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Regularity theory of Fourier integral operators with complex phases and singularities of affine fibrations

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Contents

Introduction

The present monograph deals with two different but related topics. The first one is the regularity theory of Fourier integral operators. We restrict our attention to non-degenerate operators but allow phase functions to be complex valued. The non-degeneracy means that the canonical relation of the operator is locally a graph of a diffeomorphism between cotangent bundles of two manifolds. Such operators arise naturally as propagators (solution operators) for partial differential equations. The theory of these operators is well developed and we refer the reader to excellent monographs [11], [27], [74]. Regularity properties of these operators have been under study for a long time. The L^2 boundedness of operators of zero order was established in [25], but was essentially known in different forms before (see, for example, [16]). The boundedness properties of Fourier integral operators in L^p spaces for $1 < p < \infty$ were established in [62]. The authors prove that operators of order $-(n-1)|1/p - 1/2|$ are bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$, where $n = \dim X = \dim Y$. However, in this paper the authors came up with an interesting condition called "the smooth factorization condition", which allows to improve L^p estimates given certain information on the dimension of the singular support of the Schwartz integral kernel of the operator and singularities of its wave front. All this concerns operators with real phase functions.

Fourier integral operators with complex phases appear naturally in different problems. The theory of operators with complex phases was systematically developed in [37] and [38]. It was used in [74] to describe solution operators for the Cauchy problem for partial differential operators with complex characteristics. L^2 boundedness of these Fourier integral operators of zero orders is due to [38], and to [26] in greater generality. The complex phase is used in the analysis of the oblique derivative problems ([38]) and in the description of projections to kernels and cokernels of pseudo-differential operators with non-involutive characteristics ([12]). In a way the use of complex phase functions is more natural than the real ones. First, there are no geometric obstructions like the non-triviality of Maslov cohomology class. Second, one can use a single complex phase function to give a global parameterization of a Fourier integral operator ([33]).

The first chapter of this monograph describes the L^p properties of Fourier integral operators with complex phases. We suggest a local graph type condition (L) which insures that operators of order $-(n-1)|1/p-1/2|$ with complex phases are bounded from L^p_{comp} to L^p_{loc}. If the imaginary part of the phase function is zero, that is if the phase function is real, our condition (L) is equivalent to the local graph condition for real phase functions. In Section 1.12 we propose the smooth factorization type condition (F) for the complex phase. Again, if the phase function is real valued, our condition (F) is equivalent to the smooth factorization condition of [62]. Under this condition we establish better L^p properties of operators. We provide the reader with the necessary background information on Fourier integral operators in Sections 1.1, 1.2, 1.3, as well as the overview of the regularity theory for real and complex phase functions in Sections 1.4 and 1.6.

It is convenient to use the complex domain for the analysis of the smooth factorization condition. This is the second topic of the present monograph. We will state a general problem of singularities of affine fibrations which includes the smooth factorization condition as a particular case. Let Ω be an open subset of $\mathbb{C}^n$. An affine fibration in Ω is a family of affine subspaces of Ω which locally do not intersect and whose union equals to almost the whole of Ω. These subspaces will be controlled by kernels of holomorphic mappings. To be more precise, let $A : \Omega \to \mathbb{C}^{p\times n}$ be a holomorphic matrix valued mapping. Let $k = \max_{\xi\in\Omega}$ rank $A(\xi)$ be the maximal rank of A in Ω. The set $\Omega^{(k)}$ where it is maximal, is open and dense in Ω, and on this set the mapping

$$\varkappa : \xi \mapsto \ker A(\xi)$$

is regular from $\Omega^{(k)}$ to the space of all $(n-k)$-dimensional linear subspaces of Ω. Our main condition will be that $\varkappa$ defines a fibration in $\Omega^{(k)}$, which means that for $\xi \in \Omega^{(k)}$, $\varkappa$ is constant on $\xi + \ker A(\xi)$. This setting will be made more precise as conditions (A1), (A2) in Chapter 2.

An important case occurs when $A = D\Gamma$ is the Jacobian of a holomorphic mapping $\Gamma : \Omega \to \mathbb{C}^p$. In this case Γ is constant on $\xi + \varkappa(\xi)$ for all $\xi \in \Omega^{(k)}$. This means that the level set $\Gamma^{-1}(\Gamma(\xi))$ is an affine subspace of Ω, equal to $\xi + \varkappa(\xi)$. Such fibrations will be called Jacobian. Conditions (A1), (A2) will be called (ΓA1), (ΓA2) in this case. If Γ itself is a gradient of a holomorphic mapping $\phi : \Omega \to \mathbb{C}$, the fibration will be called a gradient fibration. In this case $A(\xi) = D^2\phi(\xi)$ is the Hessian of ϕ. In our applications ϕ will be the complex analytic extension of the phase function of a Fourier integral operator when the phase function is real analytic.

In Chapter 2 we will study these fibrations and especially their singular sets. It turns out that the mapping $\varkappa$ is meromorphic and we can use methods of complex analytic geometry. All the background information will be provided in Section 2.5.

Chapter 3 is complementary to Chapter 2 and there we will study fibrations of gradient type in both real and complex setting. In Chapter 4 we will apply results of Chapters 1 and 2 to derive further estimates for analytic Fourier integral operators. Those are operators whose phase function is real analytic. Based on the estimates for the set of singularities for corresponding fibrations in Chapter 2, we will show that the smooth factorization type condition (F) holds automatically in a number of cases.

Finally, in Chapter 5 the analysis will be applied to several problems. In Section 5.1 we will describe several applications of the regularity theory of Fourier integral operators. We will go on to discuss regularity properties of solutions of hyperbolic equations in Section 5.2. In Section 5.4 we will allow the characteristic roots of a partial pseudo-differential equation to be complex. However, in order to be able to apply the theory of Fourier integral operators with complex phases, we will assume that the imaginary part of characteristic roots is non-negative. Then, according to [74], the Cauchy problem is well posed and its propagator is a Fourier integral operator with complex phase. We will derive estimates for fixed-time solutions of these equations in L^p spaces. We will also discuss the improvements when the smooth factorization type condition (F) is satisfied. In some cases it is satisfied automatically, for example in $\mathbb{R}^4$ or $\mathbb{R}^5$, when coefficients of the operator may depend on time, but not on the other variables. As another application, we will briefly discuss L^p estimates for the oblique derivative problem in Section 5.5.

The present monograph is based on the author's doctoral thesis. However, there only operators with real phases were investigated, meanwhile the emphasis of this monograph is on operators with complex phases. Parts of this book have appeared in several papers. Chapter 4 is a complex valued phase version of [52], where analytic operators with real phases were considered. Some results of Section 5.2 have appeared in [53] and in [55]. Section 1.11 has appeared in [54]. A survey of the regularity theory of operators with real phases has appeared in [56]. However, Chapter 2 presents a more general problem (A1), (A2) for holomorphic matrix valued functions. Some of its results were announced in [57] and [58]. We would like to mention the paper [59], which is related to the L^p estimates under the failure of the factorization condition. However, we did not feel it would take an integral part in this book and we mention it only briefly in Remark 1.12.4. The results on the complex phase were announced in [60]. Above all, in this book we tried to emphasize the geometric role played by the affine fibrations in the regularity theory, especially in the case of complex phase functions in Section 1.12.

Finally, it is a pleasure for me to thank several people who have contributed in one way or another to the appearance of this work. First, I would like to thank Hans Duistermaat for all the support which I have had from him during my years at the Utrecht University as his graduate student. His influence on my understanding of Fourier integral operators and my work can not be overestimated. I would like to thank departments of mathematics of Utrecht University, the Johns Hopkins University and University of Edinburgh, and finally the Imperial College, where I was able to continue to work on this project. I am also grateful to Chris Sogge, Andreas Seeger, Anders Melin, Ari Laptev and Yuri Safarov for interesting discussions about real and complex phases.

Chapter 1

Fourier integral operators

In this chapter we will discuss Fourier integral operators. The exposition will include both real and complex phase functions. We will provide essential definitions and point to the references for the general theory. Section 1.4 gives an overview of the regularity properties of Fourier integral operators with real phase functions. In Section 1.6 we will give a brief introduction to the regularity properties of operators with complex phases. However, we will not give proofs in these sections as we will prove more general statements for complex phases later. These sections are informally written and can be read independently.

Section 1.1 contains the necessary definitions from the theory of Fourier integral operators with real phases. Section 1.2 gives backgrounds on the use of complex phase functions. It also gives several references to more detailed sources. In Section 1.3 we recall the necessary facts from the relevant function theory.

In Section 1.5 we describe the smooth factorization condition for Fourier integral operators with real phases and how it enters the regularity theory. This is where the relation with affine fibrations of the following chapters plays a role and some discussion of it can be found in Section 1.5.2. Section 1.7 contains the L^p-estimates for Fourier integral operators with complex phases. We introduce condition (L) which plays a role of the local graph condition in the analysis. The proof of these estimates is given in Section 1.9. Estimates in L^p spaces imply estimates in other function spaces in Section 1.10. In Section 1.11 we consider the question of the sharpness of the estimates. To determine the order of the best possible estimate it is sufficient to consider real phases. An application of the stationary phase method gives the best possible orders for all dimensions of the singular support of the operator. As a consequence, we also derive a representation formula for bounded elliptic operators of small negative orders.

An analogue of the smooth factorization condition for complex phases is formulated in Section 1.12 (condition (F)). There we also establish the improved L^p estimates under this condition. These estimates are best possible in view of the arguments of Section 1.11. In the case of the real phase functions, results of Sections 1.7 and 1.12 imply results of [62] on the regularity properties of Fourier integral operators.

In the sequel, our primary concern will be the local analysis. However, global constructions often help to give a better insight in problems at hand. The intrinsic global characterization of Fourier integral operators was systematically developed in [25], [9], [27]. Excellent expositions of the theory can be found in [14], [15]. Many important to us notions are described in [35]. The global theory is based on constructions of the symplectic geometry.

1.1 Fourier integral operators with real phases

We start by recalling several relevant notions from the symplectic geometry. Let M be a smooth real manifold. A form ω is called *symplectic* on M if it is a 2-form on M such that $d\omega = 0$ and such that for each $m \in M$ the bilinear form ω_m is antisymmetric and non-degenerate on T_mM. The pairs (T_mM, ω_m) and (M, ω) are called the *symplectic vector space* and the *symplectic manifold*, respectively. Let X be a smooth real manifold of dimension n. The canonical symplectic form σ on the cotangent bundle T^*X of X can be introduced as follows. Let $\pi : T^*X \ni (x,\xi) \mapsto x \in X$ be the canonical projection. Then for $(x,\xi) \in T^*X$ the mappings $D\pi_{(x,\xi)} : T_{(x,\xi)}(T^*X) \to T_xX$ and $\xi : T_xX \to \mathbb{R}$ are linear. Their composition

$$\alpha_{(x,\xi)} = \xi \circ D\pi_{(x,\xi)} \tag{1.1.1}$$

is a 1-form on T^*X. Its derivative $\sigma = d\alpha$ is called the *canonical 2-form on* T^*X and it follows that σ is symplectic. This form corresponds to the form $\sum_{j=1}^n dp_j \wedge dq_j$ in mechanics, with possible change of sign. It can be shown that any symplectic form takes the latter form in *symplectic coordinates.* The same objects can be introduced on complex analytic manifolds.

A submanifold Λ of T^*X is called *Lagrangian* if $(T_{(x,\xi)}\Lambda)^\sigma = T_{(x,\xi)}\Lambda$, where

$$(T_{(x,\xi)}\Lambda)^\sigma = \{p \in T_{(x,\xi)}(T^*X) : \sigma(p,q) = 0 \ \forall q \in T_{(x,\xi)}\Lambda\}.$$

In particular, this implies $\dim \Lambda = n$. A submanifold Λ of $T^*X\backslash 0 = \{(x,\xi) \in T^*X : \xi \neq 0\}$ is called *conic* if $(x,\xi) \in \Lambda$ implies $(x,\tau\xi) \in \Lambda$ for all $\tau > 0$. Let $\Sigma \subset X$ be a smooth submanifold of X of dimension k. Its *conormal bundle* in T^*X is defined by

$$N^*\Sigma = \{(x,\xi) \in T^*X : x \in \Sigma, \xi(\delta x) = 0, \ \forall \delta x \in T_x\Sigma\}.$$

The proofs of the subsequent statements can be found in [14], [15], [27]. We mainly follow [11].

Proposition 1.1.1. (1) *Let $\Lambda \subset T^*X\backslash 0$ be a closed submanifold of dimension n. Then Λ is a conic Lagrangian manifold if and only if the form α in (1.1.1) vanishes on Λ.*

(2) *Let $\Sigma \subset X$ be a submanifold of dimension k. Then its conormal bundle $N^*\Sigma$ is a conic Lagrangian manifold.*

(3) *Let* $\Lambda \subset T^*X\backslash 0$ *be a conic Lagrangian manifold and let* $D\pi_{(x,\xi)} : T_{(x,\xi)}\Lambda \to T_xX$ *have constant rank equal to* k *for all* $(x,\xi) \in \Lambda$*. Then each* $(x,\xi) \in \Lambda$ *has a conic neighborhood* Γ *such that*

(a) $\Sigma = \pi(\Lambda \cap \Gamma)$ *is a smooth submanifold of* X *of dimension* k*,*

(b) $\Lambda \cap \Gamma$ *is an open subset of* $N^*\Sigma$*.*

In the sequel we will mainly deal with conic Lagrangian manifolds and we will need their local representations. For this purpose, we consider a local trivialization $X \times (\mathbb{R}^n\backslash 0)$ of $T^*X\backslash 0$, where we can assume X to be an open set of dimension n. However, in the sequel we will also need a slight generalization of it, so that we allow the dimensions of X and the fibers differ. Thus, let Γ be a cone in $X \times (\mathbb{R}^N\backslash 0)$. A smooth function $\phi : X \times (\mathbb{R}^N\backslash 0) \to \mathbb{R}$ is called a *phase function*, if it is homogeneous of degree one in θ and has no critical points as a function of (x,θ): $\phi(x,\tau\theta) = \tau\phi(x,\theta)$ for $\tau > 0$ and $d_{(x,\theta)}\phi(x,\theta) \neq 0$ for all $(x,\theta) \in X \times (\mathbb{R}^N\backslash 0)$. A phase function is called *non-degenerate in* Γ if $(x,\theta) \in \Gamma, d_\theta\phi(x,\theta) = 0$ imply that $d_{(x,\theta)}\frac{\partial\phi(x,\theta)}{\partial\theta_j}$ are linearly independent for $j = 1,\dots,N$.

Proposition 1.1.2. (1) *Let* Γ *be a cone in* $X \times (\mathbb{R}^N\backslash 0)$ *and let* ϕ *be a non-degenerate phase function in* Γ*. Then there exists an open cone* $\widetilde{\Gamma} \supset \Gamma$ *such that the set*

$$C_\phi = \{(x,\theta) \in \widetilde{\Gamma} : d_\theta\phi(x,\theta) = 0\}$$

is a smooth conic submanifold of $X \times (\mathbb{R}^N\backslash 0)$ *of dimension* n*. The mapping*

$$T^\phi : C_\phi \ni (x,\theta) \mapsto (x, d_x\phi(x,\theta)) \in T^*X\backslash 0$$

is an immersion, commuting with the multiplication with positive real numbers in the fibers. Let us denote $\Lambda_\phi = T^\phi(C_\phi)$*.*

(2) *Let* Λ *be a submanifold of* $T^*X\backslash 0$ *of dimension* n*. Then* Λ *is a conic Lagrangian manifold if and only if every* $(x,\xi) \in \Lambda$ *has a conic neighborhood* Γ *such that* $\Lambda \cap \Gamma = \Lambda_\phi$ *for some non-degenerate phase function* ϕ*.*

Naturally, the cone condition for Λ corresponds to the homogeneity of ϕ.

We give the following definition for the completeness, although we will not use it in the sequel. Let Λ be a closed conic Lagrangian submanifold of $T^*X\backslash 0$. A distribution u is called the *Lagrangian distribution of order* m *associated to* Λ, $u \in I^m(X,\Lambda)$, if

$$\prod_{j=1}^n P_j u \in {}^\infty H^{loc}_{-m-n/4}(X), \tag{1.1.2}$$

whenever $P_j \in \Psi^1(X)$ are properly supported pseudo-differential operators whose principal symbols $p_j(x,\xi)$ vanish on Λ and ${}^\infty H^{loc}_{-m-n/4}$ is the localization of the usual Besov space. First, a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the Besov space ${}^{\infty}H_\sigma(\mathbb{R}^n)$, if $\hat{u} \in L^2_{loc}(\mathbb{R}^n)$ and

$$\begin{aligned} \|u\|_{{}^{\infty}H_\sigma(\mathbb{R}^n)} &= \left(\int_{|\xi|\leq 1} |\hat{u}(\xi)|^2 d\xi\right)^{1/2} \\ &+ \sup_{j\geq 0}\left(\int_{2^j\leq|\xi|\leq 2^{j+1}} |2^{\sigma j}\hat{u}(\xi)|^2 d\xi\right)^{1/2} < \infty. \end{aligned}$$

For a smooth manifold X of dimension n the space ${}^{\infty}H^{loc}_\sigma(X)$ is defined to be the space of all $u \in \mathcal{D}'(X)$ such that $(\psi u)\circ \varkappa^{-1}$ is in ${}^{\infty}H_\sigma(\mathbb{R}^n)$ whenever $\Omega \subset X$ is a coordinate patch with coordinates $\varkappa$ and $\psi \in C_0^\infty(\Omega)$. More details can be found in [27, 25.1] and [65, 6.1].

Let now X, Y be open in $\mathbb{R}^n$ and let Φ be a non-degenerate phase function. Let $a \in S^m_\rho(X \times Y \times \mathbb{R}^N)$ be a *symbol of type ρ and order m*, which means that $1/2 \leq \rho \leq 1$, $a \in C^\infty(X \times Y \times \mathbb{R}^N)$, and for every compact subset K of $X \times Y$ and any multi-indices α, β holds

$$|\partial^\beta_{x,y}\partial^\alpha_\xi a(x,\theta)| \leq C(\alpha,\beta,K)(1+|\theta|)^{m-\rho|\alpha|+(1-\rho)|\beta|}$$

for all $(x,y) \in K$ and $\theta \in \mathbb{R}^N\backslash 0$. Operators of the form

$$Tu(x) = \int_Y\int_{\mathbb{R}^N} e^{i\Phi(x,y,\theta)}a(x,y,\theta)u(y)d\theta dy. \tag{1.1.3}$$

are called *Fourier integral operators.* Expression (1.1.3) can be understood in the classical sense if $m+N<0$ and $u \in C_0^\infty(X)$, when the integral is absolutely convergent. The integral kernel of (1.1.3) is equal to

$$K(x,y) = \int_{\mathbb{R}^N} e^{i\Phi(x,y,\theta)}a(x,y,\theta)d\theta. \tag{1.1.4}$$

According to Proposition 1.1.2, the set

$$\Lambda_\phi = \{(x,y,d_x\phi(x,y,\theta), d_y\phi(x,y,\theta)) : d_\theta\phi(x,y,\theta)=0\} \tag{1.1.5}$$

is a conic Lagrangian submanifold of $T^*(X\times Y)\backslash 0$ of dimension $2n$. It follows that for $\rho=1$, the kernel (1.1.4) is a Lagrangian distribution in $X\times Y$ associated to Λ_Φ of order $\mu = m - n/2 + N/2$, $K \in I^\mu(X\times Y, \Lambda_\Phi)$. Conversely, any Lagrangian distribution $K \in I^\mu(X\times Y,\Lambda)$ can be microlocally written in the form (1.1.4) modulo C^∞, with $\rho = 1$ (cf. [65, Theorem 6.1.4]). The kernel (1.1.4) is also called the *Fourier integral distribution.* In general, $I^\mu_\rho(X,\Lambda)$ will denote the space of Fourier integral distributions consisting of Fourier integral distributions

$$u(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\theta)}a(x,\theta)d\theta, \tag{1.1.6}$$

with Λ locally equal to Λ_ϕ as in Proposition 1.1.2, $a \in S^m_\rho(X\times\mathbb{R}^N)$, $\rho > 1/2$, and $\mu = m - n/4 + N/2$.

The behavior of the integral (1.1.3) can be independent of some of the variables θ and the order of T is taken to be $\mu = m + (N-n)/2$, which is the order of its Lagrangian distribution (in this case we have $X \times Y$ instead of X). The following theorem describes the family of phase functions corresponding to the same Fourier integral distribution (1.1.4) (Theorem 2.3.4 in [11]). We formulate it for the general case of $I^\mu_\rho(X,\Lambda)$ as in (1.1.6), rather than for a particular case $I^\mu_\rho(X \times Y, \Lambda)$.

Theorem 1.1.3. *Suppose $\phi(x,\theta)$ and $\widetilde{\phi}(x,\widetilde{\theta})$ are non-degenerate phase functions at $(x_0,\theta_0) \in X \times (\mathbb{R}^N \backslash 0)$ and at $(x_0,\widetilde{\theta}_0) \in X \times (\mathbb{R}^{\widetilde{N}} \backslash 0)$, respectively. Let Γ and $\widetilde{\Gamma}$ be open conic neighborhoods of (x_0,θ_0) and $(x_0,\widetilde{\theta}_0)$ such that $T_\phi : C_\phi \to \Gamma_\phi$ and $T_{\widetilde{\phi}} : C_{\widetilde{\phi}} \to \Gamma_{\widetilde{\phi}}$ are injective, respectively. If $\Lambda_\phi = \Lambda_{\widetilde{\phi}}$, then any Fourier integral distribution, defined by the phase function ϕ and an amplitude $a \in S^m_\rho(X \times \mathbb{R}^N), \rho > 1/2$, with* ess supp *$a$ contained in a sufficiently small conic neighborhood of (x_0,θ_0), is equal to a Fourier integral distribution defined by the phase function $\widetilde{\phi}$ and an amplitude $\widetilde{a} \in S^{m+\frac{1}{2}(N-\widetilde{N})}_\rho(X \times \mathbb{R}^{\widetilde{N}})$.*

In particular, the phase function Φ of the operator T in (1.1.3) can be always written in the form

$$\Phi(x,y,\xi) = \langle x,\xi\rangle - \psi(y,\xi),$$

with some function ψ and $\xi \in \mathbb{R}^n$. This fact will be often used in the sequel. The function ψ (as well as Φ) is also called the generating function for Λ_Φ.

Thus, the notion of Fourier integral operator becomes independent of a particular choice of a phase function associated to the Lagrangian manifold Λ. The set

$$C = \Lambda' = \{((x,\xi),(y,\eta)) \in T^*X \times T^*Y : (x,y,\xi,-\eta) \in \Lambda\}$$

is a conic Lagrangian manifold in $T^*X\backslash 0 \times T^*Y \backslash 0$ with respect to the symplectic structure $\sigma_X \oplus -\sigma_Y$ and it is called a *homogeneous canonical relation from T^*Y to T^*X*. The space of integral operators with distributional kernels in $I^\mu_\rho(X \times Y, \Lambda)$ will be denoted by $I^\mu_\rho(X,Y;C)$ and it is the space of *Fourier integral operators associated to the canonical relation Λ'*.

1.2 Fourier integral operators with complex phases

In this section we will discuss the use of complex valued phase functions in the theory of Fourier integral operators. We will give only an overview as a reader can consult [74], [14], [27] for details.

As before, Fourier integral operators are operators which can be represented microlocally in the form

$$Tu(x) = \int_Y \int_{\mathbb{R}^n} e^{i\Phi(x,y,\theta)} a(x,y,\theta) u(y) d\theta dy. \tag{1.2.1}$$

The integral makes sense only if $\operatorname{Im}\Phi \geq 0$. The rest of the properties of the phase function are similar to the real ones. Let $V \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N \backslash \{0\}$ be a conic set. A smooth in V function $\Phi = \Phi(x, y, \theta)$ is called a *regular phase function of positive type*, if

(i) Φ has no critical points: $d\Phi \neq 0$ on V.

(ii) Φ is positive homogeneous of degree one in θ: $\Phi(x, y, t\theta) = t\Phi(x, y, \theta)$ for $t > 0$.

(iii) $d(\partial\Phi/\partial\theta_1), \ldots, d(\partial\Phi/\partial\theta_N)$ are linearly independent over $\mathbb{C}$ on $C_{\Phi\mathbb{R}} = \{(x, y, \theta) \in V : \ \Psi'_\theta = 0\}$.

(iv) $\operatorname{Im}\Phi(x, y, \theta) \geq 0$ on V.

In order to set up the calculus of such operators one uses the notion of almost analytic continuation.

A function $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ is called almost analytic in $U_\mathbb{R} = U \cap \mathbb{R}^n$, if f satisfies the Cauchy–Riemann equation in $U_\mathbb{R}$ of infinite order, that is $\bar{\partial} f$ and all of its derivatives vanish in $U_\mathbb{R}$. In a natural way one defines an almost analytic extension of a real manifold X requiring that corresponding coordinate functions are almost analytic in X. The positivity of the canonical relation C means that

$$i^{-1}\left(\sigma_X(u, \bar{u}) - \sigma_Y(v, \bar{v})\right) \geq 0$$

for all $(u, v) \in C$, where σ_X and σ_Y are the standard symplectic forms on T^*X and T^*Y, lifted to their almost analytic extensions. Let us give more details now.

Let Ω be an open subset of $\mathbb{R}^n$ and let $\rho : \Omega \to \mathbb{R}$ be a non-negative Lipschitz function. A function $f : \Omega \to \mathbb{R}$ is called *ρ-flat on Ω* if for every compact set $K \subset \Omega$ and for every integer $N \geq 0$ there exist a constant $C = C(K, N) > 0$ such that $|f(x)| \leq C|\rho(x)|^N$, for all $x \in K$.

This notion defines an equivalence relation on the space of function on Ω. Thus, two functions f and g are called *ρ-equivalent* if $f - g$ is ρ-flat on Ω. A function is called *flat* on a compact set $K \subset \Omega$ if it is ρ-flat with $\rho(x) = \operatorname{dist}(x, K)$.

Let function $f \in C^\infty(\Omega)$ be ρ-flat. Then all its derivatives $D^\alpha f$ are also ρ-flat. It follows that f is flat on K if and only if $D^\alpha f(x) = 0$ for all $x \in K$ and all α.

Let now O be an open subset of $\mathbb{C}^n$ and let K be a closed subset of O. A function $f \in C^\infty(O)$ is called *almost analytic* on K if for $j = 1, \ldots, n$ the functions $\partial_j f$ are flat on K. For a set $O \subset \mathbb{C}^n$ by $O_\mathbb{R}$ we denote the intersection $O \cap \mathbb{R}^n$. On the other hand, for an open set $\Omega \subset \mathbb{R}^n$ we denote $\widetilde{\Omega} = \Omega + i\mathbb{R}^n \subset \mathbb{C}^n$ and will identify Ω with $\widetilde{\Omega} \cap \{\operatorname{Im} z = 0\}$.

Each function $f \in C^\infty(O_\mathbb{R})$ defines an equivalence class of almost analytic functions on $O_\mathbb{R}$, which consists of functions in $C^\infty(O)$ which are almost analytic on $O_\mathbb{R}$, modulo functions which are flat on $O_\mathbb{R}$. Any representative of this class is called an *almost analytic continuation* of f in O.

Let O be an open subset of $\mathbb{C}^n$, M a smooth submanifold of codimension $2k$ of O, K a closed subset of O. Then M is called *almost analytic* on K if every point z_0 of K has an open neighborhood U in O in which there exist k complex smooth functions $f_1, \ldots, f_k$, almost analytic on $K \cap U$, and such that in the set U, M is defined by the equations $f_1(z) = \ldots = f_k(z) = 0$, and differentials $df_1, \ldots, df_k$ are $\mathbb{C}$-linearly independent. We say that two almost analytic submanifolds M_1 and M_2 of O are equivalent, if they have the same dimension, and the same intersection $M_{\mathbb{R}}$ with $\mathbb{R}^n$, and locally $f_j - g_j$ are flat functions on $M_{\mathbb{R}}$, where M_1 and M_2 are defined by f_j and g_j, respectively. Thus, a real manifold Ω defines an equivalence class of almost analytic manifolds in $\widetilde{\Omega}$. A representative of this class is called an *almost analytic continuation* of Ω in $\mathbb{C}^n$.

For further definitions in this section we will mostly follow [14]. Let now M be a real symplectic manifold of dimension $2n$, and let $\widetilde{M}$ be its almost analytic continuation in $\mathbb{C}^{2n}$. Let $\Lambda \subset \widetilde{M}$ be an almost analytic submanifold containing the real point $\rho_0 \in M$, and let (x, ξ) be be real symplectic coordinates in a neighborhood $W \subset \mathbb{R}^{2n}$ of the point ρ_0. Let $(\widetilde{x}, \widetilde{\xi})$ be almost analytic continuation of the coordinates (x, ξ) in W so that $(\widetilde{x}, \widetilde{\xi})$ map $\widetilde{W}$ diffeomorphically onto an open subset in $\mathbb{C}^{2n}$. Let g be an almost analytic function such that $\operatorname{Im} g \geq 0$ in $\mathbb{R}^n$, and such that Λ is defined in a neighborhood of ρ_0 by the equations $\widetilde{\xi} = \partial g(x)/\partial \widetilde{x}$, $\widetilde{x} \in \mathbb{C}^n$. An almost analytic manifold Λ satisfying this property in some real symplectic coordinate system at every real point is called a *positive Lagrangian manifold.* An almost analytic manifold $\Lambda \subset \widetilde{M}$ is called a *strictly positive Lagrangian manifold* if $\dim \Lambda_{\mathbb{R}} = 2n$ and $\Lambda_{\mathbb{R}}$ is a submanifold in M, $\sigma_\alpha|_{\Lambda_\alpha} \sim 0$ for all local representatives Λ_α of Λ and for all local almost analytic continuations σ_α of the symplectic form σ on $T_\rho(\widetilde{M})$, and if $i^{-1}\sigma(v, \bar{v}) > 0$ for all $v \in T_\rho(\Lambda) \backslash \widetilde{T_\rho(\Lambda_{\mathbb{R}})}$, $\rho \in \Lambda_{\mathbb{R}}$, where $\widetilde{T_\rho(\Lambda_{\mathbb{R}})}$ is the complexification of $T_\rho(\Lambda_{\mathbb{R}})$. The local coordinates $\widetilde{x}, \widetilde{\xi}$ on Λ are almost analytic continuations of local coordinates (x, ξ) on $\Lambda_{\mathbb{R}}$.

Proposition 1.2.1. *Let M, Λ and W be as before. If $(\widetilde{y}, \widetilde{\eta})$ is another almost analytic continuation of coordinates in $\widetilde{W}$ and Λ is defined by the equation $\widetilde{\eta} = H(\widetilde{y})$ in a neighborhood of the point ρ_0, then Λ is locally equivalent to the manifold $\widetilde{\eta} = \partial h(\widetilde{y})/\partial \widetilde{y}$, $\widetilde{y} \in \mathbb{C}^n$, where h is an almost analytic function and $\operatorname{Im} h \geq 0$ in $\mathbb{R}^n$.*

Furthermore, let Φ be a regular phase function of the positive type that is defined in a conical neighborhood. Let $\widetilde{\Phi}$ be an almost analytic homogeneous continuation of Φ in a canonical neighborhood in $\mathbb{C}^n \times \mathbb{C}^n \times (\mathbb{C}^N \backslash 0)$. Let

$$C_{\widetilde{\Phi}} = \{(\widetilde{x}, \widetilde{y}, \widetilde{\theta}) \in \mathbb{C}^n \times \mathbb{C}^n \times (\mathbb{C}^N \backslash 0) : \partial \widetilde{\Phi}(\widetilde{x}, \widetilde{y}, \widetilde{\theta})/\partial \widetilde{\theta} = 0\}.$$

Then the image $\Lambda_{\widetilde{\Phi}}$ of the set $C_{\widetilde{\Phi}}$ under the map

$$C_{\widetilde{\Phi}} \ni (\widetilde{x}, \widetilde{y}, \widetilde{\theta}) \mapsto \left(\widetilde{x}, \frac{\partial \widetilde{\Phi}(\widetilde{x}, \widetilde{y}, \widetilde{\theta})}{\partial \widetilde{x}}, \frac{\partial \widetilde{\Phi}(\widetilde{x}, \widetilde{y}, \widetilde{\theta})}{\partial \widetilde{y}}\right) \in \mathbb{C}^n \times (\mathbb{C}^n \backslash 0) \times (\mathbb{C}^n \backslash 0)$$

is a local conical positive Lagrangian manifold. Further, $\Lambda_{\Phi_{\mathbb{R}}}$ *is the image of* $C_{\Phi_{\mathbb{R}}}$, *and when* $\widetilde{\Phi}$ *is replaced by an equivalent almost analytic continuation, the manifold* $\Lambda_{\widetilde{\Phi}}$ *is replaced by an equivalent conical positive Lagrangian manifold.*

A manifold $\Lambda \subset T^*\widetilde{(X \times Y)}\backslash 0$ is called a *positive canonical relation* if $\Lambda' = \{(x,\xi,y,\eta) : (x,\xi,y,-\eta) \in \Lambda\}$ is a closed conical positive Lagrangian manifold in $T^*\widetilde{(X \times Y)}\backslash 0$ and $\Lambda_{\mathbb{R}} \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$. The set of Fourier integral operators $I^\mu_\rho(X,Y;\Lambda)$ with positive canonical relation Λ, $\rho > 1/2$, with symbols in $S^\mu_\rho(X \times Y \times \mathbb{R}^n)$ is defined as for the operators with real phase functions. There is a usual definition of the principal symbol and the calculus of such operators holds as well. We can refer the reader to [37] and [74] for the details.

1.3 Spaces of functions

In this section we will briefly discuss function spaces, which will be used throughout this monograph. Let X be a smooth manifold with a measure dx. For $1 \le p < \infty$ by $L^p(X)$ we will denote the usual space (of the equivalence classes) of measurable functions f on X with finite norm $||f||_p = (\int_X |f|^p dx)^{1/p}$. For $0 < p < 1$ this expression fails to be a norm and the substitute for $L^p(X)$ in the analysis of singular integrals are Hardy spaces $H^p(X)$. The general theory of complex and real variable versions of Hardy spaces can be found in [67], [70], [18], [68], where one can also find proofs of subsequent statements of this section. Since our interest are the local properties, we restrict to the real Euclidean case of $H^p(\mathbb{R}^n)$.

Let $\mathcal{S}$ be the Schwartz space of smooth rapidly decreasing functions, equipped with a countable family of seminorms

$$||\phi||_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta \phi(x)|.$$

Let $\Phi \in \mathcal{S}$ and for $t > 0$ define $\Phi_t(x) = t^{-n}\Phi(x/t)$. Then for a distribution f the convolutions $f * \Phi_t$ are smooth and one defines the maximal operator

$$M_\Phi f(x) = \sup_{t>0} |(f * \Phi_t)(x)|.$$

Let $\mathcal{F}$ be a finite collection of seminorms on $\mathcal{S}$ and one defines

$$\mathcal{S}_\mathcal{F} = \{\Phi \in \mathcal{S} : ||\Phi||_{\alpha,\beta} \le 1 \text{ for all } ||\cdot||_{\alpha,\beta} \in \mathcal{F}\}.$$

The maximal operator associated to the family $\mathcal{S}_\mathcal{F}$, determining the approximation of the identity, is now defined by

$$\mathcal{M}_\mathcal{F} f(x) = \sup_{\Phi\in\mathcal{S}_\mathcal{F}} M_\Phi f(x).$$

For the definition of the Hardy space $H^p(\mathbb{R}^n)$, we take the space of functions, satisfying one of the following equivalent conditions.

Proposition 1.3.1. *Let f be a distribution and let $0 < p \leq \infty$. Then the following conditions are equivalent:*

(1) *There is a function $\Phi \in \mathcal{S}$ with $\int \Phi dx \neq 0$ so that $M_\Phi f \in L^p(\mathbb{R}^n)$.*

(2) *There is a collection $\mathcal{F}$ so that $\mathcal{M}_\mathcal{F} f \in L^p(\mathbb{R}^n)$.*

The expression $||f||_{H^p} = ||M_\Phi f||_{L^p}$ can be taken to be the norm of $H^p(\mathbb{R}^n)$. Note, that it is equivalent to $||\mathcal{M}_\mathcal{F}||_{L^p}$ and it is actually a norm only if $p \geq 1$. For $0 < p < 1$, the topology of H^p can be defined by the metric $d(f,g) = ||f-g||^p_{H^p}$.

In the case $1 < p \leq \infty$, a simple argument shows that $H^p(\mathbb{R}^n)$ coincide with the Lebesgue spaces $L^p(\mathbb{R}^n)$. However, already for $p = 1$ one only has $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. On the other hand, if $f \in L^1_{comp}(\mathbb{R}^n)$ satisfies the moment condition $\int f dx = 0$ (which is in fact necessary for f to belong to $H^1(\mathbb{R}^n)$), then $f \in L^q(\mathbb{R}^n)$ for any $q > 1$ implies $f \in H^1(\mathbb{R}^n)$. One often makes use of an atomic decomposition of Hardy spaces, similar to the classical Calderón-Zygmund decomposition. For $0 < p \leq 1$, an H^p *atom* is a function a such that

(1) a is supported in a ball B,

(2) $|a| \leq |B|^{-1/p}$ almost everywhere,

(3) $\int x^\alpha a(x) dx = 0$ for all α with $|\alpha| \leq n(p^{-1} - 1)$.

An H^p atom belongs to $H^p(\mathbb{R}^n)$ with uniform bound and to $L^p(\mathbb{R}^n)$ with

$$\int |a(x)|^p dx \leq 1,$$

which follows from (1) and (2). One has the following characterization of H^p in terms of H^p atoms.

Proposition 1.3.2. *Let $0 < p \leq 1$.*

(1) *Let a_k be a collection of H^p atoms and let $\lambda_k \in \mathbb{C}$ satisfy $\sum_k |\lambda_k|^p < \infty$. Then the series*

$$f = \sum_k \lambda_k a_k \tag{1.3.1}$$

converges distributionally, its sum f belongs to $H^p(\mathbb{R}^n)$, and

$$||f||_{H^p} \leq c \left(\sum_k |\lambda_k|^p \right)^{1/p}.$$

(2) *Let $f \in H^p(\mathbb{R}^n)$. Then f can be written as a sum of H^p atoms as in (1.3.1), which converges in H^p norm. Moreover,*

$$\left(\sum_k |\lambda_k|^p \right)^{1/p} \leq c||f||_{H^p}.$$

Let X be an open subset of $\mathbb{R}^n$. For $0 < \gamma < 1$ the Lipschitz (Hölder) space Lip (X, γ) consists of functions f for which there exists a constant A such that $|f(x)| \leq A$ almost everywhere and

$$\sup_x |f(x-y) - f(x)| \leq A|y|^\gamma$$

holds for all y with $x - y \in X$. Minimal A satisfying these two inequalities can be taken to be the norm of f (cf. [66], [75], [4]). The Hardy space H^1 plays an important role in the complex interpolation method.

Proposition 1.3.3. *Let T_z be a family of linear operators on $\mathbb{R}^n$, parameterized by complex z with $0 \leq$ Re $(z) \leq 1$. Suppose that for all simple (step-) functions f, g, vanishing outside a set of finite measure, the map*

$$z \mapsto \int_{\mathbb{R}^n} (T_z f) g dx$$

is bounded and analytic in the open strip $0 <$ Re $(z) < 1$ and is continuous in its closure. Suppose that $||T_z f||_{L^1} \leq C_0 ||f||_{H^1}$ for Re $(z) = 0$ and $||T_z f||_{q_1} \leq C_1 ||f||_{p_1}$ for Re $(z) = 1$. Then also

$$||T_t f||_{q_t} \leq C_0^{1-t} C_1^t ||f||_{p_t} \tag{1.3.2}$$

with p_t and q_t defined by $1/p_t = (1-t) + t/p_1$ and $1/q_t = (1-t) + t/q_1$.

The proof is based on the duality between H^1 and BMO ([18]).

Proposition 1.3.4 (Hardy-Littlewood-Sobolev). *For every $0 < \gamma < n$, $1 < p < q < \infty$ and $1/q = 1/p - (n-\gamma)/n$, there exists a constant A_{pq} such that*

$$||f * (|y|^{-\gamma})||_{L^q} \leq A_{pq} ||f||_{L^p}.$$

There is a similar result in Hardy spaces ([68, III.5.21]).

Proposition 1.3.5. *The operator $I_\alpha f = f * (|y|^{-\gamma})$ allows an analytic extension on the set $-n(p^{-1} - 1) \leq$ Re $\gamma < n$, when $\int x^\alpha f(x) dx = 0$ hold for $|\alpha| \leq n(p^{-1} - 1)$ and $p \leq 1$. For every $0 < p < q < \infty$ and $1/q = 1/p - (n-\gamma)/n$, there exists a constant A_{pq} such that*

$$||f * (|y|^{-\gamma})||_{H^q} \leq A_{pq} ||f||_{H^p}.$$

The same result holds for $q = \infty$, $p \leq 1$ and $0 < p = q < \infty$, Re $\gamma = n$.

The development of the complex Hardy space theory can be traced in [78] for $\mathbb{C}$ and in [13], [66], [70] for $\mathbb{C}^n$. The real theory in terms of maximal operators and Calderon-Zygmund decomposition can be found in [18]. Applications of Hardy spaces to several problems of the theory of singular integral operators appeared already in [23]. The general theory and applications can be found in [66], [70] and [68].

1.4 Overview of the regularity theory for the real phase

In this section we will give a brief informal overview of the regularity properties of Fourier integral operators. In the sequel by the continuity in L^p (or from L^p to L^p) we will always understand the continuity (of a linear operator) from L^p_{comp} to L^p_{loc}. Because L^p results may depend on the geometric structure of the wave front of the operator, we will discuss several related problems from the singularity theory of the wave fronts. Singularities arising in the L^p theory are a particular case of the singularities of affine fibrations, discussed in Chapters 2 and 3.

Let T be a Fourier integral operator with real phase. As in Section 1.1, this means that locally T is of the form

$$Tu(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^N} e^{i\Phi(x,y,\theta)} a(x,y,\theta) u(y) d\theta dy, \tag{1.4.1}$$

where $a \in S^\nu(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^N)$ is a symbol of order ν. The phase function Φ is smooth, real valued, non-degenerate, and positively homogeneous of degree one in θ. The Schwartz integral kernels of operators of the form (1.4.1) are Lagrangian distributions (or Fourier integral distributions). The wave front of the Lagrangian distribution of an operator T defines a geometric invariant for the operator T. Indeed, the set

$$WF(T) = \{(x, d_x\Phi(x,y,\theta), y, d_y\Phi(x,y,\theta)) : \, d_\theta\Phi(x,y,\theta) = 0\} \tag{1.4.2}$$

in the cotangent bundle $T^*(\mathbb{R}^n \times \mathbb{R}^m)$ does not depend on the choice of a phase function Φ. If $T^*(\mathbb{R}^n \times \mathbb{R}^m)$ is equipped with its standard symplectic form, then the conic set in (1.4.2) becomes a Lagrangian submanifold of the cotangent bundle $T^*(\mathbb{R}^n \times \mathbb{R}^m)$. The converse statement is one of the main results of the global theory of Fourier integral operators. It can be formulated more conveniently in the manifold setting. Let X and Y be real smooth manifolds of dimensions n and m, respectively, and *in this paper we will assume that* $n = m$. Let σ_X and σ_Y denote the canonical symplectic forms on T^*X and T^*Y, respectively. Let C be a conic Lagrangian submanifold of the cotangent bundle $T^*X\backslash 0 \times T^*Y\backslash 0$ equipped with the symplectic form $\sigma_X \oplus -\sigma_Y$. Then C defines a family of Fourier integral operators T with $WF(T)' = C$, locally of the form (1.4.1). The set C is called the canonical relation. If we fix an order ν of a symbol a in (1.4.1), the family of operators T is denoted by $I^\mu(X,Y;C)$, where $\mu = \nu + (N-n)/2$. Let us give now several important examples of Fourier integral operators. If we take $n = m = N$, $a = 1$, and

$$\Phi(x,y,\theta) = \langle x - y, \theta \rangle, \tag{1.4.3}$$

then the right hand side of (1.4.1) is a composition of the Fourier transform and its inverse in $\mathbb{R}^n$, and T is the identity operator in this case. If a phase function Φ of an operator T is given by (1.4.3) and its symbol a is polynomial in θ, then

T defines a partial differential operator with symbol a. If a phase function Φ of T is given by (1.4.3) and its symbol $a \in S^\nu$ is arbitrary, then T defines a pseudo-differential operator with symbol a. The space of pseudo-differential operators of order μ is denoted by Ψ^μ. The solution operator to the wave equation has the phase function of the form $\Phi(x,y,\theta) = \langle x-y,\theta\rangle + |\theta|$. In Section 1.5.3 we will give more examples of convolution operators and in Chapter 5 the solution operators to the Cauchy problem for hyperbolic equations will be regarded as Fourier integral operators as well.

In this monograph we will be interested in the continuity properties of the described operators in various function spaces. The best behavior is exhibited by pseudo-differential operators. Thus, pseudo-differential operators $P \in \Psi^0$ of zero order are continuous (as linear operators) from L^p to L^p for all $1 < p < \infty$. Moreover, a pseudo-differential operator $P \in \Psi^\mu$ of order $\mu \in \mathbb{R}$ can be extended to a continuous operator from the Sobolev space L^p_k to $L^p_{k-\mu}$ for all $k \in \mathbb{R}$, $k \geq \mu$, and $1 < p < \infty$. Similar results hold in Lipschitz spaces, operators $P \in \Psi^\mu$ are continuous from $\operatorname{Lip}(\gamma)$ to $\operatorname{Lip}(\gamma-\mu)$ for all $\gamma > \mu$. However, the phase function Φ for pseudo-differential operators is of the form (1.4.3) and its canonical relation C is equal to the conormal bundle to the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. For the Fourier integral operators the structure of the canonical relation C is much more complicated and their continuity properties depend on the geometric structure of the corresponding canonical relations.

Let $T \in I^0(X,Y;C)$ be a Fourier integral operator of zero order, associated to the canonical relation C. Let $\pi_{X\times Y}, \pi_X, \pi_Y$ be the canonical projections:

$$\begin{array}{ccccc} T^*X\backslash 0 & \xleftarrow{\pi_X} & C & \xrightarrow{\pi_Y} & T^*Y\backslash 0. \\ & & \Big\downarrow{\scriptstyle \pi_{X\times Y}} & & \\ & & X\times Y & & \end{array} \tag{1.4.4}$$

It turns out that the continuity properties of a Fourier integral operator T rely heavily on singularities of the projections $\pi_{X\times Y}, \pi_X, \pi_Y$. Projections π_X, π_Y can be diffeomorphic only simultaneously and in this case for every $\lambda_0 = (x_0,\xi_0,y_0,\eta_0) \in C$ there exists a symplectomorphism χ (a diffeomorphism preserving the symplectic structure) in a neighborhood of the point $(y_0,\eta_0) \in T^*Y\backslash 0$ such that in a neighborhood of λ_0, the canonical relation C has the form

$$\{(x,\xi,y,\eta) : (x,\xi) = \chi(y,\eta)\}. \tag{1.4.5}$$

In this case, C is locally equal to the graph of a canonical symplectic transformation and is called a local canonical graph or just a local graph. It is clear that C being a canonical graph implies that the projections π_X, π_Y are diffeomorphic from C to $T^*X\backslash 0$ and $T^*Y\backslash 0$, respectively. In particular, this implies $n = m$. The converse is also true. In fact, assume that, say, $\pi_Y : C \to T^*Y\backslash 0$ is a local diffeomorphism. Then, the canonical relation is locally of the form (1.4.5) in (y,η) coordinates and the condition of C to be Lagrangian for $\sigma_X \oplus -\sigma_Y$ implies that $\sigma_X \oplus -\sigma_Y$ vanishes on C and $\sigma_Y = \chi^*(\sigma_X)$. The latter means that χ^* is a symplectomorphism and C is a canonical graph.

In this monograph we will be interested in applications to the hyperbolic partial differential equations, where canonical relations are local canonical graphs. Such operators arise as solution operators of hyperbolic Cauchy problems. In this case, the mapping $\pi_X|_C \circ \pi_Y^{-1}$ is equal to χ in (1.4.5) and defines a local diffeomorphism from $T^*Y\backslash 0$ to $T^*X\backslash 0$. It follows that dimensions of X and Y coincide. Operators $T \in I^0(X,Y;C)$ with a local canonical graph C are continuous in L^2 ([16], [25], [27]). The proof is based on the fact that the canonical relation of $T \circ T^*$ is the conormal bundle of the diagonal in $X \times X$, and, therefore, it is a pseudo-differential operator of order 0 and hence bounded on L^2. In general, for $1 < p < \infty$, $p \neq 2$, Fourier integral operators of order zero need not be bounded on L^p.

From the point of view of the L^p continuity of Fourier integral operators, pseudo-differential operators and operators arising as solution operators to the wave equation are two opposite cases. The phase function of the latter has the form $\langle x-y, \xi\rangle + |\xi|$ in $\mathbb{R}^n$, and Littman ([34]) has shown that the corresponding operators $T \in I^\mu(X,Y;C)$ are not bounded in L^p when $\mu > -(n-1)|1/p - 1/2|$. The L^p properties of solutions to hyperbolic Cauchy problems for the equations of the wave type have been studied in many papers ([67], [44], [40], [3]). Lipschitz and L^p estimates for the wave equation on compact manifolds were derived in [8] and some results for hyperbolic equations are in [71]. General results on L^p continuity were obtained in [62]. Let us describe them in more detail. Operators $T \in I^\mu(X,Y;C)$ are bounded from L^p_{comp} to L^p_{loc} if $\mu \leq -(n-1)|1/p - 1/2|$ and this order is sharp if T is elliptic and $d\pi_{X\times Y}|_C$ has full rank, equal to $2n-1$, anywhere. These conditions hold for operators arising as solutions to strictly hyperbolic Cauchy problems in some cases. There is the same loss of order by $(n-1)|1/p - 1/2|$ in Sobolev and in Lipschitz spaces (in Lipschitz spaces $p = \infty$). If we denote $\alpha_p = (n-1)|1/p - 1/2|$, then operators $T \in I^\mu(X,Y;C)$ are continuous from L^p_α to $L^p_{\alpha-\alpha_p-\mu}$ and from Lip (α) to Lip $(\alpha - \alpha_\infty - \mu)$. The proof of the L^p boundedness is based on the complex interpolation method. Having the L^2 boundedness of zero order operators, the problem reduces to showing that operators of order $-(n-1)/2$ are locally bounded from the Hardy space H^1 to L^1.

