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Algebraic cycles and toplogy of real algebraic varieties

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1991 Mathematics Subject Classification: 14Cxx (Algebraic cycles and subschemes), 14P25 (Topology of real algebraic varieties), 55N91 (Equivariant homology and cohomology), 57S17 (Finite transformation groups).

ISBN 90 6196 493 8
NUGI-code: 811

Printed in the Netherlands

Editorial Preface

This CWI Tract contains the PhD dissertation of Dr. Joost van Hamel, as it has been defended at the University of Leiden on May 26, 1997. The research of this thesis has been performed within the framework of the Dutch Research School "Thomas Stieltjes Institute for Mathematics". The Stieltjes Institute has awarded Dr. Van Hamel's thesis with a prize, as being the best 1997 thesis written in this Research School. We congratulate Dr. Van Hamel with this prize, and appreciate that we have obtained his permission to include his thesis into the CWI Tract Series.

The editors

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Introduction

Geometrically, an algebraic variety X defined over the real numbers can be viewed as a complex algebraic variety with an involution given by complex conjugation. The study of the topology of X is therefore a study of the topology of its set of complex points $X(\mathbf{C})$ together with the action of $G = \mathbf{Z}/2$ determined by this involution. Of course we are particularly interested in the set of fixed points of this G-action, since this is the set of real points $X(\mathbf{R})$, which has in fact a natural structure of a real algebraic variety in the sense of [BCR], being locally biregularly isomorphic to an irreducible algebraic subset of $\mathbf{R}^n$. A Zariski-closed subset of $X(\mathbf{R})$ of dimension k has a fundamental class, hence it represents a homology class in the group $H_k(X(\mathbf{R}), \mathbf{Z}/2)$. The subgroup of $H_k(X(\mathbf{R}), \mathbf{Z}/2)$ consisting of classes represented by k-dimensional Zariski-closed subsets will be denoted by $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$. The main theme of this work will be the following question.

Question 1. *Given an algebraic variety X defined over $\mathbf{R}$ and a natural number k, which classes in $H_k(X(\mathbf{R}), \mathbf{Z}/2)$ are represented by k-dimensional Zariski-closed subsets?*

This question has been studied before by several authors, and it is related to other important problems in real algebraic geometry, like the approximation of C^∞ mappings by regular mappings and the approximation of C^∞ hypersurfaces by real algebraic hypersurfaces; see for example the survey [BK2], which also contains an extensive bibliography.

The leading principle in this work is that in order to understand $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$ it is crucial to understand the topology of the complex subvarieties of $X_{\mathbf{C}} = X \otimes \mathbf{C}$ and the way complex conjugation acts on these subvarieties. Therefore, we will consider the group $\mathscr{Z}_k(X)$ of algebraic cycles of dimension k on X, as defined in [Fu], which can be identified with the fixed part under complex conjugation of the group $\mathscr{Z}_k(X_{\mathbf{C}})$ of complex algebraic k-cycles on $X_{\mathbf{C}}$. Borel and Haefliger have defined the cycle maps

$$\mathrm{cl}^{\mathbf{R}} : \mathscr{Z}_k(X) \to H_k(X(\mathbf{R}), \mathbf{Z}/2),$$

and

$$\mathrm{cl}^{\mathbf{C}} : \mathscr{Z}_k(X_{\mathbf{C}}) \to H_{2k}(X(\mathbf{C}), \mathbf{Z})$$

by sending a subvariety $V \subset X$ defined over $\mathbf{R}$ (resp. a complex subvariety $V \subset X_{\mathbf{C}}$) to the homology class represented by $V(\mathbf{R})$ (resp. $V(\mathbf{C})$). In particular, $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$ is the image of $\mathscr{Z}_k(X)$ under $\mathrm{cl}^{\mathbf{R}}$.

Geometrically, the connection between the real cycle map $\mathrm{cl}^{\mathbf{R}}$ and the restriction of the complex cycle map $\mathrm{cl}^{\mathbf{C}}$ to the subgroup $\mathscr{Z}_k(X) = \mathscr{Z}_k(X_{\mathbf{C}})^G$ of cycles defined over $\mathbf{R}$ is obvious. If it would be possible to find a connection between $\mathrm{cl}^{\mathbf{R}}$ and $\mathrm{cl}^{\mathbf{C}}$ on the level of homology, we would be able to apply the considerable knowledge that exists on the complex cycle map to the study of the real cycle map.

Question 2. *Is there a homological way of relating* $\mathrm{cl}^{\mathbf{R}}$ *and* $\mathrm{cl}^{\mathbf{C}}$*?*

Loosely speaking, we ask for a homological equivalent of 'taking fixed points'. As far as I know, it has been only recently that real algebraic geometers came to realize that such a construction is actually possible. Several different, but related techniques were developed independently around the beginning of the previous decade. When I started my research, I was aware of the work of F. Mangolte in [Ma1], and an observation made in [CTS]. Mangolte concentrates on the case of nonsingular, simply connected complex surfaces with $G = \{1, \sigma\}$ acting via an antiholomorphic involution. For such a surface X he was able to show that every homology class in the subgroup $M = \{\gamma \in H_2(X(\mathbf{C}), \mathbf{Z})\colon \sigma(\gamma) = -\gamma\}$ is actually represented by a topological cycle c with the property that $\sigma(c) = -c$. Taking the fixed point set of c then defines a mapping $M \to H_1(X(\mathbf{R}), \mathbf{Z}/2)$. Since complex conjugation reverses the orientation of the set of complex points of a curve defined over $\mathbf{R}$, the image of the complex cycle map $\mathrm{cl}^{\mathbf{C}}$ restricted to the group $\mathscr{Z}_1(X) = \mathscr{Z}_1(X_{\mathbf{C}})^G$ is contained in M, so the composite mapping $\varphi \circ \mathrm{cl}^{\mathbf{C}}$ is defined on $\mathscr{Z}_1(X)$, and it follows from the definition that $\varphi \circ \mathrm{cl}^{\mathbf{C}}$ coincides with $\mathrm{cl}^{\mathbf{R}}$. This construction has the advantage that it is very concrete, but it is hard to generalize to an arbitrary surface X or to cycles of arbitrary dimension on a higher-dimensional variety.

On the other hand, Remark 2.3.5 in [CTS] implies that for any nonsingular variety X defined over $\mathbf{R}$, the cycle map in étale cohomology with coefficients in $\mathbf{Z}/2$, composed with a certain natural mapping from the étale cohomology of X into the cohomology of $X(\mathbf{R})$, coincides with the real cycle map in cohomology (which is derived from $\mathrm{cl}^{\mathbf{R}}$ using Poincaré duality). Also, the cycle map in étale cohomology composed with the natural mapping into the cohomology of $X(\mathbf{C})$ gives the complex cycle map (modulo 2). In other words, both the complex and the real cycle map factorize via étale cohomology, which provides, at least for nonsingular varieties, a positive answer to Question 2.

However, this answer is not completely satisfactory, since the étale topology is much harder to work with than the Euclidean topology. From [Ni1] I learned that the étale cohomology of X with coefficients in $\mathbf{Z}/2$ coincides with equivariant cohomology $H^*(X(\mathbf{C}); G, \mathbf{Z}/2)$ of the G-space $X(\mathbf{C})$ in the sense of Borel ([Bo1]) and Grothendieck ([Gr, Ch. V]). Equivariant cohomology, with integral coefficients or coefficients modulo 2, had already proven to be an effective tool in real algebraic geometry (cf. [Kr1], [Si], [Ni1]), which is not really surprising, since it was originally constructed in order to establish connections between the cohomology of a G-space and the cohomology of its fixed point set. Hence it was clear that the natural thing to do was to define for any nonsingular variety X over $\mathbf{R}$ of dimension n an *equivariant cycle map*

$$\mathrm{cl} : \mathscr{Z}_k(X) \to H^{2n-2k}(X(\mathbf{C}); G, \mathbf{Z}(n-k))$$

where $\mathbf{Z}(n-k)$ denotes the constant sheaf $\mathbf{Z}$ with a twisted G-action (see Section III.8). This twist is needed, since complex conjugation reverses the orientation on a complex manifold of odd dimension. The definition of cl itself is in fact straightforward, and it follows from the definitions that the composition of the equivariant cycle map with the natural mapping

$$e^{2n-2k}\colon\ H^{2n-2k}(X(\mathbf{C}); G, \mathbf{Z}(n-k)) \to H^{2n-2k}(X(\mathbf{C}), \mathbf{Z})$$

coincides with the cohomological version of the complex cycle map $\mathrm{cl}^{\mathbf{C}}$. Also the existence of a natural mapping β from the equivariant cohomology of $X(\mathbf{C})$ into the cohomology of $X(\mathbf{R})$ is a well-known fact. The hard part is the proof that the composition of the equivariant cycle map cl with the mapping

$$\beta^{n-k}\colon\ H^{2n-2k}(X(\mathbf{C}); G, \mathbf{Z}(n-k)) \to H^{n-k}(X(\mathbf{R}), \mathbf{Z}/2)$$

induced by β coincides with the cohomological version of $\mathrm{cl}^{\mathbf{R}}$. For this fact I originally only obtained a satisfactory proof in the case $n-k=1$; in the other cases it depended too much on rather messy computations.

Then I found the paper [Kr2], containing the same equivariant cycle map cl, with for $n-k=1$ the same proof for the compatibility between $\beta^{n-k} \circ \mathrm{cl}$ and $\mathrm{cl}^{\mathbf{R}}$ as I had. The case of higher codimension was reduced to the case of codimension 1 using rather nontrivial techniques from algebraic geometry. Hence, at least for a nonsingular variety X, Question 2 was settled in the context of ordinary topology. Nevertheless, both for theoretical and practical purposes the answer was still unsatisfactory, since ideally we should have for any *topological* G-space $Z \subset X(\mathbf{C})$ representing a class $[Z]$ in the equivariant cohomology of $X(\mathbf{C})$, that the image of $[Z]$ in $H^*(X(\mathbf{R}), \mathbf{Z}/2)$ under the mapping β is essentially the fundamental class represented by $Z^G \subset X(\mathbf{R})$. The algebro-geometric nature of Krasnov's proof made it impossible to generalize it to the topological situation, apart from some very special cases (see [Kr3, § 1.3]).

This problem, and also the idea that a good cycle map should have its image in homology rather than cohomology, led me to the construction of equivariant Borel-Moore homology, as can be found in Section III.1. I found a natural analogue to the mapping β, a homomorphism ρ from the equivariant homology of X into the ordinary homology of $X(\mathbf{R})$ (see Section III.7), and I obtained the following, purely topological result (see Section III.7.1).

Theorem. *Let X be a cohomology manifold over $\mathbf{Z}/2$ of dimension n with an action of $G = \mathbf{Z}/2$ and let $\mu_X \in H_n(X; G, \mathbf{Z}/2)$ be the equivariant fundamental class of X. For any connected component $V \subset X^G$ of cohomological dimension d, the image of $\rho(\mu_X) \in H_*(X^G, \mathbf{Z}/2)$ under the projection $H_*(X^G, \mathbf{Z}/2) \to H_d(V, \mathbf{Z}/2)$ is the fundamental class of V.*

The definition of an equivariant cycle map then is completely analogous to the definition of the complex cycle map. Since the mapping ρ is covariantly functorial, it is easy to deduce from the above theorem that the following diagram is commutative for any algebraic variety X defined over $\mathbf{R}$ (see Section IV.1).

$$\begin{tikzcd}
\mathscr{Z}_k(X) \arrow[r, "\mathrm{cl}^{\mathbf{C}}"] \arrow[d, "\mathrm{cl}^{\mathbf{R}}"'] \arrow[dr, "\mathrm{cl}"] & H_{2k}(X(\mathbf{C}), \mathbf{Z}) \\
H_k(X(\mathbf{R}), \mathbf{Z}/2) & H_{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \arrow[l, "\rho_k"'] \arrow[u, "e"']
\end{tikzcd}$$

This completely answers Question 2 by purely topological methods.

In particular, it follows that $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$ is contained in the image of ρ_k. Since in general ρ_k does not map surjectively onto $H_k(X(\mathbf{R}), \mathbf{Z}/2)$, this gives a restriction on H_k^{alg} that only depends on the G-equivariant topology of $X(\mathbf{C})$. This fact is exploited in several concrete situations. In Example IV.3.1 the image of ρ_k is determined for an abelian variety X defined over $\mathbf{R}$, giving in many cases highly nontrivial restrictions on the dimension of $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$. In Chapter V the mapping ρ_1 is used in order to determine the group $H_1^{\mathrm{alg}}(Y(\mathbf{R}), \mathbf{Z}/2)$ for a real Enriques surface Y. Since in that case $H^2(Y, \mathscr{O}_Y) = 0$ we actually have that $H_1^{\mathrm{alg}}(Y(\mathbf{R}), \mathbf{Z}/2)$ coincides with the image of ρ_1. A close examination of the equivariant homology of a real Enriques surface and the mapping ρ_1, due to F. Mangolte and myself, the gives the following result (see Theorem V.1.2 for more details).

Theorem. *Let Y be a real Enriques surface with $Y(\mathbf{R}) \neq \emptyset$. If all connected components of the real part $Y(\mathbf{R})$ are orientable, then*

$$H_1^{\mathrm{alg}}(Y(\mathbf{R}), \mathbf{Z}/2) = H_1(Y(\mathbf{R}), \mathbf{Z}/2).$$

Otherwise,

$$\dim H_1^{\mathrm{alg}}(Y(\mathbf{R}), \mathbf{Z}/2) = \dim H_1(Y(\mathbf{R}), \mathbf{Z}/2) - 1.$$

Finally, Chapter VI is based on the research I did before learning about equivariant cohomology. In the original paper on which this chapter is based, the main result

(Theorem VI.1.5, see below) is proven using well-known, but rather nontrivial facts from algebraic geometry. The present version uses equivariant cohomology, which not only shortens the proof considerably, it also embeds this result in a topological framework, clarifying both the role of the special topological structure and of the special algebraic structure of the varieties involved. The objects of study are real algebraic cycles of codimension 1 on a complex algebraic variety X, or, to be more precise, algebraic cycles on the *underlying real algebraic structure* $X_{\mathbf{R}}$ of X. Here $X_{\mathbf{R}}$ is *not* an algebraic variety over $\mathbf{R}$, but it is a real algebraic variety in the sense of [BCR]; in other words, $X_{\mathbf{R}}$ is locally isomorphic to an algebraic subset of $\mathbf{R}^n$. The set of points of $X_{\mathbf{R}}$ is the set of complex points of X, and the real algebraic structure is obtained by considering an affine open subset $U \subset X$ as a real algebraic set via the identification of $\mathbf{C}^N$ with $\mathbf{R}^{2N}$. In particular, if X is of (complex) dimension d, then $X_{\mathbf{R}}$ has dimension $2d$ as a real algebraic variety. This construction allows us, for example, to ask which classes in the first homology group of a complex curve C are represented by real algebraic curves on C. The main result of Chapter VI is the following theorem.

Theorem VI.1.5. *Let X be a complete, nonsingular irreducible algebraic variety over $\mathbf{C}$. The Albanese mapping $\alpha_X : X \to \mathrm{Alb}\, X$ induces an isomorphism*

$$\alpha_X^* : H^1_{\mathrm{alg}}(\mathrm{Alb}(X)_{\mathbf{R}}, \mathbf{Z}/2) \xrightarrow{\sim} H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2).$$

This result, together with the Lefschetz Theorem on hyperplane sections and Bertini's Theorem, allows us to construct nonsingular complex algebraic varieties X of any dimension ≥ 2 with prescribed cohomology in codimension 1 and prescribed dimension of $H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$ (see Theorem VI.3.2). On the other hand, a method due to J. Huisman for the computation of the group H^1_{alg} of a real abelian variety then gives a method of computing $H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$ in terms of the period matrix of X. Using these computations, the following results are obtained.

Corollary VI.3.8. *Let X be a complete nonsingular irreducible complex algebraic variety with*

$$H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2) = H^1(X, \mathbf{Z}/2).$$

Then $H^1(X, \mathbf{Z}/2) = H^1(X, \mathbf{Z}) \otimes \mathbf{Z}/2$ and $\mathrm{Alb}\, X$ is of CM-type.

Corollary VI.3.10. *There are, up to isomorphism, only countably many nonsingular irreducible projective complex curves C with $H^1_{\mathrm{alg}}(C_{\mathbf{R}}, \mathbf{Z}/2) = H^1(C, \mathbf{Z}/2)$.*

Apart from the main questions concerning the cycle maps, there are some other questions concerning algebraic cycles and the topology of algebraic varieties defined over $\mathbf{R}$ that are studied in this work. First of all, there is the problem that if we consider algebraic k-cycles of X and on $X_{\mathbf{C}}$ modulo rational equivalence, then we obtain the Chow groups $CH_k(X)$ and $CH_k(X_{\mathbf{C}})$, and we see that the natural identification $\mathscr{Z}_k(X) = \mathscr{Z}_k(X_{\mathbf{C}})^G$ gives a mapping $CH_k(X) \to CH_k(X_{\mathbf{C}})^G$, but this mapping need not be injective or surjective. There are examples of pairs of cycles defined over $\mathbf{R}$ which are not rationally equivalent over $\mathbf{R}$, but which are rationally equivalent over $\mathbf{C}$.

On the other hand, there are examples of complex k-cycles on a variety over $\mathbf{R}$ that are rationally equivalent to their complex conjugate without being equivalent to a cycle that is actually defined over $\mathbf{R}$ (see Section I.2).

Question 3. *What is the kernel and what is the cokernel of the natural mapping $CH_k(X) \to CH_k(X_{\mathbf{C}})^G$?*

In general this question is very hard. However, we will see that when X is nonsingular and complete a good knowledge of the equivariant cohomology of X can shed light on this question; see Section IV.5.

Remark. This monograph is a slight revision of my PhD Thesis from 1997. I have corrected some errors and misprints, updated the references, and in a few cases added a remark about later developments.

CHAPTER I

Real algebraic cycles

In this chapter the basic theory concerning algebraic cycles on varieties defined over **R** will be reviewed. Apart from the notion of *real algebraic equivalence* (Definition 1.5), everything is classical.

1. Definitions

In complex algebraic geometry there are two equivalent ways of defining an algebraic variety. The classical definition of a complex projective algebraic variety is an irreducible Zariski-closed subset of a complex projective space. In the language of schemes, a complex projective algebraic variety is a projective integral scheme over the complex numbers. Over the real numbers these two definitions yield entirely different concepts. The classical definition gives rise to a projective real algebraic variety in the sense of [BCR]. Here we will mostly study varieties in the scheme-theoretic sense. In order to avoid confusion, we will not call these varieties real algebraic varieties.

Definition 1.1. Let K be a field of characteristic zero. An *algebraic variety over a field* K is a reduced, separated scheme of finite type over K. If $\overline{K}$ is an algebraic closure of K and $X_{\overline{K}} = X \times_K \overline{K}$ is irreducible, X is said to be geometrically irreducible.

For us, the field K will always be the field of real numbers **R** or the field of complex numbers **C**. Note that a variety over K is not by definition irreducible, since irreducibility behaves badly under field extensions; it is for example very easy to find an irreducible variety X defined over **R** such that $X_{\mathbf{C}}$ is not irreducible. We will be mostly interested in geometrically irreducible varieties, but this property is not included in the definition of a variety, since we will need varieties that are not geometrically irreducible when working with algebraic cycles.

Let $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \{1, \sigma\}$ be the Galois group of $\mathbf{C}/\mathbf{R}$. The G-action on **C** induces a **C**-antilinear G-action on $X_{\mathbf{C}}$. Indeed, a good way of looking at an algebraic variety over $\mathbf{R}$ is considering it as a complex variety with a $\mathbf{C}$-antilinear (or 'anti-holomorphic') G-action. For example, if Y is another algebraic variety defined over $\mathbf{R}$, the set of morphisms $\mathrm{Hom}_{\mathbf{R}}(X, Y)$ defined over $\mathbf{R}$ is bijective to the set of G-equivariant morphisms $\mathrm{Hom}_{G,\mathbf{C}}(X_{\mathbf{C}}, Y_{\mathbf{C}})$ defined over $\mathbf{C}$.

The set of complex points $X(\mathbf{C})$ of X will be equipped with the euclidean topology. The canonical G-action on $X(\mathbf{C})$, which will sometimes be called the *complex conjugation on* $X(\mathbf{C})$, is continuous and its set of fixed points is $X(\mathbf{R})$, the set of real points (or the *real part*) of X. Although $X(\mathbf{R})$ has the natural structure of a real algebraic variety in the sense of [BCR], we will consider it as a topological space with the euclidean topology. In particular, $X(\mathbf{R})$ can have several connected components, even if it is irreducible for the Zariski-topology. Note that there is a bijection between the quotient $X(\mathbf{C})/G$ and the set of closed points of X. In other words, a closed point P of X with a complex residue field determines a pair of complex points in $X(\mathbf{C})$.

By $\mathscr{Z}_k(X)$ we denote the free abelian group generated by the k-dimensional irreducible closed subvarieties of X. An element of $\mathscr{Z}_k(X)$ is called an *algebraic cycle of dimension* k. Any closed subscheme $S \subset X$ of pure dimension k gives rise to a cycle consisting of the subvarieties of X associated to the irreducible components of S counted with geometric multiplicity (see [Fu, 1.5]). By abuse of notation this cycle will be denoted by $S \in \mathscr{Z}_k(X)$. Sometimes it is more convenient to have a grading corresponding to codimension. Then we write $\mathscr{Z}^k(X) = \mathscr{Z}_{n-k}(X)$, where n is the dimension of X. The G-action on $X_{\mathbf{C}}$ induces a G-action on $\mathscr{Z}_k(X_{\mathbf{C}})$, and base change from $\mathbf{R}$ to $\mathbf{C}$ induces an isomorphism $\mathscr{Z}_k(X) \xrightarrow{\sim} \mathscr{Z}_k(X_{\mathbf{C}})^G$ for any algebraic variety X over $\mathbf{R}$.

A proper morphism $f\colon X \to Y$ induces a *proper push-forward* $f_*\colon \mathscr{Z}_k(X) \to \mathscr{Z}_k(Y)$, and a flat morphism $f\colon X \to Y$ induces a *flat pull-back* $f^*\colon \mathscr{Z}^k(Y) \to \mathscr{Z}^k(X)$ (see [Fu, §§1.4 and 1.7] for the definitions). Note that the base change isomorphism $\mathscr{Z}_k(X) \xrightarrow{\sim} \mathscr{Z}_k(X_{\mathbf{C}})^G$ is in fact the flat pull-back associated to the canonical projection $\pi\colon X_{\mathbf{C}} \to X$. On the other hand, the proper push-forward associated to π gives a mapping $\pi_*\colon \mathscr{Z}_k(X_{\mathbf{C}}) \to \mathscr{Z}_k(X)$. We have that $\pi_* \circ \pi^*\colon \mathscr{Z}_k(X) \to \mathscr{Z}_k(X)$ is multiplication by 2 and $\pi^* \circ \pi_*\colon \mathscr{Z}_k(X_{\mathbf{C}}) \to \mathscr{Z}_k(X_{\mathbf{C}})$ is the mapping $(1+\sigma)$, where $\sigma \in G$ is the non-trivial element.

Definition 1.2. Let X be an algebraic variety over field K of characteristic zero. The subgroup $\mathscr{Z}_k^{\mathrm{rat}}(X) \subset \mathscr{Z}_k(X)$ of cycles *rationally equivalent to zero* is the subgroup generated by the algebraic cycles of the form $V_0 - V_\infty$, where $V \subset X \times \mathbf{P}^1_K$ is a subvariety that is flat, of relative dimension k over $\mathbf{P}^1_K$.

Of course, two cycles $a, b \in \mathscr{Z}_k(X)$ are said to be *rationally equivalent* if $a - b \in \mathscr{Z}_k^{\mathrm{rat}}(X)$. It is important to realize that the injection $\pi^*\colon \mathscr{Z}_k^{\mathrm{rat}}(X) \hookrightarrow \mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}})^G$ need not be surjective; see Example 2.3. It easily follows from the definitions that proper push-forward and flat pull-back homomorphisms always respect rational equivalence (see [Fu, T. 1.4, Th. 1.7]). In particular, $\pi^* \circ \pi_*$ maps $\mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}})$ into itself. Since $\pi^* \circ \pi_*$ restricted to $\mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}})^G$ is multiplication by 2, the quotient $\mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}})^G/\pi^*\mathscr{Z}_k^{\mathrm{rat}}(X)$ is purely 2-torsion.

There is another, equivalent, way of defining rational equivalence which can be very useful, especially in the codimension 1 case. Let $Z \subset X$ be an irreducible closed subvariety of codimension 1. Let $\mathscr{O}_{Z,X}$ be the local ring of Z on X. For an element $r \in \mathscr{O}_{Z,X}$, let $\mathrm{ord}_Z(r)$ be the length of $\mathscr{O}_{Z,X}/(r)$ as an $\mathscr{O}_{Z,X}$-module (see [Fu, § 1.2]). When X is normal then $\mathrm{ord}_Z(r)$ can be interpreted as the order of vanishing of r along Z. Now let $K(X)$ be the function field of X. Write $f \in K(X)^*$ as $f = a/b$ for some $a, b \in \mathscr{O}_{Z,X}$ and define

$$\mathrm{ord}_Z(f) = \mathrm{ord}_Z(a) - \mathrm{ord}_Z(b).$$

This gives a well-defined homomorphism $K(X)^* \to \mathbf{Z}$ and we get a homomorphism $\mathrm{div} : K(X)^* \to \mathscr{Z}_k(X)$ by putting

$$\mathrm{div}(f) = \sum_Z \mathrm{ord}_Z(f)Z,$$

where Z ranges over all irreducible closed subvarieties of X of codimension 1. The cycle $\mathrm{div}(f)$ is called the *divisor of f*. Fulton defines $\mathscr{Z}^1_{\mathrm{rat}}$ to be the image of the mapping div, and for $k > 1$ he defines $\mathscr{Z}^k_{\mathrm{rat}}(X)$ to be the image of the homomorphism

$$\bigoplus_{W \hookrightarrow X} \mathscr{Z}^1_{\mathrm{rat}}(W) \to \mathscr{Z}^k(X),$$

where W ranges over the irreducible closed subvarieties of X of codimension $k-1$. This definition coincides with Definition 1.2 by [Fu, Prop. 1.6].

If we would replace the base curve $\mathbf{P}^1_{\mathbf{R}}$ in Definition 1.2 by an arbitrary nonsingular curve C over $\mathbf{R}$, and instead of the points 0 and ∞ we would take two real points $t_0, t_1 \in C(\mathbf{R})$, this would give a definition of algebraic equivalence that is compatible with Fulton's definition. However, if t_0 and t_1 are in different connected components of $C(\mathbf{R})$, both intuitively and for any practical purposes the fibres V_{t_0} and V_{t_1} cannot be considered to be in the same 'continuous' family of subvarieties defined over $\mathbf{R}$. Therefore it is of little use trying to take into account the ground field $\mathbf{R}$, and it seems more natural to say that two cycles over $\mathbf{R}$ are algebraically equivalent if and only they are algebraically equivalent over $\mathbf{C}$.

Definition 1.3. Let $X_{\mathbf{C}}$ be an algebraic variety defined over $\mathbf{C}$. The subgroup $\mathscr{Z}_k^{\mathrm{alg}}(X_{\mathbf{C}}) \subset \mathscr{Z}_k(X_{\mathbf{C}})$ of cycles *algebraically equivalent to* 0 is the subgroup generated by cycles of the form $V_{t_0} - V_{t_1}$, where t_0 and t_1 are closed points on a nonsingular curve C over $\mathbf{C}$ and $V \subset X_{\mathbf{C}} \times C$ is a subvariety that is flat of relative dimension k over C.

For an algebraic variety X over $\mathbf{R}$, we define

$$\mathscr{Z}_k^{\mathrm{alg}}(X) = \mathscr{Z}_k(X) \cap \mathscr{Z}_k^{\mathrm{alg}}(X_{\mathbf{C}}).$$

Remark 1.4. If T is a nonsingular variety over $\mathbf{C}$ of dimension n, and $V \subset X_{\mathbf{C}} \times T$ is a subvariety which is flat over T of relative dimension dimension k, then for any pair of closed points t_0, $t_1 \in T$ the cycle $V_{t_0} - V_{t_1}$ is algebraically equivalent to zero, since we can connect any two closed points on T by a chain of locally closed embeddings of nonsingular curves.

An analogous result holds if V is any family of k-cycles over T, but then one cannot just take the scheme-theoretic fibres V_{t_0} and V_{t_1} — which need not even be of dimension k —, so a more elaborate concept like Fulton's Gysin morphisms is needed; see [Fu, Ch. 10]. In particular, we have by [Fu, Ex. 10.3.2], that for cycles over $\mathbf{C}$ our definition coincides with Fulton's definition.

Note that there actually exist algebraic varieties over $\mathbf{R}$ for which Fulton's definition of algebraic equivalence gives an equivalence relation different from the equivalence relation defined here; one of them is in fact the curve of Example 2.3.

The above discussion suggests a third equivalence relation under which the members of a 'continuous' family of algebraic cycles over $\mathbf{R}$ are equivalent.

Definition 1.5. Let X be an algebraic variety defined over $\mathbf{R}$. The subgroup $\mathscr{Z}_k^{\mathbf{R}\text{-alg}}(X) \subset \mathscr{Z}_k(X)$ of cycles *real algebraically equivalent to* 0 is the subgroup generated by cycles of the form $V_{t_0} - V_{t_1}$, where t_0 and t_1 are points in the same connected component of the real part $C(\mathbf{R})$ of a nonsingular curve C over $\mathbf{R}$, and $V \subset X_{\mathbf{R}} \times C$ is a closed subvariety that is flat, of relative dimension k over C.

It is not hard to check that proper push-forward and flat pull-back respect algebraic equivalence and real algebraic equivalence. Slightly more subtle is the fact that the mapping $\pi_*\colon \mathscr{Z}_k(X_{\mathbf{C}}) \to \mathscr{Z}_k(X)$ sends a complex cycle that is algebraically equivalent to zero to a cycle that is real algebraically equivalent to zero.

Lemma 1.6. *Let X be a geometrically irreducible algebraic variety defined over* $\mathbf{R}$, *and let* $k \geq 0$. *Let* $\pi_*\colon \mathscr{Z}_k(X_{\mathbf{C}}) \to \mathscr{Z}_k(X)$ *be the proper push-forward associated to the canonical projection* $\pi\colon X_{\mathbf{C}} \to X$. *Then* π_* *maps* $\mathscr{Z}_k^{\mathrm{alg}}(X_{\mathbf{C}})$ *into* $\mathscr{Z}_k^{\mathbf{R}\text{-alg}}(X)$.

Proof. Let C be a complex curve, let t_0, t_1 be closed points of C and let $V \subset X_{\mathbf{C}} \times C$ be as in Definition 1.3. We will show that $\pi_* V_{t_0}$ is real algebraically equivalent to $\pi_* V_{t_1}$.

Let C^σ be the conjugate curve, i.e., the scheme C with structure morphism given by $C \to \operatorname{Spec} \mathbf{C} \xrightarrow{\sigma} \operatorname{Spec} \mathbf{C}$. Then $C \times C^\sigma$ is a quasi-projective variety over $\mathbf{C}$ with a canonical $\mathbf{C}$-antilinear involution, so by descent theory there is a quasi-projective variety $C_{\mathscr{W}}$ defined over $\mathbf{R}$, such that $(C_{\mathscr{W}})_{\mathbf{C}} = C \times C^\sigma$. In fact, $C_{\mathscr{W}}$ is the *Weil restriction* of C (see also Section VI.1); in particular, $C_{\mathscr{W}}(\mathbf{R})$ is homeomorphic to $C(\mathbf{C})$, hence connected.

Consider the subvariety $V' = V \times C^\sigma \subset X_{\mathbf{C}} \times C \times C^\sigma$. Then, assuming for simplicity $\sigma(V') \neq V'$, there is a subvariety $V'' \subset X \times C_{\mathscr{W}}$, such that

$$V'' \otimes \mathbf{C} = V' \cup \sigma(V').$$

Clearly, V'' is flat over $C_{\mathscr{W}}$ since $V' \cup \sigma(V')$ is flat over $C \times C^{\sigma}$. For $i = 0, 1$, we define $s_i = (t_i, t_i^{\sigma}) \in C(\mathbf{C}) \times C^{\sigma}(\mathbf{C})$. Then s_i is invariant under σ, so $s_i \in C_{\mathscr{W}}(\mathbf{R})$. From the fact that $(V'' \otimes \mathbf{C})_{s_i} = V_{t_i} \cup \sigma(V_{t_i}) \subset X_{\mathbf{C}}$, we see that $(V'' \otimes \mathbf{C})_{s_i}$ is the cycle $\pi^* \circ \pi_* V_{t_i} \in \mathscr{Z}_k(X_{\mathbf{C}})$, so $V''_{s_i} = \pi_* V_{t_i} \in \mathscr{Z}_k(X)$ for $i = 0, 1$. Since $C_{\mathscr{W}}$ is nonsingular and $C_{\mathscr{W}}(\mathbf{R})$ is connected, this means that $\pi_*(V_{t_0} - V_{t_1})$ is real algebraically equivalent to zero. □

Clearly we have

$$\mathscr{Z}_k^{\mathrm{rat}}(X) \subset \mathscr{Z}_k^{\mathbf{R}\text{-alg}}(X) \subset \mathscr{Z}_k^{\mathrm{alg}}(X) \subset \mathscr{Z}_k(X).$$

The *Chow group in dimension k* is defined to be

$$CH_k(X) = \mathscr{Z}_k(X)/\mathscr{Z}_k^{\mathrm{rat}}(X).$$

The elements of $CH_k(X)$ are called *cycle classes*, and the class of a cycle $a \in \mathscr{Z}_k(X)$ is denoted by $[a] \in CH_k(X)$. The subgroup $\mathscr{Z}_k^{\mathrm{alg}}(X)/\mathscr{Z}_k^{\mathrm{rat}}(X)$ of cycle classes algebraically equivalent to zero is denoted by $CH_k^{(0)}(X)$. The subgroup $\mathscr{Z}_k^{\mathbf{R}\text{-alg}}(X)/\mathscr{Z}_k^{\mathrm{rat}}(X) \subset CH_k^{(0)}(X)$ of cycle classes real algebraically equivalent to zero is denoted by $CH_k^{(0)\mathbf{R}}(X)$.

For cycles of codimension k modulo rational equivalence we also use the notation $CH^k(X)$. The group $\mathscr{Z}^1(X)/\mathscr{Z}^1_{\mathrm{alg}}(X)$ is known as the *Néron-Severi group* of X and denoted by $NS(X)$. Since proper push-forward f_* and flat pull-back f^* respect rational equivalence, they induce homomorphisms of Chow groups which are denoted by f_* (resp. f^*) as well. One of the properties of the Chow groups that are important in applications, the existence of an *intersection product*

$$CH^i(X) \otimes CH^j(X) \to CH^{i+j}(X)$$

whenever X is nonsingular, will only play a minor role in the present work.

2. The Galois action

In this section we will investigate the mapping

$$\pi^* \colon\ CH_k(X) \to CH_k(X_{\mathbf{C}})^G$$

for an algebraic variety X defined over $\mathbf{R}$. The main technical tool will be the cohomology $H^*(G, M)$ of the group $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \{1, \sigma\}$ with coefficients in a G-module M (an abelian group with a G-action). Here it will be sufficient to know that $H^*(G, M)$ is functorial in M, it transforms short exact sequences into long exact sequences and it can be computed as the cohomology of the complex

$$M \xrightarrow{1-\sigma} M \xrightarrow{1+\sigma} M \xrightarrow{1-\sigma} M \xrightarrow{1+\sigma} \cdots \tag{1}$$

(see also Section III.7).

First we will study the kernel of the mapping $\pi^*\colon CH_k(X) \to CH_k(X_{\mathbf{C}})^G$. Note that the homomorphism $\pi_* \circ \pi^*$ is multiplication by 2, so the kernel of π^* is contained in the 2-torsion of $CH_k(X)$. It is easy to see that the kernel of π^* is isomorphic to the cokernel of the map

$$\mathscr{Z}_k^{\mathrm{rat}}(X) \to \mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}})^G.$$

Unfortunately, for most k these two groups are quite mysterious; the codimension 1 case is an exception.

Proposition 2.1. *Let X be a normal, geometrically irreducible variety over $\mathbf{R}$. The kernel of the mapping $\pi^*\colon CH^1(X) \to CH^1(X_{\mathbf{C}})^G$ is isomorphic to $H^1(G, \mathscr{O}(X_{\mathbf{C}})^*)$.*

Proof. Since $X_{\mathbf{C}}$ is normal and irreducible, we have a short exact sequence

$$0 \to \mathscr{O}(X_{\mathbf{C}})^* \to K(X_{\mathbf{C}})^* \xrightarrow{\mathrm{div}} \mathscr{Z}^1_{\mathrm{rat}}(X_{\mathbf{C}}) \to 0, \tag{2}$$

where $\mathscr{O}(X_{\mathbf{C}})$ denotes the ring of regular functions on $X_{\mathbf{C}}$ and $K(X_{\mathbf{C}})$ denotes the field of rational functions on $X_{\mathbf{C}}$. The short exact sequence induces a long exact sequence

$$\cdots \to (K(X_{\mathbf{C}})^*)^G \to \mathscr{Z}^1_{\mathrm{rat}}(X_{\mathbf{C}})^G \to H^1(G, \mathscr{O}(X_{\mathbf{C}})^*) \to H^1(G, K(X_{\mathbf{C}})^*) \to \cdots$$

Now $H^1(G, K(X_{\mathbf{C}})^*) = 0$ by Hilbert's Theorem 90 and the image of $(K(X_{\mathbf{C}})^*)^G \to \mathscr{Z}^1_{\mathrm{rat}}(X_{\mathbf{C}})^G$ is precisely the image of $\mathscr{Z}_k^{\mathrm{rat}}(X) \to \mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}})^G$. $\square$

As a corollary we obtain the following well-known and important result.

Corollary 2.2. *If X is a complete, normal, geometrically irreducible variety over $\mathbf{R}$, then $\pi^*\colon CH^1(X) \to CH^1(X_{\mathbf{C}})^G$ is injective.*

Proof. If X is complete, then $\mathscr{O}(X_{\mathbf{C}})^* = \mathbf{C}^*$, so it follows from Hilbert's Theorem 90 that $H^1(G, \mathscr{O}(X_{\mathbf{C}})^*) = 0$. $\square$

The following example shows that the completeness condition is important.

Example 2.3. Let $S \subset \mathbf{A}^2_{\mathbf{R}}$ be the nonsingular affine plane curve defined by the equation $x^2 + y^2 = 1$. Then S is isomorphic to the projective line $\mathbf{P}^1_{\mathbf{R}}$ minus one closed point with complex residue field. It follows that any $f \in K(S)^*$ having no poles on S has an even number of zeroes, so a single real point on S is not rationally equivalent to zero over $\mathbf{R}$. In fact, we easily see that $CH^1(S) = \mathbf{Z}/2$. On the other hand, $S_{\mathbf{C}}$ is isomorphic to the complex projective line minus two points, so $CH^1(S_{\mathbf{C}}) = 0$.

Since the restriction of the mapping $\pi^* \circ \pi_*$ to $CH_k(X_{\mathbf{C}})^G$ is multiplication by 2, the cokernel of the map $\pi^*\colon CH_k(X) \to CH_k(X_{\mathbf{C}})^G$ is purely 2-torsion. The short exact sequence of G-modules

$$0 \to \mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}}) \to \mathscr{Z}_k(X_{\mathbf{C}}) \to CH_k(X) \to 0$$

gives a long exact sequence

$$\cdots \to \mathscr{Z}_k(X_{\mathbf{C}})^G \to CH_k(X)^G \to H^1(G, \mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}})) \to H^1(G, \mathscr{Z}_k(X_{\mathbf{C}})) \to \cdots$$

The image of the mapping $\mathscr{Z}_k(X_{\mathbf{C}})^G \to CH_k(X)^G$ in the long exact sequence is the image of π^*, and it is easily seen that $H^1(G, \mathscr{Z}_k(X_{\mathbf{C}})) = 0$, so the cokernel of π^* is isomorphic to $H^1(G, \mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}}))$. Again, this description is only really useful in codimension 1.

Lemma 2.4. *Let X be a normal, geometrically irreducible variety over $\mathbf{R}$. Then the cokernel of $\pi^*\colon CH^1(X) \to CH^1(X_{\mathbf{C}})^G$ is isomorphic to the kernel of the canonical homomorphism $H^2(G, \mathscr{O}(X_{\mathbf{C}})^*) \to H^2(G, K(X_{\mathbf{C}})^*)$.*

PROOF. From the long exact sequence associated to the short exact sequence (2) and the fact that $H^1(G, K(X_{\mathbf{C}})^*) = 0$, we see that the sequence

$$0 \to H^1(G, \mathscr{Z}_k^{\mathrm{rat}}(X_{\mathbf{C}})) \to H^2(G, \mathscr{O}(X_{\mathbf{C}})^*) \to H^2(G, K(X_{\mathbf{C}})^*)$$

is exact. □

Corollary 2.5. *Let X be a complete, normal, geometrically irreducible variety over $\mathbf{R}$. If -1 is not the sum of two squares in $K(X)^*$, then $\pi^*\colon CH^1(X) \to CH^1(X_{\mathbf{C}})^G$ is surjective. Otherwise the cokernel of π^* is isomorphic to $\mathbf{Z}/2$.*

PROOF. Since X is complete, $\mathscr{O}(X_{\mathbf{C}}) = \mathbf{C}$, so $H^2(G, \mathscr{O}(X_{\mathbf{C}})^*) = \mathbf{R}^*/\mathbf{R}^*_{>0} = \mathbf{Z}/2$ is generated by the class of -1. The class of -1 is zero in $H^2(G, K(X_{\mathbf{C}})^*)$ if and only if there is a function $h \in K(X_{\mathbf{C}})^*$ such that $h \cdot h^\sigma = -1$. In other words, writing $h = f + ig$, we see that the homomorphism $H^2(G, \mathscr{O}(X_{\mathbf{C}})^*) \to H^2(G, K(X_{\mathbf{C}})^*)$ is zero if and only if there are $f, g \in K(X)^*$ such that $f^2 + g^2 = -1$. □

Combining the above results on injectivity and surjectivity of π^* we get the following well-known and important result.

Theorem 2.6. *Let X be a complete, normal, geometrically irreducible variety over $\mathbf{R}$ with a nonsingular real point. Then $\pi^*\colon CH^1(X) \to CH^1(X_{\mathbf{C}})^G$ is an isomorphism.*

PROOF. This follows from Corollary 2.2 and Corollary 2.5 since $X(\mathbf{R})$ is Zariski-dense in X, so there are no $f, g \in K(X)^*$ such that $f^2 + g^2 = -1$. □

There are many examples of complete, normal, geometrically irreducible varieties over $\mathbf{R}$ with $X(\mathbf{R}) = \emptyset$ and $CH^1(X) \neq CH^1(X_{\mathbf{C}})^G$. The easiest example is probably the projective conic C given by the equation

$$x^2 + y^2 + z^2 = 0.$$

Since $C_{\mathbf{C}}$ is isomorphic to the complex projective line, $CH^1(X_{\mathbf{C}}) = CH^1(X_{\mathbf{C}})^G \simeq \mathbf{Z}$ is generated by a single point, but all divisors on C defined over $\mathbf{R}$ have even degree, so the cokernel of the mapping $\pi^*\colon CH^1(C) \to CH^1(C_{\mathbf{C}})^G$ is isomorphic to $\mathbf{Z}/2$.

Observe that the above does *not* imply that π^* induces an isomorphism $\mathcal{NS}(X) \xrightarrow{\sim} \mathcal{NS}(X_{\mathbf{C}})^G$ whenever X satisfies the conditions of the theorem (see also [Si, § I.4]). We will come back to this question in Section IV.5.

3. Divisors and line bundles

An algebraic cycle of codimension 1 is also known as a *Weil divisor.* If X is a nonsingular irreducible variety over **R**, a closed subvariety $Z \subset X$ of codimension 1 is locally defined by one equation, so Z can be given by a collection (U_i, f_i) where the U_i form an open covering of X and the f_i are nonzero elements of $\mathscr{O}(U_i)$ such that on $U_{ij} = U_i \cap U_j$ we have $f_i|_{U_{ij}} = h_{ij} f_j|_{U_{ij}}$ for some unit $h_{ij} \in \mathscr{O}^*(U_{ij})$. Since the f_i are unique up to multiplication by an element of $\mathscr{O}^*(U_i)$, we see that a Weil divisor determines a section of the sheaf $\mathscr{K}_X^*/\mathscr{O}_X^*$, where $\mathscr{K}_X$ is the (constant) sheaf of rational functions on X. For an arbitrary irreducible variety X over **R** we say that an element of $\Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*)$ is a *Cartier divisor.* A Cartier divisor is said to be *principal* if it is in the image of the canonical map $\Gamma(X, \mathscr{K}_X^*) \to \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*)$. We have a homomorphism

$$\operatorname{div} : \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*) \to \mathscr{Z}^1(X)$$

which is defined in the following way. Let (U_i, f_i) be a collection as above, representing a Cartier divisor D. For an irreducible subvariety $Z \subset X$ of codimension 1 define $\operatorname{ord}_Z(d) = \operatorname{ord}_Z(f_i)$, where the index i is chosen such that $U_i \cap Z$ is nonempty. Then we put

$$\operatorname{div}(D) = \sum_Z \operatorname{ord}_Z(D) Z.$$

When X is nonsingular, the mapping div is the inverse of the homomorphism $\mathscr{Z}^1(X) \to \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*)$ defined above. The composite mapping $\Gamma(X, \mathscr{K}_X^*) \to \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*) \xrightarrow{\operatorname{div}} \mathscr{Z}^1(X)$ is precisely the mapping div defined in the previous section, and the short exact sequence

$$0 \to \mathscr{O}_X^* \to \mathscr{K}_X^* \to \mathscr{K}_X^*/\mathscr{O}_X^* \to 0$$

induces an exact sequence

$$\Gamma(X, \mathscr{K}_X^*) \to \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*) \to H^1(X, \mathscr{O}_X^*) \to 0$$

because $\mathscr{K}_X^*$ is flasque (it is a constant sheaf on the space X with the Zariski topology). This implies that the mapping div induces a homomorphism

$$H^1(X, \mathscr{O}_X^*) \to CH^1(X)$$

which is an isomorphism if X is nonsingular; in that case we may use the term *divisor class* for an element in $CH^1(X)$ (or $H^1(X, \mathscr{O}_X^*)$) without any danger of confusion.

The group $H^1(X, \mathscr{O}_X^*)$ may also be interpreted as the *Picard group* of X, denoted $\operatorname{Pic}(X)$, which is the group of isomorphism classes of invertible sheaves on X with the tensor product as group operation. This interpretation is also valid for a singular variety X. More geometrically, $\operatorname{Pic}(X)$ is the group of isomorphism classes of line bundles on X over **R**. Here a *line bundle* on X over **R** is a variety $\mathscr{L}$ and a morphism $p\colon \mathscr{L} \to X$

such that we have an open cover $\{U_i\}$ of X and isomorphisms $\psi_i\colon p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbf{A}^1_{\mathbf{R}}$ having the property that $\psi_j \circ \psi_i^{-1}$ is a *linear* automorphism of $(U_i \cap U_j) \times \mathbf{A}^1_{\mathbf{R}}$ (see [Ha, Exc. II.5.18]). After base change to $\mathbf{C}$ a line bundle $p\colon \mathscr{L} \to X$ gives rise to a complex algebraic line bundle $p\colon \mathscr{L}_{\mathbf{C}} \to X_{\mathbf{C}}$ with a $\mathbf{C}$-antilinear involution $\sigma\colon \mathscr{L}_{\mathbf{C}} \to \mathscr{L}_{\mathbf{C}}$ compatible with the involution σ on $X_{\mathbf{C}}$, and it is well-known that, conversely, any such pair $(\mathscr{L}_{\mathbf{C}}, \sigma)$ actually comes from a line bundle $\mathscr{L}$ on X defined over $\mathbf{R}$. Two line bundles $\mathscr{L}$, $\mathscr{L}'$ on X defined over $\mathbf{R}$ are isomorphic if and only if there is an isomorphism of complex line bundles $f\colon \mathscr{L}_{\mathbf{C}} \xrightarrow{\sim} \mathscr{L}'_{\mathbf{C}}$ that is equivariant for the Galois action, i.e., $\sigma \circ f = f \circ \sigma$.

Definition 3.1. Let $X_{\mathbf{C}}$ be an algebraic variety defined over $\mathbf{C}$. The subgroup $\operatorname{Pic}^0(X_{\mathbf{C}}) \subset \operatorname{Pic}(X_{\mathbf{C}})$ of isomorphism classes of line bundles *algebraically equivalent to* 0 is the subgroup generated by line bundles of the form $\mathscr{L}_{t_0} \otimes \mathscr{L}_{t_1}^{-1}$, where t_0 and t_1 are closed points on a nonsingular irreducible variety T and $\mathscr{L}$ is a line bundle on $X_{\mathbf{C}} \times T$.

Here $\mathscr{L}_{t_i}$ is of course the restriction of $\mathscr{L}$ to the fibre $(X_{\mathbf{C}})_{t_i}$, which we identify with $X_{\mathbf{C}}$. For an algebraic variety X over $\mathbf{R}$ we define $\operatorname{Pic}^0(X) = (\pi^*)^{-1}\operatorname{Pic}^0(X_{\mathbf{C}})$. Of course, we define the group

$$\operatorname{Pic}^{0_{\mathbf{R}}}(X) \subset \operatorname{Pic}^0(X)$$

of isomorphism classes of line bundles *real algebraically equivalent to zero* as the subgroup generated by line bundles of the form $\mathscr{L}_{t_0} \otimes \mathscr{L}_{t_1}^{-1}$, where t_0 and t_1 are real points in the same connected component of the real part $T(\mathbf{R})$ of a nonsingular variety T and $\mathscr{L}$ is a line bundle on $X \times T$ defined over $\mathbf{R}$.

Lemma 3.2. *Let X be an algebraic variety defined over $\mathbf{R}$. Then the homomorphism*

$$\operatorname{div}\colon \operatorname{Pic}(X) \to CH^1(X)$$

maps $\operatorname{Pic}^0(X)$ *into* $CH^1_{(0)}(X)$ *and* $\operatorname{Pic}^{0_{\mathbf{R}}}(X)$ *into* $\mathscr{Z}^1_{\mathbf{R}\text{-alg}}(X)/\mathscr{Z}^1_{\mathrm{rat}}(X)$.

PROOF. Any two points on a nonsingular irreducible variety over $\mathbf{C}$ can be connected by a chain of nonsingular irreducible curves, and for any two points t_1, t_2 in a connected component of the real part $T(\mathbf{R})$ of a nonsingular irreducible variety T defined over $\mathbf{R}$ we can find (using the Stone–Weierstrass Theorem and Bertini's Theorem) a nonsingular curve $C \subset T$ such that t_1, t_2 are in the same connected component of $C(\mathbf{R})$. This means that the proof is easily reduced to the statement that if $K = \mathbf{C}$, (resp. $\mathbf{R}$), X is a variety over K, C is a nonsingular curve over K and $\mathscr{L} \in \operatorname{Pic}(X \times C)$, then fixing $t_0, t_1 \in C(K)$, there is a subscheme $V \subset X \times C$ of pure dimension $n = \dim X$ that is flat over C, and such that $[V_{t_i}] = \operatorname{div}\mathscr{L}_{t_i}$ in $CH^1(X_{t_i})$ for $i = 0, 1$.

