CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
Telephone + 31 - 20 592 9333
Telefax + 31 - 20 592 4199
URL http://www.cwi.nl

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Probabilistic and analytical aspects of the umbral calculus

A. Di Bucchianico

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Alessandro Di Bucchianico

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Contents

Chapter 1

Introduction

This introduction consists of two parts. The first part is a historical introduction to the subject of the tract. The second part briefly describes the contents of the chapters of this tract (more elaborate descriptions of the chapters can be found in the introductions of the chapters). The aim of this tract is to study the probabilistic and analytic aspects of the Umbral Calculus. Therefore, the contents of this tract have little overlap with the existing books on Umbral Calculus ([60, 161, 163, 134, 209, 202]), which mainly stress the combinatorial and algebraic aspects of the Umbral Calculus. An interesting book with the same emphasis as the present tract is [91].

I have tried to make this tract as self-contained as possible. I have added Mathematical Reviews references to the items in bibliography at the end of this tract.

Historical introduction

There are quite a number of well-known sequences of polynomials, e.g. those attached to the names of Hermite, Legendre, Laguerre and many others. These sequences can be described in several ways. E.g., they can be described by generating functions, as solutions to differential equations, by orthogonality relations or by recurrence relations. The subject of this tract is a class of sequences of polynomials $(q_n)_{n\in\mathbb{N}}$ defined by the following functional equations

$$q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y) \qquad (n = 0, 1, \dots) \tag{1.1}$$

A sequence of polynomials that satisfies (1.1) is called a sequence of polynomials of convolution type. These sequences are closely related to the sequences of polynomials of binomial type introduced by Rota (see [162] and [210]), i.e.

sequences of polynomials $(p_n)_{n\in\mathbb{N}}$ satisfying

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y) \qquad (n = 0, 1, \ldots) \tag{1.2}$$

The sequence $(x^n)_{n\in\mathbb{N}}$ is of binomial type by the Binomial Theorem, which explains the name binomial type. Obviously, $(q_n)_{n\in\mathbb{N}}$ is of convolution type if and only if $(n!\, q_n)_{n\in\mathbb{N}}$ is of binomial type. Thus these two types are essentially the same. I have chosen to work with sequences of polynomials of convolution type instead of sequences of polynomials of binomial type because convolution is a fundamental operation in analysis and probability theory. The binomial convolution appearing in (1.2) has advantages when dealing with certain combinatorial problems (see [162]).
An extension of the class of sequences of polynomials of binomial/convolution type is the class of Sheffer sequences $(s_n)_{n\in\mathbb{N}}$, whose convolution type version is defined by

$$s_n(x+y) = \sum_{k=0}^{n} s_k(x)\, q_{n-k}(y) \qquad (n = 0, 1, \ldots) \tag{1.3}$$

for some fixed sequence $(q_n)_{n\in\mathbb{N}}$ of convolution type. The class of Sheffer sequences includes (amongst others) the Hermite, Bernoulli and Laguerre polynomials (more examples can be found in [29, 202, 235]).
The history of Sheffer sequences goes back to 1880 when Appell studied sequences $(a_n)_{n\in\mathbb{N}}$ of polynomials satisfying $Da_n = n\, a_{n-1}$ (D is the differentiation operator). Appell showed that these sequences satisfy

$$a_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} a_k(x)\, y^{n-k} \qquad (n = 0, 1, \ldots) \tag{1.4}$$

These sequences are called Appell sequences nowadays (see [32, Chapter 6], [72], [202, Chapter 4], [210, Section 13] or [215]). The Hermite polynomials form an Appell sequence.
The next major step was taken by Sheffer, whose work on difference equations led him in 1939 to generalize the Appell polynomials (see [217] or [72, p. 25]). Sheffer called his generalization polynomial sets of type zero; they are the Sheffer sequences defined by (1.3). The same class of polynomials was introduced in 1941 by Steffensen [227] (see also [220, 219, 227, 228, 229, 226]). There do exist even more general classes of polynomials such as Brenke sequences. These classes will not be considered in this tract; the interested reader is referred to the papers [8, 7, 20, 28, 29, 33, 36, 39, 40, 44, 49, 56, 57, 107, 108, 121, 122, 176].
A very elegant theory of Sheffer sequences is due to Rota and co-authors (see [162, 210]). These two papers are part of a series of papers on combinatorics, namely: [208, 69, 162, 109, 9, 81, 80, 210, 82, 30]. The Rota theory uses linear operators on the vector space of polynomials (cf. [209, Foreword]) and is therefore of a purely algebraic nature. It also provides a rigorous foundation

for the Umbral Calculus (also called Blissard Calculus, see e.g. [23, 113, 185]). The systematic nature of the Rota theory easily yields numerous identities for special polynomials (see [202, Chapter 4] or [210]). The Rota theory of operators rests on earlier work by Pincherle, Steffensen, Toscano and Curry [70, 72, 180, 246]).

An extended and polished form of the Rota theory can be found in [202]. There have been several attempts to generalize the Rota theory. In particular, Roman has extended the Rota theory to include many sequences of polynomials that are not Sheffer sequences, e.g. Jacobi polynomials (see [201]) or q-polynomials (see [202, Section 6.4] or [203]). Roman remarks in [202, Section 6.1] that his ideas go back to 1936 ([246]), but he forgets to mention the work of Viskov (see [242]). Viskov has even an extension of the Rota theory including all sequences of polynomials (see [243, 244], cf. [139, 138, 154]). Cholewinski has adapted the Rota theory to Bessel functions (see [60]). Grabiner has extended the Roman theory to classes of entire functions (see [111, 112]). An application of operator calculus to hypergeometric functions can be found in [240].

For other generalizations of the Rota theory, see [13, 15, 16, 17, 18, 19, 34, 36, 37, 39, 40, 53, 54, 63, 51, 52, 103, 104, 119, 133, 136, 135, 149, 145, 147, 148, 146, 150, 158, 177, 178, 186, 197, 196, 204, 238, 239, 247, 248].

There is a wide range of applications of the Rota theory, e.g. statistics ([164, 168, 170]), combinatorics ([79, 114, 137, 162, 167, 169, 174, 173, 175, 189, 193, 194, 195, 198, 208, 223, 249, 250]), approximation theory ([123, 129, 241, 159, 219, 234]), recurrence relations ([41, 166, 165, 171, 206, 205]), physics ([25, 26, 89, 88, 91, 115, 116, 117, 252]), algebraic topology ([187, 188, 191, 190, 192]) and stochastic processes ([50, 222, 224]).

A survey of the Umbral Calculus with over 400 references can be obtained in electronic form through the Electronic Journal of Combinatorics:

http://ejc.math.gatech.edu:8080/Journal/Surveys/index.html

Contents of this tract

Chapter 2 is an introduction to the Rota Umbral Calculus. It is shown that if $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type, then $q_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$ for some sequence $(g_n)_{n\in\mathbb{N}}$ with $g_0 = 0$. This aspect is only implicitly present in the work of Rota. However, it will be shown in Chapter 2 that the coefficient sequence $(g_n)_{n\in\mathbb{N}}$ is important for the theory of polynomials of convolution type. The section on cross sequences and Steffensen sequences brings together several results scattered in the literature.

Chapter 3 contains a miscellany of applications of the Umbral Calculus. Topics covered include finite probability distributions, combinatorial identities, exponential families, approximation operators, orthogonal polynomials, semigroups of probability measures, and integral representations of shift-invariant operators. These sections are partly based on [75, 77].

Chapter 4 starts with some general Banach algebra theory. This theory is used to obtain a new, unified treatment of existence problems for logarithms. This treatment is applied to polynomials of convolution type and yields analytic results on the generating function of sequences of polynomials of convolution type. Moreover, a two-sided analogue of polynomials of convolution type is introduced and studied. This section extends the results of [73].
The first sections of Chapter 5 consider central limit theorems for the coefficients of polynomials of convolution type as in [48, 55, 96, 97, 222, 224]. Results by Stam ([222, 224] are extended to the case of non-negative coefficients. The last section of Chapter 5 concerns an application of the theory of Chapter 4 to the theory of infinitely divisible probability measures on $\mathbb{N}$. It is shown that the Banach algebra techniques used in the literature (in particular, those by Chover, Ney and Wainger [61]) can be simplified considerably. Moreover, we give a simple proof of a result by Embrechts and Hawkes [87] on subexponential sequences.

Chapter 2

Umbral Calculus

This chapter is an introduction to Rota's Umbral Calculus as presented in [210]. For reasons explained in the introduction, we use polynomials of convolution type instead of binomial type.
Rota and his co-authors used operators together with formal power series in their papers [162, 210]. Although a rigorous foundation of formal power series exists (see e.g. [172]), we prefer to use operators only to develop the basic theory (see Section 2.2). More important, however, is our emphasis in this chapter on the coefficient sequence (Definition 2.1.11). The coefficient sequence is only implicitly present in [210]. Another feature of our approach is the use of elementary operator methods.

In Section 2.1 we study the system of convolution equations that defines the polynomials of convolution type. Sections 2.2 and 2.3 are an introduction to delta operators and polynomials of convolution type. The concept of polynomials of convolution type is generalized to Sheffer polynomials in Section 2.4 and to cross and Steffensen sequences in Section 2.5. Examples are included to illustrate the theory; systematic presentations of examples can be found in [202, Chapter 4] and [235].

Contents of Chapter 2

Notation and conventions

The degree of a polynomial, notation: $\deg p$, is defined as usual, however, the degree of a nonzero constant is defined to be zero and the degree of the zero polynomial is defined to be -1.
$\mathbb{N}$ is defined to be the set $\{0, 1, 2, \ldots\}$.
The vector space of polynomials with coefficients in some fixed commutative ring, is denoted by $\mathcal{P}$. The commutative rings that we will use are the reals, the integers and the complex numbers.

2.1 A convolution equation

In this section we study the following system of equations:

$$f_n(x+y) = \sum_{k=0}^{n} f_k(x)\, f_{n-k}(y) \qquad (n = 0, 1, \ldots), \tag{2.1}$$

where each f_n ($n \in \mathbb{N}$) is defined on a semigroup $\mathcal{S}$ and takes values in a commutative ring $\mathcal{R}$. This general setting enables us to prove the necessary results for all semigroups of interest to us (the reals, the positive reals, the natural numbers, etc.) at the same time.

These equations come up at several places:

- transition probabilities of stochastic processes ([126]): let $N(t)_{t\geq 0}$ be a stationary stochastic process with independent increments. If $f_n(x) = P(N(x) = n)$, then (2.1) follows by conditioning $N(x+y)$ on $N(x)$. For further references, see [1, pp. 111-116] and [3, Chapter 12])
- semigroups of convolution operators on sequence spaces: let $(T_t)_{t>0}$ be a semigroup of convolution operators on some Banach space X of one-sided sequences. Then $(T_t x)_n = \sum_{k=0}^{n} f_k(t)\, x_{n-k}$ for some sequence of functions $(f_n)_{n\in\mathbb{N}}$ and (2.1) follows from the semigroup property.
- combinatorics: let $f_n(x)$ denote the number of functions with some specified property (*e.g.*, injectivity) from an n-element set to an x-element set. Then (2.1) follows by partitioning an $x+y$-element set into two disjoint sets (see [162]).

All bounded solutions for $\mathcal{S} = \mathcal{R} = (0, \infty)$ have been determined in [126], where (2.1) is related to transition probabilities of a stationary stochastic process with independent increments. It turns out that the solutions are given by so-called compound Poisson processes. A general study of the system of equations (2.1) was undertaken by Aczél and collaborators (see e.g. [2], [3, Chapter 12],[4]). We give a less general self-contained treatment which suffices for our purposes.

Definition 2.1.1 *A sequence $(f_n)_{n\in\mathbb{N}}$ of functions on a semigroup S and taking values in a commutative ring $\mathcal{R}$ is* **a sequence of functions of convolution type** *if $(f_n)_{n\in\mathbb{N}}$ satisfies (2.1) for all $n \in \mathbb{N}$ and all $x, y \in S$. If each f_n is a polynomial, then $(f_n)_{n\in\mathbb{N}}$ is* **a sequence of polynomials of convolution type**.

The interpretation of (2.1) in terms of transition probabilities of compound Poisson processes suggests that f_n must be of the form $e^{-\lambda x} \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$. Before we continue to determine the general solution of (2.1), we define the numbers g_n^{k*} and give some properties.

Definition 2.1.2 *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ and $\beta = (\beta_n)_{n\in\mathbb{N}}$ be sequences in a commutative ring $\mathcal{R}$. The convolution $\alpha * \beta$ is the sequence defined by $(\alpha * \beta)_n := \sum_{k=0}^{n} \alpha_k \beta_{n-k}$.*
If $k \in \mathbb{N}$, then α^{k*} is defined recursively as follows: $\alpha^{0*} := (\delta_{0n})_{n\in\mathbb{N}}$ (δ_{0n} is the Kronecker delta) and $\alpha^{(k+1)*} := \alpha^{k*} * \alpha$.
For sake of brevity, we will write α_n^{k*} instead of $(\alpha^{k*})_n$.

Remarks 2.1.3 Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a sequence in a commutative ring $\mathcal{R}$.
a) If $\sum_{n=0}^{\infty} \alpha_n z^n$ is a formal power series, then α_n^{k*} is the coefficient of z^n in $(\sum_{n=0}^{\infty} \alpha_n z^n)^k$. In other words,

$$\alpha_n^{k*} = \sum_{i_1+\cdots+i_k=n} \alpha_{i_1} \cdots \alpha_{i_k}.$$

b) It follows directly from Definition 2.1.2 that $\alpha^{1*} = \alpha$ and $\alpha^{2*} = \alpha * \alpha$.
c) Note that the convolution operation is commutative and associative. Associativity implies $\alpha^{i*} * \alpha^{j*} = \alpha^{(i+j)*}$ for all $i, j \in \mathbb{N}$. In particular, taking $j = k - i$, we obtain

$$\alpha_n^{k*} = \sum_{n=0}^{\infty} \alpha_m^{i*} \alpha_{n-m}^{(k-i)*}, \ (0 \le i \le k). \tag{2.2}$$

d) Let us prove the useful fact that α_n^{k*} is a polynomial in $\alpha_0, \ldots, \alpha_n$ for all $k \ge 1$ and all $n \in \mathbb{N}$. We proceed by induction on k. The statement holds for $k = 1$. Suppose by induction that the statement is true at k. Then $\alpha_n^{(k+1)*} = \sum_{n=0}^{\infty} \alpha_m \alpha_{n-m}^{k*}$ is a polynomial in $\alpha_0, \ldots, \alpha_n$. This completes the proof.
It follows that, if $\beta_0, \beta_1, \ldots, \beta_N$ is a finite sequence in $\mathcal{R}$, then β_n^{k*} is well-defined for $n \le N$ and all $k \in \mathbb{N}$.
e) Formula (2.1) is a system of convolution equations, because it says that $(f_n(x+y))_{n\in\mathbb{N}}$ is the convolution of the sequences $(f_n(x))_{n\in\mathbb{N}}$ and $(f_n(y))_{n\in\mathbb{N}}$. If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of functions of convolution type, then $(f_n)_{n\in\mathbb{N}}$, defined by $f_n(x) := \sum_{k=0}^{n} a_k(x) b_{n-k}(x)$ for all $x \in S$, is also a sequence of functions of convolution type, since convolution is an associative and commutative operation.

Remark 2.1.4 If $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers such that $\sum_{n=0}^{\infty}\alpha_n = 1$, then α_n^{k*} has the following probabilistic interpretation. Let X_i, $i = 1, 2, \ldots$ be independent identically distributed random variables on some probability space $(\Omega, \mathcal{P}, \mathcal{F})$ with $P(X_i = n) = \alpha_n$ for $i = 1, 2, \ldots$ and $n \in \mathbb{N}$. Define $S_k := X_1 + \cdots + X_k$ $(k = 1, 2, \ldots)$. Using Remark 2.1.3a, it is easy to see that $P(S_k = n) = \alpha_n^{k*}$. Suppose $\alpha_0 = 0$. It follows from $P(S_k = k) = P(X_1 =, X_2 = \ldots = X_k = 1)$ that $\alpha_k^{k*} = (\alpha_1)^k$. If $k > n$, then $P(S_k = n) = 0$, since $P(X_i \geq 1) = 1$ for $i = 1, 2, \ldots$. Hence, $\alpha_n^{k*} = 0$ if $k > n$.

Formula (2.2) can be interpreted as conditioning on S_i, i.e.

$$P(S_k = n) = \sum_{n=0}^{\infty} P(S_k = n \cap S_i = m) = \sum_{n=0}^{\infty} P(S_i = m)\, P(S_{k-i} = n - m).$$

The following lemma shows that the two properties mentioned in Remark 2.1.4 are also true under a more general condition.

Lemma 2.1.5 *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in a commutative ring such that $\alpha_0 = 0$. Then:*

a) $\alpha_n^{k*} = 0$ *if* $k > n$ $(k, n \in \mathbb{N})$.

b) $\alpha_n^{n*} = (\alpha_1)^n$ *for all* $n \in \mathbb{N}$.

c) α_n^{k*} *is a polynomial in* $\alpha_1, \ldots, \alpha_{n-1}$ *for* $2 \leq k \leq n$ $(k, n \in \mathbb{N})$.

Proof: a) We apply induction on k. The statement is true for $k = 0$. Suppose by induction that the statement is true at k. Then $\alpha_{n-m}^{k*} = 0$ for $k + 1 > n$ and $m \geq 1$. Hence, $\alpha_n^{(k+1)*} = \sum_{n=0}^{\infty} \alpha_m\, \alpha_{n-m}^{k*} = \alpha_0\, \alpha_n^{k*} = 0$ since $\alpha_0 = 0$.
b) We apply induction on n. The statement is true for $n = 0$, since $\alpha_0^{0*} = 1$ by definition. Suppose by induction that $\alpha_n^{n*} = (\alpha_1)^n$. It follows from a) and $\alpha_0 = 0$ that $\alpha_{n+1}^{(n+1)*} = \sum_{m=0}^{n+1} \alpha_m\, \alpha_{n+1-m}^{n*} = \alpha_1\, \alpha_n^{n*} = (\alpha_1)^{n+1}$.
c) We proceed by induction on k. The statement is true for $k = 2$, since $\alpha_n^{2*} = \sum_{i=0}^{n} \alpha_i\, \alpha_{n-i}$. Suppose by induction that the statement is true at k. Then Formula (2.2) yields $\alpha_n^{(k+1)*} = \sum_{n=0}^{\infty} \alpha_m\, \alpha_{n-m}^{k*}$, which equals $\sum_{m=1}^{n-k} \alpha_m\, \alpha_{n-m}^{k*}$ since $\alpha_0 = 0$ and $\alpha_{n-m}^{k*} = 0$ for $m > n - k$ by a). □

We are now ready to derive the general form of sequences of functions of convolution type (Definition 2.1).

Lemma 2.1.6 *Let $g_0, g_1, \ldots, g_N$ be elements of a commutative ring $\mathcal{R}$. Let $h_0, h_1, \ldots, h_N$ be functions from a semigroup $\mathcal{S}$ to $\mathcal{R}$ such that for fixed $x, y \in \mathcal{S}$,*

$$h_n(x + y) = \sum_{k=0}^{n} h_k(x)\, h_{n-k}(y)$$

for all $n = 0, 1, \ldots, N$. *Define* $f_n(x) := \sum_{k=0}^{n} g_n^{k*}\, h_k(x)$ $(n = 0, 1, \ldots, N)$. *Then*

$$f_n(x+y) = \sum_{k=0}^{n} f_k(x)\, f_{n-k}(y)$$

for all $0 \leq n \leq N$.

Proof: This follows from direct substitution and the following form of (2.2):

$$\sum_{k=0}^{n} g_k^{i*}\, g_{n-k}^{j*} = g_n^{(i+j)*}.$$

□

Lemma 2.1.7 *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of functions of convolution type from a semigroup* $\mathcal{S}$ *to a commutative ring* $\mathcal{R}$.

a) If $f_0 = 0$, *then* $f_n = 0$ *for all* $n \in \mathbb{N}$.

b) If $\mathcal{S} = (0, \infty)$ *and* $f_0(x) = 0$ *for some* x, *then* $f_n = 0$ *for all* $n \in \mathbb{N}$.

Proof: a) Follows directly by induction on n.
b) We apply induction on n. It follows immediately from $f_0(x+y) = f_0(x)\, f_0(y)$ that $f_0(t) = 0$ for all $t \geq x$ and that $f_0(x/2) = 0$. Iterating this argument yields $f_0(t) = 0$ for all $t > 0$. Suppose by induction that we proved that $f_m = 0$ for all $m < n$. Then $f_n(t) = f_n(\frac{1}{2}t + \frac{1}{2}t) = \sum_{k=0}^{n} f_k(\frac{1}{2}t)\, f_{n-k}(\frac{1}{2}t) = 0$ for all $t > 0$. □

Theorem 2.1.8 *Let* $\mathcal{S} \subset \mathbb{C}$ *be a semigroup and let* $\mathcal{R}$ *be a subset of* $\mathbb{C}$ *that is closed under addition and a group with respect to multiplication. Suppose* $\mathcal{A}$ *is an algebra of functions* $\mathcal{S} \to \mathcal{R}$ *such that*

- *the only non-zero solutions in* $\mathcal{A}$ *to the equation* $f(x+y) = f(x) + f(y)$ *are* $f(x) = cx$ *with* $c \in \mathcal{R}$.
- *the only non-zero solutions in* $\mathcal{A}$ *to the equation* $f(x+y) = f(x)\, f(y)$ *are* $f(x) = e^{ax}$ *with* $a \in \mathcal{R}$.

Then $(f_n)_{n\in\mathbb{N}}$ *is a sequence of functions of convolution type in* $\mathcal{A}$ *if and only if there exist* $a \in \mathcal{R}$ *and a sequence* $(g_n)_{n\in\mathbb{N}}$ *in* $\mathcal{R}$ *with* $g_0 = 0$, *such that*

$$f_n(x) = e^{ax} \sum_{k=0}^{n} g_n^{k*}\, \frac{x^k}{k!}$$

for all $n \in \mathbb{N}$ *and all* $x \in \mathcal{S}$.

Proof: '⇐' This follows from Lemma 2.1.6 with $h_n(x) = x^n/n!$.
'⇒' If $f_0 = 0$, then $f_n = 0$ for all $n \in \mathbb{N}$ by Lemma 2.1.7a and the theorem holds with $g_n = 0$ for all $n \in \mathbb{N}$. We therefore assume that $f_0 \neq 0$.

Since $f_0(x+y) = f_0(x)\, f_0(y)$ for all $x, y \in \mathcal{S}$ and $f_0 \in \mathcal{A}$, there exists $a \in \mathcal{R}$ such that $f_0(x) = e^{ax}$. Define $(p_n)_{n \in \mathrm{N}}$ by $p_n(x) := e^{-ax} f_n(x)$. Note that $(p_n)_{n \in \mathrm{N}}$ is a sequence of functions of convolution type in $\mathcal{A}$ too.
We now use induction on n in order to show that there exists a sequence $(g_n)_{n \in \mathrm{N}}$ in $\mathcal{R}$ such that $p_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$ for all $n \in \mathbb{N}$. Note that convolutions of finite sequences are well-defined by Remark 2.1.3d.
If $n = 0$, then $p_0(x) = 1 = g_0^{0*} \frac{x^0}{0!}$ for all x. Since $p_1 \in \mathcal{A}$ and $p_1(x+y) = p_0(x)\, p_1(y) + p_1(x)\, p_0(y) = p_1(x) + p_1(y)$, it follows that $p_1(x) = p_1(1)\, x$. So $p_1(x) = g_1^{0*} \frac{x^0}{0!} + g_1^{1*} \frac{x^1}{1!} = g_1 x$, if g_1 is defined to be $p_1(1)$.
Suppose that we have $g_0, g_1, \dots, g_{n-1}$ $(n > 1)$ in $\mathcal{R}$ such that $g_0 = 0$ and $p_m(x) = \sum_{k=0}^{m} g_m^{k*} \frac{x^k}{k!}$ for $m < n$. It follows from (2.1) that p_n is a solution of the following linear functional equation in p:

$$p(x+y) - p(x) - p(y) = \sum_{k=1}^{n-1} p_k(x)\, p_{n-k}(y)$$

for all $x, y \in \mathcal{S}$.
It follows from Lemma 2.1.6 that p, defined by $p(x) := \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$, is a solution in $\mathcal{A}$ of this functional equation. Thus $(p_n - p)(x+y) = (p_n - p)(x) + (p_n - p)(y)$ for all $x, y \in \mathcal{S}$. Hence, there exists $c \in \mathcal{R}$ such that $(p_n - p)(x) = cx$ for all $x \in \mathcal{S}$. Set $g_n := c$. Since g_n^{k*} $(1 < k \leq n)$ can be expressed in terms of $g_1, g_2, \dots, g_{n-1}$ by Remark 2.1.3d, we have $p_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$. □

Theorem 2.1.9 *The assumptions of Theorem 2.1.8 are satisfied in the following cases:*

- $\mathcal{A} =$ *measurable functions,* $\mathcal{S} = \mathbb{N}, \mathbb{Z}, \mathbb{R}, (0, \infty)$ *or* $\mathbb{C}$, *and* $\mathcal{R} = \mathbb{R}, (0, \infty)$ *or* $\mathbb{C}$
- $\mathcal{A} =$ *locally bounded functions,* $\mathcal{S} = \mathbb{N}, \mathbb{Z}, \mathbb{R}, (0, \infty)$ *or* $\mathbb{C}$, *and* $\mathcal{R} = \mathbb{R}, (0, \infty)$ *or* $\mathbb{C}$
- $\mathcal{A} =$ *polynomials,* $\mathcal{S} = \mathbb{N}, \mathbb{Z}, \mathbb{R}, (0, \infty)$ *or* $\mathbb{C}$, *and* $\mathcal{R} = \mathbb{R}, (0, \infty)$, *or* $\mathbb{C}$.

Proof: It follows from [3, Remark below Theorem 4, p. 56], [3, Proposition 1, p. 53 and remark below Theorem 4, p. 56] and [3, Proposition 1, p. 53 and remark below Theorem 4,p. 56] that the assumptions of Theorem 2.1.8 are satisfied in these cases.

Remarks 2.1.10 a) Note that the proof of Theorem 2.1.8 shows that Theorem 2.1.8 also holds for finite sequences of functions of convolution type (cf. Remark 2.1.3d).
b) It follows from Theorem 2.1.8 that if $(q_n)_{n \in \mathrm{N}}$ is a sequence of polynomials of convolution type, then $g_n = q_n'(0)$. Thus $(q_n)_{n \in \mathrm{N}}$ determines $(g_n)_{n \in \mathrm{N}}$. Conversely, $(g_n)_{n \in \mathrm{N}}$ determines $(q_n)_{n \in \mathrm{N}}$ by Lemma 2.1.6. Hence, there is a one-to-one correspondence between sequences $(q_n)_{n \in \mathrm{N}}$ of polynomials of convolution type and sequences $(g_n)_{n \in \mathrm{N}}$ with $g_0 = 0$.

c) It follows from Theorem 2.1.8 that the convolution property of coefficients of polynomials of binomial type mentioned in [207, Proposition 4.3]) is nothing else than formula (2.2) in disguise.
d) It is possible to give a purely algebraic proof of Theorem 2.1.8 in case $(f_n)_{n\in\mathrm{N}}$ is a sequence of polynomials. The method of proof is similar to the idea employed in the proof of Theorem 2.1.12c.

We now present some general properties of sequences of polynomials of convolution type.

Definition 2.1.11 *Let $(q_n)_{n\in\mathrm{N}}$ be a sequence of polynomials of convolution type. The* **coefficient sequence** *of $(q_n)_{n\in\mathrm{N}}$ is the sequence $(g_n)_{n\in\mathrm{N}}$ such that $q_n(x) = \sum_{k=0}^{n} g_n^{k*}\, \frac{x^k}{k!}$.*

The following theorem describes the interplay between a sequence of polynomials of convolution type and its coefficient sequence. Other results have been obtained by Niederhausen, see [164, 167].

Theorem 2.1.12 *Let $(q_n)_{n\in\mathrm{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathrm{N}}$. Then:*

a) $\deg q_n \leq n$ *for all* $n \in \mathbb{N}$.

b) $\deg q_n = n$ *for all* $n \in \mathbb{N}$ *if and only if* $g_1 \neq 0$.

c) if $g_1 = 0$, then $\deg q_n \leq [n/2]$ *for all $n \in \mathbb{N}$. In this case,* $\deg q_n = [n/2]$ *for all $n \in \mathbb{N}$ if and only if $g_2 \neq 0$ and $g_3 \neq 0$.*

d) the following formal generating function relation holds:

$$\sum_{n=0}^{\infty} q_n(x)\, t^n = e^{x\, g(t)}, \tag{2.3}$$

where $g(t) = \sum_{n=0}^{\infty} g_n\, t^n$.

e) $q_0 = 1$ and $q_n(0) = 0$ for $n \geq 1$.

Proof: a) This follows directly from Theorem 2.1.8.
b) This follows directly from Theorem 2.1.8 and Lemma 2.1.5b.
c) We apply induction on n. It follows from Theorem 2.1.8 that the statement is true for $n = 0$ and $n = 1$.
Suppose by induction that $\deg q_m \leq [m/2]$ for all $m < n$ $(n \geq 2)$. If $\deg q_n > [n/2]$, then $q_n(x) = \sum_{k=0}^{N} a_k x^k$ $(N > [n/2],\ a_N \neq 0)$. Moreover, (2.1) yields

$$\sum_{k=0}^{N} a_k (2x)^k = q_n(2x) = \sum_{k=0}^{n} q_k(x) q_{n-k}(x).$$

Using the induction hypothesis and the inequality $\left[\frac{k}{2}\right] + \left[\frac{n-k}{2}\right] \leq \left[\frac{n}{2}\right]$ $(0 \leq k \leq n)$, we see that the coefficient of x^N on the left-hand side equals $a_N 2^N$, whereas the coefficient of x^N on the right-hand side equals $2a_N$. This leads to $N = 1$ since $a_N \neq 0$, which contradicts $N > [n/2] \geq 1$. We conclude that $\deg q_n \leq [n/2]$. This proves the first assertion.
For the second assertion, we note that the last line of Remark 2.1.3a easily yields $g_{2k}^{k*} = (g_2)^k$ and $g_{2k+1}^{k*} = k\,(g_2)^{k-1}\,g_3$, since $g_0 = g_1 = 0$. Hence, $\deg q_n = [n/2]$ for all $n \in \mathbb{N}$ if and only if $g_2 \neq 0$ and $g_3 \neq 0$.
d) Using Theorem 2.1.8 and Lemma 2.1.5a we have

$$\sum_{n=0}^{\infty} q_n(x)\, t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_n^{k*}\, \frac{x^k}{k!}\, t^n = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} g_n^{k*}\, t^n\right) \frac{x^k}{k!} = e^{x\, g(t)}.$$

e) By definition, $g_0^{0*} = 1$. Thus Theorem 2.1.8 yields $q_0 = 1$. It follows from Lemma 2.1.5a that $g_n^{0*} = 0$ for $n \geq 1$. Hence, $q_n(0) = 0$ for $n \geq 1$ by Theorem 2.1.8. □

Remarks 2.1.13
a) Theorem 2.1.12c yields the following extension of Lemma 2.1.5a: if $g_0 = g_1 = 0$, then $g_n^{k*} = 0$ for $k > [n/2]$.
b) An example of a sequence of polynomials of convolution type with $\deg q_n = [n/2]$ is the sequence of polynomials defined by the generating function

$$\sum_{n=0}^{\infty} q_n(x)\, z^n = e^{x(\log(1+z)-z)}.$$

These polynomials appear in combinatorics (see [199, p. 73]). It follows from Theorem 2.1.12c that $\deg q_n = [n/2]$ for all $n \in \mathbb{N}$.
c) It is possible to extend Theorem 2.1.12c to the case $g_0 = \ldots = g_k = 0$. As an illustration, let us consider the following example due to Daniel Loeb. Fix $k \in \mathbb{N}$. Take $g_n = \delta_{nk}$ for all $n \in \mathbb{N}$. Then $q_n(x) = \frac{x^{n/k}}{n/k!}$ if k divides n, and 0 otherwise.

We conclude this section with an extension of a theorem due to Markowsky (see [154, Theorem 4.4]).

Theorem 2.1.14 *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in a commutative ring $\mathcal{R}$ such that $\alpha_0 = 1$. Then for each sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{R}\setminus\{0\}$ there exists a unique sequence of polynomials of convolution type $(q_n)_{n\in\mathbb{N}}$ such that $q_n(x_n) = \alpha_n$ for all $n \in \mathbb{N}$.*

Proof: Uniqueness is clear, since $q_n(x_n) = \alpha_n$ determines the values of $q_n(k\,x_n)$ for all $k \in \mathbb{N}$ (use (2.1)). Existence can be shown inductively as follows. By Theorem 2.1.8, it suffices to find a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ with $g_0 = 0$ such that $q_n(x) = \sum_{k=0}^{n} g_n^{k*}\, \frac{x^k}{k!}$ for all $n \in \mathbb{N}$. Suppose g_k has been found for each $k < n$. By Lemma 2.1.5c, g_n^{k*} is a polynomial in $g_1, \ldots, g_{n-1}$ for $2 \leq k \leq n$. This means that we can choose g_n such that $\sum_{k=0}^{n} g_n^{k*}\, \frac{x_n^k}{k!} = \alpha_n$. This proves existence. □

2.2 Basic polynomials and delta operators

In this section we link polynomials of convolution type to a certain kind of linear operators on polynomials, following and extending the exposition of the Umbral Calculus in [162, 210]. Our emphasis on operator methods yields simpler proofs than in [162] and [210]. The key theorems of this section are Theorems 2.2.15, 2.2.17 and 2.2.19 which describe the relation between delta operators and polynomials of convolution type.
Most proofs are new or modifications of existing proofs.

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with $\deg q_n = n$ for all $n \in \mathbb{N}$. Then $(q_n)_{n\in\mathbb{N}}$ is a basis for the vector space $\mathcal{P}$ of polynomials with coefficients in some field $\mathcal{K}$ of characteristic zero. Hence there exists a unique linear operator Q on $\mathcal{P}$ with $Qq_n = q_{n-1}$, $n \geq 1$ and $Qq_0 = 0$. The Rota Umbral Calculus is based on this operator. It turns out that this operator is shift-invariant (see Definition 2.2.2 and Theorem 2.2.19).

Definition 2.2.1 *The* **shift-operator** E^a *is defined by* $(E^a p)(x) := p(x+a)$ $(p \in \mathcal{P})$.

Definition 2.2.2 *An operator T on $\mathcal{P}$ is called* **shift-invariant** *if* $E^a T = T E^a$ *for all a.*

Examples 2.2.3 Examples of shift-invariant operators include:

a) the identity operator I.

b) the differentiation operator D.

c) the operators E^a of Definition 2.2.1.

d) the forward difference operator $E^1 - I$.

e) the backward difference operator $I - E^{-1}$.

f) the Abel operators DE^a.

g) the Laguerre operator L , defined by

$$(Lp)(x) := -\int_0^\infty e^{-t}\, p'(x+t)\, dt$$

h) the Bernoulli operator J , defined by

$$(Jp)(x) := \int_x^{x+1} p(t)\, dt$$

Remark 2.2.4 If S is an invertible shift-invariant operator on $\mathcal{P}$, then its inverse S^{-1} is also shift-invariant, since $S^{-1}E^a = S^{-1}E^a S S^{-1} = S^{-1} S E^a S^{-1} = E^a S^{-1}$ for all a.

Definition 2.2.5 *A linear operator Q on $\mathcal{P}$ is called a* **delta operator** *if Q is shift-invariant and Qx is a nonzero constant.*

Examples 2.2.6 Examples of delta operators include b, d, e, f and g from Examples 2.2.3, but not a, c and h.

It is a remarkable fact that every linear shift-invariant operator has a Taylor-like expansion in terms of an arbitrary delta operator (see Theorem 2.2.22). We start by proving this expansion theorem for the differentiation operator D, because this yields simple proofs for properties of shift-invariant operators.

Theorem 2.2.7 ([210, Theorem 2]) *Let D be the differentiation operator and define $q_n(x) = \frac{x^n}{n!}$ for all $n \in \mathbb{N}$. Then T is a linear shift-invariant operator on $\mathcal{P}$ if and only if*

$$T = \sum_{k=0}^{\infty} (Tq_k)(0)\, D^k.$$

Proof: '$\Leftarrow$' Note that the infinite sum is in fact a finite sum when applied to a polynomial and thus is a well-defined operator on $\mathcal{P}$. Shift-invariance of T follows from shift-invariance of D.
'$\Rightarrow$' Since $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$, it suffices to verify the result for Tq_n for all $n \in \mathbb{N}$. Using the Binomial Theorem, we obtain $(Tq_n)(a) = (E^a T\, q_n)(0) = (TE^a\, q_n)(0) = \left(\sum_{k=0}^{n} q_{n-k}(a)\, Tq_k\right)(0) = \left(\sum_{k=0}^{\infty} (Tq_k)(0) D^k q_n\right)(a)$ for all $n \in \mathbb{N}$ and all a.

Examples 2.2.8
a) Consider the shift-invariant operator E^a. Theorem 2.2.7 yields

$$E^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} D^k = e^{aD}.$$

Hence, $p(x+a) = (E^a p)(x) = \sum_{k=0}^{\infty} (D^k\, p)(x)\, \frac{a^k}{k!}$ for all $p \in \mathcal{P}$, which is Taylor's Formula.
b) Consider the Laguerre operator of Example 2.2.3e. Since for $k \geq 1$ we have

$$L\left(\frac{x^k}{k!}\right)(0) = -\int_0^{\infty} e^{-t}\, \frac{t^{k-1}}{(k-1)!}\, dt \;=\; -1,$$

it follows that $L = -\sum_{k=0}^{\infty} D^k = D(D-I)^{-1}$.
c) Consider the Bernoulli operator of Example 2.2.3f. Since

$$J\left(\frac{x^k}{k!}\right)(0) = \int_0^1 \frac{t^k}{k!}\, dt = \frac{1}{(k+1)!},$$

it follows that $J = \sum_{k=0}^{\infty} \frac{D^k}{(k+1)!}$.

We now derive some corollaries from Theorem 2.2.7. The first corollary is an extension of [210, Propositions 1 and 2, p. 687]. Recall that the degree of a nonzero constant is defined to be zero and that the degree of the zero polynomial is defined to be -1.

Corollary 2.2.9 *a) If T is a linear shift-invariant operator on $\mathcal{P}$, then there exists a non-negative integer $n(T)$ such that* $\deg Tp = \max\{-1, \deg p - n(T)\}$ *for all $p \in \mathcal{P}$. The null space of T equals the set of polynomials with degree less than $n(T)$.*
b) If Q is a delta operator, then $\deg Qp = \max\{-1, \deg(p) - 1\}$ *and the null space of Q equals the set of constant polynomials.*

Proof: a) By Theorem 2.2.7, we have $T = \sum_{k=0}^{\infty} a_k D^k$ for some sequence $(a_n)_{n\in\mathbb{N}}$. It follows from $\deg D^k p = \max\{-1, \deg(p) - k\}$ that if we set $n(T) := \min\{k \in \mathbb{N} \;:\; a_k \neq 0\}$, then $\deg Tp = \max\{-1, \deg(p) - n(T)\}$ for all $p \in \mathcal{P}$. Thus $Tp = 0$ if and only if $\deg p < n(T)$.
b) By definition, $Q\,x$ is a nonzero constant. Thus a) implies that $\deg Qp = \max\{-1, \deg(p) - 1\}$ for all polynomials $p \in \mathcal{P}$. □

Remarks 2.2.10 a) The converse of Corollary 2.2.9a is not true. Fix $m \in \mathbb{N}$. We construct a linear, non shift-invariant operator T on $\mathcal{P}$ such that $\deg Tp = \max\{-1, \deg(p) - m\}$ for all $p \in \mathcal{P}$.
Define a linear operator T on $\mathcal{P}$ by $Tx^k := 0$ if $k < m$, $Tx^m := 1$, $Tx^{m+1} := \frac{1}{2}x$ and $Tx^k := x^{k-m}$ if $k \geq m+2$. Clearly $\deg Tp = \max\{-1, \deg(p) - m\}$ for all $p \in \mathcal{P}$. Then $(TE^1)x^{m+1} = T(x+1)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} T\,x^k = T\left(x^{m+1} + (m+1)x^m\right) = \frac{1}{2}x + m + 1$ and $(E^1 T)\,x^{m+1} = \frac{1}{2}(x+1)$. Thus $m = -\frac{1}{2}$, which is impossible since $m \in \mathbb{N}$. We conclude that T is not shift-invariant.
For more information on the structure of linear shift-invariant operators on $\mathcal{P}$, see Remark 2.2.22.
b) Erik Thomas has pointed out to me that Corollary 2.2.9 can be used to find all translation-invariant linear subspaces of $\mathcal{P}$. A linear subspace L of $\mathcal{P}$ is translation-invariant if $E^a L \subset L$ for all a. Translation-invariant linear subspaces are important in harmonic analysis.
The following result is somewhat stronger: if L is a linear subspace and $E^b L \subset L$ for some $b \neq 0$, then L is either one of the trivial subspaces $\{0\}$ or $\mathcal{P}$, or there exists $n \in \mathbb{N}$ such that $L = \mathcal{P}_n$, where $\mathcal{P}_n$ is the set of all polynomials with degree not exceeding n. The proof runs as follows: let $p \in L$ be arbitrary and let m be the degree of p. Consider the linear shift-invariant operator $E^b - I$. Since $(E^b - I)\,x = b \neq 0$, it follows that $E^b - I$ is a delta operator. It follows from Corollary 2.2.9b that $\deg\left((E^b - I)^k\right)p = m - k$ for $0 \leq k \leq m$. By linearity, $\mathcal{P}_m \subset L$. Suppose $L \neq \{0\}$ and define $n := \sup\{\deg p | p \in L\}$. The above argument yields that $L = \mathcal{P}_n$ if $n < \infty$ and that $L = \mathcal{P}$ if $n = \infty$.
Conversely, $\mathcal{P}_n$ is translation-invariant for each $n \in \mathbb{N}$.

Corollary 2.2.11 ([210, Corollary 1]) *Let T be a linear shift-invariant operator on $\mathcal{P}$. Then the following are equivalent:*

a) T is invertible.

b) $T1 \neq 0$.

c) $\deg p = \deg Tp$ for all $p \in \mathcal{P}$.

Proof: 'a $\Rightarrow$ b' The null space of an invertible linear operator consists of 0 only, so $T1 \neq 0$.
'b $\Rightarrow$ c' Since $T1 \neq 0$, it follows from Corollary 2.2.9a that $\deg p = \deg Tp$ for all $p \in \mathcal{P}$.
'c $\Rightarrow$ a' It suffices to prove that T is injective and surjective. If $p, q \in \mathcal{P}$ and $p \neq q$, then $T(p-q) \neq 0$ since $\deg(p-q) \geq 0$. Moreover, $\deg p = \deg Tp$ implies that $(Tx^n)_{n \in N}$ is a basis for $\mathcal{P}$. Hence, T is surjective. □

Corollary 2.2.12 ([210, Corollary 4]) *Any two linear shift-invariant operators on $\mathcal{P}$ commute.*

Proof: All linear shift-invariant operators can be represented as a formal power series in the differentiation operator D by Theorem 2.2.7. Since the action of these operators on a polynomial only involves finitely many terms of their expansions, the result follows. □

The polynomials $q_n(x) = \frac{x^n}{n!}$ appeared in the proof of Theorem 2.2.7. These polynomials have the properties $q_0 = 1$, $Dq_n = q_{n-1}$ and $q_n(0) = 0$ for $n > 0$. Moreover, they are of convolution type by the Binomial Formula. We will now show that for every delta operator there exists a sequence of polynomials with analogous properties (see Theorems 2.2.15, 2.2.17 and 2.2.19).

Definition 2.2.13 *Let Q be a delta operator. A sequence $(q_n)_{n \in \mathbb{N}}$ of polynomials is a* **basic sequence** *for Q if:*

1. $q_0 = 1$
2. $q_n(0) = 0$ *if* $n \geq 1$
3. $Qq_n = q_{n-1}$ *if* $n \geq 1$.

Remarks 2.2.14 a) It follows from (1), (3) and Corollary 2.2.9b that $\deg q_n = n$ for all $n \in \mathbb{N}$.
b) Note that properties (1) and (2) of Definition 2.2.13 are satisfied by each sequence $(q_n)_{n \in \mathbb{N}}$ of polynomials of convolution type by Theorem 2.1.12e.
c) If $(q_n)_{n \in \mathbb{N}}$ is a sequence of polynomials of convolution type with $\deg q_1 = 1$ and T is a linear operator on $\mathcal{P}$ such that $Tq_n = q_{n-1}$for $n \geq 1$, then T is shift-invariant since $TE^y q_n = T\left(\sum_{k=0}^n q_{n-k}(y)\, q_k\right) = \sum_{k=1}^n q_{n-k}(y)\, q_{k-1} = \sum_{h=0}^{n-1} q_{n-1-h}(y)\, q_h = E^y q_{n-1} = E^y T q_n$. Hence, by linearity, $TE^y = E^y T$.

Theorem 2.2.15 ([210, Proposition 3]) *There is a unique basic sequence for every delta operator.*

Proof: Let Q be an arbitrary delta operator. It follows from Theorem 2.2.7 and Corollary 2.2.9b that there exists a sequence $(\alpha_n)_{n\in\mathrm{N}}$ with $\alpha_1 \neq 0$ such that $Q = \sum_{k=1}^{\infty} \alpha_k D^k$. By Remark 2.2.14a, we must construct polynomials q_n of degree n. By (1) of Definition 2.2.13, $q_0 = 1$. Suppose by induction that $q_{n-1} = \sum_{k=0}^{n-1} a_{n-1,k}x^k$ has been constructed. Since $\deg q_n = n$, q_n must be of the form $\sum_{k=0}^{n} a_{n,k}x^k$. Because $q_n(0) = 0$ by (3) of Definition 2.2.13, $a_{n,0}$ must be zero. Substitution of $Q = \sum_{k=1}^{\infty} \alpha_k D^k$ into $Qq_n = q_{n-1}$ and comparing coefficients yields the following system of equations:

$$\begin{aligned}
a_{n-1,n-1} &= \alpha_1 n a_{n,n} \\
a_{n-1,n-2} &= \alpha_1 (n-1) a_{n,n-1} + \alpha_2 n(n-1) a_{n,n} \\
\vdots \quad &\quad \vdots \\
a_{n-1,1} &= \alpha_1 .2 a_{n,2} + \alpha_2 .2.3 a_{n,3} + \cdots + \alpha_{n-1} n! a_{n,n}
\end{aligned}$$

Because $\alpha_1 \neq 0$ this system of equations has a unique solution. This proves uniqueness and existence. □

Explicit formulas for the calculation of basic sequences will be discussed in Section 2.3.

Examples 2.2.16
a) The differentiation operator D has basic sequence $\left(\frac{x^n}{n!}\right)_{n\in\mathrm{N}}$.

b) The forward difference operator $E^1 - I$ has basic sequence $\left(\binom{x}{n}\right)_{n\in\mathrm{N}}$, where

$$\binom{x}{n} := \frac{x(x-1)\dots(x-n+1)}{n!}$$

are the lower factorials.
c) The backward difference operator $I - E^{-1}$ has basic sequence $\left(\binom{x+n-1}{n}\right)_{n\in\mathrm{N}}$, where

$$\binom{x+n-1}{n} := \frac{x(x+1)\dots(x+n-1)}{n!}$$

are the upper factorials .
d) The Abel operator DE^a has basic sequence $\left(\frac{x(x-na)^{n-1}}{n!}\right)_{n\in\mathrm{N}}$, the Abel polynomials .

Theorem 2.2.17 ([210, Theorem 1]) *The basic sequence of a delta operator is a sequence of polynomials of convolution type.*

Proof: Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathrm{N}}$. According to Definition 2.1.1 we have to prove

$$q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y) \tag{2.4}$$

for all $n \in \mathbb{N}$ and all x, y (cf. Remark 2.1.10b). We proceed by induction on n. The case $n = 0$ is trivial because $q_0 = 1$.
Suppose by induction that (2.4) has been proved for $m < n$. Fix y. It follows from Definition 2.2.5 that $QE^y q_n = E^y Q q_n = E^y q_{n-1}$. Hence,

$$Q\left(E^y q_n - \sum_{j=0}^{n} q_j q_{n-j}(y)\right) = E^y q_{n-1} - \sum_{j=1}^{n} q_{j-1}\, q_{n-j}(y) =$$

$$E^y q_{n-1} - \sum_{k=0}^{n} q_k\, q_{n-1-k}(y) = 0.$$

Corollary 2.2.9b implies that $E^y q_n - \sum_{k=0}^{n} q_k\, q_{n-k}(y)$ is a constant. So $q_n(x+y) = c + q_k(x)\, q_{n-k}(y)$. Taking $x = 0$ yields $c = 0$, since $q_n(0) = 1$ for $n \geq 1$. Because y was arbitrary, we obtain $q_n(x+y) = \sum_{k=0}^{n} q_k(x)\, q_{n-k}(y)$ for all x, y. □

Remark 2.2.18 Theorem 2.2.17 shows that the polynomials appearing in Examples 2.2.16 are of convolution type. This yields the following formulas:

a)
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(the well-known Binomial Formula).

b)
$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}$$

(this is the Vandermonde convolution formula, see e.g. [200, p. 8]).

c)
$$\binom{x+y+n-1}{n} = \sum_{k=0}^{n} \binom{x+n-1}{k} \binom{y+n-k-1}{n-k}$$

This formula is equivalent to the Vandermonde convolution formula, since $\binom{x+k-1}{k} = (-1)^k \binom{-x}{k}$.

d)
$$(x+y)(x+y-na)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} x\,(x-ka)^{k-1}\, y\,(y-(n-k)a)^{n-k-1}$$

(this is the Abel generalization of the Binomial Formula, see e.g. [200, p. 18]).

