CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
Telephone + 31 - 20 592 9333
Telefax + 31 - 20 592 4199
URL http://www.cwi.nl

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Nonlinear $\mathcal{H}_\infty$ control: the singular case

W.C.A. Maas

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Preface

Nonlinear $\mathcal{H}_\infty$ control has been an important research topic during the last 5 years. Motivated by the state-space solution to the linear $\mathcal{H}_\infty$ problem derived at the end of the eighties several authors have contributed to a nonlinear extension of these ideas. In most of this literature the so called regular nonlinear $\mathcal{H}_\infty$ control problem is considered. Fundamental to this theory is a certain regularity assumption. When this assumption is violated we talk about the singular nonlinear $\mathcal{H}_\infty$ control problem. In this book two approaches to this more general problem are presented. This monograph tries to bring the reader up to date with respect to the state-space solution of the nonlinear $\mathcal{H}_\infty$ control problem, with emphasis on the singular case.

The required background is some knowledge of (nonlinear) control theory as well as some understanding of the state-space solution to the linear $\mathcal{H}_\infty$ control problem. We tried to keep this book accessable for both researchers in the related fields as to graduate students by briefly recapitulating most of the important theory and providing the reader with an extensive reference list.

This book is the result of my work as a Ph.D. student at the Department of Applied Mathematics of the University of Twente in The Netherlands during the last 4 years. Needless to say several people have either directly or indirectly contributed to this book. Without claiming to be exhaustive I want to mention some of them explicitly. I would like to thank Arjan van der Schaft who gave me a perfect introduction to the area of nonlinear $\mathcal{H}_\infty$ control. Henk Nijmeijer I thank for sparking my interest in nonlinear control theory. Many thanks also to Morten Dalsmo, Harry Trentelman and Carsten Scherer for there contributions to this monograph.

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Chapter 1

Introduction

This book deals with the singular $\mathcal{H}_\infty$ problem for nonlinear systems. In this first chapter a brief overview is given of the state space approach to $\mathcal{H}_\infty$ theory both for linear and nonlinear systems. In particular we pay attention to the two approaches used to solve the linear singular $\mathcal{H}_\infty$ problem. Finally, the organization of the book is clarified and some standard notations are recalled or introduced.

1.1 A brief history

System and control theory is a relatively new research field. The essential feature of system theory is the concept of system. A system is described as a process seen as a part of reality. The interconnection of this process with the outer world is described by external signals. This description has proved to be useful to model and control all kinds of processes.

One of the most powerful techniques in system and control theory is undoubtedly feedback. Roughly speaking a feedback is a mechanism which corrects the inputs of the process based on knowledge of the outputs. A fundamental problem of theoretical and practical interest in control theory is the design of feedbacks (controllers) that lead to a desirable performance not only for our nominal process, but also for the process under all kinds of disturbances. Over the years several approaches to this problem have been developed. One approach that was studied in the sixties and the seventies is known as the linear quadratic Gaussian (LQG) design method. In this method some statistical (Gaussian) properties are ascribed to the disturbances. Another approach which originates back to the beginning of the eighties ([Za 81]) is called $\mathcal{H}_\infty$ optimization. This method can be viewed as a worst-case design methodology in the frequency domain. In fact $\mathcal{H}_\infty$ stands for the space of complex functions which are bounded and analytic in the closed right half of the complex plane. This space is called Hardy space, after the British mathematician Godfrey H. Hardy (1877-1947). The $\mathcal{H}_\infty$ optimal control theory was developed in response to the need for a synthesis procedure which explicitly incorporates modelling errors and external disturbances whose statistical nature is unknown.

Although originally this $\mathcal{H}_\infty$ optimization method was formulated in the frequency domain it can also be described in the time domain. The $\mathcal{H}_\infty$ norm from the exogenous disturbance inputs to the to-be-controlled variables in the frequency domain, used to describe the control objectives, is equal to the L_2-induced norm for the time-domain versions , under the constraint of internal stability. A breakthrough in the $\mathcal{H}_\infty$ theory for linear systems was the state-space solution to the $\mathcal{H}_\infty$ problem (see for instance [DGKF 89], [KPR 88], [Sch 91], [St 92]). The state of a system summarizes the information about the system and the external signals acting on the system insofar as relevant for the future behavior. In the state space approach to system theory we distinguish two different classes of feedbacks, namely state feedback and measurement feedback. State feedback can be used when the state can be measured completely. In other cases we can only use measurement feedback based on a possibly disturbed function of the state. Both the state feedback solution and the measurement feedback solution to the $\mathcal{H}_\infty$ problem are described using Riccati equations. The tools used to derive these results are familiar from the linear quadratic (LQ) and the linear quadratic Gaussian (LQG) problem. These tools are related to the notion of dissipativity (see [Wi 71]). The property of finite L_2-induced norm of a stable system, also called finite L_2-gain, can be characterized as the dissipativity of the system with respect to a certain supply rate.

The suboptimal $\mathcal{H}_\infty$ problem can also be formulated as a two player, zero sum linear quadratic differential game, where the disturbances are considered as the maximizing player whose goal it is to maximize a certain cost criterium, while the controls denote the minimizing player whose goal it is to minimize the same cost criterion ([BB 90], [BO 82]).

Both the theory of dissipative systems and the theory of differential games have been also developed for nonlinear systems ([Wi 72], [BO 82]). Essential in these theories are Hamilton-Jacobi equations which extend the Riccati equations used in the linear theory. Therefore it was not surprising that after the state-space solution of the $\mathcal{H}_\infty$ problem for linear systems had been found ([DGKF 89]) the approach was sought to be extended to nonlinear systems (see [vdS 91], [vdS 92], [IsAs 92], [BHW 93]). The solution of the nonlinear state feedback $\mathcal{H}_\infty$ problem was described using a Hamilton-Jacobi inequality. The nonlinear measurement feedback $\mathcal{H}_\infty$ problem is up to now not completely understood. Nevertheless sufficient conditions for the existence of controllers of a specific form solving the regular measurement feedback $\mathcal{H}_\infty$ problem have been derived ([vdS 93], [IsAs 92]). Also necessary conditions for solvability of the problem have been given.

Most of the results are concerned with the *regular* $\mathcal{H}_\infty$ problem for nonlinear systems which are affine in the inputs and the disturbances. Recently an extension of some of these results to general nonlinear systems have been made ([IsKa 95], see also [BHW 93]). The regularity of the $\mathcal{H}_\infty$ problem is concerned with certain rank conditions on the feed through matrices. When these regularity assumptions are violated we talk about the *singular* $\mathcal{H}_\infty$ problem. Singular $\mathcal{H}_\infty$ problems naturally arise when considering certain robustness problems such as parameter uncertainty and multiplicative uncertainty ([HiPr 86], [St 92]).

For linear systems the solution of the regular $\mathcal{H}_\infty$ problem has been extended to the singular case in two different ways. The two approaches are the cheap control approach (see citekpz87, [Pe 87a], [Pe 87b], [KPR 88], [ZK 88], [KPZ 90]) and the geometric approach ([StTr 90], [Sch 91], [St 91], [St 92]), which will be briefly described in the next section.

Clearly one of the open problems in the nonlinear $\mathcal{H}_\infty$ theory is the singular nonlinear $\mathcal{H}_\infty$ problem as considered in the present book. The aim of this monograph is to treat the singular nonlinear $\mathcal{H}_\infty$ control problem by generalizing the concepts and ideas used in the two approaches to the singular $\mathcal{H}_\infty$ problem for linear systems.

1.2 Two approaches to the singular $\mathcal{H}_\infty$ problem

Here we briefly describe the two different methods for solving the singular $\mathcal{H}_\infty$ problem for linear systems.

In the first approach the singular state feedback $\mathcal{H}_\infty$ problem is solved using a cheap control approach. This approach is based on regularization of the system such that the regularity assumptions are met, and the solution of the regular $\mathcal{H}_\infty$ problem can be applied. The method is similar to the regularization of the singular linear optimal control problem, commonly called 'cheap control' approach. This disturbance attenuation approach was studied in the end of the eighties (see [KPZ 87], [Pe 87a], [Pe 87b], [KPR 88], [ZK 88], [KPZ 90]). The existence of a state feedback solution to the problem is characterized by the solvability of a parameterized Riccati equality. Similar to the regular case an explicit state feedback can be given which solves the singular $\mathcal{H}_\infty$ problem once a solution to the parameterized Riccati equality is found.

A second approach uses ideas from the geometric approach to linear system theory (see [Wo 79], [BaMa 92]). The notion of strongly controllable subspace ([Ha 83]) is used to decompose the state of the system. Correspondingly the singular $\mathcal{H}_\infty$ problem can be decomposed into a regular $\mathcal{H}_\infty$ subproblem and an almost disturbance decoupling problem with stability ([Wi 81], [Wi 82], [Tr 86]). Using these ideas the solvability of the singular $\mathcal{H}_\infty$ problem can be characterized by Linear Matrix Inequalities (LMI) ([StTr 90], [St 91], [St 92], [Sch 91]). The tools used in this approach are similar to the geometric tools used to solve the singular LQ problem for linear systems.

1.3 Organization of this book

The main part of this book is concerned with the singular nonlinear state feedback $\mathcal{H}_\infty$ problem. The two approaches to solve the singular $\mathcal{H}_\infty$ problem, the cheap control approach and the geometric approach, are considered in the Chapters 3 and 5 respectively. An overview of all the chapters follows:

Chapter 2: The L_2-gain and the nonlinear $\mathcal{H}_\infty$ control problem

In this chapter we consider general nonlinear systems. Most of the material covered in this chapter is known. The notion of L_2-gain is introduced. Several characterizations of the finite L_2-gain property are given. Relevant results from the theory on dissipative systems, and about invariant manifolds and Hamiltonian systems are recapitulated.

In the second part of this chapter the $\mathcal{H}_\infty$ problem is defined for general nonlinear systems. Also some results on the regular state feedback $\mathcal{H}_\infty$ problem are recalled. A brief exposition of a way to solve the regular measurement feedback $\mathcal{H}_\infty$ problem is given. The method for finding a compensator which solves the $\mathcal{H}_\infty$ problem is provided by application of the the worst case certainty equivalence principle. This principle consists in first solving the state feedback problem and then replacing the actual state by the estimate of the state corresponding to the worst possible disturbance which is compatible with the applied input and the resulting output. This method will be explained in more detail at the end of Chapter 3 where it is used to construct a compensator which solves the singular nonlinear $\mathcal{H}_\infty$ problem.

Chapter 3: The singular $\mathcal{H}_\infty$ problem: a cheap control approach

The singular nonlinear $\mathcal{H}_\infty$ problem will be solved for nonlinear systems which are affine in the inputs and the disturbances. The results are an extension of the linear results derived using the cheap control approach for linear systems ([KPZ 87], [Pe 87a], [Pe 87b], [KPR 88], [ZK 88], [KPZ 90]). Part of these results have already appeared in [MvdS 94] and [MvdS 96].

In the first section the linear results are briefly recapitulated. The main part of this chapter is concerned with the singular nonlinear state feedback $\mathcal{H}_\infty$ problem. A sufficient, and under an extra assumption also necessary, condition for the existence of a state feedback which solves the L_2-gain problem is given in terms of the solvability of a parameterized Hamilton-Jacobi inequality. This Hamilton-Jacobi inequality corresponds to a regular $\mathcal{H}_\infty$ state feedback problem for a regularized version of the system. Another way to consider this singular problem is by linking it to the same problem for its linearization. It is shown that the state feedback $\mathcal{H}_\infty$ control problem for the linearization is solvable if and only if the state feedback $\mathcal{H}_\infty$ control problem for the nonlinear system is locally solvable. Finally in the third section we describe a way to solve the singular nonlinear measurement feedback $\mathcal{H}_\infty$ problem using the worst case certainty equivalence principle.

Chapter 4: The $\mathcal{H}_\infty$ almost disturbance decoupling problem

In Chapter 4 the almost disturbance decoupling problem is considered for affine nonlinear systems without direct feedthrough from the inputs to the to-be-controlled outputs. These results are an extension of the results considered in the publications [MT 95] and [MRST 94] for single-input single-output (SISO) systems. This chapter is instrumental for the next chapter where the singular $\mathcal{H}_\infty$ problem is decomposed into a regular $\mathcal{H}_\infty$ problem and an almost disturbance decoupling problem. The $\mathcal{H}_\infty$ almost disturbance decoupling problem considered in this chapter is a special case of a singular $\mathcal{H}_\infty$ problem, since solvability of the $\mathcal{H}_\infty$ almost disturbance decoupling problem implies that for every disturbance attenuation level $\gamma > 0$ we can solve the singular $\mathcal{H}_\infty$ problem.

Chapter 5: The singular $\mathcal{H}_\infty$ problem: a geometric approach

In this chapter we partially extend the results from the geometric approach to the linear singular $\mathcal{H}_\infty$ problem to nonlinear systems. In the first section a recapitulation of the linear results is given. In order to extend these results to nonlinear systems we need to introduce some notions from geometric nonlinear system theory. The maximal conditioned invariant distribution containing the input vector fields is introduced. This distribution constitutes a nonlinear extension of the strongly controllable subspace ([HaSi 83], [Ha 83]) used in the linear $\mathcal{H}_\infty$ theory ([StTr 90], [St 92]). Similar to the results for linear systems we can decompose the singular nonlinear $\mathcal{H}_\infty$ problem into two problems, a regular $\mathcal{H}_\infty$ problem for one subsystem and an $\mathcal{H}_\infty$ almost disturbance decoupling problem for another subsystem. In this way sufficient conditions for the solvability of the singular problem can be given. The solvability of the regular $\mathcal{H}_\infty$ subproblem will be shown to be also necessary for the solvability of the singular $\mathcal{H}_\infty$ problem. For a special class of nonlinear systems we derive necessary and sufficient conditions.

Finally a factorization idea is explored to derive conditions for solvability of the singular $\mathcal{H}_\infty$ problem. In this method we reduce the nonlinear $\mathcal{H}_\infty$ control problem to the nonlinear $\mathcal{H}_\infty$ control problem for an auxiliary system, which is easier to solve.

Chapter 6: Examples

In this chapter two examples are considered to clarify and illustrate the ideas and results from the Chapters 3, 4 and 5. First we consider the model of the orientation of a rigid body. For this rigid body model we solve the tracking problem in two different ways. First the cheap control approach is used. This method comes down to the search for a solution to the parameterized Hamilton-Jacobi inequality. The second solution is constructed using the geometric approach from Chapter 5.

As a second example the nonlinear model of the inverted pendulum on a cart is considered. We pay attention to the selection of the parameter in the cheap control approach. Finally the two feedbacks constructed using the two methods are compared.

Chapter 7: Robust stabilization under gain-bounded uncertainties

Some applications of the singular $\mathcal{H}_\infty$ theory are explained. These applications concern robustness problems like robust stabilization under parameter uncertainties and multiplicative perturbations.

Chapter 8: Conclusions

In this chapter we recapitulate the two methods for solving the singular state feedback $\mathcal{H}_\infty$ problem for nonlinear systems. We also point out the merits and the drawbacks of the two methods. Finally some remarks are made about open problems that may be investigated in the future.

Appendix A: Notions from differential geometry

In the appendix some notions and concepts from differential geometry which are used in this monograph are briefly recapitulated.

1.4 Notation

Throughout this monograph we use a fairly standard notation.
We denote for $x \in \mathbb{R}^n$ by $\|x\|^2$ or $x^T x$ the squared n-dimensional Euclidean norm. We say that $z : (0, \infty) \to \mathbb{R}^m$ is in $L_2(t_0, t_1)$ if

$$\int_{t_0}^{t_1} \|z(\tau)\|^2 \,\mathrm{d}\tau < \infty.$$

A function is C^k if the function is k times continuously differentiable. A function is called smooth if it is C^∞.
By

$$V_x(x)$$

we denote the n-dimensional row-vector of partial derivatives of a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, and by

$$V_x^T(x)$$

we denote the transposed column-vector. The Hessian matrix of the function V will be denoted by

$$V_{xx}(x).$$

Consider

$$\dot{x} = f(x, d)$$

where $d \in \mathbb{R}^q$, and x are local coordinates for a smooth state space manifold M. We denote by $x(t_1) = \varphi(t_1, t_0, x_0, d)$ the state vector at time t_1 with initial condition $x(t_0) = x_0$ and input $d : [t_0, t_1] \to \mathbb{R}^q$.

The set of complex numbers is denoted as $\mathbb{C}$. By $\mathbb{C}^-$, $\mathbb{C}^0$ and $\mathbb{C}^+$ we mean the subsets of $\mathbb{C}$ that have nonzero negative, zero, and nonzero positive real part respectively. For an $n \times n$ matrix A we denote the set of eigenvalues by $\sigma(A)$. By $\overline{\sigma}(A)$ we denote the maximal eigenvalue of the matrix A.
We end definitions, remarks, lemmas, theorems, etc., by □
while we end the proofs by ■

Chapter 2

The L_2-gain, and the nonlinear $\mathcal{H}_\infty$ control problem

Originally the $\mathcal{H}_\infty$ control problem was formulated as a design problem in the frequency domain ([Za 81]). However it can be also naturally formulated in the time-domain because for asymptotically stable linear systems the $\mathcal{H}_\infty$-norm of the transfer matrix from inputs to outputs is equal to the L_2-induced norm from the input time functions to the output time functions. This L_2-induced norm for a stable linear system is commonly called the L_2-gain of the system. Also for non-linear systems this L_2-gain can be defined and plays a crucial role in the studies of the so-called nonlinear $\mathcal{H}_\infty$ optimal control problem ([BHW 93], [IsAs 92], [vdS 91], [vdS 92]); probably the terminology nonlinear L_2-gain optimal control problem would be more appropriate. Earlier the notion of finite L_2-gain of nonlinear systems was studied in the context of input-output stability and dissipativity for nonlinear systems ([HiMo 76], [Wi 72]). Later these results were extended by using a geometric approach based on invariant manifolds of Hamiltonian systems ([vdS 91], [vdS 92]).

In the first section of this chapter we will recapitulate those ingredients from these theories which are important for this book. In the second section the nonlinear $\mathcal{H}_\infty$ control problem is introduced and under a regularity assumption the known solution ([vdS 92], [vdS 93]) to the state feedback $\mathcal{H}_\infty$ problem is recalled for general nonlinear systems.

2.1 The L_2-gain of a nonlinear system

Consider nonlinear systems of the following state space form

$$\begin{aligned} \dot{x} &= f(x,d) \\ z_j &= h_j(x), \qquad j=1,\ldots,p \end{aligned} \tag{2.1}$$

where $x=(x_1,\ldots,x_n)$ are local coordinates for a smooth (C^∞) state space manifold denoted by M, the disturbances are denoted by $d \in \mathbb{R}^q$ and $z \in \mathbb{R}^p$ are the outputs. The functions f and h are C^k, with $k \geq 1$ (at least continuously differentiable). Furthermore we assume that the system has an equilibrium in $(x,d)=(0,0)$, i.e., $f(0,0)=0$, and without loss of generality $h_j(0)=0$, $j=1,\ldots,p$.
For simplicity of notation we shall abbreviate the outputs as

$$z = h(x).$$

Following the literature we define the L_2-gain of the nonlinear system (2.1) in the following way ([Vi 93], [vdS 93]).

Definition 2.1 Let γ be a fixed non-negative constant. The system (2.1) is said to have *L_2-gain less than or equal to γ* if for all $x \in M$ there exists a constant $K(x)$, $0 \leq K(x) < \infty$, with $K(0)=0$, such that the following inequality holds

$$\int_0^t \|z(\tau)\|^2 \mathrm{d}\tau \leq \gamma^2 \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + K(x_0) \tag{2.2}$$

for all disturbances $d \in L_2(0,t)$ and all $t \in [0,T]$, with $[0,T)$ any open interval in which the corresponding solutions of the differential equation $\dot{x}=f(x,d)$ exist, with $z(\tau)=h(\varphi(\tau,0,x_0,d))$ denoting the output of (2.1) resulting from d for initial state $x(0)=x_0$.

The system has *L_2-gain less than γ* if there exists some $0 \leq \tilde{\gamma} < \gamma$ such that the system (2.1) has L_2-gain less than or equal to $\tilde{\gamma}$. The *L_2-gain is equal to γ* if it has L_2-gain less than or equal to γ and not less than γ. □

We can also define the finite L_2-gain property in another way. Define the *available storage* ([Wi 72]) as

$$V^a(x) = -\inf_d \frac{1}{2} \int_0^t \left(\gamma^2 \|d(\tau)\|^2 - \|z(\tau)\|^2\right) \mathrm{d}\tau$$

where the infimum is taken over all $d \in L_2(0, t)$ and all $t \geq 0$, and where $z(\tau)$ denotes the response of the system (2.1) to a disturbance d and an initial condition $x(0) = x$.

Now condition (2.2) is equivalent to saying that V^a is finite for every $x \in M$ and $V^a(0) = 0$. A third equivalent way to define the finite L_2-gain property is by stating that the system (2.1) has L_2-gain less than or equal to γ if the system (2.1) is *dissipative* with respect to the *supply rate* $s(d, z) = \frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|z\|^2$, in the sense that there exists a solution $V \geq 0$ (called a *storage function*) to the *integral dissipation inequality*

$$V(x(t_1)) - V(x(t_0)) \leq \frac{1}{2}\int_{t_0}^{t_1} (\gamma^2\|d(\tau)\|^2 - \|z(\tau)\|^2)\mathrm{d}\tau, \quad V(0) = 0 \quad (2.3)$$

for all $t_1 \geq t_0$ and all $d \in L_2(t_0, t_1)$. If V^a is finite for all x then it follows (see [Wi 72]) that V^a satisfies (2.3), while it is clear that $V^a(x) \geq 0$ and $V^a(0) = 0$. Furthermore V^a is the minimal function $V : M \to \mathbb{R}^+$ which satisfies this integral dissipation inequality, i.e., the minimal *storage function*. Correspondingly $2V^a(x)$ is the minimal constant $K(x)$ for which the inequality (2.2) holds.

If there exists a *continuously differentiable* solution V to the integral dissipation inequality (2.3) then this V is immediately seen to be a solution to the *differential dissipation inequality*

$$V_x(x)f(x, d) \leq \frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|z\|^2, \quad V(0) = 0 \quad (2.4)$$

for all $d \in \mathbb{R}^q$ and all $x \in M$. Since the Hessian at $x = 0$, $d = 0$ of

$$V_x(x)f(x, d) - \frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}\|z\|^2$$

with respect to d is equal to $-\gamma^2 I$ it follows that at least locally near the origin we can find the worst case disturbance with respect to the inequality (2.4) given by

$$d_{\max}(x) = \arg\max_d \left(V_x(x)f(x, d) - \frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}h^T(x)h(x) \right). \quad (2.5)$$

Locally the differential dissipation inequality (2.4) is equivalent to the *Hamilton-Jacobi inequality*

$$V_x(x)f(x, d_{\max}(x)) - \frac{1}{2}\gamma^2\|d_{\max}(x)\|^2 + \frac{1}{2}h^T(x)h(x) \leq 0, \quad V(0) = 0. \quad (2.6)$$

Summarizing:

Theorem 2.2 *The system (2.1) has L_2-gain less than or equal to γ if and only if there exists a solution $V : M \to \mathbb{R}^+$ to the integral dissipation inequality (2.3) for all $t_1 \geq t_0$, all $d \in L_2(t_0, t_1)$ and all $x \in M$.*

Further, there exists a non-negative C^1-solution to the integral dissipation inequality (2.3) if and only if there exists a non-negative C^1-solution to the differential dissipation inequality (2.4) for all $d \in \mathbb{R}^q$.

There exists a local non-negative C^1-solution to the differential dissipation inequality (2.4) if and only if locally there exists a non-negative C^1-solution to the Hamilton-Jacobi inequality (2.6). □

Remark 2.3 When the system (2.1) is affine in d

$$\begin{aligned} \dot{x} &= f(x) + \textstyle\sum_{j=1}^m g_j(x)d_j = f(x) + g(x)d \\ z &= h(x) \end{aligned} \tag{2.7}$$

with $g(x)$ an $n \times m$ matrix, then the worst case disturbance $d_{\max}(x)$ is globally defined and given by

$$d_{\max}(x) = \frac{1}{\gamma^2} g^T(x) V_x^T(x),$$

and hence the differential dissipation inequality for the system (2.7) is (globally) equivalent to the Hamilton-Jacobi inequality

$$V_x(x)f(x) + \frac{1}{2}\frac{1}{\gamma^2} V_x(x)g(x)g^T(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0, \quad V(0) = 0.$$

□

Another way to approach the finite L_2-gain property is by considering the Hamiltonian system corresponding to the finite L_2-gain for the system (2.1). Before doing so we define stability of the equilibrium for these systems. Two notions of stability will be used throughout this monograph.

Definition 2.4 The equilibrium point $x = 0$ is said to be a *stable* equilibrium for the system (2.1), with $d = 0$, if for every neighborhood W of 0 there exists a neighborhood $\tilde{W}$ of the origin such that for every initial condition $x_0 \in \tilde{W}$ the solution $\varphi(t, 0, x_0, 0)$ $(d = 0)$ belongs to W for all times $t \geq 0$, i.e.,

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 \text{ such that } \|x_0\| < \delta(\varepsilon) \Rightarrow \|\varphi(t, 0, x_0, 0)\| < \varepsilon, \qquad \forall t \geq 0.$$

□

Definition 2.5 The equilibrium point $x = 0$ is said to be a *locally asymptotically stable* equilibrium for the system (2.1), with $d = 0$, if 0 is stable and there exists a neighborhood $\bar{W}$ of the origin such that all solutions $\varphi(t, 0, x_0, 0)$ with $x_0 \in \bar{W}$ converge to 0 if $t \to \infty$, i.e. there exists a $\mu > 0$ such that

$$\|x_0\| < \mu \Rightarrow \lim_{t\to\infty} \varphi(t, 0, x_0, 0) = 0. \tag{2.8}$$

The equilibrium is said to be *globally asymptotically stable* if it is stable and all solutions $\varphi(t, 0, x_0, 0)$ converge to 0. □

Sometimes we will just say that $\dot{x} = f(x, 0)$ is locally or globally asymptotically stable.
There is a close connection between the property of finite L_2-gain and asymptotic stability.

Definition 2.6 The system (2.1) is called *zero-state observable* if for any trajectory such that $d(t) \equiv 0$, $z(t) \equiv 0$ implies $x(t) \equiv 0$. □

Theorem 2.7 *([HiMo 76], [vdS 92]) Assume the system (2.1) is zero-state observable. Suppose there exists a C^1-solution $V \geq 0$ to either (2.3), (2.4) or (2.6). Then $V(x) > 0, x \neq 0$, and the free system $\dot{x} = f(x, 0)$ is locally asymptotically stable. Furthermore, assume that V is* proper *(i.e., for each $c > 0$ the set $\{x \in M \,|\, 0 \leq V(x) \leq c\}$ is compact), then $\dot{x} = f(x, 0)$ is globally asymptotically stable.* □

The finite L_2-gain property can be also considered from a Hamiltonian viewpoint. I mention the main results without proofs. Details and proofs can be found in [AbMa 78], [vdS 91] and [vdS 92].

Consider the system (2.1), where the functions f,h are C^k-functions, with $k \geq 2$. Define the *pre-Hamiltonian* $K_\gamma : T^*M \times \mathbb{R}^q \to \mathbb{R}$ as

$$K_\gamma(x, p, d) := p^T f(x, d) - \frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}h^T(x)h(x)$$

with (x, p) the natural coordinates for the $2n$-dimensional cotangent bundle T^*M (state x and co-state p). We note that (2.4) can be rewritten as

$$K_\gamma(x, V_x^T(x), d) \leq 0, \qquad V(0) = 0.$$

The maximizing disturbance $d^*(x, p) = \arg\max_d K_\gamma(x, p, d)$ can be substituted into the pre-Hamiltonian K_γ, which yields the *Hamiltonian* C^k-function $H_\gamma : T^*M \to \mathbb{R}$ given by

$$H_\gamma(x, p) := K_\gamma(x, p, d^*(x, p)).$$

Note that by definition of $d_{\max}(x)$, see (2.5), $d^*(x, V_x^T(x)) = d_{\max}(x)$ and hence the Hamilton-Jacobi inequality (2.6) can be written as

$$H_\gamma(x, V_x^T(x)) \leq 0, \qquad V(0) = 0.$$

Corresponding to the Hamiltonian function H_γ we consider the C^{k-1} *Hamiltonian vector field* X_{H_γ} on T^*M given by

$$\begin{aligned} \dot{x}_i &= \frac{\partial H_\gamma}{\partial p_i}(x, p) \\ \dot{p}_i &= -\frac{\partial H_\gamma}{\partial x_i}(x, p), \qquad i = 1, \ldots, n \end{aligned}$$

with equilibrium $(x, p) = (0, 0)$. Then $V : M \to \mathbb{R}$ is a (not necessarily nonnegative) solution to

$$H_\gamma(x, V_x^T(x)) = 0, \qquad V(0) = 0, \qquad V_x(0) = 0 \tag{2.9}$$

if and only if the n-dimensional submanifold

$$N = \{(x, p) \in T^*M | p^T = V_x(x)\}$$

is an *invariant manifold* of X_{H_γ} through $(0, 0)$, i.e., $X_{H_\gamma}(x, p)$ is tangent to N at every point $(x, p) \in N$.

Now suppose that the linearization of X_{H_γ} at $(0, 0)$ given by the *Hamiltonian matrix*

$$DX_{H_\gamma}(0, 0) = \begin{bmatrix} \frac{\partial^2 H_\gamma}{\partial x \partial p} & \frac{\partial^2 H_\gamma}{\partial p^2} \\ -\frac{\partial^2 H_\gamma}{\partial x^2} & -\frac{\partial^2 H_\gamma}{\partial p \partial x} \end{bmatrix} (0, 0)$$

has no eigenvalues on the imaginary axis, i.e., the vector field X_{H_γ} is *hyperbolic*.

From the fact that $DX_{H_\gamma}(0, 0)$ is a Hamiltonian matrix it follows that it has n eigenvalues in $\mathbb{C}^-$ and n eigenvalues in $\mathbb{C}^+$, and therefore there exists an n-dimensional *stable invariant manifold* N^- consisting of all points in T^*M converging to $(0, 0)$ along the flow of X_{H_γ}. Furthermore N^- is tangent at $(0, 0)$ to the stable eigenspace of $DX_{H_\gamma}(0, 0)$ and N^- generates a C^k solution V^- of the

Hamilton-Jacobi equality (2.9) if and only if N^- is *projectable* on M, i.e., N^- is diffeomorphic to M under the C^{k-1} projection $\pi : T^*M \to M$, where π denotes the *canonical projection* $\pi : (x, p) \mapsto x$.

This solution V^- is not necessary non-negative. However when we additionally assume that the vector field $f(x, 0)$ of the system (2.1) is globally asymptotically stable then V^- is nonnegative. Furthermore the minimal storage function V^- is equal to the available storage V^a. Summarizing:

Theorem 2.8 *Consider the system (2.1), with f, h C^k functions, with $k \geq 2$. Assume that the C^{k-1} vector field X_{H_γ} is hyperbolic in $(0, 0)$ and that N^- is projectable on M. Then there exists a unique C^k function $V^- : M \to \mathbb{R}$ such that:*

(i)

$$N^- = \left\{(x, p) \in T^*M | p^T = V_x^-(x)\right\};$$

(ii)

$$H_\gamma(x, (V_x^-)^T(x)) = 0, \qquad V(0) = 0, \qquad V_x(0) = 0;$$

(iii) $\dot{x} = f(x, d^*(x, (V_x^-)^T(x)))$ *is globally asymptotically stable on M.*

Moreover if we also assume that $f(x, 0)$ is globally asymptotically stable then also:

(iv) $V^- \geq 0$, *and* $V^- = V^a$;

(v) the system (2.1) has L_2-gain less than or equal to γ.

□

The assumption about the *global* projectability of the submanifold N^- is hard to verify. By considering the linearization of (2.1) at the origin we can give an easily verifiable condition for the *local* projectability of the submanifold N^-.

Local instead of global projectability in Proposition 2.8 will lead to a local solution V^- and therefore to the notion of local L_2-gain of a system (2.1).

Definition 2.9 Let γ be a fixed non-negative constant. The system (2.1) is said to have *locally L_2-gain less than or equal to γ* if there exist a neighborhood

$W \subset M$ of the origin and constants $K(x)$, $0 \leq K(x) < \infty$, with $K(0) = 0$, such that

$$\int_0^t \|z(\tau)\|^2 \mathrm{d}\tau \leq \gamma^2 \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + K(x_0)$$

for all $t \in [0, T]$, with $[0, T)$ any open interval in which the corresponding solutions of the differential equation $\dot{x} = f(x, d)$ exists, and for all $d \in L_2(0, t)$ and $x(0) \in W$ such that the state trajectories do not leave the neighborhood W.

The system has *locally L_2-gain less than* γ if there exists some $0 \leq \tilde{\gamma} < \gamma$ such that the system (2.1) has locally L_2-gain less than or equal to $\tilde{\gamma}$. The *local L_2-gain is equal to* γ if it has local L_2-gain less than or equal to γ and not less than γ (Note that the local L_2-gain need not be defined). □

The linearization of (2.1) at the origin is given by

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{d} \\ \bar{z} &= H\bar{x} \end{aligned} \tag{2.10}$$

where $\bar{x} \in \mathbb{R}^n$, $\bar{d} \in \mathbb{R}^q$, $\bar{z} \in \mathbb{R}^p$ and the matrices F, G and H are defined as:

$$F = \frac{\partial f}{\partial x}(0,0), \quad G = \frac{\partial f}{\partial d}(0,0), \quad H = \frac{\partial h}{\partial x}(0).$$

Then it can be shown that the Hamiltonian matrix $DX_{H_\gamma}(0,0)$ can be written in terms of the matrices F, G, and H.

Lemma 2.10 *The Hamiltonian matrix $DX_{H_\gamma}(0,0)$ is given by*

$$DX_{H_\gamma}(0,0) = \begin{pmatrix} F & \frac{1}{\gamma^2}GG^T \\ -H^T H & -F^T \end{pmatrix}. \tag{2.11}$$

Proof Since $d^*(x,p)$ is the solution of $\frac{\partial K_\gamma}{\partial d} = 0$

$$p^T \frac{\partial f}{\partial d}(x, d^*(x,p)) = \gamma^2 \left(d^*(x,p)\right)^T, \qquad \forall x, p.$$

Differentiation of this equality with respect to x in $p = 0$ leads to

$$\frac{\partial d^*}{\partial x}(x, 0) = 0, \qquad \forall x$$

and the derivative with respect to p in $(x,p) = (0,0)$ is given by

$$\frac{\partial d^*}{\partial p}(0,0) = \frac{1}{\gamma^2}\left(\frac{\partial f}{\partial d}(0,0)\right)^T.$$

Hence

$$\begin{aligned}
\frac{\partial H_\gamma}{\partial p}(x,p) &= f^T(x,d^*(x,p)) + p^T\frac{\partial f}{\partial d}(x,d^*(x,p))\frac{\partial d^*}{\partial p}(x,p) \\
&\quad -\gamma^2\left(d^*(x,p)\right)^T\frac{\partial d^*}{\partial p}(x,p) \\
&= f^T(x,d^*(x,p)), \\
\frac{\partial H_\gamma}{\partial x}(x,p) &= p^T\frac{\partial f}{\partial x}(x,d^*(x,p)) + p^T\frac{\partial f}{\partial x}(x,d^*(x,p))\frac{\partial d^*}{\partial x}(x,p) \\
&\quad -\gamma^2\left(d^*(x,p)\right)^T\frac{\partial d^*}{\partial x}(x,p) + h^T(x)h(x) \\
&= p^T\frac{\partial f}{\partial x}(x,d^*(x,p)) + h^T(x)h(x).
\end{aligned}$$

Furthermore

$$\begin{aligned}
\frac{\partial^2 H_\gamma}{\partial x\partial p}(0,0) &= \frac{\partial f}{\partial x}(0,d^*(0,0)) = \frac{\partial f}{\partial x}(0,0) = F, \\
\frac{\partial^2 H_\gamma}{\partial p^2}(0,0) &= \frac{\partial f}{\partial d}(0,d^*(0,0))\frac{\partial d^*}{\partial p}(0,0) \\
&= \frac{1}{\gamma^2}\frac{\partial f}{\partial d}(0,0)\left(\frac{\partial f}{\partial d}(0,0)\right)^T = \frac{1}{\gamma^2}GG^T, \\
\frac{\partial^2 H_\gamma}{\partial x\partial p}(0,0) &= \left(\frac{\partial f}{\partial x}(0,d^*(0,0))\right)^T = F^T, \\
\frac{\partial^2 H_\gamma}{\partial x^2}(0,0) &= \left(\frac{\partial h}{\partial x}(0)\right)^T\left(\frac{\partial h}{\partial x}(0)\right) = H^T H.
\end{aligned}$$

■

So the first condition in Theorem 2.8 about the hyperbolicity of the vector field X_{H_γ} simply amounts to checking that the Hamiltonian matrix (2.11) does not have purely imaginary eigenvalues.

For the linearized system (2.10) the following result is well known.

Theorem 2.11 *Consider the linearized system (2.10). Assume (F, G) is stabilizable and the Hamiltonian matrix (2.11) does not have purely imaginary eigenvalues. Then there exists a unique symmetric solution P which satisfies:*

(i) the algebraic Riccati equation

$$F^T P + PF + \frac{1}{\gamma^2}PGG^T P + H^T H = 0; \qquad (2.12)$$

(ii)

$$\sigma\left(F + \frac{1}{\gamma^2} G G^T P\right) \subset \mathbb{C}^-; \tag{2.13}$$

(iii)

$$\text{span}\begin{pmatrix} I_n \\ P \end{pmatrix} = \text{stable eigenspace of} \begin{pmatrix} F & \frac{1}{\gamma^2} G G^T \\ -H^T H & -F^T \end{pmatrix}.$$

If we also assume that F is asymptotically stable then moreover:

(iv) $P \geq 0$;

(v) the system (2.10) has L_2-gain less than γ.

□

Thus if (F, G) is stabilizable then the stable eigenspace of $DX_{H_\gamma}(0,0)$ can be parameterized by the $\bar{x}$ coordinates and since N^- is tangent at $(0,0)$ to this stable eigenspace there exists a neighborhood W of the origin in M where the stable invariant manifold N^- is locally projectable, i.e., $N^- \cap T^*W$ is projectable on W. Now we can derive the following local version of Theorem 2.8.

Theorem 2.12 *Consider the system (2.1), with f, h C^k functions, with $k \geq 2$, and its linearization (2.10). Assume that the matrix (2.11) has no eigenvalues on the imaginary axis and that (F, G) is stabilizable. Then there exists a neighborhood W of 0 in M and a unique C^k function $V^- : W \to \mathbb{R}$ such that:*

(i)

$$N^- = \left\{(x, p) \in T^*M \mid p^T = V_x^-(x),\ x \in W\right\};$$

(ii)

$$H_\gamma(x, (V_x^-)^T(x)) = 0, \qquad V(0) = 0, \qquad \forall x \in W;$$

(iii) $\dot{x} = f(x, d^*(x, (V_x^-)^T(x)))$ *is (locally) asymptotically stable on W.*

Moreover if we assume that $\dot{x} = f(x, 0)$ is asymptotically stable on a possibly smaller neighborhood W (this is implied by F asymptotically stable), then also on W:

(iv) $V^- \geq 0$ *and* $V^- = V^a$*;*

(v) the system (2.1) has locally L_2*-gain less than* γ*.*

□

If the nonlinear system (2.1) has local L_2-gain less than (or equal to) γ it is easy to prove that also the linearization (2.10) has L_2-gain less than (or equal to) γ (see [vdS 92]). Furthermore if the linearization has L_2-gain less than γ and is asymptotically stable then the matrix (2.11) has no eigenvalues on the imaginary axis ([Sch 91]).

Together with Theorem 2.11 and Theorem 2.12 this leads to the following corollary.

Corollary 2.13 *Consider the nonlinear system (2.1) together with its linearization (2.10). Assume* F *is asymptotically stable. Then the following statements are equivalent:*

(i) the linear system (2.10) has L_2*-gain less than* γ*;*

(ii) there exists a symmetric solution $P \geq 0$ *to (2.12), (2.13);*

(iii) the nonlinear system (2.1) has locally L_2*-gain less than* γ*.*

□

2.2 The $\mathcal{H}_\infty$ control problem for a nonlinear system

Consider nonlinear systems in state space form

$$\begin{aligned} \dot{x} &= f(x,u,d) \\ y &= g(x,d) \\ z &= h(x,u) \end{aligned} \tag{2.14}$$

with two sets of inputs u and d, and two sets of outputs y and z. Here $u \in \mathbb{R}^m$ denotes the vector of *control inputs*, $d \in \mathbb{R}^q$ are the *exogenous inputs* (disturbances to be rejected and/or reference signals to be tracked), $y \in \mathbb{R}^r$ are the *measured outputs* and $z \in \mathbb{R}^p$ are the *to-be-controlled outputs* which could be tracking errors or cost variables. Finally the state x are local coordinates for a smooth

n-dimensional manifold M. The functions f, g, h are C^k, with $k \geq 2$. The system (2.14) is assumed to have an equilibrium at the origin, i.e., $f(0, 0, 0) = 0$, and without loss of generality also $g(0, 0) = 0$ and $h(0, 0) = 0$.

The main topic of this book will be the state feedback $\mathcal{H}_\infty$ problem. In this problem it is assumed that $g(x, d) = x$ and the goal is to construct a static state feedback such that the closed-loop system with (2.14) has L_2-gain less than (or equal to) a certain constant γ.

Definition 2.14 *Nonlinear state feedback L_2-gain optimal control problem*: find, if existing, the smallest value $\gamma^* \geq 0$ such that for any $\gamma > \gamma^*$ there exists a state feedback

$$u = l(x), \qquad l(0) = 0 \tag{2.15}$$

such that the L_2-gain of the closed loop system (2.14), (2.15) has L_2-gain less than or equal to γ (from d to z). □

Definition 2.15 *Nonlinear state feedback $\mathcal{H}_\infty$ optimal control problem*: find, if existing, the smallest value $\gamma^* \geq 0$ such that for any $\gamma > \gamma^*$ there exists a state feedback

$$u = l(x), \qquad l(0) = 0$$

such that the closed loop system has L_2-gain less than or equal to γ, and the origin is locally asymptotically stable. □

In the definition of the L_2-gain problem only the L_2-gain is considered, a priori without stability considerations. The main reason for introducing these two problems is that we want to consider internal stability separately. As seen in Theorem 2.7 certain stability properties can be derived from the finite L_2-gain property.

In Chapter 3 also the measurement feedback case shall be considered briefly. In this problem it is assumed that we can only measure a function of both state and disturbance, i.e., $y = g(x, d)$.

Definition 2.16 *Nonlinear measurement feedback $\mathcal{H}_\infty$ optimal control problem*: find, if existing, the smallest value $\gamma^* \geq 0$ such that for any $\gamma > \gamma^*$ there exists a compensator

$$\begin{aligned} \dot{\xi} &= k(\xi, y) \\ u &= m(\xi, y) \end{aligned} \tag{2.16}$$

with $k(0,0) = 0$ and $m(0,0) = 0$, such that the closed loop system (2.14), (2.16) has L_2-gain (from d to z) less than or equal to γ, and the origin is locally asymptotically stable. □

In most of the literature about nonlinear $\mathcal{H}_\infty$ theory it is assumed that the mapping h is in some sense injective from u to z. The $\mathcal{H}_\infty$ problem under this regularity assumption is often referred to as the *regular $\mathcal{H}_\infty$ problem* (see e.g. [BHW 93], [IsAs 92], [IsKa 95], [vdS 91], [vdS 92], [vdS 93]). The main aim of this monograph is to drop this regularity assumption. When this regularity assumption is violated the $\mathcal{H}_\infty$ problem is called *singular*. Sometimes to emphasize the more general character of this $\mathcal{H}_\infty$ problem it is called the *general $\mathcal{H}_\infty$ problem.*

In the next subsection some of the results derived for the regular $\mathcal{H}_\infty$ problem are recapitulated.

Also the linearization of the nonlinear system (2.14) around the origin shall be considered, denoted as

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{u} + E\bar{d} \\ \bar{y} &= M\bar{x} + N\bar{d} \\ \bar{z} &= H\bar{x} + K\bar{u} \end{aligned} \tag{2.17}$$

where $\bar{u} \in \mathbb{R}^m$, $\bar{x} \in \mathbb{R}^n$, $\bar{d} \in \mathbb{R}^q$, $\bar{y} \in \mathbb{R}^r$, $\bar{z} \in \mathbb{R}^p$ and the matrices F, G, E, M, N, H and K are defined as:

$$F = \frac{\partial f}{\partial x}(0,0,0), \quad G = \frac{\partial f}{\partial u}(0,0,0), \quad E = \frac{\partial f}{\partial d}(0,0,0),$$
$$M = \frac{\partial g}{\partial x}(0,0), \quad N = \frac{\partial g}{\partial d}(0,0), \quad H = \frac{\partial h}{\partial x}(0,0), \quad K = \frac{\partial h}{\partial u}(0,0).$$

The regular $\mathcal{H}_\infty$ problem corresponds to the matrix K having full column rank, i.e., rank $K = m$.

2.2.1 The regular state feedback $\mathcal{H}_\infty$ problem for nonlinear systems

In this subsection we consider systems of the form (2.14) with $g(x,d) = x$ and which satisfy the following regularity assumption

Assumption 1 The derivative of h with respect to u has full column rank at the origin, i.e.,

$$\text{rank}\left(\frac{\partial h}{\partial u}(0,0)\right) = m.$$

□

We seek for a nonlinear static state feedback

$$u = l(x), \qquad l(0) = 0 \tag{2.18}$$

such that the closed-loop of (2.14) with (2.18) has L_2-gain less than or equal to γ from d to z.

This L_2-gain optimal control problem can be viewed as a two player, zero sum differential game, where u is called the minimizing player which goal it is to minimize the cost criterium

$$\frac{1}{2}\int_0^t \left(\|h(x(\tau), u(\tau))\|^2 - \gamma^2 \|d(\tau)\|^2\right) \mathrm{d}\tau$$

for every t, and d is called the maximizing player whose goal it is to maximize the same cost criterium. The pre-Hamiltonian function associated with this game for the system (2.14) is a function $K_\gamma : T^*M \times \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}$ defined as

$$K_\gamma(x, p, d, u) := p^T f(x, u, d) + \frac{1}{2}\|h(x, u)\|^2 - \frac{1}{2}\gamma^2\|d\|^2.$$

Under the regularity Assumption 1 this pre-Hamiltonian K_γ has a unique saddle point with respect to u and d in a neighborhood of the origin, i.e., there exist unique functions $d^* = d^*(x, p)$ and $u^* = u^*(x, p)$ defined around $(x, p) = (0, 0)$, satisfying

$$\begin{aligned} \frac{\partial K_\gamma}{\partial d}(x, p, d^*(x, p), u^*(x, p)) &= 0 \\ \frac{\partial K_\gamma}{\partial u}(x, p, d^*(x, p), u^*(x, p)) &= 0 \end{aligned}$$

with $d^*(0, 0) = 0$ and $u^*(0, 0) = 0$, satisfying the saddle point condition

$$K_\gamma(x, p, d, u^*) \le K_\gamma(x, p, d^*, u^*) \le K_\gamma(x, p, d^*, u) \tag{2.19}$$

for every (x, p, d, u) around the origin. The existence of these functions can easily be seen by calculating the Hessian of K_γ with respect to d and u at the origin. This Hessian equals

$$\begin{pmatrix} -\gamma^2 I & 0 \\ 0 & \left(\frac{\partial h}{\partial u}(0, 0)\right)^T \left(\frac{\partial h}{\partial u}(0, 0)\right) \end{pmatrix}.$$

Because of Assumption 1 local existence of the functions $d^*(x, p)$ and $u^*(x, p)$ now follows from the Implicit Function Theorem.

Substituting the saddle point d^*, u^* into the pre-Hamiltonian K_γ leads to the Hamiltonian $H_\gamma : T^*M \to \mathbb{R}$ defined as

$$H_\gamma(x, p) := K_\gamma(x, p, d^*(x, p), u^*(x, p)). \tag{2.20}$$

The following result can be easily deduced from Theorem 2.2 and the saddle point condition (2.19).

Theorem 2.17 *Consider the system (2.14) with $y = x$, under the Assumption 1. Let $\gamma > 0$. Suppose there exists a local C^s $(k \geq s \geq 1)$ solution $V \geq 0$ to*

$$H_\gamma(x, V_x^T(x)) \leq 0 \tag{2.21}$$

with $V(0) = 0$. Then the C^{s-1} state feedback

$$u = u^*(x, V_x^T(x)) \tag{2.22}$$

locally solves the state feedback L_2-gain control problem, with constant γ, for the system (2.14).

Conversely, suppose there exists a state feedback (2.18) which solves the state feedback L_2-gain optimal control problem in the sense that there exists a differentiable solution $V \geq 0$ to the Hamilton-Jacobi inequality (2.6) for the closed-loop system (2.14), (2.18), then V is also a solution of (2.21) and hence also the feedback (2.22) leads to a closed-loop system which has L_2-gain less than or equal to γ. □

Proof The first part of the theorem follows from Theorem 2.2 and (2.19). For the converse statement we know there exists a solution V to

$$K_\gamma(x, V_x^T(x), d, l(x)) \leq 0$$

for all d. Then by taking $u = l(x)$ and $p^T = V_x(x)$ in the saddle point condition (2.19) it follows that

$$H_\gamma(x, V_x^T(x)) \leq K_\gamma(x, V_x^T(x), d^*(x, V_x^T(x)), l(x)) \leq 0.$$

■

Remark 2.18 Consider the following system which is affine in the inputs u and the disturbances d

$$\begin{aligned} \dot{x} &= f(x) + \textstyle\sum_{j=1}^m g_j(x)u_j + \sum_{i=1}^q e_i(x)d_i \\ z &= h(x) + \textstyle\sum_{j=1}^m k_j(x)u_j \end{aligned}$$

In shorthand notation we will write this system as

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + e(x)d \\ z &= h(x) + k(x)u \end{aligned} \tag{2.23}$$

with $g(x)$, $e(x)$, $k(x)$ matrices consisting of the corresponding columns. For the system (2.23) the saddle point solution $d^*(x, p)$ and $u^*(x, p)$ is given by

$$\begin{aligned} d^*(x, p) &= \frac{1}{\gamma^2} e^T(x)p \\ u^*(x, p) &= -\left(k^T(x)k(x)\right)^{-1}\left(g^T(x)p + k^T(x)h(x)\right) \end{aligned}$$

where the maximizing d is globally defined, and the minimizing u is globally defined provided the matrix $k^T(x)k(x)$ is non-singular for all x. The Hamilton-Jacobi inequality (2.21) for the system (2.23) is then given by

$$\begin{aligned} V_x(x)f(x) + \frac{1}{2}h^T(x)h(x) + \frac{1}{2}\frac{1}{\gamma^2}V_x(x)e(x)e^T(x)V_x^T(x) \\ -\frac{1}{2}\left(V_x(x)g(x) + h^T(x)k(x)\right)\left(k^T(x)k(x)\right)^{-1}\left(g^T(x)V_x^T(x) + k^T(x)h(x)\right) \\ \leq 0 \end{aligned}$$

with $V(0) = 0$. □

We see that V solving the Hamilton-Jacobi inequality (2.21) is a candidate *Lyapunov function* for the closed loop system, and from Theorem 2.7 we deduce the following stability result:

Theorem 2.19 *Assume that the closed loop system (2.14), (2.22) is zero-state observable. Suppose there exists a C^1 solution $V \geq 0$ to (2.21). Then $V > 0$, $x \neq 0$, and the free system $\dot{x} = f(x, u^*(x, V_x^T(x)), 0)$ is locally asymptotically stable, and globally asymptotically stable if V is proper.* □

Using the Hamiltonian and the machinery explained in the last section we can easily derive the following results.

Theorem 2.20 *Define the C^k Hamiltonian*

$$\begin{aligned} H_\gamma^l(x, p) &= K_\gamma(x, p, d_l^*(x, p), l(x)) \\ &= p^T f(x, l(x), d_l^*(x, p)) + \frac{1}{2}h^T(x, l(x))h(x, l(x)) \\ &\quad -\frac{1}{2}\gamma^2\|d_l^*(x, p)\|^2 \end{aligned} \tag{2.24}$$

where

$$d_l^*(x, p) = \arg\max_d K_\gamma(x, p, d, l(x)).$$

Assume that the C^{k-1} vector field X_{H_γ} is hyperbolic at $(0, 0)$ and that the stable invariant manifold N^- of X_{H_γ} is projectable on M. Then there exists a unique C^k function $V^- : M \to \mathbb{R}$ such that:

(i)

$$N^- = \{(x, p) \in T^*M | p^T = V_x^-(x)\};$$

(ii)

$$H_\gamma(x, (V_x^-)^T(x)) = 0, \qquad V(0) = 0, \qquad V_x(0) = 0;$$

(iii) $\dot{x} = f(x, l(x), d_l^*(x, (V_x^-)^T(x)))$ *is globally asymptotically stable on M.*

Moreover if we additionally assume that $\dot{x} = f(x, l(x), 0)$ is globally asymptotically stable then:

(iv) $V^- \geq 0$*;*

(v) the closed-loop system (2.14), (2.18) has L_2-gain less than γ.

□

Consider now the linearization of the nonlinear system (2.14), (2.17), with $\bar{y} = \bar{x}$. The next result is well known (see [St 92], [Sch 91]).

Theorem 2.21 *Consider the linearized system (2.17), with $\bar{y} = \bar{x}$, and assume the system (F, G, H, K) has no invariant zeros on the imaginary axis, i.e.,*

$$\operatorname{rank} \begin{pmatrix} F - j\omega I & G \\ H & K \end{pmatrix} = n + m, \qquad \forall \omega \in \mathbb{R}. \qquad (2.25)$$

Then the following statements are equivalent:

(i) there exists a feedback law $\bar{u} = L\bar{x}$ such that after applying this feedback to the system (2.17) the closed-loop system has L_2-gain less than γ and is asymptotically stable;

(ii) there exists a solution $P \geq 0$ *to*

$$\begin{aligned} F^T P + PF + \frac{1}{\gamma^2} PEE^T P & \\ - \left(PG + H^T K\right)\left(K^T K\right)^{-1}\left(K^T H + G^T P\right) + H^T H &= 0 \end{aligned} \tag{2.26}$$

satisfying

$$\sigma\left(F - G\left(K^T K\right)^{-1}\left(K^T H + G^T P\right) + \frac{1}{\gamma^2} EE^T P\right) \subset \mathbb{C}^-. \tag{2.27}$$

□

The Hamiltonian matrix corresponding to the Hamiltonian (2.20), denoted by $DX_{H_\gamma}(0,0)$, is given by

$$DX_{H_\gamma}(0,0) = \begin{pmatrix} F - G\left(K^T K\right)^{-1} K^T H & -G\left(K^T K\right)^{-1} G^T + \frac{1}{\gamma^2} EE^T \\ -H^T\left(I - K\left(K^T K\right)^{-1} K^T\right) H & -F^T + H^T K\left(K^T K\right)^{-1} G^T \end{pmatrix}.$$

Similarly as before, this matrix can be used to prove local results by considering the linearization.