The subscheme $V \subset X \times C$ is constructed in the following way. We can find a Cartier divisor D given by a collection (U_i, f_i), such that the image of D in $H^1(X \times C, \mathscr{O}^*)$ equals $\mathscr{L}$. Without loss of generality, we may assume that no f_i is zero when restricted to X_{t_0} or X_{t_1}. This implies that if we write $W = \operatorname{div}(D)$, viewed as a subscheme of $X \times C$, then the fibres W_{t_0} and W_{t_1} are of pure dimension $n-1$, and $[W_{t_i}] = \operatorname{div} \mathscr{L}_{t_i}$ in $CH^1(X_{t_i})$ for $i = 0, 1$. Now W need not be flat over C, but if we take $V \subset W$ to be the closed subscheme consisting of the irreducible components that project dominantly onto C, then $V_{t_i} = W_{t_i}$ for $i = 0, 1$ and V is flat over C. □

4. The Picard variety

Let X be a nonsingular irreducible variety over $\mathbf{R}$. For any variety T over $\mathbf{R}$ we define $\operatorname{Pic}^0(X/\mathbf{R})(T)$ to be the group $\operatorname{Pic}^0(X \times T)/p_1^* \operatorname{Pic}^0(T)$, where $p_1 \colon X \times T \to T$ is the projection. Observe that for any $x \in X(\mathbf{R})$ we have a canonical isomorphism between $\operatorname{Pic}^0(X/\mathbf{R})(T)$ and the subgroup of $\operatorname{Pic}^0(X \times T)$ consisting of isomorphism classes of invertible sheaves that are trivial when restricted to $\{x\} \times T$.

The pull-back associated to a morphism $S \to T$ of algebraic varieties defined over $\mathbf{R}$ induces a homomorphism $\operatorname{Pic}^0(X/\mathbf{R})(T) \to \operatorname{Pic}^0(X/\mathbf{R})(S)$, so $\operatorname{Pic}^0(X/\mathbf{R})$ defines a functor from the category of algebraic varieties over $\mathbf{R}$ to the category of abelian groups. If X is a complete nonsingular irreducible variety defined over $\mathbf{R}$ and $X(\mathbf{R}) \neq \emptyset$, this functor is known to be representable; there is a variety P over $\mathbf{R}$, which is in fact irreducible, and an isomorphism of functors

$$\operatorname{Pic}^0(X/\mathbf{R})(T) \simeq \operatorname{Hom}(T, P).$$

More concretely, this means that if we fix an $x_0 \in X(\mathbf{R})$, there is a *universal line bundle* $\mathscr{P}_X$ defined over $\mathbf{R}$ on $X \times P$ which is trivial when restricted to $\{x_0\} \times P$. The line bundle $\mathscr{P}_X$ is irreducible in the sense that for any variety T over $\mathbf{R}$, and any line bundle $\mathscr{L}$ on $X \times T$ defined over $\mathbf{R}$ that is trivial when restricted to $\{x_0\} \times T$, there is a unique morphism $\varphi \colon T \to P$ of varieties over $\mathbf{R}$, such that $(\mathrm{id} \times \varphi)^* \mathscr{P}_X$ is isomorphic to $\mathscr{L}$. We call P the *Picard variety* of X and $\mathscr{P}_X$ the *Poincaré bundle* of X. By abuse of notation, the Picard variety of X will be denoted by $\operatorname{Pic}^0(X/\mathbf{R})$ as well.

It is well-known that $\operatorname{Pic}^0(X/\mathbf{R})$ is an *abelian variety* over $\mathbf{R}$ of dimension $g = \dim_{\mathbf{R}} H^1(X, \mathscr{O}_X)$). (In fact, it is easily deduced from the properties of the Picard functor that $\operatorname{Pic}^0(X/\mathbf{R})$ is a complete commutative group variety). Since the group $\operatorname{Pic}^0(X)$ is isomorphic to the group of real points of $\operatorname{Pic}^0(X/\mathbf{R})$, it follows from well-known facts on the structure of real abelian varieties that

$$\operatorname{Pic}^0(X) \simeq (\mathbf{R}/\mathbf{Z})^g \oplus (\mathbf{Z}/2)^h,$$

where h is the number of connected components of $\operatorname{Pic}^0(X/\mathbf{R})(\mathbf{R})$. This result in itself could have been derived by more elementary means than using the representability of the Picard functor (see Section IV.4), but the lemma below uses the representability in its full power.

Lemma 4.1. *Let X be a complete, nonsingular variety over* $\mathbf{R}$ *with* $X(\mathbf{R}) \neq \emptyset$. *Let* P_0 *be the connected component of* $\mathrm{Pic}^0(X/\mathbf{R})(\mathbf{R})$ *containing* 0. *The canonical isomorphism*

$$\mathrm{Pic}^0(X/\mathbf{R})(\mathbf{R}) \xrightarrow{\sim} \mathrm{Pic}^0(X)$$

induces an isomorphism

$$P_0 \xrightarrow{\sim} \mathrm{Pic}^{0_{\mathbf{R}}}(X)$$

Proof. The existence of the Poincaré bundle $\mathscr{P}_X$ on $X \times \mathrm{Pic}^0(X/\mathbf{R})$ implies that P_0 maps to $\mathrm{Pic}^{0_{\mathbf{R}}}(X)$. The surjectivity of the mapping $P_0 \to \mathrm{Pic}^{0_{\mathbf{R}}}(X)$ follows from the fact that $\mathscr{P}_X$ is the universal line bundle. Namely, let T be a nonsingular irreducible variety T over $\mathbf{R}$, let $\mathscr{L}$ be a line bundle on $X \times T$, let t_0, t_1 be any pair of points in the same connected component of $T(\mathbf{R})$. We have that $\mathscr{L}' = \mathscr{L} \otimes p_1^* \mathscr{L}_{t_0}^{-1} \in \mathrm{Pic}^0(X/\mathbf{R})(T)$, where $p_1 \colon X \times T \to X$ is the projection. Now the morphism from T into the Picard variety $\mathrm{Pic}^0(X/\mathbf{R})$ induced by $\mathscr{L}'$ sends t_0 to 0 and t_1 to some other point in P_0, since the induced mapping $T(\mathbf{R}) \to \mathrm{Pic}^0(X/\mathbf{R})(\mathbf{R})$ is continuous. Hence $\mathscr{L}'_{t_1} = \mathscr{L}_{t_1} \otimes \mathscr{L}_{t_0}^{-1}$ corresponds to a point in P_0. □

As a corollary of the above lemma we get the real analogue of the fact that $CH_k^{(0)}(X_{\mathbf{C}})$ is divisible for any $k \geq 0$ and any irreducible variety $X_{\mathbf{C}}$ over $\mathbf{C}$.

Corollary 4.2. *For any irreducible variety X over* $\mathbf{R}$ *the group* $CH_k^{(0)_{\mathbf{R}}}(X)$ *is divisible.*

Proof. Let U be a nonsingular irreducible curve over $\mathbf{R}$, let t_0, t_1 be two points in the same connected component of $U(\mathbf{R})$, and let $V \subset X \times U$ be a subvariety which is flat of relative dimension k over U. We will show that for any integer m there is a cycle class $\gamma \in CH_k(X)$ such that $m\gamma = [V_{t_0} - V_{t_1}]$.

Let $C \supset U$ be the nonsingular projective closure of U, and let $W \subset X \times C$ be the Zariski-closure of V. Then W is flat over C of relative dimension k. Let $p_1 \colon X \times C \to X$ and $p_2 \colon X \times C \to C$ be the projections. Since p_1 restricted to W is proper, it induces a homomorphism $(p_1)_* \colon CH_k(W) \to CH_k(X)$. Clearly $(p_1)_*[W_{t_0} - W_{t_1}] = [V_{t_0} - V_{t_1}]$, but $[W_{t_0} - W_{t_1}] = p_2^*[t_0 - t_1]$ and by Lemma 4.1 we have a class $\gamma' \in CH^1(C)$ such that $m\gamma' = [t_0 - t_1]$, hence $m(p_1)_* p_2^* \gamma' = [V_{t_0} - V_{t_1}]$. □

Corollary 4.3. *Let X be a geometrically irreducible algebraic variety over* $\mathbf{R}$. *For any $k \geq 0$ the homomorphism*

$$\pi_* \colon CH_k^{(0)}(X_{\mathbf{C}}) \to CH_k^{(0)_{\mathbf{R}}}(X)$$

is surjective.

Proof. Since the mapping $\pi_* \circ \pi^* \colon CH_k^{(0)_{\mathbf{R}}}(X) \to CH_k^{(0)_{\mathbf{R}}}(X)$ is multiplication by 2, this follows immediately from Corollary 4.2. □

5. Cycle maps

Maps from the groups of algebraic cycles on a variety X into appropriate homology groups or cohomology groups associated to X are usually called *cycle maps*. In the case of a variety X over $\mathbf{R}$, Borel and Haefliger defined in [BH] two cycle maps, one into the Borel-Moore homology of the set of complex points, and one into the Borel-Moore homology of the real part. The use of Borel-Moore homology means that we can work with varieties that are not necessarily complete. A definition of Borel-Moore homology and an extensive treatment of its properties can be found in Section III.1 (take G to be the trivial group). Here I will list a few important properties.

Throughout this work the Borel-Moore homology (originally called *homology with closed supports*) of a locally compact space X with coefficients in a (noetherian, commutative) ring A will be denoted by $H_*(X, A)$. Since we do not use ordinary homology (also called *homology with compact supports*), this notation should not give any confusion; besides, the two theories coincide when X is compact. Borel-Moore homology is covariant with respect to proper mappings and contravariant with respect to open embeddings. If $U \subset X$ is an open subset, we have a long exact sequence

(3) $\cdots \to H_k(X - U, A) \to H_k(X, A) \to H_k(U, A) \to H_{k-1}(X - U, A) \to \cdots$

For any closed subset $Z \subset X$ we have a cap product

$$\begin{array}{ccccc} H^p_Z(X, A) & \otimes & H_q(X, A) & \to & H_{q-p}(Z, A) \\ \omega & \otimes & \gamma & \mapsto & \omega \cap \gamma \end{array},$$

where $H^p_Z(X, A)$ is cohomology with supports in Z, also denoted by $H^k(X, X - Z; A)$. If X is an n-dimensional manifold (not necessarily compact or connected), and either X is oriented or $A = \mathbf{Z}/2$, then X has a *fundamental class* $\mu_X \in H_n(X, A)$ such that cap product with A induces the Poincaré duality isomorphism

$$H^k_Z(X, A) \xrightarrow{\sim} H_{n-k}(Z, A).$$

In particular, we have

$$H_k(\mathbf{R}^n, A) = \begin{cases} A & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

If Z is a nonsingular variety over $\mathbf{C}$ or over $\mathbf{R}$ of dimension k, then $Z(\mathbf{C})$ is a $2k$-dimensional manifold and the complex structure induces a natural orientation on $Z(\mathbf{C})$. Hence $Z(\mathbf{C})$ has a fundamental class $\mu_{Z(\mathbf{C})} \in H_{2k}(Z(\mathbf{C}), \mathbf{Z})$. If Z is actually a closed subvariety of a variety X, the inclusion $Z \hookrightarrow X$ induces a homomorphism in homology, and we can define $\mathrm{cl}^{\mathbf{C}}(Z) \in H_{2k}(X(\mathbf{C}), \mathbf{Z})$ to be the image of $\mu_{Z(\mathbf{C})}$. Similarly, if Z is a nonsingular variety over $\mathbf{R}$ of dimension k, $Z(\mathbf{R})$ is a k-dimensional manifold. However, $X(\mathbf{R})$ need not be orientable, and even if it is orientable there is no canonical orientation. Hence $Z(\mathbf{R})$ only has a fundamental class $\mu_{Z(\mathbf{R})} \in H_k(Z(\mathbf{R}), \mathbf{Z}/2)$. If Z is actually a closed subvariety of a variety X over $\mathbf{R}$, we define $\mathrm{cl}^{\mathbf{R}}(Z) \in H_k(X(\mathbf{R}), \mathbf{Z}/2)$ to be the image of $\mu_{Z(\mathbf{R})}$ under the homomorphism induced by the inclusion.

In order to extend the definition of $\mathrm{cl}^{\mathbf{C}}$ and $\mathrm{cl}^{\mathbf{R}}$ to arbitrary subvarieties of X, it is necessary to have fundamental classes for $Z(\mathbf{C})$ and $Z(\mathbf{R})$ when Z is singular. Let Z_{s} be the singular locus of Z. Then $\dim Z_{\mathrm{s}} \leq k-1$, where k is the dimension of Z, hence $H_p(Z_{\mathrm{s}}(\mathbf{C}), \mathbf{Z}) = 0$ for $p > 2k-2$. Putting $Z_{\mathrm{r}} = Z - Z_{\mathrm{s}}$, we see from the long exact sequence (3) that the restriction $H_{2k}(Z(\mathbf{C}), \mathbf{Z}) \to H_{2k}(Z_{\mathrm{r}}(\mathbf{C}), \mathbf{Z})$ is an isomorphism, and we define $\mu_{Z(\mathbf{C})}$ to be the inverse image of $\mu_{Z_{\mathrm{r}}(\mathbf{C})}$.

When we apply the long exact sequence (3) to the embedding $Z_{\mathrm{r}}(\mathbf{R}) \subset Z(\mathbf{R})$, we get an exact sequence

$$0 \to H_k(Z(\mathbf{R}), \mathbf{Z}/2) \to H_k(Z_{\mathrm{r}}(\mathbf{R}), \mathbf{Z}/2) \to H_{k-1}(Z_{\mathrm{s}}(\mathbf{R}), \mathbf{Z}/2) \to \cdots$$

It follows that if there is a class $\mu_{Z(\mathbf{R})}$ which restricts to $\mu_{Z_{\mathrm{r}}(\mathbf{R})}$, this class is unique. However, $H_{k-1}(Z_{\mathrm{s}}(\mathbf{R}), \mathbf{Z}/2)$ need not be zero, so we still have to prove that $\mu_{Z(\mathbf{R})}$ actually exists. It should be said that if $Z_{\mathrm{r}}(\mathbf{R}) = \emptyset$, we define $\mu_{Z_{\mathrm{r}}(\mathbf{R})} = 0$ and $\mu_{Z(\mathbf{R})} = 0$. When $Z_{\mathrm{r}}(\mathbf{R}) \neq \emptyset$, Borel and Haefliger consider the normalization $\varphi\colon \tilde{Z} \to Z$. The normality of $\tilde{Z}$ implies that the singular locus $\tilde{Z}_{\mathrm{s}}$ is of dimension $\leq k-2$, hence the above long exact sequence shows that $\mu_{\tilde{Z}(\mathbf{R})}$ exists. Then $\varphi_* \mu_{\tilde{Z}(\mathbf{R})} \in H_k(Z(\mathbf{R}), \mathbf{Z}/2)$ is the fundamental class of $Z(\mathbf{R})$.

Thus we have, for any algebraic variety X over $\mathbf{R}$, homomorphisms

$$\mathrm{cl}^{\mathbf{C}}_{X_{\mathbf{C}}} : \mathscr{Z}_k(X_{\mathbf{C}}) \to H_{2k}(X(\mathbf{C}), \mathbf{Z}), \tag{4}$$

$$\mathrm{cl}^{\mathbf{C}}_{X} : \mathscr{Z}_k(X) \to H_{2k}(X(\mathbf{C}), \mathbf{Z}), \tag{5}$$

and

$$\mathrm{cl}^{\mathbf{R}}_{X} : \mathscr{Z}_k(X) \to H_k(X(\mathbf{R}), \mathbf{Z}/2). \tag{6}$$

It can be checked, that $\mathscr{Z}_k^{\mathrm{alg}}(X_{\mathbf{C}}) \subset \operatorname{Ker} \mathrm{cl}^{\mathbf{C}}_{X_{\mathbf{C}}}$, (see for example [Fu, 19.1.1]), hence $\mathscr{Z}_k^{\mathrm{alg}}(X) \subset \operatorname{Ker} \mathrm{cl}^{\mathbf{C}}_{X}$. Similarly, $\mathscr{Z}_k^{\mathbf{R}\text{-alg}}(X) \subset \operatorname{Ker} \mathrm{cl}^{\mathbf{R}}_{X}$ (see also Lemma IV.1.2). In particular, $\mathrm{cl}^{\mathbf{C}}$ and $\mathrm{cl}^{\mathbf{R}}$ are well-defined on $CH_k(X)$. When confusion is unlikely, we will omit X and $X_{\mathbf{C}}$ from the notation of the cycle maps. We define

$$\mathrm{cl}^{\mathbf{C}}(\mathscr{Z}_k(X_{\mathbf{C}})) = H_{2k}^{\mathrm{alg}}(X(\mathbf{C}), \mathbf{Z}),$$

and

$$\mathrm{cl}^{\mathbf{R}}(\mathscr{Z}_k(X)) = H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2).$$

In the survey [BK2] the reader can find references to the literature concerning the groups $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$. Very important for a better understanding of the groups $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$ is the fact that both $\mathrm{cl}^{\mathbf{C}}$ and $\mathrm{cl}^{\mathbf{R}}$ factor via a third cycle map

$$\mathrm{cl}_X : \mathscr{Z}_k(X) \to H_{2k}(X(\mathbf{C}); G, \mathbf{Z})$$

into equivariant Borel-Moore homology associated to the action of $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on X. The theory of equivariant Borel-Moore homology will be developed in the next two chapters, and the equivariant cycle map will be defined in Chapter IV.

Chapter II

Equivariant sheaves and Verdier duality

For the definition in Chapter III of equivariant Borel-Moore homology and the study of the connections with the existing equivariant sheaf cohomology theory as defined in [Gr, Chap. V], we need some technical preparation. Although it is very well possible to define equivariant Borel-Moore homology and equivariant cohomology by more elementary means, for example using simplicial or singular chain complexes, it turns out to be complicated to prove that these theories have the desired properties. In particular, cup product and cap product then give calculations which are difficult to handle. Therefore I have chosen to develop the theory in the framework of derived categories. There it is much easier to use the fact that both equivariant Borel-Moore homology and equivariant sheaf cohomology are rather straightforward generalizations of the corresponding nonequivariant theories.

Since for a large part of the theory it does not matter whether the transformation group G is $\mathbf{Z}/2$ or another group, I have chosen not to restrict to the case $G = \mathbf{Z}/2$. It should be stressed, however, that although the theory developed here is well-defined for any group, it only works well for discrete transformation groups, since in any application the results of Section III.6 will be needed, and there it is crucial that G acts properly discontinuously on X.

1. G-sheaves

Let G be a group and let X be a topological space with a left action of G. This means that for every $g \in G$ we have an automorphism $\varphi_g = \varphi_g^X$ of X with the conditions:

$$\varphi_1 = \mathrm{id}$$

$$\varphi_g \circ \varphi_h = \varphi_{gh} \text{ for every } g, h \in G.$$

We will often use the notation $g \cdot x$ for $\varphi_g(x)$, when $x \in X$. The quotient mapping will be denoted by

$$\pi \colon X \to X/G,$$

and the set $\{x \in X \colon g \cdot x = x \text{ for all } g \in G\}$ of fixed points will be denoted by $X^G \subset X$.

A sheaf on X will always be a sheaf of abelian groups. A *G-sheaf* $\mathcal{F}$ on X will be a sheaf on X with a G-action compatible with the G-action on X. In other words, we have for every $g \in G$ an isomorphism

$$\alpha_g^{\mathcal{F}} \colon \mathcal{F} \to \varphi_g^* \mathcal{F}$$

such that

$$\alpha_1^{\mathcal{F}} = \mathrm{id}$$
$$\varphi_h^*(\alpha_g^{\mathcal{F}}) \circ \alpha_h^{\mathcal{F}} = \alpha_{gh}^{\mathcal{F}} \text{ for every } g, h \in G.$$

Observe that on the presheaf level $\alpha_g^{\mathcal{F}}$ is given by isomorphisms $\alpha_g^{\mathcal{F}}(U) \colon \mathcal{F}(U) \to \mathcal{F}(g \cdot U)$, with the property that $\alpha_1^{\mathcal{F}} = \mathrm{id}$ and $\alpha_g^{\mathcal{F}}(h \cdot U) \circ \alpha_h^{\mathcal{F}}(U) = \alpha_{gh}^{\mathcal{F}}(U)$ for every $g, h \in G$.

For a commutative ring A, which we always assume to be unitary, it is clear what we mean by a *G-sheaf of A-modules* on a G-space X (or *G-A-module on* X for short). A homomorphism of G-A-modules on X is a homomorphism $h \colon \mathcal{E} \to \mathcal{F}$ of sheaves of A-modules on X such that

$$\varphi_g^*(h) \circ \alpha_g^{\mathcal{E}} = \alpha_g^{\mathcal{F}} \circ h.$$

We then say that h is *equivariant*.

The category of sheaves of A-modules on a space X will be denoted by $A\text{-}\mathfrak{Mod}(X)$, and the category of G-A-modules on a G-space X by $A\text{-}\mathfrak{Mod}_G(X)$. If X is a point we get categories $A\text{-}\mathfrak{Mod}$ (resp. $A\text{-}\mathfrak{Mod}_G$) of A-modules (resp. G-A-modules). All these categories are abelian categories with infinite direct sums and products. We use the notation $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ for homomorphisms in the categories without G-action and $\mathrm{Hom}_G(\mathcal{M}, \mathcal{N})$ for the equivariant homomorphisms.

2. Equivariant functors

Let X and Y be two G-spaces, and let A and B be commutative rings. An additive functor $F \colon A\text{-}\mathfrak{Mod}(X) \to B\text{-}\mathfrak{Mod}(Y)$ is said to be an *equivariant functor* if it comes together with a collection of functor isomorphisms

$$F \circ (\varphi_g^X)^* \simeq (\varphi_g^Y)^* \circ F$$

indexed by $g \in G$, satisfying the obvious associativity relations.

An equivariant functor $F\colon A\text{-}\mathfrak{Mod}(X) \to B\text{-}\mathfrak{Mod}(Y)$ induces in a natural way a functor $A\text{-}\mathfrak{Mod}_G(X) \to B\text{-}\mathfrak{Mod}_G(Y)$, which we will denote by F as well. For example, if F is covariant, then for G-A-module $\mathscr{M}$ on X, the G-action on $F(M)$ is defined by

$$\alpha_g^{F(\mathscr{M})} = F(\alpha_g^{\mathscr{M}}).$$

Similarly, if $F\colon A\text{-}\mathfrak{Mod}(X) \times A\text{-}\mathfrak{Mod}(Y) \to A\text{-}\mathfrak{Mod}(Z)$ is a bifunctor, then we say that F is equivariant if we have an isomorphism of bifunctors

$$F((\varphi_g^X)^*(-), (\varphi_g^Y)^*(-)) = (\varphi_g^Z)^* \circ F.$$

If F is equivariant, it induces a bifunctor

$$A\text{-}\mathfrak{Mod}_G(X) \times A\text{-}\mathfrak{Mod}_G(Y) \to A\text{-}\mathfrak{Mod}_G(Z).$$

If F is covariant in both variables, the G-action on $F(\mathscr{M}, \mathscr{N})$ is given by

$$\alpha_g = F(\alpha_g^{\mathscr{M}}, \alpha_g^{\mathscr{N}});$$

if F is contravariant in the first, and covariant in the second variable, then the G-action on $F(\mathscr{M}, \mathscr{N})$ is given by

$$\alpha_g = F(\varphi_g^*(\alpha_{g^{-1}}^{\mathscr{M}}), \alpha_g^{\mathscr{N}}).$$

Let $f\colon X \to Y$ be a continuous mapping. Then we have the *direct image functor*

$$f_*\colon A\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}(Y),$$

which sends a sheaf of A-modules $\mathscr{E}$ on X to the sheaf of A-modules on Y given by $f_*\mathscr{F}(U) = \mathscr{E}(f^{-1}(U))$. The *inverse image functor*

$$f^*\colon A\text{-}\mathfrak{Mod}(Y) \to A\text{-}\mathfrak{Mod}(X),$$

sends a sheaf of A-module $\mathscr{F}$ on Y to the canonical sheaf associated to the presheaf of A-modules on X given by $U \mapsto \varinjlim_{V \supset f(U)} \mathscr{F}(V)$, where V ranges over the open sets on Y containing U.

If $f\colon X \to Y$ is a morphism of G-spaces (i.e., a continuous equivariant mapping), then f_* and f^* are in a canonical way equivariant functors, so they induce covariant functors

$$f_*\colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G(Y)$$

and

$$f^*\colon A\text{-}\mathfrak{Mod}_G(Y) \to A\text{-}\mathfrak{Mod}_G(X).$$

When $f\colon X \to \mathbf{pt}$ is the constant mapping form a G-space X to a point, then f_* is the *global sections functor*

$$\Gamma(X, -)\colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G,$$

and f^*M is the *constant sheaf associated to the G-A-module M*; we will usually write M instead of f^*M.

Recall that a continuous mapping $f\colon X \to Y$ between locally compact spaces is *proper* if it is closed and the inverse image of every compact subset of Y is compact. If $f\colon X \to Y$ is an arbitrary continuous morphism between locally compact spaces, and $\mathscr{F}$ is a sheaf on X, then the presheaf

$$U \mapsto \{s \in \mathscr{F}(f^{-1}U)\colon \text{ supp}(s) \text{ is proper over } U\}$$

is a subsheaf of $f_*\mathscr{F}$, called the *direct image with proper supports* of $\mathscr{F}$ under f, and it is denoted by $f_!\mathscr{F}$. This defines a functor

$$f_!\colon A\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}(Y).$$

If f is a morphism of G-spaces, the functor $f_!$ is canonically equivariant, so it induces a functor

$$f_!\colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G(Y).$$

Observe that if f is the constant mapping, then $f_!$ is the functor of *global sections with compact support*

$$\Gamma_c(X,-)\colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G.$$

Remark 2.1. If $j\colon W \to X$ is the inclusion of a locally closed G-subspace and $\mathscr{F}$ is a G-sheaf on W, then

$$(j_!\mathscr{F})(U) = \{s \in \mathscr{F}(U \cap W)\colon \text{ supp}(s) \text{ is closed in U}\}.$$

When X is not locally compact, this defines of $j_!$ for this special kind of mappings. For a sheaf $\mathscr{F}$ on X we often write

$$\mathscr{F}_W = j_!j^*\mathscr{F};$$

it is the sheaf with stalk $\mathscr{F}_x$ for $x \in W$ and stalk 0 at $x \notin W$.

If $\mathscr{M}$ and $\mathscr{N}$ are two sheaves of A-modules on X, then the sheaf $\mathscr{H}om_A(\mathscr{M},\mathscr{N})$ is defined by

$$U \mapsto \operatorname{Hom}_A(j^*\mathscr{M}, j^*\mathscr{N}),$$

where $j\colon U \hookrightarrow X$ is the inclusion. The A-module structure on $\mathscr{N}$ induces the A-module structure on $\mathscr{H}om(\mathscr{M},\mathscr{N})$. This construction is contravariantly functorial in the first variable and covariantly functorial in the second variable, so we get a bifunctor

$$\mathscr{H}om\colon A\text{-}\mathfrak{Mod}(X) \times A\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}(X).$$

If G acts on X, the natural isomorphisms

$$\varphi_g^*\mathscr{H}om(\mathscr{M},\mathscr{N}) \simeq \mathscr{H}om(\varphi_g^*\mathscr{M}, \varphi_g^*\mathscr{N})$$

make $\mathscr{H}om$ an equivariant bifunctor, so it induces a bifunctor

$$\mathscr{H}om(-,-)\colon A\text{-}\mathfrak{Mod}_G(X) \times A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G(X),$$

which is contravariant in the first variable, covariant in the second variable and left exact in both variables.

Similarly we consider Hom as a bifunctor

$$A\text{-}\mathfrak{Mod}_G(X) \times A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G.$$

We have an isomorphisms of bifunctors

$$\Gamma(X, \mathcal{H}om_A(-,-)) \simeq \mathrm{Hom}_A(-,-) \tag{7}$$

and

$$\mathrm{Hom}_G(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}(\mathcal{M}, \mathcal{N})^G, \tag{8}$$

where L^G is the A-module of G-invariant elements of a G-A-module L. The functor sending L to L^G will be denoted by

$$\Gamma^G\colon A\text{-}\mathfrak{Mod}_G \to A\text{-}\mathfrak{Mod}.$$

The tensor product $\mathcal{M} \otimes_A \mathcal{N}$ of two sheaves of A-modules on a space X is the sheaf associated to the presheaf

$$U \mapsto \mathcal{M}(U) \otimes_A \mathcal{N}(U).$$

This defines a bifunctor

$$A\text{-}\mathfrak{Mod}(X) \times A\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}(X)$$

which is covariant in both variables and equivariant if X is a G-space, so we can extend it to a bifunctor

$$A\text{-}\mathfrak{Mod}_G(X) \times A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G(X).$$

2.1. Adjunction properties

In this section we will study the adjoints of various functors defined above. Recall, that if $S\colon \mathfrak{A} \to \mathfrak{B}$ and $T\colon \mathfrak{B} \to \mathfrak{A}$ are covariant functors, then we say that S is the *left adjoint functor* of T and T is the *right adjoint functor* of S if there is a natural transformation of bifunctors

$$\mathrm{Hom}_{\mathfrak{A}}(M, T(N)) \simeq \mathrm{Hom}_{\mathfrak{B}}(S(M), N).$$

In order to extend the known adjunction properties of the functors defined above to the equivariant setting we will use the following lemma.

Lemma 2.2. *Let X and Y be G-spaces, let A and B be commutative rings and let*

$$S\colon A\text{-}\mathfrak{Mod}(X) \to B\text{-}\mathfrak{Mod}(Y)$$

be a covariant equivariant functor with right adjoint

$$T\colon B\text{-}\mathfrak{Mod}(Y) \to A\text{-}\mathfrak{Mod}(X),$$

which is assumed to be equivariant as well. The following three statements are equivalent:

(i) *For every G-A-module $\mathscr{M}$ on X and every G-B-module $\mathscr{N}$ on Y the canonical isomorphism*

$$\mathrm{Hom}(S(\mathscr{M}),\mathscr{N}) \simeq \mathrm{Hom}(\mathscr{M}, T(\mathscr{N}))$$

induces an isomorphism

$$\mathrm{Hom}_G(S(\mathscr{M}),\mathscr{N}) \simeq \mathrm{Hom}_G(\mathscr{M}, T(\mathscr{N}))$$

(ii) *The adjunction morphism $\mathscr{M} \to T(S(\mathscr{M}))$ is equivariant for every G-A-module $\mathscr{M}$ on X.*

(iii) *The adjunction morphism $S(T(\mathscr{N})) \to \mathscr{N}$ is equivariant for every G-B-module $\mathscr{N}$ on Y.*

PROOF. (i) $\Rightarrow$ (ii). Since the adjunction morphism $\psi_{\mathscr{M}}\colon \mathscr{M} \to T(S(\mathscr{M}))$ is the image of $\mathrm{id}_{\mathscr{M}} \in \mathrm{Hom}(S(\mathscr{M}), S(\mathscr{M}))$ under the isomorphism $\mathrm{Hom}(S(\mathscr{M}), S(\mathscr{M})) \to \mathrm{Hom}(\mathscr{M}, T(S(\mathscr{M}))$, and $\mathrm{id}_{\mathscr{M}}$ is in $\mathrm{Hom}_G(S(\mathscr{M}), S(\mathscr{M}))$, the hypothesis implies that $\psi_{\mathscr{M}}$ is in $\mathrm{Hom}_G(\mathscr{M}, T(S(\mathscr{M})))$.

(ii) $\Rightarrow$ (i). Since $\mathrm{Hom}_G(-,-) = \mathrm{Hom}(-,-)^G$, it is sufficient to prove that for a G-A-module $\mathscr{M}$ on X and a G-B-module $\mathscr{N}$ on Y the natural isomorphism

$$\mathrm{Hom}(S(\mathscr{M}),\mathscr{N}) \xrightarrow{\sim} \mathrm{Hom}(\mathscr{M}, T(\mathscr{N}))$$

respects the G-action on both groups. Since T is an equivariant functor, the natural mapping

$$\mathrm{Hom}(S(\mathscr{M}),\mathscr{N}) \to \mathrm{Hom}(T(S(\mathscr{M})), T(\mathscr{N}))$$

respects the G-action. By hypothesis the adjunction morphism $\mathscr{M} \to T(S(\mathscr{M}))$ is equivariant, so the induced mapping

$$\mathrm{Hom}(T(S(\mathscr{M})), T(\mathscr{N})) \to \mathrm{Hom}(\mathscr{M}, T(\mathscr{N}))$$

is equivariant, hence the composite isomorphism

$$\mathrm{Hom}(S(\mathscr{M}),\mathscr{N}) \to \mathrm{Hom}(T(S(\mathscr{M})), T(\mathscr{N})) \to \mathrm{Hom}(\mathscr{M}, T(\mathscr{N})),$$

respects the G-action.

(i) $\Leftrightarrow$ (iii) As above. $\square$

Now we will prove some adjunction formulas.

Proposition 2.3. *For any group G, any commutative ring A, and any morphism of G-spaces $f\colon X \to Y$, the functor*

$$f^*\colon A\text{-}\mathfrak{Mod}_G(Y) \to A\text{-}\mathfrak{Mod}_G(X)$$

is left adjoint to the functor

$$f_*\colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G(Y).$$

PROOF. Since f^* and f_* are adjoint when considered as functors between $A\text{-}\mathfrak{Mod}(X)$ and $A\text{-}\mathfrak{Mod}(Y)$, we only have to prove that the adjunction homomorphism $\psi\colon \mathscr{M} \to f_*f^*\mathscr{M}$ is equivariant, but if $\mathscr{P}$ denotes the G-presheaf $U \mapsto \lim_{V\supset f(U)} \mathscr{M}(V)$ on X, with the canonical mapping $\theta\colon \mathscr{P} \to f^*\mathscr{M}$, then $f_*\mathscr{P}$ is just $\mathscr{M}$ and $\psi = f_*(\theta)$, which shows that ψ is equivariant. □

Remark 2.4. If $i\colon W \hookrightarrow X$ is the inclusion of a locally closed subspace, then, as in the nonequivariant setting (see [Iv, Prop. II.6.6]), we have a right adjoint

$$i^!\colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G(W)$$

to the functor

$$i_!\colon A\text{-}\mathfrak{Mod}_G(W) \to A\text{-}\mathfrak{Mod}_G(X),$$

defined by

$$i^!\mathscr{F} = i^*\mathscr{H}om(i_!A, \mathscr{F}).$$

If W is open, then $i^! = i^*$. For arbitrary morphisms f see Section 5.

Proposition 2.5. *Let G be any group, let X be a G-space, let A be a commutative ring and let B be a commutative A-algebra. The functor*

$$B \underset{A}{\otimes} -\colon A\text{-}\mathfrak{Mod}_G(X) \to B\text{-}\mathfrak{Mod}_G(X)$$

is left adjoint to the restriction of scalars

$$B\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}_G(X).$$

PROOF. The functor

$$B \underset{A}{\otimes} -\colon A\text{-}\mathfrak{Mod}(X) \to B\text{-}\mathfrak{Mod}_G(X)$$

is left adjoint to the restriction of scalars $B\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}(X)$ (cf. [Ve, App. 1, § 1.4]). Both functors are clearly equivariant and for a G-A-module $\mathscr{M}$ on X the adjunction homomorphism

$$\mathscr{M} \to B \underset{A}{\otimes} \mathscr{M}$$

is given by

$$\begin{array}{rcl} \mathscr{M}(U) & \to & B \otimes_A \mathscr{M}(U) \\ m & \mapsto & 1 \otimes m, \end{array}$$

so it is clearly equivariant. □

Proposition 2.6. *Let G be any group, let X be a space on which G acts trivially, and let A be a commutative ring. Then the functor*

$$\mathscr{H}om_G(A, -)\colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}(X)$$

is right adjoint to the inclusion

$$A\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}_G(X).$$

PROOF. The claim is that for every sheaf of A-modules $\mathscr{M}$ on X and every G-A-module $\mathscr{N}$ on X there is a natural isomorphism

$$\mathrm{Hom}_G(\mathscr{M},\mathscr{N}) \simeq \mathrm{Hom}(\mathscr{M},\mathscr{H}om_G(A,\mathscr{N})),$$

where on the left hand side, $\mathscr{M}$ is equipped with the trivial G-action. The isomorphism follows from the obvious fact that

$$\mathrm{Hom}_G(\mathscr{M}(U),\mathscr{N}(U)) \simeq \mathrm{Hom}(\mathscr{M}(U),\mathscr{N}(U)^G)$$

for any open $U \subset X$. □

Corollary 2.7. *Let G be any group, let X be a G-space, and let A be a commutative ring. The functor*

$$\pi_*^G \colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}(X/G)$$

is right adjoint to the functor

$$\pi^* \colon A\text{-}\mathfrak{Mod}(X/G) \to A\text{-}\mathfrak{Mod}_G(X).$$

PROOF. This follows from Proposition 2.6 and Proposition 2.3, since

$$\pi_*^G = \mathscr{H}om_G(A,-)\circ \pi_*.$$

□

Proposition 2.8. *Let G be any group, let A be a commutative ring and let X be a G-space on which G acts trivially. The functor*

$$A \underset{G}{\otimes} - \colon A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}(X)$$

is left adjoint to the inclusion

$$A\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}_G(X).$$

PROOF. It is not hard to check, that for any G-sheaf of A-modules $\mathscr{M}$ and any sheaf of A-modules $\mathscr{N}$ with trivial G-action we have a natural isomorphism

$$\mathrm{Hom}_G(\mathscr{M},\mathscr{N}) \simeq \mathrm{Hom}_A(A \underset{G}{\otimes} \mathscr{M},\mathscr{N});$$

it also follows from [Ve, App. 1, § 1.4], since we can consider a G-sheaf of A-modules on X as an $A[G]$-module. □

Finally, we will define for any G-space X two functors which are the left and right adjoints of the forgetful functor $A\text{-}\mathfrak{Mod}_G(X) \to A\text{-}\mathfrak{Mod}(X)$.

Definition 2.9. Let $\mathscr{F}$ be a sheaf of A-modules on a G-space X. The *induced G-A-module* $\mathrm{Ind}^G \mathscr{F}$ is the sheaf of A-modules

$$\bigoplus_{g\in G} \varphi_g^* \mathscr{F},$$

the *co-induced* G*-*A*-module* $\operatorname{Coind}^G \mathscr{F}$ will be the sheaf of A-modules

$$\prod_{g\in G} \varphi_g^* \mathscr{F}$$

and both are endowed with the obvious G-action.

Proposition 2.10. *Let G be any group, let A be any ring, and let X be a G-space. Then the functor*

$$\operatorname{Ind}^G : A\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}_G(X)$$

is left adjoint to the forgetful functor, and the functor

$$\operatorname{Coind}^G : A\text{-}\mathfrak{Mod}(X) \to A\text{-}\mathfrak{Mod}_G(X)$$

is right adjoint to the forgetful functor.

Proof. Easy. □

Remark 2.11. Observe that if $\mathscr{F}$ is a G-A-module on X, the collection indexed by $g \in G$ of composed homomorphisms

$$\mathscr{H}om_A(A[G], \mathscr{F}) \xrightarrow{(i_{g^{-1}}, \mathrm{id})} \mathscr{H}om_A(A, \mathscr{F}) \simeq \mathscr{F} \xrightarrow{\alpha_g^{\mathscr{F}}} \varphi_g^* \mathscr{F}$$

defines an isomorphism of G-A-modules

$$\mathscr{H}om_A(A[G], \mathscr{F}) \xrightarrow{\sim} \operatorname{Coind}^G \mathscr{F}.$$

Here $i_g \colon A \hookrightarrow A[G]$ is defined to be injection $a \mapsto ag$. The collection of composed homomorphisms

$$\varphi_g^* \mathscr{F} \xrightarrow{\varphi_g^*(\alpha_{g^{-1}}^{\mathscr{F}})} \mathscr{F} \xrightarrow{i_g \otimes \mathrm{id}} A \underset{A}{\otimes} \mathscr{F} \hookrightarrow A[G] \underset{A}{\otimes} \mathscr{F}$$

defines an isomorphism

$$\operatorname{Ind}^G \mathscr{F} \xrightarrow{\sim} A[G] \underset{A}{\otimes} \mathscr{F}.$$

3. Derived functors

In this section the notions of derived categories and derived functors are recalled. A full treatment of the subject can be found in [Ha, Ch. I], or [Iv, Ch. XI].

Let $\mathfrak{A}$ be an abelian category and let $\mathscr{P}^\bullet$ be a complex

$$\dots \longrightarrow \mathscr{P}^{n-1} \xrightarrow{\partial^{n-1}} \mathscr{P}^n \xrightarrow{\partial^n} \mathscr{P}^{n+1} \xrightarrow{\partial^{n+1}} \mathscr{P}^{n+2} \longrightarrow \dots$$

We say that $\mathscr{P}^\bullet$ is *bounded* if $\mathscr{P}^n$ is nonzero for only a finite number of indices n. We call $\mathscr{P}^\bullet$ *bounded below* if $\mathscr{P}^n = 0$ for all n smaller than some $n_0 \in \mathbf{Z}$ and *bounded above* if $\mathscr{P}^n = 0$ for all n larger than some $n_0 \in \mathbf{Z}$. The category of complexes in $\mathfrak{A}$ will be denoted by $C(\mathfrak{A})$ and the categories of bounded, bounded below, and bounded above complexes in $\mathfrak{A}$ will be denoted by $C^b(\mathfrak{A})$, $C^+(\mathfrak{A})$, and $C^-(\mathfrak{A})$, respectively. We use the notation $C^*(\mathfrak{A})$ for any of these four categories. We will often silently identify the category $\mathfrak{A}$ with the full subcategory of $C^*(\mathfrak{A})$ consisting of complexes which are concentrated in degree zero. Let $K^*(\mathfrak{A})$ be the homotopy category corresponding to $C^*(\mathfrak{A})$.

The pth *translation functor* $\mathsf{T}^p\colon C^*(\mathfrak{A}) \to C^*(\mathfrak{A})$ is defined for any $p \in \mathbf{Z}$ by

$$\mathsf{T}^p(\mathscr{P}^\bullet)^n = \mathscr{P}^{n+p},$$
$$\partial^n_{T^p(\mathscr{P}^\bullet)} = (-1)^p \partial^{n+p}_{\mathscr{P}},$$

and $\mathsf{T}^p(f)$ is given by

$$\mathsf{T}^p(f)^n = f^{n+p}\colon \mathscr{P}^{n+p} \to \mathscr{Q}^{n+p}$$

for a morphism $f\colon \mathscr{P}^\bullet \to \mathscr{Q}^\bullet$. Normally we use the notations $\mathscr{P}^\bullet[p]$ and $f[p]$ instead of $\mathsf{T}^p(\mathscr{P}^\bullet)$, and $\mathsf{T}^p(f)$. It should be noted that $\mathsf{T} = \mathsf{T}^1$ is a shift to the left; for example, if $\mathscr{P}^\bullet$ is a complex concentrated in degree p, then $\mathscr{P}^\bullet$ should be written as $\mathscr{P}^\bullet = P[-p]$, where $P = \mathscr{P}^p$. The functors T^p are compatible with homotopy, so they induce endofunctors $K^*(\mathfrak{A}) \to K^*(\mathfrak{A})$, for which we will use the same notations.

The nth homology object of a complex $\mathscr{P}^\bullet$ is defined to be

$$H^n(\mathscr{P}^\bullet) = \operatorname{Ker} \partial^n / \operatorname{Im} \partial^{n-1}.$$

Observe that $H^n(\mathscr{P}^\bullet[p]) = H^{n+p}(\mathscr{P}^\bullet)$. We get for every $n \in \mathbf{Z}$ a functor

$$H^n\colon C^*(\mathfrak{A}) \to \mathfrak{A}.$$

Since $H^n(f) = H^n(g)$ if $f,\ g\colon \mathscr{P}^\bullet \to \mathscr{Q}^\bullet$ are homotopic, the functors H^n are well-defined on the homotopy category $K^*(\mathfrak{A})$. We say that $f\colon \mathscr{P}^\bullet \to \mathscr{Q}^\bullet$ is a *quasi-isomorphism* if

$$H^n(f)\colon H^n(\mathscr{P}^\bullet) \to H^n(\mathscr{Q}^\bullet)$$

is an isomorphism for every $n \in \mathbf{Z}$.

For a functor $F\colon \mathfrak{A} \to \mathfrak{B}$ it is clear how to extend F to $C^*(\mathfrak{A})$ and $K^*(\mathfrak{A})$; for bifunctors like Hom and $\otimes$ there is something more to be said. For simplicity we will assume that $\mathfrak{A}$ has infinite direct sums and infinite direct products. If $\mathscr{P}^\bullet$ and $\mathscr{Q}^\bullet$ are complexes of objects in $\mathfrak{A}$, and we set $\mathscr{K}^{-p,q} = \operatorname{Hom}(\mathscr{P}^p, \mathscr{Q}^q)$, then the differentials of $\mathscr{P}^\bullet$ induce $\partial^{p,q}_I\colon \mathscr{K}^{p,q} \to \mathscr{K}^{p+1,q}$, and the differentials of $\mathscr{Q}^\bullet$ induce $\partial^{p,q}_{II}\colon \mathscr{K}^{p,q} \to \mathscr{K}^{p,q+1}$, so we get a commutative diagram $\mathscr{K}^\bullet$ of which the nth row $(\mathscr{K}^{\bullet,n}, \partial^{\bullet,n}_I)$ ($n \in \mathbf{Z}$ fixed) and the nth column $(\mathscr{K}^{n,\bullet}, \partial^{n,\bullet}_{II})$ are complexes for every $n \in \mathbf{Z}$. We call such a complex a *double complex*. There are various ways to transform a double complex like this into a simple complex. In this case, the right way to do it is by taking

$$\operatorname{Hom}^n(\mathscr{P}^\bullet, \mathscr{Q}^\bullet) = \prod_{p+q=n} \mathscr{K}^{p,q} = \prod_p \operatorname{Hom}(\mathscr{P}^{-p}, \mathscr{Q}^{-p+n})$$

and defining the differential ∂^n by

$$\partial^n|_{\mathscr{K}^{p,q}} = (-1)^{p+q+1}\partial_I^{p,q} + \partial_{II}^{p,q}.$$

The choice of signs is the same as in [Ve, App. 1] and [Iv, Def. I.4.3], but different from [Ha, Sec. I.6]. It is clear this construction is functorial in $\mathscr{P}^\bullet$ and $\mathscr{Q}^\bullet$. Observe that we have a canonical isomorphism

$$H^n(\mathrm{Hom}^\bullet(\mathscr{P}^\bullet, \mathscr{Q}^\bullet)) \simeq \mathrm{Hom}_{K(\mathfrak{A})}(\mathscr{P}^\bullet, \mathscr{Q}^\bullet[n]).$$

On the category $A\text{-}\mathfrak{Mod}_G(X)$, we not only have the bifunctor Hom_G into $A\text{-}\mathfrak{Mod}$, but we have a whole collection of similar functors, like Hom, $\mathscr{H}om$, etc. We define the corresponding functors of complexes in exactly the same way as $\mathrm{Hom}_G^\bullet$.

If we have a tensor product in our category then the tensor product $\mathscr{P}^\bullet \otimes \mathscr{Q}^\bullet$ is defined using the double complex

$$\mathscr{K}^{p,q} = \mathscr{P}^p \otimes \mathscr{Q}^q,$$

with $\partial_I^{p,q} = \partial_{\mathscr{P}}^p \otimes \mathrm{id}$, and $\partial_{II} = \mathrm{id} \otimes \partial_{\mathscr{Q}}^q$. The associated simple complex will now be defined by

$$(\mathscr{P}^\bullet \otimes \mathscr{Q}^\bullet)^n = \bigoplus_{p+q=n} \mathscr{P}^p \otimes \mathscr{Q}^q,$$

and the differential is given by

$$\partial^n|_{\mathscr{K}^{p,q}} = \partial_I^{p,q} + (-1)^p \partial_{II}^{p,q}.$$

This tensor product is only commutative up to sign. By this we mean that the obvious mappings

$$s^{p,q}\colon\ \mathscr{P}^p \otimes \mathscr{Q}^q \to \mathscr{Q}^q \otimes \mathscr{P}^p$$

do *not* induce a mapping of complexes $\mathscr{P}^\bullet \otimes \mathscr{Q}^\bullet \to \mathscr{Q}^\bullet \otimes \mathscr{P}^\bullet$, since the $s^{p,q}$ do not commute with the differentials. Instead, we need the collection

$$(-1)^{pq} s^{p,q}\colon\ \mathscr{P}^p \otimes \mathscr{Q}^q \to \mathscr{Q}^q \otimes \mathscr{P}^p$$

in order to define an isomorphism

$$\varsigma\colon\ \mathscr{P}^\bullet \otimes \mathscr{Q}^\bullet \to \mathscr{Q}^\bullet \otimes \mathscr{P}^\bullet.$$

The collection of mappings

$$(-1)^{(p-m)n}\colon\ \mathscr{P}^p \otimes \mathscr{Q}^q \to \mathscr{P}^p \otimes \mathscr{Q}^q$$

defines for any $m, n \in \mathbf{Z}$ an isomorphism

$$\tau_{m,n}\colon\ \mathscr{P}^\bullet[m] \otimes \mathscr{Q}^\bullet[n] \to \mathscr{P}^\bullet \otimes \mathscr{Q}^\bullet[m+n].$$

and the following diagram is commutative:

(9)
$$\begin{array}{ccc} \mathscr{P}^\bullet[m] \otimes \mathscr{Q}^\bullet[n] & \xrightarrow{\tau_{m,n}} & (\mathscr{P}^\bullet \otimes \mathscr{Q}^\bullet)[m+n] \\ \Big\downarrow{\varsigma} & & \Big\downarrow{\varsigma[m+n]} \\ \mathscr{Q}^\bullet[n] \otimes \mathscr{P}^\bullet[m] & \xrightarrow{\tau_{n,m}} & (\mathscr{Q}^\bullet \otimes \mathscr{P}^\bullet)[m+n] \end{array}$$

For any m, m', n, $n' \in \mathbf{Z}$ such that $m+n = m'+n'$ and any homomorphism $f\colon \mathscr{P}^\bullet \to \mathscr{R}^\bullet[r]$ the following diagram is commutative.

(10)
$$\begin{array}{ccc} \mathscr{P}^\bullet[m] \otimes \mathscr{Q}^\bullet[n] & \xrightarrow{f[m]\otimes \mathrm{id}} & \mathscr{R}^\bullet[m+r] \otimes \mathscr{Q}^\bullet[n] \\ \Big\downarrow{\tau_{m,n}} & & \Big\downarrow{\tau_{m+r,n}} \\ (\mathscr{P}^\bullet \otimes \mathscr{Q}^\bullet)[m+n] & & (\mathscr{R}^\bullet \otimes \mathscr{Q}^\bullet)[m+n+r] \\ \Big\downarrow{\tau_{m',n'}} & & \Big\downarrow{\tau_{m'+r,n'}} \\ \mathscr{P}^\bullet[m'] \otimes \mathscr{Q}^\bullet[n'] & \xrightarrow{f[m']\otimes \mathrm{id}} & \mathscr{R}^\bullet[m'+r] \otimes \mathscr{Q}^\bullet[n'] \end{array}$$

From diagrams (9) and (10), we see that the following diagram is commutative as well.

