CWI Tracts

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CWI Tract 57

Exponential type calculus for linear delay equations

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Preface

My interest in delay equations started with a course on delay equations by Odo Diekmann in the autumn of 1981 at the University of Amsterdam. My own research in this area started with the thesis of Stephan van Gils and continued over the years to result in my thesis in 1988. I am indebted to many persons and institutes. Especially, the Centre for Mathematics and Computer Science my "second home" from 1982 to 1988; the Mathematical Institute of the University of Leiden and especially prof. dr. ir. L.A. Peletier for offering me the possibility to become an assistent at this institute while continuing my research in delay equations; and my family.

Finally, I thank Odo Diekmann for all his support over the years.

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CHAPTER 1. INTRODUCTION

In this monograph expansions of solutions of linear delay equations in terms of generalized eigenfunctions are considered. First we recall these expansions for linear ordinary differential equations.

Consider the system described by the equation

$$\dot{x}(t) = ax(t) - dx(t) = (a-d)x(t), \tag{1.1}$$

where

$$x := \text{amount of some substance;}$$
$$a := \text{creation rate;}$$
$$d := \text{decay rate.}$$

The equation (1.1) is autonomous - i.e. translation invariant - so suppose that (1.1) governs the system for $t \geq 0$, then an initial condition $x_0 \in \mathbf{R}$ uniquely determines, via $x(0) = x_0$, the future state $x(t)$ at time t.

Next, consider more generally the linear autonomous system of equations

$$\dot{x} = Ax, \tag{1.2}$$

where A is a $n \times n$-matrix with constant coefficients. The general solution is given by

$$x(t) = e^{At}x_0, \tag{1.3}$$

where $x_0 \in \mathbf{R}^\mathbf{n}$ and

$$e^{At} := \sum_{j=0}^{\infty} \frac{A^j}{j!}t^j. \tag{1.4}$$

The matrix e^{At} has the form

$$\sum_{j=1}^{n} P_j(t)e^{\lambda_j t}, \tag{1.5}$$

where the P_j are $n \times n$-matrix-valued polynomials and the λ_j are eigenvalues of the matrix A; that is, the λ_j are the roots of the characteristic equation

$$\det(zI - A) = 0. \tag{1.6}$$

The coefficients of the matrix valued functions P_j are determined from the generalized eigenvectors v corresponding to the eigenvalue λ_j, i.e. the solutions of the equation

$$(\lambda_j I - A)^m v = 0 \tag{1.7}$$

for some m. As a consequence of the above representation of the solutions of equation (1.2), complete information is obtained from the eigenvalues of the matrix A.

Next, consider the method of Laplace transformation. Laplace transformation of the equation (1.2) yields

$$L\{x\}(z) = (zI - A)^{-1} x_0, \tag{1.8}$$

where

$$L\{x\}(z) = \int_0^\infty e^{-zt} x(t) dt. \tag{1.9}$$

Choose γ such that all the zeros of

$$\det(zI - A) = 0 \tag{1.10}$$

lie to the left of the line $\Re(z) = \gamma$. From the inversion formula for the Laplace transform we obtain the following representation:

$$x(t) = \frac{1}{2\pi i} \int_{\Re(z)=\gamma} e^{zt} (zI - A)^{-1} x_0 dz \qquad \text{for} \quad t > 0 \tag{1.11}$$

of the solution x of equation (1.2) with initial condition $x(0) = x_0$. Remark that the so-called characteristic matrix of equation (1.2)

$$(zI - A)^{-1} = \frac{\operatorname{adj}(zI - A)}{\det(zI - A)} \tag{1.12}$$

is a rational matrix-valued function that is $O(\frac{1}{|z|})$ on large circles in the complex plane. Hence, from Cauchy's residue theorem and the representation (1.11) we deduce

$$x(t) = \sum_{j=1}^{n} \operatorname*{Res}_{z=\lambda_j} e^{zt} (zI - A)^{-1} x_0, \tag{1.13}$$

where λ_j is a zero of $\det(zI - A) = 0$. A staightforward calculation of Laurent series shows that the residues are exactly of the form (1.5). Consequently, the solution $x(t)$ of (1.2) with initial condition $x(0) = x_0$ can be represented as a finite sum of residues

$$x(t) = \sum_{j=1}^{n} p_j(t)e^{\lambda_j t}, \tag{1.14}$$

where $p_j(t) = P_j(t)x_0$ is an n-vector-valued polynomial in t.

The solution x of equation (1.2) is a function of t and the initial condition x_0. To study the flow defined by the solutions of equation (1.2), define the family of bounded linear operators

$$T(t)x_0 = e^{At}x_0. \tag{1.15}$$

So $T(t_0)$ maps the state at time $t = 0$ onto the state at time $t = t_0$. From this definition the following properties are clear:

(i) $T(0) = I$;

(ii) For all $t_1, t_2 \in \mathbf{R}$: $\quad T(t_1 + t_2) = T(t_1)T(t_2)$;

(iii) $T(t)x \to T(t_0)x$ as $t \to t_0$,
(uniformly in x with $\|x\| \leq 1$).

After this introduction to linear systems of the form (1.2), we introduce more realism in the model (1.1) by taking some aspect of age into account. Suppose that (1.1) describes the evolution of some collection of individuals and assume that an individual that was born at time $t = t_0$ can take part in the reproduction process for $t \geq t_0 + h$, where h denotes the time lag, or delay in the system (and where the decay in a time interval of length h is incorporated by an adaptation of the birth parameter a).

Thus we obtain an autonomous linear differential difference equation

$$\dot{x}(t) = ax(t-h) - dx(t). \tag{1.16}$$

The first question that comes to mind is the following: what is the initial value problem for equation (1.16)? More specifically, what is the minimum amount of initial data that must be specified in order for equation (1.16) to define a unique function for $t \geq 0$. Clearly,

$$\dot{x}(t) = \begin{cases} \text{given function} - dx(t) & \text{for} \quad 0 \leq t \leq h, \\ ax(t-h) - dx(t) & \text{for} \quad t \geq h. \end{cases} \tag{1.17}$$

will do.

Hence, the initial condition that must be specified is a function on the whole interval $[-h, 0]$. We shall formulate equation (1.16) as the following initial value problem

$$\begin{aligned} \dot{x}(t) &= ax(t-h) - dx(t) \qquad \text{for} \quad t \geq 0, \\ x(t) &= \varphi(t) \qquad \text{for} \quad -h \leq t \leq 0, \end{aligned} \tag{1.18}$$

where φ is a given continuous function.

Note that from (1.17) that we can explicitly calculate the solution x of (1.18) by the method of steps (i.e. we use φ such that the solution x on $[0, h]$ satisfies the linear inhomogeneous ordinary differential equation

$$\dot{x}(t) = a\varphi(t-h) - dx(t) \text{ with } x(0) = \varphi(0),$$

which can be solved by means of the variation-of-constants formula. And then we use the solution on $[0, h]$ to calculate the solution on $[h, 2h]$ in the same manner, etc.)

Define the state space to be $\mathcal{C} = C[-h, 0]$. Provided with the supremum norm

$$||\varphi||_u = \sup_{-h \leq t \leq 0} |\varphi(t)|,$$

the state space $\mathcal{C}$ becomes a Banach space.

The next goal is to understand the geometric behaviour of the solutions of equation (1.18) when they are interpreted as orbits in the state space $\mathcal{C}$. Define the state of the solution x at time t_1 by

$$x_{t_1}(\tau) := x(t_1 + \tau; \varphi) \qquad \text{for} \quad -h \leq \tau \leq 0. \tag{1.19}$$

Then we can define a family of bounded linear operators $T(t) : \mathcal{C} \to \mathcal{C}$ by

$$T(t)\varphi := x_t(\,\cdot\,; \varphi). \tag{1.20}$$

Frequently we will not express the φ dependence in our notation and simply write $T(t)\varphi := x_t$.

From the uniqueness of solutions of equation (1.18), it is obvious that the family of operators $T(t)$ has the group property

$$T(t_1)T(t_2) = T(t_1 + t_2).$$

However, the operator $T(t)$ is only defined for $t \geq 0$ and there will be no extension of $\{ T(t) \}_{t \geq 0}$ to a group of operators defined on $\mathbf{R}$. This can be seen from the property that the solution becomes more smooth with increasing t, i.e. $T(nh)\varphi$ is n-times differentiable. Therefore, a possible definition of

$T(t)$ for negative t would impose additional smoothness conditions on the elements of the state space.

The family of operators $\{T(t)\}_{t\geq 0}$ is called a $\mathcal{C}_0$-semigroup of operators, since

(i) $T(0) = I$;

(ii) For all $t_1, t_2 \geq 0$: $\quad T(t_1 + t_2) = T(t_1)T(t_2)$;

(iii) For every $\varphi \in C$:

$$\lim_{t\downarrow 0} \|T(t)\varphi - \varphi\|_u = 0$$

(translation along a continuous function is continuous).

With every $\mathcal{C}_0$-semigroup $\{T(t)\}$ on a Banach space X we can associate an infinitesimal generator A defined by

$$A\varphi = \lim_{t\downarrow 0} \frac{1}{t}\,[T(t)\varphi - \varphi] \tag{1.21}$$

for all $\varphi \in \mathcal{D}(A)$, that is, for all $\varphi \in X$ for which the limit exists in the norm topology of X. In the example (1.15) the infinitesimal generator is just the $n \times n$-matrix A and hence, a bounded operator from $\mathbf{R}^n$ into $\mathbf{R}^n$. In case $\{T(t)\}$ is defined by (1.20) the operator A is unbounded:

$$A\varphi = \dot{\varphi} \tag{1.22}$$

for all $\varphi \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \{\varphi \in C \mid \dot{\varphi} \in C,\ \dot{\varphi}(0) = a\varphi(-h) - d\varphi(0)\}\,. \tag{1.23}$$

Remark that all information about the particular equation appears in the domain of A!

To obtain the characteristic equation for (1.18) we must try to find nontrivial solutions of equation (1.18) of the form

$$x(t) = e^{\lambda t}x_0. \tag{1.24}$$

If we substitute expressions of this type into the delay differential equation we arrive at

$$\lambda e^{\lambda t}x_0 = ae^{\lambda t}e^{-\lambda h}x_0 - de^{\lambda t}x_0. \tag{1.25}$$

Hence, $x(t) = e^{\lambda t}x_0$ is a solution of (1.18) if and only if

$$\lambda - ae^{-\lambda h} + d = 0. \tag{1.26}$$

This is a transcendental equation and it has infinitely many solutions located in some left half plane $\Re(z) \leq \gamma$. The "infinity" reflects the fact that the state space is infinite dimensional. Hence, it is not obvious that the solutions of equation (1.18) can be obtained as linear combinations of the generalized eigenfunctions. Even without discussing the question of representation of solutions in terms of series, it is not obvious that the asymptotic behaviour of the solutions of equation (1.18) is determined by the solutions of the characteristic equation (1.26). We shall attack these problems through the Laplace transform.

The transformed equation (1.18) reads

$$L\{x_{-h}\}(z) = \frac{e^{-zh}\left\{\varphi(0) + (z+d)\int_{-h}^{0} e^{-zt}\varphi(t)dt\right\}}{z - ae^{-zh} + d}, \tag{1.27}$$

where

$$L\{x_{-h}\}(z) = \int_0^\infty e^{-zt}x(t-h)dt. \tag{1.28}$$

Choose γ_0 such that all the zeros of

$$z - ae^{-zh} + d = 0$$

lie to the left of the line $\Re(z) = \gamma_0$. From the inversion formula for the Laplace transform we obtain the following representation:

$$x(t-h) = \frac{1}{2\pi i}\int_{\Re(z)=\gamma_0} e^{zt}L\{x_{-h}\}(z)dz \qquad \text{for} \quad t \geq 0 \tag{1.29}$$

of the solution of equation (1.18).

From equation (1.27) it is not difficult to see that the residues of

$$e^{zt}L\{x_{-h}\}(z)$$

are just generalized eigenfunctions and to find expansions as linear combinations of these generalized eigenfunctions we can shift the contour $\Re(z) = \gamma$ to the left. Using simple contour integration and the Riemann-Lebesgue lemma one can prove

$$x(t) = \sum_{\Re(\lambda_j)>\gamma} p_j(t)e^{\lambda_j t} + o(e^{\gamma t}) \qquad \text{as} \quad t \to \infty, \tag{1.30}$$

where for each j, the polynomial p_j is an n-vector polynomial.

Hence the asymptotic behaviour of the solutions of the equation (1.18) is indeed determined by the roots of the characteristic equation (1.26). From the results of Bellman and Cooke [4] about the growth properties of (1.26), one can actually prove that the series in (1.30) converges to the solution for $t > h$, i.e.

$$\lim_{\gamma \to -\infty} \|T(t)\varphi - \sum_{\Re(\lambda_j) > \gamma} p_j(t + \cdot)e^{\lambda_j(t+\cdot)}\|_u = 0 \qquad \text{for} \quad t > h. \tag{1.31}$$

See Banks and Manitius [2], Levinson and McCalla [27] for the details.

So for scalar equations the state space approach provides, embodied in (1.30) and (1.31), the natural generalization of the finite dimensional theory for linear autonomous differential equations. We emphasize, however, that in case we are dealing with systems of delay equations peculiar phenomena may occur. For example, consider the system of differential difference equations

$$\dot{x}(t) = Ax(t-h) - Dx(t), \tag{1.32}$$

where now A, D denote $n \times n$-matrices. The exponential estimate (1.30) still holds and if A is non-singular, Banks and Manitius [2] proved that (1.31) also holds. However, if A is singular, the system (1.32) has solutions that are identically zero after finite time. The existence of these solutions implies that (1.31) can not hold for $t \geq h$.

The asymptotic behaviour (1.30) will be the central theme of our work. Consider the following class of linear autonomous equations, so-called retarded functional differential equations (RFDE),

$$\begin{aligned} \dot{x}(t) &= \int_0^h d\zeta(\theta)x(t-\theta) \qquad \text{for} \quad t \geq 0, \\ x_0 &= \varphi, \end{aligned} \tag{1.33}$$

where $\varphi \in \mathcal{C} = C[-h, 0]$ and ζ is a $n \times n$ -matrix-valued function that belongs to NBV$[0, h]$, i.e. each element ζ_{ij} of ζ is of bounded variation, satisfies $\zeta_{ij}(0) = 0$ and is continuous from the left.

For this class of equations, we shall carefully analyse both the behaviour of the sums

$$\sum_{\Re(\lambda_j) > \gamma} p_j(t)e^{\lambda_j t}$$

and of the remainder term

$$o(e^{\gamma t}) \qquad \text{as} \quad \gamma \to -\infty$$

in (1.30). To study the remainder term in (1.30), we first consider the concept of small solutions: a solution x of (1.33) is called a small solution if for every $k \in \mathbf{R}$

$$\lim_{t\to\infty} e^{kt}x(t) = 0.$$

Let $\{T(t)\}$ denote the semigroup associated with (1.33). Let

$$\mathcal{N}(T(t)) = \{\varphi \in \mathcal{C} \mid T(t)\varphi = 0\}$$

denote the null space of $\{T(t)\}$ and let

$$\mathcal{R}(T(t)) = \{\varphi \in \mathcal{C} \mid \exists \chi \in \mathcal{C} : T(t)\chi = \varphi\}$$

denote the range of $\{T(t)\}$. Define the ascent α of $\{T(t)\}$ by

$$\alpha = \inf \{t \mid \forall \epsilon > 0 : \mathcal{N}(T(t)) = \mathcal{N}(T(t+\epsilon))\}. \tag{1.34}$$

Let δ denote the ascent of the adjoint semigroup $\{T^*(t)\}$. The following results are due to Henry [20]:

(i) $\alpha, \delta \leq nh$;

(ii) Small solutions are in the kernel of $T(\alpha)$;

(iii) The closure of the generalized eigenspace $\mathcal{M}_\mathcal{C}$ equals

$$\overline{\mathcal{M}_\mathcal{C}} = \overline{\mathcal{R}(T(\delta))}.$$

Note that because of (iii), the condition $\delta = 0$ implies completeness of the system of generalized eigenfunctions - i.e. $\overline{\mathcal{M}_\mathcal{C}} = \mathcal{C}$. Also note that $\mathcal{R}(T(t))$ "decreases" with increasing time t because the solution becomes more smooth, but that the closure of the range $\overline{\mathcal{R}(T(t))}$ becomes stable after finite time.

The following questions were the motivation for a further study of the linear autonomous RFDE (1.33).

QUESTION I. Does $\alpha = \delta$ hold?

QUESTION II. Is there an explicit characterization of the ascents α, δ in terms of the kernel ζ such that completeness can be verified easily?

QUESTION III. Is there an explicit characterization of the closure of the generalized eigenspace $\mathcal{M}_\mathcal{C}$ such that it can be verified when the semigroup $\{T(t)\}$ acts injectively on $\overline{\mathcal{M}_\mathcal{C}}$?

QUESTION IV. Does the state space decomposition

$$\mathcal{C} = \overline{\mathcal{M}_\mathcal{C}} \oplus \mathcal{N}\big(T(\alpha)\big) \quad \text{hold?}$$

And in addition: Is $\overline{\mathcal{M}_\mathcal{C}}$ the proper state space restriction when completeness fails?

QUESTION V. Recall the asymptotic estimate (1.30). What are the conditions on the initial condition φ so that all information about the solution $x(\,\cdot\,;\varphi)$ is contained in the series expansion

$$\sum_{\Re(\lambda_j)>\gamma} p_j(t)e^{\lambda_j t} \qquad \text{as} \quad \gamma \to -\infty?$$

Since the initial condition φ is given, it is not necessary to expand φ itself in a series and one can ask for conditions on φ such that

$$\lim_{\gamma\to-\infty} \|T(t)\varphi - \sum_{\Re(\lambda_j)>\gamma} p_j(t+\,\cdot\,)e^{\lambda_j(t+\,\cdot\,)}\|_u = 0 \qquad \text{for} \quad t > h. \tag{1.35}$$

In this tract we shall discuss and answer the above questions. For Question V, only a partial answer is available. Further research is needed to study summation techniques to see, in case (1.35) does not hold, in which sense the series of residues corresponding to a solution still contains all the information about this solution.

For our analysis we shall exploit the close connection between delay equations and Volterra convolution equations and develop an exponential type calculus that yields detailed information about the solution from the properties of the Laplace transform of the solution.

The organization of this tract is as follows. Chapter 2 contains a short course on Riemann-Stieltjes integrals and Chapter 3 recalls some basic facts about Laplace transformation. In Chapter 4, we introduce an exponential type calculus for a class of entire functions which will turn out to be very useful when deriving, in Chapter 7, convergence criteria for the Fourier type expansion

$$\sum_{\Re(\lambda_j)>\gamma} p_j(t)e^{\lambda_j t} \qquad \text{as} \quad \gamma \to -\infty$$

of the solution of (1.33).

In Chapter 5 we study the asymptotic behaviour of entire functions of the form

$$F(z) = z^n + \int_0^{\tau_1} e^{-zt}d\eta_1(t)z^{n-1} + \cdots + \int_0^{\tau_n} e^{-zt}d\eta_n(t), \tag{1.36}$$

where $\eta_i \in \mathrm{SBV}[0, \tau_i]$ for $1 \leq i \leq n$, i.e. $\eta_i \in \mathrm{NBV}[0, \tau_i]$ with the additional property that there exists a constant b_{ij} such that

$$\lim_{t \uparrow b_{ij}} \zeta_{ij}(t) \neq \zeta_{ij}(b_{ij}) = \zeta(h),$$

i.e. the coefficients ζ_{ij} of ζ jump before they become constant.

Since $\eta_i \in \mathrm{SBV}[0, \tau_i]$ we can associate with F a Newton polygon and study the asymptotic behaviour of F using this Newton polygon. The results extend and generalize results obtained by Bellman and Cooke [4].

We also study the behaviour of F when we drop the jump condition and assume that $\eta_i \in \mathrm{NBV}[0, \tau_i]$. In this case the Newton polygon does not control the behaviour of F and we can not use the special structure of F anymore. However, we can apply classical complex analysis to derive estimates for F which turn out to be sufficient for the applications in Chapter 7.

In Chapter 6, we study the Volterra convolution equation

$$x - \zeta * x = f, \tag{1.37}$$

where f is a continuous function defined on $[0, \infty)$ that is constant on $[h, \infty)$ and the kernel ζ is a $n \times n$-matrix-valued function that belongs to $\mathrm{NBV}[0, h]$. First we obtain an analytic continuation for the Laplace transform of x

$$L\{x\}(z) = \Delta^{-1}(z)\big(f(0) + z \int_0^h e^{-zt}(f(t) - f(h))dt\big), \tag{1.38}$$

where $\Delta^{-1}(z)$ denotes the inverse of the characteristic matrix

$$\Delta(z) = zI - \int_0^h e^{-zt} d\zeta(t). \tag{1.39}$$

From this analytic continuation we derive the asymptotic estimates

$$x(t) = \sum_{\Re(\lambda_j) > \gamma} p_j(t) e^{\lambda_j t} + o(e^{\gamma t}) \qquad \text{as} \quad t \to \infty, \tag{1.40}$$

where the summation extends over the zeros λ_j of $\det \Delta(z)$ in the right half plane $\Re(z) > \gamma$ and

$$p_j(t) e^{\lambda_j t} = \operatorname*{Res}_{z = \lambda_j} e^{zt} \Delta^{-1}(z)\big(f(0) + z \int_0^h e^{-zt}(f(t) - f(h))dt\big). \tag{1.41}$$

In Chapter 7 and Chapter 8 we study the behaviour of the remainder term in the asymptotic expansion (1.40). First we restrict ζ to the class $\mathrm{SBV}[0,h]$ and study conditions on f under which

$$x(t) = \lim_{\gamma \to -\infty} \sum_{\Re(\lambda_j) > \gamma} p_j(t) e^{\lambda_j t} \qquad \text{for} \quad t > 0, \tag{1.42}$$

uniformly on compact sets. Next we show that for $\zeta \in \mathrm{SBV}[0,h]$ the series expansion always converges after finite time.

For arbitrary $\zeta \in \mathrm{NBV}[0,h]$ we can not prove such results. The reason is that for $\zeta \in \mathrm{SBV}[0,h]$ the exponential type calculus controls the exponential growth of (1.38) and the jump condition on ζ controls the polynomial growth of (1.38). If we drop the jump condition on ζ we can not control this polynomial growth anymore. Therefore, conditions on f such that (1.42) holds are difficult to formulate. This also implies that it is not clear whether the series (1.42) converges after finite time. We can, however, state simple conditions on f such that the solution x can be given by a limit of convergent series expansions. To formulate these conditions, let $\mathcal{F}$ denote the Banach space of all continuous functions defined on $[0,\infty)$ that are constant on $[h,\infty)$ provided with supremum norm. We first characterize $\overline{\mathcal{A}}_{\mathcal{F}}$, the closure of the set of all forcing functions $f \in \mathcal{F}$ such that the solution $x = x(\,\cdot\,; f)$ of the Volterra convolution equation (1.37) has a backward continuation over $(-\infty, 0]$, i.e. for all $\sigma \in [0,\infty)$ there exists an $g \in \mathcal{F}$ such that

$$x(-\sigma + \,\cdot\,) - \zeta * x(-\sigma + \,\cdot\,) = g. \tag{1.43}$$

The characterization of $\overline{\mathcal{A}}_{\mathcal{F}}$ can be formulated as follows: $f \in \overline{\mathcal{A}}_{\mathcal{F}}$ if and only if the analytic continuation of the Laplace transform of x has no exponential growth in the left half plane, i.e. in formula (1.38) the exponential type of the numerator is less than or equal to the exponential type of the denominator.

In Chapter 8 we study the remainder term in (1.40) and we give a characterization of the set of solutions x such that

$$x(t) = o(e^{\gamma t}) \qquad \text{as} \quad t \to \infty \tag{1.44}$$

for all $\gamma \in \mathbf{R}$. From (1.44) it follows that $L\{x\}$ is entire. An application of the Paley-Wiener theorem now yields Henry's theorem on small solutions: there exists some finite $\alpha \geq 0$ such that

$$x(t) = 0 \qquad \text{for} \quad t \geq \alpha. \tag{1.45}$$

The main result of this chapter will be an explicit expression for α solely in terms of the kernel ζ. An application of the results of Chapter 7 and 8 yields the following result: *Backward continuation is unique within* $\overline{\mathcal{A}}_{\mathcal{F}}$.

In Chapter 9 we apply the results of Chapter 7 to the fundamental solution and give a series expansion for $x(\,\cdot\,; f)$ of (1.37) under weaker conditions on the forcing function f then used in Chapter 7.

In the Chapters 10 and 11 we give an introduction to the semigroup approach for RFDEs

$$\begin{aligned} \dot{x}(t) &= \int_0^h d\zeta(\theta)x(t-\theta) \qquad \text{for} \quad t \geq 0, \\ x_0 &= \varphi, \end{aligned} \tag{1.46}$$

where $\varphi \in \mathcal{C} = C[-h,0]$ and we consider the method of spectral projections developed by Hale [18] and others.

Since for this class of problems the spectral projections have a finite dimensional range, the method yields the state of a RFDE to be decomposed into the sum of a projection onto a finite-dimensional generalized eigenmanifold in the state space and a residual term (recall the asymptotic estimate (1.30)). We shall investigate whether or not the finite-dimensional projection of the state of a RFDE converges to its infinite-dimensional state as the generalized eigenmanifold is extended to include the infinite set of all generalized eigenfunctions. For $\zeta \in \mathrm{SBV}[0,h]$, we prove that this is true at least for $t \geq nh$, i.e. for every $\varphi \in \mathcal{C}$

$$\lim_{\gamma \to -\infty} \|T(t)\varphi - \sum_{\Re(\lambda_j) > \gamma} P_{\lambda_j}\varphi\|_u = 0 \tag{1.47}$$

for $t \geq nh$, uniformly on compact sets, where $P_{\lambda_j} : \mathcal{C} \to \mathcal{M}_{\lambda_j}$ denotes the spectral projection with respect to λ_j.

For general $\zeta \in \mathrm{NBV}[0,h]$ this result is not clear. However, we can prove that the closure $\overline{\mathcal{M}_{\mathcal{C}}}$ of the generalized eigenmanifold is a $\{\,T(t)\,\}$-invariant subspace on which the $\mathcal{C}_0$-semigroup $\{\,T(t)\,\}$ defined by translation acts injectively. This answers a question raised by Hale in his book [18; 3.2].

Chapter 12 deals with perturbed dual semigroups and applies the results developed by Clément, Diekmann, Gyllenberg, Heijmans and Thieme [6]. These results present a new variation-of-constants formula and yield a new interpretation of the equivalence between delay equations and Volterra convolution equations which can be used instead of the results of Chapter 10. Moreover, we can associate with the dual semigroup $\{\,T^*(t)\,\}$ on $\mathcal{C}^* \cong \mathrm{NBV}[0,h]$ a Volterra convolution equation. This makes it possible to apply the theory developed in Chapter 6, 7 and 8 to the restriction of $\{\,T^*(t)\,\}$ to the norm-closed invariant subspace on which $\{\,T^*(t)\,\}$ is strongly-continuous. From this result we obtain

$$\alpha = \delta, \tag{1.48}$$

where α denotes the ascent of $\{T(t)\}$ and δ denotes the ascent of $\{T^*(t)\}$. Finally a combination of all results derived so far yield the following "almost" decomposition

$$\mathcal{C} = \overline{\overline{\mathcal{M}_\mathcal{C}} \oplus \mathcal{N}(T(\alpha))}. \tag{1.49}$$

In Chapter 13, we present some examples related to our results. From these examples it will become clear that the techniques presented in this tract to prove our results can be applied directly to the examples and that usually we obtain stronger results in this manner . We emphasize that from the practical point of view the characterization of $\overline{\mathcal{A}_\mathcal{F}}$ is very important since it yields, at least for differential-difference equations, relations for $\overline{\mathcal{M}_\mathcal{C}}$ or $\overline{\mathcal{A}_\mathcal{F}}$. This means that one can analyse the convergence properties of the spectral projection series when the state φ is restricted to $\overline{\mathcal{M}_\mathcal{C}}$ and this results in much stronger convergence results.

Notation and Terminology

Let $\mathbf{R}_+$ denote the set of nonnegative real numbers and let $M(\mathbf{R})$ denote the space of $n \times n$-matrices with elements in $\mathbf{R}$. Let $|\cdot|$ denote the Euclidean norm on $\mathbf{R}^n$, $\mathbf{C}^n$ and $M(\mathbf{R})$, where in the last case this norm is defined by the sum of the Euclidean norms of the matrix elements.

Introduce weigthed function spaces of Lebesgue measurable functions defined on $\mathbf{R}_+$: let $p \geq 1$ be a natural number, let $\gamma \in \mathbf{R}_+$ and let $L^p(\mathbf{R}_+;\gamma)$ denote the space of functions defined on $\mathbf{R}_+$ with values in $M(\mathbf{R})$ such that

$$t \mapsto f(t)e^{-\gamma t}$$

is L^p-integrable provided with the norm

$$\begin{aligned}\|f\|_{p,\gamma} &= \|f(\cdot)e^{-\gamma\,\cdot}\|_p \\ &= \Big(\int_{\mathbf{R}_+} |f(t)e^{-\gamma t}|^p dt\Big)^{\frac{1}{p}}.\end{aligned}$$

Let $C_0(\mathbf{R}_+;\gamma)$ denote the space of continuous functions f defined on $\mathbf{R}_+$ with values in $M(\mathbf{R})$ such that

$$f(t)e^{-\gamma t} \to 0 \qquad \text{as} \quad t \to \infty$$

provided with the norm

$$\|f\|_{u,\gamma} = \sup_{t \in \mathbf{R}_+} |f(t)e^{-\gamma t}|.$$

Chapter 2. Functions of bounded variation

In line with general usage in the study of delay equations we shall work with kernels of bounded variation. A partition $P(x)$ of $[0,x]$ is a finite ordered set $P = \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$ such that $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N = x$. The width of the partition is

$$\mu(P) = \max_{1 \le j \le N} (\sigma_j - \sigma_{j-1}) .$$

Let f be a given function, the total variation function $V(f)$ is defined by

$$V(f)(x) = \sup_{P(x)} \sum_{j=1}^{N} |f(\sigma_j) - f(\sigma_{j-1})|, \tag{2.1}$$

where the supremum is taken over all partitions $P(x)$. In general, for $0 \le x \le y < \infty$,

$$0 \le V(f)(x) \le V(f)(y) \le \infty. \tag{2.2}$$

If $V(f)$ is a bounded function, then (2.2) implies that

$$T(f) = \lim_{x \to \infty} V(f)(x) \tag{2.3}$$

exists and is finite. In that case we say that f is of bounded variation, in short $f \in \mathrm{BV}$ and we call $T(f)$ the total variation of f. A complex function f is called of bounded variation if and only if $\Re(f) \in \mathrm{BV}$ and $\Im(f) \in \mathrm{BV}$. A vector-valued function f is called of bounded variation if and only if all components of f are of bounded variation. If both g and h are non-decreasing bounded functions then $f = g - h$ is of bounded variation. Actually the following result, see Titchmarsh [37], shows that this property can be used to give an equivalent definition.

THEOREM 2.1. *If $f : \mathbf{R}_+ \to \mathbf{R}^n$ is of bounded variation, then f can be expressed in the form*

$$f = g - h, \tag{2.4}$$

where both g and h are non-decreasing bounded functions.

This characterization has some immediate consequences.

COROLLARY 2.2. *The sum, difference and product of two functions of bounded variation are of bounded variation.*

COROLLARY 2.3. *If $f : \mathbf{R}_+ \to \mathbf{R}^n$ is of bounded variation, then*

$$f(\tau+) = \lim_{\sigma \downarrow \tau} f(\sigma) \quad \text{exists for } \tau \in [0, \infty) \tag{2.5}$$

and

$$f(\tau-) = \lim_{\sigma \uparrow \tau} f(\sigma) \quad \text{exists for } \tau \in (0, \infty). \tag{2.6}$$

Moreover, the set of discontinuities of f is at most countable.

We call a function $f \in \mathrm{BV}$ normalized if $f(0) = 0$ and f is continuous from the left at every point of $\mathbf{R}_+$. The class of these functions will be denoted by $\mathrm{NBV}[\mathbf{R}_+]$.

A complex function f on $\mathbf{R}_+$ is said to be absolutely continuous if to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$\sum_{j=1}^{N} (\beta_j - \alpha_j) < \delta \quad \text{implies} \quad \sum_{j=1}^{N} |f(\beta_j) - f(\alpha_j)| < \epsilon, \tag{2.7}$$

whenever $(\alpha_1, \beta_1), \cdots, (\alpha_N, \beta_N)$ are disjoint intervals. Note that every absolutely continuous function is uniformly continuous and that an absolutely continuous function is locally of bounded variation, i.e. of bounded variation on bounded intervals. The connection between functions of bounded variation and absolutely continuous functions is expressed by the following theorem due to Lebesgue, see for example [33; 8.17, 8.18]

THEOREM 2.4. *If f and g are in $L^1(\mathbf{R}_+)$ such that*

$$f(x) = \int_0^x g(t)dt. \tag{2.8}$$

Then $f \in \mathrm{NBV}[\mathbf{R}_+]$ and f is absolutely continuous with

$$f'(x) = g(x) \quad \text{a.e.}$$

On the other hand if $f \in \mathrm{NBV}[\mathbf{R}_+]$, then f is almost everywhere differentiable with $f' \in L^1(\mathbf{R}_+)$ and

$$f = f_a + f_d + f_s, \tag{2.9}$$

where the absolutely continuous part of f is denoted by f_a, the discrete singular part is denoted by f_d and the continuous singular part is denoted by f_s.

