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Probability and lattices

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(editors)

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Contents

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Preface

This tract consists of six papers, which are listed here with their original preprint data. The first, fourth and fifth paper have been revised. The third paper is followed by a supplement of comments and corrections.

1. W. VERVAAT (1988): Random upper semicontinuous functions and extremal processes. Report MS-R8801, CWI, Amsterdam.
2. T. NORBERG (1990): On the convergence of probability measures on continuous posets. Report 1990-08, Dept Math., Chalmers U. of Technology, Göteborg.
3. G. GERRITSE (1985): Lattice-valued semicontinuous functions. Report 8532, Dept Math., Cath. U., Nijmegen.
4. W. VERVAAT (1988): Spaces with vaguely upper semicontinuous intersection. Report 88-30, Dept Math. and Inf., Delft U. of Technology, Delft.
5. T. NORBERG & W. VERVAAT (1989): Capacities on non-Hausdorff spaces. Report 1989-11, Dept Math., Chalmers U. of Technology, Göteborg.
6. H. HOLWERDA (1993): A note on Fell- and epicompactness. Report 9323, Dept Math., Cath. U., Nijmegen.

These papers were written in two streams of research. The first sprang from the need for a general qualitative theory of extremal processes in the 1980 research that finally resulted in the paper O'Brien, Torfs & Vervaat (1990). For extremal processes, since long a topic of active research, an abstract definition surprisingly did not exist. The first paper in this tract provides one, and its formalism has been adopted in the field soon after its prepublication in 1988.

The second stream started with work of Norberg on random capacities in the spirit of the theory of random measures as developed by his thesis advisor Olav Kallenberg (Norberg (1986)). The fifth paper in this tract generalizes the topic to non-Hausdorff spaces, a generality demanded by developments in the first stream.

An extremal process now is regarded as a random variable with values in a topological space of sup measures, or equivalently, of upper semicontinuous functions. They are topological lattices, an area of active research since the 70s. In fact, if the time domain is locally compact (but not necessarily Hausdorff), they are a major example of continuous lattices (cf. Gierz et al. (1980)). This connection is already present in the first and third paper in a fresh and rudimentary form, but more prominently and digested in the second, fifth and sixth. For a uniform theory with many isomorphisms it is essential to regard the Hausdorff property as incidental and to consider unprohibitedly T_0 spaces.

The editor discovered this in the first paper, and the resulting attitude permeates all other papers in this tract, in particular the fifth and sixth. The second paper extends the values of random variables to the more general continuous partially ordered sets, the third paper the values of semicontinuous functions to lattices. The fourth paper explores the separation condition (satisfied by all Hausdorff spaces) that renders the intersection of closed sets upper semicontinuous, and consequently also the infimum of upper semicontinuous functions. The sixth paper places the compactness of the spaces of closed sets and semicontinuous functions in the context of the best related results for continuous lattices.

Capacities appear at the end of the first paper and dominate the fifth. There are two reasons for this. On the one hand, capacities can be regarded as upper semicontinuous functions on the space of open sets provided with a non-Hausdorff topology. That is how they appear in the first paper, and are studied again in part of the fifth. On the other hand, the space of capacities contains all kinds of interesting subspaces, as the measures, the upper semicontinuous functions and the closed sets (cf. Vervaat (1988)).

In hindsight, capacities turn out to be the most natural framework for the spaces considered in this tract. The research in this spirit is more recent, and the most important initiatives can be found in O'Brien & Vervaat (1991, 1993). A new aspect in the latter papers is that capacities just seem to be made for topologizing the theory of large deviations.

The editor wishes to thank all who have contributed to the production of this tract, the other authors for their contributions, Henk Holwerda and Bart Gerritse (different from the author of the third paper) for their many remarks and corrections, and the editorial and technical staff of CWI for the physical production.

Villeurbanne, November 1993, WIM VERVAAT

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Postscript

Wim Vervaat died suddenly on January 31, 1994. At the time of his death he had almost finished the work for the present tract. Only in the first and lengthy paper by his hand a considerable number of editorial corrections still had to be carried out.
In the meantime some of the preprints referred to in this tract appeared in final form. Here are the updated references:

Preface, [5]: G.L. O'Brien & W. Vervaat (1995): Compactness in the theory of large deviations. *Stoch. Processes and their Applications* **57** 1–10.

First paper, [35]: H. Holwerda & W. Vervaat (1996): Lattice of capacities, and related topologies. *Statistica Neerlandica* **50** 306–324.

The latter is part of a special issue of Statistica Neerlandica in memory of Wim Vervaat, which also contains a complete list of his publications.

Nijmegen, February 1997, Henk Holwerda

Random Upper Semicontinuous Functions and Extremal Processes[1]

Wim Vervaat[2]

All functions take their values in the extended real line. Spaces of upper semicontinuous (usc) functions on a topological space E are considered and topologized in different ways. Convergence in distribution of random usc functions is characterized for one topology, the sup vague topology. In a canonical way, usc functions correspond to sup measures, union-sup homomorphisms on the open sets of E. Random sup measures are interpreted as extremal processes. By identifying closed subsets of E with their indicator functions we make them a subspace of the usc functions. Consequently, the basics of random closed sets are part of the theory. A function on E is usc iff its hypograph is closed in the product space of domain and range, which establishes another relation between the usc functions on E and the closed subsets of a space, this time different from E. The natural bijections between all these spaces or subsets of them turn out to be lattice isomorphisms, and homeomorphisms if the spaces are provided with the sup vague topology. All spaces are sup vaguely Hausdorff if E is locally quasicompact, but E need not be Hausdorff itself. In fact, it is better to allow E being non-Hausdorff for a smooth theory. At the end of the paper, the developed theory is applied to capacities as a common framework for vague convergence of Radon measures and sup vague convergence of usc functions.

Keywords & Phrases: upper semicontinuous functions, sup measures, spaces of closed sets, hypographs, sup vague topology, sup narrow topology, hypo topology, locally quasicompact spaces, extremal processes, random upper semicontinous functions, random closed sets, convergence in distribution, vague convergence of capacities, Radon measures as capacities.
Mathematics Classification:
Primary: 54B20, 54D45, 60B05, 60B10.
Secondary: 06B35, 28E99, 54H12, 60G99, 60K99.

0. Introduction

The original reason for the research leading to the present paper was the necessity of formalizing the notion 'extremal process' in probability theory. What came out of it turned out to be a common framework for random closed sets, parts of optimization theory, theory of hyperspaces in set topology, and extremal processes as intended. Substantial parts of this paper could be classified as set topology, and to a lesser extent as lattice theory, rather than probability theory.

Extremal processes have come up in the probabilistic literature in the following way. Let $(\xi_k)_{k=-\infty}^{\infty}$ be a sequence of real-valued random variables (for instance independent and identically distributed, but nothing is actually assumed). Set for subsets A of $\mathbb{R}$

$$M_n(A) := \Big(\bigvee_{k:k/n \in A} X_k - b_n \Big)/a_n,$$

where $a_n > 0$ and b_n are 'normalizing constants'. In the older probabilistic literature extremal processes were limits in distribution of $M_n([0,t])$ as $n \to \infty$, regarded as random functions of t (see for instance LAMPERTI (1964), DWASS (1964), RESNICK & RUBINOWITCH (1973)). In the more recent literature the idea gradually broke through that M_n should be regarded as a random set function, for instance on the intervals (PICKANDS (1971), MORI & OODAIRA (1976), MORI (1977), RESNICK (1986, 1987)). However, the full consequence of this idea has not been drawn by these authors, because the special cases considered by them allow a nice and concise description as functionals of point processes in the plane, which aspect attracted the focus of their attention. The point process approach turns out to be too narrow in the study of stationary self-similar extremal processes by O'BRIEN, TORFS & VERVAAT (1990), which forced these authors to define extremal processes as random set functions with certain properties. For a related approach, see NORBERG (1987).

