CWI Tracts

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CWI Tract 3

System theoretic description of physical systems

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INTRODUCTION

The basic motivation underlying this monograph is two-fold. Firstly, it is thought that the framework of mathematical system theory can contribute to the modelling of dynamical systems as encountered in physics. Secondly, it is thought that system (and control) theory can benefit from a closer study of the natural structures possessed by physical systems.

With respect to the first aspect we note that the commonly used axiomatic frameworks in physics are not really aimed to deal with external influences exerted on a dynamical system. Indeed, normally attention is confined to the description of isolated systems or systems undergoing an external force that is a function of the state of the system and which therefore can be incorporated into the system. A paradigmatic example are conservative mechanical systems where one supposes that the external forces can be derived from a potential function. It is felt that especially in applications this restriction to essentially isolated systems entails quite a loss of generality (see BROCKETT (1977), WILLEMS (1979)).

In mathematical system theory the approach to describing dynamical systems has been entirely different. The possibility of exerting forces on a system is a basic notion and a dynamical system is above all viewed as an entity which transforms input signals (for instance external forces) into output signals. Since system theory finds its roots in engineering this set-up is quite natural if one for example identifies inputs with control variables and outputs with observations.

The dichotomy between the approach of (mathematical) physics and of system theory, grosso modo between the description of a dynamical system by a set of ordinary differential equations, or by an input-output map, is, however, not as large as one may think. This is due to the fact that during the last decades mathematical system theory has recognized the notion of *state* as a central issue in the modelling of dynamical systems. The state of a system at every moment contains all the information about the *past* input-output behavior which is relevant for the *future* input-output behavior. Consequently, the time evolution of the state due to initial conditions and external inputs has become crucial in the system theoretic description of a dynamical system. An important offshoot of this development is the fact that one has reached a clear and precise picture of the concept of state. The state has to be rich enough to explain the input-output behavior, but

on the other hand it is natural to look for a "minimal" state explaining the external behavior. This issue of minimality, together with the question of uniqueness of a state and its constructibility from input-output data has given rise to an elaborate theory about the notion of state. We feel that system theory can therefore contribute to a more fundamental formulation of the concept of state in physics. Indeed a system theorist feels ill at ease at the often unprecise and loose use of the word state in physics (e.g. in thermodynamics). Since both physicist and system theorist are convinced of the importance of the concept of state, a rapprochement between system theory and physics should be fruitful.

With respect to the second aspect - the benefits for system theory stemming from a closer study of the physical structure of systems - we note that the study of general and especially *linear* systems has predominated the development of system theory over the last decades. As a consequence there has been a tendendy to neglect the natural structures imposed by the physical character of a system. Furthermore some of the constructions used in control theory do not have a clear physical interpretation when applied to systems with a specific physical structure. Apart from being unsatisfactory from a theoretical point of view this may be also a serious drawback in applications. Especially for the study of nonlinear (control) systems it seems advantageous to use explicitly the physical structure of the system under consideration (see CROUCH(1981)).

The first two chapters of this monograph are devoted to general system theoretic models for the description and analysis of physical systems. The main characteristic is that the usual framework of mathematical system theory is enlarged by not requiring a priori that the external variables are split into inputs (causes) and outputs (effects). As is argued in WILLEMS (1979) the identification of which external variables constitute causes and which effects, is often not immediately clear in physical systems and should be regarded as a representation question. This results in a general system theoretic framework for physical systems, where "physical" is interpreted in a very broad sense. Section 1.1 gives some intuitive background for this framework, and Section 1.2 is devoted to a study of these ideas for set-theoretic dynamical systems. Especially the notion of state and related issues become very transparent in this set-theoretic context. In Chapter 2 we deal with two important classes of systems: finite dimensional linear and (smooth) nonlinear systems. With respect to linear systems we concentrate

mainly on a "geometric" description of the external behavior in the "frequency domain". Using an idea of Martin & Hermann we associate to a linear system a vector bundle over the complex projective line, whose invariants can be identified with the controllability indices. Furthermore there is another, dual, vector bundle, whose invariants are the observability indices. The second part of Chapter 2 contains a unified treatment of nonlinear systems with external variables from a differential geometric point of view, with particular emphasis on the use of distributions and foliations as a generalization of the "geometric (state space) theory" of linear systems. In Section 2.2.4 we present a new approach to the nonlinear realization problem.

The central piece of the monograph is Chapter 3 where we treat Hamiltonian systems from a system theoretic point of view. After the general Chapters 1 and 2 this chapter is the cornerstone for the desired rapprochement between system theory and physics. We argue that especially in the case of Hamiltonian systems, mathematical system theory can contribute to the formalization of the notion of external force. If we want to regard Newton's second law and the Euler-Lagrange equations with external forces as basic laws of mechanics, and external work as a primary concept, then the framework of (Hamiltonian) vectorfields on (symplectic) manifolds is clearly inadequate. We note that historically the science of mechanics has been the paradigm of a physical science where external forces figure in a prominent way, and that only in the last century the external forces have been removed from the Euler-Lagrange and Hamilton equations. It will appear that Hamiltonian systems with external forces and observations have nice system theoretic properties, and that especially nonlinear Hamiltonian systems form an elegant subclass of the set of all nonlinear systems. We remark that the system theoretic interests in Hamiltonian systems are not restricted to physical systems, since Hamiltonian systems come up naturally in various places such as optimal control, filtering, signal processing etc.

Another contribution of system theory to the analysis of physical systems is the treatment of symmetries and conservation laws as given in Chapter 4. From a system theoretic point of view it is natural to start with a definition of *external* symmetries, i.e. symmetries of the external behavior of a system. Then the question arises whether and how such an external symmetry is related to an *internal* symmetry, i.e. a symmetry in the dynamics of the state. The system theoretic definition of a conservation law is also more general than the usual notion of a function that remains constant along the trajectories of the system. For in-

stance, conservation of energy means that the change in the amount of energy stored in the system is equal to the externally supplied energy (e.g. by external work). We note that this generalized conception of conservation law is often implicitly used in physics, think for instance of the use of conservation of momentum and kinetic energy by collisions of particles. The major result we obtain is an extension of a classical theorem of Noether and can be expressed by saying that also in this generalized setting a symmetry for a Hamiltonian system corresponds to a conservation law and vice versa. The second part of Chapter 4 is devoted to the notion of time-reversibility. Again, time-reversibility is firstly defined in terms of the external behavior of a system. In fact the definition is particularly appealing since we only require that the external behavior, i.e. the set of possible external trajectories, is invariant under change of time direction. To this external time-reversibility there corresponds a dynamic time-reversibility of the evolution of the state. Roughly speaking, the set of state trajectories is invariant under time-reversal, modulo change of sign of the "velocities". When applied to Hamiltonian systems we arrive at the important class of time-reversible Hamiltonian systems, where the internal energy is the sum of a kinetic and a potential energy term. Furthermore for linear time-reversible Hamiltonian systems we are able to solve the so-called *synthesis* problem. We give a procedure to construct a time-reversible Hamiltonian system with a specified external behavior. The required elements are (in an electrical network context) capacitors, inductors and transformers, and in mechanical context masses, springs and (for small deviations) levers, or another mechanical analogue of transformers.

Chapter 5 is devoted to a class of physical systems which is very similar to the class of Hamiltonian systems on the level of definition, namely gradient systems. The characteristic properties of gradient systems are however less clear than in the case of Hamiltonian systems. This is mirrored as it were in the fact that some of the system theoretic results which we proved for Hamiltonian systems are no longer true for nonlinear gradient systems. Chapter 5 therefore shows that Hamiltonian systems form indeed a particularly nice subclass of (nonlinear) systems and that from a system theoretic point of view gradient systems are more complex than Hamiltonian systems.

Finally in Chapter 6 we give some initial ideas about the application of the theory of Hamiltonian systems to the area of (nonlinear) optimal control.

We have confined ourselves in this monograph to the treatment of fundamental system theoretic aspects of physical and in particular Hamiltonian and gradient systems. To return to the beginning of this Introduction we believe that the systematic study of systems with a given physical structure is also of help for the treatment of control and synthesis problems for such systems. Indeed this remains a large area for further research.

Advices and warnings for the reader.

Although this monograph has a pyramidical structure it is not necessary to read everything from page 1 to the end at a stretch. Especially Chapter 2 may be for some readers a bottleneck which can be better postponed till after reading Chapters 3, 4 and 5. Furthermore it is recognized that the interests of the readers may differ considerably. We have therefore distinguished between three categories of readers:

1. The reader primarily interested in Hamiltonian and gradient systems is advised to read Section 1.1, to skim over Section 1.2 and to take a brief glance at the first definitions of Sections 2.1 and 2.2. Then one should start with the reading of Chapter 3 and continue with Chapters 4 and 5. Occasionally one will notice some references to especially Chapter 2 to which one can return if necessary.
2. For the reader who is especially interested in nonlinear system theory, the real reading starts in Section 2.2, with a rather elaborate treatment of nonlinear systems including controlled invariance and a new approach to the nonlinear realization problem, and can be continued to the end (with possible deletion of the "linear" sections).
3. The reader who wants to confine him or herself to the treatment of the material in a mainly linear context, is advised to read Chapter 1, Section 2.1 of Chapter 2 and Sections 3.1, 3.1.4, 3.2.2, 3.5, 3.5.1, 4.1.3, 4.2.3, 4.2.4, 5.1 and 5.2.2.

CHAPTER 1

SYSTEMS WITH EXTERNAL VARIABLES

1.1. Some remarks on the description of physical systems

In this section we want to give some intuitive background and motivation for the mathematical framework that we will use in the rest of this monograph for the description of physical systems. In first instance, the word "physical" can be interpreted in a broad sense. It means that we do not want to enter the possible axiomatics of some "general system theory", and that we confine ourselves on the intuitive level to systems which are normally considered in disciplines as physics, chemistry and parts of biology and economics. Later on, in Chapters 3, 4 and 5 "physical" will be interpreted in a much more narrow sense (i.c. Hamiltonian and gradient systems).

Very roughly speaking, a mathematical description of a physical system consists of a set of variables which in a certain way represent the "empirical characteristics" of the system. Moreover, the mathematical model should give (preferably quantitative) relationships between these variables. Especially important is usually the description of the dynamical behavior of the system, i.e. the way the system evolves in time. For this purpose the variables are seen as functions of time and the mathematical model has to give information about how these variables evolve. When we talk here about the description of systems, we will mainly be referring to this dynamical behavior.

Systems with external variables

What kind of procedure should we follow in trying to describe a physical system? The first step we have to make is to look at the system as an entity distinguished from the outside world. We have to make clear what belongs to the system and what we do not want to include in it. After this separation between system and environment has been accomplished, we have, roughly speaking, the following three possibilities to describe the system.

The first one is that we consider the system as actually *isolated* from the outside world, or at least that for all purposes of accuracy we may

regard the system as isolated. The paradigmatic example of this possibility is our solar system. Indeed this can be regarded as a world on its own. However, it is hard to find down-to-earth and real (i.e. not idealized) systems which have this same strictly isolated behavior although it may be in many instances a reasonable assumption.

A second possibility is to regard the part of the outside world which may influence the system under consideration as nearly constant in time when compared to the dynamical behavior of our system (if it is really constant we could have described the system as isolated). The usual procedure is then to include into the mathematical model a set of *parameters* which represent this external influence and are supposed to be slowly varying in time. Indeed, a large part of mathematics dealing with the description of (dynamical) systems is at least partly concerned with or motivated by this type of modelling. We mention perturbation theory, bifurcation theory and the theory of structural stability.

The third possibility is to try to really include the connections of the system with the outside world *into* the description of the system. The system is therefore, so to say, not regarded as an isolated "box", but as a "box" together with the "wires" connecting it to the rest of the world. This third possibility we will call the *system theoretic description of a physical system*. Of course this goes along with a changing point of view. One does not try to isolate the system "at all costs", but one is especially interested in the continuous interplay of the system and its environment. Since this environment is considered as "unknown", we have to study the set of *all* dynamical behaviors which can occur at the boundary of the system (the wires of the box), i.e. all behaviors which are compatible with the system under consideration. This whole set is called the *external behavior* of the system. We should, however, mention that for real systems there may be a very large amount of connections with the outside world, where as also in a system theoretic description we will normally only treat a small number of them and neglect the rest. Hence the same type of questions as arising in the first and second possibility also exists in a system theoretic description. However, we have at least on a conceptual level a way to deal with the influences from and on the outside world. This seems to be an important advantage of the third possibility.

There is another argument in favor of the system theoretic description. In disciplines like physics and chemistry it has been a very successful approach to consider a system as composed of smaller and simpler subsystems

which are much easier to describe. Indeed the success story of physics seems to be partly based on its concentration on the study of simple and idealized systems. Afterwards the large real system can then be "understood" in terms of the simple systems which constitute the large system. In fact in celestial mechanics a breakthrough, made by Newton, was to consider the solar system as composed of the heavenly bodies, each forming a system on its own, governed by a simple law (Newton's second law), and undergoing forces from the other systems and on its turn exerting forces on them. This approach, called "tearing", gives us the system as a (sometimes complicated) interconnection of all kinds of relatively simple systems. To study the whole system we can study these simple systems separately. But then we should also include in their description their external behavior (i.e. the way in which they can influence and can be influenced by the outside world), since this will be needed in order to determine the behavior of the whole system. The procedure is thus as follows. Tear the system into simple subsystems. Study the systems together with their external behavior. Then interconnect the simple systems again with each other.
For example, given an electrical circuit, we can first study the behavior of its elements (capacitors, inductances, resistances, and so on) out of which the circuit is composed. Then by interconnecting these elements in accordance with Kirchhoff's laws one can obtain the original circuit again.

This brings us to another point in favor of the system theoretic approach, which has its roots in technical applications and engineering. Instead of studying the behavior of a complicated system by tearing it, we go the other way around and we want to *construct* a system with a specified behavior, out of simple *building blocks*. This leads to the so-called *synthesis problem: which* building blocks should we use and *how* should we interconnect them in order to achieve a system with a specified behavior. Clearly to tackle this problem we need a theory of systems which also includes their external behavior.

A more general argument for the system theoretic description, also originating from engineering, has to do with the attitude to consider a system as a *device*. Usually, this goes together with the so-called input-output framework, to which we return later on. One looks at a system as a device which transforms inputs (controls) into outputs. The external behavior of the device is exactly this relationship between input functions and output functions. Clearly, this external behavior of the device is

what really counts in applications.

Summarizing, we want to study systems which may be connected with other systems. Therefore, we consider the system as separated from the outside world, but we also incorporate in its description the external behavior of the system. We will assume that this external behavior is given by specifying the possible evolutions in time of a set of variables, which we will call the *external variables*.

The notion of state

Apart from connections with other systems there is still another, maybe even more fundamental reason to study the external behavior of a system. This has to do with the notion of *state*. Intuitively the state of a system should contain the whole memory of the system. Knowledge of the system at a certain instance of time, together with the knowledge of all future external influences should totally determine the future dynamical behavior of the system. Hence, in the case that the system is isolated, the state of the system is all one needs to know in order to predict the future (single) behavior of the system. The usual mathematical structure for this last situation is a set of first-order differential equations in the state variables. Partial differential equations can be seen as first-order differential equations on an infinite-dimensional state space, and many other mathematical descriptions are also variations on this theme.

Of course, this type of modelling presupposes that one knows which variables constitute the state of the system. In many situations, however, a physical system is actually given by a set of "phenomenological" laws, describing the external behavior of the system and not involving the state variables. A simple example is the law for ideal gases PV = constant, which gives the relation between the two external variables P (pressure) and V (volume). A state of the system consists of the positions and velocities of all particles involved. Another simple example is Newton's second law $F = m\ddot{q}$ which is a dynamical compatibility relation between the two external variables F (force) and q (position) as functions of time. The state of this system consists of the position and the velocity, or the position and the momentum. Hence in this case the state can be very easily constructed from the knowledge of the external variables as functions of time, but does not explicitely enter the law $F = m\ddot{q}$. We can also consider a (large) electrical

network, described by compatibility relations on the voltages and currents on some wires emanating from the network. These compatibility relations do not have to involve the state variables, which are the voltages or currents of (a subset of) the circuit elements *inside* the network. We see that there can be two reasons for giving the system as a set of compatibility relations ("laws") on the external variables, not involving the state variables:
(i) The state of the system can be very complex, while the external behavior is (relatively) simple.
(ii) The state of the system is not *accessible* to us; we cannot measure what is going on inside the system.
This second reason goes along with the so-called "black-box" description of a system. We can only observe (or we only care about) what comes into the box and what goes out of it. From an experimental point of view it can be argued that descriptions of physical systems are in first instance always "black box" descriptions.

Concluding we can say that in many cases the external behavior of a system should be actually taken as the *starting point* for the description of a system. If we want to know the state of the system we should be able to deduce it from the observations of the external behavior. This is in system theory called the *Realization Problem*: How do we construct from the external behavior
(i) a set of variables which is rich enough to be called the state of the system, and
(ii) the equations governing the evolution of the state ?
Since we only want to construct a state which "explains" the external behavior it is of course possible that we end up with a state which does not correspond to the "real physical state" of the system. In the case of a mechanical system we might take instead of the natural state, i.e. the positions and velocities (or momenta) of the particles *another* set of variables which is in one-to-one correspondence with it (Notice that we already mentioned two possibilities for a natural state: positions and velocities, or positions and momenta). For thermodynamic systems it is always possible to find a set of variables which is much smaller than the set of the positions and velocities of all the particles involved, but which on a more axiomatic level can be called the state since it contains all the memory about the external behavior. An extreme example is an ideal gas satisfying PV = constant. This system does not have memory, and hence we do not need a state. The "real physical state" will be *non-minimal*, a notion we will explain

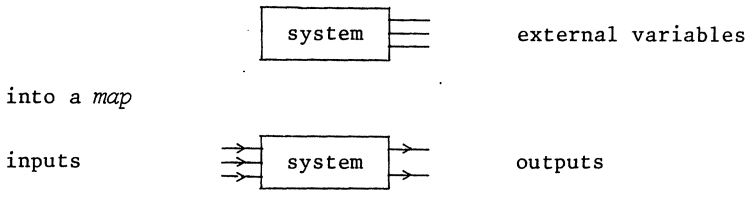
later on. Of course the loss of physical interpretation of the state variables which may occur can be a serious drawback for the theory. In the case of Hamiltonian and gradient systems (see Chapters 3 and 5) we will try to combine these notions of a "minimal" and a "physical" state, to end up with a minimal state which is also physically interpretable. The approach which will be taken can be compared with the use of generalized coordinates in classical mechanics.

Finally we remark that we restrict ourselves to the description of deterministic systems. In many cases it is of course necessary to take into account *uncertainty* about the observational data and the parameters of our models, and "*identification*" will be a central issue. In this context we remark that also in the case of systems with external variables we need a theory which gives information about the validity of our mathematical models, if some parameters are subject to uncertainty (this has much to do with the notion of *structural stability*).

The input-output point of view

We now look at a special, but important case of a system with external variables. We suppose that it is possible to split the set of external variables into two sets, such that the external behavior of the system consists of "arbitrary" (apart from reasonable smoothness assumptions) functions of time in the first set of variables and such that the functions of time in the second set of variables are uniquely determined by the functions in the first set. We call the variables in the first set the *inputs* and the variables in the second set the *outputs*. Moreover we suppose that the output at every time is already determined by the corresponding input function up to this time.

We have thus introduced a *non-anticipating* relationship among the external variables, and we can interpret the inputs as *causes* and the outputs as *effects*. Pictorially we have specified the *relation*



where the arrows stand for the "causal" relationship. This type of system with external variables is called an *input-output* system, and is the usual starting point in system (and control) theory. Whereas in the case of a general system with external variables we stress the idea of the existence of dynamical compatibility relations among the external variables, it is in the input-output case more natural to see the system as something which *transforms*, or maps, input functions into output functions. This point of view is in many instances very useful, especially for control purposes, and we will use it quite often in the sequel.

However, we like to stress that for physical systems the input-output point of view has disadvantages since it is a) often not a priori clear which external variables are the inputs, and which are the outputs, b) sometimes undesirable or even illogical to distinguish between inputs and outputs, c) possible that a global separation of external variables into inputs and outputs cannot be achieved (for instance in the case of nonlinear systems, see Chapter 2.2), and d) the identification of what are inputs and what are outputs and the corresponding causal structure that is induced seems to be itself a modeling question.

Consider for example an electrical network with some external currents and voltages. It is not a priori clear whether the currents should be treated as inputs and the voltages as outputs, or vice versa. Moreover for the description of the dynamical behavior of the network it is not necessary to make this distinction between inputs and outputs, so why impose this extra structure? Finally it is then a *theorem*, which can be proved within a general framework of (passive) networks with the voltages and currents of the external ports as external variables, that there always exists an input-output representation. However, this input-output representation is in general *hybrid*, which means that the voltages of a part of the external ports and the currents of the other ports are the inputs and the remaining external voltages and currents are the outputs.

1.2. Set-theoretic mathematical models

In this section we give a mathematical framework for the description of physical systems, which constitutes a formal setting for the intuitive ideas sketched before in Section 1.1. Such a framework will be given here on a purely set-theoretical level, and only later on in Chapter 2 we add more structure, e.g. differentiability and linearity, to this framework.

This section closely follows the lines of WILLEMS (1979).

DEFINITION 1.1 Let W be a set, called the set of *external variables* (i.e. the set in which the external variables take their values). An *external dynamical system* Σ_e on W is a subset of $W^{\mathbb{R}}$ (all functions from $\mathbb{R}$ to W).

So an external dynamical system simply consists of a set of functions from the time-axis $\mathbb{R}$ to W, or a set of trajectories in W. The idea is that W consists of the variables which are being modelled and that Σ_e expresses the (physical) laws among them, i.e. the compatibility relations which the different variables, in order for them to exist simultaneously, need to satisfy.

We will only consider (for simplicity) *time-invariant* systems. Intuitively these are systems which are homogeneous in time. Formally, we require that $S_T\Sigma_e = \Sigma_e$ for all $T \in \mathbb{R}$ with S_T the shift operator $(S_T w)(t) := w(t-T)$. To define the *state* of a system we need to introduce some notions about relations. Let R be a relation on (i.e. a subset of) A×B, with A and B sets. We define R_A as the projection of R on A and R_B as the projection of R on B. The relation R is said to be a product relation if $R = R_A \times R_B$. In this case we say that R is a *rectangle* with respect to A and B. If $(x_1,x_2) \in R$, with $x_1 \in A$ and $x_2 \in B$, and R is a rectangle, x_1 and x_2 are *independent* in a set theoretical sense. Now let R be a relation on (a subset of) A × B × C. We call $R_{\{x_2=b\}} := \{(x_1,x_3) \mid (x_1,b,x_3) \in R\}$ the relation R conditioned by $\{x_2=b\}$. We say that x_1 and x_3 are independent given x_2, or that x_2 *splits* x_1 and x_3, if, for all $b \in B$, $R_{\{x_2=b\}}$ is a product relation on A × C. These notions can be easily generalized to relations on sets of the form $\prod_{i\in I} A_i$, with I an arbitrary index set, and we use them to give

DEFINITION 1.2 Let W be the set of external variables and let X be a set called the state space. A *dynamical system in state space form* is defined as a subset Σ_i (i from "internal") of $(X\times W)^{\mathbb{R}}$ which satisfies the axiom:

x(t) splits $\{x(\tau),w(\tau);\tau<t\}$ and $\{x(\tau),w(\tau);\tau\geq t\}$ for every $t \in \mathbb{R}$.
We denote an element of $(X\times W)^{\mathbb{R}}$ by (x,w), with $x: \mathbb{R} \to X$ and $w: \mathbb{R} \to X$.
Then the axiom above is equivalent to:

$$\{(x_1,w_1)\in\Sigma_i, (x_2,w_2)\in\Sigma_i \text{ and } x_1(t)=x_2(t)\} \implies$$
$$\{(x,w)\in\Sigma_i, \text{ where } (x,w)(\tau) := \begin{cases} (x_1,w_1)(\tau), & \tau<t \\ (x_2,w_2)(\tau), & \tau\geq t \end{cases}\}$$

Notice that the above definition really formalizes the intuitive idea of state: it says that the present state is all one needs to know about the past in order to be able to specify all possible future trajectories.

DEFINITION 1.3 Let Σ_i be a dynamical system in state space form on X×W. Then $\Sigma_e := \{w \mid \exists x \text{ such that } (x,w) \in \Sigma_i\}$ is called its *external behavior*. We denote this by $\Sigma_i \Longrightarrow \Sigma_e$. If Σ_e is given and if Σ_i is such that $\Sigma_i \Rightarrow \Sigma_e$, then Σ_i is called a *(state space) realization* of Σ_e.

Let $w \in W^{\mathbb{R}}$. Then w^- and w^+ denote the restrictions of w to $(-\infty,0)$ and $[0,\infty)$ respectively. We write $w = w^- \cdot w^+$. The product $\cdot$ is called the concatenation product. Similarly we define for $x \in X^{\mathbb{R}}$, x^- and x^+. For a dynamical system Σ_e we denote $\Sigma_e^- := \{w^- \mid w \in \Sigma_e\}$ and $\Sigma_e^+ := \{w^+ \mid w \in \Sigma_e\}$. For a system in state space form Σ_i with state space X we define

$$\Sigma_i(a) := \{(x,w) \in \Sigma_i \mid x(0)=a\}$$
$$\Sigma_i^-(a) := \{(x^-,w^-) \mid (x,w) \in \Sigma_i(a)\} \qquad a \in X$$
$$\Sigma_i^+(a) := \{(x^+,w^+) \mid (x,w) \in \Sigma_i(a)\}$$

Analogously we define

$$\Sigma_e(a) := \{w \in W^{\mathbb{R}} \mid \exists x \text{ such that } (x,w) \in \Sigma_i \text{ and } x(0)=a\}$$
$$\Sigma_e^-(a) := \{w^- \mid w \in \Sigma_e(a)\}$$
$$\Sigma_e^+(a) := \{w^+ \mid w \in \Sigma_e(a)\}$$

Obviously $\Sigma_i = \bigcup_{a \in X} \Sigma_i(a)$, $\Sigma_e = \bigcup_{a \in X} \Sigma_e(a)$ and by the axiom of Definition 1.2, $\Sigma_i(a) = \Sigma_i^-(a) \cdot \Sigma_i^+(a)$ and $\Sigma_e(a) = \Sigma_e^-(a) \cdot \Sigma_e^+(a)$ (with of course, $\Sigma_e^-(a) \cdot \Sigma_e^+(a) := \{w_1^- \cdot w_2^+ \mid w_1^- \in \Sigma_e^-(a),\ w_2^+ \in \Sigma_e^+(a)\}$, and $\Sigma_i^-(a) \cdot \Sigma_i^+(a)$ defined similarly). Therefore a realization allows one to write Σ_e, viewed as a subset of the product set $W^{(-\infty,0)} \times W^{[0,\infty)}$, as a union $\bigcup_{a \in X} \Sigma_e^-(a) \cdot \Sigma_e^+(a)$ of rectangular subsets. It can be seen that the converse also holds, i.e. a union of rectangular subsets forming Σ_e yields a realization of Σ_e.

We now use these ideas in order to obtain some specific realizations of Σ_e. The producedure is based on a natural generalization of *Nerode equivalence*. This notion was originally defined in the context of input-output systems and we give the precise formulation of it later on when we treat input-output systems.

Define the relation R^+ on Σ_e by

$\{w_1 R^+ w_2\}: \Longleftrightarrow \{w_1^- \cdot w^+ \in \Sigma_e \Longleftrightarrow w_2^- \cdot w^+ \in \Sigma_e\}$

i.e. $w_1 R^+ w_2$ if w_1^- and w_2^- have equal futures.

Now take $X^+ := \Sigma_e (\text{mod } R^+)$ and define the state at time 0 corresponding to a $w \in \Sigma_e$ to be the equivalence class (i.e. element of X^+) to which w belongs. Time invariance then allows one to define the whole state trajectory corresponding to this w. This defines a realization denoted by Σ_i^+.

Similarly we define for $w_1, w_2 \in \Sigma_e$

$\{w_1 R^- w_2\}: \Longleftrightarrow \{w^- \cdot w_1^+ \in \Sigma_e \Longleftrightarrow w^- \cdot w_2^+ \in \Sigma_e\}$

i.e. $w_1 R^- w_2$ if w_1^+ and w_2^+ have equal pasts.

Define analogously $X^- := \Sigma_e (\text{mod } R^-)$, and obtain as above a realization denoted by Σ_i^-. Furthermore we can define $\{w_1 R^{\pm} w_2\} := \{w_1 R^+ w_2 \text{ and } w_1 R^- w_2\}$ i.e. w_1 and w_2 have equal futures and pasts. The realization resulting from $R^{\pm}$ is denoted by $\Sigma_i^{\pm}$.

The realizations constructed above are in some sense the canonical realizations. They also have the nice properties defined below.

DEFINITION 1.4 A realization Σ_i of Σ_e, with state space X, is called *externally induced* if there exists a map $f : \Sigma_e \to X$ such that $\{(x,w) \in \Sigma_i\} \Longrightarrow \{x(0)=f(w)\}$. It is called past (future) externally induced if there exists a map $f^- : \Sigma_e^- \to X$ ($f^+ : \Sigma_e^+ \to X$) such that $\{(x,w) \in \Sigma_i\} \Longrightarrow \{x(0)=f^-(w^-)\}$ ($\{x(0)=f^+(w^+)\}$).

It is easy to see that Σ_i^+ is past externally induced, Σ_i^- is future externally induced and that $\Sigma_i^{\pm}$ is externally induced.

DEFINITION 1.5 A realization Σ_i of Σ_e with state space X is called *minimal* if the following conditions are satisfied:

(i) there does *not* exist a subset $X' \subset X$ consisting of more than one point such that $\bigcup_{a \in X'} \Sigma_e(a)$ is a rectangle (as a subset of $W^{(-\infty,0)} \cdot W^{[0,\infty)}$).

(ii) there does not exist a subset $X' \subset X$, strictly smaller than X, such that $\Sigma_e = \bigcup_{a \in X'} \Sigma_e(a)$.

Remark: If Σ_i is externally induced, then condition (i) implies condition (ii). Proof: there exists $f : \Sigma_e \to X$ such that $\{(x,w) \in \Sigma_i\} \Longrightarrow \{x(0)=f(w)\}$. Suppose now that $\Sigma_e = \bigcup_{a \in X'} \Sigma_e(a)$, for $X' \subset X$. Then necessarily Im $f \subset X'$. However, this implies $\Sigma_e(a) = \emptyset$ $\forall a \in X \backslash X'$ and hence condition (i) is not satisfied.

Intuitively, minimality of a realization means that one cannot make the state space smaller, while retaining the same external behavior. It can be proven (see WILLEMS (1979)) that the two canonical realizations Σ_i^+ and Σ_i^- are minimal.

We call two realizations Σ_i^1, Σ_i^2 of Σ_e with respective state spaces X_1 and X_2 equivalent, denoted $\Sigma_i^1 \sim \Sigma_i^2$, if there exists a bijection $S : X_1 \to X_2$ such that $\{(x_1,w)\in\Sigma_i^1\} \Longleftrightarrow \{(Sx_1,w)\in\Sigma_i^2\}$ with $(Sx_1)(t) := S(x_1(t))$. It would be very nice if we could prove that minimal realizations are *unique*, in the sense that if Σ_i^1 and Σ_i^2 are two minimal realizations of Σ_e then Σ_i^1 is equivalent to Σ_i^2. However, in general this is not true (counterexamples can be found in WILLEMS (1979)). The case that minimal realizations are indeed unique can be characterized as follows.

THEOREM 1.6 Let Σ_e be a time-invariant system. All minimal realizations of Σ_e are equivalent if and only if one of the following equivalent conditions holds

(i) $\Sigma_i^+ = \Sigma_i^-$

(ii) $\Sigma_i^+ = \Sigma_i^- = \Sigma_i^\pm$

(iii) $R^+ = R^-$

(iv) $\Sigma_i^\pm$ is minimal

(v) There exists a realization Σ_i of Σ_e with state space X such that for all $a',a'' \in X$ with $a' \neq a''$ it holds that $\Sigma_e^-(a') \cap \Sigma_e^-(a'') = \emptyset$, and $\Sigma_e^+(a') \cap \Sigma_e^+(a'') = \emptyset$.

Moreover, if all minimal realizations are equivalent they are all past and future externally induced.

PROOF We first prove that conditions (i) to (v) are equivalent. It is easy to see that $R^+ = R^- \Longleftrightarrow \Sigma_i^+ = \Sigma_i^-$ and that $\Sigma_i^+ = \Sigma_i^- \Longleftrightarrow \Sigma_i^+ = \Sigma_i^- = \Sigma_i^\pm$. Therefore (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii). (ii) $\Longleftrightarrow$ (iv) is trivial and we prove that if $\Sigma_i^\pm$ is minimal then $\Sigma_i^\pm = \Sigma_i^- = \Sigma_i^+$. Because of the definition of $R^\pm$, every set $\Sigma_e(a)$ with $a \in \Sigma_i^+$ can be written as $\Sigma_e(a) = \bigcup_{\text{some } x\in\Sigma_i^\pm} \Sigma_e(x)$. Since $\Sigma_e(a)$ is a rectangle and $\Sigma_i^\pm$ is minimal this implies $\Sigma_e(a) = \Sigma_e(x)$ for one $x \in \Sigma_i^\pm$. Hence $\Sigma_i^\pm = \Sigma_i^+$. Similarly $\Sigma_i^\pm = \Sigma_i^-$. For the proof of (ii) $\Longleftrightarrow$ (v) we note that $\Sigma_i^+ = \Sigma_i^- = \Sigma_i^\pm$ if and only if Σ_e can be written as

$\Sigma_e = \bigcup_{\alpha\in A} B_\alpha^- \cdot B_\alpha^+$ with $B_{\alpha_1}^- \cap B_{\alpha_2}^- = \emptyset$ and $B_{\alpha_1}^+ \cap B_{\alpha_2}^+ = \emptyset$ for all $\alpha_1 \neq \alpha_2 \in A$, and with $\bigcup_{\alpha\in A} B_\alpha^- = \Sigma_e^-$ and $\bigcup_{\alpha\in A} B_\alpha^+ = \Sigma_e^+$. Of course a union $\bigcup_{\alpha\in A} B_\alpha^- \cdot B_\alpha^+$ as above defines a realization of Σ_e with state space A. Concluding (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v).

If all minimal realizations are equivalent then trivially condition (i) holds. We now prove that if conditions (iv) and (v) hold then an arbitrary realization Σ_i is equivalent to $\Sigma_i^{\pm}$. For this we only have to prove that $\Sigma_e(a) := \Sigma_e^-(a)\cdot\Sigma_e^+(a)$ is a subset of an element of $X^{\pm}$, for every a in the state space of Σ_i, since by minimality of $\Sigma_i^{\pm}$ this implies that $\Sigma_i \sim \Sigma_i^{\pm}$. Let now $w_1, w_2 \in \Sigma_e(a)$. Since also $w_1^- \cdot w_2^+ \in \Sigma_e(a)$ it follows that there is at least one element of Σ_e^+, i.e. w_2^+, such that $w_1^- \cdot w_2^+$ and $w_2^- \cdot w_2^+ \in \Sigma_e$. Using the fact that $\Sigma_e = \bigcup_{\alpha\in A} B_\alpha^- \cdot B_\alpha^+$ with B_α^- and B_α^+ as above this implies that $w_1 R^+ w_2$. Similarly we prove $w_1 R^- w_2$. For the last statement we note that if all minimal realizations are equivalent they are all equivalent to Σ_i^+ and Σ_i^- which are past (respectively future) externally induced. □

As we saw in the proof, condition (v) in Theorem 1.6 is actually a condition on the structure of Σ_e, namely that Σ_e can be written as a union $\bigcup_{\alpha\in A} B_\alpha^- \cdot B_\alpha^+$, where the B_α^-'s and the B_α^+'s are non-overlapping. We can give some other useful characterizations for uniqueness of minimal realizations which are based on this observation:

THEOREM 1.7 The following five conditions are equivalent:

(i) $\Sigma_e = \bigcup_{\alpha\in A} B_\alpha^- \cdot B_\alpha^+$ with $B_{\alpha_1}^- \cap B_{\alpha_2}^- = \emptyset$ and $B_{\alpha_1}^+ \cap B_{\alpha_2}^+ = \emptyset$ for all $\alpha_1 \neq \alpha_2 \in A$.

(ii) There exists a realization Σ_i of Σ_e which is past as well as future externally induced.

(iii) For every $w_1, w_2 \in \Sigma_e$ it holds that if w_1 and w_2 "have one common future" then their whole futures are equal, i.e. $\{\exists w\in\Sigma_e$ such that $w_1^- \cdot w^+ \in \Sigma_e$ and $w_2^- \cdot w^+ \in \Sigma_e\} \Longrightarrow \{w_1 R^+ w_2\}$

(iv) For every $w_1, w_2 \in \Sigma_e$ it holds that if w_1 and w_2 "have one common past", then their whole pasts are equal, i.e. $\{\exists w\in\Sigma_e$ such that $w^- \cdot w_1^+ \in \Sigma_e$ and $w^- \cdot w_2^+ \in \Sigma_e\} \Longrightarrow \{w_1 R^- w_2\}$.

(iv) All minimal realizations are equivalent.

PROOF (i)$\Longleftrightarrow$(v) was proved in Theorem 1.6. If $\Sigma_e = \bigcup_{\alpha\in A} B_\alpha^- \cdot B_\alpha^+$ as above then this clearly defines a realization (with state space A) which is past as well as future externally induced. Conversely if Σ_i with state space X is past externally induced, then $\Sigma_e^-(a_1) \cap \Sigma_e^-(a_2) = \emptyset$ for every $a_1 \neq a_2 \in X$. If Σ_i is also future externally induced then $\Sigma_e^+(a_1) \cap \Sigma_e^+(a_2) = \emptyset$ for every $a_1 \neq a_2 \in X$. Hence (i)$\Longleftrightarrow$(ii).

Now we prove (iii) $\Longrightarrow$ (iv). The proof of (iv)$\Longrightarrow$ (iii) is similar, while it is clear that (i)$\Longleftrightarrow$(iii) + (iv).

Assume that (iii) is satisfied. Let $w \in \Sigma_e$ be such that $w^- \cdot w_1^+ \in \Sigma_e$ and $w^- \cdot w_2^+ \in \Sigma_e$. Suppose $\tilde{w} \in \Sigma_e$ is such that $\tilde{w}^- \cdot w_1^+ \in \Sigma_e$. We have to prove that also $\tilde{w}^- \cdot w_2^+ \in \Sigma_e$. Now $\tilde{w}^- \cdot w_1^+ \in \Sigma_e$ together with $w^- \cdot w_1^+ \in \Sigma_e$ implies that $\tilde{w}$ and w have one common future. Therefore $\tilde{w} R^+ w$. Since $w^- \cdot w_2^+ \in \Sigma_e$, this implies $\tilde{w}^- \cdot w_2^+ \in \Sigma_e$. □

An important class of systems that satisfy the conditions of Theorem 1.6 and 1.7 is the class of linear systems, which we will treat in Chapter 2. It is trivial to see that also *autonomous* systems have equivalent minimal realizations. For completeness we give

DEFINITION 1.8 Let Σ_e be an external system on W. Σ_e is called *autonomous* if there exists a bijection $h : \Sigma_e^- \longrightarrow \Sigma_e^+$ such that $w \in \Sigma_e \Longleftrightarrow w^+ = h(w^-)$.

Let now Σ_i^1 and Σ_i^2 be two equivalent realizations of an external system Σ_e. The following Proposition gives sufficient conditions for the *uniqueness* of the equivalence mapping between Σ_i^1 and Σ_i^2.

PROPOSITION 1.9 Let Σ_i^1 and Σ_i^2 be equivalent realizations of Σ_e with state space X_1, respectively X_2. Let $S : X_1 \to X_2$ be an equivalence mapping. If Σ_i^1 and Σ_i^2 are externally induced realizations, i.e. there exist maps $f_1 : \Sigma_e \to X_1$ and $f_2 : \Sigma_e \to X_2$ as in Definition 1.4, *and* if f_1 and f_2 are surjective, then S is unique. In particular, if all minimal realizations of Σ_e are equivalent, and Σ_i^1 and Σ_i^2 are minimal realizations, then S is unique.

PROOF Let $w \in \Sigma_e$, then there exist x_1 and x_2 such that $(x_1,w) \in \Sigma_i^1$ and $(x_2,w) \in \Sigma_i^2$. It is clear that $S : X_1 \to X_2$ has to satisfy $S(x_1(0)) = x_2(0)$. Hence S is uniquely determined on Im f_1. If all minimal realizations of Σ_e are equivalent, then by Theorem 1.6 they are all externally induced. □

Finally we want to make a closer study of the subclass of *input-output* systems.

DEFINITION 1.10 Let Σ_e be an external system on W. Σ_e is called an *external input-output system*, if there exist sets U and Y such that $W = Y \times U$, and a map $F : U^{\mathbb{R}} \to Y^{\mathbb{R}}$ which satisfies $\{u_1(t)=u_2(t), t\leq\tau\} \Longrightarrow \{(Fu_1)(t)=(Fu_2)(t), t\leq\tau\}$ for all $u_1, u_2 \in U^{\mathbb{R}}$, such that

$$\Sigma_e = \{(y,u) \mid y=Fu,\ u\in U^{\mathbb{R}}\}.$$

In a realization theory of input-output systems it is natural to restrict the possible realizations to the subclass of past externally induced realizations. Since $u_1(t) = u_2(t)$, $t \leq \tau$ implies $(Fu_1)(t) = (Fu_2)(t)$, $t \leq \tau$, F induces a function $F^- : U^{(-\infty,0)} \longrightarrow Y^{(-\infty,0)}$. Hence an element $w^- \in \Sigma_e^-$ is given by (F^-u^-,u^-) with $u^- \in U^{(-\infty,0)}$. If a realization Σ_i is past externally induced, there exists a function $f^- : \Sigma_e^- \to X$ such that $\{(x,w)\in\Sigma_i\} \Longrightarrow \{(x(0)=f^-(w^-)\}$. Then in the input-output case we may define $\tilde{f}^- : U^{(-\infty,0)} \to X$ by $\tilde{f}^-(u^-) := f^-(F^-u^-,u^-)$ and hence $\{(x,w)\in\Sigma_i\} \Longrightarrow \{x(0)=\tilde{f}^-(u^-)\}$. Therefore past externally induced can be better called *past input* induced. Furthermore we notice that the equivalence relation R^+ in this case amounts to the classical Nerode equivalence, which is defined as a relation on the past input functions. Two past inputs u_1^- and u_2^- are called Nerode equivalent if $(Fu_1^- \cdot u^+)(t) = (Fu_2^- \cdot u^+)(t)$ for every $t \geq 0$ and every $u \in U^{\mathbb{R}}$.

Within the class of past externally induced systems (not necessarily input-output) the following holds

THEOREM 1.11 Let Σ_e be an external system. Then: all minimal past externally induced systems are equivalent and they are actually all equivalent to Σ_i^+.

PROOF Let Σ_i be a minimal past externally induced system with state space X. Let $\Sigma_e(x^+) \cap \Sigma_e(a) \neq \emptyset$ for $x^+ \in X^+$ and $a \in X$. We will prove that $\Sigma_e(a) \subset \Sigma_e(x^+)$, and hence that $\Sigma_e(x^+) = \bigcup_{a\in X'} \Sigma_e(a)$, for a subset $X' \subset X$. Since Σ_i is minimal this implies $\Sigma_e(a) = \Sigma_e(x^+)$ and therefore $\Sigma_i \sim \Sigma_i^+$. Consider all $\tilde{w}^- \in \Sigma_e^-(x^+) \cap \Sigma_e^-(a)$. Since Σ_i and Σ_i^+ are past externally induced it holds that if $\tilde{w}^- \cdot w^+ \in \Sigma_e$ then also $\tilde{w}^- \cdot w^+ \in \Sigma_e(x^+)$ and

$\tilde{w}^- \cdot w^+ \in \Sigma_e(a)$. Therefore $\Sigma_e^+(x^+) = \Sigma_e^+(a)$. By definition of Σ_i^+ this implies $\Sigma_e(a) \subset \Sigma_e(x^+)$. □

Remark: For input-output systems this is in fact the classical theorem on Nerode equivalence. Notice however that classically the existence of a *groundstate* is assumed (see Proposition 1.18).

A minimal past externally induced realization of an input-output system is not necessarily equivalent to other minimal realizations that are *not* past externally induced. In fact we can use Theorem 1.7 to state

THEOREM 1.12 Let Σ_e be an input-output system on $Y \times U$ given by $F : U^{\mathbb{R}} \to Y^{\mathbb{R}}$ as before. Let Σ_i be a minimal past externally induced realization with state space X. Then all minimal realizations of Σ_e are equivalent if and only if for every $u_1, u_2 \in U^{\mathbb{R}}$

$\{u_1^+ = u_2^+$ and $y_2^+ = y_2^+\}$ (with $y_1 = Fu_1$, $y_2 = Fu_2$)

$\Longrightarrow \{x_1(0) = x_2(0)\}$ (with $x_1(0) = \tilde{f}^-(u_1^-)$, $x_2(0) = \tilde{f}^-(u_2^-)$).

Proof ($\Longrightarrow$) Let $w_1 = (y_1, u_1)$ and $w_2 = (y_2, u_2)$. Let $u_1^+ = u_2^+$ and $y_1^+ = y_1^+$, then $w_1^+ = w_2^+$ and hence w_1 and w_2 have one common future. By Theorem 1.7 (condition (iii)) this implies $w_1 R^+ w_2$. Since Σ_i is minimal and past externally induced $\Sigma_i \sim \Sigma_i^+$ (Theorem 1.11). Therefore $w_1 R^+ w_2$ implies $x_1(0) = x_2(0)$. ($\Longleftarrow$) Let w_1 and $w_2 \in \Sigma_e$, with $w_1 = (Fu_1, u_1)$ and $w_2 = (Fu_2, u_2)$. Suppose there exists $w = (Fu, u)$ such that $w_1^- \cdot w^+ \in \Sigma_e$ and $w_2^- \cdot w^+ \in \Sigma_e$. This implies by assumption $x_1(0) = x_2(0)$, where $x_1(0) = \tilde{f}^-(u_1^-)$ and $x_2(0) = \tilde{f}^-(u_2^-)$. Since $\Sigma_i \sim \Sigma_i^+$ this gives $w_1 R^+ w_2$. Hence condition (iii) of Theorem 1.7 is satisfied. □

The property expressed in Theorem 1.12 has an immediate system theoretic interpretation. It means that, given a system at time 0, if we apply the same input function to the system in state x_1 and to the system in state x_2, the resulting output functions distinguish between the two states, i.e. if the output functions are equal then the two states are actually equal. This is the strongest version of the notion of *observability*.

DEFINITION 1.13 Let Σ_e be an input-output system on $Y \times U$. Let Σ_i be a realization of Σ_e with state space X. Then Σ_i is called *observable* if for every two states $a_1 \neq a_2 \in X$ there exists an inputfunction $u : [0,\infty) \to U$ such

that the output functions $y_1 : [0,\infty) \to Y$ corresponding to $x_1(0) = a_1$ and $y_2 : [0,\infty) \to Y$ corresponding to $x_2(0) = a_1$ are different. Σ_i is called *uniformly observable* if for every input function $u : [0,\infty) \to U$ and every $a_1 \neq a_2 \in X$ the corresponding output functions $y_1, y_2 : [0,\infty) \to Y$ are different.
We obtain

THEOREM 1.14 Let Σ_e be an input-output system and let Σ_i be a past externally induced realization with state space X. Then:
(i) Σ_i minimal $\iff$ Σ_i observable.
(ii) All minimal realizations of Σ_e are equivalent $\iff$ Σ_i is uniformly observable.

PROOF (i) ($\Longrightarrow$) If Σ_i is minimal, then $\Sigma_i \sim \Sigma_i^+ . \Sigma_i^+$ is clearly observable.
($\Longleftarrow$) Σ_i past externally induced means $\Sigma_e^-(a_1) \cap \Sigma_e^-(a_2) = \emptyset$ for every $a_1 \neq a_2 \in X$. Σ_i observable means that $\Sigma_e^+(a_1) \neq \Sigma_e^+(a_2)$ for every $a_1 \neq a_2 \in X$, and $\Sigma_e(a) \neq \emptyset \quad \forall a \in X$. This implies that if $\bigcup_{a \in X' \subset X} \Sigma_e^-(a) \cdot \Sigma_e^+(a)$ is a rectangle, then X' has to consist of one element. Therefore since Σ_i is past externally induced, Σ_i is minimal.
(ii) This statement follows from Theorem 1.12. □

From a physical point of view the property of uniform observability is very desirable. It means that if we do not know the state of the system we can perform an arbitrary "experiment" (i.e. apply an arbitrary known input function) on the system, and the outcome of the experiment will allow us to deduce the state of the system. Therefore we obtained the following conclusion:
The, often implicitly made, assumption of the existence of a unique state space model of a physical system (up to a change of coordinates - equivalence- and barring irrelevant elements in the model-minimality-) is, although in general false, equivalent to the assumption that every future experiment determines the original state of the system.

Besides minimality and observability there is another basic notion in system theory, which is usually called controllability or reachability, and has to do with the internal dynamics of a realization.

DEFINITION 1.15 Let Σ_i be a dynamical system with state space X. Let $x_0 \in X$. Σ_i is *reachable* from x_0 if $\forall a \in X$ there exists $t \geq 0$ and $(x,w) \in \Sigma_i$ such

that $x(0) = x_0$ and $x(t) = a$. It is said to be connected if $\forall x_0$ and x_1 in X there exists $(x,w) \in \Sigma_i$ and $t \geq 0$ such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

We will now show that past externally induced realizations of an input-output system Σ_e enjoy one of these reachability properties if Σ_e satisfies an extra condition:

DEFINITION 1.16 Let Σ_e be an external system. Σ_e has *finite time memory* if there exists $\Delta \geq 0$ such that $\{w(\tau);\ t-\Delta \leq \tau < t\}$ splits $\{w(\tau);\tau < t-\Delta\}$ and $\{w(\tau);\tau \geq t\}$ (if $\Delta = 0$ the system is said to be *memoryless*).

PROPOSITION 1.17 Let Σ_e be an external input-output system with finite time memory. Let Σ_i be a past externally induced and minimal realization. Then Σ_i is connected.

PROOF We will prove that Σ_i^+ is connected. It follows from the finite time memory that the state at $t = 0$ is determined by the past input on the time interval $[-\Delta,0)$. Therefore for every x_0 and x_1 in X there exist input functions u_0 and u_1 on $[-\Delta,0)$ such that the state at time 0 is x_0, respectively x_1. Now define the shift of u_1 by $\overline{u}_1(t) := u_1(t-\Delta)$, for $t \in [0,\Delta)$. Then we just have to concatenate u_0 on $[-\Delta,0)$ with u_1 on $[0,\Delta)$ to obtain the state x_1 at $t = \Delta$. □

Classically, another assumption is added to the definition of an external input-output system. It is assumed that there exists a $t_0 \in \mathbb{R}$ and an element $u^* \in U$ such that every input function satisfies $u(t) = u^*$ for every $t \leq t_0$. (The intuitive idea is that we ask that the input functions on "$t = -\infty$" are "zero"). Since $y = Fu$, there also exists a constant $y^* \in Y$ such that $y(t) = y^*$, $t \leq t_0$. If a realization Σ_i with state space X is past externally induced this implies that there exists an $x^* \in X$ such that $x(t) = x^*$, $t \leq t_0$. This x^* is called the *ground state* of the system, and we denote the realization by (Σ_i,x^*).

PROPOSITION 1.18 Let (Σ_i,x^*) be a past externally induced minimal realization of $\Sigma_e = \{(Fu,u) \mid u(t)=u^*,\ t \leq t_0\}$. Then Σ_i is reachable from x^*.

PROOF Since $u(t) = u^*$, $t \leq t_0$, every $(x,(Fu,u)) \in \Sigma_i$ is such that $x(t_0) = x^*$. Because $\Sigma_i \sim \Sigma_i^+$ there exists for every $a \in X$ an $(x,(Fu,u)) \in \Sigma_i$, such $x(0) = a$. □

Notes and References for Chapter 1

Section 1.2 is based on a paper bij WILLEMS (1979), see also WILLEMS & VAN DER SCHAFT (1982). Theorems 1.7, 1.12, 1.14 seem not to have been stated explicitly before. A good reference for the formalization of the input-output point of view is KALMAN, FALB & ARBIB (1969, especially Chapter 10), where we can also find a treatment of the classical theorem on Nerode equivalence (see the Remark after Theorem 1.11).

CHAPTER 2

SMOOTH DYNAMICAL SYSTEMS

In this chapter we shall add more mathematical structure to the general definition of a dynamical system with external variables as given in Chapter 1.

First of all the systems will have a *differentiable* structure, i.e. the state space X and the space of external variables W will be smooth manifolds, and the dynamical behavior of the system will be described by a set of first-order differential equations. Another major assumption will be the finite-dimensionality of W and in particular of X. Grosso modo, we shall look at systems which can be described in the form

$$\dot{x} = g(x,u)$$
$$w = h(x,u)$$

with $x \in X$, the state space, $w \in W$ the space of external variables, and g and h smooth functions. The variable u is an auxiliary variable and parametrizes in every state x the possible infinitesimal evolutions of x, i.e. $\dot{x}$, and the possible values of the external variables.

In the first part of the chapter we shall treat the special case of *linear* systems. Then X and W are (finite-dimensional) vector spaces, g and h are linear functions and we obtain systems of the form

$$\dot{x} = Ax + Bu$$
$$w = Cx + Du$$

where A, B, C and D are linear mappings.

We shall also deal with differentiable *external* dynamical systems. Again W is a smooth manifold, and our basic point of view will be to define an external system by a set of (implicit) *high order* differential equations in w

$$P(w,\dot{w},\ldots,w^{(k)}) = 0$$

So the external system is determined by a set of compatibility relations

between the external variables and their time-derivatives up to a certain order. An example is an electrical network whose external behavior is defined by a set of equations involving the external currents and voltages and their derivatives. Moreover we require that the high order differential equations are smooth equations, so the physical laws describing Σ_e should be of a differentiable nature.

For the linear case $P_0 w + P_1 \dot{w} + ... + P_k w^{(k)} = 0$, with P_i constant matrices, we shall be able to give a fairly satisfactory treatment of such external systems, including the realization problem. We shall make much use of the symbolic calculus, which enables us to replace the expression " $\frac{d}{dt}$ " by the variable s, with $s \in \mathbb{C}$. This is of course strongly related to the Laplace transform, and is in system theory usually called the " frequency-domain " approach (in contrast with the equations $P_0 w + ... + P_k w^{(k)} = 0$ and $\dot{x} = Ax + Bu$, $w = Cx + Du$ in the " time-domain "). For the nonlinear case such a theory is still in its embryotic stage, and we shall only give a general approach together with some problems and partial results.

2.1 Linear systems

2.1.1. Linear systems in state space form

The first class of dynamical systems with external variables which we consider are the *finite dimensional linear time-invariant systems*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ w &= Cx + Du \end{aligned} \tag{2.1}$$

with $x \in X := \mathbb{R}^n$, the state space, $w \in W := \mathbb{R}^q$, the space of external variables, and $u \in U := \mathbb{R}^m$, the space of " inputs ", parametrizing the possible velocities $\dot{x}$ and external values w in every state x. A, B, C and D are linear mappings (matrices) of appropriate dimensions. For the solution of the differential equation $\dot{x} = Ax + Bu$ to be well-defined we need some smoothness assumptions. Formally we define the dynamical system in state space form Σ_i generated by (2.1) as

$$\begin{aligned} \Sigma_i := \{(x,w): \mathbb{R} \to X \times W \mid & x \text{ is absolutely continuous and } \exists u \in L_{1oc} \\ & \text{such that } \dot{x}(t) = Ax(t) + Bu(t) \text{ for almost every } t \in \mathbb{R}, \\ & \text{and } w(t) = Cx(t) + Du(t) \text{ for every } t \in \mathbb{R}\} \end{aligned} \tag{2.2}$$

Here L_{loc} denotes the locally integrable vector-valued functions on $\mathbb{R}$. We denote equations (2.1) by $\Sigma(A, B, C, D)$ and Σ_i as in (2.2) by $\Sigma_i(A, B, C, D)$.

An important subclass is given by systems $\Sigma(A, B, C, D)$ where $D = \begin{pmatrix} \bar{D} \\ I_m \end{pmatrix}$ and $C = \begin{pmatrix} \bar{C} \\ 0_m \end{pmatrix}$, with $\bar{D}:\mathbb{R}^m \to \mathbb{R}^{q-m}$ and $\bar{C}:\mathbb{R}^n \to \mathbb{R}^{q-m}$ (assume $q \geq m$). If we write correspondingly $w = (w_1, w_2)$ with $w_2 \in \mathbb{R}^m$, $w_1 \in \mathbb{R}^{q-m}$ we obtain from (2.1)

$$(2.3) \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ w_1 &= \bar{C}x + \bar{D}u \\ w_2 &= u \end{aligned}$$

We now identify w_2 and u, denote $y := w_1$, define Y (the *output space*) as $\mathbb{R}^p$ with $p := q-m$, to obtain

$$(2.4) \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= \bar{C}x + \bar{D}u \end{aligned}$$

which we call a *linear input-output system* with input space U, output space Y and space of external variables $W = Y \times U$. Concluding, in the case of a linear input-output system the input variables u, which are in equations (2.1) more or less *auxiliary* variables, can be identified with a part of the external variables.

We notice that $\Sigma_i(A, B, C, D)$ remains unchanged by applying the transformation $u \to u + Fx$, with $F:X \to U$ an arbitrary linear mapping. Also non-singular transformations in U, $u \longmapsto Ru$, $\det R \neq 0$, do not alter $\Sigma_i(A,B,C,D)$. This can be seen as follows. The above transformations transform (A,B,C,D) in the following way

$$(2.5) \qquad (A, B, C, D) \xrightarrow[\det R \neq 0]{F,R} (A + BF, BR, C + DF, DR)$$

(We call this class of transformations the *feedback* transformations, and we say that $\Sigma(A + BF, BR, C + DF, DR)$ is *feedback equivalent* to $\Sigma(A, B, C, D)$). Then $\Sigma_i(A + BF, BR, C + DF, DR) = \{(x,w):\mathbb{R} \to X \times W \mid x \text{ is a.c. and } \exists\ u \in L_{loc}$ such that $\dot{x}(t) = (A + BF)x(t) + BRu(t)$ a.e., and $w(t) = (C + DF)x(t) + DRu(t)$, $\forall t \in \mathbb{R}\} = \{(x,w):\mathbb{R} \to X \times W \mid x \text{ a.c.}, \exists u \in L_{loc}, \dot{x}(t) = Ax(t) + B(Fx(t) + Ru(t))$ a.e., $w(t) = Cx(t) + D(Fx(t) + Ru(t))$, $\forall t \in \mathbb{R}\}$

$= \{(x,w):\mathbb{R}\to X\times W \mid x \text{ a.c.}, \exists v \in L_{1oc}, \dot{x}(t) = Ax(t) + Bv(t), \text{a.e.}, w(t) = Cx(t) + Dv(t), \forall t \in \mathbb{R}\} = \Sigma_i(A,B,C,D)$.

PROPOSITION 2.1 Let $\Sigma(A, B, C, D)$ be a linear system (2.1) with Ker $D \subset$ Ker B. Then $\Sigma(A,B,C,D)$ is feedback equivalent to an input-output system (2.4).

PROOF Suppose first that Ker $D = 0$. Consider Im $D \subset W$ and take coordinates (w_1,w_2) for W, $w_2 \in \mathbb{R}^m$, $w_1 \in \mathbb{R}^{q-m}$ such that Im $D = \{(0,w_2) \mid w_2 \in \mathbb{R}^m\}$. Then by taking appropriate coordinates for U (i.e. applying R) we obtain

$$\begin{aligned} \dot{x} &= Ax + Bu \\ w_1 &= C_1x \\ w_2 &= C_2x + u \end{aligned}$$

where we have written $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ corresponding to the basis (w_1,w_2). Now apply feedback with $F = -C_2$. Then $C + DF = \begin{pmatrix} C_1 \\ 0 \end{pmatrix}$ and hence we have obtained (2.4) (even with $\bar{D} = 0$; observe that if we had only allowed partitions $w = (w_1,w_2)$ of $W = \mathbb{R}^q$ consisting of the standard basis vectors of $\mathbb{R}^q$ we would only have been able to give a partition in which $D = \begin{pmatrix} D_1 \\ 1_m \end{pmatrix}$. Then D_1 becomes $\bar{D}$). If Ker $D \neq 0$, then since Ker $D \subset$ Ker B, we can eliminate Ker D from U, and take as new inputspace $U^1 := U_{/\text{Ker } D}$. □

We now summarize some results concerning minimality, observability and controllability from WILLEMS (1979). Let us denote the external behavior of $\Sigma_i(A, B, C, D)$ by $\Sigma_e(A, B, C, D)$, i.e. $\Sigma_e(A, B, C, D) := \{w:\mathbb{R}\to W \mid (x,w) \in \Sigma_i(A, B, C, D)\}$. We will now give the conditions on (A,B,C,D) such that $\Sigma_i(A,B,C,D)$ is a *minimal* realization of $\Sigma_e(A,B,C,D)$ (Definition 1.5). In order to do this we need to introduce a concept from the geometric theory of linear systems. The *maximal output nulling subspace*, V^*, is defined as

$$\begin{aligned} V^* := \{&x_0 \in X \mid \exists u \in L_{1oc} \text{ such that the trajectory } w \\ &\text{generated by } \dot{x} = Ax + Bu,\ w = Cx + Du,\ x(0) = x_0, \\ &\text{satisfies } w(t) = 0,\ \forall t \in \mathbb{R}\} \end{aligned}$$

The space V^* is easily computed from (A, B, C, D) (see WONHAM (1979) for algorithms and other applications of V^*).

THEOREM 2.2 (for a proof see WILLEMS (1979, 1983)). $\Sigma_i(A, B, C, D)$ is a minimal realization of $\Sigma_e(A, B, C, D)$ if and only if $V^* = 0$ and

Ker D ∩ Ker B = 0.

It can be easily proved that a linear input-output system (2.4) is observable if and only if it is uniformly observable (Definition 1.13), and that observability is equivalent to the following condition on the pair $(\overline{C},A)$: the only subspace $V \subset X$ such that $AV \subset V$ and $V \subset \text{Ker } \overline{C}$, is the zero-space $V = 0$.

We obtain the following corollary of Theorem 2.2.

COROLLARY 2.3 Let $\Sigma_i(A, B, \overline{C}, \overline{D})$ be an input-output system. Then Σ_i is minimal $\Longleftrightarrow$ Σ_i is observable (or as we will also say: $(\overline{C},A)$ is observable).

PROOF $\text{Ker } [C\ D] = \text{Ker} \begin{pmatrix} \overline{C} & \overline{D} \\ 0 & I_m \end{pmatrix} = \{ \begin{pmatrix} x \\ 0 \end{pmatrix} | x \in \text{Ker } \overline{C} \}$.

Therefore a trajectory (x,u): $\mathbb{R} \to X \times U$ contained in $\text{Ker}[C\ D]$ is of the form $(x,0)$ with $x(t) \in \text{Ker } \overline{C}$ for every t, and satisfying $\dot{x} = Ax$ (since $u = 0$). However: Σ observable $\Longleftrightarrow$ there exists no nonzero trajectory $x: \mathbb{R} \to X$ satisfying $\dot{x} = Ax$ and contained in $\text{Ker } \overline{C}$. Of course $\text{Ker } D = \text{Ker } \begin{bmatrix} 0 \\ I_m \end{bmatrix} = 0$. □.

It is well-known that a system $\Sigma_i(A, B, C, D)$ is *connected* (Definition 1.15) if and only if the following condition on the pair (A,B) holds: the only subspace $V \subset X$ such that $AV \subset V$ and $\text{Im } B \subset V$, is X itself. Usually we say in this case that $\Sigma_i(A, B, C, D)$ is *controllable* or *reachable* (or that (A,B) is controllable). We again note that minimality does *not* in general imply controllability. This is not surprising since *autonomous* linear systems $\dot{x} = Ax$, $y = Cx$ are also included in our definition (2.1), and are minimal if and only if (C,A) is observable.

2.1.2. External linear systems

External linear systems can be defined in the following elegant way. Let W be a finite-dimensional vector space, i.e. $W = \mathbb{R}^q$. Then an *external linear system* on W is given by a *linear* subspace $\Sigma_e \subset W^{\mathbb{R}}$; i.e. if the functions w_1 and w_2: $\mathbb{R} \to W$ belong to Σ_e then also $\alpha_1 w_1 + \alpha_2 w_2$ belongs to Σ_e, for every α_1, $\alpha_2 \in \mathbb{R}$.

We can prove the following general facts.

THEOREM 2.4 Let $\Sigma_e \subset W^{\mathbb{R}}$ be a linear external system.
Then i) X^+ and X^- (see after Definition 1.3) are vector spaces
ii) all minimal realizations of Σ_e are equivalent

PROOF (i) Since Σ_e is linear, the zero function $0 \in W^{\mathbb{R}}$ belongs to Σ_e.

Let $w_1 R^+ 0$, then also $\alpha w_1 R^+ 0$ for every $\alpha \in \mathbb{R}$. If $\alpha = 0$, this is trivial, and if $\alpha \neq 0$ and $0^- . w^+ \in \Sigma_e$ then since Σ_e is linear also $0^- . \frac{1}{\alpha} w^+ \in \Sigma_e$. Because $w_1 R^+ 0$ this implies $w_1^- . \frac{1}{\alpha} w^+ \in \Sigma_e$, and hence $\alpha w_1^- . w^+ \in \Sigma_e$. Therefore $\alpha w_1 R^+ 0$.
Let now $w_1 R^+ 0$ and $w_2 R^+ 0$. We will prove that $(w_1 + w_2) R^+ 0$. Indeed since $w_1 R^+ 0$ and $w_2 R^+ 0$, it follows from above that also $2w_1 R^+ 0$ and $2w_2 R^+ 0$. Let now $0^- . w^+ \in \Sigma_e$. Then also $2w_1^- . w^+ \in \Sigma_e$ and $2w_2^- . w^+ \in \Sigma_e$. Since Σ_e is linear $(2w_1 + 2w_2)^- . 2w^+ \in \Sigma_e$, and also $(w_1 + w_2)^- . w^+ \in \Sigma_e$. Hence $(w_1 + w_2) R^+ 0$.
Concluding: R^+ is a linear relation on the linear space Σ_e and therefore $X^+ = \Sigma_e (\text{mod } R^+)$ is a vector space. Analogously X^- is a vector space.
(ii) We will prove that Σ_e satisfies condition (ii) of Theorem 1.7. By linearity it is sufficient to prove that if $w_1^- . 0^+ \in \Sigma_e$ (i.e. w_1 and 0 have one common future), and $0^- . w_2^+ \in \Sigma_e$ then also $w_1^- . w_2^+ \in \Sigma_e$. This is obvious since Σ_e is linear. □.

In the sequel we shall give some examples of external linear systems. First we introduce some notation.

Let $\mathbb{R}[s]$ denote as usual the real polynomials in the indeterminate s, $\mathbb{R}^n[s]$ the n-dimensional vectors of real polynomials and $\mathbb{R}^{n_1 \times n_2}[s]$ the $(n_1 \times n_2)$ matrices of real polynomials. An element of $\mathbb{R}(s)$, the rational functions, is said to be (strictly) proper if the degree of its denominater is (strictly) larger than the degree of its numerator. Similar definitions hold for vectors and matrices of rational functions.

The following cases are clearly examples of external linear systems.

Case 1

$\Sigma_e(P) := \{w: \mathbb{R} \to W \mid w \in L_{1oc} \text{ and } P(\frac{d}{dt})w = 0\}$

Here P(s) is an element of $\mathbb{R}^{p \times q}[s]$ $(W = \mathbb{R}^q)$

and $P(\frac{d}{dt})w = 0$ is to be interpreted in the sense of distributions.

Case 2

$\Sigma_e(P,R) = \{w: \mathbb{R} \to W \mid w \in L_{1oc}, \exists \text{ distribution } \xi \text{ such that}$

$P(\frac{d}{dt})w = R(\frac{d}{dt})\xi$, with equality

in the sense of distributions$\}$

Case 3

Let $D \in \mathbb{R}^{p \times p}[s]$, $N \in \mathbb{R}^{p \times m}[s]$, det D(s) unequal to the zero polynomial, and $D^{-1}(s)N(s)$ a proper rational matrix. Consider now the set of differen-

tial equations $D(\frac{d}{dt})\ y = N\ (\frac{d}{dt})u$ (equality in the sense of distributions). This is obviously a system of the type $\Sigma_e(P)$ with $W = \mathbb{R}^{p+m}$, $w = (y,u)$ and $P = [D\vdots -N]$. It has a special form because the first p components can be called *outputs* and the last m *inputs*. We call the proper matrix $G(s) := D^{-1}(s)N(s)$ its *transfer matrix*. Note, however, that also responses not explainable by inputs, but entirely due to initial conditions may occur in $\Sigma_e([D\vdots -N])$. Later on we will come back to this case.

Case 4

$\Sigma_e(A,B,C,D)$ as defined before.

We note that we can always reduce $\Sigma_e(P,R)$ as in Case 2 to $\Sigma_e(\tilde{P})$ for a suitable $\tilde{P}$:

PROPOSITION 2.5 Let $\Sigma_e(P,R)$ be given. Then there exists $\tilde{P}$ such that $\Sigma_e(\tilde{P}) = \Sigma_e(P,R)$ (for a proof see WILLEMS (1983)).

Moreover we have

THEOREM 2.6 Let $\Sigma_e(P)$ be given. Then there exists a *minimal* $\Sigma_i(A,B,C,D)$ such that $\Sigma_e(A,B,C,D) = \Sigma_e(P)$, i.e. $\Sigma_i(A,B,C,D)$ is a *minimal realization* of $\Sigma_e(P)$.

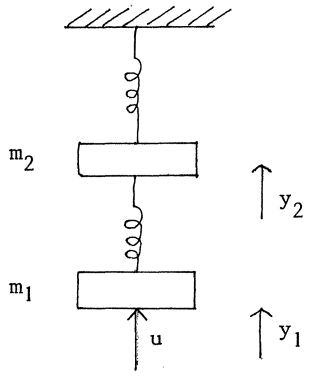
Remark. The proof is given in WILLEMS (1983) where one can also find a procedure to explicitely construct (A,B,C,D).

Concluding, all the above cases can be realized as the external behavior of a linear system in state space form $\Sigma(A,B,C,D)$. Therefore they have the common characteristic of a *finite dimensional* state space. We note, however, that it is easy to construct examples of external linear systems which have a minimal realization with an infinite dimensional state space. Consider for instance $\Sigma_e := \{w:\mathbb{R}\to W \mid w \text{ is periodic with period } 1\}$. Then Σ_e is clearly linear while $w_1 R^+ w_2 \Leftrightarrow w_1|_{[-1,0)} = w_2|_{[-1,0)}$ ($|_{[-1,0)}$ denotes restriction to $[-1,0)$). Therefore $X^+ = [-1,0)^{\mathbb{R}}$ which is an infinite-dimensional space. It would be interesting to characterize in a simple way the external linear systems which have a minimal realization with a finite dimensional state space. In the sequel we shall mainly consider external linear systems as in Case 1. Let us give a simple example which can be kept in mind while reading the next section.

Example (see also ROSENBROCK (1970, pp 38-40))

Let two masses m_1 and m_2 be connected to springs with spring constants k_1

and k_2. The displacements of the masses are y_1 and y_2. A force u is applied to the lower mass



The external variables are y_1, y_2 and u and the external linear system as in Case 1 is given by the equations $m_1\ddot{y}_1 = u - k_1(y_1 - y_2)$, $m_2\ddot{y}_2 = k_1(y_1 - y_2) - k_2y_2$.

2.1.3. External linear systems in the frequency domain

In this section we make a closer study of the linear systems whose external behavior is given as in Case 1 of 2.1.2, that is

$\Sigma_e(P) = \{w: \mathbb{R}\to W \mid w \in L_{loc},\ P(\frac{d}{dt})w = 0\}$ with $w \in W = \mathbb{R}^q$ and $P(s) \in \mathbb{R}^{p\times q}[s]$. We use symbolic calculus to transform these equations to the form $P(s)w(s) = 0$, $s \in \mathbb{C}$. Of course this transformation corresponds to the Laplace transform of w, with zero initial conditions. We do not go into details, and use the " symbolic " approach (which can be justified).

We note that $\Sigma_e(P)$ remains unchanged by pre-multiplication of P(s) with a unimodular matrix $U(s) \in \mathbb{R}^{p\times q}[s]$ (unimodular means det U(s) = constant $\neq 0$). Let P(s) have rank r, for almost every $s \in \mathbb{C}$, then we know (WOLOVICH (1974, Theorem 2.5.11))) that we can find a unimodular U(s) such that $U(s)P(s) = \begin{pmatrix}\tilde{P}(s)\\ 0\end{pmatrix}$, with $\tilde{P}(s) \in \mathbb{R}^{r\times q}[s]$ *surjective* for almost every $s \in \mathbb{C}$. Therefore, without loss of generality, we make the standing assumption that P(s) is surjective for almost every $s \in \mathbb{C}$.

Furthermore we have

PROPOSITION 2.7 Let $P(s) \in \mathbb{R}^{p\times q}[s]$, and surjective for almost every $s \in \mathbb{C}$. Then there exist $R(s) \in \mathbb{R}^{p\times p}[s]$ and $P'(s) \in \mathbb{R}^{p\times q}[s]$ such that $P(s) = R(s)P'(s)$, and P'(s) surjective for every $s \in \mathbb{C}$, and rank R(s) = p for almost every $s \in \mathbb{C}$.

Before proving this proposition we have to give some definitions. If the polynomial matrices K, I and J satisfy K(s) = I(s)J(s) then I(s) is called a left divisor of K(s) and K(s) is called a right multiple of I(s). The *greatest common left divisor* (g.c.l.d.) of two polynomial matrices $K_1(s)$ and $K_2(s)$ (with the same number of rows) is a common left divisor which is a right multiple of every common left divisor of $K_1(s)$ and $K_2(s)$. We call K_1 and K_2 *left coprime* if a g.c.l.d. of K_1 and K_2 is unimodular. Left coprimeness of K_1 and K_2 is also equivalent to $[K_1(s) \vdots K_2(s)]$ surjective for every $s \in \mathbb{C}$.

PROOF of Proposition 2.7 Write $P(s) = [M(s) \vdots N(s)]$, with $M \in \mathbb{R}^{p\times p}[s]$ and $N \in \mathbb{R}^{p\times(q-p)}[s]$. Find a greatest common left divisor R(s) of M(s) and N(s). Then $M(s) = R(s)M_1(s)$, $N(s) = R(s)N_1(s)$ for some M_1 and N_1 and moreover M_1 and N_1 are left coprime or equivalently $[M_1(s) \vdots N_1(s)]$ surjective, $\forall s \in \mathbb{C}$. Define $P^1(s) = [M_1(s) \vdots N_1(s)]$. □

Remark: There are actually algorithms to construct a g.c.l.d. of two polynomial matrices, based on a division algorithm (see WOLOVICH (1974)).

Motivated by the above proposition, we first look at external systems $\Sigma_e(P)$, with P(s) surjective $\forall s \in \mathbb{C}$. Only later on we show how we can treat the case that $P(s) = R(s)P'(s)$, with R(s) and P'(s) as above. Let now P(s) be surjective for every $s \in \mathbb{C}$. We denote by $W_{\mathbb{C}}$ the complexification of W (i.e. if $W = \mathbb{R}^q$, then $W_{\mathbb{C}} = \mathbb{C}^q$). Since the set of solutions w(s), with $w(s) \in W_{\mathbb{C}}$, of $P(s)w(s) = 0$, is invariant under pre-multiplication by a unimodular matrix U(s), we see that the solution set of $P(s)w(s) = 0$ is characterized by the kernel of P(s), which is for every $s \in \mathbb{C}$ a linear subspace of $W_{\mathbb{C}}$. Hence we have arrived at the study of the following *geometrical object*: *for every* $s \in \mathbb{C}$ *there is a linear subspace of* $W_{\mathbb{C}}$, *given by ker P(s).*

Since P(s) is surjective, $\forall s \in \mathbb{C}$, the above subspaces Ker P(s) all have equal dimension q-p. If P(s) would be only surjective for almost every $s \in \mathbb{C}$, and if we write P(s) = R(s)P'(s) as above, then the geometrical objects Ker P(s) and Ker P'(s), for every $s \in \mathbb{C}$, are equal except for those $s_0 \in \mathbb{C}$ such that $\det R(s_0) = 0$. In these points the dimension of Ker P(s) suddenly jumps while dim Ker P'(s) remains constant. Later on we see that these values of s actually are the *uncontrollable* eigen values of a minimal realization of P(s).

Let us now denote by Grass the set of all (q-p)- dimensional subspaces of $W_{\mathbb{C}}(= \mathbb{C}^q)$. We obtain

PROPOSITION 2.8. Let $P \in \mathbb{R}^{p\times q}[s]$, and surjective $\forall s \in \mathbb{C}$. Then the map $\mathbb{C} \to$ Grass, defined by $s \mapsto$ Ker P(s) is an *algebraic* map. Furthermore the following set $\bar{E}(P(s)) := \{(s,v) \mid s \in \mathbb{C},\ v \in \text{Ker } P(s) \subset W_{\mathbb{C}}\}$ is an *algebraic vector bundle* over $\mathbb{C}$.

PROOF For every $s_0 \in \mathbb{C}$ we can find a p×p submatrix M(s) of P(s) such that $\det M(s_0) \neq 0$. Therefore in a Zariski open neighborhood U of s_0, $\det M(s) \not\equiv 0$. Then for every $s \in U$ we can actually solve the equations $P(s)v = 0$, $v \in \mathbb{C}$, using a version of Cramer's rule (i.e. we may divide by det M(s)). This gives a map $\mathbb{C} \to$ Grass, which is clearly algebraic. Moreover we obtain a trivialization of $\bar{E}(P(s))$ above U, namely $\pi^{-1}(U)$ is isomorphic to U×Ker $P(s_0)$, with $\pi : \bar{E}(P(s)) \to \mathbb{C}$ defined by $(s,v) \in \bar{E}(P(s)) \to s \in \mathbb{C}$. Therefore $\bar{E}(P(s))$ is an algebraic vector bundle over $\mathbb{C}$. □

From a mathematical point of view, and also from a system theoretic one, as will become clear, it is advantageous to make one more abstraction. Namely we can look at Ker P(s) also as a vector bundle over $\mathbb{P}^1$, the complex projective line (which can be thought of as $\mathbb{C}$ together with the point at infinity). Let (s,t) be homogeneous coordinates for $\mathbb{P}^1$ and embed $\mathbb{C}$ in the usual way into $\mathbb{P}^1$ by identifying $s \in \mathbb{C}$ with $(s,1) \in \mathbb{P}^1$. Then "$s = \infty$" is identified with the point $(1,0) \in \mathbb{P}^1$, i.e. $t = 0$. We state

THEOREM 2.9 Let $P \in \mathbb{R}^{p\times q}[s]$, and surjective $\forall s \in \mathbb{C}$. Then:

(i) The algebraic map $\mathbb{C} \to$ Grass, given by $s \mapsto$ Ker P(s), can be uniquely extended to an algebraic map $\mathbb{P}^1 \to$ Grass.

(ii) Let $V(\infty)$ be the element of Grass which is attached to "$s = \infty$", i.e. to $(1,0) \in \mathbb{P}^1$. Then $E(P(s)) := \{(s,1),v) \mid s \in \mathbb{C},\ v \in \text{Ker } P(s)\} \cup \{((1,0),v) \mid v \in V(\infty)\}$ is an algebraic vector bundle over $\mathbb{P}^1$.

For the proof of this theorem we make use of the notion of row properness of a polynomial matrix.

DEFINITION 2.10 Let $P \in \mathbb{R}^{p\times q}[s]$, and surjective for almost every $s \in \mathbb{C}$. Then P is called *row proper* if $P(s) = \text{diag}(s^{k_i})P_r + L(s)$, where $\text{diag}(s^{k_i})$ is the p×p-matrix with s^{k_i} ($k_i \in \mathbb{N}$) on the (i,i)-th place and zero elsewhere

and $L(s) \in \mathbb{R}^{p\times q}[s]$ is such that the degree of the i-th row of L(s) (i.e. the highest power of s occurring in the i-th row) is strictly less than k_i, and P_r is a constant p×q-matrix which is *surjective*.

PROPOSITION 2.11 Let $P(s) \in \mathbb{R}^{p\times q}[s]$, and surjective for almost every $s \in \mathbb{C}$. Then there exists a unimodular $U(s) \in \mathbb{R}^{p\times q}[s]$ such that U(s)P(s) is row proper.

PROOF We can proceed in the same way as in the proof of WOLOVICH (1974, Theorem 2.5.7), where the proposition is proved (in a constructive way) for square matrices P(s). □

We are now able to give

PROOF of Theorem 2.9

(i) By Proposition 2.11 we may assume that P(s) is row proper, i.e. $P(s) = \mathrm{diag}(s^{k_i})P_r + L(s)$, as above, with P_r surjective. Then it follows that $\mathrm{diag}(s^{-k_i})P(s) = P_r + \tilde{L}(s^{-1})$ where $\tilde{L}(s^{-1})$ is a matrix consisting of polynomials in s^{-1} with no constant terms, i.e. $\tilde{L}(0) = 0$. The substitution $t = s^{-1}$ gives $\mathrm{diag}(t^{k_i})P(\frac{1}{t}) = P_r + \tilde{L}(t)$.
Therefore for every $t \neq 0$, $\ker P(\frac{1}{t})$ is equal to $\mathrm{Ker}(P_r+\tilde{L}(t))$.
We now define $V(\infty) := \mathrm{Ker}\ P_r$, and we only have to prove that $\mathrm{Ker}\ P_r = \lim_{t\to 0} \mathrm{Ker}(P_r+\tilde{L}(t))$ for every path $t \to 0$, and where the limit is taken with respect to the Grassmann topology of Grass. Since $\tilde{L}(0) = 0$ and $\mathrm{rank}\ (P_r+\tilde{L}(t)) = \mathrm{rank}\ P_r$, for every t small, this is clear because we can explicitly solve $\mathrm{Ker}(P_r+\tilde{L}(t))$ as in the proof of Proposition 2.8. This also shows that the in this way extended map $\mathbb{P}^1 \to$ Grass is algebraic.
(ii) For (finite) $s \in \mathbb{C}$ we have given in the proof of Proposition 2.8 a local trivialization of E(P(s)). In the neighborhood of $s = \infty$ (or $(1,0)\in\mathbb{P}^1$) a local trivialization is given by solving $\ker(P_r+\tilde{L}(t))$. □

Remark 1. Let $P(s) = P_k s^k + P_{k-1}s^{k-1} + \ldots + P_0$ with k the highest power of s present in P(s). If P_k is surjective we can of course immediately define $V(\infty) := \mathrm{Ker}\ P_k$. However it can be seen that we can *not* bring a general P(s) into this form by pre-multiplication with a unimodular matrix.

