CWI Tracts

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CWI Tract 5

Intertwining functions on compact Lie groups

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CHAPTER 0

INTRODUCTION

At the end of the nineteenth century special functions, such as Jacobi polynomials, were studied mainly by using analytic methods. Starting off from one defining property one obtained series expansions, functional equations, orthogonality relations, integral representations, differential equations, etc. by analytic manipulation. Note that each of these expansions and relations represents a property that could itself be used as a definition. A summary of many such results, proved by analytic methods, can be found in ERDÉLY [5].

For instance an addition formula for Gegenbauer polynomials was obtained by Gegenbauer in 1893 in a purely analytic way, cf. [5,3.15(19)]. But no such formula for Jacobi polynomials of general order was known. The first proof of an addition formula for Jacobi polynomials was given by KOORNWINDER (cf.[19]) by using group theoretic interpretations for Jacobi polynomials. See also VILENKIN & ŠAPIRO [30] for this subject. For certain values of the parameters Jacobi polynomials can be interpreted as complex spherical harmonics: restrictions of bihomogeneous polynomials of a certain bidegree to the sphere $S^{2n-1} \subset \mathbb{C}^n$. Now the addition formula for those values of the parameters for which this interpretation holds follows from (highly nontrivial) analysis on the sphere, and the general case is proved by using differentiation and analytic continuation. This example shows the strength of the combination of group theory and special functions. (Surprisingly, the case of Legendre polynomials was originally also treated by means of "group theory": Legendre's proof of the addition formula used potential theory, cf. ASKEY [1]).

Let (U,K) be a *Riemannian symmetric pair* of the compact type of rank one. That is, U is a compact connected semisimple Lie group, K a closed subgroup of U such that there exists an involutive automorphism θ of U with $(U_\theta)_0 \subset K \subset U_\theta$, and the -1 eigenspace of $d\theta$ in $\mathfrak{u}$ (the Lie algebra of U) contains a one dimensional maximal abelian subalgebra. If (U,K) is a Riemannian symmetric pair, then the homogeneous space U/K is a Riemannian symmetric space. In general, the dimension of a maximal abelian subalgebra in the -1 eigenspace of $d\theta$ in $\mathfrak{u}$ is called the *rank* of U/K.

Let $\mathbb{D}(U/K)$ be the algebra of all U-invariant differential operators on U/K. A function φ on U/K is called a *spherical function* if φ satisfies the following conditions:

(0.1)

(1) $\varphi(eK) = 1$,

(2) φ is left K-invariant,

(3) $D\varphi = \lambda_D \varphi$ for each $D \in \mathbb{D}(U/K)$ $(\lambda_D \in \mathbb{C})$.

If U/K has rank one, then $\mathbb{D}(U/K)$ consists of all polynomials in the Laplace-Beltrami operator on U/K, where the Laplace-Beltrami operator is the analogue of the Laplacian for a symmetric space. Thus a spherical function on a Riemannian symmetric space of rank one is a K-invariant eigenfunction of the Laplace-Beltrami operator.

Now CARTAN [3] proved that if U/K has rank one, then the spherical functions on U/K can be considered as *Jacobi polynomials*. These are polynomials, orthogonal on the interval $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. In the case of spherical functions on a rank one symmetric space the parameters α and β are certain half integers. The orthogonality follows from the fact that the spherical functions are also matrix coefficients of certain finite dimensional representations of U, and these matrix coefficients are orthogonal with respect to the invariant measure on U/K (the "orthogonality relations of Schur"). In this model the weight function thus corresponds to the invariant measure on U/K, and this measure gave rise to Jacobi polynomials. In this way one obtains a tool to prove formulas for Jacobi polynomials which extends the above mentioned method of complex spherical harmonics.

Besides the rank one symmetric spaces, it was known for two more examples of Riemannian symmetric spaces of the compact type that the spherical functions gave rise to orthogonal polynomials: The Koornwinder polynomials on certain symmetric spaces of rank two, cf. KOORNWINDER [20], and Grassmann manifolds of general rank, cf. JAMES & CONSTANTINE [18].

Koornwinder polynomials are orthogonal polynomials in two variables on the region $\Omega := \{(\xi,\eta) \in \mathbb{R}^2 \mid \eta > 0,\ 1-\xi+\eta > 0,\ \xi^2-4\eta < 0,\ 0 < \xi < 2\}$ with respect to the weight function $\eta^{\alpha}(1-\xi+\eta)^{\beta}(\xi^2-4\eta)^{\gamma}$. Then for certain values of the parameters α,β,γ the Koornwinder polynomials are spherical functions on compact symmetric spaces of rank two with a root system of type BC_2, where the restricted roots $\alpha_1, 2\alpha_1$ and α_2 have multiplicities $2\alpha-2\beta$, $2\beta+1$ and $2\gamma+1$, respectively.

James and Constantine proved that the spherical functions on the Grassmann manifold $O(p+q)/O(p)\times O(q)$ can be considered as orthogonal polynomials on the region $\Omega := \{(y_1,\dots,y_p) \in \mathbb{R}^p \mid 1 \geq y_1 \geq \dots \geq y_p \geq 0\}$ with respect to the weight function $\Pi_{i=1}^{p}(1-y_i)^{\frac{1}{2}(q-p-1)} y_i^{-\frac{1}{2}} \Pi_{i<j}(y_i-y_j)$. (In fact they even proved more, but that subject will be discussed later.)

In [31] Vretare generalized these results to Riemannian symmetric spaces of the compact type of general rank. Let U/K be a Riemannian symmetric space of the compact type of rank ℓ. Then the spherical functions on U/K can be considered as orthogonal polynomials in ℓ variables. For the proof the structure theory for compact Lie groups was needed, and the orthogonality was obtained by means of a translation of the Schur orthogonality relations. Since these results were the basis for this thesis we shall briefly review Vretare's method here.

So let $\mathfrak{u}$ be as before the Lie algebra of U. Write, by abuse of notation, also θ for the differential of θ. Thus θ is an involution of $\mathfrak{u}$. Let $\mathfrak{k}$ be the +1 eigenspace of θ in $\mathfrak{u}$, then $\mathfrak{k}$ is the Lie algebra of K, and let $i\mathfrak{p}$ be the -1 eigenspace of θ in $\mathfrak{u}$. Then $\mathfrak{u}$ decomposes as $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Let $\mathfrak{g}_c$ be the natural complexification of $\mathfrak{u}$, and put $\mathfrak{g} := \mathfrak{k} + \mathfrak{p}$. Then $\mathfrak{g}$ is a real Lie algebra for which the corresponding Lie group is noncompact; $\mathfrak{g}$ is called the *dual* of $(\mathfrak{u},\theta)$.

For $X \in \mathfrak{g}$ define the linear operator ad(X) on $\mathfrak{g}$ by

$$\mathrm{ad}(X)Y := [X,Y] \qquad (Y\in\mathfrak{g}).$$

The bilinear form B_θ on $\mathfrak{g} \times \mathfrak{g}$ defined by

$$B_\theta(X,Y) := -\mathrm{tr}(\mathrm{ad}X\,\mathrm{ad}\theta Y) \qquad (X,Y\in\mathfrak{g})$$

defines an inner product on $\mathfrak{g}$. Choose a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$, then the linear operators ad(X) $(X\in\mathfrak{a})$ on $\mathfrak{g}$ are symmetric, hence they can be simultaneously diagonalized. Therefore, for a real linear form λ on $\mathfrak{a}$, put

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid \operatorname{ad}(H)X = \lambda(H)X \quad \text{for all} \quad H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\lambda \neq (0)$ and $\lambda \neq 0$, λ is called a *root* of the pair $(\mathfrak{g},\mathfrak{a})$. Let Σ be the set of all roots of the pair $(\mathfrak{g},\mathfrak{a})$. Σ is called a *root system*.

A set of roots $\{\alpha_1,\ldots,\alpha_\ell\}$ in a root system Σ is called a *base* of Σ if $\{\alpha_1,\ldots,\alpha_\ell\}$ is a basis of span(Σ) such that each root $\beta \in \Sigma$ can be written as $\beta = \Sigma_{i=1}^{\ell} m_i\alpha_i$ ($m_i \in \mathbb{Z}$) with either all m_i nonnegative or all m_i nonpositive.

The restriction of B_θ to $\mathfrak{a}$ induces an inner product on $\mathfrak{a}^*$, which we shall denote by $(\cdot,\cdot)$. Choose a base $\{\alpha_1,\ldots,\alpha_\ell\}$ of Σ, and let $\mu_1,\ldots,\mu_\ell \in \mathfrak{a}^*$ be such that $(\mu_i,\alpha_j) = 0$ if $i \neq j$, and $(\mu_i,\alpha_i)(\alpha_i,\alpha_i)^{-1} = 2$ or 1 according to whether $2\alpha_i$ is a root or not. (If α is a root, then the only possible multiples of α which are also roots are $\pm\frac{1}{2}\alpha, \pm\alpha, \pm 2\alpha$). Let $\leq$ be the partial ordering on $\mathfrak{a}^*$ defined by $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1 = \Sigma_{i=1}^{\ell} m_i\alpha_i$ ($m_i \in \mathbb{Z}$) with all m_i nonnegative ($\lambda_1,\lambda_2 \in \mathfrak{a}^*$).

Let π be a finite dimensional irreducible representation of $\mathfrak{u}$ in a vector space V. For any $\lambda \in \mathfrak{a}^*$ put

$$V_\lambda := \{v \in V\colon \pi(H)v = \lambda(H)v \text{ for all } H \in i\mathfrak{a}\},$$

where $\lambda(iH) := i\lambda(H)$ ($H \in \mathfrak{a}$). If $V_\lambda \neq (0)$ λ is called a *(restricted) weight* of π, and V_λ is then called the weight subspace corresponding to the weight λ. Because ad($i\mathfrak{a}$) acts in a semisimple way on V_λ we have $V = \Sigma_{\lambda\in\mathfrak{a}^*} V_\lambda$ (direct sum).

The representation π of U is said to be of *class* 1 if there exists a nonzero vector $e \in V$ which is left fixed by K, i.e. $\pi(k)e = e$ for all $k \in K$. By a theorem of Cartan-Helgason (cf. WARNER [33, Theorem 3.3.1.1]) the representations of class 1 are parametrized by their highest weight, and precisely all $\lambda = \Sigma_{i=1}^{\ell} m_i\mu_i$ ($m_i \in \mathbb{Z}$) with all m_i nonnegative do occur as highest weights. Here highest is meant to be with respect to the partial ordering $\leq$. We shall identify the set of all $\Sigma_{i=1}^{\ell} m_i\mu_i$ as above with the lattice $\mathbb{Z}_+^{\ell}$ of all ℓ-tuples $(m_1,\ldots,m_\ell)$ of nonnegative integers m_i.

We shall now indicate how the spherical functions can be considered as matrix coefficients of representations of U of class 1. For $\lambda \in \mathbb{Z}_+^{\ell}$ let π_λ be the corresponding representation of class 1. Let $(\cdot|\cdot)$ be an inner product in the representation space $V(\lambda)$ according to which π_λ is unitary. The K-fixed vector $e \in V(\lambda)$ is unique up to a constant factor. Choose it such that

$(e|e) = 1$. Then the function φ_λ on U defined by

(0.2) $\qquad \varphi_\lambda(x) := (e|\pi_\lambda(x)e) \qquad (X \in U)$

is a spherical function on U. Here we identify functions on U/K with right K-invariant functions on U. Moreover, if φ is a spherical function on U, then there exists a $\lambda \in \mathbb{Z}_+^\ell$ such that $\varphi = \varphi_\lambda$.
Put $A := \exp \mathfrak{a}$. Then because of the *Cartan decomposition*

(0.3) $\qquad U = KAK$

the spherical functions are completely determined by their restriction to A. By means of an induction process with respect to a total ordering on $\mathbb{Z}_+^\ell$ (which we shall not specify here) we obtain that the spherical function φ_λ ($\lambda = \Sigma\, m_i\mu_i \in \mathbb{Z}_+^\ell$) is a polynomial in the "lowest" spherical functions $\varphi_{\mu_1},\ldots,\varphi_{\mu_\ell}$. Hence $\varphi_\lambda \circ F^{-1}$ is a polynomial on $\Omega := F(\mathfrak{a})$, where F is defined by

(0.4) $\qquad F(H) := (\varphi_{\mu_1}(\exp H),\ldots,\varphi_{\mu_\ell}(\exp H)) \qquad (H \in i\mathfrak{a})$.

Because of the fact that for $\lambda_1,\lambda_2 \in \mathbb{Z}_+^\ell$ $\varphi_{\lambda_1} = \varphi_{\lambda_2}$ if and only if π_{λ_1} and π_{λ_2} are equivalent (which is the case if and only if $\lambda_1 = \lambda_2$) the orthogonality for the φ_λ will follow from the Schur orthogonality relations for different representations of U. This gives the following weight function on Ω:

(0.5) $$w(F(H)) := \Big| \prod_{\alpha\in\Sigma^+} \sin^{m_\alpha}\alpha(iH) \prod_{\substack{\alpha\in\Sigma^+\\ 2\alpha\notin\Sigma^+}} \sin^{-1}\alpha(iH)\Big| \quad (H\in i\alpha).$$

Here $\Sigma^+ \subset \Sigma$ is the positive system defined by $\beta \in \Sigma^+$ if $\beta = \Sigma_{i=1}^\ell m_i\alpha_i$ $(m_i \in \mathbb{Z})$ with all m_i nonnegative $(\beta\in\Sigma)$, and $m_\alpha := \dim \mathfrak{g}_\alpha$ is the multiplicity of $\alpha \in \Sigma$. Observe that the first part in the right hand side of (0.5) is just the Jacobian which occurs in the integral formula for the Cartan decomposition (0.3), cf. HELGASON [10, Proposition X.1.19]. The second part in the right hand side of (0.5) is the Jacobian of the mapping $F\colon \mathfrak{a} \to \mathbb{C}^\ell$ defined by (0.4).

Now Vretare obtained the following result.

THEOREM 0.1 (VRETARE [31]). *The spherical functions on* U/K *can be considered as orthogonal polynomials with respect to the positive weight function* w, *defined on the region* Ω.

Via Theorem 0.1 one obtains, for each Dynkin diagram, a set of orthogonal polynomials, labeled by a (discrete) set of parameters, namely the multiplicities of the roots. By letting the parameters take arbitrary real values one obtains highly nontrivial examples of families of orthogonal polynomials in several variables with group theoretic interpretations as spherical functions for certain values of the parameters. Except in the rank one and rank two case these polynomials have hardly been studied yet. A start was made in VRETARE [32].

Thus from the point of view of special functions Vretare's result is quite useful. The method we mentioned for Koornwinder's proof of the addition formula for Jacobi polynomials works in a more general context. A standard method of proving explicit formulas for orthogonal polynomials of the above mentioned type is to consider first those values of the parameters for which group theoretic interpretations can be given, for instance as spherical functions. For these values of the parameters the whole machinery of (say) spherical functions is available and it may yield a proof of the desired formula. The general result then often follows by a process of analytic continuation, using Carlson's theorem (cf. TITCHMARSH [28, p.186]).

However, in many cases the distribution of parameter values admitting a spherical function interpretation does not allow an analytic continuation to all parameter values. Therefore it is desirable to find group theoretic interpretations of more general nature for special functions.

An obvious generalization of a spherical function is obtained if one replaces the K-biinvariance by left-K-, right-H-invariance. Here (U,H) is (another) Riemannian symmetric pair of the compact type, with an involutive automorphism σ such that $(U_\sigma)_0 \subset H \subset U_\sigma$, and σ and θ commute. The left-K-, right-H-invariant functions on U which are matrix coefficients of some irreducible finite dimensional representation of U (or, equivalently, which are eigenfunctions of all left-U-, right-H-invariant differential operators on U, cf. Theorem 4.3) are called *intertwining functions*. An indication that intertwining functions might also be considered as orthogonal polynomials is the above mentioned article of James and Constantine. Their proof is not only valid for the spherical functions on $O(p+q)/O(p)\times O(q)$, but also for the intertwining functions on $O(p')\times O(q')\backslash O(p+q)/O(p)\times O(q)$ $(p'+q'=p+q)$: The intertwining functions are orthogonal polynomials on the region Ω as before (here it is assumed that $p \le p'$) with respect to the weight function $\Pi_{i=1}^{p} (1-y_i)^{\frac{1}{2}(q-p'-1)} y_i^{\frac{1}{2}(p'-p-1)} \Pi_{i<j} (y_i-y_j)$.

In this thesis, intertwining functions on a compact Lie group are proved to be orthogonal polynomials indeed. Our result contains Vretare's result (Theorem 0.1) as a special case. Also the line of proof is roughly the same as in the original proof for spherical functions, see [31]. Still this generalization is far from a routine excercise: the details of the proof turn out to be much more involved than in [31]; many difficulties of an algebraic nature arise. This corresponds to many new phenomena which occur when a complex semisimple Lie algebra is studied with two commuting involutions instead of one. Some of the results obtained in this way may have their use elsewhere.

To conclude this introduction we treat the example of spherical functions on a rank one symmetric space. This gives the above cited result of CARTAN [3] that all spherical functions on a rank one symmetric space are Jacobi polynomials.

EXAMPLE 0.2 (*the rank one case*). Assume dim $\mathfrak{a} = 1$. Let $\Sigma = \{(-2\alpha), -\alpha, \alpha, (2\alpha)\}$, $\Sigma^+ = \{\alpha, (2\alpha)\}$. Let $H_0 \in \mathfrak{a}$ be such that $\alpha(H_0) = 1$. Then $\mu = k\alpha$, with $k = 1$ if $2\alpha \notin \Sigma$, $k = 2$ if $2\alpha \in \Sigma$, generates the lattice $\mathbb{Z}^1$. We shall consider the spherical functions as polynomials in the variable

$$(0.6) \qquad y := \cos k\theta.$$

Since $\varphi_\mu = a \cos k\theta + b$, with $a,b \in \mathbb{R}$ such that $a+b = 1$, the weight function in the variable $\cos k\theta$ equals w (cf.(0.5)) up to a constant factor. By abuse of notation we shall denote this weight function by w as well. Thus the weight function becomes

$$(0.7) \qquad w(\cos k\theta) = \left|\frac{\sin^{m_\alpha}\theta \, \sin^{m_{2\alpha}} 2\theta}{\sin k\theta}\right|.$$

If $2\alpha \notin \Sigma$, i.e. $m_{2\alpha} = 0$, then (0.7) becomes

$$(0.8) \qquad w(\cos\theta) = (1-\cos\theta)^{\frac{1}{2}(m_\alpha-1)} (1+\cos\theta)^{\frac{1}{2}(m_\alpha-1)},$$

and if $2\alpha \in \Sigma$, i.e. $m_{2\alpha} > 0$, then (0.7) becomes

$$(0.9) \qquad w(\cos 2\theta) = (1-\cos 2\theta)^{\frac{1}{2}(m_\alpha+m_{2\alpha}-1)} (1+\cos\theta)^{\frac{1}{2}(m_{2\alpha}-1)}.$$

Thus, via the transformation $y := \cos\theta$ in (0.8) and $y := \cos 2\theta$ in (0.9) we get that in the rank one case the spherical functions can be considered as Jacobi polynomials of order $(\frac{1}{2}(m_\alpha-1), \frac{1}{2}(m_\alpha-1))$ (or Gegenbauer polynomials) if $2\alpha \notin \Sigma$, and of order $(\frac{1}{2}(m_\alpha+m_{2\alpha}-1), \frac{1}{2}(m_{2\alpha}-1))$ if $2\alpha \in \Sigma$.

REMARK 0.3. Because of the fact that $m_{2\alpha} = 1,3$, or 7 if $2\alpha \in \Sigma$ (cf. for instance WARNER [33, Appendix 1.1.3.1]), Example 0.2 gives group theoretic interpretations for Jacobi polynomials of order $(\frac{1}{2}m, \frac{1}{2}m)$, $(\frac{1}{2}(m+1), 0)$, $(\frac{1}{2}(m+3), 1)$ and $(\frac{1}{2}(m+7), 3)$. Here m is a certain nonnegative integer. As will be seen in chapter II intertwining functions yield group theoretic interpretations for Jacobi polynomials of order $(\frac{1}{2}m, \frac{1}{2}n)$, where m and n are nonnegative integers.

CHAPTER 1

REAL SEMISIMPLE LIE ALGEBRAS WITH TWO INVOLUTIONS

Let g be a noncompact real semisimple Lie algebra, let g_c be a complexification of g. Let σ be an involution of g, not necessarily a Cartan involution. Then there exists a Cartan involution θ of g such that σ and θ commute, cf. LOOS [23, p.153]. By abuse of notation we will use σ and θ for the extensions of σ and θ to g_c.

Let $g = k + p$ be the decomposition of g in +1 and −1 eigenspaces of θ. Then this decomposition is a Cartan decomposition. Let $g = h + q$ be the decomposition of g in +1 and −1 eigenspaces of σ.

Since $\sigma\theta = \theta\sigma$ we have the following direct sum decomposition

$$g = k \cap h + k \cap q + p \cap h + p \cap q. \tag{1.1}$$

Let $u := k + ip$ be a compact real form of g_c (cf. HELGASON [13]) and put $h^0 := k \cap h + i(p \cap h)$, $q^0 := k \cap q + i(p \cap q)$. Then the decomposition of u in +1 and −1 eigenspaces of σ is given by $u = h^0 + q^0$. Put $g^0 := h^0 + iq^0$, then g^0 is a real form of g_c, and $g^0 = h^0 + iq^0$ is a Cartan decomposition of g^0. If $\sigma \neq \mathrm{id}$ g^0 is a noncompact real form of g_c. See FLENSTED-JENSEN [8,§2] for this duality.

Let $a_{pq} \subset p \cap q$ be a maximal abelian subalgebra. Note that a_{pq} consists of semisimple elements. Choose $a_{ph} \subset p \cap h$ such that $a_p := a_{pq} + a_{ph}$ is maximal abelian in p. Choose $a_{kq} \subset k \cap q$ such that $a_q := a_{pq} + a_{kq}$ is maximal abelian in q.

<u>LEMMA 1.1</u>. $[a_{ph}, a_{kq}] = (0)$.

<u>PROOF</u>. Let $X \in a_{ph}$, $Y \in a_{kq}$. Then $[X,Y] \in p \cap q$. Let $H \in a_{pq}$, then $[H,[X,Y]] = [[Y,H],X] + [[H,X],Y] = 0$. Since a_{pq} is maximal abelian in

$p \cap q$ this implies that $[X,Y] \in a_{pq}$. Hence

(1.2) $\quad \mathrm{ad}(X)(a_{kq}) \subset a_{pq}$.

But a_p is abelian, so we have

(1.3) $\quad \mathrm{ad}(X)(a_{pq}) = (0)$.

Now the fact that ad(X) is semisimple together with (1.2) and (1.3) implies $\mathrm{ad}(X)(a_{kq}) = (0)$, hence $[a_{ph}, a_{kq}] = (0)$. □

COROLLARY 1.2. *There exists* $a_{kh} \subset k \cap h$ *such that* $a := a_{pq} + a_{ph} + a_{kq} + a_{kh}$ *is a Cartan subalgebra of* g.

PROOF. By [13, Lemma VI.3.2] it is enough to show that there exists $a_{kh} \subset k \cap h$ which is abelian, such that $a_{pq} + a_{ph} + a_{kq} + a_{kh}$ is maximal abelian in g. But by Lemma 1.1 $a_{pq} + a_{ph} + a_{kq}$ is maximal abelian in $p \cap q + p \cap h + k \cap q$, hence can be extended to a maximal abelian subalgebra of g. □

Put $a_k := a_{kq} + a_{kh}$, $a_h := a_{ph} + a_{kh}$. Then $a = a_p + a_k$, and also $a = a_q + a_h$. Let Φ denote the set of roots of the pair (g_c, a_c). Then $\Phi \subset (ia_k + a_p)^*$. Via the Killing form $ia_k + a_p$ can be identified with its dual. In particular, this yields an inner product $(\cdot,\cdot)$ on $(ia_k + a_p)^*$. Let Σ_p denote the set of roots of the pair (g, a_p), and let Σ_q denote the set of roots of the pair $(g^0, a_{pq} + ia_{kq})$. It is well known that Σ_p and Σ_q are root systems.

Let Σ_{pq} denote the set of roots of the pair (g, a_{pq}), then Σ_{pq} satisfies the axioms of a root system, cf. [26, Theorem 5]. Its elements consist of all nonzero restrictions of roots in Σ_p (or, equivalently, Σ_q) to a_{pq}.

