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Characterizations of Banach spaces not containing l^1

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Preface

These notes cover the material I presented in a series of lectures at the Indian Statistical Institute in Calcutta, India, in the winter of 1987-88. The invitation to lecture there left me free to select more or less any topic I fancied. It so happened that at the time I was interested in a particular problem that involved the question whether l^1 could be embedded in a certain kind of Banach space. When I looked through the literature to supplement my rather superficial knowledge of l^1-embeddability, I became very impressed with the beauty and depth of the work in this area, and so I decided to present a survey of this in my course. An extra reason why this choice of subject seemed appropriate to me, was that I knew my audience to have a strong background in measure theory. So they would especially appreciate the measure-theoretical work of D.H. Fremlin and M. Talagrand that I intended to use as a basis for a large part of my course.

Before describing the contents of these notes let me first try to put the subject in perspective, historically as well as mathematically.

A desirable result in any structure theory is to show that the objects of study can be decomposed into "elementary" components that are more easily investigated. In the particular case of Banach spaces for instance, it would be nice to know if *every Banach space contains an isomorphic copy of one of the elementary Banach spaces* l^p, $1 \leq p < \infty$, *or* c_0. Let us call this *conjecture I.* Actually this problem was open until 1974, when *B. Tsirelson* constructed a counterexample ([95]). At around the same time another related conjecture was put to rest, this time by R.C. James and, independently, by J. Lindenstrauss. To motivate this, let us observe that l^1 is the only one among the above-mentioned "elementary" spaces with a non-separable dual. So it is quite natural to conjecture that *if a separable space has a non-separable dual, then it must contain a copy of* l^1 *(conjecture II).* In 1974 R.C. James constructed his famous counterexample to this conjecture, now known as the *James tree space* JT ([46]). This space JT is an ingenious variation of another famous space, also due to R.C. James but much older, the *classical James space* J([45]). This J has the remarkable property of being linearly isometric to its bidual, but not reflexive. In fact the canonical image of J in its bidual has codimension one. Whereas J consists of sequences, i.e. functions on $\mathbb{N}$, essentially JT is obtained from J by substituting a tree for the "base space" $\mathbb{N}$. Independently J. Lindenstrauss constructed a function space analogue of J, the James function space JF ([51]). This space also refuted conjecture II: it is separable with non-separable dual, but $l^1 \not\subset JF$. Somewhat later a third counterexample was built by J. Hagler, the space JH([34]).

Returning to conjecture I (which is false), observe that among the elementary spaces, the l^p, $1<p<\infty$, are the only reflexive ones. In fact they are even uniformly convex, a much stronger property than reflexivity. Tsirelson's example was reflexive, but not uniformly convex. So this still left open the possibility that every uniformly convex space (hence every superreflexive space) would contain an isomorphic copy of some l^p, $1<p<\infty$. However, this was also disproved in the same year 1974, by T. Figiel and W.B. Johnson ([24]). Their counterexample was a uniformly convex modification of Tsirelson's space that contained no l^p.

Since Tsirelson's space is reflexive, it leaves intact the following weakening of conjecture I: *every Banach space contains c_0, l^1 or an infinite-dimensional reflexive space (conjecture I^*).* In fact conjecture I^* is still open (see e.g. [39] for some recent work establishing the truth of conjecture I^* for a certain class of Banach spaces). In this connection we ought to mention also that conjecture I^* has long been known to be correct for Banach spaces with unconditional bases, by another fundamental result due to R.C. James ([44]). This brings us naturally to yet another famous and still open problem, namely, whether *every Banach space contains a subspace with an unconditional basis* (we hesitate to call this a conjecture). Clearly, if this were true, then conjecture I^* would have a positive answer, by the above-mentioned result of R.C. James. Finally, coming back to the (false) conjecture II, observe that if conjecture I^* is correct, then so is the following *conjecture II^** (which is open, as far as we know): *if each of the separable subspaces of a Banach space X has a non-separable dual, then X contains a copy of l^1.*

What we hope to have made clear by the preceding remarks is that it is important in the larger framework of the structure theory of Banach spaces, to be able to decide whether a given Banach space does or does not contain an isomorphic copy of l^1. Let us now describe the progress that has been made on this question since the negative results of 1974 (i.e. the subject of this book).

The starting point of these developments was the famous Rosenthal l^1-theorem (also proved in 1974): every bounded sequence in every Banach space either has a weak Cauchy subsequence or a subsequence equivalent to the unit vector basis of l^1 ([72]). An immediate corollary of this (and of the Eberlein-Smulian theorem) is that conjecture I^* is correct for weakly sequentially complete Banach spaces. In particular, when applied to the weakly sequentially complete space $L^1[0,1]$ this corollary yields a new proof of a known result of M.I. Kadec and A. Pelczynski ([48]) that a subspace of $l^1[0,1]$ is either reflexive or contains a copy of l^1. Rosenthal's l^1-theorem also led to a host of new characterizations of Banach spaces (not) containing l^1 (see Ch. 4). Most of these were proved by Rosenthal himself ([73]) and some in collaboration with E. Odell ([62]). Roughly, these results express that if a separable space X is "not too non-reflexive", then X does not contain l^1 (and conversely). Being "not too non-reflexive" can be interpreted in several ways. It may be read as "card $X=$ card X^{**}", but it can also take the form of various w^*-convergence and w^*-density properties of bounded sets in X^{**} that cannot be satisfied if X^{**} is too large in relation to X.

The original proof of Rosenthal's theorem used combinatorial techniques mostly, with Ramsey's theorem figuring in the background (see [13]). Soon afterwards another and eventually more fruitful approach emerged. The basic idea was to look at elements of X and of X^{**} as functions on the dual ball $B(X^*)$, equipped with its w^*-topology. Clearly the elements of X are continuous on $(B(X^*),w^*)$. One of Rosenthal's characterizations says that $l^1 \not\subset X$ iff $B(X)$ is w^*-sequentially dense in $B(X^{**})$. So in this case the elements of X^{**} are first class Baire functions on $(B(X^*),w^*)$ (note that the topology of pointwise convergence on $B(X^*)$ corresponds to the w^*-topology on X^{**}). The converse is also true. This example suggests a more general type of question: given a topological space T (e.g. $T=(B(X^*),w^*)$) and a uniformly bounded set of functions $Z \subset C(T)$ (e.g. $Z=B(X)$), under what conditions does the pointwise closure of Z in $\mathbb{R}^T$ consist of first class Baire functions? Or of universally measurable functions (i.e. functions measurable with respect to every Radon measure on T)? When is Z sequentially dense in its closure, or sequentially relatively compact? It turns out that these and other similar questions can be nicely answered in this general context provided T is either compact, or a Polish space. In the special case when $T=(B(X^*),w^*)$ and $Z=B(X)$ one so obtains measure-theoretical and topological characterizations of Banach spaces (not) containing l^1.

Rosenthal himself initiated this approach to the l^1-embedding problem ([73]). It was carried forward by J. Bourgain, D.H. Fremlin and M. Talagrand in their fundamental paper [8], and then further perfected by M. Talagrand in his memoir [92]. As we have just seen there are measure-theoretical and topological questions at stake here that transcend the Banach space context and deserve to be studied in their own right. This we do in the first three chapters of these notes. Banach spaces do not appear until Chapter 4. There we apply the general theory to the specific case $T=(B(X^*),w^*)$, $Z=B(X)$, to obtain the various characterizations indicated above.

In Chapter 5 and 6 we discuss a group of characterizations centered around the Pettis integral and due chiefly to K. Musial and R. Haydon. The most important ones are that X contains no copy of l^1 iff (i) X^* has the Radon-Nikodym property for the Pettis integral (this is called the weak Radon-Nikodym property), and iff (ii) X^* has the Krein-Milman property for w^*-compact convex sets (i.e. every w^*-compact convex subset of X^* is the norm closed convex hull of its extreme points). Both these characterizations should be set off against the well-known fact that X is Asplund (i.e. every separable subspace of X has a separable dual) iff (i) X^* has the RNP (for the Bochner integral), and iff (ii) X^* has the KMP (i.e. each *norm* closed convex subset of X^* is the closed convex hull of its extreme points). Especially in the non-separable case the proofs of these characterizations are not so easy. Here we have chosen to include all technical details in the main text, rather that relegating them to the Appendices, as we have done with some other technicalities less central to our considerations

Roughly then, the difference between Asplund spaces and spaces not containing l^1 is that between the Bochner and the Pettis integral, or between the KMP and its restriction to w^*-compact convex sets. That there really *is* a difference, is shown by any one of the spaces JT, JF and JH. We have chosen to present JT here (Chapter 8), because it seemed to fit in best with the results already proved, and because a very satisfactory treatment was already available in [51].

Finally, Chapter 7 is devoted to two major results about strong regularity, due to J. Bourgain and W. Schachermayer, respectively. The first one states that X fails to contain l^1 iff X^* is strongly regular (i.e. closed bounded convex sets in X^* admit small combinations of slices). Although the notion of strong regularity was formally defined only recently, this result is already essentially contained in the unpublished lecture notes [6] by J. Bourgain. The second result by W. Schachermayer is quite recent and shows the relevance of strong regularity to the still unsolved problem whether the KMP implies the RNP: if X is strongly regular, then the KMP implies the RNP ([83]). When combined with J. Bourgain's result a new proof is obtained for the much older theorem of R.E. Huff and P.D. Morris ([42]) that for dual Banach spaces the KMP and RNP are equivalent.

After this brief outline it should be pointed out that there are many more characterizations of spaces (not) containing l^1. Some of them we have stated with only little comment in the Notes at the end of each chapter. Including and proving them all would have taken too much space and time. Finally, let us mention that almost all of chapters 1 to 6 (and much more!) can be found in M. Talagrand's superb memoir [92]. In fact this is compulsory reading for anyone who really wants to pursue this subject. But clearly, it can hardly be recommended as a first introduction.

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October 1988

Chapter 0

Preliminaries on Banach spaces

We have written these notes with a reader in mind who has taken the usual basic courses in functional analysis, topology and measure theory, but who possesses no special knowledge of any of these fields. Such a reader cannot be expected to know everything that will be needed. In particular there are some facts from measure theory and topology, not usually taught in standard courses, that play a role in our development. For the convenience of the reader we have collected these in several appendices, with full proofs. In the present chapter we review what, ideally, the reader should know about Banach spaces. In so doing we also establish our terminology and notation. For the most part the material is standard and can be found in the general references listed at the end of this chapter. Special results will be used only occasionally, so there is no need to worry if one is not familiar with some of them. We shall formulate them clearly, and indicate where proofs can be found.

Let X be a real Banach space. We use the notation $B(X)$ for the unit ball $\{x \in X : \|x\| \leqslant 1\}$ and $S(X)$ for the unit sphere $\{x \in X : \|x\| = 1\}$. The *dual* or *conjugate* space X^* is the vector space of all continuous linear forms on X. We denote the value an element $x^* \in X^*$ takes at $x \in X$ by $\langle x, x^* \rangle$. X^* is a Banach space when provided with the dual norm $\|x^*\| := \sup \{\langle x, x^* \rangle : x \in B(X)\}$ $(x^* \in X^*)$. Similarly, one defines the second dual or *bidual* X^{**} as $(X^*)^*$. More generally, the n^{th} dual $X^{(n)}$ is the dual of the $(n-1)^{st}$ dual $X^{(n-1)}$ $(n = 1, 2, ...)$. Elements of X^*, $X^{**}, X^{***}, \cdots$ are written $x^*, x^{**}, x^{***}, \ldots,$ respectively. Each $x \in X$ gives rise to an element $\pi_X x \in X^{**}$ defined as follows:

$$\langle x^*, \pi_X x \rangle := \langle x, x^* \rangle \qquad (x^* \in X^*).$$

It is easily verified that the map $\pi_X : X \to X^{**}$ so defined is a linear isometry. If it is surjective, then X is called *reflexive*. R.C. James ([45]) has constructed a Banach space, called the James space J, which is linearly isometric to its bidual J^{**} without being reflexive. In fact the range of the canonical map π_J has codimension 1 in J^{**} (see [E] for a detailed account).

An elementary fact that we shall need is that the adjoint $\pi_X^* : X^{***} \to X^*$ of the canonical embedding π_X acts as the inverse of π_{X^*} on $\pi_{X^*} X^*$. In formula:

$$\pi_X^* \circ \pi_{X^*} = 1_{X^*} \quad (:= \text{identity on } X^*).$$

The proof is a simple application of the definitions of π_X and π_{X^*} : for all $x \in X$ and $x^* \in X^*$ we have

$$\langle x, \pi_X^* \circ \pi_{X^*} x^* \rangle = \langle \pi_X x, \pi_{X^*} x^* \rangle = \langle x^*, \pi_X x \rangle = \langle x, x^* \rangle .$$

We shall often identify X with the subspace $\pi_X X$ of X^{**} without even mentioning the map π_X.

Two topologies different from the norm topology are of fundamental importance. The *weak topology* on X, denoted $\sigma(X,X^*)$, makes X into a locally convex topological vector space. By definition a base of 0-nbhds consists of all sets of the form

$$V(0;x_1^*, \ldots, x_n^*;\epsilon) := \{x \in X: |\langle x, x_i^* \rangle| < \epsilon, \quad i=1, \ldots, n\},$$

where $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in X^*$ and $\epsilon > 0$ are arbitrary. For a dual space X^* we have, in addition to the weak topology $\sigma(X^*,X^{**})$, a second, generally weaker topology $\sigma(X^*,X)$, called the *weak* topology*. A 0-nbhd base is given by the sets

$$V(0;x_1, \ldots, x_n;\epsilon) := \{x^* \in X^*: |\langle x_i, x^* \rangle| < \epsilon, i=1, \ldots, n\}.$$

If X is reflexive then the weak and the weak* topologies on X^* clearly coincide. Convergence in the weak -, resp. weak* topology is denoted by $\xrightarrow{w}$, respectively $\xrightarrow{w^*}$. Often when X is understood we denote $\sigma(X,X^*)$ and $\sigma(X^*,X)$ by w, resp. w^*. Observe that whenever $\dim X = \infty$, each weak 0-nbhd $V(0;x_1^*, \ldots, x_n^*;\epsilon)$ contains the nontrivial subspace $\bigcap_{i=1}^{n} \ker x_i^*$. In particular no weak 0-nbhd is norm bounded, so $\sigma(X,X^*)$ differs from the norm topology. Another consequence of this observation is that the weak closure of $S(X)$, denoted $w\text{-}cl\, S(X)$, contains the origin. It is easily seen that every element of X^* is w-continuous on X. An important but slightly less trivial fact is that an element $x^{**} \in X^{**}$ is w^*-continuous on X^* iff it belongs to X $(=\pi_X X)$.

By a *subspace* of X we shall always mean a closed linear subspace. If Y is such a subspace, then the weak topology $\sigma(X,X^*)$ induces on Y the weak topology $\sigma(Y,Y^*)$. This is so because every $y^* \in Y^*$ extends to an element of X^*, by the Hahn-Banach theorem. Another consequence of the Hahn-Banach theorem is *Goldstine's theorem* ([C]):$B(X)$ is w^*-dense in $B(X^{**})$. The latter set is w^*-compact. In fact *Alaoglu's theorem* (which is essentially a corollary of Tychonoff's theorem) says that the unit ball $B(X^*)$ of any dual space X^* is w^*-compact. In particular the sets $\{x^*: \|x^*\| \leq c\}$, $c>0$, are w^*-closed, i.e. the dual norm $\|\cdot\|$ on X^* is w^*-l.s.c. (= w^*-lower-semi-continuous). In fact this property characterizes dual norms: a norm on X^* equivalent to the given (dual) norm on X^*, is the dual of a (necessarily equivalent) norm on X iff it is w^*-l.s.c. An important corollary of Goldstine's and Alaoglu's theorems is that X is reflexive iff its unit ball $B(X)$ is w-compact (use the fact that the w^*-topology on X^{**} induces the w-topology on its subspace X). Equivalently, X is reflexive iff every bounded set in X is relatively w-compact.

A subset $A \subset B(X^*)$ is called a *norming* set if $\|x\| = \sup\{|\langle x,x^*\rangle| : x^* \in A\}$ for every $x \in X$. If X is separable, then X has a *norming sequence.* Indeed, if (x_n) is dense in $S(X)$, and if for each $n \in \mathbb{N}$ an $x_n^* \in X^*$ is selected so that $\langle x_n, x_n^*\rangle = \|x_n^*\| = 1$ (Hahn-Banach theorem), then $\{x_n^* : n \in \mathbb{N}\}$ is norming. Even for some non-separable Banach spaces norming sequences (x_n^*) exist. E.g. if $X = l^\infty$ (:= the bounded real functions on $\mathbb{N}$, with the sup norm), take $x_n^* :=$ evaluation in n ($n = 1,2,\ldots$). A set $A \subset X$ (respectively, $A \subset X^*$) is called *total* (resp. *w^*-total*) if $sp\,A$(:= the linear span of A) is dense in X (resp. w^*-dense in X^*). Equivalently, this means that A *separates the points* of X^* (resp. X), i.e. $\langle x,x_1^*\rangle = \langle x,x_2^*\rangle$ for every $x \in A$ (resp. $\langle x_1,x^*\rangle = \langle x_2,x^*\rangle$ for every $x^* \in A$) implies $x_1^* = x_2^*$ (resp. $x_1 = x_2$). A norming set $A \subset B(X)$ is clearly w^*-total. An important fact is that for separable X the topological space $(B(X^*),w^*)$ is *metrizable* (besides being compact). Indeed, if (x_n) is dense in $B(X)$, then

$$d(x^*y^*) := \sum_{n=1}^{\infty} 2^{-n}.\frac{|\langle x_n, x^*-y^*\rangle|}{1+|\langle x_n,x^*-y^*\rangle|} \quad (x^*,y^* \in X^*)$$

is easily seen to be a metric on X^*. For nets $(x_\alpha^*) \subset X^*$ one verifies without difficulty that $x_\alpha^* \xrightarrow{w^*} x^*$ implies $d(x_\alpha^*,x^*) \to 0$, so that the d-topology is weaker that the w^*-topology. Since it is Hausdorff, and $(B(X^*),w^*)$ is compact, the d-topology coincides with the w^*-topology on $B(X^*)$.

The next three results we mention are not so simple to prove. Nevertheless they are standard fare. The *Eberlein-Smulian theorem* ([E]) says that a subset $A \subset X$ is relatively w-compact (in X) iff every sequence in A has a w-convergent subsequence. Furthermore, by the *Krein-Smulian theorem* ([J]), a subspace Y of a dual Banach space X^* is w^*-closed iff $Y \cap B(X^*)$ is w^*-closed. As a corollary, we have that an element $x^{**} \in X^{**}$ belongs to X (equivalently, x^{**} is w^*-continuous) iff its restriction to $B(X^*)$ is w^*-continuous (take $Y = \ker x^{**}$). Finally, let $K \subset X$ be closed bounded and convex. Then an element $x^* \in X^*$ need not attain its sup on K, unless K is w-compact. But the $x^* \in X^*$ that do, lie dense in X^*. This is the *Bishop-Phelps theorem* ([B]), ([17]). In particular, taking $K = B(X)$, the set of elements of X^* that "attain their norm", is dense in X^*.

A sequence (x_n) in X is called a (*Schauder*) *basis* for X iff for every $x \in X$ there exists a unique sequence $(\alpha_n) \subset \mathbb{R}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ (where this series is supposed to converge in norm). Necessary and sufficient for a sequence $(x_n) \subset X$ to be a basis for X (see [G]), is that $x_n \neq 0$ ($n = 1,2,\ldots$), $[x_n]_{n=1}^\infty = X$ (where $[x_n]_{n=1}^\infty$ denotes the closed linear span of the sequence (x_n)), and that there exists a constant $M < \infty$ so that

$$\|\sum_{i=1}^{n} \alpha_i x_i\| \leq M \|\sum_{i=1}^{n+m} \alpha_i x_i\| \text{ for all } n,m \in \mathbb{N} \text{ and all } \alpha_1,\ldots,\alpha_{n+m} \in \mathbb{R}.$$

The minimal M with this property is called the *basis constant*. A basis is called *monotone* if $M=1$, and *normalized* if $\|x_n\|=1$ $(n=1,2,\ldots)$. Of course the coefficients α_n in the expansions $x=\sum_{n=1}^{\infty}\alpha_n x_n$ depend linearly on x. In fact $x\to\alpha_n(x)$ is a bounded linear functional, usually denoted by x_n^*. Hence for every basis (x_n) there is an associated sequence of *coefficient functionals* $(x_n^*)\subset X^*$, and we have

$$x=\sum_{n=1}^{\infty}\langle x,x_n^*\rangle x_n \quad (x\in X).$$

A basis (x_n) for X is called *boundedly complete* if for every sequence $\alpha_n\subset\mathbb{R}$, $\sum_{i=1}^{\infty}\alpha_i x_i$ converges whenever $(\sum_{i=1}^{n}\alpha_i x_i)_{n=1}^{\infty}$ is bounded. In most sequence spaces, such as l^p, $1\leqslant p<\infty$, (to be discussed below) the unit vectors (e_n) form a normalized, monotone, boundedly complete basis. A typical basis that fails to be boundedly complete, is the standard basis in c_0 (:= the null sequences, with the sup norm). A fact to be noted is that a Banach space X with a boundedly complete monotone basis (x_n) is isometrically isomorphic to a dual space ([G], Prop. 1.b.4). In fact, if $(x_n^*)\subset X^*$ is the associated sequence of coefficient functionals, and if $Z:=[x_n^*]$, then it is not very hard to see that

$$X\ni x\longrightarrow(\pi_X x)|_Z\in Z^*$$

maps X isometrically onto Z^*.

Let X,Y be Banach spaces and let $T:X\to Y$ be a bounded linear map. We call T an *isomorphism into* (or an *embedding*) if T is a linear homeomorphism onto TX, and an *isometric isomorphism into* (or an *isometric embedding*) if, in addition, T preserves the norm. Y is called (*isometrically*) *isomorphic* to X if Y is the range of such an (isometric) isomorphism from X. Notation: $X\simeq Y$ (resp. $X\cong Y$). We call a linear map $T:X\to Y$ a *quotient map* if, in addition to being a *continuous surjection*, it maps int $B(X)$ $(:=\{x:\|x\|<1\})$ onto int $B(Y)$. In this case $Y\cong X/kerT$. Of course by the open mapping theorem, if $T:X\to Y$ is a continuous surjection, then $Y\simeq X/kerT$. The adjoint $T^*:Y^*\to X^*$ of a bounded linear $T:X\to Y$ is bounded, and also w^*-w^*-continuous (i.e. continuous for the respective w^*-topologies on X^* and Y^*). In particular, since $B(Y^*)$ is w^*-compact by Alaoglu's theorem, it follows that $T^*B(Y^*)$ is w^*-compact. Observe that the second adjoint $T^{**}:X^{**}\to Y^{**}$ is also w^*-w^*-continuous, and, moreover, satisfies $T^{**}|_X=T$ (as usual, we identify X and Y with $\pi_X X$ and $\pi_Y Y$ respectively). In fact, by Goldstine's theorem T^{**} is uniquely determined by these two properties. Let us also note the following duality: T is an (isometric) embedding iff T^* is a continuous surjection (a quotient map); T is a continuous surjection (quotient map) iff T^* is an (isometric) embedding. If Y is a subspace of X and if one applies these observations to the identity embedding T from Y into X, and to the quotient map $X\to X/Y$, one finds the canonical isometries

$$Y^* \cong X^*/Y^\perp,\ (X/Y)^* \cong Y^\perp \subset X^*,\ Y^{**} \cong Y^{\perp\perp} \subset X^{**}.$$

(Here the "annihilator" $Y^\perp$ is the subspace $\{x^* \in X^* : \langle y,x^*\rangle = 0$ for every $y \in Y\}$, and $Y^{\perp\perp} = (Y^\perp)^\perp$.) It should be remarked here also that the canonical isometric embeddings $(X/Y)^* \hookrightarrow X^*$, $Y^{**} \hookrightarrow X^{**}$ are homeomorphisms (into) for the respective w^*-topologies. A bounded linear map $T:X \to Y$ is called (weakly) compact if $TB(X)$ is relatively *(weakly) compact* in Y. It is well known that $T:X \to Y$ is (w-)compact iff $T^*:Y^* \to X^*$ is (w-)compact. For a proof, see e.g. [92].

For any set Γ and $1 \leqslant p < \infty$, $l^p(\Gamma)$ is the space of real functions x on Γ such that $\sum_{\gamma \in \Gamma} |x(\gamma)|^p < \infty$, with norm $\|x\|_p := (\sum_{\gamma \in \Gamma} |x(\gamma)|^p)^{1/p}$. $l^\infty(\Gamma)$ denotes the space of all bounded real functions x on Γ, with $\|x\|_\infty := \sup |x(\gamma)|$, and $c_0(\Gamma)$ is the subspace of $l^\infty(\Gamma)$ consisting of all x such that $\{\gamma \in \Gamma : |x(\gamma)| > \epsilon\}$ is finite for every $\epsilon > 0$. We write l^p, l^∞, c_0 instead of $l^p(\mathbb{N})$, $l^\infty(\mathbb{N})$, $c_0(\mathbb{N})$. Furthermore, l_n^p denotes $l^p(\{1,...,n\})$. It is well known that $c_0(\Gamma)^* \cong l^1(\Gamma)$, and $l^p(\Gamma)^* \cong l^q(\Gamma)$ for $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For K compact Hausdorff, $C(K)$ is the space of all continuous real-valued functions f on K, with $\|f\| := \sup_K |f|$. We usually write C for $C([0,1])$. By the *Riesz representation theorem* $C(K)^* \cong M(K)$, the space all Radon (:= regular Borel-) measures μ on K, with $\|\mu\| := |\mu|(K)$, where $|\mu|$ denotes the variation of μ. An element of $M(K)$ is multiplicative on $C(K)$ iff it equals δ_x (:= the Dirac measure at x) for some $x \in K$. When (Ω,Σ,μ) is a measure space, $L^p(\mu) = L^p(\Omega,\Sigma,\mu)$, $1 \leqslant p < \infty$, is the space of μ-measurable functions f on Ω such that $\int_\Omega |f|^p d\mu < \infty$, with $\|f\|_p := (\int_\Omega |f|^p d\mu)^{1/p}$. $L^\infty(\mu)$ is the space of μ-measurable, μ-essentially bounded functions on Ω, with norm $\|f\|_\infty := \operatorname{ess\,sup}_\Omega |f|$. When (Ω,Σ,μ) is the Lebesgue measure space $[0,1]$ we write L^p and L^∞ for $L^p(\mu)$ and $L^\infty(\mu)$. $L^p(\mu)^* \cong L^q(\mu)$ for $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. $(L^\infty(\mu))^*$ consists of all finitely additive measures of bounded variation that vanish on the ideal of the μ-null sets ([D]). A subset $\Phi \subset L^1(\mu)$ is called *uniformly integrable* if $\lim_{\mu E \to 0} \sup_{f \in \Phi} \int_E |f| d\mu = 0$, i.e. if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mu E < \delta$ implies that $\int_E |f| d\mu < \epsilon$ for every $f \in \Phi$. The importance of uniform integrability derives from the fact that a set $\Phi \subset L^1(\mu)$ *is relatively weakly compact iff it is bounded and uniformly integrable* ([A], [D]). It is easily seen that $\Phi \subset L^1(\mu)$ is bounded and uniformly integrable iff $\lim_{\lambda \to \infty} \sup_{f \in \Phi} \int_{\{|f| > \lambda\}} |f| d\mu = 0$. (Some authors use the term "equi-integrable" for this last property.) $L^\infty(\mu)$ is not only a Banach space, but also a (commutative) C^*-algebra. Hence by the *Gelfand-Naimark theorem* there exists a compact Hausdorff space Δ (the maximal ideal space of $L^\infty(\mu)$, or the *Stone space* of (Ω,μ)) such that $L^\infty(\mu)$ is isometrically algebra isomorphic to $C(\Delta)$ (see [I]). This representation of $L^\infty(\mu)$ is sometimes convenient, especially when elements of $L^\infty(\mu)^*$ have to be considered. These are finitely additive measures on Ω, but can also be regarded as

Radon measures on Δ. In the case of $l^\infty = l^\infty(\mathbb{N})$ it is simpler to identify l^∞ directly with $C(\beta(\mathbb{N}))$, where $\beta\mathbb{N}$ is the Cech-Stone compactification of $\mathbb{N}$.

We now review some important properties of l^1 that will be taken for granted elsewhere in these notes. First of all,

every separable Banach space X is (isometric to) a quotient of l^1.

For the proof one simply takes a dense sequence (x_n) in $B(X)$ and defines $T: l^1 \to X$ by $T((\alpha_n)) := \sum_{n=1}^{\infty} \alpha_n x_n$ $((\alpha_n) \in l^1)$. Then clearly $\|T\| \leqslant 1$. Since $TB(l^1)$ contains all x_n, we have $\overline{TB(l^1)} = B(X)$, so T is a quotient map. The same proof shows that even if X is not separable, it is still a quotient of $l^1(\Gamma)$ for suitably large Γ. Another well-known property of l^1 is that

weakly convergent sequences are norm convergent (equivalently: *weak Cauchy sequences are norm Cauchy).*

Indeed, if for contradiction we assume that some sequence $(x^{(n)}) \subset l^1$ satisfies $\|x^{(n)}\| = 1$ $(n = 1,2,\ldots)$ and $x^{(n)} \xrightarrow{w} 0$, then one can show by passing to a subsequence and applying a standard perturbation argument (see [F] for details) that without loss of generality we may in addition assume that these $x^{(n)}$ have pairwise disjoint supports $S_n := \{k \in \mathbb{N} : x_k^{(n)} \neq 0\}$. Now if we define $y = (y_k) \in l^\infty$ by $y_k :=$ sign $x_k^{(n)}$ whenever $k \in S_n$, and $y_k = 0$ if $k \notin \bigcup_{n=1}^{\infty} S_n$, then $\langle x^{(n)}, y \rangle = \|x^{(n)}\| = 1$ for all $n \in \mathbb{N}$, contradicting the assumption that $x^{(n)} \xrightarrow{w} 0$.

This result immediately implies that the sequence (e_n) of unit vectors in l^1 has no w-Cauchy subsequence. On the other hand every bounded sequence $(x^{(n)})$ in $c_0(\Gamma)$ has a w-Cauchy subsequence. To see this, note that the union of the supports of the $x^{(n)}$ is countable, so that a diagonal procedure will produce a subsequence that converges "pointwise". Recalling now that $c_0(\Gamma)^* \cong l^1(\Gamma)$, it is easily seen that such a subsequence actually is w-Cauchy. In fact on bounded sets of $c_0(\Gamma)$, pointwise convergence is the same as weak convergence. Similarly, on bounded subsets of $l^\infty(\Gamma)$, w^*-convergence equals pointwise convergence. In particular, we may conclude from these observations that

l^1 *cannot be embedded in any* $c_0(\Gamma)$.

We shall call a bounded sequence $(x_n) \subset X$ an l^1*-sequence* if there exists a constant $c > 0$ such that

$$c \sum_{i=1}^{n} |\alpha_i| \leqslant \|\sum_{i=1}^{n} \alpha_i x_i\| \text{ for all } n \in \mathbb{N} \text{ and all } \alpha_1, \ldots, \alpha_n \in \mathbb{R}.$$

Observe that in fact we then have, putting $C := \sup \|x_i\|$,

$$c\sum_{i=1}^{n}|\alpha_i|\leqslant\|\sum_{i=1}^{n}\alpha_i x_i\|\leqslant C\sum_{i=1}^{n}|\alpha_i| \text{ for all } n\in\mathbb{N} \text{ and all } \alpha_1,\ldots,\alpha_n\in\mathbb{R}.$$

These inequalities say that the map T defined on $sp(e_n)\subset l^1$ by $T(\sum_{i=1}^{n}\alpha_i e_i):=\sum_{i=1}^{n}\alpha_i x_i$ is an isomorphism. Therefore it extends uniquely to an embedding $T:l^1\to X$. We have now proved the non-trivial half of the following statement:

l^1 embeds in X iff X contains an l^1-sequence.

An immediate consequence is that

if X is isomorphic to a quotient of Y, and l^1 embeds in X, then l^1 embeds in Y.

Indeed, if $T:Y\to X$ is a continuous surjection, and if (x_n) is an l^1-sequence in X, then any bounded "lifting" of (x_n), i.e. every bounded sequence $(y_n)\subset Y$ such that $Ty_n=x_n$ $(n=1,2,\ldots)$, will clearly be an l^1-sequence in Y. The above assertion obviously generalizes: $l^1(\Gamma)$ embeds in Y whenever $l^1(\Gamma)$ embeds in X.

A large part of these notes will be directly or indirectly concerned with the delicate question whether or not l^1 embeds in a given space X. There is also the related question whether l^1 can be embedded in X as a *complemented subspace* (a subspace is called *complemented* if it is the range of a bounded linear projection). The answer to this last question is much easier and has been known for quite some time. It follows in a fairly straighforward way from the basic sequences techniques developed by C. Bessaga and A. Pelczynski. We refer the interested reader to (e.g.) the discussion in [C], in particular, to Th. 10 on p. 48.

We call a Banach space X *injective* (or a *$\mathcal{P}_1$-space*) if the Hahn-Banach extension theorem holds for bounded linear operators into X (rather than for bounded linear maps into $\mathbb{R}$), i.e. if for every Banach space Z and for every bounded linear operator $T:Y\to X$ defined on a subspace Y of Z, there exists a bounded linear $\tilde{T}:Z\to X$ so that $\tilde{T}|_Y=T$ and $\|\tilde{T}\|=\|T\|$. The simplest examples of injective Banach spaces are l^∞ and $L^\infty(\mu)$ and, more generally, every $C(K)$ with K compact and *extremally disconnected* (this means that the closure of every open subset of K is open). The proof that these $C(K)$ are injective is essentially the same as that of the classical Hahn-Banach theorem, if one uses the fact that $C(K)$, K compact, is *order complete* iff K is extremally disconnected (a Banach lattice is order complete if each order bounded subset has a least upper bound). See [G] for details.

At one point we shall need to know also that $C^{**}=C[0,1]^{**}$ is injective.

Probably the easiest way to see this is to appeal to Kakutani's theorems on abstract L_p- spaces and abstract M-spaces (see e.g. [G]). Since C is an abstract M-space, C^* is an abstract L^1-space, hence isometric to some concrete $L^1(\mu)$. But then $C^{**} \cong L^\infty(\mu)$, and therefore injective.

Recall that a point x of a convex set K is called *extreme* if $K \setminus \{x\}$ is convex. The *Krein-Milman theorem* says that if K is a compact convex subset of a locally convex space (l.c.s.) then there is an abundance of extreme points. In fact K can be recovered from its extreme points by taking their closed convex hull: $K = \overline{co}$ ext K. If K is n-dimensional, then we even have $K = co$ ext K and, moreover, every point of K can be written as a convex combination of no more than $n+1$ points (this is an old result of *Caratheodory*). There is a kind of converse to the Krein-Milman theorem, known as *Milman's theorem.* It says that $\overline{\text{ext}\,K}$ is the smallest among the closed subsets F of K with the property that $\overline{co}F = K$. In other words, if $\overline{co}F = K$ ($F \subset K$ closed), then ext $K \subset F$. The Krein-Milman theorem can be sharpened considerably if K is metrizable. A Radon probability measure μ on K is said to *represent* a point $x \in K$ if $\int_K f d\mu = f(x)$ for all $f \in A(K)$, where $A(K)$ denotes the set of all affine continuous functions on K. In this case one also calls x the *barycenter* (or *resultant*) of μ. E.g. if $x = \frac{1}{2}x_1 + \frac{1}{2}x_2$ ($x_1, x_2 \in K$), then x is represented by $\frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2}$. Clearly a point may have many representing measures. It is now an easy exercise to show that the following statement is equivalent to the Krein-Milman theorem: every $x \in K$ (K compact convex in a l.c.s.) is the barycenter of a measure μ supported by $\overline{\text{ext}\,K}$. *Choquet's theorem* strengthens this assertion considerably when K is metrizable: *every $x \in K$ is represented by a Radon probability supported by ext K* (rather than $\overline{\text{ext}\,K}$). There is also a version of this theorem for non-metrizable K, but we shall not need this (see [H]).

Let Σ be a σ-algebra of subsets of set Ω. A map $F:\Sigma \to X$, X a Banach space, is called a (*vector*) *measure* if $F(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty FE_n$ whenever $E_n \in \Sigma$ ($n = 1,2,\ldots$) and $E_n \cap E_m = \varnothing$ if $n \neq m$. (The series $\sum_{n=1}^\infty FE_n$ is meant to converge in norm.) If this equality is required only for finite disjoint unions, F is called a *finitely additive (f.a.) vector measure.* Suppose we are also given a nonnegative measure μ on Σ. Then F is called *μ-continuous*, or absolutely continuous with respect to μ (notation: $F << \mu$) if $\lim_{\mu E \to 0} FE = 0$. If F is a vector measure (i.e. countably additive), then F is μ-continuous iff $FE = 0$ whenever $\mu E = 0$ ($E \in \Sigma$). A *f.a.* vector measure F is countably additive iff $\lim_{n \to \infty} FE_n = 0$ for every sequence $(E_n) \subset \Sigma$ such that $E_n \downarrow \varnothing$ (this means $E_{n+1} \subset E_n$, $n = 1,2,\ldots$, and $\bigcap_{n=1}^\infty E_n = \varnothing$). An immediate consequence of this is that a *f.a.* F is c.a. (= countably additive) if F is μ-continuous.

The *variation* $|F|$ of a *f.a.* F is defined by $|F|(E):=\sup\sum_{i=1}^{n}\|FE_i\|$, where the sup is taken over all finite partitions $\{E_1,\ldots,E_n\}$ of E into sets of Σ. If $|F|(\Omega)<\infty$, then F is called a measure of *bounded variation.* $|F|$ is always *f.a.*, and is c.a. iff F is.

A function $f:\Omega\to X$ is said to be *weakly μ-measurable* (or *scalarly μ-measurable*) if $\langle f(\cdot),x^*\rangle$ is μ-measurable for every $x^*\in X^*$. A more restrictive notion is that of *strong-* or *Bochner μ-measurability*. By definition f is *strongly μ-measurable* if there exists a sequence (f_n) of X-valued simple functions so that $\lim_{n\to\infty} f_n=f$ μ a.e. One can show that $f:\Omega\to X$ is strongly μ-measurable iff f is *μ-essentially separably valued* (i.e. $f(\Omega\setminus N)$ is separable for some μ-null set N) and Borel measurable (i.e. $f^{-1}B$ is μ-measurable for every Borel set $B\subset X$). The famous *Pettis measurability theorem* says that for μ-essentially separably valued f, strong and weak measurability are the same.

For simple functions $f=\sum_{i=1}^{n}x_i\chi_{E_i}$ $(x_i\in X,\ E_i\in\Sigma)$ the *Bochner integral* $(B)\int_\Omega fd\mu$ is defined to be $\sum_{i=1}^{n}x_i\mu E_i$. Clearly the triangle inequality implies that for each such f,

$$\|(B)\int_\Omega fd\mu\|\leqslant\int_\Omega\|f\|d\mu=:\|f\|_1,$$

so the map $f\to(B)\int_\Omega fd\mu\in X$ is a contraction. If one now completes the space of simple functions equipped with the norm $\|\cdot\|_1$, one arrives at the space $L_X^1(\mu)$ of X-valued *Bochner integrable* functions. The Bochner integral $(B)\int_\Omega fd\mu$ of $f\in L_X^1(\mu)$ is defined by extending the above contraction to the completion $L_X^1(\mu)$. More concretely, for $f\in L_X^1(\mu)$ one chooses a sequence (f_n) of simple functions so that $\int_\Omega\|f-f_n\|d\mu\to 0$ and defines $(B)\int_\Omega fd\mu:=\lim_{n\to\infty}(B)\int_\Omega f_nd\mu$ (this limit exists and is independent of the choice of (f_n)). The inequality

$$\|(B)\int_\Omega fd\mu\|\leqslant\int_\Omega\|f\|d\mu=\|f\|_1 \text{ persists for } f\in L_X^1(\mu).$$

A strongly measurable f belongs to $L_X^1(\mu)$ iff $\int_\Omega\|f\|d\mu<\infty$.

If $f\in L_X^1(\mu)$ then the formula

$$FE:=(B)\int_E fd\mu:=(B)\int_\Omega\chi_E\cdot fd\mu$$

is easily seen to define a μ-continuous vector measure of bounded variation (in fact $|F|(E)=\int_E\|f\|d\mu$, $E\in\Sigma$). X is said to have the *RNP (= Radon Nikodym property) with respect to* (Ω,Σ,μ) if, conversely, every μ-continuous X-valued measure F of bounded variation is of this form. The integrand f is then called the *RN derivative* of F. X has the *RNP* if it has the RNP with respect to every finite measure space. It is known that X has the RNP iff X has the RNP with respect to the Lebesgue measure space [0,1]. For details on vector measures and much more information the reader should look at [D]. Much is known about spaces with the RNP. They have been characterized in terms of martingales and their geometry is known in great detail (see [B], [D], [17]). Let us mention, by way of example, that X has the RNP iff every uniformly bounded X-valued martingale defined on the Lebesgue measure space [0,1], converges a.e. Since $\mathbb{R}$ has the RNP by the classical Radon Nikodym theorem, in particular every uniformly bounded real-valued martingale on [0,1] converges a.e. This is the well-known *martingale convergence theorem.*

Vector measures are an important tool in the study of operators on function spaces. E.g. if $T: L^\infty(\mu) \to X$ is a bounded linear operator, then $FE := T\chi_E \in X$ $(E \in \Sigma)$ defines a $f.a.$ vector measure. Since T is determined by its values on the characteristic functions χ_E, the $f.a$ measure F in fact represents T. Various properties of F are reflected in those of T and vice versa (see [D], Ch. VI). We mention here one result of this type of analysis because we shall need it in Chapter 4. Let T be as above. Then either T is weakly compact, or T acts as an isomorphism on some subspace of $L^\infty(\mu)$ isometric to l^∞. Since l^∞ is non-separable, the last possibility is excluded if X is separable. Hence: *every bounded $T: L^\infty(\mu) \to X$ is weakly compact if X is separable.* In particular, since l^∞ is itself an $L^\infty(\mu)$-space, *every separable quotient of l^∞ is reflexive* ([D]).

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Chapter I

Fremlin's subsequences theorem

This chapter is devoted to a remarkable result of Fremlin's that expresses a certain dichotomy for sequences of measurable functions: a sequence of real-valued measurable functions on a perfect measure space (Ω, Σ, μ) either has a "good" subsequence or a "bad" one (or both). Here "good" means "μ a.e. convergent" and "bad" stands for "all pointwise cluster points are non-measurable". At the heart of the proof of this theorem are certain facts on measures of free filters on $\mathbb{N}$ (considered as subsets of $\{0, 1\}^{\mathbb{N}}$) which we shall develop first. For elementary background information on filters and on perfect measure spaces the reader should consult Appendices *A* and *F*.

Although more generality is possible we shall consider filters on $\mathbb{N}$ only. Every subset $A \subset \mathbb{N}$ may be identified with the point $\chi_A \in \{0, 1\}^{\mathbb{N}}$. Hence a filter $\mathcal{F}$ on $\mathbb{N}$ identifies with the subset $\{\chi_A : A \in \mathcal{F}\}$ of $\{0, 1\}^{\mathbb{N}}$. We shall make these identifications without notational distinction, so A and $\mathcal{F}$ will be considered as points, respectively subsets of $\{0, 1\}^{\mathbb{N}}$ whenever this is convenient. Points of $\{0, 1\}^{\mathbb{N}}$ will generally be denoted as (ϵ_n), where $\epsilon_n = 0$ or 1. On $\{0, 1\}^{\mathbb{N}}$ we put the (completed) product measure $\mu := (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_0)^{\mathbb{N}}$ (δ_x is the Dirac measure at x). We are interested in the μ-measure of certain filters $\mathcal{F}$. Before we state a result on this, let us make some preliminary remarks. Consider the map $\phi: \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ defined by

$$\phi(\epsilon_1, \epsilon_2, \ldots) := (1-\epsilon_1,\ 1-\epsilon_2, \ldots).$$

The following facts will be needed.

Remark 1.1
(i) ϕ and ϕ^{-1} are measurable and preserve μ-measure.
(ii) ϕ maps a subset $A \subset \mathbb{N}$ to its complement $\mathbb{N} \setminus A$. Hence $\phi(\mathcal{F}) \cap \mathcal{F} = \varnothing$ whenever $\mathcal{F}$ is a filter on $\mathbb{N}$. $\mathcal{F}$ is an ultrafilter iff $\phi(\mathcal{F}) \cup \mathcal{F} = \{0, 1\}^{\mathbb{N}}$. These facts are direct consequences of the elementary properties of filters discussed in Appendix *F*.
(iii) If $\mathcal{F}$ is free, then $\mathbb{N} \setminus \{n\} \in \mathcal{F}$ for every $n \in \mathbb{N}$, so $\mathbb{N} \setminus F \in \mathcal{F}$ for every finite $F \subset \mathbb{N}$. Hence if $\epsilon' = (\epsilon'_n)$ differs from $\epsilon = (\epsilon_n)$ in only finitely many coordinates, then $\epsilon \in \mathcal{F}$ iff $\epsilon' \in \mathcal{F}$. Formulated in probabilistic terms this means that $\mathcal{F}$ (and also its complement) is a tail event, i.e. for each $k \in \mathbb{N}$ $\mathcal{F}$ has the form $\mathcal{F} = \{0, 1\}^k \times G_k$, where $G_k \subset \prod_{n=k+1}^{\infty} \{0, 1\}$. $\square$

Proposition 1.2. *Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Then*

(i) $\mu_* \mathcal{F} = 0$,
(ii) $\mu^* \mathcal{F} \in \{0, 1\}$,
(iii) $\mu^* \mathcal{F} = 1$ *whenever* $\mathcal{F}$ *is an ultrafilter.*

PROOF Let us first recall that a measurable tail event has μ-measure 0 or 1. The same is true for the inner and outer measures of the (generally non-measurable) tail events $\mathcal{F}$ and its complement, since it is easily verified that

$$\mu_* \mathcal{F} = \sup \{\mu B : B \subset \mathcal{F} \text{ a measurable tail event}\}$$

and , by complementation (or directly)

$$\mu^* \mathcal{F} = \inf\{\mu B : B \supset \mathcal{F} \text{ a measurable tail event}\}.$$

So $\mu^* \mathcal{F}$, $\mu_* \mathcal{F} \in \{0, 1\}$ and in particular (ii) is proved.

To prove (i), note that by Remark 1.1 (ii) we have $\phi(\mathcal{F}) \cap \mathcal{F} = \varnothing$. Since ϕ is measure-preserving,

$$1 = \mu\{0, 1\}^{\mathbb{N}} \geqslant \mu_* \phi(\mathcal{F}) + \mu_* \mathcal{F} = 2\mu_* \mathcal{F}.$$

So $\mu_* \mathcal{F} \leqslant \frac{1}{2}$, hence $\mu_* \mathcal{F} = 0$ since we have just proved that $\mu_* \mathcal{F} \in \{0, 1\}$.

Finally, in the case of an ultrafilter we have $\phi(\mathcal{F}) \cup \mathcal{F} = \{0, 1\}^{\mathbb{N}}$ by Remark 1.1 (ii) and this implies

$$1 = \mu\{0, 1\}^{\mathbb{N}} \leqslant \mu^* \phi(\mathcal{F}) + \mu^* \mathcal{F} = 2\mu^* \mathcal{F}.$$

Therefore $\mu^* \mathcal{F} \geqslant \frac{1}{2}$ and so $\mu^* \mathcal{F} = 1$, by (ii). $\square$

COROLLARY 1.3 *A free filter is measurable iff* $\mu^* \mathcal{F} = 0$ *and non-measurable iff* $\mu^* \mathcal{F} = 1$. *Free ultrafilters are always non-measurable.*

PROOF: obvious. $\square$

REMARK 1.4 It is well known that the map $[0, 1] \to \{0, 1\}^{\mathbb{N}}$ that sends $t \in [0, 1]$ to the sequence of its dyadic coefficients establishes an isomorphism between the measure spaces $[0, 1]$ (with the Lebesgue measure) and $\{0, 1\}^{\mathbb{N}}$ (with the complete measure μ). Each free ultrafilter $\mathcal{F} \subset \{0, 1\}^{\mathbb{N}}$ then corresponds to a non-Lebesgue measurable subset of [0,1]. There are 2^c such sets because the cardinality of $\beta\mathbb{N} \setminus \mathbb{N}$ is 2^c and the free ultrafilters $\mathcal{F}$ on $\mathbb{N}$ are in 1-1 correspondence with the points $t \in \beta\mathbb{N} \setminus \mathbb{N}$, via the map $\mathcal{F} \to t := \cap \{\overline{A}^{\beta\mathbb{N}} : A \in \mathcal{F}\}$ (see e.g. [30]). $\square$

For the proof of Fremlin's theorem we need a refinement of Prop. 1.2 (iii). Let $o < a < 1$ and let μ_a be the completed product measure $(a\,\delta_1 + (1-a)\,\delta_0)^{\mathbb{N}}$ on $\{0, 1\}^{\mathbb{N}}$. By the same argument sketched above for the special case $a = \frac{1}{2}$ one shows that $\mu_a^* \mathcal{F}$, $\mu_{a*} \mathcal{F} \in \{0, 1\}$ for any $0 < a < 1$ when $\mathcal{F}$ is free. It can also be proved that $\mu_{a*} \mathcal{F} = 0$ but we shall not need this here. Part (iii) of Prop. 1.2 also generalizes: $\mu_a^* \mathcal{F} = 1$ when $\mathcal{F}$ is a free ultrafilter. We do need this last fact, but only for $a = 2^{-k}$, $k \in \mathbb{N}$, which simplifies the proof somewhat.

PROPOSITION 1.5 *Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$ and let $a = 2^{-k}$ for some $k \in \mathbb{N}$. Then $\mu_a^* \mathcal{F} = 1$.*

PROOF. Let us put $K := \{0, 1\}^{\mathbb{N}}$ and let us denote the points of K^k by $(\epsilon^i)_{i=1}^k$, where $\epsilon^i = (\epsilon_n^i)$. Consider the map $\psi : K^k \to K$ defined by

$$\psi((\epsilon^i)) := (\prod_{i=1}^{k} \epsilon_n^i)$$

, or alternatively,

$$\psi(A_1, \ldots, A_k) := \bigcap_{i=1}^{k} A_i \qquad (A_i \subset \mathbb{N},\ i = 1,.,k).$$

It is easy to verify that $\psi(\mu^k) = \mu_a$. Let us also observe that $\psi(\mathcal{F}^k) \subset \mathcal{F}$. Hence

$$\mu_a^* \mathcal{F} = (\psi\mu^k)^* \mathcal{F} \geqslant (\psi\mu^k)^* (\psi(\mathcal{F}^k)) \geqslant (\mu^k)^* \mathcal{F}^k$$

(for the last inequality see the trivial half of the proof of Prop. A.7). Since $\mu^* \mathcal{F} = 1$ by Prop. 1.2 (iii), it is now an easy exercise (cf. Cor. C. 3) to show that $(\mu^k)^* \mathcal{F}^k = 1$ also. This completes the proof. □

One final point needs to be explained before we can state Fremlin's theorem. Let Φ be a set of functions defined on a set Ω and taking their values in a topological space T. Then Φ can be identified with a subset of T^Ω. The *topology τ_p of pointwise convergence* (or the pointwise topology, for short) on Φ is by definition the topology that Φ inherits from T^Ω when the latter space is equipped with the product topology. A *τ_p-cluster point f* of Φ is an element of T^Ω satisfying $f \in \overline{\Phi \setminus \{f\}}$ (closure in T^Ω). A *cluster point of a sequence* (f_n) is a cluster point of the set $\{f_n : n \in \mathbb{N}\}$.

THEOREM 1.6 *Let (Ω, Σ, μ) be a perfect probability space and let (f_n) be a sequence of $\overline{\mathbb{R}}$-valued measurable functions on Ω ($\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$). Then either*
(i) (f_n) has a μ a.e. convergent subsequence or
(ii) (f_n) has a subsequence all of whose τ_p-cluster points are non-measurable (hence in particular this subsequence is τ_p-discrete).

We shall first look at the special case of the Rademacher sequence (r_n) on [0, 1], with the Lebesgue measure λ. Here the situation is simpler and in fact motivates the proof of Th. 1.6. Basically the general case is handled by reducing it to a situation that displays the essential features of the Rademacher system.

Let us recall that $r_n(t) := sgn \sin 2^n \pi t$ $(0 \leqslant t \leqslant 1; n \in \mathbb{N})$. Clearly (r_n) has no λ a.e. convergent subsequence, by Lebesgue's theorem and the fact that $\|r_n - r_m\|_1 = 1$ whenever $n \neq m$. Therefore, if the theorem is to be true, (r_n) should have no measurable τ_p-cluster points. This we now show.

Suppose r is a cluster point of (r_n). Then clearly $r^+ := \max(r, 0)$ (resp. $r^- := \max(-r, 0)$) is a cluster point of (r_n^+) (resp. (r_n^-)). We shall show that

$$\lambda^* \{r^+ = 1\} = \lambda^* \{r^- = 1\} = 1,$$

which proves that r is non-measurable. For reasons of symmetry it suffices to show that $\lambda^* \{r^+ = 1\} = 1$. For this we consider the (measurable) map

$$[0, 1] \ni t \to \psi(t) := (r_n^+(t)) \in \{0, 1\}^{\mathbb{N}}.$$

The independence of the r_n^+ easily implies that $\psi\lambda = \mu := (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\mathbb{N}}$. By Remark F.8 (iii) the τ_p-cluster point r^+ can be represented as $r^+ = \tau_p\text{-}\lim_{\mathfrak{F}} r_n^+$, where $\mathfrak{F}$ is a free ultrafilter on $\mathbb{N}$. Now $r^+ = \lim_{\mathfrak{F}} r_n^+$ implies $r^+ \in \tau_p\text{-}cl\{r_n^+ : n \in A\}$ for each $A \in \mathfrak{F}$, and therefore

$$r^+(t) \in \overline{\{r_n^+(t) : n \in A\}} \text{ for every } A \in \mathfrak{F} \text{ and every } t \in [0, 1].$$

Consequently

$$r^+(t) = 1 \text{ whenever } \{n \in \mathbb{N} : r_n^+(t) = 1\} \in \mathfrak{F}.$$

Regarding $\mathfrak{F}$ as a subset of $\{0, 1\}^{\mathbb{N}}$ again, this can be restated as follows in terms of ψ:

$$r^+(t) = 1 \text{ whenever } \psi(t) \in \mathfrak{F}$$

or

$$\{r^+ = 1\} \supset \psi^{-1}\mathfrak{F}.$$

Let us observe now that $\lambda^* (\psi^{-1}\mathfrak{F}) = \mu^* \mathfrak{F}$ by Prop. A. 7, since $\psi\lambda = \mu$ and λ is perfect. We now conclude that

$$\lambda^* \{r^+ = 1\} \geqslant \lambda^*(\psi^{-1}\mathfrak{F}) = \mu^* \mathfrak{F} = 1,$$

where the last equality follows from Prop. 1.2 (iii)

We are now fully prepared for the

Proof of Theorem 1.6 First of all we may assume that μ has no atoms. Indeed, every f_n is constant on every atom and there are at most countably many atoms, so if we can show that the f_n restricted to the (perfect!) atomless part of Ω have the required subsequence then at most a diagonal procedure (in case (i)) is needed to produce the right subsequence on the full space Ω.

Secondly, we may assume the f_n are valued in [0,1], since $\overline{\mathbb{R}}$ is homeomorphic to [0,1]. Furthermore, by passing to a subsequence if necessary, we may suppose that $f_n \xrightarrow{w} f$ in $L^2(\mu)$. Replacing (f_n) by $(f_n - f)$, we then have $f_n \xrightarrow{w} 0$. (Note that the "new" f_n are now valued in [-1,1]). Passing to a further subsequence let us also suppose that $f_n^+ \xrightarrow{w} g$ in $L^2(\mu)$. Then necessarily $f_n^- \xrightarrow{w} g$.

There are now two cases to be considered. If $g = 0$ (μ a.e.), then $|f_n| \xrightarrow{w} 0$. This easily implies $\|f_n\|_1 \to 0$, and so by standard reasoning (f_n) has a μ a.e.

convergent subsequence in this case. If on the other hand $g \neq 0$ (note that $g \geqslant 0$ μ a.e.) then, as we shall show below, (f_n) has a subsequence without any measurable cluster points.

Let $k \in \mathbb{N}$ (k will be fixed in a moment). By passing to yet another subsequence we may suppose that

$$(\chi_{\{f_n \geqslant 2^{-k}\}}) \text{ and } (\chi_{\{f_n \leqslant -2^{-k}\}})$$

both converge weakly in $L^2(\mu)$, say to g_k and g'_k, respectively. Since

$$f_n^+ \leqslant \chi_{\{f_n \geqslant 2^{-k}\}} + 2^{-k} \text{ and } f_n^- \leqslant \chi_{\{f_n \leqslant -2^{-k}\}} + 2^{-k}$$

for all $n \in \mathbb{N}$, it follows by passing to the weak limit as $n \to \infty$ that

$$g \leqslant g_k + 2^{-k} \text{ and } g \leqslant g'_k + 2^{-k} \quad \mu \text{ a.e.} \tag{1}$$

Now let us fix k so large that $A := \{g > 2^{-k+1}\}$ has positive μ-measure. Then (1) implies

$$g_k, g'_k > 2^{-k} \text{ on } A, \ \mu(A) > 0.$$

We now restrict all functions to A (and normalize the restriction μ_A of μ to A for convenience) and show that the restricted functions f_n have a subsequence with no μ_A-measurable cluster point on A (then neither do the original functions have a μ-measurable cluster point).