Remark 2. Notice that we have also given above a way to "homogenize" the equations $P(s)w = 0$, i.e. we can write this set of equations as *homogeneous* equations in the coordinates (s,t) for $\mathbb{P}^1$. First we make $P(s)$ row proper by premultiplication with a unimodular matrix $U(s)$ (i.e. by row operations). Then $U(s)P(s) = \mathrm{diag}(s^{k_i})P_r + L(s)$ as above. Now define

$$P(s,t) := \mathrm{diag}(s^{k_i})P_r + L(s,t)$$

where $L(s,t)$ is constructed from $L(s)$ by multiplication by powers of t such that every term in the i-th row has degree k_i.

Example: Return to the example at the end of section 2.12:

$$m_1\ddot{y}_1 = u - k_1(y_1-y_2)$$
$$m_2\ddot{y}_2 = k_1(y_1-y_2) - k_2y_2$$

with $w = (y_1,y_2,u) \in W = \mathbb{R}^3$. The polynomial matrix $P(s)$ is equal to

$$P(s) = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{pmatrix} s^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} s + \begin{pmatrix} k_1 & -k_2 & -1 \\ -k_1 & k_1+k_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} m_1s^2+k_1 & -k_1 & -1 \\ -k_1 & m_2s^2+k_1+k_2 & 0 \end{pmatrix}$$

Then: $\dim \ker P(s) = 1$, for every $s \in \mathbb{C}$ if and only if $k_1 \neq 0$. If $k_1 = 0$, then the points $s_0 \in \mathbb{C}$ such that $m_2s_0^2 + k_2 = 0$, are the points with $\dim \ker P(s_0) = 2$. From physical considerations it is clear that the system is not controllable if $k_1 = 0$. Let us assume $k_1 \neq 0$. If $m_1 \neq 0$ and $m_2 \neq 0$ then $P(s)$ is row proper and $V(\infty) = \mathrm{Ker} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \end{pmatrix} = \mathrm{span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (see Remark 1 above).

If $m_1 = 0$ and $m_2 \neq 0$, then $P(s)$ is still row proper and $P_r = \begin{pmatrix} k_1 & -k_2 & -1 \\ 0 & m_2 & 0 \end{pmatrix}$. Therefore $V(\infty) = \mathrm{span} \begin{pmatrix} 1 \\ 0 \\ k_1 \end{pmatrix}$.

Algebraic vectorbundles over $\mathbb{P}^1$ have some nice properties. The following theorem is implied by a theorem of Grothendieck (an elementary proof can be found in HAZEWINKEL & MARTIN (1982)).

THEOREM 2.12 Every algebraic vectorbundle over $\mathbb{P}^1$ is isomorphic to the direct sum of line bundles (i.e. vector bundles with one-dimensional fibers). Up to isomorphisms a line bundle is fully characterized by an integer $\kappa \in \mathbb{Z}$. Hence the isomorphism classes of algebraic vectorbundles over $\mathbb{P}^1$ are in one-to-one correspondance with sets of integers $\kappa_1 \leq \ldots \leq \kappa_m$, with $\kappa_i \in \mathbb{Z}$ the integers belonging to the line bundles in the decomposition.

Actually the integers κ_i above can be interpreted in many ways. Topologically they are called the *Chern numbers*, but as we see later on they are also sometimes called Kronecker indices, or controllability or observability indices, dependent on the way the vectorbundle arises.

We can easily construct line bundles with a certain positive Chern number $\kappa \geq 0$. Define $P(s) : \mathbb{C}^2 \to \mathbb{C}$ as $P(s) := [s^\kappa : -1]$. Then $\mathrm{Ker}P(s) = \mathrm{span} \begin{pmatrix} 1 \\ s^\kappa \end{pmatrix}$ and $V(\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It can be checked that $E(P(s))$ has Chern number κ. (In order to construct line bundles with negative Chern bundles in this way, we have to embed $\mathbb{C}$ into $\mathbb{P}^1$ in another way, namely by identifying $s \in \mathbb{C}$ with $(1,s) \in \mathbb{P}^1$ and then do the same construction as above for $-\kappa$). We see that we can actually generate every line bundle (with positive Chern number) by constructing a polynomial matrix $P(s)$, surjective for every $s \in \mathbb{C}$, and taking $E(P(s))$. Because of Theorem 2.9 we therefore arrive at the following, somewhat fancy, conclusion:
Every algebraic vectorbundle over $\mathbb{P}^1$ with positive Chern numbers is isomorphic to a vectorbundle $E(P(s))$, with $P(s)$ surjective for every $s \in \mathbb{C}$. Hence the class of algebraic vectorbundles over $\mathbb{P}^1$ is equal to the class of external linear systems $\Sigma_e(P)$, $P(s)$ surjective $\forall s \in \mathbb{C}$.
(we also note that we could have replaced "algebraic" by "holomorphic", since holomorphic over $\mathbb{P}^1$ is necessarily algebraic, see HAZEWINKEL & MARTIN (1982)).

Remark: Let $\Sigma_e(P)$ be an external system, with $P(s)$ surjective for every $s \in \mathbb{C}$. Then $\Sigma_e(P)$ corresponds to an algebraic vectorbundle $E(P(s))$. For algebraic vectorbundles over $\mathbb{P}^1$ we have defined an *equivalence* relation, namely $E(P_1(s))$ is equivalent to $E(P_2(s))$ if $E(P_1(s))$ is *isomorphic* to $E(P_2(s))$. Isomorphic means that there exists a bundle isomorphism from $E(P_1(s))$ to $E(P_2(s))$, i.e.

$$U(s)(\mathrm{Ker}\ P_1(s)) = \mathrm{Ker}\ P_2(s)$$

where $U(s)$ may depend on $s \in \mathbb{P}^1$ and is algebraic. One could wonder what kind of mapping from $\Sigma_e(P_1)$ to $\Sigma_e(P_2)$ corresponds to such a bundle isomorphism from $E(P_1(s))$ to $E(P_2(s))$. In the case of vectorbundles corresponding to (minimal) state space realizations of $\Sigma_e(P_1)$, respectively $\Sigma_e(P_2)$, we shall answer this question later on (see (2.7), also the Remark after Theorem 2.18).

There is another way of constructing line bundles over $\mathbb{P}^1$, which is actually equivalent to giving a state space realization $\Sigma_e(A,B,C,D)$ of $\Sigma_e(P)$. If we define the $\kappa\times\kappa$-matrix $A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & \ldots & . & 0 \end{pmatrix}_{\kappa\times\kappa}$ and the $\kappa\times 1$ vector $B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{\kappa\times 1}$

then $P_1(s) := [sI-A \vdots -B] = \begin{pmatrix} s-1 & & 0 \\ & \ddots & \\ 0 & & s-1 \end{pmatrix}_{\kappa\times(\kappa+1)}$ satisfies $\operatorname{Ker} P_1(s) = \operatorname{span} \begin{pmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^\kappa \end{pmatrix}$

and $V_1(\infty) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$. Now $E(P_1(s))$ is isomorphic to $E(P(s))$, with $P(s) = [s^\kappa \vdots -1]$ as above. From $E(P_1(s))$ to $E(P(s))$ the isomorphism is given by the constant map $F := \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & . & \ldots & 1 \end{pmatrix}_{2\times(\kappa+1)} : \mathbb{C}^{\kappa+1} \to \mathbb{C}^2$, and from $E(P(s))$ to $E(P_1(s))$ the isomorphism is given by $\begin{pmatrix} 1 & & 0 \\ s & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ s^{\kappa-1} & & 0 \\ 0 & & 1 \end{pmatrix}_{(\kappa+1)\times 2} : \mathbb{C}^2 \to \mathbb{C}^{\kappa+1}$

In system theoretic language (A,B,C,D) with A and B as above and $C := \begin{pmatrix} 1 & \ldots & 0 \\ 0 & \ldots & 0 \end{pmatrix}_{2\times\kappa}$, $D := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (such that $F = [C \vdots D]$) is a state space realization of the external system given by $P_1(s) : s^k y(s) = u(s)$, $w = (y,u) \in \mathbb{C}^2$.

For a general external system $\Sigma_e(P)$ we state

THEOREM 2.13 Let $P \in \mathbb{R}^{p\times q}[s]$, with $P(s)$ surjective $\forall s \in \mathbb{C}$. $\Sigma_i(A,B,C,D)$ is a realization of $\Sigma_e(P)$, i.e. $\Sigma_e(A,B,C,D) = \Sigma_e(P)$ if and only if $[C \vdots D]$ maps the vectorbundle $E([sI-A \vdots -B]$ onto $E(P(s))$, or equivalently

$$\text{Ker } P(s) = [C\vdots D]\ \text{Ker}[sI-A\vdots -B],\ \forall s \in \mathbb{C}$$

and $$V(\infty) = \text{Im } D \quad (V(\infty) = \lim_{s\to\infty} \text{Ker } P(s)).$$

Moreover $\Sigma_i(A,B,C,D)$ is a minimal realization if and only if $[C\vdots D]$is an isomorphism between the two vectorbundles $E(P(s))$ and $E([sI-A\vdots -B])$, or equivalently

$$\begin{pmatrix} sI-A & -B \\ C & D \end{pmatrix} \text{ injective } \forall s \in \mathbb{C}, \text{ and } D \text{ injective.}$$

Furthermore, since $P(s)$ is surjective for every $s \in \mathbb{C}$, a minimal realization $\Sigma_i(A,B,C,D)$ of $\Sigma_e(P)$ is necessarily controllable.

PROOF $\Sigma_i(A,B,C,D)$ is a realization of $\Sigma_e(P)$ if and only if $\Sigma_e(P) = \{w:\mathbb{R} \to W \mid \exists x:\mathbb{R} \to X$, absolutely continuous, $\exists u : \mathbb{R} \to U$, $u \in L_{loc}$, such that $\dot{x}(t) = Ax(t) + Bu(t)$, a.e., and $w(t) = Cx(t) + Du(t)$, $\exists t \in \mathbb{R}\}$. Hence, using symbolic calculus

$$\begin{aligned} &\{w(s) \mid P(s)w(s)=0\} = \{w(s) \mid \exists x(s) \text{ and } u(s) \text{ such that} \\ &(sI-A)x(s)-Bu(s) = 0,\ w(s) = Cx(s)+Du(s)\} \\ &=\{w(s) \mid \exists x(s) \text{ and } u(s) \text{ such that } \begin{pmatrix} x(s) \\ u(s) \end{pmatrix} \in \text{Ker}[sI-A\vdots -B] \\ &\text{and } w(s) = [C\vdots D]\begin{pmatrix} x(s) \\ u(s) \end{pmatrix}\}\ . \end{aligned}$$

Therefore $\text{Ker } P(s) = [C\vdots D]\ \text{Ker}[sI-A\vdots -B]$, $\forall s \in \mathbb{C}$. $[C\vdots D]$ is an isomorphism between $E(P(s))$ and $E([sI-A\vdots -B])$

$\Longleftrightarrow$ $[C\vdots D]$ is injective restricted to $E([sI-A\vdots -B])$

$\Longleftrightarrow$ $[C\vdots D]$ injective restricted to $\text{Ker}[sI-A\vdots -B]$, $\forall s \in \mathbb{C}$, and D is injective (since $E([sI-A\vdots -B]$ at $s = \infty$ is equal to span $\{\begin{pmatrix} 0 \\ u \end{pmatrix}, u \in U\}$).

$\Longleftrightarrow$ $\begin{pmatrix} sI-A & -B \\ C & D \end{pmatrix}$ injective $\forall s \in \mathbb{C}$, and D injective.

Furthermore it can be easily be proven that $V^* = 0$ (Theorem 2.2) if and only if $\begin{pmatrix} sI-A & -B \\ C & D \end{pmatrix}$ injective $\forall s \in \mathbb{C}$ (see KAILATH (1980,7.6)). Hence by Theorem 2.2, minimality of $\Sigma_i(A,B,C,D)$ is equivalent to the conditions given above. Finally, since $\dim \text{Ker } P(s) = q-p$, $\forall s \in \mathbb{C}$ and $[C\vdots D]$ is injective restricted to $\text{Ker}\,[sI-A\vdots -B]$ (if $\Sigma_i(A,B,C,D)$ is minimal) $\dim \text{Ker}[sI-A\vdots -B] = q-p$, $\forall s \in \mathbb{C}$.

This implies that $[sI-A\vdots-B]$ is surjective for every $s \in \mathbb{C}$, which is the so-called Hautus test for controllability of (A,B) (see HAUTUS (1969)). □

Remark 1: The class of algebraic vectorbundles over $\mathbb{P}^1$ (with positive Chern numbers) is therefore (up to isomorphisms) equal to the class of *controllable* linear systems $\Sigma(A,B,C,D)$.

Remark 2: Since D is injective if $\Sigma_i(A,B,C,D)$ is a minimal realization, $\Sigma_i(A,B,C,D)$ is feedback equivalent to an input-output system (see Proposition 2.1). We shall come back to this point.

Of course the direct sum of m line bundles with Chern numbers $\kappa_1 \geq \ldots \geq \kappa_m \geq 0$ can be realized by taking

$$(2.6) \qquad A = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{pmatrix}_{\kappa_i \times \kappa_i}\right) \quad \text{and} \quad B = \mathrm{diag}\left(\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}_{\kappa_i \times 1}\right)$$

On the other hand, as we saw earlier if $\Sigma_i(A,B,C,D)$ is a realization of $\Sigma_e(P)$, then also $\Sigma_e(A+BF,BR,C+DF,DR) = \Sigma_e(P)$, for every F and R, with $\det R \neq 0$. It is easy to see that we can also allow for state space transformations $S : X \to X$, with $\det S \neq 0$. In fact, if $\Sigma_i(A,B,C,D)$ is a minimal realization of $\Sigma_e(P)$, then $\Sigma_i(A',B',C',D')$ is also a minimal realization of $\Sigma_e(P)$ *if and only if* $(A',B',C',D') = (S(A+BF)S^{-1}, SBR^{-1}, (C+DF)S^{-1}, DR^{-1})$ for a certain S,R and F with $\det S \neq 0$, $\det R \neq 0$ (WILLEMS (1979)). We shall call this group of transformations (S,F,R) as above the *Brunovsky group*. Now it is known (WONHAM (1979,5.7)) that for every controllable pair (A,B) there exist (S,F,R), $\det S \neq 0$, $\det R \neq 0$ such that

$$S(A+BF)S^{-1} = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & \ldots & \ldots & 0 \end{pmatrix}_{\kappa_i \times \kappa_i}\right)$$

and

$$SBR^{-1} = \mathrm{diag}\left(\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}_{\kappa_i \times 1}\right)$$

for certain $\kappa_1 \geq \ldots \geq \kappa_m \geq 0$, $\kappa_i \in \mathbb{N}$ and $m = \dim U$, i.e. the form (2.6). We call this form the *Brunovsky normal form* of the pair (A,B), and the integers κ_i are called the *controllability indices* of (A,B), or also the *Kronecker indices* of the pencil $[sI-A\vdots-B]$(see GANTMACHER (1959)). Since minimal realizations of $\Sigma_e(P)$, with P(s) surjective for every $s \in \mathbb{C}$, are automatically controllable, we see that for realizing $\Sigma_e(P)$ we can take A and B already in Brunovsky normal form, with the controllability indices κ_i of

(A,B) equal to the Chern numbers of E(P(s)). Then we only have to construct a constant matrix$[C:D]$ with $C : \mathbb{R}^n \to \mathbb{R}^q$ and $D : \mathbb{R}^m \to \mathbb{R}^q$, where n and m are defined by $m := q-p$ and $n := \kappa_1 + \ldots + \kappa_m$. Therefore giving a realization of $\Sigma_e(P)$ amounts to replacing the subbundle E(P(s)) of $\mathbb{P}^1 \times \mathbb{C}^q$ by an isomorphic vectorbundle $E([sI-A:-B])$ which is a subbundle of a higher dimensional space $\mathbb{P}^1 \times \mathbb{C}^n \times \mathbb{C}^m$. Furthermore we notice that *after* embedding the vectorbundle over $\mathbb{P}^1$ in this higher dimensional space, the isomorphism class is given by *constant* linear maps. More precisely if $E([sI-A_1:-B_1])$ is isomorphic to $E([sI-A_2:-B_2])$, then (A_1,B_1) and (A_2,B_2) are necessarily of the same dimension, say A_i n×n and B_i n×m, i = 1,2, and there exists a constant linear map $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ such that

$$(2.7) \qquad H(E([sI-A_2:-B_2]) = E([sI-A_1:-B_2]).$$

This follows from Kronecker theory (GANTMACHER (1959)), or the theory around the Brunovsky normal form, since $E([sI-A_2:-B_2])$ isomorphic to $E([sI-A_1:-B_1])$ $\Longleftrightarrow$ controllability indices of (A_1,B_1) and (A_2,B_2) are equal .
Therefore there exists an element (S, F, R) of the Brunovsky group such that

$$A_1 = S(A_2+B_2F)S^{-1}$$
$$B_1 = SB_2R^{-1}.$$

Then H as above equals the map $\begin{pmatrix} S & 0 \\ -RF & R \end{pmatrix}$
(we can easily check that $\mathrm{Ker}(sI-A_1:-B_1)H = \mathrm{Ker}(sI-A_2:-B_2)$).

As we already noticed, $\Sigma_e(P)$, with P(s) surjective $\forall s \in \mathbb{C}$, can always be realized by a linear *input-output* system. In fact, geometrically we can see this as follows.
Define $U := \lim_{s\to\infty}(\mathrm{Ker}\ P(s)) = V(\infty)$ (= Ker P_r if P(s) is row proper). Then U is a linear subspace of $W = \mathbb{R}^q$ with dimension $m := q-p$. Given a minimal realization $\Sigma(A,B,C,D)$ of $\Sigma_e(P)$ we then know that U = Im D. Hence we may identify the input space of the realization $\Sigma(A,B,C,D)$ with the subspace U of W (since D is injective). Define furthermore Y as an arbitrary p-dimensional subspace of W, *complementary* to U, i.e. $W = Y \oplus U$. We may now fix $D : \mathbb{R}^m \to W = \mathbb{R}^p \times \mathbb{R}^m$ as $D = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$, and by feedback transformations $C \mapsto C + DF$ we can obtain $C : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{R}^m$ ($\mathbb{R}^n = X$ is the state space) in the form $C = \begin{pmatrix} \bar{C} \\ 0 \end{pmatrix}$.

So we have written Ker $P(s) = [C\ D]$ $(\mathrm{Ker}[sI-A\vdots -B])$ with $C = \begin{pmatrix}\overline{C}\\ 0\end{pmatrix}$ and $D = \begin{pmatrix}0\\ I\end{pmatrix}$, corresponding to a splitting $W = Y \times U$. We call $(A,B,\overline{C})$ as above an *input-output realization* of $\Sigma_e(P)$.

What kind of freedom exists in constructing a minimal input-output realization of $\Sigma_e(P)$? This is answered in

THEOREM 2.14 Let $(A,B,\overline{C})$ be a minimal input-output realization of $\Sigma_e(P)$. Then all other minimal input-output realizations of $\Sigma_e(P)$ may be obtained by applying the following transformations to $(A,B,\overline{C})$

(i) $(A,B,\overline{C}) \longmapsto (SAS^{-1}, SB, \overline{C}S^{-1})$ $S : \mathbb{R}^n \to \mathbb{R}^n$, $\det S \neq 0$

(ii) $B \longmapsto BR^{-1}$ $R : \mathbb{R}^m \to \mathbb{R}^m$, $\det R \neq 0$

(iii) $\overline{C} \longmapsto T\overline{C}$ $T : \mathbb{R}^p \to \mathbb{R}^p$, $\det T \neq 0$

(iv) $A \longmapsto A + BH\overline{C}$ $H : \mathbb{R}^p \to \mathbb{R}^m$

(i.e. output feedback transformations)

PROOF We have fixed U by setting $U = V(\infty)$. Therefore the only freedom which is left are coordinate transformations on U. This is the group of transformations (ii). If we have chosen a complementary subspace Y, such that $W = Y \oplus U$, then of course coordinate transformations on Y are allowed. This yields group (iii). Now there is freedom in choosing a complementary subspace Y. Given a subspace Y such that $W = Y \oplus U$, another subspace $Y' \subset W$ is also complementary to U if and only if the projection of Y' along U on Y is equal to Y, and $\dim Y' = \dim Y = p$. Then in a basis corresponding to $W = Y \oplus U$, Y' has the form $Y' = \{\begin{pmatrix}y\\ Hy\end{pmatrix} \mid y \in Y\}$. Therefore the transformation from a basis corresponding to $Y \oplus U$ to the basis corresponding to $Y' \oplus U$ is given by $\begin{pmatrix}I_p & 0\\ H & I_m\end{pmatrix}$, and hence in the new basis corresponding to $Y' \oplus U$ we have that $u_{new} = u_{old} + Hy_{old}$. This is exactly the group of transformations (iv). Finally group (i) are the basis transformations in the state space and is classical. It is clear that no other transformations are allowed in maintaining an input-output realization $(A,B,\overline{C})$. □

Remark 1. We notice that after we have chosen Y complementary to U, the Jordan structure of A in a minimal input-output realization $(A,B,\overline{C})$ is uniquely determined (since only coordinate transformations $S : \mathbb{R}^n \to \mathbb{R}^n$ are allowed). Furthermore we see that the eigenvalues of A are exactly given by those $s_0 \in \mathbb{C}$ such that $\mathrm{Ker}\, P(s_0) \cap Y \neq 0$!

Remark 2. If we apply feedback F to $(A,B,\overline{C})$ then $C = \begin{pmatrix}\overline{C}\\ 0\end{pmatrix}$ is changed into

$C' = \begin{pmatrix}\overline{C}\\0\end{pmatrix} + \begin{pmatrix}0\\I\end{pmatrix}F$. We see that Im $C' \cap U = 0$, if and only if F is of the form $F = HC$. Hence if we apply feedback which is not output feedback then we no longer have an input-output realization.

It is also easy to see how we can end up with input-output systems *with* feed-through term $\overline{D}$, as in equation (2.4). Take an *arbitrary* m-dimensional linear subspace U of W. Then we can take a subspace Y, complementary to $V(\infty) = \lim_{s\to\infty} \ker P(s)$, such that the projection of U along Y on $V(\infty)$ equals $V(\infty)$ (we only have to take Y complementary to $V(\infty)$ *and* U). In a basis corresponding to $Y \times V(\infty)$, U is of the form $U = \{\begin{pmatrix}\overline{D}v\\v\end{pmatrix} | v\in V(\infty)\}$ for a certain matrix $\overline{D}$, and with respect to this input space U and output space Y we obtain a realization $\dot{x} = Ax + Bu$, $y = \overline{C}x + \overline{D}u$.

We have shown that, since a minimal realization $\Sigma(A,B,C,D)$ of $\Sigma_e(P)$ has the property that D is injective, we can always give an input-output realization of $\Sigma_e(P)$. This can also be seen in another way, since we will show that $\Sigma_e(P)$ can always be written as $\Sigma_e([D\vdots -N]$ with D(s) and N(s) as in Case 3 of Section 2.1.2.

THEOREM 2.15 Let $P \in \mathbb{R}^{p\times q}[s]$, surjective for every $s \in \mathbb{C}$. There exists a unimodular U(s) such that U(s)P(s) is row proper, i.e. $U(s)P(s) = \mathrm{diag}(s^{k_i})P_r + L(s)$ as before (Definition 2.10). Choose bases for $W = \mathbb{R}^q$ and for $\mathbb{R}^p$ such that $P_r = [I_p \vdots 0]$. Write in this basis $U(s)P(s) = [D(s)\vdots -N(s)]$, with $D \in \mathbb{R}^{p\times p}[s]$, $N \in \mathbb{R}^{p\times(q-p)}[s]$. Then det D(s) is unequal to the zero polynomial, and $G(s) := D^{-1}(s)N(s)$ is strictly proper.

PROOF Since U(s)P(s) is row proper and we have taken bases such that $P_r = [I_p 0]$, it is clear that also D(s) is row proper, i.e. $D(s) = \mathrm{diag}(s^{k_i})D_r + L_D(s)$, where in fact $D_r = I_p$. Therefore det D(s) is unequal to the zero polynomial. We now have to show that every entry $g_{ij}(s)$ of $G(s) = D^{-1}(s)N(s)$ is strictly proper. We can use Cramer's rule to solve for $g_{ij}(s)$, indeed $g_{ij}(s) = \det D^{ij}(s)/\det D(s)$ where $D^{ij}(s)$ is the matrix obtained by replacing the i-th column of D(s) by the j-th column of N(s). Then we can write $D^{ij}(s) = \mathrm{diag}(s^{k_i})D_r^{ij} + L^{ij}(s)$ with $D_r{}^{ij}$ the same matrix as $D_r = I_p$ except for the i-th column which is zero, since the ℓ-th entry of the j-th column of N(s) has degree strictly less than k_ℓ. Therefore D_r^{ij} is singular. It follows that the degree of det D(s) is $\sum_{i=1}^{m} k_i$, while the degree of det $D^{ij}(s)$ is strictly less than $\sum_{i=1}^{m} k_i$, and therefore $g_{ij}(s)$ is strictly proper. □

Hence given P(s), we first have to make P(s) row proper, then define U:= ker P_r and Y an (arbitrary) complement of U in W. If we write P(s) = $[D(s)\vdots -N(s)]$ corresponding to W = Y × U then $\{w(s)\,|\,P(s)w(s)=0\} = \{w(s)=(y(s),u(s))\,|\,D(s)y(s)=N(s)u(s)\}$. Moreover we can write this system also as y(s) = G(s)u(s), and since $[D(s)\vdots -N(s)]$ is surjective $\forall s \in \mathbb{C}$, $G(s) = D^{-1}(s)N(s)$ is a *left coprime* factorization of G(s). If we denote the inverse Laplace transform of G(s) by $\tilde{G}(t)$, $t \in \mathbb{R}$, and if we assume that $w(t) = (y(t),u(t)) = (0,0)$, $\forall t \leq t_0$ (see Proposition 1.18) then $\Sigma_e(P)$ is also given as

$$\Sigma_e(P) = \{(y,u): \mathbb{R} \to Y\times U \,|\, u\in L_{1oc}, u(t)=0, t\leq t_0, y(t)= \int_{-\infty}^{t} \tilde{G}(t-\tau)u(\tau)dt\}.$$

This external input-output system is in fact the usual starting point for linear system theory (see KALMAN, FALB & ARBIB (1969))

In the sequel we will make use of a *dual* way to give the transfer matrix of an input-output system. We need the following

LEMMA 2.16 (for a proof see WILLEMS (1983)). Let $P \in \mathbb{R}^{p\times q}[s]$, and surjective $\forall s \in \mathbb{C}$. Then there exists a polynomial matrix $Q \in \mathbb{R}^{p\times q}[s]$ (with m:= q-p) such that Ker P(s) = Im Q(s), $\forall s \in \mathbb{C}$. In particular Q(s) is injective $\forall s \in \mathbb{C}$.

Now a polynomial matrix Q which is injective $\forall s \in \mathbb{C}$ can be made *column proper* by post multiplication with a unimodular matrix (a polynomial matrix Q(s) is called column proper if $Q^T(s)$ is row proper, hence this is already proved in Proposition 2.11), i.e.

$$Q(s) = Q_C \operatorname{diag}(s^{k_i}) + L(s)$$

with Q_C injective and the degree of the j-th column of L(s) strictly less than k_i. Then of course $\lim_{s\to\infty}$ Im Q(s) is well-defined (Theorem 2.9) and equal to Im Q_C. Therefore if Ker P(s) = Im Q(s), $\forall s \in \mathbb{C}$, we obtain that Ker P_r = Im Q_C. Hence we can take bases such that $P_r = [I_p\, 0]$ and $Q_C = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$. If we write correspondingly to these bases $P(s) = [D(s)\vdots -N(s)]$ and $Q(s) = \begin{pmatrix} N_1(s) \\ D_1(s) \end{pmatrix}$, then $D(s)N_1(s) = N(s)D_1(s)$ or $D^{-1}(s)N(s) = N_1(s)D_1^{-1}(s)$. The factorization $G(s) = N_1(s)D^{-1}(s)$ is called the right coprime factorization of G(s), and the external linear system can also be written as

$\{(y(s),u(s))\,|\,\exists z(s)$ such that $D_1(s)z(s) = u(s)$, and $y(s) = N_1(s)z(s)\}$.

We define the *observability indices* of a pair $(\overline{C},A)$ as the controllability indices of $(A^T,\overline{C}^T)$. Given a vectorbundle E(P(s)) and $\Sigma(A,B,\overline{C})$ a minimal input-output realization of $\Sigma_e(P)$, we have seen that the controllability indices of (A,B) are equal to the Chern numbers of E(P(s)). Dually we obtain

THEOREM 2.17 Let $P \in \mathbb{R}^{p\times q}[s]$, surjective $\forall s \in \mathbb{C}$. Let $\Sigma(A,B,\overline{C})$ be a minimal input-output realization of $\Sigma_e(P)$. The controllability indices of (A,B) are equal to the Chern numbers of E(P(s)). Now define the algebraic vectorbundle $(E(P(s)))^\perp$ over $\mathbb{P}^1$ by setting

$$(E(P(s)))^\perp := \{((s,1),v) \mid s\in\mathbb{C}, v\in(\text{Ker } P(s))^\perp\} \cup$$
$$\{((1,0),v) \mid v\in(V(\infty))^\perp\}$$

where $\perp$ denotes orthogonal complement with respect to an inner product on $W = \mathbb{R}^q$. Then the observability indices of $(\overline{C},A)$ are equal to the Chern numbers of $(E(P(s)))^\perp$.

Remark: The freedom in choosing an orthogonal complement corresponds to the freedom we have in choosing an output space Y complementary to U. It can easily be seen that the observability indices of $(\overline{C},A)$ (as well as the controllability indices of (A,B)) are invariant with respect to the transformations given in Theorem 2.14.

PROOF Write $P(s) = [D(s)\vdots -N(s)]$ and $Q(s) = \begin{pmatrix} -N_1(s) \\ D_1(s) \end{pmatrix}$ as above, i.e. Ker P(s) = Im Q(s) and $P_r = [I_p\, 0]$ and $Q_C = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$. Then $(\text{Ker}[D(s)\vdots -N(s)])^\perp =$ $= (\text{Im} \begin{pmatrix} -N_1(s) \\ D_1(s) \end{pmatrix})^\perp = \text{Ker}[-N_1^T(s)\vdots D_1^T(s)]$. The transfer matrix of this last input-output system is then given by $(D_1^T(s))^{-1}N_1^T(s)$ (notice that the inputs and outputs have changed places). However $G(s) = D^{-1}(s)N(s) = N_1(s)D_1^{-1}(s)$ and hence the transfer matrix equals $G^T(s)$. If G(s) has a minimal realization $\Sigma(A,B,\overline{C})$, then $G^T(s)$ has a minimal realization $(A^T,\overline{C}^T,B^T)$. Therefore the Chern numbers of $(E(P(s)))^\perp$ are equal to the controllability indices of $(A^T,\overline{C}^T)$, which are equal to the observability indices of $(\overline{C},A)$. □

Theorem 2.17 also gives us a way to compute the observability and controllability indices of a minimal realization of $\Sigma_e(P)$ directly from P(s):

THEOREM 2.18 Let $P \in \mathbb{R}^{p\times q}[s]$, surjective $\forall s \in \mathbb{C}$. Let $Q \in \mathbb{R}^{q\times m}[s]$, m:= q-p, injective $\forall s \in \mathbb{C}$, such that Ker P(s) = Im Q(s) (Lemma 2.16). Make P(s)

row proper by premultiplication with a unimodular matrix (row operations), and $Q(s)$ column proper by post multiplication with a unimodular matrix (column operations). Let $(A,B,\bar{C})$ be a minimal input-output realization of $\Sigma_e(P)$. Then

(i) degrees of rows of $P(s)$ = Chern numbers of $(E(P(s)))^{\perp}$ = observability indices of $(\bar{C},A)$

(ii) degrees of columns of $Q(s)$ = Chern numbers of $E(P(s))$ = controllability indices of (A,B).

Remark: If $P_1(s)$ and $P_2(s)$ are row proper and $U(s)P_1(s) = P_2(s)$ for a unimodular $U(s)$, then the row degrees of $P_1(s)$ and $P_2(s)$ are equal. Therefore the row degrees do not depend on how $P(s)$ is made row proper. Similarly for column degrees.

<u>PROOF</u> We only prove (ii), (i) is similar. If $\Sigma(A,B,\bar{C})$ is a minimal input-output realization, then $\Sigma(A,B,C,D)$ with $C = \binom{\bar{C}}{0}$ and $D = \binom{0}{I}$ satisfies $\operatorname{Ker} P(s) = [C\ D]\operatorname{Ker}[sI-A \vdots -B]$. Now (A,B) can be brought into Brunovsky normal form by applying a transformation $H = \begin{pmatrix} S & 0 \\ -RF & R \end{pmatrix}$ to $X \times U$, $\det S \neq 0$, $\det R \neq 0$ (see equation 2.7). Let now (A,B) be the Brunovsky normal form of $(\tilde{A},\tilde{B})$, i.e.

$$\tilde{A} = \operatorname{diag}\begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & \cdot & \cdot & 0 \end{pmatrix}_{\kappa_i \times \kappa_i}, \quad \tilde{B} = \operatorname{diag}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{\kappa_i \times 1}$$

with κ_i the controllability indices of (A,B). Then it is easy to check that $\operatorname{Ker}[sI-\tilde{A} \vdots -\tilde{B}]$ is given by $\operatorname{Im} \tilde{Q}(s)$, with

$$\tilde{Q}(s) := \begin{pmatrix} 1 & 0 & 0 \\ s & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ s^{\kappa_1 - 1} & \cdot & \cdot \\ 0 & 1 & \vdots \\ \vdots & s & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & s^{\kappa_2 - 1} \cdots\cdots & \vdots \\ \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & 0 \cdots\cdots & \cdot \\ \vdots & \vdots & 1 \\ \vdots & \vdots & s \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots\cdots & s^{\kappa_m - 1} \\ s^{\kappa_1} & 0 \cdots\cdots & 0 \\ 0 & s^{\kappa_2} & \cdot \\ \vdots & 0 & \vdots \\ \vdots & \cdot & s^{\kappa_m} \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \mathbb{R}^n = X \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \mathbb{R}^m = U \\ \\ \end{matrix}$$

It is clear that $\tilde{Q}(s)$ is column proper with column degrees κ_i and if we write $\tilde{Q}(s) = \text{diag}\,(s^{\kappa_i})Q_c + L(s)$ as before, then

$$Q_c = \begin{pmatrix} 0_{n\times m} \\ I_m \end{pmatrix}$$

Therefore Ker$[sI-A\vdots -B]$ is given by Im $\begin{pmatrix} S & 0 \\ -RF & R \end{pmatrix} \tilde{Q}(s)$, which is still column proper with the same column degrees, and Q_c is changed into $\begin{pmatrix} 0_{n\times m} \\ R \end{pmatrix}$. Then it can be seen that $Q(s) := [C\ D] \begin{pmatrix} S & 0 \\ -RF & R \end{pmatrix} \tilde{Q}(s)$ with $C = \begin{pmatrix} \bar{C} \\ 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 \\ I \end{pmatrix}$, is also column proper, with the same column degrees κ_i and of course Im $Q(s)$ = Ker $P(s)$, $\forall s \in \mathbb{C}$. □

Example: Let us return to our mechanical system $m_1\ddot{y}_1 = u - k_1(y_1-y_2)$, $m_2\ddot{y}_2 = k_1(y_1-y_2) - k_2y_2$ with

$$P(s) = \begin{pmatrix} m_1s^2+k_1 & -k_1 & -1 \\ -k_1 & m_2s^2+k_1+k_2 & 0 \end{pmatrix}$$

Let us assume that $k_1 \neq 0$, $m_1 \neq 0$ and $m_2 \neq 0$. $P(s)$ is row proper and the degrees of the first and second row are two. Therefore the observability indices of a minimal realization $(A,B,\bar{C})$ are 2 and 2. The kernel $Q(s)$ of $P(s)$ is given by

$$\begin{pmatrix} m_2s^2+k_1+k_2 \\ k_1 \\ m_1m_2s^4+(k_1m_1+k_2m_1+k_1m_2)s^2+k_1k_2 \end{pmatrix}$$

The column degree is four, which is equal to the controllability index. Let us denote $D(s) := \begin{pmatrix} m_1s^2+k_1 & -k_1 \\ -k_1 & m_2s^2+k_1+k_2 \end{pmatrix}$ and $N(s) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then $P(s) = [D(s)\vdots -N(s)]$, det $D(s) \neq 0$ and

$$G(s) := D^{-1}(s)N(s) = \frac{1}{(m_1s^2+k_1)(m_2s^2+k_1+k_2)-k_1^2} \begin{pmatrix} m_2s^2+k_1+k_2 & k_1 \\ k_1 & m_1s^2+k_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \frac{-1}{m_1m_2s^4+((k_1+k_2)m_1+k_1m_2)s^2+k_1k_2} \begin{pmatrix} m_2s^2+k_1+k_2 \\ k_2 \end{pmatrix}$$

is the transfer matrix (from u to $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$).

A minimal realization $(A,B,\overline{C})$ is

$$A = \begin{pmatrix} 0 & 0 & 1/m_1 & 0 \\ 0 & 0 & 0 & 1/m_2 \\ -k_1 & -k_1 & 0 & 0 \\ k_1 & -(k_1+k_2) & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \overline{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Remark: Let $P \in \mathbb{R}^{p\times q}[s]$ and $Q \in \mathbb{R}^{q\times m}[s]$ as in Theorem 2.18, with P row proper and Q column proper. Write $Q(s) = \mathrm{diag}(s^{k_i})Q_c + L(s)$ as before. Consider now a polynomial matrix $U(s) \in \mathbb{R}^{q\times q}[s]$, such that U(s)Q(s) is again proper. Write $U(s)Q(s) = \mathrm{diag}(s^{\kappa'_i})Q'_c + L'(s)$ as before. If U(s) satisfies

a) $\{\kappa_1,\ldots,\kappa_m\} = \{\kappa'_1,\ldots,\kappa'_m\}$

b) $\mathrm{Im}\ Q_c = \mathrm{Im}\ Q'_c$

then U(s) is called column degree preserving and can be interpreted as *feedback*. This can be seen as follows: Because $\mathrm{Im}\ Q_c = \mathrm{Im}\ Q'_c$, the input spaces of the external linear systems corresponding to Q(s) and U(s)Q(s) are equal. By Theorem 2.18 the column degrees of Q(s) and U(s) are the controllability indices of a minimal state space realization with state space X and input space U. Therefore there exists a Brunovsky transformation $\begin{pmatrix} S & 0 \\ -RF & R \end{pmatrix}$ on X×U which carries a minimal realization of Q(s) over in a minimal realization of U(s)Q(s). Conversely to every Brunovsky transformation there corresponds a column degree preserving U(s). This result was for input-output systems in fact first stated in HAUTUS & HEYMANN (1978), in the following equivalent formulation. Consider the submodule Ω of $\mathbb{R}^q(s)$ generated by the columns of Q(s). Then feedback corresponds to an $\mathbb{R}[s]$-homomorphism from Ω to another submodule of $\mathbb{R}^q[s]$ of the same rank, which is *degree preserving* (HAUTUS & HEYMANN (1978)). Furthermore it is noticed that such degree preserving homomorphisms can be related to bicausal isomorphisms from $\mathbb{R}^m[s^{-1}]$ to itself.

Dually we note that if $U(s) : W \to W$ is such that the row degrees of P(s)U(s) are equal to the row degrees of P(s) and $\lim_{s\to\infty} \mathrm{Ker}\ P(s)U(s) = \lim_{s\to\infty}$

Ker P(s), then U(s) corresponds to *output injection*. This is a transformation dual to the Brunovsky transformation: if $(A,B,\overline{C})$ is a minimal realization of $\Sigma_e(P)$, then it is changed into $(A',B',\overline{C}')$ with $B' = B$, $A' = S(A+H\overline{C})S^{-1}$, $\overline{C}' = T\overline{C}S^{-1}$, where $S : X \to X$, $T : Y \to Y$ with $\det S \neq 0$, $\det T \neq 0$ and $H : Y \to X$.

Finally, we want to way a few words about the case that P(s) is not surjective for *every* $s \in \mathbb{C}$, but only surjective for almost every $s \in \mathbb{C}$. First we note that in this case (contrary to the situation P(s) surjective $\forall s \in \mathbb{C}$), not all information about $\Sigma_e(P)$ is contained in the geometrical object $\{\text{Ker } P(s),\ s\in\mathbb{C}\}$. Take for instance $W = \mathbb{R}$ and consider $p_1(s) := s$ and $p_2(s) := s^2$. Clearly $\ker p_1(s) = \ker p_2(s)$, $\forall s \in \mathbb{C}$, in fact $\ker p_1(s) = \ker p_2(s) = 0$, $\forall s \neq 0$ and $\ker p_1(0) = \ker p_2(0) = W$. However, $\Sigma_e(p_1) = \{w \mid \frac{dw}{dt} = 0\}$ and $\Sigma_e(p_2) = \{w \mid \frac{d^2w}{dt^2} = 0\}$.
As we already remarked, the $s \in \mathbb{C}$ where the dimension of Ker P(s) suddenly jumps are exactly the non-controllable eigenvalues of a minimal realization of $\Sigma_e(P)$:

PROPOSITION 2.19 Let $P \in \mathbb{R}^{p\times q}[s]$, and surjective for almost every $s \in \mathbb{C}$. Let $\Sigma_i(A,B,C,D)$ be a minimal realization. Then: P(s) is surjective for all $s \in \mathbb{C}$ $\Longleftrightarrow$ $\Sigma_i(A,B,C,D)$ is controllable. Furthermore, write $P(s) = R(s)P'(s)$ as in Proposition 2.7. Then the controllable part of $\Sigma_i(A,B,C,D)$ is a minimal realization of $\Sigma_e(P')$, while the $s_0 \in \mathbb{C}$ such that $\det R(s_0) = 0$ are exactly the uncontrollable eigenvalues of A.

PROOF In Theorem 2.13 we already proved that if P(s) is surjective $\forall s \in \mathbb{C}$, then $\Sigma_i(A,B,C,D)$ is controllable. Conversely if $\Sigma_i(A,B,C,D)$ is controllable, then by the Hautus test $\text{Ker}[sI-A\vdots-B]$ has constant dimension $\forall s \in \mathbb{C}$. Since $[C\vdots D]$ maps $\text{Ker}[sI-A\vdots-B]$ isomorphically onto Ker P(s) by Theorem 2.13, it follows that P(s) is surjective $\forall s \in \mathbb{C}$. After a basis transformation on X (the state space) we can always write A and B as

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

such that (A_1,B_1) is controllable. The eigenvalues of A_3 are the uncontrollable eigenvalues. It is clear that outside the eigenvalues of A_3, $\text{Ker}[sI-A\vdots-B] = \text{Ker}[sI-A_1\vdots-B_1]$. Hence $\Sigma_i(A_1,B_1,C_1,D)$, with C_1 restriction of C to the controllable subspace of X, is a minimal (and controllable) realization

of $\Sigma_e(P')$. We also see that the dimension of Ker[sI-A⋮-B] indeed jumps for s an eigenvalue of A_3. Since [C⋮D] is an isomorphism between E(P(s)) and E([sI-A⋮-B]) these s are exactly the points s_0 such that det $R(s_0) = 0$. □

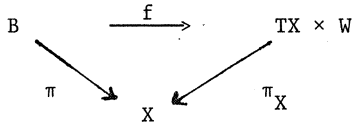
2.2 Nonlinear systems

2.2.1 Nonlinear systems in state space form

In this section we give the definition of a smooth nonlinear system (in state space form), which will be used in the sequel. Furthermore we show how this general definition can be specialized to various important subclasses of smooth nonlinear systems. Smooth will always mean C^∞, although at this stage the definitions can be easily extended to the C^k-case.

A smooth nonlinear system will consist of the following ingredients. W, the set of external variables, is a smooth manifold. Also the state space X is a smooth manifold. There is a smooth fiber bundle B over X, so $B \xrightarrow{\pi} X$, with π the bundle projection. Finally a smooth map $f : B \to TX \times W$ (TX is the tangent bundle of X) is given such that the following diagram

(2.8)



commutes (π_X is the usual projection of TX on X).

DEFINITION 2.20 A smooth *nonlinear system* consists of smooth manifolds X and W, a smooth fiber bundle $B \xrightarrow{\pi} X$, and a smooth map $f : B \to TX \times W$ such that (2.8) commutes. It is denoted by $\Sigma(X,W,B,f)$, or shortly Σ.

Let $x = (x_1,\ldots,x_n)$ be coordinates for X (n-dimensional). Then we can take coordinates $(x,u) = (x_1,\ldots,x_n,u_1,\ldots,u_m)$ for B ((n+m)-dimensional). Such coordinates for B are called *fiber respecting*. Take coordinates $w = (w_1,\ldots,w_q)$ for W (q-dimensional). Then locally Definition 2.20 amounts to

$$(2.9) \qquad \begin{cases} \dot{x} = g(x,u) \\ w = h(x,u) \end{cases}$$

where we have split $f : B \to TX \times W$ as $f = (g,h)$, with $g : B \to TX$ and $h : B \to W$. Furthermore we abuse notation by writing $g(x,u) \in TX$ as $(x,g(x,u))$.

A smooth nonlinear system $\Sigma(X,W,B,f)$ as above yields the following dynamical system in state space form (Definition 1.2):

(2.10) $\Sigma_i(X,W,B,f) := \{(x,w) : \mathbb{R} \to X \times W \mid x \text{ absolutely continuous and } (\dot{x}(t),w(t) \in f(\pi^{-1}(x(t))) \text{ a.e.}\}$

Furthermore the external behaviour of $\Sigma(X,W,B,f)$ is given by (Definition 1.3):

(2.11) $\Sigma_e(X,W,B,f) := \{w : \mathbb{R} \to W \mid \exists x : \mathbb{R} \to X \text{ such that } (x,w) \in \Sigma_i\}$

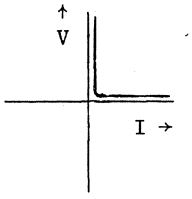
We denote this by $\Sigma \Longrightarrow \Sigma_i \Longrightarrow \Sigma_e$.

Definition 2.20 differs in two major aspects from more usual starting points in nonlinear system theory:

I. Instead of inputs and outputs we use a set of external variables W.

II. We use a state-dependent input space (namely the fibers of B).

With respect to I we refer to Chapter 1 for motivation. Furthermore we notice that, contrary to the linear case, there are situations where splitting of external variables into inputs and outputs is a delicate issue. Consider the (nearly) ideal diode given by the following I-V characteristic:



For $I > 0$ it is natural to regard I as the input and V as the output, while for $V > 0$ it is natural to see V as the input and I as the output. Around (0,0) an input-output description could be given in the scattering variables (I-V,I+V). So only *locally* we can define natural inputs and outputs, and they really differ in the three regions mentioned. In this example it is still possible to define a *global* input-output representation, namely by using the scattering variables. However, one can easily imagine a situation where it is simply not possible to give such a global split of external variables into inputs and outputs (think for instance of an I-V characteristic which is a closed curve in $\mathbb{R}^2$; a sort of hysteresis loop).

With respect to aspect II we note that a bundle $B \xrightarrow{\pi} X$ is a mathematical generalization of a product structure $B = X \times U$, with U a smooth manifold. By taking coordinates $u = (u_1,\ldots,u_m)$ for U we obtain in this case for the first part of equations (2.9):

(2.12) $\dot{x} = g(x,u), \quad x \in X$

where now for every $u \in U$, $g(\cdot,u)$ is a globally defined vectorfield on X. So if we take $B = X \times U$ we adopt the framework of a *family* of vectorfields on X, parametrized by $u \in U$. This framework, although it is what is normally used in nonlinear system theory, might be too narrow for control theoretic purposes, as is illustrated by the following example. Consider a particle moving on the surface of a sphere S^2 which can be controlled to move in any direction. If we model this by taking $B = X \times U$ (in this case $S^2 \times \mathbb{R}^2$), then the dynamics of the particle are of the form

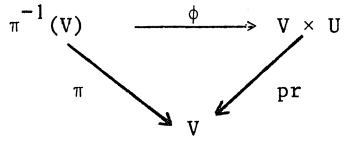
(2.13) $\dot{x} = u_1 Z_1(x) + u_2 Z_2(x)$

with Z_1 and Z_2 globally defined vectorfields on S^2, and u_1 and u_2 the controls. However, vectorfields on S^2 always have at least one equilibrium point, say $Z_1(x_1) = Z_2(x_2) = 0$, for a certain $x_1, x_2 \in S^2$. Hence, in x_1 and x_2 we can steer in at most *one* direction. Therefore if we insist on taking $B = X \times U$ we cannot describe the above situation! The dilemma is solved by taking B a nontrivial bundle, i.e. not a product $X \times U$. In fact we can take B isomorphic to TS^2.

Remark: If B is a nontrivial bundle over X, we can still define a state *independent* input space U in the following way (see TAKENS (1976)). Take U equal to the space of all (sufficiently smooth) *sections* of the bundle B, i.e. $U = \{s:X \to B \mid \pi \circ s = id\}$. Notice however, that U is in general infinite dimensional.

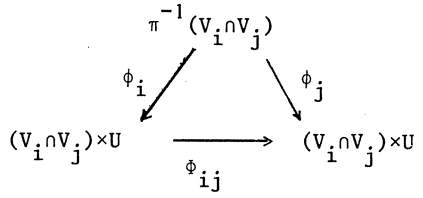
We wish to be somewhat more detailed about what we called "fiber respecting" coordinates. Since $B \xrightarrow{\pi} X$ is a fiber bundle there exists a covering set of locally trivializing charts; i.e. a collection $\{V_i\}_{i \in I}$ of open neighborhoods of X, with $X = \bigcup_{i \in I} V_i$, such that for every $i \in I$, $\pi^{-1}(V_i)$ is isomorphic to $V_i \times U$, where U is the so-called standard fiber of B. More precisely, for every chart V there exists a diffeomorphism $\phi : \pi^{-1}(V) \to V \times U$ such that

(2.14)



commutes (pr means projection on V). Moreover if V_i and V_j are two charts with diffeomorphisms ϕ_i, respectively ϕ_j, as above, and $V_i \cap V_j \neq \emptyset$, then the map Φ_{ij} such that the diagram

(2.15)



commutes, is a diffeomorphism. If V is a trivializing chart with diffeomorphism ϕ and if $x_0 \in V$ and $u_0 \in U$, we can take local coordinates $x = (x_1, \ldots x_n)$ around x_0, and $u = (u_1, \ldots, u_m)$ around u_0. Then $(x_1 \circ \phi, \ldots, x_n \circ \phi, u_1 \circ \phi, \ldots, u_m \circ \phi)$ is a set of local coordinates for B around $\phi^{-1}(x_0, u_0)$. Such coordinates are called *fiber respecting* and if no confusion is possible, we will omit the isomorphism ϕ and simply speak about the coordinates $(x_1, \ldots, x_n, u_1, \ldots, u_m)$ for B. Given a trivialization $\pi^{-1}(V) \simeq V \times U$ ($\simeq$ means isomorphic) we can interpret $g : B \to TX$ restricted to $\pi^{-1}(V)$ as a family of vectorfields on V parametrized by $u \in U$. In fact the trivialization induces a map

$$g : V \times U \to TV \tag{2.16}$$

and $g(\cdot, u)$ is for every $u \in U$ a vectorfield on V (Notice the abuse of notation. Formally we should have written $g \circ \phi^{-1} : V \times U \to TV$). In general there are many trivializations above a locally trivializing chart V. Consider two isomorphisms $\phi_1 : \pi^{-1}(V) \to V \times U$ and $\phi_2 : \pi^{-1}(V) \to V \times U$, such that (2.14) commutes. If we again take local coordinates x around x_0 and u around u_0, then the resulting fiber respecting coordinates

$$(x, u^1) := (x_1 \circ \phi_1, \ldots, x_n \circ \phi_1, u_1 \circ \phi_1, \ldots, u_m \circ \phi_1) \quad \text{and}$$
$$(x, u^2) := (x_1 \circ \phi_2, \ldots, x_n \circ \phi_2, u_1 \circ \phi_2, \ldots, u_m \circ \phi_2)$$

are different in the input (u) coordinates. There exists a diffeomorphism $\alpha : \pi^{-1}(V) \to \pi^{-1}(V)$ such that $u^2 = \alpha(x, u^1)$, in fact $\alpha = \phi_1^{-1} \circ \phi_2$. This map α can be interpreted as *feedback*. The two trivializations also induce two different sets of vectorfields $g^1(\cdot, u^1)$ and $g^2(\cdot, u^2)$, as in (2.16), which are feedback related to each other, i.e.

$$g^2(\cdot, \alpha(\cdot, u^1)) = g^1(\cdot, u^1) \tag{2.17}$$

Notice that this also motivates an alternative way to look at feedback. Namely we can define feedback as a *bundle isomorphism* $\alpha : B \to B$, i.e. a diffeomorphism α such that $B \xrightarrow{\alpha} B$ commutes.

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & B \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & X & \end{array}$$

Another related point we stress is that if B is not trivial, i.e. not $X \times U$, it is not clear what we mean by *constant inputs*. Any local trivialization $\pi^{-1}(V) \simeq V \times U$ defines sections u = constant locally above V. To define *globally* what are "constant" inputs, we need the notion of a global (integrable) *connection* on B. We return to this in section 2.2.3.

We define a nonlinear input-output system as the following specialization of Definition 2.20. First we give the notion of a *pullback bundle*. Let $k : M \to N$ be a smooth map between two manifolds, and let $\tilde{B} \xrightarrow{\tilde{\pi}} N$ be a fiber bundle. Then we define the pullback bundle $k^*\tilde{B}$ as the following fiber bundle over M:

$$k^*\tilde{B} = \{(x,b) \mid x \in M, b \in \tilde{B}, k(x) = \tilde{\pi}(b)\} \tag{2.18}$$

This gives immediately the commutative diagram

$$\begin{array}{ccc} k^*\tilde{B} & \xrightarrow{\tilde{k}} & \tilde{B} \\ {\scriptstyle k^*\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ M & \xrightarrow[k]{} & N \end{array} \tag{2.19}$$

with $k^*\tilde{\pi}$ and $\tilde{k}$ defined by $k^*\tilde{\pi}(x,b) = x$ and $\tilde{k}(x,b) = b$, for $(x,b) \in k^*\tilde{B}$ (and hence $k(x) = \tilde{\pi}(b)$).

DEFINITION 2.21 A smooth nonlinear input-output system in state space form is given by a smooth fiber bundle $\tilde{B} \xrightarrow{\tilde{\pi}} Y$, a smooth manifold X and smooth maps $h : X \to Y$ and $g : h^*\tilde{B} \to TX$ such that the diagram

$$\begin{array}{ccccc} \tilde{B} & \xleftarrow{\tilde{h}} & h^*\tilde{B} & \xrightarrow{g} & TX \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle h^*\tilde{\pi}} & \swarrow{\scriptstyle \pi_X} & \\ Y & \xleftarrow{h} & X & & \end{array} \tag{2.20}$$

commutes. We call Y the outputspace and h the output map. The system is denoted by $\Sigma(X,\tilde{B},Y,g,h)$.

Definition 2.21 specializes Definition 2.20 in the following sense. W is a fiber bundle $\tilde{B}$ over Y. Furthermore we have defined B:= $h^*\tilde{B}$ and f : B → TX×W (with W=$\tilde{B}$) by f:= $(g,\tilde{h})$. If we choose coordinates $y = (y_1,\ldots,y_p)$ for Y (p-dimensional) and fiber respecting coordinates $(y,u)= (y_1,\ldots,y_p,u_1,\ldots,u_m)$ for $\tilde{B}$ ((p+m)-dimensional) and coordinates $x = (x_1,\ldots,x_n)$ for X then $\Sigma(X,\tilde{B},Y,g,h)$ is locally given by

$$(2.21) \qquad \begin{cases} \dot{x} = g(x,u) \\ y = h(x) \end{cases}$$

Notice the following consequences of Definition 2.21:

(i) the outputs are "intrinsically" defined (they are elements of Y)

(ii) we have identified the inputspaces (the fibers of $h^*\tilde{B}$) with the fibers of $\tilde{B}$, and hence the inputs can be identified with a part of the external variables.

(iii) the inputspaces are no longer *state* dependent, but only output dependent

(iv) W can be seen as the space of outputs and inputs (although the inputs are not "intrinsically" defined, since they are elements of the fibers of $\tilde{B}$).

If in the above definition we would take B trivial, $\tilde{B} = Y \times U$, then the inputs *are* intrinsically defined. Also $B = h^*\tilde{B}$ is trivial in this case.

The fiber respecting coordinates (y,u) as we used above are of course obtained by taking a local trivialization $\tilde{\pi}^{-1}(\tilde{V}) \simeq \tilde{V}\times U$ of $\tilde{B}$, with $\tilde{V}$ an open neighborhood on Y. A local trivialization of $\tilde{B}$ induces a local trivialization of $B = h^*\tilde{B}$, namely $\pi^{-1}(h^{-1}(\tilde{V})) \simeq h^{-1}(\tilde{V}) \times U$ (with $\pi:=h^*\tilde{\pi}$). Let now $y_0 \in \tilde{V}$ and $x_0 \in h^{-1}(y_0)$ and $u_0 \in U$. Then we can take local coordinates y around y_0, x around x_0 and u around u_0. With the above trivializations of $\tilde{B}$ and B, this yields fiber respecting coordinates (y,u) for $\tilde{B}$ and (x,u) for B. In these coordinates $\tilde{h}$: B → $\tilde{B}$ is given by

$$(2.22) \qquad \tilde{h}(x,u) = (h(x),u)$$

We call a local trivialization of B as above and the resulting fiber respecting coordinates (x,u) *output-induced*. If $\pi^{-1}(\tilde{V}) \xrightarrow{\phi_1} V\times U$ and $\pi^{-1}(\tilde{V}) \xrightarrow{\phi_2} V\times U$ are different trivializations, then we obtain two different sets of fiber-respecting coordinates (y,u^1) and (y,u^2) for $\tilde{B}$, and correspondingly different output-induced coordinates (x,u^1) and (x,u^2) for B. As before there exists a bundle isomorphism $(y,u^1) \longrightarrow (y,u^2=\tilde{\alpha}(y,u^1))$ from $\tilde{\pi}^{-1}(\tilde{V})$ to itself. This induces a bundle isomorphism from $\pi^{-1}(h^{-1}(\tilde{V}))$ to itself,

given by $(x,u^1) \longrightarrow (x,u^2=\alpha(x,u^1))$, where α satisfies

$$(2.23) \qquad \alpha(x,u) = \tilde{\alpha}(h(x),u)$$

Hence the feedback transformations from one set of output induced fiber respecting coordinates of B to another, are exactly the *output feedback* transformations.

Of course there are many intermediate situations between Definition 2.20 and Definition 2.21 which are also of some interest. Let $\Sigma(X,W,B,f)$ be a nonlinear system, and write again $f = (g,h)$ with $g : B \to TX$ and $h : B \to W$. We can distinguish the following cases

I. Assume that $h : B \to W$ restricted to the fibers of B is an immersion. In other words $(\frac{\partial h}{\partial v})$ is injective (denote fiber respecting coordinates for B by (x,v)). Then by the implicit function theorem we can locally define coordinates $w = (y,u)$ for W, in which coordinates h can be written as $h(x,v) = (\bar{h}(x,v),v)$. Hence we can locally identify v and u and write (2.9) as

$$(2.24) \qquad \begin{cases} \dot{x} = g(x,u) \\ y = \bar{h}(x,u) \end{cases} \qquad w = (y,u)$$

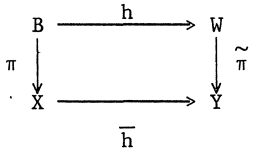
We call (2.24) a local *input-output representation with feedthrough term.*

II. Assume, apart from $(\frac{\partial h}{\partial v})$ injective, that $V + \ker dh$ is an involutive distribution of constant dimension on B (V is the vertical tangentspace of B; $V(x,v) := \{Z \in T_{(x,v)}B \mid \pi_* Z = 0\}$). Then h_*V is an involutive distribution of constant dimension on W(this will be proved in Lemma 2.27), and hence at least locally, we can factor out W by the leaves of the foliation generated by h_*V. We can take local coordinates $w = (y,u)$ for W such that the leaves are given by setting y equal to a constant. In these coordinates h can be written as $h(x,v) = (\bar{h}(x),v)$. After identifying v and u we obtain

$$(2.25) \qquad \begin{cases} \dot{x} = g(x,u) \\ y = \bar{h}(x) \end{cases} \qquad w = (y,u)$$

We call (2.25) a local *input-output representation.*

III. Suppose that W is a fiberbundle over an output manifold Y with projection $\tilde{\pi}$, and that $h : B \to W$ is a *bundle isomorphism*, i.e. there exists $\bar{h} : X \to Y$ such that



commutes

Moreover, assume that h restricted to the fibers of B is an isomorphism. Then it is easy to see that B actually is diffeomorphic to $\bar{h}^*W$, and that we have arrived at Definition 2.21.

IV. Finally, if $(\frac{\partial h}{\partial v})$ is *not* injective, then there are variables in the fibers of B which can affect the internal state behaviour via the equation $\dot{x} = g(x,v)$, but which cannot be directly identified with some of the external variables. Hence even a *local* input-output representation (with feed-through term) is not possible. We shall come back to this in Section 2.2.4.

An important *subclass* of nonlinear input-output systems is formed by the systems where the inputs enter the equations in a *linear* way. This can be formalized as follows

DEFINITION 2.22 A nonlinear input-output system $\Sigma(X,\tilde{B},Y,g,h)$ is called *affine* if B is a vectorbundle and the map $g : h^*\tilde{B} \to TX$ is an affine bundle morphism, i.e. g restricted to the fibers is an affine map.

If we take local fiber respecting coordinates $(x,u) = (x_1,\ldots,x_n, u_1,\ldots,u_m)$ for B, such that $u = (u_1,\ldots,u_m)$ are affine coordinates for the fibers of B (which are linear spaces), then Definition 2.21 yields that g(x,u) can be written as

$$g(x,u) = A(x) + \sum_{i=1}^{m} u_i B_i(x)$$

where A(x) and $B_i(x)$, i=1,...,m are locally defined vectorfields on X. We can define $\Delta_0(x) :=$ span $\{B_1(x),\ldots,B_m(x)\}$ and $\Delta(x) := A(x)+\Delta_0(x)$. This yields an *affine* distribution Δ on X and a distribution Δ_0 on X such that $\Delta_0 = \Delta-\Delta := \{Z_1-Z_2 \mid Z_1,Z_2 \in \Delta\}$.

We define another type of an affine system in the following way:

DEFINITION 2.23 An *affine control system* consists of an affine distribution Δ on X, and a map h : X → Y with Y the output manifold. It is denoted by $\Sigma(X,\Delta,Y,h)$.

An affine control system is an example of a nonlinear system $\Sigma(X,W,B,f)$ (Definition 2.20) as can be seen in the following way. Define the bundle B

over X as the subbundle of TX given by the affine distribution Δ(in every $x \in X$ there is given the affine subspace $\Delta(x) \subset T_xX$). Define an affine map $g : B \to TX$ such that Im $g = \Delta$. Then take the set of external variables equal to Y, and define $\tilde{h} : B \to W$ as $\tilde{h} = h \circ \pi$, with π the projection of B onto X. Set $f = (g,\tilde{h})$. In local coordinates y for Y, and x for X, $\Sigma(X,\Delta,Y,h)$ is given given by

$$\begin{aligned} \dot{x} &= A(x) + \sum_{i=1}^{m} u_i B_i(x) \\ y &= h(x) \end{aligned} \tag{2.26}$$

for certain (locally) defined vectorfields A and B_i on X.
Remark: Notice that if there exists a bundle $\tilde{B}$ above Y such that $B = h^*\tilde{B}$, then we can also define $W = h^*\tilde{B}$ and we have arrived at an affine nonlinear input-output system.

Actually we can associate with every nonlinear system $\Sigma(X,W,B,f)$ an affine control system by the following construction, which will be frequently used in the sequel.

DEFINITION 2.24 Let $\Sigma(X,W,B,f)$ be a nonlinear system. Write $f = (g,h)$ with $g : B \to TX$ and $h : B \to W$. Define

$$\Delta^e(x,v) := \{Z \in T_{(x,v)}B \mid \pi_* Z = g(x,v)\} \quad \text{and}$$
$$\Delta_0^e(x,v) := \{Z \in T_{(x,v)}B \mid \pi_* Z = 0\}$$

for $(x,v) \in B$. Then $\Delta_0^e = \Delta^e - \Delta^e$ and $\Sigma(B,\Delta^e,W,h)$ is an affine control system, with state space B and output manifold W. $\Sigma(B,\Delta^e,W,h)$ is called the *extended system* of $\Sigma(X,W,B,f)$ and is also denoted by $\Sigma^e(X,W,B,f)$.

If in local fiber respecting coordinates (x,v) for B and w for W, $\Sigma(X,W,B,f)$ is given by

$$\begin{aligned} \dot{x} &= g(x,v) \\ w &= h(x,v) \end{aligned}$$

then the extended system $\Sigma^e(X,W,B,f)$ can be written as

$$\begin{cases} \dot{x} = g(x,v) \\ \dot{v} = u \end{cases} \qquad w = h(x,v) \tag{2.27}$$

with (x,v) the new state, and u the new input.

Remark: Taking the extended system $\Sigma^e(X,W,B,f)$ of a nonlinear system $\Sigma(X,W,B,f)$, amounts to "integrating the input one time" (i.e. $\dot{v} = u$). Therefore, if we look at the dynamical systems in state space form associated to Σ and Σ^e (see equation (2.10)), we add one degree of differentiability in the transition from Σ to Σ^e.

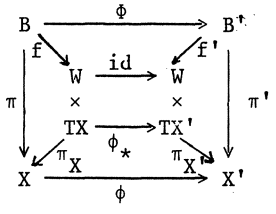
2.2.2 Minimality, observability and controllability

In this section we treat the properties of "minimality","observability" and "controllability" for the smooth nonlinear systems that we defined in the previous section. Again smooth will always mean C^∞, although the definitions and results can be often extended to the C^k-case. Furthermore, in first instance we shall remain within the "regular category". This means that we assume that the smooth maps, distributions etc. have constant rank or constant dimension. From a differential geometric point of view this adds very much to the clarity of the exposition. Later on we leave this regular category with the introduction of the observability codistribution and controllability distribution, which do not necessarily have constant dimensions. We remark that in some instances the results can be more elegantly stated by assuming (real) analyticity of the system. The reason is that in the analytic case singularities (of the (co)distributions) are easier to handle.

We give the following definition for minimality of a smooth nonlinear system in state space form.

DEFINITION 2.25 Let $\Sigma(X,W,B,f)$ and $\Sigma'(X',W,B',f')$ be two nonlinear systems. Then $\Sigma' \leq \Sigma$ if there exist surjective submersions $\phi : X \to X'$, $\Phi : B \to B'$ such that the diagram

(2.28)



commutes. Σ is called *equivalent* to Σ' ($\Sigma \sim \Sigma'$) if Φ and ϕ are diffeomorphisms. We call Σ *minimal* if $\{\Sigma' \leq \Sigma \Longrightarrow \Sigma' \sim \Sigma\}$.

Remark 1. It is clear that this definition formalizes the same idea as ex-

pressed in Definition 1.5 for general systems, but now within the "category of smooth nonlinear systems": Σ is minimal if there does not exist a Σ' smaller than Σ and explaining the same external behavior.

Remark 2. In the linear case we have given conditions on $\Sigma(A,B,C,D)$ which are necessary and sufficient for minimality of $\Sigma_i(A,B,C,D)$ (see Theorem 2.2). In the nonlinear case we have that minimality of $\Sigma_i(X,W,B,f)$ implies minimality of $\Sigma(X,W,B,f)$ as above. However, the converse is in general *not* true, and it seems hard to impose conditions on $\Sigma(X,W,B,f)$ stronger than the conditions of Definition 2.25, which ensure the minimality of $\Sigma(X,W,B,f)$.

Remark 3. If Σ is equivalent to Σ', then the diffeomorphisms Φ and ϕ as in (2.28) need not be uniquely determined, even if Σ is minimal. Recall that in the linear case (see Section 2.1.1) Φ and ϕ are indeed unique, if Σ is minimal (this is the so-called state space isomorphism theorem (see BROCKETT (1970))).

Remark 4. Definition 2.25 is an example of our general approach to remain within a smooth and *regular* (i.e. constant ranks and dimensions) category. For instance we could have strengthened the definition of minimality by allowing that Φ and ϕ are *not* submersive at isolated points.

From a differential geometric point of view it is useful to investigate what a global definition like Definition 2.25 amounts to *locally*.

For this we need the notion of a (co)distribution. A C^∞-distribution, or simply distribution, on X is a map $p \to \Delta(p)$, where $\Delta(p) \subset T_pX$ is a linear subspace of T_pX. Moreover for any $p \in X$ there exists a neighborhood U and there are C^∞ vectorfields $Z_1,\ldots,Z_k$ such that span $\{Z_1(q),\ldots,Z_k(q)\} = \Delta(q)$ for every $q \in U$. A distribution D is *involutive* if $[Z_1,Z_2] \in D$ for every Z_1,Z_2 vectorfields in D ($[,]$ is the Lie bracket). A distribution D is regular if it is involutive and it has constant dimension, i.e. dim D(p) does not depend on $p \in X$. A C^∞-codistribution, or simply codistribution, on X is a map $p \to \Delta(p)$, where $\Delta(p) \subset T_p^*X$ is a linear subspace of T_p^*X, such that locally there exist C^∞ one-forms $\theta_1,\ldots,\theta_\ell$ with $\Delta(q) = \text{span}\{\theta_1(q),\ldots,\theta_\ell(q)\}$. A codistribution P is *involutive* if for every $\theta \in P$, there exists a $\bar{\theta} \in P$ and an arbitrary one-form α such that $d\theta = \alpha \wedge \bar{\theta}$. A codistribution P is regular if it is involutive and it has constant dimension.

We shall now define the *prolongation* of a *general* (co)distribution (see also YANO & ISHIHARA (1973)). Afterwards we shall see that these definitions become much simpler for *regular* (co)distributions. First we need to define the prolongation of vectorfields, functions and one-forms. Let Z be a vectorfield on X. Z defines an one-parameter group $Z_t: X \to X$, $t \in \mathbb{R}$ and small (Z_t is the integral flow of Z). Then $(Z_t)_*$: TX → TX is the one-parameter group of a vectorfield on TX, which we denote by $\dot{Z}$. Given a function $f : X \to \mathbb{R}$, we define the function $\dot{f} : TX \to \mathbb{R}$, by $\dot{f}(v) := df(v)$, $v \in TX$. Furthermore, let θ be a one-form on X, then we define the one-form $\dot{\theta}$ on TX by setting $\dot{\theta}(Z) := Z(\theta)$, for a vectorfield Z on TX (the expression $Z(\theta)$ makes sense, since we can look at θ as a *function* from TX to $\mathbb{R}$). Let now P be a codistribution on X. Locally P is given as span $\{\theta_1,\ldots,\theta_\ell\}$ with θ_i 1-forms on X. Then define locally the codistribution $\dot{P}$ on TX by setting $\dot{P} :=$ span $\{\pi^*\theta_1,\ldots,\pi^*\theta_\ell,\dot{\theta}_1,\ldots,\dot{\theta}_\ell\}$ (with π the natural projection of TX on X). Given a codistribution P on X, we define the (C^∞-)distribution Ker P by Ker P = {C^∞ vectorfields Z on X such that $\theta(Z) = 0$, for each $\theta \in P$}. Conversely if D is a distribution on X we define the (C^∞-) codistribution Ann D (the annihilator of D) by Ann D = {C^∞ one-forms θ on $X \mid \theta(Z) = 0$ for each $Z \in D$}. In general $D \subset \mathrm{Ker}(\mathrm{Ann}\, D)$ and $P \subset \mathrm{Ann}(\mathrm{Ker}\, P)$. The prolongation of a distribution D on X is the distribution $\dot{D}$ defined by $\dot{D} = \mathrm{Ker}\, \dot{P}$, with $P = \mathrm{Ann}(D)$. Notice that $\dot{Z} \in \dot{D}$ if Z is a vectorfield contained in D.

If the (co-)distributions are regular, the above definitions become much simpler. Let P be a regular ℓ-dimensional codistribution on X, then by Frobenius' theorem (SPIVAK (1970)), there exist local coordinates $(x_1,\ldots,x_n)$ for X such that P = span $\{dx_1,\ldots,dx_\ell\}$. Hence $\dot{P}$ = span $\{dx_1,\ldots,dx_\ell, d\dot{x}_1,\ldots,d\dot{x}_\ell\}$ (we omit π^*). It is easy to see that $(x_1,\ldots,x_n,\dot{x}_1,\ldots,\dot{x}_n)$ are local coordinates for TX. Furthermore if P is regular, then D = Ker P is regular and P = Ann(Ker P). Also if D is a regular distribution, then P = Ann(D) is regular and D = Ker(Ann D). If D is regular there exist local coordinates $(x_1,\ldots,x_n)$ such that D = span $\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\}$. Then Ann D = span $\{dx_{k+1},\ldots,dx_n\}$ and hence $\dot{D}$ is given by

$$\dot{D} = \mathrm{span}\ \{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k},\frac{\partial}{\partial \dot{x}_1},\ldots,\frac{\partial}{\partial \dot{x}_k}\}$$

Now return to Definition 2.25. Let $\Sigma' \leq \Sigma$ and consider diagram (2.28). Because Φ and ϕ are submersions they define the following regular distributions on B, respectively X:

$$E := \{Z \in TB \mid \Phi_* Z = 0\}$$
$$D := \{Z \in TX \mid \phi_* Z = 0\} \tag{2.29}$$

Furthermore it can be seen that

$$\dot{D} = \{Z \in TTX \mid (\phi_*)_* Z = 0\}. \tag{2.30}$$

We notice that diagram (2.28) consists of three subdiagrams (write f = (g,h) and f' = (g',h')).

a.
$$\begin{array}{ccc} B & \xrightarrow{\Phi} & B' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ W & \xrightarrow[id]{} & W \end{array}$$

b.
$$\begin{array}{ccc} B & \xrightarrow{\Phi} & B' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow[\phi]{} & X' \end{array}$$

c.
$$\begin{array}{ccc} B & \xrightarrow{\Phi} & B' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ TX & \xrightarrow[\phi_*]{} & TX' \end{array}$$

Remark: Actually there is a fourth subdiagram

$$\begin{array}{ccc} TX & \xrightarrow{\phi_*} & TX' \\ \downarrow & & \downarrow \\ X & \xrightarrow[\phi]{} & X' \end{array}$$

but this commutes by the definition of ϕ_*.

A close inspection shows that:

$$\begin{aligned} &\text{subdiagram a) commutes} &&\Longrightarrow && E \subset \ker dh \\ &\quad " \quad \text{b)} \quad " &&\Longrightarrow && D = \pi_* E \\ &\quad " \quad \text{c)} \quad " &&\Longrightarrow && g_* E \subset \dot{D} \end{aligned} \tag{2.31}$$

Motivated by this we give

DEFINITION 2.26 Let $\Sigma(X,W,B,f)$ with $f = (g,h)$ be a nonlinear system. Σ is called *locally minimal* if, when there exist regular distributions E on B and D on X satisfying

(i) $E \subset \ker dh$

(ii) $\pi_* E = D$

(iii) $g_* E \subset \dot{D}$

then necessarily E and D are the zero distributions.

If $\Sigma(X,W,B,f)$ is not minimal then there exists a $\Sigma'(X',W,B',f')$ and Φ and ϕ, not both diffeomorphisms, such that (2.28) commutes. We saw in (2.31) that E and D as defined in (2.29) then satisfy conditions (i), (ii), (iii). Moreover E and D are not both zero. Hence Σ is also not locally minimal. Therefore we conclude: Σ locally minimal $\Longrightarrow$ Σ minimal. The converse statement is in general not true. If Σ is not locally minimal, then there exist non-trivial E and D satisfying conditions (i), (ii) and (iii). Since E and D are regular, they generate a foliation of B, respectively of X. The leaves of these foliations are the maximal integral manifolds of E, respectively D. Therefore set-theoretically we can define B' as the space of leaves of E, and X' as the set of leaves of D, together with projections $\Phi : B \to B'$ and $\phi : X \to X'$. However, we cannot always give B' and X' the structure of differentiable manifolds such that Φ and ϕ are submersions (as one says, regular distributions can a priori only locally be factored out). Hence we cannot deduce that Σ is not minimal.

Remark 1. It is clear that minimality of $\Sigma_i(X,W,B,f)$ implies also local minimality of $\Sigma(X,W,B,f)$. The reasoning above shows that if we want to characterize minimality of Σ_i in a differential geometric way, local minimality of Σ is a better candidate than minimality of Σ(however presumably still not strong enough since it is possible that an involutive, but not regular, distribution can be factored out (in a set-theoretical sense)).

Remark 2. In the case of *analytic* systems we should require that E and D as above are *analytic* distributions. Since involutive analytic distributions always define a foliation (also if they do not have constant dimensions), they can be factored out in a set-theoretical sense. Therefore we only have to ask that E and D in Definition 2.26 are involutive. Correspondingly, in Definition 2.25 Φ and ϕ should be analytic, and only surjective.

There is an equivalent way of formulating local minimality in terms of the extended system. For this we need the following

LEMMA 2.27 Let D_1 and D_2 be two regular distributions on a manifold M. Suppose that $D_1 + D_2$ is again regular. Then there exist local coordinates $(x_1,\ldots,x_n)$ for M such that

$$D_1 \cap D_2 = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\} \qquad k \leq n$$

$$D_1 = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_{k+1}},..,\frac{\partial}{\partial x_\ell}\} \qquad k \leq \ell \leq n$$

$$D_2 = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_{\ell+1}},\ldots,\frac{\partial}{\partial x_m}\} \qquad \ell \leq m \leq n$$

PROOF Notice that $D_1 + D_2$ regular is equivalent to $D_1 + D_2$ involutive and $D_1 + D_2$ or $D_1 \cap D_2$ of constant dimension. By factoring out $D_1 \cap D_2$ we may assume that $D_1 \cap D_2 = 0$. Then we refer to RESPONDEK (1982). □

Remark: Sometimes we actually need the following weaker result: there exists a basis $Z_1,\ldots,Z_\ell$ of vectorfields of D_1 such that $[Z_i,D_2] \subset D_2$, $i = 1,..,\ell$. This can in fact be proved for *arbitrary* D_1 and regular D_2 such that D_1, D_2 and $D_1 \cap D_2$ have all constant dimension and $[D_1,D_2] \subset D_1 + D_2$ (see NIJMEIJER (1981), ISIDORI et al.(1981)).

Let us define for two (possibly affine) distributions D_1 and D_2 on a manifold M, $[D_1,D_2]$ as the distribution on M given by $[D_1,D_2]$ = span $\{[Y,Z] | Y \in D_1, Z \in D_2\}$, i.e. the distribution spanned by all Lie-brackets of vectorfields in D_1 and D_2.

THEOREM 2.28 Let $\Sigma(X,W,B,f)$, with $f = (g,h)$, be a nonlinear system and let $\Sigma(B,\Delta^e,W,h)$ be its extended system (Definition 2.24). Then: $\Sigma(X,W,B,f)$ is locally minimal $\Longleftrightarrow$ there does not exist a nonzero regular distribution E on B such that

(i) $E \subset \ker dh$

(ii) $[\Delta^e,E] \subset E+\Delta_0^e$

(iii) $E \cap \Delta_0^e$ has constant dimension

(Recall the definitions of Δ^e and Δ_0^e from Definition 2.24).

PROOF We prove that: {there exist regular distributions E on B and D on X, not both zero, such that (i) $E \subset \ker dh$, (ii) $\pi_* E = D$, (iii) $g_* E \subset \dot{D}$}
$\Longleftrightarrow$ {there exists a nonzero regular distribution E on B such that (i) $E \subset \ker dh$, (ii) $[\Delta^e,E] \subset E+\Delta_0^e$, (iii) $E \cap \Delta_0^e$ has constant dimension}.
($\Longleftarrow$) $[\Delta^e,E] \subset E+\Delta_0^e$ implies that $[\Delta_0^e,E] \subset E+\Delta_0^e$. Therefore Δ_0^e and E are

regular distributions on B such that $\Delta_0^e + E$ is involutive and $\Delta_0^e \cap E$ has constant dimension. By applying Lemma 2.27 we can choose coordinates $(x_1,\ldots,x_n,u_1,\ldots,u_m)$ for B such that

$$\Delta_0^e = \text{span}\{\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_m}\}$$

$$E\cap\Delta_0^e = \text{span}\{\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_\ell}\} \quad , \ell \le m$$

$$E = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k},\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_\ell}\}$$

Furthermore it is easily seen that these coordinates are fiber respecting. Then $D:= \pi_* E = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\}$ is a well-defined distribution on X (we abuse notation by looking at x_i as coordinate functions on X). In these coordinates Δ^e is given by $\Delta^e(x,u) = \sum_{j=1}^{n} g_j(x,u)\frac{\partial}{\partial x_j} + \Delta_0^e(x,u)$, with $g = (g_1,\ldots,g_n)$. $[\Delta^e,E] \subset E + \Delta_0^e$ yields

$$(2.32) \qquad (\frac{\partial g_j}{\partial x_i}) = 0 \quad \begin{matrix} i=1,\ldots,k \\ j=k+1,\ldots,n \end{matrix} \quad \text{and} \quad (\frac{\partial g_j}{\partial u_i}) = 0 \quad \begin{matrix} i=1,\ldots,\ell \\ j=k+1,\ldots,n \end{matrix}$$

However, in these coordinates $g_* E \subset \dot{D}$ is given by the same expressions.