It is most convenient to regard an extremal process M as a random $\bar{\mathbb{R}}$-valued function ($\bar{\mathbb{R}} := [-\infty, \infty]$) on the open sets in $\mathbb{R}$ such that M has with probability 1 (wp 1) the following property:

$$M\Big(\bigcup_{j\in J} G_j\Big) = \bigvee_{j\in J} M(G_j) \tag{01}$$

for each collection $(G_j)_{j\in J}$ of open sets in $\mathbb{R}$. More formally, let E be a topological space, $\mathcal{G} = \mathcal{G}(E)$ the collection of its open sets, and let m be a $\mathbb{I}$-valued

[1] The first version of this paper was completed June 1982, while the author was visiting the School of Operations Research and Industrial Engineering and the Center for Applied Mathematics at Cornell University, supported by a NATO Science Fellowship from the Netherlands Organization for the Advancement of Pure Research (ZWO) and a Fulbright-Hays travel grant. The second version was written in 1987 at the Centre for Mathematics and Computer Science at Amsterdam, whose hospitality and support are gratefully acknowledged. It is the version referred to in the literature up to now (1993). In the present and final version errors have been corrected and references updated. However, the author did not aim at a complete face-lift to the present state of knowledge.

[2] Wim Vervaat died early 1994. His last work address was Université Claude Bermard Lyon 1. This part of CWI Tract 110 was finalized by Henk Holwerda.

function on $\mathcal{G}$. Here $\mathbb{I}$ is a fixed compact subinterval of $\bar{\mathbb{R}}$, $\mathbb{I} = \bar{\mathbb{R}}$ in the previous application, but $\mathbb{I} = [0, 1]$ for convenience in most of the present paper. Let us call m a *sup measure* if

$$m(\bigcup_{j \in J} G_j) = \bigvee_{j \in J} m(G_j)$$

for each collection $(G_j)_{j \in J}$ in $\mathcal{G}$. Then an extremal process is just a random sup measure. In order to give this definition sense, it is necessary to make SM, the collection of all sup measures, a measurable space. For the notion of convergence in distribution of extremal processes, SM must be made a topological space, preferably with the measurable structure derived from the topological. It is exactly this what the present paper is about.

The following duality, established in Sections 1 and 2, plays a key role in the theory. If m is a function on $\mathcal{G}(E)$, then its *sup derivative* is the function $d^\vee m$ on E defined by

$$d^\vee m(t) := \inf\{m(G) : G \in \mathcal{G}, t \in G\}.$$

If f is a function on E, then its *sup integral* $i^\vee f = f^\vee$ is the function on $\mathcal{G}$ defined by

$$f^\vee(G) := \bigvee_{t \in G} f(t) \text{ for } G \in \mathcal{G}.$$

It turns out that sup measures correspond one-to-one to upper semicontinuous (usc) functions on E by $d^\vee$ and $i^\vee$. Let $US = US(E)$ be the collection of all usc functions on E (here and in the sequel all functions are assumed to be $\mathbb{I}$-valued unless stated otherwise). We now can topologize SM by topologizing US or vice versa. It turns out that for E locally compact with countable base SM becomes compact (Section 4) and metric (Section 5) with the following notion of *sup vague convergence* (Section 3):

$$m_n \to m \text{ in } SM \text{ iff } \begin{cases} \limsup m_n(K) \le m(K) \text{ for } K \in \mathcal{K}, \\ \liminf m_n(G) \ge m(G) \text{ for } G \in \mathcal{G}, \end{cases} \tag{0.2}$$

where $\mathcal{K} = \mathcal{K}(E)$ is the collection of compact sets in E. In fact, the right hand side need be required only for subcollections like bases (Section 5).

If E is locally compact with countable base, then the Borel field of SM is the smallest that makes the evaluations $m \mapsto m(A)$ measurable for all $A \in \mathcal{G}$ or all $A \in \mathcal{K}$ (Section 11), and extremal processes (= SM-valued random variables) M_n converge in distribution to M iff all finite-dimensional distributions of the values of M_n at the compact balls in E converge to those of M, where the balls B in E must be restricted to those with $\mathbb{P}[M(B) = M(\text{int}B)] = 1$ (Section 12). The finite-dimensional distributions of the values of the extremal processes on $\mathcal{G}$ can be characterized by requiring (0.1) to hold wp1 separately for each countable collection (G_j) in $\mathcal{G}$ (Section 13). In Section 14 measurability and semicontinuity of the actions of taking suprema and infima in SM are investigated.

Before discussing the connections with the literature, we first indicate some relations with spaces of closed sets. Let $\mathcal{F}(E)$ be the collection of all closed sets in E, and let 1_A for $A \subset E$ be the indicator function of A: $1_A(t) := 1$ if $t \in A, 0$ if $t \in E \backslash A$. Then $F \mapsto 1_F$ maps $\mathcal{F}$ one-to-one into $US(E)$. So we can identify (random) closed sets in E with (random) $\{0,1\}$-valued usc functions or (random) $\{0,1\}$-valued sup measures. Moreover, $\mathcal{F}(E)$ is topologized by the relative topology of the sup vague topology on $US(E)$ in its image under $F \mapsto 1_F$. We call this the sup vague topology on $\mathcal{F}(E)$. On the other hand, a function on E is usc iff its *hypograph*

$$\operatorname{hypo} f := \{(t,x) \in E \times (0,1] : x \leq f(t)\}$$

is closed in $E \times (0,1]$. So 'hypo' maps $US(E)$ one-to-one into $\mathcal{F}(E \times (0,1])$. Consequently, any topology on $\mathcal{F}(E \times (0,1])$ determines a relative topology on $US(E)$, and it turns out (Section 7) that the sup vague topology on $\mathcal{F}(E\times(0,1])$ generates the sup vague topology on $US(E)$ (for this reason often called the *hypo topology*). So it is a matter of taste on which space one wants to define the sup vague topology first. The present paper starts with US (or rather SM), in contrast with most of the literature. The reason is the duality between SM and US, which seems to have been unnoticed so far, but plays a crucial role here.

Actually, the map 'hypo' has even much nicer properties, which become visible only if one is willing to consider non-Hausdorff spaces. Let $(0,1]\!\uparrow$ denote the space $(0,1]$ provided with the *upper topology*, whose nontrivial open sets are $(x,1]$ for $0 < x < 1$. Then 'hypo' turns out to be an order preserving bijection between $US(E)$ and $\mathcal{F}(E\times(0,1]\!\uparrow)$ (Section 1), and a homeomorphism between the spaces provided with the sup vauge topology (Section 7). So if $E^* = E \times (0,1]\!\uparrow$, then $US(E)$ and $\mathcal{F}(E^*)$ are homeomorphic, whereas in the previous paragraph, with $E^* = E \times (0,1]$, $US(E)$ is only homeomorphic to a subspace of $\mathcal{F}(E^*)$.

This observation compels us to considering non-Hausdorff spaces E from the beginning. It turns out that $US(E)$ and $\mathcal{F}(E)$ are sup vaguely quasicompact (qcompact) whatever is E (Section 4). Here quasicompactness refers to the finite open subcover property, without Hausdorffness. It turns out that $US(E)$ and $\mathcal{F}(E)$ are in addition Hausdorff (hence compact) in case E is locally qcompact but not necessarily Hausdorff (Section 4). In non-Hausdorff spaces things are not as one is used to (qcompact sets need not be closed, an intersection of two qcompact sets need not be qcompact, lattice-isomorphic topologies need not come from homeomorphic spaces, etc.), and this environment is explored in Sections 8 and 9.