The following theorem is a converse to Theorems 2.2.15 and 2.2.17.

Theorem 2.2.19 ([210, Theorem 1]) *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type such that* $\deg q_1 = 1$. *Then there exists a unique delta operator Q with basic sequence $(q_n)_{n\in\mathbb{N}}$.*

Proof: By Theorem 2.1.8, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $q_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$ for all $n \in \mathbb{N}$. Thus $g_1 \neq 0$, because $\deg q_1 = 1$ and $q_1(x) = g_1 x$. By Theorem 2.1.12b, $\deg q_n = n$ for all $n \in \mathbb{N}$. Therefore $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$. Since $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$, there exists a unique linear operator Q on $\mathcal{P}$ such that $Qq_n = q_{n-1}$ $(n \geq 1)$ and $Qq_0 = 0$. Since $\deg q_1 = 1$, it follows that Qx is a nonzero constant. Shift-invariance of Q follows from Remark 2.2.14c. □

Remarks 2.2.20 a) It is essential in Theorem 2.2.19b that $\deg q_1 = 1$. If $\deg q_1 \neq 1$, then $q_1 = 0$ by Theorem 2.1.8 and no delta operator Q with $Qq_1 = q_0$ can exist by Corollary 2.2.9b since $q_0 = 1$.
b) Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. As remarked in the beginning of this chapter, the Rota theory of polynomials of convolution type depends on the delta operator Q that maps q_n to q_{n-1}. Yang remarks in [253] that the linear operator on $\mathcal{P}$ that maps q_n to q_{n+1} (the so-called Roman shift) is in some cases more useful than the delta operator. For more information on the Roman shift, see [202, Section 3.6].

We conclude this section with the general Expansion Theorem (cf. Theorem 2.2.7).

Theorem 2.2.21 (Polynomial Expansion Theorem) *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Then*

$$p = \sum_{k=0}^{\infty} (Q^k p)(0)\, q_k$$

for all $p \in \mathcal{P}$.

Proof : Let $p \in \mathcal{P}$ be arbitrary and let n be the degree of p. By Remark 2.2.14a, there exist constants c_k such that $p = \sum_{k=0}^{n} c_k\, q_k$. It follows that $Q^r p = \sum_{k=r}^{n} c_k\, q_{k-r}$ for $0 \leq r \leq n$. Evaluating at zero yields $c_r = (Q^r p)(0)$ since $q_k(0) = 0$ for $k \geq 1$. Hence, $p = \sum_{k=0}^{\infty} (Q^k p)(0)\, q_k$. □

Theorem 2.2.22 (Operator Expansion Theorem, [210, Theorem 2]) *Let T be a linear shift-invariant operator on $\mathcal{P}$ and let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Let $(g_n)_{n\in\mathbb{N}}$ be the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$. Then:*

a) $T = \sum_{k=0}^{\infty} (Tq_k)(0) Q^k$

b) In particular, if $(g_n)_{n\in\mathbb{N}}$ is the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$, then $D = \sum_{n=0}^{\infty} g_n Q^n$ and $Q = \sum_{n=0}^{\infty} \bar{g}_n D^n$ where $\sum_{n=0}^{\infty} \bar{g}_n t^n$ is the composition inverse of the formal power series $\sum_{n=0}^{\infty} g_n t^n$.

Proof: a) Let $p \in \mathcal{P}$ be arbitrary with degree n. Applying Lemma 2.2.21 to $E^y p$, we obtain $TE^y p = \sum_{k=0}^{n} (Q^k E^y p)(0)\, Tq_k = \sum_{k=0}^{n} (Q^k p)(y)\, Tq_k$. Hence, $(Tp)(y) = (E^y Tp)(0) = (TE^y p)(0) = \sum_{k=0}^{n} (Tq_k)(0) \left(Q^k p\right)(y) = \sum_{k=0}^{\infty} (Tq_k)(0) \left(Q^k p\right)(y)$ for all y. This completes the proof, since p is arbitrary.
b) It follows from $q_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$ that $(Dq_n)(0) = g_n$ for all $n \in \mathbb{N}$. Thus a) yields $D = \sum_{n=0}^{\infty} (Dq_n)(0) Q^n = \sum_{n=0}^{\infty} g_n Q^n$. Since $g_0 = 0$, the formal power series $\sum_{n=0}^{\infty} g_n\, t^n$ has a compositional inverse (see e.g. [172]).

Remarks 2.2.23 a) We implicitly used the Isomorphism Theorem 2.3.1 in the proof of Theorem 2.2.22b.
b) Fix an arbitrary delta operator Q with basic sequence $(q_n)_{n\in\mathbb{N}}$. We know from Remark 2.2.14a that $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$. Let T be an arbitrary linear shift-invariant operator on $\mathcal{P}$. Consider the infinite matrix $(a_{ij})_{i,j}$ with entries a_{ij}, where $Tq_j = \sum_{i=0}^{\infty} a_{ij}\, q_i$. Theorem 2.2.22a yields

$$Tq_j = \sum_{n=0}^{\infty} (Tq_n)(0) Q^n q_j = \sum_{n=0}^{j} (Tq_n)(0) q_{j-n} = \sum_{i=0}^{j} (Tq_{j-i})(0) q_i.$$

Hence, $a_{ij} = (Tq_{j-i})(0)$. Thus $a_{i,j} = a_{i+k,j+k}$ for all $k \in \mathbb{N}$, i.e. T is a Toeplitz operator on $\mathcal{P}$.
c) There also exists Operator Expansion Theorems for more general operators than shift-invariant operators. The coefficients of these expansions are polynomials in x rather than constants (see [76, 139]).

Examples 2.2.24 a) We want to expand the differentiation operator D in powers of the forward difference operator $E^1 - I$. The basic sequence of $E^1 - I$ is $\left(\binom{x}{n}\right)_{n\in\mathbb{N}}$, so

$$D = \sum_{k=0}^{\infty} \left(D \binom{x}{k} \right)(0)\, (E^1 - I)^k = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} (E^1 - I)^k.$$

This is a classical formula for numerical differentiation.
b) Consider the shift operator E^a. Expanding E^a in powers of $E^1 - I$ yields

$$E^a = \sum_{k=0}^{\infty} \binom{a}{k} (E^1 - I)^k.$$

This is Newton's forward difference interpolation formula.

We conclude this section with a few remarks on linear functionals. In [202, 207] the Umbral Calculus is presented in terms of linear functionals instead of linear operators as in this section. The following theorem describes the relationship between linear functionals and operators on $\mathcal{P}$.

Theorem 2.2.25 *a) The map $p \to (Tp)(0)$ is a linear functional on $\mathcal{P}$ for every linear operator T on $\mathcal{P}$.*

b) If Λ is a linear functional on $\mathcal{P}$, then there exists a linear shift-invariant operator Q on $\mathcal{P}$ such that $\Lambda p = (Qp)(0)$.

Proof: a) This follows directly from the linearity of T.
b) Let Q be the shift-invariant operator defined by

$$Qp := \sum_{k=0}^{\infty} \Lambda\left(\frac{x^k}{k!}\right) D^k.$$

An easy calculation shows that $\Lambda x^n = (Q\, x^n)(0)$ for all $n \in \mathbb{N}$. Since the powers x^n span $\mathcal{P}$, this completes the proof. □

2.3 Explicit formulas for polynomials of convolution type

In this section we derive some explicit formulas for polynomials of convolution type and we discuss the problem of connection coefficients.
Whereas we avoided the use of formal power series in Section 2.2, we can hardly do so in this section. The reason is that formal power series makes computations considerably easier.

We start with showing that the ring of formal power series and the ring of linear shift-invariant operators on $\mathcal{P}$ are isomorphic.

Theorem 2.3.1 (Isomorphism Theorem = [210, Theorem 3]) *Let Q be any delta operator on $\mathcal{P}$. The map Λ_Q defined by*

$$\Lambda_Q\left(\sum_{k=0}^{\infty} a_k\, t^k\right) := \sum_{k=0}^{\infty} a_k\, Q^k$$

is an isomorphism between the ring of formal power series and the ring of linear shift-invariant operators on $\mathcal{P}$.

Proof: It is clear from Theorem 2.2.22a that Λ_Q is linear and injective. It follows from the Expansion Theorem 2.2.22a that Λ_Q is surjective. Hence, we only need to show that $\Lambda_Q(fg) = \Lambda_Q(f)\,\Lambda_Q(g)$ for all formal power series f and g. Let $f(t) = \sum_{k=0}^{\infty} a_k\, t^k$ and $g(t) = \sum_{k=0}^{\infty} b_k\, t^k$ be arbitrary formal power series. Let $(q_n)_{n\in\mathbb{N}}$ be the basic sequence of Q (Theorem 2.2.15). By Remark 2.2.14, $(q_n)_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}$, hence it suffices to prove $\Lambda_Q(fg)\, q_n = \Lambda_Q(f)\,\Lambda_Q(g)\, q_n$ for all $n \in \mathbb{N}$. Since $\left(c_k\, Q^k\right) q_n = \sum_{k=0}^{n} c_k\, q_{n-k}$, we have $\Lambda_Q(fg)\, q_n = ((a*b)*q)_n$ and $\Lambda_Q(f)\,\Lambda_Q(g)\, q_n = (a*(b*q))_n$, where $a = (a_n)_{n\in\mathbb{N}}$, $b = (b_n)_{n\in\mathbb{N}}$ and $q = (q_n)_{n\in\mathbb{N}}$. Thus $\Lambda_Q(fg)\, q_n = \Lambda_Q(f)\,\Lambda_Q(g)\, q_n$ follows from associativity of the convolution operation (Remark 2.1.3c). □

The proof of the next corollary shows once more that Theorem 2.3.1 is useful (cf. Remark 2.2.23a). Corollary 2.3.2 will be used in Section 3.3 for computing moments of discrete distributions.

Corollary 2.3.2 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$ and let $(g_n)_{n\in\mathbb{N}}$ be the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$. Let g be the formal power series defined by $g(t) := \sum_{n=0}^{\infty} g_n t^n$. Then $\sum_{k=0}^{\infty} k\, q_k(\alpha)\, Q^k = \alpha\, E^{\alpha}\, g'(Q)\, Q$.*

Proof: Fix an arbitrary α. Using the formal generating function of Theorem 2.1.12d we obtain $\sum_{k=0}^{\infty} k\, q_k(\alpha)\, t^k = t\, \frac{d}{dt}\left(\sum_{k=0}^{\infty} q_k(\alpha)\, t^k\right) = t\, \frac{d}{dt}\, e^{\alpha\, g(t)} = \alpha\, e^{\alpha\, g(t)}\, g'(t)\, t$. Theorem 2.3.1 now yields

$$\sum_{k=0}^{\infty} k\, q_k(\alpha)\, Q^k = \alpha\, e^{\alpha\, g(Q)}\, g'(Q)\, Q = \alpha\, E^{\alpha}\, g'(Q)\, Q$$

(the last equality follows from Theorem 2.2.22b and Example 2.2.8a). □

We now present explicit formulas for basic sequences of delta operators. Formulas a) through d) of Theorem 2.3.6 were already known to Steffensen (see [227, Sections 2 and 3]; see also [210, Theorem 4]).

Definition 2.3.3 *If T is a linear operator on $\mathcal{P}$, then its Pincherle derivative T' is defined by $T' := T\,\underline{\mathrm{x}} - \underline{\mathrm{x}}\,T$ where the linear operator $\underline{\mathrm{x}}$ is defined by $(\underline{\mathrm{x}}\,p)(x) := x\,p(x)$ for all x and all polynomials $p \in \mathcal{P}$.*

The Pincherle derivative was introduced by Pincherle in [179, Section 56].

We now derive some elementary properties of the Pincherle derivative.

Lemma 2.3.4 *a) If $T = \sum_{k=0}^{\infty} a_k\, D^k$, then $T' = \sum_{k=0}^{\infty} k\, a_k\, D^{k-1}$.*

b) The Pincherle derivative of a linear shift-invariant operator on $\mathcal{P}$ is a linear shift-invariant operator on $\mathcal{P}$.

c) The Pincherle derivative of a delta operator is an invertible shift-invariant operator on $\mathcal{P}$.

d) If T and S are linear shift-invariant operators on $\mathcal{P}$, then $(T\,S)' = T'\,S + T\,S'$.

Proof: a) Since $\underline{\mathrm{x}}$ is a linear operator on $\mathcal{P}$, it suffices to prove a) for the polynomials $\frac{x^n}{n!}$. We have

$$T'\, \frac{x^n}{n!} := (T\,\underline{\mathrm{x}} - \underline{\mathrm{x}}\,T)\, \frac{x^n}{n!} = (n+1) \sum_{k=0}^{\infty} a_k\, D^k\, \frac{x^{n+1}}{(n+1)!} - x \sum_{k=0}^{\infty} a_k\, D^k\, \frac{x^n}{n!} =$$

$$(n+1)\sum_{k=0}^{n+1}\frac{x^{n+1-k}}{(n+1-k)!}-\sum_{k=0}^{n}(n+1-k)\,a_k\,\frac{x^{n+1-k}}{(n+1-k)!}=$$

$$\sum_{k=0}^{n+1}k\,a_k\,\frac{x^{n+1-k}}{(n+1-k)!}=\sum_{i=0}^{n}(i+1)\,a_{i+1}\,D^i\,\frac{x^n}{n!}.$$

Hence, $T'=\sum_{i=0}^{\infty}(i+1)\,a_{i+1}\,D^i$, since $\underline{\mathbf{x}}$ is a linear operator on $\mathcal{P}$.
b) This follows directly from a) and Theorem 2.2.7.
c) By Theorem 2.2.7 and Corollary 2.2.9b we have $Q=\sum_{k=0}^{\infty}b_k\,D^k$ with $b_1\neq 0$. We get from a) that $Q'=\sum_{i=0}^{\infty}(i+1)\,b_{i+1}\,D^i$. Hence, Q' is invertible by Corollary 2.2.11.
d) This follows from $(T\,S)'=T\,S\,\underline{\mathbf{x}}-\underline{\mathbf{x}}\,T\,S=(T\,S\,\underline{\mathbf{x}}-T\,\underline{\mathbf{x}}\,S)+(T\,\underline{\mathbf{x}}\,S-\underline{\mathbf{x}}\,T\,S)=T\,S'+T'\,S$. □

Lemma 2.3.5 ([210, Proposition 4]) *For every delta operator Q there exists a unique invertible shift-invariant operator U on $\mathcal{P}$ such that $Q=DU$.*

Proof: By Theorem 2.2.7 and Corollary 2.2.9b, we have $Q=\sum_{k=1}^{\infty}b_k\,D^k$ with $b_1\neq 0$. Define U by $U:=\sum_{k=0}^{\infty}b_{k+1}\,D^k$, so $Q=DU$. The invertibility of U follows from Corollary 2.2.11b, since $b_1\neq 0$. Uniqueness of U follows from the expansion of Q and U in powers of D. □

The operator U that appears in the statement of Theorem 2.3.6 is the operator whose existence is assured by Lemma 2.3.5.

Theorem 2.3.6 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$ and let $(g_n)_{n\in\mathbb{N}}$ be the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$. Let U be the unique invertible shift-invariant operator such that $Q=DU$. Then the following formulas hold for $n\geq 1$:*

a) $n!\,q_n=(Q'\,U^{-n-1})\,(x^n)$

b) $n!\,q_n=(U^{-n})\,(x^n)-(U^{-n})'\,(x^{n-1})$

c) $n!\,q_n=(\underline{\mathbf{x}}\,U^{-n})\,(x^{n-1})$

d) $n\,q_n=(x(Q')^{-1})\,q_{n-1}$ *(***Rodrigues Formula***)*

e) $n\,q_n(x)=x\,\sum_{k=0}^{n}k\,g_k\,q_{n-k}(x)$.

Proof: Since $D'=I$, we have $Q'\,U^{-n-1}\,x^n=(D\,U)'\,U^{-n-1}\,x^n=((D'\,U+D\,U')\,U^{-n-1})\,x^n=((U+D\,U')\,U^{-n-1})\,x^n=(U^{-n}+D\,U'\,U^{-n-1})\,x^n=U^{-n}\,x^n+U'\,U^{-n-1}\,D\,x^n=U^{-n}\,x^n-(U^{-n})'\,x^{n-1}=U^{-n}\,x^n-(U^{-n}\,\underline{\mathbf{x}}-\underline{\mathbf{x}}\,U^{-n})\,x^{n-1}=(\underline{\mathbf{x}}\,U^{-n})\,x^{n-1}$, so the right-hand sides of a), b) and c) are identical. Since Q has a unique basic sequence by Theorem 2.2.15, it suffices to note that $\left(\underline{\mathbf{x}}\,U^{-n}\,x^{n-1}\right)(0)=0$ and $\left(Q\,Q'\,U^{-n-1}\right)\frac{x^n}{n!}=\left(D\,U\,Q'\,U^{-n-1}\right)\frac{x^n}{n!}=\left(Q'\,U^{-n}\,D\right)\frac{x^n}{n!}=\left(Q'\,U^{-n}\right)\frac{x^{n-1}}{(n-1)!}$ for $n\geq 1$. This proves a), b) and c).

By Lemma 2.3.4c, Q' is invertible. Thus it follows from a) that $\frac{x^{n-1}}{(n-1)!} = \left((Q')^{-1}U^n\right) q_{n-1}(x)$ for $n \geq 2$. By c),

$$nq_n(x) = \left(\underline{\mathbf{x}}\, U^{-n}\right)\frac{x^{n-1}}{(n-1)!} = \left(\underline{\mathbf{x}}\, U^{-n}(Q')^{-1}U^n\right) q_{n-1}(x) = \left(\underline{\mathbf{x}}\,(Q')^{-1}\right) q_{n-1}(x)$$

for $n \geq 2$. This proves d), since the case $n = 1$ follows from Lemma 2.3.4a and Theorem 2.2.22b.
In order to prove e) we write $n\frac{q_n(x)}{x} = \sum_{k=0}^{n-1} c_k\, q_k(x)$. Using d) and Lemma 2.2.21 we obtain

$$c_k = \left(Q^k\, n\,\frac{q_n(x)}{x}\right)(0) = \left(Q^k\,(Q')^{-1}\, q_{n-1}\right)(0) = \left((Q')^{-1}\, q_{n-1-k}\right)(0) =$$

$$(n-k)\left(\frac{q_{n-l}(x)}{x}\right)(0) = (n-k)\, g_{n-k}.$$

This completes the proof. □

Remark 2.3.7 It follows from the Rodrigues Formula that the Roman shift, i.e. the linear operator that takes q_n to q_{n+1} can be explicitly expressed as $(n+1)\, x\, (Q')^{-1}$ (cf. Remark 2.2.20b). The name Rodrigues Formula comes from the theory of orthogonal polynomials (see e.g. [58, 186]). An example of a classical Rodrigues Formula can be found in Example 2.3.8e.

Examples 2.3.8 We consider the delta operators of Examples 2.2.6 and use Theorem 2.3.6 to calculate the corresponding basic sequences (cf. Examples 2.2.16).
a) Consider the differentiation operator D. It is clear that $D' = I$ and that $U = I$, since $D = DI$. Thus Theorem 2.3.6a yields $q_n(x) = \frac{x^n}{n!}$.
b) Consider the forward difference operator $E^1 - I$. Then $\left(E^1 - I\right)' = \left(E^1\right)' = \left(e^D\right)'$ (use Theorem 2.2.22a) $= e^D$ (use Lemma 2.3.4a) $= E^1$. Thus Theorem 2.3.6d yields $q_n(x) = \frac{x}{n}\, E^{-1}\, q_{n-1}(x)$. Since $q_0 = 1$, induction on n yields $q_n(x) = \binom{x}{n} := \frac{x(x-1)\ldots(x-n+1)}{n!}$.
c) Consider the backward difference operator $I - E^{-1}$. In the same way as in b) we now find that $q_n(x) = \binom{x+n-1}{n} := \frac{x(x+1)\ldots(x+n-1)}{n!}$.
d) Consider the Abel operator DE^a for some fixed a. Obviously $U = E^a$, so $U^{-n} = E^{-na}$ for all $n \in \mathbb{N}$. Thus Theorem 2.3.6c yields $q_n(x) = \frac{x(x-na)^{n-1}}{n!}$.
e) Consider the Laguerre operator L of Example 2.2.3g. We will show that the basic sequence of the Laguerre operator is the sequence of Laguerre polynomials $L_n^{(-1)}$. We know from Example 2.2.8b that $L = -\sum_{k=0}^{\infty} D^k = D\,(D-I)^{-1}$, hence $U = (D-I)^{-1}$ in this case. Thus Theorem 2.3.6c yields $q_n(x) = \frac{x^n}{n!}(D-I)^n\, x^{n-1} = \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1}\frac{x^k}{k!}$. Since $e^x\, D\left(e^{-x}\, p\right) = e^x\left(e^{-x}\, p' - e^{-x} p\right) = (D-I)\,(p)$, we may write

$$q_n(x) = \frac{x^n}{n!}\, e^x\, D^n\left(e^{-x}\, x^{n-1}\right)$$

which is the classical Rodrigues formula for the Laguerre polynomials $L_n^{(-1)}$. The formula $L\,q_n = q_{n-1}$ is the recurrence formula $q_n' = q_{n-1}' - q_{n-1}$, since $L = D\,(D-I)^{-1}$. Since $L' = -(D-I)^{-2}$, Theorem 2.3.6d yields $nq_n(x) = -x\,(D-I)^2\,q_{n-1}(x)$.

We conclude this section with a discussion of umbral operators. Umbral operators play an important role in the connection-constant problem which will be discussed below.

Umbral operators were introduced by Rota to give a rigorous foundation to the so-called classical Umbral Calculus (also called Symbolic Calculus or Blissard Calculus). For more information on umbral operators, see [210, pp. 705-706], [102], [130], [202]. For more information on the classical Umbral Calculus we refer to [23, 113].

Definition 2.3.9 *An* **umbral operator** T *is a linear operator on* $\mathcal{P}$ *such that there exist basic sequences* $(r_n)_{n\in\mathbb{N}}$ *and* $(v_n)_{n\in\mathbb{N}}$ *with* $T\,r_n = v_n$ *for all* $n \in \mathbb{N}$.

It is important to have basic sequences in Definition 2.3.9, since this implies $\deg r_n = \deg v_n = n$ for all $n \in \mathbb{N}$ by Remark 2.2.14. Hence, both $(r_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are bases for $\mathcal{P}$.

Some important properties of umbral operators are listed in Theorem 2.3.11, which is an extension of [210, Proposition 1]). The following theorem is important for our proof of Theorem 2.3.11.

Theorem 2.3.10 *Let* Q *be a delta operator with basic sequence* $(q_n)_{n\in\mathbb{N}}$ *and let the sequence* $(p_n)_{n\in\mathbb{N}}$ *be given by* $p_n = \sum_{k=0}^n a_{n,k}\,q_k$. *Then* $(p_n)_{n\in\mathbb{N}}$ *is a sequence of polynomials of convolution type if and only if there exists a sequence* $(\gamma_n)_{n\in\mathbb{N}}$ *such that* $\gamma_0 = 0$ *and* $a_{n,k} = \gamma_n^{k*}$ *for all* k *and* n. *Moreover,* $\gamma_n = (Q\,p_n)(0)$.

Proof: '$\Leftarrow$' This follows from Lemma 2.1.6.

'$\Rightarrow$' We construct the sequence $(\gamma_n)_{n\in\mathbb{N}}$ by induction. Set $\gamma_0 = 0$. It follows from $p_1(0) = 0$ and Theorem 2.1.12b that either $p_1 = 0$ or $\deg p_1 = \deg q_1 = 1$. Hence, there is a unique γ_1 such that $p_1 = \gamma_1\,q_1$. Suppose by induction that γ_k has been constructed for $k < n$ such that $p_m = \gamma_m^{k*}\,q_k$ for $m < n$. Since $\gamma_0 = 0$, Lemma 2.1.5c yields that γ_n^{k*} is a polynomial in $\gamma_1, \ldots, \gamma_{n-1}$ for $2 \le k \le n$. Thus we can choose γ_n such that $p_n(1) = \sum_{k=0}^n \gamma_n^{k*}\,q_k(1)$. It follows from Lemma 2.1.6 that $p_n(m) = \sum_{k=0}^n \gamma_n^{k*}\,q_k(m)$ for all $m \in \mathbb{N}$. Thus $p_n = \sum_{k=0}^n \gamma_n^{k*}\,q_n$, since p_n and $\sum_{k=0}^n \gamma_n^{k*}\,q_k$ are polynomials.

The last statement follows from $Q\,q_n = q_{n-1}$ and $q_n(0) = 0$ for $n \ge 1$. □

In [103], Garsia and Joni study equivalence classes whose elements are sequences of polynomials $(p_n)_{n\in\mathbb{N}}$ of the form $p_n = \sum_{k=0}^n \gamma_n^{k*}\,q_n$, where $(q_n)_{n\in\mathbb{N}}$ is an arbitrary fixed sequence of polynomials. Representations of the form $q_n = \sum_{k=0}^n \gamma_n^{k*}\,\binom{x+k-1}{k}$ are used in Section 5.3 in the context of renewal theory. This idea is due to Stam ([222]).

The following theorem describes the basic properties of umbral operators. As an introduction to parts d), e) and g), we let Q and P be delta operators with basic sequence $(q_n)_{n\in\mathbb{N}}$, $(p_n)_{n\in\mathbb{N}}$ respectively. Let T be the umbral operator that maps r_n to q_n for all $n \in \mathbb{N}$. This leads to the following commutative diagram.

$$\begin{tikzcd} q_n \arrow[r, "Q"] & q_{n-1} \arrow[d, "T"] \\ p_n \arrow[u, "T^{-1}"] \arrow[r, "P"] & p_{n-1} \end{tikzcd}$$

We immediately read off that $P = T\,Q\,T^{-1}$.

Theorem 2.3.11 *Let T be an umbral operator. Then:*

a) T is invertible.

b) T is shift-invariant if and only if $T = I$.

c) If $(p_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of polynomials of convolution type, then $(T\,p_n)_{n\in\mathbb{N}}$ is also of convolution type.

d) If $(q_n)_{n\in\mathbb{N}}$ is the basic sequence of the delta operator Q, then $(T\,q_n)_{n\in\mathbb{N}}$ is the basic sequence of the delta operator $T\,Q\,T^{-1}$.

e) If Q is a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$, then $T\,Q^n\,T^{-1} = P^n$, where P is the delta operator of the basic sequence $(T\,q_n)_{n\in\mathbb{N}}$.

f) The map $S \longrightarrow T\,S\,T^{-1}$ is an automorphism of the sequence of linear shift-invariant operators on $\mathcal{P}$.

g) The map $Q \longrightarrow T\,Q\,T^{-1}$ is an automorphism of the sequence of delta operators on $\mathcal{P}$.

Proof: Let $(r_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be basic sequences such that $T\,r_n = v_n$. Let R and V be the delta operators of $(r_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ respectively.
a) Since $\deg\, r_n = \deg\, v_n = n$ for all $n \in \mathbb{N}$, T is invertible by Corollary 2.2.11.
b) If T is shift-invariant, then Corollary 2.2.12 yields $R\,v_n = R\,T\,r_n = T\,R\,r_n = T\,r_{n-1} = v_{n-1}$ for $n \geq 1$. Hence, $r_n = v_n$ for all $n \in \mathbb{N}$, since both $(r_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are basic sequences for R.
c) By Theorem 2.3.10, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_0 = 0$ and $p_n = \sum_{k=0}^{n} \gamma_n^{k*}\, r_k$. Thus $T\,p_n = \sum_{k=0}^{n} \gamma_n^{k*}\, v_k$ and Theorem 2.3.10 implies that $(T\,p_n)_{n\in\mathbb{N}}$ is of convolution type.
d) We know from c) that $(T\,q_n)_{n\in\mathbb{N}}$ is of convolution type. Since $q_1 = g_1\,x$, we have $\deg(T\,q_1) = 1$. Thus $(T\,q_n)_{n\in\mathbb{N}}$ is a basic sequence by Theorem 2.2.19b. Because $(T\,Q\,T^{-1})\,(T\,q_n) = T\,q_{n-1}$ for $n \geq 1$, $T\,Q\,T^{-1}$ is a linear shift-invariant operator on $\mathcal{P}$ by Theorem 2.2.19a. Moreover, $T\,Q\,T^{-1}\,x$ is a nonzero constant, since $\deg(T\,q_1) = 1$. Hence, $T\,Q\,T^{-1}$ is a delta operator.
e) This follows from d) and $T\,Q\,T^{-1} = T\,Q^n\,T^{-1}$.

f) Let S be an arbitrary linear shift-invariant operator on $\mathcal{P}$. By Theorem 2.2.22a, there exist constants a_n such that $S = \sum_{n=0}^{\infty} a_n D^n$. Then e) yields $T S T^{-1} = \sum_{n=0}^{\infty} a_n D^n$. Hence, $T S T^{-1}$ is a linear shift-invariant operator on $\mathcal{P}$. Injectivity of $S \longrightarrow T S T^{-1}$ follows from a). We now want to prove surjectivity. Let W be an arbitrary shift-invariant operator on $\mathcal{P}$. Since T^{-1} is also an umbral operator, it follows that $S := T^{-1} W T$ is a linear shift-invariant operator that satisfies $T S T^{-1} = W$.
g) It follows from a) and d) that $Q \longrightarrow T Q T^{-1}$ is an injective homomorphism of the sequence of delta operators in itself. Surjectivity follows as in the proof of f). □

For a probabilistic interpretation of umbral operators we refer to Section 3.5.

Now that we know how to calculate basic sequences, we are ready to discuss the problem of connection coefficients. The problem of connection coefficients consists of finding numbers $a_{n,k}$ such that $p_n = \sum_{k=0}^{n} a_{n,k} q_k$ where $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are sequences of polynomials with $\deg p_n = \deg q_n = n$ for all $n \in \mathbb{N}$. Note that the connection coefficients are the coefficients of the basis change $(p_n)_{n\in\mathbb{N}}$ to $(q_n)_{n\in\mathbb{N}}$ (cf. [94]).

If $(q_n)_{n\in\mathbb{N}}$ is a basic sequence and p is an arbitrary polynomial, then the connection coefficients can be calculated with Lemma 2.2.21.

Example 2.3.12 Consider the polynomials $\binom{n\,x}{n}$ which are not of convolution type (e.g, the convolution identity of Definition 2.1.1 is not satisfied for $n = 2$ and $y = \frac{1}{2}$). Since

$$\left((E^1 - I)^k \binom{n\,x}{n}\right)(0) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \binom{n\,j}{n},$$

Lemma 2.2.21 yields the following expansion in terms of the basic polynomials $\binom{x}{n}$ of Example 2.2.16b:

$$\binom{n\,x}{n} = \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \binom{n\,j}{n}\right] \binom{x}{k}.$$

If both $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are sequences of polynomials of convolution type, then the Rota theory gives the following elegant answer (cf. [162, p. 202]).

Theorem 2.3.13 *Let P and Q be delta operators with basic sequences $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ respectively. Let T be the umbral operator defined by $T q_n := \frac{x^n}{n!}$ for all $n \in \mathbb{N}$. Then the constants $a_{n,k}$ $(k, n \in \mathbb{N})$, defined by $p_n := \sum_{k=0}^{n} a_{n,k} q_k$, are uniquely determined as follows. The polynomials r_n, defined by $r_n(x) := \sum_{k=0}^{n} a_{n,k} \frac{x^k}{k!}$, are the basic polynomials of the delta operator $T P T^{-1}$. Moreover, if $P = \sum_{i=1}^{\infty} a_i Q^i$, then $T P T^{-1} = \sum_{i=1}^{\infty} a_i D^i$.*

Proof: It follows from Theorem 2.3.11d that $T\,P\,T^{-1}$ is a delta operator with basic sequence $(T\,p_n)_{n\in\mathbb{N}}$. Since $r_n = T\,p_n$ for all $n \in \mathbb{N}$, $(r_n)_{n\in\mathbb{N}}$ is the basic sequence of $T\,P\,T^{-1}$. For the last statement, note that $T\,P\,T^{-1} = D$ by Theorem 2.3.11d, since $T\,q_n = \frac{x^n}{n!}$. The last statement now follows from Theorem 2.3.11e. □

A different description of the connection coefficients is given in Theorem 2.3.10. Other descriptions of connection coefficients can be found in [99].

Examples 2.3.14 a) We want to express the lower factorials in terms of upper factorials of Example 2.2.16c, i.e. we want to calculate coefficients $a_{n,k}$ such that $\binom{x}{n} = \sum_{k=0}^{n} a_{n,k} \binom{x+k-1}{k}$. We apply Theorem 2.3.13 with $P = E^1 - I$, $Q = I - E^{-1}$ (of course, we could also apply Lemma 2.2.21). Let T be the umbral operator defined by $T\binom{x+n-1}{n} = \frac{x^n}{n!}$ for all $n \in \mathbb{N}$. Theorem 2.2.22a yields $P = \sum_{n=0}^{\infty} \left(P\binom{x+n-1}{n}\right)(0)\,Q^n = \sum_{n=1}^{\infty} Q^n$. Hence, it follows from Theorems 2.3.11d and 2.3.11e that

$$T\,P\,T^{-1} = \sum_{n=1}^{\infty} (T\,Q\,T^{-1})^n = \sum_{n=1}^{\infty} D^n = D\,(I - D)^{-1}.$$

Thus the coefficients $a_{n,k}$ are the coefficients of the polynomials $q_n(-x)$, where $(q_n)_{n\in\mathbb{N}}$ are the Laguerre polynomials of Example 2.3.8e.
Another relation between these polynomials is $\binom{x}{n} = (-1)^n \binom{-x+n-1}{n}$.
b) We want to derive duplication formulas for the Laguerre polynomials q_n of Example 2.3.8e. Fix α and define polynomials p_n by $p_n(x) := q_n(\alpha\,x)$ for all x. Let W be the umbral operator defined by $W\,x^n := \alpha^n\,x^n$. Note that $W\,q_n = p_n$. It follows from Theorem 2.3.11d that $(p_n)_{n\in\mathbb{N}}$ is the basic sequence of the delta operator P, defined by $P := W\,L\,W^{-1} = \alpha^{-1}\,D\,\left(\alpha^{-1}\,D - I\right)^{-1}$. Theorem 2.3.13 yields that the connection coefficients of $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are the coefficients of the basic sequence of the delta operator $T\,P\,T^{-1}$, where T is the umbral operator defined by $T\,q_n := \frac{x^n}{n!}$ for all $n \in \mathbb{N}$. By Theorem 2.2.22a,

$$D = \sum_{k=0}^{\infty} (D\,q_k(0))\,L^k = \sum_{k=1}^{\infty} (-1)^k\,L^k = L\,(L - I)^{-1},$$

since $q_n(x) = \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{x^k}{k!}$ (see Example 2.3.8e). Hence,

$$P = \alpha^{-1}\,L\,\left(I - (1 - \alpha^{-1})\,L\right)^{-1}$$

and the last statement of Theorem 2.3.13 yields

$$T\,P\,T^{-1} = \alpha^{-1} \sum_{n=1}^{\infty} (1 - \alpha^{-1})^{n-1}\,D^n = D\,\left(\alpha I + (1 - \alpha)\,D\right)^{-1}.$$

It follows from Theorem 2.3.6c that the basic sequence $(r_n)_{n\in\mathbb{N}}$ of $T\,P\,T^{-1}$ is given by

$$\frac{x}{n}\,(\alpha I\,+\,(1-\alpha)\,D)^n\,\frac{x^{n-1}}{n-1!}\;=\frac{x}{n}\sum_{k=0}^{n}\binom{n}{k}\alpha^k\,(1-\alpha)^{n-k}\,D^{n-k}\,\frac{x^{n-1}}{n-1!}=$$

$$\sum_{k=1}^{n}\binom{n}{k}\alpha^k\,(1-\alpha)^{n-k}\,\frac{x^{k-1}}{k-1!}=\sum_{k=1}^{n}\binom{n-1}{k-1}\alpha^k\,(1-\alpha)^{n-k}\,\frac{x^k}{k!}.$$

Putting everything together yields the following duplication formula for the Laguerre polynomials of Example 2.3.8e:

$$q_n(\alpha\,x)=\sum_{k=1}^{n}\binom{n-1}{k-1}\alpha^k\,(1-\alpha)^{n-k}\,q_k(x)\qquad\qquad (n\geq 1).$$

2.4 Sheffer sequences

Most properties of a basic sequence $(q_n)_{n\in\mathbb{N}}$ essentially depend only on the property $Q\,q_n = q_{n-1}$ (cf. Definition 2.2.13). Thus it seems plausible that the theory of basic sequences can be extended under weaker conditions. This is indeed the case, as the theory of Sheffer sequences shows (see [202, Chapter 2] or [210, Section 5]). In this section we will slightly generalize the Rota notion of Sheffer sequence.

Definition 2.4.1 *Let Q be a delta operator. A sequence of polynomials $(s_n)_{n\in\mathbb{N}}$ is called a* **wide sense Sheffer sequence for** *Q if:*

1. *s_0 is constant*
2. *$Q\,s_n = s_{n-1}$, $n = 1, 2, \ldots$.*

If moreover $s_0 \neq 0$, then $(s_n)_{n\in\mathbb{N}}$ is called a **strict sense Sheffer sequence for** *Q.*

The definition of Sheffer sequence in [210] is (apart from a factor $n!$) what we have called strict sense Sheffer sequence.
Note that if $(s_n)_{n\in\mathbb{N}}$ is a Sheffer sequence in the strict sense then, by Corollary 2.2.9b, deg $s_n = n$ for all $n \in \mathbb{N}$.

Theorem 2.4.2 *Let Q be a delta operator. A sequence $(w_n)_{n\in\mathbb{N}}$, which is not identically zero, is a wide sense Sheffer sequence for Q if and only if there exist an $N \in \mathbb{N}$ and a strict sense Sheffer sequence $(s_n)_{n\in\mathbb{N}}$ for Q such that $w_n = 0$ for $n < N$ and $w_n = s_{n-N}$ for $n \geq N$.*

Proof: '$\Leftarrow$' Clearly w_0 is constant and $Q\,w_n = w_{n-1}$ for all $n \geq 1$.
'$\Rightarrow$' If w_0 is a nonzero constant, then there is nothing to prove. Assume that $w_0 = 0$. Let $N := \min\{n : w_n \neq 0\}$. Then w_N is a (nonzero) constant by

Corollary 2.2.9b since $Q\,w_N = w_{N-1} = 0$. Define $(s_n)_{n\in\mathbb{N}}$ by $s_n := w_{n+N}$. Then $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence for Q. □

Theorem 2.4.2 seems to indicate that the notion of wide sense Sheffer sequence is not very useful. However, we will see nice applications of this notion in the proofs of Corollary 2.4.10 and Theorem 3.3.2.

It follows from Theorems 2.2.19 and 2.4.2 that a sequence of polynomials can be a Sheffer sequence of either type for at most one delta operator.

Examples 2.4.3 a) Strict sense Sheffer polynomials for the differentiation operator D are called Appell polynomials. They were studied by Appell in [10]. Examples of Appell polynomials include the Hermite polynomials H_n, defined by

$$\sum_{n=0}^{\infty} H_n(x)\, z^n = \exp\left(xz - \frac{1}{2} z^2\right),$$

and the Bernoulli polynomials B_n, defined by

$$\sum_{n=0}^{\infty} B_n(x)\, z^n = \frac{z}{e^z - 1} e^{x\,z}.$$

It follows directly from their generating functions or from Theorem 2.4.4d that these polynomials are Appell polynomials, i.e. $D\,H_n = H_{n-1}$ and $D\,B_n = B_{n-1}$ for $n \geq 1$. We will see at the end of this section that the Hermite and Bernoulli polynomials belong to the class of Wick polynomials, which is a subclass of the Appell polynomials.
b) The Laguerre polynomials of order α are strict sense Sheffer sequences for the Laguerre operator of Example 2.2.3g. The Laguerre polynomials of Example 2.3.8e are the Laguerre polynomials of order $\alpha = -1$ (cf. [202, p. 108]).

Both types of Sheffer sequences satisfy a convolution-like equation (see Theorem 2.4.4b below).

Theorem 2.4.4 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Then the following are equivalent:*

a) $(w_n)_{n\in\mathbb{N}}$ is a wide sense Sheffer sequence for Q.

b) $w_n(x+y) = \sum_{k=0}^{n} w_k(x)\, q_{n-k}(y)$ *for all $n \in \mathbb{N}$ and all x, y.*

c) $w_n = \sum_{k=0}^{n} w_k(0)\, q_{n-k}$ *for all $n \in \mathbb{N}$.*

d) there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $w_n = \sum_{k=0}^{n} a_k\, q_{n-k}$ *for all $n \in \mathbb{N}$.*

Proof: a $\Rightarrow$ b' Fix an arbitrary x. Applying the Polynomial Expansion Theorem 2.2.21 to $E^x\, w_n$ we obtain

$$E^x\, w_n = \sum_{i=0}^{\infty} \left(Q^i\, E^x\, w_n\right)(0)\, q_i = \sum_{i=0}^{n} w_{n-i}(x)\, q_i,$$

since $\deg w_n \le n$. Hence, it follows that

$$w_n(x+y) = \sum_{i=0}^{n} w_{n-i}(x)\, q_i(y) = \sum_{k=0}^{n} w_k(x)\, q_{n-k}(y)$$

for all x and y, since x is arbitrary.
'b $\Rightarrow$ c' This follows by setting $x = 0$.
'c $\Rightarrow$ d' Take $a_k := w_k(0)$.
'd$\Leftarrow$a' Note that w_0 is constant because $w_0 = a_0\, q_0 = a_0$. If $n \ge 1$, then $Q\, w_n = Q\left(\sum_{k=0}^{n} a_k\, q_{n-k}\right) = \sum_{k=0}^{n-1} a_k\, q_{n-1-k} = w_{n-1}$. Hence, $(w_n)_{n\in\mathrm{N}}$ is a wide sense Sheffer sequence for Q. □

Remarks 2.4.5 a) Theorem 2.4.4 also holds for strict sense Sheffer sequences if we add the condition $w_0 \neq 0$ to b) and c) and if we add the condition $a_0 \neq 0$ to d).
b) If $(q_n)_{n\in\mathrm{N}}$ is merely a sequence of polynomials of convolution type instead of a basic sequence (cf. Remark 2.2.20a), then b), c) and d) of Theorem 2.4.4 are still equivalent and Corollary 2.4.6 below also holds.

Corollary 2.4.6 *Let $(w_n)_{n\in\mathrm{N}}$ be a wide sense Sheffer sequence for the delta operator Q with basic sequence $(q_n)_{n\in\mathrm{N}}$. Let $(g_n)_{n\in\mathrm{N}}$ be the coefficient sequence of $(q_n)_{n\in\mathrm{N}}$. Then the following formal generating function identity holds:*

$$\sum_{n=0}^{\infty} w_n(x)\, t^n = \left(\sum_{n=0}^{\infty} w_n(0)\, t^n\right)\; \mathit{exp}\; \left(x \sum_{n=0}^{\infty} g_n\, t^n\right).$$

Proof: This follows directly from Theorems 2.1.13d and 2.4.4c. □

The following theorem describes the difference between wide sense and strict sense Sheffer sequences (of a delta operator Q with basic sequence $(q_n)_{n\in\mathrm{N}}$) in terms of the linear operator A on $\mathcal{P}$, defined by $A\, q_n := s_n$. It follows directly from Theorem 2.2.22a that $A = \sum_{k=0}^{\infty} s_k(0)\, Q^k$ (cf. the proof of [210, Corollary 1]). We also give a description of strict sense Sheffer sequences in terms of delta operators and functionals in the style of [202, 207]. We first need a lemma.

Lemma 2.4.7 *Let Λ be a linear functional such that $\Lambda\, 1 \neq 0$ and let Q be a delta operator on $\mathcal{P}$. There exists a unique sequence of polynomials $(p_n)_{n\in\mathrm{N}}$ with $\deg p_n = n$ for all $n \in \mathbb{N}$ such that $\Lambda\, Q^k\, p_n = \delta_{nk}$ for all $k, n \in \mathbb{N}$, where δ_{nk} denotes the Kronecker delta.*

Proof: Existence follows in the same way as in the proof of Theorem 2.2.15. In order to prove uniqueness, consider another sequence $(\widetilde{p}_n)_{n\in\mathbb{N}}$ such that $\Lambda\, Q^k\, p_n = \Lambda\, Q^k\, \widetilde{p}_n$ for all $k, n \in \mathbb{N}$. Suppose there is an $n \in \mathbb{N}$ such that $p_n \neq \widetilde{p}_n$. Let ℓ be the degree of $p_n - \widetilde{p}_n$. Then $Q^\ell\,(p_n - \widetilde{p}_n)$ is a non-zero constant, which contradicts $\Lambda\, Q^k\,(p_n - \widetilde{p}_n) = 0$. □

Theorem 2.4.8 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of polynomials and define the linear operator A on $\mathcal{P}$ by $A\, q_n := s_n$ for all $n \in \mathbb{N}$. Then:*

a) $(s_n)_{n\in\mathbb{N}}$ is a wide sense Sheffer sequence for Q if and only if A is shift-invariant.

*b) (**[210, Proposition 1]**) $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence for Q if and only if A is shift-invariant and invertible.*

c) $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence for Q if and only if there exists a linear functional Λ on $\mathcal{P}$ such that $\Lambda\, 1 \neq 0$ and $\Lambda\, Q^k\, s_n = \delta_{nk}$ for all $k, n \in \mathbb{N}$, where δ_{nk} denotes the Kronecker delta.

d) If $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence, then $\Lambda p = A^{-1} p(0)$ for all $p \in \mathcal{P}$, where A is as in a).

Proof: a) '⇒' Since $(q_n)_{n\in\mathbb{N}}$ is of convolution type, we have for all y

$$A\, E^y\, q_n = A\left(\sum_{k=0}^{n} q_k(y)\, q_{n-k}\right) = \sum_{k=0}^{n} q_k(y)\, s_{n-k} = E^y\, s_n = E^y\, A\, q_n.$$

Hence, by linearity, $A\, E^y = E^y\, A$ for all y.
'⇐' Corollary 2.2.9a and $s_0 = A\, q_0 = A\, 1$ together imply that s_0 is constant. Using Corollary 2.2.12 we see that $Q\, s_n = Q\, A\, q_n = A\, Q\, q_n = A\, q_{n-1} = s_{n-1}$ for $n \geq 1$.
b) '⇒' Shift-invariance follows from a). By Corollary 2.2.9b, deg $s_n = n$ for all $n \in \mathbb{N}$. Hence, A is invertible by Corollary 2.2.11.
'⇐' We need only prove that $s_0 \neq 0$ because of a). This follows from Corollary 2.2.11 and $s_0 = A\, q_0$, since A is invertible.
c) '⇒' Define the linear functional Λ by $\Lambda\, s_n = \delta_{0n}$. Because s_0 is a nonzero constant, we have $\Lambda\, 1 \neq 0$. Moreover, since shift-invariant operators commute by Corollary 2.2.12, it follows that $\Lambda\, Q^k\, s_n = \Lambda\, Q^k\, A\, q_n = \Lambda\, A\, Q^k\, q_n = \delta_{0,n-k} = \delta_{nk}$.
'⇐' Define the polynomials r_n by $r_n := Q\, s_{n+1}$ $(n \in \mathbb{N})$. Then $\Lambda Q^k\,(Q\, s_{n+1}) = \delta_{k+1,n+1} = \delta_{k,n}$. By the uniqueness part of Lemma 2.4.7, we have $Q\, s_{n+1} = s_n$ for all $n \in \mathbb{N}$. Thus $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence.
It follows from $\Lambda\, Q^k\, s_n = \Lambda\, s_{n-k} = \delta_{nk}$ with $k = 0$ that $\Lambda\, s_n = \delta_{0n}$. Since $A^{-1}\, s_n(0) = q_n(0) = \delta_{0n}$ by Definition 2.2.13 and $\deg s_n = n$ for all $n \in \mathbb{N}$, the results follows. □
The operator A of the above theorem is called **invertible operator**.