For a real linear form λ on $a_p + ia_k$ (i.e. for λ in the real span of Φ) put $(\tau_1\lambda)(X) := -\lambda(\theta X)$, $(\tau_2\lambda)(X) := -\lambda(\sigma X)$ $(X \in a_p + ia_k)$. Now
$\tilde{\lambda} := \frac{1}{2}(\lambda + \tau_1\lambda)$ gives the restriction of λ to a_p,
$\approx\!\lambda := \frac{1}{2}(\lambda + \tau_2\lambda)$ the restriction of λ to a_q, and
$\hat{\lambda} := \frac{1}{4}(\lambda + \tau_1\lambda + \tau_2\lambda + \tau_1\tau_2\lambda)$ the restriction of λ to a_{pq}.

REMARK 1.3. It follows from the above that we have the following situation: g_c is a complex semisimple Lie algebra, θ and σ are two commuting involutions of g_c. $g_c = k_c + p_c$ is the decomposition of g_c with respect to θ,

$\mathfrak{g}_c = \mathfrak{h}_c + \mathfrak{q}_c$ the decomposition of $\mathfrak{g}_c$ with respect to σ, and $\mathfrak{a}_c = (\mathfrak{a}_{pq})_c + (\mathfrak{a}_{ph})_c + (\mathfrak{a}_{kq})_c + (\mathfrak{a}_{kh})_c$ is a CSA (:= Cartan Subalgebra) of $\mathfrak{g}_c$ such that $(\mathfrak{a}_{pq})_c$ is maximal abelian in $\mathfrak{p}_c \cap \mathfrak{q}_c$, $(\mathfrak{a}_{ph})_c + (\mathfrak{a}_{pq})_c$ is maximal abelian in $\mathfrak{p}_c$, and $(\mathfrak{a}_{kq})_c + (\mathfrak{a}_{pq})_c$ is maximal abelian in $\mathfrak{q}_c$. For $\alpha \in \Phi$ define $\tau_1\alpha := -\alpha\circ\theta$, $\tau_2\alpha := -\alpha\circ\sigma$. This complex setting simplifies calculations in concrete examples since all the real forms introduced in the beginning of this chapter are avoided. Of course, in this setting we need to know that $\mathfrak{g}_c$ has a compact real form $\mathfrak{u}$ which is θ-, and σ-invariant such that $\mathfrak{u} \cap \mathfrak{a}_c$ is a CSA of $\mathfrak{u}$. The existence of such a compact real form is assured by the following theorem.

THEOREM 1.4. *Let $\mathfrak{g}_c$ be a complex semisimple Lie algebra. Let $\theta_1,\dots,\theta_n$ be commuting involutions on $\mathfrak{g}_c$, and let $\mathfrak{a}_c$ be a $(\theta_1,\dots,\theta_n)$-invariant CSA of $\mathfrak{g}_c$. Then $\mathfrak{g}_c$ has a $(\theta_1,\dots,\theta_n)$-invariant compact real form $\mathfrak{u}$ such that $\mathfrak{u} \cap \mathfrak{a}_c$ is a CSA of $\mathfrak{u}$.*

PROOF. Choose root vectors $X_\alpha \in \mathfrak{g}_c^\alpha$ according to HUMPHREYS [17, Proposition 25.2]. That is, for all $\alpha,\beta,\alpha+\beta \in \Phi$:

$$[X_\alpha, X_{-\alpha}] = H_\alpha,$$

$$[X_\alpha, X_\beta] = c_{\alpha,\beta}X_{\alpha+\beta},$$

where $c_{\alpha,\beta}$ satisfies $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$, and $H_\alpha \in \mathfrak{a}_c$ is chosen according to [17, Proposition 8.3]. Let $\beta-r\alpha,\dots,\beta+q\alpha$ be the α-string through β. Then ([17, Proposition 25.2]):

$$c_{\alpha,\beta}^2 = q(r+1)\,\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)} \quad . \tag{1.4}$$

For an involution θ on $\mathfrak{g}_c$ put $(\theta\lambda)(X) := \lambda(\theta X)$ $(X\in\mathfrak{a}_c, \lambda\in\mathfrak{a}_c^*)$. Then (1.4) implies

$$c_{\theta\alpha,\theta\beta}^2 = c_{\alpha,\beta}^2.$$

Let $\kappa_\alpha \in \mathbb{C}$ be defined by $\theta X_\alpha = \kappa_\alpha X_{\theta\alpha}$. Then $\kappa_{\theta\alpha} = \kappa_{-\alpha} = (\kappa_\alpha)^{-1}$, and by the definition of $c_{\alpha,\beta}$:

$$\kappa_\alpha\kappa_\beta c_{\theta\alpha,\theta\beta} = \kappa_{\alpha+\beta}c_{\alpha,\beta} \qquad (\alpha,\beta,\alpha+\beta\in\Phi).$$

Let $\kappa_{i,\alpha}$ be the κ_α corresponding to θ_i $(i = 1,\dots,n)$, and for $i_j = 0,1$ let $\kappa_\alpha^{i_1,\dots,i_n}$ be the κ_α corresponding to the involution $\theta_1^{i_1}\dots\theta_n^{i_n}$. Put

(1.5) $$\mu_\alpha := \prod_{i_1,\dots,i_n=0,1} \kappa_\alpha^{i_1,\dots,i_n}.$$

Then $\mu_\alpha\mu_{-\alpha} = 1$, $\mu_{\alpha+\beta} = \pm\mu_\alpha\mu_\beta$, and

$$\mu_{\theta_i\alpha} = \frac{\mu_\alpha}{(\kappa_{i,\alpha})^{2^n}} .$$

Put $Y_\alpha := |\mu_\alpha|^{-(\frac{1}{2})^n} X_\alpha$. Then for all $\alpha,\beta,\alpha+\beta \in \Phi$

$$[Y_\alpha,Y_{-\alpha}] = H_\alpha,$$

$$[Y_\alpha,Y_\beta] = c_{\alpha,\beta}Y_{\alpha+\beta},$$

and

$$\theta_i Y_\alpha = \frac{\kappa_{i,\alpha}}{|\kappa_{i,\alpha}|} Y_{\theta_i\alpha}.$$

Now $\{iH_\alpha \mid \alpha \in \Phi\} \cup \{zY_\alpha - \bar{z}Y_{-\alpha} \mid \alpha \in \Phi,\ z \in \mathbb{C}\}$ span a compact real form $\mathfrak{u}$ of $\mathfrak{g}_c$ (cf.[22, Corollary 2.4]). $\mathfrak{u}$ is θ_i-invariant:

$$\theta_i(zY_\alpha - \bar{z}Y_{-\alpha}) = z\frac{\kappa_{i,\alpha}}{|\kappa_{i,\alpha}|} Y_\alpha - \bar{z}\frac{\kappa_{i,-\alpha}}{|\kappa_{i,-\alpha}|} Y_{-\alpha}$$

$$= z\frac{\kappa_{i,\alpha}}{|\kappa_{i,\alpha}|} Y_\alpha - \overline{z\frac{\kappa_{i,\alpha}}{|\kappa_{i,\alpha}|}} Y_{-\alpha},$$

since $\frac{\kappa_{i,\alpha}}{|\kappa_{i,\alpha}|}$ has absolute value 1. □

LEMMA 1.5. *Let $\alpha \in \Phi$, and suppose $\hat{\alpha} = 0$. Then $\tilde{\alpha} = 0$ or $\tilde{\tilde{\alpha}} = 0$.*

PROOF. Suppose $\alpha \in \Phi$ is such that $\hat{\alpha} = 0$, and $\tilde{\alpha} \neq 0$, $\tilde{\tilde{\alpha}} \neq 0$. Let $0 \neq X_\alpha \in \mathfrak{g}_c$ be such that $[X,X_\alpha] = \alpha(X)X_\alpha$ for all $X \in \mathfrak{a}_c$. By the decomposition (1.1) we can write

(1.6) $$X_\alpha = X_{kh} + X_{kq} + X_{ph} + X_{pq},$$

with $X_{kh} \in \mathfrak{k}_c \cap \mathfrak{h}_c$, etc. Let $X_0 \in \mathfrak{a}_{pq}$. Then $\alpha(X_0) = 0$, hence

$$0 = [X_0,X_\alpha] = [X_0,X_{kh}] + [X_0,X_{kq}] + [X_0,X_{ph}] + [X_0,X_{pq}].$$

The decomposition (1.6) being direct, this forces $[X_0,X_{pq}] = 0$. But $\mathfrak{a}_{pq}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$, hence $X_{pq} \in \mathfrak{a}_{pq}$. Now let $X_1 \in \mathfrak{a}_{ph}$ be such that $\alpha(X_1) \neq 0$. Then $[X_1,X_\alpha] = \alpha(X_1)X_\alpha$, hence

$$[X_1,X_{kh}] + [X_1,X_{kq}] + [X_1,X_{ph}] = \alpha(X_1)(X_{kh}+X_{kq}+X_{ph}+X_{pq}). \tag{1.7}$$

Thus $\alpha(X_1)X_{kq} = 0$, thus $X_{kq} = 0$. But, again by (1.7), this implies that $X_{pq} = 0$.

Let $X_2 \in \mathfrak{a}_{kq}$ be such that $\alpha(X_2) \neq 0$. Then

$$[X_2,X_{kh}] + [X_2,X_{ph}] = \alpha(X_2)(X_{kh}+X_{ph}).$$

Hence $\alpha(X_2)X_{kh} = \alpha(X_2)X_{ph} = 0$, thus $X_{kh} = X_{ph} = 0$. Thus we have $X_{kq} = X_{kh} = X_{ph} = X_{pq} = 0$, thus $X_\alpha = 0$. Contradiction. □

<u>THEOREM 1.6</u>. *Choose a positive system* Σ^+_{pq}. *There exist positive systems* Σ^+_p, Σ^+_q *and* Φ^+ *such that for all* $\alpha \in \Phi$:

$$\begin{array}{ccccc} & & \tilde{\alpha} \in \Sigma^+_p & & \\ & \nearrow & & \searrow & \\ \hat{\alpha} \in \Sigma^+_{pq} & & & & \alpha \in \Phi^+ \\ & \searrow & & \nearrow & \\ & & \approx{\alpha} \in \Sigma^+_q & & \end{array} \tag{1.8}$$

<u>PROOF</u>. Choose a lexicographic ordering on the dual of $\mathfrak{a}_p + i\mathfrak{a}_k$ with respect to the decomposition $\mathfrak{a}_{pq} + \mathfrak{a}_{ph} + i\mathfrak{a}_{kq} + i\mathfrak{a}_{kh}$, and choose positive systems Σ^+_p, Σ^+_q and Φ^+ with respect to this ordering. These positive systems satisfy (1.8) because of Lemma 1.5. □

<u>REMARK 1.7</u>. Corollary 1.2 and Theorem 1.6 were also stated (without proof) in OSHIMA [25].

<u>REMARK 1.8</u>. It is a natural question whether all the efforts in this chapter are worthwile, that is, if there exist triples $(\mathfrak{g},\mathfrak{k},\mathfrak{h})$ such that $\mathfrak{a}_{ph}$ and $\mathfrak{a}_{kq}$ are both non-trivial. An example of such a triple is given by: $\mathfrak{g} = \mathfrak{sl}(n;\mathbb{C})$, $\mathfrak{k} = \mathfrak{su}(n)$, $\mathfrak{h} = \mathfrak{s}(\mathfrak{gl}(p;\mathbb{C})\times\mathfrak{gl}(n-p;\mathbb{C}))$, with $p \leq \frac{1}{2}n$. Then $\sigma X = JXJ$, with $J = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$, where the first p entries are $+1$, and $\theta X = -X^*$.

Let 0_{ij} denote the (ixj) matrix with only zeros as entries and put $q := n-p$, $k := q-p$. Then we can choose:

$$\mathfrak{a}_{pq} = \left\{ \begin{pmatrix} 0_{pp} & T & 0_{pk} \\ T & & 0_{qq} \\ 0_{kp} & & \end{pmatrix} : T = \mathrm{diag}(t_1,\ldots,t_p),\ t_i \in \mathbb{R} \text{ for all } i \right\},$$

$$\mathfrak{a}_{p} = \left\{ \begin{pmatrix} S & T & 0_{pk} \\ T & S & 0_{pk} \\ 0_{kp} & 0_{kp} & Y \end{pmatrix} : T = \mathrm{diag}(t_1,\ldots,t_p),\ S = \mathrm{diag}(s_1,\ldots,s_p), \right.$$

$$\left. Y = \mathrm{diag}(y_1,\ldots,y_k);\ t_i,s_i,y_i \in \mathbb{R} \text{ for all } i, \sum_{j=1}^{p} 2s_j + \sum_{j=1}^{k} y_j = 0 \right\}$$

and

$$\mathfrak{a}_{q} = \left\{ \begin{pmatrix} 0_{pp} & Z & 0_{pk} \\ Z & & 0_{qq} \\ 0_{kp} & & \end{pmatrix} : Z = \mathrm{diag}(z_1,\ldots,z_p),\ z_i \in \mathbb{C} \text{ for all } i \right\}.$$

As a last result in this chapter we mention the following theorem. In fact it states that the triple (Φ,τ_1,τ_2) is independent of the choice of $\mathfrak{a}_c$. It was proved by Loek Helminck, and the proof can be found in [14].

<u>THEOREM 1.9</u>. *Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two CSA's of $\mathfrak{g}$ such that their intersections with $\mathfrak{p} \cap \mathfrak{q}$, $\mathfrak{p}$ and $\mathfrak{q}$ are maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$, $\mathfrak{p}$ and $\mathfrak{q}$, respectively. Then $\mathfrak{a}$ and $\mathfrak{a}'$ are conjugate under* $\mathrm{Int}(\mathfrak{k}\cap\mathfrak{h})$.

CHAPTER 2

REPRESENTATIONS OF K,H-CLASS 1

Let G_c be a simply connected Lie group with Lie algebra $\mathfrak{g}_c$. Let G, K,H,H^0 and U be analytic subgroups of G_c with Lie algebras $\mathfrak{g},\mathfrak{k},\mathfrak{h},\mathfrak{h}^0$ and $\mathfrak{u}$, respectively. We shall, analogous to WARNER [33], identify a finite dimensional representation of G_c with its restriction to G or U.

Let Λ be the weight lattice corresponding to Φ. Let $\Lambda^+ \subset \Lambda$ denote the set of dominant weights (for a choice of Φ^+ as in Theorem 1.6). For $\lambda \in \Lambda^+$ let π_λ denote the finite dimensional irreducible representation of G with highest weight λ. Let $V(\lambda)$ denote the representation space of π_λ, and for $\mu \in \Lambda$ let the weight space in $V(\lambda)$ with weight μ be denoted by $V(\lambda)_\mu$. Finally, let $(\cdot|\cdot)$ denote a U-invariant inner product in $V(\lambda)$.

<u>DEFINITION 2.1</u>. π_λ is said to be of K-*class* 1 if there exists a nonzero K-fixed vector $e_K \in V(\lambda)$, that is such that $\pi_\lambda(k)e_K = e_K$ for all $k \in K$. π_λ is said to be of K,H-*class* 1 if there exist both a nonzero K-fixed vector e_K and a nonzero H-fixed vector e_H.

(e_K and e_H, if they exist, are unique up to a constant factor.) The next theorem gives the generalization of Theorem 3.3.1.1 of WARNER [33]. For spherical functions this theorem seems to go back to Cartan, see also HELGASON [11]. Let us agree to use the convention to extend $\alpha \in \Sigma_p$ to all of $\mathfrak{a}$ by rendering it trivial on $i\mathfrak{a}_k$, and similarly for $\alpha \in \Sigma_q$.

<u>THEOREM 2.2</u>. *Let* $\lambda \in \Lambda^+$. *Then* π_λ *is a representation of* K,H-*class* 1 *if and only if*

(1) $$\lambda|_{\mathfrak{a}_h \cup \mathfrak{a}_k} = 0$$

(2) $$\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \textit{for all } \alpha \in \Sigma_p \cup \Sigma_q.$$

PROOF. This theorem follows immediately by applying [33, Theorem 3.3.1.1] twice: once for the pair (G,K) and one for the pair (G^0,H^0), where G^0 is the analytic subgroup of G_c with Lie algebra $\mathfrak{g}^0$. □

A root system Σ with an involution τ is called *normal* if, for all $\alpha \in \Sigma$, $\alpha - \tau\alpha \notin \Sigma$.

LEMMA 2.3. *Let* $\alpha \in \Phi, \hat{\alpha} \neq 0$. *Then*

$$\frac{(\tilde{\alpha},\tilde{\alpha})}{(\hat{\alpha},\hat{\alpha})} \text{ and } \frac{(\approx\!\alpha,\approx\!\alpha)}{(\hat{\alpha},\hat{\alpha})} = 1,2 \text{ or } 4.$$

If value 4 is attained, then $2\hat{\alpha} \in \Sigma_{pq}$.

PROOF. We shall prove the lemma by considering all possible values of

$$m_1 := \frac{(\tilde{\alpha},\tilde{\alpha})}{(\hat{\alpha},\hat{\alpha})} \text{ and } m_2 := \frac{(\approx\!\alpha,\approx\!\alpha)}{(\hat{\alpha},\hat{\alpha})}.$$

First, consider the exceptional cases $\alpha = \tau_1\alpha$, $\alpha = \tau_2\alpha$ and $\alpha = \tau_1\tau_2\alpha$. If $\alpha = \tau_1\alpha$, then $\alpha = \tilde{\alpha}$, $\approx\!\alpha = \hat{\alpha}$, thus $m_2 = 1$,

$$m_1 = \frac{(\alpha,\alpha)}{(\approx\!\alpha,\approx\!\alpha)} = 1,2 \text{ or } 4$$

and, in case of value $4, 2\approx\!\alpha \in \Sigma_q$, by HELGASON [13, Lemma VII.8.4]. If $\alpha = \tau_2\alpha$ the lemma follows by a similar reasoning. If $\alpha = \tau_1\tau_2\alpha$, then $\tilde{\alpha} = \hat{\alpha} = \approx\!\alpha$ and the lemma is obvious. Also $\alpha = -\tau_1\alpha, -\tau_2\alpha$ or $-\tau_1\tau_2\alpha$ implies $\hat{\alpha} = 0$.

Thus, because of the fact that (Φ,τ_j) is a normal root system ([33, Lemma 1.3.6]) only the cases

$$-1 < \frac{(\alpha,\tau_j\alpha)}{(\alpha,\alpha)} \leq 0$$

are left. This leads to the following table, using $(\tilde{\alpha},\tilde{\alpha}) = \frac{1}{2}((\alpha,\alpha)+(\alpha,\tau_1\alpha))$, $(\approx\!\alpha,\approx\!\alpha) = \frac{1}{2}((\alpha,\alpha)+(\alpha,\tau_2\alpha))$, and $(\hat{\alpha},\hat{\alpha}) = \frac{1}{4}((\alpha,\alpha)+(\alpha,\tau_1\alpha)+(\alpha,\tau_2\alpha)+(\alpha,\tau_1\tau_2\alpha))$.

	$\frac{(\alpha,\tau_1\alpha)}{(\alpha,\alpha)}$	$\frac{(\alpha,\tau_2\alpha)}{(\alpha,\alpha)}$	$\frac{(\alpha,\tau_1\tau_2\alpha)}{(\alpha,\alpha)}$	$\frac{(\tilde{\alpha},\tilde{\alpha})}{(\alpha,\alpha)}$	$\frac{(\approx\alpha,\approx\alpha)}{(\alpha,\alpha)}$	$\frac{(\hat{\alpha},\hat{\alpha})}{(\alpha,\alpha)}$
1.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
2.	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$
3.	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$
4.	$-\frac{1}{2}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$
5.	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
6.	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	0
7.	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
8.	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
9.	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0
10.	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
11.	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
12.	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$

Because of Lemma 1.5 cases 6, 9 and 10 are impossible, and case 12 is impossible because $(\hat{\alpha},\hat{\alpha}) < 0$. Thus the lemma follows if we have proved that case 2 does not occur.

So assume that there exists an $\alpha \in \Phi$ such that $(\alpha,\tau_1\alpha) = (\alpha,\tau_2\alpha) = 0$, $(\alpha,\tau_1\tau_2\alpha) = \frac{1}{2}(\alpha,\alpha)$. Because $\alpha \neq \tau_1\tau_2\alpha$ this implies $\beta := \alpha-\tau_1\tau_2\alpha \in \Phi$. Thus $\hat{\beta} = 0$, hence $\tilde{\beta} = 0$ or $\approx\beta = 0$, by Lemma 1.5. But $(\tilde{\beta},\tilde{\beta}) = (\approx\beta,\approx\beta) = \frac{1}{2}(\alpha,\alpha)$, a contradiction. Thus $m_1,m_2 = 1,2$ or 4.

By the above table it is clear that if value 4 is attained, then either $(\alpha,\tau_j\alpha) = -\frac{1}{2}(\alpha,\alpha) < 0$, hence $\gamma := \alpha+\tau_j\alpha \in \Phi$ $(j = 1,2)$, or $(\alpha,\tau_1\tau_2\alpha) = -\frac{1}{2}(\alpha,\alpha) < 0$, hence $\gamma := \alpha+\tau_1\tau_2\alpha \in \Phi$. In all these cases $0 \neq \hat{\gamma} = 2\hat{\alpha} \in \Sigma_{pq}$. □

So we can skip 2,6,9,10 and 12 from the table in the proof of Lemma 2.3. Since 11 can be killed by exactly the same method as 2 (cf. the proof of Lemma 2.3), we are left with the following possibilities for $\alpha \in \Phi$ with $\pm\alpha \neq \tau_1\alpha, \tau_2\alpha$ or $\tau_1\tau_2\alpha$:

	$\frac{(\alpha,\tau_1\alpha)}{(\alpha,\alpha)}$	$\frac{(\alpha,\tau_2\alpha)}{(\alpha,\alpha)}$	$\frac{(\alpha,\tau_1\tau_2\alpha)}{(\alpha,\alpha)}$	$\frac{(\tilde\alpha,\tilde\alpha)}{(\alpha,\alpha)}$	$\frac{(\approx\alpha,\approx\alpha)}{(\alpha,\alpha)}$	$\frac{(\hat\alpha,\hat\alpha)}{(\alpha,\alpha)}$
1.	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
2.	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$
3.	$-\frac{1}{2}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$
4.	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
5.	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
6.	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

TABLE I

Put, for $\alpha \in \Sigma_{pq}$:

$$(2.1) \qquad c(\alpha) := \max_{\substack{\beta\in\Phi \\ \beta=\alpha}} \left\{\frac{(\tilde\beta,\tilde\beta)}{(\alpha,\alpha)}, \frac{(\approx\beta,\approx\beta)}{(\alpha,\alpha)}\right\}.$$

Now Lemma 2.3 has the following corollary.

COROLLARY 2.4. *Let* $\alpha \in \Sigma_{pq}$. *Then* $c(\alpha) = 1,2$ *or* 4. *If* $c(\alpha) = 4$ *then* $2\alpha \in \Sigma_{pq}$.

COROLLARY 2.5. *Condition* (2) *in Theorem* 2.2 *can be replaced by:*

$$(2') \qquad \frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in c(\alpha)\mathbb{Z} \quad \textit{for all} \quad \alpha \in \Sigma_{pq}.$$

We shall now give an example for Σ_{pq} such that $c(\alpha) = 2$ for all $\alpha \in \Sigma_{pq}$.

