the same choices of subspaces as in the proof of Theorem 4.1. Since $\mathcal{M} = W_{H \oplus H}(\mathbf{T}) \subset C_{H \oplus H}(\mathbf{T})$, this algebra satisfies Axioms (A3) and (A4). Further-more, the right multiplicative diagonal of $I - g^* g$, where g is the triangular extension of ϕ, equals $\hat{\delta}_0$ in Theorem III.4.1, which clearly is equal to M_r^{-1}. But then the theorem follows directly from Theorem II.3.4. $\square$

Using again the strictly monotone function "log det" for the matrix case the classical maximum entropy principle (see [26]), which identifies the triangular extension as the unique extension ψ that maximizes the entropy integral

$$\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \log \det (I - \psi(e^{i\theta})^* \psi(e^{i\theta})) d\theta ,$$

appears as a corollary of Theorem 4.3.

With obvious modifications the results in this section carry over to the strictly contrac-tive extension problem for operator functions in the Wiener class whose values are opera-tors from a Hilbert space H into another Hilbert space $\tilde{H}$ (choose $\mathcal{A} = W_{\tilde{H}}(\mathbf{T})$, $\mathcal{B} = \mathcal{C}^* = W_{H,\tilde{H}}(\mathbf{T})$ and $\mathcal{D} = W_H(\mathbf{T})$).

For matrix Wiener algebras formula (4.6) appeared earlier in [1], [25] and [26] (see also [20]). In the operator Wiener algebra the existence of a strictly contractive extension under the assumption that $\|\Lambda\| < 1$, with Λ as in (4.1), was established earlier in [1].

COMMENTS (Part A)

The results in this part are collected from the papers [66], [38], [39] and [40]. Chapter 1 is a rewritten version of the paper [66]. Chapter 2 contains the general results from the papers [38], [39] and [40]. Chapter 3 contains the applications of the band method concerning the finite operator algebra and the operator Wiener algebra from [38], [39] and [40].

PART B
MINIMAL RANK EXTENSIONS

In this part minimal rank extension problems are treated. It consists of Chapters 4 and 5. Chapter 4 concerns the matrix case and Chapter 5 the operator case.

CHAPTER IV: MATRICES

This chapter treats the minimal rank extension problem for matrices. In Sections 1 to 5 the given part is of triangular type. More general patterns appear in Section 7. In Section 1 a formula for the minimal lower rank is derived. Section 2 characterizes the case when there is only one minimal rank extension. In Section 3 a description is given of the set of all minimal rank extensions. Section 4 deals with the Toeplitz case. In Section 5 the minimal lower rank of the lower triangular part of an invertible matrix is described in terms of a lower triangular part of its inverse. In Section 6 the partial realization problem from mathematical systems theory appears as a minimal rank extension problem of special type. Section 7 discusses minimal rank extensions relative to general patterns.

IV.1. The minimal lower rank

Consider the following "partially defined matrix"

$$\mathcal{A} = \begin{bmatrix} A_{11} & & & \\ A_{21} & A_{22} & & ? \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}. \tag{1.1}$$

This chapter concerns the question how to fill in the unknown entries (denoted by ?) such that the rank of the full matrix is as small as possible. In this section we derive a formula for this minimal lower rank in terms of the given data.

Let us introduce some terminology. By a *partially defined matrix* we mean a matrix of which some entries are specified elements of the complex plane $\mathbb{C}$ and the remaining entries are free to be chosen from $\mathbb{C}$. A set of matrices

$$\mathcal{A} = \{ A_{ij} \mid 1 \leq j \leq i \leq n \}, \tag{1.2}$$

where A_{ij} is of size $\nu_i \times \mu_j$ ($\nu_i, \mu_j \geq 0$), is called a *lower triangular part*. We identify such a lower triangular part with the partially defined (block) matrix (1.1). A strictly lower triangular part, where also the diagonal elements are not specified, appears when $\mu_1 = 0$ and $\nu_n = 0$. Let $B = \begin{bmatrix} B_{ij} \end{bmatrix}_{i,j=1}^{n}$ be a block matrix, where B_{ij} is of size $\nu_i \times \mu_j$. The block matrix B is called an *extension* of $\mathcal{A}$ if $B_{ij} = A_{ij}$, $1 \leq j \leq i \leq n$. The

minimal lower rank $\mathcal{A}(\mathcal{A})$ of $\mathcal{A}$ is defined to be the number

$$\mathcal{A}(\mathcal{A}) := \min\{ \text{ rank } B \mid B \text{ is an extension of } \mathcal{A} \}. \qquad (1.3)$$

An extension B for which the minimum in (1.3) is attained is called a *minimal rank extension* of $\mathcal{A}$.

THEOREM 1.1. *Let* $\mathcal{A} = \{ A_{ij} \mid 1 \leq j \leq i \leq n \}$ *be a given lower triangular part. The minimal lower rank of* $\mathcal{A}$ *is given by*

$$\mathcal{A}(\mathcal{A}) = \sum_{p=1}^{n} \text{ rank } A^{(p,p)} - \sum_{p=1}^{n-1} \text{ rank } A^{(p+1,p)},$$

where

$$A^{(p,q)} = \begin{bmatrix} A_{p1} & \cdots & A_{pq} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nq} \end{bmatrix}. \qquad (1.4)$$

Proof. We consider the partially defined block matrix (1.1) as a scalar matrix of size $N \times M$, where $N = \sum \nu_i$ and $M = \sum \mu_i$. Let c_i denote the i-th scalar column of (1.1) $(i = 1,\ldots,M)$. For a column c of (1.1) we denote by $(c)_g$ the specified (given) part of the column c. For $p = 1,\ldots,n$ choose an index set $J_p \subset \{ 1 + \sum_{j=1}^{p-1} \mu_j, \ldots, \mu_p + \sum_{j=1}^{p-1} \mu_j \}$ such that $\{ (c_i)_g \mid i \in J_p \}$ is a linearly independent set of columns and

$$\text{span}\{ (c_i)_g \mid i \in J_p \} \dotplus \text{Im } A^{(p,p-1)} = \text{Im } A^{(p,p)}.$$

The symbol $\dotplus$ denotes a direct sum. The matrix $A^{(1,0)}$ should be understood as the zero matrix of size $N \times 0$. Note that $\#J_p = \text{rank } A^{(p,p)} - \text{rank } A^{(p,p-1)}$. Here and in the sequel $\#I$ stands for the number of elements of the set I. Choose arbitrary complex numbers for the unspecified entries in the columns

$$\{ c_i \mid i \in J := \bigcup_{j=1}^{n} J_j \}.$$

The unspecified entries in the other columns of (1.1) can always be chosen in such a way that these columns are in the span of $\{ c_i \mid i \in J \}$, and hence $\mathcal{A}(\mathcal{A}) \leq \#J$. On the other hand it is obvious that for any choice of complex numbers for the unknown entries in the colums $\{ c_i \mid i \in J \}$ these columns will be linearly independent. Hence $\mathcal{A}(\mathcal{A}) \geq \#J$. But then

$$\ell(\mathcal{A}) = \#J = \sum_{p=1}^{n} \#J_p$$

and the theorem is proved. $\square$

Note that the proof gives a way to construct a minimal rank extension. The row analogue of this construction will now be stated explicitely.

Let r_i denote the i-th scalar row of (1.1) ($i = 1,...,N$). For a row r of (1.1) we denote by $(r)_g$ the specified part of the row r. Choose for $p = 1,...,n$ an index set $I_p \subset \{1+\sum_{j=1}^{p-1} \nu_j, \ldots, \nu_p + \sum_{j=1}^{p-1} \nu_j\}$ such that $\{(r_i)_g \mid i \in I_p\}$ is a linearly independent set of rows and

$$\text{span}\{ (r_i)_g^T \mid i \in I_p \} \dotplus \text{Im } A^{(p+1,p)T} = \text{Im } A^{(p,p)T}.$$

The superscript ^{T} denotes a transpose. The matrix $A^{(n+1,n)}$ should be understood as the zero matrix of size $0 \times M$. Note that $\#I_p = \text{rank } A^{(p,p)} - \text{rank } A^{(p+1,p)}$ and $\sum_{p=1}^{n} \#I_p = \ell(A)$. Choose arbitrary complex numbers for the unspecified entries in r_i, $i \in I := \bigcup_{p=1}^{n} I_p$. Let r be a row which still has unspecified entries. The specified part $(r)_g$ of r is a linear combination of the corresponding parts of the rows r_i, $i \in I$. Let us denote this as

$$(r)_g = (\sum_{i \in I} \lambda_i r_i)_g.$$

Then the row r should be chosen as

$$r = \sum_{i \in I} \lambda_i r_i.$$

All minimal rank extensions can be obtained in this way.

This construction of a minimal rank extension yields the following corollary.

COROLLARY 1.2. *Let* $\mathcal{A} = \{ A_{ij} \mid 1 \le j \le i \le n \}$ *be a given lower triangular part. Let* $1 \le p,q \le n$. *If* $B = \begin{bmatrix} B_{ij} \end{bmatrix}_{i,j=1}^{n}$ *is a minimal rank extension of* $\mathcal{A}$, *then* $\hat{B} = \begin{bmatrix} B_{ij} \end{bmatrix}_{i=p,j=1}^{n \quad q}$ *is a minimal rank extension of the partially defined matrix*

$$
\begin{bmatrix}
A_{p1} & \cdots & A_{p,p} & & ? \\
\vdots & & \vdots & \ddots & \\
A_{q,1} & \cdots & A_{q,p} & \cdots & A_{qq} \\
\vdots & & \vdots & & \vdots \\
A_{n1} & \cdots & A_{np} & \cdots & A_{nq}
\end{bmatrix}.
\tag{1.5}
$$

Conversely, if $D = \left[D_{ij} \right]_{i=p,j=1}^{n \quad q}$ *is a minimal rank extension of* (1.5), *then there exists a*

minimal rank extension $B = \left[B_{ij} \right]_{i,j=1}^{n}$ *of* $\mathcal{A}$ *such that*

$$
B_{ij} = D_{ij} \, , p \; \leqq \; i \; \leqq \; n \, , 1 \; \leqq \; q \; \leqq \; n.
$$

Proof. First assume that $q = n$. The corollary states that if B is a minimal rank extension of (1.1) and we leave out rows from the top in B, then the remainder is a minimal rank extension of the corresponding given part. This as well as the converse statement follows directly from the construction given above. The same we can do with columns which we leave out from the right (the $p = 1$ case). But then we may also perform these reductions one after the other and obtain the corollary. $\square$

As one may expect, Corollary 1.2 does not remain true when one deletes rows and columns at random. For instance, consider the following two partially defined matrices:

$$
\begin{bmatrix} 1 & 0 & ? \\ 0 & 1 & 1 \end{bmatrix} , \begin{bmatrix} 0 & 0 & ? \\ 1 & 0 & 1 \end{bmatrix} .
\tag{1.6}
$$

The matrix on the left hand side in (1.6) is extended to a minimal rank extension by choosing $? = 1$, but this is a wrong choice when one only considers the last two columns. This shows that the analogue of the first statement of Corollary 1.2 fails in this case. The matrix on the right hand side in (1.6) is extended to a minimal rank extension only by choosing $? = 0$, but when one starts with a minimal rank extension for the last two columns one could have chosen the unspecified entry differently.

As we shall see later (in Section IV.6) the minimal rank extension problem for matrices is related to the notion of partial realization, which was introduced by R.E. Kalman in [51] and [52]. The minimal lower rank defined here may be viewed as a generalization of the rank of a partial behaviour (Hankel) matrix appearing in [52]. The main Lemma in [52] (which in our terminology concerns the case $n = 2$, $\nu_1 = 1$ and $\mu_2 = 1$) may be viewed as a first step in the direction of the construction of a minimal rank extension in this section.

IV.2. Uniqueness

Let $\mathscr{A} = \{ A_{ij} \mid 1 \leq j \leq i \leq n \}$ be a given lower triangular part. We say that $\mathscr{A}$ is
lower unique when $\mathscr{A}$ has only one minimal rank extension. The partially defined matrix on
the right hand side of (1.6) provides an example of a lower triangular part that is lower
unique and the matrix on the left hand side of (1.6) does not.

THEOREM 2.1. *Let* $\mathscr{A} = \{ A_{ij} \mid 1 \leq j \leq i \leq n \}$, *where* A_{ij} *is of size* $\nu_i \times \mu_j$.
Assume $\nu_1 > 0$ *and* $\mu_n > 0$. *Then the following are equivalent.*

(i) $\mathscr{A}$ *is lower unique;*

(ii) $\operatorname{rank} A^{(p,p)} = \operatorname{rank} A^{(p+1,p)} = \operatorname{rank} A^{(p+1,p+1)}$, $p = 1,\dots,n-1$;

(iii) *the numbers* $\operatorname{rank} A^{(p,q)}$, $1 \leq q \leq p \leq n$, *are all equal;*

(iv) $\operatorname{rank} A_{n1} = \ell(\mathscr{A})$.

Here $A^{(p,q)}$ *is defined in* (1.4) *and* $\ell(\mathscr{A})$ *is the minimal lower rank of* $\mathscr{A}$..

Note that for the case $\nu_n = 0$ or $\mu_1 = 0$ (i.e., when the given part is strictly lower tri-
angular) Theorem 2.1 states that $\mathscr{A}$ is lower unique if and only if $\ell(\mathscr{A}) = 0$, or,
equivalently, when all A_{ij}, $1 \leq j \leq i \leq n$, are zero. Of course, this case is of minor
interest.

We shall use the following lemma.

LEMMA 2.2. *Let* B *be a matrix. Then there exist an injective matrix* F *and a surjec-
tive matrix* G *such that* $B = FG$. *In that case* $\operatorname{rank} B = \operatorname{rank} F = \operatorname{rank} G$. *Moreover, if*
$\{\hat{F},\hat{G}\}$ *is another pair of matrices satisfying the above conditions then there is an invertible
matrix* S *such that* $F = \hat{F}S$ *and* $G = S^{-1}\hat{G}$.

Proof. The first two statements are evident, so let us prove the last statement. We
have $B = FG = \hat{F}\hat{G}$ with F and $\hat{F}$ injective and G and $\hat{G}$ surjective matrices. Let $G^{(-1)}$
and $\hat{G}^{(-1)}$ be right inverses of G and $\hat{G}$, respectively, and let $F^{(-1)}$ and $\hat{F}^{(-1)}$ be left
inverses of F and $\hat{F}$, respectively. Define $S := \hat{G}G^{(-1)} = \hat{F}^{(-1)}F$ and
$T := F^{(-1)}\hat{F} = G\hat{G}^{(-1)}$. Then $ST = \hat{F}^{(-1)}FG\hat{G}^{(-1)} = \hat{F}^{(-1)}\hat{F}\hat{G}\hat{G}^{(-1)} = I$. In the same
way $TS = I$. Furthermore, $\hat{F}S = \hat{F}\hat{G}G^{(-1)} = FGG^{(-1)} = F$ and $G = T\hat{G}$, proving the
lemma. $\square$

Proof of Theorem 2.1. Clearly, $\operatorname{rank} A_{n1} \leq \operatorname{rank} A^{(p,q)} \leq \ell(\mathscr{A})$, $1 \leq q \leq p \leq n$.
So (iv) implies (iii). The implication (iii) $\Rightarrow$ (ii) is also trivial. Let us prove (ii) $\Rightarrow$ (i).
Assume that (ii) holds. Let $B = \left[B_{ij} \right]_{i,j=1}^{n}$ and $C = \left[C_{ij} \right]_{i,j=1}^{n}$ be minimal rank

extension of $\mathcal{A}$. Note that Theorem 1.1 implies that

$$\text{rank } B = \text{rank } C = \ell(\mathcal{A}) = \text{rank } A^{(1,1)} = \text{rank } A^{(n,n)}.$$

Write $B = FG$ and $C = \hat{F}\hat{G}$ with F and $\hat{F}$ injective and G and $\hat{G}$ surjective. Denote

$$F = \text{col } \left[F_j\right]_{j=1}^{n} := \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} , G = \text{row } \left[G_j\right]_{j=1}^{n} := \begin{bmatrix} G_1 & \cdots & G_n \end{bmatrix} , \quad (2.1)$$

$$\hat{F} = \text{col } \left[\hat{F}_j\right]_{j=1}^{n} , \hat{G} = \text{row } \left[\hat{G}_j\right]_{j=1}^{n}.$$

Note that

$$F_n G = \text{row } \left[B_{nj}\right]_{j=1}^{n} = A^{(n,n)} = \text{row } \left[D_{nj}\right]_{j=1}^{n} = \hat{F}_n \hat{G}.$$

Since G and $\hat{G}$ are surjective, we get that rank $F_n = $ rank $A^{(n,n)} = $ rank $\hat{F}_n$. But then F_n and $\hat{F}_n$ are injective, and hence by Lemma 2.2 there exists an invertible matrix S such that $F_n = \hat{F}_n S$ and $G = S^{-1}\hat{G}$. In particular, $G_1 = S^{-1}\hat{G}_1$. Furthermore,

$$FG_1 = A^{(1,1)} = \hat{F}\hat{G}_1 = \hat{F}SG_1. \quad (2.2)$$

Since rank $F = $ rank $A^{(1,1)}$ and F is injective, we get that G_1 is surjective. But then (2.2) gives that $F = \hat{F}S$. Hence $B = FG = \hat{F}SS^{-1}\hat{G} = C$, proving the lower uniqueness of $\mathcal{A}$.

Assume that (ii) does not hold. Then at least one of the sets $J_2, \ldots, J_n$ and $I_1, \ldots, I_{n-1}$ introduced in Section 1 is not empty. Thus the construction in Section 1 shows that there are entries in (1.1) which are free to choose in the complex plane when making a minimal rank extension of (1.1). (Here we use that $\nu_1, \mu_n > 0$.) But then $\mathcal{A}$ is not lower unique. This proves (i) $\Rightarrow$ (ii).

We finish by proving the implication (ii) $\Rightarrow$ (iv). Assume that (ii) holds. Note that Theorem 1.1 gives that $\ell(\mathcal{A}) = $ rank $A^{(p,p)}, p = 1, \ldots, n$. Suppose that rank $A_{n1} < \ell(\mathcal{A})$. Then rank $A_{n1} < $ rank $A^{(p,p)}$ for all $1 \leq p \leq n$. Consider the partially defined matrix

$$\begin{bmatrix} A_{11} & ? & \dots & ? \\ \vdots & \vdots & & \vdots \\ A_{n-1,1} & ? & \dots & ? \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}. \tag{2.3}$$

By Theorem 1.1 the minimal lower rank of (2.3) is given by

$$\text{rank } A^{(1,1)} + \text{rank } A^{(n,n)} - \text{rank } A_{n1} > \text{rank } A^{(1,1)} = \ell(\mathscr{A}).$$

Since any extension of $\mathscr{A}$ is an extension of (2.3), we get a contradiction. □

Note that in the case when $\mathscr{A}$ is lower unique one may construct a minimal rank extension of $\mathscr{A}$ as follows. Let S_i be such that

$$A_{i1} = S_i A_{n1} , \; i = 1,\dots,n. \tag{2.4}$$

Then $B := \text{col} \left[S_i \right]_{i=1}^{n} \text{row} \left[A_{ni} \right]_{i=1}^{n}$ is the unique minimal rank extension of $\mathscr{A}$. Indeed, since $\text{rank } A_{n1} = \text{rank col} \left[A_{i1} \right]_{i=1}^{n}$, there exist $S_1, \dots , S_n$ such that (2.4) holds. Furthermore, $\text{rank } B = \text{rank } A^{(n,n)} = \ell(\mathscr{A})$. Finally, if $j \leq i$,

$$\begin{bmatrix} I & -S_i \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{i1} & A_{ij} \\ A_{n1} & A_{nj} \end{bmatrix} = \begin{bmatrix} 0 & A_{ij} - S_i A_{nj} \\ A_{n1} & A_{nj} \end{bmatrix}$$

has rank equal to $\text{rank } A_{n1}$, and hence $A_{ij} = S_i A_{nj}$.