Corollary 2.4.9 *Let $(s_n)_{n\in\mathrm{N}}$ be a strict sense Sheffer sequence for the delta operator Q with basic sequence $(q_n)_{n\in\mathrm{N}}$ and invertible operator A. Let $(g_n)_{n\in\mathrm{N}}$ be the coefficient sequence of $(q_n)_{n\in\mathrm{N}}$ and let g be the formal power series defined by $g(t) := \sum_{n=0}^{\infty} g_n\, t^n$. Then the following formal generating function identity holds:*

$$\sum_{n=0}^{\infty} s_n(x)\, t^n = f(g(t))\, e^{x\, g(t)},$$

where $A = f(D)$.

Proof: Define $s(t) := \sum_{n=0}^{\infty} s_n(x)\, t^n$. It follows from Theorem 2.2.22a that $A = \sum_{k=0}^{\infty} s_k(0)\, Q^k$. Hence, by the Isomorphism Theorem 2.3.1, we have $A = s(Q)$. Since $g_0 = 0$, the formal power series is invertible (w.r.t. to composition, cf. [172]). Hence, there exists a formal power series f such that $s = f \circ g$. By Theorem 2.2.22b, we have $A = f(g(Q)) = f(D)$. The result now follows from Corollary 2.4.6. □

Corollary 2.4.10 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathrm{N}}$.*

a) The sequence $(s_n)_{n\in\mathrm{N}}$ defined by $s_n(x) := (n+1)\,\frac{q_{n+1}(x)}{x}$ $(x \neq 0)$ and $s_n(0) := (n+1)\,(q_{n+1})'(0)$ is a strict sense Sheffer sequence.

b) The sequence $(w_n)_{n\in\mathrm{N}}$ defined by $w_n(x) := n\,\frac{q_n(x)}{x}$ $(x \neq 0)$ and $w_n(0) := n\, q_n'(0)$, is a wide sense Sheffer sequence.

c) The sequence $(s_n)_{n\in\mathrm{N}}$ defined by $s_n := (q_{n+1})'$ is a strict sense Sheffer sequence.

d) The sequence $(w_n)_{n\in\mathrm{N}}$ defined by $w_n(x) := q_n'$ is a wide sense Sheffer sequence.

*e) (***Niederhausen***) The sequence $(s_n)_{n\in\mathrm{N}}$ defined by*

$$s_n(x) := \frac{x - an - b}{x - b}\, q_n(x - b)$$

is a strict sense Sheffer sequence.

Proof: a) Recall that $q_n(0) = 0$ for $n \geq 1$ by Theorem 2.1.12e). Then $(s_n)_{n\in\mathrm{N}}$ is a strict sense Sheffer sequence by Theorem 2.4.8b, since $s_n = (Q')^{-1}\, q_n$ by Theorem 2.3.6d.
b) This follows from a) and Theorem 2.4.2.
d) It follows from Theorem 2.4.8a that $(w_n)_{n\in\mathrm{N}}$ is a wide sense Sheffer sequence.
c) By Theorem 2.2.22b, $D\, q_{n+1} = \sum_{k=0}^{n+1} g_k\, q_{n-k}$. Thus $s_0 = g_1 \neq 0$ and $(s_n)_{n\in\mathrm{N}}$ is a strict sense Sheffer sequence by Theorem 2.4.4 or Remark 2.4.5.
e) First note that $(E^{-b}\, q_n)_{n\in\mathrm{N}}$ is a strict sense Sheffer sequence by Theorem 2.4.8b. By b) and Theorem 2.4.8a, $\left(n\frac{q_n(x-b)}{x-b}\right)_{n\in\mathrm{N}}$ is a wide sense Sheffer

sequence. Since linear combinations of wide sense Sheffer sequences are wide sense Sheffer, the decomposition

$$\frac{x-an-b}{x-b}\, q_n(x-b) = q_n(x-b) - \frac{an}{x-b}\, q_n(x-b)$$

shows that $(s_n)_{n\in\mathbb{N}}$ is a wide sense Sheffer sequence. A closer look at the decomposition reveals that $\deg s_n = n$, thus $(s_n)_{n\in\mathbb{N}}$ is even a strict sense Sheffer sequence. □

We now extend the Expansion Theorems 2.2.21 and 2.2.22 to strict sense Sheffer sequences.

Theorem 2.4.11 *Let $(s_n)_{n\in\mathbb{N}}$ be a strict sense Sheffer sequence with delta operator Q and let A be the linear operator on $\mathcal{P}$ defined by $A\,q_n := s_n$.*

a) For all $p \in \mathcal{P}$, we have

$$p = \sum_{k=0}^{\infty} (A^{-1}\, Q^k p)(0)\, s_k.$$

b) If T is a linear shift-invariant operator, then

$$T = \sum_{k=0}^{\infty} (T s_k(0))\, A^{-1}\, Q^k$$

Proof: a) Apply Theorem 2.2.21 to $p = A\left(A^{-1}\, p\right)$ and use shift-invariance.
b) Apply Theorem 2.2.22 to $T = A^{-1}\,(A\,T)$ and use shift-invariance. □

Theorem 2.4.8 enables us to generalize Theorem 2.3.10 to strict sense Sheffer sequences. A generalization to wide sense Sheffer sequences is not possible (see Remark 2.4.13).

Theorem 2.4.12 *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Let $(s_n)_{n\in\mathbb{N}}$ be a strict sense Sheffer sequence for Q and let A be the linear operator on $\mathcal{P}$ defined by $A\,q_n := s_n$. The following are equivalent for a sequence $(r_n)_{n\in\mathbb{N}}$ of polynomials:*

a) $(r_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence and there exists a basic sequence $(p_n)_{n\in\mathbb{N}}$ such that $r_n = A\,p_n$ for all $n \in \mathbb{N}$.

b) there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_0 = 0$ and $\gamma_1 \neq 0$ such that $r_n = \sum_{k=0}^{n} \gamma_n^{k*}\, s_k$ for all $n \in \mathbb{N}$.

Proof: 'a ⇒ b' Since $(p_n)_{n\in\mathbb{N}}$ is a basic sequence, Theorem 2.3.10 yields the existence of a sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_0 = 0$ such that $p_n = \sum_{k=0}^{n} \gamma_n^{k*}\, q_k$ for all $n \in \mathbb{N}$. Since $\deg p_1 = 1$ (Remark 2.2.14), we have $\gamma_1 \neq 0$. Since $A\,q_n = s_n$ for all $n \in \mathbb{N}$, it follows that $r_n = A\,p_n = \sum_{k=0}^{n} \gamma_n^{k*}\, s_k$ for all $n \in \mathbb{N}$.

'b ⇒ a' Define polynomials p_n ($n \in \mathbb{N}$) by $p_n := \sum_{k=0}^{n} \gamma_n^{k*}\, q_k$. Since $\gamma_0 = 0$ and $\gamma_1 \neq 0$, it follows from Theorem 2.3.10 and Theorem 2.2.19 that $(p_n)_{n\in\mathbb{N}}$ is a basic sequence. Moreover, it is obvious that $A\,p_n = r_n$ for all $n \in \mathbb{N}$ since $A\,q_n = s_n$. It follows from Theorem 2.4.8b that $(r_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence. □

Remark 2.4.13 There exists no analogue of Theorem 2.4.12 for wide sense Sheffer sequences $(r_n)_{n\in\mathbb{N}}$. First of all, it is necessary that $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence: if $(q_n)_{n\in\mathbb{N}}$ is not a basic sequence, then the operator A need not exist (cf. Remark 2.1.13b). Suppose $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence and $r_n = \sum_{k=0}^{n} \gamma_n^{k*}\, s_k$ for all $n \in \mathbb{N}$. If $\gamma_1 \neq 0$, then $(r_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence by Theorem 2.4.12. If $\gamma_1 = 0$, then the proof of Theorem 2.1.12c yields that $\deg r_n \leq [n/2]$ for all $n \in \mathbb{N}$. It follows from Theorem 2.4.2 that $(r_n)_{n\in\mathbb{N}}$ cannot be a wide sense Sheffer sequence.

As a corollary to Theorem 2.4.12, we now derive a Rodrigues Formula for strict sense Sheffer sequences (cf. Theorem 2.3.6d). This form of the Rodrigues Formula is due to Avramjonok (see [13]).

Theorem 2.4.14 (Avramjonok) *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Let $(s_n)_{n\in\mathbb{N}}$ be a strict sense Sheffer sequence for Q and let A be the linear operator on $\mathcal{P}$ defined by $A\,q_n := s_n$ for all $n \in \mathbb{N}$. Then we have*

$$n\, s_n(x) = \left(x\, (Q')^{-1} + (Q')^{-1}\, A'\, A^{-1}\right)\, s_{n-1}(x). \tag{2.5}$$

Proof: By Theorem 2.3.6d, we have $n\, q_n(x) = x\, (Q')^{-1}\, q_{n-1}(x)$ for all $n \geq 1$ and all x. Writing $q_k = A^{-1}\, A q_k$ ($k = n-1, n$), we obtain $n\, s_n(x) = A\, x\, (Q')^{-1}\, A^{-1}\, s_{n-1}(x)$. By the definition of Pincherle derivative, we may write $A\, x = x\, A + A'$. Substituting this into the expression for $n\, s_n(x)$, we obtain the result. □

We conclude this section with a probabilistic subclass of Appell polynomials.

Definition 2.4.15 *Let X be a random variable with finite moments of all orders. The* **Wick** *polynomial sequence associated to X is the unique sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials satisfying:*

1. *$D\, p_n = p_{n-1}$ for $n \geq 1$*
2. *$E\, p_n(X) = \delta_{0n}$,*

where D denotes the differentiation operator and δ_{0n} the Kronecker delta.

Since Λ, defined by $\Lambda(p) := E\, p(X)$ is a linear functional on $\mathcal{P}$ such that $\Lambda 1 = 1$, it follows from Lemma 2.4.7 with $Q = D$ and Theorem 2.4.8 that $(p_n)_{n\in\mathbb{N}}$ is a (well-defined) strict sense Sheffer sequence.

Wick polynomials occur in quantum mechanics and in probability theory. In the latter case, they are used for noncentral limit theorems (see e.g. [12, 106]).

Theorem 2.4.16 *Let X be a random variable with distribution function F such that its moment generating function $\int_{-\infty}^{\infty} e^{zt}\, dF(t)$ is analytic on some disc in the complex plane. Then the Wick polynomials $(p_n)_{n\in\mathbb{N}}$ associated to X possess the following generating function:*

$$\sum_{n=0}^{\infty} p_n(X)\, z^n = \frac{e^{zx}}{\displaystyle\int_{-\infty}^{\infty} e^{zt}\, dF(t)}.$$

Proof: let A be the linear operator on $\mathcal{P}$ defined by $A\,xfacn := p_n$ for all $n \in \mathbb{N}$. First note that by Theorem 2.2.22a, we have $A = \sum_{k=0}^{\infty} p_k(0)\, Q^k$. and By Theorems 2.2.7 and 2.4.10d, we have

$$A^{-1} = \sum_{n=0}^{\infty} \left(A^{-1}\frac{x^n}{n!}\right)(0)\, D^n = \sum_{n=0}^{\infty} \Lambda\left(\frac{x^n}{n!}\right) D^n = \sum_{n=0}^{\infty} E\left(\frac{x^n}{n!}\right) D^n.$$

The result now follows from Theorem 2.3.1 and Corollary 2.4.6, since under the conditions of the theorem the moment generating function equals $\sum_{n=0}^{\infty} E(X^n)\, \frac{z^n}{n!}$. □

Examples 2.4.17

1. If X is distributed according to the standard normal distribution, then its moment generating function equals $e^{\frac{1}{2}z^2}$. Thus the Wick polynomials for the standard normal distribution are the Hermite polynomials (cf. Example 2.4.3a).

2. If X is distributed according to the uniform distribution on $[0, 1]$, then its moment generating function equals $(e^z - 1)/z$. Thus the Wick polynomials for the uniform distribution on $[0, 1]$ are the Bernoulli polynomials (cf. Example 2.4.3a).

The standard theory of Wick polynomials can be derived easily from the theory of this section (cf [12, 106]).

We conclude this section by remarking that Al-Salam and Verma have generalized Sheffer sequences by considering sequences of polynomials satisfying $Qs_n = s_{n-r}$ ($r \in \mathbb{N}$) for a delta operator Q (see [7]).

2.5 Cross sequences and Steffensen sequences

In the previous section we extended the notion of basic sequence by relaxing one of the defining properties. In this section we extend the notion of basic sequence by adding an extra parameter. This extra parameter comes in naturally for basic sequences connected to probability distributions. E.g., for the Poisson-Charlier polynomials this extra parameter is the parameter of the underlying Poisson distribution and for the Hermite polynomials it is the variance of the underlying zero-mean normal distribution. This section unites and extends the results of [183], [202, Section 5.3], [210, Section 8] and [38]. New is the introduction of semigroups of shift-invariant operators.

In this section we assume that the polynomials are defined on $\mathbb{R}$ and have real coefficients. We also assume that each sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ satisfies $\deg p_n = n$. Unlike the previous section, there is no use in considering *weak* and *strong* versions of cross or Steffensen sequences.

Definition 2.5.1 *A sequence of polynomials* $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *is said to be a* **cross sequence** *if*

a) $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *is a sequence of polynomials for fixed* λ.

b)

$$q_n^{[\lambda+\mu]}(x+y) = \sum_{k=0}^{n} q_k^{[\lambda]}(x)\, q_{n-k}^{[\mu]}(y) \tag{2.6}$$

for all $n \in \mathbb{N}$ *and all* $x, y, \lambda, \mu \in \mathbb{R}$.

It is obvious that any sequence of polynomials with (formal) generating function of the form

$$\sum_{n=0}^{\infty} q_n^{[\lambda]}(x)\, t^n = e^{\lambda\, h(t)}\, A(t)\, e^{x\, g(t)}$$

is a cross sequence. Also note that $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ is Sheffer for fixed λ and that $\left(q_n^{[\lambda]}(x)\right)_{n\in\mathbb{N}}$ is a cross sequence in the variable λ with parameter x.

We now wish to give a characterization of cross sequences in terms of shift-invariant operators. Since this involves (semi-)groups of shift-invariant operators, we digress a little bit by studying these semigroups.

Definition 2.5.2 *A family* $(T_t)_{t>0}$ *of linear shift-invariant operators on* $\mathcal{P}$ *is a semigroup if* $T_{s+t} = T_s\, T_t$ *for all* $s, t > 0$.

Theorem 2.5.3 *If* $(T_t)_{t>0}$ *is a* **semigroup** *of linear shift-invariant operators on* $\mathcal{P}$*, then* T_t *is invertible for all* $t > 0$ *and hence,* $(T_t)_{t>0}$ *can be extended to a group* $(T_t)_{t\in\mathbb{R}}$.

Proof: It follows from Corollary 2.2.9a that there exist non-negative integers $n(t)$ such that $\deg(T_t p) = \max\{-1, \deg(p) - n(t)\}$ for all $t > 0$ and all $p \in \mathcal{P}$. The semigroup property implies that $n(s+t) = n(t) + n(s)$ for all $s, t > 0$. Hence, $n(t) = 0$ for all $t > 0$, since $n(t)$ is integer-valued for all $t > 0$. Thus T_t is invertible for all $t > 0$ by Corollary 2.2.11. Define $T_0 := I$ and $T_t := (T_{-t})^{-1}$

for all $t < 0$. Then $(T_t)_{t\in\mathbb{R}}$ is a group, since obviously $T_{s+t} = T_s\, T_t$ for all $s, t \in \mathbb{R}$. □

We now expand the operators T_t into powers of D as in Theorem 2.2.7 and study the coefficients of these expansions.

Theorem 2.5.4 *Let $(T_t)_{t>0}$ be a semigroup of linear shift-invariant operators on $\mathcal{P}$ and let the functions $a_n (n \in \mathbb{N})$ be defined by $T_t = \sum_{n=0}^{\infty} a_n(t)\, D^n$ for all $t > 0$ and all $n \in \mathbb{N}$. Then:*

a) *the sequence $(a_n)_{n\in\mathbb{N}}$ is a sequence of functions of convolution type.*

b) *if $(a_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions, then there exists a linear shift-invariant operator T on $\mathcal{P}$ such that $T_t = e^{tT}$ for all $t > 0$ and $(T_t)_{t>0}$ can be extended to a group $(T_t)_{t\in\mathbb{R}}$.*

Proof: a) This follows from

$$\sum_{n=0}^{\infty} a_n(s+t)\, D^n = T_{s+t} = T_s\, T_t =$$

$$\sum_{m=0}^{\infty} a_m(s) D^m \sum_{r=0}^{\infty} a_r(t)\, D^r = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k(s)\, a_{n-k}(t)\, D^n$$

for all $s, t > 0$.
b) By Theorem 2.1.8, the measurability of a_n implies that there exist an $a \in \mathbb{R}$ and a sequence of real numbers $(g_n)_{n\in\mathbb{N}}$ such that $a_n(t) = e^{at} \sum_{k=0}^{n} g_n^{k*} \frac{t^k}{k!}$. Define the linear shift-invariant operator T on $\mathcal{P}$ by $T := aI + \sum_{k=1}^{\infty} g_k\, D^k$. Then

$$e^{tT} = \exp\left(taI + t\sum_{r=0}^{\infty} g_r\, D^r\right) = \left(\sum_{k=0}^{\infty} \frac{t^k\, a^k\, I^k}{k!}\right) \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{r=0}^{\infty} g_r^{m*}\, D^r\right) =$$

$$e^{at}\left(\sum_{r=0}^{\infty}\sum_{m=0}^{\infty} \frac{{g_r}^{m*}\, t^m}{m!}\, D^r\right) = e^{at}\sum_{r=0}^{\infty} a_r(t)\, D^r = T_t.$$

Moreover, we may extend $(T_t)_{t>0}$ to a group $(T_t)_{t\in\mathbb{C}}$ by setting $T_t := e^{tT}$ for all $t \in \mathbb{C}$. □

The operator T of Theorem 2.5.4 is called **infinitesimal generator** of the semigroup $(T_t)_{t>0}$ in standard semigroup theory.

The following theorem describes when the functions a_n of the above theorem are measurable.

Theorem 2.5.5 *Let $(T_t)_{t>0}$ be a semigroup of linear shift-invariant operators on $\mathcal{P}$ and let the functions $a_n (n \in \mathbb{N})$ be defined by $T_t = \sum_{n=0}^{\infty} a_n(t)\, D^n$ for all $t > 0$ and all $n \in \mathbb{N}$. Then the following are equivalent:*

a) the functions a_n are continuous.

b) the functions a_n are measurable.

c) $\lim_{t\downarrow 0}\left(\frac{T_t-I}{t}p\right)(x) = Tp(x)$ *for all* $p \in \mathcal{P}$ *and all* $x \in \mathbb{R}$, *where* T *is the infinitesimal generator of* $(T_t)_{t>0}$.

d) $\lim_{t\downarrow 0}(T_t p)(x) = p(x)$ *for all* $p \in \mathcal{P}$ *and all* $x \in \mathbb{R}$.

Proof: '$a \Rightarrow b$' This is trivial, since continuous functions are measurable.
'$b \Rightarrow c$' By Theorem 2.5.4, we have

$$\lim_{t\downarrow 0} \frac{T_t - I}{t} p(x) = \lim_{t\downarrow 0} \frac{e^{tT} - I}{t} p(x) = \lim_{t\downarrow 0} \frac{1}{t} \sum_{n=1}^{\infty} (tT)^n \, p(x) = T\, p(x),$$

since there are only finitely many nonzero terms in the summation.
'$c \Rightarrow d$' This is trivial.
'$d \Rightarrow a$' Note that $(T_t\, x^n)(0) = \sum_{k=0}^{n} a_k(t)\,(D^k\, x^n)(0) = a_n(t)$ for all $t > 0$ and all $n \in \mathbb{N}$. Hence, $\lim_{t\downarrow 0} a_n(t)$ exists for all $n \in \mathbb{N}$. Now Theorem 2.5.4a and Remark 2.1.10g yield that a_n is continuous for all $n \in \mathbb{N}$. □

We now return to cross sequences. The following theorem characterizes cross sequences as the orbit of a basic sequence under a group of shift-invariant operators.

Theorem 2.5.6 ([210]) *A sequence* $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *is a cross sequence if and only if there exists a delta operator* Q *with basic sequence* $(q_n)_{n\in\mathbb{N}}$ *and a group of shift-invariant operators* $(T_t)_{t\in\mathbb{R}}$ *such that* $q_n^{[\lambda]} = T_\lambda\, q_n$.

Proof: '$\Rightarrow$' If $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ is a cross sequence, then it follows from (2.6) and Theorem 2.2.19 that $\left(q_n^{[0]}\right)_{n\in\mathbb{N}}$ is of convolution type with delta operator Q, say. Moreover, by setting $\mu = 0$ in (2.6) we see that for fixed λ, $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ is a Sheffer sequence with delta operator Q and invertible operator T_λ, say. In order to show that $(T_t)_{t\in\mathbb{R}}$ is a group of linear operators, it suffices to show that $T_{\lambda+\mu}\, q_n = T_\lambda\, T_\mu\, q_n$. By Theorem 2.4.4 and Remark 2.4.5a, we have $q_n^{[\lambda]}(x) = \sum_{k=0}^{n} q_{n-k}^{[\lambda]}(0)\, q_k(x)$. Since $T_\lambda\, q_n = q_n^{[\lambda]}$, it follows from Theorem 2.2.22a that $T_\lambda = \sum_{k=0}^{\infty} q_k^{[\lambda]}(0)\, Q^k$. Now $T_{\lambda+\mu}\, q_n = T_\lambda\, T_\mu\, q_n$ follows from (2.6) with $x = 0$.
'$\Leftarrow$' First note that since T_λ is an invertible shift-invariant operator, $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence for fixed λ. Hence,

$$E^y\, q_n^{[\lambda]}(x) = \sum_{k=0}^{n} q_k^{[\lambda]}(y)\, q_{n-k}(x).$$

Applying the shift-invariant operator T_μ to both sides of the last equation, we obtain that $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ is a cross sequence. □

Theorem 2.5.7 *Let* $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *be a cross sequence. Then each polynomial* p *can be expanded as*

$$p = \sum_{n=0}^{\infty} (T_\lambda\, Q)\,(0)\, q_n^{[\lambda]} \tag{2.7}$$

where Q *and* $(q_n)_{n\in\mathbb{N}}$ *are as in Theorem 2.5.6.*

Proof: This follows directly from Theorem 2.4.11, since $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ is a Sheffer sequence for fixed λ. □

Examples 2.5.8 Examples of cross sequences include:

a) (**Hermite polynomials**): $Q = D$, $T_\lambda = e^{-\frac{1}{2}\lambda D^2}$ (see [202, pp.87-97] for more details). The more general cross sequence with $T_\lambda = e^{-\lambda D^m}$ is studied in [183]. Their formulas follow directly from the results of Section 2.4 and 2.5. E.g., the Rodrigues Formula for Sheffer sequences (Theorem 2.4.14) yields

$$\begin{aligned} s_n(x) &= x\, s_{n-1}(x) + m\,\lambda\, D^{m-1}\, e^{\lambda D^m}\, x^{n-1} \\ &= x\, s_{n-1}(x) + m\,\lambda\, D^{m-1}\, s_{n-1}(x) \\ &= x\, s_{n-1}(x) + m\,\lambda\, s_{n-m}(x), \end{aligned}$$

which is [183, Formula (5.2)].

b) (**Bernoulli polynomials**): $Q = D$, $T_\lambda = (e^D - 1)/D)^{-\lambda} = (D/(e^D - 1))^\lambda$ (see [202, pp. 93-100] for more details).

c) (**Euler polynomials**): $Q = D$, $T_\lambda = ((e^D + 1)/2)^{-\lambda}$ (see [202, pp. 101-106] for more details.

d) (**Poisson-Charlier polynomials**): $Q = e^D - 1$, $T_\lambda = e^{-\lambda\,\left(e^D - 1\right)}$. This differs a factor λ^n from the ordinary definition of Poisson-Charlier polynomial (see [202, pp. 119-122] for more details).

e) (**actuarial polynomials**): $Q = \log(1 - D)$, $T_\lambda = (1 - D)^\lambda$ (see [202, pp. 123-125] for more details).

Definition 2.5.9 *A sequence of polynomials* $\left(s_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *is said to be a* **Steffensen sequence** *if*

a) $\left(s_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *is a sequence of polynomials for fixed* λ.

b) there exists a basic sequence $(q_n)_{n\in\mathbb{N}}$ *such that*

$$s_n^{[\lambda+\mu]}(x+y) = \sum_{k=0}^{n} s_k^{[\lambda]}(x)\, q_{n-k}^{[\mu]}(y) \tag{2.8}$$

for all $n \in \mathbb{N}$ *and all* $x, y, \lambda, \mu \in \mathbb{R}$.

Theorem 2.5.10 *Let* $\left(s_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *be a sequence of polynomials for fixed* λ. *Then the following are equivalent:*

a) $\left(s_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *is a Steffensen sequence*

b) there exists a cross sequence $\left(q_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *and an invertible shift-invariant operator* A *such that* $s_n^{[\lambda]} = Aq_n^{[\lambda]}$ *for all* $n \in \mathbb{N}$

c) there exists a group $(T_t)_{t\in\mathbb{R}}$ *of linear shift-invariant operators and a Sheffer sequence* $(s_n)_{n\in\mathbb{N}}$ *such that* $s_n^{[\lambda]} = T_\lambda\, s_n$.

Proof: '$a \Rightarrow b$' Setting $\lambda = x = 0$ in (2.8) we see that $s_n^{[\mu]} = \sum_{k=0}^{n} s_k^{[0]}(0)\, q_{n-k}^{[\mu]}$. Note that $s_0^{[0]} \neq 0$, since $\left(s_n^{[0]}\right)_{n\in\mathbb{N}}$ is Sheffer. Hence, $s_n^{[\mu]} = Aq_n^{[\lambda]}$, where A is the invertible linear shift-invariant operator defined by $A := \sum_{k=0}^{\infty} s_k^{[0]}(0)\, Q^k$.
'$b \Rightarrow c$' This follows directly from Theorem 2.5.6.
'$c \Rightarrow a$' By Theorem 2.5.6, we have

$$s_n^{[\lambda+\mu]}(x+y) = E^y A q_n^{[\lambda+\mu]}(x) = E^y A\left(\sum_{k=0}^{n} q_k^{[\lambda]}(x)\, q_{n-k}^{[\mu]}\right)(0) = \sum_{k=0}^{n} q_{n-k}^{[\mu]}\, s_k^{[\lambda]}(x).$$

This concludes the proof. □

Example 2.5.11 As an example of a Steffensen sequence we mention the Laguerre polynomials $Q = D/(D-I)$, $A = I - D$, $T_\lambda = (I-D)^\lambda$ (see [202, pp. 108-113] for more details).

Theorem 2.5.12 *Let* $\left(s_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *be a Steffensen sequence. Then each polynomial* p *can be expanded as*

$$p = \sum_{n=0}^{\infty} (A\, T_\lambda\, Q)\,(0)\, s_n^{[\lambda]} \tag{2.9}$$

where Q *and* $(q_n)_{n\in\mathbb{N}}$ *are as in Theorem 2.5.10.*

Proof: This follows directly form Theorem 2.4.11, since $\left(s_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ is a Sheffer sequence for fixed λ. □

Theorem 2.5.13 ([38]) *If* $\left(s_n^{[\lambda]}\right)_{n\in\mathbb{N}}$ *is a Steffensen sequence and* σ_n *is a sequence of real numbers, then* $\left(s_n^{[\sigma_n]}\right)_{n\in\mathbb{N}}$ *is a Sheffer sequence if and only if there exists real numbers* α *and* β *such that* $\sigma_n = \alpha + \beta\, n$.

Proof: By Theorem 2.5.10, there exists a Sheffer sequence $(s_n)_{n\in\mathbb{N}}$ for some delta operator Q and a group of linear shift-invariant operators $(T_t)_{t\in\mathbb{R}}$ such that $s_n^{[\lambda]} = T_\lambda\, s_n$. Hence, $Qs_n^{[\alpha+\beta\, n]} = QT_{\alpha+\beta\, n} = T_{\alpha+\beta\, n}s_{n-1} = T_\beta\, s_{n-1}^{[\alpha+\beta\,(n-1)]}$. In other words, $\left(s_n^{[\alpha+\beta\, n]}\right)_{n\in\mathbb{N}}$ is Sheffer for the delta operator $T_{-\beta}\, Q$. For the converse, see [38]. $\square$

Example 2.5.14 For the Laguerre polynomials $\left(L_n^{[\lambda]}\right)_{n\in\mathbb{N}}$, it is easy to compute that $L_n^{[x-n]}(\lambda)$ is the n^{th} Poisson-Charlier polynomial of Example 2.5.8d.

Chapter 3

Applications of the Umbral Calculus

In this chapter we present a miscellany of new applications and new results concerning Umbral Calculus and polynomials of convolution type.
In Section 3.1 all sequences of polynomials of convolution type are determined such that $q_n(1) = c$ for all $n \geq 1$. Section 3.2 shows how the theory of Chapter 2 yields identities with binomial coefficients. Moreover, a new proof of a result by G. Labelle on polynomials of convolution type is given. In Section 3.3 probability distributions arising from polynomials of convolution type are studied. The calculation of moments of these distributions (which is of importance for approximation theory) will be calculated using the operator methods of Chapter 2. A simplified proof of the classification of orthogonal Sheffer polynomials is presented in Section 3.4. In Section 3.5 polynomials of convolution type are related to semigroups of probability measures. It transpires that in this context umbral composition can be interpreted as subordination. It is shown in Section 3.6 that each shift-invariant operator can be written as an integral operator. As a corollary, a characterization of Sheffer sequences due to Sheffer is obtained. This representation is shown to be connected with moment problems. Finally, in Section 3.7 we study natural exponential families from an umbral point of view. It is shown that the variance function of a natural exponential family is intimately related to the delta operator of its associated Sheffer sequence. In fact, we will see that the classification of natural exponential families with quadratic variance function coincides with the classification of orthogonal Sheffer polynomials of Section 3.4. We will also see how natural exponential families are related to exponential operators appearing in approximation theory.

Contents of Chapter 3

3.2 Applications to combinatorial identities.

3.3 Discrete probability distributions.

3.4 Orthogonal Sheffer polynomials

3.5 Moment sequences

3.6 Shift-invariant operators and integral operators

3.7 Natural exponential families

3.1 Polynomials with $q_n(1) = c$ for $n \geq 1$.

In this section we determine all sequences $(q_n)_{n\in\mathbb{N}}$ of polynomials of convolution type such that $q_n(1) = c$ for $n \geq 1$. Recall from Theorem 2.1.14 that polynomials of convolution type are determined by the numbers $q_n(1)$.

Theorem 3.1.1 *If $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type such that $q_n(1) = c$ for $n \geq 1$, then*

$$q_n(x) = \sum_{k=0}^{n} \binom{x}{k} (c-1)^k \binom{x+n-k}{n-k} .$$

Proof: First note that there exists a unique sequence of polynomials of convolution type such that $q_n(1) = c$ for $n \geq 1$ by Theorem 2.1.14 with $x_n = 1$ for all $n \in \mathbb{N}$. Both $\left(\binom{x}{n}(c-1)^n\right)_{n\in\mathbb{N}}$ and $\left(\binom{x+n}{n}\right)_{n\in\mathbb{N}}$ are of convolution type (see Remark 2.2.18), so their convolution is also of convolution type by Remark 2.1.3e. Since $\sum_{k=0}^{\infty}(c-1)^k\binom{1+n-k}{n-k} = c$ for all $n \geq 1$, the theorem follows. □

Remarks 3.1.2 : a) Another way of proving Theorem 3.1.1 is to use generating functions:

$$\sum_{n=0}^{\infty} q_n(1)\, z^n = 1 + c\sum_{n=0}^{\infty} z^n = 1 + \frac{c\,z}{1-z} = (1+(c-1)\,z)\,\frac{1}{1-z}.$$

The theorem now follows by observing that

$$\sum_{n=0}^{\infty} \binom{x+n-1}{n} (c-1)^n\, z^n = (1+(c-1)\,z)^x$$

and

$$\sum_{n=0}^{\infty} \binom{x+n-1}{n} z^n = \left(\frac{1}{1-z}\right)^x .$$

b) It follows from a) that

$$\sum_{n=0}^{\infty} q_n(x)\, z^n = \left(\frac{1+(c-1)\,z}{1-z}\right)^x = (1-f(z))^{-x}$$

with $f(z) = \dfrac{cz}{1+(c-1)\,z}$. If $0 < c < 1$, then f is a probability generating function and $(n!\,q_n)_{n\in\mathrm{N}}$ is a sequence of polynomials of binomial type with the renewal property in the terminology of [224].

Examples 3.1.3 : a) If $c = 1$, then $q_n(x) = \binom{x+n-1}{n}$ (see Example 2.2.16c).
b) If $c = 2$, then $q_n(x) = \sum_{k=0}^{n} \binom{x}{k}\binom{x+n-k}{n-k}$, the Mittag-Leffler polynomials (see [202, p. 75-76] note that the calculation on p. 76 contains an error).

3.2 Applications to combinatorial identities

In this section an identity for convolutions of sequences of numbers (in some field of characteristic zero $\mathcal{K}$) will be derived. Moreover, the theory of polynomials of convolution type of Chapter 2 will be used to calculate convolutions of such sequences. This will yield combinatorial identities.

Theorem 3.2.1 *Let $(\beta_n)_{n\in\mathrm{N}}$ be a sequence in some field of characteristic zero. The r-fold convolution of the sequence $\left(\frac{1}{n+1}\,\beta_n^{(n+1)*}\right)_{n\in\mathrm{N}}$ is the sequence $\left(\frac{r}{n+r}\,\beta_n^{(n+r)*}\right)_{n\in\mathrm{N}}$. In particular, the following holds for $1 \le i, j \le n$:*

$$\frac{i+j}{n+i+j}\,\beta_n^{(n+i+j)*} = \sum_{m=0}^{n} \frac{i}{m+i}\,\beta_m^{(m+i)*}\,\frac{j}{n-m+j}\,\beta_{n-m}^{(n-m+j)*}.$$

Proof: If $\beta_0 = 0$, then $\beta_n^{m*} = 0$ for all $m > n$ by Lemma 2.1.5a. Therefore we may suppose that $\beta_0 \neq 0$. Consider the operator $T := \sum_{r=0}^{\infty} \beta_r\,D^r$ on $\mathcal{P}$. It follows from Corollary 2.2.11 that T is invertible. Define $U := T^{-1}$ and consider the delta operator $Q := D\,U$ with basic sequence $(q_n)_{n\in\mathrm{N}}$. It follows from Theorem 2.1.8 that $q_m(x) = \sum_{k=0}^{m} g_m^{k*}\,\frac{x^k}{k!}$. Since $U^{-m} = \sum_{r=0}^{\infty} \beta_r^{m*}\,D^r$, Theorem 2.3.6c yields

$$q_m(x) = \frac{x}{m}\,U^{-m}\,\frac{x^{m-1}}{(m-1)!} = \frac{x}{m}\sum_{r=0}^{m-1} \beta_r^{m*}\,D^r\,\frac{x^{m-1}}{(m-1)!} =$$

$$\sum_{r=0}^{m-1} \beta_r^{m*}\,\frac{x^{m-1-r}}{(m-1-r)!} = \frac{x}{m}\sum_{k=1}^{m} \beta_{m-k}^{m*}\,\frac{x^{k-1}}{(k-1)!} = \sum_{k=1}^{m} \frac{k}{m}\,\beta_{m-k}^{m*}\,\frac{x^k}{k!}.$$

Comparing coefficients of q_m yields $g_m^{k*} = \beta_{m-k}^{m*}$ for $1 \le k \le m$. Setting $k = r$ and $m = n + r$ yields the first statement.

The second statement follows from the first statement and Remark 2.1.14c. □

Using Lagrange inversion, Steutel derived a similar convolution identity and some extensions (see [232]).

Theorem 3.2.1 in itself yields interesting identities. However, more insight can be obtained by relating Theorem 3.2.1 to a non-abelian group structure on the set of sequences of polynomials of convolution type introduced by G. Labelle (see [140, Proposition 1]). We first need a lemma. The proof of Lemma 2.2.2 was shown to the author by Piet Bruinsma.

Lemma 3.2.2 *Let P be a polynomial in two variables with coefficients in some field $\mathcal{K}$ of characteristic zero such that $P(l,m) = 0$ for all $l,m \in \mathbb{N}$. Then $P = 0$.*

Proof: Denote the coefficients of P by a_{ij}, i.e. $P(x,y) = \sum_{i,j=0}^{n} a_{ij}\, x^i\, y^j$. Consider the matrix P, defined by $P(i,j) := a_{ij}$, acting on $\mathcal{K}^{n+1}$. For each $l,m \in \mathbb{N}$, we have $(1,l,\ldots,l^n)\, P\,((1,m,\ldots,m^n)^t) = 0$. It follows directly from Vandermonde's determinant that the vectors $(1,l,\ldots,l^n)^t$, $l = 1,\ldots,n$, are a basis for $\mathcal{K}^{n+1}$. Hence, $(1,m,\ldots,m^n)^t$ belongs to the kernel of P for each $m \in \mathbb{N}$. Therefore P is the zero matrix and $P = 0$. □

The following theorem was proved for basic sequences in [210, Proposition 4]. An extension to sequences of polynomials of convolution type was given in [140, Proposition 2]. We present here a new proof based on Theorem 3.2.1.

Theorem 3.2.3 *If $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type, then for all a both $\left(\frac{x}{x+na}\, q_n(x+na)\right)_{n\in\mathbb{N}}$ and $\left(\frac{x}{x+na}\, q_n(-x-na)\right)_{n\in\mathbb{N}}$ are sequences of polynomials of convolution type.*

Proof: If $a = 0$, then there is nothing to prove. If $q_0 = 0$, then the result follows from Lemma 2.1.7. Suppose $q_0 \neq 0$ and $a \neq 0$. It follows from (2.1) that $(r_n)_{n\in\mathbb{N}}$, defined by $r_n(x) := q_n(ax)$ for all $n \in \mathbb{N}$, is a sequence of polynomials of convolution type. It follows from Theorem 2.1.8 that $p_n(x) := \dfrac{x}{x+n} r_n(x+n)$ is a polynomial in x for all $n \in \mathbb{N}$. Define $\beta_n := r_n(1)$ for all $n \in \mathbb{N}$. It follows from (2.1) that $\beta_n^{k*} = r_n(k)$ for all $k,n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. Define the polynomial P in two variables by $P(x,y) := p_n(x+y) - \sum_{j=0}^{n} p_j(x)\, p_{n-j}(y)$. It follows from Theorem 3.2.1 that $P(l,m) = 0$ for all $l,m \in \mathbb{N}$. By Lemma 3.2.2, $P = 0$. Since n was arbitrary, it follows that $(p_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type. By Remark 2.1.10b we have for all u,v:

$$\frac{u+v}{u+v+n}\, q_n(a(u+v)+an) =$$

$$\sum_{k=0}^{n} \frac{u}{u+k}\, q_k(au+ak)\, \frac{v}{v+n-k}\, q_{n-k}(av+a(n-k)).$$

Multiplying the numerators and denominators of the above identity with a and taking $x = ua$ and $y = va$, we obtain for all x,y:

$$\frac{x+y}{x+y+na}\, q_n(x+y+an) =$$

$$\sum_{k=0}^{n} \frac{x}{x+ka}\, q_k(x+ak)\, \frac{y}{y+(n-k)a}\, q_{n-k}(y+a(n-k)).$$

Thus, $\left(\frac{x}{x+na}\, q_n(x+na)\right)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type. For the second statement note that $(u_n)_{n\in\mathbb{N}}$, defined by $u_n(x) := q_n(-x)$ for all $n \in \mathbb{N}$, is a sequence of polynomials of convolution type by Theorem 2.1.8 and Remark 2.1.10b. Applying the first statement to $(u_n)_{n\in\mathbb{N}}$ instead of $(q_n)_{n\in\mathbb{N}}$ yields the second statement. □

Examples 3.2.4 a) If $q_n(x) = \frac{x^n}{n!}$, then $\frac{x}{x+na}\, q_n(x+na) = x\,(x+na)^{n-1}/n!$, the n^{th} Abel polynomial.
If $q_n(x) = \binom{x}{n}$, then $\frac{x}{x+na}\, q_n(x+na)$ is called the n^{th} Gould polynomial.

Remark 3.2.5 It is possible to derive Theorem 3.2.3 from [210, Proposition 4] in case the coefficients are real or complex. This proof is due to Aart Stam (private communication). Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. If $g_1 \neq 0$, then $\deg q_n = n$ for all $n \in \mathbb{N}$ by Theorem 2.1.12a and the result follows from Theorem 2.2.19 and [210, Proposition 4]. If $g_1 = 0$, then we define a sequence of complex numbers $(h_n)_{n\in\mathbb{N}}$ by $h_n := g_n$ if $n \neq 1$ and $h_1 := \varepsilon (\varepsilon \neq 0)$. Then $(h_n)_{n\in\mathbb{N}}$ is the coefficient sequence of a sequence $(r_n)_{n\in\mathbb{N}}$ of polynomials of convolution type with $\deg r_n = n$ for all $n \in \mathbb{N}$. Letting ε go to zero and applying Lemma 2.1.5c, we obtain the desired result.

For examples of identities arising from Theorem 3.2.3, see Examples 3.2.7.

We now use Theorem 2.3.10 to calculate convolutions of scalar sequences.

Theorem 3.2.6 *Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence in some field of characteristic zero with $\beta_0 = 0$ such that the following holds: there exists sequences of polynomials of convolution type $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ such that $\beta_n = a_{n,1}$ for all $n \in \mathbb{N}$, where the numbers $a_{n,k}$ are defined by $p_n = \sum_{k=0}^{n} a_{n,k}\, q_k$. Then $\beta_n^{k*} = a_{n,k}$ for all $k, n \in \mathbb{N}$.*

Proof: This follows directly from Theorem 2.3.10. □

Examples 3.2.7 a) Consider the basic sequence $\left(\binom{x}{n}\right)_{n\in\mathbb{N}}$ of Example 2.2.16b. By definition ([199, p. 33]), $\binom{x}{n} = \sum_{k=0}^{n} \frac{k!}{n!}\, s(n,k) \frac{x^k}{k!}$, where the numbers $s(n,k)$ are the Stirling numbers of the first kind. If $g_n := s(n,1)/n! = (-1)^{n-1}/n$, then $g_n^{k*} = s(n,k)$ by Theorem 3.2.6. Thus Remark 2.1.3c yields after some simplifications:

$$\binom{k}{i}\, s(n,k) = \sum_{m=0}^{n} \binom{n}{m}\, s(m,i)\, s(n-m, k-i).$$

Similar identities can be obtained for the signless Stirling numbers of the first kind and the Stirling numbers of the second kind.

Applying Theorem 3.2.3 we obtain the following identity for Gould polynomials (see e.g., [210, Section 12]):

$$\frac{x+y}{x+y-an}\binom{x+y-an}{n} =$$

$$\sum_{k=0}^{n} \frac{x}{x-ak}\binom{x-ak}{k}\frac{y}{y-a(n-k)}\binom{y-a(n-k)}{n-k}.$$

A similar identity holds for the upper factorials of Example 2.2.16c.
The sequence of polynomials $\left(\frac{x}{x-an}\binom{x-an}{n}\right)_{n\in\mathbb{N}}$ is the basic sequence of the operator $E^a(E-I)$ (cf. [210, Section 12]). Since

$$\left(E^a\left(E-I\right)\right)^k\left(\binom{x-an}{n}\right)(0) = \binom{-a(n-k)}{n-k},$$

Lemma 2.2.21 yields

$$\sum_{k=0}^{n} \frac{x}{x-ak}\binom{x-ak}{k}\binom{-a(n-k)}{n-k} = \binom{x-an}{n}.$$

The above identity can also be obtained by noting that $\left(\binom{x-an}{n}\right)_{n\in\mathbb{N}}$ is a Sheffer sequence for the delta operator $E^a(E-I)$ (Definition 2.4.1) and by applying Theorem 2.4.4. For a generalization of this identity, see [210, p. 736].
b) Consider the basic sequence $\left(\frac{x^n}{n!}\right)_{n\in\mathbb{N}}$. Then Theorem 3.2.3 yields another proof of the Abel generalization of the Binomial Formula (cf. Remark 2.2.18d).
c) Consider $g(z) = \frac{1}{2}\left(1-(1-4z)^{\frac{1}{2}}\right)$, the generating function of the Catalan numbers $C_n = \frac{k}{n}\binom{2n-2}{n-1}$ $(n \geq 1)$. It is not easy to calculate convolutions of the Catalan numbers directly. Consider the compositional inverse of g. This is $f(z) = z - z^2$. Let Q be the delta operator $D - D^2$. It follows from Theorem 2.2.22b that Q is the delta operator of the basic polynomials $(q_n)_{n\in\mathbb{N}}$ whose coefficient sequence is the sequence of Catalan numbers. Theorem 2.3.6 yields

$$q_n(x) = \frac{x}{n!}\left(I-D\right)^{-n}x^{n-1} = \frac{x}{n!}\sum_{k=0}^{n-1}\binom{-n}{k}(-1)^k D^k x^{n-1} =$$

$$\frac{1}{n}\sum_{k=0}^{n-1}\binom{n+k-1}{k}\frac{1}{(n-1-k)!}x^{n-k} = \sum_{i=0}^{n}\binom{2n-i-1}{n-i}.$$

Thus, $C_n^{k*} = \frac{k}{n}\binom{2n-k-1}{n-k}$ for $1 \leq k \leq n$ (cf. the approach in [174]). Note that the Catalan numbers are related to the Bessel polynomials introduced by Krall and Frink (see [202, sect. 4.1.7, pp. 78-79]).

3.3 Discrete probability distributions

In this section we study discrete probability distributions that arise from a general construction with polynomials of convolution type. These distributions arise as conditional distributions and in approximation theory (references are given below). We will show how to use Umbral Calculus for computing moments of these distributions. The approaches of [74] and [241] are presented.

Let $(q_k)_{k\in\mathbb{N}}$ be a sequence of polynomials of convolution type with real coefficients. Fix n, α, and β such that

1. $n \geq 1$
2. $q_n(\alpha+\beta) \neq 0$
3. $(q_k(\alpha)\, q_{n-k}(\beta))/(q_n(\alpha+\beta) \geq 0$
4. $\alpha \neq 0$
5. $\beta \neq 0$.

Denote by $P_n^{\alpha,\beta}$ the probability distribution on $\{0, 1, ..., n\}$ with

$$P_n^{\alpha,\beta}\{k\} = \frac{q_k(\alpha)\, q_{n-k}(\beta)}{q_n(\alpha+\beta)} \tag{3.1}$$

Examples 3.3.1 Examples of probability distributions of the form (3.1) include:

a) If $q_k(x) = \frac{x^k}{k!}$ and $\alpha, \beta \in \mathbb{N}$, then

$$P_n^{\alpha,\beta}\{k\} = \binom{n}{k}\left(\frac{\alpha}{\alpha+\beta}\right)^k\left(\frac{\beta}{\alpha+\beta}\right)^{n-k}.$$

Hence, $P_n^{\alpha,\beta}$ is the binomial distribution with parameters n and $\alpha/(\alpha+\beta)$.

b) If $q_n(x) = \binom{x}{n}$ and $\alpha, \beta \in \mathbb{N}$, then

$$P_n^{\alpha,\beta}\{k\} = \frac{\binom{\alpha}{k}\binom{\beta}{n-k}}{\binom{\alpha+\beta}{n}}.$$

Hence, if α, β are positive integers, then $P_n^{\alpha,\beta}$ is the hypergeometric distribution with parameters n, α, and β.

c) If $q_n(x) = \binom{x+n-1}{n}$ and $\alpha, \beta \in \mathbb{N}$, then

$$P_n^{\alpha,\beta}\{k\} = \frac{\binom{\alpha+k-1}{k}\binom{\beta+n-k-1}{n-k}}{\binom{\alpha+\beta+n-1}{n}}.$$

Hence, $P_n^{\alpha,\beta}$ is the Pólya-Eggenberger distribution (see [128, Chapter 9, Section 4]).

d) If $q_n(x) = x\,(x - an)^{n-1}$ $(a < 0)$ and $\alpha, \beta > 0$, then $P_n^{\alpha,\beta}$ is the quasi-binomial distribution (see [65]).