1.5 Overview of the low rank conditions

1.5.1 Smooth factorization condition

In general, the projection $\pi_{X\times Y}$ satisfies inequalities

$$n \leq \text{ rank } d\pi_{X\times Y}|_C \leq 2n-1, \tag{1.5.1}$$

because C is conic. The boundary cases are pseudo-differential operators with no loss of smoothness and solutions to strictly hyperbolic partial differential equations with the loss of $(n-1)/2$ derivatives. An important ingredient influencing the best order for L^p continuity is the rank in (1.5.1). Here it is essential that $p \neq 2$ because the L^2 results do not depend on the rank in (1.5.1). Now, we will formulate the important result due to [62] for the operators in $I^\mu_\rho(X,Y;C)$, which assures the improved L^p regularity under the following condition. The canonical relation C will be said to satisfy *the smooth factorization condition*, if there exists k, $0 \le k \le n-1$, such that $\pi_{X\times Y}$ can be locally factored by fiber–preserving homogeneous maps on C of constant rank $n+k$. More precisely, this means that for every $\lambda_0 = (x_0, \xi_0, y_0, \eta_0) \in C$ there is a conic neighborhood U_{λ_0} of λ_0 in C, and a smooth map $\pi_{\lambda_0} : C \cap U_{\lambda_0} \to C$, homogeneous of degree 0, such that

$$\begin{aligned} \operatorname{rank} d\pi_{\lambda_0} &\equiv n+k, \\ \pi_{X\times Y}|_{C\cap U_{\lambda_0}} &= \pi_{X\times Y} \circ \pi_{\lambda_0}. \end{aligned} \tag{1.5.2}$$

Under the condition that C is a local graph and under the smooth factorization condition, the operators in $I^\mu_\rho(X,Y;C)$, $1/2 \le \rho \le 1$, are bounded from L^p_{comp} to L^p_{loc} provided $1 < p < \infty$ and $\mu \le (-k + (n-k)(1-\rho))|1/p - 1/2|$.

The smooth factorization condition is not necessary for operators $T \in I^\mu$ with $\mu \le -k|1/p - 1/2|$ to be continuous in L^p. Several examples show that the continuity is possible when the factorization condition fails ([59]).

The relaxation of the smooth factorization condition is an open problem. The best result would be to show that operators $T \in I^\mu(X,Y;C)$ are bounded from L^p_{comp} to L^p_{loc} provided $\mu \le -k|1/p-1/2|$, $1 < p < \infty$, and $\operatorname{rank} d\pi_{X\times Y}|_C \le n+k$. Note that the proof of this result for $k = n-1$ is based on the continuity result from H^1 to L^1 for operators of the order $-(n-1)/2$. One can assume that these operators are also weakly continuous in L^1. However, this problem is still open. The factorization condition is interesting in its own right and it allows fascinating generalizations, which we will discuss in Chapters 2 and 3.

The smooth factorization condition is trivially satisfied in two cases: pseudo-differential operators with $k = 0$ and the maximal rank case with $k = n-1$, where π_{λ_0} is the projection along the conical direction. It turns out, that under some natural conditions on the canonical relation C, corresponding to the most important cases, it is extremely difficult to exhibit the failure of the smooth factorization condition.

Now, we will describe the geometric meaning of this condition. On a smooth submanifold Σ^∞ of the set $\Sigma = \pi_{X\times Y}(C)$ the canonical relation can be found back as the conormal bundle $N^*\Sigma^\infty$ of Σ^∞. The set Σ is the singular support of the Schwartz integral kernel of the operator T. The conormal bundle $N^*\Sigma^\infty$ of $\Sigma^\infty \subset X \times Y$ is defined by

$$\begin{aligned} N^*\Sigma^\infty \;=\; &\{(x,\xi,y,\eta) \in T^*(X\times Y) :\ (x,y) \in \Sigma^\infty, \\ &\xi(\delta x) + \eta(\delta y) = 0,\ \forall(\delta x, \delta y) \in T_{(x,y)}\Sigma^\infty\}. \end{aligned} \tag{1.5.3}$$

Thus, the diagram

$$\begin{array}{ll} N^*\Sigma^\infty & \subset C \\ \quad\big\downarrow & \pi_{X\times Y} \\ \Sigma^\infty & \end{array} \tag{1.5.4}$$

defines a smooth local fibration over Σ^∞ with affine fibers. In these terms the factorization condition becomes equivalent to the condition that the smooth fibration of $N^*\Sigma^\infty$ allows a smooth extension to C. This smooth extension is defined by the levels of the mapping π_{λ_0}. In Chapters 2 and 3 we will analyze this property with different degrees of smoothness of fibrations. However, the theory becomes much more subtle if we assume that the described fibration over Σ^∞ is analytic. The analyticity assumption is quite natural and is almost always satisfied. In fact, if a critical point of a phase function has finite order of degeneracy, then the phase function is actually even a polynomial in a suitably chosen coordinate system in a neighborhood of the critical point. On the other hand, there are very few infinitely degenerate critical points, because the coefficients of the Taylor expansion of such phase function have to satisfy infinitely many independent algebraic equations. Another reason is that the study of Fourier integral operators is often reduced to the asymptotics of the oscillatory integrals corresponding to the associated Lagrangian distributions. However, the asymptotic behavior (as $\lambda \to \infty$) of the oscillatory integrals $\int e^{\lambda\phi}\psi$ depends essentially on the first terms of Taylor expansion, namely the power of the highest order term of the asymptotic expansion corresponds to the first non-vanishing term of the Taylor expansion ([9], [2]). In particular, the analyticity assumption is satisfied for the propagation operators of hyperbolic partial differential equations with analytic coefficients. The properties of the oscillatory integrals with analytic phase functions were studied in, for example, [2]. The corresponding geometric constructions can be found in [27], [14]. The singularities of wave fronts are analyzed in [1]. The smooth factorization condition is called *the holomorphic factorization condition* if all mappings in (1.5.2) are analytic (and hence holomorphic after the continuation to a complex domain). We will show, that under the analyticity assumption the holomorphic factorization condition holds in the most important cases. Some results of this type can be found in [52] and [53]. There are alternative presentations of Lagrangian distributions which make use of complex-valued phase functions. Fourier integral operators associated to such functions have been studied in [37] and global representations of Lagrangian distributions were obtained in [33]. A survey of the main theory is in [14].

The converse sharpness results for the wave-type equations can be found in [40], [44], [62] for the case when rank $d\pi_{X\times Y}|_C = 2n-1$ somewhere. For essentially homogeneous symbols (S^μ_ρ with $\rho = 1$) we will generalize the sharpness results to arbitrary ranks. We will show that the order $-k|1/p - 1/2|$ is sharp for all elliptic Fourier integral operators provided rank $d\pi_{X\times Y}|_C \le n + k$. As a consequence it follows that elliptic operators of small negative orders which are continuous in L^p or from L^p to L^q can be obtained as a composition of pseudo-differential operators with Fourier integral operators induced by a smooth coordinate change. Some results of this type appeared in [54] and we will describe them in Section 1.11. There, we will also mention the case of an arbitrary ρ where the sharpness of the orders for general elliptic operators under a rank restriction condition for the projection $\pi_{X\times Y}|_C$ is not settled in general. In Chapter 4 we will provide examples of the failure of the factorization condition in general and for the canonical relations corresponding to the translation invariant operators in $\mathbb{R}^n$.

Now we will briefly describe the smooth factorization condition in terms of the phase function of a Fourier integral operator. *From now on, we will replace the frequency variable θ by ξ in the cases when the dimensions of X, Y and the frequency space Ξ coincide.* By the equivalence-of-phase-function theorem (Theorem 1.1.3 below), we can assume that a phase function Φ of an operator $T \in I^\mu(X,Y;C)$ is of the form

$$\Phi(x,y;\xi) = \langle x,\xi\rangle - \phi(y,\xi).$$

Therefore, the corresponding wave front is given by

$$\Lambda_\Phi = \{(\nabla_\xi\phi(y,\xi),\xi,y,\nabla_y\phi(y,\xi))\}$$

and $C = \Lambda'_\Phi = \{(x,\xi,y,-\eta) : (x,\xi,y,\eta) \in \Lambda_\Phi\}$. The local graph condition is equivalent to

$$\det \phi''_{y\xi}(y,\xi) \neq 0 \tag{1.5.5}$$

on the support of the symbol of the operator T. The mapping

$$\gamma(y,\xi) = Y \times \Xi \ni (y,\xi) \mapsto (\nabla_\xi\phi(y,\xi),\xi,y,\nabla_y\phi(y,\xi)) \in T^*X \times T^*Y$$

defines a diffeomorphism from $Y \times \Xi$ to C. The level sets of the mapping $\pi_{X\times Y} : C \to X \times Y$ correspond to the kernels of the linear mapping $d\pi_{X\times Y}|_C$, or to the kernels of the mapping $d\pi_{X\times Y} \circ d\gamma$. It is straightforward that

$$\ker d\pi_{X\times Y} \circ d\gamma(y,\xi) = (0, \ker \frac{\partial^2\phi}{\partial\xi^2}(y,\xi)).$$

Therefore, fibration (1.5.5) reduces to the fibration defined by the kernels $\ker\phi''_{\xi\xi}(y,\xi)$, or by the level sets of the mapping $(y,\xi) \mapsto \nabla_\xi\phi(y,\xi)$ on the set where the rank of $\phi''_{\xi\xi}$ is maximal. A simple example of the failure of the factorization condition is already possible in $\mathbb{R}^3$. The fibers of the function

$$\phi(y,\xi) = \langle y,\xi\rangle + \frac{1}{\xi_3}(y_1\xi_1 + y_2\xi_2)^2, \tag{1.5.6}$$

(i.e. the level sets of $\nabla\phi(y,\xi)$ with respect to ξ) are straight lines with the slope equal to y_2/y_1. It is clear that the corresponding fibration is not continuous at zero. In the case when a phase function is real analytic we will concentrate on its complex extension and on the holomorphic mapping

$$(y,\xi) \mapsto \nabla_\xi\phi(y,\xi).$$

Let us discuss the corresponding singularity problem in terms of affine fibrations in more detail.

1.5.2 Parametric fibrations

First let us reduce the general manifold setting to open sets in $\mathbb{R}^n$. It does not restrict the generality since we are interested in local properties of operators. Note that sets X and Y have equal dimensions which follows from the condition that the canonical relation is a local graph. In general, the canonical relation of a Fourier integral operator in $T^*X \times T^*Y$ can be regarded as a smooth family of Lagrangian submanifolds of $T^*\mathbb{R}^n$ parameterized by points in $T^*\mathbb{R}^n$. Let us first show that from this point of view the ranks of the projection to the base space differ by n. It follows from the equivalence-of-phase-function theorem (Theorem 1.1.3) that a homogeneous canonical relation has the form

$$\Lambda = \{(\nabla_\xi \phi(y,\xi), \xi, y, \nabla_y \phi(y,\xi))\} \tag{1.5.7}$$

in a neighborhood of a point $(x_0, \xi_0, y_0, \eta_0)$. The generating function ϕ satisfies

$$\nabla_\xi \phi(y_0, \xi_0) = x_0, \nabla_y \phi(y_0, \xi_0) = \eta_0.$$

The phase function of an operator $T \in I^\mu_\rho(X, Y; \Lambda)$ has the form

$$\Phi(x, y, \xi) = \langle x, \xi \rangle - \phi(y, \xi) \tag{1.5.8}$$

and $\Lambda = \Lambda_\Phi$ is locally the collection of the points

$$\{(\nabla_\xi \phi(y,\xi), \xi, y, \nabla_y \phi(y,\xi))\}.$$

The set in (1.5.7) will be often denoted by Λ_Φ, with phase function Φ as in (1.5.8). Microlocally, the Schwartz kernel of operator T has the form

$$K(x,y) = \int_{\mathbb{R}^n} e^{i(\langle x,\xi\rangle - \phi(y,\xi))} b(x,y,\xi) d\xi \tag{1.5.9}$$

with some symbol $b \in S^\mu_\rho$. Therefore, locally, modulo smooth terms, the Schwartz kernels of operators in $I^\mu_\rho(X, Y; \Lambda')$ are finite sums of kernels of the form (1.5.9) with symbols $b \in S^\mu_\rho$. Each of the kernels has this form in its own coordinate system.

Therefore, we will assume that X and Y are open sets in $\mathbb{R}^n$. The local graph condition is equivalent to the condition

$$\det \phi''_{y\xi}(y,\xi) \neq 0 \tag{1.5.10}$$

on the support of the symbol of T. In this local coordinate system, the projection $\pi_{X\times Y}|_\Lambda$ takes the form

$$\pi_{X\times Y}|_\Lambda : (\nabla_\xi \phi(y,\xi), \xi, y, \nabla_y \phi(y,\xi)) \mapsto (\nabla_\xi \phi(y,\xi), y). \tag{1.5.11}$$

The level sets of the projection $\pi_{X\times Y}|_\Lambda$ can be parameterized by points ξ, for which the gradient $\nabla_\xi \phi(y,\xi)$ is constant. On the other hand, in view of Proposition 1.1.1 an open submanifold Λ_0 of the canonical relation Λ is equal to the conormal bundle $N^*\Sigma_0$ of a smooth submanifold Σ_0 of the set

$\Sigma = \pi_{X\times Y}(\Lambda)$. It follows that the level set of the projection $\pi_{X\times Y}|_\Lambda$ in Λ_0 at the point $(x_0, \xi_0, y_0, \eta_0)$ is a linear subspace of $T^*_{(x_0,y_0)}(X \times Y)$. By (1.5.11), these linear subspaces are parameterized by the pairs $(\xi, \nabla_y \phi(y,\xi))$. Therefore, points ξ in the level set of $\nabla_\xi \phi(y,\xi)$ form a linear subspace of the n–dimensional frequency space Ξ. It is clear how this subspace depends on (y,ξ):

Lemma 1.5.1. *The mapping $\gamma : Y \times \Xi \to T^*X \times T^*Y$ defined by*

$$\gamma(y,\xi) = (\nabla_\xi \phi(y,\xi), \xi, y, \nabla_y \phi(y,\xi))$$

is a diffeomorphism from $Y \times \Xi$ to Λ. For every $y \in Y$, the restriction $\xi \mapsto \gamma(y,\xi)$ is a diffeomorphism from Ξ to $\Lambda \cap (\mathbb{R}^n \times \mathbb{R}^n \times \{y\} \times \mathbb{R}^n)$, with the inverse given by the projection $(x,\xi,y,\eta) \mapsto \xi$.

Therefore, the linear mapping $d\pi_{X\times Y}|_\Lambda$ is isomorphic to $d\pi_{X\times Y} \circ d\gamma|_{Y\times\Xi}$ and, in particular, their kernels are isomorphic. The latter is

$$\ker d\pi_{X\times Y} \circ d\gamma|_{Y\times\Xi}(y,\xi) =$$
$$\{(\delta y, \delta\xi) : \tfrac{\partial^2\phi}{\partial\xi^2}(y,\xi)\delta\xi + \tfrac{\partial^2\phi}{\partial y\partial\xi}(y,\xi)\delta y = 0, \delta y = 0\}.$$

We obtain

$$\ker d\pi_{X\times Y} \circ d\gamma|_{Y\times\Xi}(y,\xi) = (0, \ker \frac{\partial^2\phi}{\partial\xi^2}(y,\xi)). \tag{1.5.12}$$

We proved the following characterization of the projection in terms of the phase function:

Theorem 1.5.2. *Let a local graph Λ be defined by the generating phase function*
$\Phi(x,y,\xi) = \langle x,\xi\rangle - \phi(y,\xi)$. Then, for every $0 \le k \le n-1$, the following statements are equivalent.

(1) rank $d\pi_{X\times Y}|_{\Lambda_\Phi} \le n + k$.

(2) rank $d\pi_{X\times Y} \circ d\gamma|_{\{y\}\times\Xi} \le k$ *for all $y \in Y$ and γ from Lemma 1.5.1.*

(3) rank $\frac{\partial^2\phi}{\partial\xi^2}(y,\xi) \le k$ *for all $y \in Y$ and $\xi \in \Xi$.*

Remark 1.5.3. By the linearity of the level sets of the projection $\pi_{X\times Y}|_\Lambda$, it follows that the equality

$$\pi^{-1}_{X\times Y}(\pi_{X\times Y}(\lambda)) \cap \Lambda = \ker d\pi_{X\times Y}|_\Lambda(\lambda)$$

holds locally at every $\lambda \in \Lambda$. On the other hand, if we use linearity of the level sets of the mapping $\nabla_\xi\phi(y,\xi)$ and the isomorphism in (1.5.12), we get that the level sets of the mapping

$$\xi \mapsto \nabla_\xi \phi(y,\xi)$$

are defined by the kernels of the matrix $D^2_{\xi\xi}\phi(y,\xi)$. The same conclusion follows if we apply the argument of Section 3 to the function ϕ directly.

The level sets of $\pi_{X\times Y}|_\Lambda$ correspond to the level sets of the gradient $\nabla_\xi\phi(y,\xi)$ with respect to ξ. In particular, they are disjoint and are extendible to an open set Ω in $\mathbb{C}^n$, provided that ϕ is analytic. Therefore, we have fibrations in $\Omega\cap\mathbb{R}^n$ and Ω by the level sets (with respect to ξ) of the gradient $\nabla_\xi\phi(y,\xi)$ and its holomorphic extension.

Remark 1.5.4. If the maximal rank of $d\pi_{X\times Y}|_\Lambda$ equals $n+k$, we denote by $\Lambda^{(k)}$ the set on which it is attained. In terms of fibrations, the fibration by linear subspaces in Remark 1.5.3 is also defined by the smooth mapping

$$\varkappa_{\mathbb{R}} : \gamma^{-1}(\Lambda^{(k)}) \ni (y,\xi) \mapsto \ker \frac{\partial^2\phi}{\partial\xi^2}(y,\xi) \in \mathbb{G}_{n-k}(\mathbb{R}^n). \tag{1.5.13}$$

The factorization condition (1.12.2) is equivalent to the condition that the mapping $\varkappa_{\mathbb{R}}\circ\gamma^{-1}$ is locally smoothly extendible from $\Lambda^{(k)}$ to Λ. This extension corresponds to the fibration by the kernels $\ker d\pi_{\lambda_0}$, where π_{λ_0} is as in (1.12.2). In terms of fibrations, the set of essential singularities of the mapping $\varkappa_{\mathbb{R}}$ is equal to $\Omega^{\rm sing}\cap(\mathbb{R}^n\times\mathbb{R}^n)$, where $\Omega^{\rm sing}$ is the set of essential singularities of the complex extension of the mapping $\varkappa_{\mathbb{R}}$.

The following example shows that the factorization condition is not trivial. In view of Remarks 1.5.3 and 1.5.4, by Theorem 1.5.2 the factorization condition reduces to the study of the inequality $\ker\phi''_{\xi\xi}(y,\xi)\leq k$. In this case we have $\operatorname{rank} d\pi_{X\times Y}|_{\Lambda_\Phi}\leq n+k$ by Theorem 1.5.2, (3). The function

$$\phi(y,\xi)=\langle y,\xi\rangle+\frac{1}{\xi_n}\sum_{i=2}^{k+1}(y_1\xi_1+y_i\xi_i)^2$$

satisfies the necessary rank condition in a neighborhood of $\xi_n=1$. On the other hand, we have

$$\nabla_\xi\phi(y,\xi)=y+\frac{2}{\xi_n}\begin{pmatrix} y_1\sum_{i=2}^{k+1}(y_1\xi_1+y_i\xi_i)\\ y_2(y_1\xi_1+y_2\xi_2)\\ \vdots\\ y_{k-1}(y_1\xi_1+y_{k-1}\xi_{k-1})\\ 0\\ \vdots\\ 0\\ \frac{1}{\xi_n}\sum_{i=2}^{k+1}(y_1\xi_1+y_i\xi_i)^2\end{pmatrix}. \tag{1.5.14}$$

For points y with $y_i\neq 0$, $1\leq i\leq k+1$, the level sets in (1.5.14) correspond to the level sets if the mappings $y_1\xi_1+y_i\xi_i$, $2\leq i\leq k+1$. The direction of the level sets is now determined by fractions y_i/y_1, $2\leq i\leq k+1$, which are not continuous at $y=0$. We obtain

Example 1.5.5. Let $1\leq k\leq n-2$ and let $x,y,\xi\in\mathbb{R}^n$. The function

$$\Phi(x,y,\xi)=\langle x-y,\xi\rangle-\frac{1}{\xi_n}\sum_{i=2}^{k+1}(y_1\xi_1+y_i\xi_i)^2$$

satisfies the rank condition

$$\text{rank } d\pi_{X\times Y}|_{\Lambda_\Phi} \leq n+k$$

and defines a local canonical graph Λ_Φ, for which the fibration by the level sets of the mapping $\pi_{X\times Y}|_{\Lambda_\Phi}$ does not allow a continuous extension over $y=0$.

Remark 1.5.6. In the case of $k=0$, the operators are conormal and their analysis can be reduced to the analysis of pseudo-differential operators by composing them with Fourier integral operators induced by a smooth coordinate transformation (see Section 1.11.2). For such operators, the factorization condition is trivially satisfied. The case of $k=n-1$ correspond to the inequality rank $d\pi_{X\times Y}|_\Lambda \leq 2n-1$, when the factorization condition is satisfied because Λ is conic. In this case, one can take π_{λ_0} in (1.12.2) to be the projection in the conic direction.

Remark 1.5.7. The failure of the factorization condition implies that the singular support Σ of an operator T can not be a smooth manifold. Indeed, if Σ is a smooth manifold, then $\Lambda = N^*\Sigma$ and the factorization condition holds. Conversely, the fact that Σ is not smooth does not imply the failure of the factorization condition. For example, let us take a semicubic parabola G in $\mathbb{R}^3$:

$$G = \{x \in \mathbb{R}^3 : x_1^2 = x_2^3, x_3 = 0\}$$

and define Σ by $\Sigma = \{(x,y) \in \mathbb{R}^6 : x-y \in G\}$. It is straightforward that a corresponding phase function Φ has the form

$$\Phi(x,y,\xi) = \langle x-y,\xi\rangle - \frac{4}{27}\frac{\xi_2^3}{\xi_1^2}.$$

It is clear that Σ is not smooth at zero. Meanwhile, the factorization condition holds for Φ, which will become clear in Section 4.2.

In Chapter 3, we will discuss fibrations in the real setting in the classes C^m. For now, we restrict ourselves to one example.

Example 1.5.8. For $k \in \mathbb{N}$ the function

$$\Phi(x,y,\xi) = \langle x-y,\xi\rangle - \frac{1}{\xi_n}(y_1^{k+1}y_2^{k+1}\xi_1 + (y_1^2+y_2^2)\xi_2)^2, \quad y \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n$$

satisfies condition $\text{rank} D_\xi^2\Phi \leq 1$ and defines a local canonical graph, for which the fibration by level sets of the function $\nabla_\xi\Phi$ is continuously extendible over $y=0$. Moreover, this extension is locally C^{2k-1}, but not C^{2k}.

1.5.3 Operators, commuting with translations

Here we want to give an intuitive argument for the L^p continuity under the factorization condition for the translation invariant operators. Let X, Y be open sets in $\mathbb{R}^n$. Let an operator $T \in I^\mu(X,Y;\Lambda')$ commute with translations.

It follows that T is a convolution operator with some distribution w. This distribution w is a Lagrangian distribution of order μ, associated to some real conic Lagrangian manifold $C \subset T^*X$. Let C satisfy the smooth factorization condition. This condition means that the rank of the projection $\pi : C \to \mathbb{R}^n$ does not exceed k, is equal to k at points in general position, and the projection π can be factored in a composition

$$\pi = \alpha \circ \beta,$$

where β is a fibration from Λ to some k-dimensional smooth manifold M and α is a smooth mapping from M to $\mathbb{R}^n$. The mapping α induces the pullback α^* from $C^\infty(\mathbb{R}^n)$ to $C^\infty(M)$ by $\alpha^* f = f \circ \alpha$ and the pushforward α_* from $(C^\infty)'(M)$ to $(C_0^\infty)'(\mathbb{R}^n)$ by $\alpha_* g = g \circ \alpha^*$. Let μ be a smooth positive compactly supported measure in M. Then the distribution $v = \alpha_* \mu$ defines a Fourier integral distribution with the Lagrangian manifold C, obtained from Λ by fixing an arbitrary point in T^*X. The pullback α_* maps measures in M to measure in $\mathbb{R}^n$, hence v is also a measure in $\mathbb{R}^n$. In order to see that $v = \alpha_* \mu$ defines a Fourier integral distribution we write v in some local coordinates z in M:

$$\begin{aligned} v(x) &= (2\pi)^{-n} \int\int e^{i\langle x-y,\xi\rangle} v(y) dy d\xi \\ &= (2\pi)^{-n} \int\int e^{i\langle x-y,\xi\rangle} (\alpha_*\mu)(y) dy d\xi \\ &= (2\pi)^{-n} \int\int \alpha_y^*(e^{i\langle x-y,\xi\rangle})(z)\mu(z) dz d\xi \\ &= (2\pi)^{-n} \int\int e^{i\langle x-\alpha(z),\xi\rangle} \mu(z) dz d\xi. \end{aligned}$$

As a candidate for the distribution w above we can take the Fourier integral distribution v. One readily checks that the order of v as a Fourier integral distribution is equal to $-k/2$. For the convolution operator with v we have the estimate

$$||Tf||_{L^p} = ||v * f||_{L^p} \leq v(\mathbb{R}^n)||f||_{L^p},$$

based on the translation invariance of the L^p norm. Letting p go to one we get the continuity of operators of order $-k/2$ in L^p spaces, which corresponds to the expected orders.

1.6 Fourier integral operators with complex phase functions

The theory of Fourier integral operators with complex valued phase functions was systematically developed in [37] and good expositions of the theory can be also found in [74], [14], [27]. An approach using positive conic ideals instead of almost analytic extensions is presented in Sections 25.4 and 25.5 in [27]. The L^2 continuity was established in [38] in the case when C is the graph of a positive complex canonical transformation and then was generalized in [26] to more general canonical relations. In the present chapter, under a local graph type condition we will establish L^p estimates for operators with complex valued phase functions. These results extend results of [62] to the complex case as well as results of [38] to the case of L^p with any $1 < p < \infty$. The result is sharp if the imaginary part of the phase function vanishes or is small (see [62], [65], [54] for the arguments in the real case). However, if it becomes large, the regularity properties can be improved and we will give some conditions for it as well.

Let us note here that the use of complex valued phase function is advantageous in many situations. It is well known that canonical relations of Fourier integral operators do not allow global parameterizations with a single real valued phase function due to the non-triviality of Maslov cohomology class. However, it was shown in [33] that such global parameterizations become possible if one allows phase functions to be complex valued. In this way our estimates allow to apply the regularity theory of Fourier integral operators to certain global problems.

Let $C \subset (T^*X\backslash 0) \times \widetilde{(T^*Y\backslash 0)}$ be a smooth complex positive homogeneous canonical relation, closed in $T^*(X \times Y)\backslash 0$. As usual, the tilde denotes an almost analytic extension of the corresponding space.

The L^2 continuity of operators in $I^0(X, Y; C)$ was established in [38] in the case when C is the graph of a positive complex canonical transformation and then was generalized in [26] to the following formulation. Assume that for every γ in the real subset $C_{\mathbb{R}}$ of C the maps from $(T_\gamma C)_{\mathbb{R}}$ to the tangent spaces of T^*X and T^*Y are injective. Then the operators in $I^0(X, Y; C)$ are L^2 continuous. Moreover, one can also assume that locally the maps $C_{\mathbb{R}} \to T^*X\backslash 0$ and $C_{\mathbb{R}} \to T^*Y\backslash 0$ from the real subset $C_{\mathbb{R}}$ are injective. In this case the operator $T \circ T^*$ belongs to $\Psi^0_{1/2}$, which imply the L^2 continuity of T.

The proof of L^p estimates in Theorem 1.7.1 consists of a standard complex interpolation argument between the L^2 continuity of zero order operators and the continuity from H^1 to L^1 of operators of order $-(n-1)/2$ in Theorem 1.7.2. In order to establish the latter it is sufficient to obtain uniform bounds for all atoms due to the atomic decomposition of H^1. If the support of an atom is large, the estimates follow easily from the Cauchy-Schwartz inequality. However, if the support is small, we need an additional almost orthogonality argument. Applying a refined Littlewood-Paley decomposition to the operators, it is possible to replace the phase function by its linear approximation on every dyadic region in the frequency space. Then we have to take into account the fact that the phase function is complex valued. It turns out that we can replace it by a non-degenerate real valued phase function in the integration by parts argument and still have the desired estimates for the kernels. The singular support of the integral kernel can also be replaced by the singular support of the corresponding operator with a real valued phase function and the exceptional set is constructed for this new operator. Let us note here that we can deduce certain L^p properties of operators by applying corresponding results for operators with real valued phase functions. Below we give an argument, from which it follows that operators with complex phase of order μ are continuous from L^p_{comp} to L^p_{loc}, provided $\mu \leq -(n-1/2)|1/p - 1/2|$. This estimate follows from the observation that we can always regard an operator with complex phase as an operator with real phase of the same order and of type $1/2$. However, we will see that the bound above is never sharp, even for elliptic operators. This is the reason why a more refined analysis is necessary, and the local graph type assumption (1.7.2) in Theorem 1.7.1 will be used to relate dyadic peaces of the operator to some operators with real phase, for which we can perform the integration by parts procedure.

By the standard duality argument we get the estimates in Lipschitz space. We also establish the continuity from L^2 to L^p by reducing the operators to fractional integrals. This result is applied to analyze the regularity properties of a parametrix in Section 5.5. We also derive L^p–L^q properties of Fourier integral operators with complex phase and these properties are in general sharp as well. In Section 1.7 we consider the case of I^μ with complex phases and in Section 1.12 we consider the general case I^μ_ρ, $1/2 \leq \rho 1$, and the smooth factorization type condition. In Section 1.10 we discuss L^p-L^q estimates as well as other spaces.

1.7 Estimates for operators with complex phases

Let X and Y be smooth manifolds of dimension n and let $C \subset (T^*X\backslash 0) \times \widetilde{(T^*Y\backslash 0)}$ be a complex positive homogeneous canonical relation. Let $T \in I^\mu(X, Y; C)$ be a Fourier integral operator of order μ. As usual, modulo C^∞ we can write T locally as an integral operator with the kernel

$$A(x,y) = \int_{\mathbb{R}^N} e^{i\Phi(x,y,\theta)} a(x,y,\theta) d\theta, \tag{1.7.1}$$

where a is a symbol of order $\mu + (n-N)/2$ and Φ is a smooth regular phase function of positive type. By the equivalence-of-phase-function theorem, we can always write the kernel of a Fourier integral operator T in the form (1.7.1) with $N = n$. Therefore, in this monograph without loss of generality we will always assume $N = n$. By the symbol class S^m in this section we will always mean the symbol class $S^m_{1,0}$ unless we state otherwise. The notation $a(x,y,\theta) \in S^m$ means that it is a smooth function, satisfying

$$|\partial^\alpha_{x,y} \partial^\beta_\theta a(x,y,\theta)| \leq C_{\alpha\beta} (1 + |\theta|)^{m-|\beta|}.$$

Let $C \subset (T^*X\backslash 0) \times \widetilde{(T^*Y\backslash 0)}$ be a smooth complex positive homogeneous canonical relation which is closed in $T^*\widetilde{(X \times Y)}\backslash 0$. Let Φ be a regular phase function of positive type, locally parameterizing C. Our main assumption will be that there exists $\tau \in \mathbb{R}$, such that

(L) $\operatorname{Re}\Phi + \tau \operatorname{Im}\Phi$ *defines a local graph in* $T^*(X \times Y)\backslash 0$.

Locally, this means that

$$\det \partial_x \partial_\theta (\operatorname{Re}\Phi + \tau \operatorname{Im}\Phi) \neq 0, \quad \det \partial_y \partial_\theta (\operatorname{Re}\Phi + \tau \operatorname{Im}\Phi) \neq 0 \tag{1.7.2}$$

on the support of the symbol a. Condition (1.7.2) is an analogue of the local graph condition. For example, if $\operatorname{Im}\Phi = 0$, it is equivalent to the local graph condition for C saying that the projections from $C_{\mathbb{R}}$ to T^*X and T^*Y are locally diffeomorphic. In general, as it will be clear from Lemma 1.8.1, assumption (1.7.2) with $\tau \in \mathbb{R}$ is equivalent to the same assumption with some $\tau \in \mathbb{C}$. Indeed, the determinants in (1.7.2) are polynomial in τ, which means that if (1.7.2) holds for some τ, it also holds for all but finitely many τ in $\mathbb{C}$. *Therefore, (L) is equivalent to the existence of $\tau \in \mathbb{C}$ such that (1.7.2) hold on the support of the symbol of a.*

We should also note that condition (1.7.2) is invariant in the following sense. Given an operator $T \in I^\mu(X,Y;C)$, according to [33], there exists a *single* complex valued phase function Φ, which parameterizes C globally. Such Φ does not depend on a particular choice of the coordinate system in (x,y,θ)-variables. Note finally, that in the general case with $N \geq n$ condition (1.7.2) should be replaced by condition that the rank of the matrix $\partial_x\partial_\theta(\text{ Re } \Phi + \tau \text{ Im } \Phi)$ equals n.

Theorem 1.7.1. *Let $C \subset (T^*X\backslash 0) \times (\widetilde{T^*Y\backslash 0})$ be a smooth complex positive homogeneous canonical relation which is closed in $T^*(\widetilde{X \times Y})\backslash 0$. Let Φ be a regular phase function of positive type, locally parameterizing C. Assume that condition (L) holds. Let $\mu \leq -(n-1)|1/p-1/2|$, $1 < p < \infty$. Then the operators in $I^\mu(X,Y;C)$ are continuous from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$.*

Note that the conditions of Theorem 1.7.1 imply the L^2 continuity of $I^0(X,Y;C)$ (as in [26, Theorem 3.5]), which we will freely use in the rest of the paper. Let us, however, give a more direct argument for L^2 continuity.

There is a simple argument showing certain L^p properties. It also shows that Theorem 1.7.1 does not not follow from its real valued counterpart. Using assumption (L), we can always write the operator $T \in I^\mu(X,Y;C)$ in the form

$$Tf(x) = \int e^{i(\text{ Re } \Phi+\tau \text{ Im } \Phi)} a(x,y,\theta) e^{-(1+i\tau) \text{ Im } \Phi} f(y) d\theta dy, \tag{1.7.3}$$

with real $\tau \in \mathbb{R}$ and with the phase function $\text{ Re } \Phi + \tau \text{ Im } \Phi$ defining a real canonical relation which is a local graph (condition (L)). The symbol of this operator is

$$a(x,y,\theta) e^{-(1+i\tau) \text{ Im } \Phi(x,y,\theta)}$$

and it belongs to $S^\mu_{1/2}$ if $a \in S^\mu$. This follows from the fact that $e^{-(1+i\tau) \text{ Im } \Phi} \in S^0_{1/2}$. See Lemma 5.3 in [38] for the detailed argument. Regarding T as an Fourier integral operator with real phase in $I^\mu_{1/2}$, we get that T is bounded on L^p when $\mu \leq -(n-1/2)|1/p-1/2|$, according to Section 1.5.1. However, this order is not as good as the one in Theorem 1.7.1. In order to improve this order in Theorem 1.7.1, we need to avoid the reduction to a real phase function and perform the almost orthogonality argument in the complex case. Thus, for the proof we will make a construction similar to the real case as in [62], making some necessary modifications. However, this argument already shows the L^2 boundedness of operators of order zero. Indeed, according to (1.7.3), we can view T in Theorem 1.7.1 as a Fourier integral operator with real phase, of order zero and type 1/2. It now follows from, for example [20], that T is bounded on L^2.

One can simplify the kernel (1.7.1) in the following way (see also the proof of Theorem 3.5 in [26]). Let $\gamma = (x_0, \xi_0, y_0, \eta_0) \in C_{\mathbb{R}}$ and let $\Phi(x, y, \theta)$ be a regular phase function of positive type at $(x_0, \xi_0, y_0, \theta_0)$. The corresponding canonical relation will be denoted by C_Φ. Then $\xi_0 = \Phi'_x(x_0, y_0, \theta_0)$ and $\eta_0 = -\Phi'_y(x_0, y_0, \theta_0)$ are real. Assume that ξ_0 and η_0 are nonzero and that the critical point of $\Phi(x, y, \theta) + \langle y, \eta\rangle$ as a function of y, θ is non-degenerate at (y_0, θ_0), $x = x_0$, $\eta = \eta_0$. Let $\phi(x, \eta)$ be the critical value of an almost analytic extension of $\Phi(x, y, \theta) + \langle y, \eta\rangle$. Thus means that $\phi(x, \eta) = \Phi(x, y, \theta) + \langle y, \eta\rangle$ at the points where $\partial\Phi/\partial y = -\eta$ and $\partial\Phi/\partial\theta = 0$. The almost analytic continuation of C_Φ is

$$\widetilde{C_\Phi} = \left\{ \left(x, \frac{\partial \Phi}{\partial x}, y, -\frac{\partial \Phi}{\partial y} \right),\ \frac{\partial \Phi}{\partial \theta} = 0 \right\} \tag{1.7.4}$$

and $\phi(x, \eta) - \langle y, \eta\rangle$ defines the complex canonical relation

$$\left\{ \left(x, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial \eta}, \eta \right) \right\}. \tag{1.7.5}$$

At the critical points $\partial\Phi/\partial\theta = 0$ the canonical relations (1.7.4) and (1.7.5) are equal. According to Lemma 2.1 of [37] the function ϕ is of positive type and by Theorem 4.2 of [37] for classical symbols the kernel (1.7.1) can be given by

$$A(x, y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\eta) - \langle y, \eta\rangle)} a(x, y, \eta)\, d\eta, \tag{1.7.6}$$

where $a \in S^\mu$ and ϕ is of positive type. Localizing by partitions of unity, may assume that the symbol a is compactly supported in x and y. By the complex interpolation techniques, we will show that Theorem 1.7.1 follows from the following endpoint result.

Theorem 1.7.2. *Let $C \subset (T^*X\backslash 0) \times \widetilde{(T^*Y\backslash 0)}$ be a smooth complex positive homogeneous canonical relation satisfying the conditions of Theorem 1.7.1. Let $T \in I^{-(n-1)/2}(X, Y; C)$ be an operator with the integral kernel locally given by (1.7.6). Then T is continuous from H^1 to L^1.*

Let us make some remarks about the sharpness of the orders above. In the case $\operatorname{Im}\Phi = 0$, the operator has a real phase function and the estimates in Theorem 1.7.1 are sharp if the rank of the canonical projection from $C \subset T^*(X \times Y)$ to $X \times Y$ has maximal rank equal to $2n-1$ at some point. However, when $\operatorname{Im}\Phi$ becomes strictly positive and large, we can make certain improvements using representation (1.7.3) of T. Thus, in general one can expect the improved estimates since the symbol in (1.7.3) contains an exponent to the negative power. For example, if $\operatorname{Im}\phi$ in (1.7.6) behaves like $|\eta|$, T regarded as an operator with real valued phase function, is a smoothing operator. In general, if $m \geq 0$, $1/2 \leq \rho \leq 1$ are such that

$$e^{-\operatorname{Im}\phi} \in S^{-m}_{\rho,1-\rho},$$

then T is a bounded operator from L^p_{comp} to L^p_{loc}, provided $\mu \leq -(n-\rho)|1/p-1/2|+m$, $1<p<\infty$. This follows easily from the real phase results. Indeed, according to Lemma 1.8.1 below, there is $0<\epsilon<1$, such that $\Psi = \text{Re } \Phi + \epsilon i \text{ Im } \Phi$ is a non-degenerate phase function of positive type and such that C_Ψ satisfies the conditions of Theorem 1.7.1. It follows, that we can regard T as a composition of an operator with phase function Ψ and symbol of order μ, with a pseudo-differential operator of order $-m$. Since this pseudo-differential operator is bounded from L^p to L^p_m, we obtain the statement.

1.8 A relation between real and complex phases

We begin with an observation relating our operators to the real Lagrangian distributions.

Lemma 1.8.1. *Let $\Lambda \subset \widetilde{T^*X}\backslash 0$ be a closed conic positive Lagrangian manifold. Let $\phi(x,\theta) \in C^\infty(\Gamma), \Gamma \subset X \times (\mathbb{R}^N\backslash 0)$, be a non-degenerate phase function which generates Λ locally near $(x_0,\theta_0) \in (C_\phi)_\mathbb{R}$. Then the matrix*

$$\text{Re } (\phi''_{\theta x}, \phi''_{\theta\theta}) + \mu \text{ Im } (\phi''_{\theta x}, \phi''_{\theta\theta}) \tag{1.8.1}$$

has rank N at (x_0,θ_0) for all $\mu \in \mathbb{C}$ in the complement of a discrete subset of $\mathbb{C}$.

Suppose, in addition, that $N \geq n$ and that there exists $\tau \in \mathbb{C}$ such that

$$\text{rank } (\text{ Re } \phi''_{x\eta} + \tau \text{ Im } \phi''_{x\eta}) = n.$$

Then the rank of

$$\text{Re } \phi''_{x\eta} + \mu \text{ Im } \phi''_{x\eta} \tag{1.8.2}$$

is maximal, equal to n, for all $\mu \in \mathbb{C}$ in the complement of a discrete subset of $\mathbb{C}$.

Proof. The fact that ϕ is non-degenerate means that the matrix

$$\text{Re } (\phi''_{\theta x}, \phi''_{\theta\theta}) + \mu \text{ Im } (\phi''_{\theta x}, \phi''_{\theta\theta})$$

has rank N at (x_0,θ_0) for $\mu = i$. The rank condition is an algebraic condition in μ which implies the statement. For the second part, the rank condition for (1.8.2) is algebraic and open, $n \leq N$, therefore we obtain the second statement.

Remark 1.8.2. *The argument of Lemma 1.8.1 can be used in order to show that there exists a real conic Lagrangian manifold Λ_0, containing $\Lambda_\mathbb{R}$, such that $I^\mu_\rho(X,\Lambda) \subset I^\mu_{1/2}(X,\Lambda_0)$ for every $\rho > 1/2$.*

In order to see it one finds a non-degenerate matrix in (1.8.1) with real τ and writes $u \in I^\mu_\rho(X,\Lambda)$ as

$$u(x) = \int e^{i(\text{ Re } \phi + \tau \text{ Im } \phi)} a(x,\theta) e^{-(1+i\tau) \text{ Im } \phi} d\theta. \tag{1.8.3}$$

Then the phase function in the first exponent in (1.8.3) defines a real Lagrangian manifold Λ_0, meanwhile

$$e^{-(1+i\tau)\,\mathrm{Im}\,\phi} \in S^0_{1/2}. \tag{1.8.4}$$

Let us make some observations, clarifying the symplectic aspects of the problem. Let $C \subset (T^*X\backslash 0) \times (\widetilde{T^*Y\backslash 0})$ be a smooth complex positive homogeneous canonical relation such that for every $\gamma \in C_{\mathbb{R}}$ the projections $(T_\gamma C)_{\mathbb{R}} \to T(T^*X)$ and $(T_\gamma C)_{\mathbb{R}} \to T(T^*Y)$ are bijective. It is sufficient to restrict ourselves to the linear case first. Let S_1, S_2 be real symplectic finite dimensional vector spaces with complexifications $S_{1\mathbb{C}}, S_{2\mathbb{C}}$, and with complexified symplectic forms σ_1 and σ_2. Let $\Lambda \subset S_{1\mathbb{C}} \oplus S_{2\mathbb{C}}$ be a linear positive canonical relation from S_2 to S_1. It means that Λ is a linear complex subspace of $S_{1\mathbb{C}} \oplus S_{2\mathbb{C}}$, Lagrangian with respect to the difference $\sigma_1 - \sigma_2$, where σ_1 and σ_2 are lifted to $S_1 \oplus S_2$, and such that

$$i^{-1}\left(\sigma_1(X,\bar{X}) - \sigma_2(Y,\bar{Y})\right) \geq 0,\ (X,Y) \in \Lambda.$$

Suppose that the projections of $\Lambda_{\mathbb{R}} = \Lambda \cap (S_1 \oplus S_2)$ to S_1 and S_2 are bijective. Let

$$\lambda_1 = \{X \in S_{1\mathbb{C}} :\ (X,0) \in \Lambda\},\ \lambda_2 = \{X \in S_{2\mathbb{C}} :\ (0,X) \in \Lambda\}, \tag{1.8.5}$$

let

$$S_{12} = \{\,\mathrm{Re}\, X : X \in \lambda_1\},\ S_{22} = \{\,\mathrm{Re}\, X : X \in \lambda_2\}$$

and let S_{11} and S_{21} be symplectically orthogonal complements of S_{12} and S_{22} in S_1 and S_2, respectively. Then we have $S_1 = S_{11} \oplus S_{12}$ and $S_2 = S_{21} \oplus S_{22}$. It follows from Lemma 3.3 in [26] that the decomposition

$$\Lambda = \Lambda_0 \oplus \lambda_1 \oplus \lambda_2$$

is unique, where λ_1 (λ_2) is a strictly positive (negative) Lagrangian plane in $S_{12\mathbb{C}}$ ($S_{12\mathbb{C}}$), respectively, and Λ_0 is the graph of a symplectic isomorphism from $S_{21\mathbb{C}}$ to $S_{11\mathbb{C}}$. The planes λ_1 and λ_2 in this decomposition are given by (1.8.5). If we assume that $\lambda_j = \lambda_{j\mathbb{R}} \oplus i\lambda_{j\mathbb{R}}$ for $j = 0$ or 1, we get $\Lambda_{\mathbb{R}} = \Lambda_{0\mathbb{R}} \oplus \lambda_{1\mathbb{R}} \oplus \lambda_{2\mathbb{R}}$. The set $\lambda_{1\mathbb{R}}$ consists of $X \in S_1$ with $(X,0) \in \Lambda$. However, in this case $(X,0) \in \Lambda_{\mathbb{R}}$ and hence $X = 0$. The same argument applies to λ_2 and we obtain that $\lambda_1 = \lambda_2 = \{0\}$ and $\Lambda = \Lambda_0$ is the graph of a symplectic isomorphism from $S_{2\mathbb{C}}$ to $S_{1\mathbb{C}}$. This would also imply that C is locally a graph of a symplectomorphism from $\widetilde{T^*X}$ to $\widetilde{T^*Y}$.

1.9 Proofs

Let us now prove Theorem 1.7.2. For a function $f \in H^1(\mathbb{R}^n)$ we will use its atomic decomposition

$$f = \sum_Q \alpha_Q a_Q,$$

similar to the one described in Section 1.3, where $\sum|\alpha_Q|$ is comparable to $||f||_{H_1}$ and the atoms $a_Q \in H^1$ satisfy

(1) Every a_Q is supported in the cube $Q \subset \mathbb{R}^n$.

(2) $||a_Q||_\infty \leq |Q|^{-1}$.

(3) The cancellation property $\int_Q a(y)dy = 0$.

Therefore it is sufficient to establish the uniform estimate $||Ta_Q||_{L^1} \leq C$ for all atoms a_Q with small cubes Q. In fact, if the sidelength of Q is larger than one, the Cauchy-Schwartz inequality and L^2 continuity of T imply

$$||Ta_Q||_{L^1} \leq C|Q|^{1/2}||Ta_Q||_{L^2} \leq C'|Q|^{1/2}||a_Q||_{L^2},$$

which is uniformly bounded by the second property of the atoms.

The idea of the proof for small cubes Q is to make a dyadic decomposition of T and to replace T by an operator with a complex linear phase function of positive type. For every such operator we make integration by parts replacing the complex phase function by a non-degenerate real phase function according to Lemma 1.8.1.

For every $\lambda = 2^j$ we consider the set η_λ^ν of points on the unit sphere $\mathbb{S}^{n-1}$, $|\eta_\lambda^\nu| = 1$, such that $|\eta_\lambda^\nu - \eta_\lambda^{\nu'}| \geq \lambda^{-1/2}$ if $\nu \neq \nu'$, and for every $\eta \in \mathbb{S}^{n-1}$ there exists a η_λ^ν with $|\eta - \eta_\lambda^\nu| \leq \lambda^{-1/2}$. Let $N(\lambda)$ be the maximal number of points η_λ^ν for a fixed λ. It it easy to see that

$$N(\lambda) = O(\lambda^{(n-1)/2}). \tag{1.9.1}$$

The points η_λ^ν define a set of roughly equally distributed directions in the phase space. Let

$$\Gamma_\lambda^\nu = \{\eta : |\eta/|\eta| - \eta_\lambda^\nu| \leq 2\lambda^{-1/2}\}$$

be the corresponding conical decomposition of the η-space. Let χ_λ^ν be an associated partition of unity (see [68, IX.4.4]). The functions χ_λ^ν are homogeneous of degree 0 in η such that

(i) Every χ_λ^ν is supported in Γ_λ^ν.

(ii) For all $\eta \neq 0$ and all λ holds $\sum_\nu \chi_\lambda^\nu(\eta) = 1$.

(iii) $|\partial_\eta^\alpha \chi_\lambda^\nu(\eta)| \leq A_\alpha \lambda^{|\alpha|/2}|\eta|^{-|\alpha|}$.