The next theorem explains the importance of the class NBV[$\mathbf{R}_+$] and makes it possible to apply abstract integration theory. To formulate the result, recall the definition of a Borel measure. A Borel measure is a measure μ defined on the σ-ring generated by the compact subsets of $\mathbf{R}$ and such that $\mu(C) < \infty$ for every compact subset C of $\mathbf{R}$. The proof of the theorem can be found in [33; 8.14].

THEOREM 2.5. *There exists a one-to-one correspondence between elements of* NBV[$\mathbf{R}_+$] *and Borel measures on* $\mathbf{R}_+$ *expressed by*

$$f(x) = \mu_f([0, x]), \tag{2.10}$$

where $f \in$ NBV[$\mathbf{R}_+$] *and* μ_f *is a Borel measure. The above correspondence is one-to-one because of the normalization of* f, *i.e.* $f(0) = 0$ *and* f *is continuous from the left.*

Let f and g be integrable. For those $t \in \mathbf{R}_+$ such that

$$\int_0^t |f(t-s)g(s)|ds < \infty,$$

define the convolution of f and g by

$$f * g(t) = \int_0^t f(t-s)g(s)ds. \tag{2.11}$$

The following theorem [33; 7.14] shows the existence of $f * g$ in $L^1(\mathbf{R}_+)$.

THEOREM 2.6. *Suppose* $f, g \in L^1(\mathbf{R}_+)$. *Then* $f * g \in L^1(\mathbf{R}_+)$ *and*

$$||f * g||_1 \leq ||f||_1 ||g||_1. \tag{2.12}$$

Because of Theorem 2.5 we can extend the convolution to functions of bounded variation. Let $\alpha, \beta \in$ NBV[$\mathbf{R}_+$] and define the Riemann-Stieltjes convolution by

$$\alpha * \beta = \int_0^t \alpha(t-s)d\beta(s). \tag{2.13}$$

To give a meaning to this formula, let μ_α and μ_β denote the corresponding Borel measures. From [33; Exercise 7.5] it follows that the convolution of the Borel measures μ_α and μ_β

$$\mu_\alpha * \mu_\beta([0,t]) = \int_{\mathbf{R}_+} \mu_\alpha([0, t-s])d\mu_\beta([0, s]) \tag{2.14}$$

defines a Borel measure on $\mathbf{R}_+$. Therefore, the convolution $\alpha * \beta$ defines an element of NBV$[\mathbf{R}_+]$. Furthermore, the Riemann-Stieltjes convolution is commutative and extends the ordinary convolution (2.11) in the following sense: Let α, β be absolutely continuous functions. Then we can write the corresponding Borel measures as follows

$$d\mu_\alpha = f dm$$

and

$$d\mu_\beta = g dm,$$

where m denotes the Lebesgue measure and $f, g \in L^1(\mathbf{R}_+)$. Hence, the convolution $\alpha * \beta$ is absolutely continuous and

$$d\mu_{\alpha*\beta} = d(\mu_\alpha * \mu_\beta) = f * g dm. \tag{2.15}$$

In the sequel we shall use the convention that $*$ denotes the Riemann-Stieltjes convolution if and only if it arises in the context of Riemann-Stieltjes integrals, which we are going to define next. Let φ and f be two complex-valued functions on $[a, b] \subset [0, \infty)$. For any partition P we introduce the sum

$$S(\varphi, P, f) = \sum_{j=1}^{N} \varphi(\tau_j) \left(f(\sigma_j) - f(\sigma_{j-1}) \right), \tag{2.16}$$

where $\sigma_{j-1} \leq \tau_j \leq \sigma_j$. Suppose a complex constant $A \in \mathbf{C}$ exists such that for any $\epsilon > 0$ there exists an $\delta = \delta(\epsilon) > 0$ such that

$$|A - S(\varphi, P, f)| < \epsilon \tag{2.17}$$

for all partitions P with width $\mu(P) < \delta$ and any choice of the "intermediate" points τ_j. Then we will call φ Riemann-Stieltjes integrable with respect to f (or in short $\varphi \in \mathrm{S}(f)$) over $[a, b]$ and we shall write

$$A = \int_a^b \varphi df. \tag{2.18}$$

Let φ be continuous and $f \in$ NBV$[a, b]$. By using upper- and lower sums for (2.16) we clearly see that the Riemann-Stieltjes integral exists. In general we can use Theorem 2.5 to give a unique meaning to (2.18). Let $f \in$ NBV$[a, b]$ and let μ_f denote the Borel measure corresponding to f. An application of the Radon-Nikodym theorem [33; 6.12] asserts that there exists a Borel function h with $|h| = 1$ such that

$$d\mu_f = hd|\mu_f|, \tag{2.19}$$

where $|\mu|$, the total variation measure of μ, is a positive Borel measure. As a consequence, we can define integration with respect to a Borel measure μ_f by the formula

$$\int_{\mathbf{R}_+} \varphi d\mu_f = \int_{\mathbf{R}_+} \varphi h d|\mu_f|, \tag{2.20}$$

where φ is any Borel measurable function on $\mathbf{R}_+$.

The extension of the definition of the Riemann-Stieltjes integral by means of a Lebesgue integral makes it possible to apply results from abstract integration theory such as: Lebesgue's dominated convergence theorem and the Fubini theorem. The following theorem recalls the main results. Define $L^1(\mu)$ to be the collection of all complex measurable functions φ on $\mathbf{R}_+$ for which

$$\int_{\mathbf{R}_+} |\varphi| d\mu < \infty. \tag{2.21}$$

The members of $L^1(\mu)$ are called Lebesgue integrable functions (with respect to the Borel measure μ).

THEOREM 2.7. *If f and g belong to* NBV$[\mathbf{R}_+]$ *and if μ_f denotes the Borel measure corresponding to f. Then*

(i) Suppose $\{\varphi_j\}$ is a sequence of complex measurable functions on $\mathbf{R}_+$ such that

$$\varphi(x) = \lim_{j\to\infty} \varphi_j(x) \tag{2.22}$$

exists for every $x \in \mathbf{R}_+$. If there exists a function $\chi \in L^1(\mu_f)$ such that for every j

$$|\varphi_j(x)| \leq \chi(x) \quad \text{a.e.} \tag{2.23}$$

with respect to μ_f, then $\varphi \in L^1(\mu_f)$ and

$$\lim_{j\to\infty} \int_{\mathbf{R}_+} |\varphi - \varphi_j| df = 0. \tag{2.24}$$

(ii) Let φ be a Borel measurable function on $\mathbf{R}_+ \times \mathbf{R}_+$. Suppose that

$$\int_{\mathbf{R}_+} |df(x)| \int_{\mathbf{R}_+} |\varphi(x,y)| dg(y) < \infty,$$

then

$$\int_{\mathbf{R}_+} df(x) \int_{\mathbf{R}_+} \varphi(x,y) dg(y) = \int_{\mathbf{R}_+} dg(x) \int_{\mathbf{R}_+} \varphi(x,y) df(y). \tag{2.25}$$

(iii) If φ is a continuous bounded function on $\mathbf{R}_+$. Then for all finite intervals [a, b]

$$\int_a^b \varphi df = \varphi(b)f(b) - \varphi(a)f(a) - \int_a^b f d\varphi. \tag{2.26}$$

Moreover,

$$\int_a^b \varphi df \leq \sup_{x\in[a,b]} \{|\varphi(x)|\}(V(f)(b) - V(f)(a)). \tag{2.27}$$

Define the subclass NBV$[a,b]$ of NBV$[\mathbf{R}_+]$ by

$$\mathrm{NBV}[a,b] = \{f \in \mathrm{NBV}[\mathbf{R}_+] : f(t) = 0 \text{ for } t \leq a,\ f(t) = f(b) \text{ for } t \geq b\}$$

and use for $f \in \mathrm{NBV}[a,b]$ the following convention

$$\int_a^b \varphi df = \int_{\mathbf{R}_+} \varphi d\mu_f.$$

Because of Theorem 2.7 (iii), for every $f \in \mathrm{NBV}[a,b]$ the mapping

$$\varphi \mapsto \int_a^b \varphi df \tag{2.28}$$

defines a continuous linear functional on $C([a,b])$. The Riesz representation theorem [33; 6.19] shows that every continuous linear functional can be thus represented.

THEOREM 2.8. *Let Λ be a continuous linear functional on $C[a,b]$. There exists a unique $f \in \mathrm{NBV}[\mathbf{R}_+]$ such that for all $\varphi \in C[a,b]$*

$$\Lambda(\varphi) = \langle\varphi, \Lambda\rangle = \int_a^b \varphi df \tag{2.29}$$

and $||\Lambda|| = T(f)$.

In order to facilitate the application to delay equations we shall adopt two peculiar conventions. We shall write $\int df\varphi$ instead of $\int \varphi df$. If now φ is a $\mathbf{C}^n$-valued function we shall think of the values of φ as column-vectors and the values of f as row vectors and still write $\int df\varphi$ to denote

$$\sum_{j=1}^n \int df_j\varphi_j = \sum_{j=1}^n \int \varphi_j df_j. \tag{2.30}$$

Analogously every continuous linear mapping from $C([a,b];\mathbf{C}^n)$ into $\mathbf{C}^n$ can be uniquely represented by

$$\varphi \mapsto \int_a^b d\zeta\varphi, \tag{2.31}$$

where ζ is a $n \times n$-matrix whose elements belong to NBV$[a,b]$. For reasons which will become clear later, when we discuss the duality between delay- and renewal equations, we shall take as domain of definition for the kernel ζ not the interval $[-h,0]$ but the mirror image under time reversal $[0,h]$.

In most of the examples we shall encounter the case that the matrix-valued function ζ consists of finitely many jumps and an absolutely continuous part. According to the representation (2.9) the discrete singular part of ζ then corresponds exactly of all the jumps of ζ and the continuous singular part of ζ is zero.
Next consider a linear system of autonomous retarded functional differential equations (RFDE).

EXAMPLE 2.9. Consider

$$\dot{x}(t) = \int_0^h d\zeta(\theta)x(t-\theta) \qquad \text{for} \quad t \geq 0, \tag{2.32}$$

satisfying the initial condition

$$x(t) = \varphi(t) \qquad \text{for} \quad -h \leq t \leq 0, \tag{2.33}$$

where the matrix-valued function ζ belongs to NBV$[0,h]$ and the initial condition φ is a given continuous function, in short $\varphi \in \mathcal{C} = C[-h,0]$. In the study of the behaviour of the solution of the above system of RFDE's it turns out to be useful to rewrite the problem as a Volterra convolution integral equation (or, as it is frequently called, a renewel equation).

We split up the integral to separate the part involving the known φ from the part involving the unknown x:

$$\begin{aligned}\dot{x}(t) &= \int_0^t d\zeta(\theta)x(t-\theta) + \int_t^h d\zeta(\theta)\varphi(t-\theta)\\ &= -\int_0^t d_\theta\zeta(t-\theta)x(\theta) - \int_{-h}^0 d_\theta\zeta(t-\theta)\varphi(\theta)\end{aligned}$$

(recall that ζ is defined to be constant on $[h,\infty)$).
Next we integrate from 0 to t and obtain

$$x(t) - \varphi(0) = -\int_0^t \int_0^\sigma d_\theta\zeta(\sigma-\theta)x(\theta)d\sigma - \int_0^t \int_{-h}^0 d_\theta\zeta(\sigma-\theta)\varphi(\theta)d\sigma.$$

So, because of Theorem 2.7 (ii)

$$
\begin{aligned}
x(t)-\varphi(0) &= -\int_0^t d_\theta \int_\theta^t \zeta(\sigma-\theta)d\sigma x(\theta) - \int_{-h}^0 d_\theta \int_0^t \zeta(\sigma-\theta)d\sigma\varphi(\theta) \\
&= -\int_0^t \zeta(t-\theta)x(\theta)d\theta + \int_{-h}^0 (\zeta(t-\theta)-\zeta(-\theta))\varphi(\theta)d\theta.
\end{aligned}
$$

We summarize the end result of our manipulations as follows. The solution x of (2.32) satisfies the renewal equation

$$x - \zeta * x = f, \tag{2.34}$$

where by definition

$$f(t) = \varphi(0) + \int_{-h}^0 (\zeta(t-\theta)-\zeta(-\theta))\varphi(\theta)d\theta. \tag{2.35}$$

REMARKS 2.10.

(i) The so-called forcing function f defined by (2.35) is constant for $t \geq h$.

(ii) The forcing function f defined by (2.35) is absolutely continuous. In fact for $\varphi \in \mathcal{C}$

$$\dot{f}(t) = \int_t^h d\zeta(\theta)\varphi(t-\theta) \tag{2.36}$$

is well-defined and even of bounded variation.

(iii) The formula (2.35) makes perfect sense if $\varphi(0)$ is given as an element of $\mathbf{R}^n$ while $\varphi(\theta)$ for $-h \leq \theta \leq 0$ is given as an integrable function. Moreover, Delfour and Manitius [8] proved that the mapping defined by (2.36) has a continuous extension to a mapping from $L^1[-h,0] \to L^1[-h,0]$. So f is still absolutely continuous, although there is no explicit formula for $\dot{f}$ anymore.

(iv) Partial integration shows that the derivative of the solution of the linear autonomous RFDE (2.32) also satisfies a renewal equation of the form

$$\dot{x} - \zeta * \dot{x} = h,$$

where h is defined on $[0,\infty)$ and is constant on the interval $[h,\infty)$. See Chapter 12 for detailed results about the close connection between delay- and renewal equations.

Chapter 3. The Laplace transform

The Laplace transform of a function g defined on $\mathbf{R}_+$ is given by the infinite integral

$$L\{g\}(z) = \lim_{T \to \infty} \int_0^T e^{-zt} g(t) dt, \tag{3.1}$$

where the parameter z takes complex values. For each value of z for which the limit exists, a value $L\{g\}(z)$ is defined. If g is a σ_0-exponentially bounded function, i.e.

$$|g(t)| \leq Ce^{\sigma_0 t} \quad \text{a.e.}$$

Then for z in the half plane $\Re(z) > \sigma_0$

$$\begin{aligned} |L\{g\}(z)| &\leq \lim_{T \to \infty} \int_0^T e^{-\Re(z)t} |g(t)| dt \\ &\leq \frac{C}{\Re(z) - \sigma_0}, \end{aligned} \tag{3.2}$$

or in words: the infinite integral (3.1) converges absolutely for z in the right half plane $\Re(z) > \sigma_0$. Moreover, in this half plane, the complex function $L\{g\}$ is bounded and depends analytically on the parameter z.

In this chapter we state some properties of the Laplace transform and its complex inversion formula. A thorough introduction to the fundamental properties of the Laplace transform and its applications can be found in Doetsch [13] and Widder [41]. The following proposition indicates why the Laplace transform is so useful.

Proposition 3.1. *If f and g are σ_0-exponentially bounded functions on $\mathbf{R}_+$. Then*

(i) If $L\{f\}(z) = L\{g\}(z)$ for $\Re(z) > \sigma_0$, then

$$f(t) = g(t) \quad \text{a.e.;}$$

(ii) For $t_0 \geq 0$ define $f_{t_0}(t) = f(t+t_0)$, then

$$L\{f_{t_0}\}(z) = e^{zt_0}L\{f\}(z) - e^{zt_0}\int_0^{t_0} e^{-zt}f(t)dt \qquad \text{for} \quad \Re(z) > \sigma_0;$$

(iii) For $a \geq 0$,

$$L\{e^{at}f\}(z) = L\{f\}(z-a) \qquad \text{for} \quad \Re(z) > \sigma_0 + a;$$

(iv)

$$L\{f * g\}(z) = L\{f\}L\{g\} \qquad \text{for} \quad \Re(z) > \sigma_0;$$

(v) Suppose

$$h(t) = h(0) + \int_0^t f(s)ds,$$

then $L\{h\}$ converges absolutely in the half plane $\Re(z) > \max(\sigma_0, 0)$ and

$$L\{h\}(z) = \frac{1}{z}\left(h(0) + L\{f\}(z)\right). \tag{3.3}$$

Because of the properties listed above we can apply Laplace transformation to linear functional differential equations in x, involving derivatives and differences, to arrive at linear algebraic equations involving only $L\{x\}$. We shall next present an inversion formula for the Laplace transform, i.e. a formula which gives the function x in terms of the Laplace transform $L\{x\}$. Depending on the application we will use one of the following complex inversion formulas which are special cases of general complex inversion formulas that can be found in Widder [41; 7.3-5] and Doetsch [13; 24.4].

THEOREM 3.2. *Let g be a σ_0-exponentially bounded function that is locally of bounded variation. Then for $\gamma > \sigma_0$ and $t > 0$ we have the inversion formula*

$$\frac{g(t+) + g(t-)}{2} = \lim_{\omega\to\infty} \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} e^{zt}L\{g\}(z)dz. \tag{3.4}$$

For $t = 0$ we have

$$\frac{g(0+)}{2} = \lim_{\omega\to\infty} \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} L\{g\}(z)dz. \tag{3.5}$$

REMARK 3.3. The convergence of the complex line integral (3.4) does not necessarely imply the existence of the infinite integral

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}L\{g\}(z)dz. \tag{3.6}$$

The limit in (3.4) is called the principal value and will be denoted by

$$\frac{1}{2\pi i}\int\limits_{L(\gamma)} e^{zt}L\{g\}(z)dz = \lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} e^{zt}L\{g\}(z)dz. \tag{3.7}$$

where $L(\gamma)$ denotes the complex line $\Re(z)=\gamma$.

We will mainly use the following application of Theorem 3.2.

COROLLARY 3.4. *If g is a σ_0-exponentially bounded function that is absolutely continuous for $t\geq t_0$. Then for $\gamma>\sigma_0$ and $t>t_0$*

$$g(t)=\frac{1}{2\pi i}\int\limits_{L(\gamma)} e^{zt}L\{g\}(z)dz. \tag{3.8}$$

From the formula (3.8) it follows that the value of the complex line integral is independent of the choice of $\gamma>\sigma_0$. This can also be shown directly using complex integration. We shall demonstrate this in detail since the techniques that are used here will be used repeatly in the sequel. Define $\Gamma_N(\gamma,\gamma')$ to be the closed contour in the complex plane, which is composed of four straight lines and connects the points $\gamma-iN,\gamma'-iN,\gamma'+iN,\gamma+iN$. Since $L\{g\}$ is analytic in the half plane $\Re(z)>\sigma_0$, we can apply the Cauchy theorem to obtain for $\gamma'>\gamma$

$$\frac{1}{2\pi i}\int\limits_{L(\Gamma_N(\gamma,\gamma'))} e^{zt}L\{g\}(z)dz=0. \tag{3.9}$$

Hence, by taking the limit $N\to\infty$

$$\frac{1}{2\pi i}\int\limits_{L(\gamma)} e^{zt}L\{g\}(z)dz=\frac{1}{2\pi i}\int\limits_{L(\gamma')} e^{zt}L\{g\}(z)dz, \tag{3.10}$$

if

$$\lim_{N\to\pm\infty}\left|\int_{\gamma+iN}^{\gamma'+iN} e^{zt}L\{g\}(z)dz\right|=0. \tag{3.11}$$

But this is a direct consequence of the next lemma, the so-called Riemann-Lebesgue lemma, applied to $L\{g\}$.

LEMMA 3.5. *If f belongs to $L^1(\mathbf{R}_+)$, then*

$$\lim_{N\to\pm\infty}\left|\int\limits_{\mathbf{R}_+} e^{iNt}f(t)dt\right|=0. \tag{3.12}$$

The proof can be found in Hewitt and Stromberg [21; 21.39].

If $L\{g\}$ possesses an analytic continuation to the left of $\Re(z) = \gamma$ we may be able to obtain information about the asymptotic behaviour of $g(t)$ for $t \to \infty$, by shifting the contour of integration $L(\gamma)$ to the left and taking account of the singularities we encounter. To follow this approach we first have to analyse the class of analytic functions involved in the analytic continuation of $L\{g\}$. This we shall do in the next chapter.

Chapter 4. The exponential type calculus

In this chapter we study the zero distribution and growth properties of a class of entire vector valued functions, which is large enough to include all the entire functions appearing in the sequel.

Definition 4.1. An entire function $F : \mathbf{C} \to \mathbf{C}$ is of order 1 if and only if

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = 1, \tag{4.1}$$

where

$$M(r) = \max_{0\leq\theta\leq 2\pi} \left\{ |F(re^{i\theta})| \right\}. \tag{4.2}$$

An entire function of order 1 is of exponential type if and only if

$$\limsup_{r\to\infty} \frac{\log M(r)}{r} = \mathrm{E}(F), \tag{4.3}$$

where $0 \leq \mathrm{E}(F) < \infty$. In that case $\mathrm{E}(F)$ is called the exponential type of F. A vector-valued function $F = (F_1, \ldots, F_n) : \mathbf{C} \to \mathbf{C}^n$ will be called an entire function of exponential type if and only if the components F_j of F are entire functions of order 1 that are of exponential type. Furthermore, the exponential type will be defined by

$$\mathrm{E}(F) = \max_{1\leq j\leq n} \mathrm{E}(F_j). \tag{4.4}$$

Next, we define two classes of entire functions that will be studied in this chapter.

Definition 4.2. Let $\mathbf{P}$ denote the class of entire functions which can be represented by a polynomial of finite degree with coefficients that are a finite Laplace transform of a function of NBV[$\mathbf{R}_+$]. Let $\mathbf{P}_l$ denote the subset of $\mathbf{P}$ such that $F \in \mathbf{P}_l$ can be represented by

$$F(z) = \sum_{j=0}^{l} \int_0^{\tau_j} e^{-zt} d\eta_j(t) z^{l-j}, \tag{4.5}$$

where for $j = 1, \ldots, l$, we have $\eta_j \in \mathrm{NBV}[0, \tau_j]$ and $\eta_j' \notin \mathrm{NBV}[0, \tau_j]$. Define the following subclass of NBV[a, b]

$$\mathrm{SBV}[a,b] = \{ f \in \mathrm{NBV}[a,b] \mid \exists t : t \leq b \text{ and } f(t-) \neq f(t) = f(b) \},$$

and define the subclass $\mathbf{J}$ of $\mathbf{P}$ by the class of entire functions which can be represented by a polynomial of finite degree with coefficients that are a finite Laplace transform of a function of SBV[a, b]. Let $\mathbf{J}_l$ denote the subset of $\mathbf{J}$ such that $F \in \mathbf{J}_l$ can be represented by

$$F(z) = \sum_{j=0}^{l} \int_0^{\tau_j} e^{-zt} d\eta_j(t) z^{l-j}, \tag{4.6}$$

where for $j = 0, \ldots, l$ we have $\eta_j \in \mathrm{SBV}[0, \tau_j]$.

REMARK 4.3. Partial integration of the coefficients shows that in general the representation for an element F of $\mathbf{P}$ is not unique. The restricted classes $\mathbf{P}_l$, $\mathbf{J}$ and $\mathbf{J}_l$ are such that the representation (4.5) or (4.6) is indeed unique. For these classes we can define the degree of F, denoted by $\deg(F) = l$, as the highest power of z in the representation (4.5) or (4.6). The introduction of the restricted class $\mathbf{J}$ is needed to derive lower bounds for $|F|$, see Theorem 4.6 and Chapter 5. In general lower bounds are very difficult to obtain. For example, let η be N-times differentiable and such that

$$\eta(t) = \begin{cases} t^N & \text{for } t \leq \frac{1}{4}, \\ (1-t)^N & \text{for } \frac{3}{4} \leq t \leq 1. \end{cases} \tag{4.7}$$

Then

$$\begin{aligned} \int_0^1 e^{-zt} d\eta(t) &= \frac{1}{z^{N-1}} \int_0^1 e^{-zt} \eta^{(N)}(t) dt \\ &= \frac{e^{-z}}{z^{N-1}} \int_{-1}^0 e^{-zt} \eta^{(N)}(t+1) dt. \end{aligned}$$

We start with a special case of the Paley-Wiener theorem [5; 6.9.1].

THEOREM 4.4. *If F is an entire function which is uniformly bounded in the closed right half plane $\Re(z) \geq 0$. Then F is of exponential type τ and L^2-integrable along the imaginary axis if and only if*

$$F(z) = \int_0^{\tau} e^{-zt}\varphi(t)dt, \tag{4.8}$$

where $\varphi \in L^2[0,\tau]$ and φ does not vanish a.e. in any neighbourhood of τ.

Next we collect some consequences of the Paley-Wiener Theorem 4.4 in a lemma.

LEMMA 4.5. *If F and G are elements of $\mathbf{P}$. Then*

(i) F is an entire function of order 1 with finite exponential type and

$$\mathrm{E}(F) = \max_j \tau_j ; \tag{4.9}$$

(ii) The product $F \cdot G$ belongs to $\mathbf{P}$;

(iii) The subclass $\mathbf{J}$ of $\mathbf{P}$ is closed under multiplication.

PROOF. From the Paley-Wiener Theorem 4.4 we derive

$$\mathrm{E}\left(\int_0^{\tau_j} e^{-zt} d\eta_j(t)\right) = \tau_j .$$

So, the first property is a direct consequence of the definitions. Property (ii) follows directly from the Laplace transform property

$$\int_0^{\tau_1} e^{-zt} d\eta_1(t) \int_0^{\tau_2} e^{-zt} d\eta_2(t) = \int_0^{\tau_1+\tau_2} e^{-zt} d\eta_1 * \eta_2(t),$$

where $\eta_1 \in \mathrm{NBV}[0,\tau_1]$, $\eta_2 \in \mathrm{NBV}[0,\tau_2]$ and $\eta_1 * \eta_2$ denotes the Riemann-Stieltjes convolution (2.13). An application of (2.14) states that the convolution of two functions of bounded variation is again of bounded variation. Furthermore, if η_1 and η_2 belong to $\mathrm{SBV}[0,\tau_2]$, then $\eta_1 * \eta_2 \in \mathrm{SBV}[0,\tau_1+\tau_2]$, since

$$\eta_1 * \eta_2(t) = \eta_1(\tau_1) \int_0^{\tau_2} d\eta_2(t) \qquad \text{for} \quad t \geq \tau_1 + \tau_2$$

and

$$\eta_1 * \eta_2(\tau_1 + \tau_2-) = \eta_1(\tau_1-) \int_0^{\tau_2} d\eta_2(t).$$

This shows that $\mathbf{J}$ is closed under multiplication and the proof of the lemma is complete. ∎

To study the zero distribution of elements of $\mathbf{P}$ we will first study the zero distribution of elements of $\mathbf{J}_0$. For the following result, the restriction to $\mathbf{J}_0$ is necessary. See Young [43] for additional information.

THEOREM 4.6. *If F is an element of $\mathbf{J}$ of degree zero, i.e.*

$$F(z) = \int_0^\tau e^{-zt} d\eta(t). \tag{4.10}$$

Then there exist constants $\gamma < 0$, m, and M such that all zeros of F are in the right half plane $\Re(z) > \gamma$ and such that in the left half plane $\Re(z) \leq \gamma$

$$m|e^{-\tau z}| \leq |F(z)| \leq M|e^{-\tau z}|. \tag{4.11}$$

Besides, for an appropriate choice of m, the estimate (4.11) also holds in each finite strip $\gamma < \Re(z) < \gamma'$ as long as one stays bounded away from the zeros of F.

PROOF. From Theorem 2.6 (iii) it follows that in the left half plane $\Re(z) \leq \gamma$

$$|\int_0^\tau e^{-zt} d\eta(t)| \leq |e^{-z\tau}| V(\eta)(\tau). \tag{4.12}$$

This proves the upper bound in (4.11). To prove the lower bound choose $\delta > 0$ such that the variation of η over the interval $[\tau - \delta, \tau)$ is smaller then ϵ. Rewrite

$$F(z) = (\eta(\tau) - \eta(\tau-))e^{-\tau z} + \int_0^{\tau-\delta} e^{-zt} d\eta(t) + \int_{\tau-\delta}^{\tau-} e^{-zt} d\eta(t) \tag{4.13}$$

and apply the estimate (4.12) to the integrals at the right hand side. Then we obtain for γ sufficiently small in the left half plane $\Re(z) < \gamma$

$$\begin{aligned} |F(z)| &\geq |e^{-\tau z}| \left(|\eta(\tau) - \eta(\tau-)| - V(\eta)(\tau - \delta)|e^{z\delta}| - \epsilon\right) \\ &\geq m|e^{-\tau z}|. \end{aligned} \tag{4.14}$$

Next we want to prove that for each $\epsilon > 0$ there exists a number $m > 0$ such that in the strip $\gamma < \Re(z) < \gamma'$

$$|F(z)| \geq m|e^{-\tau z}|,$$

outside the circles of radius ϵ centered at the zeros of F. Denote the zeros of F by $\lambda_1, \lambda_2, \ldots$ and suppose that such a constant m does not exist. Then there exist a positive constant ϵ and a sequence $z_1, z_2, z_3, \ldots$ of points that lie inside the vertical strip $\gamma < \Re(z) < \gamma'$, but outside the disks $|z - \lambda_j| \leq \epsilon$ such that

$$\lim_{j\to\infty} F(z_j) = 0. \tag{4.15}$$

Write $z_j = x_j + iy_j$. Since $\sup_j |x_j| < \max(|\gamma|, |\gamma'|)$, we may suppose without loss of generality that

$$\lim_{j\to\infty} x_j = \bar{x}.$$

Now define a sequence $\{F_j\}$ of entire functions by

$$F_j(z) = F(z + iy_j).$$

Since F is bounded in an neighbourhood $\mathcal{U}$ of the closed strip $\gamma \leq \Re(z) \leq \gamma'$, the sequence $\{F_j\}$ is uniformly bounded on $\mathcal{U}$. So from Montel's theorem [7; VII.2.9], the sequence $\{F_j\}$ forms a normal family. Accordingly, there exists a subsequence $\{F_{j_k}\}$ that converges uniformly on compact subsets of the strip $\gamma < \Re(z) < \gamma'$ to a limit function $\overline{F}$. Since

$$F_j(x_j) \to 0 \qquad \text{as} \quad j \to \infty,$$

it follows that

$$\overline{F}(\bar{x}) = 0.$$

Since $\overline{F}$ does not vanish identically, Hurwitz's theorem [7; VII.2.5] implies that all but a finite number of the F_j must have a zero inside the disk $\{z : |z - \bar{x}| < \frac{\epsilon}{2}\}$. But this contradicts the fact that $|z_j - \lambda_k| > \epsilon$ for all j and k and the theorem is proved. ∎

LEMMA 4.7. *If F is an element of $\mathbf{P}_0$, i.e.*

$$F(z) = \int_0^T e^{-zt}\, d\eta(t). \tag{4.16}$$

Then in any right half plane $\Re(z) > \gamma$

$$|F(z)| = O(1) \qquad \text{as} \quad |z| \to \infty.$$

PROOF. From Theorem 2.6 (iii) it follows that in the right half plane $\Re(z) > \gamma$

$$\left|\int_0^T e^{-zt}\, d\eta(t)\right| \leq \max(1, e^{-\gamma\tau}) V(\eta)(\tau). \tag{4.17}$$

∎

The following lemma is a direct consequence of the above lemma.

LEMMA 4.8. *Suppose $F \in z^n + \mathbf{P}_{n-1}$, i.e.*

$$F(z) = z^n + \sum_{k=1}^{n} \int_0^{\tau_k} e^{-zt} d\eta_k(t) z^{n-k}. \tag{4.18}$$

Then F has at most finitely many zeros in any given right half plane $\Re(z) > \gamma$. In particular, there exists a γ_0 such that F has no zeros in the right half plane $\Re(z) > \gamma_0$.

PROOF. For a zero z_j of F we have

$$|z_j|^n = |\sum_{k=1}^{n} \int_0^{\tau_k} e^{-z_j t} d\eta_k(t) z_j^{n-k}|. \tag{4.19}$$

Fix $\gamma \in \mathbf{R}$ and suppose that there exists an infinite sequence of zeros z_j of F with $\Re(z_j) > \gamma$ and $|z_j| \to \infty$, then because of Lemma 4.7, equation (4.19) yields

$$1 \leq O(\frac{1}{|z_j|}) \qquad \text{as} \quad |z_j| \to \infty,$$

which gives a contradiction. The proof of the lemma is completed by noting that the zeros of F cannot have a finite density point since F is entire. ∎

For elements of class $z^n + \mathbf{P}_{n-1}$ we next describe the connection between the asymptotic properties of the distribution of the zeros and of the growth properties of the function itself.

THEOREM 4.9. *If F be an element of $z^n + \mathbf{P}_{n-1}$ and if $\lambda_1, \lambda_2, \ldots$ denote the zeros of F in the left half plane $\Re(z) < 0$. Then*

$$\sum_{j=1}^{\infty} \Re(\frac{1}{\lambda_j}) \quad \textit{converges}. \tag{4.20}$$

PROOF. Since F is an entire function of finite exponential type, the Lindelöf theorem [5; 2.10.1] states that the sums

$$S(r) = \sum_{|z_j| \leq r} \frac{1}{z_j} \quad \text{are bounded}, \tag{4.21}$$

where $(z_j)_{j \geq 1}$ denote the zeros of F different from zero. From Lemma 4.8 it follows that F has at most a finite number of zeros in the right half plane $\Re(z) > 0$. Since η is real, the sum

$$\sum_{j=1}^{\infty} \frac{1}{z_j}$$

actually converges. Therefore, the sum

$$\sum_{j=1}^{\infty} \Re\left(\frac{1}{\lambda_j}\right) \quad \text{converges.}$$

∎

The next theorem describes the behaviour of elements of **P** in the left half plane $\Re(z) < 0$.