We now calculate the first moment of the distribution $P_n^{\alpha,\beta}$ defined above. By definition, the first moment of $P_n^{\alpha,\beta}$ equals

$$\sum_{k=0}^{n} k\, P_n^{\alpha,\beta}\{k\} = \sum_{k=0}^{n} k\, \frac{q_k(\alpha)\, q_{n-k}(\beta)}{q_n(\alpha+\beta)}.$$

Theorem 3.3.2 *Let $P_n^{\alpha,\beta}$ be the probability distribution defined in (3.1). Then the first moment of $P_n^{\alpha,\beta}$ equals $n\alpha/(\alpha+\beta)$.*

Proof: We give two proofs.
First proof: define the wide sense Sheffer sequence $(w_k)_{k\in\mathbb{N}}$ by $w_k(x) := k\,x^{-1}\,q_k(x)$ (see Example 2.4.10a). Applying Theorem 2.4.4b to $(w_k)_{k\in\mathbb{N}}$ yields $\sum_{k=0}^{n} k\,\alpha^{-1}\,q_k(\alpha)\,q_{n-k}(\beta) = (\alpha+\beta)^{-1}\,n\,q_n(\alpha+\beta)$. Hence, the first moment equals $n\,\alpha/(\alpha+\beta)$.
Second proof: define the linear operator T on $\mathcal{P}$ by $Tq_m := \sum_{k=0}^{m} k\,q_k(\alpha)\,q_{m-k}$ for all $m \in \mathbb{N}$. Then the first moment of $P_n^{\alpha,\beta}$ equals $(Tq_n)(\beta)/q_n(\alpha+\beta)$. It follows from the Expansion theorem 2.2.22 that $T = \sum_{m=0}^{\infty}(Tq_m)(0)\,Q^m = \sum_{m=0}^{\infty} m\,q_m(\alpha)\,Q^m$, which equals $\alpha\,E^{\alpha}\,g'(Q)\,Q$ by Corollary 2.3.2. We therefore have $(Tq_n)(\beta) = (\alpha\,E^{\alpha}\sum_{k=0}^{n} k\,g_k\,q_{n-k})\,(\beta)$. By Theorem 2.3.6e, the first moment of $P_n^{\alpha,\beta}$ equals $n\,\alpha/(\alpha+\beta)$. □

Remark 3.3.3 The formula for the first moment of $P_n^{\alpha,\beta}$ is formula 17 of [140]. The proof in [140] uses formal generating functions. The proofs of Theorem 3.3.2 are therefore new proofs of this formula.

If $\sum_{n=0}^{\infty} g_n\,z^n$ has a positive radius of convergence, $g_n \geq 0$ for all $n \in \mathbb{N}$ and both α and β are non-negative real numbers, then a probabilistic proof of Theorem 3.3.2 is possible. Take $\theta > 0$ such that $\sum_{n=0}^{\infty} g_n\,\theta^n < \infty$. Let X and Y be independent random variables with $P(X=k) = \theta^k\,q_k(\alpha)\,e^{-\alpha g(\theta)}$ and $P(Y=h) = \theta^h\,q_h(\beta)\,e^{\beta g(\theta)}$. It follows from Theorem 2.1.12d that $\sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} P(Y=k) = 1$. Then

$$P(X=k \mid X+Y=n) = \frac{P(X=k)\,P(Y=n-k)}{P(X+Y=n)} = \frac{q_k(\alpha)\,q_{n-k}(\beta)}{q_n(\alpha+\beta)}.$$

Suppose α and β are rational. Then there exist $r, s \in \mathbb{N}$ and $M \in \mathbb{R}$ such that $\alpha = r/M$ and $\beta = s/M$. Define random variables $X_i (i = 1, \ldots, r+s)$ by $P(X_i = k) = \theta^k\,q_k(1/M)\,e^{-g(\theta)/M}$. The convolution property of the polynomials q_k implies that X has the same distribution as $X_1 + \ldots + X_r$ and that Y has the same distribution as $X_{r+1} + \ldots + X_{r+s}$. Since $E(X_1 + \ldots + X_{r+s} \mid X+Y=n) = E(X+Y \mid X+Y=n) = n$, we have $E(X \mid X+Y=n) = rn/(r+s) = \alpha n/(\alpha+\beta)$. The general case where α and β are real follows from a continuity argument.

A characterization of probability distributions of the type $P(X = k \mid X + Y = n)$ can be found in [110]. There are examples of polynomials of convolution type such that $q_k(\alpha)\, q_{n-k}(\beta))/q_n(\alpha+\beta) = P(X = k \mid X + Y = n)$ yields known distributions. For the Abel polynomials $x\,(x - an)^{n-1}/n!$ see [65], for the Gould polynomials from Example 3.2.4b) (cf. [210, Section 12]), see [125]. For applications of these probability distributions, we refer to [64, 66, 67].

We can now calculate the second moment of $P_n^{\alpha,\beta}$. As is often the case with discrete distributions, it is easier to calculate descending factorial moments than moments (cf. [128, p. 19]). We use the idea of the second proof of Theorem 3.3.2.

Theorem 3.3.4 *Let $P_n^{\alpha,\beta}$ be the probability distribution defined in 3.1. Then the second factorial moment of $P_n^{\alpha,\beta}$ equals*

$$\frac{\alpha}{q_n(\alpha+\beta)} \left(E^\alpha\, Q^2 \left\{g''(Q) + \alpha\,(g'(Q))^2\right\} q_n\right)(\beta). \tag{3.2}$$

Proof: Define the linear operator V on $\mathcal{P}$ by $V\,q_n := \sum_{k=0}^n k(k-1)\, q_k(\alpha)\, q_{n-k}$ for all $n \in \mathbb{N}$. Then the second descending factorial moment of $P_n^{\alpha,\beta}$ equals $((V q_n)(\beta))/q_n(\alpha+\beta)$. It follows from the Expansion Theorem 2.2.22 that

$$V = \sum_{n=0}^{\infty} (V\,q_n)(0)\, Q^n = \sum_{n=0}^{\infty} n(n-1)\, q_n(\alpha)\, Q^n.$$

The formal generating formula 2.1.12d yields

$$\sum_{n=0}^{\infty} n(n-1)\, q_n(\alpha)\, z^n = z^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2}\, e^{\alpha g(z)} = \alpha\, z^2\, e^{\alpha g(z)} \left\{g''(z) + \alpha\,(g'(z))^2\right\}.$$

It follows from the Isomorphism Theorem 2.3.1 that

$$V = \alpha\, Q^2\, e^{\alpha g(Q)} \left\{g''(Q) + \alpha\,(g'(Q))^2\right\}.$$

Since $g(Q) = D$ by Theorem 2.2.22b, we have $e^{\alpha g(Q)} = E^\alpha$ by Example 2.2.8a. Putting everything together yields the result. □

The following method of calculating moments is adapted from [241], where it is used in the context of approximation operators (cf. [182]). These operators are defined for continuous functions on $[0, 1]$ by

$$(L_n\, f)(x) = \frac{1}{q_n(1)} \sum_{k=0}^{n} q_k(x)\, q_{n-k}(1-x)\, f(k/n),$$

where $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type with $q_n(x) \geq 0$ on $[0, 1]$. For obvious reasons, it is important to calculate the action of these operators for $f(x) = x^m$ $(m \in \mathbb{N})$, which is nothing but computing the moments of the distribution defined by (3.1). More information on Umbral Calculus and approximation theory can be found in Subsection 3.7.2.

The Manole approach in [241] rests on the following lemmas. Note that Formula (3.3) only holds for $m \leq n$ (cf. [241]).

Lemma 3.3.5 ([241, Lemma 1]) *Let Q be a delta operator with basic sequence $(q_n)_{n\in\mathbb{N}}$. Then we have for all $m \leq n$:*

$$(P\,Q)^m = \sum_{k=0}^{n} S(m,k)\,P^k\,Q^k, \tag{3.3}$$

where $P = x\,(Q')^{-1}$ and $S(m,k)$ denotes the Stirling number of the second kind.

Proof: It suffices to check that both sides of (3.3) agree when applied to q_n for all $n \in \mathbb{N}$.
First note that by Theorem 2.3.6d, we have $P\,q_n = (n+1)\,q_{n+1}$ for all $n \in \mathbb{N}$. Hence, $(P\,Q)^m\,q_n = n^m\,q_n$. By definition, the connection constants for expressing x^m in terms of the lower factorials[1] $(x)_n = x\,(x-1)\ldots(x-k+1)$ are the Stirling numbers of the second kind, i.e. $x^m = \sum_{k=0}^{m} S(m,k)\,(x)_k$. Since for $k \leq m \leq n$ we have $P^k\,Q^k\,q_n = P^k\,q_{n-k} = (n)_k\,q_n$, it follows that

$$(P\,Q)^m\,q_n = n^m\,q_n = \left(\sum_{k=0}^{m} S(m,k)\,(x)_k\right) q_n = \sum_{k=0}^{m} S(m,k)\,P^k\,Q^k q_n.$$

This concludes the proof, since n was arbitrary. □

Lemma 3.3.6 ([241, Lemma 2]) *Let $P_n^{\alpha,\beta}$ be the probability distribution defined in 3.1. Then the ℓ^{th} moments of $P_n^{\alpha,\beta}$ equals*

$$\frac{1}{q_n(\alpha+\beta)} \sum_{k=0}^{\ell} S(\ell,k)\,P^k\,E^\beta\,q_n(\alpha+\beta). \tag{3.4}$$

Proof: Let T be the linear operator on $\mathcal{P}$ defined by $T := x\,(Q')^{-1}\,Q$. It follows from Theorem 2.3.6d (Rodrigues Formula) that $T\,q_n = n\,q_n$. Hence,

$$\sum_{k=0}^{n} k^\ell\,q_k(\alpha)\,q_{n-k}(\beta) = \left(T^\ell \sum_{k=0}^{n} q_{n-k}(\beta)\,q_k\right)(\alpha) = \left(T^\ell\,E^\beta\,q_n\right)(\alpha).$$

Now note that $T = P\,Q$, where $P := x\,(Q')^{-1}$. Substituting Formula (3.3) into the last expression, we obtain the result. □

Formula (3.4) is too general to be used for direct computations. The following theorem presents a simplification of the special $m = 2$ of Formula 3.4, which is suitable for computations. Note that P is not shift-invariant and does not commute with shift-invariant operators.

[1] There is a disagreement in notation for the lower factorials between combinatorics and special functions. We follow the convention of combinatorics.

Theorem 3.3.7 (Manole ([241])) *Let $P_n^{\alpha,\beta}$ be the probability distribution defined in 3.1. Then the second moment of $P_n^{\alpha,\beta}$ equals*

$$n^2 \frac{\alpha}{\alpha+\beta} - \frac{\alpha\,\beta\,(Q')^{-2}\, q_{n+2}(\alpha+\beta)}{q_n(\alpha+\beta)}. \tag{3.5}$$

Proof: It follows from $x^2 = x + x(x-1)$ that the Stirling numbers of the second kind satisfy $S(2,0) = 0$ and $S(2,1) = S(2,2) = 1$. In the rest of the proof we will repeatedly use the Rodrigues Formula (Theorem 2.3.6d). Since shift-invariant operators commute by Corollary 2.2.12, we may rewrite $P\,E^\beta$ as $x\,E^\beta\,(Q')^{-1}$ which yields

$$P\,E^\beta\, q_{n-j}(x) = (n-j+1)\,\frac{x}{x+\beta}\, q_{n-j+1}(x+\beta). \tag{3.6}$$

Writing $\frac{x}{x+\beta} = 1 - \frac{\beta}{x+\beta}$, we obtain the following convenient version of (3.6):

$$P\,E^\beta\, q_{n-j}(x) = (n-j+1)\,E^\beta\, q_{n-j+1}(x) - \beta\,E^\beta\,(Q')^{-1}\, q_{n-j}(x) \tag{3.7}$$

Using first (3.7) and then (3.7), we find that

$$\begin{aligned} P^2\,E^\beta\, q_{n-2}(x) &= P\left((n-1)\,E^\beta\, q_{n-1}(x) - \beta\,E^\beta\,(Q')^{-1}\, q_{n-2}(x)\right) \\ &= n\,(n-1)\,\frac{x}{x+\beta}\, q_n(x+\beta) - x\beta\,E^\beta\,(Q')^{-2}\, q_{n-2}(x) \\ &= n\,(n-1)\,\frac{x}{x+\beta}\, q_n(x+\beta) - x\beta\,(Q')^{-2}\, q_{n-2}(x+\beta). \end{aligned}$$

Substituting the above into (3.4), we obtain the desired result. □

Examples 3.3.8 We now compute the second (factorial) moments of the probability distributions discussed in Examples 3.3.1.

a) For the binomial distribution with parameters n and $\alpha/(\alpha+\beta)$, we have $q_k(x) = \frac{x^k}{k!}$, $Q = D$ and $g(z) = z$. Thus Formula (3.2) yields that the second descending factorial moment of $P_n^{\alpha,\beta}$ equals $n(n-1)\,(\alpha/(\alpha+\beta)))^2$.

b) For the hypergeometric distribution with parameters n, α and β, we have $q_k(x) = \binom{x}{k}$, $Q = E^1 - I$ and $g(z) = \log(1+z)$. It follows that $g'(Q) = (I+Q)^{-1} = (E^1)^{-1} = E^{-1}$ and that $g''(Q) = -(I+Q)^{-2} = -(E^1)^{-2} = -E^{-2}$. Thus Formula (3.2) yields that the second descending factorial moment of $P_n^{\alpha,\beta}$ equals

$$\alpha \binom{\alpha+\beta}{n}^{-1} \left(\left(-E^{\alpha-2} + \alpha\,E^{\alpha-2}\right) Q^2 \binom{x}{n}\right)(\beta) =$$

$$\alpha\,(\alpha-1) \binom{\alpha+\beta}{n}^{-1} \binom{\alpha+\beta-2}{n-2} = n\,(n-1)\,\frac{\alpha\,(\alpha-1)}{(\alpha+\beta)\,(\alpha+\beta-1)}.$$

c) For the Pólya-Eggenberger distribution , we have $q_k(x) = \binom{x+n-1}{k}$, $Q = I - E^{-1}$ and $g(z) = -\log(1-z)$. It follows that $g'(Q) = (I-Q)^{-1} = (E^{-1})^{-1} = E^1$

and that $g''(Q) = (I - Q)^{-2} = (E^1)^{-2} = E^{-2}$. Thus Formula (3.2) yields that the second descending factorial moment of $P_n^{\alpha,\beta}$ equals

$$n\,(n-1)\,\frac{\alpha\,(\alpha+1)}{(\alpha+\beta)\,(\alpha+\beta+1)}.$$

d) For the quasi-binomial distribution we have $q_k(x) = x\,(x - ak)^{k-1}/k!$ and $Q = D\,E^a$. Since there does not exist a closed formula for $g(z)$, we cannot use Formula (3.2). By Lemma 2.3.4a, we have $Q' = e^{a\,D}\,(1 + a\,D) = E^a\,(1 + a\,D)$ and thus

$$\begin{aligned}
(Q')^{-2}\,q_{n-2} &= (Q')^{-1}\,(n-1)\,\frac{q_{n-1}}{x}\\
&= (Q')^{-1}\,\frac{(x-(n-1)a)^{n-2}}{(n-2)!}\\
&= e^{-a\,D}\sum_{j=0}^{n-2}(-a)^j\,D^j\,\frac{(x-(n-1)a)^{n-2}}{(n-2)!}\\
&= (-a)^{n-2}\sum_{i=0}^{n-2}\frac{(-a\,(x-na))^i}{i!}.
\end{aligned}$$

Now Formula (3.5) yields that the second moment of $P_n^{\alpha,\beta}$ equals

$$n^2\,\frac{\alpha}{\alpha+\beta} - n!\,\frac{\alpha\,\beta}{(\alpha+\beta)\,(\alpha+\beta-na)^{n-1}}\,(-a)^{n-2}\sum_{i=0}^{n-2}\frac{(-a\,(x-na))^i}{i!}.$$

The trivial relation $E(X\,(X-1)) = E(X^2) - E(X)$, the above formula immediately yields that the second factorial moment of $P_n^{\alpha,\beta}$ equals

$$n(n-1)\,\frac{\alpha}{\alpha+\beta} - n!\,\frac{\alpha\,\beta}{(\alpha+\beta)\,(\alpha+\beta-na)^{n-1}}\,(-a)^{n-2}\sum_{i=0}^{n-2}\frac{(-a\,(x-na))^i}{i!}.$$

3.4 Orthogonal Sheffer polynomials

In this section we show how to use the Umbral Calculus for finding all orthogonal Sheffer polynomials[2]. By Favard's Theorem (see e.g. [58, Theorem 4.4]), a polynomial sequence $(s_n)_{n\in\mathbb{N}}$ is orthogonal if and only if it satisfies the following three-term recurrence relation for $n \geq 0$:

$$s_{n+1}(x) = (a_n\,x - b_n)\,s_n(x) - c_n\,s_{n-1}(x), \tag{3.8}$$

where $s_1 = 0$, s_0 is a non-zero constant, and $c_n\,a_n\,a_{n-1} > 0$ for $n \geq 1$. We must be careful with normalizations when dealing with Sheffer sequences, because if

[2]All Sheffer polynomials are orthogonal in the sense that they are orthogonal with respect to some functional (see [210, Section 9]).

$(s_n)_{n\in\mathrm{N}}$ is Sheffer, then $(\lambda_n s_n)_{n\geq 0}$ need not be Sheffer. Hence, we must use the non-monic form (3.8).
The following theorem shows the relation between the three-term recurrence relation (3.8) and differential equations for the delta operator and shift-invariant operator of a Sheffer sequence. These differential equations are to be understood in the Pincherle sense (cf. Definition 2.3.3).

Theorem 3.4.1 *Let $(s_n)_{n\in\mathrm{N}}$ be a Sheffer sequence with delta operator Q and invertible operator A. If $(s_n)_{n\in\mathrm{N}}$ satisfies the three-term recurrence relation (3.8), then*

$$Q' = \frac{1}{a_0} I + \left(\frac{b_1}{a_1} - \frac{b_0}{a_0}\right) Q + \left(\frac{b_2}{a_2} - \frac{c_1}{a_1}\right) Q^2 \tag{3.9}$$

$$A' = \frac{b_0}{a_0} A + \frac{c_1}{a_0} A\,Q. \tag{3.10}$$

If conversely $Q' = d_1 + d_2 Q + d_3 Q^2$ and $A' = d_4 A + d_5 AQ$, then $(s_n)_{n\in\mathrm{N}}$ satisfies the following three-term recurrence relation:

$$(d_3 + d_5)\, s_{n-1} + d_2 s_n + d_1 s_{n+1}$$

$$s_{n+1}(x) = (a_n\, x - b_n)\, s_n(x) - c_n\, s_{n-1}(x)$$

Proof: We make extensive use of the Operator Expansion Theorem 2.4.11 and of the fact that $q_n(0) = 0$ for $n \geq 1$, where $(q_n)_{n\in\mathrm{N}}$ is the basic sequence of Q. In particular,

$$T' = \sum_{k=0}^{\infty} \left[T' s_k\right]_{x=0} AQ^k$$

where T' is the Pincherle derivative of the operator T. Notice that

$$\left[T'p\right]_{x=0} = \left[Txp - xTp\right]_{x=0} = \left[T\mathrm{x}p\right]_{x=0}. \tag{3.11}$$

It will be convenient to adopt the convention that $q_k = 0$ and $s_k = 0$ for $k < 0$. Assume that $(s_n)_{n\in\mathrm{N}}$ satisfies the three-term recurrence relation (3.8). We first apply (3.11) to $T = A'$, which yields

$$\begin{aligned}
A' &= \sum_{k=0}^{\infty} \left[Axs_k\right]_{x=0} A\,Q^k \\
&= \sum_{k=0}^{\infty} \frac{1}{a_k} \left[A\left(s_{k+1} + b_k\, s_k + c_k s_{k-1}\right)\right]_{x=0} AQ^k \\
&= \sum_{k=0}^{\infty} \frac{1}{a_k} \left(q_{k+1}(0) + b_k\, q_k(0) + c_k\, q_{k-1}(0)\right) A\,Q^k \\
&= \frac{b_0}{a_0} A + \frac{c_1}{a_0} A\,Q.
\end{aligned}$$

We now apply (3.11) to $T = (AQ)'$.

$$\begin{aligned}
AQ' + A'Q &= (AQ)' \\
&= \sum_{k=0}^{\infty} \left[(AQ)'\, s_k\right]_{x=0} A\, Q^k \\
&= \sum_{k=0}^{\infty} \left[AQ\, x s_k\right]_{x=0} A\, Q^k \\
&= \sum_{k=0}^{\infty} \frac{1}{a_k} \left[AQ\,(s_{k+1} + b_k s_k + c_k s_{k-1})\,\right]_{x=0} AQ^k \\
&= \frac{1}{a_0} A + \sum_{k=1}^{\infty} \frac{1}{a_k} \left(q_k(0) + b_k q_{k-1}(0) + c_k q_{k-2}(0)\right) AQ^k \\
&= \frac{1}{a_0}\, A + \frac{b_1}{a_1}\, AQ + \frac{c_2}{a_2}\, AQ^2.
\end{aligned}$$

Now we eliminate A' in the last equation by using the invertibility of A and Formula (3.10):

$$\begin{aligned}
AQ' + A'Q &= \frac{1}{a_0}\, A + \frac{b_1}{a_1}\, AQ + \frac{c_2}{a_2}\, AQ^2 \\
AQ' &= \frac{1}{a_0}\, A + \left(\frac{b_1}{a_1} - \frac{b_0}{a_0}\right) AQ + \left(\frac{b_2}{a_2} - \frac{c_1}{a_1}\right) AQ^2 \\
Q' &= \frac{1}{a_0}\, I + \left(\frac{b_1}{a_1} - \frac{b_0}{a_0}\right) Q + \left(\frac{b_2}{a_2} - \frac{c_1}{a_1}\right) Q^2.
\end{aligned}$$

Conversely, assume that $Q' = d_1 + d_2 Q + d_3 Q^2$ and $A' = d_4 A + d_5 AQ$. We apply the Polynomial Expansion Theorem 2.4.11a to $x s_n$, which yields

$$\begin{aligned}
x s_n &= \sum_{k=0}^{n+1} \left[AQ^k x s_n\right]_{x=0} s_k \\
&= \sum_{k=0}^{n+1} \left[(AQ^k)' s_n\right]_{x=0} s_k \\
&= \sum_{k=0}^{n+1} \left\{\left((A'Q^k) + k\, AQ^{k-1} Q'\right) s_n(0)\right\} s_k \\
&= \sum_{k=0}^{n+1} \left\{A' s_{n-k}(0) + AQ' s_{n-k+1}(0)\right\} s_k \\
&= \sum_{k=0}^{n+1} \{(d_4 A + d_5 AQ)\, s_{n-k}(0) + \\
&\qquad \left(d_1 A + d_2 AQ + d_3 AQ^2\right) s_{n-k+1}(0)\big\}\, s_k
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{n+1} \{d_4\, q_{n-k}(0) + d_5\, q_{n-k-1}(0) + \\
&\quad d_1\, q_{n-k+1}(0) + d_2\, q_{n-k}(0) + d_3\, q_{n-k-1}(0)\}\, s_k \\
&= \frac{d_4}{n!}\, s_n + \frac{d_5}{(n-1)!}\, s_{n-1} + \frac{d_1}{(n+1)!}\, s_{n+1} + \frac{d_2}{n!}\, s_n + \frac{d_3}{(n-1)!}\, s_{n-1} \\
&= (d_3 + d_5)\, s_{n-1} + d_2 s_n + d_1 s_{n+1}
\end{aligned}$$

Since $d_1 \neq 0$, this means that $(s_n)_{n\in\mathbb{N}}$ satisfies a three-term recurrence relation of the form (3.8). □

Solving the differential equations for Q and A, we find all orthogonal Sheffer polynomials. Thus, we have a new proof of the Meixner classification of orthogonal Sheffer polynomials (see [157] for the original proof, cf. [5, 57, 100, 132, 141, 206, 202]). The advantage of our proof is that it is a constructive proof based on first principles of the Umbral Calculus. A generalization of the Meixner classification was obtained by Al-Salam in [6], where it is shown that the Meixner result remains true even if we consider the more general class of polynomials with generating function $e^{Q(x,t)}$, where $Q(x,t)$ is a polynomial in x and a power series in t.

It is an open problem to determine which Sheffer sequences are orthogonal with respect to a Borel measure in the complex plane, cf. [210, p. 751]. Two such sequences are the polynomials $x^n/n!$, which are orthogonal with respect to arc length on the unit circle, and the lower factorial polynomials (see [153] and references therein). All Sheffer sequences orthogonal on the unit circle have been classified by Kholodov (see [132]).

3.5 Moment systems

In the work of Feinsilver (e.g. [89, 90, 91]; see also [117]) sequences $(p_n)_{n\in\mathbb{N}}$ of polynomials appear that satisfy

$$p_n(t) = \int_{-\infty}^{\infty} x^n \, d\mu_t(x), \tag{3.12}$$

where $(\mu_t)_{t\geq 0}$ is a convolution semigroup of probability measures (usually induced by a stochastic process with stationary independent increments). It follows directly from the Binomial Formula that Formula (3.12) implies

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y). \tag{3.13}$$

In this section we will study when a sequence $(p_n)_{n\in\mathbb{N}}$ satisfying (3.13) admits a representation of the form (3.12). We will show that representations of the form (3.12) are related to umbral operators (cf. Section 2.3) and to moment sequences of infinitely divisible probability measures. Moreover, the relation between umbral composition and subordination of probability measures is pointed out. Representations of the form (3.12) by *groups* of complex Borel measures or by convolution semigroups of probability measures with support in $[0, \infty)$ are also studied. At the end of this section we present some explicit examples.

This section is based on [75].

Definition 3.5.1 *Let $(\mu_t)_{t\geq 0}$ be a collection of complex Borel measures on the real line.*

If $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geq 0$ (where $*$ denotes convolution), then $(\mu_t)_{t\geq 0}$ is a said to be a **convolution semigroup**.

If $\int_{-\infty}^{\infty} f(x)\ d\mu_t(x)$ is a measurable function of t for each bounded continuous function f on the real line, then $(\mu_t)_{t\geq 0}$ is said to be **weakly measurable**.

If $\int_{-\infty}^{\infty} f(x)\ d\mu_t(x)$ converges to $\int_{-\infty}^{\infty} f(x)\ d\mu_0(x)$ for each continuous function f on the real line with compact support and $\mu_t(\mathbb{R})$ converges to $\mu_0(\mathbb{R})$ as t goes to zero, then $(\mu_t)_{t\geq 0}$ is said to be **weakly continuous**.

Lemma 3.5.2 *Let μ and ν be probability measures on the real line. If $\mu * \nu$ has finite moments of all orders, then both μ and ν have finite moments of all orders.*

Proof: The definition of Lebesgue integrals implies that $\mu * \nu$ has finite *absolute* moments of all orders. Let r be a positive integer. By the definition of convolution,

$$\int_{-\infty}^{\infty} |t|^r\ d(\mu * \nu)(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x+y|^r\ d\mu(x)d\nu(y) < \infty.$$

Hence, $\int_{-\infty}^{\infty} |x+y|^r\ d\mu(x)) < \infty$, μ-a.e. in y. Since

$$|x|^r \leq 2^r|x+y|^r + 2^r|y|^r,$$

it follows that

$$\int_{-\infty}^{\infty} |x|^r\ d\mu(x) < \infty.$$

Likewise we see that ν has a finite absolute moment of order r. □

The following theorem shows that convolution semigroups of measures generate umbral operators (cf. Definition 2.3.9).

Theorem 3.5.3 *Let $(\mu_t)_{t\geq 0}$ be a weakly measurable convolution semigroup of complex Borel measures on the real line having finite moments of all orders and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. If $\mu_t(\mathbb{R}) = 1$ for all nonnegative t, then $(q_n)_{n\in\mathbb{N}}$ is also a sequence of polynomials of convolution type, where*

$$q_n(t) = \int_{-\infty}^{\infty} p_n(x)\, d\mu_t(x).$$

Proof: The sequence $(q_n)_{n\in\mathbb{N}}$ is well-defined since $(\mu_t)_{t\geq 0}$ has finite moments of all orders. We verify that $(q_n)_{n\in\mathbb{N}}$ is of convolution type:

$$q_n(s+t) = \int_{-\infty}^{\infty} p_n(x)\, d\mu_{s+t}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(x+y)\, d\mu_s(x)\, d\mu_t(y) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{k=0}^{n} p_k(x)\, p_{n-k}(y)\, d\mu_s(x)\, d\mu_t(y) = \sum_{k=0}^{n} q_k(s)\, q_{n-k}(t).$$

If $p_0(0) = 0$, then $p_n = 0$ for all n and there is nothing to prove. If $p_0(0) \neq 0$, then $p_0 = 1$ and we have $q_0 = 1$ because $\mu_t(\mathbb{R}) = 1$. Moreover, since $(\mu_t)_{t\geq 0}$ is weakly measurable, the functions q_n are Borel measurable. It now follows from Theorem 2.1.8 that $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials. □

Remark 3.5.4 The linear operator U, defined by $Up(t) := \int_{-\infty}^{\infty} p(x)\, d\mu_t(x)$, is an umbral operator in the sense of Definition 2.3.9. We will see in Corollary 3.5.11 which umbral operators can be represented in this way.

The following two theorems describe necessary and sufficient conditions for the existence of convolution semigroups of probability measures.

Definition 3.5.5 *A complex-valued function f on the real line is* **negative definite** *if*

$$\sum_{k=1}^{n} c_k = 0 \textit{ implies } \sum_{j,k=1}^{n} c_j\bar{c}_k f(s_j - s_k) \leq 0$$

for all nonnegative sequences $s_1, \ldots, s_n$ and all complex sequences $c_1, \ldots, c_n$.

A real-valued C^∞-function f on $(0,\infty)$ is said to be a **Bernstein function** *if*

$$f \geq 0 \textit{ and } (-1)^k D^k f \geq 0 \textit{ for all integers } k \geq 1$$

Theorem 3.5.6 *If a continuous real-valued function f on the real line is negative definite and satisfies $f(0) = 0$, then there exists a weakly continuous semigroup $(\mu_t)_{t\geq 0}$ of probability measures on the real line such that for all real y*

$$e^{-t\, f(y)} = \int_{-\infty}^{\infty} e^{-ixy}\, d\mu_t(x)$$

Proof: This follows from [24, Chapter 2, Theorem 8.3 and Corollary 8.6]. □.

The following theorem is the analogue of Theorem 3.5.6 for convolution semigroups of probability measures on $[0,\infty)$.

Theorem 3.5.7 *If a real-valued function f on $(0,\infty)$ is a Bernstein function and satisfies $f(0) = 0$, then there exists a weakly continuous semigroup $(\mu_t)_{t\geq 0}$ of probability measures on $[0,\infty)$ such that for all positive y*

$$e^{-t\, f(y)} = \int_{0}^{\infty} e^{-xy}\, d\mu_t(x)$$

Proof: See [24, Chapter 2, Theorem 9.18 and Remark 9.19]. □

We now investigate which sequences of polynomials of convolution type admit a representation of the form (3.12), i.e. can be represented as moment systems in the terminology of Feinsilver (see [89, 91, 117]). Our first theorem relates negative definiteness of $-g$ to representability as moment system.

Theorem 3.5.8 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. Suppose that*

1. $\sum_{n=0}^{\infty} g_n z^n$ *has a positive radius of convergence*
2. *the function $x \mapsto -g(-ix)$ has a continuous, negative definite extension to the real line.*

Then there exists a weakly continuous convolution semigroup $(\mu_t)_{t\geq 0}$ of probability measures on the real line such that for $t \geq 0$

$$q_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x)$$

Proof: It follows from $g_0 = 0$, condition 2) and Theorem 3.5.6 that there exists a weakly continuous semigroup $(\mu_t)_{t\geq 0}$ of probability measures such that

$$\int_{-\infty}^{\infty} e^{-ixy}\, d\mu_t(x) = e^{t\, g(-iy)}$$

for all real y. It follows from condition 1) that there exists $r > 0$ such that g is analytic for $|z| < r$. Thus by [151, Theorem 7.1.1]

$$\int_{-\infty}^{\infty} e^{zx}\, d\mu_t(x) = e^{t\, g(z)}$$

for $|z| < r$. Since g is analytic, it follows from [151, Corollary 1 to Theorem 2.3.1] that each μ_t has finite moments of all orders. Moreover, [151, Corollary 2 to Theorem 2.3.1]) yields

$$e^{t\, g(z)} = \int_{-\infty}^{\infty} e^{zx}\, d\mu_t(x) = \sum_{n=0}^{\infty} \left(\int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x) \right) z^n$$

for $|z| < r$. It now follows from Theorem 2.1.12d that $q_n(t) = \displaystyle\int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x)$.
□

We will see in Theorem 4.4.10 that condition 1) is equivalent to: there exists $r > 0$ such that for all $t > 0$

$$\sum_{n=0}^{\infty} |q_n(t)| r^n < \infty.$$

Of course, there is a corresponding result for probability measures on $[0, \infty)$.

Theorem 3.5.9 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. Suppose that*

1. $\sum_{n=0}^{\infty} g_n z^n$ *has a positive radius of convergence*

2. *the function $x \mapsto -g(-x)$ has an extension to a Bernstein function on $(0,\infty)$.*

Then there exists a weakly continuous convolution semigroup $(\mu_t)_{t\geq 0}$ of probability measures on $[0,\infty)$ such that for $t\geq 0$

$$q_n(t) = \int_0^\infty \frac{x^n}{n!}\, d\mu_t(x)$$

Proof: The proof is analogous to the proof of Theorem 3.5.8 (use Theorem 3.5.7 instead of Theorem 3.5.6). □

A sequence $(q_n)_{n\in\mathbb{N}}$ of polynomials of convolution type is determined by the numbers $q_n(1)$ by Theorem 2.1.14. The following theorem gives a necessary and sufficient condition on the numbers $n!\,q_n(1)$ for the representation of Theorem 3.5.8 to hold.

Theorem 3.5.10 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. There exists a weakly continuous convolution semigroup $(\mu_t)_{t\geq 0}$ of probability measures on the real line such that $q_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x)$ for $t\geq 0$ if and only if $(n!\,q_n(1))_{n\in\mathbb{N}}$ is the moment sequence of an infinitely divisible probability measure on the real line.*

Proof: "$\Rightarrow$" This follows from $n!\,q_n(1) = \int_{-\infty}^{\infty} x^n\, d\mu_1(x)$, since μ_1 is clearly infinitely divisible.
"$\Leftarrow$" Let μ be an infinitely divisible probability measure with moment sequence $(n!\,q_n(1))_{n\in\mathbb{N}}$. By [93, Chapter 9, Section 5, Theorem 2], there exists a weakly continuous convolution semigroup of probability measures $(\mu_t)_{t\geq 0}$ on the real line such that $\mu_1 = \mu$. It follows from Lemma 3.5.2 that each probability measure μ_t has finite moments of all orders. Thus Theorem 3.5.3 implies that the sequence $(h_n)_{n\in\mathbb{N}}$, defined by $h_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x)$ is a sequence of polynomials of convolution type. This sequence is determined by the numbers $h_n(1)$ by Theorem 2.1.14. Hence, $h_n = q_n$ for all n, since $h_n(1) = q_n(1)$. □

As a corollary we now describe when an umbral operator (cf. Definition 2.3.9) can be represented as an integral operator. This representation differs from the integral operator representation for shift-invariant operators in Section 3.6, since umbral operators are *never* shift-invariant (except for the identity operator) by Theorem 2.3.11b.

Corollary 3.5.11 *Let U be an umbral operator. Then there exists a semigroup $(\mu_t)_{t\geq 0}$ of probability measures on the real line such that*

$$(Up)(t) = \int_{-\infty}^{\infty} p(x)\ d\mu_t(x)$$

if and only if $((U\, x^n)(1))_{n\in\mathbb{N}}$ is the moment sequence of an infinitely divisible probability measure.

Proof: Define $q_n = U\frac{x^n}{n!}$. Since U is an umbral operator, it follows that $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type. The result now follows from Theorem 3.5.10. □

The following theorem relates umbral operators to subordination of convolution semigroups of probability measures. For another relation between umbral operators and subordination, see [222].

Theorem 3.5.12 *Let $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ be weakly measurable convolution semigroups of probability measures and let $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be the associated sequences of polynomials of convolution type, i.e. $p_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\ d\mu_t(x)$ and $q_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\ d\nu_t(x)$. Let U be the umbral operator that maps $\frac{x^n}{n!}$ to p_n and define polynomials r_n by $r_n = Uq_n$. Then*

$$r_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\rho_t(x)$$

where $(\rho_t)_{t\geq 0}$ is the convolution semigroup subordinated to $(\nu_t)_{t\geq 0}$ by means of $(\mu_t)_{t\geq 0}$.

Proof: This follows from

$$\begin{aligned} r_n(t) &= Uq_n(t) \\ &= \int_{-\infty}^{\infty} q_n(x)\ d\mu_t(x) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y^n}{n!}\ d\nu_x(y)\ d\mu_t(x) \end{aligned}$$

where $\displaystyle\int_{-\infty}^{\infty} d\mu_t(x)\ \nu_x$ is the probability measure resulting from subordinating to $(\nu_t)_{t\geq 0}$ by means of $(\mu_t)_{t\geq 0}$ (cf. [24, Section 9.20]). □

The following theorem states when a sequence of polynomials of convolution type is generated by a *group* of complex Borel measures on the real line.

Definition 3.5.13 *A group $(\mu_t)_{t\in\mathbb{R}}$ of probability measures on the real line is said to be* **strongly continuous** *if the operators $f \mapsto f * \mu_t$ form a strongly continuous group on $L^1(-\infty,\infty)$.*
A complex Borel measure μ on the real line is said to be **invertible** *, if there exists a complex Borel measure ν on the real line such that $\mu * \nu = \delta_0$, where δ_0 is the point mass at 0.*

Theorem 3.5.14 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. There exists a strongly continuous group of complex Borel measures $(\mu_t)_{t\in\mathbb{R}}$ such that*

$$q_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x)$$

if and only if $(n!\, q_n(1))_{n\in\mathbb{N}}$ is the moment sequence of an invertible complex Borel measure on the real line with total mass one.

Proof: '$\Rightarrow$' This follows from $n!\, q_n(1) = \int_{-\infty}^{\infty} x^n\, d\mu_1(x)$. Note that μ_1 is invertible, since $(\mu_t)_{t\in\mathbb{R}}$ is a group.
'$\Leftarrow$' Let μ be an invertible complex Borel measure with moment sequence $(n!\, q_n(1))_{n\in\mathbb{N}}$. It follows from [95] there exists a strongly continuous convolution group of complex Borel measures $(\mu_t)_{t\in\mathbb{R}}$ on the real line such that $\mu_1 = \mu$ and $\mu_t(\mathbb{R}) = 1$. It follows from Theorem 3.5.3 that $(h_n)_{n\in\mathbb{N}}$, defined by $h_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x)$, is a sequence of polynomials of convolution type. This sequence is determined by the numbers $h_n(1)$ by Theorem 2.1.14. Hence, $h_n = q_n$ for all n, since $h_n(1) = q_n(1)$. □

We conclude this section with examples of sequences of polynomials of convolution type that are moment systems.

Examples 3.5.15 Explicit examples of sequences of polynomials of convolution type that are moment systems include:

1. Take $\mu_t = \delta_t$. Then

$$q_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\delta_t(x) = \frac{t^n}{n!}$$

2. Take $\mu_t = \sum_{k=0}^{\infty} e^{-t}\, \frac{t^k}{k!}\, \delta_k$ (**Poisson semigroup**). Then

$$q_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x) = \frac{1}{n!} \sum_{k=0}^{\infty} e^{-t}\, \frac{t^k}{k!}\, k^n = \frac{e^{-t}}{n!} \sum_{k=0}^{\infty} \frac{t^k\, k^n}{k!}$$

The ratio test yields that this series converges absolutely for all real t. The series

$$e^{-t} \sum_{k=0}^{\infty} \frac{t^k\, k^n}{k!}$$

is known as the Dobinski Formula for the exponential polynomials (see [202, p. 66] and [152]).

3. Take $d\mu_t(x) = 1_{(0,\infty)}(x)\, \frac{1}{\Gamma(t)}\, x^{t-1}\, e^{-x}\, dx$ (**Gamma-semigroup**). Then

$$q_n(t) = \int_{-\infty}^{\infty} \frac{x^n}{n!}\, d\mu_t(x) = \int_0^{\infty} \frac{x^{n+t-1}}{n!\, \Gamma(t)}\, e^{-x} dx = \frac{\Gamma(n+t)}{\Gamma(t)\, n!} = \binom{t+n-1}{n}$$

4. Take $d\mu_t(x) = \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/2t}\, dx$ (**Brownian semigroup**).

   Then $q_n(t) = 0$ if n is odd and for n even we have

$$q_n(t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \frac{x^n}{n!}\, e^{-x^2/2t}\, dx = \frac{t^{n/2}}{2^{n/2}\,(n/2)!}$$

   Note that the degree of q_n is less than n.

3.6 Shift-invariant operators and integral operators.

Theorems 2.2.7, 2.2.22 and 2.4.11b show that shift-invariant operators can be represented by power series. In this section we prove that each shift-invariant operator can be represented as a random shift, i.e. as an integral operator. This representation will be used to give a new proof of a characterization theorem for Sheffer polynomials (due to Sheffer) which characterizes Sheffer sequences as shifted moments of a complex Borel measure on the real line. This section is based on [75].

The following lemma is essential.

Lemma 3.6.1 (Boas) *For each sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers, there exist infinitely many complex Borel measures μ on the real line such that*

$$a_n = \int_{-\infty}^{\infty} y^n\, d\mu(y).$$

(**Pólya**) *Among these measures are discrete measures and absolutely continuous measures.*

Proof: Boas and Pólya stated their results for real sequences (see [27] and [181]). Our lemma follows immediately from their results by considering real and imaginary parts. □

If μ is a complex Borel measure on the real line having finite moments of all orders, then

$$(Tp)(x) = \int_{-\infty}^{\infty} p(x+y)\, d\mu(y)$$

defines a linear shift-invariant operator on the space of polynomials. The converse is also true as the following theorem shows (cf. [223, formula 13].

Theorem 3.6.2 *If T is a linear shift-invariant operator, then there exists a complex Borel measure μ on the real line such that for all polynomials p*

$$(Tp)(x) = \int_{-\infty}^{\infty} p(x+y)\, d\mu(y).$$

Proof: By the Expansion Theorem 2.2.7, we have

$$T = \sum_{k=0}^{\infty} \left(T \frac{x^k}{k!} \right) (0) D^k.$$

By Lemma 3.6.1, there exists a complex Borel measure μ on the real line such that $\left(T x^k\right)(0) = \int_{-\infty}^{\infty} y^k \, d\mu(y)$. Define the linear shift-invariant operator V by $(Vp)(x) = \int_{-\infty}^{\infty} p(x+y)\, d\mu(y)$ for all polynomials p. Then

$$\left(V \frac{x^k}{k!} \right) (0) = \int_{-\infty}^{\infty} \frac{y^k}{k!} \, d\mu(y) = \left(T \frac{x^k}{k!} \right) (0.)$$

It follows from the Expansion Theorem 2.2.7 that $T = V$. □

Note that if $T \neq 0$ is a non-invertible shift-invariant operator, then there does not exist a non-negative Borel measure such that $(Tp)(x) = \int_{-\infty}^{\infty} p(x+y)\, d\mu(y)$ for all polynomials p. Indeed, this would imply $T1 = 0$, since $T1 \neq 0$ is equivalent to invertibility. Hence, $\mu(\mathbb{R}) = 0$. Since $\mu \neq 0$ it follows that μ cannot be a non-negative Borel measure.

Examples 3.6.3 We present some explicit examples of representations of linear shift-invariant operators. Let δ_0 be the point mass at 0. Recall that the measure μ of Theorem 3.6.2 is not unique.

Linear shift-invariant operator	Measure μ of Theorem 3.6.2
Identity operator	δ_0
Laguerre operator	$\delta_0 - e^{-t}\, dt$
Weierstrass operator	$\frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt$

The Weierstrass operator is the invertible operator of the Hermite polynomials (cf. Example 2.4.3a.

Remark 3.6.4 The proof of Theorem 3.6.2 show that the existence of absolutely continuous measures for the identity operator, the differentiation operator, and the shift operators E^a, are equivalent to the following moment problems:

1. identity operator: $\displaystyle\int_{-\infty}^{\infty} d\mu(y) = 1, \int_{-\infty}^{\infty} y^n \, d\mu(y) = 0$ for $n > 0$.

2. differentiation operator: $\displaystyle\int_{-\infty}^{\infty} y \, d\mu(y) = 1, \int_{-\infty}^{\infty} y^n \, d\mu(y) = 0$ for $n \neq 1$.

3. shift-operator E^a: $\displaystyle\int_{-\infty}^{\infty} y^n \, d\mu(y) \; = a^n$.

As Erik Thomas showed to me (private communication), it is possible to solve these moment problems explicitly using the theory of tempered distributions.

We now use Theorem 3.6.2 to prove a characterization theorem for Sheffer polynomials due to Sheffer (see [218, Theorem 2]).

Theorem 3.6.5 (Sheffer) *A sequence $(s_n)_{n\in\mathbb{N}}$ of polynomials is a strict sense Sheffer sequence with basic sequence $(q_n)_{n\in\mathbb{N}}$ if and only if there exists a complex Borel measure μ on the real line such that $\mu(\mathbb{R}) \neq 0$ and*

$$s_n(x) = \int_{-\infty}^{\infty} q_n(x+y)\, d\mu(y)$$

for all $n \in \mathbb{N}$.

Proof: '$\Leftarrow$' The operator A, defined by $(Ap)(x) = \int_{-\infty}^{\infty} p(x+y)\, d\mu(y)$ is an invertible linear shift-invariant operator on $\mathcal{P}$. Clearly $s_n = Aq_n$ and hence, deg $s_n = n$. Moreover, $s_0 \neq 0$ since $q_0 = 1$ and $\mu(\mathbb{R}) \neq 0$. Thus $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence by Theorem 2.4.8b.
'$\Rightarrow$' If $(s_n)_{n\in\mathbb{N}}$ is a strict sense Sheffer sequence, then the linear operator A, defined by $Aq_n = s_n$, is shift-invariant by Theorem 2.4.8b. Thus Theorem 3.6.2 yields a complex Borel measure μ such that $(Ap)(x) = \int_{-\infty}^{\infty} p(x+y)\, d\mu(y)$ for all polynomials p. In particular, $s_n(x) = Aq_n(x) = \int_{-\infty}^{\infty} q_n(x+y)\, d\mu(y)$ for all n. Moreover, $\mu(\mathbb{R}) = s_0(0) \neq 0$. □

The formal moment generating function of the measure μ in Theorem 3.6.5 is equal to the formal power series $\sum_{n=0}^{\infty} s_n(0)\, t^n$ (see [218, Corollary on p. 742]). The following integral representation for Hermite polynomials

$$H_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(x+y)^n}{n!}\, e^{-y^2/2}\, dy$$

is an illustration of Theorem 3.6.5.

We now show that linear functionals can be represented by integrals (cf. Theorem 2.4.8).

Theorem 3.6.6 *Let Λ be a linear functional on $\mathcal{P}$. Then there exists a complex Borel measure μ on the real line such that*

$$\Lambda p = \int_{-\infty}^{\infty} p(x)\, d\mu(x)$$

for all polynomials p.

Proof: Define $(a_n)_{n\in\mathbb{N}}$ by $\Lambda \frac{x^n}{n!} := a_n$. Define the linear operator T on $\mathcal{P}$ by $T := \sum_{n=0}^{\infty} a_n D^n$. Since $(T \frac{x^n}{n!})(0) = a_n$ for all $n \in \mathbb{N}$, we conclude that $\Lambda p = (Tp)(0)$ for all $p \in \mathcal{P}$. The theorem now follows from Theorem 3.6.2 □

The above theorem can be used to prove another characterization theorem for Sheffer polynomials. This theorem was proved by Thorne for Appell polynomials (see [236]) and extended by Sheffer to Sheffer polynomials (see [218, p. 744]).