To simplify the notation, let us put

$$\Omega := A, \ \mu := \frac{\mu_A}{\|\mu_A\|}, \ a := 2^{-k} \text{ and } g := g_k|_A, g' := g'_k|_A, f_n := f_n|_A.$$

Then

$$\chi_{\{f_n \geqslant a\}} \xrightarrow{w} g, \ \chi_{\{f_n \leqslant -a\}} \xrightarrow{w} g' \text{ in } L^2(\mu), \ g, g' > a \text{ on } \Omega.$$

We have now reached a situation roughly resembling that in the special case discussed earlier (the functions $\chi_{\{f_n \geqslant a\}}$ and $\chi_{\{f_n \leqslant -a\}}$ should be compared to r_n^+ and r_n^-, respectively). We could try to proceed as in the special case and use the functions $\chi_{\{f_n \geqslant a\}}$ to produce a map ψ from Ω to $\{0, 1\}^{\mathbb{N}}$. The problem is that the sets $X_n := \{f_n \geqslant a\}$ are not independent and therefore $\psi\mu$ will fail to be a product measure. However, the fact that $\chi_{\{f_n \geqslant a\}} \xrightarrow{w} g$ and $g > a$ on Ω enables us to remedy this situation by passing to a subsequence of subsets $U_n \subset X_n = \{f_n \geqslant a\}$ which is independent. Specifically, we construct inductively a subsequence (n_k) of $\mathbb{N}$ and sets $U_k \in \Sigma$ so that

(i) $U_k \subset X_{n_k}$ $(k = 1, 2, \ldots)$,

(ii) $P(U_k \mid U_1, \ldots, U_{k-1}) = a$ $(k = 1, 2, \ldots)$.

(Here P denotes conditional probability. Explicitly (ii) means that for each A in the σ-algebra Σ_{k-1} generated by $U_1, \ldots, U_{k-1}$ we have $\mu(U_k \cap A) = a\,\mu A$.)

For the choice of n_1 and U_1, observe that $\chi_{\{f_n \geqslant a\}} \xrightarrow{w} g$ implies

$\mu X_n \to \int_\Omega g d\mu > a$. Now choose n_1 so that $\mu X_{n_1} > a$ and then select $U_1 \subset X_{n_1}$, $U_1 \in \Sigma$ with $\mu U_1 = a$ (remember that μ has no atoms). Suppose now that $n_1 < \ldots < n_{k-1}$ and $U_1, \ldots, U_{k-1}$ have been properly constructed. Let A be any atom of the (finite) Σ-algebra Σ_{k-1}. Then $\chi_{\{f_n \geqslant a\}} \xrightarrow{w} g$ implies

$$\mu(X_n \cap A) = \int_\Omega \chi_{\{f_n \geqslant a\}} \cdot \chi_A d\mu \to \int_\Omega g \chi_A d\mu = \int_A g d\mu > a\mu A.$$

Therefore we may choose $n_k > n_{k-1}$ so that

$$\mu(X_{n_k} \cap A) > a\mu A \text{ for every atom } A \text{ of } \Sigma_{k-1}.$$

Finally, for each such A let $U_A \subset X_{n_k} \cap A$, $U_A \in \Sigma$ be such that $\mu(U_A) = a\mu A$. Then the union of the U_A's is clearly a correct choice for U_k. This completes the induction.

We now pass to the subsequence (f_{n_k}). For simplicity of notation we relabel the f_{n_k} and X_{n_k} as f_n and X_n. Then we have $U_n \subset X_n$ $(n = 1, 2, \ldots)$. We are now ready to finish the proof by showing that a τ_p-limit f of (f_n) along any free ultrafilter $\mathfrak{F}$ satisfies $\mu^*\{f \geqslant a\} = 1$. Since the same argument can be used after one more passage to a subsequence (use the sets $\{f_n \leqslant -a\}$) to show that both $\mu^*\{f \geqslant a\} = 1$ and $\mu^*\{f \leqslant -a\} = 1$, the conclusion is that f is non-measurable.

We use the sets U_n to define

$$\Omega \ni t \to \psi(t) := (\chi_{U_n}(t)) \in \{0, 1\}^{\mathbb{N}}.$$

The important thing to notice is that the independence condition $P(U_k \mid U_1, \ldots, U_{k-1}) = a$ means precisely that $\psi\mu = \mu_a$ $(:= (a\delta_1 + (1-a)\delta_0)^{\mathbb{N}})$. Let $t \in \Omega$ be such that $\{n \in \mathbb{N} : t \in U_n\} \in \mathfrak{F}$. Then the larger set $\{n \in \mathbb{N} : t \in X_n\} = \{n \in \mathbb{N} : f_n(t) \geqslant a\}$ also belongs to $\mathfrak{F}$, so $f(t) \geqslant a$. Considering $\mathfrak{F}$ as a subset of $\{0, 1\}^{\mathbb{N}}$ again, this means that

$$f(t) \geqslant a \text{ whenever } \psi(t) \in \mathfrak{F},$$

or

$$\{f \geqslant a\} \supset \psi^{-1}\mathfrak{F}.$$

The same argument we used in the special case (this time relying on Prop. 1.5) now yields

$$\mu^*\{f \geqslant a\} \geqslant \mu^*(\psi^{-1}\mathfrak{F}) = \mu_a^*\mathfrak{F} = 1. \qquad \square$$

NOTES Theorem 1.6 was first proved in [26]. The proof presented here follows M. Talagrand's memoir [92]. The preliminary material on measurability of filters was developed by M. Talagrand in [91] (in greater generality). In that paper he also proves that the condition in Fremlin's theorem that the measure space be perfect, is essential (cf. the notes to Ch. 5).

Chapter II

Stable sets of measurable functions

Let (Ω, Σ, μ) be a complete finite measure space and let $M(\mu)$ denote the set of all μ-measurable real-valued function on Ω. In this chapter we shall introduce and study μ-stable subsets of $M(\mu)$. These are sets that satisfy an explicit criterion that implies their relative τ_p-compactness in $M(\mu)$. The converse implication is not true (cf. Th. 2.7), although in some sense μ-stability is close to relative τ_p-compactness in $M(\mu)$, as is illustrated by Propositions 2.4 and 2.5. Let us emphasize that we regard $M(\mu)$ as a subset of $\mathbb{R}^\Omega$, so no identification of μ a.e. equal functions is made. Also *all subsets Z of $M(\mu)$ to be considered in this chapter will be assumed to be τ_p-bounded*, i.e. relatively compact in $\mathbb{R}^\Omega$. When we say that Z is relatively τ_p-compact in $M(\mu)$, we mean that τ_p-cl $Z \subset M(\mu)$.

Before we can give the definition of μ-stability we must take a closer look at what it means for a subset $Z \subset M(\mu)$ not to be relatively τ_p-compact (in $M(\mu)$) i.e. to have a non-measurable τ_p-cluster point. Let us observe first that if $A \notin \Sigma$, then there exists a $B \in \Sigma$ such that $\mu B > 0$ and

$$\mu^* (A \cap B) = \mu B \text{ and } \mu_* (A \cap B) = 0. \tag{1}$$

Indeed, choose $E, F \in \Sigma$ so that $E \subset A \subset F$ and $\mu E = \mu_* A < \mu^* A = \mu F$. Then (1) is satisfied with $B := F \setminus E$.

Next we prove a simple characterization of non-measurable functions that is the key to the definition of μ-stability.

Lemma 2.1 *Let $f: \Omega \to \mathbb{R}$ be non-measurable. Then there exist numbers $\alpha < \beta$ and a $B \in \Sigma$ with $\mu B > 0$ such that*

$$\mu^* (B \cap \{f<\alpha\}) = \mu^* (B \cap \{f>\beta\}) = \mu B. \tag{2}$$

Proof Choose $\gamma \in \mathbb{R}$ so that $\{f \leqslant \gamma\} \notin \Sigma$. Then select $E \in \Sigma$ so that

$$E \supset \{f \leqslant \gamma\} \text{ and } \mu E = \mu^* \{f \leqslant \gamma\} \quad (>0).$$

Then

$$\mu^* (E \cap \{f>\gamma\}) = \mu^* (E \cap (\bigcup_{n=1}^{\infty} \{f>\gamma+\frac{1}{n}\})) > 0$$

and therefore there exists an $n \in \mathbb{N}$ so that with $\beta := \gamma + \frac{1}{n}$ we have $\mu^* (E \cap \{f>\beta\}) > 0$. Now choose $F \in \Sigma$ so that

$$F \supset E \cap \{f>\beta\} \text{ and } \mu F = \mu^*(E \cap \{f>\beta\}).$$

Then $B := E \cap F$ satisfies $\mu B > 0$ since $B \supset E \cap \{f>\beta\}$. It is now easily seen from the definitions of E and F that for any α such that $\gamma<\alpha<\beta$ we have (2). Indeed, $\mu_*(B \cap \{f \geqslant \alpha\}) \leqslant \mu_*(B \cap \{f>\gamma\}) \leqslant \mu_*(E \cap \{f>\gamma\}) = 0$ and $\mu_*(B \cap \{f \leqslant \beta\}) \leqslant \mu_*(F \setminus (E \cap \{f>\beta\})) = 0$. □

Suppose now that a subset $Z \subset M(\mu)$ has a non-measurable τ_p-cluster point h. By the lemma we may choose numbers $\alpha<\beta$ and an $A \in \Sigma$, $\mu A > 0$ so that the sets

$$U := \{h<\alpha\} \cap A \text{ and } V := \{h>\beta\} \cap A$$

satisfy $\mu^* U = \mu^* V = \mu A$. It is now a consequence of h being in the τ_p-closure of Z that

$$\forall k, l \in \mathbb{N} \quad U^k \times V^l \subset \bigcup_{f \in Z} \{f<\alpha\}^k \times \{f>\beta\}^l. \tag{3}$$

Let us note now that by Corollary C. 3 we have $\mu^*_{k+l}(U^k \times V^l) = (\mu A)^{k+l}$ (for convenience we now write μ_k instead of μ^k for the product measure). So (3) implies

$$\forall k, l \in \mathbb{N} \quad \mu^*_{k+l} \bigcup_{f \in Z} ((\{f<\alpha\}^k \times \{f>\beta\}^l) \cap A^{k+l}) = (\mu A)^{k+l}. \tag{4}$$

DEFINITION 2.2 Let $Z \subset M(\mu)$ be τ_p-bounded. A set $A \in \Sigma$ with $\mu A > 0$ for which there are numbers $\alpha<\beta$ such that (4) holds is called a *critical set* (for Z). Z is called *μ-stable* (or stable when μ is understood) if there exists no critical set for Z. More explicitly, Z is μ-stable if for all $A \in \Sigma$, $\mu A > 0$ and for all $\alpha<\beta$ there exist $k, l \in \mathbb{N}$ such that

$$\mu^*_{k+l} \bigcup_{f \in Z} ((\{f<\alpha\}^k \times \{f>\beta\}^l) \cap A^{k+l}) < (\mu A)^{k+l}. \quad \square$$

The above argument shows that the existence of a non-measurable τ_p-cluster point of Z implies the existence of a critical set. Thus *stable sets $Z \subset M(\mu)$ are relatively τ_p-compact in $M(\mu)$.* Note also that subsets of stable sets and τ_p-closures of stable sets $Z \subset M(\mu)$ are stable again (the union appearing in (4) is the same whether taken over Z or over the τ_p-closure of Z.)

It is not in general true that relatively τ_p-compact subsets $Z \subset M(\mu)$ are stable. We shall show in Prop. 2.4 however that stable = relatively τ_p-compact for countable Z if the measure space is reasonable, i.e. perfect. The further restriction that μ is a Radon measure on a compact Hansdorff space T and that $Z \subset C(T)$ allows an even sharper conclusion (Prop. 2.5). The final result of this section (Theorem 2.7) shows that in a sense stable sets are "small", namely totally bounded for the (pseudo-metric) topology of convergence in measure. This suggests that generally relatively τ_p-compact sets are too "big" to be stable. An example will be discussed in the Notes of a later chapter.

We begin our discussion with a lemma that will be instrumental in producing non-measurable cluster points.

LEMMA 2.3 *Let* (Ω, Σ, μ) *be a perfect atomless probability space and let* (f_n) *be a sequence of* μ*-measurable functions. Let us also suppose that for each* $k \in \mathbb{N}$ *we are given a* μ_k*-measurable subset* $G_k \subset \Omega^k$ *with* $\mu_k G_k = 1$. *Then there exists a collection* $\{U_\iota : \iota \in I\}$ *of disjoint subsets of* Ω *and a partition* $I = I_1 \cup I_2$, $I_1 \cap I_2 = \varnothing$ *of* I *so that*

(i) $\forall k \in \mathbb{N}\ \forall \iota_1, \ldots, \iota_k \in I$ *distinct* $[t_i \in U_{\iota_i} (i=1, \ldots, k) \Rightarrow t = (t_1, \ldots, t_k) \in G_k]$

(ii) $\mu^*(\bigcup_{I_1} U_\iota) = \mu^*(\bigcup_{I_2} U_\iota) = 1$,

(iii) $\forall \iota \in I\ \forall n \in \mathbb{N}\ f_n$ *is constant on* U_ι.

PROOF

a) We first assume that $\Omega = [0,1]$ and μ is an atomless Borel measure on [0,1]. In this special case we show that we may take each U_ι to be a singleton (so that (iii) becomes redundant). Denoting $\bigcup_{I_1} U_\iota$ and $\bigcup_{I_2} U_\iota$ by U_1 and U_2 respectively, we then have two disjoint subsets $U_1, U_2 \subset [0,1]$ with $\mu^* U_1 = \mu^* U_2 = 1$ such that $t = (t_1, \ldots, t_k) \in G_k$ for any choice of finitely many distinct $t_1, \ldots, t_k \in U := U_1 \cup U_2$.

Let $\{B_\alpha : \alpha < \omega_1\}$ be an enumeration of the Borel subsets of [0,1] with positive measure such that $B_\alpha = B_{\alpha+1}$ whenever α is even. (ω_1 denotes the first uncountable ordinal; note that we are using the continuum hypothesis here, since the cardinality of the enumerated set is *c*.) We may assume that each G_k is invariant for all permutations of the coordinates. (If necessary replace G_k by $G'_k := \bigcap_\sigma \{(t_{\sigma(1)}, \ldots, t_{\sigma(k)}) : (t_1, \ldots, t_k) \in G_k\}$, where the intersection is over all permutations σ of $\{1, \ldots, k\}$. Note that $G'_k \subset G_k$ and that $\mu_k G'_k = 1$ also.) For $t = (t_1, \ldots, t_p) \in [0,1]^p$ and $u = (u_1, \ldots, u_k) \in [0,1]^k$, $p,k \in \mathbb{N}$, we use the following notation:

$$t \circ u := (t_1, \ldots, t_p, u_1, \ldots, u_k) \in [0,1]^{p+k}.$$

Now for all $k,p \in \mathbb{N}$ and every $t \in [0,1]^p$ we set

$$G_k^t := \{u \in [0,1]^k : t \circ u \in G_{p+k}\}.$$

Next we define inductively points $t_\alpha \in [0,1]$ for all $\alpha < \omega_1$ so that the following conditions are satisfied:

(iv) $t_\alpha \in B_\alpha$,

(v) $t_\alpha \neq t_{\alpha'}$ if $\alpha \neq \alpha'$,

(vi) for every $p \in \mathbb{N}$ and for every $t \in [0,1]^p$ whose coordinates are distinct and belong to $\{t_\alpha : \alpha < \omega_1\}$ we have

(vi)$_1$ $t \in G_p$,

(vi)$_2$ for all $k \in \mathbb{N}$ G_k^t is measurable and $\mu_k(G_k^t) = 1$.

The argument needed to properly define $t_0 \in B_0 \cap G_1$ so that (vi)$_2$ is satisfied is a special case of that used in the induction step, so we omit it. Suppose now that for some $\alpha_0 < \omega_1$ the t_α, $\alpha < \alpha_0$, have been defined so that (iv) and (v) are satisfied and $(\mathrm{vi})_1$ and $(\mathrm{vi})_2$ hold for all $p\in\mathbb{N}$ and all $t\in[0,1]^p$ whose (distinct) coordinates belong to $\{t_\alpha:\alpha<\alpha_0\}$. Now fix such $p\in\mathbb{N}$ and $t\in[0,1]^p$ and define

$$A_t:=\{x\in[0,1]: t\circ x\in G_{p+1} \text{ and } \forall k\in\mathbb{N}\ G_k^{t\circ x} \text{ is measurable and } \mu_k G_k^{t\circ x}=1\}.$$

We claim that A_t is measurable and $\mu A_t=1$. For this it suffices to prove that for each fixed $k\in\mathbb{N}$

$$A_{t,k}:=\{x\in[0,1]: t\circ x\in G_{p+1} \text{ and } G_k^{t\circ x} \text{ is measurable with } \mu_k G_k^{t\circ x}=1\}$$

is measurable and $\mu A_{t,k}=1$. But for all choices of $x\in[0,1]$ and of $u=(u_1,\ldots,u_k)\in[0,1]^k$ we have that

$$u\in G_k^{t\circ x} \text{ iff } x\circ u\in G_{k+1}^t$$

and

$$t\circ x\in G_{p+1} \text{ iff } x\in G_1^t.$$

Since by the induction hypothesis G_1^t and G_{k+1}^t are measurable with $\mu(G_1^t)=\mu_{k+1}G_{k+1}^t=1$, it now follows from Fubini's theorem that $A_{t,k}$ is measurable with $\mu A_{t,k}=1$. Therefore also $A_t=\bigcap_{k=1}^{\infty}A_{t,k}$ is measurable with $\mu A_t=1$. So far we have considered a fixed $t\in[0,1]^p$ with distinct coordinates in $\{t_\alpha:\alpha<\alpha_0\}$. Since there are only countably many such t's, also $\bigcap_t A_t$ has measure 1. We now pick $t_{\alpha_0}\in(\bigcap_t A_t)\cap B_{\alpha_0}$ and distinct from the countably many $t_\alpha,\alpha<\alpha_0$ already selected, which is possible because μ is atomless. The t_α, $\alpha\leqslant\alpha_0$ clearly satisfy (iv) and (v). The fact that $(\mathrm{vi})_1$ and $(\mathrm{vi})_2$ are satisfied for every $p\in\mathbb{N}$ and for every $t\in[0,1]^p$ with distinct coordinates in $\{t_\alpha:\alpha\leqslant\alpha_0\}$ is clear from the construction and from the assumption that the G_k are permutation invariant. This completes the construction of the t_α, $\alpha<\omega_1$.

We now put

$$U_1:=\{t_\alpha:\alpha \text{ odd}\},\ U_2:=\{t_\alpha:\alpha \text{ even}\} \text{ and } U:=U_1\cup U_2.$$

Since $t_\alpha\in B_\alpha$ and $B_\alpha=B_{\alpha+1}$ for α even, both U_1 and U_2 meet every B_α, so $\mu^* U_1=\mu^* U_2=1$. It is clear that for every $k\in\mathbb{N}$ and distinct $t_1,\ldots,t_k\in U$ we have $(t_1,\ldots,t_k)\in G_k$.

b) Let us now consider the general case of a perfect atomless probability space (Ω,Σ,μ). By shrinking G_k a bit if necessary we may suppose that each G_k belongs to the σ-algebra Σ_k generated by the product sets $A_1\times\cdots\times A_k\subset\Omega^k$, with $A_i\in\Sigma\,(i=1,\ldots,k)$. Fix $k\in\mathbb{N}$. If D denotes any countable collection of such product sets, and $\Sigma(D)$ is the σ-algebra generated by D, then $\bigcup_D\Sigma(D)$ (union over all D) is a σ-algebra. It therefore coincides with Σ_k. Hence $G_k\in\Sigma(D)$ for some D. Repeating this argument for each $k\in\mathbb{N}$, the conclusion follows that there is a countable set $\{A_n:n\in\mathbb{N}\}\subset\Sigma$ so that each G_k belongs to Σ_0^k, where Σ_0 is the σ-algebra generated by the A_n, $n=1,2,\ldots$. By suitably enlarging the set $\{A_n:n\in\mathbb{N}\}$, but keeping it countable, we may in

addition assume
(a) that Σ_0 has no atoms and
(b) that the A_n separate points that are separated by the f_n (i.e. if $f_n(t_1) \neq f_n(t_2)$ for some $n \in \mathbb{N}$, then there is an A_m with $t_1 \in A_m$, $t_2 \notin A_m$).

Now define $\phi : \Omega \to [0,1]$ by $\phi := \sum_{n=1}^{\infty} 3^{-n} \chi_{A_n}$ and for each $k \in \mathbb{N}$ let $\phi^k : \Omega^k \to [0,1]^k$ be the product map. Obviously each ϕ^k is measurable with respect to $\Sigma^k (k = 1,2,...)$. In fact more is true. The special nature of ϕ easily implies that $\phi^{-1}\mathfrak{B}([0,1]) = \Sigma_0$ and similarly $(\phi_k)^{-1}\mathfrak{B}([0,1]^k) = \Sigma_0^k$. $(k = 1,2,...)$ ($\mathfrak{B}([0,1]^k)$ denotes the σ-algebra of the Borel subsets of $[0,1]^k$.) Let us denote the ϕ-image of μ by ν. Then $\phi^k \mu^k = \nu^k$. It follows from the preceding observation and from (a) that ν, hence also ν^k, has no atoms. Furthermore there exists for each k a Borel set $G'_k \subset [0,1]^k$ with $(\phi^k)^{-1} G'_k = G_k$, so $\nu^k G'_k = \mu^k G_k = 1$.

By what we have proved under a) for the space $([0,1],\nu)$ there are two disjoint sets $U'_1, U'_2 \subset [0,1]$ with $\nu^* U'_1 = \nu^* U'_2 = 1$ so that $t' = (t'_1, \ldots, t'_k) \in G'_k$ for every choice of finitely many distinct $t'_1, \ldots, t'_k \in U' := U'_1 \cup U'_2$. If $\{u'_\iota : \iota \in I_1\}$ and $\{u'_\iota : \iota \in I_2\}$ are enumerations of U'_1 and U'_2, respectively, with $I_1 \cap I_2 = \varnothing$, let us define

$$U_\iota := \phi^{-1} u'_\iota \text{ for } \iota \in I := I_1 \cup I_2.$$

Clearly these sets satisfy the requirements of the lemma, where (ii) follows from Prop. A.7 and (iii) from (b) above (note that ϕ separates points that are separated by the f_n). □

We now apply Lemma 2.3 to show that for sequences on perfect measure spaces stability is the same as relative τ_p-compactness.

PROPOSITION 2.4 *Let (Ω, Σ, μ) be a perfect probability space and suppose (f_n) is a relatively τ_p-compact sequence in $M(\mu)$. Then (f_n) is μ-stable.*

PROOF. Suppose not. Then there exist an $A \in \Sigma$ with $\mu A > 0$ and numbers $\alpha < \beta$ so that

$$\forall k, l \in \mathbb{N} \quad \mu_{k+l}(H_{k,l} \cap A^{k+l}) = (\mu A)^{k+l},$$

where

$$H_{k,l} := \bigcup_{n=1}^{\infty} \{f_n < \alpha\}^k \times \{f_n > \beta\}^l \quad (k,l = 1,2,...).$$

(Observe that the $H_{k,l}$ are measurable since (f_n) is countable; in the uncountable case they may not be and Lemma 2.3 is useless.) It is important to notice at this point that the critical set A is necessarily atomless, since for any atom $B \subset A$ the sets $H_{k,l}$ are clearly disjoint from B^{k+l} while $\mu_{k+l} B^{k+l} > 0$.

We now apply Lemma 2.3 to the measure space (A, Σ_A, μ_A) (note that μ_A is perfect by Prop. A. 2), with

$$G_{2k} := H_{k,k} \cap A^{2k},\ G_{2k+1} = A^{2k+1}\quad (k=1,2,...).$$

Let the U_ι, $\iota \in I = I_1 \cup I_2$ be as in Lemma 2.3. The conclusion is that whenever $F_1, F_2 \subset A$ are sets of k points each ($k \in \mathbb{N}$ arbitrary), the points of F_i being chosen from distinct U_ι with $\iota \in I_i$ ($i=1,2$), then there is a function f_{F_1,F_2} in the sequence (f_n) such that

$$f_{F_1,F_2} < \alpha \text{ on } F_1 \text{ and } f_{F_1,F_2} > \beta \text{ on } F_2.$$

By Lemma 2.3 (iii) these inequalities hold not just on F_i ($i=1,2$), but on the unions of the U_ι from which the points of F_i were chosen. If we partially order the set of pairs (F_1,F_2) by $(F_1,F_2) \leqslant (F'_1,F'_2)$ iff $F_1 \subset F'_1$ and $F_2 \subset F'_2$, then (f_{F_1,F_2}) is a net. It is now clear from the above that the net (f_{F_1,F_2}), hence the sequence (f_n), has a τ_p-cluster point f satisfying

$$f \leqslant \alpha \text{ on } \bigcup_{I_1} U_\iota \text{ and } f \geqslant \beta \text{ on } \bigcup_{I_2} U_\iota.$$

f is non-measurable by (ii) of Lemma 2.3, contradicting the relative τ_p-compactness of (f_n) in $M(\mu)$. $\square$

We now specialize further and consider a compact Radon measure space (T,Σ,μ). If $Z \subset M(\mu)$ consists of continuous functions, then the sets $\bigcup_{f\in Z}\{f<\alpha\}^k \times \{f>\beta\}^l$ are open, hence measurable for the Radon extension $\mu_{k+l,R}$ of μ_{k+l} discussed in Appendix C. This fact can be used to prove the following result.

Proposition 2.5 *Let T be compact Hausdorff, μ a Radon measure on T, and let $Z \subset C(T)$ be τ_p-bounded. Then the following are equivalent.*
(i) *Z is relatively τ_p-compact in $M(\mu)$,*
(ii) *every countable subset of Z is μ-stable,*
(iii) *every countable subset of Z is relatively τ_p-compact in $M(\mu)$.*

Proof.
(i) $\Rightarrow$ (iii) is trivial and (iii) $\Rightarrow$ (ii) is a consequence of Prop. 2.4 (recall that μ is perfect by Prop. A.4). We now prove (ii) $\Rightarrow$ (i).

Suppose for contradiction that the τ_p-closure of Z contains a non-measurable h. Then by Lemma 2.1 there exist an $A \in \Sigma$ with $\mu A > 0$ and numbers $\alpha < \beta$ such that the sets

$$U := \{h<\alpha\} \cap A \text{ and } V := \{h>\beta\} \cap A$$

satisfy $\mu^* U = \mu^* V = \mu A$. This implies by Cor. C.3 that $\forall k,l \in \mathbb{N}$ $\mu^*_{k+l,R}(U^k \times V^l) = (\mu A)^{k+l}$. Note also that

$$U^k \times V^l \subset \bigcup_{f\in Z} \{f<\alpha\}^k \times \{f>\beta\}^l \quad (k,l=1,2,...),$$

by the definition of U and V and the fact that $h \in \tau_p\text{-cl } Z$. The sets $\bigcup_{f\in Z}\{f<\alpha\}^k \times \{f>\beta\}^l$ being open, hence $\mu_{k+l,R}$-measurable, we therefore have

$$\mu_{k+l,R}\,((\bigcup_{f\in Z}\{f<\alpha\}^k\times\{f>\beta\}^l)\cap A^{k+l})=(\mu A)^{k+l}\quad(k,l=1,2\cdots).$$

By regularity, for each pair $k,l\in\mathbb{N}$ there exists a σ-compact set $C_{k,l}$ contained in $\bigcup_{f\in Z}(\{f<\alpha\}^k\times\{f>\beta\}^l)\cap A^{k+l}$ satisfying $\mu_{k+l,R}\,C_{k,l}=(\mu A)^{k+l}$. Since the $f\in Z$ are continuous there is a countable subset $Z_{k,l}\subset Z$ such that

$$C_{k,l}\subset\bigcup_{f\in Z_{k,l}}(\{f<\alpha\}^k\times\{f>\beta\}^l\cap A^{k+l})\quad(k,l=1,2,...).$$

The set on the right is clearly in the product σ-algebra Σ_{k+l}. Putting $Z':=\bigcup_{k,l\in\mathbb{N}}Z_{k,l}$ and observing that Z' is countable, we conclude that

$$\forall k,l\in\mathbb{N}\ \ \mu_{k+l}(\bigcup_{f\in Z'}\{f<\alpha\}^k\times\{f>\beta\}^l)\cap A^{k+l})=(\mu A)^{k+l}.$$

This means that the countable subset $Z'\subset Z$ is not stable, contradicting the assumption. □

We now come to the main result of this section: stable sets are relatively compact in $M(\mu)$ for the (pseudo-metric) topology τ_m of convergence in measure. This remarkable result is true without any assumption on the measure space. The proof uses a tool that is explained in the next lemma.

LEMMA 2.6. *Let (Ω,Σ,μ) be a probability space and let Γ be any subset of Σ. If f is a weak $L^2(\mu)$-cluster point of $\{\chi_A:A\in\Gamma\}$ and if $B:=\{f>0\}$, then*

$$\forall k\in\mathbb{N}\ \ \mu_k^*\bigcup_{A\in\Gamma}(A\cap B)^k=(\mu B)^k.$$

PROOF. Fix $k\in\mathbb{N}$. Observe that the function $f^{(k)}$ on Ω^k defined by

$$f^{(k)}(t_1,\ldots,t_k):=\prod_{i=1}^{k}f(t_i)\quad(t_1,\ldots,t_k\in\Omega)$$

is a weak cluster point in $L^2(\mu_k)$ of the set $\{\chi_{A^k}:A\in\Gamma\}$. (This is because the functions g of the form $g(t_1,\ldots,t_k)=\prod_{i=1}^{k}g_i(t_i)\ \ (t_1,\ldots,t_k\in\Omega)$ with $g_1,\ldots,g_k\in L^2(\mu)$ are total in $L^2(\mu_k)$.) Since obviously $\{f^{(k)}>0\}=B^k$ we see now that it suffices to prove the lemma for $k=1$.

For this let $C\in\Sigma$ with $C\subset B$ and $\mu C>0$ be arbitrary. Then $\int_C fd\mu=\int_\Omega f\chi_C d\mu>0$. Also $\int_C fd\mu$ is in the closure of $\{\int_\Omega\chi_A\cdot\chi_C d\mu:A\in\Gamma\}$, so $\mu(A\cap C)>0$ for some $A\in\Gamma$. Thus $\bigcup_{A\in\Gamma}(A\cap B)$ intersects every $C\subset B$ with $\mu C>0$. This proves the assertion. □

THEOREM 2.7 *Let (Ω,Σ,μ) be a probability space and let $Z\subset M(\mu)$ be μ-stable. Then the identity map $(Z,\tau_p)\to(Z,\tau_m)$ is continuous. In particular Z is totally bounded for τ_m (and $\overline{Z}^{\tau_p}=\overline{Z}^{\tau_m}$ τ_m-compact).*

Proof Suppose not. Then there is a net $(f_\alpha)_{\alpha\in I}$ in Z that converges to some $g\in Z$ pointwise, but not for τ_m. Passing to a subnet if necessary, we may then assume that for some $\lambda>0$

$$\int_\Omega |f_\alpha-g|\wedge 1 d\mu\geqslant\lambda \text{ for all } \alpha\in I.$$

One more passage to a subnet now yields either

$$\int_\Omega (f_\alpha-g)^+\wedge 1 d\mu\geqslant\tfrac{1}{2}\lambda \text{ or } \int_\Omega (f_\alpha-g)^-\wedge 1 d\mu\geqslant\tfrac{1}{2}\lambda \text{ for all } \alpha\in I.$$

Let us suppose we are in the first case and let us pass to a further subnet in order to achieve that $((f_\alpha-g)^+\wedge 1)_{\alpha\in I}$ converges weakly in $L^2(\mu)$, say to $h\in L^2(\mu)$. Then of course

$$\int_\Omega h d\mu\geqslant\tfrac{1}{2}\lambda>0.$$

Now let us fix $a>0$ so that $\mu\{h>3a\}>0$. Next let us choose a set $A\in\Sigma$ with $\mu A>0$ and a $c\in\mathbb{R}$ so that

$$A\subset\{h>3a\} \text{ and } A\subset\{c-a\leqslant g<c\}. \tag{5}$$

We claim that A is a critical set, contradicting the stability of Z. We prove this in two steps.

Step 1. First we show that

$$\text{every } w\text{-cluster point of } \chi_{\{f_\alpha\geqslant c+a\}} \text{ is } \geqslant a \text{ on } A. \tag{6}$$

For the proof of this, let $B\subset A$, $B\in\Sigma$ with $\mu B>0$ be arbitrary. Since w-$\lim\limits_\alpha (f_\alpha-g)^+\wedge 1=h$ and $\int_B h d\mu>3a\mu B$ it follows that there is an $\alpha_0\in I$ so that

$$\int_B (f_\alpha-g)^+\wedge 1 d\mu>3a\mu B \text{ for all } \alpha\geqslant\alpha_0. \tag{7}$$

Let us observe next that we have

$$\mu(\{f_\alpha\geqslant g+2a\}\cap B)\geqslant a\mu B \text{ for } \alpha\geqslant\alpha_0. \tag{8}$$

Indeed, this follows from (7):

$$\begin{aligned}3a\mu B<\int_B (f_\alpha-g)^+\wedge 1 d\mu&=\int_{\{f_\alpha\geqslant g+2a\}\cap B}(f_\alpha-g)^+\wedge 1 d\mu+\\&\quad+\int_{\{f_\alpha<g+2a\}\cap B}(f_\alpha-g)^+\wedge 1 d\mu\\&\leqslant\mu(\{f_\alpha\geqslant g+2a\}\cap B)+2a\mu B\quad(\alpha\geqslant\alpha_0).\end{aligned}$$

Since $c-a\leqslant g$ on B by (5), (8) implies that for $\alpha\geqslant\alpha_0$,

$$\int_B\chi_{\{f_\alpha\geqslant c+a\}}d\mu=\mu(\{f_\alpha\geqslant c+a\}\cap B)\geqslant\mu(\{f_\alpha\geqslant g+2a\}\cap B)\geqslant a\mu B=\int_B a d\mu.$$

The conclusion (6) is now immediate, since $B\subset A$ was arbitrary.

STEP 2. We now fix $k,l\in\mathbb{N}$ and prove that

$$\mu^*_{k+l}\bigcup_{f\in Z}(\{f\leqslant c\}^k\times\{f\geqslant c+a\}^l)\cap A^{k+l}=(\mu A)^{k+l}. \tag{9}$$

This shows that A is critical for Z, contradicting the stability assumption. Fix $s=(s_1,\ldots,s_k)\in A^k$ and put

$$Z':=\{f\in Z: f(s_i)\leqslant c \text{ for } i=1,\ldots,k\}. \tag{10}$$

Since by (5) we have $g(s_i)<c$ for $i=1,\ldots,k$, the set Z' is a τ_p-neighborhood of g relative to Z and therefore $f_\alpha\in Z'$ for sufficiently large $\alpha\in I$. It follows now from the preceding lemma and from (6) that

$$\mu^*_l\bigcup_{f\in Z'}(\{f\geqslant c+a\}^l\cap A^l)=(\mu A)^l. \tag{11}$$

What we have proved now is the following: let W denote the union appearing in (9). Then for each $s=(s_1,\ldots,s_k)\in A^k$ the section

$$W_s:=\{(t_1,\ldots,t_l)\in A^l:(s_1,\ldots,s_k,t_1,\ldots,t_l)\in W\}$$

satisfies $\mu^*_l\,W_s=(\mu A)^l$ (this is (11). Clearly this implies (9). □

NOTES The notion of μ-stability has its origins in work of D.H. FREMLIN and M. TALAGRAND ([28], [91]). As far as we know it was first systematically studied by M. TALAGRAND in [92], although it seems that in the background D.H. Fremlin has contributed much in the form of unpublished notes (see the references in [92]). Much effort is spent in [92] to prove results under assumptions weaker that the continuum hypothesis. We have taken as much from [92] as we need for a thorough discussion of the l^1 embedding theorems (this is our main concern), with total disregard for such subtleties as weakening the continuum hypothesis.

Chapter III

Topologically stable sets of continuous functions

Everywhere in this chapter (unless mention is made of the contrary) T *is either a compact or a Polish space* and $Z \subset C(T)$ a (uniformly) bounded set of continuous functions. We know what it means for Z to be μ-stable, for a Radon measure μ on T. We may also consider sets that are μ-stable for every Radon measure μ. It turns out that for sets Z of continuous functions this notion of "joint" stability has a topological formulation: topological stability. The main theorems of this section express a certain dichotomy: if Z is topologically stable then Z is "nice", i.e. relatively compact in many different senses; if not, then Z contains an l^1-sequence, which implies various forms of non-compactness.

DEFINITION 3.1. A closed set $L \subset T$, $L \neq \emptyset$ is called *topologically critical (t-critical)* for Z if there exist numbers $\alpha < \beta$ such that

$$\forall k,l \in \mathbb{N} \; \Big(\bigcup_{f \in Z} \{f<\alpha\}^k \times \{f>\beta\}^l\Big) \cap L^{k+l} \text{ is dense in } L^{k+l}. \tag{1}$$

Z is called *topologically stable (t-stable)* if no t-critical sets exist. □

Observe that subsets of t-stable sets are t-stable again and that $\bar{Z} := \tau_p\text{-cl}\, Z$ (taken in $C(T)$) is t-stable if Z is.

It is easy to see that t-stability implies joint stability for all Radon measures. The converse will be proved later (cf. Cor. 3.6).

LEMMA 3.2. *If Z is t-stable then it is μ-stable for every Radon measure μ on T.*

PROOF. Suppose for contradiction that $A \in \Sigma$, $\mu A > 0$, is a critical set for some Radon measure μ, so that some numbers $\alpha < \beta$:

$$\forall k,l \in \mathbb{N} \; \mu^*_{k+l} \bigcup_{f \in Z} ((\{f<\alpha\}^k \times \{f>\beta\}^l) \cap A^{k+l}) = (\mu A)^{k+l}. \tag{2}$$

Since $\mu A = \sup\,\{\mu K : K \subset A \text{ compact}\}$, we may suppose that A is compact. Let us further assume, as we clearly may, that A is self supported. Observe that then also A^{k+l} is self supported, relative to μ_{k+l} for all $k,l \in \mathbb{N}$. But now (2) implies that

$$\forall k,l \in \mathbb{N} \; \bigcup_{f \in Z} ((\{f<\alpha\}^k \times \{f>\beta\}^l \cap A^{k+l}) \text{ is dense in } A^{k+l}.$$

So A is t-critical for Z, contradicting the hypothesis. □

The plan for the rest of this chapter is now as follows. We first study sets Z that are not t-stable. The main result here will be that such Z contain l^1-sequences, and are not relatively τ_p-compact in $M(\mu)$, hence not μ-stable, and not even totally bounded in $L^1(\mu)$, for certain Radon measures on T (cf. Th. 2.7). We then prove the main result for t-stable sets Z: each sequence in Z has a τ_p-convergent subsequence. From this other nice properties will follow. The last part of this chapter is devoted to another compactness result: for Polish T the space $\mathfrak{B}_1(T)$ of bounded first class Baire functions on T is "angelic" for the pointwise topology (see Def. 3.12).

§ 1. The non t-stable case.

An essential tool in the study of non t-stable sets is the following notion of independence.

Definition 3.3.

(i) A (finite or infinite) *sequence of pairs* (A_n,B_n) of subsets of a set Ω (no topology) is called *independent* if for each pair of finite disjoint subsets $P,Q \subset \mathbb{N}$ we have

$$\left[\bigcap_{n\in P} A_n\right] \cap \left[\bigcap_{n\in Q} B_n\right] \neq \emptyset.$$

(ii) A (finite or infinite) *sequence of functions* (f_n) on Ω is called *independent* on A (where A is a subset of Ω) if there exist numbers $\alpha<\beta$ such that the sequence of pairs $(\{f_n<\alpha\}\cap A,\ \{f_n>\beta\}\cap A)$ is independent. In case we want to specify α and β we say that (f_n) is *(α,β)-independent* on A. If A is not mentioned, independence means independence on Ω. Clearly independence on A implies independence. □

Example. Let (r_n) be the sequence of the Rademacher functions on $[0,1]=T$. Put $A_n:=\{r_n=1\}$, $B_n:=\{r_n=-1\}$. Then (A_n,B_n) is independent in the sense of (i). Taking e.g. $\alpha=-\frac{1}{2}$, $\beta=\frac{1}{2}$ we see that (r_n) is an independent sequence in the sense of (ii). □

Proposition 3.4 *If Z is not t-stable, then Z contains an independent sequence.*

Proof. Let $L\subset T$ be a t-critical set and let $\alpha<\beta$ be such that (1) is satisfied. The key to the inductive proof below is the following reformulation of (1):

$$\left.\begin{array}{l}\text{For every } n\in\mathbb{N} \text{ and for every } n\text{-tuple } U_1,\ldots,U_n \text{ of non-empty open}\\ \text{subsets of } L \text{ there exists an } f\in Z \text{ that on each } U_i\ (i=1,\ldots,n) \text{ attains}\\ \text{values } <\alpha \text{ and values } >\beta.\end{array}\right\} \quad (3)$$

To see that (3) follows from (1) it suffices to observe that (given $n\in\mathbb{N}$ and $U_1,\ldots,U_n$) we clearly have

$$U_1 \times \cdots \times U_n \times U_1 \times \cdots \times U_n \cap \left[\bigcup_{f \in Z} \{f<\alpha\}^n \times \{f>\beta\}^n \right] \neq \varnothing .$$

Obviously also (3) implies (1). The easy proof is left to the reader.

The construction of the independent sequence is now easy. For $n=1$ take $U_1 = L$. Then by (3) there is an $f_1 \in Z$ such that $U_1 \cap \{f_1<\alpha\} \neq \varnothing$ and $U_1 \cap \{f_1>\beta\} \neq \varnothing$. Suppose $f_1, \ldots, f_n$ have been selected so that $(f_i)_{i=1}^n$ is (α,β)-independent on L. To choose f_{n+1} we apply (3) to the 2^n-tuple of non-empty open subsets $U_P \cap L$, where

$$U_P := \left[\bigcap_{k \in P} \{f_k<\alpha\} \right] \cap \left[\bigcap_{k \notin P} \{f_k>\beta\} \right] \text{ for every } P \subset \{1, \ldots, n\}.$$

(Observe that $U_P \cap L \neq \varnothing$ by the induction hypothesis.) Let $f_{n+1} \in Z$ be as in (3) for these $U_P \cap L$. Then both $\{f_{n+1}<\alpha\}$ and $\{f_{n+1}>\beta\}$ meet each $U_P \cap L$, i.e. $(f_i)_{i=1}^{n+1}$ is (α,β)-independent on L. This completes the induction and the proof. □

PROPOSITION 3.5.
(i) *if (f_n) is a (uniformly) bounded independent sequence of function on any set Ω (no topology), then (f_n) is an l^1-sequence for the sup norm.*
(ii) *if T is Polish or compact and $Z \subset C(T)$ is not t-stable, then there exists a Radon measure μ on T such that $L^1(\mu)$ is isometric to $L^1 := L^1[0,1]$ (notation: $L^1(\mu) \cong L^1$) and so that Z is not totally bounded in $L^1(\mu)$.*

PROOF. (i): Let $\alpha<\beta$ be such that (f_n) is (α,β)-independent. Since the sequence (f_n) is bounded, it will be an l^1-sequence if we can show that for every finite sequence $\alpha_1, \ldots, \alpha_k$ we have

$$\left\| \sum_{i=1}^{k} \alpha_i f_i \right\| \geqslant \tfrac{1}{2}(\beta-\alpha) \sum_{i=1}^{k} |\alpha_i|. \tag{4}$$

We distinguish two cases.

CASE 1: $(\alpha+\beta) \sum_{i \leqslant k} \alpha_i \geqslant 0$.

Putting $P := \{i \leqslant k : \alpha_i \geqslant 0\}$ and $Q := \{i \leqslant k : \alpha_i < 0\}$ we then have by the (α,β)-independence of (f_n) that

$$\left[\bigcap_{i \in Q} \{f_i<\alpha\} \right] \cap \left[\bigcap_{i \in P} \{f_i>\beta\} \right] \neq \varnothing .$$

For any t in this intersection,

$$\sum_{i=1}^{k} \alpha_i f_i(t) \geqslant \beta \sum_{i \in P} \alpha_i + \alpha \sum_{i \in Q} \alpha_i = \frac{\alpha+\beta}{2} \sum_{i=1}^{k} \alpha_i + \frac{\beta-\alpha}{2} \sum_{i=1}^{k} |\alpha_i| \geqslant$$

$$\geqslant \tfrac{1}{2}(\beta-\alpha) \sum_{i=1}^{k} |\alpha_i|, \text{ proving (4).}$$

CASE 2: $(\alpha+\beta) \sum_{i \leqslant k} \alpha_i < 0$.

If we replace the α_i by $-\alpha_i$ we are in case 1 and it follows that for some $s \in T$,

$$-\sum_{i=1}^{k} a_i f_i(s) \geq \tfrac{1}{2}(\beta-\alpha) \sum_{i=1}^{k} |a_i|, \text{ again proving (4).}$$

(ii): Let us first assume that T is compact. By Prop. 3.4 Z contains a sequence (f_n) which is (α,β)-independent for some $\alpha<\beta$. Let us observe that the sets $\{f_n \leq \alpha\} \cup \{f_n \geq \beta\}$ $(n=1,2,\ldots)$ satisfy the finite intersection property. Hence

$$K := \bigcap_{n=1}^{\infty} (\{f_n \leq \alpha\} \cup \{f_n \geq \beta\}) \neq \emptyset \quad \text{(and compact).}$$

We now define a map $K \xrightarrow{h} \{0,1\}^{\mathbb{N}}$ with components h_n by

$$h_n(t) := \begin{cases} 0 & \text{if } f_n(t) \leq \alpha, \\ 1 & \text{if } f_n(t) \geq \beta. \end{cases} (n=1,2,\ldots)$$

Since each h_n is continuous and (by the independence of (f_n)) hK is dense in $\{0,1\}^{\mathbb{N}}$, h is a surjection. Letting ν denote the product measure $(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\mathbb{N}}$ on $\{0,1\}^{\mathbb{N}}$, we know from Prop. B. 1 that there is a Radon probability μ on K (which may be regarded as a Radon measure on T) such that $h\mu = \nu$ and with the additional property that $L^1(\mu) \cong L^1(\nu) \cong L^1$ (L^1 denotes $(L^1[0,1],\lambda)$; the isometry $L^1(\nu) \cong L^1$ is standard). Since for $m \neq n$ the set $K \cap \{f_n \leq \alpha\} \cap \{f_m \geq \beta\}$ is the preimage under h of $\{(\epsilon_k) \in \{0,1\}^{\mathbb{N}} : \epsilon_n = 0$ and $\epsilon_m = 1\}$, we have $\mu\{f_n \leq \alpha\} \cap \{f_m \geq \beta\} = \frac{1}{4}$. It immediately follows that $\|f_n - f_m\|_1 \geq \frac{1}{4}(\beta - \alpha)$, so that (f_n) is not totally bounded in $L^1(\mu)$. Neither is Z.

In the case that T is Polish only minor modifications are needed. We again choose an independent sequence in Z as in Prop. 3.4, but with a little more care. At each step in the induction process we determine a closed set $A_n \subset T$ so that (for some complete metric) A_n intersects each of the U_P with $P \subset \{1, \ldots, n\}$ in a non-empty set of diameter $< \frac{1}{n}$ (simply intersect each U_P with a suitable ball). We then proceed with the sets $U_P \cap A_n$ (rather than U_P) to define the next f_{n+1}. Returning to the present proof, we now define

$$K := \bigcap_{n=1}^{\infty} (\overline{\{f_n < \alpha\} \cup \{f_n > \beta\} \cap A_n}).$$

Then K is closed and non-empty and for each $n \in \mathbb{N}$ can be covered by finitely many $\frac{1}{n}$-balls. Hence K is compact and the preceding proof can be repeated to produce a measure μ on K with the required properties. This μ can then be regarded as a Radon measure on T. $\square$

COROLLARY 3.6. *If $Z \subset C(T)$ fails to be t-stable, then there exists a Radon probability μ on T with $L^1(\mu) \cong L^1$, and sequence $(f_n) \subset Z$ such that (f_n) is not* *relatively τ_p-compact in $M(\mu)$. (In particular Z is not jointly stable for all Radon measures.)*

PROOF. By the proof of Prop. 3.5 (ii) there exists a Radon measure μ on T so that $L^1(\mu) \cong L^1$, and a sequence (f_n) in Z so that (f_n) is not totally bounded in $L^1(\mu)$. Theorem 2.7 now implies that (f_n) is not μ-stable. Hence by Prop. 2.4 (f_n) is not relatively τ_p-compact in $M(\mu)$ since μ is perfect by Prop. A.4. Alternatively, one might use Fremlin's theorem 1.6 for the last conclusion: since (f_n) is not $L^1(\mu)$-totally bounded, some subsequence has no μ a.e. convergent subsequence (by Lebesgue's theorem) and therefore all the τ_p-cluster points of some further subsequence lie outside $M(\mu)$. $\square$

§ 2. THE *t*-STABLE CASE.

Before we can treat this case some preliminaries must be dealt with. For any topological space T we denote by $\mathfrak{B}_1(T)$ the set of first class Baire functions (see Appendix E) and by $\mathfrak{B}_r(T)$ the set of all functions f on T having the property that for each closed $L \subset T$ the restriction f_L of f to L has a point of continuity. A classical result of Baire (see Th. E.1) states that $\mathfrak{B}_1(T) \subset \mathfrak{B}_r(T)$ whenever T has the property that every one of its closed subsets is of the 2^{nd} category in itself (example: compact, or complete metric T). If T Polish, so in particular if T is compact metric, then $\mathfrak{B}_1(T) = \mathfrak{B}_r(T)$.

From now on we restore the convention that T *always denotes a Polish or a compact space.*

Here is a characterization of $\mathfrak{B}_r(T)$ with some "stability flavor".

LEMMA 3.7. *The following are equivalent for a function f on T.*

(i) $f \in \mathfrak{B}_r(T)$,

(ii) *for every non-empty closed $L \subset T$ and for all numbers $\alpha < \beta$ the sets $L \cap \{f < \alpha\}$ and $L \cap \{f > \beta\}$ are not both dense in L.*

PROOF.

(i) $\Rightarrow$ (ii): if $\overline{L \cap \{f<\alpha\}} = \overline{L \cap \{f>\beta\}} = L$ then f_L has no continuity point.

(ii) $\Rightarrow$ (i): let $L \subset T$ be an arbitrary closed set and let $((\alpha_n, \beta_n))$ be an enumeration of all pairs of rationals (α, β) with $\alpha < \beta$. For each $n \in \mathbb{N}$ consider the sets

$$A_n := L \cap \{f < \alpha_n\} \text{ and } B_n := L \cap \{f > \beta_n\}.$$

Observe that each $L_n := \overline{A}_n \cap \overline{B}_n$ is nowhere dense in L. (If $U \subset L_n$, U relatively open in L, then $\overline{U} = \overline{A_n \cap U} = \overline{B_n \cap U}$, so $f_{\overline{U}}$ has no continuity point, contrary to the assumption.) Now it follows from Baire's category theorem that $\bigcap_{n=1}^{\infty} (L \setminus L_n) = L \setminus \bigcup_{n=1}^{\infty} L_n =: G$ is a dense G_δ in L. It is clear that f_L is continuous in every point of G (every discontinuity point of f_L must be in some L_n). $\square$

The next lemma makes clear how (ii) above relates to stability.

LEMMA 3.8. *If $Z \subset C(T)$ is t-stable, then Z is relatively τ_p-compact in $\mathfrak{B}_r(T)$ $(=\mathfrak{B}_1(T)$ if T Polish).*

PROOF. Suppose for contradiction that h is in the τ_p-closure of Z, but $h \notin \mathfrak{B}_r(T)$. Then by Lemma 3.7 there are a closed subset $L \subset T$ and numbers $\alpha < \beta$ such that

$$\overline{L \cap \{h < \alpha\}} = \overline{L \cap \{h > \beta\}} = L.$$

But this clearly implies that

$$\forall k,l \in \mathbb{N} \bigcup_{f \in Z} ((\{f < \alpha\}^k \times \{f > \beta\}^l) \cap L^{k+l}) \text{ is dense in } L^{k+l},$$

contradicting the t-stability of Z. □

We now prove a rather general result that will be helpful again when we discuss the fact that $\mathfrak{B}_1(T)$ is τ_p-angelic for Polish T. The result says that for T Polish, and (f_n) a sequence that is relatively τ_p-compact in $\mathfrak{B}(T)$ (= the Borel functions on T), every pointwise cluster point of (f_n) in $\mathfrak{B}(T)$ is the τ_p-limit of a subsequence. Observe that for t-stable $Z \subset C(T)$, pointwise cluster points are automatically in $\mathfrak{B}(T)$ by the above lemma, so that for t-stable sequences in $C(T)$ (T Polish), the τ_p-closure coincides with the τ_p-sequential closure.

PROPOSITION 3.9. *Let T be Polish and let $D \subset \mathfrak{B}(T)$ be countable and relatively τ_p-compact in $\mathfrak{B}(T)$. Then every τ_p-cluster point of D in $\mathfrak{B}(T)$ is the limit of a subsequence of D.*

PROOF. Let $f \in \mathfrak{B}(T)$ be a τ_p-cluster point of $D = (f_n)$. We may assume $f = 0$, and $f_n \geq 0$, considering $(|f_n - f|)$ instead of (f_n).

STEP 1. We show in this step that it is enough to prove the assertion under the following extra assumption:

$$D \text{ is a } t\text{-stable sequence in } C(T). \tag{5}$$

Let us consider the map

$$T \ni t \longrightarrow F(t) := (f_n(t)) \in \mathbb{R}^{\mathbb{N}}.$$

Put $S := FT$. Since F is Borel by Lemma D. 11 (i), S is analytic (Prop. D. 12). Let e_n $(n = 1,2,...)$ be the n^{th} coordinate function on $S \subset \mathbb{R}^{\mathbb{N}}$. Then $e_n \circ F = f_n$ $(n = 1,2,...)$. Let us now note the trivial fact that the map $g \to g \circ F$ is a homeomorphic embedding of $\mathbb{R}^S$ into $\mathbb{R}^T$ (for the product topologies). The following consequences are immediate:

(i) the τ_p-cluster points of (f_n) are precisely the functions of the form $g \circ F$, where $g \in \mathbb{R}^S$ is a τ_p-cluster point of (e_n);
(ii) the τ_p-convergence of a subsequence (f_{n_k}) is equivalent to the τ_p-

convergence of the corresponding subsequence (e_{n_k}) on S.

Let us also recall (Cor. D. 13) that

(iii) a function g on S belongs to $\mathfrak{B}(S)$ iff $g \circ F \in \mathfrak{B}(T)$.

The fact that S is analytic means there is a Polish space R and a continuous surjection $G: R \to S$. Notice that the functions $e_n \circ G$ are continuous on R and that the above statements (i), (ii) and (iii) also hold with T, F and (f_n) replaced by R, G and $(e_n \circ G)$, respectively. Finally let us observe that $(e_n \circ G) \subset C(R)$ is t-stable by Cor. 3.6, since all its τ_p-cluster points are Borel by (i), (iii) and the assumption. We have now proved that assumption (5) is justified.

STEP 2. Let $\mathfrak{F}$ be the Fréchet filter on D, i.e. the filter generated by the countable basis consisting of the sets $F_n := \{f_k : k \geqslant n\}$, or any other filter on D with a countable basis and 0 among its τ_p-cluster points. we now prove the following claim: *for every closed subset $L \subset T$ and for every $\epsilon > 0$ there is a filter $\mathfrak{F}' \supset \mathfrak{F}$ with a countable basis and also having 0 as a τ_p-cluster point, and a non-empty open $U \subset L$ such that every τ_p-cluster point g of $\mathfrak{F}'$ satisfies $g \leqslant \epsilon$ on U.*

To prove this let (t_n) be a dense sequence in T. Fix $L \subset T$ closed, and $\epsilon > 0$. Since (f_n) is t-stable, L is not t-critical for (f_n), so there exists a k-tuple of non-empty open subsets $U_1, \ldots, U_k$ of L such that *no* f_n takes values $< \frac{\epsilon}{2}$ and $> \epsilon$ on *each* U_i $(i = 1, \ldots, k)$. In formula:

$$U_1 \times \cdots \times U_k \times U_1 \times \cdots \times U_k \cap \Big(\bigcup_{n=1}^{\infty} \{f_n < \tfrac{\epsilon}{2}\}^k \times \{f_n > \epsilon\}^k\Big) = \varnothing$$

(see the proof of Prop. 3.4). Now choose p so large that $\{t_1, \ldots, t_p\} \cap U_i \neq \varnothing$ for $i = 1, \ldots, k$. It follows that for each f_n we have the following implication:

$$[\forall 1 \leqslant j \leqslant p \;\; f_n(t_j) < \tfrac{\epsilon}{2}] \Rightarrow [\exists 1 \leqslant i \leqslant k \;\; f_n \leqslant \epsilon \text{ on } U_i]. \tag{6}$$

Observe now that

$$V := \{g : g(t_j) < \tfrac{\epsilon}{2} \text{ for } j = 1, \ldots, p\}$$

is a τ_p-nbhd of 0. (6) says that $V \cap D$ is covered by the finitely many sets $V_i \cap D$, $i = 1, \ldots, k$, where $V_i := \{g : g \leqslant \epsilon \text{ on } U_i\}$. Since 0 is a τ_p-cluster point of $\mathfrak{F}$, we have $0 \in \overline{F \cap V}$ for each $F \in \mathfrak{F}$ (the bar denotes τ_p-closure of course). The filter property (F2) now implies that for *some* $i_0 \leqslant k$ we have $0 \in \overline{F \cap V_{i_0}}$ for every $F \in \mathfrak{F}$. In particular $F \cap V_{i_0} \neq \varnothing$ for all $F \in \mathfrak{F}$. Now let $\mathfrak{F}'$ be the filter generated by the sets $F \cap V_{i_0}$ with $F \in \mathfrak{F}$. Clearly $\mathfrak{F}'$ has a countable basis again, $\mathfrak{F}' \supset \mathfrak{F}$, and, by construction, has 0 among its τ_p-cluster points. Finally, every τ_p-cluster point g of $\mathfrak{F}'$ satisfies $g \leqslant \epsilon$ on U_{i_0}, since $g \in \overline{V}_{i_0} = V_{i_0}$.