THEOREM 4.10. *If F is an element of* **P**. *Then for almost all $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$*

$$\lim_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r} = -\mathrm{E}(F)\cos\theta. \tag{4.22}$$

For every fixed $\theta_0 \in (0, \frac{\pi}{2})$, there exists a sequence r_j, such that $r_j \to \infty$ and

$$\lim_{j\to\infty} \frac{\log|F(r_j e^{i\theta})|}{r_j} = -\mathrm{E}(F)\cos\theta,$$

uniformly in $\frac{\pi}{2} + \theta_0 \leq \theta \leq \frac{3\pi}{2} - \theta_0$.

PROOF. Since F has at most polynomial growth on the imaginary axis we know that

$$\int_1^{\infty} \frac{\log|F(\pm iy)|}{y^2} dy \quad \text{exists}$$

and we can apply the Ahlfors-Heins theorem [5; 7.26]. ∎

As a corollary we have a very easy calculus for exponential types of elements of **P**.

COROLLARY 4.11. *If F_1 and F_2 are elements of* **P**. *Then*

$$\mathrm{E}(F_1F_2) = \mathrm{E}(F_1) + \mathrm{E}(F_2). \tag{4.23}$$

The following theorem characterizes the zero distribution of an element of $z^n + \mathbf{P}_{n-1}$.

THEOREM 4.12. *If F is an element of $z^n + \mathbf{P}_{n-1}$. Then the zero distribution of F has the following properties.*

(i) The zeros λ_j of F are located in a left half plane $\Re(z) < \gamma_0$ and have a density in ordinary sense:

$$\lim_{r\to\infty} \frac{n(r)}{r} = \frac{\mathrm{E}(F)}{\pi}, \tag{4.24}$$

where $n(r)$ denotes the number of zeros satisfying $|\lambda_j| < r$;

(ii) The zeros λ with

$$\frac{\pi}{2} + \epsilon < \arg(\lambda) < \frac{3\pi}{2} - \epsilon,$$

where $0 < \epsilon < \frac{\pi}{2}$, have a density equal to zero, i.e.

$$\lim_{r\to\infty} \frac{n^*(r)}{r} = 0, \tag{4.25}$$

where $n^*(r)$ denotes the number of zeros satisfying $|\lambda| < r$ and

$$\frac{\pi}{2} + \epsilon < \arg(\lambda) < \frac{3\pi}{2} - \epsilon.$$

PROOF. The first statement of (i) was proved in Lemma 4.8. Because of Theorem 4.10 and the Lebesgue dominated convergence theorem we have for r tending to infinity

$$r^{-1} \int_0^{2\pi} \log|F(re^{i\theta})|d\theta \to -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{E}(F)\cos\theta d\theta = 2\mathrm{E}(F).$$

From Jensen's formula [7; XI.1.2]

$$\int_0^r \frac{n(s)}{s} ds = \frac{1}{2\pi} \int_0^{2\pi} \log|F(re^{i\theta})|d\theta - \log|\frac{F^{(m)}}{m!}| + m\log r, \tag{4.26}$$

where F has a zero at $z = 0$ of multiplicity m, it follows that

$$r^{-1} \int_0^r \frac{n(s)}{s} ds \to \frac{\mathrm{E}(F)}{\pi} \qquad \text{as} \quad r \to \infty. \tag{4.27}$$

For every $k > 1$ we have

$$\begin{aligned} n(r)\log k &\le \int_r^{kr} \frac{n(s)}{s} ds \\ &= \int_0^{kr} \frac{n(s)}{s} ds - \int_0^r \frac{n(s)}{s} ds. \end{aligned} \tag{4.28}$$

A combination of (4.27) and (4.28) implies that for every $\epsilon > 0$ we can choose r so large that

$$n(r) < \frac{k-1}{\log k} \left(\frac{\mathrm{E}(F)}{\pi} + \epsilon \right) r. \tag{4.29}$$

Taking the limit $k \downarrow 1$ we obtain

$$n(r) \leq \left(\frac{\mathrm{E}(F)}{\pi} + \epsilon\right) r.$$

The proof that

$$n(r) \geq \left(\frac{\mathrm{E}(F)}{\pi} - \epsilon\right) r,$$

follows similar lines.

In order to prove (ii) we recall that

$$\sum_{j=1}^{\infty} |\Re(\frac{1}{\lambda_j})| \quad \text{converges.}$$

So, it follows that the sum

$$\sum\nolimits^{*} |\frac{1}{\lambda_j}| \leq \frac{1}{\cos\epsilon} \sum\nolimits^{*} |\Re(\frac{1}{\lambda_j})| \quad \text{converges,} \tag{4.30}$$

where $\sum^*$ denotes summation over the zeros λ_j with

$$\frac{\pi}{2} + \epsilon < \arg(\lambda) < \frac{3\pi}{2} - \epsilon.$$

Hence

$$\lim_{r\to\infty} \frac{n^*(r)}{r} = 0.$$

∎

Chapter 5. The asymptotic behaviour

In this chapter we discuss the asymptotic behaviour of elements of $z^n + \mathbf{P}_{n-1}$ and $z^n + \mathbf{J}_{n-1}$ in more detail. The applications we have in mind concern the behaviour of quotients

$$\frac{F}{G}, \tag{5.1}$$

where $F \in \mathbf{P}_{n-1}$ and $G \in z^n + \mathbf{P}_{n-1}$ in the left half plane $\Re(z) < 0$. We shall describe two possible approaches, the first one uses the Newton polygon - or distribution diagram - and exploits the specific form of elements of $\mathbf{J}_{n-1}$. The second approach uses complex analysis to show that the behaviour of (5.1) on large semi-circles in the left half plane is completely controlled by the behaviour of the Blaschke product

$$\prod_{j=1}^{\infty} \frac{1 - \frac{z}{\lambda_j}}{1 + \frac{z}{\lambda_j}}, \tag{5.2}$$

where the product is taken over all zeros of G in the left half plane $\Re(z) < 0$.

Our results give the asymptotic behaviour for a larger class of entire functions than considered by Bellman and Cooke [4]; however, the price we have to pay is that for this larger class of entire functions the asymptotic chains of zeros can not be given explicitly anymore and, in general, Theorem 4.12 is all we can say about the asymptotic location of the zeros.

Consider

$$H(z, w) = z^n + w^{l_1} z^{n-1} + \ldots + w^{l_n}, \tag{5.3}$$

where $l_0 = 0$ and for $j = 1, \ldots, n$ the exponents l_j are nonnegative real numbers. Assign to every term of (5.3) a point A_j with coordinates $(l_j, n - j)$.

Definition 5.1. The Newton polygon associated with H and denoted by $\mathrm{N}(H)$, is defined by the polygon determined by the upper convex envelope of

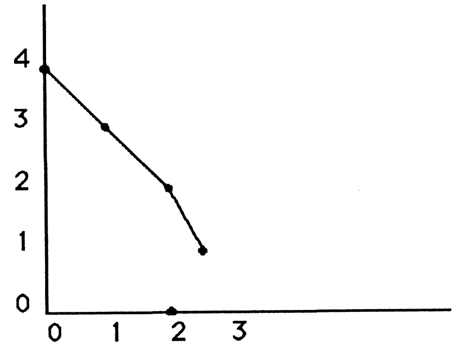

FIGURE 5.1. The Newton polygon of $z^4 + wz^3 + w^2z^2 + w^{\frac{5}{2}}z + w^2$.

the set of points $\{A_j : j = 0, 1, \ldots, n\}$. The upper convex property implies that the slopes of the line segments of the Newton polygon are negative and decrease.

The Newton polygon consists of a finite number of segments L_k with endpoints $(A_{j_{k-1}}, A_{j_k})$ and slopes α_k, where $k = 1, 2, \ldots, m$. We are going to prove that corresponding to each endpoint A_{j_k} there is a domain $W_k \subset \mathbf{C} \times \mathbf{C}$ so that

$$\bigcup_{k=1}^{m} W_k = \mathbf{C} \times \mathbf{C}$$

and

$$|H(z, w)| \geq C|z|^{n-j_k}|w|^{l_{j_k}} \tag{5.4}$$

for $(z, w) \in W_k$ with $|z|$ and $|wz^{\alpha_k}|$ sufficiently large.

EXAMPLE 5.2. Consider

$$H(z, w) = z^4 + wz^3 + w^2z^2 + w^{\frac{5}{2}}z + w^2.$$

From Figure 5.1 we see that the Newton polygon of H has two line segments, L_1 and L_2 with respectively, $\alpha_1 = -1$ and $\alpha_2 = -2$.
Define the following domains

$$W_1 = \{\, (z, w) \in \mathbf{C} \times \mathbf{C} : |z|^4 > C_1|w|^2|z|^2 \,\}$$

and in the complement of this set

$$W_2 = \{ (z,w) \in W_1^c : |w^2 z^2| > C_2 |w^{\frac{5}{2}} z| \}$$

and finally,

$$W_3 = \{ W_1 \cup W_2 \}^c.$$

We can choose C_1 sufficiently large so that for $(z,w) \in W_1$,

$$\begin{aligned} |H(z,w)| &\geq \big| \, |z|^4 - C_1^{-\frac{1}{2}}|z|^4 - C_1^{-1}|z|^4 - C_1^{-\frac{5}{4}}|z|^{\frac{7}{2}} - C_1^{-1}|z|^2 \, \big| \\ &\geq C|z|^4. \end{aligned} \tag{5.5}$$

For $(z,w) \in W_2$, we first consider the terms on the line segment L_1

$$z^4 + wz^3 + w^2z^2 = w^2z^2\big((w^{-1}z)^2 + w^{-1}z + 1 \big).$$

Since $(z,w) \in W_1^c$, we can choose $|w^{-1}z|^{-1} = |wz^{\alpha_1}|$ sufficiently large so that

$$|z^4 + wz^3 + w^2z^2| \geq \frac{1}{2}|w^2z^2|.$$

Consequently for $(z,w) \in W_2$ with $|wz^{\alpha_1}|$ and C_2 sufficiently large,

$$\begin{aligned} |H(z,w)| &\geq \big| \, |z^4 + wz^3 + w^2z^2| - |w^{\frac{5}{2}}z| - |w|^2 \, \big| \\ &\geq \Big| \frac{1}{2}|w^2z^2| - C_2^{-1}|w^2z^2| - C_2^{-1}|w^{\frac{3}{2}}z| \Big| \\ &\geq C|w^2z^2|. \end{aligned} \tag{5.6}$$

Finally for $(z,w) \in W_3$,

$$|z|^4 \leq C_1|w^2z^2| \leq C_1C_2|w^{\frac{5}{2}}z|.$$

Again we first consider the terms on the line segment L_2 to derive for $|w^{-\frac{1}{2}}z|^{-2} = |wz^{\alpha_2}|$ sufficiently large,

$$\begin{aligned} |w^2z^2 + w^{\frac{5}{2}}z| &\geq |w^{\frac{5}{2}}z| \, \big| \, 1 - |w^{-\frac{1}{2}}z| \, \big| \\ &\geq \frac{1}{2}|w^{\frac{5}{2}}z|. \end{aligned}$$

Therefore for $(z,w) \in W_3$ with $|z|$ and $|wz^{\alpha_2}|$ sufficiently large,

$$\begin{aligned} |H(z,w)| &\geq \big| \, |w^2z^2 + w^{\frac{5}{2}}z| - |z|^4 - |w||z|^3 - |w|^2 \, \big| \\ &\geq |w^{\frac{5}{2}}z| \, \Big| \frac{1}{2} - C_1|z||w|^{-\frac{1}{2}} - C_2|z||w|^{-1} - |z|^{-1}|w|^{-\frac{1}{2}} \Big| \\ &\geq C|w^{\frac{5}{2}}z|. \end{aligned} \tag{5.7}$$

With this example in mind we state and prove the general result. Let m denote the number of vertices on the Newton polygon and define the following domains

$$W_0 = \{ (z,w) \in \mathbf{C} \times \mathbf{C} : |z|^n > C_0 |z|^{n-j_1} |w|^{l_{j_1}} \}$$

and for $k = 1, 2, \ldots, m-1$

$$W_k = \{ (z,w) \in \{\cup_{j=0}^{m-1} W_j\}^c : |z|^{n-j_k} |w|^{l_{j_k}} > C_k |z|^{n-j_{k+1}} |w|^{l_{j_{k+1}}} \}$$

and

$$W_m = \{ \bigcup_{j=1}^{m} W_j \}^c.$$

We then have the following theorem

THEOREM 5.3. *Suppose that $(l_{j_k}, n-j_k)$ is any vertex of the Newton polygon of $H(z,w)$. Then there are constants $C_k, C > 0$ such that*

$$|H(z,w)| \geq C |z|^{n-j_k} |w|^{l_{j_k}}$$

for $(z,w) \in W_k$ with $|z|$ and $|wz^{\alpha_k}|$ sufficiently large, bounded away from the zeros of $H(z,w)$.

PROOF. Although technical, the proof of this theorem is straightforward. We first give a lower bound for the sum of terms of $H(z,w)$ corresponding to vertices on a segment L_k provided that $|wz^{\alpha_k}|$ is large, bounded away from the zeros of $H(z,w)$. Then we show for $(z,w) \in W_k$ with $|z|$ sufficiently large, that this lower bound is in fact a lower bound for $|H(z,w)|$ itself. For the terms of $H(z,w)$ corresponding to the vertices on a segment L_k the powers of w and the powers of z are proportional, i.e.

$$\alpha_k (l_{j_i} - l_{j_k}) = (j_k - j_i).$$

Therefore, we obtain

$$\Big| \sum_{(i,l_i) \in L_k} z^{n-i} w^{l_i} \Big| \geq |z|^{n-j_k} |w|^{l_{j_k}} (1 - O(|q|)), \tag{5.8}$$

where q is defined by

$$q = (wz^{\alpha_k})^{-1}. \tag{5.9}$$

For $(z,w) \in W_{k-1}^c$,

$$1 \geq C_{k-1} |wz^{\alpha_k}|^{l_{j_k} - l_{j_{k-1}}}.$$

Since by construction $l_{j_k} - l_{j_{k-1}} > 0$, we derive that $|q|$ can be made arbitrarily small for $|wz^{\alpha_k}|$ sufficiently large. So

$$\Big| \sum_{(i,l_i) \in L_k} z^{n-i} w^{l_i} \Big| \geq \frac{1}{2} |z|^{n-j_k} |w|^{l_{j_k}}$$

for $(z,w) \in W_k$ with $|wz^{\alpha_k}|$ sufficiently large, bounded away from the zeros of $H(z,w)$.
To derive a lower bound for $|H(z,w)|$ for $(z,w) \in W_k$ and $|z|$ sufficiently large we rewrite

$$\begin{aligned} |H(z,w)| &\geq \Big|\, \Big| \sum_{(i,l_i)\in L_k} z^{n-i} w^{l_i} \Big| - \sum_{(i,l_i)\notin L_k} z^{n-i} w^{l_i} \Big| \\ &\geq |z|^{n-j_k} |w|^{l_{j_k}} \Big| \frac{1}{2} - \sum_{(i,l_i)\notin L_k} z^{j_k - i} w^{l_i - l_{j_k}} \Big| \\ &\geq |z|^{n-j_k} |w|^{l_{j_k}} \Big| \frac{1}{2} - \sum_{(i,l_i)\notin L_k} \left(|w||z|^{\beta_i}\right)^{l_i - l_{j_k}} \Big|, \end{aligned} \tag{5.10}$$

where

$$\beta_i = \frac{j_k - i}{l_i - l_{j_k}}. \tag{5.11}$$

To complete the estimate we have to consider three cases:

I $i < j_k$;

II $i > j_k$ and $k < m$;

III $i > j_m$.

Case i. For $i < j_k$ we have by construction

$$l_i < l_{j_k}$$

and

$$\beta_i > \alpha_k.$$

Hence for $|wz^{\alpha_k}|$ sufficiently large, the terms

$$\left(|w||z|^{\beta_i} \right)^{l_i - l_{j_k}},$$

where $(i,l_i) \notin L_k$ and $i < j_k$ can be made arbitrarily small.

Case ii. For $i > j_k$ we have by construction

$$l_{j_{k+1}} - l_{j_k} > 0$$

and

$$\alpha_{k+1} > \beta_i.$$

Since for $(z,w) \in W_k$

$$1 > C_k |wz^{\alpha_{k+1}}|^{l_{j_{k+1}} - l_{j_k}},$$

we derive for C_k sufficiently large, that the terms

$$\left(|w||z|^{\beta_i} \right)^{l_i - l_{j_k}},$$

where $(i,l_i) \notin L_k$ and $i > j_k$ can be made arbitrarily small.

CASE III. For $i > j_m$ we have by construction

$$l_i - l_{j_m} < 0$$

and

$$\beta_i > 0.$$

Hence we conclude that for $|z|$ sufficiently large the terms

$$\left(|w||z|^{\beta_i} \right)^{l_i - l_{j_m}},$$

where $(i, l_i) \notin L_m$ and $i > j_m$ can be made arbitrarily small. This completes the proof of the theorem. ∎

As an application we consider the case $w = e^{-z}$. So

$$H(z, e^{-z}) = z^n + e^{-l_1 z} z^{n-1} + e^{-l_2 z} z^{n-2} + \ldots + e^{-l_n z}. \tag{5.12}$$

In this case the domains W_k can be easily depicted in the complex plane. The left boundary of the W_k can be represented by

$$|z| = C_k |e^{\alpha_k^{-1} z}| = C_k e^{\alpha_k^{-1} \Re(z)},$$

where by construction of the Newton polygon α_k^{-1} increases as k increases. And since this boundary asymptotically behaves like the exponential function

$$y = C_k e^{\alpha_k^{-1} x},$$

the boundaries do not intersect each other for $|z|$ sufficiently large.
And we can formulate the following corollary to Theorem 5.3.

COROLLARY 5.4. *Suppose that $(l_{j_k}, n - j_k)$ is any vertex of the Newton polygon of $H(z, e^{-z})$. Then there is a constant $C_k > 0$ such that*

$$|H(z, e^{-z})| \geq C_k |z|^{n - j_k} |e^{-l_{j_k} z}| \tag{5.13}$$

for $z \in W_k$ with $|z|$ sufficiently large, bounded away from the zeros of $H(z, e^{-z})$.

Recall the above derived estimates and make the following observation: for $k = 1, 2, \ldots, n$

$$\left| \frac{z^{n-k} e^{-l_k z}}{H(z, e^{-z})} \right| = O(1) \quad \text{as} \quad z \to \infty, \tag{5.14}$$

bounded away from the zeros of $H(z, e^{-z})$.

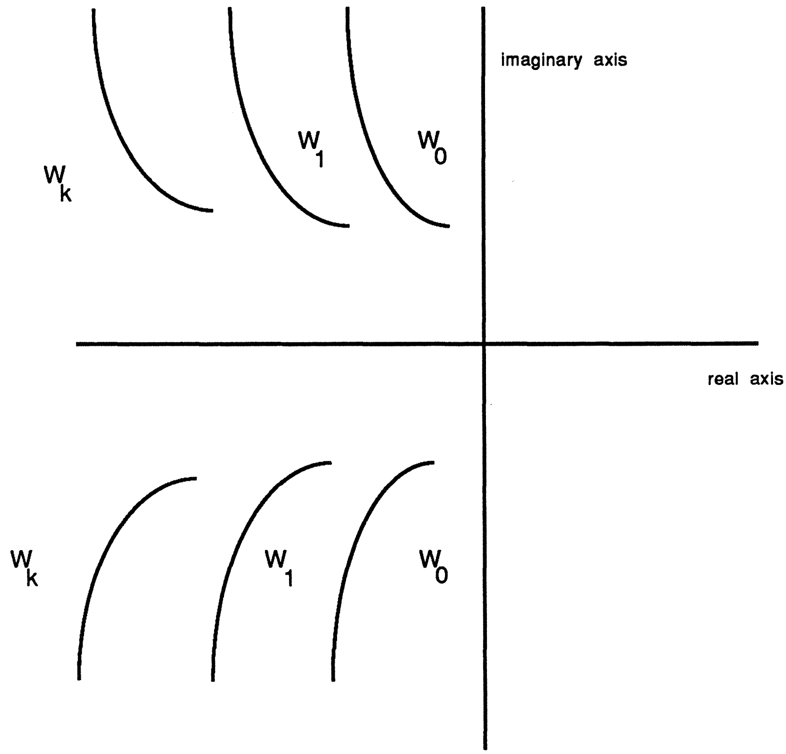

FIGURE 5.2. The domains W_k in the complex plane for $|z|$ large.

For a general element $F \in z^n + \mathbf{J}_{n-1}$ the situation is only a little bit different, since we can use the estimates proved in Theorem 4.6. To formulate the result we represent $F \in z^n + \mathbf{J}_{n-1}$ by

$$F(z) = z^n + \int_0^{\tau_1} e^{-zt} d\eta_1(t) z^{n-1} + \cdots + \int_0^{\tau_n} e^{-zt} d\eta_n(t) \tag{5.15}$$

and associate with F the exponential polynomial

$$H(z, e^{-z}) = z^n + e^{-\tau_1 z} z^{n-1} + \cdots + e^{-\tau_n z}. \tag{5.16}$$

Recall from Theorem 4.6 that because of the jump condition on η_k, we can find appropriate constants m_k, M_k, and $\gamma_k < 0$ so that

$$m_k |e^{-\tau_k z}| \leq \left| \int_0^{\tau_k} e^{-zt} d\eta_k(t) \right| \leq M_k |e^{-\tau_k z}| \tag{5.17}$$

for $\Re(z) < \gamma_k$.

From observation (5.14) and estimate (5.17) we obtain the following corollary.

COROLLARY 5.5. *If $F \in z^n + \mathbf{J}_{n-1}$ is defined by (5.15) and if $H(z, e^{-z})$ is defined by (5.16). Then there exist appropriate constants m, M, and $\gamma < 0$ such that*

$$m \leq \left|\frac{F(z)}{H(z, e^{-z})}\right| \leq M$$

in the left half plane $\Re(z) < \gamma$ with $|z|$ sufficiently large, bounded away from the zeros of $H(z, e^{-z})$.

From the proof of theorem 4.12 it follows that the zeros of F have a density. Therefore we can construct a sequence of contours C_l contained in the left half plane $\Re(z) < 0$ so that

(i) If $z \in C_l$, then $|z| \to \infty$ as $l \to \infty$;

(ii) The zeros of F are bounded away from the contours C_l.

We shall study the behaviour of the following class of meromorphic functions on the contours C_l

$$\frac{G}{F}, \tag{5.18}$$

where $G \in \mathbf{P}_{n-1}$ and $F \in z^n + \mathbf{J}_{n-1}$.

First we associate a Newton polygon with F.

DEFINITION 5.6. Let $F \in \mathbf{P}_n$ and represent F by

$$F(z) = \sum_{j=0}^{n} \int_0^{\tau_j} e^{-zt} d\eta_j(t) z^{n-j}. \tag{5.19}$$

The Newton polygon of F will be defined to be the Newton polygon of H, the exponential polynomial

$$H(z, e^{-z}) = e^{-\tau_0 z} z^n + e^{-\tau_1 z} z^{n-1} + \ldots + e^{-\tau_n z} \tag{5.20}$$

associated with F.

Let $\mathrm{N}(G)$ denote the Newton polygon associated with G and let $\mathrm{N}(F)$ denote the Newton polygon associated with F. We write $\mathrm{N}(G) \leq \mathrm{N}(F)$ to denote that the set enclosed by $\mathrm{N}(G)$ and the vertical lines $x = 0$ and $x = \mathrm{E}(G)$, is contained in the set enclosed by $\mathrm{N}(F)$ and the vertical lines $x = 0$ and $x = \mathrm{E}(F)$. Using this notation we can formulate the following corollary.

COROLLARY 5.7. *If $G \in \mathbf{P}_n$ and $F \in z^n + \mathbf{J}_{n-1}$ are such that*

$$\mathrm{N}(G) \leq \mathrm{N}(F).$$

Then

$$|\frac{G(z)}{F(z)}| = O(1) \tag{5.21}$$

for $z \in C_l$ as $l \to \infty$.

PROOF. Let $H(z, e^{-z})$ denote the exponential polynomial (5.20) associated with F and represent $G \in \mathbf{P}_{n-1}$ by

$$G(z) = \int_0^{\sigma_1} e^{-zt} d\nu_1(t) z^{n-1} + \cdots + \int_0^{\sigma_n} e^{-zt} d\nu_n(t).$$

Since $G \in \mathbf{P}_{n-1}$, Theorem 4.6 now only yields an upper estimate for the coefficients of G:

$$\left|\int_0^{\sigma_k} e^{-zt} d\nu_k(t)\right| \leq M_k |e^{-\sigma_k z}| \tag{5.22}$$

for $z \in C_l$ and for some constant M_k. Since

$$\mathrm{N}(G) \leq \mathrm{N}(F),$$

we derive from observation (5.14) and the estimate (5.22) that

$$|\frac{G(z)}{H(z, e^{-z})}| = O(1) \text{ for } z \in C_l \text{ as } l \to \infty. \tag{5.23}$$

From Corollary 5.5 we have for $z \in C_l$

$$|\frac{F(z)}{H(z, e^{-z})}| \geq m \tag{5.24}$$

and hence a combination of (5.23) and (5.24) gives the desired result. ∎

If we replace the Newton polygon inequality by the exponential type inequality $\mathrm{E}(G) \leq \mathrm{E}(F)$ the above result is no longer true.

COROLLARY 5.8. *Suppose $G \in \mathbf{P}_n$ and $F \in z^n + \mathbf{J}_{n-1}$ are such that*

$$\mathrm{E}(G) \leq \mathrm{E}(F).$$

Then

$$|\frac{G(z)}{F(z)}| = O(|z|^n) \tag{5.25}$$

for $z \in C_l$ as $l \to \infty$.

PROOF. If $\mathrm{E}(G) \leq \mathrm{E}(F)$ then

$$\mathrm{N}(G) \leq \mathrm{N}(\overline{F}),$$

where $\overline{F}(z) = z^n F(z)$. Hence we can apply Corollary 5.7 to G and $\overline{F}$. ∎

The remaining part of this chapter will be devoted to the study of the behaviour of

$$\frac{G}{F}, \tag{5.26}$$

where $G \in \mathbf{P}_{n-1}$ and $F \in z^n + \mathbf{P}_{n-1}$.
In this case the jump condition does not hold anymore for the coefficients of F. From Remark 4.3, we derive that we can not control the lower bounds for the coefficients of F anymore. Hence apart from the behaviour of F on the real- and imaginary axis, the specific form (4.5) of F yields no additional information. To derive some results in this general case we shall use complex analysis, since we only use the behaviour of (5.26) on the real- and imaginary axis, the results are less sharp then Corollary 5.8. The result we are going to prove states: If $\mathrm{E}(G) < \mathrm{E}(F)$, then (5.26) has only polynomial growth on the contours C_l, but now the degree of the polynomial growth on the contours C_l depends on l and can be arbitrarily large.

We need some lemmata before we can prove the theorem.

LEMMA 5.9. *If $(\lambda_j)_{j\geq 1}$ are the zeros of $F(z)$ in the left half plane $\Re(z) < 0$ and the infinite Blaschke product $B(z)$ is defined by*

$$B(z) = \prod_{j=1}^{\infty} \frac{1 - \frac{z}{\overline{\lambda}_j}}{1 + \frac{z}{\lambda_j}}. \tag{5.27}$$

Then $B(z)$ converges uniformly on compact sets bounded away from the points $\{-\overline{\lambda}_j\}$.

PROOF. Let $B_m(z)$ denote the finite Blaschke product

$$B_m(z) = \prod_{j=1}^{m} \frac{1 - \frac{z}{\overline{\lambda}_j}}{1 + \frac{z}{\lambda_j}}$$

and let $d(z)$ denote the minimal distance between z^{-1} and the points $-\overline{\lambda}_j^{-1}$, i.e.

$$d(z) = \inf_{j\geq 1} \left|\frac{1}{z} + \frac{1}{\overline{\lambda}_j}\right|.$$

Recall from Theorem 4.9 that

$$\sum_{j=1}^{\infty} |\Re(\frac{1}{\lambda_j})| \text{ converges,}$$

Since we can estimate

$$\begin{aligned}|\frac{1-\frac{z}{\lambda_j}}{1+\frac{z}{\lambda_j}}| &\leq 1+|\frac{1-\frac{z}{\lambda_j}}{1+\frac{z}{\lambda_j}}-1| \\ &\leq \exp\{-\frac{2}{d(z)}\Re(\frac{1}{\lambda_j})\},\end{aligned} \tag{5.28}$$

we can find for every $\epsilon > 0$ an $N \in \mathbf{N}$ such that for $N < k \leq l$

$$|B_l(z) - B_k(z)| \leq \epsilon$$

and so, the lemma is proved. ∎

LEMMA 5.10. *If $G \in \mathbf{P}_{n-1}$ and $F \in z^n + \mathbf{P}_{n-1}$ and if P is defined by*

$$P(z) = \frac{G(z)B(z)}{F(z)}. \tag{5.29}$$

Then P is an analytic function of exponential type in the left half plane $\Re(z) < 0$.

PROOF. The function P is clearly analytic in the left half plane $\Re(z) < 0$. Moreover, by applying the argument presented in the proof of Theorem 6.4.5 of Boas [5], the function P is of exponential type in $\Re(z) < 0$. This can be seen by writing the Hadamard factorisation for F

$$F(z) = Ae^{bz}\prod_{j=1}^{\infty}(1-\frac{z}{w_j})e^{\frac{z}{w_j}},$$

where $(w_j)_{j\geq 1}$ denote the zeros of F. Because of Theorem 4.9 we can rewrite B as follows

$$B(z) = \prod_{j=1}^{\infty}\frac{(1-\frac{z}{\lambda_j})e^{\frac{z}{\lambda_j}}}{(1+\frac{z}{\lambda_j})e^{-\frac{z}{\lambda_j}}}\prod_{j=1}^{\infty}e^{2\Re(\frac{1}{\lambda_j})z}.$$

Hence

$$\frac{F(z)}{B(z)} = Ae^{cz}\prod_{j=1}^{\infty}(1-\frac{z}{z_j})e^{\frac{z}{z_j}}$$

for some c and where z_j denotes the sequence of finitely many zeros of F in the right half plane $\Re(z) > 0$, followed by the points $\{-\bar{\lambda}_j\}$. Therefore in the left half plane $\Re(z) < 0$

$$\frac{F(z)}{B(z)}$$

is an analytic function of order at most one [5; 2.6.5] and because of the Lindelöf theorem [5; 2.10.1] of exponential type. Since the quotient of two functions of exponential type is of exponential type provided it is analytic, the proof of the lemma is complete. ∎

LEMMA 5.11. *If P is defined by (5.29) and if* $\mathrm{E}(G) < \mathrm{E}(F)$. *Then*

$$|P(z)| = O(\frac{1}{|z|})$$

as $|z| \to \infty$ *in the left half plane* $\Re(z) < 0$.

PROOF. Recall that $G \in \mathbf{P}_{n-1}$ and $F \in z^n + \mathbf{P}_{n-1}$. Since $|B(iy)| = 1$ we obtain, because of the Riemann-Lebesgue Lemma 3.5

$$|P(iy)| = O(\frac{1}{|y|}). \tag{5.30}$$

Next, we estimate $|P(z)|$ on the negative real axis. Since $|B(-x)| \leq 1$, the condition $\mathrm{E}(G) < \mathrm{E}(F)$ and Theorem 4.10 imply that

$$\lim_{x \to \infty} \log \frac{|P(-x)|}{x} \leq 0. \tag{5.31}$$

An application of Boas [5; 6.2.4] now yields

$$|P(z)| = O(\frac{1}{|z|}) \qquad \text{as} \quad |z| \to \infty$$

in the whole left half plane $\Re(z) < 0$. ∎

Finally we shall estimate $|B(z)^{-1}|$ in the left half plane $\Re(z) < 0$.