Theorem 3.6.7 (Thorne-Sheffer) *A sequence $(s_n)_{n\in\mathbb{N}}$ of polynomials such that $\deg s_n = n$ is a Sheffer sequence if and only if there exist a delta operator Q and a complex Borel measure μ on the real line such that $\mu(\mathbb{R}) \neq 0$, μ has finite moments of all orders, and*

$$\int_{-\infty}^{\infty} \left(Q^k s_n\right)(x)\, d\mu(x) = \delta_{kn}$$

Proof: '$\Rightarrow$' By Theorem 2.4.8c, there exist an invertible linear functional Λ (i.e., $\Lambda 1 \neq 0$) and a delta operator Q such that $\Lambda Q^k s_n = \delta_{nk}$. By Theorem 3.6.6, there exists a complex Borel measure μ on the real line such that $\Lambda p = \int_{-\infty}^{\infty} p(y)\, d\mu(y)$. Hence,

$$\int_{-\infty}^{\infty} \left(Q^k s_n\right)(x)\, d\mu(x) = \delta_{kn}.$$

'$\Leftarrow$' This follows directly from Theorem 2.4.8c, since $p \mapsto \int_{-\infty}^{\infty} p(x)\ d\mu(x)$ is an invertible linear functional on $\mathcal{P}$. $\square$

3.7 Exponential families

Exponential families of probability measures play a traditional role in statistics (dating back to the thirties) because of their nice estimation properties (see e.g. [221]). However, recently exponential families appear as the cornerstone of the important class of generalized linear models (see [78] for an excellent introduction). In [160], Morris studied natural exponential families on the real line. He showed that there are six classes of natural exponential families with quadratic variance function (i.e. where the variance is a polynomial of degree at most two). In this section we study natural exponential families in light of expansions of their density function in terms of Sheffer polynomials. The delta operator of the associated Sheffer sequence will be shown to relate directly to the variance function of natural exponential family. Using slightly different terminology, Feinsilver proved [89, Chapter 4] that a natural exponential family has a quadratic variance function if and only if the corresponding Sheffer polynomials are orthogonal. This result immediately follows from our approach to natural exponential families (cf. Section 3.4). It is interesting to note that the Morris classification was discovered a few years earlier in approximation theory by May (see [156] and for generalizations [124]). We discuss the relation between exponential families and exponential approximation operators. We also indicate how our approach differs from the approach in [124, 156].

This section is based on [77].

3.7.1 Natural exponential families

We begin by recalling the definition of a natural exponential family. Our notation closely follows [144].
Let ν be a measure on the real line. We assume that ν is not concentrated in one point.
Let the Laplace transform of ν be given by

$$L(\theta) = \int_{-\infty}^{\infty} e^{x\theta}\, d\nu(x). \tag{3.14}$$

We define Θ to be the interior of the set $\{\theta \in \mathbb{R} \mid L(\theta) < \infty\}$. If Θ is non-empty, then the **natural exponential family** generated by ν is the set of probability distributions of the form

$$P_\theta(A) = \int_A e^{x\theta - k(\theta)}\, d\nu(x), \tag{3.15}$$

where k is the **cumulant** of ν, i.e. $k(\theta) := \log L(\theta)$, and $\theta \in \Theta$. We will see later that different ν may generate the same natural exponential family. Since $k(\theta) = \log L(\theta)$, we have

$$e^{k(\theta)} = \int_{-\infty}^{\infty} e^{x\theta}\, d\nu(x). \tag{3.16}$$

It follows by differentiating (3.16) with respect to θ that

$$k'(\theta) = \int_{-\infty}^{\infty} x\, dP_\theta(x). \tag{3.17}$$

Differentiating (3.16) twice with respect to θ and using (3.17), we obtain

$$k''(\theta) = \int_{-\infty}^{\infty} (x - k'(\theta))^2\, dP_\theta(x). \tag{3.18}$$

Let M_ν be the range of k', i.e. $M_\nu = k'(\Theta)$. Since k is strictly convex on Θ by the Hölder inequality[3], $k' : \Theta \longrightarrow M_\nu$ is a bijection. Its inverse will be denoted by

$$\psi : M_\nu \to \Theta.$$

This means that we may reparametrize the densities with respect to ν in (3.15) as

$$\varphi(m, x) = e^{x\psi(m) - k(\psi(m))}. \tag{3.19}$$

Using the reparametrization of (3.19), we now come to the following important definition.

[3] If $0 < \lambda < 1$, then $k(\lambda\theta + (1-\lambda)\xi) = \log\left(\int_{-\infty}^{\infty} e^{\lambda\theta x} e^{(1-\lambda)\xi x}\, d\nu(x)\right) < \log\left(\left(\int_{-\infty}^{\infty} e^{\theta x}\, d\nu(x)\right)^{\lambda} \left(\int_{-\infty}^{\infty} e^{\xi x}\, d\nu(x)\right)^{1-\lambda}\right) = \lambda k(\theta) + (1-\lambda)k(\xi)$. Note that the inequality is strict, since ν is not concentrated in one point.

Definition 3.7.1 *Let $\{P_\theta \mid \theta \in \Theta\}$ be the natural exponential family generated by a measure ν. The function $V_\nu : M_\nu \to \mathbb{R}$ defined by $V_\nu(m) = \int_{-\infty}^{\infty}(x - m)^2 \varphi(m,x)\, d\nu(x)$ is called the* **variance function** *of $\{P_\theta \mid \theta \in \Theta\}$.*

A natural exponential family is uniquely determined by its variance function together with the domain of the variance function ([160]). In the theory of generalized linear models, the variance function is called the **link function** ([78]). The link function is essential for estimating purposes.
Before we continue, we give an example in order to illustrate the notions introduced above.

Example (Poisson family) Consider a Poisson distribution with parameter θ, i.e. $\Pr(k) = \dfrac{e^{-\theta}\,\theta^n}{n!}$ for $n = 0,1,2,\ldots$. Writing $\dfrac{e^{-\theta}\,\theta^n}{n!} = \dfrac{e^{n\log\theta}}{n!\,e^{e^{\log\theta}}}$, we see that $\{P_\theta \mid \theta \in (0,\infty)\}$ is a natural exponential family generated by the discrete measure $\nu\{n\} = 1/n!$, $n = 0,1,2,\ldots$ where P_θ is Poisson$(\log\theta)$ distributed. An easy calculation shows that $k(\theta) = e^\theta$, $\Theta = \mathbb{R}$, $\psi(m) = \log m$, $M_\nu = (0,\infty)$, and $V_\nu(m) = m$. We see that the standard change from θ to the so-called natural parameter $\log\theta$ (cf. [78] is nothing but our reparametrization (3.19).
The following lemma is crucial to our approach.

Lemma 3.7.2 *If $(P_\theta \mid \theta \in \Theta)$ is a natural exponential family, then there exist a real number t and a natural exponential family $\{\widetilde{P}_\theta \mid \theta \in \widetilde{\Theta}\}$ generated by a measure μ such that*

1. $P_\theta(A) = \widetilde{P}_\theta(A + t)$
2. $\displaystyle\int_{-\infty}^{\infty} x\, d\mu(x) = 0$
3. $0 \in \widetilde{\Theta}$
4. $\widetilde{k}'(0) = 0$
5. $V_\mu(m) = V_\nu(m + t)$.

Proof: First note that $\{P_\theta \mid \theta \in \Theta\}$ is also generated by the measure $e^{\theta_0 x}\, d\nu(x)$ for any $\theta_0 \in \Theta$ and that the corresponding parameter set $\widetilde{\Theta}$ equals $\Theta - \theta_0$. In particular, $0 \in \widetilde{\Theta}$. Now define the measure μ by $d\mu(x) = d\nu(x + k'(0))$. It follows from (3.17) that $\widetilde{k}'(0) = 0$. Moreover, easy calculations shows that (1) and (5) hold with $t = k'(0)$. □

Example (Poisson family continued) Let μ be the measure obtained by shifting the generating measure ν one unit ($= k'(0)$) to the left, i.e. $\mu\{n\} = 1/(n+1)!$, $n = -1,0,1,2,\ldots$. An easy calculation yields that μ is of mean zero, $\widetilde{\Theta} = \mathbb{R}$, $\widetilde{k}'(\theta) = e^\theta - 1$, $M_\mu = (-1,\infty)$, $\widetilde{k}'(0) = 0$, and $V_\mu(m) = m + 1$.

Thus we may and will assume without loss of generality that $(P_\theta \mid \theta \in \Theta)$ satisfies the extra conditions of the above Lemma. By well-known properties of Laplace transforms, k and ψ are analytic functions in a neighbourhood of zero. Hence, we may expand (3.19) into a power series in m for $m \in M_\nu$. It follows from (3.15) and (3.17) that $k'(0) = 0$. Moreover, since ν is not concentrated in one point, it follows from (3.18) that $k''(0) \neq 0$. Thus, $\psi(0) = 0$ and $\psi'(0) \neq 0$, which implies that s_n is a polynomial of degree exactly n. The **associated Sheffer polynomials** of a natural exponential family are the polynomials $(s_n)_{n\in\mathrm{N}}$ defined by

$$\varphi(m,x) = \sum_{n=0}^{\infty} s_n(x)\, m^n, \tag{3.20}$$

where φ is defined by (3.19).

The following theorem relates the variance function of a natural exponential family to the delta operator of its associated Sheffer sequence.

Theorem 3.7.3 *Let $\{P_\theta \mid \theta \in \Theta\}$ be a natural exponential family generated by a measure ν with associated Sheffer sequence $(s_n)_{n\in\mathrm{N}}$ (thus we assume without loss of generality that the extra conditions of Lemma 3.7.2 hold). Let $Q = q(D)$ be the delta operator and $A = f(D)$ be the invertible operator of $(s_n)_{n\in\mathrm{N}}$. Then $q'(D) = V_\nu(q(D))$ and $f'(D) = q(D)\, f(D)$. Moreover, f is the Laplace transform of ν.*

Proof: It follows from equations (3.19) and (3.20) and Corollary 2.4.9 that $q(D) = \psi^{-1}(D) = k'(D)$. Thus, by equation (3.18) and the definition of variance function, we arrive at $q'(D) = k''(D) = V_\nu(k'(D))$. For the second statement, note that by Corollary 2.4.9 we have $f(D) = e^{k(D)}$. Hence, $f'(D) = k'(D)\, e^{k(D)} = q(D)\, f(D)$. The last statement follows from $k(\theta) = \log L(\theta)$ and Equation (3.14). □

We now are ready to prove the classification result mentioned in the introduction. The original proof is in [89]), another proof of this result can be found in [143, Theorem 4.1]. The merit of our proof is that it explains why the result is true.

Theorem 3.7.4 (Feinsilver) *The variance function of a natural exponential family is quadratic if and only if the associated Sheffer polynomials are orthogonal.*

Proof: Combine Theorems 3.7.3 and 3.4.1. □

3.7.2 Natural exponential families and approximation theory

In this subsection we show that exponential families appear in disguise in approximation theory[4]. An important consequence of this is that the results of [156, 123, 124] are of importance for the statistics literature (in particular, it turns out that many of the results in [160] were predated by the above-mentioned papers). We will also see how our approach differs from the approach in [123, 124, 156] (apart from different terminology).

We begin with recalling the basics of exponential-type approximation operators, following the exposition in [156] (see also [123, 124]). We slightly change the notation in order to be able to compare directly.

Let $W(\lambda, m, x)$ be the kernel of an exponential-type operator, i.e. $W(\lambda, m, x)$ is a generalized function[5] such that

$$\begin{aligned} W(\lambda, m, x) &\geq 0 && (3.21)\\ \int_{-\infty}^{\infty} W(\lambda, m, x)\, dx &= 1 && (3.22)\\ \frac{\partial}{\partial m} W(\lambda, m, x) &= \frac{\lambda}{p(m)} W(\lambda, m, x)\,(x-m), && (3.23) \end{aligned}$$

where p is analytic and positive on an interval on the real line. The corresponding positive approximation operator is defined by

$$(S_\lambda f)(t) = \int_{-\infty}^{\infty} W(\lambda, t, x)\, f(x)\, dx. \tag{3.24}$$

It is shown in [124, Corollary 3.2] that any solution of the partial differential equation (3.23) (together with the normalization condition (3.22)) is of the form

$$W(\lambda, m, x) = \exp\left(\lambda \int_c^m \frac{x-y}{p(y)}\, dy\right) C(\lambda, x). \tag{3.25}$$

The normalization condition (3.22) yields that $\exp\left(\lambda \int_c^{g(m)} y/p(y)\, dy\right)$ is the Laplace transform of $C(\lambda, x)$, where $g(m) = \int_c^m 1/p(y)\, dy$. In other words, for fixed λ, the $W(\lambda, m, x)$ form a natural exponential family generated by $d\nu(x) = C(\lambda, x)$ (cf. formulas (3.15) and (3.19)) such that $\psi(m) = \int_c^m \lambda/p(y)\, dy$.

[4]On April 29, 1992, Gérard Letac delivered a beautiful lecture on natural exponential families at a one-day conference in Leuven, Belgium. This subsection arose out of remarks made on that occasion by Mourad Ismail.

[5]In fact, one would like to say that $W(\lambda, m, x)$ is the density function of a random variable. However, since we don't want to exclude random variables with discrete parts, we have to resort to generalized functions.

Conversely, let $\{P_\theta \,|\, \theta \in \Theta\}$ be the natural exponential family generated by a measure ν. Consider the reparametrization (3.19). Define the functions $W(\lambda, m, x)$ by

$$W(\lambda, m, x) := e^{x\psi(\lambda m) - k(\psi(\lambda m))}. \tag{3.26}$$

Since $\psi = k^{-1}$, it follows that

$$\psi'(m) = 1/(k''(\psi(m))$$

and

$$(k(\psi(m))' = m/(k''(\psi(m)).$$

Hence, the functions $W(\lambda, m, x)$ defined by (3.26) satisfy

$$\frac{\partial}{\partial m} W(\lambda, m, x) = \frac{\lambda}{k''(\psi(m))}(x - m).$$

Note that $k''(\psi)$ is the variance function of $\{P_\theta \,|\, \theta \in \Theta\}$ by (3.19) (i.e. it is the p appearing in (3.23)).

We have thus obtained a complete correspondence between kernels of exponential-type approximation operators and natural exponential families. Hence, we have shown that the classification problems for exponential-type approximation operators and natural exponential families are equivalent. However, we now want to point some differences between the approach in [123, 124, 156] and our approach. Cast in our terminology, Ismail and May expand the moment generating function of ν and invert Laplace transforms, while we expand the densities with respect to ν in (3.15) and solve differential equations. As a consequence, the polynomial sequences that correspond to approximation operators as in [123] differ from our polynomial sequences. Thus although the classifications yield the same probability distributions, they yield different associated polynomial sequences.

3.7.3 Quadratic variance functions

In this section we use the results of the previous sections to present the classification for natural exponential families with quadratic variance function in full detail.

Theorem 3.7.3 tells us that we must solve the differential equations (3.9) and (3.10) in order to obtain all exponential families with quadratic variance function. Note that since $Q = q(D)$ is a delta operator, we must have $q(0) = 0$ and $q'(0) \neq 0$. Since $A' = A\,Q$ and A is invertible, it follows that $\log(A^{-1}\,A') = Q$. Hence,

$$a(D) = \exp\,(\int q(D)\,dD). \tag{3.27}$$

Note that the integration constant must be equal to zero, since $a(t)$ is the Laplace transform of a mean zero distribution.

Under these conditions, we find the following natural exponential families:

Normal distribution

If the variance function is constant, then $Q' = \alpha$ with $\alpha > 0$. Hence, $Q = \alpha D$. Now (3.27) yields that $A = e^{\alpha D^2/2}$. Thus, the corresponding natural exponential family is generated by a normal distribution with mean zero and variance α for $\alpha > 0$. The associated Sheffer polynomials are the Hermite polynomials of variance α ([202, p. 87 ff.]).

Poisson distribution

If the variance function is a polynomial of degree one, then $Q' = \alpha + \beta Q$. Thus, $Q = \alpha \left(\frac{e^{\beta D} - 1}{\beta} \right)$ and $A = \exp \left(\frac{\alpha}{\beta} \left(\frac{1}{\beta} e^{\beta D} - D \right) \right)$. Thus, the corresponding family is the Poisson family. The associated Sheffer polynomials are the Poisson-Charlier polynomials (see Section 2.5 or cite[p. 119 ff.]Rom9).

Gamma distribution

If the variance function is a polynomial of degree two with two identical roots, then $Q' = \alpha (Q - \beta)^2$. Hence, $Q = \beta \frac{D}{D + 1/(\alpha\beta)}$ and $A = e^{\beta D} \, (1 + \alpha\beta D)^{1/\alpha}$. Thus, the corresponding natural exponential family is the gamma distribution family. The associated Sheffer polynomials are the Laguerre polynomials of variance α ([202, p. 108 ff.]).

Binomial distribution

If the variance function has two different positive roots, then the corresponding natural exponential family is the binomial distribution family. The associated Sheffer polynomials are the Krawtchouk polynomials ([202, p. 125-126]).

Negative binomial distribution

If the variance function has two different negative roots, then the corresponding natural exponential family is the negative binomial distribution family. The associated Sheffer polynomials are a subclass of the Meixner polynomials of the first kind ([202, p. 125-126]).

Hyperbolic distribution

If the variance function has two complex conjugate roots, then the associated Sheffer polynomials are a subclass of the Meixner polynomials of the second kind ([202, p. 126]). The corresponding natural exponential family is generated by the hyperbolic distribution (see [144]).

Conclusion

A few final remarks on generalizations are in order. The approach of this section is not restricted to natural exponential families with quadratic variance function. For example, it could be used to obtain a classification of natural exponential families with cubic variance function as in [144] (in [124] no attempt is being made to obtain a complete classification). Letac and Mora state that it seems hard to obtain classifications of natural exponential families with higher order polynomial variance functions. In light of our approach, this is probably related to the fact that the differential equations $Q' = V(Q)$ are hard (resp. impossible) to solve explicitly when V is a polynomial of degree more than three (resp. four).

A more interesting direction is to generalize our approach to natural exponential families generated by multivariate distributions ([59, 129, 143, 14]).

Chapter 4

Banach algebras

Existence of logarithms of functions is needed in several parts of mathematics. E.g., in the theory of entire functions of a complex variable one needs that if f is a non-vanishing entire function, then there exists an entire function g such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$. In probability theory the following analogous result is essential for the theory of infinitely divisible probability measures: if f is a non-vanishing complex-valued continuous function on $\mathbb{R}$, then there exists a continuous function g such that $f(x) = e^{g(x)}$ for all $x \in \mathbb{R}$ (see e.g. [62, Chapter 7]).

These results are well-known, but their proofs use ad-hoc methods. The following theorem is not well-known and no elementary proof is known: if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is such that $\sum_{n=0}^{\infty} |a_n| < \infty$ and $f(z) \neq 0$ for $|z| \leq 1$, then there exists a function $g(z) = \sum_{n=0}^{\infty} b_n z^n$ such that $\sum_{n=0}^{\infty} |b_n| < \infty$ and $f(z) = e^{g(z)}$ for all $|z| \leq 1$. We will apply this theorem in Section 4.4, where convergence problems concerning polynomials of convolution type will be studied. The theorem is also useful in prediction theory, see [21, Theorem 4.1] or [237, Theorem 6].

In this chapter a unified approach is presented to these and related results. The approach, which seems to be new, uses only elementary Banach algebra techniques and is presented in Section 4.1. Sections 4.2 and 4.3 contain applications of the results of Section 4.1. E.g., the three results mentioned above are Theorems 4.2.11, 4.2.7 and 4.2.2. We apply the results of Sections 4.2 and 4.3 to obtain new analytical results on polynomials of convolution type in Section 4.4 and central limit theorems in Chapter 5. The reader should consult [111] or the survey [112] for other applications of Banach algebra theory to polynomials of convolution type. Finally, in Section 4.5 a two-sided analogue of functions of convolution type is introduced and studied.

This chapter is an extended version of [73].

Contents of Chapter 4

4.1 General Banach algebra techniques

The purpose of this section is to set up the Banach algebra machinery for the approach mentioned in the introduction of this chapter. At the end of this section, a new discussion of the Arens-Royden theorem can be found.

We begin with a review of the basics of Banach algebra theory. Chapter 18 of [213] is recommended as a quick introduction to Banach algebra theory.

A **Banach algebra** $\mathcal{B}$ is a complex Banach space that also possesses a multiplication (it is important that $\mathcal{B}$ is a vector space over $\mathbb{C}$, see [212, Remarks 10.4]). This multiplication must obey the distributive and associative law and must satisfy the inequality $\|xy\| \leq \|x\| \, \|y\|$ for all $x, y \in \mathcal{B}$. A Banach algebra $\mathcal{B}$ such that $xy = yx$ for all $x, y \in \mathcal{B}$ is said to be commutative. An important example of a commutative Banach algebra is the Banach algebra $\mathcal{C}(\mathcal{K})$ of continuous complex-valued functions on a compact Hausdorff space $\mathcal{K}$; addition and multiplication are defined pointwise. The norm of $\mathcal{C}(\mathcal{K})$ is the supremum norm.

An element $u \in \mathcal{B}$ is called **unit element** or **identity** of $\mathcal{B}$ if $xu = ux = x$ for all $x \in \mathcal{B}$ and $\|u\| = 1$ (the last requirement can be weakened, see e.g. [212, Theorem 10.2]). It follows from elementary algebra that at most one unit element exists. If a Banach algebra has no unit element, then a unit element can be adjoined (see [142] for a detailed account of the relations between a Banach algebra without unit element and the Banach algebra obtained by adjoining a unit element).
Let $\mathcal{B}$ be a Banach algebra with unit element u. An element $x \in \mathcal{B}$ is **invertible** if there exists a $y \in \mathcal{B}$ such that $xy = yx = u$. The set of all invertible elements of $\mathcal{B}$ will be denoted by inv $\mathcal{B}$. We equip inv $\mathcal{B}$ with the norm topology inherited from $\mathcal{B}$. Equipped with this topology inv $\mathcal{B}$ is a topological group.
Continuous linear functionals are important in the theory of Banach spaces . Their role is taken over by complex homomorphisms in the Banach algebra case. A **complex homomorphism** of a Banach algebra is a not identically zero continuous linear multiplicative map from the Banach algebra into $\mathbb{C}$. The set of complex homomorphisms of a Banach algebra $\mathcal{B}$ is called the **maximal**

ideal space of $\mathcal{B}$ and will be denoted by $\mathcal{M}$. The name maximal ideal space is explained by the fact that there is a one-to-one correspondence between maximal ideals of $\mathcal{B}$ and null spaces of complex homomorphisms of $\mathcal{B}$ (see [212, Theorem 11.15]). If $\mathcal{B}$ is a commutative Banach algebra with unit element, then $\mathcal{M} \neq \emptyset$ (see [142, Theorem 3.3.2]). We will consider elements of the maximal ideal space as complex homomorphisms. We equip the maximal ideal space $\mathcal{M}$ with the Gelfand topology, i.e. the topology of pointwise convergence. This makes $\mathcal{M}$ into a compact Hausdorff space ([212, Theorem 11.9a]). Complex homomorphisms are useful in deciding the invertibility of an element as can be seen from Theorem 4.1.2.

The following theorem gives an explicit example of a maximal ideal space.

Theorem 4.1.1 *Let $\mathcal{K}$ be a compact Hausdorff space. The complex homomorphisms of $\mathcal{C}(\mathcal{K})$ are the point evaluations on $\mathcal{K}$. Moreover, the maximal ideal space of $\mathcal{C}(\mathcal{K})$ with its Gelfand topology is homeomorphic to $\mathcal{K}$.*

Proof: See [212, Example 11.13a] or [83, Proposition 2.3]. □

Theorem 4.1.2 *Let $\mathcal{B}$ be a Banach algebra with unit element. Then $x \in inv\ \mathcal{B}$ if and only if $\Lambda(x) \neq 0$ for all $\Lambda \in \mathcal{M}$.*

Proof: See [212, Theorem 11.5c] or [213, Theorem 18.17c]. □

Let $\mathcal{B}$ be a Banach algebra with unit element u. Define **exp**$\mathcal{B}$ to be the subset of $\mathcal{B}$ consisting of those $x \in \mathcal{B}$ such that $x = e^y$ for some $y \in \mathcal{B}$. Here e^y is defined by $e^y := \sum_{n=0}^{\infty} \frac{y^n}{n!}$ for all $y \in \mathcal{B}$, where $e^0 := u$. It is easy to see that this series converges in the norm topology for all $y \in \mathcal{B}$.

Remarks 4.1.3 Let $\mathcal{B}$ be a Banach algebra with unit element u.

a) It follows from $e^y e^{-y} = u$ for all $y \in \mathcal{B}$ (see [35, Lemma 1.4.1]), that exp $\mathcal{B} \subset$ inv $\mathcal{B}$.
b) If $x \in \mathcal{B}$ and $\|x - u\| < 1$, then $x \in$ exp $\mathcal{B}$ (see [35, Lemma 1.4.2]). In particular, $x \in$ inv $\mathcal{B}$.
c) It holds true that inv $\mathcal{B}$ is open in $\mathcal{B}$ (see [212, Theorem 10.12]).
d) The following inequality holds true for all $x \in \mathcal{B}$ and all $t \in \mathbb{C}$:

$$\|e^{tx}\| = \|\sum_{k=0}^{\infty} \frac{t^k x^k}{k!}\| \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} \|x\|^k = e^{|t|\,\|x\|}$$

Let T be a topological space. A subset U of T is **connected** if $U = O_1 \cup O_2$ where O_1 and O_2 are disjoint open subsets of U, then $O_1 = \emptyset$ or $O_2 = \emptyset$ (see e.g. [84, Chapter 5]. A **component of** U is a connected subset of U which is not contained in a larger connected subset of U. Note that components are relatively closed ([84, Chapter 5, Theorem 3.2]), that components of open sets in a locally connected space ([84, Chapter 5, Definition 4.1]) are open ([84, Chapter 5, Theorem 4.2]) and that continuous images of connected sets are connected ([84, Chapter 5, Theorem 1.4]). Note that a union of non-disjoint connected sets is again connected ([84, Chapter 5, Theorem 1.5]).

We saw above that if $\mathcal{B}$ is a Banach algebra with unit element, then inv $\mathcal{B}$ is a topological group with the relative norm topology. Let $\mathcal{G}_1$ be the component of inv $\mathcal{B}$ that contains the unit element of $\mathcal{B}$.

Theorem 4.1.4 *Let $\mathcal{B}$ be a commutative Banach algebra with unit element u. Then exp $\mathcal{B} = \mathcal{G}_1$. In particular, exp $\mathcal{B}$ is closed in inv $\mathcal{B}$.*

Proof: An elementary proof of the first statement can be found in [35, Theorem 1.4.3]. The second statement follows from the first statement, since components are relatively closed. □

Theorem 4.1.4 is difficult to use, since there is no general way to calculate $\mathcal{G}_1$. For the algebras that will be discussed in Section 4.2, this problem will be solved by using the following theorem.

Definition 4.1.5 *Let T be a topological space and $a, b \in T$. A* **path from** *a* **to** *b* **in** *T is a continuous function $f : [0,1] \to T$ such that $f(0) = a$ and $f(1) = b$.*

Theorem 4.1.6 *Let $\mathcal{B}$ be a commutative Banach algebra with unit element u. Then $x \in$ exp $\mathcal{B}$ if and only if $x \in$ inv $\mathcal{B}$ and there is path f in inv $\mathcal{B}$ from αu to x for some $\alpha \in \mathbb{C} \setminus \{0\}$.*

Proof: '$\Rightarrow$' By definition there exists $y \in \mathcal{B}$ such that $x = e^y$. Then F, defined by $F(t) := e^{ty}$, is a path in inv $\mathcal{B}$ from u to $e^y = x$.
'$\Leftarrow$' Let g be an arbitrary path in $\mathbb{C} \setminus \{0\}$ from 1 to α . Then h, defined by $h(t) := g(t)\,u$, is a path in inv $\mathcal{B}$ from u to αu. Therefore F, defined by $F(t) := h(2t)$ for $0 \le t \le \frac{1}{2}$ and $F(t) := f(2t-1)$ for $\frac{1}{2} \le t \le 1$, is a path in inv $\mathcal{B}$ from u to x. It follows from the continuity of F that $F([0,1])$ is connected. Hence $F([0,1]) \cup \mathcal{G}_1$ is connected, since $u \in F([0,1]) \cap \mathcal{G}_1$. Because $\mathcal{G}_1$ is the largest connected subset of inv $\mathcal{B}$ that contains u, we have $F([0,1]) \subset \mathcal{G}_1$. It follows in particular that $x = F(1) \in \mathcal{G}_1$, hence $x \in$ exp $\mathcal{B}$ by Theorem 4.1.4.
□

Remarks 4.1.7 a) Theorem 4.1.6 implies that $\mathcal{G}_1$ is a path-component of inv $\mathcal{B}$ (see [84, Section 5.5]). This also follows from Theorem 4.1.4, since inv $\mathcal{B}$ is locally path-connected (because it is an open subset of a normed linear space) and in a locally path-connected space components and path-components coincide ([84, Chapter 5, Theorem 5.5]). Note that in general path-connectedness is stronger than connectedness ([84, Chapter 5, Theorem 5.3]).
b) Theorems 4.1.4 and 4.1.6 are false when $\mathcal{B}$ is not commutative. E.g., let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Then inv $\mathcal{B}(\mathcal{H})$ is connected ([212, Theorem 12.37]), but need not be equal to exp $\mathcal{B}(\mathcal{H})$ ([212, Theorem 12.38]).

The following theorem describes the form of the components of inv $\mathcal{B}$. Recall that $x\exp\mathcal{B} := \{xy : y \in \exp\mathcal{B}\}$.

Theorem 4.1.8 *Let $\mathcal{B}$ be a commutative Banach algebra with unit element u. Then the components of inv $\mathcal{B}$ are of the form $x\exp\mathcal{B}$ with $x \in$ inv $\mathcal{B}$.*

Proof: Let C be a component of inv $\mathcal{B}$. Take an arbitrary element $x \in C$. The continuous image $x^{-1}C$ of C is connected and contains u. Hence, $x^{-1}C \subset \exp\mathcal{B}$ which implies that $C \subset x\exp\mathcal{B}$. In order to prove the other inclusion, note that $x\exp\mathcal{B}$ is a connected subset of inv $\mathcal{B}$ (since it is the continuous image of a connected set by Theorem 4.1.4) that contains x. Hence, $x\exp\mathcal{B} \subset C$. □

In order to obtain further results on exp $\mathcal{B}$ and inv $\mathcal{B}$, we need to discuss the Gelfand transform. This is necessary in those cases where Theorem 4.1.6 does not help us. The idea behind the Gelfand transform is to transfer problems in a Banach algebra (e.g., the calculation of $\mathcal{G}_1$) to a canonically associated Banach algebra of the form $\mathcal{C}(\mathcal{K})$, i.e. a Banach algebra of continuous functions. The advantage of this procedure is that Banach algebras of continuous functions are simpler to work with.

If $x \in \mathcal{B}$, then we define a continuous function $\widehat{x}$ on $\mathcal{M}$ (the maximal ideal space of $\mathcal{B}$ by $\widehat{x}(\Lambda) := \Lambda x$ for all $\Lambda \in \mathcal{M}$. The function $\widehat{x}$ is called the **Gelfand transform** of x. Note that the Gelfand topology is the weakest topology that makes all functions $\widehat{x}$ continuous (see [142, Corollary 3.3.1]). The Gelfand transform maps $\mathcal{B}$ onto a subalgebra $\mathcal{B}$ of $\mathcal{C}(\mathcal{M})$. The image of the algebra $\mathcal{B}$ under the Gelfand transform, equipped with the supremum norm, need not be a closed subalgebra of $\mathcal{C}(\mathcal{M})$.

When $\mathcal{B} = L^1(\mathbb{R})$, then the Gelfand transform is nothing but the Fourier transform (see e.g. [213, Chapter 18]). However, this is not a typical example for the sequel since $L^1(\mathbb{R})$ is a Banach algebra without unit.

The following two theorems show that the Gelfand transform is useful for our purposes.

Theorem 4.1.9 *Let $\mathcal{B}$ be a commutative Banach algebra with unit element u and let x be an arbitrary element of $\mathcal{B}$. Then $x \in$ inv $\mathcal{B}$ if and only if $\widehat{x} \in$ inv $\mathcal{C}(\mathcal{M})$.*

Proof: Note that $\mathcal{M}$ with its Gelfand topology is a compact Hausdorff space ([212, Theorem 11.9a]). It follows from Theorems 4.1.1 and 4.1.2 that $\widehat{x} \in$ inv $\mathcal{C}(\mathcal{M})$ if and only if $\widehat{x}(\Lambda) \neq 0$ for all $\Lambda \in \mathcal{M}$. By Theorem 4.1.2, $x \in$ inv $\mathcal{B}$ if and only if $\Lambda(x) \neq 0$ for all $\Lambda \in \mathcal{M}$. The theorem now follows from the definition of $\widehat{x}$. □

The usual proof of Theorem 4.1.9 (see e.g. [83, Proposition 2.34]) uses the correspondence between maximal ideals and complex homomorphisms.

The following theorem is an analogue of Theorem 4.1.9 for exp $\mathcal{B}$.

Theorem 4.1.10 *Let $\mathcal{B}$ be a commutative Banach algebra with unit element u and let x be an arbitrary element of $\mathcal{B}$. Then $x \in exp\ \mathcal{B}$ if and only if $\widehat{x} \in exp\ \mathcal{C}(\mathcal{M})$.*

Proof: '$\Rightarrow$' Let $y \in \mathcal{B}$ be such that $x = e^y$. Since the Gelfand transform is continuous, it follows that $\widehat{x} = e^{\widehat{y}}$. Hence, $\widehat{x} \in \exp \mathcal{C}(\mathcal{M})$.
'$\Leftarrow$' See [101, Chapter 3, Corollary 6.2]. □

The proof of Theorem 4.1.10 in [101] uses holomorphic functions. Since the statement of Theorem 4.1.10 is a topological statement (cf. Theorem 4.1.4), it seems appropriate to prove Theorem 4.1.10 in a purely topological way. Unfortunately, I have not been able to find a topological proof of Theorem 4.1.10.

We are now able to say something more about the Gelfand transform. The following corollary shows that the image of a Banach algebra under the Gelfand transform is a special kind of subalgebra of $\mathcal{C}(\mathcal{M})$ (cf. [212, Theorem 10.18]).

Corollary 4.1.11 *Let $\mathcal{B}$ be a commutative Banach algebra with unit element u. The Gelfand transform maps distinct components of inv $\mathcal{B}$ into distinct components of inv $\mathcal{C}(\mathcal{M})$.*

Proof: Let $y, z \in$ inv $\mathcal{B}$ arbitrary. Suppose that $\widehat{y}$ and $\widehat{z}$ are in the same component of inv $\mathcal{C}(\mathcal{M})$. An application of Theorem 4.1.8 to the Banach algebra $\mathcal{C}(\mathcal{M})$ yields that $\widehat{y} \in \widehat{z} \exp \mathcal{C}(\mathcal{M})$, so $\widehat{(yz^{-1})} = \widehat{y}\, \widehat{z}^{-1} \in \exp \mathcal{C}(\mathcal{M})$. It follows from Theorem 4.1.10 that $yz^{-1} \in \exp \mathcal{B}$, so $y \in z \exp \mathcal{B}$. Hence, y and z are in the same component of inv $\mathcal{B}$ by Theorem 4.1.8. □

We conclude this section with a discussion of the Arens-Royden Theorem on the structure of inv $\mathcal{B}$/exp $\mathcal{B}$. It follows directly from Theorem 4.1.8 that there exists a one-to-one correspondence between inv $\mathcal{B}$/exp $\mathcal{B}$ and the components of inv $\mathcal{B}$. The Arens-Royden Theorem says that the algebraic quotient group inv $\mathcal{B}$/exp $\mathcal{B}$ (with multiplication as binary operation) is isomorphic to $\mathcal{H}^1(\mathcal{M}, \mathbb{Z})$, the first Čech cohomology group of $\mathcal{M}$. Thus the Arens-Royden Theorem expresses inv $\mathcal{B}$/exp $\mathcal{B}$ in terms of the maximal ideal space $\mathcal{M}$ of $\mathcal{B}$. The original proofs of Arens and Royden can be found in [11] and [211]; another (elegant) proof is in [101, Corollary 7.4, p. 91]. Following a suggestion of Douglas (see [83, Chapter 2]) , we state the theorem in terms of $\pi^1(\mathcal{M})$, the first cohomotopy group of $\mathcal{M}$ (definition below). It is proved in [120, Chapter 11, Theorem 7.1] that $\pi^1(\mathcal{M})$ and $\mathcal{H}^1(\mathcal{M}, \mathbb{Z})$ are isomorphic.

Definition 4.1.12 *Let $\mathcal{K}$ be a topological space and let $f, g : \mathcal{K} \to V \subset \mathbb{C}$ be continuous functions. A* **homotopy in V of** *f* **with** *g is a continuous function $H : [0,1] \times \mathcal{K} \to V$ such that $H(0,z) = f(z)$ and $H(1,z) = g(z)$ for all $z \in \mathcal{K}$. If moreover $\mathcal{K}$ is compact and Hausdorff, then the* **first cohomotopy group** *$\pi^1(\mathcal{K})$ is defined to be the group of homotopy equivalence classes of continuous maps from $\mathcal{K}$ to $\{z \ : \ |z| = 1\}$. The group operation of $\pi^1(\mathcal{K})$ is pointwise multiplication.*

We now prove the special case $\mathcal{B} = C(\mathcal{K})$ of the Arens-Royden Theorem. This special case is due to Bruschlinsky and Eilenberg (see [43] and [85]). Our proof is inspired by the proof of [83, Theorem 2.18].

Theorem 4.1.13 *If $\mathcal{K}$ is a compact Hausdorff space, then inv $\mathcal{C}(\mathcal{K})/$exp $\mathcal{C}(\mathcal{K})$ and $\pi^1(\mathcal{K})$ are isomorphic groups.*

Proof: We define a homomorphism H from inv $\mathcal{C}(\mathcal{K})/\exp\ \mathcal{C}(\mathcal{K})$ to $\pi^1(\mathcal{K})$ as follows. Let $[f]$ be an element of inv $\mathcal{C}(\mathcal{K})/\exp\ \mathcal{C}(\mathcal{K})$. Then $H([f])$ is defined to be the homotopy equivalence class of $f/|f|$, where f is any representative of $[f]$. Note that H is a well-defined homomorphism, since it follows from Theorem 4.1.6 that the elements of inv $\mathcal{C}(\mathcal{K})/\exp\ \mathcal{C}(\mathcal{K})$ are homotopy equivalence classes of continuous maps from $\mathcal{K}$ into $\mathbb{C} \setminus 0$. Since it is obvious that H is surjective, it only remains to prove that H is injective. Suppose that $H([f]) = H([g])$. Let f, g be representatives of $[f], [g]$ respectively. Since f and $f/|f|$ are homotopic, it follows that f and g are homotopic. Hence, $[f] = [g]$. □

We conclude this section with a few words on the general case of the Arens-Royden Theorem. In view of Theorem 4.1.13, it suffices to show that

$$\text{inv } \mathcal{B}/\exp \mathcal{B} \cong \text{ inv } \mathcal{C}(\mathcal{M})/\exp \mathcal{C}(\mathcal{M}),$$

where $\mathcal{M}$ is the maximal ideal space of $\mathcal{B}$. It follows from Corollary 4.1.11 that the canonical map induced by the Gelfand transform is an injective homomorphism from inv $\mathcal{B}/\exp\ \mathcal{B}$ into inv $\mathcal{C}(\mathcal{M})/\exp\ \mathcal{C}(\mathcal{M})$. The difficult point is to prove surjectivity. For a proof of surjectivity using holomorphic calculus, see [101, Chapter 3, Theorem 7.2].

For more information on cohomology and Banach algebras, we refer to the survey articles by Johnson and Taylor in [251].

4.2 Algebras with contractible maximal ideal space

In this section Theorem 4.1.6 will be applied to some explicit Banach algebras. We thus obtain among other things a simple proof of a theorem due to Borsuk (see Theorem 4.2.8).

Notation $\mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}$; $\overline{\mathcal{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

For later use we prove the following uniqueness lemma.

Lemma 4.2.1 *a) Let $\mathcal{K}$ be a connected topological space. Suppose that g and h are complex-valued continuous functions on $\mathcal{K}$ such that $g(a) = h(a)$ for some $a \in \mathcal{K}$ and that $e^{g(z)} = e^{h(z)}$ for all $z \in \mathcal{K}$. Then $g(z) = h(z)$ for all $z \in \mathcal{K}$.*

b) Let $\mathcal{B}$ be a Banach algebra of absolutely summable sequences with component-wise addition and convolution as multiplication. If $x, y \in \mathcal{B}$ and $x = e^y$, then y_n is uniquely determined for $n \geq 1$ and y_0 is determined by $x_0 = e^{y_0}$. In particular, if $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$ and there exists $h \in \mathcal{B}$ with $h_0 = 0$ such that $(q_n(t_0))_{n\in\mathbb{N}} = e^{t_0 h}$, then $h_n = t_0 g_n$ for all $n \in \mathbb{N}$.

Proof: a) It follows from $e^{g(z)} = e^{h(z)}$ that $g(z) = h(z) + 2\pi i k(z)$ with $k(z) \in \mathbb{Z}$. Hence, k is a continuous integer-valued function on $\mathcal{K}$ with $k(a) = 0$. Since the continuous image of $\mathcal{K}$ is connected, we must have $k = 0$.
b) Define $\gamma_0 := 0$ and $\gamma_n := y_n$ for $n \geq 1$. Then $x = e^y$ is equivalent to $x_0 = e^{y_0}$ and $x_n = e^{y_0} \sum_{k=0}^n \frac{\gamma_n^{k*}}{k!}$ for $n \geq 1$. We will now show by induction on n that y_n is uniquely determined by $x_0, \ldots, x_n$. The case $n = 1$ is clear, since $x_1 = e^{y_0}\gamma_1 = x_0 y_1$ (note that $x_0 \neq 0$). Suppose by induction that the statement is true at n. It follows from Lemma 2.1.5b and Lemma 2.1.5c that γ_{n+1}^{k*} is a polynomial in $\gamma_1, \ldots, \gamma_n$ with coefficients not depending on $(\gamma_n)_{n\in\mathbb{N}}$ for $2 \leq k \leq n+1$. Since $\gamma_n = y_n$ for $n \geq 1$, the induction hypothesis implies that y_{n+1} is uniquely determined by $x_0, \ldots, x_{n+1}$. Because $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type, we have $q_n(x) = \sum_{k=0}^n g_n^{k*} \frac{x^k}{k!}$ for all $n \in \mathbb{N}$ and $g_0 = 0$ by Theorem 2.1.8. Since $h_0 = 0$ and $(2_n)_{n\in\mathbb{N}} q t_0 = e^{t_0 h}$, the argument used above yields $h_n = t_0 g_n$ for all $n \in \mathbb{N}$. □

4.2.1 Algebras of summable sequences

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying $\alpha_0 = 1$ and $\alpha_{n+m} \leq \alpha_n \alpha_m$ for all $n, m \in \mathbb{N}$. Let $\ell_1(\alpha)$ be the Banach algebra of all complex sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $\|x\|_{1,\alpha} := \sum_{n=0}^\infty \alpha_n |x_n| < \infty$. Addition in $\ell_1(\alpha)$ is defined component-wise, multiplication is defined to be convolution.
The complex homomorphisms of $\ell_1(\alpha)$ are of the form $\Lambda_z(x) = \sum_{n=0}^\infty x_n z^n$ with $|z| \leq e^\rho$, where $\rho := \lim_{n\to\infty} n^{-1} \log \alpha_n$ (see [105, Section 19, pp. 116-120]). If $\rho = -\infty$, then the only complex homomorphism of $\ell_1(\alpha)$ is $\Lambda_0(x) = x_0$.
The unit element of $\ell_{1,\alpha}$ is the sequence $(1, 0, 0, \ldots)$.
If $\alpha_n = 1$ for all $n \in \mathbb{N}$, then $\ell_1(\alpha)$ is the usual Banach algebra ℓ_1 of absolutely summable sequences.

Theorem 4.2.2 *$\{x \in \ell_1(\alpha) : \sum_{n=0}^\infty x_n z^n \neq 0 for\ all\ |z| \leq e^\rho\} = inv\ \ell_1(\alpha) = exp\ \ell_1(\alpha)$.*

Proof: The first equality follows from Theorem 4.1.2.
For the second equality we only need to prove inv $\ell_1(\alpha) \subset$ exp $\ell_1(\alpha)$ by Remark 4.1.3a. Let $x \in$ inv $\ell_1(\alpha)$ be arbitrary. Then $\sum_{n=0}^\infty x_n z^n \neq 0$ for all $|z| \leq e^\rho$. Hence, in particular $x_0 \neq 0$. Define $f : [0,1] \longrightarrow \ell_1(\alpha)$ by $f(t) := (t^n x_n)_{n\in\mathbb{N}}$. It follows from dominated convergence that $\lim_{t\to s} \|f(t) - f(s)\|_{1,\alpha} = 0$. Hence, f is a path in inv $\ell_1(\alpha)$ from $x_0 u$ to x. Theorem 4.1.6 now yields $x \in$ exp $\ell_1(\alpha)$. □

For an extension of Theorem 4.2.2, see Theorem 4.3.9.

4.2.2 Algebras of continuous functions

Let $\mathcal{K}$ be a compact Hausdorff space. Denote by $\mathcal{C}(\mathcal{K})$ the Banach algebra of continuous functions on $\mathcal{K}$ with pointwise addition and multiplication (see [212, Example 11.13a]). The norm on $\mathcal{C}(\mathcal{K})$ is the supremum norm, denoted by $\|f\|_\infty$. The unit element of $\mathcal{C}(\mathcal{K})$ is the function that is identically one.
We will prove that if $\mathcal{K}$ is contractible to a point (see Definition 4.2.3 below), then an analogue of Theorem 4.2.2 holds for $\mathcal{C}(\mathcal{K})$.

Definition 4.2.3 *Let $\mathcal{K}$ be a topological space. A* **contraction** *of $\mathcal{K}$ to $z_0 \in \mathcal{K}$ is a continuous mapping $H : [0,1] \times \mathcal{K} \to \mathcal{K}$ such that $H(0,z) = z_0$ and $H(1,z) = z$ for all $z \in \mathcal{K}$. If there exists a contraction of $\mathcal{K}$ to some point of $\mathcal{K}$, then $\mathcal{K}$ is said to be* **contractible**.

Examples 4.2.4 a) Any disc $z \in \mathbb{C} : |z| \leq r$ is contractible: take $H(t,z) = tz$.
b) If $a, b \in \mathbb{R}$, then $[a,b]$ is contractible: take $H(t,x) = a + t(x-a)$.
c) The set $\mathbb{C} \setminus (-\infty, 0]$ is contractible: take $H(t, re^{i\varphi}) := (tr + 1 - t)e^{it\varphi}$.