(11)
$$\begin{array}{ccc} \mathscr{Q}^\bullet[n] \otimes \mathscr{P}^\bullet[m] & \xrightarrow{\mathrm{id}\otimes f[m]} & \mathscr{Q}^\bullet[n] \otimes \mathscr{R}^\bullet[m+r] \\ \Big\downarrow{(-1)^{nm}\tau_{n,m}} & & \Big\downarrow{(-1)^{n(m+r)}\tau_{n,m+r}} \\ (\mathscr{Q}^\bullet \otimes \mathscr{P}^\bullet)[m+n] & & (\mathscr{Q}^\bullet \otimes \mathscr{R}^\bullet)[m+n+r] \\ \Big\downarrow{(-1)^{n'm'}\tau_{n',m'}} & & \Big\downarrow{(-1)^{n'(m'+r)}\tau_{n',m'+r}} \\ \mathscr{Q}^\bullet[n'] \otimes \mathscr{P}^\bullet[m'] & \xrightarrow{\mathrm{id}\otimes f[m']} & \mathscr{Q}^\bullet[n'] \otimes \mathscr{R}^\bullet[m'+r] \end{array}$$

The *derived category* $D^*(\mathfrak{A})$ corresponding to $K^*(\mathfrak{A})$ is constructed by formally inverting all quasi-isomorphisms in $K^*(\mathfrak{A})$. This means, for example, that a diagram

$$\mathscr{P}^\bullet \xLeftarrow{f} \mathscr{X}^\bullet \xrightarrow{g} \mathscr{Q}^\bullet$$

in $C^*(\mathfrak{A})$, with f a quasi-isomorphism, represents a morphism from $\mathscr{P}^\bullet$ to $\mathscr{Q}^\bullet$ in $D^*(\mathfrak{A})$. A morphism in $D^*(\mathfrak{A})$ will also be called a *quasi-morphism* of complexes and it will be denoted by an arrow $\mathscr{P}^\bullet \dashrightarrow \mathscr{Q}^\bullet$, unless it is obvious that it is represented by a morphism $\mathscr{P}^\bullet \to \mathscr{Q}^\bullet$ in $C^*(\mathfrak{A})$. The homology functors $H^n\colon C^*(\mathfrak{A}) \to \mathfrak{A}$ and the translation functors $\mathsf{T}^n C^*(\mathfrak{A}) \to C^*(\mathfrak{A})$ induce in the obvious way functors $H^n\colon D^*(\mathfrak{A}) \to \mathfrak{A}$ and $\mathsf{T}^n D^*(\mathfrak{A}) \to D^*(\mathfrak{A})$.

We will use that $D^*(\mathfrak{A})$ is a so-called *triangulated category*. A *triangle* in $D^*(\mathfrak{A})$ is a sequence of quasi-morphisms

$$\mathscr{P}^\bullet \overset{f}{\dashrightarrow} \mathscr{Q}^\bullet \overset{g}{\dashrightarrow} \mathscr{R}^\bullet \overset{h}{\dashrightarrow} \mathscr{P}^\bullet[1], \tag{12}$$

that is quasi-isomorphic to a sequence

$$\tilde{\mathscr{P}}^\bullet \to \tilde{\mathscr{Q}}^\bullet \to \tilde{\mathscr{R}}^\bullet \to \tilde{\mathscr{P}}^\bullet[1],$$

obtained in the usual way from a split short exact sequence

$$0 \to \tilde{\mathscr{P}}^\bullet \to \tilde{\mathscr{Q}}^\bullet \to \tilde{\mathscr{R}}^\bullet \to 0$$

in $C^*(\mathfrak{A})$. It can be shown that any short exact sequence of complexes in $C^*(\mathfrak{A})$

$$0 \to \mathscr{P}^\bullet \xrightarrow{f} \mathscr{Q}^\bullet \xrightarrow{g} \mathscr{R}^\bullet \to 0 \tag{13}$$

gives rise to a (uniquely defined) triangle, and any morphism of short exact sequences induces a morphism of triangles. Finally, applying the homology functor to a triangle gives a long exact sequence

$$\cdots \xrightarrow{H^{n-1}(h)} H^n(\mathscr{P}^\bullet) \xrightarrow{H^n(f)} H^n(\mathscr{Q}^\bullet) \xrightarrow{H^n(g)} H^n(\mathscr{R}^\bullet) \xrightarrow{H^n(h)} H^{n+1}(\mathscr{P}^\bullet) \xrightarrow{H^{n+1}(f)} \cdots$$

If $F\colon \mathfrak{A} \to \mathfrak{B}$ is a covariant additive functor between abelian categories, F induces functors $C^*(\mathfrak{A}) \to C^*(\mathfrak{B})$ and $K^*(\mathfrak{A}) \to K^*(\mathfrak{B})$, which will be denoted by F as well. If F is exact, we can even extend it to the derived categories, and we will use the notation

$$F\colon D^*(\mathfrak{A}) \to D^*(\mathfrak{B}),$$

but in general this is not possible. Under some conditions, however, often only satisfied when $D^*(\mathfrak{A}) = D^+(A)$ or $D^b(A)$, we can construct the *right derived functor*

$$RF\colon D^*(\mathfrak{A}) \to D(\mathfrak{B}),$$

which has the properties that it transforms triangles into triangles and that it comes with a morphism

$$\xi\colon Q \circ F \to RF \circ Q$$

of functors $K^*(\mathfrak{A}) \to D(\mathfrak{B})$, where $Q\colon K^*(\mathfrak{B}) \to D^*(\mathfrak{B})$ is the localization functor. It is characterized by the fact that RF is universal with respect to these properties (see [Ha, Sec. I.5]).

If the right derived functor of F is defined on $D^*(\mathfrak{A})$, we write

$$R^pF(\mathscr{P}^\bullet) = H^p(RF\mathscr{P}^\bullet)$$

for any complex $\mathscr{P}^\bullet$ in $D^*(\mathfrak{A})$. Observe that the fact that RF transforms triangles into triangles implies that a short exact sequence of complexes (13) induces a long exact sequence

$$\cdots \to R^pF(\mathscr{P}^\bullet) \to R^pF(\mathscr{Q}^\bullet) \to R^pF(\mathscr{R}^\bullet) \to R^{p+1}F(\mathscr{P}^\bullet) \to \cdots \tag{14}$$

If F is left exact, which is true for all functors F for which we will construct RF, the universal property of ξ implies, that for any complex $\mathscr{P}^\bullet$ in $C^*(\mathfrak{A})$ such that $\mathscr{P}^n = 0$ for $n < 0$, we have a canonical isomorphism

$$R^0F(\mathscr{P}^\bullet) = F(H^0(\mathscr{P}^\bullet)).$$

For a right exact covariant additive functor F it makes in general more sense to construct a *left derived functor*

$$LF\colon D^*(\mathfrak{A}) \to D(\mathfrak{B}),$$

which, if it exists, is the universal functor that transforms triangles into triangles and comes with a morphism of functors

$$\xi\colon LF \circ Q \to Q \circ F.$$

In particular, if $\mathscr{P}^\bullet$ is a complex in $C^*(\mathfrak{A})$ with $\mathscr{P}^n = 0$ for $n > 0$, and F is right exact then

$$L_0F(\mathscr{P}^\bullet) = F(H^0(\mathscr{P}^\bullet)),$$

where we use the 'homological' notation

$$L_pF(\mathscr{P}^\bullet) = H_p(LF(\mathscr{P}^\bullet)) = H^{-p}(LF(\mathscr{P}^\bullet)).$$

The short exact sequence (13) induces a long exact sequence

$$\cdots \to L_pF(\mathscr{P}^\bullet) \to L_pF(\mathscr{Q}^\bullet) \to L_pF(\mathscr{R}^\bullet) \to L_{p-1}F(\mathscr{P}^\bullet) \to \cdots \tag{15}$$

Proposition 3.1. *Let $\mathfrak{A}$, $\mathfrak{B}$ be abelian categories, let $F\colon \mathfrak{A} \to \mathfrak{B}$ be an additive functor. Suppose we have a collection $\mathscr{C}$ of objects from $\mathfrak{A}$ satisfying the following properties:*

(i) *Every object of $\mathfrak{A}$ admits an injection into an element of $\mathscr{C}$.*
(ii) *If*

$$0 \to M \to M' \to M'' \to 0$$

is a short exact sequence, with $M \in \mathscr{C}$, then

$$M' \in \mathscr{C} \Leftrightarrow M'' \in \mathscr{C}$$

(iii) *F carries short exact sequences of objects of $\mathscr{C}$ into short exact sequences.*

Then the right derived functor RF is defined on $D^+(\mathfrak{A})$.

Proof. See [Ha, Corollary I.5.3]. The first two conditions ensure that the derived category $D^+(\mathfrak{C})$ of the full subcategory $\mathfrak{C}$ of $\mathfrak{A}$ that has the elements of $\mathscr{C}$ as its objects, is equivalent to $D^+(\mathfrak{A})$. By the last condition F takes quasi-isomorphism between complexes in $C^+(\mathfrak{C})$ to quasi-isomorphisms in $C^+(\mathfrak{B})$, so F is well-defined on $D^+(\mathfrak{C})$.

To be more specific, we get the following recipe for computing $RF(\mathscr{P}^\bullet)$ for a bounded below complex $\mathscr{P}^\bullet$ of objects from $\mathfrak{A}$. Take a quasi-isomorphism $i\colon \mathscr{P}^\bullet \to \mathscr{J}^\bullet$ into a complex of objects from $\mathscr{C}$. Then $RF(\mathscr{P}^\bullet)$ is $F(\mathscr{J}^\bullet)$ and $\xi(F)\colon F(\mathscr{P}^\bullet) \to RF(\mathscr{P}^\bullet)$ is $F(i)$. $\square$

In particular, if $\mathfrak{A}$ has *enough injectives*, i.e., if every object in $\mathfrak{A}$ can be injected into an injective object, then for every additive functor F the right derived functor RF is defined on $D^+(\mathfrak{A})$.

Lemma 3.2. *Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a left exact additive functor between abelian categories and assume that there is a collection $\mathscr{C}$ that satisfies the conditions of Proposition 3.1. Let $\mathscr{C}'$ be the collection of objects M in $\mathfrak{A}$ such that $R^p F(M) = 0$ for $p > 0$. Then $\mathscr{C} \subset \mathscr{C}'$ and $\mathscr{C}'$ satisfies the conditions of Proposition 3.1.*

Proof. From the construction of RF, we see that for any $M \in \mathscr{C}$, we have $R^p F(M) = 0$ for $p > 0$, so $\mathscr{C} \subset \mathscr{C}'$, which implies that $\mathscr{C}'$ satisfies condition (i) of Proposition 3.1. Conditions (ii) and (iii) can be checked by writing down the long exact sequences associated to the short exact sequences. □

An object M in $\mathfrak{A}$ such that $R^p F(M) = 0$ for $p > 0$ is called *F-acyclic*.

Lemma 3.2 will allow us under certain conditions to extend the definition of RF from $D^+(\mathfrak{A})$ to the whole derived category $D(\mathfrak{A})$. A sufficient condition is that F has *finite cohomological dimension on* $\mathfrak{A}$, i.e., there is an $n \in \mathbf{N}$, such that $R^i F(A)$ for every object A in $\mathfrak{A}$ and every $i > n$.

Proposition 3.3. *Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be an additive functor between abelian categories such that we have a collection $\mathscr{C}$ for F as in Proposition 3.1. If F has finite cohomological dimension on $\mathfrak{A}$, then RF can be extended to $D(\mathfrak{A})$.*

Proof. See again [Ha, Corollary I.5.3]. The idea of the proof is that under the hypothesis on F, we have for any complex of objects $\mathscr{P}^\bullet$ from $\mathfrak{A}$ a quasi-isomorphism into a complex $\mathscr{J}^\bullet$ of F-acyclic objects. Then Lemma 3.2 allows us to define $RF(\mathscr{P}^\bullet) = F(\mathscr{J}^\bullet)$. □

We have similar methods for the construction of left derived functors: just 'reverse the arrows'. However, the categories we are going to work with usually do not have enough projectives, so even for bounded above complexes it is much more work to prove the existence of the left derived functor associated a specific functor.

3.1. Composition of derived functors and spectral sequences

Proposition 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories and assume that $\mathfrak{A}$ has enough injectives. Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor, and let $\mathscr{P}^\bullet$ be a bounded below complex of objects in $\mathfrak{A}$. Then we have a spectral sequence*

$$E_2^{p,q} = R^p F(H^q(\mathscr{P}^\bullet)) \Rightarrow R^{p+q} F(\mathscr{P}^\bullet).$$

This construction is functorial in $\mathscr{P}^\bullet$.

Proof. Well-known. □

Corollary 3.5. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be abelian categories and assume that $\mathfrak{A}$ and $\mathfrak{B}$ have enough injectives. Let $T : \mathfrak{A} \to \mathfrak{B}$ and $S : \mathfrak{B} \to \mathfrak{C}$ be covariant left exact functors, and let $\mathscr{P}^\bullet$ be a bounded below complex of objects in $\mathfrak{A}$. Then we have a spectral sequence*

$$E_2^{p,q} = R^p S(R^q T(\mathscr{P}^\bullet)) \Rightarrow R^{p+q} S(RT(\mathscr{P}^\bullet)).$$

This construction is functorial in $\mathscr{P}^\bullet$.

Proof. Let $i\colon \mathscr{P}^\bullet \to \mathscr{I}^\bullet$ be an injection into a bounded below complex of injectives. Then on the one hand $R^{p+q} S(RT(\mathscr{P}^\bullet)) = R^{p+q} S(T(\mathscr{I}^\bullet))$, and on the other hand $R^p S(R^q T(\mathscr{P}^\bullet)) = R^p S(H^q(T(\mathscr{I}^\bullet)))$, so we can apply Proposition 3.4. □

Proposition 3.6. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be abelian categories. Let $T : \mathfrak{A} \to \mathfrak{B}$ and $S : \mathfrak{B} \to \mathfrak{C}$ be covariant left exact functors. Assume that RT is defined on $D^*(\mathfrak{A})$, that RS is defined on $D^*(\mathfrak{B})$ and that RT maps $D^*(\mathfrak{A})$ into $D^*(\mathfrak{B})$. If there is a collection $\mathscr{C}$ of T-acyclic objects in $\mathfrak{A}$, such that every complex in $C^*(\mathfrak{A})$ admits an injection into a complex of objects from $\mathscr{C}$, and such that T maps objects in $\mathscr{C}$ to S-acyclic objects, then we have a canonical isomorphism*

$$R(S \circ T) \simeq RS \circ RT.$$

Proof. This follows from the construction of derived functors, cf. [Ha, Prop. I.5.4]. □

The isomorphism of derived functors $R(S \circ T) \simeq RS \circ RT$ induces an isomorphism $R^{p+q} S(RT(-)) \simeq R^{p+q}(S \circ T)$, hence under the conditions of Proposition 3.6, we have by Corollary 3.5 a spectral sequence

$$E_2^{p,q} = R^p S(R^q T(\mathscr{P}^\bullet)) \Rightarrow R^{p+q}(S \circ T)(\mathscr{P}^\bullet) \tag{16}$$

for any bounded below complex $\mathscr{P}^\bullet$ of objects in $\mathfrak{A}$.

In this context the following lemma is often very useful.

Lemma 3.7. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two abelian categories. Let $S : \mathfrak{A} \to \mathfrak{B}$ be a functor having a right adjoint functor $T : \mathfrak{B} \to \mathfrak{A}$. Then*

(i) *S is right exact and T is left exact.*
(ii) *If T is exact, then S takes projective objects in $\mathfrak{A}$ to projective objects in $\mathfrak{B}$.*
(iii) *If S is exact, then T takes injective objects in $\mathfrak{B}$ to injective objects in $\mathfrak{A}$.*

Proof. Easy category theory. □

4. Derived functors for G-sheaves

In this section we will construct the derived functors of the functors introduced in Section 2. We will use the notation $D(X, A)$ instead of $D(A\text{-}\mathfrak{Mod}(X))$, and $D_G(X, A)$ instead of $D(A\text{-}\mathfrak{Mod}_G(X))$.

Most of the derived functors we are going to define will be right derived functors defined on the category of bounded below complexes, so the following lemma makes things easy.

Lemma 4.1. *Let G be any group, let A be any ring and let X be any G-space. Then the category $A\text{-}\mathfrak{Mod}_G(X)$ has enough injectives.*

Proof. The category $A\text{-}\mathfrak{Mod}(X)$ has enough injectives, so for a G-A-module $\mathcal{F}$ on X, we can find a an injective A-module $\mathcal{I}$ on X and a nonequivariant injection $\mathcal{F} \hookrightarrow \mathcal{I}$. This gives rise to an equivariant injection $\mathcal{F} \hookrightarrow \operatorname{Coind}^G \mathcal{I}$, and $\operatorname{Coind}^G \mathcal{I}$ is injective by Proposition 2.10 and Lemma 3.7. $\square$

This means that for any group G and any equivariant continuous mapping $f\colon X \to Y$ between G-spaces, we have the derived functors

$$\begin{aligned} Rf_*\colon D_G^+(X,A) &\to D_G^+(Y,A),\\ R\pi_*^G\colon D_G^+(X,A) &\to D_G^+(X/G,A),\\ R\Gamma(X,-)\colon D_G^+(X,A) &\to D_G^+(A\text{-}\mathfrak{Mod}),\\ R\Gamma^G(X,-)\colon D_G^+(X,A) &\to D^+(A\text{-}\mathfrak{Mod}). \end{aligned}$$

Proposition 4.2. *Let A be a ring, let G be a group and let $f\colon X \to Y$ and $f'\colon Y \to Z$ be equivariant continuous mappings of G-spaces. Then*

$$R(f'_* \circ f_*) = Rf'_* \circ Rf_*.$$

Proof. This follows by Proposition 3.6 from Proposition 2.3 and Lemma 3.7. $\square$

If X and Y are locally compact, we have the derived functor

$$Rf_!\colon D_G^+(X,A) \to D_G^+(Y,A),$$

and if $f_!$ (resp. $\Gamma_c(X,-)$) is of finite cohomological dimension, then by Proposition 3.3 we can extend $Rf_!$ (resp. $R\Gamma_c$), to the whole derived category. If $\Gamma_c(X,-)$ has cohomological dimension n on $\mathfrak{Ab}(X)$, we say that X *has cohomological dimension n*. If $\Gamma(X,-)$ has cohomological dimension n on $\mathfrak{Ab}(X)$, we say that X *has strict cohomological dimension n*. If X has strict cohomological dimension n, then X has cohomological dimension $\leq n$ (see [Ve, Exp. 2, Th. 4.1]).

A useful collection of sheaves satisfying the properties of Proposition 3.1 for the functors $\Gamma_c(X,-)$ and $f_!$ is the collection of c-soft, resp. $f_!$-soft sheaves.

Definition 4.3. A sheaf $\mathcal{F}$ on a locally compact space X is *c-soft* if for every compact subspace $i\colon K \hookrightarrow X$ the restriction mapping $\Gamma(X,\mathcal{F}) \to \Gamma(K, i^*\mathcal{F})$ is surjective. A sheaf $\mathcal{F}$ is *$f_!$-soft*, for a mapping $f\colon X \to Y$ of locally compact spaces, if for every $y \in Y$, the restriction of $\mathcal{F}$ to $f^{-1}(y)$ is c-soft.

If $\mathcal{F}$ is c-soft, then $\mathcal{F}$ is $f_!$-soft for any f, and an $f_!$-soft sheaf is $f_!$-acyclic.

Proposition 4.4. *Let A be a ring, let G be a group and let $f\colon X \to Y$ and $f'\colon Y \to Z$ be equivariant mappings of locally compact G-spaces. Then*

$$R(f'_! \circ f_!) = Rf'_! \circ Rf_!.$$

PROOF. This follows from Proposition 3.6 and the fact that $f_!$ transforms c-soft sheaves into c-soft sheaves (see [Ve, Exp. 3, Prop. 1.4]). □

The bifunctors we defined for G-sheaves give rise to pairings between derived categories. For example, if $\mathscr{P}^\bullet$ is a complex of G-A-modules, we have the right derived functor in one variable

$$R\mathrm{Hom}_G(\mathscr{P}^\bullet, -)\colon D_G^+(X, A) \to D(A\text{-}\mathfrak{Mod}).$$

Moreover, this construction is functorial in $\mathscr{P}^\bullet$, and for any complex of injectives $\mathscr{I}^\bullet$ in $A\text{-}\mathfrak{Mod}_G(X)$, a quasi-isomorphism $\mathscr{P}^\bullet \to \mathscr{Q}^\bullet$ in $C_G(X, A)$ induces an isomorphism $\mathrm{Hom}_G(\mathscr{Q}^\bullet, \mathscr{I}^\bullet) \to \mathrm{Hom}_G(\mathscr{P}^\bullet, \mathscr{I}^\bullet)$, so we get a bifunctor

$$R\mathrm{Hom}_G(-,-)\colon D_G(X, A) \times D_G^+(X, A) \to D(\mathfrak{Ab}).$$

Since $R\mathrm{Hom}_G(\mathscr{P}^\bullet, -)$ is an ordinary right derived functor, it transforms triangles into triangles. When $\mathscr{C}^\bullet$ is a bounded below complex of G-A-modules on X, then $R\mathrm{Hom}_G(-, \mathscr{C}^\bullet)$ transforms triangles into triangles as well. A triangle

$$\mathscr{P}^\bullet \dashrightarrow \mathscr{Q}^\bullet \dashrightarrow \mathscr{R}^\bullet \dashrightarrow \mathscr{P}^\bullet[1]$$

is transformed into a triangle

$$\begin{aligned} R\mathrm{Hom}_G(\mathscr{R}^\bullet, \mathscr{C}^\bullet) \dashrightarrow R\mathrm{Hom}_G(\mathscr{Q}^\bullet, \mathscr{C}^\bullet) \dashrightarrow R\mathrm{Hom}_G(\mathscr{P}^\bullet, \mathscr{C}^\bullet) \dashrightarrow& \\ \dashrightarrow R\mathrm{Hom}_G(\mathscr{R}^\bullet, \mathscr{C}^\bullet)[1].& \end{aligned}$$

In the same way we get for any group G, any commutative ring A and any G-space X the following bifunctors:

$$\begin{aligned} R\mathrm{Hom}(-,-)\colon& D_G(X, A) \times D_G^+(X, A) \to D_G(A) \\ R\mathscr{H}\!om_G(-,-)\colon& D_G(X, A) \times D_G^+(X, A) \to D(X/G, A) \\ R\mathscr{H}\!om(-,-)\colon& D_G(X, A) \times D_G^+(X, A) \to D_G(X, A) \end{aligned}$$

Proposition 4.5. *Let A be a ring, let G be a group. Then for any $\mathscr{M}^\bullet$, $\mathscr{N}^\bullet$ in $D_G^+(X, A)$ we have an isomorphism*

$$R\mathrm{Hom}_G(\mathscr{M}^\bullet, \mathscr{N}^\bullet) \simeq R\Gamma^G\, R\mathrm{Hom}(\mathscr{M}^\bullet, \mathscr{N}^\bullet)$$

functorial in $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$.

PROOF. This follows from Proposition 3.6 if we take $S = \Gamma^G$, $T = R\mathrm{Hom}(\mathscr{M}^\bullet, -)$, and for $\mathscr{C}$ the injective objects in $A\text{-}\mathfrak{Mod}_G(X)$ of the form $\mathrm{Ind}^G(\mathscr{I})$, where $\mathscr{I}$ is injective in $A\text{-}\mathfrak{Mod}(X)$. □

For the definition of the left derived functor of the tensor product we cannot use projective resolutions, since in general $A\text{-}\mathfrak{Mod}_G(X)$ does not have enough projectives. Instead we use *flat* G-A-modules. Let $\mathscr{F}$ be a G-sheaf of A-modules on X, then we say that $\mathscr{F}$ is flat if the functor $\mathscr{F} \otimes -$ is exact. Observe that $\mathscr{F}$ is flat as a G-A-module if and only if it is flat as a sheaf of A-modules. We know that the category $A\text{-}\mathfrak{Mod}(X)$ has enough flats; there is for every sheaf of A-modules $\mathscr{M}$ on X a flat sheaf of A-modules $\mathscr{F}$ mapping surjectively onto $\mathscr{M}$. If $\mathscr{M}$ is a G-sheaf, the surjection $\mathscr{F} \to \mathscr{M}$ induces an equivariant surjection $\operatorname{Ind}_G \mathscr{F} \to \mathscr{M}$. Clearly $\operatorname{Ind}_G \mathscr{F}$ is flat, so $A\text{-}\mathfrak{Mod}_G(X)$ has enough flats. A reasoning as in [Ha, §II.4] shows that we can extend the tensor product on complexes to a pairing

$$\begin{array}{cccccc} -\overset{L}{\otimes}-\colon & D_G^-(X,A) & \times & D_G^-(X,A) & \to & D_G^-(X,A) \\ & \mathscr{P}^\bullet & \times & \mathscr{Q}^\bullet & \mapsto & \mathscr{P}^\bullet \overset{L}{\underset{A}{\otimes}} \mathscr{Q}^\bullet \end{array}.$$

We also obtain for any homomorphism of commutative rings $A \to B$ the left derived functor

$$B \overset{L}{\otimes} -\colon D_G^-(X,A) \to D_G^-(X,B).$$

4.1. Adjunction properties

Proposition 4.6 (Yoneda). *Let $\mathfrak{A}$ be an abelian category having enough injectives. Then for any $\mathscr{P}^\bullet \in D(\mathfrak{A})$, $\mathscr{Q}^\bullet \in D^+(\mathfrak{A})$, we have a canonical isomorphism*

$$R^n\operatorname{Hom}(\mathscr{P}^\bullet, \mathscr{Q}^\bullet) \simeq \operatorname{Hom}_{D(\mathfrak{A})}(\mathscr{P}^\bullet, \mathscr{Q}^\bullet[n]).$$

Proof. See [Ha, Th. I.6.4]. $\square$

Taking $n = 0$ in the above proposition, we see that an equality

$$R\operatorname{Hom}(RV(\mathscr{P}^\bullet), \mathscr{Q}^\bullet) = R\operatorname{Hom}(\mathscr{P}^\bullet, RW(\mathscr{Q}^\bullet))$$

implies that the derived functors RV and RW are adjoint.

Proposition 4.7. *Let G be a group, A a commutative ring, and $f\colon X \to Y$ a morphism of G-spaces. We have for any complex $\mathscr{P}^\bullet$ in $D_G(X,A)$ and any complex $\mathscr{Q}^\bullet$ in $D_G^+(Y,A)$ isomorphisms*

$$\begin{aligned} R\operatorname{Hom}_G(f^*\mathscr{P}^\bullet, \mathscr{Q}^\bullet) &\simeq R\operatorname{Hom}_G(\mathscr{P}^\bullet, Rf_*\mathscr{Q}^\bullet), \\ R\operatorname{Hom}(f^*\mathscr{P}^\bullet, \mathscr{Q}^\bullet) &\simeq R\operatorname{Hom}(\mathscr{P}^\bullet, Rf_*\mathscr{Q}^\bullet), \\ Rf_*R\mathscr{H}om_G(f^*\mathscr{P}^\bullet, \mathscr{Q}^\bullet) &\simeq R\mathscr{H}om_G(\mathscr{P}^\bullet, Rf_*\mathscr{Q}^\bullet), \\ Rf_*R\mathscr{H}om(f^*\mathscr{P}^\bullet, \mathscr{Q}^\bullet) &\simeq R\mathscr{H}om(\mathscr{P}^\bullet, Rf_*\mathscr{Q}^\bullet), \end{aligned}$$

functorial in $\mathscr{P}^\bullet$ and $\mathscr{Q}^\bullet$.

PROOF. We will prove the equality

$$R\mathrm{Hom}_G(f^*\mathscr{P}^\bullet, \mathscr{Q}^\bullet) \simeq R\mathrm{Hom}_G(\mathscr{P}^\bullet, Rf_*\mathscr{Q}^\bullet);$$

the rest goes in a similar way. Take an injective resolution $\mathscr{Q}^\bullet \to \mathscr{I}^\bullet$ of $\mathscr{Q}^\bullet$. Then we have a quasi-isomorphism

$$R\mathrm{Hom}_G(f^*\mathscr{P}^\bullet, \mathscr{Q}^\bullet) \simeq \mathrm{Hom}^\bullet_G(f^*\mathscr{P}^\bullet, \mathscr{I}^\bullet).$$

On the other hand $Rf_*(\mathscr{Q}^\bullet)$ is quasi-isomorphic to $f_*\mathscr{I}^\bullet$, and $f_*\mathscr{I}^\bullet$ consists of injective G-A-modules on Y, since f_* transforms injectives into injectives by Proposition 2.3 and Lemma 3.7. In other words,

$$R\mathrm{Hom}_G(\mathscr{P}^\bullet, Rf_*\mathscr{Q}^\bullet) \simeq \mathrm{Hom}^\bullet_G(\mathscr{P}^\bullet, f_*\mathscr{I}^\bullet).$$

The isomorphism

$$\mathrm{Hom}^\bullet_G(f^*\mathscr{P}^\bullet, \mathscr{I}^\bullet) \simeq \mathrm{Hom}^\bullet_G(\mathscr{P}^\bullet, f_*\mathscr{I}^\bullet)$$

follows immediately from Proposition 2.3. □

Proposition 4.8. *Let G be a group, let A be a commutative ring, and let X be a G-space. We have for any complex $\mathscr{Q}^\bullet$ in $D^+_G(X, A)$ and any complex $\mathscr{P}^\bullet$ in $D(X/G, A)$ an isomorphism*

$$R\mathrm{Hom}_G(\pi^*\mathscr{P}^\bullet, \mathscr{Q}^\bullet) \simeq R\mathrm{Hom}(\mathscr{P}^\bullet, R\pi^G_*(A, \mathscr{Q}^\bullet)).$$

In particular, we have

$$R\Gamma(X, \mathscr{Q}^\bullet) \simeq R\Gamma(X/G, R\pi^G_*\mathscr{Q}^\bullet),$$

and for $M^\bullet$ in $D(A)$ and $N^\bullet$ in $D^+_G(A)$ we have

$$R\mathrm{Hom}_G(M^\bullet, N^\bullet) \simeq R\mathrm{Hom}(M^\bullet, R\Gamma^G N^\bullet).$$

The isomorphisms are functorial in $\mathscr{P}^\bullet$ and $\mathscr{Q}^\bullet$ (resp. $M^\bullet$ and $N^\bullet$).

PROOF. The first equation is proven as Proposition 4.7, using Corollary 2.7 instead of Proposition 2.3. The other two equations are special cases of the first one. □

Proposition 4.9. *Let G be a group, A a commutative ring, and let X be a G-space. We have for any complex $\mathscr{P}^\bullet$ in $D(X, A)$ and any complex $\mathscr{Q}^\bullet$ in $D^+_G(X, A)$ isomorphisms*

$$R\mathrm{Hom}(\mathscr{P}^\bullet, \mathscr{Q}^\bullet) \simeq R\mathrm{Hom}_G(\mathrm{Ind}^G\mathscr{P}^\bullet, \mathscr{Q}^\bullet),$$

and

$$R\mathrm{Hom}(\mathscr{Q}^\bullet, \mathscr{P}^\bullet) \simeq R\mathrm{Hom}_G(\mathscr{Q}^\bullet, \mathrm{Coind}^G\mathscr{P}^\bullet),$$

which are functorial in $\mathscr{P}^\bullet$ and $\mathscr{Q}^\bullet$.

PROOF. This follows from Proposition 2.10. □

5. Verdier duality

Poincaré duality is in the Grothendieck-Verdier formalism one of the consequences of the existence of a right adjoint functor $f^!$ to the derived functor $Rf_!$. The existence of $f^!$ is not as obvious as the existence of adjoints to other derived functors, since $f_!$ itself has in general no right adjoint. In the nonequivariant setting Verdier has constructed $f^!$ for any mapping $f: X \to Y$ of locally compact spaces such that $f_!$ has finite cohomological dimension (see [Ve]). The generalization to the equivariant situation is straightforward. A short description following [Iv, Sec. VIII.3] and [Bo2, §V.7] is included for sake of completeness and because one of the intermediate results will be needed in the next section.

Lemma 5.1. *Let G be a group, let A be a commutative ring, and let $f: X \to Y$ be a morphism of locally compact G-spaces. Let $\mathscr{M}$ be a G-A-module on Y and let $\mathscr{N}$ be a G-A-module on X. The natural homomorphism of G-sheaves*

$$\mathscr{M} \underset{A}{\otimes} f_!\mathscr{N} \to f_!(f^*\mathscr{M} \underset{A}{\otimes} \mathscr{N}),$$

is an isomorphism if $\mathscr{M}$ is flat.

Proof. The homomorphism $\mathscr{M} \otimes_A f_!\mathscr{N} \to f_!(f^*\mathscr{M} \otimes_A \mathscr{N})$ is defined using the adjunction morphism $\mathscr{M} \to f_*f^*\mathscr{M}$ and the obvious equivariant mapping

$$f_*f^*\mathscr{M} \underset{A}{\otimes} f_!\mathscr{N} \to f_!(f^*\mathscr{M} \underset{A}{\otimes} \mathscr{N}).$$

The statement now follows from [Ve, Exp. 3, Lemme 4.4.1]. □

Lemma 5.2. *Let G be a group, let A be a commutative ring, and let $f: X \to Y$ be a morphism of locally compact G-spaces, such that $f_!$ has finite cohomological dimension. Let $\mathscr{M}$ and $\mathscr{N}$ be any G-sheaves of A-modules on X, such that $\mathscr{N}$ is $f_!$-soft and either $\mathscr{M}$ or $\mathscr{N}$ is flat, then $\mathscr{M} \otimes \mathscr{N}$ is $f_!$-soft.*

Proof. See [Bo2, Prop. V.6.5.]. □

Proposition 5.3. *Let G be a group, let A be a commutative ring, and let $f: X \to Y$ be a morphism of locally compact G-spaces, such that $f_!$ has finite cohomological dimension. For any complex $\mathscr{Q}^\bullet$ in $D_G(X, A)$ and any complex $\mathscr{P}^\bullet$ in $D_G^-(Y, A)$ we have a natural isomorphism*

$$\mathscr{P}^\bullet \overset{L}{\otimes} Rf_!(\mathscr{Q}^\bullet) \simeq Rf_!(f^*\mathscr{P}^\bullet \overset{L}{\otimes} \mathscr{Q}^\bullet).$$

Proof. Take a flat resolution $\mathscr{F}^\bullet \to \mathscr{P}^\bullet$ of $\mathscr{P}^\bullet$ and an $f_!$-soft resolution $\mathscr{Q}^\bullet \to \mathscr{C}^\bullet$ of $\mathscr{Q}^\bullet$. Then

$$\mathscr{P}^\bullet \overset{L}{\otimes} Rf_!(\mathscr{Q}^\bullet) \simeq \mathscr{F}^\bullet \otimes f_!\mathscr{C}^\bullet,$$

and by Lemma 5.2,

$$Rf_!(f^*\mathscr{P}^\bullet \overset{L}{\otimes} \mathscr{Q}^\bullet) \simeq f_!(f^*\mathscr{F}^\bullet \otimes \mathscr{C}^\bullet),$$

so we can apply Lemma 5.1. □

Corollary 5.4. *Let G be a group, let A be a commutative ring, and let $f: X \to Y$ be a morphism of locally compact G-spaces, such that $f_!$ has finite cohomological dimension. Then for any flat and $f_!$-soft sheaf $\mathcal{N}$, the sheaf $f_!\mathcal{N}$ is flat.*

PROOF. Flatness of $f_!\mathcal{N}$ is equivalent to the property that for any sheaf of A-modules $\mathcal{M}$ on Y and any $n > 0$, we have $H_n(\mathcal{M} \otimes^L f_!\mathcal{N}) = 0$, but this is a trivial consequence of the fact that $H_n(\mathcal{M} \otimes^L f_!\mathcal{N}) = H^{-n}(Rf_!(f^*\mathcal{M} \otimes^L \mathcal{N})) = R^{-n}f_!(f^*\mathcal{M} \otimes \mathcal{N})$, which follows from Proposition 5.3 and Lemma 5.2. □

A standard c-soft and flat resolution of the constant sheaf A on X is the Godement resolution $A \to \mathcal{C}^\bullet$ (see [Iv, II.3.6]), which we give the obvious G-action. If A is noetherian and $f_!$ has cohomological dimension n, the truncation $\tau_{\leq n}\mathcal{C}^\bullet$ is a finite $f_!$-soft and flat resolution of A (cf. [Iv, Prop. VI.1.3]).

The following result will be needed in Section III.1.

Corollary 5.5. *Let G be a group, let A be a noetherian commutative ring and let B be a commutative A-algebra. Let $f: X \to Y$ be a morphism of locally compact G-spaces, such that $f_!$ has finite cohomological dimension. Then we have for any bounded above complex $\mathcal{P}^\bullet$ of G-A-modules on X a natural isomorphism*

$$B \overset{L}{\underset{A}{\otimes}} Rf_!(\mathcal{P}^\bullet) \simeq Rf_!(B \overset{L}{\underset{A}{\otimes}} \mathcal{P}^\bullet)$$

in $D^-_G(X, B)$.

PROOF. Let $\mathcal{F}^\bullet \to \mathcal{P}^\bullet$ be a flat resolution, and let $A \to \mathcal{K}^\bullet$ be a finite $f_!$-soft and A-flat resolution. Then $\mathcal{Q}^\bullet = \mathcal{F}^\bullet \otimes \mathcal{K}^\bullet$ is a bounded above $f_!$-soft and A-flat complex quasi-isomorphic to $\mathcal{P}^\bullet$. Since $Rf_!(B \otimes^L_A \mathcal{P}^\bullet) \simeq f_!(B \otimes_A \mathcal{Q}^\bullet)$ by Lemma 5.2, and $B \otimes^L_A Rf_!(\mathcal{P}^\bullet) \simeq B \otimes_A f_!\mathcal{Q}^\bullet$ by Corollary 5.4, the statement follows from Lemma 5.1. □

Observe that the isomorphism involved is an isomorphism in the derived category of G-B-modules, hence Corollary 5.5 is not just a special case of Proposition 5.3.

Now we can get down to the actual construction of the functor $f^!$. For a flat and soft A-module $\mathcal{K}$ on X and any A-module $\mathcal{J}$ on Y the presheaf

$$U \mapsto \mathrm{Hom}(f_!\mathcal{K}_U, \mathcal{J})$$

is a sheaf which we denote by $f^!(\mathcal{K}, \mathcal{J})$. This construction is functorial in both variables and equivariant as a bifunctor. Hence $f^!(\mathcal{K}, \mathcal{J})$ is equipped with a canonical G-action if $\mathcal{K}$ is a flat and soft G-A-module on X and $\mathcal{J}$ is a G-A-module on Y.

Lemma 5.6. *Let G be a group and let A be a commutative ring. Let $f: X \to Y$ be an equivariant mapping between locally compact G-spaces, such that $f_!$ has finite cohomological dimension. For any* *flat and soft G-A-module $\mathscr{K}$ on X, any G-A-module $\mathscr{F}$ on X and any G-A-module $\mathscr{J}$ on Y we have an isomorphism*

$$\mathscr{H}om(f_!(\mathscr{K}\otimes\mathscr{F}),\mathscr{J}) \to f_*\mathscr{H}om(\mathscr{F}, f^!(\mathscr{K},\mathscr{J}))$$

of G-A-modules on Y, which is functorial in $\mathscr{F}$, $\mathscr{K}$ and $\mathscr{J}$.

Proof. For any U open in X and any V open in Y there is a canonical bilinear pairing

$$\mathscr{F}(U)\times f_!\mathscr{K}_U(V) \to f_!(\mathscr{F}\otimes\mathscr{K})(V)$$

which is compatible with the mapping $\mathscr{F}(U)\to\mathscr{F}(g\cdot U)$, the mapping $f_!\mathscr{K}_U(V)\to f_!\mathscr{K}_{g\cdot U}(g\cdot V)$ and the mapping $f_!(\mathscr{F}\otimes\mathscr{K})(V)\to f_!(\mathscr{F}\otimes\mathscr{K})(g\cdot V)$ induced by the G-actions on $\mathscr{F}$ and $\mathscr{K}$. This pairing gives us a homomorphism

$$\psi(U,V)\colon \operatorname{Hom}(f_!(\mathscr{F}\otimes\mathscr{K})(V),\mathscr{J}(V)) \to \operatorname{Hom}(\mathscr{F}(U),\operatorname{Hom}(f_!\mathscr{K}_U(V),\mathscr{J}(V))).$$

When we vary U and V, we see that the $\psi(U,V)$ are compatible with the mappings induced by the restriction mappings of $\mathscr{F}$, $\mathscr{K}$ and $\mathscr{J}$, so the collection $\{\psi(U,V)\}$ defines a homomorphism

$$\psi\colon \mathscr{H}om(f_!(\mathscr{F}\otimes\mathscr{K}),\mathscr{J}) \to f_*\mathscr{H}om(\mathscr{F}, f^!(\mathscr{K},\mathscr{J})),$$

which can be checked to be equivariant. It is shown in [Bo2, V.7.14] that the homomorphism ψ is an isomorphism. □

Corollary 5.7. *Let $f\colon X\to Y$ and $\mathscr{K}$ be as in Lemma 5.6. The functor $f^!(\mathscr{K},-)$ is left exact and transforms injective G-A-modules on Y into injective G-A-modules on X.*

Proof. Since the functor $f_!(-\otimes\mathscr{K})$ is exact by Lemma 5.2, this follows from Lemma 3.7. □

Assume the commutative ring A is noetherian and fix a quasi-isomorphism $A\to\mathscr{K}^\bullet$ of the constant G-sheaf A on X into a bounded complex $\mathscr{K}^\bullet$ of $f_!$-soft and flat G-A-modules. For a complex of G-A-modules $\mathscr{J}^\bullet$ we define the complex $f^!(\mathscr{K}^\bullet,\mathscr{J}^\bullet)$ following the same conventions as in the definition of the complex $\operatorname{Hom}^\bullet$. Since A-$\mathfrak{Mod}_G(Y)$ has enough injectives, the right derived functor

$$f^!\colon D^+_G(Y,A)\to D^+_G(X,A)$$

of $f^!(\mathscr{K}^\bullet,-)$ is defined by sending a complex $\mathscr{N}^\bullet$ of G-A-modules on Y to $f^!(\mathscr{K}^\bullet,\mathscr{I}^\bullet)$, where $\mathscr{I}^\bullet$ is an injective complex quasi-isomorphic to $\mathscr{N}^\bullet$. If we have another $f_!$-soft and flat resolution $A\to\mathscr{L}^\bullet$, then $f^!(\mathscr{L}^\bullet,\mathscr{I}^\bullet)$ is quasi-isomorphic to $f^!(\mathscr{K}^\bullet,\mathscr{I}^\bullet)$, so the definition does not depend on the choice of $\mathscr{K}^\bullet$.

Theorem 5.8. *Let G be a group and let A be a noetherian commutative ring. Let $f\colon X\to Y$ be an equivariant mapping between locally compact G-spaces such that $f_!$ has finite cohomological* *dimension. We have for any complex* $\mathcal{M}^\bullet$ *in* $D_G(X, A)$ *and any complex* $\mathcal{N}^\bullet$ *in* $D_G^+(Y, A)$ *isomorphisms*

$$R\mathcal{H}om(Rf_!\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq Rf_* R\mathcal{H}om(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet),$$
$$R\mathrm{Hom}(Rf_!\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq R\mathrm{Hom}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet),$$

and

$$R\mathrm{Hom}_G(Rf_!\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq R\mathrm{Hom}_G(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet),$$

which are functorial in $\mathcal{M}^\bullet$ *and* $\mathcal{N}^\bullet$.

Proof. Let $\mathcal{N}^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism of $\mathcal{N}^\bullet$ into a complex of injectives $\mathcal{I}^\bullet$, and let $\mathcal{K}^\bullet$ be as above. Then

$$\mathcal{H}om^\bullet(f_!(\mathcal{M}^\bullet \otimes \mathcal{K}^\bullet), \mathcal{I}^\bullet) \simeq R\mathcal{H}om(Rf_!\mathcal{M}^\bullet, \mathcal{N}^\bullet).$$

On the other hand, $f^!(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ is an injective complex, so by Corollary 5.7

$$\mathcal{H}om^\bullet(\mathcal{M}^\bullet, f^!(\mathcal{K}^\bullet, \mathcal{I}^\bullet)) \simeq R\mathcal{H}om(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet).$$

Since $\mathcal{H}om^\bullet(\mathcal{M}^\bullet, f^!(\mathcal{K}^\bullet, \mathcal{I}^\bullet))$ consists of flabby sheaves by [Gr, Prop. 4.1.3],

$$f_*\mathcal{H}om^\bullet(\mathcal{M}^\bullet, f^!(\mathcal{K}^\bullet, \mathcal{I}^\bullet)) \simeq Rf_* R\mathcal{H}om(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet).$$

Hence the first isomorphism is a consequence of Lemma 5.6; the other two isomorphisms follow easily. □

Remark. In [BL] J. Bernstein and V. Lunts construct a category which plays the role of the derived category of G-sheaves on a space with an action of a Lie group G, together with all the usual functors and Verdier duality. For a finite group G, they show in Section 8 that their construction is equivalent to the derived category of the category of G-sheaves. They remark that even in this case their construction might be useful, since is 'not abolutely clear' how to obtain Verdier duality directly. As we have seen in this section, the generalization from the non-equivariant setting is in fact quite straightforward.

Chapter III

Equivariant homology and cohomology

In [Gr, Chap. V] equivariant cohomology was defined as the natural equivariant analogue of ordinary sheaf cohomology $H^n(X, \mathscr{F}) = R^n\Gamma(X, \mathscr{F})$. Similarly, the sheaf-theoretic definition of Borel-Moore homology $H_n(X, M) = R^{-n}\mathrm{Hom}(R\Gamma_c A, M)$ has an obvious equivariant analogue. In Sections 2, 3, and 4 we see that the standard properties of homology and cohomology, including cup product, cap product and Poincaré duality, come out automatically, since the formalism of derived functors and the results of the previous chapter allow us to copy all constructions and proofs directly from the nonequivariant case. The Hochschild-Serre spectral sequence (also known as the Borel-Serre spectral sequence), is introduced in Section 5. It is one of the main tools for determining the equivariant (co)homology from the G-action on the non-equivariant (co)homology groups. The connection between the equivariant (co)homology of a G-space X and the (co)homology of the fixed point set X^G is developed in Sections 6 and 7. In particular, the mapping ρ mentioned in the introduction is defined in Section 7 for $G = \mathbf{Z}/p$, and the main result of this chapter is Theorem 7.4. Some extra information about the case $G = \mathbf{Z}/2$ is gathered in Section 8, and we conclude with a series of examples of spaces with an involution in Section 9.

The results concerning equivariant *co*homology in this chapter are well-known; they are included for convenience and completeness. The definition of equivariant Borel-Moore homology, the mapping ρ and all related results are new.

1. Definitions

Definition 1.1. Let G be a group, let A be a commutative ring, let X be a G-space. Then for a sheaf of G-A-modules $\mathscr{F}$ on X we define the *nth equivariant cohomology group* *of X with coefficients in $\mathscr{F}$* by

$$H^n(X; G, \mathscr{F}) = R^n\Gamma^G(X, \mathscr{F}),$$

For an equivariant locally closed subset $V \subset X$ we define *equivariant cohomology with support in V* by

$$H^n_V(X; G, \mathscr{F}) = R^n\mathrm{Hom}_G(A_V, \mathscr{F}).$$

Observe that for a G-A-module M we have

$$H^n(\mathbf{pt}; G, M) = H^n(G, M)$$

(cohomology of groups). It can be checked that our definition of equivariant cohomology coincides for reasonable spaces with the more geometric definition of Borel (see [Bo2]) if we give G the discrete topology.

Definition 1.2. Let G be a group acting on a locally compact space X of finite cohomological dimension, let A be a commutative ring, and let M be a G-A-module. Then we define

$$H_n(X; G, M) = R^{-n}\mathrm{Hom}_G(R\Gamma_c A, M)$$

We call $H_n(X; G, M)$ the *n-th equivariant Borel-Moore homology group with coefficients in M.*

For a G-A-module M, we have that $H_n(\mathbf{pt}; G, M) = H^{-n}(G, M)$, so equivariant Borel-Moore homology can be considered as a mix of ordinary Borel-Moore homology and group *co*homology. In particular, even for compact spaces, equivariant Borel-Moore homology does not coincide with equivariant homology as defined, for example, in [Br, VII.7], which should be considered as a mix of ordinary homology and group homology. Probably the most striking difference between equivariant Borel-Moore homology and other equivariant homology theories is that $H_n(X; G, M)$ need not be zero for $n < 0$.

It should be noted that if B is a commutative A-algebra, then a G-B-module is in a natural way a G-A-module as well, so the notations for cohomology and homology are a bit ambiguous. For example, it might make a difference for the computation of $H^p_V(X; G, \mathbf{Z}/2)$ and $H_p(X; G, \mathbf{Z}/2)$ whether we consider $\mathbf{Z}/2$ as a $\mathbf{Z}/2$-module or as a $\mathbf{Z}$-module. However, it is well-known that an $R\mathrm{Hom}_{G,B}(B_V, -)$-acyclic B-module is also an $R\mathrm{Hom}_{G,A}(A_V, -)$-acyclic A-module. This implies that $R\mathrm{Hom}_{G,B}(B_V, \mathscr{F}) \simeq R\mathrm{Hom}_{G,A}(A_V, \mathscr{F})$ for any G-B-module $\mathscr{F}$ on X.

For Borel-Moore homology the situation is similar, since we have in $D(A)$ a well-known adjunction formula

$$R\mathrm{Hom}_{G,A}(R\Gamma_c(X, A), M) \simeq R\mathrm{Hom}_{G,B}(B \overset{L}{\underset{A}{\otimes}} R\Gamma_c(X, A), M),$$

that gives gives by Corollary II.5.5 a quasi-isomorphism

$$R\mathrm{Hom}_{G,A}(R\Gamma_c(X, A), M) \simeq R\mathrm{Hom}_{G,B}(R\Gamma_c(X, B), M)$$

for any G-B-module M. Hence, we get the same result, whether we compute $H_n(X; G, M)$ over B or over A. Even when G is trivial, this does not seem to be as widely known as the corresponding statement in cohomology. In particular, for any locally compact space X of cohomological dimension $n < \infty$ (over $\mathbf{Z}$), and any abelian group M we see that $H_k(X; M) = 0$ if $k < 0$ or $k > n$, since we may assume the base ring to be $\mathbf{Z}$ and then the proof of the second part of [Iv, Prop. IX.1.6] is easily adapted to homology with coefficients in M. As far as I know, there is no proof in the literature of the vanishing of Borel-Moore homology in negative degrees for such general X and M (see for example [Bo2, V.7.2]).

2. Functoriality and long exact sequences

When $f\colon X \to Y$ is a continuous equivariant mapping of G-spaces, and $i\colon W \hookrightarrow Y$ is the inclusion of an equivariant locally closed subspace, then the adjunction morphism $\mathscr{F} \to Rf_* f^* \mathscr{F}$ induces for G-A-module $\mathscr{F}$ on Y a quasi-morphism $R\mathrm{Hom}_G(i_! A, \mathscr{F}) \dashrightarrow R\mathrm{Hom}_G(i_! A, Rf_* f^* \mathscr{F})$. Since $R\mathrm{Hom}_G(i_! A, Rf_* f^* \mathscr{F})$ is quasi-isomorphic to the complex $R\mathrm{Hom}_G(f^* i_! A,\ f^* \mathscr{F})$ by Proposition II.4.7 and $f^* \circ i_! = j_! \circ f^*$, where $j\colon f^{-1}(W) \hookrightarrow X$ is the inclusion of the inverse image of W, we get a quasi-morphism

$$(17) \qquad R\mathrm{Hom}_G(i_! A, \mathscr{F}) \dashrightarrow R\mathrm{Hom}_G(j_! A,\ f^* \mathscr{F}).$$

This induces for any $n \in \mathbf{Z}$ the *pull-back* homomorphism

$$f^*\colon\ H^n_W(Y; G, \mathscr{F}) \to H^n_{f^{-1}(W)}(X; G, f^* \mathscr{F}),$$

which is clearly functorial in $\mathscr{F}$ and has the property that

$$(f \circ g)^* = g^* \circ f^*.$$

It is not hard to check that if $j\colon U \hookrightarrow X$ is the inclusion of an open subspace, and $V \subset U$ is locally closed, then the restriction

$$j^*\colon\ H^n_W(X; G, \mathscr{F}) \to H^n_V(U; G, j^* \mathscr{F})$$

is an isomorphism.