Theorem 2.22 *Consider the system (2.14), with* $y = x$*, and its linearization given by (2.17), with* $\bar{y} = \bar{x}$*. Assume that (2.25) is satisfied. Then the following statements are equivalent:*

(i) there exists a feedback law $\bar{u} = L\bar{x}$ *such that after applying this feedback to the system (2.17) the closed-loop system has* L_2*-gain less than* γ *and is asymptotically stable;*

(ii) there exists a solution $P \geq 0$ *to (2.26) satisfying (2.27);*

(iii) there exists a neighborhood $W \subset M$ *of 0 and a nonlinear state feedback* $u = l(x)$ *defined on* W*, such that* $F + GL$*, with* $L = \frac{\partial l}{\partial x}(0)$*, is asymptotically stable and the closed-loop system (2.14), (2.18) has locally* L_2*-gain less than* γ *on* W *(i.e., the nonlinear state feedback* $\mathcal{H}_\infty$ *problem is solvable on* W*).*

□

Finally we note that if the system (2.14) is not influenced by disturbances then the $\mathcal{H}_\infty$ problem turns into an optimal control problem in the sense that the Hamiltonian H_γ does not depend on γ, and is equal to the (optimal) Hamiltonian for the optimal control problem of minimizing

$$\frac{1}{2}\int_0^\infty \|h(x(t))\|^2 \, dt.$$

In this case feedback solutions can be found by applying nonlinear optimal control theory ([LeMa 67]).

2.2.2 The regular measurement feedback $\mathcal{H}_\infty$ problem for nonlinear systems

Consider systems of the form

$$\begin{aligned} \dot{x} &= f(x, u, d_1) \\ y &= g(x) + d_2 \\ z &= h(x, u) \end{aligned} \tag{2.28}$$

satisfying the regularity Assumption 1. We want to construct a compensator of the form

$$\begin{aligned} \dot{\xi} &= k(\xi, y) \\ u &= m(\xi, y) \end{aligned}$$

which solves the measurement feedback $\mathcal{H}_\infty$ problem.

A useful method for finding such a compensator is the worst case certainty equivalence principle. This principle consists in solving first the state feedback problem and then replacing the actual state x in the feedback by the estimated state corresponding to the worst possible disturbance which is compatible with the applied input and the resulting output (see [DBB 93], [BO 82], [BB 90]).

The state feedback problem for the system (2.28), with $y = x$, is solved in Subsection 2.2.1. To construct the estimation of the state corresponding to the worst possible disturbance compatible with the applied input and the resulting output we have to solve a maximization problem with constraints. This maximization problem is hard to solve for general nonlinear systems. The special structure of the measurement equation of the system (2.28) however makes it possible to rewrite this constrained maximization problem as an unconstrained problem ([vdS 93]).

More details will be given in Chapter 3 where this method is used to construct a controller using a regularized version of the affine nonlinear system.

Chapter 3

The singular $\mathcal{H}_\infty$ problem: a cheap control approach

In this chapter the singular $\mathcal{H}_\infty$ problem is considered from a cheap control point of view. In a sense this comes down to regularizing the singular $\mathcal{H}_\infty$ problem. For linear systems this way of solving the state feedback $\mathcal{H}_\infty$ problem has been studied in the end of the eighties ([KPZ 87], [Pe 87a], [Pe 87b], [KPR 88], [ZK 88], [KPZ 90]). The basic tool in this approach is a parameterized algebraic Riccati equation. This approach can be extended to nonlinear systems. We prove that the solvability of the singular state feedback $\mathcal{H}_\infty$ problem can be characterized by the solvability of a parameterized Hamilton-Jacobi inequality. This Hamilton-Jacobi inequality also corresponds to the regular $\mathcal{H}_\infty$ problem for a regularized version of our system. This second viewpoint leads to a slightly more conservative feedback for the singular $\mathcal{H}_\infty$ problem.

First the results for linear systems shall be recapitulated. After that the main part of this chapter is devoted to the nonlinear extension of this theory in which also the connection with the $\mathcal{H}_\infty$ problem for the linearization of the nonlinear system shall be explained. Finally the measurement feedback $\mathcal{H}_\infty$ problem is considered in the last section. In this section we use the more conservative feedback corresponding to the regular $\mathcal{H}_\infty$ problem for a regularized version of our nonlinear system in order to be able to apply the worst case certainty equivalence principle.

3.1 Linear disturbance attenuation

We consider linear systems of the form (2.17), with $\bar{y} = \bar{x}$,

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{u} + E\bar{d} \\ \bar{z} &= H\bar{x} + K\bar{u} \end{aligned} \tag{3.1}$$

Here we assume that there exists no direct feedthrough from the disturbances $\bar{d}$ to the to-be-controlled variables $\bar{z}$. The theory, however, can be directly extended to systems with a direct feedthrough from $\bar{d}$ to $\bar{z}$.

The following matrices are defined.

Definition 3.1 Suppose $\text{rank}(K) = m_1 \leq m$. Let $U \in \mathbb{R}^{p\times m_1}$ and $\Pi \in \mathbb{R}^{m_1\times m}$ be any matrices such that:

- $\text{rank}(U) = \text{rank}(\Pi) = m_1$;
- $K = U\Pi$;
- $\Pi\Pi^T = I_{m_1}$.

Let $\Phi \in \mathbb{R}^{(m-m_1)\times m}$ be such that:

- $\Phi\Pi^T = 0$;
- $\Phi\Phi^T = I_{m-m_1}$ (Φ is void if $m_1 = m$).

Define

$$\begin{aligned} \Sigma &:= \Pi^T \left(U^T U\right)^{-1} \Pi \\ &= \Pi^T \left(\Pi K^T K \Pi^T\right)^{-1} \Pi. \end{aligned}$$

□

It should be noted that matrices U, Π and Φ as in Definition 3.1 always exist.

Now the aim is to find a feedback $\bar{u} = L\bar{x}$ for the system (3.1) such that the resulting closed-loop system has L_2-gain from $\bar{d}$ to $\bar{z}$ less then or equal to a constant γ, and such that the closed-loop system $F + GL$ is asymptotically stable. The solvability of this problem can be characterized by the following property in terms of a parameterized algebraic Riccati equation.

Definition 3.2 Let Φ and Σ be as in Definition 3.1. The system (3.1) is said to satisfy the *Algebraic Riccati Equation (ARE) with constant* γ if there exists a positive definite matrix Q and an $\varepsilon > 0$ such that there exists a positive definite solution P to

$$\begin{aligned}\left(F - G\Sigma K^T H\right)^T P + P\left(F - G\Sigma K^T H\right) + \frac{1}{\gamma^2} PEE^T P \qquad\qquad (3.2)\\ -PG\Sigma G^T P - \frac{1}{\varepsilon} PG\Phi^T\Phi G^T P + H^T\left(I - K\Sigma K^T\right) H + \varepsilon Q = 0.\end{aligned}$$

□

The following lemma shows that the existence of a positive definite solution P of ARE (3.2) does not depend on the particular choice of Q.

Lemma 3.3 *([ZK 88]) Suppose there exists a positive definite matrix* $Q \in \mathbb{R}^{n\times n}$ *and a constant* $\varepsilon > 0$ *such that the algebraic Riccati equation (3.2) has a positive definite solution. Then given any positive definite* $\tilde{Q} \in \mathbb{R}^{n\times n}$ *there exists a constant* $\varepsilon^* > 0$ *such that the ARE (3.2) with* Q *replaced by* $\tilde{Q}$ *has a positive definite solution* P *for all* $\varepsilon \in (0, \varepsilon^*]$. □

For the solvability of the singular $\mathcal{H}_\infty$ problem for the system (3.1) the following can be stated ([ZK 88]).

Theorem 3.4 *Consider the system (3.1). Let* $\gamma > 0$. *Then the following statements are equivalent:*

(i) there exists a linear feedback $\bar{u} = L\bar{x}$ *such that* $F + GL$ *is asymptotically stable and the closed-loop system (3.1) with this feedback has* L_2*-gain less than* γ*;*

(ii) the system (3.1) satisfies the Algebraic Riccati Equation with constant γ *(3.2).*

Moreover if $P > 0$ *is a solution of the ARE (3.2) for some* $Q > 0$ *and constant* $\varepsilon > 0$ *then the feedback*

$$L = -\left(\frac{1}{2\varepsilon}\Phi^T\Phi + \Sigma\right) G^T P - \Sigma K^T H \qquad (3.3)$$

leads to a closed-loop system which has L_2*-gain from* $\bar{d}$ *to* $\bar{z}$ *less than* γ*, and* $F + GL$ *is stable.* □

Furthermore, if there exists a positive definite solution P of (3.2), then there also exists a stabilizing solution of (3.2) (see [KPZ 90]).

Theorem 3.5 *Suppose for $Q > 0$ there exists an $\varepsilon^* > 0$ such that (3.2) has a positive definite solution P_ε for every $\varepsilon \in (0, \varepsilon^*]$. Then for every $\varepsilon \in (0, \varepsilon^*)$ there also exists a stabilizing solution $\tilde{P}_\varepsilon > 0$ for (3.2), i.e., there exists a solution $\tilde{P}_\varepsilon > 0$ to (3.2) with the additional property that*

$$F - G\Sigma K^T H + \frac{1}{\gamma^2} KK^T \tilde{P}_\varepsilon - G\Sigma G^T \tilde{P}_\varepsilon - \frac{1}{\varepsilon} G\Phi^T \Phi G^T \tilde{P}_\varepsilon$$

is asymptotically stable. □

3.2 Singular nonlinear state feedback $\mathcal{H}_\infty$ control

To keep the ideas behind the results transparent and to avoid complex notation we will restrict attention to nonlinear systems of the form (2.14), with $y = x$, which are affine in the disturbances d and the inputs u,

$$\Sigma \begin{cases} \dot{x} &= f(x) + g(x)u + e(x)d \\ z &= h(x) + k(x)u \end{cases} \tag{3.4}$$

We try to extend the linear results described in the last section to solve the singular state feedback $\mathcal{H}_\infty$ problem. So we want to find a nonlinear static feedback

$$u = l(x), \qquad l(0) = 0 \tag{3.5}$$

such that the resulting closed-loop system has L_2-gain less than or equal to γ, i.e., cf. Definition 2.1: for every $x \in M$ there exists a constant $K(x)$, $0 \leq K(x) < \infty$, with $K(0) = 0$, such that

$$\int_0^t \|z(x(\tau))\|^2 d\tau \leq \gamma^2 \int_0^t \|d(\tau)\|^2 d\tau + K(x(0))$$

for all $d \in L_2[0, t]$ and all $t \geq 0$, with $x(\tau)$ denoting the response of (3.4), (3.5) for initial condition $x(0) = x$.

We make the following constant rank assumption.

Assumption 2 There exists a neighborhood W of the origin such that

$$\text{rank } k(x) = m_1 \leq m, \qquad \forall x \in W.$$

□

From Assumption 2 it follows that at least locally we can make similar definitions as in the linear case (see Definition 3.1).

Definition 3.6 Define an $(m \times (m - m_1))$-matrix $\beta_2(x)$ such that:

- im $\beta_2(x) = \ker k(x)$;
- $\beta_2^T(x)\beta_2(x) = I_{m-m_1}$;

on a neighborhood $\overline{W}$ of the origin, which may be smaller than W.
Then define an $(m \times m_1)$-matrix $\beta_1(x)$ such that:

- $\beta_1^T(x)\beta_2(x) = 0$;
- $\beta_1^T(x)\beta_1(x) = I_{m_1}$.

For notational reasons also introduce

$$\psi(x) = \beta_1(x)\left(\beta_1^T(x)k^T(x)k(x)\beta_1(x)\right)^{-1}\beta_1^T(x).$$

□

Note that matrices $\beta_1(x)$ and $\beta_2(x)$ as in Definition 3.6 always exist. Furthermore note that from Definition 3.6 it follows that

$$\begin{pmatrix} \beta_1(x) & \beta_2(x) \end{pmatrix}^{-1} = \begin{pmatrix} \beta_1^T(x) \\ \beta_2^T(x) \end{pmatrix}. \tag{3.6}$$

Remark 3.7 In Definition 3.6 the matrices are defined in a way which is slightly different from Definition 3.1. Nevertheless Definition 3.6 is an extension of Definition 3.1. This can be shown by comparing the present Definition 3.6 with the following straightforward extension of Definition 3.1 to the nonlinear setting:
Define a $(p \times m_1)$-matrix $\nu(x)$, a $(m_1 \times m)$-matrix $\beta_1^T(x)$ and a $((m - m_1) \times m)$-matrix $\beta_2^T(x)$ such that:

- rank $\nu(x) =$ rank $\beta_1^T(x) = m_1$;
- $k(x) = \nu(x)\beta_1^T(x)$;
- $\beta_1^T(x)\beta_1(x) = I_{m_1}$;
- $\beta_2^T(x)\beta_1(x) = 0$;

- $\beta_2^T(x)\beta_2(x) = I_{m-m_1}$.

Let now $\beta_1(x)$ and $\beta_2(x)$ be defined according to Definition 3.6. Then define $\nu(x)$ as

$$\nu(x) = k(x)\beta_1(x).$$

It follows that (using (3.6)):

$$\begin{aligned} k(x) &= k(x)I_m \\ &= k(x)\begin{pmatrix} \beta_1(x) & \beta_2(x) \end{pmatrix}\begin{pmatrix} \beta_1^T(x) \\ \beta_2^T(x) \end{pmatrix} \\ &= \begin{pmatrix} k(x)\beta_1(x) & 0 \end{pmatrix}\begin{pmatrix} \beta_1^T(x) \\ \beta_2^T(x) \end{pmatrix} \\ &= \nu(x)\beta_1^T(x). \end{aligned}$$

Hence $\nu(x)$, $\beta_1^T(x)$ and $\beta_2^T(x)$ satisfy the alternative definition given in this remark.

Conversely, if we define $\nu(x)$, $\beta_1^T(x)$ and $\beta_2^T(x)$ according to this alternative definition then $v = \beta_2(x)w$ for some w implies that

$$k(x)v = \nu(x)\beta_1^T(x)v = \nu(x)\beta_1^T(x)\beta_2(x)w = 0.$$

Therefore $v \in \ker k(x)$ and $\operatorname{im}\beta_2(x) \subset \ker k(x)$. Finally because both $\operatorname{im}\beta_2(x)$ and $\ker k(x)$ have dimension $m - m_1$, we have that $\operatorname{im}\beta_2(x) = \ker k(x)$. □

Remark 3.8 As can be seen from Remark 3.7 for a linear system the matrices Π and Φ in Definition 3.1 correspond to the matrices β_1^T, respectively β_2^T. □

3.2.1 Singular $\mathcal{H}_\infty$ control

To clarify the definitions and the choice of our assumptions we use Definition 3.6 to rewrite the system Σ. We apply to Σ the preliminary feedback

$$u = \beta(x)v$$

where

$$\beta(x) = \begin{pmatrix} \beta_1(x) & \beta_2(x) \end{pmatrix}, \tag{3.7}$$

which leads to the transformed system

$$\begin{array}{rcl} \dot{x} & = & f(x) + g_1(x)v_1 + g_2(x)v_2 + e(x)d \\ z & = & h(x) + \nu(x)v_1 \end{array} \tag{3.8}$$

where $v_1 = \beta_1^T(x)u$ can be seen as the regular part of the inputs and $v_2 = \beta_2^T(x)u$ as the singular part. The matrices $g_1(x)$ and $g_2(x)$ are given by

$$\begin{array}{rcl} g_1(x) & = & g(x)\beta_1(x) \\ g_2(x) & = & g(x)\beta_2(x) \end{array} \tag{3.9}$$

and $\nu(x)$ has full column rank. In [MvdS 96] we started with systems which are already in the special form (3.8).

Now we search for a feedback of the form

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} l_1(x) \\ l_2(x) \end{pmatrix} = l(x) \tag{3.10}$$

which solves the nonlinear state feedback $\mathcal{H}_\infty$ control problem for the system (3.8). Because the matrix $\beta(x)$ in (3.7) is invertible this state feedback $\mathcal{H}_\infty$ control problem for the transformed system is solvable with constant γ if and only if the state feedback $\mathcal{H}_\infty$ control problem for the original system Σ is solvable with the same constant γ. Moreover the feedback (3.10) solves the state feedback $\mathcal{H}_\infty$ control problem with constant γ for the system (3.8) if and only if the feedback

$$u = \beta(x)l(x)$$

solves the state feedback $\mathcal{H}_\infty$ control problem with constant γ for Σ.

So we try to find a feedback (3.10) such that the L_2-gain for the closed-loop system (3.8), (3.10) is less than (or equal to) γ.

The pre-Hamiltonian corresponding to this problem is given by

$$\begin{array}{rcl} K_\gamma(x, p, d, v) & = & p^T\left(f + g_1 v_1 + g_2 v_2 + ed\right) \\ & & +\frac{1}{2}\left(h + \nu v_1\right)^T\left(h + \nu v_1\right) - \frac{1}{2}\gamma^2\|d\|^2. \end{array}$$

As in the regular $\mathcal{H}_\infty$ problem considered in Subsection 2.2.1 the maximizing disturbance d^* and the minimizing (optimal) input v_1^* can be calculated (see also Remark 2.18) as:

$$\begin{array}{rcl} d^*(x, p) & = & \frac{1}{\gamma^2}e^T(x)p; \\ v_1^*(x, p) & = & -\left(\nu^T(x)\nu(x)\right)^{-1}\left(g_1^T(x)p + \nu^T(x)h(x)\right). \end{array}$$

For the singular part of the inputs v_2 we take an arbitrary function of x and p

$$v_2(x, p) = \alpha(x, p).$$

Substitution of these input and disturbance functions into the pre-Hamiltonian K_γ leads to the Hamiltonian $H_\gamma : T^*M \to \mathbb{R}$ given by

$$\begin{aligned}
H_\gamma(x,p) &= p^T\left(f(x) + g_1(x)v_1^*(x,p) + g_2(x)\alpha(x,p) + e(x)d^*(x,p)\right) \\
&\quad + \frac{1}{2}\left(h(x) + \nu(x)v_1^*(x,p)\right)^T\left(h(x) + \nu(x)v_1^*(x,p)\right) \\
&\quad - \frac{1}{2}\gamma^2\|d^*(x,p)\|^2 \\
&= p^T\left(f(x) - g_1(x)\left(\nu^T(x)\nu(x)\right)^{-1}\nu^T(x)h(x) + g_2(x)\alpha(x,p)\right) \\
&\quad + \frac{1}{2}\frac{1}{\gamma^2}p^Te(x)e^T(x)p - \frac{1}{2}p^Tg_1(x)\left(\nu^T(x)\nu(x)\right)^{-1}g_1^T(x)p \\
&\quad + \frac{1}{2}h^T(x)\left(I_p - \nu(x)\left(\nu^T(x)\nu(x)\right)^{-1}\nu^T(x)\right)h(x)
\end{aligned}$$

which in the original vector fields $g(x)$ and $k(x)$ can be rewritten as

$$\begin{aligned}
&H_\gamma(x,p) \\
&= p^T\left(f(x) - g(x)\beta_1(x)\left(\beta_1^T(x)k^T(x)k(x)\beta_1(x)\right)^{-1}\beta_1^T(x)k^T(x)h(x)\right) \\
&\quad + p^Tg(x)\beta_2(x)\alpha(x,p) + \frac{1}{2}\frac{1}{\gamma^2}p^Te(x)e^T(x)p \\
&\quad - \frac{1}{2}p^Tg(x)\beta_1(x)\left(\beta_1^T(x)k^T(x)k(x)\beta_1(x)\right)^{-1}\beta_1^T(x)g^T(x)p \\
&\quad + \frac{1}{2}h^T(x)h(x) \\
&\quad - \frac{1}{2}h^T(x)k(x)\beta_1(x)\left(\beta_1^T(x)k^T(x)k(x)\beta_1(x)\right)^{-1}\beta_1^T(x)k^T(x)h(x) \\
&= p^T\left(f(x) - g(x)\psi(x)k^T(x)h(x) + g(x)\beta_2(x)\alpha(x,p)\right) \\
&\quad + \frac{1}{2}\frac{1}{\gamma^2}p^Te(x)e^T(x)p - \frac{1}{2}p^Tg(x)\psi(x)g^T(x)p \\
&\quad + \frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x).
\end{aligned}$$

Now the following result can be stated.

Theorem 3.9 *Consider the nonlinear system Σ. Let $\gamma > 0$. Suppose there exists a non-negative C^1-solution V to the Hamilton-Jacobi inequality*

$$\begin{aligned} V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x) + g(x)\beta_2(x)\alpha(x, V_x^T(x))\right) & \\ +\frac{1}{2}V_x(x)\left(\frac{1}{\gamma^2}e(x)e^T(x) - g(x)\psi(x)g^T(x)\right)V_x^T(x) & \\ +\frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) & \leq 0 \end{aligned} \tag{3.11}$$

with $V(0) = 0$. Then the feedback

$$\begin{aligned} u = \varphi(x) \;=\; & -\psi(x)\left(g^T(x)V_x^T(x) + k^T(x)h(x)\right) \\ & +\beta_2(x)\alpha(x, V_x^T(x)) \end{aligned} \tag{3.12}$$

locally solves the state feedback L_2-gain control problem with constant γ for the system Σ. □

Proof This result follows by applying Theorem 2.2 to the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\varphi(x) + e(x)d \\ z &= h(x) + k(x)\varphi(x) \end{aligned}$$

where $\varphi(x)$ is taken as the feedback defined in (3.12).

As an alternative approach we could also look at the closed-loop system (3.8) together with the feedback

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\left(\nu^T(x)\nu(x)\right)^{-1}\left(g_1^T(x)V_x^T(x) + \nu^T(x)h(x)\right) \\ \alpha(x, V_x^T(x)) \end{pmatrix}.$$

Again using Theorem 2.2 it follows that the closed-loop system has L_2-gain less than or equal to γ. Then also our original system Σ together with the feedback

$$\begin{aligned} u &= \begin{pmatrix} \beta_1(x) & \beta_2(x) \end{pmatrix} v \\ &= \begin{pmatrix} \beta_1(x) & \beta_2(x) \end{pmatrix} \begin{pmatrix} -\left(\nu^T(x)\nu(x)\right)^{-1}\left(g_1^T(x)V_x^T(x) + \nu^T(x)h(x)\right) \\ \alpha(x, V_x^T(x)) \end{pmatrix} \\ &= -\psi(x)\left(g^T(x)V_x^T(x) + k^T(x)h(x)\right) + \beta_2(x)\alpha(x, V_x^T(x)) \end{aligned}$$

leads to a closed-loop system which has L_2-gain less than or equal to γ. ■

To compare this result with the results for linear systems we will make a somewhat arbitrary assumption with respect to the singular part of the feedback, $\alpha(x, p)$. From now on we take similar to the linear case

$$\alpha(x,p) = -\frac{1}{2\varepsilon} g_2^T(x) p = -\frac{1}{2\varepsilon}\beta_2^T(x) g^T(x) p.$$

Theorem 3.9 specialized for this specific choice of α reads as:

Corollary 3.10 *Consider the nonlinear system Σ. Let $\gamma > 0$. Suppose there exists a constant $\varepsilon > 0$ such that there exists a non-negative C^1-solution V to the Hamilton-Jacobi inequality*

$$\begin{aligned} V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) + \frac{1}{2}\frac{1}{\gamma^2} V_x(x)e(x)e^T(x)V_x^T(x) \\ -\frac{1}{2}V_x(x)\left(g(x)\psi(x)g^T(x) + \frac{1}{\varepsilon}g(x)\beta_2(x)\beta_2^T(x)g^T(x)\right)V_x^T(x) \\ +\frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) \leq 0 \end{aligned} \tag{3.13}$$

with $V(0) = 0$. Then the feedback

$$\begin{aligned} u = \varphi(x) &= -\left(\frac{1}{2\varepsilon}\beta_2(x)\beta_2^T(x) + \psi(x)\right) g^T(x) V_x^T(x) \\ &\quad -\psi(x)k^T(x)h(x) \end{aligned} \tag{3.14}$$

locally solves the state feedback L_2-gain control problem with constant γ for the system Σ. □

From this corollary we can easily derive the following result. This result is the nonlinear extension of the sufficiency part of Theorem 3.4 as far as only L_2-gains are considered.

Corollary 3.11 *Consider the nonlinear system Σ. Let $\gamma > 0$. Suppose there exist constants $\varepsilon, \mu > 0$ such that there exists a non-negative C^1-solution V to the Hamilton-Jacobi inequality*

$$\begin{aligned} V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) \\ +\frac{1}{2}\left(\frac{1}{\gamma^2} + \mu\right) V_x(x)e(x)e^T(x)V_x^T(x) \\ -\frac{1}{2}V_x(x)\left(g(x)\psi(x)g^T(x) + \frac{1}{\varepsilon}g(x)\beta_2(x)\beta_2^T(x)g^T(x)\right)V_x^T(x) \\ +\frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) \leq 0 \end{aligned} \tag{3.15}$$

with $V(0) = 0$. *Then there exists a constant* $\overline{\gamma} < \gamma$ *such that the feedback (3.14) locally solves the state feedback* L_2*-gain control problem with constant* $\overline{\gamma}$ *for the system* Σ. □

Proof From inequality (3.15) it follows that there exists a constant $0 \leq \tilde{\gamma} < \gamma$ such that (3.13) is satisfied with γ replaced by $\tilde{\gamma}$. For instance take $\tilde{\gamma} = \frac{\gamma}{1+\mu\gamma}$. Then by Theorem 3.9 the closed-loop system has L_2-gain less than or equal to $\tilde{\gamma}$ and therefore less than γ. ■

Remark 3.12 The Hamilton-Jacobi inequality (3.15) is the extension of the algebraic Riccati equation (3.1). For linear systems of the form (3.1) the inequality (3.15) comes down to the existence of positive constants ε, $\mu > 0$ and a positive definite solution P to

$$\begin{aligned} \tilde{Q} \; &:= \; P\left(F - G\Sigma K^T H\right) + \left(F - G\Sigma K^T H\right)^T P + \left(\frac{1}{\gamma^2} + \mu\right) PEE^T P \\ &\quad -PG\Sigma G^T P - \frac{1}{\varepsilon} PG\Phi^T \Phi G^T P + H^T\left(I_p - K\Sigma K^T\right) H \\ &\leq \; 0. \end{aligned}$$

Then this P is also a solution to (3.2) for

$$Q = \frac{\mu}{\varepsilon} PEE^T P - \frac{1}{\varepsilon}\tilde{Q} > 0.$$

On the other hand if there exists a positive definite solution P to (3.2) for a certain $Q > 0$ then by Finsler's Theorem (see [KPZ 90], [Pe 87b] and [Ja 77] for a proof) we can choose $\mu > 0$ such that

$$\mu PEE^T P < \varepsilon Q$$

and for this μ P is a solution to (3.15). □

In Corollaries 3.10 and 3.11 the feedback (3.14) consists of a singular and a regular part. The gain of the singular feedback part is parameterized by ε. We could of course also apply feedbacks of the form (3.14) where the gain $\frac{1}{2\varepsilon}$ for the singular part is replaced by a larger gain $\eta \geq \frac{1}{2\varepsilon}$.

Corollary 3.13 *Consider the nonlinear system Σ. Let $\gamma > 0$. Suppose there exists a constant $\varepsilon > 0$ such that there exists a non-negative C^1-solution V to the Hamilton-Jacobi inequality (3.13) with $V(0) = 0$. Then for $\eta \geq \frac{1}{2\varepsilon}$ the state feedback L_2-gain control problem with constant γ is locally solvable for the system Σ by the feedback*

$$\begin{aligned} u = \varphi(x) &= -\left(\eta\beta_2(x)\beta_2^T(x) + \psi(x)\right) g^T(x) V_x^T(x) \\ &\quad -\psi(x)k^T(x)h(x). \end{aligned} \tag{3.16}$$

□

Proof The result follows by applying Theorem 2.2 to the closed-loop system of Σ with the feedback (3.16), together with the following inequality

$$\begin{aligned} & V_x(x)\left(f(x) + g(x)\varphi(x)\right) + \frac{1}{2}\frac{1}{\gamma^2} V_x(x)e(x)e^T(x)V_x^T(x) \\ & \quad + \frac{1}{2}\left(h(x) + k(x)\varphi(x)\right)^T \left(h(x) + k(x)\varphi(x)\right) \\ = \; & V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) + \frac{1}{2}\frac{1}{\gamma^2} V_x(x)e(x)e^T(x)V_x^T(x) \\ & -\eta V_x(x)g(x)\beta_2(x)\beta_2^T(x)g^T(x)V_x^T(x) \\ & -\frac{1}{2}V_x(x)g(x)\psi(x)g^T(x)V_x^T(x) + \frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) \\ \leq \; & V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) + \frac{1}{2}\frac{1}{\gamma^2} V_x(x)e(x)e^T(x)V_x^T(x) \\ & -\frac{1}{2\varepsilon}V_x(x)g(x)\beta_2(x)\beta_2^T(x)g^T(x)V_x^T(x) \\ & -\frac{1}{2}V_x(x)g(x)\psi(x)g^T(x)V_x^T(x) + \frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) \\ \leq \; & 0 \end{aligned}$$

for all $\eta \geq \frac{1}{2\varepsilon}$. ■

Corollary 3.14 *Consider the nonlinear system Σ. Let $\gamma > 0$. Suppose there exist constants $\varepsilon, \mu > 0$ such that there exists a non-negative C^1-solution V to the Hamilton-Jacobi inequality (3.15). Then there exists a constant $\overline{\gamma} < \gamma$ such that the feedback (3.16) locally solves the nonlinear state feedback L_2-gain control problem with constant $\overline{\gamma}$ for the system Σ for every $\eta \geq \frac{1}{2\varepsilon}$.* □

Proof The proof follows easily along the lines of the proofs of Corollary 3.11 and Corollary 3.13. ■

The converse result of Corollary 3.10 is more involved. Assume there exists a feedback $u = l(x)$, $l(0) = 0$, such that the closed-loop system of Σ with this feedback has L_2-gain less than (or equal to) γ. When does there exists a solution $V \geq 0$ to the Hamilton-Jacobi inequality (3.13) or (3.15)?

We start by making the following assumption on the closed-loop system resulting from applying $u = l(x)$, $l(0) = 0$.

Assumption 3 The L_2-gain from d to $\beta_2^T(x)l(x)$ is finite, i.e., there exists a constant $N > 0$ such that for all $x \in M$ there exists a constant $\tilde{K}(x)$, $0 \leq \tilde{K}(x) < \infty$, with $\tilde{K}(0) = 0$ such that

$$\int_0^t \|\beta_2^T(x(\tau))l(x(\tau))\|^2 \mathrm{d}\tau \leq N \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + \tilde{K}(x)$$

for all $t > 0$ and all $d \in L_2(0, t)$, where $x(\tau)$ is the solution of the state equation of the closed loop system Σ with the feedback $u = l(x)$. □

Then the following converse result can be stated.

Theorem 3.15 *Consider the system Σ. Suppose that the feedback $u = l(x)$ with $l(0) = 0$ solves the state feedback L_2-gain control problem with $\overline{\gamma} < \gamma$ for the system Σ with a differentiable storage function $K(x)$ and Assumption 3 is satisfied with a differentiable storage function $\tilde{K}(x)$. Then there exists a solution V of (3.15) for certain ε, $\mu > 0$, and hence also the feedback (3.16) results in a closed-loop system which has L_2-gain less than γ.* □

Proof The system

$$\begin{aligned} \dot{x} &= f(x) + g(x)l(x) + e(x)d \\ z &= h(x) + k(x)l(x) \end{aligned}$$

has L_2-gain less than γ. We rewrite this system as done in the beginning of this section (see system (3.8)), using the notation defined in (3.9). Then also the system

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)l_1(x) + g_2(x)l_2(x) + e(x)d \\ z &= h(x) + u(x)l_1(x) \end{aligned}$$

with $l_1(x) = \beta_1^T(x)l(x)$ and $l_2(x) = \beta_2^T(x)l(x)$ has L_2-gain less than γ. Hence there exists a constant $\nu > 0$ such that for all $x \in M$ there exists a differentiable function $K(x)$, $0 \leq K(x) < \infty$, with $K(0) = 0$, such that

$$\int_0^t \|z(x(\tau))\|^2 \mathrm{d}\tau \leq (\gamma^2 - \nu) \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + K(x(0)) \tag{3.17}$$

holds for all $d \in L_2[0, t]$ and for all $t \geq 0$. On the other hand from Assumption 3 it follows that there exists a constant $N > 0$ such that for every $x \in M$ there exists a differentiable $\tilde{K}(x)$, $0 \leq \tilde{K}(x) < \infty$, with $\tilde{K}(x(0)) = 0$, such that

$$\begin{aligned} \int_0^t \|\beta_2^T(x(\tau))l(x(\tau))\|^2 \mathrm{d}\tau &= \int_0^t \|l_2(x(\tau))\|^2 \mathrm{d}\tau \\ &\leq N \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + \tilde{K}(x(0)). \end{aligned} \tag{3.18}$$

Combining the inequalities (3.17) and (3.18) one obtains

$$\begin{aligned} &\int_0^t \left(\|z(x(\tau))\|^2 + \varepsilon \|l_2(x(\tau))\|^2\right) \mathrm{d}\tau \\ &\qquad \leq (\gamma^2 - \beta) \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + K(x(0)) + \varepsilon \tilde{K}(x(0)) \end{aligned}$$

for $0 < \varepsilon < \nu/N$ and $\beta > 0$ sufficiently small. According to Assumption 3 there exists for $\varepsilon, \mu > 0$ sufficiently small a solution $V \geq 0$ (for instance $V(x) = K(x) + \varepsilon \tilde{K}(x)$) to

$$\begin{aligned} &V_x(x)\left(f(x) + g_1(x)l_1(x) + g_2(x)l_2(x)\right) \\ &+\frac{1}{2}\left(\frac{1}{\gamma^2} + \mu\right) V_x(x)e(x)e^T(x)V_x^T(x) + \frac{1}{2}\|z(x)\|^2 + \frac{1}{2}\varepsilon\|l_2(x)\|^2 \leq 0 \end{aligned}$$

with $V(0) = 0$.

Finally we use a completion of the squares argument

$$\begin{aligned} &V_x(x)\Big(f(x) - g_1(x)\left(v^T(x)v(x)\right)^{-1}\left(g_1^T(x)V_x^T(x) + u^T(x)h(x)\right) \\ &\quad -\frac{1}{\varepsilon}g_2(x)g_2^T(x)V_x^T(x)\Big) \\ &\leq V_x(x)\left(f(x) + g_1(x)l_1(x) + g_2(x)l_2(x)\right) \\ &\quad +V_x(x)g_1(x)\left(-\left(v^T(x)v(x)\right)^{-1}\left(g_1^T(x)V_x^T(x) + v^T(x)h(x)\right) - l_1(x)\right) \\ &\quad +V_x(x)g_2(x)\left(-\frac{1}{\varepsilon}g_2^T(x)V_x^T(x) - l_2(x)\right) \end{aligned}$$

$$
\begin{aligned}
\leq\ & -\frac{1}{2}\left(\frac{1}{\gamma^2}+\mu\right) V_x(x)e(x)e^T(x)V_x^T(x) - \frac{1}{2}h^T(x)h(x) \\
& -\frac{1}{2}\left\| \left(\nu^T(x)\nu(x)\right)^{\frac{1}{2}} l_1(x) \right. \\
& \qquad \left. + \left(\nu^T(x)\nu(x)\right)^{-\frac{1}{2}} \left(g_1^T(x)V_x^T(x) + \nu^T(x)h(x)\right) \right\|^2 \\
& -\frac{1}{2}\left\| l_2(x) + \frac{1}{\varepsilon}g_2^T(x)V_x^T(x) \right\|^2 - \frac{1}{2\varepsilon}V_x(x)g_2(x)g_2^T(x)V_x^T(x) \\
& -\frac{1}{2}V_x(x)g_1(x)\left(\nu^T(x)\nu(x)\right)^{-1} g_1^T(x)V_x^T(x) \\
& +\frac{1}{2}h^T(x)\nu(x)\left(\nu^T(x)\nu(x)\right)^{-1}\nu^T(x)h(x)
\end{aligned}
$$

from which it can be concluded that

$$
\begin{aligned}
& V_x(x)\left(f(x) - g_1(x)\left(\nu^T(x)\nu(x)\right)^{-1}\nu^T(x)h(x)\right) \\
& \qquad +\frac{1}{2}\left(\frac{1}{\gamma^2}+\mu\right) V_x(x)e(x)e^T(x)V_x^T(x) \\
& -\frac{1}{2}V_x(x)\left(g_1(x)\left(\nu^T(x)\nu(x)\right)^{-1} g_1^T(x) + \frac{1}{\varepsilon}g_2(x)g_2^T(x)\right)V_x^T(x) \\
& \qquad +\frac{1}{2}h^T(x)\left(I_p - \nu(x)\left(\nu^T(x)\nu(x)\right)^{-1}\nu^T(x)\right)h(x) \ \leq\ 0.
\end{aligned}
$$

Substitution of g_1 and g_2 now leads to

$$
\begin{aligned}
& V_x(x)\left(f(x) - g(x)\beta_1(x)\left(\beta_1^T(x)k^T(x)k(x)\beta_1(x)\right)^{-1}\beta_1^T(x)k^T(x)h(x)\right) \\
& \quad +\frac{1}{2}h^T(x)\left(I_p - k(x)\beta_1(x)\left(\beta_1^T(x)k^T(x)k(x)\beta_1(x)\right)\beta_1^T(x)k^T(x)\right)h(x) \\
& \quad +\frac{1}{2}V_x(x)\left(\left(\frac{1}{\gamma^2}+\mu\right)e(x)e^T(x) - \frac{1}{\varepsilon}g(x)\beta_2(x)\beta_2^T(x)g^T(x)\right)V_x^T(x) \\
& \quad -\frac{1}{2}V_x(x)g(x)\beta_1(x)\left(\beta_1^T(x)k^T(x)k(x)\beta_1(x)\right)\beta_1^T(x)g^T(x)V_x^T(x) \\
=\ & V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) \\
& +\frac{1}{2}\left(\frac{1}{\gamma^2}+\mu\right)V_x(x)e(x)e^T(x)V_x^T(x) \\
& -\frac{1}{2}V_x(x)g(x)\left(\psi(x) + \frac{1}{\varepsilon}\beta_2(x)\beta_2^T(x)\right)g^T(x)V_x^T(x) \\
& +\frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) \leq 0
\end{aligned}
$$

and the conclusion follows from Corollary 3.14. ■

Remark 3.16 Corollary 3.14 together with Theorem 3.15 gives under Assumption 3 a necessary and sufficient condition for the solvability of the state feedback L_2-gain control problem with constant less than γ. The question arises under what conditions a feedback of the form (3.16) will satisfy Assumption 3. A partial answer to this question will be given in the next subsection (see Corollary 3.20). □

Remark 3.17 For linear systems Assumption 3 is already implied by the fact that the feedback is assumed to be internally stabilizing. Thus Theorem 3.15 is a generalization of Theorem 3.4 to the nonlinear setting. □

3.2.2 Regularized $\mathcal{H}_\infty$ control

A different but similar way to attack the singular $\mathcal{H}_\infty$ problem is considered in this subsection. Let $\varepsilon > 0$. We consider the following *regularized* version of the system Σ:

$$\Sigma_r \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + e(x)d \\ z_r & = & \begin{pmatrix} h(x) + k(x)u \\ \sqrt{\varepsilon}\beta_2^T(x)u \end{pmatrix} \end{array} \right. \tag{3.19}$$

Using Assumption 3 we can state the following:

Theorem 3.18 *Let $u = l(x)$, $l(0) = 0$, be a feedback for the system Σ. Then we have the following implications regarding the statements (i) and (ii) below. The statement (ii) implies (i), and under Assumption 3 the statements (i) and (ii) are equivalent.*

(i) The closed loop system Σ with static state feedback $u = l(x)$ has L_2-gain less than γ.

(ii) For ε sufficiently small the closed loop system Σ_r with static state feedback $u = l(x)$ has L_2-gain less than γ.

□

Proof *(i)* $\Rightarrow$ *(ii)* By Assumption 3 there exists a constant $N > 0$ such that for all $x \in M$ there exists a constant $\tilde{K}(x)$, $0 \leq \tilde{K}(x) < \infty$, with $\tilde{K}(0) = 0$ such that

$$\int_0^t \|\beta_2^T(x(\tau))l(x(\tau))\|^2 \mathrm{d}\tau \leq N \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + \tilde{K}(x(0))$$

for all $t > 0$ and all $d \in L_2(0, t)$.

From Definition 2.14 we know that there exists a constant $\delta > 0$ such that for every $x \in M$ there exists a constant $K(x)$, $0 \leq K(x) < \infty$, with $K(0) = 0$, such that

$$\int_0^t \|z(x(\tau))\|^2 \mathrm{d}\tau \leq (\gamma^2 - \delta) \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + K(x(0))$$

for all $d \in L_2(0, t)$ and all $t \geq 0$.

Now take $\varepsilon > 0$ such that $\varepsilon N < \delta$. Then some $\mu > 0$ can be found such that, with bias term $M(x) := K(x) + \varepsilon \tilde{K}(x)$, the following inequality holds

$$\begin{aligned} &\int_0^t (\|z(x(\tau))\|^2 + \varepsilon \|\beta_2^T(x(\tau))l(x(\tau))\|^2) \mathrm{d}\tau \\ \leq\ & (\gamma^2 - \mu) \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + M(x(0)) \end{aligned}$$

for all $t > 0$ and all $d \in L_2(0, t)$. Hence $u = l(x)$ combined with Σ_r leads to a closed loop system which has L_2-gain from d to z_r less than γ.

For proving *(ii)* $\Rightarrow$ *(i)* we note that if $u = l(x)$ solves the suboptimal problem for the system Σ_r then it also solves the suboptimal problem for Σ because

$$\int_0^t \|z_r(\tau)\|^2 \mathrm{d}\tau \geq \int_0^t \|z(\tau)\|^2 \mathrm{d}\tau.$$

■

Based on Theorem 3.18 we will search for a state feedback which makes the L_2-gain for the system Σ_r less than γ. Since Σ_r is a regular system we can find the min-max solution for this $\mathcal{H}_\infty$ problem. The pre-Hamiltonian $K_\gamma : T^*M \times \mathbb{R}^q \times \mathbb{R}^m$ corresponding to this problem is

$$\begin{aligned} K_\gamma(x, p, d, u) &= p^T(f(x) + g(x)u + e(x)d) \\ &\quad -\frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}\|z\|^2 + \frac{1}{2}\varepsilon\|\beta_2^T(x)u\|^2 \end{aligned}$$

which has saddle point solution:

$$\begin{aligned} d^* &= \frac{1}{\gamma^2} e^T(x) p, \\ u^* &= -\left(\frac{1}{\varepsilon}\beta_2(x)\beta_2^T(x) + \psi(x)\right) g^T(x) p - \sigma(x) k^T(x) h(x). \end{aligned} \tag{3.20}$$

Substitution of this saddle point solution into the pre-Hamiltonian K_γ leads to the Hamiltonian $H_\gamma(x,p) = K_\gamma(x,p,d^*(x,p),u^*(x,p))$ given as

$$\begin{aligned} H_\gamma(x,p) &= p^T\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) + \frac{1}{2}\frac{1}{\gamma^2} p^T e(x) e^T(x) p \\ &\quad -\frac{1}{2} p^T g(x)\left(\psi(x) + \frac{1}{\varepsilon}\beta_2(x)\beta_2^T(x)\right) g^T(x) p \\ &\quad +\frac{1}{2} h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right) h(x). \end{aligned}$$

Theorem 3.19 *Consider the nonlinear system Σ. Let $\gamma > 0$. Suppose there exists a C^1 solution $V \geq 0$ to the Hamilton-Jacobi inequality (3.13). Then the state feedback*

$$\begin{aligned} u = l(x) &= -\left(\frac{1}{\varepsilon}\beta_2(x)\beta_2^T(x) + \psi(x)\right) g^T(x) V_x^T(x) \\ &\quad -\psi(x)k^T(x)h(x) \end{aligned} \tag{3.21}$$

locally solves the state feedback L_2-gain problem with constant γ for the system Σ. □

Proof The closed loop system Σ_r, (3.21) is given by

$$\begin{aligned} \dot{x} &= f(x) - g(x)\left(\frac{1}{\varepsilon}\beta_2(x)\beta_2^T(x) + \psi(x)\right) g^T(x) V_x^T(x) \\ &\quad -g(x)\sigma(x)k^T(x)h(x) + e(x)d \\ z_r &= \begin{pmatrix} h(x) + k(x)u \\ -\frac{1}{\sqrt{\varepsilon}}\beta_2^T(x) g^T(x) V_x^T(x) \end{pmatrix} \end{aligned}$$

which has by Theorem 2.2 and Theorem 3.18 L_2-gain less than or equal to γ. ■

Based on this Theorem we can give the following (partial) answer to the question posed in Remark 3.16.

Corollary 3.20 *Consider the system Σ. Assume there exists a solution $V \geq 0$ to the Hamilton-Jacobi inequality (3.13). Then for any $\eta \geq \frac{1}{\varepsilon}$ the feedback (3.16) satisfies Assumption 3.* □

Proof For a fixed $\eta \geq \frac{1}{\varepsilon}$ look at the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + e(x)d, \\ z &= \begin{pmatrix} h(x) + k(x)u \\ \frac{1}{\sqrt{\eta}}\beta_2^T(x)u \end{pmatrix}. \end{aligned}$$

The Hamiltonian for the $\mathcal{H}_\infty$ problem for this system is given by

$$\begin{aligned} H_\gamma(x,p) &= p^T\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) + \frac{1}{2}\frac{1}{\gamma^2}p^Te(x)e^T(x)p \\ &\quad -\frac{1}{2}p^Tg(x)\left(\psi(x) + \eta\beta_2(x)\beta_2^T(x)\right)g^T(x)p \\ &\quad +\frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x). \end{aligned}$$

Hence for $\eta \geq \frac{1}{\varepsilon}$:

$$\begin{aligned} & V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) + \frac{1}{2}\frac{1}{\gamma^2}V_x(x)e(x)e^T(x)V_x^T(x) \\ &\quad -\frac{1}{2}V_x(x)g(x)\psi(x)g^T(x)V_x^T(x) \\ &\quad -\frac{1}{2}\eta V_x(x)g(x)\beta_2(x)\beta_2^T(x)g^T(x)V_x^T(x) \\ &\quad +\frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) \\ &\leq V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) + \frac{1}{2}\frac{1}{\gamma^2}V_x(x)e(x)e^T(x)V_x^T(x) \\ &\quad -\frac{1}{2}V_x(x)g(x)\psi(x)g^T(x)V_x^T(x) \\ &\quad -\frac{1}{2}\frac{1}{\varepsilon}V_x(x)g(x)\beta_2(x)\beta_2^T(x)g^T(x)V_x^T(x) \\ &\quad +\frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) \\ &\leq 0. \end{aligned}$$

It follows that the gain from d to $\sqrt{\eta}\beta_2^T(x)g^T(x)V_x^T(x)$ is less than or equal to γ. Hence the gain from d to $\eta\beta_2^T(x)g^T(x)V_x^T(x)$ is less than or equal to $\gamma\sqrt{\eta}$. ■

Until now we have not considered the stability of the closed loop system. The following theorem can be easily obtained from Theorem 2.7.

Theorem 3.21 *Suppose there exists a solution $V \geq 0$ to (3.13). Assume the closed-loop system of Σ with a feedback of the form (3.16) for $\eta \geq \frac{1}{2\varepsilon}$ is zero-state observable. Then $V(x) > 0$ for $x \neq 0$ and the closed loop system (with $d(t) \equiv 0$) is locally asymptotically stable.*

Assume additionally that V is proper, then the closed loop system is globally asymptotically stable. □

3.2.3 The nonlinear system versus its linearization

In this section we consider the link between the solvability of the state feedback L_2-gain and $\mathcal{H}_\infty$ control problems (see Definitions 2.14 and 2.15) for the nonlinear system Σ and the solvability of these problems for its linearization. So we consider the linearization of the nonlinear system Σ around the equilibrium $x = 0$:

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{u} + E\bar{d} \\ \bar{z} &= H\bar{x} + K\bar{u} \end{aligned} \tag{3.22}$$

where $\bar{u} \in \mathbb{R}^m$, $\bar{x} \in \mathbb{R}^n$, $\bar{d} \in \mathbb{R}^q$, $\bar{z} \in \mathbb{R}^p$ and the matrices F, G, K, H and M are defined as:

$$F = \frac{\partial f}{\partial x}(0), \quad G = g(0), \quad E = e(0), \quad H = \frac{\partial h}{\partial x}(0), \quad K = k(0).$$

Straightforwardly from Chapter 2 the following results are obtained.

Theorem 3.22 *Suppose the L_2-gain of (3.4), (3.5) is less than γ, and assume $F + GL$ with $L = \frac{\partial l}{\partial x}(0)$ is asymptotically stable, then there exists a neighborhood W of 0 and a smooth function $V \geq 0$ on W satisfying (3.15).*

Alternatively, assume $f + gl$ is globally asymptotically stable. Define the Hamiltonian

$$\begin{aligned} H_\gamma(x, p) &= p^T\left(f(x) + g(x)l(x)\right) + \frac{1}{\gamma^2}p^T e(x)e^T(x)p \\ &\quad + \frac{1}{2}\left(h(x) + k(x)l(x)\right)^T\left(h(x) + k(x)l(x)\right) \end{aligned}$$

and suppose X_{H_γ} is hyperbolic at $(0, 0)$, and its stable invariant manifold is diffeomorphic to M under the canonical projection $\pi : T^*M \to M$. Then there exists a global solution $V \geq 0$ to (3.15). □

Theorem 3.23 *Let $\gamma > 0$. Suppose there exists a smooth feedback $u = l(x)$, with $l(0) = 0$, for Σ such that the L_2-gain of the nonlinear system Σ, (3.5) is less than (or equal to) γ. Then the linear feedback $\bar{u} = L\bar{x}$, with $L = \frac{\partial l}{\partial x}(0)$, for (3.22) results in the linear closed loop system*

$$\begin{aligned}\dot{\bar{x}} &= (F+GL)\,\bar{x} + E\bar{d} \\ \bar{z} &= (H+KL)\,\bar{x}\end{aligned} \tag{3.23}$$

which also has L_2-gain less than (or equal to) γ. □

Proof The linearization of Σ with (3.5) is equal to (3.23) ([vdS 92]). ■

For the linearization (3.22) the matrices described in Definition 3.1 are given by

$$U = \nu(0) = k(0)\beta_1(0), \quad \Sigma = \psi(0), \quad \Phi = \beta_2^T(0).$$

Remark 3.24 If the state feedback L_2-gain control problem with gain less than or equal to γ is solvable for the nonlinear system in the sense that for some $\varepsilon > 0$ there exists a differential non-negative solution V to (3.13) then we know from Corollary 3.10 that (3.14) is a solution of the state feedback L_2-gain control problem for the nonlinear system Σ. From Theorem 3.23 it follows that then the linear feedback

$$\begin{aligned}\bar{u} &= -\left(\frac{1}{2\varepsilon}\beta_2(0)\beta_2^T(0) + \psi(0)\right) g^T(0)V_{xx}^T(0) - \psi(0)k^T(0)\frac{\partial h}{\partial x}(0) \\ &= -\left(\frac{1}{2\varepsilon}\Phi^T\Phi + \Sigma\right) G^T P^T \bar{x} - \Sigma K^T H\bar{x}\end{aligned} \tag{3.24}$$

solves the state feedback L_2-gain optimal control problem for the linearized system (3.22) and

$$P = V_{xx}(0)$$

is a solution of

$$\begin{aligned}&\left(F - G\Sigma K^T H\right)^T P + P\left(F - G\Sigma K^T H\right) + \frac{1}{\gamma^2} PEE^T P \\ &-PG\Sigma G^T P - \frac{1}{\varepsilon} PG\Phi^T\Phi G^T P + H^T\left(I - K\Sigma K^T\right) H \;\leq\; 0\end{aligned}$$

On the other hand when the state feedback L_2-gain control problem with gain less than γ is solvable in the sense that for some $\varepsilon > 0$ and $\mu > 0$ there exists a differentiable non-negative solution V to (3.15) then again the feedback

(3.14) solves the problem and the feedback (3.24) solves the state feedback $\mathcal{H}_\infty$ control problem with gain less than γ for the linearization (3.22). Furthermore similar to Remark 3.12 there can be chosen a matrix Q such that $P = V_{xx}(0)$ is a solution to (3.2). □

Furthermore the following connection between the nonlinear system and its linearization can be stated.

Theorem 3.25 *Consider the linearized system (3.22). Let $\gamma > 0$. Suppose there exists a feedback $\bar{u} = L\bar{x}$ such that the L_2-gain of the closed loop system (from $\bar{d}$ to $\bar{z}$) is* less than *γ and the closed loop system is asymptotically stable.*
Then there exists a neighborhood W of x_0 and a smooth function $V \geq 0$ defined on W such that V is a solution of the Hamilton-Jacobi inequality (3.15). Furthermore, for $\eta \geq \frac{1}{2\varepsilon}$, the feedback (3.16) locally solves the state feedback L_2-gain problem with constant less than γ for the system Σ. □

Proof The L_2-gain of the closed loop linearized system is less than γ. By definition this means that there exists a constant $\tilde{\gamma} < \gamma$ for which the closed-loop system has L_2-gain less than or equal to $\tilde{\gamma}$. This implies that for every $\hat{\gamma}$ which satisfies $\tilde{\gamma} < \hat{\gamma} < \gamma$ we have that the closed-loop system has L_2-gain less than $\hat{\gamma}$. Hence by the Theorems 3.4 and 3.5 there exists for every $Q > 0$ a constant $\varepsilon^* > 0$ and a stabilizing solution $P_\varepsilon > 0$ of the ARE

$$\begin{aligned}(F - G\Sigma K^T H)^T P + P(F - G\Sigma K^T H) + \left(\frac{1}{\gamma^2} + \nu\right) PEE^T P & \\ -PG\Sigma G^T P - \frac{1}{\varepsilon} PG\Phi^T \Phi G^T P + H^T\left(I - K\Sigma K^T\right) H + \varepsilon Q &= 0\end{aligned}$$

for some $\kappa > 0$ and $\varepsilon \in (0, \varepsilon^*]$. Now take the Hamiltonian

$$\begin{aligned}H_\gamma(x, p) := \; & p^T\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) \\ & + \frac{1}{2}\left(\frac{1}{\gamma^2} + \kappa\right) p^T e(x) e^T(x) p \\ & - \frac{1}{2} p^T g(x)\psi(x) g^T(x) p - \frac{1}{2\varepsilon} p^T g(x)\beta_2(x)\beta_2^T(x) g^T(x) p \\ & + \frac{1}{2} h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right) h(x) + \varepsilon q(x)\end{aligned}$$

where q is an arbitrary function $q : M \to \mathbb{R}^+$ which satisfies

$$q(0) = 0, \quad \frac{\partial q}{\partial x}(0) = 0, \quad \frac{\partial^2 q}{\partial x^2}(0) = Q > 0.$$

Then the linearization of X_{H_γ} at $(0, 0)$ is given by the Hamiltonian matrix

$$DX_{H_\gamma}(0,0) = \tag{3.25}$$

$$\begin{pmatrix} F - G\Sigma K^T H & (\frac{1}{\gamma^2} + \kappa)EE^T - G\Sigma G^T - \frac{1}{\varepsilon}G\Phi^T\Phi G^T \\ -H^T\left(I_p - K\Sigma K^T\right)H - \varepsilon Q & -\left(F - G\Sigma K^T H\right)^T \end{pmatrix}.$$

Then $P_\varepsilon = P_\varepsilon^T$ is a solution of (3.2) if and only if

$$\begin{pmatrix} F - G\Sigma K^T H & (\frac{1}{\gamma^2} + \kappa)EE^T - G\Sigma G^T - \frac{1}{\varepsilon}G\Phi^T\Phi G^T \\ -H^T\left(I_p - K\Sigma K^T\right)H - \varepsilon Q & -\left(F - G\Sigma K^T H\right)^T \end{pmatrix}\begin{bmatrix} I \\ P_\varepsilon \end{bmatrix}$$

$$= \begin{bmatrix} I \\ P_\varepsilon \end{bmatrix}\left(F + (\frac{1}{\gamma^2} + \kappa)EE^T P_\varepsilon - G\Sigma G^T P_\varepsilon - \frac{1}{\varepsilon}G\Phi^T\Phi G^T P_\varepsilon\right)$$

and thus

$$\text{span}\begin{bmatrix} I \\ P_\varepsilon \end{bmatrix} = \text{stable eigenspace of}$$

$$\begin{pmatrix} F - G\Sigma K^T H & (\frac{1}{\gamma^2} + \kappa)EE^T - G\Sigma G^T - \frac{1}{\varepsilon}G\Phi^T\Phi G^T \\ -H^T\left(I_p - K\Sigma K^T\right)H - \varepsilon Q & -\left(F - G\Sigma K^T H\right)^T \end{pmatrix}$$

for some ε. Thus the Hamiltonian matrix (3.25) does not have imaginary eigenvalues.

