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Dynamic Boolean models

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Abstract

Consider an ordinary Boolean model, that is, a homogeneous Poisson point process in $\mathbb{R}^d$, where the points are all centres of random balls with i.i.d. radii. Now let these points move around according to i.i.d. stochastic processes. It is not hard to show that at each fixed time t we again have a Boolean model with the original distribution. Hence if the original model is supercritical then, for any t, the probability of having an unbounded occupied component at time t equals 1. We show that under mild conditions on the dynamics (e.g. for Brownian motion) we can interchange the quantifiers in the above statement, namely: if the original model is supercritical, then the probability of having an unbounded occupied component for all t simultaneously equals 1. Roughly analogous statements are valid for the subcritical regime, under some further mild conditions. © 1997 Elsevier Science B.V.

1. Introduction

Classical percolation models are usually static, i.e. there is no time parameter involved. One way of introducing the concept of time is first-passage percolation, see for instance Kesten (1984). Häggström et al. (1997) have introduced a dynamical percolation model where the role of time is completely different. Since the setup in our paper is related to that in the paper just mentioned, we start with a brief description of some of their results. Fix some $p \in [0,1]$ and suppose $G=(V,E)$ is a countably infinite, locally finite graph, each edge (bond) of which is open with probability p and closed with probability $1-p$, independently of all the other edges. Write Φ_p for this product measure. One of the questions in percolation theory is whether the subgraph formed by the open edges of G has an infinite connected component (cluster). Defining $\mathscr{C}$ to be the event that there exists such an infinite cluster, we have that for some critical probability $p_c = p_c(G) \in [0,1]$,

$$\Phi_p(\mathscr{C}) = \begin{cases} 1 & \text{for } p > p_c, \\ 0 & \text{for } p < p_c. \end{cases}$$

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In the dynamical version of Häggström et al. (1997) the edge-configuration at time 0 is distributed as Φ_p, and from then on each edge, independently of all other edges, changes its status (open or closed) according to a stationary continuous-time two-state Markov chain. Thus the edge-configuration is time-stationary, with distribution Φ_p for any fixed time $t \geqslant 0$. We shall write $\boldsymbol{P}_p$ for the probability measure governing this process and assume the underlying probability space is large enough for all our purposes.

If we denote by $\mathscr{C}(t)$ the event that this process exhibits an infinite cluster at time t then

$$\boldsymbol{P}_p(\mathscr{C}(t)) = \begin{cases} 1 & \text{for } p > p_c, \\ 0 & \text{for } p < p_c, \end{cases}$$

for any $t \geqslant 0$, and moreover, by Fubini's Theorem,

$$\begin{cases} \boldsymbol{P}_p(\mathscr{C}(t) \text{ occurs for Lebesgue-a.e. } t) = 1 & \text{for } p > p_c, \\ \boldsymbol{P}_p(\neg\mathscr{C}(t) \text{ occurs for Lebesgue-a.e. } t) = 1 & \text{for } p < p_c, \end{cases}$$

where $\neg A$ denotes the complement of the event A.

In the spirit of Fukushima's work on quasi-everywhere properties of Brownian motion (Fukushima, 1984), Alexander's work on simultaneous uniqueness (Alexander, 1995) and others (e.g. Le Gall, 1992 and Shepp, 1972), it is natural to ask whether the quantifier 'for a.e. t' in the above statements can be replaced by 'for every t'. Häggström et al. show (among many other things) that the answer to this is affirmative, that is, for any graph G,

$$\begin{cases} \boldsymbol{P}_p(\mathscr{C}(t) \text{ occurs for every } t) = 1 & \text{if } p > p_c, \\ \boldsymbol{P}_p(\neg\mathscr{C}(t) \text{ occurs for every } t) = 1 & \text{if } p < p_c. \end{cases}$$

In this paper we will consider a continuum percolation process known as the (Poisson) Boolean model. Let ρ be a (strictly) positive random variable. Consider a homogeneous Poisson point process in $\mathbb{R}^d$ ($d \geqslant 1$) with intensity $\lambda > 0$. Suppose that centred at each point we place a closed (Euclidean) ball, in such a way that the radius of each ball has the same distribution as ρ and that these radii are independent of each other and of the positions of the Poisson points. This is the Boolean model, which we denote by $X_{\lambda,\rho}$. (For a more formal description, see the general reference for continuum percolation, Meester and Roy (1996). The law of this process is denoted by $\boldsymbol{P}_{\lambda,\rho}$.