There are related developments in many fields of mathematics. Here we indicate them only globally. More detailed comments are made at the end of each section. Furthermore, we do not discuss the special case that E is compact and metric, in which case $\mathcal{F}(E)$ is equal to the space $\mathcal{K}(E)$ of compact sets, metrized by the Hausdorff distance. This is a classical topic in topology. Random closed sets (= $\mathcal{F}(E)$-valued random variables) are the subject of a monograph by MATHERON (1975), which was developed further by SALINETTI

& WETS (1981, 1986) and NORBERG (1984, 1986), with random usc functions appearing in the 1986 papers. Usc functions appear as images with grey levels in SERRA (1982), who attributes this idea to MATHERON in the early 1970s (personal communication).

Random closed sets appear in the equivalent shape of 'measurable closed multifunction' in the optimization literature (ROCKAFELLAR (1976), CASTAING & VALADIER (1977), KLEIN & THOMPSON (1984)). Similarly, random *lower* semicontinuous functions can be identified with 'normal integrands' in the optimization literature (ROCKAFELLAR (1976)). The two viewpoints were conciliated by SALINETTI & WETS (1981, 1986). A whole system of convergence notions ('Γ- and G-limits') was developed for variational analysis by DE GIORGI & FRANZONI (1975), DE GIORGI (1977, 1979) and BUTTAZZO (1977). The sup vague convergence notions in $US(E)$ and $\mathcal{F}(E)$ are particular cases of this. Applications in mathematical economics can be found in DEBREU (1966, 1974) and HILDENBRAND (1974).

The following topological literature is relevant for $\mathcal{F}(E)$ and $US(E)$. A convergence concept corresponding to the sup vague topology in $\mathcal{F}(E)$ for locally compact E was studied by CHOQUET (1948) and KURATOWSKI (1966), and actually has its traces in the beginning of this century. The topology itself was studied first by FELL (1962), and later on by DIXMIER (1968) and MATHERON (1975). Spaces of usc functions were already considered by MOSCO (1969) and BUTTAZZO (1977).

Spaces of subsets of a given topological space ('hyperspaces') are a topic of study in set topology. Classical references are MICHAEL (1951) and the more recent monograph by NADLER (1978). However, our sup vague topology does not occur at all in these references. For a possible reason, see our discussion in 4.6. In contrast to this, the recent monograph by KLEIN & THOMPSON (1984) also treats the sup vague topology, motivated by applications in economics and optimization.

The spaces $US(E)$ and $\mathcal{F}(E)$ (with reverse order) play a central role as examples of continuous lattices in GIERZ ET AL. (1980). Actually, a whole chapter in this monograph is devoted to a general and abstract theory of spaces of lower semicontinuous functions, which appear there in the shape of 'Scott continuous functions'. See also MISLOVE (1982) for a fast introduction. Only in the last decade locally qcompact spaces (not necessarily Hausdorff) have been studied, exclusively in the context of lattice theory. See GIERZ ET AL. (1980) and HOFMANN & MISLOVE (1981).

Sup measures are a special case of semilattice homomorphisms in GIERZ ET AL. (1980) and of semigroup-valued measures in SION (1973). The terms 'sup derivative', 'sup integral' and 'sup vague topology' remind us that we are dealing here with the 'minimax analogue' of calculus and analysis of Radon measures, in the sense of CUNINGHAME-GREEN (1979), who developed a matrix calculus and spectral theory with addition $+$ replaced by $\vee$ and multiplication replaced by $+$.

There are two topics in the paper that have not been mentioned yet. Initially, mainly in Sections 3 and 4, a whole class of topologies on SM and US

are introduced, by replacing $\mathcal{K}$ in (0.2) by some general class of sets $\mathcal{B}$, and the resulting topologies are called the sup $\mathcal{B}$ topologies. The major reason is that in this way we obtain results as well for the sup $\mathcal{F}$ or 'sup narrow' topology, which is favorite in classical set topology.

Another development is that sup vague convergence of sup measures and vague convergence of (additive) Radon measures can be put into one common framework of vague convergence of *capacities*, monotone set functions on the qcompact sets with certain semicontinuity properties. Sections 15 and 16 complement the pioneering paper by NORBERG (1986).

The author has tried to make this paper self-contained, which entails that part of the results is not new. There are several reasons for this. The results elsewhere are often formulated in the context of other fields of mathematics, and therefore not easily understandable for probabilists. And even where the formulations in the literature are more familiar, the approach in the present paper is rather different. For instance, the basics of random closed sets appear as side results of a more general theory of random usc functions, so that this paper can serve as an alternative to the introduction in MATHERON's (1975) monograph. Furthermore, the generality of non-Hausdorff spaces permeates the paper from the beginning.

There is more in probability theory than extremal processes that can benefit from a self-contained and direct introduction to random usc functions. Pointwise ordered pairs of lsc and usc functions $(-f, g \in US, f \leq g)$ form a space which is a compactification of the function space $C(E)$. By considering this compactification or related ones the proof of Donsker's theorem can be interpreted in a new way (cf. VERVAAT (1981), LENSTRA (1985) and SALINETTI & WETS (1986)). Furthermore, the most natural context for processes of random closed sets is a generalization of extremal processes, whose values are no longer in $\bar{\mathbb{R}}$, but in a lattice L, for instance $L = \mathcal{F}(E')$ for some other space E'. Lattice-valued usc functions are investigated in GERRITSE (1985), BEER (1987) and HOLWERDA (1993a), and the corresponding probability theory is being developed by NORBERG. Part of the basic theory has already been dealt with by GIERZ ET AL. (1980).

The prerequisites for the present paper are the measure theoretical foundations of probability theory, convergence in distribution in Polish spaces and basics of set topology.

0.1.. Notations and Conventions

All functions are $\mathbb{I}$-valued, unless stated otherwise; $\mathbb{I}$ is a compact subinterval of $\bar{\mathbb{R}} := [-\infty, \infty]$, for convenience $\mathbb{I} = [0, 1]$ in the present paper, but $\mathbb{I} = \bar{\mathbb{R}}$ is more appropriate for applications.

E is a topological space. No separation axioms are assumed in general. In many places all or part of the regularity conditions show up: E is locally quasicompact with countable base. In particular these are assumed in the probabilistic sections 11, 12 and 13. A subset K of E is *quasicompact (qcompact)* if each open cover contains a finite subcover; if K is in addition Hausdorff, then K is *compact*. For $A \subset E$ its *saturation* sat$A$ is the intersection of all open sets containing A; if $A = \mathrm{sat}A$, then A is *saturated.*

$\mathcal{G}$	:=	{open sets};
$\mathcal{F}$	:=	{closed sets};
$\mathcal{K}$	:=	{qcompact sets};
$\mathcal{Q}$	:=	{saturated qcompact sets};
$\mathcal{G}_0$	:=	base of open sets;
$\mathcal{K}_0$	:=	base-like collection of qcompact sets;
US	:=	{upper semicontinuous functions on E};
SM	:=	{ sup measures on $\mathcal{G}$} (cf. §2);

When E varies and the dependence on E becomes relevant, we write $\mathcal{G}(E)$, $US(E)$, etc. In this paper, F is always a closed set, G an open set, K a qcompact set, Q a saturated qcompact set. In proofs these qualities are not always mentioned. Let $A \subset E$. Then:

1_A is the indicator function of A:

$1_A(t) := 1$ for $t \in A, 0$ for $t \in E \backslash A$;

clos$A$ is the closure of A;

int$A$ is the interior of A;

sat A is the saturation of A as defined above (cf. also 1.7);

sqc A is the smallest qcompact set containing A if it exists (§8);

$A^c := E \backslash A$ is the complement of A.

Moreover,

hypo f is the hypograph of f (§1);

$d^\vee m$ is the sup derivative of m (§2);

$i^\vee f = f^\vee$ is the sup integral of f (§2).

Convergence in distribution of random variables is denoted by $\to_d$, equality in distribution by $=_d$.

lsc	=	lower semicontinuous;
usc	=	upper semicontinuous;
rv	=	random variable;
wp1	=	with probability one;
qcompact	=	quasicompact.