Then the stable invariant manifold N^- of X_{H_γ} through $(0, 0)$ is n-dimensional and is tangent at $(0, 0)$ to

$$\text{span}\begin{bmatrix} I \\ P_\varepsilon \end{bmatrix}$$

for $\varepsilon \in (0, \varepsilon^*)$.

Furthermore locally around 0 the manifold N^- is given as

$$N^- = \left\{\left(x, p = \frac{\partial^T V_\varepsilon}{\partial x}(x)\right)\middle|\, x \text{ around } 0\right\}$$

where V_ε is a (local) solution of the Hamilton-Jacobi equation

$$\begin{aligned} & V_x(x)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right) \\ & + \frac{1}{2}\left(\frac{1}{\gamma^2} + \kappa\right)V_x(x)e(x)e^T(x)V_x^T(x) \\ -\frac{1}{2}V_x(x) & \left(g(x)\psi(x)g^T(x) + \frac{1}{\varepsilon}g(x)\beta_2(x)\beta_2^T(x)g^T(x)\right)V_x^T(x) \\ & + \frac{1}{2}h^T(x)\left(I_p - k(x)\psi(x)k^T(x)\right)h(x) \quad \le \quad 0 \end{aligned}$$

with $V_{xx}(0) = P$.
From Theorem 2.12 $V_\varepsilon \geq 0$ because

$$F - G\Sigma K^T H - G\Sigma G^T P_\varepsilon - \frac{1}{\varepsilon} G\Phi^T \Phi G^T P_\varepsilon$$

is asymptotically stable. Then the result follows from Corollary 3.14. ∎

Finally summarizing some of these results we can state the following extension of Theorem 2.22.

Theorem 3.26 *Consider the nonlinear system Σ and its linearization (3.22). Then the following statements are equivalent:*

(i) there exists a linear feedback $\bar{u} = L\bar{x}$ such that after applying this feedback to the system Σ the closed loop system has L_2-gain less than γ and $F + GL$ is asymptotically stable;

(ii) there exists a positive definite solution P_ε to the Algebraic Riccati Equation

$$\begin{aligned} (F - G\Sigma K^T H)^T P + P(F - G\Sigma K^T H) + \frac{1}{\gamma^2} PEE^T P & \\ -PG\left(\Sigma + \frac{1}{\varepsilon}\Phi^T\Phi\right) G^T P + H^T(I - K\Sigma K^T)H + \varepsilon Q &= 0 \end{aligned}$$

for all $Q > 0$ and for some $\varepsilon > 0$, which also satisfies

$$\sigma\left(F + \frac{1}{\gamma^2} EE^T P - G\Sigma(G^T P + K^T H) - \frac{1}{\varepsilon} G\Phi^T\Phi G^T P\right) \subset \mathbb{C}^-;$$

(iii) there exists a neighborhood $W \subset M$ of 0, and a nonlinear state feedback $u = l(x)$ defined on W, such that $F + GL$, with $L = \frac{\partial l}{\partial x}(0)$, is asymptotically stable and the closed loop system of this nonlinear feedback and the system Σ has locally L_2-gain less than γ on W.

□

3.3 Singular nonlinear measurement feedback $\mathcal{H}_\infty$ control

It is possible to extend the state feedback results to the measurement feedback problem by using the worst case certainty equivalence principle. Consider systems of the form:

$$\Sigma^m \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + e(x)d_1 \\ y & = & c(x) + d_2 \\ z & = & h(x) + k(x)u \end{array} \right. \tag{3.26}$$

We search for a dynamic affine nonlinear compensator

$$\begin{array}{rcl} \dot{\xi} & = & w(\xi) + p(\xi)y \\ u & = & q(\xi) \end{array} \tag{3.27}$$

with $w(0) = 0$ and $q(0) = 0$, such that the closed loop system has L_2-gain less than γ.

For application of this worst case certainty equivalence principle we again use a regularized version of our system Σ^m, given by

$$\Sigma_r^m \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + e(x)d_1 \\ y & = & c(x) + d_2 \\ \bar{z} & = & \begin{pmatrix} h(x) + k(x)u \\ \sqrt{\varepsilon}\beta_2^T(x)u \end{pmatrix} \end{array} \right. \tag{3.28}$$

To be able to derive a result similar to Theorem 3.18 we make the following assumption.

Assumption 4 Let (3.27) be a compensator for Σ^m. We assume that the L_2-gain from d_1, d_2 to $\beta_2^T(x)q(\xi)$ is finite, i.e., there exists a constant $N > 0$ such that for all $x \in M$, $\xi \in M_c$ there exists a constant $\tilde{K}_c(x, \xi)$, $0 \leq \tilde{K}_c(x, \xi) < \infty$, with $\tilde{K}(0, 0) = 0$ such that

$$\begin{aligned} &\int_0^t \|\beta_2^T(x(\tau))q(\xi(\tau))\|^2 \mathrm{d}\tau \\ &\quad \leq N \int_0^t \left(\|d_1(\tau)\|^2 + \|d_2(\tau)\|^2\right) \mathrm{d}\tau + \tilde{K}_c(x(0), \xi(0)) \end{aligned}$$

for all $t > 0$ and all $d_1,\ d_2 \in L_2(0, t)$, where $x(\tau)$ is the solution of the state equation of the closed loop system of Σ^m together with the compensator (3.27). □

Then the following result can be stated.

Theorem 3.27 *Let (3.27) be a compensator for Σ^m. We have the following implications regarding the statements (i) and (ii) below. Statement (ii) implies (i), and under Assumption 4 the statements (i) and (ii) are equivalent.*

(i) The closed loop system consisting of Σ^m with the compensator (3.27) has L_2-gain less than γ.

(ii) For ε sufficiently small the closed loop system consisting of Σ_r^m with the compensator (3.27) has L_2-gain less than γ.

□

The proof is similar to the proof of Theorem 3.18 and will be omitted.

Now we search our compensator among the set of compensators which result in a finite L_2-gain from d_1, d_2 to $\beta_2^T(x)q(\xi)$.

We will do so by applying the worst case certainty equivalence principle to the regularized system Σ_r^m. This principle consists in first solving the state feedback problem and then replacing the state by an estimated state. A sufficient condition for the worst case certainty equivalence principle to hold is that there exists a saddle point solution to this state feedback problem. This is the reason for considering a regularized version of the system Σ^m. Now a brief exposition of the construction of the compensator based on the worst-case certainty equivalence principle will be given (for more details see [BB 90], [vdS 93]).

We start by considering the $\mathcal{H}_\infty$ problem on a finite time horizon, i.e., we consider the L_2-gain on some fixed finite interval $[T_1, T_2]$. This amounts to the max-min solution of the performance criterion

$$\frac{1}{2}\int_{T_1}^{T_2} \big(\varepsilon\|\beta_2^T(x)u\|^2 + (h(x)+k(x)u)^T(h(x)+k(x)u) \\ -\gamma^2 d_1^T d_1 - \gamma^2 d_2^T d_2\big)\,\mathrm{d}t \qquad (3.29)$$

where the control $u(t)$, $t \in [T_1, T_2]$, is only allowed to depend on $y(\tau)$ with $T_1 \le \tau \le t$. This problem can be split into two parts.

First we look at the state feedback $\mathcal{H}_\infty$ control problem, considered in Subsection 3.2 for the infinite horizon case. This leads in the finite horizon case to the non-stationary Hamilton-Jacobi equation (compare with (3.15))

$$V_t(x,t) + V_x(x,t)\left(f(x) - g(x)\psi(x)k^T(x)h(x)\right)$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\frac{1}{\gamma^2}+\delta\right) V_x(x,t)e(x)e^T(x)V_x^T(x,t) \\
&-\frac{1}{2}V_x(x,t)\left(g(x)\psi(x)g^T(x)+\frac{1}{\varepsilon}g(x)\beta_2(x)\beta_2^T(x)g^T(x)\right)V_x^T(x,t) \\
&+\frac{1}{2}h^T(x)\left(I_p-k(x)\psi(x)k^T(x)\right)h(x) = 0 \\
&V(x,T_2) = 0
\end{aligned}
$$

with resulting suboptimal state feedback

$$
u(t) = -\left(\frac{1}{\varepsilon}\beta_2(x)\beta_2^T(x)+\psi(x)\right)g^T(x)V_x^T(x,t)-\psi(x)k^T(x)h(x). \tag{3.30}
$$

Secondly, let $\tau \in [T_1, T_2]$, and let $\bar{u}(t)$ and $\bar{y}(t)$, $t \in [T_1, \tau]$ be a given pair of inputs and corresponding measured output trajectories of the system Σ_r^m. Then we look for the maximizing solution $x(T_1)$ and $d_1(t)$, $d_2(t)$ of the performance criterion

$$
\begin{aligned}
\frac{1}{2}\int_{T_1}^{\tau} &\left(\varepsilon\|\beta_2^T(x)\bar{u}\|^2+(h(x)+k(x)\bar{u})^T(h(x)+k(x)\bar{u})\right. \\
&\left.-\gamma^2 d_1^T d_1-\gamma^2 d_2^T d_2\right)\mathrm{d}t+V(x(\tau),\tau)
\end{aligned} \tag{3.31}
$$

which satisfies the constraint that the measured output equals $\bar{y}(t)$. We assume that this maximization problem has a unique solution for every $\tau \in [T_1, T_2]$. Then we define for every $\tau \in [T_1, T_2]$

$$
\begin{aligned}
\tilde{u}(\tau) &= -\left(\frac{1}{\varepsilon}\beta_2(\hat{x}(\tau))\beta_2^T(\hat{x}(\tau))+\psi(\hat{x}(\tau))\right)g^T(\hat{x}(\tau))V_x^T(\hat{x}(\tau),\tau) \\
&\quad -\psi(\hat{x}(\tau))k^T(\hat{x}(\tau))h(\hat{x}(\tau))
\end{aligned} \tag{3.32}
$$

where $\hat{x}(\cdot)$ is the state trajectory corresponding to the maximizing solution of the performance criterion (3.31). Now $\hat{x}(\cdot)$ depends on $\bar{u}(\cdot)$ and by (3.32) we have defined a causal mapping from $\bar{u}(t)$ to $\tilde{u}(t)$, $t \in [T_1, T_2]$. Denote the fixed point of this mapping by $\hat{u}(t)$. This fixed point only depends on $\bar{y}$, in a causal way. Now

$$
\begin{aligned}
\hat{u}(t) &= -\left(\frac{1}{\varepsilon}\beta_2(\hat{x}(t))\beta_2^T(\hat{x}(t))+\psi(\hat{x}(t))\right)g^T(\hat{x}(t))V_x^T(\hat{x}(t),t) \\
&\quad -\psi(\hat{x}(t))k^T(\hat{x}(t))h(\hat{x}(t))
\end{aligned} \tag{3.33}
$$

is the solution of the considered optimization problem (3.29) (see [BB 90]). This is called the *worst case certainty equivalence solution.*

We still have to solve the maximization problem of the performance criterion (3.31) under the constraint that the measured output equals $\bar{y}(t)$. Since the disturbance d_2 fully influences the observations y we can substitute $d_2 = c(x) - \bar{y}$ into the performance criterion such that the constraint is automatically satisfied. Now we have simplified the constrained maximization problem to the unconstrained maximization of the performance criterium (3.31) with $d_2 = c(x) - \bar{y}$. This can be done in a classical way by first maximizing the criterion (3.31) under the constraint that the final value of the state $x(\tau)$ equals x, and after that maximizing the solution of this problem with respect to x. The second maximization with respect to x is equal to the maximum of $S(x,\tau) = V(x,\tau) - W(x,\tau)$ where $W \geq 0$ satisfies

$$\begin{aligned} W_t(x,t) + W_x(x,t)\left(f(x) + g(x)\bar{u}(t)\right) \\ +\frac{1}{2}\frac{1}{\gamma^2}W_x(x,t)e(x)e^T(x)W_x^T(x,t) \\ +\frac{1}{2}\left(h(x)+k(x)\bar{u}(t)\right)^T(x)\left(h(x)+k(x)\bar{u}(t)\right) - \frac{1}{2}\gamma^2 c^T(x)c(x) \\ +\gamma^2 c^T(x)\bar{y}(t) - \frac{1}{2}\gamma^2\|\bar{y}(t)\|^2 + \frac{1}{2}\varepsilon\|\beta_2^T(x)\bar{u}(t)\|^2 &= 0, \\ W(x,T_1) &= 0. \end{aligned} \tag{3.34}$$

Assume that this maximum is determined by $S_x(\hat{x}(t),t) = 0$ and that the Hessian is non-degenerate. Then the state equation for $\hat{x}(\cdot)$ can be found by differentiation of this equality (see [vdS 93]).

The resulting compensator which solves the *infinite* horizon $\mathcal{H}_\infty$ problem can be found by letting $T_2 \to \infty$ while imposing that $x(t) \to 0$ for $t \to \infty$ and $T_1 \to -\infty$ while $x(t) \to 0$ for $t \to -\infty$.

A finite dimensional approximation to the constructed nonlinear controller which locally solves the $\mathcal{H}_\infty$ problem is given by

$$\begin{aligned} \dot{\xi} &= f(\xi) - g(\xi)\left(\frac{1}{\varepsilon}\beta_2(\xi)\beta_2^T(\xi) + \sigma(\xi)\right)g^T(\xi)V_\xi^T(\xi) \\ &\quad -g(\xi)\psi(\xi)k^T(\xi)h(\xi) + \frac{1}{\gamma^2}e(\xi)e^T(\xi)V_\xi^T(\xi) \\ &\quad +\gamma^2\left[W_{\xi\xi}(\xi) - V_{\xi\xi}(\xi)\right]^{-1}\frac{\partial c^T}{\partial \xi}(\xi)\left(y(t) - c(\xi)\right) \\ u &= -\left(\frac{1}{\varepsilon}\beta_2(\xi)\beta_2^T(\xi) + \psi(\xi)\right)g^T(\xi)V_\xi^T(\xi) - \psi(\xi)k^T(\xi)h(\xi) \end{aligned} \tag{3.35}$$

where $V(\xi)$ is a solution of (3.15) with equality and $W(\xi)$ is a solution of the stationary version of (3.34) ($\bar{u}(t) = 0$; $\bar{y}(t) = 0$)

$$\begin{aligned} W_x(x) f(x) + \frac{1}{2}\frac{1}{\gamma^2} W_x(x)e(x)e^T(x)W_x^T(x) & \\ + \frac{1}{2}h^T(x)h(x) - \frac{1}{2}\gamma^2 c^T(x)c(x) &= 0 \\ W(0) &= 0 \end{aligned} \tag{3.36}$$

such that:

$$\begin{aligned} & f - g\left(\frac{1}{\varepsilon}\beta_2\beta_2^T + \psi\right) g^T V_x^T - g\psi k^T h + \frac{1}{\gamma^2} ee^T V_x^T \\ & \text{is exponentially stable;} \\ & -\left(f + \frac{1}{\gamma^2} ee^T W_x^T\right) \text{ is exponentially stable;} \\ & W_{xx}(x) > V_{xx}(x), \qquad \forall x. \end{aligned} \tag{3.37}$$

Hence we have the following result.

Theorem 3.28 *Consider the system Σ^m, and suppose there exist a constant $\varepsilon > 0$ and solutions $V \geq 0$, $W \geq 0$ to (3.15) with equality, respectively (3.36), satisfying (3.37). Then the closed loop system Σ^m, (3.35) has locally L_2-gain less than γ.* □

Proof The proof is based on linearization of the closed-loop system and combining the results about the so called central controller (see [DGKF 89]) and Theorem 2.22 (see [vdS 93]). ■

Also a converse result can be obtained, invoking Assumption 4.

Theorem 3.29 *Suppose the $\mathcal{H}_\infty$ suboptimal control problem for Σ^m with constant $\bar{\gamma} < \gamma$ is solvable by a compensator (3.27) satisfying Assumption 4 in the sense that there exists a constant $\mu > 0$ and a C^1-solution $V(x, \xi) \geq 0$ to the corresponding Hamilton-Jacobi inequality*

$$\begin{aligned} & V_x(x,\xi)\left(f(x) + g(x)q(\xi)\right) + V_\xi(x,\xi)\left(w(\xi) + p(\xi)c(x)\right) \\ & \quad + \frac{1}{2}\left(\frac{1}{\gamma^2} + \mu\right) V_x(x,\xi)e(x)e^T(x)V_x^T(x,\xi) \\ & \quad + \frac{1}{2}\left(\frac{1}{\gamma^2} + \mu\right) V_\xi(x,\xi)p(\xi)p^T(\xi)V_\xi(x,\xi) \end{aligned} \tag{3.38}$$

$$
\begin{aligned}
+\frac{1}{2}\left(h(x)+k(x)q(\xi)\right)^T\left(h(x)+k(x)q(\xi)\right)& \\
+\frac{1}{2}\varepsilon q^T(\xi)\beta_2(x)\beta_2^T(x)q(\xi) &\leq 0, \\
V(0,0) &= 0.
\end{aligned}
$$

Furthermore assume that $V_\xi(x,\xi)=0$ has a C^1-solution $\xi=F(x)$, $F(0)=0$, with $F: M \to M_c$. Then there exist nonnegative solutions $P(x)$ and $W(x)$ to the Hamilton-Jacobi inequality (3.15) respectively

$$
\begin{aligned}
W_x(x)f(x)+\frac{1}{2}\frac{1}{\tilde{\gamma}^2}W_x(x)e(x)e^T(x)W_x^T(x) & \\
+\frac{1}{2}h^T(x)h(x)-\frac{1}{2}\tilde{\gamma}^2c^T(x)c(x) &\leq 0, \\
W(0) &= 0,
\end{aligned}
\tag{3.39}
$$

where $\frac{1}{\tilde{\gamma}^2}=\frac{1}{\gamma^2}+\mu$ satisfying the coupling condition

$$
V(x)\leq W(x) \tag{3.40}
$$

near 0. □

Proof Define

$$
P(x):=V(x,F(x))\geq 0
$$

then substitution of $\xi=F(x)$ into (3.38) yields

$$
\begin{aligned}
& P_x(x)\left(f(x)+g(x)q(F(x))\right)+\frac{1}{2}\left(\frac{1}{\gamma^2}+\mu\right)P_x(x)e(x)e^T(x)P_x^T(x) \\
& \quad +\frac{1}{2}\left(h(x)+k(x)q(F(x))\right)^T\left(h(x)+k(x)q(F(x))\right) \\
& \quad +\frac{1}{2}\varepsilon q^T(F(x))\beta_2(x)\beta_2^T(x)q(F(x)) \\
= {} & P_x(x)\left(f(x)-g(x)\psi(x)k^T(x)h(x)\right) \\
& \quad +\frac{1}{2}\left(\frac{1}{\gamma^2}+\mu\right)P_x(x)e(x)e^T(x)P_x^T(x) \\
& \quad -\frac{1}{2}P_x(x)\left(g(x)\psi(x)g^T(x)+\frac{1}{\varepsilon}g(x)\beta_2(x)\beta_2^T(x)g^T(x)\right)P_x^T(x) \\
& \quad +\frac{1}{2}h^T(x)\left(I_p-k(x)\psi(x)k^T(x)\right)h(x)
\end{aligned}
$$

$$+\frac{1}{2}\varepsilon \left\| \beta_2^T(x)q(F(x)) + \frac{1}{\varepsilon}\beta_2^T(x)g^T(x)P_x^T(x) \right\|^2$$

$$+\frac{1}{2}\left\| \left(\nu^T(x)\nu(x)\right)^{\frac{1}{2}} \beta_1^T(x)q(F(x)) \right.$$

$$\left. + \left(\nu^T(x)\nu(x)\right)^{-\frac{1}{2}} \left(\beta_1^T(x)g^T(x)P_x^T(x) + \nu^T(x)h(x)\right) \right\|^2$$

while $P(0) = V(0,0) = 0$. Thus we see that $P(x)$ is a solution to the Hamilton-Jacobi inequality (3.15). For a solution W to (3.39) we define

$$W(x) := V(x,0) \geq 0.$$

Substitution of $\xi = 0$ into (3.38) and completion of the squares yields

$$W_x(x)f(x) + \frac{1}{2}\frac{1}{\tilde{\gamma}^2}W_x(x)e(x)e^T(x)W_x^T(x) + \frac{1}{2}h^T(x)h(x)$$

$$+\frac{1}{2}\frac{1}{\tilde{\gamma}^2}\left\|\tilde{\gamma}^2 c(x) + p^T(0)V_\xi^T(x,0)\right\|^2 - \frac{1}{2}\tilde{\gamma}^2 c^T(x)c(x) \leq 0$$

with $W(0) = 0$, and consequently W is a solution of (3.39).

Finally since $V(x,\xi) \geq 0$ and $V_\xi(x, F(x)) = 0$ it necessarily follows that $P(x) = V(x, F(x)) \leq V(x,0) = W(x)$, at least for x near zero. ■

The measurement feedback $\mathcal{H}_\infty$ control problem even in the regular case is not completely solved. As described in this subsection sufficient and necessary conditions for the singular case can be given but a full characterization of the problem has not been derived yet. In this subsection we only considered singularities with respect to the inputs. The $\mathcal{H}_\infty$ problem with singular measurements in which the direct feedthrough from disturbances to measurements is not surjective is an open problem.

Chapter 4

The $\mathcal{H}_\infty$ almost disturbance decoupling problem

A special case of a $\mathcal{H}_\infty$ optimal control problem is the almost disturbance decoupling problem (ADDP) in which the goal is to make the L_2-gain from disturbances to the to-be-controlled variables arbitrary small by applying a (possibly high) gain feedback. For single-input single-output affine nonlinear systems this problem is considered and sufficient conditions for the solvability are given in [MRST 94] and [MT 95]. In this chapter we make an extension of these results to the multi-input multi-output case. To be able to extend the SISO results we assume that a certain decoupling condition is satisfied.

The results of this chapter are instrumental for the next chapter where the singular $\mathcal{H}_\infty$ problem is solved in a geometric way by decomposing via a state transformation the $\mathcal{H}_\infty$ problem into a regular $\mathcal{H}_\infty$ problem together with an $\mathcal{H}_\infty$ almost disturbance decoupling problem.

4.1 Almost disturbance decoupling for affine nonlinear systems

Consider affine nonlinear MIMO systems with m inputs and p outputs ($m \geq p$) of the form

$$\begin{array}{rcl} \dot{x} &=& f(x) + \sum_{j=1}^{m} g_j(x)u_j + \sum_{i=1}^{q} e_i(x)d_i \\ z &=& h(x) \end{array} \tag{4.1}$$

where again $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth manifold M, $z \in \mathbb{R}^p$, and the functions f, g_j, e_i and h are smooth.

Parallel to the L_2-gain and $\mathcal{H}_\infty$ problems defined in section 2.2 the following two problems are defined.

Definition 4.1 The *L_2-gain almost disturbance decoupling problem* is solvable for (4.1) if there exists a smooth parameterized state feedback control

$$u = u(x, k)$$

such that there exists a constant $K(x)$, $0 \leq K(x) < \infty$, $K(0) = 0$, such that for every t, $0 \leq t \leq T$,

$$\int_0^t z^T(\tau)z(\tau)\mathrm{d}\tau \leq \frac{1}{k}\int_0^t d^T(\tau)d(\tau)\mathrm{d}\tau + K(x_0)$$

for the closed-loop system with initial condition $x(0) = x_0$, and for every disturbance function $d(\tau) \in L_2(0, t)$, with $[0, T)$ any open interval on which the corresponding solution exists. □

Definition 4.2 The *$\mathcal{H}_\infty$ almost disturbance decoupling problem* is solvable for (4.1) if the L_2-gain almost disturbance decoupling problem is solvable by a feedback $u = u(x, k)$, $u(0, k) = 0$, $\forall k \in \mathbb{R}^+$, while the origin is globally asymptotically stable for the closed-loop dynamics

$$\dot{x} = f(x) + g(x)u(x, k)$$

for every k (large enough). □

Let us define the strong control and the disturbance characteristic indices for (4.1), which indicate the amount of differentiations before the specific output components are influenced by the control respectively the disturbance.

Definition 4.3 The *strong control characteristic indices* $\rho_1, \ldots, \rho_p$ for the system (4.1) are defined such that:

$$L_{g_j}L_f^r h_l(x) = 0, \qquad 0 \leq r \leq \rho_l - 2, 1 \leq j \leq m, \forall x;$$
$$L_{g_j}L_f^{\rho_l - 1} h_l(x) \neq 0, \qquad \text{for some } j, \ \forall x.$$

If $L_{g_j}L_f^i h_l(x) = 0$, $\forall i, j$ and $\forall x \in \mathbb{R}^n$, then $\rho_l = \infty$. □

It should be noted that the strong characteristic indices do not always exists. This is due to the fact that we require the functions $L_{g_j}L_f^{\rho_l-1}h_l$ to be unequal to zero *for all* x. We call this strong control characteristic indices $\rho_1, \ldots, \rho_p$ *well defined* if ρ_l exists and is finite for all $l = 1, \ldots, p$.

Definition 4.4 The *disturbance characteristic indices* $\nu_1, \ldots, \nu_p$ for the system (4.1) are defined such that:

$$
\begin{aligned}
L_{e_i}L_f^r h_l(x) &= 0, && 0 \le r \le \nu_l - 2,\ 1 \le i \le q,\ \forall x;\\
L_{e_i}L_f^{\nu_l-1} h_l(x) &\neq 0, && \text{for some } i, \text{ and for some } x.
\end{aligned}
$$

If $L_{e_i}L_f^j h_l(x) = 0$, $\forall i, j$ and $\forall x \in \mathbb{R}^n$, then $\nu_l = \infty$. □

Assume that $\rho_1, \ldots, \rho_p$ are well defined. We introduce the $(p \times m)$ *decoupling matrix* $A(x)$:

$$
A(x) = \begin{pmatrix} L_{g_1}L_f^{\rho_1-1}h_1(x) & \cdots & L_{g_m}L_f^{\rho_1-1}h_1(x) \\ \vdots & & \vdots \\ L_{g_1}L_f^{\rho_p-1}h_p(x) & \cdots & L_{g_m}L_f^{\rho_p-1}h_p(x) \end{pmatrix}. \tag{4.2}
$$

Assume that this decoupling matrix has full row rank. Then we know that if $\nu_i > \rho_i$ holds for all i the disturbance decoupling problem is solvable by a regular state feedback (see [Is 89], [NvdS 90], [MT 95]), or otherwise stated the L_2-gain from disturbance to output can be made zero by applying a proper state feedback. This problem is no longer solvable when for some i we have that $\nu_i \le \rho_i$. In this case it is interesting to look at the almost disturbance decoupling problem. In this chapter we consider the case where $\nu_i \le \rho_i$ for all i. Because we assume that the decoupling matrix $A(x)$ has full row rank it is not difficult to extend the results to the case where for some i it holds that $\nu_i > \rho_i$.

We describe the results only for the case where the number of inputs is equal to the number of outputs ($m = p$). For the situation $m > p$ assuming that the strong controllability indices are well defined and the decoupling matrix $A(x)$ has full *row* rank we can at least locally always select p independent columns of this matrix $A(x)$ and apply the results using as decoupling matrix the $(p \times p)$ matrix consisting of these p independent columns together with the corresponding inputs.

We start by the following straightforward extension of the results for SISO systems.

Theorem 4.5 *Let for the system (4.1) the following assumptions be satisfied:*

(i) $m = p$;

(ii) $\rho_1, \dots, \rho_p$ *are well defined;*

(iii) the decoupling matrix $A(x)$ *has full rank;*

(iv) the distribution $G = \{g_1, \dots, g_m\}$ *is involutive;*

(v) $\mathrm{d}\left(L_{e_j} L_f^i h_l\right) \in \mathrm{span}\left\{\mathrm{d}h_l, \mathrm{d}\left(L_f h_l\right), \dots, \mathrm{d}\left(L_f^i h_l\right)\right\}$, $\nu_l - 1 \leq i \leq \rho_l - 1$, $1 \leq j \leq q$, $1 \leq l \leq p$, $\forall x \in \mathbb{R}^n$;

(vi) the vector fields

$$\tilde{f} = f - gA^{-1} \begin{pmatrix} L_f^{\rho_1} h_1 \\ \vdots \\ L_f^{\rho_p} h_p \end{pmatrix}, \qquad \tilde{g} = gA^{-1}$$

are complete (see Appendix A).

Then the L_2*-gain almost disturbance decoupling problem is solvable.* □

Proof The matrix $A(x)$ can be written as

$$A(x) = \begin{pmatrix} \mathrm{d}L_f^{\rho_1-1} h_1(x) \\ \vdots \\ \mathrm{d}L_f^{\rho_p-1} h_p(x) \end{pmatrix} \begin{pmatrix} g_1(x) & \cdots & g_m(x) \end{pmatrix}.$$

Because $m = p$ it follows from the non-singularity of this matrix that the m-vectors $g_1(x), \dots, g_m(x)$ are linearly independent. Hence the distribution G is constant dimensional.

Therefore by *(iv)*, *(vi)* a global version of Frobenius' Theorem yields a global change of coordinates (see [MRS 89] and the references in there)

$$\begin{aligned} q_{11} &= h_1(x), \\ &\vdots \\ q_{1\rho_1} &= L_f^{\rho_1-1} h_1(x), \\ q_{21} &= h_2(x), \end{aligned}$$

$$
\begin{aligned}
&\vdots\\
q_{2\rho_2} &= L_f^{\rho_2-1}h_2(x),\\
&\vdots\\
&\vdots\\
q_{p1} &= h_p(x),\\
&\vdots\\
q_{p\rho_p} &= L_f^{\rho_p-1}h_p(x),\\
q_{\rho+1} &= \varphi_{\rho+1}(x),\\
&\vdots\\
q_n &= \varphi_n(x)
\end{aligned}
\tag{4.3}
$$

with $\rho = \sum_{i=1}^{m}$ and $\varphi_i(x)$, $\rho+1 \le i \le n$, $\varphi_i(0) = 0$, such that

$$\langle \mathrm{d}\varphi_i, G\rangle = 0$$

together with the state feedback

$$v = A(x)u + \begin{pmatrix} L_f^{\rho_1}h_1(x) \\ \vdots \\ L_f^{\rho_p}h_p(x) \end{pmatrix} \tag{4.4}$$

which globally transform (4.1) into

$$
\begin{aligned}
\dot{q}_{11} &= q_{12} + \sum_{i=1}^{q} L_{e_i}h_1(x)d_i = q_{12} + W_{11}^T(q)d,\\
&\vdots\\
\dot{q}_{1\rho_1} &= v_1 + \sum_{i=1}^{q} L_{e_i}L_f^{\rho_1-1}h_1(x)d_i = v_1 + W_{1\rho_1}^T(q)d,\\
\dot{q}_{21} &= q_{22} + \sum_{i=1}^{q} L_{e_i}h_2(x)d_i = q_{22} + W_{21}^T(q)d,\\
&\vdots\\
\dot{q}_{2\rho_2} &= v_2 + \sum_{i=1}^{q} L_{e_i}L_f^{\rho_2-1}h_2(x)d_i = v_2 + W_{2\rho_2}^T(q)d,
\end{aligned}
$$

$$
\begin{aligned}
&\vdots \\
&\vdots \\
\dot{q}_{p1} &= q_{p2} + \sum_{i=1}^{q} L_{e_i} h_p(x) d_i = q_{p2} + W_{p1}^T(q)d, \\
&\vdots \\
\dot{q}_{p\rho_p} &= v_p + \sum_{i=1}^{q} L_{e_i} L_f^{\rho_p - 1} h_p(x) d_i = v_p + W_{p\rho_p}^T(q)d, \\
\dot{q}_r &= \varphi(q) + \Psi^T(q)d, \\
z_l &= q_{l1}, \qquad 1 \le l \le p
\end{aligned}
\tag{4.5}
$$

with $q_r = (q_{\rho+1}, \ldots, q_n)$ and $d = (d_1, \ldots, d_q)$. Because of condition *(v)* the system (4.5) takes the following form (see [MRST 94])

$$
\begin{aligned}
\dot{q}_{11} &= q_{12} + W_{11}^T(q_{11})d, \\
&\vdots \\
\dot{q}_{1\rho_1} &= v_1 + W_{1\rho_1}^T(q_{11}, \ldots, q_{1\rho_1})d, \\
\dot{q}_{21} &= q_{22} + W_{21}^T(q_{21})d, \\
&\vdots \\
\dot{q}_{2\rho_2} &= v_2 + W_{2\rho_2}^T(q_{21}, \ldots, \rho_{2\rho_2})d, \\
&\vdots \\
&\vdots \\
\dot{q}_{p1} &= q_{p2} + W_{p1}^T(q_{p1})d, \\
&\vdots \\
\dot{q}_{p\rho_p} &= v_p + W_{p\rho_p}^T(q_{p1}, \ldots, q_{p\rho_p})d, \\
\dot{q}_r &= \varphi(q) + \Psi^T(q)d, \\
z_l &= q_{l1}, \quad 1 \le l \le p.
\end{aligned}
\tag{4.6}
$$

Now the first ρ variables of the system are separated in p blocks for which we can apply the construction of the state feedback used in the proof in [MT 95] or [MRST 94].

In other words we can consider p systems

$$\Sigma_i \begin{cases} \dot{q}_{i1} = q_{i2} + W_{i1}^T(q_{i1})d, \\ \vdots \\ \dot{q}_{i\rho_i} = v_i + W_{i\rho_i}(q_{i1}, \dots, q_{i\rho_i})d, \\ z_i = q_{i1} \end{cases} \tag{4.7}$$

for $1 \leq i \leq p$.
Construct for each of these systems a state feedback control

$$v_i = q^*_{i,\rho_i+1}(q_{i1}, \dots, q_{i\rho_i}, k)$$

such that

$$\int_0^T z_i^T(\tau)z_i(\tau)\mathrm{d}\tau \leq \frac{\rho_i}{k}\int_0^T d^T(\tau)d(\tau)\mathrm{d}\tau$$

for all $T \geq 0$.
Then it follows that

$$\int_0^T z^T(\tau)z(\tau)\mathrm{d}\tau \leq \frac{\rho}{k}\int_0^T d^T(\tau)d(\tau)\mathrm{d}\tau.$$

■

Remark 4.6 The stability properties of the zero-dynamics

$$\dot{q}_r = \varphi(0, q_r) + \Psi^T(0, q_r)d$$

do not influence the L_2-gain result. In case we also require the closed-loop system to be internally stable then the stability of the zero-dynamics has to be taken into account as will be shown in Theorem 4.10. □

As in the SISO case ([MRST 94], [MT 95]) also another set of sufficient conditions for the solvability of the L_2-gain almost disturbance decoupling problem can be formulated.

Theorem 4.7 *Assume that for the system (4.1) the following is satisfied:*

(i) $m = p$;

(ii) $\rho_1, \ldots, \rho_p$ *are well defined;*

(iii) the decoupling matrix $A(x)$ *has full rank;*

(iv) the vector fields

$$\tilde{f} = f - gA^{-1}\begin{pmatrix} L_f^{\rho_1} h_1 \\ \vdots \\ L_f^{\rho_p} h_p \end{pmatrix}, \qquad \tilde{g} = gA^{-1}$$

are complete.

Also assume that after applying the feedback

$$u = A^{-1}(x)v$$

to the system (4.1) the following conditions hold:

(v) $\mathcal{G}_{\rho-1} = \operatorname{span}\left\{\tilde{g}_j, \operatorname{ad}_f \tilde{g}_j, \ldots, \operatorname{ad}_f^{\rho_j-1}\tilde{g}_j;\ j = 1, \ldots, m\right\}$ *is involutive and has constant dimension* $\rho = \sum_{i=1}^m \rho_i$;

(vi) $\operatorname{ad}_{e_i}\mathcal{G}_j^l \subset \mathcal{G}_j^l$, *with* $\mathcal{G}_j^l = \operatorname{span}\left\{\tilde{g}_l, \operatorname{ad}_f\tilde{g}_l, \ldots, \operatorname{ad}_f^j \tilde{g}_l\right\}$, $1 \le i \le q$, $0 \le j \le \rho_l - 2$, $1 \le l \le m$.

Then the L_2*-gain almost disturbance decoupling problem is solvable.* □

Proof We can globally define a change of coordinates (4.3) with $\varphi_i(x)$, $\rho+1 \le i \le n$, $\varphi_i(0) = 0$, such that

$$\langle \mathrm{d}\varphi_i, \mathcal{G}_{\rho-1}\rangle = 0$$

together with the feedback (4.4) which globally transform (4.1) into

$$\dot{q}_{11} = q_{12} + \sum_{i=1}^{q} L_{e_i}h_1(x)d_i = q_{12} + W_{11}^T(q)d,$$

$$\vdots$$

$$
\begin{aligned}
\dot{q}_{1\rho_1} &= v_1 + \sum_{i=1}^{q} L_{e_i} L_f^{\rho_1 - 1} h_1(x) d_i = v_1 + W_{1\rho_1}^T(q) d, \\
\dot{q}_{21} &= q_{22} + \sum_{i=1}^{q} L_{e_i} h_2(x) d_i = q_{22} + W_{21}^T(q) d, \\
&\vdots \\
\dot{q}_{2\rho_2} &= v_2 + \sum_{i=1}^{q} L_{e_i} L_f^{\rho_2 - 1} h_2(x) d_i = v_2 + W_{2\rho_2}^T(q) d, \\
&\vdots \\
&\vdots \\
\dot{q}_{p1} &= q_{p2} + \sum_{i=1}^{q} L_{e_i} h_p(x) d_i = q_{p2} + W_{p1}^T(q) d, \\
&\vdots \\
\dot{q}_{p\rho_p} &= v_p + \sum_{i=1}^{q} L_{e_i} L_f^{\rho_p - 1} h_p(x) d_i = v_p + W_{p\rho_p}^T(q) d, \\
\dot{q}_r &= \varphi(q_{11}, q_{21}, \ldots, q_{p1}, q_r) + \Psi^T(q) d, \\
z_l &= q_{l1}, \quad 1 \le l \le p.
\end{aligned}
\tag{4.8}
$$

In q-coordinates we have

$$
\mathcal{G}_{\rho-1} = \operatorname{span}\left\{ \frac{\partial}{\partial q_{i1}}, \ldots, \frac{\partial}{\partial q_{i\rho_i}}; 1 \le i \le p \right\}
$$

and

$$
\mathcal{G}_j^l = \operatorname{span}\left\{ \frac{\partial}{\partial q_{l(\rho_l - j)}}, \ldots, \frac{\partial}{\partial q_{l\rho_l}} \right\}
$$

for $0 \le j \le \rho_l - 1$. Then conditions *(vi)* imply

$$
\begin{aligned}
W_{li}^T(q) &= W_{li}^T(q_{l1}, \ldots, q_{li}, q_r) \\
& \qquad\qquad\qquad\qquad 1 \le i \le \rho_l,\ 1 \le l \le p. \\
\Psi^T(q) &= \Psi^T(q_{11}, q_{21}, \ldots, q_{p1}, q_r)
\end{aligned}
$$

And therefore (4.8) becomes

$$
\begin{aligned}
\dot{q}_{11} &= q_{12} + W_{11}^T(q_{11}, q_r) d, \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\dot{q}_{1\rho_1} &= v_1 + W_{1\rho_1}^T(q_{11}, \ldots, q_{1\rho_1}, q_r)d, \\
\dot{q}_{21} &= q_{22} + W_{21}^T(q_{21}, q_r)d, \\
&\vdots \\
\dot{q}_{2\rho_2} &= v_2 + W_{2\rho_2}^T(q_{21}, \ldots, \rho_{2\rho_2}, q_r)d, \\
&\vdots \\
&\vdots \\
\dot{q}_{p1} &= q_{p2} + W_{p1}^T(q_{p1}, q_r)d, \\
&\vdots \\
\dot{q}_{p\rho_p} &= v_p + W_{p\rho_p}^T(q_{p1}, \ldots, q_{p\rho_p}, q_r)d, \\
\dot{q}_r &= \varphi(q_{11}, q_{21}, \ldots, q_{p1}, q_r) + \Psi^T(q_{11}, q_{21}, \ldots, q_{p1}, q_r)d, \\
z_l &= q_{l1}, \quad 1 \le l \le p.
\end{aligned}
\tag{4.9}
$$

Again the first ρ variables of the system have been separated in p blocks, for each of which we can apply the construction of the state feedback used in the proof of Theorem 4.5. ∎

The conditions *(v)* in Theorem 4.5 and *(vi)* in Theorem 4.7 result in the strict block structure described in the proofs. On the other hand, a closer inspection of the proof shows that these conditions can be weakened. First we weaken the conditions of Theorem 4.5.

Theorem 4.8 *Assume that for the system (4.1) the following is satisfied:*

(i) $m = p$;

(ii) $\rho_1, \ldots, \rho_p$ *are well defined;*

(iii) the decoupling matrix $A(x)$ *has full rank;*

(iv) the distribution $G = \{g_1, \ldots, g_m\}$ *is involutive;*

(v) $\mathrm{d}\left(L_{e_j} L_f^i h_l\right) \in \mathrm{span}\left\{\mathrm{d}h_s, \mathrm{d}\left(L_f h_s\right), \ldots, \mathrm{d}\left(L_f^{\max(i,\rho_s-1)} h_s\right)\right\}$, *for any* $s = 1, \ldots, p$, $\nu_l - 1 \le i \le \rho_l - 1$, $1 \le j \le q$, $1 \le l \le p$, $\forall x \in \mathbb{R}^n$;

(vi) the vector fields

$$\tilde{f} = f - gA^{-1}\begin{pmatrix} L_f^{\rho_1}h_1 \\ \vdots \\ L_f^{\rho_p}h_p \end{pmatrix}, \qquad \tilde{g} = A^{-1}g$$

are complete.

Then the L_2-gain almost disturbance decoupling problem is solvable. □

Proof As in the proof of Theorem 4.5 the distribution G is involutive and constant dimensional, and hence we can again define a change of coordinates (4.3) with $\varphi_i(x)$, $\rho + 1 \le i \le n$, $\varphi_i(0) = 0$, such that

$$\langle \mathrm{d}\varphi_i, G\rangle = 0.$$

Together with the state feedback (4.4) this change of coordinates globally transforms (4.1) into (4.5).

With condition *(v)* the system (4.1) with the feedback (4.4) is of the form

$$\begin{array}{rcl}
\dot{q}_{11} & = & q_{12} + W_{11}^T(\tilde{q}_1)d, \\
& \vdots & \\
\dot{q}_{1\rho_1} & = & v_1 + W_{1\rho_1}^T(\tilde{q}_1, \ldots, \tilde{q}_{\rho_1})d, \\
\dot{q}_{21} & = & q_{22} + W_{21}^T(\tilde{q}_1)d, \\
& \vdots & \\
\dot{q}_{2\rho_2} & = & v_2 + W_{2\rho_2}^T(\tilde{q}_1, \ldots, \tilde{q}_{\rho_2})d, \\
& \vdots & \\
& \vdots & \\
\dot{q}_{p1} & = & q_{p2} + W_{p1}^T(\tilde{q}_1)d, \\
& \vdots & \\
\dot{q}_{p\rho_p} & = & v_p + W_{p\rho_p}^T(\tilde{q}_1, \ldots, \tilde{q}_{\rho_p})d, \\
\dot{q}_r & = & \varphi(q) + \Psi^T(q)d, \\
z_l & = & q_{l1}, \qquad 1 \le l \le p
\end{array} \tag{4.10}$$

where $\tilde{q}_j$ is the set of coordinates given by

$$\tilde{q}_j = \left\{q_{kj}, \text{ for all } k = 1, \ldots, p \text{ for which } j \leq \rho_k\right\}.$$

Then we can construct a storage function and a static feedback for this system (4.10) which leads to closed loop system which has L_2-gain less than or equal to $\frac{1}{k}$ by using the following algorithm:

Step 1 Define

$$q_{i2}^*(\tilde{q}_1, k) = -q_{i1} - \frac{1}{4}kq_{i1}W_{i1}^T(\tilde{q}_1)W_{i1}(\tilde{q}_1) \tag{4.11}$$

for $i = 1, \ldots, p$, and consider

$$V_1 = \frac{1}{2}\sum_{l=1}^{p} q_{l1}^2.$$

Its time derivative with respect to the closed loop system (4.10) using (4.11) is given by

$$\begin{aligned}
\dot{V}_1 &= \sum_{l=1}^{p}\left(-q_{l1}^2 - \frac{1}{4}kq_{l1}^2W_{l1}^TW_{l1} + q_{l1}W_{l1}^Td - \frac{1}{k}d^Td\right.\\
&\qquad \left. + \frac{1}{k}d^Td\right)\\
&= \sum_{l=1}^{p}\left(-q_{l1}^2 - k\left\|\frac{1}{2}q_{l1}^2W_{l1} - \frac{1}{k}d\right\|^2 + \frac{1}{k}d^Td\right)\\
&\leq \sum_{l=1}^{p}\left(-q_{l1}^2 + \frac{1}{k}d^Td\right).
\end{aligned}$$

Then define the index sets:

$$\begin{aligned}
I_j &= \{s \mid \rho_s > j\};\\
\mathcal{J}_j &= \{s \mid \rho_s = j\};\\
\mathcal{K}_j &= \{s \mid \rho_s \leq j\}.
\end{aligned}$$

Finally we set

$$\begin{aligned}
v_i &= q_{i2}^*(\tilde{q}_1, k),\\
\hat{q}_{i1} &= q_{i1}
\end{aligned}$$

for all $i \in \mathcal{J}_1$.

Step $s+1$ Assume that for a given index s, $1 \leq s \leq \max_i \rho_i$, for the system

$$\begin{aligned} \dot{q}_{l1} &= q_{l2} + W_{l1}^T(\tilde{q}_1)d \\ &\vdots \\ \dot{q}_{l\rho_l} &= v_l + W_{l\rho_l}^T(\tilde{q}_1, \ldots, \tilde{q}_{\rho_l})d, \qquad \forall l \in \mathcal{K}_s \\ \dot{q}_{l1} &= q_{l2} + W_{l1}^T(\tilde{q}_1)d \\ &\vdots \\ \dot{q}_{ls} &= v_l + W_{ls}^T(\tilde{q}_1, \ldots, \tilde{q}_s)d, \qquad \forall l \in I_s \end{aligned}$$

there exists for each $l \in I_s$ s functions

$$q_{lj}^* = q_{lj}^*(\tilde{q}_1, \ldots, \tilde{q}_{j-1}, k), \qquad 2 \leq j \leq s+1, \quad l \in I_s$$

with $q_{lj}^*(0, \ldots, 0, k) = 0$ such that in the new coordinates

$$\begin{aligned} \bar{q}_{l1} &= q_{l1} \\ \bar{q}_{lj} &= q_{lj} - q_{lj}^*(\tilde{q}_1, \ldots, \tilde{q}_{j-1}, k), \qquad 2 \leq j \leq s, \quad l \in I_s \end{aligned}$$

the function

$$V_s = \frac{1}{2}\sum_{l \in I_s}\sum_{j=1}^{s} \bar{q}_{lj}^2 + \frac{1}{2}\sum_{l \in \mathcal{K}_s}\sum_{j=1}^{\rho_l} \hat{q}_{lj}^2 \tag{4.12}$$

has time derivative, with $q_{l(s+1)} = q_{l(s+1)}^*$ ($l \in I_s$), satisfying the inequality

$$\dot{V}_s \leq -\sum_{l \in I_s}\sum_{j=1}^{s} \bar{q}_{lj}^2 - \sum_{l \in \mathcal{K}_s}\sum_{j=1}^{\rho_l} \hat{q}_{lj}^2 + \frac{c}{k}\|d\|^2$$

with

$$c = (\# I_s)\, s + \sum_{l \in \mathcal{K}_s} \rho_l$$

where $\# I_s$ indicates the number of indices in the set I_s.

Then we consider the function

$$V_{s+1} = \frac{1}{2}\sum_{l \in I_s}\sum_{j=1}^{s+1} \bar{q}_{lj}^2 + \frac{1}{2}\sum_{l \in \mathcal{K}_s}\sum_{j=1}^{\rho_l} \hat{q}_{lj}^2$$

where

$$\bar{q}_{l(s+1)} = q_{l(s+1)} - q^*_{l(s+1)}(\tilde{q}_1, \ldots, \tilde{q}_s, k), \qquad l \in I_s.$$

When $q_{l(s+2)} = q^*_{l(s+2)}$ for all $l \in I_s$ in (4.10), we have that

$$\begin{aligned}\dot{V}_{s+1} \le & -\sum_{l\in I_s}\sum_{j=1}^{s}\bar{q}_{lj}^2 - \sum_{l\in\mathcal{K}_s}\sum_{j=1}^{\rho_l}\hat{q}_{lj}^2 + \frac{c}{k}\|d\|^2 \\ & + \sum_{l\in I_s}\bar{q}_{l(s+1)}\left[\bar{q}_{ls} + W^T_{l(s+1)}d - \sum_{i=1}^{s}\frac{\partial q^*_{l(s+1)}}{\partial\tilde{q}_s}\dot{\tilde{q}}_s + q^*_{l(s+2)}\right]\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}\sum_{i=1}^{s}\frac{\partial q^*_{l(s+1)}}{\partial\tilde{q}_s}\dot{\tilde{q}}_s = & \sum_{j\in I_s}\sum_{i=1}^{s}\frac{\partial q^*_{l(s+1)}}{\partial q_{ji}}\left(q_{j(i+1)} + W^T_{ji}d\right) \\ & + \sum_{j\in\mathcal{K}_s}\sum_{i=1}^{\rho_j}\frac{\partial q^*_{l(s+1)}}{\partial q_{ji}}\left(q_{j(i+1)} + W^T_{ji}d\right) \\ & + \sum_{j\in\mathcal{K}_s}\frac{\partial q^*_{l(s+1)}}{\partial q_{j\rho_j}}\left(v_j(\tilde{q}_1, \ldots, \tilde{q}_{\rho_j}, k) + W^T_{j\rho_j}d\right).\end{aligned}$$

Now we define the following functions:

$$\begin{aligned}\alpha_l(\tilde{q}_1, \ldots, \tilde{q}_{s+1}, k) = & \ \bar{q}_{ls} - \sum_{j\in I_s}\sum_{i=1}^{\rho_j}\frac{\partial q^*_{l(s+1)}}{\partial q_{ji}}q_{j(i+1)} \\ & - \sum_{j\in\mathcal{K}_s}\sum_{i=1}^{\rho_j}\frac{\partial q^*_{l(s+1)}}{\partial q_{ji}}q_{j(i+1)} \\ & - \sum_{j\in\mathcal{K}_s}\frac{\partial q^*_{l(s+1)}}{\partial q_{j\rho_j}}v_j(\tilde{q}_1, \ldots, \tilde{q}_{\rho_j}, k); \\ \beta_l(\tilde{q}_1, \ldots, \tilde{q}_{s+1}) = & \ W_{l(s+1)} - \sum_{j\in I_s}\sum_{i=1}^{s}\frac{\partial q^*_{l(s+1)}}{\partial q_{ji}}W_{ji} \\ & - \sum_{j\in\mathcal{K}_s}\sum_{i=1}^{\rho_j}\frac{\partial q^*_{l(s+1)}}{\partial q_{ji}}W_{ji}; \\ q^*_{l(s+2)}(\tilde{q}_1, \ldots, \tilde{q}_{s+1}, k) = & \ -\alpha_l - \bar{q}_{l(s+1)} - \frac{1}{4}k\bar{q}_{l(s+1)}\beta_l^T\beta_l;\end{aligned}$$

with $l \in I_s$. Then (4.13) becomes

$$\begin{aligned}
\dot{V}_{s+1} &\le -\sum_{l\in I_s}\sum_{j=1}^{s+1}\bar{q}_{lj}^2 - \sum_{l\in \mathcal{K}_s}\sum_{j=1}^{\rho_l}\hat{q}_{lj}^2 + \frac{c}{k}\|d\|^2 + \frac{\# I_s}{k}\|d\|^2 \\
&\quad + \sum_{l\in I_s}\bar{q}_{l(s+1)}\left[\beta_l^T d - \frac{1}{4}k\bar{q}_{l(s+1)}\beta_l^T\beta_l\right] - \frac{\# I_s}{k}\|d\|^2 \\
&= -\sum_{l\in I_s}\sum_{j=1}^{s+1}\bar{q}_{lj}^2 - \sum_{l\in \mathcal{K}_s}\sum_{j=1}^{\rho_l}\hat{q}_{lj}^2 - \sum_{l\in I_s}k\left\|\frac{1}{2}\bar{q}_{l(s+1)}\beta_l - \frac{1}{k}d\right\|^2 \\
&\quad + \frac{c+\# I_s}{k}\|d\|^2 \\
&\le -\sum_{l\in I_s}\sum_{j=1}^{s+1}\bar{q}_{lj}^2 - \sum_{l\in \mathcal{K}_s}\sum_{j=1}^{\rho_l}\hat{q}_{lj}^2 + \frac{c+\# I_s}{k}\|d\|^2.
\end{aligned}$$

Define

$$\left.\begin{aligned}
\hat{q}_{li} &= \bar{q}_{li}, \qquad 1 \le i \le s+1 \\
v_l &= q_{l(s+2)}^*(\tilde{q}_1, \ldots, \tilde{q}_{s+1}, k)
\end{aligned}\right. \qquad l \in \mathcal{J}_{s+1}.$$

Then from (4.13) we can conclude that

$$\dot{V}_{s+1} \le -\sum_{l\in I_{s+1}}\sum_{j=1}^{s+1}\bar{q}_{lj}^2 - \sum_{l\in \mathcal{K}_{s+1}}\sum_{j=1}^{\rho_l}\hat{q}_{lj}^2 + \frac{c}{k}\|d\|^2$$

where

$$\begin{aligned}
c &= (\# I_s)s + \sum_{l\in\mathcal{K}_s}\rho_l + \# I_s \\
&= (\# I_{s+1})(s+1) + \sum_{l\in\mathcal{K}_{s+1}}\rho_l.
\end{aligned}$$

(end of step $s+1$)

After $\max_i \rho_i$ steps we have that

$$V_{\max_i \rho_i} = \frac{1}{2}\sum_{l=1}^{p}\sum_{j=1}^{\rho_l}\hat{q}_{lj}^2$$

while

$$\begin{aligned}
\dot{V}_{\max_i \rho_i} &\le -\sum_{l=1}^{p}\sum_{j=1}^{\rho_l}\hat{q}_{lj}^2 + \frac{\rho}{k}\|d\|^2 \\
&= -\sum_{l=1}^{p} q_{l1}^2 - \sum_{l\in I_1}\sum_{j=1}^{\rho_l}\hat{q}_{lj}^2 + \frac{\rho}{k}\|d\|^2 \\
&\le -\|z\|^2 + \frac{\rho}{k}\|d\|^2
\end{aligned}$$

which implies that

$$V_{\max_i \rho_i}(q(t)) - V_{\max_i \rho_i}(q(0)) \le \int_0^t \left(-\|z(\tau)\|^2 + \frac{\rho}{k}\|d(\tau)\|^2\right) \mathrm{d}\tau,$$

and because $V_{\max_i \rho_i} \ge 0$

$$\int_0^t \|z(\tau)\|^2 \mathrm{d}\tau \le \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + V_{\max_i \rho_i}(q(0)).$$

■

Weakening of the conditions of Theorem 4.7 leads to:

Theorem 4.9 *Assume that for the system (4.1) the following is satisfied:*

(i) $m = p$;

(ii) $\rho_1, \ldots, \rho_p$ *are well defined;*

(iii) the decoupling matrix $A(x)$ *has full rank;*

(iv) the vector fields

$$\tilde{f} = f - gA^{-1}\begin{pmatrix} L_f^{\rho_1} h_1 \\ \vdots \\ L_f^{\rho_p} h_p \end{pmatrix}, \qquad \tilde{g} = gA^{-1}$$

are complete.