The random balls of $X_{\lambda,\rho}$ occupy a region in $\mathbb{R}^d$. Analogous to the bond percolation case above, let $\mathscr{C}$ denote the event that the occupied region has an unbounded connected component. It is well known that there exists a critical intensity $\lambda_c = \lambda_c(\rho) \geqslant 0$ such that

$$\boldsymbol{P}_{\lambda,\rho}(\mathscr{C}) = \begin{cases} 1 & \text{if } \lambda > \lambda_c, \\ 0 & \text{if } \lambda < \lambda_c. \end{cases}$$

When $d \geqslant 2$ we know that $\lambda_c < \infty$.

Equivalently, writing U for the component of the occupied region that contains the origin ($U = \emptyset$ if the origin is not in the occupied region), it can be shown that λ_c can also be written as

$$\lambda_c(\rho) = \sup\{\lambda : \boldsymbol{P}_{\lambda,\rho}(d(U) < \infty) = 1\},$$

where $d(\cdot)$ denotes diameter. Writing ℓ for Lebesgue measure and $\#(U)$ for the number of balls in U, we define the following additional critical intensities:

$$\lambda_T(\rho) = \sup\{\lambda : \boldsymbol{E}_{\lambda,\rho}(\ell(U)) < \infty\},$$

$$\lambda_H(\rho) = \sup\{\lambda : \boldsymbol{P}_{\lambda,\rho}(\ell(U) < \infty) = 1\},$$

$$\lambda_{\#}(\rho) = \sup\{\lambda : \boldsymbol{P}_{\lambda,\rho}(\#(U) < \infty) = 1\},$$

where $\boldsymbol{E}_{\lambda,\rho}$ denotes the expectation operator with respect to $\boldsymbol{P}_{\lambda,\rho}$. Menshikov and Sidorenko (1987) showed that all these critical intensities are equal when ρ is bounded above. For general ρ, the inequality $\lambda_T(\rho) \leqslant \lambda_H(\rho)$ is obvious. Before continuing, we prove that $\lambda_H(\rho) = \lambda_c(\rho) = \lambda_{\#}(\rho)$ for any ρ, since this fact does not seem to appear anywhere in the literature. The following preliminary lemma is related to a result of Meester et al. (1994).

Lemma 1.1. *Let ρ be any positive random variable. For each $r > 0$ with $\alpha_r := \boldsymbol{P}(\rho \geqslant r) > 0$, let ρ_r be a random variable with distribution*

$$\boldsymbol{P}(\rho_r \in \cdot) = \boldsymbol{P}(\rho \in \cdot \mid \rho \geqslant r).$$

Then we have $\lambda_c(\rho_r) \to \lambda_c(\rho)$ as $r \to 0$. Moreover, this conclusion remains true if the critical intensity λ_c is replaced by λ_T or λ_H.

Proof. We prove only the λ_c result; the proofs for the other critical intensities are identical. First note that $0 < r_1 < r_2$ implies by a simple coupling argument that $\lambda_c(\rho) \geqslant \lambda_c(\rho_{r_1}) \geqslant \lambda_c(\rho_{r_2})$; hence $\lambda_c(\rho_r)$ tends to some limit $L \leqslant \lambda_c(\rho)$ as $r \to 0$. If $X_{\alpha_r\lambda,\rho_r}$ percolates, then so does $X_{\lambda,\rho}$, again by a coupling argument. This yields $\lambda_c(\rho) \leqslant \lambda_c(\rho_r)/\alpha_r$, which converges to L as $r \to 0$. Therefore, $L = \lambda_c(\rho)$. $\square$

Proposition 1.2. *For any positive random variable ρ we have $\lambda_H(\rho) = \lambda_c(\rho) = \lambda_{\#}(\rho)$.*

Proof. It is clear that when U is bounded, its Lebesgue measure is finite. This yields $\lambda_H(\rho) \geqslant \lambda_c(\rho)$. For the reverse inequality, first consider ρ_r. In this case, the radius random variable is bounded below by r, and it is simple to see that whenever the component of the origin is unbounded its Lebesgue volume must be infinite. Therefore $\lambda_H(\rho_r) \leqslant \lambda_c(\rho_r)$ for all $r > 0$ and hence $\lambda_H(\rho) \leqslant \lambda_c(\rho)$ follows by Lemma 1.1. For the second equality, note that when U is unbounded, $\#(U)$ has to be infinite; if U is bounded, the local finiteness of the Poisson process implies that $\#(U) < \infty$. $\square$

We introduce dynamics into the Boolean model by letting the balls move around. Let $(\boldsymbol{W}(t): t \geqslant 0)$ be a stochastic process taking values in $\mathbb{R}^d$, with $\boldsymbol{W}(0) = \boldsymbol{0}$. We denote by $(X_{\lambda,\rho}(t): t \geqslant 0)$ the process where the balls of $X_{\lambda,\rho}$ move independently (of each other and of the initial configuration) so that the displacement of the centre of any ball from its original position is distributed as W. As in Häggström et al. (1997) a time-stationarity condition is satisfied. The proof requires a little more work in our case, but as it is a standard application of the Mapping Theorem for Poisson processes, we shall omit it.