LEMMA 5.12. *For every $\sigma > 0$ and each l sufficiently large, there are constants C and $m = m(l)$ so that on the contour C_l we have*

$$|\frac{1}{B(z)}| \leq C|z|^m e^{-\sigma \Re(z)}. \tag{5.32}$$

PROOF. To provide the estimate we follow the method used by Ahlfors-Heins [1]. Consider the Green's function relative to $\Re(z) < 0$ with pole at λ_j

$$g(z, \lambda_j) = \log |\frac{z + \bar{\lambda}_j}{z - \lambda_j}|. \tag{5.33}$$

It then follows that

$$\log|B(z)| = \sum_{j=1}^{\infty} g(z, \lambda_j).$$

Since $z \in C_l$ is bounded away from the zeros $(\lambda_j)_{j\geq 1}$ we can define

$$\epsilon = \epsilon(l) = \min_{j\geq 1} \left\{ \ |z - \lambda_j|^2 : z \in C_l \right\}.$$

Fix l, let $z \in C_l$ and write

$$\frac{1}{B(z)} = \prod_{j\in N_1} \frac{1 + \frac{z}{\overline{\lambda}_j}}{1 - \frac{z}{\lambda_j}} \prod_{j\in N_2} \frac{1 + \frac{z}{\overline{\lambda}_j}}{1 - \frac{z}{\lambda_j}}, \tag{5.34}$$

where

$$N_1 = N_1(l) = \left\{ j \in \mathbf{N} : |z - \lambda_j|^2 \leq \frac{1}{4}|\lambda_j|^2 \text{ for } z \in C_l \right\}$$

and

$$N_2 = N_2(l) = \left\{ j \in \mathbf{N} : |z - \lambda_j|^2 > \frac{1}{4}|\lambda_j|^2 \text{ for } z \in C_l \right\}.$$

First consider the set N_1. The condition $|z - \lambda_j|^2 \leq \frac{1}{4}|\lambda_j|^2$ holds if and only if

$$|\frac{z}{\lambda_j} - 1|^2 \leq \frac{1}{4}.$$

Since this condition is definitely not satisfied for

$$|z| < \frac{1}{2}|\lambda_j| \quad \text{or} \quad |z| > \frac{3}{2}|\lambda_j|$$

we see that the set $N_1 = N_1(l)$ is finite and that for $j \in N_1$

$$\frac{1}{2}|\lambda_j| \leq |z| \leq \frac{3}{2}|\lambda_j|.$$

Next, rewrite (5.33)

$$\begin{aligned} g(z, \lambda_j) &= \frac{1}{2}\log\left(1 + 4\frac{\Re(z)\Re(\lambda_j)}{|z - \lambda_j|^2}\right) \\ &\leq 2\frac{\Re(z)\Re(\lambda_j)}{|z - \lambda_j|^2}. \end{aligned} \tag{5.35}$$

Hence for $j \in N_2$, equation (5.35) implies

$$g(z, \lambda_j) \leq 2\Re(z)\Re(\frac{1}{\lambda_j}). \tag{5.36}$$

Now choose N_0 such that

$$\sum_{j=N_0}^{\infty} \Re(\frac{1}{\lambda_j}) > -\frac{1}{2}\sigma,$$

because of (5.36)

$$|\prod_{j \in N_2} \frac{1+\frac{z}{\lambda_j}}{1-\frac{z}{\lambda_j}}| \leq \prod_{j \in N_3} \frac{1+\frac{z}{\lambda_j}}{1-\frac{z}{\lambda_j}} e^{-\sigma\Re(z)}, \tag{5.37}$$

where

$$N_3 = \{j \in N_2 : j \leq N_0\}.$$

Hence for l sufficiently large, the finite sets N_1 and N_3 are both contained in the finite set

$$N_4 = N_4(l) = \{j \in \mathbf{N} : \frac{1}{2}|\lambda_j| < \min_{z \in C_l} |z| \}.$$

Therefore we can estimate

$$\frac{1}{B(z)} \leq \prod_{j \in N_4} \frac{1+\frac{z}{\lambda_j}}{1-\frac{z}{\lambda_j}} e^{-\sigma\Re(z)},$$

where the finite product over N_4 can be estimated by

$$\begin{aligned} |\prod_{j \in N_4} \frac{1+\frac{z}{\lambda_j}}{1-\frac{z}{\lambda_j}}| &\leq \prod_{j \in N_4} |z|\left|\frac{\frac{\lambda_j}{z} + \frac{\lambda_j}{\lambda_j}}{\epsilon}\right| \\ &\leq |z|^k (\frac{3}{\epsilon})^k \\ &\leq C|z|^m, \end{aligned} \tag{5.38}$$

for some constants C and m and where k denotes the cardinality of N_4. This proves the lemma. ∎

So we have proved the following theorem.

THEOREM 5.13. *If $G \in \mathbf{P}_{n-1}$ and $F \in z^n + \mathbf{P}_{n-1}$ are such that*

$$\mathrm{E}(G) < \mathrm{E}(F).$$

Then for each sufficiently large l, there are constants C and $m = m(l)$ so that on the contour C_l

$$|\frac{G(z)}{F(z)}| \leq C|z|^m. \tag{5.39}$$

Since m growths with l, this result gives no uniform bound for m like in the case $F \in z^n + \mathbf{J}_{n-1}$ where we could estimate $m \leq n$.

It is an interesting question to study the minimal conditions on

$$F(z) = z^n + \int_0^{\tau_1} e^{-zt} d\eta_1(t) z^{n-1} + \cdots + \int_0^{\tau_n} e^{-zt} d\eta_n(t),$$

such that m in the above theorem can be uniformly bounded. Is it for example sufficient for the continuous singular parts of all η_j to be zero?

In the sequel we shall encounter two types of situations, one where we need the uniform bound on m and in that case we have to restrict ourselves to the $z^n + \mathbf{J}_{n-1}$ class and one where we do not need this uniform bound and where we can use the $z^n + \mathbf{P}_{n-1}$ class.

CHAPTER 6. VOLTERRA CONVOLUTION EQUATIONS

In this chapter we shall use Laplace transform methods to investigate the asymptotic behaviour of the solution of a Volterra convolution (or, renewal) equation

$$x - \zeta * x = f, \tag{6.1}$$

where ζ is a matrix-valued function on $\mathbf{R}_+$ which belongs to NBV$[0, h]$.

We start with the following "existence of a unique solution" result for equation (6.1).

THEOREM 6.1. *If γ is sufficiently large and if $f \in L^1(\mathbf{R}_+; \gamma)$. Then the Volterra convolution equation (6.1) has a unique solution $x \in L^1(\mathbf{R}_+; \gamma)$ and*

$$x = f - R * f, \tag{6.2}$$

where the so-called resolvent R belongs to $L^1(\mathbf{R}_+; \gamma)$ and is defined as the unique (matrix-valued) solution of both

$$R = R * \zeta - \zeta \quad \textit{and} \quad R = \zeta * R - \zeta. \tag{6.3}$$

PROOF. If $y(t) = x(t)e^{-\gamma t}$. Then

$$y = \tilde{\zeta} * y + \tilde{f},$$

where

$$\tilde{\zeta}(t) = \zeta(t)e^{-\gamma t}$$

and

$$\tilde{f}(t) = f(t)e^{-\gamma t}.$$

Choose γ so large that

$$\int_0^\infty |\tilde{\zeta}(t)|dt = \int_0^\infty |\zeta(t)|e^{-\gamma t}dt < 1$$

and define a mapping

$$Q : L^1(\mathbf{R}_+) \to L^1(\mathbf{R}_+)$$

by

$$Qy = \tilde{\zeta} * y + \tilde{f}.$$

From Theorem 2.6 it follows that the convolution of two elements of $L^1(\mathbf{R}_+)$ belongs to $L^1(\mathbf{R}_+)$ and the L^1-norm is less than or equal to the product of the L^1-norms. Hence

$$\|Qy_1 - Qy_2\|_1 \leq \|\tilde{\zeta}\|_1\|y_1 - y_2\|_1 \tag{6.4}$$

and consequently Q is a contraction. It follows that Q has a unique fixed point, which can be obtained by successive approximations. The same proof applies to equation (6.3). The expansion

$$-\tilde{R} = \tilde{\zeta} + \tilde{\zeta} * \tilde{\zeta} + \tilde{\zeta} * \tilde{\zeta} * \tilde{\zeta} + \ldots$$

where $\tilde{R}(t) = R(t)e^{-\gamma t}$ converges in $L^1(\mathbf{R}_+)$ and shows that

$$\tilde{R} \in L^1(\mathbf{R}_+)$$

and

$$\tilde{R} * \tilde{\zeta} = \tilde{\zeta} * \tilde{R}.$$

Hence

$$R * \zeta = \zeta * R.$$

Finally, if x satisfies (6.1) then

$$\begin{aligned} R * x &= R * \zeta * x + R * f \\ &= R * x + \zeta * x + R * f. \end{aligned}$$

So

$$\zeta * x = -R * f.$$

Substituting this identity into (6.1) we obtain (6.2). ∎

Our interest in renewal equations comes from the close connection between renewal and delay equations presented in Example 2.9. Given this connection, the classes of forcing functions we have to consider are rather special, namely constant for $t \geq h$, see Remark 2.10. Recall (2.14) to derive that the convolution of an element of $L^1(\mathbf{R}_+;\gamma)$ and an element of NBV$[\mathbf{R}_+]$ is absolutely continuous. Hence from equation (6.3), we conclude that the resolvent is locally of bounded variation. Now we can reformulate Theorem 6.1 in the form which we will use in the sequel.

COROLLARY 6.2. *If $f \in L^1(\mathbf{R}_+;\gamma)$ such that f is constant for $t \geq h$. Then the Volterra convolution equation has a unique solution in $L^1(\mathbf{R}_+;\gamma)$ that is absolutely continuous for $t \geq h$.*

Let $\mathcal{F}$ denote the Banach space of continuous functions on $\mathbf{R}_+$ that are constant on the interval $[0, \infty)$ provided with the supremum norm. We shall associate with (6.1) a family of operators $\{S(t)\}$ acting on $\mathcal{F}$ such that

$$x_t = \zeta * x_t + S(t)f, \tag{6.5}$$

where $x_t(\,\cdot\,) = x(t+\,\cdot\,)$. Since

$$x(t+s) = \int_0^{t+s} \zeta(\theta)x_t(s-\theta)d\theta + f(t+s),$$

we obtain

$$\big(S(t)f\big)(s) = f(t+s) + \int_0^t \zeta(t+s-\theta)x(\theta)d\theta. \tag{6.6}$$

Using representation (6.2) and definition (6.6) for $S(t)$ the following theorem can be proved [10], consult Definition 10.2 for the definition of a $\mathcal{C}_0$-semigroup.

THEOREM 6.3. *The family of operators $\{S(t)\}$ is a $\mathcal{C}_0$-semigroup and the infinitesimal generator B of $\{S(t)\}$ is defined by*

$$Bf = \dot{f} + \zeta(\,\cdot\,)f(0)$$

with

$$\mathcal{D}(B) = \{f \in \mathcal{F} : \dot{f} + \zeta(\,\cdot\,)f(0) \in \mathcal{F}\}.$$

In Chapter 12 we shall return to the semigroup approach for renewal equations. For now we shall only use the semigroup property

$$S(t_1+t_2)f = S(t_1)s(t_2)f. \tag{6.7}$$

From representation (6.2) and equation (6.6) it follows that

$$\mathcal{R}(S(h)) \subset \mathcal{D}(B).$$

Moreover, $BS(h)f$ is locally of bounded variation for every $f \in \mathcal{F}$.

Since, in this chapter, we are interested in the large time behaviour of solutions of equation (6.1) the semigroup property (6.7) implies that - without loss of generality - the forcing function f can be restricted to $\mathcal{D}(B)$, the domain of the generator. Hence from Theorem 6.3, the solution $x(\,\cdot\,; f)$ of (6.1) is absolutely continuous for $t \geq 0$.

Because of the above remarks and the properties of the Laplace transform, listed in Proposition 3.1, we can Laplace transform the renewal equation (6.1) to obtain in some right half plane $\Re(z) > \gamma$:

$$L\{x\} = L\{\zeta\}L\{x\} + L\{f\}. \tag{6.8}$$

Since ζ and f belong to NBV$[0,h]$, we can rewrite equation (6.8) such that it makes sense in the whole complex plane

$$\left(zI - \int_0^h e^{-zt}d\zeta(t)\right) L\{x\}(z) = f(0) + \int_0^h e^{-zt}df(t). \tag{6.9}$$

Equation (6.9) yields the analytic continuation to the whole complex plane for $L\{x\}$. The possible singularities of $L\{x\}$ are the singular points of the so-called characteristic matrix of the renewal equation (6.1) defined by

$$\Delta(z) = zI - \int_0^h e^{-zt}d\zeta(t). \tag{6.10}$$

In order to be able to apply the Laplace inversion formula we first have to analyse the inverse $\Delta^{-1}(z)$ of the characteristic matrix.

LEMMA 6.4. *The determinant of the characteristic matrix $\Delta(z)$, can be written as follows*

$$\det \Delta(z) = z^n - \sum_{j=1}^{n} \int_0^{jh} e^{-zt}d\eta_j(t)z^{n-j}. \tag{6.11}$$

So $\det \Delta(z)$ *is an entire function and has exponential type*

$$\mathrm{E}(\det \Delta(z)) \leq nh. \tag{6.12}$$

PROOF. The results follow directly from the exponential type calculus presented in Chapter 4. ∎

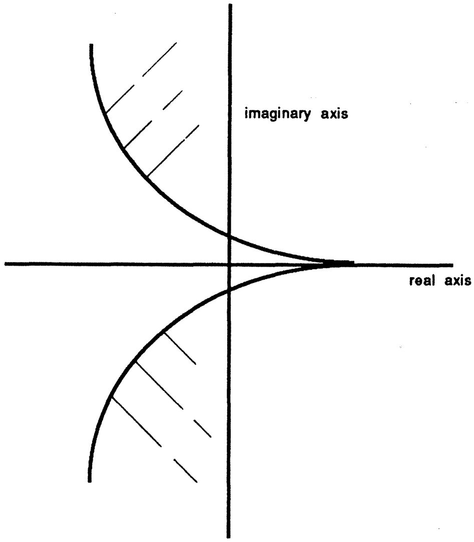

FIGURE 6.1. In the hatched domain $|z| > C_1 e^{-h\Re(z)}$.

Because of the above lemma we can apply Theorem 4.12 to obtain information about the location of the zeros of $\det \Delta(z)$. The following lemma describes the set W_0, introduced in Chapter 5, for $\det \Delta(z)$.

LEMMA 6.5. *There exist constants $C_1, C_2 > 0$ so that*

$$\left|\det \Delta(z)\right| \geq C_2 |z|^n$$

for $|z| \geq C_1 |e^{-hz}|$.

PROOF. From the representation (6.11) we obtain the estimate

$$|\det \Delta(z)| \geq \left||z|^n - |\sum_{j=1}^{n} \int_0^{jh} e^{-zt} d\eta_j(t) z^{n-j}|\right|. \tag{6.13}$$

For

$$e^{-h\Re(z)} \leq \frac{1}{C_1}|z|$$

with C_1 sufficiently large,

$$\begin{aligned} |\sum_{j=1}^{n} \int_0^{jh} e^{-zt} d\eta_j(t) z^{n-j}| &\leq \sum_{j=1}^{n} \frac{1}{C_1^j} \int_0^{jh} |d\eta_j(t)||z|^n \\ &\leq \frac{1}{2}|z|^n. \end{aligned}$$

Hence, the result follows. ∎

The above lemma has an easy corollary.

COROLLARY 6.6. *The entire function* $\det\Delta(z)$ *has no zeros in the domain*

$$\{z : |z| > C_1 e^{-h\Re(z)}\}$$

for C_1 *sufficiently large. Consequently, there are only finitely many zeros in each strip*

$$-\infty < \gamma_1 < \Re(z) < \gamma_2 < \infty$$

and $\det\Delta(z)$ *has a zero free right half plane* $\Re(z) > \gamma$.

Now we turn to a representation for $\Delta^{-1}(z)$. Rewrite

$$\Delta^{-1}(z) = \frac{\operatorname{adj}\Delta(z)}{\det\Delta(z)}, \tag{6.14}$$

where $\operatorname{adj}\Delta(z)$ denotes the matrix of cofactors of $\Delta(z)$, i.e. the coefficients of $\operatorname{adj}\Delta(z)$ are the $(n-1)\times(n-1)$ subdeterminants of $\Delta(z)$. Because of the exponential type calculus presented in Chapter 4 we have the following representation for the cofactors:

$$(\operatorname{adj}\Delta(z))_{ij} = \delta_{ij} z^{n-1} + \sum_{k=1}^{n-1} \int_0^{kh} e^{-zt} d\eta_{ijk}(t) z^{n-1-k}, \tag{6.15}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Also

$$\mathrm{E}\big((\operatorname{adj}\Delta(z))_{ij}\big) \leq (n-1)h.$$

Rewrite equation (6.9) as follows

$$L\{x\}(z) = \frac{\operatorname{adj}\Delta(z)}{\det\Delta(z)}\big(f(0) + \int_0^h e^{-zt} df(t)\big). \tag{6.16}$$

On account of Corollary 6.6 we can choose γ such that $\det\Delta(z)$ has no zeros in the right half plane $\Re(z) > \gamma$. Hence, the Laplace transform $L\{x\}$ is analytic in this half plane. So, from Corollary 3.4 and the remarks made about shifting the contour, we obtain the following representation for the solution $x = x(\,\cdot\,; f)$ of the renewal equation (6.1)

$$x(t) = \frac{1}{2\pi i} \int_{L(\gamma)} e^{zt} \frac{\operatorname{adj}\Delta(z)}{\det\Delta(z)}\big(f(0) + \int_0^h e^{-zt} df(t)\big)\, dz \qquad \text{for} \quad t > 0. \tag{6.17}$$

Next we derive the asymptotic behaviour of $x(t)$ as $t \to \infty$, by shifting the contour of integration to the left. First we analyse the singularities of

$$H(z,t) = e^{zt} \frac{\operatorname{adj}\Delta(z)}{\det\Delta(z)}\big(f(0) + \int_0^h e^{-zt} df(t)\big). \tag{6.18}$$

Clearly the only singularities are poles of finite order, given by the zeros of $\det\Delta(z)$.

LEMMA 6.7. *If λ_j is a zero of* $\det\Delta(z)$ *of order m_{λ_j}, then the residue of $H(z,t)$ for $z=\lambda_j$ equals*

$$\operatorname*{Res}_{z=\lambda_j} H(z,t) = p_j(t)e^{\lambda_j t}, \tag{6.19}$$

where p_j is a polynomial in t of degree less than or equal to $(m_{\lambda_j}-1)$.

PROOF. We calculate the coefficient of $(z-\lambda_j)^{-1}$ in the Laurent expansion of $H(z,t)$ in a neighbourhood of $z=\lambda_j$,

$$\Delta^{-1}(z) = \frac{\operatorname{adj}\Delta(z)}{\det\Delta(z)} = A_{-m_{\lambda_j}}(z-\lambda_j)^{-m_{\lambda_j}} + \cdots + M_0 + \cdots$$

$$f(0) + \int_0^h e^{-zt}df(t) = \sum_{k=0}^{\infty} a_k(z-\lambda_j)^k,$$

$$e^{zt} = e^{\lambda_j z}e^{(z-\lambda_j)z} = e^{\lambda_j t}\sum_{k=0}^{\infty}\frac{t^k}{k!}(z-\lambda_j)^k.$$

Since the residue in $z=\lambda_j$ equals the coefficient of the $(z-\lambda_j)^{-1}$-term of the Laurent series of $H(z,t)$ in a neighbourhood of $z=\lambda_j$, a multiplication of the above series expansions yields the desired result. ∎

Denote the zeros of $\det\Delta(z)$ by $\lambda_1, \lambda_2, \ldots$. On account of Corollary 6.6 we can define a sequence $\{\gamma_l\}$ such that the number of zeros of $\det\Delta(z)$ with real part strictly between γ_l and γ equals l. Define $\Gamma(\gamma,\gamma_l)$ to be the closed contour in the complex plane, which is composed of four straight lines and connects the points $\gamma_l - iN, \gamma - iN, \gamma + iN$, and $\gamma_l + iN$, where N is larger than $\max_{1\le j\le l}|\Im(\lambda_j)|$.

From the above lemma and the Cauchy theorem of residues we obtain

$$\frac{1}{2\pi i}\int_{\Gamma(\gamma,\gamma_l)} H(z,t)dz = \sum_{j=1}^{l} p_j(t)e^{\lambda_j t}. \tag{6.20}$$

In order to be able to shift the contour $L(\gamma)$ to $L(\gamma_l)$ we have to derive estimates for

$$|H(\sigma+i\omega;t)| \quad \text{for large values of } |\omega|.$$

LEMMA 6.8. *If $-\infty < \gamma_l < \gamma < \infty$. Then*

$$\lim_{|z|\to\infty} |H(z,t)| = 0$$

uniformly in $\gamma_l \le \Re(z) \le \gamma$.

PROOF. Fix γ_l and γ. From Lemma 6.5, we obtain for $|\omega|$ sufficiently large

$$|\det \Delta(z)| \geq C_1 |z|^n,$$

where $\gamma_l \leq \Re(z) \leq \gamma$ and $|\Im(z)| \geq |\omega|$. From Theorem 4.7 and the representation for $\mathrm{adj}\Delta(z)$, we derive that on the horizontal lines:

$$|\Im(z)| = C \geq |\omega| \qquad \text{for} \quad \gamma_l \leq \Re(z) \leq \gamma,$$

we have

$$|\mathrm{adj}\Delta(z)| \leq C_2 |z|^{n-1}.$$

Hence, because of equation (6.18), we obtain

$$|H(z,t)| \leq e^{\gamma t} \frac{C_3}{|z|}. \tag{6.21}$$

∎

From equation (6.20) and the above lemma we obtain by taking the limit $N \to \infty$ in (6.20) that

$$x(t) = \sum_{j=1}^{l} p_j(t) e^{\lambda_j t} + \frac{1}{2\pi i} \int\limits_{L(\gamma_l)} H(z,t) dz. \tag{6.22}$$

So it remains to prove estimates for the remainder integral.

THEOREM 6.9. *For the remainder integral in the expansion (6.22) the following asymptotic estimate*

$$\left|\frac{1}{2\pi i} \int\limits_{L(\gamma_l)} H(z,t)\right| = o(e^{\gamma_l t}) \qquad \text{as} \quad t \to \infty \tag{6.23}$$

holds.

PROOF. Introduce the following notation

$$\overline{H}(z) = \frac{\mathrm{adj}\Delta(z)}{\det \Delta(z)} \big(f(0) + \int_0^h e^{-zt} df(t)\big). \tag{6.24}$$

Then

$$\frac{1}{2\pi i} \int\limits_{L(\gamma_l)} H(z,t) dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\omega} \overline{H}(\gamma_l + i\omega) d\omega.$$

So it suffices to prove that

$$\int_{-\infty}^{\infty} e^{it\omega}\overline{H}(\gamma_l + i\omega)d\omega \to 0 \qquad \text{as} \quad t \to \infty. \tag{6.25}$$

From the Riemann-Lebesgue Lemma 3.5, we derive that for every fixed N, the integral

$$\int_{-N}^{N} e^{it\omega}\overline{H}(\gamma_l + i\omega)d\omega \to 0 \qquad \text{as} \quad t \to \infty.$$

Hence to prove (6.25) it suffices to show that the integral

$$\int_{-\infty}^{\infty} e^{it\omega}\overline{H}(\gamma_l + i\omega)d\omega \tag{6.26}$$

converges uniformly in t with t larger than some fixed value T_0.

To show the uniform convergence of (6.24) for $t > T_0$, we use the representations for $\mathrm{adj}\Delta(z)$ and $\det\Delta(z)$ to derive that on the line $L(\gamma_l)$

$$\mathrm{adj}\Delta(z) = z^{n-1}I + O(z^{n-2})$$

and

$$\det\Delta(z) = z^n + O(z^{n-1})$$

as $|\Im(z)| \to \infty$. Hence on the line $L(\gamma_l)$

$$\overline{H}(z) = \frac{1}{z}\big(f(0) + \int_0^h e^{-zt}df(t)\,w\big) + O(z^{-2}).$$

Since the contribution of the last term to the integral converges absolutely it suffices to prove the uniform convergence for $t > T_0$ of

$$\int_N^{\infty} \frac{e^{it\omega}}{\gamma_l + i\omega}d\omega$$

and

$$\int_N^{\infty}\int_0^h \frac{e^{i(t-s)\omega}e^{-\gamma_l s}}{\gamma_l + i\omega}df(s)d\omega.$$

But this follows easily using partial integration. For example

$$\begin{aligned}
\Big|\int_N^{\infty}\int_0^h \frac{e^{i(t-s)\omega}e^{-\gamma_l s}}{\gamma_l + i\omega}df(s)d\omega\Big| &\le \Big|\int_0^h \frac{e^{i(t-s)\omega}e^{-\gamma_l s}}{i(t-s)}df(s)\frac{1}{\gamma_l + i\omega}\Big|_N^{\infty}\Big| \\
&\quad + \Big|\int_N^{\infty}\int_0^h \frac{e^{i(t-s)\omega}e^{-\gamma_l s}}{i(t-s)}df(s)\frac{d\omega}{(\gamma_l + i\omega)^2}\Big| \\
&\le \frac{C(\gamma_l)}{T_0 - h}\Big(\frac{1}{N - \gamma_l} + \int_N^{\infty}\frac{1}{|\gamma_l + i\omega|^2}d\omega\Big).
\end{aligned}$$

Consequently, the integral (6.26) converges uniformly in t on $[T_0, \infty)$ for $T_0 > 0$ and this completes the proof of the theorem. ∎

REMARK 6.10. Note that the difficulties in the above proof arise because the integrals involved do not converge absolutely.

COROLLARY 6.11. *Fix some* $\gamma \in \mathbf{R}$ *such that* $\det \Delta(z) \neq 0$ *on* $L(\gamma)$. *Then we have the following asymptotic expansion for the solution* x *of the renewal equation (6.1)*

$$x(t) = \sum_{\Re(\lambda_j)>\gamma} p_j(t)e^{\lambda_j t} + o(e^{\gamma t}) \qquad \text{as} \quad t \to \infty. \tag{6.27}$$

The question whether the remainder integral (6.23) converges to zero for all $t > 0$ as $\gamma \to -\infty$ - i.e. the solution x has a convergent series expansion - is far from trivial and will be studied in the next chapter.

REMARK 6.12. If we define the kernel ζ to be zero for negative t, then we can consider the equation (6.1) as a Wiener-Hopf equation with symbol

$$A(\zeta) = I - \int_0^\infty e^{-zt}\zeta(t)dt,$$

which is analytic for $\Re(z) > 0$. Now we can use the scaling arguments from the proof of Theorem 6.1 and apply the asymptotic expansions obtained by Fel'dman, see Corollary 2.1 of the Appendix in [15] to prove Corollary 6.11. Here we shall exploit the special form of ζ and define the characteristic matrix associated with (6.1) by $\Delta(z) = zA(\zeta)$. This defines an entire matrix valued function and makes it possible to analyse the remainder term in the asymptotic expansion (6.27) in more detail which we shall do in the next chapters.

Chapter 7. Fourier type series expansions of solutions

In chapter 6 we studied the large time behaviour of solutions of a Volterra convolution equation by deriving an asymptotic estimate for the remainder integral in equation (6.22). In the first part of this chapter we shall consider the behaviour of the remainder integral as function of t and γ_l and derive sufficient conditions such that the sum of the residues in equation (6.22) converges to the solution when the summation is extended over all singularities. In order to be able to apply the results of Chapter 5 we have to restrict the kernel ζ to $\mathrm{SBV}[0,h]$. In the second part of this chapter there will be no restriction on the kernel ζ and we shall give a complete characterization of the closure of the set of all forcing functions f such that the solution $x(\,\cdot\,;f)$ of the Volterra convolution equation is defined on the whole $\mathbf{R}$. The characterization of this set, which is closely related to the structure of the set of solutions of the Volterra convolution equation, has important applications (see Chapter 11).

So first consider the following class of Volterra convolution equations

$$x - \zeta * x = f, \tag{7.1}$$

where $f \in \mathcal{F}$, the supremum normed Banach space of continuous functions on $\mathbf{R}_+$ that are constant on $[h,\infty)$ and ζ is a matrix-valued element of $\mathrm{SBV}[0,h]$. Finally, choose the "delay" h as sharp as possible, i.e. such that at least one of the ζ_{ij}'s has a jump at h.

From Theorem 6.1, we derive that equation (7.1) has a unique solution $x(\,\cdot\,;f)$ as an element of $C_0(\mathbf{R}_+;\gamma)$, where γ is chosen sufficiently large. Since the solution $x(\,\cdot\,;f)$ of equation (7.1) is an element of $L^1(\mathbf{R}_+;\gamma)$, we can Laplace transform the equation to obtain for $\Re(z) > \gamma$

$$L\{x\}(z) = \Delta^{-1}(z)\Big(f(h) + z\int_0^h e^{-zt}(f(t) - f(h))dt\Big), \tag{7.2}$$

where $\Delta(z)$ denotes, as before, the characteristic matrix

$$\Delta(z) = zI - \int_0^h e^{-zt} d\zeta(t). \tag{7.3}$$

The expression at the right hand side of (7.2) yields an analytic continuation of $L\{x\}$ to the whole complex plane. We denote this analytic continuation by $H_f(z)$.

For general $f \in \mathcal{F}$, the solution $x(\,\cdot\,; f)$ is only for $t > h$ locally of bounded variation, so from the Laplace inversion Theorem 3.2 we only have an integral representation for the solution for $t > h$

$$x(t; f) = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-\omega i}^{\gamma+\omega i} e^{zt} H_f(z) dz. \tag{7.4}$$

Thus to obtain an integral representation for the solution $x(t; f)$ for $t > 0$, we have to restrict the class of forcing functions to $f \in \mathcal{F}$ such that f is locally of bounded variation. From the semigroup property (6.7)

$$x(h + \,\cdot\,; f) = x(\,\cdot\,; S(h)f).$$

Since $S(h)f$ is locally of bounded variation, the restricted class of forcing functions is large enough to cover the solutions $x(h + \,\cdot\,; f)$ with $f \in \mathcal{F}$. So in the sequel we assume that $f \in \mathcal{F}$ such that f is locally of bounded variation. Then $H_f(z)$ can be represented as follows

$$H_f(z) = \frac{P_f(z)}{\det \Delta(z)}, \tag{7.5}$$

where

$$P_f(z) = \mathrm{adj}\Delta(z)\big(f(0) + \int_0^h e^{-zt} df(t)\big). \tag{7.6}$$

Therefore, $P_f(z)$ is a vector-valued element of $\mathbf{P}_{n-1}$ and $\det \Delta(z)$ is an element of $z^n + \mathbf{J}_{n-1}$.

A series expansion of $x(\,\cdot\,; f)$ will now be established directly from the integral representation (7.4). To calculate the complex line integral over $L(\gamma)$, we construct the following sequence of closed contours $\Gamma_{\omega_l}(\gamma, \gamma_l)$ composed of four straight lines, which connects the points $\gamma - i\omega_l, \gamma + i\omega_l, \gamma_l + i\omega$, and $\gamma_l - i\omega_l$, where ω_l is chosen such that the zeros of $\det \Delta(z)$ are bounded away from the contours and such that

$$(\omega_l^2 + \gamma_l^2)^{\frac{1}{2}} = C_1 e^{-\gamma_l h}, \tag{7.7}$$

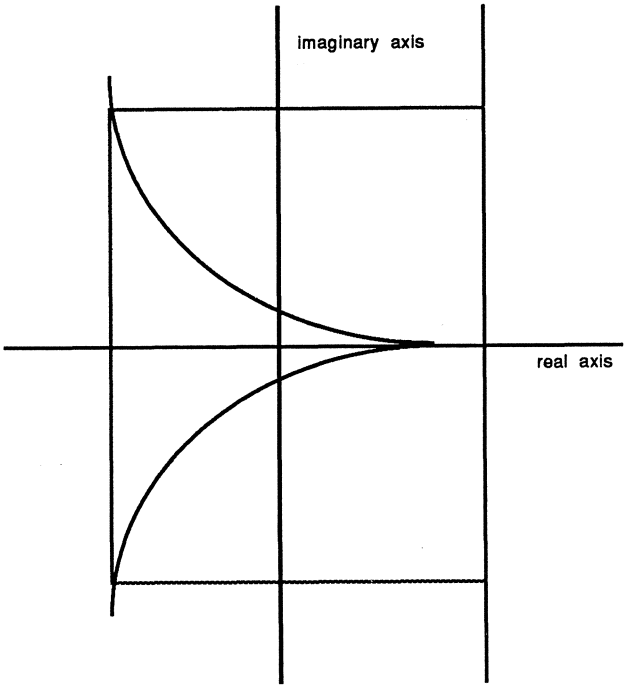

FIGURE 7.1. The contour $\Gamma_{\omega_l}(\gamma, \gamma_l)$.

where C_1 is "defined" by Lemma 6.5 and γ_l is chosen such that $\det \Delta(z) \neq 0$ on $L(\gamma_l)$.
Hence from Cauchy's residue theorem applied to the contour $\Gamma_{\omega_l}(\gamma, \gamma_l)$, we deduce

$$\frac{1}{2\pi i} \int_{\Gamma_{\omega_l}(\gamma,\gamma_l)} e^{zt} H_f(z) dz = \sum_{\Re(\lambda_j) > \gamma_l} p_j(t) e^{\lambda_j t}, \tag{7.8}$$

where the summation at the right hand side of (7.8) is finite.

The series obtained by taking the limit $\gamma_l \to -\infty$ at the right hand side of (7.8) is called the Fourier type series - or exponential series - expansion of the solution $x(\,\cdot\,; f)$.

In general one cannot expect that the Fourier type series expansion converges to the solution. For example, small solutions - i.e. solutions that tend to zero faster than any exponential - have an entire Laplace transform. Therefore from equation (7.8) small solutions have a series expansion in which all terms are zero. In the next chapter we shall characterize the set of all small solutions of the Volterra convolution equation (6.1) and present necessary and sufficient conditions for the existence of small solutions.