Also assume that after applying the feedback

$$u = A^{-1}(x)v$$

to the system (4.1) the following conditions hold:

(v) $\mathcal{G}_{\rho-1} = \text{span}\left\{g_j, \text{ad}_f g_j, \ldots, \text{ad}_f^{\rho_j-1} g_j;\ j = 1, \ldots, m\right\}$ *is involutive and has constant dimension* $\rho = \sum_{i=1}^{p} \rho_i$*;*

(vi) $\text{ad}_{e_i} \mathcal{G}_j \subset \mathcal{G}_j$, $1 \le i \le q$, $0 \le j \le \max_s \rho_s$,
with $\mathcal{G}_j = \text{span}\left\{g_l, \text{ad}_f g_l, \ldots, \text{ad}_f^{j+\rho_l-\max_s \rho_s} g_l,\ 1 \le l \le m\right\}$.

Then the L_2-gain almost disturbance decoupling problem is solvable. □

Proof It follows from *(i)-(v)* that we can globally define a change of coordinates as in the proof of Theorem 4.7 which leads to (4.5).
In q-coordinates we have that

$$\mathcal{G}_j = \text{span}\left\{\frac{\partial}{\partial q_{i(\max_s \rho_s - j)}}, \ldots, \frac{\partial}{\partial q_{i\rho_i}};\ i = 1, \ldots, p\right\}$$

for $0 \le j \le \max_s \rho_s - 1$. Then condition *(vi)* leads to the system form

$$\begin{aligned}
\dot{q}_{11} &= q_{12} + W_{11}^T(\tilde{q}_1, q_r)d, \\
&\ \vdots \\
\dot{q}_{1\rho_1} &= v_1 + W_{1\rho_1}^T(\tilde{q}_1, \ldots, \tilde{q}_{\rho_1}, q_r)d, \\
\dot{q}_{21} &= q_{22} + W_{21}^T(\tilde{q}_1, q_r)d, \\
&\ \vdots \\
\dot{q}_{2\rho_2} &= v_2 + W_{2\rho_2}^T(\tilde{q}_1, \ldots, \tilde{q}_{\rho_2}, q_r)d, \\
&\ \vdots \\
&\ \vdots \\
\dot{q}_{p1} &= q_{p2} + W_{p1}^T(\tilde{q}_1, q_r)d, \\
&\ \vdots \\
\dot{q}_{p\rho_p} &= v_p + W_{p\rho_p}^T(\tilde{q}_1, \ldots, \tilde{q}_{\rho_p}, q_r)d, \\
\dot{q}_r &= \varphi(\tilde{q}_1, q_r) + \Psi^T(\tilde{q}_1, q_r)d, \\
z_l &= q_{l1}, \quad 1 \le l \le p
\end{aligned}$$

where

$$\tilde{q}_j = \left\{q_{kj}, \text{ for all } k = 1, \ldots, p \text{ for which } j \le \rho_k\right\}.$$

Now the proof continues in the same way as the proof of Theorem 4.8 using the there described algorithm. ■

It should be remarked that several slightly different sets of conditions can be formulated replacing conditions *(vi)* in Theorem 4.9, all leading to a structure on which an algorithm similar to the one used in the proof of Theorem 4.8 can be applied.

Finally it can be shown that if we additionally assume the zero-dynamics to be independent of d, namely

$$\dot{q}_r = \varphi(0, q_r),$$

then asymptotic stability of the origin $q_r = 0$ for the zero-dynamics implies that the $\mathcal{H}_\infty$ almost disturbance decoupling problem is solvable.

Theorem 4.10 *If, in addition to the conditions (i)-(vi) of Theorem 4.7 or of Theorem 4.9, system (4.1) is such that also:*

(vii) the zero dynamics are independent of d and globally asymptotically stable;

then the $\mathcal{H}_\infty$ almost disturbance decoupling problem is solvable. □

Proof Combine the SISO result from [MRST 94] with Theorem 4.7 respectively Theorem 4.9. Details are left to the reader. ■

Chapter 5

The singular $\mathcal{H}_\infty$ problem: a geometric approach

For linear systems a second approach to tackle the singular $\mathcal{H}_\infty$ problem is discussed in [StTr 90], [St 92] and [Sch 91]. In this geometric approach the system is decomposed using a state transformation based on the strongly controllable subspace. For the decomposed system it is possible to split the singular $\mathcal{H}_\infty$ problem into a regular state feedback $\mathcal{H}_\infty$ problem for the first subsystem, having as inputs part of the states of the second subsystem, and an almost disturbance decoupling problem for the second subsystem to 'track' the constructed feedback solution of the regular state feedback problem of the first subsystem.

In this chapter we will show that a similar approach can be applied to the nonlinear singular $\mathcal{H}_\infty$ problem. We first recapitulate the state feedback $\mathcal{H}_\infty$ solution for linear systems. Instead of constructing the strongly controllable subspace as is done in [St 92] we extend the linear system by adding the inputs and the disturbances as extra state variables and construct for this extended system the minimal conditioned invariant subspace containing the input vector fields. The projection of this subspace onto the state space of the original linear system is equal to the strongly controllable subspace and thus is equivalent to the approach of [StTr 90], [St 92]. The advantage of the present construction, however, is that it admits a direct generalization to the nonlinear case, as will be explained in the second section. This nonlinear decomposition will be also used in Subsection 5.2.4 to derive sufficient conditions for the solvability of the nonlinear singular $\mathcal{H}_\infty$ problem similar to the linear case. Part of these conditions are proved to be necessary for the solvability of the singular $\mathcal{H}_\infty$ problem.

The almost disturbance decoupling problem for nonlinear systems is not completely understood. In particular there are no necessary and sufficient (geometric) conditions known for the solvability of the almost disturbance decoupling problem. Therefore we were not able to derive a full characterization for the singular $\mathcal{H}_\infty$ problem. However for special classes of systems necessary and sufficient conditions for the solvability of the singular $\mathcal{H}_\infty$ problem will be derived.

Finally we apply a factorization idea to derive sufficient conditions. In this method the singular $\mathcal{H}_\infty$ problem is reduced to the singular $\mathcal{H}_\infty$ problem for an auxiliary system, which is easier solvable.

For affine nonlinear systems some of these ideas are also considered in the paper [As 94]. The decomposition in this paper, however, is not a generalization of the decomposition used in the linear case. Therefore even for linear systems this method will not lead to necessary and sufficient conditions.

5.1 Singular linear $\mathcal{H}_\infty$ control by state feedback

In this section the state feedback $\mathcal{H}_\infty$ results from [St 92], [StTr 90] are recapitulated for linear systems of the form

$$\overline{\Sigma}\begin{cases} \dot{\bar{x}} &= F\bar{x} + G\bar{u} + E\bar{d} \\ \bar{z} &= H\bar{x} + K\bar{u} \end{cases} \tag{5.1}$$

For details and proofs the reader is referred to [St 92], where also the extension to linear systems containing direct feedthrough terms from disturbances to outputs may be found. Instead of immediately introducing the strongly controllable subspace as in [St 92] we construct this subspace in two steps. First we extend the system $\overline{\Sigma}$ by including both the inputs and the disturbances as extra state components. For this extended system we calculate the minimal conditioned invariant subspace containing the image of the input matrix. In the second step we project this subspace onto the state space of the original system $\overline{\Sigma}$. In the first subsection we introduce the notion of conditioned invariance for linear systems without direct feedthrough from the inputs to the outputs are (the case $K = 0$). We decompose systems of this form using the minimal conditioned invariant subspace containing the image of the input matrix. In the second subsection we see that the extended system is of this special form. Furthermore the projection of the minimal conditioned invariant subspace including the input matrix for the extended system onto the state space of the original system $\overline{\Sigma}$ is equal to the strongly controllable subspace used in [St 92].

5.1.1 State transformation for linear systems with $K = 0$

In this subsection our attention is restricted to systems of the form

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{u} + E\bar{d} \\ \bar{z} &= H\bar{x} \end{aligned} \tag{5.2}$$

Conditioned invariance

Basic notions from the geometric approach to linear system theory are the notions of controlled invariance and conditioned invariance ([Ha 83], [HaSi 83], [Wo 79], [BaMa 92]). In this subsection our attention is focussed on the second notion.

Definition 5.1 A subspace $\mathcal{V}$ of $\mathbb{R}^n$ is said to be an *invariant subspace* for the matrix F if

$$F\mathcal{V} \subset \mathcal{V}.$$

The subspace $\mathcal{V}$ is said to be a *conditioned invariant subspace* with respect to the pair (F, H) if there exists a linear mapping B such that

$$(F + BH)\mathcal{V} \subset \mathcal{V}.$$

□

So a subspace $\mathcal{V}$ is called conditioned invariant if there exists an output injection matrix B which renders the subspace $\mathcal{V}$ invariant for the matrix $F + BH$. The property of conditioned invariance can also be characterized in the following way.

Lemma 5.2 *A subspace $\mathcal{V}$ of $\mathbb{R}^n$ is a conditioned invariant subspace for the pair (F, H) if and only if*

$$F\left(\mathcal{V} \cap \ker H\right) \subset \mathcal{V}.$$

□

Now we introduce a subspace which will play a key role in this section.

Definition 5.3 Consider the system (5.2). Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then there exists a *minimal conditioned invariant subspace* containing the subspace $\mathcal{W}$, denoted by $\mathcal{S}^*(\mathcal{W})$, defined as the smallest subspace $\mathcal{V}$ of $\mathbb{R}^n$ such that:

(i) $\mathcal{V}$ is conditioned invariant;

(ii) $\mathcal{W} \subset \mathcal{V}$.

□

An explicit recursive algorithm to calculate $\mathcal{S}^*(\mathcal{W})$ can be given.

Lemma 5.4 *Consider the system (5.2). Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then the minimal conditioned invariant subspace containing $\mathcal{W}$, $\mathcal{S}^*(\mathcal{W})$, is the limit of the increasing sequence of subspaces $\mathcal{S}_i(\mathcal{W})$ generated by the recursive algorithm*

$$\begin{aligned} \mathcal{S}_1(\mathcal{W}) &:= \mathcal{W} \\ \mathcal{S}_{i+1}(\mathcal{W}) &:= \mathcal{W} + F\left(\mathcal{S}_i(\mathcal{W}) \cap \ker H\right), \qquad i = 1, 2, \ldots. \end{aligned}$$

□

State transformation

We transform the state of the system (5.2) by using the minimal conditioned invariant subspace containing $\operatorname{im}G$, $\mathcal{S}^*(\operatorname{im}G)$. In order to do so we rewrite $\mathbb{R}^n = \bar{X}_1 \oplus \bar{X}_2 \oplus \bar{X}_3$ with $\bar{X}_2 = \mathcal{S}^*(\operatorname{im}G) \cap \ker H$, $\bar{X}_2 \oplus \bar{X}_3 = \mathcal{S}^*(\operatorname{im}G)$ and $\bar{X}_1$ arbitrary. Let $(\xi_1, \ldots, \xi_\nu, \xi_{\nu+1}, \ldots, \xi_\mu, \xi_{\mu+1}, \ldots, \xi_n)$ be a basis of $\mathbb{R}^n$ such that $\xi_{\nu+1}, \ldots, \xi_\mu$ is a basis of $\mathcal{S}^*(\operatorname{im}G) \cap \ker H$ and $\xi_{\nu+1}, \ldots, \xi_n$ a basis of $\mathcal{S}^*(\operatorname{im}G)$.

Then the linear mappings F, G and H with respect to this basis have the following matrix form

$$F = \begin{pmatrix} F_{11} & 0 & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ G_2 \\ G_3 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & 0 & H_3 \end{pmatrix}.$$

If we accordingly decompose the state x and the matrix E as

$$\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix}, \qquad E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

then in the new basis the system (5.2) takes the form

$$\dot{\bar{x}}_1 = F_{11}\bar{x}_1 + F_{13}\bar{x}_3 + E_1\bar{d} \tag{5.3}$$

$$\begin{pmatrix} \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{pmatrix} = \begin{pmatrix} F_{22} & F_{23} \\ F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} + \begin{pmatrix} G_2 \\ G_3 \end{pmatrix}\bar{u} + \begin{pmatrix} F_{21} \\ F_{31} \end{pmatrix}\bar{x}_1 + \begin{pmatrix} E_2 \\ E_3 \end{pmatrix}\bar{d} \tag{5.4}$$

$$\bar{z} = H_1\bar{x}_1 + H_3\bar{x}_3 \tag{5.5}$$

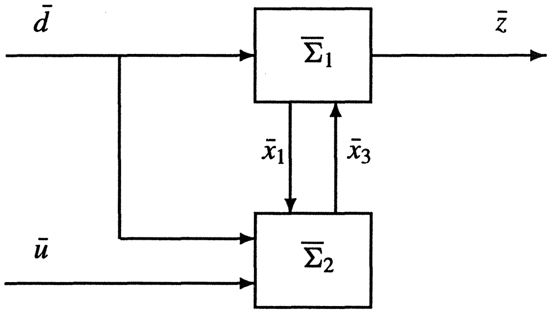

Figure 5.1: system (5.2) after state transformation

This arrangement already suggests to write this system as an interconnection of two subsystems. The first subsystem $\overline{\Sigma}_1$ is given by (5.3) and (5.5), and has as state space $\bar{\mathcal{X}}_1$, input space $\bar{\mathcal{X}}_3$ and output space $\mathbb{R}^p$. The second subsystem $\overline{\Sigma}_2$ is given by (5.4) having state space $\bar{\mathcal{X}}_2 \oplus \bar{\mathcal{X}}_3$, input space $\mathbb{R}^m$ and output space $\bar{\mathcal{X}}_3$ (see Figure 5.1).

The system $\overline{\Sigma}_1$ has the nice property that H_3 has full column rank. Indeed, let $\bar{x}_3$ be such that $H_3\bar{x}_3 = 0$. Then the vector $\bar{x} = (\ 0^T \quad 0^T \quad \bar{x}_3^T\ )$ is an element of $\bar{X}_2 = \mathcal{S}^* \cap \ker H$, and hence $\bar{x}_3 = 0$.

5.1.2 State transformation for linear system with $K \neq 0$

Now we consider the system $\overline{\Sigma}$, with $K \neq 0$. We construct an extended system by adding the inputs $\bar{u}$ and disturbances $\bar{d}$ as extra state components:

$$\overline{\Sigma}_e \begin{cases} \dot{\bar{x}} = F\bar{x} + G\bar{u} + E\bar{d} \\ \dot{\bar{u}} = \bar{v} \\ \dot{\bar{d}} = \bar{w} \\ \bar{z} = H\bar{x} + K\bar{u} \end{cases} \tag{5.6}$$

having state $\bar{x}_e^T = (\bar{x}^T \quad \bar{u}^T \quad \bar{d}^T)$. We can rewrite $\overline{\Sigma}_e$ more compactly as:

$$\overline{\Sigma}_e \begin{cases} \dot{\bar{x}}_e &= F_e \bar{x}_e + G_e \bar{v} + E_e \bar{w} \\ \bar{z} &= H_e \bar{x}_e \end{cases} \tag{5.7}$$

where

$$F_e = \begin{pmatrix} F & G & E \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_e = \begin{pmatrix} 0 \\ I_m \\ 0 \end{pmatrix},$$

$$E_e = \begin{pmatrix} 0 \\ 0 \\ I_q \end{pmatrix}, \quad H_e = (H \quad K \quad 0).$$

It is clear that $\overline{\Sigma}_e$ is a system of the form considered in Subsection 5.1.1. Thus we can compute the minimal conditioned invariant subspace as in Subsection 5.1.1.

Now we calculate for the system $\overline{\Sigma}_e$ the minimal conditioned invariant distribution containing $\mathrm{im} G_e$, $\mathcal{S}_e^*(\mathrm{im} G_e)$, using the algorithm from Lemma 5.4. Again this distribution is used to rewrite the state space $\mathbb{R}^{n+m+q} = \bar{\mathcal{X}}_{1e} \oplus \bar{\mathcal{X}}_{2e} \oplus \bar{\mathcal{X}}_{3e}$ with $\bar{\mathcal{X}}_{2e} = \mathcal{S}_e^*(\mathrm{im} G_e) \cap \ker H_e$, $\bar{\mathcal{X}}_{2e} \oplus \bar{\mathcal{X}}_{3e} = \mathcal{S}_e^*(\mathrm{im} G_e)$ and $\bar{\mathcal{X}}_{1e}$ arbitrary. For the extended system $\overline{\Sigma}_e$ the subspace $\mathrm{im} G_e$ is given by

$$\mathrm{im} G_e = \mathrm{span}\{v_1, \dots, v_m\}$$

where $v_1, \dots, v_m$ is any basis of the input space $\mathbb{R}^m$. For later convenience we will choose $v_1, \dots, v_m$ such that $v_1, \dots, v_{m_1}$ is a basis of $\ker K$. It follows from the algorithm and the special structure of the matrix F_e that the subspace $\bar{\mathcal{X}}_{1e}$ can be chosen such that

$$\mathrm{span}\left\{\bar{d}_1, \dots, \bar{d}_q\right\} \subset \bar{\mathcal{X}}_{1e}.$$

Finally let $(\xi_1, \dots, \xi_\nu, \xi_{\nu+1}, \dots, \xi_\mu, \xi_{\mu+1}, \dots, \xi_n)$ be a basis of $\mathbb{R}^n$ such that:

$$\begin{aligned} \mathrm{span}\left\{\xi_{\nu+1}, \dots, \xi_\mu\right\} + \mathrm{span}\left\{v_1, \dots, v_{m_1}\right\} &= \mathcal{S}_e^*(\mathrm{im} G_e) \cap \ker H_e; \\ \mathrm{span}\left\{\xi_{\nu+1}, \dots, \xi_n\right\} + \mathrm{span}\left\{v_1, \dots, v_m\right\} &= \mathcal{S}_e^*(\mathrm{im} G_e). \end{aligned}$$

Then the subspaces $\bar{\mathcal{X}}_{1e}$, $\bar{\mathcal{X}}_{2e}$ and $\bar{\mathcal{X}}_{3e}$ are given by:

$$\begin{aligned} \bar{\mathcal{X}}_{1e} &= \mathrm{span}\left\{\xi_1, \dots, \xi_\nu, \bar{d}_1, \dots, \bar{d}_q\right\}; \\ \bar{\mathcal{X}}_{2e} &= \mathrm{span}\left\{\xi_{\nu+1}, \dots, \xi_\mu, v_1, \dots, v_{m_1}\right\}; \\ \bar{\mathcal{X}}_{3e} &= \mathrm{span}\left\{\xi_{\mu+1}, \dots, \xi_n, v_{m_1+1}, \dots, v_m\right\}. \end{aligned}$$

In this new basis for the state space $\mathbb{R}^n$ and the input space $\mathbb{R}^m$ the mappings F_e and H_e have the following special form

$$F_e = \left(\begin{array}{ccc|cc|c} F_{11} & 0 & F_{13} & 0 & G_{12} & E_1 \\ F_{21} & F_{22} & F_{23} & G_{21} & G_{22} & E_2 \\ F_{31} & F_{32} & F_{33} & G_{31} & G_{32} & E_3 \\ \hline & 0 & & & 0 & 0 \\ \hline & 0 & & & 0 & 0 \end{array}\right),$$
$$H_e = \left(\begin{array}{ccc|cc|c} H_1 & 0 & H_3 & 0 & K_2 & 0 \end{array}\right)$$

and the original system $\overline{\Sigma}$ takes the form

$$\dot{\bar{x}}_1 = F_{11}\bar{x}_1 + \begin{pmatrix} F_{13} & G_{12} \end{pmatrix}\begin{pmatrix} \bar{x}_3 \\ \bar{u}_2 \end{pmatrix} + E_1\bar{d} \tag{5.8}$$

$$\begin{pmatrix} \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{pmatrix} = \begin{pmatrix} F_{22} & F_{23} \\ F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} + \begin{pmatrix} G_{21} \\ G_{31} \end{pmatrix}\bar{u}_1 + \begin{pmatrix} G_{22} \\ G_{32} \end{pmatrix}\bar{u}_2 + \begin{pmatrix} F_{21} \\ F_{31} \end{pmatrix}\bar{x}_1 + \begin{pmatrix} E_2 \\ E_3 \end{pmatrix}\bar{d} \tag{5.9}$$

$$\bar{z} = H_1\bar{x}_1 + \begin{pmatrix} H_3 & K_2 \end{pmatrix}\begin{pmatrix} \bar{x}_3 \\ \bar{u}_2 \end{pmatrix} \tag{5.10}$$

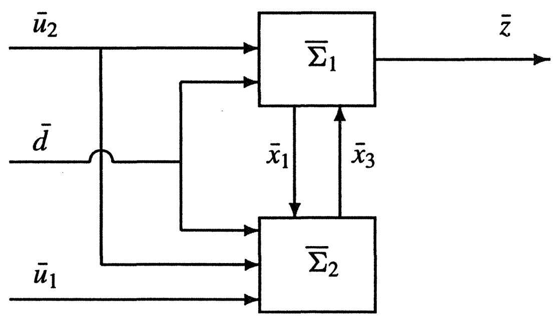

Figure 5.2: system $\overline{\Sigma}$ after input and state transformation

Again this system can be seen as an interconnection of two systems. The first subsystem $\overline{\Sigma}_1$ is given by (5.8) and (5.10) with state $\bar{x}_1$, inputs $(\bar{x}_3, \bar{u}_2, \bar{d})$ and outputs $\bar{z}$, and the second subsystem $\overline{\Sigma}_2$ is given by (5.9) with state $(\bar{x}_2, \bar{x}_3)$, inputs $(\bar{u}_1, \bar{x}_1, \bar{u}_2, \bar{d})$ and outputs $\bar{x}_3$ (see Figure 5.2). As in 5.1.1 it follows that the matrix $\begin{pmatrix} H_3 & K_2 \end{pmatrix}$ is injective.

5.1.3 The quadratic matrix inequality

An important role in the linear theory for the singular $\mathcal{H}_\infty$ problem is played by the *quadratic matrix inequality*

$$F_\gamma(P) := \begin{pmatrix} F^T P + PF + \frac{1}{\gamma^2} PEE^T P + H^T H & H^T K + PG \\ K^T H + G^T P & K^T K \end{pmatrix} \geq 0.$$

Symmetric solutions P to this quadratic matrix inequality can be proved (see [St 92]) to be of the form

$$P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the new basis. We define $\mathcal{G}(s)$ as the transfer matrix

$$\mathcal{G}(s) = H\,(sI - F)^{-1}\,G + K$$

and the $(n \times (n+m))$ controllability pencil $L_\gamma(P,s)$ by

$$L_\gamma(P,s) := \begin{pmatrix} sI - F - \frac{1}{\gamma^2} EE^T P & -G \end{pmatrix}.$$

Then we recapitulate the following result ([St 92]).

Theorem 5.5 *The following statements are equivalent:*

(i) there exists a symmetric solution $P \geq 0$ *to* $F_\gamma(P) \geq 0$ *such that*

$$\text{rank } F_\gamma(P) = \text{rank}_{\mathbb{R}(s)}\, \mathcal{G}$$

and

$$\text{rank} \begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} = n + \text{rank}_{\mathbb{R}(s)}\, \mathcal{G} \qquad \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+$$

where $\text{rank}_{\mathbb{R}(s)}\, \mathcal{G}$ *is the* normal rank *for the polynomial matrix* $\mathcal{G}(s)$*, with entries from* $\mathbb{R}(s)$*, defined by*

$$\text{rank}_{\mathbb{R}(s)}\, \mathcal{G} = \max \left\{ \text{rank}\, \mathcal{G}(s) \,|\, s \in \mathbb{C} \right\};$$

(ii) there exists a symmetric solution $P_1 \geq 0$ *to*

$$P_1 F_{11} + F_{11}^T P_1 + \frac{1}{\gamma^2} P_1 E_1 E_1^T P_1 - P_1 G_{12} \left(K_2^T K_2\right)^{-1} G_{12}^T P_1 + H_1^T H_1$$

$$- \begin{pmatrix} H_3^T H_1 + F_{13}^T P_1 \\ K_2^T H_1 + G_{12}^T P_1 \end{pmatrix}^T \begin{pmatrix} H_3^T H_3 & H_3^T K_2 \\ K_2^T H_3 & K_2^T K_2 \end{pmatrix}^{-1} \begin{pmatrix} H_3^T H_1 + F_{13}^T P_1 \\ K_2^T H_1 + G_{12}^T P_1 \end{pmatrix}$$

$$= 0$$

satisfying

$$\sigma\Bigg(F_{11} + \frac{1}{\gamma^2} E_1 E_1^T P_1 - \begin{pmatrix} F_{13}^T \\ G_{12}^T \end{pmatrix}^T \begin{pmatrix} H_3^T H_3 & H_3^T K_2 \\ K_2^T H_3 & K_2^T K_2 \end{pmatrix}^{-1} \begin{pmatrix} H_3^T H_1 + F_{13}^T P_1 \\ K_2^T H_1 + G_{12}^T P_1 \end{pmatrix}\Bigg) \subset \mathbb{C}^-.$$

□

It should be noted that statement *(ii)* in Theorem 5.5 is equivalent to the existence of a feedback law

$$\begin{pmatrix} \bar{x}_3 \\ \bar{u}_2 \end{pmatrix} = L\bar{x}_1$$

which solves the regular $\mathcal{H}_\infty$ state feedback problem for the subsystem $\overline{\Sigma}_1$ with inputs $(\bar{x}_3, \bar{u}_2)$ (see Theorem 2.21).

5.1.4 The state feedback $\mathcal{H}_\infty$ control problem

Solvability of the strict suboptimal $\mathcal{H}_\infty$ problem for the linear system $\overline{\Sigma}$ can be characterized as follows (compare with Theorem 2.21).

Theorem 5.6 *Consider* $\overline{\Sigma}$. *Let* $\gamma > 0$. *Assume the system* (F, G, H, K) *has no invariant zeros on the imaginary axis (see Theorem 2.21). Then the following statements are equivalent:*

(i) there exists a feedback $\bar{u} = L\bar{x}$ *such that after applying this feedback to the system* $\overline{\Sigma}$ *the closed loop system has* L_2*-gain less than* γ *and is asymptotically stable;*

(ii) there exists a non-negative, symmetric solution P to the quadratic matrix inequality $F_\gamma(P) \geq 0$ such that

$$\text{rank } F_\gamma(P) = \text{rank}_{\mathbb{R}(s)} \mathcal{G}_{ci}$$

and

$$\text{rank} \begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} = n + \text{rank}_{\mathbb{R}(s)} \mathcal{G}_{ci}, \qquad \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+.$$

□

Together with Theorem 5.5 this means that the solvability of the singular $\mathcal{H}_\infty$ control problem is fully characterized by the solvability of the regular $\mathcal{H}_\infty$ control problem for the subsystem $\overline{\Sigma}_1$. Before we give a sketch of the proof of Theorem 5.6 we first recapitulate the following result about the solvability of the almost disturbance decoupling problem with internal stability for linear systems of the form $\overline{\Sigma}$ (see [Tr 86], [Wi 81], [Wi 82], [St 92]).

Theorem 5.7 *Consider the system $\overline{\Sigma}$. Let $\mathcal{T}$ and $\mathcal{V}_g$ denote the strongly controllable subspace and the stabilizable weakly unobservable subspace respectively. Then the following statements are equivalent:*

(i)

$$\mathcal{V}_g + \mathcal{T} = \mathbb{R}^n;$$

(ii)

$$\text{rank} \begin{pmatrix} sI - F & -G \\ H & K \end{pmatrix} = n + \text{rank} \begin{pmatrix} H & K \end{pmatrix}, \qquad \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+;$$

(iii) for all $\varepsilon > 0$ there exists a feedback $\bar{u} = L\bar{x}$ for $\overline{\Sigma}$ such that the closed-loop system has L_2-gain less than or equal to ε and is asymptotically stable (the almost disturbance decoupling problem with internal stability is solvable for $\overline{\Sigma}$).

□

Sketch of the proof of Theorem 5.6 For details we refer to [St 92].

(ii) $\Rightarrow$ *(i)* Statement *(ii)* implies that the regular $\mathcal{H}_\infty$ problem for the subsystem $\overline{\Sigma}_1$ is solvable (Theorem 5.5 and Theorem 2.21). Then the state $\bar{x}_3$ can be transformed by subtracting the optimal solution of the regular subproblem. So when this new variable is equal to zero we have perfect tracking of the solution for Σ_1. Now we transform the state of the system $\overline{\Sigma}_2$ such that part of the state is equal to the tracking error between $\bar{x}_3$ and the $\bar{x}_3$-feedback solution of the regular $\mathcal{H}_\infty$ problem for the subsystem $\overline{\Sigma}_1$. This transformed system will be denoted by $\overline{\Sigma}_2^T$. Both $\overline{\Sigma}_2$ and $\overline{\Sigma}_2^T$ are strongly controllable which means that the state space is equal to the strongly controllable subspace. By Theorem 5.7 this implies that the almost disturbance decoupling problem with stability is solvable for $\overline{\Sigma}_2^T$. Combining these two results leads to the solvability of the $\mathcal{H}_\infty$ problem of the complete system $\overline{\Sigma}$.

(i) $\Rightarrow$ *(ii)* From statement *(i)* it follows that (F, G) is stabilizable and for the system $\overline{\Sigma}$ there exists a $\delta < \gamma$ such that for all $\bar{d} \in L_2(0,\infty)$ there exists an $\bar{u} \in L_2(0,\infty)$ such that $\bar{x} \in L_2(0,\infty)$ and

$$\int_0^\infty \|\bar{z}(\tau)\|^2 \mathrm{d}\tau \leq \delta^2 \int_0^\infty \|\bar{d}(\tau)\|^2 \mathrm{d}\tau \tag{5.11}$$

then it follows that $\bar{x}_1 \in L_2(0,\infty)$ and $\bar{x}_3 \in L_2(0,\infty)$. Together with (5.11) this implies that the regular $\mathcal{H}_\infty$ problem for the subsystem $\overline{\Sigma}_1$ is solvable (details see [St 92]), which via Theorem 2.21 and 5.5 leads to the statement *(ii)*. ■

5.1.5 Factorization approach

The sufficiency part of the result in Theorem 5.6 can also be proved in a different way. Factorize the matrix $F_\gamma(P)$ corresponding to the quadratic matrix inequality

$$F_\gamma(P) = \begin{pmatrix} H_P^T \\ K_P^T \end{pmatrix} \begin{pmatrix} H_P & K_P \end{pmatrix}$$

for certain matrices H_P and K_P. Define the new system

$$\overline{\Sigma}_P \begin{cases} \dot{\bar{x}} &= \left(F + \frac{1}{\gamma^2} E E^T P\right) \bar{x} + G\bar{u} + E\bar{d}_p \\ \bar{z}_p &= H_P \bar{x} + K_P \bar{u} \end{cases} \tag{5.12}$$

where $\bar{d}_p = \bar{d} - \frac{1}{\gamma^2} E^T P \bar{x}$. Then the following result can be proved ([St 92]).

Lemma 5.8 *Consider $\overline{\Sigma}$ and $\overline{\Sigma}_P$. Let P satisfy the condition (ii) of Theorem 5.6 and let L be a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$. Then the following statements are equivalent:*

(i) the closed loop system consisting of $\overline{\Sigma}$ with the feedback $\bar{u} = L\bar{x}$ has L_2-gain from $\bar{d}$ to $\bar{z}$ less than γ and is asymptotically stable;

(ii) the closed loop system consisting of $\overline{\Sigma}_P$ with the feedback $\bar{u} = L\bar{x}$ has L_2-gain from $\bar{d}_p$ to $\bar{z}_p$ less than γ and is asymptotically stable.

□

Sketch of the proof The L_2-gain part of the result can be proved using the following completion of the squares argument:

$$
\begin{aligned}
&\|\bar{z}_p\|^2 - \gamma^2\|\bar{d}_p\|^2 \\
&= \bar{x}^T H_P^T H_P \bar{x} + 2\bar{x}^T H_P^T K_P \bar{u} + \bar{u}^T K_P^T K_P \bar{u} - \gamma^2\|\bar{d}\|^2 + 2\bar{x}^T P E \bar{d} \\
&\quad - \frac{1}{\gamma^2}\bar{x}^T P E E^T P \bar{x} \\
&= \begin{pmatrix} \bar{x}^T & \bar{u}^T \end{pmatrix} F_\gamma(P) \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} - \gamma^2\|\bar{d}\|^2 + 2\bar{x}^T P E \bar{d} - \frac{1}{\gamma^2}\bar{x}^T P E E^T P \bar{x} \\
&= 2\bar{x}^T P E \bar{d} + \bar{x}^T H^T H \bar{x} + 2\bar{x}^T H^T K \bar{u} + 2\bar{x}^T P G \bar{u} + \bar{u}^T K^T K \bar{u} \\
&= \|\bar{z}\|^2 - \gamma^2\|\bar{d}\|^2 + 2\bar{x}^T P \left(F\bar{x} + G\bar{u} + E\bar{d}\right) \\
&= \|\bar{z}\|^2 - \gamma^2\|\bar{d}\|^2 + 2\bar{x}^T P \dot{\bar{x}}.
\end{aligned}
$$

Now we take the integral from 0 to ∞ of both sides of this equality:

$$
\begin{aligned}
&\int_0^\infty \|\bar{z}_p(\tau)\|^2 - \gamma^2\|\bar{d}_p(\tau)\|^2 \mathrm{d}\tau \\
&= \int_0^\infty \|\bar{z}(\tau)\|^2 - \gamma^2\|\bar{d}(\tau)\|^2 \mathrm{d}\tau + \lim_{t\to\infty} 2\bar{x}^T(t) P \bar{x}(t) - 2\bar{x}^T(0) P \bar{x}(0).
\end{aligned}
$$

For $\bar{x}(0) = 0$ and assuming that the system is asymptotically stable, implying that $\lim_{t\to\infty} \bar{x}(t) = 0$, then it follows that the L_2-gains from disturbance to output of the systems $\overline{\Sigma}$ and $\overline{\Sigma}_P$ are equal. Therefore application of a feedback $\bar{u} = L\bar{x}$ to both systems leads to the same L_2-gain. ■

Condition *(ii)* from Theorem 5.6 can be rewritten as

$$\operatorname{rank}\begin{pmatrix} sI - F - \frac{1}{\gamma^2}EE^TP & -G \\ H_P & K_P \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} H_P & K_P \end{pmatrix}, \quad \forall s \in \mathbb{C}^0 \cup \mathbb{C}^+$$

which implies solvability of the almost disturbance decoupling problem for the system $\overline{\Sigma}_P$ (see Theorem 5.7).

5.2 Singular $\mathcal{H}_\infty$ problem for general nonlinear systems

In this section we try to extend the linear result explained in the previous section to general nonlinear systems of the form

$$\Sigma \begin{cases} \dot{x} &= f(x, u, d) \\ y &= x \\ z &= h(x, u) \end{cases} \tag{5.13}$$

In the first two subsections we define a state transformation for these systems based on a minimal conditioned invariant distribution. Similar to the linear case (Section 5.1) we first define the notion of conditioned invariance for affine nonlinear systems without a direct feedthrough from the inputs to the outputs. As in the linear case this step is instrumental in the second subsection where we construct an extended system from the system Σ by adding the inputs and the disturbances as extra state components. This extended system is affine and does not have a direct feedthrough from the new defined inputs to the outputs, which makes it possible to apply the results from the first subsection.

The rest of the chapter is concerned with finding sufficient and necessary conditions for the solvability of the singular $\mathcal{H}_\infty$ problem. These conditions are in terms of the solvability of a regular $\mathcal{H}_\infty$ problem for a subsystem of Σ and an $\mathcal{H}_\infty$ almost disturbance decoupling problem. Sometimes it will be necessary to make extra assumptions on the structure of the system Σ in order to derive such conditions. For instance in Subsection 5.2.6 where necessary and sufficient conditions are derived and in Subsection 5.2.7 to apply a factorization approach.

5.2.1 State transformation for affine nonlinear systems without direct feedthrough from inputs to outputs

First we restrict our attention to nonlinear systems of the form

$$\begin{array}{rcll} \dot{x} &=& f(x) + \sum_{j=1}^{m} g_j(x)u_j + \sum_{i=1}^{q} e_i(x)d_i & \\ z_l &=& h_l(x), & l = 1, \ldots, p \end{array} \tag{5.14}$$

Conditioned invariance

Also for nonlinear systems the notions of invariance, controlled invariance and conditioned invariance can be defined (see [IKGM 81], [MRS 94], [vdS 85], [vdS 87]). In this book the following definitions are used

Definition 5.9 A distribution $\mathcal{D}$ is said to be an *invariant distribution* under the dynamics (5.14) if

$$\begin{array}{rcll} [f, \mathcal{D}] &\subset& \mathcal{D}, & \\ [g_j, \mathcal{D}] &\subset& \mathcal{D}, & j = 1, \ldots, m. \end{array}$$

□

Define

$$\ker \mathrm{d}h = \bigcap_{l=1}^{p} \ker \mathrm{d}h_l.$$

Definition 5.10 A distribution $\mathcal{D}$ is said to be a *conditioned invariant distribution* under the dynamics (5.14) if

$$\begin{array}{rcll} [f, \mathcal{D} \cap \ker \mathrm{d}h] &\subset& \mathcal{D}, & \\ [g_j, \mathcal{D} \cap \ker \mathrm{d}h] &\subset& \mathcal{D}, & j = 1, \ldots, m. \end{array}$$

□

For linear systems the second conditions are automatically satisfied because in that case the vectors g_j are constant and the distribution is generated by constant vectors from a linear subspace while the Lie bracket of constant vector fields is zero. It should be noted that these definitions of invariance and conditioned invariance are independent from the disturbance vector fields. Therefore for the calculation of these distributions we can consider the system (5.14) without the disturbances. The definition of the notion of conditioned invariant distribution is an extension of the characterization of a conditioned invariant subspace in Lemma 5.2.

Similar to the linear case we can define for any distribution $\mathcal{W}$ the *minimal involutive conditioned invariant distribution* containing $\mathcal{W}$. An explicit algorithm to calculate this distribution is given by

Lemma 5.11 *Consider the system (5.14). Then the minimal involutive conditioned invariant distribution containing $\mathcal{W}$, denoted $\mathcal{S}^*(\mathcal{W})$, is given by*

$$\mathcal{S}^*(\mathcal{W}) = \bigcup_{k\geq 1} \overline{\mathcal{S}}_k(\mathcal{W})$$

where $\overline{\mathcal{D}}$ indicates the involutive closure of the distribution $\mathcal{D}$, and the increasing sequence $\mathcal{S}_k(\mathcal{W})$ is generated by the following recursive algorithm

$$\begin{aligned} \mathcal{S}_1(\mathcal{W}) &= \mathcal{W} \\ \mathcal{S}_{k+1}(\mathcal{W}) &= \overline{\mathcal{S}}_k(\mathcal{W}) + [f, \overline{\mathcal{S}}_k(\mathcal{W}) \cap \ker \mathrm{d}h] \\ &\quad + \sum_{j=1}^{m} [g_j, \overline{\mathcal{S}}_k(\mathcal{W}) \cap \ker \mathrm{d}h] \qquad k = 1, 2, \ldots. \end{aligned}$$

□

Remark 5.12 For constructing just the minimal conditioned invariant distribution containing $\mathcal{W}$ instead of the minimal *involutive* conditioned invariant distributions we can use the recursive algorithm without involutive closures.

□

State transformation

Inspired by the linear theory we will use the minimal involutive conditioned invariant distribution which contains the input vector fields g_j, which we will refer to as $\mathcal{S}^*$, to construct a state transformation for the system (2.27). This distribution is according to Lemma 5.11 generated by the algorithm:

$\mathcal{S}^*$ algorithm

$$\mathcal{S}_1 = \mathrm{span}\{g_1, \ldots, g_m\}$$

$$
\begin{aligned}
\mathcal{S}_{k+1} &= \overline{\mathcal{S}}_k + \left[f, \overline{\mathcal{S}}_k \cap \ker \mathrm{d}h\right] + \sum_{j=1}^{m} \left[g_j, \overline{\mathcal{S}}_k \cap \ker \mathrm{d}h\right] \qquad k = 1, 2, \ldots \\
\mathcal{S}^* &= \bigcup_{k \geq 1} \overline{\mathcal{S}}_k
\end{aligned}
$$

□

By construction $\mathcal{S}^*$ is involutive and conditioned invariant, i.e.,

$$
\left[f, \mathcal{S}^* \cap \ker \mathrm{d}h\right] \subset \mathcal{S}^*, \tag{5.15}
$$

$$
\left[g_j, \mathcal{S}^* \cap \ker \mathrm{d}h\right] \subset \mathcal{S}^*, \qquad j = 1, \ldots, m. \tag{5.16}
$$

and is the minimal conditioned invariant involutive distribution which contains the input vector fields g_j.

If the distributions $\mathcal{S}^*$ and $\mathcal{S}^* \cap \ker \mathrm{d}h$ are constant dimensional then we can make the following decomposition of the state space of (5.14). Define $\mathcal{X}_2 := \mathcal{S}^* \cap \ker \mathrm{d}h$, $\mathcal{X}_2 \oplus \mathcal{X}_3 = \mathcal{S}^*$ and $\mathcal{X}_1$ arbitrary such that $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 = M$.

Using Frobenius' Theorem ([NvdS 90], [Is 89]) it follows that similar to the linear case there exist local coordinates $(\xi_1, \ldots, \xi_\nu, \xi_{\nu+1}, \ldots, \xi_\mu, \xi_{\mu+1}, \ldots, \xi_n)$ such that:

$$
\begin{aligned}
\mathcal{X}_1 &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_\nu}\right\}; \\
\mathcal{X}_2 &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_{\nu+1}}, \ldots, \frac{\partial}{\partial \xi_\mu}\right\}; \\
\mathcal{X}_3 &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_{\mu+1}}, \ldots, \frac{\partial}{\partial \xi_n}\right\}.
\end{aligned}
$$

In these new coordinates the system (5.14) takes the following form

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_3) & & + \textstyle\sum_{i=1}^{q} e_{i1}(x_1, x_2, x_3) d_i \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) & + \textstyle\sum_{j=1}^{m} g_{j2}(x_1, x_2, x_3) u_j & + \textstyle\sum_{i=1}^{q} e_{i2}(x_1, x_2, x_3) d_i \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) & + \textstyle\sum_{j=1}^{m} g_{j3}(x_1, x_2, x_3) u_j & + \textstyle\sum_{i=1}^{q} e_{i3}(x_1, x_2, x_3) d_i \\
z_l &= h(x_1, x_3), & & l = 1, \ldots, p
\end{aligned}
$$

or in shorthand notation

$$
\dot{x}_1 = f_1(x_1, x_3) + e_1(x_1, x_2, x_3) d \tag{5.17}
$$

$$
\dot{x}_2 = f_2(x_1, x_2, x_3) + g_2(x_1, x_2, x_3) u + e_2(x_1, x_2, x_3) d \tag{5.18}
$$

$$
\dot{x}_3 = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3) u + e_3(x_1, x_2, x_3) d \tag{5.19}
$$

$$
z = h(x_1, x_3) \tag{5.20}
$$

Remark 5.13 The above form can be easily verified by using the implications:

$$
\begin{array}{rcll}
[f, \mathcal{X}_2] \subset \mathcal{X}_2 \oplus \mathcal{X}_3 & \Rightarrow & \frac{\partial f_1}{\partial x_2} \equiv 0; & \\
g_j \in \mathcal{X}_2 \oplus \mathcal{X}_3 & \Rightarrow & g_{j1} \equiv 0, & j = 1, \ldots, m; \\
\mathcal{X}_2 \subset \ker \mathrm{d}h & \Rightarrow & \frac{\partial h_l}{\partial x_2} \equiv 0, & l = 1, \ldots, p.
\end{array}
$$

□

As in the linear case the system (5.14) can be viewed as an interconnection of two general nonlinear systems. The first subsystem, denoted by Σ_1, has state x_1, inputs (x_2, x_3, d) and outputs (z, x_1) and is given by (5.17), (5.20), and the second subsystem Σ_2 has state (x_2, x_3), inputs (x_1, u, d) and outputs (x_2, x_3) and is given by (5.18), (5.19) (see Figure 5.3).

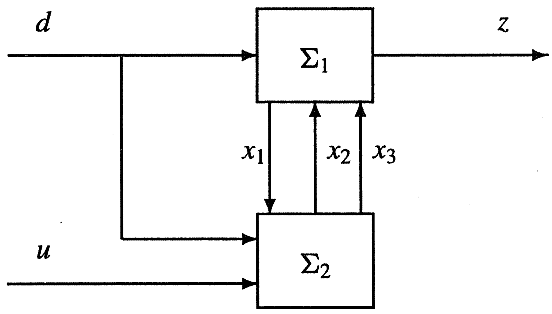

Figure 5.3: system (5.14) after state transformation

It should be noted that compared with the decomposition of the linear system (5.2) there exists an extra connection, namely the state components x_2 from the subsystem Σ_2 influence the subsystem Σ_1. This is due to the fact that for nonlinear systems the disturbance vector fields are state dependent while they are not taken into account in the construction of the distribution $\mathcal{S}^*$. A way to eliminate this extra interconnection is to use a stronger notion of invariance and conditioned invariance. This comes down to including the Lie-brackets with the disturbance vector fields e_i in the $\mathcal{S}^*$-algorithm at the expense of a possibly larger distribution $\mathcal{S}^*$. For the final state transformation of the system Σ this extension is not necessary as will be shown in the next subsection.

Comparing with the linear case we deduce that also in the nonlinear case the mapping h has an injectivity property.

Lemma 5.14 *Assume that $\mathcal{S}^*$ and $\mathcal{S}^* \cap \ker \mathrm{d}h$ are constant dimensional. Then the Jacobian of the mapping h from x_3 to z has full column rank and thus the mapping h from x_3 to z is locally injective around every point $p \in M$.* □

Proof As shown earlier there exist coordinates $(\xi_1, \ldots, \xi_\nu, \xi_{\nu+1}, \ldots, \xi_\mu, \xi_{\mu+1}, \ldots, \xi_n)$ such that:

$$\begin{aligned} \mathcal{X}_2(p) &= \mathcal{S}^*(p) \cap \ker \mathrm{d}h(p) \\ &= \operatorname{span}\left\{ \left.\frac{\partial}{\partial \xi_{\nu+1}}\right|_p, \ldots, \left.\frac{\partial}{\partial \xi_\mu}\right|_p \right\}; \\ \mathcal{X}_2(p) \oplus \mathcal{X}_3(p) &= \mathcal{S}^*(p) \\ &= \operatorname{span}\left\{ \left.\frac{\partial}{\partial \xi_{\nu+1}}\right|_p, \ldots, \left.\frac{\partial}{\partial \xi_n}\right|_p \right\}. \end{aligned}$$

By definition of $\mathcal{X}_2$ and $\mathcal{X}_3$

$$\left.\frac{\partial h}{\partial \xi_j}\right|_p = \mathrm{d}h|_p \left.\frac{\partial}{\partial \xi_j}\right|_p \neq 0, \qquad j = \mu+1, \ldots, n.$$

Suppose now there exist constants $\alpha_{\mu+1}, \ldots, \alpha_n$ such that

$$\begin{aligned} &\alpha_{\mu+1} \left.\frac{\partial h}{\partial \xi_{\mu+1}}\right|_p + \cdots + \alpha_n \left.\frac{\partial h}{\partial \xi_n}\right|_p \\ &= \mathrm{d}h|_p \left(\alpha_{\mu+1} \left.\frac{\partial}{\partial \xi_{\mu+1}}\right|_p + \cdots + \alpha_n \left.\frac{\partial}{\partial \xi_n}\right|_p \right) = 0 \end{aligned}$$

then

$$\alpha_{\mu+1} \left.\frac{\partial}{\partial \xi_{\mu+1}}\right|_p + \cdots + \alpha_n \left.\frac{\partial}{\partial \xi_n}\right|_p \in \ker \mathrm{d}h(p)$$

which in turn implies that there exist $\alpha_1, \ldots, \alpha_\mu$ such that

$$\alpha_{\mu+1} \left.\frac{\partial}{\partial \xi_{\mu+1}}\right|_p + \cdots + \alpha_n \left.\frac{\partial}{\partial \xi_n}\right|_p = \alpha_1 \left.\frac{\partial}{\partial \xi_1}\right|_p + \cdots + \alpha_\mu \left.\frac{\partial}{\partial \xi_\mu}\right|_p,$$

and because the vectors $\{\frac{\partial}{\partial \xi_1}|_p, \ldots, \frac{\partial}{\partial \xi_n}|_p\}$ are independent it follows that all α_i are equal to zero. Hence $\frac{\partial h}{\partial x_3}|_p$ has full column rank, and by the Inverse function theorem this implies that h from x_3 to z is locally injective. ■

5.2.2 State transformation for general nonlinear systems

In this subsection we will use the results from Subsection 5.2.1 to define a state transformation for the original system Σ.

In order to do so we define similar to the linear case, an extended system Σ_e by adding the inputs u and the disturbances d to the state:

$$\Sigma_e \begin{cases} \dot{x} &= f(x, u, d) \\ \dot{u} &= v \\ \dot{d} &= w \\ z &= h(x, u) \end{cases} \tag{5.21}$$

The extended system is affine in the new inputs v and the new disturbances w. We rewrite the system Σ_e with state $x_e^T = (\ x^T \quad u^T \quad d^T\ )$ as

$$\Sigma_e \begin{cases} \dot{x}_e &= f_e(x_e) + \sum_{j=1}^m g_{je}(x_e)v_j + \sum_{i=1}^q e_{ie}(x_e)w_i \\ z &= h_e(x_e) \end{cases} \tag{5.22}$$

where the vector fields are given by:

$$f_e(x_e) = \begin{pmatrix} f(x, u, d) \\ 0 \\ 0 \end{pmatrix}, \qquad g_{je}(x_e) = Id_{n+j},$$
$$e_{ie}(x_e) = Id_{n+m+i}, \qquad h_e(x_e) = h(x, u)$$

with Id_r the r-th identity vector, i.e., the $(n + m + q)$-vector with all elements zero except an element equal to one on the r-th position.

The system Σ_e is clearly of the form (5.14). So we can apply the state transformation based on the minimal conditioned invariant distribution containing the input vector fields as described in Subsection 5.2.1. The system Σ_e has a special structure which results in some nice properties for this state transformation. We start by making the following assumption

Assumption 5 The $(p \times m)$-matrix

$$\frac{\partial h}{\partial u}(x, u)$$

has constant column rank equal to $m - m_1$ at least locally around the origin. □

First we construct the minimal conditioned invariant distribution containing the input vector fields, denoted by $\mathcal{S}_e^*$, by applying the $\mathcal{S}^*$-algorithm described in Subsection 5.2.1. The distributions defined in the previous subsection for the system Σ applied to the system Σ_e will be marked by an extra subscript e, i.e., $\mathcal{X}_{1e}$ etc..