IV.3. The set of minimal rank extensions

In this section we describe the set of all minimal rank extensions of a given lower triangular part. Before stating the main result, let us consider the following example. All minimal rank extensions of

$$\begin{bmatrix} 1 & 0 & \\ 0 & 1 & 1 \end{bmatrix}$$

are the matrices

$$\begin{bmatrix} x_1 & x_2 & x_1 x_3 + x_2 \\ 1 & 0 & x_3 \\ 0 & 1 & 1 \end{bmatrix},$$

where x_1, x_2 and x_3 are complex numbers which one can choose freely. It is clear that there is a one-one correspondence between the extensions and the triples (x_1,x_2,x_3). More precisely, in this case the set of minimal rank extensions is a manifold diffeomorphic to $\mathbb{C}^3$.

THEOREM 3.1. *Let* $\mathcal{A} = \{ A_{ij} \mid 1 \leq j \leq i \leq n \}$, *where* A_{ij} *is of size* $\nu_i \times \mu_j$. *The set* $\mathcal{M}(\mathcal{A})$ *of all minimal rank extensions of* $\mathcal{A}$ *is a manifold diffeomorphic to* $\mathbb{C}^k$, *where*

$$k = k(\mathcal{A}) = \sum_{j=1}^{n-1} \sum_{i=1}^{j} \nu_i (\varrho_{j+1,j+1} - \varrho_{j+1,j}) + \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} \mu_i (\varrho_{j,j} - \varrho_{j+1,j}) \qquad (3.1)$$

$$- \sum_{j=1}^{n-1} \sum_{i=j}^{n-1} (\varrho_{j,j} - \varrho_{j+1,j})(\varrho_{i+1,i+1} - \varrho_{i+1,i})$$

and

$$\varrho_{p,q} = \operatorname{rank} A^{(p,q)} = \operatorname{rank} \begin{bmatrix} A_{p1} & \cdots & A_{pq} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nq} \end{bmatrix}.$$

More precisely, there are polynomials $p_{ij} : \mathbb{C}^k \to \mathbb{C}$ *such that the matrix polynomial*

$$P = \left[p_{ij} \right]_{i=1,j=1}^{N \quad M} : \mathbb{C}^k \to \mathcal{M}(\mathcal{A})$$

is a diffeomorphism. Moreover, the polynomials p_{ij} *may be chosen such that the variables* x_r, $r = 1,...,k$, *appearing in the polynomials* p_{ij}, *have an exponent of degree 0 or 1.*

Proof. Choose index sets I_p and J_p, $p = 1,...,n$ as in Section 1. Choose for the unspecified entries in c_i, $i \in J := \bigcup_{p=1}^{n} J_p$, arbitrary complex numbers. Since all the columns of (the partially defined matrix belonging to) $\mathcal{A}$ can be made to be in the span of $\{ c_i \mid i \in J \}$ one obtains a partially defined matrix $\hat{\mathcal{A}}$ with the same minimal rank. Choose now arbitrary complex numbers for the still unspecified entries in r_i, $i \in I := \bigcup_{p=1}^{n} I_p$. Because of the choice of the rows r_i, $i \in I$, and since the minimal rank of $\hat{\mathcal{A}}$ equals $\ell(\mathcal{A}) = \#I$, once again one does not increase the minimal rank. Suppose that we have chosen arbitrary complex numbers for these $k(\mathcal{A})$ entries (referred to as the $k(\mathcal{A})$ variables) in these columns and rows.

Introduce the following partially defined matrix

$$\mathcal{B} = \begin{bmatrix} B_{10} & B_{11} & & & \\ B_{20} & B_{21} & B_{22} & & ? \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ B_{n0} & B_{n1} & B_{n2} & \cdots & B_{nn} \\ B_{n+1,0} & B_{n+1,1} & B_{n+1,2} & \cdots & B_{n+1,n} \end{bmatrix},$$

where col $\left[B_{i0} \right]_{i=1}^{n+1}$ consists of the columns c_i, $i \in J$, and col $\left[B_{ip} \right]_{i=1}^{n+1}$ consists of

the columns c_i, $i \in \{1+\sum_{i=1}^{p-1}\mu_i, \ldots, \mu_p+\sum_{i=1}^{p-1}\mu_i\}\backslash J_p$ for $p = 1,\ldots,n$. Analogously, row

$\left[B_{n+1,j} \right]_{j=0}^{n}$ consists of the rows r_i, $i \in I$, and row $\left[B_{pj} \right]_{j=0}^{n}$ consists of the rows r_i,

$i \in \{1+\sum_{i=1}^{p-1}\nu_i, \ldots, \nu_p+\sum_{i=1}^{p-1}\nu_i\}\backslash I_p$ for $p = 1,\ldots,n$. Note that the order of scalar

columns or scalar rows within one block column or block row, respectively, is irrelevant.
We have that $\ell(\mathcal{B}) = \ell(\mathcal{A})$.

Let us prove that rank $B_{n+1,0} = \ell(\mathcal{A})$. For this consider the 2×2 extension problem

$$\mathcal{C} = \begin{bmatrix} C_{11} & ? \\ C_{21} & C_{22} \end{bmatrix},$$

where $C_{11} = \text{col} \left[B_{i0} \right]_{i=1}^{n}$, $C_{21} = B_{n+1,0}$ and $C_{22} = \text{row} \left[B_{n+1,i} \right]_{i=1}^{n}$. Now

$$\text{rank} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = \#J = \ell(\mathcal{A}) = \#I = \text{rank} \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}.$$

Since $\mathcal{C}$ has a extension of rank equal to $\ell(\mathcal{B}) = \ell(\mathcal{A})$, it follows that $\ell(\mathcal{C}) = \ell(\mathcal{A})$ and
hence

$$\text{rank } B_{n+1,0} = \text{rank } C_{21} = \text{rank} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} + \text{rank} \begin{bmatrix} C_{21} & C_{22} \end{bmatrix} - \ell(\mathcal{C}) = \ell(\mathcal{A}).$$

We may conclude that for all fully specified submatrices of $\mathcal{B}$ containing $B_{n+1,0}$ the
rank is equal to $\ell(\mathcal{A}) = \ell(\mathcal{B})$. Hence, following Theorem 2.1, $\mathcal{B}$ has a unique minimal
rank extension, and, furthermore, by the remark at the end of the previous section the unk-
nown entries B_{ij}, $1 \leq i < j \leq n$, are given by

$$B_{ij} = S_i \begin{bmatrix} B_{nj} \\ B_{n+1,j} \end{bmatrix}, \tag{3.2}$$

where S_i is a solution of

$$\begin{bmatrix} B_{i0} & B_{i1} \end{bmatrix} = S_i \begin{bmatrix} B_{n0} & B_{n1} \\ B_{n+1,0} & B_{n+1,1} \end{bmatrix}. \tag{3.3}$$

Since rank $B_{n+1,0} = \ell(\mathscr{A}) =$ rank $\begin{bmatrix} B_{n0} & B_{n1} \\ B_{n+1,0} & B_{n+1,1} \end{bmatrix}$, we get that (3.3) is equivalent with

$$\begin{bmatrix} B_{i0} & B_{i1} \end{bmatrix} = S_i \begin{bmatrix} I & B_{n0}B_{n+1,0}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B_{n+1,0} & B_{n+1,1} \end{bmatrix}.$$

Since we know that (3.3) is solvable, we get that

$$S_i \begin{bmatrix} I & B_{n0}B_{n+1,0}^{-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} * & B_{i0}B_{n+1,0}^{-1} \end{bmatrix},$$

where $*$ may be chosen arbitrarily. Let us choose 0 for $*$. Then

$$S_i = \begin{bmatrix} 0 & B_{i0}B_{n+1,0}^{-1} \end{bmatrix}. \tag{3.4}$$

With (3.2) one concudes that the entries of $B_{ij} = B_{i0}B_{n+1,0}^{-1}B_{n+1,j}$, $1 \leq i < j \leq n$, are expressed in terms of the $k(\mathscr{A})$ variables via a rational expression. Since the $k(\mathscr{A})$ variables may vary over the whole complex plane, these expressions should in fact be polynomial. Using the fact that each x_r, $r = 1,...,k(\mathscr{A})$, appears at most once in the matrices B_{i0}, $B_{n+1,0}$, and $B_{n+1,j}$, we get that each scalar entry in B_{ij}, being a linear combination of determinants of submatrices of $B_{n+1,0}$, does not contain powers of x_r higher than one. $\square$

Note that for $\nu_1 > 0$ and $\mu_n > 0$, the number $k(\mathscr{A}) = 0$ if and only if

$$\varrho_{ii} = \varrho_{i+1,i} = \varrho_{i+1,i+1} \quad , i = 1,...,n-1,$$

which should be the case because of Theorem 2.1.

Let b_{ij} denote the (i,j)th scalar entry of B. Since $p_{ij}(\mathbb{C}^k)$ is either all of $\mathbb{C}$ or a singleton, the set $\{ b_{ij} \mid B \in \mathscr{M}(\mathscr{A}) \}$ is either the whole complex plane or a singleton.

IV.4. The Toeplitz case

This section concerns minimal rank extension problems for lower triangular parts with a Toeplitz structure. The extension will also be required to be Toeplitz. By a *Toeplitz (strict) lower triangular part* we mean a set $\mathcal{T} = \{ A_{ij} \mid 0 \le j < i \le n \}$ of matrices of sizes $\nu \times \mu$ such that $A_{i+1,j+1} = A_{ij}$ for all admissible indices (i,j). We shall also write $\mathcal{T}$ as $\mathcal{T} = \{ A_p \mid p = -n,\dots,-1 \}$, where $A_p := A_{-p,0}$. The partially defined matrix associated with $\mathcal{T}$ is

$$
\begin{bmatrix}
? & ? & \dots & ? & ? \\
A_{-1} & ? & \dots & ? & ? \\
A_{-2} & A_{-1} & & ? & ? \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{-n+1} & A_{-n+2} & \cdots & ? & ? \\
A_{-n} & A_{-n+1} & \cdots & A_{-1} & ?
\end{bmatrix} .
\tag{4.1}
$$

By adding a empty first column and an empty last row one may see $\mathcal{T}$ as a lower triangular part of the type considered in (1.2). Therefore we may speak of a (minimal rank) extension of $\mathcal{T}$ and the minimal lower rank of $\mathcal{T}$. A block matrix $B = \begin{bmatrix} B_{ij} \end{bmatrix}_{i,j=0}^{n}$ is called a *Toeplitz minimal rank extension* of $\mathcal{T}$ if B is a minimal rank extension of $\mathcal{T}$ and in addition B is Toeplitz, i.e., $B_{i+1,j+1} = B_{ij}$ for all admissible indices (i,j). The set of all Toeplitz minimal rank extensions of $\mathcal{T}$ will be denoted by $\mathcal{M}_T(\mathcal{T})$. At first sight it is not clear whether $\mathcal{M}_T(\mathcal{T}) \ne \varnothing$. The next theorem shows, among other things, that $\mathcal{M}_T(\mathcal{T}) \ne \varnothing$. The latter result (among all minimal rank extensions of a Toeplitz lower triangular part there is a Toeplitz one) may be deduced as well from the minimal degree formula in [37], and is suggested in [52].

THEOREM 4.1. *Let* $\mathcal{T} = \{ A_p \mid p = -1,\dots,-n \}$ *be a Toeplitz lower triangular part, with A_p of size* $\nu \times \mu$. *Then* $\mathcal{M}_T(\mathcal{T})$ *is a manifold diffeomorphic to* $\mathbb{C}^m$, *where*

$$
m = m(\mathcal{T}) = \sum_{p=1}^{n} (\varrho_{p,p} - \varrho_{p,p-1})(\varrho_{p,p} - \varrho_{p+1,p}),
\tag{4.2}
$$

with for $1 \le q \le p \le n$,

$$
\varrho_{p,q} = \operatorname{rank} A^{(p,q)} = \operatorname{rank} \begin{bmatrix} A_{-p} & \cdots & A_{-p+q-1} \\ \vdots & & \vdots \\ A_{-n} & \cdots & A_{-n+q-1} \end{bmatrix} ,
$$

and $\varrho_{1,0} = 0 = \varrho_{n+1,n}$. *More precisely, there exist polynomials* $p_{ij} : \mathbb{C}^m \to \mathbb{C}$, $1 \leq i \leq (n+1)\nu$, $1 \leq j \leq (n+1)\mu$, *such that the matrix polynomial*

$$P = \left[p_{ij} \right]_{i=1, \quad j=1}^{(n+1)\nu \ (n+1)\mu} : \mathbb{C}^m \to \mathcal{M}_T(\mathcal{T})$$

is a diffeomorphism.

We shall not prove Theorem 4.1 in detail, but give a procedure to construct all Toeplitz minimal rank extensions which explains along the way the main idea of the proof.

To describe the construction, we view (4.1) as an $(n+1)\nu \times (n+1)\mu$ matrix of which some entries are specified (the ones in A_p, $p < 0$) and some are unspecified, i.e., these entries are free variables over $\mathbb{C}$. As in Section IV.1 let c_i denote the i-th scalar column and r_i the i-th row of the $(n+1)\nu \times (n+1)\mu$ matrix. The specified (given) part of a column c or row r we denote by $(c)_g$ and $(r)_g$, respectively. Choose an index set $J_1 \subset \{1,...,\mu\}$ such that $\{ (c_i)_g \mid i \in J_1 \}$ is a basis for Im $A^{(1,1)}$. Furthermore, choose inductively for $p = 2,...,n$ an index set $J_p \subset \{(p-1)\mu+1, \ldots, p\mu\}$ such that $J_p - \mu$ $(:= \{j - \mu \mid j \in J_p\}) \subset J_{p-1}$, $\{ (c_i)_g \mid i \in J_p \}$ is a linearly independent set of columns and

$$\text{span}\{ (c_i)_g \mid i \in J_p \} \dotplus \text{Im} A^{(p,p-1)} = \text{Im} A^{(p,p)}.$$

So compared to the sets J_p that we obtained in Section IV.1, these sets J_p have the additional property that $J_p - \mu \subset J_{p-1}$. It is possible to choose $J_1, \ldots, J_n$ in such a way because $\mathcal{T}$ is Toeplitz. Next we make index sets I_p, $p = 1,...,n$, for the rows. Choose $I_n \subset \{n\nu+1,...,(n+1)\nu\}$ such that $\{ (r_i)_g^T \mid i \in I_n \}$ is a basis for Im $A^{(n,n)T}$. Furthermore, choose inductively for $p = n-1,...,1$ an index set $I_p \subset \{p\nu+1, \ldots, (p+1)\nu\}$ such that $I_p + \nu \subset I_{p+1}$, $\{ (r_i)_g \mid i \in I_p \}$ is a linearly independent set of rows and

$$\text{span}\{ (r_i)_g^T \mid i \in I_p \} \dotplus \text{Im} A^{(p+1,p)T} = \text{Im} A^{(p,p)T}.$$

In [52] (see also [12]) a similar procedure of picking out column and row indices is described in the setting of Hankel matrices.

Let us make the following picture of a block matrix:
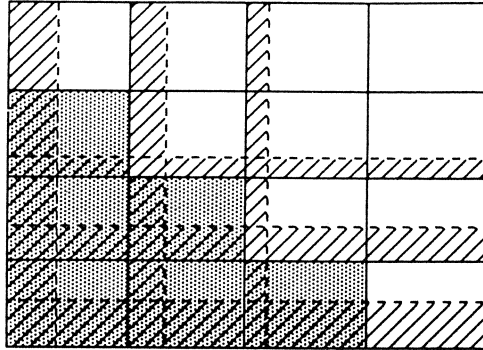
Figure 1

The grey strict lower triangular part corresponds to the specified part. The cross-hatched columns and rows correspond to the columns c_i, $i \in J := \bigcup_{p=1}^{n} J_p$, and rows r_i,

$i \in I := \bigcup_{p=1}^{n} I_p$. Any (not necessarily block Toeplitz) minimal rank extension can now be obtained by choosing freely the entries in the cross-hatched part in the upper triangle. But we want to find a block *Toeplitz* minimal rank extension $B = \left[B_{j-i} \right]_{i,j=0}^{n}$ of (4.1). Let us start by constructing B_0. For this we look at all the places where B_0 should appear and we consider only those entries of B_0 which are free to choose in each of these places. For those entries the choice is free. Make an arbitrary choice for these entries. We get the following picture for B_0.
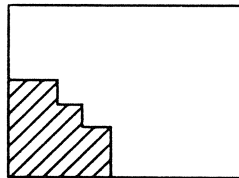
Figure 2

The cross-hatched part stands for the entries which we have just chosen. We claim that the rest of B_0 is uniquely determined. Let us illustrate this on the 4×4 example

$$B = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 \\ B_{-1} & B_0 & B_1 & B_2 \\ B_{-2} & B_{-1} & B_0 & B_1 \\ B_{-3} & B_{-2} & B_{-1} & B_0 \end{bmatrix},$$

which corresponds to Figure 1. Partition B_0 as a 4×4 block matrix suggested by Figure 2. So $B_0 = \left[\Sigma_{ij} \right]_{i,j=1}^{4}$, and the matrices Σ_{21}, Σ_{31}, Σ_{41}, Σ_{32}, Σ_{42} and Σ_{43} correspond to the cross-hatched part. Note that Σ_{11} is of size $(\nu - \#I_3) \times \#J_3$, Σ_{22} is of size $(\#I_3 - \#I_2) \times (\#J_2 - \#J_3)$, Σ_{33} is of size $(\#I_2 - \#I_1) \times (\#J_1 - \#J_2)$ and Σ_{44} is of size $\#I_1 \times (\mu - \#J_1)$. Now consider the first block column of B and leave out the $\nu - \#I_1$ scalar rows corresponding to $\left[\Sigma_{ij} \right]_{i=1,j=1}^{3 \quad 4}$. So one has

$$\begin{bmatrix} \left[\Sigma_{41} \ \Sigma_{42} \ \Sigma_{43} \right] & \Sigma_{44} \\ B_{-1} & \\ B_{-2} & \\ B_{-3} & \end{bmatrix} =: \begin{bmatrix} \tilde{\Sigma} & \Sigma_{44} \\ B_{-1}^{(1)} & B_{-1}^{(2)} \\ B_{-2}^{(1)} & B_{-2}^{(2)} \\ B_{-3}^{(1)} & B_{-3}^{(2)} \end{bmatrix},$$

with Σ_{44} unspecified. Since

$$\mathrm{rank} \begin{bmatrix} \tilde{\Sigma} \\ B_{-1}^{(1)} \\ B_{-2}^{(1)} \\ B_{-3}^{(1)} \end{bmatrix} = \mathrm{rank} \begin{bmatrix} B_{-1}^{(1)} \\ B_{-2}^{(1)} \\ B_{-3}^{(1)} \end{bmatrix} = \mathrm{rank} \begin{bmatrix} B_{-1} \\ B_{-2} \\ B_{-3} \end{bmatrix} = \#J_1 ,$$

Theorem 2.1 yields that a minimal rank extension for this part is uniquely determined. Thus Σ_{44} is uniquely determined. Now omit the first block row, and repeat the reasoning for the first two block columns of

$$B = \begin{bmatrix} B_{-1} & B_0 & B_1 & B_2 \\ B_{-2} & B_{-1} & B_0 & B_1 \\ B_{-3} & B_{-2} & B_{-1} & B_0 \end{bmatrix}.$$

This determines $\left[\Sigma_{33} \ \Sigma_{34} \right]$. Proceeding this way we see that all $\{ \Sigma_{ij} \}_{i \leq j}$ are uniquely determined. Thus when making a Toeplitz minimal rank extension B all the entries in B_0 in the cross-hatched part in Figure 2 are free to choose and all other entries of B_0 are

determined by that choice. This principle also works for the construction of the other B_p, $p \geqq 0$. By counting the number of entries which are free to choose one finds the number $m(\mathcal{T})$.