Such family can be constructed by taking a smooth non-negative function $\beta(u)$, supported in $|u| \leq 2$, and equal to one in $|u| \leq 1$. Then one defines

$$\beta_\lambda^\nu(\eta) = \beta(\lambda^{1/2}[\eta/|\eta| - \eta_\lambda^\nu])$$

and

$$\chi_\lambda^\nu = \beta_\lambda^\nu \left(\sum_\nu \beta_\lambda^\nu\right)^{-1}. \tag{1.9.2}$$

Let θ_λ be a dyadic decomposition in the phase space. Let θ be a smooth function supported in the interval $(1/4, 1)$ such that $\sum_l \theta(2^{-l}s) = 1$ for all $s > 0$. Let $\theta_\lambda(\eta) = \theta(\lambda^{-1}\eta)$. Let $\theta_0 = 1 - \sum_{l>0} \theta_{2^l}$.

Lemma 1.9.1. *Let $a^\nu_\lambda(x, y, \eta) = \chi^\nu_\lambda(\eta)\theta_\lambda(\eta)a(x, y, \eta)$. Define*

$$A^\nu_\lambda(x, y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\eta)-\langle y,\eta\rangle)} a^\nu_\lambda(x, y, \eta) d\eta.$$

Then for all $y, y' \in \mathbb{R}^n$ we have

$$\int |A^\nu_\lambda(x, y)| dx \leq C\lambda^{-(n-1)/2}, \quad \text{and} \tag{1.9.3}$$

$$\int |A^\nu_\lambda(x, y) - A^\nu_\lambda(x, y')| dx \leq C|y - y'|\lambda \cdot \lambda^{-(n-1)/2}. \tag{1.9.4}$$

Proof. By a rotation we assume that $\eta_1 = \eta^\nu_\lambda$ and $\eta' = (\eta_2, \ldots, \eta_n)$ is perpendicular to η^ν_λ. Denote

$$r(\eta) = \phi(x, \eta) - \langle \phi'_\eta(x, \eta^\nu_\lambda), \eta \rangle.$$

Then the original phase function is

$$\begin{aligned} \phi(x, \eta) - \langle y, \eta \rangle &= \langle \phi'_\eta(x, \eta^\nu_\lambda) - y, \eta \rangle (\phi(x, \eta) - \langle \phi'_\eta(x, \eta^\nu_\lambda), \eta \rangle) \\ &= \langle \phi'_\eta(x, \eta^\nu_\lambda) - y, \eta \rangle + r(\eta). \end{aligned}$$

We have the following estimates for $r(\eta)$:

$$\left| \left(\frac{\partial}{\partial \eta_1} \right)^N r(\eta) \right| \leq C_N \lambda^{-N}, \tag{1.9.5}$$

$$|(\nabla_{\eta'})^N r(\eta)| \leq C_N \lambda^{-N/2}, \tag{1.9.6}$$

if $N \geq 1$ and $\eta \in \operatorname{supp} a^\nu_\lambda(x, y, \eta)$. Estimates (1.9.5) and (1.9.6) hold for $\operatorname{Re} r$ and $\operatorname{Im} r$ in view of their homogeneity, since the first two terms in the Taylor expansion vanish at η^ν_λ. It follows then that (1.9.5), (1.9.6), also hold for $r(\eta)$.

We can rewrite A^ν_λ as

$$A^\nu_\lambda(x, y) = \int_{\mathbb{R}^n} e^{i\langle \phi'_\eta(x,\eta^\nu_\lambda)-y,\eta\rangle} b^\nu_\lambda(x, y, \eta) d\eta, \tag{1.9.7}$$

where $b^\nu_\lambda(x, y, \eta) = e^{ir(\eta)} \chi^\nu_\lambda(\eta)\theta_\lambda(\eta)a(x, y, \eta)$. Define the self adjoint operator

$$L^\nu_\lambda = \left(I - \lambda^2 \frac{\partial^2}{\partial \eta_1^2} \right) (I - \lambda \langle \nabla_{\eta'}, \nabla_{\eta'} \rangle). \tag{1.9.8}$$

The fact that $a \in S^{-(n-1)/2}$ and the choice of χ^ν_λ imply that

$$|(L^\nu_\lambda)^N a^\nu_\lambda(x,y,\eta)| \le C_N \lambda^{-(n-1)/2} \tag{1.9.9}$$

and in view of (1.9.5) and (1.9.6) the same estimate holds for $b^\nu_\lambda(x,y,\eta)$. Integration by parts implies

$$A^\nu_\lambda(x,y) = H^\nu_\lambda(x,y) \int_{\mathbb{R}^n} (L^\nu_\lambda)^N b^\nu_\lambda(x,y,\eta) e^{i\langle \phi'_\eta(x,\eta^\nu_\lambda)-y,\eta\rangle} d\eta, \tag{1.9.10}$$

where

$$H^\nu_\lambda(x,y) = \left(1+\lambda^2|(\phi'_\eta(x,\eta^\nu_\lambda)-y)_1|^2\right)^{-N} \left(1+\lambda|(\phi'_\eta(x,\eta^\nu_\lambda)-y)'|^2\right)^{-N}. \tag{1.9.11}$$

We will need the following

Lemma 1.9.2. *Let $z,\alpha,\beta \in \mathbb{R}$ and let $\mu \in \mathbb{R}$ be such that $|\mu| \le 1/4$. Then*

$$1+\lambda|z-\alpha-i\beta|^2 \ge 1/2(1+\lambda(z-\alpha-\mu\beta)^2)$$

for every $\lambda \ge 0$.

Proof. The case with $\mu = 0$ is easy. Assume that $\mu > 0$. We have

$$(z-\alpha-\mu\beta)^2 + (1+\mu^2)\beta^2 - |z-\alpha-i\beta|^2 = 2\mu\beta(\mu\beta+\alpha-z),$$

which implies

$$(z-\alpha-\mu\beta)^2 + (1+\mu^2)\beta^2 \le |z-\alpha-i\beta|^2 + 2\mu|\beta||z-\alpha-\mu\beta|.$$

It follows that

$$\begin{aligned} &1+\lambda|z-\alpha-i\beta|^2 \ge \\ &\quad 1+\lambda\left[(1+\mu^2)\beta^2 + (z-\alpha-\mu\beta)^2 - 2\mu|\beta||z-\alpha-\mu\beta|\right]. \end{aligned}$$

The statement would follow from the inequality

$$2\mu|\beta||z-\alpha-\mu\beta| \le \frac{1}{2}\left((1+\mu^2)\beta^2 + (z-\alpha-\mu\beta)^2\right). \tag{1.9.12}$$

Let's consider two cases. First, assume that $\mu|\beta| \le \frac{1}{4}|z-\alpha-\mu\beta|$. Then

$$2\mu|\beta||z-\alpha-\mu\beta| \le \frac{1}{2}(z-\alpha-\mu\beta)^2,$$

implying (1.9.12). On the other hand, if $\mu|\beta| \ge \frac{1}{4}|z-\alpha-\mu\beta|$, we get

$$\frac{(1+\mu^2)\beta^2}{2\mu|\beta|} \ge \frac{1+\mu^2}{2\mu^2}\mu|\beta| \ge \frac{1+\mu^2}{2\mu^2}\frac{1}{4}|z-\alpha-\mu\beta| \ge 2|z-\alpha-\mu\beta|$$

when $\mu \leq 1/4$. Thus, (1.9.12) follows. The case $\mu < 0$ is similar. The proof of the lemma is complete.

We now apply Lemma 1.9.2 to $(\phi'_\eta(x,\eta^\nu_\lambda) - y)$ with $\mu \leq 1/4$. It follows that

$$H^\nu_\lambda(x,y) \leq 2^{-2N} C_N (1 + \lambda^2 |(\text{ Re } \phi'_\eta(x,\eta^\nu_\lambda) + \mu \text{ Im } \phi'_\eta(x,\eta^\nu_\lambda) - y)_1|^2)^{-N} \tag{1.9.13}$$

$$\left(1 + \lambda |(\text{ Re } \phi'_\eta(x,\eta^\nu_\lambda) + \mu \text{ Im } \phi'_\eta(x,\eta^\nu_\lambda) - y)'|^2\right)^{-N}$$

for any $\mu \leq 1/4$. We fix $0 < \mu < 1/4$ for which the matrices in Lemma 1.8.1 have maximal ranks (we write μ rather than τ to emphasize that μ is real). Then the mapping

$$x \mapsto \text{ Re } \phi'_\eta(x,\eta^\nu_\lambda) + \mu \text{ Im } \phi'_\eta(x,\eta^\nu_\lambda)$$

is non-degenerate, implying that $\int |H^\nu_\lambda(x,y)| dx \leq C\lambda^{-(n+1)/2}$ if $N > n/2$. Since the support of $b^\nu_\lambda(x,y,\eta)$ has volume at most $\lambda \cdot \lambda^{(n-1)/2}$ and estimate (1.9.9) holds for b^ν_λ, the integral in (1.9.10) is bounded by λ. Estimate (1.9.3) now follows. Differentiation in y introduces a factor bounded by λ, so we get

$$\int |\nabla_y A^\nu_\lambda(x,y)| dx \leq C\lambda \cdot \lambda^{-(n-1)/2},$$

which implies (1.9.4).

In order to take into account the singular support of the operator, we define an exceptional set with respect to the real valued phase function $\psi = \text{ Re } \phi + \mu \text{ Im } \phi$ for a fixed $\mu \in \mathbb{R}$ as above. Let the atom a_Q be supported in the cube Q centered at y_0 and with sidelength δ. For every η^ν_λ define a rectangle

$$R^\nu_\lambda = \{x : |y_0 - \psi'_\eta(x,\eta^\nu_\lambda)| \leq c\lambda^{-1/2}, |\pi^\nu_\lambda(y_0 - \psi'_\eta(x,\eta^\nu_\lambda))| \leq c\lambda^{-1}\}, \tag{1.9.14}$$

where π^ν_λ is the orthogonal projection in the direction of η^ν_λ. Let $N_Q = \bigcup_\nu R^\nu_{\delta^{-1}}$. Then because the mapping $\psi'_\eta(x,\eta)$ has a non-vanishing Jacobian in x for every η, we get

$$|N_Q| \leq C_1 N(\delta^{-1})(\delta^{-1})^{-(n+1)/2} \leq C_2\delta.$$

Lemma 1.9.3. *Let Q and δ be as above. Then for every $y \in Q$ and $\lambda \geq \delta^{-1}$ holds*

$$\int_{\mathbb{R}^n \setminus N_Q} |A^\nu_\lambda(x,y)| dx \leq C\lambda^{-1}\delta^{-1}\lambda^{-(n-1)/2}. \tag{1.9.15}$$

Proof. We will argue similar to [68, IX.4.7] with a difference that we replace the phase function by its real modification. According to the choice of the set η^ν_λ, there exists a unit vector $\eta^\varkappa_{\delta^{-1}}$ such that $|\eta^\nu_\lambda - \eta^\varkappa_{\delta^{-1}}| \leq \delta^{1/2}$. Since $N_Q = \bigcup_\nu R^\nu_{\delta^{-1}}$, $x \notin N_Q$ implies $x \notin R^\varkappa_{\delta^{-1}}$, which in view of (1.9.14) means

$$\delta^{-1/2}|y_0 - \psi'_\eta(x,\eta^\varkappa_{\delta^{-1}})| + \delta^{-1}|\pi^\varkappa_{\delta^{-1}}(y_0 - \psi'_\eta(x,\eta^\varkappa_{\delta^{-1}}))| \geq c. \tag{1.9.16}$$

We have $|y-y_0|\le\delta$ because $y\in Q$, and since c can be assumed sufficiently large, applying triangle inequality to (1.9.16) and multiplying by $\lambda\delta$, we obtain

$$\lambda|(y-\psi'_\eta(x,\eta^\nu_\lambda))_1|+\lambda^{1/2}|(y-\psi'_\eta(x,\eta^\nu_\lambda))'|\ge\tilde{c}\lambda\delta \tag{1.9.17}$$

when $\lambda\ge\delta^{-1}$. Now we can perform the integration by parts argument as in (1.9.10) and insert (1.9.17) into (1.9.3). We use the estimates

$$(1+\alpha^2)^{-N}(1+\beta^2)^{-N}\le 2^{2N}(1+\alpha+\beta)^{-2N},\ \alpha,\beta\ge 0,$$

with $\alpha=\lambda|(y-\psi'_\eta(x,\eta^\nu_\lambda))_1|$ and $\beta=\lambda^{1/2}|(y-\psi'_\eta(x,\eta^\nu_\lambda))'|$, and

$$(1+\lambda|(y-\psi'_\eta(x,\eta^\nu_\lambda))_1|+\lambda^{1/2}|(y-\psi'_\eta(x,\eta^\nu_\lambda))'|)^{-1}\le C\lambda^{-1}\delta^{-1}$$

in order to get

$$\begin{aligned}H^\nu_\lambda(x,y)\le{}& C\lambda^{-1}\delta^{-1}(1+\lambda|(y-\psi'_\eta(x,\eta^\nu_\lambda))_1|\\ &+\ \lambda^{1/2}|(y-\psi'_\eta(x,\eta^\nu_\lambda))'|)^{1-2N},\ \lambda\ge\delta^{-1},\end{aligned} \tag{1.9.18}$$

from (1.9.11). Using inequality (1.9.10) and the fact that $\psi''_{x\eta}$ is non-degenerate, we get

$$\begin{aligned}&\textstyle\int_{\mathbb{R}^n\setminus N_Q}|A^\nu_\lambda(x,y)|dx\le\\ &C\lambda\cdot\lambda^{-1}\delta^{-1}\textstyle\int(1+\lambda^2|(y-x)_1|^2)^{1-2N}(1+\lambda|(y-x)'|^2)^{1-2N}dx.\end{aligned}$$

With $2N-1>n$ we obtain (1.9.15).

The final estimate in our preparation is

Lemma 1.9.4. *Let T be as in Theorem 1.7.2. Then the uniform estimate*

$$\int_{N_Q}|Ta_Q|dx\le C$$

holds for all cubes Q with sidelength bounded by one.

Proof. The proof is relied on the fact that the operator $T\circ(I-\Delta)^{(n-1)/4}$ belongs to $I^0(X,Y;C')$ and, therefore, is bounded on L^2,

$$||Ta_Q||_2\le C||(I-\Delta)^{-(n-1)/4}a_Q||_2. \tag{1.9.19}$$

The rest of the argument is standard. The Hardy-Littlewood-Sobolev inequality for fractional integrals yields

$$||(I-\Delta)^{-(n-1)/4}a_Q||_2\le C||a_Q||_{p_n}, \tag{1.9.20}$$

where $p_n=2n/(2n-1)$. By the second property of atoms we have $||a_Q||_{p_n}\le|Q|^{-1/2n}$. Applying the Hölder inequality to Ta_Q, (1.9.19) and (1.9.20), we obtain,

$$||Ta_Q||_{L^1(N_Q)}\le C|Q|^{1/2n}||a_Q||_{p_n}\le C.$$

End of the proof of Theorem 1.7.2. We set $\lambda = 2^l$ now. Let

$$A_{2^l}(x,y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\eta)-\langle y,\eta\rangle)} a_{2^l}(x,y,\eta)d\eta,$$

where $a_\lambda(x,y,\eta) = \theta_\lambda(\eta)a(x,y,\eta)$. Then $a_{2^l}(x,y,\eta) = \sum_\nu a^\nu_{2^l}(x,y,\eta)$ and because of (1.9.1) with $\lambda = 2^l$, the estimates of Lemma 1.9.1 and Lemma 1.9.3 imply

$$\int |A_{2^l}(x,y)|dx \le C, \; y \in \mathbb{R}^n, \tag{1.9.21}$$

$$\int |A_{2^l}(x,y) - A_{2^l}(x,y')|dx \le C|y-y'|2^l, \; y,y' \in \mathbb{R}^n, \tag{1.9.22}$$

$$\int_{\mathbb{R}^n\setminus N_Q} |A_{2^l}(x,y)|dx \le C2^{-l}\delta^{-1}, \; y \in B, \; 2^{-l} \le \delta. \tag{1.9.23}$$

Writing $T_{2^l}a(x) = \int A_{2^l}(x,y)a(y)dy$, we have a decomposition $Ta_Q = \Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{2^l \le \delta^{-1}} T_{2^l}a_Q; \; \Sigma_2 = \sum_{2^l > \delta^{-1}} T_{2^l}a_Q. \tag{1.9.24}$$

For the first sum we can use the cancellation property for the atom a_Q, $\int a(y)dy = 0$, to obtain $T_{2^l}a_Q(x) = \int_Q [A_{2^l}(x,y) - A_{2^l}(x,y_0)]a_Q(y)dy$. Using (1.9.22), we get

$$\int |T_{2^l}a_Q(x)|dx \le C2^l\delta||a_Q||_1,$$

and

$$\int_{\mathbb{R}^n} |\Sigma_1(x)|dx \le \sum_{2^l \le \delta^{-1}} \int_{\mathbb{R}^n} |T_{2^l}a_Q(x)|dx \le C \sum_{2^l \le \delta^{-1}} 2^l\delta \le C.$$

For the second sum we can use (1.9.23) to obtain

$$\begin{aligned} \int_{\mathbb{R}^n\setminus N_Q} |\Sigma_2(x)|dx &\le \textstyle\sum_{2^l > \delta^{-1}} \int_{\mathbb{R}^n\setminus N_Q} |T_{2^l}a_Q(x)|dx \\ &\le C\left(\textstyle\sum_{2^l > \delta^{-1}} 2^{-l}\delta^{-1}\right) ||a_Q||_1 \\ &\le C. \end{aligned}$$

The estimates for Σ_1 and Σ_2 yield $||Ta_Q||_{L^1(\mathbb{R}^n\setminus N_Q)} \le C$, which, together with Lemma 1.9.4, imply $\int_{\mathbb{R}^n} |Ta_Q(x)|dx \le C$ and the statement of Theorem 1.7.2.

Proof of Theorem 1.7.1. The adjoint operator T^* belongs to $I^0(Y,X;(\bar{C}^{-1})')$ and it is clear that $\bar{C}^{-1}$ satisfies the assumptions of Theorem 1.7.1 as well. Therefore we may assume that $1 < p < 2$ and for $2 < p < \infty$ take the adjoint operators. Now the statement follows by complex interpolation. In fact, one can define the analytic family of Fourier integral operators T_z with kernels

$$A_z(x,y) = \int_{\mathbb{R}^n} e^{i(\phi(x,\eta)-\langle y,\eta\rangle)} a(x,y,\eta)(1+|\eta|^2)^{z/2} d\eta.$$

The L^2 continuity of zero order order operators implies

$$||T_z f||_{L^2} \leq C_z ||f||_{L^2}, \quad \mathrm{Re}\, z = (n-1)(1/p-1/2),$$

meanwhile Theorem 1.7.2 means

$$||T_z f||_{L^1} \leq C_z ||f||_{H^1}, \quad \mathrm{Re}\, z = (n-1)(1/p-1).$$

The bounds C_z depend only on finitely many derivatives of symbols and hence the constants C_z have at most polynomial growth in $|z|$. The complex interpolation techniques of Proposition 1.3.3 imply the theorem.

Finally, for the sake of the factorization condition (F) in Section 1.12 let us prove a slightly stronger version of Lemma 1.9.2.

Lemma 1.9.5. *Let $z,\alpha,\beta \in \mathbb{R}$. Then for every $0 < \mu_0 < 1/\sqrt{3}$ there exists $\delta > 0$ such that for every $\mu \in \mathbb{R}$ with $|\mu| \leq \mu_0$ holds*

$$1 + \lambda|z-\alpha-i\beta|^2 \geq \delta(1+\lambda(z-\alpha-\mu\beta)^2)$$

for every $\lambda \geq 0$.

Proof. The case with $\mu = 0$ is easy. Assume that $\mu > 0$. Let us indicate the difference with the proof of Lemma 1.9.2. Let $\delta = 1 - 2\mu_0/\sqrt{1+\mu_0^2}$. The condition $0 < \mu_0 < 1/\sqrt{3}$ implies $1 > \delta > 0$. The statement of the Lemma would follow from

$$\begin{aligned} &1 + \lambda|z-\alpha-i\beta|^2 \geq \\ &\qquad 1 + \lambda\left[(1+\mu^2)\beta^2 + (z-\alpha-\mu\beta)^2 - 2\mu|\beta||z-\alpha-\mu\beta|\right] \end{aligned}$$

and from the inequality

$$2\mu|\beta||z-\alpha-\mu\beta| \leq (1-\delta)\left[(1+\mu^2)\beta^2 + (z-\alpha-\mu\beta)^2\right]. \tag{1.9.25}$$

The first inequality is the same as in Lemma 1.9.2. For the second one let us consider two cases. First, assume that $\mu|\beta| \leq \frac{1-\delta}{2}|z-\alpha-\mu\beta|$. Then

$$2\mu|\beta||z-\alpha-\mu\beta| \leq (1-\delta)(z-\alpha-\mu\beta)^2,$$

implying (1.9.25). On the other hand, if $\mu|\beta| \geq \frac{1-\delta}{2}|z-\alpha-\mu\beta|$, we get

$$\frac{(1+\mu^2)\beta^2}{2\mu|\beta|} \geq \frac{1+\mu^2}{2\mu^2}\mu|\beta| \geq \frac{1+\mu^2}{2\mu^2}\frac{1-\delta}{2}|z-\alpha-\mu\beta| \geq \frac{1}{1-\delta}|z-\alpha-\mu\beta|,$$

provided that $\frac{1+\mu^2}{2\mu^2}\frac{1-\delta}{2} \geq \frac{1}{1-\delta}$. This is equivalent to $(1-\delta)^2 \geq \frac{4\mu^2}{1+\mu^2}$, which is an increasing function of $\mu > 0$, but $(1-\delta)^2 \geq \frac{4\mu_0^2}{1+\mu_0^2}$ due to our choice of δ. Thus, (1.9.25) follows. The case $\mu < 0$ is similar. The proof of the lemma is complete.

1.10 Estimates in other spaces

As consequence of Theorem 1.7.1 we obtain estimates in other function spaces.

Corollary 1.10.1. *Let C be as in Theorem 1.7.1 and let $T \in I^0(X,Y;C)$. Then*

(i) *T is continuous from $(L^p_{\alpha+(n-1)/|1/p-1/2|})_{comp}(Y)$ to $(L^p_\alpha)_{loc}(X)$ for every $\alpha \in \mathbb{R}$, $1 < p < \infty$.*

(ii) *T is continuous from* $\mathrm{Lip}\,(\alpha + (n-1)/2)_{comp}(Y)$ *to* $\mathrm{Lip}\,(\alpha)_{loc}(X)$ *for every $\alpha \in \mathbb{R}$.*

(iii) *T is continuous from $(L^2_{\alpha+n(1/2-1/q)})_{comp}(Y)$ to $(L^q_\alpha)_{loc}(X)$ for every $\alpha \in \mathbb{R}$ and $2 \le q$.*

The first part follows from Theorem 1.7.1 and properties of pseudo-differential operators. The second part follows from Theorem 1.7.2 and the duality argument (see [62], [68]). For the third part we write

$$T = (I-\Delta)^{\mu/2} \circ (I-\Delta)^{-\mu/2} \circ T,$$

so T is continuous from L^2 to L^q when $(I-\Delta)^{\mu/2}$ is. According to Hardy-Littlewood-Sobolev theorem there is the loss of $n(1/2-1/q)$ derivatives (Proposition 1.3.4). Statement (iii) of Corollary 1.10.1 can be generalized to L^p–L^q continuity:

Theorem 1.10.2. *Let C be as in Theorem 1.7.1. Let $1 < p \le q \le 2$. Let $\mu \le 1/q - n/p + (n-1)/2$. Then the operators in $I^\mu(X,Y;C)$ are continuous from $L^p_{comp}(Y)$ to $L^q_{loc}(X)$. The dual statement holds for $2 \le p \le q < \infty$.*

Let us first note that the the statement of this Theorem is in general sharp. For example, if $\mathrm{Im}\,\Phi = 0$, then according to the remarks above, T satisfies the local graph condition. If $\mu > 1/q - n/p + (n-1)/2$, T is elliptic, and the rank of the canonical projection from the canonical relation to the base space equals $2n-1$, then there exists a function $f \in L^p_{comp}$ such that $Tf \notin L^q_{loc}$. Such f can be constructed uniformly for all T (see Section 1.11). The statement of Theorem 1.10.2 follows from Theorem 1.7.1. Indeed, the statement would follow by a standard complex interpolation argument between Theorem 1.7.1 and the fact that operators of order $-n/2$ are continuous from the Hardy space H^1 to L^2. To show the latter, let $S \in I^{-n/2}(X,Y;C)$ be a Fourier integral operator with complex phase of order $-n/2$. We can write S as

$$S = S \circ (I-\Delta)^{n/4} \circ (I-\Delta)^{-n/4},$$

which is a composition of $(I-\Delta)^{-n/4}$ with a Fourier integral operator of zero order, with complex phase satisfying assumptions of Theorem 1.7.1. This latter operator is therefore continuous in L^2, meanwhile $(I-\Delta)^{-n/4}$ is continuous from H^1 to L^2 by the Hardy space version of the Hardy-Littlewood-Sobolev theorem (see Proposition 1.3.5).

Let use now make some remarks concerning the case of real phase functions.

Remark 1.10.3. Let $T \in I^\mu$ be microlocally given by

$$Tf(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)} a(x,y,\xi) f(y)\, dy\, d\xi,$$

where Φ is real valued. Let $1 < p, q < \infty$. Then by interpolation the operator $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous provided $\mu \leq n(1/q - 1/p)$ and $p \leq 2 \leq q$. For $p = 1$ the statement holds if we replace $L^1(\mathbb{R}^n)$ by the Hardy space $H^1(\mathbb{R}^n)$.

However, in the case rank $D^2_{\xi\xi}\Phi = n - 1$ this statement can be improved. Let $1 < p, q < \infty$ and let p', q' denote their conjugates. Then operator T is continuous from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ provided

$$\mu \leq \frac{1}{q} - \frac{n}{p} + \frac{n-1}{2}, \quad q \leq p',$$
$$\mu \leq \frac{n}{q} - \frac{1}{p} - \frac{n-1}{2}, \quad q \geq p';$$

The statement extends to the case of $p = 1$ if we replace $L^1(\mathbb{R}^n)$ by the Hardy space $H^1(\mathbb{R}^n)$.

The proof of Remark 1.10.3 follows by the complex interpolation method between Theorem 1.7.1 and the fact that under conditions of Remark 1.10.3 the operator T is continuous from $H^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ provided $\mu = -(n+1)/2$. The sharpness of the bound for μ in Remark 1.10.3 is shown in [68, IX.6.16] for operators with the phase function $\Phi(x,y,\xi) = \langle x - y, \xi \rangle + |\xi|$. See also Section 1.11.

Remark 1.10.4. It follows from Proposition 1.3.5 with $p = 1$ and $q = 2$ that operators of order $-n/2$ are continuous from H^1 to L^2.

As a consequence, we get a statement for pseudo-differential operators.

Proposition 1.10.5. *Let $P \in \Psi^\mu(X)$ and let $1 < p \leq q < \infty$. Then $P : L^p_{comp}(X) \to L^q_{loc}(X)$ is continuous, when $\mu \leq -n(1/p - 1/q)$.*

Proof. Let us give a direct proof without using the interpolation technique. The case $p = q$ is well known and it follows, for example, from Theorem 1.12.1 with $k = 0$ and $\mu \leq 0$. Let us assume now that $p < q$. The operator $(I - \Delta)^{-\mu/2} \circ P$ is continuous in L^p and the statement reduces to the properties of the operator $(I - \Delta)^{\mu/2}$. Its principal symbol is homogeneous of degree μ and its integral kernel has degree $-n - \mu$ in $|x - y|$. Therefore, the operator $(I - \Delta)^{\mu/2}$ is of the form

$$I_\gamma f(x) = \int_{\mathbb{R}^n} |x - y|^{-\gamma} f(y)\, dy$$

with $\gamma = n + \mu$. For $\mu < 0$, it follows from Proposition 1.3.4 that I_γ is continuous from L^p to L^q provided $1/q = 1/p + \mu/n$. This completes the proof of Proposition 1.10.5.

1.11 Sharpness of the estimates

In this section we will discuss the sharpness of the orders of L^p-bounded Fourier integral operators in $I^\mu(X,Y;C)$ with real phases for different ranks of the projection $d\pi_{X\times Y}|_C$. We will show that the orders for the general L^p–L^q continuity depend on the rank of $d\pi_{X\times Y}|_C$ and that the orders derived in this chapter are sharp. For the question of the sharpness of the orders it is sufficient to consider operators with real phase functions.

Let X and Y be smooth paracompact n–dimensional manifolds and let $T \in I^\mu_\rho(X,Y;C)$ be a Fourier integral operator with a real canonical relation C. In the sequel we will concentrate on the essentially homogeneous case ($\rho = 1$). However, for $1/2 \le \rho < 1$, the sharpness of the order $\mu = -(n-\rho)|1/p - 1/2|$ for the continuity in L^p can be shown by the following example. Let Σ be a manifold given by the set of points (x', x'', y', y'') with $x' = y'$ and $|x''-y''| = 1$. Let C be the conormal bundle $N^*\Sigma$ of Σ. Let $\Psi(\xi)$ be a smooth function, homogeneous of degree zero for large ξ, supported in the truncated cone

$$\{\xi = (\xi', \xi'') \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1} : |\xi'| \ge 1, |\xi''|/2 \le |\xi'| \le 2|\xi''|\},$$

and equal to one in an open truncated subcone. Let T be a convolution operator with the kernel

$$K(x,y) = \int \Psi(\xi) e^{i|\xi|^{1-\rho}} |\xi|^\mu e^{i[\langle x-y,\xi\rangle + |\xi'|]} d\xi.$$

One can readily check that $T \in I^\mu_\rho(\mathbb{R}^n, \mathbb{R}^n; C)$. Let $\hat{g}_\sigma(\xi) = (1+|\xi|^2)^{-\sigma/2}$. Then $g_\sigma \in L^p$ for $\sigma > n(1-1/p)$. Acting by T on g_σ and applying the stationary phase method in polar coordinates for ξ', one can show that $Tg_\sigma \notin L^p_{loc}$ for σ converging to $n(1-1/p)$, and $\mu > -(n-\rho)|1/p-1/2|$. Asymptotic expansions for such operators were derived in [77] and the analysis was used in [62] to show that the order μ is sharp in Theorem 1.12.1.

1.11.1 Essentially homogeneous case

In the essentially homogeneous case with $\rho = 1$ one can show that the order $\mu = -k|1/p - 1/2|$ is sharp for arbitrary L^p continuous elliptic operators with real phases satisfying the condition

$$\operatorname{rank} d\pi_{X\times Y}|_C \le n + k. \tag{1.11.1}$$

It turns out that in order to check that an operator is not continuous in L^p it suffices to let T act on functions f with point singularities in L^p. In this case the singularities of Tf can appear only in directions transversal to some fixed k–dimensional manifold in X.

Theorem 1.11.1. *Let the real canonical relation C be a local canonical graph such that the inequality* $\operatorname{rank} d\pi_{X\times Y}|_C \le n+k$ *holds with $0 \le k \le n-1$, and the rank $n+k$ is attained somewhere. Then elliptic operators $T \in I^\mu(X,Y;C)$ are not bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$, provided $\mu > -k|1/p - 1/2|$ and $1 < p < \infty$.*

Let us note at once that by the equivalence-of-phase-function theorem it is sufficient to consider operators T in $\mathbb{R}^n$ with kernels, locally given by

$$K(x,y) = \int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \phi(y,\xi)]} b(x,y,\xi)d\xi, \tag{1.11.2}$$

with symbols $b \in S^\mu$ supported in compact sets with respect to x and y. The local graph condition means that the phase function ϕ satisfies

$$\det \phi''_{y\xi} \neq 0 \tag{1.11.3}$$

on the support of b. Locally Λ has the form $\{(\nabla_\xi \phi, \xi, y, \nabla_y \phi)\}$. We can assume that $1 < p \leq 2$. For $2 < p$ it suffices to consider the adjoint operator T^* and the statement follows from the result for the conjugate index $p' = p/(p-1) < 2$.

The set $C_0 = \{\lambda \in C : \operatorname{rank} d\pi_{X\times Y}|_C(\lambda) = n + k\}$ is not empty and is open in C. Let $\lambda_0 = (x_0, \xi_0, y_0, \eta_0) \in C_0$. Consider a family of functions $f_s(y) = ((I-\Delta)^{-s/2}\delta_{y_0})(y)$ for a fixed value of $y_0 \in Y$. Let K_{-s} be the integral kernel of $(I-\Delta)^{-s/2}$. Then we have

$$f_s(y) = \int K_{-s}(y,z)\delta_{y_0}(z)dz = K_{-s}(y,y_0).$$

Using standard estimates for Schwartz kernels of pseudo-differential operators ([64], [11], [68]), we get that $|K_{-s}(y,y_0)| \leq C|y-y_0|^{-n+s}$ in some local coordinate system. This means that $f_s \in L^p_{loc}$ if and only if $s > n(1-1/p)$.

Let $\Sigma = \pi_{X\times Y}(C \cap U)$, where $U \subset C_0$ is a neighborhood of λ_0. Because the rank of $\pi_{X\times Y}$ is constant in U, the set $\Sigma \subset X \times Y$ is a smooth k-dimensional manifold, given by equations $h_j(x,y) = 0$, $1 \leq j \leq n-k$, in a neighborhood of y_0. The vectors $\nabla h_1, \dots, \nabla h_{n-k}$ are linearly independent on Σ. Then, microlocally, C is the conormal bundle of Σ, and the phase function of the operator T assumes the form

$$\Phi(x,y,\lambda) = \sum_{j=1}^{n-k} \lambda_j h_j(x,y).$$

Since a composition with a pseudo-differential operator does not change the canonical relation, we get that $T \circ (I-\Delta)^{-s/2} \in I^{\mu-s}(X,Y;C)$. It follows that $Tf_s(x) = T \circ (I-\Delta)^{-s/2}(\delta_{y_0})(x)$, which in local coordinates can be written as

$$\begin{aligned} Tf_s(x) &= \textstyle\int_{\mathbb{R}^n}(\int_{\mathbb{R}^{n-k}} e^{i\sum \lambda_j h_j(x,y)} a(x,\bar\lambda)\delta_{y_0}(y)d\bar\lambda)dy \\ &= \textstyle\int_{\mathbb{R}^{n-k}} e^{i\langle \bar\lambda, \bar h(x,y_0)\rangle} a(x,\bar\lambda)d\bar\lambda \\ &= (2\pi)^{n-k}(\mathcal{F}^{-1}a)(x, \bar h(x,y_0)), \end{aligned} \tag{1.11.4}$$

where $\bar\lambda$ and $\bar h$ are vectors with components λ_j and h_j, respectively, and $\mathcal{F}^{-1}$ is the inverse Fourier transform. The symbol $a \in S^{\mu-s+k/2}(\mathbb{R}^{n-k})$ is obtained from the symbol of the operator $T \circ (I-\Delta)^{-s/2}$ by using the stationary phase method, where we can eliminate k variables. The inverse Fourier transform of the symbol a with respect to the second variable is

$$(2\pi)^{n-k}(\mathcal{F}^{-1}a)(x,\zeta) = \int_{\mathbb{R}^{n-k}} e^{i\langle\lambda,\zeta\rangle} a(x,\lambda)\hat{\delta}_0(\lambda)d\lambda = P\delta_0(\zeta) = K(\zeta,0),$$

where $P \in \Psi^m(\mathbb{R}^{n-k})$ is a pseudo-differential operator of order $m = n-s+k/2$ with symbol $a(x,\lambda) \in S^m$ and K is the integral kernel of P. The function $K(\zeta,0)$ is equivalent to $|\zeta|^{-(n-k)-m}$. For the set $\Sigma_{y_0} = \{x : (x,y_0) \in \Sigma\}$ we have $\mathrm{dist}(x,\Sigma_{y_0}) \approx |\bar{h}(x,y_0)|$, and, therefore,

$$(2\pi)^{n-k}(\mathcal{F}^{-1}a)(x,\bar{h}(x,y_0)) \sim |\mathrm{dist}(x,\Sigma_{y_0})|^{-(n-k)-(\mu-s+k/2)},$$

locally uniformly in x. It follows from (1.11.4) that the function Tf_s is smooth in Σ_{y_0}. Hence $Tf_s \notin L^p_{loc}(\mathbb{R}^n)$ if and only if

$$p(n-k+\mu-s+k/2) \geq n-k. \tag{1.11.5}$$

For s it means that $s \leq \mu + (n-k)(1-1/p) + k/2$. Thus, in order to have $f_s \in L^p_{loc}$ and $Tf_s \notin L^p_{loc}$, it is sufficient to have the inequality

$$n(1-1/p) < \mu + (n-k)(1-1/p) + k/2,$$

which is equivalent to $\mu > -k|1/p - 1/2|$.

Concerning $L^p - L^q$ continuity we have:

Corollary 1.11.2. *Let T and C satisfy conditions of Theorem 1.11.1 and let $1 < p,q < \infty$. Then T is not bounded from $L^p_{comp}(Y)$ to $L^q_{loc}(X)$, provided $\mu > n(1/q - 1/p) - k(1/q - 1/2)$.*

To prove this we can apply the same argument as in the proof of Theorem 1.11.1. The only difference is that now we do not need to assume $p \leq 2$ and inequality (1.11.5) is replaced by

$$q(n-k+\mu-s+k/2) \geq n-k.$$

Remark 1.11.3. In the case when $k = n-1$ and $1 < p \leq q \leq 2$, the statement of Corollary 1.11.2 complements the statement of Remark 1.10.3.

By duality we also have a statement, analogous to the one in Theorem 1.11.6, for indices $2 < p \leq q < \infty$.

Remark 1.11.4. The operator T in Theorem 1.11.1 is not bounded as a linear operator in Sobolev spaces $L^p_\alpha \to L^p_{\alpha-k|1/p-1/2|-\mu}$, $1 < p < \infty$.

1.11.2 A representation formula for continuous operators of small negative orders

According to Proposition 1.10.5 pseudo-differential operators of zero order are continuous in L^p for $1 < p < \infty$. Now we will show that all L^p continuous elliptic Fourier integral operators with real phases can be obtained from pseudo-differential operators by a composition with operators, induced by a smooth coordinate change. A smooth mapping $\sigma : X \to Y$ induces the pullback $\sigma^* : C^\infty(Y) \to C^\infty(X)$, defined by $(\sigma^* f)(x) = f(\sigma(x))$. It is not difficult to see that σ^* is a Fourier integral operator with phase function $\langle \sigma(x) - y, \eta\rangle$. The canonical relation of σ^* is equal to the graph of the mapping $\widetilde{\sigma} : T^*X\backslash 0 \to T^*Y\backslash 0$, where $\widetilde{\sigma}(x,\xi) = (\sigma(x), -({}^t D\sigma_x)^{-1}(\xi))$.

Theorem 1.11.5. *Let $1 < p < \infty$, $p \neq 2$, and $0 \geq \mu > -|1/p - 1/2|$. Let C be a real local canonical graph. Then an elliptic operator $T \in I^\mu(X,Y;C)$ is continuous from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$ if and only if there exist pseudo-differential operators $P \in \Psi^\mu(X), Q \in \Psi^\mu(Y)$ such that $T = P \circ \sigma^* = \sigma^* \circ Q$, where σ^* is the pullback by a smooth coordinate change from X to Y.*

Proof. The pullback σ^* is continuous in L^p. Pseudo differential operators P and Q of order $\mu \leq 0$ are continuous in L^p. Therefore, T is also continuous in L^p. Conversely, let k be such that $n + k = \max_{\lambda \in C}$ rank $d\pi_{X\times Y}|_C(\lambda)$. Then $n - k$ is the maximal dimension of the set $\Sigma = \pi_{X\times Y}(C) \subset X \times Y$. In view of Theorem 1.11.1 and L^p continuity of T, it is necessary that $k = 0$. This means that rank $d\pi_{X\times Y}|_C \equiv n$ and Σ is a smooth n–dimensional submanifold of $X \times Y$. The rank of the differential $d\pi_X|_\Sigma$ of the projection $\pi_X : X \times Y \to X$ is equal to n because C is a local canonical graph. Surjectivity of $d\pi_X|_\Sigma$ and condition $\dim \Sigma = n$ imply that $\pi_X|_\Sigma$ is a diffeomorphism and Σ can be locally parameterized by $\Sigma = \{(x, \sigma(x))\}$, for some diffeomorphism $\sigma : X \to Y$. It follows that the canonical relation of the pullback σ^* is the conormal bundle of Σ, which is C'. Therefore, an operator Q in $T = \sigma^*_+ \circ Q$ must be pseudo-differential, since its canonical relation is the conormal bundle to the diagonal in $X \times Y$. A similar argument in Y implies that the formula in the theorem holds for an operator P with the same mapping σ, since the canonical relation of σ^* equals C'.

Theorem 1.11.6. *Let $1 < p \leq q < 2$ and $-n(1/p - 1/q) \geq \mu > -(1/q - 1/2) - n(1/p - 1/q)$. Let $T \in I^\mu(X,Y;C)$ be elliptic and let C be a real local canonical graph. Then T is continuous from $L^p_{comp}(Y)$ to $L^q_{loc}(X)$ if and only if there exist pseudo-differential operators $P \in \Psi^\mu(X), Q \in \Psi^\mu(Y)$, such that $T = P \circ \sigma^* = \sigma^* \circ Q$, where σ^* is the pullback by some diffeomorphism $\sigma : X \to Y$.*

The continuity of T follows from the continuity from L^p to L^q of pseudo-differential operators of order $-n(1/p - 1/q)$ (Proposition 1.10.5).

Note also that Corollary 1.11.2 with $k = n$ implies the sharpness of the orders in Proposition 1.10.5 for elliptic operators P.

1.12 Smooth factorization condition for complex phases

In this section we discuss an analogue of the smooth factorization condition for complex phase functions. The results of this section were announced in [61].

As we mentioned in Section 1.5, the L^p estimates can be improved if the rank $d\pi_{X\times Y}$ restricted to the real wave front of the operator does not attain $2n-1$. The exact statement is in the beginning of Subsection 1.5.1.

The similar situation happens in the case of complex phase functions. Let us recall first the factorization condition for the real non-degenerate phase function $\Psi(x,y,\theta)$. Let Λ_Ψ be locally defined by

$$\begin{aligned}\Lambda_\Psi &= \{(x,y,d_x\Psi(x,y,\theta),d_y\Psi(x,y,\theta)) : d_\theta\Psi(x,y,\theta)=0\}\\ &\subset T^*(X\times Y).\end{aligned} \tag{1.12.1}$$

The smooth factorization condition for Ψ can be formulated as follows. Suppose that there exists a number k, $0\leq k\leq n-1$, such that for every $\lambda_0=(x_0,\xi_0,y_0,\eta_0)\in\Lambda_\Psi$, there exist a conic neighborhood $U_{\lambda_0}\subset\Lambda_\Psi$ of λ_0 and a smooth homogeneous of zero order map $\pi_{\lambda_0}:\Lambda_\Psi\cap U_{\lambda_0}\to\Lambda_\Psi$ with constant rank rank $d\pi_{\lambda_0}=n+k$, for which holds

$$\pi_{X\times Y}|_{\Lambda_\Psi}=\pi_{X\times Y}|_{\Lambda_\Psi}\circ\pi_{\lambda_0}. \tag{1.12.2}$$

Recall, that under this assumption, operators $T\in I^\mu_\rho(X,Y;\Lambda_\Psi)$ with $1/2\leq\rho\leq1$ are L^p-bounded if $1<p<\infty$ and $\mu\leq-(k+(n-k)(1-\rho))|1/p-1/2|$ (see Section 1.5.1).

Let now Φ be a non-degenerate complex phase function. Let Φ satisfy the local graph type condition (L) of Section 1.7 for some $\tau\in\mathbb{R}$. Recall that it then satisfies (L) for all but finitely many $\tau\in\mathbb{R}$ and it satisfies (1.7.2) for all but finitely many $\tau\in\mathbb{C}$. Our *smooth factorization type condition* (F) *for* Φ will be the following condition

(F) *There exists a real $|\tau|<1/\sqrt{3}$ such that condition* (L) *holds with this τ and, moreover, the real phase function $\Psi=\operatorname{Re}\Phi+\tau\operatorname{Im}\Phi$ satisfies the real smooth factorization condition (1.12.2) with some k, $0\leq k\leq n-1$.*

Note that if the phase function Φ is real, condition (F) is just the smooth factorization condition for the real valued phase function. The reason to impose condition $|\tau|<1/\sqrt{3}$ is technical and is due to Lemma 1.9.5. Under condition (F) we have

Theorem 1.12.1. *Let $C\subset(\widetilde{T^*X\backslash0\times T^*Y\backslash0})$ be a smooth positive homogeneous canonical relation which is closed in $T^*(\widetilde{X\times Y})\backslash0$. Let Φ be a regular phase function of positive type, locally parameterizing C. Assume that Φ satisfies the smooth factorization type condition* (F). *Let $\mu\leq-(k+(n-k)(1-\rho))|1/p-1/2|$ with $1<p<\infty$ and $1/2\leq\rho\leq1$. Then operators $T\in I^\mu_\rho(X,Y;C)$ are continuous from L^p_{comp} to L^p_{loc}.*

Operators $T\in I^0_\rho(X,Y;C)$ are continuous in L^2 and this result does not depend on the factorization condition. Indeed, the argument of Section 1.7 using formula (1.7.3) reducing the situation to the operators in $I^0_{1/2}$ with real phases (see also Remark 1.8.2), shows that operators in $I^2_\rho(X,Y;C)$ with C as in Theorem 1.12.1 are bounded in L^2.

As usual, we can assume that the kernel of T is given by (1.7.1). Proof of Theorem 1.12.1 is based on the complex interpolation method (as in Proposition 1.3.3) applied to the continuity in L^2 and in the Hardy space H^1, for which we have

Theorem 1.12.2. *Let C satisfy conditions of Theorem 1.12.1, let its Schwartz integral kernel be given by* (1.7.1), *and let the number N of frequency variables be equal to n. Assume that the symbol $a \in S_\rho^{k/2+(n-k)(1-\rho)/2}$ in* (1.7.1) *vanishes outside a compact set in $\mathbb{R}^n_x \times \mathbb{R}^n_y$. Then T is continuous from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

Let us indicate briefly some ideas behind the proof of the above theorem. Let us begin with pointing out the difference with the proof of Theorem 1.7.1. Let a be a symbol of order $-(n-1)|1/p-1/2|$ and type $(1,0)$, and we assume that $1 < p \leq 2$. Let T_z be a family of Fourier integral operators with Lagrangian distributions

$$A_z(x,y) =$$
$$\int_{\mathbb{R}^n} e^{i\Phi(x,y,\theta)} a(x,y,\theta)(1+|\theta|^2)^{\frac{1}{2}[(n-1)(1/p-1/2)-(n-1)/2+z(n-1)/2]} d\theta.$$

The continuity of zero order operators in L^2 implies

$$||T_z f||_{L^2} \leq C_z ||f||_{L^2}, \ \mathrm{Re}\, z = 1,$$

and Theorem 1.12.2 implies

$$||T_z f||_{L^1} \leq C_z ||f||_{H^1}, \ \mathrm{Re}\, z = 0.$$

Constants C_z depend on finitely many derivatives of the symbol and, therefore, have polynomial growth in $|z|$. Let us apply Proposition 1.3.3 with $t = 2(1-1/p)$. We have $1/p_t = 1/q_t = 1-t/2$, $(n-1)(1/p-1/2)-(n-1)/2+t(n-1)/2 = 0$, so that $T_t = T$ and Theorem 1.12.1 with $k = n-1$ and $\rho = 1$ follows from Proposition 1.3.3. Similarly, Theorem 1.12.1 follows from Theorem 1.12.2 if we take $a \in S_\rho^{-(k+(n-k)(1-\rho))|1/p-1/2|}$ and kernels

$$\begin{aligned} A_z(x,y) \ &= \ \textstyle\int_{\mathbb{R}^n} e^{i\Phi(x,y,\theta)} a(x,y,\theta) \\ &\quad (1+|\theta|^2)^{\frac{1}{2}[-(k+(n-k)(1-\rho))|1/p-1/2|+(z-1)[k+(n-k)(1-\rho)]/2]} d\theta. \end{aligned}$$

To prove Theorem 1.12.2, it is convenient to use an atomic decomposition $f = \sum_k \lambda_k a_k$, as in Proposition 1.3.2. If a ball B from the definition of an atom contains the unit cube, the Cauchy–Schwartz inequality implies

$$||Ta_k||_{L^1} \leq C|\ \mathrm{supp}\ a_k|^{1/2} ||Ta_k||_{L^2} \leq C'|\ \mathrm{supp}\ a_k|^{1/2} ||a_k||_{L^2},$$

which is uniformly bounded in view of the second property in the definition of atoms. Therefore, Proposition 1.3.2 reduces the statement of Theorem 1.12.2 to proving the uniform bound $||Ta_k||_{L^1} \leq C$ for all atoms supported in sufficiently small balls. This estimate can be obtained using a number of decompositions in the frequency space. However, decomposition in the phase space will be related to the real phase function $\Psi = \text{Re}\,\Phi + \tau\,\text{Im}\,\Phi$ satisfying the smooth factorization type condition (F). First one uses a conic decomposition of the phase space and applies a dyadic decomposition to each cone. Using the corresponding partition of unity, it is possible to replace the phase function of T by a linear function, in each domain, so that integration by parts yields L^1 estimates. However, to get the estimate outside the exceptional set N_Q one needs to use the L^2-boundedness of certain pseudo-differential operators.

The proof of Theorem 1.12.2 is similar to the proof of Theorem 1.7.2. However, some modifications are necessary. There are also certain additional complications related to the fact that the phase function is complex. We will indicate the difference with the proof of Theorem 1.7.2.