Lemma 5.15 *Consider the system Σ_e. Assume Assumption 5 is satisfied and $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker \mathrm{d}h$ are constant dimensional. Then the state transformation defined in Subsection 5.2.1 has the following properties:*

(i)

$$\mathcal{G}_e := \operatorname{span}\left\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\right\} \subset \mathcal{S}_e^* = \mathcal{X}_{2e} \oplus \mathcal{X}_{3e};$$

(ii) $\mathcal{X}_{1e}$ can be chosen such that

$$\operatorname{span}\left\{\frac{\partial}{\partial d_1}, \ldots, \frac{\partial}{\partial d_q}\right\} \subset \mathcal{X}_{1e};$$

(iii) a state dependent input transformation of the form $u = \beta(x, v)$ can be found such that in the new coordinates $v = (v_1, \ldots, v_{m_1}, v_{m_1+1}, \ldots, v_m)$ the distributions $\mathcal{X}_{2e}$ and $\mathcal{X}_{3e}$ can be chosen as follows:

$$\begin{aligned} \operatorname{span}\left\{\frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_{m_1}}\right\} &\subset \mathcal{S}_e^* \cap \ker \mathrm{d}h = \mathcal{X}_{2e}; \\ \operatorname{span}\left\{\frac{\partial}{\partial v_{m_1+1}}, \ldots, \frac{\partial}{\partial v_m}\right\} &\subset \mathcal{X}_{3e}. \end{aligned}$$

□

Proof Using the $\mathcal{S}^*$-algorithm for Σ_e we see that

$$\mathcal{S}_{1e} = \operatorname{span}\left\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\right\} = \overline{\mathcal{S}}_{1e}$$

and because $\mathcal{S}^* = \cup_{l \geq 1} \overline{\mathcal{S}}_{le}$ property *(i)* is proved. Because of Assumption 5 we can find a state dependent input transformation $u = \beta(x, v)$ in such a way that for the first m_1 ($\leq m$) components of the new $v = (v_1, \ldots, v_m)$ the following holds

$$\frac{\partial}{\partial v_j} \in \overline{\mathcal{S}}_{1e} \cap \ker \mathrm{d}h \;\Leftrightarrow\; \frac{\partial}{\partial v_j} \in \ker \mathrm{d}h \;\Leftrightarrow\; \frac{\partial h_l}{\partial v_j} \equiv 0, \qquad l = 1, \ldots, p. \tag{5.23}$$

Hence *(iii)* is proved.
Furthermore

$$\begin{aligned}
\mathcal{S}_{2e} &= \overline{\mathcal{S}}_{1e} + \left[f(x,u,d)\frac{\partial}{\partial x}, \overline{\mathcal{S}}_{1e} \cap \ker \mathrm{d}h \right] + \sum_{j=1}^{m} \left[\frac{\partial}{\partial u_j}, \overline{\mathcal{S}}_{1e} \cap \ker \mathrm{d}h \right] \\
&= \overline{\mathcal{S}}_{1e} + \left[f(x,u,d)\frac{\partial}{\partial x}, \overline{\mathcal{S}}_{1e} \cap \ker \mathrm{d}h \right] \\
&= \operatorname{span}\left\{ \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_m} \right\} + \sum_{j=1}^{m_1} \left[f(x,\beta(x,v),d)\frac{\partial}{\partial x}, \frac{\partial}{\partial v_j} \right].
\end{aligned}$$

It can be easily seen that the Lie-bracket

$$\left[k(x,v,d)\frac{\partial}{\partial x}, \frac{\partial}{\partial v_j} \right]$$

for an arbitrary function k is always of the form $\tilde{k}(x,v,d)\frac{\partial}{\partial x}$. Therefore the distributions $\mathcal{S}_{2e}$ and $\overline{\mathcal{S}}_{2e}$ only contains the vector fields $\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}$ and vector fields of the form

$$\tilde{k}(x,u,d)\frac{\partial}{\partial x}.$$

The same can be proved for $\mathcal{S}_{3e}$, $\overline{\mathcal{S}}_{3e}$, etc..
So $\mathcal{X}_{1e}$ can be chosen such that *(ii)* holds. ∎

We already assumed that the distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker \mathrm{d}h$ are constant dimensional. Furthermore we want to be able to project these distributions on the state space M of the original system Σ, i.e., a distribution $\mathcal{D}$ on $M \times \mathbb{R}^m \times \mathbb{R}^q$ is called *projectable* on M by the canonical projection $\pi : M \times \mathbb{R}^m \times \mathbb{R}^q \to M$ given by $\pi : x_e \mapsto x$ ($x_e := (x,u,d)$) if for all $\kappa_1, \kappa_2 \in M \times \mathbb{R}^m \times \mathbb{R}^q$ we have

$$\pi(\kappa_1) = \pi(\kappa_2) \Rightarrow \frac{\partial \pi}{\partial x_e}(\kappa_1)\,(\mathcal{D}(\kappa_1)) = \frac{\partial \pi}{\partial x_e}(\kappa_2)\,(\mathcal{D}(\kappa_2)).$$

The next lemma gives conditions for the projectability of these two distributions.

Lemma 5.16 *Assume that the distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker \mathrm{d}h$ are constant dimensional. The distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker \mathrm{d}h$ are projectable on M (by the canonical projection $\pi : M \times \mathbb{R}^m \times \mathbb{R}^q \to M$ given by $\pi(x,u,d) = x$) if* *and only if the following inclusions hold:*

$$\left[\frac{\partial}{\partial d_j}, \mathcal{S}_e^*\right] \subset \mathcal{S}_e^*, \qquad j = 1, \ldots, q; \qquad (5.24)$$

$$\left[\frac{\partial}{\partial v_i}, \mathcal{S}_e^* \cap \ker \mathrm{d}h\right] \subset \mathcal{S}_e^* \cap \ker \mathrm{d}h + \mathcal{G}_e, \qquad i = m_1 + 1, \ldots, m.(5.25)$$

□

Proof From [MRS 94] it follows that we have to prove that:

$$\left[\frac{\partial}{\partial u_j}, Q\right] \subset Q + \mathcal{G}_e + \mathcal{D}_e, \qquad j = 1, \ldots, m; \qquad (5.26)$$

$$\left[\frac{\partial}{\partial d_i}, Q\right] \subset Q + \mathcal{G}_e + \mathcal{D}_e, \qquad i = 1, \ldots, q; \qquad (5.27)$$

holds for $Q = \mathcal{S}_e^*$ and $Q = \mathcal{S}_e^* \cap \ker \mathrm{d}h$, where the distribution $\mathcal{D}_e$ is defined as

$$\mathcal{D}_e = \operatorname{span}\left\{\frac{\partial}{\partial d_1}, \ldots, \frac{\partial}{\partial d_q}\right\}.$$

Since $\mathcal{S}_e^*$ is involutive and $\mathcal{G}_e \subset \mathcal{S}_e^*$, (5.26) is satisfied for $Q = \mathcal{S}_e^*$. Furthermore from the proof of Lemma 5.15 it follows that $\mathcal{S}_e^*$ is spanned by the vector fields $\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}$ and by vector fields of the form $\tilde{k}(x, u, d)\frac{\partial}{\partial x}$ for arbitrary functions $\tilde{k}$. Then it follows that

$$\left[\frac{\partial}{\partial d_i}, X\right] \notin \mathcal{D}_e$$

for all vector fields $X \in \mathcal{S}_e^*$. Hence (5.24) holds if and only if (5.27) is satisfied for $Q = \mathcal{S}_e^*$.

The vector fields $\frac{\partial}{\partial v_i}$ for $i = 1, \ldots, m_1$ are elements of $\mathcal{S}_e^* \cap \ker \mathrm{d}h$. The involutivity of $\mathcal{S}_e^* \cap \ker \mathrm{d}h$ together with (5.25) implies that (5.26) holds for $Q = \mathcal{S}_e^* \cap \ker \mathrm{d}h$.
By (5.24):

$$\left[\frac{\partial}{\partial d_j}, \mathcal{S}_e^* \cap \ker \mathrm{d}h\right] \subset \mathcal{S}_e^*, \qquad j = 1, \ldots, q. \qquad (5.28)$$

Take an arbitrary vector field X from $\mathcal{S}_e^* \cap \ker \mathrm{d}h$. Then

$$\sum_{i=1}^{n} \frac{\partial h_l}{\partial x_i} X_i + \sum_{r=1}^{m} \frac{\partial h_l}{\partial u_r} X_{n+r} = 0, \qquad l = 1, \ldots, p$$

and it follows that

$$\begin{aligned}
\mathrm{d}h_l\left[\frac{\partial}{\partial d_j}, X\right] &= \mathrm{d}h_l\left(\sum_{i=1}^{n}\frac{\partial X_i}{\partial d_j}\frac{\partial}{\partial x_i}+\sum_{r=1}^{m}\frac{\partial X_{r+n}}{\partial d_j}\frac{\partial}{\partial u_r}\right)\\
&= \sum_{i=1}^{n}\frac{\partial h_l}{\partial x_i}\frac{\partial X_i}{\partial d_j}+\sum_{r=1}^{m}\frac{\partial h_l}{\partial u_r}\frac{\partial X_{r+n}}{\partial d_j}\\
&= \frac{\partial}{\partial d_j}\left(\sum_{i=1}^{n}\frac{\partial h_l}{\partial x_i}X_i+\sum_{r=1}^{m}\frac{\partial h_l}{\partial u_r}X_{r+n}\right)\\
&= 0 \qquad\qquad l=1,\ldots,p
\end{aligned}$$

or in coordinate free notation

$$\mathrm{d}h_l\left[\frac{\partial}{\partial d_j}, X\right]=\frac{\partial}{\partial d_j}(X(h_l))-X\left(\frac{\partial}{\partial d_j}(h_l)\right)=0 \qquad l=1,\ldots,p$$

because $X\in\ker \mathrm{d}h_l$ and h_l does not depend on d. Hence

$$\left[\frac{\partial}{\partial d_j}, \mathcal{S}_e^*\cap\ker\mathrm{d}h\right]\subset\ker\mathrm{d}h$$

and together with (5.28) this proves (5.27) for $\mathcal{S}_e^*\cap\ker\mathrm{d}h$. ■

Assume that the distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^*\cap\ker\mathrm{d}h$ are constant dimensional and that the conditions (5.24) and (5.25) from Lemma 5.16 are satisfied. Then the distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^*\cap\ker\mathrm{d}h$ are projectable on the state manifold M by the canonical projection $\pi: M\times\mathbb{R}^m\times\mathbb{R}^q\to M$ onto involutive distributions on M. Therefore we can find local coordinates written as $(\xi_1,\ldots,\xi_\nu,\xi_{\nu+1},\ldots,\xi_\mu,\xi_{\mu+1},\ldots,\xi_n)$ for M such that:

$$\begin{aligned}
\pi_*\left(\mathcal{S}_e^*\cap\ker\mathrm{d}h\right) &= \operatorname{span}\left\{\frac{\partial}{\partial\xi_{\nu+1}},\ldots,\frac{\partial}{\partial\xi_\mu}\right\};\\
\pi_*\mathcal{S}_e^* &= \operatorname{span}\left\{\frac{\partial}{\partial\xi_{\nu+1}},\ldots,\frac{\partial}{\partial\xi_n}\right\}.
\end{aligned}$$

Furthermore by condition (5.24) there exists a basis of $\mathcal{S}_e^*$ which does not depend on the disturbances d. Then we transform the inputs u by a state dependent transformation $u=\beta(x,v)$ such that for the first m_1 components of the new coordinates $v=(v_1,\ldots,v_{m_1},v_{m_1+1},\ldots,v_m)$ we have that

$$\operatorname{span}\left\{\frac{\partial}{\partial v_1},\ldots,\frac{\partial}{\partial v_{m_1}}\right\}=\mathcal{S}_e^*\cap\ker\mathrm{d}h\cap\mathcal{G}_e.$$

Then the distributions $\mathcal{X}_{1e}$, $\mathcal{X}_{2e}$ and $\mathcal{X}_{3e}$ can be chosen in the following way:

$$\begin{aligned}
\mathcal{X}_{1e} &= \text{span}\left\{\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_\nu}, \frac{\partial}{\partial d_1}, \ldots, \frac{\partial}{\partial d_q}\right\}; \\
\mathcal{X}_{2e} &= \text{span}\left\{\frac{\partial}{\partial \xi_{\nu+1}}, \ldots, \frac{\partial}{\partial \xi_\mu}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_{m_1}}\right\}; \\
\mathcal{X}_{3e} &= \text{span}\left\{\frac{\partial}{\partial \xi_{\mu+1}}, \ldots, \frac{\partial}{\partial \xi_n}, \frac{\partial}{\partial v_{m_1+1}}, \ldots, \frac{\partial}{\partial v_m}\right\}.
\end{aligned}$$

In these new coordinates the vector field $f_e(x_e)$ and the function $h_e(x_e)$ have the following special structure

$$f_e(x_e) = \begin{pmatrix} f_1(x_1, x_3, u_2, d) \\ f_2(x_1, x_2, x_3, u_1, u_2, d) \\ f_3(x_1, x_2, x_3, u_1, u_2, d) \\ \hline 0 \\ \hline 0 \end{pmatrix}, \qquad h_e(x_e) = h(x_1, x_3, u_2, d).$$

The projections of the distributions $\mathcal{X}_{1e}$, $\mathcal{X}_{2e}$ and $\mathcal{X}_{3e}$ on the state space M will be indicated by $\mathcal{X}_1^P$, $\mathcal{X}_2^P$ and $\mathcal{X}_3^P$. In these new coordinates for the state space $M = \mathcal{X}_1^P \oplus \mathcal{X}_2^P \oplus \mathcal{X}_3^P$ and the input space $\mathbb{R}^m$ the original system Σ takes the following form

$$\Sigma \begin{cases} \dot{x}_1 &= f_1(x_1, x_3, u_2, d) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, u_1, u_2, d) \\ \dot{x}_3 &= f_3(x_1, x_2, x_3, u_1, u_2, d) \\ z &= h(x_1, x_3, u_2) \end{cases} \tag{5.29}$$

where the derivative of h with respect to (x_3, u_2) has full column rank (see Lemma 5.14).

Again the system Σ can be viewed as the interconnection of two general nonlinear systems Σ_1 and Σ_2. The first subsystem Σ_1 with state x_1, inputs (x_3, u_2, d) and output (z, x_1) and another system Σ_2 with state (x_2, x_3), inputs (x_1, u_1, u_2, d) and output x_3 (see Figure 5.4).

$$\Sigma_1 \quad \begin{cases} \dot{x}_1 &= f_1(x_1, x_3, u_2, d) \\ z &= h(x_1, x_3, u_2) \end{cases}$$

$$\Sigma_2 \quad \begin{cases} \dot{x}_2 &= f_2(x_1, x_2, x_3, u_1, u_2, d) \\ \dot{x}_3 &= f_3(x_1, x_2, x_3, u_1, u_2, d) \end{cases}$$

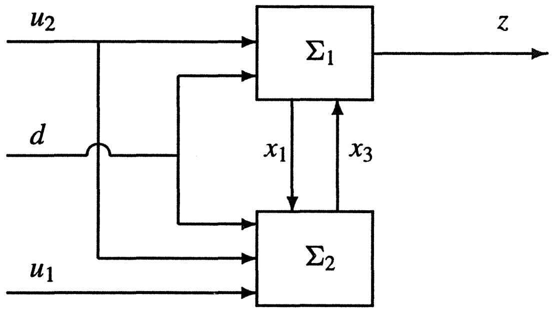

Figure 5.4: system Σ after input and state transformation

5.2.3 Special cases

This subsection will be concerned with the two extreme cases of the general $\mathcal{H}_\infty$ problem for general nonlinear systems considered in this section.

Regular $\mathcal{H}_\infty$ problem

First of all we will look at the regular $\mathcal{H}_\infty$ problem. So we will consider systems Σ which satisfy Assumption 1. These systems clearly satisfy Assumption 5, with m_1 equal to zero. Therefore there is no need to transform the inputs.

The extended system Σ_e in this case satisfies

$$\mathcal{G}_e \cap \ker \mathrm{d}h = 0.$$

Hence the $\mathcal{S}^*$-algorithm applied to Σ_e stops after one step and locally around the origin we have that:

$$\begin{aligned} \mathcal{S}_e^* &= \mathcal{G}_e = \operatorname{span}\left\{\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_m}\right\}; \\ \mathcal{S}_e^* \cap \ker \mathrm{d}h &= 0. \end{aligned}$$

These distributions are clearly projectable. There is no need for a transformation of the inputs and the conditions (5.24) and (5.25) for the projectability are satisfied:

$$\begin{aligned} \left[\frac{\partial}{\partial d_j}, \mathcal{G}_e\right] &= 0 \in \mathcal{G}_e, \qquad j = 1, \dots, q; \\ \left[\frac{\partial}{\partial u_i}, 0\right] &= 0 \in \mathcal{G}_e, \qquad i = 1, \dots, m. \end{aligned}$$

The distributions $\mathcal{X}_{1e}$, $\mathcal{X}_{2e}$ and $\mathcal{X}_{3e}$ for the system Σ_e are given by:

$$\begin{aligned}
\mathcal{X}_{1e} &= \operatorname{span}\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial d_1},\dots,\frac{\partial}{\partial d_q}\right\};\\
\mathcal{X}_{2e} &= 0;\\
\mathcal{X}_{3e} &= \operatorname{span}\left\{\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_m}\right\}.
\end{aligned}$$

So the projected spaces are $\mathcal{X}_1^P = M$, $\mathcal{X}_2^P = 0$ and $\mathcal{X}_3^P = 0$, and the regular subsystem Σ_1 is equal to the complete system Σ.

Full singular $\mathcal{H}_\infty$ problem

In the full singular case we assume that the output mapping h does not depend on the inputs u. So $h(x,u) = h(x)$ and automatically Assumption 5 is satisfied with $m_1 = m$. In this special case we can see that for the extended system Σ_e it follows that

$$\mathcal{G}_e \subset \mathcal{S}_e^* \cap \ker \mathrm{d}h$$

and the projectability condition (5.25) is void. Again no transformation of the input is necessary.

The remaining assumptions are that the distributions $\mathcal{S}_e$ and $\mathcal{S}_e \cap \ker \mathrm{d}h$ are constant dimensional and that condition (5.24) is satisfied, which implies projectability of the distributions onto the state space of the original system Σ. Then we can find local coordinates $(\xi_1,\dots,\xi_\nu,\xi_{\nu+1},\dots,\xi_\mu,\xi_{\mu+1},\dots,\xi_n)$ such that the distributions $\mathcal{X}_{1e}$, $\mathcal{X}_{2e}$ and $\mathcal{X}_{3e}$ for the extended system Σ_e are given by:

$$\begin{aligned}
\mathcal{X}_{1e} &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_1},\dots,\frac{\partial}{\partial \xi_\nu},\frac{\partial}{\partial d_1},\dots,\frac{\partial}{\partial d_q}\right\};\\
\mathcal{X}_{2e} &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_{\nu+1}},\dots,\frac{\partial}{\partial \xi_\mu},\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_m}\right\};\\
\mathcal{X}_{3e} &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_{\mu+1}},\dots,\frac{\partial}{\partial \xi_n}\right\}.
\end{aligned}$$

The system Σ after the state transformation is of the form

$$\Sigma\left\{\begin{aligned}
\dot{x}_1 &= f_1(x_1,x_3,d)\\
\dot{x}_2 &= f_2(x_1,x_2,x_3,u,d)\\
\dot{x}_3 &= f_3(x_1,x_2,x_3,u,d)\\
z &= h(x_1,x_3)
\end{aligned}\right.$$

In this case there is no direct influence of the inputs on the regular subsystem Σ_1.

5.2.4 The nonlinear state feedback $\mathcal{H}_\infty$ problem: sufficient conditions

Using the transformations described in the previous subsections the nonlinear system Σ can be seen as the interconnection of two subsystems Σ_1 and Σ_2. Inspired by the linear theory described in the beginning of this chapter we will split the singular $\mathcal{H}_\infty$ problem for the system Σ into two problems. The first part is a $\mathcal{H}_\infty$ problem for the subsystem Σ_1 with inputs (x_3, u_2), outputs z and disturbances d. Because the Jacobian of the mapping h from the inputs (x_3, u_2) to the outputs z has full column rank this problem is regular. So for this problem we can use the regular $\mathcal{H}_\infty$ theory described in Chapter 2. The solution of this regular state feedback $\mathcal{H}_\infty$ problem consists of a trajectory of part of the state of the subsystem Σ_2, namely x_3, and an optimal state feedback for the inputs u_2. In the second step we take the inputs u_2 equal to the feedback solution of the regular subproblem and try to track the state components x_3 along the solution of the regular problem by applying a suitable feedback for u_1 to the subsystem Σ_2.

From Chapter 2 we know that the regular strictly suboptimal $\mathcal{H}_\infty$ problem for the subsystem Σ_1 with inputs (x_3, u_2), disturbances d and outputs z is solvable if there exists for some $\overline{\gamma} < \gamma$ a local solution $V \geq 0$ to the Hamilton-Jacobi inequality

$$H_{\overline{\gamma}}(x_1, V_{x_1}^T(x_1)) \leq 0, \qquad V(0) = 0$$

where the Hamiltonian $H_{\overline{\gamma}} : T^*\mathcal{X}_1^P \to \mathbb{R}$ is given by

$$H_{\overline{\gamma}}(x_1, p_1) = K_{\overline{\gamma}}(x_1, p_1, d^*(x_1, p_1), x_3^*(x_1, p_1), u_2^*(x_1, p_1))$$

with $K_\gamma : T^*\mathcal{X}_1^P \times \mathbb{R}^q \times \mathcal{X}_3^P \times \mathbb{R}^{m-m_1} \to \mathbb{R}$ the pre-Hamiltonian corresponding to this problem, that is

$$K_{\overline{\gamma}}(x_1, p_1, d, x_3, u_2) = p_1^T f_1(x_1, x_3, u_2, d) + \frac{1}{2}\|h(x_1, x_3, u_2)\|^2 - \frac{1}{2}\overline{\gamma}^2\|d\|^2$$

and $x_3^*(x_1, p_1)$, $u_2^*(x_1, p_1)$ and $d^*(x_1, p_1)$ is the unique solution of

$$\begin{aligned}
\frac{\partial K_{\overline{\gamma}}}{\partial d}(x_1, p_1, d^*(x_1, p_1), x_3^*(x_1, p_1), u_2^*(x_1, p_1)) &= 0 \\
\frac{\partial K_{\overline{\gamma}}}{\partial x_3}(x_1, p_1, d^*(x_1, p_1), x_3^*(x_1, p_1), u_2^*(x_1, p_1)) &= 0 \\
\frac{\partial K_{\overline{\gamma}}}{\partial u_2}(x_1, p_1, d^*(x_1, p_1), x_3^*(x_1, p_1), u_2^*(x_1, p_1)) &= 0
\end{aligned}$$

with $x_3^*(0,0)=0$, $u_2^*(0,0)=0$ and $d^*(0,0)=0$.

Then the feedback

$$\begin{aligned} x_3 &= x_3^*(x_1, V_{x_1}^T(x_1)), \\ u_2 &= u_2^*(x_1, V_{x_1}^T(x_1)) \end{aligned}$$

applied to the subsystem Σ_1 leads to a closed loop system which has L_2-gain less than or equal to $\overline{\gamma} < \gamma$.

In the second step we choose the inputs u_2 equal to the optimal one

$$u_2 = u_2^*(x_1, V_{x_1}^T(x_1))$$

and we try to track the optimal state trajectory calculated in the first step for the state components x_3. In order to do so we introduce a new state variable for Σ_2 equal to the tracking error

$$q_3 := x_3 - x_3^*(x_1, V_{x_1}^T(x_1)). \tag{5.30}$$

This state transformation leads to the following form of the system Σ, which we will refer to as Σ^{tr}:

$$\Sigma^{\text{tr}} \begin{cases} \dot{x}_1 &= f_1(x_1, q_3 + x_3^*, u_2^*, d) \\ \dot{x}_2 &= f_2(x_1, x_2, q_3 + x_3^*, u_1, u_2^* d) \\ \dot{q}_3 &= \tilde{f}_3(x_1, x_2, q_3 + x_3^*, u_1, u_2^*, d) \\ z &= h(x_1, q_3 + x_3^*, u_2^*) \end{cases}$$

where

$$\begin{aligned} x_3^* &= x_3^*(x_1, V_{x_1}^T(x_1)); \\ u_2^* &= u_2^*(x_1, V_{x_1}^T(x_1)); \\ \tilde{f}_3(x_1, x_2, x_3, u_1, u_2, d) &= f_3(x_1, x_2, x_3, u_1, u_2, d) \\ &\quad -\frac{\mathrm{d}x_3^*}{\mathrm{d}x_1}(x_1, V_{x_1}^T(x_1)) f_1(x_1, x_3, u_2, d). \end{aligned}$$

The transformed system Σ^{tr} is still the interconnection of two subsystems namely (see Figure 5.5):

$$\Sigma_1^{\text{tr}} \begin{cases} \dot{x}_1 &= f_1(x_1, q_3 + x_3^*, u_2^*, d) \\ z &= h(x_1, q_3 + x_3^*, u_2^*) \end{cases}$$

$$\Sigma_2^{\text{tr}} \begin{cases} \dot{x}_2 &= f_2(x_1, x_2, q_3 + x_3^*, u_1, u_2^* d) \\ \dot{q}_3 &= \tilde{f}_3(x_1, x_2, q_3 + x_3^*, u_1, u_2^*, d) \end{cases}$$

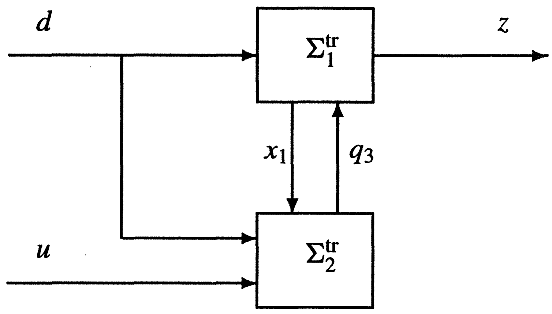

Figure 5.5: system Σ^{tr}

It should be noted that $q_3 = 0$ is the a solution of the regular $\mathcal{H}_\infty$ state feedback problem for the subsystem Σ_1, with inputs q_3.

Parallel to the linear theory we want to solve the $\mathcal{H}_\infty$ almost disturbance decoupling problem for the transformed subsystem Σ_2^{tr} with disturbances (x_1, d), inputs u_1 and outputs q_3. Because we use x_1 as a disturbance input for the system Σ_2^{tr} we have to make an extra assumption on the input-to-state stability of the system Σ_1^{tr}.

Assumption 6 Assume that the L_2-gain from q_3 and d to x_1 for the system Σ_1^{tr} is finite, or otherwise stated, there exist constants $M_1, M_2 > 0$ and for every x_1 there exists a constant $M(x_1)$, $0 \leq M(x_1) < \infty$, $M(0) = 0$, such that

$$\int_0^t \|x_1(\tau)\|^2 \mathrm{d}\tau \leq M_1 \int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau + M_2 \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + M(x_1(0)) \tag{5.31}$$

for all $d, q_3 \in L_2(0, t)$ and for all $t \geq 0$. □

Also other assumptions can be made to ensure input-to-state stability (see for instance [SW 94] for necessary and sufficient conditions for input-to-state stability).

Theorem 5.17 *Consider the system Σ. Suppose that for the extended system Σ_e the distributions S_e^* and $S_e^* \cap \ker \mathrm{d}h$ are constant dimensional, that Assumption 5 is satisfied and that (5.24) and (5.25) hold. Assume there exists for some $\overline{\gamma} < \gamma$ a solution $V \geq 0$ of*

$$H_{\overline{\gamma}}(x_1, V_{x_1}^T(x_1)) \leq 0 \tag{5.32}$$

with $V(0) = 0$. Additionally assume that Assumption 6 holds.
Then there exists a constant k^* such that the nonlinear state feedback L_2-gain control problem with constant $\overline{\gamma} < \gamma$ is solvable locally for Σ by the feedback

$$\begin{aligned} u_1 &= \tilde{\beta}(x_1, x_2, x_3, k) = \beta(x_2, x_3 - x_3^*(x_1, V_{x_1}^T(x_1)), k) \\ u_2 &= u_2^*(x_1, V_{x_1}^T(x_1)) \end{aligned}$$

for $k > k^*$ if the parameterized feedback

$$u_1 = \beta(x_2, q_3, k)$$

solves the L_2-gain almost disturbance decoupling problem with constant $\frac{1}{k}$ for Σ_2^{tr}. □

Before we prove this theorem we will state the following observation.

Lemma 5.18 *Consider the system Σ. Suppose that for the extended system Σ_e the distributions S_e^* and $S_e^* \cap \ker \mathrm{d}h$ are constant dimensional, that Assumption 5 is satisfied and that (5.24) and (5.25) hold. Assume there exists for some $\overline{\gamma} < \gamma$ a solution $V \geq 0$ of*

$$H_{\overline{\gamma}}(x_1, V_{x_1}^T(x_1)) \leq 0 \tag{5.33}$$

with $V(0) = 0$. Then locally around the origin there exists constants $N, \varepsilon > 0$ and a constant $R(x_1)$, $0 \leq R(x_1) < \infty$, with $R(0) = 0$ such that

$$\int_0^t \|z(\tau)\|^2 \mathrm{d}\tau \leq N \int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau + (\gamma^2 - \varepsilon) \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + R(x_1(0)) \tag{5.34}$$

for all $d, q_3 \in L_2(0, t)$ and $x_1(0)$ such that the resulting state trajectories stay in a neighborhood of the origin. □

Proof Consider the pre-Hamiltonian

$$\begin{aligned} L(x_1, p, d, q_3) := {} & p^T f_1(x_1, q_3 + x_3^*, u_2^*, d) \\ & + \frac{1}{2} h^T(x_1, q_3 + x_3^*, u_2^*) h(x_1, q_3 + x_3^*, u_2^*) - \frac{1}{2} N \|q_3\|^2 - \frac{1}{2} \overline{\gamma}^2 \|d\|^2. \end{aligned}$$

The Hessian of L with respect to d and q_3 in $(x_1, p, d, q_3) = (0, 0, 0, 0)$ is given by

$$\begin{pmatrix} -\overline{\gamma}^2 I & 0 \\ 0 & -NI + \left(\frac{\partial h}{\partial x_3}(0,0,0)\right)^T \left(\frac{\partial h}{\partial x_3}(0,0,0)\right) \end{pmatrix},$$

where I is the identity matrix with dimension equal to the dimension of q_3. For N sufficiently large, i.e., larger than the largest eigenvalue of the matrix

$$\left(\frac{\partial h}{\partial x_3}(0,0,0)\right)^T \left(\frac{\partial h}{\partial x_3}(0,0,0)\right),$$

this matrix is negative definite. Since

$$L(x_1, p, d, q_3) = K_{\overline{\gamma}}(x_1, p, d, q_3 + x_3^*(x_1, p), u_2^*(x_1, p)) - \frac{1}{2}N\|q_3\|^2$$

it follows that the following equalities have a unique solution given by $q_3 = 0$, $d = d^*(x_1, p)$

$$\begin{aligned} \frac{\partial L}{\partial q_3} &= \frac{K_{\overline{\gamma}}}{\partial x_3} - N(x_3 - x_3^*(x_1, p))^T = 0 \\ \frac{\partial L}{\partial d} &= \frac{K_{\overline{\gamma}}}{\partial d} = 0 \end{aligned}$$

because $\frac{\partial K_{\overline{\gamma}}}{\partial x_3} = 0$ and $\frac{\partial K_{\overline{\gamma}}}{\partial d} = 0$ has a unique solution equal to $x_3 = x_3^*(x_1, p)$ $(q_3 = 0)$ and $d = d^*(x_1, p)$ when $u_2 = u_2^*(x_1, p)$. Thus $q_3 = 0$, $d = d^*(x_1, p)$ is the maximum of the pre-Hamiltonian L for N sufficiently large and

$$L(x_1, p, d, q_3) \leq K_{\overline{\gamma}}(x_1, p, d^*(x_1, p), x_3^*(x_1, p), u_2^*(x_1, p)).$$

Hence $V \geq 0$ is a non-negative solution of the Hamilton-Jacobi inequality

$$L(x_1, V_{x_1}^T(x_1), d, q_3) \leq 0.$$

This proves the inequality (5.34) for:

$$\begin{aligned} N &> \overline{\sigma}\left(\left(\frac{\partial h}{\partial x_3}(0,0,0)\right)^T \left(\frac{\partial h}{\partial x_3}(0,0,0)\right)\right); \\ \varepsilon &< \gamma^2 - \overline{\gamma}^2; \\ R(x_1) &\geq 2V(x_1); \end{aligned}$$

where $\overline{\sigma}(P)$ denotes the largest eigenvalue of the symmetric matrix P. ■

Proof of Theorem 5.17 The distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker dh$ for the extended system Σ_e are constant dimensional and involutive, Assumption 5 is satisfied and the conditions (5.24) and (5.25) hold. Therefore we can apply the state transformation described in Subsection 5.2.2. The solution of the Hamilton-Jacobi inequality (5.32) is used to transform the state x_3 according to (5.30) which results in the interconnected systems Σ_1^{tr} and Σ_2^{tr}. Now we take $u_2 = u_2^*(x_1, V_{x_1}^T(x_1))$. We assumed that the feedback $u_1 = \beta(x_2, q_3, k)$ solves the $\mathcal{H}_\infty$ almost disturbance decoupling problem (see Definition 4.2) for Σ_2^{tr} with output q_3 and disturbances x_1 and d. Hence there exists a constant $K(x_2, q_3)$, $0 \leq K(x_2, q_3) < \infty$, $K(0,0) = 0$, such that

$$\int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau \leq \frac{1}{k}\int_0^t \left(\|d(\tau)\|^2 + \|x_1(\tau)\|^2\right) \mathrm{d}\tau + K(x_2(0), q_3(0)) \quad (5.35)$$

for all $d, x_1 \in L_2(0,t)$ and all t. Let z_{q_3} denote the output of the subsystem Σ_1^{tr} resulting from disturbance d, initial state $x_1(0)$ and input q_3. Then from Assumption 6 and Lemma 5.18 it follows that there exist constants $N, \varepsilon, M_1, M_2 > 0$,and constants $R(x_1), M(x_1), 0 \leq R(x_1), M(x_1) < \infty$, $R(0) = 0$, $M(0) = 0$, such that locally:

$$\begin{aligned}\int_0^t \|z_{q_3}(\tau)\|^2 \mathrm{d}\tau &\leq N\int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau \quad (5.36)\\ &\quad +(\gamma^2 - \varepsilon)\int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + R(x_1(0));\\ \int_0^t \|x_1(\tau)\|^2 \mathrm{d}\tau &\leq M_1\int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau \quad (5.37)\\ &\quad +M_2\int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + M(x_1(0)).\end{aligned}$$

Combining the inequalities (5.35) and (5.37), letting $k > M_1$, results in

$$\begin{aligned}\int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau &\leq \frac{1+M_2}{k-M_1}\int_0^t \|d(\tau)\|^2 \mathrm{d}\tau\\ &\quad + \frac{k}{k-M_1}\left(M(x_1(0)) + K(x_2(0), q_3(0))\right)\end{aligned}$$

which together with (5.36) leads to

$$\begin{aligned}\int_0^t \|z_{q_3}(\tau)\|^2 \mathrm{d}\tau &\leq \left(N\left(\frac{1+M_2}{k-M_1}\right) + \gamma^2 - \varepsilon\right)\int_0^t \|d(\tau)\|^2 \mathrm{d}\tau \quad (5.38)\\ &\quad + \frac{Nk}{k-M_1}\left(M(x_1(0)) + K(x_2(0), q_3(0))\right) + R(x_1(0)).\end{aligned}$$

Therefore if we choose

$$k^* \geq \frac{N}{\varepsilon}(1 + M_2) + M_1 > M_1$$

then the L_2-gain from d to z_{q_3} is less than or equal to γ. ■

Theorem 5.17 only states an L_2-gain result. For local stability of the closed-loop system we have to make extra assumptions.

Theorem 5.19 *Assume all assumptions from Theorem 5.17 are satisfied. Additionally assume that Assumption 6 is satisfied with a proper, differentiable function M. Then there exists a constant k^* such that the nonlinear state feedback L_2-gain control problem with constant $\overline{\gamma} < \gamma$ is locally solvable for Σ by the feedback*

$$\begin{aligned} u_1 &= \tilde{\beta}(x_1, x_2, x_3, k) = \beta(x_2, x_3 - x_3^*(x_1, V_{x_1}^T(x_1)), k) \\ u_2 &= u_2^*(x_1, V_{x_1}^T(x_1)) \end{aligned}$$

for $k > k^*$, and the closed-loop system is stable if the parameterized feedback

$$u_1 = \beta(x_2, q_3, k)$$

solves the L_2-gain almost disturbance decoupling problem with constant $\frac{1}{k}$ for Σ_2^{tr}, with a proper, nonnegative C^1 storage function. □

Proof We take the disturbance d equal to zero. Then combining the inequalities (5.35) and (5.37) leads to the next two inequalities for x_1 and q_3:

$$\begin{aligned} \int_0^t \|x_1(\tau)\|^2 \mathrm{d}\tau &\leq \frac{k}{k - M_1}(M_1 K(x_2(0), q_3(0)) + M(x_1(0)))\,; \quad (5.39) \\ \int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau &\leq \frac{k}{k - M_1}\left(K(x_2(0), q_3(0)) + \frac{1}{k}M(x_1(0))\right). \end{aligned}$$

Furthermore there exists a proper solution $V \geq 0$ to

$$\begin{aligned} & V(x_2(t_1), q_3(t_1)) - V(x_2(t_0), q_3(t_0)) \\ & \leq \frac{1}{2}\int_{t_0}^{t_1}\left(\frac{1}{k}\|x_1(\tau)\|^2 - \|q_3(\tau)\|^2\right)\mathrm{d}\tau \\ & \leq \frac{1}{2}\int_{t_0}^{t_1}\frac{1}{k}\|x_1(\tau)\|^2 \mathrm{d}\tau \end{aligned}$$

for all $t_0 \leq t_1$. Now we take $t_0 = 0$ and $t_1 = t$ and use the inequality (5.39) leading to the existence of a proper solution $V \geq 0$ to

$$\begin{aligned} & V(x_2(t), q_3(t)) - V(x_2(0), q_3(0)) \\ & \leq \frac{1}{2}\frac{1}{k - M_1}(M_1 K(x_2(0), q_3(0)) + M(x_1(0))) \end{aligned}$$

for all t. So for all t the function $V(x_2(t), q_3(t)) - V(x_2(0), q_3(0))$ is bounded by a function depending on the initial states. Because V is proper it follows that the x_2 and q_3 dynamics are stable. In the same way the stability of the x_1-dynamics can be proved using the properness of M. The L_2-gain result follows from Theorem 5.17. ∎

In Theorem 5.19 we only obtained a stability result and no asymptotic stability. An obvious way to ensure local asymptotic stability is provided by the following theorem.

Theorem 5.20 *Assume all assumptions from Theorem 5.17 are satisfied. Additionally assume that Assumption 6 is satisfied with a differentiable storage function $M(x_1)$. Then there exists a constant k^* such that the nonlinear state feedback $\mathcal{H}_\infty$ control problem is locally solvable with constant $\overline{\gamma} < \gamma$ for Σ by the feedback*

$$\begin{aligned} u_1 &= \tilde{\beta}(x_1, x_2, x_3, k) = \beta(x_2, x_3 - x_3^*(x_1, V_{x_1}^T(x_1)), k) \\ u_2 &= u_2^*(x_1, V_{x_1}^T(x_1)) \end{aligned} \tag{5.40}$$

for $k > k^*$ if the parameterized feedback

$$u_1 = \beta(x_2, q_3, k)$$

solves the L_2-gain almost disturbance decoupling problem with constant $\frac{1}{k}$ for Σ_2^{tr}, with a differentiable storage function, and the closed-loop system consisting of Σ together with the feedback (5.40) is zero-state observable. □

Proof The proof for the L_2-gain result follows along the lines of the proof of Theorem 5.17 where we take for the storage function $R(x_1)$ a differentiable function satisfying $R(x_1) \geq 2V(x_1)$ (see the proof of Lemma 5.18). Furthermore there exists a differentiable function $K(x_2, q_3)$ satisfying (5.35). Then the inequality (5.38) is satisfied with a differentiable storage function and the function

$$\frac{2Nk}{k - M_1}(M(x_1) + K(x_2, q_3)) + 2R(x_1)$$

is a solution of the Hamilton-Jacobi inequality for the closed loop system. Finally, Theorem 2.7 implies that the closed loop system is asymptotically stable. ∎

In the linear case the $\mathcal{H}_\infty$ almost disturbance decoupling problem for Σ_2^{tr} is known to be solvable because the state space of Σ_2^{tr} is equal to the strongly controllable subspace (see [Wi 81], [Wi 82], [Tr 86]). For nonlinear systems such a characterization of the solvability of the singular $\mathcal{H}_\infty$ problem by the solvability of the regular $\mathcal{H}_\infty$ problem for Σ_1 can not be derived.

Therefore there is no need to decompose the singular $\mathcal{H}_\infty$ problem in this way. We could also replace solving the L_2-gain almost disturbance decoupling problem for Σ_2^{tr} with disturbances x_1 and d, and outputs q_3 by solving the same problem for the complete system Σ^{tr} with as disturbances only d, and outputs q_3. In this way there is no need for an extra assumption like Assumption 6, and Theorem 5.17 can be reformulated as follows.

Theorem 5.21 *Consider the system Σ. Suppose that for the extended system Σ_e the distributions S_e^* and $S_e^* \cap \ker \mathrm{d}h$ are constant dimensional. Assume there exists for some $\overline{\gamma} < \gamma$ a solution $V \geq 0$ to the Hamilton-Jacobi inequality (5.32) with $V(0) = 0$.*
Then there exists a constant k^* such that the nonlinear state feedback L_2-gain control problem with constant $\overline{\gamma} < \gamma$ is solvable for Σ by the feedback

$$\begin{aligned} u_1 &= \tilde{\beta}(x_1, x_2, x_3, k) = \beta(x_1, x_2, x_3 - x_3^*(x_1, V_{x_1}^T(x_1)), k) \\ u_2 &= u_2^*(x_1, V_{x_1}^T(x_1)) \end{aligned}$$

for $k > k^*$ if the parameterized feedback

$$u_1 = \beta(x_1, x_2, q_3, k) \tag{5.41}$$

solves the L_2-gain almost disturbance decoupling problem with constant $\frac{1}{k}$ for Σ^{tr}. □

Proof As in the proof of Theorem 5.17 the assumptions make it possible to apply the described state transformations resulting in the interconnection of the systems Σ_1^{tr} and Σ_2^{tr}. ¿From the solvability of the almost disturbance decoupling problem we have that for the closed loop system Σ^{tr}, (5.41) there exists a constant $K(x_1, x_2, q_3)$, $0 \leq K(x_1, x_2, q_3) < \infty$, $K(0, 0, 0) = 0$ such that

$$\int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau \leq \frac{1}{k} \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + K(x_1(0), x_2(0), q_3(0)) \tag{5.42}$$

for all $d \in L_2(0, t)$ and all t. Let z_{q_3} denote the output of the subsystem Σ_1^{tr} resulting from disturbance d, initial state $x_1(0)$ and input q_3.

Then from Lemma 5.18 it follows that there exist constants $N, \varepsilon > 0$,and a constant $R(x_1)$, $0 \le R(x_1) < \infty$, $R(0) = 0$ such that locally

$$\int_0^t \|z_{q_3}(\tau)\|^2 \mathrm{d}\tau \le N \int_0^t \|q_3(\tau)\|^2 \mathrm{d}\tau + (\gamma^2 - \varepsilon) \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau + R(x_1(0)). \tag{5.43}$$

Combining the inequalities (5.42) and (5.43), results in

$$\begin{aligned} \int_0^t \|z_{q_3}(\tau)\|^2 \mathrm{d}\tau &\le \left(\frac{N}{k} + \gamma^2 - \varepsilon\right) \int_0^t \|d(\tau)\|^2 \mathrm{d}\tau \\ &\quad + N K(x_1(0), x_2(0), q_3(0)) + R(x_1(0)). \end{aligned} \tag{5.44}$$

Therefore if we choose

$$k^* \ge \frac{N}{\varepsilon}$$

then the L_2-gain from d to z_{q_3} is less than or equal to γ. ■

To solve the state feedback $\mathcal{H}_\infty$ problem we can give the following sufficient conditions

Theorem 5.22 *Assume the assumptions in Theorem 5.21 are satisfied. Then there exists a constant k^* such that the nonlinear state feedback $\mathcal{H}_\infty$ control problem with constant $\overline{\gamma} < \gamma$ is solvable for Σ by the feedback*

$$\begin{aligned} u_1 &= \tilde{\beta}(x_1, x_2, x_3, k) = \beta(x_1, x_2, x_3 - x_3^*(x_1, V_{x_1}^T(x_1)), k) \\ u_2 &= u_2^*(x_1, V_{x_1}^T(x_1)) \end{aligned}$$

for $k > k^*$ if the parameterized feedback

$$u_1 = \beta(x_1, x_2, q_3, k)$$

solves the $\mathcal{H}_\infty$ almost disturbance decoupling problem with constant $\frac{1}{k}$ for Σ^{tr}. □

Proof The L_2-gain result follows from Theorem 5.21 and closed loop stability is implied by the solvability of the $\mathcal{H}_\infty$ almost disturbance decoupling problem for the complete system. ■

5.2.5 The regular $\mathcal{H}_\infty$ subproblem

In the previous subsection only sufficient conditions for the solvability of the singular $\mathcal{H}_\infty$ problem were given. It is interesting to look whether these conditions are also necessary. We will prove under an extra regularity assumption that the solvability of the regular $\mathcal{H}_\infty$ subproblem for the subsystem Σ_1 is indeed necessary. Again we assume that the distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker \mathrm{d}h$ are constant dimensional, that Assumption 5 is satisfied and that the conditions (5.24) and (5.25) hold in order to be able to define the state and input transformation defined in Subsection 5.2.2.

Suppose that there exists a feedback

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} l_1(x_1, x_2, x_3) \\ l_2(x_1, x_2, x_3) \end{pmatrix} = l(x) \tag{5.45}$$

which solves the state feedback L_2-gain control problem with constant γ for the system Σ, with a differentiable, non-negative storage function V. Then this function V is a solution to the Hamilton-Jacobi inequality

$$\begin{aligned} V_{x_1}(x) f_1(x_1, x_3, l_2(x), d) + V_{x_2}(x) f_2(x_1, x_2, x_3, l_1(x), l_2(x), d) & \\ + V_{x_3}(x) f_3(x_1, x_2, x_3, l_1(x), l_2(x), d) & \\ + \frac{1}{2} h^T(x_1, x_3, l_2(x)) h(x_1, x_3, l_2(x)) - \frac{1}{2}\gamma^2 d^T d & \leq 0 \end{aligned} \tag{5.46}$$

for all x and d. Then we look at solutions to the next two equations

$$V_{x_2}(x_1, x_2, x_3) = 0, \qquad V_{x_3}(x_1, x_2, x_3) = 0.$$

Assume there exists a differentiable solution to these equalities given by

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} F_2(x_1) \\ F_3(x_1) \end{pmatrix} =: F(x_1), \qquad F(0) = 0.$$

Remark 5.23 If the Hessian of V with respect to x_2 and x_3 is non-singular at the origin then at least locally existence and uniqueness of such a solution is assured by the Implicit Function Theorem. □

Theorem 5.24 *Let $\gamma > 0$. Suppose that the distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker \mathrm{d}h$ are constant dimensional, Assumption 5 is satisfied and that both the conditions (5.24) and (5.25) hold. Suppose there exists a feedback (5.45) which solves the* *state feedback L_2-gain control problem for the system Σ, with constant γ and a differentiable storage function $V \geq 0$. Additionally assume that the equations*

$$V_{x_2}(x_1, x_2, x_3) = 0, \qquad V_{x_3}(x_1, x_2, x_3) = 0 \tag{5.47}$$

have a differentiable solution

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} F_2(x_1) \\ F_3(x_1) \end{pmatrix} = F(x_1), \qquad F(0) = 0.$$

Then the feedback

$$\begin{aligned} x_3 &= F_3(x_1) \\ u_2 &= l_2(x_1, F_2(x_1), F_3(x_1)) \end{aligned} \tag{5.48}$$

solves the state feedback L_2-gain control problem,with constant γ, for the system Σ_1 with a differentiable storage function given by

$$P(x_1) = V(x_1, F_2(x_1), F_3(x_1)).$$

□

Proof Substitution of $x_2 = F_2(x_1)$ and $x_3 = F_3(x_1)$ in the inequality (5.46), using the equations (5.47), leads to the following inequality for $P(x_1)$:

$$P_{x_1}(x_1) f_1(x_1, x_3, u_2, d) + \frac{1}{2} h^T(x_1, x_3, u_2) h(x_1, x_3, u_2) - \frac{1}{2}\frac{1}{\gamma^2} d^T d \leq 0 \tag{5.49}$$

for all x_1, d, when x_3 and u_2 are given by (5.48).
This means that the feedback (5.48) solves the state feedback L_2-gain control problem, with constant γ, for Σ_1 with storage function P (see Theorem 2.2). ■

A similar result can be proved for the solvability of the state feedback $\mathcal{H}_\infty$ control problem.

Theorem 5.25 *Let $\gamma > 0$. Suppose that the distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker \mathrm{d}h$ are constant dimensional, Assumption 5 is satisfied and that both the conditions (5.24) and (5.25) hold. Suppose there exists a feedback (5.45) which solves the state feedback $\mathcal{H}_\infty$ control problem for the system Σ, with constant γ and a* *differentiable storage function* $V(x) > 0$ $(x \neq 0)$*. Additionally assume that the equations*

$$V_{x_2}(x_1, x_2, x_3) = 0, \qquad V_{x_3}(x_1, x_2, x_3) = 0 \tag{5.50}$$

have a differentiable solution

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} F_2(x_1) \\ F_3(x_1) \end{pmatrix} = F(x_1), \qquad F(0) = 0.$$

Then the feedback (5.48) solves the state feedback $\mathcal{H}_\infty$ *control problem,with constant* γ*, for the system* Σ_1 *with a differentiable storage function given by*

$$P(x_1) = V(x_1, F_2(x_1), F_3(x_1)).$$

□

Proof The L_2-gain result follows from Theorem 5.24. Because $V(x) > 0$ for $x \neq 0$ it follows that $P(x_1) > 0$ for $x_1 \neq 0$ which implies that the closed-loop system Σ_1, (5.48) is locally asymptotically stable. ■

The saddle point solution of the pre-Hamiltonian $K_\gamma(x_1, p_1, d, x_3, u_2)$ corresponding to the L_2-gain problem for the subsystem Σ_1 is $x_3^* = x_3^*(x_1, p)$, $u_2^* = u_2^*(x_1, p)$ and $d^* = d^*(x_1, p)$. This solution satisfies the saddle point condition

$$K_\gamma(x_1, p, d, x_3^*, u_2^*) \leq K_\gamma(x_1, p, d^*, x_3^*, u_2^*) \leq K_\gamma(x_1, p, d^*, x_3, u_2)$$

for every (x_1, p, d, x_3, u_2). For the specific case that p is equal to $P_{x_1}(x_1)$ and x_3 equal to $F_3(x_1)$ it follows from the right inequality of the saddle point condition that also the feedback

$$\begin{aligned} x_3 &= x_3^*(x_1, P_{x_1}^T(x_1)) \\ u_2 &= u_2^*(x_1, P_{x_1}^T(x_1)) \end{aligned} \tag{5.51}$$

solves the state feedback L_2-gain control problem, with constant γ, for the system Σ_1. Summarizing:

Theorem 5.26 *Assume all conditions of Theorem 5.24 are satisfied. Then the feedback (5.51) solves the state feedback* L_2*-gain control problem, with constant* γ*, for the system* Σ_1 *with differentiable storage function*

$$P(x_1) = V(x_1, F_2(x_1), F_3(x_1)).$$

□

Proof The proof follows by using the saddle point condition as explained above, together with the inequality (5.49) derived from Theorem 5.24. ■

Until now little is known about the structure of the feedback solution (5.45) to the singular L_2-gain control problem. The following result gives, under the worst case disturbance, some idea of the structure of this feedback.

Theorem 5.27 *Assume all conditions of Theorem 5.24 are satisfied. Additionally assume that:*

$$\begin{aligned} F_3(x_1) &= x_3^*(x_1, P_{x_1}^T(x_1)); \\ l_2(x_1, F_2(x_1), F_3(x_1)) &= u_2^*(x_1, P_{x_1}^T(x_1)). \end{aligned}$$

Furthermore assume that (5.46) holds with equality along $(x_1, F_2(x_1), F_3(x_1))$ *and that the Hessian of* V *with respect to* x_2 *and* x_3 *is non-singular. Then the feedback (5.45) satisfies the following equality on* $(x_1, F_2(x_1), F_3(x_1))$

$$\begin{aligned} &\begin{pmatrix} F_{2x_1}(x) \\ F_{3x_1}(x) \end{pmatrix} f_1(x_1, x_3, l_2(x), d_{\max}(x)) \\ &= \begin{pmatrix} f_2(x_1, x_2, x_3, l_1(x), l_2(x), d_{\max}(x)) \\ f_3(x_1, x_2, x_3, l_1(x), l_2(x), d_{\max}(x)) \end{pmatrix} \end{aligned} \tag{5.52}$$

where $d_{\max}(x)$ *is the worst case disturbance corresponding to the inequality (5.46), given by*

$$\begin{aligned} d_{\max}(x) = \arg\max_d & \left\{ \frac{1}{2} h^T(x_1, x_3, l_2(x)) h(x_1, x_3, l_2(x)) \right. \\ & + V_{x_1}(x) f_1(x_1, x_3, l_2(x), d) \\ & + V_{x_2}(x) f_2(x_1, x_2, x_3, l_1(x), l_2(x), d) \\ & \left. + V_{x_3}(x) f_3(x_1, x_2, x_3, l_1(x), l_2(x), d) - \frac{1}{2}\gamma^2 d^T d \right\}. \end{aligned}$$

□

Proof (5.46) can be written as

$$\begin{aligned} V_{x_1} f_1(x_1, x_3, l_2, d_{\max}) + V_{x_2} f_2(x_1, x_2, x_3, l_1, l_2, d_{\max}) & \\ + V_{x_3} f_3(x_1, x_2, x_3, l_1, l_2, d_{\max}) + & \\ \frac{1}{2} h^T(x_1, x_3, l_2) h(x_1, x_3, l_2) - \frac{1}{2}\gamma^2 d_{\max}^T d_{\max} &\leq 0 \end{aligned} \tag{5.53}$$

with equality along $(x_1, F_2(x_1), F_3(x_1))$. Differentiation with respect to x_2 of the left side of this inequality leads to

$$\begin{aligned}
&V_{x_1x_2} f_1 + V_{x_1} \frac{\partial f_1}{\partial u_2} \frac{\partial l_2}{\partial x_2} + V_{x_1} \frac{\partial f_1}{\partial d} \frac{\partial d_{\max}}{\partial x_2} + \\
&V_{x_2x_2} f_2 + V_{x_2} \frac{\partial f_2}{\partial x_2} + V_{x_2} \frac{\partial f_2}{\partial u_1} \frac{\partial l_1}{\partial x_2} + V_{x_2} \frac{\partial f_2}{\partial u_2} \frac{\partial l_2}{\partial x_2} + V_{x_2} \frac{\partial f_2}{\partial d} \frac{\partial d_{\max}}{\partial x_2} + \\
&V_{x_3x_2} f_3 + V_{x_3} \frac{\partial f_3}{\partial x_2} + V_{x_3} \frac{\partial f_3}{\partial u_1} \frac{\partial l_1}{\partial x_2} + V_{x_3} \frac{\partial f_3}{\partial u_2} \frac{\partial l_2}{\partial x_2} + V_{x_3} \frac{\partial f_3}{\partial d} \frac{\partial d_{\max}}{\partial x_2} + \\
&\frac{\partial}{\partial x_2} \left(\frac{1}{2} h^T h \right) - \frac{\partial}{\partial x_2} \left(\frac{1}{2} \gamma^2 d_{\max}^T d_{\max} \right).
\end{aligned}$$

Hence along $(x_1, F_2(x_1), F_3(x_1))$, using the maximizing property of $d_{\max}$, and the minimizing property of l_2 $(= u_2^*)$, it follows that

$$V_{x_1x_2} f_1 + V_{x_2x_2} f_2 + V_{x_3x_2} f_3 = 0.$$

The same holds for differentiation with respect to x_3 along $(x_1, F_2(x_1), F_3(x_1))$, where we use apart from the maximizing respectively minimizing properties of $d_{\max}$ and u_2^* also the minimizing property of F_3 $(= x_3^*)$.
Therefore along $(x_1, F_2(x_1), F_3(x_1))$ also

$$V_{x_1x_3} f_1 + V_{x_2x_3} f_2 + V_{x_3x_3} f_3 = 0.$$

Furthermore the equalities

$$\begin{aligned}
V_{x_2}(x_1, F_2(x_1), F_3(x_1)) &= 0, \\
V_{x_3}(x_1, F_2(x_1), F_3(x_1)) &= 0
\end{aligned}$$

hold and differentiation of these equalities along $(x_1, F_2(x_1), F_3(x_1))$ with respect to x_1 leads to

$$\begin{aligned}
V_{x_1x_2} + V_{x_2x_2} F_{2x_1} + V_{x_3x_2} F_{3x_1} &= 0, \\
V_{x_1x_3} + V_{x_2x_3} F_{2x_1} + V_{x_3x_3} F_{3x_1} &= 0.
\end{aligned}$$

Hence in vector notation we have along $(x_1, F_2(x_1), F_3(x_1))$

$$\begin{aligned}
\begin{pmatrix} V_{x_1x_2} \\ V_{x_1x_3} \end{pmatrix} + \begin{pmatrix} V_{x_2x_2} & V_{x_3x_2} \\ V_{x_2x_3} & V_{x_3x_3} \end{pmatrix} \begin{pmatrix} F_{2x_1} \\ F_{3x_1} \end{pmatrix} &= 0, \\
\begin{pmatrix} V_{x_1x_2} \\ V_{x_1x_3} \end{pmatrix} f_1 + \begin{pmatrix} V_{x_2x_2} & V_{x_3x_2} \\ V_{x_2x_3} & V_{x_3x_3} \end{pmatrix} \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} &= 0.
\end{aligned}$$

It follows from the non-singularity of the Hessian of V with respect to x_2 and x_3 that along $(x_1, F_2(x_1), F_3(x_1))$

$$\begin{pmatrix} F_{2x_1}(x) \\ F_{3x_1}(x) \end{pmatrix} f_1(x_1, x_3, l_2(x), d_{\max}) = \begin{pmatrix} f_2(x_1, x_2, x_3, l_1(x), l_2(x), d_{\max}) \\ f_3(x_1, x_2, x_3, l_1(x), l_2(x), d_{\max}) \end{pmatrix}.$$

■

Remark 5.28 Condition (5.52) is equivalent to the submanifold

$$S = \{(x_1, x_2, x_3) \in M | x_2 = F_2(x_1), x_3 = F_3(x_1)\}$$

being invariant under the dynamics

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_3, l_2, d_{\max}), \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, l_1, l_2, d_{\max}), \\ \dot{x}_3 &= f_3(x_1, x_2, x_3, l_1, l_2, d_{\max}). \end{aligned}$$

□

5.2.6 The $\mathcal{H}_\infty$ almost disturbance decoupling problem

In this subsection we will restrict ourself to systems of a special form. One of the important restrictions is the triangular structure of the state dependency of the disturbance vector fields. These restrictions are imposed in order to be able to apply the sufficiency results for the solvability of the almost disturbance decoupling problem derived in Chapter 4.