If $W \subset X$ is an equivariant locally closed subspace, and V is an equivariant *closed* subspace of W, then the canonical mapping $A_W \to A_V$ induces a natural homomorphism

$$(18) \qquad H^n_V(X; G, \mathscr{F}) \to H^n_W(X; G, \mathscr{F})$$

for any G-A-module $\mathscr{F}$ on X.

If $f\colon X \to Y$ is a proper mapping of locally compact spaces, then $f_! = f_*$, so by composing the adjunction morphism $A \to f_* f^* A = f_! A$ with the canonical mapping $f_! A_X \to Rf_! A$, we get a quasi-morphism $A \to Rf_! A$, which induces a quasi-morphism

$R\Gamma_c(Y,A) \to R\Gamma_c(Y,Rf_!A) \simeq R\Gamma_c(X,A)$, hence for any G-A-module M of a quasi-morphism

$$R\mathrm{Hom}_G(R\Gamma_c(X,A),M) \to R\mathrm{Hom}_G(R\Gamma_c(Y,A),M).$$

This defines for any $n \in \mathbf{Z}$ a homomorphism

$$f_*\colon H_n(X;G,M) \to H_n(Y;G,M), \tag{19}$$

which we call the *proper push-forward by* f. The construction is clearly functorial in M, and

$$(f \circ g)_* = f_* \circ g_*.$$

If $j\colon U \to X$ is the inclusion of an *open* equivariant subspace, we have a natural endomorphism $j_!A \to A$ which induces in a similar way a *restriction homomorphism*

$$j^*\colon H_n(X;G,M) \to H_n(U;G,M). \tag{20}$$

A short exact sequence

$$0 \to \mathscr{M} \to \mathscr{N} \to \mathscr{P} \to 0$$

of sheaves of G-A-modules on X gives us for every locally closed subspace $V \subset X$ a long exact sequence

$$\begin{aligned} \cdots \to H^n_V(X;G,\mathscr{M}) \to H^n_V(X;G,\mathscr{N}) \to H^n_V(X;G,\mathscr{P}) \to& \\ \to H^{n+1}_V(X;G,\mathscr{M}) \to H^{n+1}_V(X;G,\mathscr{N}) \to \cdots& \end{aligned} \tag{21}$$

which is a special case of the long exact sequence (14). A morphism of short exact sequences induces a morphism of long exact sequences and the pull-back morphisms f^* induce for any equivariant mapping $f\colon X \to Y$ a morphism of long exact sequences.

Similarly, assuming X to be locally compact of finite cohomological dimension, a short exact sequence of G-A-modules

$$0 \to M \to N \to P \to 0$$

gives rise to a long exact sequence

$$\cdots \to H_n(X;G,M) \to H_n(X;G,N) \to H_n(X;G,P) \to H_{n-1}(X;G,M) \to \cdots \tag{22}$$

A morphism of short exact sequences induces a morphism of long exact sequences and the push-forward morphisms f_* induce for any equivariant mapping $f\colon X \to Y$ a morphism of long exact sequences.

If $i\colon Z \hookrightarrow X$ is the inclusion of a *closed* equivariant subspace into an arbitrary G-space X, then for any sheaf $\mathscr{F}$ of G-A-modules on X the kernel of the canonical mapping $\mathscr{F} \to \mathscr{F}_Z$ is precisely the image of the natural morphism $\mathscr{F}_U \to \mathscr{F}$, where

$j\colon U \hookrightarrow X$ is the inclusion of the complement of Z. In other words, we get a short exact sequence

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Z \to 0, \tag{23}$$

which gives a long exact sequence

$$\begin{aligned} \cdots \to H^n(X; G, \mathscr{F}_U) \to H^n(X; G, \mathscr{F}) \xrightarrow{i^*} H^n(Z; G, i^*\mathscr{F}) \to \\ \to H^{n+1}(X; G, \mathscr{F}_U) \to \cdots \end{aligned} \tag{24}$$

where we use the fact that $H^n(X; G, \mathscr{F}_Z) \simeq H^n(Z; G, i^*\mathscr{F})$ by Proposition II.4.7.

If for any sheaf of G-A-modules $\mathscr{E}$ we apply the derived functor $R\mathrm{Hom}_G(-, \mathscr{E})$ to the triangle in $D_G(X, A)$ associated to the short exact sequence (23) with $\mathscr{F} = A$, we get a long exact sequence

(25)
$$\cdots \to H^n_Z(X; G, \mathscr{E}) \to H^n(X; G, \mathscr{E}) \to H^n(U; G, j^*\mathscr{E}) \to H^{n+1}_Z(X; G, \mathscr{E}) \to \cdots$$

where $R^n\mathrm{Hom}_G(A_U, \mathscr{E}) = H^n(U; G, j^*\mathscr{E})$, since we have by Remark II.2.4 a canonical isomorphism $R^n\mathrm{Hom}_G(A_U, \mathscr{E}) \simeq R^n\mathrm{Hom}_G(A, j^*\mathscr{E}))$. It can be checked that the mapping $H^n(X; G, \mathscr{E}) \to H^n(U; G, \mathscr{E})$ in the long exact sequence is the homomorphism j^* defined above.

Applying $R\mathrm{Hom}_G(R\Gamma_c(X, -), M)$ to the triangle associated to (23) with $\mathscr{F} = A$, we get for any bounded below complex M of G-A-modules a long exact sequence

(26)
$$\cdots \to H_n(Z; G, M) \xrightarrow{i_*} H_n(X; G, M) \xrightarrow{j^*} H_n(U; G, M) \to H_{n-1}(Z; G, M) \to \cdots$$

3. Cup product and cap product

Let $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$ be bounded complexes of G-A-modules on X. Let $\mathscr{F}^\bullet$ and $\mathscr{P}^\bullet$ be bounded above complexes of flat G-A-modules on X. Since A-$\mathfrak{Mod}_G(X)$ has enough flats, enough injectives and a tensor product, we have by [GH, Prop. 2.1] an A-bilinear pairing

(27)
$$R^p\mathrm{Hom}_G(\mathscr{F}^\bullet, \mathscr{M}^\bullet) \times R^q\mathrm{Hom}_G(\mathscr{P}^\bullet, \mathscr{N}^\bullet) \to R^{p+q}\mathrm{Hom}_G(\mathscr{F}^\bullet \otimes \mathscr{P}^\bullet, \mathscr{M}^\bullet \otimes \mathscr{N}^\bullet)$$

which is functorial in $\mathscr{F}^\bullet$, $\mathscr{P}^\bullet$, $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$, associative, and symmetric up to the usual sign $(-1)^{pq}$. In particular, if V and W are equivariant locally closed subspaces of X, then we can take $\mathscr{F}^\bullet$ to be the single sheaf A_V and $\mathscr{P}^\bullet$ to be the single sheaf A_W in degree 0 and we get the *cup product*

$$
\begin{array}{lcccc}
H_V^p(X;G,\mathscr{M}) & \times & H_W^q(X;G,\mathscr{N}) & \to & H_{V\cap W}^{p+q}(X;G,\mathscr{M}\otimes\mathscr{N}) \\
\omega & \times & \eta & \mapsto & \omega\cup\eta.
\end{array} \tag{28}
$$

which is functorial in X, V, W, $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$, associative and symmetric up to the usual sign $(-1)^{pq}$, by which we mean that

$$\omega\cup\eta = (-1)^{pq}\eta\cup\omega$$

if $\omega\in H_V^p(X;G,\mathscr{M})$ and $\eta\in H_W^q(X;G,\mathscr{N})$. Using the cup product we give $H_V^*(X;G,\mathscr{M})$ the structure of a right $H^*(X;G,A)$-algebra, hence also the structure of a right $H^*(G,A)$-algebra (since we have the canonical mapping $H^*(G,A)\to H^*(X;G,A)$ induced by the constant mapping $X\to\mathbf{pt}$).

For the definition of the pairing (27) we use the fact that by Proposition II.4.6 a class in $R^p\mathrm{Hom}_G(\mathscr{F}^\bullet,\mathscr{M}^\bullet)$ can be represented by a quasi-morphism $\alpha\colon \mathscr{F}^\bullet \dashrightarrow \mathscr{M}^\bullet[p]$, and a class in $R^q\mathrm{Hom}(\mathscr{P}^\bullet,\mathscr{N}^\bullet)$ by a quasi-morphism $\beta\colon \mathscr{P}^\bullet \dashrightarrow \mathscr{N}^\bullet[q]$. Now the product of α and β is represented by the composition

$$\mathscr{F}^\bullet\otimes\mathscr{P}^\bullet \xrightarrow{\mathrm{id}\otimes\beta} \mathscr{F}^\bullet\otimes\mathscr{N}^\bullet[q] \xrightarrow{\alpha\otimes\mathrm{id}} \mathscr{M}^\bullet[p]\otimes\mathscr{N}^\bullet[q] \xrightarrow{\tau_{p,q}} (\mathscr{M}^\bullet\otimes\mathscr{N}^\bullet)[p+q].$$

Here $\alpha\otimes\mathrm{id}$ is defined in the following way. Let α be represented by a diagram

$$\mathscr{F}^\bullet \Longleftarrow \mathscr{Q}^\bullet \longrightarrow \mathscr{M}^\bullet[p]$$

with $\mathscr{Q}^\bullet$ flat. The flatness of $\mathscr{F}^\bullet$ implies that $\mathscr{F}^\bullet\otimes\mathscr{N}^\bullet[q]\to\mathscr{Q}^\bullet\otimes\mathscr{N}^\bullet[q]$ is a quasi-isomorphism and $\alpha\otimes\mathrm{id}$ is the quasi-morphism represented by the diagram

$$\mathscr{F}^\bullet\otimes\mathscr{N}^\bullet[q] \Longleftarrow \mathscr{Q}^\bullet\otimes\mathscr{N}^\bullet[q] \longrightarrow \mathscr{M}^\bullet[p]\otimes\mathscr{N}^\bullet[q].$$

If $\mathscr{F}^\bullet=\mathscr{N}^\bullet=A$, the product of α and β is just the composite quasi-morphism $\alpha[q]\circ\beta$, as can be seen from the commutative diagram

$$
\begin{array}{ccccc}
A\otimes\mathscr{P}^\bullet & \xrightarrow{\mathrm{id}\otimes\beta} & A\otimes A[q] & \xrightarrow{\alpha\otimes\mathrm{id}} & \mathscr{M}^\bullet[p]\otimes A[q] \\
\downarrow & & \downarrow{\scriptstyle \tau_{0,q}} & & \downarrow{\scriptstyle \tau_{p,q}} \\
\mathscr{P}^\bullet & \xrightarrow{\beta} & A[q] & \xrightarrow{\alpha[q]} & \mathscr{M}^\bullet[p+q]
\end{array} \tag{29}
$$

When X is locally compact of finite cohomological dimension we can define a cap product between homology and cohomology in a similar way. Let $\mathscr{F}^\bullet$ and $\mathscr{P}^\bullet$ again be bounded above complexes of flat G-A-modules on X, and let $M^\bullet$ and $N^\bullet$ be bounded complexes of G-A-modules. We will define a pairing

$$
\begin{aligned}
&R^p\mathrm{Hom}_G(R\Gamma_c(X,\mathscr{F}^\bullet),M^\bullet)\times R^q\mathrm{Hom}_G(\mathscr{P}^\bullet,N^\bullet) \\
&\qquad\to R^{p+q}\mathrm{Hom}_G(R\Gamma_c(\mathscr{F}^\bullet\otimes\mathscr{P}^\bullet),M^\bullet\otimes N^\bullet).
\end{aligned} \tag{30}
$$

Taking $\mathscr{F}^\bullet = A_V$, and $\mathscr{P}^\bullet = A_W$, with V and W as above, we obtain the *cap product*

$$(31)\qquad \begin{array}{ccccc} H_p(V;G,M) & \times & H^q_W(X;G,N) & \to & H_{p-q}(V\cap W;G,M\otimes N) \\ \gamma & \times & \omega & \mapsto & \gamma\cap\omega. \end{array}$$

The cap product is functorial in X, V, W, M and N.

For the definition of the pairing (30) we embed $\mathscr{F}^\bullet$ in a complex $\mathscr{C}^\bullet$ of flat and c-soft G-A-modules on X. Since $\Gamma_c(X,-)$ transforms flat and c-soft A-modules into flat A-modules by Corollary II.5.4, we see that $\Gamma_c(X,\mathscr{C}^\bullet)$ represents $R\Gamma_c(X,\mathscr{C}^\bullet)$ and that it consists of flat A-modules. We define the product of a class in $R^p\mathrm{Hom}_G(R\Gamma_c(X,\mathscr{F}^\bullet),M^\bullet)$ represented by a quasi-morphism $\alpha\colon \Gamma_c(X,\mathscr{C}^\bullet) \dashrightarrow M^\bullet[p]$ and a class in $R^q\mathrm{Hom}(\mathscr{P}^\bullet,N^\bullet)$ represented by a quasi-morphism $\beta\colon \mathscr{P}^\bullet \dashrightarrow N^\bullet[q]$ to be the class represented by the composite quasi-morphism

$$\Gamma_c(X,\mathscr{C}^\bullet\otimes\mathscr{P}^\bullet) \overset{\beta'}{\dashrightarrow} \Gamma_c(X,\mathscr{C}^\bullet\otimes N^\bullet[q]) \simeq$$
$$\simeq \Gamma_c(X,\mathscr{C}^\bullet)\otimes N^\bullet[q] \overset{\alpha\otimes\mathrm{id}}{\dashrightarrow} M^\bullet[p]\otimes N^\bullet[q] \overset{\tau_{p,q}}{\dashrightarrow} (M^\bullet\otimes N^\bullet)[p+q],$$

where β' is the image of β under the exact functor $\Gamma_c(X,\mathscr{C}^\bullet\otimes -)$.

Remark 3.1. If $\mathscr{F}^\bullet = A$ and $N^\bullet = A$, then a diagram similar to (29) shows that the product of α and β is represented by the composite quasi-morphism $\alpha[q]\circ\beta'$.

From the definitions it is clear that if $\gamma\in H_p(V;G,M^\bullet)$, $\omega\in H^q_W(X;G,N^\bullet)$ and $\omega\in H^{q'}_{W'}(X;G,N'^\bullet)$, then

$$(32)\qquad (\gamma\cap\omega)\cap\omega' = \gamma\cap(\omega\cup\omega),$$

and we give $H_*(X;G,M)$ the structure of a right $H^*(X;G,A)$-module, hence also the structure of a right $H^*(G,A)$-module using the cap product.

If $f\colon X\to Y$ is a proper equivariant mapping between locally compact spaces of finite cohomological dimension, and V, W are locally closed subspaces of Y, then for $\gamma\in H_p(f^{-1}(V);G,M^\bullet)$ and $\omega\in H^q_W(Y;G,N^\bullet)$ we have the *projection formula*

$$(33)\qquad f_*\gamma\cap\omega = f_*(\gamma\cap f^*\omega).$$

This follows from the commutativity of the following diagram, where $i\colon V\hookrightarrow Y$, $j\colon W\hookrightarrow Y$, $i'\colon f^{-1}(V)\hookrightarrow X$ and $j'\colon f^{-1}(W)\hookrightarrow X$, denote the inclusions.

$$\begin{tikzcd}
R\Gamma_c(Y,A_V\otimes A_W) \arrow[r, dashed, "R\Gamma_c(\mathrm{id}\otimes\omega)"] \arrow[d] & R\Gamma_c(Y,A_V\otimes N^\bullet[q]) \arrow[d] \\
R\Gamma_c(X,A_{f^{-1}(V)}\otimes A_{f^{-1}(W)}) \arrow[r, dashed, "R\Gamma_c(\mathrm{id}\otimes f^*\omega)"] & R\Gamma_c(X,A_{f^{-1}(V)}\otimes N^\bullet[q]) \arrow[d, "\tau_{p,q}\circ(\gamma\otimes\mathrm{id})"] \\
 & (M^\bullet\otimes N^\bullet)[p+q]
\end{tikzcd}$$

If X is a *compact* space of finite cohomological dimension, the constant mapping $f\colon X \to \mathbf{pt}$ is proper, so it induces a homomorphism

$$f_*\colon H_0(X; G, A) \to H_0(\mathbf{pt}; G, A).$$

Since $H_0(\mathbf{pt}; G, A) = H^0(G, A) = A$, the mapping f_* induces a canonical homomorphism

$$\deg\colon H_0(X; G, A) \to A,$$

which enables us to define for any $k \in \mathbf{Z}$ a *cap product pairing*

$$\begin{array}{rcccl} \langle -, - \rangle\colon H_k(X; G, A) & \times & H^k(X; G, A) & \to & A \\ \gamma & \times & \omega & \mapsto & \deg(\gamma \cap \omega). \end{array} \tag{34}$$

Remark 3.2. The constructions in Sections 2 and 3 also work for the groups $H^p_V(X; G, \mathscr{F}^\bullet) = R^p\mathrm{Hom}_G(A_V, \mathscr{F}^\bullet)$ (often referred to as *hypercohomology groups*), where $\mathscr{F}^\bullet$ is a bounded below complex of G-A-modules on X. The sole exception is the cup product

$$H^p_V(X; G, \mathscr{M}^\bullet) \times H^q_W(X; G, \mathscr{N}^\bullet) \to H^{p+q}_{V\cap W}(X; G, \mathscr{M}^\bullet \otimes \mathscr{N}^\bullet),$$

which we have defined for *bounded* complexes $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$, using the pairing (27). However, a closer look at the definition of the product (27) reveals that it already works when $\mathscr{M}^\bullet$ is bounded below and only $\mathscr{N}^\bullet$ is bounded on both sides. Hence we can consider $H^*_V(X; G, \mathscr{F}^\bullet)$ as a graded right $H^*(X; G, A)$-module and as a graded right $H^*(G, A)$-module.

4. Equivariant Poincaré duality

Let G be a group, let A be a commutative noetherian ring, and let X be an n-*dimensional cohomology manifold* over A with a G-action. The definition of a cohomology manifold can be found in [Bo1, Ch. I]; its dimension is the cohomological dimension as defined in Section II.4. The prime example is a (not necessarily compact) n-dimensional topological manifold. We define the *orientation sheaf* $\mathscr{O}r_X(A)$ of X with coefficients in A to be the sheaf associated to the presheaf

$$U \mapsto H_n(U, A)$$

(here, as always, $H_n(U, A)$ denotes Borel-Moore homology). In fact, this presheaf is already a sheaf, locally isomorphic to A. It has a natural G-action induced by the G-action on X.

We say that X is A-*orientable* if $\mathscr{O}r_X(A)$ is isomorphic to the constant sheaf A (in the category A-$\mathfrak{Mod}(X)$), and an A-*orientation* of X is an isomorphism

$$\psi\colon A \xrightarrow{\sim} \mathscr{O}r_X(A)$$

of sheaves of A-modules on X. Any cohomological manifold has a unique $\mathbf{Z}/2$-orientation. An orientation of differentiable manifold X in the usual sense induces a unique $\mathbf{Z}$-orientation $\mathbf{Z} \xrightarrow{\sim} \mathcal{O}r_X(\mathbf{Z})$ and vice versa. If a group G acts on X, then we say that the G-action *preserves the A-orientation* if ψ is an isomorphism of G-sheaves.

Assume that X is an n-dimensional cohomology manifold with a G-action. Denoting the constant mapping by $f\colon X \to \mathbf{pt}$, we have a canonical quasi-isomorphism $f^!A \xrightarrow{\sim} \mathcal{O}r_X A[n]$ of G-A-modules on X (cf. [Ve, Exp. 5]). Since we have by Verdier duality (Theorem II.5.8) a canonical isomorphism

$$R^p\mathrm{Hom}_G(A, f^!A) \simeq H_{-p}(X; G, A),$$

for every $p \in \mathbf{Z}$, this means that

$$\begin{aligned} H^p(X; G, \mathcal{O}r_X A) \simeq R^p\mathrm{Hom}_G(A, f^!A[-n]) &= \\ = R^{p-n}\mathrm{Hom}_G(A, f^!A) &\simeq H_{n-p}(X; G, A). \end{aligned}$$

If the G-action preserves the A-orientation, the equivariant isomorphism $A \xrightarrow{\sim} \mathcal{O}r_X A$ induces an isomorphism $H^p(X; G, A) \xrightarrow{\sim} H_{n-p}(X; G, A)$. More generally the A-orientation gives for every closed G-subspace $i\colon Z \hookrightarrow X$ an isomorphism

$$H^p_Z(X; G, A) \xrightarrow{\sim} R^p\mathrm{Hom}_G(i_!A, f^!A[-n]) \simeq H_{n-p}(Z; G, A).$$

Conceptually it is better to describe these isomorphisms in terms of the cap product.

Definition 4.1. Let A be a noetherian commutative ring, let G be a group and let X be an n-dimensional A-oriented manifold with an A-orientation preserving action of G. The *equivariant fundamental class* $\mu_X \in H_n(X; G, A)$ of X is the image of the A-orientation $A \xrightarrow{\sim} \mathcal{O}r(A)$ under the canonical isomorphism $R^0\mathrm{Hom}_G(A, \mathcal{O}r(A)) \simeq H_n(X; G, A)$.

It is clear from the definition that $e(\mu_X) \in H_n(X; A)$ is the usual fundamental class of X. Hence the image of μ_X under the composite homomorphism

$$H_n(X; G, A) \xrightarrow{e} H_n(X; A) \longrightarrow \varinjlim_{U \ni x} H_n(U; A) \simeq A$$

is a generator for any point $x \in X$, where the limit is taken over the open neighbourhoods of x. Conversely, if a class $\nu \in H_n(X; G, A)$ has this property, then ν induces an A-orientation of X which is preserved by the action of G and for which ν is the fundamental class.

Theorem 4.2 (Poincaré duality). *Let A be a noetherian commutative ring, let G be a group and let X be an A-oriented n-dimensional cohomology manifold with an A-orientation preserving G-action. Cap product with the equivariant fundamental class $\mu_X \in H_n(X; G, A)$ gives for any closed G-subspace $Z \subset X$ and any G-A-module M an isomorphism*

$$H^m_Z(X; G, M) \xrightarrow{\mu_X \cap} H_{n-m}(Z; G, M).$$

PROOF. Let $\alpha\colon\ Rf_!A \dashrightarrow A[-n]$ be the quasi-morphism corresponding to μ_X, where $f\colon\ X \to \mathbf{pt}$ is the constant mapping. Then tensoring with M gives a quasi-morphism $\alpha'\colon\ Rf_!M \dashrightarrow M[-n]$. The image of α' under the isomorphism

$$\Psi\colon\ \mathrm{Hom}_{D_G^+(A\text{-}\mathfrak{Mod})}(Rf_!M, M[-n]) \xrightarrow{\sim} \mathrm{Hom}_{D_G^+(X,A)}(M, f^!M[-n])$$

is a quasi-isomorphism $\Psi(\alpha')\colon\ M \overset{\sim}{\dashrightarrow} f^!M[-n]$. This means that composition with $\Psi(\alpha'[m])$ defines an isomorphism

$$\begin{aligned} H_Z^m(X; G, M) = \mathrm{Hom}_{D_G^+(X,A)}(A_Z, M[m]) \to \\ \to \mathrm{Hom}_{D_G^+(X,A)}(A_Z, f^!M[m-n]) = H_{n-m}(Z; G, M). \end{aligned}$$

On the other hand, composition with $\Psi(\alpha'[m])$ corresponds to cap product with μ_X, by Remark 3.1 and the commutativity of the following diagram.

$$\begin{tikzcd} \mathrm{Hom}_{D_G^+(X,A)}(j_!A, M[m]) \arrow[r, "\circ\Psi(\alpha'[m])"] \arrow[d, "Rf_!"] & \mathrm{Hom}_{D_G^+(X,A)}(j_!A, f^!M[m-n]) \arrow[d, "\Psi^{-1}"] \\ \mathrm{Hom}_{D_G^+(A)}(R(f\circ j)_!A, Rf_!M[m]) \arrow[r, "\circ\alpha'[m]"] & \mathrm{Hom}_{D_G^+(A)}(R(f\circ j)_!A, M[m-n]) \end{tikzcd}$$

where $A \to \mathscr{C}^\bullet$ is a c-soft flat resolution of A, and j is the inclusion $W \hookrightarrow X$. □

Remark 4.3. Using Poincaré duality we can transfer constructions in homology to cohomology and vice versa. We can for example define a *Gysin map*

$$f_!\colon\ H^{n-k}(X; G, A) \to H^{m-k}(Y; G, A)$$

for any proper mapping $f\colon\ X \to Y$ of cohomological manifolds with an A-orientation preserving G-action, where n is the dimension of X and m is the dimension of Y. As in the nonequivariant case, $f_!$ is determined by the equation

$$\mu_Y \cap f_!\omega = f_*(\mu_X \cap \omega). \tag{35}$$

When X is a compact cohomological manifold of dimension n with an A-orientation preserving G-action, the cap product pairing (34) can be transformed to the *cup product pairing*

$$\begin{array}{ccccc} H^{n-k}(X; G, A) & \times & H^k(X; G, A) & \to & A \\ \omega & \times & \omega' & \mapsto & \langle\omega, \omega'\rangle, \end{array} \tag{36}$$

which is related to the cap product pairing by the formula

$$\langle\omega, \omega'\rangle = \langle\mu_X \cap \omega, \omega'\rangle. \tag{37}$$

Similarly, for X as above, the cap product pairing can be transformed to the *intersection pairing* in Borel-Moore homology

$$\begin{array}{ccccc} H_k(X; G, A) & \times & H_{n-k}(X; G, A) & \to & A \\ \gamma & \times & \gamma' & \mapsto & \langle\gamma, \gamma'\rangle, \end{array} \tag{38}$$

which is related to the cap product pairing by the formula

$$\langle \gamma, \mu_X \cap \omega \rangle = \langle \gamma, \omega \rangle \tag{39}$$

for $\gamma \in H_k(X; G, A)$, and $\omega \in H_{n-k}(X; G, A)$. Note that both the cup product pairing and the intersection pairing depend on the choice of an A-orientation (hence of the fundamental class μ_X).

5. The Hochschild-Serre spectral sequences

Proposition 5.1 (Hochschild-Serre spectral sequence for cohomology). *Let G be any group, let X be a G-space, let W be a locally closed G-subspace of X. Then for any bounded below complex $\mathscr{F}$ of G-A-modules on X there is a spectral sequence*

$$E_2^{p,q} = H^p(G, H^q_W(X, \mathscr{F})) \Rightarrow H^{p+q}_W(X; G, \mathscr{F}),$$

functorial in X and $\mathscr{F}$.

Proof. The existence of the spectral sequence is by Corollary II.3.5 a consequence of the isomorphism of functors

$$R\mathrm{Hom}_G(A_W, -) = R\Gamma^G \circ R\mathrm{Hom}(A_W, -)$$

(Proposition II.4.5).

By functoriality in X we mean that a mapping of G-spaces $f\colon X' \to X$ induces a commutative 'diagram'

$$\begin{array}{ccc} H^p(G, H^q_W(X, \mathscr{F})) & \Rightarrow & H^{p+q}_W(X; G, \mathscr{F}) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ H^p(G, H^q_W(X', f^*\mathscr{F})) & \Rightarrow & H^{p+q}_W(X'; G, f^*\mathscr{F}) \end{array}$$

By functoriality in $\mathscr{F}$ we mean that a mapping $\mathscr{E} \to \mathscr{F}$ of complexes of G-A-modules on X induces a commutative 'diagram'

$$\begin{array}{ccc} H^p(G, H^q_W(X, \mathscr{E})) & \Rightarrow & H^{p+q}_W(X; G, \mathscr{E}) \\ \downarrow & & \downarrow \\ H^p(G, H^q_W(X, \mathscr{F})) & \Rightarrow & H^{p+q}_W(X; G, \mathscr{F}) \end{array}$$

Both diagrams are an immediate consequence of the functorial origin of the spectral sequence. □

The filtration of $H^*_W(X; G, \mathscr{F})$ associated to the Hochschild-Serre spectral sequence will be denoted by F^*. We have

$$0 = F^{-1} H^n_W(X; G, \mathscr{F}) \subset F^1 H^n_W(X; G, \mathscr{F}) \subset \dots \\ \subset F^n H^n_W(X; G, \mathscr{F}) = H^n_W(X; G, \mathscr{F})$$

and

$$F^q H^n_W(X; G, \mathscr{F}) / F^{q-1} H^n_W(X; G, \mathscr{F}) = E^{n-q,q}_\infty.$$

The edge morphism

$$e\colon H^p_W(X; G, \mathscr{F}) \to H^p_W(X, \mathscr{F})^G$$

in degree p of the Hochschild-Serre spectral sequence is the homomorphism induced by the canonical mapping

$$R\mathrm{Hom}_G(A_W, \mathscr{F}) \to R\mathrm{Hom}(A_W, \mathscr{F}).$$

It is compatible with cup product:

$$e(\omega) \cup e(\eta) = e(\omega \cup \eta) \tag{40}$$

for any $\omega \in H^p_V(X; G, \mathscr{M})$ and any $\eta \in H^q_W(X; G, \mathscr{N})$, where V, W are arbitrary locally closed G-subspaces of X and $\mathscr{M}$, $\mathscr{N}$ are arbitrary bounded below complexes of G-A-modules on X.

Proposition 5.2 (Hochschild-Serre spectral sequence for homology). *Let G be any group, let X be locally compact a G-space, of finite cohomological dimension. Then for any bounded below complex M of G-A-modules there is a spectral sequence*

$$E^2_{p,q} = H^{-p}(G, H_q(X, M)) \Rightarrow H_{p+q}(X; G, M),$$

functorial in X and M.

Proof. The existence of the spectral sequence is by Corollary II.3.5 a consequence of the functor isomorphism

$$R\mathrm{Hom}_G(R\Gamma_c(X, A), -) = R\Gamma^G \circ R\mathrm{Hom}(R\Gamma_c(X, A), -)$$

(Proposition II.4.5). □

The edge morphism

$$e\colon H_p(X; G, M) \to H_p(X, M)^G$$

of the Hochschild-Serre spectral sequence is just the homomorphism induced by the canonical mapping

$$R\mathrm{Hom}_G(R\Gamma_c(X, A), M) \to R\mathrm{Hom}(R\Gamma_c(X, A), M).$$

It is compatible with cap product:

$$e(\gamma) \cap e(\omega) = e(\gamma \cap \omega), \tag{41}$$

hence for a compact G-space X we have

$$\langle \gamma, \omega \rangle = \langle e(\gamma), e(\omega) \rangle \tag{42}$$

for $\gamma \in H_k(X; G, A)$ and $\omega \in H^k(X; G, A)$, and when X is a compact cohomological manifold of dimension n with an A-orientation preserving G-action, we have

$$\langle \omega, \omega' \rangle = \langle e(\omega), e(\omega') \rangle \tag{43}$$

for $\omega \in H^{n-k}(X; G, A)$ and $\omega' \in H^k(X; G, A)$, and

$$\langle \gamma, \gamma' \rangle = \langle e(\gamma), e(\gamma') \rangle \tag{44}$$

for $\gamma \in H_k(X; G, A)$, and $\gamma' \in H_{n-k}(X; G, A)$, since e maps the euivariant fundamental class of X to the ordinary fundamental class of X.

The filtration of $H_*(X; G, M)$ associated to the Hochschild-Serre spectral sequence will be denoted by F_*. If d is the cohomological dimension of X, we have We have

$$0 = F_{d+1} H_n(X; G, \mathscr{F}) \subset F_d H_n(X; G, \mathscr{F}) \subset \cdots \\ \subset F_n H_n(X; G, \mathscr{F}) = H_n(X; G, \mathscr{F})$$

and

$$F_q H_n(X; G, \mathscr{F}) / F_{q-1} H_n(X; G, \mathscr{F}) = E^{\infty}_{n-q,q}.$$

Remark 5.3. If X is an A-oriented n-dimensional cohomology manifold with an A-orientation preserving G-action, then of course the Poincaré duality isomorphism

$$H^m_Z(X; G, M) \to H_{n-m}(Z; G, M).$$

of Theorem 4.2 corresponds to an isomorphism between the spectral sequence

$$E_2^{p,q} = H^p(G, H^q_Z(X, M)) \Rightarrow H^{p+q}_Z(X; G, M)$$

and the spectral sequence

$$E^2_{p,q} = H^{-p}(G, H_q(Z, M)) \Rightarrow H_{p+q}(Z; G, M),$$

which is on the E_2-level given by the collection of isomorphisms

$$H^p(G, H^q_Z(X, M)) \xrightarrow{\sim} H^p(G, H_{n-q}(Z, M))$$

induced by the Poincaré duality isomorphisms $H^q_Z(X, M) \xrightarrow{\sim} H_{n-q}(Z, M)$.

6. Localization

One of the main purposes of equivariant (co)homology, is establishing connections between the ordinary (co)homology groups of a G-space, of the fixed point set and of the quotient space. We will first study the extreme cases of fixed point free actions and trivial actions, and then prove a localization theorem of Borel-Atiyah-Segal type for equivariant Borel-Moore homology. The base ring A will be assumed to be noetherian and commutative throughout this section. In order not to overload the notation, cohomology with supports in a locally closed subspace will not be mentioned, but everything generalizes without any problem.

A central role is played by the Leray spectral sequence

$$E_2^{p,q} = H^p(X/G, R^q\pi_*^G\mathscr{F}) \Rightarrow H^{p+q}(X;G,\mathscr{F}).$$

We will now define a class of G-actions for which the sheaves $R^q\pi_*^G\mathscr{F}$ are easy to describe. Recall that if x is a point of a G-space X, then the *stabilizer* G_x of x is the subgroup of G leaving x fixed, i.e. $G_x = \{g \in G : g\cdot x = x\}$. The G-action is said to be *free* if $G_x = \{1\}$ for every $x \in X$. A G-action on X is called *properly discontinuous* if for every $x \in X$ the stabilizer G_x is finite and there is a neighbourhood U_x of x such that $g\cdot U_x \cap U_x = \emptyset$ for any $g \in G \setminus G_x$. In particular, a finite group action on a Hausdorff space X is properly discontinuous. If G acts properly discontinuously on a space X, then X/G is at a point y locally homeomorphic to the finite quotient U_x/G_x for an arbitrary $x \in \pi^{-1}(y)$. This implies that if X is locally compact of cohomological dimension n, then X/G is locally compact of cohomological dimension n by [Bo1, Prop. III.5.1], and if X is locally paracompact of strict cohomological dimension n, then X/G is locally paracompact of strict cohomological dimension n by [Qu, Prop. A.11].

Lemma 6.1. *Let X be a space with a properly discontinuous action of a group G. Let $y \in X/G$ and $x \in \pi^{-1}(y) \subset X$. Then for any G-A-module $\mathscr{F}$ on X and any $q \in \mathbf{Z}$ the stalk $(R^q\pi_*^G\mathscr{F})_y$ is isomorphic to $H^q(G,(\pi_*\mathscr{F})_y) \simeq H^q(G_x,\mathscr{F}_x)$.*

Proof. See [Gr, Th. 5.3.1]. □

From the above lemma and the Leray spectral sequence we see that for a group G acting properly discontinuously and freely on X the edge morphism

$$H^*(X/G;\pi_*^G\mathscr{F}) \to H^*(X;G,\mathscr{F})$$

is an isomorphism. In particular, if M is an A-module with a trivial G-action we have for such a G-space a canonical isomorphism

$$H^*(X/G;M) \simeq H^*(X;G,M).$$

Using Verdier duality this easily generalizes to homology.

Proposition 6.2. *Let X be a locally compact space of finite cohomological dimension with a properly discontinuous free G-action. For every A-module M with trivial G-action and every $p \in \mathbf{Z}$ there is a natural isomorphism*

$$H_p(X/G;M) \xrightarrow{\sim} H_p(X;G,M).$$

Proof. Since $\pi\colon X \to X/G$ is a locally trivial covering, we have a canonical isomorphism of derived functors $\pi^* \xrightarrow{\sim} \pi^!$. Since π_*^G is exact by Lemma 6.1, the derived functor $R\pi_*^G \circ \pi^*$ is canonically isomorphic to the functor $\pi_*^G \circ \pi^*$, which is the identity functor $\mathrm{id}_{X/G}$ on $D(X/G, A)$. Hence we obtain an isomorphism $\mathrm{id}_{X/G} \xrightarrow{\sim} R\pi_*^G \circ \pi^!$, which induces for any A-module M an isomorphism

$$R\Gamma(X/G, f^! M) \xrightarrow{\sim} R\Gamma(X/G, R\pi_*^G \pi^! f^! M) \simeq R\Gamma^G(X, (\pi \circ f)^! M),$$

where $f\colon X/G \to \mathbf{pt}$ denotes the constant mapping. By Verdier duality this gives an isomorphism $H_p(X/G, M) \xrightarrow{\sim} H_p(X; G, M)$. □

If a group G acts trivially on X and $\mathscr{F}$ is a sheaf of A-modules on which G acts trivially as well, we have a natural mapping $R\mathrm{Hom}(A, \mathscr{F}) \to R\mathrm{Hom}_G(A, \mathscr{F})$, which induces for every $p \in \mathbf{Z}$ a homomorphism $H^p(X; \mathscr{F}) \to H^p(X; G, \mathscr{F})$. By abuse of notation we will denote the image of a class $\omega \in H^p(X; \mathscr{F})$ by the same symbol ω. If G is finite and A is a field, the above homomorphism and the cup product induce an isomorphism of graded A-modules

$$H^*(X; \mathscr{F}) \otimes H^*(G, A) \to H^*(X; G, \mathscr{F}).$$

This follows from the Künneth type formula

$$H^n(X; G, \mathscr{F}) \simeq \bigoplus_{p+q=n} \mathrm{Hom}(H_q(G, A), H^p(X; \mathscr{F}))$$

(see [Gr, Th. 4.4.1]) and the fact that the cup product corresponds to the composition mapping

$$\bigoplus_{p+q=n} \mathrm{Hom}(A, H^p(X; \mathscr{F})) \otimes \mathrm{Hom}(H_q(G, A), A) \to \bigoplus_{p+q=n} \mathrm{Hom}(H_q(G, A), H^p(X; \mathscr{F})),$$

which is an isomorphism, since $H_p(G, A)$ has finite dimension because G is finite.

In equivariant Borel-Moore homology a similar fact holds true. If G acts trivially on X, and M is an A-module on which G acts trivially as well, we have a natural mapping

$$R\mathrm{Hom}(R\Gamma_c(X, A), M) \to R\mathrm{Hom}_G(R\Gamma_c(X, A), M),$$

which induces for every $p \in \mathbf{Z}$ a homomorphism $H_p(X; M) \to H_p(X; G, M)$; the image of a nonequivariant class γ will again be denoted by γ.

Proposition 6.3. *Let G be a finite group acting trivially on a locally compact space X of finite cohomological dimension. If A is a field and M is an A-module with a trivial G-action, the natural homomorphism $H_p(X; M) \to H_p(X; G, M)$ and the cap product induces an isomorphism of graded A-modules*

$$H_*(X; M) \otimes H^*(G, A) \xrightarrow{\sim} H_*(X; G, M).$$

PROOF. Since G acts trivially on X and M, we have a composite isomorphism

$$R\mathrm{Hom}_G(R\Gamma_c(X,A),M) \simeq R\mathrm{Hom}_G(A, R\mathrm{Hom}(R\Gamma_c(X,A),M)) \simeq$$
$$\simeq R\mathrm{Hom}(A \overset{L}{\underset{G}{\otimes}} A, R\mathrm{Hom}(R\Gamma_c(X,A),M)),$$

and we obtain for every $n \in \mathbf{Z}$ a Künneth type isomorphism

$$H_n(X;G,M) \simeq \bigoplus_{p-q=n} \mathrm{Hom}(H_q(G,A), H_p(X;M)).$$

A closer analysis of the cap product (cf. Remark 3.1) shows that the mapping

$$H_*(X;M) \otimes H^*(G,A) \to H_*(X;G,M)$$

corresponds to the composition mapping

$$\bigoplus_{p-q=n} \mathrm{Hom}(A, H_p(X,M)) \otimes \mathrm{Hom}(H_q(G,A),A) \to$$
$$\bigoplus_{p-q=n} \mathrm{Hom}(H_q(G,A), H_p(X,M)),$$

which is an isomorphism, since each $H_q(G,A)$ is finite dimensional. □

Now we will consider a localization theorem of Borel-Atiyah-Segal type for equivariant Borel-Moore homology. Suppose G acts properly discontinuously on X. Let S be a multiplicative subset of the centre of $H^*(G,A)$, where A can be any noetherian commutative ring. For every $x \in X$ the inclusion $G_x \hookrightarrow G$ induces a homomorphism $H^*(G,A) \to H^*(G_x,A)$. We define for $\eta \in H^*(G,A)$ the closed subspace

$$X^\eta = \{x \in X : \eta \text{ does not map to } 0 \text{ in } H^*(G_x,A)\}$$

and for a subset $S \subset H^*(G,A)$ we define $X^S = \bigcap_{\eta\in S} X^\eta$.

A localization theorem for equivariant cohomology of Borel-Atiyah-Segal type states that under certain conditions on X and G the inclusion $X^S \hookrightarrow X$ induces an isomorphism

$$S^{-1}H^*(X;G,M) \xrightarrow{\sim} S^{-1}H^*(X^S;G,M)$$

(see for example [Hs, §III.2]). Here we will avoid one of the technical conditions on the G-action by proving that $S^{-1}H^*(X;G,M) \to S^{-1}H^*(X^\eta;G,M)$ is an isomorphism for any $\eta \in S$. When G is finite, which is the case we will be interested in, this does not make a difference, since then we always have an $\eta \in S$ such that $X^\eta = X^S$.

In order to prove the analogue for equivariant Borel-Moore homology, we will use Verdier duality. This means that we need a version of the localization theorem valid for equivariant hypercohomology (see Remark 3.2).

Theorem 6.4. *Let G be a group acting properly discontinuously on a locally paracompact space X of finite strict cohomological dimension. Let A be a noetherian commutative ring, let S be a multiplicative subset of the centre of $H^*(G, A)$ and let $\eta \in S$. For any bounded below complex of G-A-modules $\mathscr{F}^\bullet$, the inclusion $i\colon X^\eta \hookrightarrow X$ induces an isomorphism*

$$S^{-1}H^*(X; G, \mathscr{F}^\bullet) \xrightarrow{\sim} S^{-1}H^*(X^\eta; G, i^*\mathscr{F}^\bullet).$$

Proof. The proof is essentially the same as the proof of [Hs, Th. III.1']. We disregard the trivial case by assuming $\eta \notin H^0(G, A)$. In order to simplify notation we assume that η is homogeneous of degree d. By the hypercohomology version of the long exact sequence (24) it is sufficient to prove that $S^{-1}H^*(X; G, \mathscr{F}^\bullet) = 0$ for any bounded below complex of sheaves with $\operatorname{supp}(\mathscr{F}^\bullet) \cap X^\eta = \emptyset$.

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(X/G, R^q\pi_*^G\mathscr{F}^\bullet) \Rightarrow H^{p+q}(X; G, \mathscr{F}^\bullet),$$

and observe that cup product with η is visible at the E_2-level as the mapping $H^p(X/G, R^q\pi_*^G\mathscr{F}^\bullet) \to H^p(X/G, R^{q+d}\pi_*^G\mathscr{F}^\bullet)$ induced by a coefficient mapping

$$\varphi\colon R^q\pi_*^G\mathscr{F}^\bullet \to R^{q+d}\pi_*^G\mathscr{F}^\bullet.$$

Restricted to the stalk at a point $y \in X/G$, the mapping φ is the homomorphism of groups $H^q(G_x, \mathscr{F}_x^\bullet) \to H^{q+d}(G_x, \mathscr{F}_x^\bullet)$ given by cup product with the image of η in $H^d(G_x, A)$, where x is any point in the fibre $\pi^{-1}(y)$. In other words, the assumption $\operatorname{supp}(\mathscr{F}^\bullet) \cap X^\eta = \emptyset$ implies that φ is zero.

This need not imply that cup product with η is zero in $H^*(X; G, \mathscr{F})$. Nevertheless, since X/G has finite strict cohomological dimension, there is an N such that $H^p(X/G; R^q\pi_*^G\mathscr{F}^\bullet) = 0$ for all $p > N$. The general theory of spectral sequences then tells us that the mapping

$$H^k(X; G, \mathscr{F}^\bullet) \to H^{k+(N+1)d}(X; G, \mathscr{F}^\bullet)$$

given by cup product with η^{N+1} is zero. □

Theorem 6.5. *Let G be a group acting properly discontinuously on a locally compact space X of finite strict cohomological dimension. Let A be a noetherian commutative ring, let S be a multiplicative subset in the centre of $H^*(G, A)$ and let $\eta \in S$. For any G-A-module M the inclusion $i\colon X^\eta \hookrightarrow X$ induces an isomorphism*

$$S^{-1}H_*(X^\eta; G, M) \xrightarrow{\sim} S^{-1}H_*(X; G, M).$$

Proof. By the long exact sequence (26) it is sufficient to prove that $S^{-1}H_*(X - X^\eta; G, M) = 0$, but this follows from Theorem 6.4, since by Verdier duality we have that $S^{-1}H_*(X - X^\eta; G, M) = S^{-1}H^*(X - X^\eta; G, f^!M)$, where $f\colon X - X^\eta \to \mathbf{pt}$ is the constant mapping. □

7. Transformation groups of prime order

In this section we will study the equivariant cohomology and homology of a space X with an action of $G = \mathbf{Z}/p$, the cyclic group of prime order p. In order to be able to apply the results of the previous section, we put some cohomological conditions on X. A *nice, finite dimensional G-space* is a locally compact G-space of finite strict cohomological dimension. For example, a finite dimensional locally finite CW-complex with a G-action is a nice, finite dimensional G-space.

Let p be a prime and let σ be the generator of $G = \mathbf{Z}/p$. Let A be a commutative ring and let $\nu = 1 + \sigma + \sigma^2 + \cdots + \sigma^{p-1}$ be the *norm element* of the group ring $A[G]$. Then for any G-A-module M we have the following canonical Γ^G-acyclic resolution of M:

(45)

$$M \longrightarrow \mathrm{Hom}(A[G], M) \xrightarrow{1-\sigma} \mathrm{Hom}(A[G], M) \xrightarrow{\nu} \mathrm{Hom}(A[G], M) \xrightarrow{1-\sigma} \cdots$$

Hence $H^*(G, M)$ is the homology of the complex of abelian groups

$$M \xrightarrow{1-\sigma} M \xrightarrow{\nu} M \xrightarrow{1-\sigma} \cdots \tag{46}$$

In particular, $H^*(G, M)$ is periodic: we have for any $k \geq 0$ a surjection $H^k(G, M) \to H^{k+2}(G, M)$, which is an isomorphism when $k > 0$. With some more effort it can be deduced from the resolution (45) that as a graded ring

$$H^*(G, \mathbf{Z}/p) \simeq \begin{cases} \mathbf{Z}/p[\eta] & \text{if } p = 2, \\ \mathbf{Z}/p[\zeta, \eta^2]/(\zeta^2) & \text{if } p \text{ is odd.} \end{cases}$$

Here η and ζ have degree 1 and η^2 has degree 2. Hence Proposition 6.3 gives for any nice finite dimensional space X an isomorphism of groups

$$h\colon\ H_k(X^G; G, \mathbf{Z}/p) \simeq \bigoplus_{m \geq k} H_m(X; \mathbf{Z}/p)$$

and we have an analogous isomorphism of groups in cohomology. When $p \neq 2$ these isomorphisms do not behave well with respect to cup product and cap product, since ζ is sent to 1, whereas $\zeta^2 = 0$.

Therefore we take the maximal ideal

$$\mathfrak{m} = \begin{cases} (\eta - 1) & \text{if } p = 2, \\ (\zeta, \eta^2 - 1) & \text{if } p \text{ is odd,} \end{cases}$$

of $H^*(G, \mathbf{Z}/p)$. By Proposition 6.3, we have an isomorphism

$$H_*(X^G; G, \mathbf{Z}/p)/\mathfrak{m} \simeq H_*(X^G; \mathbf{Z}/p), \tag{47}$$

and in cohomology we have the corresponding isomorphism

(48) $$H^*(X^G; G, \mathbf{Z}/p)/\mathfrak{m} \simeq H^*(X^G; \mathbf{Z}/p).$$

These isomorphisms do preserve cup product and cap product. The following proposition gives the main reason why $\mathbf{Z}/p$-actions are so much easier to handle than actions of other groups.

Proposition 7.1. *With notations as above, we have for any nice, finite-dimensional G-space X canonical isomorphisms*

$$H_*(X; G, \mathbf{Z}/p)/\mathfrak{m} \simeq H_*(X^G; \mathbf{Z}/p)$$

and

$$H^*(X; G, \mathbf{Z}/p)/\mathfrak{m} \simeq H^*(X^G; \mathbf{Z}/p).$$

Proof. Let $S = H^*(G, \mathbf{Z}/p) - \mathfrak{m}$. The inclusion $i\colon X^G \hookrightarrow X$ induces by Theorem 6.5 an isomorphism

$$S^{-1}H_*(X^G; G, \mathbf{Z}/p) \xrightarrow{\sim} S^{-1}H_*(X; G, \mathbf{Z}/p),$$

hence an isomorphism

$$H_*(X^G; G, \mathbf{Z}/p)/\mathfrak{m} \xrightarrow{\sim} H_*(X; G, \mathbf{Z}/p)/\mathfrak{m} = H_*(X^G; \mathbf{Z}/p).$$

The corresponding statement in cohomology follows from Theorem 6.4. □

In other words, reduction modulo $\mathfrak{m}$ gives a homomorphism of rings

$$\beta\colon H^*(X; G, \mathbf{Z}/p) \to H^*(X^G; \mathbf{Z}/p),$$

and a homomorphism of groups

$$\rho\colon H_*(X; G, \mathbf{Z}/p) \to H_*(X^G; \mathbf{Z}/p),$$

such that, by the projection formula (33),

$$\rho(\gamma) \cap \beta(\omega) = \rho(\gamma \cap \omega).$$

Of course we have for any equivariant mapping $f\colon X \to X'$ that

$$\beta \circ f^* = f^* \circ \beta,$$

if f is proper we also have

$$\rho \circ f_* = f_* \circ \rho,$$

and when $j\colon U \hookrightarrow X$ is the inclusion of an open subspace, then

$$\rho \circ j^* = j^* \circ \rho.$$

For a compact G-space X the above equalities imply that we have

(49) $$\langle \rho(\gamma), \beta(\omega) \rangle = \langle \gamma, \omega \rangle$$

for any $\gamma \in H_k(X; G, \mathbf{Z}/p)$ and any $\omega \in H^k(X; G, \mathbf{Z}/p)$.