Proposition 1.3. (i) *For any process W, the distribution of $X_{\lambda,\rho}(t)$ is constant in time.*
(ii) *If $(W(s+\cdot)-W(s))$ has the same distribution for any $s \geqslant 0$, then*

$$(X_{\lambda,\rho}(t)\colon t \geqslant 0)$$

is time-stationary.

Let $\mathscr{C}(t)$ be the event that the occupied region associated with $X_{\lambda,\rho}(t)$ contains an unbounded component. Assuming for the moment that the distribution of $X_{\lambda,\rho}(t)$ is constant in time, so is $\boldsymbol{P}_{\lambda,\rho}(\mathscr{C}(t))$. Therefore

$$\boldsymbol{P}_{\lambda,\rho}(\mathscr{C}(t)) = \begin{cases} 1 & \text{if } \lambda > \lambda_c(\rho), \\ 0 & \text{if } \lambda < \lambda_c(\rho), \end{cases}$$

for any fixed $t \geqslant 0$, and again

$$\begin{cases} \boldsymbol{P}_{\lambda,\rho}(\mathscr{C}(t) \text{ occurs for Lebesgue-a.e. } t) = 1 & \text{if } \lambda > \lambda_c(\rho) \\ \boldsymbol{P}_{\lambda,\rho}(\neg\mathscr{C}(t) \text{ occurs for Lebesgue-a.e. } t) = 1 & \text{if } \lambda < \lambda_c(\rho) \end{cases}$$

follows from Fubini's theorem. Our central question is whether the 'almost every t' of the above statements can be replaced by 'every t'.

Theorem 1.4. *Suppose W is a.s. continuous at 0 and that $(W(s+\cdot)-W(s))$ has the same distribution for any $s \geqslant 0$. If $\lambda > \lambda_H(\rho) = \lambda_c(\rho)$ then $\boldsymbol{P}_{\lambda,\rho}(\mathscr{C}(t))$ occurs for every $t) = 1$.*

Theorem 1.5. *Suppose W satisfies the conditions of Theorem 1.4, and suppose in addition that*

$$E(\rho^{2d}) < \infty \tag{1}$$

and

$$\boldsymbol{E}_{\lambda,\rho}\left(\left(\max_{0 \leqslant s \leftarrow t} |W(s)|\right)^{2d}\right) < \infty \tag{2}$$

for all $t > 0$. If $\lambda < \lambda_T(\rho)$ then $\boldsymbol{P}_{\lambda,\rho}(\neg\mathscr{C}(t)$ occurs for every $t) = 1$.

The proofs of Theorems 1.3 and 1.4 are given in Sections 3 and 4 respectively.

Remarks

1. The conditions on W above are clearly satisfied when W is Brownian motion.
2. If in Theorem 1.5 we additionally suppose that $\boldsymbol{E}_{\lambda,\rho}(\#(U)) < \infty$, then we do not need condition (1). This follows from a result of Hall (1985).

3. It is not immediately obvious that the events in Theorems 1.4 and 1.5 are measurable. The argument here, however, is similar to the discrete case; we refer the reader to Section 2 of Häggström et al. (1997) for the details.
4. Recall that $\lambda_T(\rho) = \lambda_H(\rho)$ when ρ is bounded. Therefore in this case we may replace $\lambda_T(\rho)$ by $\lambda_H(\rho)$ in Theorem 1.4; but when ρ is not bounded there is possibly a gap between these critical values.
5. If $E(\rho^d) = \infty$, then the whole space is occupied a.s. for any choice of $\lambda > 0$ (see Proposition 3.1 in Meester and Roy (1996)). It is not hard to see that in this case, $\boldsymbol{P}_{\lambda,\rho}$ (the whole space is covered for every t) $= 1$.

As an immediate corollary we have:

Corollary 1.6. *Let $\boldsymbol{W}$ be Brownian motion and suppose that ρ is bounded above. Then, if $\lambda < \lambda_c(\rho)$ we have*

$$\boldsymbol{P}_{\lambda,\rho}(\neg\mathscr{C}(t) \text{ occurs for every } t) = 1,$$

and if $\lambda > \lambda_c(\rho)$ we have

$$\boldsymbol{P}_{\lambda,\rho}(\mathscr{C}(t) \text{ occurs for every } t) = 1.$$

Before we go on, we fix some notation: $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, and $M_t(\boldsymbol{W}) := \max_{0\leqslant s\leqslant t}|\boldsymbol{W}(s)|$. $S_r(\boldsymbol{x})$ is the Euclidean sphere $\{y \in \mathbb{R}^d\colon |\boldsymbol{x} - y| \leqslant r\}$; for convenience we write S_r for $S_r(\boldsymbol{0})$. Lebesgue measure in $\mathbb{R}^d$ is denoted ℓ, and V_r is defined to equal $\ell(S_r)$. Given a random element Y we write μ_Y for the distribution of Y.