0.2.. Contents

1. Upper Semicontinuous Functions

In the present section we collect some results about real-valued upper semicontinuous functions, many of them well-known. All functions are defined on a topological space E, without any separation axiom assumed, and take their values in some compact interval $\mathbb{I}$ in the extended real line. In many probabilistic applications $\mathbb{I} = [0, \infty]$ or $\mathbb{I} = [-\infty, \infty]$. For convenience we fix $\mathbb{I} = [0, 1]$ in the present paper. We write $\mathbb{I}' := \mathbb{I} \setminus \inf \mathbb{I}$, so $\mathbb{I}' = (0, 1]$. For functions $f : E \to \mathbb{I}$ we define the *hypograph* of f by

$$\operatorname{hypo} f := \{(t, x) \in E \times \mathbb{I}' : x \leq f(t)\}.$$

By $[-\infty, \infty]\downarrow$ we denote the set $[-\infty, \infty]$ provided with the *lower topology*, whose nontrivial open sets are $[-\infty, x)$ for $x \in (-\infty, \infty]$. A subset A provided with the relative lower topology is denoted $A\downarrow$. Similar conventions apply to the *upper topology* on $[-\infty, \infty]$: $[-\infty, \infty]\uparrow$ has nontrivial open sets $(x, \infty]$. Observe that a nonempty subset of $A\downarrow$ is quasicompact (qcompact) iff it contains its supremum. Quasicompactness refers to the finite open subcover property. A compact set is both quasicompact and Hausdorff.

1.1. Definition. *Let E be a topological space. A function $f : E \to \mathbb{I}$ is* ***upper semicontinuous (usc)*** *at $t \in E$ if*

$$f(t) = \bigwedge_{\text{open } G \ni t} \bigvee_{u \in G} f(u).$$

A function $f : E \to \mathbb{I}$ is usc if it is usc at all $t \in E$. A function $f : E \to \mathbb{I}$ is ***lower semicontinuous (lsc)*** *if $1 - f$ is usc.*

1.2. Theorem. *The following are equivalent:*

(i) *f is usc;*

(ii) hypo$f$ *is closed in* $E \times \mathbb{I}'$;
(iii) $f : E \to \mathbb{I}\!\downarrow$ *is continuous, i.e.,* $f^{\leftarrow}[0,x)$ *is open for all* $x \in \mathbb{I}'$.

1.3. COROLLARIES. The first two corollaries are based on independent observations about functions with closed hypographs that will play a role in the proof of the theorem.
(a) Let $A \subset E$. Then 1_A has a closed hypograph iff A is closed. (From (ii). Observe that clos hypo $1_A = \text{hypo}\, 1_{\text{closA}}$.) Similarly, $x1_A \vee y1_E$ with $x > y$ has a closed hypograph iff A is closed.
(b) If $(f_j)_{j \in J}$ is a collection of functions with closed hypographs then $\bigwedge_{j\in J} f_j$ has a closed hypograph. If, moreover, J is finite, then $\bigvee_{j\in J} f_j$ has a closed hypograph. (From (ii). Observe that hypo $\bigwedge = \bigcap$ hypo, hypo $\bigvee = \bigcup$ hypo, the latter only for finite collections.)
(c) If f is usc and E is qcompact, then f has a maximum. (From (iii). Observe that $f(E)$ in $\mathbb{I}\!\downarrow$ is qcompact.)

Before proving Theorem 1.2, it is useful to make the following observation. Set

$$f_G := (\bigvee_{u \in G} f(u))1_E \vee 1_{E\setminus G}$$

for open $G \subset E$, and

$$f^* := \bigwedge_G f_G.$$

Then Definition 1.1 tells us that f is usc iff $f = f^*$.

1.4. LEMMA. clos hypo f = hypof^*.

PROOF. Obviously $f \leq f^*$, so hypo $f \subset$ hypof^*. Furthermore, hypo f^* is closed by the observations in Corollaries 1.3(a,b). So it is sufficient to prove hypo $f^* \subset$ clos hypo$f$. To this end, consider $(t, x) \notin$ clos hypo$f$. Then there is an open $G \ni t$ and a real $y < x$ such that $G \times (y, 1]$ does not intersect hypo f. Hence $f(u) \leq y < x$ for $u \in G$, so $f^*(t) \leq y < x$, i.e., $(t, x) \notin$ hypo f^*. □

PROOF OF THEOREM 1.2.
(i) ⇒ (iii). If $t \in f^{\leftarrow}[0,x)$, then $\bigwedge_{\text{open}G\ni t} \bigvee_{u\in G} f(u) = f(t) < x$, so $\bigvee_{u\in G} f(u) < x$ for some open $G \ni t$. So $t \in G \subset f^{\leftarrow}[0,x)$, which proves $f^{\leftarrow}[0,x)$ to be open.
(iii) ⇒ (ii). We have in general:

$$f = \bigwedge_{x \in \mathbb{I}'} (x1_E \vee 1_{f^{\leftarrow}[x,1]}). \tag{1.1}$$

From (iii) and the independent hypo observations in Corollaries 1.3(a,b) we see that hypo f is closed.
(ii) ⇒ (i). By Lemma 1.4 we have $f = f^*$ if hypo f is closed. □

Let $US = US(E)$ be the set of all $\mathbb{I}$-valued usc functions on E. We want to characterize US as a whole. First a notation. By $\mathcal{F} = \mathcal{F}(E)$ we denote the family of closed sets in E.

1.5. THEOREM. (a) *US is the smallest class of $\mathbb{I}$-valued functions on E that contains $x1_F$ for $x \in \mathbb{I}'$, $F \in \mathcal{F}$ and is closed for arbitrary infima and finite suprema.*
(b) *US is the smallest class of $\mathbb{I}$-valued functions on E that contains*

$$x1_F \vee y1_E \text{ for } x, y \in \mathbb{I}, x \geq y,\ F \in \mathcal{F},$$

and is closed for arbitrary infima.

PROOF. (b) Follows from (1.1), Theorem 1.2 and Corollaries 1.3(a,b).
(a) Follows from (b) and Corollary 1.3(b).

The space $US(E)$ is a complete lattice (all subsets have infima and suprema) with the infimum being pointwise infimum and the supremum of $(f_j)_j$ being $(\bigvee_j f_j)^*$. The space $\mathcal{F}(E)$ is a complete lattice with the infimum being intersection and the supremum being closure of the union.

1.6. THEOREM. *The map* hypo *is a lattice isomorphism from $US(E)$ onto $\mathcal{F}(E \times \mathbb{I}'\!\uparrow)$.*

PROOF. The family of closed sets in $E \times \mathbb{I}'\!\uparrow$ is the smallest class that contains

$$F \times (0, x] = \text{hypo}\, x1_F \text{ for } x \in \mathbb{I}', F \in \mathcal{F}(E)$$

and is closed for arbitrary intersections and finite unions. Apply Theorem 1.5(a). □

From Theorem 1.6 we learn that $US(E)$ can be examined by considering $\mathcal{F}(E^*)$ for $E^* = E \times \mathbb{I}'\!\uparrow$. However, E^* is not Hausdorff. This motivates us to maintain a generality beyond Hausdorffness in the present paper. In non-Hausdorff spaces the following notion will be useful.

1.7. DEFINITION. *The **saturation** of a set $A \subset E$ is the set*

$$\text{sat}A := \bigcap_{\text{open}G \supset A} G.$$

*If $A = \text{sat}A$, then A is said to be **saturated**.*

All sets in E are saturated iff E is T_1 (cf. §3), in particular if E is Hausdorff. Note that $u \in \text{sat}\{t\} =: \text{sat}t$ iff $t \in \text{clos}\{u\} =: \text{clos}u$. More generally, we have $\text{clos}u \bigcap \text{sat}A \neq \emptyset$ iff $\text{clos}u \cap A \neq \emptyset$. Applying this for $A = \bigcup_j A_j$ we find $\text{sat} \bigcup_j A_j = \bigcup_j \text{sat}A_j$. The intersection of saturated sets is saturated, as in general $\text{sat} \bigcap_j A_j \subset \bigcap_j \text{sat}A_j \subset \text{sat} \bigcap_j \text{sat}A_j$.