In later sections we will often compose ρ with the projection $H_*(X^G, \mathbf{Z}/p) \to H_k(X^G, \mathbf{Z}/p)$. Then we will denote the resulting mapping by

$$\rho_k\colon\ H_*(X; G, \mathbf{Z}/p) \to H_j(X^G, \mathbf{Z}/p).$$

Similarly, we define ρ_{even} and ρ_{odd} to be the mapping ρ followed by the projection into the subgroup $H_{\text{even}}(X^G, \mathbf{Z}/p)$ (resp. $H_{\text{odd}}(X^G, \mathbf{Z}/p)$) of $H_*(X^G, \mathbf{Z}/p)$ generated by the homogeneous classes of even (resp. odd) degree. In cohomology the similar conventions are followed; we write $\beta^j\colon\ H^*(X; G, \mathbf{Z}/p) \to H^j(X^G, \mathbf{Z}/p)$, β^{even} and β^{odd}.

Note that for odd p the grading modulo 2 is preserved by ρ and β. In other words,

$$\begin{aligned} \rho\,[H_{\text{even}}(X; G, \mathbf{Z}/p)] &= H_{\text{even}}(X^G; \mathbf{Z}/p), \\ \rho\,[H_{\text{odd}}(X; G, \mathbf{Z}/p)] &= H_{\text{odd}}(X^G; \mathbf{Z}/p), \end{aligned}$$

and similarly for β. When $p = 2$, the mappings ρ and β do *not* preserve the grading modulo 2.

This is one of the reasons that it is often useful to take coefficients in $\mathbf{Z}$, since we have

$$H^*(G, \mathbf{Z}) \simeq \mathbf{Z}[\eta^2]/(p\eta^2)$$

for any prime p, so taking $\mathfrak{m}' = (1 - \eta^2)$ we get by Theorem 6.5 isomorphisms

$$H_*(X; G, \mathbf{Z})/\mathfrak{m}' \simeq H_*(X^G; G, \mathbf{Z})/\mathfrak{m}'$$

and

$$H^*(X; G, \mathbf{Z})/\mathfrak{m}' \simeq H^*(X^G; G, \mathbf{Z})/\mathfrak{m}'$$

which do preserve the grading modulo 2 for all primes p. The interpretation of $H_*(X^G; G, \mathbf{Z})/\mathfrak{m}'$ and $H^*(X^G; G, \mathbf{Z})/\mathfrak{m}'$, is slightly more difficult since $\mathbf{Z}$ is not a field. In particular, the Bockstein homomorphism

$$\delta\colon\ H_n(X^G; G, \mathbf{Z}/p) \to H_{n-1}(X^G; G, \mathbf{Z}/p)$$

associated to the short exact sequence

$$0 \to \mathbf{Z}/p \to \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0. \tag{50}$$

comes into play.

Lemma 7.2. *Let p be a prime, let $G = \mathbf{Z}/p$, let h be the isomorphism (7), and let X be a nice, finite dimensional G-space. For any $k \in \mathbf{Z}$ there is a commutative diagram*

$$\begin{array}{ccc} H_k(X^G; G, \mathbf{Z}) & \xrightarrow{h'} & H_k(X^G; \mathbf{Z}) \oplus H_{k+2}(X^G; \mathbf{Z}/p) \oplus H_{k+4}(X^G; \mathbf{Z}/p) \oplus \cdots \\ \Big\downarrow{\scriptstyle \bmod p} & & \Big\downarrow{\scriptstyle \psi} \\ H_k(X^G; G, \mathbf{Z}/p) & \xrightarrow{h} & H_k(X^G, \mathbf{Z}/p) \oplus H_{k+1}(X^G, \mathbf{Z}/p) \oplus H_{k+2}(X^G, \mathbf{Z}/p) \oplus \cdots \end{array}$$

where the mapping h' is a natural isomorphism of groups, and the homomorphism ψ sends a class $\gamma_k + \gamma_{k+2} + \gamma_{k+4} + \cdots$ to the class

$$\gamma_k \bmod p + \delta(\gamma_{k+2}) + \gamma_{k+2} + \delta(\gamma_{k+4}) + \gamma_{k+4} + \cdots .$$

PROOF. Since G acts trivially on X^G, we have that

$$H_k(X^G; G, \mathbf{Z}) \simeq R^{-k}\mathrm{Hom}(R\Gamma_c(X^G, \mathbf{Z}), R\Gamma^G\mathbf{Z}).$$

Now $R\Gamma^G\mathbf{Z}$ is the complex (46) with $M = \mathbf{Z}$, so $R\Gamma^G\mathbf{Z}$ splits, i.e., we have a quasi-isomorphism $\tilde{h}'\colon\ R\Gamma^G\mathbf{Z} \xrightarrow{\sim} H^*(G, \mathbf{Z})$. This induces an isomorphism

$$H_k(X^G; G, \mathbf{Z}) \xrightarrow{\sim} H_k(X^G; H^0(G, \mathbf{Z})) \oplus H_{k+2}(X^G; H^2(G, \mathbf{Z})) \oplus \cdots ,$$

which we choose to be the isomorphism h' in the commutative diagram. When we replace $\mathbf{Z}$ by $\mathbf{Z}/p$ in the above discussion, we get the isomorphism h instead of h'.

Thus, the commutativity of the diagram can be checked on the level of the coefficients of $R^{-k}\mathrm{Hom}(R\Gamma_c(X^G, \mathbf{Z}), -)$ in the derived category of abelian groups. It is sufficient to prove that the quasi-morphism $\tilde{\psi} = \tilde{h} \circ (\bmod p) \circ (\tilde{h}')^{-1}$ that makes the following diagram commutative, does induce ψ.

$$\begin{array}{ccc} R\Gamma^G\mathbf{Z} & \xrightarrow{\tilde{h}'} & H^*(G, \mathbf{Z}) \\ \Big\downarrow{\scriptstyle \bmod p} & & \Big\downarrow{\scriptstyle \tilde{\psi}} \\ R\Gamma^G\mathbf{Z}/p & \xrightarrow{\tilde{h}} & H^*(G, \mathbf{Z}/p) \end{array}$$

The resolution (45) applied to $\mathbf{Z}$ and $\mathbf{Z}/p$ enables us to write down a representative for the quasi-morphism $\tilde{s}$, and when we do this, we see that $\tilde{\psi} = \tilde{\psi}_1 + \tilde{\psi}_2$, where

$$\tilde{\psi}_1\colon\ H^*(G, \mathbf{Z}) \to H^{\mathrm{even}}(G, \mathbf{Z}/p)$$

is just the reduction mod p map, and

$$\tilde{\psi}_2\colon\ H^*(G, \mathbf{Z}) \to H^{\mathrm{odd}}(G, \mathbf{Z}/p)$$

restricted to $H^{2k}(G, \mathbf{Z})$ is a non-trivial quasi-morphism

$$\mathbf{Z}/p[-2k] = H^{2k}(G, \mathbf{Z}) \dashrightarrow H^{2k-1}(G, \mathbf{Z}/p) = \mathbf{Z}/p[1 - 2k]$$

when $k > 0$, and zero on $H^0(G, \mathbf{Z})$. In fact, $\tilde{\psi}_2$ restricted to $H^{2k}(G, \mathbf{Z})$ is a the quasi-morphism $\mathbf{Z}/p \dashrightarrow \mathbf{Z}/p[1]$ associated to the short exact sequence (50), hence $\tilde{\psi}$ induces ψ. $\square$

By abuse of notation we also write ρ for the composite mapping

$$H_*(X; G, \mathbf{Z}) \to H_*(X; G, \mathbf{Z}/p) \xrightarrow{\rho} H_*(X^G; \mathbf{Z}/p)$$

and β for the composite mapping

$$H^*(X; G, \mathbf{Z}) \to H^*(X; G, \mathbf{Z}/p) \xrightarrow{\beta} H^*(X^G; \mathbf{Z}/p).$$

Corollary 7.3. *Let X and G be as in Lemma 7.2, let $\mathfrak{m}'$ be the maximal ideal*
$$(1-\eta^2) \subset H^*(G,\mathbf{Z}).$$
If $p \neq 2$, the homomorphism ρ induces isomorphisms
$$H_{\text{even}}(X;G,\mathbf{Z})/\mathfrak{m}' \xrightarrow{\sim} H_{\text{even}}(X^G;\mathbf{Z}/p),$$
$$H_{\text{odd}}(X;G,\mathbf{Z})\mathfrak{m}' \xrightarrow{\sim} H_{\text{odd}}(X^G;\mathbf{Z}/p).$$
If $p=2$, the homomorphism ρ induces isomorphisms
$$H_{\text{even}}(X;G,\mathbf{Z})/\mathfrak{m}' \xrightarrow{\sim} \{\gamma+\delta(\gamma)\colon \gamma \in H_{\text{even}}(X^G;\mathbf{Z}/p)\},$$
$$H_{\text{odd}}(X;G,\mathbf{Z})/\mathfrak{m}' \xrightarrow{\sim} \{\gamma+\delta(\gamma)\colon \gamma \in H_{\text{odd}}(X^G;\mathbf{Z}/p)\},$$
where δ is the Bockstein homomorphism associated to the short exact sequence (50).

Proof. This follows from Lemma 7.2 and the definition of ρ. □

There are obvious analogues of Lemma 7.2, Corollary 7.3 in cohomology; see [Kr4, Th. 1.2] .

7.1. The equivariant fundamental class and the fixed point set

If X is an n-dimensional cohomology manifold over $\mathbf{Z}/p$ with an action of $G=\mathbf{Z}/p$, then X^G is a cohomology manifold over $\mathbf{Z}/p$ as well, and X^G is $\mathbf{Z}/p$-orientable if X is $\mathbf{Z}/p$-orientable (see [Bo1, Th. V.2.2]). We will now see that the mapping ρ connects the fundamental class of X to the fundamental class of X^G.

Theorem 7.4. *Let p be a prime. Let X be a $\mathbf{Z}/p$-oriented n-dimensional cohomological manifold with an action of $G=\mathbf{Z}/p$ and let $\mu_X \in H_n(X;G,\mathbf{Z}/p)$ be the equivariant fundamental class of X. For any connected component $V \subset X^G$ of dimension d, the image of $\rho(\mu_X) \in H_*(X^G;\mathbf{Z}/p)$ under the projection $H_*(X^G;\mathbf{Z}/p) \to H_d(V;\mathbf{Z}/p)$ is a fundamental class of V.*

Proof. Cap product with μ_X gives an isomorphism
$$H^*(X;G,\mathbf{Z}/p) \xrightarrow{\sim} H_*(X;G,\mathbf{Z}/p),$$
so it induces an isomorphism
$$H^*(X;G,\mathbf{Z}/p)/\mathfrak{m} \xrightarrow{\sim} H_*(X;G,\mathbf{Z}/p)/\mathfrak{m},$$
hence cap product with the restriction γ_V of $\rho(\mu_X)$ to V gives an isomorphism
$$H^*(V;\mathbf{Z}/p) \xrightarrow{\sim} H_*(V;\mathbf{Z}/p),$$
which means that the projection of γ_V to $H_d(V;\mathbf{Z}/p)$ is a fundamental class of the cohomology manifold V. □

Note that the theorem does not claim that $\rho(\mu_X)$ is the $\mathbf{Z}/p$-fundamental class of X^G (the sum of the fundamental classes of all connected components). The following example is a counterexample to that assertion.

Example 7.5. Let X be the complex projective plane and let $G = \mathbf{Z}/2$ act on X via the complex conjugation. Then X^G is the real projective plane and $H_k(X^G, \mathbf{Z}/2) \simeq \mathbf{Z}/2$ for $k = 0, 1, 2$. Since the natural orientation of X is preserved by the involution, X has an equivariant fundamental class $\mu_X \in H_4(X; G, \mathbf{Z})$. Let us show that $\rho(\mu_X) = \mu_{X^G} + \delta(\mu_{X^G})$, where δ is the Bockstein homomorphism associated to (50).

It follows from Corollary 7.3 that $\rho(\mu_X) = \mu_{X^G} + \delta(\mu_{X^G}) + \gamma$ for some $\gamma \in H_0(X^G; \mathbf{Z}/2)$. Let $f\colon X \to \mathbf{pt}$ be the constant mapping. Then $f_*(\mu_X) = 0$, hence $f_*(\rho(\mu_X)) = f_*(\gamma) = 0$, and since X^G is connected, this implies $\gamma = 0$. We conclude that

$$\rho(\mu_X) = \mu_{X^G} + \delta(\mu_{X^G}) \neq \mu_{X^G},$$

since $\delta(\mu_{P^2(\mathbf{R})}) \in H_1(P^2(\mathbf{R}), \mathbf{Z}/2)$ is not zero.

Remark 7.6. The above proof of Theorem 7.4 was inspired by the proof of [AP, Prop. 5.3.7]. In fact, the result of Allday and Puppe is sufficient to prove the above theorem for the case of X being complete. My original version consisted of checking that for every $x \in V$ the image of $\rho(\mu_X)$ is nonzero in

$$\varinjlim_{x \in W} H_*(V; \mathbf{Z}/p) = \varinjlim_{x \in W} H_d(V; \mathbf{Z}/p) = \mathbf{Z}/p,$$

where W ranges over the open neighbourhoods of x in V. When X is locally homeomorphic to $\mathbf{R}^n$ with a linear G-action, this is easily done by considering a small equivariant open ball around x in X and using the Hochschild-Serre spectral sequence. In the general case the use of this spectral sequence involves certain subtleties, but these subtleties have already been dealt with in the proof of the fact that X^G is a cohomology manifold.

Let us have a closer look at the connection between equivariant Poincaré duality on X and the usual Poincaré duality on X^G. Let $\theta_X \in H^*(X; G, \mathbf{Z}/p)$ be the *equivariant Thom class* of X^G in X, the class defined by the equation

$$i_*\mu_{X^G} = \mu_X \cap \theta_X,$$

where $\mu_{X^G} = \sum_i \mu_{V_i}$, the sum of the $\mathbf{Z}/p$-fundamental classes of all connected components of X^G. Then $\rho(\mu_X) \cap \beta(\theta_X) = \mu_{X^G}$ in $H_*(X^G; \mathbf{Z}/p)$ and $\beta(\theta_X)$ is invertible in $H^*(X^G; \mathbf{Z}/p)$, so

$$\rho(\mu_X) = \mu_{X^G} \cap \beta(\theta_X)^{-1}. \tag{51}$$

This means that the following diagram is commutative.

$$\begin{tikzcd} H^*(X; G, \mathbf{Z}/p) \arrow[r, "\beta"] \arrow[d, "\mu_X \cap"'] & H^*(X^G, \mathbf{Z}/p) \arrow[r, "\beta(\theta_X)^{-1} \cup"] \arrow[d, "\rho(\mu_X) \cap"'] & H^*(X^G, \mathbf{Z}/p) \arrow[dl, "\mu_{X^G} \cap"] \\ H_*(X; G, \mathbf{Z}/p) \arrow[r, "\rho"] & H_*(X^G, \mathbf{Z}/p) & \end{tikzcd} \tag{52}$$

If $\rho(\mu_X) \neq \mu_{X^G}$, as in Example 7.5, then so $\beta(\theta_X)$ is not the identity element $1 \in H^0(X; G, \mathbf{Z}/p)$

Remark 7.7. For the Gysin map

$$f_!\colon H^{n-k}(X; G, A) \to H^{m-k}(Y; G, A)$$

induced by a proper mapping of cohomology G-manifolds $f\colon X \to Y$, we see from (35) and (51) that

$$\beta(f_!\omega) = \beta(\theta_Y) \cup f_!(\beta(\theta_X)^{-1} \cap \beta(\omega)),$$

where $f_!\colon H^*(X^G;\mathbf{Z}/p) \to H^*(Y^G;\mathbf{Z}/p)$ is the usual Gysin map defined by the equation

$$\mu_{Y^G} \cap f_!\omega = f_*(\mu_{X^G} \cap \omega).$$

Similarly, the cup product pairing (36) with coefficients in $\mathbf{Z}/p$ is not completely compatible with β:

$$\langle\omega, \omega'\rangle = \langle\beta(\theta_X)^{-1} \cup \beta(\omega), \beta(\omega')\rangle. \tag{53}$$

Also the intersection pairing (38) in homology with coefficients in $\mathbf{Z}/p$ is not completely compatible with ρ:

$$\langle\gamma, \gamma'\rangle = \langle\rho(\gamma) \cap \beta(\theta_X)^{-1}, \beta(\gamma')\rangle. \tag{54}$$

8. Topological spaces with an involution

In the rest of this work the results of the present chapter will be applied exclusively to topological spaces with an involution. Therefore it is worth studying the case $G = \mathbf{Z}/2$ in greater detail. In particular the Hochschild-Serre spectral sequence will be subject to a closer examination, but first it will be shown how to define the equivariant fundamental class with integral coefficients for a manifold with an orientation reversing involution.

When X is an oriented manifold on which $G = \mathbf{Z}/2$ acts via an orientation reversing involution, the orientation sheaf as defined in Section 4 will not be isomorphic to the constant G-sheaf $\mathbf{Z}$, so X will not have an equivariant fundamental class with coefficients in $\mathbf{Z}$. However, we can overcome this difficulty by 'twisting' the G-action on the sheaf $\mathbf{Z}$.

Definition 8.1. Let $G = \mathbf{Z}/2 = \{1, \sigma\}$. For any $k \in \mathbf{Z}$ the G-module $\mathbf{Z}(k)$ is defined to be the group $\mathbf{Z}$ with G-action given by

$$\sigma(z) = (-1)^k z.$$

For an arbitrary G-module M and any $k \in \mathbf{Z}$ we define

$$M(k) = M \otimes \mathbf{Z}(k).$$

Of course, $\mathbf{Z}(2m+k)$ can (and will) be identified to $\mathbf{Z}(k)$, and $\mathbf{Z}(0)=\mathbf{Z}$, so we could have confined ourselves to defining the G-module $\mathbf{Z}(1)$, but in the notation it is often convenient to have $\mathbf{Z}(k)$ defined for every k. For example, we have for every k, $l \in \mathbf{Z}$ a canonical isomorphism of G-modules

$$\mathbf{Z}(k)\otimes\mathbf{Z}(l)\simeq\mathbf{Z}(k+l).$$

From the complex (45) we see that for $n\geq 0$

$$H^n(G,\mathbf{Z}(1))=\begin{cases}0 & \text{if } n \text{ is even}\\ \mathbf{Z}/2 & \text{if } n \text{ is odd}\end{cases}$$

The generator of $H^1(G,\mathbf{Z}(1))$ will be denoted by η, a notation justified by the fact that the reduction modulo 2 mapping

$$H^1(G,\mathbf{Z}(1))\to H^1(G,\mathbf{Z}/2)$$

sends η to the class $\eta\in H^1(G,\mathbf{Z}/2)$ defined in the previous section.

By Section 3 we have for every k, l a cup product

$$H^*(X;G,\mathbf{Z}(k))\otimes H^*(X;G,\mathbf{Z}(l))\to H^*(X;G,\mathbf{Z}(k+l))$$

and a cap product

$$H_*(X;G,\mathbf{Z}(k))\otimes H^*(X;G,\mathbf{Z}(l))\to H_*(X;G,\mathbf{Z}(k-l)),$$

where the choice for writing $k+l$ and $k-l$, respectively, is for aesthetic reasons only. As in the previous section we denote the composite mapping

$$H_*(X;G,\mathbf{Z}(k))\to H_*(X;G,\mathbf{Z}/2)\xrightarrow{\rho} H_*(X^G,\mathbf{Z}/2)$$

by ρ, and the composite mapping

$$H^*(X;G,\mathbf{Z}(k))\to H^*(X;G,\mathbf{Z}/2)\xrightarrow{\beta} H^*(X^G,\mathbf{Z}/2)$$

by β, and we have that

$$\beta(\omega\cup\omega')=\beta(\omega)\cup\beta(\omega') \tag{55}$$

and

$$\rho(\gamma\cap\omega)=\rho(\gamma)\cap\beta(\omega). \tag{56}$$

for any $\omega\in H^*(X;G,\mathbf{Z}(k))$, any $\omega'\in H^*(X;G,\mathbf{Z}(l))$ and any $\gamma\in H_*(X;G,\mathbf{Z}(m))$.

Let $\mathfrak{m}'=(1-\eta^2)$ be the maximal ideal of $H^*(G,\mathbf{Z})$ defined in Section 7. Obviously, cup product with $\eta\in H^1(G,\mathbf{Z}(1))$ induces an isomorphism

$$H^*(X;G,\mathbf{Z}(1))/\mathfrak{m}'\xrightarrow{\sim} H^*(X;G,\mathbf{Z})/\mathfrak{m}',$$

and cap product with η induces an isomorphism

$$H^*(X;G,\mathbf{Z}(1))/\mathfrak{m}'\xrightarrow{\sim} H^*(X;G,\mathbf{Z})/\mathfrak{m}'.$$

Since cap product with η changes the parity of the degree, we have by Corollary 7.3 that

$$\rho[H_{\mathrm{even}}(X; G, \mathbf{Z}(1))] = \{\gamma + \delta(\gamma) \colon \gamma \in H_{\mathrm{odd}}(X^G; \mathbf{Z}/2)\}, \tag{57}$$

$$\rho[H_{\mathrm{odd}}(X; G, \mathbf{Z}(1))] = \{\gamma + \delta(\gamma) \colon \gamma \in H_{\mathrm{even}}(X^G; \mathbf{Z}/2)\}. \tag{58}$$

A similar formula in cohomology holds for β (cf. [Kr4]).

In Chapter V we will also need to know the relation between the Bockstein homomorphism

$$\delta_k^{(m)} \colon H_k(X; G, \mathbf{Z}/2) \to H_k(X; G, \mathbf{Z}(m))$$

associated to the short exact sequence

$$0 \to \mathbf{Z}(m) \to \mathbf{Z}(m) \to \mathbf{Z}/2 \to 0$$

and the Bockstein homomorphism

$$\delta \colon H_n(X^G; G, \mathbf{Z}/p) \to H_{n-1}(X^G; G, \mathbf{Z}/p)$$

associated to the short exact sequence (50). Reasoning, for example, as in the proof of Lemma 7.2 we find that

$$\rho_{\mathrm{even}}(\delta_k^{(m)}(\gamma)) = \rho_{\mathrm{even}}(\gamma) + \delta(\rho_{\mathrm{odd}}(\gamma)) \qquad \text{if } m + k \text{ is even}, \tag{59}$$

$$\rho_{\mathrm{odd}}(\delta_k^{(m)}(\gamma)) = \rho_{\mathrm{odd}}(\gamma) + \delta(\rho_{\mathrm{even}}(\gamma)) \qquad \text{if } m + k \text{ is odd}. \tag{60}$$

Now we come back to the question of defining an equivariant fundamental class with integral coefficients when the G-action reverses the $\mathbf{Z}$-orientation. Formally, we will say that when X is an n-dimensional $\mathbf{Z}$-oriented cohomology manifold with an action of $G = \mathbf{Z}/2$, then the G-action *reverses the* $\mathbf{Z}$*-orientation* if the orientation $\psi \colon \mathbf{Z} \xrightarrow{\sim} \mathscr{O}r_X(\mathbf{Z})$ (see Section 4) induces an isomorphism of G-sheaves

$$\psi \colon \mathbf{Z}(1) \xrightarrow{\sim} \mathscr{O}r_X(\mathbf{Z}).$$

Then the equivariant fundamental class $\mu_X \in H_n(X; G, \mathbf{Z}(1))$ of X is the image of ψ under the canonical isomorphism

$$\begin{aligned} R^0\mathrm{Hom}_G(\mathbf{Z}(1), \mathscr{O}r_X(\mathbf{Z})) &\simeq R^{-n}\mathrm{Hom}_G(Rf_!\mathbf{Z}(1), \mathbf{Z}) \simeq \\ &\simeq R^{-n}\mathrm{Hom}_G(Rf_!\mathbf{Z}, \mathbf{Z}(1)) = H_n(X; G, \mathbf{Z}(1)), \end{aligned}$$

where $f \colon X \to \mathbf{pt}$ is the constant mapping. In this case cap product with μ_X gives an isomorphism

$$H_Z^i(X; G, M) \xrightarrow{\mu_X \cap} H_{n-i}(Z; G, M(1))$$

for any closed G-subspace $Z \subset X$, any $i \in \mathbf{Z}$, and any G-module M.

For many questions concerning the topology of a space with an involution it is important to know whether the Hochschild-Serre spectral sequences are trivial or not.

In real algebraic geometry, a variety X defined over $\mathbf{R}$ is called a *GM-variety* if the spectral sequence

$$E_2^{p,q} = H^p(G, H^q(X(\mathbf{C}), \mathbf{Z}/2) \Rightarrow H^{p+q}(X(\mathbf{C}); G, \mathbf{Z}/2)$$

for $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \mathbf{Z}/2$ is trivial (i.e., $E_2 = E_\infty$), and X is called a $\mathbf{Z}$-*GM-variety* if the spectral sequence

$$E_2^{p,q} = H^p(G, H^q(X(\mathbf{C}), \mathbf{Z}) \Rightarrow H^{p+q}(X(\mathbf{C}); G, \mathbf{Z}/2)$$

is trivial. The notation '*GM*', introduced in [Kr1], is often pronounced as 'Galois-Maximal'. In situations where the Galois-action on the set of complex points is considered, ($\mathbf{Z}$-)*GM*-varieties are often the ideal case (see for example Corollary 8.3, Corollary IV.5.2), but not every variety is ($\mathbf{Z}$-)*GM*; see Section 9, Example IV.5.9 and Theorem V.2.9.

Going back to the general situation of a topological space X with an action of $G = \mathbf{Z}/2$, we have that the periodicity of the cohomology of G greatly simplifies the structure of the Hochschild-Serre spectral sequence. For equivariant cohomology this has been elaborated in [Kr1]; here the same will be done for equivariant Borel-Moore homology. The important fact will be that for any abelian group N with an action of $G = \{1, \sigma\}$ we have that the cup product with $\eta \subset H^2(G, \mathbf{Z}(1))$ induces for $i \geq 0$ a surjection

$$H^i(G, N) \xrightarrow{\cup\eta} H^{i+1}(G, N(1)); \tag{61}$$

that is an isomorphism if $i > 0$.

On the E^2-level of the spectral sequences

$$E^2_{p,q} = H^{-p}(G, H_q(X, M)) \Rightarrow H_{p+q}(X; G, M)$$

and

$$E(1)^2_{p,q} = H^{-p}(G, H_q(X, M(1))) \Rightarrow H_{p+q}(X; G, M(1))$$

the mapping

$$H_{p+q}(X; G, M) \xrightarrow{\cap\eta} H_{p+q-1}(X; G, M(1))$$

is given by

$$H^{-p}(G, H_q(X, M)) \xrightarrow{\cup\eta} H^{-p+1}(G, H_q(X, M(1))).$$

It follows that when X is a locally compact space of finite cohomological dimension, and $n \geq 0$, the cap product with η^n induces for every $k \in \mathbf{Z}$ and every $q \geq k$ a surjection

$$F_q H_k(X; G, M) \to F_q H_{k-n}(X; G, M(-n)),$$

which is an isomorphism if $k < 0$. In particular, taking $q = k$, we get that the cap product with η^n defines for $q \geq 0$ a surjection

$$H_q(X; G, M) \twoheadrightarrow F_q H_{q-n}(X; G, M(-n)) \tag{62}$$

and for $q < 0$ an isomorphism

$$H_q(X; G, M) \xrightarrow{\sim} H_{q-n}(X; G, M(-n)). \tag{63}$$

We deduce the following result, which is the direct analogue in homology of Lemmas 2.1 and 3.2 in [Kr1] (see Section 7 for notations and terminology).

Lemma 8.2. *Let X be a nice, finite dimensional topological space with an action of $G = \mathbf{Z}/2$, and let $q > 0$. The mappings*

$$\rho\colon\ H_{-q}(X; G, \mathbf{Z}/2) \to H_*(X^G, \mathbf{Z}/2),$$

$$\rho_{\text{even}}\colon\ H_{-q}(X; G, \mathbf{Z}(q)) \to H_{\text{even}}(X^G, \mathbf{Z}/2),$$

and

$$\rho_{\text{odd}}\colon\ H_{-q}(X; G, \mathbf{Z}(q+1)) \to H_{\text{odd}}(X^G, \mathbf{Z}/2),$$

are isomorphisms.

Proof. The isomorphism (63) with $M = \mathbf{Z}/2$ implies that for $q > 0$ the inclusion

$$H_{-q}(X; G, \mathbf{Z}/2) \hookrightarrow H_*(X; G, \mathbf{Z}/2)$$

induces an isomorphism

$$H_{-q}(X; G, \mathbf{Z}/2) \xrightarrow{\sim} H_*(X; G, \mathbf{Z}/2)/\mathfrak{m}$$

so for coefficients in $\mathbf{Z}/2$ the result follows from Proposition 7.1. For integral coefficients the isomorphism (63) implies that the composite mappings

$$H_{-q}(X; G, \mathbf{Z}(q)) \xrightarrow{\cap\eta^q} H_{-2q}(X; G, \mathbf{Z}) \to H_{\text{even}}(X; G, \mathbf{Z})/\mathfrak{m}'$$

and

$$H_{-q}(X; G, \mathbf{Z}(q+1)) \xrightarrow{\cap\eta^{q+1}} H_{-(2q+1)}(X; G, \mathbf{Z}) \to H_{\text{odd}}(X; G, \mathbf{Z})/\mathfrak{m}'$$

are isomorphisms so the result follows from Corollary 7.3. □

As a corollary we obtain the homological analogue of Theorems 2.3 and 3.3 in [Kr1].

Corollary 8.3. *Let X be a nice, finite dimensional topological space with an action of $G = \mathbf{Z}/2$.*

(i) *We have an inequality*

$$\dim H_*(X^G, \mathbf{Z}/2) \leq \dim H^1(G, H_*(X, \mathbf{Z}/2))$$

which is an equality if and only if the Hochschild-Serre spectral sequence

$$E^2_{p,q} = H^{-p}(G, H_q(X, \mathbf{Z}/2)) \Rightarrow H_{p+q}(X; G, \mathbf{Z}/2)$$

is trivial.

(ii) *We have an inequality*

$$\dim H_{\text{even}}(X^G, \mathbf{Z}/2) \leq \dim H^1(G, H_{\text{odd}}(X, \mathbf{Z})) + \dim H^2(G, H_{\text{even}}(X, \mathbf{Z}))$$

which is an equality if and only if the Hochschild-Serre spectral sequence

$$E^2_{p,q} = H^{-p}(G, H_q(X, \mathbf{Z})) \Rightarrow H_{p+q}(X; G, \mathbf{Z})$$

is trivial.

(iii) *We have an inequality*

$$\dim H_{\text{odd}}(X^G, \mathbf{Z}/2) \leq \dim H^1(G, H_{\text{even}}(X, \mathbf{Z})) + \dim H^2(G, H_{\text{odd}}(X, \mathbf{Z}))$$

which is an equality if and only if the Hochschild-Serre spectral sequence

$$E^2_{p,q} = H^{-p}(G, H_q(X, \mathbf{Z})) \Rightarrow H_{p+q}(X; G, \mathbf{Z})$$

is trivial.

PROOF. This follows immediately from Lemma 8.2, the structure of the E^2-term of the Hochschild-Serre spectral sequence and the periodicity of the cohomology of G. □

We can also use the mappings

$$H^{-p}(G, H_q(X, M)) \xrightarrow{\cup\eta} H^{-p+1}(G, H_q(X, M(1))).$$

to deduce that for any G-module M the differentials of the Hochschild-Serre spectral sequence converging to $H_{p+q}(X; G, M)$ are completely determined by the differentials with source $E^r_{0,q}$ of the spectral sequence

$$E^2_{p,q} = H^{-p}(G, H_q(X, M)) \Rightarrow H_{p+q}(X; G, M)$$

and the differentials with source $E(1)^r_{0,q}$ of the spectral sequence

$$E(1)^2_{p,q} = H^{-p}(G, H_q(X, M(1))) \Rightarrow H_{p+q}(X; G, M(1)).$$

In particular we have the following useful result that will be used in Section V.2.

Lemma 8.4. *Let X be a locally compact space of finite cohomological dimension with an action of $G = \mathbf{Z}/2$, and let M be a G-module. The following conditions are equivalent.*

(i) *The Hochschild-Serre spectral sequence*

$$E^2_{p,q} = H^{-p}(G, H_q(X, M)) \Rightarrow H_{p+q}(X; G, M)$$

is trivial.

(ii) *The Hochschild-Serre spectral sequence*

$$E^2_{p,q} = H^{-p}(G, H_q(X, M(1))) \Rightarrow H_{p+q}(X; G, M(1))$$

is trivial.

(iii) *The edge morphisms*

$$e\colon H_q(X; G, M) \to H_q(X, M)^G$$

and

$$e\colon H_q(X; G, M(1)) \to H_q(X, M(1))^G$$

are surjective for every $q \geq 0$.

PROOF. This follows from the discussion above. □

Finally, we will record a few technical lemmas for later use.

Lemma 8.5. *Let X be a nice, finite-dimensional, compact connected space with an action of $G = \mathbf{Z}/2$, and let $A = \mathbf{Z}$ or $\mathbf{Z}/2$. The differentials of the spectral sequence*

$$E^2_{p,q} = H^{-p}(G, H_q(X, A)) \Rightarrow H_{p+q}(X; G, A)$$

having source $E^r_{p,0}$ are trivial for any $p \leq 0$, $r \geq 2$, if and only if $X^G \neq \emptyset$.

PROOF. Suppose $X^G \neq \emptyset$. Then there is an equivariant mapping $f\colon \mathbf{pt} \to X$, which induces an isomorphism $H_0(\mathbf{pt}, A) \xrightarrow{\sim} H_0(X, A) = A$, since X is compact and connected. The statement is now easily proved by examining the morphism of spectral sequences induced by f.

Conversely, if $X^G = \emptyset$, then $H_{-2}(X; G, A) = 0$, so in particular

$$E^\infty_{-2,0} = 0 \neq E^2_{-2,0} = H^2(G, H_0(X, A)) = \mathbf{Z}/2$$

Since for any $r \geq 2$ the differentials with target $E^r_{-2,0}$ have source $E^r_{r-2,1-r} = 0$, there is a non-trivial differential with source $E^r_{-2,0}$ for some $r \geq 2$. □

Lemma 8.6. *Let X be a connected cohomological manifold of dimension d with an action of $G = \mathbf{Z}/2$, and let $A = \mathbf{Z}$ or $\mathbf{Z}/2$. The differentials of the spectral sequence*

$$E^2_{p,q} = H^{-p}(G, H_q(X, A)) \Rightarrow H_{p+q}(X; G, A)$$

having target $E^r_{p,d}$ are trivial for any $p \leq 0$, $r \geq 2$, if and only if $X^G \neq \emptyset$.

PROOF. Since the natural mapping $H^p(G, H_d(X, \mathbf{Z})) \to H^p(G, H_d(X, \mathbf{Z}/2)) = \mathbf{Z}/2$ is injective for $p > 0$, it is sufficient to prove the statement for $A = \mathbf{Z}/2$. Let $\mu_X \in H_d(X; G, \mathbf{Z}/2)$ be the equivariant fundamental class. Then $E^2_{p,d} \neq E^\infty_{p,d}$ if and only if $\mu_X \cap \eta^p = 0$ in $H_{d-p}(X; G, \mathbf{Z}/p)$. By Theorem 7.4, we have that $\rho(\mu_X) = 0$ if and only if $X^G = \emptyset$, which implies by Proposition 7.1 that $\mu_X \cap \eta^p \neq 0$ for $p > d$ if and only if $X^G \neq \emptyset$. □

For the lemmas below we introduce for a compact G-space X the notation

$$H_*(X^G,\mathbf{Z}/2)^0 = \mathrm{Ker}\{f_*\colon H_*(X^G,\mathbf{Z}/2) \to H_*(\mathbf{pt},\mathbf{Z}/2)\},$$

where f_* is induced by the constant mapping $X \to \mathbf{pt}$. In other words, $H_*(X^G,\mathbf{Z}/2)^0$ is the direct sum of the groups $H_i(X^G,\mathbf{Z}/2)$ for $i>0$ plus the subgroup of $H_0(X^G,\mathbf{Z}/2)$ generated by pairs of points. Observe that $\rho[H_q(X;G,\mathbf{Z}/2)] \subset H_*(X^G,\mathbf{Z}/2)^0$ for every $q>0$, since $f_*\circ\rho = \rho\circ f_*$ and $H_q(\mathbf{pt};G,\mathbf{Z}/2)=0$ for $q>0$. We also write

$$H_{\mathrm{even}}(X^G,\mathbf{Z}/2)^0 = H_{\mathrm{even}}(X^G,\mathbf{Z}/2)\cap H_*(X^G,\mathbf{Z}/2)^0.$$

Lemma 8.7. *Let X be a nice, finite-dimensional compact connected G-space with $X^G \neq \emptyset$. The mapping*

$$\rho\colon H_2(X;G,\mathbf{Z}/2) \to H_*(X^G,\mathbf{Z}/2)^0$$

is surjective if and only if the composite mapping

$$H_1(X;G,\mathbf{Z}/2) \xrightarrow{e_1} H_1(X,\mathbf{Z}/2)^G \xrightarrow{\cup\eta^2} H^2(G,H_1(X,\mathbf{Z}/2))$$

is zero.

Proof. Using Lemma 8.5, we see from the Hochschild-Serre spectral sequence and Lemma 8.2 that ρ induces an isomorphism

$$F_1H_0(X;G,\mathbf{Z}/2) \xrightarrow{\sim} H_*(X^G,\mathbf{Z}/2)^0,$$

where F_* is the filtration corresponding to the Hochschild-Serre spectral sequence. The lemma now follows from the fact that in the following commutative diagram the rows are exact and the vertical arrows are surjective.

$$\begin{array}{ccccc}
F_2H_0(X;G,\mathbf{Z}/2) & \longrightarrow & F_1H_0(X;G,\mathbf{Z}/2) & \longrightarrow & H^1(G,H_1(X,\mathbf{Z}/2)) \\
{\scriptstyle\cap\eta^2}\big\uparrow & & {\scriptstyle\cap\eta}\big\uparrow & & {\scriptstyle\cap\eta}\big\uparrow \\
H_2(X;G,\mathbf{Z}/2) & \xrightarrow{\cap\eta} & H_1(X;G,\mathbf{Z}/2) & \xrightarrow{e} & H_1(X,\mathbf{Z}/2)^G
\end{array}$$

□

Lemma 8.8. *Let X be a nice, finite-dimensional compact connected G-space. The mapping*

$$\rho_{\mathrm{even}}\colon H_2(X;G,\mathbf{Z}) \to H_{\mathrm{even}}(X^G,\mathbf{Z}/2)^0$$

is surjective if and only if the composite mapping

$$H_1(X;G,\mathbf{Z}(1)) \xrightarrow{e_1} H_1(X,\mathbf{Z}(1))^G \xrightarrow{\cup\eta^2} H^2(G,H_1(X,\mathbf{Z}(1))$$

is zero.

Proof. Similar to the proof of Lemma 8.7. □

Lemma 8.9. *Let X be a nice, finite-dimensional, connected G-space with $X^G \neq \emptyset$. The mapping*

$$\rho_{\mathrm{odd}}\colon H_1(X; G, \mathbf{Z}) \to H_{\mathrm{odd}}(X^G, \mathbf{Z}/2)$$

is surjective. Moreover, the mapping

$$\rho_{\mathrm{odd}}\colon H_2(X; G, \mathbf{Z}(1)) \to H_{\mathrm{odd}}(X^G, \mathbf{Z}/2)$$

is surjective if and only if the composite mapping

$$H_1(X; G, \mathbf{Z}) \xrightarrow{e_1} H_1(X, \mathbf{Z})^G \xrightarrow{\cup\eta^2} H^2(G, H_1(X, \mathbf{Z}))$$

is zero.

PROOF. Similar to the proof of Lemma 8.7. □

9. Examples

The examples of spaces with an involution collected in this section serve several purposes: they illustrate the rather abstract theory developed in this chapter, they should give an idea of the techniques involved in calculating equivariant cohomology, and they will be used in later chapters. The reader will not be surprised that all but one have their roots in real algebraic geometry, and even this one (Example 9.1) will be used later (in Example IV.5.9).

When the Hochschild-Serre spectral sequence converging to $H_q(X; G, \mathbf{Z})$ is nontrivial, a diagram of the level(s) having nontrivial differentials will be given. In such diagrams all trivial differentials will be omitted, and all groups that are known to be trivial will be denoted by 0. For every $n \in \mathbf{Z}$ the groups $E^r_{p,q}$ with $p + q = n$, which are all subquotients of the group $H_n(X; G, \mathbf{Z})$, will be connected with a dotted line. Note that in all examples the G-space under consideration is compact and connected, so $H_0(X, \mathbf{Z}) = \mathbf{Z}$ with trivial G-action, hence

$$H^k(G, H_0(X, \mathbf{Z})) = \begin{cases} \mathbf{Z} & \text{if } k = 0 \\ 0 & \text{if } k > 0 \text{ is odd} \\ \mathbf{Z}/2 & \text{if } k > 0 \text{ and } k \text{ is even.} \end{cases}$$

Example 9.1. Let X be the circle S^1 with a fixed point free involution. We then have that the involution is orientation preserving so G acts trivially on $H_2(X, \mathbf{Z})$. In particular, for every $k \geq 0$ we have

$$H^k(G, H_1(X, \mathbf{Z})) \simeq \begin{cases} \mathbf{Z} & \text{if } k = 0 \\ 0 & \text{if } k \text{ is odd} \\ \mathbf{Z}/2 & \text{if } k > 0 \text{ and } k \text{ is even.} \end{cases}$$

From Lemma 8.5 and Lemma 8.6 we see that the E^2-level of the Hochschild-Serre spectral sequence is given by the diagram below.

$$
\begin{tikzcd}
 & 0 & H^2(G, H_1(X,\mathbf{Z})) & 0 & H_1(X,\mathbf{Z})^G \\
 & 0 & H^2(G, H_0(X,\mathbf{Z})) \arrow[ull] & 0 & H_0(X,\mathbf{Z})^G \arrow[ull]
\end{tikzcd}
$$

We see that $H_n(X; G, \mathbf{Z}) \simeq \mathbf{Z}$ if $n = 0$ or 1, and $H_n(X; G, \mathbf{Z})$ is zero for $n < 0$, which is not surprising in view of Proposition 6.2, since X/G is again homeomorphic to S^1.

Example 9.2. Let X be a nonsingular projective curve defined over $\mathbf{R}$ of genus g, and assume that $X(\mathbf{R})$ has $s > 0$ connected components. This means that $X(\mathbf{R})$ is homeomorphic to a disjoint union of s copies of S^1. Note that $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ acts via an orientation reversing involution on the Riemann surface $X(\mathbf{C})$, so for $k \geq 0$ we have

$$H^k(G, H_2(X(\mathbf{C}),\mathbf{Z})) \simeq \begin{cases} 0 & \text{if } k \text{ is even} \\ \mathbf{Z}/2 & \text{if } k \text{ is odd} \end{cases}$$

It is well-known that

$$H_1(X(\mathbf{C}),\mathbf{Z})^G \simeq \mathbf{Z}^g \simeq H_1(X(\mathbf{C}),\mathbf{Z}(1))^G.$$

Since $X(\mathbf{R}) \neq \emptyset$, the Hochschild-Serre spectral sequence is trivial by Lemma 8.5 and Lemma 8.5. In other words, X is a $\mathbf{Z}$-*GM*-variety. Counting dimensions Corollary 8.3 gives us that

$$H^1(G, H_1(X(\mathbf{C}),\mathbf{Z})) \simeq (\mathbf{Z}/2)^{s-1} \tag{64}$$

$$H^2(G, H_1(X(\mathbf{C}),\mathbf{Z})) \simeq (\mathbf{Z}/2)^{s-1} \tag{65}$$

Combining this information we get that

$$H_q(X(\mathbf{C}); G, \mathbf{Z}) \simeq \begin{cases} 0 & \text{if } q \geq 2, \\ \mathbf{Z}^g \oplus \mathbf{Z}/2 & \text{if } q = 1, \\ \mathbf{Z} \oplus (\mathbf{Z}/2)^{s-1} & \text{if } q = 0, \\ (\mathbf{Z}/2)^s & \text{if } q < 0. \end{cases}$$

The mapping $\rho\colon H_q(X(\mathbf{C}); G, \mathbf{Z}) \to H_*(X(\mathbf{R}), \mathbf{Z}/2)$ can be used to get some more insight in the structure of $H_q(X(\mathbf{C}); G, \mathbf{Z})$. Let us, as an example, consider

$H_0(X(\mathbf{C}); G, \mathbf{Z})$. We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & F_1 H_0(X(\mathbf{C}); G, \mathbf{Z}) \arrow[r] \arrow[d, "\rho"] & H_0(X(\mathbf{C}); G, \mathbf{Z}) \arrow[r] \arrow[d, "\rho"] & H_0(X(\mathbf{C}), \mathbf{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_0(X(\mathbf{R}), \mathbf{Z}/2)^0 \arrow[r] & H_0(X(\mathbf{R}), \mathbf{Z}/2) \arrow[r] & \mathbf{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Note that

$$F_1 H_0(X(\mathbf{C}); G, \mathbf{Z}) \xrightarrow{\sim} H^1(G, H_1(X(\mathbf{C}), \mathbf{Z}))$$

and

$$H_0(X(\mathbf{R}), \mathbf{Z}/2)^0 \xrightarrow{\sim} (\mathbf{Z}/2)^{s-1},$$

so the isomorphism (64), which we obtained purely by counting dimensions, is in fact *canonically* given by the isomorphism

$$F_1 H_0(X(\mathbf{C}); G, \mathbf{Z}) \xrightarrow{\sim} H_0(X(\mathbf{R}), \mathbf{Z}/2)^0$$

in the diagram. More geometrically we can describe this isomorphism in the following way. It can be checked that every class in $\gamma \in F_1 H_0(X(\mathbf{C}); G, \mathbf{Z})$ is of the form

$$\gamma = [\lambda] \cap \eta,$$

where $\eta \in H^1(G, \mathbf{Z}(1))$ is the nontrivial class, and $[\lambda] \in H_1(X(\mathbf{C}); G, \mathbf{Z}(1))$ is the fundamental class of an oriented loop $\lambda \subset X(\mathbf{C})$ on which G acts via an orientation reversing involution. Then the fixed point set of λ consists of two points, P_1 and P_2 in $X(\mathbf{R})$, and $\rho(\gamma) = \rho(\lambda) = [P_1] + [P_2] \in H_0(X(\mathbf{R}), \mathbf{Z}/2)$; see Figure 1 for an example of the situation on a curve of genus 3 with $X(\mathbf{R})$ having 2 connected components.

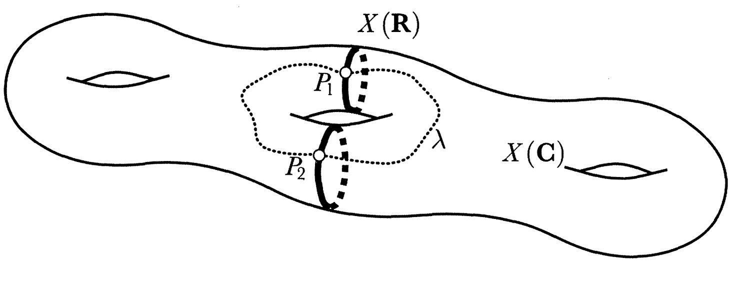

FIGURE 1

Example 9.3. Let X be a nonsingular, geometrically irreducible, projective curve over $\mathbf{R}$ of genus g with $X(\mathbf{R}) = \emptyset$. Then the $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$-space $X(\mathbf{C})$ depends, up to equivariant homeomorphisms, only on the genus g.

If g is even, $X(\mathbf{C})$ is homeomorphic to the topological surface depicted in Figure 2 with the G-action given by reflection in the origin.

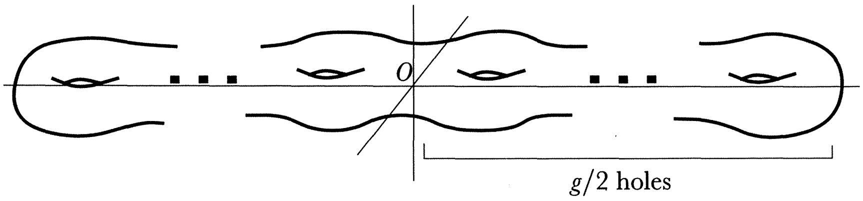

FIGURE 2. A curve of even genus without real points

We see that $H_1(X(\mathbf{C}),\mathbf{Z})^G \simeq \mathbf{Z}^g$ and $H^k(G,H_1(X(\mathbf{C}),\mathbf{Z})) = 0$ for $k > 0$, so for $r \neq 3$ every differential on the E^r-level of the Hochschild-Serre spectral sequence converging to $H_*(X(\mathbf{C});G,\mathbf{Z})$ is trivial, since either the source or the target is zero. The groups $H^k(G,H_2(X(\mathbf{C}),\mathbf{Z}))$ and $H^k(G,H_0(X(\mathbf{C}),\mathbf{Z}))$ are as in the case of a curve with $X(\mathbf{R})$ nonempty (see the previous example), and it follows from Lemma 8.5 (or Lemma 8.5) that on the E^3-level the Hochschild-Serre spectral sequence is given by the following diagram.

$$\begin{tikzcd}
 & H^3(G,H_2(X,\mathbf{Z})) & 0 & H^1(G,H_2(X,\mathbf{Z})) & 0 \\
 {} & 0 & 0 & 0 & H_1(X,\mathbf{Z})^G \\
 & 0 & H^2(G,H_0(X,\mathbf{Z})) \arrow[ull] & 0 & H_0(X,\mathbf{Z})^G \arrow[uulll]
\end{tikzcd}$$

If g is odd, on the other hand, then $X(\mathbf{C})$ is equivariantly homeomorphic to the surface depicted in Figure 3 with the G-action again given by reflection in the origin.

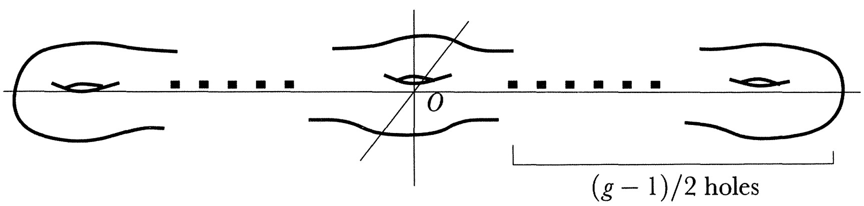

FIGURE 3. A curve of odd genus without real points

We then have $H_1(X(\mathbf{C}),\mathbf{Z})^G \simeq \mathbf{Z}^g$, like in the case g even, but now we see from the picture that for $k > 0$ we have

$$H^k(G,H_1(X(\mathbf{C}),\mathbf{Z})) \simeq \mathbf{Z}/2,$$

and using the fact that $H_q(X(\mathbf{C}),\mathbf{Z}) = 0$ for $q < 0$ since $X(\mathbf{R}) = \emptyset$, we see that all nontrivial differentials of the Hochschild-Serre spectral sequence occur at the E^2-level, which is given by the diagram below. Note the difference with the case when g is even.

$$\begin{tikzcd}
H^3(G,H_2(X,\mathbf{Z})) & 0 & H^1(G,H_2(X,\mathbf{Z})) & 0 \\
H^3(G,H_1(X,\mathbf{Z})) & H^2(G,H_1(X,\mathbf{Z})) & H^1(G,H_1(X,\mathbf{Z})) \arrow[ull] & H_1(X,\mathbf{Z})^G \\
0 & H^2(G,H_0(X,\mathbf{Z})) & 0 & H_0(X,\mathbf{Z})^G \arrow[ull]
\end{tikzcd}$$

Example 9.4. Let X be a complete, nonsingular, geometrically irreducible surface defined over $\mathbf{R}$ such that $H_1(X(\mathbf{C}),\mathbf{Z}) = 0$. By the non-degeneracy of the intersection product in homology, and the fact that $H_3(X(\mathbf{C}),\mathbf{Z})$ is torsion-free, we have that $H_3(X(\mathbf{C}),\mathbf{Z}) = 0$ as well. Note that the action of $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on $X(\mathbf{C})$ preserves the orientation. Using Lemma 8.5 and Lemma 8.6 it is easy to see that X is a $\mathbf{Z}$-*GM*-variety if and only if $X(\mathbf{R}) \neq \emptyset$. Note in particular that if $X(\mathbf{R}) \neq \emptyset$, we have for any $q \leq 0$ an isomorphism

$$H_{2q+1}(X(\mathbf{C});G,\mathbf{Z}) \simeq H^1(G,H_2(X(\mathbf{C}),\mathbf{Z})), \tag{66}$$

whereas ρ induces for any $q \leq 0$ an isomorphism

$$H_{2q+1}(X(\mathbf{C});G,\mathbf{Z}) \simeq H_1(X(\mathbf{R}),\mathbf{Z}/2). \tag{67}$$

These isomorphisms have played an important role in the study of real rational surfaces (i.e. nonsingular projective surfaces X defined over $\mathbf{R}$ such that $X_{\mathbf{C}}$ is birationally isomorphic to $\mathbf{P}^2_{\mathbf{C}}$) and real K3-surfaces (see for example [Si] and [Ma2]).