2. The supercritical phase

In this section we present a proof of Theorem 1.4.

We first demonstrate that it is sufficient to prove Theorem 1.4 for ρ bounded away from zero. Suppose $\lambda > \lambda_c(\rho)$. According to Lemma 1.1, there exists $r > 0$ with $\lambda > \lambda_c(\rho_r)/\alpha_r$. Therefore if we have proved Theorem 1.4 for ρ bounded below it follows that

$$\boldsymbol{P}_{\alpha_r\lambda,\rho_r}(\mathscr{C}(t) \text{ occurs for every } t) = 1.$$

Now observe that we can couple $X_{\alpha_r\lambda,\rho_r}(t)$ and $X_{\lambda,\rho}(t)$ for all $t \geqslant 0$ simultaneously such that the occupied region in $X_{\alpha_r\lambda,\rho_r}(t)$ is contained in the occupied region of $X_{\lambda,\rho}(t)$; indeed, we can couple so that at time 0 the balls of $X_{\alpha_r\lambda,\rho_r}$ form a subset of the balls of $X_{\lambda,\rho}$, and corresponding balls perform the same movements. This yields

$$\boldsymbol{P}_{\lambda,\rho}(\mathscr{C}(t) \text{ occurs for every } t) = 1,$$

as desired.

So suppose now $\lambda > \lambda_c(\rho)$ and $\rho \geqslant r_0 > 0$. Our aim is to find a Boolean model X on the same probability space, and a time $t > 0$ such that

(i) X percolates almost surely, and
(ii) the occupied region of X is contained in the occupied region of $X_{\lambda,\rho}(s)$ for all $0 \leqslant s \leqslant t$.

This then yields $\boldsymbol{P}_{\lambda,\rho}(\mathscr{C}(s)$ occurs for all $0 \leqslant s \leqslant t) = 1$ and therefore it follows from Proposition 1.3 that, for each $s_0 \geqslant 0$,

$$\boldsymbol{P}_{\lambda,\rho}(\mathscr{C}(s) \text{ occurs for all } s \in [s_0, s_0 + t]) = 1.$$

The proof is then complete since $\mathbb{R}^+$ is a countable union of intervals of length t.

The desired t and X are chosen as follows. Firstly, choose $\alpha < 1$ such that $\alpha^d \lambda > \lambda_c(\rho)$. By a.s. continuity of $\boldsymbol{W}$ at 0, and monotone convergence, there exists $t > 0$ such that

$$\pi_t := \boldsymbol{P}(M_t(\boldsymbol{W}) \leqslant (1-\alpha) r_0)$$

satisfies $\pi_t \alpha^d \lambda > \lambda_c(\rho)$. Next, colour the (labelled) Poisson points $(\boldsymbol{x}, r)$ of $X_{\lambda,\rho}$: a point is coloured blue if the motion associated with it satisfies $M_t(\boldsymbol{W}) \leqslant (1-\alpha) r_0$; otherwise, it is coloured red. Since the motions performed by different points are independent (of each other and of the positions of the points), the same is true for the colouring. Therefore, the set of blue points is itself a Boolean model, namely $X_{\pi_t \lambda, \rho}$. The spheres $S_r(\boldsymbol{x})$ represented by these points are those that during the time period $[0, t]$ never move further than $(1-\alpha) r_0 \leqslant (1-\alpha) r$ away from their starting position; thus, each point on the circumference of such a sphere remains at a distance at least αr from the original centre $\boldsymbol{x}$. Now we set $X := X_{\pi_t \lambda, \alpha\rho}$, i.e. X is derived from $X_{\pi_t \lambda, \rho}$ by multiplying the radii of all balls by a factor α. This X clearly satisfies (ii) above. To see that it also satisfies (i), note that a simple scaling argument shows that $\boldsymbol{P}_{\pi_t \lambda, \alpha\rho}(\mathscr{C}) = \boldsymbol{P}_{\pi_t \alpha^d \lambda, \rho}(\mathscr{C})$, which equals 1 because of our choice of t. $\square$

3. The subcritical phase

In the supercritical phase, the idea was to find a Boolean model X and a time $t > 0$, such that X has an unbounded component almost surely, and X is simultaneously 'dominated' by $X_{\lambda,\rho}(s)$ for all $0 \leqslant s \leqslant t$. For the subcritical phase we apply a similar line of attack. Here we wish to find a Boolean model with no unbounded component, which simultaneously dominates $X_{\lambda,\rho}(s)$ for all $0 \leqslant s \leqslant t$.