STEP 3. The rest of the proof consists in a clever exploitation of what we have proved in step 2. By transfinite induction we shall construct for some countable ordinal α_0 a strictly decreasing transfinite sequence $(L_\alpha)_{\alpha \leqslant \alpha_0}$ of closed subsets $L_\alpha \subset T$ with $L_{\alpha_0} = \emptyset$ and an increasing sequence of filters $(\mathfrak{F}_\alpha)_{\alpha \leqslant \alpha_0}$ on D, each with a countable basis, each having 0 among its cluster points, and so that all cluster points of $\mathfrak{F}_\alpha$ are $\leqslant \epsilon$ on $T \setminus L_\alpha (\epsilon > 0$ given). The result of this exhaustion argument is *a filter* $\mathfrak{F}_{\alpha_0} \supset \mathfrak{F}$ *with a countable basis and cluster point 0, and having the property that all of its cluster points are* $\leqslant \epsilon$ *on T (since* $L_{\alpha_0} = \emptyset$*).*

For the proof let us start with $\mathfrak{F}_0 := \mathfrak{F}$ and $L_0 := T$. Assume now that $(L_\alpha)_{\alpha<\beta}$ and $(\mathfrak{F}_\alpha)_{\alpha<\beta}$ have been properly defined for some countable β. If $\beta = \alpha + 1$ we apply the argument of step 2 with $\mathfrak{F} = \mathfrak{F}_\alpha$ and $L = L_\alpha$ and find a non-empty open $U \subset L_\alpha$ and a filter $\mathfrak{F}_{\alpha+1} \supset \mathfrak{F}_\alpha$ with cluster point 0 and all its cluster points $\leqslant \epsilon$ on U (hence on $U \cup (T \setminus L_\alpha)$, since cluster points of $\mathfrak{F}_{\alpha+1}$ are cluster points of $\mathfrak{F}_\alpha$). Now put $L_{\alpha+1} := L_\alpha \setminus U$. If β is a limit ordinal, choose an increasing sequence of ordinals α_n so that $\beta = \sup \alpha_n$. For each $n \in \mathbb{N}$ let $(F_k^n)_{k=1}^\infty$ be a decreasing countable basis for $\mathfrak{F}_{\alpha_n}$ and put $F_k := \bigcap_{n=1}^{k} F_k^n$. Clearly the filter $\mathfrak{F}_\beta$ generated by $(F_n)_{n=1}^\infty$ contains each $\mathfrak{F}_\alpha, \alpha < \beta$, has a countable basis, has 0 among its cluster points, and each cluster point of $\mathfrak{F}_\beta$, being a cluster point of each $\mathfrak{F}_\alpha$, $\alpha<\beta$, is $\leqslant \epsilon$ on $T \setminus \bigcap_{\alpha<\beta} L_\alpha$. It remains to put $L_\beta := \bigcap_{\alpha<\beta} L_\alpha$.

To conclude the proof let us observe that the Polish space T has a countable basis and therefore no uncountable transfinite sequence of strictly decreasing closed sets exists. Thus $L_\alpha = \emptyset$ for some countable α. Let α_0 be the first such ordinal.

Step 4. We now repeatedly apply the result of step 3 for each ϵ of the form $\epsilon = \frac{1}{n}$, finding an increasing sequence $(\mathfrak{F}_n)$ of filters with countable bases, such that each $\mathfrak{F}_n$ has 0 among its cluster points and so that all cluster points of $\mathfrak{F}_n$ are $\leqslant \frac{1}{n}$ on T $(n = 1,2,...)$. Again denoting by $(F_k^n)_{k=1}^\infty$ a decreasing countable basis of $\mathfrak{F}_n$ $(n = 1,2,...)$, let us choose a subsequence $f_{n_k})$ of (f_n) such $f_{n_k} \in \bigcap_{n=1}^{k} F_k^n$ $(k = 1,2,...)$ Then each cluster point of (f_{n_k}) is a cluster point of every $\mathfrak{F}_n$ and therefore $\leqslant \frac{1}{n}$ on T for every n. Hence 0 is the *only* cluster point of (f_{n_k}) and therefore (f_{n_k}) converges to 0 pointwise on T. So the proof is finished. □

We are now able to complete our analysis of t-stable sets $Z \subset C(T)$:

Proposition 3.10 *As always let T be either compact or Polish and let* $Z \subset C(T)$ *be t-stable. Then every sequence* (f_n) *in Z has a* τ_p*-convergent subsequence. In fact every* τ_p*-cluster point of* (f_n) *is the* τ_p*-limit of a subsequence.*

Proof. By the observation preceding Prop. 3.9 all we have to do is reduce the compact case to the Polish one. Let (f_n) be a sequence in Z and let us assume that T is compact. Consider the map $F: T \to \mathbb{R}^{\mathbb{N}}$ defined by $F(t) := (f_n(t))_{n=1}^{\infty}$ ($t \in T$) and put $S := FT$. Then S is compact metric, hence Polish. The argument in step 1 of the previous proof shows that it suffices to show that every τ_p-cluster point of the sequence (e_n) of coordinate functions on S is the τ_p-limit of a subsequence, since $f_n = e_n \circ F$. Again by the observation preceding Prop. 3.9 all we have to do is show that (e_n) is t-stable. For contradiction suppose that $L \subset S$ is t-critical (hence compact) for (e_n) and let $\alpha < \beta$ be as in (1). By an easy application of Zorn's lemma there a minimal compact $M \subset T$ with $FM = L$, i.e. such that $M' \subsetneq M$, M' compact implies $FM' \subsetneq L$. We claim that M is t-critical for (f_n), contradicting the fact that (f_n) is t-stable. Indeed, for any k-tuple of non-empty open sets $U_1, \ldots, U_k \subset M$ we have by the minimality of M that each FU_i contains a non-empty open subset $V_i \subset L$ ($i = 1, \ldots, k$). Since we are assuming that L is t-critical for (e_n). some e_n takes values $< \alpha$ and $> \beta$ on each V_i (see (3)). This implies that the corresponding $f_n = e_n \circ F$ takes values $< \alpha$ and $> \beta$ on each U_i. Because $U_1, \ldots, U_k$ were arbitrary we have now proved that M is t-critical for (f_n), a contradiction. □

It is now time to summarize our results.

THEOREM 3.11. *Let $Z \subset C(T)$ be bounded (T compact or Polish). Consider the following properties:*

(i) *Z does not contain an l^1-sequence,*
(ii) *Z does not contain an independent sequence,*
(iii) *each sequence in Z has a pointwise convergent subsequence,*
(iv) *Z is relatively τ_p-compact in $\mathfrak{B}_r(T)$ ($= \mathfrak{B}_1(T)$ if T Polish),*
(v) *for each Radon measure μ on T, Z is relatively τ_p-compact in $M(\mu)$ (i.e. all τ_p-cluster points of Z in $\mathbb{R}^T$ are universally measurable),*
(vi) *for each Radon measure μ on T, Z is μ-stable,*
(vii) *for each Radon measure μ on T such that $L^1(\mu) \cong L^1$, Z is totally bounded in $L^1(\mu)$,*
(viii) *Z is t-stable.*

The properties (ii) - (viii) are equivalent. If T is compact, then also (i) is equivalent to (ii)-(viii). In the Polish case this is not generally true, but (i) implies the other properties.

PROOF.
(viii) $\Rightarrow$ (iii): Prop. 3.10.

(iii) $\Rightarrow$ (i) (for compact T): bounded sequences in $C(T)$ are τ_p-Cauchy iff they are weakly Cauchy, by Lebesgue's theorem. Since the unit vectors in l^1 obviously have no weakly Cauchy subsequence, Z cannot contain an l^1-sequence. To see that this implication fails in general for Polish T, take $T = \mathbb{N}$ and let Z be any independent $\{-1, +1\}$-valued sequence on $\mathbb{N}$. Then (f_n) is an l^1-sequence by Prop. 3.5 (i) and clearly has a τ_p-convergent subsequence (use a diagonal procedure).

(i) $\Rightarrow$ (ii): Prop. 3.5 (i).

(ii) $\Rightarrow$ (viii): Prop. 3.4.

(viii) $\Rightarrow$ (iv): Lemma 3.8.

(iv) $\Rightarrow$ (v): it clearly suffices to show that $\mathfrak{B}_r(T) \subset M(\mu)$ for every Radon measure μ. Suppose $f \notin M(\mu)$ for some μ. Then there exist an $A \in \Sigma_\mu$, $\mu A > 0$, and numbers $\alpha < \beta$ such that

$$\mu^*(\{f < \alpha\} \cap A) = \mu^*(\{f > \beta\} \cap A) = \mu A, \text{ by Lemma 2.1.}$$

By shrinking A a bit if necessary (use the regularity of μ) we may suppose that A is compact and self-supported. But then $\{f < \alpha\} \cap A$ and $\{f > \beta\} \cap A$ are both dense in A, so Lemma 3.7 shows that $f \notin \mathfrak{B}_r(T)$.

(v) $\Rightarrow$ (viii): Cor. 3.6.

(viii) $\Rightarrow$ (vi): Lemma 3.2.

(vi) $\Rightarrow$ (vii): Th. 2.7.

(vii) $\Rightarrow$ (viii): Prop. 3.5 (ii). $\square$

§ 3. $\mathfrak{B}_1(T)$ IS ANGELIC FOR POLISH T.

For the remainder of this chapter T will be Polish. We shall prove that $\mathfrak{B}_1(T)$ is angelic for the pointwise topology. Prop. 3.9 is only a partial result in this direction.

DEFINITION 3.12. A topological space (T, τ) is called *angelic* if every relatively countably compact subset $A \subset T$ is
(i) relatively compact in T and
(ii) every $t \in \bar{A}$ is the limit of a sequence in A. $\square$

The most familiar example of an angelic space is a metric space. But the metric spaces do not exhaust the class of angelic spaces. A generally non-metrizable example is that of a normed space with its weak topology (no proof). Without proof we mention the following facts (which we shall not need):

I. Subspaces of angelic spaces are angelic.
II. If (T, τ) is angelic and τ' is finer that τ and regular, then (T, τ') is angelic.
III. If (T, τ) is angelic then for subsets $A \subset T$,
(rel.) countably compact $=$ (rel.) sequentially compact $=$ (rel.) compact.

We now prove the main result in this §.

Theorem 3.13. *Let T be Polish. Then $\mathfrak{B}_1(T)$ is angelic for the topology τ_p.*

Proof.
1) Let $Z \subset \mathfrak{B}_1(T)$ be relatively countably τ_p-compact (we drop the convention that Z is uniformly bounded). Suppose for contradiction that Z is not relatively compact. Then the pointwise closure $\overline{Z}$ (taken in $\mathbb{R}^T$) contains some $f \notin \mathfrak{B}_1(T)$ (it should be observed that Z is pointwise bounded, so that it is relatively compact in $\mathbb{R}^T$). Recall that $\mathfrak{B}_1(T) = \mathfrak{B}_r(T)$. By Lemma 3.7 there is a closed $L \subset T$ and there are numbers $\alpha < \beta$ such that $\overline{L \cap \{f < \alpha\}} = \overline{L \cap \{f > \beta\}} = L$. Let (U_n) be an open basis for L and for each $n \in \mathbb{N}$ choose points $t_n, t_n' \in U_n$ such that $f(t_n) < \alpha$ and $f(t_n') > \beta$. Since $f \in \overline{Z}$ a diagonal procedure will produce a sequence $(f_k) \subset Z$ such that

$$\lim_{k\to\infty} f_k(t_n) = f(t_n) \text{ and } \lim_{k\to\infty} f_k(t_n') = f(t_n') \text{ for all } n \in \mathbb{N}.$$

Now by the relative countable compactness of Z the sequence (f_k) has a τ_p-cluster point $g \in \mathfrak{B}_1(T)$. Obviously g must coincide with f in all points t_n, t_n' $(n = 1,2,\ldots)$. Therefore $g(t_n) = f(t_n) < \alpha$ and $g(t_n') = f(t_n') > \beta$ for all $n \in \mathbb{N}$. This implies that g_L has no continuity point, contradicting the fact that $g \in \mathfrak{B}_1(T) = \mathfrak{B}_r(T)$.

2) We now prove the second defining property of angelic spaces: if $Z \subset \mathfrak{B}_1(T)$ is relatively (countably) compact in Z and $f \in \overline{Z}$, then f is the τ_p-limit of a sequence in Z. Since $\mathfrak{B}_1(T) \subset \mathfrak{B}(T)$ it suffices by Prop. 3.9 to show that $f \in \overline{D}$ for some countable subset $D \subset Z$. We may assume without loss of generality that $f = 0$. Let us fix $m \in \mathbb{N}$ and let us consider the map

$$F_m : \mathfrak{B}_1(T) \to \mathfrak{B}_1(T^m)$$

defined by $F_m(g)(t_1, \ldots, t_m) = |g(t_1)| + \ldots + |g(t_m)|$ $(g \in \mathfrak{B}_1(T))$ Note that T^m is Polish again. Clearly F_m is continuous for the respective pointwise topologies, so since $F_m 0 = 0$, 0 is a τ_p-cluster point of $F_m Z$. Let E denote the set of all functions in $\mathfrak{B}_1(T^m)$ that are cluster points of countable subsets of $F_m Z$. We now fix $\epsilon > 0$ and construct inductively *a countable ordinal α_0 and a strictly decreasing transfinite sequence $(L_\alpha^m)_{\alpha \leqslant \alpha_0}$ of subsets of T^m with $L_{\alpha_0}^m = \varnothing$ and for each $\alpha \leqslant \alpha_0$ an $F_m f_\alpha \in E$ such that $F_m f_\alpha < \epsilon$ on $L_\alpha \setminus L_{\alpha+1}$.*

Let us start with $L_0^m := T^m$. If D' is a countable dense subset of T^m then surely, since 0 is a cluster point of $F_m Z$, there is an $F_m f_0 \in E$ such that $F_m f_0 = 0$ on D'. Since $F_m f_0 \in \mathfrak{B}_1(T^m) = \mathfrak{B}_r(T^m)$ it has a continuity point. It follows that $F_m f_0 < \epsilon$ on some non-empty open $U \subset T^m$. Put $L_1 := T \setminus U$.

Suppose now that $L_{\alpha+1}$ and $F_m f_\alpha$ have been properly constructed for all $\alpha < \beta$, where β is some countable ordinal. If $\beta = \alpha + 1$ let D' be dense in $L_{\alpha+1}$ and as before choose $F_m f_{\alpha+1} \in E$ so that $F_m f_{\alpha+1} = 0$ on D'. Since $F_m f_{\alpha+1} \in \mathfrak{B}_1(T^m) = \mathfrak{B}_r(T^m)$ its restriction $(F_m f_{\alpha+1})|_{L_{\alpha+1}}$ has a continuity point. Again this implies that $F_m f_{\alpha+1} < \epsilon$ on some non-empty open $U \subset L_{\alpha+1}$. Put $L_{\alpha+2} := L_{\alpha+1} \setminus U$. If β is a limit ordinal we put $L_\beta := \bigcap_{\alpha<\beta} L_\alpha$ and argue the same way with a countable dense set $D' \subset L_\beta$ to define $F_m f_\beta$ and $L_{\beta+1}$. Now since T^m is Polish we must have $L_\alpha = \emptyset$ for some countable α. Letting α_0 be the first such ordinal, the construction is finished.

Considering the union of the countably many countable subsets of $F_m Z$ of which the respective $F_m f_\alpha$, $\alpha < \alpha_0$, are cluster points, we find a countable set $D_{(m,\epsilon)} \subset Z$ with the property that for each m-tuple $t = (t_1, \ldots, t_m) \in T^m$ there is an $f \in D(m, \epsilon)$ with $(F_m f)(t) < \epsilon$, implying that

$$|f(t_i)| < \epsilon \quad \text{for } i = 1, \ldots, m.$$

Observe now that $m \in \mathbb{N}$ and $\epsilon > 0$ were arbitrary. Let us form $D(m, \frac{1}{k})$ for all $m, k \in \mathbb{N}$. Then $D := \bigcup_{m,k=1}^{\infty} D(m, \frac{1}{k}) \subset Z$ is countable and has the property that for all $k, m \in \mathbb{N}$ and every $(t_1, \ldots, t_m) \in T^m$ there exists an $f \in D$ such that $|f(t_i)| < \frac{1}{k}$ for $i = 1, \ldots, m$. This means that 0 is in the τ_p-closure of D and the proof in finished. $\square$

NOTES Many of the ideas underlying the results in this chapter can be traced back to H.P. ROSENTHAL ([72], [73]). Subsequent perfection of them by (among others) J. BOURGAIN, D.H. FREMLIN and M. TALAGRAND ([8]) culminated in the main Theorem 3.11. The fact that $\mathfrak{B}_1(T)$ is angelic for Polish T was also proved in [8]. Although some theorems remain difficult, many of the original proofs were considerably simplified by M. TALAGRAND in his memoir [92].

Chapter IV

Some characterizations of B-spaces not containing l^1

We shall now characterize in many ways those Banach spaces in which l^1 cannot be embedded. It turns out that once the right framework is chosen these characterizations are relatively simple corollaries of the theory developed in the preceding chapters.

We first consider separable spaces.

THEOREM 4.1. *For a separable Banach space X the following are equivalent.*
(1) *X contains no subspace isomorphic to l^1,*
(2) *X is w^*-sequentially dense in X^{**},*
(3) *card X^{**} = card X,*
(4) *every bounded sequence in X has a weak Cauchy subsequence,*
(5) *every bounded sequence in X^{**} has a weak * convergent subsequence,*
(6) *every bounded subset of X is weakly sequentially dense in its weak closure,*
(7) *every bounded subset of X^{**} is weak * sequentially dense in its w^* closure,*
(8) *X^* contains no subspace isomorphic to $L^1:=L^1[0,1]$,*
(9) *X^* contains no subspace isomorphic to $l^1(\Gamma)$ for any uncountable Γ,*
(10) *$C:=C[0,1]$ is not isomorphic to a quotient of X,*
(11) *X^* contains no subspace isomorphic to C^*.*

PROOF All these equivalences are fairly simple consequences of the deep results proved in chapter 3 once the right framework is chosen. Let T be the unit ball of X^*, equipped with its w^*-topology. Then T is compact by Alaoglu's theorem, and also Polish since X is assumed to be separable. We now regard the elements of X^{**} as (bounded) functions on T. i.e. we identify X^{**} with a subspace of $\mathbb{R}^T$. Notice that under this identification the w^*-topology of X^{**} corresponds to the topology τ_p of pointwise convergence on T. Furthermore the elements of X are in $C(T)$, so in particular norm bounded sets in X correspond to uniformly bounded subsets $Z\subset C(T)$. In the proof that follows we shall repeatedly switch from one point of view to the other without saying so or indicating it by cumbersome notation.

(1) $\Leftrightarrow$ (4): this is the equivalence (i) $\Leftrightarrow$ (iii) of Theorem 3.11, applied to every bounded set in X.

(1) $\Rightarrow$ (7): by Goldstine's theorem the w^*-closure of the unit ball $B(X)$ is $B(X^{**})$. Regarding the elements of X^{**} as functions on T, this implies that every $x^{**}\in X^{**}$ is in the τ_p-closure of a bounded set, say Z, in $C(T)$. Now by

the assumption (1) and the equivalence (i) $\Leftrightarrow$ (iv) of Th. 3.11, Z is relatively τ_p-compact in $\mathfrak{B}_r(T)$. Also $\mathfrak{B}_r(T) = \mathfrak{B}_1(T)$ since T is Polish. Thus $X^{**} \subset \mathfrak{B}_1(T)$. Now Th. 3. 13 comes in: $(\mathfrak{B}_1(T), \tau_p)$ is angelic, so every relatively τ_p-compact subset of $\mathfrak{B}_1(T)$ is sequentially dense in its closure. This applies in particular to bounded subsets of X^{**}, since these are relatively w^*-compact in X^{**} (Alaoglu), hence a fortiori relatively τ_p-compact in $\mathfrak{B}_1(T)$.

(7) $\Rightarrow$ (6): trivial, since the w^*-topology on X^{**} induces the weak topology on X (regarded as canonically embedded in X^{**}).

(7) $\Rightarrow$ (5): a bounded sequence $(x_n^{**}) \subset X^{**}$ either has a constant subsequence (in which case the assertion is trivial), or a weak *-cluster point (by Alaoglu's theorem). In the second case (7) yields a subsequence w^*-convergent to this cluster point.

(5) $\Rightarrow$ (4): trivial (cf. the proof of (7) $\Rightarrow$ (6)).

(7) $\Rightarrow$ (2): follows from Golstine's theorem since $X^{**} = \bigcup_{n=1}^{\infty} nB(X^{**})$.

(2) $\Rightarrow$ (3): first observe that any separable X has cardinality c (unless $X = \{0\}$ of course). Now let (x_n)be dense in X. Since X is w^*-sequentially dense in X^{**}, so is (x_n). Thus there are no more elements in X^{**} than there are subsequences of (x_n). Therefore card $X^{**} = c$.

(7) $\Rightarrow$ (9): suppose for contradiction that $l^1(\Gamma)$ is isomorphic to a subspace of X^*, for some uncountable Γ. Then the adjoint T of the isomorphic embedding $l^1(\Gamma) \to X^*$ is a w^*-w^*-continuous bounded surjection from X^{**} onto $l^\infty(\Gamma) \cong l^1(\Gamma)^*$. Hence $T^{-1}(B(l^\infty(\Gamma)))$ is w^*-closed. By the open mapping theorem, for suitably large α the w^*-compact set $K := T^{-1}(B(l^\infty(\Gamma))) \cap \alpha B(X^{**})$ satisfies $TK = B(l^\infty(\Gamma))$. Now the unit ball $B(c_0(\Gamma))$ is w^*-dense (Goldstine) but not w^*-sequentially dense in $B(l^\infty(\Gamma))$, since clearly every w^*-cluster point of a sequence in $c_0(\Gamma)$ must have countable support in Γ. It follows now that $(T|_K)^{-1}(B(c_0(\Gamma))$ is bounded in X^{**} but not w^*-sequentially dense in its w^*-closure, since this closure in w^*-compact and therefore is mapped *onto* $B(l^\infty(\Gamma))$.

(9) $\Rightarrow$ (11): it suffices to observe that the set $\{\delta_x : x \in [0,1]\}$ of Dirac measure spans a subspace of C^* isomorphic to $l^1([0,1])$.

(11) $\Rightarrow$ (10): clear.

(1) $\Rightarrow$ (8): Suppose for contradiction that $L^1 \subset X^*$. Then L^∞ is isomorphic to a quotient of X^{**}. Let $T: X^{**} \to L^\infty$ be a w^*-w^*-continuous surjection. Using Goldstine's theorem again, as well as the open mapping theorem, we see that there exists an $M > 0$ so that the w^*-closure of the set $W := MT(\pi B(X))$

contains $B(L^\infty)$, where $\pi: X \to X^{**}$ denotes the canonical embedding. It now suffices to prove that W contains an l^1-sequence (f_n), since any preimage under $T \circ \pi$ of such a sequence in $MB(X)$ will then be an l^1-sequence in X, contradicting the assumption (1). We shall in fact construct an independent sequence in W (this is enough by Prop. 3.5 (i)). All that matters for this construction is the fact that w^* cl $W \supset B(L^\infty)$. We start the induction with any $f_1 \in W$ such that $\lambda\{f_1 > \frac{1}{2}\} > 0$ and $\lambda\{f_1 < -\frac{1}{2}\} > 0$. Clearly such f_1 is available in W by w^*-density (e.g. approximate $\chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}$ closely with respect to the L_1-functions $\chi_{[0,\frac{1}{2}]}$ and $\chi_{[\frac{1}{2},1]}$). Suppose now that $f_1, \dots, f_n \in W$ have been constructed so that for each $P \subset \{1, \dots, n\}$,

$$\lambda\Big(\big(\bigcap_P A_k\big) \cap \big(\bigcap_Q B_k\big)\Big) > 0, \textit{ where}$$

$$A_k := \{f_k > \tfrac{1}{2}\},\ B_k := \{f_k < -\tfrac{1}{2}\},\ Q := \{1, \dots, n\} \setminus P.$$

Put $T_P := (\bigcap_P A_k) \cap (\bigcap_Q B_k)$. Since $\lambda T_P > 0$ we may choose

$$U_P, W_P \subset T_P,\ U_P \cap W_P = \varnothing \text{ so that } \lambda U_P = \lambda W_P = \tfrac{1}{2}\lambda T_P$$

for every $P \subset \{1, \dots, n\}$. Now consider the function $\phi \in L^\infty$ defined as

$$\phi(t) = \begin{cases} +1 & \text{if } t \in \bigcup_P U_P, \\ -1 & \text{if } t \in \bigcup_P W_P, \\ 0 & \text{elsewhere.} \end{cases}$$

A sufficiently close w^*-approximation $f_{n+1} \in W$ to ϕ with respect to the finitely many functions $x_{U_P}, x_{W_P} \in L^1$ will then satisfy

$$\lambda(\{f_{n+1} > \tfrac{1}{2}\} \cap T_P) > 0,\ \lambda(\{f_{n+1} < -\tfrac{1}{2}\} \cap T_P) > 0\ \forall P \subset \{1, \dots, n\}.$$

It is now clear that for each subset $P' \subset \{1, \dots, n+1\}$ we have

$$\lambda\Big(\big(\bigcap_{P'} A_n\big) \cap \big(\bigcap_{Q'} B_n\big)\Big) > 0, \text{ where } Q' = \{1, \dots, n+1\} \setminus P'.$$

This completes the inductive definition of the independent sequence.

What we have shown so far is:

$$\begin{tikzcd}
 & 4 & 5 \arrow[l] & & \\
8 & 1 \arrow[l] \arrow[u] \arrow[r] & 7 \arrow[u] \arrow[r] \arrow[d] \arrow[dr] & 6 & \\
10 & 11 \arrow[l] & 9 \arrow[l] & 2 \arrow[r] & 3
\end{tikzcd}$$

It remains to show that (3), (6), (8) and (10) fail to hold whenever (1) fails, i.e. for every separable X containing a copy of l^1.

$\neg(1) \Rightarrow \neg(3)$: let $Y \subset X$, $Y \simeq l^1$. Since $Y^{**} \simeq (l^1)^{**}$ canonically embeds in X^{**}, it suffices to show card $(l^1)^{**} > c$. This is easy. Recall first that every separable Banach space, so in particular C, is a quotient of l^1. Thus C^* embeds in $(l^1)^*$. Now C^* contains a subspace isometric to $l^1([0,1])$ (namely the closed span of the Dirac measures). Thus $l^1([0,1])$ embeds in $(l^1)^*$, so $(l^1)^{**}$ maps onto $l^\infty([0,1]) \cong l^1([0,1])^*$. Since card $l^\infty([0,1]) = 2^c$, it follows that card $(l^1)^{**} \geqslant 2^c$. (It is easily seen that card $(l^1)^{**} = 2^c$.)

$\neg(1) \Rightarrow \neg(6)$: assume $Y \subset X$, $Y \simeq l^1$. Since the weak topology $\sigma(X, X^*)$ induces $\sigma(Y, Y^*)$, it suffices to produce a bounded set $B \subset l^1$ that is not weakly sequentially dense in its w-closure. Recall that in l^1 weakly convergent sequences are norm convergent. Now take $B = \{x \in l^1 : \|x\| = 1\}$. Then $0 \in w$-cl B (this holds in any infinite-dimensional Banach space, since every weak 0-nbhd contains a non-trivial subspace), but 0 is not the weak limit of a sequence of unit vectors.

$\neg(1) \Rightarrow \neg(10)$: this proof is based on a result of Pelczynski ([65]) that says that whenever a *separable* space W contains a copy U of C then U contains a subspace V (depending on W) that is isomorphic to C and complemented in W. We shall not prove this result here, as the tools needed to do this are wholly unrelated to the subject matter of these notes. If we assume it we can finish the proof quickly. Again let $Y \subset X$, $Y \simeq l^1$. Since every separable space is a quotient of l^1, there is a surjection $T: Y \to C$. Let us embed C isometrically in l^∞ (or in any other injective space). Then the surjection $T: Y \to C$ can be extended to a map $\tilde{T}: X \to l^\infty$. X being separable, $W := \overline{\tilde{T}X}$ is separable and of course contains C. So by Pelczynski's result quoted above, there is a subspace $V \subset C$ isomorphic to C and complemented in W. If P denotes any bounded projection from W onto V, then $P \circ \tilde{T}$ is the desired surjection of X onto $V (\simeq C)$.

$\neg(1) \Rightarrow \neg(8)$: if l^1 embeds in X, then by the preding proof there is a surjection of X onto C. But then C^* embeds in X^*, so it remains to observe that L^1 can be identified (isometrically) with the subspace of C^* consisting of the λ-continuous Radon measures. $\square$

Some of the above equivalences are true also for non-separable X. Most of them fail in the general case, however. In the next result we compare (1) to the other properties in the non-separable case.

PROPOSITION 4.2. *Let X be a (not necessarily separable) Banach space. Then*
(i) *(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ 8 $\Leftrightarrow$ (11).*
(ii) *(1) is implied by each of (2), (5), (6), (7) and (9), but the converse implications are false in general, except possibly (1) $\Rightarrow$ (6).*
(iii) *(1) $\Rightarrow$ (3), (3) $\Rightarrow$ (1) and (10) $\Rightarrow$ (1) are generally false, but (1) $\Rightarrow$ (10) is true.*

PROOF. (i) If X is non-separable then $T := B(X^*)$ with the w^*-topology is not

Polish, but it is still compact. So the equivalence (i) $\Leftrightarrow$ (iii) in Th. 3.11 and therefore (1) $\Leftrightarrow$ (4) remains true.

We shall now show (1) $\Rightarrow$ (8) $\Rightarrow$ (11) $\Rightarrow$ (1). The proof we gave of (1) $\Rightarrow$ (8) did not use separability, so remains valid in the non-separable case. (8) $\Rightarrow$ (11) is clear, since C^* contains L^1 isometrically. We now show that (11) $\Rightarrow$ (1).

Suppose for contradiction that $Y \subset X$, $Y \simeq l^1$. Since C is separable, there is a surjection $T: Y \to C$. Let π be the canonical embedding of C in C^{**}. Since C^{**} is injective there is a bounded linear map $\tilde{T}: X \to C^{**}$ which extends T, more precisely $\tilde{T}|Y = \pi \circ T$. Let $\pi_1 : C^* \to C^{***}$ be the canonical embedding. We now claim that $\tilde{T}^* \circ \pi_1$ embeds C^* in X^*.

$$
\begin{tikzcd}
X^* & & & C^{***} \arrow[lll, "\tilde{T}^*"'] \arrow[dl, "\pi^*"'] & C^* \arrow[l, "\pi_1"'] \\
X \arrow[drrr, "\tilde{T}"] & & C^* & & \\
Y \arrow[u, phantom, "\cup"] \arrow[rr, "T"] & & C \arrow[r, "\pi"] & C^{**} &
\end{tikzcd}
$$

The proof depends on the following two facts.

(a) $\tilde{T}B(X)$ contains a multiple $r\pi B(C)$, $r > 0$. This is clear since T is a surjection, hence open.

(b) $\pi^* \circ \pi_1 = 1_{C^*}$. This is generally true when C is replaced by any Banach space.

Now observe that for $\mu \in C^*$ we have

$$\|\tilde{T}^* \pi_1 \mu\| = \sup_{x \in B(X)} |\langle x, \tilde{T}^* \pi_1 \mu \rangle| = \sup_{x \in B(X)} |\langle \tilde{T}x, \pi_1 \mu \rangle|$$

(a) (b)

$$\geqslant r \sup_{y \in B(C)} |\langle \pi y, \pi_1 \mu \rangle| = r \sup_{y \in B(C)} |\langle y, \pi^* \pi_1 \mu \rangle| = r\|\mu\|$$

(ii) Suppose that we have (5), (6) or (7) for some non-separable X and that (1) fails, so that there exists $Y \subset X$, $Y \simeq l^1$. The fact to be noticed is that Y^{**} can be identified with the w^*-closed subspace $Y^{\perp\perp} \subset X^{**}$ and that the w^*-topology on X^{**} induces on $Y^{\perp\perp}$ the topology that corresponds (under the map that identifies $Y^{\perp\perp}$ with Y^{**}) with the w^*-topology of Y^{**}. Since each of (5), (6) and (7) fails for Y (by Th. 4.1, since Y is separable), it therefore also fails for X. Contradiction.

(2) $\Rightarrow$ (1): this implication requires a little argument. Let $Y \subset X$ be any subspace. We shall prove that (2) holds for Y also. Taking Y separable, we then infer from Th. 4.1 that Y contains no copy of l^1. So neither does X, since Y is arbitrary.

To prove the claim, let us identify Y^{**} with the subspace $Y^{\perp\perp} \subset X^{**}$ and let $y^{**} \in Y^{\perp\perp}$. By our assumption (2), $y^{**} = w^*\text{-}\lim x_n$ with $(x_n) \subset X$. Then actually $y^{**} = w^*\text{-}\lim y_n$ with $(y_n) \subset Y$, as we now show. Let us suppose for simplicity that $\|y^{**}\| = 1$. It suffices to prove that

$$d(B(Y), \overline{co}\,\{x_n, x_{n+1}, \dots\}) = 0 \text{ for every } n \in \mathbb{N}. \tag{*}$$

Indeed, once (*) is proved we can choose elements $y_n \in B(Y)$ and $b_n \in co\{x_n, x_{n+1}, \ldots\}$ $(n=1,2,\ldots)$ so that $\|y_n - b_n\| \to 0$. Observe that $y^{**}=w^*$-lim x_n implies $y^{**}=w^*$-lim b_n. Since $\|y_n - b_n\| \to 0$ we then conclude $y^{**}=w^*$-lim y_n.

Let us suppose now that (*) fails, so that for some $N \in \mathbb{N}$

$$d(B(Y),\ K)>0, \text{ where } K=\overline{co}\{x_N, x_{N+1}, \ldots\}.$$

The Hahn-Banach theorem then supplies an $x^* \in X^*$ such that

$$\sup_{B(Y)} x^* < \inf_K x^* = \inf_{n \geq N} \langle x_n, x^* \rangle.$$

Since $\|y^{**}\|=1$ Goldstine's theorem implies that $\langle x^*, y^{**}\rangle \leq \sup_{B(Y)} x^*$. Hence $\langle x^*, y^{**}\rangle < \inf_{n \geq N} \langle x_n, x^*\rangle$. But this last inequality contradicts the fact that $y^{**}=w^*$-lim x_n.

(9) $\Rightarrow$ (1): this is due to the possibility of "lifting" $l^1(\Gamma)$. If $Y \subset X$, $Y \simeq l^1$, then there is a surjection $T: X^* \to (l^1)^*$. By Th. 4.1 $(l^1)^*$ contains a copy of $l^1(\Gamma)$, Γ uncountable. If $(e_\gamma)_{\gamma \in \Gamma}$ denotes the transfinite "sequence" in $(l^1)^*$ corresponding to the unit vectors in $l^1(\Gamma)$, then clearly any bounded set of elements $x^*_\gamma \in X^*$ with $Tx^*_\gamma = e_\gamma$ $(\gamma \in \Gamma)$ spans a copy of $l^1(\Gamma)$ in X^*.

Finally, the falsity of each of (1) $\Rightarrow$ (2), (5), (7), (9) can be shown with a single example, namely $X=c_0(\Gamma)$, $\Gamma=[0,1]$. Since every separable subspace of $c_0(\Gamma)$ is contained in a copy of c_0, and c_0 has no subspaces isomorphic to l^1, (1) holds for $c_0(\Gamma)$. To see that (2) fails, note that w^*-limits of sequences in $c_0(\Gamma)$ all vanish off a countable subset of Γ. These elements therefore fail to fill up $c_0(\Gamma)^{**} \cong l^\infty(\Gamma)$. The same argument shows that $B(c_0(\Gamma))$ is not w^*-sequentially dense in its w^*-closure $B(l^\infty(\Gamma))$, i.e. (7) fails. To see the failure of (5), observe that on bounded sets in $l^\infty(\Gamma)=c_0(\Gamma)^{**}$, w^*-convergence means pointwise convergence on Γ.Then the Rademacher functions (r_n) constitute an example of a sequence in $l^\infty(\Gamma)$ without pointwise $(=w^*$-) convergent subsequence. Finally, since $c_0(\Gamma)^* \cong l^1(\Gamma)$, (9) fails.

(iii) To see that (3) $\Rightarrow$ (1) fails, consider an X of the form $X=Z \oplus l^1$, where Z is a reflexive space of large cardinality (e.g. $Z=l^2(\Gamma)$ for large Γ). On the other hand simple calculations show that card $l^\infty(\Gamma)=2^{\mathrm{card}\Gamma}>$ card $\Gamma=$ card $c_0(\Gamma)$ whenever card $\Gamma \geq c$. So (1) $\Rightarrow$ (3) also fails.

It is clear from what we have proved earlier that (1) $\Rightarrow$ (10) ((1) $\Rightarrow$ (8) and (8) $\Rightarrow$ (11) $\Rightarrow$ (10)). On the other hand it is known that every separable quotient of l^∞ must be reflexive (cf. Ch. 0). Hence C is not isomorphic to a quotient of $l^\infty=(l^1)^*$, although l^1 is a subspace of l^∞ (any independent bounded sequence of functions on $\mathbb{N}$ spans a copy of l^1 in l^∞). So (10) $\Rightarrow$ (1) fails. $\square$

REMARK 4.3. It is interesting to note that $c_0(\Gamma)$, whatever Γ, is not a counterexample for (1) $\Rightarrow$ (6). The proof of this assertion is an elementary exercise we leave to the reader. $\square$

NOTES Many people have had a hand in the results of this chapter. The earliest contributions were made by A. PELCZYNSKI. In [64] he proved the equivalence of (1) with (8), (9), (10) and (11) in the separable case, albeit under a special assumption (we do not spell it out here, as it turned out to be irrelevant). A few years later J. HAGLER ([33]) was able to remove this special condition, and also to extend (8) and (11) to the non-separable case. Although, as we have seen, (1) $\Rightarrow$ (9) is false in the absence of separability, J. Hagler did prove the following non-separable version of (1) $\Rightarrow$ (9): if X^* contains a copy of $l^1(\Gamma)$ and if the cardinality of Γ is larger than the dimension of X [: = the least cardinal number of a set whose closed linear span is X], then X contains a copy of l^1. This should be compared to the example $X = c_0(\Gamma)$ we gave to disprove (1) $\Rightarrow$ (9) in general: dim $c_0(\Gamma)=$ card Γ, so Hagler's condition fails here.

A big jump ahead was made when H.P. ROSENTHAL proved his famous l^1-theorem ([72]) (extended by L. DOR to the complex case a little later in [15]): every bounded sequence in an arbitrary Banach space either has a w-Cauchy subsequence or an l^1-subsequence. This implies (1) $\Leftrightarrow$ (4). Observe that, conversely, Rosenthal's theorem follows when in Theorem 3.11 we apply the equivalence (i) $\Leftrightarrow$ (iii) to a countable set $Z \subset C(T)$. The approach of H.P. ROSENTHAL in [72] was rather combinatorial in nature, involving essentially Ramsey's theorem. A detailed account of the "Ramsey" approach to the l^1 problem can be found in [13]. Shortly after [72] the measure-topological approach via first class Baire functions (initiated by H.P. Rosenthal himself) began to emerge in [73], [62] and [74]. In [62] the characterizations (2), (3) and (5) were derived for separable X, and in [74] also (6) and (7) made their appearance. In fact (6) and (7) are partly due to J. BOURGAIN, D.H. FREMLIN and M. TALAGRAND ([8]). Motivated by some open questions in [73], they initiated a deep study of the various function spaces involved in the l^1 problem, especially with regard to compactness properties. We have already mentioned their result that $\mathfrak{B}_1(T)$ is angelic for Polish T (Theorem 3. 13).

The characterization (7) prompted H.P. ROSENTHAL ([73]) to ask the following question. Suppose $B(X^*)$ is not w^*-sequentially compact, does this imply that X constans a copy of $l^1(\Gamma)$ for uncountable Γ? By (7) $\Leftrightarrow$ (8) this is true if $X \simeq Y^*$ for separable Y. The answer is negative in general. J. Hagler and E. ODELL ([36]) have constructed an X with non-w^*-sequentially compact dual ball in which even l^1 does not embed. For more on this, see [36], [35] and Chapter 13 in [13].

The list of equivalences in Theorem 4.1 is far from complete. We mention here a few possible additions.

(12) For every $x^{**} \in X^{**}$ and for every w^*-compact subset $A \subset X^*$ the

restriction $x^{**}|_A$ has a point of w^*-continuity.

The equivalence (1) $\Leftrightarrow$ (12) was first explicitly stated by E. and P. SAAB in [80]. In our setup it is an immediate consequence of (iv) $\Leftrightarrow$ (i) in Th. 3. 11. No separability is required. In the same paper [80] several other equivalences were deduced. We state two of them without further comment. Explaining them would take us too far afield and, in any case, would be rather pointless without recourse to the parallel results on Asplund spaces (cf. e.g. [60], [18]) to compare them with.

(13) All bounded sets in X^* are w^*-dentable in (X^*,w) (i.e. all bounded sets in X^* admit w^*-slices that are abitrarity "small" in the sense of the weak topology).

(14) All bounded sets in X^* are w^*-scalarly dentable (i.e. for every bounded set $A \subset X^*$ and for every $x^{**} \in X^{**}$ there is a w^*-slice of A on which x^{**} has arbitrarily small oscillation).

The next equivalent property comes directly from (v) in Theorem 3. 11. We shall come back to it in Chapter 6, where a different proof will be given.

(15) The identity map $(B(X^*),w^*) \to X^*$ is universally scalarly measurable (i.e. for every Radon measure on $(B(X^*),w^*)$ every $x^{**} \in X^{**}$ is μ-measurable).

There are also several characterizations related to the Dunford-Pettis property that we have neglected to mention in the main text. The first is due to E. Odell (see [73])

(16) Every Dunford-Pettis operator from X into any other Banach space Y is compact.

[A *Dunford-Pettis operator* is an operator that sends w-Cauchy sequences to norm Cauchy sequences] Note that (16) follows immediately from (4). In [22] G. EMMANUELE manipulates this result of E. Odell to show that (1) is also equivalent to each of the following two properties.

(17) For every Banach space Y with the Dunford-Pettis property every operator $T: Y \to X^*$ is Dunford-Pettis.

[Y has the *Dunford-Pettis property* iff for every space Z every w-compact $T: Y \to Z$ is a Dunford-Pettis operator]. Observe that (17) $\Rightarrow$ (8), since L^1 has the Dunford-Pettis property (see [14]), and an isomorphic embedding of L^1 into any space is clearly not a Dunford-Pettis operator. A weaker version of (17) had earlier been proved by H. FAKHOURY to be equivalent to (1) (see [74]).

(18) Every subset $K \subset X^*$ such that $\lim_{n\to\infty} \sup_{x^*\in K} |\langle x_n, x^*\rangle| = 0$ for every w-null sequence $(x_n) \subset X$, is necessarily relatively compact.

We end the list with three characterizations whose special nature sets them apart from the others (this also shows up in the proofs, which we omit). Especially the first one, due to B. MAUREY ([53], see also [75]) is a beauty. It is true only for separable X.

(19) l^1 embeds in a separable space X iff X^{**} contains an element $x^{**} \neq 0$ so that $\|x^{**} + x\| = \|x^{**} - x\|$ for all $x \in X$.

Whereas (18) characterizes the compact subsets of X^* in terms of whether l^1 embeds in X or not, the next result, due to R.G. BILYEU and P.W. LEWIS ([4]), is about compact subsets of X. We say that *uniform Gateaux differentiability characterizes compactness* in Y provided a set $K \subset Y$ is relatively compact iff there exists an $x \in Y$ so that $\lim_{t\to 0} \dfrac{\|x+ty\| - \|x\|}{t}$ exists uniformly for $y \in K$.

Now the result is as follows:

(20) l^1 embeds in X iff there exists an infinite-dimensional subspace $Y \subset X$ and an equivalent norm $|||\cdot|||$ on Y so that uniform Gateaux differentiability characterizes compactness in $(Y, |||\cdot|||)$.

For the final characterization we need some notation. For Banach spaces X and Y we denote by $C_{wb}(X,Y)$ the space of all (not necessarily linear) functions f from X to Y such that for each bounded subset B of X the restriction $f|_B$ is continuous from (B,w) to $(Y, \|\cdot\|)$. Also let $C_{wsc}(X,Y)$ denote the set of function $f: X \to Y$ that are sequentially continuous from (X,w) to $(Y, \|\cdot\|)$. Clearly $C_{wb}(X,Y) \subset C_{wsc}(X,Y)$. R.M. ARON, J. DIESTEL and A.K. RAJAPPA ([2]) proved:

(21) l^1 embeds in X iff $C_{wb}(X,Y) \neq C_{wsc}(X,Y)$ for every Banach space Y.

[There is a strong dichotomy here: J. FERRERA, J. GOMEZ GIL and J.G. LLAVONA showed earlier in [23] that if $l^1 \not\subset X$ then $C_{wb}(X,Y) = C_{wsc}(X,Y)$ for *all* Y.]

Chapter V

The Pettis integral and the weak Radon-Nikodym property

This chapter should be viewed as a short intermezzo in preparation for Chapter 6. There we shall prove several more characterizations of spaces not containing l^1. All of these involve the Pettis integral, and in this chapter we develop the necessary background needed to understand them.

§ 1. Elementary facts about the Pettis integral.

Let (Ω,Σ,μ) be a probability space, X a Banach space and let a map $\phi:\Omega\to X$ be given.

DEFINITION 5.1. We say that
(i) ϕ is *scalarly measurable* if $\langle\phi,x^*\rangle$ is measurable for every $x^*\in X^*$.
(ii) ϕ is *scalarly* L^1 if $\langle\phi,x^*\rangle\in L^1(\mu)$ for every $x^*\in X^*$.
(iii) ϕ is *scalarly bounded* if $\langle\phi,x^*\rangle\in L^\infty(\mu)$ for every $x^*\in X^*$. In the case of a function $\phi:\Omega\to X^*$ into a dual space we say that ϕ is (i) w^*-*scalarly measurable*, (ii) w^*-*scalarly* L^1, (iii) w^*-*scalarly bounded* if $\langle x,\phi\rangle$ (rather than $\langle\phi,x^{**}\rangle$) satisfies the respective conditions above for every $x\in X(=\pi X)$. $\square$

Given a $\phi:\Omega\to X$ which is *scalarly* L^1, let us consider the map

$$X^*\ni x^* \xrightarrow{S} \langle\phi,x^*\rangle\in\mathbb{R}^\Omega \text{ or } L^1(\mu).$$

It is elementary to check that S, regarded as a map into $L^1(\mu)$ has a closed graph, so that it is bounded by the closed graph theorem. Let us observe also that S, when considered as a map into $\mathbb{R}^\Omega$, is continuous for the w^*-topology on X^* and the topology of pointwise convergence on Ω. Using the w^*-compactness of $B(X^*)$, we see therefore that the set

$$Z_\phi := \{\langle\phi,x^*\rangle : \|x^*\|\leqslant 1\},$$

being the S-image of $B(X^*)$, is
(i) a τ_p-compact set of measurable functions $(\subset\mathbb{R}^\Omega)$ and
(ii) a bounded set in $L^1(\mu)$.

We shall usually not specify whether Z_ϕ is considered as a subset of $\mathbb{R}^\Omega$ or of $L^1(\mu)$. It should be clear from the context.

The adjoint $S^* =: T$ maps $L^\infty(\mu)$ into X^{**} and is defined by the formula

$$\langle x^*, Tf\rangle = \int_\Omega f\cdot\langle\phi, x^*\rangle d\mu \quad (f\in L^\infty(\mu),\ x^*\in X^*). \tag{1}$$

T is called the *Dunford operator* associated with ϕ, and we occasionally denote it by T_ϕ.

$$\begin{array}{ccc} L^\infty(\mu)\ni f & \xrightarrow{\quad T=S^*\quad} & Tf\in X^{**} \\ L^1(\mu)\ni\langle\phi,x^*\rangle & \xleftarrow{\quad S\quad} & x^*\in X^* \\ \Omega & \xrightarrow{\quad \phi\quad} & X \end{array}$$

DEFINITION 5.2. A scalarly L^1 function $\phi:\Omega\to X$ is called *Pettis integrable* if its associated Dunford operator T maps $L^\infty(\mu)$ into X (rather than X^{**}). In that case $T(\chi_E)$, where $E\in\Sigma$, is called the *Pettis integral* of ϕ over E. Notation: $T(\chi_E)=(P)\int_E\phi d\mu$. □

REMARK 5.3. Clearly $T(L^\infty(\mu))\subset X$ is equivalent to $T\chi_E\in X$ for every $E\in\Sigma$ (since T is bounded). The (P)-integral $(P)\int_E\phi d\mu$, if it exists, is the "weak" integral of ϕ over E. It is the unique element of X that satisfies

$$\langle (P)\int_E\phi d\mu, x^*\rangle = \int_E\langle\phi,x^*\rangle\, d\mu \quad \forall x^*\in X^*$$

(take $f=\chi_E$ in (1)). □

We shall see many (P)-integrable functions in the course of this chapter. Here is a simple example of a scalarly L^1 function that is not (P)-integrable.

EXAMPLE 5.4. Let $\phi:[0,1]\to c_0$ be defined by

$$\phi(t) := (n\chi_{(0,\frac{1}{n}]}(t))_{n=1}^\infty \quad (t\in[0,1]).$$

For every $x^*=(\xi_n)\in l^1(\cong c_0^*)$ we have that $\langle\phi,x^*\rangle = \sum_{n=1}^\infty n\,\xi_n\,\chi_{(0,\frac{1}{n}]}\in L^1(\lambda)$ (λ = Lebesgue measure). However, the element $T_\phi(\chi_{[0,1]})\in c_0^{**}\cong l^\infty$ that maps $(\xi_n)=x^*$ to $\int_0^1\langle\phi,x^*\rangle d\lambda = \sum_{n=1}^\infty\xi_n$ is not w^*-continuous. In fact it is given by the element $(1,1,1,\ldots)\in l^\infty\setminus c_0$, so $(P)\int_0^1\phi d\lambda$ does not exist. □

Several facts about the (P)-integral are immediate consequences of the criterion formulated in the next proposition.

PROPOSITION 5.5. *Let $\phi:\Omega\to X$ scalarly L^1. Then the following are equivalent:*
(i) *ϕ is (P)-integrable,*
(ii) *the canonical map $\mathbb{R}^\Omega\supset Z_\phi\to Z_\phi\subset L^1(\mu)$ that sends each function in Z_ϕ to its equivalence class in $L^1(\mu)$, is pointwise-to-weak continuous.*

PROOF. For the purpose of this proof, let us regard Z_ϕ as a subset of $\mathbb{R}^\Omega$, and let us give the name i to the canonical map that sends each $\langle\phi,x^*\rangle\in Z_\phi$ to its equivalence class in $L^1(\mu)$. Then $S:X^*\to L^1(\mu)$, restricted to $B(X^*)$, decomposes as follows:

$$\begin{array}{ccccc} S:(B(X^*),w^* & \longrightarrow & (Z_\phi,\tau_p) & \xrightarrow{\quad i \quad} & (L^1(\mu),w) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ x^* & & \langle\phi,x^*\rangle & & i(\langle\phi,x^*\rangle) \end{array}$$

As we already observed $x^*\to\langle\phi,x^*\rangle$ maps the w^*-compact set $B(X^*)$ continuously onto the τ_p-compact set Z_ϕ, and therefore it is a quotient map (in the topological sense). This implies that i is τ_p-to-w-continuous iff $S|_{B(X^*)}$ is w^*-to-w-continuous. This last condition implies that for every $f\in L^\infty(\mu)$ the null space $\ker Tf$ intersects $B(X^*)$ in a w^*-closed set. By the Krein-Smulian theorem then $\ker Tf$ is w^*-closed, i.e. $Tf\in X$. So $TL^\infty(\mu)\subset X$. Conversely $TL^\infty(\mu)\subset X$ is equivalent to S being w^*-to-w-continuous, so it implies the w^*-to-w continuity of $S|_{B(X^*)}$. $\square$

Before stating some consequences of Prop. 5.5 let us introduce, given a $\phi:\Omega\to X$ that is scalarly L^1, a new function Φ that is related to the variation of the (P)-integral. Observe that all elements $\langle\phi,x^*\rangle\in Z_\phi$ satisfy $|\langle\phi,x^*\rangle|\leqslant\|\phi\|$ and that $\|\phi\|$ is a finite function. However, in general $\|\phi\|$ is not measurable. This is not a serious problem. We claim that there exists a measurable function $\Phi:\Omega\to\mathbb{R}^+$ satisfying the following three properties:

(i) $\Phi\leqslant\|\phi\|$ everywhere on Ω (so Φ is finite),
(ii) $|\langle\phi,x^*\rangle|\leqslant\Phi$ μ a.e. for every $x^*\in B(X^*)$,
(iii) if (ii) holds for some other measurable function Φ' instead of Φ, then $\Phi\leqslant\Phi'$ μ.a.e.

To define Φ, simply select a sequence $(x_n^*)\subset B(X^*)$ such that $\int_\Omega|\langle\phi,x_n^*\rangle|d\mu\to\sup\{\int_\Omega|\langle\phi,x^*\rangle|d\mu:\|x^*\|\leqslant1\}$ (note that this sup is finite since Z_ϕ is L^1-bounded) and put $\Phi:=\sup_n|\langle\phi,x_n^*\rangle|$. Then $\int_\Omega\Phi d\mu\geqslant\sup\{\int_\Omega|\langle\phi,x^*\rangle|d\mu:\|x^*\|\leqslant1\}$. The proof that this Φ satisfies the requirements is now straightforward. (Put in abstract terms, we have established the existence of a least upper bound for $Z_\phi/_\sim$ in the Riesz space $M(\mu)/_\sim$, where $\sim$ denotes identification of μ a.e. equal functions.)

Here are some consequences of Prop. 5.5.

PROPOSITION 5.6. *Let $\phi:\Omega\to X$ be (P)-integrable. Then*

(i) *ϕ is equi-scalarly L^1, i.e. Z_ϕ is bounded and uniformly integrable as a subset of $L^1(\mu)$,*
(ii) *S is weakly compact,*
(iii) *T is weakly compact,*
(iv) *the X-valued set function F on Σ defined by $F(E):=T\chi_E=(P)\int_E\phi d\mu$ is a measure (i.e. countably additive). Furthermore, F is μ-continuous and has σ-*

finite variation $|F|$.

PROOF. By Prop. 5.5 (ii) and the τ_p-compactness of $Z_\phi \subset \mathbb{R}^\Omega$, the set $Z_\phi \subset L^1(\mu)$ is weakly compact (hence bounded). It is well known that the relatively weakly compact sets in $L^1(\mu)$ are precisely the bounded and uniformly integrable ones. This proves (i) , and also (ii), since $Z_\phi \subset L^1(\mu)$ is the S-image of $B(X^*)$. (iii) follows from (ii) since adjoints of weakly compact maps are weakly compact (and conversely).

For the proof of (iv), observe first that $E \to T(\chi_E)$ is certainly finitely additive. The countable additivity is then equivalent to the assertion that $E_n \downarrow \varnothing$ implies $FE_n = T\chi_{E_n} \to 0$ ($E_n \in \Sigma$; $E_n \downarrow \varnothing$ means (E_n) decreasing and $\bigcap_{n=1}^{\infty} E_n = \varnothing$). But this is clear from the obvious μ-continuity of F:

$$\|FE\| = \|T\chi_E\| = \sup_{\|x^*\|\leqslant 1} \langle T\chi_E, x^*\rangle = \sup_{\|x^*\|\leqslant 1} \int_E \langle \phi, x^*\rangle d\mu \to 0$$

as $\mu E \to 0$, by the uniform integrability of Z_ϕ.

Recall that the variation $|F|$ of the X-valued measure $F:\Sigma \to X$ is the non-negative set function defined by $|F|(E) := \sup \sum_{i=1}^{n} \|FE_i\|$ ($E \in \Sigma$), where the sup is taken over all finite partitions $\{E_1, \ldots, E_n\}$ of E with $E_i \in \Sigma$, $i = 1, \ldots, n$. It is well known and easy to prove that the countable additivity of F implies that of $|F|$. To show that $|F|$ is σ-finite, observe that for all $E \in \Sigma$ and $x^* \in B(X^*)$ we have

$$|\langle FE, x^*\rangle| \leqslant \int_E |\langle \phi, x^*\rangle| d\mu \leqslant \int_E \Phi d\mu, \tag{2}$$

where Φ is the (finite!) function defined prior to Prop. 5.6. Putting $\Omega_n := \{n-1 \leqslant \Phi < n\}$, $n = 1, 2, \ldots$, it follows from (2) that $|F|(\Omega_n) \leqslant n\mu\Omega_n$, so that $|F|$ is σ-finite. □

It is not true in general that $T = T_\phi$ is compact for a (P)-integrable ϕ (equivalently, the range $\{(P)\int_E \phi d\mu : E \in \Sigma\}$ of the Pettis integral need not be relatively compact). A mild condition on the measure space, however, is enough to guarantee this.