($\Longrightarrow$) Since $\pi_* E = D$ is a regular distribution on X, we can choose coordinates $(x_1,\ldots,x_n)$ for X and $(x_1,\ldots,x_n,u_1,\ldots,u_m)$ for B such that $D = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\}$, and $E = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k},\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_\ell}\}$ with $k \le n$, $\ell \le m$. Because $\Delta_0^e = \text{span}\{\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_m}\}$, this implies $[\Delta_0^e,E] \subset \Delta_0^e + E$.

As above (2.32) $g_* E \subset \dot{D}$ yields that $[\Delta^e,E] \subset E+\Delta_0^e$. □

Remark: In fact it can easily be proved that an affine control system $\Sigma(X,\Delta,Y,h)$ (Definition 2.23) is locally minimal if and only if there does not exist a nonzero regular distribution D on X such that (i) $D \subset \ker dh$, (ii) $[\Delta,D] \subset D+\Delta_0$ and (iii) $D \cap \Delta_0$ has constant dimension (see also Section 2.2.3, Theorem 2.56). Therefore we proved in Theorem 2.28 that $\Sigma(X,W,B,f)$ is locally minimal if and only if $\Sigma^e(X,W,B,f)$ is locally minimal.

Theorem 2.28 enables us to give a conceptual "*algorithm*" to check local minimality. Define $(\Delta^e)^{-1}(E+\Delta_0^e)$ as the distribution spanned by all vector-fields Z on B such that $[\Delta^e,Z] \subset E+\Delta_0^e$. Then define a sequence of distributions $\{E^m\}$, $m = 0,1,\ldots$, by setting

$$E^0 := \ker dh$$
$$E^m := E^{m-1} \cap (\Delta^e)^{-1}(E^{m-1}+\Delta_0^e) \qquad m = 1,2,\ldots$$

We can prove (see NIJMEIJER (1980,1981,Th.4.1), ISIDORI et al(1981 b)) that if $\dim(\ker dh) = k$ then $\lim_{m\to\infty} E^m = E^k$ and if E^k has constant dimension, then E^k is the maximal regular distribution E which satisfies $[\Delta^e,E] \subset E+\Delta_0^e$ and $E \subset \ker dh$. Therefore if $E^k \cap \Delta_0^e$ has constant dimension, then Theorem 2.28 implies that $\Sigma(X,W,B,f)$ is locally minimal if and only if $E^k = 0$.

We now direct our attention to nonlinear input-output systems. First we notice that the definition of equivalence (Definition 2.25) becomes much simpler.

PROPOSITION 2.29 Let $\Sigma_1(X_1,\tilde{B},Y,g_1,h_1)$ and $\Sigma_2(X_2,\tilde{B},Y,g_2,h_2)$ be two nonlinear input-output systems. Choose local fiber respecting coordinates (y,u) for $\tilde{B}$. Choose fiber respecting coordinates (x_1,u) for $h_1{}^*\tilde{B}$ and (x_2,u) for $h_2{}^*\tilde{B}$, output induced by (y,u), such that Σ_1 and Σ_2 are given by

$$\Sigma_1 : \dot{x}_1 = g_1(x_1,u), \quad y = h_1(x_1)$$
$$\Sigma_2 : \dot{x}_2 = g_2(x_2,u), \quad y = h_2(x_2)$$

Then $\Sigma_1 \sim \Sigma_2$ if and only if there exists a diffeomorphism $\phi : X_1 \to X_2$ such that

(i) $\phi_* g_1(x,u) = g_2(\phi(x),u)$ and

(ii) $h_1(x) = h_2 \circ \phi(x)$

If Σ_1 and Σ_2 are *affine* input-output systems, they are locally given by

$$\Sigma_1 : \dot{x}_1 = A^1(x_1) + \sum_{i=1}^m u_i B_i^1(x_1), \qquad y = h_1(x_1)$$

$$\Sigma_2 : \dot{x}_2 = A^2(x_2) + \sum_{i=1}^m u_i B_i^2(x_2), \qquad y = h_2(x_2).$$

Then $\Sigma_1 \sim \Sigma_2$ if and only if there exists a diffeomorphism $\phi : X_1 \to X_2$ such that

(i) $\phi_* A^1 = A^2$, $\phi_* B_i^1 = B_i^2$, $i = 1,\ldots,m$

(ii) $h_1 = h_2 \circ \phi$

We call ϕ the *equivalence mapping*.

PROOF $\Sigma_1 \sim \Sigma_2$ if and only if there exist diffeomorphisms Φ and ϕ such that (2.28) in Definition 2.25 commutes. In particular

$$\begin{array}{ccc} h_1^{*}\tilde{B} & \xrightarrow{\Phi} & h_2^{*}\tilde{B} \\ \tilde{h}_1 \downarrow & & \downarrow \tilde{h}_2 \\ \tilde{B} & \xrightarrow[\mathrm{id}]{} & \tilde{B} \end{array} \quad \text{and} \quad \begin{array}{ccc} h_1^{*}\tilde{B} & \xrightarrow{\Phi} & h_2^{*}\tilde{B} \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X_1 & \xrightarrow[\phi]{} & X_2 \end{array}$$

commute . In the output induced fiber respecting coordinates (x_1,u) for $h_1^{*}\tilde{B}$ and (x_2,u) for $h_2^{*}\tilde{B}$ as above, this yields that $\Phi(x,u) = (\phi(x),u)$. The rest follows easily. □

In the sequel we want to show that for nonlinear input-output systems, local minimality is equivalent to some kind of observability (compare Theorem 1.14 and Corollary 2.3) First we state a preliminary lemma.

LEMMA 2.30 Let $\Sigma(X,\tilde{B},Y,g,h)$ be a nonlinear input-output system. Let $\pi^{-1}(V) \xrightarrow[\phi_1]{} V\times U$ and $\pi^{-1}(V) \xrightarrow[\phi_2]{} V\times U$ be two output induced trivializations of $B = h^{*}\tilde{B}$, resulting in two sets of coordinates (x,u^1) and (x,u^2) for $\pi^{-1}(V)$ and two maps $g^1 : V\times U \to TV$ and $g^2 : V\times U \to TV$. Let P be a regular codistribution on V such that $dh \subset P$. Then (with £ the Lie-derivative):

$$\{\pounds_{g^1(\cdot,u^1)} P\subset P, \forall u^1\in U\} \iff \{\pounds_{g^2(\cdot,u^2)} P\subset P, \forall u^2\in U\}$$

PROOF We know that there exists a map $\tilde{\alpha}$ (output feedback) such that $u^2 = \tilde{\alpha}(h(x),u^1)$ and $g^2(\cdot,\tilde{\alpha}(h(\cdot),u^1)) = g^1(\cdot,u^1)$. Since P is regular we can choose coordinates $x = (x_1,\ldots,x_n)$ for X such that $P = \operatorname{span}\{dx_1,\ldots,dx_k\}$, $k \le n$ (Frobenius). Suppose now $\pounds_{g^2(\cdot,u^2)} P\subset P, \forall u^2\in U$.

Then for $i = 1,\ldots,k$, $\pounds_{g^1(\cdot,u^1)} dx_i = dg_i^1(\cdot,u^1) = dg_i^2(\cdot,\tilde{\alpha}(h(\cdot),u^1)) =$

$$\sum_{j=1}^{n} \frac{\partial g_i^2}{\partial x_j}(\cdot,\tilde{\alpha}(h(\cdot),u^1))dx_j + \sum_{j=1}^{n}\sum_{r=1}^{m}\sum_{\ell=1}^{p} \frac{\partial g_i^2}{\partial u_r}\frac{\partial \tilde{\alpha}_r}{\partial y_\ell}\frac{\partial h_\ell}{\partial x_j}(\cdot)\,dx_j \qquad (2.33)$$

with $g^1 = (g_1^1,\ldots,g_n^1), g^2 = (g_1^2,\ldots,g_n^2), h = (h_1,\ldots,h_p)$ and $\tilde{\alpha} = (\tilde{\alpha}_1,\ldots,\tilde{\alpha}_m)$.

Since $\pounds_{g^2(\cdot,u^2)} P\subset P$, the first term in (2.33) belongs to P. The second term is contained in $dh = \operatorname{span}\{dh_1,\ldots,dh_p\}$ and hence also belongs to P. □

Then we define the following notion of observability:

DEFINITION 2.31 Let $\Sigma(X,\tilde{B},Y,g,h)$ be a nonlinear input-output system. Σ is called *locally weakly observable* if there does not exist a regular codistribution P on X, with $P \neq T^*X$, such that

i) $dh \subset P$ (i.e. $h^*(\tilde{T}^*Y) \subset P$)

ii) $\pounds_{g(\cdot,u)} P \subset P$, $\forall u \in U$

on every output-induced trivialization $\pi^{-1}(V) \simeq V \times U$, with $g : V \times U \to TV$ defined by the trivialization.

Remark: In order to define (local weak) observability we have to choose input-coordinates; i.e. we have to specify the sections u = constant of B. By Lemma 2.30 it follows that every output-induced trivialization yields the same observability properties. Also, output-induced trivializations are natural since they identify the inputs in the fibers of B with the inputs in the fibers of $\tilde{B}$, a part of the external variables. If we take trivializations of B which are *not* output induced then the observability properties will change in general (we apply general feedback, instead of output feedback).

We can state the following, already announced

THEOREM 2.32 Let $\Sigma(X,\tilde{B},Y,g,h)$ be a nonlinear input-output system. Then:
Σ locally minimal $\Longleftrightarrow$ Σ locally weakly observable

PROOF ($\Longrightarrow$) Let (x,u) be output-induced fiber respecting coordinates for $B = h^*\tilde{B}$. Suppose that Σ is not locally weakly observable. Then there exists a regular codistribution $P \neq T^*X$ such that (i) $dh \subset P$, (ii) $\pounds_{g(\cdot,u)} P \subset P$, $\forall u$. Hence D:= ker P is a nonzero regular distribution on X which satisfies (i) $D \subset \ker dh$, (ii) $[g(\cdot,u),D] \subset D$, $\forall u$. Now we can lift D in a unique way to a regular distribution D_ℓ on B, such that the integral manifolds of D_ℓ are contained in the sets $\tilde{h}^{-1}(\bar{y},\bar{u})$, with $(\bar{y},\bar{u}) \in \tilde{B}$. In particular the integral manifolds of D_ℓ are contained in the sections u = constant of B. Then (i) yields $D_\ell \subset \ker \tilde{dh}$ and (ii) implies $g_* D_\ell \subset \dot{D}$. Therefore Σ is not locally minimal.

($\Longleftarrow$) Suppose Σ is not locally minimal. Then there exist regular distributions E on B and D on X such that $g_* E \subset \dot{D}$, $\pi_* E = D$ and $E \subset \ker \tilde{dh}$. It follows from $E \subset \ker \tilde{dh}$, that we can find output-induced fiber respecting coordinates (x,u) for B such that

$$E = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\}\ ,\ k < n$$

Then $D = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\}$ (where we now interpret x_i as functions on X!) and $g_*E \subset \dot{D}$ yields $[g(\cdot,u),D] \subset D, \forall u$. Furthermore $E \subset \ker \tilde{dh}$ implies $D \subset \ker dh$ Then $P := \{\theta \in T^*X \mid \theta(Z)=0, \forall Z \in D\}$ satisfies $dh \subset P$ and $\pounds_{g(\cdot,u)}P \subset P$. Also P is regular and has dimension strictly less than dim X = n. Hence Σ is not locally weakly observable. □

Usually local weak observability is defined in a more intrinsically control-theoretic way (HERMANN-KRENER (1977)), by requiring the following property, which we shall call Property I:

For every $x \in X$ there exists an open neighborhood V of x such that for every $x' \in V$, $x' \neq x$, there exists an input function u such that if we apply u to Σ, then the output functions corresponding to the initial conditions x(0) = x and x(0) = x' are different, while the state space trajectories remain in V.

To understand the connection between Definition 2.31 and Property I we introduce the *observability codistribution*.

DEFINITION 2.33 Let $\Sigma(X,\tilde{B},Y,g,h)$ be a nonlinear input-output system. Let $\pi^{-1}(V) \simeq V\times U$ be an output induced trivialization of $B = h^*\tilde{B}$. Let us denote by $\mathcal{G}$ the linear space of smooth functions on V containing all (finite) linear combinations of functions of the form

$$\pounds_{g(\cdot,u_1)}\pounds_{g(\cdot,u_2)}\cdots\pounds_{g(\cdot,u_k)}h, \text{ with } u_i \in U \text{ arbitrary} \tag{2.34}$$

where $g : V\times U \to TV$ is defined by the trivialization. Then $\mathcal{G}$ generates a codistribution O on V by setting

$$O(x) := \text{span}\{dk(x) \mid k \in \mathcal{G}\},\ x \in V. \tag{2.35}$$

This codistribution is called the *observability codistribution*.

The following proposition shows that the definition of the observability codistribution is *independent* of the output-induced trivialization of $\pi^{-1}(V)$ (compare Lemma 2.30).

PROPOSITION 2.34 Let $\pi^{-1}(V) \xrightarrow{\phi_1} V\times U$ and $\pi^{-1}(V) \xrightarrow{\phi_2} V\times U$ be two output induced trivializations, resulting in two sets of coordinates (x,u^1) and (x,u^2) for $\pi^{-1}(V)$ and two maps $g^1 : V\times U \to TV$ and $g^2 : V\times U \to TV$. Define

for both trivializations the observability codistributions 0^1, respectively 0^2 on V. Then $0^1 = 0^2$.

PROOF We know that there exists a map $\tilde{\alpha}$ (output feedback) such that $u^2 = \tilde{\alpha}(h(x),u^1)$ and $g^2(\cdot,\tilde{\alpha}(h(\cdot),u^1)) = g^1(\cdot,u^1)$. Therefore $\pounds_{g^1(\cdot,u^1)} h = \pounds_{g^2(\cdot,\tilde{\alpha}(h(\cdot),u^1))} h$ and hence $d\pounds_{g^1(\cdot,u^1)} h = d(\pounds_{g^2(\cdot,\tilde{\alpha}(h(\cdot),u^1))} h) =$

$$d(\pounds_{g^2(\cdot,u^2)} h) + \frac{\partial g^2}{\partial u^2}(\cdot,u^2)\,\frac{\partial \tilde{\alpha}}{\partial y}\,dh$$

($\frac{\partial \tilde{\alpha}}{\partial y}$ means differentiation w.r.t. to the first components).

Since $\frac{\partial g^2}{\partial u^2}(\cdot,u^2)\,\frac{\partial \tilde{\alpha}}{\partial y}\,dh$ is contained in dh, it follows that $d\pounds_{g^1(\cdot,u^1)} h \in 0^2$.

By induction to the number of Lie-derivatives in (2.34) it is easy to prove that $0^2 \subset 0^1$. By the same argument $0^1 \subset 0^2$. Hence $0^1 = 0^2$. □

Remark: The vectorspace G *does* depend on the trivialization.

It also follows from Proposition 2.34 that the observability codistribution 0 is *globally* defined (we can take an *atlas* of trivializing charts V). Moreover 0 is *involutive* and is the *smallest* codistribution that contains dh and is invariant under g, i.e. $\pounds_{g(\cdot,u)} 0 \subset 0$ for every u, on an output induced trivialization. However 0 does not necessarily have *constant* dimension. Therefore we go outside the *regular* category in which we worked so far. The connections between Definition 2.31, Property I and 0 are explained in

THEOREM 2.35 Let $\Sigma(X,\tilde{B},Y,g,h)$ be a nonlinear input-output system. Then:

(i) $\dim 0(x) = \dim X \quad \forall x \in X \Longrightarrow \Sigma$ has property I

(ii) Σ has property I $\Longrightarrow \dim 0(x) = \dim X$ for x in an open and dense subset of X

(iii) $\dim 0(x) = \dim X$ for a certain $x \in X \Longrightarrow \Sigma$ is locally weakly observable.

PROOF (i) and (ii) are proved in HERMANN-KRENER (1977). The proof of (iii) is given as follows: Let Σ be not locally weakly observable. Then there exists a regular codistribution P satisfying the conditions of Definition 2.31, and $P \neq T^*X$. It is clear that $0(x) \subset P(x)$ for every $x \in X$. Hence $\dim 0(x) < \dim X$, $\forall x \in X$. □

Remark 1. If O has constant dimension then: Σ has property I $\Longleftrightarrow$ dim O = dim X $\Longleftrightarrow$ Σ is locally weakly observable $\Longleftrightarrow$ Σ locally minimal.

Remark 2. An interesting question is the following: Let dim O(x) < dim X for every $x \in X$. Does this imply that Σ is not locally minimal? In the *analytic* case the answer is affirmative (see NIJMEIJER (1982 b)). Since O is in this case a codistribution generated by analytic functions, O has constant dimension on an open and dense subset of X. Define D as the *analytic* annihilator of O. Then D has constant dimension and hence is regular. Moreover it satisfies $[g(\cdot,u),D] \subset D$, $\forall u \in U$ and $D \subset \ker dh$.

Remark 3. In the analytic case Property I and dim O(x) = dim X for every $x \in X$ are equivalent if Σ is also locally weakly controllable (see Definition 2.53, HERMANN-KRENER (1977)).

Remark 4. Of course, if dim $O(x_0) = k$, then dim $O(x) \geq k$ for every x in some open neighborhood of x_0.

For nonlinear systems which have a local input-output representation with feedthrough term, as defined in Case I, equation (2.24), we can also define some kind of observability and show that it is equivalent to local minimality.

DEFINITION 2.36 Let $\Sigma(X,W,B,f)$, with $f = (g,h)$, be a nonlinear system such that h restricted to the fibers is an immersion. Then Σ is called *locally distinguishable* if there does not exist a regular codistribution P on X, unequal to T^*X, such that for every local input-output representation with feedthrough term

$$\begin{aligned} \dot{x} &= g(x,u) \\ y &= \bar{h}(x,u) \qquad w = (y,u) \end{aligned} \tag{2.36}$$

P satisfies

(i) $\quad f_{g(\cdot,u)}P \in P$

(ii) $\quad d_x\bar{h}(\cdot,u) \subset P$

($d_x\bar{h}(\cdot,u)$ are the one-forms on X obtained by differentiating $\bar{h}(x,u)$ with respect to x).

Analogously to Lemma 2.30 we can prove that conditions (i) and (ii) do *not* depend on the coordinates (x,v) for B and (y,u) for W such that h is

given as $h(x,v) = (\bar{h}(x,v),v)$. We obtain

THEOREM 2.37. Let $\Sigma(X,W,B,f)$, with $f = (g,h)$, be a nonlinear system, such that h restricted to the fibers is an immersion. Then:

$$\Sigma \text{ locally minimal} \iff \Sigma \text{ is locally distinguishable}$$

PROOF ($\Longrightarrow$) Let P be a regular codistribution on X which satisfies $\pounds_{g(\cdot,u)}P \subset P$ for all u, and $d_x\bar{h}(\cdot,u) \in P$ for all u for every local input-output representation with feedthrough term of Σ. Notice that because h restricted to the fibers is an immersion, ker dh is a distribution on B such that $\Delta_0^e \cap \ker dh = 0$. Furthermore it is easy to see that $\pi_{*(x,u)}\ker dh(x,u) \supset D(x)$, for every $(x,u) \in B$, where D is the regular distribution $D = \ker P$. Therefore we can lift D to a regular distribution D_ℓ on B contained in ker dh. Then $g_*D_\ell \subset \dot{D}$ and $D_\ell \subset \ker dh$. By local minimality of Σ this implies that D_ℓ and D are zero, or $P(x) = T_x^*X$, for every $x \in X$.

($\Longleftarrow$) If Σ is not locally minimal, then there exist nonzero regular E and D such that $g_*E \subset \dot{D}$, $\pi_*E = D$ and $E \subset \ker dh$. If we take coordinates (x,v) for B and (y,u) for W such that $h(x,v) = (\bar{h}(x,v),v)$, this implies that $D \subset d_x\bar{h}(\cdot,v)$, and $[g(\cdot,v),D] \subset D$ for all v. Hence Σ is not locally distinguishable. □

Analogous to Definition 2.33,we can also define an observability codistribution for nonlinear systems with local input-output representations with feedthrough term.

DEFINITION 2.38 Let $\Sigma(X,W,B,f)$ be a nonlinear system, with $f = (g,h)$, such that h restricted to the fibers is an immersion. Take coordinates $w=(y,u)$ for W and (x,u) for B such that $h(x,u) = (\bar{h}(x,u),u)$. In these coordinates the extended system $\Sigma^e(X,W,B,f)$ is given by

$$\begin{aligned} \dot{x} &= g(x,u) \\ \dot{u} &= v \qquad \text{with v the new input} \\ w &= h(x,u) \end{aligned} \tag{2.37}$$

Define G^e as the linear space of functions on this coordinate neighborhood V of B containing all finite linear combinations of functions of the form

$$\pounds_{Z_1}\pounds_{Z_2}\cdots\pounds_{Z_k}h \tag{2.38}$$

with $Z_i = g(x,u)\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial u_j}$, $j = 1,\ldots,m$.

Define the codistribution 0^e on V by setting

(2.39) $\quad 0^e(x,u) := \{\text{span}\{dk(x,u) \mid k \in G^e\}$

This codistribution is called the *(extended) observability codistribution.*

Analogous to Proposition 2.34 we can prove that 0^e does *not* depend on which coordinates for B and (y,u) for W we take as long as $h(x,u) = (\bar{h}(x,u),u)$. Notice that the coordinates for B which are allowed are in a certain sense output-induced. Thus 0^e is a *globally* defined codistribution on B, which is involutive but does not necessarily have constant dimension. We derive the following

PROPOSITION 2.39 Let $\Sigma(X,W,B,f)$ with $f = (g,h)$ and let h restricted to the fibers be an immersion. Then: dim $0^e(x,u)$ = dim B for a certain $(x,u) \in B$ $\Longrightarrow$ Σ is locally distinguishable ($\Longleftrightarrow$ Σ is locally minimal). Moreover if dim $0^e(x,u)$ = constant, then: dim 0^e = dim B $\Longleftrightarrow$ Σ locally distinguishable ($\Longleftrightarrow$ Σ locally minimal).

PROOF Suppose that Σ is not locally minimal. Then there exist regular E and D, with $E \neq 0$, such that $\pi_* E = D$, $g_* E \subset \dot{D}$ and $E \subset \ker dh$. From these properties it follows that $E \subset \ker 0^e$ and hence dim $0^e(x,u) <$ dim B everywhere. By Theorem 2.37, Σ locally distinguishable $\Longleftrightarrow$ Σ locally minimal. □

Analogous to Chapter 1, Definition 1.13, we shall now define a kind of *uniform* observability. First we deal with nonlinear input-output systems.

DEFINITION 2.40 Let $\Sigma(X,\tilde{B},Y,g,h)$ be a nonlinear input-output system. Σ is called *uniformly locally weakly observable* if there does not exist a regular codistribution P on X, with $P \neq T^*X$, such that for every trivializing chart V of $h^*\tilde{B}$ there exists an output-induced trivialization $\pi^{-1}(V) \simeq V \times U$ ($\pi: h^*\tilde{B} \to X$ projection) such that for at least one $\bar{u} \in U$

i) $\pounds_{g(\cdot,\bar{u})} P \subset P$

ii) $dh \subset P$

with $g : V \times U \to TV$ defined by the trivialization

Remark 1. Locally we require that the *autonomous* systems

$$\text{(2.40)} \qquad \begin{aligned} \dot{x} &= g(x,\bar{u}) \\ y &= h(x) \end{aligned} \qquad \bar{u} \in U,\ \bar{u} \text{ fixed}$$

corresponding to an output-induced trivialization are locally weakly observable for *every* $\bar{u} \in U$.

Remark 2. Let $\pi^{-1}(V) \xrightarrow{\phi_1} V\times U$ be a local trivialization which gives rise to the system $\dot{x} = g(x,u)$, $y = h(x)$. If $\pi^{-1}(V) \xrightarrow{\phi_2} V\times U$ is another trivialization, then this corresponds to output feedback $v = \alpha(h(x),u)$. In the same way as in Lemma 2.30, we can prove that if the autonomous systems $\dot{x} = g(x,u)$, $y = h(x)$, $\bar{u} \in U$, corresponding to the first trivialization are all locally weakly observable, then the autonomous systems corresponding to the second trivialization are also locally weakly observable.

If $\Sigma(X,W,B,f)$ has only local input-output representations with feedthrough term, we give

DEFINITION 2.41 Let $\Sigma(X,W,B,f)$, with $f = (g,h)$ be a nonlinear system, and let h restricted to the fibers be an immersion. Σ is called *uniformly locally distinguishable* if there does not exist a regular codistribution P on X, with $P \neq T^*X$, such that for every coordinate neighborhood of B, with coordinates (x,v) for B and (y,u) for W such that $h(x,v) = (\bar{h}(x,v),v)$ there exists at least one $\bar{u}$ with the property that in these coordinates

i) $\mathfrak{L}_{g(\cdot,u)}P \subset P$

ii) $d_x h(\cdot,\bar{u}) \subset P$.

Remark: Locally we require that the *autonomous* systems

$$\begin{aligned} \dot{x} &= g(x,\bar{u}) \\ y &= h(x,\bar{u}) \end{aligned}$$

are locally weakly observable for *every* $\bar{u}$. If (x',v') and (y',u') are other coordinates for the same neighborhood such that still $h(x',v') = (\bar{h}(x',v'),v')$ then local weak observability of $\dot{x} = g(x,\bar{u})$, $y = \bar{h}(x,\bar{u})$ for all $\bar{u}$, implies local weak observability of $\dot{x}' = g'(x',\bar{u}')$, $y' = \bar{h}'(x',\bar{u}')$ for all $\bar{u}'$ (again change of coordinates corresponds to output feedback).

We recall from Chapter 1 that a minimal (past externally induced) realization Σ_i of a general external system Σ_e is uniformly observable if and only if all minimal realizations of Σ_e are equivalent (Theorem 1.14). The

following conjecture is therefore reasonable. Let $\Sigma(X,W,B,f)$ be a system with a local input-output representation with feedthrough term. Then Σ is uniformly locally distinguishable if and only if all locally minimal realizations $\Sigma'(X',W',B',f')$ are equivalent (see Definition 2.25).

However, making conjectures in this context is much easier than proving them, and only in Section 2.2.3 we shall briefly return to the notion of uniform observability.

Finally we want to define "controllability" for nonlinear systems. There are several possibilities but the notion which we shall use is *strong accessibility*. Our choice is motivated by the following arguments:
(i) Strong accessibility can be defined in the same differential geometric style as local minimality and local weak observability.
(ii) The definition of strong accessibility is closest to the definition of controllability for linear systems (see after Corollary 2.3).
(iii) Strong accessibility is "dual" to local weak observability or local distinguishability, in a sense which will be made clear in Chapter 3, Theorems 3.19 and 3.31.
However, we remark that also the notion of *local weak controllability* (definition 2.53) has a direct differential geometric interpretation and has its own merits.

First we define strong accessibility for *affine* control systems (in a way which is slightly different from SUSSMANN & JURDJEVIC (1972)).

DEFINITION 2.42 Let $\Sigma(X,\Delta,Y,h)$ be an affine control system. Σ is called *strongly accessible* if there does not exist a regular distribution D, unequal to TX, such that

i) $\Delta_0 \subset D$ $\qquad (\Delta_0 := \Delta - \Delta)$

ii) $[\Delta,D] \subset D$ $\qquad$ (D is invariant under Δ).

Remark: If $D \neq 0$, then $[\Delta,D] \subset D$ already implies $\Delta_0 \subset D$.

As in the case of local weak observability, strong accessibility is usually defined in a more control theoretic way by requiring the following property, which we shall call Property II:

Let $\dot{x} = g(x,u)$ be the state space equations (in local coordinates) of a nonlinear system $\Sigma(X,W,B,f)$. Define $R(T,x_0)$ as the set of points that are reachable from x_0 in exactly time T:

$R(T,x_0) := \{x_1 \in X \mid \exists$ state trajectory $x(t)$, generated by g, for a certain input function u, such that $x(0) = x_0$ and $x(T) = x_1\}$.
Then Σ has property II if for all $x_0 \in X$ and for all $T > 0$ the set $R(T,x_0)$ has a nonempty interior.

The connection between Definition 2.42 and Property II is provided by the *controllability distribution*:

DEFINITION 2.43 Let $\Sigma(X,\Delta,Y,h)$ be an affine control system with $\Delta(x) = A(x) + \text{span}\{B_1(x),\ldots,B_m(x)\}$. Define $\bar{F}$ as the linear space of vectorfields on X containing all linear combinations of the elements $ad_A^i B_j$, $i = 0,1,\ldots$, $j = 1,\ldots,m$, and their Lie-brackets ($ad_A^i B_j$ is recursively defined by $ad_A^0 B_j = B_j$ and $ad_A^{i+1} B_j = [A,ad_A^i B_j]$, $i \geq 0$). Furthermore, define the distribution C by setting $C(x) = \text{span}\{Z(x) \mid Z \in \bar{F}\}$.

It can be readily seen that C does not depend on the way we represent Δ as $\Delta(x) = A(x) + \text{span}\{B_1(x),\ldots,B_m(x)\}$ (contrary to $\bar{F}$, which *does* depend on the representation). Hence C is a globally defined distribution on X. C is involutive but not necessarily of constant dimension. It is clear that C is the smallest involutive distribution containing Δ_0 and invariant under Δ (i.e. $[\Delta,C] \subset C$).

We derive (compare Theorem 2.35)

THEOREM 2.44 Let $\Sigma(X,\Delta,Y,h)$ be an affine control system. Then:
i) $\dim C(x) = \dim X$, $\forall x \in C \Longrightarrow$ Σ has property II
ii) Σ has property II $\Longrightarrow$ $\dim C(x) = \dim X$ for x in an open and dense subset of X
(iii) $\dim C(x) = \dim X$ for a certain $x \in X \Longrightarrow$ Σ is strongly accessible.

PROOF i) and ii) can be proved by easy adaptations of the arguments of HERMANN & KRENER (1977), where these statements are proved for local weak controllability. To prove iii) let us assume that Σ is not strongly accessible. Then there exists a regular distribution $D \neq TX$ as in Definition 2.42. It is clear that $C(x) \subset D(x)$, for every $x \in X$. Therefore $\dim C(x) < \dim X$, for every $x \in X$. □

Remark 1. If C has *constant* dimension then:
Σ has property II $\Longleftrightarrow$ $\dim C = \dim X$ $\Longleftrightarrow$ Σ is strongly accessible.

Remark 2. (compare Remark 2 after Theorem 2.35) It is not totally clear

when dim C(x) < dim X, for every $x \in X$, implies that Σ is *not* strongly accessible.

Remark 3. In the analytic case Property II implies that dim C(x) = dim X for every x.

For general nonlinear systems we define strong accessibility in the following way.

DEFINITION 2.45 Let $\Sigma(X,W,B,f)$, with $f = (g,h)$, be a nonlinear system. Σ is called *strongly accessible* if there does not exist a regular distribution D on X which satisfies (Recall the definition of $\dot{D}$ after Definition 2.25)

$$g_*(\pi_*^{-1}(D)) \subset \dot{D}$$

Of course we have to prove that for affine control systems Definitions 2.43 and 2.45 are equivalent. We shall prove more, by using the *extended* system (Definition 2.24).

THEOREM 2.46 Let $\Sigma(X,W,B,f)$ be a nonlinear system. Then $\Sigma(X,W,B,f)$ is strongly accessible (Definition 2.45) if and only if its extended system $\Sigma^e(X,W,B,f)$, an affine control system, is strongly accessible (Definition 2.43). If $\Sigma(X,W,B,f)$ is already an affine control system, then Σ is strongly accessible in the sense of Definition 2.43 if and only if Σ is strongly accessible in the sense of Definition 2.45.

PROOF Take fiber respecting coordinates (x,v) for B. The state space equations of Σ are given by

$$\dot{x} = g(x,v)$$

while Σ^e (the extended system) is given by

$$\begin{aligned} \dot{x} &= g(x,v) \\ \dot{v} &= u \qquad\qquad u \text{ the new input.} \end{aligned}$$

If $\Sigma^e(X,W,B,f)$ is not strongly accessible, then there exists a regular $E \neq TB$ on B such that $\Delta_0^e \in E$ and $[\Delta^e,E] \subset E$. Hence $D := \pi_* E$ is a regular distribution on X, and in fact $E = \pi_*^{-1}(D)$. In the same way as in Theorem 2.28 we can prove that $[\Delta^e,E] \subset E$ implies that $g_*(\pi_*^{-1}(D)) \subset \dot{D}$. Hence Σ is not strongly accessible. Conversely, let Σ be not strongly accessible. Then there exists a regular $D \neq TX$ such that $g_*(\pi_*^{-1}(D)) \subset \dot{D}$. This implies

$[\Delta^e, \pi_*^{-1}(D)] \subset \pi_*^{-1}(D)$, and hence Σ^e is not strongly accessible.

To prove the last statement we make the following observations. Let $\Sigma(X,W,B,f)$ be already affine, i.e. state space equations

$\dot{x} = A(x) + \sum_{i=1}^{m} v_i B_i(x)$ with $\Delta_0(x) = \text{span}\{B_1(x),\ldots,B_m(x)\}$ and

$\Delta(x) = A(x) + \Delta_0(x)$.

Then the extended system Σ^e is given by

$$\dot{x} = A(x) + \sum_{i=1}^{m} v_i B_i(x)$$
$$\dot{v} = u$$

i.e. $\Delta_0^e(x,v) = \text{span}\{\frac{\partial}{\partial v_1},\ldots,\frac{\partial}{\partial v_m}\}$ and

$$\Delta^e(x,v) = \sum_{k=1}^{n} (A^k(x) \frac{\partial}{\partial x_k} + \sum_{i=1}^{m} v_i B_i^k(x) \frac{\partial}{\partial x_k}) + \Delta_0^e(x,v)$$

(with A^k and B_i^k the components of A and B_i).

Let now D be a regular distribution on X such that $\Delta_0 \subset D$ and $[\Delta,D] \subset D$. Then $\pi_*^{-1}(D)$ satisfies $\Delta_0^e \subset \pi_*^{-1}(D)$ and $[\Delta^e, \pi_*^{-1}(D)] \subset \pi_*^{-1}(D)$. Conversely, if E is a regular distribution on B such that $\Delta_0^e \subset E$, then $D := \pi_* E$ is a regular distribution on X. If E satisfies $[\Delta^e,E] \subset E$, then

$$[\sum_{k=1}^{n} A^k(x) \frac{\partial}{\partial x_k} + \sum_{i=1}^{m} v_i B_i^k(x) \frac{\partial}{\partial x_k}, E] \subset E$$

and we obtain $[A,D] \subset D$ and $B_i \in D$, $i = 1,\ldots,m$. Hence $[\Delta,D] \subset D$ and $\Delta_0 \subset D$. □

Since a general nonlinear system is strongly accessible if and only if its extended system is strongly accessible we can characterize strong accessibility by the controllability distribution of its extended system:

DEFINITION 2.47 Let $\Sigma(X,W,B,f)$ be a nonlinear system. Let (x,v) be fiber respecting coordinates for B, such that the state space equations of Σ^e are given by

$$\dot{x} = g(x,v)$$
$$\dot{v} = u$$

Define F^e as the linear space over $\mathbb{R}$ of vectorfields on B containing all

linear combinations of the vectorfields $ad^i_{g(x,v)\frac{\partial}{\partial x}} \frac{\partial}{\partial v_j}$, $i = 0,1,\ldots$, $j = 1,\ldots,m$ and their Lie-brackets.

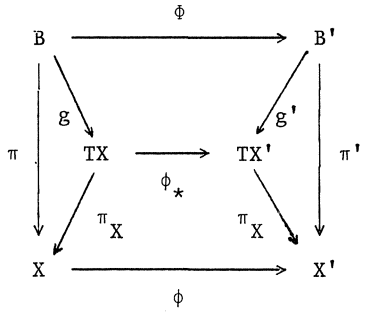
Furthermore define the (*extended*) *controllability distribution* C^e on B by setting $C^e(x,v) = \text{span}\ \{ Z(x,v) \mid Z \in F^e \}$.

Again C^e does *not* depend on the representation of $\Sigma^e(X,W,B,f)$, i.e. C^e does not depend on the choice of fiber respecting coordinates for B. Statement iii) of Theorem 2.44 also holds for C^e, and for statements i) and ii) we remark that is has been proved (VAN DER SCHAFT (1982c)) that if $\Sigma^e(X,W,B,f)$ has property II, then also $\Sigma(X,W,B,f)$ has property II. Conversely if $\Sigma(X,W,B,f)$ has property II, and the fibers of B are *connected*, then also $\Sigma^e(X,W,B,f)$ has property II.

Just like local minimality has a global counterpart-minimality-, there is also a global version of strong accessibility. For this we introduce the notion of a *quotient system*.

DEFINITION 2.48 Let $\Sigma(X,W,B,f)$ and $\Sigma'(X',W',B',f')$ be nonlinear systems with $f = (g,h)$ and $f' = (g',h')$. $\Sigma'(X',W',B',f')$ is called a *quotient system* of $\Sigma(X,W,B,f)$ if there exist surjective submersions Φ and ϕ such that

(2.41)



commutes.

The above definition is reminiscent of the definition of minimality (Definition 2.25). The difference is that we do not require compatibility of h and h'. Now if $\Sigma'(X',W',B',f')$ is such that $B' = X'$ (such a system is called *autonomous*, see also Definition 1.9), then the regular distribution $D := \{Z \in TX \mid \phi_* Z = 0\}$ satisfies $g_*(\dot{\pi}_*^{-1}(D)) \subset \dot{D}$ and Σ is not strongly accessible (if $\dim X' \geq 1$).

We saw in Theorem 2.44 that the strong accessibility properties for an affine control system (or affine input-output system; this makes no

difference since we only consider the state space equations), are characterized by the linear space of vectorfields $\bar{F}$ or the distribution C, given by $\bar{F}$. Furthermore, by Theorem 2.46, in the case of a general nonlinear system strong accessibility is determined by $\bar{F}^e$ or C^e of its extended (affine) system. Especially if $\bar{F}$ and C do not have constant dimension, it is of interest to go into more detail about the construction of $\bar{F}$ and C (and similarly $\bar{F}^e$ and C^e). Let therefore

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x) \tag{2.42}$$

be a local representation of an affine distribution Δ(with $\Delta_0(x)$:= span $\{B_1(x),\ldots,B_m(x)\}$, $\Delta(x) = A(x) + \Delta_0(x)$). Given vectorfields $Z_1,\ldots,Z_k$ on X we denote by $(Z_1,\ldots,Z_k)$ the linear subspace of V(X), i.e. the linear space (over $\mathbb{R}$) of all vectorfields on X, spanned by $Z_1,\ldots,Z_k$.

CONSTRUCTION 2.49 Define $F_0 := (B_1,\ldots,B_m)$ and $\Gamma := A + (B_1,\ldots,B_m)$ (an affine subspace of V(X)). Define furthermore $F_k := [\Gamma,F_{k-1}] + F_{k-1}$, $k \geq 1$. Then $F_k \subset F_{k+1}$, $\forall k \geq 0$. Define $\bar{F} := \bigcup_{k\geq 0} F_k$.

It follows from the Jacobi-identity, that $\bar{F}$ is a Lie subalgebra of V(X). This is proved as follows. An element in $\bar{F}$ consists of linear combinations of expressions of the form

$$* \qquad [Z_1,[Z_2,\ldots[Z_k,B_i]..]] \qquad i = 1,\ldots,m$$

with Z_j, $j = 1,\ldots,k$, equal to A or B_i, $i = 1,\ldots,m$.
Consider now the Lie-bracket of two such expressions:

$$** \qquad [[Z_1,[Z_2,\ldots,[Z_k,B_i]\ldots]],[Y_1,[Y_2,\ldots[Y_\ell,B_i]\ldots]]]$$

This is by the Jacobi-identity equal to:

$$[Z_1,[[Z_2,\ldots,[Z_k,B_i]\ldots]],[Y_1,[Y_2,\ldots,[Y_\ell,B_i]..]]]]$$

$$- [[Z_2,\ldots,[Z_k,B_i]..],[Z_1,[Y_1,[Y_2,\ldots[Y_\ell,B_i]..]]]].$$

Therefore by repeated use of the Jacobi-identity, it follows that we can write ** as a linear combination of expressions of the form *.

It is now clear that $\bar{F}$ is the same as $\bar{F}$ defined in Definition 2.43. Notice that $\bar{F}$ can be characterized in the following two ways. $\bar{F}$ is the

smallest linear subspace (over $\mathbb{R}$) of V(X) containing $B_1,\ldots,B_m$ and invariant under taking Lie-brackets with respect to A and $B_1,\ldots,B_m$. On the other hand $\mathcal{F}$ is also the smallest Lie subalgebra of V(X) containing $B_1,\ldots,B_m$ and invariant under A (or equivalently, invariant under A and $B_1,\ldots,B_m$). It follows that the controllability distribution C can also be characterized in two ways. C is the smallest distribution that contains $B_1,\ldots,B_m$ and is invariant under taking Lie-brackets with $A,B_1,\ldots,B_m$, *and* C is the smallest *involutive* distribution containing $B_1,\ldots,B_m$ and invariant under A.

The distribution C can be also constructed in the following way:

CONSTRUCTION 2.50 Let Δ_0 and Δ be the distribution, respectively affine distribution as above. Define $\Delta_k = [\Delta,\Delta_{k-1}]$, $k \geq 1$. Then $\Delta_k \subset \Delta_{k+1}$, $k \geq 0$. Define $C(x) = \bigcup_{k\geq 0} \Delta^k(x)$.

Again it follows from the Jacobi-identity that C is involutive. Then it is clear that C equals C as defined in Definition 2.43. Notice that if dim X = n, then $\Delta^k(x) = \Delta^{n-1}(x)$ for all $k \geq n-1$ (cf. NIJMEIJER (1980, 1982 a)). This constitutes a striking difference with Construction 2.49, where F_k may be bigger than F_{k-1} for arbitrary large k. Note also that the Δ_k are independent of the representation (2.42) of Δ, while the F_k are *dependent*.

We can also give analogous constructions for $\mathcal{G}$ and the observability codistribution O (Definition 2.33), and similarly for $\mathcal{G}^e$ and O^e (Definition 2.38). Let $h : X \to Y$ and take coordinates for Y, such that the coordinate functions of h are $h_1,\ldots,h_p$. Given functions $r_1,\ldots,r_k$ on X, we denote by $(r_1,\ldots,r_k)$ the linear subspace of C(X), i.e. the linear space (over $\mathbb{R}$) of all C^∞ functions on X, spanned by $r_1,\ldots,r_k$.

CONSTRUCTION 2.51 Define $G_0 := (h_1,\ldots,h_p)$, and $G_k := L_\Gamma G_{k-1} + G_{k-1}$, $k \geq 1$, with Γ as above. Then

$$G_k \subset G_{k+1},\ k \geq 0. \text{ Define } \mathcal{G} := \bigcup_{k\leq 0} G_k$$

Notice that $\mathcal{G}$ is the smallest linear subspace of C(X) that is invariant under $A,B_1,\ldots,B_m$ and contains $h_1,\ldots,h_p$. The observability codistribution O can be also constructed in the following way.

CONSTRUCTION 2.52 Let D_0 be the codistribution defined by $D_0(x) :=$ span $\{dh_1(x),\ldots,dh_p(x)\}$. Define the sequence of codistributions ($k \geq 1$):

$$D_k(x) := (\mathcal{L}_A D_{k-1})(x) + (\mathcal{L}_{B_1} D_{k-1})(x) + \ldots + (\mathcal{L}_{B_m} D_{k-1})(x) + D_{k-1}(x)$$

Then $D_k \subset D_{k+1}$, $k \geq 0$. Define $O(x) = U_{k\geq 0} D_k(x)$.

Notice that Construction 2.51 depends on the representation (2.42) of Δ as well as on the coordinatization of Y. Construction 2.52 is independent of the coordinatization of Y, but still dependent on the representation (2.42).

For completeness we give for affine systems (see HERMANN & KRENER (1977) for more information).

DEFINITION 2.53 An affine (control or input-output) system with affine distribution satisfies the rank condition for *local weak controllability* if span $\{C(x),\Delta(x)\} = T_x X$ for every $x \in X$ (i.e. if $\Delta(x) = A(x) + \Delta_0(x)$, this amounts to span $\{A(x),C(x)\} = T_x X$).

Finally we stress the striking similarity in the conditions of "minimality" and "controllability" for linear and nonlinear systems. A nonlinear system $\Sigma(X,W,B,f)$, with $f = (g,h)$ is locally minimal if there do not exist regular distributions E and D such that (i) $E \subset \ker dh$, (ii) $\pi_* E = D$, (iii) $g_* E \subset \dot{D}$, with E nonzero.

In the linear case D is replaced by a linear subspace $V \subset X$, and E is a linear subspace of the form $T + U' \subset X \times U$, with T of the form $T = \{(x,Fx) \mid x \in V$, for a $F : X \to U\}$ and U' a linear subspace of U. Hence condition (ii) is satisfied. Condition (i) amounts to (with $h(x,u) = Cx + Du$) $T \subset \text{Ker } [C:D]$ and $U' \subset \text{Ker } D$, while (iii) yields (with $g(x,u) = Ax + Bu$) $[A:B]T \subset V$ and $BU' \subset V$. Since $T = \{(x,Fx) \mid x \in V\}$, the first inclusion equals $(A+BF)V \subset V$. Hence if $V \neq 0$, then V is an output nulling subspace, and therefore $\Sigma(A,B,C,D)$ is not minimal, and if $V = 0$, but $U' \neq 0$, then, since $BU' \subset V = 0$, $U' \subset \text{Ker } B \cap \text{Ker } D$, and hence $\Sigma(A,B,C,D)$ is also not minimal (Theorem 2.2).

For strong accessibility we look at regular distributions D such that $g_*(\pi_*^{-1}(D)) \subset \dot{D}$. In the linear case D is again replaced by $V \subset X$, and $\pi_*^{-1}(D)$ becomes $V \times U \subset X \times U$. Then $g_*(\pi_*^{-1}(D)) \subset \dot{D}$ yields $[A:B]V \times U \subset V$, or $AV + \text{Im}B \subset V$. This is equivalent to $AV \subset V$ and $\text{Im}B \subset V$. If $V \subsetneq X$, then (A,B) is not controllable (see after Corollary 2.3).

2.2.3 Controlled invariance

The conditions for local minimality and strong accessibility which we gave in the previous section are directly related to the notion of controlled invariance. This concept has proved to be a basic tool in the solution of various control and synthesis problems (WONHAM (1979)). We shall concentrate on its connections with the preceding pages (for more information see ISIDORI et al(1981 a), HIRSCHORN (1981) and NIJMEIJER & VAN DER SCHAFT (1982 b)).

Consider a nonlinear system $\Sigma(X,W,B,f)$, in local coordinates given by $\dot{x} = g(x,u)$, $w = h(x,u)$. We shall only look at the state space equations $\dot{x} = g(x,u)$. A regular distribution D on X is said to be *invariant* with respect to g (in this trivialization of B!), if $[g(\cdot,u),D] \subset D$, for every u. Of course this means that the integral flow of $\dot{x} = g(x,u)$ leaves the foliation of X, induced by D, invariant. D is called *controlled invariant* if there exists a C^∞ feedback $v = \alpha(x,u)$ (with $\frac{\partial \alpha}{\partial u}$ nonsingular) such that D is invariant with respect to the modified dynamics $\dot{x} = \tilde{g}(x,v)$, where $\tilde{g}(x,\alpha(x,u)) := g(x,u)$, i.e. $[\tilde{g}(\cdot,v),D] \subset D$, for every v.

As we saw before, locally feedback can be interpreted as the choice of a local trivialization of B. Therefore we arrive at the following

DEFINITION 2.54 Let $\Sigma(X,W,B,f)$, with $f = (g,h)$, be a nonlinear system. Let D be a regular distribution on X. D is called *locally controlled invariant* if for every $x \in X$ there exists an open neighborhood V of x and a trivialization $\pi^{-1}(V) \simeq V \times U$ such that $[g(\cdot,u),D] \subset D$, for every $u \in U$, with $g : V \times U \to TV$ corresponding to the trivialization.

Remark: Later on we shall also define *global controlled invariance*.

We now give necessary and sufficient conditions for a distribution to be locally controlled invariant.

THEOREM 2.55 Let $\Sigma(X,W,B,f)$, with $f = (g,h)$, be a nonlinear system, and let D be a regular distribution on X. Assume that $\Delta_0^e \cap g_*^{-1}(\dot{D})$ has constant dimension. Then D is locally controlled invariant if and only if

$$g_*(\pi_*^{-1}(D)) \subset \dot{D} + g_*(\Delta_0^e).$$

PROOF Let $\pi^{-1}(V) \simeq V \times U$ be a local trivialization, and let $(x,u) = (x_1,\ldots,x_n,u_1,\ldots,u_m)$ be corresponding fiber respecting coordinates for $\pi^{-1}(V)$. Since D is regular, we can also take the coordinates $x = (x_1,\ldots,x_n)$

such that $D = \text{span}\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\}$, $k \leq n$.

Write correspondingly g in components: $g = (g_1,\ldots,g_n)$. For simplicity of notation we denote $x^1 = (x_1,\ldots,x_k)$, $x^2 = (x_{k+1},\ldots,x_n)$, $g^1 = (g_1,\ldots,g_k)$ and $g^2 = (g_{k+1},\ldots,g_n)$. Then $g_*(\pi_*^{-1}(D)) \subset \dot{D} + g_*(\Delta_0^e)$ is equivalent to

$$\frac{\partial g^2}{\partial x^1}(x,u) \subset \text{Im}\,\frac{\partial g^2}{\partial u}(x,u), \text{ for every } (x,u). \tag{2.43}$$

I.e. there exist m-vectors $m_i(x,u)$, $i = 1,\ldots,k$, such that

$$\frac{\partial g^2}{\partial x_i}(x,u) + \frac{\partial g^2}{\partial u}(x,u).m_i(x,u) = 0 \qquad i = 1,\ldots,k. \tag{2.44}$$

Let P be the annihilating codistribution of D (D=ker P). Then $g^*\dot{P}$ is a codistribution on B. Because P is involutive, also $g^*\dot{P}$ is involutive. In the same coordinates as above $P = \text{span}\{dx_{k+1},\ldots,dx_n\}$, and $\dot{P} = \text{span}\ \{dx_{k+1},\ldots,dx_n,\ d\dot{x}_{k+1},\ldots,d\dot{x}_n\}$.

Hence $g^*\dot{P} = \text{span}\{dx_{k+1},\ldots,dx_n,dg_{k+1},\ldots,dg_n\} =$

$$\text{span}\{dx_{k+1},\ldots,dx_n,\ \sum_{i=1}^{k}\frac{\partial g_{k+1}}{\partial x_i}dx_i + \sum_{i=1}^{m}\frac{\partial g_{k+1}}{\partial u_i}du_i,\ldots,\ \sum_{i=1}^{k}\frac{\partial g_n}{\partial x_i}dx_i + \sum_{i=1}^{m}\frac{\partial g_n}{\partial u_i}du_i\}$$

Because of (2.44) it can now be seen that $\ker g^*\dot{P} = \text{span}\{\frac{\partial}{\partial x_i} + m_i(x,u)\frac{\partial}{\partial u}$, $i = 1,\ldots,k\} + \ker\frac{\partial g^2}{\partial u}\,du$. Note that $\ker\frac{\partial g^2}{\partial u} = \Delta_0^e \cap g_*^{-1}(\dot{D})$, which by assumption has constant dimension. Hence $g^*\dot{P}$ is a *regular* codistribution on B, while $\pi_*(\ker g^*\dot{P})$ is a regular distribution on X, equal to $\ker P = D$. It can now be proved that there exists a k-dimensional regular distribution E on $\pi^{-1}(V)$, contained in $\ker g^*\dot{P}$, such that $\pi_*E = D$ and $\dim E = \dim D$ (locally we can factor out the fibers of B by the regular distribution $\ker\frac{\partial g^2}{\partial u}\,du$; since $\ker\frac{\partial g^2}{\partial u}\,du \subset \ker g^*P$, $\ker g^*\dot{P}$ projects then well to a regular distribution). Moreover we can extend E to an n-dimensional regular distribution H on $\pi^{-1}(V)$ such that $\pi_*H = TX$ and $E \subset H$. The maximal integral manifolds of H form a transversal foliation of B. This defines a trivialization $\pi^{-1}(V) \xrightarrow{\phi} V\times U \xrightarrow{pr} U$ (pr is projection on U, along V), by requiring that the sets $(pr \circ \phi)^{-1}(c)$, c constant, are the leaves of this foliation. By definition $g_*(\ker g^*\dot{P}) \subset \ker\dot{P} = \dot{D}$, and hence $g_*E \subset \dot{D}$. Therefore the trivialization defined by H makes D invariant. □

Remark. If $\Delta_0^e \cap g_*^{-1}(\dot{D}) = 0$, the vectors $m_i(x,u)$ in (2.44) are uniquely determined. Since $g^*\dot{P}$ is involutive they satisfy a set of partial differential equations. These equations are called the *integrability conditions*, since satisfying them is a necessary and sufficient condition for the (local) existence of a map $\alpha(x,u)$ such that

$$\frac{\partial \alpha}{\partial x_i}(x,u) = m_i(x,\alpha(x,u)), \qquad i = 1,\ldots,k. \tag{2.45}$$

This map α is exactly the feedback which is required to make D invariant. If $\Delta_0^e \cap g_*^{-1}(\dot{D}) \neq 0$, we can take arbitrary coordinates on this part of the fibers in order to make D invariant. This is worked out in detail in NIJMEIJER & VAN DER SCHAFT (1982 b).

Concluding, if D is locally controlled invariant, then, if we define the codistribution P by ker P = D and the distribution E on B by $E = \ker g^*\dot{P}$, we obtain

$$\pi_* E = D \text{ and } g_* E \subset \dot{D}.$$

The relation with local minimality and strong accessibility is clear. If E as above also satisfies $E \subset \ker dh$, then, if $E \neq 0$, $\Sigma(X,W,B,(g,h))$ is not locally minimal (Definition 2.26). If $E = \pi_*^{-1}(D)$ then, if $D \neq 0$, Σ is not strongly accessible (Definition 2.45). For affine systems, with Δ an affine distribution on X, Theorem 2.55 amounts to (as can easily be checked): Let D be a regular distribution on X, such that $\Delta_0 \cap D$ has constant dimension. Then D is locally controlled invariant if and only if $[\Delta,D] \subset D+\Delta_0$. We remark that we can therefore rephrase Theorem 2.28 in the following way: $\Sigma(X,W,B,f)$ is locally minimal iff its extended system has no nonzero locally controlled invariant distribution contained in ker dh.

We also give

THEOREM 2.56 Let $\Sigma(X,W,B,(g,h))$ be a nonlinear system, and let $\Sigma^e(B,\Delta^e,W,h)$ be its extended system. Then: A regular distribution D on X, such that $g_*^{-1}(\dot{D}) \cap \Delta_0^e$ has constant dimension, is locally controlled invariant if and only if there exists a regular distribution E on B, with $\pi_* E = D$, and hence dimension $E \cap \Delta_0^e$ constant, which is locally controlled invariant with respect to $\Sigma(B,\Delta^e,W,h)$.

PROOF If D is locally controlled invariant and $g_*^{-1}(\dot{D}) \cap \Delta_0^e$ has constant dimension, then $E := \ker g^*\dot{P}$ is a regular distribution and satisfies $\pi_* E = D$ and $g_* E \subset \dot{D}$. This is equivalent to $[\Delta^e,E] \subset \Delta_0^e + E$ (see Theorem 2.28). Conversely if $[\Delta^e,E] \subset E + \Delta_0^e$ and $E \cap \Delta_0^e$ has constant dimension,

then (Theorem 2.28) $\pi_* E$ is a regular distribution D and $g_* E \subset \dot{D}$. □

Finally we want to say something about *global* controlled invariance. For the definition of local controlled invariance we required that on every trivializing chart V we can take a trivialization $\pi^{-1}(V) \simeq V \times U$ that makes D invariant. We also saw that this is equivalent to the existence of an n-dimensional regular distribution H on $\pi^{-1}(V)$, with $\pi_* H = TV$, such that $g_*(\pi_*^{-1}(D) \cap H) \subset \dot{D}$. Now, for global controlled invariance we need that all these locally defined distributions H can be nicely put together to a distribution on B.

DEFINITION 2.57 Let $\Sigma(X,W,B,f)$ be a nonlinear system and let D be a regular distribution on X. D is called *globally controlled invariant* if there exists a regular n-dimensional distribution H on B such that

(i) $\pi_* H = TX$

(ii) $g_*(\pi_*^{-1}(D) \cap H) \subset \dot{D}$

We remark that an n-dimensional distribution H satisfying $\pi_* H = TX$ determines what is called an *integrable connection* on B. We also note that the existence of an integrable connection on B puts some restrictions on the structure of B. We shall not further elaborate this point (in fact, a fiberbundle B allows an integrable connection if and only if B is isomorphic to a fiber bundle with discrete structure group, see CAMACHO & LINS NETO (1979, Ch.5, Th. 4), see also NIJMEIJER & VAN DER SCHAFT (1982 b, Definition 3.2)).

We have defined (local) controlled invariance by requiring that, after applying feedback, the modified dynamics leave the distribution invariant for *all* inputs. This requirement might be too strong and we could be content if the foliation induced by D is invariant for only a *part* of the inputs. We call this degenerate (local) controlled invariance and we formalize it as follows (only for the most degenerate case):

DEFINITION 2.58 Let $\Sigma(X,W,B,f)$ be a nonlinear system, and let D be a regular distribution on X. D is called degenerate locally controlled invariant, if for every trivializing chart V there exists a section $s : V \to \pi^{-1}(V)$ (i.e. $\pi \circ s = id$), such that

$$[g(\cdot, s(\cdot)), D] \subset D.$$

Remark: Degenerate *global* controlled invariance can be defined by requiring

a global section $s : X \to B$.

If F is a leaf of D, contained in V, then s induces a submanifold s(F) of B. The following proposition shows the close relationship between degenerate controlled invariance and uniform observability (Definitions 2.40 and 2.41).

PROPOSITION 2.59 Let $\Sigma(X,W,B,f)$ be a nonlinear system with local input-output representations with feedthrough term. Then: Σ is uniformly locally distinguishable if and only if there does not exist a nonzero regular distribution D which is degenerate locally controlled invariant and such that $T(s(F)) \subset \ker dh$ for every leaf F of D. (T(s(F)) is the tangent space of s(F)).

PROOF If Σ is not uniformly locally distinguishable, then there exist coordinates (x,v) for B and (y,u) for W such that the local representation (2.24) $\dot{x} = g(x,u)$, $y = \overline{h}(x,u)$ is not locally weakly observable for a certain $\overline{u}$, i.e. there exists a regular D, contained in $\ker d_x\overline{h}(\cdot,\overline{u})$ such that $[g(\cdot,\overline{u}),D] \subset D$. Now define $s(x) = (x,\overline{u})$, then D is degenerate locally controlled invariant. The converse can be proved in the same way. □

2.2.4 Nonlinear Realization Theory

In this section we try to generalize some ideas of the realization theory for linear systems to the case of nonlinear systems. Analogous to the linear case (see Sections 2.1.2, 2.1.3) we look at external dynamical systems described by a set of higher-order *nonlinear* implicit differential equations

$$(2.46) \qquad P_i(w,\dot{w},\ddot{w},\ldots,w^{(k)}) = 0, \qquad i = 1,\ldots,p$$

where $w^{(j)}$ denotes the j-th order derivative of a function $w : \mathbb{R} \to W$ ($\mathbb{R}$ is the time-axis). We shall assume that the P_i's are smooth (C^∞) equations, and that W, the set of external variables, is a smooth manifold. Hence we assume that the physical laws which govern the external system are of a smooth nature.

Equations (2.46) can also be given in a coordinate-free way. Given an arbitrary manifold M, we can define in an intrinsic way the higher-order tangent bundles of M (see for instance YANO-ISHIHARA (1973)). We denote by T_kM the k-th order tangent bundle of M. The first order tangent bundle T_1M is just the ordinary tangent bundle TM. If M has local coordinates x, then T_kM will have coordinates denoted by $(x,\dot{x},\ldots,x^{(k)})$. If $s : (-\varepsilon,\varepsilon) \to M$ is a C^k-map, then s induces an element of T_kM, which in the above coordinates is given by $(s(0),\frac{ds}{dt}(0), \frac{d^2s}{dt^2}(0),\ldots,\frac{d^ks}{dt^k}(0))$. Hence, coordinate-free the equations (2.46) yield a subset of T_kW which we denote by P. We assume that P is actually a submanifold. Therefore the external dynamical system is generated by a smooth submanifold

$$(2.47) \qquad P \subset T_kW.$$

In order to define $\Sigma_e(P)$, the external dynamical system corresponding to P, we have to worry about the smoothness assumptions that we want to impose on the functions $w : \mathbb{R} \to W$ belonging to $\Sigma_e(P)$. In the linear case this problem is easily solved by taking w locally integrable and satisfying $P(\frac{d}{dt})w = 0$ in the sense of distributions (see section 2.1.2). However, for the nonlinear case this is not so easy anymore. For simplicity we therefore define the *smooth* external system $\tilde{\Sigma}_e(P)$:

DEFINITION 2.60 Let $P \subset T_kW$ be a smooth submanifold. Then the smooth external system $\tilde{\Sigma}_e(P)$ is given by $\{w:\mathbb{R}\to W \mid w \text{ is } C^\infty, (w(t),\dot{w}(t),\ldots,w^k(t)) \in P, \forall t\}$.

Given a nonlinear system $\Sigma(X,W,B,f)$ (see Definition 2.20) we can also define the smooth versions of $\Sigma_i(X,W,B,f)$ and $\Sigma_e(X,W,B,f)$ (see (2.10), (2.11)):

$$
\begin{aligned}
\tilde{\Sigma}_i(X,W,B,f) &:= \{(x,w):\mathbb{R} \to W \mid x \text{ and } w\ C^\infty, \\
&\qquad (x(t),w(t)) \in f(\pi^{-1}(x(t))), \forall t\} \\
\tilde{\Sigma}_e(X,W,B,f) &:= \{w:\mathbb{R} \to W \mid \exists x \text{ such that } (x,w) \in \tilde{\Sigma}_i(X,W,B,f)\}.
\end{aligned} \tag{2.48}
$$

The Smooth Realization Problem is now as follows:
Given $P \subset T_kW$, find a nonlinear system $\Sigma(X,W,B,f)$ such that $\tilde{\Sigma}_e(P) = \tilde{\Sigma}_e(X,W,B,f)$.

In this section we shall discuss a general approach to the solution of the Smooth Realization Problem. At the end of the section we give an illustrative example of this approach.

Consider $T_kW \times \mathbb{R}$, with $\mathbb{R}$ the time-axis. Then on $T_kW \times \mathbb{R}$ the following one-forms are defined, called the Cartan forms,

$$
\begin{aligned}
\theta^1 &= dw - \dot{w}\, dt \\
\theta^2 &= d\dot{w} - \ddot{w}\, dt \\
&\vdots \\
\theta^k &= dw^{(k-1)} - w^{(k)} dt
\end{aligned} \tag{2.49}
$$

(of course by θ^i, $i = 1,\ldots,k$ we mean a *set* of one-forms, for instance θ^1 is $dw_1 - \dot{w}_1 dt,\ldots,dw_q - \dot{w}_q dt$, if $w = (w_1,\ldots,w_q)$). We can restrict the Cartan-forms to $P \times \mathbb{R}$ to obtain a set of one-forms $\bar{\theta}_1,\ldots,\bar{\theta}_k$ defined on $P \times \mathbb{R}$. Now consider in every point $(x,t) \in P\times\mathbb{R}$, with $x \in P$ and $t \in \mathbb{R}$, the linear subspace $V(x,t)$ of $T_{(x,t)}(P\times\mathbb{R}) = T_xP \times T_t\mathbb{R}$ given by the kernel of $\bar{\theta}_1,\ldots,\bar{\theta}_k$:

$$
V(x,t) := \{Z \in T_{(x,t)}(P\times\mathbb{R}) \mid \bar{\theta}^i{}_{(x,t)}(Z) = 0,\ i=1,\ldots,k\}. \tag{2.50}
$$

Then there are two possibilities for $V(x,t)$:

I. $V(x,t) \subset T_xP\times 0$

II. $V(x,t) \not\subset T_xP\times 0$

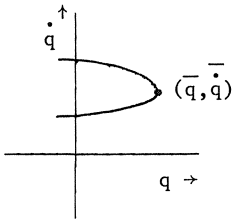
Furthermore it is clear that it only depends on x if $V(x,t)$ satisfies possibility I or II (this is because the equations (2.46) do not depend on t).

If possibility I holds for $x \in P$, then $V(x,t)$ does not contain a vector Z of the form

$$
Z = Y_x \times \frac{\partial}{\partial t}\Big|_t \tag{2.51}
$$

with $Y_x \in T_xP$ and $\frac{\partial}{\partial t}\Big|_t$ the tangent vector to $\mathbb{R}$. This implies that we *cannot* define a solution of the higher-order differential equations with initial condition x. This phenomenon already arises in implicit first-order differential equations. Consider for instance (consult TAKENS (1976 a) for more information) the submanifold $P \subset T\mathbb{R}$ given by the following graph ($T\mathbb{R}$ has coordinates $(q,\dot{q})$)

(2.52)



Then V(x,t), with $x \in P$, is the kernel of $dq - \dot{q}\,dt$ restricted to P. It is easy to see that in $x = (\bar{q},\dot{\bar{q}})$, $V(x,t) = \text{span}\{\frac{\partial}{\partial \dot{q}}\}$, so $x = (\bar{q},\dot{\bar{q}})$ satisfies possibility I. All other points of P satisfy possibility II. In the point $x = (\bar{q},\dot{\bar{q}})$ we cannot define a (differentiable) solution of the implicit differential equation.

Consider now a point $x \in P$ for which possibility II holds. Then V(x,t) has the form

$$V(x,t) = \Delta_0(x) + \text{span}\{A(x) + \frac{\partial}{\partial t}\} \tag{2.53}$$

with $A(x) \in T_xP$ and $\Delta_0(x)$ a linear subspace of T_xP. By $\text{span}\{A(x)+\frac{\partial}{\partial t}\}$ we denote the one-dimensional subspace of $T_xP \times T_t\mathbb{R} \simeq T_xP \times \mathbb{R}$ generated by $A(x) + \frac{\partial}{\partial t}$. Therefore if for *every* $x \in P$ possibility II holds, we can define a *distribution* Δ_0 on P and a *vectorfield* A on P such that for every x, V(x,t) satisfies (2.53). Furthermore we can define an *output map* $h : P \to Y$, by taking the output manifold equal to W and h the projection of T_kW on W, restricted to $P \subset T_kW$. Summarizing:

PROPOSITION 2.61 Let $P \subset T_kW$ be a submanifold. Suppose that for every $x \in P$ possibility II holds. Then we can define an affine control system (Definition 2.23) $\Sigma(X,\Delta,Y,h)$, with $X = P$, $\Delta(x) = A(x) + \Delta_0(x)$, $Y = W$ and h the natural projection of $P \subset T_kW$ on W. Furthermore the *smooth external behavior* $\tilde{\Sigma}_e(X,\Delta,Y,h)$ of $\Sigma(X,\Delta,Y,h)$, i.e. $\tilde{\Sigma}_e(X,\Delta,Y,h) := \{w:\mathbb{R}\to W \mid w \text{ is } C^\infty$ and $\exists x : \mathbb{R} \to X$, x is C^∞, such that $\dot{x}(t) \in \Delta(x(t))$ and $w(t) = h(x(t))\}$ is equal to $\tilde{\Sigma}_e(P)$.

Of course $\Sigma(X,\Delta,Y,h)$ as in Proposition 2.61 does not have to be (locally) minimal. However we can apply the "algorithm" as given after Theorem 2.28, i.e. we define the sequence of distributions on X

$$(2.54)\qquad \begin{aligned} E^0 &:= \ker dh \\ E^m &:= E^{m-1} \cap \Delta^{-1}(E^{m-1}+\Delta_0),\ m = 1,2,\ldots \end{aligned}$$

with $\Delta^{-1}(E^{m-1}+\Delta_0)$ the distribution spanned by all the vectorfields Z such that $[\Delta,Z] \subset E^{m-1} + \Delta_0$. Then $\lim_{m\to\infty} E^m = E^k$ if $\dim(\ker dh) = k$, and $E := E^k$ is an involutive distribution which satisfies

$$(2.55)\qquad [\Delta,E] \subset E + \Delta_0, \text{ and } E \subset \ker dh$$

(in fact E is the maximal distribution with these properties). If we now *assume* that E has constant dimension, then at least locally we can factor X out by the foliation induced by E, to obtain a smaller state space X'. Moreover the affine distribution Δ projects well to an affine distribution Δ' on X', and since $E \subset \ker dh$, we can define $h' : X' \to W$ such that $h' \circ \pi = h$ (π is projection of X onto X'). Summarizing:

PROPOSITION 2.62 Let $P \subset T_kW$ be a submanifold. Suppose that for every $x \in P$ possibility II holds. By Proposition 2.61 we can define an affine control system $\Sigma(X,\Delta,Y,h)$ such that $\tilde{\Sigma}_e(X,\Delta,Y,h) = \tilde{\Sigma}_e(P)$. Suppose that E as constructed in (2.54) has constant dimension and that X can be globally factored out by the leaves of foliation induced by E. Then we can construct a locally minimal affine control system $\Sigma'(X',\Delta',Y,h')$ such that $\tilde{\Sigma}_e'(X',\Delta',Y,h') = \tilde{\Sigma}_e(P)$.

We now treat the case that there exist points $x \in P$ for which possibility I holds. We shall only propose a tentative approach. First of all we simply omit all the points $x \in P$ for which possibility I holds (of course from the viewpoint of applications this can be a severe restriction). Then we *assume* that the set of points $x \in P$ for which possibility II holds forms a smooth submanifold N of P. Let $V(x,t) = \Delta_0(x) + \text{span}\{A(x) + \frac{\partial}{\partial t}\}$, for every $x \in N$, with $\Delta_0(x) \subset T_xP$ and $A(x) \in T_xP$ for every $x \in N$. We can distinguish between two cases:

Case I: Let $\Delta_0(x) = \text{span}\{B_1(x),\ldots,B_m(x)\}$, $B_i(x) \in T_xP$, for every $x \in N$. Suppose that for every $x \in N$ we can find feedback functions $\alpha_1(x),\ldots,\alpha_m(x)$,

$x \in N$, such that $\overline{A}(x) := A(x) + \sum_{i=1}^{m} \alpha_i(x) B_i(x)$ is contained in $T_x N$ for every $x \in N$. Define furthermore $\overline{\Delta}_0(x) := \Delta_0(x) \cap T_x N$ for every $x \in N$. Then we can define an affine control system $\Sigma(\overline{X},\overline{\Delta},\overline{Y},\overline{h})$ by setting $\overline{X} = N$, $\overline{\Delta}(x) = \overline{A}(x) + \overline{\Delta}_0(x)$, $\overline{Y} = W$ and $\overline{h}$ the projection of $T_k W$ on W restricted to N. Then $\tilde{\Sigma}_e(\overline{X},\overline{\Delta},\overline{Y},\overline{h}) = \tilde{\Sigma}_e(P)$.

Case II: Suppose that there exist points $x \in N$, for which we *cannot* find feedback functions such that $\overline{A}(x) \in T_x N$. In this case we should look for a submanifold $M \subset N$ such that for every $x \in M$ we can find feedback functions such that $\overline{A}(x) \in T_x M$. Then we can again define an affine control system $\Sigma(\overline{X},\overline{\Delta},\overline{W},\overline{h})$ with state space $\overline{X} = M$. Furthermore $\tilde{\Sigma}_e(\overline{X},\overline{\Delta},\overline{W},\overline{h}) \subset \tilde{\Sigma}_e(P)$.

In fact, in Case II we should look for a *maximal* submanifold M with the above property. Of course it is not clear whether such a maximal submanifold exists. We remark that we can apply the same procedure as before (Proposition 2.62) to obtain a locally minimal affine control system with the same external behavior as $\Sigma(\overline{X},\overline{\Delta},\overline{W},\overline{h})$.

From now on we assume that we have a locally minimal affine control system $\Sigma(X,\Delta,W,h)$ such that $\tilde{\Sigma}_e(X,\Delta,W,h) = \tilde{\Sigma}_e(P)$. Now it can be easily seen that it is still possible that there exists an affine control system $\Sigma'(X',\Delta',W,h')$, such that $\dim X' < \dim X$ and $\tilde{\Sigma}_e'(X',\Delta',W,h') = \tilde{\Sigma}_e(P)$. Consider for instance the two minimal systems

(2.56)

I.		II.	
$\dot{x} = u$		$w = u$	
$w = x$			
x and w	C^∞	w	C^∞

Then the smooth external behavior of system I as well as of system II is equal to the set of all C^∞ functions $w : \mathbb{R} \to W$. This phenomenon arises because we are considering *smooth* external behaviors. In fact if we look at $\tilde{\Sigma}_i(X,W,B,f) := \{(x,w) : \mathbb{R} \to X \times W \mid x \text{ and } w \text{ are } C^\infty \text{ and } (x(t),w(t)) \in f(\pi^{-1}(x(t)))\}$ (compare (2.10)), then $x \in X$ does not satisfy the axiom of *state* (Definition 1.2). The reason is that if $x_1, x_2 : \mathbb{R} \to X$ are C^∞ and $x_1(0) = x_2(0)$, then $x_1^- \cdot x_2^+$ is not necessarily C^∞.

The rest of this Section is devoted to a closer study of this phenomenon. We restrict ourselves to a *local* study. In fact we strive to a (local) reduction procedure which yields a system that has the same smooth external behavior as $\Sigma(X,\Delta,W,h)$ and has a state space of minimal dimension.

Suppose that $\Delta_0 = \Delta - \Delta$ is a *regular* distribution on X. Then (locally) we can factor out X by Δ_0 to obtain X_1, i.e. $X \xrightarrow{\pi} X_1$, with $\ker \pi_* = \Delta_0$. Taking coordinates x_1 for X_1 and (x_0,x_1) for X, and writing $\Delta(x) = A(x) + \text{span}\{B_1(x),\ldots,B_m(x)\}$ we obtain

$$\begin{aligned} \dot{x}_0 &= A^0(x_0,x_1) + \sum_{i=1}^{m} u_i B_i^0(x_0,x_1) \\ \dot{x}_1 &= A^1(x_0,x_1) \end{aligned} \tag{2.57}$$

By feedback (2.57) is equivalent to

$$\begin{aligned} \dot{x}_0 &= \tilde{u} \\ \dot{x}_1 &= A^1(x_0,x_1) \end{aligned} \tag{2.58}$$

with $\tilde{u} = (\tilde{u}_1,\ldots,\tilde{u}_m)$ the new input. This suggests reducing (2.58) by taking x_0 as the new input $v = (v_1,\ldots,v_m)$, i.e.

$$\dot{x}_1 = A^1(x_1,v). \tag{2.59}$$

Coordinate-free equations (2.59) are the state-space equations of a nonlinear system $\Sigma'(X',W',B',f')$ with $B' = X$, $X' = X_1$ and $f' = (g',h')$ with $g'(x_1,v) = A^1(x_1,v)$ (Notice that the transition from (2.58) to (2.59) is exactly the converse of the way we constructed the extended system in Definition 2.24). Now we also want to include the output map h in our considerations. Suppose that not only Δ_0 is regular, but also $\ker dh \cap \Delta_0$ is regular. Then we can (locally) factor out X by $\ker dh \cap \Delta_0$, i.e. $X \xrightarrow{\pi_1} X_1$ with $\ker \pi_{1*} = \ker dh \cap \Delta_0$. Furthermore, since Δ_0 is regular, $\pi_{1*}\Delta_0$ is a regular distribution on X_1 (Lemma 2.27). Hence we can locally factor out X_1 by $\pi_{1*}\Delta_0$, i.e. $X_1 \xrightarrow{\pi_2} X_2$ with $\ker \pi_{2*} = \pi_{1*}\Delta_0$. Let x_2 be coordinates for X_2, (x_1,x_2) for X_1 and (x_0,x_1,x_2) for X. Then we obtain

$$\begin{aligned} \begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \end{pmatrix} &= \begin{pmatrix} A^0(x_1,x_1,x_2) \\ A^1(x_0,x_1,x_2) \end{pmatrix} + \sum_{i=1}^{m} u_i \begin{pmatrix} B_i^0(x_1,x_1,x_2) \\ B_i^1(x_0,x_1,x_2) \end{pmatrix} \\ \dot{x}_2 &= A^2(x_0,x_1,x_2) \\ w &= h(x_0,x_1,x_2) \end{aligned} \tag{2.60}$$

(actually h only depends on x_1 and x_2). By feedback (2.60) is equivalent to

$$
\begin{aligned}
\dot{x}_0 &= \tilde{u}_0 \\
\dot{x}_1 &= \tilde{u}_1 \\
\dot{x}_2 &= A^2(x_0,x_1,x_2) \\
w &= h(x_0,x_1,x_2)
\end{aligned}
\tag{2.61}
$$

with $\tilde{u} = (\tilde{u}_0,\tilde{u}_1)$ the new input. Hence we can define new inputs $v = (v_0,v_1)$ by setting $v_0 = x_0$, $v_1 = x_1$, to obtain

$$
\begin{aligned}
\dot{x}_2 &= A^2(x_2,v_0,v_1) \\
w &= h(x_2,v_0,v_1)
\end{aligned}
\tag{2.62}
$$

Therefore locally we can define a nonlinear system $\Sigma'(X',W',B',f')$ with $X' = X_2$, $W' = W, B' = X$ and $f' = (g',h')$ such that $g'(x_2,v_0,v_1) = A^2(x_2,v_0,v_1)$ and $h'(x_2,v_0,v_1) = h(x_2,v_0,v_1)$. Furthermore it is clear that locally

$$
\tilde{\Sigma}_e(X,\Delta,W,h) = \tilde{\Sigma}_e'(X',W,B',f'). \tag{2.63}
$$

Hence we have reduced $\Sigma(X,\Delta,W,h)$ to a system with a smaller state space and the same smooth external behavior.

Of course the above reduction procedure may be again applied to $\Sigma'(X',W',B',f')$. Then we need that $g'(x_2,v_0,v_1)$ defines a regular distribution on X_2. More precisely we have to consider the sets

$$
g'(\pi_1^{-1}(x_1,x_2)) \subset T_{x_2}X_2.
$$

Indeed suppose that g' is of the form

$$
g'(x_2,v_0,v_1) = A'(x_2,v_1) + \sum_{i=1}^{k} v_0^i B_i'(x_2) \tag{2.64}
$$

with $v_0 = (v_0^1,\ldots,v_0^k)$. This is equivalent to the assumption that $g'(\pi_1^{-1}(x_1,x_2))$ is for every (x_1,x_2) an affine subspace of $T_{x_2}X_2$ such that the *linear* subspace $\Delta_0' := g'(\pi_1^{-1}(x_1,x_2)) - g'(\pi_1^{-1}(x_1,x_2))$ does not depend on x_1. Now the question is if $\Delta_0' = \text{span}\{B_1',\ldots,B_k'\}$ is a *regular* distribution on X_2. If Δ_0 is regular then we can factor out X_2 by this distribution to obtain a smaller state space X_3, i.e. $X_2 \xrightarrow{\pi_3} X_3$, with $\ker \pi_{3*} = \Delta_0'$. Denote coordinates for X_3 by x_3, and let $x_2 = (x_2',x_3)$. Then (2.62) is feedback equivalent to

$$
\begin{aligned}
\dot{x}_2' &= \tilde{v}_0 \\
\dot{x}_3 &= A'^3(x_3, x_2', v_1) \\
y &= h(x_3, x_2', v_1)
\end{aligned}
\tag{2.65}
$$

with $\tilde{v}_0 = (\tilde{v}_0^1, \ldots, \tilde{v}_0^k)$ the new input and A'^3 the component of A' in the x_3-direction. Notice that we have used the fact that h does not depend on v_0. Now we can take x_2' as the new input, and we obtain a nonlinear system with state x_3 and inputs x_2' and v_1. Hence we have (locally) obtained a nonlinear system $\Sigma''(X'',W,B'',f'')$ such that $\tilde{\Sigma}_e''(X'',W,B'',f'') = \tilde{\Sigma}_e'(X',W,B',f') = \tilde{\Sigma}_e(X,\Delta,W,h)$.

The reduction procedure from $\Sigma'(X',W,B',f')$ to $\Sigma''(X'',W,B'',f'')$, with $f' = (g',h)$ may terminate because of the following three reasons:

I. h is an immersion restricted to the fibers of B'.
II. g' does not define a distribution Δ_0' on X'
III. the distribution Δ_0' is not regular.

If h is an immersion restricted to the fibers of B', then we have obtained a system with a local input-output representation with feedthrough term. Therefore if in every step of the reduction procedure g' defines a regular distribution Δ_0' then the procedure is continued till we have a system with a local input-output representation with feedthrough term. If g' does *not* define a regular distribution Δ_0' on X' a serious problem arises, since $g'(\pi_1^{-1}(x_1,x_2)) - g'(\pi_1^{-1}(x_1,x_2))$ may *contain* a regular distribution on X', by which we can factor out. Therefore it is possible that we can apply a "partial" reduction procedure. We shall not consider this problem.

Finally we illustrate the reduction procedure by the following example, considered in a different context in FREEDMAN & WILLEMS (1978).