It again suffices to prove Theorem 1.5 for ρ bounded below by some $r_0 > 0$. The proof of this is just as simple as in the supercritical regime and is omitted. In this section we assume that ρ is bounded below by $r_0 > 0$ and that the hypotheses of Theorem 1.5 hold.

For $t \geqslant 0$ let $R(t) = R_\rho(t)$ be defined as $R_\rho(t) := \rho + M_t(\boldsymbol{W})$, where the two terms in the r.h.s. are considered as *independent* random variables. In particular, $R_\rho(0) = \rho$. Conditions (1) and (2) now reduce to $\boldsymbol{E}_{\lambda,\rho}(R(t)^{2d}) < \infty$ for all $t \geqslant 0$. Consider the

Boolean models $X_{\lambda,R(t)}$ for $t \geq 0$. We can couple $X_{\lambda,\rho}(s)$, $0 \leq s \leq t$, and $X_{\lambda,R(t)}$ in the obvious way (as in the previous section), such that if $X_{\lambda,R(t)}$ does not percolate, then neither do the models $X_{\lambda,\rho}(s)$. So it suffices for Theorem 1.5 to prove the following proposition.

Proposition 3.1. *Suppose $\lambda < \lambda_T(\rho)$. Then there exists $t > 0$ such that $\boldsymbol{P}_{\lambda,R(t)}(\mathscr{C}) = 0$.*

For the proof of this proposition, we need some additional concepts and notation.

An event A is said to be *increasing* if the following is true: whenever a realisation is in A and we add Poisson points (with associated balls), the resulting configuration will still be in A. We say that an event A *lives on a set* $U \subset \mathbb{R}^d$ if it is measurable with respect to the points in U (and their associated balls); i.e. it is possible to decide whether or not A occurs by just looking at the Poisson points in U and their associated balls. For two increasing events A and B we say that 'A and B occur disjointly' if there exist two disjoint sets of Poisson points such that any configuration which contains the first set of points (with their associated balls) is in A and any configuration which contains the second set of points is in B. We write this event as $A \square B$. More details can be found in van den Berg (1996) or Meester and Roy (1996). The following inequality, proved in van den Berg (1996), is a continuum version of the standard BK inequality.

Lemma 3.2. *Suppose U is a bounded measurable set in $\mathbb{R}^d$ and A and B are two increasing events living on U. Then $\boldsymbol{P}_{\lambda,\rho}(A \square B) \leq \boldsymbol{P}_{\lambda,\rho}(A)\boldsymbol{P}_{\lambda,\rho}(B)$.*

Let $\boldsymbol{x}$, $\boldsymbol{y}$ be points in $\mathbb{Z}^d$ and n a positive integer. We denote by Q_x the cube $\boldsymbol{x} + [0,1)^d$. For ω a realisation of an arbitrary Boolean model $X_{\lambda,\sigma}$, we say that $\omega \in (\boldsymbol{x} \overset{n}{\sim} \boldsymbol{y})$ if there are distinct balls $B_1, \ldots, B_n$ such that the sets $Q_x \cap B_1$, $Q_y \cap B_n$ and $B_i \cap B_{i+1}$ (for $i = 1, \ldots, n-1$) are all nonempty. Let $E_{\lambda,\sigma}(n)$ be the expected number of points $\boldsymbol{x} \in \boldsymbol{Z}^d$ for which $(\boldsymbol{0} \overset{n}{\sim} \boldsymbol{x})$ occurs in $X_{\lambda,\sigma}$.

We define the event $(\boldsymbol{x} \overset{n}{\sim} *)$ by

$$(\boldsymbol{x} \overset{n}{\sim} *) = \bigcup_{\boldsymbol{y} \in \boldsymbol{Z}^d} (\boldsymbol{x} \overset{n}{\sim} \boldsymbol{y}). \tag{3}$$

It is intuitively clear that if $\boldsymbol{x}$ percolates in ω (i.e. the occupied component containing $\boldsymbol{x}$ is unbounded) then $\omega \in (\boldsymbol{x} \overset{n}{\sim} *)$ for every $n \geq 1$, and we omit the elementary proof of this fact.

We now proceed with the proof of Proposition 3.1. We first state our key lemma, Lemma 3.3 below, and then show how the proposition follows from it. The rest of the section is then devoted to the proof of this lemma.