If $X(\mathbf{R}) = \emptyset$, then $H_q(X(\mathbf{C});G,\mathbf{Z}) = 0$ for $q < 0$, so we see that the only nontrivial differentials of the Hochschild-Serre spectral sequence converging to $H_*(X(\mathbf{C});G,\mathbf{Z})$ can be found on the E^3-level, which is given by the following diagram.

$$\begin{tikzcd}
0 & H^2(G,H_4(X,\mathbf{Z})) & 0 & H_4(X,\mathbf{Z})^G \\
0 & 0 & 0 & 0 \\
H^3(G,H_2(X,\mathbf{Z})) & H^2(G,H_2(X,\mathbf{Z})) & H^1(G,H_2(X,\mathbf{Z})) & H_2(X,\mathbf{Z})^G \\
0 & 0 & 0 & 0 \\
0 & H^2(G,H_0(X,\mathbf{Z})) & 0 & H_0(X,\mathbf{Z})^G \arrow[uulll]
\end{tikzcd}$$

The final example, inspired by [Si, §V.4], is a surface which has real points, but which is not **Z**-*GM*.

Example 9.5. Let X be a nonsingular, projective, geometrically irreducible surface defined over $\mathbf{R}$ with $X(\mathbf{R}) \neq \emptyset$ admitting a *ruling* defined over $\mathbf{R}$. By this we mean a ruling in the strict sense, not a birational ruling, so there is a nonsingular, projective, geometrically irreducible curve C defined over $\mathbf{R}$, and a morphism $p\colon X \to C$ defined over $\mathbf{R}$, such that for *every* complex point $P \in C(\mathbf{C})$ the fibre X_P is isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ (cf. [BPV, §V.4]). This implies that for any real point $P \in C(\mathbf{R})$ the fibre X_P is either isomorphic to $\mathbf{P}^1_{\mathbf{R}}$ or to the real conic $\{x_0^2 + x_1^2 + x_2^2 = 0\}$, which has no real points. Clearly we have that for any two points P, P' lying in the same connected component of $C(\mathbf{R})$, that the fibres X_P and $X_{P'}$ are isomorphic over $\mathbf{R}$. Hence $X(\mathbf{R})$ is an S^1-bundle over some, but not necessarily all connected components of $C(\mathbf{R})$. In fact, it is not hard to construct for any triple (g, s, t) satisfying the conditions $g+1 \geq s \geq t \geq 0$ an explicit ruling $p\colon X \to C$ defined over $\mathbf{R}$, with the genus of C equal to g, the number of connected components of $C(\mathbf{R})$ equal to s and the number of connected components of $X(\mathbf{R})$ equal to t.

Using the fact that the projection p induces isomorphisms

$$p^*\colon\ H^1(X(\mathbf{C}),\mathbf{Z}) \xrightarrow{\sim} H^1(X(\mathbf{C}),\mathbf{Z})$$

and

$$p_*\colon\ H_1(X(\mathbf{C}),\mathbf{Z}) \to H_1(C(\mathbf{C}),\mathbf{Z})$$

we have by Poincaré duality and Example 9.2 that

$$\begin{aligned} H^1(G, H_1(X(\mathbf{C},\mathbf{Z})) &\simeq (\mathbf{Z}/2)^{s-1} \simeq H^1(G, H_3(X(\mathbf{C}),\mathbf{Z})) \\ H^2(G, H_1(X(\mathbf{C},\mathbf{Z})) &\simeq (\mathbf{Z}/2)^{s-1} \simeq H^2(G, H_3(X(\mathbf{C}),\mathbf{Z})). \end{aligned}$$

Since $H_2(X(\mathbf{C}),\mathbf{Z}) \simeq \mathbf{Z}^2$, we have that

$$4s - 2 \leq \dim H^1(G, H_*(X,\mathbf{Z})) + \dim H^2(G, H_*(X,\mathbf{Z})) \leq 4s.$$

By the above description of $X(\mathbf{R})$ we have that $\dim H_*(X(\mathbf{R}),\mathbf{Z}/2) = 4t$. It follows from Corollary 8.3 that X is a **Z**-*GM*-variety if and only if $s = t$.

In the case $t < s$ this result does not completely determine the Hochschild-Serre spectral sequence. For example, it does not give the information we will need in Example IV.5.3. In order to determine the spectral sequence completely, we will use some more geometry and the methods developed in earlier sections. First, we will consider the G-action on the group $H_2(X(\mathbf{C}),\mathbf{Z})$, which is generated by the fundamental class F of a fibre and the image S of the fundamental class of $C(\mathbf{C})$ under a section $C_{\mathbf{C}} \to X_{\mathbf{C}}$ defined over $\mathbf{C}$. Observe that we can represent F by the fibre X_P over a real point $P \in C(\mathbf{R})$, hence $\sigma(F) = -F$. On the other hand, the intersection product in homology gives $\langle F, F\rangle = 0$ and $\langle F, S\rangle = 1$. Since $\langle F, \sigma(S)\rangle = \langle \sigma(F), S\rangle = -1$, this means that F and $(1-\sigma)S$ are two linearly independent G-anti-invariant elements in $H_2(X(\mathbf{C}),\mathbf{Z})$. Since $H_2(X(\mathbf{C}),\mathbf{Z}) \simeq \mathbf{Z}^2$, this means that G acts on $H_2(X(\mathbf{C}),\mathbf{Z})$ via multiplication by (-1).

If $t < s$, we can find a $P \in C(\mathbf{R})$ such that $X_P(\mathbf{R}) = \emptyset$. Taking the equivariant fundamental class $[X_P] \in H_2(X(\mathbf{C}); G, \mathbf{Z}(1))$ represented by $X_P(\mathbf{C})$, we see that on the one hand $e([X_P]) = F$, but on the other hand $\rho([X_P]) = 0$ by Theorem 7.4. This means that, still assuming $t < s$, the class of the section S is not in the image of the edge morphism

$$e\colon\ H_2(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H_2(X(\mathbf{C}),\mathbf{Z}(1))^G,$$

since an element $\Sigma \in e^{-1}(S)$ would satisfy $\langle [X_P], \Sigma\rangle = 1$, by equation (44) but this contradicts equation (54).

The cokernel of the edge morphism

$$e\colon\ H_1(X(\mathbf{C}); G, \mathbf{Z}) \to H_1(X(\mathbf{C}),\mathbf{Z})^G$$

is isomorphic to $(\mathbf{Z}/2)^{s-t}$. Using the projection $p\colon\ X \to C$, it can be shown that this cokernel is generated by the fundamental classes of (non-equivariant) loops $\lambda \subset X(\mathbf{C})$ such that p induces a homeomorphism between λ and a connected component of $C(\mathbf{R})$ *not* dominated by a connected component of $X(\mathbf{R})$. Similarly, it can be shown that the cokernel of the edge morphism

$$e\colon\ H_1(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H_1(X(\mathbf{C}),\mathbf{Z}(1))^G$$

is isomorphic to $(\mathbf{Z}/2)^{s-t}$, and generated by the fundamental classes of (non-equivariant) loops $\lambda \subset X(\mathbf{C})$ such that p induces a homeomorphism between λ and a loop $\lambda_C \subset C(\mathbf{C})$ on which G acts via an orientation reversing involution, such that at least one of the two fixed points of λ_C is contained in a connected component of $C(\mathbf{R})$ *not* dominated by a connected component of $X(\mathbf{R})$.

Chapter IV

The equivariant cycle map

Using the theory developed in Chapters II and III we will define the equivariant cycle map into Borel-Moore homology, and we will show that both the real and the complex cycle map of Section I.5 factorize via this equivariant cycle map. For nonsingular varieties, Poincaré duality transforms the homological cycle map into a cohomological cycle map, and in Section 2 it is shown that this map coincides with Krasnov's equivariant cycle map. In Section 3 we will see that the factorization of the real cycle map via the equivariant cycle map puts severe restrictions on the groups $H_k^{\mathrm{alg}}(X(\mathbf{R}),\mathbf{Z}/2)$. In Section 4 the kernel and image of the equivariant cycle map are analyzed for the case of divisors on nonsingular complete varieties, and in Section 5 we will see how we can use equivariant homology for studying Question 3 of the Introduction.

1. Definition and basic properties

Let V be an algebraic variety over $\mathbf{R}$ of dimension k. If V is nonsingular, then $V(\mathbf{C})$ is a $2k$-dimensional topological manifold with a natural orientation induced by the complex structure. Since $V(\mathbf{C})$ is locally equivariantly homeomorphic to an open (not necessarily connected) G-subset of $\mathbf{C}^k$, the G-action preserves the orientation if k is even and it reverses the orientation if k is odd. Hence the orientation determines a fundamental class $\mu_V \in H_{2k}(V(\mathbf{C}); G, \mathbf{Z}(k))$. If V is singular, we proceed as in [BH] (see Section I.5). Let $V_{\mathrm{s}}(\mathbf{C}) \subset V(\mathbf{C})$ be the set of singular points, and let $V_{\mathrm{r}}(\mathbf{C})$ be the complement of $V_{\mathrm{s}}(\mathbf{C})$ in $V(\mathbf{C})$. Then $V_{\mathrm{s}}(\mathbf{C})$ is of cohomological dimension $\leq 2k-2$, and we see from the Hochschild-Serre spectral sequence that $H_n(V_{\mathrm{s}}(\mathbf{C}); G, \mathbf{Z}(k))$ is zero for $n > 2k-2$, so the exact sequence

$$(68)\quad \cdots \to H_{2k}(V_s(\mathbf{C}); G, \mathbf{Z}(k)) \to H_{2k}(V(\mathbf{C}); G, \mathbf{Z}(k)) \to \\ \to H_{2k}(V_r(\mathbf{C}); G, \mathbf{Z}(k)) \to H_{2k-1}(V_s(\mathbf{C}); G, \mathbf{Z}(k)) \to \cdots$$

shows that

$$H_{2k}(V(\mathbf{C}); G, \mathbf{Z}(k)) \simeq H_{2k}(V_r(\mathbf{C}); G, \mathbf{Z}(k)),$$

and we define $\mu_V \in H_{2k}(V(\mathbf{C}); G, \mathbf{Z}(k))$ to be the inverse image of the fundamental class of $V_r(\mathbf{C})$.

Let $i\colon V \hookrightarrow X$ be the inclusion of a k-dimensional subvariety. We define

$$\mathrm{cl}_X(V) = i_*\mu_V \in H_{2k}(V(\mathbf{C}); G, \mathbf{Z}(k)),$$

which gives us for every k an *equivariant cycle map*

$$\mathrm{cl}_X : \mathscr{Z}_k(X) \to H_{2k}(V(\mathbf{C}); G, \mathbf{Z}(k)).$$

The image of cl_X is denoted by

$$H_{2k}^{\mathrm{alg}}(X(\mathbf{C}); G, \mathbf{Z}/2),$$

and we omit the X in the notation cl_X if confusion is unlikely.

Theorem 1.1. *Let X be an algebraic variety defined over $\mathbf{R}$. Let $\mathrm{cl}^{\mathbf{C}} : \mathscr{Z}_k(X) \to H_{2k}(X(\mathbf{C}), \mathbf{Z})$ and $\mathrm{cl}^{\mathbf{R}} : \mathscr{Z}_k(X) \to H_k(X(\mathbf{R}), \mathbf{Z}/2)$ be the usual cycle maps into Borel-Moore homology. For every $k \geq 0$ the following diagram is commutative*

$$\begin{tikzcd}
\mathscr{Z}_k(X) \arrow[r, "\mathrm{cl}^{\mathbf{C}}"] \arrow[d, "\mathrm{cl}^{\mathbf{R}}"'] \arrow[dr, "\mathrm{cl}"] & H_{2k}(X(\mathbf{C}), \mathbf{Z}) \\
H_k(X(\mathbf{R}), \mathbf{Z}/2) & H_{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \arrow[l, "\rho_k"'] \arrow[u, "e"']
\end{tikzcd}$$

Proof. It follows from the definitions that for any k-dimensional variety V over $\mathbf{R}$ we have $e(\mu_V) = \mu_{V(\mathbf{C})}$, so the functoriality of the edge morphism e implies that $\mathrm{cl}^{\mathbf{C}} = e \circ \mathrm{cl} : \mathscr{Z}_k(X) \to H_{2k}(X(\mathbf{C}), \mathbf{Z})$.

In order to prove that $\rho_k \circ \mathrm{cl} = \mathrm{cl}^{\mathbf{R}}$ it is sufficient to prove that for any k-dimensional variety V over $\mathbf{R}$ we have $\rho_k(\mu_V) = \mu_{V(\mathbf{R})}$. From the commutative diagram

$$\begin{tikzcd}
H_{2k}(V(\mathbf{C}); G, \mathbf{Z}(k)) \arrow[r] \arrow[d, "\rho_{2k}"'] & H_{2k}(V_r(\mathbf{C}); G, \mathbf{Z}(k)) \arrow[d, "\rho_{2k}"'] \\
H_k(V(\mathbf{R}), \mathbf{Z}/2) \arrow[r] & H_k(V_r(\mathbf{R}), \mathbf{Z}/2)
\end{tikzcd}$$

and Theorem III.7.4 we see that $\rho_k(\mu_V)$ maps to the fundamental class of $V_r(\mathbf{R})$ under the restriction mapping, hence $\rho_k(\mu_V)$ is the fundamental class of $V(\mathbf{R})$. $\square$

A straightforward equivariant adaptation of Lemma 19.1.2 in [Fu] shows that for a proper morphism $f\colon X \to Y$ of algebraic varieties over $\mathbf{R}$

$$\mathrm{cl}_Y \circ f_* = f_* \circ \mathrm{cl}_X, \tag{69}$$

where $f_*\colon \mathscr{Z}_k(X) \to \mathscr{Z}_k(Y)$ is the proper push-forward.

For the following lemma, recall that $\mathscr{Z}_k^{\mathbf{R}\text{-alg}}(X)$ denotes the group of k-cycles that are real algebraically equivalent to zero (see Definition I.1.5).

Lemma 1.2. *Let X be an algebraic variety defined over $\mathbf{R}$. For every $k \geq 0$ the equivariant cycle map $\mathrm{cl}: \mathscr{Z}_k(X) \to H_{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$ vanishes on $\mathscr{Z}_x^{\mathbf{R}\text{-alg}}(X)$.*

Proof. Let C and $V \subset X \times C$ be as in Definition I.1.5, and let t_0 and t_1 be two points in the same connected component of $C(\mathbf{R})$. V is reduced and irreducible. Let $L \subset C(\mathbf{R})$ be a closed line segment connecting t_0 and t_1. Let $V_L = f^{-1}(L) \subset V(\mathbf{C})$, then the mapping $\varphi\colon V_L \to X(\mathbf{C})$ induced by the projection $V \to X$ is proper and $\varphi_* \mathrm{cl}_{V_L}(V_{t_i}) = \mathrm{cl}_X(V_{t_i})$, where $\mathrm{cl}_{V_L}(V_{t_i}) \in H_{2k}(V_L; G, \mathbf{Z}(k))$ denotes, of course, the image of $\mathrm{cl}(V_{t_i}) \in H_{2k}(V_{t_i}; G, \mathbf{Z}(k))$ under the mapping induced by the inclusion $V_{t_i} \hookrightarrow V_L$. It is therefore sufficient to prove that $\mathrm{cl}_{V_L}(V_{t_0}) = \mathrm{cl}_{V_L}(V_{t_1})$.

Let $[t_i] \in H^2_{\{t_i\}}(C(\mathbf{C}); G, \mathbf{Z}(1))$ be the class that maps to the class of $t_i \in H_0(t_i; G, \mathbf{Z})$ under the Poincaré duality mapping

$$H^2_{\{t_i\}}(C(\mathbf{C}); G, \mathbf{Z}(1)) \xrightarrow{\mu_{C(\mathbf{C})}\cap} H_0(t_i; G, \mathbf{Z}).$$

Using the fact that the edge morphism $e_{2k}\colon H_{2k}(V_{t_i}(\mathbf{C}); G, \mathbf{Z}(k)) \to H_{2k}(V_{t_i}(\mathbf{C}), \mathbf{Z})$ is injective, we see from [Fu, Lemma 19.1.3] that $\mathrm{cl}(V_{t_i}) = \mu_V \cap f^*[t_i]$ for $i \in \{0,1\}$. This implies that

$$\mathrm{cl}_{V_L}(V_{t_i}) = \mu_V \cap f^*\omega_i,$$

where ω_i is the image of $[t_i]$ under the mapping

$$H^2_{\{t_i\}}(C(\mathbf{C}); G, \mathbf{Z}(1)) \to H^2_L(C(\mathbf{C}); G, \mathbf{Z}(1)).$$

Now $\omega_0 = \omega_1$ since L is connected, and it follows that $\mathrm{cl}_{V_L}(V_{t_0}) = \mathrm{cl}_{V_L}(V_{t_1})$. $\square$

Since $\mathscr{Z}_k^{\mathrm{rat}}(X) \subset \mathscr{Z}_k^{\mathbf{R}\text{-alg}}(X)$ the lemma implies in particular, that the cycle map is well-defined modulo rational equivalence, so we get a homopmorphism

$$CH_k(X) \to H_{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)),$$

which we denote by cl as well.

2. The equivariant cycle map into cohomology

If X is nonsingular of dimension n, Poincaré duality enables us to define an equivariant cycle map cl^X into cohomology. For every $k \geq 0$ we define cl^X by the following diagram.

$$\begin{tikzcd}
\mathscr{Z}^k(X) \arrow[r, "\mathrm{cl}^X"] \arrow[d, equal] & H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \arrow[d, "\mu_X\cap"] \\
\mathscr{Z}_{n-k}(X) \arrow[r, "\mathrm{cl}_X"] & H_{2(n-k)}(X(\mathbf{C}); G, \mathbf{Z}(k))
\end{tikzcd}$$

It is easy to check that this definition is equivalent to the original definition of the cycle map in equivariant cohomology given in [Kr2].

If we put $\mathrm{cl}^X_{\mathbf{R}} = \beta^k \circ \mathrm{cl}^X$, we get the following diagram.

$$\begin{tikzcd}
\mathscr{Z}^k(X) \arrow[r, "\mathrm{cl}^X_{\mathbf{R}}"] \arrow[d, equal] & H^k(X(\mathbf{R}), \mathbf{Z}/2) \arrow[d, "\mu_{X(\mathbf{R})}\cap"] \\
\mathscr{Z}_{n-k}(X) \arrow[r, "\mathrm{cl}^{\mathbf{R}}_X"] & H_{n-k}(X(\mathbf{R}), \mathbf{Z}/2)
\end{tikzcd} \tag{70}$$

Using the fact that $\mathrm{cl}^{\mathbf{R}}_X = \rho_k \circ \mathrm{cl}_X$ by Theorem 1.1, and that $\rho_m(\mathrm{cl}_X(V))$ is zero in $H_m(X(\mathbf{R}), \mathbf{Z}/2)$ for $m > k = \dim V$, the commutativity of this diagram follows from the compatibility of ρ and β with cap-product.

As was mentioned in the Introduction, the result that $\beta^k \circ \mathrm{cl}^X$ is the usual real cycle map in cohomology is originally due to V.A. Krasnov ([Kr4, Th. 4.2]). However, the proof given here is quite different, in the sense that it does not use any involved techniques from algebraic geometry. On the other hand, Krasnov's approach does give a precise description of the total image $\beta(\mathrm{cl}^X(Z)) \in H^*(X(\mathbf{R}), \mathbf{Z}/2)$ in terms of the Steenrod squares of $\mathrm{cl}^X_{\mathbf{R}}(Z)$; I have not considered this question here.

Since cl_X is well-defined on $CH_*(X)$, we see that cl^X is well-defined on $CH^*(X)$. Moreover, by [Kr2, Prop. 2.1.3] we have

$$\mathrm{cl}^X([V] \cdot [W]) = \mathrm{cl}^X(V) \cup \mathrm{cl}^X(W),$$

hence

$$\mathrm{cl}_X([V] \cdot [W]) = \mathrm{cl}_X(V) \cap \mathrm{cl}^X(W) = \mathrm{cl}_X(V) \cdot \mathrm{cl}_X(W)$$

for cycles V, W on X.

3. Topological restrictions on the image of the real cycle map

In this section we will see that the factorization of the real cycle map via the equivariant cycle map puts severe restrictions on its image $H^{\mathrm{alg}}_k(X(\mathbf{R}), \mathbf{Z}/2)$. It follows from Theorem 1.1 that $H^{\mathrm{alg}}_k(X(\mathbf{R}), \mathbf{Z}/2)$ is contained in the image of the mapping

$$\rho_k\colon H_{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H_k(X(\mathbf{R}), \mathbf{Z}/2),$$

which is in many cases *not surjective*, as can be seen from the example below. Since the mapping ρ_k is compatible with equivariant homeomorphisms, this is a restriction of a purely *topological* nature. For example, we will see that for any abelian variety A defined over $\mathbf{R}$ of which $A(\mathbf{R})$ is not connected, the image of $H_2(A(\mathbf{C}); G, \mathbf{Z}(1))$ under ρ_1 is not the full group $H_1(A(\mathbf{R}), \mathbf{Z}/2)$. This means, that there is a proper subgroup $M \subsetneq H_1(A(\mathbf{R}), \mathbf{Z}/2)$, which does not depend on the algebraic structure of A, and which contains $H_1^{\mathrm{alg}}(A(\mathbf{R}), \mathbf{Z}/2)$. Or, to be precise, for any algebraic variety Y defined over $\mathbf{R}$ admitting an equivariant *homeomorphism*

$$\varphi\colon A(\mathbf{C}) \xrightarrow{\sim} Y(\mathbf{C}),$$

we have that $H_1^{\mathrm{alg}}(Y(\mathbf{R}), \mathbf{Z}/2) \subset \varphi_* M$.

It should be stressed, however, that this restriction does depend on the topology of $X(\mathbf{C})$ as a G-space, not only on the topology of $X(\mathbf{R})$. Consider, for example, a K3-surface X with $X(\mathbf{R})$ homeomorphic to a pair of tori. Then $X(\mathbf{R})$ is homeomorphic to the real part $A(\mathbf{R})$ of an abelian surface A of which $A(\mathbf{R})$ has 2 connected components. However, for such X we have that $H_2(X(\mathbf{C}); G, \mathbf{Z}(1))$ maps surjectively onto $H_1(X(\mathbf{R}), \mathbf{Z}/2)$ (cf. Example III.9.4), so there is no topological restriction on the subgroup H_1^{alg}. In fact, it follows from [Ma2, Th. 5.7] that we may choose an X as above actually having $H_1^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2) = H_1(X(\mathbf{R}), \mathbf{Z}/2)$. Of course, H_1^{alg} is very small for a sufficiently general K3-surface X_g over $\mathbf{R}$ with $X_g(\mathbf{R})$ as above (see [Ma2, Cor. 5.8] for a more precise statement), but contrary to the case of the abelian surface, this has nothing to do with the topology of X_g, but only with its algebraic structure.

Example 3.1. By definition, an *abelian variety* X over $\mathbf{R}$ is a complete geometrically irreducible group variety defined over $\mathbf{R}$. In particular, $X(\mathbf{R}) \neq \emptyset$ since $X(\mathbf{R})$ contains the zero-element. Let X be an abelian variety over $\mathbf{R}$ of dimension n. It is a classical fact from the theory of real abelian varieties that $X(\mathbf{C})$ is equivariantly homeomorphic to a product of the form

$$T_1 \times \cdots \times T_{n-s} \times C_1^{(1)} \times \cdots \times C_s^{(1)} \times C_1^{(2)} \times \cdots \times C_s^{(2)},$$

where each T_i is a torus $S^1 \times S^1$ with G acting on T_i via exchanging the factors, each $C_i^{(1)}$ is a circle with trivial G-action and each $C_i^{(2)}$ is a copy of the unit circle in the complex plane with the G-action given by complex conjugation. In particular, $X(\mathbf{R})$ is homeomorphic to the disjoint union of 2^s copies of the n-torus.

It follows from a Künneth-type theorem that $H_*(X(\mathbf{C}); G, \mathbf{Z}/2)$ is isomorphic to the tensor product

$$\begin{aligned} &H_*(T_1; G, \mathbf{Z}/2) \otimes \cdots \otimes H_*(T_{n-s}; G, \mathbf{Z}/2) \otimes H_*(C_1^{(1)}; G, \mathbf{Z}/2) \otimes \cdots \\ &\qquad \cdots \otimes H_*(C_s^{(1)}; G, \mathbf{Z}/2) \otimes H_*(C_1^{(2)}; G, \mathbf{Z}/2) \otimes \cdots \otimes H_*(C_s^{(2)}; G, \mathbf{Z}/2). \end{aligned}$$

For $i = 1, \ldots, n-s$, let $\psi_i \in H_0(T_i; G, \mathbf{Z}/2)$ be the class of a point in T_i^G and let $\tau_i \in H_2(T_i; G, \mathbf{Z}(1))$ be the fundamental class. For $i = 1, \ldots, s$, and $j = 1, 2$ let

$\varphi_i^{(j)} \in H_0(C_i^{(j)}; G, Z(j-1))$ be the class of a point in the fixed point set of $C_i^{(j)}$ and let $\gamma_i^{(j)} \in H_1(C^{(j)}; G, \mathbf{Z}(j-1))$ be the fundamental class of $C_i^{(j)}$. Then for all i we have that $\rho(\psi_i)$ is the class of a point, $\rho(\tau_i)$ is the fundamental class of T_i^G. For all i, j we have that $\rho(\varphi_i^{(j)})$ is the class of a point and $\rho(\gamma_i^{(j)})$ is the fundamental class of the fixed point set of $C_i^{(j)}$. Hence for all i the image of $H_*(T_i; G, \mathbf{Z}/2)$ under ρ is generated by $\rho(\psi_i)$ and $\rho(\tau_i)$; the image of $H_*(C^{(j)}; G, \mathbf{Z}/2)$ under ρ is generated by $\rho(\varphi_i^{(j)})$ and $\rho(\gamma_i^{(j)})$for all i, j.

We deduce that a basis for $\rho_k[(H_{2k}(X(\mathbf{C}); G, \mathbf{Z}/2))]$ is given by the image under ρ of the set of classes of the form

$$t_1 \times \cdots \times t_{n-s} \times c_1^{(1)} \times \cdots \times c_s^{(1)} \times c_1^{(2)} \times \cdots \times c_s^{(2)} \cap \eta^D,$$

where $D \geq 0$, η is the nontrivial element in $H^1(G, \mathbf{Z}(1))$, $t_i = \psi_i$ or τ_i and $c_i^{(j)} = \varphi_i$ or $\gamma_i^{(j)}$ subject to the following restrictions. Let A be the number of indices i such that $t_i = \tau_i$, let B be the number of indices i such that $c_i^{(1)} = \gamma_i^{(1)}$, and let C be the number of indices i such that $c_i^{(2)} = \gamma_i^{(2)}$. We require that $2A + B + C - D = 2k$ and $A + B = k$.

This implies that for any $k \geq 0$ we have that

$$\dim_{\mathbf{Z}/2} \rho_k(H_{2k}(X(\mathbf{C}); G, \mathbf{Z}/2)) = \sum_{B=0}^{k} \sum_{C=B}^{s} \binom{n-s}{k-B} \binom{s}{B} \binom{s}{C},$$

which gives an upper bound

$$\dim_{\mathbf{Z}/2} H_k^{\mathrm{alg}}(X(\mathbf{R}); \mathbf{Z}/2) \leq \sum_{B=0}^{k} \sum_{C=B}^{s} \binom{n-s}{k-B} \binom{s}{B} \binom{s}{C},$$

that is highly non-trivial when $k \gg 0$ and $s \gg 0$, since

$$\dim_{\mathbf{Z}/2} H_k(X(\mathbf{R}); \mathbf{Z}/2) = 2^s \binom{n}{k}.$$

In the preceding example our knowledge of the structure of the G-space $X(\mathbf{C})$ enabled us to give a geometrical construction of a basis for the image of ρ_k. In general this is not so simple. However, for a complete nonsingular X we may use the homological intersection products on the equivariant homology of $X(\mathbf{C})$ and the ordinary homology of $X(\mathbf{R})$. The we can put topological restrictions on the subgroup $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2)$ by constructing certain classes in the homology group of complementary dimension. In order to make this precise, we need some definitions.

Definition 3.2. Let X be an algebraic variety defined over $\mathbf{R}$. For any $k \geq 0$ we define the subgroup of $H_k(X(\mathbf{R}), \mathbf{Z}/2)$ of *potentially algebraic homology classes* to be

$$H_k^{(\leq 1/2)}(X(\mathbf{R}), \mathbf{Z}/2) = \rho_k(\{\gamma \in H_{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \colon \rho_m(\gamma) = 0 \text{ for } m > k\}).$$

We also define

$$H^k_{(\geq 1/2)}(X(\mathbf{R}),\mathbf{Z}/2) = \beta^k\left(\{\omega \in H^{2k}(X(\mathbf{C});G,\mathbf{Z}(k))\colon\ \beta^m(\omega)=0 \text{ for } m<k\}\right),$$
$$H_k^{(<1/2)}(X(\mathbf{R}),\mathbf{Z}/2) = \rho_k\left(\{\gamma \in \mathrm{Tor}\,(H_{2k}(X(\mathbf{C});G,\mathbf{Z}(k)))\ :\ \rho_m(\gamma)=0 \text{ for } m>k\}\right),$$

and

$$H^k_{(>1/2)}(X(\mathbf{R}),\mathbf{Z}/2) = \beta^k\left(\{\omega \in \mathrm{Tor}\,(H^{2k}(X(\mathbf{C});G,\mathbf{Z}(k)))\ :\ \beta^m(\omega)=0 \text{ for } m<k\}\right).$$

Here $\mathrm{Tor}\,(M)$ denotes the torsion subgroup of an abelian group M.

The notation $H_k^{(\leq 1/2)}$ is chosen to indicate that intuitively $H_k^{(\leq 1/2)}(X(\mathbf{R}),\mathbf{Z}/2)$ should be thought of as the classes representable by k-dimensional subspaces of $X(\mathbf{R})$ that are fixed point sets of oriented G-equivariant subspaces of $X(\mathbf{C})$ of dimension at least $2k$. When X is geometrically irreducible and nonsingular of dimension n, then Poincaré duality gives isomorphisms $H^k_{(\geq 1/2)}(X(\mathbf{R}),\mathbf{Z}/2) \simeq H_{n-k}^{(\leq 1/2)}(X(\mathbf{R}),\mathbf{Z}/2)$ and $H^k_{(>1/2)}(X(\mathbf{R}),\mathbf{Z}/2) \simeq H_{n-k}^{(<1/2)}(X(\mathbf{R}),\mathbf{Z}/2)$.

Note that for $k=0,1$ the condition

$$\beta^m(\omega)=0 \text{ for } m<k$$

is automatically satisfied by every $\omega \in H^{2k}(X(\mathbf{C});G,\mathbf{Z}(k))$, and, similarly, when n is the dimension of X, and $k=n$ or $n-1$, the condition

$$\rho_m(\gamma)=0 \text{ for } m>k$$

is automatically satisfied by every $\gamma \in H_{2k}(X(\mathbf{C});G,\mathbf{Z}(k))$, hence we have for $k=0,1$ that

$$H_k^{(\leq 1/2)}(X(\mathbf{R}),\mathbf{Z}/2) = \rho_{n-k}H_{2(n-k)}(X(\mathbf{C});G,\mathbf{Z}(n-k)), \tag{71}$$

and

$$H^k_{(\geq 1/2)}(X(\mathbf{R}),\mathbf{Z}/2) = \beta^k H^{2k}(X(\mathbf{C});G,\mathbf{Z}(k)). \tag{72}$$

Proposition 3.3. *Let X be a geometrically irreducible algebraic variety over $\mathbf{R}$.*

(i) *For any $k \geq 0$ we have*

$$H_k^{\mathrm{alg}}(X(\mathbf{R}),\mathbf{Z}/2) \subset H_k^{(\leq 1/2)}(X(\mathbf{R}),\mathbf{Z}/2).$$

(ii) *If X is complete, $H_k^{\mathrm{alg}}(X(\mathbf{R}),\mathbf{Z}/2)$ is orthogonal to $H^k_{(>1/2)}(X(\mathbf{R}),\mathbf{Z}/2)$ for the cap product pairing.*

(iii) *If X is complete, nonsingular and of dimension n, $H_k^{\mathrm{alg}}(X(\mathbf{R}),\mathbf{Z}/2)$ is orthogonal to $H_{n-k}^{(<1/2)}(X(\mathbf{R}),\mathbf{Z}/2)$ for the intersection pairing in homology.*

PROOF. (i) It is easily seen from the definition of the equivariant cycle map that any $\gamma \in H_{2k}^{\mathrm{alg}}(X(\mathbf{C}); G, \mathbf{Z}(k))$ has the property that $\rho_m(\gamma) = 0$ for $m > k$, so the statement follows from the fact that $\mathrm{cl}^{\mathbf{R}} = \rho_k \circ \mathrm{cl}$.

(ii) For $\gamma \in H_{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$ and $\omega \in \mathrm{Tor}\left(H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))\right)$ we have $\langle \gamma, \omega \rangle = 0$, so $\langle \rho(\gamma), \beta(\omega) \rangle = 0$ by equation (49). Now observe that if $\rho_m(\gamma) = 0$ for $m > k$ and $\beta^m(\omega) = 0$ for $m < k$, then $\langle \rho(\gamma), \beta(\omega) \rangle = \langle \rho_k(\gamma), \beta^k(\omega) \rangle$, hence $H_k^{(\leq 1/2)}(X(\mathbf{R}), \mathbf{Z}/2)$ is orthogonal to $H^k_{(>1/2)}(X(\mathbf{R}), \mathbf{Z}/2)$ for the cap product pairing, and the statement follows from the previous statement.

(iii) This follows immediately from the previous statement and the definition of the intersection pairing in homology.

□

The third part of the proposition can be thought of as a topological generalization of the following result of W. Kucharz.

Corollary 3.4 (W. Kucharz). *For any complete, nonsingular, geometrically irreducible variety X over $\mathbf{R}$ of dimension n the subgroup $H_k^{\mathrm{alg}}(X(\mathbf{R}), \mathbf{Z}/2) \subset H_k(X(\mathbf{R}), \mathbf{Z}/2)$ is orthogonal to $\mathrm{cl}^{\mathbf{R}}\left(\mathscr{Z}_{n-k}^{\mathrm{alg}}(X)\right) \subset H_{n-k}(X(\mathbf{R}), \mathbf{Z}/2)$ with respect to the intersection product in homology.*

PROOF. Since the real cycle map $\mathrm{cl}^{\mathbf{R}}$ sends $\mathscr{Z}_{n-k}^{\mathrm{alg}}(X)$ into $H_{n-k}^{(<1/2)}(X(\mathbf{R}), \mathbf{Z}/2)$, this is an immediate consequence of Proposition 3.3. □

Remark 3.5. It should be said that a direct proof of Corollary 3.4, using the compatibility modulo 2 between the real cycle map and the intersection product of algebraic cycles is not hard either (see [Ku, Th. 3]). In fact, in Kucharz' approach the group $\mathscr{Z}_{n-k}^{\mathrm{alg}}(X)$ can be replaced by the group $\mathscr{Z}_{n-k}^{\mathrm{num}}(X)$ of cycles *numerically equivalent to zero* (see [Fu, Def. 19.1]). Observe that $\mathrm{cl}^{\mathbf{R}}$ sends $\mathscr{Z}_{n-k}^{\mathrm{num}}(X)$ into $H_{n-k}^{(<1/2)}(X(\mathbf{R}), \mathbf{Z}/2)$ if $\mathscr{Z}_{n-k}^{\mathrm{num}}(X)$ is the kernel of the cycle map $\mathscr{Z}_k \to H_k(X(\mathbf{C}), \mathbf{Q})$. This is a famous conjecture (cf. [Fu, 19.3.2]).

The results of Section 4 imply that in fact not only for the trivial cases $k = 0$, n we have an equality $\mathrm{cl}^{\mathbf{R}}\left(\mathscr{Z}_{n-k}^{\mathrm{alg}}(X)\right) = H_{n-k}^{(<1/2)}(X(\mathbf{R}), \mathbf{Z}/2)$, but also for case $k = 1$, if $H^2(X(\mathbf{C}), \mathbf{Z})$ torsionfree (or if we replace algebraic equivalence by numerical equivalence, since then $\mathscr{Z}_{n-k}^{\mathrm{num}}(X)$ is the kernel of the cycle map $\mathscr{Z}_k \to H_k(X(\mathbf{C}), \mathbf{Q})$ by [Mat]). Hence, for these k Proposition 3.3 is not really an improvement over Corollary 3.4. However, in the intermediate dimensions the image of $\mathscr{Z}_{n-k}^{\mathrm{alg}}(X)$ under $\mathrm{cl}^{\mathbf{R}}$ is not known in general. In the case of an arbitrary abelian variety X, for example, the only thing we can be sure about (for $k \neq n$) is that the image of $\mathscr{Z}_{n-k}^{\mathrm{alg}}(X)$ in $H_{n-k}(X(\mathbf{R}), \mathbf{Z}/2)$ contains the intersection of k different classes from $\mathrm{cl}^{\mathbf{R}}\left(\mathscr{Z}_{n-1}^{\mathrm{alg}}(X)\right) = H_{n-1}^{(<1/2)}(X(\mathbf{R}), \mathbf{Z}/2)$. Therefore, when X is an abelian variety such that the real part has 2^s connected components, Corollary 3.4 only gives (for $k \neq n$) an upper bound

$$\dim_{\mathbf{Z}/2} H_k^{\mathrm{alg}}(X(\mathbf{R});\mathbf{Z}/2) \leq 2^s\binom{n}{k} - \binom{s}{k},$$

as was derived in [Ku, Th. 3]. For large s and k this bound is considerably weaker then the upper bound given in Example 3.1.

4. The cycle map in codimension 1

In complex algebraic geometry the cycle map for divisors on a complete nonsingular variety X is well-understood. The kernel consists exactly of the cycles algebraically equivalent to zero and the image consists of the subgroup of Hodge classes in the second integral cohomology group. In this section we will see that for a complete nonsingular variety over $\mathbf{R}$ the equivariant cycle map has similar properties. All results are straightforward corollaries of a result by V.A. Krasnov that the equivariant cycle map can be interpreted as the connecting morphism in the long exact sequence associated to an equivariant version of the well known short exponential sequence of complex geometry.

Let us fix a complete, nonsingular, geometrically irreducible variety X over $\mathbf{R}$. Then $X(\mathbf{C})$ can be considered as a complex analytic manifold with an antiholomorphic involution, and $\mathcal{O}_{\mathrm{an}}$, the sheaf of germs of analytic functions on $X(\mathbf{C})$, has the natural structure of a G-sheaf. We have a short exact sequence of G-sheaves

$$0 \to \mathbf{Z}(1) \xrightarrow{\times 2\pi \mathrm{i}} \mathcal{O}_{\mathrm{an}} \xrightarrow{\exp} \mathcal{O}^*_{\mathrm{an}} \to 0$$

which induces a long exact sequence

$$\begin{aligned} (73)\quad \cdots \to H^1(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H^1(X(\mathbf{C}); G, \mathcal{O}_{\mathrm{an}}) \to \\ \to H^1(X(\mathbf{C}); G, \mathcal{O}^*_{\mathrm{an}}) \to H^2(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H^2(X(\mathbf{C}); G, \mathcal{O}_{\mathrm{an}}) \to \cdots \end{aligned}$$

By [Kr2, Prop. 1.1.1] we may identify $H^1(X(\mathbf{C}); G, \mathcal{O}^*_{\mathrm{an}})$ with the group of equivariant isomorphism classes of holomorphic line bundles on $X(\mathbf{C})$ with an antiholomorphic involution compatible with the involution on the base space. By the GAGA-principle this group is isomorphic to the group of equivariant isomorphism classes of complex algebraic line bundles on $X_{\mathbf{C}}$ with a $\mathbf{C}$-antilinear involution compatible with the complex conjugation on $X_{\mathbf{C}}$, and from the remarks made in Section I.3 we then see that we have a canonical isomorphism

$$H^1(X(\mathbf{C}); G, \mathcal{O}^*_{\mathrm{an}}) \xrightarrow{\sim} \mathrm{Pic}(X).$$

We also have a natural isomorphism $\mathrm{Pic}(X)$ with $CH^1(X)$, and the connecting morphism of the long exact sequence (73) then corresponds to the equivariant cycle map cl : $CH^1(X) \to H^2(X(\mathbf{C}); G, \mathbf{Z}(1))$ by [Kr2, Prop. 1.3.1]. In order to describe the image, we define the subgroup

$$H^{1,1}(X(\mathbf{C}); G, \mathbf{Z}(1)) \subset H^1(X(\mathbf{C}); G, \mathbf{Z}(1))$$

to be the inverse image of the Hodge $(1,1)$-classes $H^{1,1}(X(\mathbf{C}), \mathbf{C})$ under the composite mapping

$$H^2(X(\mathbf{C}); G, \mathbf{Z}(1)) \xrightarrow{e} H^2(X(\mathbf{C}), \mathbf{Z}(1)) \xrightarrow{\times 2\pi i} H^2(X(\mathbf{C}), \mathbf{C}).$$

Theorem 4.1 (V.A. Krasnov). *If X is a complete, nonsingular, geometrically irreducible variety over $\mathbf{R}$, the equivariant cycle map in codimension 1 fits into an exact sequence*

$$\begin{aligned} 0 \to H^1(X(\mathbf{C}), \mathbf{Z}(1))^G \to H^1(X(\mathbf{C}), \mathscr{O}_{\mathrm{an}})^G \to \mathrm{Pic}(X) \\ \to H^{1,1}(X(\mathbf{C}); G, \mathbf{Z}(1)) \to 0. \end{aligned}$$

Proof. In view of the above discussion it is sufficient to prove that the image of $H^1(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H^1(X(\mathbf{C}); G, \mathscr{O}_{\mathrm{an}})$ is precisely $H^1(X(\mathbf{C}), \mathbf{Z}(1))^G$ and the kernel of $H^2(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H^2(X(\mathbf{C}); G, \mathscr{O}_{\mathrm{an}})$ is precisely $H^{1,1}(X(\mathbf{C}); G, \mathbf{Z}(1))$. This is easily deduced from the analogous statements in complex geometry since the edge morphisms of the Hochschild-Serre spectral sequence induce for every $k \geq 0$ an isomorphism

$$H^k(X(\mathbf{C}); G, \mathscr{O}_{\mathrm{an}}) \xrightarrow{\sim} H^k(X(\mathbf{C}), \mathscr{O}_{\mathrm{an}})^G$$

and the edge morphism

$$H^1(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H^1(X(\mathbf{C}), \mathbf{Z}(1))^G$$

is surjective. □

Corollary 4.2. *Let X be a complete, nonsingular, geometrically irreducible variety over $\mathbf{R}$.*

(i) *The kernel of the equivariant cycle map*

$$\mathrm{cl} : \mathscr{Z}^1(X) \to H^1(X(\mathbf{C}); G, \mathbf{Z}(1))$$

is precisely the group $\mathscr{Z}^1_{\mathbf{R}\text{-alg}}(X)$ of cycles real algebraically equivalent to zero.

(ii) *The group $\mathscr{Z}^1_{\mathbf{R}\text{-alg}}(X)/\mathscr{Z}^1_{\mathrm{rat}}(X)$ is the largest divisible subgroup of $CH^1(X)$. We have an isomorphism*

$$\mathscr{Z}^1_{\mathbf{R}\text{-alg}}(X)/\mathscr{Z}^1_{\mathrm{rat}}(X) \simeq (\mathbf{R}/\mathbf{Z})^q,$$

where $q = \dim_{\mathbf{R}} H^1(X, \mathscr{O}_X)$.

Proof. Since $H^1(X(\mathbf{C}), \mathbf{Z})$ is a lattice in $H^1(X(\mathbf{C}), \mathscr{O}_{\mathrm{an}})$, we have that the image of $H^1(X(\mathbf{C}), \mathbf{Z}(1))^G$ is a lattice in $H^1(X(\mathbf{C}), \mathscr{O}_{\mathrm{an}})^G \simeq H^1(X, \mathscr{O}_X)$. Hence, writing $P = H^1(X(\mathbf{C}), \mathscr{O}_{\mathrm{an}})^G / H^1(X(\mathbf{C}), \mathbf{Z}(1))^G$, considered as subgroup of $\mathrm{Pic}(X)$, it is sufficient to show that the image of $\mathscr{Z}^1_{\mathbf{R}\text{-alg}}(X)$ in $\mathrm{Pic}(X)$ is P.

Comparing the equivariant exponential sequence with the complex exponential sequence, it is not hard to check that the image of P under the base change homomorphism $\pi^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X_{\mathbf{C}})$ is $(1+\sigma)\operatorname{Pic}^0(X_{\mathbf{C}})$, and that π^* is injective on P (this last fact also follows from Corollary I.2.2). Since $(1+\sigma)\operatorname{Pic}^0(X_{\mathbf{C}})$ is the image of the composite mapping $\operatorname{Pic}^0(X_{\mathbf{C}}) \xrightarrow{\pi_*} \operatorname{Pic}(X) \xrightarrow{\pi^*} \operatorname{Pic}(X_{\mathbf{C}})$, we see that $P = \pi_*(\operatorname{Pic}^0(X_{\mathbf{C}}))$, which is precisely $\mathscr{Z}^1_{\mathbf{R}\text{-alg}}(X)/\mathscr{Z}^1_{\text{rat}}(X)$ by Lemma I.1.6. □

Remark 4.3. The above result shows that for varieties over **R** the group of cycles real algebraically equivalent to zero plays a rôle similar to the role played by the group of cycles algebraically equivalent to zero in complex algebraic geometry.

On a theoretic level Theorem 4.1 and Theorem 1.1 completely solve the question of determining $H^1_{\text{alg}}(X(\mathbf{R}),\mathbf{Z}/2)$ for a nonsingular complete variety X over **R**. To be precise, we obtain a description of the image of the cycle map $\mathrm{cl}^{\mathbf{R}}$ in codimension 1 purely in terms of equivariant cohomology and the Hodge decomposition.

Corollary 4.4. *When X is a complete, nonsingular, geometrically irreducible variety over* **R**,

$$H^1_{\text{alg}}(X(\mathbf{R}),\mathbf{Z}/2) = \beta^1\left[H^{1,1}(X(\mathbf{C});G,\mathbf{Z}(1))\right].$$

In particular, if $H^2(X,\mathscr{O}_X)=0$, we have

$$H^1_{\text{alg}}(X(\mathbf{R}),\mathbf{Z}/2) = \beta^1\left[H^2(X(\mathbf{C});G,\mathbf{Z}(1))\right],$$

hence

$$H^{\text{alg}}_{n-1}(X(\mathbf{R}),\mathbf{Z}/2) = \rho_{n-1}\left[H_{2n-2}(X(\mathbf{C});G,\mathbf{Z}(n-1))\right],$$

where n is the dimension of X.

PROOF. Immediate from Theorem 4.1. The statement on homology follows from diagram (70). □

When X is a real rational surface, Corollary 4.4 and Lemma III.8.9 imply the result of R. Silhol that $H_1^{\text{alg}} = H_1$ ([Si, Th. III.3.4]). For real Enriques surfaces the image of ρ_1 is determined in Chapter V, but for most varieties X very little is known about the mapping ρ_1 (or β^1). Moreover, when $H^2(X,\mathscr{O}_X)\neq 0$, we not only need to know the mapping β^1, but we also need a convenient description of $H^{1,1}(X(\mathbf{C});G,\mathbf{Z}(1))$ in order to describe $H^1_{\text{alg}}(X(\mathbf{R}),\mathbf{Z}/2)$. When X is an abelian variety over **R**, J. Huisman shows how to compute H^1_{alg} in [Hu2]. Although he does not use the language of equivariant cohomology, his results are not hard to translate. For arbitrary X with $H^2(X,\mathscr{O}_X)\neq 0$, a precise description of $H^1_{\text{alg}}(X(\mathbf{R}),\mathbf{Z}/2)$ will be very hard to give. It should be possible, however, to use Corollary 4.4 in order to describe the behaviour of H^1_{alg} under deformations and to generalize the results for real K3-surfaces obtained by F. Mangolte in [Ma2].

5. Divisors over C and divisors over R

Using the description of the structure of $\mathrm{Pic}(X)$ given in Theorem 4.1 we are able to answer the questions of Section I.2 concerning the mapping $\pi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbf{C}})^G$ in terms of the equivariant cohomology of X. Recall, that for a complete, nonsingular, geometrically irreducible variety X over $\mathbf{R}$ the mapping $\pi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbf{C}})^G$ is an isomorphism when $X(\mathbf{R})$ is nonempty. However, the induced mapping $\pi^*\colon \mathcal{NS}(X) \to \mathcal{NS}(X_{\mathbf{C}})^G$ need not be surjective. This problem will be treated first. Another problem, the surjectivity of $\pi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbf{C}})^G$ when $X(\mathbf{R})$ is empty, was already solved in Corollary I.2.5 in terms of the algebra of the function field of X. In this section Propositions 5.6 and 5.7 give a criterion of a topological-analytical nature which is far less elegant but easier to apply in general. Moreover, the approach of Propositions 5.6 and 5.7 might also be useful determining the image of $\pi^*\colon CH^k(X) \to CH^k(X_{\mathbf{C}})$ for $k > 1$. It must be said, however, that apart from the case of zero-cycles this remains extremely complicated for $k > 1$, not the least because then even the structure of $CH^k(X_{\mathbf{C}})$ is very mysterious.

Proposition 5.1. *When X is a complete, nonsingular, geometrically irreducible variety over $\mathbf{R}$,*

$$\pi^* \mathcal{NS}(X) = \mathcal{NS}(X_{\mathbf{C}})^G \cap e(H^2(X(\mathbf{C}); G, \mathbf{Z}(1))).$$

Proof. We have that $\mathcal{NS}(X_{\mathbf{C}}) \simeq H^{1,1}(X(\mathbf{C}), \mathbf{Z})$ and, by Theorem 4.1, $\pi^* \mathcal{NS}(X)$ is the image of $H^{1,1}(X(\mathbf{C}); G, \mathbf{Z}(1))$ under the edge morphism e. □

In particular, $\mathcal{NS}(X) \simeq \mathcal{NS}(X_{\mathbf{C}})^G$ when the edge morphism e maps surjectively onto $H^2(X(\mathbf{C}), \mathbf{Z}(1))^G$, so we get the following result, which was already conjectured by R. Silhol (private communication).