Let $Y = U = \mathbb{R}^m$, and let $W = Y \times U$ with coordinates $w = (y,u)$. Consider the external system described by the equations

$$
P_i(y,u,\dot{y},\dot{u}) := \dot{y}_i - a_i(u,y,\dot{u}) = 0, \quad i = 1,\ldots,m \tag{2.66}
$$

for certain smooth functions a_i. We abbreviate (2.66) as $\dot{y} - a(u,y,\dot{u}) = 0$. Equations (2.62) determine in every point $w \in W$ a subset of T_wW, and hence determine a subset $P \subset TW$. Consider the Cartan-forms $dw - \dot{w}\,dt$, i.e. $dy_i - \dot{y}_i dt$ and $du_i - \dot{u}_i dt$, $i = 1,\ldots,m$. If we restrict these forms to $P \times \mathbb{R}$ we obtain $dy_i - a_i(u,y,\dot{u})dt$ and $du_i - \dot{u}_i dt$, $i = 1,\ldots,m$. The kernel of

these forms is equal to

$$(2.67)\qquad \text{span}\ \{\frac{\partial}{\partial \dot{u}_i}\ ,\ \sum_{j=1}^{m} \dot{u}_j\frac{\partial}{\partial u_j} + a_j(y,u,\dot{u})\frac{\partial}{\partial y_j} + \frac{\partial}{\partial t}\ ,\quad i = 1,\ldots,m\}$$

Therefore for every $x \in P$ possibility II holds. Hence we may apply Proposition 2.61 and take $X = P$. It is clear that $(y,u,\dot{u})$ are coordinates for P. The output map $h : P \to W$ simply is $h(y,u,\dot{u}) = (y,u)$. It is clear that we can always factor out the regular distribution $\text{span}\ \{\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_m}\}$ on P. After this reduction the new state space X' can be taken equal to W, with coordinates (y,u). The equations of $\Sigma'(X',W,B',f')$ are

$$(2.68)\qquad \begin{aligned} \dot{u} &= v \\ \dot{y} &= a(y,u,v) \\ w &= (y,u) \end{aligned}$$

with $v = (v_1,\ldots,v_m)$ the input. The question is: When can we reduce $\Sigma'(X',W,B',f')$ with equations (2.68) to a system $\Sigma''(X'',W,B'',f'')$ by the reduction procedure ? First of all we need that we can write (see (2.64))

$$(2.69)\qquad \begin{pmatrix} v \\ a(y,u,v) \end{pmatrix} = \begin{pmatrix} 0 \\ b(y,u) \end{pmatrix} + \begin{pmatrix} I_m \\ c(y,u) \end{pmatrix} v$$

for a certain smooth map $b : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, and a certain $m\times m$-matrix $c(y,u)$, whose coefficients are smooth functions of (y,u). We denote the i-th column of $c(y,u)$ by $c_i(y,u)$. Notice that in this example

$$(2.70)\qquad \Delta_0'(y,u) = \text{span}\left\{ \begin{pmatrix} e_1 \\ c_1(y,u) \end{pmatrix}\ ,\ldots,\ \begin{pmatrix} e_m \\ c_m(y,u) \end{pmatrix} \right\}$$

with e_i the i-th basis vector of $\mathbb{R}^m$. Therefore Δ_0' always has constant dimension. Hence we only need that Δ_0' is *involutive*. This is equivalent to

$$(2.71)\qquad [\frac{\partial}{\partial u_i} + c_i(y,u)\frac{\partial}{\partial y}, \frac{\partial}{\partial u_j} + c_j(y,u)\frac{\partial}{\partial y}\ ] = 0$$

for all $i,j = 1,\ldots,m$ (with $\frac{\partial}{\partial y} = (\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_m})$).
Equations (2.71) are equivalent to

$$(2.72)\qquad \frac{\partial c_i}{\partial u_j} - \frac{\partial c_j}{\partial u_i} + c_j\frac{\partial c_i}{\partial y} - c_i\frac{\partial c_j}{\partial y} = 0,\ i,j = 1,\ldots,m$$

Now(2.72) are the necessary and sufficient conditions for the existence of

a smooth map $k : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that (see SPIVAK (1970)):

$$\frac{\partial k}{\partial u_i}(z,u) = c_i(k_i(z,u),u), \text{ with } z,u \in \mathbb{R}^m. \tag{2.73}$$

Furthermore we can take k in such a way that the map $(z,u) \longmapsto (k(z,u),u)$ is a diffeomorphism. Define

$$\ell(z,u) := \left(\frac{\partial k}{\partial z}(z,u)\right)^{-1} b(k(z,u),u) \tag{2.74}$$

(with b as in (2.69)). Then the following system

$$\begin{aligned} \dot{z} &= \ell(z,u) \\ y &= k(z,u) \\ u &= u, \qquad w = (y,u) \end{aligned} \tag{2.75}$$

is the reduced system $\Sigma''(X'',W,B'',f'')$. The output map of (2.75) is $(z,u) \longmapsto (k(z,u),u)$. This is clearly an immersion with respect to u, so no further reduction is possible. System (2.75) is an input-output representation with feedthrough term and the smooth external behavior of (2.75) is equal to $\Sigma_e(P)$.

Notice furthermore that since $(z,u) \longmapsto (k(z,u),u)$ is a diffeomorphism, we may as well take (z,u) as coordinates for W (instead of (y,u)). In these coordinates we obtain the input-output system (Definition 2.22)

$$\begin{aligned} \dot{z} &= \ell(z,u) \\ z &= z \\ u &= u \end{aligned} \tag{2.76}$$

Summarizing we have the following conclusion (also obtained in FREEDMAN & WILLEMS (1978)):

There exists an input-output system, which has the same smooth external behavior as the external system (2.66) if and only if $a(y,u,\dot{u})$ is of the form $b(y,u) + c(y,u)\dot{u}$, with the columns of c(y,u) satisfying (2.72).

Notes and References for Chapter 2

Sections 2.1.1 and 2.1.2 are mainly based on WILLEMS (1979) (see also WILLEMS (1983)). For more information about the description of systems of the form $D(\frac{d}{dt})y(t) = N(\frac{d}{dt})u(t)$, or in the frequency domain $D(s)y(s) = N(s)u(s)$, we refer to WOLOVICH (1974) and ROSENBROCK (1970).

The basic idea of Section 2.1.3 is due to MARTIN & HERMANN (1978), see also HERMANN (1979,1980). These authors associate to a transfer matrix $G(s) = D^{-1}(s)N(s)$ an algebraic vector bundle over $\mathbb{P}^1(\mathbb{C})$, and make the observation that the Chern numbers of this vector bundle are equal to the controllability indices of a minimal realization of G(s). For a concise treatment of algebraic vectorbundles over $\mathbb{P}^1(\mathbb{C})$ we refer to HAZEWINKEL & MARTIN (1982). The "geometric" treatment of output feedback as a choice of the output space in the space $U \times Y$ also appears in BROCKETT & BYRNES (1981). In the text we defined the dual bundle $E(P(s))^{\perp}$ by taking for every $s \in \mathbb{C}$ the orthogonal complement of Ker P(s) with respect to an (arbitrary) inner product on W. Of course we can also define $E(P(s))^{\perp}$ as a subbundle of $\mathbb{P}^1 \times W^*_{\mathbb{C}}$ (with W^* the dual space of W).

Sections 2.2.1 and 2.2.2 are mainly based on VAN DER SCHAFT (1982 c). Definition 2.20 is due to WILLEMS (1979), while the idea to give a coordinate free description of $\dot{x} = g(x,u)$ by using a fiber bundle B over the state space X is due to BROCKETT (1977, 1980), see also TAKENS (1976). This "bundle approach" is also elaborated in NIJMEIJER & VAN DER SCHAFT (1982 a,b) and NIJMEIJER (1980). For related results on minimality and equivalence of nonlinear input-output systems (in a somewhat different setting) we refer to SUSSMANN (1977), HERMANN & KRENER (1977) and BROCKETT (1980). The definition of local weak observability (in the sense of Property I), and the corresponding rank condition of the observability codistribution is due to HERMANN & KRENER (1977). For other approaches to uniform observability (especially for affine nonlinear systems) we refer to NIJMEIJER (1982 b), GAUTHIER & BORNARD (1981). The existence of universal inputs, i.e. input functions which distinguish between every two states x_1 and x_2, has been investigated in SUSSMANN (1979). The definition of strong accessibility (in a different setting) is due to ELLIOTT (1971) and SUSSMANN & JURDJEVIC (1972). The notion of local weak controllability appears in LOBRY (1970) and is further explored in HERMANN & KRENER (1977), and other papers.

Section 2.2.3 is mainly based on NIJMEIJER & VAN DER SCHAFT (1982, b,a, also 1983 b). The generalization of the concept of controlled invariance for linear systems to affine nonlinear systems is due to ISIDORI, KRENER, GORI-GIORGI & MONACO (1981 a), and HIRSCHORN (1981). The relations between controlled invariance, observability and local weak controllability for affine systems are treated in ISIDORI et al. (1981 a).

With respect to Section 2.2.4 we draw attention to previous work on realization theory for nonlinear systems from an input-output point of view. In JAKUBCZYK (1980) a minimal smooth realization of an input-output system is constructed by using Nerode-equivalence (see Chapter 1). Furthermore there has been done important work on the realization of input-output relations which can be described by Volterra series, see for instance BROCKETT (1976), KRENER & LESIAK (1978), CROUCH (1981 a), and in another direction FLIESS (e.g. 1981) (see also LAMNABHI (1982)).

CHAPTER 3

HAMILTONIAN SYSTEMS

"That", said Merry, pointing with his hand, "that is the line of the Withywindle..... The Withywindle is said to be the queerest part of the whole wood - the centre from which all the queerness comes, as it were".

J.R.R. Tolkien: The Lord of the Rings

3.1. Introduction

In the previous chapter we have given a system theoretic framework for the study of linear and (smooth) nonlinear systems. Within these both classes we now narrow our scope to what we shall call *Hamiltonian* systems. The distinctive feature of a Hamiltonian system will be the existence of a so-called *symplectic structure* on the state space as well as on the space of external variables. Furthermore the equations of the system have to be compatible with both symplectic structures.

From a mathematical point of view we know that the symplectic structure is rich enough to define a useful mathematical "category", in which all operations, transformations etc. respect the symplectic structure. We show in this chapter that also the system theoretic concepts as introduced in the previous chapters nicely fit into this symplectic framework.

With the study of Hamiltonian systems we enter the realm of classical mechanics. Actually the Hamiltonian formalism lies at the basis of a much larger part of theoretical physics, as for example electro-magnetism and also quantum mechanics, the non-classical part of physics *par excellence*. Most treatments of Hamiltonian systems in the modern mathematical and physical literature deal only with what is called *analytical mechanics*. This part of mechanics confines itself to the study of mechanical systems without external influences. For instance (see SANTILLI (1978,1983)) the equations that are customarily referred to as the Euler-Lagrange equations, and Hamilton's equations are *not* the equations originally conceived by Lagrange

and Hamilton. The latter include *external forces* , i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + F_i$$

and only since the beginning of this century have the external terms been removed. As a consequence of the predominance of analytical mechanics, the recent formalization of classical mechanics in terms of symplectic geometry (see the text books of ARNOLD (1978) and ABRAHAM & MARSDEN (1978)) has only been carried out for systems without external forces. In fact to our knowledge there have been very few attempts to include on a fundamental level external forces in a modern mathematical theory of Hamiltonian systems (however we like to mention some work of Tulczyjew and co-workers, e.g. TULCZYJEW & KIJOWSKI (1979)). Of course there are strong historical reasons why the study of external forces in modern classical mechanics and theoretical physics is not very fashionable, and we have more to say about that in Section 3.2.2.

For completeness we mention that an approach other than ours to the formalization of external forces in mechanical systems is provided by the calculus of variations. The starting point is then in fact the variational derivation of the Euler-Lagrange equations with external forces (TAKENS (1977), see also HERMANN (e.g. 1982), GODBILLON (1969)). In our approach we treat, roughly spreaking, Lagrangian systems as a specialization of Hamiltonian systems (see Section 3.6), and we obtain the Euler-Lagrange equations with external forces as the basic example of a Lagrangian system.

Another element, apart from the external forces, which we shall introduce in the study of Hamiltonian systems is the formalization of the notion of partial *observations* made on a system. This is a typical system theoretic concern and seems even more absent in the physics literature than the conceptualization of external forces. In fact we give a framework for Hamiltonian systems in which the (generalized) forces and observations are in the same way dual to each other as the (generalized) positions and momenta are dual to each other.

In the following sections 3.1.1 till 3.1.4 we give some motivation for the definition of a Hamiltonian system as given in Section 3.2.

In 3.1.1 we speak about memoryless (in this mechanical context better called *static*) Hamiltonian systems. Actually there has been some flourishing of the use of symplectic geometry, especially the notion of Lagrangian submanifolds, in the treatment of various physical systems with a "reciprocal" or "symmetric" character. We shall give some illustrative examples.

After this we are heading for a definition of a *dynamical Hamiltonian* system. In section 3.1.2 we give the equations of an electrical network, consisting of (nonlinear) capacitors and inductors as an example of a dynamical system which we call Hamiltonian. In section 3.1.3 we give a short review of the definitions of a Hamiltonian vectorfield in terms of symplectic geometry, including the formulation as a Lagrangian submanifold of the tangent bundle which is advocated in recent works.

Finally in section 3.1.4 we briefly sketch some motivation stemming from linear system theory. We show that if a transfer matrix possesses a certain symmetry, then a minimal realization of this transfer matrix is also endowed with a special structure and can be called a (linear) Hamiltonian system.

We now give a very simple but paradigmatic example of what we shall call a Hamiltonian system.

Consider a point mass m with position q_1, influenced by a force F_1. According to Newton's second law, the relation between q_1 and F_1 as functions of time is given by

$$m\ddot{q}_1 = F_1 \tag{3.1}$$

Note that we see F_1 as a basic variable and that (3.1) expresses a compatibility relation between forces and positions. Hence we have an external (linear) system (see section 2.1.2)

$$\begin{aligned}\Sigma_e := \{&(q_1(\cdot),F_1(\cdot)):\mathbb{R} \to \mathbb{R}^2 \mid (q_1(\cdot),F_1(\cdot))\in L_{1oc} \text{ and} \\ &m\ddot{q}_1 = F_1, \text{ with equality in the sense of distributions}\}\end{aligned} \tag{3.2}$$

In fact Σ_e is an external input-output system with input $u_1 = F_1$. A minimal realization of Σ_e is given by

$$\begin{aligned}\dot{q}_1 &= \frac{1}{m} p_1 \\ \dot{p}_1 &= u_1\end{aligned} \qquad , \; y_1 = q_1 \tag{3.3}$$

i.e. a linear input-output system $\Sigma(A,B,C)$ (see equation 2.4) with $A = \begin{pmatrix} 0 & \frac{1}{m} \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $C = (1\ 0)$. Any definition of a Hamiltonian system surely ought to include systems (3.2) and (3.3). The basic observation is that the state space (q_1,p_1) can be seen as a symplectic space with the usual symplectic form $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then A as above is a Hamiltonian matrix, i.e. A satisfies $A^TJ + JA = 0$, and B and C are related as $B^TJ = C$. Furthermore we notice that the space of inputs and outputs (y_1,u_1) can be also seen as a symplectic space with the symplectic form $J^e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Next we look at another mechanical system. Consider a particle attached to a spring with spring constant k. Assume that we can control the position q_2 of the particle. We take as output the force F_2 exerted by the spring on the particle, i.e. the force that we experience if we control the particle in a certain position. This yields the static system $F_2 = -kq_2$, which can be also written as

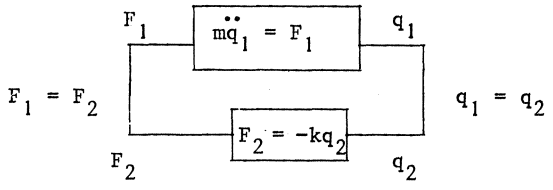
$$(3.4) \qquad F_2 = -\frac{dV}{dq_2}(q_2)$$

with $V(q_2) = \frac{1}{2}kq_2^2$ the potential energy. We regard (3.4) as a static Hamiltonian system with input q_2 and output F_2. Equation (3.4) defines a Lagrangian submanifold in the (q_2,F_2)-space with generating function $V(q_2)$ (see Section 3.1.1). Instead of the potential energy $\frac{1}{2}kq_2^2$ corresponding to a linear spring we can take an arbitrary potential energy function $V(q_2)$. Notice also that (3.4) is an example where external forces are not necessarily inputs.

Finally we can *interconnect* the Hamiltonian systems (3.3) and (3.4) by setting

$$(3.5) \qquad q_1 = q_2 \ , \ F_1 = F_2$$

(this can be regarded as Newton's third law) :



We shall see that the interconnection (3.5) is a particularly simple example of what we call a Hamiltonian interconnection. This will be dealt with in Section 3.2.1. The system resulting from the interconnection has the form (setting $q=q_1=q_2$) :

$$(3.6) \qquad m\ddot{q} + kq = 0 \quad , \text{ or } , \quad \frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

This constitutes a Hamiltonian vectorfield, or as we shall say an autonomous (i.e. no inputs) Hamiltonian system. As outputs we could take the position q, or the position q together with $-\frac{dV}{dq}(q) = -kq$, which is now the *internal* force.

3.1.1 Static Hamiltonian systems; reciprocity.

First we develop the mathematical preliminaries which will be used for the formalization of the examples of static Hamiltonian systems given in this section. For details we refer to ARNOLD (1978), ABRAHAM & MARSDEN (1978).

Let M be a manifold provided with a 2-form ω which is

(i) nondegenerate, i.e. for every $Y \in T_xM$, with $Y \neq 0$, there exists a $Z \in T_xM$ such that $\omega_x(Y,Z) \neq 0$

(ii) closed, i.e. $d\omega = 0$.

Such an ω is called a *symplectic form*, and (M,ω) is called a *symplectic manifold*. It follows from (i) that M is necessarily even-dimensional, say $\dim M = 2n$. From (i) and (ii) it can be deduced that there exist local coordinates $q_1,..,q_n,p_1\cdots,p_n$ for M such that

$$(3.7) \qquad \omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

This is known as Darboux's theorem (ARNOLD (1978), ABRAHAM & MARSDEN (1978)). The coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ for which (3.7) holds are called *canonical* or symplectic.

The prototype of a symplectic manifold is a cotangent bundle. Let T^*Q be a cotangent bundle, with $\dim Q = n$. Then we can define a natural 1-form θ on T^*Q as follows. Take $\alpha \in T^*Q$, $X \in T_\alpha(T^*Q)$ and let π denote the projection of T^*Q on Q. Then set

$$(3.8) \qquad \theta_\alpha(X) = \theta(\alpha)(X) := \alpha(\pi_* X)$$

If $(q_1,\ldots,q_n)$ are (arbitrary) coordinates for Q we can define *natural coordinates* $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ (fiber respecting for the bundle T^*Q) by

letting $(\bar{q}_1,\ldots,\bar{q}_n,\bar{p}_1,\ldots,\bar{p}_n)$ correspond to the one-form $\sum_{i=1}^{n} \bar{p}_i dq_i$ on Q in the point $(\bar{q}_1,\ldots,\bar{q}_n) \in Q$. In such coordinates we have that

$$\theta = \sum_{i=1}^{n} p_i dq_i \tag{3.9}$$

Finally we define $\omega := d\theta$, i.e. in natural coordinates $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$, which is a symplectic form on T^*Q. Moreover we see that the natural coordinates are also canonical.

The basic notion in the description of static Hamiltonian systems is that of a Lagrangian submanifold.

DEFINITION 3.1 Let (M,ω) be a symplectic manifold, with dim M = 2n. A submanifold $N \subset M$ is called *Lagrangian* if
(i) $\omega|_N = 0$, i.e. $\omega_x(Y,Z) = 0$, for every $x \in N$, $Y \in T_xN$ and $Z \in T_xN$.
(ii) dim N = n

It can be proved that (i) implies that dim $N \leq n$. An important property of a Lagrangian submanifold is the following

THEOREM 3.2 (for a proof see ABRAHAM & MARSDEN (1978)) Let N be a Lagrangian submanifold of (M,ω). Let $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ be canonical coordinates for M on a coordinate neighbourhood $U \subset M$. Then for every $x \in U$ we can find an open neighborhood V of x, with $V \subset U$, and a function S defined on $N \cap V$ such that $N \cap V$ is given by

$$q_i = \frac{\partial S}{\partial p_i} \quad , \quad p_j = -\frac{\partial S}{\partial q_j} \tag{3.10}$$

where the index i ranges through a part of the set $\{1,\ldots,n\}$ and the index j through the complementary part (S is only a function of the variables p_i and q_j with respect to which S is differentiated in (3.10)).

The function S as in (3.10) is called a generating function of N. We shall now give some typical examples of static Hamiltonian systems which illustrate the relationship with notions like reciprocity and potentiality. It will be shown that a natural mathematical framework is the use of Lagrangian submanifolds. The following treatment is taken nearly verbatim from KIJOWSKI & TULCZYJEW (1979).

Consider a particle in 3-dimensional space Q, subject to an external force field. The position of the particle will be described by coordinates $q = (q_1,q_2,q_3)$. We assume that for every configuration q there exists a unique force $F = (F_1,F_2,F_3)$ exerted by the external field. Hence there exist functions ϕ_i such that

$$F_j = \phi_j(q), \qquad j = 1,2,3. \tag{3.11}$$

If an external mechanism is used to control the position of the particle, then an infinitesimal displacement from a position q to $q + \delta q := (q_1+\delta q_1, q_2+\delta q_2, q_3+\delta q_3)$ requires the mechanism to perform a virtual work

$$A = \sum_{i=1}^{3} F_i \delta q_i \tag{3.12}$$

with $F_i = \phi_i(q)$ (Of course -F is just the force that the controlling mechanism has to exert to maintain the configuration q). If we define the 1-form $\phi = \sum_{j=1}^{3} \phi_j dq_j$ on Q, then $A = -\phi(\delta q)$. Consider now a displacement from one configuration q^1 to another configuration q^2, along a path γ. The work performed by the mechanism equals

$$A(\gamma) = -\int_{\gamma} \phi. \tag{3.13}$$

If $A(\gamma)$ only depends on the endpoints of γ, we can define a *potential* function on Q:

$$V(q) = -\int_{q^0}^{q} \phi \tag{3.14}$$

with q^0 some fixed (reference) configuration. We obtain that

$$\phi = -dV, \text{ or equivalently, } \phi_j = -\frac{\partial V}{\partial q_j}, \quad j = 1,2,3. \tag{3.15}$$

Equations (3.15) express the *potentiality* of the system. A 1-form ϕ such that there exists a function V with $\phi = -dV$ is called *exact*. On the other hand, if $\phi = -dV$, then $d\phi = -d(dV) = 0$, or equivalently

$$\frac{\partial \phi_i}{\partial q_j} = \frac{\partial \phi_j}{\partial q_i} \qquad i,j = 1,2,3. \tag{3.16}$$

We can interpret (3.16) in the following way. Let the system be displaced

in such a way as to increment the coordinate q_j by an infinitesimal value ε without changing the remaining coordinates. The (first-order) increment of the i-th component F_i of the force caused by this displacement is $\frac{\partial F_i}{\partial q_j}\varepsilon$. If on the other hand the coordinate q_i is incremented by ε then $\frac{\partial F_j}{\partial q_i}\varepsilon$ is the corresponding first-order increment of F_j. Equations (3.16) imply that the two first-order increments are equal! This property is called *reciprocity*. As we saw the property of reciprocity is equivalent to the condition that $d\phi = 0$. We call a 1-form ϕ with $d\phi = 0$ *closed*. If ϕ is exact, then ϕ is also closed. If on the other hand ϕ is closed, then (by Poincaré's lemma) there exists at least locally a function V satisfying $\phi = -dV$ (if Q is simply connected V exists globally). Hence locally: *reciprocity* $\Longleftrightarrow$ *potentiality*.

We can also describe the above situation with the aid of the cotangent bundle T^*Q, whose coordinates are $(q,F) = (q_1,q_2,q_3,F_1,F_2,F_3)$. Since the force F is a function of q, we have a 3-dimensional submanifold $N \subset T^*Q$:

$$N = \{(q_1,q_2,q_3,\phi_1(q),\phi_2(q),\phi_3(q)\}. \tag{3.17}$$

The natural 1-form θ on T^*Q is given by $\theta = \sum_{i=1}^{3} F_i dq_i$. An infinitesimal displacement from q to $q + \delta q$ yields an infinitesimal change from (q,F) to $(q+\delta q, F+\delta F)$. The virtual work equals

$$A = -\theta_{(q,F)}((\delta q, \delta F)). \tag{3.18}$$

A finite displacement from q^1 to q^2 along a path γ in Q results in a finite displacement from $(q^1,F^1) \in N$ to $(q^2,F^2) \in N$, along a path $\overline{\gamma}$ in N. The work performed equals

$$A(\gamma) = -\int_{\overline{\gamma}} \theta \tag{3.19}$$

If the work only depends on the endpoints q^1 and q^2, we can define a (potential) function V on N by

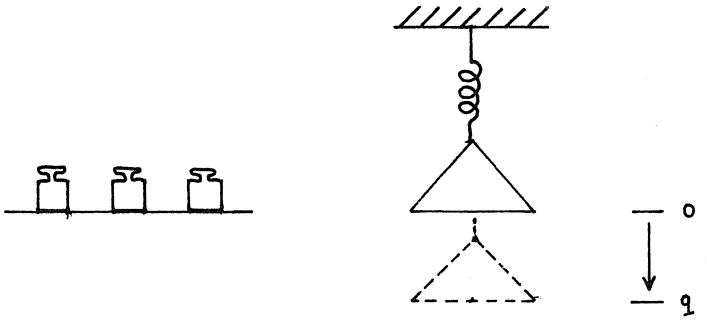
$$V(q,F) = -\int_{(q^0,F^0)}^{(q,F)} \theta \text{ , with } (q,F) \in N \text{ and } (q^0,F^0) \text{ a (reference) point in N.} \tag{3.20}$$

Formula (3.20) is equivalent to

$$-dV = \theta|_N \tag{3.21}$$

This means that θ *restricted* to N is exact. Therefore the cotangent bundle interpretation of potentiality is that $\theta|_N$ is exact. To find the cotangent bundle interpretation of reciprocity, we define the natural symplectic form $\omega = d\theta$ on T^*Q. Reciprocity means that $\theta|_N$ is closed, i.e. $d(\theta|_N) = 0$, or equivalently $\omega|_N = d\theta|_N = 0$. Hence reciprocity is equivalent to saying that N is a Lagrangian submanifold of (T^*Q,ω).

A particularly simple example of the foregoing situation is a particle attached to a spring with one degree of freedom. Then $q \in Q \subset \mathbb{R}$, $F = -kq$, with k the spring constant and $N = \{(q,-kq) \mid q \in Q\} \subset T^*\mathbb{R} = \mathbb{R}^2$. Since a 1-dimensional submanifold of $T^*\mathbb{R}$ is always Lagrangian, the system is reciprocal and in fact the generating function of N is $V(q) = \frac{1}{2}kq^2$, the *internal energy* of the spring (since V is globally defined we have in fact potentiality). We shall now show for this special example that instead of taking q as the input (control) and F as the output, we can also regard F as the input and q as the output. We consider a scale suspended in the gravitational field attached to the spring with constant k. The force applied to the system is controlled by placing weights on the scale. The weights are stored at the level of the equilibrium position if there are no weights on the scale.



The force and the position are in the relation $F = -kq$. If the force is increased from F to $F + \delta F$ by transferring a weight δF from the storage to the scale then the performed virtual work equals

$$\overline{A} = -q\delta F. \tag{3.22}$$

If the weight is changed from 0 to F then the total work equals

$$\text{(3.23)} \qquad \overline{V} = -\int_0^F q\delta F = -\frac{1}{k}\int_0^F F dF = -\frac{1}{2k}F^2.$$

This work is spent on the internal energy of the spring:

$$\text{(3.24)} \qquad V = \frac{1}{2}kq^2 = \frac{1}{2k}F^2$$

and on changing the gravitational energy of the weights by

$$\text{(3.25)} \qquad \psi = qF = -\frac{1}{k}F^2.$$

Hence $\overline{V} = -V - \psi$, and we see that we can write $N = \{(q,-\frac{dV}{dq}) \,|\, q\in\mathbb{R}\}$ also as $N = \{(\frac{d\overline{V}}{dF},F) \,|\, F\in\mathbb{R}\}$.

Return now to the more general situation with $Q = \mathbb{R}^3$, and assume that the Lagrangian submanifold $N \subset T^*Q$ as in (3.17) can also be parametrized by $F = (F_1,F_2,F_3)$. We shall see that the same construction as above goes through for the general case. Locally we can define the 1-form $\overline{\theta} := \sum_{i=1}^{3} q_i dF_i$. Then $\theta + \overline{\theta} = d\psi$, with $\psi = \sum_{i=1}^{3} q_i F_i$.

The integral $\overline{A}(\overline{\gamma}) = -\int_{\overline{\gamma}}\overline{\theta}$ along a path $\overline{\gamma}$ contained in N is the work performed, and only depends on the endpoints. Therefore we can (locally) define

$$\text{(3.26)} \qquad \overline{V}(q,F) = \int_{(q^0,F^0)}^{(q,F)} \overline{\theta}\,, \quad \text{for } (q,F) \text{ and } q^0,F^0) \text{ in } N.$$

Obviously, $d\overline{V} = -\overline{\theta}|_N$ and if the reference state (q^0,F^0) is the same as the one used in (3.20) we obtain

$$\text{(3.27)} \qquad \overline{V}(q,F) = -\int_{(q^0,F^0)}^{(q,F)} \overline{\theta} = \int_{(q^0,F^0)}^{(q,F)} \theta - d\psi = -V(q,F) - \psi|_N$$

and N is also given by $N = \{(-\frac{\partial\overline{V}}{\partial F_i},F_i) \,|\, F_i\in\mathbb{R},\ i=1,2,3\}$.

The relation between V and $\overline{V}$ as in 3.27 is called the *Legendre transformation.*

We see that for the notion of reciprocity it does not make a difference how the submanifold N is parametrized in canonical coordinates of T^*Q. Also we do not use the cotangent bundle structure of T^*Q but only the fact that T^*Q possesses a symplectic form. We conclude with the following

DEFINITION 3.3 Let W, the set of external variables, be a symplectic manifold with symplectic form ω^e. A memoryless (see Definition 1.16) external system Σ_e on W is given by a submanifold $N \subset W$ such that $\Sigma_e = \{w \in W^{\mathbb{R}} \mid w(t) \in N \ \forall t \in \mathbb{R}\}$. A memoryless system is *static Hamiltonian* if N is a Lagrangian submanifold of (W, ω^e).

Above we saw some examples of a static Hamiltonian system with $W = T^*Q$. In the literature one can find numerous other examples (see for references ABRAHAM & MARSDEN (1978), KIJOWSKI & TULCZYJEW (1979)). In fact the equations of nonlinear capacitors or inductors as treated in the next section are also examples. Furthermore Kirchhoff's laws of interconnecting voltages and currents can be viewed as a (special) example of a Lagrangian submanifold of the space of voltages and currents (see for more information Section 3.2.1). We close this section with another mechanical example.

Example: Consider a mass m attached to a rope which is swept around with constant angular velocity ω. Assume that we can control the length r of the rope, if the mass is attached to the end, or the position r where the mass is attached to the rope. Take the generating function equal to the kinetic energy of the mass due to the rotation, i.e. $\frac{1}{2}m\omega^2r^2$. If r is the input, then we obtain the output $y = -m\omega^2 r$, i.e. the centripetal force.

3.1.2 Nonlinear LC-networks

After Newton's second law as treated in Section 3.1, this will be the first example of what we call a dynamical Hamiltonian system. Our treatment of an LC-network is strongly influenced by work of BRAYTON (1978), see also BRAYTON & MOSER (1964), to which we also refer for more details.

A nonlinear inductor is given by a constitutive relation

$$i = f(\phi) \tag{3.28}$$

with i the *current*, ϕ the *magnetic flux* and f a smooth function. Since i and ϕ are one-dimensional, we can construct a function $S(\phi)$ such that

$$i = f(\phi) = \frac{dS}{d\phi}(\phi). \tag{3.29}$$

Equivalently, we can look at an inductor as a submanifold $N = \{(\phi, f(\phi)) \mid \phi \in \mathbb{R}\}$ of $\mathbb{R}^2 = \{(\phi, i) \mid \phi \in \mathbb{R}, i \in \mathbb{R}\}$. Since N is 1-dimensional, N is a Lagrangian submanifold of $\mathbb{R}^2$ with its natural symplectic form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

and therefore an S as in (3.29) exists (Theorem 3.2).

If we *interconnect* inductors, satisfying Kirchhoff's laws, the interconnected system is still reciprocal, i.e. a Lagrangian submanifold of a (higher-dimensional) space (BRAYTON (1978)). Indeed, let L be an inductive n_1-port,i.e. a set of inductors connected to each other in some way satisfying Kirchhoff's laws with n_1 external channels. The corresponding set of external variables are the fluxes and currents of the n_1 external channels. Suppose that L can be parametrized by $(\phi_1,\ldots,\phi_{n_1})$, with ϕ_j the magnetic flux on the j-th channel. Then there exists (locally) a function $S(\phi_1,\ldots,\phi_{n_1})$, the *magnetic energy*, such that

$$i_j = \frac{\partial S}{\partial \phi_j}, \qquad j = 1,\ldots,n_1 \tag{3.30}$$

with i_j the current of the j-th channel.

A nonlinear capacitor is given by a constitutive relation

$$v = g(q) \tag{3.31}$$

with v the *voltage*, q the *charge* and g a smooth function. Also a capacitor can be regarded as a Lagrangian submanifold $N = \{(q,g(q))\,|\,q\in\mathbb{R}\}$ of $\mathbb{R}^2 = \{(q,v)\,|\,q,v\in\mathbb{R}\}$ with its natural symplectic form. Moreover there exists a function T(the generating function of N) such that

$$v = g(q) = \frac{dT}{dq}(q). \tag{3.32}$$

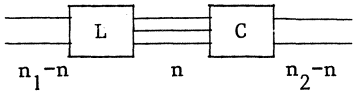
Again, interconnections satisfying Kirchhoff's laws leave the system reciprocal. If C is a capacitive n_2-port (a set of capacitors interconnected to each other with n_2 external channels), and if we assume that C can be parametrized by $(q_1,\ldots,q_{n_2})$, the charges of the external channels, then there exists (locally) a function $T(q_1,\ldots,q_{n_2})$, the *electric energy*, such that

$$v_j = \frac{\partial T}{\partial q_j}, \qquad j = 1,\ldots,n_2. \tag{3.33}$$

From (3.30) and (3.33) it follows that L as well as C can be regarded as static Hamiltonian systems (Definition 3.3).

Let us now connect the first n channels of L and C with each other ($n \le n_1$, $n \le n_2$):

(3.34)



i.e.

$$(3.35) \qquad i_j = -\frac{dq_j}{dt}\,, \quad v_j = \frac{d\phi_j}{dt}\,, \quad j = 1,\ldots,n$$

(if two channels are connected, the sign of the currents is reversed; this is a standard convention in circuit theory). Furthermore define the *energy* $H(\phi_1,\ldots,\phi_{n_2},q_1,\ldots,q_{n_2}) := S(\phi_1,\ldots,\phi_{n_1}) + T(q_1,\ldots,q_{n_2})$. Then we obtain from (3.30) and (3.33)

$$(3.36) \qquad \begin{cases} \dfrac{d\phi_j}{dt} = \dfrac{\partial H}{\partial q_j} \\ \dfrac{dq_j}{dt} = -\dfrac{\partial H}{\partial \phi_j} \end{cases} \qquad j = 1,\ldots,n$$

$$\begin{cases} i_{n+k} = \dfrac{\partial H}{\partial \phi_{n+k}} & k = 1,\ldots,n_1-n \\ v_{n+k} = \dfrac{\partial H}{\partial q_{n+k}} & k = 1,\ldots,n_2-n \end{cases}$$

We call this a Hamiltonian system with state space $(\phi_1,\ldots,\phi_n,q_1,\ldots,q_n)$ and space of external variables $(\phi_{n+1},\ldots,\phi_{n_1},q_{n+1},\ldots,q_{n_2},i_{n+1},\ldots,i_{n_1},$ $v_{n+1},\ldots,v_{n_2})$.

Notice that the assumptions on the parametrizations of L and C, made in (3.30) and (3.33), are *not* essential with respect to the external variables on the external channels which are *not* connected. If for instance C is parametrized by $(q_1,\ldots,q_n,v_{n+1},\ldots,v_{n_2})$, then there exists a function $\overline{T}$ depending on these variables such that

$$
\begin{aligned}
v_j &= \frac{\partial \overline{T}}{\partial q_j}, && j = 1,\ldots,n \\
q_{n+k} &= -\frac{\partial \overline{T}}{\partial v_{n+k}}, && k = 1,\ldots,n_2-n
\end{aligned}
\tag{3.37}
$$

(see Theorem 3.2; $\overline{T}$ is obtained by taking the Legendre transformation of T with respect to the variables $(q_{n+1},\ldots,q_{n_2})$). If we now define $\overline{H} := S + \overline{T}$, we obtain after the interconnection (3.35), the following Hamiltonian system

$$
\begin{cases}
\dfrac{d\phi_j}{dt} = \dfrac{\partial \overline{H}}{\partial q_j} \\
\dfrac{dq_j}{dt} = -\dfrac{\partial \overline{H}}{\partial \phi_j}
\end{cases}
\qquad j = 1,\ldots,n
$$

$$
\begin{cases}
i_{n+k} = \dfrac{\partial \overline{H}}{\partial \phi_{n+k}} & k = 1,\ldots,n_1-n \\
q_{n+k} = -\dfrac{\partial \overline{H}}{\partial v_{n+k}} & k = 1,\ldots,n_2-n
\end{cases}
\tag{3.38}
$$

In equations (3.36) we can regard $(\phi_{n+1},\ldots,\phi_{n_1},q_{n+1},\ldots,q_{n_2})$ as the *inputs* and $(i_{n+1},\ldots,i_{n_1},v_{n+1},\ldots,v_{n_2})$ as the *outputs*, while in (3.38) $(\phi_{n+1},\ldots,\phi_{n_1},v_{n+1},\ldots,v_{n_2})$ are the inputs and the remaining coordinates the outputs.

3.1.3 Hamiltonian vectorfields

Let (M,ω) be a symplectic manifold. Let $H : M \to \mathbb{R}$ be a smooth function. Since ω is nondegenerate we can define a vectorfield X_H on M by setting

$$\omega(X_H,-) = -dH \tag{3.39}$$

Let $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ be canonical coordinates, i.e. $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. Then (3.39) implies that X_H is given by

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \tag{3.40}$$

which gives the familiar Hamilton equations

$$\text{(3.41)} \qquad \begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \qquad i = 1,\ldots,n$$

We call X_H a globally Hamiltonian vectorfield. There are two, equivalent, ways of defining a *locally* Hamiltonian vectorfield. The first, most common, way is to say that a vectorfield X on (M,ω) is locally Hamiltonian if $\pounds_X\omega = 0$, where $\pounds_X$ is the Lie-derivative with respect to X. Since $\pounds_X\omega = (d\omega)(X,-) + d(\omega(X,-))$ and $d\omega = 0$, this is equivalent to $d(\omega(X,-)) = 0$. Therefore by Poincaré's lemma there exists, at least locally, a function $H : M \to \mathbb{R}$ such that $\omega(X,-) = -dH$.

The other approach makes use of the concept of a Lagrangian submanifold. If (M,ω) is a symplectic manifold, also TM has a canonically defined symplectic form, denoted by $\dot{\omega}$, which is defined in the following way. Since ω is nondegenerate, it defines a bundle isomorphism $\alpha : TM \longrightarrow T^*M$ by setting $\alpha(X) = \omega_x(X,-)$ for $X \in T_xM$. Now T^*M is a cotangent bundle and therefore has a natural symplectic form $\tilde{\omega}$. Then $\alpha^*\tilde{\omega}$ is a symplectic form on TM which we denote by $\dot{\omega}$. If $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$, one can check that $\dot{\omega}$ is given by $\dot{\omega} = \sum_{i=1}^{n} d\dot{p}_i \wedge dq_i + dp_i \wedge d\dot{q}_i$ (where $\dot{q}_i$ and $\dot{p}_i$ are the functions on TM defined by $\dot{q}_i(v) = dq_i(v)$, $\dot{p}_i(v) = dp_i(v)$ for $v \in TM$). It can now be seen (ABRAHAM & MARSDEN (1978, Prop. 5.3.2) that $\pounds_X\omega = 0$ is equivalent to

DEFINITION 3.4 Let X be a vectorfield on (M,ω). Then X is a locally *Hamiltonian vectorfield* if graph $X \subset TM$ is a Lagrangian submanifold of $(TM,\dot{\omega})$.

Let now X be a locally Hamiltonian vectorfield and let $(q,p) = (q_1,\ldots,q_n,p_1,\ldots,p_n)$ be canonical coordinates for (M,ω). Since graph X is a Lagrangian submanifold of $(TM,\dot{\omega})$ parametrized by (q,p), it has (locally) a generating function H(q,p) and is given by

$$\text{(3.42)} \qquad \text{graph } X = \{(q_1,\ldots,q_n,p_1,\ldots,p_n,\dot{q}_1 = \frac{\partial H}{\partial p_1},\ldots,\dot{q}_n = \frac{\partial H}{\partial p_n}, \dot{p}_1 = -\frac{\partial H}{\partial q_1},\ldots,\dot{p}_n = -\frac{\partial H}{\partial q_n}\}$$

Hence we have again obtained the Hamilton equations (3.41).

Usually we omit the words "locally" or "globally" and just speak about a Hamiltonian vectorfield when we actually mean a locally or globally Hamiltonian vectorfield.

Because we need it later on we also define the *Poissonbracket*. Let $F,G : M \to \mathbb{R}$ be two smooth functions. By (3.39) they define Hamiltonian vectorfields X_F, respectively X_G on M. The Poissonbracket $\{F,G\}$ is the smooth function on M defined by

$$\{F,G\} = \omega(X_F,X_G). \tag{3.43}$$

In local canonical coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ this gives

$$\{F,G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} \right) = X_F(G). \tag{3.44}$$

Furthermore, one can check that

$$[X_F,X_G] = X_{\{F,G\}} . \tag{3.45}$$

Equation (3.45) can be interpreted as follows. Let C(M) be the set of smooth functions on M. Endowed with the Poissonbracket (3.43) C(M) forms a Lie algebra, called the Poisson algebra. Consider on the other hand the set of all globally Hamiltonian vectorfields $V_H(M)$. By (3.45) this is a Lie subalgebra of V(M), the set of all smooth vectorfields on M. Then the map $\alpha : C(M) \to V(M)$, defined by $F \mapsto X_F$ is by (3.45) a Lie algebra morphism, and α : C(M) (modulo constant functions) $\to V_H(M)$ is a Lie algebra isomorphism.

3.1.4 Hamiltonian transfermatrices

Let G(s) be a m×m transfer matrix (see Case 3 after Theorem 2.4) enjoying the symmetry property

$$\Xi G(s) = G^T(-s)\Xi \tag{3.46}$$

with Ξ a signature matrix (i.e. a nonsingular matrix whose only nonzero elements are +1 or -1 on the diagonal). Such a G(s) is called a *Hamiltonian transfer matrix*. Let $\Sigma(A,B,C,D)$ (see Section 2.1.1) be a minimal realization of G(s)

$$\begin{aligned} \dot{x} &= Ax + Bu \quad , x \in X \\ y &= Cx + Du \quad , u \in U = \mathbb{R}^m, \; y \in Y = \mathbb{R}^m \end{aligned} \tag{3.47}$$

i.e. $G(s) = D + C(Is-A)^{-1}B$. Since $\Xi G(s) = G^T(-s)\Xi$ we obtain $\Xi D + \Xi C(Is-A)^{-1}B = D^T\Xi - B^T(Is+A^T)^{-1}C^T\Xi$, and hence $(-A^T, -C^T\Xi, B^T, D^T\Xi)$ is also a minimal realization of $G(s)$. Since minimal linear realizations are equivalent, there exists a unique nonsingular $J : X \to X$ such that (BROCKETT & RAHIMI (1972))

$$
\begin{aligned}
B^T &= \Xi C J^{-1} \\
-C^T\Xi &= JB \\
-A^T &= JAJ^{-1} \\
\text{and also } \Xi D &= D^T\Xi \ .
\end{aligned}
\tag{3.48}
$$

Because also $-J^T$ satisfies the equations (3.48) instead of J and the solution of (3.48) is unique, it follows that $J = -J^T$. Hence the bilinear form $(x,y) \to x^TJy$, $x,y \in X$, is a symplectic form on X, and X is necessarily even-dimensional, say $\dim X = 2n$. By Darboux's theorem there exist (canonical) bases of X such that

$$
J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{3.49}
$$

We call a system $\Sigma(A,B,C,D)$ satisfying (3.48) a *linear Hamiltonian input-output system* with feedthrough term. Notice that $-A^T = JAJ^{-1}$ is equivalent to $A^TJ + JA = 0$, i.e. A is a Hamiltonian matrix. We have obtained the following

PROPOSITION 3.5 {there exists a Hamiltonian realization of $G(s)$} $\Longleftrightarrow$ {$G(s)$ is Hamiltonian} $\Longleftrightarrow$ {a minimal realization of $G(s)$ is Hamiltonian.}

We notice that the transfer matrix corresponding to Newton's second law $m\ddot{q} = F$ equals $\frac{1}{ms^2}$ and hence trivially satisfies $G(s) = G^T(-s)$. A minimal Hamiltonian realization of $\frac{1}{ms^2}$ is given in (3.3). Also the transfer-matrix of a linear LC-network is Hamiltonian (with Ξ depending on the input-output parametrization). This will be shown inter alia, in Section 4.2.2. Linear Hamiltonian systems will be further treated in Section 3.5.

3.2 Hamiltonian systems; general definitions

Recall the definition of a smooth nonlinear system $\Sigma(X,W,B,f)$ as given in Definition 2.20. $\Sigma(X,W,B,f)$ is given by the commutative diagram

$B \longrightarrow TX \times W$, and if we take local coordinates x for X, (x,u) for B, and w

$\pi \searrow_X \swarrow \pi_X$

for W then the system is given by

$$\begin{aligned} \dot{x} &= g(x,u) \\ w &= h(x,u) \end{aligned} \tag{3.50}$$

with $f: B \longrightarrow TX \times W$ written as $f = (g,h)$.

In order to define a *Hamiltonian* smooth nonlinear system we require that

i) X is a symplectic manifold with symplectic form ω. In this case we denote the state space by M, to emphasize that the state space is a symplectic manifold. From now on dim M = 2n.

ii) W, the space of external variables, is a symplectic manifold with symplectic form ω^e (dim W=2m).

iii) $f : B \longrightarrow TM \times W$ is an *imbedding*, and hence f(B) is a (C^∞-)submanifold of TM × W.

Remark : If $\Sigma(X,W,B,f)$ is locally minimal and f has constant rank, we can actually *prove* that f is an immersion (notice that the regular distribution E:= ker df on B satisfies conditions (i) to (iii) of Definition 2.26, with $\pi_*(\ker df) = 0$).

Recall from Section 3.1.3 that ω induces the symplectic form $\dot{\omega}$ on TM. Hence we can define the symplectic form $\Omega := \pi_1^* \dot{\omega} - \pi_2^* \omega^e$ on TM × W (π_1 and π_2 denote the projections of TM × W on TM, respectively W).

DEFINITION 3.6 $\Sigma(M,W,B,f)$ with M and W symplectic manifolds as above is called a *(full) Hamiltonian system* if f(B) is a Lagrangian submanifold of $(TM \times W, \Omega)$.

Remark : Definition 3.6 generalizes Definition 3.3 (static Hamiltonian systems) as well as Definition 3.4 (Hamiltonian vectorfields). Definition 3.4 deals with Lagrangian submanifolds of $(TM, \dot{\omega})$, parametrized by M, and Definition 3.3 with Lagrangian submanifolds of (W, ω^e). In Definition 3.6 we look at Lagrangian submanifolds of TM × W, parametrized by B. We notice that the definition of a Hamiltonian system only depends on the submanifold f(B), and not on f and B separately.

In local coordinates Definition 3.6 amounts to

PROPOSITION 3.7. Let $\Sigma(M,W,B,f)$ be a (full) Hamiltonian system. Let $(q_1,\ldots,q_n,p_1,\ldots,p_n) = (q,p)$ be canonical coordinates for M and $(v_1,\ldots,v_m,z_1,\ldots,z_m)$ canonical coordinates for W. Then f(B) is locally parametrized by (q,p) and m coordinate functions $(v_i)_{i\in I_1}$ and $(z_i)_{i\in I_2}$, with $I_1 \cup I_2 = \{1,\ldots,m\}$ and $I_1 \cap I_2 = \emptyset$. Denote these m coordinate functions by $u = (u_1,\ldots,u_m)$ and denote the remaining coordinate functions for W by $y_1,\ldots,y_m$ (in such a way that $\omega^e = \sum_{j=1}^{m} c_j\, du_j \wedge dy_j$ with $c_j = \pm 1$). Then locally there exists a function $H(q_1,\ldots,q_n,p_1,\ldots,p_n,u_1,\ldots,u_m)$ such that f(B) is given by

$$
\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i}(q,p,u) && i = 1,\ldots,n \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i}(q,p,u) && \\
y_j &= -c_j \frac{\partial H}{\partial u_j}(q,p,u) && j = 1,\ldots,m
\end{aligned}
\tag{3.51}
$$

PROOF: Since f(B) is a Lagrangian submanifold of $(TM\times W,\Omega)$ f(B) can be locally parametrized by 2n + m of the 4n + 2m canonical coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n,\dot{p}_1,\ldots,\dot{p}_n,\dot{q}_1,\ldots,\dot{q}_n,v_1,\ldots,v_m,z_1,\ldots,z_m)$ of $(TM\times W,\Omega)$ in the specific way as stated in Theorem 3.2. From the commutativity of

$$
\begin{array}{ccc}
B & \xrightarrow{f} & TM \times W \\
& {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_M} & \\
& M &
\end{array}
\qquad \text{(see 2.8)}
$$

it follows that $(q,p) = (q_1,\ldots,q_n,p_1,\ldots,p_n)$ are coordinates on f(B). We can choose m additional coordinates on f(B) from the coordinates $(v_1,\ldots,v_m,z_1,\ldots,z_m)$. These additional coordinates are called $(u_1,\ldots,u_m)$, and the remaining m coordinates of $(v_1,\ldots,v_m,z_1,\ldots,z_m)$ are called $(y_1,\ldots,y_m)$ such that $\omega^e = \sum_{j=1}^{m} c_j du_j \wedge dy_j$. By theorem 3.2 there exists locally a generating function H(q,p,u) of f(B), and (3.51) results. □

Note that the equations of a LC-network (3.36) and (3.38) are examples of (3.51). The situation that not all the c_j's are +1 or -1 corresponds in this case to a so-called *hybrid* representation of the network.

We now show that equations (3.51) are in fact a *local input-output representation with feedthrough term* for $\Sigma(M,W,B,f)$ (see Section 2.2.1).

Since f is an imbedding, f is an immmersion, and since $f : B \longrightarrow TM \times W$ is given as $(x,u) \mapsto (x,g(x,u),h(x,u))$, actually f *restricted to the fibers of B* is an immersion.
Moreover:

PROPOSITION 3.8. Let $\Sigma(M,W,B,f)$ be full Hamiltonian with $f = (g,h)$. If f restricted to the fibers is an immersion, then h restricted to the fibers is an immersion.

PROOF: Consider (3.51). Since f restricted to the fibers is an immersion, the dimension of the fibers of B is m. Because for fixed $x \in M$, dim $h(\pi^{-1}(x)) = m$, it then follows that h restricted to the fibers is an immersion. □

Hence for a full Hamiltonian system, h restricted to the fibers is an immersion and we can obtain a local input-output representation with feedthrough term by taking coordinates (x,v) for B and (y,u) for W such that $h(x,v) = (\bar{h}(x,v),v)$. Notice however that we restrict the coordinatizations (y,u) for W to (semi-) canonical coordinates, i.e. $\omega^e = \sum_{j=1}^{m} c_j du_j \wedge dy_j$, $c_j = \pm 1$.

We shall now extend Definition 3.6 to what we call *degenerate* Hamiltonian systems. We need the following notions

DEFINITION 3.9. Let (M,ω) be a symplectic manifold with $N \subset M$ a submanifold. Define for every $x \in N, (T_xN)^{\perp} := \{X \in T_xM \mid \omega_x(X,Y) = 0, \forall Y \in T_xN\}$. N is called *coisotropic* if $(T_xN)^{\perp} \subset T_xN$, $\forall x \in N$, and *isotropic* if $T_xN \subset (T_xN)^{\perp}$, $\forall x \in N$.

Let dim M = 2n. It can be proved that if N is coisotropic (isotropic) then dim $N \geq n$ (dim $N \leq n$). Therefore a Lagrangian submanifold is a coisotropic (isotropic) submanifold of minimal (maximal) dimension (i.e.n), and N is a Lagrangian submanifold if and only if $(T_xN)^{\perp} = T_xN$ $\forall x \in N$ (see ABRAHAM & MARSDEN (1978)).

DEFINITION 3.10. Let $\Sigma(M,W,B,f)$ be a full Hamiltonian system, with $f = (g,h)$. Let $K \subset W$ be a coisotropic submanifold, called the *restriction manifold*, such that $h^{-1}(K) \subset B$ is again a fiber bundle over M. Then $\Sigma(M,W,h^{-1}(K),f_r)$, with f_r the restriction of f to $h^{-1}(K)$, is called a *degenerate Hamiltonian system*, which we also denote by $\Sigma(M,W,B,f,K)$

Remark: Note that $f(h^{-1}(K))$ is an isotropic submanifold of $TM \times W$.

Intuitively, a degenerate Hamiltonian system is a Hamiltonian system with less than m inputs, i.e. in equations (3.51) some of the u_j, $j = 1,\ldots,m$, are constant (at the end of this section we give some examples). To make this idea rigorous we need some preliminary propositions.

PROPOSITION 3.11. Let (M_1,ω_1) and (M_2,ω_2) be symplectic manifolds. Let $L \subset M_1 \times M_2$ be a Lagrangian submanifold of $(M_1 \times M_2, \Omega := \pi_1^*\omega_1 - \pi_2^*\omega_2)$, with π_1 and π_2 the projections on M_1 respectively M_2. Suppose that $L_1 := \pi_1(L)$ and $L_2 := \pi_2(L)$ are submanifolds of M_1 and M_2. Then

(i) L_1 and L_2 are coisotropic

(ii) If L_1 or L_2 is Lagrangian $\Rightarrow$ L_1 and L_2 are Lagrangian $\Leftrightarrow$ L is equal to $L_1 \times L_2$

PROOF (i) Let $x_1 \in L_1$ and $X \in T_{x_1}M_1$ such that $\omega_1(X,Y) = 0$, $\forall Y \in T_{x_1}L_1$. There exists $x_2 \in M_2$ such that $(x_1,x_2) \in L$. Define $X \times 0$ as the element of $T_{(x_1,x_2)}(M_1 \times M_2)$ such that $\pi_{1*}(X \times 0) = X$ and $\pi_{2*}(X \times 0) = 0$. Then for every $Z \in T_{(x_1,x_2)}L$ we obtain $\Omega(X \times 0, Z) = \omega_1(X, \pi_{1*}Z) = 0$, since $\pi_{1*}Z \in T_{x_1}L_1$. Hence $X \times 0 \in (T_{(x_1,x_2)}L)^{\perp}$. Since L is Lagrangian this implies that $X \times 0 \in T_{(x_1,x_2)}L$ and therefore $X \in T_{x_1}L_1$. Hence L_1 is coisotropic.

(ii) Let L_2 be Lagrangian. Let $x_1 \in L_1$ and $X_1 \in T_{x_1}L_1$. There exists $x_2 \in M_2$ and $X_2 \in T_{x_2}M_2$ such that $(x_1,x_2) \in L$ and $X_1 \times X_2 \in T_{(x_1,x_2)}L$, where $X_1 \times X_2$ satisfies $\pi_{1*}\, X_1 \times X_2 = X_1$ and $\pi_{2*}X_1 \times X_2 = X_2$. Then for every $Z \in T_{(x_1,x_2)}L$

$$0 = \Omega(X_1 \times X_2, Z) = \omega_1(X_1, \pi_{1*}Z) - \omega_2(X_2, \pi_{2*}Z)$$

and since L_2 is Lagrangian this yields $\omega_1(X_1, \pi_{1*}Z)$ for every $Z \in T_{(x_1,x_2)}L$. Hence $T_{x_1}L_1$ is isotropic. By part(i) L_1 is coisotropic. Therefore L_1 is Lagrangian. It is easy to see that $L_1 \times L_2$ is then Lagrangian. This implies that $L = L_1 \times L_2$. Conversely if $L = L_1 \times L_2$ is Lagrangian it is easy to check that both L_1 and L_2 are Lagrangian. □

For the rest of this section we *assume* that $h(B) \subset W$ is a submanifold of W and that g(B) is a submanifold of TM. Then we conclude from Proposition 3.11 that if $\Sigma(M,W,B,f)$ is a full Hamiltonian system, then $\pi_2(f(B)) = h(B)$ is a *coisotropic* submanifold.

Therefore if the restriction manifold K in Definition 3.10 is equal to h(B) (or contains h(B)), then no extra constraints are imposed on the system and $\Sigma(M,W,B,f) = \Sigma(M,W,B,f,K)$. If $\dim h(B) < \dim W$ we say that $\Sigma(M,W,B,f)$

contains *memoryless* parts. The extreme case is that h(B) is Lagrangian. Then by Proposition 3.11 (ii), g(B) is a Lagrangian submanifold of TM. Hence we have a Hamiltonian vectorfield on M, together with a static Hamiltonian system on W. If dim h(B) = dim W we call $\Sigma(M,W,B,f)$ a *regular* system.

The following theorem shows that locally we can reduce every full Hamiltonian system to a regular system.

THEOREM 3.12. (see for a proof ABRAHAM & MARSDEN (1978, p.416)). Let N be a coisotropic submanifold of (W,ω^e). Then the distribution $D(x) := (T_xN)^\perp$, $x \in N$ is a regular distribution on N. Therefore locally we can define a manifold $\bar{W}$ and a C^∞ surjective submersion pr : N $\longrightarrow \bar{W}$, such that ker $pr_* = D$. Moreover there exists a symplectic form $\bar{\omega}^e$ on $\bar{W}$, such that $pr^*\bar{\omega}^e = \omega^e|_N$.

Hence if dim h(B) < dim W we define $(\bar{W},\bar{\omega}^e)$ such that dim pr$\circ$h(B) = $\bar{W}$. Then $\Sigma(M,\bar{W},B,\bar{f})$, with $\bar{f} = (g,\bar{h})$ and $\bar{h} = pr \circ h$ is regular.

Finally, the following lemma enables us to give local expressions, similar to(3.51) for degenerate Hamiltonian systems.

LEMMA 3.13. Let $K \subset (W,\omega^e)$ be coisotropic with dim W = 2m and dim K = m + k ($k \leq m$). Then there exist canonical coordinates $(v_1,\ldots,v_m,z_1,\ldots,z_m)$ for W such that K is given by $z_{k+1} = \ldots = z_m = 0$.

PROOF: There exist (m-k) independent functions $G_{k+1},\ldots,G_m$ such that locally K is given by $G_{k+1} = \ldots = G_m = 0$. Define $D(x) := (T_xK)^\perp$, $x \in K$. Then D is a regular distribution on K (Theorem 3.12), with dimension m-k. Let $X_{G_{k+1}},\ldots,X_{G_m}$ be the Hamiltonian vectorfields corresponding to $G_{k+1},\ldots,G_m$. Then $\omega(X_{G_{k+j}},Z) = -dG_{k+j}(Z) = 0$, $j = 1,\ldots,m-k$, for every vectorfield Z on K. Therefore D is spanned by $X_{G_{k+1}},\ldots,X_{G_m}$. Since K is coisotropic $0 = \omega(X_{G_{k+i}},X_{G_{k+j}}) = \{G_{k+i},G_{k+j}\}$, for every $i,j = 1,\ldots,m-k$. Hence $G_{k+1},\ldots,G_m$ are a set of *partial canonical* coordinates. By Darboux's theorem (see the proof given in ARNOLD (1978)), we can extend the set $(G_{k+1},\ldots,G_m)$ to a set $(F_1,\ldots,F_m,G_1,\ldots,G_m)$ of canonical coordinates. □

With the aid of Lemma 3.13 the following proposition can be proved (the proof of the linear analogue will be worked out in full detail in Proposition 3.41).

PROPOSITION 3.14. Let $\Sigma(M,W,B,f)$ be a regular full Hamiltonian system and $\Sigma(M,W,B,f,K)$ a degenerate Hamiltonian system. Let dim K = m+k. Take canonical coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ for M. Then there exists coordinates $(y_1,\ldots,y_m,u_1,\ldots,u_m)$ for W and locally a function H(q,p,u) such that $\omega^e = \sum_{j=1}^{m} c_j du_j \wedge dy_j$, $c_j = \pm 1$, and $f(h^{-1}(K))$ is locally given as

$$
\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i}(q,p,u) \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i}(q,p,u) \qquad (3.52) \\
y_j &= -c_j \frac{\partial H}{\partial u_j}(q,p,u) \qquad c_j = \pm 1,\ j = 1,\ldots,m
\end{aligned}
$$

and $u_{k+1} = \ldots = u_m = 0$

Of course the most degenerate situation occurs if the restriction set K is Lagrangian. Then we obtain (3.52) with $u_1 = \ldots = u_m = 0$ and therefore the fibers of the bundle $h^{-1}(K)$ are discrete. If we assume that the fibers consist of exactly one point, we have in fact obtained an *autonomous* system (see Definition 1.9)

$$
\begin{aligned}
q_i &= \frac{\partial \bar{H}}{\partial p_i}(q,p) \qquad i = 1,\ldots,n \\
p_i &= -\frac{\partial \bar{H}}{\partial q_i}(q,p) \qquad (3.53) \\
y_j &= h_j(q,p) \qquad j = 1,\ldots,m
\end{aligned}
$$

where $\bar{H}(q,p) := H(q,p,o)$ and $h_j(q,p) := -\frac{\partial H}{\partial u_j}(q,p,o)$

(Notice that Im h (with $h = (h_1,\ldots,h_m)$) is contained in K).

We close this section with some examples of Hamiltonian systems (more examples appear later on; especially in Sections 3.3, 3.4 and 3.5).

EXAMPLE 1 Consider k point masses m_i, $i = 1,\ldots,k$, in $\mathbb{R}^3$. Denote their positions by $q^i := (q_1^i, q_2^i, q_3^i)$, and their momenta by $p^i := (p_1^i, p_2^i, p_3^i)$. The masses attract each other according to the inverse square law, and the gravitational potential is given by $V(q^1,\ldots,q^k) := \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}$

Furthermore the kinetic energy is $K(p^1,\ldots,p^k) := \sum_{i=1}^{k} \frac{1}{2m_i} |p^i|^2$

We assume that we observe the positions of the first ℓ point masses ($\ell \leq k$) and that we can exert forces on the masses which are observed. Hence the state space is an open set contained in $(q^1,\ldots,q^k,p^1,\ldots,p^k) = \mathbb{R}^{6k}$ (we have to exclude the points with $q^i = q^j$, for some $i \neq j$, since V is there not defined), while W, the set of external variables, is an open set contained in $(q^1,\ldots,q^\ell,F^1,\ldots F^\ell) = \mathbb{R}^{6\ell}$ ($F^i=(F_1^i,F_2^i,F_3^i)$ is the external force on the i-th pointmass). We obtain the full Hamiltonian system

$$
\begin{aligned}
\dot{q}_j^i &= \frac{\partial \bar{H}}{\partial p_j^i} && i = 1,\ldots,k\ ,\ j = 1,2,3 \\
\dot{p}_j^i &= -\frac{\partial \bar{H}}{\partial q_j^i} + F_j^i && i = 1,\ldots,\ell\ ,\ j = 1,2,3 \qquad (3.54) \\
\dot{p}_j^i &= -\frac{\partial \bar{H}}{\partial q_j^i} && i = \ell+1,\ldots,k\ ,\ j = 1,2,3 \\
y_j^i &= q_j^i && i = 1,\ldots,\ell\ ,\ j = 1,2,3
\end{aligned}
$$

with the internal energy $\bar{H}(q^1,\ldots,q^k,p^1,\ldots,p^k) := V(q^1,\ldots,q^k) + K(p^1,\ldots,p^k)$. The generating function $H(q,p,u)$, as in (3.51), is in this case
$H(q^1,\ldots,q^k,p^1,\ldots,p^k,F^1,\ldots,F^\ell) := \bar{H}(q^i,\ldots,q^k,p^1,\ldots,p^k) - \sum_{i=1}^{\ell} \langle q^i,F^i\rangle$

EXAMPLE 2 Consider again k point masses m_i, $i = 1,\ldots,k$, but assume now that the inputs are the positions of the first ℓ pointmasses ($\ell<k$), i.e. $u = (q^1,\ldots,q^\ell)$. The state space consists now of the positions and momenta of the last $k-\ell$ point masses and is therefore an open set contained in $\mathbb{R}^{6(k-\ell)}$. The generating function for the full Hamiltonian system is given by:
$H(q^{\ell+1},\ldots,q^k,p^{\ell+1},\ldots,p^k,q^1,\ldots,q^\ell) := V(q^1,\ldots,q^k) + \sum_{i=\ell+1}^{k} \frac{1}{2m_i}|p^i|^2$
This yields

$$
\begin{aligned}
\dot{q}_j^i &= \frac{\partial H}{\partial p_j^i} && i = \ell+1,\ldots,k \\
\dot{p}_j^i &= -\frac{\partial H}{\partial q_j^i} && j = 1,2,3 \qquad (3.55) \\
y_j^i &= -\frac{\partial H}{\partial q_j^i} && i = 1,\ldots,\ell \\
& && j = 1,2,3
\end{aligned}
$$

Hence the outputs y^i equal the *forces* exerted on the first ℓ point masses, from which the positions can be controlled.

EXAMPLE 3 Take again k point masses m_i, and assume that the input is now the *mass* of the first particle m_1. Assume furthermore that the position q^1 of this particle is fixed and hence $p^1 = 0$. The generating function is $H = \sum_{i<j} \frac{m_i m_j}{|q^i - q^j|} + \sum_{i=2}^{k} \frac{1}{2m_i}|p^i|^2$ with $u = m_1$, and we obtain

$$\dot{q}^i_j = \frac{\partial H}{\partial q^i_j} \qquad \begin{matrix} i = 2,\ldots,k \\ j = 1,2,3 \end{matrix}$$

$$\dot{p}^i_j = -\frac{\partial H}{\partial q^i_j} \tag{3.56}$$

$$y = -\frac{\partial H}{\partial m_1} = -\sum_{i=2}^{k} \frac{m_i}{|q^1 - q^i|}$$

The output y is (minus) the potential energy due to the interactions of the masses $m_2, m_3, \ldots, m_k$ with a unit mass located at q^1.

The above examples yield also examples of *degenerate* Hamiltonian systems. In Example 1 we can assume that we observe ℓ point masses but exert forces only on a *part* of the ℓ point masses, or that the exerted force on some of the ℓ point masses is a potential force derived from a potential that is a function of the positions of the ℓ point masses. In Example 2 we may assume that we can only control some of the positions of the first ℓ masses. The use of degenerate Hamiltonian systems becomes also clear in the next section.

3.2.1. Hamiltonian interconnections

In this section we define Hamiltonian interconnections and show how a Hamiltonian interconnection of Hamiltonian systems yields a (degenerate) Hamiltonian system. In Section 3.1 we already encountered a (very simple) example of a Hamiltonian interconnection, namely Newton's third law $q_1 = q_2$, $F_1 = F_2$ yielding the autonomous Hamiltonian system (3.6). Also in Section 3.1.2 we saw how by interconnecting electrical circuit elements in a "Hamiltonian way" (in this case in accordance with Kirchhoff's laws) the interconnected network is again Hamiltonian (or "reciprocal").

DEFINITION 3.15. Let (W_i,ω_i^e), i = 1,...,k, be symplectic manifolds. Then $(W_1\times\ldots\times W_k, \pi_1^*\omega_1^e+\ldots+\pi_k^*\omega_k^e)$ is a symplectic manifold (π_i are the projections of $W_1\times\ldots\times W_k$ on W_i). An *interconnection* of $(W_i)_{i=1,\ldots k}$ is a submanifold $I \subset W_1\times\ldots\times W_k$. We call an interconnection *(full) Hamiltonian* if I is a Lagrangian submanifold.

EXAMPLE: (see (3.1) to (3.6)). Let $W_1 = \{(q_1,F_1)\} = \mathbb{R}^2$ and $W_2 = \{(F_2,q_2)\}=\mathbb{R}^2$, with the natural symplectic form $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ on $\mathbb{R}^2$. Then the subspace of $W_1 \times W_2$, defined by $q_1 = q_2$ and $F_1 = F_2$ is a Lagrangian subspace.

Remark: A Hamiltonian interconnection as above is in the literature also called a canonical or symplectic relation (SNIATYCKI & TULCZYJEW (1972a)).

If the symplectic manifolds W_i are cotangent bundles T^*Y_i, with θ_i^e the natural 1-forms, we can give a stronger version:

DEFINITION 3.16. Let (T^*Y_i,θ_i^e), i = 1,...,k be cotangent bundles. Then $(T^*Y_1\times\ldots\times T^*Y_k,\pi_1^*\theta_1^e+\ldots+\pi_k^*\theta_k^e)$ is again a cotangent bundle with a natural 1-form (π_i projections on T^*Y_i). An interconnection $I \subset T^*Y_1\times\ldots\times T^*Y_k$ is called *Lagrangian* if I is a Lagrangian submanifold, and also $\pi_1^*\theta_1^e+\ldots+\pi_k^*\theta_k^e$ restricted to I is zero.

To see what this last definition amounts to, we assume for simplicity that we have only two manifolds $W_1 = T^*Y_1$ and $W_2 = T^*Y_2$, of dimension $2m_1$, respectively $2m_2$. In natural coordinates $\theta_1^e = \sum_{j=1}^{m_1} u_i^1\,dy_i^1$ and $\theta_2^e = \sum_{j=1}^{m_2} u_j^2\,dy_j^2$. Assume furthermore that $m_1 \geq m_2$ and that the interconnection $I \subset T^*Y_1 \times T^*Y_2$ can be parametrized by (y_i^1,u_j^2), i = 1,...,m_1, j = 1,...,m_2. Then there exists a smooth map $\phi : Y_1 \longrightarrow Y_2$ such that (see ABRAHAM & MARSDEN (1978, Exercise 3.2F, 5.2B), BRAYTON (1978)) $I = \{(y^1,u^1,y^2,u^2) \in T^*Y_1 \times T^*Y_2 \mid y^2 = \phi(y^1),\ u^1 = -\frac{\partial\phi}{\partial y^1}\,u^2\}$ Hence I is, what is called in electrical network theory, nonmixing, i.e. y^2 is only related to y^1, and u^2 is only related to u^1. Moreover we see that the relation between u^2 and u^1 is *linear* in every y. An even more special form of I arises when $W = Y_1 \times U_1$, $W_2 = Y_2 \times U_2$ and not only $\sum_{i=1}^{m_1} u_i^1\,dy_i^1 + \sum_{j=1}^{m_2} u_j^2\,dy_j^2$, but also the dual 1-form $\sum_{i=1}^{m_1} y_i^1\,du_i^1 + \sum_{j=1}^{m_2} y_j^2\,du_j^2$ is zero restricted to I.

It follows that the interconnection I is in this case necessarily totally linear, i.e. there exists a matrix A such that $y^2 = Ay^1$ and $u^1 = -A^T u^2$. These are exactly the reciprocal interconnections encountered in (linear) electrical network theory (in fact such an interconnection can be realized by a set of transformers, see Section 4.2.4, Proposition 4.34). If the only elements of A are 0,1 or -1 then these equations are exactly Kirchhoff's laws (with for instance $(y^1,y^2) = (v^1,v^2)$ the voltages and $(u^1,u^2) = (i^1,i^2)$ the currents). We refer to BRAYTON (1978) for more details and results about interconnections, especially for electrical networks.

Finally we define a more general kind of interconnection.

DEFINITION 3.17. Let (W_i,ω_i^e), $i = 1,\ldots,k$, be symplectic manifolds. An interconnection I is called *degenerate Hamiltonian* if I is a coisotropic submanifold (see Definition 3.9) of $(W_1\times\ldots\times W_k, \pi_1^* \omega_1^e + \ldots + \pi_k^* \omega_k^e)$.

Definition 3.17 allows, contrary to Definition 3.15, interconnections where not "half of the variables are linked to the other half of the variables". Indeed, since I is coisotropic the distribution $I^\perp(x) := (T_x I)^\perp$, $x \in I$, is a regular distribution on I (Theorem 3.12). Therefore I can be (locally) factored out by $I^\perp$ to obtain a new symplectic manifold, which can be interpreted as the set of those variables which are *not* interconnected.

The next theorem shows that, under regularity assumptions, a degenerate Hamiltonian interconnection on the spaces of external variables of dynamical Hamiltonian systems yields a new (degenerate) Hamiltonian system.

THEOREM 3.18. Let $\Sigma_i(M_i,W_i,B_i,f_i=(g_i,h_i))$, $i = 1,\ldots,k$ be full Hamiltonian systems. Define $g := (g_1,\ldots,g_k)$: $B_1\times\ldots\times B_k \longrightarrow TM_1\times\ldots\times TM_k$ and $h := (h_1,\ldots,h_k)$: $B_1\times\ldots\times B_k \longrightarrow W_1\times\ldots\times W_k$. Let $I \subset W_1\times\ldots\times W_k$ be a degenerate Hamiltonian interconnection such that $h^{-1}(I) \subset B_1\times\ldots\times B_k$ is a bundle over $M_1\times\ldots\times M_k$. Then $\Sigma(M_1\times\ldots\times M_k, W_1\times\ldots\times W_k, B_1\times\ldots\times B_k, f := (g,h), I)$ is a degenerate Hamiltonian system.

PROOF: We note that a product of fiber bundles is itself a fiber bundle above the product of the basis spaces. The rest follows from Definition 3.10. □

Remark: It also follows that a Hamiltonian interconnection of degenerate Hamiltonian systems results in a degenerate Hamiltonian system.

3.2.2. About forces

"Denn man muss die Gegenstände schon in ziemlich hohem Grade kennen, wenn man die Regel angeben will, wie sich eine Wissenschaft von ihnen zu Stande bringen lasse"

I. Kant : Kritik der reinen Vernunft

In the previous sections we used without hesitation the word *force*. While in older works on classical mechanics the notion of force is treated as one of the basic concepts of mechanics, nowadays the idea of force has somewhat of a dubious reputation. Only in the more technical literature the study of forces still has an important place.

There are many historical reasons for the fact that the concept of force is not very fashionable anymore. We feel that it is necessary to say at least a few words about it, eventhough we do not claim any expertise on it. For more thorough statements we refer to books on the history of classical mechanics, and the philosophy of science (see for instance DIJKSTERHUIS (1950), JAMMER (1957)).

A general reason for the peculiar position of force in classical mechanics is the emphasis that has been laid on the description of the behavior of *isolated* systems. In this case the present forces are functions of the configuration and/or velocity variables and are therefore really *internal* forces. Especially if these internal forces are *conservative*, then they can be easily incorporated into the system by adding a potential function to the internal energy. Therefore if one concentrates on the description of isolated systems, one can give a formulation of the behavior of the system by using only the configuration variables and their time-derivatives. It is tempting and has the air of rationality to totally disregard the notion of force, being a notion that can only obscure the mathematical description of the system. The extreme position is then to identify mechanics with the study of second-order differential equations (POINCARÉ (1905, pp 89-110). Even for isolated mechanical systems we are of the opinion that this point of view does not do justice to the science of mechanics. Newton's contribution to celestial mechanics was not only to give a second-order differential

equation, which solutions satisfy Kepler's laws, but also to show how this equation is obtained. In the first place he showed that the law $F = m\ddot{q}$, which is the basic equation for the description of mechanical systems on earth, can be also used for a description of the orbits of the planets. Furthermore he defined the gravitational force, according to the inverse square law, and then *he set F equal to this gravitational force*. Thus Newton explained the resulting second-order differential equation by showing how it is constructed from simple sub systems satisfying basic laws (for a more elaborate system theoretic treatment we refer to WILLEMS (1979)). In this context we remark that if there are *several* internal forces present in a system, one usually loses information about the structure of the system by only considering the *sum* of these internal forces (this point was made by Kirchhoff (see JAMMER (1957, p 223)). Describing the system as an interconnection of subsystems with external forces can therefore be very useful. Of course the tendency to consider only isolated systems is in sharp contrast with the attitude in technical applications of trying to *prescribe* the behavior of the system. Then a framework which cannot deal with forces on a fundamental level is totally inadequate. It seems that this "engineering attitude" did not have much influence on the mainstream of classical mechanics, at least not on its theoretical developments. For instance the study of celestial mechanics, the paradigmatic example of an isolated system, has influenced the mathematical theory of mechanics much more, with problems like the stability of the solar system. We remark that contrary to mechanics in *thermodynamics* the engineering aspect *did* have an important impact on the theoretical developments.

The role of force in physics has furthermore been obscured by two more or less related issues. The first is the notion of *causality*. It has been assumed that by using the word force in the description of the dynamical behavior of a system, one says something about the *cause* of motion. Indeed, forces are identified with causal explanations. Now it has become a generally shared conviction that the search for causes should not be a part of the science of mechanics. This diminished popularity of causality in science has also influenced the status of force. We remark however that by adopting the notion of force, one is not obliged to take a real cause/effect point of view. In the same way as we can sometimes split the variables of an external system into variables which we can call inputs and other variables which we can call outputs (see Chapter 1), we can sometimes split the external varia-

bles of a mechanical system into forces and positions. This only implies something about the structure of the compability relations describing the external behavior of the system.

The second issue that obscures the notion of force has to do with the nature of gravitation. Ever since the rise of classical mechanics a prototype of force has been the gravitational force. Therefore the historical developments in the treatment of gravitation have had as immediate counterparts changes in opinion about the notion of force. The somewhat mystical character of gravitational forces has been partly responsible for the flowering as well as the descent of the role of force in science.

We are of the opinion that, despite the historical burden that lies on the notion of force, forces deserve a fundamental place in a mathematical theory of mechanics, if one wants to cope with the (from a practical point of view very plausible) possibility of exerting forces on a system, and if one wants to include *statics* in such a theory. We remark that the theory of statics provides a theory of *measuring* forces. Hence from an operational point of view statics can underly a theory of dynamics.

Of course a cornestone in any theory of force is the interpretation of Newton's second law. If we do not want to allow forces as basic entities in the science of mechanics we are obliged to interpret F = ma as a mere definition of force. At most F = ma can be viewed as a methodological rule for investigating dynamical systems (see NAGEL (1961, pp.153-203)). It seems that at least Newton himself saw "his second law" as a synthetic statement, expressing a relation between two basic variables, the exerted force and the acceleration, with the mass m assumed to be defined in an independent way (DIJKSTERHUIS (1950, p.520)).

3.2.3. Controllability and observability

We observed that a Hamiltonian system $\Sigma(M,W,B,f)$ has a local input-output representation with feedthrough term (see Proposition 3.7 and 3.8), namely (3.51). Hence we can define the *extended observability codistribution* O^e (Definition 2.38) which characterizes the local distinguishability properties of a representation (3.51) (and since local minimality is equivalent to local distinguishability, it also characterizes the local minimality properties).

Furthermore we can always define the *extended controllability distribution* C^e (Definition 2.43), which characterizes the strong accessibility

properties. We shall now show that for a full Hamiltonian system 0^e and C^e are *isomorphic*.

THEOREM 3.19. Let $\Sigma(M,W,B,f)$ be a full Hamiltonian system with local representation

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q,p,u) \qquad i = 1,\ldots,n \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q,p,u) \qquad (3.57) = (3.51) \\ y_j &= -c_j \frac{\partial H}{\partial u_j}(q,p,u) \end{aligned}$$

with $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$ and $\omega^e = \sum_{j=1}^{m} c_j du_j \wedge dy_j$

We can regard (q,p,u) as fiber respecting coordinates for B. Denote the vectorfield $\dot{q}_i = \frac{\partial H}{\partial p_i}(q,p,u)$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}(q,p,u)$ by X_H. Define the linear space of functions G^e as the space which contains the functions $u_1,\ldots,u_m$, $\frac{\partial H}{\partial u_1},\ldots,\frac{\partial H}{\partial u_m}$ and is invariant under taking Lie derivatives with respect to the vectorfields X_H and $\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_m}$. Define the codistribution 0^e on B by $0^e(q,p,u) = \text{Span}\{dk(q,p,u) \mid k \in G^e\}$ (see Definition 2.38). Define the distribution C^e on B as the smallest distribution which contains the vectorfields $\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_m}$ and is invariant under taking Lie derivatives with respect to the vectorfields X_H and $\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_m}$. Define an isomorphism α: TB $\longrightarrow$ T^*B (on this coordinate neighborhood) by $\alpha(\frac{\partial}{\partial x_i}) := \omega(\frac{\partial}{\partial x_i},\text{—})$ $i = 1,\ldots,n$ and $\alpha(\frac{\partial}{\partial u_i}) = du_i$. Then:

$$\alpha(C^e) = 0^e$$

PROOF: Rewrite $\frac{\partial H}{\partial u_i} = \pounds_{\frac{\partial}{\partial u_i}} H$. It is easy to see that $d_x(\pounds_{\frac{\partial}{\partial u_i}} H) = \pounds_{\frac{\partial}{\partial u_i}}(d_x H)$ (d_x means differentiation w.r.t. x).
Write $0^e = \{du_1,\ldots,du_m, d(\frac{\partial H}{\partial u_1}),\ldots,d(\frac{\partial H}{\partial u_m}) + \text{invariance under } X_H \text{ and } \frac{\partial}{\partial u_i}\}$

Then $0^e = \{du_1,\ldots,du_m, \pounds_{\frac{\partial}{\partial u_1}} d_x H,\ldots,\pounds_{\frac{\partial}{\partial u_m}} d_x H + \text{invariance under } X_H \text{ and } \frac{\partial}{\partial u_i}\}$

Furthermore

$$C^e = \{\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_m} + \text{invariance under } X_H \text{ and } \frac{\partial}{\partial u_i}\}$$

$$= \{\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_m}, \pounds_{\frac{\partial}{\partial u_1}} X_H,\ldots,\pounds_{\frac{\partial}{\partial u_m}} X_H + \text{invariance under } X_H \text{ and } \frac{\partial}{\partial u_i}\}$$

Make now the following observations:

a) $\alpha(\frac{\partial}{\partial u_i}) = du_i$

b) $\alpha(X_H) = d_x H$

c) $\alpha(\pounds_{\frac{\partial}{\partial u_i}} X_H) = \pounds_{\frac{\partial}{\partial u_i}} \alpha(X_H) = \pounds_{\frac{\partial}{\partial u_i}} d_x H$

d) $\pounds_{X_H} \pi^* \omega = 0$ and $\pounds_{\frac{\partial}{\partial u_i}} \pi^* \omega = 0$ (with π projection $B \longrightarrow X$)

Therefore X_H and $\frac{\partial}{\partial u_i}$ are Hamiltonian vectorfields with respect to the degenerate symplectic form $\pi^*\omega$ on B. Since Lie brackets of Hamiltonian vectorfields are again Hamiltonian, C^e is generated by Hamiltonian vectorfields.

e) Take an arbitrary Hamiltonian vectorfield Z in C^e. Then

$$\alpha(\pounds_{X_H} Z) = \pounds_{X_H} \alpha(Z) \ , \qquad \text{since } \pounds_{X_H} \omega = 0$$

$$\alpha(\pounds_{\frac{\partial}{\partial u_i}} Z) = \pounds_{\frac{\partial}{\partial u_i}} \alpha(Z) \ , \quad \text{since } \pounds_{\frac{\partial}{\partial u_i}} \omega = 0$$

These observations yield easily that $\alpha(C^e) = 0^e$. □

Remark: In fact α respects the "structure"of C^e and 0^e. This will be more explicitly stated in the affine case (Theorem 3.31).