Lemma 3.3. *Suppose $\lambda < \lambda_T(\rho)$. Then there exist $t > 0$ and $n \geq 1$ with $E_{\lambda,R(t)}(n) < 1$.*

Proof of Proposition 3.1. The proof is based on a suitable adaptation of results by Hammersley (1957) and van den Berg and Kesten (see Corollary 3.18 in van den Berg and Kesten (1985)).

We will show that for t as in the above lemma, $\boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{m}{\sim} *)$ converges to zero as $m \to \infty$. In light of the remark following (3) this yields

$$\boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \text{ percolates}) = 0.$$

Hence by stationarity,

$$\boldsymbol{P}_{\lambda,R(t)}(\boldsymbol{x} \text{ percolates}) = 0$$

for any $\boldsymbol{x} \in \mathbb{Q}^d$ (the set of points with rational coordinates). The result then follows since

$$\mathscr{C} = \bigcup_{\boldsymbol{x} \in \mathbb{Q}^d} (\boldsymbol{x} \text{ percolates}).$$

First, suppose that for some Boolean model and some $m, n \geq 1$ the event $(\mathbf{0} \overset{n+m}{\sim} *)$ occurs, that is, there are distinct balls $B_1, \ldots, B_{m+n}$ such that the sets $Q_0 \cap B_1$ and $B_i \cap B_{i+1}$ (for $i = 1, \ldots, m+n-1$) are all nonempty. Since the Q_z partition $\mathbb{R}^d$, there exists some $z \in \boldsymbol{Z}^d$ such that $Q_z \cap B_n \cap B_{n+1} \neq \emptyset$, thus the events $(z \overset{m}{\sim} *)$ and $(\mathbf{0} \overset{n}{\sim} z)$ occur disjointly. So we have

$$(\mathbf{0} \overset{m+n}{\sim} *) \subset \bigcup_{z \in \mathbb{Z}^d} ((\mathbf{0} \overset{n}{\sim} z) \,\square\, (z \overset{m}{\sim} *)). \tag{4}$$

Now let t and n be as in Lemma 3.3. By (4),

$$\boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{m+n}{\sim} *) \leq \sum_{z \in \mathbb{Z}^d} \boldsymbol{P}_{\lambda,R(t)}((\mathbf{0} \overset{n}{\sim} z) \,\square\, (z \overset{m}{\sim} *)). \tag{5}$$

Next we apply the BK inequality Lemma 3.2 to the r.h.s of (5). Strictly speaking we cannot do this immediately since the events here do not live on bounded sets. However, the procedure to overcome this difficulty is rather standard: one has to approximate the events in the above expression by events which do live on bounded subsets of the space. This then yields

$$\boldsymbol{P}_{\lambda,R(t)}((\mathbf{0} \overset{n}{\sim} z) \,\square\, (z \overset{m}{\sim} *)) \leq \boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{n}{\sim} z) \boldsymbol{P}_{\lambda,R(t)}(z \overset{m}{\sim} *). \tag{6}$$

Further, by stationarity, the last factor in (6) equals $\boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{m}{\sim} *)$. Combining (5) and (6) and using the definition of $E_{\lambda,R(t)}(n)$, we have

$$\boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{m+n}{\sim} *) \leq \boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{m}{\sim} *) E_{\lambda,R(t)}(n). \tag{7}$$

But $E_{\lambda,R(t)}(n) < 1$, thus $\boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{m}{\sim} *)$ decays geometrically to zero as $m \to \infty$. □

Remark. Hammersley (1957) and van den Berg and Kesten (1985) use, instead of the lattice analogue of the event $(\mathbf{0} \overset{n}{\sim} *)$, the event that there is an open path between $\mathbf{0}$ and some vertex at distance at least n from $\mathbf{0}$. However, the continuum analogue of the latter appeared not suitable for our purpose.

It remains now only to prove Lemma 3.3. The proof proceeds by a series of five further lemmas.

Lemma 3.4. *Let $X_{\lambda,\sigma}$ be any Boolean model and let $c > 0$. Let $\Sigma_{\lambda,\sigma}(c)$ denote the sum of the volumes of those $X_{\lambda,\sigma}$-balls which intersect S_c. Then $E(\Sigma_{\lambda,\sigma}(c)) = \lambda E(V_\sigma V_{\sigma+c})$.*

Proof. Define $f: \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ by $f(x,r) = V_r \mathbf{1}\{r \geqslant |x| - c\}$. Then considering $X_{\lambda,\sigma}$ as a Poisson process in $\mathbb{R}^d$ labelled by radii, $\Sigma_{\lambda,\sigma}(c)$ is the sum of the values of f over all the (labelled) Poisson points. Thus, by Campbell's Theorem (see Kingman (1993)).