First, in the proof of Theorem 1.12.2 it will be more convenient to us to interchange the roles of x and y. Thus, as before, we may assume that the integral kernel of the operator T in Theorem 1.12.2 is given by

$$A(x,y) = \int_{\mathbb{R}^n} e^{i(\langle x,\xi\rangle - \phi(y,\xi))} a(x,y,\xi)d\xi,$$

with some symbol $a \in S^\mu_\rho$. Thus, we have to show that operators with such kernels are continuous from H^1 to L^1, if $a \in S_\rho^{-m(k,\rho)}$ is supported in a narrow cone in ξ-space, and $m(k,\rho) = k/2 + (n-k)(1-\rho)/2$.

Let $\tau \in \mathbb{R}$ be such that condition (L) is satisfied and let

$$\psi(y,\xi) = \text{Re}\,\phi(y,\xi) + \tau\,\text{Im}\,\phi(y,\xi).$$

Since ψ satisfies the real local graph condition, we may assume that there exists a k-dimensional submanifold $S_k(y)$ of $\mathbb{S}^{n-1}\cap\Gamma$ for some narrow cone Γ, $S_k(y)$ varies smoothly with y, and such that $\mathbb{S}^{n-1}\cap\Gamma$ is parameterized by $\bar\xi = \bar\xi_y(u,v)$, for (u,v) in a bounded open set $U\times V$ near $(0,0)\in\mathbb{R}^k\times\mathbb{R}^{n-k-1}$. Furthermore, $\bar\xi_y(u,v)\in S_k(y)$ if and only if $v=0$, and

$$\psi'_\xi(y,\bar\xi_y(u,v)) = \psi'_\xi(y,\bar\xi_y(u,0)).$$

Let us now describe constructions similar to those of the proof of Theorem 1.7.2 in Section 1.9. We keep in mind that later we will set $\lambda = 2^l$. Thus, let u^ν_λ be a collection of points in U, such that $|u^\nu_\lambda - u^{\nu'}_\lambda| \geq C_0\lambda^{-1/2}$ for $\nu\neq\nu'$, and such that U is covered by balls centered at u^ν_λ and radius $C_1\lambda^{-1/2}$. Let $N_k(\lambda)$ be the maximum of such points for a fixed λ. It is easy to see that $N_k(\lambda) = O(\lambda^{k/2})$.

Let $\xi^\nu_\lambda = \bar\xi_y(u^\nu_\lambda,0)$ and let $x^\nu_\lambda(y) = \psi'_\xi(y,\xi^\nu_\lambda)$. Let the atom a_Q be supported in the cube Q centered at some y_0 with sidelength δ. For sufficiently large M let $R^y_{\lambda\nu}$ be the set of all x such that

$$|\langle x - \psi'_\xi(y,\xi^\nu_\lambda), e'\rangle| \leq M\lambda^{-1/2},$$

for all vectors e' tangential to $S_k(y)$ at ξ^ν_λ, and

$$|\langle x - \psi'_\xi(y,\xi^\nu_\lambda), e''\rangle| \leq M\lambda^{-\rho},$$

for all unit vectors e'' which are normal to $S_k(y)$ at ξ_λ^ν. Thus, $R_{\lambda\nu}^y$ is a rectangle with k sides of length of order $\lambda^{-1/2}$ and $n-k$ sides of length of order $\lambda^{-\rho}$.

Let

$$N_Q = \bigcup_{y\in Q} \bigcup_{\nu=1}^{N_k(\lambda)} R_{\lambda\nu}^y.$$

Then

$$|N_Q| \le C|Q|^{\rho(1-k/n)}.$$

The argument of Lemma 1.9.4 gives

$$||Ta_Q||_{L^1(N_Q)} = \int_{N_Q} |Ta_Q|dx \le C.$$

Indeed, $T\circ(I-\Delta)^{-m(k,\rho)/2} \in I_\rho^0(X,Y;C)$ is bounded in L^2. Therefore, by the Hardy–Littlewood–Sobolev inequality,

$$||Ta_Q||_2 \le C||(I-\Delta)^{-m(k,\rho)/2}a_Q||_2 \le C||a_Q||_{p_n},$$

where $p_n = \frac{2n}{(2-\rho)n+k\rho}$. By the second property of atoms, we have $||a_Q||_{p_n} \le |Q|^{-1+1/p_n}$. Hence by the Hölder inequality we obtain

$$||Ta_Q||_2 \le C|Q|^{\frac{-\rho(n-k)}{2n}}.$$

Finally, by the Cauchy–Schwartz inequality, we get

$$\int_{N_Q} |Ta_Q|dx \le C|N_Q|^{1/2}||Ta_Q||_2 \le C|Q|^{(n-k)\rho/n}\cdot|Q|^{-\rho(n-k)/n} \le C.$$

Note that to derive this estimate we used the size of N_Q which is determined by the fact that the real phase function ψ defines a local graph, that is $\det\psi''_{y\xi} \ne 0$.

Now we need an estimate outside the exceptional set N_Q. Let $\widetilde{\chi}_\lambda^\nu$ be a partition of unity in u-coordinates, such that, as before for the partition in (1.9.2),

$$||D_u^\alpha \widetilde{\chi}_\lambda^\nu||_\infty = O(\lambda^{|\alpha|/2}).$$

Define the partition of unity χ_λ^ν on Γ corresponding to the directions ξ_λ^ν, by setting

$$\chi_\lambda^\nu(s\bar{\xi}(u,v)) = \widetilde{\chi}_\lambda^\nu(u),\ \ s>0.$$

Then we define a_λ^ν similar to Section 1.9, by

$$a_\lambda^\nu(x,y,\xi) = \chi_\lambda^\eta(\xi)\theta_\lambda(\xi)a(x,y,\xi),$$

and

$$\begin{aligned} A_\lambda^\nu(x,y) &= \int_{\mathbb{R}^n} e^{i(\langle x,\xi\rangle-\phi(y,\xi))}a_\lambda^\nu(x,y,\xi)d\xi \\ &= \int_{\mathbb{R}^n} e^{i\langle x-\phi'_\xi(y,\xi_\lambda^\nu),\xi\rangle}b_\lambda^\nu(x,y,\xi)d\xi, \end{aligned}$$

where $b_\lambda^\nu(x,y,\xi) = e^{i(\langle\phi'_\xi(y,\xi_\lambda^\nu)-\phi(y,\xi),\xi\rangle)}a_\lambda^\nu(x,y,\xi)$.

As before, the estimate for $||Ta_Q||_{L^1(\mathbb{R}^n\setminus N_Q)}$ will follow from the estimate

$$\int_{\mathbb{R}^n\setminus N_Q} |A^\nu_\lambda(x,y)|dx \le C\lambda^{-k/2}(\lambda\delta)^{-1}, \tag{1.12.3}$$

when $\lambda \ge \delta^{-1}$, and where δ is the sidelength of Q, and if for all $y, y' \in \mathbb{R}^n$ we would have

$$\int |A^\nu_\lambda(x,y) - A^\nu_\lambda(x,y')|dx \le C|y-y'|\lambda \cdot \lambda^{-k/2}, \tag{1.12.4}$$

which will be used when $\lambda < \delta^{-1}$.

By a rotation, we may assume that every $\xi \in \Gamma$ splits into $\xi = (\xi', \xi'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ where ξ'' is normal to $S_k(y)$ at $\xi^\nu_\lambda = \bar{\xi}(u^\nu_\lambda, 0)$. Define the operator

$$L^\nu_\lambda = \left(I - \langle \lambda^{1/2}\nabla_{\xi'}, \lambda^{1/2}\nabla_{\xi'}\rangle\right)\left(I - \langle \lambda^\rho\nabla_{\xi''}, \lambda^\rho\nabla_{\xi''}\rangle\right).$$

The fact that $a \in S^{-m(r,\rho)}_\rho$ implies that

$$|(L^\nu_\lambda)^N a^\nu_\lambda(x,y,\xi)| \le C_N\lambda^{-m(k,\rho)},$$

and, as before, the same estimate holds for b^ν_λ. Recalling that $x^\nu_\lambda(y) = \phi'_\xi(y,\xi^\nu_\lambda) - x$, integration by parts yields

$$\begin{aligned} A^\nu_\lambda(x,y) &= (1 + |\lambda^\rho(\phi'_\xi(y,\xi^\nu_\lambda) - x)''|^2)^{-N} \times \\ &\times \ (1 + |\lambda^{1/2}(\phi'_\xi(y,\xi^\nu_\lambda) - x)'|^2)^{-N} K^\nu_{\lambda N}(x,y), \end{aligned} \tag{1.12.5}$$

where

$$K^\nu_{\lambda N}(x,y) = \int (L^\nu_\lambda)^N b^\nu_\lambda(x,y,\xi) e^{i\langle x - \phi'_\xi(y,\xi^\nu_l),\xi\rangle} d\xi. \tag{1.12.6}$$

Note that similar to Section 1.9, we always use Lemma 1.9.5 instead of Lemma 1.9.2 in the integration by parts argument to get rid of the complex phase, by replacing ϕ by $\psi = \operatorname{Re}\phi + \tau \operatorname{Im}\phi$ with τ for which condition (F) holds. All such estimates are similar to the one in (1.9.13). Condition (F) guarantees that $\mu_0 < 1/\sqrt{3}$ in Lemma 1.9.5. The rest of the proof is similar to [62], but some changes are necessary in order to take into consideration the imaginary part of the phase function.

If we dilate $R^y_{\lambda\nu}$ be a factor $C(\lambda\delta)^{-1}$ the resulting rectangle will still be contained in N_Q. Hence we can use the Cauchy–Schwartz inequality in x'' to obtain for $y \in Q$

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^n\setminus N_Q} |A^\nu_\lambda(x,y)|dx \le \\ &C_N\lambda^{-\rho(n-k)/2} \textstyle\int_{|x'|<C\delta^{1/2}} \left((1 + |\lambda^{1/2}(\phi'_\xi(y,\xi^\nu_l) - x)'|^2\right)^{-N} \times \\ &\times \left(\textstyle\int |K^\nu_{\lambda N}(x',x'',y)|^2 dx''\right)^{1/2} dx' + \\ &C_N\lambda^{-\rho(n-k)/2}(\lambda\delta)^{(2N-n+k)\rho/2} \textstyle\int \left((1 + |\lambda^{1/2}(\phi'_\xi(y,\xi^\nu_l) - x)'|^2\right)^{-N} \times \\ &\times \left(\textstyle\int |K^\nu_{\lambda N}(x',x'',y)|^2 dx''\right)^{1/2} dx'. \end{aligned} \tag{1.12.7}$$

For fixed x' and ξ' let the pseudo-differential operator $S^\nu_{\lambda N}(x',\xi')$ be given by

$$[S^\nu_{\lambda N}(x',\xi')f]\,(x'') = \lambda^{m(k,\rho)}\int (L^\nu_\lambda)^N b^\nu_\lambda(x',x'',y,\xi',\xi'')\hat{f}(\xi'')e^{i\langle x'',\xi''\rangle}d\xi'',$$

which is of order 0 and type $(\rho, 1-\rho)$. Let g be a smooth function, defined on $\mathbb{R}^{n-k}$ with the property that $\hat{g}(\xi'') = 1$ for $|\xi''| \leq 8$, which vanishes for $|\xi''| \geq 16$. Define $\hat{g}_\lambda = \hat{g}(\lambda^{-1}\cdot\,)$. Then

$$K^\nu_{\lambda N}(x,y) = \lambda^{-m(k,\rho)}\int [S^\nu_{\lambda N}(x',\xi')f_\lambda]\,(x'')e^{i\langle x',\xi'\rangle}d\xi', \tag{1.12.8}$$

where $\hat{f}_\lambda(\xi'') = \hat{g}_\lambda(\xi'')e^{-i\langle\phi'_\xi(y,\xi^\nu_l),\xi\rangle}\sigma^\nu_\lambda(\xi',\xi'')$, where σ^ν_λ is a smooth bounded cutoff function in a narrow cone around ξ^ν_λ, $\sigma^\nu_\lambda(\xi) = 1$ on supp χ^ν_λ and equal to 0 outside the cone twice the support of χ^ν_λ. Function $\hat{f}_\lambda$ depends on ξ' as a parameter.

The condition $\mathrm{Im}\,\Phi \geq 0$ means $\mathrm{Im}\,\phi \leq 0$ since $\Phi(x,y,\xi) = \langle x,\xi\rangle - \phi(y,\xi)$. Suppose that $\mathrm{Im}\,\phi(y,\xi^\nu_\lambda) < 0$. Then by the Euler identity $\mathrm{Im}\,\langle\phi'_\xi(y,\xi^\nu_\lambda),\xi\rangle$ is equal to $\mathrm{Im}\,\phi(y,\xi^\nu_\lambda)$ and is strictly negative at $\xi = \xi^\nu_\lambda$. Therefore, if λ is sufficiently large, then $\xi \in$ supp σ^ν_λ implies that ξ and ξ^ν_λ are sufficiently close, so that $\mathrm{Im}\,\langle\phi'_\xi(y,\xi^\nu_\lambda),\xi\rangle < 0$ for all $\xi \in$ supp σ^ν_λ. In fact, for sufficiently large λ we can always choose partition u^ν_λ such that $\mathrm{Im}\,\phi(y,\xi^\nu_\lambda) < 0$, unless $\mathrm{Im}\,\phi(y,\xi)$ equals to zero in an open set around some $\xi = \xi^\nu_\lambda$. If λ is large, we can assume that this set contains supp σ^ν_λ. But in this case, $\mathrm{Im}\,\langle\phi'_\xi(y,\xi^\nu_\lambda),\xi\rangle$ is 0 since the derivative with respect to ξ is 0 at ξ^ν_λ. It follows that in all cases for sufficiently large λ we have $\mathrm{Im}\,\langle\phi'_\xi(y,\xi^\nu_\lambda),\xi\rangle \leq 0$ for all $\xi \in$ supp σ^ν_λ.

This implies that

$$||f_\lambda||_{L^2(\mathbb{R}^{n-k})} = ||\hat{f}_\lambda||_{L^2(\mathbb{R}^{n-k})} \leq C||\hat{g}_\lambda||_{L^2(\mathbb{R}^{n-k})} \leq C\lambda^{n-k}.$$

Observe now that $S^\nu_{\lambda N}(x',\xi') = 0$ for ξ' outside a set of measure bounded by $C\lambda^{k/2}$. Finally, since pseudo-differential operators of order 0 and type $(\rho, 1-\rho)$ are bounded on L^2, we get

$$\begin{aligned}&\left(\int |K^\nu_{\lambda N}(x',x'',y)|^2dx''\right)^{1/2} \leq \\ &C\lambda^{-m(k,\rho)}\lambda^{k/2}||f_\lambda||_{L^2(\mathbb{R}^{n-k})} \\ &\leq C\lambda^{(n-k)\rho/2}\end{aligned} \tag{1.12.9}$$

for large λ. The argument on the sign of $\mathrm{Im}\,\langle\phi'_\xi(y,\xi^\nu_\lambda),\xi\rangle$ depends on y, but we have $y \in Q$ and the support of the symbol $a(x,y,\xi)$ is compact in (x,y), so we obtain (1.12.9) uniformly if λ is larger than some absolute constant λ_0.

Performing the x'-integration in (1.12.7) yields the estimate (1.12.3), for $\lambda > \delta^{-1}$. The same argument also yields

$$\int |\nabla_y A^\nu_\lambda(x,y)|dx \leq C\lambda\cdot\lambda^{-k/2},$$

where the extra factor λ is introduced because of the differentiation with respect to y. But this inequality implies (1.12.4). The rest of the proof proceeds similarly to the end of the proof of Theorem 1.7.2 after Lemma 1.9.4. The only difference is when the sum there is over $l > 0$ such that $\lambda = 2^l < \lambda_0$. The support of $\hat{f}_\lambda$ is bounded in ξ'' and $S^\nu_{\lambda N}(x', \xi')$ vanishes for ξ' outside a set of bounded measure. This implies that the estimate (1.12.9) is still valid. Moreover, there are only finitely many $l > 0$ with $2^l < \lambda_0$, and each of the integrals in Σ_1 is multiplied by some constant, this constant can be chosen uniform with respect to the small cubes Q since the symbol $a(x, y, \xi)$ is compactly supported in (x, y). This completes the proof of Theorem 1.7.2.

Proposition 1.12.3. *Let the canonical relation C satisfy conditions of Theorem 1.12.1. Let $1 < p \le q \le 2$ and let $\mu \le -n/p + (n-k)/q + k/2$. Then operators $T \in I^\mu(X, Y; C)$ are continuous from $L^p_{comp}(Y)$ to $L^q_{loc}(X)$. The dual statement holds for $2 \le p \le q < \infty$.*

The statement follows by the interpolation method between the statements of Theorem 1.12.1 and Remark 1.10.4. Similarly, we get that operators from I^μ are continuous from $\mathrm{Lip}\,(\alpha - k/2 - \mu)$ to $\mathrm{Lip}\,(\alpha)$ for all $\alpha \in \mathbb{R}$.

Remark 1.12.4. Let us note that the factorization condition is not necessary for operators in Theorem 1.12.1 to be bounded. Thus, in [59] it is shown that Fourier integral operators with phases as in Example 1.5.5 of Section 1.5.2 and of order of Theorem 1.12.1 are still L^p-continuous. For example, if the phase function of an operator $T \in I^\mu(\mathbb{R}^3, \mathbb{R}^3, C)$ is given by

$$\Phi(x, y, \xi) = \langle x - y, \xi \rangle + \frac{1}{\xi_3}(y_1\xi_1 + y_2\xi_2)^2$$

in a conic neighborhood of $\xi_3 = 1$ away from $\xi = 0$, then the factorization condition fails for C. However, operators T are still bounded from L^p_{comp} to L^p_{loc}, provided that $1 < p < \infty$ and $\mu \le -|1/p - 1/2|$. Note that in this case $k = 1$ and the order $-|1/p - 1/2|$ can not be improved according to Section 1.11.

Chapter 2

Affine fibrations

2.1 Introduction

The singularity theory of the fibrations and the fibers of the regular mappings have been under study for a very long time. One of the early references is [49]. Many general results can be found, for example, in [32], [51], [63]. One is often interested in the singularities of fibrations and of the fibers themselves. In this monograph we will study the following classes of fibrations. We assume that the fibers are regular and, moreover, that locally they have a very simple form of affine spaces. Let Ω be an open subset of $\mathbb{C}^n$. The fibers are $(n-k)$-dimensional affine subspaces of Ω, equal to the kernels $\ker A(\xi)$ of a holomorphic matrix valued mapping $A : \Omega \to \mathbb{C}^{p\times n}$, on the set where the rank of $A(\xi)$ is maximal and equal to some $k \le n-1$. It follows that the fibration is regular on an open dense subset of Ω, i.e. the fibers do not intersect and behave analytically with respect to their position. We will analyze its singularities in the complement of this regular set.

A particular case arises when the matrices are Jacobi matrices of a regular (holomorphic) mapping Γ. In this case $A = D\Gamma$ and the regular fibers can be defined as level sets of this holomorphic mapping Γ. The direction of the fibers is the same as the direction of the kernels of the Jacobi matrix $D\Gamma$. In such way we are led to consider a more general case of the fibrations defined by the shifted kernels of the Jacobi matrix, for which the fibers of the mapping become the ruled analytic varieties constituted by the kernels.

Problems of this type arise in the analysis of the regularity properties of Fourier integral operators and hyperbolic equations, see [62], although there they are formulated in a very different form in the real domain. However, the analysis in $\mathbb{C}^n$ allows to draw many conclusions in the real domain as well. In particular, some results of this chapter will be applied to establish the L^p-regularity properties of Fourier integral operators, which turn out to be sharp ([52], [54]). As a consequence, in [53], sharp L^p-regularity is derived for solutions of hyperbolic Cauchy problems in four dimensions. The results of this chapter extend some of the analysis of [54] to the general case of fibrations defined by kernels of arbitrary holomorphic matrix valued functions. Here it will appear in subsequent chapters.

In general, the affineness assumption holds generically in the study of Lagrangian manifolds, which can be viewed as closures of smooth (analytic) conormal bundles. Regular fibers in this case are the fibers of a conormal bundle. Results of this chapter imply, in particular, that certain singularities are impossible in lower dimensions.

First, in Section 2.2 we formulate the problem in the invariant case corresponding to the Fourier integral operators commuting with translations. In this case the generating function is $\phi(y,\xi) = \langle y,\xi\rangle - H(\xi)$. According to the theory of Fourier integral operators (Section 1.5.2), functions $H : V \to \mathbb{R}$ on an open subset V of $\mathbb{R}^n$ have the following property. The maximal rank k of the Hessian $D^2H(\xi)$ is strictly less than n, and points ξ where $\operatorname{rank} D^2H(\xi) = k$ is maximal form an open set U. One of the interesting properties of the gradient $\Gamma : \xi \mapsto \nabla H(\xi)$ is that for every point $\xi \in U$ the level set $\Gamma^{-1}(\Gamma(\xi))$ locally in a neighborhood of ξ coincides with the affine space $\xi + \ker D^2H(\xi)$. If H is real analytic, then the holomorphic extension Γ of the gradient ∇H has the same property in some open neighborhood of the open set V in $\mathbb{C}^n$. This property motivates the study of holomorphic mappings Γ with properties (A1), (A2) formulated below.

2.2 Fibrations with affine fibers

2.2.1 Affine fibrations setting

Now we will give a precise formulation of the first problem. Let A be a holomorphic mapping from a connected open subset Ω of $\mathbb{C}^n$ to $\mathbb{C}^{p\times n}$, with $p, n \in \mathbb{N}$. Assume that for some $k \le n-1$ holds

(A1) $\max_{\xi\in\Omega} \operatorname{rank} A(\xi) = k$.

We will be interested in the kernels of $A(\xi)$ and thus the condition $k \le n-1$ is natural. The set Ω can be decomposed into disjoint union of the sets $\Omega^{(i)}$ of the points $\xi \in \Omega$ with $\operatorname{rank} A(\xi) = i$, $i = 0, \dots, k$. Then the set $\Omega\backslash\Omega^{(k)}$ where $\operatorname{rank} A(\xi) < k$ is an analytic subset of Ω without interior points and in the open dense subset $\Omega^{(k)}$ the mapping

$$\varkappa : \xi \mapsto \ker A(\xi)$$

is holomorphic from $\Omega^{(k)}$ to the Grassmann manifold $\mathbb{G}_{n-k}(\mathbb{C}^n)$, where $\mathbb{G}_{n-k}(\mathbb{C}^n)$ consists of all $(n-k)$-dimensional subspaces of $\mathbb{C}^n$. Let us denote by Ω^{sing} the subset of $\xi \in \Omega\backslash\Omega^{(k)}$ such that *$\varkappa$ can not be extended to a holomorphic mapping* $U \to \mathbb{G}_{n-k}(\mathbb{C}^n)$ on any open neighborhood U of ξ in Ω. Thus, the mapping $\varkappa$ is regular at $\xi \in \Omega^{(k)}$ and such points will be called *regular*. Its complement $\Omega\backslash\Omega^{(k)}$ will be called *the exceptional set*. It consists of *the removable singularities* at $\Omega\backslash(\Omega^{(k)} \cup \Omega^{\text{sing}})$ and *essential singularities* (or simply *singularities*) at Ω^{sing}.

An additional assumption which we will make is that *the affine subspaces* $\xi + \varkappa(\xi)$ *define a local fibration in* Ω. It means that locally there is exactly one fiber through a regular point, or that the regular fiber is the same for all points contained in it. In other words, $\varkappa(\xi + \zeta) = \varkappa(\xi)$ for all $\xi \in \Omega^{(k)}$, $\zeta \in \varkappa(\xi)$, such that $\xi + \zeta \in \Omega^{(k)}$. For the later convenience we formulate it here in the following form:

(A2) For every $\xi \in \Omega^{(k)}$ and $\eta \in \Omega$ with $\eta \in (\xi + \varkappa(\xi)) \cap \Omega^{(k)}$, the subspaces $\varkappa(\xi)$ and $\varkappa(\eta)$ are equal.

If $(\xi + \varkappa(\xi)) \cap \Omega$ is connected (which certainly is the case if Ω is convex or if we require that for any affine subspace L in Ω of codimension k the set $L \cap \Omega$ is connected), property (A2) is global in $\Omega^{(k)}$. Thus, for simplicity we will assume that Ω is convex, which is not restrictive because we will analyze only local singularities. The mapping $\varkappa$ is holomorphically extendible to the open subset $\Omega \backslash \Omega^{\text{sing}}$ containing $\Omega^{(k)}$, and we will denote this extension also by $\varkappa$. Then, for every $\xi \in \Omega \backslash \Omega^{\text{sing}}$, $\varkappa$ is also constant on $(\xi + \varkappa(\xi)) \cap \Omega$. And, if $\xi, \eta \in \Omega \backslash \Omega^{\text{sing}}$, then $(\xi + \varkappa(\xi)) \cap (\eta + \varkappa(\eta)) \cap (\Omega \backslash \Omega^{\text{sing}})$ is empty, or, if not, $\varkappa(\xi) = \varkappa(\eta)$. This property extends (A2) from $\Omega^{(k)}$ to $\Omega \backslash \Omega^{\text{sing}}$. This makes it possible to define the following equivalence relation in $\Omega \backslash \Omega^{\text{sing}}$. The relation $\xi \sim \eta$ if $\eta \in \xi + \varkappa(\xi)$ and $\varkappa(\xi) = \varkappa(\eta)$ defines an equivalence relation in $\Omega \backslash \Omega^{\text{sing}}$, the factor space $\Omega \backslash \Omega^{\text{sing}} / \sim$ is a smooth analytic space of dimension k and the projection $\xi \mapsto (\xi + \varkappa(\xi)) \cap (\Omega \backslash \Omega^{\text{sing}})$ is an analytic submersion. In this sense $(\xi + \varkappa(\xi)) \cap (\Omega \backslash \Omega^{\text{sing}})$, $\xi \in \Omega \backslash \Omega^{\text{sing}}$, define a smooth fibration of $\Omega \backslash \Omega^{\text{sing}}$.

The simplest singular fibration can be defined for $\Omega = \mathbb{C}^n$ by taking for one dimensional fibers open rays starting from zero. Clearly, such fibration is analytic for all $\xi \in \mathbb{C}^n$, $\xi \neq 0$, and the singular set $\Omega^{\text{sing}} = \{0\}$. However, it turns out that it is impossible to construct a holomorphic mapping A, for which the described lines would be the kernels of A.

Condition (A2) can be regarded as a definition of the fibrations in Ω. The linearity of the fibers is, therefore, essential. Let us give now an important example related to the theory of Fourier integral operators. Let X be a smooth (analytic) manifold of dimension n and let T^*X denote the cotangent bundle of X. Let Λ be a (conic) analytic Lagrangian submanifold of T^*X. Let $\pi : T^*X \to X$ be the canonical projection. It follows that $\pi(\Lambda)$ is semi-analytic as the image of an analytic set Λ under a proper mapping π. Let Σ denote its regular part. Manifold Σ is smooth and its conormal bundle

$$N^*\Sigma = \{(x, \xi) : x \in \Sigma, \xi(\delta x) = 0 \text{ for all } \delta x \in T_x\Sigma\}$$

is a conic Lagrangian submanifold of T^*X, densely contained in Λ. It follows from the Poincaré lemma that every $(x, \xi) \in \Lambda$ has a conic neighborhood C such that $\Lambda \cap C$ is locally equal to the set of points

$$\{(\nabla\phi(\xi), \xi)\},$$

where ϕ is a generating function for Λ. The detailed proof of this can be found in [11, 3.7]. In this notation the fibers of $d\pi|_\Lambda$ correspond to the kernels of the

Hessian $D^2\phi$ and the fibration in Λ is defined by the mapping $A(\xi) = D^2\phi(\xi)$. The regular set $\Omega^{(k)}$ is the set of points ξ from $N^*\Sigma$. This example will be discussed in more detail in the next chapters.

2.2.2 Jacobian affine fibrations setting

A particular case of (A1), (A2) happens when we assume that A is a Jacobian of a holomorphic vector valued mapping $\Gamma : \Omega \to \mathbb{C}^p$. However, this generalizes the previous example, where $\Gamma = \nabla\phi$. In the case of a general Γ the matrix $A = D\Gamma$ is not necessarily symmetric as in the case when it is a Hessian of a holomorphic function ϕ as above. However, later we will derive a number of results without using the symmetricity. Relations between ϕ and the fibration $\varkappa$ defined by $A = D^2\phi$ will be explored in Chapter 3 in more detail. As consequence, we will derive there several properties of fibrations in the Lagrangian manifolds (as above) as well as related properties of Fourier integral operators, for which ϕ is related to their phase functions. Thus, for future convenience we formulate condition (A1) in this case as

(ΓA1) The matrix $A(\xi)$ is the Jacobian, $A(\xi) = D\Gamma(\xi)$, for all $\xi \in \Omega$, of a holomorphic mapping $\Gamma : \Omega \to \mathbb{C}^p$. For this A holds $\max_{\xi\in\Omega}$ rank $A(\xi) = k$.

Here, if $\Gamma = (\Gamma_1, \ldots, \Gamma_p)$, then $A_{ij}(\xi) = \partial_{\xi_j}\Gamma_i(\xi)$.

Condition (A2) can now be formulated in terms of Γ, both locally and globally in Ω. Condition (A2) means that the mapping $\varkappa = \ker D\Gamma$ is constant along $\varkappa(\xi)$ for each $\xi \in \Omega^{(k)}$. It then follows that Γ is constant on $\varkappa(\xi)$ (Proposition 2.8.2). Because $\xi \in \Omega^{(k)}$, the level set $\Gamma^{-1}(\Gamma(\xi))$ is a smooth analytic manifold of dimension $n-k$, locally at ξ, locally containing the $(n-k)$-dimensional affine space $\xi + \varkappa(\xi)$. It then follows that $\xi + \varkappa(\xi)$ itself is the level set of Γ, locally at ξ. Condition (A2) is, therefore, equivalent to

(ΓA2) For every $\xi \in \Omega^{(k)}$ the affine subspace $\xi + \varkappa(\xi)$ is locally (at ξ) equal to $\Gamma^{-1}(\Gamma(\xi))$, the fiber of Γ passing through ξ.

The global version of the condition (ΓA2) is the following

(ΓA2$'$) For every $\xi \in \Omega^{(k)}$ the affine subspace $(\xi+\varkappa(\xi))\cap\Omega$ is equal to $\Gamma^{-1}(\Gamma(\xi))$, the fiber of Γ passing through ξ.

As before, $\varkappa$ can be extended to $\Omega\backslash\Omega^{\text{sing}}$. In view of our assumption that Ω is convex, for every $\xi \in \Omega\backslash\Omega^{\text{sing}}$, Γ is constant on $(\xi + \varkappa(\xi)) \cap \Omega$. And, if $\xi, \eta \in \Omega\backslash\Omega^{\text{sing}}$, then $(\xi + \varkappa(\xi)) \cap (\eta + \varkappa(\eta)) \cap (\Omega\backslash\Omega^{\text{sing}})$ is empty, or, if not, $\varkappa(\xi) = \varkappa(\eta)$. As before, $\varkappa$ induces an equivalence relation in $\Omega\backslash\Omega^{\text{sing}}$ and a smooth fibration of $\Omega\backslash\Omega^{\text{sing}}$ by $(\xi + \varkappa(\xi)) \cap (\Omega\backslash\Omega^{\text{sing}})$, $\xi \in \Omega\backslash\Omega^{\text{sing}}$. The mapping Γ factorizes through this fibration (see Corollary 2.8.4 for the details). Therefore, conditions (A2), (ΓA2) and (ΓA2$'$) are equivalent. A further characterization of this property will be given in Proposition 2.8.2.

The assumption (ΓA2) of the linearity of the fibers is essential for the analysis. Thus, we will show that for $k \leq 2$ the set Ω^{sing} of singular points is empty, which even for $k = 1$ is not the case if we restrict to the assumption (ΓA1) only (Example 2.8.1). However, in a number of important cases, (ΓA2) is satisfied, especially if Γ is the gradient of a holomorphic function ϕ, which then, because $D^2\phi(\xi) = D\Gamma(\xi)$ has rank $\leq k < n$, necessarily is a solution of the Monge-Ampère equation $D^2\phi(\xi) \equiv 0$.

We will be interested in the properties of the singular set Ω^{sing}. It is shown in [52] and in Chapter 1 that for all $1 \leq k \leq n-1$ there exist analytic families $\Gamma_y(\xi)$ of mappings satisfying (ΓA1), (ΓA2) for every y in an open subset of $\mathbb{C}^m$, $m \geq 2$, for which the singular set Ω^{sing} is not empty. The construction there is similar to the σ-processes centered at some point of Ω^{sing} ([63]). This problem will be formulated in the next section. However, if such family Γ_y consists of only one mapping Γ, the theory becomes much more subtle. In [52] the partial results of this chapter have been applied to establish the L^p-regularity properties for the translation invariant Fourier integral operators (for more discussion see Chapter 4), which are in fact sharp (Section 1.11). In general, the gradient of an analytic phase function corresponding to a Fourier integral operator in a space of dimension $n+1$ satisfies conditions (ΓA1), (ΓA2) (Theorem 4.2.3). With this approach one can interpret the set Ω as a section of the closure of an analytic conormal bundle and Ω^{sing} as the set of the essential singularities for the canonical projection in this closure. Again, this leads to a description of the wave front sets of a class of Fourier integral operators (Chapter 4) and to improvements in the regularity theory of the hyperbolic partial differential equations (Sections 5.2 and 5.4).

There is a number of interesting problems related to fibrations arising in this way. Thus, for a given fibration $\varkappa_0$ in an open dense set Ω_0 in Ω, we would like to determine whether there exists a holomorphic mapping Γ satisfying conditions (ΓA1), (ΓA2), such that the fibration defined by $\varkappa$ coincides with the fibration defined by $\varkappa_0$ in $\Omega_0 \cap \Omega^{(k)}$. The construction of a phase function ϕ for a given $\Gamma(\xi) = \nabla_\xi\phi(\xi)$ leads to further complications. In Chapter 3 we will derive necessary and sufficient conditions in terms of a system of partial differential equations with coefficients corresponding to a given fibrations. However, the regularity (analyticity) of the fibration does not immediately imply that solutions of the constructed system of differential equations are sufficiently regular. Moreover, this system depends on the choice of a local coordinate system in the Grassmannian. It would be interesting to obtain an invariant description of the results of Chapter 3 as well as their generalization to spaces of higher dimensions. The results of this chapter (for example, Theorems 2.4.2 – 2.4.5) describe possible dimensions of the set Ω^{sing} under conditions (A1), (A2). They also give some understanding of its structure. It would be interesting to investigate its further properties, especially in the case of gradient fibrations. For example, in Section 3.4, we will give examples of fibrations of gradient type for which Ω^{sing} is not empty. However, in all our examples, the set Ω^{sing} is affine and its dimension is equal to $n-2$. It is not clear whether the condition $\dim \Omega^{\mathrm{sing}} = n-2$ is necessary for the fibrations of gradient type.

2.3 Formulation for the parametric fibrations

2.3.1 Affine fibrations setting

Let us briefly consider a more general problem allowing y dependence. Let A be a holomorphic mapping from an open connected set $\Omega \subset \mathbb{C}^m \times \mathbb{C}^n$ to $\mathbb{C}^{p\times n}$. The dependence of $A = A(y,\xi)$ on y can be assumed to be smooth. In this case all the results are smooth in y and holomorphic in ξ.

Let $k \leq n-1$ be the maximal rank of $A(y,\xi)$ in Ω,

(R1) $\max_{(y,\xi)\in\Omega} \operatorname{rank} A(y,\xi) = k$.

Let $\Omega^{(k)}$ be the set of points (y,ξ) where the maximal rank k is attained. It is open and dense in Ω. The mapping

$$\varkappa : (y,\xi) \mapsto \ker A(y,\xi)$$

is holomorphic from $\Omega^{(k)}$ to the Grassmannian $\mathbb{G}_{n-k}(\mathbb{C}^n)$ of $(n-k)$-dimensional linear subspaces of $\mathbb{C}^n$. Let Ω^{sing} denote the set of the essential singularities of $\varkappa$, i.e. the set of points $(y,\xi) \in \Omega$ such that the mapping $\varkappa$ does not allow a holomorphic extension to any neighborhood of (y,ξ) in Ω. It is clear that the sets $\Omega^{(k)}$ and Ω^{sing} are disjoint. As before, the following condition guarantees that kernels of A define a fibration with respect to ξ:

(R2) *For every $(y,\xi) \in \Omega^{(k)}$ and $(y,\eta) \in \Omega^{(k)}$ with $\eta \in (\xi + \varkappa(y,\xi))$, the affine spaces $\varkappa(y,\xi)$ and $\varkappa(y,\eta)$ are equal.*

Similar to conditions (A1), (A2), condition (R2) means that $\varkappa$ defines a local holomorphic fibration in $\Omega\backslash\Omega^{\text{sing}}$. Example (1.5.6) with $A = D^2\phi$ shows that in general fibrations can have essential singularities. If we fix a value of y or if we take A independent of y, then conditions (R1), (R2) are equivalent to conditions (A1), (A2) above. It turns out that simple examples as in (1.5.6) are impossible in the problem (A1), (A2) and the analysis becomes more interesting. For example, one of the necessary conditions for an analytic set Ω^{sing} to be the set of the essential singularities of a mapping $\varkappa$ associated to a holomorphic mapping A is the following dimension estimate:

$$k-1 \leq \dim_\xi \Omega^{\text{sing}} \leq n-2,$$

provided that $A(y,\xi) = A(\xi)$ is constant in y, and Ω^{sing} stands for the set of the essential singularities of $\varkappa$ associated to $A(\xi)$, as in the previous section.

2.3.2 Jacobian affine fibration setting

Here, as before we assume that there is a holomorphic mapping Γ from an open connected set $\Omega \subset \mathbb{C}^m \times \mathbb{C}^n$ to $\mathbb{C}^p$, such that $A = D\Gamma$. Let $k \leq n-1$ be the maximal rank of the Jacobian $D_\xi\Gamma(y,\xi)$ in Ω,

(ΓR1) $\max_{(y,\xi)\in\Omega} \operatorname{rank} D_\xi\Gamma(y,\xi) = k$.

The following condition guarantee the linearity of the level sets of Γ with respect to ξ:

(ΓR2) *For every* $(y, \xi) \in \Omega^{(k)}$ *the affine space* $(y,\xi)+(0,\varkappa(y,\xi))$ *locally coincides with the level set* $\Gamma^{-1}(\Gamma(y,\xi))$ *through the point* (y,ξ).

Again, if we fix a value of y or if we take Γ independent of y, then conditions (R1), (R2) are equivalent to conditions (A1), (A2) above. In this case the analysis becomes even more interesting. For example, one of the necessary conditions for an analytic set Ω^{sing} to be the set of the essential singularities of a mapping $\varkappa$ associated to a holomorphic mapping Γ is the following dimension estimate:

$$\max\{k-1, n-k+1\} \leq \dim_\xi \Omega^{\text{sing}} \leq n-2,$$

provided that $\Gamma(y,\xi) = \Gamma(\xi)$ is independent of y, and Ω^{sing} stands for the set of the essential singularities of $\varkappa$ associated to $A(\xi) = D\Gamma(\xi)$. In particular, Ω^{sing} can not contain isolated points.

In view of the similarity of two problems described above *we will use the same notations in their analysis. In order to eliminate any confusion, we will always consider problem* (A1), (A2) *unless we explicitly state otherwise.*

2.4 Main results

Our main results are Theorems 2.4.1–2.6.4, 2.8.3, 2.9.2, which however employ the notation introduced later. Theorem 2.7.7 with the upper bound on the dimension of Ω^{sing} is quite standard (in the spirit of [51]). It turns out, that the singularities are removable if the fibers are of codimension 1 or 2, i.e. if $k=1$ or $k=2$. In particular, it follows that for $n=3$ the singular set is always empty, which corresponds to the fact that in 4-dimensional space, playing an important role in the applications to the theory of strictly hyperbolic equations in physics, the translation invariant Fourier integral operators satisfy the assumption for the regularity in Chapter 1. Summarizing the main results, we state them here in a form avoiding the notation introduced later.

Theorem 2.4.1. *Let* A *satisfy* (A1), (A2). *Suppose that* Ω^{sing} *is not empty. Then the singular set* Ω^{sing} *is an analytic subset of* Ω *and for every* $\xi \in \Omega^{\text{sing}}$ *holds*

$$k-1 \leq \dim_\xi \Omega^{\text{sing}} \leq n-2.$$

Under the Jacobian condition (ΓA1), we have a stronger statement.

Theorem 2.4.2. *Let* Γ *satisfy* (ΓA1), (ΓA2). *Suppose* Ω^{sing} *is not empty. Then the following holds:*

(i) *The singular set* Ω^{sing} *is an analytic subset of* Ω *and for every* $\xi \in \Omega^{\text{sing}}$ *we have*

$$\max\{k-1, n-k+1\} \leq \dim_\xi \Omega^{\text{sing}} \leq n-2.$$

In particular, $3 \leq k \leq n-1$, *and* $n \geq 4$.

(ii) *Let* $\xi \in \Omega^{\text{sing}}$ *be a point in the smooth (regular) component of* Ω^{sing}. *Let* $\xi = \lim_{i\to\infty} \xi_i$, $\xi_i \in \Omega^{(k)}$, *and* $\mathbb{G}_{n-k}(\mathbb{C}^n) \ni \varkappa = \lim_{i\to\infty} \varkappa(\xi_i)$. *Then* $\varkappa \subset T_\xi \Omega^{\text{sing}}$.

As a consequence we get that in the case $k \leq 2$ all singularities are removable:

Theorem 2.4.3. *Let* Γ *satisfy* (ΓA1), (ΓA2). *Let* $k \leq 2$. *Then the singular set* Ω^{sing} *is empty.*

The estimates on the dimension of Ω^{sing} in (i) of Theorem 2.4.2 and Theorem 2.4.3 together with the statement of Theorem 2.4.2, (ii), are sharp, in a sense that all the intermediate dimension can be exemplified with even entire mappings in $\mathbb{C}^n$:

Theorem 2.4.4. *For every* $3 \leq k \leq n-1$ *and* $2 \leq d \leq \min\{k-1, n-k+1\}$ *there exist a holomorphic mapping* $\Gamma : \mathbb{C}^n \to \mathbb{C}^n$ *satisfying* (ΓA1), (ΓA2), *and such that* $\dim \Omega^{\text{sing}} = n - d$. *Moreover,* Γ *can be chosen such that* $\Omega \backslash \Omega^{(k)} = \Omega^{\text{sing}}$.

Note, that similar results remain valid if $\mathbb{C}^p$ is replaced by an arbitrary analytic space. Finally, we would like to formulate precisely the relation for the mappings with affine fibers and properties (A1), (A2).

Theorem 2.4.5. *Let* $\Gamma : \Omega \to \mathbb{C}^p$ *be holomorphic,* Ω *open subset of* $\mathbb{C}^n$. *Let* Ω_0 *be open and dense in* Ω *and suppose that for every* $\xi \in \Omega_0$ *the fiber* $\Gamma^{-1}(\Gamma(\xi))$ *through* ξ *is an affine subspace of* Ω *of codimension* k, $k \leq n-1$. *Then* Γ *satisfies properties* (ΓA1), (ΓA2), *and* $\Omega^{\text{sing}} \subset \Omega \backslash \Omega_0$.

There are different ways to formulate the problems in a more general setting. One way is to observe that $\varkappa$ is a meromorphic mapping and work in the category of meromorphic mappings. Another way is to define the set $\text{Mor}(X, Y)$ of morphisms from the analytic set X to the (compact) analytic set Y as follows. The morphism χ belongs to $\text{Mor}(X, Y)$ if there is an analytic subset C of X such that locally $\dim C < \dim X$ and such that there is a holomorphic mapping $\widetilde{\chi} : X \backslash C \to Y$. Then χ consists of the pair $(C, \widetilde{\chi})$. However, our analysis will sometimes rely on the fact that the fibration $\varkappa$ is related to a holomorphic mapping Γ (as in (ΓA1), (ΓA2)) which is fixed in this monograph. Thus, if we chose to work with morphisms which are smooth on a dense open part of Ω, a number of results would be lost. This is the reason for us to avoid the general category language and to restrict to the local properties of $\varkappa$.

2.5 Methods of complex analytic geometry

In this section, we will review some notions and facts of complex analytic geometry which will be frequently used in the sequel. Following [51], let us define meromorphic mappings.

Definition 2.5.1. A mapping $\tau : X \to Y$ between complex manifolds X and Y is called *meromorphic*, if the following three conditions hold.

(1) For every $x \in X$ the image set $\tau(x) \subset Y$ is non-empty and compact in Y.

(2) The graph of the mapping τ, that is the set all pairs $(x, y) \in X \times Y$ such that $y \in \tau(x)$, is a connected complex analytic subset of $X \times Y$ of dimension equal to the dimension of X.

(3) There exist a dense subset X^* of X, such that for every $x \in X^*$ the image set $\tau(x)$ consists of a single point.

One of the important tools for the analysis of the structure of analytic sets are dimension estimates. Let M be a complex n-dimensional manifold and let $E \subset M$ be an arbitrary subset of M. Dimension of E is defined by

$$\dim E = \sup\{\dim F : F \subset E\},$$

where supremum is taken over all smooth submanifolds of M contained in E. If the set on the right hand side is empty, we assume that the supremum is $-\infty$. If E is a submanifold or an analytic subset of M, then this notion of dimension coincides with the standard, used in the analytic geometry. If π is a projection from the Cartesian product to one of the sets, then the upper bound for the dimension of the preimage is given by the following theorem.

Theorem 2.5.2. *Let $E \subset M \times N$, where M, N are complex manifolds. Let $\pi : E \to M$ be the natural projection. Assume that for $k \in \mathbb{N}$ holds*

$$\dim \pi^{-1}(z) \leq k, \ \forall z \in \pi(E).$$

Then $\dim E \leq k + \dim \pi(E)$.

By a *globally analytic subset of the manifold M* we mean any set of the form

$$Z(f_1, \dots, f_k) = \{z \in M : f_1(z) = \dots = f_k(z) = 0\},$$

where $f_1, \dots, f_k$ are some holomorphic functions on M. A subset Z of the manifold M is called *analytic* if every point of M has an open neighborhood U such that the set $Z \cap U$ is globally analytic in U. Analytic subsets of open sets in M are called *locally analytic in M*. In particular, a set Z is analytic in M if and only if it is locally analytic and closed.

Theorem 2.5.3. (1) *Let $V \subset M$ and $W \subset N$ be non-empty subsets of manifolds M and N, respectively. Then the product $V \times W$ is (locally) analytic in $M \times N$ if and only if both V and W are (locally) analytic in M and N, respectively.*

(2) *(The Analytic Graph Theorem) Every continuous mapping $f : M \to N$ with analytic graph is holomorphic.*

(3) *Every proper analytic subset Z of a connected manifold M is nowhere dense and its complement $M \backslash Z$ is open and connected.*

If $V \subset M$ and $W \subset N$ are locally analytic subsets, then the mapping $f : V \to W$ is called *holomorphic* if every point in V has an open neighborhood U in M such that $f|_{U \cap V}$ is the restriction of a holomorphic mapping from U to N.

Theorem 2.5.4. *Let $k \in \mathbb{N}$ and let f be a holomorphic mapping defined above. Then*

(1) $\dim V \geq k + \dim f(V)$, *if* $\dim f^{-1}(w) \geq k$ *for all* $w \in f(V)$.

(2) $\dim V \leq k + \dim f(V)$, *if* $\dim f^{-1}(w) \leq k$ *for all* $w \in f(V)$.

The proof of the first part is based on the reduction of the estimate to a subset Z of the image $f(V)$ of dimension equal to $\dim f(V)$. So, in this case, the set $f(V)$ can be semi-analytic. The second estimate is based on Theorem 2.5.2. If the level sets are of the same dimension, we have

Corollary 2.5.5. *Let $Z \subset W$ be a locally analytic subset of N. If* $\dim f^{-1}(w) = k$ *holds for all $w \in Z$, then* $\dim f^{-1}(Z) = k + \dim Z$.

For $z \in V$ let $l_z f$ denote the germ of the fiber of f

$$l_z f = \left(f^{-1}(f(z))\right)_z$$

at the point z.

Theorem 2.5.6. (1) (semicontinuity) *The mapping*

$$V \ni z \mapsto \dim l_z f$$

is upper semicontinuous: for all $a \in V$ the inequality $\dim l_z f \leq \dim l_a f$ *holds in some neighborhood of a.*

(2) *The inequality* $\dim l_z f \geq \dim_z V - \dim f(V)$ *holds for all $z \in V$.*

(3) (Cartan–Remmert). *For every $k \in \mathbb{N}$ the set $\{z \in V : \dim l_z f \geq k\}$ is analytic.*

The set E is called *analytically constructible* if there exist analytic sets V and W such that $E = V \backslash W$.

Theorem 2.5.7. *Let $V, W \subset M$ be analytic sets. Then the closures of any connected component of the set $V \backslash W$, any open and closed set in $V \backslash W$, and, in particular, the set $\overline{V \backslash W}$, are unions of some simple components of V not contained in W, and so they are analytic. The family of the connected components of the set $V \backslash W$ is locally finite.*

We will also need the following

Proposition 2.5.8. *If V is an analytically constructible set, then the set $\bar{V}\backslash V$ is nowhere dense in $\bar{V}$.*

Proofs of the above statements can be found in [32] in II.1.4, II.3.4, V.1.1, II.3.6, V.3.2, V.3, IV.2.10. The last proposition is a remark on p. 250 in [32]. By [51] and [32, V.5.1] we have:

Theorem 2.5.9 (Remmert's Proper Mapping Theorem). *If $f : X \to Y$ is a proper holomorphic mapping of analytic spaces, then its range $f(X)$ is analytic in Y. Consequently, the image of each analytic subset of X is an analytic subset of Y.*

Let us briefly review some other techniques. A subset Z of a complex manifold M is called *thin*, if it is closed, nowhere dense, and for every open set Ω in M every holomorphic function on $\Omega\backslash Z$ which is locally bounded near $\Omega \cap Z$ extends to a holomorphic function on Ω. An important statement which will be of use to us is the following ([32, II.3.5]):

Proposition 2.5.10. *Every nowhere dense analytic subset of a complex manifold M is thin in M.*

If E is a nowhere dense subset of a non-empty set F, where both E and F are analytically constructible in M, then $\dim E < \dim F$. In fact, as in [32, p. 253], we have

Proposition 2.5.11. *For E and F as above, E is nowhere dense in F if and only if $\dim_z E < \dim_z F$ for all $z \in E$.*

If M is connected, we have the following property ([32, II.3.6]).