COROLLARY 4.2. *Let* $n \geqq 2$. *The Toeplitz lower triangular part* $\mathcal{T}$ *has a unique Toeplitz minimal rank extension if and only if for some* $p \in \{1,...,n-1\}$ *we have that*

$$\operatorname{rank} A^{(p,p)} = \operatorname{rank} A^{(p+1,p)} = \operatorname{rank} A^{(p+1,p+1)}. \tag{4.3}$$

Proof. If $\mathit{r}(\mathcal{T}) = 0$ the statement is trivial, so suppose that $\mathit{r}(\mathcal{T}) \neq 0$. The situation that $\mathcal{T}$ has an unique block Toeplitz minimal rank extension corresponds to the situation when $m(\mathcal{T}) = 0$ (use Theorem 4.1). Since the numbers $\varrho_{q,q} - \varrho_{q,q-1} = \#J_q$ form a descending sequence, and the numbers $\varrho_{q,q} - \varrho_{q+1,q} = \#I_q$ form an ascending sequence, we have that $m(\mathcal{T}) = 0$ if and only if $\max\{q \mid \varrho_{q,q} - \varrho_{q,q-1} \neq 0\} < \min\{q \mid \varrho_{q,q} - \varrho_{q+1,q} \neq 0\}$. If this is the case, one may choose in (4.3) the integer p equal to $\max\{q \mid \varrho_{q,q} - \varrho_{q,q-1} \neq 0\}$. Conversely, if (4.3) holds, then $\varrho_{p,p} = \varrho_{p+1,p} = \varrho_{p+1,p+1}$ and

$$\max\{q \mid \varrho_{q,q} - \varrho_{q,q-1} \neq 0\} \leqq p < \min\{q \mid \varrho_{q,q} - \varrho_{q+1,q} \neq 0\}. \quad \square$$

Note the difference between this result and the uniqueness result in Section IV.2, where an equation like (4.3) is required for *all* relevant p.

Since $\mathbb{C}$ is an algebraically closed field the set $p_{ij}(\mathbb{C}^m)$ is either the whole complex plane or a singleton. If a_{ij} denotes the (i,j)th scalar entry of a matrix A, then the set $\{a_{ij} \mid A \in \mathcal{M}_T(\mathcal{T})\}$ is either a singleton or the whole complex plane.

IV.5. Minimal lower rank and inverses

In this section we show that the minimal lower rank of the lower triangular part of an invertible block matrix may be computed in terms of the minimal lower rank of the strictly lower triangular part of its inverse.

THEOREM 5.1. *Let* $T = \left[T_{ij} \right]_{i,j=1}^{n}$ *be an invertible block matrix with* T_{ij} *of size* $\nu_i \times \mu_j$. *So* $\sum \nu_i = \sum \mu_i =: N$. *The inverse of* T *is partitioned according to the partitioning of* T: $T^{-1} = \left[S_{ij} \right]_{i,j=1}^{n}$, *where* S_{ij} *is of size* $\mu_i \times \nu_j$. *Put*

$$\mathcal{T} = \{ T_{ij} \mid 1 \leqq j \leqq i \leqq n \}, \quad \mathcal{S} = \{ S_{ij} \mid 1 \leqq j < i \leqq n \}. \tag{5.1}$$

Then

$$\rho(\mathcal{T}) + \rho(\mathcal{S}) = N.$$

Proof. If $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is of size $(r_1 + r_2) \times (s_1 + s_2)$ and

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} := \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}$$

is of size $(s_1 + s_2) \times (r_1 + r_2)$, then from Theorem 1.1 in [9] we obtain that

$$\text{rank } C + s_2 = \text{rank } \gamma + r_2.$$

Applying this on the identity

$$T^{-1} = \begin{bmatrix} * & * \\ T^{(p,q)} & * \end{bmatrix}^{-1} = \begin{bmatrix} * & * \\ S^{(q+1,p-1)} & * \end{bmatrix},$$

we obtain that

$$\text{rank } T^{(p,q)} + \sum_{j=q+1}^{n} \mu_j = \text{rank } S^{(q+1,p-1)} + \sum_{j=p}^{n} \nu_j.$$

Here $T^{(p,q)}$ and $S^{(p,q)}$ are assumed to be zero when p or q is not in $\{1,\ldots,n\}$. But then

$$\sum_{p=1}^{n} \text{rank } T^{(p,p)} + \sum_{p=1}^{n-1} \sum_{j=p+1}^{n} \mu_j = \sum_{p=2}^{n-1} \text{rank } S^{(p+1,p-1)} + \sum_{p=1}^{n} \sum_{j=p}^{n} \nu_j \qquad (5.2)$$

and

$$\sum_{p=1}^{n-1} \text{rank } T^{(p+1,p)} + \sum_{p=1}^{n-1} \sum_{j=p+1}^{n} \mu_j = \sum_{p=1}^{n-1} \text{rank } S^{(p+1,p)} + \sum_{p=1}^{n-1} \sum_{j=p+1}^{n} \nu_j. \qquad (5.3)$$

Subtracting (5.3) from (5.2) gives together with Theorem 1.1 that

$$\rho(\mathcal{T}) = -\rho(\mathcal{S}) + \sum_{j=1}^{n} \nu_j ,$$

and the theorem is proved. $\square$

In the special case when $S_{ij} = 0$ for all $1 \leq j < i \leq n$ Theorem 5.1 states that $\rho(\mathcal{T}) = N$. In other words, in that case changes in the strict upper triangular part of T does not spoil the invertibilty of T. This result may also easily be derived by using Kramer's rule. A more interesting corollary of Theorem 5.1 concerns the following result of E. Asplund [4].

COROLLARY 5.2. *Let* $T = \left[t_{ij} \right]_{i,j=1}^{n}$ *be an invertible* $n \times n$ *scalar matrix and*

denote $T^{-1} = \left[s_{ij} \right]_{i,j=1}^{n}$. *Let* $p \in \{0,...,n-1\}$, *and let* $\mathcal{F}$ *denote the lower triangular*

part $\mathcal{F} = \{ t_{ij} \mid 1 \leq j < i+p \leq n \}$. *Then* $\ell(\mathcal{F}) = p$ *if and only if* $s_{ij} = 0$, $j < i-p$,

and $s_{j+p,j} \neq 0$, $j = 1,...,n-p$.

Proof. First let $p = 0$. Then $\ell(\mathcal{F}) = 0$ if and only if T is upper triangular, but this holds if and only if S is upper triangular, and since S is invertible its diagonal elements can not be zero. Next, let $p \in \{0,...,n-1\}$. View T as a $(n-p+1) \times (n-p+1)$ block matrix where the first p columns of T and its last p rows are taken together. In S this corresponds to taking together the first p rows and the last p columns. Applying now Theorem 5.1 we get that $\ell(\mathcal{F}) = p$ if and only if the partially defined matrix

$$\begin{bmatrix} s_{p+1,1} & & & ? \\ s_{p+2,1} & s_{p+2,2} & & \\ \vdots & \vdots & \ddots & \\ s_{n1} & s_{n2} & \cdots & s_{n,n-p} \end{bmatrix}$$

has minimal lower rank equal $n-p$, which is precisely its order. Use now the $p = 0$ case to see that this can only happen when $s_{ij} = 0$, $j < i-p$ and $s_{j+p,j} \neq 0$, $j = 1,...,n-p$. $\square$

With Corollary 5.2 together with its upper triangular analogue (which one may obtain by reversing the order of the rows and columns) one can describe those scalar matrices whose inverses are band matrices. We will not do this here since it would involve new notations. In this way one may recover results from [8] and [60].

IV.6. Connections with the partial realization problem

Let $M_1, \cdots, M_n$ be a given finite sequence of $\nu \times \mu$ matrices. A system $\Sigma = (A, B, C)$ of matrices, where A, B and C are matrices of sizes $l \times l$, $l \times \mu$ and $\nu \times l$, respectively, is called a *realization* of $M_1, \cdots, M_n$ if

$$C A^{i-1} B = M_i, \quad i = 1,...,n.$$

The space $\mathbb{C}^l$ on which A acts is called the *state space* of the realization Σ. If $\Sigma = (A, B, C)$ is a realization of $M_1, \cdots, M_n$ and S is invertible, then $(S^{-1}AS, S^{-1}B, CS)$ is also a realization of $M_1, \cdots, M_n$, which is called *similar* to the

realization Σ. Note that similar realizations have the same state space dimension. Following [37], we call the smallest possible state space dimension of a realization of $M_1, \cdots, M_n$ the *degree* of $M_1, \cdots, M_n$ A realization of $M_1, \cdots, M_n$ is called *minimal* if its state space dimension is equal to the degree of $M_1, \cdots, M_n$. The problem of partial realization, which was introduced by R.E. Kalman in [51] and [52], consists of finding all minimal realizations of $M_1, \cdots, M_n$.

Let us make the connection with the minimal rank extension problem considered in Section 4. Consider the partially defined matrix

$$\begin{bmatrix} ? & ? & \ldots & ? & ? \\ M_n & ? & \ldots & ? & ? \\ M_{n-1} & M_n & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_2 & M_3 & \ldots & ? & ? \\ M_1 & M_2 & \ldots & M_n & ? \end{bmatrix}. \tag{6.1}$$

There is a 1-1 correspondence between the set of non-similar minimal realizations of $M_1, \cdots, M_n$ and the set of Toeplitz minimal rank extensions of (6.1). In one direction the correspondence is simple: If $\Sigma = (A, B, C)$ is a minimal realization of $M_1, \cdots, M_n$, then the matrix $\left[CA^{j-i+n}B \right]_{i,j=0}^{n}$ is a Toeplitz minimal rank extension of (6.1), and similar minimal realizations give the same Toeplitz minimal rank extension. The other direction is more involved. For this we refer to [12] and [37], where instead of the Toeplitz the Hankel version is considered. Thus Theorem IV.4.1 yields the following corollary.

COROLLARY 6.1. *Let* $M_1, \cdots, M_n$ *be given* $\nu \times \mu$ *matrices. The set*
$\mathcal{M}(M_1, \cdots, M_n) \subset \prod_{i=1}^{\infty} \mathbb{C}^{\nu \times \mu}$, *defined by*

$$\{ (CA^{i-1}B)_{i=0}^{\infty} \mid (A, B, C) \text{ is a minimal realization of } M_1, \cdots, M_n \},$$

is a manifold diffeomorphic to $\mathbb{C}^{m(\mathcal{M})}$. *Here* $\mathcal{M}$ *is the Toeplitz lower triangular part corresponding to* (6.1) *and* $m(\mathcal{M})$ *is defined in* (4.2). *The diffeomorphism acting from* $\mathbb{C}^{m(\mathcal{M})}$ *onto* $\mathcal{M}(M_1, \cdots, M_n)$ *may chosen to be a polynomial.*

The case when there is only one minimal realization (up to similarity) corresponds to the case when $m(\mathcal{M}) = 0$. With Corollary 4.2 one sees that this happens if and only if for some $p \in \{1, \ldots, n-1\}$

$$\text{rank } M^{(p,p)} = \text{rank } M^{(p+1,p)} = \text{rank } M^{(p+1,p+1)},$$

where

$$M^{(p,q)} = \begin{bmatrix} M_{n-p+1} & \cdots & M_{n-p+q} \\ \vdots & & \vdots \\ M_1 & \cdots & M_q \end{bmatrix}.$$

This is a somewhat other version of the characterization of uniqueness of minimal realizations (up to similarity) given in Theorem 2.1(iii) in [12].

IV.7. General patterns

Up to now we only considered minimal rank extensions for partially defined matrices with given entries in a triangular form. In this section we consider the case when the given entries are not necessarily located in a triangular part. Let us introduce some notations. Let J be a *pattern*, i.e., a subset of $\{1,...,n\} \times \{1,...,m\}$. Let A_{ij}, $(i,j) \in J$, be given block matrices of size $\nu_i \times \mu_j$. We call a block matrix $B = \left[B_{ij} \right]_{i=1,j=1}^{n \quad m}$, with B_{ij} of size $\nu_i \times \mu_j$, an *extension* of $\mathcal{A} = \{ A_{ij} \mid (i,j) \in J \}$ if $B_{ij} = A_{ij}$, $(i,j) \in J$. The *minimal rank* of $\mathcal{A}$ is defined by

$$\rho(\mathcal{A}) := \min\{ \text{rank } B \mid B \text{ is an extension of } \mathcal{A} \}, \tag{7.1}$$

and all extensions of $\mathcal{A}$ which attain the minimum in (7.1) are called *minimal rank extensions* of $\mathcal{A}$. We shall call the pattern J *triangular* if there exist permutations σ on $\{1,...,n\}$ and τ on $\{1,...,m\}$ such that

$$J_{\sigma\tau} := \{ (\sigma(i),\tau(j)) \mid (i,j) \in J \}$$

has the property that $(i,j) \in J_{\sigma\tau}$ implies

$$\{ (k,r) \mid i \leq k \leq n , 1 \leq r \leq j \} \subset J_{\sigma\tau}.$$

Since the minimal rank extension problem does not change when one permutes rows and columns, Theorem 1.1 gives a formula for the minimal rank in the case that J is triangular. In fact, in that case the minimal rank is determined by the ranks of certain fully specified submatrices. The following example shows that for a general pattern the number $\rho(\mathcal{A})$ depends, in general, upon more data.

Consider the following partially defined matrices

$$\begin{bmatrix} ? & 1 & 1 \\ 1 & ? & 1 \\ 1 & 1 & ? \end{bmatrix}, \quad \begin{bmatrix} ? & 1 & 1 \\ 1 & ? & 1 \\ 1 & 2 & ? \end{bmatrix}. \tag{7.2}$$

In both matrices all fully specified submatrices have rank 1, but the matrix on the left hand side in (7.2) has minimal rank 1 while the other has minimal rank 2.

The question arises for which patterns the minimal rank is determined by the ranks of fully specified submatrices. Let us be more precise. We call $R \subset \{1,...,n\} \times \{1,...,m\}$ a *rectangular* pattern if R is of the form $R = I \times \bar{I}$. A pattern J is called *rank determined* if for given $\mathscr{A} = \{ A_{ij} \mid (i,j) \in J \}$ and $\hat{\mathscr{A}} = \{ \hat{A}_{ij} \mid (i,j) \in J \}$, with A_{ij} and $\hat{A}_{ij}$ of size $\nu_i \times \mu_j$, the minimal ranks $r(\mathscr{A})$ and $r(\hat{\mathscr{A}})$ are equal as soon as

$$\text{rank } \left[A_{ij} \right]_{(i,j) \in R} = \text{rank } \left[\hat{A}_{ij} \right]_{(i,j) \in R}$$

for all rectangular patterns $R \subset J$.

A convenient way to describe matrix patterns is via bipartite graphs. Given a pattern J, the corresponding (undirected) *bipartite graph* $G(J)$ has vertices $\{v_1, \ldots, v_n, u_1, \ldots, u_m\}$, and (v_i, u_j) is an edge in $G(J)$ if and only if $(i,j) \in J$. A bipartite graph G is called *chordal* if there are no minimal cycles of length ≥ 6. In a bipartite graph all minimal cycles have even length. See, e.g., [44] for further properties of chordal bipartite graphs. Note that the bipartite graph corresponding to the pattern of the partially defined matrices in (7.2) is not chordal.

THEOREM 7.1. *If the pattern J is rank determined, then the corresponding bipartite graph $G(J)$ is chordal.*

Proof. Suppose that $G(J)$ is a $2k$-cycle, $k \geq 3$. Applying a permutation and leaving out fully unspecified rows and columns a partially defined matrix with pattern J looks like

$$\begin{bmatrix} a_1 & b_1 & ? & ? & \ldots & ? & ? \\ ? & a_2 & b_2 & ? & \ldots & ? & ? \\ . & . & . & . & & . & . \\ ? & ? & ? & ? & \ldots & a_{k-1} & b_{k-1} \\ b_k & ? & ? & ? & \ldots & ? & a_k \end{bmatrix}. \tag{7.3}$$

Let a_i, b_i be non-zero numbers. Then the ranks of all fully specified submatrices of (7.3) are equal to 1. On the other hand one easily checks that the minimal rank of (7.3) is 1 if $\prod a_i = \prod b_i$ and 2 otherwise. Thus J is not rank determined.

Suppose that $G(J)$ is not chordal. Then $G(J)$ contains a $2k$-cycle, for some $k \geq 3$. This means that any partially defined matrix corresponding to J contains a submatrix as in (7.3). Choosing the given entries outside this submatrix equal to 0, one may use the same reasoning as above in order to show that J is not rank determined. $\square$

It remains an open question whether the converse of Theorem 7.1 is true.

CONJECTURE 7.2. *Let J be a pattern for which the corresponding bipartite graph is chordal. Then J is rank determined. Moreover, if $\mathcal{A} = \{ A_{ij} \mid (i,j) \in J \}$ is a given part, then*

$$\mathscr{l}(\mathcal{A}) = \max_T \mathscr{l}(\{ A_{ij} \mid (i,j) \in T \}), \tag{7.4}$$

where the maximum is taken over all triangular $T \subset J$.

Note that (7.4) is true if $=$ is replaced by $\geq$.