We see that for full Hamiltonian systems controllability and observability is characterized by one codistribution (or distribution). For later use we give

DEFINITION 3.20. Let $\Sigma(M,W,B,f)$ be a full Hamiltonian system. Then

a) Σ satisfies the *minimality rank condition* (M.R.C) if dim $0^e(q,p,u)$ = dim B for a certain $(q,p,u) \in B$.
(or equivalently dim $C^e(q,p,u)$ = dim B somewhere)

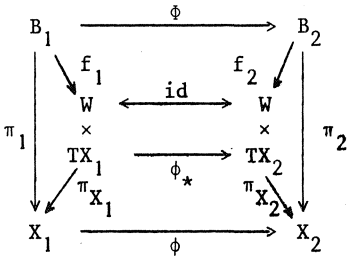
b) Σ satifies the *strong minimality rank condition* (S.M.R.C) if dim 0^e = dim B on an open and dense subset of B.

Remark: If Σ satisfies the minimality rank condition, then Σ is strongly accessible (Theorem 2.46) as well as locally distinguishable as well as locally minimal (Proposition 2.39). In applications later on we shall sometimes only use that Σ is strongly accessible.

3.2.4 Equivalent Hamiltonian systems

In Definition 2.20 we called two nonlinear systems $\Sigma^1(X_1,W,B_1,f_1)$ and $\Sigma^2(X_2,W,B_2,f_2)$ *equivalent* if there exist diffeomorphisms $\phi : X_1 \to X_2$ and $\Phi : B_1 \to B_2$ such that the diagram

(3.58)



commutes.

We now show that equivalence of Hamiltonian systems implies equivalence in the "category of Hamiltonian systems", i.e. ϕ is a symplectomorphism.

THEOREM 3.21 Let $\Sigma^1(M_1,W,B_1,f_1)$ and $\Sigma^2(M_2,W,B_2,f_2)$ be full Hamiltonian systems, with (M_1,ω_1) and (M_2,ω_2) symplectic manifolds. Let Σ^1 and Σ^2 be equivalent and let the equivalence be given by $\phi : M_1 \to M_2$ and $\Phi : B_1 \to B_2$ as in (3.58). Assume that Σ^1 and Σ^2 satisfy the minimality rank condition. Assume furthermore that $\phi^*\omega_2 - \omega_1$ has constant rank. Then $\phi^*\omega_2 = \omega_1$, i.e. ϕ is a symplectomorphism.

PROOF Since $f_1(B_1)$ is a Lagrangian submanifold of $(TM_1 \times W, \pi_1^*\dot{\omega}_1 - \pi_2^*\omega^e)$, we have that $g_1^*\dot{\omega}_1 = h_1^*\omega^e$. Analogously $g_2^*\dot{\omega}_2 = h_2^*\omega^e$. Because (3.58) commutes, $f_1(B_1)$ is mapped by ϕ_* and id in a bijective way onto $f_2(B_2)$. Therefore $\Sigma(M_2,W,B_2,\tilde{f}_2)$, with $\tilde{f}_2 = (\phi_* \circ g_1 \circ \Phi^{-1}, id \circ h_1 \circ \Phi^{-1})$ is a Hamiltonian system. Hence $(\Phi^{-1})^* g_1^* (\phi_*)^* \dot{\omega}_2 = (\Phi^{-1})^* h_1^* (id)^* \omega^e$ which implies $g_1^* (\phi_*)^* \dot{\omega}_2 = h_1^* \omega^e$. Together with $g_1^* \dot{\omega}_1 = h_1^* \omega^e$, this yields $g_1^* ((\phi_*)^* \dot{\omega}_2 - \dot{\omega}_1) = 0$.

Define $\Omega = \phi^*\omega_2 - \omega_1$, then we obtain $g_1^* \dot{\Omega} = 0$. Since Ω is closed and by assumption has constant rank there exist coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ for M_1 such that (ABRAHAM & MARSDEN (1978, Theorem 5.1.3))

$$\Omega = \sum_{i=1}^{k} dq_i \wedge dp_i, \quad k \leq n$$

(rank $\Omega = 2k$). Denote the (regular) distribution ker Ω by D, then

$$D = \text{span}\left\{\frac{\partial}{\partial q_{k+1}},\ldots,\frac{\partial}{\partial q_n}, \frac{\partial}{\partial p_{k+1}},\ldots,\frac{\partial}{\partial p_n}\right\} .$$

Write correspondingly to these coordinates $g_1 = (g_{q_1},\ldots,g_{q_n},g_{p_1},\ldots,g_{p_n})$ then $g_1^* \dot{\Omega} = 0$ yields $\sum_{i=1}^{k} (dg_{p_i} \wedge dq_i - dg_{q_i} \wedge dp_i) = 0$ which implies that

$$\frac{\partial g_{p_i}}{\partial u_j} = \frac{\partial g_{q_i}}{\partial u_j} = 0 \qquad \begin{matrix} i = 1,\ldots,k \\ j = 1,\ldots,m \end{matrix} \qquad \text{and}$$

$$\frac{\partial g_{p_i}}{\partial q_j} = \frac{\partial g_{p_i}}{\partial p_j} = \frac{\partial g_{q_i}}{\partial q_j} = \frac{\partial g_{q_i}}{\partial p_j} = 0 \qquad \begin{matrix} i = 1,\ldots,k \\ j = k+1,\ldots,n \end{matrix}$$

These are however the local expressions for $g_{1*}(\pi_*^{-1}(D)) \subset \dot{D}$! Since $\Sigma^1(M_1,W_1,B_1,f_1)$ satisfies MRC and is therefore strongly accessible (Theorem 3.19), D = ker Ω is necessarily TM_1, hence $\Omega = 0$ or equivalently $\phi^*\omega_2 = \omega_1$. □

Remark: If we assume that Σ satisfies the *strong minimality rank condition* we can omit the regularity assumption that $\phi^*\omega_2 - \omega_1$ has constant rank. There is always an open and dense set of points of M_1 which have a neighborhood on which $\phi^*\omega_2 - \omega_1$ has constant rank. Then by the same arguments as in the proof of Theorem 3.21 we can prove that on all these neighborhoods $\phi^*\omega_2 - \omega_1 = 0$. Hence by continuity $\phi^*\omega_2 = \omega_1$ everywhere.

Note that Theorem 3.21 expresses that the symplectic structure on the state space of a full Hamiltonian system satisfying (S)MRC is itself a *structural invariant*.

3.3 Affine Hamiltonian systems

In this section we treat a subclass of nonlinear Hamiltonean systems, which encompasses most of the examples encountered in mechanics. This subclass has a mathematical structure that is much easier to handle than the structure of general Hamiltonian systems.

Consider a nonlinear input-output system $\Sigma(X,\tilde{B},Y,g,h)$ (Definition 2.21). In order to define a Hamiltonian input-output system we require that

(i) X is a symplectic manifold (M,ω)

(ii) $\tilde{B} = T^*Y$. In section 3.1.1 we saw that external forces (inputs) can be naturally considered as elements of the fibers of the cotangent bundle over the manifold of positions (outputs). Such an element α of a fiber of T^*Y is a linear function on the tangent vectors $\dot{y}$ of Y in that same point. Therefore $\alpha(\dot{y})$ (force times velocity) is defined and represents the *instantaneous external work* performed on the system. Furthermore, $\tilde{B} = T^*Y$ is a symplectic manifold with the natural symplectic form on T^*Y.

(iii) Σ is a Hamiltonian system in the sense of Definition 3.6, i.e. $f(h^*(T^*Y))$ is a Lagrangian submanifold of $TM \times T^*Y$ with symplectic form $\pi_1^* \dot{\omega} - \pi_2^* \omega^e$ (f equals $(g,\tilde{h})$, π_1 and π_2 are the projections on TM and T^*Y).

The following proposition shows that such a Hamiltonian input-output system is automatically an *affine* input-output system (Definition 2.22).

PROPOSITION 3.22 Let $\Sigma(M,T^*Y,Y,g,h)$ be a Hamiltonian input-output system. Then Σ is an *affine* input-output system.

PROOF Let $(y,u) = (y_1,\ldots,y_m,u_1,\ldots,u_m)$ be natural coordinates for T^*Y and let (x,u) be output induced fiber respecting coordinates for h^*B, such that $\tilde{h} : h^*B \to T^*Y$ is given by $\tilde{h}(x,u) = (h(x),u)$. Because Σ is a Hamiltonian system, $f(h^*B)$, with $f = (g,\tilde{h})$, is a Lagrangian submanifold of $TM \times T^*Y$. This yields

$$g^*\dot{\omega} = \tilde{h}^*\omega^e. \tag{3.59}$$

Substituting $\tilde{h}(x,u) = (h(x),u)$ in (3.59) yields

$$\omega(\frac{\partial g}{\partial u_i}(x,u),-) = dh_i(x), \qquad i = 1,\ldots,m \tag{3.60}$$

where $h : M \to Y$ is equal to $h = (h_1,\ldots,h_m)$. Since ω is nondegenerate, (3.60) implies that $\frac{\partial g}{\partial u_i}(x,u)$ does not depend on u. Hence $g(x,u)$ is affine

in u, and there exist locally vectorfields A and B_i on M, i = 1,...m, such that $g(x,u) = A(x) + \sum_{i=1}^{m} u_i B_i(x)$ and $B_i = \frac{\partial g}{\partial u_i}(x,u)$.

By (3.60) B_i satisfies $\omega(B_i,-) = dh_i(x)$, i = 1,...,m. Finally (3.59) yields $\sum_{i=1}^{m} dA_{n+i} \wedge dq_i + dp_i \wedge dA_i = 0$, with $A = (A_1,\ldots,A_n,A_{n+1},\ldots,A_{2n})$ in canonical coordinates $x = (q_1,\ldots,q_n,p_1,\ldots,p_n)$ for M. Hence $d(\omega(A,-)) = 0$, and A is Hamiltonian. □

To simplify notation we give the following concise definition of an affine Hamiltonian system, which we shall use in the sequel.

DEFINITION 3.23 Let (M,ω) be a symplectic manifold, denoting the state space. Let Y be the output (observation) manifold. Define $\Omega := \pi_1^* \omega - \pi_2^* \omega^e$, with ω^e the natural symplectic form on T^*Y (π_1 and π_2 projections of $TM \times T^*Y$ onto TM and T^*Y). An *affine Hamiltonian system* is given by a submanifold $L \subset TM \times T^*Y$ such that

(i) L can be parametrized by the coordinates of M and the fibers of T^*Y

(ii) L is a Lagrangian submanifold of $(TM \times T^*Y, \Omega)$

(iii) The value of the Y-coordinates of a point on L is only a function of the M-coordinates of this point.

We denote the system by $\Sigma(M,T^*Y,L)$.

Remark: Notice that the input bundle B is totally suppressed in the above definition.

PROPOSITION 3.24 Let $\Sigma(M,T^*Y,L)$ be an affine Hamiltonian system. Then in local coordinates the system is given by

$$\begin{aligned} \dot{x} &= X_H(x) - \sum_{i=1}^{m} u_i X_{C_i}(x) \\ y_i &= C_i(x) \qquad\qquad i = 1,\ldots,m \end{aligned} \tag{3.61}$$

with x local coordinates for M, $y = (y_1,\ldots,y_m)$ local coordinates for Y and $u = (u_1,\ldots,u_m)$ the corresponding natural coordinates for the fibers of T^*Y. We call H the *energy* function and C_i the *observation* (or *output*) functions.

PROOF: Because of i) and iii) the generating function of L with respect to the symplectic form Ω has the form $H(x) - \sum_{i=1}^{m} u_i C_i(x)$. Therefore the $\dot{x}$-coordinates of points of L are given by $\dot{x} = X_H(x) - \sum_{i=1}^{m} u_i X_{C_i}(x)$, and the y-coordinates are equal to $y_i = C_i(x)$, $i = 1,\ldots,m$. □

We see that an affine input-output system $\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$, $y_i = C_i(x)$, $i = 1,\ldots,m$, on (M,ω) is Hamiltonian if A is a locally Hamiltonian vectorfield ($\pounds_A\omega=0$), and B_i are Hamiltonian vectorfields satisfying $\omega(B_i,-) = dC_i$, $i = 1,\ldots,m$. Another way to look at equations (3.61) is to start from a (locally) Hamiltonian vectorfield $A = X_H$ on (M,ω), to add an observation map $C : M \longrightarrow Y$, and to define the input vectorfields (the directions in which we can exert external forces) as the Hamiltonian vectorfields with Hamilton functions $-C_i$, where in coordinates for Y, $C = (C_1,\ldots,C_m)$. This expresses the idea that the possibilities of influencing the system correspond to adding to the Hamiltonian H a function that only depends on the observations.

In a certain sense, an affine Hamiltonian system can be viewed as a first-order approximation of a general Hamiltonian system. This can be seen as follows. Let $\Sigma(M,W,B,f)$ be a Hamiltonian system, and let $K \subset W$ be a Lagrangian restriction manifold resulting in an autonomous degenerate Hamiltonian system $\Sigma(M,W,B,f,K)$ (see Definition 3.10). In coordinates we have equation (3.53):

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(q,p,0) = \frac{\partial \bar{H}}{\partial p_i}(q,p) \qquad i = 1,\ldots,n$$

$$p_i = -\frac{\partial H}{\partial q_i}(q,p,0) = -\frac{\partial \bar{H}}{\partial q_i}(q,p)$$

$$y_i = -\frac{\partial H}{\partial u_j}(q,p,0) =: h_j(q,p) \qquad j = 1,\ldots,m$$

where $\bar{H}(q,p) := H(q,p,0)$. Since K is Lagrangian we know (ABRAHAM & MARSDEN (1978, Theorem 5.3.18)) that K has an open neighborhood in W, which is symplectomorphic to the cotangent bundle T^*K, with its natural symplectic form, in such a way that $K \subset W$ is mapped onto the zero-section of T^*K. Therefore we define Y:= K, and construct an affine Hamiltonian system as the Lagrangian submanifold of $TM \times T^*Y$ that corresponds to the equations

$$\dot{x} = X_{\bar{H}}(x) - \sum_{j=1}^{m} u_j X_{h_j}(x)$$

(3.62) $$y_j = h_j(x) \qquad \text{with } x = (q,p)$$

The generating function of this Lagrangian submanifold is

$$\bar{H}(q,p) - \sum_{i=1}^{m} u_i h_j(q,p) = H(q,p,0) + \sum_{i=1}^{m} u_i \frac{\partial H}{\partial u_i}(q,p,0) \tag{3.63}$$

i.e. the first order approximation (with respect to u) of H(p,q,u).

We now return to the local representation (3.61). First observe that by choosing other coordinates $(y_1', \ldots, y_m')$ for Y one obtains another set of natural coordinates $(u_1', \ldots, u_m')$, and that the vectorfields $X_{C_1}, \ldots, X_{C_m}$ are changed into $X_{C_1'}, \ldots, X_{C_m'}$, where span $\{X_{C_1'}(x), \ldots, X_{C_m'}(x)\}$ = span $\{X_{C_1}(x), \ldots, X_{C_m}(x)\}$. Hence this corresponds to a state dependent (actually *output* dependent) transformation of the input space above each $x \in M$.

We consider the following type of *feedback*.

DEFINITION 3.25 Let $\Sigma(M,T^*Y,L)$ be an affine Hamiltonian system. Hamiltonian feedback for Σ corresponds to a Lagrangian submanifold $F \subset T^*Y$, which can be parametrized by Y, i.e. F is the graph of a closed one-form β on Y. Hence locally there exists a function $P : Y \longrightarrow \mathbb{R}$ such that $\beta = dP$. The Hamiltonian feedback is given by the *output* feedback $v = \tilde{\alpha}(y,u) = \frac{\partial P}{\partial y}(y) + u$.

Remark: Consider the static Hamiltonian system on T^*Y given by F (Definition 3.3). Define a Hamiltonian interconnection with $\Sigma(M,T^*Y,L)$ by identifying $(y,u) \in T^*Y$ with $(y,-u) \in T^*Y$. The resulting system is an autonomous Hamiltonian system with Hamiltonian H + P∘C, if H is the energy function of $\Sigma(M,T^*Y,L)$, and C the observation map.

We can prove

THEOREM 3.26 Let $\Sigma(M,T^*Y,L)$ be an affine Hamiltonian system in local coordinates given by

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$$

(3.64) $$y_i = C_i(x) \qquad i = 1, \ldots, m$$

with $\mathfrak{L}_{A}\omega = 0$, and hence locally $A = X_H$, and $\omega(B_i,-) = dC_i$, i.e. $B_i = -X_{C_i}$.

Let $u \longmapsto v := \alpha(x,u)$ be a feedback for (3.64). The system after feedback is again an affine Hamiltonian system

$$\begin{aligned} \dot{x} &= \tilde{A}(x) + \sum_{i=1}^{m} v_i \tilde{B}_i(x) \\ y_i &= C_i(x) \qquad\qquad i = 1,\ldots,m \end{aligned} \tag{3.65}$$

if and only if α is a Hamiltonian feedback, i.e. if there exists (locally) a function $P : Y \longrightarrow \mathbb{R}$ such that $\tilde{A}$ and $\tilde{B}_i$ satisfy

i) $\tilde{B}_i = B_i$, $i = 1,..,m$

ii) $\tilde{A} = X_{\tilde{H}}$, with $\tilde{H} = H + P \circ C$

PROOF: It is clear that if α is a Hamiltonian feedback, then the resulting system (3.65) is again affine Hamiltonian. Let now α be a feedback such that (3.65) is affine Hamiltonian. Because the system after feedback must be again affine, $\alpha(x,u)$ has the form $\alpha(x,u) = \nu(x) - K(x)u$ with ν a vector and $K(x)$ a matrix. Since $\omega(\tilde{B}_i,-) = dC_i$, it follows that $\tilde{B}_i = B_i$, $i = 1,\ldots,m$. Hence $K(x) = I$ (the $m \times m$ identity matrix). Take output-induced fiber respecting coordinates (x,u) for the vector bundle h^*T^*Y. Then feedback amounts to changing the sections $u = $ constant into new sections given by $\alpha(x,u) = v = $ constant (see Chapter 2). In the Hamiltonian case these new sections of B have to satisfy the condition that the images of these sections under $\tilde{h}$ in T^*Y are Lagrangian submanifolds and therefore have dimension $m = \dim Y$. This implies that $\alpha(x,u)$ can only depend on $C(x)$ and u, and therefore there exists an outputfeedback $v = \tilde{\alpha}(y,u)$ such that $\alpha(x,u) = \tilde{\alpha}(C(x),u)$. Also $\tilde{\alpha}$ is such that (y,v) are canonical coordinates (i.e. $\omega^e = \Sigma dv_i \wedge dy_i$), since the sections in T^*Y defined by $v = $ constant have to be Lagrangian. Consider now the new zero-section $v = 0$. This has a generating function $P : Y \longrightarrow \mathbb{R}$. If in the old coordinates (y,u) the system had the generating function $H(x) - \sum_{i=1}^{m} u_i C_i(x)$, then it follows that in the new coordinates (y,v), the system has the generating function $H(x) + P(C(x)) - \sum_{i=1}^{m} v_i C_i(x)$. Therefore $\tilde{A} = X_{\tilde{H}}$, with $\tilde{H} = H + P \circ C$. □

Notice that the total class of transformations which is allowed in order that (3.61) remains affine Hamiltonian is exactly equal to the class of transformations characterized in the following

PROPOSITION 3.27 (for a proof, see ABRAHAM & MARSDEN (1978, Exercise 3.2F)). Let ϕ: $T^*Y \longrightarrow T^*Y$ be such that

(i) ϕ maps fibers of T^*Y onto fibers

(ii) $\phi^*\omega^e = \omega^e$ (ω^e the natural symplectic from on T^*Y)

Then there exist a diffeomorphism $\psi : Y \longrightarrow Y$ and a closed one-form β on Y such that $\phi = T_\beta \circ \psi^*$, where T_β denotes fiberwise translation by β.

The transformation ψ corresponds to changing the input vectorfields X_{C_i} into $X_{C_i'}$ such that span $\{X_{C_1}(x),\ldots,X_{C_m}(x)\}$ = span $\{X_{C_1'}(x),\ldots,X_{C_m'}(x)\}$, while T_β corresponds to Hamiltonian feedback.

The definition of a degenerate Hamiltonian system (Definition 3.10) reduces in the affine case to

DEFINITION 3.28 Let $\Sigma(M,T^*Y,L)$ be an affine Hamiltonian system. Let $P \subset T^*Y$, the restriction manifold, be a regular codistribution on Y. Assume that $L' := L \cap (TM \times P)$ is a submanifold of $TM \times T^*Y$. Then we call $\Sigma(M,T^*Y,L')$ a degenerate affine Hamiltonian system.

We obtain the easily proved analogue of Proposition 3.24.

PROPOSITION 3.29 Let $\Sigma(M,T^*Y, L'=L\cap(TM\times P))$ be a degenerate affine Hamiltonian system. Since P is regular, there exist local coordinates $(y_1,\ldots,y_m)$ for Y such that P = span $\{dy_1,\ldots,dy_k\}$, $k \leq m$. Let $(u_1,\ldots,u_m)$ be corresponding natural coordinates for the fibers of T^*Y. Then the system is given by

$$\begin{aligned} \dot{x} &= X_H(x) - \sum_{i=1}^{k} u_i X_{C_i}(x) \\ y_i &= C_i(x) \qquad\qquad i = 1,\ldots,m \end{aligned} \tag{3.66}$$

with $H(x) - \sum_{i=1}^{k} u_i C_i(x)$ the generating function of L.

Remark: We can easily extend Definition 3.28 to the case that P is an affine codistribution, i.e. $P = \beta + \tilde{P}$, with β a 1-form on Y and $\tilde{P}$ a codistribution. We then require that $\tilde{P}$ is regular and that β is closed. There exist coordinates $(y_1,\ldots,y_m)$ for Y such that $\tilde{P}$ = span $\{dy_1,\ldots,dy_k\}$, and locally there exists a function $V : Y \longrightarrow \mathbb{R}$ such that $dV = \beta$. Then the system is given by

$$\dot{x} = X_{H+V\circ C}(x) - \sum_{i=1}^{k} u_i X_{C_i}(x),\ y_i = C_i(x),\ i = 1,\ldots,m.$$

3.3.1 Controllability and observability

We shall specialize Theorem 3.19 to the case of an affine Hamiltonian system on (M,ω):

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x),\quad y_i = C_i(x) \qquad i = 1,\ldots,m$$

with $\pounds_A \omega = 0$, $\omega(B_i,-) = dC_i$, $i = 1,\ldots,m$.

Recall from Chapter 2 that the strong accessibility and local weak observability properties of the system are characterized by respectively the controllability distribution C and the observability codistribution O (see Definitions 2.33, 2.43). One way to generate C (Construction 2.49) is to define $\Gamma := A + (B_1,\ldots,B_m)$, $F_0 := (B_1,\ldots,B_m)$ and $F_k := [\Gamma,F_{k-1}] + F_{k-1}$, $k \geq 1$. Then the linear subspace F of $V(M)$ given by $F := \bigcup_{k\geq 0} F_k$ is such that $C(x) = \{Z(x) \mid Z \text{ vectorfield in } F\}$. For the construction of O we define (Construction 2.51) $G_0 := (C_1,\ldots,C_m)$, and $G_k := \pounds_\Gamma G_{k-1} + G_{k-1}$, $k \geq 1$. Then $G := \bigcup_{k\geq 0} G_k$ (a linear subspace of $C(M)$) satisfies $O(x) = \text{span}\,\{dg(x) \mid g \in G\}$. For an affine Hamiltonian system this last construction becomes particularly nice. Since $\pounds_A \omega = 0$, there exists (locally) an $H : M \longrightarrow \mathbb{R}$ such that $A = X_H$. We derive

PROPOSITION 3.30 Define $K := H + (C_1,\ldots,C_m)$ (an affine subspace of $C(M)$). Then the G_k's defined above satisfy $G_k = \{K,G_{k-1}\} + G_{k-1}$, with $\{\ ,\ \}$ the Poisson bracket on M.

PROOF: Elements of G_k are linear combinations of functions of the form

$$\pounds_{f_1} \pounds_{f_2} \cdots \pounds_{f_r} C_j,\ r \leq k, \text{ with } f_i = A \text{ or } f_i = B_\ell,\ \ell = 1,\ldots,m. \tag{3.67}$$

The Poisson bracket $\{N_1,N_2\}$ satisfies $\{N_1,N_2\} = \omega(X_{N_1},X_{N_2}) = X_{N_1}(N_2)$, for two functions N_1,N_2 on M (see 3.43)). Therefore, since $A = X_H$ and $B_i = -X_{C_i}$, the expressions (3.67) equal

$$\pm \{h_1,\{h_2,\{h_3,\ldots,\{h_r,C_j\}\ldots\}, \text{ with } h_i = H \text{ or } h_i = C_\ell,\ \ell = 1,\ldots,m. \tag{3.68}$$

□

We know that F is a Lie sub algebra of the algebra V(M) of vectorfields on M (with respect to the Lie bracket), see Construction 2.49. It follows from Proposition 3.30 that in the same way G is a Lie sub algebra of the algebra C(M) of functions on M (with respect to the Poisson bracket). (Sketch of the proof: the Poisson bracket of two expressions of the form (3.68) can, by repeated use of the Jacobi-identity for the Poisson bracket, be written as a linear combination of expressions of the form (3.68)). This suggests the following

THEOREM 3.31 Let $\dot{x} = X_H(x) - \sum_{i=1}^{m} u_i X_{C_i}(x)$, $y_i = C_i(x), i = 1,\ldots,m$, be an affine Hamiltonian system, with F_k and G_k as above. Then the map α: C(M) $\longrightarrow$ V(M), defined by $\alpha(N) = X_N$ is an isomorphism between G_k (modulo $\mathbb{R}$) and F_k, for all $k \geq 0$. Hence α is an isomorphism between G (mod $\mathbb{R}$) and F.

PROOF: It is easy to see that α maps constant functions to the zero vectorfield. Therefore we shall omit for brevity the suffix (modulo $\mathbb{R}$). By induction: For $k = 0$ the statement is immediate because $F_0 = (-X_{C_1},\ldots,-X_{C_m})$ and $G_0 = (C_1,\ldots,C_m)$. Suppose it is true for k-1. We shall prove it for k. Now $G_k = \{K,G_{k-1}\} + G_{k-1}$. By the induction assumption G_{k-1} is mapped isomorphically onto F_{k-1}, and hence we only have to prove that $\{K,G_{k-1}\}$ is mapped onto $[\Gamma,F_{k-1}]$. We have

$$\{K,G_{k-1}\} = \{H+G_0,G_{k-1}\} = \{H,G_{k-1}\} + \{G_0,G_{k-1}\}$$

and

$$[\Gamma,F_{k-1}] = [A+F_0,F_{k-1}] = [A,F_{k-1}] + [F_0,F_{k-1}]$$

Because $\alpha(\{N_1,N_2\}) = [X_{N_1},X_{N_2}]$ (see(3.45)) it easily follows that, since $\alpha(H) = A$ and $\alpha(G_0) = F_0$, $\alpha(\{H,G_{k-1}\}) = [A,F_{k-1}]$ and $\alpha(\{G_0,G_{k-1}\}) = [F_0,F_{k-1}]$) and therefore $\alpha(\{K,G_{k-1}\}) = [\Gamma,F_{k-1}]$. □

COROLLARY 3.32 0 and C are isomorphic, with the isomorphism $\beta \longrightarrow X_\beta$ given by $\omega(X_\beta,-) = \beta$, if β is a one-form on M.

Analogously to Section 3.2.3, Definition 3.20, we define

DEFINITION 3.33 Let $\Sigma(M,T^*Y,L)$ be an affine Hamiltonian system. Σ satisfies the *minimality rank condition* (MRC) if dim $O(x)$ = dim M for at least one $x \in M$. Σ satisfies the *strong minimality rank condition* (SMRC) if dim $O(x)$ = dim M on an open and dense subset of M.

Remark 1: Of course if dim $O(x)$ = dim M, then necessarily dim $O(q)$ = dim M for all q in some neighborhood of x.

Remark 2: Recall from Chapter 2:
Σ satisfies MRC $\Longrightarrow$ Σ locally weakly observable and strongly accessible.
If dim $O(x)$ = dim $C(x)$ = constant, then:
Σ satisfies property I $\Longleftrightarrow$ Σ locally weakly observable $\Longleftrightarrow$ dim $O(x)$ = dim M $\Longleftrightarrow$ dim $C(x)$ = dim M $\Longleftrightarrow$ Σ is strongly accessible $\Longleftrightarrow$ Σ satisfies property II.

We note that the second possibility for generating C, i.e. by defining the *distributions* $\Delta_0(x) := \text{span}\{B_1(x),\ldots,B_m(x)\}$, $\Delta(x) := A(x) + \Delta_0(x)$, $\Delta_k := [\Delta,\Delta_{k-1}]$, $k \geq 1$ (Construction 2.50), can also be related, via the map α, to a sequence of subsets of C(M). This goes as follows. Let $N_1,\ldots,N_k$ be functions on M. Take all (smooth) functions on M that can be written as "functions of N_i", i.e. all functions of the forms $\phi \circ (N_1,\ldots,N_k) : M \longrightarrow \mathbb{R}$, with $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}$. This generates a linear subspace of C(M), which we denote by $\langle N_1,\ldots,N_k\rangle$. Notice that $\{dN(x) \mid N \in \langle N_1,\ldots,N_k\rangle\} =$ span $\{dN_1(x),\ldots,dN_k(x)\}$. Define $E_0 := \langle C_1,\ldots,C_m\rangle$, $E := H + \langle C_1,\ldots,C_m\rangle$ and $E_k := \{E,E_{k-1}\}$, $k \geq 1$. Then one sees that α maps E_k (modulo $\mathbb{R}$) isomorphically onto Δ_k, $k \geq 0$.

In the rest of this section we briefly sketch how we can regard affine Hamiltonian systems from a different, more algebraic, point of view. Assume that the local representation (3.61) is *global*, i.e. $H : M \longrightarrow \mathbb{R}$ and $C_i : M \longrightarrow \mathbb{R}$ are globally defined functions on M. Then the system is characterized by H and $C_1,\ldots,C_m$, and their Poisson bracket relations, in fact the linear spaces G_k as above. Hence from an abstract point of view we can *identify* the system with the Poisson algebra G, structured and generated by H and $C_1,\ldots,C_m$. The next step is to *forget* that this algebra is realized as a Poisson algebra of functions on M. Then we arrive at an *abstract algebra* G with the same algebraic relations as the original Poisson algebra. This is indeed the same idea that has proved to be useful for providing a transition from classical mechanics to quantum mechanics.

Roughly speaking, in this case one tries to realize a Poisson algebra of functions on a symplectic manifold, which in some way corresponds to a classical mechanical system, as an isomorphic algebra of (unitary) operators on a Hilbert space. This last algebra is called the algebra of *observables* (a somewhat confusing terminology in our context). The problem to construct this algebra of observables is called the quantization problem. We remark that the quantization problem is very delicate. For instance, if we consider the Poisson algebra of all smooth functions on $T^*\mathbb{R}^n$, then a "reasonable" quantization is not possible (see e.g. ABRAHAM & MARSDEN (1978, 5.4)). Therefore one takes a sub algebra of the algebra of all smooth functions on $T^*\mathbb{R}^n$, containing if possible the configuration and momentum variables and the Hamiltonian H, and tries to realize this subalgebra as an algebra of observables.

It might be of interest to study the quantization properties of a Poisson algebra G as above, and to relate these properties to the system theoretic properties of the affine Hamiltonian system corresponding to G. Also one might hope that this study gives rise to a systematic way of including external force fields in a description of quantum mechanical systems.

Without entering the physical implications, we shall give a simple example which illustrates the mathematical possibilities of a (formal) quantization of G. The generating function of the affine Hamiltonian system consisting of a mass m attached to a spring and influenced by an external force u is $H(q,p,u) = \frac{p^2}{2m} + \frac{1}{2}kq^2 - uq$ (k is the spring constant; $(q,p)\in T^*\mathbb{R}$). The corresponding Poisson algebra G is simply the linear space of functions on $T^*\mathbb{R}$ spanned by q,p and 1. We can formally quantize G by assigning to q the operator q. (multiplication by q), to p the operator $\frac{1}{i}\frac{\partial}{\partial q}$ (both are operators on the Hilbert space $L^2(\mathbb{R},\mathbb{C})$), and to 1 the identity operator on $L^2(\mathbb{R},\mathbb{C})$. This quantization also quantizes the "Hamiltonian" H(q,p,u). The Schrödinger equation corresponding to the quantized H(p,q,u) is (set $\hbar$=1)

$$i\,\frac{\partial\phi}{\partial t}(t,q) = -\frac{1}{2m}\frac{\partial^2\phi}{\partial q^2} + \frac{1}{2}kq^2\phi - uq\phi$$

with $\phi \in L^2(\mathbb{R},\mathbb{C})$. This can be considered as the equation of a quantum mechanical description of a particle in an oscillator well, which is also subject to a uniform external force field, whose overall strength and direction is an arbitrary function of time u(·) (see also TARN, GARNG HUANG,

CLARK (1980)).

The Hilbert space $L^2(\mathbb{R},\mathbb{C})$ has a natural symplectic structure and (see ABRAHAM & MARSDEN (1978, Proposition 5.5.5) the Hamilton function of the Hamiltonian vectorfield on $L^2(\mathbb{R},\mathbb{C})$ that corresponds to the operator $q\cdot$, is the function $L^2(\mathbb{R},\mathbb{C}) \longrightarrow \mathbb{R}$ given by $\phi \longmapsto \int_{\mathbb{R}} iq\,\phi\bar{\phi}dq$, i.e. the *expectation value* of the observable q. Therefore the "classical output" q $(=-\frac{\partial H}{\partial u}(q,p,u))$ is replaced by the expectation value of the observable q.

3.3.2. Equivalent affine Hamiltonian systems and reduction of the state space.

In Theorem 3.21 we proved that equivalent "minimal" Hamiltonian systems are necessarily symplectomorphic. For affine Hamiltonian systems the situation simplifies considerably. First of all we recall from Chapter 2 that two affine input-output systems

$$\Sigma_1 : \dot{x}_1 = A^1(x_1) + \sum_{i=1}^{m} u_i\, B_i^1(x_1),\ x_1 \in X_1\ ,\ y_j = C_j^1(x_1)\quad ,\quad j = 1,\ldots,p$$

$$\Sigma_2 : \dot{x}_2 = A^2(x_2) + \sum_{i=1}^{m} u_i\, B_i^2(x_2),\ x_2 \in X_2\ ,\ y_j = C_j^2(x_2)\quad ,\quad j = 1,\ldots,p$$

are equivalent, if and only if there exists a diffeomorphism $\phi : X_1 \longrightarrow X_2$ such that

$$\begin{aligned} &\phi_* A^1 = A^2 \\ &\phi_* B_i^1 = B_i^2 \qquad i = 1,\ldots,m \\ &C_j^1 = \phi^* C_j^2 \qquad j = 1,\ldots,p \end{aligned} \tag{3.69}$$

We note that (3.69) implies that $\phi_* F_k^1 = F_k^2$ and $G_k^1 = \phi^* G_k^2$, for all k, where F_k^i and G_k^i, i = 1,2 are the linear subspaces of $V(X_i)$, respectively $C(X_i)$, as defined in Chapter 2, Constructions 2.49 and 2.51 (see also Section 3.3.1).

For affine Hamiltonian systems we now obtain

PROPOSITION 3.34:
Let $\Sigma_1(M_1,T^*Y,L_1)$ and $\Sigma_2(M_2,T^*Y,L_2)$ be affine Hamiltonian systems with state spaces (M_1,ω_1), respectively (M_2,ω_2). Let

$$\dot{x}_1 = X_H^1(x_1) - \sum_{i=1}^{m} u_i X_{C_i}^1(x_1) \quad , \quad y_i = C_i^1(x_1) \quad , \quad i = 1,\ldots,m$$

and

$$\dot{x}_2 = X_H^2(x_2) - \sum_{i=1}^{m} u_i X_{C_i}^2(x_2) \quad , \quad y_i = C_i^2(x_2) \quad , \quad 1 = 1,\ldots,m$$

be local representations of Σ_1 and Σ_2, with $(y_1,\ldots,y_m,u_1,\ldots,u_m)$ natural coordinates for T^*Y. Let Σ_1 and Σ_2 be equivalent, with equivalence mapping $\phi : M_1 \longrightarrow M_2$. Then:

i) $X_f \in \ker \phi^*\omega_2 - \omega_1$, for all $f \in G^1 = \bigcup_{k\geq 0} G_k^1$

ii) If Σ_1 satisfies the minimality rank condition (Definition 3.33) and rank $\phi^*\omega_2 - \omega_1$ is constant, then $\phi^*\omega_2 = \omega_1$

iii) If Σ_1 satisfies the strong minimality rank condition (Definition 3.33), then $\phi^*\omega_2 = \omega_1$

iv) If $\phi^*\omega_2 = \omega_1$, then $\phi^*H^2 = H^1 + c$, with c a constant

PROOF i) We have $\phi^*\omega_2(X_{C_i}^1,-) = \omega_2(\phi_* X_{C_i}^1, \phi_* -) = \omega_2(X_{C_i}^2, \phi_* -) =$

$= -dC_i^2(\phi_* -) = -d(\phi^* C_i^2) = -dC_i^1 = \omega_1(X_{C_i}^1,-)$, for all $i = 1,\ldots,m$. Hence

$X_{C_i}^1 \in \ker \phi^*\omega_2 - \omega_1$, $i = 1,\ldots,m$. Furthermore $\pounds_{X_H^1}(\phi^*\omega_2-\omega_1) =$

$= \pounds_{X_H^1}\phi^*\omega_2 + \pounds_{X_H^1}\omega_1 = \phi^*(\pounds_{\phi_* X_H^1}\omega_2) = \phi^*\pounds_{X_H^2}\omega_2 = 0$, and hence

$0 = \pounds_{X_H^1}((\phi^*\omega_2-\omega_1)(X_{C_i}^1,-)) = (\phi^*\omega_2-\omega_1)(\pounds_{X_H^1}X_{C_i}^1,-) = \phi^*\omega_2 - \omega_1(X_{\{H^1,C_i^1\}},-)$.

By induction this implies that $X_f \in \ker \phi^*\omega_2 - \omega_1$, for every $f \in G^1$.

ii) and iii) Since $X_f \in \ker \phi^*\omega_2 - \omega_1$, $\forall f \in G^1$, $\dim \ker(\phi^*\omega_2-\omega_1)(x) \geq$ $\dim O(x)$, for each $x \in M$. Hence if $\dim O(x_0) = \dim M$, then $\ker(\phi^*\omega_2-\omega_1)(x_0) =$ $= T_{x_0}M$. If $\operatorname{rank}(\phi^*\omega_2-\omega_1)$ is constant, this implies $\phi^*\omega_2 = \omega_1$. If $\dim O(x) =$ $\dim M$ on an open and dense subset of M, then $\phi^*\omega_2 - \omega_1$ is zero on this open and dense subset. Hence by continuity $\phi^*\omega_2 = \omega_1$

iv) This follows from $\phi_* X_H^1 = X_H^2$ and $\phi^*\omega_2 = \omega_1$. □

We now consider affine Hamiltonian systems which do not satisfy the minimality rank condition. Under the assumption that the controllability distribution C and thus the observability codistribution O have *constant* dimension, we show that, if one extra condition is satisfied, we can reduce the system to a locally minimal system with the same external behavior, which is again Hamiltonian. This extra condition is explained in

PROPOSITION 3.35 Let $\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$, $y_j = C_j(x)$, $j = 1,\ldots,p$ be an affine input-output system on X (not necessarily Hamiltonian). Suppose that there exists an $x_0 \in X$ such that $A(x_0) \in C(x_0)$, where C is the controllability distribution, which has *constant* dimension. Then an integral manifold Q of C through x_0 has the following properties:

i) $A(x) \in T_xQ$, for every $x \in Q$.

ii) Since also $B_i(x) \in T_xQ$, for each $x \in Q$, we can restrict the system $\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x)$, $y_j = C_j(x)$ to an affine input-output system on Q.

The controllability distribution of this system is equal to TQ.

PROOF: It is clear that $[A,C] \subset C$. Therefore if $A(x_0) \in C(x_0)$ and Q is an integral manifold of C through x_0, then $A(x) \in C(x) = T_xQ$ for each $x \in Q$ (Otherwise A would not leave the integral manifold of C invariant). □

We can regard a submanifold $Q \subset X$ as in Proposition 3.35 as the "controllable" (or"reachable") part of the system with groundstate x_0. Next we can factor out Q by the "non-observable" part of the system. In the Hamiltonian case we obtain:

THEOREM 3.36 Let

$$\dot{x} = A(x) + \sum_{i=1}^{m} u_i B_i(x) \quad , \quad y_i = C_i(x) \tag{3.70}$$

be an affine Hamiltonian system on (M,ω). Suppose that C has constant dimension strictly less than dim M. Assume that there exists an $x_0 \in M$ such that $A(x_0) \in C(x_0)$. Let Q be an integral manifold of C through x_0. Assume that the distribution $C \cap \ker O$ on Q has constant dimension. Then there exists a manifold N and a surjective submersion $\pi : Q \longrightarrow N$ such that $\ker \pi_* = C \cap \ker O|_Q$. Moreover N is a symplectic manifold with symplectic form $\bar{\omega}$ such that $\pi^*\bar{\omega} = \omega|_Q$. On N we can define an affine Hamiltonian

system

$$\dot{\bar{x}} = \bar{A}(\bar{x}) + \sum_{i=1}^{m} u_i \bar{B}_i(\bar{x}), \quad y_i = \bar{C}_i(\bar{x}) \tag{3.71}$$

which has a controllability distribution equal to TN, such that the external behavior of (3.71) is equal to the external behavior of (3.70) restricted to Q.

PROOF: First we show that *locally* Q can be factored out as above. Notice that since $\omega(C,-) = 0$ by Corollary 3.32 ker 0 equals the distribution $C^{\perp}$ (with $\perp$ orthogonal complement with respect to ω, i.e. $C^{\perp}(x) = \{Z \in T_x M \mid \omega_x(Z,Y)=0$ for all $Y \in C(x)\}$. Therefore $C \cap C^{\perp}$ is involutive. Since by assumption dim $C \cap \ker 0$ = constant, $C \cap C^{\perp}$ is regular on Q. Hence we can locally factor out Q by the leaves of $C \cap C^{\perp}$ and obtain a manifold N and a submersion π: Q $\longrightarrow$ N such that $\ker \pi_* = C \cap C^{\perp}$. It also follows (ABRAHAM & MARSDEN (1978, Theorem 5.3.23) that N has a unique symplectic form $\bar{\omega}$ such that $\pi^*\bar{\omega} = \omega|_Q$. Because $[A,C^{\perp}] \subset C^{\perp}$ and $[B_i,C^{\perp}] \subset C^{\perp}$, i = 1,...,m, the vectorfields A and B_i project under π to vectorfields $\bar{A}$, respectively $\bar{B}_i$ on N, i.e. $\pi_* A = \bar{A}$, $\pi_* B_i = \bar{B}_i$. Since $C^{\perp} \subset \ker dC_i$, i = 1,...,m, there exist functions $\bar{C}_i$ on N such that $\pi^*\bar{C}_i = C_i$. The equalities $\omega(B_i,-) = dC_i$ then imply $\bar{\omega}(\bar{B}_i,-) = d\bar{C}_i$, i = 1,...,m. Furthermore on Q, $\omega(A,-) = \pi^*(\bar{\omega}(\bar{A},-))$ and therefore $\bar{\omega}(\bar{A},-)$ is closed, or equivalently $\bar{A}$ is locally Hamiltonian on N. We now refer to HERMANN & KRENER (1977, Theorem 3.9) to conclude that the external behavior of the system on N as defined above is the same as the external behavior of (3.70) restricted to Q, and that the controllability distribution of the system on N is equal to TN. Moreover, this last theorem asserts that since (3.70) is strongly accessible on Q we can *globally* factor out Q by $C \cap C^{\perp}$. Hence the local constructions above hold globally. □

Another way to look at Theorem 3.36 is to consider the Poisson algebra G of (3.61). We can extend G to a bigger Poisson algebra $\tilde{G}$ by defining

$$\tilde{G} = \{\tilde{g} \in C(M) \mid \tilde{g} = \phi \circ (g_1,\ldots,g_k), \text{ for a certain smooth function } \phi : \mathbb{R}^k \longrightarrow \mathbb{R}, \text{ and } g_i \in G\} \tag{3.72}$$

Notice that $0(x) = \{d\tilde{g}(x) \mid \tilde{g} \in \tilde{G}\} = \text{span}\{dg(x) \mid g \in G\}$.

We have the following

THEOREM 3.37 Let dim $O(x)$ = constant. Then there exist (locally) independent functions $f_1,\ldots,f_k,h_1,\ldots,h_k,\ell_1,\ldots,\ell_m$ ($m+2k\leq 2n$) on M, contained in $\tilde{G}$, such that

$$
\begin{aligned}
&\{f_i,f_j\} = \{h_i,h_j\} = \{\ell_i,\ell_j\} = 0 && \forall i,j\\
&\{f_i,h_j\} = \delta_{ij} && i,j = 1,\ldots,k\\
&\{\ell_i,f_j\} = \{\ell_i,h_j\} = 0 && \forall i,j
\end{aligned}
\tag{3.73}
$$

and $O(x) = \text{span}\,\{df_1(x),\ldots,df_k(x),dh_1(x),\ldots,dh_k(x),d\ell_1(x),\ldots,d\ell_m(x)\}$.

For a proof we refer to LIE & ENGEL (1890) (see also HERMANN (1976)). By Darboux's theorem (ARNOLD (1978)), we can extend the above set of functions to 2n independent functions which are *symplectic* coordinate functions. Now it is clear how the manifold N, constructed in Theorem 3.36 can be interpreted. Indeed, a local coordinate system for N is given by $(f_1,\ldots,f_k,h_1,\ldots,h_k)$ with f_i and h_i as in (3.73), and $\bar{\omega}$ equals $\bar{\omega} = \sum_{i=1}^{k} df_i \wedge dh_i$.

In Theorem 3.36 we had to assume that there exists an $x_0 \in M$ with $A(x_0) \in C(x_0)$. There are two important cases where this certainly happens. Let $A = X_H$.

i) Assume that dH(x) vanishes somewhere.

ii) Consider the functions ℓ_i, $i = 1,\ldots,m$ in (3.73). It is clear that $\{g,\ell_i\} = 0$, for every $g \in G$. Now assume that also $\{H,\ell_i\} = 0$, $i = 1,\ldots,m$. Then since $X_H(\ell_i) = 0$, X_H is tangent to every manifold of the form $\ell_i = c_i$, c_i constants. However these manifolds contain the integral manifolds of C and are exactly equal to them if C is coisotropic (see also Chapter 4).

3.4. The rigid body with external torques

A well-known and much studied example of a Hamiltonian vectorfield are the equations of a rigid body spinning around its center of mass. In this section we consider the situation that there are also external torques acting on the rigid body. We shall investigate if this results in an (affine) Hamiltonian system as treated before (Section 3.3).

The equations of a rigid body are given as follows. Let $S = (a_1,a_2,a_3)$ be a set of axes fixed in the rigid body. Let R be the (3×3)-matrix which expresses the orientation of the rigid body axes S with respect to a set of inertial axes $Q = (e_1,e_2,e_3)$. Thus if x is a column 3-vector with respect to the inertial axes Q, then Rx is that same vector with respect to the rigid body axes S. It follows that R satisfies $R^TR = I_3$, and if we only consider right handed sets of axes det R = 1, so $R \in SO(3)$. Let furthermore $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ be the vector of angular velocities with respect to the rigid body axes, i.e. ω_i is the angular velocity around a_i. Let J be a (3×3) symmetric positive definite matrix. J is called the *inertia* matrix. We call the eigenvectors of J the principal axes, and the eigenvalues the principal moments of inertia. Denote by $S(\omega)$ the skew-symmetric matrix $S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$

Then the equations of the rigid body are (ARNOLD (1978))

$$\begin{aligned} \dot{R} &= S(\omega)R \\ J\dot{\omega} &= S(\omega)J\omega \end{aligned} \tag{3.74}$$

Let now b_1, b_2 and b_3 be another set of axes fixed in the rigid body, and suppose that we can exert external torques around these axes b_i. In practice such external torques can be for instance realized by attaching two identical but opposed gas jets at the end of every axis b_i. This yields (see CROUCH & BONNARD (1980))

$$\begin{aligned} \dot{R} &= S(\omega)R \\ J\dot{\omega} &= S(\omega)J\omega + u_1b_1 + u_2b_2 + u_3b_3 \end{aligned} \tag{3.75}$$

with u_i, i = 1,2,3, the controls (inputs).

If b_1, b_2 and b_3 are independent we call (3.75) the rigid body with *three controls*. If $b_3 = 0$, we have only two torques and we speak about the rigid body with *two controls*. Finally $b_2 = b_3 = 0$ gives the rigid body with *one control*. We shall now consider the question if it is possible to regard these three cases as (affine) Hamiltonian systems (notice that we have not yet defined output maps).

First we give some mathematical preliminaries which are needed to give a coordinate free description of (3.75). The state space of a rotating rigid

body is $T^*SO(3)$. There are two ways of identifying $T^*SO(3)$ with $SO(3) \times so^*(3)$ ($so(3)$ is the Lie algebra of $SO(3)$, and $so^*(3)$ is the dual Lie algebra). The first corresponds to giving the equations in *body-coordinates* as in (3.75) (i.e. in coordinates with respect to the axes S fixed in the rigid body), while the second results in equations in space-coordinates (i.e. with respect to Q). Consider the left translation $L_g: SO(3) \longrightarrow SO(3)$, $g \in SO(3)$, defined by $L_g h = gh$, for $h \in SO(3)$. Then we define the isomorphism $\lambda: T^*SO(3) \longrightarrow SO(3) \times T_e^* SO(3)$ ($e \in SO(3)$ is the identity matrix) by setting

$$\lambda(\alpha) = (g, (T_e L_g)^* \alpha) \tag{3.76}$$

with $g = \pi(\alpha)$ (π projection of $T^*SO(3)$ on $SO(3)$) and $T_e L_g$ the differential of L_g in $e \in SO(3)$. Since $so(3) = T_e SO(3)$ and $so^*(3) = T_e^* SO(3)$, λ is an isomorphism $T^*SO(3) \longrightarrow SO(3) \times so^*(3)$. This isomorphism λ corresponds to choosing body coordinates (to obtain the equations in space coordinates we have to consider the right translation $R_g h = hg$). Since $T^*SO(3)$ is a cotangent bundle, it has a canonical 1-form θ, and a symplectic form $\omega := d\theta$. Hence $(\lambda^{-1})^* \theta =: \theta_B$ is a 1-form on $SO(3) \times so^*(3)$ and $\omega_B := d\theta_B = (\lambda^{-1})^* \omega$ is a symplectic form on $SO(3) \times so^*(3)$. Now consider the symmetric positive definite bilinear form J on $so(3) = T_e SO(3)$. By left translation we can extend J to a left-invariant Riemannian metric on $SO(3)$, denoted by $\langle \, , \, \rangle_J$. Furthermore J induces a quadratic function $\frac{1}{2} p^T J^{-1} p$ on $so^*(3)$, with $p \in so^*(3)$. By left translation we can extend this quadratic function to a left-invariant function $K : T^*SO(3) \longrightarrow \mathbb{R}$. We call K the *kinetic energy* (of course K and $\langle \, , \, \rangle_J$ are immediately related, see ABRAHAM & MARSDEN (1978, Section 4.5.2)). The dynamics of a rigid body without external torques is now given by the Hamiltonian vectorfield X_H on $T^*SO(3)$ (with symplectic form ω), where $H : T^*SO(3) \longrightarrow \mathbb{R}$ is equal to K. Equations (3.74) are the expressions of X_H in body coordinates.

In order to include the external torques we need to say something more about so(3). It is well known that a basis of so(3) is given by the matrices

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.77}$$

Furthermore we can identify so(3) with $\mathbb{R}^3$ by defining $j : \mathbb{R} \longrightarrow so(3)$ as $x = x_1 e_1 + x_2 e_2 + x_3 \overset{j}{\longmapsto} X = x_1 E_1 + x_2 E_2 + x_3 E_3$, with (e_1, e_2, e_3) the

standard basis of $\mathbb{R}^3$. If we make $\mathbb{R}^3$ into a Lie algebra by taking as operation the vector product $\times$, then j is actually a Lie algebra isomorphism. In the same way we can identify $so^*(3)$ with $(\mathbb{R}^3)^*$, and a basis of so(3) is given by the (row vectors) $f_1, f_2, f_3 \in (\mathbb{R}^3)^*$ such that $f_i(e_j) = \delta_{ij}$, i,j = 1,2,3. We can extend $E_i \simeq e_i$ by left translation to left invariant vectorfields X_1, X_2 and X_3 on SO(3), while f_i can be extended to left invariant one-forms $\theta_1, \theta_2, \theta_3$ on SO(3). It follows from the commutation relations (3.77) that

$$[X_1,X_2] = X_3 \quad , \quad [X_2,X_3] = X_1 \quad , \quad [X_3,X_1] = X_2$$

(3.78)

$$d\theta_1 = \theta_2 \wedge \theta_3 \ , \ d\theta_2 = \theta_3 \wedge \theta_1, \quad d\theta_3 = \theta_1 \wedge \theta_2$$

Let us write (3.75) as (with $x = (R, J\omega)$)

$$\dot{x} = A(x) + u_1 B_1(x) + u_2 B_2(x) + u_3 B_3(x) \tag{3.79}$$

where A and B_1, B_2, B_3 are the vectorfields on $SO(3) \times so^*(3)$ given by (with $R \in SO(3)$ and $J\omega \in so^*(3)$)

$$A(R,J\omega) = \begin{pmatrix} S(\omega)R \\ S(\omega)J\omega \end{pmatrix} \quad , \quad B_i(R,J\omega) = \begin{pmatrix} 0 \\ b_i \end{pmatrix} \quad , \ i = 1,2,3 \tag{3.80}$$

The symplectic form $\omega_B = d\theta_B$ on $SO(3) \times so^*(3)$ has the following explicit expression (ABRAHAM & MARSDEN (1978, Proposition 4.4.1)). Let $(g,\mu) \in SO(3) \times so^*(3)$ and $(v,\rho),(w,\sigma) \in T_{(g,\mu)}(SO(3) \times so^*(3)) = T_g SO(3) \times so^*(3)$ (where we have identified $T_\mu so^*(3)$ and $so^*(3)$). Then :

$$\omega_B(g,\mu)((v,\rho),(w,\sigma)) = \rho(T_g L_{g^{-1}} w) - \sigma(T_g L_{g^{-1}} v) - \mu([T_g L_{g^{-1}} v, T_g L_{g^{-1}} w]) \tag{3.81}$$

It is a matter of calculation (see ABRAHAM & MARSDEN (1978)) to see that, with $R \in SO(3)$ and $p \in so^*(3)$,

$$\omega_B(R,p)(A(R,p),-) = -d(\tfrac{1}{2} p^T J^{-1} p) \tag{3.82}$$

Moreover it follows from (3.81) that

$$\omega_B(g,\mu)((0,p),(w,\sigma)) = \rho(T_g L_{g^{-1}} w) \tag{3.83}$$

Therefore we obtain, for $R \in SO(3)$ and $p \in so^*(3)$, that

$$\omega_B(R,p)(B_i(R,p),-) = b_i^T \quad , \quad i = 1,2,3 \tag{3.84}$$

where we have identifed $b_i^T \in (\mathbb{R}^3)^*$ with an element of $so^*(3)$. By the isomorphism λ (3.76) between $SO(3) \times so^*(3)$ and $T^*SO(3)$ we can regard A, B_1, B_2 and B_3 also as vectorfields on $T^*SO(3)$. Then we obtain from (3.82) and (3.84) that (with ω the symplectic form on $T^*SO(3)$)

$$\begin{aligned} &\omega(A,-) = -dK \\ &\omega(B_i,-) = b_i^1 \theta_1 + b_i^2 \theta_2 + b_i^3 \theta_3 \quad , \quad i = 1,2,3 \end{aligned} \tag{3.85}$$

where $b_i = \begin{pmatrix} b_i^1 \\ b_i^2 \\ b_i^3 \end{pmatrix}$, $i = 1,2,3$.

Therefore A is a Hamiltonian vectorfield on $T^*SO(3)$. However, since $d\theta_i \neq 0$, $i = 1,2,3$ (by 3.78), the vectorfields B_i , $i = 1,2,3$, are *not* Hamiltonian vectorfields. This implies that we *cannot* consider (3.75) as the state space equations of an affine Hamiltonian system. On the other hand we can still try to transform equations (3.75) into an equivalent system which is Hamiltonian. Indeed we may allow for a state dependent change of the inputspace:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = N(R,J\omega) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{3.86}$$

with $N(R,J\omega)$ a nonsingular 3×3 matrix. Then (3.86) transforms (3.79) into

$$\dot{x} = A(x) + v_1 \tilde{B}_1(x) + v_2 \tilde{B}_2(x) + v_3 \tilde{B}_3(x) \tag{3.87}$$

where span $\{\tilde{B}_1(x), \tilde{B}_2(x), \tilde{B}_3(x)\}$ = span$\{B_1(x), B_2(x), B_3(x)\}$ for each $x \in SO(3) \times so^*(3)$.

Therefore the problem is as follows. Can we find a basis consisting of *Hamiltonian* vectorfields for the distribution $\Delta_o(x)$ = span $\{B_1(x), B_2(x), B_3(x))\}$? Consider first the rigid body with three contr ls. Define Y:= SO(3), and the output map $C : M \longrightarrow Y$, with $M = T^*SO(3)$, as the canonical projection of $T^*SO(3)$ onto SO(3). Then define the affine Hamiltonian system $\Sigma_3(M, T^*Y, L)$ by taking as generating function for $L \subset TM \times T^*Y$ the function $K = \sum_{i=1}^{3} u_i C_i$, where $C = (C_1, C_2, C_3)$ in local coordinates for $Y = SO(3)$, and (u_1, u_2, u_3) the corresponding natural coordinates for the fibers of $T^*SO(3)$. Using the isomorphism λ, we can also take $M = SO(3) \times so^*(3)$ instead of $M = T^*SO(3)$. Since span $\{X_{C_1}(x), X_{C_2}(x), X_{C_3}(x)\}$ =

span $\{B_1(x),B_2(x),B_3(x)\}$, the state space equations of $\Sigma_3(M,T^*Y,L)$ are indeed equivalent to equations (3.75) or (3.79). Hence the rigid body with three controls can be formulated as an affine Hamiltonian system.

Next consider the rigid body with two controls, which are external torques around the axes b_1 and b_2. Let P be the linear subspace of $so^*(3) \cong (\mathbb{R}^3)^*$ spanned by b_1^T and b_2^T. Then by left translation we obtain a left invariant codistribution P on SO(3). Since the distribution D := ker P is one-dimensional, D and hence P are regular. It can be seen that SO(3) can be globally factored out by the leaves of the foliation generated by D, and that the quotient manifold is equal to S^2 (the two-dimensional sphere). In fact we obtain a *fiber bundle* $SO(3) \xrightarrow{p} S^2$, with $\ker p_* = D$ and the fibers diffeomorphic to S^1. For instance if D is spanned by X_1, then $p : SO(3) \longrightarrow S^2$ is simply the projection of $R \in SO(3)$ onto its first column (or its first row). Therefore we define the output manifold $Y := S^2$, and the output map $C : M \longrightarrow Y$, with $M = T^*SO(3)$, as $C := p \circ \pi$ (π is the projection of $T^*SO(3)$ onto SO(3)). We obtain an affine Hamiltonian system $\Sigma_2(M, T^*Y,L)$, with L given by its generating function $K = \sum_{i=1}^{2} u_i C_i(x)$, with $C = (C_1,C_2)$ in local coordinates for $Y = S^2$ and (u_1,u_2) the corresponding natural coordinates for the fibers of T^*S^2. It is easy to see that $\text{span}\{X_{C_1}(x),X_{C_2}(x)\} = \text{span}\{B_1(x),B_2(x)\}$. Hence also the rigid body with two controls can be formulated as an affine Hamiltonian system. Notice that the rigid body with two controls can be also considered as a *degenerate* Hamiltonian system (Definition 3.28) by defining Y = SO(3) and P as above.

Finally we consider the rigid body with one control, i.e. one external torque around the axis b_1. Let D be the two-dimensional subspace of so(3) given by $D := \{x \in \mathbb{R}^3 \tilde{=} so(3), b_1^T x = 0\}$. From the commutation relations (3.78) it can be easily seen that so(3) does not possess a two-dimensional subalgebra. Hence the left invariant distribution D on SO(3) given by left translation of $D \subset so(3)$ and the left invariant codistribution P on SO(3) generated by left translation of b_1^T are *not* involutive. Hence we cannot define a 1-dimensional output manifold Y and an output map $C : M \longrightarrow Y$ such that $\text{span}\{B_1(x)\} = \text{span}\{X_C(x)\}$. Therefore the rigid body with one control *cannot* be formulated as a (full or degenerate) Hamiltonian system.

We summarize the above discussion in

THEOREM 3.38 Consider the rigid body with external torques (3.75). Then

a) The rigid body with three controls can be formulated as an affine Hamiltonian system $\Sigma_3(M,T^*Y,L)$, with $M = T^*SO(3)$ and $\underline{Y} = SO(3)$.
b) The rigid body with two controls can be formulated as an affine Hamiltonian system $\Sigma_2(M,T^*Y,L)$, with $M = T^*SO(3)$ and $Y = S^2$, *or* as a degenerate affine Hamiltonian system with $Y = SO(3)$.
c) The rigid body with one control cannot be formulated as a (full or degenerate) affine Hamiltonian system.

We remark that the controllability properties of (3.75) have been investigated in CROUCH & BONNARD (1980) and BAILLIEUL (1981). In fact

a) The rigid body with three controls has a regular controllability distribution C_3, and $\dim C_3 = \dim T^*SO(3)$.
b) The rigid body with two controls has a regular controllability distribution C_2. Moreover $\dim C_2 = \dim T^*SO(3) - 2$ if and only if (with $S(x) = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$)

$$S(x)J^{-1}\,x \subset \operatorname{span}\{b_1,b_2\}, \text{ for each } x \in \operatorname{span}\{b_1,b_2\} \tag{3.88}$$

If (3.88) does not hold, then $\dim C_2 = \dim T^*SO(3)$.

By Theorem 3.31 it follows that the affine Hamiltonian system corresponding to the rigid body with three controls is always locally weakly observable, while the affine Hamiltonian system $\Sigma_2(T^*SO(3), T^*S^2, L)$ corresponding to the rigid body with two controls is locally weakly observable if and only if (3.88) does not hold.

3.5 Linear Hamiltonian systems

Consider a linear system in state space form $\Sigma(A,B,C,D)$ (see Section 2.1):

$$\dot{x} = Ax + Bu \quad , \quad x \in X \quad , \quad u \in U \tag{3.89}$$
$$= (2.1) \qquad w = Cx + Du \quad , \quad w \in W$$

A necessary condition for Σ in order to be a *linear Hamiltonian system* is that X and W are *symplectic linear spaces*. This means that there exist

nondegenerate skew-symmetric bilinear forms J and J^e on X, respectively W. By Darboux's theorem there exist bases of X and W, such that in these bases $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and $J^e = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$ (X and W are necessarily even-dimensional; say dim X = 2n and dim W = 2m). Such bases are called canonical (or symplectic). In the next theorem we translate the conditions of Definition 3.6 (the definition of full Hamiltonian systems) to linear systems.

THEOREM 3.39 Let $\Sigma(A,B,C,D)$ be a linear system (3.89). Assume that $\begin{pmatrix} B \\ D \end{pmatrix}$ is injective. Let (X,J) and (W,J^e) be linear symplectic spaces. Then $\Sigma(A,B,C,D)$ is (*full*) *Hamiltonian* if and only if A,B,C and D satisfy

$$\begin{aligned} & A^TJ + JA - C^TJ^eC = 0 \\ & B^TJ \qquad - D^TJ^eC = 0 \\ & D^TJ^eD \qquad\qquad = 0 \quad , \text{ and rank } D = m \end{aligned} \tag{3.90}$$

Moreover, if $\Sigma(A,B,C,D)$ is full Hamiltonian, then there exists a feedback transformation $A \longrightarrow A + BF$, $C \rightarrow C + DF$ (see Section 2.1.1), and a canonical basis $w = (y_1,\ldots,y_m,u_1,\ldots,u_m)$ of W such that in these coordinates the feedback transformed system $\Sigma(A',B',C',D')$ satisfies

$$\begin{aligned} & A'^TJ + JA' = 0 \\ & B'^TJ \qquad = \bar{C}' \end{aligned} \tag{3.91}$$

and

$$C' = \begin{pmatrix} \bar{C}' \\ 0_m \end{pmatrix}, \quad D' = \begin{pmatrix} 0_m \\ I_m \end{pmatrix}$$

We call $\Sigma(A,B,\bar{C})$ satisfying equations (3.91) a *linear Hamiltonian input-output system*. Hence a full Hamiltonian system is feedback equivalent to a Hamiltonian input-output system.

Remark 1 : $\begin{pmatrix} B \\ D \end{pmatrix}$ injective is the linear translation of the condition that $f : B \longrightarrow TM \times W$ is an embedding. Compare also Theorem 2.2.

Remark 2: Recall from Section 2.1.1 that feedback transformations do not change the dynamical system in state space form $\Sigma_i(A,B,C,D)$.

PROOF of Theorem 3.40 : Notice that the symplectic form $\dot{J}$ on TX is given by $\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$. According to Definition 3.6

$V = \{ \begin{pmatrix} x \\ Ax+Bu \\ Cx+Du \end{pmatrix} | x\in X, u\in U\}$ has to be a Lagrangian subspace of $(TX\times W, \dot{J}\oplus -J^e)$,

i.e. $(y^T \; y^TA^T+v^TB^T \; y^TC^T+v^TD^T) \begin{pmatrix} 0 & J & 0 \\ J & 0 & 0 \\ 0 & 0 & -J^e \end{pmatrix} \begin{pmatrix} x \\ Ax+Bu \\ Cx+Du \end{pmatrix} = 0$

$\forall x,y \in X$ and $\forall u, v \in U$. This yields equations (3.90). Notice that $B^TJ - D^TJ^eC = 0$ implies that Ker D $\subset$ Ker B. Since $\begin{pmatrix} B \\ D \end{pmatrix}$ is injective, this gives that D is injective and that dim U = rank D. Because V is Lagrangian, and therefore has dimension 2n + m, it follows that rank D = m, and that Im D is a Lagrangian subspace of (W,J^e). Hence we can choose a basis for W such that $D' = \begin{pmatrix} 0_m \\ I_m \end{pmatrix}$ and $J^e = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$. By applying feedback u = Fx + v, i.e. $A \longmapsto A + BF$, $C \longmapsto C + DF$, we can bring C into the form $C' = \begin{pmatrix} \bar{C} \\ 0_m \end{pmatrix}$.

This implies that $C'^TJ^eC = 0$, and that $D'^TJ^eC' = [0 \;\; I_m] \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \begin{pmatrix} \bar{C}' \\ 0 \end{pmatrix} = \bar{C}'$. This yields (3.91). □

Remark: A matrix A satisfying $A^TJ + JA = 0$ is called a Hamiltonian matrix.

In Theorem 3.39 we have chosen a canonical basis for W such that $D = \begin{pmatrix} 0_m \\ I_m \end{pmatrix}$. If, on the other hand, we start from a *fixed* canonical basis $(y_1,\ldots,y_m,z_1,\ldots,z_m)$ the situation is as follows. Since Im D is Lagrangian, Im D can be parametrized by a set of basis vectors $\{y_i\}_{i\in I_1} \cup \{z_j\}_{j\in I_2}$, with $I_1 \cup I_2 = \{1,\ldots,m\}$ and $I_1 \cap I_2 = \emptyset$ (Theorem 3.2). Then construct an m × m signature matrix Ξ by setting the k-th diagonal element equal to +1 if $k \in I_2$, and equal to -1 if $k \in I_1$ (remember that the off-diagonal elements of Ξ are zero). In the permuted basis $(r_1,\ldots,r_m,p_1,\ldots,p_m)$ defined by

$$\begin{cases} r_i = y_i, & \text{if } i \in I_2 \\ r_i = z_i, & \text{if } i \in I_1 \end{cases} \quad \text{and} \quad \begin{cases} p_i = z_i, & \text{if } i \in I_2 \\ p_i = y_i, & \text{if } i \in I_1 \end{cases}$$

Im D is then of the form Im $\begin{pmatrix} \bar{D}' \\ I_m \end{pmatrix}$ for a certain $\bar{D}'$. Therefore there exists a non singular R : U ⟶ U such that $DR = \begin{pmatrix} \bar{D}' \\ I_m \end{pmatrix}$. Moreover, since Im D is Lagrangian, $\bar{D}'$ satisfies $\Xi\,\bar{D}' = (\bar{D}')^T\,\Xi$. By applying feedback we can bring

C into the form $\begin{pmatrix} \bar{C} \\ 0_m \end{pmatrix}$. Then the equations (3.90) result in

$$
\begin{aligned}
& A'^{T}J + JA' = 0 \\
(3.92) \qquad & B'^{T}J = \Xi\bar{C}' \\
& \Xi\bar{D}' = \bar{D}'^{T}\Xi
\end{aligned}
$$

We call (3.92) a linear Hamiltonian system with feedthrough term. Note that (3.92) are exactly the same expressions as we obtained in equations (3.48) in Section 3.1.4.

Theorems 3.20 and 3.33 about the relation between observability and controllability for Hamiltonian systems have a very simple analogue in the linear case. Let V be a linear subspace of (X,J). Denote by $V^{\perp}$ the orthogonal complement with respect to J, i.e. $V^{\perp} = \{x \in X \mid x^{T}Jv = 0, \forall v \in V\}$. Then we derive: $AV \subset V$ and $\text{Im } B \subset V \iff (AV)^{\perp} \supset V$ and $(\text{Im } B)^{\perp} \supset V^{\perp}$. One verifies that $(AV)^{\perp} = A^{-1}(V^{\perp})$, since $A^{T}J + JA = 0$, and that $(\text{Im } B)^{\perp} = \text{Ker } \bar{C}$, since $B^{T}J = \bar{C}$. Hence : $(AV)^{\perp} \supset V^{\perp}$ and $(\text{Im } B)^{\perp} \supset V^{\perp} \iff AV^{\perp} \subset V^{\perp}$ and $V^{\perp} \subset \text{Ker } \bar{C}$. Concluding:

$\exists V \neq X$ with $AV \subset V$ and $\text{Im } B \subset V$ if and only if
$\exists V^{\perp} \neq 0$ with $AV^{\perp} \subset V^{\perp}$ and $V^{\perp} \subset \text{Ker } C$,
and hence, $(\bar{C},A)$ is observable if and only if (A,B) is controllable.

The definition of a *degenerate* Hamiltonian system (Definition 3.10) simplifies considerably in the linear case. Recall that a full Hamiltonian system is called *regular* if $\dim h(B) = \dim W$. Therefore linear Hamiltonian systems (3.90) are regular if and only if $[C \vdots D]$ is surjective, or equivalently if $\bar{C}$ in (3.91) is surjective. For the definition of a degenerate Hamiltonian system $\Sigma(M,W,B,f,K)$, with K the coisotropic restriction manifold, we required that $h^{-1}(K)$ is a bundle over M. In the linear case, K is a coisotropic subspace of W, and $[C \vdots D]^{-1}(K)$ is a bundle over X if and only if for each $x \in X$ there exists a $u \in U$ such that $Cx + Du \in K$. Equivalently, $[C \vdots D]^{-1}(K)$ is a bundle over X if and only if $\text{Im } C \subset K + \text{Im } D$. If (3.90) is regular, then $\text{Im } C \subset K + \text{Im } D$ if and only if $K + \text{Im } D = W$. To simplify notation we only consider the regular case:

LEMMA 3.40 (Compare Lemma 3.13) Let $K \subset (W,J^{e})$ be a coisotropic subspace (i.e. $K^{\perp} \subset K$). Let Im D be a Lagrangian subspace, such that $K + \text{Im } D = W$.

Then there exists a canonical basis $(a_1,\ldots,a_m,b_1,\ldots,b_m)$ of W such that

i) $K = \text{span}\{a_1,\ldots,a_m,b_1,\ldots,b_k\}$, for some $k \leq m$

ii) Im D can be parametrized by $b_1,\ldots,b_m$, i.e. there exists a matrix $\bar{D}$ such that in the above basis $\text{Im } D = \text{Im}\begin{pmatrix}\bar{D}\\ I_m\end{pmatrix}$.

PROOF: First notice that ii) is equivalent to $\text{Im } D \cap \text{span}\{a_1,\ldots,a_m\} = 0$. Now take an arbitrary basis $a_{k+1},\ldots,a_m$ for $K^\perp$. Then, since $K + \text{Im } D = W$, $K^\perp \cap (\text{Im } D)^\perp = K^\perp \cap \text{Im } D = 0$ (because Im D is Lagrangian). Therefore $\text{Im } D \cap \text{span}\{a_{k+1},\ldots,a_m\} = 0$. By Darboux's theorem (ABRAHAM & MARSDEN (1978, Proposition 3.1.2)) we can choose independent vectors $b_{k+1},\ldots,b_m$ such that $b_{k+i}^T J^e b_{k+j} = 0$, $b_{k+i}^T J^e a_{k+j} = \delta_{ij}$, $i,j = 1,\ldots,m-k$. Necessarily $b_{k+1} \notin (K^\perp)^\perp = K$, $i = 1,\ldots,m-k$. Furthermore we can choose independent vectors $a_1,\ldots,a_k$ such that $\text{span}\{a_1,\ldots,a_k\} \subset (\text{span}\{a_{k+1},\ldots,a_m,b_{k+1},\ldots,b_m\})^\perp$, $a_i^T J^e a_j = 0$, $i,j = 1,\ldots,k$ and $a_i \in K$. It can be also seen that we can choose $b_{k+1},\ldots,b_m,a_1,\ldots,a_k$ in such a way that $\text{Im } D \cap \text{span}\{a_1,\ldots,a_m\} = 0$. Finally we can choose independent vectors $b_1,\ldots,b_k$ satisfying $\text{span}\{b_1,\ldots,b_k\} \subset \text{span}\{a_{k+1},\ldots,a_m,b_{k+1},\ldots,b_m\}^\perp$ and $b_i^T J^e b_j = 0$, $b_i^T J^e a_j = \delta_{ij}$, $i,j = 1,\ldots,k$, and moreover $b_i \in K$. □

We obtain the following analogue of Proposition 3.14

PROPOSITION 3.41 Let $\Sigma(A,B,C,D)$ be a full and regular Hamiltonian system. Let the restriction manifold be a coisotropic subspace $K \subset W$, such that $K + \text{Im } D = W$. If $\dim K = m + k$ $(k \leq m)$, we can choose a canonical basis $w = (y_1,\ldots,y_m,u_1,\ldots,u_m)$ for W, in which the degenerate Hamiltonian system has the form

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= \bar{C}x + \bar{D}u \qquad u_{k+1} = \ldots = u_m = 0\end{aligned}$$

with $A^TJ + JA = 0$, $B^TJ = \bar{C}$, $\bar{D} = \bar{D}^T$.

PROOF: By Lemma 3.40 we can choose a canonical basis $(a_1,\ldots,a_m,b_1,\ldots,b_m)$ of W such that $K = \text{span}\{a_1,\ldots,a_m,b_1,\ldots,b_k\}$ and $\text{Im } D = \text{Im}\begin{pmatrix}\bar{D}\\ I_m\end{pmatrix}$ for some $m \times m$ matrix $\bar{D}$. Since Im D is Lagrangian, $\bar{D}^T = \bar{D}$. Then define the output space $Y = \text{span}\{a_1,\ldots,a_m\}$ and the input space $U' = \text{span}\{b_1,\ldots,b_m\}$. By feedback we can bring C into the form $\begin{pmatrix}\bar{C}\\ 0\end{pmatrix}$. Since $\text{Im } D = \text{Im}\begin{pmatrix}\bar{D}\\ I_m\end{pmatrix}$ there

exists a nonsingular $R : U \longrightarrow U$ such that $D = \begin{pmatrix} \bar{D} \\ I_m \end{pmatrix} R$. Now identify U' and R(U). Then since $K = \text{span}\{a_1,\ldots,a_m,b_1,\ldots,b_k\}$, it follows that the allowed inputs $u \in U'$ satisfy $u_{k+1} = \ldots = u_m = 0$. □

3.5.1 Realization theory for linear Hamiltonian systems

In Sections 2.1.2 and 2.1.3 we studied linear external systems of the form $\Sigma_e(P) = \{w:\mathbb{R} \longrightarrow W \mid w \in L_{1oc}$ and $P(\frac{d}{dt})\, w = 0$ in the sense of distributions$\}$ with $W = \mathbb{R}^q$ and $P(s) \in \mathbb{R}^{p\times q}[s]$, i.e. a $p \times q$ matrix consisting of polynomials in the indeterminate s. We saw that we can associate with $\Sigma_e(P)$ a geometrical object, by considering for every $s \in \mathbb{C}$ the linear subspace of $W_{\mathbb{C}} = \mathbb{C}^q$ (the complexification of W) defined by Ker P(s). In particular, if dim Ker P(s) does not depend on s, this geometrical object is an algebraic vector bundle over $\mathbb{C}$, which can be uniquely extended to an algebraic vector bundle over $\mathbb{P}^1$ (the complex projective line), see Theorem 2.9. Furthermore, $\Sigma_e(P)$ is in this case uniquely determined by this bundle over $\mathbb{P}^1$ (which we baptized as E (P(s))) *together* with its embedding in the trivial bundle $\mathbb{P}^1 \times W_{\mathbb{C}}$. The condition that dim Ker P(s) is constant is equivalent to the controllability of a minimal realization $\Sigma(A,B,C,D)$ of $\Sigma_e(P)$ (Theorem 2.13).

We shall now formulate conditions on the structure of a vector bundle E (P(s)) which are necessary and sufficient in order that $\Sigma_e(P)$ can be realized by a linear system $\Sigma(A,B,C,D)$ which is *Hamiltonian*. Later on we show how these conditions can be translated to conditions directly on the structure of $\Sigma_e(P)$.

We have to assume that W, the external space, is a symplectic space (W,J^e), with dim W = 2m. Then we can introduce on $W_{\mathbb{C}}$ a "Hermitian symplectic form" $\omega^e_{\mathbb{C}}$ as follows:

$$\omega^e_{\mathbb{C}}(w_1,w_2) := \bar{w}_1^T J^e w_2 \quad , \quad w_1,w_2 \in W_{\mathbb{C}} \tag{3.93}$$

where $\bar{}$ denotes the complex conjugate. We call a complex subspace $V \subset W_{\mathbb{C}}$ a Hermitian Lagrangian subspace of $(W_{\mathbb{C}},\omega^e_{\mathbb{C}})$ if

i) $\omega^e_{\mathbb{C}}(v_1,v_2) = 0$, for every $v_1,v_2 \in V$

ii) the complex dimension of V is m.

Analogous to the "real" symplectic case (ABRAHAM & MARSDEN (1978, Propositions 5.3.2, 5.3.3) we can prove that the maximal complex dimension of a subspace V satisfying i) is m. Let us now define an external linear Hamiltonian system.

DEFINITION 3.42 Let $P(s) \in \mathbb{R}^{m\times 2m}[s]$. Let Ker P(s) have constant dimension m, so E(P(s)) is an algebraic vector bundle over $\mathbb{P}^1$. Then $\Sigma_e(P)$ is an external Hamiltonian system if for every $s = i\omega$, $\omega \in \mathbb{R}$, Ker P(s) is a Hermitian Lagrangian subspace of $(W_{\mathbb{C}}, \omega^e_{\mathbb{C}})$.

THEOREM 3.43 $\Sigma_e(P)$ has a Hamiltonian realization $\Sigma(A,B,C,D) \iff \Sigma_e(P)$ is an external linear Hamiltonian system $\iff$ a minimal realization of $\Sigma_e(P)$ is a Hamiltonian system.