To prove the convergence of the Fourier type series expansion to the solution in $C_0(\mathbf{R}_+;\gamma)$, it suffices to prove that the absolute value of the line integrals

$$\int_{\gamma+i\omega_l}^{\gamma_l+i\omega_l}, \quad \int_{\gamma_l-i\omega_l}^{\gamma-i\omega_l}, \quad \text{and} \quad \int_{\gamma_l-i\omega_l}^{\gamma_l+i\omega_l} \tag{7.9}$$

of $e^{zt}H_f(z)$, tend to zero as $\gamma_l \longrightarrow -\infty$, uniform for $t \geq \epsilon$. The following lemma shows that along the vertical lines of the contour this is always the case.

LEMMA 7.1. *If $f \in \mathcal{F}$ is such that f is locally of bounded variation. Then for $z \in C_l$ we have*

$$\Big|\int_{\gamma+i\omega_l}^{\gamma_l+i\omega_l} e^{zt}H_f(z)dz\Big| = o(e^{\gamma(t-\epsilon)}) \qquad \textit{as} \quad l \longrightarrow \infty \tag{7.10}$$

and

$$\Big|\int_{\gamma_l-i\omega_l}^{\gamma-i\omega_l} e^{zt}H_f(z)dz\Big| = o(e^{\gamma(t-\epsilon)}) \qquad \textit{as} \quad l \longrightarrow \infty, \tag{7.11}$$

uniform for $t \geq \epsilon$.

PROOF. Recall the representation for H_f derived in (7.5). Since $P_f \in \mathbf{P}_{n-1}$ and $\det\Delta(z) \in z^n + \mathbf{J}_{n-1}$ and since the contour lies in the domain

$$|z| \geq C_1 e^{-h\Re(z)}.$$

An application of Lemma 6.5 yields

$$|H_f(z)| \leq \frac{C}{|z|}\Big|\int_0^h e^{-zt}df(t)\Big| + O(\frac{1}{|z|^2}) \qquad \text{for} \quad z \in C_l \tag{7.12}$$

as $l \longrightarrow \infty$ and where C denotes some constant. So we can estimate the integral (7.10)

$$\Big|\int_{\gamma+i\omega_l}^{\gamma_l+i\omega_l} e^{zt}H_f(z)dz\Big| \leq Ce^{\gamma t'}\int_{\gamma_l}^{\gamma} |\chi_l(x)|dx, \tag{7.13}$$

where $t = t' + \epsilon$ and

$$\chi_l(x) = \frac{e^{\epsilon x}}{(x^2+\omega_l^2)^{\frac{1}{2}}}\Big|\int_0^h e^{-i\omega_l t}e^{-xt}df(t)\Big|, \tag{7.14}$$

where $\gamma_l \leq x \leq \gamma$.

From the Riemann-Lebesgue Lemma 3.5

$$\chi_l(x) \to 0 \qquad \text{as} \quad l \to \infty$$

pointwise in x. Since

$$e^{-xt} \leq (x^2 + \omega_l^2)^{\frac{1}{2}},$$

we can estimate

$$|\chi_l(x)| \leq M e^{\epsilon x}$$

for some constant M and $\gamma_l \leq x \leq \gamma$. Therefore we can apply the Lebesgue dominated convergence theorem to conclude that

$$\int_{\gamma_l}^{\gamma} |\chi_l(x)| dx \to 0 \qquad \text{as} \quad l \to \infty. \tag{7.15}$$

So from (7.13) and (7.15), we find for $t \geq \epsilon$

$$\Big| \int_{\gamma + i\omega_l}^{\gamma_l + i\omega_l} e^{zt} H_f(z) dz \Big| = o(e^{\gamma(t-\epsilon)}) \qquad \text{as} \quad l \to \infty.$$

The proof that

$$\Big| \int_{\gamma_l - i\omega_l}^{\gamma - i\omega_l} e^{zt} H_f(z) dz \Big| = o(e^{\gamma(t-\epsilon)}) \qquad \text{as} \quad l \to \infty, \tag{7.16}$$

follows similar lines and the lemma is proved. ∎

COROLLARY 7.2. *If $f \in \mathcal{F}$ is such that f is locally of bounded variation. Then the Fourier type series expansion converges to the solution if and only if*

$$\lim_{l \to \infty} \Big| \int_{\gamma_l - i\omega_l}^{\gamma_l + i\omega_l} e^{zt} H_f(z) dz \Big| = 0. \tag{7.17}$$

As an application of Corollary 5.7 we shall first state sufficient conditions on the forcing function f so that the Fourier type series expansion converges to the solution $x(\,\cdot\,; f)$.

THEOREM 7.3. *Suppose $f \in \mathcal{F}$ such that f is locally of bounded variation and such that the Newton polygon inequality*

$$\mathrm{N}(z \operatorname{adj} \Delta(z)(f(0) + \int_0^h e^{-zt} df(t))) \leq \mathrm{N}(\det \Delta(z)) \tag{7.18}$$

holds componentwise. Then for all $\epsilon > 0$ the solution $x(\epsilon + \cdot\,; f)$ of equation (7.1) is represented by a $C_0(\mathbf{R}_+; \gamma)$-convergent Fourier type series.

PROOF. On account of Corollory 7.2 it remains to analyse

$$\Big|\int_{\gamma_l - i\omega_l}^{\gamma_l + i\omega_l} e^{zt} H_f(z)dz\Big|. \tag{7.19}$$

An application of Corollary 5.7 yields

$$|H_f(z)| = O(\frac{1}{|z|}) \tag{7.20}$$

for $z \in L(\gamma_l)$ and l sufficiently large. Therefore,

$$\Big|\int_{\gamma_l - i\omega_l}^{\gamma_l + i\omega_l} e^{zt} H_f(z)dz\Big| \leq C \log\big(\omega_l + (\omega_l^2 + \gamma_l^2)^{\frac{1}{2}}\big) e^{\gamma_l \epsilon} e^{\gamma_l (t-\epsilon)} \tag{7.21}$$

and since ω_l satisfies equation (7.7) the right hand side tends to zero as l tends to infinity, uniform for $t \geq \epsilon$. ∎

In contrast with the additional z in the Newton polygon condition (7.18) we can also shift in time to prove (7.17).

THEOREM 7.4. *Suppose $f \in \mathcal{F}$ such that f is locally of bounded variation and such that the Newton polygon inequality*

$$\mathrm{N}\big(\mathrm{adj}\Delta(z)(f(0) + \int_0^h e^{-zt} df(t))\big) \leq \mathrm{N}\big(\det \Delta(z)\big) \tag{7.22}$$

holds componentwise. Then for all $\epsilon > 0$ the solution $x(h + \epsilon + \cdot\,; f)$ of equation (7.1) is represented by a $C_0(\mathbf{R}_+; \gamma)$-convergent Fourier type series.

PROOF. In this case, an application of Corollary 5.7 yields

$$|H_f(z)| = O(1) \tag{7.23}$$

for $z \in L(\gamma_l)$ and l sufficiently large. Therefore,

$$\Big|\int_{\gamma_l - i\omega_l}^{\gamma_l + i\omega_l} e^{zt} H_f(z)dz\Big| \leq C\omega_l e^{\gamma_l h} e^{\gamma_l \epsilon} e^{\gamma_l (t-h-\epsilon)}, \tag{7.24}$$

and since ω_l satisfies equation (7.7) the right hand side tends to zero as l tends to infinity, uniform for $t \geq h + \epsilon$. ∎

We shall see shortly that from the theoretical point of view the importance of the following theorem lies in the fact that the Fourier type series does converge to the solution after finite and in fact uniformly bounded time.

THEOREM 7.5. *If f be an element of $\mathcal{F}$. Then for all $\epsilon > 0$ the solution $x(nh+\epsilon+\cdot\,;f)$ is represented by a $C_0(\mathbf{R}_+;\gamma)$-convergent Fourier type series.*

PROOF. Recall from (6.15)

$$(\mathrm{adj}\Delta(z))_{ij} = \delta_{ij}z^{n-1} + \sum_{k=1}^{n-1}\int_0^{kh} e^{-zt}d\eta_{ijk}(t)z^{n-k-1}. \tag{7.25}$$

Since for $z \in L(\gamma_l)$ and l sufficiently large we always have that

$$|\det\Delta(z)| > C_1|z|^n,$$

where C_1 is "defined" by Lemma 6.5, we derive the following estimate for $|H_f(z)e^{nh\gamma_l}|$

$$|H_f(z)e^{nh\gamma_l}| \le O(\frac{1}{|\gamma_l+\omega|}) \qquad \text{as} \quad |\omega| \to \infty. \tag{7.26}$$

Therefore,

$$|\int_{\gamma_l-i\omega_l}^{\gamma_l+i\omega_l} e^{zt}H_f(z)dz| \le C\log(\omega_l + (\omega_l^2+\gamma_l^2)^{\frac{1}{2}})e^{\gamma_l\epsilon}e^{\gamma_l(t-nh-\epsilon)}$$

and since ω_l satisfies equation (7.7) the right hand side tends to zero as l tends to infinity, uniform for $t \ge nh+\epsilon$. ∎

The conditions in the above theorems are sufficient conditions for convergence. In concrete applications one might want to use variants of these conditions. For example, remark that if

$$\mathrm{N}(z\mathrm{adj}\Delta(z)) \le \mathrm{N}(\det\Delta(z)), \tag{7.27}$$

then

$$|\frac{\mathrm{adj}\Delta(z)}{\det\Delta(z)}| = O(|z|^{-1})$$

for $z \in L(\gamma_l)$ and l sufficiently large. Therefore, for every $f \in \mathcal{F}$ such that f is locally of bounded variation the solution $x(h+\epsilon+\cdot\,;f)$ is represented by a $C_0(\mathbf{R}_+;\gamma)$-convergent Fourier type series for every $\epsilon > 0$. Furthermore, to prove sharp results in concrete examples we have to apply the techniques rather than the theorems developed in this section. See the examples in Chapter 13.

In the results so far we controlled the behaviour of $H_f(z)$ by imposing conditions on the Newton polygon or by shifting along the solution. Next, we will study what occurs with the Fourier type series expansion if we impose additional smoothness conditions on the forcing function f. We start with a definition.

DEFINITION 7.6. Let $\mathcal{E}$ denote the subspace of all $f \in F$ such that the following exponential type condition holds

$$\mathcal{E} = \left\{ f \in \mathcal{F} : \mathrm{E}\big(\mathrm{adj}\Delta(z)(f(h) + z \int_0^h e^{-zt}(f(t) - f(h))dt\big) \leq \mathrm{E}\big(\det\Delta(z)\,\big)\right\}.$$

Let B denote the infinitesimal generator of the $\mathcal{C}_0$-semigroup $\{S(t)\}$ associated with the Volterra convolution equation (7.1). Recall from Theorem 6.3

$$Bf = \dot{f} + \zeta(\,\cdot\,)f(0) \tag{7.28}$$

for $f \in \mathcal{D}(B)$, where

$$\mathcal{D}(B) = \left\{ f \in F : \dot{f} + \zeta(\,\cdot\,)f(0) \in \mathcal{F} \right\}.$$

Note that for the solution $x(\,\cdot\,; f)$ of (7.1) we have

$$\dot{x} \in C_0\big(\mathbf{R}_+;\gamma\big) \quad \text{if and only if} \quad f \in \mathcal{D}(B).$$

The next lemma will be the key in the following results.

LEMMA 7.7. *The linear subspace $\mathcal{E}$ is invariant under the resolvent operator*

$$R(\lambda, B) = (\lambda I - B)^{-1} \tag{7.29}$$

with $\lambda \in \rho(B)$.

PROOF. Choose $f \in \mathcal{F}$ and suppose that $g = R(\lambda, B)f$. Then

$$f = \lambda g - Bg.$$

Since

$$Bg = \dot{g} + \zeta(\,\cdot\,)g(0),$$

we obtain

$$\begin{aligned}
\mathrm{adj}\Delta(z)z\int_0^\infty e^{-zt}(Bg)(t)dt &= \mathrm{adj}\Delta(z)z\int_0^h e^{-zt}\dot{g}(t)dt \\
&\quad + \mathrm{adj}\Delta(z)zg(0) - \det\Delta(z)g(0) \\
&= \mathrm{adj}\Delta(z)z\big(e^{-zh}g(h) + z\int_0^h e^{-zt}g(t)dt\big) \\
&\quad - \det\Delta(z)g(0).
\end{aligned}$$

Hence

$$\mathrm{adj}\Delta(z)z\int_0^\infty e^{-zt}(Bg)(t)dt = \mathrm{adj}\Delta(z)z^2\int_0^\infty e^{-zt}g(t)dt - \det\Delta(z)g(0).$$

Thus

$$(\lambda - z)\operatorname{adj}\Delta(z)z\int_0^\infty e^{-zt}g(t)dt = \operatorname{adj}\Delta(z)z\int_0^\infty e^{-zt}f(t)dt \\ - \det\Delta(z)g(0).$$

Consequently,

$$\mathrm{E}\big(\operatorname{adj}\Delta(z)z\int_0^\infty e^{-zt}g(t)dt\big) \le \mathrm{E}\big(\operatorname{adj}\Delta(z)z\int_0^\infty e^{-zt}f(t)dt\big) \\ + \mathrm{E}\big(\det\Delta(z)\big).$$

Thus we have proved

$$f \in \mathcal{E} \text{ if and only if } g = R(\lambda, B)f \in \mathcal{E}.$$

∎

We then have the following theorem.

THEOREM 7.8. *Suppose $f \in \mathcal{D}(B^n) \cap \mathcal{E}$ such that $B^n f$ is locally of bounded variation. Then for every $\epsilon > 0$ the solution x of (7.1) has a Fourier type expansion*

$$x(\epsilon + \cdot\,; f) = \sum_{j=1}^{\infty} p_j(t)e^{\lambda_j t} \tag{7.30}$$

in $C_0(\mathbf{R}_+;\gamma)$.

PROOF. From Lemma 7.7 and repeated integration by parts we obtain

$$\operatorname{adj}\Delta(z)z\int_0^\infty e^{-zt}f(t)dt = \frac{1}{z^n}\big[\operatorname{adj}\Delta(z)(B^n f(0) + \int_0^h e^{-zt}dB^n f(t)) \\ + \det\Delta(z)p(z)\big],$$

where p is a polynomial of degree $n-1$. So define

$$G(z) = \operatorname{adj}\Delta(z)(B^n f(0) + \int_0^h e^{-zt}dB^n f(t)) + \det\Delta(z)p(z)$$

and

$$F(z) = z^n \det\Delta(z).$$

Since

$$\mathrm{N}(zG(z)) \le \mathrm{N}(F(z)), \tag{7.31}$$

we can apply the proof of Theorem 7.3 to arrive at the desired result. ∎

Define

$$\alpha = \max_{f\in\mathcal{F}} \{ \mathrm{E}(\mathrm{adj}\Delta(z)z\int_0^\infty e^{-zt}f(t)dt\ ) - \mathrm{E}(\det\Delta(z))\ \}. \tag{7.32}$$

In the next chapter we shall prove that $S(\alpha)f = 0$ if and only if $x(\,\cdot\,;f)$ is a small solution.

The next corollary shows that if f is sufficiently smooth then the series expansion converges to the solution as soon as all small solutions have disappeared. Compare this result with Theorem 7.5 where we had to shift over nh to compensate the lack of smoothness.

COROLLARY 7.9. *Suppose $f \in \mathcal{D}(B^n)$ such that $B^n f$ is locally of bounded variation. Then*

$$x(\alpha+\epsilon+\,\cdot\,;f) = \sum_{j=1}^{\infty} p_j(t)e^{\lambda_j t} \tag{7.33}$$

in $C_0(\mathbf{R}_+;\gamma)$.

PROOF. As in the proof as Theorem 7.8 define

$$G(z) = e^{\alpha z}\left(\mathrm{adj}\Delta(z)z(B^n f(0) + \int_0^h e^{-zt}dB^n f(t)) + \det\Delta(z)p(z)\right)$$

and

$$F(z) = z^n \det\Delta(z).$$

So we have $\mathrm{N}(zG(z)) \le N(F(z))$ and we can apply the proof of Theorem 7.3. ∎

When $\mathrm{E}(\det\Delta(z))$ is maximal - i.e. $\mathrm{E}(\det\Delta(z)) = nh$ - we can combine the above theorems to obtain the following result. Bank and Manitius [2] have studied this case when the kernel ζ is given by a finite sum of jumps.

COROLLARY 7.10. *If $\mathrm{E}(\det\Delta(z)) = nh$ and if $f \in \mathcal{D}(B)$ such that Bf is locally of bounded variation. Then*

$$x(\epsilon+\,\cdot\,;f) = \sum_{j=1}^{\infty} p_j(t)e^{\lambda_j t} \tag{7.34}$$

in $C_0(\mathbf{R}_+;\gamma)$.

PROOF. Since $\mathrm{E}(\det\Delta(z)) = nh$, the points $(0,n)$ and $(nh,0)$ lie on the Newton polygon of $\det\Delta(z)$. Hence

$$\mathrm{N}(e^{-zh}\mathrm{adj}\Delta(z)) \le \mathrm{N}(\det\Delta(z)).$$

From Theorem 6.3 it follows that for $f \in \mathcal{D}(B)$

$$z\mathrm{adj}\Delta(z)z\int_0^\infty e^{-zt}f(t)dt = \mathrm{adj}\Delta(z)z\int_0^\infty e^{-zt}(Bf)(t)dt + \det\Delta(z)f(0).$$

Since Bf is locally of bounded variation and

$$\mathrm{E}(\int_0^\infty e^{-zt}B(t)dtg) \leq h,$$

the condition (7.18) is satisfied. ∎

In the remaining part of this chapter, we turn to the complete characterization of the closure of the set of all forcing functions such that the solution $x(\cdot\,;f)$ is defined on the whole real line. There will be no restrictions on the kernel anymore. We assume that ζ is a matrix-valued element of NBV$[0,h]$ and consider the Volterra convolution equation

$$x - \zeta * x = f, \tag{7.35}$$

where $f \in \mathcal{F}$.

DEFINTION 7.11. Let $\mathcal{A}_{\mathcal{F}}$ denote the subspace of all forcing functions $f \in \mathcal{F}$ such that the solution $x(\cdot\,;f)$ is defined on the whole real line, i.e. for every $\sigma > 0$ there exists a forcing function $g \in \mathcal{F}$ such that $S(\sigma)g = f$.

The jump condition on the kernel ζ when $\zeta \in$ SBV$[0,h]$ implies that

$$\left|e^{\alpha z}H_f(z)\right| = O(|z|^n). \tag{7.36}$$

Thus, we see that Theorem 7.6 covers the worst case that actually can happen when $\zeta \in$ SBV$[0,h]$. For general $\zeta \in$ NBV$[0,h]$ it is not clear whether there will be a finite power N such that

$$\left|e^{\alpha z}H_f(z)\right| = O(|z|^N). \tag{7.37}$$

Therefore it is not clear whether the Fourier type series expansion converges to the solution after finite time. The only result we have in this direction is Theorem 5.12 and this result turns out to be sufficient for the characterization of $\overline{\mathcal{A}}_{\mathcal{F}}$.

We will prove the following characterization of the closure of $\mathcal{A}_{\mathcal{F}}$ in $\mathcal{F}$.

THEOREM 7.12. *The closure of $\mathcal{A}_{\mathcal{F}}$ in $\mathcal{F}$ equals*

$$\overline{\mathcal{A}}_{\mathcal{F}} = \mathcal{E}.$$

We divide the proof of Theorem 7.12 into two theorems. First we prove the following inclusion.

THEOREM 7.13. $\mathcal{E} \subset \overline{\mathcal{A}}_{\mathcal{F}}$.

PROOF. Let f be an element of $\mathcal{E}$. Recall from Lemma 7.7 that $\mathcal{E}$ is invariant under the resolvent $R(\lambda, B)$. Since $\mathcal{D}(B^\infty)$ is dense in $\mathcal{F}$, we can choose a sequence f_j such that

$$f_j \in \mathcal{E} \cap \mathcal{D}(B^\infty)$$

and

$$f_j \to f \qquad \text{as} \quad j \to \infty.$$

So Theorem 6.1 implies that

$$x(\,\cdot\,; f_j) \to x(\,\cdot\,; f) \qquad \text{as} \quad j \to \infty$$

in $C_0(\mathbf{R}_+; \gamma)$.

From Lemma 7.7 and repeated integration by parts we see that for every k we can represent H_{f_j} as

$$H_{f_j}(z) = \frac{P_{B^k f_j}(z)}{z^k \det \Delta(z)} + \frac{p_k(z)}{z^k}, \tag{7.38}$$

where p_k is a polynomial of degree $k-1$. Furthermore, since $f_j \in \mathcal{E}$,

$$\mathrm{E}(e^{\epsilon z} P_{B^k f_j}(z)) < \mathrm{E}(\det \Delta(z)).$$

Therefore, on account of Theorem 5.13 we can, for every l, find a constant $m = m(l, \det \Delta(z))$ such that for every k

$$\left|\frac{e^{\epsilon z} P_{B^k f_j}(z)}{\det \Delta(z)}\right| \leq C|z|^m \qquad \text{for} \quad z \in L(\gamma_l). \tag{7.39}$$

For every l we can choose $k = m+1$ such that

$$|e^{\epsilon z} H_{f_j}| \leq \frac{C}{|z|} \tag{7.40}$$

for $z \in L(\gamma_l)$. On account of Corollary 7.2 it suffices to analyse

$$\left|\int_{\gamma_l - i\omega_l}^{\gamma_l + i\omega_l} e^{zt} H_{f_j}(z) dz\right|. \tag{7.41}$$

Because of (7.40) we can estimate

$$\left|\int_{\gamma_l - i\omega_l}^{\gamma_l + i\omega_l} e^{zt} H_{f_j}(z) dz\right| \leq C \log\left(\omega_l + (\omega_l^2 + \gamma_l^2)^{\frac{1}{2}}\right) e^{\gamma_l \epsilon} e^{\gamma_l (t-\epsilon)}.$$

Since ω_l satisfies equation (7.7) the right hand side tends to zero as l tends to infinity, uniform for $t > \epsilon$. Therefore, for every f_j and $\epsilon > 0$ we have the following Fourier type series expansion

$$x(\epsilon + \cdot\,; f_j) = \sum_{k=1}^{\infty} p_k^{(j)} e^{\lambda_k t}$$

in $C_0(\mathbf{R}_+;\gamma)$. Let $f_{j,\epsilon}$ denote the forcing function corresponding to the solution $x(\epsilon + \cdot\,; f_j)$, i.e.

$$f_{j,\epsilon} = S(\epsilon) f_j.$$

Recall Theorem 6.3 for the definition of the $\mathcal{C}_0$-semigroup $\{S(t)\}$ and conclude from the $\mathcal{C}_0$-semigroup property that

$$\|S(\epsilon)f_j - f_j\|_u \to 0 \qquad \text{as} \quad \epsilon \downarrow 0. \tag{7.42}$$

Therefore we can construct a subsequence

$$\{\tilde{f}_j\} \text{ of } \{S(\epsilon)f_j\},$$

such that $x(\cdot\,;\tilde{f}_j)$ has a $C_0(\mathbf{R}_+;\gamma)$-convergent Fourier type series and $\tilde{f}_j$ converges to f in $\mathcal{F}$ as $j \to \infty$. Since finite Fourier type series expansions are well defined on the whole real axis, we clearly have that $\tilde{f}_j \in \overline{\mathcal{A}_{\mathcal{F}}}$ and so $f \in \overline{\mathcal{A}_{\mathcal{F}}}$. ∎

THEOREM 7.14. *$\mathcal{E}$ is a closed subspace of $\mathcal{F}$.*

PROOF. Let $\{f_k\} \subset \mathcal{E}$ be a sequence of uniformly bounded forcing functions such that $f_k \to f$ in $\mathcal{F}$. We are going to prove that f belongs to $\mathcal{E}$ as well. Since f and all f_k are constant on the interval $[h, \infty)$, the convergence of f_k to f in $\mathcal{F}$ implies that uniformly on compact sets

$$z\int_0^{\infty} e^{-zt} f_k(t)dt \to z\int_0^{\infty} e^{-zt} f(t)dt \qquad \text{as} \quad k \to \infty. \tag{7.43}$$

Let τ denote $\mathrm{E}(\det \Delta(z))$. Assume that f does not belong to $\mathcal{E}$. Then

$$\mathrm{E}\big(e^{\tau z}\operatorname{adj}\Delta(z) z\int_0^{\infty} e^{-zt} f(t)dt\big) = \sigma > 0. \tag{7.44}$$

Since the forcing functions f_k are uniformly bounded and belong to $\mathcal{E}$, an application of the exponential type calculus yields the existence of a constant M such that for all k

$$\big|e^{(\tau+\frac{\sigma}{2})z}\operatorname{adj}\Delta(z) z\int_0^{\infty} e^{-zt} f_k(t)dt\big| \le M e^{\frac{\sigma}{2}\Re(z)}|z|^n \tag{7.45}$$

in the left half plane $\Re(z) < 0$. And from (7.43) and (7.45) we obtain for every $R > 0$ and for all $r \leq R$ the following estimate on the negative real axis

$$\left|e^{(\tau+\frac{\sigma}{2})z}\operatorname{adj}\Delta(z)z\int_0^\infty e^{-zt}f(t)dt\right| \leq Me^{-r\frac{\sigma}{2}}|r|^n. \tag{7.46}$$

On the other hand it follows from the Ahlfors-Heins Theorem 4.10 and equation (7.44) that for every $\epsilon > 0$ there exists a sequence r_k such that $r_k \to \infty$ and on the negative real axis

$$e^{(\frac{\sigma}{2}-\epsilon)r_k} \leq \left|e^{(\tau+\frac{\sigma}{2})z}\operatorname{adj}\Delta(z)z\int_0^\infty e^{-zt}f(t)dt\right|, \tag{7.47}$$

where $z = -r_k$. Now choose $\epsilon = \frac{\sigma}{4}$. A combination the equations (7.46) and (7.47) yields

$$e^{\frac{\sigma}{4}r_k} \leq Me^{-\frac{\sigma}{2}r_k}|r_k|^n,$$

which yields a contradiction for r_k sufficiently large. Hence $\sigma = 0$, and f belongs to $\mathcal{E}$. ∎

PROOF OF THEOREM 7.12. Only the inclusion $\overline{\mathcal{A}_\mathcal{F}} \subset \mathcal{E}$, remains to be proved. From the semigroup property

$$x(\,\cdot\,; S(\alpha)f) = x(\alpha + \,\cdot\,; f) \quad \text{for every} \quad f \in \mathcal{F}.$$

Hence,

$$H_{S(\alpha)f}(z) = e^{\alpha z}H_f(z) + \int_{-\alpha}^0 e^{-zt}x(t+\alpha; f)dt, \tag{7.48}$$

where H_f is defined by (7.5). From the definition of α, see equation (7.32), it follows that for every $f \in \mathcal{F}$

$$\mathrm{E}(e^{\alpha z}P_f(z)) \leq \mathrm{E}(\det\Delta(z)).$$

So, because of (7.48),

$$\mathrm{E}(P_{S(\alpha)f}(z)) \leq \mathrm{E}(\det\Delta(z)).$$

Therefore, we arrive at the following inclusion

$$\mathcal{R}\big(S(\alpha)\big) \subset \mathcal{E}. \tag{7.49}$$

By definition of $\mathcal{A}_\mathcal{F}$

$$\mathcal{A}_\mathcal{F} \subset \mathcal{R}\big(S(t)\big) \text{ for every } t \geq 0. \tag{7.50}$$

Consequently, we have the following sequence of inclusions

$$\mathcal{A}_\mathcal{F} \subset \mathcal{R}\big(S(\alpha)\big) \subset \mathcal{E}.$$

But $\mathcal{E}$ is closed and therefore we have proved the inclusion $\overline{\mathcal{A}_\mathcal{F}} \subset \mathcal{E}$. ∎

To conclude this chapter we present a very important corollary of the characterization obtained above. We say that $f \in \mathcal{F}$ has a backward continuation over σ if there exists a $g \in \mathcal{F}$ such that

$$x(\sigma + \cdot\,; g) = x(\cdot\,; f). \tag{7.51}$$

A backward continuation does not always exist. Moreover, is not unique (recall the existence of small solutions). But based on the characterization of the small solutions, see Chapter 8, we can prove the following result: If $f \in \overline{\mathcal{A}}_{\mathcal{F}}$ such that f has a (finite) backward continuation. Then this backward continuation is unique. This result is a direct consequence of Theorem 8.2 and the following corollary.

COROLLARY 7.15. *If f is a non-zero element of $\overline{\mathcal{A}}_{\mathcal{F}}$. Then the solution $x(\cdot\,; f)$ of equation (7.35) cannot tend to zero faster than every exponential - i.e. cannot be a small solution.*

PROOF. Assume that the solution $x(\cdot\,; f)$ tends to zero faster than every exponential. Then $L\{x\}$ is an entire function. Because of equation (7.2) and Theorem 7.12, we have that $L\{x\}$ is an entire function of zero exponential type. It also follows from equation (7.2) and the Paley-Wiener Theorem 4.4 that $L\{x\}$ is L^2-integrable along the imaginary axis. Hence, another application of the Paley-Wiener Theorem 4.4 shows that $x(\cdot\,; f)$ is identically zero, which is a contradiction to the fact that f is a non-zero element of $\overline{\mathcal{A}}_{\mathcal{F}}$. ∎

EXAMPLE 7.16. Consider the Volterra convolution equation with characteristic matrix

$$\Delta(z) = \begin{pmatrix} z & 1 & -e^{-z} \\ -e^{-z} & z & 0 \\ 0 & 0 & z \end{pmatrix}. \tag{7.52}$$

Then the inverse of the characteristic matrix becomes

$$\Delta^{-1}(z) = \frac{1}{z(z^2 + e^{-z})} \begin{pmatrix} z^2 & -z & ze^{-z} \\ ze^{-z} & z^2 & e^{-2z} \\ 0 & 0 & z^2 + e^{-z} \end{pmatrix}. \tag{7.53}$$

Now we can use Theorem 7.12 to describe the set $\overline{\mathcal{A}}_{\mathcal{F}}$. From Definition 7.6 it follows that we have to solve the system of equations

$$\begin{aligned} \mathrm{E}\big(z^2(zL\{f_1\}(z)) - z(zL\{f_2\}(z)) + ze^{-z}(zL\{f_3\}(z))\big) &\le 1 \\ \mathrm{E}\big(ze^{-z}(zL\{f_1\}(z)) + z^2(zL\{f_2\}(z)) + e^{-2z}(zL\{f_3\}(z))\big) &\le 1 \\ \mathrm{E}\big((z^2 + e^{-z})(zL\{f_3\}(z))\big) &\le 1, \end{aligned} \tag{7.54}$$

where $\mathrm{E}(\det \Delta(z)) = 1$ and for $1 \le j \le 3$

$$zL\{f_j\}(z) = f_j(1) + z\int_0^1 e^{-zt}(f_j(t) - f_j(1))dt.$$

The first and third equation of (7.54) show that f_3 is constant. The second equation of (7.54) yields

$$\overline{\mathcal{A}}_{\mathcal{F}} = \big\{f \in \mathcal{F} : f_1(t) = c_1 t + c_2 \text{ and } f_3(t) = c_1, \text{ where } c_1, c_2 \in \mathbf{R}\big\}.$$

Chapter 8. Small solutions

In this chapter we shall give a characterization of the smallest possible time t_0 such that all small solutions vanish for $t \geq t_0$. This characterization of t_0 is needed in order to establish the results concerning completeness of the system of generalized eigenfunctions which we will present in Chapter 10.

Consider the Volterra convolution equation of Chapter 6.

$$x - \zeta * x = f, \tag{8.1}$$

where ζ is a matrix-valued element of $\mathrm{NBV}[0, h]$ and f is an element of $\mathcal{F}$.

Let $\Delta(z)$ denote the characteristic matrix function

$$\Delta(z) = zI - \int_0^h e^{-zt} d\zeta(t). \tag{8.2}$$

The function $\det \Delta(z)$ is an entire function of exponential type less than or equal to nh. Define ϵ by

$$\mathrm{E}(\det \Delta(z)) = nh - \epsilon. \tag{8.3}$$

Let $\mathrm{adj}\Delta(z)$ denote the matrix of cofactors of $\Delta(z)$. Since the cofactors $C_{ij}(z)$ are $(n-1) \times (n-1)$-subdeterminants of $\Delta(z)$, it follows that the exponential type of the cofactors is less than or equal to $(n-1)h$. Define σ by

$$\max_{1 \leq i,j \leq n} \mathrm{E}(C_{ij}) = (n-1)h - \sigma. \tag{8.4}$$

Definition 8.1. A small solution x of (8.1) is a solution x such that

$$\lim_{t \to \infty} e^{kt} x(t) = 0 \tag{8.5}$$

for all $k \in \mathbf{R}$.

We can now state and prove a sharp version of Henry's theorem on small solutions [20] for the Volterra convolution equation (8.1).

THEOREM 8.2. *All small solutions of (8.1) vanish for $t \geq \epsilon - \sigma$ and $\epsilon - \sigma$ is the smallest possible time with this property.*

As an application of this result we prove

THEOREM 8.3. *There are no small solutions if and only if* $\mathrm{E}(\det \Delta(z))$ *equals* nh.