$$
\begin{aligned}
E(\Sigma_{\lambda,\sigma}(c)) &= \lambda \int_{x \in \mathbb{R}^d} \int_{r > 0} f(x,r)\, \mathrm{d}\mu_\sigma(r)\, \mathrm{d}x \\
&= \lambda \int_{r > 0} \int_{|x| \leqslant r + c} V_r\, \mathrm{d}x\, \mathrm{d}\mu_\sigma(r) \\
&= \lambda \int_{r > 0} V_r V_{r+c}\, \mathrm{d}\mu_\sigma(r) \qquad \square
\end{aligned}
$$

Lemma 3.5. *Let $\mathscr{H}$ be a countable collection of balls in $\mathbb{R}^d$. Write $H \subset \mathbb{R}^d$ for the union of all balls in $\mathscr{H}$, and let $\nu(H)$ be the number of cubes Q_z ($z \in \mathbb{Z}^d$) with $H \cap Q_z \neq \emptyset$. If $r_0 > 0$ is a lower bound for the radii of the balls in $\mathscr{H}$ then there exists some constant $K = K(r_0, d)$ such that*

$$
\nu(H) \leqslant K\ell(H) \leqslant K \sum_{B \in \mathscr{H}} \ell(B). \tag{8}
$$

Proof. Let $\tilde{Q}_z$ denote the 'augmented cube' $z + [-1,2)^d$. It is clear that if $Q_z \cap H$ is nonempty then $\ell(\tilde{Q}_z \cap H) \geqslant M(r_0, d)$, where $M(r_0,d)$ is some positive constant. Hence

$$
\begin{aligned}
\nu(H) &= \sum_{z: Q_z \cap H \neq \emptyset} 1 \\
&\leqslant \sum_{z \in \mathbb{Z}^d} \frac{\ell(\tilde{Q}_z \cap H)}{M(r_0,d)} \\
&= \frac{3^d}{M(r_0,d)} \ell(H),
\end{aligned}
$$

where the last line follows since the $\tilde{Q}_z$ cover each point of $\mathbb{R}^d$ exactly 3^d times.

Lemma 3.6. *Suppose $\lambda < \lambda_T(\rho)$. Then $E_{\lambda,\rho}(n)$ converges to zero as $n \to \infty$.*

Proof. We have

$$
E_{\lambda,\rho}(n) = \boldsymbol{E}_{\lambda,\rho}\left(\sum_{x \in \mathbb{Z}^d} \mathbf{1}(\mathbf{0} \overset{n}{\sim} x) \right) \leqslant \boldsymbol{E}_{\lambda,\rho}(\nu(U)\mathbf{1}(\#(U) \geqslant n)).
$$

But by assumption, $\boldsymbol{E}_{\lambda,\rho}(\ell(U))$ is finite and hence, by (8), so is $\boldsymbol{E}_{\lambda,\rho}(\nu(U))$. Moreover, $\lambda < \lambda_T(\rho) \leqslant \lambda_H(\rho) = \lambda_\#(\rho)$ by Lemma 1.2 and therefore $\#(U)$ is also a.s. finite. Thus, the result follows by dominated convergence. $\square$

Lemma 3.7. *$E_{\lambda,R(t)}(n)$ is finite for all positive λ, n and all $t \geqslant 0$.*

Proof. By an argument similar to that for Proposition 3.1 above,

$$\begin{aligned} E_{\lambda,R(t)}(n+1) &= \sum_{x \in \mathbb{Z}^d} P_{\lambda,R(t)}(\mathbf{0} \overset{n+1}{\sim} x) \\ &\leqslant \sum_{x,z \in \mathbb{Z}^d} P_{\lambda,R(t)}(\mathbf{0} \overset{1}{\sim} z) P_{\lambda,R(t)}(z \overset{n}{\sim} x) \\ &= \sum_{z \in \mathbb{Z}^d} P_{\lambda,R(t)}(\mathbf{0} \overset{1}{\sim} z) \sum_{x \in \mathbb{Z}^d} P_{\lambda,R(t)}(\mathbf{0} \overset{n}{\sim} x). \end{aligned}$$

Thus

$$E_{\lambda,R(t)}(n+1) \leqslant E_{\lambda,R(t)}(1) E_{\lambda,R(t)}(n),$$

and by induction we then find

$$E_{\lambda,R(t)}(n) \leqslant (E_{\lambda,R(t)}(1))^n.$$

But according to (8) and Lemma 3.4, for some constant K,

$$\begin{aligned} E_{\lambda,R(t)}(1) &\leqslant K\boldsymbol{E}(\Sigma_{\lambda,R(t)}(\sqrt{d})) = \lambda K \boldsymbol{E}(V_{R(t)} V_{R(t)+\sqrt{d}}) \\ &\leqslant \lambda K \boldsymbol{E}((V_{R(t)+\sqrt{d}})^2). \end{aligned}$$