PROOF: Let $\Sigma_e(P)$ be an external Hamiltonian system. Ker P(s) at "$s = \infty$", denoted by $V(\infty)$, is uniquely determined as $\lim_{s\to\infty}$ Ker P(s) for an arbitrary path $s \to \infty$ in $\mathbb{C}$. Hence $V(\infty) = \lim_{\omega\to\infty}$ Ker $P(i\omega) = \lim_{\omega\to -\infty}$ Ker $P(i\omega)$. Since Ker P(s) is a Hermitian Lagrangian subspace for every s on the imaginary axis, and $V(\infty)$ is a *real* subspace of $W_{\mathbb{C}}$, and hence a subspace of W, we obtain that $V(\infty)$ is a Lagrangian subspace of (W,J^e). Then we can take a Lagrangian subspace Y of (W,J^e) that is complementary to $U := V(\infty)$. Furthermore we can choose a canonical basis $(a_1,\ldots,a_m,b_1,\ldots,b_m)$ of W such that Y = span $\{a_1,\ldots,a_m\}$ and U = span$\{b_1,\ldots,b_m\}$. By Lemma 2.16 we can find $m \times m$ polynomial matrices D(s) and N(s) such that corresponding to this basis Ker $P(s) = \mathrm{Im} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$, $\forall s \in \mathbb{C}$, and $G(s) := N(s)\, D^{-1}(s)$ is a strictly proper transfer matrix. Since the coefficients of P(s) are *real*, we notice that if $w \in W_{\mathbb{C}}$ satisfies $P(i\omega)\, w = 0$, then $P(-i\omega)\, \bar{w} = 0$. Because Ker P(s) is Hermitian Lagrangian on the imaginary axis we therefore conclude that $w_1^T J^e w_2 = 0$ for every $w_1 \in$ Ker $P(-i\omega)$ and $w_2 \in$ Ker $P(i\omega)$. This yields

$$(v^T N^T(-i\omega)\ v^T D^T(-i\omega)) \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \begin{pmatrix} N(i\omega)z \\ D(i\omega)z \end{pmatrix} = 0,\ \forall v,\ z \in \mathbb{R}^m,\ \forall \omega \in \mathbb{R},$$

or

$$-N^T(-i\omega)\, D^T(i\omega) + D^T(-i\omega)\, N(i\omega) = 0 \text{ , for each } \omega \in \mathbb{R}. \tag{3.94}$$

Since (3.94) is a meromorphic expression we obtain by analyticity

$$-N^T(-s)\, D^T(s) + D^T(-s)\, N(s) = 0 \text{ , for each } s \in \mathbb{C} \tag{3.95}$$

or equivalently

$$G(s) = G^T(-s) \text{ , for each } s \in \mathbb{C} \tag{3.96}$$

Now we can apply Proposition 3.5 to conclude that a minimal input-output realization $\Sigma(A,B,\bar{C})$ of $G(s)$ is Hamiltonian i.e. $A^TJ + JA = 0$, $B^TJ = \bar{C}$, where (X,J) is a symplectic space. Since by Theorem 3.39 every Hamiltonian realization $\Sigma(A,B,C,D)$ of $\Sigma_e(P)$ is equivalent to a Hamiltonian input-output realization of $\Sigma_e(P)$ we have obtained the desired conclusion. □

Remark: It can easily be seen that the set of controllability indices of a Hamiltonian system is equal to the set of observability indices. By Theorem 2.18 this implies that the Chern numbers of $E(P(s))$ and $(E(P(s))^\perp$ are equal if $\Sigma_e(P)$ is an external Hamiltonian system.

In the case of degenerate Hamiltonian systems we obtain

PROPOSITION 3.44 Let $\Sigma_e(P)$ be an external Hamiltonian system. Let $K \subset W$ be a coisotropic subspace of (W,J^e) such that $K + V(\infty) = W$. Then a minimal realization of $\Sigma_e^K(P) := \{w \in \Sigma_e(P) \text{ and } w \in K\}$ is a degenerate Hamiltonian system (see Theorem 3.41).

PROOF: A minimal realization of $\Sigma_e(P)$ is Hamiltonian. Then apply Theorem 3.41. □

In Theorem 3.43 the conditions for an external Hamiltonian system were given in the "frequency-domain", i.e. as conditions on the structure of $E(P(s))$. We can also give conditions in the "time-domain":

THEOREM 3.45 Let $P \in \mathbb{R}^{m\times 2m}[s]$, and surjective for every $s \in \mathbb{C}$. Then $\Sigma_e(P)$ is an external Hamiltonian system if and only if for all $w_1, w_2 \in \Sigma_e(P)$ with compact support (i.e. w_1 and w_2 are zero outside a compact interval of $\mathbb{R}$)

$$(3.97) \qquad \int_{-\infty}^{+\infty} w_1^T(t)\, J^e w_2(t)\, dt = 0$$

PROOF: (only if). Let $\Sigma(A,B,C,D)$ be a minimal full Hamiltonian realization of $\Sigma_e(P)$, with state space (X,J). Let (x_1,w_1) and (x_2,w_2) be elements of $\Sigma_i(A,B,C,D)$. Then one easily checks that

$$(3.98) \qquad \int_{t_1}^{t_2} w_1^T(t)\, J^e w_2(t)dt = x_1^T(t_2)Jx_2(t_2) - x_1^T(t_1)Jx_2(t_1)$$

for all $t_1 \le t_2$. Suppose that w_1 and w_2 have support inside the interval (t_1,t_2). Then by minimality it follows (Theorem 2.2) that $x_1(t_1) = x_2(t_1) = x_1(t_2) = x_2(t_2) = 0$, and therefore(3.97) holds.

(if) Let $\int_{-\infty}^{+\infty} w_1^T(t)\, J^e w_2(t)\, dt = 0$, for all $w_1, w_2 \in \Sigma_e(P)$

with compact support. Fourier transformation yields

$$(3.99) \qquad \int_{-\infty}^{+\infty} w_1^T(-i\omega)\, J^e w_2(i\omega)\, d\omega = 0$$

Since dim Ker $P(s) = m$, $\forall s \in \mathbb{C}$, a minimal realization of $\Sigma_e(P)$ is controllable, and we can split w into inputs $u \in \mathbb{R}^m$ and outputs $y \in \mathbb{R}^m$, such that $y(i\omega) = G(i\omega)\, u(i\omega)$, where G(s) is the transfer matrix corresponding to P(s). Because u is an input we can reach every frequency ω. Hence it follows from (3.99) that $w_1^T(-i\omega)\, J^e w_2(i\omega) = 0$, for every $\omega \in \mathbb{R}$ and for every $w_1(i\omega)$, $w_2(i\omega) \in$ Ker $P(i\omega)$. Since dim Ker $P(i\omega) = m$ for each $\omega \in \mathbb{R}$, this implies that Ker $P(i\omega)$ is a Hermitian Lagrangian subspace of $(W_{\mathbb{C}}, \omega^e_{\mathbb{C}})$ for every $i\omega$ on the imaginary axis. □

Remark 1: We can take a canonical basis $w = (y,u)$ for $W = \mathbb{R}^{2m}$, such that y are the outputs and u are the inputs. Then (3.97) is equivalent to

$$(3.100) \qquad \int_{-\infty}^{+\infty} (u_1^T(t) y_2(t) - y_1^T(t) u_2(t))dt = 0$$

for every (y_1,u_1) and (y_2,u_2) with compact support. If y is for instance the position q, and u the external force F, then this yields

$$(3.101) \qquad \int_{-\infty}^{+\infty} F_1^T(t) q_2(t) dt = \int_{-\infty}^{+\infty} F_2^T(t) q_1(t) dt$$

for every (q_1,F_1) and (q_2,F_2) with compact support. Equation (3.101) is of course highly suggestive, but we do not know any physical interpretation of it. On the other hand we remark that for *reciprocal* systems (see WILLEMS (1974), DAY (1971)) we can obtain expressions similar to (3.101) (however including a time-reversal !), which are just as hard to interpret.

Remark 2: Note that if dim Ker P(s) is not constant in Theorem 3.45 (i.e. if a minimal realization of $\Sigma_e(P)$ is *not* controllable), then condition (3.97) is not sufficient to conclude that a *minimal* realization of $\Sigma_e(P)$ is full or degenerate Hamiltonian. Consider for instance the autonomous system on (X,J)

$$\dot{x} = Ax \ , \ y = Cx, \ \ w = (y,u) \ , \ u = 0$$

This system is degenerate Hamiltonian if and only if $C^T J^e C = 0$ and

$A^TJ + JA = 0$. However if $C^TJ^eC = 0$ and A is *arbitrary*, then $w_1^T(t)J^ew_2(t) = 0$ for every w_1, w_2 in the external behavior of the system. Therefore if $C^TJ^eC = 0$ then condition (3.97) is always satisfied. We can interpret this, roughly speaking, as follows. We have to *violate* the internal interconnections of the system which result in the restriction set u = 0 (a Lagrangian subspace of W), and after these internal interconnections have been broken, we can apply condition (3.97) to conclude whether the system is Hamiltonian or not. This is very similar to the use of *variational principles* in the treatment of interconnected systems. Then one considers variations that do not satisfy the interconnection constraints. Indeed, equation (3.97) can be considered as a sort of variational principle. We return to this later on in Section 3.8.

We give some illustrative examples of linear Hamiltonian systems

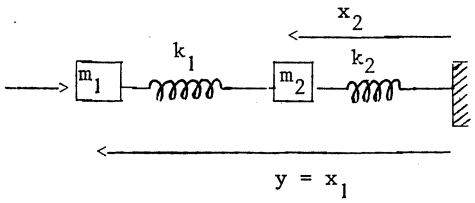
EXAMPLE 1 Newton's second law $F = m\ddot{q}$ defines an external linear system $\Sigma_e(P)$, with $P(s) = (ms^2 \vdots -1)$. It is easy to see that Ker $P(i\omega)$ is a Hermitian Lagrangian subspace of $\mathbb{C}^2$. A minimal realization $\Sigma(A,B,\bar{C})$ is given by

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}0 & 1/m\\ 0 & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} u$$

$$y = (1 \quad 0)\begin{pmatrix}q\\ p\end{pmatrix} \qquad \text{with } u = F$$

Equation (3.97) or (3.101) yields that $\int_{-\infty}^{+\infty} F_1(t)q_2(t)dt = \int_{-\infty}^{+\infty} q_1(t)F_2(t)dt$ for all (q_1,F_1), $(q_2,F_2) \in \Sigma_e(P)$ with compact support.

EXAMPLE 2 (compare the example at the end of Section 2.1.2) Consider two masses m_1 and m_2, attached to springs with spring constants k_1 and k_2



Take as external variables the force exerted on the first mass and the position y of the first mass. We obtain the equations

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ m_1\dot{x}_1 \\ m_2\dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/m_1 & 0 \\ 0 & 0 & 0 & 1/m_2 \\ -k_1 & k_1 & 0 & 0 \\ k_1 & -(k_1+k_2) & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ m_1\dot{x}_1 \\ m_2\dot{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u \ , \ y = x_1$$

i.e. a linear Hamiltonian system.

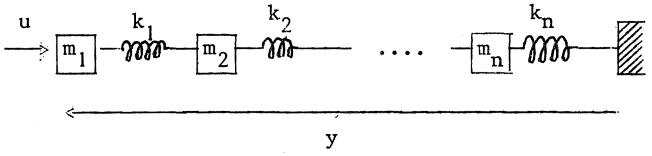
The transfer function from u to y equals

$$g(s) = \frac{m_2 s^2 + (k_1+k_2)}{m_1 m_2 s^4 + (m_2 k_1 + m_1(k_1+k_2))s^2 + k_1 k_2} \ .$$

Hence $g(s) = g(-s)$ and Σ_e is given by

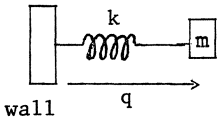
$$m_1 m_2 \, y^{(4)} + (m_2 k_1 + m_1(k_1+k_2))y^{(2)} + k_1 k_2 y = m_2 u^{(2)} + (k_1+k_2)u.$$

In the same way we can prove that n coupled masses



with input u the force on the first mass and output y the position of the first mass form a linear Hamiltonian system. In fact, one can prove that *every* external Hamiltonian system with one input and one output (i.e. an external system corresponding to a transfer *function* satisfying g(s) = g(-s)) such that the internal energy of a minimal realization is positive (this can be expressed as a condition on g(s), namely $(\sigma+i\omega)g(\sigma+i\omega) + (\sigma-i\omega)g(\sigma-i\omega) \geq 0 \ \forall \sigma \geq 0$ and $\forall \omega \in \mathbb{R}$) , can be realized by such a Hamiltonian system by selecting the values of the masses $m_1,\ldots,m_n$ and the spring constants $k_1,\ldots,k_n$ (n is equal to the number of poles of g(s)), see BROCKETT (1977), WILLEMS (1972)).

EXAMPLE 3 Consider a mass attached to a spring k

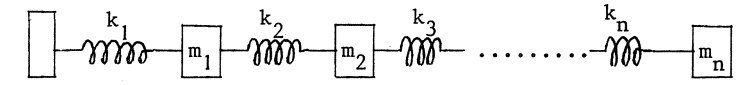


Assume that the wall is movable, and that the velocity v_w of the wall can

be controlled, i.e. $u = v_w$. Let p be the momentum of the mass (with respect to the inertial frame) Then the equations are

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ -k & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u$$

If we define the output $y = \bar{C}\begin{pmatrix} q \\ p \end{pmatrix}$ such that $B^T J = \bar{C}$, then $\bar{C} = (0 \quad 1)$, and hence the output y equals the momentum p. This is a linear Hamiltonian system. Notice that $\dot{y}$ is the *reaction* force. Again we can extend this example to n coupled masses, driven by a movable wall



with u the velocity of the wall and y the momentum of the first mass m_1 (with respect to the inertial frame). By selecting $m_1,\ldots,m_n$ and $k_1,\ldots,k_n$ we can again realize every transfer function g(s) satisfying $g(s) = g(-s)$ and $(\sigma+i\omega)g(\sigma+i\omega) + (\sigma-i\omega)g(\sigma-i\omega) \geq 0$, $\forall \sigma \geq 0$, $\forall \omega \in \mathbb{R}$.

EXAMPLE 4 Consider a mass m in $\mathbb{R}^3$ attached to a spring k, with electrical charge e and subject to a magnetic field given by a vector potential $A = (A_1,A_2,A_3)$ (i.e. B, the magnetic field, equals rot A). The system has a Lagrangian function (see Section 3.6)

$$L(q,v) = \frac{1}{2} m |v|^2 + \frac{e}{c} \sum_{i=1}^{3} v_i A_i + \frac{1}{2} \sum_{i=1}^{3} k q_i^2$$

with $v = (v_1,v_2,v_3)$ the velocity. Therefore, since $p_i = \frac{\partial L}{\partial \dot{q}_i}$, $\dot{p}_i = \frac{\partial L}{\partial q_i}$

we obtain $\begin{aligned} p_i &= mv_i + \frac{e}{c} A_i \\ \dot{p}_i &= kq_i \end{aligned}$ $\qquad i = 1,2,3$

If we consider the components A_1,A_2,A_3 of A as the inputs u_1, u_2 and u_3, then, since $v_i = \frac{1}{m} p_i - \frac{e}{cm} A_i$, we obtain the following system

$$\frac{d}{dt} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0_3 & \frac{1}{m} I_3 \\ -kI_3 & 0_3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} -\frac{e}{cm} I_3 \\ 0_3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Calculating $\bar{C}$ from $B^T J = \bar{C}$ yields

$$\bar{C} = (0_3 \ \frac{e}{cm} I_3)$$

and hence the output (y_1,y_2,y_3) is given by $y_i = \frac{e}{cm} p_i$.

Of course, the above examples yield examples of linear degenerate Hamiltonian systems by setting some of the inputs (or a linear combination of the inputs) equal to zero.

3.6 Lagrangian systems and the Euler-Lagrange equations

The Hamiltonian formalism yields an elegant description of classical meachnics in terms of first-order differential equations in the canonical q- and p- variables. From a mathematical point of view there is a perfect duality between those q- and p-variables, which suggests that the p-variables in the same way as the q-variables can be regarded as basic variables. In many applications however the Lagrangian formulation of classical mechanics, with q and $\dot{q}$ as state variables and the dynamics as second-order differential equations in q is certainly more obvious, and the p-variables are not a priori given but are *constructed* by means of the Legendre transformation. Furthermore the Lagrangian and Hamiltonian approach are not fully equivalent, due to possible degeneracies in this Legendre transformation. In a certain sense one could say that the Hamiltonian framework is for some cases unnecessarily abstract.

In this section we give the formulation of Lagrangian systems with external forces, culminating in the *Euler-Lagrange* equations. First we state some mathematical preliminaries. Consider a phase space T^*Q, endowed with its natural 1-form θ. Then define a 1-form $\dot{\theta}_L$ on $T(T^*Q)$ as follows (see TULCZYJEW (1974)). θ is a 1-form on T^*Q so we can also regard θ as a function $\hat{\theta} : T(T^*Q) \longrightarrow \mathbb{R}$. Furthermore for an arbitrary manifold N we can define a natural involution $\sim$ on TTN. Let $(x,\dot{x})$ be coordinates for TN, and let us denote coordinates for TTN by $(x,\dot{x},\delta x,\delta\dot{x})$. Then $\sim$ is given by $(x,\dot{x},\delta x,\delta\dot{x}) \longrightarrow (x,\delta x,\dot{x},\delta\dot{x})$ (see TULCZYJEW (1974)). Now we define $\dot{\theta}_L$ by

(3.102) $\dot{\theta}_L(X) := \tilde{X}(\hat{\theta})$, for any vector field X on $T(T^*Q)$

In natural coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ for T^*Q, it can be checked that $\dot{\theta}_L = \sum_{i=1}^{n} \dot{p}_i dq_i + p_i \, d\dot{q}_i$.

Apart from $\dot{\theta}_L$ we define another natural 1-form on $T(T^*Q)$, which we denote by $\dot{\theta}_H$. Since $d\theta$ is a symplectic form on T^*Q we obtain an isomorphism $T(T^*Q) \xrightarrow{\alpha} T^*(T^*Q)$, given by $\alpha(X) = d\theta(X,-)$ for $X \in T(T^*Q)$. Because $T^*(T^*Q)$ has a natural 1-form Θ, $\alpha^* \Theta =: \dot{\theta}_H$ is a 1-form on $T(T^*Q)$. In the same local coordinates as above $\dot{\theta}_H$ is given by

$$(3.103) \qquad \dot{\theta}_H = \sum_{i=1}^{n} \dot{p}_i \, dq_i - \dot{q}_i \, dp_i$$

Notice that $d\dot{\theta}_L = d\dot{\theta}_H$. Moreover $d\dot{\theta}_L = d\dot{\theta}_H$ is a symplectic form on $T(T^*Q)$, and is equal to $\dot{\omega}$ if $\omega = d\theta$ (see Section 3.1.3). Finally we note that there exists a canonical isomorphism between $T(T^*Q)$ and $T^*(TQ)$). In local coordinates this isomorphism is given by $(q,p,\dot{q},\dot{p}) \longrightarrow (q,\dot{q},p,\dot{p})$ (see TULCZYJEW (1974)). Therefore we can also look at $\dot{\theta}_L$ and $\dot{\theta}_H$ as one-forms on $T^*(TQ)$.

First we define Lagrangian vector fields (compare Definition 3.4).

DEFINITION 3.46 A Lagrangian submanifold V of $T(T^*Q)$ is a Lagrangian vectorfield if V can be parametrized by TQ. Then there exists (locally) a function $L : TQ \to \mathbb{R}$ such that $\dot{\theta}_L$ restricted to V is equal to dL. L is called the *Lagrangian* function.

Remark: If V can be also parametrized by T^*Q, then V is also a Hamiltonian vectorfield. In this case there exists (locally) a function $H : T^*Q \longrightarrow \mathbb{R}$, such that restricted to V, $\dot{\theta}_H = -dH$.

In local coordinates $\dot{\theta}_L = dL$ yields

$$\sum_{i=1}^{n} \dot{p}_i \, dq_i + p_i d\dot{q}_i = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \ , \text{ or}$$

$$(3.104) \qquad \begin{aligned} p_i &= \frac{\partial L}{\partial \dot{q}_i} \\ \dot{p}_i &= \frac{\partial L}{\partial q_i} \end{aligned} \qquad i = 1,\ldots,n$$

and therefore $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$, $i = 1,\ldots,n$

i.e. the *Euler-Lagrange* equations *without* external forces. On the other hand $\dot{\theta}_H = -dH$ yields $\sum_{i=1}^{n} \dot{p}_i \, dq_i - \dot{q}_i \, dp_i = \sum_{i=1}^{n} - \frac{\partial H}{\partial q_i} dq_i - \frac{\partial H}{\partial p_i} dp_i$, or

$$(3.105) \qquad \begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= - \frac{\partial H}{\partial q_i} \end{aligned} \qquad i = 1,\ldots,n$$

i.e. the Hamilton equations.

The definition of a Lagrangian *system* now becomes

DEFINITION 3.47 A Lagrangian system is given by a Lagrangian submanifold $V \subset T(T^*Q) \times T^*Y$ (with symplectic form $\pi_1^* d\theta_L - \pi_2^* d\theta^e$, where π_1 and π_2 are the projections on $T(T^*Q)$ and T^*Y, and θ^e is the natural 1-form on T^*Y), which can be parametrized by TQ and the fibers of T^*Y. Moreover the value of the Y-coordinates of a point of V only depends on its TQ-coordinates.

Remark: Note that this is the same definition as for an *affine Hamiltonian* system (Definition 3.23), except that V is not parametrized by T^*Q and the fibers, but by TQ and the fibers.

In local coordinates we obtain

PROPOSITION 3.48 Let $(q_1,\ldots,q_n)$ be local coordinates for Q. Then locally a Lagrangian system has the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = -\sum_{j=1}^{m} u_j \left(\frac{d}{dt}\frac{\partial C_j}{\partial \dot{q}_i} - \frac{\partial C_j}{\partial q_i}\right) \qquad i = 1,\ldots,n$$

(3.106)

$$y_j = C_j(q,\dot{q}) \qquad j = 1,\ldots,m$$

where L and $C_1,..,C_m$ are functions from TQ to $\mathbb{R}$, and $(y_1,\ldots,y_m,u_1,\ldots,u_m)$ are natural coordinates for T^*Y.

PROOF: V has a generating function of the form

$L(q,\dot{q}) + \sum_{j=1}^{m} u_j C_j(q,\dot{q})$, i.e.

$$\sum_{i=1}^{n} \dot{p}_i dq_i + p_i d\dot{q}_i + \sum_{j=1}^{m} y_j du_j = d\left(L(q,\dot{q}) + \sum_{j=1}^{m} u_j C_j(q,\dot{q})\right).$$

This yields : $p_i = \frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{m} u_i \frac{\partial C_j}{\partial \dot{q}_i}$

$$\dot{p}_i = \frac{\partial L}{\partial q_i} + \sum_{j=1}^{m} u_i \frac{\partial C_j}{\partial q_i} \quad , \quad y_j = C_j(q,\dot{q})$$

or: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{m} u_j \frac{\partial C_j}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + \sum_{j=1}^{m} u_i \frac{\partial C_j}{\partial q_i}$, $i = 1,\ldots,n$ □

The Euler-Lagrange equations are a simple special case of (3.106). We assume that the output functions $C_j : TQ \to \mathbb{R}$, $j = 1,\ldots,m$, are given by $C_j = q_j$, $j = 1,\ldots,m$. Then (3.106) yields

$$
\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \qquad , \quad i = 1,\ldots,m \\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad , \quad i = m+1,\ldots,n \\
&\qquad\qquad y_i = q_i \qquad , \quad i = 1,\ldots,m
\end{aligned}
\tag{3.107}
$$

i.e. the celebrated *Euler-Lagrange equations with external forces.*

3.7 Internal energy and external work

In this section we give definitions for internal energy and external work in the case of *affine* Hamiltonian systems. Later on we make some remarks about the general Hamiltonian case.

Let us consider an affine Hamiltonian system $\Sigma(M,T^*Y,L)$ locally given by

$$
\begin{aligned}
&\dot{x} = X_H(x) - \sum_{i=1}^{m} u_i X_{C_i}(x) \\
&y_i = C_i(x) \qquad\qquad i = 1,\ldots,m
\end{aligned}
\tag{3.108}
$$

$= (3.61)$

with $(y_1,\ldots,y_m,u_1,\ldots,u_m)$ natural coordinates for T^*Y. First we notice that on every neighborhood $U \subset M$ where a representation (3.108) holds, the function $H : U \longrightarrow \mathbb{R}$ in (3.108) is *uniquely* determined by $\Sigma(M,T^*Y,L)$ up to a constant factor. The reason is that we have taken *natural* coordinates for T^*Y, and therefore have fixed the zero-section of T^*Y (compare Theorem 3.26). Hence we can uniquely (up to a constant factor) define the *internal energy* of $\Sigma(M,T^*Y,L)$ on U as $H : U \longrightarrow \mathbb{R}$.

Let us now proceed to the definition of external work:

DEFINITION 3.49 Let $\Sigma_e(M,T^*Y,L)$ be the external behavior of $\Sigma(M,T^*Y,L)$ (see (2.11)). Let $w \in \Sigma_e(M,T^*Y,L)$ and let $\bar{w} : [0,T] \longrightarrow T^*Y$ be the restriction of w to $[0,T]$ $(T \geq 0)$. Then the *external work* performed on the system, during the external trajectory $\bar{w}$, is

$$
\text{External work} = \int_{\bar{w}} \theta^e \tag{3.109}
$$

with θ^e the natural 1-form on T^*Y and $\int_{\bar{w}} \theta^e$ the integral of θ^e along the trajectory $\bar{w}$ in T^*Y.

We apply Definition 3.49 to the following two cases:

i) $C_j = q_j$, the usual case

ii) $C_j = p_j$, see for instance Examples 3 and 4 in Section 3.5.1.

(In a certain sense these choices of C_j are the "canonical" choices, from which other output functions can be constructed). In both cases Definition 3.49 is very satisfactory.

In case i) $y_j = q_j$ and thus $u_j = F_j$ (the external force). Then θ^e is locally given by $\sum_{j=1}^{m} F_j dq_j$ and $\int_{\bar{w}} \theta^e$ equals

$$\int_{\bar{w}} \sum_{j=1}^{m} F_j dq_j = \int_0^T \sum_{j=1}^{m} \bar{F}_j(t)\, \dot{\bar{q}}_j(t)dt \tag{3.110}$$

if $\bar{w}(t) = (\bar{q}(t),\bar{F}(t))$, $t \in [0,T]$.

In case ii) $y_j = p_j$ and thus $u_j = v_j$ (the (extra) velocity ; see Example 3, Section 3.5.1). Hence $\theta^e = \sum_{j=1}^{m} v_j dp_j$ and $\int_{\bar{w}} \theta^e$ equals

$$\int_{\bar{w}} \sum_{j=1}^{m} v_j dp_j = \int_0^T \sum_{j=1}^{m} \bar{v}_j(t)\, \dot{\bar{p}}_j(t)dt \tag{3.111}$$

if $\bar{w}(t) = (\bar{p}(t),\bar{v}(t))$. Therefore (3.111) is the integral of force ($\dot{p}$) times velocity (v). In case i) we have obtained : external force $\times$ (internal) velocity, while case ii) can, roughly speaking, be interpreted as (extra) velocity $\times$ (internal) force.

We now derive a general expression for the external work $\int_{\bar{w}} \theta^e$.

Let (y,u) be natural coordinates for T^*Y. Then $\bar{w}$ can be written as $\bar{w}(t) = (\bar{y}(t),\bar{u}(t))$, $t \in [0,T]$. Let $\bar{x} : [0,T] \to M$ be such that $(\bar{x},\bar{w}) \in \Sigma_i(M,T^*Y,L)$. Now the external work equals

$$\int_0^T \sum_{i=1}^{m} \overline{u_i}(t)\, \dot{\bar{y}}_i(t)dt = \int_0^T \sum_{i=1}^{m} \overline{u_i}(t)\{H - \sum_{j=1}^{m} \overline{u_j}(t)C_j, C_i\}(\bar{x}(t))dt \tag{3.112}$$

$$= \int_0^T \sum_{i=1}^{m} \overline{u_i}(t)\, \{H,C_i\}(\bar{x}(t))dt - \int_0^T \sum_{i,j=1}^{m} \overline{u_i}(t)\overline{u_j}(t)\{C_i,C_j\}(\bar{x}(t))dt$$

where we used that $\dot{\bar{y}}_i(t) = (X_H - \sum_{j=1}^{m} \bar{u}_j(t)X_{C_j})(C_i(\bar{x}(t))) =$

$$= \{H,C_i\}(\bar{x}(t)) - \sum_{j=1}^{m} \bar{u}_j(t)\,\{C_j,C_i\}(\bar{x}(t)).$$

Because $\{C_i,C_j\} = -\{C_j,C_i\}$ the last term of (3.112) vanishes and we obtain

$$H(\bar{x}(T)) - H(\bar{x}(0)) = \sum_{i=1}^{m} \int_0^T \bar{u}_i(t)\ \{H,C_i\}(\bar{x}(t))dt \tag{3.113}$$

If we try to give a reasonable definition of internal energy and external work for general Hamiltonian systems $\Sigma(M,W,B,f)$ we run into some serious difficulties. We only make the following remarks:

i) The internal energy H for an affine Hamiltonian system is uniquely determined because the symplectic manifold T^*Y has a well-defined zero section. This is not the case if W is a general symplectic manifold.

ii) Consequently, if W is a general symplectic manifold, then the splitting of the w-variables into inputs and outputs as in (3.51) is no longer unique. Furthermore there does not exist a natural 1-form θ^e on W.

iii) In Example 2 in Section 3.2 we saw that there exist Hamiltonian systems with as input u the position of a particle and as output y the force exerted on this particle. In this case it seems that the external work equals $\dot{u}$ times y.

iv) Return to the nonlinear LC-networks of Section 3.1.2. There the internal energy of the network depends *instantaneously* on the inputs. Therefore the internal energy is no longer a function of the *state*, but of the state *and* the inputs.

3.8 Variational principles, realization theory for Hamiltonian systems and the inverse problem of the calculus of variations

In this section we first show that the generalized "variational principle" (3.100), which we proved for linear Hamiltonian systems, can be extended to the nonlinear case. Consider a Hamiltonian system $\Sigma(M,W,B,f)$. Let $\Sigma_i(M,W,B,f)$ (see (2.10)) be the corresponding dynamical system in state space form (Definition 1.2), and let $\Sigma_e(M,W,B,f)$ (see (2.11)) be its external behavior. To avoid technical difficulties we assume throughout this section that $\Sigma_i(M,W,B,f)$ consists of functions $(x,w) : \mathbb{R} \longrightarrow M \times W$ that are at least C^1. First we define *variations*. Let $(\bar{x},\bar{w}) \in \Sigma_i(M,W,B,f)$ and assume that there exists a family of functions $(x(t,\varepsilon), w(t,\varepsilon))$, $\varepsilon \in \mathbb{R}$ and small, such that

i) $(x(\cdot,\varepsilon), w(\cdot,\varepsilon)) \in \Sigma_i(M,W,B,f)$, for every ε

ii) $(x(t,\varepsilon), w(t,\varepsilon))$ is at least C^1 in ε (and by assumption also C^1 in t)

iii) $(x(\cdot,0), w(\cdot,0)) = (\bar{x},\bar{w})$

Then we define the variation $(\delta x,\delta w)$ of $(\bar{x},\bar{w})$ by:

$$(3.114) \qquad (\delta x(t), \delta w(t)) = \left(\frac{\partial x}{\partial \varepsilon}(t,0), \frac{\partial w}{\partial \varepsilon}(t,0)\right)$$

Therefore $(\delta x,\delta w)$ is a function from $\mathbb{R}$ to $TM \times TW$ such that $(\delta x(t),\delta w(t)) \in T_{\bar{x}(t)}M \times T_{\bar{w}(t)}W$.

We obtain (compare Theorem 3.45)

THEOREM 3.50 Let $\Sigma(M,W,B,f)$ be a full or degenerate Hamiltonian system. Then $\Sigma_i(M,W,B,f)$ satisfies

$$(3.115) \quad \int_{t_1}^{t_2} \omega^e_{w(t)}(\delta_1 w(t),\delta_2 w(t))\, dt =$$

$$\omega_{x(t_2)}(\delta_1 x(t_2),\delta_2 x(t_2)) - \omega_{x(t_1)}(\delta_1 x(t_1),\delta_2 x(t_1))$$

for all $t_1 \le t_2 \in \mathbb{R}$, for all $(x,w) \in \Sigma_i$ and for all variations $(\delta_1 x,\delta_1 w)$ and $(\delta_2 x,\delta_2 w)$ of (x,w) with respect to Σ_i.

PROOF: We prove that for every $t \in [t_1,t_2]$

$$(3.116) \qquad \frac{d}{dt}\, \omega_{x(t)}(\delta_1 x(t),\delta_2 x(t)) = \omega^e_{w(t)}(\delta_1 w(t),\delta_2 w(t))$$

where $\frac{d}{dt}$ denotes differentiation along the trajectory $x(\cdot)$ in M. We know that there exists an input function $u : [t_1,t_2] \longrightarrow U$ such that $x(t) = (q(t),p(t))$ is a solution of

$$(3.117) \qquad \begin{aligned} \dot{q}(t) &= \frac{\partial H}{\partial p}(q(t),p(t),u(t)) \\ \dot{p}(t) &= -\frac{\partial H}{\partial q}(q(t),p(t),u(t)) \end{aligned}$$

Now (3.117) are the equations of a time-varying Hamiltonian vectorfield on M, which we denote by $X_{H(t)}$. Then $\pounds_{X_{H(t)}} \omega = 0$, and hence

$$(3.118) \quad \frac{d}{dt}\, \omega_{x(t)}(\delta_1 x(t),\delta_2 x(t)) = (\pounds_{X_{H(t)}} \omega)(\delta_1 x(t),\delta_2 x(t))$$

$$+ \omega_{x(t)}\left(\frac{d}{dt}(\delta_1 x(t)),\delta_2 x(t)\right) + \omega_{x(t)}\left(\delta_1 x(t), \frac{d}{dt}(\delta_2 x(t))\right)$$

$$= \omega_x(t)\left(\frac{d}{dt}(\delta_1 x(t)),\delta_2 x(t)\right) + \omega_x(t)\left(\delta_1 x(t), \frac{d}{dt}(\delta_2 x(t))\right).$$

If $(x,w) : \mathbb{R} \longrightarrow M \times W$ is an element of Σ_i, then we can define a function $(x,\dot{x},w) : \mathbb{R} \longrightarrow TM \times W$, by setting $\dot{x}(t) = \frac{dx(t)}{dt}$. It is easy to see that a variation $(\delta x,\delta w)$ of (x,w) gives a variation $(\delta x,\delta\dot{x},\delta w)$ of $(x,\dot{x},w)$. Then by our smoothness assumptions:

$$\frac{d}{dt}\,(\delta_1 x(t)) = \delta_1(\frac{d}{dt}\,x(t)) = \delta_1\dot{x}(t)$$

$$\frac{d}{dt}\,(\delta_2 x(t)) = \delta_2(\frac{d}{dt}\,x(t)) = \delta_2\dot{x}(t)$$

Therefore (3.118) yields

$$(3.119)\quad \frac{d}{dt}\,\omega_{x(t)}\,(\delta_1 x(t),\delta_2 x(t)) = \omega_{x(t)}(\delta_1\dot{x}(t),\delta_2 x(t)) + \omega_{x(t)}\,(\delta_1 x(t),\delta_2\dot{x}(t)).$$

It is a matter of calculation to see that the right hand side equals $\dot{\omega}_{(x(t),\dot{x}(t))}\,((\delta_1 x(t),\delta_1\dot{x}(t)),\,(\delta_2 x(t),\delta_2\dot{x}(t))$. Since $\Sigma(M,W,B,f)$ is Hamiltonian it follows that $\pi_1^*\dot{\omega} - \pi_2^*\omega^e = 0$ restricted to $f(B)$ (with π_1 and π_2 projections of $TM \times W$ onto TM and W). Therefore

$$(3.120)\quad \frac{d}{dt}\,\omega_{x(t)}(\delta_1 x(t),\delta_2 x(t)) = \dot{\omega}_{(x(t),\dot{x}(t))}((\delta_1 x(t),\delta_1\dot{x}(t)),\,(\delta_2 x(t),\delta_2\dot{x}(t))) = \omega^e(\delta_1 w(t),\delta_2 w(t))$$

□

Remark: Conversely, if (3.115) holds we can prove that $\pi_1^*\dot{\omega} - \pi_2^*\omega^e = 0$ restricted to $f(B)$, *if* for every $(x,w) \in \Sigma_i$ there exists a sufficiently large number of variations of (x,w). Hence if we also know that $f(B)$ is a submanifold of $TM \times W$ with dimension equal to $\dim M + \frac{1}{2}\dim W$, then we can conclude that $f(B)$ is Lagrangian, and therefore $\Sigma(M,W,B,f)$ is a full Hamiltonian system (compare Theorem 3.45).

We notice that if we only consider variations $(\delta x,\delta w)$ of (x,w) which have *compact* support then (3.115) yields

$$(3.121)\quad \int_{t_1}^{t_2} \omega^e{}_{w(t)}\,(\delta_1 w(t),\,\delta_2 w(t))dt = 0$$

for every $t_1 \leq t_2$ and for all variations with support contained in (t_1,t_2). Equation (3.121) can be regarded as a generalized *variational principle* which holds for the external behavior of a Hamiltonian system. This equation seems to be crucial for what we call the *Hamiltonian realization problem*. The problem consists of finding conditions on the external behavior of a

system which imply that the system is actually Hamiltonian. More precisely:

The Hamiltonian Realization Problem :

Let Σ_e be an external system on a symplectic manifold (W,ω^e). Question: Find necessary and sufficient conditions on Σ_e in order that we can construct a Hamiltonian system $\Sigma(M,W,B,f)$ such that $\Sigma_e = \Sigma_e(M,W,B,f)$.

For the *linear* case we solved this problem in Theorem 3.45. For nonlinear systems the problem is especially hard since there exist very few results on the realization even for general nonlinear systems, without the complication due to the Hamiltonian structure. However, motivated by (3.121) and by the linear analogue we state the following

CONJECTURE: Let (W,ω^e) be a symplectic manifold. Let L be a space of sufficiently smooth functions $w : \mathbb{R} \longrightarrow W$ that can be given the structure of an (infinite dimensional) C^∞-manifold. Define the weak symplectic form Ω (see CHERNOFF & MARSDEN (1974)) on L by setting

$$(3.122) \qquad \Omega_w(\delta_1 w,\delta_2 w) = \int_{-\infty}^{+\infty} \omega^e_{w(t)} \, (\delta_1 w(t),\delta_2 w(t)) \, dt$$

for $w \in L$, and with $\delta_1 w$ and $\delta_2 w$ variations of w with respect to L with compact support. Let Σ_e be an external system on (W,ω^e) such that Σ_e is a submanifold of L. Then : Σ_e can be realized by a Hamiltonian system $\Sigma(M,W,B,f)$, if and only if Σ_e is a Lagrangian submanifold of (L,Ω).

We can also give a more restricted version of the Hamiltonian realization problem, by restricting the external systems Σ_e to systems given by high order differential equations (see Section 2.2.4).

The Restricted Hamiltonian Realization Problem:

Let Σ_e be described by a set of smooth equations $P_i(w,\dot{w},\ldots,w^{(k)}) = 0$, $i = 1,\ldots,p$, i.e. a smooth submanifold $P \subset T_k W$, with (W,ω^e) a symplectic manifold. Question : Give necessary and sufficient conditions on P in order to construct a Hamiltonian system $\Sigma(M,W,B,f)$ with $\Sigma_e(M,W,B,f) = \Sigma_e$.

We shall show that a classical problem in variational calculus and classical mechanics, namely the Inverse Problem of the Calculus of Variations is a special case of the Restricted Hamiltonian Realization Problem. Since

this Inverse Problem has in fact been solved, also the Restricted Hamiltonian Realization Problem is solved in this special case (we only need the *local* solution of the Inverse Problem; for a treatment of the global Inverse Problem we refer to TAKENS (1979)).

We sketch the Inverse Problem and its solution (an elaborate study of the Inverse Problem and its history can be found in SANTILLI (1978)). We confine ourselves to its simplest form. Let $R_i(q,\dot{q},\ddot{q}) = 0$, $i = 1,\ldots,m$, be a system of second order differential equations in $q = (q_1,\ldots,q_m)$. Assume that the $(m\times m)$-matrix $\left(\frac{\partial R_i}{\partial \ddot{q}_j}\right)$ is nonsingular everywhere. The Inverse Problem can now be stated as follows: Under which conditions on R_i, $i = 1,\ldots,m$, does there exist locally a function $L(q_1,\ldots,q_m,\dot{q}_1,\ldots,\dot{q}_m)$ such that

$$(3.123)\qquad \frac{d}{dt}\,\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = R_i(q,\dot{q},\ddot{q}) \quad , \quad i = 1,\ldots,m \quad ?$$

In other words, when are $R_i(q,\dot{q},\ddot{q}) = 0$ locally the Euler-Lagrange equations corresponding to the Lagrangian function L. Remark that since we assumed that $\left(\frac{\partial R_i}{\partial \ddot{q}_j}\right)$ is nonsingular, also $\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)$, with L satisfying (3.123), has to be nonsingular everywhere.

The solution of the Inverse Problem is as follows.
Let $q(t,\varepsilon) = (q_1(t,\varepsilon),\ldots,q_m(t,\varepsilon))$, with $\varepsilon \in \mathbb{R}$ and small, be a family of paths in $Q = \mathbb{R}^m$, such that $q_k(t,\varepsilon)$ is at least C^2 in t and C^1 in ε. Define the variations $\delta q = (\delta q_1,\ldots,\delta q_m)$ by

$$(3.124)\qquad \delta q_k(t) := \frac{\partial q_k}{\partial \varepsilon}(t,0) \quad , \quad k = 1,\ldots,m$$

Then define the variational forms $M_i(\delta q)$ of R_i, $i = 1,\ldots,m$,

$$(3.125)\qquad M_i(\delta q(t)) := \frac{\partial \bar{R}_i}{\partial \varepsilon}\bigg|_{\varepsilon=0} := \sum_{k=1}^{m} \frac{\partial R_i}{\partial q_k}(q(t,0),\dot{q}(t,0),\ddot{q}(t,0))\,\delta q_k(t) + \frac{\partial R_i}{\partial \dot{q}_k}(q(t,0),\dot{q}(t,0),\ddot{q}(t,0))\,\delta \dot{q}_k(t) + \frac{\partial R_i}{\partial \ddot{q}_k}(q(t,0),\dot{q}(t,0)\ddot{q}(t,0))\delta \ddot{q}_k(t)$$

with $\bar{R}_i(t,\varepsilon) := R_i(q(t,\varepsilon),\dot{q}(t,\varepsilon),\ddot{q}(t,\varepsilon))$.

With every system of variational forms $\tilde{M}_i$ as above we can uniquely associate a system of variational forms $\tilde{M}_i(\tilde{\delta q})$ such that there exists a function $Q(\delta q,\tilde{\delta q})$ with the property that

$$(3.126)\qquad \sum_{i=1}^{m} \tilde{\delta q}_i\, M_i(\delta q) - \delta q_i\, \tilde{M}_i(\tilde{\delta q}) = \frac{d}{dt}\, Q(\delta q,\tilde{\delta q})$$

holds for all variations δq and $\tilde{\delta} q$ of $q(\cdot,0)$ as defined in (3.124) (see SANTILLI (1978)). $\tilde{M}_i(\tilde{\delta} q)$, $i = 1,\ldots,m$, is called the *adjoint* system.

The solution of the Inverse Problem is (SANTILLI (1978)):
There exists (locally) a function $L(q,\dot{q})$ such that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = R_i$$

if and only if the variational forms M_i of R_i are self-adjoint, i.e. $M_i(\delta q) = \tilde{M}_i(\delta q)$, $i = 1,\ldots,m$. This condition is equivalent to the following equations

$$\frac{\partial R_i}{\partial \ddot{q}_k} = \frac{\partial R_k}{\partial \ddot{q}_i}$$

$$\frac{\partial R_i}{\partial \dot{q}_k} + \frac{\partial R_k}{\partial \dot{q}_i} = 2\frac{d}{dt}\frac{\partial R_i}{\partial \ddot{q}_k} \tag{3.127}$$

$$\frac{\partial R_i}{\partial q_k} - \frac{\partial R_k}{\partial q_i} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial R_i}{\partial \dot{q}_k} - \frac{\partial R_k}{\partial \dot{q}_i}\right) \quad \text{for all } i,k = 1,\ldots,m$$

(in SANTILLI (1978) these equations are attributed to Helmholtz).

Now we want to show that the Inverse Problem is a special case of the Restricted Hamiltonian Realization Problem. Take $Y := \mathbb{R}^m$ with $y = (y_1,\ldots,y_m) \in Y$, and $W := T^*Y = \mathbb{R}^m \times \mathbb{R}^m$. Let $w = (y_1,\ldots,y_m,F_1,\ldots,F_m)$ be natural coordinates for T^*Y. Instead of the equations $R_i(q,\dot{q},\ddot{q}) = 0$ we consider the system of second-order differential equations on W

$$0 = P_i(w,\dot{w},\ddot{w}) := R_i(y,\dot{y},\ddot{y}) - F_i \,, \quad i = 1,\ldots,m \tag{3.128}$$

If the Inverse Problem corresponding to R_i can be solved with a Lagrangian function L, then we obtain the following realization of the external system on W defined by P_i:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i \,, \quad i = 1,\ldots,m \tag{3.129}$$

$$y_i = q_i$$

i.e. a Lagrangian system (Section 3.6), namely the Euler-Lagrange equations with external forces. We already remarked that $\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)$ is necessarily everywhere nonsingular. Therefore (3.129) is equivalent to the Hamiltonian system

$$
(3.130)\qquad
\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} && i = 1,\ldots,m\\
\dot{p}_i &= -\frac{\partial H}{\partial q_i} + F_i\\
y_i &= q_i && i = 1,\ldots,m
\end{aligned}
$$

with $H(q_1,\ldots,q_m,\frac{\partial L}{\partial \dot{q}_1},\ldots,\frac{\partial L}{\partial \dot{q}_m}) = \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L(q,\dot{q})$

and $p_i = \frac{\partial L}{\partial \dot{q}_i}$, $i = 1,\ldots,m$.

Conclusion : Σ_e as described by (3.128) has a Hamiltonian realization (3.130). Furthermore it is easy to see that the observability codistribution of (3.130) has constant dimension 2m, and hence (3.130) is a locally minimal Hamiltonian system.

Finally we interpret equation (3.126) in our framework. Since $R_i(y,\dot{y},\ddot{y}) = F_i$, $i = 1,\ldots,m$, $M_i(\delta y)$ as in (3.125) equals δF_i. Hence (3.126) can be also written as

$$
(3.131)\qquad \sum \delta\tilde{y}_i\,\delta F_i \;-\; \delta y_i\,\delta\tilde{F}_i = \frac{d}{dt}\,Q(\delta y,\delta\tilde{y})
$$

or, if we define the symplectic form $\omega^e = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$ on $W = T^*\mathbb{R}^m$,

$$
(3.132)\qquad \omega^e((\delta y,\delta F),\,(\tilde{\delta y},\tilde{\delta F})) = \frac{d}{dt}\,Q(\delta y,\tilde{\delta y})
$$

Now we observe that (3.132) is identical to (3.116) with $\omega(\delta x,\tilde{\delta x}) = Q(\delta y,\tilde{\delta y})$.

Notes and References for Chapter 3

For an elaborate list of references on symplectic geometry and classical mechanics we refer to ABRAHAM & MARSDEN (1978). The references that we shall give are concerned with the system theoretic aspects.
Chpater 3 is mainly based on VAN DER SCHAFT (1982 d, 1982 c, 1982 b, 1983 b), see also the survey WILLEMS & VAN DER SCHAFT (1982). These papers are in turn much inspired by an innovative paper of BROCKETT (1977), and by TAKENS (1976). Another source of inspiration were some of the books or papers by HERMANN (e.g. 1968, 1973, 1974, 1976). The definition of a Hamiltonian vectorfield as a Lagrangian submanifold of TM, as in Section 3.1.3, can be found in SNIATYCKI & TULCZYJEW (1972b). The result of Section 3.1.4 is due to BROCKETT & RAHIMI (1972). Sections 3.2 and 3.2.1 are based on VAN DER SCHAFT (1982d). The expression of a Hamiltonian system in local coordinates as in (3.51) is given in BROCKETT (1977) (apart from the fact that here the output y is defined as $\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right)$ instead of $-\frac{\partial H}{\partial u}$). The treatment of degenerate Hamiltonian systems is an improved version of VAN DER SCHAFT (1982d). Section 3.2.1 is heavily influenced by BRAYTON (1971), and also HERMANN (1974). Section 3.2.3 is based on VAN DER SCHAFT (1982c). Theorem 3.21 was first proved (under stronger conditions) in BASTO GONCALVES (1980, 1981). Section 3.3 is mainly based on VAN DER SCHAFT (1982b, 1983b). Theorem 3.26 was proved in VAN DER SCHAFT (1981), while Sections 3.3.1 and 3.3.2 are partly based on VAN DER SCHAFT (1982b). Section 3.4 is motivated by CROUCH & BONNARD (1980), CROUCH (1981). Some of the results of Section 3.5 are already stated in VAN DER SCHAFT (1982d). Section 3.6 is an improved version of VAN DER SCHAFT (1982d). Definition 3.46 is in fact due to TULCZYJEW (1974). Section 3.7 is partly based on VAN DER SCHAFT (1983b). Finally Section 3.8 is partly based on VAN DER SCHAFT (1982b) and is inspired by TAKENS (1976) and SANTILLI (1978).

CHAPTER 4

SYMMETRIES, CONSERVATION LAWS AND TIME-REVERSIBILITY

4.1 Symmetries and Conservation Laws

An important topic in the theory of differential equations or more general dynamical systems and physics, is the notion of *symmetry*. Usually a symmetry is a (sometimes infinitesimal) transformation of the state space which, roughly speaking, leaves the system invariant. The existence of symmetries gives (physical) insight into the structure of the system. Furthermore the existence of symmetries is often related to the presence of *conservation laws*. These are usually seen as functions on the state space which remain constant along every solution path of the system of differential equations under consideration. Apart from their physical relevance per se, conservation laws can be used for integrating the set of differential equations that constitutes the system, since they yield first integrals (see ARNOLD (1978)).

In this section we show how we can define symmetries and conservation laws for systems with external variables as treated in Chapters 1 and 2. In sections 4.1.1, 4.1.2 and 4.1.3 we specialize these definitions to the Hamiltonian systems as defined in Chapter 3, and we show how in this case the existence of symmetries implies the existence of conservation laws, and vice versa. This generalizes a classical theorem on Hamiltonian differential equations, usually called Noether's theorem.

Let us first consider the notion of symmetry. This can already be defined for the set theoretic systems as considered in Chapter 1.

DEFINITION 4.1 Let $\Sigma_e \subset W^{\mathbb{R}}$ be a (time-invariant) external dynamical system (Definition 1.1). An *external symmetry* for Σ_e is a map $\psi : W \to W$ which leaves Σ_e invariant: if $w \in \Sigma_e$ then also $\psi(w) \in \Sigma_e$, and if $w \in \Sigma_e$ then there exists a $\tilde{w} \in \Sigma_e$ such that $\psi(\tilde{w}) = w$ ($\psi(w):\mathbb{R}\to W$ is defined in the obvious way: $(\psi(w))(t)=\psi(w(t))$). More compactly: $\psi(\Sigma_e) = \Sigma_e$.

Let $\Sigma_i \subset (X\times W)^{\mathbb{R}}$ be a dynamical system in state space form (Definition 1.2). A *symmetry* for Σ_i is a pair (ϕ,ψ), with $\phi : X \to X$ and $\psi : W \to W$, which leaves Σ_i invariant, i.e. $(\phi,\psi)\Sigma_i = \Sigma_i$.

It is clear that if (ϕ,ψ) is a symmetry for Σ_i, then ψ is an external symmetry for the external behavior Σ_e of Σ_i. We give some simple examples of (external) symmetries. The last two examples will be treated in more detail at the end of Section 4.1.2.

Example 1 Consider N particles in $\mathbb{R}^3$ in a potential field $V(q_1,\ldots,q_N)$. Suppose that we observe the positions of the first two particles, i.e. $y_1 = q_1$ and $y_2 = q_2$. Now suppose that these two particles are "identical", and that $V(q_1,q_2,\ldots,q_N) = V(q_2,q_1,\ldots,q_N)$. Then an external symmetry $\psi : \mathbb{R}^6 \to \mathbb{R}^6$ is given by $\psi(y_1,y_2) = (y_2,y_1)$. It is clear that $\psi(\Sigma_e) = \Sigma_e$. The internal map $\phi : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ such that (ϕ,ψ) is a symmetry is defined by $\phi(q_1,q_2,\ldots,q_N) = (q_2,q_1,\ldots,q_N)$. If we can also exert forces on the particles q_1 and q_2 then the external symmetry also has to include the permutation of these forces.

Example 2 Consider a mass m attached to a spring with spring constant k, say with one degree of freedom. The position of m is given by $q \in \mathbb{R}$. We observe the position, i.e. $y = q$, and we can exert a force F on m. Hence $W = \mathbb{R}^2$. The external system Σ_e is given by all force functions $F(\cdot)$ and corresponding output functions $y(\cdot) = q(\cdot)$. This external system possesses the set of external symmetries $\psi_a : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\psi_a(y,F) = (y+a,F+ka)$, $a \in \mathbb{R}$. The corresponding internal maps ϕ_a are given by $\phi_a(q,v) = (q+a,v)$, with v the velocity, i.e. translations over a.

Example 3 Consider a mass m in $\mathbb{R}^3$ subject to a potential field V and undergoing an external force F. The external variables are F and the observation of the position, i.e. $y = q$. Suppose now that V is invariant under rotations around the e_1-axis in $\mathbb{R}^3$. Then an external symmetry is given by rotating the output y around the e_1-axis and simultaneously rotating the direction of the exerted force around the e_1-axis. The internal map ϕ consists of rotating the position together with the velocity (or momentum) around the e_1-axis.

Remark: We can *extend* the above definition of (external) symmetry by allowing maps $\psi : W \times \mathbb{R} \longrightarrow W \times \mathbb{R}$ and $\phi : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$, with $\mathbb{R}$ the time axis, such that $\psi(\Sigma_e) = \Sigma_e$ and $(\phi,\psi)\Sigma_i = \Sigma_i$. Notice also that if we allow the external systems Σ_e to be *time-variant*, then the property of being time-invariant can be expressed by saying that the shift operators $(S_T w)(t) := w(t-T)$, $T \in \mathbb{R}$, are symmetries (in this generalized sense) for Σ_e.

Let now ψ be an external symmetry for Σ_e, and let Σ_i be a state space realization of Σ_e with state space X. We can pose the following basic questions:

Question 1: Does there exist a map $\phi : X \to X$ such that (ϕ,ψ) is a symmetry for Σ_i?
and if so:

Question 2: When is $\phi : X \to X$ uniquely determined by $\psi : W \to W$?

The following proposition gives a (partial) answer.

PROPOSITION 4.2 Let Σ_i be a realization of Σ_e. Let Σ_i be externally induced (Definition 1.4), i.e. there exists a map $f : \Sigma_e \to X$ such that if $(x,w) \in \Sigma_i$ then $x(0) = f(w)$. Let $\psi : W \to W$ be an external symmetry for Σ_e. Then we can define $\phi : X \to X$ such that (ϕ,ψ) is a symmetry for Σ_i if and only if $\{f(w_1)=f(w_2) \implies f\circ\psi(w_1)=f\circ\psi(w_2)$, for all $w_1,w_2\in\Sigma_e\}$. Furthermore ϕ is unique if and only if f is surjective.
If all minimal realizations of Σ_e are equivalent (see Theorems 1.6 and 1.7) then if Σ_i is a minimal realization of Σ_e, we can always define ϕ, and ϕ is unique.

PROOF ϕ has to make the following diagram comutative.

(4.1)

$$\begin{array}{ccc} \Sigma_e & \xrightarrow{\ f\ } & X \\ {\scriptstyle\psi}\downarrow & & \vdots\ \phi \\ \Sigma_e & \xrightarrow[\ f\ ]{} & X \end{array}$$

A necessary and sufficient condition for the existence of ϕ is that $f(w_1) = f(w_2)$ implies that $f\circ\psi(w_1) = f\circ\psi(w_2)$. If this condition is satisfied we define ϕ as follows. Let $x_0 \in$ Im f. Then there exists $(x,w) \in \Sigma_i$ with $x(0) = x_0$. Since $w \in \Sigma_e$, also $\psi(w) \in \Sigma_e$ and there exists an $\tilde{x}$ such that $(\tilde{x},\psi(w)) \in \Sigma_i$. Then define $\phi(x_0) = \tilde{x}(0)$. Notice that ϕ is arbitrary outside Im f and is uniquely determined on Im f. Hence if Im f = X then ϕ is unique. Suppose that all minimal realizations of Σ_e are equivalent. Let Σ_i be minimal. Since $\psi(\Sigma_e) = \Sigma_e$ also $(id,\psi)(\Sigma_i)$, with id : X → X the identity, is a minimal realization of Σ_e. Hence there exists an equivalence $\phi : X \to X$ between Σ_i and $(id,\psi)(\Sigma_i)$. Moreover ϕ is unique (Proposition 1.8). It is easy to see that (ϕ,ψ) is a symmetry for Σ_i. □

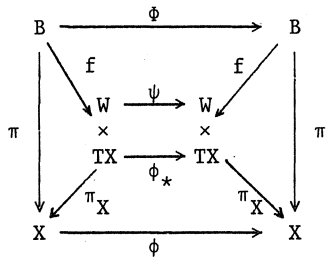
Remark: We notice that the "canonical" realizations Σ_i^+, Σ_i^- and $\Sigma_i^\pm$ (Defi-

nition 1.4) satisfy the above conditions, i.e. if ψ is an external symmetry for Σ_e, then there exist unique ϕ's such that (ϕ,ψ) is a symmetry for Σ_i^+, Σ_i^- or $\Sigma_i^\pm$. The crucial observation is that: $w_1 R^+ w_2 \iff \psi(w_1) R^+ \psi(w_2)$ and $w_1 R^- w_2 \iff \psi(w_1) R^- \psi(w_2)$.

Let us now proceed to the definition of a symmetry for smooth nonlinear systems $\Sigma(X,W,B,f)$ (Definition 2.20).

DEFINITION 4.3 A symmetry for a nonlinear system $\Sigma(X,W,B,f)$ is a triple (ϕ,ψ,Φ), such that $\phi : X \to X$, $\psi : W \to W$ and $\Phi : B \to B$ are diffeomorphisms for which the following diagram

(4.2)



commutes.

It is clear that if (ϕ,ψ,Φ) is a symmetry for $\Sigma(X,W,B,f)$ then (ϕ,ψ) is a symmetry for $\Sigma_i(X,W,B,f)$ and ψ is an external symmetry for $\Sigma_e(X,W,B,f)$, in the sense of Definition 4.1. For nonlinear input-output systems (Definition 2.21) the above definition simplifies considerably.

PROPOSITION 4.4 Let (ϕ,ψ,Φ) be a symmetry for a nonlinear input-output system $\Sigma(X,\tilde{B},Y,g,h)$. Choose local fiber respecting coordinates (y,u) for $\tilde{B}$, and (x,u) for $B = h^*\tilde{B}$, output induced by (y,u), such that Σ is given by

$$\begin{aligned} \dot{x} &= g(x,u) \\ y &= h(x) \end{aligned} \tag{4.3}$$

Then $\psi : \tilde{B} \to \tilde{B}$ has the form $\psi(y,u) = (\rho(y),\nu(y,u))$ for certain smooth maps ρ and ν. Also $\tilde{h}\circ\Phi(x,u) = \psi(\tilde{h}(x,u)) = \psi(h(x),u) = (\rho(h(x)),\nu(h(x),u))$ (recall the definition of $\tilde{h} : h^*\tilde{B} \to \tilde{B}$, i.e. $\tilde{h}(x,u) = (h(x),u)$). Moreover the following holds:

$$\begin{aligned} \phi_* g(\cdot,u) &= g(\cdot,\nu(h(\cdot),u)) \\ \rho\circ h &= h\circ\phi \end{aligned} \tag{4.4}$$

PROOF We refer to the (similar) proof of Proposition 2.29. □

Remark: Note that (4.4) expresses that ϕ maps the vectorfields $g(\cdot,u)$ onto the feedback transformed vectorfields $g(\cdot,\nu(h(\cdot),u))$. This feedback is actually *output* feedback.

For smooth nonlinear systems it is mathematically advantageous to look at *infinitesimal* symmetries.

DEFINITION 4.5 Let $\Sigma(X,W,B,f)$ be a smooth nonlinear system. An *infinitesimal* symmetry is a triple (S,T,R) with S,T and R vectorfields on X,W, respectively B, such that (S_t,T_t,R_t) is a symmetry for every $t \in \mathbb{R}$ and t small (Here S_t, T_t and R_t denote the one-parameter groups generated by S,T and R; they are at least defined locally and for small t).

The notion of infinitesimal symmetry can be expressed in the following concise way. Recall the definition of $\dot{S}$ for a vectorfield S on X (see after Definition 2.25); if S has one-parameter group S_t then $\dot{S}$ is the vectorfield on TX with one-parameter group $(S_t)_*$.

PROPOSITION 4.6 Let $\Sigma(X,W,B,f)$, with $f = (g,h)$, be a nonlinear system. Then (S,T,R) is an infinitesimal symmetry, if and only if

(i) $g_*R = \dot{S}$

(ii) $h_*R = T$

PROOF (S,T,R) is an infinitesimal symmetry if and only if (4.2) commutes for every (S_t,T_t,R_t), t small. Equivalently:

(i)' $(S_t)_* \circ g = g \circ R_t$

(ii)' $T_t \circ h = h \circ R_t$.

Differentiating (i)' and (ii)' with respect to t, in t = 0, yields (i) and (ii). □

For infinitesimal symmetries we can answer Question 2 after Definition 4.1 affirmatively.

PROPOSITION 4.7 Let $\Sigma(X,W,B,f)$ be locally minimal (Definition 2.26). Let (S_1,T,R_1) and (S_2,T,R_2) be infinitesimal symmetries. Suppose that the distributions span $\{R_1(x,u)-R_2(x,u)\}$ on B and span $\{S_1(x)-S_2(x)\}$ on X have constant dimension (i.e. 0 or1). Then $R_1 = R_2$ and $S_1 = S_2$.

PROOF By Proposition 4.6 it follows that

(i) $g_*(R_1-R_2) = \dot{S}_1-S_2$

(ii) $h_*(R_1-R_2) = 0$

Now define the distributions D on X and E on B by

$D(x) = \text{span}\ \{S_1(x)-S_2(x)\}$

$E(x,u) = \text{span}\ \{R_1(x,u)-R_2(x,u)\}$

Then E and D are by assumption of constant dimension, and clearly involutive. Moreover they satisfy

(i) $g_*E \subset \dot{D}$

(ii) $h_*E = 0$

(iii) $\pi_*E = D$

Since $\Sigma(X,W,B,f)$ is locally minimal, this implies $E = 0$, $D = 0$ and hence $R_1 = R_2$ and $S_1 = S_2$. □

Remark: We can avoid the regularity assumptions on R_1-R_2 and S_1-S_2 in the following way. It can be easily seen that $R_1-R_2 \in \ker 0^e$, with the codistribution 0^e on B as defined after Theorem 2.35. Therefore, if we assume that $\dim 0^e = \dim B$ on an open and dense subset of B (an assumption which is somewhat stronger than local minimality, Prop 2.39), then this implies that R_1-R_2 is zero on an open and dense subset of B, and hence by continuity $R_1 = R_2$ everywhere. In the case of affine input-output systems (Definition 2.22) it can be seen that $S_1-S_2 \in \ker 0$, with 0 as in Definition 2.33. Therefore if $\dim 0 = \dim X$ on an open and dense subset of X, then $S_1 = S_2$.

Hence for a locally minimal system, R and S are in fact uniquely determined by T. We now treat (external) symmetries for *linear* systems (see Section 2.1.2, 2.1.3).

DEFINITION 4.8 Let $\Sigma_e(P)$ be an external linear system with $P \in \mathbb{R}^{p\times q}[s]$, and $W = \mathbb{R}^q$. An external symmetry is a nonsingular linear map $Q : W \to W$ such that $Q(\Sigma_e(P)) = \Sigma_e(P)$.

Let $\Sigma(A,B,C,D)$ be a linear system with state space X and inputspace U. A symmetry is a triple (S,Q,T), with $S : X \to X$, $Q : W \to W$ and $T : X \times U \longrightarrow X \times U$ nonsingular linear maps, such that diagram (4.2) commutes.

PROPOSITION 4.9 Let (A,B,C,D) be a linear system

$$\dot{x} = Ax + Bu \qquad , x \in X,\ u \in U,\ w \in W$$
$$w = Cx + Du$$

Let (S,Q,T) be a symmetry. Then there exists a nonsingular linear map

$R : U \to U$, and a linear map $F : X \to U$ such that T is given by $\begin{pmatrix} S & 0 \\ F & R \end{pmatrix}$. Moreover the following identities hold

$$
\begin{aligned}
A &= S^{-1}(A+BFS^{-1})S \\
BR &= SB \\
QC &= (C+DFS^{-1})S \\
DR &= QD
\end{aligned}
\tag{4.5}
$$

Furthermore it follows that $F(\text{Ker } C) \subset \text{Ker } D$. Hence if D is injective (this is the case if $\Sigma(A,B,C,D)$ is feedback equivalent to an input-output system, cf. Proposition 2.1), then $F(\text{Ker } C) = 0$ and there exists an $H : W \to U$ such that $F = HC$.

PROOF Follows from the commutativity of (4.2). □

In the light of Proposition 4.9 we call (S,Q,F,R) as in (4.5) or (S,Q,H,R), if $F = HC$, a symmetry for a linear system.

Finally, we show that for linear systems we can answer Questions 1 and 2 after Definition 4.1 affirmatively.

THEOREM 4.10 Let $P \in \mathbb{R}^{p\times q}[s]$, with $W = \mathbb{R}^q$. Let $Q : W \to W$ be an external symmetry. This is equivalent to $\text{Ker } P(s)Q = \text{Ker } P(s)$, $\forall s \in \mathbb{C}$, or $E(P(s)Q) = E(P(s))$. Let $\Sigma(A,B,C,D)$ be a minimal realization of $\Sigma_e(P)$. Then there exist linear maps $S : X \to X$, $R : U \to U$, $H : W \to U$ with $\det S \neq 0$, $\det R \neq 0$ such that (S,Q,H,R) is a symmetry for $\Sigma(A,B,C,D)$. Moreover S,R and HC are uniquely determined by Q.

PROOF Since Q is an external symmetry, also $\Sigma(A,B,QC,QD)$ is a minimal realization of $\Sigma_e(P)$. Hence $\Sigma(A,B,C,D)$ and $\Sigma(A,B,QC,QD)$ are feedback equivalent (in a unique way). Hence there exist S,R and F satisfying (4.5). A minimal realization of $\Sigma_e(P)$ has the property that $\text{Ker } D = 0$. Therefore $F = HC$. □

We now proceed to the definition of a *conservation law*.

DEFINITION 4.11 Let $\Sigma_i \subset (X\times W)^{\mathbb{R}}$ be a dynamical system in state space form. Let $\Sigma_e \subset W^{\mathbb{R}}$ be its external behavior. Let $F_e : W \to \mathbb{R}$ be such that for every $w \in \Sigma_e$, $F_e(w(\cdot)) \in L_{1oc}$, and let $F : X \to \mathbb{R}$. We call the pair (F,F_e) a *conservation law* if

$$
F(x(t_2)) - F(x(t_1)) = \int_{t_1}^{t_2} F_e(w(\tau))d\tau \tag{4.6}
$$

holds for all $(x,w) \in \Sigma_i$ and for all $t_2 \geq t_1$.

We call F in (4.6) the *conserved quantity*. Equation (4.6) expresses that the decrease or increase of F along a trajectory x is only a function of the external trajectory w.

For smooth nonlinear systems we can take the differential version of (4.6). Recall that if $F : X \to \mathbb{R}$, then $\dot{F} : TX \to \mathbb{R}$ is defined by $\dot{F}(v) = dF(v)$, for $v \in TX$ (see after Definition 2.25).

DEFINITION 4.12 Let $\Sigma(X,W,B,f)$ with $f = (g,h)$, be a nonlinear system. Let $F_e : W \to \mathbb{R}$ and $F : X \to \mathbb{R}$ be smooth functions. Then (F,F_e) is a conservation law if

$$\dot{F} \circ g = F_e \circ h. \tag{4.7}$$

Notice that if (x,u), with $x = (x_1,\ldots,x_n)$, are fiber respecting coordinates for B, and w coordinates for W, then $\dot{F}(x,\dot{x}) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \dot{x}_i$. Hence $\dot{F} \circ g(x,u) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} g_i(x,u)$ and $\dot{F} \circ g(x,u)$ is just the time-derivative of F in x along a trajectory of the vectorfield $g(\cdot,u)$. So we can read (4.7) as $\frac{d}{dt} F(x,u) = F_e(h(x,u))$, with $\frac{d}{dt}$ the time-derivative with respect to $g(\cdot,u)$ (i.e. $\frac{dF}{dt} = \pounds_{g(\cdot,u)} F$). Instead of (4.7) we shall frequently use the suggestive notation $\frac{d}{dt}\big|_{\Sigma} F = F_e$.

Examples of conservation laws as in Definition 4.11 or 4.12 appear in many instances. Consider an *electrical network* consisting of only lossless elements (i.e. capacitors, inductors, transformers or gyrators). Take F, the conserved quantity, to be the energy stored in the capacitors and inductors. Let $V_e \in \mathbb{R}^m$ and $I_e \in \mathbb{R}^m$ be the voltages and currents at the external channels of the network. Define $F_e(V_e,I_e) = V_e^T I_e$, i.e. the supplied power. Then since the circuit is lossless $\frac{d}{dt}\big|_{\Sigma} F = F_e$.

Another example is provided by the first law of thermodynamics. We take F equal to the internal energy and F_e equal to the sum of the external instantaneous work performed on the thermodynamical system and the rate of the supplied heat. More examples of conservation laws for *Hamiltonian* systems will appear at the end of Section 4.1.2.

Notice that Definitions 4.11 and 4.12 also cover the usual definitions of a

conserved quantity for autonomous systems (i.e. no external influences). Indeed if the system is autonomous then usually $F_e(w(t)) = 0$ for every $w \in \Sigma_e$, and hence (4.6) and (4.7) yield $F(x(t_2)) = F(x(t_1))$ for all $t_2 \geq t_1$, respectively $\frac{d}{dt} F = 0$, with $\frac{d}{dt}$ the time-derivative along the state space equations of the system.

4.1.1 Symmetries and Conservation Laws for Hamiltonian systems.

In this section we show that in the case of Hamiltonian systems the existence of (infinitesimal) symmetries and of conservation laws is closely related to each other. First we define Hamiltonian (infinitesimal) symmetries.

DEFINITION 4.13 Let $\Sigma(M,W,B,f)$ be a full Hamiltonian system, with (M,ω) and (W,ω^e) symplectic manifolds. A symmetry (ϕ,ψ,Φ) for Σ is called *Hamiltonian* if $\phi^*\omega = \omega$ and $\psi^*\omega^e = \omega^e$.
An infinitesimal symmetry (S,T,R) is called *Hamiltonian* if $\pounds_S\omega = 0$ and $\pounds_T\omega^e = 0$.

So an (infinitesimal) Hamiltonian symmetry should respect the symplectic structure on M as well as on W. Actually, in the minimal case, the internal symplectic invariance on M is implied by the external symplectic invariance on W:

THEOREM 4.14 Let $\Sigma(M,W,B,f)$ be a full Hamiltonian system, which satisfies the minimality rank condition (Definition 3.20). Let (ϕ,ψ,Φ) be a symmetry such that $\psi^*\omega^e = \omega^e$. Assume that $\phi^*\omega-\omega$ has constant rank. Then $\phi^*\omega = \omega$.
Let (S,T,R) be an infinitesimal symmetry such that $\pounds_T\omega^e = 0$. Assume that for every t small, $S_t^*\omega-\omega$ has constant rank. Then $\pounds_S\omega = 0$.

PROOF Since (ϕ,ψ,Φ) is a symmetry, the commutativity of (4.2) yields that $(\phi_*,\psi)\ (f(B)) = f(B)$. Therefore $\Sigma(M,W,B,f)$, with $f = (\phi_*\circ g,\psi\circ h)$ is also a Hamiltonian system. Hence $g^*((\phi_*)^*\dot{\omega}) = h^*(\psi^*\omega^e)$ and $g^*\dot{\omega} = h^*\omega^e = h^*\psi^*\omega^e = g^*((\phi_*)^*\dot{\omega})$. This yields $g^*\dot{\Omega} = 0$, with $\Omega := \phi^*\omega - \omega$. In the same way as in the proof of Theorem 3.21 we derive that since Σ satisfies the minimality rank condition(and hence is strongly accessible) $\Omega = 0$, or $\phi^*\omega = \omega$. □

Remark: We can avoid the regularity conditions on $\phi^*\omega - \omega$ and $(S_t)^*\omega - \omega$ by requiring that Σ satisfies the strong minimality rank condition (Definition 3.20), i.e. dim O^e = dim B on an open and dense subset of B (see the Remark after Theorem 3.21).

From now on we concentrate on infinitesimal symmetries. We shall speak about symmetries when we actually mean infinitesimal symmetries.

Recall the set-up for Noether's theorem on Hamiltonian vectorfields (see ABRAHAM & MARSDEN (1978), ARNOLD (1978)). A vectorfield S on (M,ω) is called a Hamiltonian symmetry for a Hamiltonian vectorfield X_H on M if

(i) $\pounds_S\omega = 0$

(ii) $S(H) = 0$.

Then it follows from (i) that there exists (locally) a $F : M \to \mathbb{R}$ such that $S = X_F$. Moreover (ii) implies $X_H(F) = -S(H) = 0$. Hence F is a conserved quantity for X_H. Conversely, if $F : M \to \mathbb{R}$ is such that $X_H(F) = 0$, then $S = X_F$ satisfies (i) and (ii) and is a Hamiltonian symmetry.

Concluding we can state Noether's theorem as follows: if S is a Hamiltonian symmetry for X_H then there exists locally a conserved quantity F for X_H. If on the other hand F is a conserved quantity then X_F is a Hamiltonian symmetry.

Remark: Usually a restricted version of the above is called Noether's theorem. In this restricted version it is assumed that $M = T^*Q$ and that the symmetry vectorfields on T^*Q are generated by vectorfields on Q. Originally Noether's theorem was stated for vectorfields on Q which leave the *Lagrangian* invariant.

In our framework we obtain the following generalization of Noether's theorem:

THEOREM 4.15 Let $\Sigma(M,W,B,f)$ be a full Hamiltonian system. Let (S,T,R) be a Hamiltonian symmetry. Then (locally) there exists a conservation law (F,F_e), i.e. $\frac{d}{dt}|_\Sigma F = F_e$. Conversely if (F,F_e) is a conservation law, then there exists a Hamiltonian symmetry (S,T,R) such that $S = X_F$ and $T = X_{F_e}$.

PROOF Let (S,T,R) be a Hamiltonian symmetry. Since $\Sigma(M,W,B,f)$ is Hamiltonian, $g^*\dot{\omega} = h^*\omega^e$. Because S and T are Hamiltonian vectorfields, there exist (locally) functions $F : M \to \mathbb{R}$ and $F_e : W \to \mathbb{R}$ such that $S = X_F$ and $T = X_{F_e}$. Then by Proposition 4.6:

$$g^*\dot{\omega}(R,-) = h^*\omega^e(R,-) \iff \dot{\omega}(g_*R,g_*-) = \omega^e(h_*R,h_*-)$$

$\iff$

$$\dot{\omega}(\dot{S},g_*-) = \omega^e(T,h_*-) \iff d\dot{F}(g_*-) = dF_e(h_*-) \iff d(\dot{F}\circ g) = d(F_e\circ h)$$

where we have used the fact that if $\omega(S,-) = -dF$, then $\dot{\omega}(\dot{S},-) = -d\dot{F}$ (see HERMANN (1976)). By using the freedom in the choice of F_e (uniquely determined by T up to a constant) we obtain $\dot{F}\circ g = F_e \circ h$. Hence (F,F_e) is a conservation law. Conversely, let (F,F_e) be a conservation law. Define the function $\dot{F} - F_e$: $TM\times W \to \mathbb{R}$ by setting $(\dot{F}-F_e)(x,\dot{x},w) := \dot{F}(x,\dot{x}) - F_e(w)$, for $(x,\dot{x}) \in TM$ and $w \in W$. Then the fact that (F,F_e) is a conservation law is equivalent to: $\dot{F} - F_e$ restricted to $f(B)$ is zero. Define the Hamiltonian vectorfield $X_{\dot{F}-F_e}$ on $(TM\times W, \pi_1^*\dot{\omega} - \pi_2^*\omega^e)$ with Hamilton function $\dot{F} - F_e$. It is clear that $X_{\dot{F}-F_e} = (X_{\dot{F}}, X_{F_e})$, with $X_{\dot{F}}$ the Hamiltonian vectorfield on TM and X_{F_e} the Hamiltonian vectorfield on W. Since $\dot{F} - F_e$ restricted to $f(B)$ is zero we obtain $\pi_1^*\dot{\omega} - \pi_2^*\omega^e(X_k, X_{\dot{F}-F_e}) = X_k(\dot{F}-F_e) = 0$ on $f(B)$ for all Hamiltonian vectorfields X_k tangent to $f(B)$. Because $f(B)$ is Lagrangian this implies that also $X_{\dot{F}-F_e} = (X_{\dot{F}}, X_{F_e})$ is tangent to $f(B)$. Denote $S := X_F$ and $T := X_{F_e}$. Then, since $(\dot{S},T)$ is tangent to $f(B)$, we obtain for every t small:

$$((S_t)_*, T_t)(f(B)) = f(B).$$

Now we can construct a smooth one-parameter family $\Phi_t : B \to B$ such that

$((S_t)_*, T_t)\circ f = f\circ\Phi_t$, for every t small. Define the vectorfield R on B by $R(x) := \left.\frac{d\Phi_t}{dt}\right|_{t=0}(x)$.

Then (S,T,R) is a Hamiltonian symmetry. □

Remark: The proof of theorem 4.15 shows that in the definition of an infinitesimal Hamiltonian symmetry the vectorfield R on B is somewhat redundant. We can also define a Hamiltonian symmetry as a *pair* (S,T) with S and T vectorfields on M respectively W, satisfying

(i) $\pounds_S\omega = \pounds_T\omega^e = 0$

(ii) $(\dot{S},T)$ is tangent to $f(B)$ in every point of $f(B)$.

Often one does not consider a single symmetry of a dynamical system, but a *group* of symmetries. This can be also easily formalized in our framework. We do not go into details, but only give the following key observation:

THEOREM 4.16 Let $\Sigma(M,W,B,f)$ be a full Hamiltonian system. Let (S^i,T^i,R^i), $i = 1,\ldots,k$ be Hamiltonian symmetries with conservation laws (F^i,F_e^i), $i = 1,\ldots,k$. Then $([S^i,S^j],[T^i,T^j],[R^i,R^j])$ are Hamiltonian symmetries with conservation laws $(\{F^i,F^j\},\{F_e^i,F_e^j\})$ for all $i,j = 1,\ldots,k$.

PROOF Take $k = 2$. Then $(\dot{S}^1,T^1) = (X_{\dot{F}^1},X_{F_e^1})$ and $(\dot{S}^2,T^2) = (X_{\dot{F}^2},X_{F_e^2})$ are tangent to $f(B)$ and the functions $\dot{F}^1 - F_e^1$ and $\dot{F}^2 - F_e^2$ are zero restricted to $f(B)$. This yields that $\{\dot{F}^1-F_e^1,\dot{F}^2-F_e^2\} = X_{\dot{F}^1-F_e^1}(\dot{F}^2-F_e^2) = 0$ on $f(B)$ (Poisson bracket on $TM\times W$!) and that $([\dot{S}^1,\dot{S}^2],[T^1,T^2])$ is tangent to $f(B)$. □

Remark: Let P be the Poisson algebra on M generated by F^i, $i = 1,\ldots,k$. Let P_e be the Poisson algebra on W generated by F_e^i, $i = 1,\ldots,k$. Define $\alpha : P \to P_e$ such that $(F,\alpha(F))$, with $F \in P$, is a conservation law. Then it follows that α is a Poisson algebra morphism. If $\Sigma(M,W,B,f)$ satisfies the strong minimality rank condition, then α : P(modulo constant functions) $\longrightarrow P_e$ is an isomorphism. (Use Proposition 4.7 and the Remark after Proposition 4.7).

4.1.2 Symmetries and Conservation Laws for affine Hamiltonian Systems.

We now specialize the results of the preceding section to the case of affine Hamiltonian systems (see Section 3.3), to obtain more explicit formulas than is possible in the general Hamiltonian case. We already noticed in the remark after Theorem 4.15 that we can omit the vectorfield R on B in the definition of a Hamiltonian symmetry. Since we also suppressed the fiber bundle B in the definition of an affine Hamiltonian system $\Sigma(M,T^*Y,L)$ (Definition 3.23) we arrive at

DEFINITION 4.17 Let $\Sigma(M,T^*Y,L)$ be an affine Hamiltonian system. A Hamiltonian symmetry is a pair of diffeomorphisms (ϕ,ψ), with $\phi : M \to M$ and $\psi : T^*Y \to T^*Y$ such that

(i) $(\phi_*,\psi)(L) = L$

(ii) $\phi^*\omega = \omega$, $\psi^*\omega^e = \omega^e$ (ω^e the natural symplectic form on T^*Y).

An infinitesimal Hamiltonian symmetry is a pair of vectorfields (S,T), with S a vectorfield on M and T a vectorfield on W such that

(i) $(\dot{S}(z),T(z)) \in T_zL$, $\forall z \in L$ ($(\dot{S},T)$ is tangent to L)

(ii) $\pounds_S\omega = 0$, $\pounds_T\omega^e = 0$.

Remark: Analogous to Theorem 4.14 we can prove that if dim O = dim M on an open and dense subset of M(i.e. the strong minimality rank condition is satisfied) then $\psi^* \omega^e = \omega^e$ together with $(\phi_*,\psi)L = L$ (or $\pounds_T \omega^e = 0$ together with $(\dot{S}(z),T(z)) \in T_z L$) *implies* that $\phi^* \omega = \omega$ (or $\pounds_S \omega = 0$). Details can be found in VAN DER SCHAFT (1983 b).

We prove

PROPOSITION 4.18 Let (ϕ,ψ) be a Hamiltonian symmetry, then $\psi : T^*Y \to T^*Y$ is fiber respecting. Let (S,T) be an infinitesimal Hamiltonian symmetry. Then T is fiber respecting (i.e. $\pi_* T$ is a well defined vectorfield on Y, with $\pi : T^*Y \to Y$ the projection). Therefore T has locally a Hamilton function of the form

$$F_e(y,u) = \sum_{i=1}^{m} u_i K_i(y) + V(y) \tag{4.8}$$

for smooth functions K_i and V on $Y((y,u) = (y_1,\ldots,y_m,u_1,\ldots,u_m)$ are natural coordinates on T^*Y).