PROPOSITION 5.7. *Let* (Ω, Σ, μ) *be perfect and* $\phi:\Omega \to X$ *Pettis-integrable. Then* $T = T_\phi$ *is compact.*

PROOF. We show that S is compact, or equivalently that Z_ϕ is compact in $L^1(\mu)$. We first need to reduce to the case where Z_ϕ is L^∞-bounded. For this recall that Φ is finite, so that $\mu\{\Phi \geqslant n\} \to 0$ as $n \to \infty$. Z_ϕ being bounded and uniformly integrable (Prop. 5.6) it follows that $\lim_{n\to\infty} \langle \phi, x^*\rangle \chi_{\{\Phi \geqslant n\}} = 0$ in L^1-sense, uniformly for $x^* \in B(X^*)$. Hence it suffices to prove that for each $n \in \mathbb{N}$ the set $Z_\phi \cdot \chi_{\{\Phi \leqslant n\}} := \{\langle \phi, x^*\rangle \chi_{\{\Phi \leqslant n\}} : x^* \in B(X^*)\}$ is compact in $L^1(\mu)$. But now Theorem 1.6 can be used: the τ_p-compactness of Z_ϕ implies that of

$Z_\phi\cdot\chi_{\{\Phi\leq n\}}$, so by Fremlin's theorem each sequence in $Z_\phi\cdot\chi_{\{\Phi\leq n\}}$ has a μ a.e. convergent subsequence. But since $Z_\phi\cdot\chi_{\{\Phi\leq n\}}$ is L^∞-bounded, every such subsequence is also L^1-convergent, by Lebesgue's theorem. □

§ 2. The weak Radon-Nikodym property.

We have seen that each Pettis integrable function $\phi:\Omega\to X$ gives rise to an X-valued measure F: $FE:=(P)\int_E\phi d\mu$ $(E\in\Sigma)$. We are now going to consider the converse problem (for a fixed complete probability space (Ω,Σ,μ)): which X-valued measures are Pettis integrals? Prop. 5.6 shows that the question is meaningful only for μ-continuous measures of σ-finite variation.

Definition 5.8. A Banach space X is said to have the *weak Radon-Nikodym property* (WRNP) *with respect to a given complete probability space* (Ω,Σ,μ) if every μ-continuous measure $F:\Sigma\to X$ of σ-finite variation has a (P)*-integrable "derivative"* ϕ, i.e. a (P)-integrable $\phi:\Omega\to X$ satisfying

$$FE=(P)\int_E\phi d\mu \quad (E\in\Sigma).$$

We say that X has the *WRNP* if it has the WRNP with respect to every complete probability space.

If X is a dual space Y^*, we shall use the term w^* *-derivative* for every w^*-scalarly measurable function $\phi:\Omega\to Y^*$ that satisfies

$$\langle y, FE\rangle=\int_E\langle y,\phi\rangle d\mu \quad (y\in Y, E\in\Sigma). \qquad \square$$

There is really no need to consider measures of σ-finite variation. Measures of bounded variation suffice.

Lemma 5.9. *Each of the following two properties is equivalent to the WRNP for X.*

(i) *every X-valued measure F on every complete probability space* (Ω,Σ,μ) *for which there exists a constant $M<\infty$ so that $\|FE\|\leq M\mu E$ for all $E\in\Sigma$ (this implies $F\ll\mu$) has a (P)-integrable derivative.*
(ii) *for every complete probability space* (Ω,Σ,μ) *every bounded linear operator $T:L^1(\mu)\to X$ satisfies $T|_{L^\infty(\mu)}=T_\phi$ for a Pettis integrable $\phi:\Omega\to X$.*

Proof. Suppose that (i) holds and that $F:\Sigma\to X$ is a μ-continuous measure of σ-finite variation. As we have remarked earlier $|F|$ is then also a measure, i.e. countably additive. Let us write Ω as a disjoint union $\bigcup_{n=1}^{\infty}\Omega_n$ with $|F|(\Omega_n)<\infty$, $n=1,2,\ldots$. On each Ω_n the restriction of $|F|$ is then μ-continuous, since $\mu E=0$ implies $|F|E=0$ for $E\subset\Omega_n$, $E\in\Sigma$ (this condition is equivalent to μ-continuity for finite non-negative measures). It now follows from the classical Radon-

Nikodym theorem that there exists a measurable $h:\Omega\to\mathbb{R}^+$ satisfying

$$|F|(E)=\int_E h\,d\mu \quad (E\in\Sigma).$$

Now put $\Omega'_n:=\{n-1\leqslant h<n\}$ $(n=1,2,\ldots)$. Then the restriction F_n of F to Ω'_n satisfies $\|F_nE\|\leqslant n\mu E$ $(E\in\Sigma,\, E\subset\Omega'_n)$, and therefore by our assumption has a (P)-integrable derivative $\phi_n:\Omega'_n\to X$. We claim that the function $\phi:\Omega\to X$ defined by $\phi|_{\Omega'_n}:=\phi_n$ is a (P)-integrable derivative of F.

Let us first show that ϕ is scalarly L^1. Fix $x^*\in X^*$ and put $\Omega^+:=\{\langle\phi,x^*\rangle\geqslant 0\}$, $\Omega^-:=\{\langle\phi,x^*\rangle<0\}$. Then

$$\int_\Omega|\langle\phi,x^*\rangle|d\mu=\int_{\Omega^+}\langle\phi,x^*\rangle d\mu+\int_{\Omega^-}\langle\phi,-x^*\rangle d\mu.$$

Since

$$\int_{\Omega^+}\langle\phi,x^*\rangle d\mu=\sum_{n=1}^{\infty}\int_{\Omega'_n\cap\Omega^+}\langle\phi,x^*\rangle d\mu=\sum_{n=1}^{\infty}\langle F(\Omega'_n\cap\Omega^+),x^*\rangle=\langle F\Omega^+,x^*\rangle<\infty,$$

and similarly

$$\int_{\Omega^-}\langle\phi,-x^*\rangle d\mu=-\langle F(\Omega^-),x^*\rangle<\infty,$$

we conclude that $\int_\Omega|\langle\phi,x^*\rangle|d\mu<\infty$. The fact that ϕ is a (P)-integrable derivative of F now follows easily: for every $E\in\Sigma$ and every $x^*\in X^*$ we have

$$\langle F(E),x^*\rangle=\sum_{n=1}^{\infty}\langle F(E\cap\Omega'_n),x^*\rangle=\sum_{n=1}^{\infty}\int_{E\cap\Omega'_n}\langle\phi,x^*\rangle d\mu=\int_E\langle\phi,x^*\rangle d\mu,$$

where the last equality comes from Lebesgue's theorem (recall that we already know that $<\phi,x^*>\in L^1(\mu)$).

Finally, to see that (ii) is equivalent to (i), observe that there is a 1-1 correspondence between the bounded linear operators $T:L^1(\mu)\to X$ and X-valued measures $F:\Sigma\to X$ satisfying the condition $\|FE\|\leqslant M\mu E$ $(E\in\Sigma)$ for some $M<\infty$: simply put $FE:=T\chi_E$ when T is given, and when F is given, observe that T defined on the simple functions by $T(\sum_{i=1}^{n}\alpha_i\chi_{E_i}):=\sum_{i=1}^{n}\alpha_i FE_i$ is bounded, so extends uniquely to a bounded operator on $L^1(\mu)$ (with norm $\|T\|\leqslant M$). The existence of a (P)-integrable derivative ϕ for every $F:\Sigma\to X$ with $\|FE\|\leqslant M\mu E$ $(E\in\Sigma)$ for some $M<\infty$ is then clearly equivalent to the statement that for every bounded $T:L^1(\mu)\to X$ the restriction $T|_{L^\infty(\mu)}$ is a Dunford operator T_ϕ.

Let us note also that in this case (1) in § 1 extends to

$$\langle Tf,x^*\rangle=\int_\Omega f\cdot\langle\phi,x^*\rangle d\mu \quad (f\in L^1(\mu),\ x^*\in X^*). \tag{3}$$

(The condition $\|FE\|\leqslant M\mu E$ $(E\in\Sigma)$ is equivalent, via (3), with $Z_\phi\subset MB(L^\infty(\mu))$, as one readily verifies.) □

REMARK 5.10. If $F=(P)\int\phi_1 d\mu$ then a function $\phi_2:\Omega\to X$ is Pettis integrable with the same Pettis integral F iff ϕ_1 and ϕ_2 are *scalarly equivalent*, i.e.

$$\langle\phi_1,x^*\rangle = \langle\phi_2,x^*\rangle \ \ \mu \text{ a.e. for every } x^*\in X^*.$$

This does not in general imply that $\phi_1=\phi_2$ μ a.e., because the exceptional null set is allowed to vary with x^*. However, if X^* contains a total sequence (x_n^*), then we must have $\phi_1=\phi_2$ μ a.e.: if $\langle\phi_1,x_n^*\rangle=\langle\phi_2,x_n^*\rangle$ off a null set N_n $(n=1,2,...)$ and if $N=\bigcup_{n=1}^{\infty} N_n$, then $\mu N=0$ and off N we have that $\langle\phi_1,x_n^*\rangle=\langle\phi_2,x_n^*\rangle$ for all $n\in\mathbb{N}$, hence $\phi_1=\phi_2$. Such a total sequence exists whenever X is separable, but also for some non-separable spaces such as l^∞ (take $x_n^*=e_n$, the n^{th} unit vector of l^1). □

EXAMPLE 5.11. Let $\{e_t:t\in[0,1]\}$ be the standard orthonormal basis for the non-separable Hilbert space $l^2([0,1])$. Consider the map $[0,1]\ni t\xrightarrow{\phi} e_t\in l^2([0,1])$. Then ϕ is everywhere non-zero, but scalarly equivalent to the 0-function, since for every $x^*\in l^2([0,1])^*=l^2([0,1]), \langle\phi,x^*\rangle=0$ off a countable set (we are using Lebesgue measure). In particular ϕ is scalarly measurable, and scalarly L^1. Its Pettis integral exists and is identically 0. (ϕ is not Bochner - or strongly measurable, since its range is non-separable!) □

REMARK 5.12. Let X be separable and let $F:\Sigma\to X$ be a measure satisfying $\|FE\|\leqslant M\mu E$ for all $E\in\Sigma$. If $\phi:\Omega\to X$ is a (P)-integrable derivative of F then ϕ is valued μ a.e. in $MB(X)$. To see this, let (x_n^*) be a sequence of unit vectors in X^* that norms X, i.e. such that

$$\|x\| = \sup_{n\in\mathbb{N}}\langle x,x_n^*\rangle \text{ for every } x\in X.$$

Such a sequence clearly exists by the separability of X. Now the complement of $MB(X)$ is covered by the sets $\{x_n^*>M\}$, so if ϕ is not μ a.e. valued in $MB(X)$ then $\mu(\{\langle\phi,x_{n_0}^*\rangle>M\})>0$ for some $n_0\in\mathbb{N}$. Putting $E:=\{\langle\phi,x_{n_0}^*\rangle>M\}$, we then have

$$\|F(E)\|\geqslant\langle FE,x_{n_0}^*\rangle=\int_E\langle\phi,x_{n_0}^*\rangle d\mu> M\mu E,$$

contradicting our assumption. Note also that Example 5.11 shows that this is false in general for non-separable X.

By the Pettis measurability theorem ϕ, being scalarly measurable and separably valued, is strongly- or Bochner-measurable. In particular $\|\phi\|$ is measurable, and also $\int_\Omega\|\phi\|d\mu\leqslant M\mu\Omega<\infty$, so ϕ is Bochner integrable. But then $(B)\int_E\phi d\mu=(P)\int_E\phi d\mu$ $(E\in\Sigma)$, so what we have shown is that if X is separable, then the WRNP implies the (generally stronger) RNP: every μ-continuous vector measure $F:\Sigma\to X$ satisfying $\|FE\|\leqslant M\mu E$ $(E\in\Sigma)$ for some $M<\infty$, has a Bochner integrable derivative. It can be shown that also for WCG (=weakly compactly generated) spaces the RNP and the WRNP coincide. □

Further on it will become clear that there exist spaces with the WRNP that fail the RNP. E.g. the dual of the James tree space JT is such a space. A typical example of a space that fails the WRNP is l^∞. In fact we shall prove later that every dual space failing the WRNP must have l^∞ as a quotient.

EXAMPLE 5.13. l^∞ fails the WRNP.
Let μ be the usual completed product measure $(\frac{1}{2}\delta_0+\frac{1}{2}\delta_1)^{\mathbb{N}}$ on $\Omega:=\{0,1\}^{\mathbb{N}}$, and $\Sigma:=\Sigma_\mu$. Consider the l^∞-valued measure F defined by

$$FE:=(\int_E \epsilon_n d\mu)_{n=1}^\infty \quad (E\in\Sigma),$$

where ϵ_n denotes the n^{th} coordinate function on Ω. (Note that $\|FE\|\leqslant\mu E$ $(E\in\Sigma)$ and that obviously F is finitely additive. Hence F is countably additive.) We now show that F has no (P)-integrable derivative. Suppose for contradiction that ϕ is such a (P)-integrable derivative. Letting ϕ_n be the n^{th} component of ϕ, and e_n^* the n^{th} unit vector in l^1 (better: πl^1), the formula

$$\langle FE, e_n^*\rangle = \int_E\langle\phi,e_n^*\rangle d\mu \quad (E\in\Sigma)$$

must hold. This gives

$$\int_E \epsilon_n\, d\mu = \int_E \phi_n d\mu \quad (E\in\Sigma, n\in\mathbb{N}),$$

hence $\epsilon_n=\phi_n$ μ a.e. for every $n\in\mathbb{N}$. It follows that ϕ is μ a.e. equal to the identity embedding of $\{0,1\}^{\mathbb{N}}$ into l^∞ We may as well assume from now on that ϕ actually equals this identity embedding. We claim that ϕ is not scalarly measurable (ϕ is of course w^*-scalarly measurable). To see this it is convenient to regard l^∞ as $C(\beta\mathbb{N})$. Fix any $p\in\beta\mathbb{N}\setminus\mathbb{N}$ and consider the Dirac measure $\delta_p\in C(\beta\mathbb{N})^*=(l^\infty)^*$. We shall prove that $\langle\phi,\delta_p\rangle$ is not μ-measurable.

Every $t\in\Omega$ is a sequence $(\epsilon_n(t))$ of zeros and ones. Put $E_t:=\{n\in\mathbb{N}:\epsilon_n(t)=1\}$. Now $\phi(t)$ is this same sequence $(\epsilon_n(t))$. But considered as an element of $C(\beta\mathbb{N})$, it equals $\tilde{\chi}_{E_t}$, the (unique) continuous extension of χ_{E_t} from $\mathbb{N}$ to $\beta\mathbb{N}$. Now $\tilde{\chi}_{E_t}=\chi_{\overline{E}_t}$, where $\overline{E}_t$ is the closure of E_t in $\beta\mathbb{N}$, so we have

$$\langle\phi(t),\delta_p\rangle=\tilde{\chi}_{E_t}(p)=\chi_{\overline{E}_t}(p).$$

This means that

$$\langle\phi(t),\delta_p\rangle=\begin{cases}1 & \text{if } p\in\overline{E}_t,\\ 0 & \text{if } p\notin\overline{E}_t.\end{cases} \quad (t\in\Omega)$$

Note now that $\mathcal{F}:=\{E\subset\mathbb{N}:p\in\overline{E}\}$ is a free ultrafilter on $\mathbb{N}$. (This is because for every $E\subset\mathbb{N}$, $\overline{E}\cup\overline{\mathbb{N}\setminus E}=\beta\mathbb{N}$ and $\overline{E}\cap\overline{\mathbb{N}\setminus E}=\varnothing$.) If we now identify the points $t\in\Omega$ with the corresponding subsets $E_t\subset\mathbb{N}$ (as in Ch. I), then what we have proved is that $\{\langle\phi,\delta_p\rangle=1\}=\mathcal{F}$. But we know from Prop. 1.2 that $\mu_*\mathcal{F}=0$

and $\mu^* \mathcal{F} = 1$. The conclusion is therefore that $\langle \phi, \delta_p \rangle$ is not μ-measurable. □

NOTES The material on the Pettis integral in this chapter is well known and can be found e.g. in [14] and [92]. The elegant and useful criterion of Prop. 5.5 appears in [92]. In connection with Prop. 5.7 (which is due to C. STEGALL, see [28]), let us mention the following remarkable result proved in [28]: for every infinite set Γ there exists a probability space (Ω,Σ,μ) and a uniformly bounded Pettis integrable $\phi:\Omega \to l^\infty(\Gamma)$ such that the Dunford operator T_ϕ is not compact. Combining this with Prop. 5.7 one sees that this probability space (Ω,Σ,μ) cannot be perfect. But since perfectness in needed only once in the proof of Prop. 5.7, namely for the application of Fremlin's theorem, we may then conclude indirectly that Fremlin's theorem fails in general for non-perfect spaces. This was first proved by M. TALAGRAND in [91], in response to a question of D.H. FREMLIN ([26]). For a discussion of the question under what conditions (other than perfectness of the measure space) the range of a Pettis integral is compact, see [56].

The WRNP was first introduced and studied by K. MUSIAL in [55]. The assertions of Lemma 5.9 are due to him, as well as the observation that the RNP coincides with the WRNP for weakly compactly generated spaces (hence in particular for separable spaces). A martingale characterization of the WRNP can be found in [56]. N. GHOUSSOUB and E. SAAB have shown in [29] that for Banach lattices the WRNP and the RNP are identical, and also that the same is true for dual spaces X^* that are embeddable as complemented subspaces of Banach lattices. Let us observe that Remark 5.10 and the observations in Remark 5.12 show that the WRNP implies the RNP, for a given X, iff every Pettis integrable X-valued function is scalarly equivalent to a Bochner (= strongly) measurable function. This assertion also goes back to K. MUSIAL ([55]). Later on in Chapters 7 and 8 we shall see that the dual JT^* of the James tree space JT has the WRNP but fails the RNP. Hence there must exist a JT^*-valued function that is Pettis integrable but not scalarly equivalent to a strongly measurable function. An explicit example of such a function is given in [51].

Another noteworthy fact is that X possesses the WRNP iff it has the WRNP with respect to the Lebesgue measure space [0,1]. For a proof, see [92] or [57]. The analoguous assertion for the RNP is also true, and well known. The paper [41] contains some alternative proofs of elementary facts about the Pettis integral that may be enlightening.

Finally, Example 5.13 is due to C. RYLL-NARDZEWSKI (see [55]), although the fact on which it is based, viz. that the identity embedding $\{0,1\}^{\mathbb{N}} \to l^\infty$ is not scalarly measurable, goes back to W. SIERPINKI ([89], see also [85]).

Chapter VI

Characterizations of spaces not containing l^1 related to the Pettis integral

Our aim in this chapter is to give some dual characterizations of spaces not containing l^1. One way or another Pettis integration is involved in all of them. The principal ones are:
(I) $l^1 \not\subset X$ iff X^* has the WRNP,
(II) $l^1 \not\subset X$ iff every w^*-compact convex subset of X^* is the *norm*-closed convex hull of its extreme points.

Both (I) and (II) are fairly easy to prove for separable X, but the non-separable case presents serious difficulties. We concentrate on (I) first, and begin with a proof for separable X. From this it will become apparent what extra difficulties there are in the non-separable case. A summary analysis of them will then motivate our further strategy.

Now even for the separable case we need some tools. The first is the so-called lifting theorem. It says that for every complete finite measure space (Ω,Σ,μ) one can select from each equivalence class in $L^\infty(\mu)$ a bounded measurable function in a consistent manner, i.e. so that the resulting map from $L^\infty(\mu)$ to $M(\mu)$ is a linear, multiplicative, positive isometry that preserves the constants. In the formulation that follows not all properties are independent; some of them we state for emphasis only. For a discussion and a proof, see Appendix G.

PROPOSITION 6.1. *(lifting theorem)*
Let (Ω,Σ,μ) be a complete finite measure space. Then there exists a map $\rho:L^\infty(\mu)\to M(\mu)$ (called a lifting on $L^\infty(\mu)$) satisfying the following properties for every $f\in L^\infty(\mu)$:
(i) $\rho(f)=f$ *μ a.e.,*
(ii) $\|\rho(f)\| = \|f\|_\infty$ *($\|\cdot\|$ denotes the sup norm),*
(iii) $\rho(1) = 1$,
(iv) $f\geqslant 0$ *μ a.e.* $\Rightarrow \rho(f)\geqslant 0$ *everywhere,*
(v) *ρ is an algebra isomorphism into, i.e. ρ is linear and multiplicative.* □

Condition (iv) and the linearity of ρ of course imply that whenever $f\leqslant g$ μ a.e., then $\rho(f)\leqslant\rho(g)$ everywhere, so ρ preserves order.

We now show that for X^*-valued measures the existence of w^*-scalarly measurable derivatives is no problem even without separability.

PROPOSITION 6.2. *Let (Ω,Σ,μ) be a complete probability space, X a Banach space, and $F:\Sigma\to X^*$ a measure satisfying $\|FE\|\leqslant M\mu E$ $(E\in\Sigma)$ for some*

$M<\infty$. *Then* F *has a* w^**-scalarly measurable and uniformly bounded derivative* $\phi:\Omega\to X^*$, *i.e.*

$$\langle x, FE\rangle = \int_E \langle x,\phi\rangle d\mu \quad (E\in\Sigma, x\in X). \tag{1}$$

PROOF. Let $T:L^1(\mu)\to X^*$ be the bounded operator associated to F (cf. the proof of Lemma 5.9 (ii)). We shall construct ϕ with the required properties and so that

$$\langle x, Tf\rangle = \int_\Omega f\langle x,\phi\rangle d\mu \quad (f\in L^1(\mu), x\in X)$$

((1) is a particular case of this, taking $f=\chi_E$). For each fixed $x\in X$ the map

$$L^1(\mu)\ni f \longrightarrow \langle x, Tf\rangle \in\mathbb{R}$$

is a bounded linear functional on $L^1(\mu)$, so it defines an element $\phi_x\in L^\infty(\mu)$:

$$\int_\Omega f\cdot\phi_x d\mu = \langle x, Tf\rangle \quad (f\in L^1(\mu), x\in X). \tag{2}$$

Let us observe that $\|\phi_x\|_\infty = \sup_{\|f\|_1\leqslant 1} |\langle x,Tf\rangle|\leqslant\|T\|\cdot\|x\|$. If ρ is a lifting on $L^\infty(\mu)$, then for each $t\in\Omega$ the map

$$X\ni x\longrightarrow \rho(\phi_x)(t)\in\mathbb{R}$$

is linear and bounded (with norm $\leqslant\|T\|$), so it defines an element $\phi(t)\in X^*$:

$$\langle x, \phi(t)\rangle = \rho(\phi_x)\,(t) \quad (t\in\Omega, x\in X). \tag{3}$$

This defines ϕ. Note that ϕ is uniformly bounded by $\|T\|$, and, obviously, w^*-scalarly measurable. Combining (2) and (3) yields

$$\langle x,Tf\rangle=\int_\Omega f\cdot\phi_x d\mu=\int_\Omega f\cdot\rho(\phi_x)d\mu=\int_\Omega f\langle x,\phi\rangle d\mu \quad (f\in L^1(\mu),x\in X),$$

so that ϕ has the required properties. □

We are now ready to prove (I) for separable X.

PROPOSITION 6.3. For separable X the following are equivalent:
(i) $l^1\not\subset X$,
(ii) X^* *has the WRNP.*

PROOF. (i) $\Rightarrow$ (ii): Suppose $l^1\not\subset X$ and let $F:\Sigma\to X^*$ be a measure satisfying $\|FE\|\leqslant M\mu E\,(E\in\Sigma)$ for some $M<\infty$, where (Ω,Σ,μ) is any complete probability space. Then by Prop. 6.2 F has a uniformly bounded w^*-scalarly measurable derivative ϕ:

$$\langle x, FE\rangle = \int_E \langle x,\phi\rangle d\mu \quad (E\in\Sigma, x\in X). \tag{4}$$

All that needs to be done now is to show that ϕ is scalarly measurable and that

$$\langle FE, x^{**}\rangle = \int_E \langle \phi, x^{**}\rangle d\mu \quad (E\in\Sigma, x^{**}\in X^{**}). \tag{5}$$

Fix $x^{**}\in X^{**}$. By Theorem 4.1, (1) $\Leftrightarrow$ (2), there is a (bounded) sequence $(x_n)\subset X$ so that $w^*\text{-}\lim_{n\to\infty} x_n = x^{**}$. Now let $n\to\infty$ in the relation

$$\langle x_n, FE\rangle = \int_E \langle x_n, \phi\rangle d\mu \quad (n\in\mathbb{N}, E\in\Sigma).$$

This gives (5), by Lebesgue's theorem and the uniform boundedness of ϕ.

(ii) $\Rightarrow$ (i): Suppose now that $l^1\subset X$. We shall show there exists a measure $F:\Sigma\to X^*$ with no Pettis integrable derivative. For this it is enough to construct (for some complete probability space (Ω,Σ,μ)) a bounded w^*-scalarly measurable function $\phi:\Omega\to X^*$ that is not scalarly measurable. Indeed, every such ϕ is the w^*-derivative of a measure $F:\Sigma\to X^*$. To see this, define $S:X\to L^\infty(\mu)$ by $Sx := \langle x,\phi\rangle$ $(x\in X)$. Since ϕ is bounded so is S, hence the adjoint S^* restricted to $L^1(\mu)$ defines a bounded linear map $T:L^1(\mu)\to X^*$. Putting $FE := T\chi_E$ $(E\in\Sigma)$, it is clear that F is a measure and satisfies

$$\langle x, FE\rangle = \int_E \langle x,\phi\rangle d\mu \quad (x\in X, E\in\Sigma), \tag{6}$$

i.e. ϕ is a w^*-derivative of F. Suppose now that F has a Pettis integrable derivative ψ, i.e.

$$\langle FE, x^{**}\rangle = \int_E \langle \psi, x^{**}\rangle d\mu \quad (x^{**}\in X^{**}, E\in\Sigma). \tag{7}$$

Comparison of (6) and (7) shows that

$$\langle x,\phi\rangle = \langle x,\psi\rangle \quad \mu\,\text{a.e. for every } x\in X.$$

Hence $\phi=\psi$ μ a.e. by the separability of X (cf. Remark 5.10). But then ϕ is scalarly measurable, a contradiction.

Now for the existence of a w^*-scalarly measurable bounded function that is not scalarly measurable, let $T:l^1\to X$ be an embedding. Then the adjoint $T^*:X^*\to l^\infty$ is a w^*-w^* continuous surjection. Observe that the compact space $K:=\{0,1\}^{\mathbb{N}}$ can be identified, topologically, with a subset of (l^∞, w^*). We may assume without loss of generality that $K\subset T^*B(X^*)$ (simply multiply T with a constant if necessary). Now if μ is the usual product measure on K and $L\subset B(X^*)$ is a w^*-compact set satisfying $T^*L=K$, then by Prop. B.1 there is a w^*-Radon propability ν on L so that $T^*\nu=\mu$. We may of course regard ν as a w^*-Radon probability on $B(X^*)$. Now recall that in Example 5.13 (see also the Notes of Ch. 5) we have proved that the identity embedding $\phi:K\to l^\infty$ is not scalarly measurable. This means that for some $y^{**}\in(l^\infty)^*$ the function $y^{**}\circ\phi = y^{**}|_K$ is not μ-measurable. But then $T^{**}y^{**}|_L = y^{**}\circ\phi\circ T^*|_L$ fails to be ν-measurable, by Prop. A7, since $(B(X^*),\nu)$ is perfect (by Prop. A4). What we have proved now is that the (bounded) identity embedding $(B(X^*),\nu)\xrightarrow{i}X^*$ (which is obviously w^*-scalarly measurable), is not scalarly measurable (since $(T^{**}y^{**})\circ i$ is not ν-measurable).

[Note: we could have used the equivalence (i) $\Leftrightarrow$ (v) of Th. 3.11 for the last part of the proof, but the argument given here is more direct.] □

Implicit in the above proof is the following fact.

COROLLARY 6.4. *Let X be separable. Then the following are equivalent:*
(i) *X^* has the WRNP,*
(ii) *for every complete probability space (Ω,Σ,μ) every w^*-scalarly measurable $\phi:\Omega\to X^*$ is scalarly measurable.*

[Note: in the proof above we showed the equivalence of (i) and (ii) with ϕ bounded. The extension of (ii) to non-bounded ϕ is immediate if one writes $\phi=\lim_{n\to\infty}\phi\cdot\chi_{\{\|\phi\|\leq n\}}$ and observes that $\|\phi\|=\sup_{n\in\mathbb{N}}|\langle x_n,\phi\rangle|$, with (x_n) dense in $B(X)$, so that $\|\phi\|$ is measurable if ϕ in w^*-scalarly measurable.]

Now let us consider where the proof of Prop. 6.3 may break down if X is not necessarily separable. Suppose $l^1\not\subset X$ and the measure $F:\Sigma\to X^*$ satisfies $\|FE\|\leq M\mu E$ $(E\in\Sigma)$ for some $M<\infty$ (where (Ω,Σ,μ) is some complete probability space). Using Prop. 6.2 we may still find a w^*-scalarly measurable uniformly bounded derivative $\phi:\Omega\to X^*$:

$$\langle x, FE\rangle = \int_E\langle x,\phi\rangle d\mu \quad (E\in\Sigma,\, x\in X). \tag{8}$$

For simplicity let us suppose that ϕ is $B(X^*)$-valued, so that $\phi\mu$ is supported in $B(X^*)$. But now the extended formula

$$\langle FE,x^{**}\rangle = \int_E\langle\phi,x^{**}\rangle d\mu \quad (E\in\Sigma,x^{**}\in X^{**}) \tag{9}$$

may not even make sense. Indeed, x^{**} is no longer the w^*-limit of a sequence $(x_n)\subset X$, so that $x^{**}\circ\phi$ may fail to be μ-measurable.

Now let us assume for the moment that

$$\left.\begin{array}{l}\phi \text{ is } w^*\text{-Borel measurable and the } (w^*\text{-Borel}) \text{ image measure } \nu:=\phi\mu\\ \text{on } B(X^*) \text{ is } w^*\text{-Radon.}\end{array}\right\} \tag{10}$$

[It should be noted that (10) holds automatically if X is separable, in virtue of the following well-known facts:
(i) ν is a w^*-Baire measure. This is so because the σ-algebra of the w^*-Baire sets is generated by $X(=$ the collection of *linear* w^*-continuous functions on $X^*)$. We do not prove this here.
(ii) the w^*-Baire subsets of $B(X^*)$ coincide with the w^*-Borel subsets of $B(X^*)$, since $B(X^*)$ is w^*-metrizable.
(iii) every w^*-Borel measure on $B(X^*)$ is w^*-Radon, again by the metrizability

of $(B(X^*), w^*)$.]

Now, assuming (10), Theorem 3.11, (i) $\Leftrightarrow$ (v) (applied with $T=(B(X^*),w^*)$ and $Z=B(X)$) comes to the rescue. Indeed, every $x^{**}\in B(X^{**})$ is in the τ_p-closure of $B(X)$, by Golstine's theorem, so by (v) of Th. 3.11 x^{**} is ν-measurable, and therefore $\langle x^{**},\phi\rangle$ is μ-measurable, since (Ω,Σ,μ) is assumed to be complete.

Having now given sense to the formula (9), under assumption (10), the next problem is whether it is satisfied. One moment's thought shows that this question is equivalent to asking whether the (now scalarly measurable) ϕ is (P)-integrable. Indeed, suppose it is. If $G(E):=(P)\int_E\phi\, d\mu$ $(E\in\Sigma)$, then (9), and therefore also (8), hold with G replacing F. Since also (8) holds as written, it follows that $F=G$, and therefore (9) holds as written.

Elementary arguments now show that the problem whether ϕ is (P)-integrable is really the same as asking whether the identity map $(B(X^*),\nu)\xrightarrow{i}X^*$ is (P)-integrable (details will follow later). But this question is answered affirmatively by Th. 2.7 and the criterion of Prop. 5.5. Indeed, by Th. 3.11, (vi) $\Leftrightarrow$ (i), $Z=B(X)$ and therefore also its τ_p-closure $B(X^{**})$ is ν-stable. Therefore Th. 2.7 says that the identity on $B(X^{**})=Z_i$ is continuous from the τ_p-topology to the $L^1(\nu)$-topology (observe that τ_m coincides with the $L^1(\nu)$-topology on $L^\infty(\nu)$-bounded sets such as $B(X^{**})$). So in particular it is continuous for the weak topology of $L^1(\nu)$. Hence the (P)-integrability of i follows from Prop. 5.5.

Also the second part of the proof of Prop. 6.3 uses the separability of X (namely, at the point where Remark 5.10 is applied). We shall see, however, that it can be salvaged by again using Th. 3.11 (this time (i) $\Leftrightarrow$ (vii)). No new tools are needed for this.

We shall now begin a series of developments designed to show, essentially, that any X^*-valued measure (X arbitrary) has a w^*-derivative ϕ that satisfies the assumption (10).

Let (Ω,Σ,μ) be a probability space. Then $L^\infty(\mu)$ is a C^*-algebra, so by the Gelfand-Naimark theorem there exists a compact space Δ (= the maximal ideal space of $L^\infty(\mu)$) so that $L^\infty(\mu)$ is isometrically algebra isomorphic to $C(\Delta)$. As usual we denote the image (= Gelfand transform) of $f\in L^\infty(\mu)$ in $C(\Delta)$ by $\hat f$. Now the map $\hat f\to\int_\Omega f\,d\mu$ is an element of $C(\Delta)^*$, so by the Riesz representation theorem there is a Radon measure $\hat\mu$ on Δ satisfying

$$\int_\Omega f\,d\mu=\int_\Delta \hat f\,d\hat\mu \quad (f\in L^\infty(\mu)). \tag{11}$$

Since the Gelfand transform is positive and isometric, and μ is a probability, so is $\hat\mu$. Let $A\in\Sigma$. Then χ_A is an idempotent in $L^\infty(\mu)$ and therefore $\hat\chi_A$ is an idempotent in $C(\Delta)$, i.e. a continuous $\{0,1\}$ valued function. Thus $\hat\chi_A=\chi_{\hat A}$ for some clopen $\hat A\subset\Delta$. We have $\mu A=\hat\mu\hat A$, so in particular $\hat A$ is non-empty iff

$\mu A > 0$.

We now claim that $\{\hat{A} : A \in \Sigma\}$ is a (clopen) basis for the topology of Δ, so that in particular Δ is totally disconnected. Indeed, let $O \subset \Delta$ be open, and $p \in O$, both arbitrary. By Urysohn's lemma there exists an $\hat{f} \in C(\Delta)$ with

$$0 \leqslant \hat{f} \leqslant 1, \quad \hat{f}(p) = 1, \quad \hat{f} = 0 \text{ on } \Delta \setminus 0.$$

Since $\|\hat{f}\| = \|f\|_\infty = 1$ and $0 \leqslant f \leqslant 1 \quad \mu$ a.e., the set $A := \{f \geqslant \frac{1}{2}\}$ satisfies $\mu A > 0$. Also

$$f = \widehat{f\chi_A + f\chi_{\{\Omega \setminus A\}}} = \hat{f}\chi_{\hat{A}} + \widehat{f\chi_{\{\Omega \setminus A\}}}.$$

Since $\hat{f}(p) = 1$ and $\|\widehat{f\chi_{\{\Omega \setminus A\}}}\| = \|f\chi_{\{\Omega \setminus A\}}\|_\infty \leqslant \frac{1}{2}$, we must have $p \in \hat{A}$. Finally, $\frac{1}{2}\chi_A \leqslant f$, so $\frac{1}{2}\chi_{\hat{A}} \leqslant \hat{f}$ and therefore $\hat{A} \subset O$, because $\hat{f} = 0$ outside O. We have now shown that $p \in \hat{A} \subset O$, and our claim is proved.

We need two more facts that we formulate next.

LEMMA 6.5.
(i) *A $\hat{\mu}$-measurable subset of Δ is closed and self supported iff it is of the form $\hat{A}$ for some $A \in \Sigma$.*
(ii) *The σ-algebra $\Sigma_{\hat{\mu}}$ is the $\hat{\mu}$-completion of the Baire σ-algebra $\mathfrak{Ba}(\Delta)$.*

PROOF. (i): It is straightforward that each $\hat{A}$ is closed and self supported: use the fact that the sets $\hat{A}$ ($A \in \Sigma$) form a clopen basis for the topology of Δ and our earlier remark that $\hat{\mu}\hat{A} > 0$ (iff $\mu A > 0$) iff $\hat{A} \neq \varnothing$.

For the converse let $E \subset \Delta$ be clopen and self supported. By the regularity of $\hat{\mu}$ and the fact that the sets $\hat{A}$ form a clopen basis for the topology of Δ which is closed for finite unions ($A \to \hat{A}$, $A \in \Sigma$, is a Boolean algebra isomorphism !), we have

$$\hat{\mu}(\Delta \setminus E) = \sup\{\hat{\mu}\hat{A} : \hat{A} \subset \Delta \setminus E\}.$$

Choose an increasing sequence $(\hat{A}_n)$ such that $\hat{\mu}\hat{A}_n \uparrow \hat{\mu}(\Delta \setminus E)$ as $n \to \infty$. We claim that

$$\Delta \setminus E = \hat{A} \text{ (hence } E = \widehat{\Omega \setminus A}\text{), where } A := \bigcup_{n=1}^{\infty} A_n.$$

Since $\hat{\mu}\hat{A} = \mu A = \lim \mu A_n = \lim \hat{\mu}\hat{A}_n$, we have

$$\hat{\mu}\hat{A} = \lim \hat{\mu}\hat{A}_n = \hat{\mu}(\Delta \setminus E).$$

It follows from this that $\hat{A} \cap E = \varnothing$. Indeed, if not, then since E is self supported, we would have $\hat{\mu}(\hat{A} \cap E) > 0$, and this clearly contradicts

$$\hat{A}_n \subset \hat{A} \setminus E \quad (n = 1,2,\ldots) \text{ and } \hat{\mu}\hat{A}_n \to \hat{\mu}\hat{A}.$$

On the other hand $\Delta \setminus (\hat{A} \cup E)$ has $\hat{\mu}$ measure 0, and therefore, being open, it must be empty.

(ii): Let $E \subset \Delta$ be $\hat{\mu}$ measurable. By the regularity of $\hat{\mu}$ we may construct an increasing sequence of compact self supported subsets $K_n \subset E$ such that $\hat{\mu}K_n \uparrow \hat{\mu}E$ as $n \to \infty$. By (i) each K_n is of the form $\hat{A}_n$ with $A_n \in \Sigma$. Observe that $\hat{A}_n \in \mathfrak{Ba}(\Delta)$ since $\chi_{\hat{A}_n}$ is continuous. Now $B_1 = \bigcup_{n=1}^{\infty} \hat{A}_n$ is clearly a Baire set contained in E with $\hat{\mu}B_1 = \hat{\mu}E$. Similarly, by complementation , one finds a Baire set $B_2 \supset E$ with $\hat{\mu}B_2 = \hat{\mu}E$. $\square$

Now let (Ω,Σ,μ) be a complete probability space, X a Banach space, and $\phi:\Omega \to X^*$ w^*-scalarly bounded. Δ as usual will denote the (compact) maximal ideal space of $L^\infty(\mu)$ and $\hat{f} \in C(\Delta)$ the Gelfand transform of $f \in L^\infty(\mu)$. Finally, let ρ be a lifting on $L^\infty(\mu)$. We are going to define maps $\hat{\phi}$, $\hat{\rho}$ and $\rho(\phi)$, so that for every $x \in \pi X \subset X^{**}$ the following diagram commutes:

$$\begin{tikzcd}
\Delta \arrow[dr, "\hat{\phi}"] \arrow[drr, "\langle x,\phi\rangle\hat{}"] & & \\
\Omega \arrow[u, "\hat{\rho}"] \arrow[r, "\rho(\phi)"] \arrow[rr, bend right, "\rho(\langle x,\phi\rangle)"'] & (X^*,w^*) \arrow[r, "x"] & \mathbb{R}
\end{tikzcd}$$

The important facts about these maps are:

(i) $\hat{\rho}$ is Borel measurable, i.e. $\hat{\rho}^{-1}B \in \Sigma$ for every $B \in \mathfrak{B}(\Delta)$.

(ii) $\hat{\phi}$ (= the *Stonian transform* of ϕ) is continuous when X^* has its w^*-topology. Hence

(iii) $\rho(\phi)$ is w^*-Borel measurable, i.e. $\rho(\phi)^{-1}E \in \Sigma$ for every w^*-Borel set $E \subset X^*$. Also $\rho(\phi)$ is uniformly bounded.

(A) Definition of $\hat{\phi}$.

For every $x \in X$ we have, since ϕ is w^*-scalarly bounded, that $\langle x,\phi\rangle \in L^\infty(\mu)$. So $\langle x,\phi\rangle$ has a Gelfand transform $\langle x,\phi\rangle\hat{} \in C(\Delta)$. Fix $s \in \Delta$ and consider the map

$$X \ni x \longrightarrow \langle x,\phi\rangle\hat{}(s) \in \mathbb{R}$$

Clearly this map is linear, and bounded, again by the w^*-scalarly boundedness of ϕ. Therefore it defines an element of X^* which we denote $\hat{\phi}(s)$:

$$\langle x,\hat{\phi}(s)\rangle := \langle x,\phi\rangle\hat{}(s) \quad (s \in \Delta, x \in X) \tag{12}$$

We claim that $\hat{\phi}$ is continuous, for the w^*-topology on X^*. For this it suffices to show that $\langle x,\hat{\phi}(\cdot)\rangle$ is continuous on Δ for every $x \in X$. However, this is clear from (12), since $\langle x,\phi\rangle\hat{}$ is continuous.

(B) Definition of $\hat{\rho}$.

Let us fix $t \in \Delta$ and consider the map

$$\begin{array}{ccccccc} \hat{f} & \rightarrow & f & \rightarrow & \rho(f) & \rightarrow & \rho(f)(t). \\ \cap & & \cap & & \cap & & \cap \\ C(\Delta) & & L^{\infty}(\mu) & & M(\mu) & & \mathbb{R} \end{array}$$

This is a multiplicative linear functional on $C(\Delta)$, and therefore equals evaluation in a unique point of Δ that we give the name $\hat{\rho}(t)$, i.e.

$$\hat{f}(\hat{\rho}(t)) := \rho(f)(t) \quad (t \in \Omega, f \in L^{\infty}(\mu)). \tag{13}$$

This defines the map $\hat{\rho}$. Taking in particular $f = \langle x, \phi \rangle$, (13) becomes

$$\langle x, \phi \rangle\hat{}(\hat{\rho}(t)) = \rho(\langle x, \phi \rangle)(t) \quad (t \in \Omega, x \in X). \tag{14}$$

For any $f \in L^{\infty}(\mu)$, $\rho(f)$ is measurable. Hence, by (13), $\hat{f} \circ \hat{\rho}$ is measurable. Since every continuous function on Δ is of the form $\hat{f}$ with $f \in L^{\infty}(\mu)$, this proves that $\hat{\rho}$ is Baire-measurable.

Let $\hat{\rho}(\mu)$ be the $\hat{\rho}$-image of μ on the σ-algebra $\Sigma' := \{B \subset \Delta : \hat{\rho}^{-1} B \in \Sigma\}$. We just observed that Σ' contains the Baire sets. Σ' is also $\hat{\rho}(\mu)$-complete, since Σ is assumed to be μ-complete. It follows from (13) that

$$\int_{\Delta} \hat{f}\, d\hat{\rho}(\mu) = \int_{\Omega} \rho(f) d\mu = \int_{\Omega} f\, d\mu \quad (f \in L^{\infty}(\mu)).$$

Since $\int_{\Omega} f\, d\mu = \int_{\Delta} \hat{f}\, d\hat{\mu}$ by the definition of $\hat{\mu}$, this shows that

$$\int_{\Delta} \hat{f}\, d\hat{\rho}(\mu) = \int \hat{f}\, d\hat{\mu} \quad (f \in L^{\infty}(\mu)).$$

Therefore $\hat{\rho}(\mu)$ and $\hat{\mu}$ must coincide on the Baire sets of Δ. Hence the $\hat{\mu}$-completion ($=$ $\hat{\rho}(\mu)$-completion) of $\mathfrak{B}a(\Delta)$ is contained in Σ'. However, by Lemma 6.5 (ii) the $\hat{\mu}$-completion of $\mathfrak{B}a(\Delta)$ equal $\Sigma_{\hat{\mu}}$, so in particular it contains $\mathfrak{B}(\Delta)$. Conclusion: $\mathfrak{B}(\Delta) \subset \Sigma'$, i.e. $\hat{\rho}$ is Borel measurable.

(C) DEFINITION OF $\rho(\phi)$.

Fix $t \in \Omega$. The map

$$\begin{array}{ccccccc} x & \rightarrow & \langle x, \phi \rangle & \rightarrow & \rho(\langle x, \phi \rangle) & \rightarrow & \rho(\langle x, \phi \rangle)(t) \\ \cap & & \cap & & \cap & & \cap \\ X & & L^{\infty}(\mu) & & M(\mu) & & \mathbb{R} \end{array}$$

is bounded (since ϕ w^*-scalarly bounded) and linear, so defines an element $\rho(\phi)(t) \in X^*$:

$$\langle x, \rho(\phi)(t) \rangle := \rho(\langle x, \phi \rangle)(t) \quad (t \in \Omega,\ x \in X). \tag{15}$$

Combining (14) and (15) yields

$$\langle x, \phi \rangle\hat{}(\hat{\rho}(t)) = \langle x, \rho(\phi)(t) \rangle \quad (t \in \Omega, x \in X), \tag{16}$$

while (16) and (12) imply

$$\langle x, \hat{\phi}(\hat{\rho}(t))\rangle = \langle x, \rho(\phi)(t)\rangle \quad (t \in \Omega, x \in X), \tag{17}$$

proving that $\hat{\phi}\circ\hat{\rho} = \rho(\phi)$. Since $\hat{\rho}$ is Borel measurable and $\hat{\phi}$ is continuous for the w^*-topology on $\hat{X}^*$, $\rho(\phi)$ is w^*-Borel measurable. We have now proved all we have claimed for $\hat{\phi}$, $\hat{\rho}$ and $\rho(\phi)$, since it readily follows from the definition of $\rho(\phi)$ that $\rho(\phi)$ is uniformly bounded. □

Let us note an interesting consequence of these arguments. We have assumed that ϕ was w^*-scalarly bounded, i.e. for some $M<\infty$,

$$\|x\circ\phi\|_\infty \leqslant M \text{ for all } x \in B(X).$$

This does not means that ϕ is μ a.e. bounded, for the same reason we have mentioned before: the set where $\|x\circ\phi\| \leqslant M$ fails may depend on x. However, since $\rho(\phi)$ is uniformly bounded we have proved:

Corollary 6.6. *Every w^*-scalarly bounded $\phi:\Omega\to X^*$ is w^*-scalarly equivalent to a bounded w^*-Borel measurable function $\rho(\phi):\Omega\to X^*$.*
($\langle x,\rho(\phi)\rangle = \rho(\langle x,\phi\rangle) = \langle x,\phi\rangle$ μ a.e. for every $x \in X$.)

We can now strengthen Prop. 6.2 considerably, as follows.

Proposition 6.7. *Let (Ω,Σ,μ) be a complete probability space, a Banach space and $F:\Sigma\to X^*$ a measure satisfying $\|FE\| \leqslant M\mu E$ $(E\in\Sigma)$ for some $M<\infty$. Then F has a w^*-Borel measurable uniformly bounded w^*-derivative with the further property that the image measure $\phi\mu$ is w^*-Radon.*

Proof. We define ϕ exactly as in the proof of Prop. 6.2, using an arbitrary lifting ρ on $L^\infty(\mu)$:

$$\langle x,\phi(t)\rangle = \rho(\phi_x)(t) \quad (t\in\Omega, x\in X). \tag{18}$$

Now it follows from (18) that

$$\rho(\langle x,\phi\rangle) = \langle x,\phi\rangle \quad (x\in X) \tag{19}$$

(since "$\rho^2=\rho$"). Combining this with (15) gives

$$\langle x,\rho(\phi)\rangle = \langle x,\phi\rangle \quad (x\in X),$$

so that $\phi=\rho(\phi)$. But then ϕ is w^*-Borel measurable, since $\rho(\phi)$ is (see (iii) above). We also have

$$\phi = \rho(\phi) = \hat{\phi}\circ\hat{\rho} \text{ (see the diagram on p. 69).}$$

For simplicity let us assume ϕ is valued in $B(X^*)$. Setting $\nu:=\phi\mu$ (the w^*-Borel image measure on $B(X^*)$), we have

$$\nu = \hat{\phi}(\hat{\rho}(\mu)) = \hat{\phi}(\hat{\mu}).$$

This implies that ν is w^*-Radon, since $\hat{\mu}$ is Radon and $\hat{\phi}$ is continuous (for the w^*-topology on $B(X^*)$). □

We now prove (I) in its full generality, and at the same time supplement it.

THEOREM 6.8. *For any Banach space X the following are equivalent:*
(i) $l^1 \not\subset X$,
(ii) X^* *has the WRNP,*
(iii) *the canonical injection* $(B(X^*),w^*) \xrightarrow{i} X^*$ *is universally Pettis integrable, i.e. for every* w^**-Radon measure* ν *on* $(B(X^*),w^*)$ i *is* (P)*-integrable (so in particular* i *is universally scalarly measurable).*

PROOF.
(i) $\Rightarrow$ (iii): Suppose $l^1 \not\subset X$. To prove (iii) let Z denote $B(X)$, considered as a (bounded) set of continuous functions on $(B(X^*),w^*)$. Then by Th. 3.11 (v) Z is relatively τ_p-compact in $M(\nu)$ for every w^*-Radon measure ν on $B(X^*)$. Now by Goldstine's theorem the τ_p-closure of Z can be identified with $B(X^{**})$. So $B(X^{**}) \subset M(\nu)$ for every ν. This proves the universal scalar measurability of i.

For the (P)-integrability of i, fix ν and recall from Th. 3.11 (vi) that Z and therefore also its τ_p-closure $B(X^{**})$ is ν-stable. This implies by Th. 2.7 and the uniform boundedness of $B(X^{**})$ that the canonical map $B(X^{**}) \rightarrow L^1(\nu)$ is τ_p-to-norm, hence in particular τ_p-to-weak continuous. Now the criterion of Prop. 5.5 implies that i is (P)-integrable. Thus (iii) is proved.

(iii) $\Rightarrow$ (ii): To show that X^* has the WRNP let (Ω,Σ,μ) be a complete probability space and $F:\Sigma \rightarrow X^*$ a measure satisfying $\|FE\| \leqslant M\mu E$ $(E \in \Sigma)$ for some $M<\infty$. Let $\phi:\Omega \rightarrow X^*$ be a w^*-scalarly measurable derivative of F. We may assume by Prop. 6.7 that ϕ is w^*-Borel measurable and that its image measure $\nu=\phi\mu$ is w^*-Radon and supported in $B(X^*)$. Then by assumption (iii) above

$$i:(B(X^*),\nu) \rightarrow X^* \text{ is } (P)\text{-}integrable.$$

Let us now factor $\phi:\Omega \rightarrow X^*$ as follows:

$$\phi:(\Omega,\mu) \xrightarrow{\phi_1} (B(X^*),\nu) \xrightarrow{i} X^*$$

(ϕ_1 is ϕ, but considered as a map into $B(X^*)$ rather that X^*). Define $S_{\phi_1}:L^1(\nu) \rightarrow L^1(\mu)$ by $S_{\phi_1}(f):=f \circ \phi_1 (f \in L^1(\nu))$ and $T_{\phi_1}:=S^*_{\phi_1}:L^\infty(\mu) \rightarrow L^\infty(\nu)$. The Dunford operators T_ϕ and T_i now satisfy $T_\phi = T_i \circ T_{\phi_1}$ and T_ϕ maps into X^* because T_i does (since i is (P)-integrable). Thus we have proved that ϕ is (P)-integrable. Also clearly its Dunford operator T_ϕ satisfies $T_\phi \chi_E = FE$ $(E \in \Sigma)$ (since $\langle T_\phi \chi_E, x \rangle = \langle T_i \circ T_{\phi_1} \chi_E, x \rangle = \langle \chi_E, S_{\phi_1} \circ S_i x \rangle = \int_E \langle x,\phi \rangle d\mu$ $\forall x \in X$).

(ii) $\Rightarrow$ (i): If $l^1 \subset X$ then by Th. 3.11 (vii) there is a w^*-Radon measure μ on $(B(X^*),w^*)$ such that $L^1(\mu) \cong L^1$ and $Z=B(X)$ is not totally bounded in $L^1(\mu)$. Consider the canonical injection $i:(B(X^*),\mu) \rightarrow X^*$. i is w^*-scalarly measurable and as such a w^*-derivative of some measure $F:\Sigma_\mu \rightarrow X^*$ (cf. the second part of the proof of Prop. 6.3). Any possible (P)-integrable derivative ϕ of F would be w^*-equivalent to i (same reference). Since μ is w^*-Radon it is perfect (Prop. A. 4) and Prop. 5.7 now implies that S_ϕ (equivalently: T_ϕ) is compact. Clearly $S_\phi B(X^{**})$ must include $Z=B(X)$, which is not totally bounded in $L^1(\mu)$, so we have a contradiction. □

We now start working towards characterization (II) stated at the beginning of this chapter. First let us reformulate (iii) above in the language of Choquet theory. It is well known that every Radon probability μ on a compact convex subset K of a locally convex Hausdorff space has a unique *barycenter* (or *resultant*) $r_\mu \in K$ defined by the condition that

$$f(r_\mu) = \int_K f(x)d\mu(x) \text{ for all continuous affine functions.} \tag{20}$$

In particular, if K is a w^*-compact convex subset of a dual Banach space X^*, and μ a w^*-Radon measure on K, then

$$\langle x,r_\mu\rangle = \int_K \langle x,x^*\rangle d\mu(x^*) \quad \text{for all } x \in X. \tag{21}$$

It is a result of Choquet that the "barycenter formula" (20) extends to the affine functions in $\mathfrak{B}_1(K):=$ {first class Baire functions f on K}. However, there is a counterexample showing that it fails in general for affine Baire functions of the second class (cf. [66]).

Turning to the particular case (21) again, if X is w^*-sequentially dense in X^{**}, then (21) is valid for every $x^{**} \in X^{**}$ (instead of x) since in that case $X^{**} \subset \mathfrak{B}_1(K)$. One would expect this to be false without the condition that $X^{**} \subset \mathfrak{B}_1(K)$. However, if $l^1 \not\subset X$, then we shall see that the barycentric formula holds for all $x^{**} \in X^{**}$, even though X need not be w^*-sequentially dense in X^{**}. We express this by saying that every $x^{**} \in X^{**}$ "*satisfies the barycentric calculus*", i.e.

$$\langle r_\mu,x^{**}\rangle = \int_{B(X^*)} \langle x^*,x^{**}\rangle d\mu(x^*) \text{ for every } w^*\text{-Radon measure } \mu \text{ on } B(X^*). \tag{22}$$

(It is enough of course, to consider $B(X^*)$ instead of general K.)

Another interesting fact to be proved below is that the universal scalar measurability of $i:(B(X^*),w^*)\to X^*$ by itself already suffices to conclude that $l^1 \not\subset X$. The (universal) (P)-integrability apparently is an automatic consequence. Finally, the validity of the barycentric formula for all x^{**} implies property (vi) below. Also this property turns out to be equivalent to $l^1 \not\subset X$.

We summarize all this in

THEOREM 6.9. *For any Banach space X the properties (i), (ii) and (iii) of Th. 6.8 are equivalent to each of the following:*

(iv) *the canonical injection $(B(X^*),w^*)\xrightarrow{i} X^*$ is universally scalarly measurable and each $x^{**} \in X^{**}$ satisfies the barycentric calculus,*

(v) *$i:(B(X^*),w^*)\to X^*$ is universally scalarly measurable,*

(vi) *for each w^*-compact subset $A \subset X^*$, $w^*cl\ coA = \overline{co}A$ ($\overline{co}$ denotes norm* *closed convex hull).*

PROOF.
(iii) $\Rightarrow$ (iv): Let μ be a w^*-Radon measure on $B(X^*)$. Since we are assuming (iii), $(P)\int_{B(X^*)} id\mu$ exists, and by definition satisfies

$$\langle (P)\int_{B(X^*)} id\mu, x^{**}\rangle = \int_{B(X^*)} \langle x^*,x^{**}\rangle d\mu(x^*) \quad \forall x^{**}\in X^{**}. \tag{23}$$

Combining (21) and (23) yields

$$\langle x, (P)\int_{B(X^*)} id\mu\rangle = \langle x,r_\mu\rangle \quad \forall x\in X,$$

so $(P)\int_{B(X^*)} id\mu=r_\mu$. Substituting this in (23) gives (22), so each $x^{**}\in X^{**}$ satisfies the barycentric calculus.

(iv) $\Rightarrow$ (v): trivial.

(iv) $\Rightarrow$ (vi): Let $A\subset X^*$ be w^*-compact. We shall need the fact that every $x^*\in w^*\,cl\,coA$ is the barycenter of a w^*-Radon probability concentrated on A. To see this, fix $x^*\in w^*\,cl\,co\,A$ and choose a net (x^*_α) in $co\,A$ so that $x^*_\alpha \xrightarrow{w^*} x^*$. Next let μ_α represent x^*_α and be concentrated on A (if $x^*_\alpha = \sum_{i=1}^n \lambda_i x^*_i, \lambda_i\geqslant 0, \sum_{i=1}^n \lambda_i=1,\ x^*_i\in A$, simply put $\mu_\alpha := \sum_{i=1}^n \lambda_i\delta_{x^*_i}$). By the w^*-compactness of the set $P(A)$ of w^*-Radon probability measures on A, we may suppose by passing to subnet if necessary that $\mu_\alpha \xrightarrow{w^*} \mu\in P(A)$. Now taking the limit over α in the relation

$$\langle x,x^*_\alpha\rangle = \int_A \langle x,y^*\rangle d\mu_\alpha(y^*) \quad (x\in X),$$

we obtain

$$\langle x,x^*\rangle = \int_A \langle x,y^*\rangle d\mu(y^*) \quad (x\in X),$$

proving that x^* is the barycenter of μ.

Suppose now that $w^*cl\,co\,A \neq \overline{co}A$. Fix $x^*\in w^*cl\,co\,A\setminus\overline{co}\,A$ and use the Hahn-Banach theorem to find an $x^{**}\in X^{**}$ so that

$$\sup_A x^{**} < \langle x^*,x^{**}\rangle. \tag{24}$$

By the preceding paragraph there is a w^*-Radon probability on A representing x^*. Since x^{**} by assumption satisfies the barycentric calculus, we have

$$\int_A \langle y^*,x^{**}\rangle d\mu(y^*) = \langle x^*,x^{**}\rangle.$$

But this clearly contradicts (24).