1.8. THEOREM. *If f is usc and $A \subset E$, then $\bigvee_{t\in A} f(t) = \bigvee_{t \in \text{sat}A} f(t)$.*

PROOF. As $\text{sat}A = \bigcup_{t\in A} \text{sat}t$, it is sufficient to prove the theorem for $A = \{a\}$. Since $G \ni a$ implies $G \supset \text{sat}a$ for open G, we have

$$f(a) \leq \bigvee_{t \in \text{sat}a} f(t) \leq \bigwedge_{G \ni a} \bigvee_{t \in G} f(t) = f(a).$$ □

1.9. EXAMPLE. $E = \mathbb{R}\downarrow$. Then sat$t = (-\infty, t]$ for $t \in E$. The space $US(E)$ consists of all nondecreasing right-continuous $\mathbb{I}$-valued functions on $\mathbb{R}$, and can be identified with the class of all probability distribution functions on the extended real line $[-\infty, \infty]$.

1.10. LITERATURE. Most results are classical knowledge in a perhaps less classical presentation. For a lattice-theoretical approach to *lower* semicontinuous functions, see Chapter II of GIERZ & AL. (1980). The three characterizations of upper semicontinuity in Theorem 1.2 need no longer be equivalent in case the totally ordered range $\mathbb{I}$ is replaced by a more general lattice or partially ordered space. See PENOT & THÉRA (1982), GERRITSE (1985), BEER (1987) and HOLWERDA (1993a).

2. SUP MEASURES

In the present section we introduce the sup measures, which henceforth will be close companions of usc functions. By $\mathcal{G} = \mathcal{G}(E)$ we denote the class of open sets in a topological space E.

2.1. DEFINITION. (a) *The* ***sup derivative*** *of a function* $m : \mathcal{G} \to \mathbb{I}$ *is the function* $d^\vee m : E \to \mathbb{I}$ *defined by*

$$d^\vee m(t) := \bigwedge_{G \ni t} m(G) \quad \textit{for } t \in E. \tag{21}$$

(b) *The* ***sup integral*** *of a function* $f : E \to \mathbb{I}$ *is the function* $f^\vee : \mathcal{G} \to \mathbb{I}$ *defined by*

$$f^\vee(G) := \bigvee_{t \in G} f(t) \quad \textit{for } G \in \mathcal{G},$$

where $\bigvee \emptyset := 0$. *Occasionally we will write* $i^\vee f$ *instead of* $f^\vee$.

2.2. LEMMA. *Let* m *and* f *be as in Definition 2.1. Then*
(a) $d^\vee m$ *is usc*,
(b) $m \geq i^\vee d^\vee m$,
(c) $f \leq d^\vee i^\vee f$.

PROOF. (a) Note that

$$d^\vee m = \bigwedge_{G \in \mathcal{G}} (m(G) 1_E \vee 1_{E \setminus G}),$$

so $d^\vee m$ is usc by Corollaries 1.3(a,b).
(b,c) Obvious. □

2.3. REMARK. Note that Definition 1.1 can be rephrased as: f is usc iff $f = d^\vee i^\vee f =: f^*$. In Lemma 1.4 we recognize f^* as the smallest usc function larger than f, the function with clos hypo f as hypograph.

2.4. Definition. *A function $m : \mathcal{G} \to \mathbb{I}$ is called a* **sup measure** *if $m(\emptyset) = 0$ and for all collections $(G_j)_{j\in J}$ of open sets*

$$m(\bigcup_{j\in J} G_j) = \bigvee_{j\in J} m(G_j). \tag{2.2}$$

Obviously, all sup integrals are sup measures, but different f in Definition 2.1(b) may generate the same sup measure. Example: $E = \mathbb{R}, 1_{\mathbb{R}}^{\vee} = 1_{\mathbb{Q}}^{\vee} \equiv 1$ on $\mathcal{G}\backslash\{\emptyset\}$. The following theorem shows that all sup measures are sup integrals of usc functions, and that the correspondence is one-to-one, so that $i^{\vee}$ applied to usc functions and $d^{\vee}$ to sup measures are inverses of each other.

2.5. Theorem. *Let m and f be as in Definition 2.1.*
(a) *m is a sup measure iff $m = i^{\vee}d^{\vee}m$.*
(b) *If m is a sup measure, then $f = d^{\vee}m$ is the largest f and only usc f with $f^{\vee} = m$.*
(c) *If m is a sup measure, then $\bigvee_{t\in A} d^{\vee}m(t) = \bigwedge_{G\supset A} m(G)$ for all nonempty sets $A \subset E$.*
(d) *$d^{\vee} : SM \to US$ is a bijection, and its inverse is $i^{\vee}$.*

Proof. (a) The 'if' part is trivial, the 'only if' part a special case of (c) for open A.
(b) Follows from (a), Lemma 2.2(a,c) and Remark 2.3.
(c) For all $t \in A$ we have $d^{\vee}m(t) \le \bigwedge_{G\supset A} m(G)$, so $\bigvee_{t\in A} d^{\vee}m(t) \le \bigwedge_{G\supset A} m(G)$. To prove the reverse inequality, fix $x > \bigvee_{t\in A} d^{\vee}m(t)$. For each $t \in A$ there is an open $G_t \ni t$ such that $m(G_t) < x$, so $m(\bigcup_{t\in A} G_t) \le x$, implying $\bigwedge_{G\supset A} m(G) \le x$.
(d) Follows from (a). □

Let m be an increasing $\mathbb{I}$-valued function on $\mathcal{G}$ and let $\mathcal{G}_0$ be a base of $\mathcal{G}$. Obviously, $d^{\vee}m$ does not change if we restrict G to $\mathcal{G}_0$ in (2.1). Furthermore, if m is a sup measure (hence increasing), then its values on $\mathcal{G}$ are determined by its values on $\mathcal{G}_0$ and (2.2). The following theorem characterizes which functions on $\mathcal{G}_0$ can be extended to sup measures on $\mathcal{G}$.

2.6. Theorem. (Extension Theorem). *Let $\mathcal{G}_0$ be a base of $\mathcal{G}$. If m is an $\mathbb{I}$-valued function on $\mathcal{G}_0$ such that $m(\emptyset) = 0$ and (2.2) holds whenever $G_j \in \mathcal{G}_0$ for $j \in J$ and $\bigcup_{j\in J} G_j \in \mathcal{G}_0$, then m can be extended to a unique sup measure on $\mathcal{G}$ by (2.2).*

Proof. By rephrasing Definition 2.1(a) and the proofs of Lemma 2.2(a) and Theorem 2.5(a) with $\mathcal{G}_0$ instead of $\mathcal{G}$, we obtain that $d^{\vee}m$ (in its new definition) is usc, and that $m = i^{\vee}d^{\vee}m$ on $\mathcal{G}_0$. Hence the unique extension of m to $\mathcal{G}$ is $i^{\vee}d^{\vee}m$. □

If the topology $\mathcal{G}$ of E has a countable base, then (2.2) is equivalent to its restriction to countable J, whether or not restricted further to $\mathcal{G}_0$. In particular this is the case if $E = \mathbb{R}$ and $\mathcal{G}_0$ is the collection of open intervals in $\mathbb{R}$.

2.7. EXAMPLE. Let $E = \mathbb{R}\downarrow$ as in Example 1.9. Then the sup measures m can be identified via $m(-\infty, \cdot)$ with the nondecreasing left-continuous $\mathbb{I}$-valued functions on $\mathbb{R}$, and $d^\vee m$ turns out to be the right-continuous version of $m(-\infty, \cdot)$.