For linear systems necessary and sufficient conditions have been derived for the solvability of the almost disturbance decoupling problem (see [Tr 86], [Wi 81], [Wi 82]). Until now geometric necessary and sufficient conditions for the solvability of the almost disturbance decoupling problem for nonlinear systems have not been derived, and in fact it seems unlikely that this is possible. As a consequence it is not possible to derive necessary and sufficient conditions for the solvability of the singular $\mathcal{H}_\infty$ problem for general nonlinear systems similar to the linear case.

In this subsection we consider systems of the following form

$$\begin{aligned} \dot{\xi}_{11} &= \xi_{12} + W_1^T(\tilde{\xi}_1, \xi_r, u_r)d, \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
\dot{\xi}_{1\rho_1} &= u_1 + W_{1\rho_1}^T(\tilde{\xi}_1, \ldots, \tilde{\xi}_{\rho_1}, \xi_r, u_r)d, \\
\dot{\xi}_{21} &= \xi_{22} + W_{21}^T(\tilde{\xi}_1, \xi_r, u_r)d, \\
&\vdots \\
\dot{\xi}_{2\rho_2} &= u_2 + W_{2\rho_2}^T(\tilde{\xi}_1, \ldots, \tilde{\xi}_{\rho_2}, \xi_r, u_r)d, \\
&\vdots \\
&\vdots \\
\dot{\xi}_{k1} &= \xi_{k2} + W_{k1}^T(\tilde{\xi}_1, \xi_r, u_r)d, \\
&\vdots \\
\dot{\xi}_{k\rho_k} &= u_k + W_{k\rho_k}^T(\tilde{\xi}_1, \ldots, \tilde{\xi}_{\rho_k}, \xi_r, u_r)d, \\
\dot{\xi}_r &= f(\tilde{\xi}_1, \xi_r, u_r) + e(\tilde{\xi}_1, \xi_r, u_r)d, \\
z &= h(\tilde{\xi}_1, \xi_r, u_r)
\end{aligned}
\tag{5.54}
$$

where $\rho = \sum_{i=1}^k \rho_i$ and $\tilde{\xi}_j$ indicates the j-th state components of every block

$$\tilde{\xi}_j = \left\{\xi_{lj},\ \text{for all } l = 1, \ldots, k \text{ for which } j \leq \rho_l\right\}.$$

So $\tilde{\xi}_1 = (\xi_{11}, \xi_{21}, \ldots, \xi_{k1})$ etc..

The state of the system (5.54) is given by $(\tilde{\xi}_1, \ldots, \tilde{\xi}_k, \xi_r)$ and the inputs are $(u_1, \ldots, u_k, u_r)$. Furthermore the function h is assumed to be an injective mapping from $(\tilde{\xi}_1, u_r)$ to z. For the extended version of the system (5.54) the minimal conditioned invariant distribution containing the input vector fields can be calculated. It is left to the reader to prove that indeed the distributions are projectable and that the projected distributions $\mathcal{X}_1^P$, $\mathcal{X}_2^P$ and $\mathcal{X}_3^P$ are given by:

$$
\begin{aligned}
\mathcal{X}_1^P &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_r}\right\}; \\
\mathcal{X}_2^P &= \operatorname{span}\left\{\frac{\partial}{\partial \tilde{\xi}_2}, \ldots, \frac{\partial}{\partial \tilde{\xi}_{\max_{i=1}^k \rho_k}}\right\}; \\
\mathcal{X}_3^P &= \operatorname{span}\left\{\frac{\partial}{\partial \tilde{\xi}_1}\right\}.
\end{aligned}
$$

Thus the system (5.54) can be seen as an interconnection of two systems. The first subsystem Σ_1 has states ξ_r, inputs $(\tilde{\xi}_1, u_r, d)$ and outputs z. This subsystem is connected with a second subsystem, denoted by Σ_2, which has states $(\tilde{\xi}_1, \ldots, \tilde{\xi}_k)$, inputs $(u_1, \ldots, u_k, u_r, d)$ and outputs $\tilde{\xi}$ (see Figure 5.6).

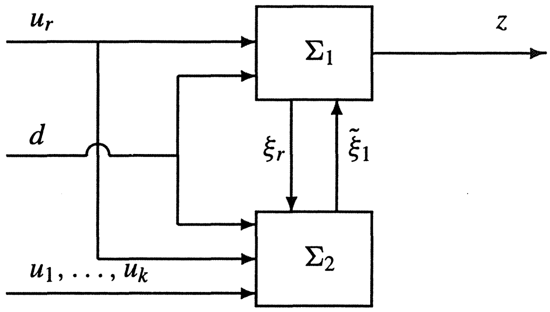

Figure 5.6: system (5.54)

For the system (5.54) we can prove the following L_2-gain result (see also Theorem 5.21).

Theorem 5.29 *Let $\gamma > 0$. Assume there exists a smooth feedback*

$$\begin{aligned} \tilde{\xi}_1 &= \alpha(\xi_r) \\ u_r &= \beta(\xi_r) \end{aligned}$$

that solves the state feedback L_2-gain control problem, with constant less than γ, for the subsystem Σ_1 Additionally assume that the vector field given by the vector

$$\Big(\xi_{12} \quad \cdots \quad \xi_{1\rho_1} \quad L_f^{\rho_1}\alpha_1(\xi_r) \quad \xi_{22} \quad \cdots \quad \xi_{2\rho_2} \quad L_f^{\rho_2}\alpha_2(\xi_r) \quad \cdots$$
$$\cdots \quad f^T(\tilde{\xi}_1, \xi_r, \beta(\xi_r)) \;\Big)^T$$

is complete. Then the state feedback L_2-gain control problem for the system (5.54) is solvable, i.e., there exists a feedback such that the closed loop system consisting of the system (5.54) together with the feedback has L_2-gain less than γ.

Conversely, suppose that the feedback

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ u_r \end{pmatrix} = \begin{pmatrix} l_1(\xi_{11}, \ldots, \xi_{1\rho_1}, \ldots, \xi_{k1}, \ldots, \xi_{k\rho_k}, \xi_r) \\ \vdots \\ l_k(\xi_{11}, \ldots, \xi_{1\rho_1}, \ldots, \xi_{k1}, \ldots, \xi_{k\rho_k}, \xi_r) \\ l_r(\xi_{11}, \ldots, \xi_{1\rho_1}, \ldots, \xi_{k1}, \ldots, \xi_{k\rho_k}, \xi_r) \end{pmatrix}$$

solves the state feedback L_2-gain control problem, with constant less than γ, for the system (5.54) with a differentiable storage function $V \geq 0$. Additionally *assume that the equations*

$$V_{\xi_{lj}}(\xi_{11},\ldots,\xi_{1\rho_1},\ldots,\xi_{k1},\ldots,\xi_{k\rho_k},\xi_r)=0, \qquad l=1,\ldots,k,\ j=1,\ldots,\rho_l$$

have a differentiable solution

$$\begin{pmatrix}\xi_{11}\\ \vdots\\ \xi_{1\rho_1}\\ \vdots\\ \xi_{k1}\\ \vdots\\ \xi_{k\rho_k}\end{pmatrix}=\begin{pmatrix}F_{11}(\xi_r)\\ \vdots\\ F_{1\rho_1}(\xi_r)\\ \vdots\\ F_{k1}(\xi_r)\\ \vdots\\ F_{k\rho_k}(\xi_r)\end{pmatrix}=:F(\xi_r), \qquad F(0)=0.$$

Then the state feedback L_2-gain control problem, with constant less than γ, for the subsystem Σ_1 is solvable with a differentiable storage function. □

Proof Choose the inputs u_r equal to the solution of the state feedback problem of the regular subsystem Σ_1

$$u_r=\beta(\xi_r),$$

and define the new output equation

$$\tilde{h}=\tilde{\xi}_1-\alpha(\xi_r).$$

The system consisting of the dynamics described in (5.54) together with this new output equation and the inputs u_r chosen as before satisfies the conditions of Theorem 4.9. Therefore the L_2-gain almost disturbance decoupling problem is solvable for this system. Finally it follows from Theorem 5.21 that the singular L_2-gain problem for the system (5.54) is solvable.
The second part of the theorem follows from Theorem 5.24. ■

Thus for systems of the form (5.54) the solvability of the singular L_2-gain problem is characterized by the solvability of the regular $\mathcal{H}_\infty$ problem for the subsystem Σ_1.

For stability results extra assumptions should be imposed; similar to the ones used for Theorem 5.22, Theorem 4.10 and Theorem 5.25. This is left to the reader.

5.2.7 Factorization approach

As described in Subsection 5.1.5 there exists a second way to derive conditions for the solvability of the singular $\mathcal{H}_\infty$ problem. Also for nonlinear systems this seems to be an elegant way to derive at least sufficient conditions for the solvability of the singular $\mathcal{H}_\infty$ problem.

To be able to extend the linear ideas to nonlinear systems we restrict the class of systems to nonlinear systems that are affine in the disturbances

$$\Sigma \begin{cases} \dot{x} &= f(x,u) + e(x,u)d \\ z &= h(x,u) \end{cases}$$

The pre-Hamiltonian corresponding to the $\mathcal{H}_\infty$ problem, with L_2-gain less than or equal to γ, for the system Σ is given by

$$K_\gamma(x,p,u,d) = p^T\left(f(x,u) + e(x,u)d\right) + \frac{1}{2}h^T(x,u)h(x,u) - \frac{1}{2}\gamma^2 d^T d.$$

The maximizing disturbance with respect to this pre-Hamiltonian is

$$d_{\max}(x,p,u) = \arg\max_d K_\gamma(x,p,u,d) = \frac{1}{\gamma^2}e^T(x,u)p.$$

As an extension of the quadratic matrix inequality used in the linear theory we introduce the *nonlinear dissipation inequality* defined as

$$\begin{aligned} F_\gamma(x,V(x),u) &= K_\gamma(x, V_x^T(x), d_{\max}(x, V_x^T(x), u), u) \\ &= V_x(x)f(x,u) + \frac{1}{2}\frac{1}{\gamma^2}V_x(x)e(x,u)e^T(x,u)V_x^T(x) \\ &\quad + \frac{1}{2}h^T(x,u)h(x,u) \;\geq\; 0. \end{aligned} \tag{5.55}$$

Remark 5.30 The relation of the nonlinear dissipation inequality with the quadratic matrix inequality can be clarified by considering the nonlinear dissipation inequality for nonlinear systems which are not only affine in the disturbances but also in the inputs

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + e(x)d \\ z &= h(x) + k(x)u \end{aligned} \tag{5.56}$$

Then the inequality (5.55) amounts to

$$\begin{aligned} F_\gamma(x,V(x),u) &= V_x(x)\left(f(x) + g(x)u\right) + \frac{1}{2}\frac{1}{\gamma^2}V_x(x)e(x)e^T(x)V_x^T(x) \\ &\quad + \frac{1}{2}\left(h(x)+k(x)u\right)^T\left(h(x)+k(x)u\right) \;\geq\; 0 \end{aligned}$$

which can equivalently be described as the positive semi-definiteness of the matrix

$$\begin{pmatrix} 2V_x(x)f(x) + \frac{1}{\gamma^2}V_x(x)e(x)e^T(x)V_x^T(x) + h^T(x)h(x) & V_x(x)g(x) + h^T(x)k(x) \\ g^T(x)V_x^T(x) + k^T(x)h(x) & k^T(x)k(x) \end{pmatrix}$$

This is clearly equivalent to the quadratic matrix inequality condition considered for linear systems in Subsection 5.1.3. □

As in Subsection 5.1.5 we define an auxiliary system based on a solution $V \geq 0$, $V(0) = 0$, to the nonlinear dissipation inequality as

$$\Sigma_P \begin{cases} \dot{x} &= \Big(f(x,u) + \frac{1}{\gamma^2}e(x,u)e^T(x,u)V_x^T(x)\Big) + e(x,u)d_p \\ z_p &= h_p(x,u) \end{cases} \tag{5.57}$$

where the new output equation is defined by

$$\begin{aligned} &\frac{1}{2}h_p^T(x,u)h_p(x,u) \\ &= F_\gamma(x, V(x), u) \\ &= V_x(x)f(x,u) + \frac{1}{2}\frac{1}{\gamma^2}V_x(x)e(x,u)e^T(x,u)V_x^T(x) + \frac{1}{2}h^T(x,u)h(x,u) \end{aligned}$$

and $d_p := d - \frac{1}{\gamma^2}e^T(x,u)V_x^T(x)$. Of course we want the new output equation to be sufficiently smooth. A sufficient condition for the local existence of a smooth h_p is provided by a generalized version of Morse's Lemma (see for Morse's Lemma for instance [JJT 86]).

Lemma 5.31 *Let f be a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$ with $f(0) = 0$, $\frac{\partial f}{\partial x}(0) = 0$, rank $\frac{\partial^2 f}{(\partial x)^2} = k \leq n$ around the origin and all eigenvalues are nonnegative. Then there exists local coordinates $q_1, \ldots, q_n$ such that in the new coordinates f can be written as*

$$f(q) = q_1^2 + \cdots + q_k^2.$$

□

Proof First we take the Taylor expansion of f around the origin

$$f(x_1, \ldots, x_n) = \frac{1}{2}\sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(0) + \frac{1}{6}\sum_{i,j,l} x_i x_j x_l \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_l}(0) + \ldots.$$

Since rank $\frac{\partial^2 f}{\partial x^2}(0) = k$ we can find, by a linear coordinate transform, new coordinates $\nu_1, \dots, \nu_n$ such that

$$f(\nu_1, \dots, \nu_n) = \nu_1^2 + \dots + \nu_k^2 + \varphi(\nu_1, \dots, \nu_n)$$

where φ consists of third order terms and higher of $\nu_1, \dots, \nu_n$, that is $\varphi(0) = 0$, $\frac{\partial \varphi}{\partial \nu}(0) = 0$, $\frac{\partial^2 \varphi}{\partial \nu^2}(0) = 0$. Hence

$$\frac{\partial^2 f}{\partial \nu^2}(\nu) = 2\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} + \frac{\partial^2 \varphi}{\partial \nu^2}(\nu)$$

around 0 and hence, since rank$\frac{\partial^2 f}{\partial x^2}(x) = k$ around the origin

$$\frac{\partial^2 \varphi}{\partial \nu_i \partial \nu_j}(\nu) = 0, \qquad i = k+1, \dots, n, \quad j = 1, \dots, n$$

around the origin.
Then

$$\frac{\partial \varphi}{\partial \nu_i}(\nu_1, \dots, \nu_n) = \int_0^1 \sum_{j=1}^n \frac{\partial^2 \varphi}{\partial \nu_i \partial \nu_j}(t\nu_1, \dots, t\nu_n)\nu_j \mathrm{d}t = 0, \qquad i = k+1, \dots, n$$

and thus φ does not depend on $\nu_{k+1}, \dots, \nu_n$ around 0.
Hence

$$f(\nu_1, \dots, \nu_n) = \nu_1^2 + \dots + \nu_k^2 + \varphi(\nu_1, \dots, \nu_k)$$

and f only depends on $\nu_1, \dots, \nu_k$.
Now we can apply Morse's Lemma (see e.g. [JJT 86]) to the function f (seen as a function on $\mathbb{R}^k$, with rank of its Hessian equal to k), and we can find coordinates $q_1, \dots, q_k$, and $q_{k+1} = \nu_{k+1}, \dots, q_n = \nu_n$ such that

$$f(q_1, \dots, q_n) = \sum_{i=1}^k q_i^2.$$

■

From Lemma 5.31 it follows that a sufficient condition for the local existence of a smooth output function $h_p(x, u)$ is that the Hessian of the nonlinear dissipation inequality $F_\gamma(x, V(x), u)$ with respect to x and u has constant rank around $(x, u) = (0, 0)$. Indeed in that case the dimension of $h_p(x, u)$ can be chosen equal to the rank of the Hessian of the nonlinear dissipation inequality $F_\gamma(x, V_x(x), u)$ where the components of h_p are chosen equal to the (x, u) coordinate representation of the corresponding q components.

The auxiliary system Σ_P will appear to be helpful for solving the $\mathcal{H}_\infty$ problem for the system Σ. Similar to the linear case we obtain the following theorem relating the L_2-gains of Σ and Σ_P.

Theorem 5.32 *Suppose there exists a solution $V \geq 0$ to the nonlinear matrix inequality $F_\gamma(x, V(x), u)$. Then for all d*

$$\begin{aligned} &\frac{1}{2}\int_0^t \left(\|z_p(\tau)\|^2 - \gamma^2\|d_p(\tau)\|^2\right) \mathrm{d}\tau \\ &= \frac{1}{2}\int_0^t \left(\|z(\tau)\|^2 - \gamma^2\|d(\tau)\|^2\right) \mathrm{d}\tau + V(x(t)) - V(x(0)). \end{aligned} \tag{5.58}$$

□

Proof This follows easily by writing out z_p and d_p in

$$\begin{aligned} \|z_p\|^2 - \gamma^2\|d_p\|^2 &= h_p^T(x,u)h_p(x,u) - \frac{1}{\gamma^2}V_x(x)e(x,u)e^T(x,u)V_x^T(x) \\ &\quad -2V_x(x)e(x,u)d - \gamma^2 d^T d \\ &= 2V_x(x)(f(x,u) + e(x,u)d) \\ &\quad +h^T(x,u)h(x,u) - \gamma^2\|d\|^2 \\ &= 2\frac{\mathrm{d}}{\mathrm{d}t}V(x) + \|z\|^2 - \gamma^2\|d\|^2. \end{aligned}$$

Taking the integral from 0 to t leads to the desired result

$$\begin{aligned} &\int_0^t \left(\|z_p(\tau)\|^2 - \gamma^2\|d_p(\tau)\|^2\right) \mathrm{d}\tau \\ &= 2\left(V(x(t)) - V(x(0))\right) + \int_0^t \left(\|z(\tau)\|^2 - \gamma^2\|d(\tau)\|^2\right) \mathrm{d}\tau. \end{aligned}$$

■

From Theorem 5.32 we can derive the following partial generalization of Theorem 5.8.

Corollary 5.33 *Suppose there exists a solution $V \geq 0$, $V(0) = 0$, to the nonlinear matrix inequality $F_\gamma(x, V(x), u) \geq 0$ and there exists a feedback which solves the state feedback $\mathcal{H}_\infty$ control problem, with constant γ, for the system Σ_P with storage function $K(x)$. Then we have the following inequality for the system Σ*

$$\int_0^\infty \|z(\tau)\|^2 \mathrm{d}\tau \leq \gamma^2 \int_0^\infty \|d(\tau)\| \mathrm{d}\tau + V(x(0)) + K(x(0))$$

for all $d \in L(0, \infty)$. □

Proof The $\mathcal{H}_\infty$ problem for the system Σ_P is solvable. So there exists a feedback such that for the closed loop system the following inequality holds

$$\frac{1}{2}\int_0^t \|z_p(\tau)\|^2 \mathrm{d}\tau \leq \gamma^2 \frac{1}{2}\int_0^t \|d_p(\tau)\| \mathrm{d}\tau + K(x(0)).$$

Using Theorem 5.32 this gives

$$\begin{aligned}
&\int_0^t \left(\|z(\tau)\|^2 - \gamma^2 \|d(\tau)\|^2\right) \mathrm{d}\tau \\
&= \int_0^t \left(\|z_p(\tau)\|^2 - \gamma^2 \|d_p(\tau)\|^2\right) \mathrm{d}\tau + 2V(x(0)) - 2V(x(t)) \\
&\leq 2V(x(0)) - 2V(x(t)) + 2K(x(0)).
\end{aligned}$$

Because of the asymptotic stability of the closed loop system $x(t) \to 0$ for $t \to \infty$. Thus if we take the limit in (5.58) for t tending to infinity then $V(x(t)) \to 0$ because $V(0) = 0$ and we obtain

$$\int_0^\infty \|z(\tau)\|^2 \mathrm{d}\tau \leq \gamma^2 \int_0^\infty \|d(\tau)\| \mathrm{d}\tau + V(x(0)) + K(x(0))$$

for all $d \in L(0, \infty)$. ■

Let us define the pre-Hamiltonian for the system Σ_P

$$\begin{aligned}
K_\gamma^P(x, p, d_p, u) &= p^T \left(f(x, u) + \frac{1}{\gamma^2} e(x, u) e^T(x, u) V_x^T(x) + e(x, u) d_p \right) \\
&\quad + \frac{1}{2} h_p^T(x, u) h_p(x, u) - \frac{1}{2} \gamma^2 d_p^T d_p.
\end{aligned}$$

With respect to the pre-Hamiltonians of the systems Σ and Σ_P a similar result as in Theorem 5.32 can be derived.

Theorem 5.34 *Suppose there exists a solution $V \geq 0$ to the nonlinear matrix inequality $F_\gamma(x, V(x), u)$. Then*

$$K_\gamma(x, p + V_x^T(x), d, u) = K_\gamma^P(x, p, d_p, u), \qquad \forall x, p, d, u.$$

□

Proof Again the result follows by substitution of the definitions of the new output function h_p and the new disturbance d_p:

$$\begin{aligned}
&K_\gamma(x, p + V_x^T(x), d, u) \\
&\quad = \; p^T\left(f(x,u) + e(x,u)d\right) + V_x(x)\left(f(x,u) + e(x,u)d\right) \\
&\qquad + \frac{1}{2}h^T(x,u)h(x,u) - \frac{1}{2}\gamma^2 d^T d \\
&\quad = \; p^T\left(f(x,u) + e(x,u)d\right) + V_x(x)e(x,u)d + \frac{1}{2}h_p^T(x,u)h_p(x,u) \\
&\qquad - \frac{1}{2}\frac{1}{\gamma^2}V_x(x)e(x,u)e^T(x,u)V_x^T(x) - \frac{1}{2}\gamma^2 d^T d \\
&\quad = \; p^T\left(f(x,u) + \frac{1}{\gamma^2}e(x,u)e^T(x,u)V_x^T(x) + e(x,u)d_p\right) \\
&\qquad + \frac{1}{2}h_p^T(x,u)h_p(x,u) - \frac{1}{2}\gamma^2 d_p^T d_p \\
&\quad = \; K_\gamma^P(x, p, d_p, u).
\end{aligned}$$

■

From this theorem we can easily derive some interesting relations between the solvability of the L_2-gain control problems for the system Σ and Σ_P.

Theorem 5.35 *Suppose there exists a solution $V \geq 0$ to the nonlinear dissipation inequality $F_\gamma(x, V(x), u)$. Additionally assume there exists a feedback $u = l(x)$ for the system Σ_P such that the closed loop system has L_2-gain less than or equal to γ in the sense that there exists a solution $W \geq 0$ to the dissipation inequality*

$$K_\gamma^P(x, W_x^T(x), d_p, l(x)) \leq 0$$

for all d_p. Then the same feedback applied to the system Σ leads to a closed loop system which has L_2-gain less than or equal to γ because $W + V$ is a solution to

$$K_\gamma(x, (W+V)_x^T(x), d, l(x)) \leq 0$$

for all d. □

Proof Use Theorem 5.34 with $p = W_x^T(x)$ and $u = l(x)$. ∎

Also a converse result can be stated.

Theorem 5.36 *Assume there exists a solution $V \geq 0$ to the Hamilton-Jacobi equality*

$$\begin{aligned} &K_\gamma(x, V_x^T(x), d_{\max}(x, V_x^T(x), u^*(x)), u^*(x)) \\ &= V_x(x) f(x, u^*(x)) + \frac{1}{2}\frac{1}{\gamma^2} V_x(x) e(x, u^*(x)) e^T(x, u^*(x)) V_x^T(x) \\ &\quad + \frac{1}{2} h^T(x, u^*(x)) h(x, u^*(x)) = 0 \end{aligned}$$

where $u^*(x)$ is defined by

$$u^*(x) = \arg\min_u K_\gamma\left(x, V_x^T(x), d_{\max}(x, V_x^T(x), u), u\right). \tag{5.59}$$

Then the feedback $u = u^*(x)$ solves the state feedback L_2-gain control problem, with constant γ, for the system Σ. Furthermore $V \geq 0$ is a solution to the nonlinear dissipation inequality and the zero function $W \equiv 0$ is a solution to

$$K_\gamma^P(x, W_x^T(x), d_p, u^*(x)) \leq 0$$

for all d_p, with equality for $d_p = 0$. □

Proof Use Theorem 5.34 with $p = W_x^T(x) = 0$ and $u = u^*(x)$, where $u^*(x)$ is defined by (5.59). ∎

Note that in Theorem 5.36 we do not assume that $u^*(x)$ is a saddle point solution of the pre-Hamiltonian corresponding to the L_2-gain control problem for the system Σ.

Theorem 5.35 can be used to simplify the L_2-gain problem for the system Σ after application of the state and input transformation as described in Subsection 5.2.2

$$\Sigma \begin{cases} \dot{x}_1 = f_1(x_1, x_3, u_2) + e_1(x_1, x_3, u_2) d \\ \dot{x}_2 = f_2(x_1, x_2, x_3, u_1, u_2) + e_2(x_1, x_2, x_3, u_1, u_2) d \\ \dot{x}_3 = f_3(x_1, x_2, x_3, u_1, u_2) + e_3(x_1, x_2, x_3, u_1, u_2) d \\ z = h(x_1, x_3, u_2) \end{cases}$$

We look at a solution $V(x_1) \geq 0$ to the Hamilton Jacobi equality for the regular $\mathcal{H}_\infty$ subproblem for the system Σ_1 with a constant γ

$$\begin{aligned} V_{x_1}(x_1) f_1(x_1, x_3^*, u_2^*) + \frac{1}{2} h^T(x_1, x_3^*, u_2^*) h(x_1, x_3^*, u_2^*) & \\ + \frac{1}{2}\frac{1}{\gamma^2} V_{x_1}(x_1) e_1(x_1, x_3^*, u_2^*) e_1^T(x_1, x_3^*, u_2^*) V_{x_1}^T(x_1) &= 0 \end{aligned} \tag{5.60}$$

where $x_3^* = x_3^*(x_1, V_{x_1}^T(x_1))$ and $u_2^* = u_2^*(x_1, V_{x_1}^T(x_1))$. Because of the saddle point condition with respect to x_3^* and u_2^* this solution V, which only depends on the state x_1 of the regular subsystem, is also a solution to the nonlinear dissipation inequality for the system Σ. This solution V to the nonlinear dissipation inequality will be used to define the new system Σ_P

$$\Sigma_P \begin{cases} \dot{x}_1 &= f_1(x_1, x_3, u_2) + \frac{1}{\gamma^2} e_1(x_1, x_3, u_2) e_1^T(x_1, x_3, u_2) V_{x_1}^T(x_1) \\ & \quad + e_1(x_1, x_3, u_2) d_p \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, u_1, u_2) \\ & \quad + \frac{1}{\gamma^2} e_2(x_1, x_2, x_3, u_1, u_2) e_1^T(x_1, x_3, u_2) V_{x_1}^T(x_1) \\ & \quad + e_2(x_1, x_2, x_3, u_1, u_2) d_p \\ \dot{x}_3 &= f_3(x_1, x_2, x_3, u_1, u_2) \\ & \quad + \frac{1}{\gamma^2} e_3(x_1, x_2, x_3, u_1, u_2) e_1^T(x_1, x_3, u_2) V_{x_1}^T(x_1) \\ & \quad + e_3(x_1, x_2, x_3, u_1, u_2) d_p \\ z &= h_p(x_1, x_3, u_2) \end{cases}$$

with $d_p = d - \frac{1}{\gamma^2} e_1^T(x_1, x_3, u_2) V_{x_1}^T(x_1)$ and the new output function h_p being a solution to of

$$\begin{aligned} & \frac{1}{2} h_p^T(x_1, x_3, u_2) h_p(x_1, x_3, u_2) \\ &= V_{x_1}(x_1) f_1(x_1, x_3, u_2) + \frac{1}{2}\frac{1}{\gamma^2} V_{x_1}(x_1) e_1(x_1, x_3, u_2) e_1^T(x_1, x_3, u_2) V_{x_1}^T(x_1) \\ & \quad + \frac{1}{2} h^T(x_1, x_3, u_2) h(x_1, x_3, u_2). \end{aligned}$$

From Theorem 5.35 we know that a sufficient condition for the solvability of the L_2-gain problem with constant γ for Σ is the solvability of the same problem for Σ_P. Presumably this problem is easier to solve for Σ_P. The following result follows directly from Theorem 5.35.

Corollary 5.37 *Assume there exists a solution $V(x_1) \geq 0$ to (5.60). Consider the systems Σ and Σ_P. The following statements are equivalent:*

(i) *the state feedback L_2-gain control problem with gain γ is solvable for Σ with a differentiable storage function $W + V$ where $W \geq 0$;*

(ii) *the state feedback L_2-gain control problem with gain γ is solvable for Σ_P with a differentiable storage function $W \geq 0$.*

□

Remark 5.38 In special cases such as described in the paper [DaMa 96] it is not necessary to consider the Hamilton-Jacobi equality (5.60), but also solutions to the Hamilton-Jacobi inequality yield similar results. □

In the linear case the conditions *(ii)* from Theorem 5.6 imply that the almost disturbance decoupling problem for Σ_P is solvable.

The output equation h_p is related to the tracking error between the solution of the regular $\mathcal{H}_\infty$ problem for the subsystem Σ_1 as defined in Subsection 5.2.2 and the actual x_3. ¿From the uniqueness of the saddle point solutions it follows that locally around $x_1 = 0$ we have that

$$h_p(x_1, x_3, u_2^*(x_1, V_{x_1}^T(x_1)) = 0 \Longleftrightarrow x_3 = x_3^*(x_1, V_{x_1}^T(x_1)).$$

So this factorization idea is closely related to the approach described in Subsection 5.2.4.

Chapter 6

Examples

In this chapter the two different methods for solving the singular $\mathcal{H}_\infty$ control problem which are discussed in this monograph are applied to two examples.

The first example is concerned with a tracking problem for a rigid body spinning around its center of mass. The rigid body model that is used includes three control torques and three torque disturbances. The model is rewritten using Euler-parameters (see [WeDe 91]). This example is mainly meant to clarify and illustrate the theory. Therefore no comparison is made between the performances of the two different controllers resulting from the cheap control and from the geometric approach. We remark that in [MD 95] the geometric approach to the singular $\mathcal{H}_\infty$ problem has been extended to the more general case in which also the movement of the center of mass is taken into account.

As a second example the inverted pendulum on a cart is considered.

6.1 Tracking of the orientation of a rigid body

The orientation of the rigid body with respect to the inertial frame (I-frame) is represented by $R \in SO(3)$. The frame fixed to the rigid body is called the B-frame. So R denote the orthonormal rotation matrix from the I-frame to the B-frame. $SO(3)$ is the Special Orthonormal Group of order 3 which is represented by the set of all 3×3 orthogonal rotation matrices. We will study the orientation of a rigid body with kinematic differential equations given by

$$\dot{R} = RS(\omega) \tag{6.1}$$

where ω is the angular velocity, in B-frame coordinates. The skew symmetric matrix $S(\omega)$ is given by

$$S(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

for $\omega^T = (\omega_1 \ \ \omega_2 \ \ \omega_3)^T$. The equations of motion of the rigid body are then given in the B-frame as

$$M\dot{\omega} = -S(\omega)M\omega + u + d \tag{6.2}$$

where $M = M^T > 0$ is the 3×3 inertia matrix of the rigid body in B-frame coordinates, $u = (\ u_1 \ \ u_2 \ \ u_3\ )^T$ are the control inputs and the torque disturbances are given by $d = (\ d_1 \ \ d_2 \ \ d_3\ )^T$.

The reference orientation in $SO(3)$ is given by the desired frame, the D-frame, represented by R_d. $R_d \in SO(3)$ is the orthogonal rotation matrix from the I-frame to the D-frame. The desired angular velocity, ${}^D\omega_d$, in D-frame coordinates is given by

$$S({}^D\omega_d) = R_d^T \dot{R}_d$$

or equivalently

$$\dot{R}_d = R_d S({}^D\omega_d). \tag{6.3}$$

We define

$$\tilde{R} := R_d^T R. \tag{6.4}$$

$\tilde{R}$ is the rotation matrix from the D-frame to the B-frame. The desired angular velocities in B-frame coordinates are then given by

$$\omega_d = \tilde{R}^T\, {}^D\omega_d.$$

We define $\tilde{\omega}$

$$\tilde{\omega} := \omega - \omega_d.$$

Differentiation of equation (6.4) gives

$$\begin{aligned} \dot{\tilde{R}} &= \dot{R}_d^T R + R_d^T \dot{R} \\ &= -S({}^D\omega_d)\tilde{R} + \tilde{R}S(\omega) \\ &= -\tilde{R}\tilde{R}^T S({}^D\omega_d)\tilde{R} + \tilde{R}S(\omega) \\ &= -\tilde{R}S(\omega_d) + \tilde{R}S(\omega) \\ &= \tilde{R}S(\tilde{\omega}) \end{aligned}$$

since $\tilde{R}^T S({}^D\omega_d)\tilde{R} = S(\omega_d)$. The error kinematics are therefore given by

$$\dot{\tilde{R}} = \tilde{R}S(\tilde{\omega}). \tag{6.5}$$

Our main goal is to keep $(\tilde{R}, \tilde{\omega}) = (I, 0)$ in the presence of the torque disturbances d. To this end we define the penalty variable

$$\begin{pmatrix} d_{SO(3)}(\tilde{R}, I) & \tilde{\omega}^T \end{pmatrix}^T \tag{6.6}$$

where $d_{SO(3)}(\tilde{R}, I)$ (see [SBE 91]) is the geodesic metric on $SO(3)$ which is given by:

$$d_{SO(3)}(\tilde{R}, I) = \frac{2}{\pi} \arccos\left(\frac{1}{2}\sqrt{1 + \mathrm{Tr}(\tilde{R})}\right) \in [0, 1]$$

for $\tilde{R} \in SO(3)$.

In the control law to be found it is desirable to have an attitude deviation vector of dimension three. To this end we use Euler parameters which are also known as unit quaternions. The rotation matrix $\tilde{R} \in SO(3)$ is parameterized by a rotation $\tilde{\phi}$ around the unit vector $\tilde{k}$ so that

$$\tilde{R} = I + \sin(\tilde{\phi})S(\tilde{k}) + (1 - \cos(\tilde{\phi}))S^2(\tilde{k}).$$

It is seen that $(\tilde{\phi}, \tilde{k})$ and $(-\tilde{\phi}, -\tilde{k})$ correspond to the same rotation matrix $\tilde{R}$. The Euler parameters corresponding to $\tilde{R}$ are given by [Hug 86]

$$\tilde{\epsilon} = \sin(\frac{\tilde{\phi}}{2})\tilde{k}, \quad \tilde{\eta} = \cos(\frac{\tilde{\phi}}{2}).$$

The Euler parameters satisfy the normalization equation

$$\tilde{\epsilon}^T\tilde{\epsilon} + \tilde{\eta}^2 = 1$$

which means that $(\tilde{\epsilon}, \tilde{\eta}) \in S^3$, where S^3 is the unit sphere in $\mathbb{R}^4$. It follows that

$$\tilde{R} = (\tilde{\eta}^2 - \tilde{\epsilon}^T\tilde{\epsilon})I + \tilde{\epsilon}\tilde{\epsilon}^T + 2\tilde{\eta}S(\tilde{\epsilon})$$

and it is seen that $(\tilde{\epsilon}, \tilde{\eta})$ and $(-\tilde{\epsilon}, -\tilde{\eta})$ correspond to the same rotation matrix $\tilde{R}$. Note that $\tilde{\epsilon} = 0 \Leftrightarrow \tilde{R} = I$. Also, $\tilde{\epsilon} = 0 \Leftrightarrow \tilde{\eta} \pm 1$ so that both $\tilde{\epsilon} = 0, \tilde{\eta} = 1$ and $\tilde{\epsilon} = 0, \tilde{\eta} = -1$ correspond to $\tilde{R} = I$.

The kinematical differential equations associated with

$$\dot{\tilde{R}} = \tilde{R}S(\tilde{\omega})$$

are given by a vector field on S^3 defined by

$$\begin{pmatrix} \dot{\tilde{\epsilon}} \\ \dot{\tilde{\eta}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tilde{\eta} I + S(\tilde{\epsilon}) \\ -\tilde{\epsilon}^T \end{pmatrix} \tilde{\omega}.$$

Let the unit quaternions $(\tilde{\epsilon}, \tilde{\eta})$ represent $\tilde{R}$. The error kinematics corresponding to the equations (6.5) are then given by

$$\begin{aligned} \dot{\tilde{\epsilon}} &= \tfrac{1}{2}(\tilde{\eta} I + S(\tilde{\epsilon}))\tilde{\omega} \\ \dot{\tilde{\eta}} &= -\tfrac{1}{2}\tilde{\epsilon}^T\tilde{\omega} \end{aligned} \tag{6.7}$$

The penalty variable defined in (6.6) can be written in Euler-parameters as

$$\begin{pmatrix} \tfrac{2}{\pi}\arccos|\tilde{\eta}| & \tilde{\omega}^T \end{pmatrix}^T$$

But we want this penalty variable to be a smooth, at least continuously differentiable, function. Therefore we use the fact that

$$\frac{2}{\pi}\|\tilde{\epsilon}\| \leq d_{SO(3)}(\tilde{R}, I) \leq \|\tilde{\epsilon}\|$$

to define as new penalty variable

$$z = \begin{pmatrix} \tilde{\epsilon}^T & \tilde{\omega}^T \end{pmatrix}^T. \tag{6.8}$$

6.1.1 Cheap control approach

In the previous section we have constructed the following model for the rigid body in Euler coordinates

$$\begin{aligned} \dot{\tilde{\epsilon}} &= \frac{1}{2}(\tilde{\eta} I + S(\tilde{\epsilon}))\,\tilde{\omega} \\ \dot{\tilde{\eta}} &= -\frac{1}{2}\tilde{\epsilon}^T\tilde{\omega} \\ \dot{\tilde{\omega}} &= -M^{-1}S(\tilde{\omega})M\,(\tilde{\omega}+\omega_d) - M^{-1}S(\omega_d)M\tilde{\omega} + M^{-1}v + M^{-1}d \\ z &= \begin{pmatrix} \tilde{\epsilon} \\ \tilde{\omega} \end{pmatrix} \end{aligned} \tag{6.9}$$

where

$$u = v + M\dot{\omega}_d + S(\omega_d)M\omega_d. \tag{6.10}$$

The model (6.9) will also be denoted by

$$\begin{aligned} \dot{x} &= f(x, \omega_d) + gv + gd \\ z &= h(x) \end{aligned} \tag{6.11}$$

where $x = \begin{pmatrix} \tilde{\epsilon}^T & \tilde{\eta}^T & \tilde{\omega}^T \end{pmatrix}^T$ is the state and f, g and h are given by

$$f(x, \omega_d) = \begin{pmatrix} \frac{1}{2}(\tilde{\eta} I + S(\tilde{\epsilon}))\tilde{\omega} \\ -\frac{1}{2}\tilde{\epsilon}^T\tilde{\omega} \\ -M^{-1}S(\tilde{\omega})M(\tilde{\omega} + \omega_d) - M^{-1}S(\omega_d)M\tilde{\omega} \end{pmatrix},$$

$$g = \begin{pmatrix} 0 \\ 0 \\ M^{-1} \end{pmatrix}, \qquad h(x) = \begin{pmatrix} \tilde{\epsilon} \\ \tilde{\omega} \end{pmatrix}.$$

It should be noted that ω_d, the desired angular velocity, is time-varying. The direct feedthrough term from the inputs $v \in \mathbb{R}^3$ to the to-be-controlled outputs $z \in \mathbb{R}^6$ is zero. Therefore Assumption 2 in Section 3.2 is clearly satisfied and $m_1 = 0$. The (3×3)-matrix $\beta_2(x)$ can be chosen equal to the identity matrix I_3 and the matrix $\beta_1(x)$ is void. This means that v is a singular input and the transformed system (3.8) is equal to the original system (6.9).

Our first aim is to find for some $\gamma, \varepsilon > 0$ a solution $V \geq 0$ of the Hamilton-Jacobi inequality (3.13) which for this system is given by

$$\begin{aligned} & H_\gamma(x, V_x^T(x)) \qquad (6.12) \\ = \; & V_x(x) f(x, \omega_d) + \frac{1}{2}\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right) V_x(x) g g^T V_x^T(x) + \frac{1}{2} h^T(x) h(x) \\ = \; & L_f V(x) + \frac{1}{2}\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right)\left(L_g V(x)\right)^2 + \frac{1}{2} h^T(x) h(x) \; \leq \; 0 \end{aligned}$$

with as equilibrium condition $V(0, 1, 0) = 0$.

Motivated by the results in ([DaEg 95]) we try the following storage function

$$V(\tilde{\epsilon}, \tilde{\eta}, \tilde{\omega}) = aT(\tilde{\omega}) + bG(\tilde{\epsilon}, \tilde{\omega}) + cP(\tilde{\eta}) \qquad (6.13)$$

where $T(\tilde{\omega})$ is the kinetic energy, $G(\tilde{\epsilon}, \tilde{\omega})$ is a cross term and $P(\tilde{\eta})$ is a non-negative function respectively given by:

$$\begin{aligned} T(\tilde{\omega}) &= \frac{1}{2}\tilde{\omega}^T M \tilde{\omega}, \\ G(\tilde{\epsilon}, \tilde{\omega}) &= \tilde{\omega}^T M \tilde{\epsilon}, \\ P(\tilde{\eta}) &= 2(1 - \tilde{\eta}). \end{aligned}$$

Since $|\tilde{\eta}| \leq 1$ by definition, $P(\tilde{\eta}) \geq 0$ and $P(\tilde{\eta}) = 0$ iff $|\tilde{\eta}| = 1$. So $V(0, 1, 0) = 0$ and moreover since

$$\begin{aligned} P(\tilde{\eta}) &= 2(1 - \tilde{\eta}) \\ &= (1 - \tilde{\eta})^2 + \tilde{\epsilon}^T \tilde{\epsilon} \\ &\geq \tilde{\epsilon}^T \tilde{\epsilon} \end{aligned}$$

we have that

$$\begin{aligned} V(\tilde{\epsilon}, \tilde{\eta}, \tilde{\omega}) &= \frac{1}{2} a\tilde{\omega}^T M\tilde{\omega} + b\tilde{\omega}^T M\tilde{\epsilon} + 2c\,(1-\tilde{\eta}) \\ &\geq \frac{1}{2} a\tilde{\omega}^T M\tilde{\omega} + b\tilde{\omega}^T M\tilde{\epsilon} + c\tilde{\epsilon}^T\tilde{\epsilon} \\ &= \frac{1}{2} y^T Q y \end{aligned}$$

where

$$y = \begin{pmatrix} \tilde{\epsilon} \\ \tilde{\omega} \end{pmatrix}, \qquad Q = \begin{pmatrix} 2cI & bM \\ bM & aM \end{pmatrix}.$$

Since the matrix Q is positive definite iff

$$\begin{aligned} a &> 0, \\ acI &> \tfrac{1}{2} b^2 M \end{aligned} \tag{6.14}$$

these conditions (6.14) are sufficient for the function V to be non-negative. Straightforward calculations lead to

$$\begin{aligned} L_f V(\tilde{\epsilon}, \tilde{\eta}, \tilde{\omega}) &= -a\tilde{\omega}^T S(\omega_d) M\tilde{\omega} + \frac{1}{2} b\tilde{\omega}^T M\,(\tilde{\eta} I + S(\tilde{\epsilon}))\,\tilde{\omega} \\ &\quad -b\tilde{\epsilon}^T S(\tilde{\omega}) M\,(\tilde{\omega} + \omega_d) - b\tilde{\epsilon}^T S(\omega_d) M\tilde{\omega} + c\tilde{\epsilon}^T\tilde{\omega} \\ L_g V(\tilde{\epsilon}, \tilde{\eta}, \tilde{\omega}) &= a\tilde{\omega}^T + b\tilde{\epsilon} \end{aligned}$$

where we have used that $\omega_d^T S(\omega_d) = 0$. Inserting these inequalities in the Hamiltonian $H_\gamma(x, V_x^T(x))$ gives

$$\begin{aligned} H_\gamma(x, V_x^T(x)) &= -a\tilde{\omega}^T S(\omega_d) M\tilde{\omega} + \frac{1}{2} b\tilde{\omega}^T M\,(\tilde{\eta} I + S(\tilde{\epsilon}))\,\tilde{\omega} \\ &\quad -b\tilde{\epsilon}^T S(\tilde{\omega}) M\,(\tilde{\omega} + \omega_d) - b\tilde{\epsilon}^T S(\omega_d) M\tilde{\omega} + c\tilde{\epsilon}^T\tilde{\omega} \\ &\quad +\frac{1}{2} a^2 \left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right) \|\tilde{\omega}\|^2 + \frac{1}{2} b^2 \left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right) \|\tilde{\epsilon}\|^2 \\ &\quad +ab\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right) \tilde{\epsilon}^T\tilde{\omega} \\ &\quad +\frac{1}{2} \|\tilde{\omega}\| + \frac{1}{2} \|\tilde{\epsilon}\|\,. \end{aligned} \tag{6.15}$$

By choosing

$$c = ab\left(\frac{1}{\varepsilon} - \frac{1}{\gamma^2}\right) \tag{6.16}$$

the two terms in H_γ which include the inner product $\tilde{\epsilon}^T\tilde{\omega}$ are eliminated. We assume that the desired angular velocity is bounded, i.e., $\|\omega_d\| \le \rho_d$ for a certain constant ρ_d. Then using Schwarz inequality, and the equalities $\|S(\tilde{\omega})\| = \|\tilde{\omega}\|$, $\|\tilde{\epsilon}\| \le 1$ and $\|\tilde{\eta}I + S(\tilde{\epsilon})\| = 1$ we see that:

$$\begin{aligned}
\left|\tilde{\omega}^T S(\omega_d)M\tilde{\omega}\right| &\le \|M\|\,\|\omega_d\|\,\|\tilde{\omega}\|^2 \le \|M\|\,\rho_d\,\|\tilde{\omega}\|^2\,;\\
\left|\tilde{\omega}^T M\,(\tilde{\eta}I + S(\tilde{\epsilon}))\,\tilde{\omega}\right| &\le \|M\|\,\|\tilde{\omega}\|^2\,;\\
\left|\tilde{\epsilon}^T S(\tilde{\omega})M\,(\tilde{\omega}+\omega_d)\right| &\le \|M\|\,\|\tilde{\omega}\|\,\|\tilde{\omega}+\omega_d\| \le \|M\|\,\|\tilde{\omega}\|^2 + \|M\|\,\rho_d\,\|\tilde{\omega}\|\,;\\
\left|\tilde{\epsilon}^T S(\omega_d)M\tilde{\omega}\right| &\le \|M\|\,\rho_d\,\|\tilde{\omega}\|\,.
\end{aligned}$$

Inserting these inequalities and the choice for the constant c into the Hamiltonian H_γ one obtains

$$\begin{aligned}
H_\gamma(x, V_x^T(x)) &\le a\,\|M\|\,\rho_d\,\|\tilde{\omega}\|^2 + \frac{3}{2}b\,\|M\|\,\|\tilde{\omega}\|^2 + 2b\,\|M\|\,\rho_d\,\|\tilde{\omega}\|\\
&\quad + \frac{1}{2}a^2\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right)\|\tilde{\omega}\|^2 + \frac{1}{2}b^2\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right)\|\tilde{\epsilon}\|^2\\
&= \left(a\,\|M\|\,\rho_d + \frac{3}{2}b\,\|M\| + \frac{1}{2}a^2\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right) + \frac{1}{2}\right)\|\tilde{\omega}\|^2\\
&\quad + 2b\,\|M\|\,\rho_d\,\|\tilde{\omega}\|\\
&\quad + \left(\frac{1}{2}b^2\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right) + \frac{1}{2}\right)\|\tilde{\epsilon}\|^2\,.
\end{aligned}$$

The terms including γ and ε are the only terms that can be made negative by choosing $\varepsilon < \gamma^2$. So following the lines of the regulation problem considered in the paper [DaEg 95] does not lead to global results for the considered tracking problem unless the desired angular velocity is equal to zero for all t ($\rho_d = 0$). However it is possible to derive a so called semi-global result since for every constant $N > 0$

$$\|\tilde{\omega}\|^2 \le N\,\|\tilde{\omega}\| \tag{6.17}$$

holds for all $\|\tilde{\omega}\| \le N$. Using the inequality (6.17) the Hamiltonian H_γ has the following upper bound on the neighborhood $\|\tilde{\omega}\| \le N$

$$\begin{aligned}
H_\gamma(x, V_x^T(x)) &\le \Bigg(aN\,\|M\|\,\rho_d + \frac{3}{2}bN\,\|M\| + 2b\,\|M\|\,\rho_d\\
&\quad + \frac{1}{2}a^2N\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right) + \frac{1}{2}N\Bigg)\|\tilde{\omega}\|\\
&\quad + \left(\frac{1}{2}b^2\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right) + \frac{1}{2}\right)\|\tilde{\epsilon}\|^2\\
&= \delta_{\tilde{\omega}}(a, b, N, \gamma, \varepsilon)\,\|\tilde{\omega}\| + \delta_{\tilde{\epsilon}}(b, \gamma, \varepsilon)\,\|\tilde{\epsilon}\|^2\,.
\end{aligned}$$

We want to choose the constants ε, a and b in such a way that this upper bound is less than or equal to zero, i.e., such that both the functions $\delta_{\tilde{\omega}}(a, b, N, \gamma, \varepsilon)$ and $\delta_{\tilde{\epsilon}}(b, N, \gamma, \varepsilon)$ are less than or equal to zero. The choice of a and b should at the same time satisfy condition (6.14), with c defined in (6.16). First of all we have to choose $\varepsilon < \gamma^2$, otherwise we are not able to make $\delta_{\tilde{\epsilon}} \leq 0$. Then both the functions $\delta_{\tilde{\omega}}(a, b, N, \gamma, \varepsilon)$ and $\delta_{\tilde{\epsilon}}(b, N, \gamma, \varepsilon)$ can be made smaller or equal to zero. When we choose

$$b^2 \geq \frac{1}{\frac{1}{\varepsilon} - \frac{1}{\gamma^2}} \tag{6.18}$$

it follows that $\delta_{\tilde{\epsilon}}(b, \gamma, \varepsilon) \leq 0$. Furthermore the function $\delta_{\tilde{\omega}}(a, b, N, \gamma, \varepsilon)$ can be rewritten as a second order function in a

$$\begin{aligned} \delta_{\tilde{\omega}}(a, b, N, \gamma, \varepsilon) &= \left(\frac{1}{2}N\left(\frac{1}{\gamma^2} - \frac{1}{\varepsilon}\right)\right)a^2 + (\|M\|\,\rho_d N)\,a \\ &\quad + \left(\frac{3}{2}bN\,\|M\| + 2b\,\|M\|\,\rho_d + \frac{1}{2}N\right). \end{aligned} \tag{6.19}$$

This function is for $\varepsilon < \gamma^2$ convex and has two zeros given by

$$\frac{\|M\|\,\rho_d N \pm \sqrt{\|M\|^2\,\rho_d^2 N^2 + 2N\left(\frac{1}{\varepsilon} - \frac{1}{\gamma^2}\right)\left(\frac{3}{2}bN\,\|M\| + 2b\,\|M\|\,\rho_d + \frac{1}{2}N\right)}}{N\left(\frac{1}{\varepsilon} - \frac{1}{\gamma^2}\right)}.$$

Only one of the two zeros is positive. Hence we choose the constant a greater or equal to

$$\frac{\|M\|\,\rho_d N + \sqrt{\|M\|^2\,\rho_d^2 N^2 + 2N\left(\frac{1}{\varepsilon} - \frac{1}{\gamma^2}\right)\left(\frac{3}{2}bN\,\|M\| + 2b\,\|M\|\,\rho_d + \frac{1}{2}N\right)}}{N\left(\frac{1}{\varepsilon} - \frac{1}{\gamma^2}\right)}. \tag{6.20}$$

Note that a satisfies

$$a^2 \geq \frac{3b\,\|M\|}{\frac{1}{\varepsilon} - \frac{1}{\gamma^2}}.$$

Together with c as in (6.16) that leads to

$$ac = a^2 b\left(\frac{1}{\varepsilon} - \frac{1}{\gamma^2}\right) \geq 3b^2\,\|M\|$$

which implies the second condition from (6.14). Hence the constants a and c defined in (6.20) and (6.16) satisfy the conditions (6.14) and so the corresponding V is non-negative.