(v) $\Rightarrow$ (i): This has already been taken care of. In the second part of the proof of Prop. 6.3 we showed (without using the separability of X) that if l^1 embeds in X, then there exists a w^*-Radon measure ν on $B(X^*)$ so that $(B(X^*),\nu)\xrightarrow{i}X^*$ is not scalarly measurable. [Note: (v) $\Leftrightarrow$ (i) is also a direct consequence of the equivalence (v) $\Leftrightarrow$ (i) in Th. 3.11.]

(vi) $\Rightarrow$ (i): Let us assume for contradiction that we have an embedding $T:l^1\to X$. Also, let $S:l^1\to C[0,1]$ be a quotient map.

$$\begin{array}{ccccc} M[0,1]= C[0,1]^* & \xrightarrow{S^*} & l^\infty=(l^1)^* & \xleftarrow{T^*} & X^* \\ C[0,1] & \xleftarrow{S} & l^1 & \xrightarrow{T} & X \end{array}$$

Then S^* is an isometric embedding and T^* a quotient map. Let $\delta:[0,1]\to C[0,1]^*=M[0,1]$ be the map that sends each $t\in[0,1]$ to the Dirac measure δ_t. Then δ is a homeomorphism into $(C[0,1]^*,w^*)$ and $A_1:=\delta([0,1])$ is the set of extreme points of the w^*-compact convex set $P:=\{\mu\in M[0,1]:\mu\geqslant 0, \|\mu\|=1\}$. Thus $P=w^*\,cl\,co\,A_1$, by the Krein-Milman theorem. But $P\neq\overline{co}\,A_1$, since clearly $\overline{co}A_1$ consists only of the purely atomic measure in P. Because $S^*:C[0,1]^*\to l^\infty=(l^1)^*$ is homeomorphic for both the norm and the w^*-topologies, $A_2:=S^*A_1$ is a w^*-compact subset of $(l^1)^*$ with $w^*cl\,co\,A_2\neq\overline{co}\,A_2$. Finally $T^*:X^*\to(l^1)^*$ is a w^*-w^*-continuous surjection. Using the open mapping theorem we see that there exists a w^*-compact preimage $A\subset X^*$ of A_2 under T^*. By the w^*-w^*-continuity of T^*, $T^*(w^*\,cl\,co\,A)=w^*\,cl\,co\,A_2$, and on the other hand, the norm continuity of T^* implies $T^*(\overline{co}\,A)\subset\overline{co}\,A_2$. Since $w^*cl\,co\,A_2\neq\overline{co}\,A_2$, we conclude that $w^*cl\,co\,A\neq\overline{co}\,A$, contradicting the assumption (vi). □

We still have not proved (II), although in one direction we are almost done. To see this, assume that $K=\overline{co}$ ext K for every w^*-compact convex $K\subset X^*$. Then it follows that (vi) in Theorem 6.9 is satisfied (and therefore $l^1\not\subset X$). This is an easy consequence of the fact that by Milman's theorem the extreme points of $w^*cl\,co\,A$ are contained in A whenever $A\subset X^*$ is w^*-compact:

$$w^*cl\,co\,A=w^*cl\,co\,(\mathrm{ext}\,w^*cl\,co\,A)=\overline{co}\,(\mathrm{ext}\,w^*cl\,co\,A)\subset\overline{co}\,A\subset w^*cl\,co\,A.$$

(Here the first equality comes from the Krein-Milman theorem, the second one from the assumption, while the inclusion immediately after that is a consequence of Milman's theorem, as explained above.)

To prove the other half of (II) new tools are needed that we now start to develop. But first let us point out how simple the proof is for separable X. Suppose $l^1\not\subset X$ and let $K\subset X^*$ be w^*-compact and convex. Let us assume for contradiction that $\overline{co}$ ext $K\underset{\neq}{\subset}K$. Then choose $x^*\in K\setminus\overline{co}\,\mathrm{ext}\,K$ and use the Hahn-Banach theorem to find an $x^{**}\in X^{**}$ so that

$$\sup_{\mathrm{ext}K}x^{**}<\langle x^*,x^{**}\rangle. \tag{25}$$

Now since we are assuming that X is separable, (K,w^*) is metrizable, so by Choquet's theorem the point x^* is represented by a probability measure μ carried by ext K, i.e.

$$\int_{\text{ext}K} \langle x,y^*\rangle d\mu(y^*) = \langle x,x^*\rangle \quad \forall x \in X. \tag{26}$$

But x^{**} is a first class w^*-Baire function by Th. 4.1, (1) $\Leftrightarrow$ (2), and therefore, as we pointed out before, (26) holds with x^{**} replacing x. But this of course contradicts (25).

We now prepare for the general case. First we need a definition. If A is a bounded subset of a Hausdorff locally convex space X then a *slice* of A is any set of the form

$$S = S(A,x^*,\alpha) := \{x \in A : \langle x,x^*\rangle > M(x^*) - \alpha\},$$

where $x^* \in X^*, \alpha > 0$ and $M(x^*) := \sup_A x^*$. Observe that $\bar{S} = \{x \in A : \langle x,x^*\rangle \geqslant M(x^*) - \alpha\}$ if A is convex and that by definition slices are always non-empty. If $x \in A$ and S is a slice containing x, then S is a weak nbhd of x in A. In fact the slices containing x form a subbasis for the weak nbhds of x relative to A. It is an important fact that if A is convex and compact and $x \in$ ext A, then x has a (weak) nbhd basis (relative to A) consisting of slices.

Lemma 6.10. *Let K be a compact convex subset of a Hausdorff l.c.s. X and let $x_0 \in$ ext K. Then the slices of K containing x_0 form a nbhd basis at x_0 relative to K (every extreme point is "strongly extreme" in Choquet's terminology).*

Proof. Observe first that on K the given topology of X coincides with the weak topology. By definition any (weak) nbhd V of x_0 in K contains a finite intersection of slices $S_i = S(K,x_i^*,\alpha_i)$ $(i = 1, \ldots, n)$. Let H_i be the closed half space $\{x \in X : \langle x,x_i^*\rangle \leqslant M(x_i^*) - \alpha_i\}$ $(i = 1, \ldots, n)$. Then $x_0 \notin H_i$ $(i = 1, \ldots, n)$, so since x_0 is extreme, $x_0 \notin co(\bigcup_{i=1}^{n} H_i \cap K)$. Notice that this last set is compact as a finite convex hull of compact convex sets $H_i \cap K$. Now by the Hahn-Banach theorem we can separate x_0 from $co(\bigcup_{i=1}^{n} H_i \cap K)$ by a closed hyperplane $H = \{x : \langle x,x^*\rangle = r\}$. Supposing that $\langle x_0,x^*\rangle > r$, as we clearly may, we then have $x_0 \in S(K,x^*,M(x^*) - r) \subset \bigcap_{i=1}^{n} S_i \subset V$. □

Proposition 6.11. *Let K be a compact convex subset of a Hausdorff l.c.s. X. Then ext K is a Baire space for the relative (weak) topology.*

Proof. Let (V_n) be a sequence of open dense subsets of ext K. We must show that $\bigcap_{n=1}^{\infty} V_n$ is dense in ext K. So let V be any non-empty open subset of ext K

and let us prove that $V\cap(\bigcap_{n=1}^{\infty} V_n)\neq\emptyset$. We may assume $V_n\downarrow$, since $V_1\cap\cdots\cap V_n$ is dense and open for every $n\in\mathbb{N}$. Let us now choose subsets $U_n (n=1,2,...)$ and U of K so that each U_n is open and dense in K, U is open in K, $U_n\downarrow$ and

$$V_n = U_n \cap \operatorname{ext} K \ (n=1,2,...), \quad V = U\cap \operatorname{ext} K.$$

[If $\underline{V'_n}$ is open in K with $V'_n\cap \operatorname{ext} K=V_n$, then put $U_n:= \overline{(K\setminus \operatorname{ext} K)}\cup V'_1\cap\cdots\cap V'_n$; U is defined similarly.]

We are now going to construct inductively a sequence of slices $S_n = S(K,x_n^*,\alpha_n)$ of K so that the following holds:

$$\begin{cases} S_1 \subset U \text{ and} \\ \overline{S}_{n+1} \subset S'_n \cap U_n, \text{ where } S'_n:=S(K,x_n^*,\frac{1}{2}\alpha_n). \end{cases}$$

To start the inductive process, choose $x\in V=U\cap \operatorname{ext} K$. By Lemma 6.10 there is a slice $S_1=S(K,x_1^*,\alpha_1)$ so that $x\in S_1\subset U$. Now suppose $S_1, S_2, \ldots, S_n$ have been constructed as required (observe that $S_1\supset S_2\supset\cdots\supset S_n$). Note that $S'_n\cap U_n\cap \operatorname{ext} K\neq\emptyset$, since $S'_n\cap \operatorname{ext} K$ is open in ext K, and $\neq\emptyset$ by the Krein-Milman theorem, and $U_n\cap \operatorname{ext} K=V_n$ is dense in ext K. Now again choose any $x\in S'_n\cap U_n\cap \operatorname{ext} K$ and apply Lemma 6.10 to this x and its nbhd $\underline{S}'_n\cap U_n$. This yields a slice $S_{n+1}=S(K,x_{n+1}^*,\alpha_{n+1})$ containing x so that $\overline{S}_{n+1}\subset S'_n\cap U_n$. This completes the induction.

It is clear that $\bigcap_{n=1}^{\infty}\overline{S}_n = \bigcap_{n=1}^{\infty} S'_n\neq\emptyset$, as $\overline{S}_n\downarrow$ and each $\overline{S}_n$ is compact. Denote $\cap\overline{S}_n=\cap S'_n$ by S. Then S is convex and compact, and $K\setminus S$ is also convex, since $S_n\downarrow$. Let x be any extreme point of S. We claim that $x\in \operatorname{ext} K$. Indeed, if not, then $x=\frac{1}{2}y+\frac{1}{2}z$, $y\neq z$, $y,z\in K$. We must have either $y\notin S$, $z\in S$ or $y\in S$, $z\notin S$, since both S and $K\setminus S$ are convex and $x\in \operatorname{ext} S$. Suppose the former, so $y\notin S$. But then $y\notin S_n$ for some $n\in\mathbb{N}$, so $\langle y,x_n^*\rangle\leq M(x_n^*)-\alpha_n$. On the other hand $\langle z,x_n^*\rangle\leq M(x_n^*)$ since $z\in K$. It follows that

$$\langle x,x_n^*\rangle = \frac{1}{2}\langle y,x_n^*\rangle + \frac{1}{2}\langle z,x_n^*\rangle\leq M(x_n^*)-\frac{1}{2}\alpha_n$$

and therefore $x\notin S'_n$, contradicting $x\in S = \bigcap_{n=1}^{\infty} S'_n$. So we have proved that $S\cap \operatorname{ext} K(= \operatorname{ext} S)\neq\emptyset$. It remains to note that $S\cap \operatorname{ext} K\subset V\cap(\bigcap_{n=1}^{\infty} V_n)$, by construction. Hence $V\cap(\cap V_n)\neq\emptyset$ and the proof is finished. □

We are now fully prepared for

THEOREM 6.12. *For any Banach space X each of the properties (i) - (vi) of*

Theorem 6.8 and 6.9 is also equivalent to
(vii) *for every w^*-compact convex subset $K \subset X^*$,*

$$K = \overline{co}\ \mathrm{ext}\ K.$$

PROOF. We already showed that (vii) implies (vi). We now prove

(i) $\Rightarrow$ (vii): Let us assume $l^1 \not\subset X$ and suppose for contradiction that $K \neq \overline{co}$ ext K for some w^*-compact convex $K \subset X^*$. By the Hahn-Banach theorem there exists an $x^{**} \in X^{**}$ so that

$$1 = \sup_K x^{**} > \sup_{\overline{co}\,\mathrm{ext}\,K} x^{**}.$$

The Bishop-Phelps theorem even allows us to assume that x^{**} attains its sup on K, i.e. the face $F := \{x^* \in K : \langle x^*, x^{**}\rangle = 1\}$ is non-empty. Put $C := w^*\, cl\, F$ and let $E :=$ ext C. Observe that x^{**} is < 1 on E, since $E \cap F = \varnothing$ (any point of $E \cap F$ would be extreme in K, contrary to the choice of x^{**}). Hence

$$E \in \bigcup_{n=1}^{\infty} w^*\, cl\, \{x^* \in E : \langle x^*, x^{**}\rangle < 1 - \frac{1}{n}\}.$$

Now E is a Baire space for the w^* topology by Prop. 6.11. It follows that for some $n \in \mathbb{N}$ the set $w^* cl\{x^* \in E : \langle x^*, x^{**}\rangle < 1 - \frac{1}{n}\}$ contains a w^*-open subset O of E. Recall now that by Th. 3.11, (i) $\Leftrightarrow$ (iv) the assumption $l^1 \not\subset X$ implies that $x^{**} \in \mathfrak{B}_r(B(X^*), w^*)$. In particular $x^{**}|w^*\, cl\, O$ must have a point of w^*-continuity. This implies the existence of a w^*-closed U of C with $(\mathrm{int}_E\, U) \cap O \neq \varnothing$ and such that the oscillation of x^{**} on U is $< \frac{1}{2n}$ ($\mathrm{int}_E\, U$ denotes the w^*-interior of U relative to E). Now observe that

$$C = w^*\, cl\, co\, E = co(w^*\, cl\, co\, U,\ w^*\, cl\, co(E \setminus U)).$$

Since $E \setminus U$ misses the w^*-open subset $(\mathrm{int}_E\, U) \cap O$ of E, it follows from Milman's theorem that $w^*\, cl\, co(E \setminus U) \underset{\neq}{\subset} C$. Hence $F \not\subset w^*\, cl\, co(E \setminus U)$, so there is an $x^* \in F$ of the form $x^* = \lambda x_1^* + (1-\lambda) x_2^*$, $x_1^* \in w^*\, cl\, co\, U$, $x_2^* \in w^*\, cl\, co(E \setminus U)$ and $\lambda > 0$. Since $1 = \langle x^*, x^{**}\rangle = \lambda\langle x_1^*, x^{**}\rangle + (1-\lambda)\langle x_2^*, x^{**}\rangle$ and $\langle x_1^*, x^{**}\rangle$, $\langle x_2^*, x^{**}\rangle \leq 1$, we shall have a contradiction once it is proved that $\langle x_1^*, x^{**}\rangle < 1$. To see this, note that $x_1 \in w^*\, cl\, co\, U$ is the barycenter of a w^*-Radon measure μ concentrated on U (see the proof of (iv) $\Rightarrow$ (vi) in Theorem 6.9). Finally recall that by (iv) of Theorem 6.9,

$$\langle x_1^*, x^{**}\rangle = \int_U \langle x^*, x^{**}\rangle d\mu(x^*).$$

We may conclude from this that $\langle x_1^*, x^{**}\rangle < 1 - \frac{1}{2n}$, since U intersects $\{x^* \in E : \langle x^*, x^{**}\rangle < 1 - \frac{1}{n}\}$ and since the oscillation of x^{**} on U is $< \frac{1}{2n}$. This finishes the proof. $\square$

NOTES Several mathematicians have contributed to the results of this chapter. The fact that X^* has the WRNP iff $l^1 \not\subset X$ was proved for separable X by K. Musial ([55]). For non-separable X the implication X^* WRNP $\Rightarrow$ $l^1 \not\subset X$ was deduced from the separable case by K. MUSIAL and C. RYLL-NARDZEWSKI ([59]), using a lifting theorem for vector measures. The reverse implication was proved by L. JANICKA ([47]) and, independently, by J. BOURGAIN (unpublished, but see [58]).

The important fact (Prop. 6.7) that $\rho(\phi)$ is w^*-Borel measurable and that it induces a w^*-Radon measure on $B(X^*)$ is due to D. SENTILLES ([87], see also [20]). A deep study of Pettis integration via the Stonian transform was made by D. SENTILLES and R.F. WHEELER in [88].

The characterizations (iii) - (vii) are all due to R. HAYDON ([38]). The equivalence of (iii) and (v) has led L.H. RIDDLE and E. SAAB ([69]) to a more general result where $(B(X^*),w^*)$ is replaced by any compact space K and i by a bounded map from K to (X^*,w^*) that is universally Lusin measurable. For more on this, see [1]. In another noteworthy development, E. SAAB has shown in [78] that for a dual space X^* to have the WRNP it suffices that every Dunford-Pettis operator $T:L^1 \rightarrow X^*$ (rather than any bounded operator) has a Pettis integrable derivative. This result has been "localized" by L.H. RIDDLE ([68]). See also [77], where E. Saab proves an interesting analogue of condition (v) characterizing w^*-compact convex sets with the RNP.

In the separable case the equivalence of (i) and (vii) is due to E. ODELL and H.P. ROSENTHAL ([62]). The preliminary results Lemma 6.10 and Prop. 6.11 were proved by G. CHOQUET, see [11]. There may be some novelty in our proof of Prop. 6.11: we have eliminated the need for Lemma 27.8 in [11] by interposing the slices S'_n in the construction. In connection with the barycentric calculus, let us mention the following result of E. ODELL and H.P. ROSENTHAL ([62]): for any X, an element $x^{**} \in X^{**}$ belongs to $\mathfrak{B}_1(B(X^*),w^*)$ iff $x^{**} = w^*\text{-}\lim x_n$ for some sequence $(x_n) \subset X$. This result is non-trivial: it says that an affine function that is the limit of a sequence of w^*-continuous functions, is in fact a limit of a sequence of w^*-continuous *affine* functions. Let us write $K:=(B(X^*),w^*)$ and let us denote the space of affine continuous functions on K by $A(K)$. More generally we may put $A_0(K):=A(K)$ and define inductively, for any ordinal α, $A_\alpha(K):=$ the set of pointwise limits of bounded sequences in $\bigcup\limits_{\beta<\alpha} A_\beta(K)$. Then $A_1(K) = \mathfrak{B}_1(K)$, as we have just seen. This result leads to a simple example where $\mathfrak{B}_1(K) \neq \mathfrak{B}_r(K)$. Let us take $X:=c_0(\Gamma)$, so $X^* = l^1(\Gamma)$, $X^{**} = l^\infty(\Gamma)$, Γ uncountable. Since $l^1 \not\subset X$, we know from Theorem 3.11 that each $x^{**} \in l^\infty(\Gamma)$ belongs to $\mathfrak{B}_r(K)$. However, each $x^{**} \in \mathfrak{B}_1(K) = A_1(K)$ is a w^*-limit of a sequence in $c_0(\Gamma)$, and therefore must have countable support. This shows that $\mathfrak{B}_1(K) \neq \mathfrak{B}_r(K)$. Returning to the general situation, let us note that generally $\mathfrak{B}_2(K) \neq A_2(K)$. This follows from the example on p. 104 in [66] that we have mentioned before ($\mathfrak{B}_2(K)$ of course denotes the set of Baire functions of the second class on K). To see this, note that functions in $A_2(K)$ satisfy the barycentric calculus. Indeed, more generally, by the Lebesgue theorem and induction over α, every $f \in \bigcup\limits_\alpha A_\alpha(K)$ is Borel and satisfies the barycentric calculus. The question has been raised whether a Borel function on K satisfying the barycentric calculus must at least belong to some $A_\alpha(K)$. Even this is not true. M. TALAGRAND ([93]) has constructed a separable Banach space X and an $x^{**} \in X^{**}$ so that $x^{**}|_K \in \mathfrak{B}_2(K)$ $(K=(B(X^*),w^*))$, $x^{**}|_K$ satisfies the barycentric calculus, but does not belong to any $A_\alpha(K)$.

We now describe some interesting characterizations of spaces not containing l^1 that were recently proved by G. GODEFROY ([32]). They are related to (and in some sense generalize, at least in the separable case) (vi) and (vii). Let C be a closed convex bounded subset of a dual space X^*. A subset $B \subset C$ is called a *boundary* of C if for every $x \in X$ there exists an $x^* \in B$ so that $\langle x,x^* \rangle = \sup_{y^* \in C} \langle x,y^* \rangle$. Note that if C is w^*-compact, then $B=$ ext$C$ is a boundary of C. In general, however, C has boundaries that do not contain ext$C$ and may even miss ext$C$ altogether. Now each of the following two properties is, for separable X, equivalent to $l^1 \not\subset X$:

(22) for every equivalent norm $|||\cdot|||$ on X^* and for every boundary B of $C := \{x^* \in X^* : |||x^*||| \leq 1\}$, we have $C = \overline{co}B$.

(23) every closed convex bounded set $C \subset X^*$ that has a boundary (i.e. is such that each $x \in X$ attains its sup on C) is w^*-compact.

Examples in [32] show that the separability of X is essential in both (22) and (23).

Finally, coming back to the WRNP, let us mention that, just like the RNP, the WRNP has been localized in recent years. One calls a set $K \subset X$ a WRNP-set if for every complete probability space (Ω,Σ,μ) every measure $F:\Sigma \to X$ with "average range" $\{\frac{FE}{\mu E} : E \in \Sigma,\ \mu E > 0\}$ contained in K, has a (P)-integrable K-valued derivative. Various characterizations of WRNP sets are known ([70], [79], [67]).

Chapter VII

KMP, RNP and strong regularity

Introduction

We have already mentioned in passing that a Banach space X has the *Radon-Nikodym property (RNP)* if for every complete finite measure space (Ω,Σ,μ) and for every X-valued measure $F:\Sigma\to X$ which is μ-continuous and of bounded variation, F has a Bochner integrable derivative ϕ:

$$FE = (B)\int_E \phi d\mu \quad (E\in\Sigma).$$

We shall not discuss this property here, but we need the equivalent formulation of the RNP in terms of dentability (this may be taken as the definition).

Let $A\subset X$ be any bounded subset. Recall from Chapter 6 that a *slice* of A is a set of the form

$$S = S(A,x^*,\alpha):=\{x\in A:\langle x,x^*\rangle > M(x^*)-\alpha\},$$

where $x^*\in X^*$, $\|x^*\|=1$, $\alpha>0$ and $M(x^*):=\sup_A x^*$. If X is a dual space and if the functional defining the slice is in the predual of X, then S is said to be a *w^*-slice*. A is *dentable* if A has small slices, i.e. if for every $\epsilon>0$ there exists a slice S of A with diam $S<\epsilon$. A fundamental result that we shall not prove here is that *X has the RNP iff every closed bounded convex $K\subset X$ is dentable.* In this case X is called *dentable.*

The definition of dentability suggests a close relationship between the RNP and the existence of extreme points for closed bounded convex sets. We say that X has the *Krein-Milman property (KMP)* if every closed bounded convex $K\subset X$ equals the closed convex hull of its extreme points: $K=\overline{co}$ ext K. It is well known and not very difficult to prove that the *RNP implies the KMP.* The converse is open in general but has been established in a variety of special cases. E.g. R. Huff and P. Morris have shown that RNP= KMP for dual spaces.

In this chapter we wish to prove two results that both involve the notion of *strong regularity.* We first define strong regularity and discuss it at some length. Thereafter the two main results will be established. The first is another characterization of spaces not containing l^1, in term of their duals:

(I) X^* is strongly regular iff l^1 does not embed in X.

The second result gives another reason why strong regularity is important:

(II) Strongly regular X with the KMP have the RNP
(so RNP = KMP for strongly regular X).

An easy corollary of these results will be the old Huff-Morris theorem mentioned above that KMP = RNP for dual spaces.

§ 1. Strong regularity

Definition 7.1. A Banach space X is called *strongly regular* if for every closed bounded convex $K \subset X$ and for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and slices $S_1, \ldots, S_n$ of K such that

$$\operatorname{diam} \frac{1}{n}(S_1 + \ldots + S_n) < \epsilon. \tag{1}$$

If X is a dual space and there are w^*-slices $S_1, \ldots, S_n$ as above, then we shall say that X is *w^*-strongly regular.* □

Remark 7.2. The above definition is not weakened if instead of (1) one requires the existence of arbitrarily small convex combinations of slices $\sum_{i=1}^{n} \lambda_i S_i$ with $\lambda_1, \ldots, \lambda_n > 0$, $\sum_{i=1}^{n} \lambda_i = 1$. This is because any such $\lambda_1, \ldots, \lambda_n$ can be simultaneously approximated by rationals $\frac{k_1}{n}, \ldots, \frac{k_n}{n}$ (n large), and because $\frac{k_i}{n} S_i = \frac{1}{n} \underbrace{(S_i + \ldots + S_i)}_{k}$ (there is no rule against repeating slices). □

To get a feeling for what strong regularity means we now first prove some easy lemmas about it that will be needed later anyway.

Lemma 7.3. *Let X be a Hausdorff l.c.s. and let $K \subset X$ be closed bounded and convex. Then every relatively weakly open subset $U \subset K$ contains a combination of slices, i.e. there exist $n \in \mathbb{N}$ and slices $S_1, \ldots, S_n$ of K so that*

$$\frac{1}{n} \sum_{i=1}^{n} S_i \subset U.$$

Proof. Let us choose $x_0 \in U$, $\epsilon > 0$ and $x_1^*, \ldots, x_k^* \in X^*$ so that $V := \{x \in X : |\langle x, x_j^* \rangle| < \epsilon \text{ for } j = 1, \ldots, k\}$ satisfies $(x_0 + 2V) \cap K \subset U$. Next we define $\Phi: X \to l_k^\infty$ by $\Phi x := (\langle x, x_j^* \rangle)_{j=1}^k$ $(x \in X)$. Then $\overline{\Phi K} =: C$ is a compact convex subset of l_k^∞, so by a classical result of Caratheodory, $C = co \operatorname{ext} C$. In particular

$$\Phi x_0 = \sum_{i=1}^{m} \lambda_i y_i \text{ with } y_i \in \operatorname{ext} C, \lambda_1, \ldots, \lambda_m > 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1.$$

By Lemma 6.10 applied to C with its norm topology, each y_i is contained in a slice T_i of C with diam $T_i<\epsilon$ $(i=1,\dots,m)$. Now put $S_i:=(\Phi^{-1}T_i)\cap K$ $(i=1,\dots,m)$. Clearly the S_i are slices of K and we claim that $\sum_{i=1}^{m}\lambda_i S_i\subset V$. Indeed, let $x_i\in S_i$ be arbitrary $(i=1,\dots,m)$. Observe that $\|\Phi x_i-y_i\|<\epsilon$ $(i=1,\dots,m)$, since $\Phi x_i\in T_i$, so that

$$\|\sum_{i=1}^{m}\lambda_i\Phi x_i-\sum_{i=1}^{m}\lambda_i y_i\|=\|\Phi(\sum_{i=1}^{m}\lambda_i x_i)-\Phi x_0\|<\epsilon.$$

But this is equivalent to saying that

$$|\langle\sum_{i=1}^{m}\lambda_i x_i-x_0,x_j^*\rangle|<\epsilon \text{ for } j=1,\dots,k,$$

i.e. $\sum_{i=1}^{m}\lambda_i x_i\in x_0+V$, hence $\sum_{i=1}^{m}\lambda_i S_i\subset x_0+V$. Finally, approximating the λ_i with rationals and repeating some S_i, as indicated is Remark 7.2, we find slices $S_1,\dots,S_n$ of K so that $\frac{1}{n}\sum_{i=1}^{n}S_i\subset(x_0+2V)\cap K\subset U$. □

Corollary 7.4. *A [dual] Banach space X is [w^*-] strongly regular if (and only if) for every closed bounded convex $K\subset X$ and for every $\epsilon>0$ there exist $n\in\mathbb{N}$ and relatively w-[w^*-] open subsets $U_1,\dots,U_n$ in K so that diam $\frac{1}{n}\sum_{i=1}^{n}U_i<\epsilon$.*

Proof. Given such $U_1,\dots,U_n$ use the preceding lemma to find $m\in\mathbb{N}$ and [w^*-] slices $S_1^{(i)},\dots,S_m^{(i)}$ of K such that

$$\frac{1}{m}\sum_{j=1}^{m}S_j^{(i)}\subset U_i\quad(i=1,\dots,n)$$

(we clearly may assume, as we have done, that these combinations of slices have a common "length" m, cf. Remark 7.2). Now

$$\frac{1}{nm}\sum_{j=1}^{m}\sum_{i=1}^{n}S_j^{(i)}\subset\frac{1}{n}\sum_{i=1}^{n}U_i,\text{ so diam}\frac{1}{nm}\sum_{j=1}^{m}\sum_{i=1}^{n}S_j^{(i)}<\epsilon.\quad□$$

In the next result Lemma 7.3 is used to prove a useful consequence of strong regularity.

Lemma 7.5. *Let X be strongly regular and let a closed bounded convex $K\subset X$, a slice $S=S(K,x^*,\alpha)$ of K, and an $\epsilon>0$ be given. Then there exist $k\in\mathbb{N}$ and slices $T_1,\dots,T_k$ of K such that*
(i) $T_i\subset S$ $(i=1,\dots,k)$,
(ii) diam $\frac{1}{k}\sum_{i=1}^{k}T_i<\epsilon$.

Proof. Consider the closed slice $\bar{S}(K,x^*,\frac{\alpha}{2})$. By strong regularity there exist slices $S_1, \ldots, S_n$ of $\bar{S}(K,x^*,\frac{\alpha}{2})$ so that diam $\frac{1}{n}\sum_{i=1}^{n} S_i < \frac{\epsilon}{2}$. Now note that each set $S_i \cap S(K,x^*,\frac{\alpha}{2})$ is non-empty and relatively weakly open in K. Hence by Lemma 7.3 there exist a $k \in \mathbb{N}$ and slices $T_j^{(i)}$ $(j=1,\ldots,k;\ i=1,\ldots,n)$ of K so that

$$\frac{1}{k}\sum_{j=1}^{k} T_j^{(i)} \subset S_i \cap S(K,x^*,\frac{\alpha}{2}) \quad (i=1,\ldots,n).$$

Put $m:=kn$ and enumerate the $T_j^{(i)}$ as $T_1, \ldots, T_m$. Then

$$\frac{1}{m}\sum_{i=1}^{m} T_i \subset \frac{1}{n}\sum_{i=1}^{n} S_i, \text{ so diam} \frac{1}{m}\sum_{i=1}^{m} T_i < \frac{\epsilon}{2} \text{ and}$$

$$\frac{1}{m}\sum_{i=1}^{m} T_i \subset S(K,x^*,\frac{\alpha}{2}). \tag{2}$$

We now claim that at least half of the T_i's are actually contained in $S = S(K,x^*,\alpha)$. To see this, let us rearrange the T_i so that, for some $k \leqslant m$,

$$T_i \subset S \quad \text{for } i=1,\ldots,k, \quad T_i \not\subset S \text{ for } i=k+1,\ldots,m.$$

Choose $x_i \in T_i (i=1,\ldots,m)$ so that $\langle x_i,x^*\rangle \leqslant M(x^*) - \alpha$ for $i=k+1,\ldots,m$. Then, by (2),

$$\frac{1}{m}\sum_{i=1}^{m} x_i \in S(K,x^*,\frac{\alpha}{2}), \text{ so } \langle \frac{1}{m}\sum_{i=1}^{m} x_i,x^*\rangle > M(x^*) - \frac{\alpha}{2}. \tag{3}$$

On the other hand

$$\langle \frac{1}{m}\sum_{i=1}^{m} x_i,x^*\rangle = \frac{1}{m}\sum_{i=1}^{k}\langle x_i,x^*\rangle + \frac{1}{m}\sum_{i=k+1}^{m} \langle x_i x^*\rangle$$

$$\leqslant \frac{1}{m}(kM(x^*) + (m-k)(M(x^*)-\alpha)). \tag{4}$$

From (3) and (4) one easily obtains $k \geqslant \frac{1}{2}m$. Hence

$$\text{diam} \frac{1}{k}\sum_{i=1}^{k} T_i \leqslant 2\,\text{diam} \frac{1}{m}\sum_{i=1}^{m} T_i < 2. \frac{\epsilon}{2} = \epsilon$$

and we are done: the slices $T_1, \ldots, T_k$ satisfy (i) and (ii). $\square$

We prove one more preparatory lemma before passing to the main results. It is elementary but important, and has nothing to do with strong regularity.

LEMMA 7.6. *Suppose $K \subset B(X)$ is closed bounded and convex. Let $x^*,y^* \in X^*$ and numbers ϵ, α and c be given such that*

$$\|x^*\| = \|y^*\| = 1, \quad 0<\epsilon<1, \quad \alpha>0 \text{ and } -1 \leqslant c \leqslant 1.$$

Then if sup $\{\langle x,y^*\rangle : x \in S(K,x^*,\frac{\epsilon}{3}\alpha)\} > c$, there is a slice T of K such that

(i) $T \subset S(K,x^*,\alpha)$ *and*
(ii) $\inf\{\langle x,y^*\rangle : x \in T\} > c - \epsilon$.

Proof. Consider the following relatively weakly open set U in K,

$$U := \{x \in K:\ x \in S(K,x^*,\tfrac{\epsilon}{3}\alpha),\ \langle x,y^*\rangle > c\}.$$

Note that by assumption $U \neq \emptyset$. From lemma 7.3 we obtain $n \in \mathbb{N}$ and slices $T_1, \ldots, T_n$ of K such that

$$\frac{1}{n}\sum_{i=1}^{n} T_i \subset U.$$

We claim that one at least of these T_i satisfies the requirements (i) and (ii). This is proved by simply estimating the number of T_i's for which either (i) or (ii) fails. Suppose $T_1, \ldots, T_{k_1}$ fail (i). Choose $x_i \in T_i$ $(i = 1, \ldots, n)$ so that $\langle x_i,x^*\rangle \leqslant M(x^*) - \alpha$ for $i = 1, \ldots, k_1$. Note that $\langle x_i,x^*\rangle \leqslant M(x^*)$ for $i = k_1+1, \ldots, n$. Then since $\frac{1}{n}\sum_{i=1}^{n} x_i \in U$,

$$M(x^*) - \frac{\epsilon}{3}\alpha < \langle \frac{1}{n}\sum_{i=1}^{n} x_i,x^*\rangle \leqslant \frac{k_1}{n}(M(x^*)-\alpha) + \frac{n-k_1}{n}M(x^*),$$

and it easily follows that $k_1 \leqslant \frac{\epsilon}{3}n$.

Next (reorder the T_i's again) suppose $T_1, \ldots, T_{k_2}$ fail (ii). Choose $0<\rho<1$ and then $x_i \in T_i$ $(i=1,\ldots,n)$ so that $\langle x_i,y^*\rangle < c - \rho\epsilon$ for $i = 1, \ldots, k_2$. Note that $\langle x_i,y^*\rangle \leqslant 1$ for $i = k_2+1, \ldots, n$ (because $K \subset B(X)$ and $\|y^*\| = 1$). Then since $\frac{1}{n}\sum_{i=1}^{n} x_i \in U$ we find

$$c < \langle \frac{1}{n}\sum_{i=1}^{n} x_i,y^*\rangle \leqslant \frac{k_2}{n}(c-\rho\epsilon) + \frac{n-k_2}{n}.$$

Since $0<\rho<1$ was arbitrary one easily obtains $k_2 < \frac{(1-c)n}{1-c+\epsilon}$. Using the fact that $1-c \leqslant 2$, it follows now that

$$k_1 + k_2 < \frac{\epsilon}{3}n + \frac{1-c}{1-c+\epsilon}n \leqslant \left(\frac{\epsilon}{3} + \frac{2}{2+\epsilon}\right)n < n.$$

The conclusion is now evident that one at least of the T_i's satisfies both (i) and (ii). $\square$

We now embark on the project of showing that if l^1 does not embed in X, then X^* is strongly regular (the converse is fairly easy). There are some technical details involved in the proof that we now take care of first.

(A) A *(bounded) tree* in a Banach space X is a (bounded) collection $T=\{x_{n,k}:n=0,1,...;\ k=1,...,2^n\}\subset X$ satisfying

$$\frac{1}{2}(x_{n+1,2k-1}+x_{n+1,2k})=x_{n,k}\quad \forall n,\forall 1\leqslant k\leqslant 2^n. \tag{5}$$

T is called a *(bounded) ϵ-tree* (where $\epsilon>0$) if in addition

$$\|x_{n+1,2k-1}-x_{n,k}\|=\|x_{n+1,2k}-x_{n,k}\|\geqslant\epsilon\quad \forall n,\forall 1\leqslant k\leqslant 2^n. \tag{6}$$

A typical example of a 1-tree contained entirely in the unit ball is found in $L^1=L^1[0,1]$:

$$f_{n,k}:=2^n\chi_{[\frac{k-1}{2^n},\frac{k}{2^n}]}\quad (n=0,1,...;\ 1\leqslant k\leqslant 2^n).$$

We shall refer to this system as the *standard tree in* L^1. Trees may be regarded as particularly simple instances of X-valued martingales on [0,1], usually called *"dyadic martingales"*. Given a tree $T=\{x_{n,k}:n=0,1,...;\ 1\leqslant k\leqslant 2^n\}$, let us define an X-valued martingale $(f_n,\Sigma_n)_{n=0}^{\infty}$ on [0,1] as follows:

$\Sigma_n:=$the finite σ-algebra generated by the intervals $[\frac{k-1}{2^n},\frac{k}{2^n}]$, $1\leqslant k\leqslant 2^n$,

$$f_n:=\sum_{k=1}^{2^n}x_{n,k}\chi_{[\frac{k-1}{2^n},\frac{k}{2^n}]}.$$

It is clear that each f_n is Σ_n-measurable ($n=0,1,...$). The *martingale equality*

$$\int_E f_{n+1}\,d\lambda=\int_E f_n\,d\lambda\quad (E\in\Sigma_n)\ (n=0,1,...) \tag{7}$$

is an immediate consequence of (5). (In the general case of X-valued martingales the integrals in (7) are supposed to be Bochner integrals. It suffices for our purposes to consider simple functions. Then the Bochner integral is the obvious one.) Observe that (f_n) is uniformly bounded iff T is bounded. Also, if T is an ϵ-tree, then, by (6), for all $n=0,1,...$

$\|f_n(t)-f_{n+1}(t)\|\geqslant\epsilon$ whenever $t\notin D:=$ the set of dyadic numbers in [0,1],

since clearly $f_n(t)-f_{n+1}(t)$ equals either $x_{n,k}-x_{n+1,2k-1}$ or $x_{n,k}-x_{n+1,2k}$ for some k depending on $t(t\notin D)$. Hence the uniformly bounded martingale (f_n) is almost everywhere divergent. (f_n) is also divergent in L^1-sense. □

(B) Suppose now that (f_n) is a uniformly bounded X^*-valued martingale on [0,1], where X^* is some dual space (we suppose the f_n's are everywhere defined). For each $t\in[0,1]$ the sequence $(f_n(t))$ is bounded in X^*, so has a w^*-cluster point (Alaoglu). Define $f(t)$ to be any such w^*-cluster point.

We claim that the function $f:[0,1]\to X^*$ so defined is w^*-scalarly measurable. To see this, consider for any $x\in X$ the real-valued uniformly bounded martingale

$$(f_n^x)_{n=1}^\infty := (\langle x, f_n(\cdot)\rangle)_{n=0}^\infty.$$

By the martingale convergence theorem (f_n^x) converges pontwise to some (measurable) function f^x outside some λ-null set N_x depending on x. Take any $t\notin N_x$. Since $f(t)$ is by definition a w^*-cluster point of $(f_n(t))$, $\langle x, f(t)\rangle$ is a cluster point of $(\langle x, f_n(t)\rangle)_{n=0}^\infty$. On the other hand $(\langle x, f_n(t)\rangle)$ converges to $f^x(t)$. Hence

$$f^x(t) = \langle x, f(t)\rangle \quad (t\notin N_x).$$

We have now proved the w^*-scalar measurability of f. □

(C) Our last preparatory remark concerns a method of generating certain X^*-valued bounded linear operators defined on a Banach space Y, given a uniformly bounded system of X^*-valued operators defined on finite-dimensional subspaces of Y. This device is often referred to as a *"Lindenstrauss compactness argument"*. Let Y be a Banach space that is the closed linear span of some sequence (y_k). Suppose that for each $n\in\mathbb{N}$ a bounded linear operator $T_n:[y_k]_{k=1}^n\to X^*$ is given with $\|T_n\|\leqslant 1$. We claim that there exists a linear operator $T:Y\to X^*$ with $\|T\|\leqslant 1$ such that T is a "cluster point" of the sequence (T_n) in the following sense: for all finite sets $\{z_1,\ldots,z_p\}\subset Y_1:=sp(y_k)$ and $\{x_1,\ldots,x_q\}\subset X$ and for every $\epsilon>0$ there exists $n\in\mathbb{N}$ so that

$$|\langle x_j, Tz_i - T_n z_i\rangle| < \epsilon \quad (i=1,\ldots,p;\, j=1,\ldots,q).$$

This is simple. First extend each T_n in some (not necessarily linear) manner to a map $\tilde{T}_n: Y_1\to X^*$ so that $\tilde{T}_n B(Y_1)\subset B(X^*)$. E.g. define

$$\tilde{T}_n y := \begin{cases} T_n y & \text{if } y\in[y_k]_{k=1}^n, \\ 0 & \text{otherwise.}\end{cases}$$

Then identify each $\tilde{T}_n$ in the obvious way with a point of $B(X^*)^{B(Y_1)}$, and equip this space with the product of the w^*-topologies. Observe that $B(X^*)^{B(Y_1)}$ is compact. Any cluster point T of $\tilde{T}_n\subset B(X^*)^{B(Y_1)}$, when regarded again as a map from $B(Y_1)$ to $B(X^*)$, obviously extends to a bounded linear map $T:Y\to X^*$ with the required properties. □

We put the last preparatory result in the form of a proposition.

PROPOSITION 7.7. *Let $f:[0,1]\to X^*$ be w^*-scalarly measurable and uniformly bounded and suppose that l^1 does not embed in X. Then*

$$\lim_{n\to\infty}\int_0^1 f.r_n\,d\lambda = 0.$$

((r_n) denotes the sequence of the Rademacher functions; $\int_0^1 f.r_n d\lambda$ is the "w^*-integral" i.e. the unique element of X^* satisfying

$$\langle x, \int_0^1 f.r_n d\lambda\rangle = \int_0^1 \langle x,f\rangle r_n d\lambda \quad \text{for all}\, x \in X;$$

existence of this w^*-integral is clear from the proof of Prop. 6.3, (ii) $\Rightarrow$ (i).)

PROOF. If the conclusion is false there exists an $\epsilon>0$ and a subsequence of (r_n) (which we continue to denote by (r_n)), so that

$$\|\int_0^1 f.r_n d\lambda\| > \epsilon \quad (n=1,2,\ldots).$$

Choose elements $x_n \in B(X)$ so that

$$\int_0^1 \langle x_n,f\rangle r_n\, d\lambda > \epsilon \quad (n=1,2,\ldots). \tag{8}$$

Now by the assumption that $l^1 \not\subset X$ and by Theorem 4.1, (4) $\Leftrightarrow$ (1), we may assume by passing to a further subsequence that (x_n) is weakly Cauchy. This implies that the uniformly bounded sequence $(\langle x_n,f\rangle)$ is pointwise convergent on $[0,1]$, and therefore L^1-convergent by Lebesgue's theorem. Let ϕ be the L^1-limit. Choose $n_0 \in \mathbb{N}$ so large that

$$\|\phi - \langle x_n,f\rangle\|_1 < \frac{\epsilon}{2} \quad \text{for} \quad n \geqslant n_0.$$

Then for all $n \geqslant n_0$ we have

$$\int_0^1 \langle x_n,f\rangle r_n d\lambda \leqslant \int_0^1 \phi r_n d\lambda + \|\langle x_n,f\rangle - \phi\|_1 < \int_0^1 \phi r_n d\lambda + \frac{\epsilon}{2}.$$

Finally observe that $\lim_{n\to\infty} \int_0^1 \phi r_n d\lambda = 0$, since $r_n \xrightarrow{w} 0$ in L^2 (or by direct argument). Hence we have a contradiction with (8). □

THEOREM 7.8. *Suppose that l^1 does not embed in X and let $B \subset X^*$ be any bounded set. Then for every $\epsilon>0$ there exist $n \in \mathbb{N}$ and non-empty relatively w^*-open subsets $U_1, \ldots, U_n \subset B$ such that*

$$\operatorname{diam} \frac{1}{n} \sum_{i=1}^{n} U_i \leqslant \epsilon.$$

PROOF. Fix $\epsilon>0$ For simplicity let us assume that $B \subset B(X^*)$. Let us suppose for contradiction that for any finite collection of non-empty relatively w^*-open subset $U_1, \ldots, U_n \subset B$ we have

$$\operatorname{diam} \frac{1}{n} \sum_{i=1}^{n} U_i > \epsilon.$$

We now define inductively elements $x_n \in X$ and non-empty relatively w^*-open $U_{n,k}$ $(n=0,1,\ldots;\ k=1,\ldots,2^n)$ in B so that for all n and k,
(i) $\|x_n\| = 1$,
(ii) $U_{n+1,2k-1} \cup U_{n+1,2k} \subset U_{n,k}$,
(iii) $\frac{1}{2^n} \sum_{k=1}^{2^n} [\inf_{x^* \in U_{n+1,2k-1}} \langle x_{n+1}, x^* \rangle - \sup_{x^* \in U_{n+1,2k}} \langle x_{n+1}, x^* \rangle] > \epsilon.$
x_0 and $U_{0,1}$ may be chosen arbitrarily. For the inductive step, suppose $x_0,\ldots,x_n$ and $U_{m,k}$ $(m=0,\ldots,n;\ k=1,\ldots,2^m)$ have been properly chosen. Then since by assumption

$$\operatorname{diam} \frac{1}{2^n} (U_{n,1} + \ldots + U_{n,2^n}) > \epsilon,$$

there exist $x_k^*, y_k^* \in U_{n,k}$ $(k=1,\ldots,2^n)$ so that

$$\|\frac{1}{2^n} \sum_{k=1}^{2^n} (x_k^* - y_k^*)\| > \epsilon.$$

Now choose $x_{n+1} \in X$ with $\|x_{n+1}\| = 1$ so that for some $\delta > 0$,

$$\sum_{k=1}^{2^n} \frac{1}{2^n} [\langle x_{n+1}, x_k^* \rangle - \langle x_{n+1}, y_k^* \rangle] > \epsilon + 2\delta.$$

If one defines, for $k=1,\ldots,2^n$,

$$U_{n+1,2k-1} := \{x^* \in U_{n,k} : \langle x_{n+1}, x_k^* - x^* \rangle < \delta\},$$
$$U_{n+1,2k} := \{x^* \in U_{n,k} : \langle x_{n+1}, x^* - y_k^* \rangle < \delta\},$$

then (ii) and (iii) are clearly satisfied.

Next we are going to define an operator $T: L^1 \to X^*$ with $\|T\| \leq 1$ so that

$$Tf_{n,k} \in \tilde{co}\, U_{n,k} \quad (n=0,1,\ldots;\ k=1,\ldots,2^n), \tag{9}$$

where $(f_{n,k})$ is the standard tree in L^1 and $\tilde{co}$ denotes w^*-closed convex hull. For this we use the device discussed in (C) above. Observe first that L^1 is the closed linear span of its standard tree. It therefore suffices to define linear operators $T_n: sp\{f_{m,k} : m=0,\ldots,n;\ k=1,\ldots,2^m\} \to X^*$ with $\|T_n\| \leq 1$ and

$$T_n f_{m,k} \in co\, U_{m,k}\ (m=0,\ldots,n;\ k=1,\ldots,2^m) \quad (n=1,2,\ldots).$$

To define T_n we start by putting

$$T_n f_{n,k} := \text{any element of } U_{n,k} \quad (k=1,\ldots,2^n).$$

Since $f_{n,1},\ldots,f_{n,2^n}$ are linearly independent and all $f_{m,k}$ $(m=0,\ldots,n; k=1,\ldots,2^m)$ are convex combinations of $f_{n,1},\ldots,f_{n,2^n}$, this choice determines a linear T_n on $sp\{f_{m,k}: m=1,\ldots,n;\ k=1,\ldots,2^m\}$. Since

$$f_{n-1,k} = \frac{1}{2} f_{n,2k-1} + \frac{1}{2} f_{n,2k} \quad (k=1,\ldots,2^{n-1}),$$

it follows from (ii) above that

$$T_n f_{n-1,k} \in \frac{1}{2} U_{n,2k-1} + \frac{1}{2} U_{n,2k} \subset co\, U_{n-1,k} \quad (k=1,\ldots,2^{n-1}).$$

Using "backward" induction from n to 0 yields

$$T_n f_{m,k} \in co\ U_{m,k} \quad (m=0,\ldots,n; k=1,\ldots,2^m). \tag{10}$$

Finally, it is a trivial exercise to show that $\|T_n\| \leqslant 1$ for each $n=0,1,\ldots$ (recall that we have assumed $B \subset B(X^*)$). A Lindenstrauss compactness argument now produces the required T ((10) evidently implies (9) for the "cluster point" T).

The tree definition and the linearity of T show that $\{Tf_{n,k}: n=0,1,\ldots; k=1,\ldots,2^n\}$ is a bounded tree in X^*, which therefore corresponds to a uniformly bounded X^*-valued dyadic martingale (g_n) on $[0,1]$, as explained in (A) above. Specifically, we have

$$g_n = \sum_{k=1}^{2^n} Tf_{n,k}\, \chi_{[\frac{k-1}{2^n}, \frac{k}{2^n}]} \quad (n=0,1,\ldots).$$

As we have seen in (B), defining $g(t)$ to be any w^*-cluster point of $(g_n(t))$, yields a uniformly bounded w^*-scalarly measurable $g:[0,1] \to X^*$. It is also immediate from (9) that

$$g(t) \in \tilde{co}\ U_{n,k} \text{ whenever } t \in [\frac{k-1}{2^n}, \frac{k}{2^n}] \text{ and } t \notin D. \tag{11}$$

This last fact will lead to a contradiction with Prop. 7.7. Indeed, for every $n=0,1,\ldots$ we have by the definition of r_n, and by (11) and (iii),

$$\int_0^1 \langle x_{n+1}, g\rangle r_{n+1}\, d\lambda \geqslant$$

$$\frac{1}{2^{n+1}} \sum_{k=1}^{2^n} [\inf_{t \in [\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}] \setminus D} \langle x_{n+1}, g(t)\rangle - \sup_{t \in [\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}] \setminus D} \langle x_{n+1}, g(t)\rangle]$$

$$\geqslant \frac{1}{2^{n+1}} \sum_{k=1}^{2^n} [\inf_{x^* \in U_{n+1,2k-1}} \langle x_{n+1}, x^*\rangle - \sup_{x^* \in U_{n+1,2k}} \langle x_{n+1}, x^*\rangle] > \frac{\epsilon}{2}.$$

So $\|\int_0^1 g.r_{n+1}\, d\lambda\| > \frac{\epsilon}{2}$ $(n=0,1,\ldots)$, contradicting Prop. 7.7. □

COROLLARY 7.9. *X^* is strongly regular iff l^1 does not embed in X.*

PROOF. Corollary 7.4 and the preceding theorem immediately imply that X^* is w^*-strongly regular if l^1 does not embed in X. Suppose now that l^1 does embed in X. Then by Prop. 4.2 ((1) $\Leftrightarrow$ (8)), X^* contains a copy of L^1. Since it is clear that strong regularity is inherited by subspaces and is invariant for isomorphism, it now suffices to show that L^1 is not strongly regular. To prove this let F be the positive face of $B(L^1)$, i.e.

$$F := \{x \in L^1 : x \geqslant 0 \text{ and } \|x\| = 1\}.$$

Clearly F is closed bounded and convex. We show that F has no small combinations of slices. Suppose $S_i = S_i(F, x_i^*, \alpha_i)$ $(i=1,\ldots,n)$ are slices of F

determined by unit vectors $x_1^*, \ldots, x_n^* \in L^\infty$. Observe that

$$M(x_i^*) = \sup_F x_i^* = \text{ess sup}\, x_i^* \quad (i=1,\ldots,n).$$

Clearly, therefore, it is possible to find, for any $\alpha>0$, so in particular for $\alpha := \min_{i=1,\ldots,n} \alpha_i$, disjoint subsets $E_1, \ldots, E_n$ of $[0,1]$ of positive measure so that

$$x_i^* > M(x_i^*) - \alpha \quad \text{on } E_i \quad (i=1,\ldots,n).$$

For each $i=1,\ldots,n$ now choose $A_i,\ B_i \subset E_i,\ A_i \cap B_i = \varnothing$ so that $\lambda A_i,\ \lambda B_i > 0$. Then

$$\frac{1}{\lambda A_i}\chi_{A_i},\ \frac{1}{\lambda B_i}\chi_{B_i} \ \in S_i \quad (i=1,\ldots,n),$$

so

$$\frac{1}{n}\sum_{i=1}^n \frac{1}{\lambda A_i}\chi_{A_i},\ \frac{1}{n}\sum_{i=1}^n \frac{1}{\lambda B_i}\chi_{B_i} \in \frac{1}{n}\sum_{i=1}^n S_i.$$

However,

$$\left\|\frac{1}{n}\sum_{i=1}^n \frac{1}{\lambda A_i}\chi_{A_i} - \frac{1}{n}\sum_{i=1}^n \frac{1}{\lambda B_i}\chi_{B_i}\right\| = 2,$$

so we have proved that diam $\frac{1}{n}\sum_{i=1}^n S_i = 2$ (obviously 2 is an upper bound). This holds for any choice of $n \in \mathbb{N}$ and slices $S_1, \ldots, S_n$ of F, so the proof is finished. □

§ 2. The equivalence of KMP and RNP for strongly regular spaces.

We begin by establishing some notation. Let F again be the positive face of $B(L^1)$:

$$F := \{f \in L^1 : f \geqslant 0 \text{ and } \|f\| = 1\}.$$

The bidual $(L^1)^{**} = (L^\infty)^*$ consists of the finitely additive signed bounded measures on the σ-algebra of Lebesgue measurable subsets of $[0,1]$ which vanish on the ideal of the λ-null sets. Alternatively, if Ω is the Stone space of $([0,1], \lambda)$ (cf. Chapter 6), each $\mu \in (L^\infty)^*$ may be regarded as a Radon measure on Ω. This is sometimes convenient, as measures are easier to handle than finitely additive measures.

For every $\mu \in \tilde{F}(:= w^* cl\, F$ in $(L^1)^{**})$, every $\epsilon \geqslant 0$ and every finite partition $P = \{A_1, \ldots, A_n\}$ of $[0,1]$ into sets A_i of positive λ-measure, we introduce the sets

$$V_{P,\epsilon}(\mu) := \{f \in F : \sum_{i=1}^n |\mu A_i - \int_{A_i} f d\lambda| \leqslant \epsilon\},$$

$$\tilde{V}_{P,\epsilon}(\mu) := \{\nu \in \tilde{F} : \sum_{i=1}^n |\mu A_i - \nu A_i| \leqslant \epsilon\}.$$

Since the simple functions $\sum_{i=1}^{n} \alpha_i \chi_{A_i}$ are dense in L^∞, it is clear that the sets $\tilde{V}_{P,\epsilon}(\mu)$ for $\epsilon>0$ form a w^*-nbhd basis of μ relative to $\tilde{F}$, and that $\tilde{V}_{P,\epsilon}(\mu)$ is the w^*-closure of $V_{P,\epsilon}(\mu)$, as the notation already suggests.

Now let $T:L^1\to X$ be a bounded linear operator. For any $\mu\in\tilde{F}$ we define the following two numbers:

$$\rho_T(\mu):=\inf\{\operatorname{diam} T\tilde{V}_{P,\epsilon}(\mu):\epsilon>0 \text{ and } P\in\mathcal{P}\},$$

where $\mathcal{P}$ denotes the set of all finite partitions of [0,1] specified above, and

$$d_T(\mu):=d(T^{**}\mu,X).$$

(Here $T^{**}\mu\in X^{**}$; X is regarded as a subspace of X^{**} and d is the norm distance in X^{**}.) It is trivial but important to observe that always

$$d_T(\mu)\leqslant\rho_T(\mu). \tag{12}$$

Indeed, choose a net (f_α) in F so that $f_\alpha \xrightarrow{w^*} \mu$ and fix $P\in\mathcal{P}$ and $\epsilon>0$. Then there exists an α_0 so that

$$f_\alpha \in V_{P,\epsilon}(\mu) \text{ for } \alpha\geqslant\alpha_0.$$

Hence, since T^{**} is w^*-w^*-continuous and the norm of $(L^1)^{**}$ is w^*-l.s.c.,

$$d_T(\mu)\leqslant\|T^{**}\mu - Tf_{\alpha_0}\| \leqslant \varliminf_{\alpha\to\infty} \|Tf_\alpha - Tf_{\alpha_0}\|\leqslant \operatorname{diam} TV_{P,\epsilon}(\mu).$$

This proves (12) since $V_{P,\epsilon}(\mu)$ was arbitrary.

The first of the two basic propositions on which the main result rests, makes use of the following technical lemma.

Lemma 7.10. *Let $P\in\mathcal{P}$, $\epsilon\geqslant0$, $\mu_1,\mu_2\in\tilde{F}$ and $\lambda_1,\lambda_2>0$ with $\lambda_1+\lambda_2=1$ be given. Then*

$$\lambda_1 \tilde{V}_{P,\epsilon}(\mu_1) + \lambda_2 \tilde{V}_{P,\epsilon}(\mu_2) = \tilde{V}_{P,\epsilon}(\mu_1\mu_1+\lambda_2\mu_2). \tag{13}$$

Proof. We first prove (13) for $\epsilon=0$. Observe that $\tilde{V}_{P,0}(\mu)$ consists of all $\nu\in\tilde{F}$ that coincide with μ on the sets of the partition $P=\{A_1,\ldots,A_n\}$. Hence the inclusion

$$\lambda_1 \tilde{V}_{P,0}(\mu_1)+\lambda_2 \tilde{V}_{P,0}(\mu_2)\subset\tilde{V}_{P,0}(\lambda_1\mu_1+\lambda_2\mu_2)$$

is trivial. The reverse inclusion is equally easy: if $\nu\in\tilde{V}_{P,0}(\lambda_1\mu_1+\lambda_2\mu_2)$ and if $\mu:=\lambda_1\mu_1+\lambda_2\mu_2$, then $\nu A_i=\mu A_i$ for $i=1,\ldots,n$. Define $\nu_j (j=1,2)$ by

$$\nu_j A:=\sum_{i=1}^{n}\frac{\mu_j A_i}{\mu A_i}\nu(A\cap A_i) \quad (j=1,2; A\subset[0,1]).$$

Then clearly $\nu_j A_i=\mu_j A_i$ $(j=1,2; i=1,\ldots,n)$, so $\nu_j\in\tilde{V}_{P,0}(\mu_j)$ $(j=1,2)$. It is also evident that $\lambda_1\nu_1+\lambda_2\nu_2=\nu$.