2.8. LITERATURE. SHILKRET (1970) investigated sup measures with emphasis on analogues of integration theorems in measure theory. This research was continued in the wider context of capacities by NORBERG (1986). Sup measures are a special case of semigroup-valued measures as studied by SION (1973). Sup measures are semilattice homomorphisms between $(\mathcal{G}, \cup)$ and $(\mathbb{I}, \vee)$, which are studied in categorical generality by GIERZ & AL. (1980). With some effort the duality between sup measures and usc functions can be related to a Galois connection (cf. GIERZ & AL. (1980, §0.3), the dual of m being $x \mapsto \text{int} f^{\leftarrow}[0, x]$ with $f = d^\vee m$). Lemma 2.2(a) and part of Theorem 2.5 have been proved previously by BUTTAZZO (1977, Lemmas (1.5) and (1.6)) and GRAF (1980, Proposition 6.1). The terminology 'sup measure', 'sup derivative', 'sup integral' indicates that we are dealing here with 'minimax' analogues of measure theory and calculus, in the sense of CUNINGHAME-GREENE (1979) (replace + by $\vee$). Theorem 2.5(a) can be seen as the analogue of the Fundamental Theorem of integral calculus, which identifies the indefinite integral as an antiderivative.

3. THE SUP TOPOLOGIES

In the present section we introduce a class of topologies on $SM = SM(E)$, the lattice of sup measures on $\mathcal{G}(E)$. By Theorem 2.5 we may identify SM with US via the bijections

$$SM(E) \underset{i^\vee}{\overset{d^\vee}{\rightleftarrows}} US(E),$$

so all topologies on SM carry over to US by declaring $d^\vee$ and $i^\vee$ homeomorphisms. The map 'ind': $\mathcal{F} \ni F \mapsto 1_F$ injects $\mathcal{F}$ into US, and each topology on US induces in this way a relative topology on $\mathcal{F}$.

Recall that the sup measures as defined in Section 2 have the open sets $\mathcal{G}$ as their domain. However, by Theorem 2.5(c) there is a canonical extension to all subsets A of E by

$$m(A) := \bigvee_{t \in A} d^\vee m(t) = \bigwedge_{G \supset A} m(G). \tag{31}$$

The right-hand side depends only on A via sat A, so

$$m(A) = m(\text{sat } A) \text{ for } A \subset E, \tag{32}$$

which result is equivalent to Theorem 1.8 by Theorem 2.5. Two classes of subsets of E will determine the topology on SM, the open sets $\mathcal{G}$ and another class $\mathcal{B}$, the ***bounding class*** of the topology. For a bounding class we require only that it contains $\emptyset$ (this condition does not matter here, but will be convenient later on when we consider $\mathcal{F}(E)$). Examples of bounding classes are:

$$\begin{array}{lllllll}
\mathcal{B} & = & \{\emptyset\}, & & & & \\
\mathcal{B} & = & \mathcal{J} & = & \mathcal{J}(E) & := & \{\text{finite subsets of } E\}, \\
\mathcal{B} & = & \mathcal{K} & = & \mathcal{K}(E) & := & \{\text{qcompact subsets of } E\}, \\
\mathcal{B} & = & \mathcal{F} & = & \mathcal{F}(E) & := & \{\text{closed subsets of } E\} \text{ as defined before}, \\
\mathcal{B} & = & \mathcal{G} & = & \mathcal{G}(E), & & \\
\mathcal{B} & = & \mathcal{F}_d & & & := & \{d\text{-bounded closed subsets of } E\},
\end{array}$$

where d is a metric that metrizes the topology of E.

3.1. DEFINITION. *The* ***sup topology*** *on SM(E) with* ***bounding class*** $\mathcal{B}$*, or the* ***sup*** $\mathcal{B}$ ***topology*** *on SM (E), is the smallest topology that makes the evaluations*

$$m \mapsto m(A) \quad usc\, for\, A \in \mathcal{B}, \quad lsc\, for\, A \in \mathcal{G}. \tag{3.3}$$

3.2. REMARKS. (a) The sets

$$\begin{aligned} &\{m \in SM : m(B) < x\},\ \{m \in SM : m(G) > x\} \text{ for} \\ &\quad B \in \mathcal{B}, G \in \mathcal{G}, x \in \mathbb{I} \end{aligned} \tag{3.4}$$

form a subbase of the sup $\mathcal{B}$ topology in SM.
(b) A net (m_n) converges sup $\mathcal{B}$ to m in SM iff

$$\begin{array}{lllll} \limsup_n m_n(B) & \le & m(B) & \text{for} & B \in \mathcal{B}, \\ \liminf_n m_n(G) & \ge & m(G) & \text{for} & G \in \mathcal{G}. \end{array} \tag{3.5}$$

For *additive* Radon measures and $\mathcal{B} = \mathcal{K}$ (3.5) is known to characterize vague convergence. Similarly, (3.5) with $\mathcal{B} = \mathcal{F}$ characterizes weak (or narrow) convergence for *additive* bounded measures, in particular probability measures. Therefore we call sup $\mathcal{K}$ convergence also *sup vague* convergence, and sup $\mathcal{F}$ convergence also *sup weak* (or *sup narrow*) convergence.
(c) If $m_n \to m$ in the sup $\{\emptyset\}$ topology, then also $m_n \to m'$ for each $m' \le m$. So the sup $\mathcal{B}$ topology is not Hausdorff for $\mathcal{B} = \{\emptyset\}$. This may happen also for other $\mathcal{B}$.
(d) The sup $\mathcal{G}$ topology on SM is relative to the product topology on $\mathbb{I}^{\mathcal{G}}$.
(e) The sup $\mathcal{B}$ topology on SM does not change if
(e1) $\mathcal{B}$ is enlarged to be closed for finite unions,
(e2) $\mathcal{B}$ is replaced by $\mathcal{B}^{\text{sat}} := \{\text{sat}B : B \in \mathcal{B}\}$.
For (e1), note that

$$\begin{aligned} &\limsup_n m_n(B_1 \cup B_2) = \limsup_n (m_n(B_1) \vee m_n(B_2)) \\ &\le \limsup_n m_n(B_1) \vee \limsup_n m_n(B_2) \le m(B_1) \vee m(B_2) = m(B_1 \cup B_2) \end{aligned}$$

if $m_n \to m$ and $B_1, B_2 \in \mathcal{B}$; (e2) follows from (3.2).

3.3. DEFINITION. *The sup* $\mathcal{B}$ *topology on* $US(E)$ *is the topology that makes the bijection* $d^{\vee}$ *between* $SM(E)$ *and* $US(E)$ *a homeomorphism. The sup* $\mathcal{B}$ *topology on* $\mathcal{F}(E)$ *is the topology that makes the injection 'ind' from* $\mathcal{F}(E)$ *into* $US(E)$ *a homeomorphism.*

3.4. PROPERTIES. (a) The sup $\mathcal{B}$ topology on $US(E)$ is the smallest that makes the sup evaluations

$$f \mapsto f^\vee(A) \text{ usc for } A \in \mathcal{B}, \text{ lsc for } A \in \mathcal{G}, \tag{3.6}$$

so $f_n \mapsto f$ in $US(E)$ iff

$$\begin{aligned} &\limsup_n f_n^\vee(B) \le f^\vee(B) \text{ for } B \in \mathcal{B}, \\ &\liminf_n f_n^\vee(G) \ge f^\vee(G) \text{ for } G \in \mathcal{G}. \end{aligned} \tag{3.7}$$

(b) The sets

$$\{F \in \mathcal{F} : F \cap B = \emptyset\} \text{ for } B \in \mathcal{B}, \quad \{F \in \mathcal{F} : F \cap G \neq \emptyset\} \text{ for } G \in \mathcal{G} \tag{3.8}$$

form a subbase for the sup $\mathcal{B}$ topology on $\mathcal{F}(E)$. Note that $\mathcal{F}$ itself belongs to it, because $\emptyset \in \mathcal{B}$. A net (F_n) converges sup $\mathcal{B}$ to F in $\mathcal{F}(E)$ iff the following implications hold:

$$\begin{aligned} F \cap B = \emptyset \quad &\Rightarrow \quad F_n \cap B = \emptyset \quad \text{for all sufficiently large } n \ (B \in \mathcal{B}), \\ F \cap G \neq \emptyset \quad &\Rightarrow \quad F_n \cap G \neq \emptyset \quad \text{for all sufficiently large } n \ (G \in \mathcal{G}). \end{aligned} \tag{3.9}$$

In set topology one usually preferred to consider $\mathcal{F}(E)$ with the sup $\mathcal{F}$ topology. In probability and optimization one considered $\mathcal{F}(E)$, and more recently also $US(E)$, with the sup $\mathcal{K}$ topology, for locally compact E. When it comes to probability in the present paper, we will restrict ourselves to the sup $\mathcal{K}$ topology on $US(E)$ for locally qcompact E (not necessarily Hausdorff).