EXAMPLE 2.6. Let $U := SU(2) \times SU(2)$. Put $J := \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, and define the involutions θ and σ by $\theta(u,v) := (v,u)$, $\sigma(u,v) := (JuJ,JvJ)$ $(u,v\in SU(2))$. Then $K = \mathrm{diag}\,(SU(2))$, $H^0 = U(1)\times U(1)$. For the maximal abelian subalgebras we choose:

$$\mathfrak{a} = \mathfrak{a}_q := \left\{A_{s,t} := \begin{pmatrix} \begin{matrix} is & 0 \\ 0 & -is \end{matrix} & 0_{22} \\ 0_{22} & \begin{matrix} it & 0 \\ 0 & -it \end{matrix} \end{pmatrix} : s,t \in \mathbb{R}\right\},$$

$$\mathfrak{a}_p = \mathfrak{a}_{pq} := \left\{ A_{s,-s} := \begin{pmatrix} \begin{matrix} is & 0 \\ 0 & -is \end{matrix} & 0_{22} \\ 0_{22} & \begin{matrix} -is & 0 \\ 0 & is \end{matrix} \end{pmatrix} \quad : s \in \mathbb{R} \right\}.$$

Denote the linear form $A_{s,t} \mapsto as + bt$ on $\mathfrak{a}$ by (a,b). Put $\alpha := (1,0)$, $\beta := (0,1)$. Then $\Phi = \{\alpha,-\alpha,\beta,-\beta\}$. Choose an ordering such that $\Phi^+ = \{\alpha,-\beta\}$. Then, because $\tau_1\gamma = -\gamma\circ\theta$, $\tau_2\gamma = -\gamma\circ\sigma$ $(\gamma\in\Phi)$, we have $\hat{\alpha} = \tilde{\alpha} = (\frac{1}{2},-\frac{1}{2}) = -\tilde{\beta} = -\hat{\beta}$. Now $(\lambda,\mu) \in \Lambda^+$ if and only if $\lambda \in \frac{1}{2}\mathbb{Z}^+$, $\mu \in \frac{1}{2}\mathbb{Z}^+$. The restricted root systems are given by $\Sigma^+_{pq} = \Sigma^+_p = \{\tilde{\alpha}\}$, $\Sigma^+_q = \Phi^+ = \{\alpha,-\beta\}$. Now:

$$\frac{((\lambda,-\lambda),(\frac{1}{2},-\frac{1}{2}))}{((\frac{1}{2},-\frac{1}{2}),(\frac{1}{2},-\frac{1}{2}))} \in \mathbb{Z}^+ \iff \lambda \in \tfrac{1}{2}\mathbb{Z}^+ .$$

But:

$$\left.\begin{matrix} \dfrac{((\lambda,-\lambda),(1,0))}{((1,0),(1,0))} \in \mathbb{Z}^+ \\[2ex] \dfrac{((\lambda,-\lambda),(0,-1))}{((0,-1),(0,-1))} \in \mathbb{Z}^+ \end{matrix}\right\} \iff \lambda \in \mathbb{Z}^+ .$$

Let Λ_{pq} be the weight lattice corresponding to Σ_{pq}, Λ^+_{pq} the set of dominant weights in Λ_{pq}. It follows from Theorem 2.2 that if π_λ is of K,H-class 1, then λ lives only on $\mathfrak{a}_{pq}$, and the set of all such λ forms a lattice. Hence there exist $\mu_1,\ldots,\mu_\ell \in \Lambda^+_{pq}$ such that π_λ is of K,H-class 1 if and only if $\lambda = \Sigma^\ell_{j=1} m_j\mu_j$ for nonnegative integers m_j. Let us identify $\lambda = \Sigma^\ell_{j=1} m_j\mu_j$ ($m_j \in \mathbb{Z}$ for $j = 1,\ldots,\ell$) with the point $(m_1,\ldots,m_\ell) \in \mathbb{Z}^\ell$. Let the set of all $(m_1,\ldots,m_\ell) \in \mathbb{Z}^\ell$ with all m_j nonnegative be denoted by $\mathbb{Z}^\ell_+$. (This is analogous to the case of representations of K-class 1, see VRETARE [31]). Thus π_λ is of K,H-class 1 if and only if $\lambda \in \mathbb{Z}^\ell_+$. In the next chapter we shall obtain explicit expressions for $\mu_1,\ldots,\mu_\ell$.

We conclude this chapter with one more example for Φ, namely one for which $(\Phi,\tau_1\tau_2)$ is not a normal root system. In this example (Σ_q,τ_1) also is not a normal root system

<u>EXAMPLE 2.7</u>. Let $\mathfrak{g}_c := \mathfrak{sl}(4,\mathbb{C})$. Put

$$J_1 := \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0_{22} \\ 0_{22} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} ,$$

$$J_2 := \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0_{22} \\ 0_{22} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix} .$$

Define the involutions θ,σ by $\theta X := J_1XJ_1$, $\sigma X := J_2XJ_2$ $(X\in\mathfrak{g}_c)$. Let $\mathfrak{a}_c := \{\mathrm{diag}(z_1,z_2,z_3,z_4)\colon z_j \in \mathbb{C}\ (j = 1,\dots,\ell),\ \Sigma_{j=1}^4\ z_j = 0\}$. Put $\alpha_{ij}\colon \mathrm{diag}(z_1,z_2,z_3,z_4) \to z_i-z_j\ (i\neq j)$. Then $\Phi = \{\alpha_{ij}\colon i,j = 1,\dots,4,\ i \neq j\}$. We have $\tau_1\alpha_{13} = \alpha_{32}$, $\tau_2\alpha_{13} = \alpha_{42}$ and $\tau_1\tau_2\alpha_{13} = \alpha_{14}$. Thus $(\alpha_{13},\tau_1\alpha_{13}) = -\frac{1}{2}(\alpha_{13},\alpha_{13})$, $(\alpha_{13},\tau_2\alpha_{13}) = 0$, and $(\alpha_{13},\tau_1\tau_2\alpha_{13}) = \frac{1}{2}(\alpha_{13},\alpha_{13})$. Thus $\alpha_{13}-\tau_1\tau_2\alpha_{13} \in \Phi$, since $(\alpha_{13},\tau_1\tau_2\alpha_{13}) > 0$. Hence $(\Phi,\tau_1\tau_2)$ is not a normal root system.

For Σ_q we get $\tilde{\tilde{\alpha}}_{13} = \frac{1}{2}(\alpha_{12}-\alpha_{34})$, $\tau_1\tilde{\tilde{\alpha}}_{13} = \frac{1}{2}(\alpha_{12}+\alpha_{34})$. Thus $\tilde{\tilde{\alpha}}_{13}-\tau_1\tilde{\tilde{\alpha}}_{13} = \alpha_{43} = (\alpha_{13}-\tau_1\tau_2\alpha_{13})^{\approx} \in \Sigma_q$, since $\alpha_{13}-\tau_1\tau_2\alpha_{13} \in \Phi$. Hence (Σ_q,τ_1) is not a normal root system.

NB. Observe that this gives an example of row 4 (and hence also of row 6) from Table I.

CHAPTER 3

THE LATTICE $\mathbb{Z}^{\ell}$

In this chapter we shall obtain epxlicit expressions for the generators of the lattice $\mathbb{Z}^{\ell}$ (see chapter 2). For this, we need to study the function $c(\alpha)$ $(\alpha\in\Sigma_{pq})$, as defined by (2.1), first. Let W be the Weyl group of Φ, W_p the Weyl group of Σ_p, W_q the Weyl group of Σ_q, and W_{pq} the Weyl group of Σ_{pq}. For $\alpha \in \Sigma_{pq}$, let s_α be the reflection corresponding to α. That is $s_\alpha(\beta) := \beta - 2\,(\beta,\alpha)/(\alpha,\alpha)\;\alpha$ $(\beta\in\mathfrak{a}^*_{pq})$.

<u>PROPOSITION 3.1</u>. *Let* $s \in W_{pq}$. *Then there exists* $w \in W$ *such that* $w|_{\mathfrak{a}^*_{pq}} = s$, *and* $w\tau_i = \tau_i w$ $(i = 1,2)$.

<u>PROOF</u>. Let $\alpha \in \Sigma_{pq}$. We shall show that there exists $w \in W$, commuting with τ_1 and τ_2, such that $w|_{\mathfrak{a}^*_{pq}} = s_\alpha$. Since the s_α $(\alpha\in\Sigma_{pq})$ generate W_{pq} this proves the proposition.

Let $\beta \in \Phi$ be such that $\hat{\beta} = \alpha$. If $\beta = \tau_1\beta, \tau_2\beta$ or $\tau_1\tau_2\beta$, then we are back in the case of one involution, and the assertion follows from WARNER [33, Lemma 1.1.3.4], since the w constructed there is easily seen to be commuting with τ_1 as well as τ_2. If $\beta \neq \tau_1\beta, \tau_2\beta$ or $\tau_1\tau_2\beta$ then β is one of the cases from Table I, since $\hat{\beta} \neq 0$. In this case we can also quite easily construct w in the same fashion. For instance, if β satisfies row 1 of Table I, put $w := s_\beta s_{\tau_1\beta} s_{\tau_2\beta} s_{\tau_1\tau_2\beta}$. Then w commutes with τ_1 and τ_2, and $w|_{\mathfrak{a}^*_{pq}} = s_\alpha$. The other cases are left to the reader. □

<u>COROLLARY 3.2</u>. $c(\alpha)$ *is* W_{pq}*-invariant.*

<u>PROOF</u>. Let $s \in W_{pq}$. Choose $w \in W$, commuting with τ_1 and τ_2 such that $w|_{\mathfrak{a}^*_{pq}} = s$. Let $\underset{\sim}{w}_{pq}$ ... $\tilde{w} := w|_{\mathfrak{a}^*_p}$, $\tilde{\tilde{w}} := w|_{\mathfrak{a}^*_q}$. Then for all $\beta \in \Phi, \hat{\beta} \neq 0$ we have $s\hat{\beta} = (w\beta)^\wedge$. Thus, if $\beta = w\gamma$, $(\tilde{\beta},\tilde{\tilde{\beta}}) = ((w\gamma)^\sim,(w\gamma)^{\approx}) = (\tilde{w}\tilde{\gamma},\tilde{\tilde{w}}\tilde{\tilde{\gamma}}) = (\tilde{\gamma},\tilde{\tilde{\gamma}})$,

and hence also $(\approx\beta,\approx\beta) = (\approx\gamma,\approx\gamma)$ and $(\hat\beta,\hat\beta) = (\hat\gamma,\hat\gamma)$. □

Let Σ^c_{pq} be defined by

$$(3.1)\qquad \Sigma^c_{pq} := \{c(\alpha)\alpha \mid \alpha \in \Sigma_{pq}\}.$$

LEMMA 3.3. *Σ^c_{pq} is a root system.*

PROOF. By Corollary 3.2 we have, for $\alpha,\beta \in \Sigma_{pq}$

$$s_{c(\alpha)\alpha}(c(\beta)\beta) = s_\alpha(c(\beta)\beta) = c(\beta)s_\alpha(\beta) = c(s_\alpha(\beta))s_\alpha(\beta) \in \Sigma^c_{pq}.$$

Choose $\tilde\alpha \in \Sigma_p$ such that $c(\alpha) = (\tilde\alpha,\tilde\alpha)/(\alpha,\alpha)$ (if $\approx\alpha \in \Sigma_q$ is such that $c(\alpha) = (\approx\alpha,\approx\alpha)/(\alpha,\alpha)$ the proof is exactly the same). Then

$$(3.2)\qquad 2\,\frac{(c(\alpha)\alpha,c(\beta)\beta)}{(c(\alpha)\alpha,c(\alpha)\alpha)} = 2\,\frac{c(\beta)}{c(\alpha)}\,\frac{(\alpha,\beta)}{(\alpha,\alpha)} = 2\;c(\beta)\,\frac{(\tilde\alpha,\beta)}{(\tilde\alpha,\tilde\alpha)}\quad .$$

By Corollary 2.4 $c(\beta) = 1,2$ or 4. If $c(\beta) = 1$, then $\beta = \tilde\beta$ $(\tilde\beta\in\Sigma_p)$, thus (3.2) equals

$$2\,\frac{(\tilde\alpha,\tilde\beta)}{(\tilde\alpha,\tilde\alpha)}\,,$$

which is an integer since Σ_p is a root system. If $c(\beta) = 2$ or 4, then (3.2) equals

$$\tfrac12 c(\beta)\left\{2\,\frac{(\tilde\alpha,\tilde\beta)}{(\tilde\alpha,\tilde\alpha)} + 2\,\frac{(\tilde\alpha,\tau_2\tilde\beta)}{(\tilde\alpha,\tilde\alpha)}\right\},$$

which is an integer since Σ_p is a root system and $\tau_2\tilde\beta \in \Sigma_p$. □

As a corollary to Lemma 3.3 we obtain the following. Let $\alpha, 2\alpha \in \Sigma_{pq}$. Then we have the following possibilities:

$$(3.3)\qquad \begin{cases} \text{(a)} & c(2\alpha)2\alpha = 2c(\alpha)\alpha \;\Rightarrow\; c(\alpha) = c(2\alpha),\\ \text{(b)} & c(2\alpha)2\alpha = c(\alpha)\alpha \;\Rightarrow\; c(\alpha) = 2c(2\alpha),\\ \text{(c)} & c(2\alpha)2\alpha = \tfrac12 c(\alpha)\alpha \;\Rightarrow\; c(\alpha) = 4c(2\alpha).\end{cases}$$

Now: π_λ is of K,H-class 1 $\iff \lambda \in \mathbb{Z}^\ell$

$$\iff \frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in c(\alpha)\mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+_{pq}$$

(by Corollary 2.5)

$$\iff \frac{(\lambda,c(\alpha)\alpha)}{(c(\alpha)\alpha,c(\alpha)\alpha)} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+_{pq}. \tag{3.4}$$

Let $(\Sigma^c_{pq})'$ be the reduced root system defined by

$$(\Sigma^c_{pq})' := \{c(\alpha)\alpha \in \Sigma^c_{pq} \mid 2c(\alpha)\alpha \notin \Sigma^c_{pq}\}. \tag{3.5}$$

Let $\{\alpha_1,\ldots,\alpha_\ell\}$ be the base of Σ_{pq} corresponding to the chosen positive system Σ^+_{pq}. Let $\beta_j \in (\Sigma^c_{pq})'$ be such that $\beta_j = c_j\alpha_j$ with $c_j > 0$ ($j = 1,\ldots,\ell$). Then for all $\beta \in (\Sigma^c_{pq})'$ we can write $\beta = \Sigma^\ell_{j=1} d_j\beta_j$, with all $d_j \in \mathbb{Z}^+$ or all $d_j \in \mathbb{Z}^-$. Thus, by [17, Theorem 10.1'], the set $\{\beta_1,\ldots,\beta_\ell\}$ forms a base of $(\Sigma^c_{pq})'$.

It follows from (3.4) that π_λ is of K,H-class 1 if and only if λ is twice a dominant weight for $(\Sigma^c_{pq})'$ (dominant with respect to the base $\{\beta_1,\ldots,\beta_\ell\}$ of $(\Sigma^c_{pq})'$), namely $c(\alpha)\alpha$ is positive in $(\Sigma^c_{pq})'$ if and only if $\alpha \in \Sigma^+_{pq}$. Thus, if Λ_c is the weight lattice corresponding to $(\Sigma^c_{pq})'$, then $\mathbb{Z}^\ell = 2\Lambda_c$. Since the Weyl group of $(\Sigma^c_{pq})'$ clearly equals W_{pq}, this implies:

<u>PROPOSITION 3.4</u>. *$\mathbb{Z}^\ell$ is W_{pq}-invariant.*

Let μ'_j be the fundamental weight corresponding to β_j. Thus μ'_j is defined by

$$2\,\frac{(\mu'_j,\beta_i)}{(\beta_i,\beta_i)} = \delta_{ij}\ .$$

Put $\mu_j := 2\mu'_j$, then the μ_j generate $\mathbb{Z}^\ell$, by the above remarks, and thus we have proved:

<u>THEOREM 3.5</u>. $\lambda \in \mathbb{Z}^\ell_+ \iff \lambda = \Sigma^\ell_{j=1} n_j\mu_j$ ($n_j \in \mathbb{Z}$, $n_j \geq 0$ for $j = 1,\ldots,\ell$).

Let $\{\alpha_1,\ldots,\alpha_\ell\}$ be the base for Σ_{pq} as above. Then the following two lemmas are obvious.

<u>LEMMA 3.6</u>. $(\mu_i,\alpha_j) \neq 0 \iff i = j$ ($i,j = 1,\ldots,\ell$).

LEMMA 3.7. *Let* $\lambda \in \mathbb{Z}^{\ell}$. *Then*

$$\lambda \in \mathbb{Z}^{\ell}_{+} \iff (\lambda,\alpha_j) \geq 0 \quad (j = 1,\ldots,\ell).$$

REMARK 3.8. Let $\beta_j \in \Phi$ be such that $\hat{\beta}_j = \alpha_j$ $(j = 1,\ldots,\ell)$. Because of the obvious fact that $\hat{\mu}_i = \mu_i$ for all i, Lemma 3.6 also implies that $(\mu_i,\beta_j) \neq 0 \iff i = j$, $(\mu_i,\tilde{\beta}_j) \neq 0 \iff i = j$, and $(\mu_i,\approx{\beta}_j) \neq 0 \iff i = j$ $(i,j = 1,\ldots,\ell)$.

LEMMA 3.9. *Let* $\nu \in \mathbb{Z}^{\ell}$. *Then there exists* $s \in W_{pq}$ *such that* $s\nu \in \mathbb{Z}^{\ell}_{+}$.

PROOF. Apply HUMPHREYS [17, Theorem 10.3(a)]. □

CHAPTER 4

INTERTWINING FUNCTIONS

From now on we shall work with the compact real form U of the simply connected complex Lie group G_c. Then K and H^0 are the analytic subgroups of U corresponding to the Lie algebras $\mathfrak{k}$ and $\mathfrak{h}^0$. Let $\mathbb{D}_0(U)$ be the algebra of differential operators on U which are left-U-, and right-H^0-invariant.

DEFINITION 4.1. Let π_λ be a representation of U of K,H^0-class 1 on $V(\lambda)$, with highest weight $\lambda \in \mathbb{Z}_+^\ell$. Let $e_K \in V(\lambda)$ be a K-fixed vector, $e_H \in V(\lambda)$ an H^0-fixed vector. Let $(\cdot|\cdot)$ be an inner product in $V(\lambda)$ according to which π_λ is a unitary representation of U. Then the function φ_λ defined by

$$\varphi_\lambda(u) := (e_K|\pi_\lambda(u)e_H) \qquad (u\in U)$$

is called an *intertwining function*.

Thus φ_λ is determined by π_λ up to a constant factor . If $(e_K|e_H) \neq 0$, then $\varphi_\lambda(e) \neq 0$ and it is convenient to normalize φ_λ such that $\varphi_\lambda(e) = 1$. If, however, $\varphi_\lambda(e) = 0$, then we fix a normalization for φ_λ. In chapter 7 this (arbitrary) normalization will be somewhat refined.

REMARK 4.2. The earliest reference for the name intertwining function is JAMES & CONSTANTINE [18], see also DUNKL [4]. The name is motivated by the following characterization: Let π_λ be an irreducible representation of U of K,H^0-class 1 in a vector space $V(\lambda)$. Then there exist continuous embeddings j_K, j_H which realize π_λ in $L^2(U/K)$ and $L^2(U/H^0)$:

$$\begin{array}{ccc} & & L^2(U/K) \\ & \overset{j_K}{\nearrow} & \\ V(\lambda) & & \\ & \underset{j_H}{\searrow} & \\ & & L^2(U/H^0) \end{array}$$

Here $(j_K v)(x) := (v|\pi_\lambda(x)e_K)$, $(j_H v)(x) := (v|\pi_\lambda(x)e_H)$. Since these realizations yield equivalent representations of U, there exists an intertwining operator for these realizations. Such an operator is given by $j_H j_K^*$. Let $f \in L^2(U/K)$. Since $j_K^* f = \int_U f(x)\pi_\lambda(x)e_K dx$ we have:

$$(j_H j_K^* f)(x) = (j_K^* f|\pi_\lambda(x)e_H) =$$

$$= \int_U f(y)(e_K|\pi_\lambda(y^{-1}x)e_H)dy = (f*\varphi_\lambda)(x).$$

Thus the mapping $A: f \mapsto f*\varphi_\lambda : L^2(U/K) \mapsto L^2(U/H^0)$ is an intertwining operator. Hence $\varphi \in C(K\backslash U/H^0)$ is an intertwining function if and only if $\dim(L^2(U/K)*\varphi) > 0$, and there is no $\varphi' \in C(K\backslash U/H^0)$ such that $(0) \neq L^2(U/K)*\varphi' \subsetneq L^2(U/K)*\varphi$.

Let dk, dh denote the Haar measures on K and H^0, respectively, normalized such that $\int_K dk = \int_{H^0} dh = 1$.

<u>THEOREM 4.3</u>. *Let* φ *be a function on* U. *The following conditions are equivalent:*

(1) *There exists a* K,H^0*-class* 1 *representation* π_λ *such that* $\varphi = \varphi_\lambda$.

(2) φ *is continuous, not identically* 0 *and there exists a* $c \neq 0$ *such that*

$$\varphi(x)\overline{\varphi(z)}\varphi(y) = c\int_K\int_{H^0} \varphi(xhz^{-1}ky)\,dh\,dk \qquad \textit{for all } x,y,z \in U.$$

(3) φ *is* C^∞, *left-*K*-, and right-*H^0*-invariant, not identically* 0 *and there exists a function* $\lambda: \mathbb{D}_0(U) \mapsto \mathbb{C}$ *such that*

$$D\varphi = \lambda(D)\varphi \qquad \textit{for all } D \in \mathbb{D}_0(U).$$

PROOF (1) ⇒ (2) $$\int_K \int_{H^0} \pi_\lambda(hz^{-1}ky)e_H dh dk =$$

$$= \int_K \frac{(\pi_\lambda(z^{-1}ky)e_H | e_H)e_H}{(e_H|e_H)} dk$$

$$= \frac{(\int_K \pi_\lambda(ky)e_H dk \,|\, \pi_\lambda(z)e_H)}{(e_H|e_H)} e_H$$

$$= \frac{(\pi_\lambda(y)e_H|e_K)}{(e_K|e_K)} \frac{(e_K|\pi_\lambda(z)e_H)}{(e_H|e_H)} e_H$$

$$= \frac{\overline{\varphi_\lambda(y)}\varphi_\lambda(z)}{(e_K|e_K)(e_H|e_H)} e_H.$$

Hence: $$\int_K \int_{H^0} (e_K|\pi_\lambda(xhz^{-1}ky)e_H) dh dk$$

$$= \frac{\overline{\overline{\varphi_\lambda(y)}\varphi_\lambda(z)}}{(e_K|e_K)(e_H|e_H)} (\pi_\lambda(x^{-1})e_K|e_H)$$

$$= \frac{\varphi_\lambda(x)\overline{\varphi_\lambda(z)}\varphi_\lambda(y)}{(e_K|e_K)(e_H|e_H)} .$$

(2) ⇒ (3) Let $\rho \in C^\infty(K\backslash U/H^0)$ be such that

$$\int_U \rho(z)\overline{\varphi(z)}dz \neq 0.$$

Then:

$$\int_U \varphi(z)\rho(yz^{-1}x)dz = \int_U \varphi(xz^{-1}y)\rho(z)dz \tag{4.1}$$

$$= \int_U \int_K \int_{H^0} \varphi(xhz^{-1}ky)\rho(z)dz dh dk$$

$$= c.\{\int_U \varphi(z)\rho(z)dz\}\varphi(x)\varphi(y). \tag{4.2}$$

Because of the fact that $\rho \in C^\infty(K\backslash U/H^0)$, (4.1) is C^∞ in x. Hence (4.2) is C^∞ in x, thus φ is C^∞. Hence for all $D \in \mathbb{D}_0(U)$:

$$c\{\int_U \overline{\varphi(z)}\rho(z)dz\}(D\varphi)(x)\varphi(y) = \int_U \varphi(z)D_x\rho(yz^{-1}x)dz$$

$$= \int_U \varphi(z)(D\rho)(yz^{-1}x)dz =$$

$$= c\{\int_U \overline{\varphi(z)}(D\rho)(z)dz\}\varphi(x)\varphi(y).$$

($D\rho$ is again left-K-, right-H^0-invariant because $D \in \mathbf{D}_0(U)$). Hence:

$$(D\phi)(x) = \frac{\int_U \overline{\varphi(z)}(D\rho)(z)dz}{\int_U \overline{\varphi(z)}\rho(z)dz}\,\varphi(x).$$

(3) ⇒ (1) Let ψ be a spherical function corresponding to the symmetric pair (U,H^0) (in our setting this means that ψ is an H^0,H^0-intertwining function). Then $D\psi = \lambda_\psi(D)\psi$ for all $D \in \mathbf{D}_0(U)$, and the λ_ψ determine ψ completely, cf. HELGASON [10, ch.X]. Let ρ be a continuous function on U. Then:

$$(\psi*\rho*\varphi)(x) = \int_U\int_U \psi(y)\rho(y^{-1}z)\varphi(z^{-1}x)dydz$$

$$= \int_U\int_U \psi(xy)\rho(y^{-1}z)\varphi(z^{-1})dydz,$$

and

$$(\psi*\rho*\varphi)(e) = \int_U (\varphi*\psi)(y)\rho(y^{-1})dy.$$

Hence $\psi*\rho*\varphi$ is again H^0-biinvariant, and belongs to the space spanned by all right-translates of ψ. Hence:

$$\psi*\rho*\varphi = \{\int_U (\varphi*\psi)(y)\rho(y^{-1})dy\}\psi.$$

Also $D(\psi*\rho*\varphi) = \lambda(D)\ (\psi*\rho*\varphi)$ for all $D \in \mathbf{D}_0(U)$.
Hence:

$$\{\int_U (\varphi*\psi)(y)\rho(y^{-1})dy\}(\lambda(D)-\lambda_\psi(D))\psi = 0. \tag{4.3}$$

Equation (4.3) is valid for all continuous ρ on U and for all spherical functions ψ. Hence $\lambda_\psi \neq \lambda$ implies that $\varphi * \psi = 0$, and thus φ belongs to the irreducible representation of U which corresponds to the spherical function ψ with $\lambda = \lambda_\psi$. □

REMARK 4.4. The implication (1) ⇒ (2) already occurs in DUNK [4].