The general case $\epsilon>0$ follows immediately once the following formula is proved for all $\mu\in\tilde{F}$:

$$\tilde{V}_{P,\epsilon}(\mu)=[\tilde{V}_{P,0}(\mu)+\epsilon B((L^\infty)^*)]\cap\tilde{F}.$$

The inclusion $\tilde{V}_{P,\epsilon}(\mu)\supset[\tilde{V}_{P,0}(\mu)+\epsilon B((L^\infty)^*)]\cap\tilde{F}$ is trivial if one recalls that the norm of $(L^\infty)^*$ is the variation norm. For the proof of the reverse inclusion it is convenient to identify $(L^1)^*=L^\infty$ with $C(\Omega)$ and $(L^1)^{**}=(L^\infty)^*$ with $M(\Omega)$. Then we are dealing with countably additive measures, so that the Hahn-Jordan decomposition theorem may be used unrestrictedly. For the proof of $\tilde{V}_{P,\epsilon}(\mu)\subset\tilde{V}_{P,0}\ (\mu)+\epsilon B((L^\infty)^*)$ it clearly suffices (by a translation) to show that every measure ν satisfying $\sum_{i=1}^{n}|\nu(A_j)|\leqslant\epsilon$ can be written as $\nu=\rho+\sigma$ with $\rho(A_j)=0$ $(j=1,\dots,n)$ and $\sigma\in\epsilon B((L^\infty)^*)$. This we now do. Let ν be as described and put $\nu_j:=\nu|_{A_j}$ $(j=1,\dots,n)$. If $\nu A_j=\nu_j^+A_j-\nu_j^-A_j>0$, define μ_j by

$$\mu_j=\left[\frac{\nu_j^-A_j}{\nu_j^+A_j}\right]\nu_j^+-\nu_j^-.$$

Then $\mu_jA_i=0$ $(i=1,\dots,n)$ and

$$\nu_j-\mu_j=\nu_j^+-\nu_j^--\left[\frac{\nu_j^-A_j}{\nu_j^+A_j}\right]\nu_j^++\nu_j^-=\left[1-\frac{\nu_j^-A_j}{\nu_j^+A_j}\right]\nu_j^+\geqslant 0,$$

hence

$$\|\nu_j-\mu_j\|=\nu_j^+A_j-\nu_j^-A_j=\nu A_j.$$

If $\nu A_j\leqslant 0$ we similarly define μ_j so that

$$\mu_jA_i=0\ (i=1,\dots,n)\ \text{ and }\ \|\nu_j-\mu_j\|=-\nu A_j.$$

Putting $\mu':=\sum_{j=1}^{n}\mu_j$ we now have $\mu'\in\tilde{V}_{P,0}\ (0)$ and

$$\|\nu-\mu'\|=\sum_{j=1}^{n}\|\nu_j-\mu_j\|=\sum_{j=1}^{n}|\nu A_j|\leqslant\epsilon,$$

so

$$\nu=\mu'+(\nu-\mu')\in\tilde{V}_{P,0}(0)+\epsilon B((L^\infty)^*),$$

as required. □

We are now ready for the first of the two basic propositions.

PROPOSITION 7.11. *Let $T:L^1\to X$ be a bounded linear operator and put $C:=\overline{TF}$, so $\tilde{C}=(TF)^\sim=T^{**}\tilde{F}$. Suppose also that $X\cap$ ext $\tilde{C}=\varnothing$. Then for every $x\in$ ext C (if any) there exists a $\mu\in\tilde{F}$ so that*

$$T^{**}\mu=x\ \text{ and }\ \rho_T(\mu)>0.$$

PROOF. Let $x \in \operatorname{ext} C$ be arbitrary. Since $x \notin \operatorname{ext} \tilde{C}$, there are $x_1^{**}, x_2^{**} \in \tilde{C}$ with $x_1^{**} \neq x_2^{**}$ so that

$$x = \frac{1}{2} x_1^{**} + \frac{1}{2} x_2^{**}.$$

Since $\tilde{C} \cap X = C$, the assumption that $x \in \operatorname{ext} C$ entails $x_i^{**} \notin X$ $(i=1,2)$. Now choose $\mu_i \in \tilde{F}$ so that $T^{**}\mu_i = x_i^{**}$ $(i=1,2)$ and put $\mu := \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. From (12) we now obtain

$$\rho_T(\mu_i) \geqslant d_T(\mu_i) > 0 \quad (i=1,2).$$

Finally, (13) implies that

$$\frac{1}{2} V_{P,\epsilon}(\mu_1) + \frac{1}{2} V_{P,\epsilon}(\mu_2) \subset V_{P,\epsilon}(\mu)$$

for all $\epsilon > 0$ and all $P \in \mathcal{P}$. From this it easily follows that

$$\rho_T(\mu) \geqslant \max\left(\frac{1}{2}\rho_T(\mu_1), \frac{1}{2}\rho_T(\mu_2)\right),$$

and therefore $\rho_T(\mu) > 0$. □

What we are after is to construct a bounded linear operator $T: L^1 \to X$ (under suitable assumptions on X) so that any $\mu \in \tilde{F}$ with $\rho_T(\mu) > 0$ must have its T^{**}-image outside X. The conclusion that can then be drawn from Prop. 7.11 is that the closed bounded convex set C introduced above has no extreme points, so that X fails the KMP. This T will be constructed inductively on increasing finite-dimensional subspaces of L^1. The next fundamental proposition enables us to arrange step by step for "pushing $T^{**}\mu$ out of X".

PROPOSITION 7.12. *Let $K \subset X$ be closed bounded and convex, let $S_1, \ldots, S_m$ be slices of K and let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ be given with $\lambda_1, \ldots, \lambda_m > 0$ and $\sum_{j=1}^m \lambda_j = 1$. Also let $E = \{x_1, \ldots, x_k\}$ be any finite set in X. Define*

$$\tau(\lambda) := \inf \operatorname{diam} \sum_{j=1}^m \lambda_j T_j,$$

where the infimum is taken over all m-tuples $(T_1, \ldots, T_m)$ of slices of K such that $T_j \subset S_j$ $(j=1, \ldots, m)$. Then for every $\epsilon > 0$ a choice $(T_1, \ldots, T_m)$ as above is possible so that

(i) diam $\sum_{j=1}^m \lambda_j T_j < \tau(\lambda) + \epsilon$,

(ii) $d(x_i, \sum_{j=1}^m \lambda_j T_j) > \frac{1}{2}\tau(\lambda) - \epsilon \quad (i=1, \ldots, k)$.

PROOF. Fix ϵ so that $0 < \epsilon < 1$. Let us assume without loss of generality that $K \subset B(X)$. Using the definition of $\tau(\lambda)$ we may choose slices $S'_j \subset S_j$ $(j=1, \ldots, m)$ satisfying diam $\sum_{j=1}^m \lambda_j S'_j < \tau(\lambda) + \epsilon$. Thus (i) will hold for any choice of $T_j \subset S'_j$ $(j=1,\dots,m)$. The problem is to satisfy (ii). For this Lemma 7.6 will be needed.

Supposing that $S'_j = S(K,x_j^*,\alpha_j)$, put

$$R_j = S(K,x_j^*,\frac{\epsilon}{6}\alpha_j) \quad (j=1,\dots,m).$$

Since diam $\sum_{j=1}^{m} \lambda_j R_j \geqslant \tau(\lambda)$, there exists by the Hahn-Banach theorem a $y^* \in X^*$ with $\|y^*\|=1$ so that

$$\sup\{\langle x-x',y^*\rangle : x,x' \in \sum_{j=1}^{m} \lambda_j R_j\} > \tau(\lambda)-\epsilon.$$

(We may clearly assume that $\tau(\lambda)-\epsilon>0$, since otherwise (ii) holds trivially.) This inequality in particular implies, upon replacing y^* by $-y^*$ if necessary, that

$$\sup\{\langle x,y^*\rangle : x \in \sum_{j=1}^{m} \lambda_j R_j\} = \sum_{j=1}^{m} \lambda_j \sup_{R_j} y^* > \langle x_1,y^*\rangle + \frac{\tau(\lambda)-\epsilon}{2}.$$

Now choose any numbers c_j such that $c_j < \sup_{R_j} y^*$ $(j=1,\dots,m)$ and

$$\sum_{j=1}^{m} \lambda_j c_j > \langle x_1,y^*\rangle + \frac{\tau(\lambda)-\epsilon}{2}. \tag{14}$$

Applying Lemma 7.6 for each $j=1,\dots,m$ (with $\epsilon/2$ rather than ϵ), we find slices T_j of K contained in $S'_j = S(K,x_j^*,\alpha_j)$ so that

$$\inf_{T_j} y^* > c_j - \epsilon/2 \quad (j=1,\dots,m). \tag{15}$$

It now follows from (14) and (15) that for each choice of $z_j \in T_j$ $(j=1,\dots,m)$ we have

$$\langle \sum_{j=1}^{m} \lambda_j z_j, y^*\rangle = \sum_{j=1}^{m} \lambda_j \langle z_j,y^*\rangle > \sum_{j=1}^{m} \lambda_j (c_j - \epsilon/2)$$
$$= \sum_{j=1}^{m} \lambda_j c_j - \epsilon/2 > \langle x_1,y^*\rangle + \frac{\tau(\lambda)}{2} - \epsilon.$$

This shows that

$$d(x_1, \sum_{j=1}^{m} \lambda_j T_j) > \frac{\tau(\lambda)}{2} - \epsilon.$$

Suppose now that for some $n<k$ we have found slices $T_j \subset S'_j$ $(j=1,\dots,m)$ so that

$$d(x_i, \sum_{j=1}^{m} \lambda_j T_j) > \frac{\tau(\lambda)}{2} - \epsilon \quad (i=1,\dots,n).$$

Then we repeat the argument above with T_j and x_{n+1} replacing S'_j and x_1, respectively, to find slices T'_j of K such that $T'_j \subset T_j$ $(j=1,\dots,m)$ and so that

$$d(x_{n+1}, \sum_{j=1}^{m} \lambda_j T'_j) > \frac{\tau(\lambda)}{2} - \epsilon.$$

Of course also

$$d(x_i, \sum_{j=1}^{m} \lambda_j T'_j) \geqslant d(x_i, \sum_{j=1}^{m} \lambda_j T_j) > \frac{\tau(\lambda)}{2} - \epsilon \text{ for } i=1,\ldots,n,$$

so the proof is completed by induction. □

We now come to the main result.

THEOREM 7.13 *Let X be a separable Banach space that is strongly regular but fails the RNP. Then there exists a bounded linear operator $T:L^1 \to X$ and an $\alpha>0$ such that (with the notation established in Prop. 7.11)*
(i) $d(x^{**},X) \geqslant \alpha$ *for every* $x^{**} \in \text{ext}\, \tilde{C}$,
(ii) $d_T(\mu) \geqslant \frac{1}{2} \rho_T(\mu)$ *for every* $\mu \in \tilde{F}$.

PROOF. Since X fails the RNP there exists a closed bounded convex non-dentable subset $K \subset X$, which we may assume to be in $B(X)$. Let $\alpha>0$ be such that every slice of K has diameter $>2\alpha$. Furthermore let (x_k) be a dense sequence in X and let us put $E_n := \{x_1, \ldots, x_n\}$ $(n=1,2,\ldots)$.

STEP 1 We shall construct a "bush" of slices of K in a rather complicated manner. It seems advisable to first start the (inductive) construction before describing the general procedure.

To begin let us choose slices $S_1, \ldots, S_{m_1}$ of K such that

$$\text{diam} \frac{1}{m_1} \sum_{j=1}^{m_1} S_j < 2^{-1},$$

(this is possible by strong regularity) and put

$$\Omega_1 := \{1, \ldots, m_1\}, \quad M_1 := m_1.$$

The slices $S_1, \ldots, S_{m_1}$ are not yet our definitive choice for the elements of the first level of the bush. The final choice, to be called $T_1, \ldots, T_{m_1}$, will be constructed from $S_1, \ldots, S_{m_1}$ by another inductive procedure. Let us denote by F_{M_1} the positive face of $B(l^1_{M_1})$, i.e.

$$F_{M_1} := \{\lambda = (\lambda_\omega)_{\omega \in \Omega_1} : \lambda_\omega \geqslant 0 \text{ and } \sum_{\omega \in \Omega_1} \lambda_\omega = 1\},$$

and let us determine a $\frac{1}{2}$-net

$$\{\lambda^{1,1}, \ldots, \lambda^{1,p_1}\} \text{ of } F_{M_1} \quad (\lambda^{1,i} = (\lambda^i_\omega)_{\omega \in \Omega_1};\ i=1,\ldots,p_1).$$

Next let us put

$$\tau(\lambda^{1,1}) := \inf \operatorname{diam} \sum_{\omega\in\Omega_1} \lambda^1_\omega R_\omega ,$$

where the infimum is taken over all M_1-tuples $(R_1, \ldots, R_{M_1})$ of slices R_ω of K such that $R_\omega \subset S_\omega$ $(\omega\in\Omega_1)$. In virtue of Prop. 7.12 we may choose slices $S^1_\omega \subset S_\omega$ $(\omega\in\Omega_1)$ so that

$$\operatorname{diam}\Big(\sum_{\omega\in\Omega_1} \lambda^1_\omega S^1_\omega\Big) < \tau(\lambda^{1,1}) + 2^{-1}$$

and

$$d(x,E_1) > \frac{1}{2}\tau(\lambda^{1,1}) - 2^{-1} \text{ for all } x \in \sum_{\omega\in\Omega_1} \lambda^1_\omega S^1_\omega .$$

After this we repeat this procedure to define slices $S^2_\omega \subset S^1_\omega$ $(\omega\in\Omega_1)$ so that

$$\operatorname{diam}\Big(\sum_{\omega\in\Omega_1} \lambda^2_\omega S^2_\omega\Big) < \tau(\lambda^{1,2}) + 2^{-1}$$

and

$$d(x,E_1) > \frac{1}{2}\tau(\lambda^{1,2}) - 2^{-1} \text{ for all } x \in \sum_{\omega\in\Omega_1} \lambda^2_\omega S^2_\omega ,$$

where now we have put

$$\tau(\lambda^{1,2}) := \inf \operatorname{diam} \sum_{\omega\in\Omega_1} \lambda^2_\omega R_\omega ,$$

the infimum being taken over all $R_\omega \subset S^1_\omega$ $(\omega\in\Omega_1)$. We continue this procedure p_1 times, arriving at slices $S^{p_1}_\omega$ $(\omega\in\Omega_1)$. These will be our final choice for the elements of the first level of the bush we are constructing:

$$T_\omega := S^{p_1}_\omega \quad (\omega\in\Omega_1).$$

We now describe the inductive procedure in general. We shall construct a sequence (m_n) of natural numbers, and (putting $\Omega_n := \{1, \ldots, m_1\} \times \{1, \ldots, m_2\} \times \cdots \times \{1, \ldots, m_n\}$, and $M_n :=$ card $\Omega_n = m_1 \cdots m_n$, $F_{M_n} :=$ positive face of $B(l^1_{M_n})$), for each $n\in\mathbb{N}$

a 2^{-n}-*net* $\{\lambda^{n,1}, \ldots, \lambda^{n,p_n}\}$ of F_{M_n} $(\lambda^{n,i} = (\lambda^i_\omega)_{\omega\in\Omega_n}; i = 1, \ldots, p_n)$

and for each $\omega\in\Omega_n$ slices

$S_\omega := S^0_\omega \supset S^1_\omega \supset \cdots \supset S^{p_n}_\omega =: T_\omega$ of K such that

$$\overline{S}_{(\omega',j)} \subset T_{\omega'} \quad (\omega'\in\Omega_{n-1};\ j = 1, \ldots, m_n), \tag{16}$$

$$\operatorname{diam} \frac{1}{m_n} \sum_{j=1}^{m_n} S_{(\omega',j)} < 2^{-n} \quad (\omega'\in\Omega_{n-1}), \tag{17}$$

$$\operatorname{diam} \sum_{\omega\in\Omega_n} \lambda^i_\omega S^i_\omega < \tau(\lambda^{n,i}) + 2^{-n} \text{ and} \tag{18}$$

$$(i=1,\ldots,p_n)$$
$$d(x, E_n) > \frac{1}{2}\tau(\lambda^{n,i}) - 2^{-n} \text{ for all } x \in \sum_{\omega\in\Omega_n} \lambda_\omega^i S_\omega^i, \tag{19}$$

where

$$\tau(\lambda^{n,i}) := \inf \operatorname{diam} \sum_{\omega\in\Omega_n} \lambda_\omega^i R_\omega,$$

the infimum being taken over all slices $R_\omega \subset S_\omega^{i-1}$ $(\omega\in\Omega_n)$.

We have already seen how m_1, Ω_1, M_1, $\{\lambda^{1,1},\ldots,\lambda^{1,p_1}\}$ and $S_\omega = S_\omega^0 \supset S_\omega^1 \supset \cdots \supset S_\omega^{p_1} =: T_\omega$ $(\omega\in\Omega_1)$ are defined. Suppose now that the construction has been carried out for $1,2,\ldots,n-1$. Then first, using Lemma 7.5, we determine $m_n \in \mathbb{N}$ and for each $\omega' \in \Omega_{n-1}$ slices $S_{(\omega',j)} = S^0_{(\omega',j)}$, $j=1,\ldots,m_n$ so that

$$\overline{S}_{(\omega',j)} \subset T_{\omega'} \quad (j=1,\ldots,m_n)$$

and

$$\operatorname{diam} \frac{1}{m_n} \sum_{j=1}^{m_n} S_{(\omega',j)} < 2^{-n}$$

(observe that m_n is chosen independently of ω'; this is clearly possible). Next we put

$$\Omega_n := \Omega_{n-1} \times \{1,\ldots,m_n\} \text{ and } M_n := \operatorname{card}\Omega_n = m_1 \ldots\ldots m_n$$

and choose a 2^{-n}-net $\{\lambda^{n,1},\ldots,\lambda^{n,p_n}\}$ for F_{M_n} $(\lambda^{n,i} = (\lambda_\omega^i)_{\omega\in\Omega_n}$, $i = 1,\ldots,p_n)$. Putting

$$\tau(\lambda^{n,1}) := \inf\{\operatorname{diam} \sum_{\omega\in\Omega_n} \lambda_\omega^1 R_\omega : R_\omega \subset S_\omega\}$$

we now select slices $S_\omega^1 \subset S_\omega$ $(\omega\in\Omega_n)$ so that

$$\operatorname{diam} \sum_{\omega\in\Omega_n} \lambda_\omega^1 S_\omega^1 < \tau(\lambda^{n,1}) + 2^{-n}$$

and

$$d(x, E_n) > \frac{1}{2}\tau(\lambda^{n,1}) - 2^{-n} \text{ for all } x \in \sum_{\omega\in\Omega_n} \lambda_\omega^1 S_\omega^1$$

(using Prop. 7.12).

Suppose now that $S_\omega^0 \supset S_\omega^1 \supset \cdots \supset S_\omega^i$ $(\omega\in\Omega_n)$ have been selected as stated, for some i, $1 \leqslant i < p_n$. To define S_ω^{i+1} $(\omega\in\Omega_n)$ put

$$\tau(\lambda^{n,i+1}) := \inf\{\operatorname{diam} \sum_{\omega\in\Omega_n} \lambda_\omega^{i+1} R_\omega : R_\omega \subset S_\omega^i\}$$

and use Prop. 7.12 again to select $S_\omega^{i+1} \subset S_\omega^i$ $(\omega\in\Omega_n)$ so that

$$\operatorname{diam} \sum_{\omega\in\Omega_n} \lambda_\omega^{i+1} S_\omega^{i+1} < \tau(\lambda^{n,i+1}) + 2^{-n}$$

and

$$d(x,E_n) > \frac{1}{2}\tau(\lambda^{n,i+1}) - 2^{-n} \text{ for all } x \in \sum_{\omega\in\Omega_n} \lambda_\omega^{i+1} S_\omega^{i+1}.$$

This completes the definition of S_ω^i ($\omega\in\Omega_n$, $i=1,\ldots,p_n$). Finally we put $T_\omega := S_\omega^{p_n}$ ($\omega\in\Omega_n$) and the construction is finished up to n.

STEP 2 We shall now define the operator $T:L^1\to X$. It will be convenient to represent the measure space $([0,1],\lambda)$ in the following manner. Let $\Delta := \prod_{n=1}^{\infty}\{1,\ldots,m_n\}$. On each set $\{1,\ldots,m_n\}$ consider the measure that assigns weight $\frac{1}{m_n}$ to every point and let m_Δ on Δ be the product measure. Needless to say the completion of (Δ,m_Δ) is measure algebra isomorphic to $([0,1],\lambda)$. If for each $\omega=(k_1,\ldots,k_n)\in\Omega_n$ we define

$$\Delta_\omega := \{(n_j)_{j=1}^\infty \in\Delta : n_j=k_j \text{ for } j=1,\ldots,n\},$$

then clearly the functions χ_{Δ_ω} ($\omega\in\Omega := \bigcup_{n=1}^{\infty}\Omega_n$) span a dense subspace of $L^1(m_\Delta)$. So it suffices to define T on this subspace. This we do by selecting arbitrary elements $x_\omega\in T_\omega$ ($\omega\in\Omega$) and putting

$$T(M_n\chi_{\Delta_\omega}) := \lim_{k\to\infty} M_{n+k}^{-1} M_n \sum_{\psi\in\Omega_{n+k}^\omega} x_\psi, \quad (n\in\mathbb{N}\ \omega\in\Omega_n), \tag{20}$$

where Ω_{n+k}^ω denotes the set of all elements of Ω_{n+k} whose first n coordinates equal those of ω.

There are two points that need to be checked. The first is that for each $\omega\in\Omega$ the limit of the sequence in (20) exists. To see this we fix $n\in\mathbb{N}$ and $\omega\in\Omega_n$ and we compare two successive terms:

$$M_{n+k+2}^{-1}M_n \sum_{\psi\in\Omega_{n+k+2}^\omega} x_\psi = M_{n+k}^{-1}M_n \sum_{\phi\in\Omega_{n+k}^\omega} \left[\frac{1}{m_{n+k+1}}\sum_{i=1}^{m_{n+k+1}}\left(\frac{1}{m_{n+k+2}}\sum_{j=1}^{m_{n+k+2}} x_{(\phi,i,j)}\right)\right],$$

$$M_{n+k+1}^{-1}M_n \sum_{\rho\in\Omega_{n+k+1}^\omega} x_\rho = M_{n+k}^{-1}M_n \sum_{\phi\in\Omega_{n+k}^\omega} \left(\frac{1}{m_{n+k+1}}\sum_{i=1}^{m_{n+k+1}} x_{(\phi,i)}\right).$$

Now observe that (16) implies that $\overline{T}_{(\omega,j)}\subset T_\omega$ ($\omega\in\Omega_{n-1}, j=1,\ldots,m_n$), so

$$\frac{1}{m_{n+k+1}}\sum_{i=1}^{m_{n+k+1}}\left(\frac{1}{m_{n+k+2}}\sum_{j=1}^{m_{n+k+2}} x_{(\phi,i,j)}\right), \frac{1}{m_{n+k+1}}\sum_{i=1}^{m_{n+k+1}} x_{(\phi,i)} \in \frac{1}{m_{n+k+1}}\sum_{i=1}^{m_{n+k+1}} T_{(\phi,i)}$$

for every $\phi\in\Omega_{n+k}^\omega$. Since diam $\frac{1}{m_{n+k+1}}\sum_{i=1}^{m_{n+k+1}} T_{(\phi,i)} < 2^{-(n+k+1)}$ by (17), and card $\Omega_{n+k}^\omega = M_{n+k}^{-1}M_n$, it follows that

$$\|M_{n+k+2}^{-1}M_n \sum_{\psi\in\Omega_{n+k+2}^\omega} x_\psi - M_{n+k+1}^{-1}M_n \sum_{\rho\in\Omega_{n+k+1}^\omega} x_\rho\| < 2^{-(n+k+1)},$$

and so the existence of the limit in (20) follows.

The second point that needs to be checked is that the definition (20) is "consistent", i.e. that for every $\omega \in \Omega_n$,

$$T(M_n \chi_{\Delta_\omega}) = \frac{1}{m_{n+1}} \sum_{j=1}^{m_{n+1}} T(M_{n+1}\, \chi_{\Delta_{(\omega,j)}}).$$

Written out, this means

$$\lim_{k\to\infty} M_{n+k}^{-1} \sum_{\psi\in\Omega^{\omega}_{n+k}} x_\psi = \frac{1}{m_{n+1}} \sum_{j=1}^{m_{n+1}} \left[\lim_{k\to\infty} M_{n+k+1}^{-1}\, M_{n+1} \sum_{\phi\in\Omega^{(\omega,j)}_{n+k+1}} x_\phi\right]$$

But this is clearly true, because we have just checked that the limits involved exist, and because

$$\frac{1}{m_{n+1}} \sum_{j=1}^{m_{n+1}} M_{n+k+1}^{-1}\, M_{n+1} \sum_{\phi\in\Omega^{(\omega,j)}_{n+k+1}} x_\phi = M_{n+k+1}^{-1}\, M_n \sum_{\psi\in\Omega^{\omega}_{n+k+1}} x_\psi$$

It is now evident that (20) can be extended to define a bounded linear operator $T:L^1(m_\Delta)\to X$ with norm $\|T\|\leqslant 1$ (this is because $K\subset B(X)$ by assumption).

Let us finally take note of the crucial fact that the formula $\overline{T}_{(\omega',j)}\subset T_{\omega'}$ ($\omega'\in\Omega_{n-1}$, $j=1,\ldots,m_n$), and definition (20) of T imply that

$$T\,F_\omega \subset T_\omega, \text{ hence } T^{**}\,\tilde{F}_\omega \subset \tilde{T}_\omega \quad (\omega\in\Omega), \tag{21}$$

where $F_\omega := \{f\in F : \int_{\Delta_\omega} f d\mu = 1\}$ and

$$\tilde{F}_\omega = w^*\, cl\, F_\omega = \{\mu\in \tilde{F} : \mu(\Delta_\omega) = 1\}.$$

Step 3 We now show that T satisfies the requirements (i),(ii).

Proof of (ii): Let us fix $\mu\in\tilde{F}$ and $n\in\mathbb{N}$ and consider the partition $P_n := \{\Delta_\omega : \omega\in\Omega_n\}$ of Δ. By the construction in Step 1 there exists a $\lambda^{n,i}\in F_{M_n}$ (for some i, $1\leqslant i\leqslant p_n$) so that

$$\sum_{\omega\in\Omega_n} |\mu(\Delta_\omega) - \lambda^i_\omega| \leqslant 2^{-n}. \tag{22}$$

Note that (21) implies

$$TV_{P_n,0}(\mu) \subset \sum_{\omega\in\Omega_n} \mu(\Delta_\omega)\, T_\omega \subset \sum_{\omega\in\Omega_n} \mu(\Delta_\omega)\, S^i_\omega\,. \tag{23}$$

Recall now from Lemma 7.10 that

$$V_{P_n,2^{-n}}(\mu) = [V_{P_n,0}(\mu) + 2^{-n}\, B(L^\infty(m_\Delta)^*)] \cap F. \tag{24}$$

From (22), (23) and (24) we obtain

$$TV_{P_n,2^{-n}}(\mu) \subset \sum_{\omega\in\Omega_n} \mu(\Delta_\omega) S^i_\omega \; + 2^{-n}\, B(X)$$

$$\subset \sum_{\omega \in \Omega_n} \lambda^i_\omega S^i_\omega + 2.2^{-n} B(X). \tag{25}$$

Hence (18) implies

$$\operatorname{diam} TV_{P_n,2^{-n}}(\mu) \leqslant \tau(\lambda^{n,i}) + 5.2^{-n},$$

and so a fortiori

$$\rho_T(\mu) \leqslant \tau(\lambda^{n,i}) + 5.2^{-n}. \tag{26}$$

On the other hand it follows from (19) and (25) that

$$d(x,E_n) \geqslant \frac{1}{2}\tau(\lambda^{n,i}) - 3.2^{-n} \text{ for all } x \in TV_{P_n,2^{-n}}(\mu)$$

and therefore also, by the convexity of $TV_{P_n,2^{-n}}(\mu)$,

$$d(x^{**},E_n) \geqslant \frac{1}{2}\tau(\lambda^{n,i}) - 3.2^{-n} \text{ for all } x^{**} \in (TV_{P_n,2^{-n}}(\mu))^{\sim}. \tag{27}$$

Now since $n \in \mathbb{N}$ was arbitrary and (x_k) is dense in X, (26) and (27) yield, since $T^{**}\mu \in \bigcap_{n=1}^{\infty} (TV_{P_n,2^{-n}}(\mu))^{\sim}$, that

$$d_T(\mu) \geqslant \frac{1}{2}\rho_T(\mu).$$

Proof of (i): Let $x^{**} \in \operatorname{ext} \tilde{C} = \operatorname{ext} T^{**}\tilde{F}$. Fix $n \in \mathbb{N}$. We claim that

$$x^{**} = T^{**}\mu \text{ for some } \mu \in \tilde{F}_{\omega_0}, \text{ where } \omega_0 \in \Omega_n.$$

Indeed, observe that the sets $\tilde{F}_\omega$ $(\omega \in \Omega_n)$ are w^*-compact and convex and that $\tilde{F} = co\{\tilde{F}_\omega : \omega \in \Omega_n\}$. Hence

$$\tilde{C} = T^{**}\tilde{F} = co\{T^{**}\tilde{F}_\omega : \omega \in \Omega_n\}.$$

x^{**} being extreme in $\tilde{C}$, we must then have $x^{**} = T^{**}\mu$ with $\mu \in \tilde{F}_{\omega_0}$ for some $\omega_0 \in \Omega_n$, proving our claim.

Next, using the construction in Step 1 again, we can select a $\lambda^{n,i}$ with $1 \leqslant i \leqslant p_n$ such that

$$\sum_{\omega \in \Omega_n} |\mu(\Delta_\omega) - \lambda^i_\omega| \leqslant 2^{-n}.$$

It then follows that $\lambda^i_{\omega_0} \geqslant 1 - 2^{-n}$, since $\mu(\Delta_{\omega_0}) = 1$. Every slice of K having diameter exceeding 2α, the definition of $\tau(\lambda^{n,i})$ now leads to the conclusion that

$$\tau(\lambda^{n,i}) > (1 - 2^{-n})2\alpha.$$

Exactly as in the proof of (ii) above, we now obtain

$$d(x^{**},E_n) > \frac{1}{2}\tau(\lambda^{n,i}) - 3.2^{-n} \text{ for all } x^{**} \in (TV_{P_n,2^{-n}}(\mu))^{\sim}.$$

Therefore, $n \in \mathbb{N}$ being arbitrary, it follows that

$$d(x^{**},X) \geqslant \alpha$$

and the proof is finished. □

COROLLARY 7.14 *If X is strongly regular, then X has the RNP iff it has the KMP. Moreover, in this case the KMP is separably determined, i.e. X has the KMP iff every separable subspace of X has the KMP. (This is unknown in general.)*

PROOF. Since the RNP implies the KMP for every Banach space, it suffices to prove that if X fails the RNP then there exists a separable subspace of X that fails the KMP. Now the RNP is well known to be separably determined, so there exists a separable $Y \subset X$ without the RNP. Of course Y is still strongly regular. It is now immediate from Prop. 7.11 and Theorem 7.13 (applied to Y) that the set $C \subset Y$ (see Prop. 7.11) has no extreme points. Thus Y fails the KMP. □

COROLLARY 7.15. *Let X^* be any dual space. Then X^* has the RNP iff X^* has the KMP. Again, the KMP is separably determined.*

PROOF. We distinguish two cases.
CASE I: l^1 does not embed in X. Then by Corollary 7.9 X^* is strongly regular, so the preceding Corollary applies.
CASE II: l^1 embeds in X. Then by Prop. 4.2 ((1) $\Leftrightarrow$ (8)) L^1 embeds in X^*. It is well known and easy to prove that the positive face F of $B(L^1)$ has no extreme points. So L^1 fails the KMP. But then L^1 also fails the RNP (we have even proved in Cor. 7.9 that F has no small combinations of slices!). Since L^1 is separable and both the KMP and the RNP are isomorphic invariants and are inherited by subspaces, we have now proved the assertion in case II: X^* fails both the RNP and the KMP, and the KMP fails even for a separable subspace. □

Let us recall that X is called an *Asplund space* if every separable subspace $Y \subset X$ has a separable dual. It is known that X is Asplund iff X^* has the RNP. The table below summarizes the situation for dual spaces. The class of spaces which are not Asplund but in which l^1 does not embed, includes such celebrated examples as the James tree space JT, the James function space JF, and the James-Hagler space JH. We already know a great deal about this class of spaces, e.g. that its elements are characterized by weakened forms of the KMP and the RNP. However, we have as yet discussed no concrete example. The next chapter will fill this gap.

X^*

	KMP	RNP	KMP for w^*compact convex sets	WRNP
X Asplund	+	+	+	+
$l^1 \not\subset X$ and X not Asplund	-	-	+	+
$l^1 \subset X$	-	-	-	-

NOTES The concept of strong regularity, although first defined explicitly by W. SCHACHERMAYER in [83], was already implicit in the work of J. BOURGAIN ([6], [7]). In fact the main result in § 1, that X^* is strongly regular iff $l^1 \not\subset X$, can already be found in [6]. All of § 2 is due to W. SCHACHERMAYER ([83]). Corollary 7.15 is an old result of R. HUFF and P.D. MORRIS ([42]). Of course their original proof is quite different. Recently H.P. ROSENTHAL ([76]) has given an integrated presentation of the work of J. BOURGAIN in [7] and that of W. SCHACHERMAYER in [83]. Some of the preliminary lemmas in § 1 were taken from [76]. Finally let us observe that Cor. 7.14 reduces the famous (and still open) problem whether the KMP implies the RNP to the question whether the KMP implies strong regularity.

Chapter VIII

The James tree space

In this final chapter we settle a point that was left open so far, namely we show that for dual spaces the weakened forms of the RNP and the KMP we have discussed previously (see the diagram on p. 103) are indeed different from the original ones. We do this by giving a concrete example of a separable Banach space whose dual is non-separable, but in which l^1 does not embed. There are several such examples and none of them is simple. We choose here the so-called James tree space JT, because the analysis of this space is easiest.

The James tree space is rooted in the classical James space J. Therefore we first recall the definition of J and its properties as far as we need them. The space J consists of all real sequences $x=(x_j)$ such that

$$\sup\left[\sum_{i=1}^{n}\left[\sum_{j=k_i}^{k_{i+1}-1} x_j\right]^2\right]^{1/2}$$

is finite, where the sup is taken over all increasing finite sequences $k_1<k_2<\cdots<k_{n+1}$ in $\mathbb{N}$. The norm $\|x\|$ is defined to be this supremum. The completeness of J is readily verified by standard arguments. Furthermore the unit vectors (e_n) form a monotone boundedly complete basis. If (e_n^*) denotes the corresponding sequence of coefficient functionals, then $J^*=[e_n^*]_{n=1}^{\infty}\oplus\mathbb{R}e^*$, where e^* is defined by $\langle x,e^*\rangle:=\sum\limits_{j=1}^{\infty}x_j$ $(x\in J)$. Note that $|\sum\limits_{j=n}^{n+m}x_j|\to 0$ as $n\to\infty$, by the finiteness of $\|x\|$, so that $\sum\limits_{j=1}^{\infty}x_j$ converges. Also $|\sum\limits_{j=1}^{n}x_j|\leqslant\|x\|$ for all $n\in\mathbb{N}$, and therefore $\|e^*\|\leqslant 1$. In fact $\|e^*\|=1$ because $\langle e_n,e^*\rangle=1=\|e_n\|$ $(n\in\mathbb{N})$. Since $\langle e_n,e^*\rangle=1$ for all $n\in\mathbb{N}$ and $\lim\limits_{n\to\infty}\langle e_n,e_m^*\rangle=0$ for all $m\in\mathbb{N}$, it follows that

$$\lim_{n\to\infty}\langle e_n,x^*\rangle \text{ exists for every } x^*\in J^*=[e_n^*]_{n=1}^{\infty}\oplus\mathbb{R}e^*$$

and that

$$\lim_{n\to\infty}\langle e_n,x^*\rangle=0 \ \text{ iff } \ x^*\in[e_n^*]_{n=1}^{\infty} \quad (x^*\in J^*). \tag{1}$$

We now come to JT, the James tree space. Put

$$T:=\{(n,k):n=0,1,\ldots;\ k=1,\ldots,2^n\}.$$

If one thinks of the elements of T as arranged in the following pattern,

$$\begin{tikzcd}[arrows=dash] & & & (0,1) \arrow[dll] \arrow[drr] & & & \\ & (1,1) \arrow[dl] \arrow[dr] & & & & (1,2) \arrow[dl] \arrow[dr] & \\ (2,1) & & (2,2) & & (2,3) & & (2,4) \end{tikzcd}$$

it will be clear why we call T a *tree*. This arrangement also suggests a natural partial order on T. A point $(n,k)\in T$ has two *immediate successors*, namely $(n+1,2k-1)$ and $(n+1,2k)$. We now say that $p<q$ $(p,q\in T)$ iff there are points $p_1,\ldots,p_m\in T$ so that $p_1=p$, $p_m=q$ and p_{j+1} is an immediate successor of p_j $(j=1,\ldots,m-1)$. Any such finite sequence $\{p_1,\ldots,p_m\}$ is called a *segment* of T. So p and q are comparable iff they lie in a common segment. A *branch* will be a maximal totally ordered subset of T. Evidently a branch is of the form

$$\{(0,1),(1,k_1),(2,k_2),\ldots\ldots,(n,k_n),\ldots\ldots\}$$

where $k_n\in\{2k_{n-1}-1,2k_{n-1}\}$ $(n=1,2,..;\ k_0:=1)$. Observe that the set of all branches has cardinality c. Sometimes we shall call the set $\{(n,1),\ldots,(n,2^n)\}$ the n^{th} *level* of the tree and shall refer to points (m,k) with $m<n$, resp. $m>n$ as lying *above* resp. *below* the n^{th} *level.*

Now let JT be the set of all real functions x on T such that

$$\|x\|:=\sup\left[\sum_{j=1}^{l}\left[\sum_{(n,k)\in S_j}x(n,k)\right]^2\right]^{1/2}<\infty,$$

where the sup is taken over all $l\in\mathbb{N}$ and all sets of pairwise disjoint segments $S_1,\ldots,S_l$. The proof that $\|\cdot\|$ is a norm and that JT equipped with this norm is complete, is standard, so we omit it. Note that $JT\subset c_0(T)$.

Another easily verified fact is that the unit vectors $e_{n,k}\in JT$, defined by $e_{n,k}(n',k'):=\delta_{n,n'}\ \delta_{k,k'}$ $((n,k),(n',k')\in T)$, form a normalized monotone boundedly complete basis of JT, when enumerated in lexicographic order. By a well-known result JT is then isometrically isomorphic to the dual of the closed linear span in JT^* of the corresponding sequence of coefficient functionals $e^*_{n,k}\in JT^*$. Thus $JT\cong Y^*$, where $Y:=[e^*_{n,k}]_{(n,k)\in T}\subset JT^*$. Hence $JT^*\cong Y^{**}$. One should also note that under this last identification the identity embedding $Y\subset JT^*$ corresponds to π_Y, the canonical embedding of Y in Y^{**}.

Using the basis $(e_{n,k})_{(n,k)\in T}$ (lexicographically ordered) we shall often represent elements $x\in JT$ as

$$x=\sum_{n=0}^{\infty}\sum_{k=1}^{2^n}t_{n,k}\,e_{n,k},\ \text{with}\ t_{n,k}\in\mathbb{R}.$$

We now discuss several types of projections on subspaces of JT. All these projections are built alike, so we first explain the general principle involved here. Let A be any subset of T and let us define P_A on the span of the $e_{n,k}$ by

$$P_A\left[\sum_n \sum_{k=1}^{2^n} t_{n,k}\, e_{n,k}\right] := \sum_n \sum_{(n,k)\in A} t_{n,k}\, e_{n,k}$$

Clearly $P_A^2 = P_A$, but generally P_A is not bounded. E.g. suppose A intersects a segment

$$S = \{(n,k_n), (n+1,k_{n+1}), \ldots, (n+2p, k_{n+2p})\}.$$

in the "odd" points:

$$A \cap S = \{(n+2i-1, k_{n+2i-1}) : i = 1, \ldots, p\}.$$

Define $x = \sum_{n'=1}^{\infty} \sum_{k'=1}^{2^{n'}} t_{n',k'}\, e_{n',k'}$ by

$$t_{n',k'} = \begin{cases} -1 & \text{if } (n',k') \in S \setminus A, \\ 1 & \text{if } (n',k') \in S \cap A, \\ 0 & \text{if } (n',k') \notin S. \end{cases}$$

It is now a simple matter to verify that $\|x\| = (2p+1)^{1/2}$, but $\|P_A x\| = p$. Now A may clearly be so chosen that the above situation occurs at different locations in T for arbitrarily large p. Then P_A is unbounded.

However, if A is such that for every segment S the intersection $A \cap S$ is again a segment, then P_A has norm 1. This is so because for every finite number of pairwise disjoint segments $S_1, \ldots, S_l$, and for every $x \in JT$,

$$\sum_{j=1}^{l} \left[\sum_{(n,k)\in S_j} (P_A x)(n,k)\right]^2 = \sum_{j=1}^{l} \left[\sum_{(n,k)\in A\cap S_j} x(n,k)\right]^2 \leqslant \|x\|^2.$$

We list now several special cases where the condition that A intersects segments in segments is obviously satisfied. Therefore all projections below have norm 1.

(i) for each fixed $(n,k) \in T$ the map

$$x = \sum_{n'=0}^{\infty} \sum_{k'=1}^{2^{n'}} t_{n',k'}\, e_{n',k'} \longrightarrow t_{n,k}\, e_{n,k}$$

is a norm 1 projection. An immediate consequence is that $\|e_{n,k}^*\| = 1$ (since $\|e_{n,k}\| = 1$).

Writing $x = \sum_{n'=0}^{\infty} \sum_{k'=1}^{2^{n'}} t_{n',k'}\, e_{n',k'}$ for the generic element of JT, the following formulas define norm 1 projections:

(ii) $P_n x := \sum_{n'=n}^{\infty} \sum_{k'=1}^{2^{n'}} t_{n',k'}\, e_{n',k'} \quad (n = 0, 1, \ldots)$,

(iii) $P_{n,k} x := \sum_{n'=1}^{\infty} \sum_{(n',k') \geqslant (n,k)} t_{n',k'}\, e_{n',k'} \quad ((n,k) \in T)$,

(iv) $P_B x := \sum_{(n',k')\in B} t_{n',k'}\, e_{n',k'}$.

Here B is any branch of T, and the summation is in the order B inherits from T. It is important to note that also the projections $I-P_n$, $I-P_{n,k}$ and $I-P_B$ have norm 1 (on the basis of the same general principle). It is also an interesting fact that for each $(n,k)\in T$ the range $P_{n,k}JT$ is isometric to JT, while for each branch B the range P_BJT is isometric to the classical James space J. (The maps that realize these isometries are the obvious ones.)

We now wish to show that for every $x\in JT$ and every $n\in\mathbb{N}$,

$$\|P_n x\|^2 = \sum_{k=1}^{2^n} \|P_{n,k}x\|^2. \tag{2}$$

Again, the reason for this is best explained in more general terms.

LEMMA 8.1. *Let finitely many finite subsets $A_1,\dots,A_p$ of T be given subject to the condition (which implies disjointness) that each segment S of T intersects at most one A_i (not necessarily in a segment). Put $A := \bigcup_{i=1}^{p} A_i$. Then for every choice of scalars $t_{n,k}$ we have*

$$\Big\|\sum_{(n,k)\in A} t_{n,k}\, e_{n,k}\Big\|^2 = \sum_{i=1}^{p} \Big\|\sum_{(n,k)\in A_i} t_{n,k}\, e_{n,k}\Big\|^2. \tag{3}$$

PROOF. Let $S_1,\dots,S_q$ be any pairwise disjoint segments. Then by assumption for every $j\in\{1,\dots,q\}$ there exists an $i(j)\in\{1,\dots,p\}$ so that $A\cap S_j = A_{i(j)}\cap S_j$, hence $A_i\cap S_j=\varnothing$ for $i\neq i(j)$. From this it follows that

$$\sum_{j=1}^{q}\left[\sum_{(n,k)\in A\cap S_j} t_{n,k}\right]^2 = \sum_{j=1}^{q}\sum_{i=1}^{p}\left[\sum_{(n,k)\in A_i\cap S_j} t_{n,k}\right]^2 \leqslant$$
$$\leqslant \sum_{i=1}^{p}\Big\|\sum_{(n,k)\in A_i} t_{n,k}\, e_{n,k}\Big\|^2,$$

where as usual a sum over an empty set is to be interpreted as 0. This proves $\leqslant$ in (3).

To see the converse, for each $i\in\{1,\dots,p\}$ let $S_1^i,\dots,S_{q_i}^i$ be pairwise disjoint segments that intersect A_i (and therefore are disjoint with $A\setminus A_i$, i.e. $A\cap S_j^i = A_i\cap S_j^i$, $j=1,\dots,q_i$). Let us consider a pair of segments $S_{j_1}^{i_1}$, $S_{j_2}^{i_2}$ $(1\leqslant j_1\leqslant q_{i_1};\ 1\leqslant j_2\leqslant q_{i_2})$ with $i_1\neq i_2$. Now the intersection $S := S_{j_1}^{i_1}\cap S_{j_2}^{i_2}$ may be non-empty but $A\cap S=\varnothing$, since otherwise at least one of $S_{j_1}^{i_1}$, $S_{j_2}^{i_2}$ would intersect more that one A_i. Now either $S_{j_1}^{i_1}\setminus S$ or $S_{j_2}^{i_2}\setminus S$ (or both) is a segment, as one easily sees. Suppose $S_{j_1}^{i_1}\setminus S$ is. Then we may replace $S_{j_1}^{i_1}$ by $S_{j_1}^{i_1}\setminus S$, to get disjointness with $S_{j_2}^{i_2}$. Repeating this argument a finite number of times, we arrive at pairwise disjoint segments $\tilde S_j^i$ $(i=1,\dots,p;\ j=1,\dots,q_i)$ which have the same intersection with A as the original S_j^i, so that in particular $A\cap\tilde S_j^i = A_i\cap\tilde S_j^i$ $(i=1,\dots,p;\ j=1,\dots,q_i)$. Now we get

$$\sum_{i=1}^{p}\sum_{j=1}^{q_i}\left[\sum_{(n,k)\in A_i\cap S_j^i} t_{n,k}\right]^2=\sum_{i=1}^{p}\sum_{j=1}^{q_i}\left[\sum_{(n,k)\in A_i\cap \tilde{S}_j^i} t_{n,k}\right]^2=$$

$$=\sum_{i=1}^{p}\sum_{j=1}^{q_i}\left[\sum_{(n,k)\in A\cap \tilde{S}_j^i} t_{n,k}\right]^2\leqslant\|\sum_{(n,k)\in A} t_{n,k}\,e_{n,k}\|^2.$$

This proves $\geqslant$ in (3) and we are done. □

Evidently (2) is an immediate consequence of Lemma 8.1, since the sets $\{(n',k'):(n',k')\geqslant(n,k)\}$, $k=1,\ldots,2^n$, satisfy the condition.

We are now going to define a map V on JT^* which is the key to the structure theorem we are about to prove. Let $x^*\in JT^*$ and let B be any branch of T. Recall that P_BJT is isometric to the classical space J and that by our earlier remarks on J, $\lim_{n\to\infty}\langle e_n,y^*\rangle$ exists for each $y^*\in J^*$ (where (e_n) denotes the unit vector basis of J), and vanishes iff $y^*\in[e_n^*]_{n=1}^{\infty}$ (see (1)). It follows from these facts that

$$(Vx^*)(B):=\lim_{(n,k)\in B}\langle e_{n,k},x^*\rangle \text{ exists for every } x^*\in JT^* \text{ and every branch } B. \tag{4}$$

Denoting by Γ the set of all branches B, (4) then defines a map $V:JT^*\to\mathbb{R}^{\Gamma}$. Clearly V is linear. But much more is true:

THEOREM 8.2. *The operator V defined by (4) is a quotient map from JT^* onto $l^2(\Gamma)$. Furthermore, ker $V=Y(:=[e_{n,k}^*])$. Hence $l_2(\Gamma)\cong Y^{**}/\pi_Y Y$.*

PROOF. We break up the proof in several assertions that we shall state as we go along. The first one is easy.

(i) *V maps JT^* into $l^2(\Gamma)$ and its norm as an element of $L(JT^*,l^2(\Gamma))$ is $\leqslant 1$.*

PROOF OF (i): Let $B_1,\ldots,B_q$ be any finite number of distinct branches. Then for some $n\in\mathbb{N}$ these branches do not intersect on or below level n, i.e. the sets $B_p\cap\{(n',k'): n'\geqslant n\}$ $(p=1,\ldots,q)$ are pairwise disjoint. Now pick any elements $(n_p,k_p)\in B_p$ with $n_p\geqslant n$ $(p=1,\ldots,q)$. Then by Lemma 8.1 we have

$$\|\sum_{p=1}^{q} t_p\,e_{n_p,k_p}\|^2=\sum_{p=1}^{q} t_p^2 \text{ for all } t_1,\ldots,t_q\in\mathbb{R}.$$

This means that $[e_{n_p,k_p}]_{p=1}^{q}$ is a q-dimensional Hilbert space with orthonormal basis $\{e_{n_p,k_p}\}_{p=1}^{q}$. Hence

$$\sum_{p=1}^{q}\langle e_{n_p,k_p},x^*\rangle^2=\|x^*|_{[e_{n_p,k_p}]}\|^2\leqslant\|x^*\|^2 \text{ for every } x^*\in JT^*. \tag{5}$$

Now for each p let (n_p,k_p) tend to infinity along the branch B_p. Then, taking limits in (5) we get, by the definition of V,

$$\sum_{p=1}^{q}(Vx^*)(B_p)^2\leqslant\|x^*\|^2 \qquad (x^*\in JT^*).$$

Since this inequality is true for any choice of distinct branches $B_1, \ldots, B_q$, we have shown that $Vx^* \in l^2(\Gamma)$ and $\|Vx^*\| \leqslant \|x^*\|$ $(x^* \in JT^*)$. □

(ii) *V is a quotient map onto $l^2(\Gamma)$.*

Proof of (ii) Since we already know that $\|V\| \leqslant 1$, it suffices, by the open mapping theorem, to show that each finitely supported element of $l^2(\Gamma)$ with norm 1 is of the form Vx^* with $\|x^*\| \leqslant 1$ (actually, then $\|x^*\|=1$). So let $\{B_1, \ldots, B_q\}$ be any finite subset of Γ and let $t_1, \ldots, t_q \in \mathbb{R}$ satisfy $\sum_{p=1}^{q} t_p^2 = 1$. We are looking for an $x^* \in JT^*$, $\|x^*\| \leqslant 1$, so that

$$(Vx^*)(B) = \begin{cases} t_p & \text{if } B = B_p \text{ for some } p \in \{1,\ldots,q\}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Again let us choose $n \in \mathbb{N}$ so that the sets $B_p \cap \{(n',k') : n' \geqslant n\}$ $(p=1, \ldots, q)$ are pairwise disjoint. We now define x^* on the span of the $e_{n,k}$ by

$$\left\langle \sum_{n'=0}^{m} \sum_{k'=1}^{2^{n'}} t_{n',k'}\, e_{n',k'}, x^* \right\rangle := \sum_{p=1}^{q} t_p \left(\sum_{\substack{(n',k') \in B_p \\ n \leqslant n' \leqslant m}} t_{n',k'} \right)$$

The verification of (6) is now straightforward, so all that needs to be checked is that $\|x^*\| \leqslant 1$. But this is immediate:

$$\left| \sum_{p=1}^{q} t_p \left(\sum_{\substack{(n',k') \in B_p \\ n \leqslant n' \leqslant m}} t_{n',k'} \right) \right| \leqslant \underbrace{\left(\sum_{p=1}^{q} t_p^2 \right)^{1/2}}_{=1} \left[\sum_{p=1}^{q} \left(\sum_{\substack{(n',k') \in B_p \\ n \leqslant n' \leqslant m}} t_{n',k'} \right)^2 \right]^{1/2}$$

$$\leqslant \left\| \sum_{n'=0}^{m} \sum_{k'=1}^{2^{n'}} t_{n',k'}\, e_{n',k'} \right\|.$$

The last inequality here is evident from definition of the norm in JT, since the sets $\{(n',k') \in B_p : n \leqslant n' \leqslant m\}$, $p=1, \ldots, q$ are pairwise disjoint segments. □

Now let N be the kernel of V, i.e.

$$N := \{x^* \in JT^* : \lim_{\substack{(n,k) \in B \\ n \to \infty}} \langle e_{n,k}, x^* \rangle = 0 \text{ for all } B \in \Gamma\}.$$

It is clear that $Y \subset N$, by (1). The proof of the reverse inclusion is less obvious. First we need to prove the following fact.

(iii) $\lim_{n \to \infty} \left(\max_{1 \leqslant k \leqslant 2^n} \|P_{n,k}^* x^*\| \right) = 0$ *for every* $x^* \in N$.

Proof of (iii) Suppose not. Then there exist $x^* \in N$, $\epsilon > 0$ and a sequence of distinct elements $(n_i,k_i)\in T$ so that

$$\|P^*_{n_i,k_i}x^*\|>\epsilon \quad (i=1,2,...). \tag{7}$$

We now first prove that the number of mutually incomparable elements among the (n_i,k_i) is bounded. Indeed, let us suppose that (n_i,k_i), $i=1,\ldots,j$ are mutually incomparable and let us use (7) to choose finitely supported elements

$$x_i\in P_{n_i,k_i}JT \text{ with } \|x_i\|=1 \text{ and } \langle x_i,x^*\rangle>\epsilon \quad (i=1,\ldots,j).$$

It is now easily seen that the condition of Lemma 8.1 is satisfied for $A_i:=\operatorname{supp} x_i$ $(i=1,\ldots,j)$. Consequently we may conclude from (3) that

$$\|\sum_{i=1}^{j}x_i\|=j^{1/2}, \text{ so } j\epsilon\leqslant\langle\sum_{i=1}^{j}x_i,x^*\rangle\leqslant\|x^*\|j^{1/2}.$$

This proves that $j\leqslant\dfrac{\|x^*\|^2}{\epsilon^2}$.

It is convenient now to introduce a notation for the set of all successors of an element $(n,k)\in T$. So let us put

$$T(n,k):=\{(n',k')\in T:(n,k)\leqslant(n',k')\}$$

Observe that mutual incomparability of $(n_1,k_1),\ldots,(n_p,k_p)$ means that the sets $T(n_i,k_i)$, $i=1,\ldots,p$ are pairwise disjoint. This is so because the predecessors of any given element from a segment, i.e. a totally ordered set.

Now, returning to the sequence $((n_i,k_i))$ defined as in (7), let $\{(n_{i_1},k_{i_1},\ldots\ldots,(n_{i_p},k_{i_p})\}$ be a system of mutually incomparable elements of maximal cardinality p. Put $n_0:=\max\limits_{j=1,\ldots,p} n_{i_j}$. Then

$$\{(n_i,k_i):n_i\geqslant n_0\}\subset\bigcup_{j=1}^{p}T(n_{i_j},k_{i_j}). \tag{8}$$

Indeed, suppose $(n_i,k_i)\notin\bigcup_{j=1}^{p}T(n_{i_j},k_{i_j})$ and $n_i\geqslant n_0$. Then $\{(n_i,k_i),(n_{i_1},k_{i_1},\ldots,(n_{i_p},k_{i_p})\}$ would consist of mutually incomparable elements (by the above criterion), contradicting the maximality of p. Now clearly (8) implies that for some $j\in\{1,\ldots,p\}$, $T(n_{i_j},k_{i_j})$ intersects the sequence $((n_i,k_i))$ in an infinite set, i.e. $\{(n_i,k_i):(n_i,k_i)\geqslant(n_{i_j},k_{i_j})\}$ is infinite. We claim that it is also totally ordered. For otherwise replacing (n_{i_j},k_{i_j}) in $\{(n_{i_1},k_{i_1},\ldots,(n_{i_p},k_{i_p})\}$ by two mutually incomparable successors, would yield an incomparable system of cardinality $p+1$, again contradicting the choice of p.