The properties of the sup $\mathcal{B}$ topologies depend strongly on the separation axioms assumed for the topology $\mathcal{G}$ on E and the interaction between $\mathcal{G}$ and the bounding class $\mathcal{B}$. Here we list the separation axioms and interaction hypotheses that occur in this paper.

3.5. SEPARATION AXIOMS FOR $\mathcal{G}$. *The space E (or rather its topology $\mathcal{G}$) is*
(a) T_0 *if for each $\{t,u\} \subset E$ there is a $G \in \mathcal{G}$ such that $\#(G \cap \{t,u\}) = 1$,*
(b) T_1 *if for each $\{t,u\} \subset E$ there are $G_1, G_2 \in \mathcal{G}$ such that $G_1 \cap \{t,u\} = \{t\}, G_2 \cap \{t,u\} = \{u\}$,*
(c) T_2 *(Hausdorff) if G_1 and G_2 in (b) can be chosen disjoint,*
(d) T_3 *if for each $t \in E$ and $F \in \mathcal{F}$ with $t \notin F$ there are disjoint $G_1, G_2 \in \mathcal{G}$ such that $t \in G_1, F \subset G_2$.*

3.6. LOCAL AXIOMS FOR $\mathcal{B}$. *The space E (or rather its topology $\mathcal{G}$) is*
(a) ***locally*** *$\mathcal{B}$ if for each $t \in E$ and each open $G \ni t$ there is a $B \in \mathcal{B}$ such that $t \in \text{int}B \subset B \subset G$,*
(b) ***internally*** *$\mathcal{B}$ for each $t \in E$ there is a $B \in \mathcal{B}$ such that $t \in \text{int}B$,*
(c) ***fragmentally*** *$\mathcal{B}$ if for each $t \in E$ and each open $G \ni t$ there is a $B \in \mathcal{B}$ such that $t \in B \subset G$.*
Synonyms for locally, internally and fragmentally $\mathcal{K}$ are locally, internally and fragmentally qcompact (compact if E is T_2).

3.7. PROPERTIES. (a) If E is locally $\mathcal{B}$, then E is internally $\mathcal{B}$ and fragmentally $\mathcal{B}$.
(b) the space E is locally $\mathcal{B}$ iff $\mathcal{B}$ contains a neighborhood base at t for each $t \in E$ iff for each open G there is a collection $\{B_j\}_{j \in J} \subset \mathcal{B}$ such that

$$G = \bigcup_{j \in J} \text{int} B_j = \bigcup_{j \in J} B_j.$$

(c) the space E is internally $\mathcal{B}$ iff $E = \bigcup_{B \in \mathcal{B}^*} \text{int } B$, in particular if $E \in \mathcal{B}$.
(d) If E is T_2, then E is locally compact iff E is internally compact. A similar equivalence does not hold for local qcompactness in absence of T_2. In particular, a qcompact E is internally qcompact by (c), but need not be locally qcompact. For an example of the latter, consider the one-point qcompactification E' of a Hausdorff space E, obtained by adding one point ∞ to E and making the complements in E' of compact sets in E to be its open neighborhoods. Then E' is qcompact, and E' is locally qcompact iff E' is Hausdorff iff E is locally compact.
(e) The space E is locally $\mathcal{G}$ and internally $\mathcal{F}$. The space E is locally $\mathcal{F}$ iff E is T_3.
(f) If $\mathcal{J} \subset \mathcal{B}$, then E is is fragmentally $\mathcal{B}$; $\mathcal{J} \subset \mathcal{K}$; $\mathcal{J} \subset \mathcal{F}$ iff E is T_1.
(g) $\mathcal{F} \subset \mathcal{K}$ iff $E \in \mathcal{K}$. If E is T_2, then $\mathcal{K} \subset \mathcal{F}$.

3.8. EXAMPLE. Let $E = \mathbb{R}\!\downarrow$ as in Examples 1.9 and 2.7. Then

$$\begin{aligned}
&\mathcal{G} = \{\emptyset, \mathbb{R}, (-\infty, t) : t \in \mathbb{R}\},\\
&\mathcal{F} = \{\emptyset, \mathbb{R}, [t, \infty) : t \in \mathbb{R}\},\\
&\mathcal{K} = \{\emptyset, A \subset \mathbb{R} : \sup A \in A\},\\
&\mathcal{F}^{\text{sat}} = \{\emptyset, \mathbb{R}\},\\
&\mathcal{K}^{\text{sat}} = \{\emptyset, (-\infty, t] : t \in \mathbb{R}\}.
\end{aligned}$$

The space $\mathbb{R}\!\downarrow$ is T_0, but not T_1, T_2, T_3 (= locally $\mathcal{F}$) or fragmentally $\mathcal{F}$. It is locally qcompact with countable base $\{\emptyset, (-\infty, t) : t \in \mathbb{Q}\}$. The spaces US and SM have been described in Examples 1.9 and 2.7. Recall that US can be identified with the probability distribution functions on $[-\infty, \infty]$.
The following characterizes $f_n \to f$ sup $\mathcal{B}$ in US for different $\mathcal{B}$:
(a) $\mathcal{B} = \{\emptyset\} : f(t-) \leq \liminf_n f_n(t-)$ for $t \in \mathbb{R}$,
(b) $\mathcal{B} = \mathcal{F}$ or $\{\emptyset, \mathbb{R}\}$: (a) and $\limsup_n f_n(\infty-) \leq f(\infty-)$,
(c) $\mathcal{B} = \mathcal{G} : \lim_n f_n(t-) = f(t-)$ for $t \in \mathbb{R}$,
(d) $\mathcal{B} = \mathcal{K}$ or $\mathcal{J}$:
$f(t-) \leq \liminf_n f_n(t-) \leq \limsup_n f_n(t) \leq f(t)$ for $t \in \mathbb{R}$
$\Leftrightarrow \lim_n f_n(t) = f(t)$ for all t where $f(t-) = f(t)$.
So US is not sup $\mathcal{B}$ Hausdorff for $\mathcal{B} = \{\emptyset\}$ or $\mathcal{F}$. It is Hausdorff, even compact for $\mathcal{B} = \mathcal{K}$ or $\mathcal{G}$, and metrizable for $\mathcal{B} = \mathcal{K}$ by a Lévy-type distance) but not for $\mathcal{B} = \mathcal{G}$. So $US(\mathbb{R}\!\downarrow)$ is sup vaguely compact and metrizable, even though $\mathbb{R}\!\downarrow$ is not Hausdorff. In the next sections we will identify the relevant properties of $\mathbb{R}\!\downarrow$ as local qcompactness with countable base.