Proposition 2.5.12. *If the complex manifold M is connected, then every proper analytic subset Z of M is nowhere dense and its complement $M\backslash Z$ is connected and open.*

An important inequality for studying dimensions of analytic sets is that for any analytic germs $A_1, \dots, A_k$ at a point a of a complex manifold M, holds

$$\text{codim } (A_1 \cap \ldots \cap A_k) \leq \text{ codim } A_1 + \ldots \text{ codim } A_k. \qquad (2.5.1)$$

The proof can be found in [32, III.4.6]. Let now f_i, $i = 1, \dots, k$, be analytic functions on a neighborhood of a such that $f_i(a) = 0$, for all $i = 1, \dots, k$. Let Z_{f_i} denote the zero set of f_i. As a consequence of (2.5.1), we get

$$\text{codim } (Z_{f_1} \cap \ldots \cap Z_{f_k}) \leq k,$$

or

$$\dim (Z_{f_1} \cap \ldots \cap Z_{f_k}) \geq n - k. \qquad (2.5.2)$$

2.6 Some properties of affine parametric fibrations

In this section we will discuss several general properties of parametric fibrations and their implications for the non-parametric case with $k = 1$. Thus, under conditions (ΓR1), (ΓR2) we have:

Theorem 2.6.1. *Let Γ satisfy conditions* (ΓR1), (ΓR2) *with $k = 1$ and let $(y, \xi) \in \Omega^{\text{sing}}$. Then the mapping $\eta \mapsto \Gamma(y, \eta)$ is constant.*

To prove this, we start with two lemmas.

Lemma 2.6.2. *Let A satisfy conditions* (R1) *and* (R2). *For $\omega \in \Omega^{\text{sing}}$ and for every k-dimensional linear subspace C of $\mathbb{C}^n$ there exists a sequence $\omega_j \in \Omega^{(k)}$ such that ω_j converges to ω as $j \to \infty$, $\varkappa(\omega_j)$ converges to $\varkappa \in \mathbb{G}_{n-k}(\mathbb{C}^n)$ as $j \to \infty$, and $\varkappa \cap C \neq \{0\}$.*

Proof. The set $G(C) = \{L \in \mathbb{G}_{n-k}(\mathbb{C}^n) : L \cap C = \{0\}\}$ is holomorphically diffeomorphic to $\mathbb{C}^{k(n-k)}$ (cf. [32, B.6.6] and [32, Prop., p.367]). Therefore, if there exist a neighborhood U of the point ω in Ω such that the image $\varkappa(U \cap \Omega^{(k)})$ is contained in a compact set in $G(C)$, then $\omega \notin \Omega^{\text{sing}}$. For the latter conclusion we use Proposition 2.5.10, which means, in particular, that every bounded holomorphic function on the complement of an analytic subset of U has a holomorphic extension to U.

Lemma 2.6.3. *Let Γ satisfy conditions* (ΓR1), (ΓR2). *For every point $(y, \xi) \in \Omega^{\text{sing}}$ and for every k-dimensional linear subspace C of $\mathbb{C}^n$ there exists a linear subspace L of C with $\dim L \geq 1$ such that for every $l \in L$ holds*

$$\Gamma(y, \xi + l) = \Gamma(y, \xi).$$

Proof. According to Lemma 2.6.2 there exists a sequence $\omega_j \to (y, \xi)$ with $\omega_j \in \Omega^{(k)}$ such that the limit $\lim_j \varkappa(\omega_j) = \varkappa$ exists and $\varkappa \cap C \neq \{0\}$. Let $L = \varkappa \cap C$. In view of condition (R2) for all $\omega_j = (y_j, \xi_j)$ and $z_j \in \varkappa(\omega_j)$ holds $\Gamma(y_j, \xi_j + z_j) = \Gamma(y_j, \xi_j)$. The statement of the lemma now follows from the continuity of Γ.

The proof of Theorem 2.6.1 follows from Lemma 2.6.3 with $k = 1$, since in this case Lemma 2.6.3 holds for an arbitrary C.

Corollary 2.6.4. *Let Γ satisfy* (ΓA1), (ΓA2) *with $k = 1$. Then the set of essential singularities Ω^{sing} is empty.*

Proof. Let $\xi \in \Omega^{\text{sing}}$. According to Theorem 2.6.1 the mapping $\eta \mapsto \Gamma(\eta)$ is constant in Ω and the matrix $D_\xi \Gamma$ vanishes. This contradicts $k = 1$.

2.7 General properties of affine fibrations

In this section we will analyze some structural properties of the singular set Ω^{sing} and establish several general properties of affine fibrations. From now on we will always assume that the mapping A satisfies (A1) and (A2), unless stated otherwise.

We start with noting that the graph of $\varkappa$ is given by

$$G = \{(\xi, L) \in \Omega \times \mathbb{G}_{n-k}(\mathbb{C}^n) : \xi \in \Omega^{(k)}, L = \varkappa(\xi)\}$$

and we also define

$$E = \{(\xi, L) \in \Omega \times \mathbb{G}_{n-k}(\mathbb{C}^n) : L \subset \ker A(\xi)\}.$$

Clearly E is a closed analytic subset of $\Omega \times \mathbb{G}_{n-k}(\mathbb{C}^n)$ and

$$G = (\Omega^{(k)} \times \mathbb{G}_{n-k}(\mathbb{C}^n)) \cap E = E \backslash \{(\xi, L) \in \Omega \times \mathbb{G}_{n-k}(\mathbb{C}^n) : \xi \in \Omega \backslash \Omega^{(k)}\},$$

which is the complement in E of the closed analytic subset $E \cap \{(\xi, L) \in \Omega \times \mathbb{G}_{n-k}(\mathbb{C}^n) : \xi \in \Omega \backslash \Omega^{(k)}\}$. Let $\bar{\varkappa}(\xi) \subset \mathbb{G}_{n-k}(\mathbb{C}^n)$ be the set of the limits of $\varkappa(\xi_j)$ as $\xi_j \to \xi$, $\xi_j \in \Omega^{(k)}$.

For $V \subset \mathbb{G}_{n-k}(\mathbb{C}^n)$ we define the set $\widetilde{V}$ by

$$\widetilde{V} = \bigcup_{L \in V} L.$$

We will use the following properties of the mapping $V \mapsto \widetilde{V}$:

Proposition 2.7.1. *Let V be an analytic subset of $\mathbb{G}_{n-k}(\mathbb{C}^n)$. Then $\widetilde{V}$ is an analytic subset of $\mathbb{C}^n$. Moreover,*

$$\dim \widetilde{V} \leq \dim V + n - k.$$

If $\dim V \geq 1$, then $\dim \widetilde{V} \geq n - k + 1$.

Proof. The incidence relation I defined by

$$I = \{(\xi, L) \in \mathbb{C}^n \times \mathbb{G}_{n-k}(\mathbb{C}^n) : \xi \in L\}$$

is an analytic (even algebraic) subset of $\mathbb{C}^n \times \mathbb{G}_{n-k}(\mathbb{C}^n)$. Let $\pi_1 : (\xi, L) \mapsto \xi$ be the projection from $\mathbb{C}^n \times \mathbb{G}_{n-k}(\mathbb{C}^n)$ to $\mathbb{C}^n$ and let V be an analytic subset of $\mathbb{G}_{n-k}(\mathbb{C}^n)$. Then

$$\widetilde{V} = \pi_1(I \cap (\mathbb{C}^n \times V)).$$

The set $\mathbb{C}^n \times V$ and, therefore, $I \cap (\mathbb{C}^n \times V)$ is analytic in $\mathbb{C}^n \times \mathbb{G}_{n-k}(\mathbb{C}^n)$, so $\widetilde{V}$ is analytic if the mapping π_1 is proper by Remmert's proper mapping theorem 2.5.9. This is indeed the case in view of the compactness of the Grassmannian $\mathbb{G}_{n-k}(\mathbb{C}^n)$.

Now we will prove the estimate of the dimension. Let $\pi_2 : (\xi, L) \mapsto L$ be the projection from $\mathbb{C}^n \times \mathbb{G}_{n-k}(\mathbb{C}^n)$ to $\mathbb{G}_{n-k}(\mathbb{C}^n)$. We have $\pi_2(I \cap (\mathbb{C}^n \times V)) = V$ and let $v \in V \subset \mathbb{G}_{n-k}(\mathbb{C}^n)$. The fiber of π_2 in v in $I \cap (\mathbb{C}^n \times V)$ is the set $\{(\xi, v) : \xi \in v\}$, so that $\dim(\pi_2^{-1}(v) \cap I \cap (\mathbb{C}^n \times V)) = \dim \widetilde{v} = n - k$. The application of Theorem 2.5.4 implies that $\dim(I \cap (\mathbb{C}^n \times V) = n-k+\dim V$. The projection π_1 can not increase the dimension and the estimate of Proposition follows.

Assume now that $\dim V \geq 1$ and $\dim \widetilde{V} = n - k$. The set $\widetilde{V}$ is analytic, therefore it has at most finite number of irreducible components at each point. By definition of $\widetilde{V}$, each of this components can be only a $(n-k)$-dimensional subspace of $\mathbb{C}^n$. But this would imply that the set V is finite, a contradiction with $\dim V = 1$. The proof is complete.

Now we can prove

Proposition 2.7.2. *The following holds:*

(i) *The set G is analytically constructible, the closure $\bar{G}$ is analytic. The set $\bar{\varkappa}(\xi)$ is analytic and connected. The set $\widetilde{\bar{\varkappa}}(\xi)$ is analytic in $\mathbb{C}^n$.*

(ii) *We have $\widetilde{\bar{\varkappa}}(\xi) \subset \ker A(\xi)$ for every $\xi \in \Omega$.*

(iii) *If $\xi \in \Omega^{\mathrm{sing}}$, then $\dim \bar{\varkappa}(\xi) \geq 1$ and $\dim \widetilde{\bar{\varkappa}}(\xi) \geq n - k + 1$. On the other hand, if $\xi \in \Omega \backslash \Omega^{\mathrm{sing}}$, then $\bar{\varkappa}(\xi)$ contains only one element $L \in \mathbb{G}_{n-k}(\mathbb{C}^n)$ and $\widetilde{\bar{\varkappa}}(\xi) = L$.*

(iv) *Moreover, if $\xi \in \Omega^{\mathrm{sing}}$ and C is an irreducible component of $\widetilde{\bar{\varkappa}}(\xi)$, then $\dim C \geq n - k + 1$.*

(v) *If $\xi \in \Omega^{(k-1)} \cap \Omega^{\mathrm{sing}}$, then $\widetilde{\bar{\varkappa}}(\xi) = \ker A(\xi)$ and $\dim \widetilde{\bar{\varkappa}}(\xi) = n - k + 1$.*

Proof. (i) We have already shown that G is analytically constructible. The closure $\bar{G}$ is analytic because the closure of any analytically constructible set is an analytic set (Theorem 2.5.7). It follows that

$$\bar{\varkappa}(\xi) = \{L \in \mathbb{G}_{n-k}(\mathbb{C}^n) : (\xi, L) \in \bar{G}\}$$

is an analytic subset of $\mathbb{G}_{n-k}(\mathbb{C}^n)$, since $\{\xi\} \times \bar{\varkappa}(\xi) = (\{\xi\} \times \mathbb{G}_{n-k}(\mathbb{C}^n)) \cap \bar{G}$ and hence analytic, implying the analyticity of $\bar{\varkappa}(\xi)$ by Theorem 2.5.3, (1).

Now we shall prove that $\bar{\varkappa}(\xi)$ is connected. Let U, V be open subsets of $\mathbb{G}_{n-k}(\mathbb{C}^n)$, $U \cap V = \emptyset$ and $\bar{\varkappa}(\xi) \subset (U \cup V)$. Let $A = \{\eta \in \Omega^{(k)} : \varkappa(\eta) \in U\}$ and $B = \{\eta \in \Omega^{(k)} : \varkappa(\eta) \in V\}$. Then A and B are disjoint open subsets of $\Omega^{(k)}$. There is an open neighborhood W of ξ, such that $W \cap \Omega^{(k)}$ is connected and $W \cap \Omega^{(k)} \subset A \cup B$. Hence $A \cap W \cap \Omega^{(k)} = \emptyset$ or $B \cap W \cap \Omega^{(k)} = \emptyset$ and it follows that $\bar{\varkappa}(\xi) \cap U = \emptyset$ or $\bar{\varkappa}(\xi) \cap V = \emptyset$. This completes the proof that $\bar{\varkappa}(\xi)$ is connected.

The analyticity of $\widetilde{\bar{\varkappa}}(\xi)$ in $\mathbb{C}^n$ follows from Proposition 2.7.1.

(ii) The inclusion (and equality) holds for $\xi \in \Omega \backslash \Omega^{\mathrm{sing}}$. Now, let $\xi \in \Omega^{\mathrm{sing}}$ and let $\varkappa_0 \in \widetilde{\bar{\varkappa}}(\xi)$. Then there exists a sequence $\xi_j \in \Omega^{(k)}$ such that ξ_j

converges to ξ in Ω and $\varkappa(\xi_j)$ converges to $\varkappa_0$ in the Grassmannian. Then we have $\varkappa(\xi_j) = \ker A(\xi_j)$ and $A(\xi_j)\varkappa(\xi_j) = 0$. Taking a limit and using the continuity of A, we get $A(\xi)\varkappa_0 = 0$, which means $\varkappa_0 \subset \ker A(\xi)$. Because this holds for any $\varkappa_0 \in \tilde{\bar{\varkappa}}(\xi)$, we obtain the statement.

(iii) If $\xi \notin \Omega^{\text{sing}}$, then $\bar{\varkappa}(\xi)$ consists of one point, so that $\dim \bar{\varkappa}(\xi) = 0$. Conversely, if $\xi \in \Omega^{\text{sing}}$ we have that for every $C \in \mathbb{G}_k(\mathbb{C}^n)$ the set

$$\mathbb{G}_{n-k}(\mathbb{C}^n)_C = \{L \in \mathbb{G}_{n-k}(\mathbb{C}^n) : L \cap C \neq \{0\}\}$$

is a hypersurface (for general n, k with singularities, analytically constructible in $\mathbb{G}_k(\mathbb{C}^n) \times \mathbb{G}_{n-k}(\mathbb{C}^n)$, see [32, p.367]), the intersection property $\bar{\varkappa}(\xi) \cap \mathbb{G}_{n-k}(\mathbb{C}^n)_C \neq \emptyset$ for every $C \in \mathbb{G}_k(\mathbb{C}^n)$ implies that $\bar{\varkappa}(\xi)$ is infinite, hence $\dim \bar{\varkappa}(\xi)$ can not be equal to zero. Proposition 2.7.1 implies that $\dim \tilde{\bar{\varkappa}}(\xi) \geq n - k + 1$.

(iv) Suppose again that $\dim C \leq n - k$. Then $C \in \mathbb{G}_{n-k}(\mathbb{C}^n)$, $C \in \bar{\varkappa}(\xi)$. The set $\bar{\varkappa}(\xi)$ is connected by (i) and contains more than one element by (iii). It follows that C belongs to the closure of a smooth part of $\bar{\varkappa}(\xi)$ of positive dimension, which implies that C is contained in an irreducible component of $\tilde{\bar{\varkappa}}(\xi)$ of dimension $\geq n - k + 1$, in contradiction with the assumption that C is an irreducible component of $\tilde{\bar{\varkappa}}(\xi)$.

(v) If $\xi \in \Omega^{(k-1)} \cap \Omega^{\text{sing}}$, then $\ker A(\xi)$ is a linear subspace of codimension $k-1$, and $\xi \in \Omega^{\text{sing}}$ implies that $\tilde{\bar{\varkappa}}(\xi)$ has codimension $k-1$. For $\eta \in \ker A(\xi)$, $\eta \neq 0$, let L denote the linear span of η. Then $L \subset \ker A(\xi)$ and there exists $C \in \mathbb{G}_k(\mathbb{C}^n)$ such that $L = \ker A(\xi) \cap C$. By Lemma 2.6.2, there exists $\varkappa_0 \in \bar{\varkappa}(\xi)$ with $\dim(\varkappa_0 \cap C) \geq 1$. Then $\varkappa_0 \cap C \subset \ker A(\xi) \cap C = L$ and, therefore, $\varkappa_0 \cap C = L$. This implies $\eta \in L = \varkappa_0 \cap C \subset \varkappa_0 \subset \tilde{\bar{\varkappa}}(\xi)$ and the equality of $\tilde{\bar{\varkappa}}(\xi)$ and $\ker A(\xi)$.

Remark 2.7.3. *The statement of Proposition 2.7.2,* (iii), *implies in particular, that for $\xi \in \Omega^{\text{sing}}$ holds $\dim_\xi(\xi + \tilde{\bar{\varkappa}}(\xi)) \geq n - k + 1$.*

Remark 2.7.4. *The graph of the mapping $\bar{\varkappa}$ is analytic in $\Omega \times \mathbb{G}_{n-k}(\mathbb{C}^n)$. The analytic graph theorem* (Theorem 2.5.3, (2)) *implies then that the following conditions are equivalent:*

(i) *$\bar{\varkappa}$ is locally bounded.*

(ii) *$\bar{\varkappa}$ is continuous.*

(iii) *$\bar{\varkappa}$ is holomorphic.*

(iv) *Ω^{sing} is empty.*

Remark 2.7.5. *For $k = n - 1$ we have $\mathbb{G}_{n-k}(\mathbb{C}^n) = \mathbb{P}(\mathbb{C}^n)$ and by Chow's theorem* ([6]) *it follows that $\bar{\varkappa}(\xi)$ must be algebraic, defined by some homogeneous polynomial equations in $\mathbb{C}^n$. For arbitrary k, using Plücker embedding $\mathbb{G}_{n-k}(\mathbb{C}^n) \to \mathbb{P}(\Lambda^{n-k}\mathbb{C}^n)$ and Chow's theorem, it follows that $\bar{\varkappa}(\xi)$ is algebraic in $\Lambda^{n-k}\mathbb{C}^n$.*

Remark 2.7.6. *If for an algebraic variety V we denote by $T_\eta V$ its Zariski tangent space at η, which is the intersection of all $\ker Df(\eta)$ with f a polynomial, which is constant on V, then it follows that, for every $\eta \in \tilde{\bar{\varkappa}}(\xi)$ such that $\xi + \eta \in \Omega$, we have $T_\eta \tilde{\bar{\varkappa}}(\xi) \subset \ker A(\xi + \eta)$.*

In view of Proposition 2.7.2 and compactness of $\mathbb{G}_{n-k}(\mathbb{C}^n)$, the mapping $\bar{\varkappa} : \Omega \to \mathbb{G}_{n-k}(\mathbb{C}^n)$ is meromorphic in a sense that it coincides with holomorphic mapping $\varkappa$ on $\Omega^{(k)}$, its graph is analytic and the values $\bar{\varkappa}(\xi)$ for $\xi \in \Omega \backslash \Omega^{(k)}$ are compact. Now we will derive an upper bound on the dimension of Ω^{sing}.

Theorem 2.7.7. *The set Ω^{sing} is an analytic subset of Ω with*

$$\dim \Omega^{\text{sing}} \leq n - 2.$$

Proof. In fact, this is a consequence of [51, p.369]. However, let us give an idea of its proof. Let π be the restriction to $\bar{G}$ of the projection $(\xi, L) \to \xi$. Then, according to a theorem of Cartan and Remmert (Theorem 2.5.6, (3)), the set Σ of the points $g \in \bar{G}$ such that the dimension of the germs at g of the fiber $\pi^{-1}(\pi(g))$ has positive dimension is an analytic subset of $\bar{G}$, so an analytic subset of $\Omega \times \mathbb{G}_{n-k}(\mathbb{C}^n)$. On the other hand, due to the compactness of $\mathbb{G}_{n-k}(\mathbb{C}^n)$, $\pi|_{\bar{G}}$ is a proper analytic mapping from $\bar{G}$ to Ω, so in view of Remmert's proper mapping theorem (Theorem 2.5.9), $\pi(\bar{G})$ is an analytic subset of Ω. However, $\pi(\bar{G} \backslash G) = \Omega^{\text{sing}}$, so Ω^{sing} is an analytic subset of Ω.

By Proposition 2.7.2 and Proposition 2.5.8, the set $\bar{G} \backslash G$ is nowhere dense in $\bar{G}$. Then, by Proposition 2.5.11, we have $\dim \bar{G} \backslash G < \dim \bar{G}$ and since every analytically constructible set is a locally finite union of analytically constructible leaves ([32, Prop.3(2),p.249]), we have $\dim \bar{G} = \dim G$. Thus, $\dim \bar{G} \backslash G \leq \dim G - 1 = n - 1$. The projection π of $\bar{G} \backslash G$ to the first factor by Proposition 2.7.2 has dimension of each fiber ≥ 1 and image equal to Ω^{sing}. By Theorem 2.5.4 we get

$$\dim \Omega^{\text{sing}} \leq \dim \bar{G} \backslash G \leq n - 2.$$

Proposition 2.7.8. *The mapping $\varkappa$ is holomorphically extendible over every point $\xi \in \Omega \backslash \Omega^{\text{sing}}$. The set $\Omega \backslash \Omega^{\text{sing}}$ is a connected subset of Ω and, therefore, $\varkappa$ allows the holomorphic extension to $\Omega \backslash \Omega^{\text{sing}}$. This extension coincides with the restriction of $\bar{\varkappa}$ to $\Omega \backslash \Omega^{\text{sing}}$. Moreover, for every $\xi \in \Omega \backslash \Omega^{\text{sing}}$ and $\eta \in (\xi + \varkappa(\xi)) \cap (\Omega \backslash \Omega^{\text{sing}})$ we have $\varkappa(\xi) = \varkappa(\eta)$.*

Proof. By definition of Ω^{sing} each point $\xi \in \Omega \backslash \Omega^{\text{sing}}$ is a removable singularity for $\varkappa$ and the mapping $\varkappa$ is extendible in a neighborhood of this point in $\Omega \backslash \Omega^{\text{sing}}$, which is an open subset of Ω. It is also connected by Proposition 2.5.12, and $\varkappa$ has the global holomorphic extension to $\Omega \backslash \Omega^{\text{sing}}$, say $\varkappa_0$. The graph of $\varkappa_0$ is between the graphs of $\varkappa$ and $\bar{\varkappa}$ in $\Omega \times \mathbb{G}_{n-k}(\mathbb{C}^n)$. It follows that the closure of the graph of $\varkappa_0$ is equal to the graph of $\bar{\varkappa}$ and this implies that $\varkappa_0$ is equal to the restriction of $\bar{\varkappa}$ to $\Omega \backslash \Omega^{\text{sing}}$. Suppose now, that ξ and η are as in the conditions of Proposition 2.7.8. Let $\xi_j \in \Omega^{(k)}$, $\xi_j \to \xi$. Then there exists a sequence $\eta_j \in (\xi_j + \varkappa(\xi_j)) \cap \Omega^{(k)}$ such that $\eta_j \to \eta$. Then $\varkappa(\eta_j) \to \varkappa(\eta)$ as $\varkappa$ is extendible over η. But $\varkappa(\xi_j) = \varkappa(\eta_j)$ because $\xi_j, \eta_j \in \Omega^{(k)}$ and (A2), so that $\varkappa(\xi) = \lim_j \varkappa(\xi_j) = \varkappa(\eta)$.

Now we will introduce the equivalence relation defined by the fibration $\varkappa$. On the regular subset $\Omega^{(k)}$ two points are in the same equivalence class if they belong to the same fiber. The closure of such equivalence relation defines a relation on Ω, for which the class of a point $\xi \in \Omega$ consists of the limit cone $\bar{\varkappa}(\xi)$. In particular, for $\xi \in \Omega' = \Omega \backslash \Omega^{\text{sing}}$ the equivalence classes are of the same dimension as for $\xi \in \Omega^{(k)}$. This is used to establish some relations with Ω^{sing} and estimate its dimension.

Proposition 2.7.9. *The set*

$$\mathcal{R} = \{(\xi, \eta) \in \Omega' \times \Omega' : \eta - \xi \in \varkappa(\xi)\}$$

is a closed analytic subset of $\Omega' \times \Omega'$, *smooth,* $\dim \mathcal{R} = 2n - k$. *Moreover,* $\mathcal{R}$ *defines an equivalence relation in* Ω', *namely*

(i) $(\xi, \xi) \in \mathcal{R}$ *for every* $\xi \in \Omega'$.

(ii) *If* $(\xi, \eta) \in \mathcal{R}$, *then* $(\eta, \xi) \in \mathcal{R}$.

(iii) *If* $(\xi, \eta) \in \mathcal{R}$ *and* $(\eta, \zeta) \in \mathcal{R}$, *then* $(\xi, \zeta) \in \mathcal{R}$.

If $\xi \in \Omega'$ *then*

$$\mathcal{R}(\xi) = \{\eta \in \Omega' : (\xi, \eta) \in \mathcal{R}\}$$

is the equivalence class of ξ *and* Ω' *is partitioned into equivalence classes. The set* $\Omega'/\mathcal{R}$ *is a smooth complex analytic manifold of dimension equal to* k.

Proof. The statements in the first sentence about $\mathcal{R}$ are obvious, with $\dim \mathcal{R} = n + (n - k) = 2n - k$. Properties (ii) and (iii) follow from Proposition 2.7.8. If $\mathcal{R}(\xi) \cap \mathcal{R}(\eta) \neq \emptyset$, then $\mathcal{R}(\xi) = \mathcal{R}(\eta)$ and Ω' is equal to the union of $\mathcal{R}(\xi)$'s. Because $\mathcal{R}$ is a closed subset of $\Omega' \times \Omega'$, the quotient space $\Omega'/\mathcal{R}$ of equivalence classes is a Hausdorff topological space, when provided with the strongest topology for which the natural projection $p : \xi \mapsto \mathcal{R}(\xi)$ is continuous. Using transversal sections we see that actually $\Omega'/\mathcal{R}$ is a smooth complex analytic manifold of dimension equal to k, since $\dim \mathcal{R}(\xi) = n - k$.

The set $\mathcal{R} \cap (\Omega^{(k)} \times \Omega^{(k)})$ is equal to the complement of

$$((\Omega \backslash \Omega^{(k)}) \times \Omega) \cup (\Omega \times (\Omega \backslash \Omega^{(k)})) \tag{2.7.1}$$

in the analytic subset

$$\{(\xi, \eta) \in \Omega \times \Omega : \eta - \xi \in \ker A(\xi)\}$$

of $\Omega \times \Omega$. Because the set (2.7.1) is an analytic subset of $\Omega \times \Omega$, it follows that $\mathcal{R}$ is analytically constructible and the closure $\bar{\mathcal{R}}$ of $\mathcal{R}$ in $\Omega \times \Omega$ is an analytic subset of $\Omega \times \Omega$ (Theorem 2.5.7). Thus, we have

Lemma 2.7.10. *The set $\mathcal{R} = \bar{\mathcal{R}} \cap (\Omega' \times \Omega')$ is analytically constructible, its closure $\bar{\mathcal{R}}$ is an analytic subset of $\Omega \times \Omega$ and*

$$\dim(\bar{\mathcal{R}} \backslash \mathcal{R}) \leq 2n - k - 1.$$

Proof. We have already shown that $\mathcal{R}$ is analytically constructible and its closure $\bar{\mathcal{R}}$ is analytic. Then the set $\mathcal{R} \cap (\Omega^{(k)} \times \Omega^{(k)})$ is dense in $\bar{\mathcal{R}}$ and Proposition 2.5.11 and 2.7.9 imply

$$\dim(\bar{\mathcal{R}} \backslash (\mathcal{R} \cap (\Omega^{(k)} \times \Omega^{(k)}))) < \dim \bar{\mathcal{R}} = 2n - k.$$

The dimension estimate of Lemma follows from it.

We have the projections

$$\pi_1 : (\xi, \eta) \mapsto \xi : \Omega \times \Omega \to \Omega, \ \pi_2 : (\xi, \eta) \mapsto \eta : \Omega \times \Omega \to \Omega,$$

which are analytic mappings. We will actually consider the restrictions of π_1 and π_2 to $\bar{\mathcal{R}}$. Note that $\pi_2(\pi_1^{-1}(\{\xi\}) \cap \bar{\mathcal{R}})$ is equal to the cone $\tilde{\bar{\varkappa}}(\xi)$ over $\bar{\varkappa}(\xi)$.

Because by Theorem 2.7.7 the set Ω^{sing} is an analytic subset of Ω, also

$$\bar{\mathcal{R}}_1^{\text{sing}} = \bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)$$

is an analytic subset of $\Omega \times \Omega$ and we have the analytic mappings $\pi_1 : \bar{\mathcal{R}}_1^{\text{sing}} \to \Omega^{\text{sing}}$, which is surjective, and $\pi_2 : \bar{\mathcal{R}}_1^{\text{sing}} \to \Omega$. We have

$$\bigcup_{\xi \in \Omega^{\text{sing}}} (\xi + \tilde{\bar{\varkappa}}(\xi)) = \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)). \tag{2.7.2}$$

Note that the dimension estimate $\dim \Omega^{\text{sing}} \leq n-2$ in Theorem 2.7.7 follows also from Lemma 2.7.10. Indeed, the set $\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega')$ is contained in $\bar{\mathcal{R}} \backslash \mathcal{R}$, so has dimension $\leq 2n - k - 1$. The fibers of π_1 from $\bar{\mathcal{R}}_1^{\text{sing}}$ onto Ω^{sing} have dimension $\geq n - k + 1$. Hence $\dim \Omega^{\text{sing}} \leq n - 2$.

Proposition 2.7.11. *The following estimates hold:*

$$n - k + 1 \leq \dim \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)), \tag{2.7.3}$$

$$\dim \Omega^{\text{sing}} + n - k + 1 \leq \dim(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)). \tag{2.7.4}$$

Proof. The estimate (2.7.3) follows immediately from (2.7.2). By Proposition 2.7.2 for each $w \in \pi_1(\bar{\mathcal{R}}_1^{\text{sing}}) = \Omega^{\text{sing}}$ the fiber $\pi_1^{-1}(w)$ restricted to $\bar{\mathcal{R}}_1^{\text{sing}}$ satisfies $\dim \pi_1^{-1}(w) \geq n - k + 1$, hence by Theorem 2.5.4 we get (2.7.4).

Now we will analyze the structure of Ω^{sing}. Let us first overview a general construction and then concentrate of the case $k = n - 1$.

By a m-dimensional projective subspace of the space $\mathbb{P} = \mathbb{P}(\mathbb{C}^n)$ we mean any subset of $\mathbb{P}(\mathbb{C}^n)$ of the form $\mathbb{P}(L)$, where $L \in \mathbb{G}_{m+1}(\mathbb{C}^n)$. For more discussion on the general structure of these sets see [32]. By $\mathbb{G}_m(\mathbb{P})$ we denote the set of all m-dimensional projective subspaces of $\mathbb{P}(\mathbb{C}^n)$, so that we have the bijection

$$\omega : \mathbb{G}_{m+1}(\mathbb{C}^n) \ni L \mapsto \mathbb{P}(L) \in \mathbb{G}_m(\mathbb{P}).$$

If L is one dimensional, we will identify L with $\mathbb{P}(L)$ and $\mathbb{G}_0(\mathbb{P})$ with $\mathbb{P}$. Via this mapping ω we can identify elements of $\mathbb{G}_m(\mathbb{P})$ with $(m+1)$-dimensional linear subspaces of $\mathbb{C}^n$. Let $\mathcal{D} = \mathbb{G}_{n-k-1}(\mathbb{P})$ and for a given $L \in \mathcal{D}$ and complementary to $\omega^{-1}(L)$ k-dimensional linear subspace H of $\mathbb{C}^n$, let $\pi_{L,H}$ be the linear projection from $\mathbb{C}^n$ onto H along L.

Let $\mathcal{D}_H = \mathbb{P}(H)$ be the space of directions in H. Then $\pi_{L,H}$ induces a holomorphic rational fiber bundle from $\mathcal{D}\backslash\{\mathbb{P}(L)\}$ onto $\mathbb{P}(H)$. The fiber $\pi_{L,H}^{-1}(M)$ over any $M \in \mathbb{P}(H)$ consists of all $K \in \mathcal{D}$ such that $K \neq \mathbb{P}(L)$ and $K \subset L+M$, so that

$$\pi_{L,H}^{-1}(M) = \mathbb{G}_{n-k-1}(\mathbb{P}(L+M))\backslash\{\mathbb{P}(L)\}.$$

Adding (the points of) $\mathbb{P}(L)$ to it, we get the spaces $\mathbb{G}_{n-k-1}(\mathbb{P}(L+M)) \cong \mathbb{G}_{n-k}(L+M)$ in $\mathbb{G}_{n-k-1}(\mathbb{P})$ as the closure of the fibers. Thus, $\pi_{L,H}$ defines an almost fibration $\mathcal{D} \to \mathbb{P}(H)$ with $(n-k)$-dimensional projective subspaces as fibers, except that all fibers intersect each other at the subspaces of L in $\mathcal{D}$.

We may assume that not for all $\xi \in \Omega^{(k)}$ we have $\varkappa(\xi) = L$. This implies that

$$\Omega_L^{(k)} = \{\xi \in \Omega^{(k)} : \varkappa(\xi) \neq L\}$$

is equal to the complement in Ω of an algebraic subset of complex codimension ≥ 1. Hence $\Omega_L^{(k)}$ is a connected, open and dense subset of $\Omega^{(k)}$ and of Ω. On $\Omega_L^{(k)}$ we have the set valued mapping $\pi_{L,H} \circ \omega \circ \varkappa : \Omega_L^{(k)} \to \mathbb{P}(H)$.

Let us do now the technically simpler case of $k = n-1$, which will turn out to be the most important case later. The above construction simplifies. We consider the projection of the fibers to some hyperplane H in $\mathbb{C}^n$, which can have its own singularities. Let $L \in \mathbb{P}(\mathbb{C}^n)$ be given and let an (n-1)-dimensional linear subspace H of $\mathbb{C}^n$ be complementary to L. Let $\pi_{L,H}$ be the linear projection from $\mathbb{C}^n$ onto H along L. The projective space $\mathbb{P}(H)$ is the space of directions in H and $\pi_{L,H}$ induces a holomorphic rational fiber bundle from $\mathbb{P}(\mathbb{C}^n)\backslash\{L\}$ onto $\mathbb{P}(H)$. The fiber $\pi_{L,H}^{-1}(M)$ over any $M \in \mathbb{P}(H)$ consists of all $K \in \mathbb{P}(\mathbb{C}^n)$ such that $K \neq L$ and $K \subset L+M$, so that

$$\pi_{L,H}^{-1}(M) = \mathbb{P}(L+M)\backslash\{L\}.$$

Adding L to it, we get the spaces $\mathbb{P}(L+M)$ in $\mathbb{P}(\mathbb{C}^n)$ as the closure of the fibers. Thus, $\pi_{L,H}$ defines an almost fibration $\mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(H)$, except that all fibers intersect each other at L in $\mathbb{P}(\mathbb{C}^n)$.

We may assume that not for all $\xi \in \Omega^{(k)}$ we have $\varkappa(\xi) = L$. As before, the set

$$\Omega_L^{(k)} = \{\xi \in \Omega^{(k)} : \varkappa(\xi) \neq L\}$$

is a connected, open and dense subset of $\Omega^{(k)}$ and of Ω. On $\Omega_L^{(k)}$ we have the holomorphic mapping $\pi_{L,H} \circ \varkappa : \Omega_L^{(k)} \to \mathbb{P}(H)$.

Let Ω_L^{sing} be the set of $\xi \in \Omega$ such that $\pi_{L,H} \circ \varkappa$ does not have a holomorphic extension to any open neighborhood of ξ. Clearly $\Omega_L^{\text{sing}} \subset \Omega^{\text{sing}}$. If $\xi \in \Omega^{\text{sing}} \backslash \Omega_L^{\text{sing}}$, then $\pi_{L,H}$ maps $\bar{\varkappa}(\xi)$ to the single value of the analytic extension of $\pi_{L,H} \circ \varkappa$ over ξ, say M, which means that $\bar{\varkappa}(\xi) \subset \mathbb{P}(L+M)$. By Proposition 2.7.2 we have $\dim \widetilde{\bar{\varkappa}}(\xi) \geq n-k+1$ and $\dim \widetilde{\mathbb{P}}(L+M) = \dim(L+M) = n-k+1 = 2$, so we get that it is open in $\mathbb{P}(L+M)$. It follows that $\bar{\varkappa}(\xi)$ is an open conic subset of $\mathbb{P}(\mathbb{C}^n)$. If $\xi_j \in \Omega^{\text{sing}}$ and $\xi_j \to \xi$ in Ω, $L_j \in \bar{\varkappa}(\xi_j)$, $L_j \to L$ in $\mathbb{P}(\mathbb{C}^n)$, then $L \in \bar{\varkappa}(\xi)$. This implies that $\xi \in \Omega^{\text{sing}}$ and $\bar{\varkappa}(\xi)$ contains all limit positions of elements of $\bar{\varkappa}(\xi_j)$, $j \to \infty$. In particular, if $0 \in \Omega^{\text{sing}} \backslash \Omega_L^{\text{sing}}$, $\bar{\varkappa}(0) = \mathbb{P}(J)$, J a 2-dimensional linear subspace of $\mathbb{C}^n$, and we choose $L' \in \mathbb{P}(\mathbb{C}^n)$ with $L' \not\subset J$, then $\xi \notin \Omega^{\text{sing}} \backslash \Omega_{L'}^{\text{sing}}$ for all ξ in a neighborhood of zero. Thus, we have proved

Proposition 2.7.12. *Let $k = n-1$. For every $\xi \in \Omega^{\text{sing}}$ there exist an open neighborhood U of ξ in Ω, such that $\Omega^{\text{sing}} \cap U = \Omega_L^{\text{sing}} \cap U$ for L in an open dense subset of $\mathbb{G}_{n-k}(\mathbb{C}^n)$.*

Now we want to establish estimates from below on the dimension of Ω^{sing}. We will need the following simple result from linear algebra, the proof of which is obvious.

Lemma 2.7.13. *Let $A \in \mathbb{C}^{p\times n}$ with $\operatorname{rank} A = k$, attained on a submatrix $A_{\mathcal{L}}^{\mathcal{M}} \in \mathbb{C}^{k\times k}$ of the rows and columns with numbers in $\mathcal{L}$ and $\mathcal{M}$ respectively, $\mathcal{L} \subset \{1,\dots,p\}$, $\mathcal{M} \subset \{1,\dots,n\}$. Then for each $1 \leq r \leq k$ and $\{\lambda_i\}_{i=1}^r \subset \mathcal{L}$ there exist $\{\mu_i\}_{i=1}^r \subset \mathcal{M}$, such that $\det A_{\lambda_i}^{\mu_i} \neq 0$, where $A_{\lambda_i}^{\mu_i} \in \mathbb{C}^{r\times r}$ is a submatrix of A, obtained by the intersection of rows λ_i with columns μ_i, $i = 1,\dots,r$.*

The following Lemma shows the ambiguity of a blowing-up in the case when its center S is not an immersed manifold. We assume that the functions defining the blowing-up do not have a common factor vanishing at the points of S. If S is a smooth manifold of codimension k, then the fibers of the blowing-up over the points of S are equal to the projective space $\mathbb{P}_{k-1}$ (cf. [32], [63]). If S is analytic, then the above procedure can be applied to the interiors of its irreducible components of different dimensions, leading to the projective spaces of different dimensions. See [24] for more details in fuller generality. For our purposes it is important that the fibers are not zero dimensional, or that the blowing-up is not holomorphically extendible over its center.

Lemma 2.7.14. *Let $f : \Omega \to \mathbb{C}^p$ be holomorphic, Ω open and connected in $\mathbb{C}^n$, $p \geq 2$, $f \not\equiv 0$. Let $S = \{\xi \in \Omega : f(\xi) = 0\}$, $\xi_0 \in S$, $\dim_{\xi_0} S \leq n-2$. Then there is no open neighborhood U of ξ_0 in Ω such that the holomorphic mapping*

$$F : \xi \mapsto \mathbb{C}f(\xi) : \Omega \backslash S \to \mathbb{P}(\mathbb{C}^p)$$

has a holomorphic extension to U.

Remark 2.7.15. *The condition* $\dim_{\xi_0} S \le n-2$ *is equivalent to the condition that* $f_1, \dots, f_p$ *do not have a common factor, which is zero at* ξ_0. *The set* $\bar{F}(\xi_0)$ *of limit values of* F *at* ξ_0 *is an algebraic subvariety of* $\mathbb{P}(\mathbb{C}^p)$ *of positive dimension.*

Proof. The argument as in Lemma 2.6.2, shows that if there exists a hyperplane H in $\mathbb{P}(\mathbb{C}^p)$ such that $\bar{F}(\xi_0) \cap H$ is empty, then F has a holomorphic extension to a neighborhood of ξ_0, which we also denote by F. The converse also holds. Then there exists an index i such that $\eta \in F(\xi_0)$, $\eta \ne 0$ imply $\eta_i \ne 0$. Because of the continuity of F at ξ_0, we get an $\epsilon > 0$ and a neighborhood U of ξ_0 in Ω such that $\xi \in U$, $\eta \in F(\xi)$ imply $|\eta_i| \ge \epsilon |\eta_j|$ for every $j \ne i$. It follows that if $\xi \in U \backslash S$ then $f_i(\xi) \ne 0$, or

$$S \cap U \subset f_i^{-1}(0) \cap U \subset S,$$

in contradiction with $\dim f_i^{-1}(0) = n-1$ and $\dim_{\xi_0} S \le n-2$. This completes the proof of Lemma.

Proposition 2.7.16. *Either* Ω^{sing} *is empty or* $\dim_\xi \Omega^{\text{sing}} \ge k-1$ *for every* $\xi \in \Omega^{\text{sing}}$.

Proof. By Lemma 2.7.13 there exist subsets $\mathcal{L}, \mathcal{M}$ of $\{1, \dots, n\}$, with k elements, such that $\Delta_{\mathcal{L}}^{\mathcal{M}}(\xi) = \det A_{ij}(\xi)_{i \in \mathcal{L}, j \in \mathcal{M}}$ is not equal to zero for all $\xi \in \Omega$. Let $Z_{\mathcal{L},\mathcal{M}}$ be the zero set of $\Delta_{\mathcal{L}}^{\mathcal{M}}$ in Ω. Then, for $\xi \in \Omega \backslash Z_{\mathcal{L},\mathcal{M}}$, we have $\eta \in \ker A(\xi)$ if and only if

$$\sum_{j \in \mathcal{M}} A_{ij}(\xi)\eta_j + \sum_{m \notin \mathcal{M}} A_{im}(\xi)\eta_m = 0, \ i \in \mathcal{L}. \tag{2.7.5}$$

Moreover, the equations (2.7.5) can be solved with respect to η_j, $j \in \mathcal{M}$ and we get that (2.7.5) is equivalent to

$$\eta_j = \sum_{m \notin \mathcal{M}} \frac{f_{\mathcal{L},\mathcal{M}}^{jm}(\xi)}{\Delta_{\mathcal{L}}^{\mathcal{M}}(\xi)} \eta_m, \ j \in \mathcal{M}, \tag{2.7.6}$$

in which $f_{\mathcal{L},\mathcal{M}}^{jm}(\xi)$ is a polynomial in the coefficients of $A(\xi)$. For each $j \in \mathcal{M}$, let $Z_{\mathcal{L},\mathcal{M}}^j$ be the common zero set of the functions $f_{\mathcal{L},\mathcal{M}}^{jm}$, $m \notin \mathcal{M}$, and $\Delta_{\mathcal{L}}^{\mathcal{M}}$, after we have divided away possible common factors. Because $\Omega \backslash Z_{\mathcal{L},\mathcal{M}}$ is dense in Ω, we get from Lemma 2.7.14 that $Z_{\mathcal{L},\mathcal{M}}^j \subset \Omega^{\text{sing}}$. The number of $m \notin \mathcal{M}$ is $n - |\mathcal{M}| = n-k$, and with $\Delta_{\mathcal{L}}^{\mathcal{M}}$ holds

$$\dim_\xi Z_{\mathcal{L},\mathcal{M}}^j \ge n - (n-k+1) = k-1$$

by (2.5.2). Hence we get the estimate $\dim_\xi \Omega^{\text{sing}} \ge k-1$.

Proposition 2.7.17. *If* $k = n-1$, *then either* Ω^{sing} *is empty or* $\dim \Omega^{\text{sing}} = n-2$. *Moreover, if* A *is an irreducible component of* Ω^{sing}, *then* $\dim A = n-2$.

Proof. The first statement follows from Proposition 2.7.16 and Theorem 2.7.7. Now, for $\mathcal{L}, \mathcal{M}$ as in the proof of Proposition 2.7.16 there exist unique numbers $1 \leq l, m \leq n$, $l \notin \mathcal{L}$, $m \notin \mathcal{M}$ and we denote $\Delta_{lm} = \Delta_{\mathcal{L}}^{\mathcal{M}}$ and $Z^k_{\mathcal{L},\mathcal{M}} = Z^k_{lm}$. The second statement will follow if we show that

$$\Omega^{\text{sing}} = \bigcup_{l,m,k:\Delta_{lm}\not\equiv 0,\ f^k_{lm}\not\equiv 0} Z^k_{lm}.$$

From the proof of Proposition 2.7.16 we see that $Z^k_{lm} \subset \Omega^{\text{sing}}$, so it is sufficient to prove that for every $\xi \in \Omega^{\text{sing}}$ there exist l, m, k such that $\Delta_{lm} \not\equiv 0$, $f^k_{lm} \not\equiv 0$ and $\xi \in Z^k_{lm}$. By Lemma 2.7.13, for every l there exists m such that $\Delta_{lm} \not\equiv 0$. Let $\xi \in \Omega^{\text{sing}}$. Then $\Delta_{lm}(\xi) = 0$, because otherwise $\xi \in \Omega^{(n-1)}$, but $\Omega^{\text{sing}} \cap \Omega^{(n-1)} = \emptyset$. The system (2.7.6) becomes

$$\eta_k = \frac{f^k_{lm}(\xi)}{\Delta_{lm}(\xi)}\eta_m, \ k \neq m.$$

If for all $k \in \mathcal{M}$ the functions $f^k_{lm}(\xi) \equiv 0$, this would imply that all $\eta_k \equiv 0$, $k \neq m$, which means that the fibration would be constant. Thus, for every m there exist l and k such that $f^k_{lm} \not\equiv 0$. The condition $\xi \notin Z^k_{lm}$ would mean that $f^k_{lm}(\xi) \neq 0$, or $\eta_m = 0$. If this is true for all m, then all $\eta_m = 0$, a contradiction with $\Omega^{\text{sing}} \neq \emptyset$. The proof is complete.

2.8 Affine fibrations of Jacobian type

In this section we will always assume that A is of the Jacobian type, $A = D\Gamma$, and that properties (ΓA1) and (ΓA2) are satisfied. First of all, we can recall Corollary 2.6.4, which states that if $k = 1$ then Ω^{sing} is empty, that is there can be no essential singularities in fibrations by hyperplanes.

Note, that the assumption (ΓA2) of Γ being constant along $\varkappa(\xi)$ is very essential. Assumptions (ΓA1) alone do not guaranty the emptiness of Ω^{sing} for $k = 1$. It is shown by the following example.

Example 2.8.1. *Let* $\Gamma(\xi) = (\xi_1^2 + \xi_2^2 + \xi_3^2, 0, 0) \in \mathbb{C}^3$. *Then*

$$\frac{\partial \Gamma}{\partial \xi}(\xi) = \begin{pmatrix} 2\xi_1 & 2\xi_2 & 2\xi_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

thus $k = 1$, $\Omega^{(1)} = \mathbb{C}^3 \backslash \{0\}$ *and for* $\xi \in \Omega^{(1)}$ *with* $\xi_1 \neq 0$, *the kernel* $\varkappa(\xi)$ *is spanned by*

$$\begin{pmatrix} -\xi_2/\xi_1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\xi_3/\xi_1 \\ 0 \\ 1 \end{pmatrix}.$$

The singular set is $\Omega^{\text{sing}} = \{0\}$.

However, the mapping Γ is not constant along $\varkappa(\xi)$. Related to the assumption (ΓA2), we have the following characterization:

Proposition 2.8.2. *For every $\xi \in \Omega^{(k)}$ the mapping Γ is constant on $(\xi + \varkappa(\xi)) \cap \Omega$, i.e.*

$$\Gamma(\zeta) = \Gamma(\xi) \tag{2.8.1}$$

for all $\zeta \in (\xi + \varkappa(\xi)) \cap \Omega$ if and only if $\varkappa(\xi)$ is constant on $(\xi + \varkappa(\xi)) \cap \Omega$, i.e.

$$\varkappa(\xi) \subset \ker D\Gamma(\zeta) \tag{2.8.2}$$

for all $\zeta \in (\xi + \varkappa(\xi)) \cap \Omega$ with the equality if $\zeta \in (\xi + \varkappa(\xi)) \cap \Omega^{(k)}$.

Proof. Differentiation of (2.8.1) with respect to the basis elements of $\varkappa(\xi)$ implies (2.8.2). For $\zeta \in \Omega^{(k)}$ the dimensions of both sides coincide, so that the equality holds. Conversely, it follows from (2.8.2) that $D\Gamma(\xi + \eta)\varkappa(\xi) = 0$ for $\eta \in \varkappa(\xi)$ and therefore $\Gamma(\xi + \eta)$ is constant along $\varkappa(\xi)$, which is (2.8.1).