Summarizing it follows that for every $\gamma, N > 0$ and for every $\varepsilon < \gamma^2$ we have constructed a local solution $V \geq 0$ to the Hamilton-Jacobi inequality (6.15) defined on the neighborhood $\|\tilde{\omega}\| \leq N$. This storage function is given by (6.13) where the constants a, b and c should be chosen according to (6.20), (6.18) and (6.16).

Now the following result can be derived from Corollary 3.13.

Theorem 6.1 *Let $\gamma, N > 0$ and choose ε, b and a such that:*

$$
\begin{aligned}
\varepsilon &< \gamma^2; \\
b &\geq \sqrt{\frac{1}{\frac{1}{\varepsilon} - \frac{1}{\gamma^2}}}; \\
a &\geq \frac{\|M\| \rho_d N + \sqrt{\|M\|^2 \rho_d^2 N^2 + 2N\left(\frac{1}{\varepsilon} - \frac{1}{\gamma^2}\right)\left(\frac{3}{2} bN \|M\| + 2b \|M\| \rho_d + \frac{1}{2}N\right)}}{N\left(\frac{1}{\varepsilon} - \frac{1}{\gamma^2}\right)}.
\end{aligned}
$$

Then the error model (6.9) combined with the state feedback

$$v = -\mu \left(a\tilde{\omega} + b\tilde{\epsilon}\right) \tag{6.21}$$

with $\mu \geq \frac{1}{2\varepsilon}$ leads to a closed loop system which has local L_2-gain less than or equal to γ on the neighborhood $\|\tilde{\omega}\| \leq N$ and which is locally asymptotically stable. □

Proof The local L_2-gain result follows from Corollary 3.13 using the storage function constructed previously. Furthermore we see that $z \equiv 0$ implies that $(\tilde{\epsilon}, \tilde{\omega}) = (0, 0)$. From the normalization equation it follows that $\tilde{\eta} \pm 1$. If we define $(\tilde{\epsilon}, \tilde{\omega}, \tilde{\eta}) = (0, 1, 0)$ as our zero-state, the closed loop system

$$
\begin{aligned}
\begin{pmatrix} \dot{\tilde{\epsilon}} \\ \dot{\tilde{\eta}} \\ \dot{\tilde{\omega}} \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}\left(\tilde{\eta} I + S(\tilde{\epsilon})\right)\tilde{\omega} \\ -\frac{1}{2}\tilde{\epsilon}^T \tilde{\omega} \\ -M^{-1}S(\tilde{\omega})M\left(\tilde{\omega}\omega_d\right) - M^{-1}S(\omega_d)M\tilde{\omega} - \mu M^{-1}\left(a\tilde{\omega} + b\tilde{\epsilon}\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ 0 \\ M^{-1} \end{pmatrix} d \\
z &= \begin{pmatrix} \tilde{\epsilon} \\ \tilde{\omega} \end{pmatrix}
\end{aligned}
$$

is zero-state observable in a neighborhood of $(\tilde{\epsilon}, \tilde{\eta}, \tilde{\omega}) = (0, 1, 0)$. Then local asymptotic stability follows from Theorem 2.7 or 3.21. ■

For our original system (6.7), (6.2) this result can be reformulated.

Corollary 6.2 *Choose γ, N, ε, a and b in the same way as in Theorem 6.1. Then the $\mathcal{H}_\infty$ problem with state feedback for the rigid spacecraft (6.7), (6.2) with penalty variable (6.8) is solved locally by the state feedback*

$$u = M\dot{\omega}_d + S(\omega_d)M\omega_d - \mu\,(a\tilde{\omega} + b\tilde{\epsilon}) \tag{6.22}$$

for every $\mu \geq \frac{1}{2\varepsilon}$ on the neighborhood $\|\tilde{\omega}\| \leq N$. □

The choice of γ, ε and N clearly influences the lower bound for the constants a and b and therefore the gain of the feedback (6.21). When the neighborhood $\|\tilde{\omega}\| \leq N$ is increased such that the solution of the $\mathcal{H}_\infty$ problem applies on a larger area around the zero-state this results in a higher lower bound for a as can be seen in (6.20). Also a choice of the parameter ε close to γ^2 or to zero will obviously lead to a higher choice for both a and b. Therefore the choice of the constants N, μ and ε depend on the specific model at hand and the desired performance.

6.1.2 Geometric approach

The model for the rigid spacecraft can also be rewritten in the following form, see (6.2) and (6.7)

$$\begin{aligned} \dot{\tilde{\epsilon}} &= \frac{1}{2}(\tilde{\eta}I + S(\tilde{\epsilon}))\,\tilde{\omega} \\ \dot{\tilde{\eta}} &= -\frac{1}{2}\tilde{\epsilon}^T\tilde{\omega} \\ \dot{\tilde{\omega}} &= -M^{-1}\,(S(\tilde{\omega}) + S(\omega_d))\,M\,(\tilde{\omega} + \omega_d) + M^{-1}v + M^{-1}d \\ z &= \begin{pmatrix} \tilde{\epsilon} \\ \tilde{\omega} \end{pmatrix} \end{aligned} \tag{6.23}$$

where

$$u = v + M\dot{\omega}_d. \tag{6.24}$$

Again for the model (6.23) we will also use the shorthand notation

$$\begin{aligned} \dot{x} &= f(x, \omega_d) + gv + gd \\ z &= h(x) \end{aligned} \tag{6.25}$$

where $x = \begin{pmatrix} \tilde{\epsilon}^T & \tilde{\eta}^T & \tilde{\omega}^T \end{pmatrix}^T$ is the state and f, G and h are given by

$$f(x, \omega_d) = \begin{pmatrix} \frac{1}{2}\,(\tilde{\eta}I + S(\tilde{\epsilon}))\,\tilde{\omega} \\ -\frac{1}{2}\tilde{\epsilon}^T\tilde{\omega} \\ -M^{-1}\,(S(\tilde{\omega}) + S(\omega_d))\,M\,(\tilde{\omega} + \omega_d) \end{pmatrix},$$

$$g = \begin{pmatrix} 0 \\ 0 \\ M^{-1} \end{pmatrix}, \qquad h(x) = \begin{pmatrix} \tilde{\epsilon} \\ \tilde{\omega} \end{pmatrix}.$$

Because the inputs $v \in \mathbb{R}^3$ do not directly influence the to-be-controlled outputs z Assumption 5 (Subsection 5.2.2) is clearly satisfied, with m_1 equal to zero (see also Subsection 5.2.7). The system (6.23) or (6.25) will be referred to as Σ. Because $m_1 = 0$ there is no need to transform the inputs. For applying the state transformation described in Chapter 5 we construct the extended system Σ_e

$$\Sigma_e \begin{cases} \dot{x}_e &= f_e(x_e, \omega_d) + g_e v + e_e d \\ z &= h_e(x_e) \end{cases}$$

where the 13 dimensional state vector x_e is equal to $\begin{pmatrix} x^T & v^T & d^T \end{pmatrix}^T$ and the functions $f_e(x_e)$, g_e, e_e and $h_e(x_e)$ are given by

$$f_e(x_e, \omega_d) = \begin{pmatrix} f(x, \omega_d) + gv + gd \\ 0 \\ 0 \end{pmatrix}, \qquad g_e = \begin{pmatrix} 0 \\ I_3 \\ 0 \end{pmatrix},$$

$$e_e = \begin{pmatrix} 0 \\ 0 \\ I_3 \end{pmatrix}, \qquad h_e(x_e) = h(x).$$

For this system Σ_e we want to calculate the minimal conditioned invariant distribution containing the input vector fields using the $\mathcal{S}^*$-algorithm.

In the first step of this algorithm we obtain

$$\mathcal{S}_{1e} = \text{span}\left\{ \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3} \right\}$$

whose involutive closure is equal to $\mathcal{S}_{1e}$ itself and is contained in ker $\mathrm{d}h$. Therefore we calculate in the second step $\mathcal{S}_{2e}$ which is given by

$$\begin{aligned} \mathcal{S}_{2e} &= \mathcal{S}_{1e} + [f_e(x_e), \mathcal{S}_{1e}] \\ &= \text{span}\left\{ \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3}, \frac{\partial}{\partial \tilde{\omega}_1}, \frac{\partial}{\partial \tilde{\omega}_2}, \frac{\partial}{\partial \tilde{\omega}_3} \right\} \\ &= \overline{\mathcal{S}}_{2e} \end{aligned}$$

and which clearly satisfies

$$\mathcal{S}_{2e} \cap \ker \mathrm{d}h = \mathcal{S}_{1e}.$$

This means that the algorithm stops after two steps and that the minimal conditioned invariant distribution containing the input vector fields for the system Σ_e is given by

$$\mathcal{S}_e^* = \text{span}\left\{\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3}, \frac{\partial}{\partial \tilde{\omega}_1}, \frac{\partial}{\partial \tilde{\omega}_2}, \frac{\partial}{\partial \tilde{\omega}_3}\right\}.$$

This distribution also satisfies the projectability condition (5.24)

$$\left[\frac{\partial}{\partial d_i}, \mathcal{S}_e^*\right] = 0 \in \mathcal{S}_e^*, \qquad i = 1, 2, 3.$$

The distributions $\mathcal{X}_{1e}$, $\mathcal{X}_{2e}$ and $\mathcal{X}_{3e}$ are then given by:

$$\begin{aligned}
\mathcal{X}_{1e} &= \text{span}\left\{\frac{\partial}{\partial \tilde{\epsilon}_1}, \frac{\partial}{\partial \tilde{\epsilon}_2}, \frac{\partial}{\partial \tilde{\epsilon}_3}, \frac{\partial}{\partial \tilde{\eta}}, \frac{\partial}{\partial d_1}, \frac{\partial}{\partial d_2}, \frac{\partial}{\partial d_3}\right\}; \\
\mathcal{X}_{2e} &= \text{span}\left\{\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3}\right\}; \\
\mathcal{X}_{3e} &= \text{span}\left\{\frac{\partial}{\partial \tilde{\omega}_1}, \frac{\partial}{\partial \tilde{\omega}_2}, \frac{\partial}{\partial \tilde{\omega}_3}\right\}.
\end{aligned}$$

Then the projections of $\mathcal{X}_{1e}$, $\mathcal{X}_{2e}$ and $\mathcal{X}_{3e}$ onto the state space of the original system are:

$$\begin{aligned}
\mathcal{X}_1^P &= \text{span}\left\{\frac{\partial}{\partial \tilde{\epsilon}_1}, \frac{\partial}{\partial \tilde{\epsilon}_2}, \frac{\partial}{\partial \tilde{\epsilon}_3}, \frac{\partial}{\partial \tilde{\eta}}\right\}; \\
\mathcal{X}_2^P &= 0; \\
\mathcal{X}_3^P &= \text{span}\left\{\frac{\partial}{\partial \tilde{\omega}_1}, \frac{\partial}{\partial \tilde{\omega}_2}, \frac{\partial}{\partial \tilde{\omega}_3}\right\}.
\end{aligned}$$

The resulting regular subsystem Σ_1 is

$$\Sigma_1 \left\{ \begin{aligned}
\dot{\tilde{\epsilon}} &= \tfrac{1}{2}\left(\tilde{\eta} I + S(\tilde{\epsilon})\right)\tilde{\omega} \\
\dot{\tilde{\eta}} &= -\tfrac{1}{2}\tilde{\epsilon}^T \tilde{\omega} \\
z &= \begin{pmatrix} \tilde{\epsilon} \\ \tilde{\omega} \end{pmatrix}
\end{aligned} \right.$$

Notice that there are no disturbances entering this subsystem. Therefore the regular $\mathcal{H}_\infty$ problem for the system Σ_1 reduces to an optimal control problem. The cost-criterium for this optimal control problem is

$$\frac{1}{2}\int_0^T \|z(t)\|^2 \mathrm{d}t,$$

which we minimize by considering $\tilde{\omega}$ as inputs to this subsystem.
The pre-Hamiltonian of this problem is given by

$$K(\tilde{\epsilon}, \tilde{\eta}, p_{\tilde{\epsilon}}, p_{\tilde{\eta}}, \tilde{\omega}) = \frac{1}{2} p_{\tilde{\epsilon}}^T (\tilde{\eta} I + S(\tilde{\epsilon}))\, \tilde{\omega} - \frac{1}{2} p_{\tilde{\eta}} \tilde{\epsilon}^T \tilde{\omega} + \frac{1}{2} \tilde{\epsilon}^T \tilde{\epsilon} + \frac{1}{2} \tilde{\omega}^T \tilde{\omega}.$$

Hence the optimal (minimizing) control $\tilde{\omega}^*$ is given by

$$\tilde{\omega}^* = \frac{1}{2} \tilde{\epsilon} p_{\tilde{\eta}} - \frac{1}{2} (\tilde{\eta} I - S(\tilde{\epsilon}))\, p_{\tilde{\epsilon}}. \tag{6.26}$$

Substituting this optimal control $\tilde{\omega}^*$ into the pre-Hamiltonian $K(\tilde{\epsilon}, \tilde{\eta}, p_{\tilde{\epsilon}}, p_{\tilde{\eta}}, \tilde{\omega})$ we see that this optimal control problem can be solved by finding a non-negative solution V to the following Hamilton-Jacobi equality

$$\begin{aligned} -\frac{1}{8} V_{\tilde{\epsilon}}(\tilde{\epsilon}, \tilde{\eta})\, (\tilde{\eta} I + S(\tilde{\epsilon}))\, (\tilde{\eta} I - S(\tilde{\epsilon}))\, V_{\tilde{\epsilon}}^T(\tilde{\epsilon}, \tilde{\eta}) - \frac{1}{8} V_{\tilde{\eta}}^2(\tilde{\xi}, \tilde{\eta}) \tilde{\epsilon}^T \tilde{\epsilon} & \\ + \frac{1}{4} V_{\tilde{\eta}}(\tilde{\epsilon}, \tilde{\eta}) \tilde{\epsilon}^T (\tilde{\eta} I - S(\tilde{\epsilon}))\, V_{\tilde{\epsilon}}^T(\tilde{\epsilon}, \tilde{\eta}) + \frac{1}{2} \tilde{\epsilon}^T \tilde{\epsilon} &= 0. \end{aligned} \tag{6.27}$$

Such a solution is quite easy to give as can be seen in the next lemma.

Lemma 6.3 *The function*

$$V(\tilde{\epsilon}, \tilde{\eta}) = 2\,(1 - \tilde{\eta}) \tag{6.28}$$

is a non-negative solution to (6.27). The corresponding optimal control is given by $\tilde{\omega}^*(\tilde{\epsilon}, \tilde{\eta}) = -\tilde{\epsilon}$. □

Proof By definition of the Euler parameters $|\tilde{\eta}| \leq 1$, and hence V is non-negative. Substitution of V into (6.27) leads to

$$-\frac{1}{8}(-2)^2 \tilde{\epsilon}^T \tilde{\epsilon} + \frac{1}{2} \tilde{\epsilon}^T \tilde{\epsilon} = 0.$$

The optimal control follows by substituting $p_{\tilde{\epsilon}} = V_{\tilde{\epsilon}}^T(\tilde{\epsilon}, \tilde{\eta}) = 0$, $p_{\tilde{\eta}} = V_{\tilde{\eta}}^T(\tilde{\epsilon}, \tilde{\eta})$ into (6.26)

$$\tilde{\omega}^*(\tilde{\epsilon}, \tilde{\eta}) = \frac{1}{2} \tilde{\epsilon}(-2) = -\tilde{\epsilon}.$$

■

Remark 6.4 The system Σ_1 is lossless ([Wi 72]). In general a system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ z &= \begin{pmatrix} h(x) \\ u \end{pmatrix} \end{aligned}$$

with $x \in M$, $u \in \mathbb{R}^m$ and $z \in \mathbb{R}^p$ is called *lossless* if there exists a non-negative function $V : M \to \mathbb{R}$ such that:

$$\begin{aligned} V_x(x)f(x) &= 0; \\ V_x(x)g(x) &= h^T(x). \end{aligned}$$

For a lossless system it is well known ([TaAr 81], [vdS 93]) that a solution to the optimal control problem with cost criterium

$$\int_0^t \|z(\tau)\|^2 \mathrm{d}\tau$$

is given by

$$u = -h(x).$$

□

Now we want to look at the $\mathcal{H}_\infty$ problem for the complete system (6.23). Because V is a solution to the Hamilton-Jacobi equality we can use the results from Subsection 5.2.7. Otherwise stated V is a solution to the nonlinear matrix inequality which for the system (6.23) reads as

$$2V_x(x)\left(f(x,\omega_d) + gv\right) + \frac{1}{\gamma^2} V_x(x) g g^T V_x^T(x) + h^T(x)h(x) \geq 0.$$

This nonlinear dissipation inequality is used to define a new output function h_p. First we note that $V_x(x)g \equiv 0$. Then we consider the equation

$$\begin{aligned} h_p^T(x)h_p(x) &= 2V_x(x)f(x,\omega_d) + h^T(x)h(x) \\ &= 2\tilde{\epsilon}^T\tilde{\omega} + \tilde{\epsilon}^T\tilde{\epsilon} + \tilde{\omega}^T\tilde{\omega} \\ &= (\tilde{\omega} + \tilde{\epsilon})^T(\tilde{\omega} + \tilde{\epsilon}). \end{aligned}$$

A smooth solution $h_p(x)$ is given by

$$h_p(x) = \tilde{\omega} + \tilde{\epsilon}.$$

This new output function leads to a new system Σ_P which in this case since the disturbances are absent, is equal to Σ except for the new output

$$z_p = h_p(x) = \tilde{\omega} + \tilde{\epsilon}. \tag{6.29}$$

Now the following result can be derived from Theorem 5.35.

Corollary 6.5 *Suppose the L_2-gain problem for the system Σ is solvable with L_2-gain less than or equal to γ in the sense that there exists a non-negative solution W to the dissipation inequality*

$$W_x(x)\left(f(x,\omega_d) + gv + gd\right) + \frac{1}{2}z_p^T z_p - \frac{1}{2}\gamma^2 d^T d \leq 0.$$

Then the L_2-gain problem for the system Σ_P is solvable with L_2-gain less than or equal to γ. □

Finally it is sufficient to prove that the $\mathcal{H}_\infty$ problem for the system Σ_P is solvable. For the system Σ_P the $\mathcal{H}_\infty$ almost disturbance decoupling problem is solvable. This can be proved by verifying the conditions from Theorem 4.7 for the system

$$\Sigma_P \left\{ \begin{array}{rcl} \dot{\tilde{\epsilon}} & = & \frac{1}{2}\left(\tilde{\eta}I + S(\tilde{\epsilon})\right)\tilde{\omega} \\ \dot{\tilde{\eta}} & = & -\frac{1}{2}\tilde{\epsilon}^T\tilde{\omega} \\ \dot{\tilde{\omega}} & = & -M^{-1}\left(S(\tilde{\omega}) + S(\omega_d)\right)M\left(\tilde{\omega}+\omega_d\right) + M^{-1}v + M^{-1}d \\ z_p & = & \tilde{\omega} + \tilde{\epsilon} \end{array} \right.$$

We have that $m = p = 3$, the strong control characteristic indices ρ_1, ρ_2, ρ_3 are all equal to 1 and the decoupling matrix $A(x)$ is equal to the positive definite, symmetric matrix M^{-1}. Furthermore the vector fields

$$\begin{pmatrix} \frac{1}{2}\left(\tilde{\eta}I + S(\tilde{\epsilon})\right)\tilde{\omega} \\ -\frac{1}{2}\tilde{\epsilon}^T\tilde{\omega} \\ \frac{1}{2}\left(\tilde{\eta}I + S(\tilde{\epsilon})\right)\tilde{\omega} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ I_3 \end{pmatrix}$$

are complete and the set $\mathcal{G}_0$ is equal to

$$\operatorname{span}\left\{\frac{\partial}{\partial\tilde{\omega}_1}, \frac{\partial}{\partial\tilde{\omega}_2}, \frac{\partial}{\partial\tilde{\omega}_3}\right\}$$

which is clearly involutive and of dimension 3. The last condition is void. So the L_2-gain almost disturbance decoupling problem for the system Σ_P is solvable. An actual feedback can be constructed following the proof of Theorem 4.5

and the SISO results derived in [MRST 94] and [MT 95]. Indeed, because the decoupling matrix is non-singular we can input-output decouple the system (see [Is 89], [NvdS 90]) by the feedback

$$\begin{aligned} v &= -M\begin{pmatrix} L_f h_1(x) \\ L_f h_2(x) \\ L_f h_3(x) \end{pmatrix} + M\tilde{v} \\ &= -M\left(\dot{\tilde{\omega}} + \dot{\tilde{\epsilon}}\right) + M\tilde{v} \qquad (6.30) \\ &= \left(S(\tilde{\omega}) + S(\omega_d)\right) M\left(\tilde{\omega} + \omega_d\right) - \frac{1}{2}M\left(\tilde{\eta}I + S(\tilde{\epsilon})\right)\tilde{\omega} + M\tilde{v}. \end{aligned}$$

This feedback together with a transformation of the state components $\tilde{\omega}$ into

$$q = \tilde{\omega} + \tilde{\epsilon}$$

leads to the system

$$\begin{aligned} \dot{\tilde{\epsilon}} &= \tfrac{1}{2}\left(\tilde{\eta}I + S(\tilde{\epsilon})\right)\tilde{\omega} \\ \dot{\tilde{\eta}} &= -\tfrac{1}{2}\tilde{\epsilon}^T\tilde{\omega} \\ \dot{q} &= \tilde{v} + M^{-1}d \\ z_p &= q \end{aligned} \qquad (6.31)$$

and we take

$$\tilde{v}_i = -q_i - \frac{1}{4}kq_i\left(\left(M_{i*}^{-1}\right)\left(M_{i*}^{-1}\right)^T\right) \qquad i = 1,\ldots,3 \qquad (6.32)$$

where M_{i*}^{-1} indicates the i-th row of the matrix M^{-1}. Now define

$$W(q) = \frac{1}{4}q^T q.$$

Then from the following completion of the squares argument (see [MT 95])

$$\begin{aligned} \dot{W} &= \sum_{i=1}^{3}\left(-\frac{1}{2}q_i^2 - \frac{1}{2}\frac{1}{4}kq_i^2\left(\left(M_{i*}^{-1}\right)\left(M_{i*}^{-1}\right)^T\right) + \frac{1}{2}q_i\left(M_{i*}^{-1}\right)d\right. \\ &\qquad \left. -\frac{1}{2}\frac{1}{k}d^T d + \frac{1}{2}\frac{1}{k}d^T d\right) \\ &= \frac{1}{2}\sum_{i=1}^{3}\left(-q_i^2 - k\left\|\frac{1}{2}q_i\left(M_{i*}^{-1}\right)^T - \frac{1}{k}d\right\|^2 + \frac{1}{k}d^T d\right) \\ &\le \sum_{i=1}^{3}\left(-\frac{1}{2}q_i^2 + \frac{1}{2}\frac{1}{k}d^T d\right) \\ &= -\frac{1}{2}q^T q + \frac{1}{2}\frac{3}{k}d^T d \end{aligned}$$

it follows that the feedback (6.32) applied to the system Σ_P leads to a closed loop system which has L_2-gain less than or equal to $\sqrt{\frac{3}{k}}$. Since k can be chosen arbitrarily large the feedback

$$\begin{aligned} u &= M\dot{\omega}_d + (S(\tilde{\omega}) + S(\omega_d))\, M\,(\tilde{\omega} + \omega_d) - \frac{1}{2} M\,(\tilde{\eta} I + S(\tilde{\epsilon}))\,\tilde{\omega} \\ &\quad - Mq - \frac{1}{4} k M q N_M \end{aligned}$$

where

$$N_M = \operatorname{diag}\left\{ \left(M_{1*}^{-1}\right)\left(M_{1*}^{-1}\right)^T, \ldots, \left(M_{3*}^{-1}\right)\left(M_{3*}^{-1}\right)^T \right\}$$

solves the $\mathcal{H}_\infty$ almost disturbance decoupling problem for the system (6.23) with L_2-gain $\sqrt{\frac{3}{k}}$, where k can be chosen arbitrary large. Summarizing:

Theorem 6.6 *Let γ be an arbitrary positive constant. Then the feedback*

$$\begin{aligned} u &= M\dot{\omega}_d + (S(\tilde{\omega}) + S(\omega_d))\, M\,(\tilde{\omega} + \omega_d) - \frac{1}{2} M\,(\tilde{\eta} I + S(\tilde{\epsilon}))\,\tilde{\omega} \\ &\quad - Mq - \frac{3}{4}\frac{1}{\gamma^2} M q N_M \end{aligned} \qquad (6.33)$$

applied to the system (6.23) leads to a closed loop system which has L_2-gain less than or equal to γ and is locally asymptotically stable. □

Proof Take as a storage function

$$V(x) + W(x) = 2\,(1 - \tilde{\eta}) + \frac{1}{4}\,(\tilde{\omega} + \tilde{\epsilon})^T\,(\tilde{\omega} + \tilde{\epsilon})$$

which is seen to be a non-negative solution to the Hamilton-Jacobi inequality

$$P_x(x)\,(f(x,\omega_d) + gu) + \frac{1}{2}\frac{1}{\gamma^2} P_x(x) g g^T P_x(x) + \frac{1}{2} z^T z \leq 0.$$

This storage function is even positive definite outside the origin because all separate terms are non-negative and when $\tilde{\epsilon} = -\tilde{\omega}$ it follows that $\tilde{\eta}^2 = 1 - \tilde{\omega}^T\tilde{\omega} \neq 1$ and hence the first term is positive. Therefore the closed loop system is locally asymptotically stable. ■

To solve the singular $\mathcal{H}_\infty$ problem we have followed the factorization approach as described in Subsection 5.2.7. Similarly we could use the results from Subsection 5.2.4, Theorem 5.21. These results also apply when we have a solution to the Hamilton-Jacobi inequality corresponding to the regular $\mathcal{H}_\infty$ problem for the subsystem Σ_1 instead of a solution to the equality.

6.2 Inverted pendulum on a cart

As a second example we shall consider the physical example of an inverted pendulum on a cart (see also [St 92]). We assume that the mass m of the pendulum is concentrated in the top. l is the length of the pendulum and M the mass of the cart. To describe the position, q and φ express the distance of the cart from some reference point, respectively the angle of the pendulum with respect to the vertical axis. The only input is the horizontal force u applied to the cart. We assume that the system is stiff and that the friction between the cart and the ground is linear with friction coefficient F. Finally g denotes the accelaration of gravity. We derive the following nonlinear model for this system:

$$(M+m)\,\ddot{q} + ml\ddot{\varphi}\cos\varphi - ml\,(\dot{\varphi})^2\sin\varphi + F\dot{q} = u; \quad (6.34)$$

$$l\ddot{\varphi} - g\sin\varphi + \ddot{q}\cos\varphi = 0. \quad (6.35)$$

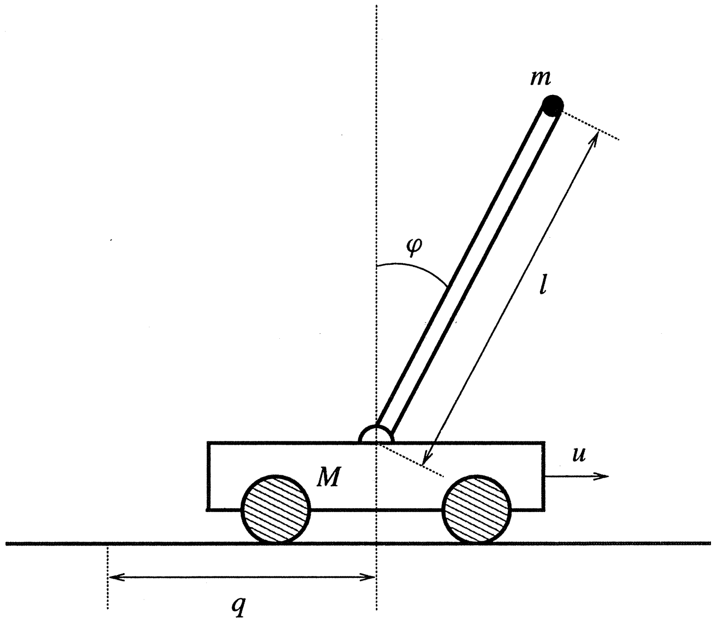

Figure 6.1: inverted pendulum on a cart

Furthermore we assume that $l \neq 0$ and $M \neq 0$. Then the equations (6.34) and (6.35) can be rewritten to the following equations:

$$\ddot{q} = -\frac{mg\sin\varphi\cos\varphi}{M+m\sin^2\varphi} + \frac{ml(\dot{\varphi})^2\sin\varphi}{M+m\sin^2\varphi} - \frac{F\dot{q}}{M+m\sin^2\varphi}$$

$$+\frac{1}{M+m\sin^2\varphi}u; \tag{6.36}$$

$$\begin{aligned}\ddot{\varphi} &= \frac{g\sin\varphi}{l}+\frac{mg\sin\varphi\cos^2\varphi}{l(M+m\sin^2\varphi)}-\frac{m(\dot{\varphi})^2\sin\varphi\cos\varphi}{M+m\sin^2\varphi}\\ &\quad +\frac{F\dot{q}\cos\varphi}{l(M+m\sin^2\varphi)}-\frac{\cos\varphi}{l(M+m\sin^2\varphi)}u.\end{aligned} \tag{6.37}$$

We define the state $x = (\ q \quad \dot{q} \quad \varphi \quad \dot{\varphi}\ )^T$. Then the system is given by

$$\dot{x} = f(x) + g(x)u \tag{6.38}$$

where

$$f(x) = \begin{pmatrix} \dot{q} \\ -\frac{mg\sin\varphi\cos\varphi}{M+m\sin^2\varphi}+\frac{ml(\dot{\varphi})^2\sin\varphi}{M+m\sin^2\varphi}-\frac{F\dot{q}}{M+m\sin^2\varphi} \\ \dot{\varphi} \\ \frac{g\sin\varphi}{l}+\frac{mg\sin\varphi\cos^2\varphi}{l(M+m\sin^2\varphi)}-\frac{m(\dot{\varphi})^2\sin\varphi\cos\varphi}{M+m\sin^2\varphi}+\frac{F\dot{q}\cos\varphi}{l(M+m\sin^2\varphi)} \end{pmatrix},$$

$$g(x) = \begin{pmatrix} 0 \\ \frac{1}{M+m\sin^2\varphi} \\ 0 \\ -\frac{\cos\varphi}{l(M+m\sin^2\varphi)} \end{pmatrix}.$$

An important question is which kind of disturbances we would like to guard against. Different kinds of disturbances will lead to different uncertainty models. We decided to guard against all fluctuations in the two differential equations (6.36) and (6.37), because we want to consider parameter uncertainties in the parameters m, F and M. Of course, this is only one of the options. We penalize the state by choosing the to-be-controlled variable z equal to the state x. Hence the perturbed model we will consider is

$$\Sigma\begin{cases} \dot{x} = f(x) + g(x)u + e(x)d \\ z = x \end{cases}$$

where the matrix $e(x)$ is a constant matrix given by

$$e(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and $d = \begin{pmatrix} d_1 & d_2 \end{pmatrix}^T$ represents the two disturbances acting on the two differential equations (6.36) and (6.37).

We will take as nominal values for the mass of the pendulum $m = 0.1\,(\mathrm{kg})$, the mass of the cart $M = 1\,(\mathrm{kg})$, the length of the pendulum $l = 1\,(\mathrm{m})$ and the friction coefficient $F = 0.1\,(\mathrm{Ns/m})$. In the next subsection we will describe how the cheap control approach can be used to find a solution to this state feedback $\mathcal{H}_\infty$ problem. Especially the choice of the parameter ε will be considered. In the second subsection we will show how the geometric approach can be applied to this problem. Finally in the Subsection 6.2.3 the feedbacks calculated using the two different approaches will be compared with respect to closed-loop asymptotic stability and parameter robustness in the parameters m, F and M for two different values of the L_2-gain γ. This example is mainly meant to illustrate the methods and results from Chapter 3 and 5, and to show the issues to be considered. Therefore no a priori design goals have been formulated.

6.2.1 Cheap control approach

We apply the theory of Chapter 3 to the system Σ. Solving the singular $\mathcal{H}_\infty$ comes down to finding a constant $\varepsilon > 0$ for which there exists a solution $V \geq 0$ to the parameterized Hamilton-Jacobi equality

$$V_x(x)f(x) + \frac{1}{2}V_x(x)\left(\frac{1}{\gamma^2}e(x)e^T(x) - \frac{1}{\varepsilon}g(x)g^T(x)\right)V_x^T(x) + \frac{1}{2}x^Tx = 0. \tag{6.39}$$

Based on an article of Lukes ([Lu 69]) an iterative procedure for calculation of higher order polynomial solutions to the Hamilton-Jacobi equality (6.39) can be derived (see [Vr 94], [Dow 93]). Computation of these solutions is done using the formula manipulation computer package Mathematica. We have chosen to calculate polynomial solutions V of (6.39) up to order 4. Correspondingly we will calculate a feedback for the system Σ being an approximation of the feedback

$$u = -\frac{1}{\varepsilon}g^T(x)V_x^T(x)$$

considered in Subsection 3.2.2. A crucial point is the selection of the constant $\varepsilon > 0$ as will be explained in the next part of this subsection.

The selection of ε

In the Mathematica procedure we first calculate the optimal γ^* for the linearized system. The L_2-bound γ for the nonlinear system is then chosen as

$$\gamma = \gamma^* + \Delta\gamma$$

where $\Delta\gamma$ indicates the distance from the optimal value γ^*.

In most cases there is a link between the selection of ε and the value γ^* of the linearized system. Furthermore both ε and $\Delta\gamma$ will probably influence the stability properties of the closed-loop system obtained from applying the approximation of the feedback. To give an idea of the influences of ε and $\Delta\gamma$ on the closed-loop stability we have chosen three values of $\Delta\gamma$ relative to γ^* and three values of ε as follows

$$\begin{array}{lll} \Delta\gamma_1 = 0.01\gamma^*, & \Delta\gamma_2 = 0.05\gamma^*, & \Delta\gamma_3 = 0.2\gamma^*; \\ \varepsilon_1 = 0.05, & \varepsilon_2 = 0.4, & \varepsilon_3 = 1. \end{array}$$

For all 9 combinations of these parameters $\Delta\gamma$ and ε we have calculated the maximal values of $q(0)$ and $\varphi(0)$ still leading to an asymptotically stable closed-loop behaviour. The results are shown in Table 6.1, 6.2 and 6.3.

ε	γ^*	γ	$q_{\max}(0)$	$\varphi_{\max}(0)$
0.05	1.03	1.04	0.030	0.004
0.4	1.65	1.67	0.076	0.007
1	2.20	2.22	0.133	0.009

Table 6.1: stability results for $\Delta\gamma_1$ $(= 0.01\gamma^*)$

ε	γ^*	γ	$q_{\max}(0)$	$\varphi_{\max}(0)$
0.05	1.03	1.08	0.30	0.045
0.4	1.65	1.73	0.80	0.075
1	2.20	2.31	1.38	0.103

Table 6.2: stability results for $\Delta\gamma_2$ $(= 0.05\gamma^*)$

From these results we see that the stability region increases when we choose the margin $\Delta\gamma$ larger at the price of an higher L_2-gain γ for the closed-loop system. On the other hand for a specific choose of $\Delta\gamma$ the L_2-gain γ can be

ε	γ^*	γ	$q_{\max}(0)$	$\varphi_{\max}(0)$
0.05	1.03	1.24	2.1	0.32
0.4	1.65	1.98	5.3	0.59
1	2.20	2.64	8.9	0.90

Table 6.3: stability results for $\Delta\gamma_3$ ($= 0.2\gamma^*$)

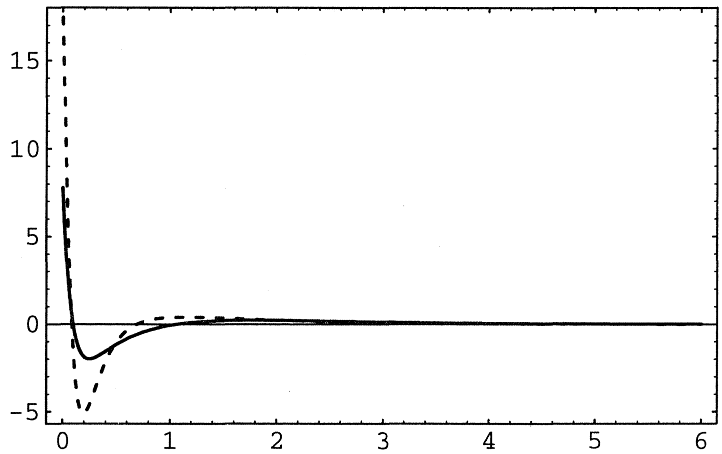

Figure 6.2: inputs for $\varepsilon = 0.05$ (dashed) and $\varepsilon = 0.4$ (solid) ($\Delta\gamma = 0.2\gamma^*$)

decreased by selecting ε smaller. The drawback of a smaller ε however is that this leads to an higher gain for the feedback as shown in Figure 6.2 where for $\Delta\gamma = 0.2\gamma^*$ the inputs u are plotted for $\varepsilon = 0.05$ and 0.4 ($x(0) = (1\ 0\ 0\ 0)$).

Hence from the experiments we have done so far it is clear that given an L_2-gain γ for the closed-loop system the selection of ε, and correspondingly $\Delta\gamma$, is a choice between two possible design objectives, namely the domain of attraction and the gain of the feedback.

6.2.2 Geometric approach

In this section we explain how the geometric approach can be applied to construct a solution to the state feedback $\mathcal{H}_\infty$ problem. The approach followed in this subsection is based on Subsection 5.2.7. First we construct a state transformation as in Subsection 5.2.2. Therefore we consider the extended system

$$\Sigma_e \begin{cases} \dot{x} &= f(x) + g(x)u + e(x)d \\ \dot{u} &= v \\ \dot{d} &= w \\ z &= x \end{cases}$$

or written in a compact form

$$\Sigma_e \begin{cases} \dot{x}_e &= f_e(x_e) + g_e(x_e)v + e_e(x_e)w \\ z &= h_e(x_e) \end{cases}$$

where $x_e = \begin{pmatrix} x & u & d \end{pmatrix}^T$ is the state and f_e, g_e, e_e and h_e are given by

$$f_e(x_e) = \begin{pmatrix} f(x) + g(x)u + e(x)d \\ 0 \\ 0 \end{pmatrix}, \quad g_e(x_e) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$e_e(x_e) = \begin{pmatrix} 0 \\ 0 \\ I_2 \end{pmatrix}, \quad h_e(x_e) = x.$$

For this system Σ_e we calculate the minimal involutive conditioned invariant distribution containing the input vector field $\frac{\partial}{\partial u}$, denoted by $\mathcal{S}_e^*$, by applying the $\mathcal{S}^*$ algorithm (see Subsection 5.2.2).

In the first step of the algorithm we obtain

$$\mathcal{S}_{1e} = \text{span}\left\{\frac{\partial}{\partial u}\right\} = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

which is clearly contained in ker dx. Therefore we calculate in the second step

$$\begin{aligned} \mathcal{S}_{2e} &= \mathcal{S}_{1e} + [f_e(x_e), \mathcal{S}_{1e}] \\ &= \text{span}\left\{\frac{\partial}{\partial u}, g(x)\frac{\partial}{\partial x}\right\} \\ &= \text{span}\left\{\frac{\partial}{\partial u}, \frac{1}{M + m\sin^2\varphi}\frac{\partial}{\partial \dot{q}} - \frac{\cos\varphi}{l(M + m\sin^2\varphi)}\frac{\partial}{\partial \dot{\varphi}}\right\} \end{aligned}$$

$$= \text{span}\left\{\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\ \frac{1}{M+m\sin^2\varphi}\\0\\ -\frac{\cos\varphi}{l(M+m\sin^2\varphi)}\\0\\0\\0\end{pmatrix}\right\}.$$

Because $\overline{S}_{2e} = S_{2e}$ and $S_{2e} \cap \ker \mathrm{d}x = S_{1e}$ we end up with $S_e^* = S_{2e}$. Clearly the distributions S_e^* and $S_e^* \cap \ker \mathrm{d}x$ are constant dimensional and satisfy the projectability conditions (see Subsection 5.2.2). The projections of the distributions S_e^* and $S_e^* \cap \ker \mathrm{d}x$ onto the state space of the original system Σ, $\mathcal{X}_2 \oplus \mathcal{X}_3$ and $\mathcal{X}_2$ respectivily, are given by:

$$\begin{aligned}
\mathcal{X}_2 &= 0;\\
\mathcal{X}_2 \oplus \mathcal{X}_3 &= \text{span}\left\{\frac{1}{M+m\sin^2\varphi}\frac{\partial}{\partial \dot{p}} - \frac{\cos\varphi}{l(M+m\sin^2\varphi)}\frac{\partial}{\partial\dot{\varphi}}\right\}\\
&= \text{span}\left\{\begin{pmatrix}0\\ \frac{1}{M+m\sin^2\varphi}\\0\\ -\frac{\cos\varphi}{l(M+m\sin^2\varphi)}\end{pmatrix}\right\}.
\end{aligned}$$

Accordingly we define new coordinates

$$\begin{aligned}
\xi_1 &= \left(\frac{l^2}{l^2+\cos^2\varphi}\right)^{\frac{1}{2}}\left(\frac{\cos\varphi}{l}\dot{q}+\dot{\varphi}\right),\\
\xi_2 &= q,\\
\xi_3 &= \varphi,\\
\xi_4 &= \left(\frac{l^2}{l^2+\cos^2\varphi}\right)^{\frac{1}{2}}\left(-\dot{q}+\frac{\cos\varphi}{l}\dot{\varphi}\right).
\end{aligned}$$

This defines a globally defined state transformation $\xi = S(x)$ since

$$\left|\frac{\partial S}{\partial x}(x)\right| = \begin{vmatrix}
0 & \left(\frac{l^2}{l^2+\cos^2\varphi}\right)^{\frac{1}{2}}\frac{\cos\varphi}{l} & * & \left(\frac{l^2}{l^2+\cos^2\varphi}\right)^{\frac{1}{2}}\\
1 & 0 & 0 & 0\\
0 & 0 & 1 & 0\\
0 & -\left(\frac{l^2}{l^2+\cos^2\varphi}\right)^{\frac{1}{2}} & * & \left(\frac{l^2}{l^2+\cos^2\varphi}\right)^{\frac{1}{2}}\frac{\cos\varphi}{l}
\end{vmatrix} = -1$$

for all x. Then we can choose the distribution $\mathcal{X}_1$ such that:

$$\begin{aligned}
\mathcal{X}_1 &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \xi_3}\right\};\\
\mathcal{X}_2 &= 0;\\
\mathcal{X}_3 &= \operatorname{span}\left\{\frac{\partial}{\partial \xi_4}\right\}.
\end{aligned}$$

Now we calculate the expression for the system Σ in this new coordinates. For simplicity we introduce the following notation

$$n(\varphi) := \left(\frac{l^2}{l^2 + \cos^2\varphi}\right)^{\frac{1}{2}}.$$

Then the system Σ is given by

$$\begin{aligned}
\dot{\xi}_1 &= n(\xi_3)\frac{g\sin\xi_3}{l} + n^3(\xi_3)\frac{\sin\xi_3}{l}\xi_1\xi_4 + n^3(\xi_3)\frac{\sin\xi_3\cos\xi_3}{l^2}\xi_4^2\\
&\quad + n(\xi_3)\frac{\cos\xi_3}{l}d_1 + n(\xi_3)d_2, && (6.40)\\
\dot{\xi}_2 &= n(\xi_3)\left(\frac{\cos\xi_3}{l}\xi_1 - \xi_4\right), && (6.41)\\
\dot{\xi}_3 &= n(\xi_3)\left(\xi_1 + \frac{\cos\xi_3}{l}\xi_4\right), && (6.42)\\
\dot{\xi}_4 &= n^{-1}(\xi_3)\frac{mg\sin\xi_3\cos\xi_3}{M + m\sin^2\xi_3} - n(\xi_3)\frac{ml\sin\xi_3\left(\xi_1 + \frac{\cos\xi_3}{l}\xi_4\right)^2}{M + m\sin^2\xi_3}\\
&\quad + \frac{F\left(\frac{\cos\xi_3}{l}\xi_1 - \xi_4\right)}{M + m\sin^2\xi_3} + n(\xi_3)\frac{g\sin\xi_3\cos\xi_3}{l^2}\\
&\quad - n^3(\xi_3)\frac{\sin\xi_3}{l}\left(\xi_1 + \frac{\cos\xi_3}{l}\xi_4\right)\xi_1\\
&\quad - n^{-1}(\xi_3)\frac{u}{M + m\sin^2\xi_3} - n(\xi_3)d_1 + n(\xi_3)\frac{\cos\xi_3}{l}d_2, && (6.43)\\
z &= \begin{pmatrix} \xi_2\\ n(\xi_3)\left(\frac{\cos\xi_3}{l}\xi_1 - \xi_4\right)\\ \xi_3\\ n(\xi_3)\left(\xi_1 + \frac{\cos\xi_3}{l}\xi_4\right)\end{pmatrix}. && (6.44)
\end{aligned}$$

This system can be seen as an interconnection of two subsystems. The first system Σ_1 has as state (ξ_1, ξ_2, ξ_3), inputs (ξ_4, d_1, d_2) and output z. The second system Σ_2 has state ξ_4, inputs $(u, \xi_1, \xi_2, \xi_3, d_1, d_2)$ and output ξ_4. As described in Subsection 5.2.4 we start by solving the regular $\mathcal{H}_\infty$ problem for the subsystem Σ_1. Note that the input ξ_4 does not appear linearly in the equations for Σ_1. For simplicity of calculations we linearize the ξ_1-dynamics with respect to the input ξ_4 around $\xi_4 = 0$. This comes down to neglecting the quadratic term in ξ_4. The (linearized) equations for Σ_1, with the state (ξ_1, ξ_2, ξ_3) denoted by η, are then given by

$$\Sigma_1 \begin{cases} \dot{\eta} &= f_1(\eta) + g_1(\eta)\xi_4 + e_1(\eta)d \\ z &= h_1(\eta) + k_1(\eta)\xi_4 \end{cases}$$

where

$$f_1(\eta) = \begin{pmatrix} n(\xi_3)\frac{g\sin\xi_3}{l} \\ n(\xi_3)\frac{\cos\xi_3}{l}\xi_1 \\ n(\xi_3)\xi_1 \end{pmatrix}, \quad g_1(\eta) = \begin{pmatrix} n^3(\xi_3)\frac{\sin\xi_3}{l}\xi_1 \\ -n(\xi_3) \\ n(\xi_3)\frac{\cos\xi_3}{l} \end{pmatrix},$$

$$e_1(\eta) = \begin{pmatrix} n(\xi_3)\frac{\cos\xi_3}{l} & n(\xi_3) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

$$h_1(\eta) = \begin{pmatrix} \xi_2 \\ n(\xi_3)\frac{\cos\xi_3}{l}\xi_1 \\ \xi_3 \\ n(\xi_3)\xi_1 \end{pmatrix}, \quad k_1(\eta) = \begin{pmatrix} 0 \\ -n(\xi_3) \\ 0 \\ n(\xi_3)\frac{\cos\xi_3}{l} \end{pmatrix}.$$

The system Σ_1 satisfies the standard $\mathcal{H}_\infty$ assumptions, namely $k_1^T(\eta)h_1(\eta) = 0$ and $k_1^T(\eta)k_1(\eta) = 1$. Therefore solving the L_2-gain problem with constant γ for this system Σ_1 comes down to finding a solution $V \geq 0$ to the following Hamilton-Jacobi equality

$$\begin{aligned} V_\eta(\eta)f_1(\eta) + \frac{1}{2}h_1^T(\eta)h_1(\eta) & \qquad (6.45) \\ +\frac{1}{2}V_\eta(\eta)\left(\frac{1}{\gamma^2}e_1(\eta)e_1^T(\eta) - g_1(\eta)g_1^T(\eta)\right)V_\eta^T(\eta) &= 0. \end{aligned}$$

Similar as in Subsection 6.2.1 an approximate 4th order polynomial solution V to this Hamilton-Jacobi equality (6.45) will be calculated. The corresponding approximate solution $\xi_4^*(\eta)$ to the regular state feedback $\mathcal{H}_\infty$ problem for the regular subsystem Σ_1 will be used to define an auxiliary system Σ_P as done in Subsection 5.2.7. It can be shown that

$$\frac{1}{2}\left(\xi_4-\xi_4^*(\eta)\right)^T\left(\xi_4-\xi_4^*(\eta)\right)=V_\eta(\eta)f_1(\eta)+\frac{1}{2}h_1^T(\eta)h_1(\eta)$$
$$+\frac{1}{2}V_\eta(\eta)\left(\frac{1}{\gamma^2}e_1(\eta)e_1^T(\eta)-g_1(\eta)g_1^T(\eta)\right)V_\eta^T(\eta).$$

Now we define the new variable

$$q=\xi_4-\xi_4^*(\eta). \tag{6.46}$$

From Subsection 5.2.7 it follows that the state feedback $\mathcal{H}_\infty$ problem with gain γ is solvable for Σ if the same problem is solvable for the auxiliary system

$$\Sigma_P\left\{\begin{array}{rcl}\dot{\eta}&=&f_1(\eta)+\frac{1}{\gamma^2}e_1(\eta)e_1^T(\eta)V_\eta^T(\eta)+g_1(\eta)\left(q+\xi_4^*(\eta)\right)+e_1(\eta)d_p\\ \dot{q}&=&f_q(\eta,q)+g_q(\eta)u+e_q(\eta)d_p\\ z&=&q\end{array}\right.$$

where f_q, g_q, e_q follows from (6.43), (6.37) and Σ_1 when we choose

$$d_p=d-\frac{1}{\gamma^2}e_1^T(\eta)V_\eta^T(\eta).$$

This singular $\mathcal{H}_\infty$ control problem can be solved by using the feedback construction derived in [MRST 94] and [MT 95] (see also Chapter 4). The applied feedback is given by

$$u=-g_q^{-1}(\eta)f_q(\eta)-g_q^{-1}(\eta)q-\frac{1}{4}\frac{1}{\gamma^2}g_q^{-1}(\eta)q\left(e_q(\eta)e_q^T(\eta)\right)$$

which together with Σ_P and hence also together with Σ leads to a closed-loop system which has L_2-gain less than or equal to γ. In fact we had to make an extra 4th order approximation of the feedback which was applied to the original system to reduce the complexity of the final differential equation. Compared to the calculation of the feedback using the cheap control approach Mathematica used quite some time to calculate an approximate feedback using this geometric approach.

6.2.3 Comparison of the two controllers

We will compare the feedbacks constructed by the cheap control approach on the one hand and the geometric approach on the other hand for two different values of γ. First we consider $\gamma = 0.8$ which is close to the optimal γ^* for the linearized system when we let ε go to zero. Therefore we have to choose ε small to be able to find an approximate solution to the Hamilton-Jacobi equality using the iterative method described in [Vr 94]. In Table 6.5 we see that the geometric approach yields a greater region of attraction. Probably this is due to the fact that in the cheap control approach we still penalize the inputs which avoids the gain of the feedback to become (too) high. The parameter robustness of the two methods is almost the same. In Table 6.4 we see that both the closed-loop systems corresponding to the two different feedbacks are very robust with respect to uncertainties in the parameters m, F and M. Some simulation results comparing the two methods are shown in Figure 6.3 and 6.4.

	m_{nom}	m_{max}	F_{nom}	F_{max}	M_{nom}	M_{max}
Cheap control approach	0.1	340	0.1	910	1	38
Geometric approach	0.1	260	0.1	920	1	100

Table 6.4: robustness with respect to parameter uncertainties in m, F and M for $\gamma = 0.8$ ($\varepsilon = 0.0001$, $x(0) = (\frac{1}{10}\ 0\ 0\ 0)$)

	$q_{\max}(0)$	$\varphi_{\max}(0)$
Cheap control approach	0.36	0.084
Geometric approach	4.0	0.68

Table 6.5: stability results for $\gamma = 0.8$ ($\varepsilon = 0.0001$)

To evaluate these results for different values of the computed L_2-gain we considered a second value of γ being $\gamma = 1.5$. As seen before this leads to an increase of the region of attraction (Table 6.7). The response for non-zero initial condition ($x(0) = (1\ 0\ 0\ 0)$) are shown in Figure 6.5. It can be seen that the response corresponding to the feedback calculated using the geometric approach is slower and the corresponding input is a bit smoother. For the robustness properties with respect to the parameters m, F and M some more realistic values are computed, as shown in Table 6.6.

Remarkable are the extreme differences in robustness between the two values of γ. These differences are not only due to the differences in the initial condition in the two experiments. For $\gamma = 1.5$ the robustness properties are still very good. Variations up to 100 times the nominal values for the param-

	m_{nom}	m_{max}	F_{nom}	F_{max}	M_{nom}	M_{max}
Cheap control approach	0.1	10	0.1	14	1	6.0
Geometric approach	0.1	2.5	0.1	9.0	1	2.1

Table 6.6: robustness with respect to parameter uncertainties in m, F and M for $\gamma = 1.5$ ($\varepsilon = 0.2$, $x(0) = (1\ 0\ 0\ 0)$)

	$q_{\text{max}}(0)$	$\varphi_{\text{max}}(0)$
Cheap control approach	1.3	0.16
Geometric approach	12	0.88

Table 6.7: stability results for $\gamma = 1.5$ ($\varepsilon = 0.2$)

eters still lead to a stable closed-loop behaviour. For $\gamma = 1.5$ we also see in Table 6.6 that the feedback constructed using the cheap control approach leads to better robustness properties than the feedback constructed with the geometric approach. Again it should be noted that this example is mainly meant to illustrate the two methods and to make some comparing remarks about possible differences. We did not pretend to make a feedback design which satisfies certain design objectives.