The result now follows since, by assumption, $\boldsymbol{E}(R(t)^{2d}) < \infty$. □

Lemma 3.8. *For any $z \in \mathbb{Z}^d$ and $n \geqslant 1$, we have*

$$\lim_{t \to 0} P_{\lambda,R(t)}(\mathbf{0} \overset{n}{\sim} z) = P_{\lambda,\rho}(\mathbf{0} \overset{n}{\sim} z).$$

Proof. Consider an arbitrary Boolean model $X_{\lambda,\sigma}$ and let $z \in \mathbb{Z}^d$ and $n \geqslant 1$. We say that an $X_{\lambda,\sigma}$-ball B is on an n-step path from $\mathbf{0}$ to z if there exists a sequence of distinct $X_{\lambda,\sigma}$-balls $B_1, \ldots, B_n$, one of which is B, such that $Q_0 \cap B_1$, $Q_z \cap B_n$ and $B_i \cap B_{i+1}$, $i = 1, \ldots, n-1$, are all nonempty. Let $\mathscr{H}_{\lambda,\sigma} = \mathscr{H}_{\lambda,\sigma}(n,z)$ denote the (random) collection of all $X_{\lambda,\sigma}$-balls which are on an n-step path from $\mathbf{0}$ to z. Thus $\{\mathscr{H}_{\lambda,\sigma} \neq \emptyset\}$ is the event that $(\mathbf{0} \overset{n}{\sim} z)$ occurs in $X_{\lambda,\sigma}$. It is straightforward to see that on $\{\mathscr{H}_{\lambda,R(1)}$ is infinite$\}$ there exists some (random) bounded subset of $\mathbb{R}^d$ that is intersected by infinitely many $X_{\lambda,R(1)}$-balls. But by Lemma 3.1 in Meester and Roy (1996), this happens with probability zero since $\boldsymbol{E}(R(t)^d) < \infty$. Therefore $\mathscr{H}_{\lambda,R(1)}$ is almost surely finite. Hence, since W is a.s. continuous at $t = 0$, we have (coupling all models so that $\mathscr{H}_{\lambda,R(t)} \supset \mathscr{H}_{\lambda,\rho}$ for all $t > 0$),

$$\sup\{t : \mathscr{H}_{\lambda,R(t)} = \mathscr{H}_{\lambda,\rho}\} > 0 \text{ a.s.}$$

So

$$\mathbf{1}\{(\mathbf{0} \overset{n}{\sim} x) \text{ in } X_{\lambda,R(t)}\} \to \mathbf{1}\{(\mathbf{0} \overset{n}{\sim} x) \text{ in } X_{\lambda,\rho}\} \text{ as } t \to 0 \text{ a.s.},$$

and the result follows. □

Proof of Lemma 3.3. By Lemma 3.6 there exists $n \geqslant 1$ such that $E_{\lambda,\rho}(n) < \frac{1}{3}$. By Lemma 3.7,

$$E_{\lambda,R(1)}(n) = \sum_{x \in \mathbb{Z}^d} \boldsymbol{P}_{\lambda,R(1)}(\mathbf{0} \overset{n}{\sim} \boldsymbol{x}) < \infty,$$

so there exists $K > 0$ such that

$$\sum_{x \in \mathbb{Z}^d \setminus S_K} \boldsymbol{P}_{\lambda,R(1)}(\mathbf{0} \overset{n}{\sim} \boldsymbol{x}) < \tfrac{1}{3}.$$

Finally, by Lemma 3.8 there exists $t \in (0,1]$ such that

$$\sum_{x \in \mathbb{Z}^d \cap S_K} \boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{n}{\sim} \boldsymbol{x}) < \tfrac{1}{3} + \sum_{x \in \mathbb{Z}^d \cap S_K} \boldsymbol{P}_{\lambda,\rho}(\mathbf{0} \overset{n}{\sim} \boldsymbol{x})$$

$$\leqslant \tfrac{1}{3} + E_{\lambda,\rho}(n).$$

Combining now these arguments, and using monotonicity in t, we have

$$E_{\lambda,R(t)}(n) = \sum_{x \in \mathbb{Z}^d \cap S_K} \boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{n}{\sim} \boldsymbol{x}) + \sum_{x \in \mathbb{Z}^d \setminus S_K} \boldsymbol{P}_{\lambda,R(t)}(\mathbf{0} \overset{n}{\sim} \boldsymbol{x}) < 1. \qquad \square$$

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