Theorem 4.2.5 *If $\mathcal{K}$ is a contractible compact Hausdorff space, then we have inv $\mathcal{C}(\mathcal{K})$ = exp $\mathcal{C}(\mathcal{K})$. In particular, if $\mathcal{K}$ is a contractible compact subset of $\mathbb{C}$ and f is a non-vanishing continuous function on $\mathcal{K}$, then there exists a continuous function g on $\mathcal{K}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathcal{K}$. Moreover, g is analytic in those points in which f is analytic. If $f(a) = 1$ for some point $a \in \mathcal{K}$, then there is unique continuous function g on $\mathcal{K}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathcal{K}$ and $g(a) = 0$.*

Proof: It follows from Remark 4.1.3a that exp $\mathcal{C}(\mathcal{K}) \subset$ inv $\mathcal{C}(\mathcal{K})$. Let $f \in$ inv $\mathcal{C}(\mathcal{K})$ be arbitrary and let H be an arbitrary contraction of $\mathcal{K}$ to $a \in \mathcal{K}$, say. By the uniform continuity of f and H, $\lim_{t\to s} \|f(H(t,.)) - f(H(s,.))\|_\infty = 0$. Hence F, defined by $F(t) := f(H(t,.))$, is a path in inv $\mathcal{C}(\mathcal{K})$ from $f(a)u$ to f. Now Theorem 4.1.6 yields $f \in$ exp $\mathcal{C}(\mathcal{K})$.
If f is analytic at z_0, then for all z sufficiently close to z_0, we have $g(z) = \zeta + \log\left\{1 + \frac{f(z)-f(z_0)}{f(z_0)}\right\}$, where log denotes the principal branch of the logarithm and ζ denotes some number such that $e^{\zeta)} = f(z_0)$. Thus g is analytic in z_0.
The last statement follows directly from Lemma 4.2.1a. □

Remarks 4.2.6 a) Theorem 4.2.5 also holds if each component of $\mathcal{K}$ is compact and contractible or if $\mathcal{K}$ is the union of an increasing sequence of compact contractible Hausdorff spaces.
b) Theorem 4.2.5 also holds for compact subsets of $\mathbb{C}$ if contractibility of $\mathcal{K}$ is weakened to connectedness of $\mathbb{C} \setminus \mathcal{K}$ ([45, Corollary 4.33]). I have not been able to find a simple proof of this result with the methods of this chapter (cf. Remark4.2.9).
The so-called topologist's sine-curve (see e.g. [245, pp. 44-45]) is an example of a compact connected subset of $\mathbb{C}$ with connected complement which is not contractible (this example was shown to me by Jan van Mill). It follows from

the Alexander Duality Theorem ([245, Chapter 11]) that if $\mathcal{K}$ is a compact contractible subset of $\mathbb{C}$, then both $\mathcal{K}$ and $\mathbb{C} \setminus \mathcal{K}$ are connected.
c) Jan van Mill also pointed out to me that if $\mathcal{K}$ is a compact connected subset of $\mathbb{C}$ with connected complement, then it follows from [31, Theorem 7.6, p. 322], that there exists a decreasing sequence $(\mathcal{K}_n)_{n\in\mathbb{N}}$ of compact contractible subsets of $\mathbb{C}$ such that $\mathcal{K} = \bigcap_{n\in\mathbb{N}} \mathcal{K}_n$. Using this result, we can easily prove the extension of Theorem 4.2.5 mentioned in b) as follows: let f be an arbitrary non-vanishing continuous function on $\mathcal{K}$. By the Tietze Extension Theorem ([213, Theorem 20.4]), f has a continuous extension F on $\mathbb{C}$. Suppose there is a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n \in \mathcal{K}_n$ and $F(z_n) = 0$ for all $n \in \mathbb{N}$. Since each $\mathcal{K}_n$ is compact, there exists a convergent subsequence $(z_{n_k})_{k\in\mathbb{N}}$ whose limit z_0 belongs to $\bigcap_{n\in\mathbb{N}} \mathcal{K}_n = \mathcal{K}$. Hence, $f(z_0) = F(z_0) = \lim_{n\to\infty} F(z_n) = 0$, which contradicts that f is non-vanishing. Thus we have shown that there exists an $N \in \mathbb{N}$ such that $F(z) \neq 0$ for all $z \in \mathcal{K}_N$. Now the result follows by applying Theorem 4.2.5 to $\mathcal{K}_N$ and F.
d) The following example shows that contractibility of $\mathcal{K}$ is not a necessary condition in Theorem 4.2.5. Let $\mathcal{K}$ be a finite set with at least two elements and equip $\mathcal{K}$ with the discrete topology. Then $\mathcal{K}$ is a compact topological space, which is not contractible. It is easy to see that inv $\mathcal{C}(\mathcal{K}) = \exp \mathcal{C}(\mathcal{K})$.
The Banach algebra ℓ_∞ of bounded complex sequences is a more sophisticated example. The norm on ℓ_∞ is the supremum norm. Addition and multiplication are defined pointwise. It follows from general properties of the Čech-Stone compactification (see e.g. [101, Theorem 8.3, p. 17] or [142, p. 90]) that the Banach algebras ℓ_∞ and $\mathcal{C}(\beta\mathbb{N})$, where $\beta\mathbb{N}$ denotes the Čech-Stone compactification of $\mathbb{N}$, are isomorphic. Note that $\beta\mathbb{N}$ is not contractible, since each $n \in \mathbb{N}$ is an isolated point of $\beta\mathbb{N}$. We will now show that inv $\mathcal{C}(\beta\mathbb{N}) = \exp \mathcal{C}(\beta\mathbb{N})$ by showing that inv $\ell_\infty = \exp \ell_\infty$. It is clear that $(x_n)_{n\in\mathbb{N}} \in \text{inv } \ell_\infty$ if and only if $\inf_{n\in\mathbb{N}} |x_n| > 0$. Let $(x_n)_{n\in\mathbb{N}} \in \text{inv } \ell_\infty$ be arbitrary. Choose $y_n \in \mathbb{C}$ such that $e^{y_n} = x_n$ and $Im y_n \in [0, 2pi]$ for all $n \in \mathbb{N}$. Then $(\text{Re } y_n)_{n\in\mathbb{N}} \in \ell_\infty$, since $0 < \inf_{n\in\mathbb{N}} |x_n| \leq \sup_{n\in\mathbb{N}} |x_n| < \infty$. Thus $(y_n)_{n\in\mathbb{N}} \in \ell_\infty$ and $(x_n)_{n\in\mathbb{N}} \in \exp \ell_\infty$.

The special case $\mathcal{K} = [a, b]$ $(a, b \in \mathbb{R})$ of Theorem 4.2.5 and the following theorem are important in probability theory, see e.g. [62, Chapter 7].

Theorem 4.2.7 *Let f be a non-vanishing continuous function on $\mathbb{R}$ such that $f(0) = 1$. Then there exists a unique continuous function g on $\mathbb{R}$ such that $f(x) = e^{g(x)}$ for all $x \in \mathbb{R}$ and $g(0) = 0$.*

Proof: It follows from Theorem 4.2.5 and Example 4.2.4b that there exists for each $n \in \mathbb{N}$ a unique continuous function g_n such that $e^{g_n} = f$ on $[-n, n]$ and $g_n(0) = 0$. By Lemma 4.2.1a, $g_n = g_m$ on $[-n, n]$ if $m > n$. Hence, the function g, defined by $g(x) := g_n(x)$ if $|x| \leq n$, is well-defined, continuous, satisfies $f(x) = e^{g(x)}$ for all $x \in \mathbb{R}$, and $g(0) = 0$. Uniqueness follows from Lemma 4.2.1a. □

If $\mathcal{K}$ is an arbitrary compact subset of $\mathbb{C}$, then the following theorem due to Borsuk (see e.g. [45, Theorem 4.24]) states which continuous functions on $\mathcal{K}$

have continuous logarithms. Our proof, which is new, follows from the simple observation that if $f, g \in \mathcal{C}(\mathcal{K})$ and there exists a homotopy in $\mathbb{C} \setminus 0$ of f with g, then there is a path in inv $\mathcal{C}(\mathcal{K})$ (with respect to the norm topology) from f to g.

For the definition of homotopy appearing in the following theorem, see Definition 4.1.12.

Theorem 4.2.8 (Borsuk) *Let $\mathcal{K}$ be a compact subset of $\mathbb{C}$ and let $f : \mathcal{K} \to \mathbb{C} \setminus 0$ be continuous. Then the following statements are equivalent:*

1. *there exists a homotopy in $\mathbb{C} \setminus 0$ of f with a constant function.*
2. *there exists a continuous function $g : \mathcal{K} \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathcal{K}$.*
3. *f has an extension to a continuous function $F : \mathbb{C} \to \mathbb{C} \setminus 0$.*

Proof: We will prove $1 \Leftrightarrow 2$ and $2 \Leftrightarrow 3$.

'$1 \Rightarrow 2$' Let H be a homotopy of f in $\mathbb{C} \setminus 0$ with a constant α. By the uniform continuity of H on $\mathcal{K}$, we have $\lim_{t \to s} \|H(t,.) - H(s,.)\|_\infty = 0$. Hence, H is a path in inv $\mathcal{C}(\mathcal{K})$ from f to αu. Now 2) follows from Theorem 4.1.6.
'$2 \Rightarrow 1$' If g is any function as in 2), then H, defined by $H(t,z) := e^{(1-t)g(z)}$, is a homotopy in $\mathbb{C} \setminus 0$ of f with the constant function 1.
'$2 \Rightarrow 3$' Let g be any function as in 2). By the Tietze Extension Theorem ([213, Theorem 20.4]), g has a continuous extension G on $\mathbb{C}$. Obviously, the function e^G is a non-vanishing continuous extension of f to $\mathbb{C}$.
'$3 \Rightarrow 2$' Since $\mathcal{K}$ is compact, there exists $r > 0$ such that $\mathcal{K} \subset \{z \in \mathbb{C} : |z| \leq r\}$. By Example 4.2.4a and Theorem 4.2.5, there exists a continuous function g such that $F(z) = e^{g(z)}$ for all $|z| \leq r$. □

Remark 4.2.9 If $\mathcal{K}$ is a compact connected subset of $\mathbb{C}$ with connected complement, then every non-vanishing continuous function on $\mathcal{K}$ satisfies 1) of Theorem 4.2.8 (Robbert Fokkink pointed out to me that this is a special case of the Alexander Duality Theorem ([245, Chapter 11]); there seems to be no direct simple proof of this special case). Hence, every non-vanishing continuous function on $\mathcal{K}$ has a continuous logarithm (cf. Remark 4.2.6b).

For topological proofs of the theorems on continuous functions in this section, see [45, Chapter IV] (uses homotopy) or [98, Chapter 1] (uses covering spaces).

4.2.3 Algebras of holomorphic functions

Let $\mathcal{A}_r$ ($r > 0$) be the Banach algebra of all continuous functions on $\{z \in \mathbb{C} : |z| \leq r\}$ that are holomorphic on $z \in \mathbb{C} : |z| < r$. The norm is the supremum norm. If $r = 1$, then we write $\mathcal{A}$ for $\mathcal{A}_1$. The algebra $\mathcal{A}$ is known as the disc algebra.

Theorem 4.2.10 *If $f \in \mathcal{A}_r$ does not vanish on $\{z \in \mathbb{C} : |z| \leq r\}$, then there is a $g \in \mathcal{A}_r$ such that $f(z) = e^{g(z)}$ for all $|z| \leq r$.*

Proof: We can use Theorem 4.2.5 or proceed as follows: the complex homomorphisms of $\mathcal{A}_r$ are point evaluations on $\{z \in \mathbb{C} : |z| \leq r\}$ ([213, proof of Theorem 18.18]). Hence $f \in \text{inv } \mathcal{A}_r$ and F, defined by $F(t)(z) := f(tz)$, is a path in inv $\mathcal{A}_r$ from $f(0)\,u$ to f. We conclude from Theorem 4.1.6 that $f \in \exp \mathcal{A}_r$. □

Using the same trick as in the proof of Theorem 4.2.7, we now extend Theorem 4.2.10 to entire functions, i.e. functions holomorphic on $\mathbb{C}$.

Theorem 4.2.11 *Let f be a non-vanishing entire function such that $f(0) = 1$. Then there exists a unique entire function g such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$ and $g(0) = 0$.*

Proof: Applying Theorem 4.2.10 to the Banach algebras $\mathcal{A}_n$ ($n \in \mathbb{N}$) and the restrictions f_n of f to $z \in \mathbb{C} : |z| \leq n$, we obtain holomorphic functions g_n on $z \in \mathbb{C} : |z| < n$ such that $g_n(0) = 0$ and $e^{g_n(z)} = f_n(z)$ for all $|z| < n$. It follows from Lemma 4.2.1a that the function g, defined by $g(z) := g_n(z)$ for $|z| \leq n$, is well-defined. Clearly g is entire, $g(0) = 0$, and $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$. Uniqueness follows from Lemma 4.2.1a. □

4.3 Algebras on the unit circle

In this section we will derive analogues of Theorem 4.2.5 for $\mathcal{C}(\mathbb{T})$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, and the Wiener algebra. These results will be used to prove Theorem 4.3.9, which is essential for Section 4.4.

Let us have a closer look at inv $\mathcal{C}(\mathbb{T})$ before stating and proving the correct analogue of Theorem 4.2.5. Note that $\mathbb{T}$ is not contractible (see [42]) and that Theorem 4.2.5 is not true for $\mathcal{K} = \mathbb{T}$. E.g, $e^{i\theta} \in \text{inv } \mathcal{C}(\mathbb{T})$, but $e^{i\theta} \notin \exp \mathcal{C}(\mathbb{T})$ (see [42]).
Let $f \in \text{inv } \mathcal{C}(\mathbb{T})$ be arbitrary. Then f can be identified with a non-vanishing continuous function on $[-\pi, \pi]$. Hence, by Theorem 4.2.5 there exists a $\varphi \in \mathcal{C}([-\pi, \pi])$ such that

$$f(e^{i\theta}) = e^{\varphi(\theta)} \text{ for all } \theta \in [-\pi, \pi].$$

Moreover, if $\varphi_1, \varphi_2 \in \mathcal{C}([-\pi, \pi])$ both satisfy the above equation, then an application of Lemma 4.2.1a to $\mathcal{K} = [-\pi, \pi]$, $a = -\pi$, $g(\theta) := \varphi_1(\theta) - \varphi_1(-\pi) + \varphi_2(-\pi)$ and $h(\theta) := \varphi_2(\theta)$ yields $\varphi_1(\pi) - \varphi_1(-\pi) = \varphi_2(\pi) - \varphi_2(-\pi)$. Thus the following notion is well-defined:

Definition 4.3.1 Let f be a non-vanishing complex-valued continuous function on $\{z \ : \ |z| = R\}$. Then **ind (f)**, the **index of** f, is defined to be $(2\pi i)^{-1}\ \varphi(\pi) - \varphi(-\pi)$ where φ is any continuous function on $[-\pi, \pi]$ satisfying $f(Re^{i\theta}) = e^{\varphi(\theta)}$ for all $\theta \in [-\pi, \pi]$.

Lemma 4.3.2 *If f and g are non-vanishing complex-valued continuous functions on $\{z \ : \ |z| = R\}$, then ind $(f) \in \mathbb{Z}$ and ind $(fg) =$ ind $(f) +$ ind (g).*

Proof: Let φ and γ be such that $f(Re^{i\theta}) = e^{\varphi(\theta)}$ and $g(Re^{i\theta}) = e^{\gamma(\theta)}$ for all $\theta \in [-\pi, \pi]$. Since $e^{\varphi(-\pi)} = e^{\varphi(\pi)}$, it follows that $\varphi(\pi) - \varphi(-\pi)$ is a multiple of $2\pi i$. Hence, ind $f \in \mathbb{Z}$.
The second statement follows from ind $(fg) = (\varphi + \gamma)(\pi) - (\varphi + \gamma)(-\pi) = \varphi(\pi) - \varphi(-\pi) + (\gamma(\pi) - \gamma(-\pi)) =$ ind $(f) +$ ind (g). □

We can now state the analogues of Theorem 4.2.5 alluded to in the introduction of this section.

Theorem 4.3.3 *Let $f \in \mathcal{C}(\mathbb{T})$ be arbitrary. Then the following are equivalent:*

a) $f \in exp\ \mathcal{C}(\mathbb{T})$.

b) $f \in inv\ \mathcal{C}(\mathbb{T})$ and for all $n \in \mathbb{N}$ there exists $g \in \mathcal{C}(\mathbb{T})$ such that $g^n = f$.

c) $f \in inv\ \mathcal{C}(\mathbb{T})$ and ind $(f) = 0$.

Proof: 'a⇒b' It follows from Remark 4.1.3a that $f \in$ inv $\mathcal{C}(\mathbb{T})$. If $f = e^h$, then $g := e^{h/n}$ satisfies $g^n = f$.
'b⇒c' Suppose ind $(f) \neq 0$. Take $n >$ |ind (f)| and let g be such that $g^n = f$. Since $n$ind $(g) =$ ind $(g^n) =$ ind (f), it follows that ind $(g) \notin \mathbb{Z}$, which is absurd.
'c⇒a' Let g be a continuous function such that $f(e^{i\theta}) = e^{g(\theta)}$ for all $\theta \in [-\pi, \pi]$. It follows from ind $(f) = 0$ that $g(\pi) = g(-\pi)$. Hence G, defined by $G(e^{i\theta}) := g(\theta)$, belongs to $\mathcal{C}(\mathbb{T})$ and $f = e^G$. □

It was mentioned in the introduction of this section that $e^{i\theta} \notin$ exp $\mathcal{C}(\mathbb{T})$. This follows directly from Theorem 4.3.3, since ind $(e^{i\theta}) = 1$.

The following theorem describes the components of inv $\mathcal{C}(\mathbb{T})$:

Theorem 4.3.4 *Define $C_k := \{f \in inv\ \mathcal{C}(\mathbb{T}) | ind\ (f) = k\}$ for all $k \in \mathbb{Z}$. The components of inv $\mathcal{C}(\mathbb{T})$ are precisely the sets C_k.*

Proof: It is clear that the sets C_k form a partition of inv $\mathcal{C}(\mathbb{T})$. Note that exp $\mathcal{C}(\mathbb{T})$ is a component by Theorem 4.1.4 and that $C_0 =$ exp $\mathcal{C}(\mathbb{T})$ by Theorem 4.3.3. Thus the theorem holds for $k = 0$. Define for each $k \in \mathbb{Z}$ the map F_k by $(F_k f)(e^{it}) := e^{itk} f(e^{it})$. It follows from ind $(e^{itk}) = k$ and ind $(fg) =$ ind $(f) +$ ind (g) that F_k maps C_0 onto C_k. Moreover, it is easy to see that F_k is an isometry. Hence, the sets C_k are homeomorphic to C_0, and we are done. □

Theorem 4.3.4 enables us to prove that $\pi_1(\mathbb{T})$, the fundamental group of $\mathbb{T}$, is isomorphic to $\mathbb{Z}$ (a theorem appearing in every introductory textbook on algebraic topology, see e.g. [155, Chapter 12, Section 5]). The fundamental group of $\mathbb{T}$ consists of all homotopy equivalence classes of continuous maps $f : [0,1] \to \mathbb{T}$ such that $f(0) = f(1) = 1$. The group operation is defined as follows: if $[f]$ and $[g]$ are homotopy equivalence classes of continuous maps with the above mentioned properties, then $[f \circ g]$ is the map defined by $(f \circ g)(t) := f(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $(f \circ g)(t) := g(2t-1)$ for $\frac{1}{2} \leq t \leq 1$. This group operation is well-defined, since it does not depend on the choice of the representatives f and g of $[f]$, $[g]$ respectively (see e.g. [155, Chapter 5]). Define $H : \pi_1(\mathbb{T}) \to \mathbb{Z}$ as follows. If $[f] \in \pi_1(\mathbb{T})$ has representative f, define $H([f]) := \text{ind}\,(F)$, where F is defined by $F(e^{i\theta}) := f(\frac{1}{2} + \theta/(2\pi))$. This definition does not depend on the choice of the representative f: if f and g are homotopic, then the corresponding functions F and G are also homotopic, hence have the same index by Theorems 4.1.6 and 4.3.4. It follows from Lemma 4.3.2 that H is a homomorphism and it follows from Theorems 4.1.6 and 4.3.4 that H is injective. Hence, H is an isomorphism, since H is clearly surjective. We conclude that the fundamental group of $\mathbb{T}$ is isomorphic to $\mathbb{Z}$.

The **Wiener algebra** $\mathcal{W}$ consists of all continuous functions on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ that can be expanded as absolutely convergent Fourier series (see [212, Example 11.13b]).
Addition and multiplication are defined pointwise; the norm is defined by $\|\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\| := \sum_{n=-\infty}^{\infty} |a_n|$. Note that the algebra $\ell_1(\mathbb{Z})$ of absolutely summable two-sided sequences is isometric to $\mathcal{W}$. The elements of $\mathcal{W}$ are precisely the Gelfand transforms of elements of $\ell_1(\mathbb{Z})$.
The complex homomorphisms of $\mathcal{W}$ are of the form $\Lambda_z(a) = \sum_{n=-\infty}^{\infty} a_n z^n$ for some $z \in \mathbb{T}$ (see e.g. [83, Theorem 2.57] or [142, Section 4.6]). Thus Theorem 4.1.2 yields that the invertible elements of $\mathcal{W}$ are precisely those elements of $\mathcal{W}$ that do not vanish on $\mathbb{T}$ (see e.g. [213, Lemma 11.6]). This is a famous theorem due to Wiener; the Banach algebra proof indicated above (due to Gelfand) was one the first successes of Banach algebra theory.
Since the canonical bijection $z \to \Lambda_z$ is a continuous map from the compact set $\mathbb{T}$ onto the compact Hausdorff set $\mathcal{M}(\mathcal{W})$ (in its Gelfand topology), it follows that $\mathbb{T}$ and $\mathcal{M}(\mathcal{W})$ are homeomorphic (cf. [212, Section 3.8]). The analogue of Theorem 4.2.5 for the Wiener algebra $\mathcal{W}$ can be found in [46]. We state this result as Theorem 4.3.5 and remark that the proof in [46] uses a special case of the deep Wiener-Lévy Theorem ([212, Theorem 10.27]).

Theorem 4.3.5 *Let $f \in \mathcal{W}$ be arbitrary. Then the following are equivalent:*

a) $f \in exp\ \mathcal{W}$.

b) $f \in inv\ \mathcal{W}$ and for all $n \in \mathbb{N}$ there exists $g \in C(\mathbb{T})$ such that $g^n = f$.

c) $f \in inv\ \mathcal{W}$ *and* $ind\ (f) = 0$.

Proof: 'a⇒b' It follows from Remark4.1.3a that $f \in \text{inv } \mathcal{W}$. If $f = e^h$, then $g := e^{h/n}$ satisfies $g^n = f$.
'b⇒c' Suppose ind $(f) \neq 0$. Take $n > |\text{ind } (f)|$ and let g be such that $g^n = f$. Since $n\,\text{ind } (g) = \text{ind } (g^n) = \text{ind } (f)$ by Lemma 4.3.2, it follows that ind $(g) \notin \mathbb{Z}$, which is absurd.
'c⇒a' First note that $f \in \text{inv } \mathcal{C}(\mathbb{T})$. By Theorem 4.3.3, there exists $h \in \mathcal{C}(\mathbb{T})$ such that $f(z) = e^{h(z)}$ for all $z \in \mathbb{T}$. Define $H : \mathcal{M}(\mathcal{W}) \to \mathbb{C}$ by $H(\Lambda_z) = h(z)$. Then $H \in \mathcal{C}(\mathcal{M}(\mathcal{W}))$, since $\mathcal{M}(\mathcal{W})$ and $\mathbb{T}$ are homeomorphic. Moreover, $\widehat{f}(\Lambda_z) = \Lambda_z(f) = f(z) = e^{h(z)} = e^{H(z)} = \left(e^H\right)(\Lambda_z)$ for all $z \in \mathbb{T}$. Thus $f \in \exp \mathcal{M}(\mathcal{W})$ and Theorem 4.1.10 yields that $f \in \exp \mathcal{W}$. □

Remark 4.3.6 The proof of c⇒a of Theorem 4.3.5 implicitly contains the trivial result that if X and Y are homeomorphic compact Hausdorff spaces, then $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are isometric Banach spaces. The converse is also true and known as the Banach-Stone Theorem (see e.g. [22; Theorem 3, p. 130]).

Theorem 4.3.7 *Define* $\mathcal{W}_k := \{f \in inv\ \mathcal{W} | ind\ (f) = k\}$ *for all* $k \in \mathbb{Z}$. *The components of inv* $\mathcal{W}$ *are precisely the sets* C_k.

Proof: The proof is identical to the proof of Theorem 4.3.3. □

We conclude this section with a result (Theorem 4.3.9) which will be essential for one of the main theorems of this chapter (Theorem 4.4.1). Theorem 4.3.9 is an extension of Theorem 4.2.2.

Lemma 4.3.8 *Let* $f, g \in inv\ \mathcal{C}(\mathbb{T})$ *be such that* $|f(z) - g(z)| < |f(z)|$ *for all* $z \in \mathbb{T}$. *Then* $ind\ (f) = ind\ (g)$.

Proof: The assumptions imply that $|1 - g(z)/f(z)| < 1$ for all $z \in \mathbb{T}$. Hence, $\|1 - g(z)/f(z)\| < 1$ since $\mathbb{T}$ is compact. By Remark 4.1.3b, $g/f \in \exp \mathcal{C}(\mathbb{T})$ and Theorem 4.3.3 yields ind $(g/f) = 0$. It follows from Lemma 4.3.2 that ind $(g) = \text{ind } (f) + \text{ind } (g/f) = \text{ind } (f)$. □

For notation of the following theorem, see Subsection 4.2.1.

Theorem 4.3.9 *Let* $x \in \ell_1(\alpha)$ *be arbitrary. Define* $\xi : \mathbb{T} \to \mathbb{C}$ *by* $\xi(z) := \sum_{n=0}^{\infty} x_n \left(ze^{\rho}\right)^n$ *for all* $z \in \mathbb{T}$. *Then the following are equivalent:*

a) $\xi(z) \neq 0$ *for all* $z \in \mathbb{T}$ *and* $ind\ \xi = 0$.

b) $x \in inv\ \ell_1(\alpha)$.

c) $x \in exp\ \ell_1(\alpha)$.

Proof: 'a $\Rightarrow$ b' Define ξ_r $(0 < r < 1)$ on $\overline{D}$ by $\xi_r(z) := \xi(rz)$. Since ξ is a non-vanishing continuous function on $\mathbb{T}$, there exists $\delta > 0$ such that $\delta < |\xi(z)|$ for all $z \in \mathbb{T}$. If $r < 1$ is close enough to 1, then $|\xi(z) - \xi_r(z)| < \delta < |\xi(z)|$ for all $z \in \mathbb{T}$. By Lemma 4.3.8, ind $(\xi_r) =$ ind $(\xi) = 0$ (restrict ξ_r to $\mathbb{T}$). Now the Argument Principle ([45, Corollary 5.86, p. 179]) yields that $\sum_{n=0}^{\infty} x_n\, z^n \neq 0$ for $|z| \leq re^{\rho}$. Since r can be arbitrarily close to 1, $\sum_{n=0}^{\infty} x_n z^n \neq 0$ for $|z| < e^{\rho}$. Hence, $x \in \text{inv}\ \ell_1(\alpha)$ by Theorem 4.2.2.
'b $\Rightarrow$ c' This follows from Theorem 4.2.2.
'c $\Rightarrow$ a' Since $x \in \exp \ell_1(\alpha)$, we have $\sum_{n=0}^{\infty} x_n\, z^n \neq 0$ for all $|z| \leq e^{\rho}$. A similar use of the Argument Principle as above yields that ind $\xi = 0$. □

4.4 Applications to polynomials of convolution type

In this section we will study the analytical behaviour of the following generating function for polynomials of convolution type (Theorem 2.1.12d):

$$\sum_{n=0}^{\infty} q_n(t)\, z^n = e^{t\, g(z)}, \tag{4.1}$$

where $g(z)$ denotes the formal power series $\sum_{k=0}^{\infty} g_k\, z^k$.
In particular, we will study absolute convergence and radius of convergence of the left-hand side of (4.1).

Notation If $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type, we will write $\psi(t,z) := \sum_{n=0}^{\infty} q_n(t)\, z^n$ whenever this series converges absolutely.
We write g for the coefficient sequence $(g_n)_{n\in\mathbb{N}}$ of $(q_n)_{n\in\mathbb{N}}$ and $q(t)$ for $(q_n)_{n\in\mathbb{N}}t$.

For the notation in the following theorem we refer to Subsection 4.2.1.

Theorem 4.4.1 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $g = (g_n)_{n\in\mathbb{N}}$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying $\alpha_0 = 1$ and $\alpha_{n+m} \leq \alpha_n\, \alpha_m$ for all $n, m \in \mathbb{N}$. Then the following are equivalent:*

a) $g \in \ell_{1,\alpha}$.

b) There exists $M > 0$ such that $\|q(t)\|_{1,\alpha} \leq e^{|t|\, M}$ for all $t \in \mathbb{C}$.

c) $\lim_{t\downarrow 0} \|q(t)\|_{1,\alpha} = 1$.

d) $\limsup_{t\downarrow 0} \|q(t)\|_{1,\alpha} < 2$.

e) There are $\delta > 0$ and $t_0 \in (0,\delta)$ such that $q(t) \in \ell_{1,\alpha}$ for all $t \in (0,\delta)$ and $\psi(t_0, z) \neq 0$ if $|z| = e^{\rho}$.

f) There is $t_0 \in \mathbb{C} \setminus \{0\}$ such that $q(t_0) \in \ell_{1,\alpha}$ and $\psi(t_0, z) \neq 0$ for all $|z| \leq e^\rho$.

g) There is $t_0 \in \mathbb{C} \setminus \{0\}$ such that $q(t_0) \in \ell_{1,\alpha}$ and $q(-t_0) \in \ell_{1,\alpha}$.

Moreover, if one of these conditions holds, then (4.1) holds and both series in (4.1) converge absolutely for all $t \in \mathbb{C}$ and all $|z| \leq e^\rho$.

Proof: '$a \Rightarrow b$' We first show that $\left(e^{t\,g}\right)_n = q_n(t)$ for all $n \in \mathbb{N}$. Since the coordinate functionals of $\ell_{1,\alpha}$ are continuous, we have $\left(e^{t\,g}\right)_n = \left(\sum_{k=0}^\infty \frac{t^k g^k}{k!}\right)_n = \sum_{k=0}^\infty \left(\frac{t^k g^k}{k!}\right)_n = \sum_{k=0}^\infty g_n^{k*} \frac{t^k}{k!} = \sum_{k=0}^n g_n^{k*} \frac{t^k}{k!} = q_n(t)$ for all $n \in \mathbb{N}$. Now b) follows from Remark 4.1.3d.
'$b \Rightarrow c$' This follows from $q_0 = 1$ and $\alpha_0 = 1$.
'$c \Rightarrow d$' This is trivial.
'$d \Rightarrow e$' It follows that there exists t_0 such that $\|q(t) - u\|_{1,\alpha} < 1$ for all $t \in [0, t_0]$. In particular, $q(t_0) \in \exp\ \ell_{1,\alpha}$ by Remark 4.1.3b. The statement now follows from Theorem 4.2.2.
'$e \Rightarrow f$' Since $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type, we have $\psi(u+v, z) = \psi(u, z)\,\psi(v, z)$ if $u, v, u+v \in [0, \delta)$ and $|z| = e^\rho$. Hence, $\psi(t, z) \neq 0$ for all $t \in (0, \delta)$ and all $|z| = e^\rho$ by [118, Theorem 4.17.1, p. 144]. Recall from Lemma 4.3.2 that the index of a non-vanishing continuous function on $\mathbb{T}$ (Definition 4.3.1) is always an integer. If ind $\psi(t, .) \neq 0$ for some $t \in (0, \delta)$, then ind $\psi(t/n, .) \notin \mathbb{Z}$ for n large enough by Lemma 4.3.2, which is impossible. Hence Theorem 4.2.2 and Theorem 4.3.9 imply that for all $t \in (0, \delta)$, $\psi(t, z) \neq 0$ if $|z| \leq e^\rho$.
'$f \Rightarrow g$' Since $q_0(t_0) \neq 0$, there exists a unique sequence $(a_k)_{k\in\mathbb{N}}$ such that $\sum_{k=0}^n q_k(t_0)\,a_{n-k} = \delta_{0n}$ for all $n \in \mathbb{N}$. By Theorem 4.2.2, $(a_k)_{k\in\mathbb{N}} \in \ell_{1,\alpha}$. Using the defining property of convolution type we see that $a_k = q_k(-t_0)$ for all $k \in \mathbb{N}$. Hence, $q(-t_0) \in \ell_{1,\alpha}$.
'$g \Rightarrow a$' It follows from the defining property of polynomials of convolution type that $q(-t_0) = q(t_0)^{-1}$. Hence, $q(t_0) \in inv\,(\ell_{1,\alpha})$ and by Theorem 4.2.2, there exists a sequence $b = (b_n)_{n\in\mathbb{N}} \in \ell_{1,\alpha}$ with $b_0 = 0$ such that $q(t_0) = e^b$. It follows from Lemma 4.2.1b that $b_n = t_0\,g_n$ for all $n \in \mathbb{N}$, which implies $g \in \ell_{1,\alpha}$.

The last statement follows from Theorem 2.1.8, since b) allows us to interchange summations. □

Remarks 4.4.2 a) It follows from the proof of '$a \Rightarrow b$' that $M = \|g\|_{1,\alpha}$ suffices in b).
b) If f) or g) hold, then (4.1) implies that they hold for all $t \in \mathbb{C}$.
c) If $g_n = 0$ for n even and $g \notin \ell_{1,\alpha}$, then $q(t) \notin \ell_{1,\alpha}$ for any $t \neq 0$ since in this case $q_n(-t) = (-1)^n\,q_n(t)$.
d) For an alternative proof of '$e \Rightarrow f$' see Remark 4.5.4a.
e) We may weaken condition d) of Theorem 4.4.1 to d': 'there is $t \in \mathbb{C} \setminus \{0\}$ such that $\|q(t)\|_{1,\alpha} < 2$', since obviously $c \Rightarrow d' \Rightarrow g$.

Corollary 4.4.3 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. If there exists $M, \delta > 0$ such that $|q_n(t)| \le M$ for all $t \in (0,\delta)$, then $\sum_{n=0}^{\infty} g_n\, z^n$ converges absolutely for $|z| < 1$.*

Proof: Fix an arbitrary r with $0 < r < 1$. Define $\alpha_n := r^n$. Since $\lim_{t\downarrow 0} q_n(t) = 0$ for $n \ge 1$ by Theorem 2.1.12e, it follows from dominated convergence that $\lim_{t\downarrow 0} \|q(t)\|_{1,\alpha} = 1$. Thus Theorem 4.4.1 $c \Rightarrow a$ implies that $\sum_{n=0}^{\infty} g_n\, z^n$ converges absolutely for $|z| \le r$. Since r was arbitrary, it follows that $\sum_{n=0}^{\infty} g_n\, z^n$ converges absolutely for $|z| < 1$. □

The converse of Corollary 4.4.3 is not true. E.g., take $g_n = n$ for all $n \in \mathbb{N}$. Then $|q_n(t)| \ge g_1\,|t| = n\,|t|$. It is an open problem to find necessary and sufficient conditions on $(g_n)_{n\in\mathbb{N}}$ that insure that $\sum_{n=0}^{\infty} |q_n(t)| < \infty$ for all $t \in \mathbb{C}$ or for all $t \in (0,\infty)$.

We now prove an analogue of Theorem 4.4.1 for strict sense Sheffer sequences (see Section 2.4). We write $s(t)$ for $(s_n(t))_{n\in\mathbb{N}}$.

The following lemma is needed for the proof of Theorem 4.4.5.

Lemma 4.4.4 *Let $x(t) = (x_n(t))_{n\in\mathbb{N}}$ $(t > 0)$ and $x = (x_n)_{n\in\mathbb{N}}$ be sequences in $\ell_{1,\alpha}$. If $\lim_{t\downarrow 0} \|x(t)\|_{1,\alpha} = \|x\|_{1,\alpha}$ and $\lim_{t\downarrow 0} x_n(t) = x_n$ for all $n \in \mathbb{N}$, then $\lim_{t\downarrow 0} \|x(t) - x\|_{1,\alpha} = 0$.*

Proof: Let $\epsilon > 0$ be arbitrary. Choose $k \in \mathbb{N}$ such that $\|x\|_{1,\alpha} - \|P_k\, x\|_{1,\alpha} < \epsilon/5$, where $P_k\, x := (x_0, x_1, \dots, x_k, 0, 0, \dots)$. Choose $s > 0$ such that $\|x(t)\|_{1,\alpha} < \|x\|_{1,\alpha}| + \epsilon/5$ and $\|P_k\, x(t)\|_{1,\alpha} > \|P_k\, x\|_{1,\alpha} - \epsilon/5$ for $0 < t < s$. If $0 < t < s$, then $\|x - x(t)\|_{1,\alpha} = \|P_k\, x - x(t)\|_{1,\alpha} + \|(I - P_k)\, x - x(t)\|_{1,\alpha} \le \epsilon/5 + \|(I - P_k)\, x\|_{1,\alpha} + \|(I - P_k)\, x(t)\|_{1,\alpha} \le \epsilon/5 + \epsilon/5 + \|x(t)\|_{1,\alpha} - \|P_k\, x(t)\|_{1,\alpha} \le 2\,\epsilon/5 + \|x\|_{1,\alpha} + \epsilon/5 - \|P_k\, x\|_{1,\alpha} + \epsilon/5 \le \epsilon$. □

Theorem 4.4.5 *Let $(s_n)_{n\in\mathbb{N}}$ be a strict sense Sheffer set for a delta operator Q with basic set $(q_n)_{n\in\mathbb{N}}$. Let $g = (g_n)_{n\in\mathbb{N}}$ be the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying $\alpha_0 = 1$ and $\alpha_{n+m} \le \alpha_n\, \alpha_m$ for all $n, m \in \mathbb{N}$. Then the following are equivalent:*

a) $g \in \ell_{1,\alpha}$ and $s(0) \in inv\, \ell_{1,\alpha}$.

b) $s(0) \in inv\, \ell_{1,\alpha}$ and there exists $M > 0$ such that $e^{-t\,M}\, \|s(0)\|_{1,\alpha} \le \|s(t)\|_{1,\alpha} \le e^{t\,M}\, \|s(0)\|_{1,\alpha}$ for $t > 0$.

c) $s(0) \in inv\, \ell_{1,\alpha}$ and $\lim_{t\downarrow 0} \|s(t)\|_{1,\alpha} = \|s(0)\|_{1,\alpha}$.

d) $g \in \ell_{1,\alpha}$ and $s(t) \in inv\, \ell_{1,\alpha}$ for some $t \in \mathbb{C} \setminus \{0\}$.

Moreover, if one of these conditions holds, then

$$\sum_{n=0}^{\infty} s_n(t)\, z^n = e^{t\, g(z)} \sum_{n=0}^{\infty} s_n(0)\, z^n$$

for all $t \in \mathbb{C}$ and all $|z| \le e^{\rho}$.

Proof: We will use that by Theorem 2.4.4, $s(t) = s(0) * q(t)$, where $*$ denotes convolution.
'$a \Rightarrow b$' It follows from Theorem 4.4.1 that $\|q(t)\|_{1,\alpha} \leq e^{t\,M}$ for all $t > 0$. Hence, $\|s(t)\|_{1,\alpha} \leq e^{t\,M} \|s(0)\|_{1,\alpha}$. For the other inequality, note that $s(0) = s(0) * q(t) * q(-t) = s(t) * q(-t)$. Hence, for $t > 0$ we have $\|s(0)\|_{1,\alpha} \leq \|s(t)\|_{1,\alpha} \|q(-t)\|_{1,\alpha} \leq \|s(t)\|_{1,\alpha}\, e^{t\,M}$.
'$b \Rightarrow c$' This is trivial.
'$c \Rightarrow d$' By Remark 4.1.3b, inv $\ell_{1,\alpha}$ is open. Since $\lim_{t\downarrow 0} s_n(t) = s_n(0)$ for all $n \in \mathbb{N}$, Lemma 4.4.4 yields $\lim_{t\downarrow 0} \|s(t) - s(0)\|_{1,\alpha} = 0$. In particular, $s(t_0) \in \text{inv } \ell_{1,\alpha}$ for some $t_0 > 0$. Moreover, $q(t_0) = s(t_0) * s(0)^{-1} \in \text{inv } \ell_{1,\alpha}$. Now Theorem 4.4.1 yields $g \in \ell_{1,\alpha}$.
'$d \Rightarrow a$' It follows from Theorem 4.4.1 that $q(-t) \in \text{inv } \ell_{1,\alpha}$. Hence, $s(0) \in \text{inv } \ell_{1,\alpha}$, since $s(0) = s(t) * q(-t)$.

The last statement follows from $s(t) = s(0) * q(t)$ (Theorem 2.4.4) and Theorem 4.4.1). □

We now return to Theorem 4.4.1. We will try to obtain convergence results on (4.1) with weaker conditions on $(g_n)_{n\in\mathbb{N}}$.

The Banach algebra $\mathcal{TA}$ consists of all one-sided sequences $(a_n)_{n\in\mathbb{N}}$ of complex numbers such that $f(z) := \sum_{n=0}^{\infty} a_n\, z^n$ is analytic on $\mathcal{D}$ and can be extended to a continuous function on $\overline{\mathcal{D}}$. Addition is defined componentwise, multiplication is defined to be convolution. The norm on $\mathcal{TA}$ is defined by $\|(a_n)_{n\in\mathbb{N}}\|_{\mathcal{TA}} := \sup_{|z|<1} |\sum_{n=0}^{\infty} a_n\, z^n|$. Note that if $(a_n)_{n\in\mathbb{N}} \in \mathcal{TA}$ and $f(z) := \sum_{n=0}^{\infty} a_n\, z^n$, then the Maximum Modulus Theorem yields $\|(a_n)_{n\in\mathbb{N}}\|_{\mathcal{TA}} = \sup_{|z|<1} |f(z)| = \sup_{|z|\leq 1} |f(z)| = \sup_{|z|=1} |f(z)|$ (here we denoted the extension of f to $\overline{\mathcal{D}}$ also by f) .The space $\mathcal{TA}$ is isometric to the disc algebra $\mathcal{A}$ studied in Subsection 4.2.3).

Theorem 4.4.6 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $g = (g_n)_{n\in\mathbb{N}}$. Then the following are equivalent:*

a) $g \in \mathcal{TA}$.

b) There are $t \in \mathbb{C}\setminus\{0\}$ and $\delta > 0$ such that $q(t) \in \mathcal{TA}$ and $|\psi(t,z)| > \delta$ for all $z \in \mathcal{D}$.

c) There exists $t \in \mathbb{C}\setminus\{0\}$ such that $q(t) \in \mathcal{TA}$ and $q(-t) \in \mathcal{TA}$.

If a), b) or c) holds, then :

d) $q(t) \in \mathcal{TA}$ for all $t \in \mathbb{C}$ and $|\psi(t,z)| > \delta(t) > 0$ for all $z \in \mathcal{D}$.

e) (4.1) holds for all $t \in \mathbb{C}$ and all $z \in \mathcal{D}$.

f) There exists $M > 0$ such $\|q(t)\|_{\mathcal{TA}} \leq e^{|t|\,M}$ for all $t \in \mathbb{C}$.

g) $\lim_{t\downarrow 0}\ \|q(t)\|_{\mathcal{TA}} = 1$.

Proof '$a \Rightarrow b$' Fix an arbitrary r with $0 < r < 1$. Define α_n ($n \in \mathbb{N}$) by $\alpha_n := r^n$. Then Theorem 4.4.1 $a \Rightarrow e$ together with the last statement of Theorem 4.4.1 imply that $\psi(t,z) = e^{t\,g(z)}$ for all $t \in \mathbb{C}$ and all z with $|z| < r$. Since r was arbitrary, it follows that $\psi(t,z) = e^{t\,g(z)}$ for all $t \in \mathbb{C}$ and all $z \in \mathcal{D}$. Hence, $q(t) \in \mathcal{TA}$ for all $t \in \mathbb{C}$ since $e^{t\,g} \in \mathcal{TA}$. The second statement of b) also follows from $\psi(t,z) = e^{t\,g(z)}$.
'$b \Rightarrow c$' Since $q_0(t) \neq 0$, there exists a unique sequence $(a_k)_{k\in\mathbb{N}}$ such that $\sum_{k=0}^{n} q_k(t)\,a_{n-k} = \delta_{0n}$ for all $n \in \mathbb{N}$. It follows Theorem 4.2.10 that $q(t) \in \exp\ \mathcal{TA}$. Hence, in particular $q(t) \in \text{inv}\ \mathcal{TA}$ and $(a_n)_{n\in\mathbb{N}} \in \mathcal{TA}$. Using the defining property of convolution type, we see that $a_k = q_k(-t)$ for all $k \in \mathbb{N}$. Hence, $q(-t) \in \mathcal{TA}$.
'$c \Rightarrow a$' It follows from Theorem 4.2.10 that $q(t) \in \exp\ \mathcal{TA}$. Thus there exists $(b_n)_{n\in\mathbb{N}} \in \mathcal{TA}$ with $b_0 = 0$ such that $\psi(t,z) = \exp\ \left(\sum_{n=0}^{\infty} b_n\,z^n\right)$ for all $z \in \mathcal{D}$. It follows from Lemma 4.2.1b that $b_n = t\,g_n$ for all $n \in \mathbb{N}$. Hence, $g \in \mathcal{TA}$.

Statements d) and e) follow from the proof of a $\Rightarrow$ b. In order to prove f), note that e) implies $\|q(t)\|_{\mathcal{TA}} \le e^{|t|\,\|g\|_{\mathcal{TA}}}$. Finally, $\liminf_{t\downarrow 0}\ \|q(t)\|_{\mathcal{TA}} \ge \liminf_{t\downarrow 0}\ |\psi(t,0)| = q_0(t) = 1$ and f) implies that $\limsup_{t\downarrow 0}\ \|q(t)\|_{\mathcal{TA}} \le 1$. Hence, $\lim_{t\downarrow 0}\ \|q(t)\|_{\mathcal{TA}} = 1$. □

Remark 4.4.7 It is an open problem whether property f) of Theorem 4.4.6 implies any of the properties a), b) or c).

The next theorem gives a sufficient condition for boundedness of the coefficient sequence $(g_n)_{n\in\mathbb{N}}$ in terms of $(q_n)_{n\in\mathbb{N}}$.

Theorem 4.4.8 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. If $\lim_{t\downarrow 0}\ q_n(t) = 0$ uniformly in $n = 1,2,\ldots$ and $\lim_{t\downarrow 0}\ \frac{q_n(t)}{t} = g_n$ uniformly in $n = 1,2,\ldots$, then $(g_n)_{n\in\mathbb{N}}$ is a bounded sequence.*

Proof: Suppose $(g_n)_{n\in\mathbb{N}}$ is unbounded. Choose δ, $0 < \delta < 1$ such that $|q_n(t)| \le 1$ and $\left|\frac{q_n(t)}{t} - g_n\right| \le 1$ for $0 < t < \delta$. Choose N such that $|g_N| > 4/\delta$. Then $|q_N(\frac{1}{2}\delta)| = \left|\frac{q_N(\frac{1}{2}\delta)}{\frac{1}{2}\delta}\,\frac{1}{2}\delta\right| = \frac{1}{2}\delta\left|g_N + \left(\frac{q_N(\frac{1}{2}\delta)}{\frac{1}{2}\delta} - g_N\right)\right| \ge \frac{1}{2}\delta|g_N| - \frac{1}{2}\delta\left|\frac{q_N(\frac{1}{2}\delta)}{\frac{1}{2}\delta} - g_N\right| \ge 2 - \frac{1}{2}\delta > 1\frac{1}{2}$. This contradicts the choice of δ. □

Note that the converse of Theorem 4.4.8 does not hold. E.g. take $g_n = 1$ for all $n \in \mathbb{N}$. Then $g_n^{2*} = n-1$ and $q_n(t) \ge \frac{1}{2}\,(n-1)\,t^2$.

We now study the relation between the radii of convergence of $\sum_{n=0}^{\infty} q_n(t)\,z^n$ and $\sum_{k=0}^{\infty} g_k\,z^k$, where $(g_n)_{n\in\mathbb{N}}$ is the coefficient sequence of $(q_n)_{n\in\mathbb{N}}$.

$$\begin{aligned}
\mathcal{R}_g &:= \text{radius of convergence of} \sum_{k=0}^{\infty} g_k\, z^k. \\
\rho_t &:= \text{radius of convergence of} \sum_{n=0}^{\infty} q_n(t)\, z^n. \\
\mathcal{N}_t &:= \{z \,:\, |z| < \rho_t \text{ and } \psi(t,z) = 0\}. \\
\nu_t &:= \inf\{|z| \,:\, z \in \mathcal{N}_t\} \text{ if } \mathcal{N}_t \neq \emptyset, \nu_t := \rho_t \text{ if } \mathcal{N}_t = \emptyset.
\end{aligned}$$

We start our discussion with some examples.

Examples 4.4.9 a) $q_n(t) = \frac{x^n}{n!}$: $\rho_t = \mathcal{R}_g = \infty$ for all $t \in \mathbb{C}$.
b) $q_n(t) = \binom{t}{n}$: $\mathcal{R}_g = 1$, $\rho_t = \infty$ for $t \in \mathbb{N}$, $\rho_t = 1$ for $t \notin \mathbb{N}$.
c) $q_n(t) = t(t-an)^{n-1}/n!$: $\rho_t = \mathcal{R}_g = (|a|e)^{-1}$ (use Stirling's Formula).

Note that $\rho_0 = \infty$ because $q_n(0) = 0$ for $n \geq 1$.
It follows from Theorem 4.4.1 $a \Rightarrow b$ that $\mathcal{R}_g \leq \rho_t$ for all $t \in \mathbb{C}$. The examples suggest that $\mathcal{R}_g = \rho_t$ for all except countably many t. Theorem 4.4.1 $g \Rightarrow a$ shows that it is not possible that both $\rho_t > \mathcal{R}_g$ and $\rho_{-t} > \mathcal{R}_g$. The following theorem shows that the zeros of the functions $\psi(t,.)$ determine $\mathcal{R}_g$. Moreover, it enables us to prove the important property stated as Theorem 4.4.10e. This property will play an important role in the rest of this section and in Section 4.5.

Theorem 4.4.10 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type. Then:*

a) If $\mathcal{R}_g = 0$, then $\rho_t = 0$ for all $t \in \mathbb{C} \setminus \{0\}$.

b) If $\mathcal{R}_g = \infty$, then $\rho_t = \infty$ for all $t \in \mathbb{C}$.

c) If $0 < \mathcal{R}_g < \infty$, then $\mathcal{R}_g = \nu_t$ for all $t \in \mathbb{C} \setminus \{0\}$. In particular, $\nu_t = \nu_s$ for all $t, s \in \mathbb{C} \setminus \{0\}$ and $\psi(t,z) \neq 0$ for all $t \in \mathbb{C}$ and all $|z| < \mathcal{R}_g$.

d) There are at most countably many $t \in \mathbb{C}$ such that $\rho_t > \mathcal{R}_g$.

e) If $|z| < \rho_t$ for uncountably many $t \in \mathbb{C}$, then $\psi(t,z) \neq 0$ for all $t \in \mathbb{C}$.