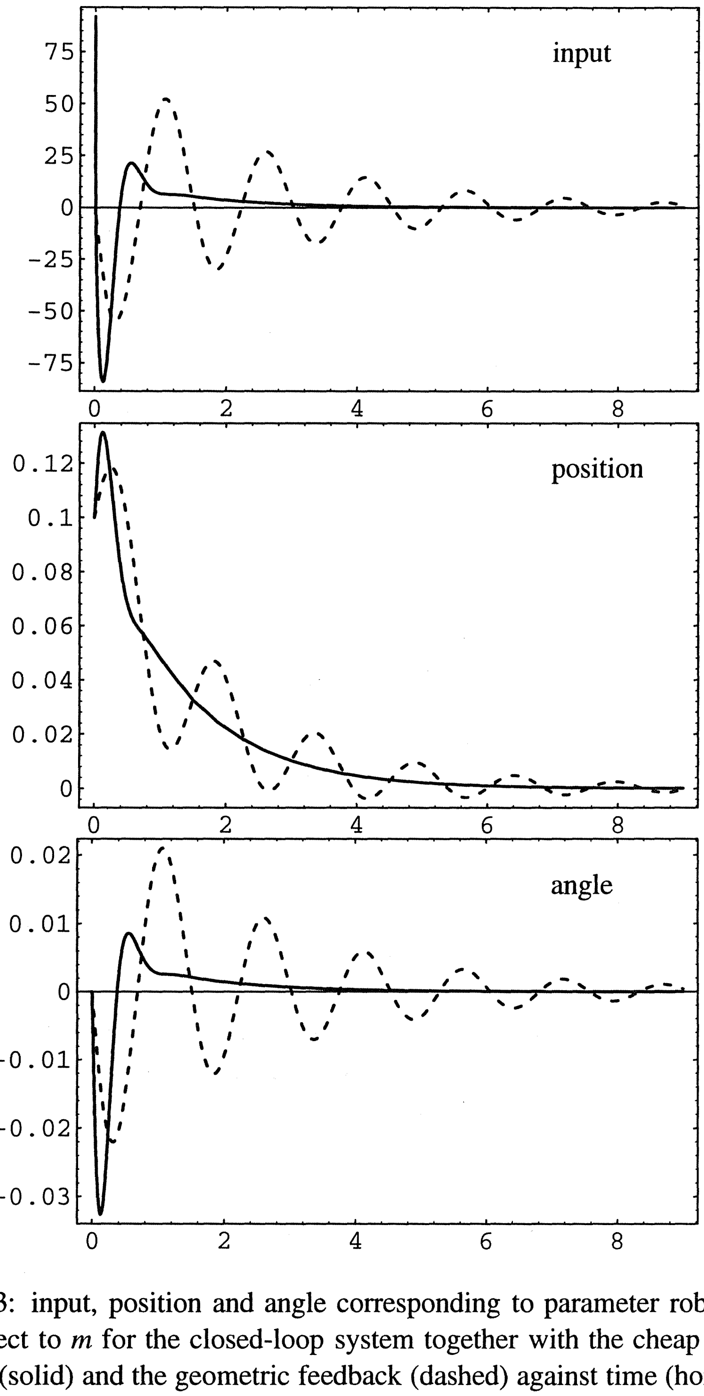

Figure 6.3: input, position and angle corresponding to parameter robustness with respect to m for the closed-loop system together with the cheap control feedback (solid) and the geometric feedback (dashed) against time (horizontal axis) for $\gamma = 0.8$ ($m = 250$ instead of the nominal value $m = 0.1$)($\varepsilon = 0.0001$, $x(0) = (\frac{1}{10}\ 0\ 0\ 0)$)

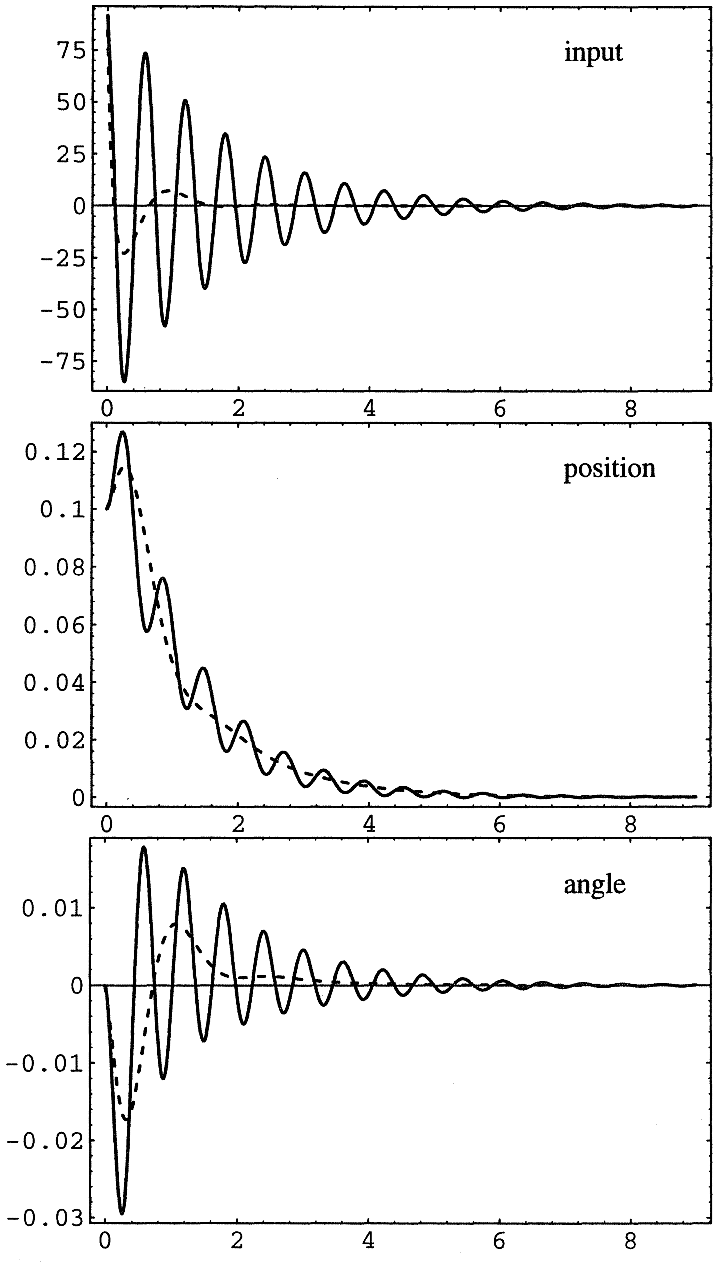

Figure 6.4: input, position and angle corresponding to parameter robustness with respect to M for the closed-loop system together with the cheap control feedback (solid) and the geometric feedback (dashed) against time (horizontal axis) for $\gamma = 0.8$ ($M = 35$ instead of the nominal value $M = 1$)($\varepsilon = 0.0001$, $x(0) = (\frac{1}{10}\ 0\ 0\ 0)$)

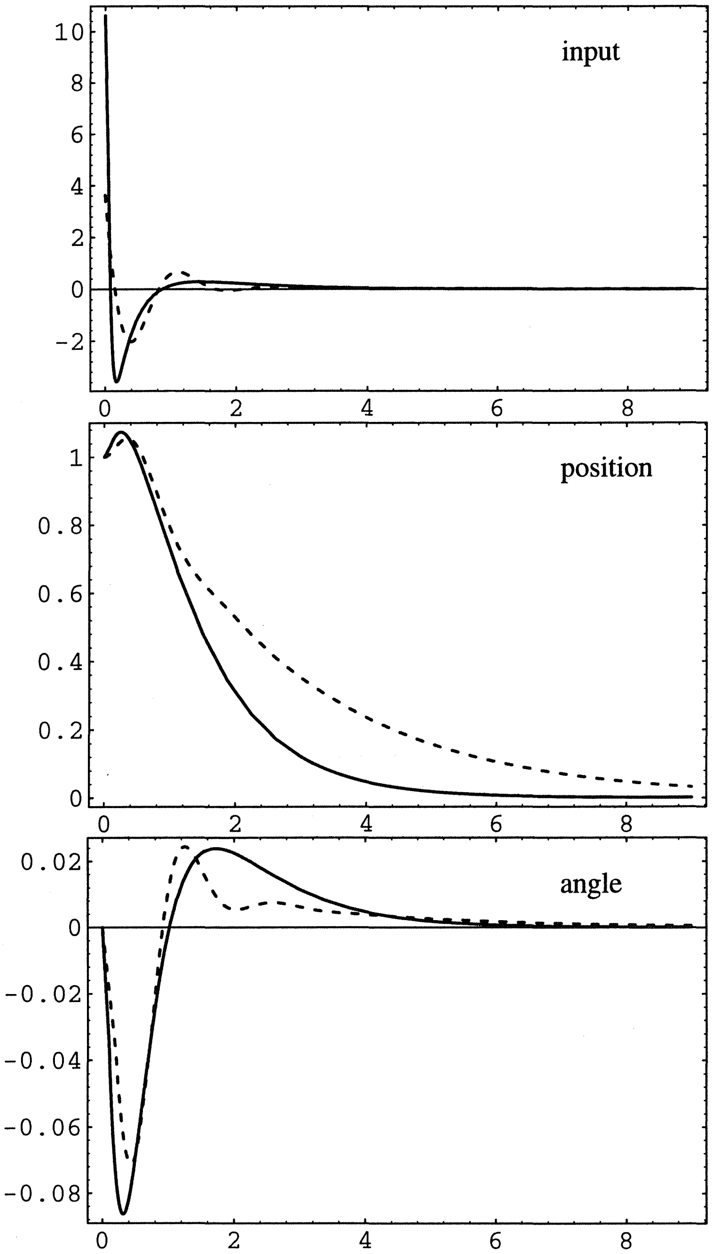

Figure 6.5: response for non-zero initial condition for the closed-loop system together with the cheap control feedback (solid) and the geometric feedback (dashed) against time (horizontal axis) for $\gamma = 1.5$ ($\varepsilon = 0.2$, $x(0) = (1\ 0\ 0\ 0)$)

Chapter 7

Robust stabilization under gain-bounded uncertainties

Several robustness problems can be described as a $\mathcal{H}_\infty$ control problem. For instance, robust stabilization under additive perturbations, parameter uncertainties, or multiplicative perturbations, are all examples of robust control problems under gain-bounded uncertainties (nonlinear systems: [vdS 95], [ST 95], [IsTa 95]; linear systems: [XS 90], [XS 92]), which can be described as $\mathcal{H}_\infty$ control problems. Also other types of uncertainty can be captured into the $\mathcal{H}_\infty$ framework such as gain-bounded and Lipschitz bounded uncertainties as in the paper [Ng 96a], or uncertainties which satisfy a certain integral functional constraint ([Ng 96b]), and robust stability and robust performance problems (see [AsGu 93]). Most of these problems, however, give rise to regular $\mathcal{H}_\infty$ control problems, for which we do not need the theory developed in this book.

In this chapter we apply the results derived in this book to two problems concerned with the stabilization of an uncertain system which do lead to singular $\mathcal{H}_\infty$ problems. The results from the Chapters 3 and 5 are applied to systems with two different types of gain-bounded uncertainty:

- Parameter uncertainties;
- Multiplicative uncertainties.

7.1 Parameter uncertainty

Consider the system

$$\dot{x} = f(x, \theta) + g(x)u \tag{7.1}$$

where x is the n-dimensional state vector in local coordinates and $u \in \mathbb{R}^m$. The matrix $\theta \in \mathbb{C}^{q \times p}$ contains the uncertain parameters. So θ is unknown and constant.

First we assume that we are able to measure the state completely. We assume the following linear dependency of f on the uncertain parameters θ

$$f(x, \theta) = f(x, \theta_{\text{nom}}) + e(x)\,(\theta - \theta_{\text{nom}})\,h(x) \tag{7.2}$$

for some known p-dimensional vector $h(x)$ and $n \times q$-matrix $e(x)$ where θ_{nom} denotes the nominal value of θ. We will denote $f(x) = f(x, \theta_{\text{nom}})$. These kind of uncertainties are considered in [HiPr 86] for linear systems and are referred to as structured perturbations.

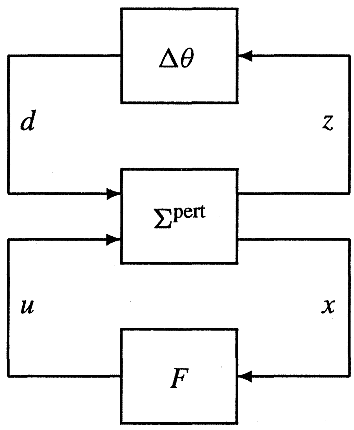

Figure 7.1: parameter perturbed system with feedback

Under assumption (7.2) the perturbed system (7.1) can be rewritten as

$$\Sigma^{\text{pert}} \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + e(x)d \\ z & = & h(x) \\ y & = & x \end{array} \right. \tag{7.3}$$

where d is given by $d = \Delta\theta z$ with $\Delta\theta = \theta - \theta_{\text{nom}}$ the constant matrix specifying the parameter uncertainties.

Then the *robust stabilization problem* is to find a feedback such that the closed loop system is L_2-stable for the largest possible class of perturbations $\Delta\theta$, where "largest" refers to the L_2-norm of $\Delta\theta$, which, since $\Delta\theta$ is constant, is the maximal singular value of $\Delta\theta$. This robust stabilization problem comes down to finding a feedback (F)

$$u = \sigma(x) \tag{7.4}$$

such that the L_2-gain of the closed loop system (7.3) and (7.4) from d to z is minimized to γ^*. Using the small-gain theorem (see e.g. [DeVi 75]) this means that the closed-loop system is L_2-stable for all perturbations $\Delta\theta$ with maximal singular value of $\Delta\theta$ strictly less than $\frac{1}{\gamma^*}$.

Note that $\Delta\theta$ is a real matrix, while the small-gain theorem allows for time-varying $\Delta\theta$ or complex $\Delta\theta$. Hence we actually obtain conservative bounds on the real perturbations $\Delta\theta$ (see [HiPr 88], [HiPr 90], [ToRy 91]).

The problem of minimizing the L_2-gain from d to $z = h(x)$ is a singular state feedback $\mathcal{H}_\infty$ optimal nonlinear optimal control problem as studied in this monograph. For the solution to the suboptimal problem using a cheap control approach we can formulate the following result.

Theorem 7.1 *Suppose there exist a constant $\varepsilon > 0$ and a solution $V \geq 0$ to*

$$\begin{array}{rcl} V_x(x)f(x) + +\frac{1}{2}h^T(x)h(x) & & \\ \frac{1}{2}V_x(x)\left[\frac{1}{\gamma^2}e(x)e^T(x) - \frac{1}{\varepsilon}g(x)g^T(x)\right]V_x^T(x) & \leq & 0 \\ V(0) & = & 0 \end{array} \tag{7.5}$$

and the closed loop system with the feedback

$$u = -\frac{1}{\varepsilon}g^T(x)V_x^T(x) \tag{7.6}$$

is zero-state observable. Then the feedback (7.6) locally asymptotically stabilizes the origin of the closed-loop system (7.3), (7.6) for every perturbation $\Delta\theta$ with maximal singular value less than $\frac{1}{\gamma}$. □

Remark 7.2 If the solution $V \geq 0$ of (7.5) is also proper then the feedback (7.6) globally asymptotically stabilizes the closed loop system. □

Under Assumption 3 (in Subsection 3.2.1) we can also formulate a converse result:

Theorem 7.3 *Suppose there exists a feedback (7.4) which locally stabilizes the closed loop system (7.3), (7.4). Assume that Assumption 3 (see Subsection 3.2.1) is satisfied. Then there exist a constant $\varepsilon > 0$ and a solution $V \geq 0$ to the Hamilton-Jacobi inequality (7.5).* □

The geometric approach to this singular state feedback $\mathcal{H}_\infty$ control problem is described in Chapter 5. As in Subsection 5.2.2 we consider the following extended version of the system Σ^{pert}

$$\Sigma_e^{\text{pert}} \begin{cases} \dot{x} &= f(x) + g(x)u + e(x)d \\ \dot{u} &= v \\ \dot{d} &= w \\ z &= h(x) \\ y &= x \end{cases}$$

For this extended system Σ_e^{pert} we construct the minimal involutive conditioned invariant distribution containing the input vector fields, denoted as $\mathcal{S}_e^*$, as described in Subsection 5.2.2. We assume that the distributions $\mathcal{S}_e^*$ and $\mathcal{S}_e^* \cap \ker dh$ are constant dimensional and that

$$\left[\frac{\partial}{\partial d_j}, \mathcal{S}_e^*\right] \subset \mathcal{S}_e^*, \qquad j = 1, \ldots, q. \tag{7.7}$$

Then as we have seen Section 5.2 there exist local coordinates such that the system Σ^{pert} is of the form

$$\Sigma^{\text{pert}} \begin{cases} \dot{x}_1 &= f_1(x_1, x_3) + e_1(x_1, x_3)d \\ \dot{x}_2 &= f_2(x_1, x_2, x_3) + g_2(x_1, x_2, x_3)u + e_2(x_1, x_2, x_3)d \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u + e_3(x_1, x_2, x_3)d \\ z &= h(x_1, x_3) \end{cases} \tag{7.8}$$

In these new coordinates the perturbed system with the to be constructed feedback has the form shown in Figure 7.2, where the subsystem Σ_1^{pert} is given by

$$\Sigma_1^{\text{pert}} \begin{cases} \dot{x}_1 &= f_1(x_1, x_3) + e_1(x_1, x_3)d \\ z &= h(x_1, x_3) \end{cases}$$

and the subsystem Σ_2^{pert} by the x_2 and x_3 dynamics of (7.8). Then as in Chapter 5 the singular $\mathcal{H}_\infty$ control problem can be split in two steps. First we consider

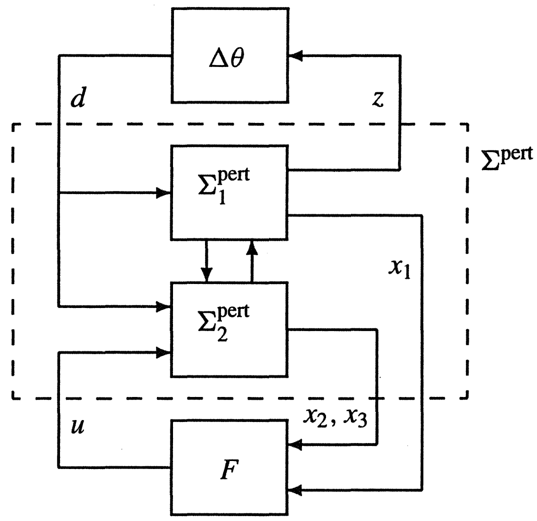

Figure 7.2: parameter perturbed system with feedback after transformation of the state

a regular L_2-gain control problem for the system Σ_1^{pert}. The pre-Hamiltonian corresponding to this problem for L_2-gain less than or equal to $\overline{\gamma}$ is given by

$$\begin{aligned} K_{\overline{\gamma}}(x_1, p_1, d, x_3) &= p_1^T(f_1(x_1, x_3, 0) + e_1(x_1, x_3)d) \\ &\quad + \frac{1}{2}h^T(x_1, x_3)h(x_1, x_3) - \frac{1}{2}\frac{1}{\overline{\gamma}^2}d^T d. \end{aligned}$$

The saddle point solution of this pre-Hamiltonian, given by $x_3 = x_3^*(x_1, p_1)$, $d^* = \frac{1}{\overline{\gamma}^2}e_1(x_1, x_3^*(x_1, p_1))e_1^T(x_1, x_3^*(x_1, p_1))p_1$, can be found by solving

$$\frac{\partial K_{\overline{\gamma}}}{\partial d} = 0, \qquad \frac{\partial K_{\overline{\gamma}}}{\partial x_3} = 0.$$

Substitution of this saddle point solution into the pre-Hamiltonian results in the Hamiltonian

$$\begin{aligned} H_{\overline{\gamma}}(x_1, p_1) &= p_1^T f_1(x_1, x_3^*(x_1, p_1)) \\ &\quad + \frac{1}{2}\frac{1}{\overline{\gamma}^2}p_1^T e_1(x_1, x_3^*(x_1, p_1))e_1^T(x_1, x_3^*(x_1, p_1))p_1 \\ &\quad + \frac{1}{2}h^T(x_1, x_3^*(x_1, p_1))h(x_1, x_3^*(x_1, p_1)). \end{aligned}$$

Then the following results can be stated (see Theorem 5.22).

Theorem 7.4 *Consider the system Σ^{pert}. Suppose that the distributions S_e^* and $S_e^* \cap \ker dh$ for the system Σ_e^{pert} are constant dimensional, and that condition (7.7) holds. Furthermore assume there exists a solution $V \geq 0$ to the Hamilton-Jacobi inequality*

$$H_{\overline{\gamma}}(x_1, V_{x_1}^T(x_1)) \leq 0$$

for some constant $\overline{\gamma} < \gamma$. Then there exists a constant k^* such that the robust stabilization problem is solvable for all disturbances $\Delta\theta$ with maximal singular value less than or equal to $\frac{1}{\gamma}$ by the feedback

$$u = \beta(x_1, x_2, x_3, k)$$

for $k > k^*$, if the parameterized feedback

$$u = \beta(x_1, x_2, x_3, k)$$

solves the $\mathcal{H}_\infty$ almost disturbance decoupling problem for the system Σ^{pert} with output $x_3 - x_3^*(x_1, V_{x_1}^T(x_1))$. □

On the other hand the factorization approach from Subsection 5.2.7 can be used to rewrite the robust stabilization problem into the following form. Assume there exists a solution $V \geq 0$, $V(0) = 0$, to the Hamilton-Jacobi equality

$$H_\gamma(x_1, V_{x_1}^T(x_1)) = 0.$$

We consider the auxiliary system

$$\Sigma_P^{\text{pert}} \left\{ \begin{array}{rcl} \dot{x}_1 &=& f_1(x_1, x_3) + \frac{1}{\gamma^2} e_1(x_1, x_3) e_1^T(x_1, x_3) V_{x_1}^T(x_1) + e_1^T(x_1, x_3) d_p \\ \dot{x}_2 &=& f_2(x_1, x_2, x_3) + \frac{1}{\gamma^2} e_2(x_1, x_2, x_3) e_1^T(x_1, x_3) V_{x_1}^T(x_1) \\ && + g_2(x_1, x_2, x_3) u + e_2(x_1, x_2, x_3) d_p \\ \dot{x}_3 &=& f_3(x_1, x_2, x_3) + \frac{1}{\gamma^2} e_3(x_1, x_2, x_3) e_1^T(x_1, x_3) V_{x_1}^T(x_1) \\ && + g_3(x_1, x_2, x_3) u + e_3(x_1, x_2, x_3) d_p \\ z &=& h_p(x_1, x_3) \end{array} \right.$$

where

$$d_p = d - \frac{1}{\gamma^2} e_1^T(x_1, x_3) V_{x_1}^T(x_1)$$

and the new output equation h_p is such that

$$\begin{aligned} & \frac{1}{2} h_p^T(x_1, x_3) h_p(x_1, x_3) \\ = \; & V_{x_1}^T(x_1) f_1(x_1, x_3) + \frac{1}{2}\frac{1}{\gamma^2} V_{x_1}^T(x_1) e_1(x_1, x_3) e_1^T(x_1, x_3) V_{x_1}(x_1) \\ & + \frac{1}{2} h^T(x_1, x_3) h(x_1, x_3). \end{aligned} \tag{7.9}$$

Recall that a smooth output equation h_p locally exists if the Hessian of the right hand part of equation (7.9) has constant rank around the origin. The following result can be stated (see Subsection 5.2.7).

Theorem 7.5 *Assume there exists a solution* $V \geq 0$ *to*

$$H_\gamma(x_1, V_{x_1}^T(x_1)) = 0.$$

Consider the systems Σ^{pert} *and* Σ_P^{pert}*. The following statements are equivalent:*

(i) the robust stabilization problem for the system Σ^{pert} *is solvable for all perturbations* $\Delta\theta$ *with maximal singular value less than or equal to* $\frac{1}{\gamma}$ *with a differentiable storage function* $W + V$, $W \geq 0$*;*

(ii) the robust stabilization problem for the system Σ_P^{pert} *is solvable for all perturbations* $\Delta\theta$ *with maximal singular value less than or equal to* $\frac{1}{\gamma}$ *with a differentiable storage function* $W \geq 0$.

□

If the full state is not available for measurements we can try to solve the robust stabilization problem under parameter uncertainties using the worst case certainty equivalence principle described in Section 3.3. Consider the following system

$$\begin{array}{rcl} \dot{x} &=& f(x,\theta) + g(x)u \\ y &=& c(x,\eta) \end{array} \tag{7.10}$$

where the uncertainty is given by the constant matrix θ and by the constant vector η. We assume that $f(x,\theta)$ satisfies (7.2) and furthermore we assume that c can be described as:

$$c(x,\eta) = c(x,\eta_{\text{nom}}) + (\eta - \eta_{\text{nom}})\,h(x) \tag{7.11}$$

where we denote $\Delta\eta := \eta - \eta_{\text{nom}}$ and $c(x) := c(x,\eta_{\text{nom}})$.

Then the perturbed system is

$$\Sigma^{\text{pert}} \left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g(x)u + e(x)d_1 \\ z &=& h(x) \\ y &=& c(x) + d_2 \end{array} \right. \tag{7.12}$$

where $d_1 = \Delta\theta z$ and $d_2 = \Delta\eta z$.

Then the *robust stabilization problem* is to find a compensator such that the closed loop system is L_2-stable for the largest possible class of perturbations $\Delta\theta$ and $\Delta\eta$, where again "largest" refers to the L_2-norm of $\Delta\theta$ and $\Delta\eta$, which, since $\Delta\theta$ and $\Delta\eta$ are constant is the maximal singular value of $\Delta\theta$ respectively $\Delta\eta$. This robust stabilization problem comes down to finding a compensator C

$$C\begin{cases} \dot{\xi} &= p(\xi, y), \qquad p(0,0)=0 \\ u &= q(\xi, y), \qquad q(0,0)=0 \end{cases} \tag{7.13}$$

such that the L_2-gain of the closed loop system (7.12) and (7.13) from d_1, d_2 to z is minimized.

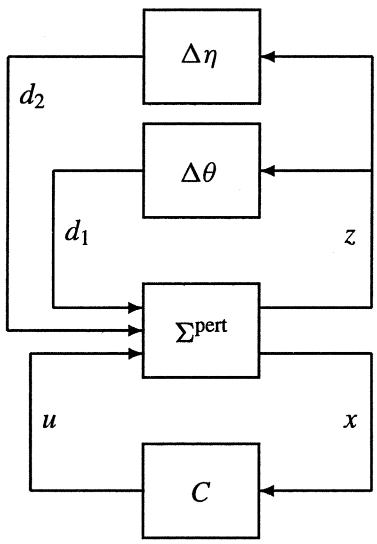

Figure 7.3: parameter perturbed system with controller

The suboptimal measurement feedback $\mathcal{H}_\infty$ control problem for the system Σ^{pert} is (as we have seen in Section 3.3) locally solvable if there exists a constant $\varepsilon > 0$ and a solution $V \geq 0$ to (7.5) with equality and a solution $W \geq 0$ to (3.36) satisfying:

$$\begin{aligned} &f - \frac{1}{\varepsilon} g g^T V_x^T + \frac{1}{\gamma^2} e e^T V_x^T \text{ is exponentially stable;} \\ &-\left(f + \frac{1}{\gamma^2} e e^T W_x^T\right) \text{ is exponentially stable;} \\ &W_{xx}(x) > V_{xx}(x), \quad \forall x. \end{aligned} \tag{7.14}$$

Hence we can state the following result.

Theorem 7.6 *Suppose there exists a constant $\varepsilon > 0$ and a solution $V \geq 0$ to (7.5) with equality and a solution $W \geq 0$ to (3.36) with satisfies (7.14). Assume that the certainty equivalence principle holds for Σ^{pert}. Then the controller*

$$\begin{aligned}
\dot{\xi} &= f(\xi) - \frac{1}{\varepsilon}g(\xi)g^T(\xi)V_\xi^T(\xi) + \frac{1}{\gamma^2}e(\xi)e^T(\xi)V_\xi^T(\xi) \\
&\quad + \gamma^T\left[W_{\xi\xi}(\xi) - V_{\xi\xi}(\xi)\right]^{-1}\frac{\partial c^T}{\partial \xi}(\xi)\,(y(t) - c(\xi)), \qquad (7.15)\\
u &= -\frac{1}{\varepsilon}g^T(\xi)V_\xi^T(\xi)
\end{aligned}$$

locally asymptotically stabilizes the closed loop system (7.12), (7.15) for every perturbation $\Delta\theta$, $\Delta\eta$ having maximal singular values less than γ. □

7.2 Multiplicative perturbations

Consider the system

$$\Sigma\begin{cases} \dot{x} &= f(x) + g(x)u \\ \bar{y} &= h(x) \end{cases} \qquad (7.16)$$

and suppose the output $\bar{y}$ is perturbed by a disturbance d. We assume that this disturbance is the output of an arbitrary nonlinear system with input $\bar{y}$:

$$\Delta\begin{cases} \dot{\varphi} &= \alpha(\varphi, \bar{y}) \\ d &= \beta(\varphi, \bar{y}) \end{cases} \qquad (7.17)$$

Then from the interconnection shown in Figure 7.4 we see that the perturbed system is given by

$$\Sigma^{\text{pert}}\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) + d \\ z &= h(x) \end{cases} \qquad (7.18)$$

Now the robust stabilization problem is, similar to the previous section, to find a compensator such that the closed loop system is L_2-stable for the largest possible class of perturbations Δ, where again "largest" refers to the L_2-norm

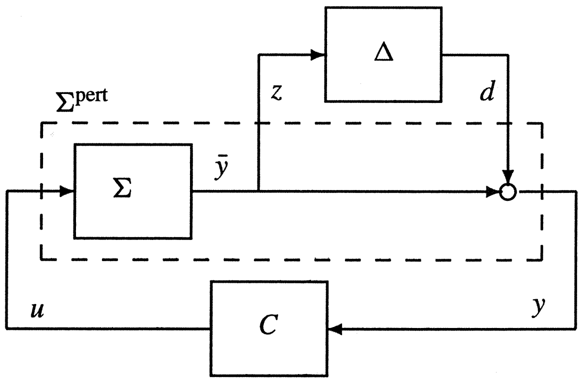

Figure 7.4: multiplicative perturbed system with controller

of Δ. This robust stabilization problem comes down to finding a compensator C

$$C\begin{cases} \dot{\xi} &= p(\xi, y) \\ u &= q(\xi, y) \end{cases} \tag{7.19}$$

such that the L_2-gain of the closed loop system (7.18), (7.19) from d to z is minimized.

In Section 3.3 we have seen that the suboptimal measurement feedback $\mathcal{H}_\infty$ control problem for the system Σ^{pert} is locally solvable if there exists a constant $\varepsilon > 0$ and a solution $V \geq 0$ to

$$\begin{aligned} V_x(x)f(x) - \frac{1}{2\varepsilon}V_x(x)g(x)g^T(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) &= 0 \\ V(0) &= 0 \end{aligned} \tag{7.20}$$

and a solution $W \geq 0$ to

$$\begin{aligned} W_x(x)f(x) + \frac{1}{2}(1-\gamma^2)h^T(x)h(x) &= 0 \\ W(0) &= 0 \end{aligned} \tag{7.21}$$

such that:

$$\begin{aligned} &f - \frac{1}{\varepsilon}gg^TV_x^T \text{ is exponentially stable;} \\ &-f \text{ is exponentially stable;} \\ &W_{xx}(x) > V_{xx}(x), \ \ \forall x. \end{aligned} \tag{7.22}$$

Summarizing:

Theorem 7.7 *Suppose there exist a constant $\varepsilon > 0$ and a solution $V \geq 0$ to (7.20) and a solution $W \geq 0$ to (7.21) which satisfies (7.22). Assume that the certainty equivalence principle holds for Σ^{pert}. Then the controller*

$$\begin{aligned} \dot{\xi} &= f(\xi) - \frac{1}{\varepsilon} g(\xi) g^T(\xi) V_\xi^T(\xi) \\ &\quad + \gamma^T \left[W_{\xi\xi}(\xi) - V_{\xi\xi}(\xi) \right]^{-1} \frac{\partial h^T}{\partial \xi}(\xi) \left(y(t) - h(\xi) \right), \\ u &= -\frac{1}{\varepsilon} g^T(\xi) V_\xi^T(\xi) \end{aligned} \tag{7.23}$$

locally asymptotically stabilizes the closed loop system (7.18), (7.17) and (7.23) for every perturbation system Δ as in (7.17), having L_2-gain less than γ. □

Example 7.8 A nonlinear system $\dot{x} = f(x) + g(x)u$, $y = h(x)$, $x \in M$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ is called *lossless* ([Wi 72]) if there exists a nonnegative function $H : M \to \mathbb{R}$, $H(0) = 0$, such that

$$\begin{aligned} H_x(x) f(x) &= 0, \\ H_x(x) g(x) &= h^T(x) \end{aligned}$$

or, equivalently, $\frac{\mathrm{d}}{\mathrm{d}t} H = u^T y$ (see also Remark 6.4).
Assume that the system Σ is lossless. We note that

$$V(x) = \sqrt{\varepsilon} H(x)$$

is a solution of the Hamilton-Jacobi equality (7.20). Furthermore for $\gamma = 1$

$$W(x) = bH(x)$$

is a solution to (7.21) for every $b \geq 0$. If we assume that $H_{xx} > 0$ for all x we also have that the last conditions of (7.22) is satisfied when $b > \sqrt{\varepsilon}$. Hence when $f - \frac{1}{\sqrt{\varepsilon}} g g^t H_x^T$ and $-f$ are exponentially stable it follows that the controller

$$\begin{aligned} \dot{\xi} &= f(\xi) - \frac{1}{\sqrt{\varepsilon}} g(\xi) h(\xi) \\ &\quad + \frac{1}{b - \sqrt{\varepsilon}} H_{\xi\xi}^{-1}(\xi) \frac{\partial h^T}{\partial \xi}(\xi) \left(y(t) - h(\xi) \right), \\ u &= -\frac{1}{\sqrt{\varepsilon}} h(\xi) \end{aligned}$$

locally asymptotically stabilizes the closed-loop system for every perturbation Δ having L_2-gain less than 1. But the assumption that $-f$ is exponentially stable is violated in this case because the system is lossless which corresponds to the fact that the linearized system has invariant zeros on the imaginary axis.

On the other hand, the static output feedback

$$u = -y$$

yields a closed-loop system which has L_2-gain less than or equal to 1. This can be seen by constructing the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) - g(x)h(x) - g(x)d \\ z &= h(x) \end{aligned} \tag{7.24}$$

for which $V(x) = H(x)$ is clearly a solution to the Hamilton-Jacobi inequality

$$V_x(x)\left(f(x) - g(x)h(x)\right) + \frac{1}{2}V_x(x)g(x)g^T(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) = 0.$$

Hence under local asymptotic stability of $\dot{x} = f(x) - g(x)h(x)$ this static output feedback also solves the robust stabilization problem for all perturbations Δ with L_2-gain less than or equal to 1. □

Chapter 8

Conclusions

In this chapter we summarize the main results derived in this monograph, and afterwards we consider some open problems associated with the subjects covered.

8.1 Summary

A standing assumption in most of the literature on the state space approach to linear or nonlinear $\mathcal{H}_\infty$ theory is a certain regularity condition. When this regularity assumption is violated the problem generalizes to the singular $\mathcal{H}_\infty$ problem. Two classes of singular $\mathcal{H}_\infty$ problems can be distinguished. The first class appears when the direct feed through from the inputs to the to-be-controlled variables is not injective and the second class of singular $\mathcal{H}_\infty$ problems appears when the direct feed through from disturbances to the measurements is not surjective. In this book we have presented two methods to solve the singular $\mathcal{H}_\infty$ problem for nonlinear systems with respect to singularities from the first class. These methods are extensions of approaches to the singular linear $\mathcal{H}_\infty$ problem derived at the end of the eighties. The first method has been considered by Khargonekar, Petersen and Zhou ([KPZ 87], [Pe 87a], [Pe 87b], [KPR 88], [ZK 88], [KPZ 90]), and is based on a regularization of the linear system such that the regular linear $\mathcal{H}_\infty$ theory can be applied ("cheap control" approach). The second method has been derived by Stoorvogel, Trentelman ([StTr 90], [St 92]) and Scherer ([Sch 91]). This method uses results from geometric system theory to decompose the system, and to split the singular $\mathcal{H}_\infty$ control problem into a regular $\mathcal{H}_\infty$ problem and an almost disturbance decoupling problem.

In Chapter 3 the cheap control approach has been investigated to solve the singular $\mathcal{H}_\infty$ problem for nonlinear systems. We have considered nonlinear systems whose dynamical equations are affine in the inputs and the disturbances. Probably most of the results can be extended to general nonlinear systems which are not affine in the inputs and the disturbances. The solvability of the singular state feedback $\mathcal{H}_\infty$ problem is characterized by the solvability of a parameterized Hamilton-Jacobi inequality which generalizes the parameterized Riccati equality from the linear theory. This Hamilton-Jacobi inequality also corresponds to a regular state feedback $\mathcal{H}_\infty$ control problem for a regularized version of the system. This connection lead to some nice properties about the feedback solution to the singular $\mathcal{H}_\infty$ problem. The connection with the linear state feedback $\mathcal{H}_\infty$ problem for the linearization has also been treated. We have proved that the singular $\mathcal{H}_\infty$ problem for the nonlinear system is locally solvable with an L_2-gain less than a certain constant γ if and only if the singular $\mathcal{H}_\infty$ problem with gain less than γ is solvable for its linearization. Finally, the singular nonlinear measurement feedback $\mathcal{H}_\infty$ problem has been solved using the worst case certainty equivalence principle. Necessary conditions generalizing the necessary and sufficient conditions for linear systems are derived. Also a set of sufficient conditions is given. At this moment we cannot expect anything, since even for regular systems a full characterization of the measurement feedback $\mathcal{H}_\infty$ problem has not been derived yet.

In Chapter 4 the sufficient conditions for the solvability of the almost disturbance decoupling problem for single-input single-output systems derived in the papers [MRST 94] and [MT 95] are extended to multi-input multi-output systems. In fact the almost disturbance decoupling problem is a special case of a singular $\mathcal{H}_\infty$ problem. The solvability of the almost disturbance decoupling problem is equal to the solvability of the singular $\mathcal{H}_\infty$ control problem for arbitrary small L_2-gain $\gamma > 0$. We have given a set of sufficient conditions for the solvability of the almost disturbance decoupling problem for affine nonlinear systems without a direct feed through from the inputs to the outputs. This set of conditions is still far from being necessary, and in fact we give some other slightly different sets of sufficient conditions as well.

We partially extend the results from the geometric approach to the linear singular $\mathcal{H}_\infty$ problem to general nonlinear systems. An important notion in this geometric approach to the singular state feedback $\mathcal{H}_\infty$ problem is the minimal involutive conditioned invariant distribution containing the input vector fields for an extended system which includes the inputs and the disturbances as extra state components. This notion extends the notion of strongly controllable subspace in the linear theory, and is used to decompose the nonlinear system. This decomposition leads to several sets of sufficient conditions for the solvability of the state feedback $\mathcal{H}_\infty$ problem. These conditions are in terms of the solvability of a regular $\mathcal{H}_\infty$ control problem for a subsystem of the original system and an almost disturbance decoupling problem for a complementary subsystem. Under certain extra assumption we proved that the solvability of the regular subproblem is also a necessary condition for the solvability of the singular $\mathcal{H}_\infty$ problem.

A promising way to attack the singular $\mathcal{H}_\infty$ problem is to define an auxiliary system based on the solvability of the Hamilton-Jacobi equality corresponding to the regular $\mathcal{H}_\infty$ subproblem. There exists a very strong relation between the solvability of the singular $\mathcal{H}_\infty$ problem for the original system and the solvability of the same problem for the auxiliary system. In the example considered in Chapter 6 it is shown that this method leads to satisfactory results. For a special class of nonlinear systems we can fully characterize the solvability of the singular problem for the complete system by the solvability of the regular problem for one of the subsystems, where the states of the complementary subsystem are considered as inputs for this subsystem.

The geometric approach is certainly more elegant and appealing than the cheap control method. In a sense the singular $\mathcal{H}_\infty$ problem is characterized by a regular $\mathcal{H}_\infty$ problem for a reduced order system. For nonlinear systems however a full characterization of the singular state feedback $\mathcal{H}_\infty$ problem has not been derived yet. This is one of the main drawbacks of this geometric method compared with the cheap control approach.

The merits and drawbacks of the two methods are also shown in Chapter 6. As a first example the model of the orientation of a rigid body is considered. For this system we treated the tracking problem which is described as a singular $\mathcal{H}_\infty$ problem. The problem is solved using the two different methods. The cheap control approach comes down to the search for a solution to the parameterized Hamilton-Jacobi inequality. Another solution is constructed using the geometric approach. The main advantage of the geometric method is that we only need to find a solution to a reduced order Hamilton-Jacobi inequality. By this fact the geometric approach can be easily extended to the general tracking problem of a rigid body model including motion of its center of mass (see [MD 95]). In the second example, the inverted pendulum on a cart, some computational issues are described.

8.2 Open problems

We mention some open problems of interest.

(i) **Almost disturbance decoupling problem**
In order to obtain a full geometric characterization of the nonlinear $\mathcal{H}_\infty$ problem, even in the state feedback case, a better understanding of the almost disturbance decoupling problem for nonlinear systems is instrumental. In the current approach a triangular structure on the disturbance vector fields is needed which is quite restrictive (see [MRST 94], [MT 95] and Chapter 4).

(ii) **Measurement feedback $\mathcal{H}_\infty$ control problem**
The dynamic output feedback $\mathcal{H}_\infty$ problem is still largely open both in the regular and singular case. Necessary conditions have been found and there is some insight in the structure of possible, compensators but a full characterization and understanding of this problem has not been derived yet. One of the problems we have not investigated is the singular $\mathcal{H}_\infty$ control problem with singular measurements in which case the direct feed through from disturbances to measurements is not surjective.

(iii) **Nonlinear robust design methodology**
The usefulness of nonlinear $\mathcal{H}_\infty$ control needs further research. The application of the theory to design problems can lead to new insight and can contribute to a better understanding of some of the open problems. It is useful to look how frequency based notions such as the sensitivity function and the well known limits of performance for linear systems can be captured in the nonlinear theory.

(iv) **Computational issues**
Further research should be done on the computation of (approximate) solutions of the Hamilton-Jacobi inequalities used in the nonlinear $\mathcal{H}_\infty$ theory.

(v) **Factorization approach**
The factorization approach considered in Subsection 5.2.7 is promising. Perhaps a better understanding of the almost disturbance decoupling problem can help to derive a full characterization of the singular state feedback $\mathcal{H}_\infty$ problem using this factorization approach.

Appendix A

Notions from differential geometry

In this appendix an overview is given of concepts from differential geometry that are used in this book. The presentation of this material largely follows [Hu 91] and [NvdS 90].

Manifold

Consider the space $\mathbb{R}^N$ with *Euclidean norm* $\|\cdot\|$ defined by

$$\|x\| = \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}.$$

A subset $U \subset \mathbb{R}^N$ is called an *open subset* of $\mathbb{R}^N$ if for every $\bar{x} \in U$ there exists an $\varepsilon > 0$ such that $\{x \in \mathbb{R}^N | \|x - \bar{x}\| < \varepsilon\} \subset U$. For $x \in \mathbb{R}^N$, an open subset $U \subset \mathbb{R}^N$ containing x is called a *neighborhood* of x. Let U be a (not necessarily open) subset of $\mathbb{R}^N$. A subset $\overline{U} \subset U$ is called a *relatively open subset* of U if there exists an open subset $\tilde{U}$ of $\mathbb{R}^N$ such that $\tilde{U} \cap U = \overline{U}$.

Let U, V be subsets of $\mathbb{R}^N$ and consider a mapping $\Phi : U \to V$. Φ is called *continuous* if the inverse image of every open subset of V is an open subset of U. If U is open, Φ is called C^k if it has continuous partial derivatives up to order k, and for $k = \infty$ the mapping is also called *smooth.* If U is not open, Φ is called C^k if for every $\bar{x} \in U$ there exists a neighborhood $\overline{U}$ of $\bar{x}$ in $\mathbb{R}^N$ and a C^k mapping $\Psi : \overline{U} \to V$ such that Ψ equals Φ on $U \cap \overline{U}$. Φ is called a C^k *diffeomorphism* if $\Phi^{-1} : V \to U$ exists and both Φ and Φ^{-1} are C^k.

A subset $M \subset \mathbb{R}^N$ is called a *manifold* of dimension n if there exist an index set I, relatively open subsets U_i $(i \in I)$ of M, open subsets V_i $(i \in I)$ of $\mathbb{R}^n$ and diffeomorphisms $\phi_i : U_i \to V_i$ $(i \in I)$, such that $\cup_{i \in I} U_i = M$. A subset N of M is called a *submanifold* of M if N itself is a manifold.

Local coordinates

For every $i \in I$, the pair (U_i, ϕ_i) is called a *local coordinate chart* on M.

For $k = 1, \ldots, n$, let r_k denote the natural *coordinate functions* on $\mathbb{R}^n$, i.e., $r_k(a_1, \ldots, a_n) = a_k$. The functions $x_i = r_i \circ \phi$ $(i = 1, \ldots, n)$ are called *local coordinate functions* and the values $x_1(p), \ldots, x_n(p)$ of a point $p \in U$ are called the *local coordinates* of p.

Tangent space, tangent bundle

For $a \in \mathbb{R}^n$, the *tangent space* $T_a\mathbb{R}^n$ at a is the set of tangent vectors to $\mathbb{R}^n$ at a. The natural basis of $T_a\mathbb{R}^n$ associated with the natural coordinate functions r_i is denoted by $\{\frac{\partial}{\partial r_1}|_a, \ldots, \frac{\partial}{\partial r_n}|_a\}$. Consider a manifold M of dimension n and let $p \in M$. Let (U, ϕ) be a local coordinate chart around p, and $x_1, \ldots, x_n$ local coordinates. For $X \in T_{\phi(p)}\mathbb{R}^n$, define $\phi_{*p}^{-1} X := \frac{\partial \phi^{-1}}{\partial a}(\phi(p))X$. Define

$$\left.\frac{\partial}{\partial x_i}\right|_p := \phi_{*p}^{-1} \left.\frac{\partial}{\partial r_i}\right|_{\phi(p)}, \qquad i = 1, \ldots, n.$$

Then the *tangent space* T_pM of M at p is defined as

$$T_pM = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p \right\}.$$

Hence $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$ is a basis of T_pM. The elements of T_pM are called *tangent vectors* at p. The set

$$TM = \{(p, X_p) | p \in M, X_p \in T_pM\}$$

is called the *tangent bundle* of M.

Vector field

A *vector field* on M is a mapping X that assigns to each $p \in M$ a tangent vector $X_p \in T_pM$. X is called a C^k vector field if for each $p \in M$ there exists a local coordinate chart (U, ϕ) around p and C^k functions $X_1, \dots, X_n$ such that for all $\bar{p} \in U$ we have

$$X(x_1(\bar{p}), \dots, x_n(\bar{p})) = \sum_{i=1}^{n} X_i(x_1(\bar{p}), \dots, x_n(\bar{p})) \left. \frac{\partial}{\partial x_i} \right|_{\bar{p}}.$$

Throughout by vector fields we will mean C^k vector fields with k sufficiently large. The vector field X in local coordinates, is often identified with the n-dimensional column vector $(X_1(x_1, \dots, x_n), \dots, X_n(x_1, \dots, x_n))^T$.

A *smooth curve* σ on M is a smooth mapping $\sigma : (a, b) \to M$, where (a, b) is an open interval of $\mathbb{R}$. For $t \in (a, b)$, let (U, ϕ) be a local coordinate chart around $\sigma(t) \in M$. Define $\dot{\sigma}(t) \in T_{\sigma(t)}M$ by the conventional limit

$$\dot{\sigma}(t) = \lim_{h \to 0} \frac{\sigma(t+h) - \sigma(t)}{h}$$

σ is called an *integral curve* of a given vector field X on M if $\dot{\sigma}(t) = X(\sigma(t))$ for all $t \in (a, b)$. In local coordinates $x_1, \dots, x_n$ this means that $\sigma(t) = (\sigma_1(t), \dots, \sigma_n(t))$ is a solution of the set of differential equations

$$\left\{ \begin{array}{rcl} \dot{\sigma}_1(t) & = & X_1(\sigma_1(t), \dots, \sigma_n(t)) \\ & \vdots & \\ \dot{\sigma}_n(t) & = & X_n(\sigma_1(t), \dots, \sigma_n(t)) \end{array} \right. \qquad t \in (a, b)$$

where X is identified with the column vector $(X_1, \dots, X_n)^T$. So, to a vector X given in local coordinates we associate in a one-to-one way the set of differential equations

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & X_1(x_1(t), \dots, x_n(t)) \\ & \vdots & \\ \dot{x}_n(t) & = & X_n(x_1(t), \dots, x_n(t)) \end{array} \right.$$

also abbreviated as $\dot{x} = X(x)$. A submanifold $N \subset M$ is called *invariant* for $\dot{x} = X(x)$ if

$$X(x) \in T_pN, \qquad \forall p \in N.$$

By the existence and uniqueness theorem for smooth differential equations it follows that for any $p \in M$ there exists an interval (a, b) of maximal length containing 0 and a unique interval curve $\sigma(t)$, $t \in (a, b)$ with $\sigma(0) = p$. If for every p we have $(a, b) = (-\infty, \infty)$, and so solutions are defined for all time t the vector field X is called *complete*.

Coordinate transform

Let now (U, ϕ) and (V, ψ) be two overlapping coordinates charts yielding a *coordinate transformation* $z = S(x)$, where x and z are local coordinates corresponding to (U, ϕ) and (V, ψ) respectively, with $S = \psi \circ \phi^{-1}$. Let $X(p) \in T_pM$, with $p \in U \cap V$, be expressed in the basis corresponding to (U, ϕ) respectively (V, ψ) as

$$X(p) = \sum_{i=1}^{n} \alpha_i(p) \left.\frac{\partial}{\partial x_i}\right|_p = \sum_{i=1}^{n} \beta_i(p) \left.\frac{\partial}{\partial z_i}\right|_p$$

then the coefficients $\alpha_i(p)$ and $\beta_i(p)$ are related as

$$\begin{pmatrix} \beta_1(p) \\ \vdots \\ \beta_n(p) \end{pmatrix} = \frac{\partial S}{\partial x}(x(p)) \begin{pmatrix} \alpha_1(p) \\ \vdots \\ \alpha_n(p) \end{pmatrix}.$$

Lie-derivative, Lie-bracket

A vector field X defines in any $p \in M$ a tangent vector $X(p)$. For $f : M \to \mathbb{R}$ this yields in any $p \in M$ the directional derivative $X(p)(f)$. Hence by varying p we obtain a smooth function $X(f)$ defined as

$$X(f)(p) := X(p)(f).$$

The function $X(f) : M \to \mathbb{R}$ will be called the *Lie-derivative* of f along X also denoted as $L_X f$.

For two vector fields X and Y on M, we define a new vector field, denoted as $[X, Y]$ and called the *Lie-bracket* of X and Y defined by (in local coordinates)

$$[X, Y] = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial Y_j}{\partial x_i} X_i - \frac{\partial X_j}{\partial x_i} Y_i \right) \frac{\partial}{\partial x_j}.$$

Cotangent bundle

The *dual space* of the tangent space T_pM, denoted by T_p^*M, is called the cotangent space, i.e., the dual space V^* of a linear space V is the set of all linear functions on V. Elements of T_p^*M are called *cotangent vectors*. Let

$$\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$$

be a basis for T_pM corresponding to local coordinates $x_1, \dots, x_n$ on M, then we denote the dual basis of T_p^*M by $dx_1|_p, \dots, dx_n|_p$. The set

$$T^*M = \left\{(p, \omega) | p \in M, \omega \in T_p^*M\right\}$$

is called the *cotangent bundle* of M. A *covector field* (or *one-form*) ω on M is a C^k-mapping that to each $p \in M$ assigns a cotangent vector. If $\omega \in T_p^*M$, then the value of ω at $X \in T_pM$ is denoted by $\langle \omega, X \rangle$.

With every continuous function h on M we can associate a covector field dh by defining

$$dh(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x)\, dx_i|_x .$$

Distributions, Frobenius

A *distribution* $\mathcal{D}$ on M is a mapping that assigns to each $p \in M$ a linear subspace of T_pM. If $X_1, \dots, X_q$ is a set of vector fields on M, then their span, denoted by span $\left\{X_1, \dots, X_q\right\}$ is the distribution defined by

$$\text{span}\left\{X_1, \dots, X_q\right\} : p \mapsto \text{span}\left\{X_1(p), \dots, X_q(p)\right\}, \qquad (p \in M). \qquad \text{(A.1)}$$

A distribution is called C^k if it is defined by (A.1) for C^k vector fields $X_1, \dots, X_q$. Throughout by distributions we will mean C^k distributions with k sufficiently large. The sum and the intersection of two distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined as:

$$\begin{aligned} \mathcal{D}_1 + \mathcal{D}_2 : p &\mapsto \mathcal{D}_1(p) + \mathcal{D}_2(p); \\ \mathcal{D}_1 \cap \mathcal{D}_2 : p &\mapsto \mathcal{D}_1(p) \cap \mathcal{D}_2(p). \end{aligned}$$

A vector field X on M is said to belong to a distribution $\mathcal{D}$, denoted by $X \in \mathcal{D}$, if for every $p \in M$ we have $X(p) \in \mathcal{D}(p)$. A distribution $\mathcal{D}_1$ is said to be contained in a distribution $\mathcal{D}_2$, denoted by $\mathcal{D}_1 \subset \mathcal{D}_2$, if every vector field belonging to $\mathcal{D}_1$ also belongs to $\mathcal{D}_2$. The *dimension* of a distribution $\mathcal{D}$ at $p \in M$ is the dimension of the linear subspace $\mathcal{D}(p)$. A distribution is called *constant dimensional* if the dimension of $\mathcal{D}(p)$ does not depend on the point $p \in M$. If $\mathcal{D}$ is a distribution of constant dimension, say k, then around any $p \in M$ there exist k independent vector fields $X_1, \dots, X_k$ such that

$$\mathcal{D}(q) = \text{span}\left\{X_1(q), \dots, X_k(q)\right\}, \qquad q \text{ near } p.$$

A distribution $\mathcal{D}$ is called *involutive* if $[X, Y] \in \mathcal{D}$ whenever $X, Y \in \mathcal{D}$. If $\mathcal{D}$ is not involutive there always exists a smallest involutive distribution containing $\mathcal{D}$. This distribution is called the *involutive closure* of $\mathcal{D}$ and is denoted by $\overline{\mathcal{D}}$.

With the covector field $\mathrm{d}h$ defined before we can define the distribution

$$\ker \mathrm{d}h(x) : p \mapsto \left\{X(p) \in T_pM | \langle \mathrm{d}h(x(p)), X(p)\rangle = 0\right\}.$$

This distribution is automatically involutive.

Theorem A.1 Frobenius theorem *(local version)*
Let $\mathcal{D}$ be an involutive constant dimensional distribution on M. Then around any $p \in M$ there exists a coordinate chart (U, ϕ) with local coordinates denoted by $x_1, \ldots, x_n$ such that:

$$\mathcal{D}(q) = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x_1}\right|_q, \ldots, \left.\frac{\partial}{\partial x_k}\right|_q \right\} = \ker\left\{ \left.\mathrm{d}x_{k+1}\right|_q, \ldots, \left.\mathrm{d}x_n\right|_q \right\}, \qquad q \in U.$$

□

There also exists a global version of the Frobenius theorem ([Spi 70]).

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