The upshot of all this is that by passing to a subsequence if necessary, we may suppose that $((n_i,k_i))$ is totally ordered. This means that there is a unique branch B_0 containing all (n_i,k_i). Let us reorder the (n_i,k_i) so that $n_i<n_{i+1}$ $(i=1,2,...)$

We observe next that

$$\lim_{i\to\infty}\|y^*-P^*_{n_i,k_i}y^*\|=\|y^*\| \text{ for every } y^*\in JT^*. \tag{9}$$

To see this, recall first that $\|I-P_{n_i,k_i}\| = \|I^*-P^*_{n_i,k_i}\|=1$, so that $\overline{\lim_{i\to\infty}}\, \|y^*-P^*_{n_i,k_i}y^*\| \leqslant \|y^*\|$. Now fix $\delta>0$ and pick $y\in sp\ \{e_{n,k}:(n,k)\in T\}$ so that $\|y\|=1$ and $\langle y,y^*\rangle > \|y^*\|-\delta$. Then for sufficiently large i we have

$$\langle y,y^*-P^*_{n_i,k_i}y^*\rangle = \langle y-P_{n_i,k_i}y,y^*\rangle = \langle y,y^*\rangle > \|y^*\|-\delta.$$

Since $\delta>0$ was arbitrary it follows now that $\underline{\lim}_{i\to\infty} \|y^*-P^*_{n_i,k_i}y^*\| \geqslant \|y^*\|$, hence (9) is proved. As a particular case of (9), we get for every $i=1,2,...$ that

$$\|P_{n_i,k_i}x^*\| = \lim_{j\to\infty} \|P^*_{n_i,k_i}x^* - P^*_{n_j,k_j}\ P^*_{n_i,k_i}x^*\| = \lim_{j\to\infty} \|P^*_{n_i,k_i}x^* - P^*_{n_j,k_j}x^*\|.$$

Using (7), we may therefore pass to a further subsequence and assume that

$$\|P_{n_i,k_i}x^* - P^*_{n_{i+1},k_{i+1}}x^*\| > \epsilon \quad \text{for all } i\in\mathbb{N}. \tag{10}$$

Since $\lim_{\substack{(n,k)\in B_0\\ n\to\infty}} \langle e_{n,k},x^*\rangle = 0$ (recall that $x^*\in N$),
it follows from the fact that $P_{B_0}JT\cong J$ and from (1) that

$$P^*_{B_0}x^* \in [e^*_{n,k}]_{(n,k)\in B_0}.$$

This implies that for sufficiently large i we have

$$\|(P^*_{n_i,k_i} - P^*_{n_{i+1},k_{i+1}})P^*_{B_0}x^*\| < \epsilon/2 \tag{11}$$

(approximate $P^*_{B_0}x^*$ with a finite linear combination of the $e^*_{n,k}$). For simplicity let us assume that (11) is valid for all $i\in\mathbb{N}$. Now let us define

$$U^*_i = (P^*_{n_i,k_i} - P^*_{n_{i+1},k_{i+1}}) - (P^*_{n_i,k_i} - P^*_{n_{i+1},k_{i+1}})P^*_{B_0} \quad (i-1,2,...).$$

The notation U^*_i is justified since the right member is clearly the adjoint of the projection

$$U_i := (P_{n_i,k_i} - P_{n_{i+1},k_{i+1}}) - P_{B_0}(P_{n_i,k_i} - P_{n_{i+1},k_{i+1}})$$

from JT onto the subspace of all elements of JT supported by the set

$$A_i := T(n_i,k_i)\setminus T(n_{i+1},k_{i+1})\setminus B_0 \quad (i=1,2,...).$$

It is easy to verify the condition of Lemma 8.1 that each segment S of T intersects at most one of the above A_i. [Sketch of proof: the only way a segment S can "enter" A_i is by passing through (n_i,k_i). But $(n_i,k_i)\in B_0$, so that S then must have been in B_0 all along.] By dualizing (3), we now get, for any $j\in\mathbb{N}$,

$$\|\sum_{i=1}^{j} U^*_i x^*\|^2 = \sum_{i=1}^{j} \|U^*_i x^*\|^2. \tag{12}$$

Observe, however, that on the one hand, by (10) and (11),

$$\|U^*_i x^*\| > \frac{\epsilon}{2} \quad (i=1,\ldots,j), \tag{13}$$

while on the other hand,

$$\|\sum_{i=1}^{j} U_i^* \| = \|(P_{n_1,k_1}^* - P_{n_{j+1},k_{j+1}}^*) - (P_{n_1,k_1}^* - P_{n_{j+1},k_{j+1}}^*)P_{B_0}^* \|$$
$$\leqslant \|P_{n_1,k_1}^*\| + \|P_{n_{j+1},k_{j+1}}^*\| + (\|P_{n_1,k_1}^*\| + \|P_{n_{j+1},k_{j+1}}^*\|)\|P_{B_0}^*\| = 4,$$

so that

$$\|\sum_{i=1}^{j} U_i^* x^* \|^2 \leqslant 16 \|x^*\|^2. \tag{14}$$

Thus, putting (12), (13) and (14) together, we have

$$j \cdot \frac{\epsilon^2}{4} \leqslant 16 \|x^*\|^2.$$

This is contradictory for large j, so the proof of (iii) is finished. □

We now come to the most delicate part of the proof, which is to show that $N \subset Y$. We assume that $Y \underset{\neq}{\subseteq} N$ and derive a contradiction. First we choose a $\delta > 0$ so small that

$$3.5 < 4(1-\delta)^2. \tag{15}$$

Next we pick $x^* \in N$ so that

$$d(x^*, Y) > 1-\delta \text{ and } \|x^*\| = 1. \tag{16}$$

Since $\lim_{n\to\infty} \|x^* - P_n^* x^*\| = \|x^*\|$ (this is proved just as (9)), we may choose $m \in \mathbb{N}$ so that

$$\|x^* - P_m^* x^*\| > 1-\delta. \tag{17}$$

Finally, we choose $\epsilon > 0$ so that

$$2^{m+2}\epsilon^2 < (1-\delta)^2. \tag{18}$$

We are now going to choose elements $x,y \in JT$ with $\|x\| = \|y\| = 1$ and satisfying

$$P_m y = 0 \quad \text{and} \quad x = P_q x \text{ for some large } q > m,$$

so that

$$\langle x+y, x^* \rangle \geqslant 2(1-\delta).$$

This last inequality implies

$$\|x+y\| \geqslant 2(1-\delta).$$

However, the choice of x and q will be made in such a way that the norm definition in JT makes this last inequality impossible.

The choice of y is easy. Simply use (17) to select a $y \in JT$ so that

$$\|y\| = 1,\ P_m y = 0 \text{ and } \langle y, x^* \rangle > 1-\delta. \tag{19}$$

The choice of q and x is more subtle. First we show

(iv) *There exists a* $q \in \mathbb{N}$ *with* $q > m$ *and an* $x \in JT$ *with*

$$\|x\| = 1,\ x = P_q x,\ \langle x, x^* \rangle > 1 - \delta \tag{20}$$

and

$$\|P_{q,j} x\| < \frac{\epsilon}{1-\delta} \quad (j = 1, \ldots, 2^q). \tag{21}$$

PROOF OF (iv) Since $x^* \in N$, (iii) enables us to choose $q > m$ so that

$$\|P^*_{q,j} x^*\| \leqslant \epsilon \quad (j = 1, \ldots, 2^q). \tag{22}$$

Note now that (16) implies

$$\|P^*_q x^*\| = \|x^* - (x^* - P^*_q x^*)\| > 1 - \delta, \text{ since } x^* - P^*_q x^* \in Y.$$

Hence, dualizing (2) we get

$$\sum_{j=1}^{2^q} \|P^*_{q,j} x^*\|^2 = \|P^*_q x^*\|^2 > (1-\delta)^2.$$

Now for each $j \in \{1, \ldots, 2^q\}$ let us choose $x_j \in JT$ so that

$$\|x_j\| = 1, \quad P_{q,j} x_j = x_j$$

and

$$\sum_{j=1}^{2^q} \langle x_j, P^*_{q,j} x^* \rangle^2 \left(= \sum_{j=1}^{2^q} \langle x_j, x^* \rangle^2\right) > (1-\delta)^2. \tag{23}$$

We now put

$$C := \left[\sum_{j=1}^{2^q} \langle x_j, x^* \rangle^2\right]^{1/2}$$

and claim that

$$x := \frac{1}{C} \sum_{j=1}^{2^q} \langle x_j, x^* \rangle x_j$$

satisfies (20) and (21).

Indeed, by (2) again, and because $\|x_j\| = 1$, $j = 1, \ldots, 2^q$, we have

$$\|x\|^2 = \frac{1}{C^2} \Big\|\sum_{j=1}^{2^q} \langle x_j, x^* \rangle x_j\Big\|^2 = \frac{1}{C^2} \sum_{j=1}^{2^q} \langle x_j, x^* \rangle^2 = 1.$$

Furthermore, by (23) and the definitions of x and C,

$$\langle x, x^* \rangle = \frac{1}{C} \sum_{j=1}^{2^q} \langle x_j, x^* \rangle^2 = C > 1 - \delta.$$

Finally $x_j = P_{q,j} x$ $(j = 1, \ldots, 2^q)$ implies that $x = P_q x$, and also, by (22), that

$$|\langle x_j, x^* \rangle| = |\langle P_{q,j} x_j, x^* \rangle| = |\langle x_j, P^*_{q,j} x^* \rangle| \leqslant \epsilon \quad (j = 1, \ldots, 2^q).$$

Hence

$$\|P_{q,j}x\| = \frac{1}{C}\,\|\langle x_j, x^*\rangle x_j\| \leqslant \frac{\epsilon}{C} < \frac{\epsilon}{1-\delta} \quad (j=1,\ldots,2^q).$$

This completes the proof of (iv). $\square$

Since $\lim\limits_{n\to\infty} P_n x = 0$, and therefore $\lim\limits_{n\to\infty} \dfrac{x-P_nx}{\|x-P_nx\|} = x$,

(iv) will also hold with x replaced by $\dfrac{x-P_nx}{\|x-P_nx\|}$ for sufficiently large $n>q$. So we may assume without loss of generality that, in addition to (20) and (21), also $P_p x = 0$ for some $p>q$.

As we observed before, the choice of x and y implies $\|x+y\| \geqslant 2(1-\delta)$, so, by (15),

$$\|x+y\|^2 \geqslant 4(1-\delta)^2 > 3.5.$$

We now finish the proof by showing

(v) $\|x+y\|^2 < 3.5.$

Proof of (v) By the choice of q,m and p we have

$$x+y = \sum_{n=0}^{p} \sum_{k=1}^{2^n} t_{n,k}\, e_{n,k}$$

and also, $t_{n,k}=0$ whenever $m<n<q$. Since $x+y$ is finitely supported, its norm is attained, so there exist pairwise disjoint segments S_j $(j=1,\ldots,l)$, all of which may be assumed to intersect the support of $x+y$, so that

$$\|x+y\|^2 = \sum_{j=1}^{l} \left[\sum_{(n,k)\in S_j} t_{n,k}\right]^2.$$

We now divide the S_j into three groups, according as they contain elements (n,k) with $n\geqslant q$ or with $n\leqslant m$, or both. Let

$S'_1,\ldots,S'_{l'}$ be the segments S_j that contain both elements (n,k) with $n\geqslant q$, and with $n\leqslant m$;

$S''_1,\ldots,S''_{l''}$ the segments S_j that contain no element (n,k) with $n\geqslant q$;

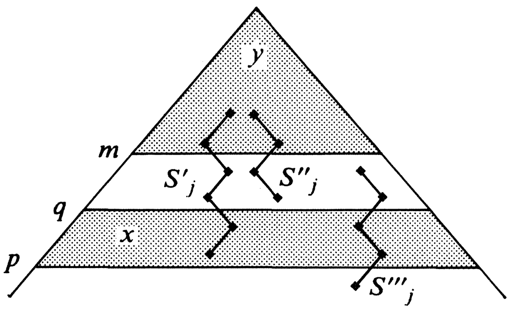
$S'''_1,\ldots,S'''_{l'''}$ the segments S_j that contain no element (n,k) with $n\leqslant m$.

In the diagram below a segment from each group is sketched. We also indicate that the supports of x and y are located between the levels q and p, and above level m, respectively.

Now we have

$$\|x+y\|^2 = \sum_{j=1}^{l'} \left[\sum_{(n,k)\in S_j'} t_{n,k}\right]^2 + \sum_{j=1}^{l''} \left[\sum_{(n,k)\in S_j''} t_{n,k}\right]^2 + \sum_{j=1}^{l'''} \left[\sum_{(n,k)\in S_j'''} t_{n,k}\right]^2$$

$$=: \quad A' \quad + \quad A'' \quad + \quad A'''. \tag{24}$$



First note that

$$A''' \leqslant \|x\|^2 = 1. \tag{25}$$

Next observe that the elementary inequality $(\alpha+\beta)^2 \leqslant 2(\alpha^2+\beta^2)$, $\alpha,\beta \in \mathbb{R}$, implies

$$A' \leqslant 2\Big(\sum_{j=1}^{l'}\Big(\sum_{\substack{(n,k)\in S_j' \\ n\leqslant m}} t_{n,k}\Big)^2 + \sum_{j=1}^{l'}\Big(\sum_{\substack{(n,k)\in S_j' \\ n\leqslant m}} t_{n,k}\Big)^2\Big)$$

$$=: 2\big(\quad A'_- \qquad + \quad A'_+ \quad\big). \tag{26}$$

Now since the segments appearing in A'_- and A'' do not intersect the support of x, we have

$$2A'_- + A'' \leqslant 2(A'_- + A'') \leqslant 2\,\|y\|^2 = 2. \tag{27}$$

Next observe that since each S'_j must pass through level m, the total number l' of the S_j' is $\leqslant 2^m$. Let us enumerate the points where the S_j' cross level q as follows:

$$(q,k_j) \in S_j',\ 1 \leqslant j \leqslant l'\,(\leqslant 2^m)$$

Now we use (21) to get

$$A'_+ = \sum_{j=1}^{l'}\Big(\sum_{\substack{(n,k)\in S_j' \\ (n,k)\geqslant(q,k_j)}} t_{n,k}\Big)^2 \leqslant \sum_{j=1}^{l'} \|P_{q,k_j}x\|^2 \leqslant 2^m.\frac{\epsilon^2}{(1-\delta)^2}. \tag{28}$$

Putting together (25), (27) and (28), and using (18), we finally get what we want:

$$\|x+y\|^2 \leqslant (2A'_- + A'') + 2A'_+ + A''' \leqslant 2 + 2^{m+1}.\frac{\epsilon^2}{(1-\delta)^2} + 1 < 2 + 0.5 + 1 = 3.5\,\square$$

COROLLARY 8.3. *JT^* is non-separable and*

$$JT^{**} \cong JT \oplus l^2(\Gamma),$$

so in particular l^1 does not embed in JT.

PROOF. We have observed earlier that card $\Gamma = c$. Hence $l^2(\Gamma)$ is non-separable. Since we have just shown that JT^* maps onto $l^2(\Gamma)$, JT^* is non-separable also. [Alternatively, one can easily show that the functionals $x_B^* \in JT^*$ $(B \in \Gamma)$ defined by $\langle x, x_B^* \rangle = \sum_{(n,k) \in B} x(n,k)$ $(x \in JT)$ satisfy $\|x_B^*\| = 1$ and $\|x_{B_1}^* - x_{B_2}^*\| \geq \sqrt{2}$ whenever $B_1, B_2 \in \Gamma$ are distinct. This suffices to prove non-separability of JT^*.]

Recall now that $Y = N$. Therefore

$$l^2(\Gamma) \cong JT^*/N \cong Y^{**}/\pi Y$$

(where $\pi: Y \to Y^{**}$ is the canonical embedding). If $\pi_1 : Y^* \to Y^{***}$ denotes the canonical embedding, then, as we have seen before, $\pi^* \circ \pi_1 = 1_{Y^*}$. Since ker $\pi^* = (\pi Y)^{\perp}$, this immediately implies that

$$Y^{***} \cong \pi_1 Y^* \oplus (\pi Y)^{\perp}.$$

Finally, using that $(\pi Y)^{\perp} \cong (Y^{**}/\pi Y)^* \cong l^2(\Gamma)$ and $Y^* \cong JT$, we get

$$JT^{**} \cong JT \oplus l^2(\Gamma).$$

Now there are many ways to see from this that l^1 does not embed in JT. The easiest is to observe that card $JT^{**} = c =$ card JT and to apply Theorem 4.1, (1) $\Leftrightarrow$ (3). $\square$

NOTES What we have called the "classical" James space J, was invented by R.C. JAMES ([45]). It is beyond the scope of these notes to discuss it in detail and to make clear why it is of such fundamental importance. A leisurely discussion of J and its properties can be found in [16]. The James tree space JT is of more recent vintage ([46]). Its main impact at the time was that it refuted the conjecture that l^1 embeds in every separable space with non-separable dual. But ever since its invention it has been a source of inspiration in many different problem areas. James deduced the fact that $l^1 \not\subset JT$ from the much stronger assertion that JT is hereditarily l^2. The proof of this is quite complicated (for a detailed discussion, see [16]). We have followed here the presentation of J. LINDENSTRAUSS and C. STEGALL ([51]). Their Theorem 8.2 has many other interesting consequences (cf. [51]). There is also a discussion in [51] of the so-called James function space JF, another example of a separable space with non-separable dual in which l^1 does not embed. A third such example is the James-Hagler space JH, due to J. HAGLER ([34]). Each of these three examples is interesting in its own right. In many respects they differ considerably and they deserve to be studied in detail. And in fact they are (see e.g. the recent paper [84]).

Appendix A

Perfect measure spaces

DEFINITION A. 1. A finite measure space (Ω,Σ,μ) is called *perfect* if for every measurable function $f:\Omega\to\mathbb{R}$ and for every set $E\subset\mathbb{R}$ such that $f^{-1}E\in\Sigma$, there exists a Borel set $B\subset E$ with $\mu(f^{-1}B)=\mu(f^{-1}E)$. $\square$

If the completion of (Ω,Σ,μ) is perfect, then so is (Ω,Σ,μ), trivially. We shall see shortly that the converse is also true. Anyway, if (Ω,Σ,μ) is complete, then Def. A.1 can be stated as follows: (Ω,Σ,μ) is perfect iff for every measurable $f:\Omega\to\mathbb{R}$ the maximal σ-algebra $\Sigma':=\{E\subset\mathbb{R}: f^{-1}E\in\Sigma\}$ for which f is $\Sigma-\Sigma'$-measurable, provided with the image measure $\nu:=f(\mu)$, is the ν-completion of $\mathfrak{B}(\mathbb{R})$ ($:=$ the Borel sets in $\mathbb{R}$).

The next few propositions describe the most important examples of perfect measure spaces we shall encounter.

PROPOSITION A.2. *Suppose* (Ω,Σ,μ) *is perfect and* $A\in\Sigma$. *Then* (A,Σ_A,μ_A) *is perfect, where*

$$\Sigma_A:=\{B\cap A : B\in\Sigma\} \text{ and } \mu_A(B\cap A):=\mu(B\cap A) \text{ for } B\in\Sigma.$$

PROOF. Suppose that $f:A\to\mathbb{R}$ is measurable and $E\subset\mathbb{R}$ is such that $f^{-1}E\in\Sigma_A$. There is nothing to prove if $E=\mathbb{R}$. If not, choose $\alpha\in\mathbb{R}\setminus E$ and extend f to a function $\tilde{f}:\Omega\to\mathbb{R}$ by putting

$$\tilde{f}(t):=\begin{cases} f(t) & \text{if } t\in A,\\ \alpha & \text{if } t\notin A.\end{cases}$$

Since $\tilde{f}$ is measurable and (Ω,Σ,μ) is perfect, there exists a Borel set $B\subset E$ such that $\mu(\tilde{f}^{-1}B)=\mu(\tilde{f}^{-1}E)$. But obviously $f^{-1}B=\tilde{f}^{-1}B$ and $f^{-1}E=\tilde{f}^{-1}E$, so $\mu_A(f^{-1}B)=\mu_A(f^{-1}E)$. $\square$

PROPOSITION A.3. *A measure space* (Ω,Σ,μ) *is perfect iff its* μ*-completion* (Ω,Σ_μ,μ) *is perfect.*

PROOF. Sufficiency is trivial, as we already observed above. So now assume that (Ω,Σ,μ) is perfect and let $f:\Omega\to\mathbb{R}$ be Σ_μ-measurable and let $E\subset\mathbb{R}$ be such that $f^{-1}E\in\Sigma_\mu$. It is easy to see (and well known) that there is a Σ-measurable $g:\Omega\to\mathbb{R}$ so that $f=g$ μ a.e., hence $\mu((f^{-1}E)\Delta(g^{-1}E))=0$. So we can choose A so that

$$A \in \Sigma, \ A \subset (f^{-1}E) \cap (g^{-1}E) \text{ and } \mu A = \mu(f^{-1}E) = \mu(g^{-1}E). \qquad (1)$$

By Prop. A.2 (A, Σ_A, μ_A) is perfect. So if g_A denotes the (Σ_A-measurable) restriction of g to A, then (since $g_A^{-1}E = A \in \Sigma_A$) there is a Borel set $B \subset E$ so that $\mu_A(g_A^{-1}B) = \mu_A(g_A^{-1}E)$, i.e. $\mu((g^{-1}B) \cap A) = \mu((g^{-1}E) \cap A)$. Using now that $f = g$ μ a.e., and (1), it easily follows from this that $\mu(f^{-1}B) = \mu(f^{-1}E)$. □

Proposition A.4. *Let Ω be a Hausdorff topological space and let μ be a finite Radon measure on Ω. Then $(\Omega, \mathfrak{B}(\Omega), \mu)$ is perfect.*

Proof. Suppose $f: \Omega \to \mathbb{R}$ is measurable and $E \subset \mathbb{R}$ satisfies $f^{-1}E \in \mathfrak{B}(\Omega)$. By regularity there exists for each $n \in \mathbb{N}$ a compact $K_n \subset f^{-1}E$ so that $\mu(f^{-1}E) - \mu(K_n) < \frac{1}{n}$. Moreover, by Lusin's theorem, we may assume that $f|_{K_n}$ is continuous, so that fK_n is compact, hence Borel. It is now evident that $B := \bigcup_{n=1}^{\infty} fK_n$ is Borel and satisfies $B \subset E$, $\mu(f^{-1}B) = \mu(f^{-1}E)$. □

We often say that μ *is perfect* when the measure space (Ω, Σ, μ) is understood.

The next result shows that the property of measurable functions $f: \Omega \to \mathbb{R}$ that characterizes perfectness of (Ω, Σ, μ), remains true for functions f valued in more general measure spaces. First, some definitions must be recalled.

Let Ω be a set, Σ a σ-algebra of subsets of Ω. Such a pair (Ω, Σ) is often called a *measurable space.* We shall say that (Ω, Σ) is *countably generated* if there is a sequence (A_n) in Σ that generates Σ, i.e. $\Sigma = \sigma(\bigcup_{n=1}^{\infty} A_n)$. We call the measurable space (Ω, Σ) *separated* if for every pair $x, y \in \Omega$ with $x \neq y$ there exists an $A \in \Sigma$ containing precisely one of x and y. A simple example of a separated countably generated measurable space is a subset A of $[0,1]$ with its Borel σ-algebra: $(A, \mathfrak{B}(A))$ (cf. Lemma D. 11 (ii)). This is essentially the only example, as we now show.

Lemma A.5. *Let (Ω, Σ) be a separated, countably generated measurable space. Then there exists a subset $A \subset [0,1]$ and an isomorphism (in the obvious sense) between (Ω, Σ) and $(A, \mathfrak{B}(A))$.*

Proof. Let $(A_n) \subset \Sigma$ be a sequence that generates Σ and define $\phi: \Omega \to [0,1]$ by $\phi(x) := \sum_{n=1}^{\infty} 3^{-n} \chi_{A_n}(x)$ $(x \in \Omega)$. We claim that $\Sigma = \phi^{-1}(\mathfrak{B})$. Indeed, $\phi^{-1}(\mathfrak{B}) \subset \Sigma$ is trivial, since ϕ is clearly measurable. On the other hand $\phi^{-1}(\mathfrak{B})$ (which is a σ-algebra) contains each A_n (exercise), and therefore $\Sigma = \sigma(\bigcup_{n=1}^{\infty} A_n)$. Next let us show that ϕ is injective: if $\phi(x) = \phi(y)$ for some $x, y \in \Omega$, then for each $A \in \phi^{-1}(\mathfrak{B})$ we have $x \in A$ iff $y \in A$. But $\phi^{-1}(\mathfrak{B}) = \Sigma$ and the sets of Σ separate the points of Ω, by assumption. So we must have $x = y$. [Note: we have shown that (Ω, Σ) is *countably separated* since the injectivity of ϕ means that the A_n $(n = 1, 2, \ldots)$ separate the points of Ω.] The assertion of the lemma is now clear if we put $A:=\phi(\Omega)$. $\square$

The example of a separated countably generated measurable space (Ω,Σ) that will suffice for all our applications is a Hausdorff topological space Ω with a countable base, provided with its Borel σ-algebra Σ.

PROPOSITION A.6. *Let (Ω_1,Σ_1) be a separated countably generated measurable space and let (Ω,Σ,μ) be perfect. Then for each measurable $f:\Omega\to\Omega_1$ (measurable here means Σ-Σ_1-measurable) and for each $E_1\subset\Omega_1$ such that $f^{-1}E_1\in\Sigma$ there exists a $B_1\in\Sigma_1$ with $B_1\subset E_1$ and $\mu(f^{-1}B_1)=\mu(f^{-1}E_1)$.*

PROOF. Let $\phi:\Omega_1\to[0,1]$ be the isomorphism defined in the proof of lemma A.5. Putting $E:=\phi E_1$ we have $(\phi\circ f)^{-1}E=f^{-1}E_1$. By the definition of perfectness there is a Borel set $B\subset E$ with $\mu((\phi\circ f)^{-1}B)=\mu((\phi\circ f)^{-1}E)$. Now $\phi^{-1}B=:B_1$ satisfies $B_1\in\Sigma_1$, $B_1\subset E_1$ and

$$\mu(f^{-1}B_1)=\mu((\phi\circ f)^{-1}B)=\mu((\phi\circ f)^{-1}E)=\mu(f^{-1}E_1). \qquad \square$$

We now state a consequence of perfectness that is needed in the proof of Fremlin's Theorem and in several other places as well.

PROPOSITION A.7. *Let (Ω,Σ,μ) be perfect an let (Ω_1,Σ_1) be a separated countably generated measurable space (e.g. Ω_1 = Hausdorff topological space with a countable base, and $\Sigma_1=\mathscr{B}(\Omega_1)$). Let $f:\Omega\to\Omega_1$ be measurable and let us define $\nu:=f(\mu)$ on Σ_1. Then for every subset $E_1\subset\Omega_1$ we have*

$$\mu_*(f^{-1}E_1)=\nu_*E_1 \text{ and } \mu^*(f^{-1}E_1)=\nu^*E_1. \tag{2}$$

Consequently, for every function $g:\Omega_1\to\mathbb{R}$ we have that

$$g\circ f \text{ is } \mu\text{-measurable iff } g \text{ is } \nu\text{-measurable.} \tag{3}$$

PROOF. It suffices to prove the first equality in (2); the second then follows by complementation. The inequality $\mu_*(f^{-1}E_1)\geq\nu_*(E_1)$ is trivial: if $B_1\subset E_1$, $B_1\in\Sigma_1$ is such that $\nu B_1=\nu_*E_1$, then $f^{-1}B_1\in\Sigma$ and $f^{-1}B_1\subset f^{-1}E_1$, so $\mu_*(f^{-1}E_1)\geq\mu(f^{-1}B_1)=\nu B_1=\nu_*E_1$.

For the proof of $\mu_*(f^{-1}E_1)\leq\nu_*E_1$ perfectness is needed. Choose $B\in\Sigma$ so that $B\subset f^{-1}E_1$ and $\mu B=\mu_*(f^{-1}E_1)$. Consider the measure space (B,Σ_B,μ_B), which is perfect by Prop. A.2, and let $f_B:B\to\Omega_1$ be the (Σ_B-measurable restriction of f to B. Since $f_B^{-1}E_1=B\in\Sigma_B$, there exists by Prop. A.6 a set $B_1\subset E_1$, $B_1\in\Sigma_1$ such that

$$\mu_B(f_B^{-1}B_1)=\mu_B(f_B^{-1}E_1)=\mu B=\mu_*(f^{-1}E_1).$$

The conclusion now follows easily:

$$\nu_*E_1\geq\nu B_1=\mu(f^{-1}B_1)\geq\mu_B(f_B^{-1}B_1)=\mu_*(f^{-1}E_1).$$

The last statement (3) is obvious: if $B\subset\mathbb{R}$ is Borel then $g^{-1}B=:E_1$ is ν-

measurable iff $(g\circ f)^{-1}B = f^{-1}E_1$ is μ-measurable, by (2). $\square$

We conclude with an example showing that not all finite measure spaces are perfect.

EXAMPLE A.8. Let (Ω,Σ,μ) be a finite measure space and let $E\subset\Omega$ be a subset that is not necessarily μ-measurable. Then E generates a measure space (E,Σ_E,μ_E), as follows:

$$\Sigma_E := \{A\cap E : A\in\Sigma\},\ \mu_E(A\cap E) := \mu(A\cap F) \text{ for } A\in\Sigma,$$

where F is any set satisfying

$$F\in\Sigma,\quad E\subset F \text{ and } \mu F = \mu^* E.$$

(Note that this definition reduces to that in Prop. A.2 if $E\in\Sigma$; observe also that μ_E is independent of the choice of F.)

Now suppose in addition that (Ω,Σ) is separated and countably generated and that E is not μ-measurable. We claim that (E,Σ_E,μ_E) is not perfect. Indeed, if it were, then by Prop. A.6 applied to the measurable identity map $E\xrightarrow{i}\Omega$ there would exist a $B\subset E$ with $B\in\Sigma$ so that $\mu_E B = \mu_E E$. But then $\mu^* E = \mu_E E = \mu_E B = \mu B \leqslant \mu_* E$, so we have a contradiction. $\square$

REMARK A.9. If (Ω,Σ,μ) is also perfect (besides being separated and countably generated) then, by Prop. A.2 and Ex. A.8, (E,Σ_E,μ_E) is perfect iff E is μ-measurable.

NOTES Perfect measure spaces were first defined by B. V. GNEDENKO and A.N. KOLMOGOROV in [31], and studied by V.V. SAZONOV ([81]). For more recent information, see [49].

Appendix B

Radon measures

Let T be a topological Hausdorff space. A *Radon measure* on T is a finite nonnegative Borel measure μ which is *regular* in the sense that $\mu A = \sup\{\mu K : K \subset A \text{ compact }\}$ for every Borel set $A \subset T$. (We shall have almost no need to consider signed or complex or even infinite Radon measures, so it is convenient to restrict the term Radon measure to finite nonnegative measures unless specified differently.) Σ_μ denotes the μ-completion of $\mathfrak{B}(T)$, the Borel σ-algebra on T.

For a Radon measure μ on T consider the union of all open μ-null sets $O \subset T$. The regularity of μ easily implies that this union is itself a null set. The (closed) complement of this largest open null set is called the *support* of μ and denoted supp μ. If $B \in \mathfrak{B}(T)$ then we define μ_B by $\mu_B(A) := \mu(A \cap B)$, $A \in \mathfrak{B}(T)$. Since $\mathfrak{B}(B) = \{A \cap B : A \in \mathfrak{B}(T)\}$, μ_B is a Radon measure on B, often called the *restriction* of μ to B. B is called *self supported* if $B =$ supp μ_B, in other words if $\mu(O \cap B) > 0$ whenever $O \subset T$ is open and $O \cap B \neq \varnothing$. μ_B can also be (and often will be) considered to be a Radon measure on T. As such it may have a larger support.

Let T_1 an T_2 be Hausdorff spaces and let $\phi : T_1 \to T_2$ be continuous. Then ϕ is Borel-measurable so if μ_1 is a Radon measure on T_1 then the image measure $\mu_2 := \phi\mu_1$ can be defined, first on $\mathfrak{B}(T_2)$, and then on Σ_{μ_2} the usual way: $\mu_2 B_2 := \mu_1(\phi^{-1} B_2)$ for $B_2 \in \Sigma_{\mu_2}$. Observe that μ_2 is also Radon since ϕ maps compact sets to compact sets.

Now let us take T_1 and T_2 compact Hausdorff and let $\phi : T_1 \to T_2$ be a continuous surjection. We denote by $C(T_i)$ $(i = 1,2)$ the Banach space of continuous real-valued functions on T_i, with the sup norm. The map $\phi^* : C(T_2) \to C(T_1)$ defined by

$$\phi^*(f_2) := f_2 \circ \phi \quad (f_2 \in C(T_2))$$

is a linear isometry into. Therefore its adjoint $\phi^{**} : M(T_1) \to M(T_2)$ is a quotient map (Here we identify $M(T_i)$, the set of signed Radon measures on T_i, with the dual space $C(T_i)^*$, via the Riesz representation theorem; under this identification the variation norm of $\mu_i \in M(T_i)$ corresponds to the dual norm of $C(T_i)^*$.)

$$
\begin{array}{lll}
M(T_1) \xrightarrow{\phi^{**}(=\phi)} M(T_2) & & \phi^{**}\ w^*-w^*\text{-continuous quotient map} \\
C(T_1) \ni \phi^*(f_2) = f_2 \circ \phi \xleftarrow{\phi^*} f_2 \in C(T_2) & & \phi^* \text{ isomorphism into} \\
T_1 \xrightarrow{\phi} T_2 & & \phi \text{ continuous surjection}
\end{array}
$$

Since the closed unit balls of $M(T_1)$ and $M(T_2)$ are w^*-compact (Alaoglu's theorem) and ϕ^{**} is a w^*-w^* continuous quotient map, ϕ^{**} maps the closed unit ball of $M(T_1)$ onto that of $M(T_2)$. In particular for every Radon probability $\mu_2 \in M(T_2)$ there is a $\mu_1 \in M(T_1)$ with $\|\mu_1\| = 1$. and $\phi^{**}\mu_1 = \mu_2$. The obvious fact that ϕ^* maps 1_{T_1}(:= the function on T_1 that is identically equal to 1) to 1_{T_2} implies that $\mu_2(T_2) = \mu_1(T_1) = 1$, so that $\mu_1 \geqslant 0$, hence a Radon probability. Of course $\phi^{**}\mu_1$ is precisely what we have called $\phi\mu_1$ (the image measure) before. We have therefore shown that each Radon probability μ_2 on T_2 is the image of a Radon probability μ_1 on T_1 (T_1, T_2 compact).

We now prove a fact that will be used sometimes in these notes: given μ_2, the Radon probability μ_1 can be so chosen that ϕ^* is an isometry *onto* when considered as a map from $L^1(\mu_2)$ to $L^1(\mu_1)$.

PROPOSITION B.1 *Let T_1, T_2 be compact Hausdorff spaces and $\phi: T_1 \to T_2$ a continuous surjection. Then every Radon probability μ_2 on T_2 is the ϕ-image of a Radon probability μ_1 on T_1 such that $\phi^*: L^1(\mu_2) \to L^1(\mu_1)$ is an isometry onto.*

PROOF. Clearly ϕ^* is an isometry into, whatever the choice of μ_1. The point of interest is the surjectivity. For this we consider the set P_1 of all Radon probabilities μ_1 with $\phi\mu_1 = \mu_2$ (μ_2 is fixed). Clearly P_1 is w^*-compact and convex. We claim that if we choose $\mu_1 \in \text{ext } P_1$ (observe that ext $P_1 \neq \varnothing$ by the Krein-Milman theorem) then ϕ^* is onto. Suppose not. Then there exists a non-zero $g \in L^\infty(\mu_1) = L^1(\mu_1)^*$ such that

$$\int_{T_1} g \cdot \phi^*(f) d\mu_1 = \int_{T_1} g \cdot (f \circ \phi) d\mu_1 = 0 \quad \forall f \in L^1(\mu_1). \tag{1}$$

We may assume $-1 \leqslant g \leqslant 1$. Let us write

$$\mu_1 = \frac{1}{2}(1+g)\mu_1 + \frac{1}{2}(1-g)\mu_1 \tag{2}$$

(1) states that the measure $g\mu_1$ has ϕ-image 0. Hence each of the (non-negative!) Radon measure $(1 \pm g)\mu_1$ has ϕ-image μ_2. Also each of them is a probability, since $\int_{T_1} g d\mu_1 = \int_{T_1} g \cdot (1_{T_2} \circ \phi) d\mu_1 = 0$, by (1). Thus $(1 \pm g)\mu_1 \in P_1$ and (2) contradicts the fact that $\mu_1 \in \text{ext } P_1$. $\square$

REMARK B.2. The result is intuitively clear. Consider e.g. the special case where μ_2 is a discrete measure. Let $\{t_2\}$ be an atom of μ_2 and suppose that $B_1 := \phi^{-1}\{t_2\}$ is not a singleton. Then all functions $\phi^* f$ are constant on B_1. So if ϕ^* is to be surjective, every μ_1-measurable function should be constant μ_1 a.e. on B_1. This can be achieved by concentrating $(\mu_1)_{B_1}$ in one point, say $t_1 \in B_1 : (\mu_1)_{B_1} = \mu_2\{t_2\} \cdot \delta_{t_1}$.

NOTES All material here is standard, except Prop. B.1, which was observed by M. TALAGRAND in [92].

Appendix C

Products of Radon measures

Let T_1 and T_2 be compact Hausdorff spaces and let μ_1 and μ_2 be Radon probability measures on T_1 and T_2, respectively. We shall denote by $\mathfrak{B}_1$, $\mathfrak{B}_2$ the respective Borel σ-algebras and by Σ_{μ_1}, Σ_{μ_2} their μ_1-, respectively μ_2-completions. On $\mathfrak{B}_1 \times \mathfrak{B}_2$ ($:=$ the product σ-algebra) we consider the product measure $\mu := \mu_1 \times \mu_2$. Let Σ_μ denote the μ-completion of $\mathfrak{B}_1 \times \mathfrak{B}_2$. Obviously Σ_μ equals the μ-completion of $\Sigma_{\mu_1} \times \Sigma_{\mu_2}$. There is also the σ-algebra $\mathfrak{B}$ of the Borel sets of $T := T_1 \times T_2$ (with the product topology). We have $\mathfrak{B}_1 \times \mathfrak{B}_1 \subset \mathfrak{B}$ but it is known that in general $\mathfrak{B} \not\subset \Sigma_\mu$. An example of this situation can be found in [27]. However, continuous functions on T are measurable with respect to $\mathfrak{B}_1 \times \mathfrak{B}_2$ so we may consider the positive linear functional

$$f \to \int_T f \, d\mu \quad (f \in C(T)).$$

By the Riesz representation theorem there is a Radon measure μ_R on $\mathfrak{B}$ satisfying

$$\int_T f \, d\mu_R = \int_T f \, d\mu. \tag{1}$$

It is easy to see from the uniqueness clause in the Riesz representation theorem that

(i) μ_R coincides with μ on $\mathfrak{B}_1 \times \mathfrak{B}_2$
(apply (1) to functions depending on one coordinate only, to show that $\mu_R(B_1 \times T_2) = \mu(B_1 \times T_2)$ and $\mu_R(T_1 \times B_2) = \mu(T_1 \times B_2)$ for all Borel sets $B_1 \subset T_1, B_2 \subset T_2$);

(ii) μ_R is uniquely determined by μ
(the corresponding functional is uniquely determined by μ, see (1)).

Thus μ_R is the unique Radon extension of μ from $\mathfrak{B}_1 \times \mathfrak{B}_2$ to $\mathfrak{B}$, or from Σ_μ to Σ_{μ_R}, if Σ_{μ_R} denotes the μ_R-completion of $\mathfrak{B}$.

We now derive an explicit formula for μ_R. Notation: for $B \subset T_1 \times T_2$ and $t_1 \in T_1$, $B_{t_1} := \{t_2 \in T_2 : (t_1, t_2) \in B\}$ and similarly $B_{t_2} := \{t_1 \in T_1 : (t_1, t_2) \in B\}$ for $t_2 \in T_2$.

Proposition C.1. *For every $B \in \mathfrak{B}$,*

$$\mu_R(B) = \int_{T_1} \mu_2(B_{t_1}) d\mu_1(t_1) = \int_{T_2} \mu_1(B_{t_2}) d\mu_2(t_2).$$

Proof. By symmetry it suffices to prove the first formula. Let us consider the collection

$$\Sigma_0 := \{B \in \mathfrak{B}: \int_{T_1} \mu_2(B_{t_1}) d\mu_1(t_1) \text{ is well-defined}\}$$

(by "well defined" we mean that B_{t_1} is μ_2-measurable for μ_1 a.e. t_1 and that $t_1 \to \mu_2(B_{t_1})$ is μ_1-measurable).

Observe that

(i) Σ_0 is a Dynkin system (i.e. $T \in \Sigma_0$; $A, B \in \Sigma_0$ and $A \subset B$ implies $B \setminus A \in \Sigma_0$; $A_n \in \Sigma_0$ $(n = 1,2,...)$ and $A_n \uparrow$ implies $\bigcup_{n=1}^{\infty} A_n \in \Sigma_0$).

(ii) Σ_0 contains the compact sets (for B compact B_{t_1} is compact, hence Borel and $t_1 \to \mu_2(B_{t_1})$ is u.s.c. (= upper-semi-continuous) hence Borel).

Since the compact sets generate the Borel σ-algebra and are closed for (finite) intersections, a well-known theorem ([12], Th. 1.6.1) states that $\Sigma_0 = \mathfrak{B}$. It is now clear that

$$\mu'(B) := \int_{T_1} \mu_2(B_{t_1}) d\mu_1(t_1) \quad (B \in \mathfrak{B})$$

defines a Borel measure extending μ.

We must show that $\mu' = \mu_R$. For this it is enough to prove that $\mu'K = \mu_R K$ for compact K. In fact it suffices to show that

$$\mu'K \geqslant \mu_R K \quad \forall \text{compact } K. \tag{2}$$

Indeed, since $\mu'T = \mu_R T$, it then follows that

$$\mu'(T \setminus K) \leqslant \mu_R(T \setminus K) \quad \forall \text{ compact } K.$$

Suppose for contradiction that $\mu'(T \setminus K) < \mu_R(T \setminus K)$ for some compact K. Then by the regularity of μ_R there is a compact $L \subset T \setminus K$ so that

$$\mu'L \leqslant \mu'(T \setminus K) < \mu_R L \leqslant \mu_R(T \setminus K),$$

contradicting (2) with $K = L$.

For the proof of (2) observe that the u.s. continuity of $t_1 \to \mu_2(K_{t_1})$ and the regularity of μ_1 easily imply (exercise!) that

$$\mu'K = \inf\{\mu'B : B \supset K,\ B \in \mathfrak{B}_1 \times \mathfrak{B}_2\} \tag{3}$$

(we may even restrict the infimum to the finite unions of open rectangles containing K). Since $\mu_R B = \mu' B$ for all $B \in \mathfrak{B}_1 \times \mathfrak{B}_2$ we conclude from (3) that $\mu_R K \leqslant \mu' K$ and the proof is finished. □

Remark C.2. The above discussion can be extended to finite products $\mu_1 \times \cdots \times \mu_k$ of Radon measures μ_i on compact T_i $(i = 1,...,k)$. The existence of the unique Radon measure $(\mu_1 \times \cdots \times \mu_k)_R$ on $T := T_1 \times \cdots \times T_k$ extending $\mu_1 \times \cdots \times \mu_k$ follows exactly as before. To show that it is given by a Fubini type formula as in Prop. C.1, it is best to use induction, observing that the formula

$$((\mu_1 \times \cdots \times \mu_l)_R \times (\mu_{l+1} \times \cdots \times \mu_k)_R)_R = (\mu_1 \times \cdots \times \mu_k)_R \qquad (4)$$

holds for all $l,k \in \mathbb{N}$ with $1 \leqslant l < k$. $\square$

COROLLARY C.3. *Let T_i be compact Hausdorff and μ_i a Radon probability on T_i $(i=1,\ldots,k)$. If $X_i \subset T_i$ satisfies $\mu_i^* X_i = 1$ $(i=1,\ldots,k)$ then*

$$(\mu_1 \times \cdots \times \mu_k)_R^* \, (X_1 \times \cdots \times X_k) = 1. \qquad (5)$$

In particular $(\mu_1 \times \cdots \times \mu_k)^* \, (X_1 \times \cdots \times X_k) = 1.$

PROOF. We use induction on k. Formula (4) takes care of the induction step from k to $k+1$ $(k \geqslant 2)$, so it suffices to prove the result for $k=2$. Let B be any Borel set in $T_1 \times T_2$ disjoint with $X_1 \times X_2$. Then the formula in Prop. C.1 immediately yields $\mu_R B = 0$. $\square$

NOTES The fact that the completion of the product σ-algebra of two Radon measures may not contain the Borel sets was proved by D.H. FREMLIN in [27]. Corollary C.3 (for $k=2$) appears in [92], with a very sketchy proof.

Appendix D

Polish spaces and analytic sets

DEFINITION D.1. A topological space is called *Polish* if it is homeomorphic to a separable complete metric space. □

It is known that a subspace of a Polish space T is Polish iff it is a G_δ in T. We only need:

LEMMA D.2. *Every closed subspace and every open subspace of a Polish space T is Polish.*

PROOF The first assertion is obvious. Let O be open in T and let d be a complete metric for T. We define

$$\rho(t,t') := d(t,t') + \left|\frac{1}{d(t,T\setminus O)} - \frac{1}{d(t',T\setminus O)}\right| \quad (t,t'\in O).$$

Clearly ρ is a metric on O and it is readily seen that a sequence $(t_n)\subset O$ converges for d to a point $t\in O$ iff this is the case for ρ, since $t\to\frac{1}{d(t,T\setminus O)}$ is d-continuous on O. So the d- and ρ- topologies on O coincide. To see the completeness of (O,ρ), let $(t_n)\subset O$ be ρ-Cauchy. Then (t_n) is d-Cauchy since $d\leqslant\rho$, so d-lim $t_n=t$ for some $t\in T$. If we can show that $t\in O$, then ρ-lim $t_n=t$, by our earlier observation. But this is simple: if $t\notin O$ then $\frac{1}{d(t_n,T\setminus O)}\to\infty$, and this is incompatible with $\lim_{n,m\to\infty}\rho(t_n,t_m)=0$. □

EXAMPLE D.3. For theoretical reasons the space $\mathbb{N}^{\mathbb{N}}$ of all sequences of natural numbers is an important example of a Polish space. Of course each factor $\mathbb{N}$ has the discrete (metric!) topology and $\mathbb{N}^{\mathbb{N}}$ the product topology. For each choice of $n_1,\ldots,n_k\in\mathbb{N}$ let us define

$$A(n_1,\ldots,n_k) := \{(m_i)\in\mathbb{N}^{\mathbb{N}} : m_i=n_i \text{ for } i=1,\ldots,k\}.$$

The sets $A(n_1,\ldots,n_k)$ form a countable base for the topology of $\mathbb{N}^{\mathbb{N}}$. □

The reason for introducing $\mathbb{N}^{\mathbb{N}}$ is the following fact.

PROPOSITION D.4. *Each non-empty Polish space T is a continuous image of* $\mathbb{N}^{\mathbb{N}}$.

PROOF. Let d be a complete metric on T. We shall define inductively subsets

$F(n_1, \ldots, n_k)$ of T indexed by finitely many $n_1, \ldots, n_k \in \mathbb{N}$ so that the following conditions hold:

(i) $F(n_1, \ldots, n_k)$ is closed and non-empty,

(ii) diam $F(n_1, \ldots, n_k) \leqslant \frac{1}{k}$ $(k=1,2,\ldots)$,

(iii) $T = \bigcup_{n_1=1}^{\infty} F(n_1)$ and, more generally,

(iv) $F(n_1, \ldots, n_{k-1}) = \bigcup_{n_k=1}^{\infty} F(n_1, \ldots, n_{k-1}, n_k)$ for every choice of $n_1, \ldots, n_{k-1} \in \mathbb{N}$.

The possibility of defining such sets is immediate from the fact that each closed subset of T is a countable union of closed subsets with diameter $\leqslant \epsilon$ ($\epsilon > 0$ arbitrary).

We now define $f : \mathbb{N}^{\mathbb{N}} \to T$ as follows. Given $(n_k)_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ we take $f((n_k))$ to be the unique point in $\bigcap_{k=1}^{\infty} F(n_1, \ldots, n_k)$. Properties (i), (ii) and the completeness of d imply that such a unique point exists for any choice of $(n_k)_{k=1}^{\infty}$. Moreover, the singletons $\bigcap_{k=1}^{\infty} F(n_1, \ldots, n_k)$ cover T, by (iii) and (iv), so that f is surjective. Finally, to see the continuity of f, note that, by (ii), the oscillation of f is $\leqslant \frac{1}{k}$ on the nbhd $A(n_1, \ldots, n_k)$ of any given $(n_k) \in \mathbb{N}^{\mathbb{N}}$ $(k=1,2,\ldots)$.
□

DEFINITION D.4. A subset of a Polish space T is called *analytic* if it is a continuous image of some Polish space S. □

It will become apparent in a moment that every Borel subset of a Polish space is analytic. There are analytic sets which fail to be Borel, however (we shall not prove this), but if both A and its complement are analytic, then A is Borel. Before proving these assertions we collect some elementary facts about analytic sets in the next lemma.

LEMMA D.5.

(i) *every closed and every open subset of a Polish space is analytic,*

(ii) *if $A_1, A_2, \ldots$ are analytic subsets of a Polish space T then $\bigcap_{k=1}^{\infty} A_k$ and $\bigcup_{k=1}^{\infty} A_k$ are also analytic.*

PROOF. (i) follows trivially from Lemma D.2.

(ii): for every $k \in \mathbb{N}$ let S_k be Polish and $f_k : S_k \to A_k$ a continuous surjection. Then $S := \prod_{k=1}^{\infty} S_k$ is Polish. Let D be the subset of S consisting of all $(s_k)_{k=1}^{\infty}$ with $f_1(s_1) = f_2(s_2) = \ldots\ldots$ Then D is closed in S, so Polish itself. Now $\bigcap_{k=1}^{\infty} A_k$ is the image of D under the continuous map that sends $(s_k)_{k=1}^{\infty} \in D$ to the common value $f_1(s_1) = f_2(s_2) = \cdots$.

To prove the assertion about unions, let f_k and S_k be as above. Now let us form the disjoint union S' of the S_k. Then S' is Polish. [Sketch of proof: let d_k be a complete metric on S_k, $k=1,2,...$, and assume each d_k has values <1. Then define d on S' by $d(x,y):=d_k(x,y)$ if $x,y\in S_k$ for some $k\in\mathbb{N}$, and $=1$ otherwise.] The map $f: S'\to\bigcup_{k=1}^{\infty}A_k$ that agrees with f_k on S_k ($k=1,2,...$) is continuous and surjective, so the proof is finished. □

We are now prepared for

PROPOSITION D.6. *Every Borel subset of a Polish space T is analytic.*

PROOF. Let $\mathfrak{A}$ be the collection of all subsets $A\subset T$ such that both A and $T\setminus A$ are analytic. By Lemma D.5 (i) $\mathfrak{A}$ contains the closed (and the open) sets. By definition, if $A\in\mathfrak{A}$ then $T\setminus A\in\mathfrak{A}$. Finally, whenever $A_k\in\mathfrak{A}$ ($k=1,2,...$), then $\bigcup_{k=1}^{\infty}A_k$ and $T\setminus\bigcup_{k=1}^{\infty}A_k=\bigcap_{k=1}^{\infty}(T\setminus A_k)$ are both analytic by Lemma D.5 (ii), so $\bigcup_{k=1}^{\infty}A_k\in\mathfrak{A}$. We have now proved that $\mathfrak{A}$ is a σ-algebra containing the closed sets, so $\mathfrak{A}$ contains the Borel sets and we are done. □

REMARK D.7. It will become clear in a moment that actually $\mathfrak{A}=\mathfrak{B}$: if A and $T\setminus A$ are both analytic, then A is Borel. □

Let us note the following consequence of Prop. D.4 and the definition of analyticity:

COROLLARY D.8. *Every analytic set is a continuous image of $\mathbb{N}^{\mathbb{N}}$.*

The result mentioned in Remark D.7 is a corollary of the following "separation theorem" for analytic sets.

PROPOSITION D.9. *If A_1,A_2 are disjoint analytic subsets of a Polish space T, then A_1 and A_2 can be separated by Borel sets, i.e. there exist disjoint Borel sets $B_1,B_2\subset T$ such that $A_1,\subset B_1$ and $A_2\subset B_2$.*

PROOF. The proof is based on the following simple observation: if $(E_n)_{n=1}^{\infty}$ and $(F_m)_{m=1}^{\infty}$ are two sequences of subsets of T such that for each pair $(n,m)\in\mathbb{N}\times\mathbb{N}$ the sets E_n and F_m (are disjoint and) can be separated by Borel sets, then so can $\bigcup_{n=1}^{\infty}E_n$ and $\bigcup_{m=1}^{\infty}F_m$. Indeed, by assumption there exist for each pair (n,m) Borel sets $C_{n,m},D_{n,m}$ with

$$C_{n,m}\cap D_{n,m}=\varnothing,\ E_n\subset C_{n,m},\ F_m\subset D_{n,m}.$$

Then $C:=\bigcup_{n=1}^{\infty}\bigcap_{m=1}^{\infty}C_{n,m}$ and $D:=\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}D_{n,m}$ are Borel and satisfy

$$C \cap D = \varnothing, \quad \bigcup_{n=1}^{\infty} E_n \subset C, \quad \bigcup_{m=1}^{\infty} F_m \subset D.$$

Now recall that by Cor. D.8 there are continuous maps $f, g : \mathbb{N}^{\mathbb{N}} \to T$ so that $f(\mathbb{N}^{\mathbb{N}}) = A_1$ and $g(\mathbb{N}^{\mathbb{N}}) = A_2$. Let us suppose for contradiction that A_1 and A_2 cannot be separated (by Borel sets). Let $A(n_1, \ldots, n_k)$ be defined as in Example D.3. Then

$$A_1 = \bigcup_{n_1=1}^{\infty} f(A(n_1)), \quad A_2 = \bigcup_{m_1=1}^{\infty} g(A(m_1)),$$

so by the above observation there are $\bar{n}_1, \bar{m}_1 \in \mathbb{N}$ so that

$$f(A(\bar{n}_1)) \text{ and } g(A(\bar{m}_1)) \text{ cannot be separated.}$$

Repeating this argument we can define inductively sequences $(\bar{n}_k), (\bar{m}_k) \in \mathbb{N}^{\mathbb{N}}$ so that for each $k \in \mathbb{N}$,

$$f(A(\bar{n}_1, \ldots, \bar{n}_k)) \text{ and } g(A(\bar{m}_1, \ldots, \bar{m}_k)) \text{ cannot be separated.} \tag{1}$$

Observe now that by the disjointness of A_1 and A_2 we have

$$f((\bar{n}_k)) \neq g((\bar{m}_k)).$$

Let U_1, U_2 be disjoint nbhds of these respective points in T. Then by the continuity of f and g we must have

$$f(A(\bar{n}_1, \ldots, \bar{n}_k)) \subset U_1, \ g(A(\bar{m}_1, \ldots, \bar{m}_k)) \subset U_2$$

for sufficiently large $k \in \mathbb{N}$ (recall that the sets $A(\bar{n}_1, \ldots, \bar{n}_k)$ form a nbhd basis of $(\bar{n}_k)$ in $\mathbb{N}^{\mathbb{N}}$; similarly for $(\bar{m}_k)$). This contradicts (1). □

COROLLARY D.10. *Let T be Polish and let $A \subset T$ have the property that both A and $T \setminus A$ are analytic. Then A is Borel.*

PROOF. If B_1, B_2 are Borel sets separating A and $T \setminus A$, then $A = B_1$. □

The final result we need is that analytic sets behave well under Borel maps. Some trivialities must be dealt with first.

LEMMA D.11. *Let T_1, T_2 be separable metrizable spaces. Then:*

(i) $\mathfrak{B}(T_1 \times T_2) = \mathfrak{B}(T_1) \times \mathfrak{B}(T_2)$ *(where $\mathfrak{B}(T_1) \times \mathfrak{B}(T_2)$ denotes the product σ-algebra). More generally,*
$\mathfrak{B}(\prod_{i=1}^{\infty} T_i) = \prod_{i=1}^{\infty} \mathfrak{B}(T_i)$ *when $T_i (i = 1,2,\ldots)$ is separable and metrizable.*

(ii) *For any $A \subset T_1$, $\mathfrak{B}(A) = \{B \cap A : B \in \mathfrak{B}(T_1)\}$.*

(iii) *If $f : T_1 \to T_2$ is Borel-measurable then its graph $G(f)$ belongs to $\mathfrak{B}(T_1 \times T_2)$.*

PROOF. (i): The inclusion $\mathfrak{B}(T_1) \times \mathfrak{B}(T_2) \subset \mathfrak{B}(T_1 \times T_2)$ is true for any topological spaces T_1 and T_2. The other inclusion holds e.g. whenever $T_1 \times T_2$ is Lindelöf, since then every open $O \subset T_1 \times T_2$ is a countable union of open

rectangles. The extension to countable products is immediate.
(ii): This assertion holds for arbitrary topological spaces T_1. Consider the identity embedding $i:A\to T_1$. Then $i^{-1}\mathfrak{B}(T_1)=\{B\cap A: B\in\mathfrak{B}(T_1)\}$ is a σ-algebra containing the open subsets $O\cap A$ of $A\,(O\subset T_1$ open), hence $\mathfrak{B}(A)$. On the other hand $\{B\subset T_1: i^{-1}B=B\cap A\in\mathfrak{B}(A)\}$ is a σ-algebra containing the open subsets of T_1, hence $\mathfrak{B}(T_1)$. This proves the assertion.
(iii): consider the map $F:T_1\times T_2\to T_2\times T_2$ defined by

$$F(t_1,t_2):=(f(t_1),t_2)\quad (t_1\in T_1, t_2\in T_2)$$

By (i) F is $\mathfrak{B}(T_1\times T_2)$ - $\mathfrak{B}(T_2\times T_2)$ measurable. let D be the diagonal in $T_2\times T_2$. Then D is closed, hence Borel, so $F^{-1}D=G(f)\in\mathfrak{B}(T_1\times T_2)$. □

Proposition D.12. *Let T_1,T_2 be Polish and let $f:T_1\to T_2$ be Borel measurable. If $A_1\subset T_1$ and $A_2\subset T_2$ are analytic then fA_1 and $f^{-1}A_2$ are analytic.*

Proof.
a) By lemma D.11 (ii) the restriction $F_{A_1}:A_1\to T_2$ is Borel measurable, so $G(f_{A_1})\in\mathfrak{B}(A_1\times T_2)$, by Lemma D.11 (iii). Again by Lemma D.11 (ii) we may now write $G(f_{A_1})=(A_1\times T_2)\cap B$ with $B\in\mathfrak{B}(T_1\times T_2)$. Since it is immediate the finite and even countable products of analytic sets are analytic, $A_1\times T_2$ is analytic, and so is $(A_1\times T_2)\cap B=G(f_{A_1})$, by Prop. D.6 and Lemma D.5 (ii). Hence there is a Polish space S and a continuous surjection g from S onto $G(f_{A_1})$. Composing g with the continuous projection p_2 from $T_1\times T_2$ onto T_2, we conclude that $(p_2\circ g)S=p_2\,G(f_{A_1})=fA_1$ is analytic.
b) For the second assertion we observe that by similar arguments $G(f)\cap(T_1\times A_2)$ is analytic, and therefore also $p_1G(f)=f^{-1}A_2$, where p_1 denotes the projection from $T_1\times T_2$ on T_1. □

Corollary D.13. *Let T_1,T_2 be Polish and $f:T_1\to T_2$ Borel measurable. Put $S:=fT_1$. Let $g:S\to\mathbb{R}$ be real-valued function. Then*

g is Borel iff $g\circ f$ is Borel.