The conjecture would, for instance, imply that the minimal rank of

$$\begin{pmatrix} A & B & ? \\ ? & C & D \end{pmatrix} \tag{7.5}$$

is equal to

$$\text{rank} \begin{bmatrix} B \\ C \end{bmatrix} + \max \{ \text{rank} \begin{bmatrix} A & B \end{bmatrix} - \text{rank } B , \text{ rank} \begin{bmatrix} C & D \end{bmatrix} - \text{rank } D \}.$$

The latter statement is indeed true. To see this put $r_1 = \text{rank} \begin{bmatrix} A & B \end{bmatrix} - \text{rank } B$ and $r_2 = \text{rank} \begin{bmatrix} C & D \end{bmatrix} - \text{rank } D$. Suppose $r_1 \geq r_2$. Then select r_1 linearly independent columns in A which span together with Im B the image of $\begin{bmatrix} A & B \end{bmatrix}$. Replace the corresponding columns in the $(2,1)$ entry of (7.5) by the r_2 linearly independent columns from D that together with Im C span Im $\begin{bmatrix} C & D \end{bmatrix}$. Now all the columns in D are linearly dependent of the columns on the left of D in (7.5). Thus it follows that the minimal rank of (7.5) is equal to the minimal rank of $\begin{bmatrix} A & B \\ *? & C \end{bmatrix}$, where $*$ denotes the columns we just filled in. The minimal rank of the latter is equal to rank $\begin{bmatrix} B \\ C \end{bmatrix} + r_1$.

More evidence for the correctness of Conjecture 7.2 may be found in [14].

CHAPTER V. TRIANGULAR OPERATORS

This chapter concerns minimal rank extensions of operators which act on infinite dimensional Hilbert spaces and are triangular relative to chains of orthogonal projections. Section 1 presents the main results for the case of finite chains. Preliminaries on lower triangular parts of an operator relative to arbitrary chains of projections appear in Section 2. In Section 3 formulas are given for the minimal lower rank. In Section 4 the case when there is only one minimal rank extension is characterized. Section 5 presents a construction to obtain all minimal rank extensions. In Sections 6 and 7 the results are specified for semi-infinite operator matrices and for kernels of integral operators, respectively. Section 8 treats minimal rank extensions of difference kernels and in Section 9 the connections with systems theory are made.

V.1. Finite chains

Let Z and Y be separable Hilbert spaces over $\mathbb{C}$, and let $\mathscr{P} = \{P_0, \cdots, P_n\}$ and $\mathscr{Q} = \{Q_0, \cdots, Q_n\}$ be finite chains of orthogonal projections on Z and Y, respectively. See [41], Section I.3, for the definition. We do not assume that $\mathscr{P}$ and $\mathscr{Q}$ are bordered, i.e., $\mathscr{P}$ and $\mathscr{Q}$ are not required to contain the operators 0 and I. We define the *lower triangular part of* T *relative to the chains* $\mathscr{P}$ *and* $\mathscr{Q}$ to be the operator

$$\mathscr{L}(T;\mathscr{P},\mathscr{Q}) := (I-Q_0)TP_1 + \sum_{j=2}^{n} (I-Q_{j-1})T(P_j - P_{j-1}). \tag{1.1}$$

Note that in formula (1.1) the projections P_0 and Q_n do not play any role. If $\mathscr{P}$ and $\mathscr{Q}$ are bordered (and thus $P_n = I$ and $Q_0 = 0$), then (1.1) can be rewritten as

$$\mathscr{L}(T;\mathscr{P},\mathscr{Q}) := \sum_{j=1}^{n} (I-Q_{j-1})T(P_j - P_{j-1}) =$$

$$= \sum_{j=1}^{n} \sum_{i=j}^{n} (Q_i - Q_{i-1})T(P_j - P_{j-1}).$$

The operator $\mathscr{L}(T;\mathscr{P},\mathscr{Q})$ may be represented as a lower triangular operator matrix. To see this, put $Z_0 = \text{Im } P_0$, $Z_i = \text{Im } (P_i - P_{i-1})$ $(i = 1,...,n)$, $Z_{n+1} = \text{Im } (I-P_n)$, $Y_0 = \text{Im } Q_0$, $Y_i = \text{Im } (Q_i - Q_{i-1})$ $(i = 1,...,n)$, and $Y_{n+1} = \text{Im } (I-Q_n)$. Then

$$Z = \bigoplus_{i=0}^{n+1} Z_i \ , \ Y = \bigoplus_{i=0}^{n+1} Y_i. \tag{1.2}$$

Decompose $T = \left[T_{ij} \right]_{i,j=0}^{n+1}$ relative to the decompositions in (1.2) of Z and Y. Then

$$
\mathcal{A}(T;\mathcal{P},\mathcal{Q}) =
\begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
T_{1,0} & T_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
T_{2,0} & T_{2,1} & T_{2,2} & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
T_{n-1,0} & T_{n-1,1} & T_{n-1,2} & T_{n-1,3} & \cdots & T_{n-1,n-1} & 0 & 0 \\
T_{n,0} & T_{n,1} & T_{n,2} & T_{n,3} & \cdots & T_{n,n-1} & T_{n,n} & 0 \\
T_{n+1,0} & T_{n+1,1} & T_{n+1,2} & T_{n+1,3} & \cdots & T_{n+1,n-1} & T_{n+1,n} & 0
\end{bmatrix}
. \quad (1.3)
$$

The operator $T : Z \to Y$ is said to be of *finite* $(\mathcal{P},\mathcal{Q})$-*lower rank* if there exists a finite rank operator $K : Z \to Y$ such that $\mathcal{A}(T;\mathcal{P},\mathcal{Q}) = \mathcal{A}(K;\mathcal{P},\mathcal{Q})$. In that case K is called a *finite rank extension* of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$. Note that T is of finite $(\mathcal{P},\mathcal{Q})$-lower rank if and only all the T_{ij} appearing in the representation (1.3) are of finite rank, which in turn is equivalent to saying that $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$ has finite rank. The *minimal* $(\mathcal{P},\mathcal{Q})$-*lower rank* of T is by definition the smallest possible rank of a finite rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$. In other words, the minimal $(\mathcal{P},\mathcal{Q})$-lower rank $\mathcal{l}(T;\mathcal{P},\mathcal{Q})$ of T is given by

$$
\mathcal{l}(T;\mathcal{P},\mathcal{Q}) := \min\{ \text{ rank } K \mid \mathcal{A}(T;\mathcal{P},\mathcal{Q}) = \mathcal{A}(K;\mathcal{P},\mathcal{Q})\}. \quad (1.4)
$$

All K's for which the minimum in (1.4) is attained are called *minimal rank extensions* of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$. The operator T is called $(\mathcal{P},\mathcal{Q})$-*lower unique* when $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$ has only one minimal rank extension. We have the following theorems.

THEOREM 1.1. *Let* $T : Z \to Y$, *and let* $\mathcal{P} = \{P_0, \cdots, P_n\}$ *and* $\mathcal{Q} = \{Q_0, \cdots, Q_n\}$ *be finite chains of orthogonal projections on Z and Y, respectively. Assume that T is of finite $(\mathcal{P},\mathcal{Q})$-lower rank. Then*

$$
\mathcal{l}(T;\mathcal{P},\mathcal{Q}) = \sum_{p=1}^{n} \text{rank } (I-Q_{p-1})TP_p - \sum_{p=1}^{n-1} \text{rank } (I-Q_p)TP_p . \quad (1.5)
$$

Proof. We use the decompositions (1.2) and the representation (1.3). If Z and Y are finite dimensional, the theorem follows directly from Theorem IV.1.1. For the general case make decompositions

$$
Z_i = Z_i^{(1)} \oplus Z_i^{(2)} , \; Y_i = Y_i^{(1)} \oplus Y_i^{(2)} , \; i = 0,\ldots,n+1,
$$

such that $Z_i^{(1)}$ and $Y_i^{(1)}$ are finite dimensional and relative to these decompositions T_{ij} is of the form $\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$. Since all T_{ij} in (1.3) are of finite rank, such decompositions exist. But

then the theorem follows immediately from the finite dimensional case. $\square$

An alternative proof of Theorem 1.1 may be found in [65].

THEOREM   1.2.   *Let* $T : Z \to Y$,   *and*   *let* $\mathscr{P} = \{P_0, \cdots, P_n\}$ *and*
$\mathscr{Q} = \{Q_0, \cdots, Q_n\}$ *be finite chains of orthogonal projections on* Z *and* Y, *respectively,*
with $P_n = I$ *and* $Q_0 = 0$. *Assume that* T *is of finite* $(\mathscr{P}, \mathscr{Q})$-*lower rank. Then the following*
are equivalent.

(i) T *is* $(\mathscr{P}, \mathscr{Q})$-*lower unique;*

(ii)   *the operators*

$$(I - Q_{i-1})TP_i \quad , i = 1, \ldots, n, \tag{1.6}$$

$$(I - Q_i)TP_i \quad , i = 1, \ldots, n-1, \tag{1.7}$$

all have the same rank;

(iii)   *the operators*

$$(I - Q_j)TP_i \, , 1 \leq i \leq j+1 \leq n,$$

all have the same rank;

(iv)   rank $(I - Q_{n-1})TP_1 = \ell(T; \mathscr{P}, \mathscr{Q})$.

Theorem IV.2.1 is the finite dimensional version of Theorem 2.1. Theorem 1.2 will follow as a special case of Theorem 4.1, and will not be used before. Note that the operators in (1.6) and (1.7) are exactly the ones which appear in the formula for the minimal lower rank in Theorem 1.1.

V.2. Arbitrary closed chains

2.1. Lower triangular parts and minimal rank extensions. Let Z and Y be separable Hilbert spaces over $\mathbb{C}$, and let $\mathscr{P}$ and $\mathscr{Q}$ denote closed chains of orthogonal projections on Z and Y, respectively. Let $t \mapsto P_t$ and $t \mapsto Q_t$ be parametrizations (see [41], Section V.1) of $\mathscr{P}$ and $\mathscr{Q}$ defined on the same closed subset Λ of the extended real line $\mathbb{R} \cup \{-\infty, \infty\}$. We shall refer to Λ as the *parameter set* of $\mathscr{P}$ and $\mathscr{Q}$. We do not assume that the chains $\mathscr{P}$ and $\mathscr{Q}$ contain the operators 0 and I. For a closed subset Δ of Λ we put $\mathscr{P}_\Delta = \{P_t \mid t \in \Delta\}$ and $\mathscr{Q}_\Delta = \{Q_t \mid t \in \Delta\}$. Note that also $\mathscr{P}_\Delta$ and $\mathscr{Q}_\Delta$ are closed chains on Z and Y, respectively, with parameter set Δ. A finite subset $\{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}, \alpha_n\}$ of Λ is called a *partition* of Λ if $\alpha_i < \alpha_j$ $(i < j)$, $\alpha_0 = \min \Lambda$

and $\alpha_n = \max \Lambda$. Obviously, any partition of Λ is a closed subset of Λ.

For a partition π of the parameter set Λ the operator $\mathscr{L}(T;\mathscr{P}_\pi,\mathscr{Q}_\pi)$ is defined by formula (1.1). By definition the *lower triangular part of T relative to the chains $\mathscr{P}$ and $\mathscr{Q}$* is the operator

$$\mathscr{L}(T;\mathscr{P},\mathscr{Q}) := \lim_\pi \mathscr{L}(T;\mathscr{P}_\pi,\mathscr{Q}_\pi), \qquad (2.1)$$

provided the right hand side exists. The limit in (2.1) should be understood as follows. For every $\epsilon > 0$ there exists a partition π_ϵ of Λ such that

$$\| \mathscr{L}(T;\mathscr{P},\mathscr{Q}) - \mathscr{L}(T;\mathscr{P}_\pi,\mathscr{Q}_\pi) \| < \epsilon$$

for all partitions π of Λ such that $\pi_\epsilon \subset \pi$. (Note that for finite chains the definitions in (1.1) and (2.1) coincide.) It is known ([41], Sections I.10 and III.7) that for a Hilbert-Schmidt operator T the operator $\mathscr{L}(T;\mathscr{P},\mathscr{Q})$ is well-defined when $\mathscr{P} = \mathscr{Q}$. This result also holds for $\mathscr{P} \neq \mathscr{Q}$. To see this note that $\mathscr{L}(T;\mathscr{P},\mathscr{Q})$ equals the (2,1) entry of the operator

$$\mathscr{L}\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix};\mathscr{P}\oplus\mathscr{Q},\mathscr{P}\oplus\mathscr{Q}), \qquad (2.2)$$

where $\mathscr{P}\oplus\mathscr{Q}$ is the chain on $Z\oplus Y$ given by $\mathscr{P}\oplus\mathscr{Q} = \{ P_t\oplus Q_t \mid t \in \Lambda \}$, and the operator (2.2) exists if T is Hilbert-Schmidt. In particular, if T has finite rank, then $\mathscr{L}(T;\mathscr{P},\mathscr{Q})$ exists. To illustrate the definition of the lower triangular part, let us consider the following example.

EXAMPLE 2.1. Let k be a Hilbert-Schmidt kernel defined on the square $[0,2]\times[-1,1]$, and let $K : L_2[-1,1] \rightarrow L_2[0,2]$ be the corresponding integral operator, i.e.,

$$(Kf)(t) = \int_{-1}^{1} k(t,s)f(s)ds, \quad 0 \leq t \leq 2, \ a.e..$$

Further, let P_t and Q_t be the orthogonal projections on $L_2[-1,1]$ and $L_2[0,2]$, respectively, defined by $P_t f = \chi_{[-1,t]}f$ and $Q_t f = \chi_{[0,t]}f$ ($t \in [0,1]$). (Here χ_E is the function which takes the values 1 on E and zero elsewhere.) Then, the lower part L of K relative to the chains $\mathscr{P} = \{ P_t \mid t \in [0,1] \}$ and $\mathscr{Q} = \{ Q_t \mid t \in [0,1] \}$ is the operator

$$(Lf)(t) = \int_{-1}^{\min\{t,1\}} k(t,s)f(s)ds, \quad 0 \leq t \leq 2.$$

The operator $T : Z \rightarrow Y$ is said to be of *finite $(\mathscr{P},\mathscr{Q})$-lower rank* if there exists a finite

rank operator $K : Z \to Y$ such that $\mathscr{R}(T;\mathscr{P},\mathscr{Q}) = \mathscr{R}(K;\mathscr{P},\mathscr{Q})$. (Such an operator is also called lower separable relative to the given chains of orthogonal projections, cf. [33] and also [48], [65] and [49].) In that case K is called a *finite rank extension* of $\mathscr{R}(T;\mathscr{P},\mathscr{Q})$. The *minimal $(\mathscr{P},\mathscr{Q})$-lower rank* of T is by definition the smallest possible rank of a finite rank extension of $\mathscr{R}(T;\mathscr{P},\mathscr{Q})$. In other words, the minimal $(\mathscr{P},\mathscr{Q})$-lower rank $\lambda(T;\mathscr{P},\mathscr{Q})$ of T is given by

$$\lambda(T;\mathscr{P},\mathscr{Q}) := \min\{ \text{rank } K \mid \mathscr{R}(T;\mathscr{P},\mathscr{Q}) = \mathscr{R}(K;\mathscr{P},\mathscr{Q})\}. \tag{2.3}$$

All K's for which the minimum in (2.3) is attained are called *minimal rank extensions* of $\mathscr{R}(T;\mathscr{P},\mathscr{Q})$. The operator T is called *$(\mathscr{P},\mathscr{Q})$-lower unique* when $\mathscr{R}(T;\mathscr{P},\mathscr{Q})$ has only one minimal rank extension.

The next lemma contains a few simple observations. The proof of this lemma is simple and therefore omitted.

LEMMA 2.2. *Let $T : Z \to Y$ be a $(\mathscr{P},\mathscr{Q})$-lower separable operator, and let Δ and Δ_1 be closed subsets of Λ with $\Delta \subset \Delta_1$, $\min \Delta = \min \Delta_1$ and $\max \Delta = \max \Delta_1$. Then*

(i) $\mathscr{R}(\mathscr{R}(T;\mathscr{P}_\Delta,\mathscr{Q}_\Delta);\mathscr{P}_{\Delta_1},\mathscr{Q}_{\Delta_1}) = \mathscr{R}(T;\mathscr{P}_{\Delta_1},\mathscr{Q}_{\Delta_1})$;

(ii) *if K is a finite rank extension of $\mathscr{R}(T;\mathscr{P}_\Delta,\mathscr{Q}_\Delta)$, then K is a finite rank extension of $\mathscr{R}(T;\mathscr{P}_{\Delta_1},\mathscr{Q}_{\Delta_1})$;*

(iii) $\lambda(T;\mathscr{P}_\Delta,\mathscr{Q}_\Delta) \geq \lambda(T;\mathscr{P}_{\Delta_1},\mathscr{Q}_{\Delta_1})$.

2.2. Reduction to finite chains and C-partitions. In the analysis of minimal rank extensions problems in the context of arbitrary chains reduction to finite chains plays an important role. An important tool for such a reduction is the following special class of partitions. Let $K : Z \to Y$ be a finite rank operator. Consider the maps

$$n_1 : \Lambda \to \{0,1,\cdots,\text{rank } K\}, n_2 : \Lambda \to \{0,1,\cdots,\text{rank } K\}$$

defined by

$$n_1(t) := \text{rank } (I-Q_t)K, n_2(t) := \text{rank } KP_t. \tag{2.4}$$

Note that n_1 is monotonically decreasing and n_2 is monotonically increasing. Furthermore, n_1 is right continuous and n_2 is left continuous. For instance, the right continuity of n_1 can be shown in the following way. Since $t \mapsto Q_t$ is continuous in the strong operator topology and K has finite rank, the map $t \mapsto (I-Q_t)K$ is continuous in the operator norm. Take $s \in \Lambda$. Observe that $(I-Q_s)K$ is injective on $[\text{Ker } (I-Q_s)K]^\perp$. So there exists

$t_0 \in \Lambda$, $t_0 > s$ such that $(I-Q_t)K$ is injective on $[\text{ Ker } (I-Q_s)K]^{\perp}$ for $s \leq t < t_0$. It follows that $n_1(t) \geq n_1(s)$ for $s \leq t < t_0$. On the other hand, we always have $n_1(t) \leq n_1(s)$ for $t \geq s$. Thus $n_1(t) = n_1(s)$ for $s \leq t < t_0$, which proves that n_1 is right continuous. We call a partition π of the parameter set Λ a *C-partition* for $\mathscr{P}$, $\mathscr{Q}$ and K if the functions n_1 and n_2 are continuous on $\Lambda \backslash \pi$. The C in C-partition stands for constant, referring to the fact that on the open intervals of $\Lambda \backslash \pi$ the functions n_1 and n_2 are constant. Note that for any finite rank operator K there exists a C-partition for $\mathscr{P}, \mathscr{Q}$ and K. If π is a C-partition for $\mathscr{P}, \mathscr{Q}$ and K, then any partition π' finer than π is again a C-partition for $\mathscr{P}, \mathscr{Q}$ and K. The intersection of two C-partitions for $\mathscr{P}, \mathscr{Q}$ and K is again a C-partition for $\mathscr{P}, \mathscr{Q}$ and K. Hence there is a coarsest C-partition π_0 for $\mathscr{P}, \mathscr{Q}$ and K, and it follows that the partition π of Λ is a C-partition for $\mathscr{P}, \mathscr{Q}$ and K if and only if $\pi_0 \subset \pi$.