In case of the rank drops by one, we have the following

Theorem 2.8.3. *Assume that $\xi \in \Omega^{\text{sing}}$ and that $\xi \in \Omega^{(k-1)}$, i.e.* rank $D\Gamma(\xi) = k - 1$. *Then $\Gamma(\xi + \lambda) = \Gamma(\xi)$ for all $\lambda \in \ker D\Gamma(\xi)$, $\xi + \lambda \in \Omega$.*

Proof. Denote $M = \ker D\Gamma(\xi)$. Note, that $\xi \in \Omega^{(k-1)}$ implies $\dim M = n - k + 1$. For every $\varkappa_0 = \lim_{j\to\infty} \varkappa(\xi_j)$ as $\Omega^{(k)} \ni \xi_j \to \xi$, we have that $\varkappa_0 \subset M$ by continuity of $D\Gamma$. For each one dimensional linear subspace L of M there exists a k-dimensional linear subspace C of $\mathbb{C}^n$, such that $L = M \cap C$. On the other hand $1 \le \dim(\varkappa_0 \cap C)$ and $\varkappa_0 \cap C \subset M \cap C = L$, which implies that $\varkappa_0 \cap C = L$. Hence, by Lemma 2.6.3, $\Gamma(\xi + \lambda) = \Gamma(\xi)$ for all $\lambda \in L$. Because this holds for every one dimensional subspace L of M, we get that $\Gamma(\xi + \lambda) = \Gamma(\xi)$ for all $\lambda \in M$. The proof is complete.

Proof of Theorem 2.4.5: Assume, that the properties (ΓA1) and (ΓA2) are satisfied for Γ with some $m \le n$ (instead of k). Then, by the implicit function theorem we have that the regular fibers of Γ are $(n-m)$-dimensional, and, therefore, $m = k$, and thus we have (ΓA1) and (ΓA2). For $\xi \in \Omega_0$ define $\lambda(\xi) = \Gamma^{-1}(\Gamma(\xi)) - \xi$. By the assumption λ is the mapping from Ω_0 to $\mathbb{G}_{n-k}(\mathbb{C}^n)$. The mapping Γ is constant on $(\xi + \lambda(\xi)) \cap \Omega$ and hence by Proposition 2.8.2 we get $\lambda(\xi) \subset \ker D\Gamma(\zeta)$ for $\zeta \in (\xi + \lambda(\xi)) \cap \Omega$. In particular, $\lambda(\xi) = \ker D\Gamma(\xi)$ for all $\xi \in \Omega_0 \cap \Omega^{(k)}$ and we obtain (ΓA2) on $\Omega_0 \cap \Omega^{(k)}$. The set $\Omega_0 \cap \Omega^{(k)}$ is dense in Ω and $\Omega^{(k)}$. Therefore, for every $\xi \in \Omega^{(k)}$ there exists a sequence $\xi_j \in \Omega_0 \cap \Omega^{(k)}$, convergent to ξ. The compactness of $\mathbb{G}_{n-k}(\mathbb{C}^n)$ implies the existence of a subsequence of ξ_j, such that the corresponding subsequence of $\varkappa(\xi_j)$ converges to some $\varkappa_0 \in \mathbb{G}_{n-k}(\mathbb{C}^n)$. Without loss of generality we denote this subsequence also by ξ_j. For every $\eta \in (\xi + \varkappa_0) \cap \Omega$ there exists a sequence $\eta_j \in (\xi_j + \varkappa(\xi_j)) \cap \Omega$, such that η_j converges to η. The above proof of (ΓA2) for $\Omega_0 \cap \Omega^{(k)}$ implies that Γ is constant on $\xi_j + \varkappa(\xi_j)$, and, therefore, $\Gamma(\eta_j) = \Gamma(\xi_j)$. Letting j to infinity, we get that $\Gamma(\eta) = \Gamma(\xi)$ and $\eta \in \Gamma^{-1}(\Gamma(\xi))$. The argument holds for any $\eta \in (\xi + \varkappa_0) \cap \Omega$, and we obtain $(\xi + \varkappa_0) \cap \Omega \subset \Gamma^{-1}(\Gamma(\xi))$. By the implicit function theorem the fiber $\Gamma^{-1}(\Gamma(\xi))$ is a smooth analytic submanifold of Ω of codimension k and, therefore, coincides with $\xi + \varkappa_0$ locally at ξ. On the other hand, $\varkappa$ is holomorphic at ξ, implying $\varkappa(\xi) = \varkappa_0$ and (ΓA2) for all $\xi \in \Omega^{(k)}$,

Finally, if $\xi \in \Omega^{\text{sing}}$, then Proposition 2.7.2 implies that the fiber through ξ is of codimension strictly less than k, and hence $\xi \notin \Omega_0$. The proof is complete.

The following is the consequence of Proposition 2.7.9.

Corollary 2.8.4. *Let p be the natural projection*

$$p : \Omega' \ni \xi \mapsto \mathcal{R}(\xi) \in \Omega'/\mathcal{R}.$$

The mapping $\Gamma : \xi \mapsto \Gamma(\xi)$ is constant on the $\mathcal{R}(\xi)$, so there is an analytic mapping $g : \Omega'/\mathcal{R} \to \mathbb{C}^p$, such that

$$\Gamma = g \circ p. \tag{2.8.3}$$

The mapping g is an immersion on $\Omega^{(k)}/\mathcal{R}$, that is, at each point its tangent mapping is injective, because rank $D\Gamma = k$ *implies* rank $Dg \geq k$.

Remark 2.8.5. *The factorization conclusion (2.8.3), when restricting to the real domain, implies the "factorization condition" of Section 1.5.1, especially when Ω^{sing} is empty.*

Remark 2.8.6. *If $(\xi, \eta) \in \bar{\mathcal{R}}$, then still $\Gamma(\xi) = \Gamma(\eta)$ by the continuity of Γ. So, if $\hat{\mathcal{R}}$ denote the smallest closed equivalence relation which contains $\bar{\mathcal{R}}$, then*

$$(\xi, \eta) \in \hat{\mathcal{R}} \Rightarrow \Gamma(\xi) = \Gamma(\eta).$$

If $\hat{\mathcal{R}}$ has an equivalence class with nonempty interior, then Γ is constant and $k = 0$.

Lemma 2.8.7. *We have the inclusion $\pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)) \subset \Omega \backslash \Omega^{(k)}$.*

Proof. Let $\xi \in \Omega^{\text{sing}}$, let C be an irreducible component of $\bar{\varkappa}(\xi)$ and let C^0 be its smooth part. Because $\Gamma = \Gamma(\xi)$ is constant on $\xi + \bar{\tilde{\varkappa}}(\xi)$, we have for every $\eta^0 \in C^0$ such that $\xi + \eta^0 \in \Omega$, that $T_\eta C^0 \subset \ker D\Gamma(\xi + \eta^0)$. So by Proposition 2.7.2, (iii), $\dim \ker D\Gamma(\xi + \eta^0) \geq n - k + 1$ or $\xi + \eta^0 \in \Omega \backslash \Omega^{(k)}$. Because C^0 is dense in C and Ω is open, every $\eta \in C$ can be approximated by $\eta^0 \in C^0$ such that $\xi + \eta^0 \in \Omega$, hence $\xi + \eta^0 \in \Omega \backslash \Omega^{(k)}$. Because $\Omega \backslash \Omega^{(k)}$ is closed in Ω, it follows that $\xi + C \subset \Omega \backslash \Omega^{(k)}$. Because this holds for every irreducible component C of $\bar{\tilde{\varkappa}}(\xi)$, we get $\xi + \bar{\tilde{\varkappa}}(\xi) \subset \Omega \backslash \Omega^{(k)}$ and because this holds for every $\xi \in \Omega^{\text{sing}}$, we get $\pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)) \subset \Omega \backslash \Omega^{(k)}$.

As a consequence of Proposition 2.7.11, we get

Corollary 2.8.8. *The following estimate holds:*

$$n - k + 1 \leq \dim \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)) \leq n - 1, \tag{2.8.4}$$

Proof. The estimates (2.8.4) follow immediately from (2.7.2), Lemma 2.8.7 and the fact that $\dim \Omega\backslash\Omega^{(k)} \leq n-1$.

Now we will prove some consequences of the relative position of Ω^{sing} and $\pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega))$, which we will use later to show that certain cases are impossible. In what follows we will assume that the set Ω^{sing} of essentially singular points is not empty.

Proposition 2.8.9. *The following holds:*

(i) $\Omega^{\text{sing}} \subset \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)) \subset \Omega\backslash\Omega^{(k)}$.

(ii) *If* $\Omega^{\text{sing}} = \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega))$, *then the closure* $\bar{\mathcal{R}}$ *can be decomposed into relations on* $(\Omega\backslash\Omega^{\text{sing}}) \times (\Omega\backslash\Omega^{\text{sing}})$ *and on* $\Omega^{\text{sing}} \times \Omega^{\text{sing}}$. *In formulas,*

$$\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega) \subset \Omega^{\text{sing}} \times \Omega^{\text{sing}}.$$

Moreover, in this case

$$n - k + 1 \leq \dim_\xi \Omega^{\text{sing}}$$

for every $\xi \in \Omega^{\text{sing}}$.

(iii) *If* $\Omega^{\text{sing}} \neq \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega))$ *and*

$$l = \max_{\eta \in \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega\backslash\Omega^{\text{sing}}))} \dim(\Omega^{\text{sing}} \cap (\eta + \varkappa(\eta))),$$

then

$$\dim \pi_1(\mathcal{R} \cap (\Omega^{\text{sing}} \times \Omega\backslash\Omega^{\text{sing}})) \leq k - 2 + l. \tag{2.8.5}$$

(iv) *If in addition to* (iii) *either* $\dim(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega\backslash\Omega^{\text{sing}})) = \dim(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega))$ *or* $\dim \pi_1(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega\backslash\Omega^{\text{sing}})) = \dim \Omega^{\text{sing}}$, *then*

$$\dim \Omega^{\text{sing}} + 2 \leq \dim \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)). \tag{2.8.6}$$

(v) *Let* $\Omega^{\text{sing}} \neq \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega))$. *Then for any* $\xi \in \Omega^{\text{sing}}$ *with* $\xi \notin \pi_1(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega\backslash\Omega^{\text{sing}}))$ *we have* $\xi + \bar{\bar{\varkappa}}(\xi) \subset \Omega^{\text{sing}}$. *Moreover, in this case* $k \geq 3$ *and* $\dim_\xi \Omega^{\text{sing}} \geq n - k + 1$.

Proof. (i) The first inclusion follows from the fact that if $\xi \in \Omega^{\text{sing}}$, then it is a limit point of some sequence $\zeta_i \in \Omega^{(k)}$ because of the density of $\Omega^{(k)}$ in Ω. By Proposition 2.7.9, (i), it follows that $(\zeta_i, \zeta_i) \in \mathcal{R} \subset \bar{\mathcal{R}}$ and, therefore, $(\xi, \xi) \in \bar{\mathcal{R}}$, $(\xi, \xi) \in \bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)$ and $\xi \in \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega))$. The second inclusion is Lemma 2.8.7.

(ii) The statement follows easily from definitions and the last part of (ii) from (2.7.3) and Remark 2.7.3.

(iii), (iv) Assume now that $\Omega^{\text{sing}} \neq \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega))$. To abbreviate the notation, we denote

$$\begin{array}{rcl} B & = & \bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega \backslash \Omega^{\text{sing}}), \\ C & = & \pi_2(B) = \pi_2(\bar{\mathcal{R}}_1^{\text{sing}}) \backslash \Omega^{\text{sing}}, \\ D & = & \pi_1(B). \end{array}$$

For $\eta \in C$ the fiber of $\pi_2 : B \to C$ over η consists of all (ξ, η) with $\xi \in \Omega^{\text{sing}}$ and $\xi \in \eta + \bar{\varkappa}(\eta) = \eta + \varkappa(\eta)$, i.e. $\pi_2^{-1}(\eta) \cap B = (\Omega^{\text{sing}} \cap (\eta + \varkappa(\eta)), \eta)$. Thus, by Theorem 2.5.4, we obtain

$$\dim B \leq \max_{\eta \in B} \dim(\Omega^{\text{sing}} \cap (\eta + \varkappa(\eta))) + \dim C. \tag{2.8.7}$$

We have $\pi_2^{-1}(\Omega \backslash \Omega^{\text{sing}}) = \Omega \times (\Omega \backslash \Omega^{\text{sing}})$ and the set $\Omega \backslash \Omega^{\text{sing}}$ is open in Ω, hence B is open in $\bar{\mathcal{R}}_1^{\text{sing}}$. Moreover, for the set B we have

$$\begin{array}{rcl} B & = & \bar{\mathcal{R}}_1^{\text{sing}} \cap (\Omega \times (\Omega \backslash \Omega^{\text{sing}})) = \bar{\mathcal{R}}_1^{\text{sing}} \cap \pi_2^{-1}(\Omega \backslash \Omega^{\text{sing}}) \\ & = & \bar{\mathcal{R}}_1^{\text{sing}} \backslash (\Omega^{\text{sing}} \times \Omega^{\text{sing}}), \end{array}$$

hence B is analytically constructible and $\bar{B}$ is analytic. The set $\Omega^{\text{sing}} \cap (\eta + \varkappa(\eta))$ is an analytic subset of $\varkappa(\eta)$, proper, in view of $\eta \notin \Omega^{\text{sing}}$, so $\dim(\Omega^{\text{sing}} \cap (\eta + \varkappa(\eta))) \leq n - k - 1$. Hence from (2.8.7) we also have

$$\dim B \leq n - k - 1 + \dim C. \tag{2.8.8}$$

Note that if B is also dense in $\bar{\mathcal{R}}_1^{\text{sing}}$, then $\dim B = \dim \bar{B} = \dim \bar{\mathcal{R}}_1^{\text{sing}}$ and estimates (2.7.4) and (2.8.8) imply (2.8.6). If B is not dense in $\bar{\mathcal{R}}_1^{\text{sing}}$, then B is a component of the analytic set $\bar{\mathcal{R}}_1^{\text{sing}}$ with

$$\dim B = \dim \bar{B} < \dim \bar{\mathcal{R}}_1^{\text{sing}}.$$

Now we will deduce the estimates for the dimension of Ω^{sing}. The fiber of $\pi_1 : B \to D$ over $\xi \in D$ consists of points (ξ, η) with $\eta \in (\xi + \tilde{\bar{\varkappa}}(\xi))$ and $\eta \notin \Omega^{\text{sing}}$. It follows that $\pi_1^{-1}(\xi) \cap B = (\xi, (\Omega \backslash \Omega^{\text{sing}}) \cap (\xi + \tilde{\bar{\varkappa}}(\xi)))$, so

$$\dim D + \min_{\xi \in D} \dim(\Omega \backslash \Omega^{\text{sing}}) \cap (\xi + \tilde{\bar{\varkappa}}(\xi)) \leq \dim B. \tag{2.8.9}$$

If $\xi \in \Omega^{\text{sing}}$ then $\dim \tilde{\bar{\varkappa}}(\xi) \geq n - k + 1$ by Proposition 2.7.2, (iii). The set $\bar{\varkappa}(\xi)$ does not have isolated points by the same Proposition, (i) and it follows that $\dim((\Omega \backslash \Omega^{\text{sing}}) \cap (\xi + \tilde{\bar{\varkappa}}(\xi))) \geq n - k + 1$ for $\xi \in D$, so

$$\dim D + n - k + 1 \leq \dim B. \tag{2.8.10}$$

By the first part of Proposition we get $C \subset \Omega \backslash \Omega^{(k)}$ and $\dim C \leq n - 1$. This, the estimates (2.8.10) and Theorem 2.5.4, (ii), imply

$$\begin{array}{rcl} \dim D & \leq & \dim B - (n - k + 1) \\ & \leq & \dim C - (n - k + 1) + \max_{\eta \in B} \dim(\Omega^{\text{sing}} \cap (\xi + \tilde{\bar{\varkappa}}(\xi))) \\ & \leq & k - 2 + \max_{\eta \in B} \dim(\Omega^{\text{sing}} \cap (\xi + \tilde{\bar{\varkappa}}(\xi))), \end{array}$$

which is (2.8.5) and (iii) is proved.

Combination of (2.8.10) and (2.8.8) yields $\dim D \le \dim C - 2$, which implies (2.8.6) if $\dim D = \dim \Omega^{\text{sing}}$ by $\dim C \le \dim \pi_2(\bar{\mathcal{R}}_1^{\text{sing}})$. On the other hand, if $\dim B = \dim \bar{\mathcal{R}}_1^{\text{sing}}$, then (2.8.8) and (2.7.4) yield $\dim \Omega^{\text{sing}} + 2 \le \dim C$, which again imply (2.8.6) if we use that $\dim C \le \dim \pi_2(\bar{\mathcal{R}}_1^{\text{sing}})$.

(v) Let $\xi \in \Omega^{\text{sing}} \backslash \pi_1(B)$. This means $(\bar{\mathcal{R}} \cap \pi_1^{-1}(\xi)) \cap B = \emptyset$ and in view of $\bar{\mathcal{R}} \cap \pi_1^{-1}(\xi) \subset \bar{\mathcal{R}}_1^{\text{sing}}$ we get

$$\bar{\mathcal{R}} \cap \pi_1^{-1}(\xi) \subset \bar{\mathcal{R}}_1^{\text{sing}} \backslash B \subset \bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega^{\text{sing}}).$$

On the other hand $\bar{\mathcal{R}} \cap \pi_1^{-1}(\xi) = (\xi, \xi + \tilde{\bar{\varkappa}}(\xi))$, and

$$\xi + \tilde{\bar{\varkappa}}(\xi) = \pi_2(\xi, \xi + \tilde{\bar{\varkappa}}(\xi)) \subset \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega^{\text{sing}})) \subset \Omega^{\text{sing}}.$$

Now, this implies

$$n - k + 1 \le \dim \tilde{\bar{\varkappa}}(\xi) \le \dim \Omega^{\text{sing}} \le n - 2,$$

and it is possible only if $k \ge 3$. The proof is complete.

Corollary 2.8.10. *We have proved*

$$\dim D \le \dim C - 2 \le n - 3.$$

Note, that statements (ii), (iv), (v) of Proposition 2.8.9 still hold in the case of general fibrations. The proof of (iii) makes use of Lemma 2.8.7, which uses the Jacobian structure of A.

Corollary 2.8.11. *If* $\dim \Omega^{\text{sing}} = n - 2$, *then the set* $D = \pi_1(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega \backslash \Omega^{\text{sing}}))$ *has measure zero in* Ω^{sing}, *or "the largest part" of* Ω^{sing} *consists of* $\pi_1(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega^{\text{sing}}))$.

Remark 2.8.12. *Note that the inclusion*

$$\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega \backslash \Omega^{\text{sing}}) \subset \bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)$$

is strict because for any $\xi \in \Omega^{\text{sing}}$ *the point* $(\xi, \xi) \in \bar{\mathcal{R}}$ *does not belong to the left hand side. The set* $\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega \backslash \Omega^{\text{sing}})$ *is open in* $\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)$ *and the the condition of Proposition 2.8.9,* (iv), *may fail if it is an open component of strictly lower dimension.*

As consequence for the positions of $\pi_1(B)$ and Ω^{sing}, we get

Proposition 2.8.13. *The following holds:*

(i) *If* $\pi_1(B) \ne \Omega^{\text{sing}}$, *then* $\dim_\xi \Omega^{\text{sing}} \ge n - k + 1$ *for every* $\xi \in \Omega^{\text{sing}} \backslash \pi_1(B)$ *and* $k \ge 3$.

(ii) *If* $\pi_1(B) = \Omega^{\text{sing}}$, *then* $\dim \Omega^{\text{sing}} \le n - 3$.

Proof. (i) The conditions $\xi \in \Omega^{\text{sing}}$ and $\xi \notin \pi_1(B)$ just mean that $\dim \tilde{\tilde{\varkappa}}(\xi) \geq n-k+1$ and $\xi + \tilde{\tilde{\varkappa}}(\xi) \subset \Omega^{\text{sing}}$. Hence by Theorem 2.7.7

$$n-k+1 \leq \dim_\xi \Omega^{\text{sing}} \leq n-2$$

and, therefore, also $k \geq 3$.

(ii) The mapping $\pi_1 : B \to \Omega^{\text{sing}}$ is surjective. For every $\xi \in \Omega^{\text{sing}}$ the fiber is equal to $(\xi, (\xi + \tilde{\tilde{\varkappa}}(\xi)) \cap \Omega')$. Using the same argument as in the proof of Proposition 2.8.9, (iv), we get $\dim(\xi + \tilde{\tilde{\varkappa}}(\xi)) \cap \Omega') \geq n-k+1$ and by Theorem 2.5.4 we conclude that

$$\dim B \geq \dim \Omega^{\text{sing}} + n - k + 1.$$

Next we consider the mapping π_2 from B into the analytic subset $\Omega \backslash \Omega^{(k)}$, not necessarily surjective. For each $\eta \in \pi_2(B)$ the fiber over η is equal to $((\eta + \varkappa(\eta)) \cap \Omega^{\text{sing}}, \eta)$. Because $(\eta + \varkappa(\eta)) \cap \Omega^{\text{sing}}$ is an analytic subset of $(\eta + \varkappa(\eta)) \cap \Omega$ with nonempty interior, we get $\dim(\eta + \varkappa(\eta)) \cap \Omega^{\text{sing}} \leq \dim(\eta + \varkappa(\eta)) - 1 \leq n-k-1$. Again, by Theorem 2.5.4 we conclude that

$$n-1 \geq \dim \Omega \backslash \Omega^{(k)} \geq \dim B - (n-k-1)$$

and the proof is complete.

Corollary 2.8.14. *If $k=2$, then* $\dim \Omega^{\text{sing}} \leq n-3$.

Proof. An application of Proposition 2.8.13, (i), (ii).

Theorem 2.8.15. *If $n=3$ and $k=2$, then $\Omega^{\text{sing}} = \emptyset$.*

Proof. Proposition 2.7.17 and Corollary 2.8.14 imply that Ω^{sing} is empty.

Now we turn back to the case of arbitrary $1 \leq k \leq n-1$.

Example 2.8.16. *Consider the mapping*

$$\begin{aligned}&\mathbb{C}^n \ni (x_1, \ldots, x_l, y_1, \ldots, y_{n-l}) \mapsto \\ &\quad (\sum_{i=1}^{m-1} y_i x_i + y_m \sum_{i=m}^{l} x_i, y_1, \ldots, y_l, y_{l+1}, \ldots, y_r, 0) \in \mathbb{C}^n\end{aligned}$$

with $2 \leq m \leq l \leq r \leq n-l$, $0 \in \mathbb{C}^{n-r-1}$. The singular set is given by

$$\Omega^{\text{sing}} = \{(x,y) \in \Omega : y_1 = \ldots = y_m = 0\},$$

so that

$$\dim \Omega^{\text{sing}} = n - m.$$

The dimension of the regular fibers is equal to $l - 1 + (n-l) - r = n-r-1$, hence we have

$$\operatorname{rank} D\Gamma \leq r+1.$$

The limit set $\pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega))$ is the set of $y_i = 0$, $1 \leq i \leq m$, arbitrary $x_1, \ldots, x_l$ and $y_{r+1}, \ldots, y_{n-l}$, and is equal to Ω^{sing}.

Note, that if $m = l$, then $\Omega\backslash\Omega^{(k)} = \Omega^{\text{sing}}$. As a consequence, we have the converse to Proposition 2.8.9, (ii).

Proof of Theorem 2.4.4: It is sufficient to take $r = k - 1$ and $m = d$ in Example 2.8.16 and in view of $k - 1 \leq n - d$, $d \leq n/2$ and hence $d \leq n - d$, the mapping

$$\Gamma(x_1, \dots, x_d, y_1, \dots, y_{n-d}) = (\sum_{i=1}^{d} y_d x_d, y_1, \dots, y_{k-1}, 0) \in \mathbb{C}^n$$

is well defined and satisfies the required conditions. The equality $\dim \Omega^{\text{sing}} = n - d$ follows from the fact that if not all y_i, $1 \leq i \leq d$ are equal to zero, then rank $D\Gamma$ at such point is equal to k. On the other hand, clearly the points with $y_i = 0$ for all $1 \leq i \leq d$ belong to Ω^{sing}. This finishes the proof.

We can see from Theorem 2.4.4 that the case $k = 2$ and not all dimensions of Ω^{sing} are obtained in this way. This is partly explained by Proposition 2.7.16 and is not without a reason, which we intend to show next. First we will show that the complementary intersection of Ω^{sing} with any fiber is impossible.

Theorem 2.8.17. *Let* $\dim \Omega^{\text{sing}} = k - 1$ *and let* A *be a smooth component of the same dimension. Assume that there exist a* k*-dimensional surface* S *containing* A*, transversal to* $\Omega\backslash\Omega^{(k)}$*, and* $\eta \notin \Omega^{\text{sing}}$ *such that* $\eta + \varkappa(\eta)$ *intersects* S *transversally at a point of* A*. Then* Ω^{sing} *is empty.*

Proof. Let $\delta : \Omega \to \mathbb{C}^{n-k}$ be the the determining mapping for $\varkappa(\xi)$, $\xi \in \Omega^{(k)}$. The mapping δ is given by a meromorphic mapping which takes all values in an arbitrary neighborhood of $\xi = 0$, which we assume to be in Ω^{sing}. Let A be the smooth part of Ω^{sing} with $\dim A = k - 1$, it is a complex analytic manifold with the origin in its closure. Let S be a complex analytic k-dimensional surface containing A and transversal to $\Omega\backslash\Omega^{(k)}$, as in the assumption. The fact that most of the limit directions of the fibers $\varkappa(\xi)$ are in $\pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)) \subset \Omega\backslash\Omega^{(k)}$, hence transversal to S, makes that each regular fiber $\varkappa(\xi)$, $\xi \in \Omega^{(k)}$ intersects S transversally in view of

$$\dim S + \dim \varkappa|_{\Omega^{(k)}}(\xi) = k + (n - k) = n.$$

This implies that every point of A is a point of indeterminacy of $\delta|_S$. But this is in contradiction with the fact that the points of indeterminacy of $\delta|_S$ form a set of codimension 2 in S, being the zero set of at least two analytic equations. This implies that A is empty, a contradiction.

Corollary 2.8.18. *Let* $k = n - 1$ *and let* A *be a smooth component of* Ω^{sing}*. Let* $\eta \in \pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega'))$ *and* $\xi = (\eta + \varkappa(\eta)) \cap A$*. Then* $\varkappa(\eta) \subset T_\xi A$*.*

Proof. We have $\dim A = n - 2$ by Proposition 2.7.17. If $\eta + \varkappa(\eta)$ is not tangential to A, then we have the existence of S satisfying conditions of Theorem 2.8.17 and a contradiction.

2.9 Limiting properties of the fibers

In this section we will show that in the case $\pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)) \neq \Omega^{\text{sing}}$ in Proposition 2.8.9, the intersection of the fibers with Ω^{sing} can not be transversal. As a result, we get additional estimates on the dimension of Ω^{sing}.

Proposition 2.9.1. *Let $\pi_2(\bar{\mathcal{R}} \cap (\Omega^{\text{sing}} \times \Omega)) \neq \Omega^{\text{sing}}$, A a smooth component of Ω^{sing} and assume that there exists $\xi \in A$ such that $\tilde{\bar{\varkappa}}(\xi) \not\subset T_\xi A$. Then Ω^{sing} is empty.*

Proof. Let $\xi \in \Omega^{\text{sing}}$ with $\xi + \tilde{\bar{\varkappa}}(\xi) \not\subset \Omega^{\text{sing}}$. Then because of the connectedness of $\bar{\varkappa}(\xi)$ there exist different $\varkappa_1, \varkappa_2 \in \bar{\varkappa}(\xi)$, not contained in Ω^{sing} and in $T_\xi\Omega^{\text{sing}}$. The set $\mathcal{K}_{tr}$ of all $H \in \mathbb{G}_{k+1}(\mathbb{C}^n)$ such that $H \cap \varkappa_1, H \cap \varkappa_2$ are not contained in $T_\xi\Omega^{\text{sing}}$, is open and dense in $\mathbb{G}_{k+1}(\mathbb{C}^n)$. The set

$$\mathcal{K}_0 = \{H \in \mathbb{G}_{k+1}(\mathbb{C}^n) : H \cap \varkappa_1 \neq H \cap \varkappa_2\}$$

is open and dense in $\mathbb{G}_{k+1}(\mathbb{C}^n)$. For $i = 1, 2$ the sets

$$\mathcal{K}_i = \{H \in \mathbb{G}_{k+1}(\mathbb{C}^n) : \dim H \cap \varkappa_i = 1\}$$

are open and dense in $\mathbb{G}_{k+1}(\mathbb{C}^n)$, their intersection is open and dense in $\mathbb{G}_{k+1}(\mathbb{C}^n)$ and we take $H \in \mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_0 \cap \mathcal{K}_{tr}$. Let $\eta \in \Omega^{(k)}$ be close to ξ with $\varkappa(\eta)$ close to one of $\varkappa_i$. Then, by transversality, $\dim(\eta + \varkappa(\eta)) \cap (\xi + H) = 1$. The set $\Omega^{(k)} \cap (\eta + \varkappa(\eta))$ is not empty, and hence is open and dense in $\Omega \cap (\eta + \varkappa(\eta))$. Therefore, there exists $\zeta \in \Omega^{(k)} \cap (\eta + \varkappa(\eta))$, ζ close to $(\eta + \varkappa(\eta)) \cap (\xi + H)$, such that there exists $H_0 \in \mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_0 \cap \mathcal{K}_{tr}$ with $\zeta \in H_0$. Thus, without loss of generality we may take $H = H_0$. Now, by (ΓA2), $\varkappa(\eta) = \varkappa(\zeta)$, implying that the mapping

$$\gamma = \Gamma|_{(\xi+H)\cap\Omega}$$

satisfies $\ker D\gamma(\zeta) = \varkappa(\zeta) \cap H$, which is one dimensional, and, therefore, $\operatorname{rank} D\gamma(\zeta) = \dim H - 1 = k$. Moreover, if $\theta \in \ker D\gamma(\zeta)$, then $\gamma(\theta) = \Gamma(\theta) = \Gamma(\zeta) = \gamma(\zeta)$ because $\theta \in \varkappa(\zeta)$. This means that conditions (ΓA1) and (ΓA2) are satisfied for γ. Let $\eta_j \in \Omega^{(k)}$ such that $\eta_j \to \xi$ and $\varkappa(\eta_j)$ converges to one of $\varkappa_i$.

Because the set

$$\Omega^{(k)}(H) = \{\zeta \in (\xi + H) \cap \Omega : \operatorname{rank} D\gamma(\zeta) = k\}$$

is open and dense in $(\xi + H) \cap \Omega$, we can find $\zeta_j \in \Omega^{(k)}(H)$ which are arbitrary close to $(\xi + H) \cap (\eta_j + \varkappa(\eta_j))$, from which it follows that $\varkappa(\zeta_j)$ is arbitrary close to $\varkappa(\eta_j)$. Note, that $\Omega^{(k)}(H) \subset \Omega^{(k)}$ and $\varkappa$ is constant on an open dense subset of $(\eta + \varkappa(\eta)) \cap \Omega$, $\eta \in \Omega^{(k)}$.

This proves that at the limit point ξ in $(\xi + H) \cap (\xi + \varkappa)$ we get all $H \cap \bar{\varkappa}(\xi)$ as limits of $H \cap \varkappa(\zeta_j)$, $\zeta_j \in \Omega^{(k)}(H)$, $\zeta_j \to \xi$, or ξ is in the singular set for the mapping γ by $H \in \mathcal{K}_0$, the $\varkappa_i \cap H \in \bar{\varkappa}(\xi) \cap H$ are two different limit lines for the fibration defined by γ. On the other hand, we have

$$\Omega^{\text{sing}}(\gamma) \subset \Omega^{\text{sing}}(\Gamma) \cap H,$$

implying $(\xi + \varkappa_i) \cap H \not\subset \Omega^{\text{sing}}(\gamma)$ because $\xi + \varkappa_i \not\subset \Omega^{\text{sing}}(\Gamma)$ by the choice of $\varkappa_1$, $\varkappa_2$. Moreover,

$$T_\xi \Omega^{\text{sing}}(\gamma) \subset T_\xi \Omega^{\text{sing}}(\Gamma) \cap H$$

and by $H \in \mathcal{K}_{tr}$ we get that $H \cap \varkappa_i \not\subset T_\xi \Omega^{\text{sing}}(\gamma)$. This is a contradiction with Corollaries 2.8.18, 2.8.11 and Proposition 2.7.17 applied to γ with $n = k + 1$. In case $n = 3$, this is also a contradiction with Theorem 2.8.15, which says that for $n = 3, k = 2$ the singular set is empty.

Proof of Theorem 2.4.3: For $k = 1$ this is Theorem 2.6.4. For $k = 2$ by Proposition 2.7.2, (iii), for $\xi \in \Omega^{\text{sing}}$ we have $\dim \tilde{\bar{\varkappa}}(\xi) \geq n - 1$ and, therefore, conditions of Proposition 2.9.1 are satisfied.

Finally, Theorem 2.4.2 follows from

Theorem 2.9.2. *Let Ω^{sing} be not empty. Then for every smooth point ξ in Ω^{sing} holds $\tilde{\bar{\varkappa}}(\xi) \subset T_\xi \Omega^{\text{sing}}$. Moreover, for every $\xi \in \Omega^{\text{sing}}$ we have*

$$\begin{array}{lcl} n - k + 1 & \leq & \dim_\xi \Omega^{\text{sing}}, \\ k - 1 & \leq & \dim_\xi \Omega^{\text{sing}}. \end{array}$$

Proof. The first statement follows from Proposition 2.7.16. The inclusion $\bar{\varkappa}(\xi) \subset T_\xi \Omega^{\text{sing}}$ implies $\dim_\xi \Omega^{\text{sing}} \geq \dim \tilde{\bar{\varkappa}}(\xi)$ and the first estimate follows from Proposition 2.7.2. The last estimate is Proposition 2.8.9.

Chapter 3

Affine fibrations of gradient type

In this chapter we will concentrate on different topics. First, we will introduce local coordinates in Ω and in the Grassmannian, and derive several local characterizations of the fibrations. In particular, conditions for the global extendibility of fibrations in the direction of fibers will be discussed.

Further, we will study fibrations of gradient type. These are fibration corresponding to mappings Γ for which $\Gamma = \nabla\phi$ for some holomorphic function ϕ. This case is closely related to Fourier integral operators, where ϕ would be the generating phase function with factored out conic direction. The analysis will be carried out in both complex and real settings at the same time.

Questions of reconstruction of the phase function from a given local fibrations will be discussed to a certain extent. Finally, we will present families of functions ϕ for which the smooth factorization condition fails. Note, that if the parameter dependence is allowed, examples are quite easy to construct (Section 1.5.2). However, if there is no parameter dependence, that is when the corresponding Fourier integral operator is translation invariant, the construction of examples is more complicated. We present such families in Section 3.4.

3.1 Fibrations in local coordinates

In this section we continue to always assume that Γ satisfies conditions (ΓA1), (ΓA2). Without loss of generality we can assume that the origin of the space $\mathbb{C}^n$ belongs to the open set Ω. Clearly, if the fibration $\varkappa$ is continuous at zero, that is, if $0 \notin \Omega^{\text{sing}}$, then there is a neighborhood U of the origin, disjoint from Ω^{sing}, such that the fibers through the points of U are close to the fiber $\varkappa(0)$ through zero. Therefore, there exists a k–dimensional linear subspace of $\mathbb{C}^n$, transversal to all $\varkappa(\xi)$ for $\xi \in U$. On the other hand, if $0 \in \Omega^{\text{sing}}$ and $U \subset \Omega$ is a small open neighborhood of zero, then for every $H \in \mathbb{G}_k(\mathbb{C}^n)$, either the set

$$U_H = \{\xi \in U \cap \Omega^{(k)} : \varkappa(\xi) \cap H = \{0\}\}$$

is empty, or it is open and dense in $U \cap \Omega^{(k)}$, and hence in U. Thus, for H with non-empty set U_H, we can choose a coordinate system around zero such that H is parameterized by points $(h,0) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$ and that on an open dense subset of the set $\Omega^{(k)}$ all fibers are transversal to the k-dimensional subspace of $(h,0) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$. Let $\pi_h : \mathbb{C}^n \to \mathbb{C}^k$ denote the projection to the first k coordinates. Let $\Omega_h^{(k)}$ be the set of all points $h \in \pi_h(\Omega)$ with $(h,0) \in \Omega^{(k)}$. The set of all $h \in \pi_h(\Omega)$ such that $(h,0) \in \Omega$ we denote by Ω_h.

For the simplicity in notations throughout this section we will identify $\mathbb{C}^n$ with $\mathbb{C}^{1\times n}$, so that the elements of $\mathbb{C}^n$ are the *rows* with n elements in $\mathbb{C}$. In this notation we parameterize the fibration by a matrix valued mapping $R : \Omega_h^{(k)} \subset \mathbb{C}^k \to \mathbb{C}^{(n-k)\times k}$. The fibers $\varkappa(h,0)$ can be parameterized by λ in some neighborhood of the origin in $\mathbb{C}^{n-k}$: locally we have

$$\begin{aligned} \varkappa(h,0) &= \{(h+\lambda R(h),\lambda), \lambda \in \mathbb{C}^{n-k}\} \\ &= \{(h+\textstyle\sum_{i=1}^{n-k}\lambda_i R_i(h),\lambda), \lambda \in \mathbb{C}^{n-k}\}. \end{aligned}$$

The rows of $R(h)$ will be denoted by $R_i(h) \in \mathbb{C}^k$, $i = 1,\dots,n-k$, and the components of each row $R_i(h)$ by $R_i^j(h)$, $j = 1,\dots,k$. The j-th column of $R(h)$ we denote by $R^j(h) \in \mathbb{C}^{n-k}$, $j = 1,\dots,k$, and its elements by $R_i^j(h)$. In order to avoid any confusion with this notation, we write

$$\left.\begin{aligned} R &= \overbrace{\begin{pmatrix} R_1^1 & R_1^2 & \dots & R_1^k \\ \vdots & \vdots & \dots & \vdots \\ R_{n-k}^1 & R_{n-k}^2 & \dots & R_{n-k}^k \end{pmatrix}}^{k} \Bigg\} (n-k), \\ R_l &= \begin{pmatrix} R_l^1 & R_l^2 & \dots & R_l^k \end{pmatrix}, \\ DR_l &= \begin{pmatrix} \partial_1 R_l^1 & \dots & \partial_k R_l^1 \\ \vdots & \dots & \vdots \\ \partial_1 R_l^k & \dots & \partial_k R_l^k \end{pmatrix}. \end{aligned}\right\} \quad (3.1.1)$$

Condition (ΓA2) that Γ is constant on $\varkappa$ can now be written as

$$\Gamma(h+\lambda R(h),\lambda) = \Gamma(h,0), \forall h \in U \cap \Omega_h^{(k)}, \lambda \in V, \quad (3.1.2)$$

in some neighborhood $U \times V \subset (\mathbb{C}^k \times \mathbb{C}^{n-k}) \cap \Omega$ of the origin.

Lemma 3.1.1. *Define* $\gamma : \Omega_h \to \mathbb{C}^p$ *by* $\gamma(h) = \Gamma(h,0)$. *Then*

$$\{\partial_i \Gamma(h+\lambda R(h),\lambda)\}_{i=1}^k \left[I_k + \sum_{l=1}^{n-k} \lambda_l DR_l(h) \right] = D\gamma(h). \quad (3.1.3)$$

for all $h \in \Omega_h^{(k)}$.

Proof. Differentiating Γ_j in (3.1.2) with respect to h_m, we obtain

$$\begin{aligned} \partial_m \Gamma_j(h,0) &= \tfrac{\partial}{\partial h_m}\Gamma_j(h+\lambda R(h),\lambda) \\ &= \partial_m \Gamma_j(h+\lambda R(h),\lambda)[1+\lambda\partial_m R^m(h)] \\ &+ \textstyle\sum_{1\le i\le n-k, i\ne m} \partial_i \Gamma_j(h+\lambda R(h),\lambda)[\lambda\partial_m R^i(h)]. \end{aligned}$$

In matrix notation it means (3.1.3).

The group $GL_n(\mathbb{C})$ acts on Γ in a natural way. For $A \in GL_n$ let us denote

$$\Gamma_A(\xi) = \Gamma(A\xi).$$

Proposition 3.1.2. *Let Γ satisfy conditions* (ΓA1), (ΓA2).

(1) *For every $A \in GL_n$ the mapping $\Gamma_A : A^{-1}(\Omega) \to \mathbb{C}^p$ satisfies* (ΓA1), (ΓA2). *Moreover, $\Omega^{(k)}(\Gamma_A) = A^{-1}(\Omega^{(k)}(\Gamma))$ and $\Omega^{\text{sing}}(\Gamma_A) = A^{-1}(\Omega^{\text{sing}}(\Gamma))$.*

(2) *There exists $A \in GL_n$, such that the mapping $\gamma_A : (A^{-1}(\Omega))_h \to \mathbb{C}^p$ defined by $\gamma_A(h) = \Gamma_A(h, 0)$ satisfies*

$$\operatorname{rank} D\gamma_A(h) = k$$

for h in an open dense subset of $(A^{-1}(\Omega))_h$.

Proof. Statement (1) is straightforward. Now we will prove (2). Let $i_1, \dots, i_k$ be the minimal indices, for which there exist $j_1, \dots, j_k$ and $\xi \in \Omega^{(k)}$, such that

$$\det \begin{pmatrix} \partial_{i_1}\Gamma_{j_1}(\xi) & \dots & \partial_{i_k}\Gamma_{j_1}(\xi) \\ \vdots & \dots & \vdots \\ \partial_{i_1}\Gamma_{j_k}(\xi) & \dots & \partial_{i_k}\Gamma_{j_k}(\xi) \end{pmatrix} \neq 0. \tag{3.1.4}$$

For $\tau \neq 0$ define $A_\tau \in GL_n$ by

$$A_\tau\zeta = \tau\zeta + \sum_{l=1}^{k} \zeta_l e_{i_l},$$

where e_l stands for the l-th standard unit basis vector of $\mathbb{C}^n$. By the first part of the proposition the mapping Γ_{A_τ} satisfies conditions (ΓA1), (ΓA2). Jacobian of the mapping γ_{A_τ} has the form

$$D\gamma_{A_\tau}(\zeta) = \tag{3.1.5}$$
$$\begin{pmatrix} \tau\partial_1\Gamma_1(A_\tau\zeta) + \partial_{i_1}\Gamma_1(A_\tau\zeta) & \dots & \tau\partial_k\Gamma_1(A_\tau\zeta) + \partial_{i_k}\Gamma_1(A_\tau\zeta) \\ \vdots & \vdots & \vdots \\ \tau\partial_1\Gamma_p(A_\tau\zeta) + \partial_{i_1}\Gamma_p(A_\tau\zeta) & \dots & \tau\partial_k\Gamma_p(A_\tau\zeta) + \partial_{i_k}\Gamma_p(A_\tau\zeta) \end{pmatrix}.$$

In particular, this matrix contains a block of the form

$$\begin{pmatrix} \tau\partial_1\Gamma_{j_1}(A_\tau\zeta) + \partial_{i_1}\Gamma_{j_1}(A_\tau\zeta) & \dots & \tau\partial_k\Gamma_{j_1}(A_\tau\zeta) + \partial_{i_k}\Gamma_{j_1}(A_\tau\zeta) \\ \vdots & \vdots & \vdots \\ \tau\partial_1\Gamma_{j_k}(A_\tau\zeta) + \partial_{i_1}\Gamma_{j_k}(A_\tau\zeta) & \dots & \tau\partial_k\Gamma_{j_k}(A_\tau\zeta) + \partial_{i_k}\Gamma_{j_k}(A_\tau\zeta) \end{pmatrix}. \tag{3.1.6}$$

The determinant of this block does not vanish for some $\zeta \in A_\tau^{-1}(\Omega^{(k)})$ in view of the choice of $i_1, \dots, i_k$ and $j_1, \dots, j_k$ as in (3.1.4), if τ is sufficiently small.

Finally, in view of the analyticity of Γ, the block (3.1.6) is non-degenerate on the complement of an analytic set in $\Omega^{(k)}{}_h$, implying the last statement.

In view of Proposition 3.1.2 without loss of generality we may assume that the mapping γ in (3.1.3) satisfies the non-degeneracy condition of Proposition 3.1.2, (2). Thus, for the global fibrations we get

Proposition 3.1.3. *Let $\Gamma : \mathbb{C}^n \to \mathbb{C}^p$ be an entire holomorphic function. Then for every $h \in \Omega_h^{(k)}$ the condition* $\operatorname{rank} D\gamma(h) = k$ *implies that the matrices $DR_l(h)$ are nilpotent for all $1 \le l \le n-k$.*

Proof. Suppose that some $DR_{l_0}(h)$ is not nilpotent and let $\mu \in \mathbb{C}$ be its nonzero eigenvalue. Equation (3.1.3) holds locally in h, λ, but both sides are holomorphic with respect to λ, so by the analytic continuation it holds for all $\lambda \in \mathbb{C}$, because Γ is an entire function. Substitution of $\lambda_l = -\mu^{-1}\delta_{ll_0}$ leads to the degeneracy of $D\gamma(h)$, a contradiction.

Example 3.1.4. Define $\Gamma : \mathbb{C}^4 \to \mathbb{C}^4$ by

$$\Gamma(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_1\xi_2 + \xi_3\xi_4, \xi_2, \xi_3, 0).$$

The fibration associated to Γ is global, and for the mapping Γ and its restriction

$$\gamma(h_1, h_2, h_3) = (h_1 h_2, h_2, h_3, 0)$$

holds

$$D\Gamma(\xi) = \begin{pmatrix} \xi_2 & \xi_1 & \xi_4 & \xi_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D\gamma(h) = \begin{pmatrix} h_2 & h_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proposition 3.1.3 holds for all $h \in \mathbb{C}^3$ with $h_2 \neq 0$. Localization R of the fibration $\varkappa$ satisfies the equality $D\Gamma(h,0)\left(\begin{smallmatrix} R(h) \\ 1 \end{smallmatrix}\right) = 0$, from which we conclude that $R^1(h) = -h_3/h_2$, $R^2(h) = 0$, $R^3(h) = 0$, with nilpotent matrix $DR(h) = \left(\begin{smallmatrix} 0 & h_3 h_2^{-2} & -h_2^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

Example 3.1.5. The rank of the matrix $D\Gamma$ can drop by a number larger than one at points of $\Omega \backslash \Omega^{(k)}$. For $\Gamma_1(\xi) = (\xi_1\xi_2 + \xi_3\xi_4, \xi_2^2, \xi_3^2, 0)$ and $\Gamma_2(\xi) = \frac{1}{2}((\xi_1\xi_2 + \xi_3\xi_4)^2, \xi_2^2, \xi_3^2, 0)$ we have

$$\begin{aligned} D\Gamma_1(\xi) &= \begin{pmatrix} \xi_2 & \xi_1 & \xi_4 & \xi_3 \\ 0 & 2\xi_2 & 0 & 0 \\ 0 & 0 & 2\xi_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ D\Gamma_2(\xi) &= \begin{pmatrix} \xi_2\alpha(\xi) & \xi_1\alpha(\xi) & \xi_4\alpha(\xi) & \xi_3\alpha(\xi) \\ 0 & \xi_2 & 0 & 0 \\ 0 & 0 & \xi_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

with $\alpha(\xi) = \xi_1\xi_2 + \xi_3\xi_4$. Therefore, $\operatorname{rank} D\Gamma_1|_{\xi_2=\xi_3=0} = 1$, $\operatorname{rank} D\Gamma_1|_{\xi=0} = 0$ and $\operatorname{rank} D\Gamma_2|_{\xi_2=\xi_3=0} = 0$.

3.2 Fibrations of gradient type

In what follows we prove the results simultaneously for two fields of scalars, $\mathbb{R}$ and $\mathbb{C}$, which we will denote by $\mathbb{K}$. In this notation for $m \geq 1$ we will write $C^m(\mathbb{K}^n)$ for the space $C^m(\mathbb{R}^n)$ when $\mathbb{K} = \mathbb{R}$, and for the space of holomorphic functions when $\mathbb{K} = \mathbb{C}$.

We consider the case of the fibration by codimension k hyperplanes, corresponding to the conditions (ΓA1), (ΓA2), with Γ having a gradient form, that is

$$\Gamma(\xi) = \nabla\psi(\xi) \tag{3.2.1}$$

for some function $\psi : \Omega \to \mathbb{K}$. If the mapping Γ is as in (3.2.1), the associated fibration will be called the *fibration of gradient type*. Fibrations of this type will appear in Section 4, where Γ is of the form (3.2.1) with a generating function ϕ or its non-homogeneous version ψ. In this case $D\Gamma = D^2\psi$ and fibers $\varkappa$ of the mapping Γ correspond to the level sets of the gradient $\nabla\psi$. In general, we will make some smoothness assumptions, but we note, that the following statements are valid where all the derivatives make sense, so at least at the open subset $\Omega^{(k)}$ with rank $D^2\psi = k$. We adopt notation (3.1.1), related to the mapping

$$R : \Omega_h^{(k)} \subset \mathbb{K}^k \to \mathbb{K}^{(n-k)\times k}$$

as in the previous section for the case $\Gamma = \nabla\psi$ as well.