REMARK 4.5. By the equivalence (2) ⟺ (3) in Theorem 4.3 we see that it would have changed nothing if we had replaced $\mathbb{D}_0(U)$ by the algebra of left-K-, and right-U-invariant differential operators on U. This settles the conjecture in FLENSTED-JENSEN [6] for a compact Lie group. (NB. In [6] intertwining functions are called "spherical" functions). Thus we could even replace $\mathbb{D}_0(U)$ by Z(U): the algebra of left-, and right-U-invariant differential operators on U. The proof of (2) ⇒ (3) remains the same, and the proof of (3) ⇒ (1) even becomes simpler, by using the fact that every representation is completely determined by its infinitesimal character.

We are now able to generalize some of the results in section 2 in [31] to the case of intertwining functions. Therefore, we shall consider intertwining functions on $\exp \mathfrak{a}_c$, which is no real restriction because of the generalized Cartan decomposition for U, cf. chapter 6.

Let $f_0, \ldots, f_d$ be an orthonormal basis of $V(\lambda)$, such that f_j is a weight vector of weight λ_j $(j = 0, \ldots, d)$ with $\lambda_0 = \lambda$. Then

$$(4.4) \qquad \pi_\lambda(\exp X) f_j = e^{\lambda_j(X)} f_j \qquad (X \in \mathfrak{a}_c).$$

Hence, by analytic continuation from $i\mathfrak{a}_p + \mathfrak{a}_k$ to $\mathfrak{a}_c$,

$$(4.5) \qquad \bar{\varphi}_\lambda(\exp X) = \sum_{j=0}^{d} (e_K | f_j)(e_H | f_j) e^{\lambda_j(x)} \qquad (X \in \mathfrak{a}_c).$$

We have already shown $\lambda = \Sigma_{j=1}^{\ell} m_j \mu_j$ $(m_j \in \mathbb{Z})$ with all m_j nonnegative, and the following theorem is the analogue of Theorem 2.4 in [31].

THEOREM 4.6. *Suppose that*

$$(4.6) \qquad \varphi_\lambda(\exp X) = \sum_{i=0}^{d} c_i e^{\lambda_i(X)} \qquad (X \in \mathfrak{a}_c).$$

Then $c_i \neq 0$ *implies that* $\lambda_i = \Sigma_{j=1}^{\ell} n_j \mu_j$ $(n_j \in \mathbb{Z})$.

PROOF. We have to prove $\tilde{\lambda}_i = \lambda_i$, $\tilde{\tilde{\lambda}}_i = \lambda_i$ and $(\lambda_i,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ for all $\alpha \in \Sigma_p \cup \Sigma_q$. Just like Vretare we follow the corresponding proof for the highest weight of a K,K-class 1 representation in [33, Theorem 3.3.1.1], using $P_1 := \int_K \pi_\lambda(k)dk$ (projection of $V(\lambda)$ on $\mathbb{C}e_K$), and $P_2 := \int_{H^0} \pi_\lambda(h)dh$ (projection of $V(\lambda)$ on $\mathbb{C}e_H$).
Again, if $P_1f_i \neq 0$ and $P_2f_i \neq 0$, the proof works and we obtain $\tilde{\lambda}_i = \lambda_i$, $\tilde{\tilde{\lambda}}_i = \lambda_i$ and $(\lambda_i,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ for all $\alpha \in \Sigma_p \cup \Sigma_q$. However, if $P_1f_i = 0$, then $c_i = (e_K|f_i)(e_H|f_i) = (e_K|P_1f_i)(e_H|f_i) = 0$, and in the same way we obtain that $P_2f_i = 0$ implies $c_i = 0$. □

REMARK 4.7. It is easily seen that the coefficient of the highest weight in (4.6), i.e. c_0, is nonzero, cf. the proof of Theorem 3.3.1.1 in [33].

Let $\nu \in \mathfrak{a}^*_{pq}$. If there is an intertwining function φ_λ $(\lambda\in\mathbb{Z}^\ell_+)$ such that e^ν appears with nonzero coefficient in the "series expansion" (4.5) of $\bar{\varphi}_\lambda$, then we shall call ν an *appearing weight*. Next, we introduce a partial ordering $\preceq$ on $\mathfrak{a}^*_{pq}$ by putting for $\lambda_1,\lambda_2 \in \mathfrak{a}^*_{pq}$

$$\lambda_1 \preceq \lambda_2 \quad \text{if} \quad \lambda_2-\lambda_1 = \sum_{j=1}^{\ell} m_j\alpha_j \tag{4.7}$$

with all m_j nonnegative integers. Here $\{\alpha_1,\ldots,\alpha_\ell\}$ is the base for $\mathfrak{a}^*_{pq}$ from chapter 3. Write $\lambda_1 \prec \lambda_2$ if $\lambda_1 \preceq \lambda_2$ and $\lambda_1 \neq \lambda_2$.

LEMMA 4.8. *Let* $\lambda \in \mathbb{Z}^\ell_+$.

(1) *Let* λ_i *be a weight of* π_λ. *Then* $\hat{\lambda}_i \preceq \lambda$.
(2) $\#\{\nu \in \mathbb{Z}^\ell_+ : \nu \preceq \lambda\} < \infty$.
(3) $\mathbb{Z}^\ell_+$ *is the collection of all highest weights of representations of* K,H^0*-class 1.*

PROOF. (1) $\lambda_i = \lambda-\beta_1-\ldots-\beta_k$, with all $\beta_i \in \Phi^+$ (cf. HUMPRHEYS [17, Proposition 21.3]). Hence

$$\hat{\lambda}_i = \lambda-\hat{\beta}_1-\ldots-\hat{\beta}_k = \lambda- \sum_{i=1}^{\ell} m_i\alpha_i,$$

with all m_i nonnegative integers.

(2) $(\mu_j,\alpha_i)/(\alpha_i,\alpha_i) = c_i\delta_{ij}$ with $c_i \geq 0$. Also $(\alpha_i,\alpha_j) \leq 0$ if $i \neq j$, thus $(\mu_j,\mu_j) \geq 0$. Hence $\nu \in \mathbb{Z}^\ell_+$ can be written as

$$\nu = \sum_{i=1}^{\ell} a_i\alpha_i \quad (a_i\geq 0),$$

and thus also

$$\lambda = \sum_{i=1}^{\ell} b_i \alpha_i \qquad (b_i \geq 0).$$

Now if $\nu \leq \lambda$ then $0 \leq a_i \leq b_i$, and $b_i - a_i \in \mathbb{Z}$.

(3) Already known. □

CHAPTER 5

THE ACTION OF THE WEYL GROUP

It is a natural question whether our intertwining functions are invariant under the Weyl group. However, there are some complications here. Therefore, let us introduce another root system (cf. FLENSTED-JENSEN [8,§2]). Let $\mathfrak{g}^{+\sigma\theta}$ be the (reductive) Lie algebra of fixed points of the involution $\sigma\theta$ in $\mathfrak{g}$, thus $\mathfrak{g}^{+\sigma\theta} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$. Let Σ_0 be the root system corresponding to the pair $(\mathfrak{g}^{+\sigma\theta}, \mathfrak{a}_{pq})$. Then for any root $\alpha \in \Sigma_0$ we have $\varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp X)$ $(\lambda \in \mathbb{Z}_+^\ell, X \in i\mathfrak{a}_{pq})$, cf. Remark 5.3.

Of course, $\Sigma_0 \subset \Sigma_{pq}$. Now the above question leads to two questions here: (i) Can it occur that $\Sigma_0 \neq \Sigma_{pq}$?, and (ii) If $\Sigma_0 \neq \Sigma_{pq}$, is φ_λ invariant under all s_α, $\alpha \in \Sigma_{pq}$? Both questions will be answered in this chapter.

<u>EXAMPLE 5.1</u>. Let $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{k} := \mathfrak{o}(2)$, $\mathfrak{h} := \mathfrak{o}(1,1)$. Then we have:

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : \quad a,b \in \mathbb{R} \right\},$$

$$\mathfrak{q} = \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} : \quad a,b \in \mathbb{R} \right\}.$$

Hence

$$\mathfrak{p} \cap \mathfrak{q} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : \quad a \in \mathbb{R} \right\},$$

and $\mathfrak{k} \cap \mathfrak{h} = (0)$, thus $\mathfrak{g}^{+\sigma\theta} = \mathfrak{p} \cap \mathfrak{q}$. Since $\mathfrak{p} \cap \mathfrak{q}$ is abelian we can put $\mathfrak{a}_{pq} := \mathfrak{p} \cap \mathfrak{q}$. Then $\mathfrak{a}_{pq}$ is a Cartan subalgebra for $\mathfrak{g}$, thus $\mathfrak{a}_{kq} = \mathfrak{a}_{ph} = \mathfrak{a}_{kh} = (0)$. Now $\Sigma_{pq} = \{2a,-2a\}$, whereas Σ_0 is void, $\mathfrak{g}^{+\sigma\theta}$ being abelian. Thus $\Sigma_0 \neq \Sigma_{pq}$.

Let $\alpha \in \Sigma_{pq}$. If $X \in \mathfrak{g}_\alpha$, then obviously $\sigma\theta X \in \mathfrak{g}_\alpha$. Hence, if we define $\mathfrak{g}^{-\sigma\theta}$ to be the -1 eigenspace of $\sigma\theta$ in $\mathfrak{g}$ (thus $\mathfrak{g}^{-\sigma\theta} = \mathfrak{k} \cap \mathfrak{q} + \mathfrak{p} \cap \mathfrak{h}$), we have the direct sum decomposition $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} + \mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta}$. Let $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ be such that $\sigma\theta X_\alpha = X_\alpha$ or $\sigma\theta X_\alpha = -X_\alpha$. Normalize X_α such that

$$B(X_\alpha,\theta X_\alpha) = -\frac{2}{(\alpha,\alpha)} .$$

Then $X_\alpha, \theta X_\alpha$ and $H_\alpha := -[X_\alpha,\theta X_\alpha]$ form a standard basis for a $\mathfrak{sl}(2,\mathbb{R})$. Define $A_\alpha \in \mathfrak{a}_{pq}$ by

$$B(X,A_\alpha) = \alpha(X) \qquad \text{for all } X \in \mathfrak{a}_{pq}.$$

Then $H_\alpha = 2/(\alpha,\alpha)\, A_\alpha$. Under the identification of $\mathfrak{a}_{pq}$ and $\mathfrak{a}^*_{pq}$ we have $s_\alpha X = X - 2\,\alpha(X)/\alpha(A_\alpha)\, A_\alpha$ for all $X \in \mathfrak{a}_{pq}$. Put

$$k_\alpha := \exp \tfrac{1}{2}\pi(X_\alpha+\theta X_\alpha), \tag{5.1}$$

and

$$p_\alpha := \exp \tfrac{1}{2}\pi i(X_\alpha-\theta X_\alpha). \tag{5.2}$$

Then $k_\alpha \in K$. Also

$$\mathrm{Ad}(k_\alpha)X = s_\alpha X \qquad (X\in\mathfrak{a}_{pq}),$$

and

$$\mathrm{Ad}(p_\alpha)X = s_\alpha X \qquad (X\in\mathfrak{a}_{pq}).$$

<u>PROPOSITION 5.2</u>. *If* $\mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} \neq (0)$, *then for all* $\lambda \in \mathbb{Z}^\ell_+$:

$$\varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp X) \qquad (X\in i\mathfrak{a}_{pq}).$$

<u>PROOF</u>. Let $X_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta}$. Then $\sigma\theta X_\alpha = X_\alpha$, hence $k_\alpha \in K \cap H^0$, by (5.1). Thus, by Definition 4.1, we have for all $X \in i\mathfrak{a}_{pq}$:

$$\begin{aligned}\varphi_\lambda(\exp s_\alpha X) &= (e_K|\pi_\lambda(\exp s_\alpha X)e_H) = (\pi_\lambda(k_\alpha^{-1})e_K|\pi_\lambda(\exp X)\pi_\lambda(k_\alpha^{-1})e_H)\\ &= (e_K|\pi_\lambda(\exp X)e_H) = \varphi_\lambda(\exp X). \quad \square\end{aligned}$$

REMARK 5.3. The condition $\mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} \neq (0)$ means that $\alpha \in \Sigma_0$; Thus Propositions 5.2 states that φ_λ is invariant under W_0, the Weyl group of Σ_0.

LEMMA 5.4. $p_\alpha k_\alpha = \exp(-\frac{1}{2}\pi i H_\alpha)$.

PROOF. Let U_α be the analytic subgroup of U with Lie algebra spanned by $\{X_\alpha+\theta X_\alpha,\ i(X_\alpha-\theta X_\alpha),\ iH_\alpha\}$. Then U_α is compact with SU(2) as simply connected covering group. The lemma now follows by a simple calculation in SU(2), using the identification:

$$X_\alpha + \theta X_\alpha \quad \leftrightarrow \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$i(X_\alpha-\theta X_\alpha) \quad \leftrightarrow \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$iH_\alpha \quad \leftrightarrow \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

of $X_\alpha + \theta X_\alpha$, $i(X_\alpha-\theta X_\alpha)$ and iH_α with elements of $\mathfrak{su}(2)$. □

PROPOSITION 5.5. *If* $\mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta} \neq (0)$, *then for all* $\lambda \in \mathbb{Z}_+^\ell$:

$$\varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_\alpha)) \qquad (X \in i\mathfrak{a}_{pq}).$$

PROOF. Let $X_\alpha \in \mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta}$. Then $\sigma\theta X_\alpha = -X_\alpha$, hence $k_\alpha \in K$, by (5.1), and $p_\alpha \in H^0$, by (5.2). Thus, by Definition 4.1 and Lemma 5.4, we have for all $X \in i\mathfrak{a}_{pq}$:

$$\begin{aligned}
\varphi_\lambda(\exp s_\alpha X) &= (e_K|\pi_\lambda(k_\alpha)\pi_\lambda(\exp X)\pi_\lambda(k_\alpha^{-1})e_H) \\
&= (e_K|\pi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_\alpha))\pi_\lambda(p_\alpha)e_H) \\
&= (e_K|\pi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_\alpha))e_H) \\
&= \varphi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_\alpha)). \quad \square
\end{aligned}$$

COROLLARY 5.6. *If* $\mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta} \neq (0)$ *and* $\mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} \neq (0)$, *then*

$$\frac{(\lambda_j,\alpha)}{(\alpha,\alpha)} \in 2\mathbb{Z}$$

for all appearing weights λ_j.

PROOF. By Proposition 5.2 and Proposition 5.5 we have

$$\varphi_\lambda(\exp X) = \varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_\alpha)).$$

But, by (4.3):

$$(5.3)\qquad \bar{\varphi}_\lambda(\exp X) = \sum_{j=0}^{d} c_j\, e^{\lambda_j(X)} \qquad \text{for all } X \in i\mathfrak{a}_{pq}.$$

Hence:

$$\varphi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_\alpha)) = \sum_{j=0}^{d} c_j e^{\lambda_j(X+\frac{1}{2}\pi i H_\alpha)}$$

$$(5.4)\qquad = \sum_{j=0}^{d} c_j e^{\pi i \frac{(\lambda_j,\alpha)}{(\alpha,\alpha)}} e^{\lambda_j(X)} \qquad \text{for all } X \in i\mathfrak{a}_{pq}.$$

(5.4) being equal to (5.3), the corollary follows. □

So far we have set up some symmetries for the action of the Weyl group on the function φ_λ. However, the question remains whether for $\alpha \in \Sigma_{pq}$, $\alpha \notin \Sigma_0$, $\varphi_\lambda(\exp s_\alpha X) \neq \varphi_\lambda(\exp X)$ $(X \in i\mathfrak{a}_{pq})$. Therefore, let us consider the following example.

Let $U := SU(2)$, and let H_ℓ be the irreducible $SU(2)$-module of dimension $2\ell+1$ $(\ell \in \frac{1}{2}\mathbb{Z}^+)$ with orthonormal basis ψ_n^ℓ $(n = -\ell,-\ell+1,\ldots,\ell)$ as considered in KOORNWINDER [21], see also VILENKIN [29, ch.III]. Here the ψ_n^ℓ are also weight vectors with respect to $i\mathfrak{a}_{pq}$. H_ℓ is the space of homogeneous polynomials of degree 2ℓ in two complex variables x and y, and ψ_n^ℓ is defined by $\psi_n^\ell(x,y) = \binom{2\ell}{\ell-n}^{\frac{1}{2}} x^{\ell-n} y^{\ell+n}$. Define a representation π_ℓ of U on H_ℓ by

$$\pi_\ell\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} f(x,y) := f(\alpha x+\gamma y, \beta x+\delta y),$$

then π_ℓ is a unitary irreducible representation of U, and each unitary irreducible representation of U is equivalent to some π_ℓ, cf. [29, Theorem III. 2.5.1]. Let $d\pi_\ell$ denote the differential of π_ℓ. Then (cf. Example 5.1)

$$(5.5)\qquad d\pi_\ell\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} e_K = 0,$$

$$(5.6)\qquad d\pi_\ell\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} e_H = 0.$$

Now (5.5) and (5.6) determine e_K and e_H up to a constant factor. Namely

$$d\pi_\ell\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\psi_n^\ell = -\sqrt{(\ell-n)(\ell+n+1)}\ \psi_{n+1}^\ell + \sqrt{(\ell-n+1)(\ell+n)}\,\psi_{n-1}^\ell .$$

Hence if $d\pi_\ell\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\left\{\Sigma_{n=-\ell}^{\ell} c_n^\ell \psi_n^\ell\right\} = 0$, then

$$c_{n-1}^\ell = \sqrt{\frac{(\ell-n)(\ell+n+1)}{(\ell-n+1)(\ell+n)}}\ c_{n+1}^\ell ,$$

and $c_{\ell-1}^\ell = 0 = c_{-\ell+1}^\ell$. Hence $\ell \notin \mathbb{Z}$ implies $e_K = 0$, and if $\ell \in \mathbb{Z}$

$$e_K = c.\ \sum_{\substack{n=-\ell\\ \ell-n\in 2\mathbb{Z}}}^{\ell} \sqrt{\frac{(\frac12)_{\frac12(\ell-n)}(\frac12)_{\frac12(\ell+n)}}{(\frac12(\ell-n))!\,(\frac12(\ell+n))!}}\ \psi_n^\ell . \tag{5.7}$$

The same reasoning shows that $\ell \notin \mathbb{Z}$ implies $e_H = 0$, and for $\ell \in \mathbb{Z}$

$$e_H = c.\ \sum_{\substack{n=-\ell\\ \ell-n\in 2\mathbb{Z}}}^{\ell} (-1)^{\frac12(\ell-n)} \sqrt{\frac{(\frac12)_{\frac12(\ell-n)}(\frac12)_{\frac12(\ell+n)}}{(\frac12(\ell-n))!\,(\frac12(\ell+n))!}}\ \psi_n^\ell . \tag{5.8}$$

e_K and e_H both being nonzero, ℓ has to be an integer, ie. $\ell \in \mathbb{Z}$. Now let $X \in i\mathfrak{a}_{pq}$. Then for a certain $\phi \in \mathbb{R}$ we have

$$X = \begin{pmatrix}-i\phi & 0\\ 0 & i\phi\end{pmatrix},$$

and for the spherical functions on U with respect to K we get, by (5.7):

$$(e_K|\pi_\ell(\exp X)e_K) = c.\ \sum_{\substack{n=-\ell\\ \ell-n\in 2\mathbb{Z}}}^{\ell} \frac{(\frac12)_{\frac12(\ell-n)}(\frac12)_{\frac12(\ell+n)}}{(\frac12(\ell-n))!\,(\frac12(\ell+n))!}\, e^{2in\phi}$$
$$= c.\ P_\ell(\cos 2\phi).$$

Here P_ℓ is a *Legendre polynomial*. By (5.7) and (5.8) we get the following expression for the intertwining functions on U:

$$\varphi_\ell(\exp X) = (e_K|\pi_\ell(\exp X)e_H) = c.\ \sum_{\substack{n=-\ell\\ \ell-n\in 2\mathbb{Z}}}^{\ell} \frac{(\frac12)_{\frac12(\ell-n)}(\frac12)_{\frac12(\ell+n)}}{(\frac12(\ell-n))!\,(\frac12(\ell+n))!}(-1)^{\frac12(\ell-n)} e^{2in\phi}$$
$$= c.\ P_\ell(\cos(2\phi-\tfrac12\pi)).$$

Let s be the nontrivial element of the Weyl group of Σ_{pq}, then $sX = -X$ ($X\in i\mathfrak{a}_{pq}$). Then

$$\varphi_\ell(\exp sX) = \varphi_\ell(\exp(-X)) = c.\ P_\ell(\cos(2\phi+\tfrac12\pi)).$$

Thus in this case we obtain that for $s \notin W_0, \varphi_\ell(\exp X) \neq \varphi_\ell(\exp sX)$.

LEMMA 5.7. *$\mathbb{Z}^\ell$ is the collection of all appearing weights. For given $\lambda \in \mathbb{Z}_+^\ell$ the collection of appearing weights is invariant under* W_{pq}.

PROOF. According to (4.5) we can write

$$\varphi_\lambda(\exp X) = \sum_\mu c_\mu e^{\mu(X)} \quad (X \in i\mathfrak{a}_{pq}). \tag{5.9}$$

We claim that for all $\alpha \in \Sigma_{pq}, c_\mu \neq 0$ if and only if $c_{s_\alpha\mu} \neq 0$. Indeed, if there exists an $X_\alpha \in \mathfrak{g}_\alpha$ such that $\sigma\theta X_\alpha = X_\alpha$, then $\varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp X)$, and in this case the assertion follows from (5.9). So assume that $\sigma\theta X_\alpha = -X_\alpha$. Then it follows from (5.9) that

$$\bar{\varphi}_\lambda(\exp(s_\alpha X + \tfrac{1}{2}\pi i H_\alpha)) = \sum_\mu c_\mu e^{\mu(s_\alpha X)} e^{\frac{1}{2}\pi i \mu(H_\alpha)}. \tag{5.10}$$

But, according to Lemma 5.5 $\bar{\varphi}_\lambda(\exp X) = \bar{\varphi}_\lambda(\exp(s_\alpha X + \frac{1}{2}\pi i H_\alpha))$. Thus, by (5.9) and (5.10)

$$\sum_\mu c_\mu e^{\mu(X)} = \sum_\mu c_\mu e^{\frac{1}{2}\pi i \mu(H_\alpha)} e^{(s_\alpha\mu)(X)}$$

$$= \sum_\mu c_{s_\alpha\mu} e^{\frac{1}{2}\pi i (s_\alpha\mu)(H_\alpha)} e^{\mu(X)}.$$

Hence

$$c_\mu = c_{s_\alpha\mu} \cdot e^{\frac{1}{2}\pi i (s_\alpha\mu)(H_\alpha)}. \tag{5.11}$$

Now (5.11) implies that $c_\mu \neq 0$ if and only if $c_{s_\alpha\mu} \neq 0$.