V.3. Minimal lower rank

Let Z and Y be separable Hilbert spaces over $\mathbb{C}$, and let $\mathscr{P}$ and $\mathscr{Q}$ denote closed chains of orthogonal projections on Z and Y, respectively, with common parameter set Λ. For $\gamma \in \Lambda$ denote the predecessor of γ in Λ by γ^*, i.e., put $\gamma^* := \sup\{ \alpha \in \Lambda \mid \alpha < \gamma \}$ if $\gamma \neq \min \Lambda$ and $(\min \Lambda)^* := \min \Lambda$. The points where $\gamma^* \neq \gamma$ correspond to jumps in the chains. Let T be of finite $(\mathscr{P}, \mathscr{Q})$-lower rank. Consider a partition $\pi = \{\alpha_0, \cdots, \alpha_n\}$ of Λ with a *set of intermediate points* $\tau_\pi = \{\tau_1, \cdots, \tau_n\}$. This means that $\tau_\pi \subset \Lambda$, $\alpha_{i-1} < \tau_i \leq \alpha_i$ and $\tau_i^* \neq \alpha_i$ for $i = 1,...,n$. We define the number

$$\lambda(\pi, \tau_\pi, T) := \sum_{i=1}^{n} \text{rank } (I-Q_{\tau_i})TP_{\tau_i} - \sum_{i=1}^{n-1} \text{rank } (I-Q_{\alpha_i})TP_{\alpha_i}. \tag{3.1}$$

Note that for any partition π of Λ there exist sets of intermediate points τ_π with the properties mentioned above. Indeed, if $(\alpha_{i-1}, \alpha_i) \cap \Lambda = \varnothing$, then one can choose $\tau_i = \alpha_i$; otherwise one can take for τ_i any point in the intersection of (α_{i-1}, α_i) and Λ.

THEOREM 3.1. *Let* $T : Z \to Y$ *be of finite* $(\mathscr{P}, \mathscr{Q})$-lower rank, then

$$\lambda(T; \mathscr{P}, \mathscr{Q}) = \max_{\pi, \tau_\pi} \lambda(\pi, \tau_\pi, T), \tag{3.2}$$

where the maximum is taken over all possible partitions π of the parameter set of $\mathscr{P}$ and $\mathscr{Q}$ and over all sets τ_π of intermediate points corresponding to π. Moreover, the maximum in (3.2) is attained whenever π is a C-partition for $\mathscr{P}$, $\mathscr{Q}$ and some finite rank extension of $\lambda(T; \mathscr{P}, \mathscr{Q})$ and τ_π is some set of intermediate points corresponding to π.

THEOREM 3.2. *Let T be of finite* $(\mathscr{P},\mathscr{Q})$-*lower rank, and let K be a finite rank extension of* $\mathscr{A}(T;\mathscr{P},\mathscr{Q})$. *Then*

$$\mathscr{A}(T;\mathscr{P},\mathscr{Q}) = \min\{ \mathscr{A}(K;\mathscr{P}_\pi,\mathscr{Q}_\pi) \mid \pi \text{ partition of } \Lambda \} \tag{3.3}$$

and in (3.3) *the minimum is attained whenever* π *is a C-partition for* $\mathscr{P},\mathscr{Q}$ *and K.*

In order to prove Theorem 3.1 we need the following lemma.

LEMMA 3.3. *Let* $K : Z \rightarrow Y$ *be a finite rank operator, and let* $\pi = \{\alpha_0, \cdots, \alpha_n\}$ *be a C-partition for* $\mathscr{P},\mathscr{Q}$ *and K. Fix* $i, j \in \{1,...,n\}$. *Then*

$$\text{rank } (I - Q_{\alpha_{j-1}})KP_{\alpha_i} = \text{rank } (I - Q_\delta)KP_\gamma,$$

for $\gamma, \delta \in \Lambda$ *with* $\alpha_{i-1} < \gamma \le \alpha_i$ *and* $\alpha_{j-1} \le \delta < \alpha_j$.

Proof. Let $\gamma, \delta \in \Lambda$ be such that $\alpha_{i-1} < \gamma \le \alpha_i$ and $\alpha_{j-1} \le \delta < \alpha_j$. Write $K = FG$, where $F : X \rightarrow Y$ is injective, $G : Z \rightarrow X$ is surjective and X is a (finite dimensional) Hilbert space. Since π is a C-partition for $\mathscr{P},\mathscr{Q}$ and K, and G is surjective, it follows that rank $(I - Q_\delta)F = $ rank $(I - Q_{\alpha_{j-1}})F$ and hence dim Ker $(I - Q_\delta)F = $ dim Ker $(I - Q_{\alpha_{j-1}})F$. Since Ker $(I - Q_{\alpha_{j-1}})F \subset $ Ker $(I - Q_\delta)F$, we get that Ker $(I - Q_{\alpha_{j-1}})F = $ Ker $(I - Q_\delta)F$. Analogously, Im $GP_\gamma = $ Im GP_{α_i}. Hence

$$\text{rank } (I - Q_{\alpha_{j-1}})KP_{\alpha_i} = \text{rank } (I - Q_{\alpha_{j-1}})FGP_{\alpha_i} =$$

$$\text{dim Im } GP_{\alpha_i} - \text{dim (Im } GP_{\alpha_i} \cap \text{ Ker } (I - Q_{\alpha_{j-1}})F) =$$

$$\text{dim Im } GP_\gamma - \text{dim (Im } GP_\gamma \cap \text{ Ker } (I - Q_\delta)F) =$$

$$\text{rank } (I - Q_\delta)FGP_\gamma . \quad \square$$

Note that for a finite rank extension K of $\mathscr{A}(T;\mathscr{P},\mathscr{Q})$

$$(I - Q_\gamma)TP_\delta = (I - Q_\gamma)KP_\delta , \gamma \ge \delta. \tag{3.4}$$

Proof of Theorem 3.1. Let K be a minimal rank extension of $\mathscr{A}(T;\mathscr{P},\mathscr{Q})$, $\pi = \{\alpha_0, \cdots, \alpha_n\}$ a partition of Λ and $\tau_\pi = \{\tau_1, \cdots, \tau_n\}$ a set of intermediate points belonging to π. Since K is a finite rank extension of $\mathscr{A}(K;\mathscr{P}_\pi,\mathscr{Q}_\pi)$ and rank $K = \mathscr{A}(T;\mathscr{P},\mathscr{Q}) = \mathscr{A}(K;\mathscr{P},\mathscr{Q}) \le \mathscr{A}(K;\mathscr{P}_\pi,\mathscr{Q}_\pi)$, the operator K is a minimal rank extension of $\mathscr{A}(K;\mathscr{P}_\pi,\mathscr{Q}_\pi)$. So from Theorem 1.1 we get that

$$\text{rank } K = \sum_{i=1}^{n} \text{rank } (I - Q_{\alpha_{i-1}})KP_{\alpha_i} - \sum_{i=1}^{n-1} \text{rank } (I - Q_{\alpha_i})KP_{\alpha_i}. \tag{3.5}$$

Since $(I-Q_{T_i})(I-Q_{\alpha_{i-1}})KP_{\alpha_i}P_{T_i} = (I-Q_{T_i})TP_{T_i}$, we conclude that

$$\text{rank } (I-Q_{\alpha_{i-1}})KP_{\alpha_i} \geqq \text{rank } (I-Q_{T_i})TP_{T_i} \ , \ i = 1,\ldots,n.$$

Further, $(I-Q_{\alpha_i})KP_{\alpha_i} = (I-Q_{\alpha_i})TP_{\alpha_i}$ (because of (3.4)). So from (3.5) we deduce that

$$\mathscr{A}(T;\mathscr{P},\mathscr{Q}) = \text{rank } K \geqq \lambda(\pi,\tau_\pi,T).$$

Hence $\max\limits_{\pi,\tau_\pi} \lambda(\pi,\tau_\pi,T)$ exists and is majorized by $\mathscr{A}(T;\mathscr{P},\mathscr{Q})$.

Next, in addition to our earlier hypothesis, assume that π is a C-partition for $\mathscr{P},\mathscr{Q}$ and K. Then (by Lemma 3.3 and formula (3.4))

$$\text{rank } (I-Q_{\alpha_{i-1}})KP_{\alpha_i} = \text{rank } (I-Q_{T_i})KP_{T_i} = \text{rank } (I-Q_{T_i})TP_{T_i}. \tag{3.6}$$

Using (3.5) we get that $\mathscr{A}(T;\mathscr{P},\mathscr{Q}) = \text{rank } K = \lambda(\pi,\tau_\pi,T)$ for this particular π and τ_π. $\square$

Proof of Theorem 3.2. Let K be a finite rank extension of $\mathscr{A}(T;\mathscr{P},\mathscr{Q})$. Obviously, $\mathscr{A}(T;\mathscr{P},\mathscr{Q}) = \mathscr{A}(K;\mathscr{P},\mathscr{Q}) \leqq \mathscr{A}(K;\mathscr{P}_\pi,\mathscr{Q}_\pi)$ for any partition π of the parameter set Λ. Now assume that $\pi = \{\alpha_0, \cdots, \alpha_n\}$ is a C-partition for $\mathscr{P},\mathscr{Q}$ and K, and let $\tau_\pi = \{\tau_1, \cdots, \tau_n\}$ be a set of intermediate points belonging to π. Formula (3.4) allows us to replace the operator T in the right hand side of (3.1) by K. Since rank $(I-Q_{\alpha_{i-1}})KP_{\alpha_i} = \text{rank } (I-Q_{T_i})KP_{T_i}$ (Lemma 3.3), we get that

$$\mathscr{A}(T;\mathscr{P},\mathscr{Q}) \geqq \lambda(\pi,\tau_\pi,T) = \sum_{i=1}^{n} \text{rank } (I-Q_{\alpha_{i-1}})KP_{\alpha_i} - \sum_{i=1}^{n-1}(I-Q_{\alpha_i})KP_{\alpha_i}. \tag{3.7}$$

Using that the right hand side of (3.7) equals $\mathscr{A}(K;\mathscr{P}_\pi,\mathscr{Q}_\pi)$, we obtain $\mathscr{A}(T;\mathscr{P},\mathscr{Q}) \geqq \mathscr{A}(K;\mathscr{P}_\pi,\mathscr{Q}_\pi) \geqq \mathscr{A}(K;\mathscr{P},\mathscr{Q}) = \mathscr{A}(T;\mathscr{P},\mathscr{Q})$, which completes the proof. $\square$

V.4. Uniqueness

Recall that an operator T of finite $(\mathscr{P},\mathscr{Q})$-lower rank is called $(\mathscr{P},\mathscr{Q})$-*lower unique* if $\mathscr{A}(T;\mathscr{P},\mathscr{Q})$ has only one minimal rank extension. We have the following characterization of lower uniqueness. Put $\lambda_0 = \min \Lambda$ and $\lambda_1 = \max \Lambda$.

THEOREM 4.1. *Assume that $I \in \mathscr{P}$ and $0 \in \mathscr{Q}$. Let $T : Z \to Y$ be of finite $(\mathscr{P},\mathscr{Q})$-lower rank. Then the following are equivalent.*

(i)   *T is $(\mathscr{P},\mathscr{Q})$-lower unique;*

(ii)   *the operators*

$$(I - Q_\gamma \cdot) TP_\gamma \quad (\lambda_0 < \gamma \in \Lambda , \gamma^* \ne \lambda_1) \tag{4.1}$$

$$(I - Q_\gamma) TP_\gamma \quad (\lambda_0 < \gamma \in \Lambda , \gamma \ne \lambda_1) \tag{4.2}$$

all have the same rank, r say;

(iii) *the operators*

$$(I - Q_\sigma \cdot) TP_\gamma \quad (\lambda_0 < \gamma \in \Lambda, \gamma \le \sigma \in \Lambda, \sigma^* \ne \lambda_1)$$

all have the same rank.

Furthermore, in that case the minimal $(\mathscr{P}, \mathscr{D})$-lower rank $\ell(T;\mathscr{P},\mathscr{D})$ is equal to r.

First we need some preliminaries on seperable representations. Let $K : Z \to Y$ be a operator with finite rank. A pair of operators $\{F, G\}$, where $F : X \to Y$ and $G : Z \to X$, is called a *separable representation* of K if X is a finite dimensional inner product space and $K = FG$. The space X is called the *internal space* of the representation and its dimension the *order*. Two separable representations $\{F_1, G_1\}$ and $\{F_2, G_2\}$ with internal spaces X_1 and X_2, respectively, are called *similar* if there exists an invertible operator $S : X_1 \to X_2$ such that $F_1 = F_2 S$ and $G_1 = S^{-1} G_2$. Note that two similar representations have the same order. A separable representation $\{F, G\}$ of K is called a *minimal separable representation* if among all separable representations of K the order of $\{F, G\}$ is as small as possible. It is clear that any finite rank operator $K : Z \to Y$ has a (minimal) separable representation.

LEMMA 4.2. *Let $K : Z \to Y$ be an operator with finite rank, and let $\{F, G\}$ be a separable representation of K. Then the following are equivalent:*

(i) $\{F, G\}$ *is a minimal separable representation of K;*

(ii) *the order of $\{F, G\}$ is equal to the rank of K;*

(iii) F *is injective and G is surjective.*

Moreover, if $\{F_1, G_1\}$ and $\{F_2, G_2\}$ are minimal separable representations of K, then they are similar.

Proof. The implication (iii) $\Rightarrow$ (ii) is evident. To prove (ii) $\Rightarrow$ (i) note that the order of a separable representation is always greater than or equal to the rank of K. So when equality holds the separable representation must be minimal.

Next we show (i) $\Rightarrow$ (iii). It is easy to see that K can be written as $K = F'G'$ with $F' : X' \to Y$ an injective and $G' : Z \to X'$ a surjective operator. Take for instance $X' = \text{Im } K$, $F : X' \to Y$ the inclusion and $G = K : Z \to \text{Im } K$. So, $\dim X' = \text{rank } K$.

Suppose that $F : X \to Y$ is not injective, then $\dim X' = \operatorname{rank} K \leqq \operatorname{rank} F < \dim X$. So $\{F,G\}$ is not a minimal separable representation of K. In the same way the assumption that G is not surjective leads to a contradiction.

To prove the last part assume we have $K = F_1 G_1 = F_2 G_2$ with F_1, F_2 injective and G_1, G_2 surjective operators. Let $G_1^{(-1)}$ and $G_2^{(-1)}$ be right inverses of G_1 and G_2, respectively, and let $F_1^{(-1)}$ and $F_2^{(-1)}$ be left inverses of F_1 and F_2, respectively. Define $S := G_2 G_1^{(-1)} = F_2^{(-1)} F_1$ and $T := F_1^{(-1)} F_2 = G_1 G_2^{(-1)}$. Then $ST = F_2^{(-1)} F_1 G_1 G_2^{(-1)} = F_2^{(-1)} F_2 G_2 G_2^{(-1)} = I$. In the same way $TS = I$. Furthermore, $F_2 S = F_2 G_2 G_1^{(-1)} = F_1 G_1 G_1^{(-1)} = F_1$ and $G_1 = TG_2$, proving the proposition. $\square$

Proof of Theorem 4.1. We prove (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). In order to prove (i) $\Rightarrow$ (iii) suppose that there are $\gamma_i, \sigma_i \in \Lambda$ with $\lambda_0 < \gamma_i \leqq \sigma_i$, $\sigma_i^* \neq \lambda_1$ ($i = 1,2$) and $\sigma_1 \geqq \sigma_2$ such that $\operatorname{rank} (I - Q_{\sigma_1 \cdot}) TP_{\gamma_1} \neq \operatorname{rank} (I - Q_{\sigma_2 \cdot}) TP_{\gamma_2}$. Then $\operatorname{rank} (I - Q_{\sigma_1 \cdot}) TP_{\gamma_1} \neq \operatorname{rank} (I - Q_{\sigma_1 \cdot}) TP_{\gamma_2}$ or $\operatorname{rank} (I - Q_{\sigma_1 \cdot}) TP_{\gamma_2} \neq \operatorname{rank} (I - Q_{\sigma_2 \cdot}) TP_{\gamma_2}$. So without loss of generality we may assume that either $\sigma_1 = \sigma_2$ or $\gamma_1 = \gamma_2$. We shall obtain a contradiction for the second possibility; for the first one the proof is similar.

So let us assume that $\gamma_1 = \gamma_2 =: \gamma$ and hence $\sigma_2 < \sigma_1$. Note that $\gamma \leqq \sigma_1^*$. Let A be a minimal rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$. Then $\operatorname{rank} (I - Q_{\sigma_1 \cdot}) AP_\gamma < \operatorname{rank} (I - Q_{\sigma_2 \cdot}) AP_\gamma$, and hence

$$\operatorname{rank} (I - Q_{\sigma_1 \cdot}) AP_\gamma < \operatorname{rank} AP_\gamma.$$

Put $Z_1 = P_\gamma Z$, $Z_2 = (P_{\sigma_1 \cdot} - P_\gamma) Z$, $Z_3 = (I - P_{\sigma_1 \cdot}) Z$ and $Y_1 = Q_\gamma$, $Y_2 = (Q_{\sigma_1 \cdot} - Q_\gamma) Y$, $Y_3 = (I - Q_{\sigma_1 \cdot}) Y$. Note that Z_2 and Y_2 may be trivial spaces. In fact, this happens if and only if $\gamma = \sigma_1^*$. All other spaces are nonzero. Writing $A = (A_{ij})_{i,j=1}^3$: $Z_1 \oplus Z_2 \oplus Z_3 \to Y_1 \oplus Y_2 \oplus Y_3$, we have that $\operatorname{rank} A_{31} < \operatorname{rank} (\operatorname{col}(A_{j1})_{j=1}^3)$. So there exists a (nonzero) vector $\phi \in Z_1$ such that $\operatorname{col}(A_{j1})_{j=1}^3 \phi \neq 0$ and $A_{31} \phi = 0$. Let ψ be a nonzero vector in Z_3, and define $C : Z \to Z$ by $C\psi := \phi$ and $Cv := 0$ for $v \in M$, where M is a closed linear suspace of Z with $Z_1 \oplus Z_2 \subset M$ and $Z_3 \subset \operatorname{span} \{\psi\} \oplus M$. Put $A' := A(I + C)$. Since $I + C$ is invertible (with inverse $I - C$), $\operatorname{rank} A = \operatorname{rank} A'$. Furthermore, $\mathcal{A}(A;\mathcal{P},\mathcal{Q}) = \mathcal{A}(A';\mathcal{P},\mathcal{Q})$, which one obtains from Lemma 2.2(i) with $\Delta = \{\lambda_0, \gamma, \sigma_1^*, \lambda_1\} \subset \Lambda$. Thus $\mathcal{A}(T;\mathcal{P},\mathcal{Q}) = \mathcal{A}(A';\mathcal{P},\mathcal{Q})$. So A' is a minimal rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$ which is different from A (because $AC \neq 0$). Contradiction.