Theorem 3.2.1. (1) *Let $\psi \in C^2(\mathbb{K}^n)$. The fibration by the level sets of $\nabla\psi$ is given by the mapping $R : \mathbb{K}^k \to \mathbb{K}^{(n-k)\times k}$ in a neighborhood $U \times V \subset \mathbb{K}^k \times \mathbb{K}^{n-k}$ of the origin, i.e.*

$$\nabla\psi(h + \lambda R(h), \lambda) = \nabla\psi(h, 0), \forall h \in U \cap \Omega_h^{(k)}, \lambda \in V, \tag{3.2.2}$$

if and only if

$$D^2\psi(h + \lambda R(h), \lambda) \begin{pmatrix} R^T(h) \\ I_{n-k} \end{pmatrix} = 0 \in \mathbb{K}^{n\times(n-k)}, \quad \forall h \in U \cap \Omega_h^{(k)}, \lambda \in V. \tag{3.2.3}$$

In this case ψ must be of the form (3.2.4) with

$$\phi(h) = \psi(h, 0), \ \chi_{k+l}(h) = \frac{\partial\psi}{\partial h_{k+l}}(h, 0), \ \ l = 1, \dots, n-k.$$

If $\psi \in C^3(\mathbb{K}^n)$, then ϕ and χ_{k+l} satisfy equations (3.2.5),(3.2.7) *in $\Omega_h^{(k)}$.*

(2) *Let function ψ be defined by*

$$\begin{aligned}\psi(h+\lambda R(h),\lambda) &= \phi(h) + \langle\lambda, \nabla\psi(h,0)\begin{pmatrix} R^T(h) \\ I_{n-k}\end{pmatrix}\rangle \\ &= \phi(h) + \sum_{l=1}^{n-k}\lambda_l[\sum_{j=1}^{k}\partial_j\phi(h)R_l^j(h) + \chi_{k+l}(h)],\end{aligned} \tag{3.2.4}$$

where $\phi \in C^3(\mathbb{K}^n)$ solves the system of $\frac{1}{2}(k-1)k(n-k)$ partial differential equations

$$\sum_{i=1}^{k}\frac{\partial^2\phi}{\partial h_i\partial h_j}\frac{\partial R_l^i}{\partial h_m} - \sum_{i=1}^{k}\frac{\partial^2\phi}{\partial h_i\partial h_m}\frac{\partial R_l^i}{\partial h_j} = 0,$$
$$1 \le m < j \le k, 1 \le l \le n-k \tag{3.2.5}$$

and let functions $\chi_{k+l} \in C^2(\mathbb{K}^n)$, $l = 1, \dots, n-k$, satisfy

$$\partial_j\chi_{k+l}(h) + \sum_{i=1}^{k}\partial_j\partial_i\phi(h)R_l^i(h) = 0, \tag{3.2.6}$$
$$1 \le j \le k.$$

Then ψ satisfies the factorization condition (3.2.2) locally in h, λ, for which

$$\det\left(I_k + \sum_{l=1}^{n-k}\lambda_l DR_l(h)\right) \neq 0, \tag{3.2.7}$$

Proof. (1) Differentiation of (3.2.2) with respect to λ yields

$$\begin{aligned}\frac{\partial}{\partial\lambda_i}\partial_j\psi(h+\lambda R(h),\lambda) &= \textstyle\sum_{m=1}^{k}\partial_m\partial_j\psi(h+\lambda R(h),\lambda)R_i^m(h) \\ &+ \partial_{k+i}\partial_j\psi(h+\lambda R(h),\lambda) \\ &= \nabla\partial_j\psi(h+\lambda R(h),\lambda)\begin{pmatrix} R_i^T(h) \\ \delta_{j,k+i}\end{pmatrix}.\end{aligned} \tag{3.2.8}$$

In view of (3.2.2) this is equal to zero. Because this holds for all $1 \le j \le n$ and $1 \le i \le n-k$, we get (3.2.3). Now assume (3.2.3) to hold. Differentiating $\partial_j\psi(h+\lambda R(h),\lambda)$ for $1 \le j \le n$ with respect to λ_i, we obtain equation (3.2.8). Writing this for all i, j in matrix notation, we get

$$\nabla_\lambda\nabla\psi(h+\lambda R(h),\lambda) = D^2\psi(h+\lambda R(h),\lambda)\begin{pmatrix} R^T(h) \\ I_{n-k}\end{pmatrix},$$

the latter equals zero by (3.2.3). Hence

$$\nabla\psi(h+\lambda R(h),\lambda) = \text{const}(h) = \nabla\psi(h,0),$$

which is (3.2.2). Differentiating $\psi(h+\lambda R(h),\lambda)$ with respect to λ, we obtain

$$\begin{aligned}\nabla_\lambda\psi(h+\lambda R(h),\lambda) &= \nabla\psi(h+\lambda R(h),\lambda)\begin{pmatrix} R^T(h) \\ I_{n-k}\end{pmatrix} \\ &= \nabla\psi(h,0)\begin{pmatrix} R^T(h) \\ I_{n-k}\end{pmatrix},\end{aligned}$$

where the last equality follows from (3.2.2). Thus, ψ must have the form (3.2.4).

Now let us prove (3.2.5). Writing (3.2.3) in a scalar form once again, and substituting $\lambda=0$, the equalities for $1\le j\le k$ yield

$$\sum_{i=1}^{k}\partial_i\partial_j\phi(h)R_l^i(h)+\partial_{k+l}\partial_j\psi(h,0)=0,\ 1\le l\le n-k,$$

which are equations (3.2.7) in view of $\psi\in C^2(\mathbb{K}^n)$ and $\partial_j\partial_{k+l}\psi=\partial_{k+l}\partial_j\psi$. The relations $\partial_m\partial_j\chi_{k+l}=\partial_j\partial_m\chi_{k+l}$ for $\psi\in C^3(\mathbb{K}^n)$ and (3.2.7) imply

$$\frac{\partial}{\partial h_m}\Big(\sum_{i=1}^{k}\partial_i\partial_j\phi(h)R_l^i(h)\Big)=\frac{\partial}{\partial h_j}\Big(\sum_{i=1}^{k}\partial_i\partial_m\phi(h)R_l^i(h)\Big).$$

In view of $\phi\in C^3$ these equations imply (3.2.5).

(2) Define ψ as in (3.2.4). First we observe that equations (3.2.5) guarantee the existence of $\chi_{k+l}\in C^2(\mathbb{K}^n)$ satisfying (3.2.7). Differentiating (3.2.4) with respect to h_m, $1\le m\le k$, we get

$$\begin{aligned}&\partial_m\psi(h+\lambda R(h),\lambda)(1+\lambda\partial_m R^m(h))\\ &+\sum_{1\le i\le k, i\ne m}\partial_i\psi(h+\lambda R(h),\lambda)(\lambda\partial_m R^i(h))\\ &\qquad=\frac{\partial}{\partial h_m}\psi(h+\lambda R(h),\lambda)\\ &\qquad=\partial_m\phi(h)+\sum_{l=1}^{n-k}\lambda_l\Big[\sum_{j=1}^{k}(\partial_m\partial_j\phi(h)R_l^j(h)\\ &\qquad\quad+\partial_j\phi(h)\partial_m R_l^j(h))+\partial_m\chi_{k+l}(h)\Big]\\ &\qquad=\partial_m\phi(h)+\sum_{l=1}^{n-k}\lambda_l\Big[\sum_{j=1}^{k}\partial_j\phi(h)\partial_m R_l^j(h)\Big],\end{aligned}$$

where we used definitions (3.2.4) and (3.2.7). In matrix notation we get

$$\nabla_1^k\psi(h+\lambda R(h),\lambda)(I_k+\sum_{l=1}^{n-k}\lambda_l DR_l(h))=\nabla_1^k\psi(h,0)(I_k+\sum_{l=1}^{n-k}\lambda_l DR_l(h))$$

and for h,λ satisfying (3.2.7) we get $\partial_j\psi(h+\lambda R(h),\lambda)=\partial_j\psi(h,0)$, $1\le j\le k$, independent of λ. For the other components of $\nabla\psi$ we differentiate (3.2.4) with respect to λ_l and get

$$\begin{aligned} & \sum_{j=1}^{k} \partial_j \psi(h+\lambda R(h),\lambda) R_l^j(h) + \partial_{k+l}\psi(h+\lambda R(h),\lambda) \\ = \; & \frac{\partial}{\partial \lambda_l} \psi(h+\lambda R(h),\lambda) \\ = \; & \sum_{j=1}^{k} \partial_j \phi(h) R_l^j(h) + \chi_{k+l}(h), \end{aligned}$$

the last equation follows from definition (3.2.4). This implies

$$\partial_{k+l}\psi(h+\lambda R(h),\lambda) = \chi_{k+l}(h)$$

in view of the just proved factorization property for $1 \le j \le k$. The proof is complete.

It follows from Theorem 3.2.1 that being a fibration of gradient type is equivalent to the system of equations (3.2.5) if we regard (3.2.7) as a definition of χ_{k+l}. Expression (3.2.5) in the matrix form becomes

$$\partial_m R_l(h)(D^2\phi(h))^j = \partial_j R_l(h)(D^2\phi(h))^m, \; 1 \le l \le n-k,$$

for each row R_l of R, where $(D^2\phi(h))^j$ is the j-th column of $D^2\phi(h)$. Denoting by $\mathbb{K}^{k\times k}_{\text{symm}}$ the space of symmetric $k \times k$-matrices, we conclude

Corollary 3.2.2. *Condition (3.2.5) is equivalent to the condition*

$$DR_l(h)^T D^2\phi(h) = \begin{pmatrix} \partial_1 R_l(h) \\ \vdots \\ \partial_k R_l(h) \end{pmatrix} D^2\phi(h) \in \mathbb{K}^{k\times k}_{\text{symm}}, \; \forall h \in \Omega_h^{(k)},$$
$$1 \le l \le n-k.$$

Note, that in our notation $\partial_m R_l(h)$ is the m-th column of the matrix $DR_l(h)$, and the m-th row of $DR_l(h)^T$, which is the reason for the transpose.

Remark 3.2.3. Suppose that instead of the gradient form (3.2.1) we assume that both $D\Gamma$ and $D\gamma$ are symmetric square matrices after a possible elimination or addition of dependent rows. Then the symmetricity condition of Corollary 3.2.2 follows from equation (3.1.3) of Lemma 3.1.1, after multiplication from the left by $(I_k + \sum_l \lambda_l DR_l(h))^T$ and using the symmetricity of $D\Gamma$ and $D\gamma$. However, the matrix $DR(h)^T D\gamma(h)$ in the examples of Section 3.1 need not be symmetric, in comparison with Corollary 3.2.2.

3.3 Reconstruction of the phase function

The construction of the function ψ in Theorem 3.2.1, (2), clearly depends on the singularities of the fibration R. In the case of $\mathbb{K} = \mathbb{R}$ by $C^m(U)$ we denote the usual space of m times continuously differentiable functions in U, $1 \le m \le \infty$, and analytic functions with $m = \omega$. In the complex case $\mathbb{K} = \mathbb{C}$ all these spaces coincide with the space of holomorphic functions in U.

Let Ω_0 be the set of all points in Ω satisfying condition (3.2.7):

$$\Omega_0 = \{(h, \lambda) \in \Omega \cap (\mathbb{K}^k \times \mathbb{K}^{n-k}) : \det(I_k + \sum_{l=1}^{n-k} \lambda_l DR_l(h)) \neq 0\}. \qquad (3.3.1)$$

Assume now that a fibration R is of class C^m in an open subset Ω^a of Ω_h, ϕ is in $C^{m+1}(\Omega)$ and all $\chi_{k+l} \in C^m(\Omega)$. Then the mapping

$$\Phi : \Omega \cap (\mathbb{K}^k \times \mathbb{K}^{n-k}) \ni (h, \lambda) \mapsto (\Phi_1(h, \lambda), \Phi_2(h, \lambda)) \in \mathbb{K}^k \times \mathbb{K}^{n-k}$$

with components

$$\begin{array}{rcl} \Phi_1(h, \lambda) & = & h + \sum_{l=1}^{n-k} \lambda_l R_l(h), \\ \Phi_2(h, \lambda) & = & \lambda \end{array} \qquad (3.3.2)$$

is C^m on $\pi_h^{-1}(\Omega^a) \cap \Omega$ and its Jacobian is

$$\det D\Phi(h, \lambda) = \det \begin{pmatrix} I_k + \lambda DR(h) & R(h) \\ 0 & I_{n-k} \end{pmatrix} = \det(I_k + \sum_{l=1}^{n-k} \lambda_l DR_l(h)),$$

which is not zero when $(h, \lambda) \in \Omega_0$. By the implicit function theorem the mapping Φ^{-1} is also C^m in $\Phi(\pi_h^{-1}(\Omega^a) \cap \Omega_0)$. The statement of Theorem 3.2.1, (2), implies that on Ω_0 the function ψ satisfies

$$(\psi \circ \Phi)(h, \lambda) = \phi(h) + \sum_{j=1}^{k} \partial_j \phi(h)(\Phi_1(h, \lambda) - h)^j + \sum_{l=1}^{n-k} \lambda_l \chi_{k+l}(h),$$

implying that $\psi \in C^m$ in $\Phi(\pi_h^{-1}(\Omega^a) \cap \Omega_0) \subset \Phi(\Omega)$. Thus, we arrive at

Proposition 3.3.1. *Let $R : \Omega^a \subset \Omega_h \subset \mathbb{K}^k \to \mathbb{K}^{(n-k)\times k}$ and let Ω^a be an open subset of Ω_h. Let $3 \leq m \leq \infty$ or $m = \omega$, $R \in C^m(\Omega^a)$, $\phi \in C^{m+1}(\Omega_h)$, $\chi_{k+l} \in C^m(\Omega_h)$. Let Ω_0 be given by (3.3.1) and let Φ be as in (3.3.2). Then the function ψ defined in (3.2.4) is of class C^m in $\Phi(\pi_h^{-1}(\Omega^a) \cap \Omega_0) \subset \Phi(\Omega)$.*

Assume now, the the sets $\pi_h^{-1}(\Omega^a)$, Ω and Ω_0 are locally equal, let U denote their open intersection. In particular, this is the case with $m = \omega$, when the fibration is locally analytic, given by an analytic mapping R, for which the condition (3.2.7) is locally satisfied. As it is shown in Proposition 3.1.3, it can hold even globally if all DR_l are nilpotent. According to Theorem 3.2.1, the system of partial differential equations (3.2.5) is a sufficient condition for the existence of a function ψ, for which the fibration by the level sets of $\nabla\psi$ is locally given by R. Proposition 3.3.1 implies that ψ is analytic in $\Phi(U)$.

However, the existence of a solution ϕ in C^m of system (3.2.5) of second order partial differential equations is not automatic. In the complex case ($\mathbb{K} = \mathbb{C}$) one can use Cauchy–Kovalevskaya theorem, to obtain an analytic solution ϕ in a neighborhood of a point where not all coefficients of the iterated derivatives of the highest order vanish. In this case we obtain an analytic solution even for the Cauchy problem, prescribing ϕ on some non-characteristic hypersurface.

It would be interesting to obtain generalizations of Proposition 3.3.1. In general, given a fibration defined by a mapping $R \in C^m$, is it possible to construct a function ψ, for which the fibration by the level sets of $\nabla\psi$ is locally given by R? What are the smoothness properties of such ψ if it exists and when can we guarantee its smoothness? A special case occurs if the fibration R is analytic. According to the remarks above on Cauchy–Kovalevskaya theorem, it may imply the analyticity of ϕ and χ. But ψ, constructed by the formulae of Theorem 3.2.1, can be complex valued, even restricted to the real domain. Another interesting question, corresponding to the holomorphically extended real valued analytic functions is, can ψ be chosen in such a way, that restricted to $\mathbb{R}^n$ it defines a real valued analytic function. Or whether there exists ψ such that the imaginary part of ψ has a prescribed constant sign. In the latter case one can still regard ψ as an admissible phase function (see Section 4.2).

In general, the approach suggested by Theorem 3.2.1 reduces the questions posed above to the analysis of a large system of second order partial differential equations (3.2.5). The properties of its solutions depend on the type of the system, which is determined by the sign of its determinant, which in turn is an expression in terms of the fibration R. In general, if the fibration is sufficiently smooth, then the coefficients of system (3.2.5) are sufficiently smooth as well. In this case some properties can be found in [30], [15]. For hyperbolic equations cf. [28]. For fibrations with essential singularities the coefficients of system (3.2.5) are meromorphic and are given by rational functions.

In the case of a three dimensional space with $n = 3$ the system consists of only one equation. It simplifies the analysis. Assume now that $n = 3$ and $k = 2$. By Theorem 2.4.3 for the case $\mathbb{K} = \mathbb{C}$ fibrations with affine fibers are regular (analytic) everywhere. Hence we will assume smoothness in the real case $\mathbb{K} = \mathbb{R}$ as well. System (3.2.5) for $\phi : \mathbb{K}^2 \to \mathbb{K}$ has now the form

$$\frac{\partial^2\phi}{\partial h_1^2}\frac{\partial R_1}{\partial h_2} + \frac{\partial^2\phi}{\partial h_1 \partial h_2}\left[\frac{\partial R_2}{\partial h_2} - \frac{\partial R_1}{\partial h_1}\right] - \frac{\partial^2\phi}{\partial h_2^2}\frac{\partial R_2}{\partial h_1} = 0. \tag{3.3.3}$$

Proposition 3.3.2. *(1) If the matrix $DR(h)$ has no real eigenvalues, then equation (3.3.3) is elliptic. In this case the fibration is extendible globally in $\lambda \in \mathbb{R}$, locally uniformly in h.*

(2) If the matrix $DR(h)$ has a real non-zero eigenvalue, then the determinant of the matrix $D^2\phi(h)$ vanishes.

(3) Equation (3.3.3) is elliptic, parabolic, hyperbolic, if the eigenvalues of the matrix $DR(h)$ are not real, real equal to each other, real and distinct, respectively.

(4) Let $\psi \in C^\omega(\mathbb{R}^3)$, $h \in \mathbb{R}^2$. If the determinant of the matrix $D^2\phi(h)$ vanishes, then there exists a neighborhood U of the point h such that equation (3.3.3) is parabolic in U. The same conclusion still holds if

$\psi \in C^3(\mathbb{R}^3)$ and the fibration is globally extendible in λ, locally uniformly in h.

Proof. (1) If $DR(h)$ has no real eigenvalues, then (3.2.7) is satisfied for all real λ, which implies that the fibration extends globally in λ. We denote $R_{ij}(h) = \partial_j R_i(h)$. The equation for the eigenvalues of $DR(h)$ is $\det \begin{pmatrix} R_{11} - \mu & R_{12} \\ R_{21} & R_{22} - \mu \end{pmatrix} = 0$, the discriminant of which is $d(h) = (R_{11}(h) + R_{22}(h))^2 - 4 \det DR(h)$ and our assumption means $d(h) < 0$. Now, the type of the equation (3.3.3) is defined by the sign of

$$\det \begin{pmatrix} R_{12} & \frac{1}{2}[R_{22} - R_{11}] \\ \frac{1}{2}[R_{22} - R_{11}] & -R_{21} \end{pmatrix},$$

which is equal to $-R_{12}R_{21} - 1/4(R_{22} - R_{11})^2 = -1/4(R_{22} + R_{11})^2 + \det DR = -d(h)/4$. This means that the equation (3.3.3) is elliptic.

(2) Suppose that $\mu \in \mathbb{R}$, $\mu \neq 0$, is an eigenvalue of $DR(h)$: $DR(h)v = \mu v$. Differentiating (3.2.2) with respect to h, we get

$$\begin{pmatrix} \partial_1^2 \psi(h + \lambda R(h), \lambda) & \partial_1 \partial_2 \psi(h + \lambda R(h), \lambda) \\ \partial_2 \partial_1 \psi(h + \lambda R(h), \lambda) & \partial_2^2 \psi(h + \lambda R(h), \lambda) \end{pmatrix} (I + \lambda DR(h))$$

equal to $D^2\phi(h)$. Now we apply both sides to v and take $\lambda = -\mu^{-1}$ to conclude that $D^2\phi(h)v = 0$.

(3) Follows from part (1).

(4) The continuity of $D^2\phi(h)$ implies that there exists a neighborhood U of h, such that $D^2\phi$ is nonsingular in U. The argument of Proposition 3.1.3 implies that DR is nilpotent in U, its both eigenvalues are equal to zero. The second part of the proposition implies that (3.3.3) is parabolic. This completes the proof of Proposition 3.3.2.

In the complex case ($\mathbb{K} = \mathbb{C}$) with a holomorphic fibration R, if for $i = 1$ or $i = 2$ we denote by h_i the variable such that $\partial R_i / \partial h_j$, $j \neq i$, in (3.3.3) is locally non-zero at $h_i = 0$, then by Cauchy–Kovalevskaya theorem equation (3.3.3) with holomorphic Cauchy boundary conditions $\partial_{h_i}^l \phi$, $l < 2$, is locally soluble and the solution is unique and holomorphic. In general, if the Cauchy boundary conditions are holomorphic on some non-characteristic for (3.3.3) hypersurface S in $\mathbb{C}^2$, then there exists a unique solution ϕ of (3.3.3) locally at every point of S ([27, Vol.I, Theorems IX.4.5-IX.4.7]).

In the real case ($\mathbb{K} = \mathbb{R}$) the smoothness properties of solutions of equation (3.3.3) depend on its type. Thus, if it is elliptic with smooth (real analytic) coefficients, it always has smooth (real analytic) solutions:

Theorem 3.3.3. *Let $R : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be smooth (real analytic). Assume that for every $h \in U$ the matrix $DR(h)$ has no real eigenvalues. Then there exists a smooth (real analytic) function $\psi : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$, such that the fibration defined by the level sets of $\nabla\psi$ is given by R in U.*

Proof. According to Proposition 3.3.2 equation (3.3.3) is elliptic. This implies that its solutions ψ are smooth (analytic) if the operator in (3.3.3) has smooth (analytic) coefficients, see Corollary 8.3.2 for $WF(\phi)$ and Theorem 9.5.1 for $WF_A(\phi)$ in [27], respectively. Then also all χ_{k+l} are smooth (analytic) and Proposition 3.3.1 implies the statement.

Let us discuss briefly the case when equation (3.3.3) is strictly hyperbolic. If a mapping (fibration) R is prescribed on a smooth hypersurface S which is non-characteristic for (3.3.3), then solution ϕ is in the Sobolev space $H^{(s)}$ if the Cauchy data are in the Sobolev spaces $H^{(s)}$. By the Sobolev embedding theorem applied to all $s > 0$ we get that solution ϕ is smooth if the Cauchy data are smooth on S. For real analytic fibrations R let us use the methods described in [10, VI]. Let Ω be a connected complex analytic manifold of some dimension p. Let $\mathbb{D}(\Omega)$ be the algebra of linear partial differential operators with holomorphic coefficients in Ω. Let J be a left ideal in $\mathbb{D}(\Omega)$. Then J is called *holonomic in* Ω if for every $x \in \Omega$ the common zero set in $\mathbb{C}^p$ of the principal symbols $q^x(\theta)$ at x of the $Q \in J$ (in local coordinates) is equal to zero. In particular, the holonomicity assumption for the ideal generated by a hyperbolic operator P leads to an ordinary differential equation along every curve in the complement of the complex bicharacteristic cone of the principal symbol of P and the standard analytic continuation for ordinary differential equations can be applied. Let us apply [10, VI-16, Theorem] to our situation without giving many details.

Proposition 3.3.4. *Assume that a mapping R is analytic in an open subset Ω_0 of $\mathbb{R}^2$ and that DR has real distinct eigenvalues in Ω_0. Let Ω be the complexification of Ω_0 and let ϕ be a solution of (3.3.3). Assume also that the ideal $J = \{Q \in \mathbb{D}(\Omega) : Q\phi = 0\}$ generated by equation (3.3.3) is holonomic in Ω. Then ϕ is real analytic in Ω_0 and has a holomorphic extension ϕ_γ along each curve γ in Ω starting in Ω_0. The holomorphic extension ϕ_γ also satisfies equations $Q\phi_\gamma = 0$ for all $Q \in J$.*

Let us give several corollaries of Theorem 3.2.1 for the real case $\mathbb{K} = \mathbb{R}$ and arbitrary k.

Corollary 3.3.5. (1) *Let the determinant of the matrix $D^2\phi(h)|_{h=h_0}$ vanish. Assume that $R(h_j) \to R_0$ as $h_j \to h_0$, $h_j \in \Omega_h^{(k)}$. If the determinant of the matrix*

$$\begin{pmatrix} \partial_1^2\psi(h_0+\lambda R_0,\lambda) & \dots & \partial_1\partial_k\psi(h_0+\lambda R_0,\lambda) \\ \vdots & \dots & \vdots \\ \partial_k\partial_1\psi(h_0+\lambda R_0,\lambda) & \dots & \partial_k^2\psi(h_0+\lambda R_0,\lambda) \end{pmatrix} \tag{3.3.4}$$

does not vanish for some $\lambda \in \mathbb{R}^{n-k}$, then there exists a limit

$$A = \lim_{j\to\infty} \sum_{l=1}^{n-k} \lambda_l DR_l(h_j)$$

and (-1) is an eigenvalue of the matrix A.

(2) *If for $h \in \Omega_h$ there exists $\lambda \in \mathbb{R}^{n-k}$ such that $(h, \lambda) \in \Omega$ and if -1 is an eigenvalue of the matrix $\sum_{l=1}^{n-k} \lambda_l DR_l(h)$, then the determinant of the matrix $D^2\phi(h)$ equals zero.*

(3) *If for a global fibration at least one of the matrices $DR_j(h)$, $1 \leq j \leq n-k$, has a non-zero real eigenvalue, then the determinant of the matrix $D^2\phi(h)$ equals zero.*

Proof. (1) First we note that for $\lambda = 0$ matrix (3.3.4) is equal to the singular matrix $D^2\phi(h_0)$ and thus the non-singularity of (3.3.4) implies $\lambda \neq 0$. Differentiating (3.2.2) with respect to h, we get

$$\begin{pmatrix} \partial_1^2\psi(h+\lambda R(h),\lambda) & \dots & \partial_1\partial_k\psi(h+\lambda R(h),\lambda) \\ \vdots & \dots & \vdots \\ \partial_k\partial_1\psi(h+\lambda R(h),\lambda) & \dots & \partial_k^2\psi(h+\lambda R(h),\lambda) \end{pmatrix} \times$$
$$\times (I_k + \sum_{l=1}^{n-k} \lambda_l DR_l(h)) = D^2\phi(h). \qquad (3.3.5)$$

The assumption implies then the limit of $I_k + \sum_{l=1}^{n-k} \lambda_l DR_l(h_j)$ exists and so the limit of $\sum_{l=1}^{n-k} \lambda_l DR_l(h_j)$ exists as well in view of $\lambda \neq 0$. We also see that $I_k + A$ is singular. This means that -1 is an eigenvalue of A, implying the statement.

(3) Suppose that $\mu \in \mathbb{R}$ is an eigenvalue of $DR_j(h)$ for some $1 \leq j \leq n-k$: $DR_j(h)v = \mu v$. Differentiating (3.2.2) with respect to h, we get

$$(\partial_i\partial_j\psi(h+\lambda R(h),\lambda))_{1\leq i,j\leq k} (I_k + \sum_{l=1}^{n-k} \lambda_l DR_l(h)) = D^2\phi(h).$$

Now we apply both sides to v and take $\lambda_j = -\mu^{-1}$ and $\lambda_l = 0$ for $l \neq j$. This leads to $D^2\phi(h)v = 0$. The proof of (2) is similar.

Finally, let us make several remarks for the case $k = n-1$. In this case there is only one matrix DR_1. If for some h the matrix $DR_1(h)$ does not have any real eigenvalues, then the fibration is extendible globally in $\lambda \in \mathbb{R}$ locally uniformly in h.

Corollary 3.3.6. *Let $k = n-1$ and assume that $D^2\phi(h)|_{h=h_0}$ is singular. Let $R_0 = \lim_j R(h_j)$ as $h_j \to h_0$, $h_j \in \Omega^{(k)}{}_h$. Then we have*

$$\det \begin{pmatrix} \partial_1^2\psi(h+\lambda R_0,\lambda) & \dots & \partial_1\partial_{n-1}\psi(h+\lambda R_0,\lambda) \\ \vdots & \dots & \vdots \\ \partial_{n-1}\partial_1\psi(h+\lambda R_0,\lambda) & \dots & \partial_{n-1}^2\psi(h+\lambda R_0,\lambda) \end{pmatrix} \equiv 0 \qquad (3.3.6)$$

for all λ in a neighborhood of zero. Moreover, if there exist the limit $A = \lim_j DR(h_j)$ then (3.3.6) holds for all $|\lambda| < \rho(A)^{-1}$, where $\rho(A)$ is the spectral radius of A. If the limit $\lim_j DR(h_j)$ does not exist, then (3.3.6) holds for all λ, for which it is defined.

Proof. If equality (3.3.6) does not hold then by the second part of Corollary 3.3.5 there exists the limit $A = \lim_j DR(h_j)$ and -1 is an eigenvalue of the matrix λA. Clearly, this is false for all $|\lambda| < \rho(A)^{-1}$. The last assertion follows from (3.3.5), if we use that $\det(I_{n-1} + \mu A) \neq 0$.

Note, that λ in Corollary 3.3.6 can be chosen independent of R_0, since it is continuous as a function of R_0, which takes its values in the compact Grassmannian $\mathbb{G}_1(\mathbb{C}^n)$. In the complex notation of Section 2 the statement of Corollary 3.3.6 is to say that

$$(h_0, 0) \in \Omega \backslash \Omega^{(n-1)} \Rightarrow \bar{\varkappa}(h_0, 0) \cap \Omega \subset \Omega \backslash \Omega^{(n-1)},$$

which corresponds to Lemma 2.8.7.

3.4 Existence of singular fibrations of gradient type

Theorem 2.4.2 and results of Section 2 show that the smallest dimension for which a singular fibration can exist for $\mathbb{K} = \mathbb{C}$, is $n = 4$ with $k = 3$. Let us show that in $\mathbb{C}^4$ there is a fibration by lines corresponding to a fibration of gradient type for $\Gamma = \nabla\psi$, and analytic (even polynomial) function ψ.

Example 3.4.1. Let

$$\psi(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 \xi_2^2 + (\xi_3 - \xi_2 \xi_4)^2.$$

The Hessian of the function ψ has the form

$$D^2\psi(\xi) = \begin{pmatrix} 0 & 2\xi_2 & 0 & 0 \\ 2\xi_2 & 2\xi_1 + 2\xi_4^2 & -2\xi_4 & 4\xi_2\xi_4 - 2\xi_3 \\ 0 & -2\xi_4 & 2 & -2\xi_2 \\ 0 & 4\xi_2\xi_4 - 2\xi_3 & -2\xi_2 & 2\xi_2^2 \end{pmatrix}$$

with $k = 3$. Moreover,

$$\operatorname{rank} D^2\psi|_{\xi_2=0} = 2, \quad \operatorname{rank} D^2\psi|_{\xi_2=\xi_3=0} = 2, \quad \operatorname{rank} D^2\psi|_{\xi_1=\xi_2=\xi_3=0} = 1.$$

For $\xi_2 \neq 0$ the kernel of the matrix $D^2\psi(\xi)$ is one dimensional,

$$\ker D^2\psi(\xi) = \langle (\frac{\xi_3}{\xi_2} - \xi_4,\ 0,\ \xi_2,\ 1) \rangle.$$

Therefore, for $\lambda = \xi_4$ the fibration, given by the mapping R, has coordinates $R^1(h) = \frac{h_3}{h_2}$, $R^2(h) = 0$, $R^3(h) = h_2$. The function R^1 has a essential singularity at $h_2 = h_3 = 0$ and we have $\Omega^{\text{sing}} = \{\xi_2 = \xi_3 = 0\}$.

Example 3.4.2. Functions

$$\psi(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1 \xi_2^k + (\xi_3 - \xi_2 \xi_4)^m$$

with $k, m \geq 2$ lead to fibrations with essential singularities at $\xi_2 = \xi_3 = 0$. The fibers are given by

$$\ker D^2\psi(\xi) = \langle(\frac{m}{k}\frac{(\xi_3 - \xi_2\xi_4)^{m-1}}{\xi_2^{k-1}},\ 0,\ \xi_2,\ 1)\rangle$$

and

$$R^1(h) = \frac{m}{k}\frac{h_3^{m-1}}{h_2^{k-1}}.$$

Example 3.4.3. In n-dimensional space $\xi = (x_1, \ldots, x_{n-3}, y, z, w)$ define

$$\psi(\xi) = y^2 \sum_{i=1}^{n-3} x_i + (z - yw)^2.$$

The level sets of the gradient $\nabla\psi$ have dimension $n-3$, and hence $k = 3$. One can check that $\dim \Omega^{\mathrm{sing}} = n - 2$ (this also follows from dimension estimates of Theorem 2.4.2 with $k = 3$).

Example 3.4.4. Consider

$$\psi(x, y, z, v, w, s, t) = xy^2 + sv^2 + (z - yw - vt)^2.$$

The maximal rank of the Hessian $D^2\psi$ equals $k = 5$ and its kernel is spanned by the vectors

$$(\frac{z - yw - vt}{y}, 0, 0, y, 1, 0, 0), (0, 0, 0, v, 0, \frac{z - yw - vt}{v}, 1).$$

For $y = 0, z - vt = 0$ the first vector has an essential singularity, meanwhile the second vector is continuous at $v \neq 0$. Hence $\{y = 0, z = vt\} \subset \Omega^{\mathrm{sing}}$ and $\dim \Omega^{\mathrm{sing}} = n - 2$ by Theorem 2.7.7.

These examples present singular fibrations for different dimensions of the fibers. On the other hand, the dimension of Ω^{sing} in all examples equals $n-2$. In examples 3.4.1–3.4.2 it is necessary because for $n = 4$ Theorem 2.4.3 implies $k = 3$. It follows then by Theorem 2.4.2, (1), that $\dim \Omega^{\mathrm{sing}} = n - 2$. In this case a stronger Proposition 2.7.17 holds as well. In Examples 3.4.3 and 3.4.4 with different dimension of the fibers we still have $\dim \Omega^{\mathrm{sing}} = n - 2$. However, our constructions are based on the same idea for the singularity. It would be interesting to investigate whether the condition $\dim \Omega^{\mathrm{sing}} = n - 2$ is necessary for fibrations of gradient type.

Chapter 4

Further estimates for analytic Fourier integral operators

Suppose that manifolds X and Y are real analytic. Suppose also that the phase function of a Fourier integral operator is analytic. Such Fourier integral operator will be called *analytic.* Its canonical relation is then an analytic Lagrangian manifold. For such operators the smooth factorization condition can be extended to the complex domain. Further analysis of this condition can be applied to the L^p continuity of operators in $\mathbb{R}^n$ with $n \leq 4$, and in $\mathbb{R}^n$, with an arbitrary n and additional condition rank $d\pi_{X\times Y}|_C \leq n+2$.

4.1 Complexification

In this section we will briefly describe the complexification of real sets. Since the smooth factorization type condition (F) is formulated in terms of a real valued phase function, in both settings of real and complex valued phase functions we will need the complexification of real sets. In the complex domain results of Chapters 2 and 3 become available. For a real analytic manifold M by $\widetilde{M}$ we denote its extension to a complex domain (complexification). Let Λ be a real analytic conic Lagrangian submanifold of $T^*M\backslash 0$. By Theorem 1.1.3 the generating function ϕ for a localization Λ_ϕ of the manifold Λ is real analytic. The complexification $\widetilde{\Lambda}$ of the manifold Λ in the complexification of the cotangent bundle $\widetilde{T^*M}$ is a complex analytic conic Lagrangian submanifold of $T^*\widetilde{M}\backslash 0$.

Let us first assume that variables $\widetilde{\xi}$ can be taken as local coordinates for $\widetilde{\Lambda}$ in $(\widetilde{m}_0, \widetilde{\xi}_0) \in \widetilde{\Lambda}$. The manifold $\widetilde{\Lambda}$ in a neighborhood of the point $(\widetilde{m}_0, \widetilde{\xi}_0)$ takes the form

$$\widetilde{m}_i = H_i(\widetilde{\xi}),\ 1 \leq i \leq \dim \widetilde{M},$$

where H_i are holomorphic functions, real valued for $\widetilde{\xi}$ in the real domain. Lagrangian manifold $\widetilde{\Lambda}$ is involutive and hence the Poisson bracket defines an ideal on $\widetilde{\Lambda}$, i.e. $\{f, g\} = 0$ on $\widetilde{\Lambda}$, if $f = 0$ and $g = 0$ on $\widetilde{\Lambda}$, (cf. [5, 5.4.6]). In particular,

$$\{\widetilde{m}_i - H_i(\widetilde{\xi}), \widetilde{m}_j - H_j(\widetilde{\xi})\}|_{\widetilde{\Lambda}} = 0$$

for all i, j, and, therefore,

$$\frac{\partial H_i}{\partial \widetilde{\xi}_j} = \frac{\partial H_j}{\partial \widetilde{\xi}_i}, \ 1 \leq i, j \leq \dim \widetilde{M}.$$

This means that all mixed derivatives of the form $\sum_i H_i(\widetilde{\xi}) d\widetilde{\xi}_i$ vanish. It follows that it is closed and, therefore, exact. We obtain the existence of a holomorphic function H with $\partial H / \partial \widetilde{\xi}_i = H_i$. The manifold $\widetilde{\Lambda}$ then locally assumes the form $\{(\nabla H(\widetilde{\xi}), \widetilde{\xi})\}$. The same argument can be applied to Λ in the real domain to show that Λ is locally parameterized by $(\nabla H(\xi), \xi)$ with the same function H. The same statement is true in the general setting, when variables $\widetilde{\xi}$ do not define a local coordinate system (see [5, 5.6.4] for the complex case and [11, Lemma 3.7.4] for the real case).

Let π be the standard projection from T^*M to M and let $\widetilde{\pi}$ be its complexification from $T^*\widetilde{M}$ to $\widetilde{M}$. Let $\Lambda^{(k)}$ be the open subset of Λ, on which the maximal rank k of the mapping $d\pi|_\Lambda$ is attained. By the implicit function theorem the set $\Sigma^{(k)} = \pi(\Lambda^{(k)})$ is a smooth analytic k–dimensional submanifold of M contained in $\Sigma = \pi(\Lambda)$. We also have $\Lambda^{(k)} = N^*\Sigma^{(k)}$. The complex extension $\widetilde{\Sigma^{(k)}}$ of $\Sigma^{(k)}$ in $\widetilde{M}$ is a smooth analytic manifold of complex dimension k. Its conormal bundle $N^*\widetilde{\Sigma^{(k)}}$ is a complex analytic Lagrangian submanifold of $T^*\widetilde{M}$ containing $\Lambda^{(k)}$, where we identify $T^*\widetilde{M}$ and $\widetilde{T^*M}$. Let $\widetilde{\Lambda^{(k)}}$ denote a complexification of $\Lambda^{(k)}$. The sets $\widetilde{\Lambda^{(k)}}$ and $N^*\widetilde{\Sigma^{(k)}}$ are of complex dimension n and locally coincide. Therefore, the set $N^*\widetilde{\Sigma^{(k)}}$ is open in $\widetilde{\Lambda}$ and the level sets of $\widetilde{\pi}|_{\widetilde{\Lambda^{(k)}}}$ are linear subspaces of complex dimension k of cotangent spaces of the manifold $\widetilde{M}$. The complement $\widetilde{\Lambda} \backslash \widetilde{\Lambda^{(k)}}$ is analytic and nowhere dense.

Now let us apply the described construction to the product $M = X \times Y$, where X and Y are two real analytic manifolds. Let C be a homogeneous canonical relation satisfying the local graph condition and let ϕ be its generating function as in (1.5.7) and (1.5.8). Then ϕ is real analytic and by $\widetilde{\phi}$ we denote its complex extension. Since the complex holomorphic extension is unique, the function $\widetilde{\phi}$ is a generating function for $\widetilde{C}$. Reasoning as in Section 1.5.2 we get that the level sets (fibers) of the mapping

$$\Gamma : \widetilde{\xi} \mapsto \nabla_{\widetilde{\xi}} \widetilde{\phi}(\widetilde{y}, \widetilde{\xi}) \tag{4.1.1}$$

are affine subspaces of $\mathbb{C}^n$ of codimension k, for all $(\widetilde{y}, \widetilde{\xi})$ in the open dense set where the rank of $D^2_{\widetilde{\xi}\widetilde{\xi}} \widetilde{\phi}$ is maximal, equal to $n + k$. Thus, we proved the following theorem.

Theorem 4.1.1. *Let X, Y be real analytic manifolds of dimension n and let C be a real analytic homogeneous canonical relation from $T^*Y\backslash 0$ to $T^*X\backslash 0$. Let the maximal rank of $d\pi_{X\times Y}|_C$ be equal to $n+k$. Let ϕ be a local generating function for C, as in (1.5.7), (1.5.8). Let $\widetilde{X}, \widetilde{Y}, \widetilde{C}, \widetilde{\phi}$ be complex analytic extensions of X, Y, C, ϕ, respectively. Then the mapping (4.1.1) satisfies properties* (ΓR1), (ΓR2).

4.2 Estimates for analytic Fourier integral operators

Let us introduce the following definitions.

Definition 4.2.1. A Fourier integral operator with real phase $T \in I^\mu_\rho(X, Y; C)$ is called *analytic* if the sets X, Y, C are real analytic.

Definition 4.2.2. A Fourier integral operator with complex phase $T \in I^\mu_\rho(X, Y; C)$ is called *analytic* if the sets X, Y are real analytic and if the generating phase functions for C are real analytic.

Note, that we do not require the analyticity of the symbol in the definition since we are interested in the properties of operators modulo smooth terms. It is also often convenient to assume that the symbol of T is compactly supported.

Let $f : \mathbb{R} \to \mathbb{C}$ be a real analytic function in a neighborhood of a point $\xi_0 \in \mathbb{R}$. It means that in a neighborhood of ξ_0, $f(\xi)$ equals its Taylor expansion $\sum a_{i_1\dots i_n}\xi_1^{i_1}\cdots\xi_n^{i_n}$. It follows that the series with $\operatorname{Re} a_{i_1\dots i_n}$ and $\operatorname{Im} a_{i_1\dots i_n}$ as Taylor coefficients are absolutely convergent in a neighborhood of ξ_0 as well. Therefore, f is real analytic if and only if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are real analytic.

Let a Fourier integral operator $T \in I^\mu_\rho(X, Y; C)$ with real phase and $1/2 \le \rho \le 1$ commute with translations. Here X and Y are open sets in $\mathbb{R}^n$. Invariance means that T is a convolution operator with some distribution. The theory of such operators as Fourier multipliers is well known but their L^p properties require additional study (cf. [42], [68] and Section 1.5.3). For the case $\rho = 1/2$ see [20].

Let the distribution w be a Lagrangian distribution associated with the Lagrangian manifold $\Lambda \subset T^*(\mathbb{R}^n)$, obtained from C by fixing any point in T^*Y. By Theorem 1.1.3 there exists a real valued phase function ψ and a generating function H such that Λ is locally equal to Λ_ψ with a positively homogeneous in ξ function

$$\psi(z,\xi) = \langle z, \xi\rangle - H(\xi), \tag{4.2.1}$$

where

$$\Lambda_\psi = \{(\nabla H(\xi), \xi)\}. \tag{4.2.2}$$

The Lagrangian distribution w takes the form

$$w(z) = \int e^{i\psi(z,\xi)} a(z,\xi) d\xi,$$

where $a \in S^\mu_\rho$. The operator T assumes now the form

$$Tu(x) = u * w(x) = \int\int e^{i\Psi(x,y,\xi)} a(x-y,\xi)u(y)d\xi dy \tag{4.2.3}$$

with phase function

$$\Psi(x,y,\xi) = \langle x-y,\xi\rangle - H(\xi). \tag{4.2.4}$$

We assume that

$$\text{rank } d\pi_{X\times Y}|_C \le n+k \tag{4.2.5}$$

for some $0 \le k \le n-1$ and that the rank $n+k$ is attained somewhere. Condition (3) of Theorem 1.5.2 means that

$$\text{rank } D^2H(\xi) \le k, \tag{4.2.6}$$

for all $\xi \in \Xi$. By Theorem 4.1.1 the function H has a holomorphic extension, which we also denote by H. Since H is homogeneous of degree one, we can single out a conic variable τ and write ξ as $\xi = (\theta,\tau) \in \mathbb{C}^{n-1}\times\mathbb{C}$. For $\zeta \in \mathbb{C}^{n-1}$ let us define $F(\zeta) = H(\zeta,1)$. The homogeneity of H implies $H(\theta,\tau) = \tau F(\theta/\tau)$ and

$$\begin{array}{lcl}
\nabla_\theta H(\theta,\tau) & = & \nabla F(\theta/\tau), \\
\partial_\tau H(\theta,\tau) & = & -\langle \nabla F(\theta/\tau), \theta/\tau\rangle + F(\theta/\tau), \\
D^2_{\theta\theta} H(\theta,\tau) & = & D^2F(\theta/\tau)/\tau, \\
\partial_\tau \nabla_\theta H(\theta,\tau) & = & -\langle D^2F(\theta/\tau), \theta/\tau\rangle/\tau, \\
\partial^2_\tau H(\theta,\tau) & = & (\theta/\tau)^T D^2F(\theta/\tau)(\theta/\tau)/\tau.
\end{array}$$

Writing $\zeta = \theta/\tau$, we obtain

$$D^2H(\theta,\tau) = \frac{1}{\tau}\begin{pmatrix} D^2F(\zeta) & -D^2F(\zeta)\zeta \\ -\zeta^T D^2F(\zeta) & \zeta^T D^2F(\zeta)\zeta \end{pmatrix}.$$

Therefore, rank $D^2H(\theta,\tau) =$ rank $D^2F(\theta/\tau)$ and conditions (4.2.5) and (4.2.6) are equivalent to

$$\text{rank } D^2F(\zeta) \le k.$$

The mapping F is holomorphic in some open subset Ω of $\mathbb{C}^{n-1}$. The kernels of D^2F are the complements to the conic direction in the kernels of D^2H. Theorem 4.1.1 implies

Theorem 4.2.3. *Let X be an open set in $\mathbb{R}^n$ and let Λ be a real analytic conic Lagrangian submanifold of $T^*X\backslash 0$. Let H be its generating function, i.e. let Λ be locally of the form (4.2.1), (4.2.2). Let $k \le n-2$ be such that*

$$\text{rank } D^2H \le k.$$

In a neighborhood of $\xi_n = 1$ define F by

$$F(\zeta) = H(\zeta,1).$$

Then F is real analytic and its holomorphic extension we denote by $F : \Omega \to \mathbb{C}$ as well, where Ω is an open set in $\mathbb{C}^{n-1}$. Then the mapping

$$\nabla F : \Omega \subset \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$$

satisfies conditions (ΓA1), (ΓA2).

Remark 4.2.4. Note that condition $k \leq n-2$ in Theorem 4.2.3 does not restrict the generality because for $k = n-1$ the level sets of ∇F are zero dimensional and the smooth factorization condition holds trivially.

Having established a connection with Chapter 2 we can use its results. We will apply Theorem 4.2.3 to the real valued phase function $\Psi = \operatorname{Re}\Phi + \tau \operatorname{Im}\Phi$, where Φ is a complex valued phase function for an operator $T \in I^{\mu}_{\rho}(X, Y; C)$. Let T be a translation invariant Fourier integral operator satisfying local graph type condition (L) of Section 1.7. Then locally it is of the form (4.2.3) with a complex valued phase function

$$\Phi(x, y, \xi) = \langle x - y, \xi\rangle - K(\xi),$$

where $K(\xi)$ is a positively homogeneous of degree one function with $\operatorname{Im} K(\xi) \leq 0$. Then

$$\Psi(x, y, \xi) = \langle x - y, \xi\rangle - H(\xi),$$

with $H(\xi) = \operatorname{Re} K(\xi) + \tau \operatorname{Im} K(\xi)$. As usual, we denote

$$\begin{aligned} C_\Psi &= \{(x, y, d_x\Psi(x, y, \xi), d_y\Psi(x, y, \xi)) : d_\xi\Psi(x, y, \xi) = 0\} \\ &= \{(y + \nabla_\xi H(\xi), y, \xi, -\xi)\}. \end{aligned} \tag{4.2.7}$$

If Φ is real analytic, then both Ψ and H are real analytic in ξ. The estimate $\operatorname{rank} D^2H(\xi) \leq k$ applies now to the real phase function from the smooth factorization type condition (F) and we have $\operatorname{rank} d\pi_{X\times Y}|_{C_\Psi} = n + \max_\xi \operatorname{rank} D^2H(\xi)$. As usual, we assume that the canonical relation C satisfies conditions of Theorem 1.7.1. Therefore, we obtain

Theorem 4.2.5. *Let $T \in I^{\mu}_{\rho}(X, Y; C)$ be an analytic Fourier integral operator with complex phase commuting with translations. Let Φ be its phase function and assume that it satisfies the local graph type condition* (L). *Assume that there exists a real τ with $|\tau| < 1/\sqrt{3}$, such that for $\Psi = \operatorname{Re}\Phi + \tau \operatorname{Im}\Phi$ holds*

$$\operatorname{rank} d\pi_{X\times Y}|_{C_\Psi} \leq n + k,$$

with some $0 \leq k \leq 2$. Let $\mu \leq -(k + (n-k)(1-\rho))|1/p - 1/2|, 1 < p < \infty$, $1/2 \leq \rho \leq 1$. Then T is continuous from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$.

The statement follows from Theorems 4.2.3, 2.4.3 and Theorem 1.12.1.

Theorem 4.2.6. *Let X, Y be open sets in $\mathbb{R}^n$ with $n \leq 4$ and let $T \in I^{\mu}_{\rho}(X, Y; C)$ be an analytic Fourier integral operator with complex phase commuting with translations. Let Φ be its phase function and assume that it satisfies the local* *graph type condition* (L). *Assume that there exists a real* τ *with* $|\tau| < 1/\sqrt{3}$, *such that for* $\Psi = \operatorname{Re}\Phi + \tau \operatorname{Im}\Phi$ *holds*

$$\operatorname{rank} d\pi_{X\times Y}|_{C_\Psi} \leq n + k$$

for some k, $0 \leq k \leq 3$. *Let* $\mu \leq -(k + (n-k)(1-\rho))|1/p - 1/2|, 1 < p < \infty$ *and* $1/2 \leq \rho \leq 1$. *Then* T *is continuous from* $L^p_{comp}(Y)$ *to* $L^p_{loc}(X)$.

For $k = n-1$ and $k = 0$ the statement follows from Theorem 1.12.1. For $k = 1$ it follows from Theorems 4.2.3 and 4.2.5. The last case $n = 4, k = 2$ follows from Theorems 4.2.3, 2.4.3 and 1.12.1.