Let ν be an appearing weight, then $\nu \in \mathbb{Z}^\ell$ by Theorem 4.6. Conversely, let $\nu \in \mathbb{Z}^\ell$. Then there exists $s \in W_{pq}$ such that $s\nu \in \mathbb{Z}_+^\ell$, by Lemma 3.9. Thus $c_{s\nu} \neq 0$ in the expansion (5.9) of $\bar{\varphi}_{s\nu}$ ($s\nu \in \mathbb{Z}_+^\ell$), hence by what is said above $c_\nu \neq 0$ in the expansion of $\bar{\varphi}_{s\nu}$. Hence ν is an appearing weight. □

LEMMA 5.8. *Let $\lambda_1, \lambda_2 \in \mathbb{Z}_+^\ell$. There exists a function $c_{\lambda_1,\lambda_2} : \mathbb{Z}_+^\ell \to \mathbb{C}$ such that*

$$\varphi_{\lambda_1}\varphi_{\lambda_2} = \sum_{\nu \leq \lambda_1+\lambda_2} c_{\lambda_1,\lambda_2}(\nu)\varphi_\nu, \tag{5.12}$$

and

$$c_{\lambda_1,\lambda_2}(\lambda_1+\lambda_2) \neq 0.$$

PROOF. $\pi_{\lambda_1}\otimes\pi_{\lambda_2}$ is a representation of U of highest weight $\lambda_1+\lambda_2$. Let V be the representation space of $\pi_{\lambda_1}\otimes\pi_{\lambda_2}$, then we have the following direct sum decomposition

$$(5.13)\qquad V = \sum_{j=1}^{n\,\oplus} V_j$$

with V_j irreducible. Let π_j be the representation of U on V_j. Let $e^i_K(e^i_H)$ denote the K-fixed (H^0-fixed) vector of π_{λ_i} in the representation space $V(\lambda_i)$ (i = 1,2). Then $e_K := e^1_K\otimes e^2_K$ is a K-fixed vector in V, $e_H := e^1_H\otimes e^2_H$ an H -fixed vector in V, hence $\pi_{\lambda_1}\otimes\pi_{\lambda_2}$ is of K,H^0-class 1. Thus we have

$$(5.14)\qquad e_K = \sum_{j=1}^{n} e_{K,j},$$

with $e_{K,j} \in U_j$ (j = 1,...,n). Because the decomposition (5.13) is direct the vectors $e_{K,j}$ in (5.14) are K-fixed (apply $\pi_{\lambda_1}\otimes\pi_{\lambda_2}(k)$ to both sides of (5.14) and use the directness of (5.13)). In the same way we obtain

$$(5.15)\qquad e_H = \sum_{j=1}^{n} e_{H,j},$$

with $e_{H,j} \in U_j$ and H^0-fixed vector (j = 1,...,n). Hence we have for all u ∈ U, by using (5.13), (5.14) and (5.15):

$$\varphi_{\lambda_1}(u)\varphi_{\lambda_2}(u) = (e_K|\pi_{\lambda_1}\otimes\pi_{\lambda_2}(u)e_H)$$

$$(5.16)\qquad = \sum_{j=1}^{n} (e_{K,j}|\pi_j(u)e_{H,j}).$$

If $e_{K,j} \neq 0$ and $e_{H,j} \neq 0$, then π_j is of K,H^0-class 1, hence $\pi_j = \pi_\nu$ with $\nu \in \mathbb{Z}^\ell_+$, and $\nu \prec \lambda_1+\lambda_2$, by Lemma 4.7(1). If however $e_{K,j} = 0$ or $e_{H,j} = 0$, then π_j does not occur in (5.16). Hence (5.12) follows. The fact that $c_{\lambda_1,\lambda_2}(\lambda_1+\lambda_2) \neq 0$ follows by considering the "series expansion" (4.5) of both sides of (5.12) and observing that the coefficient of $e^{\lambda_1+\lambda_2}$ is non-zero in the left-hand side. □

CHAPTER 6

THE GENERALIZED CARTAN DECOMPOSITION $U = KA_{pq}H^0$

Let notation be as in chapter 1, and put $B := \exp \mathfrak{a}_{pq}$. Then, as was proved in FLENSTED-JENSEN [7, Theorem 4.1(i)], one has the decomposition

(6.1) $\quad G = KBH.$

In the case $\sigma = \theta$ (6.1) states $G = KBK$, ie. the Cartan decomposition. Therefore we shall call (6.1) the *generalized Cartan decomposition* for G. In fact (6.1) is much older than the paper of Flensted-Jensen (it goes back to BERGER [2]), but nowadays it gets more attention because of the increased interest in harmonic analysis on pseudo-Riemannian symmetric spaces. For our intertwining functions on U, which are left-K-, and right-H^0-invariant we shall need an analogous decomposition for a compact Lie group. This generalized Cartan decomposition for U will be the main result in this chapter. But first we prove a Cartan decomposition for H^0.

Let U be a compact connected Lie group with commuting involutions θ and σ. Put $K := (U_\theta)_0$, $H := (U_\sigma)_0$.

<u>LEMMA 6.1</u>. $(H,(K\cap H)_0)$ *is a Riemannian symmetric pair.*

<u>PROOF</u>. Observe that $(K\cap H)_0 = (H_\theta)_0$. □

Lemma 6.1 enables us to use differential geometric methods, cf. eg. HELGASON [13, chapter I], for $H/(K\cap H)_0$. Therefore, introduce an H-invariant Riemannian structure on $H/(K\cap H)_0$. In this chapter only, write $\mathfrak{u} = \mathfrak{k}+\mathfrak{p} = \mathfrak{h}+\mathfrak{q}$ for the decompositions of the Lie algebra $\mathfrak{u}$ of U with respect to θ and σ respectively.

<u>LEMMA 6.2</u>. $H = (K\cap H)_0 \exp(\mathfrak{p}\cap\mathfrak{h})$.

PROOF. By Lemma 6.1 $H/(K\cap H)_0$ is a compact Riemannian symmetric space. Hence every closed, bounded subset of $H/(K\cap H)_0$ is compact, hence $H/(K\cap H)_0$ is complete, by [13, Theorem I.10.3]. Now identify $p \cap h$ with the tangent space to $H/(K\cap H)_0$ at o ($:= e(K\cap H)_0$), then it follows from [13, Proposition I.10.5] that $\mathrm{Exp}(p\cap h) = H/(K\cap H)_0$. □

Let b_{ph} be maximal abelian in $p \cap h$, and put $B_{ph} := \exp b_{ph}$.

LEMMA 6.3. $p \cap h = \bigcup_{k\in(K\cap H)_0} \mathrm{Ad}(k).b_{ph}$.

PROOF. h is a subalgebra of u, invariant under the Cartan involution θ, hence h is reductive. If h is semisimple, the lemma follows by [13, Lemma V.6.3]. So suppose h is not semisimple. Then $h = [h,h] + z(h)$ (direct sum), with $[h,h]$ semisimple and $z(h)$ the center of h ([17, Proposition 19.1]). The only part in the proof of [13, Lemma V.6.3] in which the semisimplicity of h would be used is $B|_{k\cap h\times k\cap h}$ is negative definite (here B denotes the Killing form on h), hence $B([\mathrm{Ad}(k_0)X,H],T) = 0$ for all $T \in k \cap h$ implies $[\mathrm{Ad}(k_0).X,H] = 0$ ($k_0\in K\cap H, X\in p\cap h, H\in b_{ph}$). But if h is reductive we can argue: $B([\mathrm{Ad}(k_0).X,H],T) = 0$ for all $T \in k \cap h$ implies $[\mathrm{Ad}(k_0).X,H] \in z(h) \cap [h,h] = (0)$, hence $[\mathrm{Ad}(k_0).X,H] = 0$ ($k_0\in K\cap H, X\in p\cap h, H\in b_{ph}$). Thus the proof of [13, Lemma V.6.3] also works in the case h is reductive. □

THEOREM 6.4. $H = (K\cap H)_0 B_{ph} (K\cap H)_0$.

PROOF. Let $h \in H$. Then we can write

$$(6.2)\qquad h = \ell_1 \exp X \qquad (\ell_1\in(K\cap H)_0, X\in p\cap h),$$

and

$$(6.3)\qquad X = \mathrm{Ad}(\ell_2)H_1 \qquad (\ell_2\in(K\cap H)_0, H_1\in b_{ph}),$$

because of Lemma 2.2 and Lemma 2.3, respectively. Combination of (6.2) and (6.3) yields

$$h = \ell_1 \exp(\mathrm{Ad}(\ell_2)H_1) = \ell_1\ell_2 \exp H_1\ell_2^{-1} \in (K\cap H_0)B_{ph}(K\cap H)_0. \quad □$$

Let notation be again as introduced in chapters 1,2. Let U_0 be the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $u_0 := u^{+\sigma\theta}$ ($u^{+\sigma\theta}$ and $u^{-\sigma\theta}$ are defined analogously to $g^{+\sigma\theta}$ and $g^{-\sigma\theta}$ in chapter 5). Thus

$\mathfrak{u}_0 = \mathfrak{k} \cap \mathfrak{h} + i(\mathfrak{p} \cap \mathfrak{q})$.

In the rest of this chapter we shall need Lemmas 6.1, 6.2, 6.3 and Theorem 6.4 also in the case where the pair (θ,σ) is replaced by the pair $(\theta,\sigma\theta)$. For later reference we shall state these results in a lemma. Therefore, put

$$(6.4) \qquad A_{pq} := \exp i\mathfrak{a}_{pq}.$$

Observe that $(K\cap U_0)_0 = (K\cap H^0)_0$.

LEMMA 6.5.
(1) $H^0 = \exp i(\mathfrak{p}\cap\mathfrak{h}).(K\cap H^0)$
(2) $U_0 = \exp i(\mathfrak{p}\cap\mathfrak{q}).(K\cap H^0)$
(3) $U_0 = (K\cap H^0)A_{pq}(K\cap H^0)$.

Let Exp be the exponential mapping in the space U/K.

LEMMA 6.6. *Left multiplication with* $\exp i(\mathfrak{p}\cap\mathfrak{h})$ *leaves* $\operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{h})$ *invariant.*

PROOF. $\exp i(\mathfrak{p}\cap\mathfrak{h})\exp i(\mathfrak{p}\cap\mathfrak{h}) \subseteq H^0 = \exp i(\mathfrak{p}\cap\mathfrak{h}).(K\cap H^0)$, by Lemma 6.5(1). Thus $\exp i(\mathfrak{p}\cap\mathfrak{h}) \operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{h}) \subset \operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{h})$. □

Now Lemma 6.6 has the following corollary:

COROLLARY 6.7. $\operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{h})$ *is a totally geodesic submanifold of* U/K.

NB. Note that Corollary 6.7 also follows from the fact that $i(\mathfrak{p}\cap\mathfrak{h})$ is a Lie triple system included in $i\mathfrak{p}$, as defined in [13,p.224], by using [13, Theorem IV.7.2].

LEMMA 6.8. $\operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{h})$ *is closed in* U/K.

PROOF. H^0 is closed in U, hence compact. Because of Lemma 6.5(1) we have $\operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{h}) = \pi(H^0)$, where $\pi: U \to U/K$ is the natural projection. Hence $\operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{h})$ is closed in U/K. □

PROPOSITION 6.9. $U = K \exp i(\mathfrak{p}\cap\mathfrak{q})\exp i(\mathfrak{p}\cap\mathfrak{h})$.

PROOF. We shall prove $U/K = \exp i(\mathfrak{p}\cap\mathfrak{h})\operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{q})$, which implies the proposition. Let $P \in U/K$. Let $X \in i(\mathfrak{p}\cap\mathfrak{h})$ be such that Exp X is an element of $\operatorname{Exp} i(\mathfrak{p}\cap\mathfrak{h})$ with minimal distance to P (such an X exists because of Lemma 6.8). Let $o := \pi(e)$, and put $Q := \exp(-X)P$. Then it follows from Lemma 6.6

that o is an element of Exp $i(\mathfrak{p}\cap\mathfrak{h})$ with minimal distance to Q. Let $\gamma(t) = \text{Exp}\ tY$ $(Y\in i\mathfrak{p})$ be a geodesic which realizes the minimal distance between o and Q (such a γ exists because of [13, Theorem I.10.4], U/K being a complete Riemannian manifold, cf. [13, Theorem I.10.3]). We shall prove that $Y \in i(\mathfrak{p}\cap\mathfrak{q})$, hence $P = (\exp X)Q = \exp X\ \text{Exp}\ t_0 Y \in \exp i(\mathfrak{p}\cap\mathfrak{h})\text{Exp}\ i(\mathfrak{p}\cap\mathfrak{q})$ $(t_0\in\mathbb{R})$.

Let W be an open ball around o in $i\mathfrak{p}$ of sufficient small radius such that Exp: $W \to V := \text{Exp}\ W$ is a diffeomorphism and, for any $Q_1, Q_2 \in V$, Q_1 and Q_2 can be joined by precisely one geodesic of minimal length, which lies entirely in V, cf. [13, Theorem I.9.9].

Let Q' be an element of γ lying in V between o and Q. Suppose Q' has a shorter distance to Exp $i(\mathfrak{p}\cap\mathfrak{h})$ than $d(Q',o)$ (d denoting the Riemannian metric in U/K), say to Exp Z $(Z\in i(\mathfrak{p}\cap\mathfrak{h}))$.

Then

$d(Q,\text{Exp}\ Z) \le d(Q,Q')+d(Q',\text{Exp}\ Z) < d(Q,Q')+d(Q',o) = d(Q,o)$, a contradiction, since o was the element of Exp $i(\mathfrak{p}\cap\mathfrak{h})$ with minimal distance to Q. So we may assume $Q \in V$.

V is a ball around o, hence V is σ-invariant, hence $\sigma Q \in V$. Let $\beta(t)$ be the unique geodesic in V which joins Q and σQ. Since β is unique, we have $\beta = \sigma\beta$. We claim $o \in \beta$. Namely, suppose $o \notin \beta$. Since $\beta = \sigma\beta$ there exists a $Q'' \in \beta$ such that $\sigma Q'' = Q''$, hence $\beta \cap \text{Exp}\ i(\mathfrak{p}\cap\mathfrak{h}) \ni Q''$. Now $Q'' \neq o$, since $o \notin \beta$. Let d_β be the distance between points along β, d_γ distance along γ. β minimalizes the distance between Q and σQ, and $d(Q,o) = d(\sigma Q,o)$. Hence:

$d_\beta(Q,Q'') = \frac{1}{2}d_\beta(Q,\sigma Q) < \frac{1}{2}(d_\gamma(Q,o)+d_{\sigma\gamma}(o,\sigma Q) = d_\gamma(Q,o)$, a contradiction.
Hence $o \in \beta$, hence $\beta = \gamma$.

Remember that $Y \in i\mathfrak{p}$ is such that $\gamma(t) = \text{Exp}\ tY$. Since $\beta = \gamma$, $\sigma\gamma(t) = \gamma(-t)$, hence $\sigma Y = -Y$, ie. $Y \in i(\mathfrak{p}\cap\mathfrak{q})$, which proves the proposition by the above remarks. □

<u>THEOREM 6.10</u>. (*Generalized Cartan decomposition for* U)

$$U = KA_{pq}H^0.$$

<u>PROOF</u>. Let $u \in U$. Then, by Proposition 6.9 there exists an $X \in i(\mathfrak{p}\cap\mathfrak{q})$ such that:

(6.5) $\quad u \in K\ \exp X\ \exp i(\mathfrak{p}\cap\mathfrak{h})$.

By Lemma 6.5(3) there exists an $a \in A_{pq}$ such that:

(6.6) $$\exp X \in (K\cap H^0)a(K\cap H^0).$$

Combination of (6.5) and (6.6) gives $u \in KaH^0$. □

REMARK 6.11. The above proof of the generalized Cartan decomposition also applies to the case of a noncompact semisimple Lie group. In HOOGENBOOM [16] we present a proof of the generalized Cartan decomposition for a general semisimple Lie group G.

CHAPTER 7

INTERTWINING FUNCTIONS ON THE COMPACT GROUP U

LEMMA 7.1. *Let φ be a function on* U. *Then φ is an intertwining function on* U *if and only if $\bar{\varphi}$ is an intertwining function on* U.

PROOF. This follows immediately from Theorem 4.3(2). □

REMARK 7.2. Let $\lambda \in \mathbb{Z}_+^{\ell}$, and let π_λ be the corresponding unitary representation of U of K,H^0-class 1, φ_λ the corresponding intertwining function. Then $\bar{\varphi}_\lambda$ corresponds to the contragredient representation $\check{\pi}_\lambda$ of U, which is also unitary and of K,H^0-class 1.

Let φ_λ be an intertwining function. Then $\bar{\varphi}_\lambda$ is also an intertwining function, by Lemma 7.1. Hence there exists $\lambda' \in \mathbb{Z}_+^{\ell}$ such that $\bar{\varphi}_\lambda = \text{cst}.\varphi_{\lambda'}$. Normalize the φ_λ such that $\bar{\varphi}_\lambda = \varphi_{\lambda'}$ (cf. the remarks at the beginning of chapter 4).

REMARK 7.3. Let $w_0 \in W_{pq}$ be such that $w_0(\mathbb{Z}_+^{\ell}) = -\mathbb{Z}_+^{\ell}$. Then $\lambda' = -w_0(\lambda)$.

Let du be a Haar measure on U, normalized by $\int_U du = 1$. Let φ_{λ_1} and φ_{λ_2} be intertwining functions on U. Then, because of the fact that φ_{λ_1} and φ_{λ_2} belong to different representations of U whenever $\lambda_1 \neq \lambda_2$, it follows that

$$(7.1) \qquad \int_U \varphi_{\lambda_1}(u)\bar{\varphi}_{\lambda_2}(u) = 0 \qquad (\lambda_1 \neq \lambda_2).$$

Define an inner product $(\cdot,\cdot)$ on the space of all L^2-functions on U by

$$(7.2) \qquad (\varphi,\psi) := \int_U \varphi(u)\bar{\psi}(u)du \qquad (\varphi,\psi \in L^2(U)).$$

THEOREM 7.4. *Let $\lambda,\mu \in \mathbb{Z}_+^{\ell}$. Then there exists a function $d_{\lambda,\mu}: \mathbb{Z}_+^{\ell} \to \mathbb{C}$ such that:*

(7.3) $\qquad \varphi_\mu(u)\varphi_\lambda(u) = \sum_{\substack{-\mu'\preceq\nu\preceq\mu \\ \lambda+\nu\in\mathbb{Z}_+^\ell}} d_{\lambda,\mu}(\nu)\varphi_{\lambda+\mu}(\nu) \qquad (u\in U).$

PROOF. (7.1) implies, together with Lemma 5.8, that:

(7.4) $\qquad (\varphi_\mu\varphi_\lambda,\varphi_\nu) \neq 0 \Rightarrow \nu \preceq \lambda + \mu.$

Also

(7.5) $\qquad (\varphi_\mu\varphi_\lambda,\varphi_\nu) = \overline{(\varphi_{\mu'}\varphi_\nu,\varphi_\lambda)} \neq 0 \Rightarrow \lambda \preceq \mu' + \nu.$

It follows from (7.4) and (7.5) that $c_{\mu,\lambda}(\nu) \neq 0$ implies that $-\mu' \preceq \nu-\lambda \preceq \mu$. (For the definition of $c_{\mu,\lambda}(\nu)$ see Lemma 5.8). This proves the theorem. □

Observe that the number of terms in the sum (7.3) is independent of λ, hence (7.3) can be seen as a recurrence relation for the intertwining functions.

If $\lambda = (m_1,\ldots,m_\ell) \in \mathbb{Z}_+^\ell$, denote the monomial $x_1^{m_1}\ldots x_\ell^{m_\ell}$ by x^λ. Thus we can define a *polynomial* $P(x) = \Sigma_{\nu\leq\lambda}\, \Gamma_\nu\, x^\nu (\Gamma_\nu\in\mathbb{C}$ for all $\nu)$ with $\Gamma_\lambda \neq 0$ to be of *degree* λ. For $i = 1,\ldots,\ell$ put $\varphi_i := \varphi_{\mu_i}$. For a polynomial $P(x)$ as above Lemma 5.8 implies

(7.6) $\qquad P(\varphi) = P(\varphi_1,\ldots,\varphi_\ell) = \sum_{\nu\leq\lambda} \Gamma_\nu\varphi_\nu \qquad (\Gamma_\lambda\neq 0).$

So we can speak of polynomials in the variable $\varphi = (\varphi_1,\ldots,\varphi_\ell)$, and it is clear that $P(\varphi(u)) = 0$ for all $u \in U$ implies that P is identically zero.

Let $<$ denote the *lexicographic ordering* on $\mathfrak{a}^*_{pq}$ with respect to an orthogonal basis $\{\rho,e_2,\ldots,e_\ell\}$ of $\mathfrak{a}^*_{pq}$. Here we have put $\rho := \frac{1}{2}\,\Sigma_{\alpha\in\Sigma^+_{pq}}\, m_\alpha\alpha$, where $m_\alpha = \dim \mathfrak{g}_\alpha$. (NB. any other total ordering $<$ which satisfies (1) $\lambda_1 \prec \lambda_2$ implies $\lambda_1 < \lambda_2$, and (2) $\#\{\nu \in \mathbb{Z}_+^\ell : \nu < \lambda\} < \infty$ for all $\lambda_1,\lambda_2 \in \mathbb{Z}^\ell$, $\lambda \in \mathbb{Z}_+^\ell$ will do. It is clear that the total ordering $<$ defined above satisfies conditions (1) and (2)).

The proofs of the following two theorems are taken from VRETARE [31]. For reason of completeness we shall reproduce them here.

THEOREM 7.5. *φ_λ is a polynomial of degree λ in the variable φ.*

PROOF. We prove the theorem by induction with respect to the total ordering $<$ defined above. If $\lambda = 0$ the theorem is obvious. Suppose $0 \neq \lambda \in \mathbb{Z}_+^\ell$, and the theorem is true for all $\nu \in \mathbb{Z}_+^\ell$, $\nu < \lambda$. Write $\lambda = \Sigma_{i=1}^\ell m_i\mu_i$. Since $\lambda \neq 0$ there is a j such that $m_j \neq 0$, hence $\lambda-\mu_j \in \mathbb{Z}_+^\ell$. Now, by Lemma 5.8

$$\varphi_\lambda = c.\varphi_{\mu_j}\varphi_{\lambda-\mu_j} + \sum_{\nu<\lambda} c_\nu\varphi_\nu,$$

with $c \neq 0$. This proves the theorem by the induction hypothesis. □

THEOREM 7.6. *For all* k $(1\le k\le\ell)$ *there exists a* j $(1\le j\le\ell)$ *such that* $\bar{\varphi}_k = \varphi_j$.

PROOF. Let $P(\varphi)$ denote the polynomial $\varphi_{\mu_k'}$. Then

$$\varphi_{\mu_k} = \bar{\varphi}_{\mu_k'} = \overline{P(\varphi)} = \bar{P}(\varphi_{\mu_1'},\dots,\varphi_{\mu_\ell'}).$$

If the degree of P is $\Sigma_{j=1}^\ell m_j\mu_j$ it follows that φ_{μ_k} is a polynomial of degree $\Sigma_{j=1}^\ell m_j\mu_j'$. Since $\mu_j' \in \mathbb{Z}_+^\ell$ this is possible only if $\mu_k = \mu_j'$ for some j. □

DEFINITION 7.7. Let $X \in i\mathfrak{a}_{pq}$. Define a function $F\colon i\mathfrak{a}_{pq} \to \mathbb{C}^\ell$ by

$$F(X) := (\varphi_1(\exp X),\dots,\varphi_\ell(\exp X)).$$

Let $\Omega_0 \subset \mathbb{C}^\ell$ be defined by $\Omega_0 := F(i\mathfrak{a}_{pq})$.

Theorem 7.6 implies that $\bar{\varphi}_k \in \{\varphi_1,\dots,\varphi_\ell\}$ for $1 \le k \le \ell$. Thus we can renumber the φ_j such that

$$(7.7)\qquad \bar{\varphi}_j = \begin{cases} \varphi_{j+j_0} & \text{if} \quad j = 1,\dots,j_0, \\ \varphi_{j-j_0} & \text{if} \quad j = j_0+1,\dots,2j_0, \\ \varphi_j & \text{if} \quad j = 2j_0+1,\dots,\ell. \end{cases}$$

In the rest of this monograph we shall always assume that the φ_i are numbered according to (7.7).

DEFINITION 7.8. Let $\psi\colon \mathbb{C}^\ell \to \mathbb{C}^\ell$ be given by $\psi(z_1,\dots,z_\ell) = (x_1,\dots,x_\ell)$, with x_j defined by

$$x_j := \begin{cases} \frac{1}{2}(z_{j+j_0}+z_j) & \text{if} \quad j = 1,\ldots,j_0 \quad , \\ \frac{1}{2}(z_{j-j_0}-z_j) & \text{if} \quad j = j_0+1,\ldots,2j_0, \\ z_j & \text{if} \quad j = 2j_0+1,\ldots,\ell \ . \end{cases}$$

Let $\Omega \subset \mathbb{R}^\ell$ be defined by $\Omega := \psi(\Omega_0)$.

For $\alpha \in \Sigma_{pq}$ define V_α by $V_\alpha := \{\frac{(\mu,\alpha)}{(\alpha,\alpha)} : \mu \in \mathbb{Z}^\ell\}$. Then V is an additive subgroup of $\mathbb{Z}$, hence there exists a smallest positive element in V_α. Thus the following definition makes sense.

DEFINITION 7.9. $k(\alpha) := \min\limits_{\substack{\mu\in\mathbb{Z}^\ell \\ (\mu,\alpha)\neq 0}} \left| \frac{(\mu,\alpha)}{(\alpha,\alpha)} \right|, \quad \alpha \in \Sigma_{pq}$.

LEMMA 7.10. *Let* $s \in W_{pq}$. *Then* $k(s\alpha) = k(\alpha)$.