In order to prove (iii) $\Rightarrow$ (ii) note that the set of operators considered in (ii) is a subset of the set of operators considered in (iii). To see this, take $\sigma = \gamma$ if $\lambda_0 < \gamma \in \Lambda$ and

$\gamma^* \neq \lambda_1$ (case (4.1)) and take $\sigma = \inf\{\alpha > \gamma \mid \alpha \in \Lambda\}$ if $\lambda_0 < \gamma \in \Lambda$ and $\gamma \neq \lambda_1$ (case (4.2)). Note that in the latter case $\sigma^* = \gamma \neq \lambda_1$.

We end with (ii) $\Rightarrow$ (i). Let A and A' be minimal rank extensions of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$, and let $\{F,G\}$ and $\{F',G'\}$ be minimal separable representations for A and A', respectively. Assume the ranks of the operators in (ii) are all equal to r. Theorem 3.1 yields $r = \lambda(T;\mathcal{P},\mathcal{Q}) = \text{rank } A = \text{rank } A'$. Take $\lambda_0 < \gamma \in \Lambda$, and assume that $\gamma^* \neq \lambda_1$. Put $T^{(\gamma)} = (I - Q_\gamma{}^{\cdot})TP_\gamma$. Note that

$$T^{(\gamma)} = (I - Q_\gamma{}^{\cdot})FGP_\gamma = (I - Q_\gamma{}^{\cdot})F'G'P_\gamma$$

because of formula (3.4). Since rank $T^{(\gamma)} = r$ equals the orders of the separable representations $\{(I - Q_\gamma{}^{\cdot})F, GP_\gamma\}$ and $\{(I - Q_\gamma{}^{\cdot})F', G'P_\gamma\}$ of $T^{(\gamma)}$, it follows that they are minimal separable representations of $T^{(\gamma)}$. According to Lemma 4.2, this implies that

$$(I - Q_\gamma{}^{\cdot})FS(\gamma) = (I - Q_\gamma{}^{\cdot})F' , \quad S(\gamma)^{-1}GP_\gamma = G'P_\gamma$$

for some invertible $S(\gamma)$. Furthermore, GP_γ and $G'P_\gamma$ are surjective (Lemma 4.2 (iii)). We shall prove that $S(\gamma)$ does not depend on the choice of γ.

Take $\gamma_1 > \lambda_0$ and $\gamma_2 > \lambda_0$ in Λ. Assume that $\gamma_1 < \gamma_2$ and $\gamma_2^* \neq \lambda_1$ (and hence $\gamma_1^* \neq \lambda_1$). From $\gamma_1 < \gamma_2$ it follows that

$$S(\gamma_2)^{-1}GP_{\gamma_1} = S(\gamma_2)^{-1}GP_{\gamma_2}P_{\gamma_1} = G'P_{\gamma_2}P_{\gamma_1} = G'P_{\gamma_1} = S(\gamma_1)^{-1}GP_{\gamma_1}.$$

Thus $(S(\gamma_2)^{-1} - S(\gamma_1)^{-1})GP_{\gamma_1} = 0$. But GP_{γ_1} is surjective. Hence $S(\gamma_2) = S(\gamma_1)$ and $S := S(\gamma)$ does not depend on γ.

We have now proved that

$$(I - Q_\gamma{}^{\cdot})FS = (I - Q_\gamma{}^{\cdot})F' , \quad S^{-1}GP_\gamma = G'P_\gamma \quad (\lambda_0 < \gamma \in \Lambda, \gamma^* \neq \lambda_1). \quad (4.3)$$

This implies that $S^{-1}G = G'$. For the case when $\lambda_1^* \neq \lambda_1$ this is evident. Assume $\lambda_1^* = \lambda_1$. Then there exists a sequence $\gamma_1, \gamma_2, \cdots$ in Λ such that $\gamma_n < \lambda_1$ $(n = 1,2,..)$ and $\gamma_n \uparrow \lambda_1$. But then, since $I \in \mathcal{P}$,

$$S^{-1}G = \lim_{n \to \infty} S^{-1}GP_{\gamma_n} = \lim_{n \to \infty} G'P_{\gamma_n} = G'.$$

Next we prove that $FS = F'$. Put $\alpha := \inf\{\gamma \in \Lambda \mid \gamma > \lambda_0\}$. Obviously, $\alpha^* = \lambda_0$. If $\alpha > \lambda_0$, then (4.3) with $\gamma = \alpha$ implies that $FS = F'$. Assume that $\alpha = \lambda_0$. Then there exists a sequence $\gamma_1, \gamma_2, \cdots$ in Λ such that $\gamma_n > \lambda_0$ $(n = 1,2,..)$ and $\gamma_n \downarrow \lambda_0$, which

implies that

$$FS = \lim_{n \to \infty} (I - Q_{\gamma_n} \cdot) FS = \lim_{n \to \infty} (I - Q_{\gamma_n} \cdot) F' = F'.$$

Here we use that $0 \in \mathcal{Q}$.

Hence the representations $\{F, G\}$ and $\{F', G'\}$ are similar. Thus $A = A'$, and it follows that T is $(\mathcal{P}, \mathcal{Q})$-lower unique. $\square$

When $I \notin \mathcal{P}$ or $0 \notin \mathcal{Q}$ one easily sees that T is $(\mathcal{P}, \mathcal{Q})$-lower unique if and only if $\mathcal{A}(T; \mathcal{P}, \mathcal{Q}) = 0$. This case corresponds to the "strict lower triangular case".

We end this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. The equivalence of (i), (ii) and (iii) folows directly from Theorem 4.1, as well as the implication (iii) $\Rightarrow$ (iv). The implication (iv) $\Rightarrow$ (iii) is trivial (cf. the proof of Theorem IV.2.1). $\square$

V.5. Construction

For the construction of all minimal rank extension of a given lower triangular part we need the following three theorems. The first two compare minimal rank extensions corresponding to a part of the chain to minimal rank extensions corresponding to the full chain (cf. Corollary IV.1.2). The third theorem reduces the problem of construction to the case of finite chains.

THEOREM 5.1. *Let* $\gamma \in \Lambda$, $\gamma \neq \min \Lambda$, *and put* $\Delta = (-\infty, \gamma] \cap \Lambda$. *If* K *is a minimal rank extension of* $\mathcal{A}(T; \mathcal{P}, \mathcal{Q})$, *then* KP_γ *is a minimal rank extension of* $\mathcal{A}(T; \mathcal{P}_\Delta, \mathcal{Q}_\Delta)$. *Conversely, if* K' *is a minimal rank extension of* $\mathcal{A}(T; \mathcal{P}_\Delta, \mathcal{Q}_\Delta)$, *then there exists a minimal rank extension* K *of* $\mathcal{A}(T; \mathcal{P}, \mathcal{Q})$ *such that* $K'P_\gamma = KP_\gamma$.

Proof. Let K be a minimal rank extension of $\mathcal{A}(T; \mathcal{P}, \mathcal{Q})$. Let $\pi = \{\alpha_0, \cdots, \alpha_n\}$ be a C-partition for $\mathcal{P}, \mathcal{Q}$ and K containing γ. Note that $(I - Q_{\alpha_0}) \mathcal{A}(T; \mathcal{P}, \mathcal{Q}) = \mathcal{A}(T; \mathcal{P}, \mathcal{Q})$. So $(I - Q_{\alpha_0}) K$ is a finite rank extension of $\mathcal{A}(T; \mathcal{P}, \mathcal{Q})$. It follows that rank $(I - Q_{\alpha_0}) K =$ rank K, because K is a minimal rank extension. This implies that rank $(I - Q_{\alpha_0}) KP_\gamma =$ rank KP_γ. Suppose that $\gamma = \alpha_{i_0}$. Put $\pi_1 = \{\alpha_0, \alpha_{i_0}, \alpha_{i_0+1}, \cdots, \alpha_n\}$. Since rank $K = \rho(K; \mathcal{P}, \mathcal{Q}) \leq \rho(K; \mathcal{P}_{\pi_1}, \mathcal{Q}_{\pi_1})$, the operator K is a minimal rank extension of $\mathcal{A}(K; \mathcal{P}_{\pi_1}, \mathcal{Q}_{\pi_1})$. So rank $K = \rho(K; \mathcal{P}_{\pi_1}, \mathcal{Q}_{\pi_1})$. On the other hand, since K is a minimal rank extension of $\mathcal{A}(T; \mathcal{P}, \mathcal{Q})$, we can use Theorem 3.2 to show that rank $K = \rho(K; \mathcal{P}_\pi, \mathcal{Q}_\pi)$. Thus $\rho(K; \mathcal{P}_\pi, \mathcal{Q}_\pi) - \rho(K; \mathcal{P}_{\pi_1}, \mathcal{Q}_{\pi_1}) = 0$. Using Theorem 1.1 this identity

yields

$$\text{rank } KP_{\alpha_{i_0}} = \text{rank } (I - Q_{\alpha_0})KP_{\alpha_{i_0}} =$$

$$\sum_{p=1}^{i_0} \text{rank } (I - Q_{\alpha_{p-1}})KP_{\alpha_p} + \sum_{p=1}^{i_0-1} \text{rank } (I - Q_{\alpha_p})KP_{\alpha_p} = \prec(K; \mathscr{P}_{\pi_2}, \mathscr{Q}_{\pi_2}),$$

where $\pi_2 = \{\alpha_0, \cdots, \alpha_{i_0}\}$. Since π_2 is a C-partition for $P_\Delta, \mathscr{Q}_\Delta$ and K, and K is a finite rank extension of $\mathscr{S}(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta)$, Theorem 3.2 implies $\prec(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta) = \prec(K; \mathscr{P}_{\pi_2}, \mathscr{Q}_{\pi_2})$. So rank $KP_\gamma = \prec(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta)$, and KP_γ is a minimal rank extension of $\mathscr{S}(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta)$.

Let K' be a minimal rank extension of $\mathscr{S}(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta)$. Since $\mathscr{S}(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta)P_\gamma = \mathscr{S}(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta)$, the operator $K'P_\gamma$ is also a minimal rank extension of $\mathscr{S}(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta)$. Choose an arbitrary finite rank extension A of $\mathscr{S}(T; \mathscr{P}, \mathscr{Q})$, and put $A' = K'P_\gamma + A(I - P_\gamma)$. Then A' is also a finite rank extension of $\mathscr{S}(T; \mathscr{P}, \mathscr{Q})$ and $A'P_\gamma = K'P_\gamma$. The latter identity implies rank $A'P_\gamma = \prec(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta)$. Let $\pi = \{\alpha_0, \cdots, \alpha_n\}$ be a C-partition for $\mathscr{P}, \mathscr{Q}$ and A' containing γ. Let us say $\gamma = \alpha_m$. Put $\hat{\mathscr{P}} = \{0, P_{\alpha_m}, P_{\alpha_{m+1}}, \cdots, P_{\alpha_n}\}$ and $\hat{\mathscr{Q}} = \{0, Q_{\alpha_m}, Q_{\alpha_{m+1}}, \cdots, Q_{\alpha_n}\}$. Let K be a minimal rank extension of $\mathscr{S}(A'; \hat{\mathscr{P}}, \hat{\mathscr{Q}})$. Then K is a finite rank extension of $\mathscr{S}(T; \mathscr{P}, \mathscr{Q})$, and we shall prove that K has the desired properties.

From rank $K = \prec(A'; \hat{\mathscr{P}}, \hat{\mathscr{Q}})$ and Theorem 1.1 it follows that

$$\text{rank } K = \text{rank } A'P_{\alpha_m} + \sum_{i=m+1}^{n} \text{rank } (I - Q_{\alpha_{i-1}})A'P_{\alpha_i} \qquad (5.1)$$

$$- \sum_{i=m}^{n-1} \text{rank } (I - Q_{\alpha_i})A'P_{\alpha_i}.$$

Since $\pi_1 := \{\alpha_0, \cdots, \alpha_m\}$ is a C-partition for $\mathscr{P}_\Delta, \mathscr{Q}_\Delta$ and $A'P_{\alpha_m}$ ($= K'P_{\alpha_m}$), Theorem 3.2 implies that $\prec(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta) = \prec(A'P_{\alpha_m}; \mathscr{P}_{\pi_1}, \mathscr{Q}_{\pi_1})$. So rank $A'P_{\alpha_m} = \prec(T; \mathscr{P}_\Delta, \mathscr{Q}_\Delta) = \prec(A'P_{\alpha_m}; \mathscr{P}_{\pi_1}, \mathscr{Q}_{\pi_1})$. Now use Theorem 1.1 and formula (5.1). We obtain rank $K = \prec(A'; \mathscr{P}_\pi, \mathscr{Q}_\pi) = \prec(T; \mathscr{P}, \mathscr{Q})$, where the last equality follows from Theorem 3.2. Thus K is a minimal rank extension of $\mathscr{S}(T; \mathscr{P}, \mathscr{Q})$. Further, $KP_\gamma = \mathscr{S}(K; \hat{\mathscr{P}}, \hat{\mathscr{Q}})P_\gamma = \mathscr{S}(A'; \hat{\mathscr{P}}, \hat{\mathscr{Q}})P_\gamma = K'P_\gamma$. $\square$

Analogously one proves the following theorem.

THEOREM 5.2. *Let* $\gamma \in \Lambda$, *with* $\gamma \neq \max\Lambda$, *and denote* $\Delta = [\gamma, \infty) \cap \Lambda$. *If K is a minimal rank extension of* $\mathscr{S}(T; \mathscr{P}, \mathscr{Q})$, *then* $(I - Q_\gamma)K$ *is a minimal rank extension of*

$\mathcal{A}(T;\mathcal{P}_\Delta,\mathcal{Q}_\Delta)$. *Conversely, if K' is a minimal rank extension of $\mathcal{A}(T;\mathcal{P}_\Delta,\mathcal{Q}_\Delta)$, then there exists a minimal rank extension K of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$ such that $(I-Q_\gamma)K' = (I-Q_\gamma)K$.*

Let $T : Z \rightarrow Y$ be of finite $(\mathcal{P},\mathcal{Q})$-lower rank. Note that Theorem 5.1 shows that for a minimal rank extension K of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$ the rank of KP_γ does not depend on the particular choice of K. In fact, rank $KP_\gamma = \iota(T;\mathcal{P}_\Delta,\mathcal{Q}_\Delta)$, where $\Delta = (-\infty,\gamma]\cap\Lambda$. From Theorem 5.2 we obtain rank $(I-Q_\gamma)K = \iota(T;\mathcal{P}_{\Lambda_1},\mathcal{Q}_{\Lambda_1})$, where $\Lambda_1 = [\gamma,\infty)\cap\Lambda$, for any minimal rank extension K of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$. These observations lead to the following corollary.

COROLLARY 5.3. *Let $T : Z \rightarrow Y$ be of finite $(\mathcal{P},\mathcal{Q})$-lower rank. If π a C-partition for $\mathcal{P},\mathcal{Q}$ and some minimal rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$, then π is a C-partition for $\mathcal{P},\mathcal{Q}$ and any other minimal rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$.*

The following theorem shows that the problem to find all minimal rank extensions of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$ is in fact a finite chain problem.

THEOREM 5.4. *Let $T : Z \rightarrow Y$ be of finite $(\mathcal{P},\mathcal{Q})$-lower rank. Let K be a finite rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$, and let $\pi = \{\alpha_0, \cdot\cdot\cdot, \alpha_n\}$ be a C-partition for $\mathcal{P},\mathcal{Q}$ and K. Then any minimal rank extension of $\mathcal{A}(K;\mathcal{P}_\pi,\mathcal{Q}_\pi)$ is a minimal rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$, and, conversely, any minimal rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$ is a minimal rank extension of $\mathcal{A}(K;\mathcal{P}_\pi,\mathcal{Q}_\pi)$. In particular,*

$$(I-Q_{\alpha_{i-1}})KP_{\alpha_i} = (I-Q_{\alpha_{i-1}})K'P_{\alpha_i} \quad , i = 1,...,n \quad , \tag{5.2}$$

for any minimal rank extension K' of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$.

Proof. Let K be a finite rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$ and $\pi = \{\alpha_0, \cdot\cdot\cdot, \alpha_n\}$ a C-partition for $\mathcal{P},\mathcal{Q}$ and K. From Lemma 2.2 it is clear that any finite rank extension of $\mathcal{A}(K;\mathcal{P}_\pi,\mathcal{Q}_\pi)$ is a finite rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$. Furthermore, Theorem 3.2 yields $\iota(T;\mathcal{P},\mathcal{Q}) = \iota(K;\mathcal{P}_\pi,\mathcal{Q}_\pi)$. From these two observations it follows that any minimal rank extension of $\mathcal{A}(K;\mathcal{P}_\pi,\mathcal{Q}_\pi)$ is a minimal rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$.

To prove the converse, let K' be a minimal rank extension of $\mathcal{A}(T;\mathcal{P},\mathcal{Q})$. Take $i \in \{1,...,n\}$, and put $\Delta = [\alpha_{i-1},\alpha_i] \cap \Lambda$. Denote $P'_t = P_t \mid \text{Im } P_{\alpha_i}$ and $Q'_t = Q_t \mid \text{Ker } Q_{\alpha_{i-1}}$ for $t \in \Delta$. Let $\mathcal{P}'$ and $\mathcal{Q}'$ be the chains on $\text{Im } P_{\alpha_i}$ and $\text{Ker } Q_{\alpha_{i-1}}$, respectively, given by

$$\mathcal{P}' = \{ P'_t \mid t \in \Delta\} , \mathcal{Q}' = \{ Q'_t \mid t \in \Delta\} . \tag{5.3}$$

Note that $I \in \mathcal{P}'$ and $0 \in \mathcal{Q}'$. Since π is a C-partition of Λ for $\mathcal{P}$, $\mathcal{Q}$ and K we have (by

Lemma 3.3 and formula (3.4)) that, for $\alpha_{i-1} < \gamma \in \Delta$, $\gamma^* \neq \alpha_i$,

$$\text{rank } (I - Q_{\alpha_{i-1}})KP_{\alpha_i} = \text{rank } (I - Q_{\gamma \cdot})KP_{\gamma} = \text{rank } (I - Q_{\gamma \cdot})TP_{\gamma}.$$

Similarly, for $\alpha_{i-1} < \gamma \in \Delta$, $\gamma \neq \alpha_i$

$$\text{rank } (I - Q_{\alpha_{i-1}})KP_{\alpha_i} = \text{rank } (I - Q_{\gamma})KP_{\gamma} = \text{rank } (I - Q_{\gamma})TP_{\gamma}.$$

Using Theorem 4.1. one sees that $T' = (I - Q_{\alpha_{i-1}})TP_{\alpha_i}$: Im $P_{\alpha_i} \rightarrow$ Ker $Q_{\alpha_{i-1}}$ is $(\mathscr{P}', \mathscr{Q}')$-lower unique and $(I - Q_{\alpha_{i-1}})KP_{\alpha_i}$ is a minimal rank extension of $\mathscr{R}(T; \mathscr{P}', \mathscr{Q}')$. From Theorems 5.1 and 5.2 it follows that $(I - Q_{\alpha_{i-1}})K'P_{\alpha_i}$ is also a minimal rank exten-sion of $\mathscr{R}(T'; \mathscr{P}', \mathscr{Q}')$. But then, because of the $(\mathscr{P}', \mathscr{Q}')$-lower uniqueness, (5.2) must hold true. Since (5.2) holds, K' is a finite rank extension of $\mathscr{R}(K; \mathscr{P}_\pi, \mathscr{Q}_\pi)$. With rank $K' = \mathscr{A}(T; \mathscr{P}, \mathscr{Q}) = \mathscr{A}(K; \mathscr{P}_\pi, \mathscr{Q}_\pi)$ it follows that K' is a minimal rank extension of $\mathscr{R}(K; \mathscr{P}_\pi, \mathscr{Q}_\pi)$. $\square$

The preceding theorems yield a procedure to construct minimal rank extensions. This procedure consists of three basic elements.