Proof. Necessity is obvious. For the sufficiency suppose $g\circ f$ is Borel. Let $B\in\mathfrak{B}(\mathbb{R})$ be arbitrary. Then $g^{-1}B=f(g\circ f)^{-1}B$. Since $(g\circ f)^{-1}B$ is Borel, hence analytic, $f(g\circ f)^{-1}B=g^{-1}B$ is analytic by Prop. D.12. A similar argument shows that $S\setminus g^{-1}B=f(g\circ f)^{-1}(\mathbb{R}\setminus B)$ is analytic. Now the separation theorem D.9 says that there are Borel sets $B_1,B_2\subset T_2$ so that

$$B_1\cap B_2=\varnothing,\ g^{-1}B\subset B_1,\ S\setminus g^{-1}B\subset B_2.$$

This implies that $g^{-1}B=B_1\cap S$, so $g^{-1}B\in\mathfrak{B}(S)$, by Lemma D.11 (ii). □

NOTES The above material represents the absolute minimum needed for a full understanding of the main text. Much more about analytic sets can be learned e.g. from [12], [10] and [50].

Appendix E

First class Baire functions

We consider here the class $\mathfrak{B}_1(T)$ of the Baire functions of the first class. The defining property of these functions is that they are pointwise limits of sequences of continuous functions. Below we list several other properties that are equivalent to this for Polish T, but generally different under less severe restrictions on T. The following theorem amply covers our needs.

THEOREM E.1. *Let T be a topological space. Consider the following properties for functions $f:T\to\mathbb{R}$:*

(i) *$f\in\mathfrak{B}_1(T)$, i.e. f is a pointwise limit on T of a sequence of continuous functions.*

(ii) *$f^{-1}F$ is a G_δ in T for every closed $F\subset\mathbb{R}$, or equivalently, $f^{-1}O$ is an F_σ in T for every open $O\subset\mathbb{R}$.*

(iii) *$f\in\mathfrak{B}_r(T)$, i.e. for every closed $L\subset T$ the restriction $f|_L$ has a point of continuity.*

(iv) *For every closed non-empty $L\subset T$ and for all numbers $\alpha<\beta$, $L\cap\{f<\alpha\}$ and $L\cap\{f>\beta\}$ are not both dense in L.*

We have the following implications:

(i) $\Rightarrow$ (ii): *always true.*

(ii) $\Rightarrow$ (i): *if T is a metric space.*

(iii) $\Rightarrow$ (iv): *always true.*

(iv) $\Rightarrow$ (iii): *if T is hereditarily Baire, i.e. if every closed subspace of T is a Baire space.*

(iii) $\Rightarrow$ (ii): *if T separable metric.*

(ii) $\Rightarrow$ (iv): *if T is hereditarily Baire.*

In particular $\mathfrak{B}_1(T)\subset\mathfrak{B}_r(T)$ whenever T is hereditarily Baire (so e.g. when T is compact or complete metric), and $\mathfrak{B}_1(T)=\mathfrak{B}_r(T)$ whenever T is Polish.

PROOF.

(i) $\Rightarrow$ (ii): let $O\subset\mathbb{R}$ be open and suppose $f=\tau_p\text{-}\lim f_n, f_n:T\to\mathbb{R}$ continuous $(n=1,2,\ldots)$. Put $F_n:=\{t\in\mathbb{R}:d(t,\mathbb{R}\setminus O)\geq\frac{1}{n}\}$, where d is the usual metric on $\mathbb{R}$ $(n=1,2,\ldots)$. Then it is easily checked that

$$f^{-1}O=\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty}f_k^{-1}F_n \text{ and this is an } F_\sigma.$$

(ii) $\Rightarrow$ (i) (for T metric): let us first observe that a sum $\sum_{n=1}^{\infty}f_n$ with

$\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$ and all f_n belonging to $\mathfrak{B}_1(T)$, also belongs to $\mathfrak{B}_1(T)$ (exercise). Secondly, there is no loss of generality in supposing that f is bounded, say valued in (0,1), since $\mathbb{R}$ is homeomorphic to (0,1). Now fix $n \in \mathbb{N}$ and consider the sets

$$A_k := \{\frac{k}{n} < f\} \text{ and } B_k := \{f < \frac{k+1}{n}\} \quad (k = 0, \dots, n-1).$$

All A_k and B_k are F_σ-sets by assumption. Fix k and write $A_k = \bigcup_{l=1}^{\infty} F_{k,l}$, $F_{k,l}$ closed. Then $g_k := \sum_{l=1}^{\infty} 2^{-l} \chi_{F_{k,l}}$ is in $\mathfrak{B}_1(T)$ since each $\chi_{F_{k,l}}$ is (exercise) and by the first observation. Similarly we may write $B_k = \bigcup_{l=1}^{\infty} F'_{k,l}$, $F'_{k,l}$ closed, and put $g'_k := \sum_{l=1}^{\infty} 2^{-l} \chi_{F'_{k,l}}$. Now $\{g_k > 0\} = A_k$, $\{g'_k > 0\} = B_k$. The function

$$f_k := \frac{g_k}{g_k + g'_k} \text{ is also in } \mathfrak{B}_1(T) \text{ and satisfies}$$

$$f_k = 0 \text{ on } T \setminus A_k,\ f_k = 1 \text{ on } T \setminus B_k, \text{ and } 0 < f_k < 1 \text{ elsewhere.}$$

Now clearly $\frac{1}{n}(f_1 + ... + f_n)$ approximates f on T to within $\frac{1}{n}$ and is in $\mathfrak{B}_1(T)$. It follows that f, as a uniform limit of functions in $\mathfrak{B}_1(T)$ is also in $\mathfrak{B}_1(T)$ (a uniform limit of a sequence may be represented as a sum of a series of the form Σf_n, $\Sigma \|f_n\| < \infty$, so our first observation applies).

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (iii) (for T hereditarily Baire) are proved in Lemma 3.7.

(iii) $\Rightarrow$ (ii) (for T separable metric): let us fix an open set $O \subset \mathbb{R}$ and an $\epsilon > 0$ and let us put

$$G_\epsilon := \{t \in T : d(f(t), \mathbb{R} \setminus O) \geq \epsilon\} \quad (d \text{ the usual metric on } \mathbb{R}).$$

We shall prove that there exists an F_σ-set $F_\epsilon \subset T$ satisfying

$$G_\epsilon \subset F_\epsilon \text{ and } fF_\epsilon \subset O. \tag{1}$$

This suffices because taking $\epsilon = \frac{1}{n}$, $n = 1,2,...$ it should be clear that $f^{-1}O = \bigcup_{n=1}^{\infty} G_{1/n} = \bigcup_{n=1}^{\infty} F_{1/n}$, and $\bigcup_{n=1}^{\infty} F_{1/n}$ is an F_σ.

For the proof we shall define inductively a strictly decreasing transfinite sequence of closed subsets $(K_\alpha)_{\alpha \leq \alpha_0}$ for some countable ordinal α_0, with $K_{\alpha_0} = \varnothing$ and such that whenever $\alpha < \alpha_0$ and $t \in K_\alpha \setminus K_{\alpha+1}$ then the oscillation of f on K_α in t is $< \epsilon$. More precisely this means that there exists a nbhd V_t of t so that

$$|f(t_1) - f(t_2)| < \epsilon \text{ for all } t_1, t_2 \in K_\alpha \cap V_t.$$

Start with $K_0 := T$. Suppose the K_α have been defined for all $\alpha < \beta$, β a fixed ordinal. If β is a limit ordinal, then define $K_\beta := \bigcap_{\alpha<\beta} K_\alpha$. If $\beta=\alpha+1$, then define

$K_\beta := \{t \in K_\alpha : \forall$ nbhd V_t of t $\exists\, t_1, t_2 \in V_t \cap K_\alpha$ such that $|f(t_1)-f(t_2)| \geqslant \epsilon\}$.

Observe that K_β is closed and $K_\beta \underset{\neq}{\subset} K_\alpha$ since K_β surely does not contain any of the continuity points of $f|_{K_\alpha}$. By the separability of T there is a countable ordinal α with $K_\alpha = \varnothing$. Let α_0 be the first such ordinal. We now have

$$T = \bigcup_{\alpha<\alpha_0} (K_\alpha \setminus K_{\alpha+1}).$$

By definition each $t \in G_\epsilon \cap (K_\alpha \setminus K_{\alpha+1})$ has an open nbhd V_t so that $|f(t_1)-f(t_2)| < \epsilon$ whenever $t_1, t_2 \in V_t \cap K_\alpha$. In particular it now follows from the definition of G_ϵ that

$$f(V_t \cap K_\alpha) \subset O.$$

Now define

$$H_\alpha := \bigcup \{V_t \cap K_\alpha : t \in G_\epsilon \cap (K_\alpha \setminus K_{\alpha+1})\} \quad (\alpha < \alpha_0).$$

H_α is a relatively open subset of the closed set K_α, hence an F_σ. Finally put $F_\epsilon := \bigcup_{\alpha<\alpha_0} H_\alpha$. Then F_ϵ is an F_σ since α_0 is countable, and evidently satisfies (1).

(ii) $\Rightarrow$ (iv) (for T hereditarily Baire):
suppose $\varnothing \neq L \subset T$ is closed and for contradiction suppose

$$\overline{L \cap \{f \leqslant \alpha\}} = \overline{L \cap \{f \geqslant \beta\}} = L \text{ for some numbers } \alpha < \beta.$$

Now $L \cap \{f \leqslant \alpha\}$ and $L \cap \{f \geqslant \beta\}$ are both G_δ subsets of L. Since they are dense in L, so is their intersection, since L is Baire by assumption. But this intersection is empty. Contradiction. □

NOTES The main part of Theorem E.1 is due to R. BAIRE ([3]), see also [50] and [37]). For a thorough discussion of properties (i) - (iv) and many more, see [8].

Appendix F

Filters and ultrafilters

DEFINITION F.1 A collection $\mathfrak{F}$ of subsets of a set X is called a *filter* on X if it satisfies the following conditions:
(F1) if $A \subset B \subset X$ and $A \in \mathfrak{F}$ the $B \in \mathfrak{F}$,
(F2) if $A, B \in \mathfrak{F}$ then $A \cap B \in \mathfrak{F}$,
(F3) $\varnothing \notin \mathfrak{F}$. $\square$

If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two filters on X we say that $\mathfrak{F}_1$ is *finer* that $\mathfrak{F}_2$ (or $\mathfrak{F}_2$ *coarser* the $\mathfrak{F}_1$) if $\mathfrak{F}_2 \subset \mathfrak{F}_1$. A *basis* for a filter $\mathfrak{F}$ is a subfamily $\mathfrak{B}$ of $\mathfrak{F}$ such that every $A \in \mathfrak{F}$ contains a set $B \in \mathfrak{B}$. Put differently, $\mathfrak{B}$ is a basis for $\mathfrak{F}$ iff $\mathfrak{F}$ consists of all "supersets" of sets in $\mathfrak{B}$.

For a non-empty collection $\mathfrak{B}$ of subsets of X to be a basis for a (unique) filter $\mathfrak{F}$ on X, it is necessary and sufficient that the following hold:

(FB1) for all $A, B \in \mathfrak{B}$ there is a $C \in \mathfrak{B}$ such that $C \subset A \cap B$,
(FB2) $\varnothing \notin \mathfrak{B}$.

For any collection $\mathbb{S}$ of subsets of X that satisfies the finite intersection property (FIP), i.e. all of whose finite intersections are non-empty, the collection $\mathfrak{B}$ of all finite intersections satisfies (FB1) and (FB2) and thus is a basis for a filter $\mathfrak{F}$. We say that $\mathbb{S}$ *generates* $\mathfrak{F}$, or is a *subbasis* for $\mathfrak{F}$.

A filter that is maximal, i.e. not properly contained in any finer filter, is called an *ultrafilter*. An easy application of Zorn's lemma shows that any filter is contained in some ultrafilter.

A filter $\mathfrak{F}$ is called *free* (or *non-principal)* if $\cap \mathfrak{F} = \varnothing$. An ultrafilter on X is either free or of the form $\mathfrak{F}_x := \{A \subset X : x \in A\}$ with $x \in X$. The ultrafilters $\mathfrak{F}_x$ are called *fixed* (or *principal).*

The following well-known property characterizes ultrafilters.

PROPOSITION F.2. *A filter $\mathfrak{F}$ on X is an ultrafilter iff $A \cup B \in \mathfrak{F}$ implies that either $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$.*

PROOF. Suppose $A \cup B \in \mathfrak{F}$ but $A \notin \mathfrak{F}$, $B \notin \mathfrak{F}$. Then every $C \in \mathfrak{F}$ intersects A. Indeed, if not, then $C \cap (A \cup B) = C \cap B \subset B$, and so we would have $B \in \mathfrak{F}$, contradicting the assumption. But now $\{C \cap A : C \in \mathfrak{F}\}$ is a basis for a filter finer than $\mathfrak{F}$. Actually it is strictly finer than $\mathfrak{F}$ because it contains A. So $\mathfrak{F}$ is not an ultrafilter.

Conversely, suppose the condition holds and that $\mathcal{G}$ is a filter finer that $\mathcal{F}$. If $A \in \mathcal{G}$ then $X \setminus A \notin \mathcal{G}$ so $X \setminus A \notin \mathcal{F}$. But then by the assumption $A \in \mathcal{F}$, so we have proved that $\mathcal{G} \subset \mathcal{F}$. Hence $\mathcal{F}$ is an ultrafilter. $\square$

REMARK. F.3. The last half of the proof shows that the following formally weaker condition is already sufficient (and of course necessary) for $\mathcal{F}$ to be an ultrafilter: for every $A \subset X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$. $\square$

We say that a filter $\mathcal{F}$ on a subset S of a topological space T *converges* to $t \in \overline{S}$ (notation: $\lim \mathcal{F} = t$ of $\mathcal{F} \to t$) if every neighborhood of t contains a set $A \in \mathcal{F}$. Clearly this is equivalent to saying that $\mathcal{F}$ is finer that the filter of all neighborhouds of t intersected with S. If T is Hausdorff (which we shall suppose for the rest of this section) then limits are always unique. We say that $t \in T$ is a *cluster point* of $\mathcal{F}$ if $t \in \bigcap \{\overline{A} : A \in \mathcal{F}\}$.

Topologies can be described in terms of filters. To see this let us first note the following simple fact.

PROPOSITION F.4. *If $\mathcal{F}$ is a filter on a subset S of a topological space T, then a point $t \in T$ is a cluster point of $\mathcal{F}$ iff there exists a filter $\mathcal{G}$ on S finer than $\mathcal{F}$ (hence also an ultrafilter $\mathcal{G}$ on S refining $\mathcal{F}$) that converges to t.*

PROOF. Clearly the limit of any filter is a cluster point of that filter and of any coarser one, so the sufficiency is obvious. Now let $t \in T$ be a cluster point of a filter $\mathcal{F}$ on S. Then every neighborhood U of t intersects every $A \in \mathcal{F}$. Therefore the sets $\{U \cap A : U \text{ neighborhood of } t,\ A \in \mathcal{F}\}$ form a filter basis $\mathcal{B}$. Evidently the filter $\mathcal{G}$ on S generated by $\mathcal{B}$ is finer that $\mathcal{F}$ and converges to t. $\square$

PROPOSITION F.5. *Let S be a subset of a topological space T. Then $t \in \overline{S}$ iff there exists a filter (or ultrafilter) $\mathcal{F}$ on S that converges to t.*

PROOF. Sufficiency is clear again. On the other hand if $t \in \overline{S}$ then $\{U \cap S : U \text{ nbhd of } t\}$ is a filter basis generating a filter $\mathcal{F}$ on S that converges to t. The same is true for any ultrafilter on S finer that $\mathcal{F}$. $\square$

A convergent filter has a unique cluster point, namely its limit. An ultrafilter has at most one cluster point, since by Prop. F.4 it must converge to any of its cluster points. There may not always be a cluster point, but of course in a compact space there always is. Hence

PROPOSITION F.6. *An ultrafilter on a subset of a compact space converges.* $\square$

Continuity of maps has the following formulation in terms of filters.

PROPOSITION F.7. *Let T_1, T_2 be topological spaces and let $f: T_1 \to T_2$ be a map.*

Then the following are equivalent:
(i) *f is continuous in $t \in T_1$,*
(ii) *for every filter $\mathfrak{F}$ on T_1, $\mathfrak{F} \to t$ implies $f(\mathfrak{F}) \to f(t)$,*
(iii) *for every ultrafilter $\mathfrak{F}$ on T_1: $\mathfrak{F} \to t$ implies $f(\mathfrak{F}) \to f(t)$ ($f(\mathfrak{F})$ denotes $\{f(A): A \in \mathfrak{F}\}$).*

PROOF.
(i) $\Rightarrow$ (ii): if $\mathfrak{F} \to t$, then $\mathfrak{F}$ is finer that the nbhd filter at t. This implies the assertion.
(ii) $\Rightarrow$ (iii): clear.
(iii) $\Rightarrow$ (i): if f is not continuous at t there exists a nbhd V of $f(t)$ such that $f(U) \cap (T_2 \setminus V) \neq \emptyset$ for all nbhds U of t. Thus $f^{-1}(T_2 \setminus V)$ intersects every U, so the sets $f^{-1}(T_2 \setminus V) \cap U$ form a filter basis $\mathfrak{B}$. let $\mathfrak{F}$ be any ultrafilter refining $\mathfrak{B}$. Then $\mathfrak{F} \to t$, whereas $f(\mathfrak{F}) \to f(t)$ fails. □

Occasionally we shall consider limits of indexed sets $\{t_\iota : \iota \in I\}$ in a topological space T. If $\mathfrak{F}$ is a filter on I we write $\lim_{\mathfrak{F}} t_\iota = t$ to mean: for every nbhd U of t there is an $A \in \mathfrak{F}$ such that $\{t_\iota : \iota \in A\} \subset U$.

Let us note the following:

REMARK F.8.
(i) If T is compact and $\mathfrak{F}$ is an ultrafilter on I then $\lim_{\mathfrak{F}} t_\iota$ always exists by Prop. F.6, since $\{\{t_\iota : \iota \in A\} : A \in \mathfrak{F}\}$ is easily seen to be a basis for an ultrafilter on T (use the criterion of Remark F.3).
(ii) If $t \in \overline{\{t_\iota : \iota \in I\}}$ then there is an ultrafilter $\mathfrak{F}$ on I such that $\lim_{\mathfrak{F}} t_\iota = t$. Indeed, if $\mathfrak{G}$ is an ultrafilter on $\{t_\iota : \iota \in I\}$ that converges to t (cf. Prop. F.5), then any ultrafilter $\mathfrak{F}$ on I refining $\{\{\iota \in I : t_\iota \in A\} : A \in \mathfrak{G}\}$ satisfies the requirement.
(iii) If t is a cluster point of $\{t_\iota : \iota \in I\}$, i.e. $t \in \overline{\{t_\iota : t_\iota \neq t\}}$, then there is a free ultrafilter $\mathfrak{F}$ on I so that $\lim_{\iota \in \mathfrak{F}} t_\iota = t$: let $\mathfrak{F}$ be any ultrafilter refining $\{\{\iota \in I : t_\iota \in U \setminus \{t\}\} : U \in \mathfrak{U}\}$, where $\mathfrak{U}$ is a neighborhood basis of t. □

NOTES Elementary facts about filters can be found in [5]. For their application in Banach space theory, see [40] and [90].

Appendix G

The lifting theorem

Let (Ω,Σ,μ) be a finite measure space. Each element of $L^\infty(\mu)$ is an equivalence class of functions in $M(\mu)$. So by the axiom of choice there exists a selection map that singles out one element of $M(\mu)$ from each class. It is not at all clear that this selection can be made in a "consistent" manner, i.e. so that the selection map has nice properties such as linearity, etc. The so-called lifting theorem states that this is possible provided the measure space is complete.

Let us first agree on some notation. Throughout this section (Ω,Σ,μ) will be a *complete* finite measure space. $\mathfrak{N}$ denotes the σ-ideal of μ-null sets. We shall write $f\equiv g$ for "$f=g$ μ a.e.", and $A\equiv B$, resp. $A \Subset B$ for $\mu(A\Delta B)=0$, resp. $\mu(B\setminus A)=0$. The complement $\Omega\setminus A$ of A is denoted as A^c. Finally, $M_b(\mu)$ is the space of bounded μ-measurable functions, which is a Banach space when equipped with the sup norm.

THEOREM G.1. *(The lifting theorem)*
Let (Ω,Σ,μ) be any complete finite measure space. Then there exists a map $\rho:L^\infty(\mu)\to M_b(\mu)$ (called a lifting on $L^\infty(\mu)$) with the following properties:
(i) *$\rho(f)\in f$ for every $f\in L^\infty(\mu)$,*
(ii) *ρ is linear and multiplicative,*
(iii) $\rho(1) = 1$. □

Let us observe that these properties imply:

(iv) ρ is *positive*, i.e. if $f\geqslant 0$ μ a.e., then $\rho(f)\geqslant 0$ everywhere.
[Proof: $f\equiv(\sqrt{f})^2$, so $\rho(f)=\rho(\sqrt{f})^2$ by multiplicativity.]

(v) ρ is *order preserving*, i.e. $f\leqslant g$ μ a.e. implies $\rho(f)\leqslant\rho(g)$.
[Proof: use (iv) and the linearity of ρ.]

(vi) ρ is an *isometry* from $L^\infty(\mu)$ into $M_b(\mu)$.
[Proof: since $\rho(f)\in f$, it is clear that $\|\rho(f)\|\geqslant\|f\|_\infty$. On the other hand $\pm f\leqslant\|f\|_\infty$ μ a.e., so by (v), (iii) and the linearity of ρ, $\pm\rho(f)=\rho(\pm f)\leqslant\rho(\|f\|_\infty)=\|f\|_\infty\rho(1)=\|f\|_\infty$ everywhere. Therefore $\|\rho(f)\|\leqslant\|f\|_\infty$.]

(ii) If $i:M_b(\mu)\to L^\infty(\mu)$ is the canonical map and $\tilde\rho:M_b(\mu)\to M_b(\mu)$ is defined as $\tilde\rho:=\rho\circ i$, then $\tilde\rho$ is an idempotent, i.e. $\tilde\rho^2=\tilde\rho$. Clearly an alternative definition of a lifting on $L^\infty(\mu)$ would be a map $\tilde\rho:M_b(\mu)\to M_b(\mu)$ satisfying, (i), (ii), (iii) as well as $f\equiv g\Rightarrow\tilde\rho(f)=\tilde\rho(g)$.

Suppose now that we have a lifting ρ on $L^\infty(\mu)$. Let $A\in\Sigma$. Then $\rho(\chi_A)=$

$\rho(\chi_A^2) = \rho(\chi_A)^2$ is a $\{0,1\}$-valued μ-measurable function. Hence

$$\rho(\chi_A) = \chi_{\rho(A)} \text{ for a unique } \rho(A) \in \Sigma. \tag{1}$$

The map $\Sigma \ni A \to \rho(A) \in \Sigma$ so defined clearly has all the properties enumerated in the next definition.

DEFINITION G.2. Let $\mathfrak{A}$ be a σ-algebra so that $\mathfrak{N} \subset \mathfrak{A} \subset \Sigma$. A map $\rho: \mathfrak{A} \to \mathfrak{A}$ is called a *set lifting* on $\mathfrak{A}$ if the following properties hold for all $A, B \in \mathfrak{A}$:
(a) $\rho(A) \equiv A$,
(b) $A \equiv B \Rightarrow \rho(A) = \rho(B)$,
(c) $\rho(A \cap B) = \rho(A) \cap \rho(B)$,
(d) $\rho(A^c) = \rho(A)^c$. $\square$

Further properties follow from these, such as:

(e) $\rho(A \cup B) = \rho(A) \cup \rho(B)$.
[Proof: $\rho(A \cup B) = \rho((A^c \cap B^c)^c) = \rho(A^c \cap B^c)^c = (\rho(A^c) \cap \rho(B^c))^c = (\rho(A)^c \cap \rho(B)^c)^c = \rho(A) \cup \rho(B)$.]

(f) $\rho(\varnothing) = \varnothing$ and $\rho(\Omega) = \Omega$.
[Proof: taking $B = A^c$ in (c) and then using (d), yields $\rho(\varnothing) = \varnothing$; $\rho(\Omega) = \Omega$ now follows from (d), taking $A = \varnothing$.]

For later use let us also record the following triviality:

(g) $\mathfrak{F} := \rho(\mathfrak{A})$ is an algebra and for $F_1, F_2 \in \mathfrak{F}$ we have

$$F_1 \Subset F_2 \Rightarrow F_1 \subset F_2, \text{ so } F_1 \equiv F_2 \Rightarrow F_1 = F_2.$$

[Proof: $\rho(\varnothing) = \varnothing$ together with (b) implies $\rho(N) = \varnothing$ $\forall N \in \mathfrak{N}$. Now combine this with (e).]

Another simple but useful fact is:

(h) $\mathfrak{F} := \rho(\mathfrak{A})$ has a maximal element (with respect to inclusion).
[Proof: by (g) any element of $\mathfrak{F}$ with maximal μ-measure is maximal. So choose $E_n \in \mathfrak{F}$ so that $\mu(\bigcup_{n=1}^{\infty} E_n) = \sup \{\mu(E): E \in \mathfrak{F}\}$ and put $E := \rho(\bigcup_{n=1}^{\infty} E_n)$. Then clearly $E \in \mathfrak{F}$ and its μ-measure is maximal.]

The following result now reduces the existence problem for liftings to that for set liftings.

PROPOSITION G.3. *Let $\rho: \Sigma \to \Sigma$ be a set lifting. Then there exists a unique lifting $\rho: L^\infty(\mu) \to M_b(\mu)$ so that*

$$\rho(\chi_A) = \chi_{\rho(A)} \quad \forall A \in \Sigma.$$

Proof. It should be obvious how to proceed. First we define ρ on the dense subspace of $L^\infty(\mu)$ consisting of the simple functions $f=\sum_{i=1}^{n}\alpha_i\chi_{A_i}$ by $\rho(f):=\sum_{i=1}^{n}\alpha_i\chi_{\rho(A_i)}$ We leave it to the reader to verify that this definition does not depend on the way the simple functions are represented. Next, one should check that on the subspace of the simple functions, ρ satisfies (i), (ii) and (iii), and therefore also (iv), (v) and (vi). Finally, as a bounded linear densely defined map, ρ can be uniquely extended boundedly to all of $L^\infty(\mu)$. The extension preserves multiplicativity, and also (i):

$$\text{if } f_n \xrightarrow{L^\infty(\mu)} f \text{ and } \rho(f_n) \xrightarrow{M_b(\mu)} g:=\rho(f), \text{ then}$$

$$\|f-g\|_\infty \leq \|f-f_n\|_\infty + \|f_n-\rho(f_n)\|_\infty + \|\rho(f_n)-g\|\to 0,$$

so $g=\rho(f)\in f$ $\square$

We have now reduced the problem of constructing a lifting on $L^\infty(\mu)$ to that of constructing a set lifting on Σ. The latter will be produced by a Zorn type extension argument, familiar from the proof of the Hahn-Banach theorem. We shall first prove that if $\mathfrak{A}$ is a sub σ-algebra of Σ with $\mathfrak{N}\subset\mathfrak{A}$ and if ρ is a set lifting on $\mathfrak{A}$, then, given any $A\in\Sigma\setminus\mathfrak{A}$, ρ can be extended to a lifting on the σ-algebra generated by A and $\mathfrak{A}$ (Lemma G.7). Next we consider the collection $\mathcal{C}$ of all pair $(\mathfrak{A},\rho)$, where $\mathfrak{A}$ is a σ-algebra satisfying $\mathfrak{N}\subset\mathfrak{A}\subset\Sigma$, and ρ is a set lifting on $\mathfrak{A}$, and we order this collection in the usual way. If now we can show that every chain in $\mathcal{C}$ has an upper bound in $\mathcal{C}$, then we are done. But here a slight technical difficulty arises. Given an increasing countable chain $(\mathfrak{A}_n,\rho_n)$ in $\mathcal{C}$ (this is the only interesting case, Lemma G.8), we shall define a map $\rho:\mathfrak{A}\to\mathfrak{A}$, where $\mathfrak{A}:=\sigma(\bigcup_{n=1}^{\infty}\mathfrak{A}_n)$, so that $\rho|_{\mathfrak{A}_n}=\rho_n$ $(n=1,2,\ldots)$. Unfortunately, this ρ is not in general a set lifting, because (d) may fail. The following lemma is needed to show that a slight modification of ρ (which does not affect its values on $\bigcup_{n=1}^{\infty}\mathfrak{A}_n$) will make it a set lifting.

Definition G.4. Let $\mathfrak{A}$ be a σ-algebra with $\mathfrak{N}\subset\mathfrak{A}\subset\Sigma$. A map $\rho:\mathfrak{A}\to\mathfrak{A}$ is called a *lower density* on $\mathfrak{A}$ if (a), (b), (c) and (f) are satisfied
[an *upper density* would satisfy (a), (b), (e) and (f)]. $\square$

Thus a lower density is a set lifting iff (d) holds.

Lemma G.5. *Let $\mathfrak{A}$ be a σ-algebra so that $\mathfrak{N}\subset\mathfrak{A}\subset\Sigma$. If $\rho_1:\mathfrak{A}\to\mathfrak{A}$ is a lower density on $\mathfrak{A}$, then there exists a set lifting $\rho:\mathfrak{A}\to\mathfrak{A}$ satisfying*

$$\rho_1(A)\subset\rho(A)\subset\rho_1(A^c)^c \text{ for all } A\in\mathfrak{A}. \tag{2}$$

Proof. For any $\omega\in\Omega$ consider

$$\mathfrak{F}(\omega) := \{A \in \mathfrak{A} : \omega \in \rho_1(A)\}$$

If $\omega \in \rho_1(A) \cap \rho_1(B)$, $A, B \in \mathfrak{A}$, then $\omega \in \rho_1(A \cap B)$, by (c). This means that $A \cap B \in \mathfrak{F}(\omega)$ whenever $A, B \in \mathfrak{F}(\omega)$, so that $\mathfrak{F}(\omega)$ is a filter base (note that $\varnothing \notin \mathfrak{F}(\omega)$, by (f)). Now let $\mathfrak{U}(\omega)$ be any ultrafilter on Ω refining $\mathfrak{F}(\omega)$ and let us define

$$\rho(A) := \{\omega \in \Omega : A \in \mathfrak{U}(\omega)\} \quad (A \in \mathfrak{A}).$$

Observe that by definition we have

$$\omega \in \rho_1(A) \Leftrightarrow A \in \mathfrak{F}(\omega) \text{ and } \omega \in \rho(A) \Leftrightarrow A \in \mathfrak{U}(\omega) \tag{4}$$

From this (2) and the properties (a), (b), (c) and (d) for ρ all easily follow:

(d): Using Remark F.3, we have

$$\omega \in \rho(A) \Leftrightarrow A \in \mathfrak{U}(\omega) \Leftrightarrow A^c \notin \mathfrak{U}(\omega \Leftrightarrow \omega \notin \rho(A^c).$$

This proves that $\rho(A)^c = \rho(A^c)$.

(2): Since $\mathfrak{F}(\omega) \subset \mathfrak{U}(\omega)$, (4) immediately implies that $\rho_1(A) \subset \rho(A)$. The other inclusion in (2) follows by complementation, using (d).

(a): It follows from (1) for ρ_1 that both $\rho_1(A) \equiv A$ and $\rho_1(A^c)^c \equiv (A^c)^c = A$. Therefore $\rho(A) \equiv A$, by (2).

(b): Similarly, $A \equiv B$ implies, by (a) and (b) for ρ_1, that $\rho_1(A) = \rho_1(B) \equiv A$ and $\rho_1(A^c)^c = \rho_1(B^c)^c \equiv A$. Hence (2) yield $\rho(A) \equiv \rho(B)$.

(c): $\omega \in \rho(A \cap B) \Leftrightarrow A \cap B \in \mathfrak{U}(\omega) \Leftrightarrow A \in \mathfrak{U}(\omega)$ and $B \in \mathfrak{U}(\omega)$ $\Leftrightarrow \omega \in \rho(A)$ and $\omega \in \rho(B) \Leftrightarrow$ $\omega \in \rho(A) \cap \rho(B)$, so $\rho(A \cap B) = \rho(A) \cap \rho(B)$. □

Remark G.6. In the situation of Lemma G.5, let $\mathfrak{A}_0$ be a σ-algebra such that $\mathfrak{N} \subset \mathfrak{A}_0 \subset \mathfrak{A}$ and suppose that $\rho_1|_{\mathfrak{A}_0}$ is a set lifting on $\mathfrak{A}_0$. Then $\rho_1|_{\mathfrak{A}_0} = \rho|_{\mathfrak{A}_0}$. This is immediate from (2), since for $A \in \mathfrak{A}_0$ we have $\rho_1(A) = \rho_1(A^c)^c$. □

After these preliminaries we now finally start the actual proof of the lifting theorem. The next two lemmas are the basic ingredients.

Lemma G.7. *Let $\mathfrak{A}$ be a σ-algebra such that $\mathfrak{N} \subset \mathfrak{A} \subset \Sigma$. Suppose that ρ is a set lifting on $\mathfrak{A}$ and that $A \in \Sigma \setminus \mathfrak{A}$. Then there exists a set lifting ρ' on $\mathfrak{A}' := \sigma(\mathfrak{A} \cup \{A\})$ extending ρ.*

Proof. We first need to establish a suitable way of representing the elements of $\mathfrak{A}'$. To this end, we put $\mathfrak{F} := \rho(\mathfrak{A})$ (recall that $\mathfrak{F}$ is an algebra) and define

$$\mathfrak{E}_1 := \{E \in \mathfrak{F} : E \cap A^c \in \mathfrak{N}\}, \; \mathfrak{E}_2 := \{E \in \mathfrak{F} : E \cap A \in \mathfrak{N}\}.$$

We claim that both $\mathfrak{E}_1$ and $\mathfrak{E}_2$ contain maximal elements A_1, resp. A_2. It suffices to prove this for $\mathcal{E}_1$, since the proof for $\mathcal{E}_2$ is identical. The argument is the same as for (h). All that needs to be observed is that $\rho(\bigcup_{n=1}^{\infty} E_n)\cap A^c \in \mathfrak{N}$ whenever $E_n \cap A^c \in \mathfrak{N}$ for each $n\in\mathbb{N}$.

Now put $\tilde{A}:=(A\cup A_1)\setminus A_2$ and note that $\tilde{A}\equiv A$, since $\tilde{A}\setminus A\subset A_1\cap A^c\in\mathfrak{N}$ and $A\setminus\tilde{A}=A_2\cap A\in\mathfrak{N}$. The assumption that $\mathfrak{N}\subset\mathfrak{A}$ and the fact that $A\equiv\tilde{A}$ imply that $\mathfrak{A}'=\sigma(\mathfrak{A},\{\tilde{A}\})$. Therefore we have

$$\mathfrak{A}' = \{(C\cap\tilde{A})\cup(D\cap\tilde{A}^c) : C,D\in\mathfrak{A}\}.$$

We are now ready to define ρ' on $\mathfrak{A}'$ by

$$\rho'((C\cap\tilde{A})\cup(D\cap\tilde{A}^c)) := (\rho(C)\cap\tilde{A})\cup(\rho(D)\cap\tilde{A}^c) \quad (C,D\in\mathfrak{A}).$$

The special nature of our choice for $\tilde{A}$ will now be used to show that the definition of ρ' is independent of the chosen representation. Indeed, if

$$(C\cap\tilde{A})\cup(D\cap\tilde{A}^c)\equiv(C_1\cap\tilde{A})\cup(D_1\cap\tilde{A}^c), \text{ with } C,C_1,D,D_1\in\mathfrak{A},$$

then

$$(\rho(C)\cap\tilde{A})\cup(\rho(D)\cap\tilde{A}^c) = (\rho(C_1)\cap\tilde{A})\cup(\rho(D_1)\cap\tilde{A}^c).$$

To see this, observe first that
$C\equiv\rho(C)$, $C_1\equiv\rho(C_1)$ and $C_1\cap\tilde{A}\equiv C\cap\tilde{A}$ imply $\rho(C)\cap\tilde{A}\equiv\rho(C_1)\cap\tilde{A}$. Hence

$$\mu((\rho(C)\Delta\rho(C_1))\cap\tilde{A}) = \mu((\rho(C)\Delta\rho(C_1))\cap A) = 0.$$

This means that $\rho(C)\Delta\rho(C_1)\in\mathcal{E}_2$, and therefore $\rho(C)\Delta\rho(C_1)\subset A_2$, by the maximality of A_2 in $\mathcal{E}_2$. But $\tilde{A}\cap A_2=\varnothing$, so $(\rho(C)\Delta\rho(C_1))\cap\tilde{A}=\varnothing$ or, equivalently, $\rho(C)\cap\tilde{A}=\rho(C_1)\cap\tilde{A}$. Similarly one proves that $\rho(D)\cap\tilde{A}^c=\rho(D_1)\cap\tilde{A}^c$, using this time the maximality of A_1 in $\mathcal{E}_1$ and the fact that $\tilde{A}^c\cap A_1=\varnothing$.

It is immediate from the definition of ρ' that $\rho'|_{\mathfrak{A}}=\rho$, so it remains to prove that ρ' satisfies (a), (b), (c) and (d). All these verifications are straightforward, so we omit the details. Let us only note here that the argument given above to show that ρ' is well-defined, in fact proves (b): μ a.e. sets $(C\cap\tilde{A})\cup(D\cap\tilde{A})$ and $(C_1\cap\tilde{A})\cup(D_1\cap\tilde{A})$ have the same ρ'-image. $\square$

The second basic step in the construction is given next.

LEMMA G.8. *Let $\mathfrak{N}\subset\mathfrak{A}_1\subset\mathfrak{A}_2\subset\cdots\subset\mathfrak{A}_n\subset\cdots$ be an increasing sequence of sub-σ-algebras of Σ and let set liftings ρ_n on $\mathfrak{A}_n$ be given so that $\rho_{n+1}|_{\mathfrak{A}_n}=\rho_n$ $(n=1,2,\ldots)$. Then there exists a set lifting ρ on $\mathfrak{A}:=\sigma(\bigcup_{n=1}^{\infty}\mathfrak{A}_n)$ so that $\rho|_{\mathfrak{A}_n}=\rho_n$ $(n=1,2,\ldots)$.*

PROOF. By Lemma G.5 and Remark G.6 is suffices to define a lower density on $\mathfrak{A}$ extending each ρ_n. To this end, for each $n\in\mathbb{N}$ let us denote the algebra $\rho_n(\mathfrak{A}_n)$ by $\mathfrak{F}_n$. We now define, for all $A\in\mathfrak{A}$, $k\in\mathbb{N}$ and $0<r<1$,

$$\mathfrak{R}(A,k,r) := \{E\in\mathfrak{F}_k : \mu(A\cap F)\geqslant r\mu(F) \text{ for all } F\in\mathfrak{F}_k \text{ with } F\subset E\},$$

$$\rho(A,k,r) := \cup\, \mathfrak{R}(A,k,r)\,[:= \cup \{E : E \in \mathfrak{R}(A,k,r)\}, := \varnothing \text{ if } \mathfrak{R}(A,k,r) = \varnothing.],$$

$$\rho(A) := \bigcap_{0<r<1} \bigcup_{n\in\mathbb{N}} \bigcap_{k\geqslant n} \rho(A,k,r)\ [= \bigcap_{0<r<1} \varliminf_{k\to\infty} \rho(A,k,r)].$$

We will show that ρ is a lower density on $\mathfrak{A}$ extending each ρ_n.

Observe first that each of the families $\mathfrak{R}(A,k,r)$ has a maximal element. The argument for this is that of (h) again. What needs to be checked here is that $\rho_k(\bigcup_{n=1}^{\infty} E_n) \in \mathfrak{R}(A,k,r)$ (for fixed A,k and r) whenever $E_n \in \mathfrak{R}(A,k,r)$ for each $n \in \mathbb{N}$. This is seen as follows. For every $F \subset \rho_k(\bigcup_{n=1}^{\infty} E_n)$ with $F \in \mathfrak{F}_k$ write $F_n := F \cap (E_n \setminus \bigcup_{j=1}^{n-1} E_j)$ and note that $F \equiv \bigcup_{n=1}^{\infty} F_n$. Now it follows that

$$\mu(A \cap F) = \mu(A \cap (\bigcup_{n=1}^{\infty} F_n)) = \sum_{n=1}^{\infty} \mu(A \cap F_n) \geqslant r \sum_{n=1}^{\infty} \mu(F_n) = r\mu(F).$$

Clearly this maximal set is precisely $\rho(A,k,r)$, so that in particular this proves that $\rho(A,k,r) \in \mathfrak{R}(A,k,r) \subset \mathfrak{F}_k \subset \mathfrak{A}$.

Also $\rho(A) \in \mathfrak{A}$. To see this observe that the families $\mathfrak{R}(A,k,r)$, and therefore also the sets $\rho(A,k,r,)$, for fixed A and k, are evidently a decreasing function of r. Hence the first intersection in the definition of $\rho(A)$ may be restricted to rational r.

We note next that whenever $A \in \mathfrak{F}_n$, then for every $E \in \mathfrak{R}(A,k,r)$ with $k \geqslant n$ we must have $E \Subset A$ (otherwise $0 = \mu(A \cap (E \setminus A)) \geqslant r\mu(E \setminus A) > 0$). It is also clear that $A \in \mathfrak{R}(A,k,r)$ in this case, and therefore $E \subset A$, by (g). Hence $\rho(A,k,r) = A$ for $k \geqslant n$. Since this is true for all $0<r<1$, it follows that $\rho(A) = A = \rho_n(A)$. Hence we have shown that $\rho|_{\mathfrak{A}_n} = \rho_n$ $(n = 1,2,\ldots)$.

What is left to be proved now is that ρ satisfies properties (a), (b), (c) and (f).

Proof of (f): this is clear by what we have said above:

$$\rho(\varnothing) = \rho_n(\varnothing) = \varnothing \text{ and } \rho(\Omega) = \rho_n(\Omega) = \Omega \quad (n \in \mathbb{N}).$$

Proof of (b): if $A \equiv B$, then clearly $\mathfrak{R}(A,k,r) = \mathfrak{R}(B,k,r)$ for all k and r, and so $\rho(A) = \rho(B)$.

Proof of (c): let $k \in \mathbb{N}$ and $0<r<1$ be arbitrary. We claim that for all $A, B \in \mathfrak{A}$,

$$\mathfrak{R}(A,k,\frac{r+1}{2}) \cap \mathfrak{R}(B,k,\frac{r+1}{2}) \subset \mathfrak{R}(A \cap B,k,r). \tag{5}$$

Indeed, let E belong to the left member, and let $F \subset E$, $F \in \mathfrak{F}_k$. Then

$$\mu(A \cap B \cap F) = \mu(A \cap F) + \mu(B \cap F) - \mu((A \cup B) \cap F)$$
$$\geqslant \frac{r+1}{2}\mu(F) + \frac{r+1}{2}\mu(F) - \mu(F) = r\mu(F).$$

So $E\in\mathfrak{R}(A\cap B,k,r)$. A direct consequence of (5) is now that

$$\rho(A,k,\frac{r+1}{2})\cap\rho(B,k,\frac{r+1}{2})\subset\rho(A\cap B,k,r).$$

Since this inclusion is true for all k and r, it readily follows that $\rho(A)\cap\rho(B)\subset\rho(A\cap B)$. The reverse inclusion $\rho(A\cap B)\subset\rho(A)\cap\rho(B)$ follows trivially from the fact that for fixed k and r the families $\mathfrak{R}(A,k,r)$ are increasing functions of $A\in\mathfrak{A}$, so that also $\rho(A)$ is increasing in A.

PROOF OF (a): this is somewhat complicated. We introduce the auxiliary function

$$\rho'(A):=\bigcup_{0<r<1}\bigcap_{n\in\mathbb{N}}\bigcup_{k\geqslant n}\rho(A,k,r)\ [=\bigcup_{0<r<1}\overline{\lim_{k\to\infty}}\,\rho(A,k,r)]\quad(A\in\mathfrak{A}).$$

Observe that $\rho'(A)\in\mathfrak{A}$ for all $A\in\mathfrak{A}$, since $\bigcup\limits_{0<r<1}$ can be restricted to the rationals again, by monotonicity. Also note that

$$\rho(A)=\bigcap_{0<r<1}\underline{\lim_{k\to\infty}}\,\rho(A,k,r)\subset\bigcup_{0<r<1}\overline{\lim_{k\to\infty}}\,\rho(A,k,r)=\rho'(A)\quad(A\in\mathfrak{A}).$$

Actually both $\rho(A)$ and $\rho'(A)$ are μ a.e. equal to A. More precisely we will show that for all $A\in\mathfrak{A}$:

(α) $\rho(A)\subset\rho'(A)\Subset A$ (and *therefore also* $\rho(A^c)\subset\rho'(A^c)\subset A^c$),

(β) $\rho(A)\cup\rho'(A^c)=\Omega$.

From (α) and (β) the proof of (a) is immediate .

PROOF OF (α) We already know that $\rho(A)\subset\rho'(A)$. So suppose for contradiction that $\mu(\rho'(A)\setminus A)>0$ for some $A\in\mathfrak{A}$. Then by the definition of $\rho'(A)$, for some $0<r<1$ we have

$$\mu\,((\overline{\lim_{k\to\infty}}\,\rho(A,k,r))\setminus A)=:\alpha>0.$$

Fix $\epsilon>0$. Now since $\mathfrak{A}=\sigma(\bigcup\limits_{n=1}^{\infty}\mathfrak{A}_n)$, we can choose an $n_0\in\mathbb{N}$ and a $B\in\mathfrak{A}_{n_0}$ so that

$$\mu(B\Delta((\overline{\lim_{k\to\infty}}\,\rho(A,k,r))\setminus A)<\epsilon.$$

[Proof: let $\mathfrak{A}'\subset\mathfrak{A}$ consist of all sets $A\in\mathfrak{A}$ such that for every $\epsilon>0$ there exists a $B\in\bigcup\limits_{n=1}^{\infty}\mathfrak{A}_n$ such that $\mu(A\Delta B)>\epsilon$. Then $\mathfrak{A}'$ is a σ-algebra and contains $\bigcup\limits_{n=1}^{\infty}\mathfrak{A}_n$. Hence $\mathfrak{A}'=\mathfrak{A}$.] In particular this implies

$$\mu(A\cap B\cap\overline{\lim_{k\to\infty}}\,\rho(A,k,r))<\epsilon\text{ and }\mu(B\cap\overline{\lim_{k\to\infty}}\,\rho(A,k,r))\geqslant\alpha-\epsilon.\qquad(6)$$

Now let $k\geqslant n_0$ be arbitrary. Since $B\in\mathfrak{A}_k$, we have $\rho_k(B)\in\mathfrak{F}_k$, so it follows from the fact that $\rho(A,k,r)\in\mathfrak{R}(A,k,r)$ that

$$\mu(A\cap B\cap\rho(A,k,r)) = \mu(A\cap\rho_k(B)\cap\rho(A,k,r)) \geqslant r\mu(\rho_k(B)\cap\rho(A,k,r)) = r\mu(B\cap\rho(A,k,r).$$

Writing $C_m := B\cap[\rho(A,m,r)\setminus\bigcup_{k=n_0}^{m-1}\rho(A,k,r)]$ $(m\geqslant n_0)$ and noting that $C_m\in\mathfrak{A}_m$ for $m\geqslant n_0$, we then find

$$\mu(A\cap B\cap[\bigcup_{k=m}^{\infty}\rho(A,k,r)]) = \sum_{k=m}^{\infty}\mu(A\cap C_k)\geqslant r\sum_{k=m}^{\infty}\mu(C_k)$$

$$= r\mu(B\cap[\bigcup_{k=m}^{\infty}\rho(A,k,r)]).$$

Taking the intersection over all $m\geqslant n_0$ we now get

$$\mu(A\cap B\cap\varlimsup_{k\to\infty}\rho(A,k,r))\geqslant r\mu(B\cap\varlimsup_{k\to\infty}\rho(A,k,r)).$$

Combining this with the inequalities (6) yields

$$\epsilon > \alpha - \epsilon.$$

Choosing $\epsilon>0$ small enough, this is a contradiction.

PROOF OF (β) This proof is direct. The main point is to show that

$$\rho(A,k,r)^c\subset\rho(A^c,k,1-r) \text{ for all } k\in\mathbb{N} \text{ and all } 0<r<1. \tag{7}$$

Fix $k\in\mathbb{N}$ and $0<r<1$. Since $\rho(A^c,k,1-r)$ is the maximal element of $\mathfrak{R}(A^c,k,1-r)$, it suffices to show that $\rho(A,k,r)^c\in\mathfrak{R}(A^c,k,1-r)$. We already observed that $\rho(A,k,r)^c\in\mathfrak{F}_k$, so we must show that for every $F\subset\rho(A,k,r)^c$ with $F\in\mathfrak{F}_k$ we have

$$\mu(A^c\cap F)\geqslant(1-r)\mu(F)$$

or, equivalently, that

$$\mu(A\cap F)\leqslant r\mu(F). \tag{8}$$

Now fix $F\in\mathfrak{F}_k$, $F\subset\rho(A,k,r)^c$ and use Zorn's lemma to choose a maximal sequence (E_n) of pairwise disjoint subsets of F such that

$$E_n\in\mathfrak{F}_k \text{ and } \mu(A\cap E_n)<r\mu(E_n) \quad (n=1,2,...).$$

Put $E:=\rho(\bigcup_{n=1}^{\infty}E_n)$. Then $E\in\mathfrak{F}_k$ and we claim that

$$E(\equiv\bigcup_{n=1}^{\infty}E_n)\equiv F. \tag{9}$$

Indeed, if not then $\mu(F\setminus E)>0$, so since $F\setminus E\notin\mathfrak{R}(A,k,r)$ (since $F\subset\rho(A,k,r)^c$), there would be an $E_0\subset F\setminus\bigcup_{n=1}^{\infty}E_n$ with $\mu(E_0)>0$ and $E_0\in\mathfrak{F}_k$ satisfying $\mu(A\cap E_0)<r\mu(E_0)$. But this contradicts the maximality of (E_n), so

(9) is proved. Now (8) follows: $\mu(A\cap F)=\sum_{n=1}^{\infty}\mu(A\cap E_n)<r\sum_{n=1}^{\infty}\mu(E_n)=r\mu(F)$. The proof of ($\beta$) can now be finished quickly, using (7):

$$\begin{aligned}\rho(A)^c &= \bigcup_{0<r<1}\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\rho(A,k,r)^c\\ &\subset \bigcup_{0<r<1}\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\rho(A^c,k,1-r)=\rho'(A),\end{aligned}$$

so $\rho(A)\cup\rho'(A^c)=\Omega$. $\square$

We now put Lemmas G.7 and G.8 together to produce the

PROOF OF THEOREM G.1.
We have seen previously that it suffices to construct a set lifting on Σ. To this end let us consider the collection $\mathcal{C}$ of all pairs $(\mathfrak{A},\rho)$, where $\mathfrak{A}$ is a σ-algebra with $\mathfrak{N}\subset\mathfrak{A}\subset\Sigma$, and ρ is a set lifting on $\mathfrak{A}$. $\mathcal{C}$ is not empty, since clearly $\mathfrak{A}=\mathfrak{N}$ admits a set lifting. We partially order $\mathcal{C}$ as follows:

$$(\mathfrak{A},\rho)\leqslant(\mathfrak{A}',\rho') \text{ iff } \mathfrak{A}\subset\mathfrak{A}' \text{ and } \rho'|_{\mathfrak{A}}=\rho.$$

We now verify that $\mathcal{C}$ is inductive, so that Zorn's lemma can be used to produce a maximal element in $\mathcal{C}$.

Let $\mathcal{C}_0=\{(\mathfrak{A}_\iota,\rho_\iota):\iota\in I\}$ be a chain in $\mathcal{C}$. There are now two cases to be considered:

CASE 1. $\mathcal{C}_0$ has no countable cofinal subfamily. In this case clearly $\mathfrak{A}:=\bigcup_{\iota\in I}\mathfrak{A}_\iota$ is a σ-algebra. If we now define $\rho:\mathfrak{A}\to\mathfrak{A}$ by $\rho(A):=\rho_\iota(A)$ whenever $A\in\mathfrak{A}_\iota$, then ρ is a set lifting on $\mathfrak{A}$, so then $(\mathfrak{A},\rho)\in\mathcal{C}$ and evidently is an upper bound for $\mathcal{C}_0$.
CASE 2. $\mathcal{C}_0$ has a countable cofinal subfamily $\{(\mathfrak{A}_{\iota_n},\rho_{\iota_n}):n\in\mathbb{N}\}$. Clearly we may assume that this is an increasing sequence. Now Lemma G.8 provides a lifting ρ on $\mathfrak{A}:=\sigma(\bigcup_{n\in\mathbb{N}}\mathfrak{A}_{\iota_n})=\sigma(\bigcup_{\iota\in I}\mathfrak{A}_\iota)$ which extends each ρ_{ι_n}. Hence $(\mathfrak{A},\rho)$ is an upper bound for $\mathcal{C}_0$ in $\mathcal{C}$.

Now let $(\mathfrak{A},\rho)$ be a maximal element in $\mathcal{C}$. Then $\mathfrak{A}=\Sigma$ by Lemma G.7, so we are done. $\square$

REMARK G.9. It is interesting to note that for $1\leqslant p<\infty$, $L^p(\mu)$ never admits a lifting, except in the trivial case when (Ω,Σ,μ) is purely atomic. Even linear liftings do not exist.

To be more precise, let (Ω,Σ,μ) be a complete non-purely atomic finite measure space and fix $1<p<\infty$. Then there exists no map $\rho:L^p(\mu)\to M(\mu)$ satisfying
(i) $\rho(f)\in f$ for every $f\in L^p(\mu)$,
(ii) ρ is linear,
(iii) ρ is positive, i.e. $f\geqslant 0$ μ a.e. implies $\rho(f)\geqslant 0$ everywhere,
(iv) $\rho(1)=1$.

Firstly, there is no loss of generality in supposing that (Ω,Σ,μ) is atomless, for if Ω_1 is an atomless part of Ω, then if ρ were a lifting on $L^p(\mu)$, the "restriction" of ρ to $L^p(\Omega_1,\mu|_{\Omega_1})$ would be a lifting on $L^p(\Omega_1,\mu|_{\Omega_1})$. So suppose (Ω,Σ,μ) is an atomless probability space and let ρ be as above. Then for each $n\in\mathbb{N}$ we can choose a partition $\{E_1^n,\ldots,E_n^n\}$ of Ω so that

$$E_j^n\in\Sigma \text{ and } \mu(E_j^n)=\frac{1}{n} \quad (j=1,\ldots,n).$$

If we put $A_j^n:=\{\omega\in\Omega:\rho(\chi_{E_j^n})(\omega)=1\}$, then $A_j^n\equiv E_j^n$ $(j=1,\ldots,n)$ and therefore it is evident that

$$\mu\Big(\bigcap_{n=1}^{\infty}\bigcup_{j=1}^{n}A_j^n\Big)=1.$$

Now pick any $\omega_0\in\bigcap_{n=1}^{\infty}\bigcup_{j=1}^{n}A_j^n$ and consider the positive, and therefore bounded linear functional

$$L^p(\mu)\ni f\xrightarrow{\Phi}\rho(f)(\omega_0).$$

By the choice of ω_0, for each $n\in\mathbb{N}$ we have $\rho(\chi_{E_{j_n}^n})(\omega_0)=1$ for some $j_n\in\{1,\ldots,n\}$. But then

$$1=\rho(\chi_{E_{j_n}^n})(\omega_0)=\Phi(\chi_{E_{j_n}^n})\leqslant\|\Phi\|\,\|\chi_{E_{j_n}^n}\|_p=\|\Phi\|n^{-1/p}\to 0$$

as $n\to\infty$, so we have a contradiction. □

NOTES For [0,1] with the Lebesgue measure the lifting theorem is due to J. von Neumann ([61]). Apparently he also found a proof for the general case but never wrote it down. D. Maharam ([52]) provided one, reducing the general case to that of an infinite product of unit intervals. A little later A.I. and C.I. Tulcea ([43]) gave a proof that avoided the use of any isomorphism theorem à la Maharam. They worked directly on the abstract measure space.

To those familiar with martingales the statement of Lemma G.8 (the most non-trivial item in the proof) will suggest a martingale approach. In fact, if one is interested only in linear, positive and isometric liftings (and does not care about multiplicativity), then a relatively simple martingale proof is possible, see e.g. P.A. Meyer ([54]). The reason Meyer's proof does not directly produce multiplicative liftings is that conditional expectation operators obviously fail to be multiplicative. A.I. and C.I. Tulcea got around this difficulty, essentially by combining the martingale approach with an extreme point argument to get a multiplicative lifting. Finally T. Traynor ([94]) succeeded in eliminating all sophistication from the proof by translating these ideas into pure measure theory. We have followed his presentation here, because we actually need the multiplicativity (in Ch. 6) and because Traynor's proof is in fact not much longer than that of the Tulceas.

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