Corollary 5.2. *When X is a complete, nonsingular, geometrically irreducible $\mathbf{Z}$-GM-variety,*

$$\mathcal{NS}(X) \simeq \mathcal{NS}(X_{\mathbf{C}})^G.$$

Example 5.3. Let $X \to C$ be a conic bundle over a curve as in Example III.9.5 with $X(\mathbf{R})$ nonempty, but having less connected components than the real part $C(\mathbf{R})$ of the base curve. We then have that

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{\mathbf{C}})^G,$$

but on the other hand,

$$\mathcal{NS}(X) \neq \mathcal{NS}(X_{\mathbf{C}})^G,$$

since the Néron-Severi group of $X_{\mathbf{C}}$ is isomorphic to $H^1(X(\mathbf{C}), \mathbf{Z}(1))$, whereas we saw in Example III.9.5 that the edge morphism

$$e\colon H^2(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H^2(X(\mathbf{C}), \mathbf{Z}(1))^G$$

has cokernel $\mathbf{Z}/2$. This means that the mapping

$$\mathcal{NS}(X) \to \mathcal{NS}(X_{\mathbf{C}})^G$$

has cokernel $\mathbf{Z}/2$, and in fact this cokernel is generated by the class of a section $C_{\mathbf{C}} \hookrightarrow X_{\mathbf{C}}$ defined over $\mathbf{C}$.

Now we turn to the question concerning the mapping $\pi^*\colon\ \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbf{C}})^G$ for a complete, nonsingular variety X with $X(\mathbf{R}) = \emptyset$. Then the mapping π^* is injective, by Corollary I.2.2, but by Corollary I.2.5 it is surjective if and only if -1 is not a sum of two squares in the function field of X. From Theorem 4.1 and a close inspection of the Hochschild-Serre spectral sequence of X with coefficients in $\mathbf{Z}(1)$ we can find criteria that are often easier to handle.

We will split the problem into two parts. Since $\mathrm{Pic}^0(X)$ is by definition the inverse image of $\mathrm{Pic}^0(X_{\mathbf{C}})^G$ under the mapping π^*, we see that $\pi^*\colon\ \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbf{C}})^G$ is surjective if and only if the induced mappings

$$\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathbf{C}})^G$$

and

$$\mathcal{NS}(X) = \mathrm{Pic}(X)/\,\mathrm{Pic}^0(X) \to \mathrm{Pic}(X_{\mathbf{C}})^G/\,\mathrm{Pic}^0(X_{\mathbf{C}})^G$$

are surjective.

First, let us treat the easy case, namely the case that $\mathrm{Pic}^0(X_{\mathbf{C}})^G$ is a divisible group. It is well-known that the divisibility of $\mathrm{Pic}^0(X_{\mathbf{C}})^G$ is equivalent to the condition that $H^2(G, H^1(X(\mathbf{C}),\mathbf{Z})) = 0$. Then the mapping $\pi^* \circ \pi_*\colon\ \mathrm{Pic}^0(X_{\mathbf{C}})^G \to \mathrm{Pic}^0(X_{\mathbf{C}})^G$ is surjective, since it is multiplication by 2. This means that $\mathrm{Pic}^0(X) = \mathrm{Pic}^0(X_{\mathbf{C}})^G$. On the other hand, applying cohomology of G to the short exact sequence

$$0 \to \mathrm{Pic}^0(X_{\mathbf{C}}) \to \mathrm{Pic}(X_{\mathbf{C}}) \to \mathcal{NS}(X_{\mathbf{C}}) \to 0$$

we see that $\mathrm{Pic}(X_{\mathbf{C}})^G/\,\mathrm{Pic}^0(X_{\mathbf{C}})^G \simeq \mathcal{NS}(X_{\mathbf{C}})^G$, so we obtain the following result.

Corollary 5.4. *Let X be a complete, nonsingular, geometrically irreducible variety over $\mathbf{R}$ such that $\mathrm{Pic}^0(X_{\mathbf{C}})^G$ is divisible. Then*

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{\mathbf{C}})^G$$

if and only if $\mathcal{NS}(X_{\mathbf{C}})^G = H^{1,1}(X(\mathbf{C}),\mathbf{Z}(1))^G$ is contained in the image of the edge morphism

$$e\colon\ H^2(X(\mathbf{C}); G,\mathbf{Z}(1)) \to H^2(X(\mathbf{C}),\mathbf{Z}(1))^G.$$

Proof. By the above discussion we have that $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\mathbf{C}})^G$ if and only if $\mathcal{NS}(X) = \mathcal{NS}(X_{\mathbf{C}})^G$, so the statement follows from Proposition 5.1. $\square$

Example 5.5. When X is a real rational surface with $X(\mathbf{R}) = \emptyset$, we see in Example III.9.4 that the edge morphism

$$e\colon\ H^2(X(\mathbf{C}); G,\mathbf{Z}(1)) \to H^2(X(\mathbf{C}),\mathbf{Z}(1))^G.$$

is not surjective. Since $\mathcal{NS}(X_{\mathbf{C}}) = H^2(X(\mathbf{C}),\mathbf{Z})$, Corollary 5.4 implies that

$$\mathrm{Pic}(X) \neq \mathrm{Pic}(X_{\mathbf{C}})^G,$$

and -1 is a sum of two squares in the function field of X, which gives yet another proof of the result of Parimala and Sujatha (see [PS] and [CT]).

On the other hand, for a real K3-surface X with $X(\mathbf{R}) \neq \emptyset$, we still have that

$$e\colon\ H^2(X(\mathbf{C}); G,\mathbf{Z}(1)) \to H^2(X(\mathbf{C}),\mathbf{Z}(1))^G.$$

has cokernel $\mathbf{Z}/2$, but now $H^2(X,\mathcal{O}_X) \neq 0$, so $\mathcal{NS}(X_{\mathbf{C}}) \neq H^2(X(\mathbf{C}),\mathbf{Z})$, and Corollary 5.4 does not imply that $\mathrm{Pic}(X) \neq \mathrm{Pic}(X_{\mathbf{C}})^G$. In fact, from the theory of moduli of real K3-surfaces (see [Si, Ch. VIII], or [Ma2]), we see that Corollary 5.4 implies that the set of isomorphism classes

$$\{[X]\colon\ \mathrm{Pic}(X) \neq \mathrm{Pic}(X)^G\}$$

forms a countable union of irreducible real analytic subspaces of codimension 1 in the moduli space of real nonsingular quartics in $\mathbf{P}^3_{\mathbf{R}}$ having no real points.

When $\mathrm{Pic}^0(X_{\mathbf{C}})^G$ is not divisible, the criteria we obtain are of an even more technical nature, but Examples 5.8 and 5.9 show that they are quite easily applied in concrete situations.

Proposition 5.6. *Let X be a complete, nonsingular, geometrically irreducible variety over $\mathbf{R}$. The mapping*

$$\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathbf{C}})^G$$

is surjective if and only if the differential

$$d_2^{1,1}\colon\ H^1(G,H^1(X(\mathbf{C}),\mathbf{Z}(1))) \to H^3(G,H^0(X,\mathbf{Z}(1)))$$

of the Hochschild-Serre spectral sequence is zero.

Proof. Let $F^*H^2(X(\mathbf{C}); G,\mathbf{Z}(1))$ be the filtration associated to the Hochschild-Serre spectral sequence. We have an exact sequence

$$0 \to F^1H^2(X(\mathbf{C}); G,\mathbf{Z}(1)) \xrightarrow{\ \varepsilon\ }$$
$$\to H^1(G,H^1(X(\mathbf{C}),\mathbf{Z}(1)) \xrightarrow{d_2^{1,1}} H^3(G,H^0(X(\mathbf{C}),\mathbf{Z}(1)))$$

Applying Galois cohomology to the exact sequence

$$0 \to H^1(X(\mathbf{C}),\mathbf{Z}(1)) \to H^1(X(\mathbf{C}),\mathcal{O}_{\mathrm{an}}) \to \mathrm{Pic}^0(X_{\mathbf{C}}) \to 0 \tag{74}$$

we get the exact upper row of the following diagram.

$$\begin{tikzcd}
H^1(X(\mathbf{C}),\mathcal{O}_{\mathrm{an}})^G \arrow[r] & \mathrm{Pic}^0(X_{\mathbf{C}})^G \arrow[r] & H^1(G,H^1(X(\mathbf{C}),\mathbf{Z}(1))) \arrow[r] & 0 \\
H^1(X(\mathbf{C}),\mathcal{O}_{\mathrm{an}})^G \arrow[r] \arrow[u, equal] & \mathrm{Pic}^0(X) \arrow[r] \arrow[u, "\pi^*"] & F^1H^2(X(\mathbf{C}); G\mathbf{Z}(1)) \arrow[r] \arrow[u, "\varepsilon"] & 0
\end{tikzcd}$$

The exact bottom row is derived from the exact sequence of Theorem 4.1 and it can be checked that the diagram is commutative. It follows immediately that π^* maps surjectively onto $\mathrm{Pic}^0(X_{\mathbf{C}})^G$ if and only if ε is surjective. $\square$

Proposition 5.7. *Let X be a complete, nonsingular, geometrically irreducible variety over $\mathbf{R}$. The mapping*

$$\mathcal{NS}(X) \to \mathrm{Pic}(X_{\mathbf{C}})^G / \mathrm{Pic}^0(X_{\mathbf{C}})^G$$

is surjective if and only if the differential

$$d_3^{0,2}\colon\ E_3^{0,2} \to H^3(G, H^0(X(\mathbf{C}), \mathbf{Z}(1)))$$

of the Hochschild-Serre spectral sequence is zero when restricted to

$$E_3^{0,2} \cap H^{1,1}(X(\mathbf{C}), \mathbf{Z}).$$

Here the intersection of groups is defined by considering $E_3^{0,2} \subset E_2^{0,2} = H^2(X(\mathbf{C}), \mathbf{Z}(1))^G$ as a subgroup of $H^2(X(\mathbf{C}), \mathbf{Z})$.

Proof. Applying Galois cohomology to the short exact sequence

$$0 \to \mathrm{Pic}^0(X_{\mathbf{C}}) \to \mathrm{Pic}(X_{\mathbf{C}}) \to H^{1,1}(X(\mathbf{C}), \mathbf{Z}(1)) \to 0$$

we get the exact top row of the commutative diagram

(75)

$$\begin{tikzcd}
\mathrm{Pic}^0(X_{\mathbf{C}})^G \arrow[r] & \mathrm{Pic}(X_{\mathbf{C}})^G \arrow[r] & H^{1,1}(X(\mathbf{C}), \mathbf{Z}(1))^G \arrow[r, "\delta"] & H^1(G, \mathrm{Pic}^0(X_{\mathbf{C}})) \\
\mathrm{Pic}^0(X) \arrow[r] \arrow[u, "\pi^*"] & \mathrm{Pic}(X) \arrow[r] \arrow[u, "\pi^*"] & \dfrac{H^{1,1}(X(\mathbf{C}); G, \mathbf{Z}(1))}{F^1 H^2(X(\mathbf{C}); G, \mathbf{Z}(1))} \arrow[r] \arrow[u, "\varepsilon"] & 0 \arrow[u]
\end{tikzcd}$$

where the exact bottom row is derived from the exact sequence of Theorem 4.1. The long exact sequence of Galois cohomology associated to the short exact sequence (74) gives us an isomorphism

$$\varphi\colon\ H^1(G, \mathrm{Pic}^0(X_{\mathbf{C}})) \xrightarrow{\sim} H^2(G, H^1(X, \mathbf{Z}(1)))$$

Moreover, the following diagram can be shown to be commutative.

$$\begin{tikzcd}
H^{1,1}(X(\mathbf{C}), \mathbf{Z}(1))^G \arrow[r, "\delta"] \arrow[d, equal] & H^1(G, \mathrm{Pic}^0(X_{\mathbf{C}})) \arrow[d, "\varphi"] \\
H^{1,1}(X(\mathbf{C}), \mathbf{Z}(1))^G \arrow[r, "d_2^{0,2}"] & H^1(G, H^1(X(\mathbf{C}), \mathbf{Z}(1)))
\end{tikzcd}$$

Here $d_2^{0,2}$ denotes the restriction of the differential of the Hochschild-Serre spectral sequence to $H^{1,1}(X(\mathbf{C}), \mathbf{Z}(1))^G$. Hence we have an isomorphism

$$\mathrm{Ker}\,\delta \simeq \mathrm{Ker}\,d_2^{0,2} = E_3^{0,2} \cap H^{1,1}(X(\mathbf{C}), \mathbf{Z}),$$

so the exact sequence

$$0 \to \frac{H^{1,1}(X(\mathbf{C}); G, \mathbf{Z}(1))}{F^1 H^2(X(\mathbf{C}); G, \mathbf{Z}(1))} \to E_3^{0,2} \cap H^{1,1}(X(\mathbf{C}), \mathbf{Z}) \xrightarrow{d_3^{0,2}} H^3(G, H^0(X(\mathbf{C}), \mathbf{Z}(1)))$$

implies that in diagram (75) we have $\operatorname{Im} e = \operatorname{Ker} \delta$ if and only if $d_3^{0,2}$ restricted to $E_3^{0,2} \cap H^{1,1}(X(\mathbf{C}), \mathbf{Z})$ is zero □

Example 5.8. For a nonsingular projective, geometrically irreducible curve X over $\mathbf{R}$ of genus g with $X(\mathbf{R}) = \emptyset$ the calculations of Example III.9.3 immediately give the classical result that if g is even, we have

$$\begin{aligned} \operatorname{Pic}^0(X) &= \operatorname{Pic}^0(X)^G, \\ \mathcal{NS}(X) &\neq \operatorname{Pic}(X_{\mathbf{C}})^G / \operatorname{Pic}^0(X_{\mathbf{C}})^G, \end{aligned}$$

and if g is odd we have

$$\begin{aligned} \operatorname{Pic}^0(X) &\neq \operatorname{Pic}^G(X)^G \\ \mathcal{NS}(X) &= \operatorname{Pic}(X_{\mathbf{C}})^G / \operatorname{Pic}^0(X_{\mathbf{C}})^G. \end{aligned}$$

Example 5.9. Let A be an abelian variety of dimension d defined over $\mathbf{R}$, and let X be a nontrivial principal homogeneous space for A. This means that X is a variety over $\mathbf{R}$ with $X(\mathbf{R}) = \emptyset$, admitting an isomorphism

$$\varphi \colon X_{\mathbf{C}} \xrightarrow{\sim} A_{\mathbf{C}}$$

defined over $\mathbf{C}$ with the property that the complex conjugation σ_X on $X(\mathbf{C})$ is related to the complex conjugation σ_A on $A(\mathbf{C})$ via the formula

$$\sigma_X = \varphi^{-1} \circ (\sigma_A - t_a) \circ \varphi,$$

where t_a is translation by a 2-torsion element a in the connected component of $A(\mathbf{R})$ containing zero.

It can be checked, using the universal covering $\mathbf{C}^d \to A(\mathbf{C})$, that there is a loop $\lambda \subset A(\mathbf{R})$, passing through 0 and a and stable under the translation t_a, in such a way that the restriction mapping

$$H^1(A(\mathbf{C}), \mathbf{Z})^G \to H^1(\lambda, \mathbf{Z})$$

is surjective. This means that σ_X induces a fixed-point free involution on $\varphi^{-1}(\lambda) \subset X(\mathbf{C})$, and the restriction mapping

$$H^1(X(\mathbf{C}), \mathbf{Z})^G \to H^1(\lambda, \mathbf{Z})$$

is again surjective. Letting $G = \mathbf{Z}/2$ act on $X(\mathbf{C})$ and λ via σ_X, we have a commutative diagram

$$\begin{array}{ccc} H^1(X(\mathbf{C}); G, \mathbf{Z}) & \xrightarrow{e_X} & H^1(X(\mathbf{C}), \mathbf{Z})^G \\ \downarrow & & \downarrow\!\!\downarrow \\ H^1(\lambda; G, \mathbf{Z}) & \xrightarrow{e_\lambda} & H^1(\lambda, \mathbf{Z})^G \end{array}$$

We saw in Example III.9.1 that the edge morphism e_λ is not surjective, so from the diagram we see that e_X is not surjective, and by Proposition 5.6 we may conclude that

$$\mathrm{Pic}^0(X) \neq \mathrm{Pic}^0(X_{\mathbf{C}})^G,$$

so in particular $\mathrm{Pic}(X) \neq \mathrm{Pic}(X_{\mathbf{C}})^G$, and -1 is a sum of two squares in the function field of X.

Note that $H^2(X, \mathscr{O}_X) \neq 0$, but this fact does not interfere like in the case of K3-surfaces in Example 5.4, since the difference between $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X_{\mathbf{C}})^G$ is already found on the Pic^0-level.

Remark. The paper [vH2] is based on the results in this section.

CHAPTER V

Algebraic cycles and topology of real Enriques surfaces

An algebraic variety Y over $\mathbf{R}$ will be called a *real Enriques surface* if $Y_{\mathbf{C}} = Y \otimes \mathbf{C}$ is a complex Enriques surface. When Y is a complex Enriques surface, it is well-known that $H_2(Y(\mathbf{C}), \mathbf{Z})$ is generated by the fundamental classes of the complex curves on Y, so when Y is an Enriques surface defined over $\mathbf{R}$, it is tempting to expect that similarly we have $H_1^{\mathrm{alg}}(Y(\mathbf{R}), \mathbf{Z}/2) = H_1(Y(\mathbf{R}), \mathbf{Z}/2)$. However, this equality only holds if every connected component of the real part of Y is orientable; otherwise H_1^{alg} is of codimension 1 in $H_1(Y(\mathbf{R}), \mathbf{Z}/2)$. See Theorem 1.2 for a precise statement. The equivariant cycle map plays an important role in the proof of Theorem 1.2.

The results of Section 1 also contain information about the Hochschild-Serre spectral sequences of a real Enriques surface. Section 2 is completely devoted to these spectral sequences. The necessary and sufficient conditions for a real Enriques surface Y to be a *GM*-variety (see Section III.8) were found in [DK2]; in Section 2 we will determine necessary and sufficient conditions for Y to be $\mathbf{Z}$-*GM*. See Theorem 2.9.

This chapter is based on the paper [MvH], written by F. Mangolte and myself. In that paper the results of Section 2 were also used for computing the Brauer group of a real Enriques surface, and recently, Sujatha and I have used these results for computing the Witt groups of real Enriques surfaces in the paper [SvH]. I have omitted these computations, since I felt they did not quite fit into the theme of this work.

1. Algebraic cycles

First a few more words about the topology of real Enriques surfaces. Consider the following two classification problems:

– classification of topological types of algebraic varieties Y over $\mathbf{R}$ (the manifolds $Y(\mathbf{C})$ up to equivariant diffeomorphism),
– classification of topological types of the real parts $Y(\mathbf{R})$.

For real Enriques surfaces the two classifications have been investigated by Nikulin in [Ni2]. The topological classification of the real parts was completed by Degtyarev and Kharlamov, who give in [DK1] a description of all 87 topological types. For now it will be sufficient to know that that the real part of a real Enriques surface Y need not be connected and that a connected component V of $Y(\mathbf{R})$ is either a nonorientable surface of genus ≤ 11 or it is homeomorphic to a sphere or to a torus.

Crucial in every study of real Enriques surfaces is the fact that a real Enriques surface admits an unramified double covering $X \longrightarrow Y_{\mathbf{C}}$ defined over $\mathbf{C}$ by a complex K3-surface X. Since $X(\mathbf{C})$ is simply connected, $X(\mathbf{C})$ is the universal covering space of $Y(\mathbf{C})$ and $H_1(Y(\mathbf{C}), \mathbf{Z}) = \mathbf{Z}/2$. This covering can be constructed using the canonical line bundle of $Y_{\mathbf{C}}$, which is 2-torsion in the Picard group. Since the canonical line bundle is defined over $\mathbf{R}$, the covering can be defined over $\mathbf{R}$ as well. In other words, the complex conjugation σ on $Y(\mathbf{C})$ can be lifted to the covering $X(\mathbf{C})$, and this can be done in two different ways. Hence we can give X the structure of a variety over $\mathbf{R}$ in two different ways, which we will denote by X_1 and X_2. We will denote the image of $X_1(\mathbf{R})$ in $Y(\mathbf{R})$ under the projection by Y_1 and the image of $X_2(\mathbf{R})$ by Y_2. Both Y_1 and Y_2 consist of connected components of $Y(\mathbf{R})$ and $Y(\mathbf{R})$ is the disjoint union of Y_1 and Y_2. Following [DK1] we will call Y_1 and Y_2 the two *halves* of $Y(\mathbf{R})$, even though Y_1 and Y_2 need not be homeomorphic or even have the same number of connected components. The connected components of $X_1(\mathbf{R})$ and $X_2(\mathbf{R})$ are orientable, as is the case for the real part of any real K3-surface. If a connected component of a half Y_i is orientable, then it is covered by two components of $X_i(\mathbf{R})$, which are interchanged by the covering transformation of X. A nonorientable component of Y_i is covered by one component of $X_i(\mathbf{R})$; this is the orientation covering.

The first homology group $H_1(Y(\mathbf{C}), \mathbf{Z})$ of a complex Enriques surface is isomorphic to $\mathbf{Z}/2$, and the second homology group $H_2(Y(\mathbf{C}), \mathbf{Z})$ of a complex Enriques surface is isomorphic to $\mathbf{Z}^{10} \oplus \mathbf{Z}/2$, and the torsion part is generated by the first Chern class of the canonical line bundle of Y. Since for an $H^2(Y(\mathbf{C}), \mathscr{O}_h) = 0$ (see [BPV, V.23]), we see by Corollary IV.4.4 that for a real Enriques surface Y we have that $H_1^{\mathrm{alg}}(Y(\mathbf{R}), \mathbf{Z}/2)$ is the image of the mapping

$$\alpha_2 = \rho_1 \colon\ H_2(Y(\mathbf{C}); G, \mathbf{Z}(1)) \to H_1(Y(\mathbf{R}), \mathbf{Z}/2).$$

In order to determine the image of α_2 it will be helpful to define α_1 by

$$\alpha_1 = \rho_1 \colon\ H_1(Y(\mathbf{C}); G, \mathbf{Z}) \to H_1(Y(\mathbf{R}), \mathbf{Z}/2).$$

Observe that $\alpha_2 = \alpha_1 \circ (\cap \eta)$, where η is the nontrivial element in $H^1(G, \mathbf{Z}(1))$.

Lemma 1.1. *For a real Enriques surface Y the codimension of* $\operatorname{Im}\alpha_2$ *in* $H_1(Y(\mathbf{R}),\mathbf{Z}/2)$ *does not exceed* 1.

Proof. We may assume that $Y(\mathbf{R})\neq\emptyset$. By Lemma III.8.9, α_1 is surjective. Since $\alpha_2=\alpha_1\circ(\cap\eta)$, it is sufficient to remark that we have an exact sequence

$$H_2(Y(\mathbf{C});G,\mathbf{Z}(1))\xrightarrow{\cap\eta}H_1(Y(\mathbf{C});G,\mathbf{Z})\xrightarrow{e}H_1(Y(\mathbf{C}),\mathbf{Z})^G.$$

□

Proposition 1.1. *Let Y be a real Enriques surface. A class $\gamma\in H_1(Y(\mathbf{R}),\mathbf{Z}/2)$ is contained in the image of α_2 if and only if*

$$\deg(\gamma\cap w_1(Y(\mathbf{R})))=0,$$

where $w_1(Y(\mathbf{R}))\in H^1(Y(\mathbf{R}),\mathbf{Z}/2)$ is the first Stiefel-Whitney class of $Y(\mathbf{R})$.

Proof. Again we may assume that $Y(\mathbf{R})\neq\emptyset$. Denote by Ω the subspace of $H_1(Y(\mathbf{R}),\mathbf{Z}/2)$ consisting elements γ that satisfy $\deg(\gamma\cap w_1(Y(\mathbf{R})))=0$.

If $Y(\mathbf{R})$ is orientable, $w_1(Y(\mathbf{R}))=0$ and $\Omega=H_1(Y(\mathbf{R}),\mathbf{Z}/2)$. Furthermore, we have a surjective morphism

$$H_1(X_1(\mathbf{R}),\mathbf{Z}/2)\oplus H_1(X_2(\mathbf{R}),\mathbf{Z}/2)\to H_1(Y(\mathbf{R}),\mathbf{Z}/2)$$

where the X_1 and X_2 are the two real K3-surfaces covering Y (see the beginning of this section). This morphism fits in a commutative diagram

$$\begin{tikzcd}
H_2(X_1(\mathbf{C});G,\mathbf{Z}(1))\oplus H_2(X_2(\mathbf{C});G,\mathbf{Z}(1)) \arrow[r] \arrow[d, "\alpha_2^{X_1}\oplus\alpha_2^{X_2}"'] & H_2(Y(\mathbf{C});G,\mathbf{Z}(1)) \arrow[d, "\alpha_2"] \\
H_1(X_1(\mathbf{R}),\mathbf{Z}/2)\oplus H_1(X_2(\mathbf{R}),\mathbf{Z}/2) \arrow[r] & H_1(Y(\mathbf{R}),\mathbf{Z}/2)
\end{tikzcd}$$

Here the $\alpha_n^{X_i}\colon H_n(X_1(\mathbf{C});G,\mathbf{Z}(n-1))\to H_1(X_1(\mathbf{R}),\mathbf{Z}/2)$ are defined in the same way as α_n. As $H_1(X(\mathbf{C}),\mathbf{Z})=0$ for a real K3-surface X, it follows from Lemma III.8.9, that $\alpha_2^{X_1}$ and $\alpha_2^{X_2}$ are surjective, which implies the surjectivity of α_2. In other words, $\operatorname{Im}\alpha_2=\Omega$.

Now assume that $Y(\mathbf{R})$ is nonorientable. Then $w_1(Y(\mathbf{R}))\neq 0$, and by the non-degeneracy of the cap product pairing $\operatorname{codim}\Omega=1$. First we will prove that $\operatorname{Im}\alpha_2\subset\Omega$. Let $K\in\operatorname{Pic}(Y)$ be the canonical class of Y. Then $\mathrm{cl}_{\mathbf{C}}(K)$ generates the torsion of $H^2(Y(\mathbf{C}),\mathbf{Z})$, and, as for any algebraic variety over $\mathbf{R}$, we have that $\mathrm{cl}_{\mathbf{R}}(K)=w_1(Y(\mathbf{R}))\in H^1(Y(\mathbf{R}),\mathbf{Z}/2)$. This implies that, in the notation of Section IV.3, $w_1(Y(\mathbf{R}))$ is an element of $H^1_{(>1/2)}(Y(\mathbf{R}),\mathbf{Z}/2)$ and since $\operatorname{Im}\alpha_2=H_1^{(\leq 1/2)}(Y(\mathbf{R}),\mathbf{Z}/2)$ by equation (71), the arguments of the proof of Proposition IV.3.3 show that $\operatorname{Im}\alpha_2$ is orthogonal to $w_1(Y(\mathbf{R}))$ for the intersection pairing, hence $\operatorname{Im}\alpha_2\subset\Omega$. Lemma 1.1 now gives us that $\operatorname{Im}\alpha_2=\Omega$. □

Since $H_1^{\mathrm{alg}}(Y(\mathbf{R}),\mathbf{Z}/2)=\operatorname{Im}\alpha_2$ we immediately get the following result.

Theorem 1.2. *Let Y be a real Enriques surface. A class $\gamma \in H_1(Y(\mathbf{R}),\mathbf{Z}/2)$ can be represented by an algebraic cycle if and only if*

$$\langle \gamma, w_1(Y(\mathbf{R}))\rangle = 0.$$

In particular, if every connected component of the real part $Y(\mathbf{R})$ is orientable,

$$H_1^{\mathrm{alg}}(Y(\mathbf{R}),\mathbf{Z}/2) = H_1(Y(\mathbf{R}),\mathbf{Z}/2).$$

Otherwise,

$$\dim H_1^{\mathrm{alg}}(Y(\mathbf{R}),\mathbf{Z}/2) = \dim H_1(Y(\mathbf{R}),\mathbf{Z}/2) - 1.$$

Proof. This is an immediate consequence of Proposition 1.1 and the fact that $H_1^{\mathrm{alg}}(Y(\mathbf{R}),\mathbf{Z}/2) = \operatorname{Im}\alpha_2$. □

2. Galois-Maximality

The aim of this section is to describe which Enriques surfaces are $\mathbf{Z}$-*GM*-varieties and/or *GM*-varieties in terms of the orientability of the real part and the distribution of the components over the halves. See the introduction for the definition of Galois-Maximality and Section 1 for the definition of 'halves'.

The proof of Theorem 2.9 will consist of a collection of technical results and explicit geometric constructions of equivariant homology classes. For completeness we also consider the parts of Theorem 2.9 concerning coefficients in $\mathbf{Z}/2$, although these results can be easily extracted from [DK2].

By Lemma III.8.4 it is sufficient to study the edge morphisms of the Hochschild-Serre spectral sequences with coefficients in $\mathbf{Z}/2$, $\mathbf{Z}$ and $\mathbf{Z}(1)$. In order to distinguish them easily, we will use the following notation.

$$\begin{aligned} e_k^+ &: H_k(X(\mathbf{C}); G, \mathbf{Z}) \to H_k(X(\mathbf{C}),\mathbf{Z})^G \\ e_k^- &: H_k(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H_k(X(\mathbf{C}),\mathbf{Z}(1))^G \\ e_k &: H_k(X(\mathbf{C}); G, \mathbf{Z}) \to H_k(X(\mathbf{C}),\mathbf{Z})^G \end{aligned}$$

Lemma 2.1. *Let Y be a real Enriques surface with $Y(\mathbf{R}) \neq \emptyset$. Then*

(i) *for any $p \in \{0,2,3,4\}$, $e_p^{+/-}$ is surjective onto $H_p(Y(\mathbf{C}),\mathbf{Z}(k))^G$,*

(ii) *for any $p \in \{0,3,4\}$, e_p is surjective onto $H_p(Y(\mathbf{C}),\mathbf{Z}/2)^G$.*

Proof. This is easy to see from the Hochschild-Serre spectral sequences, using Lemma III.8.5 and Lemma III.8.6. □

Corollary 2.2. *Let Y be a real Enriques surface with $Y(\mathbf{R}) \neq \emptyset$. Then Y is $\mathbf{Z}$-GM if and only if $e_1^{+/-}$ is surjective onto $H_1(Y(\mathbf{C}),\mathbf{Z}(k))^G$ for $k = 0, 1$. Moreover, Y is GM if and only if e_1 and e_2 are surjective onto $H_1(Y(\mathbf{C}),\mathbf{Z}/2)^G$, resp. $H_2(Y(\mathbf{C}),\mathbf{Z}/2)^G$.*

Lemma 2.3. *Let Y be a real Enriques surface with $Y(\mathbf{R}) \neq \emptyset$. If e_2 is not surjective onto $H_2(Y(\mathbf{C}), \mathbf{Z}/2)^G$, then e_1 is not surjective onto $H_1(Y(\mathbf{C}), \mathbf{Z}/2)^G$.*

Proof. By Poincaré duality we see that if e_2 is not surjective onto $H_2(Y(\mathbf{C}), \mathbf{Z}/2)^G$, then e^2 is not surjective onto $H^2(Y(\mathbf{C}), \mathbf{Z}/2)^G$ by the Poincaré duality isomorphism. Let us assume that e^2 is not surjective. Then it can be seen in the Hochschild-Serre spectral sequence that there exists an $\omega \in H^1(Y(\mathbf{C}); G, \mathbf{Z}/2)$ such that $e^1(\omega) \neq 0$, but $\beta(\omega) = 0$.

Now suppose e_1 is surjective onto $H_1(Y(\mathbf{C}), \mathbf{Z}/2)^G$, then there exists a $\gamma \in H_1(Y(\mathbf{C}); G, \mathbf{Z}/2)$ such that

$$\langle e_1(\gamma), e^1(\omega)\rangle \neq 0.$$

This means that $\langle \gamma, \omega \rangle \neq 0$, but this contradicts

$$\langle \gamma, \omega \rangle = \langle \rho(\gamma), \beta(\omega)\rangle = \langle \rho(\gamma) \cap 0\rangle = 0.$$

Hence e_1 is not surjective. □

Proposition 2.4. *Let Y be a real Enriques surface with $Y(\mathbf{R}) \neq \emptyset$. Then*

(i) *Y is $\mathbf{Z}$-GM if and only if e_1^+ and e_1^- are nonzero.*
(ii) *Y is GM if and only if e_1 is nonzero.*
(iii) *If e_1 is zero then e_1^+ and e_1^- are zero. In particular, if Y is $\mathbf{Z}$-GM, then Y is also GM.*

Proof. If Y is an Enriques surface,

$$H_1(Y(\mathbf{C}), \mathbf{Z}) = H_1(Y(\mathbf{C}), \mathbf{Z}/2) = \mathbf{Z}/2,$$

so $e_1^{+/-}$ and e_1 are surjective if and only if they are nonzero. By Lemma 2.3, e_2 is surjective if $e_1 \neq 0$, so we obtain the first two assertions from Corollary 2.2. The last assertion follows from the commutative diagram

$$\begin{array}{ccc} H_1(Y(\mathbf{C}); G, \mathbf{Z}(k)) & \xrightarrow{e_1^{+/-}} & H_1(Y(\mathbf{C}), \mathbf{Z}(k)) \\ \downarrow & & \downarrow \\ H_1(Y(\mathbf{C}); G, \mathbf{Z}/2) & \xrightarrow{e_1} & H_1(Y(\mathbf{C}), \mathbf{Z}/2) \end{array}$$

□

Lemma 2.5. *Let Y be a real Enriques surface with $Y(\mathbf{R}) \neq \emptyset$. Then $e_1^+ = 0$ if and only if $Y(\mathbf{R})$ is orientable.*

Proof. We know from Proposition 1.1, that α_2 is surjective if and only if $Y(\mathbf{R})$ is orientable. Since $H_1(Y(\mathbf{C}), \mathbf{Z}) = \mathbf{Z}/2$, we have that the mapping $H_1(Y(\mathbf{C}), \mathbf{Z})^G \xrightarrow{\cup \eta^2} H^2(G, H_1(Y(\mathbf{C}), \mathbf{Z}))$ is an isomorphism, so Lemma III.8.9 gives us that α_2 is surjective if and only if $e_1^+ = 0$. □

Lemma 2.6. *If the two halves Y_1 and Y_2 of a real Enriques surface Y are nonempty, then $e_1^- \neq 0$.*

PROOF. Let X be the K3-covering of $Y_{\mathbf{C}}$, let τ be the deck transformation of this covering and denote by σ_1 and σ_2 the two different involutions of $X(\mathbf{C})$ lifting the involution σ of $Y(\mathbf{C})$. Let $X_i(\mathbf{R})$ be the set of fixed points under σ_i and let p_i be a point in $X_i(\mathbf{R})$ for $i = 1, 2$.

Let l be an arc in $X(\mathbf{C})$ connecting p_1 and p_2 without containing any other point of $X_1(\mathbf{R})$ or $X_2(\mathbf{R})$. Then the union L of the four arcs l, $\sigma_1(l)$, $\sigma_2(l)$, $\tau(l)$ is homeomorphic to a circle, and we have that $\tau(L) = L$. This implies that the image λ of L in $Y(\mathbf{C})$ is again homeomorphic to a circle; we choose an orientation on λ.

Now G acts on λ via an orientation reversing involution, so λ represents a class $[\lambda]$ in $H_1(Y(\mathbf{C}); G, \mathbf{Z}(1))$. Since $X(\mathbf{C}) \to Y(\mathbf{C})$ is the universal covering, and the inverse image of λ is precisely L, hence homeomorphic to a circle, the class of λ is nonzero in $H_1(Y(\mathbf{C}), \mathbf{Z})$, so $e_1^-([\lambda]) \neq 0$. □

Lemma 2.7. *If exactly one of the halves Y_1, Y_2 of a real Enriques surface Y is empty, then $e_1 = 0$ if and only if $Y(\mathbf{R})$ is orientable.*

PROOF. If $e_1 = 0$, we have $e_1^+ = 0$ by Proposition 2.4 and then $Y(\mathbf{R})$ is orientable by Lemma 2.5. Conversely, if $Y(\mathbf{R})$ is orientable and $X_2(\mathbf{R}) = \emptyset$, then $X_1(\mathbf{R}) \to Y(\mathbf{R})$ is the trivial double covering, so it induces a surjection $H_*(X_1(\mathbf{R}), \mathbf{Z}/2)^0 \to H_*(Y(\mathbf{R}), \mathbf{Z}/2)^0$, where $H_*(-, \mathbf{Z}/2)^0$ denotes the kernel of the homomorphism induced by the constant mapping, see page 75. Since $H_1(X(\mathbf{C}), \mathbf{Z}/2) = 0$, the mapping $\rho\colon H_2(X_1(\mathbf{C}); G, \mathbf{Z}/2) \to H_*(X_1(\mathbf{R}), \mathbf{Z}/2)^0$ is surjective by Lemma III.8.7. Now the functoriality of ρ with respect to proper equivariant mappings implies that

$$\rho_2\colon H_2(Y(\mathbf{C}); G, \mathbf{Z}/2) \to H_*(Y(\mathbf{R}), \mathbf{Z}/2)$$

is surjective, and Lemma III.8.7 then gives that e_1 is zero. □

Lemma 2.8. *If exactly one of the halves Y_1, Y_2 of a real Enriques surface Y is empty, then $e_1^- \neq 0$ if and only if $Y(\mathbf{R})$ has components of odd Euler characteristic.*

PROOF. Assume $Y_2 = \emptyset$. By Lemma III.8.8, it suffices to show that

$$\rho_{\text{even}}\colon H_2(Y(\mathbf{C}); G, \mathbf{Z}) \to H_{\text{even}}(Y(\mathbf{R}), \mathbf{Z}/2)^0$$

is surjective if and only if $Y(\mathbf{R})$ has no components of odd Euler characteristic. Although $Y(\mathbf{R})$ need not be orientable, we can apply the K3-covering as in the previous lemma and prove that the image of ρ_{even} contains a basis for the subgroup $H_0(Y(\mathbf{R}), \mathbf{Z}/2) \cap H_{\text{even}}(Y(\mathbf{R}), \mathbf{Z}/2)^0$, so ρ_{even} is surjective if and only if

$$\rho_2\colon H_2(Y(\mathbf{C}); G, \mathbf{Z}) \to H_2(Y(\mathbf{R}), \mathbf{Z}/2)$$

is surjective. We will use that $H_2(Y(\mathbf{R}), \mathbf{Z}/2)$ is generated by the fundamental classes of the connected components of $Y(\mathbf{R})$.

Pick a component V of $Y(\mathbf{R})$. If V is orientable, it gives a class in $H_2(Y(\mathbf{C}); G, \mathbf{Z})$, which maps to the fundamental class of V in $H_2(Y(\mathbf{R}), \mathbf{Z}/2)$. Now assume V is nonorientable. Let $[V]$ be the fundamental class of V in $H_2(Y(\mathbf{R}), \mathbf{Z}/2)$, let $[V]_G$ be the class represented by V in $H_2(Y(\mathbf{C}); G, \mathbf{Z}/2)$, and let $\gamma = \delta([V]_G)$ be the Bockstein image in $H_1(Y(\mathbf{C}); G, \mathbf{Z}(1))$. Then $\rho_2(\gamma) = \rho_2([V]_G) = [V]$ by equation (59), so $[V]$ is in the image of $H_2(Y(\mathbf{C}); G, \mathbf{Z})$ under ρ_2 if and only if $e_1^-(\gamma) = 0$.

From the construction of γ we see that $e_1^-(\gamma) = i_*\delta([V])$, where $i\colon\ V \to Y(\mathbf{C})$ is the inclusion and $\delta([V]) \in H_1(V, \mathbf{Z})$ is the Bockstein image of $[V]$. Therefore $e_1^-(\gamma)$ can be represented by a circle λ embedded in V. Since $X(\mathbf{C}) \to Y(\mathbf{C})$ is the universal covering, $e_1^-(\gamma)$ is zero if and only if the inverse image L of λ in $X(\mathbf{C})$ has two connected components. Let W be the component of $X_1(\mathbf{R})$ covering V. Then W is the orientation covering of V and $L \subset W$. If V has odd Euler characteristic, then it is the connected sum of a real projective plane and an orientable compact surface. We see by elementary geometry that L is connected. If V has even Euler characteristic, it is the connected sum of a Klein bottle and an orientable compact surface, and we see that L has two connected components. $\square$

The main result of this section now follows immediately.

Theorem 2.9. *Let Y be a real Enriques surface with nonempty real part.*

(i) *Suppose the two halves Y_1 and Y_2 are nonempty. Then Y is GM. Moreover, Y is $\mathbf{Z}$-GM if and only if $Y(\mathbf{R})$ is nonorientable.*
(ii) *Suppose one of the halves Y_1 or Y_2 is empty. Then Y is GM if and only if $Y(\mathbf{R})$ is nonorientable. Moreover, Y is $\mathbf{Z}$-GM if and only if $Y(\mathbf{R})$ has at least one component of odd Euler characteristic.*

PROOF. By Proposition 2.4, the first part of the theorem follows from Lemma 2.5 and Lemma 2.6, and the second part of the theorem follows from Lemma 2.7 and lemma 2.8. $\square$

There are examples of all cases described in the above theorem (see [DK1, Fig. 1]).

Remark 2.10. The Hochschild–Serre spectral sequence of a real Enriques surface has been computed independently by V.A. Krasnov in [Kr5]. He applies the results only to the calculation of the Brauer group, not of H_1^{alg}. For the edge morphism e_1^- he gets a different result; in his calculations he assumes that when S is any compact nonorientable connected topological surface, and Γ is the generator of the 2-torsion in the first homology group, then $w_1(S) \cap \Gamma \neq 0$ in $H_0(S, \mathbf{Z}/2)$. However, this only holds when the Euler characteristic of S is odd.

Chapter VI

Real algebraic cycles on complex projective varieties

If we identify $\mathbf{C}^N$ with $\mathbf{R}^{2N}$, then to any (quasi-)affine complex algebraic variety V of dimension d corresponds in a natural way a real algebraic variety of dimension $2d$ in the sense of [BCR], which we denote by $V_{\mathbf{R}}$. Since each complex morphism $V \to V'$ can be considered as a real algebraic morphism $V_{\mathbf{R}} \to V'_{\mathbf{R}}$, and all complex algebraic varieties consist of affine pieces glued together, this construction extends in a natural way to arbitrary complex algebraic varieties X, and we obtain $X_{\mathbf{R}}$, the *underlying real algebraic structure of* X. In this section we will study for a nonsingular complex variety X the group $H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$, which can be considered as the subgroup of $H^1(X, \mathbf{Z}/2)$ consisting of cohomology classes represented by real Zariski-closed subspaces of X of codimension 1.

The reader will have observed that the notations in this chapter will not be completely consistent with the other chapters. The notation $X_{\mathbf{R}}$ is chosen in order to emphasize the fact that $X_{\mathbf{R}}$ is essentially nothing more than X equipped with a finer Zariski-topology. Moreover, instead of $H^1(X(\mathbf{C}), \mathbf{Z}/2)$ the slightly less precise notation $H^1(X, \mathbf{Z}/2)$ is used in order to emphasize that $H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$ is a subset of $H^1(X, \mathbf{Z}/2)$. In particular, $X_{\mathbf{R}}$ is *not* an algebraic variety defined over $\mathbf{R}$ as defined in Section I.1.

The key result will be Theorem 1.5, which states that $H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$ is isomorphic to $H^1_{\text{alg}}(\text{Alb}(X)_{\mathbf{R}}, \mathbf{Z}/2)$, where $\text{Alb}(X)$ is the Albanese variety of X. This means that we can apply the methods for computing the H^1_{alg} of real abelian varieties, as developed by J. Huisman. This will be treated in Section 2. In Section 3 the theory developed in the first two sections will be used to prove more explicit results; the most important are Theorem 3.2 and Theorem 3.6.

This chapter is based on the paper [vH1]. The present proof of Theorem 1.5 is shorter and more of a cohomological nature than the proof given in the original paper.

1. The Weil restriction

For the Proof of Theorem 1.5 we will use a *complexification* $\mathscr{X}$ of $X_{\mathbf{R}}$ (i.e., a variety $\mathscr{X}$ over $\mathbf{R}$ with $\mathscr{X}(\mathbf{R})$ biregularly isomorphic to $X_{\mathbf{R}}$ as a real algebraic variety) of a very special form. In particular, $\mathscr{X}(\mathbf{C})$ will be homeomorphic to the product $V \times V$ of a topological space V with itself, and the action of $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \{1, \sigma\}$ on $\mathscr{X}(\mathbf{C})$ will then be given by $\sigma(v, w) = (w, v)$.

Before constructing $\mathscr{X}$, we will first establish some topological facts about varieties of this form. Let Y be a nonsingular, complete variety over $\mathbf{R}$ with $Y(\mathbf{C})$ equivariantly homeomorphic to a product $V \times V$ as above. Observe that the diagonal embedding $V \hookrightarrow V \times V$ induces a homeomorphism $V \xrightarrow{\sim} V(\mathbf{R})$.

For $i = 1, 2$, let $p_i \colon Y(\mathbf{C}) \to V$ be the projection on the ith factor. Since $H^0(Y(\mathbf{C}), \mathbf{Z})$ and $H^1(Y(\mathbf{C}), \mathbf{Z})$ are torsion-free, the Künneth theorem gives us natural isomorphisms

$$H^1(Y(\mathbf{C}), \mathbf{Z}) \simeq p_1^* H^1(V, \mathbf{Z}) \oplus p_2^* H^1(V, \mathbf{Z}) \tag{76}$$

and

$$H^2(Y(\mathbf{C}), \mathbf{Z}) \simeq p_1^* H^2(V, \mathbf{Z}) \oplus \left(p_1^* H^1(V, \mathbf{Z}) \otimes p_2^* H^1(V, \mathbf{Z})\right) \oplus p_2^* H^2(V, \mathbf{Z}). \tag{77}$$

Since the G-action on $H^*(Y(\mathbf{C}), \mathbf{Z})$ exchanges $p_1^* H^*(V, \mathbf{Z})$ and $p_2^* H^*(V, \mathbf{Z})$, we see from the complex (1) that for every $k > 0$ we have

$$H^k(G, H^1(Y(\mathbf{C}), \mathbf{Z})) = 0 \tag{78}$$

and

$$H^k(G, H^2(Y(\mathbf{C}), \mathbf{Z})) \simeq H^k(G, p_1^* H^1(V, \mathbf{Z}) \otimes p_2^* H^1(V, \mathbf{Z})). \tag{79}$$

We will denote $p_1^* H^1(V, \mathbf{Z}) \otimes p_2^* H^1(V, \mathbf{Z})) \subset H^2(Y(\mathbf{C}), \mathbf{Z})$ by $\mathrm{C}^2(Y)$.

Since for every complete, nonsingular variety Y over $\mathbf{R}$ with $Y(\mathbf{R})$ nonempty the Albanese mapping

$$\alpha_Y \colon Y \to \mathrm{Alb}(Y)$$

induces a G-equivariant isomorphism

$$\alpha_Y^* \colon H^1(\mathrm{Alb}(Y)(\mathbf{C}), \mathbf{Z}) \xrightarrow{\sim} H^1(Y(\mathbf{C}), \mathbf{Z}),$$

we see from equation (78) and the description of the structure of abelian varieties over $\mathbf{R}$ given in [Si] that $\mathrm{Alb}(Y)(\mathbf{C})$ is equivariantly homeomorphic to a product of the form $V' \times V'$, and the Albanese mapping induces an isomorphism

$$\alpha_Y^* \colon \mathrm{C}^2(\mathrm{Alb}(Y)(\mathbf{C})) \to \mathrm{C}^2(Y). \tag{80}$$

This isomorphism will be a crucial one.

Lemma 1.1. *Let Y be a complete nonsingular geometrically irreducible algebraic variety defined over $\mathbf{R}$, with $Y(\mathbf{C})$ equivariantly homeomorphic to the product $V \times V$ of two copies of a topological space with the G-action given by $\sigma(v, w) = (w, v)$. The Albanese mapping induces for every $k > 0$ an isomorphism*

$$\alpha_Y^*\colon\ H^1(G, H^2(\mathrm{Alb}(Y)(\mathbf{C}), \mathbf{Z})) \xrightarrow{\sim} H^1(G, H^2(Y(\mathbf{C}), \mathbf{Z})).$$

Proof. This is an immediate consequence of the isomorphisms (79) and (80). □

Proposition 1.2. *Let Y be as in Lemma 1.1. The Albanese mapping α_Y induces an isomorphism*

$$\alpha_Y^*\colon\ H^1_{(\geq 1/2)}(\mathrm{Alb}(Y)(\mathbf{R}), \mathbf{Z}/2) \xrightarrow{\sim} H^1_{(\geq 1/2)}(Y(\mathbf{R}), \mathbf{Z}/2).$$

Proof. By Equation (72) we have that

$$H^1_{(\geq 1/2)}(Y(\mathbf{R}), \mathbf{Z}/2) = \beta^1 H^2(Y(\mathbf{C}); G, \mathbf{Z}(1)).$$

The mapping β^1 factorizes via the homomorphism

$$H^2(Y(\mathbf{C}); G, \mathbf{Z}(1)) \xrightarrow{\cup\eta} F^2 H^3(Y(\mathbf{C}); G, \mathbf{Z}),$$

which is surjective, since it is a special case of (62). Therefore it is sufficient to prove that the Albanese mapping induces an isomorphism

$$\alpha_Y^*\colon\ F^2 H^3(\mathrm{Alb}(Y)(\mathbf{C}); G, \mathbf{Z}) \xrightarrow{\sim} F^2 H^3(Y(\mathbf{C}); G, \mathbf{Z}).$$

Since $H^2(G, H^1(\mathrm{Alb}(Y)(\mathbf{C}), \mathbf{Z})) = H^2(G, H^1(Y(\mathbf{C}), \mathbf{Z})) = 0$, the horizontal arrows of the following commutative diagram are isomorphisms.

$$\begin{array}{ccc} F^2 H^3(\mathrm{Alb}(Y)(\mathbf{C}); G, \mathbf{Z}) & \longrightarrow & H^1(G, H^2(\mathrm{Alb}(Y)(\mathbf{C}), \mathbf{Z}(1))) \\ \downarrow{\scriptstyle \alpha_Y^*} & & \downarrow{\scriptstyle \alpha_Y^*} \\ F^2 H^3(Y(\mathbf{C}); G, \mathbf{Z}) & \longrightarrow & H^1(G, H^2(Y(\mathbf{C}), \mathbf{Z}(1))) \end{array}$$

Hence the proposition follows from Lemma 1.1.

□

Corollary 1.3. *With Y as in Lemma 1.1, the group $H^1_{(\geq 1/2)}(Y(\mathbf{R}), \mathbf{Z}/2)$ is precisely the image of the reduction modulo 2 mapping*

$$r\colon\ H^1(Y(\mathbf{R}), \mathbf{Z}) \to H^1(Y(\mathbf{R}), \mathbf{Z}/2).$$

In particular, $H^1_{\mathrm{alg}}(Y(\mathbf{R}), \mathbf{Z}/2) \subset \operatorname{Im} r$.