(I). **The 2×2 case.** Let $T_{11}:Z_1 \rightarrow Y_1$, $T_{21}:Z_1 \rightarrow Y_2$ and $T_{22}:Z_2 \rightarrow Y_2$ be given finite rank operators. We have to construct an operator $T_{12}:Z_2 \rightarrow Y_1$ such that rank $(T_{ij})_{i,j=1}^2$ is as small as possible. This is done as follows. Let $X_1 = \{ x \in Z_2 \mid T_{22}x \in \text{Im } T_{21} \}$ and let X_2 be a direct complement of X_1 in Z_2. Write

$$T_{22} = [T_{22}^a \quad T_{22}^b] : X_1 \dotplus X_2 \rightarrow Y_2.$$

Note that T_{22}^b is injective. Since Im $T_{22}^a \subset$ Im T_{21}, there exists an operator $S : X_1 \rightarrow Z_1$ such that $T_{21}S = T_{22}^a$. Put

$$T_{12} = [T_{11}S \quad \Sigma] : X_1 \dotplus X_2 \rightarrow Y_1,$$

where Σ is an arbitrary operator acting $X_2 \rightarrow Y$. Then the extension $(T_{ij})_{i,j=1}^2$ has rank equal to

$$\text{rank } \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} + \text{rank } \begin{bmatrix} T_{21} & T_{22} \end{bmatrix} - \text{rank } T_{21}$$

which by Theorem 1.1 is the smallest possible rank.

(II). **The case of finite chains.** Let $\mathscr{P} = \{P_0, \cdots, P_n\}$ and $\mathscr{Q} = \{Q_0, \cdots, Q_n\}$ be finite chains of projections on Z and Y, respectively. For $\nu = 1, \dots, n$ define the chain $\mathscr{P}^{(\nu)}$ on Z by $\mathscr{P}^{(\nu)} = \{P_0, \cdots, P_\nu\}$ and the chain $\mathscr{Q}^{(\nu)}$ on Y by $\mathscr{Q}^{(\nu)} = \{Q_0, \cdots, Q_\nu\}$. Let

$T : Z \to Y$ be of finite $(\mathscr{P}, \mathscr{Q})$-lower rank. One can construct a minimal rank extension of $\mathscr{A}(T; \mathscr{P}, \mathscr{Q})$ by induction as follows. We start with a minimal rank extension of $\mathscr{A}(T; \mathscr{P}^{(1)}, \mathscr{Q}^{(1)})$, for instance $(I - Q_0)TP_1$. Next, suppose that a minimal rank extension A of $\mathscr{A}(T; \mathscr{P}^{(\nu-1)}, \mathscr{Q}^{(\nu-1)})$ has been constructed. Write

$$A \mid \operatorname{Im} P_{\nu-1} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} : \operatorname{Im} P_{\nu-1} \to \operatorname{Im} Q_{\nu-1} \oplus \operatorname{Ker} Q_{\nu-1}.$$

Consider the operator

$$A' = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} : \operatorname{Im} P_{\nu-1} \oplus \operatorname{Ker} P_{\nu-1} \to \operatorname{Im} Q_{\nu-1} \oplus \operatorname{Ker} Q_{\nu-1},$$

where $A_{22} = (I - Q_{\nu-1})TP_\nu \mid \operatorname{Ker} P_{\nu-1} : \operatorname{Ker} P_{\nu-1} \to \operatorname{Ker} Q_{\nu-1}$. Using the 2×2 case we can construct an operator $A_{21} : \operatorname{Ker} P_{\nu-1} \to \operatorname{Im} Q_{\nu-1}$ such that $B := (A_{ij})_{i,j=1}^2$ has the lowest possible rank. Such a B is a minimal rank extension of $\mathscr{A}(T; \mathscr{P}^{(\nu)}, \mathscr{Q}^{(\nu)})$ (use Theorem 1.1). When $\nu = n$ a minimal rank extension for $\mathscr{A}(T; \mathscr{P}, \mathscr{Q})$ is obtained.

(III). The general case. When $\mathscr{P}$ and $\mathscr{Q}$ are arbitray chains the procedure is as follows. Let T be of finite $(\mathscr{P}, \mathscr{Q})$-lower rank. Take any finite rank K extension of $\mathscr{A}(T; \mathscr{P}, \mathscr{Q})$. Determine a C-partition π for $\mathscr{P}, \mathscr{Q}$ and K. Next make a minimal rank extension A for $\mathscr{A}(K; \mathscr{P}_\pi, \mathscr{Q}_\pi)$, using the construction outlined under (II). Theorem 5.4 yields that A is a minimal rank extension for $\mathscr{A}(T; \mathscr{P}, \mathscr{Q})$.

The minimal extension procedure sketched above also solves the problem of finding all minimal rank extensions, which was posed in [35]. Indeed, Theorem 5.4 shows that all minimal rank extensions of $\mathscr{A}(T; \mathscr{P}, \mathscr{Q})$ can be found as minimal rank extensions of $\mathscr{A}(K; \mathscr{P}_\pi, \mathscr{Q}_\pi)$, where K is some finite rank extension for $\mathscr{A}(T; \mathscr{P}, \mathscr{Q})$ and π is a C-partition for $\mathscr{P}, \mathscr{Q}$ and K. But for finite chains the problem to construct all minimal rank extensions reduces to the 2×2 case (use the explanation in (II) and Theorems 5.1 and 5.2). It is not hard to see that in the 2×2 case the construction given under (I) yields all minimal rank extensions.

V.6. Semi-infinite operator matrices

Let $T = (T_{ij})_{i,j=1}^{\infty} : l_2(Z) \to l_2(Y)$ be a (bounded linear) operator. Here $l_2(Z)$ (resp. $l_2(Y)$) stands for the space of all square summable sequences with elements in Z (resp. Y). The spaces Z and Y are given separable Hilbert spaces. An operator

$A = (A_{ij})_{i,j=1}^{\infty} : l_2(Z) \to l_2(Y)$ is called a *finite rank extension of the lower triangular part of* T if A has finite rank and $A_{ij} = T_{ij}$ for $j \leq i$. If the lower triangular part of T has a finite rank extension, then T is called *of finite lower rank*. A finite rank extension A of the lower triangular part of T is called a *minimal rank extension* if among all finite rank extensions of the lower triangular part of T the rank of A is as small as possible. The *minimal lower rank* of an operator matrix of finite lower rank is by definition the smallest possible rank of a finite rank extension of the lower triangular part of T. The operator matrix T is called *lower unique* if the lower triangular part of T has precisely one minimal rank extension.

We specify the minimal rank extension construction (given at the end of the previous section) for semi-infinite operator matrices. Let

$$T = (T_{ij})_{i,j=1}^{\infty} : l_2(Z) \to l_2(Y)$$

be of finite lower rank, and let $K = \left[K_{ij} \right]_{i,j=1}^{\infty}$ be an arbitrary finite rank extension of T. Consider for $p = 1,2,...$ the numbers $r_{(p)} = \mathrm{rank} \ (K_{ij})_{i=p,j=1}^{\infty \ \infty}$ and $r^{(p)} = \mathrm{rank} \ (K_{ij})_{i=1,j=1}^{\infty \ p}$. Choose an $n \in \mathbb{N}$ such that $r_{(n)} = r_{(p)}$ and $r^{(n)} = r^{(p)}$ for $p \geq n$. Define L_{ij}, $1 \leq j \leq i \leq n$, by setting $L_{ij} = K_{ij} \ (= T_{ij})$ for $1 \leq j \leq i \leq n-1$, $L_{nj} = \mathrm{col}(K_{ij})_{i=n}^{\infty}$ for $1 \leq j \leq n-1$ and $L_{nn} = (K_{ij})_{i,j=n}^{\infty}$. Consider

$$L = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdot & & \cdot \\ \cdot & & \cdot \ \cdot & & \cdot \\ \cdot & & & \cdot \ \cdot \ \cdot & \\ \cdot & & & & 0 \\ L_{n1} & L_{n2} & \cdots & & L_{nn} \end{bmatrix} : Z \oplus \cdots \oplus Z \oplus l_2(Z) \to Y \oplus \cdots \oplus Y \oplus l_2(Y).$$

Theorem 5.4 implies that any minimal rank extension of L (relative to the finite chain suggested by the block form of L) is a minimal rank extension of T. Conversely, any minimal rank extension of T is a minimal rank extension of L. In order to see this define the chain $\mathscr{P} = \{P_0, P_1, P_2, \cdots, P_\infty\}$ on $l_2(Z)$ as follows: let $P_0 = 0$, and P_i be the ortogonal projection upon the first i block coordinates ($i = 1,2,...$). Further, put $P_\infty = I$. Define on $l_2(Y)$ the chain $\mathscr{Q} = \{Q_0, Q_1, Q_2, \cdots, Q_\infty\}$ analogously. Then, with this choice of $\mathscr{P}$ and $\mathscr{Q}$, the definition of minimal rank extension of Section V.2 coincides with the definition given above. Furthermore, the partition $\pi = \{0,1,2, \cdots, n, \infty\}$ is a C-partition for $\mathscr{P}, \mathscr{Q}$

and K. With the observation that $L = \mathscr{A}(K;\mathscr{P}_\pi,\mathscr{Q}_\pi)$ the assertions made in this paragraph follow immediately from Theorem 5.4.

Using the chains $\mathscr{P}$ and $\mathscr{Q}$ defined above we obtain from Theorem 4.1 the following characterization of lower uniqueness.

COROLLARY 6.1. *Let* $T = (T_{ij})_{i,j=1}^\infty : l_2(Z) \to l_2(Y)$ *be of finite lower rank. Then* T *is lower unique if and only if the operators*

$$(T_{ij})_{i=p,j=1}^{\infty\ p}, \quad p = 1,2,\dots, \tag{6.1}$$

$$(T_{ij})_{i=p,j=1}^{\infty\ p-1}, \quad p = 2,3,\dots \tag{6.2}$$

all have the same rank, r say. Furthermore, in that case the lower minimal rank of T *is equal to* r.

The condition "the operators in (6.1) and (6.2) have the same rank" does not imply that T has a finite rank extension. Take for instance

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \tfrac{1}{2} & \tfrac{1}{2} & 0 & \cdots \\ \tfrac{1}{4} & \tfrac{1}{4} & \tfrac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} : l_2 \to l_2.$$

Then T is well defined since $\sum\limits_{i=0}^\infty \dfrac{i+1}{2^{2i}} < \infty$. Assume its lower triangular part has a finite rank extension. Then the minimal lower rank is 1 and for a minimal rank extension there is only one possibility, namely

$$\begin{bmatrix} 1 & 1 & 1 & \cdots \\ \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \cdots \\ \tfrac{1}{4} & \tfrac{1}{4} & \tfrac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

But clearly this matrix does not define a bounded linear operator acting on l_2.

7. Kernels of integral operators

In this section we specify the reults for integral operators. The problem which we solve here originated in [35]. Let k be an $m \times n$ matrix kernel defined on the square $[a,b] \times [a,b]$. We say that k is *of finite lower rank* if the lower triangular part k_L of k,

which is defined by

$$k_L(t,s) = k(t,s), a \leq s < t \leq b,$$

admits a *finite rank extension*, i.e., there exists a finite rank matrix kernel h on $[a,b] \times [a,b]$ such that k_L is the lower triangular part of h. Recall that for an $m \times n$ matrix kernel h on $[\alpha,\beta] \times [\gamma,\delta]$ the *rank* of h (notation: rank h) is the rank of the corresponding integral operator

$$(Hf)(t) = \int_{\gamma}^{\delta} h(t,s)f(s)ds \; , \alpha \leq t \leq \beta, \text{a.e.,}$$

which has to be considered as an operator from $L_2^n[\gamma,\delta]$ into $L_2^m[\alpha,\beta]$. Note that k is of finite lower rank if and only if k_L is the lower triangular part of a degenerate kernel. By definition a *minimal rank extension* of k_L is a finite rank extension h of k_L with the extra property that among all finite rank extensions of k_L the rank of h is as small as possible. The rank of a minimal rank extension of k_L is called the *minimal lower rank* of k. The (lower triangular part of the) kernel k is said to be *lower unique* if k_L has precisely one minimal rank extension. Note that if k is of finite lower rank the restriction of k to the rectangle $[\gamma,b] \times [a,\gamma]$ is a finite rank kernel for each $a < \gamma < b$.

We specify the construction of minimal rank extensions for the case considered here. Let k be a given kernel of finite lower rank and let L denote the integral operator corresponding to k_L:

$$(Lf)(t) = \int_{a}^{t} k(t,s)f(s)ds \; , a \leq t \leq b . \tag{7.1}$$

Let h be an arbitrary finite rank extension of k_L, and let H denote the corresponding integral operator, i.e.,

$$(Hf)(t) = \int_{a}^{b} h(t,s)f(s)ds \; , a \leq t \leq b .$$

With H and $a \leq c \leq b$ we associate the following auxiliary operators:

$$H_c : L_2^n[a,b] \to L_2^m[a,b] \; , H_c\phi = \int_{a}^{c} h(.,s)\phi(s)ds \; ,$$

$$H^c : L_2^n[a,b] \to L_2^m[a,b] \; , H^c\phi = \chi_{[c,b]}H\phi .$$

(Here $\chi_{[c,b]}$ denotes the characteristic function of the interval $[c,b]$.) Choose a partition $\pi = \{\alpha_0, \cdots, \alpha_p\}$ of $[a,b]$ such that for $j = 1,...,p$ the numbers rank H_c and rank H^c are constant for $\alpha_{j-1} < c < \alpha_j$. For $1 \leq j \leq i \leq p$ define the operator $H_{ij} : L_2^n[\alpha_{j-1},\alpha_j] \rightarrow L_2^m[\alpha_{i-1},\alpha_i]$ by

$$(H_{ij}\phi)(t) = \int_{\alpha_{j-1}}^{\alpha_j} h(t,s)\phi(s)ds \ , \alpha_{i-1} \leq t \leq \alpha_i.$$

From Theorem 5.4 it follows that a minimal rank extension of the operator

$$\begin{bmatrix} H_{11} & 0 & \ldots & & 0 \\ H_{21} & H_{22} & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & 0 \\ H_{p1} & H_{p2} & \cdots & & H_{pp} \end{bmatrix} : \bigoplus_{i=1}^{p} L_2^n[\alpha_{i-1},\alpha_i] \rightarrow \bigoplus_{i=1}^{p} L_2^m[\alpha_{i-1},\alpha_i] \qquad (7.2)$$

(relative to the finite chain suggested by its block form in (7.2)) has a kernel which is a minimal rank extension of the kernel k_L. Moreover, and all minimal rank extensions of k_L can be obtained in this way. To make a minimal rank extension of (7.2) is just a finite chain problem, and we can use (II) in Section V.5.

In order to see that indeed the above statements follow from Section V.5 one chooses the following chains of projections. For $a \leq t \leq b$, let P_t be the projection in $L_2^n([a,b])$ defined by

$$(P_t f)(s):= \begin{cases} f(s) & a \leq s \leq t, \\ 0 & t < s \leq b . \end{cases}$$

Then $\mathscr{P} = \{ P_t \mid a \leq t \leq b \}$ is a closed chain. Define $\mathscr{Q}$ in $L_2^m([a,b])$ analogously. Note that in this case $\Lambda = [a,b]$ and $\gamma^* = \gamma$ for each $\gamma \in [a,b]$. Moreover, if K is the integral operator with kernel k, then $\mathscr{L}(K;\mathscr{P},\mathscr{Q})$ is the integral operator L in (7.1) (cf. Example V.2.1).

If one specifies Theorem V.3.1 for the integral operators considered here one obtains Theorem 0.6, which gives a formula for the minimal lower rank of a kernel. Also, with the above choice of chains we derive the following characterization of uniqueness.

COROLLARY 7.1. *Assume that k is a matrix kernel on $[a,b] \times [a,b]$ of finite lower rank. Then k is lower unique if and only if for $a < \gamma < b$ the rank of the restricted kernel*

$$k_\gamma(t,s) = k(t,s), \gamma \leqq t \leqq b, a \leqq s \leqq \gamma,$$

is independent of γ. Furthermore, in that case the rank of the (unique) minimal rank extension of k_L is precisely the integer $r = $ rank k_γ.

Proof. Let K and K_γ be the integral operators corresponding to the kernels k and k_γ, respectively. It is now easy to see that $K_\gamma = (I - Q_\gamma)KP_\gamma = (I - Q_{\gamma^-})KP_\gamma$ $(a < \gamma < b)$. According to Theorem 4.1 the operator K is lower unique if and only if for each $a < \gamma < b$ the rank of K_γ is independent of γ. Since there is a 1-1 correspondence between an integral operator and its kernel, Corollary 7.1 follows directly from Theorem 4.1. $\square$

When $k(t,s) = F(t)G(s)$, $a \leqq s < t \leqq b$, with F and G analytic on $[a,b]$, then the uniqueness condition in Corollary 7.1 is fulfilled, and we have lower uniqueness (see [35]).

Without the assumption that k is of finite lower rank Corollary 7.1 does not hold true. In other words, the condition "rank k_γ is independent of γ for $a < \gamma < b$" does not imply that k has a finite rank extension. For instance, take

$$k(t,s) = \begin{cases} t^{-\frac{1}{2}} & 0 \leqq s < t < 1, \\ 0 & 0 \leqq t \leqq s \leqq 1. \end{cases}$$

Then rank $k_\gamma = 1$ for $0 < \gamma < 1$, but k is not of finite lower rank. To see this, assume k is of finite lower rank. Then, by Corollary 7.1, the lower triangular part k_L of k has a finite rank extension h of rank 1. The only possibility is the kernel h given by $h(t,s) = t^{-\frac{1}{2}}$ for $0 \leqq t \leqq 1$ and $0 \leqq s \leqq 1$. This h, however, does not define a square integrable kernel.