Proof: a) Suppose $\rho_t \neq 0$ for some $t \neq 0$. Since $\psi(t,0) = 1$ there is a δ, $0 < \delta < \rho_t$, such that $|\psi(t,z)| > 0$ for $|z| \leq \delta$. Now Theorem 4.4.1 $f \Rightarrow a$ with $\alpha_n = \delta^n$ implies that $(g_n)_{n\in\mathbb{N}} \in \ell_1(\alpha)$, hence $\mathcal{R}_g \geq \delta > 0$.
b) This follows from Theorem 4.4.1 $a \Rightarrow b$.
c) Let $t \in \mathbb{C} \setminus \{0\}$ be arbitrary. We first prove $\mathcal{R}_g \leq \nu_t$. If $|z| < \mathcal{R}_g$, then $(g_n)_{n\in\mathbb{N}} \in \ell_1(\alpha)$ with $\alpha_n = |z|^n$. It follows from Theorem 4.4.1 that $\psi(t,z) = e^{t\,g(z)} \neq 0$ for $|z| < \mathcal{R}_g$. Hence, $\mathcal{R}_g \leq \nu_t$.
The reverse inequality $\mathcal{R}_g \geq \nu_t$ follows from Theorem 4.4.1 f $\Rightarrow$ a.

d) First recall that an analytic function can have at most finitely many zeros on a compact set ([213, Corollary to Theorem 10.18, p. 226]). Suppose that $\rho_s > \mathcal{R}_g$ for some $s \in \mathbb{C}\backslash\{0\}$. We first prove that $\psi(s,.)$ has at least one zero on $|z| = \mathcal{R}_g$. If $\psi(s,z) \neq 0$ for all $|z| = \mathcal{R}_g$, then $\psi(s,z) \neq 0$ for all $|z| < \mathcal{R}_g + \eta$ for some $\eta > 0$ since $\rho_s > \mathcal{R}_g$. Thus $\nu_s > \mathcal{R}_g$, which is impossible by c. Therefore we may write $\psi(s,z) = f_s(z) \prod_{j=1}^k (1-\alpha_j z)^{r_j}$ with $|1/\alpha_j| = \mathcal{R}_g$ and $r_j \in \mathbb{N}$, $j = 1, \ldots, k$. There exists $\delta > 0$ such that $\psi(s,.)$ has finitely many zeros on $\{z : \mathcal{R}_g \leq |z| \leq \mathcal{R}_g + \delta < \rho_s\}$. Hence f_s is a non-vanishing analytic function on $|z| < \mathcal{R}_g + \delta_1$ for some $\delta_1 > 0$. Since $f_s(0) = 1$, Theorem 4.2.10 yields a unique analytic function h such that $h(0) = 0$ and $f_s(z) = e^{h(z)}$ for $|z| < \mathcal{R}_g + \delta_1$. Hence, $\psi(s,z) = \exp\left\{h(z) + \sum_{j=1}^k r_j \log(1-\alpha_j z)\right\}$ for $|z| < \mathcal{R}_g$, where log denotes the principal branch of the logarithm on $\mathbb{C}\backslash(-\infty, 0]$. By Lemma 4.2.1a and Theorem 4.4.1, $h(z) + \sum_{j=1}^k r_j \log(1-\alpha_j z) = s\,g(z)$ for $|z| < \mathcal{R}_g$. It follows that

$$\begin{aligned}
\psi(t,z) &= \exp\left\{t/s\, h(z) + t/s \sum_{j=1}^k r_j \log(1-\alpha_j z)\right\} \\
&= \exp\left(t/s\, h(z)\right) \prod_{j=1}^k (1-\alpha_j z)_{r_j\, t/s}
\end{aligned}$$

for all $t \in \mathbb{C}$ and all $|z| < \mathcal{R}_g$. We conclude from the analyticity of h on $|z| < \mathcal{R}_g + \delta_1$, that $\rho_t > \mathcal{R}_g$ if and only if $t/s \in \mathbb{N}$.
e) It follows from d) that $|z| < \mathcal{R}_g$, hence $\psi(t,z) \neq 0$ by c). □

4.5 Two-sided sequences of functions of convolution type

In this section we will study a two-sided analogue of sequences of polynomials of convolution type.

Definition 4.5.1 *Let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of Lebesgue measurable functions on $[0,\infty)$ such that not all functions q_n are identically zero. Then $(q_n)_{n\in\mathbb{Z}}$ is said to be a* **two-sided sequence of convolution type** *if*

$$\sum_{k=-\infty}^{\infty} |q_k(t)\, q_{n-k}(s)| < \infty \textit{ for all } s,t \geq 0$$

and

$$q_n(t+s) = \sum_{k=-\infty}^{\infty} q_k(t)\, q_{n-k}(s) \textit{ for all } s,t \geq 0.$$

Notation We write $\varphi(t,z) := \sum_{n=-\infty}^{\infty} q_n(t)\, z^n$ whenever this series converges absolutely. Note that $\varphi(t+s,z) = \varphi(t,z)\,\varphi(s,z)$.

Contrary to the one-sided case, due to convergence problems there seems to be no algebraic theory for two-sided sequences of convolution type. Therefore our policy is to impose several analytical conditions on $\varphi(t,z)$ and study the consequences. Note that $(J_n)_{n\in\mathbb{Z}}$, where J_n is the Bessel function of the first kind of index n, is an example of a two-sided sequence of convolution type (cf. [184, Section 62, Theorem 39]).

For later use we state the following lemma.

Lemma 4.5.2 *If $(c_n)_{n\in\mathbb{Z}}$ is a two-sided sequence of complex numbers and $(c_n\, R^n)_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ for some $R > 0$, then $\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!}\, c_n^{k*}\, R^n$ converges absolutely for all $t \in \mathbb{C}$. In particular, $\sum_{k=0}^{\infty} \frac{t^k}{k!}\, c_n^{k*}$ converges absolutely for all $t \in \mathbb{C}$.*

Proof: This follows from

$$\sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \left| \frac{t^k}{k!}\, c_n^{k*}\, R^n \right| \leq \sum_{k=0}^{\infty} \left| \frac{t^k}{k!} \sum_{n=-\infty}^{\infty} |c_n^{k*}|\, R^n \right| \leq$$

$$\sum_{k=0}^{\infty} \left| \frac{t^k}{k!} \right| \left\{ \sum_{n=-\infty}^{\infty} |c_n|\, R^n \right\}^k < \infty.$$

□

We begin with demanding $\varphi(t,.)$ to be an invertible element of the Wiener algebra $\mathcal{W}$, i.e. $\varphi(t,z) \neq 0$ for all $z \in \mathbb{T}$ (see Section 4.3).

Theorem 4.5.3 *Let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of convolution type. If $\varphi(t,.) \in$ inv $\mathcal{W}$ for all $t \geq 0$, then there exists an $h \in \mathcal{W}$ such that $\varphi(t,z) = e^{t\,h(z)}$ for all $|z| = 1$. In particular, there exists a two-sided sequence $(z_n)_{n\in\mathbb{N}} \in \ell_1(\mathbb{Z})$ such that $q_n(t) = \sum_{j=0}^{\infty} h_n^{j*}\, \frac{t^j}{j!}$.*

Proof: Define φ_1 by $\varphi_1(t,\theta) := \varphi(t, e^{i\theta})$ $(t \geq 0, \theta \in \mathbb{R})$. The measurability of $\varphi(.,z)$ and [118, Corollary to Theorem 4.17.3, p. 145] or [3, Theorem 4, p. 56] yield the existence of complex numbers $\chi(\theta)$ $(\theta \in \mathbb{R})$ such that

$$\varphi_1(t,\theta)/\varphi_1(t,0) = \exp\,(t\,\chi(\theta)) \tag{4.2}$$

We now show that ind $\varphi(t,.) = 0$ for all $t \geq 0$. Recall from Lemma 4.3.2 that the index of a non-vanishing continuous function on $\mathbb{T}$ is always an integer. If ind $\varphi(t_0,.) \neq 0$ for some $t_0 \geq 0$, then ind $\varphi(t_0/n,.) \notin \mathbb{Z}$ for n large enough by Lemma 4.3.2, which is impossible. It follows from Theorem 4.3.5 that there exist functions $\gamma(t,.) \in \mathcal{W}$ with $\gamma(t,1) = 0$ such that

$$\varphi_1(t,\theta)/\varphi_1(t,0) = \exp\ \left(\gamma(t,e^{i\theta})\right). \tag{4.3}$$

Define continuous functions $\tilde{\gamma}(t,.)$ on $\mathbb{R}$ by $\tilde{\gamma}(t,\theta) := \gamma(t,e^{i\theta})$. We get from (4.2) and (4.3):

$$\tilde{\gamma}(t,\theta) = t\,\chi(\theta) + k(t,\theta)\,2\pi i \tag{4.4}$$

with $k(t,\theta) \in \mathbb{Z}$. From (4.4) with $t = 1$ we get

$$\tilde{\gamma}(t,\theta) = t\,\tilde{\gamma}(1,\theta) + k(t,\theta) - t\,k(1,\theta)\,2\pi i \tag{4.5}$$

From (4.5), the continuity of $\gamma(t,.)$ and $\gamma(t,0) = 0$ we obtain $k(t,\theta) - t\,k(1,\theta) = 0$. Hence,

$$\varphi_1(t,\theta)/\varphi_1(t,0) = e^{(t\,\tilde{\gamma}(1,\theta))}. \tag{4.6}$$

From (4.6) and the measurability of $\varphi_1(t,0)$:

$$\varphi_1(t,\theta) = \exp\ \{at + t\,\tilde{\gamma}(1,\theta)\}. \tag{4.7}$$

Setting $h_0 := a$ and letting h_n be the n^{th} Fourier coefficient of $\gamma(1,.)$, we arrive at $\varphi(t,e^{i\theta}) = \exp\ \left\{t\sum_{n=-\infty}^{\infty} h_n\,e^{in\theta}\right\}$ with $\sum_{n=-\infty}^{\infty} |h_n| < \infty$. □

Remarks 4.5.4 a) Using Theorem 4.5.3 we can give an interesting proof of Theorem 4.4.1 $e \Rightarrow f$: extend the sequences $(q_n)_{n\in\mathbb{N}}t$ to elements of $\ell_1(\mathbb{Z})$ by setting $q_n(t) = 0$ for $n < 0$. By Theorem 4.5.3, there exists a sequence $(c_n)_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ such that

$$\sum_{n=-\infty}^{\infty} q_n(t)\,e^{in\theta} = \exp\ \left\{t\sum_{n=-\infty}^{\infty} c_n\,e^{in\theta}\right\} = \sum_{n=-\infty}^{\infty} e^{in\theta}\sum_{k=0}^{\infty}\frac{t^k}{k!}\,c_n^{k*}$$

(Lemma 4.5.2 allows us to change the order of summation). Unicity of Fourier coefficients yields:

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\,c_n^{k*} = 0 \text{ for } n < 0; \sum_{k=0}^{\infty}\frac{t^k}{k!}\,c_n^{k*} = q_n(t) \text{ for } n \geq 0 \tag{4.8}$$

Because (4.8) holds for all $t \geq 0$, we must have $c_n = 0$ for $n < 0$. Hence, $\sum_{n=0}^{\infty} q_n(t)\,z^n = \exp\ \{t\sum_{n=0}^{\infty} c_n\,z^n\} \neq 0$ for all $|z| \leq 1$.
b) It follows from $\varphi(t,e^{i\theta}) = \exp\ \left\{t\sum_{n=-\infty}^{\infty} h_n\,e^{in\theta}\right\}$ that for all $s \geq 0$

$$\lim_{t\to s}\sum_{n=-\infty}^{\infty} |q_n(t) - q_n(s)| = 0 \tag{4.9}$$

If we weaken the assumptions of Theorem 4.5.2 by allowing $\varphi(t,.)$ to vanish, then (4.9) still holds for all $s > 0$ by [118, Theorem 9.3.1, p. 280].

Notation If $0 < a < b < \infty$, then $\mathcal{A}(a,b) := \{z \in \mathbb{C} : a < |z| < b\}$ and $\overline{\mathcal{A}}(a,b) := \{z \in \mathbb{C} : a \leq |z| \leq b\}$.

We denote the Banach algebra of all Laurent series that converge absolutely on $\overline{\mathcal{A}}(a,b)$ by $\mathcal{W}_{a,b}$.— Addition and multiplication are the usual addition and multiplication of series. The norm on $\mathcal{W}_{a,b}$ is defined by $\| \sum_{n=-\infty}^{\infty} x_n z^n \| := \max \{ \sum_{n=-\infty}^{\infty} |x_n| a^n, \sum_{n=-\infty}^{\infty} |x_n| b^n \}$.
It is easy to see that $\mathcal{W}_{a,b}$ is complete and that the polynomials in z and $1/z$ are dense in $\mathcal{W}_{a,b}$.
The unit element of $\mathcal{W}_{a,b}$ is the Laurent series with $x_0 = 1$ and $x_n = 0$ for $n \neq 0$.

Lemma 4.5.5 *The complex homomorphisms of $\mathcal{W}_{a,b}$ are point evaluations on $\overline{\mathcal{A}}(a,b)$.*

Proof: Let $\Lambda \in \mathcal{M}(\mathcal{W}_{a,b})$ be arbitrary. From $\|\Lambda\| = 1$ ([212, Proposition 10.6 and Theorem 10.7]) we infer for the polynomial z that $|\Lambda(z)| \leq b$ and $|\Lambda(1/z)| \leq \|\Lambda\| \, \|z^{-1}\| = \|z^{-1}\| = a^{-1}$. Since $1/z$ is inverse to z, $|\Lambda(z)| = 1/|\Lambda(1/z)| \geq a$. Thus $\Lambda(z) = z_0$ for some $z_0 \in \overline{\mathcal{A}}(a,b)$. Hence, if p is a polynomial in z and $1/z$, then $\Lambda(p) = p(\Lambda(z)) = p(z_0)$. Since the polynomials in z and $1/z$ are dense in $\mathcal{W}_{a,b}$, we conclude that $\Lambda(f) = f(z_0)$ for every $f \in \mathcal{W}_{a,b}$. □

Theorem 4.5.6 *Let $a, b \in \mathbb{R}$ $(0 < a < b)$ and let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of convolution type. If $\varphi(t,.) \in inv\ \mathcal{W}_{a,b}$ for all $t \geq 0$, then there exists a Laurent series $\sum_{n=-\infty}^{\infty} g_n z^n \in \mathcal{W}_{a,b}$ such that*

$$\varphi(t,z) = exp \left\{ t \sum_{n=-\infty}^{\infty} g_n z^n \right\}$$

for all $z \in \overline{\mathcal{A}}(a,b)$. In particular, $q_n(t) = \sum_{k=0}^{\infty} g_n^{k*} \frac{t^k}{k!}$.

Proof: Let $r \in [a,b]$ be arbitrary. Then $\varphi(t, r\, e^{i\theta}) \neq 0$ for $\theta \in [-\pi, \pi]$. By Theorem 4.5.3 there exists $(c_n(r))_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ such that

$$\varphi(t, r\, e^{i\theta}) = \exp \left\{ t \sum_{n=-\infty}^{\infty} c_n(r)\, e^{in\theta} \right\} \tag{4.10}$$

Define $g_n(r) := c_n(r)\, r^{-n}$. Then $(g_n\, r^n)_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$. We will now prove that $g_n(r)$ does not depend on r. By Lemma 4.5.2, we may change the order of summation in (4.10) which yields $q_n(t) = \sum_{k=0}^{\infty} g_n(r)^{k*} \frac{t^k}{k!}$. Since r was arbitrary and the right-hand side series defines a holomorphic function of t, we conclude that $g_n(r)$ does not depend on r. Define $g_n := g_n(a)$. Hence q_n has the form indicated above. Moreover, $(g_n\, r^n)_{n\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ for all $r \in [a,b]$ and thus (4.10) yields $\varphi(t,z) = \exp\ \{ t \sum_{n=-\infty}^{\infty} g_n z^n \}$ for all $z \in \overline{\mathcal{A}}(a,b)$. □

We now set out to prove the analogue of Theorem 4.5.6 for the open annulus. It turns out that two-sided sequences of convolution type possess a property that is analogous to the property for polynomials of convolution type as expressed in Theorem 4.4.10e. This property is stated in Theorem 4.5.8; the above mentioned analogue of Theorem 4.5.6 is Theorem 4.5.9.

Lemma 4.5.7 *Let $(g_n)_{n\in\mathbb{Z}}$ be an arbitrary double-sided sequence of complex numbers such that $\sum_{n=-\infty}^{\infty} g_n\, z^n$ converges absolutely on the open annulus $\mathcal{A}(a,b)$. Define $q_n(t) := \sum_{k=0}^{\infty} g_n^{k*}\, \frac{t^k}{k!}$ for $t \in [0,\infty)$. Then $(q_n)_{n\in\mathbb{Z}}$ is a two-sided sequence of convolution type. If $z \in \mathcal{A}(a,b)$, then $\sum_{n=-\infty}^{\infty} q_n(t)\, z^n$ converges absolutely and does not vanish.*

Proof: It follows from Lemma 4.5.2 that q_n $(n \in \mathbb{Z})$ is well-defined and that $\sum_{n=-\infty}^{\infty} q_n(t)\, z^n$ converges absolutely on $\mathcal{A}(a,b)$. Hence,

$$\sum_{n=-\infty}^{\infty} q_n(t)\, z^n = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} g_n^{k*}\, \frac{t^k}{k!}\, z^n =$$

$$\sum_{k=0}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} g_n^{k*}\, z^n \right\} \frac{t^k}{k!} = \exp\left\{ t \sum_{n=-\infty}^{\infty} g_n\, z^n \right\} \neq 0.$$

□

Theorem 4.5.8 *Let $a, b \in \mathbb{R}$ $(0 < a < b)$ and let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of convolution type. If $\varphi(t,z)$ converges for all $z \in \mathcal{A}(a,b)$ and all $t \geq 0$, then $\varphi(t,z) \neq 0$ for all $t \geq 0$ and all $z \in \mathcal{A}(a,b)$.*

Proof: Suppose there are $t_0 > 0$ and $z_0 \in \mathcal{A}(a,b)$ such that $\varphi(t_0,z_0) = 0$. It follows from [118, Theorem 4.17.1, p. 144], that $\varphi(t,z_0) = 0$ for all $t > 0$. Choose c, d with $a \leq c < |z_0| < d \leq b$ such that $\varphi(t_0,z) \neq 0$ for all $t > 0$ and all $z \in \mathcal{A}(c,|z_0|) \bigcup \mathcal{A}(|z_0|,d)$. This is possible since the functions $\varphi(t,.)$ are analytic and not identically zero. Choose c_1, c_2, d_1 and d_2 with $c < c_1 < c_2 < |z_0|$ and $|z_0| < d_1 < d_2 < d$. An application of Theorem 4.5.6 to the functions q_n on $\overline{\mathcal{A}}(c_1,c_2)$ yields a sequence $(g_n)_{n\in\mathbb{N}}$ of complex numbers such that $q_n(t) = \sum_{k=0}^{\infty} g_n^{k*}\, \frac{t^k}{k!}$. An application of Theorem 4.5.6 to the functions q_n on $\overline{\mathcal{A}}(d_1,d_2)$ yields a sequence $(h_n)_{n\in\mathbb{N}}$ of complex numbers such that $q_n(t) = \sum_{k=0}^{\infty} h_n^{k*}\, \frac{t^k}{k!}$. Differentiating with respect to t and substituting $t = 0$, we obtain $g_n = h_n$ for all $n \in \mathbb{Z}$. This implies that $\sum_{n=-\infty}^{\infty} g_n\, z^n$ converges for all $z \in \overline{\mathcal{A}}(c_1,c_2) \bigcup \overline{\mathcal{A}}(d_1,d_2)$, hence for all $z \in \overline{\mathcal{A}}(c_1,d_2)$. It follows from Lemma 4.5.7 that $\varphi(t_0,z) \neq 0$ for all $z \in \overline{\mathcal{A}}(c_1,d_2)$, which contradicts $\varphi(t_0,z_0) = 0$. □

Theorem 4.5.9 *Let $a, b \in \mathbb{R}$ $(0 < a < b)$ and let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of convolution type. Suppose $\varphi(t,z)$ converges absolutely for all $t \geq 0$ and all $z \in \mathcal{A}(a,b)$. Then there exists a Laurent series $\sum_{n=-\infty}^{\infty} g_n\, z^n$ that absolutely converges on $\mathcal{A}(a,b)$ and satisfies $\varphi(t,z) = exp\ \left\{ t \sum_{n=-\infty}^{\infty} g_n\, z^n \right\}$ for all $t \geq 0$ and for all $z \in \mathcal{A}(a,b)$. In particular, $\varphi(t,z)$ does not vanish on $\mathcal{A}(a,b)$ and $q_n(t) = \sum_{k=0}^{\infty} g_n^{k*}\, \frac{t^k}{k!}$.*

Proof: It follows from Theorem 4.5.8 that $\varphi(t,z) \neq 0$ for all $t \geq 0$ and all $z \in \mathcal{A}(a,b)$. Applying Theorem 4.5.6 to $\mathcal{W}_{a+1/n,b-1/n}$ for all $n \in \mathbb{N}$ such that $a + 1/n < b - 1/n$, we obtain Laurent series $h_n \in \mathcal{W}_{a+1/n,b-1/n}$ such that $\varphi(t,z) = \exp\ (t\,h_n(z))$. Since $\exp\ (t\,h_n(z)) = \exp\ (t\,h_m(z))$ for all $t \in [0,\infty)$ on a circular region, $h_n(z) = h_m(z)$ for all z in their common domain by Lemma 4.2.1a. Hence, all the Laurent series h_n are identical. If we set $g := h_1$, then g converges absolutely on $\mathcal{A}(a,b)$ and $\varphi(t,z) = e^{t\,g(z)}$ for all $t \in \mathbb{C}$ and all $z \in \mathcal{A}(a,b)$. □

We denote the Banach algebra of all Laurent series that are absolutely convergent on $\mathcal{A}(a,b)$ and have a continuous extension to $\overline{\mathcal{A}}(a,b)$ by $\mathcal{L}_{a,b}$. Addition and multiplication are defined pointwise. The norm is the supremum norm of the function corresponding to the Laurent series. Since the limit of a uniformly convergent sequence of continuous (holomorphic) functions is again continuous (holomorphic), $\mathcal{L}_{a,b}$ is complete. The unit element of $\mathcal{L}_{a,b}$ is the Laurent series with $x_0 = 1$ and $x_n = 0$ for $n \neq 0$.

Lemma 4.5.10 *The complex homomorphisms of $\mathcal{L}_{a,b}$ are point evaluations on $\overline{\mathcal{A}}(a,b)$.*

Proof: It suffices to show that the polynomials in z and $1/z$ are dense in $\mathcal{L}_{a,b}$, since we can then copy the proof of Lemma 4.5.5. If $\sum_{n=-\infty}^{\infty} a_n\, z^n \in \mathcal{L}_{a,b}$, then $\sum_{n=0}^{\infty} a_n\, z^n$ and $\sum_{n=-\infty}^{-1} a_n\, z^n$ can be approximated uniformly on $\overline{\mathcal{A}}(a,b)$ by polynomials in z, polynomials in $1/z$ respectively. This shows that the polynomials in z and $1/z$ are dense in $\mathcal{L}_{a,b}$. □

Theorem 4.5.11 *Let $a,b \in \mathbb{R}$ $(0 < a < b)$ and let $(q_n)_{n\in\mathbb{Z}}$ be a two-sided sequence of convolution type. If $\varphi(t,z) \in$ inv $\mathcal{L}_{a,b}$ for all $t \geq 0$, then there exists an $h \in \mathcal{L}_{a,b}$ such that $\varphi(t,z) = e^{t\,h(z)}$ for all $t \geq 0$ and all $z \in \overline{\mathcal{A}}(a,b)$. In particular, $q_n(t) = \sum_{k=0}^{\infty} g_n^{k*}\, \frac{t^k}{k!}$.*

Proof: First note that Theorem 4.5.9 implies the existence of a Laurent series h which absolutely converges on $\mathcal{A}(a,b)$ and satisfies $\varphi(t,z) = e^{t\,h(z)}$ for all $z \in \mathcal{A}(a,b)$.
Choose c,d such that $a < c < d < b$. Consider all $0 < \lambda < 1$ such that $\lambda c > a$. Write $\varphi_\lambda(t,z) := \varphi(t,\lambda z)$ for these λ. It follows from Theorem 4.5.6 that $\varphi_\lambda(1,.) \in \exp\ \mathcal{W}_{c,b} \subset \exp\ \mathcal{L}_{c,b}$. Since $\varphi(1,.) \in$ inv $\mathcal{L}_{c,b}$ and $\lim_{\lambda\uparrow 1} \varphi_\lambda(1,.) = \varphi(1,.) in \mathcal{L}_{c,b}$, the second statement of Theorem 4.1.4 implies that $\varphi(1,.) \in \exp\ \mathcal{L}_{c,b}$. In a similar way we see that $\varphi(1,.) \in \exp\ \mathcal{L}_{a,d}$. It follows from Lemma 4.2.1a that $\varphi(1,.) \in \exp\ \mathcal{L}_{a,b}$, i.e. there exists an $H \in \mathcal{L}_{a,b}$ such that $\varphi(1,z) = e^{(z)}$ for $z \in \mathcal{A}(a,b)$. It follows from Lemma 4.2.1a that H and h differ by a constant. We conclude that $h \in \mathcal{L}_{a,b}$.
For the last statement, see the end of the proof of Theorem 4.5.6. □

Chapter 5

Central limit theorems and infinite divisibility

In this chapter we study random variables Y_n ($n \in \mathbb{N}$) with probability generating function $q_n(\lambda x)/q_n(\lambda)$, where $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. For an interpretation of the random variables Y_n in terms of a compound Poisson process, see [222].
Canfield [47, 48] proved a central limit theorem for $(Y_n)_{n\in\mathbb{N}}$ in case $g(z) := \sum_{n=0}^{\infty} g_n z^n$ belongs to a class of entire functions including polynomials (see also [55, 216]). A central limit theorem for $(Y_n)_{n\in\mathbb{N}}$ in case g has a dominant logarithmic singularity on its circle of convergence can be found in [96, 97]. Stam [225] used renewal theory to obtain a central limit theorem. Moreover, in [222] he obtained results on the asymptotic behaviour of $q_n(x)/q_n(1)$. The main purpose of this chapter is to extend the results of [224].
Applications of these central limit theorems to asymptotic enumeration can be found in [47, 48, 96, 97, 214].
This chapter is organized as follows. Section 5.1 gives some auxiliary results that will be needed for the proof of the central limit theorem in Section 5.4. In Section 5.2 we determine the asymptotics of the polynomials q_n when g converges absolutely on its circle of convergence. In Section 5.3 we introduce the renewal approach to central limit theorems of [224] and show that his central limit theorem is a special case of the results of Section 5.4. Section 5.4 contains a central limit theorem for the case that g has a dominant logarithmic singularity on its circle of convergence. Finally, Section 5.5 deals with infinitely divisible probability measures on $\mathbb{N}$. Using Sections 4.1 and 5.2, we give a new proof for a result of Embrechts and Hawkes [87] on the asymptotic behaviour of an infinitely divisible probability measure on $\mathbb{N}$ and its Lévy-measure.

Contents of chapter 5

5.1 Preliminaries

This section contains several results that will be used in the next sections. For sake of brevity, we do not state these results in their most general form.

Lemma 5.1.1 *a) If $x > 1$, then $\left(\binom{x+n-1}{n}\right)_{n\in\mathbb{N}}$ is an increasing sequence.*

b) If $0 < x < 1$, then $\left(\binom{x+n-1}{n}\right)_{n\in\mathbb{N}}$ is a decreasing sequence.

Proof: This follows from $\binom{x+n-1}{n}/\binom{x+n-2}{n-1} = \frac{x+n-1}{n} = 1 + \frac{x-1}{n}$. □

Lemma 5.1.2 *If $a_n := (\lambda \log n)^{1/2}$, then for all $\lambda, z > 0$ and all $k \in \mathbb{N}$:*

$$\lim_{n\to\infty} e^{-a_n z} \binom{\lambda\, e^{z/a_n} + n - k - 1}{n-k} \binom{\lambda + n - k - 1}{n-k}^{-1} = e^{\frac{1}{2}z^2}.$$

For fixed $z > 0$, $e^{-a_n z} \binom{\lambda\, e^{z/a_n} + n-k-1}{n-k} \binom{\lambda+n-k-1}{n-k}^{-1}$ is uniformly bounded for all $n, k \in \mathbb{N}$ with $0 \le k < n$.

Proof: We first prove the assertion for $k = 0$. Note that

$$e^{-a_n z} \binom{\lambda\, e^{z/a_n} + n - 1}{n} \binom{\lambda + n - 1}{n}^{-1} =$$

$$e^{-a_n z}\, e^{z/a_n} \prod_{j=1}^{n-1} \left(1 + \frac{\lambda}{\lambda + j} \left(e^{z/a_n} - 1\right)\right).$$

After taking logarithms it suffices to prove

$$\lim_{n\to\infty} \left\{ z/a_n - a_n\, z + \sum_{j=1}^{n-1} \log\left(1 + \frac{\lambda}{\lambda+j} \left(e^{z/a_n} - 1\right)\right)\right\} = \frac{1}{2} z^2.$$

Expanding first the logarithms and then the exponential functions into Taylor polynomials, we obtain

$$\sum_{j=0}^{n} \log\left(1 + \frac{\lambda}{\lambda+j} \left(e^{z/a_n} - 1\right)\right) =$$

$$\sum_{j=0}^{n} \frac{\lambda}{\lambda+j}\left(e^{z/a_n}-1\right)\Big) - \sum_{j=0}^{n} \frac{\frac{1}{2}}{(1+\theta_{n,j})^2}\left(\frac{\lambda}{\lambda+j}\right)^2\left(e^{z/a_n}-1\right)^2,$$

with $0 < \theta_{n,j} < \frac{\lambda}{\lambda+j}\left(e^{z/a_n}-1\right)$ for $n \to \infty$. The last term vanishes as $n \to \infty$, because $\sum_{j=0}^{\infty}\left(\frac{\lambda}{\lambda+j}\right)^2$ converges and $\lim_{n\to\infty} z/a_n = 0$. Now we expand the first term as

$$\left(z/a_n + \frac{1}{2}\,(z/a_n)^2 + \mathrm{o}\left(a_n^{-2}\right)\right)\sum_{j=0}^{n}\frac{\lambda}{\lambda+j}.$$

Since $\sum_{j=0}^{n}\frac{\lambda}{\lambda+j} = a_n^2 + \mathrm{O}(1)$, it follows that

$$\lim_{n\to\infty}\left[\left(z/a_n + \frac{1}{2}\,(z/a_n)^2 + \mathrm{o}\left(a_n^{-2}\right)\right)\sum_{j=0}^{n}\frac{\lambda}{\lambda+j}\right] - a_n\, z = \frac{1}{2}\,z^2.$$

This completes the proof for $k = 0$. Because

$$\binom{\lambda\, e^{z/a_n}+n-k-1}{n-k}\binom{\lambda+n-k-1}{n-k}^{-1} =$$

$$\frac{\lambda+n-k}{e^{z/a_n}+n-k}\binom{\lambda\, e^{z/a_n}+n-(k-1)-1}{n-(k-1)}\binom{\lambda+n-(k-1)-1}{n-(k-1)-1}^{-1},$$

induction on k yields the first assertion.
For the second assertion note that $e^{z/a_n} \le e^{z/a_{n-k}}$, $e^{-a_n z} \le e^{-a_{n-k}z}$, and that $\binom{x+n-1}{n} < \binom{y+n-1}{n}$ for $0 < x < y$. Hence, $e^{-a_n z}\binom{\lambda e^{z/a_n}+n-k-1}{n-k} \le e^{-a_{n-k}z}\binom{\lambda e^{z/a_{n-k}}+n-k-1}{n-k}$. □

Lemma 5.1.3 *For all* $\lambda \in \mathbb{C}\setminus\{-1,-2,\dots\}$, *we have* $\lim_{n\to\infty}\binom{\lambda+n-1}{n} n^{1-\lambda} = \frac{1}{\Gamma(\lambda)}$.

Proof: Since $\binom{\lambda+n-1}{n} = \frac{\Gamma(n+\lambda)}{\Gamma(n+1)\,\Gamma(\lambda)}$ for all $\lambda \in \mathbb{C}\setminus\{-1,-2,\dots\}$, the result follows from [68, sect. 27]. □

Lemma 5.1.4 *If* $\lambda > 0$ *and* $(\lambda-1)\alpha > -1$, *then*

$$\sum_{j=0}^{m}\binom{\lambda+j-}{j}^{\alpha} \le C\, m^{1+(1-\lambda)\alpha},$$

where C *depends on* λ *and* α.

Proof: Since $\lambda > 0$ and $(\lambda-1)\alpha > -1$, Lemma 5.1.3 yields $\sum_{j=0}^{m} \binom{\lambda+j-}{j}^{\alpha} = 1 + \sum_{j=0}^{m} \binom{\lambda+j-}{j}^{\alpha} \leq 1 + C_1 \sum_{j=0}^{m} j^{(\lambda-1)\alpha}$. If $(\lambda-1)\alpha \geq 0$, then

$$1 + C_1 \sum_{j=0}^{m} j^{(\lambda-1)\alpha} \leq 1 + C_1 \int_0^m t^{(\lambda-1)\alpha}\, dt \leq C_2\, m^{1+(\lambda-1)\alpha}.$$

If $-1 < (\lambda-1)\alpha < 0$, then

$$1 + C_1 \sum_{j=0}^{m} j^{(\lambda-1)\alpha} \leq 1 + C_1 \int_1^{m-+1} t^{(\lambda-1)\alpha}\, dt \leq C_3\, m^{1+(\lambda-1)\alpha}.$$

□

We conclude this section with a useful lemma on convergence of moment generating functions.

Lemma 5.1.5 *Let $a, b \in \mathbb{R}$ be arbitrary with $a < b$. If F_n $(n \in \mathbb{N})$ and F are probability distribution functions on the real line such that*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} e^{zx}\, dF_n(x) = \int_{-\infty}^{\infty} e^{zx}\, dF(x)$$

for all $z \in (a, b)$, then F_n converges weakly to F as $n \to \infty$.

Proof: If $a < 0 < b$, then the result for arbitrary distribution functions follows from the proof of [71, Theorem 3]. Suppose 0 is not an interior point of (a, b). We will reduce this case to the case $a < 0 < b$. Choose an arbitrary $\xi \in (a, b)$. Define measures dG_n by $dG_n(x) := e^{\xi x}\, dF_n(x)$ for all $n \in \mathbb{N}$ and define dG by $dG(x) := e^{\xi x}\, dF(x)$. Then $\lim_{n\to\infty} \int_{-\infty}^{\infty} e^{vx}\, dG_n(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{(v+\xi)x}\, dF_n(x) = \int_{-\infty}^{\infty} e^{(v+\xi)x}\, dF(x) = \int_{-\infty}^{\infty} e^{vx}\, dG(x)$ for all $v \in (a-\xi, b-\xi)$. Since $a - \xi < 0 < b - \xi$, it follows that $\lim_{n\to\infty} G_n(x) = G(x)$ for all continuity points x of G. Since $e^{\xi x}$ is continuous, it follows that $\lim_{n\to\infty} F_n(x) = F(x)$ for all continuity points x of F. Because F_n and F are probability distribution functions, it follows that F_n converges weakly to F as $n \to \infty$. □

5.2 Asymptotics when g converges on its circle of convergence

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. In this section we study the asymptotic behaviour of $q_n(x)/q_n(1)$ as $n \to \infty$ in case $\sum_{n=0}^{\infty} g_n\, z^n$ converges absolutely on its circle of convergence. If $g_n \geq 0$ for all $n \in \mathbb{N}$, then the polynomials q_n have nonnegative coefficients by Lemma 2.1.5, since $q_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$ by Theorem

2.1.8. Hence, $q_n(x)/q_n(1)$ is the probability generating function of a discrete random variable. Stam [222, Theorem 4] has shown that in this case the only possible limit distribution without centering and scaling is a Poisson distribution shifted 1 to the right. We will extend this result to the case where the numbers g_n need not be non-negative. Of course, in this case $q_n(x)/q_n(1)$ need not be a probability generating function. The Banach algebra approach to subexponential distributions of [61] will be used and extended.

We start with stating and extending the results from [61] needed for the sequel. Recall that $\mathbb{N} = \{0, 1, ...\}$.

Definition 5.2.1 *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that*

1. $\lim_{n\to\infty} \mu_n^{2*}/\mu_n = c$ *exists and is finite*
2. $\lim_{n\to\infty} \mu_{n+1}/\mu_n = 1/r$ *exists and is positive*
3. $\mu_n > 0$ *for all* $n \in \mathbb{N}$
4. $\sum_{n=0}^{\infty} \mu_n r^n < \infty$.

Define

$$\mathcal{U}_L := \{(\nu_n)_{n\in\mathbb{N}} \subset \mathbb{C} \mid \lim_{n\to\infty} \nu_n/\mu_n \text{ exists}\}$$

and

$$\mathcal{U}_0 := \{(\nu_n)_{n\in\mathbb{N}} \subset \mathbb{C} \mid \lim_{n\to\infty} \nu_n/\mu_n = 0\}.$$

Condition 3) of definition 5.2.1 is missing in [61]. However, the proofs in [61] are not valid unless condition 3) is added. It is not clear how to remove condition 3).
For more information on sequences satisfying the conditions of Definition 5.2.1, see [86, sect. 2]. For example, it can be shown that $c = 2\sum_{n=0}^{\infty} \mu_n r^n$ (see [86, Theorem 2.8]).

Parts a, b and c of the following theorem are taken from [61]; parts d and e are new.

Theorem 5.2.2 *a) $\mathcal{U}_L$ and $\mathcal{U}_0$ are Banach algebras when equipped with coordinatewise addition, convolution as multiplication and norm $\|\nu\| := M \sup_{n\in\mathbb{N}} \nu_n/\mu_n \,|$, where $M := u\,\mu_n^{2*}/\mu_n$. The sequence $1, 0, 0, \ldots$ is the unit element u of both $\mathcal{U}_L$ and $\mathcal{U}_0$.*

b) If Λ is a complex homomorphism of $\mathcal{U}_0$, then there exists $\lambda \in \{z \in \mathbb{C} : |z| \le r\}$ such that $\Lambda(\nu) = \sum_{n=0}^{\infty} \nu_n \lambda^n$ for all $\nu \in \mathcal{U}_0$.

c) If Λ is a complex homomorphism of $\mathcal{U}_L$, then there exists $\lambda \in \{z \in \mathbb{C} : |z| \le r\}$ such that $\Lambda(\nu) = \sum_{n=0}^{\infty} \nu_n \lambda^n$ for all $\nu \in \mathcal{U}_L$.

d) inv $\mathcal{U}_0$ = exp $\mathcal{U}_0$.

e) $\mathit{inv}\, \mathcal{U}_L = \mathit{exp}\, \mathcal{U}_L$.

Proof: a) See [61, Lemma 1].
b) See [61, Lemma 2].
c) See [61, Lemma 3].
d) The inclusion $\exp \mathcal{U}_0 \subset \mathrm{inv}\, \mathcal{U}_0$ follows from Remark 4.1.3a. Let $\nu \in \mathrm{inv}\, \mathcal{U}_0$ be arbitrary. Define $F : [0,1] \to \mathrm{inv}\, \mathcal{U}_0$ by $F(t) := (t^n \nu_n)_{n \in \mathbb{N}}$. It follows from b) and Theorem 4.1.2 that $F(t) \in \mathrm{inv}\, \mathcal{U}_0$ for all $t \in [0,1]$. We now show that F is continuous. If $s, t \in [0,1]$, then $\|F(s) - F(t)\| = M \sup_{n \in \mathbb{N}} |(s^n - t^n)\, \nu_n / \mu_n|$. Since $\lim_{n \to \infty} \nu_n / \mu_n = 0$, it follows that $\lim_{n\to\infty} \|F(s) - F(t)\| = 0$. Hence, $\nu \in \exp \mathcal{U}_0$ by Theorem 4.1.6 since $\nu_0 \neq 0$.
e) The inclusion $\exp \mathcal{U}_L \subset \mathrm{inv}\, \mathcal{U}_L$ follows from Remark 4.1.3a. Let $\nu \in \mathrm{inv}\, \mathcal{U}_L$ be arbitrary. By c) and theorem 4.1.2, $\sum_{n=0}^{\infty} \nu_n z^n \neq 0$ for all $|z| \leq r$. In particular, $\nu_0 \neq 0$. Thus, the Gelfand transform $\widehat{\nu}$ of ν belongs to $\mathrm{inv}\, \mathcal{C}(\mathcal{M})$. Since $\mathcal{M}$ is homeomorphic to $\{z \in \mathbb{C} : |z| \leq r\}$, it follows from Theorem 4.2.5 that $\hat{\nu} \in \exp \mathcal{C}(\mathcal{M})$. By theorem 4.1.10, $\nu \in \exp \mathcal{U}_L$. □

Remark 5.2.3 If $\nu \in \mathrm{inv}\, \mathcal{U}_L$ and $\lim_{n\to\infty} \nu_n/\mu_n \neq 0$, then $F : [0,1] \to \mathrm{inv}\, \mathcal{U}_0$, defined by $F(t) := (t^n \nu_n)_{n \in \mathbb{N}}$, is continuous for $0 \leq t < 1$ but discontinuous at $t = 1$. Thus the method of proof for Theorem 5.2.2d does not work for Theorem 5.2.2e.

The following theorem gives sufficient conditions for the convergence as $n \to \infty$ of $q_n(x)/q_n(1)$. If the polynomials q_n have non-negative coefficients, then $q_n(x)/q_n(1)$ is the probability generating function of a discrete random variable Y_n. Convergence of $q_n(x)/q_n(1)$ for all $x \in (0,1]$ implies convergence in distribution of the random variables Y_n by the continuity theorem for probability generating functions (see [92, sect. XI.6]).
Theorem 5.2.4 was proved in [222] for q_n with non-negative coefficients (cf. Remark 5.2.5a).

Theorem 5.2.4 *Let $(q_n)_{n \in \mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n \in \mathbb{N}}$ such that $\mathcal{R}_g$, the radius of convergence of $\sum_{n=0}^{\infty} g_n z^n$, is finite and positive. If an $x_0 \neq 0$ exists such that:*

1. $q_n(x_0) > 0$ *for all* $n \in \mathbb{N}$
2. $\lim_{n\to\infty} q_n(2\,x_0)/q_n(x_0)$ *exists and is finite*
3. $\lim_{n\to\infty} q_{n+1}(x_0)/q_n(x_0) = 1/\mathcal{R}_g$
4. $\sum_{n=0}^{\infty} q_n(x_0) \mathcal{R}_g^n < \infty$
5. $\sum_{n=0}^{\infty} q_n(x_0) z^n \neq 0$ *for* $|z| = \mathcal{R}_g$,

then $\sum_{n=0}^{\infty} |g_n| \mathcal{R}_g^n < \infty$ *and* $\lim_{n\to\infty} q_n(x)/q_n(1) = x\, e^{(x-1)g(\mathcal{R}_g)}$ *for all* $x \in \mathbb{C}$.

Proof: By definition of convolution type, the sequence $(q_n(2\,x_0))_{n\in\mathbb{N}}$ is the twofold convolution of $(q_n(x_0))_{n\in\mathbb{N}}$. Define $\mu_n := q_n(x_0)$. It follows from Theorem 5.2.2 that $\mathcal{U}_L$ and $\mathcal{U}_0$ are Banach algebras. It follows from Theorem 4.4.1 with $\alpha_n = \mathcal{R}_g^n$ that $\sum_{n=0}^{\infty} \mu_n z^n \neq 0$ for $|z| \leq \mathcal{R}_g$. Hence, $(\mu_n)_{n\in\mathbb{N}} \in \operatorname{inv} \mathcal{U}_L$ by Theorem 5.2.2c and $(\mu_n)_{n\in\mathbb{N}} \in \exp \mathcal{U}_L$ by Theorem 5.2.2e. It follows from Lemma 4.2.1 and Theorem 4.4.1 that $(g_n)_{n\in\mathbb{N}} \in \mathcal{U}_L$. Obviously, also $(x\, g_n)_{n\in\mathbb{N}} \in \mathcal{U}_L$ for all $x \in \mathbb{C}$ and therefore $(q_n(x))_{n\in\mathbb{N}} \in \mathcal{U}_L$ for all $x \in \mathbb{C}$. Now [61, Lemma 5] implies that $\lim_{n\to\infty} q_n(k)/q_n(1) = k\, e^{(k-1)g(\mathcal{R}_g)}$ for all $k \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \frac{q_n(k/m)}{q_n(1)} = \lim_{n\to\infty} \frac{q_n(k/m)}{q_n(1/m)} \frac{q_n(1/m)}{q_n(1)} = \frac{k}{m} e^{((k/m)-1)\, g(\mathcal{R}_g)}$$

for all $k, m \in \mathbb{N}$. By continuity, $\lim_{n\to\infty} q_n(x)/q_n(1) = x\, e^{(x-1)\, g(\mathcal{R}_g)}$ for all $x \geq 0$. Applying the theorem to $(q_n(ax))_{n\in\mathbb{N}}$ for suitable a with $|a| = 1$, we obtain $\lim_{n\to\infty} q_n(x)/q_n(1) = x\, e^{(x-1)g(\mathcal{R}_g)}$ for all $x \in \mathbb{C}$. □

Remark 5.2.5 a) Let us compare Theorem 5.2.4 with [222, Theorem 4]. The conditions in the Stam theorem are: $g_n \geq 0$ for all $n \in \mathbb{N}, g_1 \neq 0$ (since $(q_n)_{n\in\mathbb{N}}$ is a basic sequence; cf. Theorems 2.2.17 and 2.1.12b), $\mathcal{R}_g < \infty$, $\sum_{n=0}^{\infty} g_n \mathcal{R}_g^n < \infty$ and the existence of a nonzero limit of $q_n(x)/q_n(1)$ for $0 \leq x < 1$. In particular, these conditions imply 1), 2), 4) and 5) of Theorem 5.2.4 for $x_0 = \frac{1}{2}$. It follows from [86, Theorem 2.8 and Lemma 2.10] that 3) is also satisfied for $x_0 = \frac{1}{2}$. Hence, Theorem 5.2.4 is more general than the Stam theorem.
b) It is shown in [222, Theorem 3] that if $g_n \geq 0$ for all $n \in \mathbb{N}$, then $g(\mathcal{R}_g) = \infty$ implies $\liminf_{n\in\mathbb{N}} q_n(x)/q_n(1) = 0$ for $0 \leq x < 1$. Thus, $g(\mathcal{R}_g) < \infty$ is necessary for the existence of a nonzero limit for $q_n(x)/q_n(1)$. Since the proof of Stam uses non-negativity in an essential way, it is not clear whether the above also holds in the general case.
c) It is possible to avoid the continuity argument at the end of the proof of Theorem 5.2.4 when $g_n > 0$ for $n \geq 1$. In order to do so, first note that $\lim_{n\to\infty} q_n(x)/g_n \neq 0$ for all $x \neq 0$ (cf. the proof of Theorem 5.5.4). This allows us to write $\lim_{n\to\infty} q_n(x)/q_n(1) = \lim_{n\to\infty} q_n(x)/g_n \lim_{n\to\infty} g_n/q_n(1)$. It follows from [86, Theorem 2.9iv] that $g_n^{2*} \sim 2\,g_n$ $(n \to \infty)$. Moreover, $\lim_{n\to\infty} g_n/g_{n+1} = \mathcal{R}_g$ by [86, Theorem 2.8 and Lemma 2.10]. Now consider the Banach algebra $\mathcal{U}_L$ with $\mu_n = g_n$. The theorem now follows from [61, Formula (2)] with $\varphi(z) = e^{xz}$ (cf. [61, Remark 2]).

Example 5.2.6 We now apply Theorem 5.2.4 to the Abel polynomials $x\,(x - an)^{n-1}/n!$ with $a < 0$. It follows from Remark 2.1.10c that $g_n = (-an)^{n-1}/n!$. Thus

$$\mathcal{R}_g = \lim_{n\to\infty} \frac{g_n}{g_{n+1}} = \lim_{n\to\infty} \left(\frac{|an|^{n-1}}{n!}\right)^{-1/n} = (|a|e)^{-1}$$

and

$$\sum_{n=0}^{\infty} |g_n| \, (|a|e)^{-n} < \infty.$$

Moreover, a simple computation yields

$$\lim_{n\to\infty} q_n(x)/q_n(1) = x\, e^{-(x-1)\,a^{-1}}.$$

Together with Theorem 5.2.4 this yields $g(\mathcal{R}_g) = -a^{-1}$. Thus we obtain $\sum_{n=0}^{\infty} \frac{n^{n-1}}{n!\,e^n} = 1$.