PROOF OF THEOREM 8.2. Let x be a small solution, then $L\{x\}$ is an entire function which satisfies the equation

$$\det \Delta(z) L\{x\}(z) = \mathrm{adj}\Delta(z)\big(f(h) + z \int_0^h e^{-zt}(f(t) - f(h))dt\big). \tag{8.6}$$

Since the quotient of two functions of exponential type is again of exponential type provided it is entire, the Laplace transform $L\{x\}$ is of exponential type. From the exponential type calculus derived in Chapter 4, in particular Corollary 4.11, the right hand side of (8.6) has exponential type less than or equal to $nh - \sigma$. So from Corollary 4.11, the Laplace transform $L\{x\}$ has finite exponential type η with

$$\eta \leq nh - \sigma - (nh - \epsilon) = \epsilon - \sigma. \tag{8.7}$$

From equation (8.6) and the Paley-Wiener Theorem 4.4 it follows that $L\{x\}$ is L^2-integrable along the imaginary axis. Hence, another application of the Paley-Wiener Theorem 4.4 yields

$$\int_0^\infty e^{-zt}x(t)dt = \int_0^\eta e^{-zt}x(t)dt \tag{8.8}$$

and $x(t) = 0$ for all $t \geq \epsilon - \sigma$.

In the following we shall call vector-valued functions of the form

$$\int_0^\omega e^{-zt}\chi(t)dt, \tag{8.9}$$

where $\omega \in \mathbf{R}_+$ and $\chi \in L^2[0, \omega]$, Paley-Wiener functions. To prove the claim that $\epsilon - \sigma$ is the smallest possible time with the property that all small solutions vanish for $t \geq \epsilon - \sigma$ we are going to construct a small solution x such that $x \not\equiv 0$ in any neighbourhood of $\epsilon - \sigma$. Laplace transformation of the equation shows that it suffices to construct a Paley-Wiener function F of exponential type $\epsilon - \sigma$ such that

$$\Delta(z)F(z) = c + q(z), \tag{8.10}$$

where $c \in \mathbf{R}^n$ and q is a Paley-Wiener function of exponential type less than or equal to h.

Choose a column of the matrix function $\operatorname{adj}\Delta(z)$ such that one of the elements of this column is the cofactor of maximal exponential type given by $(n-1)h - \sigma$. Since the arguments given below can be repeated for all other columns we may assume that we can choose the first column

$$\begin{pmatrix} C_{11}(z) \\ \vdots \\ C_{n1}(z) \end{pmatrix} \tag{8.11}$$

of $\operatorname{adj}\Delta(z)$. Then

$$\Delta(z) \begin{pmatrix} C_{11}(z) \\ \vdots \\ C_{n1}(z) \end{pmatrix} = \begin{pmatrix} \det \Delta(z) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{8.12}$$

We have to consider two cases:

I $\epsilon \leq (n-1)h$;

II $(n-1)h < \epsilon \leq nh$.

CASE I. Suppose $\epsilon \leq (n-1)h$. For $1 \leq j \leq n$ define the function c_j to be the Taylor expansion of C_{j1} of order $n-1$ in 0, then the function F_j defined by

$$F_j(z) = \frac{C_{j1}(z) - c_j(z)}{z^n}, \tag{8.13}$$

is entire. Define

$$\Delta(z) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}.$$

Then for $1 \leq j \leq n$ the functions d_j are polynomials of degree n with coefficients being constants plus Paley-Wiener functions of exponential type less than or equal to h. Furthermore,

$$\Delta(z) \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \frac{1}{z^n} \begin{pmatrix} \det \Delta(z) - d_1 \\ -d_2 \\ \vdots \\ -d_n \end{pmatrix}. \tag{8.14}$$

Since $\det \Delta(z)$ is a polynomial of degree n with coefficients being constants plus Paley-Wiener functions we have by the Paley-Wiener Theorem 4.4 that the right hand side of (8.14) can be written as follows

$$c + \int_0^{nh-\epsilon} e^{-zt} h(t)dt,$$

where $c \in \mathbf{R}^n$ and $h \in L^2\big([0, nh-\epsilon]; \mathbf{R}^n\big)$. Furthermore, the cofactors are polynomials of degree $n-1$ with coefficients being constants plus Paley-Wiener functions. Hence, the entire function F is a Paley-Wiener function and by the Paley-Wiener Theorem 4.4 we have

$$F(z) = \int_0^{(n-1)h-\sigma} e^{-zt} \chi(t)dt,$$

where $\chi \in L^2\big([0,(n-1)h-\sigma]; \mathbf{R}^n\big)$. Therefore, equation (8.14) can be rewritten as follows

$$\Delta(z) \int_0^{(n-1)h-\sigma} e^{-zt}\chi(t)dt = c + \int_0^{nh-\epsilon} e^{-zt} h(t)dt. \tag{8.15}$$

So, the function χ satisfies the equation

$$x - \zeta * x = q,$$

where $\dot{q} = h$ and q is constant on $[nh-\epsilon, \infty)$. From the remarks made directly after Theorem 6.1 we obtain,

$$\dot{\chi} \in L^2[0,(n-1)h-\sigma].$$

We rewrite equation (8.15) as follows

$$\begin{aligned}
\Delta(z) &\int_0^{(n-1)h-\sigma} e^{-zt}\chi(t)dt \\
&= e^{-((n-1)h-\epsilon)z}\Delta(z) \int_0^{\epsilon-\sigma} e^{-zt}\chi((n-1)h-\epsilon+t)dt \qquad (8.16)\\
&= c + \int_0^{nh-\epsilon} e^{-zt}h(t)dt - \Delta(z)\int_0^{(n-1)h-\epsilon} e^{-zt}\chi(t)dt.
\end{aligned}$$

Since the right hand side of (8.16) has exponential type less than or equal to $nh-\epsilon$ we have by Corollary 4.11 that

$$\int_0^{\epsilon-\sigma} e^{-zt}\chi((n-1)h-\epsilon+t)dt \tag{8.17}$$

has exponential type less than or equal to h. Furthermore, since

$$\dot{\chi} \in L^2[0, (n-1)h - \sigma]$$

partial integration shows that (8.17) can be rewritten as $c + q(z)$, where $c \in \mathbf{R}^n$ and q is a Paley-Wiener function of exponential type less than or equal to h. Hence, $\chi((n-1)h - \epsilon + t)$ is a small solution so that

$$\chi((n-1)h - \epsilon + t) \not\equiv 0 \text{ in any neighbourhood of } \epsilon - \sigma.$$

CASE II. Suppose that $(n-1)h < \epsilon \leq nh$. In this case

$$\tau = \mathrm{E}(\det \Delta(z)) < h.$$

We multiply both sides of the equation (8.12) by

$$\int_0^{h-\tau} e^{-zt} dt \tag{8.18}$$

to obtain

$$\Delta(z) \begin{pmatrix} \tilde{C}_1 \\ \vdots \\ \tilde{C}_n \end{pmatrix} = \begin{pmatrix} G \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where

$$G(z) = \int_0^{h-\tau} e^{-zt} dt \det \Delta(z)$$

and

$$\tilde{C}_{j1}(z) = \int_0^{h-\tau} e^{-zt} dt\, C_{j1}(z) \qquad \text{for} \quad 1 \leq j \leq n.$$

Hence $\mathrm{E}(G) = h$ and $\mathrm{E}(\tilde{C}) = \epsilon - \sigma$. The same arguments as used in Case I but now applied to the modified function $\tilde{C}$ yield

$$\Delta(z) \int_0^{\epsilon-\sigma} e^{-zt} \tilde{\chi}(t) dt = \tilde{c} + \tilde{q}(z), \tag{8.19}$$

where $\tilde{c} \in \mathbf{R}^n$ and $\tilde{q}$ is a Paley-Wiener function with $\mathrm{E}(\tilde{q}) \leq h$. Therefore $\tilde{\chi}$ is a small solution so that

$$\tilde{\chi}((n-1)h - \epsilon + t) \not\equiv 0 \text{ in any neighbourhood of } \epsilon - \sigma.$$

∎

PROOF OF THEOREM 8.3. Because of Theorem 8.2 it suffices to prove that

$$\forall \epsilon > 0: \quad \sigma < \epsilon. \tag{8.20}$$

Suppose $\sigma = \epsilon$. We shall calculate $\mathrm{E}(\det \operatorname{adj}\Delta(z))$ in two different ways. Since $\sigma = \epsilon$ we have

$$\begin{aligned} \mathrm{E}(\det \operatorname{adj}\Delta(z)) &\leq n((n-1)h - \epsilon) \\ &= (n-1)(nh - \epsilon) - \epsilon. \end{aligned} \tag{8.21}$$

On the other hand by Corollary 4.11 we have

$$\begin{aligned} \mathrm{E}(\det \operatorname{adj}\Delta(z)) &= \mathrm{E}((\det \Delta(z))^{n-1}) \\ &= (n-1)(nh - \epsilon). \end{aligned} \tag{8.22}$$

Hence

$$(n-1)(nh - \epsilon) \leq (n-1)(nh - \epsilon) - \epsilon, \tag{8.23}$$

which is a contradiction if $\epsilon > 0$. This completes the proof of Theorem 8.3. ∎

EXAMPLE 8.4. Consider the following system of differential-difference equations

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) + x_3(t-1) \\ \dot{x}_2(t) &= x_1(t-1) + x_3(t - \frac{1}{2}) \\ \dot{x}_3(t) &= x_3(t). \end{aligned} \tag{8.24}$$

Then the characteristic matrix becomes

$$\Delta(z) = \begin{pmatrix} z & 1 & -e^{-z} \\ -e^{-z} & z & -e^{-\frac{1}{2}z} \\ 0 & 0 & z+1 \end{pmatrix} \tag{8.25}$$

with determinant

$$\det \Delta(z) = (z+1)(z^2 + e^{-z}). \tag{8.26}$$

Therefore,

$$\epsilon = 2.$$

Since the cofactor

$$\begin{aligned} C_{23}(z) &= -\begin{vmatrix} z & -e^{-z} \\ -e^{-z} & -e^{-\frac{1}{2}z} \end{vmatrix} \\ &= ze^{-\frac{1}{2}z} + e^{-2z} \end{aligned} \tag{8.27}$$

has exponential type 2, we derive that $\sigma = 0$. Therefore, from Theorem 8.2, the ascent of the system (8.24) equals two. Thus there exists a (non trivial) small solution $x = x(\,\cdot\,;\varphi)$ such that

$$\operatorname{supp}(x) = [-1, 1].$$

EXAMPLE 8.5. Consider the following system of differential-difference equations

$$\begin{aligned}\dot{x}_1(t) &= -x_2(t) + x_3(t)\\ \dot{x}_2(t) &= x_1(t-1) + x_3(t-\frac{1}{2})\\ \dot{x}_3(t) &= x_3(t).\end{aligned} \tag{8.28}$$

Then the characteristic matrix becomes

$$\Delta(z) = \begin{pmatrix} z & 1 & 1 \\ -e^{-z} & z & -e^{-\frac{1}{2}z} \\ 0 & 0 & z+1 \end{pmatrix} \tag{8.29}$$

with determinant

$$\det \Delta(z) = (z+1)(z^2 + e^{-z}). \tag{8.30}$$

Therefore,

$$\epsilon = 2.$$

Furthermore, in this case we derive $\sigma = 1$. Therefore, from Theorem 8.2, the ascent of the system (8.28) equals one. Thus all small solutions are trivial, in the sense that they are identical zero for $t \geq 0$.

Chapter 9. The resolvent

In this chapter we consider the resolvent equation (6.3) for the Volterra convolution equation (7.1). Since the resolvent equation is also a Volterra convolution equation, the methods developed in Chapter 7 to establish a Fourier type series expansion of the solution $x(\,\cdot\,;f)$ of (7.1) can be applied directly to the resolvent equation. We obtain a series expansion not for the resolvent itself, but for the so-called fundamental matrix solution U.

Define the fundamental matrix solution U by the solution of

$$\begin{aligned} &\dot{U}(t) = \int_0^t d\zeta(\theta)U(t-\theta) \qquad \text{for} \quad t > 0, \\ &U(0) = I, \text{ and } U(t) = 0 \qquad \text{for} \quad t < 0. \end{aligned} \tag{9.1}$$

We then have the following lemma.

Lemma 9.1. *For $t \geq 0$ we have*

$$U(t) = I - \int_0^t R(\tau)d\tau, \tag{9.2}$$

where R satisfies the resolvent equation

$$R = \zeta * R - \zeta. \tag{9.3}$$

Proof. It suffices to prove that U defined by (9.2) satisfies (9.1). From the resolvent equation (9.3) it follows that

$$\dot{U}(t) = -R(t) = -\zeta * R(t) + \zeta(t) = \int_0^t d\zeta(\theta)U(t-\theta)$$

with $U(0) = I$. A contraction argument, see Theorem 6.1, shows that (9.1) has a unique solution. Thus the lemma is proved. ∎

The fundamental matrix solution can be used to represent the solution $x(\cdot\,; f)$ of the Volterra convolution equation (7.1).

COROLLARY 9.2. *If x denotes the solution of (7.1). Then for every $f \in \mathcal{F}$*

$$x(t) = -\int_0^t dU_\theta(t-\theta)f(\theta) = \int_0^t U(t-\theta)df(\theta). \tag{9.4}$$

PROOF. Recall from Theorem 6.1 that $R \in L^1(\mathbf{R}_+;\gamma)$. Hence U is locally of bounded variation. By representation (6.2)

$$x(t) = f(t) - \int_0^t R(t-\theta)f(\theta)d\theta = -\int_0^t dU_\theta(t-\theta)f(\theta),$$

which is well defined since U is locally of bounded variation. An application of Theorem 2.7 (iii) now yields

$$x(t) = -\int_0^t dU_\theta(t-\theta)f(\theta) = \int_0^t U(t-\theta)df(\theta).$$

∎

To derive a Fourier type series expansion for U we first analyse the Laplace transform of U.

LEMMA 9.3.

$$L\{U\}(z) = \Delta^{-1}(z) \qquad \text{for} \quad \Re(z) > \gamma.$$

PROOF. Since $R \in L^1(\mathbf{R}_+;\gamma)$ we can apply Laplace transformation to the resolvent equation (9.3) to obtain

$$\Delta(z)L\{R\}(z) = -\int_0^h e^{-zt}d\zeta(t). \tag{9.5}$$

Since

$$\Delta(z) = zI - \int_0^h e^{-zt}d\zeta(t),$$

we have the identity

$$\Delta(z)L\{R\}(z) = \Delta(z) - zI. \tag{9.6}$$

From (9.2) and Theorem 3.1 (v) we derive

$$L\{U\}(z) = \frac{1}{z}(I - L\{R\}(z)) \tag{9.7}$$

and a combination of (9.6) and (9.7) yields the desired result

$$L\{U\}(z) = \Delta^{-1}(z).$$

∎

Note that Lemma 9.3, representation (7.2) and the properties of the Laplace transform listed in Theorem 3.1 also imply representation (9.4).

Let

$$\mathrm{N}(z\mathrm{adj}\Delta(z)) \leq \mathrm{N}(\det \Delta(z))$$

denote the condition that the Newton polygon of each of the coefficients $\mathrm{adj}\Delta(z)_{ij}$ of $\mathrm{adj}\Delta(z)$ satisfies

$$\mathrm{N}(z\mathrm{adj}\Delta(z)_{ij}) \leq \mathrm{N}(\det \Delta(z)). \tag{9.8}$$

REMARK 9.4. If $\mathrm{E}(\det \Delta(z)) = nh$, then condition (9.8) clearly is satisfied. However, the condition (9.8) need not to be true in general (see the examples in Chapter 13).

A reformulation of Theorem 7.2 now yields

THEOREM 9.5. *If $\zeta \in \mathrm{SBV}[0,h]$ is such that the Newton polygon condition (9.8) holds. Then for all $\epsilon > 0$ the fundamental matrix solution U defined by (9.1) can be represented by*

$$U(t) = \sum_{j=1}^{\infty} P_j(t) e^{\lambda_j t}, \tag{9.9}$$

in $\mathcal{C}_0\big([\epsilon,\infty);\gamma\big)$. Moreover, for $f \in \mathcal{F}$ such that f is locally of bounded variation, the solution $x(\,\cdot\,; f)$ of (7.1) can be represented by a series expansion

$$x(t;f) = f(t) + \sum_{j=1}^{\infty} \int_0^t P_j(t-\theta) e^{\lambda_j (t-\theta)} df(\theta), \tag{9.10}$$

in $\mathcal{C}_0\big([\epsilon,\infty);\gamma\big)$.

PROOF. Given the Newton polygon condition (9.8), the series representation (9.9) follows directly from the results of Chapter 7. Because of Corollary 9.2 we can represent the solution x of (7.1) by

$$x(t;f) = \int_0^t U(t-\theta) df(\theta).$$

Hence,

$$x(t+\epsilon;f) = \int_0^t U(t+\epsilon-\theta) df(\theta) + \int_t^{t+\epsilon} U(t+\epsilon-\theta) df(\theta). \tag{9.11}$$

Since the series representation for $U(t)$ converges in $\mathcal{C}_0\big([\epsilon,\infty);\gamma\big)$ we can interchange summation and integration to obtain

$$\begin{aligned}x(t+\epsilon;f) = \int_t^{t+\epsilon} & U(t+\epsilon-\theta)df(\theta)\\ &+\sum_{j=1}^{\infty}\int_0^t P_j(t+\epsilon-\theta)e^{\lambda_j(t+\epsilon-\theta)}df(\theta).\end{aligned} \tag{9.12}$$

From

$$\lim_{\epsilon\downarrow 0}\int_t^{t+\epsilon} U(t+\epsilon-\theta)df(\theta) = f(t)$$

and

$$\lim_{\epsilon\downarrow 0} x(t+\epsilon;f) = x(t;f),$$

the representation (9.10) follows from (9.12). Finally, the uniform convergence for $t>\epsilon$ follows from the uniform convergence of U in $\mathcal{C}_0\big([\epsilon,\infty);\gamma\big)$. ∎

Chapter 10. Linear autonomous delay equations

In this chapter we use the semigroup approach to study a system of linear retarded functional differential equations (RFDE)

$$\dot{x}(t) = \int_0^h d\zeta(\theta)x(t-\theta) \qquad \text{for} \quad t \geq 0, \tag{10.1}$$

satisfying the initial condition

$$x(t) = \varphi(t) \qquad \text{for} \quad -h \leq t \leq 0,$$

where the matrix-valued function ζ belongs to $\mathrm{NBV}[0,h]$ and the initial condition φ is a given element of $\mathcal{C} = C[-h,0]$, the space of continuous functions on $[-h,0]$. The general theory was first given by by Hale [17] and Krasovskii [25], see Hale's book [18] and the references given there.

First we reformulate the equivalence between the linear autonomous RFDE (10.1) and the Volterra convolution equation (6.1) described in Example 2.9 a little more abstractly. Let $\mathcal{F}$ denote the supremum normed Banach space of continuous functions on $\mathbf{R}_+$ that are constant on $[h,\infty)$ provided with the supremum norm.

THEOREM 10.1. *The linear autonomous RFDE (10.1) and the Volterra convolution equation (6.1)*

$$x - \zeta * x = f$$

are equivalent. In the sense that there exists a bounded invertible map $G : \mathcal{F} \to \mathcal{C}$ given by

$$(Gf)(t) = \big(f - R * f\big)(h+t) \qquad \textit{for} \quad -h \leq t \leq 0, \tag{10.2}$$

where R denotes the resolvent, such that the solution $x(\cdot\,;\varphi)$ of (10.1) on $\mathbf{R}_+$ satisfies the Volterra convoltion equation

$$x - \zeta * x = F\varphi,$$

where $F : \mathcal{C} \to \mathcal{F}$ is defined by

$$F = S(h)G^{-1}$$

and can be represented by

$$\big(F\varphi\big)(t) = \varphi(0) + \int_0^t \int_s^h d\zeta(\theta)\varphi(s-\theta)ds. \tag{10.3}$$

Proof. Since $R \in L^1\big(\mathbf{R}_+;\gamma\big)$ and $f \in L^\infty\big(\mathbf{R}_+;\gamma\big)$ it follows from the convolution property [21; 21.33] that G is a bounded linear mapping. Since

$$\big(Gf\big)(t) = x(h+t;f),$$

where x satisfies the Volterra convolution equation

$$x - \zeta * x = f,$$

it follows that the mapping $G^{-1} : \mathcal{C} \to \mathcal{F}$ is given by

$$\big(G^{-1}\varphi\big)(t) = \big(\varphi - \zeta * \varphi\big)(-h+t) \qquad \text{for} \quad 0 \le t \le h. \tag{10.4}$$

The same argument as above can be used to prove that G^{-1} is a continuous mapping from $\mathcal{C}$ onto $\mathcal{F}$. Since the forcing functions are constant on $[h,\infty)$ differentiation of $y(t) = x(h+t;f)$ shows that

$$\dot{y}(t) = \int_0^h d\zeta(\theta)y(t-\theta) \qquad \text{for} \quad t \ge 0$$

with

$$y_0 = Gf.$$

This proves the equivalence between the equations (6.1) and (10.1). The remaining part of the theorem is just a reformulation of Example 2.9. ∎

We shall now associate with equation (10.1) a $\mathcal{C}_0$-semigroup $\{\,T(t)\,\}$.

DEFINITION 10.2. Let $(X, \|\cdot\|)$ be a Banach space and suppose that with every $t \in \mathbf{R}_+$ is associated a bounded linear operator $T(t) : X \to X$, in such a way that

(i) T(0)=I;

(ii) For all $t_1, t_2 \in \mathbf{R}_+ : T(t_1 + t_2) = T(t_1)T(t_2)$;

(iii) For every $\varphi \in X$
$$\lim_{t \downarrow 0} \|T(t)\varphi - \varphi\| = 0.$$

Then $\{T(t)\}$ is called a strongly continuous semigroup or, in short, a $\mathcal{C}_0$-semigroup.

With every $\mathcal{C}_0$-semigroup $\{T(t)\}$ we can associate an infinitesimal generator A defined by
$$A\varphi = \lim_{t \downarrow 0} \frac{1}{t}[T(t)\varphi - \varphi] \tag{10.5}$$
for every $\varphi \in \mathcal{D}(A)$, that is, for every $\varphi \in X$ for which the limit exists in the norm topology of X. The following theorem can be found in Rudin [32].

THEOREM 10.3. *If $\{T(t)\}$ is a $\mathcal{C}_0$-semigroup. Then*

(i) For every $x \in X$ the mapping $t \mapsto T(t)x$ is continuous from $\mathbf{R}_+$ into X;

(ii) A is a closed densely defined operator on X;

(iii) For every $x \in \mathcal{D}(A)$, the orbit $T(t)x$ satisfies the differential equation
$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax. \tag{10.6}$$

Translation along the solution of (10.1) induces a $\mathcal{C}_0$-semigroup $\{T(t)\}$ defined on $\mathcal{C}$ by
$$T(t)\varphi = x(t + \cdot\,; \varphi) =: x_t \tag{10.7}$$
with infinitesimal generator
$$A\varphi = \dot{\varphi} \tag{10.8}$$
defined on
$$\mathcal{D}(A) = \left\{\varphi \in \mathcal{C} : \dot{\varphi} \in \mathcal{C} \text{ and } \dot{\varphi}(0) = \int_0^h d\zeta(\theta)\varphi(-\theta)\ \right\}.$$

Next we describe the spectrum of A. Let
$$\Delta(z) = zI - \int_0^h e^{-zt} d\zeta(t) \tag{10.9}$$

denote the characteristic matrix associated with the RFDE (10.1) and let

$$R(z, A) : \mathcal{C} \to \mathcal{D}(A)$$

denote the resolvent

$$R(z, A) = (zI - A)^{-1} \tag{10.10}$$

of A. The following theorem yields an explicit formula for the resolvent of A.

THEOREM 10.4. *If $\varphi \in \mathcal{C}$ and if $\lambda \in \mathcal{C}$ is such that* $\det \Delta(\lambda) \neq 0$. *Then $\lambda \in \rho(A)$ and $R(\lambda, A)\varphi$ is given explicitly by*

$$R(\lambda, A)\varphi = e^{\lambda t}\{\Delta^{-1}(\lambda)K(\varphi) - \int_0^t e^{-\lambda s}\varphi(s)ds\}, \tag{10.11}$$

where

$$K(\varphi) = \lambda \int_0^\infty e^{-\lambda t} F\varphi(t)dt. \tag{10.12}$$

PROOF. Let $\chi = R(\lambda, A)\varphi$. From the definition of A it follows that

$$(\lambda I - A)\chi = \varphi$$

if and only if χ satisfies the conditions

(i) $\lambda\chi - \dot{\chi} = \varphi$;

(ii) $\lambda\chi(0) - \int_0^h d\zeta(\theta)\chi(-\theta) = \varphi(0)$;

(iii) $\dot{\chi} \in \mathcal{C}$.

Define

$$\chi(t) = e^{\lambda t}\chi(0) + \int_t^0 e^{\lambda(t-s)}\varphi(s)ds, \tag{10.13}$$

where $-h \leq t \leq 0$. Then χ satisfies the conditions (i) and (iii). Also, condition (ii) becomes

$$\Delta(\lambda)\chi(0) = K(\varphi). \tag{10.14}$$

Since $\det \Delta(\lambda) \neq 0$, we can solve

$$\chi(0) = \Delta^{-1}(\lambda)K(\varphi). \tag{10.15}$$

∎

COROLLARY 10.5. *The spectrum of A is all point spectrum and is given by*

$$\sigma(A) = \mathrm{P}\sigma(A) = \{\lambda \in \mathbf{C} : \det\Delta(\lambda) \neq 0\}. \tag{10.16}$$

PROOF. Because of the proof of Theorem 10.4 we have

$$\{\lambda \in \mathbf{C} : \det\Delta(\lambda) \neq 0\} \subset \rho(A).$$

To prove the reverse inclusion choose $\lambda \in \mathbf{C}$ such that $\det\Delta(\lambda) = 0$ and define

$$\varphi(t) = e^{\lambda t}\varphi^0 \qquad \text{for} \quad -h \leq t \leq 0,$$

where $\varphi^0 \neq 0$ is an element of the nullspace of $\Delta(\lambda)$. Then

$$A\varphi = \dot{\varphi} = \lambda\varphi.$$

Therefore, we conclude that $\lambda \in \mathrm{P}\sigma(A)$. ∎

COROLLARY 10.6.

$$\mathcal{N}\big((\lambda I - A)\big) = \big\{\varphi \in \mathcal{C} : \varphi(t) = e^{\lambda t}\varphi^0 \text{ and } \varphi^0 \in \mathcal{N}\big(\Delta(\lambda)\big)\big\}.$$

Let $\varphi \in \mathcal{C}$ be fixed and consider the function $R(z, A)\varphi$ as a function of z. By Theorem 10.4 we have that $R(z, A)\varphi$ is a meromorphic function with poles λ satisfying the equation

$$\det\Delta(z) = 0.$$

This property of $R(z, A)$ makes it possible to apply [36; V.10.1].

THEOREM 10.7. *If λ is a pole of $R(z, A)$ of order m. Then for some k with $1 \leq k \leq m$*

(i) $\mathcal{N}\big((\lambda I - A)^k\big) = \mathcal{N}\big((\lambda I - A)^{k+1}\big)$;

(ii) $\mathcal{R}\big((\lambda I - A)^k\big) = \mathcal{R}\big((\lambda I - A)^{k+1}\big)$;

(iii) $\mathcal{R}\big((\lambda I - A)^k\big)$ *is closed;*

(iv) $\mathcal{C} = \mathcal{N}\big((\lambda I - A)^k\big) \oplus \mathcal{R}\big((\lambda I - A)^k\big)$;

(v) The spectral projection P_λ corresponding to the decomposition in (iv) on $\mathcal{N}\big((\lambda I - A)^k\big)$ can be represented by the contour integral

$$P_\lambda\varphi = \frac{1}{2\pi i}\int_{\Gamma_\lambda} R(z, A)\varphi\, dz, \tag{10.17}$$

where Γ_λ is a circle enclosing λ but no other point of the discrete set $\sigma(A)$.

Let $\mathcal{M}_\lambda$ denote the generalized eigenspace $\mathcal{N}\big((\lambda I - A)^m\big)$ corresponding to an eigenvalue λ of A. By Theorem 10.4 and the definition of A we have that elements of $\mathcal{M}_\lambda$ involve combinations of

$$t^l e^{\lambda t} d_l, \tag{10.18}$$

where $l = 1, 2, \dots, m$ and the constants $d_l \in \mathbf{R}^n$ satisfy a system of linear equations. So $\mathcal{M}_\lambda$ is finite dimensional and by using this system of linear equations one can construct an explicit base for $\mathcal{M}_\lambda$ that shows that the dimension of $\mathcal{M}_\lambda$ equals m_λ, the multiplicity of λ as zero of $\det \Delta(z)$, see [18] and [26]. From Theorem 10.4 it follows that $R(\lambda, A)$ is a compact operator for $\lambda \in \rho(A)$. Therefore, the spectral projections are compact and hence have finite dimensional ranges. So, this remark also shows that the spaces $\mathcal{M}_\lambda$ are finite dimensional.

Let Q_λ denote $\mathcal{R}\big((\lambda I - A)^k\big)$. Since the generator A and the $\mathcal{C}_0$-semigroup $\{\,T(t)\,\}$ commute, the linear subspaces $\mathcal{M}_\lambda$ and Q_λ are $\{\,T(t)\,\}$-invariant. Before we continue with the characterization of these $\{\,T(t)\,\}$-invariant subspaces, we first extend the equivalence between linear autonomous RFDEs and Volterra convolution equations, given by Theorem 10.1 (a similar result was proved by Banks and Manitius [2]). As a consequence of this extension we can translate the convergence results derived in Chapter 7 to results on spectral projection series for an state of the RFDE (10.1).

THEOREM 10.8. *The λ_j-th term of the Fourier type series expansion of $x(\,\cdot\,; f)$ of the Volterra convolution equation (7.1) equals the λ_j-th spectral projection of the corresponding state of the RFDE (10.1), i.e.*

$$P_{\lambda_j}(Gf)(t-h) = \operatorname*{Res}_{z=\lambda_j} \left\{ e^{zt}\Delta^{-1}(z) z \int_0^\infty e^{-zt} f(t)\,dt \right\}. \tag{10.19}$$

PROOF. From Theorem 10.4 and the representation for the spectral projection P_{λ_j} given by (10.17) it follows that

$$P_{\lambda_j}(Gf)(t-h) = \operatorname*{Res}_{z=\lambda_j} \left\{ e^{z(t-h)}\Delta^{-1}(z) z \int_0^\infty e^{-zt} FGf(t)\,dt \right\}.$$

Since

$$FGf = S(h)f = x(h + \,\cdot\,; f) - \zeta * x(h + \,\cdot\,; f),$$

we obtain

$$\begin{aligned} P_{\lambda_j}(Gf)(t-h) &= \operatorname*{Res}_{z=\lambda_j} \Big\{ e^{zt}\Delta^{-1}(z)\Big(z \int_0^\infty e^{-zt} f(t)\,dt \\ &\qquad - \Delta(z) \int_0^h e^{-zt} x(t)\,dt\Big)\Big\} \\ &= \operatorname*{Res}_{z=\lambda_j} \left\{ e^{zt}\Delta^{-1}(z) z \int_0^\infty e^{-zt} f(t)\,dt \right\}. \end{aligned}$$

∎

As a result of the above theorem, residue calculus of the Volterra convolution equation and analysis of the spectrum of the resolvent $R(z,A)$, yield the same information. The only difference is that instead of the solution x we now analyse the state $x_t = x(t+\theta)$ as a function on the interval $[-h,0]$. In Corollary 6.11 we derived an exponential estimate for the remainder term of $x(\ \cdot\,;f)$ and of course at the same time this yields an estimate for the state

$$T(t)Gf = x_t(\ \cdot\,;Gf).$$

Recall from Theorem 10.1 that the $\mathcal{C}_0$-semigroups are intertwined, i.e.

$$T(t) = GS(t)G^{-1}. \tag{10.20}$$

COROLLARY 10.9. *Let $\Lambda(\gamma)$ be the finite set of eigenvalues defined by*

$$\Lambda = \Lambda(\gamma) = \{\lambda \in \sigma(A) : \Re(\lambda) > \gamma\}.$$

Then the state space $\mathcal{C}$ can be decomposed into two closed $\{T(t)\}$-invariant subspaces $\mathcal{M}_\Lambda$ and Q_Λ

$$\mathcal{C} = \mathcal{M}_\Lambda \oplus Q_\Lambda, \tag{10.21}$$

where

$$\mathcal{M}_\Lambda = \bigoplus_{\lambda\in\Lambda} \mathcal{M}_\lambda$$

and

$$Q_\Lambda = \bigoplus_{\lambda\in\Lambda} Q_\lambda.$$

The spectral projection P_Λ on $\mathcal{M}_\Lambda$ is given by

$$P_\Lambda = \sum_{\lambda\in\Lambda} P_\lambda.$$

Besides, if

$$\varphi = P_\Lambda\varphi + (I - P_\Lambda)\varphi,$$

according to the above decomposition. Then

$$\|T(t)(I-P_\Lambda)\varphi\| \le Ke^{\gamma t}\|(I-P_\Lambda)\varphi\| \tag{10.22}$$

for some positive constant K and $t \ge 0$.