PROOF The fiber respecting property follows from the structure of the submanifold L (linear in the direction of the fibers of T^*Y). If $\psi : T^*Y \to T^*Y$ is fiber respecting and satisfying $\psi^* \omega^e = \omega^e$, then there exists a diffeomorphism $\rho : Y \to Y$ and a closed 1-form β on Y such that $\phi = T_\beta \circ \rho^*$, where T_β denotes fiberwise translation by β(see ABRAHAM & MARSDEN (1978, Exercise 5.2 B). Locally there exists a function $f : Y \to \mathbb{R}$ such that $\beta = df$. Equation (4.8) is the infinitesimal version of this theorem. □

In the sequel we shall only deal with infinitesimal symmetries. Therefore we omit for brevity the word infinitesimal. Theorem 4.15 specializes for affine Hamiltonian systems to

THEOREM 4.19 Let $\Sigma(M,T^*Y,L)$ be an affine Hamiltonian system with output map $C : M \to Y$ and local representation (see Proposition 3.24)

$$\dot{x} = X_H(x) - \sum_{i=1}^{m} u_i X_{C_i}(x)$$

$$y_i = C_i(x) \qquad i = 1,\ldots,m$$

where $(y_1,\ldots,y_m,u_1,\ldots,u_m)$ are natural coordinates and $C = (C_1,\ldots,C_m)$. Let (S,T) be a Hamiltonian symmetry. Then there exist (locally) functions $F : M \to \mathbb{R}$, $K_i : Y \to \mathbb{R}$, $i = 1,\ldots,m$, and $V : Y \to \mathbb{R}$ such that

$$(4.9)\qquad \begin{aligned} \{H,F\} &= V\circ C \\ \{C_i,F\} &= -K_i\circ C \qquad i = 1,\ldots,m \end{aligned}$$

Conversely if there exist functions $F : M \to \mathbb{R}$, $K_i : Y \to \mathbb{R}$, $i = 1,\ldots,m$ and $V : Y \to \mathbb{R}$ such that (4.9) holds, then (X_F, X_{F_e}), with $F_e(y,u) = \sum_{i=1}^{m} u_i K_i(y) + V(y)$ is a Hamiltonian symmetry.

PROOF If (S,T) is a Hamiltonian symmetry, then we know by Theorem 4.15 that there (locally) exist functions $F : M \to \mathbb{R}$, $F_e : W \to \mathbb{R}$ such that $\dot{F}\circ g = F_e\circ h$ with g given by $g(x,u) = X_H(x) - \sum_{i=1}^{m} u_i X_{C_i}(x)$ and h equal to $h(x,u) = (C(x),u)$. This yields $\{H - \sum_{i=1}^{m} u_i C_i, F\} = F_e(C(\cdot),u) \Longleftrightarrow$

$$\{H,F\} - \sum_{i=1}^{m} u_i\{C_i,F\} = F_e(C(\cdot),u).$$

Since the left hand side is affine in u, also F_e is affine. Hence there exist functions V and K_i on Y, $i = 1,\ldots,m$, such that $F_e(y,u) = \sum_{i=1}^{m} u_i K_i(y) + V(y)$ (this also follows from Proposition 4.18). Then we obtain (4.9). For the converse part we refer to the proof of Theorem 4.15. □

Remark: It can be easily deduced that $V = 0$ if and only if T is tangent to the zero-section of T^*Y.

A maybe unsatisfying feature of equations (4.9) is that we obtain $\{H,F\} = V\circ C$, instead of $\{H,F\} = 0$ as is the usual definition of a conserved quantity in the case of a Hamiltonian vectorfield. However, we shall show that by adding a potential function P, only depending on the outputs, to the internal energy H we can change $\{H,F\} = V\circ C$ into $\{H+P\circ C,F\} = 0$. (Notice that this is equivalent to applying Hamiltonian feedback, see Definition 3.25). In fact, we can prove the following more general

THEOREM 4.20 Let $\Sigma(M,T^*Y,L)$ be an affine Hamiltonian system. Let (S^i,T^i), $i = 1,\ldots,k$, $k \le m$, be Hamiltonian symmetries such that $\pi_* T^i$, $i = 1,\ldots,k$ (by Proposition 4.18 well-defined vectorfields on Y; π is the projection of T^*Y on Y) are in every point of Y linearly independent and hence nowhere zero. Let $(F^i,F_e^{\,i})$ be the corresponding conservation laws. Suppose that $\{F_e^{\,i},F_e^{\,j}\} = 0$, $i,j = 1,\ldots,k$. Then we can (locally) construct a

function $P : Y \to \mathbb{R}$ such that $\{H+P\circ C,F_e^i\} = 0$, $i = 1,\ldots,k$.

PROOF Since $\{F_e^i,F_e^j\} = 0$, also $[T^i,T^j] = 0$. This implies $[\pi_* T^i,\pi_* T^j] = 0$. $i,j = 1,\ldots,k$. Therefore we can take local coordinates $(y_1,\ldots,y_m)$ for Y such that $\pi_* T^i = \frac{\partial}{\partial y_i}$, $i = 1,\ldots,k$. Denote $v_i := F_e^i$, $i = 1,\ldots,k$. Then we have independent functions $y_1,\ldots,y_m$ and $v_1,\ldots,v_k$, $k\leq m$, such that

$$\{y_i,y_j\} = 0 \qquad i,j = 1,\ldots,m$$

$$\{v_i,v_j\} = 0 \qquad i,j = 1,\ldots,k$$

$$\{y_i,v_j\} = \delta_{ij} \qquad i = 1,\ldots,m,\ j = 1,\ldots,k$$

Therefore (Darboux's theorem, cf. ARNOLD (1978)) we can construct a complementary set of independent functions $v_{k+1},\ldots,v_m$ such that $\{v_i,v_j\} = 0$, $i = 1,\ldots,m, j = k+1,\ldots,m$ and $\{y_i,v_j\} = \delta_{ij}$, $i = 1,\ldots,m$, $j = k+1,\ldots,m$, or equivalently, $\{y_1,\ldots,y_m,v_1,\ldots,v_m\}$ are canonical coordinates. Then the submanifold of T^*Y given by $v_1 = \ldots = v_m = 0$ is Lagrangian and has therefore (locally) a generating function $P : Y \to \mathbb{R}$. Since $\Sigma(M,T^*Y,L)$ has generating function $H - \sum_{i=1}^{m} u_i C_i$ in the natural coordinates $(y_1,\ldots,y_m,$ $u_1,\ldots,u_m)$, it has generating function $H + P\circ C - \sum_{i=1}^{m} v_i C_i$ in the new canonical (but not necessarily natural) coordinates $(y_1,\ldots,y_m,v_1,\ldots,v_m)$. Because $F_e^i = v_i$, $i = 1,\ldots,k$ and therefore T^i is tangent to the section $v_1 = \ldots = v_m = 0$, it follows that $\{H+P\circ C,F^i\} = 0$, $i = 1,\ldots,k$. □

Remark: Note that if we write $C = (C_1,\ldots,C_m)$ corresponding to the y-coordinates constructed above, we obtain

$$\{C_i,F_j\} = -\delta_{ij}, \qquad i = 1,\ldots,m,\ j = 1,\ldots,k.$$

More details can be found in VAN DER SCHAFT (1983 b).

Finally we observe that if a conservation law satisfies $\{H,F\} = 0$ and $\{C_i,F\} = 0$, for all $i = 1,\ldots,m$, i.e. $F^e = 0$, then if dF has constant dimension 1, the affine Hamiltonian system $\Sigma(M,T^*Y,L)$ is *not* locally minimal. In fact the codimension 1 submanifold $F^{-1}(c)$ of M, with c a constant, can be (locally) factored out by the integral curves of the vectorfield X_F to obtain a new symplectic manifold $\bar{M}$, with $\dim \bar{M} = \dim M-2$. The affine Hamiltonian system projects (locally) to an affine Hamiltonian system $\Sigma(\bar{M},T^*Y,\bar{L})$, which has the same external behavior as $\Sigma(M,T^*Y,L)$ (cf. Theorem 3.36, see

also the discussion at the end of Section 3.3.2).

We give some illustrative examples of Hamiltonian symmetries.

Example 1 Consider Newton's second law, written as

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{m}\\ 0 & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} u, \quad y = (1\ 0)\begin{pmatrix}q\\ p\end{pmatrix} = q, \ q \in \mathbb{R}, \ p \in \mathbb{R}.$$

If $u = 0$, then p is a conserved quantity, indeed $\{\frac{1}{2m} p^2, p\} = 0$. Since $\{q,p\} = -1$ the corresponding external Hamiltonian symmetry on $T^*Y = T^*\mathbb{R}$ is given by the Hamiltonian vectorfield $\frac{\partial}{\partial y}$ with Hamilton function u. So the external behavior is invariant under translation of the output $y = q$. The total symmetry is given by $(S,T) = (\frac{\partial}{\partial q}, \frac{\partial}{\partial y})$, i.e. translations of q and y. The conservation law equals $(F,F_e) = (p,u)$, indeed $\frac{d}{dt} p = u$.

Example 2 Consider the affine Hamiltonian system

$$\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}\frac{1}{m}p\\ -\frac{dR}{dq}(q)\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} u, \ y = q, \ q \in \mathbb{R}, \ p \in \mathbb{R}$$

with R a potential function. Then $\{\frac{1}{2m} p^2 + R(q), p\} = -\frac{dR}{dq}(q)$. Since $y = q$ we can define $V : Y \to \mathbb{R}$ (with $Y = \mathbb{R}$) as $V(y) = -\frac{dR}{dy}(y)$. Because $\{q,p\} = -1$ we obtain the external Hamiltonian symmetry $\frac{\partial}{\partial y} + \frac{d^2R}{dy^2}(y)\frac{\partial}{\partial u}$ on T^*Y, with Hamilton function $u - \frac{dR}{dy}(y)$. Thus the conservation law is $(F,F_e) = (p, u - \frac{dR}{dy}(y))$ and the Hamiltonian symmetry is $(\frac{\partial}{\partial q}, \frac{\partial}{\partial y} + \frac{d^2R}{dy^2}(y)\frac{\partial}{\partial u})$. Of course the potential function P as in Theorem 4.20 is given by $-R$.

Example 3 Consider a particle in $\mathbb{R}^3$ with mass m in a potential field V, and subject to an external force u:

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i} + u, \quad y_i = q_i, \quad i = 1,2,3.$$

Suppose that $V(q)$ (or equivalently $\sum_{i=1}^{3} \frac{1}{2m} p_i^2 + V(q)$) is invariant under rotation around the e_1-axis. Then we know that the rotations around the e_1-axis generate a symmetry S on $T^*\mathbb{R}^3$, the phase space. For zero external force the angular momentum around the e_1-axis is preserved, i.e.

$$\frac{dI}{dt} = 0, \text{ with } I := \langle q \times m\dot{q}, e_1 \rangle .$$

However for a nonzero external force u we obtain

$$\frac{dI}{dt} = \langle q \times u, e_1 \rangle = \langle y \times u, e_1 \rangle .$$

Now $I := \langle y \times u, e_1 \rangle$ is a function on T^*Y. Hence (I, I_e) forms a conservation law. The corresponding external Hamilton symmetry $T := X_{Ie}$ on T^*Y is given by

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} .$$

This expresses the fact that the output corresponding to an external force whose direction is rotated around the e_1-axis, is obtained by rotating the output in the same way.

Example 4 Consider the rigid body with three controls as treated in Section 3.5. The internal energy is invariant under left translation $L_g h = gh$, $g,h \in SO(3)$. The infinitesimal generators of this left action of SO(3) on itself are the *right-invariant* vectorfields on SO(3). Since $C: T^*SO(3) \longrightarrow Y = SO(3)$ is just the projection, the Hamiltonian symmetries are of the form (R^*, R^*), with R a right-invariant vectorfield on SO(3), and R^* the induced vectorfield on $T^*SO(3)$.

In the case of the rigid body with two controls we have an output map $T^*SO(3) \xrightarrow{\pi} SO(3) \xrightarrow{\tilde{C}} S^2$. The map $\tilde{C} : SO(3) \longrightarrow S^2$ is obtained by factoring out a *left-invariant* distribution D on SO(3) (cf. Section 3.5). Now it can be easily checked that no right-invariant vectorfield R on SO(3) projects under $\tilde{C}$ to a well-defined vectorfield on S^2. Hence the usual symmetries for the rigid body do not yield any Hamiltonian symmetry for the rigid body with two controls.

We close this section by giving the definition of a symmetry for *degenerate* Hamiltonian systems (see for more details VAN DER SCHAFT (1983 b).

DEFINITION 4.21 Let $\Sigma(M, T^*Y, L = L' \cap (TM \times P))$ be an affine degenerate Hamiltonian system (Definition 3.28). A pair of vectorfields (S,T) is an infinitesimal Hamiltonian symmetry for $\Sigma(M, T^*Y, L)$ if (S,T) is an infinitesimal Hamiltonian symmetry for $\Sigma(M, T^*Y, L')$ and the vectorfield T is tangent to $P \subset T^*Y$, i.e. $T(z) \in T_z P$ for every $z \in P$.

Remark: A similar definition can be given for a general degenerate Hamiltonian system (Definition 3.10) by requiring that T is tangent to the restriction manifold K.

4.1.3 Symmetries and Conservation laws for linear Hamiltonian systems

In this section we briefly sketch the situation for linear Hamiltonian systems.

DEFINITION 4.22 Let $\Sigma_e(P)$ be an external Hamiltonian system on (W,J^e) (Definition 3.43), i.e. dim Ker P(s) = m, for each s, and Ker P(s) is for every $s = i\omega$, $\omega \in \mathbb{R}$, a Hermitian Lagrangian subspace of $(W_{\mathbb{C}},\omega^e_{\mathbb{C}})$, where $\omega^e_{\mathbb{C}}$ is the Hermitian symplectic form on $W_{\mathbb{C}}$ induced by J^e. An *external linear Hamiltonian symmetry* is an external symmetry $Q : W \to W$ such that $Q^T J^e Q = J^e$.

We obtain (compare Theorem 4.14)

THEOREM 4.23 Let $\Sigma_e(P)$ be an external Hamiltonian system and let $\Sigma(A,B,C,D)$ be a minimal Hamiltonian realization with state space (X,J). Let Q be an external Hamiltonian symmetry. Then there exists a symmetry (S,Q,H,R) for $\Sigma(A,B,C,D)$ (Theorem 4.10). Moreover $S : X \to X$ is symplectic, i.e. $S^TJS = J$.

PROOF By Theorem 3.40 we may assume that $\Sigma(A,B,C,D)$ is a minimal input-output realization of $\Sigma_e(P)$, with $C = \begin{pmatrix}\bar{C}\\0\end{pmatrix}$ and $D = \begin{pmatrix}0\\I\end{pmatrix}$ in a canonical basis of W. It follows from (4.2) that Q in this basis has the form $Q = \begin{pmatrix}Q_1 & 0\\ Q_2 & Q_3\end{pmatrix}$. Moreover since Q is symplectic $Q_3 = (Q_1^T)^{-1}$ and $Q_1^TQ_2 = Q_2^TQ_1$. It also follows from (4.2) or (4.5) that $R = (Q_1^T)^{-1}$ and that we may take $H = Q_2$. Then (4.5) yields: $SA = AS + BQ_2C$, $SBQ_1^T = B$ and $Q_1\bar{C} = \bar{C}S$. Since $\Sigma(A,B,C,D)$ is a Hamiltonian input-output realization $A^TJ + JA = 0$, $B^TJ = \bar{C}$. Therefore

(1) $S^TJB = -S^T\bar{C}^T = -\bar{C}^TQ_1^T = JBQ_1^T = JS^{-1}B$

(2) $S^TJ(AB) = -S^TA^TJB = -(A^TS^T-\bar{C}^TQ_2^TB^T)JB = -A^TS^TJB + \bar{C}^TQ_2^T\bar{C}B$.

Now by (1) $-A^TS^TJB = -A^TJS^{-1}B = JAS^{-1}B = J(S^{-1}A+S^{-1}BQ_2\bar{C}S^{-1})B = JS^{-1}(AB) + JS^{-1}BQ_2\bar{C}S^{-1}B$. Furthermore $JS^{-1}BQ_2\bar{C}S^{-1}B = JBQ_1^TQ_2Q_1^{-1}\bar{C}B = JBQ_2^T\bar{C}B = -\bar{C}^TQ_2^T\bar{C}B$. Hence $S^TJ(AB) = JS^{-1}(AB)$. By induction we can prove that $S^TJ(A^rB) = JS^{-1}(A^rB)$, for every r = 0,1,2,.... Then by controllability $S^TJ = JS^{-1}$, or $S^TJS = J$. □

Remark 1: Notice that since $Q_1\bar{C} = \bar{C}S$ we obtain $A = S^{-1}(A+BQ_2\bar{C}S^{-1})S = S^{-1}(A+BQ_2Q_1^{-1}\bar{C})S$. Because $(Q_2Q_1^{-1})^T = (Q_1^T)^{-1}Q_2^T = Q_2Q_1^{-1}$, the term

$BQ_2Q_1^{-1}\bar{C}$ is Hamiltonian feedback (Definition 3.25).

Remark 2: By using the same arguments as above, we can prove that two minimal linear Hamiltonian realizations $\Sigma_j(A_j,B_j,C_j,D_j)$ of $\Sigma_e(P)$ with state spaces (X_j,J_j), $j = 1,2$, are linked by an equivalence map $S : X_1 \to X_2$, such that $S^TJ_2S = J_1$ (compare Theorem 3.21, see also the end of Section 5.2.2).

As we already remarked, it follows by the state space isomorphism theorem (see for instance BROCKETT (1970)), that S in (S,Q,H,R) is uniquely determined by Q. Let us now assume that a *group* G of external symmetries for $\Sigma_e(P)$ is acting on W. In other words, there exists a group representation $G \xrightarrow{\rho} G\ell(W)$, such that every $Q \in \text{Im } \rho$ is an external symmetry for $\Sigma_e(P)$. If $\Sigma(A,B,C,D)$ with state space (X,J) is a minimal realization of $\Sigma_e(P)$, it follows by Theorem 4.23 that for every $Q \in \text{Im } \rho$ there exists a unique $S_Q : X \to X$, such that $S_Q^TJS_Q = J$. Since S_Q is unique, it is easy to see that $S_{Q_1Q_2} = S_{Q_1}S_{Q_2}$ if $Q_1,Q_2 \in \text{Im } \rho$. Hence we have obtained another representation $G \xrightarrow{\sigma} G\ell(X)$, which is *equivalent* to the representation ρ. Notice also that Im ρ is contained in the space of symplectic matrices on (W,J^e) and that Im σ is contained in the group of symplectic matrices on (X,J).

Finally, instead of linear Hamiltonian symmetries we can also consider *infinitesimal* linear Hamiltonian symmetries. The associated conservation laws are quadratic functions of x and w.

4.2 Time-reversibility

In this section we deal with a special kind of "symmetry", namely *time-reversibility*. This notion does not fit into the definition of symmetry given before, since it involves a change of time-direction.

After treating time-reversibility for general set-theoretic, nonlinear and linear systems, we look at time-reversible Hamiltonian systems. For linear and affine nonlinear Hamiltonian systems we are able to prove that, very roughly speaking, time-reversibility corresponds to the property that the internal energy is the sum of a "kinetic energy" and a "potential energy" and that the inputs are really external "forces". This shows that the often encountered statement that "Hamiltonian systems are neces-

sarily time-reversible" is only true for a specific, although important, subclass of Hamiltonian systems.

In analogy with Section 4.1 we define time reversibility first for set-theoretic systems (see Section 1.2). Let V be an arbitrary set. We define the *time-reversal* operator $R : V^{\mathbb{R}} \to V^{\mathbb{R}}$ by $(R(v))(t) = v(-t)$, for $v \in V^{\mathbb{R}}$.

DEFINITION 4.24 Let $\Sigma_e \subset W^{\mathbb{R}}$ be an external (time-invariant) dynamical system (Definition 1.1). Then Σ_e is *time-reversible* if $R(\Sigma_e) = \Sigma_e$.

A definition of time-reversibility for a dynamical system in state space form Σ_i is more problematic. Of course we could require that $R(\Sigma_i) = \Sigma_i$. However, this requirement is too strong, especially since we want that time-reversibility of Σ_e is more or less equivalent to "time-reversibility" of Σ_i, if Σ_i is a minimal realization of Σ_e. Therefore

DEFINITION 4.25 Let $\Sigma_i \subset (X \times W)^{\mathbb{R}}$ be a dynamical system in state space form (Definition 1.2). Let $R : (X \times W)^{\mathbb{R}} \to (X \times W)^{\mathbb{R}}$ be the time-reversal operator. Σ_i is *dynamic time-reversible* if there exists a map $\phi : X \to X$, with $\phi^2 = \mathrm{id}$ (ϕ is an *involution*), such that $R \circ (\phi, \mathrm{id})(\Sigma_i) = \Sigma_i$ (with $((\phi, \mathrm{id})(x,w))(t) := (\phi(x(t)), w(t))$). ϕ is called the *time-reversing symmetry*.

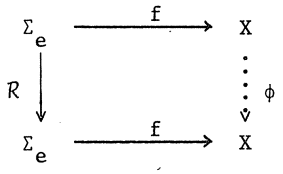
Example: Consider an autonomous system described by the equations: $m_i \ddot{q}_i + \frac{\partial V}{\partial q_i}(q_1, \ldots, q_n) = 0$, $i = 1, \ldots, n$, $q_i \in \mathbb{R}^3$, i.e. n particles in a potential field. The state is given by $(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \in \mathbb{R}^{6n}$. This system is dynamic time-reversible with time reversing symmetry $(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \longmapsto (q_1, \ldots, q_n, -\dot{q}_1, \ldots, -\dot{q}_n)$. If we define the output $y = (q_1, \ldots, q_n)$ (the positions), then the external system with external variables y is time-reversible, i.e. if $y(\cdot) : \mathbb{R} \to \mathbb{R}^{3n}$ belongs to the external behavior then also the time-reversed function $Ry(\cdot) : \mathbb{R} \to \mathbb{R}^{3n}$ is a feasible external trajectory. If we assume that we can also exert forces on the n particles then the resulting external behavior consisting of all functions $(y(\cdot), F(\cdot)) : \mathbb{R} \to \mathbb{R}^{3n} \times \mathbb{R}^{3n}$ (with $F = (F_1, \ldots, F_n)$ the external forces) is also invariant under time-reversal. So if $F(\cdot)$ yields $y(\cdot)$ for a certain state space trajectory, then there exists a state space trajectory such that the time-reversed external force $RF(\cdot)$ yields $Ry(\cdot)$.

Remark: Note that if we extend the definition of symmetry (see the remark after Definition 4.1) then time-reversibility is a symmetry in this generalized sense.

It is clear that if Σ_i is dynamic time-reversible, then the external behavior Σ_e of Σ_i is also time-reversible. The question arises when the time-reversibility of Σ_e implies dynamic time-reversibility of Σ_i.

PROPOSITION 4.26 (Compare Proposition 4.2). Let Σ_e be time-reversible, and let Σ_i be an externally induced realization of Σ_e (Definition 1.4), i.e. there exists $f : \Sigma_e \to X$ such that $(x,w) \in \Sigma_i \Longrightarrow x(0) = f(w)$. Then there exists a map $\phi : X \to X$ with $\phi^2 = id$ such $R\circ(\phi,id)\ \Sigma_i = \Sigma_i$ (i.e. Σ_i is dynamic time-reversible) if and only if $f(w_1) = f(w_2) \Longrightarrow f\circ R(w_1) = f\circ R(w_2)$, for all $w_1,w_2 \in \Sigma_e$. Furthermore ϕ is unique if and only if f is surjective. If all minimal realizations are equivalent, then minimality of Σ_i implies that we can always define ϕ, and that ϕ is unique.

PROOF ϕ has to make the following diagram commutative



A necessary and sufficient condition is that $f(w_1) = f(w_2) \Longrightarrow f\circ R(w_1) = f\circ R(w_1)$. Then ϕ is uniquely determined on Im f. Define ϕ outside Im f such that $\phi^2 = id$. We prove that ϕ inside Im f satisfies $\phi^2 = id$. Let $(x,w) \in \Sigma_i$ with $x(0) = x_0 = f(w)$. Then $\phi(x_0) = x_1$, with $x_1 = \tilde{x}(0)$ if $(\tilde{x},Rw) \in \Sigma_i$. Then $R(\phi(\tilde{x}),R(w)) = (R\circ\phi(\tilde{x}),w) \in \Sigma_i$ and hence $x_0 = \phi(x_1) = \phi^2(x_0)$. Suppose now that all minimal realizations are equivalent. Let $\Sigma_i \Longrightarrow \Sigma_e$ and Σ_i minimal. Then $R\Sigma_i$ is also a minimal realization. Hence there exists a unique $\phi : X \to X$ such that $(x,w) \in \Sigma_i \Longleftrightarrow R(\phi(x),w) \in \Sigma_i$ (see Proposition 1.8). □

Remark: We notice that $\Sigma_i^{\pm}$ satisfies the above conditions, i.e. if Σ_e is time reversible, then $\Sigma_i^{\pm}$ is dynamic time-reversible. The crucial observation is: $w_1R^+w_2 \Longleftrightarrow (Rw_1)R^-(Rw_2)$ and $w_1R^-w_2 \Longleftrightarrow (Rw_1)R^+(Rw_2)$. Notice that the other "canonical" realizations Σ_i^+ and Σ_i^- do *not* necessarily satisfy the conditions !

We now proceed to the definition of time-reversibility for nonlinear systems (Section 2.2).

DEFINITION 4.27 A smooth nonlinear system $\Sigma(X,W,B,f)$ is called *dynamic time-reversible* if there exists a diffeomorphism $\phi : M \to M$, with $\phi^2 = id$, such that

$$(4.10) \qquad (\phi_*,id)(f(B)) = (N,id)(f(B))$$

where $N : TM \to TM$ is defined by $N(x,\dot{x}) := (x,-\dot{x})$.

If $\Sigma(X,W,B,f)$ is in local coordinates given by $\dot{x} = g(x,u)$, $w = h(x,u)$, then this means that $\phi_*(g(\cdot,u) = -g(\cdot,u)$ and $h(\phi(x),u) = h(x,u)$ (compare Proposition 4.4).

For linear systems we prove that time-reversibility of the external system is equivalent to dynamic time-reversibility of a minimal realization (compare Theorem 4.10).

THEOREM 4.28 Let $P \in \mathbb{R}^{p\times q}[s]$, with $W = \mathbb{R}^q$. Then $\Sigma_e(P)$ is time-reversible if and only if $\operatorname{Ker} P(s) = \operatorname{Ker} P(-s)$, $\forall s \in \mathbb{C}$, or equivalently $E(P(s)) = E(P(-s))$. Furthermore let $\Sigma(A,B,C,D)$ be a minimal input-output realization with feedthrough term of $\Sigma_e(P)$, i.e. $D = \begin{pmatrix} \bar{D} \\ I_m \end{pmatrix}$ and $C = \begin{pmatrix} \bar{C} \\ 0 \end{pmatrix}$. Then if $\Sigma_e(P)$ is time-reversible there exists a unique nonsingular map $V : X \to X$, with $V^2 = I$ such that

$$(4.11) \qquad \begin{aligned} -A &= VAV^{-1} \\ -B &= VB \\ \bar{C} &= \bar{C}V^{-1} \end{aligned}$$

Hence $\Sigma(A,B,C,D)$ is dynamic time-reversible (with involution V). Conversely if $\Sigma(A,B,C,D)$ is dynamic time-reversible, then $\Sigma_e(P)$ is time-reversible.

PROOF It is clear that $\Sigma_e(P)$ is time-reversible if and only if $E(P(s)) = E(P(-s))$. Let $\Sigma(A,B,C,D)$ be a realization of $\Sigma_e(P)$. Then $\Sigma(-A,-B,C,D)$ is a realization of $\Sigma_e(P(-s))$. Hence if $E(P(s)) = E(P(-s))$ and $\Sigma(A,B,C,D)$ is minimal, there exist maps $V : X \to X$, $F : X \to U$, $T : U \to U$, with $\det V \neq 0$, $\det T \neq 0$ such that

$$\begin{aligned} -A &= V(A+BF)V^{-1} \\ -B &= VBT \\ C &= (C+DF)V^{-1} \\ D &= DT \end{aligned}$$

Since D is injective, it follows that $T = I$. Let now $D = \begin{pmatrix} \bar{D} \\ I_m \end{pmatrix}$ and $C = \begin{pmatrix} \bar{C} \\ 0 \end{pmatrix}$.

Then it follows from $C = (C+DF)V^{-1}$ that $F = 0$. This results in (4.11). Moreover V in (4.11) is unique. Since clearly also V^{-1} satisfies (4.11), it follows that $V^2 = I$. □

Theorem 4.28 implies a sort of normal form for a time-reversible linear system (see WILLEMS (1978)). Since $V^2 = I$ we can always find a basis for X such that $V = \begin{pmatrix} I_k & 0 \\ 0 & -I_\ell \end{pmatrix}$, with $k+\ell = n = \dim X$. Then (4.11) yields in this basis

$$(4.12) \qquad A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \quad \overline{C} = (C_1 0).$$

4.2.1 Time-reversibility for Hamiltonian systems

In this section we combine the notion of (dynamic) time-reversibility with the Hamiltonian structure of the system. We require that the time-reversing symmetry is an *anti*-symplectomorphism.

DEFINITION 4.29 Let $\Sigma(M,W,B,f)$ be a full Hamiltonian system. Σ is called *time-reversible Hamiltonian* if there exists a diffeomorphism $\phi : M \to M$ (the time-reversing symmetry) satisfying
(i) $\phi^2 = id$
(ii) $\phi^*\omega = -\omega$ (ω is the symplectic form on M)
such that $(\phi_*,id)(f(B)) = (N,id)(f(B))$ with $N : TM \to TM$ defined by $N(x,\dot{x}) = (x,-\dot{x})$.

Remark: For brevity we have omitted the word "dynamic" in the above definition.

Using local minimality, we can in fact prove that $\phi^*\omega = -\omega$ (ϕ is anti-symplectomorphism) is implied by the other conditions (compare Theorem 4.14).

THEOREM 4.30 Let $\Sigma(M,W,B,f)$ be a locally minimal full Hamiltonian system. Let Σ be dynamic time-reversible (Definition 4.27), i.e. there exists $\phi : M \to M$ with $\phi^2 = id$ and $(\phi_*,id)(f(B)) = (N,id)(f(B))$. Assume that $\phi^*\omega+\omega$ has constant rank. Then $\phi^*\omega = -\omega$.

PROOF Write $f = (g,h)$. It is clear that $\Sigma(M,W,B,\tilde{f})$ with $\tilde{f} = (N\circ\phi_*\circ g,h)$ satisfies $\tilde{f}(B) = f(B)$, and thus is also a Hamiltonian system. Hence $(N\circ\phi_*\circ g)^*\dot{\omega} = h^*\omega^e$. Together with $g^*\dot{\omega} = h^*\omega^e$ this yields $g^*((N\circ\phi_*)^*\dot{\omega}-\dot{\omega}) = 0$

or, $g^*((\phi_*)^*N^*\dot{\omega}-\dot{\omega}) = 0$. Since $N^*\dot{\omega} = -\dot{\omega}$, we obtain $g^*\dot{\Omega} = 0$, with $\Omega := \phi^*\omega + \omega$. In the same way as in the proof of Theorem 3.21 we derive that local minimality implies $\Omega = 0$, or $\phi^*\omega = -\omega$. □

Remark 1: We can again omit the constant rank condition on $\phi^*\omega + \omega$ by considering 0^e (see the remark after Theorem 3.21), and requiring that dim 0^e = dim B on an open and dense subset of B.

Remark 2: Let $\Sigma(M,W,B,f)$ be time-reversible Hamiltonian with time-reversing symmetry $\phi : M \to M$. Let $H(q,p,u)$ be a generating function for $f(B)$. Then, since $(\phi_*,id)(f(B)) = (N,id)(f(B))$ and $\phi^*\omega = -\omega$, we obtain that $H(\phi(q,p),u) = H(q,p,u)$ + constant.

Diffeomorphisms $\phi : M \to M$ satisfying $\phi^*\omega = -\omega$ and $\phi^2 = id$ have been studied in MEYER (1981), from which we summarize the following results.

THEOREM 4.31 Let (M,ω) be a symplectic manifold. Let $\phi : M \to M$ satisfy $\phi^*\omega = -\omega$ and $\phi^2 = id$. Then the set of points $p \in M$ such that $\phi(p) = p$ forms a Lagrangian submanifold Q of M. Moreover there exist an open neighborhood U of Q in M and a diffeomorphism ψ from U to an open neighborhood V of the zero-section in T^*Q such that, if we denote the natural symplectic form on T^*Q by $\bar{\omega}$ and take natural coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ for T^*Q,

i) $\psi^*\bar{\omega} = \omega$

ii) $\psi\circ\phi\circ\psi^{-1}\ (q_1,\ldots,q_n,p_1,\ldots,p_n) = (q_1,\ldots,q_n,-p_1,\ldots,-p_n)$.

We see that the existence of a time-reversing symmetry has some implications for the form of the generating function $H(q,p,u)$, which by Remark 2 has to satisfy $H(\phi(q,p),u) = H(q,p,u)$ + constant. We shall only analyze this for affine Hamiltonian systems.

4.2.2 Time-reversible affine Hamiltonian systems

Let $\Sigma(M,T^*Y,L)$ be a full affine Hamiltonian system, which in local coordinates is given by

$$\begin{aligned} \dot{x} &= X_H(x) - \sum_{i=1}^{m} u_i X_{C_i}(x) \\ y_i &= C_i(x) \qquad i = 1,\ldots,m \end{aligned} \tag{4.13}$$

with $(y_1,\ldots,y_m,u_1,\ldots,u_m)$ natural coordinates for T^*Y. Then it is easily

seen from Definition 4.29 that Σ is time-reversible Hamiltonian if and only if there exists a diffeomorphism $\phi : M \to M$ with $\phi^2 = id$ and $\phi^*\omega = -\omega$ such that

$$
\begin{aligned}
&\phi^* X_H = -X_H \\
&\phi^* X_{C_i} = -X_{C_i} \qquad (4.14) \\
&\phi^* C_i = C_i \qquad i = 1,\ldots,m
\end{aligned}
$$

Of course we can prove, analogous to Theorem 4.29 that, if dim $\mathcal{O}$ = dim M on an open and dense subset of M,i.e. Σ satisfies the strong minimality rank condition, equations (4.14) *imply* that $\phi^*\omega = -\omega$. Even the property of involutiveness of $\phi(\phi^2=id)$ is in a certain sense implied by (4.14) and local minimality:

PROPOSITION 4.32 Let $\phi : M \to M$ be a diffeomorphism satisfying (4.14). Let $\mathcal{G}$ be the Poisson algebra of the affine Hamiltonian system (4.13) (see Section 3.3.1). Then $(\phi^2)^* f = f$, $\forall f \in \mathcal{G}$. Hence if $\mathcal{O}$ ($\mathcal{O}(x)$ = span $d\mathcal{G}(x)$) has dimension equal to dim M everywhere in x, and ϕ^2 has a fixed point x_0 (i.e. $\phi^2(x_0)=x_0$), then, if M is pathconnected from x_0, $\phi^2 = id$.

PROOF By (4.14) $\phi^* C_i = C_i$, and $\phi^*\{H,C_i\} = \phi^*(\pounds_{X_H} C_i) = \pounds_{\phi_*^{-1} X_H} \phi^* C_i =$

$$= \pounds_{-X_H} C_i = -\{H,C_i\}.$$

Therefore, by induction to the number of Poisson brackets we can prove that $\phi^* f = \pm f$ for every $f \in \mathcal{G}$. Hence $(\phi^2)^* f = f$, for every $f \in \mathcal{G}$. Let now $(\phi^2)(x_0) = x_0$. Then consider the vectorfields X_f, with $f \in \mathcal{G}$. Since $(\phi^2)^* f = f$ and $\phi^*\omega = -\omega$, it follows that $(\phi^2)_* X_f = X_f$. Since $\mathcal{O}$ has full dimension we can now travel along the integral curves of X_f to every point $x \in M$, and we obtain $\phi^2(x) = x$. □

Remark: Notice that we really need the assumption of a fixed point. Consider for instance $M = T^*\mathbb{R}$ with ϕ the translation in the q-direction by $+\pi$ ($q \in \mathbb{R}$). Let $\mathcal{G}$ be the vectorspace spanned by the functions cos q, sin q and p ((q,p) natural coordinates for $T^*\mathbb{R}$). Then clearly dim $\mathcal{O}$ = dim $d\mathcal{G}$ = 2 everywhere, and $(\phi^2)^* f = f$, $\forall f \in \mathcal{G}$. However $\phi^2 \neq id$.

We conclude that for a Hamiltonian system to be time-reversible H and

C_i have to satisfy

$$\begin{aligned} &\phi^*H = H + \text{constant} \\ &\phi^*C_i = C_i, \qquad i = 1,\ldots,m \end{aligned} \tag{4.15}$$

Let us now apply Theorem 4.31 and choose canonical coordinates $(q_1,\ldots,q_n, p_1,\ldots,p_n)$ around Q (the set of fixed points of ϕ) such that $\phi(q_1,\ldots,q_n, p_1,\ldots,p_n) = (q_1,\ldots,q_n,-p_1,\ldots,-p_n)$. If $H(q,p)$ is *quadratic* in the p-coordinates, then it follows from (4.15) that H can be written as

$$H(q,p) = \tfrac{1}{2} \sum_{i,j=1}^{n} g^{ij}(q)p_ip_j + V(q)$$

for smooth functions $g^{ij} : Q \to \mathbb{R}$, satisfying $g^{ij} = g^{ji}$, i,j = 1,...,n, and a smooth function $V : Q \to \mathbb{R}$. Hence H is the sum of a potential energy V and a kinetic energy given by a "Riemannian metric" with component functions g_{ij} (with g_{ij} satisfying $\sum_{j=1}^{n} (g_{ij})(g^{jk}) = \delta_{ik}$).

These considerations lead us to the definition of the following subclass of Hamiltonian systems.

DEFINITION 4.33 Let Q be a manifold with Riemannian metric <,> (<,> is non-degenerate but not necessarily positive definite) . Let $(q_1,\ldots,q_n, p_1,\ldots,p_n)$ be natural coordinates for T^*Q. In these coordinates $(q_1,\ldots,q_n)$ for Q the metric <,> is given by smooth functions $g_{ij} : Q \to \mathbb{R}$, i,j = 1,...,n with $g_{ij} = g_{ij}$. Define the *kinetic energy* $K : T^*Q \to \mathbb{R}$ by $K(q,p) := \frac{1}{2} \sum_{i,j=1}^{n} g^{ij}p_ip_j$, with (g^{ij}) the inverse matrix of (g_{ij}). (K can be also defined in a coordinate free way, see ABRAHAM & MARSDEN (1978 , Def. 4.5.2).) Let $\tilde{V} : Q \to \mathbb{R}$ and $\tilde{C} : Q \to Y$, with component functions $\tilde{C}_i$, i = 1,...,m, be smooth maps, and denote $V := \tilde{V}\circ\pi$, $C := \tilde{C}\circ\pi$, $C_i := \tilde{C}_i\circ\pi$, with π the projection of T^*Q on Q. Define the internal energy $H := K + V$. We call the affine Hamiltonian system

$$\begin{aligned} &\dot{x} = X_H(x) - \sum_{i=1}^{m} u_iX_{C_i}(x), \qquad x \in T^*Q \\ &y_i = C_i(x) \qquad i=1,\ldots,m \end{aligned}$$

with state space $M = T^*Q$ a *simple Hamiltonian system*.

It is clear that a simple Hamiltonian system is time-reversible with

time-reversing symmetry $\phi : T^*Q \to T^*Q$ given by $(q,p) \mapsto (q,-p)$. Of course, there are many other reasons why the simple Hamiltonian systems form a natural subclass of the set of all Hamiltonian systems. In Chapter 5 we shall consider the connection between simple Hamiltonian systems and *gradient systems*.

4.2.3 Time-reversible linear Hamiltonian systems

In this section we show that for linear systems the property of time-reversibility of a Hamiltonian system is exactly equivalent to the system being a simple Hamiltonian system.

THEOREM 4.34 Let $P(s) \in \mathbb{R}^{m\times 2m}[s]$, surjective for every $s \in \mathbb{C}$. Let $W = \mathbb{R}^{2m}$ be a symplectic space with linear symplectic form J^e. Let $\Sigma_e(P)$ be an external Hamiltonian system (i.e. Ker P(s) is a complex Lagrangian subspace for every s on the imaginary axis, see Definition 3.42), and time-reversible i.e. Ker P(s) = Ker P(-s), $\forall s \in \mathbb{C}$. Then there exists a minimal realization $\Sigma(A,B,C,D)$ of $\Sigma_e(P)$ with the following properties:

i) $D = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$ and $C = \begin{pmatrix} \bar{C} \\ 0 \end{pmatrix}$ in a symplectic basis $w = (y,u)$ of (W,J^e).

ii) The state space X is even-dimensional, say $X = \mathbb{R}^{2n}$, and has a symplectic form J.

iii) There exists a symplectic basis $(q,p) = (q_1,\ldots,q_n,p_1,\ldots,p_n)$ for (X,J) such that:

$$A = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \tilde{B} \end{pmatrix}, \quad \bar{C} = (\tilde{C}\ 0) \tag{4.16}$$

with P and $Q \in \mathbb{R}^{n\times n}$, $\tilde{B} \in \mathbb{R}^{n\times m}$ and $\tilde{C} \in \mathbb{R}^{m\times n}$ and satisfying $P = P^T$, $Q = Q^T$ and $\tilde{B}^T = \tilde{C}$. We call $\Sigma(A,B,\bar{C})$ as in (4.16) a *time-reversible Hamiltonian system in normal form.*

PROOF Since $\Sigma_e(P)$ is Hamiltonian, there exists a minimal realization $\Sigma(A,B,C,D)$ with state space X such that $D = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$, $C = \begin{pmatrix} \bar{C} \\ 0 \end{pmatrix}$ in a symplectic basis for (W,J^e). Furthermore there exists a unique symplectic form J on X such that (Theorem 3.43)

$$\begin{aligned} A^TJ + JA &= 0 \\ B^TJ &= \bar{C} \end{aligned} \tag{4.17}$$

Moreover $X = \mathbb{R}^{2n}$ for a certain n. Since Ker P(s) = Ker P(-s), also $\Sigma(-A,-B,\bar{C})$ is a minimal input-output realization of $\Sigma_e(P)$. Hence (Theorem 4.28) there exists a unique nonsingular map $V : X \to X$, with $V^2 = I_n$ such that

$$(4.18) \qquad -A = VAV^{-1}, \quad -B = VB, \quad \bar{C} = \bar{C}V^{-1}$$

It follows from (4.17) and (4.18) that for all r = 0,1,2,...

$$(4.19) \qquad V^T J(A^r B) = -JV(A^r B)$$

Hence by controllability of (A,B), $V^T J = -JV$ or equivalently

$$(4.20) \qquad V^T JV = -J.$$

Now it follows from MEYER (1981, Lemma 1) that there exists a symplectic basis $(q,p) = (q_1,\ldots,q_n,p_1,\ldots,p_n)$ such that

$$(4.21) \qquad V = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

It follows from (4.17) and (4.18) that in this basis A,B and $\bar{C}$ have the required form (4.16). □

Remark 1: Note that we can even take a symplectic basis of X for which (4.16) holds, and such that $\tilde{B}$ and $\tilde{C}$ have the simple form

$$(4.22) \qquad \tilde{B} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \ \tilde{C} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \qquad k \leq m$$

This can be seen as follows. Let rank $\tilde{C}$ = k. Then there exists a nonsingular $S : \mathbb{R}^n \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^m$ such that $T\tilde{C}S^{-1} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$. Now apply the symplectic basistransformations $\bar{S} = \begin{pmatrix} S & 0 \\ 0 & (S^T)^{-1} \end{pmatrix}$ and $\bar{T} = \begin{pmatrix} T & 0 \\ 0 & (T^t)^{-1} \end{pmatrix}$ to X, respectively W. Then $A = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix}$ transforms to

$$\begin{pmatrix} 0 & SPS^T \\ -(S^T)^{-1}QS^{-1} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \tilde{B} \end{pmatrix} \text{ transforms to } \begin{pmatrix} 0 & \\ I_k & 0 \\ 0 & 0 \end{pmatrix}$$

and $C = (\tilde{C}\ 0)$ transforms to $\begin{pmatrix} I_k & 0 \\ & 0 \\ 0 & 0 \end{pmatrix}$ □

Remark 2: Of course the conditions on $\Sigma_e(P)$ to be time-reversible Hamiltonian can also be given as the following conditions on the transfermatrix G(s) associated with P(s): $G(s) = G^T(-s)$ and $G(s) = G(-s)$.

We see that the linear Hamiltonian system (4.16) has an internal energy equal to $\frac{1}{2}p^TPp + \frac{1}{2}q^TQq$. Moreover

PROPOSITION 4.35 Let $(A,B,\bar{C})$ be as in equation (4.16). Then $\Sigma(A,B,\bar{C})$ is minimal if and only if $(PQ,P\tilde{B})$ is controllable. Hence if $\Sigma(A,B,\bar{C})$ is minimal, then $\det P \neq 0$.

PROOF Since $\Sigma(A,B,\bar{C})$ is Hamiltonian we have the equivalence: (A,B) controllable $\Longleftrightarrow$ $(\bar{C},A)$ observable (this follows easily from 4.17). Therefore: $(A,B,\bar{C})$ minimal $\Longleftrightarrow$ (A,B) controllable. It is easy to see that (A,B) controllable $\Longleftrightarrow$ $(PQ,P\tilde{B})$ controllable. □

Since $\det P \neq 0$ if $\Sigma(A,B,\bar{C})$ is minimal, we can interpret the term $\frac{1}{2}p^TPp$ as the *kinetic energy* (notice however that not necessarily $P > 0$). The term $\frac{1}{2}q^TQq$ equals the potential energy, and the outputs are the *positions* and the inputs the external *forces*. Finally we state:

PROPOSITION 4.36 Let $\Sigma_i(A_i,B_i,\bar{C}_i)$, $i = 1,2$, be two minimal realizations of a time-reversible Hamiltonian system $\Sigma_e(P)$ as in (4.16), i.e.

$$A_i = \begin{pmatrix} 0 & P_i \\ -Q_i & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ \tilde{B}_i \end{pmatrix}, \quad \bar{C}_i = (\tilde{C}_i \; 0).$$

Let $K : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the unique equivalence between Σ_1 and Σ_2, i.e. $A_2 = KA_1K^{-1}$, $B_2 = KB_1$, $C_2 = C_1K^{-1}$. Then K has the form

$K = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}$, with L and $M \in \mathbb{R}^{n\times n}$ and $M = (L^T)^{-1}$. Moreover L satisfies

$$\begin{aligned} L^TP_2^{-1}L &= P_1^{-1} & L^T\tilde{B}_1 &= \tilde{B}_2 \\ L^TQ_2L &= Q_1 & \tilde{C}_1 &= \tilde{C}_2L \end{aligned} \tag{4.23}$$

PROOF Writing out $B_1, A_1B_2, \ldots$ and $B_2, A_2B_2, \ldots$, and using controllability, one sees (since for $r = 0,1,2,\ldots$ K has to satisfy $KA_1^rB = A_2^rB_2$) that $K = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}$. Because $\Sigma_i(A_i,B_i,\bar{C}_i)$ are minimal Hamiltonian, the equivalence K has to be symplectic, i.e. $K^TJK = J$ with $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. This yields $M = (L^T)^{-1}$. The rest follows easily. □

Remark 1: Note that if $\operatorname{rank} \bar{C}_i = \operatorname{rank} B_i = m$, and if we take a basis of X such that $B_i = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$ and $\bar{C}_i = (I_m \; 0)$ (this is possible by Remark 1 after

Theorem 4.34), then L has to be the identity.

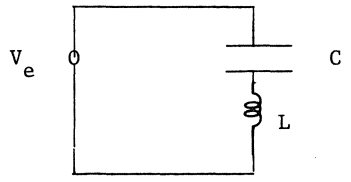
Remark 2: Let $\Sigma_i(A_i,B_i,\bar{C}_i)$, i = 1,2,be two minimal realizations, with state spaces (X_i,J_i), i = 1,2, of an external Hamiltonian system. Then the unique equivalence mapping $S : X_1 \to X_2$ satisfies $S^{-1}A_2S = A_1$ and $S^TJ_2S = J_1$. Therefore $S^T(J_2A_2)S = J_1A_1$. Since $\frac{1}{2}x_2^TJ_2A_2x$ and $\frac{1}{2}x_1^TJ_1A_1x$ are the internal energies, we can say that S leaves the internal energy of a minimal realization invariant. Equations (4.23) imply that if we consider minimal realizations of time-reversible external Hamiltonian systems, then S leaves also the kinetic energy and the potential energy invariant. We remark that for reciprocal systems (or gradient systems, see Section 5.2.2) a similar result can be stated, namely that the *difference* of the "kinetic" and "potential" energy (or "electric" and "magnetic" energy) is an invariant (WILLEMS (1972)).

4.2.4 Synthesis of linear LCT-networks

We show that a linear electrical network consisting of only inductors (L), capacitors (C) and transformers (T) is, apart from possible denegeracies, a time-reversible Hamiltonian system. Conversely, we show that every time-reversible Hamiltonian system with a positive internal energy can be realized by an LCT-network. This equivalence allows us to give necessary and sufficient conditions on a transfer matrix in order to be the driving point admittance or impedance of a LCT-network (we can also allow for hybrid representations).

First we give the simplest example of a LCT-network: a capacitor C coupled to an inductor L. There are two cases

a. *Series interconnection*



Let: V_e the external voltage
ϕ_L the magnetic flux of L
q_C the electric charge of C
I_L the current of L, V_L the voltage of L
I_C the current of C, V_C the voltage of C.

We have the constitutive relations (Compare (3.29) and (3.32))

$$\begin{aligned} \phi_L &= LI_L \\ q_C &= CV_C \end{aligned} \tag{4.24}$$

and the interconnections (Kirchhoff's laws)

$$\begin{aligned} V_e &= V_C + V_L \\ I_C &= I_L \end{aligned} \tag{4.25}$$

This yields: $\dot{q}_C = I_C = I_L = \frac{1}{L}\phi_L$

$\dot{\phi}_L = V_L = -V_C + V_e = -\frac{1}{C}q_C$

or:

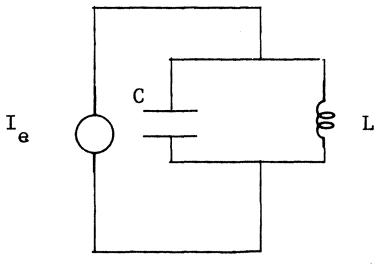
$$\frac{d}{dt}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{pmatrix}\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} V_e \tag{4.26}$$

If we define the output y by

$$y = (1\ 0)\begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = q_C \tag{4.27}$$

then (4.26) together with (4.27) is a time-reversible Hamiltonian system as in (4.16).

b. *Parallel interconnection*



I_e the external current

With the same notation as above we have the interconnection

$$\begin{aligned} V_C &= V_L \\ I_e &= I_L + I_C \end{aligned} \tag{4.28}$$

This yields

$$\frac{d}{dt}\begin{pmatrix}\phi_L\\ q_C\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{C}\\ -\frac{1}{L} & 0\end{pmatrix}\begin{pmatrix}\phi_L\\ q_C\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} I_e$$

If we define the output

$$y = (1\ 0)\begin{pmatrix}\phi_L\\ q_C\end{pmatrix} = \phi_L \tag{4.30}$$

then we have again obtained a time-reversible Hamiltonian system.

Remark: Usually the output in Case a) is taken to be $\dot{y} = I_C$, and in Case b) $\dot{y} = V_L$. We return to this later.

Let now

$$\begin{aligned}\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} &= \begin{pmatrix}0 & P\\ -Q & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} + \begin{pmatrix}0\\ \tilde{B}\end{pmatrix} u\\ y &= (\tilde{C}\ 0)\begin{pmatrix}q\\ p\end{pmatrix} \qquad \text{, with } \tilde{C}^T = \tilde{B}\end{aligned} \tag{4.31}$$

be a minimal time-reversible Hamiltonian system with positive internal energy $\frac{1}{2}p^TPp + \frac{1}{2}q^TQq$, i.e. $P \geq 0$ and $Q \geq 0$. Since (4.31) is minimal, Proposition 4.32 implies $P > 0$. We now show that (4.31) can be realized by an LCT-network. Because $P > 0$ there exists a nonsingular S with $P = SS^T$. Now apply the symplectic basis transformation $\bar{S} = \begin{pmatrix}S^{-1} & 0\\ 0 & S^T\end{pmatrix}$ to (4.31).

This yields (with different q and p)

$$\begin{aligned}\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} &= \begin{pmatrix}0 & I\\ -S^TQS & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} + \begin{pmatrix}0\\ S^T\tilde{B}\end{pmatrix} u\\ y &= \tilde{B}^TSq\end{aligned} \tag{4.32}$$

Since $S^TQS \geq 0$ we can write $S^TQS = RDR^T$, with D a diagonal matrix with diagonal elements $d_i \geq 0$, $i = 1,\ldots,n$, and $R^T = R^{-1}$. Applying the symplectic transformation $\bar{R} = \begin{pmatrix}R^{-1} & 0\\ 0 & R^T\end{pmatrix}$ one obtains

$$\begin{aligned}\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} &= \begin{pmatrix}0 & I_n\\ -D & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} + \begin{pmatrix}0\\ R^TS^T\tilde{B}\end{pmatrix} u\\ y &= \tilde{B}^TSRq\end{aligned} \tag{4.33}$$

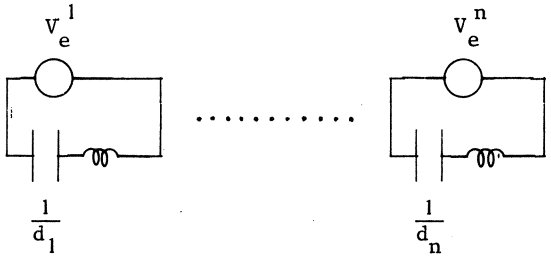
We now first realize the system

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -D & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ I_n \end{pmatrix} u \qquad (4.34)$$

$$y = q$$

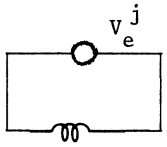
as an LC-network. This can be done as follows. If $d_i > 0$, $i = 1,\ldots,n$, we take n series interconnected pairs of one unit inductor and one capacitor $\frac{1}{d_i}$

(4.35)



with input $u = (V_e^1,\ldots,V_e^n)$ and output $y = (q^1,\ldots,q^n)$ (q^i is the charge on the i-th capacitor). If for a certain j $d_j=0$, then we replace the j-th circuit by the following circuit, consisting of only one unit inductor.

(4.36)



with the equations

$$\begin{pmatrix} \dot{q}^j \\ \dot{p}^j \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q^j \\ p^j \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} V_e^j \qquad (4.37)$$

$$y^j = q^j$$

Notice that $\dot{y}^j$ equals the current through the inductor.

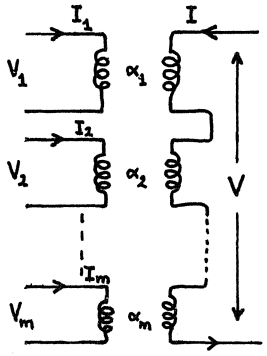
Finally, to obtain a realization of (4.33) we need *transformers*.

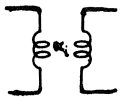
PROPOSITION 4.37 (Let $(V_1,I_1) \in \mathbb{R}^n \times \mathbb{R}^n$, $(V_2,I_2) \in \mathbb{R}^m \times \mathbb{R}^m$ be voltages and currents. Let W be an arbitrary $(n\times m)$-matrix. Then we can construct a transformer block with the transformer equations

$$V_1 = WV_2 \qquad (4.38)$$

$$I_2 = -W^T I_1$$

PROOF Consider first the case that W is a (1×m)-row vector $s = (\alpha_1,\ldots,\alpha_m)$. Then the following construction of transformers



where  denotes a transformer with ratio α_i, yields the required transformer equations

$$
\begin{aligned}
V &= \alpha_1 V_1 + \ldots + \alpha_m V_m \\
I_1 &= -\alpha_1 I \\
&\vdots \\
I_m &= -\alpha_m I
\end{aligned}
\tag{4.39}
$$

Let now r and s be two (1×m)-vectors corresponding to a construction of transformer as above, i.e.

$$
\begin{aligned}
V_2 &= sV_1 \\
I_1 &= -s^T I_2
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
V_4 &= rV_3 \\
I_3 &= -r^T I_4
\end{aligned}
\tag{4.40}
$$

Consider the interconnection

$$V = V_1 = V_3 \qquad \text{and} \qquad I = I_1 + I_2 \tag{4.41}$$

This yields

$$\begin{pmatrix} V_2 \\ V_4 \end{pmatrix} = \begin{pmatrix} s \\ r \end{pmatrix} V, \quad I = -(s^T r^T) \begin{pmatrix} I_2 \\ I_4 \end{pmatrix} \tag{4.42}$$

Therefore (2×m)-matrices W, and in the same way (n×m)-matrices W can be realized as transformer equations of a block of transformers. □

A realization of (4.31) as an LCT-network can now be completed as follows. Take the LC-network (4.35). Define the block of transformers with transformer equations

$$(4.43)\qquad \begin{aligned} V_1 &= R^T S^T \tilde{B} V_2 \\ I_2 &= -\tilde{B}^T S R I_1 \end{aligned}$$

and interconnect this with the circuit (4.35) by setting

$$\begin{aligned} V_1 &= (V_e^1, \ldots, V_e^n) \\ I_1 &= -(\dot{q}^1, \ldots, \dot{q}^n) \end{aligned}$$

Usually the input-output behavior of an electrical network is given in terms of a transfer matrix G(s) between the external voltages V_e and currents I_e. If G(s) is a map from V_e to I_e it is called an *admittance* function, and if G(s) is a map from I_e to V_e it is an *impedance* function. If G(s) is a map from a part of V_e and I_e to the complementary part of V_e and I_e we have a *hybrid* representation. Let us assume that the transfer matrix G(s) is an admittance (the other cases can be treated similarly, see VAN DER SCHAFT (1982 a)). The following theorem characterizes transfer functions of LCT networks.

THEOREM 4.38 Let G(s) be a proper transfer function. Then G(s) satisfies
(i) $G(s) = G^T(s)$
(ii) $G(s) = -G(-s)$
(iii) $G(\sigma+i\omega) + G(\sigma-i\omega) \geq 0$, for all $\sigma \geq 0$ and real ω (this property is called positive realness)
if and only if G(s) is the (driving point) admittance of an electrical network consisting of only inductors, capacitors and transformers.
The number of capacitors in such a minimal realization is less than or equal to the number of inductors. Furthermore the number of capacitors is equal to the number of inductors if and only if the McMillan degrees of G(s) and $s^{-1}G(s)$ are equal (the McMillan degree of G(s) is equal to the dimension of the state space of a minimal realization; it can be directly defined in terms of G(s), see for instance KAILATH (1980)).

PROOF (only if) Define $F(s) := s^{-1}G(s)$. Then F(s) is a strictly proper transfer matrix which satisfies

i) $F(s) = F^T(-s)$

ii) $F(s) = F(-s)$

Therefore (Remark 2 after Theorem 4.31) there exists a minimal time-reversible Hamiltonian realization $(A,B,\bar{C})$ of $F(s)$:

$$(4.45) \qquad \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{B} \end{pmatrix} u, \quad y = (\tilde{B}^T \; 0)\begin{pmatrix} q \\ p \end{pmatrix}$$

Then $\dot{y} = \tilde{B}^T\dot{q} = \tilde{B}^T P p$. Hence

$$(4.46) \qquad \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{B} \end{pmatrix} u$$

$$y = (0 \quad \tilde{B}^T P)\begin{pmatrix} q \\ p \end{pmatrix}$$

is a realization of $sF(s) = G(s)$. Notice that

$$\begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix}^T \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix} + \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} = 0 \text{ and } (0 \; \tilde{B}^T) \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix} = (0 \; \tilde{B}^T P)$$

Therefore (see WILLEMS (1972)) $\frac{1}{2}(q^T p^T)\begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{2}q^T Q q + \frac{1}{2}p^T P p$

is a storage function. Since $G(s)$ is positive real it follows that this function is positive definite. Hence $P \geq 0$, $Q \geq 0$. By minimality of (4.45) $P > 0$ (Proposition 4.35). Therefore $F(s)$ can be minimally realized by an LCT-network ((4.35) together with (4.43)) with the voltages V_e as inputs and the charges as outputs.

We note that (4.46) is controllable but not necessarily observable, and hence not necessarily a *minimal* realization of $G(s)$. In fact we can prove

LEMMA 4.39 (4.46) is a minimal realization of $G(s)$ if and only $\det Q \neq 0$.

PROOF $((0 \; \tilde{B}^T P), \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix})$ is observable if and only if $(\begin{pmatrix} 0 & -Q \\ P & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ P\tilde{B} \end{pmatrix})$ is controllable. Hence by Proposition (4.32) iff $(QP, QP\tilde{B})$ is controllable. If $(QP, QP\tilde{B})$ is controllable, then necessarily $\det Q \neq 0$. Conversely let $\det Q \neq 0$. Then $(QP, QP\tilde{B})$ controllable iff $(Q^{-1}QPQ, Q^{-1}QP\tilde{B}) = (PQ, P\tilde{B})$ controllable. Now $(PQ, P\tilde{B})$ is indeed controllable by Proposition 4.35. □

Proof of Theorem 4.38 continued

It is clear that (4.46) is a minimal realization of $G(s)$ if and only if

the McMillan degrees of G(s) and F(s) are equal. By Lemma 4.36 this is equivalent to $Q > 0$. We saw above (4.35 and 4.36) that $Q > 0$ if and only if the number of inductors is equal to the number of capacitors in a minimal realization of F(s). Suppose now that $\det Q = 0$. Then a minimal realization of G(s) can be constructed as follows. Let $\operatorname{rank} Q = k < n$. There exists a symplectic basistransformation such that $A = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix}$ has the form $\begin{pmatrix} 0 & & I_n \\ -\bar{Q} & 0 & \\ 0 & 0 & 0 \end{pmatrix}$, for a $k\times k$ diagonal matrix $\bar{Q}$ with $\det \bar{Q} \neq 0$. Write correspondingly $\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}$ with $q_1, p_1 \in \mathbb{R}^k$ and $q_2, p_2 \in \mathbb{R}^{n-k}$. We claim that

$$(4.47) \qquad \begin{pmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & I_k & 0 \\ -\bar{Q} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix} u$$

$$y = (0\ B_1^T\ B_2^T) \begin{pmatrix} q_1 \\ p_1 \\ q_2 \end{pmatrix}$$

(where $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ is the form of $\tilde{B}$ in this basis) is a minimal realization of G(s). In fact, since $\left(\begin{pmatrix} 0 & & I_n \\ -\bar{Q} & 0 & \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix}\right)$ is controllable, it is easy to see that $\left(\begin{pmatrix} 0 & I_k & 0 \\ -Q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix}\right)$ is controllable. Furthermore

$\left(\begin{pmatrix} 0 & & I_n \\ -\bar{Q} & 0 & \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix}\right)$ controllable $\Longleftrightarrow$ $\left(\begin{pmatrix} -\bar{Q} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\right)$ controllable

$\Longleftrightarrow$ (since $\det \bar{Q} \neq 0$) $\operatorname{rank} B_2 = n-k$ and $(\bar{Q}, B_1)$ controllable $\Longleftrightarrow$

(since $\det \bar{Q} \neq 0$) $\left(\begin{pmatrix} 0 & -\bar{Q} & 0 \\ I_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_1 \\ B_2 \end{pmatrix}\right)$ controllable $\Longleftrightarrow$

$\left((0\ B_1^T B_2^T), \begin{pmatrix} 0 & I_k & 0 \\ -\bar{Q} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$ observable. We notice that (4.47) amounts to the deletion of (n-k)-capacitors.

(if) Let us assume that we have a minimal electrical LCT-network. Then we do reactance extraction (see NEWCOMBE (1966)), and write

$$\begin{pmatrix} I_C \\ V_L \\ I_e \end{pmatrix} = \begin{pmatrix} 0 & Z_1 & 0 \\ -Z_1^T & 0 & -Z_2 \\ 0 & Z_2^T & 0 \end{pmatrix} \begin{pmatrix} V_C \\ I_L \\ V_e \end{pmatrix}$$

with (I_C,V_C) the currents and voltages of the capacitor block, (I_L,V_L) the currents and voltages of the inductor block and (I_e,V_e) the external currents and voltages (see for further information WILLEMS (1972)). This constitutes a minimal state space description $\Sigma(A,B,\bar{C},\bar{D})$ with

$$(4.48) \qquad A = \begin{pmatrix} 0 & -Z_1 \\ Z_1^T & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ Z_2 \end{pmatrix}, \; \bar{C} = (0 \; Z_2^T), \; \bar{D} = 0$$

From the controllability of (A,B) it follows that Z_1 is surjective, or that Z_1^T is injective. Assume first that Z_1 is square. By applying the state space transformation $\begin{pmatrix} -Z_1^{-1} & 0 \\ 0 & I \end{pmatrix}$ we obtain $A = \begin{pmatrix} 0 & I \\ -Z_1^T Z_1 & 0 \end{pmatrix}$,

$B = \begin{pmatrix} 0 \\ Z_2 \end{pmatrix}$, $\bar{C} = (0 \; Z_2^T)$. Integrating $y(\dot{z}:=y)$ yields

$$(4.49) \qquad \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -Z_1^T Z_1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ Z_2 \end{pmatrix} u$$

$$z = (Z_2^T \; 0)\begin{pmatrix} q \\ p \end{pmatrix}$$

i.e. a time-reversible Hamiltonian system.

Consider now the case that Z_1 is not square. Because Z_1 is surjective, Z_1 has more columns than rows. Let Z_1 be a $(k\times n)$-matrix $(k<n)$. Then construct a non-singular $n\times n$-matrix S such that $SZ_1^T = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$. Application of the transformation $\begin{pmatrix} I_{2n-k} & 0 \\ 0 & -S \end{pmatrix}$ gives $A = \begin{pmatrix} 0 & Z_1 S^{-1} \\ -I_k & 0 \\ 0 & \end{pmatrix}$. It is easy to see that we can construct S in such a way that $-Z_1 S^{-1}$ can be completed yo a symmetric matrix P, i.e. $\begin{pmatrix} Z_1 S^{-1} \\ V \end{pmatrix} = P$ with V a $(n-k)\times n$ matrix. Hence we have in fact obtained equations (4.46) with $Q = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$. It follows that in both cases the transfer matrix of this voltage controlled LCT-network satisfies

$G(s) = G^T(s)$ and $G(s) = -G(-s)$. Moreover transfer matrices of electrical networks (with only passive elements) are necessarily positive real. □

Remark 1: One can in fact prove that G(s) positive real, together with $G(s) = -G(-s)$ implies $G(s) = G^T(s)$ (WILLEMS (1972)). Furthermore we note that the fact that a transfer matrix satisfying conditions i), ii) and iii) corresponds to being the transfer matrix of an LCT-network is already known in the literature (see WILLEMS (1972, pp. 384,385) for a summary of this and similar results).

Remark 2: Instead of realizing G(s) satisfying conditions i), ii) and iii) by electrical elements as above we can also use *mechanical* elements. Inductors are replaced by *masses*, and capacitors become *springs*. The mechanical equivalent of a transformer should have the equations

$$F_1 = \alpha F_2$$
$$v_2 = -\alpha v_1$$

with F_1 and F_2 *forces* and v_1 and v_2 *velocities*. One can think of levers (for small deviations) or hydraulic devices.

Notes and References for Chapter 4.

The treatment of symmetries and conservation laws is an extended version of VAN DER SCHAFT (1981, 1983 b). The treatment of time-reversibility is mainly based on VAN DER SCHAFT (1983 b, 1982 a). Definitions 4.24 and 4.25 can be found in WILLEMS (1978, 1979). In WEINSTEIN (1973), MEYER (1981), see also ABRAHAM & MARSDEN (1978), a definition similar to Definition 4.29 is given for a time-reversing symmetry of a Hamiltonian *vectorfield*. The terminology "simple Hamiltonian systems" in Definition 4.33 is inspired by the definition of "simple mechanical systems" in SMALE (1970), see also ABRAHAM & MARSDEN (1978).

CHAPTER 5

GRADIENT SYSTEMS

5.1 Introduction

The precise notion of a "gradient system" is much less clear than the notion of a Hamiltonian system. One reason is that although there are many examples of systems which have a "gradient-like" behavior, convincing examples of gradient systems are not as abundant as examples of Hamiltonian systems.

Usually two aspects are thought to be characteristic for gradient systems. The first is that, while a Hamiltonian system is a prototype of a *conservative* oscillatory system, a gradient system should be a prototype of a non-oscillatory *dissipative* system. This is normally formalized by requiring the existence of a *potential function* which is decreasing along the trajectories of the system if no external energy is supplied. The second characteristic aspect of a gradient system is its symmetrical or reciprocal structure. Mathematically this is formalized by an inner product structure (or more generally a *Riemannian metric*) on the state space. We remark that for Hamiltonian systems there is also a symmetric structure present, which is in this case formalized as a *symplectic* structure.

In the Hamiltonian case the existence of the symplectic structure and the conservation of energy are intimately related. For gradient systems the connection between reciprocity and dissipativeness is much looser. For instance in electrical circuit theory one distinguishes between systems which are only dissipative and systems which are only reciprocal.

Usually some sort of stability is included in the definition of dissipativeness. Together with reciprocity this yields that the system converges, without "oscillatory" behavior, to the minima of its potential function. Our approach will be not to include the stability properties in the definition of gradient systems. Consequently we shall not demand a priori (as is usually done) that the Riemannian metric is *positive definite*, nor do we impose any structure on the potential function (note that in the Hamiltonian case we also considered *arbitrary* Hamilton functions). However we should remark that probably nearly *every vectorfield* can be written as a gradient vectorfield with respect to an *indefinite* metric (TAKENS, 1983). Therefore, if we do not require that the metric is positive definite then not much

can be said in general about the qualitative properties of a gradient vector-field. (However, even if the metric is indefinite, the definition of a *gradient system* (Definition 5.1) certainly implies a strong relation between the external variables of the system).

Concluding, we do not need positive definiteness of the metric on the level of definition, but in order to say something about the (internal) dynamical behavior of the system we need to know more about the specific form of the metric and the potential function.

A very nice example of a gradient system is provided by a linear or nonlinear electrical network consisting of capacitors, inductors and resistors (see BRAYTON & MOSER (1964), BRAYTON (1971)). Recall the notation of Section 3.1.2. An n_1-port of nonlinear capacitors C is given by (see 3.33)

$$(5.1) \qquad v_C = \frac{\partial T}{\partial q_C}(q_C)$$

with $v_C = (v_C^1,\ldots,v_C^{n_1})$ and $q_C = (q_C^1,\ldots,q_C^{n_1})$ the voltages, respectively electric charges. $T(q_C)$ is the electric energy. An n_2-port of nonlinear inductors L is given by (see 3.30)

$$(5.2) \qquad i_L = \frac{\partial S}{\partial \phi_L}(\phi_L)$$

with $i_L = (i_L^1,\ldots,i_L^{n_2})$ and $\phi_L = (\phi_L^1,\ldots,\phi_L^{n_2})$ the currents, respectively magnetic fluxes. $S(\phi_L)$ is the magnetic energy. In the same way an n-port of resistors R is given by

$$(5.3) \qquad v_R = \frac{\partial R}{\partial i_R}(i_R)$$

with $v_R = (v_R^1,\ldots,v_R^n)$ and $i_R = (i_R^1,\ldots,i_R^n)$ the voltages, respectively currents. Hybrid representations of a resistive n-port are given by

$$(5.4) \qquad v_R^a = \frac{\partial R}{\partial i_R^a}(i_R^a, v_R^b)$$

$$i_R^b = -\frac{\partial R}{\partial v_R^b}(i_R^a, v_R^b)$$

with $v_R = (v_R^a, v_R^b)$ and $i_R = (i_R^a, i_R^b)$.

Consider now the following interconnection:

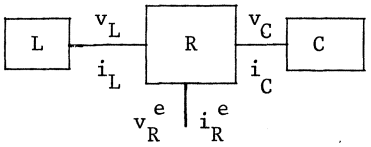
Let $v_R = (v_R^a, v_R^b, v_R^e)$ and $i_R = (i_R^a, i_R^b, i_R^e)$

and let

$$(5.5)\qquad \begin{aligned} v_R^a &= \frac{d}{dt}\phi_L = v_L \quad , \quad i_R^a = -i_L \\ i_R^b &= -\frac{d}{dt} q_C = -i_C \; , \; v_R^b = v_C \end{aligned}$$

i.e. we have interconnected the L, C and R ports:

(5.6)



where v_R^e and i_R^e are the external voltages and currents.
Now we assume that the Legendre transform (see Section 3.1.1) $\bar{T}$ of T in (5.1) exists. Then (5.1) yields

$$(5.7)\qquad q_C = \frac{\partial \bar{T}}{\partial v_C}(i_L)$$

In the same way we assume that the Legrende transform $\bar{S}$ of S in (5.2) exists. From (5.2) we obtain

$$(5.8)\qquad \phi_L = \frac{\partial \bar{S}}{\partial i_L}(i_L)$$

Differentiation of (5.7) and (5.8) gives

$$(5.9)\qquad \begin{aligned} \frac{dq_C}{dt} &= \frac{\partial^2 \bar{T}}{\partial v_C^2}(v_C)\frac{dv_C}{dt} \\ \frac{d\phi_L}{dt} &= \frac{\partial^2 \bar{S}}{\partial i_L^2}(i_L)\frac{di_L}{dt} \end{aligned}$$

Assume now that R can be parametrized by (v_R^b, i_R^a, v_R^e). By using the interconnection (5.5), together with (5.4) and (5.9), we obtain

$$(5.10\text{ a})\qquad \begin{aligned} -\frac{\partial^2 \bar{T}}{\partial v_C^2}\frac{dv_C}{dt} &= -\frac{\partial R}{\partial v_C}(v_C, i_L, v_R^e) \\ \frac{\partial^2 \bar{S}}{\partial i_L^2}\frac{di_L}{dt} &= -\frac{\partial R}{\partial i_L}(v_C, i_L, v_R^e) \end{aligned}$$

$$(5.10\text{ b})\qquad i_R^e = -\frac{\partial R}{\partial v_R^e}(v_C, i_L, v_R^e)$$

We can interpret the matrix

$$\begin{pmatrix} -\frac{\partial^2 \bar{T}}{\partial v_C^2} & 0 \\ 0 & \frac{\partial^2 \bar{S}}{\partial i_L^2} \end{pmatrix}$$

as a Riemannian metric on the state space (i_L, v_C). Then (5.10a) are for every constant v_R^e the equations of a gradient vectorfield with respect to this metric and with potential function $R(v_C, i_L, v_R^e)$. Normally $\frac{\partial^2 \bar{T}}{\partial v_C^2} > 0$ and $\frac{\partial^2 \bar{S}}{\partial i_L^2} > 0$, so the metric is *indefinite*. If there are no inductors present (an RC network), or if there are no capacitors present (an RL network), then the Riemannian metric can be taken to be positive definite.

Notice that the space of external variables (i_R^e, v_R^e) is even-dimensional and can be endowed with a *symplectic structure* such that the equations (5.10b) can be interpreted in a coordinate free way (as Lagrangian submanifolds of the space of external variables).

We call (5.10) a *gradient system with external variables*. A general definition of a gradient system including this class of examples will be given in Definition 5.1.

A *mechanical* example of a gradient system is provided by the following interpretation of Newton's second law $m\ddot{q} = F$ (compare the treatment in Section 3.1). Define $v = \dot{q}$ and write $m\ddot{q} = F$ as

$$m\dot{v} = F \tag{5.11}$$

If we define the output y equal to v, we can regard (5.11) as a gradient system. The state space is the space of velocities v and the Riemannian metric is m. Furthermore consider the memoryless (static) system

$$F' = \frac{\partial R}{\partial v'}(v') \tag{5.12}$$

which describes the force F' due to a friction depending on the velocity v'. We call (5.12) a static gradient system. (Notice that (5.12) is also a static Hamiltonian system; indeed the definition of a static gradient and that of a static Hamiltonian system coincide.) Interconnect (5.11) and (5.12) by setting

(5.13) $\quad v = v' \ , \ F = - F'$

(i.e. a Hamiltonian interconnection). We obtain

(5.14) $\quad m\dot{v} + \frac{\partial R}{\partial v}(v) = 0$

which is an *autonomous* gradient system with potential function R.

5.2 Gradient systems, affine gradient systems

In this section we give the general definition of a (nonlinear) gradient system with external variables. The definition can be given in a way that is very analogous to the definition of a general Hamiltonian system (Definition 3.6).

Let $\Sigma(X,W,B,f)$ be a smooth nonlinear system (Definition 2.20), i.e. a system that in local coordinates (x,u) for B, w for W has the form

(5.15) $\quad \dot{x} = g(x,u)$
$\quad\quad\ w = h(x,u)$

In order to define a nonlinear gradient system we require that
i) X is a Riemannian manifold with Riemannian metric $< \ , \ >$. The Riemannian metric is nondegenerate but not necessarily positive definite. To emphasize that X is a Riemannian manifold we denote the state space by $(Q,<,>)$. From now on we assume that dim Q = n.
ii) W, the space of external variables, is a symplectic manifold with symplectic form ω^e (dim W=2m).
iii) f : B ⟶ TM × W is an *imbedding*, and hence f(B) is a submanifold of TM × W.

Since $< \ , \ >$ is a nondegenerate metric on Q, it induces a bundle isomorphism α: TQ ⟶ T^*Q by setting $\alpha(X) = <X,\cdot>$, $X \in TQ$. Since T^*Q is a cotangent bundle it has a natural symplectic form ω. Then $\alpha^*\omega$ is a symplectic form on TQ (compare the definition of $\dot{\omega}$ in Section 3.1.3). Furthermore we can define the symplectic form $\Omega := \pi_1^* \alpha^* \omega - \pi_2^* \omega^e$ on TQ × W (π_1 and π_2 denote the projections of TQ × W on TQ, respectively W).

DEFINITION 5.1 $\Sigma(Q,W,B,f)$ with $(Q,<,>)$ a Riemannian manifold and (W,ω^e) a symplectic manifold is called a (full) *gradient system* if f(B) is a Lagrangian submanifold of $(TQ \times W, \Omega)$.

In local coordinates Definition 5.1 yields

PROPOSITION 5.2 Let $\Sigma(Q,W,B,f)$ be a (full) gradient system. Let $q = (q_1,\ldots,q_n)$ be coordinates for Q and let $(v_1,\ldots,v_m,z_1,\ldots,z_m)$ be canonical coordinates for W. Then f(B) is locally parametrized by q and m coordinate functions $(v_i)_{i\in I_1}$ and $(z_i)_{i\in I_2}$ with $I_1 \cup I_2 = \{1,\ldots,m\}$ and $I_1 \cap I_2 = \emptyset$. Denote these m coordinate functions by $u = (u_1,\ldots,u_m)$, and denote the remaining coordinate functions for W by $(y_1,\ldots,y_m)$ (in such a way that $\omega^e = \sum_{j=1}^{m} c_j du_j \wedge dy_j$, with $c_j = \pm 1$). Let $\langle\ ,\ \rangle$ in the local coordinates q be given by the matrix $(g_{ij}(q))$, i,j = 1,..,n , with $g_{ij} = g_{ji}$. Then locally there exists a function $V(q_1,\ldots,q_n,u_1,\ldots,u_m)$ such that f(B) is given by

$$
\begin{aligned}
\sum_{j=1}^{n} g_{ij}(q)\dot{q}_j &= -\frac{\partial V}{\partial q_i}(q,u) && i = 1,\ldots,n \\
y_j &= -c_j \frac{\partial V}{\partial u_j}(q,u) && j = 1,\ldots,m
\end{aligned}
\tag{5.16}
$$

PROOF: Let $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ be natural coordinates for T^*Q. Then $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$, and $\alpha^*\omega = \sum_{i=1}^{n} d\left(\sum_{j=1}^{n} g_{ij}(q)\dot{q}_j\right) \wedge dq_i$. For the rest of the proof we refer to the proof of Proposition 3.7. □

Analogous to Proposition 3.8 we can show that (5.16) is in fact a local input-output representation with feedthrough term of the gradient system. We see that the equations of an RLC network (5.10) are an example of (5.16).

We can define degenerate gradient systems in a manner that is totally similar to the definition of a degenerate Hamiltonian system (Section 3.2, Definition 3.10). Furthermore we can show that a *Hamiltonian interconnection* (Definition 3.15) of gradient systems results, under the same regularity conditions as in Theorem 3.18, into a degenerate gradient system. As we already remarked, the definition of a memoryless gradient system coincides with the definition of a memoryless (or *static*) Hamiltonian system (Definition 3.3).

We now immediately proceed to the definition of an affine gradient system. We require that the space of external variables W is equal to T^*Y, where Y is the output manifold. Analogous to Proposition 3.22 we can prove

that a gradient input-output system $\Sigma(M,T^*Y,Y,g,h)$ is then automatically an affine input-output system. We arrive at the following definition of an affine gradient system (compare Definition 3.23).

DEFINITION 5.3 Let $(Q,<,>)$ be a Riemannian manifold. Let Y be the output manifold. Define $\Omega := \pi_1^*\alpha^*\omega - \pi_2^*\omega^e$, with ω^e the natural symplectic form on T^*Y, π_1 and π_2 the projections of $TQ \times T^*Y$ onto TQ and T^*Y, and α and ω as above. An *affine gradient system* is given by a submanifold $L \subset TQ \times T^*Y$ such that

(i) L can be parametrized by the coordinates of Q and the fibers of T^*Y.
(ii) L is a Lagrangian submanifold of $(TQ \times T^*Y, \Omega)$.
(iii) The value of the Y-coordinates of a point on L is a function of only the Q-coordinates of this point.

We denote the system by $\Sigma(Q,T^*Y,L)$.