Remark 4.2.7. The main idea behind the last two theorems is that if X, Y are real analytic and if an analytic conic Lagrangian manifold C is associated to an operator commuting with translations, and $\operatorname{rank} d\pi_{X\times Y}|_{C_\Psi} \leq 2$, then the smooth factorization type condition (F) is satisfied.

Remark 4.2.8. Let T satisfy conditions of Theorems 4.2.5 and 4.2.6. Then it follows from Theorem 1.12.2 that T is continuous from H^1 to L^1 provided $\mu \leq -(k + (n-k)(1-\rho))/2$.

In the case when the phase function Φ is real, Theorems 4.2.5 and 4.2.6 simplify. Let us formulate them.

Corollary 4.2.9. *Let* $T \in I^\mu_\rho(X, Y; C)$ *be an analytic Fourier integral operator with real phase commuting with translations. Assume that* C *is a local graph and that*

$$\operatorname{rank} d\pi_{X\times Y}|_C \leq n + k,$$

with some $0 \leq k \leq 2$. *Let* $\mu \leq -(k + (n-k)(1-\rho))|1/p - 1/2|, 1 < p < \infty$, $1/2 \leq \rho \leq 1$. *Then* T *is continuous from* $L^p_{comp}(Y)$ *to* $L^p_{loc}(X)$.

Corollary 4.2.10. *Let* X, Y *be open sets in* $\mathbb{R}^n$ *with* $n \leq 4$ *and let* $T \in I^\mu_\rho(X, Y; C)$ *be an analytic Fourier integral operator with real phase commuting with translations. Assume that* C *is a local graph and that*

$$\operatorname{rank} d\pi_{X\times Y}|_C \leq n + k$$

for some k, $0 \leq k \leq 3$. *Let* $\mu \leq -(k + (n-k)(1-\rho))|1/p - 1/2|, 1 < p < \infty$, *and* $1/2 \leq \rho \leq 1$. *Then* T *is continuous from* $L^p_{comp}(Y)$ *to* $L^p_{loc}(X)$.

Note that these statements are sharp for the order μ in view of the Section 1.11. Finally, note that under conditions of Theorems 4.2.5 and 4.2.6 we also get L^p-L^q continuity as in Proposition 1.12.3, and that this statement is sharp for the order μ as well.

Chapter 5

Applications

In this chapter we will discuss several applications of the analysis of the preceding chapters. In Section 5.1 we give a short overview of several applications of the regularity theory of Fourier integral operators. Further, we concentrate on Cauchy problems for partial differential equations. In Section 5.2 we discuss the regularity properties of solutions to hyperbolic Cauchy problems for operators falling into the class of analytic operators of the previous chapter. In particular, this contains the class of operators with coefficients dependent on time, but not on other variables. Of course, general estimates follow from the corresponding estimates for Fourier integral operators if the solution operators are written in this form. However, the analysis of ranks of the projection $\pi_{X\times Y}$ improve the estimates. In particular, we will give several examples of hyperbolic partial differential operators with ranks of $d\pi_{X\times Y}$ restricted to the wave front being any number between n and $2n-1$.

In Section 5.4 we allow phase functions in Fourier integral operators become complex. This allows one to deal with partial pseudo-differential operators with complex characteristics. The assumption for the theory of Fourier integral operators with complex phases to apply is that the characteristics are simple and have non-negative imaginary parts. This implies that the imaginary parts of complex phase functions of solution operators are nonnegative. From this perspective we can apply estimates of Section 1.7 to derive L^p estimates for the solution of the Cauchy problem for such operators.

Finally, in Section 5.5 we briefly discuss an application of estimates of Section 1.7 to the oblique derivative problem.

5.1 Introduction

One of the main applications of the theory of Fourier integral operators is the theory of hyperbolic partial differential equations. Let

$$P(t,x,\partial_t,\partial_x) = \partial_t^m + \sum_{j=1}^{m} P_j(t,x,\partial_x)\partial_t^{m-j}$$

be a strictly hyperbolic operator of order m in a set of points $(t,x) \in \mathbb{R}^{1+n}$. The hyperbolicity means that the principal symbol $p(t,x,\tau,\xi)$ of the operator P (p is polynomial in τ or degree m and is often denoted by σ_P) has m distinct real roots τ_j in τ. In this case, the Cauchy problem

$$\begin{cases} Pu(t,x) = 0, & t \neq 0, \\ \partial_t^j u|_{t=0} = f_j(x), & 0 \leq j \leq m-1. \end{cases} \tag{5.1.1}$$

is well posed, and for small t its solution can be written as a sum of Fourier integral operators T_t^{jl}:

$$u(t,\cdot) = \sum_{j=1}^{m} \sum_{l=0}^{m-1} T_t^{jl} f_l,$$

where each operator T_t^{jl} depends smoothly on t, is of order $-l$ and depends on the root $\tau_j(t,x,\xi)$. This construction will be described in more detail later. Note that operators T_t^{jl} satisfy the local graph condition and it follows that for small t and $1 < p < \infty$ for the solution u of the Cauchy problem holds $u(t,\cdot) \in L^p_{loc}$, provided that the Cauchy data f_l are in Sobolev spaces $f_l \in (L^p_{\alpha_p - l})_{comp}$, where $\alpha_p = (n-1)|1/p - 1/2|$. Thus, there is a loss of smoothness by α_p derivatives in L^p spaces. It follows from the stationary phase method (Section 1.11) that the loss of α_p derivatives is sharp in a number of cases. For example, it is sharp for equations of the wave type with variable coefficients (when $m = 2$). It is also sharp if one imposes an additional condition that for almost all t at least one of the roots $\tau_j(t,x,\xi)$ is elliptic in ξ, i.e. $\tau_j(t,x,\xi) \neq 0$ for $\xi \neq 0$ ([62]). However, in a number of cases the order α_p can be improved. Let k denote the minimal integer for which the ranks of standard projections from the canonical relations of operators T_t^{jl} to $\mathbb{R}^n \times \mathbb{R}^n$ do not exceed $n+k$ for all j and l. These projections are discussed in more detail in Section 1.4. In the present case, projection $\pi_{X\times Y}$ in (1.5.1) satisfies the inequality $\operatorname{rank} d\pi_{X\times Y}|_{C_t^{jl}} \leq n+k$, where X and Y are (open subsets of) $\mathbb{R}^n$ and C_t^{jl} are the canonical relations of operators T_t^{jl}. It terms of phase functions it means that if Φ_j is the solution of the eikonal equation $(\partial/\partial t)\Phi_j(t,x,\xi) = \tau_j(t,x,\nabla_x\Phi_j)$ with initial condition $\Phi_j(0,x,\xi) = \langle x,\xi\rangle$, then $\operatorname{rank} \partial^2_{\xi\xi}\Phi_j(t,x,\xi) \leq k$ for all j,x,ξ. In this case one can show (Section 1.11) that the order $\alpha_p = k|1/p - 1/2|$ would be sharp, i.e. the condition that for some α holds $u(t,\cdot) \in L^p_{loc}$ for all $f_l \in (L^p_{\alpha-l})_{comp}$ implies $\alpha \geq \alpha_p = k|1/p - 1/2|$. The converse is known in a number of special cases only. For operators with constant coefficients (in particular) the converse result is proved in Chapter 5 in the case $k \leq 2$. The complete picture of the L^p properties of such operators becomes clear in the 5-dimensional space. For $3 \leq k \leq n-2$ the sharp L^p–estimates are not proved in general. In the case $k = 0$ there is no loss of smoothness for the solutions (in this case the regularity properties of solutions are the same as the regularity properties of solutions to the elliptic Cauchy problems, $\alpha_p = 0$ and, therefore, $k = 0$), the principal symbol of P has a special form. Its roots τ_j are linear in ξ (cf. Theorem 5.2.4).

In Section 5.4 we will establish L^p estimates for operators with complex characteristics.

Another important applications of Fourier integral operators are convolution operators and Radon transforms. Let X and Y be smooth manifolds and let S be a smooth submanifold of $X \times Y$. For $x \in X$, let S_x be the set of points $y \in Y$ such that $(x, y) \in S$. Let σ be a smooth measure on S. It naturally induces the measure σ_x on S_x. The Radon transform corresponding to S and σ is now defined by

$$Rf(x) = \int_{S_x} f d\sigma_x.$$

It defines an integral operator $R : C_0^\infty(Y) \to \mathcal{D}'(X)$. The Schwartz integral kernel of the operator R is a Dirac measure supported in S and equal to σ there. Because we are interested in local properties, we can assume that the set S is compact. Let $\dim S = \dim X + \dim Y - d$. Locally S is given by d equations

$$h_1(x, y) = \ldots = h_d(x, y) = 0,$$

where functions h_j are smooth and their gradients are linearly independent on S. In this way, the operator R can be regarded as a Fourier integral operator with the phase function

$$\sum_{j=1}^{d} \theta_j h_j(x, y).$$

The canonical relation of R is then equal to the conormal bundle N^*S. The local graph condition means that $\dim X = \dim Y$ and that projections π_X, π_Y from N^*S to T^*X and T^*Y are locally diffeomorphic. In this case the L^p estimates of the subsequent sections can be applied. The smooth factorization condition is satisfied for N^*S because S is a smooth manifold. Therefore, the L^p estimates follow from the general theory of Fourier integral operators. The results depend on the order of the operator R as a Fourier integral operator. Let us assume for simplicity that $Rf = f * \sigma$, where the measure σ is supported on some (smooth) submanifold Σ of $\mathbb{R}^n$. The connection between R and Fourier integral operators is mentioned in more detail in Section 1.5.3. Now, we restrict ourselves to some remarks only. The order of R as a Fourier integral operator is related to the order of decrease of the Fourier transform $\widehat{\sigma}$ at infinity. This order depends on the curvature of Σ. Let Σ be a hypersurface ($\dim \Sigma = n - 1$) with non-vanishing Gaussian curvature at all points and let $d\sigma = \psi d\mu$, where $d\mu$ is induced on S by the Lebesgue measure and $\psi \in C_0^\infty(\mathbb{R}^n)$. One can show that the Fourier transform of the measure σ satisfies the estimate $|\widehat{\sigma}(\xi)| \leq C|\xi|^{-(n-1)/2}$. It follows that R is a Fourier integral operator of order $-(n-1)/2$. There are generalizations of this estimate to $(n-d)$–dimensional manifolds Σ of finite type. Roughly speaking, Σ has type m if the order of intersection of Σ with affine hyperplanes does not exceed m. In this case holds $|\widehat{\sigma}(\xi)| \leq C|\xi|^{-1/m}$. If the manifold Σ is analytic, then, in turn, this estimate implies that Σ has type m. Indeed, otherwise $\widehat{\sigma}$ would be constant in directions orthogonal to the tangent space to Σ at the points of infinite order of tangency. The detailed discussion as well as other forms of the Radon transform can be found, for example, in [68], [20], [45], [46], [48]. For the failure of the smooth factorization condition it is necessary (but not sufficient, see Remark 1.5.7) that S is not smooth. In this case there are additional problems in determining the order of R as a Fourier integral operator since the estimates of $\widehat{\sigma}$ at infinity are more complicated.

The local graph condition holds in a number of important cases. For example, Radon transforms along hypersurfaces in $\mathbb{R}^n$ satisfy this condition. It also holds for the convolution operators with measures supported on hypersurfaces in $\mathbb{R}^n$ with non-vanishing Gaussian curvature. As an example, one has solution operators to the Cauchy problem for the wave equation (see above), where for a fixed $t > 0$, the manifold S_x consists of points $y \in \mathbb{R}^n$ with $|x - y| = t$.

Let us remark, finally, that in some applications the canonical relation fails to be a local graph. For instance, the local graph condition fails for the Radon transforms with $d > n/2$ (cf., for example, [45]). For submanifolds of codimension larger than 1 in a general position, the canonical relation C is a Whitney fold ([69], [47], [46]). Projections π_X and π_Y are also singular in applications to the diffraction theory ([72], [73]), in the theory of X-ray transforms ([19]), and deformation theory ([21], [19]). Singular Radon transforms and Fourier integral operators with densities with Calderón–Zygmund type of singularities led to the microlocal analysis of the boundedness for the degenerate type of Fourier integral operators (see [22], [20], [46]) and references therein). In many degenerate cases one looses smoothness of functions even in the L^2–case, with the loss dependent on the order of degeneracy, which can be expressed in terms of the stratification of Lagrangian ([47], [48]). A similar loss occurs in L^p norms for the Radon transforms with folding canonical relation and for other types of singularities. However, such degenerate cases fall beyond the scope of the present monograph.

5.2 Cauchy problem for a class of strictly hyperbolic equations

Results of the previous chapters can be applied to establish certain L^p estimates for the fixed time solutions of hyperbolic Cauchy problem. For more detailed discussions of the Cauchy problem for hyperbolic partial differential equations we refer to [76], [28], [11], [68], [65]. The class of operators which we consider below contains operators with constant coefficients. Solutions to the Cauchy problem for such operators are well known ([43], [29]). Results of this section have partially appeared in [53] and examples of operators are discussed in [55].

Let the operator

$$P(t, \partial_t, \partial_x) = \partial_t^m + \sum_{j=1}^{m} P_j(t, \partial_x)\partial_t^{m-j}$$

be strictly hyperbolic and have order m in $\mathbb{R} \times X$, where X is a compact n-dimensional smooth analytic manifold with $n \leq 4$. Let $p(t, \tau, \xi)$ be the principal symbol of the operator P. Strict hyperbolicity means that the polynomial

$p(t,\tau,\xi)$ has m real distinct roots in τ. The roots $\tau_j(t,\xi)$ are real, homogeneous of degree one in ξ and smooth in t. The corresponding Cauchy problem is the equation

$$\begin{cases} Pu(t,x) = 0, & t \neq 0, \\ \partial_t^j u|_{t=0} = f_j(x), & 0 \leq j \leq m-1. \end{cases} \tag{5.2.1}$$

The principal symbol p (or σ_P) of the operator P can be written as a product

$$p(t,\tau,\xi) = \prod_{j=1}^{m} (\tau - \tau_j(t,\xi)).$$

For small t a solution to the Cauchy problem (5.2.1) can be obtained in the following way. Let $\Phi_j(t,x,\xi)$ solve the eikonal equation:

$$\begin{cases} \frac{\partial}{\partial t}\Phi_j(t,x,\xi) = \tau_j(t,\nabla_x\Phi_j), \\ \Phi_j(t,x,\xi)|_{t=0} = \langle x,\xi\rangle. \end{cases} \tag{5.2.2}$$

Under the strict hyperbolicity condition the Cauchy problem (5.2.1) is well posed and modulo a smooth term its solution $u(t,x)$ can be obtained as a finite sum of elliptic Fourier integral operators, smoothly dependent on t:

$$u(t,x) = \sum_{j=1}^{m}\sum_{l=0}^{m-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i(\Phi_j(t,x,\xi)-\langle y,\xi\rangle)} a_{jl}(t,x,\xi) f_l(y)\, d\xi dy. \tag{5.2.3}$$

Canonical relations are locally generated by function Φ_j from (5.2.2), and symbols $a_{jl} \in S^{-l}$ can be found from additional transport equations [15], [11]. Expression (5.2.3) is smooth in t. Let f be a smooth function on an open set U in T^*X. Then df is a smooth one-form. A smooth vector field H_f is called *the Hamiltonian vector field for f* if the equality $\sigma(v,H_f) = \langle v, df\rangle = v(f)$ holds for all smooth vector fields v. In local coordinates at $(x,\xi) \in U$ the vector field H_f takes the form

$$H_f = \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j}.$$

For a fixed value of t the canonical relation of the solution operator in (5.2.3) has the form

$$C_t = \bigcup_{j=1}^{m} \{(x,\xi,y,\eta) : \chi_{t,j}(y,\eta) = (x,\xi)\},$$

where canonical transformations $\chi_{t,j} : T^*X\backslash 0 \to T^*X\backslash 0$ are the flows from (y,η) at $t=0$, passing through (x,ξ) at t, along the Hamiltonian vector field

$$H_j = \sum_{j=1}^{n}\left(\frac{\partial \tau_j}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial \tau_j}{\partial x_j}\frac{\partial}{\partial \xi_j}\right) = \sum_{j=1}^{n}\frac{\partial \tau_j}{\partial \xi_j}\frac{\partial}{\partial x_j}$$

in $T^*X\backslash 0$. Regarding t as one of the variables, we can write the canonical relation of the operator (5.2.3) as

$$C = \bigcup_{j=1}^{m} \{(x, \xi, y, \eta, t, \tau) : \tau = \tau_j(t, \xi), \chi_{t,j}(y, \eta) = (x, \xi)\}.$$

Using the representation and the asymptotics of $D^2_{\xi\xi}\Phi_j$ with respect to t, one can show ([62]) that the maximal rank $D^2_{\xi\xi}\Phi_j = 2n - 1$ under an additional assumption that for all t, except possibly a discrete set in $\mathbb{R}$, at least one of the roots τ_j is elliptic. In particular, this assumption holds for the wave equation with variable coefficients. In general, if k is the maximal rank of the matrices $D^2_{\xi\xi}\Phi_j$, it is possible to apply results of Section 4.2 to obtain sharp L^p estimates for the solutions of the Cauchy problem (5.2.1). Now we formulate the properties of solutions with ranks $0 \le k \le n - 1$.

Theorem 5.2.1. *Let $u(t, x)$ solve the Cauchy problem (5.2.1) and let $\Phi_j(t, x, \xi)$ solve the eikonal equation (5.2.2). Let*

$$k = \max_{x,\xi,j} \operatorname{rank} \frac{\partial^2}{\partial \xi^2} \Phi_j(t, x, \xi)$$

and let $\alpha_p = k|1/p - 1/2|$, $1 < p < \infty$. Let the Cauchy data satisfy $f_j \in L^p_{\alpha+\alpha_p-j}(X)$. Then $u(t, \cdot) \in L^p_\alpha(X)$. Moreover, if

$$k = \max_{x,\xi,j,t\in[-T,T]} \operatorname{rank} \frac{\partial^2}{\partial \xi^2} \Phi_j(t, x, \xi)$$

for a fixed $0 < T < \infty$, then

$$||u(t, \cdot)||_{L^p_\alpha} \le C_T \sum_{j=0}^{m-1} ||f_j||_{L^p_{\alpha+\alpha_p-j}}, t \in [-T, T]. \tag{5.2.4}$$

The orders $\alpha + \alpha_p - j$ can not be improved.

Proof. Let us look for the solutions to the eikonal equation (5.2.2) in the form

$$\Phi_j(t, x, \xi) = \langle x, \xi \rangle - H_j(t, \xi).$$

Then equations (5.2.2) reduce to a Cauchy problem for ordinary differential equations

$$\frac{\partial H_j}{\partial t}(t, \xi) = -\tau_j(t, \xi), \tag{5.2.5}$$

with $H_j(0, \xi) = 0$, where ξ is a parameter. Because p is strictly hyperbolic and analytic in ξ, τ_j and Φ_j must be analytic in ξ and smooth in t. The roots τ_j are homogeneous of degree one in ξ and hence functions $H_j(t, \xi)$ are analytic and homogeneous of degree one in ξ for small t. Estimates (5.2.4) now follow from Theorem 4.2.6. The sharpness follows from Theorem 1.11.1.

Remark 5.2.2. The statement of Theorem 5.2.1 can be generalized to other dimensions n of X. Under an additional assumption that $k \leq 2$, solution $u(t, \cdot)$ belongs to L^p_α and sharp estimates (5.2.4) are valid. The proof is similar to the proof of Theorem 5.2.1 and is based on Theorem 4.2.5. If X is not compact, the statement of Theorem 5.2.1 holds in L^p_{comp}.

In other function spaces we have the following statement.

Theorem 5.2.3. *In conditions of Theorem 5.2.1 the following estimates hold.*

(1) *Let* $1 < p \leq q \leq 2$. *Then*

$$||u(t, \cdot)||_{L^q_\alpha} \leq C_T \sum_{j=0}^{m-1} ||f_j||_{L^p_{\alpha+\alpha_{pq}-j}}, t \in [-T, T], \tag{5.2.6}$$

where $\alpha_{pq} = |-n/p + (n-k)/q + k/2|$. *The dual estimate holds for* $2 \leq p \leq q < \infty$.

(2) *In Lipschitz spaces* $Lip(\gamma)$ *holds*

$$||u(t, \cdot)||_{\text{ Lip } (\alpha)} \leq C_T \sum_{j=0}^{m-1} ||f_j||_{\text{ Lip } (\alpha+k/2-j)}, t \in [-T, T]. \tag{5.2.7}$$

The orders in the above statements can not be improved.

The estimates follow from corresponding theorems for Fourier integral operators from Section 4.2. The sharpness of the estimates will be considered in the following section.

The best L^p regularity properties are exhibited by operators, for which the order α_p in Theorem 5.2.1 equals zero. In this case the regularity properties of solutions are the same as for elliptic Cauchy problems, for which the solutions are given by pseudo-differential operators. This has an underlying explanation. The condition $\alpha_p = 0$ implies $k = 0$ and in the next section we will show that solution operators can be obtained from pseudo-differential operators by a composition with Fourier integral operators induced by smooth coordinate changes.

Theorem 5.2.4. *Let* $n \in \mathbb{N}$, *the order* $m \geq 2$, $1 < p < \infty$ *and* $p \neq 2$. *For the strictly hyperbolic Cauchy problem (5.2.1) the following conditions are equivalent:*

(1) *There is no loss of smoothness in* L^p *for solutions, i.e. for any Cauchy data* $f_j \in L^p_{m-j}$ *the solution* $u(t, \cdot)$ *of the Cauchy problem (5.2.1) belongs to* L^p_m.

(2) *The dimension* $n = 1$ *and the principal symbol of the operator* P *has the form*

$$\sigma_P(t, \tau, \xi) = \prod_{j=1}^{m} (\tau - \tau_j(t, \xi)) \tag{5.2.8}$$

where $\tau_j(t, \xi)$ *are linear in* $\xi \in \mathbb{R}^n$.

In both cases there exist pseudo-differential operators $Q_{jl}, S_{jl} \in \Psi^{-l}(Y)$ and diffeomorphisms $\sigma_j : X \to Y$ such that the solution to the Cauchy problem (5.2.1) has the form

$$u = \sum_{l=0}^{m-1} \sum_{j=1}^{m} (\sigma_j^* \circ Q_{jl}) f_l = \sum_{l=0}^{m-1} \sum_{j=1}^{m} (S_{jl} \circ \sigma_j^*) f_l, \tag{5.2.9}$$

where σ_j^* are the pullbacks induced by σ_j. Sobolev space estimates (5.2.4) hold with $\alpha_p = 0$.

For $p = 2$ condition (2) holds for arbitrary operators P. This is why it is necessary to exclude the case $p = 2$ from the formulation of the theorem.

Proof of Theorem 5.2.4. Assume that the homogeneous part of the highest degree of operator P is as in (5.2.8). We can differentiate equation (5.2.5) twice with respect to ξ to obtain $\partial^2_{\xi\xi}\partial_t H_j(t,\xi) = 0$. Therefore, $\partial^2_{\xi\xi}H_j(t,\xi) = \partial^2_{\xi\xi}H_j(0,\xi) = 0$, and H_j are linear in ξ. The rank k in Theorem 5.2.1 is then equal to zero, which implies the second part of the theorem, since pseudo--differential operators of order $-j$ are continuous from $(L^p_\alpha)_{comp}$ to $(L^p_{\alpha+j})_{loc}$.

Conversely, let P be as in the second part of the theorem. If T_l in (5.2.3) is an operator of order $-l$, we can apply Theorem 1.11.5 to the operator $T = T_l \circ (I - \Delta)^{l/2}$ to obtain formula (5.2.9). Moreover, we have $k = 0$, and hence $\partial^2_{\xi\xi}\tau_j(t,\xi) = -\partial_t \partial^2_{\xi\xi}H_j(t,\xi) = 0$. Therefore, τ_j must be polynomial of degree ≤ 1 in ξ. In fact, they are linear, since they are also homogeneous of order one. The condition that P is strictly hyperbolic implies that real values of linear functions $\tau_j(\xi)$ are different for $\xi \neq 0$, which is only possible when $n = 1$ when the order m of the operator is $m \geq 2$.

The Sobolev space estimates follow from the continuity of pseudo-differential operators of order $-j$ from $(L^p_\alpha)_{comp}$ to $(L^p_{\alpha+j})_{loc}$.

5.3 Monge–Ampère equation

Generating functions ϕ satisfy the following parametric Monge–Ampère equation:

$$\det \frac{\partial^2 \phi}{\partial \xi^2}(y,\xi) = 0 \tag{5.3.1}$$

for all $(y,\xi) \in Y \times \Xi$. Let us consider an invariant version of equation (5.3.1):

$$\det \frac{\partial^2 \phi}{\partial \xi^2}(\xi) = 0, \ \forall \xi \in \Omega, \tag{5.3.2}$$

which corresponds to operators commuting with translations. Such equations are called the *simplest Monge–Ampère equations*, cf. [31, 8.2]. Note that if ϕ is convex, equation (5.3.2) has the unique solution, which is zero. In general, the problem

$$\det \frac{\partial^2 \phi}{\partial \xi^2}(\xi) = f \tag{5.3.3}$$

almost everywhere in the ball $B = \{\xi \in \mathbb{R}^n : |\xi| < 1\}$, $n \geq 2$, with zero boundary conditions on ∂B and non-negative function $f \in C^2(B)$, has the unique solution in B in the class of convex functions. Solution ϕ belongs to $C^{3+\alpha}(\bar{B})$ for all $\alpha \in (0,1)$, provided $f > 0$ on $\bar{B}$. (cf. [31, 8.2.2]). In our case we have $f = 0$. Let ϕ solve equation (5.3.2). We do not require the convexity now. Using Theorems 2.4.2 and 2.4.3 we conclude

Corollary 5.3.1. *Let Ω be an open set in $\mathbb{R}^n$ (or $\mathbb{C}^n$) and let $\phi(\xi)$ be an analytic solution to the Monge–Ampère equation (5.3.2) in Ω, such that the level sets of $\nabla\phi(\xi)$ are affine for all ξ with the maximal* rank $\frac{\partial^2\phi}{\partial\xi^2}(\xi) \leq 2$. *Then the fibration by the level sets of $\nabla\phi(\xi)$ allows an analytic extension to Ω.*

5.4 Cauchy problems with complex characteristic roots

In this section we will deal with operators with complex characteristic roots. We start with first order operators. Let $A = A(x, D_x)$ be a first order pseudo-differential operator in the space of half densities on X, $D_x = -i\partial_x$. Let also $D_t = -i\partial_t$. First we consider the following Cauchy problem:

$$\begin{cases} D_t v - A(x, D_x)v = 0, \\ v(0,x) = f(x). \end{cases} \tag{5.4.1}$$

If the principal symbol $a(x,\xi)$ of A is real, Cauchy problem (5.4.1) is strictly hyperbolic and well posed. As in Section 5.2, in this case, for $1 < p < \infty$, we have the following L^p properties for the fixed time solutions of (5.4.1). Let $f \in L^p_{(n-1)|1/p-1/2|}(X)$. Then $v(t,\cdot) \in L^p_{loc}$ for all sufficiently small values of t and the loss $(n-1)|1/p - 1/2|$ is sharp when A is elliptic.

Now we consider a more general case. We assume that the principal symbol a is complex valued, such that

$$\operatorname{Im} a(x,\xi) \geq 0, \quad \forall (x,\xi) \in T^*X\backslash 0. \tag{5.4.2}$$

We also assume that A is a first order classical pseudo-differential operator. Since we are interested in local estimates, we may as well assume that X is compact. A question now is whether there is an operator solution $U(t)$ to (5.4.2), that is a continuous linear operator on $\mathcal{D}'(X;\Omega^{1/2})$ such that in X we have

$$\begin{cases} D_t U - AU = 0, \quad t \geq 0, \\ U(0) = I. \end{cases} \tag{5.4.3}$$

Such U exists for $t \geq 0$ and it is unique and smooth in t. The semigroup $t \mapsto U(t)$ is strongly continuous and $U(t)$ is a bounded linear operator on $L^2(X, \Omega^{1/2})$ (see, for example [74, XI, Theorem 2.1]). Let us briefly describe solutions of (5.4.1) and (5.4.3). Assume for the simplicity now that $U(t)$ acts on functions and not on half-densities. We define self-adjoint operators

$$A^+ = \frac{A+A^*}{2}, \quad A^- = \frac{A-A^*}{2i}.$$

The solution F of

$$\begin{cases} D_t F - A^+ F = 0, \\ F(0) = I \end{cases} \tag{5.4.4}$$

is a unitary Fourier integral operator with real phase. Setting $W = e^{-itA^+}U$ we get that W is the solution to

$$\begin{cases} \partial_t W + BW = 0, \\ W(0) = I, \end{cases} \tag{5.4.5}$$

where $B = e^{-itA^+}A^- e^{itA^+}$. By Egorov's theorem the operator B is in Ψ^1, is classical and depends smoothly on t. The principal symbol $b(t,x,\xi)$ of B is given by

$$b(t,x,\xi) = a^-(e^{-tH_{a^+}}(x,\xi)),$$

where $e^{-tH_{a^+}}$ is the Hamiltonian flow of $-a^+$. Assumption (5.4.2) implies that $b(t,x,\xi) \geq 0$ for all t and for all $(x,\xi) \in T^*X\backslash 0$. Now, according to [74, XI], the operator $W(t)$ has the kernel

$$W(t,x,y) = \int e^{i(\Phi(t,x,\xi)-\langle y,\xi\rangle)} k(t,x,\xi)d\xi, \tag{5.4.6}$$

where the amplitude k is a classical symbol of order zero, and the phase function Φ is an "approximate" solution of the eikonal equation

$$\partial_t \Phi = ib(t,x,\nabla_x\Phi), \quad \Phi(0,x,\xi) = \langle x,\xi\rangle. \tag{5.4.7}$$

Note, that one of the main difficulties here is that the solution Φ of (5.4.7) is, in general, complex valued, and it enters the equation as a variable in b as well, which is originally defined for real x and ξ only. One overcomes this by considering almost analytic extension $\widetilde{b}$ of b, this is why Φ solves (5.4.7) only approximately. However, one can control the error $\partial_t\Phi - i\widetilde{b}(t,x,\nabla_x\Phi)$ by arbitrary powers of $\operatorname{Im}\Phi$. More precisely, let $\widetilde{a}^+$ be an almost analytic extension of a^+ and let $\widetilde{b} = \widetilde{a}^- \circ e^{-tH_{\widetilde{a}^+}}$, where $e^{-tH_{\widetilde{a}^+}}$ is the Hamiltonian flow of the extension $-\widetilde{a}^+$. Then for small $t \geq 0$, microlocally, the Hamiltonian flow of $-i\widetilde{b}$ defines a positive conic almost Lagrangian manifold. The equivalence class of these manifolds microlocally yields a positive canonical relation $C'_t \subset (T^*\widetilde{X}\backslash 0) \times (T^*\widetilde{X}\backslash 0)$. The operator $W(t)$ is attached to C'_t. The operator $U(t)$ is then attached to the positive canonical relation $C_t = C_t^+ \circ C'_t$, where C_t^+ is the canonical relation of the Fourier integral operator (with real phase) e^{itA^+}. One has the usual property that e^{itA} propagates singularities only along bicharacteristics of $-a$ originating at $t=0$. For the details of these constructions we refer to [74]. Note, that (5.4.7) implies that our assumption (1.7.2) holds with $\tau = i$ and we can apply Theorem 1.7.1. It also follows that $\operatorname{Im}\Phi \geq 0$ because $b \geq 0$. Thus, we obtain

Theorem 5.4.1. *Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, and let $f \in L^p_{\alpha+(n-1)|1/p-1/2|}$ be compactly supported. Then for a fixed t the solution v of the Cauchy problem (5.4.1) satisfies $v(t,\cdot) \in (L^p_\alpha)_{loc}$.*

According to Corollary 1.10.1 we get corresponding estimates in other function spaces.

Let us now briefly explain how we can choose a complex valued phase function for $U(t)$ globally. The possibility of a global phase function is shown in [33]. Let us now concentrate on the case when a is imaginary and elliptic. Let $\sigma_A(x,\xi) = -ia(x,\xi)$, so that σ_A is real and positive. Let $u(t,x,y)$ be the fundamental solution for (5.4.1), namely let it solve the equation

$$\begin{cases} D_t u - A(x, D_x)u = 0, \\ u(0,x,y) = \delta(x-y). \end{cases} \tag{5.4.8}$$

Let $x^t(y,\eta), \xi^t(y,\eta)$ be the Hamiltonian trajectory in $T^*X\backslash 0$ generated by σ_A with initial data (y,η). Let

$$\Lambda = \{(t,\tau),(x,\xi),(y,-\eta) : \tau = -\sigma_A(y,\eta), x = x^t(y,\eta), \xi = \xi^t(y,\eta)\}$$

be a subset of $T^*\mathbb{R}\times(T^*X\backslash 0)\times(T^*X\backslash 0)$. Let $\phi(t;x;y,\eta) \in C^\infty(\mathbb{R}\times X\times T^*X\backslash 0)$ be complex valued with $\operatorname{Im}\phi \geq 0$ and homogeneous of degree 1 in η. Let F be the class of such functions satisfying conditions

(i) $\phi(t;x^t(y,\eta);y,\eta) = 0$;

(ii) $\phi'_x(t;x^t(y,\eta);y,\eta) = \xi^t(y,\eta)$;

(iii) $\det\phi''_{x\eta}(t;x^t(y,\eta);y,\eta) \neq 0$.

By Lemma 3.1 in [33] any function $\phi \in F$ gives a global parameterization of Λ. Then by Theorem 4.1 in [33] the solution u of (5.4.8) is given by a single integral

$$u(t,x,y) = \int e^{i\phi(t;x;y,\eta)} q(t;y,\eta) d_\phi(t;x;y,\eta) d\eta, \tag{5.4.9}$$

with any $\phi \in F$ and with amplitude q of order zero. Here $d_\phi \in C^\infty(\mathbb{R}\times X\times T^*X\backslash 0)$ is a 1/2-density in x, $-1/2$-density in y, homogeneous of degree 0 in η. For x close to x^t, it is defined by

$$d_\phi(t;x;y,\eta) = \exp\left(\frac{i}{4}\arg\ \det{}^2\phi''_{x\eta}(t;x;y,\eta)\right) |\det\phi''_{x\eta}(t;x;y,\eta)|^{1/2}.$$

It follows that $v(t,x) = \langle u(t,x,\cdot), f\rangle$ is a solution to (5.4.1) and it is given by a zero order Fourier integral operator of positive type with kernel (5.4.9).

In analogy to the real case, we also get L^p estimates for higher order equations. Let P be a differential–pseudo-differential operator on $[0,T]\times X$ of order m of the form

$$P = D_t^m + \sum_{j=1}^m P_j(t) D_t^{m-j}, \tag{5.4.10}$$

where X is a smooth manifold of dimension n, and for every $1 \leq j \leq m$, $P_j(t)$ is a classical pseudo-differential operator of order j on X, depending smoothly on t. The principal symbol $p_j(t, x, \xi)$ of $P_j(t)$ is a smooth function in $[0, T] \times (T^*X\backslash 0)$, complex valued, positive homogeneous of degree j in ξ. The principal symbol $p(t, \tau, x, \xi)$ of P is given by

$$P = \tau^m + \sum_{j=1}^{m} p_j(t, x, \xi)\tau^{m-j}. \tag{5.4.11}$$

Following [74, X], we make the following assumptions on P. First, we assume that P has simple characteristics. This means that for any $(x_0, \xi^0) \in T^*X\backslash 0$ and any $t_0 \in [0, T]$ the roots of the polynomial $p(t_0, \tau, x_0, \xi^0)$ in τ are distinct. Let τ_j denote these roots. By the implicit function theorem it follows that functions τ_j in the set $[0, T] \times (T^*X\backslash 0)$ are complex valued, smooth, and positive homogeneous of degree one in ξ. The principal symbol p of P can be decomposed in the product

$$p(t, \tau, x, \xi) = \prod_{j=1}^{m} (\tau - \tau_j(t, x, \xi)).$$

Our second assumption is that $\operatorname{Im} \tau_j \geq 0$ in $[0, T] \times (T^*X\backslash 0)$. We consider the following Cauchy problem for P:

$$\begin{cases} Pv = 0, & t \in [0, T], \\ \partial_t^l v(0, x) = f_l(x), & 0 \leq l \leq m - 1. \end{cases} \tag{5.4.12}$$

The assumption that the characteristics of P are simple implies that P can be factored into

$$P = L_m \cdots L_1 + R,$$

where R is a regularising operator in X, dependent smoothly on t, and $L_j = D_t - (\tau_j)_{op}(t)$, where each $(\tau_j)_{op}(t)$ is a classical pseudo-differential operator of order one with principal symbol τ_j, dependent smoothly on t. It follows from [74] that the Cauchy problem (5.4.12) is well posed and its solution is given by a composition of solution operators for problem (5.4.1) with $A = (\tau_j)_{op}$, which is a Cauchy problem for operator L_j. According to the above discussion and the composition formula for Fourier integral operators with complex phase functions, we obtain

Theorem 5.4.2. *Let P be a classical pseudo-differential operator of order m of the form (5.4.10). Assume that P has simple characteristics τ_j, which satisfy*

$$\operatorname{Im} \tau_j(t, x, \xi) \geq 0$$

in $[0, T] \times (T^*X\backslash 0)$. Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, and let $\alpha_p = (n-1)|1/p - 1/2|$. Let functions $f_l \in L^p_{\alpha+\alpha_p-l}$ be compactly supported for all l, $0 \leq l \leq m-1$. Then for a fixed t the solution v of the Cauchy problem (5.4.12) satisfies $v(t, \cdot) \in (L^p_\alpha)_{loc}$.

The orders α_p are in general sharp, because in the case of strictly hyperbolic equations with $\operatorname{Im}\tau_j = 0$, they can be shown to be optimal by application of the stationary phase method (see Section 1.11), under the condition that the projection from C_t (which is the canonical relation of the solution operator $U(t)$ of (5.4.3)) to the base space equals $2n-1$ for at least one of the problems (5.4.3) with $A = (\tau_j)_{op}$.

According to Corollary 1.10.1 and Theorem 1.10.2 we get corresponding estimates in other function spaces.

Theorem 5.4.3. *Let P be as in Theorem 5.4.2. Let $1 < p \le q \le 2$ or $2 \le p \le q < \infty$. Let $\alpha_{pq} = |1/q - n/p + (n-1)/2|$. Then for compactly supported Cauchy data $f_l \in L^p_{\alpha+\alpha_{pq}-l}$, $0 \le l \le m-1$, the fixed time solution $v(t,\cdot)$ of the Cauchy problem (5.4.12) satisfies $v(t,\cdot) \in (L^q_\alpha)_{loc}$.*

According to Theorem 1.12.1 and Proposition 1.12.3 with the smooth factorization type condition (F), we get

Corollary 5.4.4. *Let P be a classical pseudo-differential operator of order m of the form (5.4.10). Assume that P has simple characteristics τ_j, which satisfy*

$$\operatorname{Im}\tau_j(t,x,\xi) \ge 0$$

in $[0,T]\times(T^*X\backslash 0)$. Assume that the smooth factorization type condition (F) is satisfied for the solution operators for the Cauchy problem (5.4.12), with some $k \le n-1$. Let $1 < p \le q \le 2$ or $2 \le p \le q < \infty$. Let $\alpha_{pq} = |(n-k)/q - n/p + k/2|$. Then for compactly supported Cauchy data $f_l \in L^p_{\alpha+\alpha_{pq}-l}$, $0 \le l \le m-1$, the fixed time solution $v(t,\cdot)$ of the Cauchy problem (5.4.12) satisfies $v(t,\cdot) \in (L^q_\alpha)_{loc}$.

If we now use Theorem 4.2.6 and the fact that for $k \le 2$ the condition (F) is always satisfied, we automatically get a result in $\mathbb{R}^{1+n}$, for $n \le 4$.

Corollary 5.4.5. *Let P be a classical pseudo-differential operator with analytic (in ξ) symbol of order m of the form (5.4.10) in $\mathbb{R}^{1+n}$, $n \le 4$. Assume that the coefficients P_j of P may depend on t, but not on other variables. Assume that P has simple characteristics τ_j, which satisfy*

$$\operatorname{Im}\tau_j(t,x,\xi) \ge 0$$

in $[0,T]\times(T^*X\backslash 0)$. Assume that condition for the ranks of the projections for the solution operators for the Cauchy problem (5.4.12) satisfy the rank condition of Theorem 4.2.6 (then automatically $0 \le k \le 3$). Let $1 < p \le q \le 2$ or $2 \le p \le q < \infty$. Let $\alpha_{pq} = |(n-k)/q - n/p + k/2|$. Then for compactly supported Cauchy data $f_l \in L^p_{\alpha+\alpha_{pq}-l}$, $0 \le l \le m-1$, the fixed time solution $v(t,\cdot)$ of the Cauchy problem (5.4.12) satisfies $v(t,\cdot) \in (L^q_\alpha)_{loc}$.

5.5 A parametrix for a first order complex partial differential operator

In this section we consider the parametrix construction for an operator which appears naturally in the study of the oblique derivative problem (see [38]). Following their notation, let X be a compact smooth n dimensional manifold and let Γ be a smooth closed hypersurface. We assume that $X\backslash\Gamma$ is the union of two open disjoint sets X_+ and X_-, having Γ as their common boundary. Let S_c^m be the set of classical symbols of order m, which means that it is a subset of $S_{1,0}^m$ containing symbols a for which locally in every open conic V, we have a convergent asymptotic series

$$a(x,\theta) \sim \sum_{j=0}^{\infty} a_{m-j}(x,\theta),$$

where each a_{m-j} is positively homogeneous of degree $m-j$ in θ.

Let $P \in \Psi_c^1(X)$ be of the form

$$P(x,D_x) = m(x,D_x) + iQ(x,D_x), \tag{5.5.1}$$

where $m(x,\partial_x)$ is a real vector field on X and $Q \in \Psi_c^1(X)$ has a real principal symbol q. We also assume that q is of constant sign in X_+ and X_-, namely

$$q(x,\xi) \leq 0 \text{ for } x \in X_+, \;\; q(x,\xi) \geq 0 \text{ for } x \in X_-. \tag{5.5.2}$$

Let

$$K = \{x \in X : q(x,\xi) = 0 \text{ for some } \xi \neq 0\}.$$

In particular, $\Gamma \subset K$, since $q(x,\xi) = 0$ for all $x \in \Gamma$. Following Section 4 in [38], we assume that

(i) m is non-vanishing on K;

(ii) No maximal integral curve of m is entirely contained in K.

(iii) m is transversal to Γ and points into X_+.

Properties of a parametrix for P have been studied by many authors (see [36], and references in [38]). Let $\Lambda_\alpha \subset (\widetilde{T^*X\backslash 0}) \times (\widetilde{T^*Y\backslash 0})$ be a smooth family of strictly positive Lagrangian manifolds parameterized by $\alpha \in S^*X$, constructed in [38, p. 371]. Let $\pi_\alpha \in I_c^{n-1/2}(X \times X, \Lambda_\alpha)$ be a smooth family, satisfying

$$\int_{S^*X} \pi_\alpha d\alpha \equiv I.$$

This family uniquely defines the smooth family $\exp(-itP) \circ \pi_\alpha \in I_c^{n-1/2}(X \times X, \Lambda_{\alpha,t})$, where $(\alpha,t) \in (\overline{S^*X_+} \times \overline{\mathbb{R}_+}) \cup (\overline{S^*X_-} \times \overline{\mathbb{R}_-})$, $\Lambda_{\alpha,t} = \exp(tH_p)(\Lambda_\alpha)$, and $t \mapsto \exp(tH_p)(\alpha)$ is the longest real bicharacteristic strip passing through α. The existence of such π_α and $\exp(-itP) \circ \pi_\alpha$ is shown in Section 2 and

Section 4 of [38], respectively, and we may assume that $\exp(-itP) \circ \pi_\alpha = 0$ for large t. Let

$$\pi_\pm = \int_{S^* X_\pm} \pi_\alpha d\alpha$$

and let

$$E = i \int_0^{+\infty} \exp(-itP) \circ \pi_+ dt - i \int_{-\infty}^0 \exp(-itP) \circ \pi_- dt. \tag{5.5.3}$$

Note that the operators under the integral signs are complex Fourier integral operators in X_+ and X_-. It is not difficult to see that outside the characteristics of P, the unique microlocal parametrix $P^{-1} \in \Psi^{-1}(X)$ exists and it is equal to E. It is shown in [38, p.375] that E is a right parametrix for P, $P \circ E \equiv I$ and $E \circ P \equiv I - F_+$, where F_+ is a positive Fourier integral operator of order zero. It is shown in [38] that E is continuous from $(L^2_\alpha)_{comp}(X_\pm)$ to $L^2_\alpha(X)$ and from $L^2_\alpha(X)$ to $L^2_\alpha(X)$ for all s. The main difficulty with the second assertion is that the operators $\exp(-itP) \circ \pi_\pm$ are Fourier integral operators of order 0 in X and in the neighborhood of Γ one need to establish additional estimates. We extend these result to the L^p case:

Theorem 5.5.1. *The following holds:*

(i) *E is continuous from $(L^p_{\alpha+(n-1)|1/p-1/2|})_{comp}(X_\pm)$ to $L^p_\alpha(X)$ for all $\alpha \in \mathbb{R}$, $1 < p < \infty$.*

(ii) *E is continuous from $L^2_{\alpha+n(1/2-1/q)}(X)$ to $L^p_\alpha(X)$ for all $\alpha \in \mathbb{R}$, $p \geq 2$.*

Proof. (i) According to the argument of Lemma 5.6 in [38], the restriction of $\exp(-itP) \circ \pi_+$ to $\mathcal{E}'(X_+)$ is a Fourier integral operator of order 0 with classical symbol, associated to a complex canonical transformation when $t \geq 0$. Then $\exp(-itP) \circ \pi_+$ is continuous from $(L^p_{\alpha+(n-1)/|1/p-1/2|})_{comp}(X_+)$ to $L^p_\alpha(X)$ by Corollary 1.10.1, (i). Since $\exp(-itP) \circ \pi_+$ vanishes for large t, the continuity is uniform with respect to $t \geq 0$. On X_+ the operator E equals to the first term in (5.5.3), which implies the statement on X_+. The same argument for X_- implies (i).

(ii) For $\epsilon > 0$ and for any $u \in \mathcal{D}'(X)$, the singular support of $\exp(-i\epsilon P) \circ \pi_+ u$ is contained in a fixed compact subset of X_+. It follows that for $t > \epsilon > 0$ we have a decomposition

$$\exp(-itP) \circ \pi_+ \equiv (\exp(-i(t-\epsilon)P) \circ \pi_+) \circ (\exp(-i\epsilon P) \circ \pi_+). \tag{5.5.4}$$

The argument of Proposition 5.9 in [38] shows that we have a uniform continuity from $L^2_\alpha(X)$ to $L^2_\alpha(X)$ of $\exp(-i\epsilon P) \circ \pi_+$ for all $0 < \epsilon < \epsilon_0$. Thus, for a given $t > 0$ we take $0 < \epsilon < \min\{t, \epsilon_0\}$, and then decomposition (5.5.4) and Corollary 1.10.1, (iii), imply that $\exp(-itP) \circ \pi_+$ is continuous from $L^2_{\alpha+n(1/2-1/q)}(X)$ to $L^p_\alpha(X)$, since $\exp(-i(t-\epsilon)P) \circ \pi_+$ is continuous from $L^2_{\alpha+n(1/2-1/q)}(X)$ to $L^2_{\alpha+n(1/2-1/q)}(X_+)$. The proof is complete.

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Summary

The main subject of the present monograph is the regularity theory of Fourier integral operators with real and complex phases and related questions of the singularity theory of affine fibrations. These operators appear naturally in many areas of mathematics and one if often led to study their properties in various function spaces. It turns out that these properties depend on the geometric theory of Fourier integral operators. The main geometric invariant for a family of Fourier integral operators parameterized by its symbols is its wave front, which is a Lagrangian submanifold in the cotangent bundle of the base space. Geometry and singularities of this wave front are reflected in the boundedness properties of operators in L^p-spaces.

In general, L^p-spaces provide a convenient scale for the regularity theory. Properties in other spaces normally follow from L^p estimates by standard methods. The simplest case is the one of L^2-spaces when the boundedness of operators relates to the energy conservation law for hyperbolic equations.

One of the aspects of the current work is that we deal with operators with complex phase functions. The theory of such operators is well developed but their regularity has not been much studied. In a way, the use of complex phases provides a more natural approach to Fourier integral operators. In complex valued terms the geometric obstructions of the global theory with the real phase can be avoided and it is a remarkable fact that every Fourier integral operator with a real phase can be globally parameterized by a single complex phase. The sharp orders for Fourier integral operators with real phase to be bounded in L^p, are known for operators satisfying the so-called smooth factorization condition. In this monograph we extend this to the complex phase and extend existing results to the complex setting. Further this condition is analyzed in both real and complex settings.

A part of the book is devoted to the singularity theory of affine fibrations. Conditions for the continuity of Fourier integral operators are related to the singularities of affine fibrations in (subsets of) $\mathbb{C}^n$, defined by the kernels of matrix valued functions. Singularities of such fibrations are analyzed in the general case. Fourier integral operators lead to fibrations, given by the kernels of the Hessian of a phase function of the operator.

Based on the analysis of singularities for operators, commuting with translations, in a number of cases the factorization condition is shown to be satisfied, which leads to L^p estimates for operators. In the other cases, the failure of the factorization condition is exhibited by a number of examples.

Results are applied to derive L^p estimates for solutions of the Cauchy problem for hyperbolic partial differential operators. The use of the complex phase allows to treat several new examples, such as non-hyperbolic Cauchy problems for pseudo-differential equations and the oblique derivative problem.

The background information on Fourier integral operators with real and complex phases as well as the singularity theory of affine fibrations and relevant methods of complex analytic geometry is provided in the book.

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