Assume that all roots have negative real part, then we can choose $\gamma < 0$ in Corollary 10.9 and we derive exponential asymptotic stability: for all $\varphi \in \mathcal{C}$

$$\|T(t)\varphi\| \le Ke^{\gamma t}\|\varphi\| \tag{10.23}$$

for some positive constant K and negative γ.

Let $\mathcal{M}_\mathcal{C}$ denote the linear subspace generated by $\mathcal{M}_\lambda$, i.e.

$$\mathcal{M}_\mathcal{C} = \bigoplus_{\lambda\in\sigma(A)} \mathcal{M}_\lambda. \tag{10.24}$$

This linear subspace is called the generalized eigenspace of A.

DEFINITION 10.10. The generalized eigenspace $\mathcal{M}_{\mathcal{C}}$ is called complete if and only if $\mathcal{M}_{\mathcal{C}}$ is dense in $\mathcal{C}$ i.e. $\overline{\mathcal{M}_{\mathcal{C}}} = \mathcal{C}$.

Define the ascent α of the semigroup $\{T(t)\}$ by the value

$$\alpha = \inf \left\{ t \mid \forall \epsilon > 0 : \mathcal{N}\big(T(t)\big) = \mathcal{N}\big(T(t+\epsilon)\big) \right\}. \tag{10.25}$$

Recall the definitions of ϵ and σ introduced in Chapter 8:

$$nh - \epsilon = \mathrm{E}(\det \Delta(z)) \tag{10.26}$$

and

$$(n-1)h - \sigma = \max_{1 \le i,j \le n} \mathrm{E}(\mathrm{adj}\Delta(z)_{ij}). \tag{10.27}$$

An application of Theorem 8.2 and Theorem 10.1 yields the following result.

THEOREM 10.11. *The ascent α of the $\mathcal{C}_0$-semigroup $\{T(t)\}$ associated with the RFDE (10.1) is finite and is given by*

$$\alpha = \epsilon - \sigma. \tag{10.28}$$

As a result of Theorem 8.3 we have the following corollary.

COROLLARY 10.12. *The $\mathcal{C}_0$-semigroup $\{T(t)\}$ associated with the RFDE (10.1) is injective if and only if* $\mathrm{E}(\det \Delta(z)) = nh$.

Moreover, we can characterize the subspace $\mathcal{N}\big(T(\alpha)\big)$.

THEOREM 10.13.

$$\mathcal{N}\big(T(\alpha)\big) = \left\{ \varphi \in \mathcal{C} : z \mapsto R(z,A)\varphi \ \textit{ is entire} \right\}.$$

PROOF. From Theorem 10.4 it follows that only the fact

$$\varphi \in \mathcal{N}\big(T(\alpha)\big) \text{ if and only if } x(\,\cdot\,; F\varphi) \text{ is a small solution}$$

remains to be proved. But this is clear from the definitions of F and α. ∎

From the exponential estimates derived in Corollary 10.9 we can also characterize the closed subspace

$$\bigcap_{\lambda \in \sigma(A)} Q_\lambda. \tag{10.29}$$

COROLLARY 10.14.

$$\bigcap_{\lambda \in \sigma(A)} Q_\lambda = \mathcal{N}\big(T(\alpha)\big).$$

PROOF. Let $\varphi \in \mathcal{N}\big(T(\alpha)\big)$. From Theorem 10.13 and the representation (10.17) we derive for all $\lambda \in \sigma(A)$

$$P_\lambda \varphi = 0.$$

Hence

$$\varphi \in \bigcap_{\lambda \in \sigma(A)} Q_\lambda.$$

On the other hand if $\varphi \in \bigcap_{\lambda \in \sigma(A)} Q_\lambda$, then we derive from Corollary 10.9 the exponential estimate

$$\|T(t)\varphi\| \le K e^{\gamma t} \qquad \text{for} \quad t \ge 0 \tag{10.30}$$

for every $\gamma \in \mathbf{R}$ and some positive constant K. Therefore, an application of Theorem 8.2 shows that $x(\,\cdot\,; F\varphi)$ is a small solution. Thus $\varphi \in \mathcal{N}\big(T(\alpha)\big)$. ∎

From the results of this chapter we derive that, if the decomposition given in Corollary 10.9 holds with $\Lambda = \sigma(A)$, then the following state space decomposition would be true

$$\mathcal{C} = \overline{\mathcal{M}_\mathcal{C}} \oplus \mathcal{N}\big(T(\alpha)\big). \tag{10.31}$$

This state space decomposition for $\mathcal{C}$ is important since it would show that completeness holds if and only if there are no small solutions. Furthermore, this decomposition would show the procedure to follow when there do exist nontrivial small solutions, namely state space restriction to $\overline{\mathcal{M}_\mathcal{C}}$. Our main goal in the next chapter will be the study and characterization of $\overline{\mathcal{M}_\mathcal{C}}$ employing the convergence results for the Fourier type series derived in Chapter 7.

CHAPTER 11. INVARIANT SUBSPACES

In this chapter we shall study invariant subspaces for the $\mathcal{C}_0$-semigroup $\{T(t)\}$ associated with the linear autonomous RFDE

$$\begin{aligned} \dot{x}(t) &= \int_0^h d\zeta(\theta)x(t-\theta) \qquad \text{for} \quad t \geq 0, \\ x_0 &= \varphi, \end{aligned} \tag{11.1}$$

where $\varphi \in \mathcal{C} = C[-h,0]$ and $\zeta \in \mathrm{NBV}[0,h]$.

Recall from Chapter 10 that both $\mathcal{N}\big(T(\alpha)\big)$ and $\overline{\mathcal{M}_\mathcal{C}}$ are $\{\,T(t)\,\}$-invariant closed subspaces of $\mathcal{C}$. In this chapter we shall use the results of the Chapters 7 and 8 to study the subspaces $\mathcal{N}\big(T(\alpha)\big)$ and $\overline{\mathcal{M}_\mathcal{C}}$ in more detail.

To begin with we discuss the solutions of the RFDE (11.1) which are defined on $(-\infty, 0]$. Define the following $\{\,T(t)\,\}$-invariant subspaces

$$\mathcal{A}_\mathcal{C} = \big\{\varphi \in \mathcal{C} \mid \exists x : x \text{ is a solution on } (-\infty,0] \text{ with } x_0 = \varphi\big\}$$

and

$$\mathcal{A}_\mathcal{C}^b = \big\{\varphi \in \mathcal{C} \mid \exists x : x \text{ is a bounded solution on } (-\infty,0] \text{ with } x_0 = \varphi\big\}.$$

In Chapter 8, we showed that the $\mathcal{C}_0$-semigroup $\{\,T(t)\,\}$ need not to be one-to-one. To re-emphasize this point, we state the result explicitly (recall Corollary 10.12).

THEOREM 11.1. *The $\mathcal{C}_0$-semigroup $\{\,T(t)\,\}$ is one-to-one if and only if $\det\Delta(z)$ has maximal exponential type, i.e.* $\mathrm{E}(\det\Delta(z)) = nh$.

The fact that the $\mathcal{C}_0$-semigroup $\{\,T(t)\,\}$ may not be one-to-one is at the same time an annoying and an interesting feature of the theory of RFDEs and a better understanding of one-to-oneness is needed. One way to begin to understand why the map is not one-to-one is to define and study equivalence classes of initial data.

DEFINITION 11.2. We call $\varphi \in \mathcal{C}$ equivalent to $\chi \in \mathcal{C}$, notation $\varphi \sim \chi$, if there exists a $\tau \geq 0$ such that $x_\tau(\,\cdot\,;\varphi) = x_\tau(\,\cdot\,;\chi)$. If in addition $\tau \leq h$, then we call φ immediately equivalent to χ.

In contrast with the general (non autonomous) case, the following result is true for linear autonomous systems of RFDEs.

THEOREM 11.3. *For linear autonomous systems of RFDEs the equivalence classes are determined in a fixed finite time (viz. $\alpha = \epsilon - \sigma$).*

PROOF. If φ and χ belong to the same equivalence class of a linear system, then the solution x corresponding to $\varphi - \chi$ is a solution that must be zero after some finite time. It then follows from Theorem 8.2 that $x(\,\cdot\,;\varphi - \chi)$ must be identically zero for $t \geq \alpha - h$. Therefore, each equivalence class is determined in finite time. ∎

As a result of Chapter 10, in particular Corollary 10.9, we derive the following theorem.

THEOREM 11.4.

$$\mathcal{A}_{\mathcal{C}}^b \subset \bigoplus_{\Re(\lambda)\geq 0} \mathcal{M}_\lambda \quad \textit{where} \quad \lambda \in \sigma(A).$$

Hence $\mathcal{A}_{\mathcal{C}}^b$ is finite dimensional and asymptotically stable, i.e. for any bounded set $\mathcal{B} \subset \mathcal{C}$ and $\epsilon > 0$, there is a $t_0(\mathcal{B},\epsilon)$ such that

$$T(t)\mathcal{B} \subset \left\{\varphi \in \mathcal{C} : \mathrm{d}(\varphi, \mathcal{A}_{\mathcal{C}}^b) < \epsilon\right\} \qquad \text{for} \quad t \geq t_0(\mathcal{B},\epsilon), \tag{11.2}$$

where

$$\mathrm{d}(\varphi, \mathcal{A}_{\mathcal{C}}^b) = \inf_{\chi \in \mathcal{A}_{\mathcal{C}}^b} \|\varphi - \chi\|.$$

So, asymptotically, the system (11.1) is controlled by the finite dimensional subspace $\mathcal{A}_{\mathcal{C}}^b$ of $\mathcal{C}$. If we replace $\mathcal{A}_{\mathcal{C}}^b$ by $\mathcal{A}_{\mathcal{C}}$ in the above theorem, then Theorem 11.5 shows that (11.2) holds after a fixed finite time for any subset of $\mathcal{C}$. However, the price we have to pay for this stronger result is that the subspace $\mathcal{A}_{\mathcal{C}}$ is infinite dimensional and therefore difficult to characterize. It is our aim to give a complete characterization of the closure $\overline{\mathcal{A}_{\mathcal{C}}}$ of $\mathcal{A}_{\mathcal{C}}$.

THEOREM 11.5.

$$\mathcal{A}_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}} = \bigoplus_{\lambda \in \sigma(A)} \mathcal{M}_\lambda.$$

Hence $\mathcal{A}_{\mathcal{C}}$ is infinite dimensional. For any set $\mathcal{W} \subset \mathcal{C}$

$$T(t)\mathcal{W} \subset \left\{\varphi \in \mathcal{C} : \mathrm{d}(\varphi, \mathcal{A}_{\mathcal{C}}) = 0\right\} \qquad \text{for} \quad t \geq \alpha, \tag{11.3}$$

where α denotes the ascent of $\{\,T(t)\,\}$.

We divide the proof of Theorem 11.5 into two theorems.

THEOREM 11.6.

$$\mathcal{A}_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}}.$$

THEOREM 11.7.

$$\overline{\mathcal{A}}_{\mathcal{C}} = \overline{\mathcal{R}\big(T(\alpha)\big)}.$$

A result related to Theorem 11.7 states

$$\overline{\mathcal{M}}_{\mathcal{C}} = \overline{\mathcal{R}\big(T(\delta)\big)},$$

where δ is the ascent of the adjoint semigroup $T^*(t)$ and was first proved by Henry [20] using duality methods.
Here we do not need duality methods and we shall prove Theorem 11.7 by applying the results derived in Chapter 7. Theorems 11.6 and 11.7 are corollaries of the characterization of $\overline{\mathcal{A}}_{\mathcal{C}}$, which we shall prove first.

Laplace transformation of the RFDE (11.1) yields

$$\int_0^\infty e^{-zt}x(t-h)dt = \frac{Q_\varphi(z)}{\det\Delta(z)} \qquad \text{for} \quad \Re(z) > \gamma, \tag{11.4}$$

where

$$\begin{aligned} Q_\varphi(z) = e^{-zh}\big(&\mathrm{adj}\Delta(z)z\int_0^\infty e^{-zt}F\varphi(t)dt \\ &+ \det\Delta(z)\int_{-h}^0 e^{-zt}\varphi(t)dt\big). \end{aligned} \tag{11.5}$$

An application of Theorem 10.1 and Theorem 7.20 yields the following characterization.

THEOREM 11.8.

$$\overline{\mathcal{A}}_{\mathcal{C}} = \{\varphi \in \mathcal{C} : \mathrm{E}(Q_\varphi(z)) \le \mathrm{E}(\det\Delta(z))\ \}.$$

PROOF. From Theorem 10.1 we have

$$G\overline{\mathcal{A}}_{\mathcal{F}} = \overline{\mathcal{A}}_{\mathcal{C}}. \tag{11.6}$$

If $f \in \mathcal{F}$, then we derive for the solution $x(\,\cdot\,; Gf)$ of the RFDE (11.1)

$$\int_0^\infty e^{-zt}x(t-h)dt = \frac{\mathrm{adj}\Delta(z)z\int_0^\infty e^{-zt}f(t)dt}{\det\Delta(z)}. \tag{11.7}$$

Therefore, from (11.6), (11.7), Theorem 7.10, and the representation (11.4), the desired result follows. ∎

PROOF OF THEOREM 11.6. The inclusion $\mathcal{M}_{\mathcal{C}} \subset \mathcal{A}_{\mathcal{C}}$ holds by definition. To prove the remaining inclusion $\mathcal{A}_{\mathcal{C}} \subset \mathcal{M}_{\mathcal{C}}$ we are first going to prove that

$$\mathcal{A}_{\mathcal{C}} \subset \overline{\mathcal{M}_{\mathcal{C}}}.$$

From the proof of Theorem 7.11 and Theorem 10.1, it follows that

$$\{\varphi \in \mathcal{C} : \mathrm{E}(Q_\varphi(z)) \leq \mathrm{E}(\det \Delta(z))\ \} \subset \overline{\mathcal{M}_{\mathcal{C}}}. \tag{11.8}$$

Therefore, from Theorem 11.8

$$\mathcal{A}_{\mathcal{C}} \subset \overline{\mathcal{M}_{\mathcal{C}}}. \tag{11.9}$$

The elements of $\overline{\mathcal{M}_{\mathcal{C}}}$ are (infinite) series and limits of (infinite) series of the form

$$\sum p_j(t)e^{\lambda_j t}. \tag{11.10}$$

The elements of $\mathcal{A}_{\mathcal{C}}$ do have a well-defined solution on the whole real line. From (11.9) it follows that this solution can be given as a series or a limit of series of the form (11.10) and since these series involved should be well defined for negative t they cannot be infinite. Therefore, we have proved

$$\mathcal{A}_{\mathcal{C}} \subset \mathcal{M}_{\mathcal{C}}$$

and this completes the proof of Theorem 11.6. ∎

PROOF OF THEOREM 11.7. Because of the definition of $\mathcal{A}_{\mathcal{C}}$ we have

$$\mathcal{A}_{\mathcal{C}} \subset \mathcal{R}\big(T(t)\big) \qquad \text{for} \quad t \geq 0. \tag{11.11}$$

Thus we obtain the inclusion

$$\overline{\mathcal{A}_{\mathcal{C}}} \subset \overline{\mathcal{R}\big(T(\alpha)\big)}. \tag{11.12}$$

The remaining inclusion follows from Theorem 11.8. Since

$$Q_{T(\alpha)\varphi} = e^{\alpha z}\big(Q_\varphi - \int_0^\alpha e^{-zt}x(t-h)dt\big), \tag{11.13}$$

we derive from the definition of the ascent α that

$$\mathrm{E}\big(Q_{T(\alpha)\varphi}(z)\big) \leq \mathrm{E}(\det \Delta(z)). \tag{11.14}$$

Therefore,

$$\mathcal{R}\big(T(\alpha)\big) \subset \overline{\mathcal{A}_{\mathcal{C}}} \tag{11.15}$$

and the theorem is proved. ∎

Next we turn to some important applications of the characterization obtained above. We say that $\varphi \in \mathcal{C}$ has a backward continuation over σ if there exists an initial condition $\chi \in \mathcal{C}$ such that

$$T(\sigma)\chi = \varphi. \tag{11.16}$$

A backward continuation does not always exist and, moreover, is not unique. From Theorem 8.2 it follows that if a backward continuation exists over σ with $\sigma \geq \alpha$ it is unique over $\sigma - \alpha$. Clearly, for $\varphi \in \mathcal{A}_\mathcal{C}$ a backward continuation does exist and is unique. Our first result shows that this property is stable under taking the closure of $\mathcal{A}_\mathcal{C}$.

THEOREM 11.9. *Backward continuation is unique for* $\overline{\mathcal{A}_\mathcal{C}}$.

This theorem is a direct consequence of Theorem 11.8 and the following result.

THEOREM 11.10.

$$\overline{\mathcal{M}_\mathcal{C}} \cap \mathcal{N}\big(T(\alpha)\big) = \{0\}.$$

PROOF. Assume that

$$\varphi \in \overline{\mathcal{M}_\mathcal{C}} \cap \mathcal{N}\big(T(\alpha)\big).$$

Since $\varphi \in \mathcal{N}\big(T(\alpha)\big)$, the solution $x_{-h} = x(-h + \cdot\,; \varphi)$ has an entire Laplace transform

$$L\{x_{-h}\}(z) = \int_0^\alpha e^{-zt} x(t-h)\,dt. \tag{11.17}$$

On the other hand, since $\varphi \in \overline{\mathcal{M}_\mathcal{C}}$, it follows from Theorem 11.6 and Theorem 11.7 that

$$\mathrm{E}\big(L\{x_{-h}\}\big) = 0.$$

Hence, from the representation (11.17) we conclude that x_{-h} should be zero. So in particular φ should be zero. ∎

From Theorem 11.10 it follows that the direct sum

$$\overline{\mathcal{M}_\mathcal{C}} \oplus \mathcal{N}\big(T(\alpha)\big)$$

is well defined. By duality, see Chapter 12, in particular Theorem 12.5, the dual result of Theorem 11.10 reads

$$\overline{\mathcal{R}\big(T^*(\delta)\big)} \cap \mathcal{N}\big(T^*(\delta)\big) = 0,$$

where δ denotes the ascent of $\{T^*(\delta)\}$. Therefore, the direct sum $\overline{\mathcal{M}_\mathcal{C}} \oplus \mathcal{N}\big(T(\alpha)\big)$ is a dense subspace of $\mathcal{C}$. So

$$\mathcal{C} = \overline{\overline{\mathcal{M}_\mathcal{C}} \oplus \mathcal{N}\big(T(\alpha)\big)}. \tag{11.18}$$

For differential-difference equations the linear subspace $\overline{\mathcal{M}_\mathcal{C}}$ can be determined explicitly from the characterization obtained above. See the examples in Chapter 13.

The linear subspace $\overline{\mathcal{M}_\mathcal{C}}$ represents, in a certain sense, the minimal amount of initial data needed to specify a solution of the RFDE (11.1). If more initial data are specified, then the extra part belongs to $\overline{\mathcal{M}_\mathcal{C}}$ for $t \geq \alpha$. Given this observation we can recall the following question posed by Hale [18]. Does there exist a $\{T(t)\}$-invariant closed complementary subspace $\mathcal{U}$ of $\mathcal{N}(T(\alpha))$ such that

$$\mathcal{C} = \mathcal{U} \oplus \mathcal{N}(T(\alpha))? \tag{11.19}$$

From the decomposition (11.18) we derive that

$$\mathcal{U} = \overline{\mathcal{M}_\mathcal{C}} \tag{11.20}$$

and to answer this question, we shall consider backward continuations for $\overline{\mathcal{M}_\mathcal{C}}$. From Theorem 11.10 we derive, however, that backward continuation is unique for $\overline{\mathcal{M}_\mathcal{C}}$. But the decomposition (11.18) tells us that the possible backward continuation of $\varphi \in \overline{\mathcal{M}_\mathcal{C}}$ in $\mathcal{C}/\mathcal{N}(T(\alpha))$ does not necessarely belong to $\overline{\mathcal{M}_\mathcal{C}}$! In fact, the decomposition

$$\mathcal{C} = \overline{\mathcal{M}_\mathcal{C}} \oplus \mathcal{N}(T(\alpha)) \tag{11.21}$$

holds if and only if,

$$\overline{\mathcal{M}_\mathcal{C}} \cong \mathcal{C}/\mathcal{N}(T(\alpha)). \tag{11.22}$$

Or, in other words, if and only if for every $\varphi \in \overline{\mathcal{M}_\mathcal{C}}$, the possible backward continuation of φ in $\mathcal{C}/\mathcal{N}(T(\alpha))$ actually belongs to $\overline{\mathcal{M}_\mathcal{C}}$. Note that from (11.18) it follows that $\overline{\mathcal{M}_\mathcal{C}}$ is always dense in $\mathcal{C}/\mathcal{N}(T(\alpha))$ with respect to the quotient topology. The next example shows that the decomposition (11.21) does not hold in general.

EXAMPLE 11.11. Consider the scalar RFDE

$$\begin{aligned} \dot{x}(t) &= x(t-1) \qquad \text{for} \quad t \geq 0, \\ x_0 &= \varphi, \end{aligned} \tag{11.23}$$

where $\varphi \in C[-2,0]$. Note that, since the delay is chosen incorrectly the example is artificial, but indicates what happens in systems of RFDEs where more than one delay is involved.

Let $\{T(t)\}$ denote the $\mathcal{C}_0$-semigroup on $\mathcal{C} = C[-2,0]$ and let $\{\overline{T}(t)\}$ denote the $\mathcal{C}_0$-semigroup defined on $C[-1,0]$. Then, clearly

$$\mathcal{N}(T(1)) = \{\varphi \in \mathcal{C} : \operatorname{supp}(\varphi) \subset [-2,-1]\} \tag{11.24}$$

and

$$\mathcal{R}\big(T(1)\big) = \big\{\varphi \in \mathcal{C} : \overline{T}(t)\varphi(-1+\ \cdot\ ) = \varphi|_{[-1,0]}\big\}. \tag{11.25}$$

Consider an equivalence class $\varphi \in \mathcal{C}/\mathcal{N}\big(T(1)\big)$ of initial conditions. We can choose a representant $\chi \in \mathcal{C}$ for this equivalence class such that

$$\chi|_{[-1,0]} = \varphi$$

and

$$\overline{T}(1)\chi(-1+\ \cdot\ ) \neq \chi|_{[-1,0]}.$$

Then clearly $\chi \notin \overline{\mathcal{M}_\mathcal{C}} = \overline{\mathcal{R}\big(T(1)\big)}$. Hence

$$\mathcal{C} \neq \overline{\mathcal{M}_\mathcal{C}} \oplus \mathcal{N}\big(T(1)\big). \tag{11.26}$$

To show the equivalent statement for the Volterra convolution equation

$$x - \zeta * x = f, \tag{11.27}$$

where

$$\zeta(t) = \begin{cases} 0 & \text{if } t < 1; \\ 1 & \text{otherwise,} \end{cases}$$

we can use Theorem 7.12 to characterize $\overline{\mathcal{A}_\mathcal{F}} = \overline{\mathcal{R}\big(S(t)\big)}$. From this theorem it follows that

$$\overline{\mathcal{R}\big(S(t)\big)} = \big\{f \in \mathcal{F} : f(t) = f(1) \text{ for } t \geq 1\big\}. \tag{11.28}$$

Also, we clearly have

$$\mathcal{N}\big(S(t)\big) = \big\{f \in \mathcal{F} : f = x - \zeta * x,\ x \in C(\mathbf{R}_+),\ \operatorname{supp}(x) \subset [0,1]\big\}. \tag{11.29}$$

Therefore, if $f \in \overline{\mathcal{R}\big(S(t)\big)} \oplus \mathcal{N}\big(S(t)\big)$ then f is absolutely continuous on $[1,2]$. Hence

$$\mathcal{F} \neq \overline{\mathcal{R}\big(S(t)\big)} \oplus \mathcal{N}\big(S(t)\big). \tag{11.30}$$

Although the characterization of $\overline{\mathcal{M}_\mathcal{C}}$ (or $\overline{\mathcal{A}_\mathcal{C}}$) given by Theorem 11.7 is useful to determine $\overline{\mathcal{M}_\mathcal{C}}$ in concrete examples we shall also give a more abstract (equivalent) characterization. Let A denote the infinitesimal generator of $\{\,T(t)\,\}$. Recall from Theorem 10.13 the following characterization of $\mathcal{N}\big(T(\alpha)\big)$

$$\mathcal{N}\big(T(\alpha)\big) = \big\{\varphi \in \mathcal{C} : z \mapsto (zI - A)^{-1}\varphi \text{ is entire }\big\}.$$

In line with this result we have the following characterization of $\overline{\mathcal{M}_\mathcal{C}}$.

THEOREM 11.12.

$$\overline{\mathcal{M}_{\mathcal{C}}} = \{\varphi \in \mathcal{C} : z \mapsto (zI - A)^{-1}\varphi \text{ is } O(z^N) \text{ on the negative real axis } \}.$$

PROOF. From the representation for the resolvent, see Theorem 10.4, it follows that this result is a direct consequence of Theorem 11.7. ∎

The remaining part of this chapter will be devoted to a reformulation of the convergence results for the Fourier type series expansions into the RFDE framework. For the convergence results we have to restrict the class of kernels to SBV$[0, h]$ introduced in Chapter 4. Because of Theorem 10.1 and Theorem 10.8, the results will be a straightforward application of the results of Chapter 7.

Consider the following system of RFDEs

$$\begin{aligned} \dot{x}(t) &= \int_0^h d\zeta(\theta)x(t-\theta) \qquad \text{for} \quad t \geq 0, \\ x_0 &= \varphi, \end{aligned} \tag{11.31}$$

where $\varphi \in \mathcal{C}$ and $\zeta \in$ SBV$[0, h]$, i.e. all ζ_{ij} belong to SBV$[0, h]$ and there is at least one of the ζ_{ij} that jumps in h.
We then have the following convergence results for the spectral projection series.

THEOREM 11.13. *If $\varphi \in \mathcal{C}$ is such that*

$$\mathrm{N}(z\mathrm{adj}\Delta(z)(\varphi(0) + \int_0^h e^{-zt}dF\varphi(t))) \leq \mathrm{N}(\det \Delta(z)).$$

Then for every $\epsilon > 0$ the state $T(h+\epsilon)\varphi$ can be represented by a convergent spectral projection series

$$\lim_{N\to\infty} \|T(h+\epsilon)\varphi - \sum_{j=1}^N P_{\lambda_j}T(h+\epsilon)\varphi\| = 0. \tag{11.32}$$

THEOREM 11.14. *If $\varphi \in \mathcal{C}$, then for every $\epsilon > 0$ the state $T(nh+\epsilon)\varphi$ can be represented by a convergent spectral projection series*

$$\lim_{N\to\infty} \|T(nh+\epsilon)\varphi - \sum_{j=1}^N P_{\lambda_j}T(nh+\epsilon)\varphi\| = 0. \tag{11.33}$$

In general we do not know whether nh can be replaced by α in the above theorem, also we do not know whether Theorem 11.14 holds with no restriction on the kernel ζ. If

$$\mathrm{N}(z\,\mathrm{adj}\Delta(z)) \leq \mathrm{N}(\det \Delta(z)),$$

we can use the representation formula (10.3) for F to control the Newton polygon condition in Theorem 11.13 and we obtain the following corollary to Theorem 11.13.

COROLLARY 11.15. *If*

$$\mathrm{N}(z\,\mathrm{adj}\Delta(z)) \leq \mathrm{N}(\det \Delta(z))$$

and if $\varphi \in \mathcal{C}$ such that φ is locally of bounded variation. Then for every $\epsilon > 0$ the state $T(h+\epsilon)\varphi$ can be represented by a convergent spectral projection series.

REMARK 11.16. From the results of this chapter it follows that we can formulate the following necessary and sufficient condition for completeness of the system of generalized eigenfunctions:

$$\overline{\mathcal{M}_{\mathcal{C}}} = \mathcal{C} \text{ if and only if } \alpha = 0,$$

i.e. there are no small solutions.

REMARK 11.17. Delfour and Manitius introduce in their papers [8], [9], and [28] the concept of F-completeness. The generalized eigenspace $\mathcal{M}_{\mathcal{C}}$ is called F-complete if and only if

$$\overline{F\mathcal{M}_{\mathcal{C}}} = \overline{\mathcal{R}(F)},$$

where the linear operator F is defined by (10.3). From the results of this chapter it follows that we can formulate the following necessary and sufficient condition for F-completeness:

$$\overline{F\mathcal{M}_{\mathcal{C}}} = \overline{\mathcal{R}(F)} \text{ if and only if } \alpha \leq h,$$

i.e. all small solutions are in the kernel of F.

CHAPTER 12. PERTURBED DUAL SEMIGROUPS

In this chapter we are going to study the $\mathcal{C}_0$-semigroup associated with RFDE (10.1) in more detail. Recall from Chapter 10 that the RFDE (10.1) consists of a rule to extend the initial function. Starting from a given function defined on $[-h, 0]$ one obtains an extended function defined on $[-h, \infty)$ by solving the equation.

The dynamical systems point of view is to consider the given function on $[-h, 0]$ as the initial state and the part of the extended function between $t - h$ and t shifted back to the interval $[-h, 0]$ as the state at time t. Thus, there are two ingredients for the construction of the state at time t: extending the function and shifting it back. The first ingredient is specific for a particular delay equation, but the second is general, i.e. the same for all delay equations. Our approach will be to study the second ingredient first for the special case in which the extension rule is as simple as possible:

$$\begin{aligned} \dot{x}(t) &= 0 \qquad \text{for} \quad t \geq 0, \\ x_0 &= \varphi, \end{aligned} \tag{12.1}$$

where $\varphi \in \mathcal{C}$. Then we consider other extension rules as perturbations of this special case. It will turn out, that the functional analytic framework which we develop, extends and generalizes the results of Chapter 10. In particular, it associates a Volterra convolution equation with the dual semigroup $\{ T^*(t) \}$ and it is possible to formulate all the results of Chapter 11 in this framework.

It also gives a rigorous basis for the variation-of-constants formula. Recall from Chapter 9 the so-called fundamental matrix solution U. The existence of this solution is clear from the construction, but from equation (9.1) it follows that this solution corresponds to a discontinuous initial condition. Therefore, this solution does not belong to the state space $\mathcal{C}$. In Hale's book [17] we find that the variation-of-constants formula for the inhomogeneous equation

$$\begin{aligned} \dot{x}(t) &= \int_0^h d\zeta(\theta)x(t-\theta) + f(t) \qquad \text{for} \quad t \geq 0, \\ x_0 &= \varphi, \end{aligned} \tag{12.2}$$

involves the fundamental matrix solution U and reads

$$x(\varphi, f)(t) = x(\varphi, 0)(t) + \int_0^t U(t-s)f(s)ds \qquad \text{for} \quad t \geq 0. \tag{12.3}$$

Since U does not belong to $\mathcal{C}$, the formula becomes symbolic rather than functional analytic. In the functional analytic framework which we develop, however, the variation-of-constants formula has a rigorous basis in the weak $*$ sense. In this chapter we will follow the lines of a paper by Diekmann [12].

Let $\{T(t)\}$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on a Banach space X and let A denote its infinitesimal generator. The adjoint operators $\{T^*(t)\}$ form a semigroup on the dual space X^*. The semigroup $\{T^*(t)\}$ is weak$*$ continuous. But if we equip X^* with the usual norm topology, $\{T^*(t)\}$ need not be strongly continuous (unless X is reflexive). The operator A^*, the adjoint of A, is the weak$*$ generator of $\{T^*(t)\}$ but need not be densily defined. Let $X^\odot$ denote the norm closure of $\mathcal{D}(A^*)$, i.e.

$$X^\odot = \overline{\mathcal{D}(A^*)}, \tag{12.4}$$

then $X^\odot$ is the maximal invariant subspace on which $\{T^*(t)\}$ is strongly continuous. Let $\{T^\odot(t)\}$ denote the $\mathcal{C}_0$-semigroup on $X^\odot$ which is obtained by restriction of $\{T^*(t)\}$ and let $A^\odot$ denote its generator. Then $A^\odot$ is the part of A^* in $X^\odot$, i.e. the largest restriction of A^* with both domain and range in $X^\odot$ (see Hille and Phillips [20]). Repeating the same procedure we obtain a weak$*$ continuous semigroup $\{T^{\odot *}(t)\}$ on $X^{\odot *}$, the dual space of $X^\odot$, with weak$*$ generator $A^{\odot *}$. The Banach space X is called $\odot$-reflexive with respect to A if and only if X can be identified with

$$X^{\odot\odot} = \overline{\mathcal{D}(A^{\odot *})}$$

(in general X can be embedded as a subspace of $X^{\odot\odot}$). Let $\{T_0(t)\}$ be a $\mathcal{C}_0$-semigroup on X generated by A_0 and assume that X is $\odot$-reflexive with respect to A_0. We want to perturb the generator A_0 by a linear operator B, where B is bounded not as an operator from X into X, but as an operator from X into $X^{\odot *}$. To this end we consider the variation-of-constants equation for the perturbed semigroup

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(t-\tau)BT(\tau)x d\tau. \tag{12.5}$$

Here the integral has to be understood in the weak* sense, i.e.

$$\langle \int_0^t T_0^{\odot *}(t-\tau)BT(\tau)d\tau, x^{\odot}\rangle := \int_0^t \langle BT(\tau)x, T_0^{\odot}(t-\tau)x^{\odot}\rangle d\tau$$

for arbitrary $x^{\odot} \in X^{\odot}$. So in principle the integral takes values in $X^{\odot *}$ but one can show that in fact it takes values in the closed subspace $X^{\odot\odot} = X$. Within this setting the contraction argument applies with the result that (12.5) has a well-defined solution $\{T(t)\}$. Since it can be shown [8] that the spaces of strong continuity do not depend on the perturbation B we obtain, by duality and restriction, semigroups $\{T^*(t)\}$, $\{T^{\odot}(t)\}$, and $\{T^{\odot *}(t)\}$ on, respectively X^*, $X^{\odot}$, and $X^{\odot *}$. Similarly the domains of the weak* generators on the spaces X^* and $X^{\odot *}$ are independent of B. Therefore, we have the following theorem.