PROOF. $\mathbb{Z}^\ell$ is W_{pq}-invariant, by Lemma 4.5. Hence $V_{s\alpha} = V_\alpha$, thus $k(s\alpha) = k(\alpha)$. □

By using the techniques introduced in chapter 3, we have for $\alpha \in \Sigma_{pq}$:

$$k(\alpha) = \min_{\substack{\mu\in\mathbb{Z}^\ell \\ (\mu,\alpha)\neq 0}} \left| \frac{(\mu,\alpha)}{(\alpha,\alpha)} \right| = c(\alpha).\min_{\substack{\mu\in\mathbb{Z}^\ell \\ (\mu,\alpha)\neq 0}} \left| \frac{(\mu,c(\alpha)\alpha)}{(c(\alpha)\alpha,c(\alpha)\alpha)} \right|,$$

and, by (3.4),

$$\mu \in \mathbb{Z}^\ell \iff \frac{(\mu,c(\alpha)\alpha)}{(c(\alpha)\alpha,c(\alpha)\alpha)} \in \mathbb{Z} \quad \text{for all } \alpha \in \Sigma_{pq}.$$

Thus we have the following implications, by (3.3)

$$(7.8) \quad \begin{cases} \text{(a)} \quad \alpha \in \Sigma_{pq}, \frac{1}{2}\alpha, 2\alpha \notin \Sigma_{pq} \Rightarrow k(\alpha) = c(\alpha). \\ \text{(b)} \quad \alpha, 2\alpha \in \Sigma_{pq}. \text{ Then:} \\ \qquad c(\alpha) = c(2\alpha) \Rightarrow k(2\alpha) = c(2\alpha),\ k(\alpha) = 2c(\alpha) \\ \qquad c(\alpha) = 2c(2\alpha) \Rightarrow k(2\alpha) = c(2\alpha),\ k(\alpha) = c(\alpha) \\ \qquad c(\alpha) = 4c(2\alpha) \Rightarrow k(2\alpha) = 2c(2\alpha),\ k(\alpha) = c(\alpha). \end{cases}$$

Together with Corollary 2.4 (7.8) implies:

LEMMA 7.11. *For all* $\alpha \in \Sigma_{pq}, k(\alpha) = 1,2$ *or* 4.

Now, let Σ_0 be the set of all roots of the pair $(g^{+\sigma\theta}, a_{pq})$, cf. chapter 5. That is, $\Sigma_0 = \{\alpha \in \Sigma_{pq}: \mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} \neq (0)\}$. Let Σ_1 be the complement of Σ_0 in Σ_{pq}, that is $\Sigma_1 = \{\alpha \in \Sigma_{pq}: \mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} = (0)\}$. Put $\Sigma'_{pq} := \{\alpha \in \Sigma^+_{pq}: \frac{1}{2}\alpha \notin \Sigma^+_{pq}\}$, and for $i = 0,1$ put $\Sigma'_i := \Sigma_i \cap \Sigma'_{pq}$.

THEOREM 7.12. $\det dF(X) = c.\Pi_{\alpha\in\Sigma'_0} \sin k(\alpha)\alpha(iX)\ \Pi_{\alpha\in\Sigma'_1} \sin k(\alpha)(\alpha(iX)-\frac{1}{2}\pi)$.

PROOF. Theorem 4.6 and Lemma 4.7(1) imply, just as in the spherical case (cf. VRETARE [31, Lemma 3.3]), that det dF is a linear combination of exponentials e^ν, $\nu \in \mathbb{Z}^\ell$, $\nu \preceq \lambda_0$, where we have put

$$\lambda_0 = \sum_{j=1}^{\ell} \mu_j. \tag{7.9}$$

Because of Proposition 5.2 and Proposition 5.5 the function φ_λ transforms under the action of the Weyl group W_{pq} as follows ($\alpha\in\Sigma_{pq}$):

$$\begin{cases} \mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} \neq (0) \Rightarrow \varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp X) \\ \mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta} \neq (0) \Rightarrow \varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp(X+\frac{1}{2}\pi i H_\alpha)), \end{cases} \tag{7.10}$$

for all $\lambda \in \mathbb{Z}^\ell_+$, $X \in i\mathfrak{a}_{pq}$. Since F is a combination of φ_λ's, it follows that det dF transforms under the Weyl group W_{pq} as follows ($\alpha\in\Sigma_{pq}$)

$$\begin{cases} \mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta} \neq (0) \Rightarrow \det dF(s_\alpha X) = \det ds_\alpha \det dF(X) \\ \mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta} \neq (0) \Rightarrow \det dF(s_\alpha X) = \det ds'_\alpha \det dF(X) \end{cases} \tag{7.11}$$

for all $X \in i\mathfrak{a}_{pq}$. Here we have put $s'_\alpha: X \to s_\alpha X - \frac{1}{2}\pi i H_\alpha$ for $\alpha \in \Sigma_1$, $X \in i\mathfrak{a}_{pq}$, and $s'_\alpha = s_\alpha$ for $\alpha \in \Sigma_0$. Thus we have that $\det ds'_\alpha = \det ds_\alpha = -1$.
For $X \in i\mathfrak{a}_{pq}$, put

$$G(X) := \prod_{\alpha\in\Sigma'_0} \sin k(\alpha)i\alpha(X) \prod_{\alpha\in\Sigma'_1} \sin k(\alpha)i\alpha(X+\tfrac{1}{4}\pi i H_\alpha). \tag{7.12}$$

Since a linear combination of exponentials $e^\nu, \nu \in \mathbb{Z}^\ell$, $\nu \preceq \lambda_0$ is uniquely determined, up to a constant factor, by the transformation properties (7.11), we only need to prove that G(X) is also a linear combination of exponentials $e^\nu, \nu \in \mathbb{Z}^\ell$, $\nu \preceq \lambda_0$ which transforms under s_α ($\alpha\in\Sigma_{pq}$) according to (7.11).

Let $\{\alpha_1,\ldots,\alpha_\ell\}$ be the base of Σ_{pq} from chapter 3. Then we know (cf. HUMPHREYS [17]) that s_{α_j} permutes the roots in Σ'_{pq} except α_j, and

$s_{\alpha_j}\alpha_j = -\alpha_j$. Let λ_1 be defined by

$$(7.13) \qquad \lambda_1 := \sum_{\alpha\in\Sigma'_{pq}} k(\alpha)\alpha,$$

and let α_j be a simple root. Then it follows from the above that

$$s_{\alpha_j}\lambda_1 = \lambda_1 - 2k(\alpha_j)\alpha_j \qquad \text{(by Lemma 7.10)}$$

$$(7.14) \qquad = \lambda_1 - 2\frac{(\mu_j,\alpha_j)}{(\alpha_j,\alpha_j)}\alpha_j \ .$$

Also

$$(7.15) \qquad s_{\alpha_j}\lambda_0 = \lambda_0 - 2\frac{(\mu_j,\alpha_j)}{(\alpha_j,\alpha_j)}\alpha_j,$$

by Lemma 3.6. Combination of (7.14) and (7.15) yields

$$s_{\alpha_j}(\lambda_1-\lambda_0) = \lambda_1-\lambda_0 \qquad \text{for } j = 1,\ldots,\ell.$$

Thus $\lambda_1 = \lambda_0$. This implies that $G(X)$ is also a linear combination of exponentials e^{ν}, with $\nu \in \mathbb{Z}^{\ell}$, $\nu \preceq \lambda_1 = \lambda_0$. Now we only need to prove that $G(X)$ transforms under s_α (with α simple) according to (7.11), because the s_α generate W_{pq}. Therefore, let $\alpha \in \{\alpha_1,\ldots,\alpha_\ell\}$, and put $s := s_\alpha, s' := s'_\alpha$. If $\alpha \in \Sigma'_0$ then $\sin k(\alpha)i\alpha(sX) = -\sin k(\alpha)i\alpha(X)$. If $\alpha \in \Sigma'_1$ then $\sin k(\alpha)i\alpha(s'X+\frac{1}{4}\pi iH_\alpha) = -\sin k(\alpha)i\alpha(X+\frac{1}{4}\pi iH_\alpha)$.

We claim that s leaves the rest of $G(X)$ invariant. If $\alpha \in \Sigma'_0$ then, because of the fact that $\Sigma'_0 \cup -\Sigma'_0$ is a reduced root system, s permutes $\Sigma'_0\backslash\{\alpha\}$, and $s\alpha = -\alpha$ (HUMPHREYS [17, Lemma 10.2B]), and thus s also permutes Σ'_1. Thus if $\alpha \in \Sigma'_0$, then $\det dF(sX) = -\det dF(X)$, by Lemma 7.10 and the fact that $sH_\beta = H_{s\beta}$ $(\beta\in\Sigma_{pq})$.

If $\alpha \in \Sigma'_1$ then the above reasoning also applies to those $\beta \in \Sigma'_0$ for which $s\beta \in \Sigma'_0$ and to those $\beta \in \Sigma'_1$ for which $s\beta \in \Sigma'_1$. So assume $\alpha \in \Sigma'_1$ such that $s\beta = \gamma$, $\beta \in \Sigma'_1$, $\gamma \in \Sigma'_0$. Thus $\mathfrak{g}_\alpha \subset \mathfrak{g}^{-\sigma\theta}$, $\mathfrak{g}_\beta \subset \mathfrak{g}^{-\sigma\theta}$ and $\mathfrak{g}_\gamma \cap \mathfrak{g}^{+\sigma\theta} \neq (0)$. Write

$$(7.16) \qquad G(X) = \ldots \sin k(\gamma)i\gamma(X) \sin k(\beta)i\beta(X+\tfrac{1}{4}\pi iH_\beta) \ldots \ .$$

By Lemma 7.10 $k(\beta) = k(\gamma)$. If $k(\beta) = 2$ or 4, then (7.16) becomes ($k(\beta) = 2$, the case $k(\beta) = 4$ being similar):

$$\begin{aligned} G(X) &= \dots \sin 2i\gamma(X) \sin 2i\beta(X+\tfrac{1}{4}\pi i H_\beta) \dots \\ &= \dots \sin 2i\gamma(X) \sin (2i\beta(X) - \tfrac{1}{2}\pi\beta(H_\beta)) \dots \\ &= \dots -\sin 2i\gamma(X) \sin 2i\beta(X) \dots , \end{aligned}$$

because we have normalized $\beta(H_\beta) = 2$, see chapter 5. Also

$$\begin{aligned} G(s'X) &= \dots \sin 2i\gamma(sX-\tfrac{1}{2}\pi i H_\alpha) \sin 2i\beta(sX-\tfrac{1}{2}\pi i H_\alpha) \dots \\ &= \dots -\sin (2i\beta(X)-\pi\beta(H_\alpha))\sin(2i\gamma(X)+\pi\beta(H_\alpha)) \dots, \end{aligned}$$

because $\gamma(H_\alpha) = -\beta(H_\alpha)$. It follows that this part is invariant under s. So assume $k(\beta) = k(\gamma) = 1$. Then we have, because of Corollary 5.6, that $\mathfrak{g}_\gamma \subset \mathfrak{g}^{+\sigma\theta}$. Thus

$$\begin{aligned} G(s'X) &= \dots \sin i\gamma(s'X) \sin i\beta(s'X+\tfrac{1}{4}\pi i H_\beta) \dots \\ &= \dots \sin i\gamma(sX-\tfrac{1}{2}\pi i H_\alpha) \sin i\beta(sX-\tfrac{1}{2}\pi i H_\alpha+\tfrac{1}{4}\pi i H_\beta) \dots \\ &= \dots \sin is\gamma(X+\tfrac{1}{2}\pi i H_\alpha) \sin(is\beta(X+\tfrac{1}{2}\pi i H_\alpha)-\tfrac{1}{2}\pi) \dots \\ &= \dots \sin i\beta(X+\tfrac{1}{2}\pi i H_\alpha) \sin(i\gamma(X+\tfrac{1}{2}\pi i H_\alpha)-\tfrac{1}{2}\pi) \dots . \end{aligned} \tag{7.17}$$

Thus we need to prove that the expression in (7.17) is equal to (7.16). Now we claim that

$$\sin(i\beta(X)-\tfrac{1}{2}\pi\beta(H_\alpha)) = \pm \sin(i\beta(X)-\tfrac{1}{2}\pi) \tag{7.18}$$

and

$$\sin(i\gamma(X) - \tfrac{1}{2}\pi(1-\beta(H_\alpha))) = \pm \sin i\gamma(X), \tag{7.19}$$

with the same signs in (7.18) and (7.19). If we have proved (7.18) and (7.19), it follows that (7.17) equals (7.16) and the theorem will be proved. To prove (7.18) and (7.19) we proceed as follows.

Because $s\beta \neq \beta$ we have that $\beta(H_\alpha) \neq 0$. Thus, because of the definition of Σ'_{pq}, $\beta(H_\alpha) = \pm 1, \pm 2$, or ± 3 (remember that $\beta(H_\alpha) \in \mathbb{Z}$).

If $\beta(H_\alpha) = 1$, then $\sin(i\beta(X)-\tfrac{1}{2}\pi\beta(H_\alpha)) = \sin(i\beta(X)-\tfrac{1}{2}\pi)$, and $\sin(i\gamma(X)-\tfrac{1}{2}\pi(1-\beta(H_\alpha))) = \sin i\gamma(X)$, thus (7.18) and (7.19) both hold with sign +1. If $\beta(H_\alpha) = -1$ or ± 3, then the assertion follows in the same way. Now we shall prove that $\beta(H_\alpha) = \pm 2$ is impossible, thus (7.18) and (7.19) hold, which proves the theorem.

So assume $\beta(H_\alpha) = -2$, the case $\beta(H_\alpha) = 2$ being similar. Then $\Sigma_{pq} \ni \gamma = s_\alpha\beta = \beta - 2\,\beta(H_\alpha)/\alpha(H_\alpha)\,\alpha = \beta + 2\alpha$. Choose $X_\alpha \in \mathfrak{g}_\alpha$ as in chapter 5. Since $X_\alpha, \theta X_\alpha$ and H_α form a standard basis of a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, it follows from the representation theory of this Lie algebra that there exists $Y_\beta \in \mathfrak{g}_\beta$ such that $Z := (\mathrm{ad}\,X_\alpha)^2 Y_\beta \neq 0$. Then $Z \in \mathfrak{g}_{\beta+2\alpha} = \mathfrak{g}_\gamma$, hence $\sigma\theta Z = Z$ (since $\mathfrak{g}_\gamma \subset \mathfrak{g}^{+\sigma\theta}$). But $\mathfrak{g}_\alpha \subset \mathfrak{g}^{-\sigma\theta}$ and $\mathfrak{g}_\beta \subset \mathfrak{g}^{-\sigma\theta}$, thus $\sigma\theta Z = \sigma\theta[X_\alpha,[X_\alpha,Y_\beta]] = -Z$, so $Z = 0$. Contradiction. □

<u>REMARK 7.13</u>. Let $\beta \in \Phi, \hat{\beta} \neq 0$. By checking all possible values for $k(\hat{\beta})$ one sees that in the spherical case (ie. $\tau_1=\tau_2$) $2\tilde{\beta} \notin \Sigma_p$ implies that $k(\tilde{\beta}) = 1$, and $2\tilde{\beta} \in \Sigma_p$ implies that $k(\tilde{\beta}) = 2$ (remember that here $\tilde{\beta} = \hat{\beta} = \tilde{\tilde{\beta}}$ for all $\beta \in \Phi$). Thus in that case one gets

$$\lambda_1 = \sum_{\substack{\tilde{\beta}\in\Sigma_p \\ 2\tilde{\beta}\notin\Sigma_p}} \tilde{\beta},$$

which is the way in which λ_1 was originally defined by Vretare.

CHAPTER 8

THE SINGULAR SET

LEMMA 8.1. *Let* $k \in K$, $h \in H^0$ *and* $a,b \in A_{pq}$ *be such that* $b = kah$. *Then* $b^4 = ka^4k^{-1}$.

PROOF. Apply θ,σ and $\theta\sigma$ to $b = kah$ and eliminate θh and σk. This gives $a^3 = hb^3k$, or $b^3 = h^{-1}a^3k^{-1}$. Thus $b^4 = b.b^3 = kah.h^{-1}a^3k^{-1} = ka^4k^{-1}$. □

Put

$$D := \{X \in i\mathfrak{a}_{pq}: k(\alpha)\alpha(X) \in \pi i\mathbb{Z} \text{ for some } \alpha \in \Sigma_0', \quad \text{or}$$
$$k(\alpha)(\alpha(X)+\tfrac{1}{2}\pi i) \in \pi i\mathbb{Z} \text{ for some } \alpha \in \Sigma_1'\},$$

$$(8.1) \qquad A'_{pq} := A_{pq}\backslash \exp D.$$

By abuse of notation we shall denote the function on A_{pq} defined by $\exp X \mapsto F(X)$ $(X \in i\mathfrak{a}_{pq})$ also by F. Let F' denote the restriction of F to A'_{pq}. Put $M_K^* := N_K(i\mathfrak{a}_{pq})$, $M_K := C_K(i\mathfrak{a}_{pq})$, $M_{H^0}^* := N_{H^0}(i\mathfrak{a}_{pq})$, $M_{H^0} := C_{H^0}(i\mathfrak{a}_{pq})$, then $W_{pq} = M_K^*/M_K = M_{H^0}^*/M_{H^0}$.

DEFINITION 8.2. Let J be the set of all pairs (s,mh) such that $m \in M_K^*$, $h \in H^0$, $mh \in A_{pq}$ and $s = \mathrm{Ad}(m)|_{i\mathfrak{a}_{pq}}$.

Then J is a finite set, since $J \subset (W_{pq}, KH^0 \cap A_{pq})$, W_{pq} is finite by definition, and $KH^0 \cap A_{pq}$ is discrete (by Lemma 8.1) as well as compact, hence also finite. Let $j := |J|$ be the number of elements of J.

Observe that J can be given a group structure. For (s_1,m_1h_1), $(s_2,m_2h_2) \in J$ put

$$(8.2) \qquad (s_1,m_1h_1)(s_2,m_2h_2) := (s_1s_2,m_1m_2h_2h_1).$$

Since (8.2) equals $(s_1s_2, m_1(m_2h_2)m_1^{-1}(m_1h_1))$ this is well-defined. The inverse of $(s,mh) \in J$ is given by

$$(8.3) \qquad (s,mh)^{-1} := (s^{-1}, m^{-1}h^{-1}).$$

Thus (8.2) gives J a group structure. Moreover, J acts on A_{pq} in a diffeomorphic way, via

$$(8.4) \qquad (s,mh)(\exp X) := (\exp sX)mh \qquad (X \in i\mathfrak{a}_{pq}),$$

and F is invariant under this action.

Now we would like to calculate j. For this, notice that the set $M_KH^0 \cap A_{pq}$ is a group: $(m_1h_1)(m_2h_2) := m_1m_2h_2h_1$, $(mh)(m^{-1}h^{-1}) = e$, and $m^{-1}h^{-1} = m^{-1}(mh)^{-1}m \in M_KH^0 \cap A_{pq}$. Put $w := |W_{pq}|$, $k := |M_KH^0 \cap A_{pq}|$.

<u>LEMMA 8.3</u>. *Let* $s \in W_{pq}$. *There exists* $mh \in M_K^*H^0$ *such that* $(s,mh) \in J$.

<u>PROOF</u>. Let $\alpha \in \Sigma_{pq}$. As a first step we show that there exists $g \in M_K^*H^0$ such that $(s,g) \in J$. Let X_α be as in chapter 5, and let k_α, p_α be as in (5.1), (5.2). Then either $\sigma\theta X_\alpha = X_\alpha$ or $\sigma\theta X_\alpha = -X_\alpha$. If $\sigma\theta A_\alpha = X_\alpha$, then $k_\alpha \in H^0$, hence we may take $g := e = k_\alpha k_\alpha^{-1} \in M_K^*H^0 \cap A_{pq}$. If $\sigma\theta X_\alpha = -X_\alpha$, then $p_\alpha \in H^0$, hence we may take $g := \exp(\frac{1}{2}\pi i H_\alpha) = k_\alpha p_\alpha \in M_K^*H^0 \cap A_{pq}$.

Next, let $s \in W_{pq}$. Then, if $\{\alpha_1,\dots,\alpha_\ell\}$ is the base for Σ_{pq} from chapter 3, we may write $s = s_{\alpha_{i_1}} \dots s_{\alpha_{i_n}}$ for certain $i_1,\dots,i_n \in \{1,\dots,\ell\}$. For $i = 1,\dots,\ell$ let $m_ih_i \in M_KH^0 \cap A_{pq}$ be such that $(s_{\alpha_i}, m_ih_i) \in J$. Put $m := m_{i_1}\dots m_{i_n}$, $h := h_{i_n}\dots h_{i_1}$. Then $m \in M_K^*$, $h \in H^0$, and $\mathrm{Ad}(m)|_{i\mathfrak{a}_{pq}} = s$. Moreover

$$\begin{aligned} mh = {} & (m_{i_1}\dots m_{i_{n-1}})(m_{i_n}h_{i_n})(m_{i_1}\dots m_{i_{n-1}})^{-1}\cdot \\ & \cdot(m_{i_1}\dots m_{i_{n-2}})(m_{i_{n-1}}h_{i_{n-1}})(m_{i_1}\dots m_{i_{n-2}})^{-1}\cdot \\ & \cdot \dots \cdot m_{i_1}(m_{i_2}h_{i_2})m_{i_1}^{-1}\cdot m_{i_1}h_{i_1} \in A_{pq}. \end{aligned}$$

Thus $(s,mh) \in J$. □

<u>PROPOSITION 8.4</u>. $j = wk$.

<u>PROOF</u>. The mapping $(s,mh) \mapsto s\colon J \to W_{pq}$ is a surjective homomorphism. The

kernel of this homomorphism is

$$\{mh\colon m \in M_K^*,\ h \in H^0,\ mh \in A_{pq},\ \mathrm{Ad}(m)|_{i\mathfrak{a}_{pq}} = \mathrm{id}\} = M_K H^0 \cap A_{pq}.$$

Hence $|J| = |W_{pq}|.|M_K H^0 \cap A_{pq}| = wk$. □

THEOREM 8.5. *F is regular at* $a \in A_{pq}$ *if and only if* $a \in A'_{pq}$. *F' is a regular* wk-*to-one mapping of* A'_{pq} *onto an open dense subset* Ω'_0 of Ω_0.

PROOF. Regularity follows from Theorem 7.12, and A'_{pq} is open dense in A_{pq}, hence $F(A'_{pq})$ is open dense in $F(A_{pq}) = \Omega_0$. So the only thing left to prove is the fact that F' is wk-to-one. Therefore, let A''_{pq} be the set of all $a \in A_{pq}$ such that the sequence $\{a^4,a^8,a^{12},\ldots\}$ is dense in A_{pq}. Then A''_{pq} is dense in A_{pq}.

Assume $a_1 \in A''_{pq}$, $a_2 \in A_{pq}$ such that $F(a_1) = F(a_2)$. It follows from Theorem 7.5 that $F(a_1) = F(a_2)$ if and only if $\varphi_\lambda(a_1) = \varphi_\lambda(a_2)$ for all $\lambda \in \mathbb{Z}_+^\ell$. But the functions φ_λ form a complete set of functions on $K\backslash U/H^0$, thus because of Theorem 6.10 we obtain $\varphi_\lambda(a_1) = \varphi_\lambda(a_2)$ for all $\lambda \in \mathbb{Z}_+^\ell$ if and only if $k_1a_1h_1 = k_2a_2h_2$. Or, by putting $k := k_2k_1^{-1}$, $h := h_1h_2^{-1}$, $a_2 = ka_1h$. Thus, by Lemma 8.1, we obtain $a_2^4 = ka_1^4k^{-1}$ (hence $a_2 \in A''_{pq}$).

Let $X \in i\mathfrak{a}_{pq}$. Then $\mathrm{Ad}(k)X \in i\mathfrak{p}$, but also $\sigma(\mathrm{ad}(k)X) = -\mathrm{Ad}\sigma(k)X = -\mathrm{Ad}(k)X$, hence $\mathrm{Ad}(k)X \in i(\mathfrak{p}\cap\mathfrak{q})$. (The last identity follows by applying $\sigma\theta$ to $a_2^4 = ka_1^4k^{-1}$, which gives $a_2^4 = \sigma(k)a_1^4\sigma(k^{-1})$. Hence $(k^{-1}\sigma(k))a_1^4(\sigma(k^{-1})k) = a_1^4$, hence $(k^{-1}\sigma(k))a(\sigma(k^{-1})k) = a$ for all $a \in A_{pq}$, thus $\mathrm{Ad}(k)X = \mathrm{Ad}\sigma(k)X$ for all $X \in i\mathfrak{a}_{pq}$.)