5.3 Renewal theory

In the previous section we considered limit behaviour without centering or scaling of random variables with probability generating function $q_n(x)/q_n(1)$. We used the representation $q_n(x) = \sum_{k=0}^{n} g_n^{k*} \frac{x^k}{k!}$ and studied the behaviour of $\sum_{n=0}^{\infty} g_n\, z^n$. In [224] Stam introduced the idea to use the representation $q_n(x) = \sum_{k=0}^{n} f_n^{k*} \binom{x+k-1}{k}$ (cf. Theorem 2.3.10 and Example 2.2.16c) for studying limit behaviour with centering and scaling. The polynomials $\binom{x+n-1}{n}$ have interesting properties. Firstly, $\binom{x+n-1}{n}$ is the probability generating function of the number of cycles in a random permutation of $\{1,...,n\}$ and satisfies a central limit theorem (see [92, Chapter X.6b], [214, Chapter 5, Theorem 1.1] or apply Lemmas 5.1.2 and 5.1.5). Secondly, the sequence $\left(\binom{x+n-1}{n}\right)_{n\in\mathbb{N}}$ is the unique sequence of polynomials of convolution type with $q_n(1) = 1$ for all $n \in \mathbb{N}$ (see Theorem 3.1.1). The purpose of Sections 5.3 and 5.4 is to extend the results of [224] to the case where f_n is not necessarily non-negative.

The following theorem shows the connection of the Stam approach with renewal theory.

Theorem 5.3.1 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$ and let $(f_n)_{n\in\mathbb{N}}$ be the unique sequence of complex numbers such that $q_n(x) = \sum_{k=0}^{n} f_n^{k*} \binom{x+k-1}{k}$. Let $\mathcal{R}_f$, $\mathcal{R}_g$ be the radius of convergence of $f(z) = \sum_{n=0}^{\infty} f_n\, z^n$, $g(z) = \sum_{n=0}^{\infty} g_n\, z^n$ respectively. Then:*

a) $q_n(1) = \sum_{k=0}^{n} f_n^{k*}$

b) $g_n = \sum_{k=1}^{n} f_n^{k*}$

c) $f_0 = 0$; $f_n = -q_n(-1)$ *for* $n \geq 1$

d) $f_n \geq 0$ *for all* $n \in \mathbb{N} \Rightarrow g_n \geq 0$ *for all* $n \in \mathbb{N}$

e) the following formal generating function identity holds:

$$\sum_{n=0}^{\infty} q_n(x) z^n = (1 - f(z))^{-x}$$

f) $\mathcal{R}_g = \min\{|z| : |z| \le \mathcal{R}_f$ *and* $f(z) = 1\}$

g) $\sum_{n=0}^{\infty} |f_n|\, \mathcal{R}_f^n < \infty$ *and* $\sum_{n=0}^{\infty} f_n\, z^n \neq 1$ *for all* $|z| \le \mathcal{R}_f$ *if and only if* $\mathcal{R}_f = \mathcal{R}_g$ *and* $\sum_{n=0}^{\infty} |g_n\, \mathcal{R}_g^n < \infty$.

h) If there exists θ *with* $|\theta| < \mathcal{R}_f$ *such that* $\sum_{n=0}^{\infty} f_n\, \theta^n = 1$, *then* $\mathcal{R}_g < \mathcal{R}_f$ *and* $\sum_{n=0}^{\infty} |g_n\, \mathcal{R}_g^n = \infty$

i) If $\sum_{n=0}^{\infty} |f_n|\, \mathcal{R}_f^n < \infty$, $\sum_{n=0}^{\infty} f_n\, z^n \neq 1$ *for* $|z| < \mathcal{R}_f$ *and if* $\sum_{n=0}^{\infty} f_n\, \theta^n = 1$ *for some* θ *with* $|\theta| = \mathcal{R}_f$, *then* $\mathcal{R}_g = \mathcal{R}_f$ *and* $\sum_{n=0}^{\infty} |g_n|\, \mathcal{R}_g^n = \infty$.

Proof: Recall that $\left(\binom{x+n-1}{n}\right)_{n \in \mathbb{N}}$ is a sequence of polynomials of convolution type by Example 2.2.16c and Theorem 2.2.17. Thus existence and uniqueness of $(f_n)_{n\in\mathbb{N}}$ follows from Theorem 2.3.10.
a) This follows from $q_n(x) = \sum_{k=0}^{n} f_n^{k*} \binom{x+k-1}{k}$ with $x = 1$.
b) We have $g_n = (D\, q_n)(0) = \sum_{k=0}^{n} f_n^{k*}\, D^k \left(\binom{x+k-1}{k}\right)(0) = \sum_{k=1}^{n} f_n^{k*}$.
c) This follows from $\binom{k-2}{k} = 0$ for $k \ge 2 (k \in \mathbb{N})$ and $\binom{k-2}{k} = -1$ for $k = 1$.
d) This follows directly from b).
e) This follows from $\sum_{n=0}^{\infty} \binom{x+n-1}{n} z^n = \sum_{n=0}^{\infty} \binom{-x}{n} (-z)^n = (1-z)^{-x}$.
f) First suppose $\mathcal{R}_f = 0$. If $\mathcal{R}_g > 0$, then c) and Theorem 4.4.10 imply that $\mathcal{R}_f > 0$. Hence, $\mathcal{R}_f = 0$ implies $\mathcal{R}_g = 0$. Now suppose $\mathcal{R}_f > 0$. Note that $f(z) = 1 - e^{-g(z)}$ for $|z| < \min\{\mathcal{R}_f, \mathcal{R}_g)$. Hence, $\mathcal{R}_g \le \min\{|z| : |z| \le \mathcal{R}_f$ and $f(z) = 1\}$. If $f(z) \neq 1$ for $|z| \le r < \mathcal{R}_f$, then by Theorem 4.2.10 there exists a unique analytic function G on $|z| \le r$ such that $G(0) = 0$ and $f(z) = 1 - e^{-G(z)}$. It follows from Lemma 4.2.1 with $\mathcal{K} = \{z \in \mathbb{C} : |z| \le r\}$ and $a = 0$ that $G = g$. Thus, $\mathcal{R}_g \ge r$. Since r was arbitrary, if follows that $\mathcal{R}_g \ge \min\{|z| : |z| \le \mathcal{R}_f$ and $f(z) = 1\}$.
g) '$\Rightarrow$' Part f) implies that $\mathcal{R}_f = \mathcal{R}_g$. It follows from theorem 3.2.2 with $\alpha_n = \mathcal{R}_f^n$ that $(\delta_{0n} - f_n)_{n\in\mathbb{N}} \in \exp \ell_1(\alpha)$. Since $f = 1 - e^{-g}$, lemma 3.2.1 implies that $(g_n)_{n\in\mathbb{N}} \in \ell_1(\alpha)$.
'$\Leftarrow$' This follows from $f = 1 - e^{-g}$.
h) It follows from part f) that $\mathcal{R}_f < \mathcal{R}_g$ and that $\sum_{n=0}^{\infty} f_n\, \theta^n = 1$ for some θ with $|\theta| = \mathcal{R}_g$. Suppose $\sum_{n=0}^{\infty} |g_n|\, \mathcal{R}_g^n < \infty$. Then $\lim_{z\to\theta, |z|<\theta} f(z) = 1 - e^{-g(\theta)} \neq 1$, which contradicts $f(\theta) = 1$.
i) It follows from part $f)$ that $\mathcal{R}_f = \mathcal{R}_g$. Suppose $\sum_{n=0}^{\infty} |g_n|\, \mathcal{R}_g^n < \infty$. Since $\sum_{n=0}^{\infty} |f_n|\, \mathcal{R}_f^n < \infty$, we have $\lim_{z\to\theta, |z|<\theta} \sum_{n=0}^{\infty} f_n\, \theta^n = 1 - e^{-g(\theta)} \neq 1$, which contradicts $\sum_{n=0}^{\infty} f_n\, \theta^n = 1$. □

Remark 5.3.2 a) The converse of Theorem 5.3.1d is not true: consider e.g. $q_n(x) = \frac{x^n}{n!}$. Thus the set of sequences of polynomials of convolution type with $f_n \ge 0$ for all $n \in \mathbb{N}$ is a proper subset of the set of sequences of polynomials of convolution type with $g_n \ge 0$ for all $n \in \mathbb{N}$. This corresponds to the fact in probability theory that the class of compound geometric distributions is a proper subclass of the class of compound Poisson distributions.

b) From theorem 5.4.1 we see that if $f_n \geq 0$ for all $n \in \mathbb{N}$, then $(q_n(1))_{n\in\mathbb{N}}$ is an extended renewal sequence (see [Sta3, p. 185]).
c) If $f_n \geq 0$ for all $n \in \mathbb{N}$, then: '$f(\mathcal{R}_f) < 1 \Rightarrow \mathcal{R}_g = \mathcal{R}_f$ and $g(\mathcal{R}_g) < \infty$' ; '$f(\mathcal{R}_f) = 1 \Rightarrow \mathcal{R}_g = \mathcal{R}_f$ and $g(\mathcal{R}_g) = \infty$' and '$f(\mathcal{R}_f) > 1$ or $\mathcal{R}_f = \infty \Rightarrow \mathcal{R}_g < \mathcal{R}_f$ and $g(\mathcal{R}_g) = \infty$'(cf. Theorems 5.3.1g and 5.3.1h).

In [224] the usual renewal theory conditions are assumed. We now show that these conditions on $(f_n)_{n\in\mathbb{N}}$ imply that $\sum_{n=0}^{\infty} g_n\, z^n$ has a dominant logarithmic singularity on its circle of convergence. This explains why the centering and scaling constants of the central limit theorem in [224] do not depend on $(f_n)_{n\in\mathbb{N}}$ (cf. [224, p. 191, last paragraph]). An extension of the central limit theorem in [224] will be given in Section 5.4.

Theorem 5.3.3 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$ and let $(f_n)_{n\in\mathbb{N}}$ be the unique sequence of complex numbers such that $q_n(x) = \sum_{k=0}^{n} f_n^{k*} \binom{x+k-1}{k}$. Let $\mathcal{R}_f$, $\mathcal{R}_g$ be the radius of convergence of $\sum_{n=0}^{\infty} f_n\, z^n$, $\sum_{n=0}^{\infty} g_n\, z^n$ respectively. Suppose that:*

1. $0 < \mathcal{R}_g < \infty$ *and* $\lim_{r\uparrow\mathcal{R}_g} \mathit{Re}\ g(r) = +\infty$
2. $\displaystyle\sum_{n=0}^{\infty} n|q_n(-1)|\mathcal{R}_g^n < \infty$
3. $\displaystyle\sum_{n=0}^{\infty} -nq_n(-1)\mathcal{R}_g^n \neq 0$
4. $\displaystyle\sum_{n=0}^{\infty} -q_n(-1)z^n \neq 1$ *for* $|z| = 1, z \neq 1$.

Then there exists a sequence of polynomials $(r_n)_{n\in\mathbb{N}}$ of convolution type with coefficient sequence $(h_n)_{n\in\mathbb{N}}$ such that

a) $\displaystyle\mathcal{R}_g^n q_n(x) = \sum_{k=0}^{n} r_k(x)\binom{x+n-k-1}{n-k}$

b) $\displaystyle\sum_{n=0}^{\infty} |r_n(x)| < \infty$ *for all* $x \in \mathbb{C}$

c) $\displaystyle\sum_{n=0}^{\infty} |h_n| < \infty$

d) $g(\mathcal{R}_g z) = -\mathcal{L}og(1-z) + h(z)$ *for* $|z| < 1$.

Moreover, $\displaystyle\lim_{n\to\infty} q_n(1)\,\mathcal{R}_g^n = \left(\sum_{n=0}^{\infty} -n\,q_n(-1)\,\mathcal{R}_g^n\right)^{-1}$ *and* $\displaystyle\sum_{n=0}^{\infty} |g_n\,\mathcal{R}_g^n - n^{-1}| < \infty$.

Proof: First note that 2) implies $\sum_{n=0}^{\infty} |q_n(-1)| \mathcal{R}_g^n < \infty$. Hence,

$$\sum_{n=0}^{\infty} q_n(-1) \mathcal{R}_g^n = \lim_{r \uparrow \mathcal{R}_g} e^{-g(r)} = \lim_{r \uparrow \mathcal{R}_g} e^{-\text{Re } g(r)} = 0.$$

Define the sequence $(\beta_k)_{k \in \mathbb{N}}$ by $\beta_k := \sum_{i=k+1}^{\infty} -q_i(-1) \mathcal{R}_g^i$. Thus $\beta_0 = 1$ and it follows from 2) that $(\beta_k)_{k \in \mathbb{N}} \in \ell_1$. We now show that $\sum_{k=0}^{\infty} \beta_k z^k \neq 0$ for $|z| \leq 1$. If $|z| \leq 1$ and $z \neq 1$, then

$$\sum_{k=0}^{\infty} \beta_k z^k = \sum_{k=0}^{\infty} z^k \sum_{i=k+1}^{\infty} -q_i(-1) \mathcal{R}_g^i =$$

$$\sum_{i=1}^{\infty} -q_i(-1) \mathcal{R}_g^i \sum_{k=0}^{i-1} z^k = \sum_{i=1}^{\infty} -q_i(-1) \mathcal{R}_g^i \frac{z^i - 1}{z - 1} =$$

$$\frac{1}{1-z} \sum_{i=1}^{\infty} q_i(-1) \mathcal{R}_g^i (z^i - 1) = \frac{1}{1-z} \left(1 - \sum_{i=0}^{\infty} q_i(-1) (\mathcal{R}_g z)^i \right),$$

since $\sum_{i=0}^{\infty} -q_i(-1) \mathcal{R}_g^i = 0$ and $q_0 = 1$. Thus 4) implies $\sum_{k=0}^{\infty} \beta_k z^k \neq 0$ for $|z| = 1$, $z \neq 1$. Moreover, since $\sum_{i=0}^{\infty} q_i(-1) (\mathcal{R}_g z)^i = e^{-g(\mathcal{R}_g z)}$ for $|z| < 1$, we have $\sum_{k=0}^{\infty} \beta_k z^k \neq 0$ for $|z| < 1$. Finally, we get from 3) that

$$\sum_{k=0}^{\infty} \beta_k = \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} -q_i(-1) \mathcal{R}_g^i =$$

$$\sum_{i=1}^{\infty} \sum_{k=0}^{i-1} -q_i(-1) \mathcal{R}_g^i = \sum_{i=1}^{\infty} -q_i(-1) \mathcal{R}_g^i\, i = \left(1 - \sum_{i=0}^{\infty} -q_i(-1) \mathcal{R}_g^i\, i \right) \neq 0.$$

We conclude that $\sum_{k=0}^{\infty} \beta_k z^k \neq 0$ for $|z| \leq 1$. Since $\beta_0 = 1$, it follows from Theorem 4.2.2 with $\alpha_n = 1$ that there exists a sequence $(\gamma_n)_{n \in \mathbb{N}} \in \ell_1$ such that $\gamma_0 = 0$ and $\sum_{k=0}^{\infty} \beta_k z^k = \exp\left(\sum_{n=0}^{\infty} \gamma_n z^n\right)$ for $|z| \leq 1$. Define h_n by $h_n := -\gamma_n$ for all $n \in \mathbb{N}$ and write $h(z) = \sum_{n=0}^{\infty} h_n z^n$. Let $(r_n)_{n \in \mathbb{N}}$ be the sequence of polynomials of convolution type with coefficient sequence $(h_n)_{n \in \mathbb{N}}$, i.e. $r_n(x) = \sum_{k=0}^{n} h_n^{k*} \frac{x^k}{k!}$. It follows from $\sum_{k=0}^{\infty} \beta_k z^k = (1-z)^{-1} e^{-g(\mathcal{R}_g z)}$ that $e^{g(\mathcal{R}_g z)} = (1-z)^{-1} e^{h(z)}$. Since $g_0 = 0$, we have $g(\mathcal{R}_g z) = -\log(1-z) + h(z)$ for $|z| < 1$. Hence, $\mathcal{R}_g^n q_n(x) = \sum_{k=0}^{n} r_k(x) \binom{x+n-k-1}{n-k}$. This proves a) through d).

For the remaining statements, observe that $\mathcal{R}_g^n q_n(1) = \sum_{k=0}^{n} r_k(1)$. Since $\sum_{k=0}^{\infty} \beta_k z^k = \sum_{i=0}^{\infty} -q_i(-1) \mathcal{R}_g^i\, i$, it follows that

$$\lim_{n \to \infty} \mathcal{R}_g^n q_n(1) = \sum_{k=0}^{\infty} r_k(1) = e^{-\gamma(1)} = \left(\sum_{n=0}^{\infty} -n\, q_n(-1) \mathcal{R}_g^n \right)^{-1}.$$

Finally, it follows from $g(\mathcal{R}_g z) = -\log(1-z) + h(z)$ for $|z| < 1$ and $(h_n)_{n \in \mathbb{N}} \in \ell_1$ that $\sum_{n=0}^{\infty} |g_n \mathcal{R}_g^n - n^{-1}| < \infty$. □

Remark 5.3.4 a) If $(f_n)_{n\in\mathbb{N}}$ is a probability distribution and g.c.d $\{n \geq 1 : f_n \neq 0\} = 1$, then $\sum_{n=0}^{\infty} f_n\, z^n \neq 1$ for $|z| = 1$, $z \neq 1$ (see [131, Theorem 3.6.1]). Thus if $(f_n)_{n\in\mathbb{N}}$ is a probability distribution, then condition 4) of Theorem 5.3.3 is implied by aperiodicity of $(f_n)_{n\in\mathbb{N}}$, since $f_n = -q_n(-1)$ for $n \geq 1$ by Theorem 5.3.1c (cf. [224]).
b) The statement $\lim_{n\to\infty} q_n(1)\,\mathcal{R}_g^n = \left(\sum_{n=0}^{\infty} -n\, q_n(-1)\,\mathcal{R}_g^n\right)^{-1}$ is an extension of the Discrete Renewal Theorem (cf. [93, Chapter 11]).

We now apply the previous theorems to obtain a theorem similar to [224, Theorem 4] (recall that $f_n = -q_n(-1)$ for $n \geq 1$ by Theorem 5.3.1c).

Theorem 5.3.5 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. Suppose that:*

1. *the power series $g(z) = \sum_{n=1}^{\infty} g_n\, z^n$ has a positive, finite radius of convergence $\mathcal{R}_g$ and $\lim_{r\uparrow\mathcal{R}_g} Re\; g(r) = +\infty$*
2. $\sum_{n=0}^{\infty} n\,|q_n(-1)|\,\mathcal{R}_g^n < \infty$
3. $\sum_{n=0}^{\infty} -n\, q_n(-1)\,\mathcal{R}_g^n \neq 0$
4. $\sum_{n=0}^{\infty} -q_n(-1)\, z^n \neq 1$ *for* $|z| = 1, z \neq 1$.

Then $\lim_{n\to\infty} q_n(x)/q_n(1) = 0$ *for* $|x| < 1$.

Proof: It follows from Theorem 5.3.3 that $\mathcal{R}_g^n\, q_n(x) = \sum_{k=0}^{n} r_k(x)\binom{x+n-k-1}{n-k}$ with $(r_n(x))_{n\in\mathbb{N}} \in \text{inv}\ \ell_1$ for all $x \in \mathbb{C}$. If $|x| < 1$, then $\lim_{n\to\infty}\binom{x+n-1}{n} = 0$ by Lemma 5.1.3. It follows from dominated convergence that

$$\lim_{n\to\infty} \sum_{k=0}^{n} r_k(x)\binom{x+n-k-1}{n-k} = 0.$$

Moreover, $\lim_{n\to\infty} \mathcal{R}_g^n\, q_n(1) = \sum_{n=0}^{\infty} r_n(1) \neq 0$ since $(r_n(1))_{n\in\mathbb{N}} \in \text{inv}\ \ell_1$. Hence, $\lim_{n\to\infty} q_n(x)/q_n(1) = 0$ for $|x| < 1$. □

5.4 Logarithmic singularities

In this section we prove a central limit theorem for random variables $Y_n^{(\lambda)}$ with probability generating function $q_n(\lambda x)/q_n(\lambda)$, where $(q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$ such that

$\sum_{n=0}^{\infty} g_n z^n$ has a dominant logarithmic singularity on its circle of convergence. We derive our central limit theorem from the asymptotic behaviour of q_n. Similar results have been obtained by Flajolet and Soria (see [96, 97]). Contrary to Flajolet and Soria, we do not use contour integration. Instead, our method relies on simple estimates and a Banach algebra theorem from Chapter 4. Consequently, our conditions on g are different (probably incomparable) from those in [96, 97]. Note that the function R in [96, Definition on p. 169] should satisfy $R(z) = K + o\left((\log(1 - z/\rho))^{-1}\right)$ instead of $R(z) = K + o(1)$ (see [97, p. 11]).

We start with determining the asymptotic behaviour of $q_n(x)/q_n(1)$. This asymptotic behaviour will be used in Theorem 5.4.3 to obtain a central limit theorem.

Theorem 5.4.1 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of convolution type with coefficient sequence $(g_n)_{n\in\mathbb{N}}$. Suppose that the $\mathcal{R}_g$, the radius of convergence of $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is positive and finite. Define $(h_n)_{n\in\mathbb{N}}$ by $h_n := \mathcal{R}_g^n g_n - n^{-1}$ and let $(r_n)_{n\in\mathbb{N}}$ be the sequence of polynomials of convolution type with coefficient sequence $(h_n)_{n\in\mathbb{N}}$.*

If $\sum_{n=0}^{\infty} |r_n(x)| < \infty$ for all $x > 0$, then $\mathcal{R}_g^n q_n(x) \sim \dfrac{n^{x-1}}{\Gamma(x)} \sum_{k=0}^{\infty} r_k(x)$ as $n \to \infty$ for fixed $x \geq 1$. If moreover $r_n(x) = O(n^{-1})$ as $n \to \infty$ for a fixed x with $0 < x < 1$, then $\mathcal{R}_g^n q_n(x) \sim \dfrac{n^{x-1}}{\Gamma(x)} \sum_{k=0}^{\infty} r_k(x)$ as $n \to \infty$.

Proof: The definition of h_n implies that $\mathcal{R}_g^n q_n(x) = \sum_{k=0}^{n} r_k(x) \binom{x+n-k-1}{n-k}$ for all $n \in \mathbb{N}$.

I. $x \geq 1$

By Lemma 5.1.3, $\lim_{n\to\infty} \binom{x+n-k-1}{n-k} n^{1-x} = 1/\Gamma(x)$ for fixed k with $0 \leq k \leq n$. Since $x \geq 1$, $\binom{x+n-k-1}{n-k} n^{1-x} \leq \binom{x+n-1}{n} n^{1-x} \leq C_1$ by Lemmas 5.1.1a and 5.1.3. Since $\sum_{n=0}^{\infty} |r_n(x)| < \infty$, dominated convergence yields $\mathcal{R}_g^n q_n(x) \sim n^{x-1}/\Gamma(x) \sum_{k=0}^{\infty} r_k(x)$ as $n \to \infty$.

II. $0 < x < 1$

We first evaluate $\lim_{n\to\infty} n^{1-x} \sum_{k=0}^{[n/2]} \binom{x+n-k-1}{n-k}$. If $k \leq [n/2]$, then $n^{1-x} \leq 2^{1-x}(n-k)^{1-x}$ and $n^{1-x}\binom{x+n-k-1}{n-k}$ is uniformly bounded by Lemma 5.1.3. Hence, $\lim_{n\to\infty} \sum_{k=0}^{[n/2]} r_k(x) n^{1-x} \binom{x+n-k-1}{n-k} = \sum_{k=0}^{\infty} r_k(x)/\Gamma(x)$ by dominated convergence. We are done if we prove that

$$\lim_{n\to\infty} n^{1-x} \sum_{k=[n/2]+1}^{n} r_k(x) \binom{x+n-k-1}{n-k} = 0.$$

Fix an α with $1 < \alpha < (1-x)^{-1}$. Applying Lemma 5.1.4 and Hölder's inequality, we obtain

$$\begin{aligned}
n^{1-x} \sum_{k=[n/2]+1}^{n} \left| r_k(x) \binom{x+n-k-1}{n-k} \right| &\leq \\
n^{1-x} \left(\sum_{k=[n/2]+1}^{n} |r_k(x)|^{1+1/(\alpha-1)} \right)^{1-1/\alpha} \left(\sum_{j=0}^{[n/2]} \binom{x+j-1}{j}^{\alpha} \right)^{1/\alpha} &\leq \\
C_1\, n^{1-x} \left(\sum_{k=[n/2]+1}^{n} |r_k(x)|^{1/(\alpha-1)}\, |r_k(x)| \right)^{1-1/\alpha} [n/2]^{x-1+1/\alpha} &\leq \\
C_2 \left(\max_{k \geq [n/2]} |\, r_k(x)|^{1/\alpha} \right) \left(\sum_{k=[n/2]+1}^{n} |r_k(x)| \right)^{1-1/\alpha} n^{1/\alpha} &\leq \\
C_3 \left(\sum_{k=[n/2]+1}^{n} |r_k(x)| \right)^{1-1/\alpha} &= o(1)
\end{aligned}$$

as $n \to \infty$. □

If in Theorem 5.4.1, $(r_n)_{n\in\mathrm{N}}$ satisfies additional conditions as positivity or monotonicity, then Theorem 5.4.1 can be obtained from Tauber theorems (cf. [93, Chapter 8.5]).

The next theorem is a central limit theorem for a sequence of random variables $(Y_n^{(\lambda)})_{n\in\mathrm{N}}$, where $Y_n^{(\lambda)}$ has probability generating function $q_n(\lambda x)/q_n(\lambda)$. For an interpretation of $Y_n^{(\lambda)}$ in terms of a compound Poisson process, see [222]. For examples of combinatorial interpretations of $Y_n^{(1)}$, see Examples 5.4.5. We first need a lemma.

Lemma 5.4.2 *If $(q_n)_{n\in\mathrm{N}}$ is a sequence of polynomials of convolution type such that $\sum_{n=0}^{\infty} |q_n(x)| < \infty$ for all $x > 0$, then $\lim_{t\to s} |q_n(t) - q_n(s)| = 0$ for all $s > 0$.*

Proof: Consider the separable Banach algebra ℓ_1 with convolution as multiplication. Define $f : (0,\infty) \to \ell_1$ by $f(t) := (q_n(t))_{n\in\mathrm{N}}$. Since the polynomials q_n are of convolution type, we have $f(u+v) = f(u) * f(v)$ for all $u, v > 0$. If $y = (y_n)_{n\in\mathrm{N}} \in (\ell_1)^* = \ell_\infty$, then $< ft), y >= \sum_{n=0}^{\infty} q_n(t)\, y_n$. Thus, f is weakly measurable in ℓ_1 according to [118, Definition 3.5.4] and strongly measurable by [118, Corollary 2, p. 73]. The theorem now follows from [118, Theorem 9.3.1].

Theorem 5.4.3 *Let $(q_n)_{n\in\mathrm{N}}$ be a sequence of polynomials of convolution type with non-negative coefficients and with coefficient sequence $(g_n)_{n\in\mathrm{N}}$. Define*

$(h_n)_{n\in\mathbb{N}}$ by $h_n := \mathcal{R}_g^n\, g_n - n^{-1}$ and let $(r_n)_{n\in\mathbb{N}}$ be the sequence of polynomials of convolution type with coefficient sequence $(h_n)_{n\in\mathbb{N}}$. Suppose that $\mathcal{R}_g$, the radius of convergence of $g(z) = \sum_{n=0}^{\infty} g_n\, z^n$, is finite and positive and suppose that $\sum_{n=0}^{\infty} |r_n(x)| < \infty$ for all $x > 0$. Let the random variable $Y_n^{(\lambda)}$ have probability generating function $q_n(\lambda x)/q_n(\lambda)$ for each $n \in \mathbb{N}$. Then the distribution of $(Y_n^{(\lambda)} - \lambda \log n)\,(\lambda \log n)^{-1/2}$ converges to the standard normal law for all $\lambda \geq 1$. If $0 < \lambda < 1$ and $r_n(x) = O(n^{-1})$ uniformly in a real neighbourhood of λ as $n \to \infty$, then the distribution of $(Y_n^{(\lambda)} - \lambda \log n)\,(\lambda \log n)^{-1/2}$ converges to the standard normal law.

Proof: Write $a_n := (\lambda \log n)^{1/2}$. Because $Y_n^{(\lambda)}$ has probability generating function $q_n(\lambda x)/q_n(\lambda)$, $(Y_n^{(\lambda)} - \lambda \log n)\,(\lambda \log n)^{-1/2}$ has moment generating function $q_n\left(\lambda\, e^{z/a_n}\right) e^{-a_n z}/q_n(\lambda)$. By Lemma 5.1.5, it suffices to prove $\lim_{n\to\infty} q_n\left(\lambda\, e^{z/a_n}\right)\, e^{-a_n z}\,/q_n(\lambda) = e^{\frac{1}{2}z^2}$ for all $z > 0$.

Recall that in both cases ($\lambda \geq 1$ and $0 < \lambda < 1$) we have

$$\lim_{n\to\infty} n^{1-\lambda}\, \mathcal{R}_g^n\, q_n(\lambda) = \sum_{k=0}^{\infty} r_k(\lambda)$$

by Theorem 5.4.1. Thus it suffices to prove

$$\lim_{n\to\infty} n^{1-\lambda}\, \mathcal{R}_g^n\, q_n\left(\lambda\, e^{z/a_n}\right)\, e^{-a_n z} = e^{\frac{1}{2}z^2}\, \frac{1}{\Gamma(\lambda)} \sum_{k=0}^{\infty} r_k(\lambda).$$

Since $\mathcal{R}_g^n\, q_n(x) = \sum_{k=0}^{n} r_k(x)\binom{x+n-k-1}{n-k}$, we may write

$$n^{1-\lambda}\, \mathcal{R}_g^n\, q_n\left(\lambda e^{z/a_n}\right) e^{-a_n z} = n^{1-\lambda} \sum_{k=0}^{n} r_k(\lambda\, e^{z/a_n}) \binom{\lambda + n - k - 1}{n-k} \varphi_{nk}(z),$$

where

$$\varphi_{nk}(z) := e^{-a_n z} \binom{\lambda\, e^{z/a_n} + n - k - 1}{n-k} \binom{\lambda + n - k - 1}{n-k}^{-1}.$$

I. $\lambda \geq 1$

We now write

$$n^{1-\lambda} \sum_{k=0}^{n} r_k\left(\lambda\, e^{z/a_n}\right) \binom{\lambda + n - k - 1}{n-k} \varphi_{nk}(z) = T_1(n) + T_2(n),$$

where

$$T_1(n) := n^{1-\lambda} \sum_{k=0}^{n} \left\{ r_k\left(\lambda e^{z/a_n}\right) - r_k(\lambda) \right\} \binom{\lambda + n - k - 1}{n-k} \varphi_{nk}(z)$$

and

$$T_2(n) := n^{1-\lambda} \sum_{k=0}^{n} r_k(\lambda) \binom{\lambda + n - k - 1}{n - k} \varphi_{nk}(z).$$

By Lemma 5.1.2, we have $\lim_{n\to\infty} \varphi_{nk}(z) = e^{\frac{1}{2}z^2}$ for fixed k with $0 \le k \le n$. For $\lambda \ge 1$ and $0 \le k \le n$, $n^{1-\lambda} \le (n-k)^{1-\lambda}$, thus $n^{1-\lambda}\binom{\lambda+n-k-1}{n-k}$ is uniformly bounded in n and k with $0 \le k \le n$ by Lemma 5.1.3. Applying Lemma 5.1.2 and the dominated convergence theorem, we obtain $\lim_{n\to\infty} T_2(n) = e^{\frac{1}{2}z^2} \frac{1}{\Gamma(\lambda)} \sum_{k=0}^{\infty} r_k(\lambda)$. Since $\varphi_{nk}(z)$ is uniformly bounded by Lemma 5.1.2, we have

$$\lim_{n\to\infty} |T_1(n)| \le C \lim_{n\to\infty} \sum_{k=0}^{\infty} \left| r_k\left(\lambda\, e^{z/a_n}\right) - r_k(\lambda)\right| = 0$$

by Theorem 5.4.2.

II. $0 < \lambda < 1$

We first evaluate $\lim_{n\to\infty} n^{1-\lambda} \sum_{k=0}^{[n/2]} r_k(\lambda\, e^{z/a_n}) \binom{\lambda+n-k-1}{n-k} \varphi_{nk}(z)$. If $k \le [n/2]$, then $n^{1-\lambda} \le 2^{1-\lambda}\,(n-k)^{1-\lambda}$, thus $n^{1-\lambda}\binom{\lambda+n-k-1}{n-k}$ is uniformly bounded by Lemma 5.1.3. By Lemma 5.1.2, $\varphi_{nk}(z)$ is uniformly bounded. Hence,

$$\lim_{n\to\infty} \sum_{k=0}^{[n/2]} \left\{ r_k(\lambda e^{z/a_n}) - r_k(\lambda)\right\} n^{1-\lambda} \binom{\lambda + n - k - 1}{n - k} \varphi_{nk}(z) \le$$

$$\lim_{n\to\infty} \sum_{k=0}^{[n/2]} \left| r_k\left(\lambda\, e^{z/a_n}\right) - r_k(\lambda)\right| = 0$$

by Theorem 5.4.2 and

$$\lim_{n\to\infty} \sum_{k=0}^{[n/2]} r_k(\lambda)\, n^{1-\lambda} \binom{\lambda + n - k - 1}{n - k} \varphi_{nk}(z) = \sum_{k=0}^{\infty} r_k(\lambda) \frac{1}{\Gamma(\lambda)} e^{\frac{1}{2}z^2}$$

by dominated convergence. We are done if we prove that

$$\lim_{n\to\infty} n^{1-\lambda} \sum_{k=[n/2]+1}^{n} r_k\left(\lambda\, e^{z/a_n}\right) \binom{\lambda + n - k - 1}{n - k} \varphi_{nk}(z) = 0.$$

Fix an α with $1 < \alpha < (1-\lambda)^{-1}$. Applying Lemma 5.1.4 and Hölder's inequality, we obtain

$$n^{1-\lambda} \sum_{k=[n/2]+1}^{n} \left| r_k \left(\lambda\, e^{z/a_n} \right) \binom{\lambda + n - k - 1}{n - k} \right| \leq$$

$$n^{1-\lambda} \left(\sum_{k=[n/2]+1}^{n} \left| r_k \left(\lambda e^{z/a_n} \right) \right|^{1+1/(\alpha-1)} \right)^{1-1/\alpha} \left(\sum_{j=0}^{[n/2]} \binom{\lambda + j - 1}{j}^{\alpha} \right)^{1/\alpha} \leq$$

$$C_1\, n^{1-\lambda} \left(\sum_{k=[n/2]+1}^{n} \left| r_k \left(\lambda\, e^{z/a_n} \right) \right|^{1+1/(\alpha-1)} \right)^{1-1/\alpha} \left[\frac{n}{2} \right]^{\lambda-1+1/\alpha} \leq$$

$$C_2 \max_{k \geq [n/2]} \left| r_k \left(\lambda\, e^{z/a_n} \right) \right|^{1/\alpha} \left(\sum_{k=[n/2]+1}^{n} \left| r_k \left(\lambda\, e^{z/a_n} \right) \right| \right)^{1-1/\alpha} n^{1/\alpha} \leq$$

$$C_3 \left(\sum_{k=[n/2]+1}^{n} \left| r_k \left(\lambda\, e^{z/a_n} \right) \right| \right)^{1-1/\alpha} .$$

Using Theorem 5.4.2, we see that

$$\lim_{n \to \infty} \sum_{k=[n/2]+1}^{n} \left| r_k \left(\lambda\, e^{z/a_n} \right) \right| \leq$$

$$\lim_{n \to \infty} \sum_{k=[n/2]+1}^{n} \left| r_k \left(\lambda\, e^{z/a_n} \right) - r_k(\lambda) \right| + \lim_{n \to \infty} \sum_{k=[n/2]+1}^{n} |r_k(\lambda)| = 0.$$

□

Remark 5.4.4 If in Theorem 5.4.1 or 5.4.3 we have $(h_n)_{n \in \mathbb{N}} \in \ell_1$ and $h_n = O(n^{-1})$, then $r_n(x) = O(n^{-1})$ uniformly in an interval around λ ($\lambda > 0$) as $n \to \infty$ as the following proof shows. Consider the algebra O of all sequences $a \in \ell_1$ such that $|a_n| = O(n^{-1})$ with componentwise addition and convolution as multiplication. Equipped with norm $\| a \| := \| a \|_1 + u\; n|a_n|$, this algebra becomes a Banach algebra. Hence, $r_n(x) = O(n^{-1})$ for all $x \in \mathbb{C}$. We now set out to prove the uniform $O(n^{-1})$ property. Define $(b_n)_{n \in \mathbb{N}}$ by $b_n := |h_n|$ and let $(v_n)_{n \in \mathbb{N}}$ be the unique sequence of polynomials of convolution type with coefficient sequence $(b_n)_{n \in \mathbb{N}}$. Hence, if $\mu > \lambda$ and $0 < x \leq \mu$, then $r_n(x) \leq v_n(x) \leq v_n(\mu) \leq C \mu\, n^{-1}$.

Examples 5.4.5 a) It follows from Remark 5.4.4 that the following sequences of polynomials satisfy the conditions of Theorems 5.4.1 and 5.4.3 ($(g_n)_{n \in \mathbb{N}}$ is the coefficient sequence of $(q_n)_{n \in \mathbb{N}}$ and $g(z) = \sum_{n=0}^{\infty} g_n\, z^n$):

1. the derangement polynomials with $g(z) = -\log(1 - z) + z$ (this solves the open problem of [47, p. 20]). These polynomials count the number

of cycles in derangements, i.e. permutations without cycles of length one. Thus $P(Y_n^{(1)} = k)$ is the probability that a random derangement of $\{1, \ldots, n\}$ has k cycles.

2. the polynomials with $g(z) = -\log(1-z) + z + \frac{1}{2}z^2$. These polynomials count the number of connected components in 2-regular graphs. Thus $P(Y_n^{(1)} = k)$ is the probability that a random 2-regular graph with n points has k components (cf. [96, pp. 174-175]).

The other examples given in [96, pp. 173-175] also satisfy the conditions of Theorems 5.4.1 and 5.4.3.

b) Consider the Mittag-Leffler polynomials of Example 3.1.3b with $g(z) = -\log(1-z) + \log(1+z)$ (see also [202, p. 75]). Here $P(Y_n^{(\frac{1}{2})} = k)$ is the probability that a random permutation of $\{1, \ldots, n\}$ without cycles of even length has k cycles. Note that the polynomials of convolution type associated to $\log(1+z)$ are the polynomials $\binom{x}{n}$. It follows from Raabe's convergence test that $\sum_{n=0}^{\infty} \left|\binom{x}{n}\right| < \infty$ for $x > 0$. In the terminology of Theorem 5.4.3, the Mittag-Leffler polynomials are an example of a sequence of polynomials such that $\sum_{n=0}^{\infty} |r_n(x)| < \infty$ for all $x > 0$ and $\sum_{n=0}^{\infty} |h_n| = \infty$ (cf. Remark 5.4.4). The uniform $O(n^{-1})$ condition necessary for Theorem 5.4.3 follows from $n\binom{x}{n} = x \left|\frac{(x-1)\ldots(x-(n-1))}{1\ldots n-1}\right| \le x$ for $0 \le x < 1$.

c) Consider the polynomials q_n with $g(z) = z - \log(1-z^2)$. We will show that the asymptotic behaviour of q_n is different for even n and odd n. We have

$$q_{2n}(x) = \sum_{k=0}^{n} \binom{-x}{k} (-1)^k \frac{x^{2n-2k}}{(2n-2k)!}$$

and

$$q_{2n+1}(x) = \sum_{k=0}^{n} \binom{-x}{k} (-1)^k \frac{x^{2n+1-2k}}{(2n+1-2k)!}.$$

Since $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{1}{2}(e^x + e^{-x})$ and $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{1}{2}(e^x - e^{-x})$, [86, Lemma 2.2.] and Lemma 5.1.3 yield $q_{2n}(x) \sim \frac{n^{x-1}}{\Gamma(x)} \frac{1}{2}(e^x + e^{-x})$ and $q_{2n+1}(x) \sim \frac{1}{2}(e^x - e^{-x})$. In spite of the different asymptotic behaviour, there exists a central limit theorem (same proof as Theorem 5.3.2).

5.5 Infinitely divisible probability measures on $\mathbb{N}$

In this section we show that using the Banach algebra theory developed in Chapter 4 and Section 5.2, it is possible to give a more transparent proof of the main result of [87]. The proofs in [87] use Banach algebra results from [61].

Definition 5.5.1 *A probability generating function P is said to be infinitely divisible if for all $k \geq 1$ there exists a probability generating function P_k such that $P = (P_k)^k$.*

For more information on infinitely divisible probability measures, see [93, Chapter 17] and [230, 233].

We now consider infinitely divisible probability measures on $\mathbb{N}$. It follows from the Lévy-Hinčin representation (see [93, Chapter 17, Section 2] or [151, Theorem 5.5.1]; for a proof using Choquet theory see [127]) that if μ is a probability measure on $\mathbb{N}$ with infinitely divisible probability generating function, then there exists a measure ν on $\mathbb{N}$ (the **Lévy-measure**) such that

$$\sum_{n=0}^{\infty} \mu_n \, z^n = \exp \left\{ -\lambda + \sum_{k=1}^{\infty} \nu_k \, z^k \right\} \tag{5.1}$$

for all $|z| \leq 1$, $\mu_n := \mu\{n\}$, $\nu_n := \nu\{n\}$ and $\lambda := \sum_{k=1}^{\infty} \nu_k$.

As an illustration of the Banach algebra theory of Chapter 4, we now prove the Lévy-Hinčin theorem for infinitely divisible probability generating functions. For a simple real analysis proof of this theorem, see [92, Section 12.2].

Lemma 5.5.2 *Let $P(z) = \sum_{n=0}^{\infty} p_n \, z^n$ be an infinitely divisible probability generating function. Then $\sum_{n=0}^{\infty} p_n \, z^n \neq 0$ for $|z| \leq 1$.*

Proof: It follows from [151, th. 5.3.1] that $\sum_{n=0}^{\infty} p_n \, z^n \neq 0$ for $|z| = 1$. Since P is infinitely divisible, Lemma 4.3.2 yields that ind $P = 0$. Hence, the Argument Principle ([45, Corollary 5.86]) yields that $\sum_{n=0}^{\infty} p_n \, z^n \neq 0$ for $|z| \leq 1$. □

For another proof of Lemma 5.5.2, combine [151, th. 5.3.1] and [151, th. 8.4.1] (cf. [232, p. 5]).

Theorem 5.5.3 *Let $P(z) = \sum_{n=0}^{\infty} p_n \, z^n$ be an infinitely divisible probability generating function. Then there exists a sequence $(\nu_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ such that $\nu_n \geq 0$ for all $n \geq 1$ and such that $\sum_{n=0}^{\infty} \nu_n < \infty$ and $P(z) = exp \; \{\sum_{n=0}^{\infty} \nu_n \, z^n\}$ for $|z| \leq 1$.*

Proof: It follows from Lemma 5.5.2 that $\sum_{n=0}^{\infty} p_n \, z^n \neq 0$ for all $|z| \leq 1$. It follows from Theorem 4.2.2 with $\alpha_n = 1$ that there exists a sequence $(\nu_n)_{n \in \mathbb{N}} \in \ell_1$ such that $P(z) = \exp \; \{\sum_{n=0}^{\infty} \nu_n \, z^n\}$. We now set out to prove that $\nu_n \geq 0$ for $n \geq 1$. Let $(q_n)_{n \in \mathbb{N}}$ be the sequence of polynomials of convolution type with coefficient sequence $(0, \nu_1, \nu_2, \dots)$. Thus, $q_n(1) = e^{-\nu_0} \, p_n$ for all $n \in \mathbb{N}$. Since P is infinitely divisible, there exists for each integer $k \geq 2$ a sequence $(a_n)_{n \in \mathbb{N}}$ of non-negative numbers such that $a_n^{k*} = p_n$. It easily follows by induction on k that each $(a_n)_{n \in \mathbb{N}}$ is unique except for a_0. Thus infinite divisibility of P implies that $q_n(1/k) \geq 0$ for all $k, n \in \mathbb{N}$. Since $(q_n)_{n \in \mathbb{N}}$ is a sequence of polynomials of convolution type, Theorem 2.1.8 or Remark 2.1.10f implies that $\nu_n \geq 0$ for all $n \in \mathbb{N}$. □

We now give a proof of the Embrechts-Hawkes result on tails of infinitely divisible probability measures on $\mathbb{N}$.

Theorem 5.5.4 *([87, Theorem 1]) Let p be a probability measure on $\mathbb{N}$ with infinitely divisible probability generating function and Lévy measure ν. Define $\alpha_0 := -1$ and $\alpha_k := \nu_k/\lambda$ $(k \geq 1)$, where $\lambda := \sum_{k=1}^{\infty} \nu_k$. Suppose that $p_1 \neq 0$ and $\alpha_n \neq 0$ for n large enough. Then the following are equivalent:*

(i) $\alpha_n^{2*} \sim 2\,\alpha_n$ *and* $\alpha_{n+1} \sim \alpha_n$ $(n \to \infty)$

(ii) $p_n^{2*} \sim 2\,p_n$ *and* $p_{n+1} \sim p_n$ $(n \to \infty)$

(iii) $p_n \sim \lambda\,\alpha_n$ *and* $\alpha_{n+1} \sim \alpha_n$ $(n \to \infty)$.

Proof: Note that by the Lévy-Hinčin representation (5.1) and the choice $\alpha_0 = -\lambda/\lambda = -1$, we have $p = e^{\lambda\alpha}$, where $p = (p_n)_{n\in\mathbb{N}}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}}$.
'$(ii) \Rightarrow (i)$' We use the Banach algebra $\mathcal{U}_L$ of Definition 5.2.1 with $\mu_n = p_n$. First note that $p_n \neq 0$ for all $n \in \mathbb{N}$ by applying [231, Corollary on p. 813] or by using Lemma 2.1.5b to show that for $n \geq 1$ we have $e^{\lambda} p_n = \sum_{k=1}^{n} \alpha_n^{k*} \geq \alpha_1^{n*} = (\alpha_1)^n = e^{n\lambda}\,(p_1)^n/n! > 0$.
It follows from Lemma 5.5.2 that $\sum_{n=0}^{\infty} p_n\,z^n \neq 0$ for all $|z| \leq 1$. Thus $(p_n)_{n\in\mathbb{N}} \in \exp\,\mathcal{U}_L$ by Theorem 5.2.2e. Hence, $(\alpha_n)_{n\in\mathbb{N}} \in \mathcal{U}_L$. In particular, $\lim_{n\to\infty} \alpha_n/p_n = L$ exists. We now show that $L \neq 0$. If $L = 0$, then the fact that $\mathcal{U}_0$ is a Banach algebra implies that $\lim_{n\to\infty} \alpha_n^{k*}/p_n = 0$ for all $k \in \mathbb{N}$. By continuity, $\lim_{n\to\infty} p_n/p_n = 0$ because $e^{\lambda\alpha} = p$. Since this is absurd, we conclude that $L \neq 0$. Thus,

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n}\ \lim_{n\to\infty} \frac{p_{n+1}}{p_n}\ \lim_{n\to\infty} \frac{p_n}{\alpha_n} = 1.$$

It remains to prove that $\alpha_n^{2*} \sim 2\alpha_n$ $(n \to \infty)$. This follows from [86, Theorem 2.9iv].
'$(i) \Rightarrow (iii)$' Using mathematical induction on k, it follows that $\alpha_n^{k*} \sim k\,\alpha_n$ $(n \to \infty)$ (see [61, Lemma 5]). It follows from [87, Lemma 2] or [86, Theorem 2.9iii] that for each $c > 1$ there exists a positive constant A such that $\alpha_n^{k*} \leq A\,c^k\,\alpha_n$ for all $k, n \in \mathbb{N}$. Thus we may apply the dominated convergence theorem to $p_n/\alpha_n = e^{-\lambda} \sum_{k-1}^{\infty} \frac{\alpha_n^{k*}}{\alpha_n} \frac{\lambda^k}{k!}$ (this equality follows from (5.1)). We conclude that $p_n \sim \lambda\,\alpha_n$ $(n \to \infty)$.
'$(iii) \Rightarrow (ii)$' Use the real analysis proof of [87, Theorem 1]. □

Remark 5.5.5 a) The proof of '$(i) \Rightarrow (iii)$' in [87] contains some misprints, especially Formula (12).
b) For a version of Theorem 5.5.4 on $|z| \leq r$, see [87, Theorem 2].
c) The Banach algebra method works well for probability measures on $\mathbb{N}$, since the maximal ideal space of the Banach algebra involved has a simple structure (see Theorem 5.2.2c). It is possible to derive analogues of Theorem 5.5.4 for probability measures on $\mathbb{Z}$ (cf. [61, pp. 267-268]).
The Banach algebra of all complex Borel measures on $\mathbb{R}$ is much more complicated (see however [95]).

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Index

CWI TRACTS

1 D.H.J. Epema. *Surfaces with canonical hyperplane sections.* 1984.
2 J.J. Dijkstra. *Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility.* 1984.
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