Proof. Since $\mathrm{Alb}(Y)$ is an abelian variety over $\mathbf{R}$ with a connected set of real points, we see from Example IV.3.1 that the mapping

$$\beta^1\colon\ H^2(\mathrm{Alb}(Y)(\mathbf{C}); G, \mathbf{Z}(1)) \to H^1(\mathrm{Alb}(Y)(\mathbf{R}), \mathbf{Z}/2)$$

is surjective, so

$$H^1_{(\geq 1/2)}(\mathrm{Alb}(Y)(\mathbf{R}), \mathbf{Z}/2) = H^1(\mathrm{Alb}(Y)(\mathbf{R}), \mathbf{Z}/2) = H^1(\mathrm{Alb}(Y)(\mathbf{R}), \mathbf{Z}) \otimes \mathbf{Z}/2,$$

and the proof now follows from the fact that the product structure of $Y(\mathbf{C})$ implies that the Albanese mapping induces an isomorphism

$$H^1(\mathrm{Alb}(Y)(\mathbf{R}),\mathbf{Z}) \xrightarrow{\sim} H^1(Y(\mathbf{R}),\mathbf{Z}).$$

$\square$

Let us go back to the situation where X is a complete nonsingular irreducible variety defined over $\mathbf{C}$ and $X_{\mathbf{R}}$ is the underlying real algebraic structure of X. Let $X_{\mathscr{W}}$ be the *Weil restriction* of X with respect to the field extension $\mathbf{C}/\mathbf{R}$. For a full discussion of the Weil restriction in real algebraic geometry, the reader is referred to [Hu1, § 1.4]. For us it will be sufficient to know that $X_{\mathscr{W}}$ is an algebraic variety defined over $\mathbf{R}$, determined by the fact that the complexification $X_{\mathscr{W}} \otimes \mathbf{C}$ is isomorphic to the product $X \times X^\sigma$ with Galois action given by $\sigma(x,y) = (\sigma_X^{-1}(y), \sigma(x))$. Here X^σ is the *conjugate variety* of X, which is formally defined up to isomorphism by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & X^\sigma \\ \downarrow & & \downarrow \\ \mathbf{C} & \xrightarrow{\sigma} & \mathbf{C} \end{array}$$

where σ_X is an isomorphism of schemes and σ is complex conjugation. When X is projective and defined by homogeneous equations $f_1 = 0, f_2 = 0, \dots, f_n = 0$, the conjugate variety X^σ is of course defined by the equations $f_1^\sigma = 0, f_2^\sigma = 0, \dots, f_n^\sigma = 0$, where each f_i^σ is obtained by applying complex conjugation to the coefficients of f_i. We have an isomorphism of real algebraic varieties

$$X_{\mathbf{R}} \xrightarrow{\sim} X_{\mathscr{W}}(\mathbf{R})$$

given by $x \mapsto (x, \sigma_X(x))$.

Since a morphism $\varphi\colon X \to Y$ induces canonically a morphism $\varphi^\sigma\colon X^\sigma \to Y^\sigma$, it gives us a morphism $\varphi_{\mathscr{W}}$, and we see that the Weil restriction is a functor. By slight abuse of notation, the corresponding mapping $X_{\mathscr{W}}(\mathbf{R}) \to Y_{\mathscr{W}}(\mathbf{R})$ of underlying real algebraic structures will again be denoted by φ.

Since $X(\mathbf{C})$ and $X^\sigma(\mathbf{C})$ are homeomorphic, we see that Lemma 1.1, Proposition 1.2, and Corollary 1.3 apply to the case $Y = X_{\mathscr{W}}$. In particular, Corollary 3.5 in [vH1], which states that

$$H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2) \subset H^1(X,\mathbf{Z}) \otimes \mathbf{Z}/2 \subset H^1(X,\mathbf{Z}/2),$$

is a direct consequence of the *topological* product structure of $X_{\mathscr{W}}(\mathbf{C})$, whereas the proof in *ibid.* needs the fact that the product decomposition $X_{\mathscr{W}} \otimes \mathbf{C} \simeq X \times X^\sigma$ is an isomorphism of *algebraic* varieties.

Of course, this algebraic product structure is indispensable in the proof of the main theorem below, since it implies that the Künneth decomposition

$$H^2(X_{\mathscr{W}},\mathbf{Z}) \simeq p_1^* H^2(X,\mathbf{Z}) \oplus p_1^* H^1(X,\mathbf{Z}) \otimes p_2^* H^1(X^\sigma,\mathbf{Z}) \oplus p_2^* H^2(X^\sigma,\mathbf{Z}).$$

actually gives a direct sum of Hodge structures. In particular, when we write

$$\mathrm{C}^{1,1}(X_{\mathscr{W}}) = \mathrm{C}^2(X_{\mathscr{W}}) \cap H^{1,1}(X_{\mathscr{W}},\mathbf{Z}),$$

the Albanese mapping induces an isomorphism

$$\alpha^*_{X_{\mathscr{W}}}\colon \mathrm{C}^{1,1}(\mathrm{Alb}(X)_{\mathscr{W}}) \xrightarrow{\sim} \mathrm{C}^{1,1}(X_{\mathscr{W}}). \tag{81}$$

Lemma 1.4. *Let X be a complete nonsingular irreducible variety over $\mathbf{C}$. The Albanese mapping $\alpha_X\colon X \longrightarrow \mathrm{Alb}\, X$ induces an isomorphism*

$$H^1(G, H^{1,1}(\mathrm{Alb}(X)_{\mathscr{W}},\mathbf{Z})) \xrightarrow{\sim} H^1(G, H^{1,1}(X_{\mathscr{W}},\mathbf{Z})).$$

Theorem 1.5. *Let X be a complete nonsingular irreducible algebraic variety over $\mathbf{C}$. The Albanese mapping $\alpha_X\colon X \longrightarrow \mathrm{Alb}\, X$ induces an isomorphism*

$$\alpha_X^*\colon H^1_{\mathrm{alg}}(\mathrm{Alb}(X)_{\mathbf{R}},\mathbf{Z}/2) \xrightarrow{\sim} H^1_{\mathrm{alg}}(X_{\mathbf{R}},\mathbf{Z}/2).$$

Proof. Since $X_{\mathbf{R}} = X_{\mathscr{W}}(\mathbf{R})$, we have

$$H^1_{\mathrm{alg}}(X_{\mathbf{R}},\mathbf{Z}/2) = \beta^1\left[H^{1,1}(X_{\mathscr{W}}; G,\mathbf{Z}(1))\right],$$

by Theorem IV.4.1. Hence the theorem follows from Lemma 1.4 by the same arguments as used in the proof of Proposition 1.2. □

Remark 1.6. Actually, although the representability of the Picard functor is not used in the present proof of Theorem 1.5, this proof is not as far from the original proof in [vH1] as might seem at first sight. Recall, that the group of *algebraic correspondences* between two complex algebraic varieties X_1 and X_2 is defined to be the quotient group

$$\mathrm{Corr}(X_1,X_2) = \mathrm{Pic}(X_1 \times X_2)/p_1^*\,\mathrm{Pic}(X_1) \oplus p_2^*\,\mathrm{Pic}(X_2).$$

It can be checked that the complex cycle map induces an isomorphism $\mathrm{Corr}(X,X^\sigma) \xrightarrow{\sim} C^{1,1}(X)$, hence the isomorphism (81) is equivalent to the classical isomorphism

$$\mathrm{Corr}(\mathrm{Alb}\, X, \mathrm{Alb}\, X^\sigma) \simeq \mathrm{Corr}(X,X^\sigma) \tag{82}$$

of which a proof was given in *loc. cit.* using the representability of the Picard functor. Then the original proof of the statement of Theorem 1.5 proceeds by using the fact that we may identify $\mathrm{Pic}(X_{\mathscr{W}})$ with $\mathrm{Pic}(X \times X^\sigma)^G$ and that the real cycle map $\mathrm{Pic}(X \times X^\sigma)^G \longrightarrow H^1(X_{\mathscr{W}}(\mathbf{R}),\mathbf{Z}/2)$ factorizes via a surjection

$$\psi_{X_{\mathscr{W}}}\colon \mathrm{Corr}(X,X^\sigma)^G \twoheadrightarrow H^1_{\mathrm{alg}}(X_{\mathbf{R}},\mathbf{Z}/2). \tag{83}$$

(cf. [vH1]).

2. Computational methods

In this section we will see how we can compute $H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$ for a complete nonsingular irreducible complex algebraic variety X from its period lattice $\Lambda_X = H_1(X, \mathbf{Z})/\text{torsion} \subset H^0(\Omega_X)^* \simeq \mathbf{C}^g$. We adapt a method of Huisman for computing $H^1_{\text{alg}}(A(\mathbf{R}), \mathbf{Z}/2)$ for an arbitrary abelian variety A defined over $\mathbf{R}$ (see [Hu1]) to the special case of $A = X_{\mathscr{W}}$. The special structure of $X_{\mathscr{W}}$ will allow us to make some essential simplifications. Huisman's method relies on the *Appell-Humbert Theorem* describing line bundles on abelian varieties. See [Mu] or [LB] for proofs and details.

Let Λ be a lattice in $\mathbf{C}^g$. Let $\mathscr{H}(\mathbf{C}^g, \Lambda)$ be the additive group of hermitian forms H on $\mathbf{C}^g$, such that $\operatorname{Im} H$ is integral on Λ and let S^1 be the multiplicative subgroup $z \in \mathbf{C} \colon |z| = 1$ of $\mathbf{C}^*$. An *Appell-Humbert datum* for $(\mathbf{C}^g, \Lambda)$ is a pair (α, H) where $H \in \mathscr{H}(\mathbf{C}^g, \Lambda)$ and α is a homomorphism $\Lambda \to S^1$ such that if $E = \operatorname{Im} H$, then

$$\alpha(\lambda_1 + \lambda_2) = \alpha(\lambda_1)\alpha(\lambda_2)(-1)^{E(\lambda_1, \lambda_2)}.$$

Let $\operatorname{AH}(\mathbf{C}^g, \Lambda)$ be the group of Appell-Humbert data with multiplication

$$(\alpha_1, H_1)(\alpha_2, H_2) = (\alpha_1\alpha_2, H_1 + H_2).$$

Then the Appell-Humbert Theorem says, that if $\mathbf{C}^g/\Lambda$ is an abelian variety, $\operatorname{Pic}(\mathbf{C}^g/\Lambda)$ is canonically isomorphic to $\operatorname{AH}(\mathbf{C}^g, \Lambda)$. Canonically in the sense that if $f \colon \mathbf{C}^g/\Lambda \to \mathbf{C}^k/\Lambda'$ is a homomorphism of abelian varieties, induced by the $\mathbf{C}$-linear mapping $F \colon \mathbf{C}^g \to \mathbf{C}^k$, and $\mathscr{L} \in \operatorname{Pic}(\mathbf{C}^k/\Lambda')$ corresponds to (α, H), then $f^*\mathscr{L}$ corresponds to $(\alpha \circ F, H \circ (F \times F)) \in \operatorname{AH}(\mathbf{C}^g, \Lambda)$.

Using the Appell-Humbert Theorem and the isomorphism (82) we will describe $\operatorname{Corr}(X, X^\sigma)^G$ and the mapping $\psi_{X_{\mathscr{W}}} \colon \operatorname{Corr}(X, X^\sigma)^G \to H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$ induced by the real cycle map $\operatorname{cl}_{\mathbf{R}}$, for any complete, nonsingular, irreducible complex variety X. For sake of clarity, we will first determine $\operatorname{Corr}(X_1, X_2^\sigma)$ for varieties X_1, X_2 having period lattices $\Lambda_1 \subset \mathbf{C}^{g_1}$, resp. $\Lambda_2 \subset \mathbf{C}^{g_2}$. Using the canonical isomorphism $\operatorname{Corr}(X_1, X_2^\sigma) \simeq \operatorname{Corr}(\operatorname{Alb} X_1, \operatorname{Alb} X_2^\sigma)$ (of which the isomorphism (82) is a special case) we will consider $\operatorname{Corr}(X_1, X_2^\sigma)$ as subgroup of $\operatorname{AH}(\mathbf{C}^{g_1} \times \mathbf{C}^{g_2}, \Lambda_1 \times \Lambda_2^\sigma)$. From the definitions of correspondences it is immediate, that then $\operatorname{Corr}(X_1, X_2^\sigma)$ is given by

$$\begin{aligned}\{(\alpha, H) \colon\ & H((u,0),(u',0)) = H((0,v),(0,v')) = 0 \text{ for } u, u' \in \mathbf{C}^{g_1},\ v, v' \in \mathbf{C}^{g_2}, \\ & \text{and } \alpha((\lambda_1, \lambda_2)) = (-1)^{\operatorname{Im} H((\lambda_1,0),(0,\lambda_2))} \text{ for all } (\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2^\sigma\}.\end{aligned}$$

Observe that if $(\alpha, H) \in \operatorname{Corr}(X_1, X_2^\sigma)$, then α is completely determined by H.

We will now be able to obtain a convenient description of $\operatorname{Corr}(X_1, X_2^\sigma)$ in terms of matrices. Fix a basis $\mathscr{K}_1$ of Λ_1 and a basis $\mathscr{K}_2$ of Λ_2. From these bases we construct in the obvious way a basis $\mathscr{E}$ of $\Lambda_1 \times \Lambda_2^\sigma$. If multiplication by i of $\mathbf{C}^{g_j}$ is given by the matrix

M_j with respect to $\mathscr{K}_j$ considered as $\mathbf{R}$-basis of $\mathbf{C}^{g_j}$ for $j = 1,2$, then

$$\begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix}$$

is the matrix of multiplication by i in $\mathbf{C}^{g_1} \times \mathbf{C}^{g_2}$ with respect to $\mathscr{E}$.

Now if $(\alpha, H) \in \operatorname{Corr}(X_1, X_2^\sigma)$ and $E = \operatorname{Im} H$ then E is skew-symmetric and $E((\lambda_1, 0), (\lambda_1', 0)) = E((0, \lambda_2), (0, \lambda_2')) = 0$, so the matrix representation of E with respect to the basis $\mathscr{E}$, which we denote by $E_{\mathscr{E}}$, is of the form:

$$\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix},$$

where $A \in \mathbf{Z}^{2g_1 \times 2g_2}$ and A^T is its transpose. Since $E(\mathrm{i}v, \mathrm{i}w) = E(v, w)$, we also have

$$E_{\mathscr{E}} = \begin{pmatrix} M_1^T & 0 \\ 0 & -M_2^T \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix} = \begin{pmatrix} 0 & -M_1^T A M_2 \\ M_2^T A^T M_1 & 0 \end{pmatrix}.$$

This means, that if $(\alpha, H) \in \operatorname{Corr}(X_1, X_2^\sigma)$ and $E = \operatorname{Im} H$, then the representation of E with respect to $\mathscr{E}$ is of the form

$$\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$$

where $A \in \mathbf{Z}^{2g_1 \times 2g_2}$, such that $-M_1^T A M_2 = A$. On the other hand, if an E is of the above form, there is a unique pair $(\alpha, H) \in \operatorname{Corr}(X, X^\sigma)$ such that $\operatorname{Im} H = E$. Namely, we define $H(v, w) = E(\mathrm{i}v, w) + \mathrm{i}E(v, w)$ and $\alpha((\lambda_1, \lambda_2)) = (-1)^{\operatorname{Im} H((\lambda_1, 0), (0, \lambda_2))}$.

In particular, if we take $X_1 = X_2 = X$, and of course $\mathscr{K}_1 = \mathscr{K}_2 = \mathscr{K}$, $M_1 = M_2 = M$, etc., we see that $\operatorname{Corr}(X, X^\sigma)$ is isomorphic to the additive group of matrices $\{A \in \mathbf{Z}^{2g \times 2g} : -M^T A M = A\}$. Using the interpretation of M as multiplication by i, we get a description independent of the choice of $\mathscr{K}$.

Lemma 2.1. *Let X be a complete nonsingular irreducible complex algebraic variety with* $\operatorname{rank} H^1(X, \mathbf{Z}) = 2g$, *and let $\Lambda \subset \mathbf{C}^g$ be its period lattice. The group* $\operatorname{Corr}(X, X^\sigma)$ *is canonically isomorphic to the additive group $\mathscr{B}(\mathbf{C}^g, \Lambda)$ of real-valued bilinear forms $B : \mathbf{C}^g \times \mathbf{C}^g \to \mathbf{R}$, such that B assumes integral values on Λ and such that $B(\mathrm{i}v, \mathrm{i}w) = -B(v, w)$.*

In order to determine $\operatorname{Corr}(X, X^\sigma)^G$ we will investigate the action of G on $\operatorname{Pic}(\operatorname{Alb}(X)_{\mathscr{W}})$. Identifying $\operatorname{Alb}(X)_{\mathscr{W}}$ with the complex torus $\mathbf{C}^g \times \mathbf{C}^g / \Lambda \times \Lambda^\sigma$, we see, that the involution $\tau_X : X_{\mathscr{W}} \to X_{\mathscr{W}}$ is induced by the $\mathbf{C}$-antilinear mapping

$$\begin{aligned} \tau_g \colon \mathbf{C}^g \times \mathbf{C}^g &\to \mathbf{C}^g \times \mathbf{C}^g \\ (x, y) &\mapsto (\bar{y}, \bar{x}). \end{aligned}$$

Then from a simple computation we deduce that the action of $G = \{1, \sigma\}$ on $\mathrm{AH}(\mathbf{C}^g \times \mathbf{C}^g, \Lambda \times \Lambda^\sigma)$ is given by $(\alpha, H)^\sigma = (\alpha^\sigma, H^\sigma)$, where $\alpha^\sigma(\lambda) = \sigma(\alpha(\tau_g(\lambda)))$, and $H^\sigma(v, w) = \sigma(H(\tau_g(v), \tau_g(w)))$.

If I is the $2g \times 2g$ identity matrix, then τ_g is given by the matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

with respect to the basis $\mathscr{E}$, so a straightforward computation shows, that if $(\alpha, H) \in \mathrm{Corr}(X, X^\sigma)^G$, then the matrix of $E = \mathrm{Im}\, H$ with respect to $\mathscr{E}$ has the form

$$\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$$

with $A \in \mathbf{Z}^{2g \times 2g}$, such that $-M^T A M = A$ and $A^T = A$. Conversely, any such E determines a unique pair (α, H) in $\mathrm{Corr}(X, X^\sigma)^G$.

Lemma 2.2. *Let X be a complete nonsingular irreducible complex algebraic variety, with period lattice $\Lambda \subset \mathbf{C}^g$. Then $\mathrm{Corr}(X, X^\sigma)^G$ is canonically isomorphic to the subgroup $\mathscr{S}(\mathbf{C}^g, \Lambda)$ of $\mathscr{B}(\mathbf{C}^g, \Lambda)$ consisting of the bilinear forms that are symmetric.*

Remark 2.3. We can actually associate a $\mathbf{C}$-bilinear form $B'\colon \mathbf{C}^g \times \mathbf{C}^g \to \mathbf{C}$ to any $B \in \mathscr{B}(\mathbf{C}^g, \Lambda)$ by defining

$$B'(v, w) := B(\mathrm{i}v, w) + \mathrm{i}B(v, w).$$

Since every $\mathbf{C}$-bilinear form on $\mathbf{C}^g$ of which the imaginary part takes integral values on Λ is of this type, we get descriptions of $\mathrm{Corr}(X, X^\sigma)$ and of $\mathrm{Corr}(X, X^\sigma)^G$ that are even more in the spirit of the Appell-Humbert Theorem:

If X is a complete nonsingular irreducible complex algebraic variety with period lattice $\Lambda \subset \mathbf{C}^g$, then the group $\mathrm{Corr}(X, X^\sigma)$ is canonically isomorphic to the group $\mathscr{B}_{\mathbf{C}}(\mathbf{C}^g, \Lambda)$ of $\mathbf{C}$-bilinear forms B on $\mathbf{C}^g$ such that $\mathrm{Im}\, B$ takes values in $\mathbf{Z}$ on Λ. Moreover, $\mathrm{Corr}(X, X^\sigma)^G$ is canonically isomorphic to the subgroup $\mathscr{S}_{\mathbf{C}}(\mathbf{C}^g, \Lambda) \subset \mathscr{B}_{\mathbf{C}}(\mathbf{C}^g, \Lambda)$ containing all the symmetric $\mathbf{C}$-bilinear forms in $\mathscr{B}_{\mathbf{C}}(\mathbf{C}^g, \Lambda)$.

If $X \simeq \mathbf{C}^g/\Lambda$ is abelian, then the identification of $H_1(X, \mathbf{Z})$ with Λ yields, by the Universal Coefficient Theorem, an identification of $H^1(X, \mathbf{Z}/2)$ with the $\mathbf{Z}/2$-vector space $\mathrm{Hom}(\Lambda, \mathbf{Z}/2)$ of homomorphisms from Λ to $\mathbf{Z}/2$. In a similar way we identify $H^1(X_{\mathscr{W}}(\mathbf{R}), \mathbf{Z}/2)$ with the $\mathbf{Z}/2$-space $\mathrm{Hom}(\mathrm{Re}(\Lambda \times \Lambda^\sigma), \mathbf{Z}/2)$, where $\mathrm{Re}(\Lambda \times \Lambda^\sigma)$ denotes the fixed part of $\Lambda \times \Lambda^\sigma$ under τ_g.

Under these identifications the isomorphism

$$\rho_X^*\colon H^1(X_{\mathscr{W}}(\mathbf{R}), \mathbf{Z}/2) \to H^1(X, \mathbf{Z}/2),$$

induced by the canonical homeomorphism $\rho_X\colon X_{\mathbf{R}} \to X_{\mathscr{W}}(\mathbf{R})$, is the dual of the group isomorphism

$$\Lambda \ni \lambda \mapsto (\lambda, \bar{\lambda}) \in \mathrm{Re}(\Lambda \times \Lambda^\sigma).$$

Since $X_{\mathscr{W}}(\mathbf{R})$ is connected for the strong topology, we conclude from [Hu1], that if we write $\mathbf{Z}/2 = \{-1, 1\}$, then the homomorphism

$$\psi_{X_{\mathscr{W}}} : \operatorname{Corr}(X, X^\sigma)^G \to H^1(X_{\mathscr{W}}(\mathbf{R}), \mathbf{Z}/2)$$

is given by

$$\begin{array}{rcl} \operatorname{Corr}(X, X^\sigma)^G & \to & \operatorname{Hom}(\operatorname{Re}(\Lambda \times \Lambda^\sigma), \mathbf{Z}/2) \\ (\alpha, H) & \mapsto & \alpha|_{\operatorname{Re}(\Lambda\times\Lambda^\sigma)} \end{array}$$

For arbitrary complete nonsingular X we can consider

$$\operatorname{Hom}(\Lambda, \mathbf{Z}/2) \simeq \operatorname{Hom}(H_1(X, Z)/\text{torsion}, \mathbf{Z}/2) \simeq H^1(X, \mathbf{Z}) \otimes \mathbf{Z}/2$$

as a subspace of $H^1(X, \mathbf{Z}/2)$. This subspace contains $H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$ by Corollary 1.3, and we get the following recipe for the computation of $H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$.

Proposition 2.4. *Let X be a complete nonsingular irreducible complex algebraic variety with first Betti number $2g$, and let $\Lambda \subset \mathbf{C}^g$ be the period lattice of X. The composite homomorphism*

$$\mathscr{S}(\mathbf{C}^g, \Lambda) \to \operatorname{Corr}(X, X^\sigma)^G \to \operatorname{Hom}(\operatorname{Re}(\Lambda \times \Lambda^\sigma), \mathbf{Z}/2) \to \operatorname{Hom}(\Lambda, \mathbf{Z}/2)$$

is given by

$$\begin{array}{rcl} \mathscr{S}(\mathbf{C}^g, \Lambda) & \to & \operatorname{Hom}(\Lambda, \mathbf{Z}/2) \\ B & \mapsto & \{\lambda \mapsto B(\lambda, \lambda) \bmod 2\}. \end{array}$$

The image of this mapping is canonically isomorphic to $H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$.

Now suppose X is isomorphic to a product $X_1 \times \cdots \times X_n$ of (not necessarily simple) abelian varieties, we will need for later use some information on $\operatorname{Corr}(X, X^\sigma)$ in terms of the factors of the product. Let g_j be the dimension of X_j and let Λ_j be its period lattice for $j = 1, \ldots, n$. Choose a basis $\mathscr{K}_j$ of Λ_j for each j and let M_j be the matrix of multiplication by i with respect to $\mathscr{K}_j$. From the $\mathscr{K}_j$ we construct in the obvious way a basis $\mathscr{K}$ for the period lattice $\Lambda_1 \times \cdots \times \Lambda_n$ of X. It is easy to see, that we get an isomorphism

$$\mathscr{B}(\mathbf{C}^{g_1} \times \cdots \times \mathbf{C}^{g_n}, \Lambda_1 \times \cdots \times \Lambda_n) \simeq \left\{ \begin{pmatrix} B_{11} & \cdots & B_{in} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{pmatrix} : B_{ij} \in \mathbf{Z}^{2g_i \times 2g_j} \text{ and } -M_i^T B_{ij} M_j = B_{ij} \right\}$$

In particular, if all Λ_j are the same we get, with a slight abuse of notation, the following description.

Lemma 2.5. *Let $\Lambda \subset \mathbf{C}^g$ is a lattice. Then*

$$\mathscr{B}(\mathbf{C}^{ng}, \Lambda^n) = \left\{ \begin{pmatrix} B_{11} & \cdots & B_{in} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nn} \end{pmatrix} : B_{ij} \in \mathscr{B}(\mathbf{C}^g, \Lambda) \right\}$$

On the other hand, we will be interested in the case where $\operatorname{Corr}(X_i, X_j^\sigma) = 0$ for $i \neq j$. We observe, that the condition $-M_i^T B_{ij} M_j = B_{ij}$ implies that B_{ij} is the matrix associated to an element $\mathscr{L} \in \operatorname{Corr}(X_i, X_j^\sigma)$, so the following is obvious.

Lemma 2.6. *For $j = 1, \dots, n$ let X_j be a complex abelian variety of dimension g_j with period lattice Λ_j, such that $\operatorname{Hom}(X_i, X_j^\sigma) = \{0\}$ for $i \neq j$. Then*

$$\mathscr{S}(\mathbf{C}^{g_1} \times \cdots \times \mathbf{C}^{g_n}, \Lambda_1 \times \cdots \times \Lambda_n) = \left\{ \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_n \end{pmatrix} : B_j \in \mathscr{S}(\mathbf{C}^{g_j}, \Lambda_j) \right\}.$$

Finally we will compare $\mathscr{S}(\mathbf{C}^g, \Lambda_1)$ and $\mathscr{S}(\mathbf{C}^g, \Lambda_2)$ for abelian varieties X_1 and X_2 of dimension g with period lattices Λ_1, resp. Λ_2, that admit an isogeny $f\colon X_1 \to X_2$. Then f is given by a $\mathbf{C}$-linear isomorphism

$$F\colon \mathbf{C}^g \to \mathbf{C}^g$$

that maps Λ_1 into Λ_2 and induces an isomorphism

$$F \otimes \mathrm{id}\colon \Lambda_1 \otimes \mathbf{Q} \to \Lambda_2 \otimes \mathbf{Q}.$$

Also, F defines a pull-back mapping $F^*\colon \mathscr{S}(\mathbf{C}^g, \Lambda_2) \to \mathscr{S}(\mathbf{C}^g, \Lambda_1)$ given by

$$F^*(S)(v, w) = S(F(v), F(w))$$

for $S \in \mathscr{S}(\mathbf{C}^g, \Lambda_2)$, $v, w \in \mathbf{C}^g$. The following lemma is easy to check; it will be important in the next section.

Lemma 2.7. *With the above notation, the pull-back*

$$F^*\colon \mathscr{S}(\mathbf{C}^g, \Lambda_2) \to \mathscr{S}(\mathbf{C}^g, \Lambda_1)$$

induces an isomorphism

$$F^* \otimes \mathrm{id}\colon \mathscr{S}(\mathbf{C}^g, \Lambda_2) \otimes \mathbf{Q} \to \mathscr{S}(\mathbf{C}^g, \Lambda_1) \otimes \mathbf{Q}.$$

3. Concrete results

As a first application, we will show, that we can construct complex varieties with prescribed topological and real algebraic first cohomology groups. For this, we will use a variant of the Lefschetz Theorem on hyperplane sections.

Lemma 3.1. *Let X be a d-dimensional nonsingular complex subvariety of $\mathbf{P}^n_{\mathbf{C}}$ and let $H \subset \mathbf{P}^n_{\mathbf{C}}$ be a hyperplane, such that $X' = X \cap H$ is again nonsingular. If $d \geq 3$, then the inclusion $i\colon X \cap H \to X$ induces isomorphisms*

$$i^*\colon H^1(X, \mathbf{Z}) \to H^1(X', \mathbf{Z})$$
$$i^*\colon H^1(X, \mathbf{Z}/2) \to H^1(X', \mathbf{Z}/2)$$
$$i^*\colon H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2) \to H^1_{\mathrm{alg}}(X'_{\mathbf{R}}, \mathbf{Z}/2).$$

Proof. The first two homomorphisms are isomorphisms by the Lefschetz Theorem on hyperplane sections (see [Mi]). Let us to prove that the last homomorphism is an isomorphism.

Since the mapping $i_*\colon H_1(X',\mathbf{Z}) \to H_1(X,\mathbf{Z})$ is an isomorphism, we see that the induced mapping $\tilde{i}\colon \mathrm{Alb}(X') \to \mathrm{Alb}\, X$ is an isomorphism. By Theorem 1.5 this implies that $i^*\colon H^1_{\mathrm{alg}}(X_{\mathbf{R}},\mathbf{Z}/2) \to H^1_{\mathrm{alg}}(X'_{\mathbf{R}},\mathbf{Z}/2)$ is an isomorphism. □

Now we know all the obstructions for the construction of projective complex algebraic varieties of dimension ≥ 2 with prescribed first cohomology groups.

Theorem 3.2. *Given the integers d, a, g, h with $d \geq 2$, the following conditions are equivalent:*

(i) *There exists a nonsingular complex projective variety X of dimension d, such that*

$$\begin{aligned} \dim_{\mathbf{Z}/2} H^1(X,\mathbf{Z}/2) &= h \\ \dim_{\mathbf{Z}/2} H^1_{\mathrm{alg}}(X_{\mathbf{R}},\mathbf{Z}/2) &= a \\ \mathrm{rank}\ H^1(X,\mathbf{Z}) &= 2g. \end{aligned}$$

(ii) $0 \leq a \leq 2g \leq h$

Proof. (i) ⇒ (ii) This follows from Corollary 1.3 and the fact, that complex algebraic varieties always have an even first Betti number.

(ii) ⇒ (i) For $i = 0, 1, 2$, let E_i be an elliptic curve with $H^1_{\mathrm{alg}}((E_i)_{\mathbf{R}},\mathbf{Z}/2) = i$ (see [BK1]), then of course $\dim_{\mathbf{Z}/2} H^1(E_i,\mathbf{Z}/2) = \mathrm{rank}\, H^1(E_i,\mathbf{Z}) = 2$. Let Y be an Enriques surface; then it is well-known that $H^1(Y,\mathbf{Z}/2) = \mathbf{Z}/2$ and $H^1(Y,\mathbf{Z}) = 0$, so we have by Corollary 1.3 that $H^1_{\mathrm{alg}}(Y_{\mathbf{R}},\mathbf{Z}/2) = 0$. Taking the product of a right number of copies of the E_i and Y, we construct a nonsingular complex projective variety X' satisfying all the conditions on cohomology. If the dimension of X is too small, we take the product of X' and a number of copies of $\mathbf{P}^1_{\mathbf{C}}$ in order to obtain X. If the dimension is too large, Lemma 3.1 allows us to lower the dimension by repeatedly taking smooth hyperplane sections of X', which is always possible by Bertini's Theorem [Ha, Theorem II.8.18]. □

It is not known whether the statement of Theorem 3.2 with the extra condition $h = 2g$ holds for $d = 1$, i.e., for complex curves. Some results in this direction are given in [BK1]. The following example shows, that in any case there is for every g a projective complex algebraic curve C of genus g such that $H_1^{\mathrm{alg}}((C)_{\mathbf{R}},\mathbf{Z}/2) = H_1(C,\mathbf{Z}/2) = \mathbf{Z}/2^{2g}$.

Example 3.3. Fix a $g > 0$. We define C to be the nonsingular irreducible projective complex curve that has a Zariski-open subset isomorphic to the affine plane curve given by the equation

$$y^2 = x - x^{2g+2}.$$

Then C is defined over $\mathbf{R}$ and its real part $C(\mathbf{R})$ is a Zariski-closed subset of $C_{\mathbf{R}}$, homeomorphic to a circle for the strong topology. Let φ be the automorphism of C

given by $(x,y) \mapsto (\zeta_{2g+1}x, \zeta_{4g+2}y)$, where $\zeta_k = \exp(2\pi i/k)$. We put $V_0 = C(\mathbf{R})$ and $V_k = \varphi^k(V_0)$ for $k = 1, \dots, 2g$.

The two-sheeted covering $\pi\colon C \to \mathbf{P}^1_{\mathbf{C}}$ is branched in the points $1, \zeta_{2g+1}, \dots, \zeta_{2g+1}^{2g}$ and 0. From the theory of complex analytic functions we know that the two branches of $\sqrt{x - x^{2g+2}}$ in, say, $2 \in \mathbf{P}^1_{\mathbf{C}}$ can be extended analytically to $\mathbf{P}^1_{\mathbf{C}} \setminus \pi(\bigcup_{k=0}^{2g} V_k)$. In particular, $C - \bigcup_{k=0}^{2g} V_k$ is homeomorphic to two copies of the open disk. Since each V_k is homeomorphic to a circle, and $\bigcap_{k=0}^{g} V_k$ consists of one point, the theory of cell complexes tells us that the fundamental classes of $V_0, \dots, V_{2g}$ generate $H_1(C, \mathbf{Z}/2)$.

Finally, we we will investigate to what extent the structure of the endomorphism ring of an abelian variety X puts upper bounds on the dimension of $H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2)$. We have by [BK1, Th. 1.2] that a simple complex abelian variety X of dimension g with $H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2) = H^1(X, \mathbf{Z}/2)$ is of *CM-type*, i.e., its ring of endomorphisms $\mathrm{End}(X)$ is a free $\mathbf{Z}$-module of rank $2g$ (cf [Mu]).

Our aim is a generalization of this result to arbitrary abelian varieties. Recall that an arbitrary complex abelian variety will be said to be of *CM*-type if it is isogenous to a product of simple abelian varieties of *CM*-type. In order to ease the notation, we define

$$d(X) = \dim_{\mathbf{Z}/2} H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2).$$

Note that the surjection (83) implies that

$$d(X) \leq \mathrm{Corr}(X, X^\sigma), \tag{84}$$

and recall the classical fact that for any two complex abelian varieties X_1 and X_2 we have that the group $\mathrm{Corr}(X_1, X_2)$ is equal to the group $\mathrm{Hom}(X_1, X_2)$ of homomorphisms between X_1 and X_2.

If X is a simple abelian variety, then $\mathrm{rank}\,\mathrm{Hom}(X, X^\sigma)$ is either 0 or equal to $\mathrm{rank}\,\mathrm{End}(X)$, so then indeed equation (84) gives us that if $H^1_{\mathrm{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2) = H^1(X, \mathbf{Z}/2)$, then X is of *CM*-type. Much more complicated is the case when X is not simple. Then we have that X is isogenous to a product of the form

$$X_1^{\nu_1} \times (X_1^\sigma)^{\mu_1} \times \cdots \times X_n^{\nu_n} \times (X_n^\sigma)^{\mu_n},$$

where all X_i are simple abelian varieties of dimension g_i with X_i not isogenous to X_j or X_j^σ if $i \neq j$, $\mu_i = 0$ if X_i is isogenous to X_i^σ and $\nu_i \geq \mu_i$ for all i. Then we have that $\mathrm{rank}\,\mathrm{Corr}(X, X^\sigma)^G = \mathrm{rank}\,\mathrm{Corr}(\prod_i X_i^{\nu_i} \times (X_i^\sigma)^{\mu_i}, \prod_i (X_i^\sigma)^{\nu_i} \times X_i^{\mu_i})^G$, which can be very big if the ν_i are large enough, so then the above bound on $d(X)$ is not very useful. However, if X is actually *isomorphic* to such a product, we see by a direct computation, using the results of Section 2, that $d(X) \leq \sum_i \mathrm{rank}\,\mathrm{End}(X_i)$. Hence if X is not of *CM*-type, then $d(X) < 2g = \dim_{\mathbf{Z}/2} H^1(X, \mathbf{Z}/2)$.

The key result that will allow us to obtain non-trivial bounds on $d(X)$ for every X in a given isogeny class of abelian varieties not of *CM*-type, is the lemma below, which has kindly been provided by H.W. Lenstra, Jr. We will fix some notation first.

Let V be a finite dimensional vector space over $\mathbf{Q}$. Let $\Lambda \subset V$ be a finitely generated $\mathbf{Z}$-module. Let $\mathscr{S}$ be a collection of symmetric bilinear forms $B\colon V \times V \to \mathbf{Q}$, such that $B(\lambda, \lambda') \in \mathbf{Z}$ for $\lambda, \lambda' \in \Lambda$. We define

$$\begin{array}{rcl} \varepsilon\colon \mathscr{S} & \to & \operatorname{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z}/2) \\ B & \mapsto & \{\lambda \mapsto B(\lambda, \lambda) \bmod 2\}. \end{array}$$

Let $L(\varepsilon(\mathscr{S}))$ be the $\mathbf{Z}/2$-linear span of $\varepsilon(\mathscr{S})$ in $\operatorname{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z}/2)$. If K is a field containing $\mathbf{Q}$ and B is a bilinear form on V, then by B_K we denote the bilinear form $(V \otimes K) \times (V \otimes K) \to K$ given by $B_K(v \otimes k, v' \otimes k') = kk' B(v, v')$.

Lemma 3.4. *Let $K/\mathbf{Q}$ be a number field. If $U \subset V \otimes K$ is a K-linear subspace, such that $B_K(u, u) = 0$ for all $u \in U$ and all $B \in \mathscr{S}$, then* $\dim_{\mathbf{Z}/2} L(\varepsilon(\mathscr{S})) \leq \dim_{\mathbf{Q}} V - \dim_K U$.

Proof. Let $\overline{\mathbf{F}}_2$ be an algebraic closure of $\mathbf{Z}/2$. Let $\bar{\varepsilon}\colon \mathscr{S} \to \operatorname{Hom}_{\mathbf{Z}}(\Lambda, \overline{\mathbf{F}}_2)$ be the mapping induced by ε, and let $L(\bar{\varepsilon}(\mathscr{S}))$ be the linear span over $\overline{\mathbf{F}}_2$ of the image of $\bar{\varepsilon}$. Of course, $\dim_{\overline{\mathbf{F}}_2} L(\bar{\varepsilon}(\mathscr{S})) = \dim_{\mathbf{Z}/2} L(\varepsilon(\mathscr{S}))$.

Now we will construct a principal ideal domain $A \subset K$, such that K is the quotient field of A and such that the reduction map $\mathbf{Z} \to \mathbf{Z}/2$ extends to a ring homomorphism

$$\rho\colon A \to \overline{\mathbf{F}}_2.$$

We choose a prime ideal $\mathfrak{p}$ of $\mathscr{O}_K$, the ring of integers of K, that lies over $(2) \subset \mathbf{Z}$ and we take for A the localization of $\mathscr{O}_K$ at $\mathfrak{p}$ (but if $\mathscr{O}_K$ already is a principal ideal domain, we can even take $A = \mathscr{O}_K$). Then we define ρ by choosing an embedding of $A/\mathfrak{p}A$ in $\overline{\mathbf{F}}_2$.

Putting $\Lambda_A = \Lambda \otimes A$, we see that $B_K(\lambda, \lambda') \in A$ if $B \in \mathscr{S}$ and $\lambda, \lambda' \in \Lambda_A$, so the following mapping extends $\bar{\varepsilon}$.

$$\begin{array}{rcl} \varepsilon_K\colon \mathscr{S} & \to & \operatorname{Hom}_A(\Lambda_A, \overline{\mathbf{F}}_2) \\ B & \mapsto & \{\lambda \mapsto \sqrt{\rho(B_K(\lambda, \lambda))}\}, \end{array}$$

where $\sqrt{\cdot}$ is the inverse of the field automorphism $x \mapsto x^2$ of $\overline{\mathbf{F}}_2$ and the A-module structure on $\overline{\mathbf{F}}_2$ is the one induced by ρ.

The restriction map $\operatorname{Hom}_A(\Lambda_A, \overline{\mathbf{F}}_2) \to \operatorname{Hom}_{\mathbf{Z}}(\Lambda, \overline{\mathbf{F}}_2)$ induced by the inclusion $\Lambda \subset \Lambda_A$, maps $\varepsilon_K(\mathscr{S})$ bijectively onto $\bar{\varepsilon}(\mathscr{S})$, so $\dim_{\overline{\mathbf{F}}_2} L(\varepsilon_K(\mathscr{S})) = \dim_{\overline{\mathbf{F}}_2} L(\bar{\varepsilon}(\mathscr{S}))$, and we will prove the lemma by showing that

$$\dim_{\overline{\mathbf{F}}_2} L(\varepsilon_K(\mathscr{S})) \leq \dim_K V \otimes K - \dim_K U.$$

Since $U \cap \Lambda_A$ is in the kernel of all $\beta \in \operatorname{Hom}_A(\Lambda_A, \overline{\mathbf{F}}_2)$ that are of the form $\beta = \varepsilon_K(B)$ for some $B \in \mathscr{S}$, we can write ε_K as a composition

$$\mathscr{S} \to \operatorname{Hom}_A(\Lambda_A/(U \cap \Lambda_A), \overline{\mathbf{F}}_2) \to \operatorname{Hom}_A(\Lambda_A, \overline{\mathbf{F}}_2).$$

The finitely generated A-module $\Lambda_A/(U \cap \Lambda_A)$ is contained in the K-vector space $(V \otimes K)/U$, so it is torsion-free. Since A is a principal ideal domain, this implies that

$\Lambda_A/(U \cap \Lambda_A)$ is isomorphic to A^k for some $k \leq \dim_K(V \otimes K)/U$, and we see that

$$\dim_{\overline{\mathbf{F}}_2} L(\varepsilon_K(\mathscr{S})) \leq \dim_{\overline{\mathbf{F}}_2} \operatorname{Hom}_A(\Lambda_A/(U \cap \Lambda_A), \overline{\mathbf{F}}_2) \leq \dim_K(V \otimes K)/U,$$

which finishes the proof. $\square$

Proposition 3.5. *Let X be a complex abelian variety of dimension g, isogenous to a product*

$$X_1^{\nu_1} \times (X_1^\sigma)^{\mu_1} \times \cdots \times X_n^{\nu_n} \times (X_n^\sigma)^{\mu_n},$$

with notations and conventions as above. Let

$$\gamma_i = \begin{cases} 2\nu_i g_i & \text{if } X_i \text{ is not isogenous to } X_i^\sigma \\ \nu_i & \text{if } 0 < \operatorname{rank} \operatorname{Hom}(X_i, X_i^\sigma) < 2g_i \\ 0 & \text{if} \operatorname{rank} \operatorname{Hom}(X_i, X_i^\sigma) = 2g_i \end{cases}$$

Then $d(X) \leq 2g - \sum_i \gamma_i$.

Proof. Let $\Lambda \subset \mathbf{C}^g$ be the period lattice of X. By Proposition 2.4 and the previous lemma, it is sufficient to find a number field K and a K-vector space $U \subset \Lambda \otimes K$ of dimension $\sum_i \gamma_i$ such that $B_K(u, u) = 0$ for all $u \in U$ and all $B \in \mathscr{S}(\mathbf{C}^g, \Lambda)$.

The isogeny induces an isomorphism $\Lambda \otimes \mathbf{Q} \simeq \prod_i (\Lambda_i^{\nu_i} \times (\Lambda_i^\sigma)^{\mu_i}) \otimes \mathbf{Q}$, and by Lemma 2.7 an isomorphism $\mathscr{S}(\mathbf{C}^g, \Lambda) \otimes \mathbf{Q} \simeq \mathscr{S}(\mathbf{C}^g, \prod_i \Lambda_i^{\nu_i} \times (\Lambda_i^\sigma)^{\mu_i}) \otimes \mathbf{Q}$, where Λ_i is the period lattice of X_i for $i = 1, \dots, n$. By Lemma 2.6, we have

$$\mathscr{S}(\mathbf{C}^g, \prod_i \Lambda_i^{\nu_i} \times (\Lambda_i^\sigma)^{\mu_i}) = \left\{ \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_n \end{pmatrix} : B_j \in \mathscr{S}(\mathbf{C}^{(\nu_i+\mu_i)g_i}, \Lambda_i^{\nu_i} \times (\Lambda_i^\sigma)^{\mu_i}) \right\},$$

so if we have number fields K_i and K_i-vector spaces $U_i \subset (\Lambda_i^{\nu_i} \times (\Lambda_i^\sigma)^{\mu_i}) \otimes K_i$ of dimension γ_i such that $B_{K_i}(u, u) = 0$ for all $u \in U_i$ and all $B \in \mathscr{S}(\mathbf{C}^{(\nu_n+\mu_n)g_n}, \Lambda_n^{\nu_n} \times (\Lambda_n^\sigma)^{\mu_n})$, we prove the proposition by taking for K a number field containing all K_i and for U the product $\prod_i U_i \otimes_{K_i} K$.

Let us fix an $i \in \{1, \dots, n\}$ with $\gamma_i \neq 0$. If X_i is not isogenous to X_i^σ, then $\operatorname{Hom}(X_i, X_i^\sigma) = \{0\}$, since X_i is simple, so $\operatorname{Corr}(X_i, X_i^\sigma) = \{0\}$, hence $\mathscr{B}(\mathbf{C}^{g_i}, \Lambda_i) = \{0\}$ by Lemma 2.1. Therefore, by Lemma 2.5, $\mathscr{B}(\mathbf{C}^{\nu_i g_i}, \Lambda_i^{\nu_i}) = \{0\}$, and we may take $K_i = \mathbf{Q}$ and $U_i = \Lambda_i^{\nu_i} \otimes \mathbf{Q}$.

If $\operatorname{Hom}(X_i, X_i^\sigma) \neq \{0\}$, then X_i is isogenous to X_i^σ, so $\mu_i = 0$. In this case we deduce from Lemma 2.5, that we can take $U_i = W^{\nu_i} \subset \Lambda_i^{\nu_i} \otimes K_i$ if we have a number field K_i and a one-dimensional subspace $W \subset \Lambda_i \otimes K_i$ such that $B_{K_i}(w, w') = 0$ for any $B \in \mathscr{B}(\mathbf{C}^{g_i}, \Lambda_i)$ and any $w, w' \in W$. In order to find such a K_i and W, we take a $\mathbf{Z}$-basis $\{B_1, \dots, B_k\}$ of $\mathscr{B}(\mathbf{C}^{g_i}, \Lambda_i)$ with $k = \operatorname{rank} \operatorname{Corr}(X_i, X_i^\sigma) = \operatorname{rank} \operatorname{Hom}(X_i, X_i^\sigma)$. Observe that we just need one nonzero element $w \subset \Lambda_i \otimes \overline{\mathbf{Q}}$ satisfying the homogeneous quadratic equations $B_1(w, w) = 0, \dots, B_k(w, w) = 0$, where $\overline{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$. Since $k < 2g_i = \dim_{\overline{\mathbf{Q}}} \Lambda_i \otimes \overline{\mathbf{Q}}$, such a w exists. $\square$

The bounds in the above proposition are far from being sharp in many cases, but they are sufficient for generalizing the result from [BK1], that for a complex elliptic curve E without complex multiplication we have that $H^1_{\text{alg}}(E_{\mathbf{R}}, \mathbf{Z}/2) \neq H^1(E, \mathbf{Z}/2)$ (see Th. 1.7 in *loc. cit.*).

Theorem 3.6. *If X is an complex abelian variety with*

$$H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2) = H^1(X, \mathbf{Z}/2)$$

then X is of CM-type.

Proof. Immediate from Proposition 3.5. □

Corollary 3.7. *There are, up to isomorphism, only countably many complex abelian varieties X having $H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2) = H^1(X, \mathbf{Z}/2)$.*

Proof. From the description of the isogeny classes of simple complex abelian varieties of *CM*-type in [Mu] we easily deduce that there are only countably many complex abelian varieties X of *CM*-type. □

Corollary 3.8. *Let X be a complete, nonsingular complex algebraic variety with*

$$H^1_{\text{alg}}(X_{\mathbf{R}}, \mathbf{Z}/2) = H^1(X, \mathbf{Z}/2).$$

Then $H^1(X, \mathbf{Z}/2) = H^1(X, \mathbf{Z}) \otimes \mathbf{Z}/2$ and Alb X *is of CM-type.*

Proof. The first part follows from Corollary 1.3 and the second part follows from Theorem 1.5. □

If X is a curve, then $H^1(X, \mathbf{Z}/2)$ is always equal to $H^1(X, \mathbf{Z}) \otimes \mathbf{Z}/2$ and the Albanese variety of X is the *Jacobian variety* Jac X, so the previous corollary proves Conjecture 1.15 of [BK1].

Corollary 3.9. *Let C be an irreducible nonsingular projective complex curve with*

$$H^1_{\text{alg}}(C_{\mathbf{R}}, \mathbf{Z}/2) = H^1(C, \mathbf{Z}/2).$$

h Then Jac C *is of CM-type.*

Using the Torelli Theorem and the fact, that any complex abelian variety X admits, up to isomorphism, only a finite number of principal polarizations (see [NN]), we get for complex curves the analogue of Corollary 3.7.

Corollary 3.10. *There are, up to isomorphism, only countably many irreducible nonsingular projective complex curves C with $H^1_{\text{alg}}(C_{\mathbf{R}}, \mathbf{Z}/2) = H^1(C, \mathbf{Z}/2)$.*

For curves of genus 2 a stronger result than Corollary 3.9 is obtained by a direct use of Proposition 3.5.

Corollary 3.11. *Let C be an irreducible nonsingular projective complex curve of genus 2 with*

$$\dim_{\mathbf{Z}/2} H^1_{\text{alg}}(C_{\mathbf{R}}, \mathbf{Z}/2) > 2.$$

Then Jac C *is of CM-type.*

PROOF. Either Jac C is simple or it is isogenous to the product of two elliptic curves $E_1 \times E_2$. In the first case we apply inequality (84) and the fact that if X is a simple abelian variety of dimension g, then the rank of $\mathrm{End}(X)$ divides $2g$, hence the rank of $\mathrm{Hom}(X, X^\sigma)$ divides $2g$. In the second case it is well-known, that we may choose $E_1 = E_2$, since we can construct a nontrivial element $\omega \in \mathrm{Corr}(E_1, E_2)$ using the image of $C \subset \mathrm{Jac}\, C$ in $E_1 \times E_2$, and ω induces an isogeny between E_1 and E_2. Then the result follows from Proposition 3.5. $\square$

The existence of infinitely many non-isomorphic curves C of genus 2 with $d(C) = 4$ is proven in [BK1]. Together with the corollary, this implies that the set of isomorphism classes of irreducible nonsingular projective complex curves of genus 2 with $d(C) > 2$ is countably infinite. However, no examples of the case $d(C) = 3$ are known for genus 2.

Using similar methods as in Example 3.3, we see that a nonsingular projective complex curve C determined by an equation of the form $y^2 = (x^3 - 1)(x^3 - a)$ with $0 < a < 1$, has $d(C) \geq 2$. Since different values of a give non-isomorphic curves, we see that the set of isomorphism classes of irreducible nonsingular projective complex curves C of genus 2 with $d(C) = 2$ is uncountable, so the statment of Corollary 3.11 does no longer hold if '> 2' is replaced by '≥ 2'.

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Notations

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