Moreover, $\mathrm{Ad}(k)X$ centralizes $i\mathfrak{a}_{pq}$. Namely $\mathrm{Ad}(a_2^4)\mathrm{Ad}(k)X = \mathrm{Ad}(k)\mathrm{Ad}(a_1^4)X =$ $= \mathrm{Ad}(k)X$, hence $\mathrm{Ad}(a)\mathrm{Ad}(k)X = \mathrm{Ad}(k)X$ for all $a \in A_{pq}$ and all $X \in i\mathfrak{a}_{pq}$. Thus $[Y,\mathrm{Ad}(k)X] = 0$ for all $X,Y \in i\mathfrak{a}_{pq}$. Thus $k \in M_K^*$, and $kh = ka_1^{-1}k^{-1}a_2 \in A_{pq}$. So, if $a_1,a_2 \in A''_{pq}$, $k \in K$, $h \in H^0$, then $a_2 = ka_1h$ if and only if $k \in M_K^*$ and $kh \in A_{pq}$.

Now, let $a_1,a_2 \in A''_{pq}$, $k_1k_2 \in K$, $h_1,h_2 \in H^0$ be such that $a_2 = k_1a_1h_1 =$ $= k_2a_1h_2$. Put $k := k_2^{-1}k_1$, $h := h_1h_2^{-1}$, then $ka_1h = a_1$, thus $ka_1^4k^{-1} = a_1^4$, by Lemma 8.1. Thus $kak^{-1} = a$ for all $a \in A_{pq}$, hence $\mathrm{Ad}(k)X = X$ for all $X \in i\mathfrak{a}_{pq}$. Thus $\mathrm{Ad}(k_1)|_{i\mathfrak{a}_{pq}} = \mathrm{Ad}(k_2)|_{i\mathfrak{a}_{pq}}$, thus $k_1h_1 = k_2h_2$.

Thus F is a j-to-one mapping of A''_{pq} onto $F(A''_{pq})$. We shall now prove that F is a j-to-one mapping of A'_{pq} onto $F(A'_{pq})$. $F(A''_{pq})$ is dense in $F(A'_{pq})$ because A''_{pq} is dense in A'_{pq}.

Let $y \in F(A'_{pq})$. Assume $|(F')^{-1}(y)| > j$, $x_1,\ldots,x_{j+1} \in (F')^{-1}(y)$. Then there is an open neighbourhood V of y, and disjunct open neighbourhoods U_i of x_i $(i = 1,\dots,j+1)$ such that $F\colon U_i \to V$ is a homeomorphism. But there is a $z \in V \cap F(A''_{pq})$, thus $F^{-1}(z) \subset A''_{pq}$, and $|F^{-1}(z)| \geq j+1$. Contradiction.

Assume $|(F')^{-1}(y)| < j$, ie. $(F')^{-1}(y) = \{x_1,\dots,x_t\}$, $t < j$. Again, take V open neighbourhood of y, and U_i open neighbourhood of x_i $(i = 1,\dots,t)$ such that $F\colon U_i \to V$ is a homeomorphism. By the action (8.3) J acts on A_{pq} in a diffeomorphic way, and $F\circ j = j$, hence $j(A'_{pq}) = A'_{pq}$ $(j \in J)$. Let $y_n \to y$, with $y_n \in V \cap (A''_{pq})$. Let $z_n \in U_1$ be such that $F(z_n) = y_n$. There is a $j_n \in J$ such that $j_n.z_n \notin U_1 \cup\dots\cup U_t$, because $J.z_n$ has cardinality $j > t$, and is mapped to y_n, since F is injective on each U_i $(i = 1,\dots,t)$. Hence there is a subsequence $j_0.z_{i_n}$, with $j_0 \in J$ fixed (because J is finite), $z_{i_n} \to x_1$, and $j_0.z_{i_n} \to j_0.x_1 \notin U_1 \cup\dots\cup U_t$ (since $A_{pq}\backslash U_1 \cup\dots\cup U_t$ is closed), with $F(j_0.x_1) = F(x_1)$, and $j_0.x_1 \in A'_{pq}$ since $x_1 \in A'_{pq}$. Contradiction.

Thus $|(F')^{-1}(y)| = j = wk$, by Proposition 8.4. □

CHAPTER 9

AN INTEGRAL FORMULA FOR THE GENERALIZED CARTAN DECOMPOSITION

In chapter 6 we have proved the decomposition $U = KA_{pq}H^0$. For the non-compact analogue of this decomposition, ie. $G = KBH$ (for notations, see chapter 6), FLENSTED-JENSEN [8] gives an integral formula. Since our treatise of the analogue of this formula for U is mainly based on his ideas, we shall summarize the results from [8, section 2] here. For $\alpha \in \Sigma_{pq}$, put $p_\alpha := \dim(\mathfrak{g}_\alpha \cap \mathfrak{g}^{+\sigma\theta})$, $q_\alpha := \dim(\mathfrak{g}_\alpha \cap \mathfrak{g}^{-\sigma\theta})$. Put

(9.1) $$\delta_0(X) := \Big| \prod_{\alpha\in\Sigma^+_{pq}} \mathrm{sh}^{p_\alpha}\alpha(X)\,\mathrm{ch}^{q_\alpha}\alpha(X) \Big|, \qquad X \in \mathfrak{a}_{pq}.$$

Put $L' := K \cap H, M' := C_{L'}(\mathfrak{a}_{pq})$. Then, with a suitable normalization of the involved measures, we have the following integral formula ([8, Theorem 2.6]):

(9.2) $$\int_G f(g)dg = \mathrm{vol}(L'/M') \int_K \int_{\mathfrak{a}^+_{pq}} \int_H f(k\exp Xh)\,\delta_0(X)\,dh\,dX\,dk, \qquad f \in C_c(G)$$

We shall now give the analogue of (9.2) for U. Therefore, put $L := K \cap H^0$, $M := C_L(i\mathfrak{a}_{pq})$. Define a mapping $\Phi := K/M \times A_{pq} \to U/H^0$ by

(9.3) $$\Phi(kM,a) := kaH^0, \qquad k \in K,\ a \in A_{pq}.$$

Normalize measures as follows:

(9,4) $$\int_U du = \int_K dk = \int_{H^0} dh = \int_L d\ell = \int_M dm = \int_{A_{pq}} da = 1.$$

The Killing form on $\mathfrak{u}$ induces invariant measures on U/H^0, K/M, L/M and $i\mathfrak{a}_{pq}$. Let the corresponding Riemannian measures be denoted by duH^0, dkM, $d\ell M$ and dX, respectively. Let $\mathfrak{l},\mathfrak{m}$ be the Lie algebras of L,M, respectively. Let $\mathfrak{l}'$ be the orthogonal complement (with respect to the Killing form) of $\mathfrak{m}$ in

ℓ. Then, just as in the noncompact case, we have to calculate $|\det d\Phi_{(eM,a)}|$, where $d\Phi_{(eM,a)}: \ell' + (k\cap q) + i\alpha_{pq} \to d\tau(a)(k\cap q+i(p\cap q))$ is the Jacobi matrix. Here τ is defined by $\tau(u)xH^0 := uxH^0$ for $u,x \in U$. Because of the fact that for $X \in i\alpha_{pq}$ $\exp X = e$ implies $\alpha(X) \in 2\pi i\mathbb{Z}$ for all $\alpha \in \Sigma_{pq}$, the following definition makes sense.

DEFINITION 9.1. $\delta(\exp X) := |\prod_{\alpha\in\Sigma^+_{pq}} \sin^{p_\alpha}\alpha(iX)\cos^{q_\alpha}\alpha(iX)|$, $X \in i\alpha_{pq}$.

LEMMA 9.2. $|\det d\Phi_{(eM,a)}| = \delta(a)$.

PROOF. Let q_0 be the dimension of the zerospace of ad $i\alpha_{pq}$ in $i(p\cap h)$, and r_0 the dimension of the zerospace of ad $i\alpha_{pq}$ in $k \cap q$. Choose ON (:= orthonormal) bases as follows:

$$T^1_\alpha,\dots,T^{p_\alpha}_\alpha \quad (\alpha\in\Sigma^+_{pq}) \quad \text{of } \ell',$$
$$Y^1_\alpha,\dots,Y^{p_\alpha}_\alpha \quad (\alpha\in\Sigma^+_{pq}) \quad \text{of } i(p\cap q\cap\alpha^\perp_{pq}),$$
$$X^1_\alpha,\dots,X^{p_\alpha}_\alpha(\alpha\in\Sigma^+_{pq}),\ X^1_0,\dots,X^{q_0}_0 \quad \text{of } i(p\cap h),$$

and

$$Z^1_\alpha,\dots,Z^{p_\alpha}_\alpha(\alpha\in\Sigma^+_{pq}),Z^1_0,\dots,Z^{r_0}_0 \quad \text{of } k\cap q$$

such that

$$\mathrm{ad}(X)T^j_\alpha = -\alpha(iX)Y^j_\alpha,$$
$$\mathrm{ad}(X)Y^j_\alpha = \alpha(iX)T^j_\alpha,$$
$$\mathrm{ad}(X)X^j_\alpha = -\alpha(iX)Z^j_\alpha,$$

and

$$\mathrm{ad}(X)Z^j_\alpha = \alpha(iX)X^j_\alpha$$

for all $X \in i\alpha_{pq}$. Choose an ON basis $\{X_1,\dots,X_\ell\}$ of $i\alpha_{pq}$. We shall calculate the matrix of $d\Phi_{(eM,a)}$ with respect to the ON basis

$$T^1_\alpha,\dots,T^{p_\alpha}_\alpha(\alpha\in\Sigma^+_{pq}),Z^1_\alpha,\dots,Z^{p_\alpha}_\alpha(\alpha\in\Sigma^+_{pq}),Z^1_0,\dots,Z^{q_0}_0,\ X_1,\dots,X_\ell$$

of $\ell' + (k\cap q) + i\alpha_{pq}$ and the ON basis

$$Y_\alpha^1,\ldots,Y_\alpha^{p_\alpha}(\alpha\in\Sigma_{pq}^+),\ Z_\alpha^1,\ldots,Z_\alpha^{q_\alpha}(\alpha\in\Sigma_{pq}^+),Z_0^1,\ldots,Z_0^{q_0},X_1,\ldots,X_\ell$$

of $q^0 = i(p\cap q\cap a_{pq}^\perp) + k\cap q + ia_{pq}$. It is clear that $d\Phi_{(eM,a)}(X_j) = d\tau(a)(X_j)$. If $Y \in k \cap m^\perp$ $d\Phi_{(eM,a)}(Y)$ follows from differentiation of the 1-parameter curve

$$t \mapsto \pi(\exp tY\exp X) = \exp X.\pi(\exp(te^{-\mathrm{ad}X}Y)),$$

where $\pi\colon U \to U/H^0$ denotes the canonical projection, and $X \in ia_{pq}$ is such that $a = \exp X$. Thus

$$d\Phi_{(eM,a)}(Y) = d\tau(\exp X)\tfrac{1}{2}(e^{-\mathrm{ad}X}Y - e^{\mathrm{ad}X}\sigma Y).$$

Hence

$$d\Phi_{(eM,a)}(T_\alpha^j) = d\tau(\exp X)\sin\alpha(iX)Y_\alpha^j,$$

$$d\Phi_{(eM,a)}(Z_\alpha^j) = d\tau(\exp X)\cos\alpha(iX)Z_\alpha^j,$$

and

$$d\Phi_{(eM,a)}(Z_0^j) = d\tau(\exp X)Z_0^j,$$

which proves the lemma. □

Let $(A_{pq})_r$ be the set of elements in A_{pq} such that Φ is regular at (eM,a). That is

(9.5) $$(A_{pq})_r = \{\exp X\colon X \in ia_{pq},\ \alpha(X) \notin \pi i\mathbb{Z} \text{ if } p_\alpha \neq 0,$$
$$\alpha(X) + \tfrac{1}{2}\pi i \notin \pi i\mathbb{Z} \quad \text{if } q_\alpha \neq 0 \text{ for all } \alpha \in \Sigma_{pq}^+\}.$$

Thus $A''_{pq} \subset (A_{pq})_r \subset A'_{pq}$. Let the image of $K/M \times (A_{pq})_r$ under Φ, which is an open dense subset of U/H^0 (by Theorem 6.10), be denoted by $(U/H^0)_r$. Let again wk be the number of elements of J, with J as in Definition 8.2, cf. Proposition 8.4. Let $j_1 := (s_1,m_1h_1)$, $j_2 := (s_2,m_2h_2) \in J$ ($m_i\in M_K^*, h_i\in H^0$, $s_i = \mathrm{Ad}(m_i)|_{ia_{pq}}$). Then

(9.6) $$j_1 = j_2 \Longleftrightarrow m_2^{-1}m_1 \in M \text{ and } h_2 = (m_2^{-1}m_1)h_1.$$

Thus there is a well-defined action of J on $K/M \times A_{pq}$ via

$$(9.7) \qquad (\mathrm{Ad}(m)|_{i\mathfrak{a}_{pq}}, mh).(kM,a) := (km^{-1}M, mah).$$

If $m \in M_{\mathbb{K}}^{*}$ is such that $(s,mh) \in J$, then m normalizes M, thus (9.6) implies that (9.7) is well-defined.

It is clear that $\Phi \circ j = j$ for all $j \in J$.

LEMMA 9.3. *Φ is a regular* wk*-to-one mapping of* $K/M \times (A_{pq})_r$ *onto* $(U/H^0)_r$.

PROOF. Regularity follows from Lemma 9.2, and the open dense subset $(U/H^0)_r$ is by definition the image of $K/M \times (A_{pq})_r$. So the only thing left to prove is the fact that Φ is wk-to-one. So assume $a_1 \in A''_{pq}$, $a_2 \in A_{pq}$, $k_1, k_2 \in K$ be such that $\Phi(k_1M,a_1) = \Phi(k_2M,a_2)$. Then for certain $h_1,h_2 \in H^0$ we have $k_1a_1h_1 = k_2a_2h_2$. Thus, just as in the proof of Theorem 8.5, it follows that $a_2 \in A''_{pq}$ and $a_2 = j.a_1$ for a certain $j \in J$. Hence $(k_2M,a_2) = j.(k_1M,a_1)$ and $\Phi(k_1M,a_1)$ has exactly wk pre-images. Now the extension from A''_{pq} to $(A_{pq})_r$ can be done by a reasoning similar to the extension from A''_{pq} to A'_{pq}, cf. proof of Theorem 8.5 (see HOOGENBOOM [16, Proposition 4.5] for full details). This proves the lemma. □

THEOREM 9.4. *Let* $f \in C(U)$. *Then, with the normalization of measures* (9.4),

$$(9.8) \qquad \int_{A_{pq}} \delta(a)da \int_U f(u)du = \int_K \int_{A_{pq}} \int_{H^0} f(kah)\delta(a)dhdadk.$$

PROOF. From what is said above, it follows that we have the following expressions:

$$(9.9) \qquad \int_{U/H^0} f_1(uH^0)duH^0 = \frac{\gamma}{wk} \int_{A_{pq}} \int_{K/M} f_1(kaH^0)\delta(a)dkMda$$

$$\left(\gamma = \frac{1}{\mathrm{vol}(A_{pq})},\ f_1 \in C(U/H^0)\right),$$

$$(9.10) \qquad \mathrm{vol}(U/H^0) \int_U f_2(u)du = \int_{U/H^0} \left(\int_{H^0} f_2(uh)dh\right)duH^0 \qquad (f_2 \in C(U));$$

$$(9.11) \qquad \mathrm{vol}(K/M) \int_K f_3(k)dk = \int_{K/M} \left(\int_M f_3(km)dm\right)dkM \qquad (f_3 \in C(K)).$$

Now (9.9), (9.10) and (9.11) imply (cf. HELGASON [10, p.384]) that for all $f \in C(U)$:

$$\mathrm{vol}(U/H^0) \int_U f(u)\,du = \frac{\gamma}{wk}\,\mathrm{vol}(K/M) \int_{A_{pq}} \int_K \int_{H^0} f(kah)\,\delta(a)\,dh\,dk\,da.$$

(9.8) follows by substitution of $f \equiv 1$. □

REMARK 9.5. The evaluation of $\int_A \delta(a)\,da$ leads to integrals of Selberg-type. See MACDONALD [24] for some explicit values and some conjectured values for integrals of this type.

REMARK 9.6. In chapter 11 we shall derive some restrictions on the multiplicities p_α and q_α in connection with $k(\alpha)$. By using these results one obtains quite easily $(A_{pq})_r = A'_{pq}$.

CHAPTER 10

INTERTWINING FUNCTIONS ON U AS ORTHOGONAL POLYNOMIALS

In this chapter we shall prove the analogue of Theorem 3.6 in VRETARE [31] for intertwining functions. That is, we show that the intertwining functions on U may be considered as orthogonal polynomials on a region in $\mathbb{R}^{\ell}$ (namely Ω, cf. Definition 7.8), with respect to a certain positive weight function. This weight function is given in the following definition.

DEFINITION 10.1. Let the positive *weight function* w on Ω be given by:

$$w(\psi(F(X))) := \Big| \prod_{\alpha\in\Sigma^+_{pq}} \sin^{p_\alpha}\alpha(iX)\, \cos^{q_\alpha}\alpha(iX)\,.$$

$$\prod_{\alpha\in\Sigma'_0} \sin^{-1}k(\alpha)\alpha(iX) \prod_{\alpha\in\Sigma'_1} \sin^{-1}k(\alpha)(\alpha(iX)-\tfrac{1}{2}\pi)\Big|, \quad X \in i\mathfrak{a}_{pq}.$$

LEMMA 10.2. *For* $f \in C(\Omega_0)$ *we have*

$$\int_{\Omega} f(\psi^{-1}(x))w(x)\,dx = c\int_{U} f(\varphi_1(u),\ldots,\varphi_{\ell}(u))\,du.$$

PROOF. As the proof of Lemma 3.5 in [31]. The complements of A'_{pq} in A_{pq} and of Ω' in Ω are sets of measure zero (here $\Omega' = \psi(\Omega'_0)$, cf. Theorem 8.5). The lemma now follows from Theorem 7.12, Theorem 8.5 and Theorem 9.6. □

THEOREM 10.3. *The mapping* $P \to P\circ\psi\circ F$ *is an isomorphism of the algebra of polynomials on* Ω *onto the algebra of functions on* A_{pq} *spanned by the intertwining functions such that the orthogonal polynomial* $P\circ\psi$ *of degree* $\lambda \in \mathbb{Z}^{\ell}_+$ *with respect to the weight function* w *is mapped onto the intertwining function* φ_λ.

PROOF. According to Theorem 7.5 we have that φ_λ is a polynomial of degree λ in the variable $\varphi = (\varphi_1,\ldots,\varphi_{\ell})$. Hence φ_λ is a polynomial of degree λ in the variable $\psi(\varphi)$. Denote this polynomial by P_λ, that is $P_\lambda(\psi(\varphi(u))) = \varphi_\lambda(u)$. The orthogonality follows from Lemma 10.2 and the orthogonality relations of Schur ((7.1)). □

REMARK 10.4. It follows from Theorem 10.3 that for certain symmetric spaces of rank two the orthogonal polynomials considered in SPRINKHUIZEN-KUYPER [27] can be considered as intertwining functions for certain values of the parameters α,β,γ. For this topic, see also VRETARE [32]. In [32] generalizations of the Koornwinder polynomials from [27] to more variables are proved to be intertwining functions on symmetric spaces of higher rank for certain values of the parameters. Vretare's treatment of intertwining functions, however, is an ad hoc approach for the spaces

$$SO(p) \times SO(n-p)\backslash SO(n)/SO(q) \times SO(n-q),$$

$$S(U_p \times U_{n-p})\backslash SU(n)/S(U_q \times U_{n-q}),$$

and

$$Sp(p) \times Sp(n-p)\backslash Sp(n)/Sp(q) \times Sp(n-q).$$

In the first of these three cases the measure $w(x)dx$ becomes the measure on the squares of the cosines of the critical angles, as considered in JAMES & CONSTANTINE [18, formula (6.2)].

Let again $\mathbb{D}_0(U)$ be the algebra of left-U-, right-H^0-invariant differential operators on U. Let $\delta'(\Omega)$ denote the *radial part* of the Laplace-Beltrami operator on U/H^0, acting on a K-invariant function $f \in C^\infty(U/H^0)$ (which we shall denote by $f \in C^\infty(K\backslash U/H^0)$). Now the polynomials we have constructed in Theorem 10.3 can be characterized in yet another way, namely as *eigenfunctions of* $\delta'(\Omega)$. Remeber (cf. HELGASON [12]) that for a non-compact Lie group G a function φ, which has a certain convergent series expansion which is regular at ∞, is an eigenfunction of all invariant differential operators on G if and only if it is an eigenfunction of $\delta'(\Omega)$. See HOOGENBOOM [15] for an application of this theorem. For a compact Lie group we have the following analogue of this theorem: orthogonal polynomials which are spherical functions on a compact Lie group are characterized by the fact that they are of the form $\Sigma_{\nu\leq\lambda}\,\Gamma_\nu(\lambda)e^{i\nu(X)}$ and the fact that they are eigenfunctions of $\delta'(\Omega)$. This result can easily be generalized to intertwining functions.

Therefore, let us first calculate $\delta'(\Omega)$. Choose a basis $X_1,\ldots,X_\ell$ of $i\mathfrak{a}_{pq}$ such that $B(X_i,X_j) = \delta_{ij}$, where $B(\cdot,\cdot)$ denotes the Killing form on $\mathfrak{u}$. Let the function δ on A_{pq} be as in Definition 9.1. For $\alpha \in \Sigma_{pq}$ let m_α be the multiplicity of α in $\mathfrak{g}$. Thus $m_\alpha = p_\alpha + q_\alpha$. Put $\rho := \frac{1}{2}\sum_{\alpha\in\Sigma_{pq}^+} m_\alpha\alpha$. Let A_α be defined as in chapter 5, and define A_ρ by $B(X,A_\rho) = \rho(X)$ for all $X \in \mathfrak{a}_{pq}$.

<u>LEMMA 10.5</u>. $\delta'(\Omega) = \Sigma_{j=1}^{\ell} X_j^2 + 2iA_\rho + \Sigma_{\alpha\in\Sigma_{pq}^+} (p_\alpha(e^{2i\alpha}-1)^{-1}-q_\alpha(e^{2i\alpha}+1)^{-1})A_\alpha$.

<u>PROOF</u>. (See also [7, formula (4.12)] and [8, p.307]). According to Theorem 6.10 we have $U = KA_{pq}H^0$. Let $f \in C^\infty(K\backslash U/H^0)$. Observe that according to Theorem 9.4 we have

$$(10.1) \qquad \int_{U/H^0} f(x)\,dx = c.\int_{A_{pq}} f(a)\delta(a)\,da.$$

Then it follows from HELGASON [12, Theorem I.2.11] that

$$(10.2) \qquad (\delta'(\Omega)f)(a) = \delta^{-\frac{1}{2}}\circ\Delta(\delta^{\frac{1}{2}}f)(a)-\delta^{-\frac{1}{2}}\circ\Delta(\delta^{\frac{1}{2}})(a),$$

where Δ is the Laplace-Beltrami operator on A_{pq}. Thus

$$(10.3) \qquad \delta'(\Omega) = \delta^{-\frac{1}{2}}\circ\Delta\circ\delta^{\frac{1}{2}} - \delta^{-\frac{1}{2}}\circ\Delta(\delta^{\frac{1}{2}}).$$

But if $\{X_1,\ldots,X_\ell\}$ is an orthonormal basis of $i\mathfrak{a}_{pq}$, then we have

$$\Delta = \sum_{j=1}^{\ell} X_j^2.$$

Thus (10.3) becomes

$$(10.4) \qquad \delta'(\Omega) = \sum_{j=1}^{\ell} \delta^{-\frac{1}{2}}\circ X_j^2\circ\delta^{\frac{1}{2}} - \sum_{j=1}^{\ell} \delta^{-\frac{1}{2}}\circ X_j^2(\delta^{\frac{1}{2}}),$$

or, by a simple calculation

$$(10.5) \qquad \delta'(\Omega) = \sum_{j=1}^{\ell} X_j^2 + 2\sum_{j=1}^{\ell} \delta^{-\frac{1}{2}}\circ X_j(\delta^{\frac{1}{2}})\circ X_j.$$

Substitution of $\delta(\exp X) = \left|\prod_{\alpha\in\Sigma_{pq}^+} \sin^{p_\alpha}\alpha(iX)\cos^{q_\alpha}\alpha(iX)\right|$ $(X \in i\mathfrak{a}_{pq}$, cf.