The restriction to matrix kernels in this section is not essential. The results also hold for an operator-valued kernel $k(t,s) : Z \to Y$, $a \leqq t \leqq b$, $a \leqq s \leqq b$, of finite lower rank, where Z and Y are separable Hilbert spaces.

Independently, G. Peeters [58], [59] has found another procedure to construct a minimal rank extension (or, in fact, a lower separable representation) for the lower triangular part of a kernel.

8. Difference kernels

In this section we consider minimal rank extensions of difference kernels. Let k be an $m \times n$ matrix kernel defined on the square $[a,b] \times [a,b]$. We call k a *difference kernel* if a function $h : [a-b, a+b] \to \mathbb{C}^{m \times n}$ exists such that

$$k(t,s) = h(t-s) \ , a \leqq s \ , t \leqq b \ , \text{a.e.}.$$

THEOREM 8.1. *Let* k *be a* $m \times n$ *matrix difference kernel on* $[a,b] \times [a,b]$ *of finite lower rank. Then* k_L *has a unique minimal rank extension. If this extension is continuous, then it is a difference kernel.*

Without loss of generality $a = 0$ and $b = 1$. We shall view an integral operator with a difference kernel as a Toeplitz operator relative to any "equidistant decomposition". By this we mean the following Let P_t and Q_t ($t \in [0,1]$) be the projection on $L_2^n[0,1]$ and $L_2^m[0,1]$, respectively, defined by

$$P_t f = \chi_{[0,t]} \ , Q_t f = \chi[0,t]f.$$

Then $\mathscr{P} = \{ P_t \mid t \in [0,1] \}$ and $\mathscr{Q} = \{ Q_t \mid t \in [0,1] \}$ are closed chains of orthogonal projections. Let

$$\mathscr{P}_N = \{P_0, P_{\frac{1}{N}}, \ \cdots \ , P_{\frac{N-1}{N}}, P_1\} \ , \mathscr{Q}_N = \{Q_0, Q_{\frac{1}{N}}, \ \cdots \ , Q_{\frac{N-1}{N}}, Q_1\} \ .$$

If $g(t,s)$, $0 \leqq s, t \leqq 1$, is a difference kernel, and G is the corresponding integral operator, we may write G in the following way

$$G = \left[G_{ij} \right]_{i,j=1}^{N} \ , G_{ij} = (Q_{\frac{i}{N}} - Q_{\frac{i-1}{N}}) G (P_{\frac{j}{N}} - P_{\frac{j-1}{N}}).$$

We will view G as a Toeplitz operator from $(L_2^n[0,\frac{1}{N}])^N$ to $(L_2^m[0,\frac{1}{N}])^N$.

PROPOSITION 8.2. *Let* k *be a* $m \times n$ *difference kernel defined on* $[a,b] \times [a,b]$ *of finite lower rank, and let* k_γ *denote the restriction*

$$k_\gamma(t,s) = k(t,s) \ , a \leqq s \leqq \gamma \ , \gamma \leqq t \leqq b.$$

Then rank k_γ *is independent of* $\gamma \in (a,b)$.

Proof. Without loss of generality $a = 0$ and $b = 1$. Let K denote the integral operator with kernel k, and let $\pi = \{\alpha_0, \alpha_1, \ \cdots \ , \alpha_r\}$ be a C-partition of $\mathscr{P}$, $\mathscr{Q}$ and a finite

rank extension of $\mathscr{A}(K;\mathscr{P},\mathscr{Q})$. Lemma V.3.3 implies that for $\alpha_i < \gamma < \alpha_{i+1}$ the rank of $(I-Q_\gamma)KP_\gamma$ is independent of γ ($i = 0,...,r-1$). Fix $i \in \{1,...,r-1\}$ and let us show that for some $\gamma \in (\alpha_{i-1},\alpha_i)$ and $\delta \in (\alpha_i,\alpha_{i+1})$

$$\text{rank } (I-Q_\gamma)KP_\gamma = \text{rank } (I-Q_\delta)KP_\delta. \tag{8.1}$$

Choose N such that $\dfrac{1}{N} < \tfrac{1}{3}\min\{\alpha_i-\alpha_{i-1} \mid i = 1,...,r \}$ and $\{\dfrac{1}{N},...,\dfrac{N-1}{N}\}\cap\pi = \varnothing$. Consider $\mathscr{A}(K,\mathscr{P}_N\backslash\{P_1\},\mathscr{Q}_N\backslash\{Q_0\})$, which is a strictly lower triangular part. We view this lower triangular part as a Toeplitz operator. Let $\gamma = \max\{ \dfrac{j}{N} \mid \dfrac{j}{N} < \alpha_i \}$ and $\delta = \gamma+\dfrac{1}{N}$. Note that γ and δ are in the right intervals. Consider

$$(I-Q_\gamma)KP_\gamma = \left[(I-Q_\gamma)KP_{\gamma-\frac{1}{N}} \quad (I-Q_\gamma)K(P_\gamma-P_{\gamma-\frac{1}{N}}) \right].$$

Since $\gamma,\gamma-\dfrac{1}{N} \in (\alpha_{i-1},\alpha_i)$ Lemma 3.3 yields that its rank equals the rank of $(I-Q_\gamma)KP_{\gamma-\frac{1}{N}}$. Hence

$$\text{Im } (I-Q_\gamma)K(P_\gamma-P_{\gamma-\frac{1}{N}}) \subset \text{Im } (I-Q_\gamma)KP_{\gamma-\frac{1}{N}}.$$

Multiplying by $Q_{\frac{N-1}{N}}$ on the left we obtain that

$$\text{Im } (Q_{\frac{N-1}{N}}-Q_\gamma)K(P_\gamma-P_{\gamma-\frac{1}{N}}) \subset \text{Im } (Q_{\frac{N-1}{N}}-Q_\gamma)KP_{\gamma-\frac{1}{N}}.$$

Using the Toeplitz structure this implies that

$$\text{Im } (I-Q_\delta)K(P_\delta-P_\gamma) \subset \text{Im } (I-Q_\delta)K(P_\gamma-P_{\frac{1}{N}})$$

which in turn is contained in $\text{Im } (I-Q_\delta)KP_\gamma$. Thus

$$\text{rank } (I-Q_\delta)KP_\delta = \text{rank } \left[(I-Q_\delta)KP_\gamma \quad (I-Q_\delta)K(P_\delta-P_\gamma) \right] \tag{8.2}$$

$$= \text{rank } (I-Q_\delta)KP_\gamma.$$

Analogously, one proves that

$$\text{rank } (I-Q_\gamma)KP_\gamma = \begin{bmatrix} (Q_\delta-Q_\gamma)KP_\gamma \\ (I-Q_\delta)KP_\gamma \end{bmatrix} = \text{rank } (I-Q_\delta)KP_\gamma. \tag{8.3}$$

Now (8.3) and (8.2) together give (8.1). Thus rank $(I-Q_\gamma)KP_\gamma$ is independent of $\gamma \in [0,1]\backslash\pi$.

Suppose there is an $\alpha_j \in \pi$ $(j \in \{1,...,r-1\})$ such that

$$\text{rank } (I-Q_{\alpha_j})KP_{\alpha_j} \neq \text{rank } (I-Q_\eta)KP_\eta \tag{8.4}$$

for $\eta \in [0,1]\backslash\pi$. Fix $\alpha_j < \eta < \min\{\alpha_{j+1},\alpha_j+\tfrac{1}{3}(\alpha_j-\alpha_{j-1})\}$. Lemma 3.3 gives that

$$\text{rank } (I-Q_\eta)KP_{\alpha_j} = \text{rank } (I-Q_{\alpha_j})KP_{\alpha_j},$$

and hence

$$\text{rank } (I-Q_\eta)KP_\eta > \text{rank } (I-Q_\eta)KP_{\alpha_j}, \tag{8.5}$$

Put $\beta = \eta-\alpha_j$ and consider

$$(I-Q_{\alpha_j-\beta})KP_{\alpha_j-\beta} = \left[(I-Q_{\alpha_j-\beta})KP_{\alpha_j-2\beta} \quad (I-Q_{\alpha_j-\beta})K(P_{\alpha_j-\beta}-P_{\alpha_j-2\beta}) \right].$$

Because of Lemma 3.3 (and $\alpha_j-\beta$, $\alpha_j-2\beta \in (\alpha_{j-1},\alpha_j)$) we have that

$$\text{rank } (I-Q_{\alpha_j-\beta})KP_{\alpha_j-\beta} = \text{rank } (I-Q_{\alpha_j-\beta})KP_{\alpha_j-2\beta}.$$

So

$$\text{Im } (I-Q_{\alpha_j-\beta})K(P_{\alpha_j-\beta}-P_{\alpha_j-2\beta}) \subset \text{Im } (I-Q_{\alpha_j-\beta})KP_{\alpha_j-2\beta}.$$

Multiplying on the left with $Q_{1-2\beta}$ gives

$$\text{Im } (Q_{1-2\beta}-Q_{\alpha_j-\beta})K(P_{\alpha_j-\beta}-P_{\alpha_j-2\beta}) \subset \text{Im } (Q_{1-2\beta}-Q_{\alpha_j-\beta})KP_{\alpha_j-2\beta}.$$

Using the Toeplitz structure we get that

$$\text{Im } (I-Q_{\alpha_j+\beta})K(P_{\alpha_j+\beta}-P_{\alpha_j}) \subset \text{Im } (I-Q_{\alpha_j+\beta})KP_{\alpha_j}.$$

But then

$$\text{rank } (I-Q_\eta)KP_\eta = \text{rank } \left[(I-Q_\eta)KP_{\alpha_j} \quad (I-Q_\eta)K(P_\eta-P_{\alpha_j}) \right] = \text{rank } (I-Q_\eta)KP\alpha_j.$$

This contradicts (8.5). Thus (8.4) cannot hold, and the proposition is proved. $\square$

We need operator analogs of results in Section IV.4.

Let $T_j : Z \to Y$, $j = -1,...,-n+1$, be finite rank operators, and consider all $A = \left[A_{j-i} \right]_{i,j=0}^{n-1} : Z^n \to Y^n$, with $A_j = T_j$, $j = -1,...,-n+1$. Such an A with lowest

possible rank we shall call a *Toeplitz minimal rank extension* of T, where

$$T = \left[T_{ij} \right]_{i,j=0}^{n-1} , T_{ij} := \begin{cases} T_{j-i} , j < i ; \\ 0 \quad , \text{elsewhere}. \end{cases}$$

LEMMA 8.3. *Let T be as above. Then the rank of a Toeplitz minimal rank extension of T is equal to*

$$\sum_{p=1}^{n} \text{rank } T^{(p,p)} - \sum_{p=1}^{n-1} \text{rank } T^{(p+1,p)} ,$$

where

$$T^{(p,q)} = \text{rank} \begin{pmatrix} T_{-p} & \cdots & T_{-p+q-1} \\ \vdots & & \vdots \\ T_{-n} & \cdots & T_{-n+q-1} \end{pmatrix} .$$

Moreover, there exists only one Toeplitz minimal rank extension of T if and only if for some $p \in \{1,...,n-1\}$

$$\text{rank } T^{(p,p)} = \text{rank } T^{(p+1,p)} = \text{rank } T^{(p+1,p+1)}, \tag{8.6}$$

Proof. If Z and Y are finite dimensional the lemma follows directly from Theorem IV.4.1 and Corollary IV.4.2. For the general case make decompositions

$$Z = Z^{(1)} \oplus Z^{(2)} , Y = Y^{(1)} \oplus Y^{(2)} ,$$

such that $Z^{(1)}$ and $Y^{(1)}$ are finite dimensional and that relative to these decompositions T_i is of the form $\begin{bmatrix} K_i & 0 \\ 0 & 0 \end{bmatrix}$. Since all T_i are of finite rank this can be done. First note that the formula for rank A follows. Suppose now that T has only one Toeplitz minimal rank extension. Then also K, where

$$K = \left[K_{ij} \right]_{i,j=0}^{n-1} , K_{ij} := \begin{cases} K_{j-i} , j < i ; \\ 0 \quad , \text{elsewhere}. \end{cases}$$

has only one Toeplitz minimal rank extension. But then for some $p \in \{1,...,n-1\}$ we have that

$$\text{rank } K^{(p,p)} = \text{rank } K^{(p+1,p)} = \text{rank } K^{(p+1,p+1)},$$

and thus (8.6) holds.

Suppose that (8.6) holds. Using the 2×2 case of Theorem V.1.2 and the restriction results Theorems 5.1 and 5.2 one sees that A_0 is uniquely determined when making a Toeplitz minimal rank extension $A = \left[A_{j-i} \right]_{i,j=0}^{n-1}$ for T. With the same reasoning the uniqueness of $A_1, \ldots, A_{n-1}$ follows. $\square$

Proof of Theorem 8.1. Without loss of generality $a = 0$ and $b = 1$. Let k be as in the theorem. Using Corollary 7.1 and Proposition 8.2 we obtain that k_L has only one minimal rank extension.

Suppose that the unique minimal rank extension h of k_L is continuous. Let H denote the integral operator with kernel h. Note that $\{0,1\}$ is a C-partition for $\mathscr{P}$, $\mathscr{Q}$ and H. Let $p > 1$ and consider

$$\mathscr{R}(H;\mathscr{P}_{2^p}\backslash\{P_1\},\mathscr{Q}_{2^p}\backslash\{Q_0\}). \tag{8.7}$$

Lemmas 3.3 and 8.3 give that this lower triangular part has a unique Toeplitz minimal rank extension. Since H is an extension and its rank equals the minimal lower rank of (8.7), this unique Toeplitz minimal rank extension must be equal to H. Apparently, H is Toeplitz relative to the decompositions corresponding to $\mathscr{P}_{2^p}$ and $\mathscr{Q}_{2^p}$ for any p. Since the kernel h of H is continuous, we obtain that h is a difference kernel. This completes the proof. $\square$

V.9. Connections with systems theory

Consider the time variant causal system

$$\theta \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \ , a \leqq t \leqq b, \\ y(t) = C(t)x(t) \qquad\qquad , a \leqq t \leqq b, \\ x(a) = 0. \end{cases} \tag{9.1}$$

Here $A(t)$, $B(t)$ and $C(t)$ are matrices of size $r \times r$, $r \times n$ and $m \times r$, respectively. We assume that $A(t)$, as a function of t, is integrable on $[a,b]$, and that $B(.)$ and $C(.)$ are square integrable. The matrix function $A(.)$ is called the *main coefficient* and the number r is called the *state space dimension* of the system. To simplify the notation we denote the

system (9.1) by $\theta = (A(t),B(t),C(t))$. The *impulse response matrix function*(see [50], Section 9.1) of the system θ is given by

$$h(t,s) = C(t)U(t)U(s)^{-1}B(s) \quad a \leqq s < t \leqq b,$$

where $U(t)$ is the fundamental operator of the system, i.e., $U(t)$ is the unique absolutely continuous solution of the matrix differential equation

$$\dot{U}(t) = A(t)U(t), \quad a \leqq t \leqq b, U(a) = I_r.$$

Here I_r denotes the identity matrix of order r. Obviously, the impulse response matrix function is the lower triangular part of a finite rank matrix kernel. The converse statement is also true (cf., [34], Section I.4).

Let h ($= h_L$) be a kernel of finite lower rank. A causal time-variant system θ is said to be a *realization* of h if the impulse response matrix function of θ is equal to h. A realization θ of h is called *minimal* if among all realizations of h the state space dimension of θ is as small as possible. Two realizations $\theta = (A(t),B(t),C(t))$ and $\bar{\theta} = (\bar{A}(t),\bar{B}(t),\bar{C}(t))$ are said to be *similar* if there exists an absolutely continuous square matrix function $S(.)$ such that $S(t)$ is invertible for $a \leqq t \leqq b$ and

$$\bar{A}(t) = S(t)A(t)S(t)^{-1} + \dot{S}(t)S(t)^{-1},$$

$$\bar{B}(t) = S(t)B(t), \quad \bar{C}(t) = C(t)S(t)^{-1},$$

almost everywhere on $a \leqq t \leqq b$. Two minimal realizations of a given impulse response matrix function do not have to be similar. E.g., the systems $\theta_1 = (0,\chi_{[\frac{1}{2},1]},1)$ and $\theta_2 = (0,\chi_{[\frac{1}{2},1]},\chi_{[\frac{1}{2},1]})$ on the time interval $[0,1]$, are two systems which have the same impulse response matrix function but which are not similar (see also [35]). The following theorem gives the necessary and sufficient conditions on an impulse response matrix function h in order that a minimal realization of h is unique up to similarity.

THEOREM 9.1. *Let* $h(t,s)$, $a \leqq s < t \leqq b$, *be an impulse response matrix function. Then a minimal realization of* h *is unique up to similarity if and only if for* $a < \gamma < b$ *the kernels*

$$h_\gamma(t,s) = h(t,s), \gamma \leqq t \leqq b, a \leqq s \leqq b$$

all have the same (finite) rank r, say. Furthermore, in that case the state space dimension of a minimal realization of h *is equal to* r.

Let h ($= h_L$) be an impulse response matrix function. Recall (see [35]) the following

connections. The state space dimension of a minimal realization of h is equal to the minimal lower rank of h. Moreover, there is a one-one correspondence between the similarity classes of minimal realizations of h and the set of minimal rank extensions of h. Consequently, when there is only one minimal rank extension of h, then all minimal realizations of h are similar. This last remark together with Corollary 7.1 proves Theorem 9.1.

It is a classical result (see [50]) that for the time invariant case (i.e., A, B and C constant) minimal realizations of the same impulse response matrix function h are similar. In that case one obtains an analytic difference kernel h, and thus Theorem 9.1 yields that all minimal realizations of the impulse response matrix function of a time invarant causal system are similar even in the class of time variant causal systems.

Theorem 9.1 and its upper triangular analogue can also be used to answer the question of uniqueness up to similarity of SB-minimal realizations (see [35]) of integral operators with a semi-separable kernel.

COMMENTS (Part B)

The results in this part are collected from the papers [48], [65], [67] and [14]. Section V.8 did not appear before.

The minimal lower rank formula in Section IV.1 and its corollary appeared earlier in [65]. The proofs and the construction, described in this section, are in the spirit of [67]. The uniqueness result in Section IV.2 was obtained in [48]. The description of the set of all minimal rank extensions in Section IV.3 can be found in [67], as well as the results on the Toeplitz case (Section IV.4) and its connection to the partial realization problem (Section IV.6). Theorem IV.5.1 is hidden in Corollary 1.4 in [65] and the results concerning the general patterns can be found in [14]. The results in Chapter IV, with the exception of the remark in the last paragraph of Section IV.4 and of Theorem 7.1, remain true when one considers matrices over an arbitrary field (in stead of $\mathbb{C}$). Theorem 7.1 is true for any nontrivial field, and the remark in the last paragraph of Section IV.4 is true for any algebraically closed field.

Chapter V, with the exception of Section V.8, is based on the papers [48] and [65].

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