In order to give a local expression of an affine gradient system we first define *gradient vectorfields*. Let $(Q,<,>)$ be a Riemannian manifold. Let $V : Q \longrightarrow \mathbb{R}$ be a smooth function. The vectorfield Z_V on Q, defined by $<Z_V,-> = -dV$ is the (global) gradient vectorfield with potential function V. If $d<Z,-> = 0$ for a vectorfield Z on Q, then by Poincaré's lemma there exists (locally) a function $V : Q \longrightarrow \mathbb{R}$ such that $Z = Z_V$. Z is called a (local) gradient vectorfield. It can be easily seen that Z is a local gradient vectorfield if and only if graph $Z \subset TQ$ is a Lagrangian submanifold of $(TQ,\alpha^*\omega)$, where as above $\alpha: TQ \longrightarrow T^*Q$ is given by $\alpha(X) = <X,->$, and ω is the natural symplectic form on T^*Q. (Compare Definition 3.4). Usually we shall omit the prefix "local" or "global".

PROPOSITION 5.4 (Compare Proposition 3.24). Let $\Sigma(Q,T^*Y,L)$ be an affine Hamiltonian system. Then in local coordinates the system is given by

$$\begin{aligned} \dot{q} &= Z_V(q) - \sum_{i=1}^{m} u_i Z_{C_i}(q) \\ y_i &= C_i(q) \qquad\qquad i = 1,\ldots,m \end{aligned} \tag{5.17}$$

with q local coordinates for Q, $y = (y_1,\ldots,y_m)$ local coordinates for Y and $u = (u_1,\ldots,u_m)$ the corresponding natural coordinates for the fibers of T^*Y. We call V the *internal potential* and C_i the observation (or output) functions.

PROOF: (Compare Proposition 3.24). Because of i) and iii) the generating function L with respect to the symplectic form Ω on $TQ \times T^*Y$ has the form $-V(q) + \sum_{i=1}^{m} u_i C_i(q)$. Then (5.17) results. □

Remark By choosing natural coordinates for T^*Y, the internal potential function V is uniquely determined up to a constant (compare the definition of internal energy in Section 3.7).

Gradient feedback for an affine gradient system is defined analogous to Hamiltonian feedback (Definition (3.25), i.e. locally there should exist a function $P : Y \longrightarrow \mathbb{R}$ such that $V = \tilde{\alpha}(y,u) = \frac{\partial P}{\partial y}(y) + u$. We can prove, just as in Theorem 3.26, that the only feedback which transforms any affine gradient feedback into another affine gradient system is gradient feedback. The transformed system is of the form

$$\begin{aligned} \dot{q} &= Z_{V+P}(q) - \sum_{i=1}^{m} v_i Z_{C_i}(q) \\ y_i &= C_i(q) \qquad\qquad i = 1,\ldots,m \end{aligned} \tag{5.18}$$

where $(y_1,\ldots,y_m,v_1,\ldots,v_m)$ are now *symplectic* (not necessarily natural) coordinates for T^*Y.

5.2.1 Controllability, observability and equivalence of affine gradient systems

Most of the results in this section will be of a *negative* nature. We show that the nice results on controllability and observability for Hamiltonian systems, roughly speaking controllability implies observability and vice versa (Theorems 3.19 and 3.31), do not hold for affine gradient systems. Also the result that "minimal" Hamiltonian systems which are equivalent are necessarily "symplectomorphic" (Theorem 3.21 and Proposition 3.34) does not have an analogue for affine gradient systems. A fortiori similar results do not hold for general nonlinear gradient systems (Definition 5.1).

First we give a neat characterization of *observability* for affine systems, very similar to the Hamiltonian case. Recall the construction of G and the observability codistribution 0 (Definition 2.33 and Construction 2.51). We define for a local representation (5.17) $G_0 := (C_1,\ldots,C_m)$ ($(C_1,\ldots,C_m)$ denotes the linear subspace over $\mathbb{R}$ of $C(Q)$), and

$G_k := f_\Gamma G_{k-1} + G_{k-1}$, $k \geq 1$ (with $\Gamma = Z_V + (Z_{C_1},\ldots,Z_{C_m})$, an affine subspace over $\mathbb{R}$ of $V(Q)$). Then $G := U_{k\geq 0} G_k$ satisfies $O(x) = \text{span}\{dg(x) \mid g \in G\}$. Recall the definition of the Poisson bracket. Let K and L be functions on (M,ω), then the Poisson bracket is given by $\{K,L\} = \omega(X_K, X_L)$ (see (3.43)). Analogously, we define the *Beltrami* bracket of two smooth functions K and L on $(Q,<\ ,\ >)$ by setting (HERMANN(1968))

$$(5.19) \qquad [\![K,L]\!] := \langle Z_K, Z_L \rangle$$

It is clear that the Beltrami bracket $[\![K,L]\!]$ is again a smooth function on Q. Notice that although the Beltrami bracket is defined in a very similar way as the Poisson bracket, their properties are quite different. While the Poisson bracket is anti-symmetric, the Beltrami bracket is symmetric. Moreover, contrary to the Poisson bracket, the Beltrami bracket does not satisfy the Jacobi-identity.

Analogously to Proposition 3.30 we obtain

PROPOSITION 5.5 Define $R := V + (C_1,\ldots,C_m)$ (an affine subspace of $C(Q)$. Then:

$$G_k = [\![R, G_{k-1}]\!] + G_{k-1}, \quad k \geq 1.$$

Hence $G = U_{k\geq 0} G_k$ is the smallest linear subspace (over $\mathbb{R}$) of $C(Q)$, which contains $C_1,\ldots,C_m$ and is invariant under taking Beltrami brackets with respect to V and C_i, $i = 1,\ldots,m$.

PROOF Elements of G_k are linear combinations of functions of the form

$$(5.20) \qquad f_{f_1} f_{f_2} \ldots\ldots f_{f_j} C_i, \quad j \leq k, \ i = 1,\ldots,m$$

with $f_r = A$ or $f_r = B_\ell$, $r = 1,\ldots,j$, $\ell = 1,\ldots,m$. The Beltrami bracket $[\![K,L]\!]$ satisfies $[\![K,L]\!] = \langle Z_K, Z_L \rangle = -dK(Z_L) = -Z_L(K)$, for $K,L \in C(Q)$. Therefore the expressions (5.20) are equal to

$$(5.21) \qquad \pm[\![h_1,[\![h_2,\ldots[\![h_j,C_i]\!],\ldots,]\!], j\leq k, \ i=1,\ldots,m \text{ with } h_r = V \text{ or } h_r = C_\ell, \ r = 1,\ldots,j, \ \ell = 1,\ldots,m. \qquad \square$$

Remember that in the Hamiltonian case we have proven that $G = U_{k\geq 0} G_k$ is a Poisson algebra by using the Jacobi-identity for the Poisson bracket. Since the Beltrami bracket does not satisfy the Jacobi-identity we cannot use the same argument to conclude that G is an algebra of functions on Q with respect to the Beltrami bracket. In fact this remains an open problem (we conjecture that G is indeed an algebra).

We know that the controllability properties of an affine gradient system are characterized by the *Lie algebra* F of vectorfields on Q (see Construction 2.49). Hence if G is indeed an algebra it is clear that F and G cannot be isomorphic, as was the case for Hamiltonian systems (Theorem 3.31). This raises some doubts about the existence of a relationship between observability and controllability for gradient systems. Indeed, we can easily construct *counterexamples*, where the (affine) gradient system is locally weakly observable but not strongly accessible (see VAN DER SCHAFT (1982 c, p.353)).

We now direct attention to the issue of *equivalence* of gradient systems. It would be desirable to have conditions which guarantee that equivalent gradient systems are *isomorphic*, i.e. if $(Q_i,<,>_i)$, $i = 1,2$, are the state spaces of two gradient systems and $\phi : Q_1 \to Q_2$ is an equivalence mapping (see Proposition 2.29), then $\phi^*<,>_2 = <,>_1$. This would imply that the *curvature* of the state space of a gradient system, defined by the Riemannian metric, is an invariant. In Section 5.1 we saw that a nonlinear electrical RLC network can be modelled as a gradient system, and that the Riemannian metric can be constructed from the constitutive relations of the capacitors and inductors (see (5.10)). Therefore an isometry between two RLC networks yields a structural similarity of the networks. In fact it has been conjectured by VARAIYA (1971) that some kind of controllability and/or observability implies that an equivalence mapping between two RLC networks is automatically an isometry. However, only under very restrictive assumptions on the system it has been possible to prove this conjecture (for instance it can be proved for *linear* systems, see Section 5.2.2). In BASTO GONCALVES (1981) a counterexample is given for the conjecture in the general affine case. In this reference it is shown that if the equivalence mapping preserves the *connections* defined by the Riemannian metrics, then it is indeed an isometry. This seems to be the strongest result that is obtainable in the general case. In Section 5.3 we shall state another conjecture for obtaining an isometry between gradient systems, by associating to a gradient system a (simple) Hamiltonian system.

5.2.2 Linear gradient systems

In this section we briefly summarize some results about linear gradient systems. Let $\Sigma(A,B,C,D)$ be a linear system (Section 2.1), with state space X (dim $X=n$). A Riemannian metric on X is given by a nonsingular sym-

metric n×n matrix G. It can be easily seen that $\alpha^*\omega$ as in Definition 5.1 is the symplectic form on $T^*X = \mathbb{R}^{2n}$ given by the matrix $\begin{pmatrix} 0 & -G \\ G & 0 \end{pmatrix}$. We obtain (compare Theorem 3.40)

THEOREM 5.6 Let $\Sigma(A,B,C,D)$ be a linear system with state space X and space of external variables W. Assume that $\begin{bmatrix} B \\ D \end{bmatrix}$ is injective. Let G be a nondegenerate symmetric form on X (dim X = n), and let J^e be a symplectic form on W (dim W=2m). Then $\Sigma(A,B,C,D)$ is a full gradient system if and only if

$$\begin{aligned} &GA - A^TG + C^TJ^eC = 0 \\ &GB \qquad + C^TJ^eD = 0 \\ &D^TJ^eD \qquad\quad = 0, \quad \text{and rank } D = m \end{aligned} \tag{5.22}$$

Moreover if $\Sigma(A,B,C,D)$ is a full gradient system, then there exists a canonical basis w = (y,u) for W and a feedback transformation $A \mapsto A + BF$, $C \mapsto C + DF$, such that the transformed system $\Sigma(A',B',C',D')$ in this basis for W satisfies

$$GA' = A'^TG, \quad GB' = (\bar{C}')^T \tag{5.23}$$

with $C' = \begin{pmatrix} \bar{C}' \\ 0 \end{pmatrix}$ and $D' = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$. We call $\dot{x} = A'x + B'u$, $y = \bar{C}'x$, with (A',B',C') as in (5.23) a linear *input-output gradient* system. Hence a full gradient system is feedback equivalent (see Section 2.1.1) to an input-output gradient system.

PROOF By Definition 5.1 the linear subspace

$V = \{\begin{pmatrix} x \\ Ax+Bu \\ Cx+Du \end{pmatrix} \quad x\in X=\mathbb{R}^n, \ u\in U\}$ has to be a Lagrangian subspace of $\mathbb{R}^{2n}\times\mathbb{R}^{2m}$ endowed with the symplectic form $\begin{pmatrix} 0 & -G & 0 \\ G & 0 & 0 \\ 0 & 0 & -J^e \end{pmatrix}$. Hence $(y^T, y^TA^T+v^TB^T,$ $y^TC^T+v^TD^T)$ $\begin{pmatrix} 0 & -G & 0 \\ G & 0 & 0 \\ 0 & 0 & -J^e \end{pmatrix}$ $\begin{pmatrix} x \\ Ax+Bu \\ Cx+Du \end{pmatrix}$ $= 0$ for every $x,y \in \mathbb{R}^n$, $u,v \in U$.

This yields equations (5.22). Notice that $GB + C^TJ^eD = 0$ implies that Ker D ⊂ Ker B. Since $\begin{bmatrix} B \\ D \end{bmatrix}$ is injective, this yields that D is injective and that rank D = dim U. Because V is Lagrangian, and therefore has dimension 2n+m, it follows that rank D = m and that D is a Lagrangian subspace of

(W,J^e). Hence we can choose canonical coordinates $w = (y,u)$ for W such that $D = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$. Furthermore we can apply a feedback transformation $A \to A+BF$, $C \to C+DF$, such that C is transformed into $C' = \begin{pmatrix} \bar{C}' \\ 0 \end{pmatrix}$ in this basis for W. Then

$$GB' + (\bar{C}'0)^T \begin{pmatrix} 0 & -I_m \\ I_m & 0^m \end{pmatrix} \begin{pmatrix} 0 \\ I_m \end{pmatrix} = 0$$

yields $GB' = (\bar{C}')^T$. Moreover $C'^T J^e C' = 0$. Hence equations (5.23) result. □

Remark: Instead of *choosing* a canonical basis for W such that $D = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$, we can start from a *fixed* canonical basis for W. This results in formula's which are completely similar to the equations (3.92).

Analogous to the Hamiltonian case (Section 3.5.1) we can develop a realization theory for linear gradient systems. Let (W,J^e) be a linear symplectic space. Then J^e induces a symplectic form $J_{\mathbb{C}}^e$ on $W_{\mathbb{C}}$ by setting $J^e(v,w) = v^T J^e w$, for $v,w \in W_{\mathbb{C}}$ (compare (3.39)). We give (compare Definition 3.43):

DEFINITION 5.7 Let (W,J^e) be a linear symplectic space, with dim W = 2m, and let $P \in \mathbb{R}^{p\times 2m}[s]$. Then $\Sigma_e(P)$ is an *external gradient* system if for every $s \in \mathbb{C}$ Ker P(s) is a Lagrangian subspace of $(W_{\mathbb{C}},J_{\mathbb{C}}^e)$.

THEOREM 5.8 Let $\Sigma_e(P)$ be an external gradient system. Let $\Sigma(A,B,C,D)$ be a minimal realization of $\Sigma_e(P)$ with state space X. Then there exists a unique nondegenerate symmetric form G on X such that $\Sigma(A,B,C,D)$ is a full gradient system. Conversely, if $\Sigma(A,B,C,D)$ is a full gradient system (not necessarily minimal) then there exists a P such that $\Sigma_e(P) = \Sigma_e(A,B,C,D)$ and $\Sigma_e(P)$ is an external gradient system.

PROOF Let $\Sigma_e(P)$ be an external gradient system. Then $V(\infty) := \lim_{s\to\infty} \text{Ker } P(s)$ is a Lagrangian subspace of $(W_{\mathbb{C}},J_{\mathbb{C}}^e)$. Since $V(\infty)$ is a real subspace of $W_{\mathbb{C}}$, it is a Lagrangian subspace of (W,J^e). Then we can take a Lagrangian subspace Y of (W,J^e) which is complementary to $U := V(\infty)$. Furthermore we can choose a canonical basis $w = (y,u)$ for W, such that U is spanned by $(0,u)$ and Y is spanned by $(y,0)$. By Lemma 2.16 we can find polynomial matrices D(s) and N(s) such that in this basis Ker $P(s) = \text{Im } \begin{pmatrix} N(s) \\ D(s) \end{pmatrix}$, $\forall s \in \mathbb{C}$, and $G(s) := N(s)D^{-1}(s)$ is a strictly proper transfer matrix. Since $\text{Im } \begin{pmatrix} N(s) \\ D(s) \end{pmatrix}$ is Lagrangian we obtain

$$(N^T(s)D^T(s))\begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}\begin{pmatrix} N(s) \\ D(s) \end{pmatrix} = 0,\ \forall s \in \mathbb{C},\ \text{or } -N^T(s)D(s) + D^T(s)N(s) = 0.$$

Equivalently $G(s) = G^T(s)$. Let now $\Sigma(A,B,\bar{C})$ be a minimal input-output realization of $G(s)$, i.e. $D = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$ and $C = \begin{pmatrix} C \\ 0 \end{pmatrix}$, with state space X. Since $G(s) = G^T(s)$ also $\Sigma(A^T, \bar{C}^T, B^T)$ is a minimal realization. By the state space isomorphism theorem there exists a unique nonsingular map $G : X \to X$ such that $A^T = GAG^{-1}$, $\bar{C}^T = GB$, $B^T = \bar{C}G^{-1}$. Moreover $D = D^T$. Obviously, also G^T satisfies these equations. Hence $G = G^T$. Since by Theorem 5.6 every gradient system is equivalent to an input-output gradient system, we have obtained the desired conclusion. Finally it is easy to see that if $\Sigma(A,B,C,D)$ is a full gradient system, then $[C\ D](\text{Ker } [sI-A\vdots -B])$ is for every $s \in \mathbb{C}$ a Lagrangian subspace of $(W_{\mathbb{C}}, J_{\mathbb{C}}^e)$. Hence there exists a P such that $\Sigma_e(A,B,C,D) = \Sigma_e(P)$ and $\Sigma_e(P)$ is an external gradient system. □

Remark: Let $P \in \mathbb{R}^{p\times 2m}[s]$ such that $\Sigma_e(P)$ is an external gradient system. Then it follows that rank $P(s) = m$ for every $s \in \mathbb{C}$. Hence we can find a unimodular matrix $U(s) \in \mathbb{R}^{p\times p}[s]$ such that $U(s)P(s) = \begin{pmatrix} P'(s) \\ 0 \end{pmatrix}$ (see Section 2.1.3) and $P'(s) \in \mathbb{R}^{m\times 2m}[s]$ surjective for every s. Furthermore we notice that if $\Sigma(A,B,\bar{C})$ is an input-output gradient system then the transfer matrix $G(s) = \bar{C}(Is-A)^{-1}B$ satisfies $G(s) = G^T(s)$. Conversely if $G(s) = G^T(s)$ then there exists a minimal input-output gradient system that realizes $G(s)$. For more information about transfer matrices $G(s)$ satisfying $G(s) = G^T(s)$ we refer to WILLEMS (1972).

Remark 2: It easily follows from (5.23) that the set of controllability indices of a linear gradient system is equal to the set of observability indices. Therefore if $\Sigma_e(P)$ is an external gradient system, then the Chern numbers of the bundles $E(P(s))$ and $(E(P(s)))^{\perp}$ (see Section 2.1.3) are equal.

In Section 5.2.1 we remarked that for nonlinear gradient systems observability and controllability are not equivalent, and that equivalent nonlinear gradient systems need not be isomorphic. For *linear* gradient systems however these statements *do* hold (compare Remark 2 above):

THEOREM 5.9 Let $\Sigma(A,B,\bar{C})$ be an input-output gradient system.
Then: Σ is observable $\iff$ Σ is controllable.
Let $\Sigma(A_i,B_i,C_i,D_i)$ with state space X_i and inner product G_i, i = 1,2, be

two minimal realizations of an external gradient system. Let S: $X_1 \longrightarrow X_2$ be the equivalence mapping. Then $S^T G_2 S = G_1$.

PROOF: Let $V \subset X$ and let $V^\perp := \{x \in X \mid x^T G v = 0, \ \forall v \in V\}$. Then, since $A^T G = GA$, $AV \subset V \iff V^\perp \subset (AV)^\perp = A^{-1}(V^\perp) \iff AV^\perp \subset V^\perp$, and, since $GB = \bar{C}^T$, $(\mathrm{Im}\, B)^\perp = \mathrm{Ker}\, \bar{C}$. Therefore : $\{\exists V \subset X,\ AV \subset V,\ V \supset \mathrm{Im}\, B\} \iff \{\exists V^\perp \subset X, AV^\perp \subset V^\perp, V \subset \mathrm{Ker}\, \bar{C}\}$, and hence controllability $\iff$ observability.
For the second part we remark that by Theorem 5.6 we may assume that $\Sigma(A_i,B_i,C_i,D_i)$ are input-output realizations, i.e. $C_i = \begin{pmatrix} C_i \\ 0 \end{pmatrix}$ and $D_i = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$. Then S: $X_1 \longrightarrow X_2$ satisfies : $A_2 = SA_1S^{-1}$, $B_2 = SB_1$, $\bar{C}_2 = \bar{C}_1 S^{-1}$ (compare Theorem 4.23). Therefore

i) $S^T G_2 B_2 = S^T \bar{C}_2^T = \bar{C}_1^T = G_1 B_1 = G_1 S^{-1} B_2$

ii) $S^T G_2 (A_2 B_2) = S^T A_2^T G_2 B_2 = A_1 S^T G_2 B_2 = A_1^T G_1 S^{-1} B_2 = G_1 A_1 S^{-1} B_2 = G_1 S^{-1} (A_2\ B_2)$

and hence by induction $S^T G_2 (A_2^r B_2) = G_1 S^{-1} (A_2^r B_2)$ for r = 0,1,2,.. . By controllability this yields $S^T G_2 = G_1 S^{-1}$, or $S^T G_2 S = G_1$. □

5.3 Relationships between gradient systems and Hamiltonian systems.

Although the dynamical behavior of a Hamiltonian system is rather different of that of a gradient system, in fact often even opposite, it is worth while to look for parallels between the gradient and Hamiltonian framework.

From a mathematical point of view it is clear that the definition of a gradient system is very similar to the definition of a Hamiltonian system. Also from a physical viewpoint the symmetry structures imposed by the Hamiltonian framework on the one hand and the gradient framework on the other hand are certainly not mutually exclusive. Indeed, one could hope that it might be possible to *combine* the notions of Hamiltonian and gradient systems. For instance a theory of *interconnecting* Hamiltonian and gradient systems could create a framework to treat dissipative mechanical systems. In fact in electrical network theory there have been some attempts in this direction (see for some references CROUCH (1981)).

In the sequel we shall only give a relation between (affine) gradient and Hamiltonian systems which seems interesting from a system theoretic point of view. We show how we can associate with every affine gradient system

a *simple* Hamiltonian system as considered in Definition 4.33 and vice versa.

Let $\Sigma(Q,T^*Y,L_G)$ be an affine gradient system with an output map $\tilde{C} : Q \longrightarrow Y$. Q is a Riemannian manifold with Riemannian metric $< , >$. In local coordinates the system is given by

$$\begin{aligned} \dot{q} &= Z_{\tilde{V}}(q) - \sum_{i=1}^{m} u_i Z_{\tilde{C}_i}(q) , \quad q \in Q \\ y_i &= \tilde{C}_i(q) \qquad\qquad i = 1,\ldots,m \end{aligned} \tag{5.24}$$

with $\tilde{C} = (\tilde{C}_1,\ldots,\tilde{C}_m)$ and $\tilde{V} : Q \longrightarrow \mathbb{R}$ the internal potential. Now define as in Definition 4.33 the *kinetic energy* $K: T^*Q \longrightarrow \mathbb{R}$ by $K(q,p) := \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} p_i p_j$, with (g^{ij}) the inverse matrix of (g_{ij}), the functions that represent the Riemannian matric in a basis $q = (q_1,\ldots,q_n)$. Of course $(q,p) = (q_1,\ldots,q_n,p_1,\ldots,p_n)$ are natural coordinates for T^*Q. Furthermore define $V := \tilde{V} \circ \pi$, $C := \tilde{C} \circ \pi$, $C_i := \tilde{C}_i \circ \pi$, with π the projection of T^*Q on Q. Define the *internal energy* $H := K + V$. Then

$$\begin{aligned} \dot{x} &= X_H(x) - \sum_{i=1}^{m} u_i X_{C_i}(x) \qquad x \in T^*Q \\ y_i &= C_i(x) \qquad\qquad i = 1,\ldots,m \end{aligned} \tag{5.25}$$

is a simple Hamiltonian system with state space T^*Q (see Definition 4.33). We denote (5.25) by $\Sigma(T^*Q,T^*Y,L_H)$. It is clear that conversely to every simple Hamiltonian system there corresponds an affine gradient system.

Let us first investigate what this connection means for *linear* gradient systems. Let $\Sigma(A,B,\bar{C})$ be a linear input-output gradient system with an inner product G on the state space X. Define $Q := -GA$, then $Q^T = Q$ and the system is also given by

$$\begin{aligned} G\dot{q} &= - Qq + GBu \\ y &= \bar{C}q \qquad , \text{ with } \bar{C}^T = GB \end{aligned} \tag{5.26}$$

The associated simple Hamiltonian system is (set $P := G^{-1}$)

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} &= \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{C}^T \end{pmatrix} u \\ y &= (\bar{C} \quad 0)q \end{aligned} \tag{5.27}$$

Let F_G be the transfer matrix of (5.26) and F_H the transfer matrix of (5.27). Then it follows by the matrix inversion lemma (e.g. GOODWIN & PAYNE (1977)) that $F_H(s) = \bar{C}(s^2 I + PQ)^{-1} P\bar{C}^T = \bar{C}(s^2 I + G^{-1}Q)^{-1} B = F_G(s^2)$.

On the other hand if $F_G(s) = F_G^T(s)$, and if we define $F_H(s) := F_G(s^2)$, then it follows that $F_H(-s) = F_H(s) = F_H^T(-s)$, and hence F_H is the transfer matrix of a simple (or time-reversible) Hamiltonian system. A simple example (with Q=0) is provided by Newton's second law written as $m\dot{v} = F$, with output $y = v$. Then $F_G(s) = \frac{1}{ms}$, and $F_H(s) = \frac{1}{ms^2}$ is the transfer function of $m\ddot{q} = F$, with $y = q$. Notice that if $Q = 0$, then $F_H(s) = F_G(s^2) = \frac{1}{s} F_G(s)$.

In Proposition 4.35 we proved that (5.27) is minimal if and only if $(PQ, P\bar{C}^T)$ is controllable. However $(PQ, P\bar{C}^T)$ is controllable if and only if (5.26) is minimal (use Theorem 5.9). For *nonlinear* gradient and simple Hamiltonian systems we obtain the following result:

THEOREM 5.10 Let $\Sigma(Q, T^*Y, L_G)$ be a gradient system such that $V = 0$. Then: The observability codistribution O_G of $\Sigma(Q, T^*Y, L_G)$ has everywhere dimension equal to dim Q $\iff$ The observability codistribution O_H of $\Sigma(T^*Q, T^*Y, L_H)$ has everywhere dimension equal to dim T^*Q.

PROOF Let $\Sigma(Q, T^*Y, L_G)$ and $\Sigma(T^*Q, T^*Y, L_H)$ have the local representations (5.24) and (5.25) with $V = 0$. The key observation is that for arbitrary functions N_1, N_2: $Q \longrightarrow \mathbb{R}$ the following identity holds

$$\{\{K, N_1 \circ \pi\}, N_2 \circ \pi\} = [\![N_1, N_2]\!] \circ \pi \tag{5.28}$$

(π: $T^*Q \longrightarrow Q$ the projection). Indeed

$$\{K, N_1 \circ \pi\} = \{\tfrac{1}{2} \sum_{i,j=1}^{n} g^{ij} p_i p_j,\ N_1 \circ \pi\} = \sum_{i,j=1}^{n} g^{ij} p_i \frac{\partial N_1}{\partial q_j}$$

and therefore

$$\{\{K, N_1 \circ \pi\}, N_2 \circ \pi\} = \sum_{i,j=1}^{n} g^{ij} \frac{\partial N_2}{\partial q_i} \frac{\partial N_1}{\partial q_j} = Z_{N_1}(N_2) \circ \pi = [\![N_1, N_2]\!] \circ \pi.$$

For notational brevity we shall omit π in the sequel. So a function $N: Q \longrightarrow \mathbb{R}$ will also be viewed as a function $N: T^*Q \longrightarrow \mathbb{R}$ that only depends on the q-variables. Now assume that dim O_G = dim Q. As we know (Proposition 5.5), $O_G(q) = \text{span}\{dg(q) \mid g \in \mathcal{G}_G\}$, where $\mathcal{G}_G$ is the smallest

linear subspace of C(Q) that contains $\tilde{C}_1,\ldots,\tilde{C}_m$ and is invariant under taking Beltrami brackets with $\tilde{C}_1,\ldots,\tilde{C}_m$ (remember that V=0). On the other hand $O_H(q,p)$ = span $\{dg(q,p)\,|\,g\in G_H\}$, where G_H is the smallest linear subspace of $C(T^*Q)$ that contains $C_1,\ldots,C_m$, and is invariant under taking Poisson brackets with $C_1,\ldots,C_m$ and K (Proposition 3.30).
Notice that since we omit π, $C_i = \tilde{C}_i$, i = 1,...,m. It is clear from (5.28) that

(5.29) $\qquad G_G \subset G_H$

Therefore, since dim O_G = dim Q, O_H has dimension equal to dim Q in the "dq-direction". Furthermore it is clear that $\{K,G_G\} \subset G_H$, and also that functions in $\{K,G_G\}$ are *linear* in the p-variables. Taking differentials of functions in $\{K,G_G\}$ then yields that O_H has dimension equal to dim Q in the "dp-direction". Hence dim O_H = dim T^*Q.
Conversely assume that dim O_H = dim T^*Q. Functions in G_H are of the form

(5.30) $\qquad \{N_1,\{N_2,\{\ldots\{N_k,C_i\},\ldots\}$

with N_j equal to C_i, i = 1,...,m, or equal to K. Denote the number of times K appears in (5.30) by α, and denote the number of times that one of the C_i, i = 1,...,m, appears in (5.30) by β. Now notice that by (5.28):

(i) if $\alpha > \beta$, then (5.30) is quadratic or of higher order in the p-variables (or zero).
(ii) if $\alpha < \beta-1$, then (5.30) is zero.
(iii) if $\alpha = \beta$, then (5.30) is linear in the p-variables (or zero).
(iv) if $\alpha = \beta-1$, then (5.30) is actually a function on Q (does not depend on the p-variables).

Let f be an expression (5.30) with $\alpha > \beta$. Then f can be neglected for the dimension of O_H since df is zero on the zero-section of T^*Q. Therefore only categories (iii) and (iv) count. Let now f be a function (5.30) in category (iii). Then df calculated in points on the zero-section is zero or is of the form h(q)dp. Hence category (iii) does not count for the construction of O_H in the dq-direction at points of the zero section. Therefore since dim O_H = dim T^*Q, span $\{dg(q)\,|\,g$ expression in category (iv)$\}$= T^*_qQ, for every $q \in Q$.
We shall now show that

(5.31) $\qquad G_G$ = {all functions in category (iv)}
It is clear that an element of G_G belongs to category (iv).

On the other hand, by repeated use of the Jacobi-identity one can see that the expression (5.30), with $\alpha = \beta - 1$, can be written as linear combinations of expressions of the form

(a) $\{\{F_1,K\},\{\{F_2,K\},\ldots,\{\{F_\ell,K\},C_i\}\ldots.\}$ or

(b) $\{F_1,\{K,\{F_2,\{K,\ldots\{F_\ell,\{K,C_i\}\}\ldots\}$

with F_i equal to one of the Cj's. It follows immediately from (5.28) that the expressions (a) and (b) can be written as Beltrami brackets of the functions C_i, $i = 1,\ldots,m$. □

It remains an open problem if Theorem 5.10 also holds for *general* affine gradient systems, i.e. if V is arbitrary.

An example for Theorem 5.10 is provided by the rigid body with external torques as treated in Section 3.5. The associated gradient system for the rigid body with *three* (independent) controls is given by

$$\begin{aligned} \dot{q} &= -u_1 Z_{\tilde{C}_1}(q) - u_2 Z_{\tilde{C}_2}(q) - u_3 Z_{\tilde{C}_3}(q) \quad , \; q \in SO(3) \\ y_i &= \tilde{C}_i(q) \quad , \; i = 1,2,3 \end{aligned} \tag{5.32}$$

where $\tilde{C} = (\tilde{C}_1,\tilde{C}_2,\tilde{C}_3) : SO(3) \to SO(3)$ is the identity mapping. Of course this system has an observability codistribution with dimension 3 = dim SO(3).

For the rigid body with *two* (independent) controls we obtain the associated gradient system

$$\begin{aligned} \dot{q} &= -u_1 Z_{\tilde{C}_1}(q) - u_2 Z_{\tilde{C}_2}(q) \\ y_1 &= \tilde{C}_1(q) \; , \; y_2 = \tilde{C}_2(q) \end{aligned} \tag{5.33}$$

where $\tilde{C} = (\tilde{C}_1,\tilde{C}_2) : SO(3) \longrightarrow S^2$ and (y_1,y_2) are local coordinates for S^2. In Section 3.5 we gave the necessary and sufficient conditions in order that the observability codistribution of the Hamiltonian system corresponding to the rigid body with two controls has dimension equal to 6 = dim $T^*SO(3)$. By Theorem 5.10 these conditions are also necessary and sufficient in order that the observability codistribution of (5.33) has dimension 3.

Finally we shall use the connection between gradient and simple Hamiltonian systems to state a conjecture about the equivalence between gradient systems (see Section 5.2.1). Let us say that we call equivalent affine gradient systems *isomorphic* if the equivalence mapping

ϕ: $(Q_1,<\ ,\ >_1) \longrightarrow (Q_2,<\ ,\ >_2)$ is an isometry, i.e. $\phi^*< \ ,\ >_2 = <\ ,\ >_1$. Analogously we say that two equivalent affine Hamiltonian systems are isomorphic if the equivalence mapping ϕ: $(M_1,\omega_1) \longrightarrow (M_2,\omega_2)$ is a symplectomorphism, i.e. $\phi^*\omega_2 = \omega_1$. Then we state:

CONJECTURE 5.11 Let $\Sigma_G i = \Sigma(Q_i,T^*Y,L_G^i)$, i = 1,2 be affine gradient systems, and let $\Sigma_H i = \Sigma(T^*Q_i,T^*Y,L_H^i)$, i = 1,2, be their associated Hamiltonian systems. Suppose that the observability codistributions of $\Sigma_G i$ and $\Sigma_H i$ have dimension equal to dim Q_i, respectively T^*Q_i, i = 1,2. Then: $\Sigma_G 1$ and $\Sigma_G 2$ are isomorphic $\Longleftrightarrow$ $\Sigma_H 1$ and $\Sigma_H 2$ are isomorphic.

Remark : It follows from Proposition 3.34 that $\Sigma_H 1$ and $\Sigma_H 2$ are isomorphic if and only if they are equivalent.

Actually one direction in Conjecture 5.11, namely ($\Rightarrow$), can easily be proven. Let $\Sigma_G 1$ and $\Sigma_G 2$ be isomorphic. Then the equivalence mapping ϕ: $Q_1 \longrightarrow Q_2$ satisfies

i) $\phi_* Z_{\tilde{V}}1 = Z_{\tilde{V}}2$

ii) $\phi_* Z_{\tilde{C}_i}1 = Z_{\tilde{C}_i}2$ $\qquad i = 1,\ldots,m$

iii) $\tilde{C}_i^1 = \phi^*\tilde{C}_i^2$ $\qquad i = 1,\ldots,m$

iv) $\phi^*< \ ,\ >_2 = <\ ,\ >_1$

where $\tilde{V}^1 - \sum_{i=1}^m u_i\tilde{C}_i^1$ and $\tilde{V}^2 - \sum_{i=1}^m u_i\tilde{C}_i^2$ are the (local) generating functions of $\Sigma_G 1$ and $\Sigma_G 2$. Then ψ: $= (\phi^*)^{-1}$: $T^*Q_1 \longrightarrow T^*Q_2$ will be an equivalence between $\Sigma_H 1$ and $\Sigma_H 2$. This can be seen as follows. From i) and iv) it follows that $\phi^*\tilde{V}^2 = \tilde{V}^1$. Define $V^i = \tilde{V}^i \circ \pi_i$ (π_i:$T^*Q_i \longrightarrow Q_i$ projections), i = 1,2. Then $\psi^*V^2 = V^1$. Furthermore iv) implies that $\psi^*K^2 = K^1$. Hence $\psi^*(K^2+V^2) = K^1 + V^1$ or $\psi^*H^2=H^1$. Moreover iii) yields $\psi^*C_j^2 = C_j^1$, if we define $C_j^i = \tilde{C}_j^i \circ \pi_i$, j = 1,...,m , i = 1,2. Therefore ψ is an equivalence. Finally it is clear that $\psi^*\omega^2 = \omega^1$, with ω^i the natural symplectic forms on T^*Q^i, i = 1,2.

Notes and References for Chapter 5

Definition 5.1 is taken from VAN DER SCHAFT (1982 d). The use of Beltrami brackets in the context of affine gradient systems is due to CROUCH (1981). The association between gradient systems and simple Hamiltonian systems was first made in BROCKETT (1977), see also CROUCH (1981), and worked out in the linear case in VAN DER SCHAFT (1982 a). The conjecture in BROCKETT (1977) that "controllability" of a gradient system might be equivalent to "controllability" of the associated Hamiltonian system seems not to be true in general. However there may be a strong connection between the "observability" of both systems. Conjecture 5.11 is inspired by work of BASTO GONCALVES (1981). For a broader discussion of gradient and related systems we refer to CROUCH (1981).

CHAPTER 6

OPTIMAL CONTROL AND HAMILTONIAN SYSTEMS

In this chapter we wish to indicate some connections between optimal control and the theory of Hamiltonian systems we treated so far. Especially we want to relate the so-called *Maximum Principle* of optimal control to the framework of Hamiltonian systems.

We shall briefly sketch the basic idea of the Maximum Principle. For thorough treatments we refer to the classic PONTRYAGIN, BOLTYANSKII, GAMKRELIDZE & MISCHENKO (1962), and the more recent book by FLEMING & RISHEL (1975). For simplicity we only treat the so-called free terminal point problem.

Consider a control system described by the state space equations

$$\dot{x} = g(x,u) \qquad x \in X,\ u \in U. \tag{6.1}$$

Usually X is equal to $\mathbb{R}^n$ or an open subset of $\mathbb{R}^n$. Furthermore usually the assumption is made that the input space U is a *closed* subset of $\mathbb{R}^m$. Let now t_0 and t_1 be the fixed initial and terminal times, and let $x_0 \in X$ be the given fixed initial condition. Furthermore let $L : X \times U \to \mathbb{R}$ and $\phi : X \to \mathbb{R}$ be given real functions. Assume now that we wish to minimize the expression

$$\int_{t_0}^{t_1} L(x(t),u(t))dt + \phi(x(t_1)) \tag{6.2}$$

over the (measurable) control functions $u : [t_0,t_1] \to U$. Here $x : [t_0,t_1] \to X$ denotes the solution of (6.1) corresponding to $u : [t_0,t_1] \to U$ and the initial condition $x(t_0) = x_0$. Of course certain smoothness assumptions have to be made about the functions g,L and ϕ, and the control functions that are allowed; we refer to the references mentioned above. We call L the running cost and ϕ the terminal cost, and the problem of minimizing (6.2) the *optimal control problem.*

In order to solve the optimal control problem the Maximum Principle tells us to introduce the "Hamiltonian" function $H : X \times \mathbb{R}^n \times U \to R$ given by

$$H(x,p,u) := L(x,u) + p^T g(x,u) \tag{6.3}$$

with $p \in \mathbb{R}^n$ the *co-state*, and to consider the following set of differential equations

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t),p(t),u(t)) = g(x,(t),u(t))$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t),p(t),u(t)) = \tag{6.4}$$
$$-\frac{\partial L}{\partial x}(x(t),u(t)) - p^T(t)\frac{\partial g}{\partial x}(x(t),u(t))$$

with the (mixed) boundary conditions

$$x(t_0) = x_0$$
$$p(t_1) = -\frac{\partial \phi}{\partial x}(x(t_1)) \tag{6.5}$$

where $x(t_1)$ is the solution at time t_1 of (6.1) for a certain control function $u(\cdot)$ and the fixed initial condition x_0.

Then the following holds:
A necessary condition in order that a (measurable) control function $u^* : [t_0,t_1] \to U$ is *optimal*, i.e. minimizes (6.2), is that for every $t \in [t_0,t_1]$

$$\max_{u \in U} H(x(t),p(t),u) = H(x(t),p(t),u^*(t)) \tag{6.6}$$

where $(x(\cdot),p(\cdot))$ is the solution of (6.4) with $u(\cdot) = u^*(\cdot)$ and boundary conditions (6.5).

So the Maximum Principle leads to the following *optimization* problem: Find for every $(x,p) \in X \times \mathbb{R}^n$ a $u^* \in U$ such that

$$\max_{u \in U} H(x,p,u) = H(x,p,u^*) \tag{6.7}$$

In order to relate the Maximum Principle and especially equations (6.4) and (6.7) to the theory of Hamiltonian systems we make the following (rather severe) assumptions:

1. We assume that the input space U is a manifold (without boundaries).
2. We assume that the function H(x,p,u) in (6.3) is a smooth function of all its arguments.

Assumption 1 is serious, since it excludes that U is a closed subset of $\mathbb{R}^m$, unequal to $\mathbb{R}^m$ and with non-empty interior. The case that U is a manifold with corners is actually the most interesting one in applications.

Furthermore as pointed out in PONTRYAGIN et al. (1962, Chapter V), if U is an open subset of $\mathbb{R}^m$ then the optimal control problem is in fact equal to the so-called Bolza problem in the classical theory of the Calculus of Variations. Moreover the Maximum Principle is in this case more or less equivalent to the classical Weierstrass conditions.

With respect to Assumption 2 we remark that we really only need that $H(x,p,u)$ is C^1 in all its arguments. However, since we always assumed in our treatment of Hamiltonian systems that the generating function H is C^∞ we shall also assume that H in (6.3) is C^∞.

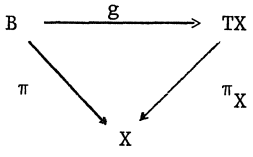
Remark that under Assumptions 1 and 2 the optimization problem (6.7) implies the following first order necessary condition

$$\frac{\partial H}{\partial u}(x,p,u^*) = 0 \tag{6.8}$$

In the sequel we show that we can associate with every optimal control problem a Hamiltonian system as in Definition 3.6. Moreover we show that by applying the first order condition (6.8) the Hamiltonian system reduces, very roughly speaking, to a Hamiltonian vectorfield on T^*X, the (x,p)-space (in fact we only obtain a Hamiltonian vectorfield if we assume some strong regularity conditions). Furthermore if $x^* : [t_1,t_2] \to X$ is an optimal trajectory resulting from an optimal control $u^*(\cdot)$ then there exists a $p^* : [t_1,t_2] \to \mathbb{R}^n$ such that $(x^*(\cdot), p^*(\cdot))$ is a solution of this Hamiltonian vectorfield satisfying the boundary conditions (6.5).

Let us take a nonlinear system $\Sigma(X,W,B,(g,h))$. We forget about the external variables $w \in W$, so we only consider

(6.9)



or in local coordinates (x,u) for B

$$\dot{x} = g(x,u) \tag{6.10}$$

We call (6.9) simply a *control system* $\Sigma(X,B,g)$. Let furthermore $L : B \to \mathbb{R}$ be a smooth function. Firstly we assume that B is trivial, i.e. $B = X \times U$, with U an m-dimensional manifold. (Notice that in the usual setting of the optimal control problem as in (6.1) and (6.2) we also assumed that $B = X \times U$)

Now $M := T^*X$ has a natural symplectic form $\omega = d\theta$ (see Section 3.1.1). Furthermore (see Section 3.1.3) TM is a symplectic manifold with symplectic form $\dot{\omega}$. Define the *space of external variables* $W = T^*U$. Then W has a natural symplectic form $\omega^e = d\theta^e$. Therefore $T(T^*X) \times T^*U$ is a symplectic manifold with symplectic form $\Omega := \pi_1^* \dot{\omega} + \pi_2^* \omega^e$ (π_1 and π_2 are the projections on $T(T^*X)$, respectively T^*U). Then define the function $H : T^*X \times U \to \mathbb{R}$ by

$$H(x,p,u) = L(x,u) + p^T g(x,u) \tag{6.11}$$

((x,p) are natural local coordinates for T^*X). H(x,p,u) is the generating function (see Theorem 3.2) of the Lagrangian submanifold $N \subset (T(T^*X) \times T^*U), \Omega)$ given by the equations

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i}(x,p,u) \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i}(x,p,u) \qquad i = 1,\ldots,n \\ y_j &= -\frac{\partial H}{\partial u_j}(x,p,u) \qquad j = 1,\ldots,m \end{aligned} \tag{6.12}$$

with $(u,y) = (u_1,\ldots,u_m,y_1,\ldots,y_m)$ natural coordinates for T^*U. Concluding:

PROPOSITION 6.1 A control system $\Sigma(X,X\times U,g)$, together with a smooth function $L : X \times U \to \mathbb{R}$ defines a (full) Hamiltonian system (see Definition 3.6) $\Sigma(T^*X, T^*U, T^*X\times U, f_H)$ where f_H is defined by (6.12), i.e. $f_H = (g_H, h_H)$ with $g_H : T^*X\times U \to T(T^*X)$ given by $g_H(x,p,u) = (x,p, \frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x})$ and $h_H : T^*X\times U \to T^*U$ given by $h_H(x,p,u) = (u,-\frac{\partial H}{\partial u})$.

Remark: An interesting but open question is the interpretation of the "outputs" y_j in (6.12)!

If B is *not* a trivial bundle, the situation becomes more complicate. In fact in this case we cannot define a Hamiltonian system in the *strict* sense of Definition 3.6, but only something which is very close to it. Although we shall restrict ourselves in the sequel to trivial bundles B, we briefly give this construction.

Let $\Sigma(X,B,g)$ be a control system with B an arbitrary fiber bundle. Let $p : T^*X \to X$ be the canonical projection, then the pull back bundle p^*B is a bundle over T^*X. Moreover denote the canonical projection from $T(T^*X)$ to

to T^*X by ρ, then $\rho^*(T^*B)$ is a bundle over $T(T^*X)$. Let $\omega = d\theta$ be the natural symplectic form on T^*X. Then $\alpha : T(T^*X) \to T^*(T^*X)$, defined by $\alpha(Z) = d\theta(Z,-)$, $Z \in T(T^*X)$, is an isomorphism. So we obtain

$$
\begin{array}{ccccc}
\bar{B}:=(\alpha^{-1})^*\rho^*T^*B & \xleftarrow{\tilde{\alpha}} & \rho^*T^*B & \xrightarrow{\tilde{\rho}} & T^*B \\
\downarrow & & \downarrow & & \downarrow \\
T^*(T^*X) & \xleftarrow[\alpha]{} & T(T^*X) & \xrightarrow[\rho]{} & T^*X
\end{array}
\tag{6.13}
$$

Since p^*B is a bundle over T^*X, $T^*(p^*B)$ is a bundle over $T^*(T^*X)$. Then the identity mapping $id : T^*(T^*X) \to T^*(T^*X)$ induces a bundle isomorphism

$$
\begin{array}{ccc}
\bar{B} & \xrightarrow{\tilde{id}} & T^*(p^*B) \\
\downarrow & & \downarrow \\
T^*(T^*X) & \xrightarrow[id]{} & T^*(T^*X)
\end{array}
\tag{6.14}
$$

So we can identify $\bar{B}$ and $T^*(p^*B)$. Now $T^*(p^*B)$ and $T^*(T^*X)$ have natural 1-forms Θ_B, respectively Θ_X. By identification of $\bar{B}$ and $T^*(p^*B)$, we can also interpret Θ_B as a 1-form on $\bar{B}$. Then $\tilde{\alpha}^*\Theta_B$ and $\alpha^*\Theta_X$ are 1-forms on ρ^*T^*B, respectively $T(T^*X)$. Let (x,p) be natural coordinates for T^*X and let (x,u,p,y) be natural coordinates for T^*B then it can be easily checked that

$$
\begin{aligned}
\tilde{\alpha}^*\Theta_B &= \sum_{i=1}^{n} \dot{p}_i dq_i - \dot{q}_i dp_i + \sum_{i=1}^{m} y_i du_i \\
\tilde{\alpha}^*\Theta_X &= \sum_{i=1}^{n} \dot{p}_i dq_i - \dot{q}_i dp_i
\end{aligned}
\tag{6.15}
$$

We define a function $H : p^*B \to \mathbb{R}$ by

$$
H = \tilde{p}^*L + p^T g(x,u) \tag{6.16}
$$

Then H is the generating function of a Lagrangian submanifold N of $(\rho^*T^*B, d\tilde{\alpha}^*\Theta_B)$. N is given by

$$
\begin{aligned}
\dot{x}_i &= \frac{\partial H}{\partial p_i}(x,p,u) \\
\dot{p}_i &= -\frac{\partial H}{\partial x_i}(x,p,u) \qquad i = 1,\ldots,n \\
y_j &= -\frac{\partial H}{\partial u_j}(x,p,u) \qquad j = 1,\ldots,m
\end{aligned}
\tag{6.17}
$$

Concluding: we do not have a Hamiltonian system in the strict sense of Definition 3.6. Instead of a *product* structure $T(T^*X)\times T^*U$, we have a bundle structure $\rho^*T^*B \to T(T^*X)$. If U is the standard fiber of B, then the standard fiber of ρ^*T^*B is T^*U.

From now on we shall assume that $B = X \times U$. Thus we can associate with every control system with running cost L a Hamiltonian system. In (6.8) we observed that a first order necessary condition for optimality of u^* is that $\frac{\partial H}{\partial u}(x,p,u^*) = 0$. We obtain

PROPOSITION 6.2 Let $\Sigma(X,X\times U,g)$ be a control system, with dim X = n, dim U im U = m. Let $L : X \times U \to \mathbb{R}$ be the running cost. Consider the associated Hamiltonian system $\Sigma(T^*X,T^*U,T^*X\times U,f_H)$ as in Proposition 6.1, with generating function $H(x,p,u) = L(x,u) + p^T g(x,u)$. Then

(i) If the map $\frac{\partial H}{\partial u}(x,p,u) : \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}^m$ has rank m for every point (x,p,u) in $(\frac{\partial H}{\partial u})^{-1}(0)$, then $(\frac{\partial H}{\partial u})^{-1}(0)$ is an immersed 2n-dimensional submanifold of $T^*X \times U$. Moreover $V := g_H((\frac{\partial H}{\partial u})^{-1}(0))$ is an immersed Lagrangian submanifold of $(TT^*X,\dot{\omega})$.

(ii) If the m×m matrix $(\frac{\partial^2 H}{\partial u_i \partial u_j})$ has full rank in every point, then V can be parametrized by T^*X. Hence V is locally the graph of a Hamiltonian vectorfield $X_{H_{opt}}$ on T^*X.

Remark: Of course rank $(\frac{\partial^2 H}{\partial u_i \partial u_j})$ = m implies that the rank of the map $\frac{\partial H}{\partial u}$ is equal to m.

PROOF i) It is well known that rank $(\frac{\partial H}{\partial u})$ = m on $(\frac{\partial H}{\partial u})^{-1}(0)$ is a sufficient condition in order that $(\frac{\partial H}{\partial u})^{-1}(0)$ is an immersed submanifold of $T^*X \times U$ of dimension 2n. For the associated Hamiltonian system it holds that

$$g_H^*\dot{\omega} = h_H^*\omega^e \tag{6.18}$$

Since $h_H^*\omega^e$ restricted to $(\frac{\partial H}{\partial u})^{-1}(0)$ is zero, it then follows that $g_H((\frac{\partial H}{\partial u})^{-1}(0))$ is an immersed Lagrangian submanifold of $(T(T^*X),\dot{\omega})$.

ii) If rank $(\frac{\partial^2 H}{\partial u_i \partial u_j})$ = m, then the equation $\frac{\partial H}{\partial u}(x,p,u^*) = 0$ has locally a

solution $u^* = u^*(x,p)$. Therefore V is locally given as

$$V = \{(x,p, \frac{\partial H}{\partial p}(x,p,u^*(x,p)), -\frac{\partial H}{\partial x}(x,p,u^*(x,p)))\} \tag{6.19}$$

Hence V is locally the graph of the Hamiltonian vectorfield

$$\dot{x} = \frac{\partial H}{\partial p}(x,p,u^*(x,p))$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x,p,u^*(x,p))$$

on T^*X, with a locally defined Hamiltonian function $H_{opt}(x,p) := H(x,p,u^*(x,p))$. □

Proposition 6.2 can be interpreted in the following way. Consider first the case that rank $(\frac{\partial^2 H}{\partial u_i \partial u_j}) = m$. Then we obtain (locally) a Hamiltonian vectorfield $X_{H_{opt}}$ on T^*X. The projection on X of the solution curves of this Hamiltonian vectorfield form a set of curves on X which by the Maximum Principle contains the optimal trajectories $x^*(\cdot)$. In fact if rank $(\frac{\partial^2 H}{\partial u_i \partial u_j}) = m$, then we can locally construct the Legendre transform (see Section 3.1.3) of H(x,p,u) with respect to u. If this function is denoted by $\bar{H}(x,p,y)$ then the Hamiltonian vectorfield $X_{H_{opt}}$ is given by

$$\begin{aligned} \dot{x} &= \frac{\partial \bar{H}}{\partial p}(x,p,0) \\ \dot{p} &= -\frac{\partial \bar{H}}{\partial x}(x,p,0). \end{aligned} \tag{6.21}$$

(We see that in a certain sense the optimal control case is *dual* to the case of an autonomous Hamiltonian system (see 3.53)). In the first case we set $y = 0$, while in the second case $u = 0$).

Secondly if we only have that the rank of the map $\frac{\partial H}{\partial u}$ is m, then we obtain an immersed Lagrangian submanifold V of $(T(T^*X),\dot{\omega})$. Now such a Lagrangian submanifold can be viewed as an *implicit* Hamiltonian differential equation $P(x,p,\dot{x},\dot{p}) = 0$. If $\pi : T(T^*X) \to T^*X$ is the natural projection, then we know that in the points $(x,p,\dot{x},\dot{p}) \in V$ where π restricted to V does not have maximal rank, the solution of this implicit differential equation cannot be defined (see also Section 2.2.4). When projected from T^*X onto X this phenomenon can cause singularities and non-uniqueness of the optimal trajectories

of the optimal control problem. We conclude that implicit Hamiltonian differential equations corresponding to Lagrangian submanifolds of $(T(T^*X),\dot{\omega})$, which do not project well onto T^*X, arise in a natural way in the solution of optimal control problems.

In the rest of this section we only wish to make some remarks. First of all we observed that optimal control problems give rise to Lagrangian submanifolds of $(T(T^*X),\dot{\omega})$. Conversely, every Lagrangian submanifold of $(T(T^*X),\dot{\omega})$ can, roughly speaking, be generated by an optimal control problem. For this we need

THEOREM 6.3 Let (T^*Q,ω) be a cotangent bundle. Let $N \subset T^*Q$ be a Lagrangian submanifold. Let $\tilde{P} \in N$ and $\pi(\tilde{P}) = P \in Q$ (π is projection on Q). Then there exists a neighborhood V of P, some number $k \in \mathbb{N}$, a neighborhood W of 0 in $\mathbb{R}^k$ and a function $F : V\times W \to \mathbb{R}$ such that the rank of the map $\frac{\partial F}{\partial u}$ on $(\frac{\partial F}{\partial u})^{-1}(0)$ is k and $\{(q,\frac{\partial F}{\partial q}(q,u)) \mid (q,u) \in (\frac{\partial F}{\partial u})^{-1}(0)\}$ is a neighborhood of $\tilde{P}$ in N ((q,u) are coordinates for V×W).

PROOF See WALL (1977), also KLOK (1982). □

Theorem 6.3 yields the following corollaries

COROLLARY 6.4 Let N be a Lagrangian submanifold of $(TM,\dot{\omega})$, with (M,ω) a symplectic manifold. Then there exists a $k \in \mathbb{N}$ and locally a $H : M\times\mathbb{R}^k \to \mathbb{R}$ such that locally N is equal to

$$\{(q,p,\frac{\partial H}{\partial p}(q,p,u),-\frac{\partial H}{\partial q}(q,p,u)) \mid (q,p,u) \in (\frac{\partial H}{\partial u})^{-1}(0)\}.$$

COROLLARY 6.5 Let N be a Lagrangian submanifold of $(TM\times W,\pi_1^*\dot{\omega}-\pi_2^*\omega^e)$, with (M,ω) and (W,ω^e) symplectic manifolds. Let (q,p) be symplectic coordinates for M and (y,u) for W. Then locally there exists a $k \in \mathbb{N}$ and H(q,p,u,v), $v \in \mathbb{R}^k$ such that locally N is equal to

$$\{(q,p,u,\frac{\partial H}{\partial p}(q,p,u,v),-\frac{\partial H}{\partial q}(q,p,u,v),-\frac{\partial H}{\partial u}(q,p,u,v)) \mid (q,p,u,v) \in (\frac{\partial H}{\partial v})^{-1}(0)\}.$$

Very roughly, we can interpret Corollary 6.4 by saying that every Lagrangian submanifold of $(TM,\dot{\omega})$ is generated as the solution of an optimal control problem. Of course this is not entirely correct since the functions H(q,p,u) that result from an optimal control problem are arbitrary except that they are *affine* in the p-variables (notice that the Hamiltonian functions H(q,p,u) corresponding to *affine* Hamiltonian systems are arbitrary

except that they are affine in the u-variables). In the same way, Corollary 6.5 shows that every Lagrangian submanifold of TM × W results from a sort of *partial* optimal control problem. Remember that in Definition 3.6 we did not define a Hamiltonian system as an *arbitrary* Lagrangian submanifold of TM × W, but as a Lagrangian submanifold that is parametrized by a bundle B over M, implying that coordinates for M are also coordinates for this submanifold.

Finally we make a remark about an application of the theory of symmetries and conservation laws as treated in Chapter 4 to the optimal control problem. Let $\Sigma(X,X\times U,g)$ be a control system, with running cost $L : X\times U \to \mathbb{R}$. Denote by Σ_{opt} the associated Hamiltonian system (Proposition 6.1). Suppose that Σ_{opt} has a Hamiltonian symmetry (S,T), with S a vectorfield on T^*X and T a vectorfield on T^*U (see Definition 4.13, also the Remark after Theorem 4.15). Let (F,F_e) be the corresponding conservation law. Suppose now that in every point of the zero section of T^*U, the vectorfield T is tangent to the zero-section. Equivalently, $\frac{\partial F_e}{\partial u}(u,0) = 0$ if (u,y) are natural coordinates for T^*U. Then S is a symmetry for the Hamiltonian vectorfield $X_{H_{opt}}$ in the sense that $[S,X_{H_{opt}}] = 0$. In particular if $F_e(u,0) = 0\ \forall\ u$, we obtain that $X_{H_{opt}}(F) = 0$. Hence we can use the associated Hamiltonian system in order to find symmetries and conservation laws for the resulting Hamiltonian vectorfield $X_{H_{opt}}$. This yields on its turn information about the optimal trajectory $x^*(\cdot)$.

Notes and references for Chapter 6

The idea of applying the theory of symmetries and conservation laws to optimal control can also be found in BROCKETT (1981). We remark that under the assumptions made in Chapter 6, the first order necessary condition (6.8) can be also derived using classical variational methods. This is not surprising since we already remarked that under our assumptions optimal control problems can be treated as classical variational problems (with dynamical constraints). See for more information BUS (1982), and PONTRYAGIN et al. (1962).

CONCLUDING REMARKS

In Chapters 1 and 2 we have laid down a framework for the study of systems with external variables in three subsequent cases: set-theoretic, linear and nonlinear systems. The basic issues covered were: definition and properties of external systems, definition and properties of a system in state space form and the notion of state, the relation between external systems and systems in state space form including the realization problem, and the related topics of minimality, observability and controllability. It is shown that these issues (except for observability) can be satisfactorily treated without specializing first to input-output systems. In fact, notions like state and minimality of a realization are in the general case even more transparent than in the input-output case.

The treatment of (external) linear systems as a vector bundle over $\mathbb{P}^1$ embedded in a trivial vector bundle over $\mathbb{P}^1$ (giving also rise to a *dual* vector bundle) seems to provide a promising alternative approach to some aspects of linear system theory, especially for dealing with properties which are related to taking the limit $s \to \infty$, $s \in \mathbb{C}$ (see also MARTIN & HERMANN (1978)).

In the nonlinear part of Chapter 2 it is shown that the geometric approach to linear systems in state space form as advocated for instance in WONHAM (1979) can also be profitably used in the nonlinear case (with distributions or foliations instead of linear subspaces). We are of the opinion that such a theory provides a useful *basic* framework for the study of nonlinear systems (see also BROCKETT (1980), ISIDORI et al(1981 a)). This framework can also be used for the solution of control problems like disturbance decoupling and non-interacting control (see e.g. NIJMEIJER (1983)). However, much remains to be done in nonlinear control theory. Especially lacking is a well-developed theory of stability and stabilizability (by feedback) of nonlinear control systems. Finally in Section 2.2.4 we have proposed a new approach to the nonlinear realization problem by starting with nonlinear higher order differential equations. This seems to be a large area for further research.

In Chapter 3, the central chapter of this monograph, we have given a unified treatment of Hamiltonian systems with external variables. We suggest two major open research problems. The first one is the so-called Hamiltonian realization problem, see Section 3.8 and the conjecture given there. The second one is the explicit use of the structure of Hamiltonian systems for

the solution of control and synthesis problems. For instance one cannot escape the feeling that more can be said about the controllability properties of Hamiltonian systems. The fact that the controllability distribution is generated by a Poisson algebra of functions on the state space should give more detailed information about the reachable sets of the system. Also a closer study of the stability properties of Hamiltonian systems seems promising. We remark that in VAN DER SCHAFT (1983 a) a preliminary result has been obtained on the disturbance decoupling problem for (linear) Hamiltonian systems. It is an interesting question how this result can be extended to the nonlinear case. Another area of research is the explicit formulation of the Hamiltonian structure which seems to underly the nonlinear filtering problem (see also the preliminary remarks about quantization at the end of Section 3.3.1).

An open problem is how the theory of symmetries and conservation laws as treated in Chapter 4 can be used for control theoretic purposes. At least it is clear that the existence of a conservation law implies a certain structure of the Poisson algebra of an affine Hamiltonian system. For instance in the case of the rigid body with external torques there should be some connection between the (non-)existence of symmetries, the controllability of the system and the Lie group structure of the configuration space.

In Chapter 5 we have given the basic definitions of a gradient system. We have shown that gradient systems are, at least from a system theoretic viewpoint, more complex than Hamiltonian systems. It seems therefore useful to study (subclasses of) gradient systems firstly in a more concrete context (see for instance CROUCH (1981)). Also attention could be directed to the related class of *dissipative* systems. The described connection between gradient systems and (simple) Hamiltonian systems remains intriguing.

Finally it is an open question in how far the theory of Hamiltonian systems can contribute to the understanding of optimal control problems for nonlinear systems. At least we have been able to give a simple proof how under regularity assumptions the first-order necessary condition on the Hamiltonian occurring in the Maximum Principle results in a (possibly implicit) Hamiltonian differential equation on the space of states and co-states. This is dealt with in Chapter 6.

REFERENCES

R.A. ABRAHAM & J.E. MARSDEN (1978), *Foundations of mechanics* (2nd edition). Benjamin/Cummings, Reading, Mass.

V.I. ARNOLD (1978), *Mathematical methods of classical mechanics* (translation of the 1974 Russian edition), Springer, New York

J. BAILLIEUL (1981), *Controllability and observability of polynomial dynamical systems*, Nonlinear Analysis, Theory, Methods and Appl., Vol 5, pp. 543-552.

J.A. BASTO GONCALVES (1980), *Equivalence of gradient systems*, Control Theory Centre Report 84, Un. of Warwick

_____ (1981), *Nonlinear controllability and observability with applications to gradient systems*, Ph. D. Thesis, Un. of Warwick.

R.K. BRAYTON (1971), *Nonlinear reciprocal networks*, SIAM-AMS Proc. Vol III pp. 1-15, AMS, Providence

R.K. BRAYTON & J.K. MOSER (1964), *A theory of nonlinear networks*, Quart. Appl. Math., Vol 22, Part I pp. 1-33, Part II pp. 81-104.

R.W. BROCKETT (1970), *Finite dimensional linear systems*, Wiley, New York.

_____ (1976), *Volterra series and geometric control theory*, Automatica, Vol 12, pp. 167-176.

_____ (1977), *Control theory and analytical mechanics*, pp. 1-46 of *Geometric control theory* (Eds C. Martin and R. Hermann), Vol VII of Lie Groups : History, Frontiers and Applications, Math Sci Press, Brookline.

_____ (1980), *Global descriptions of nonlinear control problems; vector bundles and nonlinear control theory*, Notes for a CBMS conference, manuscript

_____ (1981), Book review Foundations of Mechanics by R. Abraham and J.E. Marsden, IEEE Trans Automatic Control, Vol AC-26, pp. 977-978.

R.W. BROCKETT & C.I. BYRNES (1981), *Multivariable Nyquist criteria, root loci, and pole placement, a geometric viewpoint*, IEEE Trans. Automatic Contr., Vol AC-26, pp. 271-284

R.W. BROCKETT & A. RAHIMI (1972), *Lie algebras and linear differential equations*, Ord. Diff. Equations (Ed. L. Weiss), Academic, New York.

J.C.P. BUS (1982), *The Lagrange multiplier rule on manifolds and optimal control of nonlinear systems*, Preprint Mathematical Centre, Amsterdam.

C. CAMACHO & A. LINS NETO (1979), *Teoria geométrica das folheacões*, IMPA, Rio de Janeiro

P.R. CHERNOFF & J.E. MARSDEN (1974), *Properties of infinite dimensional Hamiltonian systems*, Lect. Notes in Math 425, Springer, Berlin.

P.E. CROUCH (1981), *Geometric structures in systems theory*, Proc. I.E.E. Vol. 128, Pt.D. No.5, pp. 242-252

P.E. CROUCH (1981a), *Dynamical realizations of finite Volterra series*, SIAM J. Control & Optimization, Vol 19, pp. 177-202.

P.E. CROUCH & B. BONNARD (1980), *An appraisal of linear analytic systems theory with applications to attitude control*, European Space Agency Contract Report, E.S.T.E.C. Contract 3771/78/NL/AK(SC).

W.A. DAY (1971), *Time-reversal and the symmetry of the relaxation function of a linear viscoelastic material*, Archive for Rat. Mech. and An. Vol 40, pp. 155-159.

E.J. DIJKSTERHUIS (1950), *De mechanisering van het wereldbeeld*, Meulenhoff, Amsterdam.

D. ELLIOTT (1971), *A consequence of controllability*, J. Diff. Eqns, Vol 10, pp. 364-370.

W.H. FLEMING & R.W. RISHEL (1975), *Deterministic and stochastic optimal control*, Springer, Berlin, New York.

M. FLIESS (1981), *Fonctionnelles causales non linéaires en indéterminées non commutatives*, Bull. Soc. Math. France, Vol 109, pp. 3-40.

M.I. FREEDMAN & J.C. WILLEMS (1978), *Smooth representation of systems with differentiated inputs*, IEEE Trans. Automatic Control, Vol AC-23, pp. 16-20.

F.R. GANTMACHER (1959), *The theory of matrices*, Chelsea, New York.

J.P. GAUTHIER & G. BORNARD (1981), *Observability for any u(t) of a class of nonlinear systems*, IEEE Trans. Automatic Control, Vol AC-26, pp 922-926.

C. GODBILLON (1969), *Géometrie différentielle et mécanique analytique*, Hermann, Paris.

M. HAZEWINKEL & C.F. MARTIN (1982), *A short elementary proof of Grothendieck's theorem on algebraic vector bundles over the projective line*, Journ. of Pure and Applied Algebra, Vol 25, pp. 207-211.

M.L.J. HAUTUS & M. HEYMANN (1978), *Linear feedback, an algebraic approach*, SIAM J. Control & Opt. Vol 16, pp. 83-105.

R. HERMANN (1968), *Differential geometry and the calculus of variations*, Academic, New York.

_____ (1973), *Geometry, physics and systems*, Dekker, New York.

_____ (1974), *Geometric structure of systems-control theory and physics*, Part A, Math Sci Press, Brookline.

_____ (1976), *The geometry of nonlinear differential equations, Bäcklund transformations and solitons*, Math Sci Press, Brookline.

_____ (1979), *Cartanian geometry, nonlinear waves, and control theory, Part A*, Math Sci Press, Brookline.

_____ (1980), *Cartanian geometry, nonlinear waves, and control theory, Part B*, Math Sci Press, Brookline.

_____ (1982), *The differential geometric structure of general mechanical systems from the Lagrangian point of view*, Journ. Math. Physics, Vol 23, pp. 2077-2089.

R. HERMANN & A.J. KRENER (1977), *Nonlinear controllability and observability*, IEEE Trans. Automatic Control, Vol AC-22, pp. 728-740.

R.W. HIRSCHORN (1981), *(A,B)-invariant distributions and disturbance decoupling of nonlinear systems*, SIAM J. Control & Opt. Vol 19, pp. 1-19.

A. ISIDORI , A.J. KRENER, C. GORI-GIORGI & S. MONACO (1981a), *Nonlinear decoupling via feedback : A differential geometric approach*, IEEE Trans. Automatic Control, Vol AC-26, pp. 331-345.

_____ (1981b), *Locally (f,g) invariant distributions*, Systems & Control Letters, Vol 1, pp. 12-15.

B. JAKUBCZYK (1980), *Existence and uniqueness of realizations of nonlinear systems*, SIAM J. Control & Optimization, Vol 18, pp. 455-471.

M. JAMMER (1957), *Concepts of force ; a study in foundations of dynamics*, Harvard University Press, Cambridge Mass.

T. KAILATH (1980), *Linear systems*, Prentice-Hall, Englewood Cliffs, N.J.

R.E. KALMAN, P.L. FALB & M. ARBIB (1969), *Topics in mathematical system theory*, McGraw-Hill, New York.

J. KIJOWSKI & W.M. TULCZYJEW (1979), *A symplectic framework for field theories*, Lect. notes in physics 107, Springer, Berlin.

F. KLOK (1982), *Generic singularities of the exponential map of Riemannian manifolds*, Report ZW-8022, University of Groningen.

A.J. KRENER & C.M. LESIAK (1978), *The existence and uniqueness of Volterra series for nonlinear systems*, IEEE Trans. Automatic Control, Vol AC-23, pp. 1090-1095.

S. LIE & F. ENGEL (1890), *Theorie der transformationsgruppen*, Vol II, Chelsea, New York (1970 ; first edition 1890).

M. LAMNABHI (1982), *A new symbolic calculus for the response of nonlinear systems*, Systems & Control Letters, Vol 2, pp. 154-162.

C. LOBRY (1970), *Contrôlabilité des systèmes nonlinéaires*, SIAM J. Control & Opt., Vol 8, pp. 573-605.

C. MARTIN & R. HERMANN (1978), *Applications of algebraic geometry to systems theory : the McMillan degree and Kronecker indices of transfer functions as topological and holomorphic system invariants*, SIAM J. Control & Opt., Vol 16, pp. 743-753.

K.R. MEYER (1981), *Hamiltonian systems with a discrete symmetry*, Journ. Diff.Eqns, Vol 41, pp. 228-238.

E. NAGEL (1961), *The structure of science*, Routledge & Kegan Paul Ltd, London.

R.W. NEWCOMB (1966), *Linear multiport synthesis*, McGraw-Hill, New York.

H. NIJMEIJER (1980), *Controlled invariant distributions for affine systems on manifolds*, Preprint Mathematical Centre, Amsterdam.

_____ (1981), *Controlled invariance for affine control systems*, Int. Journal Control, Vol 34, pp. 825-833.

_____ (1982a), *Controllability distributions for nonlinear control systems*, Systems & Control Letters, Vol 2, pp. 122-129.

_____ (1982b), *Observability of a class of nonlinear systems : A geometric approach*, Preprint Mathematical Centre, Amsterdam, also Ricerche di Automatica, Vol 11, pp. 1-19 (1981).

_____ (1983), *Nonlinear multivariable control, a geometric approach*, Doctoral thesis, Rijksuniversiteit Groningen, the Netherlands.

H. NIJMEIJER & A.J. VAN DER SCHAFT (1982a), *Controlled invariance by static output feedback for nonlinear systems*, Systems & Control Letters, Vol 2, pp. 39-47.

_____ (1982b), *Controlled invariance for nonlinear systems*, IEEE Trans. Automatic Control, Vol AC-27, pp. 904-914.

_____ (1983a), *Controlled invariance for nonlinear systems ; two worked examples*, to appear in IEEE Trans. Automatic Control.

_____ (1983b), *The disturbance decoupling problem for nonlinear control systems*, IEEE Trans. Automatic Control, Vol AC-28, pp. 621-623.

H. POINCARÉ (1905), *Science and hypothesis*, edition Dover (1952), New York.

L.S. PONTRYAGIN, V.G. BOLTYANSKII, R.V. GAMKRELIDZE & E.F. MISCHENKO (1962), *The mathematical theory of optimal processes*, Wiley, New York.

W. RESPONDEK (1982), *On decomposition of nonlinear control systems*, Systems & Control Letters, Vol 1, pp. 301-308.

H.H. ROSENBROCK (1970), *State-space and multivariable theory*, Nelson, London.

R.M. SANTILLI (1978), *Foundations of theoretical mechanics I*, Springer, New York.

_____ (1983), *Foundations of theoretical mechanics II*, Springer, New York.

A.J. VAN DER SCHAFT (1981), *Symmetries and conservation laws for Hamiltonian systems with inputs and outputs : A generalization of Noether's theorem*, Systems & Control Letters, Vol 1 pp. 108-115.

_____ (1982a), *Time-reversible Hamiltonian systems*, Systems & Control Letters, Vol 1, pp. 295-300.

_____ (1982b), *Controllability and observability for affine nonlinear Hamiltonian systems*, IEEE Trans. Automatic Control, Vol AC-27, pp. 490-492.

_____ (1982c), *Observability and controllability for smooth nonlinear systems*, SIAM Journal Control & Opt. Vol 20, pp. 338-354.

_____ (1982d), *Hamiltonian dynamics with external forces and observations*, Mathematical Systems Theory, Vol 15, pp. 145-168.

_____ (1983a), *Disturbance decoupling by observation feedback for Hamiltonian systems*, Systems & Control Letters, Vol 2, pp.

_____ (1983b), *Symmetries, conservation laws and time reversibility for Hamiltonian systems with external forces*, Journal of Mathematical Physics, Vol 24, pp. 2095-2101.

S. SMALE (1970), *Topology and mechanics (I, II)*, Inventiones Math. Vol 10, pp. 305-331, Vol 11, pp. 45-64.

J. SNIATYCKI & W.M. TULCZYJEW (1972a), *Generating forms of Lagrangian submanifolds*, Indiana Univ. Math. J. Vol 22, pp. 267-275.

_____ (1972b), *Canonical formulation of Newtonian dynamics*, Ann. Inst. H. Poincaré, Vol 16(1) : pp. 23-27.

M. SPIVAK (1970), *A comprehensive introduction to differential geometry*, Vol I, Publish or Perish, Boston.

H.J. SUSSMANN (1977), *Existence and uniqueness of minimal realizations of nonlinear systems*, Math. Systems Theory, Vol 10, pp. 263-284.

_____ (1979), *Single-input observability of continuous-time systems*, Math. Systems Theory, Vol 12, pp. 371-393.

H.J. SUSSMANN & V. JURDJEVIC (1972), *Controllability of nonlinear systems*, J. Differential Eqns, Vol 12, pp. 95-116.

F. TAKENS (1976), *Variational and conservative systems*, Report ZW-7603, University of Groningen.

_____ (1976a), *Constrained equations : A study of implicit differential equations and their discontinuous solutions*, pp. 143-235 in *Structural stability, the theory of catastrophes and applications in the sciences*, Lect. notes in Math. 525, Springer, Berlin.

_____ (1977), *Symmetries, conservation laws and variational principles*, in Palis, J. and do Carmo, M. (eds) *Geometry and Topology*, Lect. Notes Math. No. 597, Springer, New York.

_____ (1979), *A global version of the inverse problem of the calculus of variations*, J. Diff. Geometry, Vol 14, pp. 543-562.

_____ (1983), Personal communication.

T.J. TARN, GARNG HUANG & J.W. CLARK (1980), *Modelling of quantum mechanical control systems*, Mathematical Modelling, Vol 1, pp. 109-121.

W.M. TULCZYJEW (1974), *Hamiltonian systems, Lagrangian systems and the Legendre transformation*, Symp. Math. Vol 14, pp. 247-258.

P. VARAIYA (1971), *Equivalent nonlinear networks*, SIAM-AMS Proc. Vol III, pp. 141-147, AMS, Providence.

G.E. WALL (1977), *Geometric properties of generic differentiable manifolds*, Lect Notes in Math. 597, pp. 707-774, Springer, Berlin.

A. WEINSTEIN (1973), *Lagrangian submanifolds and Hamiltonian systems*, Am. Math. Vol 98, pp. 377-410.

J.C. WILLEMS (1972), *Dissipative dynamical systems*, Part I & Part II, Arch. Rational Mechanics and Analysis, Vol 45, pp. 321-392.

_____ (1974), *Consequences of a dissipation inequality in the theory of dynamical systems*, pp. 193-217 in *Physical structure in systems theory* (Eds. van Dixhoorn, Evans), Academic, New York.

_____ (1978), *Time reversibility in deterministic and stochastic dynamical systems*, pp. 318-326 of *Recent developments in variable structure systems, economics and biology* (Eds R.R. Mohler and A. Ruberti), Lect. Notes in Economics and Math. Systems Vol 162, Springer, Berlin.

_____ (1979), *System theoretic models for the analysis of physical systems*, Richerche di Automatica, Vol 10, pp. 71-106.

_____ (1983), *Input/output and state space representations of finite-dimensional linear time-invariant systems*, Linear Algebra and its Applications, Vol 50, pp. 581-608.

J.C. WILLEMS & A.J. VAN DER SCHAFT (1982) *Modeling of dynamical systems using external and internal variables with applications to Hamiltonian systems*, Proc. of the Int.Sem. on Mathematical Theory of Dynamical Systems and Microphysics, Academic, New York.

W.A. WOLOVICH (1974), *Linear multivariable systems*, Springer, New York.

W.M. WONHAM (1979), *Linear multivariable control : a geometric approach* (2nd edition), Springer, New York.

K. YANO & S. ISHIHARA (1973), *Tangent and cotangent bundles*, Dekker, New York.

SUBJECT INDEX

(numbers of pages on which definitions are given have been underlined)

SYMBOL INDEX

y	output variable	14
Z_V	gradient vectorfield with potential V	220
Δ	affine distribution	52
Δ_0	distribution consisting of input vectorfields	52
θ	one-form, sometimes natural one-form	56,99
θ^e	natural one-form on T^*Y	120
$\dot{\theta}$	prolongation of one-form	56
π	$(\pi:B\to X)$ bundle projection	45
π_X	$(\pi_X:TX\to X)$ projection	45
Σ	system, see $\Sigma(X,W,B,f)$, $\Sigma(A,B,C,D)$ etc.	
Σ_e	external (dynamical) system	8
$\Sigma_e(P)$	external system corresponding to P(s), or to $P \subset T_k W$	24 83
Σ_i	dynamical system in state space form	8
$\Sigma(A,B,C,D)$	linear system (finite-dimensional)	21
$\Sigma(A,B,\bar{C})$	linear input-output system	21
$\Sigma(M,T^*Y,L)$	affine Hamiltonian system	130
$\Sigma(M,W,B,f)$	Hamiltonian system	112
$\Sigma(Q,T^*Y,L)$	affine gradient system	220
$\Sigma(Q,W,B,f)$	gradient system	218
$\Sigma(X,W,B,f)$	smooth nonlinear system	45
$\Sigma(X,\tilde{B},Y,g,h)$	nonlinear input-output system	49
$\Sigma(X,\Delta,Y,h)$	affine control system	52
$\Sigma^e(X,W,B,f)$	extended system	53
ω	symplectic form	99
ω^e	symplectic form on W	105,112
$\dot{\omega}$	prolongation of ω	109
$\mathbb{P}^1$	complex projective line	28
$\mathbb{R}[s]$	real polynomials in s	24
$\mathbb{R}^{n_1\times n_2}[s]$	$n_1 \times n_2$ polynomial matrices	24
$\mathbb{R}(s)$	real rational function in s	24
$\cdot^e$	$(F^e,G^e,C^e,0^e,\Delta^e,\Delta_0^e)$, with respect to the extended system	
$[\ ,\ ]$	Lie bracket	55
$\pounds$	Lie derivative	62
$\{\ ,\ \}$	Poisson bracket	110
$[\![\ ,\ ]\!]$	Beltrami bracket	222
$<\ ,\ >$	Riemannian metric	199,218

MATHEMATICAL CENTRE TRACTS

1 T. van der Walt. *Fixed and almost fixed points.* 1963.

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4 G. de Leve. *Generalized Markovian decision processes, part II: probabilistic background.* 1964.

5 G. de Leve, H.C. Tijms, P.J. Weeda. *Generalized Markovian decision processes, applications.* 1970.

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13 H.A. Lauwerier. *Asymptotic expansions.* 1966, out of print; replaced by MCT 54.

14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

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23 T.J. Dekker, W. Hoffmann. *ALGOL 60 procedures in numerical algebra, part 2.* 1968.

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26 European Meeting 1968. *Selected statistical papers, part I.* 1968.

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37 J.W. de Bakker, G.A. Blaauw, A.J.W. Duijvestijn, E.W. Dijkstra, P.J. van der Houwen, G.A.M. Kamsteeg-Kemper, F.E.J. Kruseman Aretz, W.L. van der Poel, J.P. Schaap-Kruseman, M.V. Wilkes, G. Zoutendijk. *MC-25 Informatica Symposium.* 1971.

38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

39 H. Bavinck. *Jacobi series and approximation.* 1972.

40 H.C. Tijms. *Analysis of (s,S) inventory models.* 1972.

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43 F.H. Ruymgaart. *Asymptotic theory of rank tests for independence.* 1973.

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46 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 1: the language.* 1973.

47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

52 P.C. Baayen (ed.). *Topological structures.* 1974.

53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

55 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 1: theory of designs, finite geometry and coding theory.* 1974.

56 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 2: graph theory, foundations, partitions and combinatorial geometry.* 1974.

57 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 3: combinatorial group theory.* 1974.

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78 E. de Jonge, A.C.M. van Rooij. *Introduction to Riesz spaces.* 1977.

79 M.C.A. van Zuijlen. *Emperical distributions and rank statistics.* 1977.

80 P.W. Hemker. *A numerical study of stiff two-point boundary problems.* 1977.

81 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 1.* 1976.

82 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 2.* 1976.

83 L.S. van Benthem Jutting. *Checking Landau's "Grundlagen" in the AUTOMATH system.* 1979.

84 H.L.L. Busard. *The translation of the elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?), books vii-xii.* 1977.

85 J. van Mill. *Supercompactness and Wallman spaces.* 1977.

86 S.G. van der Meulen, M. Veldhorst. *Torrix I, a programming system for operations on vectors and matrices over arbitrary fields and of variable size.* 1978.

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