Definition 9.1) in (10.5) gives

$$(10.6)\qquad \delta'(\Omega) = \sum_{j=1}^{\ell} X_j^2 + 2iA_\rho + 2i \sum_{\alpha\in\Sigma_{pq}^+} (p_\alpha(e^{2i\alpha}-1)^{-1} - q_\alpha(e^{2i\alpha}+1)^{-1})A_\alpha. \quad \square$$

We shall need a slightly different version of (10.6). For $\alpha \in \Sigma_{pq}^+$, write

$$(10.7)\qquad (e^{2i\alpha}-1)^{-1} = \sum_{k=1}^{\ell} e^{-2ik\alpha},$$

$$(10.8)\qquad (e^{2i\alpha}+1)^{-1} = \sum_{k=1}^{\ell} (-1)^{k-1} e^{-2ik\alpha},$$

where (10.7) and (10.8) are to be evaluated in $X \in -(i\mathfrak{a}_{pq}^+)$, for reason of convergence. Here $\mathfrak{a}_{pq}^+$ is the positive Weyl chamber in $\mathfrak{a}_{pq}$ corresponding to the base $\{\alpha_1,\ldots,\alpha_\ell\}$ of Σ_{pq}. Thus (10.6) becomes

$$(10.9)\qquad \delta'(\Omega) = \sum_{j=1}^{\ell} X_j^2 + 2iA_\rho + 2i \sum_{\alpha\in\Sigma_{pq}^+} (p_\alpha \sum_{k=1}^{\infty} e^{-2ik\alpha} + q_\alpha \sum_{k=1}^{\infty} (-1)^k e^{-2ik\alpha}) A_\alpha.$$

Let $\lambda \in \mathbb{Z}_+^\ell$. By Theorem 10.3 there exists a polynomial P_λ, which is of the form

$$(10.10)\qquad P_\lambda(X) = \sum_{\nu\preceq\lambda} \Gamma_\nu(\lambda)e^{i\nu(X)} \qquad (X\in i\mathfrak{a}_{pq}),$$

with $\Gamma_\lambda(\lambda) \neq 0$, such that P_λ is an intertwining function on $K\backslash U/H^0$. Thus, by Theorem 4.3, P_λ is an eigenfunction of all left-U-,right-H^0-invariant differential operators on U. In particular, this means that P_λ is an eigenfunction of $\delta'(\Omega)$. By making use of the expression (10.9) for $\delta'(\Omega)$ we can calculate the eigenvalue of P_λ under $\delta'(\Omega)$.

<u>LEMMA 10.6</u>. $\delta'(\Omega)P_\lambda = -(\lambda,\lambda+2\rho)P_\lambda$.

<u>PROOF</u>. Let μ be the eigenvalue of P_λ under $\delta'(\Omega)$. Thus

$$(10.11)\qquad \delta'(\Omega)P_\lambda = \mu P_\lambda.$$

Substitution of (10.9) and (10.10) in (10.11) leads to the following recursion formula for $\Gamma_\nu(\lambda)$ (here we write Γ_ν for $\Gamma_\nu(\lambda)$).

$$(10.12)\qquad -(\mu+(\nu,\nu+2\rho))\Gamma_\nu = 2 \sum_{\alpha\in\Sigma_{pq}^+} \sum_{k\geq 1} (p_\alpha+(-1)^k q_\alpha)(\nu+2k\alpha,\alpha)\Gamma_{\nu+2k\alpha},$$

where k runs over all integers ≥ 1 for which $\nu + 2k\alpha \in \mathbb{Z}_+^\ell$. But, by (10.10), it is clear that $\nu \succ \lambda$ implies $\Gamma_\nu = 0$. Now substitute $\nu = \lambda$ in (10.12). Then the right-hand side of (10.12) becomes zero, by the above remarks. Thus the left-hand side of (10.12) becomes zero, but $\Gamma_\lambda \neq 0$, thus $\mu + (\lambda,\lambda+2\rho) = 0$, hence $\mu = -(\lambda,\lambda+2\rho)$. □

REMARK 10.7. If $G = G_1 \times G_1$ and $K = H = \mathrm{diag}(G_1)$, then (10.12) reduces to Freudenthal's formula, cf. [17, Theorem 22.3].

In the case of spherical functions on a noncompact semisimple Lie group the above calculation is due to HARISH-CHANDRA [9]. Actually, it is not too hard to compute the eigenvalue of P_λ under $\delta'(\Omega)$ directly, cf. eg. HUMPHREYS [17, Excercise 23.4]. However, in the following we shall need the recursion relation for Γ_ν which was obtained in the proof of Lemma 10.6 ((10.12)). We shall now give the characterization of intertwining functions as eigenfunctions of $\delta'(\Omega)$. Let P be a polynomial of the form

$$(10.13) \qquad P(X) = \sum_{\nu\leq\lambda} \Gamma'_\nu\, e^{i\nu(X)} \qquad (X\in i\mathfrak{a}_{pq}),$$

with $\Gamma'_\lambda \neq 0$. Assume P transforms under W_{pq} according to Proposition 5.2 and Proposition 5.5.

THEOREM 10.8. *P is the restriction to A_{pq} of an intertwining function on U if and only if $\delta'(\Omega)P = \mu'P$ for some $\mu' \in \mathbb{C}$.*

PROOF. The "only if" part follows from Theorem 4.3, hence we only need to prove that $\delta'(\Omega)P = \mu'P$ for some $\mu' \in \mathbb{C}$ implies that P is the restriction to A_{pq} of an intertwining function on U. As in the proof of Lemma 10.6 it follows from (10.9), (10.13) and the fact that P is an eigenfunction of $\delta'(\Omega)$ that the coefficients Γ'_ν satisfy a recursion relation of the form (10.12). Thus

$$(10.14) \qquad -(\mu'+(\nu,\nu+2\rho))\Gamma'_\nu = 2 \sum_{\alpha\in\Sigma^+_{pq}} \sum_{k\geq 1} (p_\alpha+(-1)^k q_\alpha)(\nu+2k\alpha,\alpha)\Gamma'_{\nu+2k\alpha}.$$

Again as in the proof of Lemma 10.6, (10.14) implies that $\mu' = -(\lambda,\lambda+2\rho)$. But then the coefficients Γ_ν for P_λ, and Γ'_ν for P satisfy the same recursion relation (10.12). Since (10.12) determines the Γ_ν, and hence P_λ up to a constant factor, P must be equal to P_λ up to multiplication by a constant. □

CHAPTER 11

EXAMPLE: THE CASE dim $\mathfrak{a}_{pq} = 1$

As a final example we shall treat the case dim $\mathfrak{a}_{pq} = 1$ here. This is a direct generalization of Example 0.2 from the introduction.

So assume $\mathfrak{a}_{pq}$ has dimension one. Let $\Sigma_{pq} = \{(-2\alpha),-\alpha,\alpha,(2\alpha)\}$, $\Sigma^+_{pq} = \{\alpha,(2\alpha)\}$. Let $X_0 \in \mathfrak{a}_{pq}$ be such that $\alpha(X_0) = 1$. Then $\mu_1 := k(\alpha)\alpha$ generates the lattice $\mathbb{Z}^1$, and we get for $\theta \in \mathbb{R}$:

$$(11.1) \qquad \varphi_{k(\alpha)\alpha}(\exp i\theta X_0) = \begin{cases} a\cos(\theta+\frac{1}{2}\pi) + b & \text{if } p_\alpha = 0 \text{ and } k(\alpha) = 1, \\ a\cos k(\alpha)\theta + b & \text{if } p_\alpha > 0 \text{ or } k(\alpha) > 1, \end{cases}$$

where $a,b \in \mathbb{R}$ are such that $a+b = 1$. Again as in Example 0.2 we shall consider the intertwining functions as polynomials in the variable

$$(11.2) \qquad y := \begin{cases} \cos(\theta+\frac{1}{2}\pi) & \text{if } p_\alpha = 0 \text{ and } k(\alpha) = 1 \\ \cos k(\alpha)\theta & \text{if } p_\alpha > 0 \text{ or } k(\alpha) > 1. \end{cases}$$

Clearly the weight function in the variable y equals w up to a constant factor. By abuse of notation we shall denote this weight function by w as well. Thus the weight function (cf. Definition 10.1) becomes

$$(11.3) \qquad w(\cos k(\alpha)\theta) = \left| \frac{\sin^{p_\alpha}\theta \cos^{q_\alpha}\theta \sin^{p_{2\alpha}}2\theta \cos^{q_{2\alpha}}2\theta}{\sin k(\alpha)\theta} \right|$$

if $p_\alpha > 0$ or $k(\alpha) > 1$ (remember that $p_\alpha > 0$ implies that $\alpha \in \Sigma_0'$, and $k(\alpha) > 1$ implies that $|\sin k(\alpha)(\alpha+\frac{1}{2}\pi)| = |\sin k(\alpha)\alpha|$), and

$$(11.4)\qquad w(\cos(\theta+\tfrac{1}{2}\pi)) = \left|\frac{\cos^{q_\alpha}\theta\sin^{p_{2\alpha}}2\theta\,\cos^{q_{2\alpha}}2\theta}{\sin(\theta+\frac{1}{2}\pi)}\right|$$

if $p_\alpha = 0$ and $k(\alpha) = 1$ (remember that $p_\alpha = 0$ implies that $\alpha \in \Sigma_1'$).

In the following lemma Σ_{pq} may be of general rank.

LEMMA 11.1. *Let* $\alpha \in \Sigma_{pq}$. *Then* $k(\alpha) = 1$ *if and only if* $2\alpha \notin \Sigma_{pq}$ *and* $p_\alpha = 0$ *or* $q_\alpha = 0$.

PROOF. Assume $k(\alpha) = 1$. If $2\alpha \in \Sigma_{pq}$, then $(\mu,2\alpha)/(2\alpha,2\alpha) \in \mathbb{Z}$ for all $\mu \in \mathbb{Z}^\ell$, hence $(\mu,\alpha)/(\alpha,\alpha) \in 2\mathbb{Z}$ for all $\mu \in \mathbb{Z}^\ell$. If $p_\alpha > 0$ and $q_\alpha > 0$, then $(\mu,\alpha)/(\alpha,\alpha) \in 2\mathbb{Z}$ for all $\mu \in \mathbb{Z}^\ell$ by Corollary 5.6 and Lemma 5.7.

Conversely, suppose $2\alpha \notin \Sigma_{pq}$, and $p_\alpha = 0$ or $q_\alpha = 0$. By (7.8) we have $k(\alpha) = c(\alpha)$, hence it suffices to show that for $\beta \in \Phi$, $\hat\beta \neq 0, \mathfrak{g}_{\hat\beta} \subset \mathfrak{g}^{+\sigma\theta}$ or $\mathfrak{g}_{\hat\beta} \subset \mathfrak{g}^{-\sigma\theta}$ implies $\tilde\beta = \hat\beta = \tilde{\tilde\beta}$.

Therefore, let $0 \neq X_{\tilde\beta} \subset \mathfrak{g}_{\tilde\beta}$. Then $\sigma\theta X_{\tilde\beta} = \varepsilon X_{\tilde\beta}$, with $\varepsilon = \pm 1$, and $[X,X_{\tilde\beta}] = \tilde\beta(X)X_{\tilde\beta}$ for all $X \in \mathfrak{a}_p$. In particular, take $X \in \mathfrak{a}_{ph}$, and apply $\sigma\theta$. This gives $[-X,\varepsilon X_{\tilde\beta}] = \tilde\beta(X)\varepsilon X_{\tilde\beta}$, hence $[X,X_{\tilde\beta}] = -\tilde\beta(X)X_{\tilde\beta}$ for all $X \in \mathfrak{a}_{ph}$. But $X_{\tilde\beta} \neq 0$, hence $\tilde\beta(X) = 0$ for all $X \in \mathfrak{a}_{ph}$, hence $\tilde\beta = \hat\beta$. By a similar reasoning we prove $\tilde{\tilde\beta} = \hat\beta$. □

Hence, if $k(\alpha) = 1$:

$$(11.5)\qquad w(\cos\theta) = c.\sin^{p_\alpha-1}\theta = c.(1-\cos^2\theta)^{\frac{1}{2}p_\alpha-\frac{1}{2}} \quad \text{if } p_\alpha \neq 0,$$

$$(11.6)\qquad w(\cos(\theta+\tfrac{1}{2}\pi)) = c.\sin^{q_\alpha-1}(\theta+\tfrac{1}{2}\pi) = c.(1-\cos^2(\theta+\tfrac{1}{2}\pi))^{\frac{1}{2}q_\alpha-\frac{1}{2}} \quad \text{if } q_\alpha \neq 0.$$

Observe that, via the substitution $y := \cos\theta$ in (11.5) and $y := \cos(\theta+\frac{1}{2}\pi)$ in (11.6), (11.5) and (11.6) both give Jacobi polynomials: orthogonal polynomials on $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ ($\alpha,\beta \in \mathbb{R}$, $\alpha,\beta > -1$). If $k(\alpha) = 2$ we have:

$$(11.7)\qquad w(\cos 2\theta) = c.|\cos^{q_{2\alpha}}2\theta|(1-\cos 2\theta)^{\frac{1}{2}p_\alpha+\frac{1}{2}p_{2\alpha}-\frac{1}{2}}(1+\cos 2\theta)^{\frac{1}{2}q_\alpha+\frac{1}{2}p_{2\alpha}-\frac{1}{2}}.$$

Note that (11.7) gives rise to Jacobi polynomials if and only if $q_{2\alpha} = 0$. Fortunately the following proposition holds. Again this proposition is valid for Σ_{pq} for general rank.

PROPOSITION 11.2. *Let* $\alpha \in \Sigma_{pq}$. *If* $k(\alpha) = 2$ *then* $q_{2\alpha} = 0$.

(Observe that this proposition is a corollary from Proposition 11.6 below.) Thus we obtain the following weight function:

$$(11.8)\qquad w(\cos 2\theta) = c.(1-\cos 2\theta)^{\frac{1}{2}p_\alpha+\frac{1}{2}p_{2\alpha}-\frac{1}{2}}(1+\cos 2\theta)^{\frac{1}{2}q_\alpha+\frac{1}{2}p_{2\alpha}-\frac{1}{2}} \quad \text{if } k(\alpha) = 2.$$

The case $k(\alpha) = 4$ can be treated by the following proposition, which again holds for Σ_{pq} of general rank.

PROPOSITION 11.3. *Let* $\alpha \in \Sigma_{pq}$. *If* $k(\alpha) = 4$ *then* $p_\alpha = q_\alpha$.

PROOF. Since $k(\alpha) = 4$, $2\alpha \in \Sigma_{pq}$. (If $2\alpha \notin \Sigma_{pq}$, then $k(\alpha) = c(\alpha)$, by (7.8). But $c(\alpha) = 4$ implies $2\alpha \in \Sigma_{pq}$, by Lemma 2.3). By (7.8) we have moreover $(c(\alpha), c(2\alpha)) = (4,1)$, $(4,2)$ or $(2,2)$. We will next show that if $\beta \in \Phi$, $\hat{\beta} = \alpha$, then $\tau_1\tau_2\beta \neq \beta$. So suppose not.

Suppose $c(\alpha) = 4$. Then there exists $\gamma \in \Phi$ such that $\hat{\gamma} = \alpha$ and $(\tilde{\gamma},\tilde{\gamma}) = 4(\alpha,\alpha) = 4(\tilde{\beta},\tilde{\beta})$ (or $(\tilde{\tilde{\gamma}},\tilde{\tilde{\gamma}}) = 4(\tilde{\tilde{\beta}},\tilde{\tilde{\beta}})$). But then $(\tilde{\gamma},\tau_1\tilde{\gamma}) = -\frac{1}{2}(\tilde{\gamma},\tilde{\gamma})$, and $2(\tilde{\gamma},\tilde{\beta})/(\tilde{\gamma},\tilde{\gamma}) = \frac{1}{2} \notin \mathbb{Z}$. Contradiction.

Suppose $c(2\alpha) = 2$. Then there exists $\gamma \in \Phi$ such that $\hat{\gamma} = 2\alpha$ and $(\tilde{\gamma},\tilde{\gamma}) = 2(2\alpha,2\alpha) = 8(\alpha,\alpha) = 8(\tilde{\beta},\tilde{\beta})$ (or $(\tilde{\tilde{\gamma}},\tilde{\tilde{\gamma}}) = 8(\tilde{\tilde{\beta}},\tilde{\tilde{\beta}})$). But then $(\tilde{\gamma},\tilde{\beta}) = 0$, hence $2(\alpha,\alpha) = (\hat{\gamma},\tilde{\beta}) = (\tilde{\gamma}+\tau_2\tilde{\gamma},\tilde{\beta}) = 0$. Contradiction.

Hence, if $\beta \in \Phi$, $\hat{\beta} = \alpha$ then $\tau_1\tau_2\beta \neq \beta$, thus $\tau_1\tau_2 g_\beta = g_{\tau_1\tau_2\beta} \neq g_\beta$. Thus the collection $\{\beta \in \Phi : \hat{\beta} = \alpha\}$ is a disjoint union $\cup_{i=1}^n \{\beta_i, \tau_1\tau_2\beta_i\}$, and each pair $\beta_i, \tau_1\tau_2\beta_i$ gives rise to a one-dimensional rootspace of α in $g^{+\sigma\theta}$ and a one-dimensional rootspace of α in $g^{-\sigma\theta}$. □

Hence the weight function becomes:

$$(11.9)\qquad w(\cos 4\theta) = c.(1-\cos 4\theta)^{\frac{1}{2}p_\alpha+\frac{1}{2}p_{2\alpha}-\frac{1}{2}}(1+\cos 4\theta)^{\frac{1}{2}q_{2\alpha}-\frac{1}{2}} \quad \text{if } k(\alpha) = 4.$$

By (11.5), (11.6), (11.8) and (11.9) we obtain the following theorem, which generalizes Cartan's result for spherical functions (cf. Example 0.2).

THEOREM 11.4. *If* $\dim \mathfrak{a}_{pq} = 1$, *then the intertwining functions on* U *can be considered as Jacobi polynomials of order* $(\frac{1}{2}m,\frac{1}{2}n)$, *where* m,n *are nonnegative integers.*

As a corollary to the previous results in this chapter, together with Proposition 11.6 below, we obtain that $k(\alpha)$ and $k(2\alpha)$ are completely determined by $p_\alpha, q_\alpha, p_{2\alpha}, q_{2\alpha}$. Hence Σ'_{pq} together with the multiplicities completely determines the weight function w.

In Lemma 11.5 and Proposition 11.6 below Σ_{pq} may be of general rank.

LEMMA 11.5. *Let* $\alpha \in \Sigma_{pq}$. *If* $k(\alpha) = 4$ *and* $p_{2\alpha}q_{2\alpha} = 0$, *then* $p_{2\alpha} = 0$.

PROOF. As in the proof of Lemma 11.1 we obtain $c(2\alpha) = 1$, hence $c(\alpha) = 4$ by (7.8). There exists $\tilde{\gamma} \in \Sigma_p$ (or $\approx{\gamma} \in \Sigma_q$) such that $\hat{\gamma} = \alpha$ and $(\tilde{\gamma},\tilde{\gamma}) = 4(\alpha,\alpha)$. For γ we now have the following possibilities:

(i) $\tau_1\gamma = \gamma$, $\approx{\gamma} = \alpha$. Thus $\alpha, 2\alpha \in \Sigma_q$ (since $c(2\alpha) = 1$). Hence, by [33, Appendix 1.1.3], $2\alpha \in \Phi$. Let $0 \neq X \in \mathfrak{g}_\gamma$, then $0 \neq [X,\tau_1\tau_2X] \in \mathfrak{g}_{2\alpha} \cap \mathfrak{g}^{-\sigma\theta}$. Hence $q_{2\alpha} > 0$, thus $p_{2\alpha} = 0$.

(ii) γ satisfies row 2 of Table I. Then $\gamma + \tau_1\tau_2\gamma \in \Phi$, $(\gamma+\tau_1\tau_2\gamma)^\wedge = 2\alpha$. Let $0 \neq X \in \mathfrak{g}_\gamma$, then $0 \neq [X,\tau_1\tau_2X] \in \mathfrak{g}_{\gamma+\tau_1\tau_2\gamma} \cap \mathfrak{g}^{-\sigma\theta}$. Hence $q_{2\alpha} > 0$, thus $p_{2\alpha} = 0$.

(iii) γ satisfies row 5 of Table I. But then $\gamma + \tau_2\gamma \in \Phi$, $(\gamma+\tau_2\gamma)^\approx = \gamma + \tau_2\gamma \neq 2\alpha$, and $(\gamma+\tau_2\gamma)^\wedge = 2\alpha$, hence $c(2\alpha) > 1$. Contradiction. □

PROPOSITION 11.6. *Let* $\alpha \in \Sigma_{pq}$. *Then* $q_{2\alpha} \neq 0$ *if and only if* $k(\alpha) = 4$.

PROOF. Since $k(\alpha) = 4$, the "if" part follows by Lemma 11.5. So we only need to prove the "only if" part here. Assume $q_{2\alpha} > 0$, and define $A_\alpha, H_\alpha, A_{2\alpha}$ and $H_{2\alpha}$ as in chapter 5. Then $A_{2\alpha} = 2A_\alpha$, and $H_{2\alpha} = \frac{1}{2}H_\alpha$. Since $q_{2\alpha} > 0$ we have for all $\lambda \in \mathbb{Z}_+^\ell$, $X \in i\mathfrak{a}_{pq}$ by Proposition 5.5:

$$\varphi_\lambda(\exp s_{2\alpha}X) = \varphi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_{2\alpha})) = \varphi_\lambda(\exp(X+\tfrac{1}{4}\pi i H_\alpha)). \tag{11.10}$$

But $s_\alpha = s_{2\alpha}$, hence (11.10) implies

$$\varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp(X+\tfrac{1}{4}\pi i H_\alpha)). \tag{11.11}$$

We shall now consider two cases, $p_\alpha > 0$ and $q_\alpha > 0$.

(i) $p_\alpha > 0$. Then, by Proposition 5.2, for all $\lambda \in \mathbb{Z}_+^\ell$, $X \in i\mathfrak{a}_{pq}$

$$\varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp X). \tag{11.12}$$

Combination of (11.11) and (11.12) yields

$$\varphi_\lambda(\exp X) = \varphi_\lambda(\exp(X+\tfrac{1}{4}\pi i H_\alpha)).$$

As in the proof of Corollary 5.6 this implies

$$\tfrac{1}{2}\,\frac{(\lambda_j,\alpha)}{(\alpha,\alpha)} \in 2\mathbb{Z}$$

for all appearing weights λ_j, hence $\frac{(\mu,\alpha)}{(\alpha,\alpha)} \in 4\mathbb{Z}$ for all $\mu \in \mathbb{Z}^{\ell}$.
(ii) $q_\alpha > 0$. Then, by Proposition 5.5, for all $\lambda \in \mathbb{Z}_+^{\ell}$, $X \in i\mathfrak{a}_{pq}$

$$\varphi_\lambda(\exp s_\alpha X) = \varphi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_\alpha)). \tag{11.13}$$

Combination of (11.11) and (11.13) implies

$$\varphi_\lambda(\exp(X+\tfrac{1}{2}\pi i H_\alpha)) = \varphi_\lambda(\exp(X+\tfrac{1}{4}\pi i H_\alpha)).$$

Again as in the proof of Proposition 5.6 this implies

$$\tfrac{1}{2}\,\frac{(\lambda_j,\alpha)}{(\alpha,\alpha)} \in 2\mathbb{Z}$$

for all appearing weights λ_j, hence $\frac{(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\mu \in \mathbb{Z}^{\ell}$. □

Now, by Lemma 11.1, Proposition 11.3 and Proposition 11.6 we obtain the following table, which is valid for Σ_{pq} of general rank. Here * means nonzero.

m_α	$p_\alpha \cdot q_\alpha$	$p_{2\alpha}$	$q_{2\alpha}$	$c(\alpha)$	$c(2\alpha)$	$k(\alpha)$	$k(2\alpha)$
*	0	0	0	1	–	1	–
*	0	*	0	1	1	2	1
*	*	0	0	2	–	2	–
*	*	*	0	2	1	2	1
*	*	0	*	4	1	4	2
*	*	*	*	$\left\{\begin{matrix}2\\4\end{matrix}\right\}$	2	4	2

Table II

The following corollary shows how $k(\alpha)$ depends upon $p_\alpha, q_\alpha, p_{2\alpha}$ and $q_{2\alpha}$.

COROLLARY 11.7. *Let* $\alpha \in \Sigma_{pq}$.

a. $2\alpha \notin \Sigma_{pq}$. *Then*: $p_\alpha q_\alpha = 0 \Rightarrow k(\alpha) = 1$
$p_\alpha q_\alpha > 0 \Rightarrow k(\alpha) = 2$.

b. $2\alpha \in \Sigma_{pq}$. *Then*: $q_{2\alpha} = 0 \Rightarrow k(\alpha) = 2$
$q_{2\alpha} > 0 \Rightarrow k(\alpha) = 4$.

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SUBJECT INDEX

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