THEOREM 12.1. *The operator $Ax = A_0^{\odot *}x + Bx$ with*

$$\mathcal{D}(A) = \{x \in \mathcal{D}(A_0^{\odot *}) : A_0^{\odot *}x + Bx \in X\}$$

is the generator of a $\mathcal{C}_0$- semigroup $\{T(t)\}$ on X and the variation-of-constants formula (12.5) holds.

Next, let $\{T_0(t)\}$ be the $\mathcal{C}_0$-semigroup generated by the equation

$$\dot{x} = 0,$$
$$x_0 = \varphi,$$

considered as a delay equation on $\mathcal{C} = C[-h, 0]$. We shall first show that $\mathcal{C}$ is $\odot$-reflexive with respect to A_0.

The semigroup $\{T_0(t)\}$ is given by

$$\big(T_0(t)\varphi\big)(\theta) = \begin{cases} \varphi(t+\theta) & \text{for } t+\theta \le 0, \\ \varphi(0) & \text{for } t+\theta \ge 0, \end{cases} \tag{12.6}$$

on $X = \mathcal{C}$ and is generated by

$$A_0\varphi = \dot{\varphi}$$

with

$$\mathcal{D}(A_0) = \{\varphi \in \mathcal{C} : \dot{\varphi} \in \mathcal{C} \text{ and } \dot{\varphi}(0) = 0\}. \tag{12.7}$$

Let X^* be represented by NBV$[0, h]$ with the pairing

$$\langle f, \varphi\rangle = \int_0^h df(\tau)\varphi(-\tau).$$

Then

$$\big(T_0^*(t)f\big)(\sigma) = f(t+\sigma) \qquad \text{for} \quad \sigma \geq 0. \tag{12.8}$$

So

$$A_0^* f = \dot{f}$$

with

$$\mathcal{D}(A_0^*) = \big\{ f \in \mathrm{NBV}[0,h] : \dot{f} \in \mathrm{NBV}[0,h] \big\}. \tag{12.9}$$

Hence

$$\begin{aligned} X^{\odot} &= \overline{\mathcal{D}(A_0^*)} \\ &= \mathbf{R}^n \oplus AC_0 \\ &= \Big\{ f : f(t) = c + \int_0^t g(\tau)d\tau,\ g \in L^1[\mathbf{R}_+],\ \operatorname{supp}(g) \subset [0,h] \Big\}. \end{aligned}$$

It is sometimes convenient to work with the couple (c,g) to represent f. This amounts to representing $X^{\odot}$ by $\mathbf{R}^n \times L^1[0,h]$, with norm

$$\|(c,g)\| = |c| + \|g\|_1.$$

In these coordinates we have

$$T_0^{\odot}(t)(c,g) = \Big(c + \int_0^t g(\tau)d\tau, g(t+\,\cdot\,)\Big). \tag{12.10}$$

So the infinitesimal generator is defined by

$$A_0^{\odot}(c,g) = (g(0), \dot{g})$$

with

$$\mathcal{D}(A_0^{\odot}) = \big\{ (c,g) \in \mathbf{R}^n \times L^1[0,h] : \dot{g} \in L^1[0,h] \big\}. \tag{12.11}$$

Next we take the representation $X^{\odot *} = M_\infty = \mathbf{R}^n \times L^\infty[-h,0]$ with norm

$$\|(\alpha,\varphi)\| = \sup\big\{ |\alpha|, \|\varphi\|_\infty \big\},$$

and pairing

$$\langle (c,g), (\alpha,\varphi) \rangle = c\alpha + \int_0^h g(\tau)\varphi(-\tau)d\tau.$$

It follows that $T_0^{\odot *}(t)$ is the shift of the α-extended φ:

$$T_0^{\odot *}(t)(\alpha,\varphi) = (\alpha, \varphi_t),$$

where

$$\varphi_t(\theta) = \begin{cases} \varphi(t+\theta) & \text{for } t+\theta \leq 0, \\ \alpha & \text{for } t+\theta > 0. \end{cases}$$

So the infinitesimal generator is defined by

$$A_0^{\odot *}(\alpha, \varphi) = (0, \dot{\varphi})$$

with

$$\mathcal{D}(A_0^{\odot *}) = \{ (\alpha, \varphi) \in \mathbf{R}^n \times L^\infty[0,h] : \varphi \in \mathrm{Lip}(\alpha) \}, \qquad (12.12)$$

where $\mathrm{Lip}(\alpha)$ denotes the class of elements of L^∞ that contain a Lipschitz continuous function which assumes the value α at $\theta = 0$. Finally,

$$\begin{aligned} X^{\odot\odot} &= \overline{\mathcal{D}(A_0^{\odot *})} \\ &= \{ (\alpha, \varphi) \in \mathbf{R}^n \times L^\infty[0,h] : \varphi \in \mathrm{C}(\alpha) \}, \end{aligned}$$

where $\mathrm{C}(\alpha)$ denotes the class of elements of L^∞ that contain a continuous function which assumes the value α at $\theta = 0$. We can embed X into $X^{\odot *}$ and clearly, we can identify this embedding of X with $X^{\odot\odot}$. Therefore, we conclude that $\mathcal{C}$ is $\odot$-reflexive with respect to A_0 (a fact which also can be deduced from the compactness of $(\lambda I - A_0)^{-1}$ with $\lambda \in \rho(A_0)$). So far we have used the semigroup $\{T_0(t)\}$ to construct a dual space $X^{\odot *} = M_\infty$ in which $X = \mathcal{C}$ lies embedded. Next we are going to perturb the generator by changing the rule for the extension of the function. Define $B : \mathcal{C} \to M_\infty$ by

$$B\varphi = (\langle \zeta, \varphi \rangle, 0). \qquad (12.13)$$

From Theorem 12.1 it follows that for a given kernel ζ we have a $\mathcal{C}_0$-semigroup $\{T(t)\}$ on $\mathcal{C}$ generated by the operator A defined by

$$A\varphi = \dot{\varphi}$$

with domain

$$\mathcal{D}(A) = \{ \varphi \in \mathcal{C} : \dot{\varphi} \in \mathcal{C} \text{ and } \dot{\varphi}(0) = \langle \zeta, \varphi \rangle \}.$$

Recall from (10.8) that the same operator A generates the $\mathcal{C}_0$-semigroup associated with the RFDE (10.1) and consequently this semigroup can be given by the variation-of-constants formula (12.5). In fact all the results derived in Chapter 10 are straightforward applications of the functional analytic framework which we developed above.

Next we shall further study the specific perturbation B given by (12.13) and derive that solving the abstract variation-of-constants formula (12.5) is equivalent to solving a renewal equation. This result gives a beter understanding of the equivalence result presented in Theorem 10.1.

Let r_j^* be the j-th row of ζ and $r_j^{\odot *} = (e_j, 0)$, where e_j denotes the j-th unit column vector in $\mathbf{R}^n$. It is convenient to combine these into matrices

$$\begin{aligned} r^* &= \zeta, \\ r^{\odot *} &= (I, 0). \end{aligned}$$

Then

$$Bx = \sum_{j=1}^{n} \langle r_j^*, x \rangle r_j^{\odot *}. \tag{12.14}$$

Let Q denote a $n \times n$-matrix valued function with entries

$$q_{jk}(t) = \langle r_j^*, \int_0^t T_0^{\odot *}(\tau) r_k^{\odot *} d\tau \rangle.$$

A simple estimate shows that Q is Lipschitz continuous. As a consequence we have a representation of the form

$$Q(t) = \int_0^t K(\tau) d\tau. \tag{12.15}$$

From

$$T_0^{\odot *}(t) r^{\odot *} = (I, H(t + \cdot) I), \tag{12.16}$$

we derive that

$$\int_0^t T_0^{\odot *}(t) r^{\odot *} d\tau = \max(0, t + \cdot) I. \tag{12.17}$$

Hence,

$$Q(t) = \int_0^t d\zeta(\tau)(t - \tau). \tag{12.18}$$

Therefore,

$$K = \zeta. \tag{12.19}$$

Now define the n-vector $y(t) = (y_1(t), \ldots, y_n(t))$ by

$$y_j(t) = \langle r_j^*, T(t) x \rangle \qquad \text{for} \quad 1 \le j \le n, \tag{12.20}$$

where $T(t)x$ is the solution of equation (12.5). Then equation (12.5), relation (12.18) and a little technical calculation (to avoid undefined expressions) imply that y satisfies the renewal equation

$$y = h + \zeta * y, \tag{12.21}$$

where the vector-valued function $h = (h_1, \dots, h_n)$ is given by

$$h_j(t) = \langle r_j^*, T_0(t)x \rangle \qquad \text{for} \quad 1 \le j \le n. \tag{12.22}$$

Conversely, given any solution y of (12.21) with h of the form (12.22) we can recover $T(t)x$ from

$$T(t)x = T_0(t)x + \sum_{j=1}^{n} \int_0^t T_0^{\odot *}(t-\tau) r_j^{\odot *} y_j(\tau) d\tau. \tag{12.23}$$

It appears that solving equation (12.5) is reduced to solving (12.21)

Substituting (12.16) into (12.23) yields

$$\big(T(t)\varphi\big)(\theta) = \big(T_0(t)\varphi\big)(\theta) + \int_0^{\max\{0,t+\theta\}} y(\tau) d\tau. \tag{12.24}$$

So if we define

$$x(t\,;\varphi) = \big(T(t)\varphi\big)(0), \tag{12.25}$$

then

$$y(t) = \dot{x}(t\,;\varphi).$$

Moreover, equation (12.24) then implies that for $t + \theta \ge 0$

$$\begin{aligned} \big(T(t)\varphi\big)(\theta) &= \varphi(0) + \int_0^{t+\theta} y(\tau) d\tau \\ &= x(t+\theta;\varphi). \end{aligned} \tag{12.26}$$

Finally,

$$h(t) = \int_t^h d\zeta(\tau)\varphi(t-\tau) + \zeta(t)\varphi(0). \tag{12.27}$$

Therefore (12.21) reads

$$\dot{x}(t) = \int_0^t \zeta(\tau)\dot{x}(t-\tau) d\tau + \int_t^h d\zeta(\tau)\varphi(t-\tau) + \zeta(t)\varphi(0). \tag{12.28}$$

Thus from Theorem 2.7 (iii)

$$\dot{x}(t) = \int_0^h d\zeta(\theta) x(t-\theta) \qquad \text{for} \quad t \ge 0 \tag{12.29}$$

with initial condition

$$x_0 = \varphi. \tag{12.30}$$

And once more we arrive at the conclusion that the semigroup $\{T(t)\}$ is the solution semigroup corresponding to (12.29)-(12.30).

To conclude this chapter we will show how the present approach yields a suitable interpretation of the dual semigroup rather directly. And as an application of this interpretation we will prove the dual version of Theorem 11.10.

Let $B^* : X^{\odot} \to X^*$ denote the adjoint of B. The semigroup $\{T^{\odot}(t)\}$ satisfies the "adjoint" variation-of-constants formula

$$T^{\odot}(t)x^{\odot} = T_0^{\odot}(t)x^{\odot} + \int_0^t T_0^{\odot *}(t-\tau)B^*T^{\odot}(\tau)x^{\odot}\,d\tau. \tag{12.31}$$

So if

$$B^*x^{\odot} = \sum_{j=1}^{n} r_j^* \langle x^{\odot}, r_j^{\odot *}\rangle, \tag{12.32}$$

then the vector-valued function $z = (z_1, \ldots, z_n)$ defined by

$$z_j(t) = \langle T^{\odot}(t)x^{\odot}, r_j^{\odot *}\rangle \qquad \text{for} \quad 1 \le j \le n, \tag{12.33}$$

satisfies the "adjoint" renewal equation

$$z = \zeta^T * z + g, \tag{12.34}$$

where ζ^T denotes the transposed kernel and the forcing function

$$g = (g_1, \ldots, g_n)$$

is defined by

$$g_j(t) = \langle T_0^{\odot}(t)x^{\odot}, r_j^{\odot *}\rangle \qquad \text{for} \quad 1 \le j \le n. \tag{12.35}$$

The action of $r^{\odot *}$ corresponds to taking the limit from above in zero. So, from (12.8)

$$g(t) = \big(T_0^{\odot}(t)f\big)(0+) = f(t+) = f(t). \tag{12.36}$$

Therefore, according to (12.31) and (12.33), we have

$$\big(T_0^{\odot}(t)f\big)(\sigma) = f(t+\sigma) + \int_0^t \zeta^T(t-\tau+\sigma)z(\tau)\,d\tau. \tag{12.37}$$

On the other hand, from the results of Chapter 6 we may start from the renewal equation

$$z = \zeta^T * z + f, \tag{12.38}$$

where f is absolutely continuous and constant on $[h, \infty)$ and define a semigroup $\{ S(t;\zeta^T) \}$ by

$$z_t = \zeta^T * z_t + S(t;\zeta^T)f. \tag{12.39}$$

Thus a straightforward computation shows that $T^{\odot}(t) = S(t;\zeta^T)$. That is, we can associate with $\{ T^{\odot}(t) \}$ the Volterra convolution equation (12.38). And therefore, we can apply the results derived for the Volterra convolution equation in Chapter 6,7, and 8 to the $\mathcal{C}_0$-semigroup $\{ T^{\odot}(t) \}$.

Let δ denote the ascent of $\{ T^{\odot}(t) \}$, i.e.

$$\delta = \inf\{ t : \forall \epsilon > 0 : \mathcal{N}(T^{\odot}(t+\epsilon)) = \mathcal{N}(T^{\odot}(t)) \}.$$

From Theorem 8.2 it follows that δ is finite and

$$\delta = \epsilon(\zeta^T) - \sigma(\zeta^T),$$

where

$$\epsilon(\zeta^T) = nh - \mathrm{E}(\det \Delta(z;\zeta^T)) \tag{12.40}$$

and

$$\sigma(\zeta^T) = (n-1)h - \max_{1 \le i,j \le n} \mathrm{E}(\mathrm{adj}\Delta(z;\zeta^T)_{ij}). \tag{12.41}$$

Since,

$$\epsilon(\zeta^T) = \epsilon(\zeta) = \epsilon$$

and

$$\sigma(\zeta^T) = \sigma(\zeta) = \sigma,$$

we obtain from Theorem 8.2 the following theorem.

THEOREM 12.2. *The ascents of* $\{ T(t) \}$ *and* $\{ T^{\odot}(t) \}$ *are equal.*

Since all results of Chapter 7 apply to the Volterra convolution equation (12.38) we can state the dual version of Theorem 11.10.

THEOREM 12.3.

$$\overline{\mathcal{R}(T^{\odot}(\delta))} \cap \mathcal{N}(T^{\odot}(\delta)) = \{0\}.$$

As a corollary we have the equivalent result for $\{ T^*(t) \}$ with respect to the weak$*$ topology.

COROLLARY 12.4.

$$\overline{\mathcal{R}(T^*(\delta))}^{w*} \cap \mathcal{N}(T^*(\delta)) = \{0\}.$$

PROOF. Since $\overline{\mathcal{R}(T^*(\delta))}^{w*}$ and $\mathcal{N}(T^*(\delta))$ are invariant under the resolvent operator $(\lambda I - A^*)$ with $\lambda \in \rho(A^*)$, the intersection

$$\overline{\mathcal{R}(T^*(\delta))}^{w*} \cap \mathcal{N}(T^*(\delta)) \tag{12.42}$$

is invariant under the resolvent operator. Moreover, this invariance property implies that the ascent of $\{T^*(t)\}$ equals δ as well. So if the intersection (12.42) is not trivial we can assume that the intersection within $\mathcal{D}(A^*)$ is not trivial. Since weak$*$-closed implies norm-closed, we obtain

$$\overline{\mathcal{R}(T^{\odot}(\delta))} = X^{\odot} \cap \overline{\mathcal{R}(T^*(\delta))}^{w*} \tag{12.43}$$

and also

$$\mathcal{N}(T^{\odot}(\delta)) = X^{\odot} \cap \mathcal{N}(T^*(\delta)). \tag{12.44}$$

So, the corollary follows from Theorem 12.3. ∎

Finally, as an application of the above results we prove the following theorem.

THEOREM 12.5.

$$\mathcal{C} = \overline{\overline{\mathcal{M}_{\mathcal{C}}} \oplus \mathcal{N}(T(\alpha))}.$$

PROOF. We shall prove that the set of all bounded functionals on $\mathcal{C}$ that vanish on $\overline{\mathcal{M}_{\mathcal{C}}} \oplus \mathcal{N}(T(\alpha))$ is trivial. Notation

$$(\overline{\mathcal{M}_{\mathcal{C}}} \oplus \mathcal{N}(T(\alpha)))^{\perp} = \{0\}. \tag{12.45}$$

Since $\alpha = \delta$, we have by Theorem 11.6

$$\overline{\mathcal{M}_{\mathcal{C}}} = \overline{\mathcal{R}(T(\delta))}. \tag{12.46}$$

From the equalities

$$(\mathcal{R}(T(\delta)))^{\perp} = \mathcal{N}(T^*(\delta)) \tag{12.47}$$

and

$$(\mathcal{N}(T(\delta)))^{\perp} = \overline{\mathcal{R}(T^*(\delta))}^{w*}, \tag{12.48}$$

we derive

$$\begin{aligned}(\overline{\mathcal{M}_{\mathcal{C}}} \oplus \mathcal{N}(T(\alpha)))^{\perp} &= (\mathcal{R}(T(\delta)))^{\perp} \cap (\mathcal{N}(T(\delta)))^{\perp} \\ &= \mathcal{N}(T^*(\delta)) \cap \overline{\mathcal{R}(T^*(\delta))}^{w*}. \end{aligned} \tag{12.50}$$

Therefore, the theorem follows from Corollary 12.4. ∎

CHAPTER 13. EXAMPLES

In this tract we have studied linear delay equations through the Laplace transform. We first considered the asymptotic estimates and then the Fourier type (or spectral projection) series expansions for solutions (or states). Finally, we characterized the closure of the generalized eigenspace. We emphasize that from the practical point of view this last characterization is very important since it yields, at least for differential-difference equations, relations for $\overline{\mathcal{M}_{\mathcal{C}}}$ or $\overline{\mathcal{A}_{\mathcal{F}}}$. This means that one can analyse the convergence properties of the spectral projection series when the state φ is restricted to $\overline{\mathcal{M}_{\mathcal{C}}}$ and this results in much stronger convergence results.

Given a linear autonomous delay equation with $\mathcal{C}_0$-semigroup $\{T(t)\}$ the idea is to follow the procedure: First we use the exponential type calculus of Chapter 4 to calculate the ascent α. When $\alpha = 0$, completeness of the system of generalized eigenfunctions holds and $\mathcal{C}$ is the proper state space to study the equation. When $\alpha > 0$ - i.e. there exist small solutions - restrict the state space $\mathcal{C}$ to $\overline{\mathcal{M}_{\mathcal{C}}}$, or equivalently, the forcing space $\mathcal{F}$ to $\overline{\mathcal{A}_{\mathcal{F}}}$ and find the relations for $\overline{\mathcal{A}_{\mathcal{F}}}$.

Next we analyse the analytic continuation of the Laplace transform of the solution and use the relations for $\overline{\mathcal{A}_{\mathcal{F}}}$ to analyse the convergence properties of the Fourier type series expansion when the forcing function f is restricted to $\overline{\mathcal{A}_{\mathcal{F}}}$.

Consider the following example.

EXAMPLE 13.1. (The delayed friction force model).

$$\ddot{x}(t) + a\dot{x}(t) + b\dot{x}(t-h) + cx(t) = 0. \tag{13.1}$$

To reduce this system to a first order system introduce the variables

$$\begin{cases} x_1 = x, \\ x_2 = \dot{x}. \end{cases} \tag{13.2}$$

Then (13.1) becomes

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -cx_1(t) - ax_2(t) - bx_2(t-h). \end{aligned} \tag{13.3}$$

The characteristic matrix of (13.3) is given by

$$\Delta(z) = \begin{pmatrix} z & -1 \\ c & z + a + be^{-hz} \end{pmatrix} \tag{13.4}$$

and

$$\det \Delta(z) = z^2 + (a + be^{-hz})z + c. \tag{13.5}$$

The matrix of cofactors equals

$$\operatorname{adj}\Delta(z) = \begin{pmatrix} z + a + be^{-hz} & 1 \\ -c & z \end{pmatrix}. \tag{13.6}$$

Hence

$$\mathrm{N}(z\operatorname{adj}\Delta(z)) \leq \mathrm{N}(\det \Delta(z)) \tag{13.7}$$

and we can apply Corollary 11.15 to conclude that for every $\epsilon > 0$ and $\varphi \in \mathcal{C}$ such that φ is locally of bounded variation the state $T(h+\epsilon)\varphi$ can be represented by a convergent spectral projection series

$$\lim_{N\to\infty} \|T(h+\epsilon)\varphi - \sum_{j=1}^{N} P_{\lambda_j} T(h+\epsilon)\varphi\| = 0, \tag{13.8}$$

where $\lambda_1, \ldots$ denote the zeros of $\det \Delta(z)$. From (13.5) and (13.6) we deduce that the ascent α of the $\mathcal{C}_0$-semigroup $\{\, T(t) \,\}$ associated with (13.3) equals

$$\alpha = h. \tag{13.9}$$

Indeed,

$$\mathcal{N}\big(T(h)\big) = \big\{\varphi \in \mathcal{C} : \varphi_1(0) = 0 \text{ and } \varphi_2 \equiv 0\big\}. \tag{13.10}$$

To characterize $\overline{\mathcal{M}_{\mathcal{C}}}$ we first characterize $\overline{\mathcal{A}_{\mathcal{F}}}$ (and then use that $G\overline{\mathcal{A}_{\mathcal{F}}} = \overline{\mathcal{M}_{\mathcal{C}}}$, where G is defined by (10.2)). Because of Theorem 7.12 we have to solve

$$\begin{aligned} \mathrm{E}((z + a + be^{-zh})zL\{f_1\}(z) + zL\{f_2\}(z)) &\leq h, \\ \mathrm{E}(-czL\{f_1\}(z) + z^2L\{f_2\}(z)) &\leq h, \end{aligned} \tag{13.11}$$

where for $1 \leq j \leq 2$

$$zL\{f_j\}(z) = f_j(h) + z\int_0^h e^{-zt}(f_j(t) - f_j(h))dt.$$

Hence

$$\overline{\mathcal{A}}_{\mathcal{F}} = \{ f \in \mathcal{F} : f_1(t) = f_1(h) \text{ for } t \geq 0 \}. \tag{13.12}$$

To find $\mathcal{N}(S(h))$ we have to solve the following equation

$$\Delta(z)L\{x\}(z) = zL\{f\}(z),$$

where x is a small solution of the system (13.3). Thus we have to solve the system

$$\begin{aligned} z\int_0^\infty e^{-zt}x_1(t)dt &= z\int_0^\infty e^{-zt}f_1(t)dt, \\ c\int_0^\infty e^{-zt}x_1(t)dt &= z\int_0^\infty e^{-zt}f_2(t)dt, \end{aligned} \tag{13.13}$$

where $x_1(t) = 0$ for $t \geq h$. Hence

$$\mathcal{N}(S(h)) = \{ f \in \mathcal{F} : f_1(h) = 0 \text{ and } f_2(t) = c\int_0^t f_1(s)ds \text{ for } t \leq h \}.$$

Therefore we have proved the following decomposition

$$\mathcal{F} = \mathcal{N}(S(h)) \oplus \overline{\mathcal{A}}_{\mathcal{F}} \tag{13.14}$$

given by

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t) - f_1(h) \\ c\int_0^t f_1(s)ds \end{bmatrix} + \begin{bmatrix} f_1(h) \\ f_2(t) - c\int_0^t f_1(s)ds \end{bmatrix}. \tag{13.15}$$

Or, equivalently,

$$\mathcal{C} = \mathcal{N}(T(h)) \oplus \overline{\mathcal{M}}_{\mathcal{C}}.$$

Thus we can restrict the state space $\mathcal{C}$ to $\overline{\mathcal{M}}_{\mathcal{C}}$ to obtain a minimal invariant subspace for (13.3). Next we will analyse what we can say about the convergence of the Fourier type series when the forcing function f is restricted to $\overline{\mathcal{A}}_{\mathcal{F}}$. From Theorem 7.3 and the characterization (13.12) we conclude that for $f \in \overline{\mathcal{A}}_{\mathcal{F}}$ such that f is locally of bounded variation the Fourier type series always converges. Therefore we derive that for $\varphi \in \overline{\mathcal{M}}_{\mathcal{C}}$ and $\epsilon > 0$ the state $T(h+\epsilon)\varphi$ can be represented by a convergent spectral projection series, i.e. we do not need the locally of bounded variation condition on φ. Furthermore, if we assume the locally of bounded variation condition on φ, then the state $T(\epsilon)\varphi$ can be represented by a convergent spectral projection series for every $\epsilon > 0$.

Similarly one can discuss the following example. Here one needs the techniques rather than the theorems of Chapter 7.

EXAMPLE 13.2. (The delayed restoring force model).

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + cx(t-h) = 0. \tag{13.16}$$

To reduce this system to a first order system introduce the variables

$$\begin{cases} x_1 = x, \\ x_2 = \dot{x}. \end{cases}$$

Then (13.16) becomes

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -bx_1(t) - cx_1(t-h) - ax_2(t) \end{aligned} \tag{13.17}$$

The characteristic matrix of (13.17) is given by

$$\Delta(z) = \begin{pmatrix} z & -1 \\ b + ce^{-hz} & z + a \end{pmatrix} \tag{13.18}$$

and

$$\det \Delta(z) = z^2 + az + b + ce^{-hz}. \tag{13.19}$$

The matrix of cofactors equals

$$\operatorname{adj}\Delta(z) = \begin{pmatrix} z + a & 1 \\ -b - ce^{-hz} & z \end{pmatrix}. \tag{13.20}$$

Remark that in this example (7.22) or (7.27) do not hold and we can not apply the results of Chapter 7 directly. Therefore, we shall first characterize $\overline{\mathcal{A}}_{\mathcal{F}}$. From (13.19) it follows that Lemma 6.5 holds for $|z| > C_1|e^{-zh/2}|$. Hence, we can choose the contours C_l such that (7.7) becomes

$$(\omega_l^2 + \gamma_l^2)^{\frac{1}{2}} = C_1 e^{-\gamma_l h/2}. \tag{13.21}$$

From Theorem 7.12 it follows that we can characterize $\overline{\mathcal{A}}_{\mathcal{F}}$:

$$\overline{\mathcal{A}}_{\mathcal{F}} = \{ f \in \mathcal{F} : f_1(t) = f_1(h) \text{ for } t \geq 0 \} \tag{13.22}$$

and as in the above example we can prove the following decomposition

$$\mathcal{F} = \mathcal{N}\big(S(h)\big) \oplus \overline{\mathcal{A}}_{\mathcal{F}}.$$

Thus we can restrict the forcing space $\mathcal{F}$ to $\overline{\mathcal{A}}_{\mathcal{F}}$ to obtain a minimal invariant subspace. Next we will analyse what we can say about the convergence of the Fourier type series when the forcing function f is restricted to $\overline{\mathcal{A}_{\mathcal{F}}}$. Then

$$|H_f(z)| = O(|z|) \qquad \text{for} \quad z \in C_l \tag{13.23}$$

as $l \to \infty$. Hence from (13.21), the proof of Theorem 7.4 and the characterization for $\overline{\mathcal{A}_{\mathcal{F}}}$ we conclude that the Fourier type series converges uniform for $t \geq \frac{1}{2}h + \epsilon$, if f is locally of bounded variation and uniform for $t \geq h + \epsilon$ for arbitrary $f \in \overline{\mathcal{A}_{\mathcal{F}}}$.

Therefore we derive that for $\varphi \in \overline{\mathcal{M}_{\mathcal{C}}}$ and $\epsilon > 0$ the state $T(h+\epsilon)\varphi$ can be represented by a convergent spectral projection series. Furthermore, if we assume in addition the locally of bounded variation condition on φ, then the state $T(\frac{1}{2}h+\epsilon)\varphi$ can be represented by a convergent spectral projection series for every $\epsilon > 0$.

Now consider an example which has non-trivial small solutions.

EXAMPLE 13.3. Consider the following differential-difference equation

$$\begin{aligned} \dot{x}_1(t) &= x_1(t-1) - x_2(t-\frac{1}{4}) + x_2(t-1), \\ \dot{x}_1(t) &= -x_1(t-1) - x_1(t-\frac{1}{4}) - x_2(t-1). \end{aligned} \tag{13.24}$$

The characteristic matrix of (13.24) is given by

$$\Delta(z) = \begin{pmatrix} z - e^{-z} & e^{-\frac{1}{4}z} - e^{-z} \\ e^{-\frac{1}{4}z} + e^{-z} & z + e^{-z} \end{pmatrix} \tag{13.25}$$

and

$$\det \Delta(z) = z^2 - e^{-\frac{1}{2}z}. \tag{13.26}$$

The matrix of cofactors equals

$$\mathrm{adj}\Delta(z) = \begin{pmatrix} z + e^{-z} & -(e^{-\frac{1}{4}z} - e^{-z}) \\ -(e^{-\frac{1}{4}z} + e^{-z}) & z - e^{-z} \end{pmatrix}. \tag{13.27}$$

From (13.26) and (13.27) we deduce that the ascent α of the $\mathcal{C}_0$-semigroup $\{T(t)\}$ associated with (13.24) equals

$$\alpha = \frac{3}{2}. \tag{13.28}$$

Thus the system (13.24) has non-trivial small solutions and the Newton polygon condition (7.29) does not hold. Therefore we shall first charcterize $\overline{\mathcal{A}_{\mathcal{F}}}$. From (13.19) it follows that Lemma 6.5 holds for $|z| > C_1|e^{-zh/2}|$. Hence, we can choose the contours C_l such that (7.7) becomes

$$(\omega_l^2 + \gamma_l^2)^{\frac{1}{2}} = C_1 e^{-\gamma_l h/4}. \tag{13.29}$$

To characterize $\overline{\mathcal{A}_{\mathcal{F}}}$ we have to solve the system

$$\begin{aligned} &\mathrm{E}((z+e^{-z})zL\{f_1\}(z)-(e^{-\frac{1}{4}z}-e^{-z})zL\{f_2\}(z)) \leq \frac{1}{2}, \\ &\mathrm{E}(-(e^{-\frac{1}{4}z}+e^{-z})zL\{f_1\}(z)+(z-e^{-z})zL\{f_2\}(z)) \leq \frac{1}{2}, \end{aligned} \tag{13.30}$$

where for $1 \leq j \leq 2$

$$zL\{f_j\}(z) = f_j(1) + z\int_0^1 e^{-zt}(f_j(t)-f_j(1))dt.$$

Hence

$$\overline{\mathcal{A}_{\mathcal{F}}} = \{f \in \mathcal{F} : f_1 = -f_2 \text{ and } f_1(t) = f_1(\tfrac{1}{4}) \text{ for } t \geq \tfrac{1}{4}\}. \tag{13.31}$$

Next we will analyse what we can say about the convergence of the Fourier type series when the forcing function f is restricted to $\overline{\mathcal{A}_{\mathcal{F}}}$. Then

$$|H_f(z)| = O(|z|) \qquad \text{for} \quad z \in C_l \tag{13.32}$$

as $l \to \infty$. Hence from (13.21), the proof of Theorem 7.4 and the characterization for $\overline{\mathcal{A}_{\mathcal{F}}}$ we conclude that the Fourier type series converges uniform for $t \geq \frac{1}{4}h+\epsilon$, if f is locally of bounded variation and uniform for $t \geq \frac{1}{2}h+\epsilon$ for arbitrary $f \in \overline{\mathcal{A}_{\mathcal{F}}}$.

Therefore we derive that for $\varphi \in \overline{\mathcal{M}_{\mathcal{C}}}$ and $\epsilon > 0$ the state $T(\frac{1}{2}h+\epsilon)\varphi$ can be represented by a convergent spectral projection series. Furthermore, if we assume the locally of bounded variation condition on φ, then the state $T(\frac{1}{4}h+\epsilon)\varphi$ can be represented by a convergent spectral projection series for every $\epsilon > 0$.

Note that in the above examples for arbitrary $\varphi \in \mathcal{C}$, the spectral projection series for the state $T(t_0)\varphi$ converges for $t_0 > \alpha$ in Example 13.1, for $t_0 > \alpha + \frac{1}{2}h$ in Example 13.2 and for $t_0 > \alpha + \frac{1}{4}h$ in Example 13.3. Of course we do not know how sharp our estimtes are in the above examples, but the results indicate that in general one can not replace nh by α in Theorem 7.5.

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