3.9. LITERATURE. Sup $\mathcal{B}$ topologies for various $\mathcal{B}$ were considered by ARENS & DUGUNDJI (1951), POPPE (1965, 1966) and MRÓWKA (1970). The sup $\mathcal{F}$ or sup narrow topology in $\mathcal{F}$ has been a major topic in set topology (e.g. MICHAEL (1951) and NADLER (1978)). More common names are 'Vietoris' or 'finite' topology. A convergence concept in $\mathcal{F}$ that is topologized by the sup $\mathcal{K}$ or sup vague topology in case E is locally qcompact is known since long, in fact as a combination of notions of upper and lower limits in $\mathcal{F}$ (cf. CHOQUET (1948), MRÓWKA (1958), KURATOWSKI (1966, §29), BERGE (1963, §I.9) and FLACHSMEYER (1964)). The first discussion about the sup vague topology in $\mathcal{F}$ as a topology is given by FELL (1962), followed by DIXMIER (1968), and MATHERON (1975) for E locally compact with countable base. Both the sup narrow and sup vague topologies are treated by KLEIN & THOMPSON (1984). Local axioms as in 3.6 were considered by CEDER (1961).

If E is metric and metrized by d, then $\mathcal{K}$ is metrized by the *Hausdorff metric*

$$\rho(K_1, K_2) := \bigvee_{t\in K_1} d(t, K_2) \vee \bigvee_{t\in K_2} d(t, K_1)$$

(establishing distance ∞ between $\emptyset$ and nonempty sets). If E is compact, then $\mathcal{F}$ and $\mathcal{K}$ coincide, and the sup vague and sup narrow topologies are the same and generated by the Hausdorff distance. The space $\mathcal{K}$ with the Hausdorff distance is classical and will not be discussed here.

The sup vague or 'hypo' (cf. Section 7) topology on US has been considered by several authors. Some of them start with the convergence concept: $f_n \to f$ in US iff for each $t \in E$ we have $\limsup_n f_n(t_n) \leq f(t)$ for all sequences $(t_n) \to t$ in E, and $\lim_n f_n(t_n) = f(t)$ for some sequence $(t_n) \to t$ in E (E locally compact with countable base). This is the case with DE GIORGI & FRANZONI (1975), BUTTAZZO (1977), who in fact study a more general collection of upper and lower limits in topology ('Γ- and G-limits'), which is also considered by DE GIORGI (1977, 1979). Other authors start with the embedding 'hypo': $US(E) \to \mathcal{F}(E \times \mathbb{I}')$, like MOSCO (1969) for convex functions, BEER (1982) for compact E and SALINETTI & WETS (1986). Characterizations (3.6) and (3.7) of the sup vague topology in US also occur in SALINETTI & WETS (1986) and NORBERG (1986). Only NORBERG (1986) and the present paper take it as starting point.

The sup vague topologies in US and $\mathcal{F}$ are a special case of the Lawson topology in continuous lattices (with reverse order) in case E is locally qcompact, cf. Th.II.4.7 and Ch.III of GIERZ ET AL. (1980).

4. GENERAL PROPERTIES OF THE SUP TOPOLOGIES

We assume $SM(E)$, $US(E)$ and $\mathcal{F}(E)$ provided with a sup $\mathcal{B}$ topology for some bounding class $\mathcal{B}$. Note that $\mathcal{F}$ is a subspace of US after identification with its image under 'ind'. We start with examining this subspace.

4.1. THEOREM. *The range* $\text{ind}(\mathcal{F})$ *is sup* $\mathcal{B}$ *closed in US iff E is locally* $\mathcal{B}$.

PROOF. Let E be locally $\mathcal{B}$. If $f \in US\backslash\text{ind}\mathcal{F}$, then $f(t) \in (0,1)$ for some $t \in E$. Select G such that $t \in G$ and $f^\vee(G) < 1$. Then select $B \in \mathcal{B}$ with $t \in \text{int}B \subset B \subset G$. The basic open set

$$\{g \in US : g^\vee(B) < 1,\ g^\vee(\text{int}B) > 0\}$$

contains f, but does not intersect $\text{ind}\mathcal{F}$. This proves that $US\backslash\text{ind}\mathcal{F}$ is open.

Conversely, assume that $US\backslash\text{ind}\mathcal{F}$ is open. By Remark 3.2.(e) we may also assume that $\mathcal{B}$ is closed for finite unions and consists of saturated sets. Consider

$$f := \frac{1}{2}1_{G\cap\operatorname{clos} t} \vee 1_{G^c}.$$

Then $f \in US$. We investigate when generic basic open sets

$$U = \{g \in US : g^\vee(B_i) < x_i \text{ for } 1 \le i \le m,\ g^\vee(G_j) > y_j \text{ for } 1 \le j \le n\}$$

contain f and do not interesect $\text{ind}\mathcal{F}$. For the latter, the actual values of x_i and y_j do not matter and may be replaced by extreme values. So we need to consider only

$$U = \{g \in US :\ g^\vee(B) < 1,\quad g^\vee(G_j) > 0 \text{ for } 1 \le j \le n\}.$$

We have $U \cap \text{ind}\mathcal{F} = \emptyset$ iff $G_j \subset B$ for some j, say $j = k$ (if existing, consider $1_{\cup_j \operatorname{clos} t_j}$ with $t_j \in G_j\backslash B$ for $j = 1, 2\ldots, n$, and note that $\operatorname{clos} t_j \cap B = \emptyset$ iff $t_j \notin B$ because B is saturated). We have $f \in U$ iff $B \subset G$ and $G_j \cap (G^c \cup \operatorname{clos} t) \ne \emptyset$ for all j. Considering $j = k$ we find $G_k \subset B \subset G$ and $G_k \cap \operatorname{clos} t \ne \emptyset$, so $t \in G_k \subset B \subset G$. We have proved that E is locally $\mathcal{B}$. □

4.2. THEOREM. *The following are equivalent:*
(i) *SM and US are qcompact;*
(ii) *$\mathcal{F}$ is qcompact;*
(iii) *$\mathcal{B} \subset \mathcal{K}$.*

4.3. THEOREM. (a) *SM, US and $\mathcal{F}$ are T_0.*
(b) *If E is fragmentally $\mathcal{B}$, then SM, US and $\mathcal{F}$ are T_1.*
(c) *If E is locally $\mathcal{B}$, then SM, US and $\mathcal{F}$ are T_2.*
(d) *If $\mathcal{F}$ is T_2, then E is internally $\mathcal{B}$.*

4.4. COROLLARIES. (a) The spaces SM, US and $\mathcal{F}$ are sup vaguely (= sup $\mathcal{K}$) qcompact for general E, and moreover compact if E is locally qcompact. If E is T_2 and SM, US or $\mathcal{F}$ is sup vaguely compact, then E is internally compact, so locally compact by Property 3.7(d). So if E is T_2, then each of SM, US and $\mathcal{F}$ is compact iff E is locally compact.
(b) The spaces SM, US and $\mathcal{F}$ are sup weakly (= sup $\mathcal{F}$) qcompact iff E is qcompact, and sup weakly compact if E is qcompact and T_3 (= locally $\mathcal{F}$).

PROOF OF THEOREM 4.2. By (3.4) each closed set in US is an intersection of finite unions of

$$\{f : f^\vee(B) \ge x\},\ \{f : f^\vee(G) \le y\} \text{ for } B \in \mathcal{B},\ G \in \mathcal{G},\ x, y \in \mathbb{I}.$$

By Alexander's subbase theorem (KELLEY (1955, p.139)) US is qcompact iff for each instance of

$$\begin{aligned}&\bigcap_{i\in I}\{f : f^\vee(B_i) \ge x_i\} \cap \bigcap_{j\in I}\{f : f^\vee(G_j) \le y_j\} =: \\ &\bigcap_{i\in I} F_{1,i} \cap \bigcap_{j\in J} F_{2,j} = \emptyset\end{aligned} \tag{4.1}$$

the same holds true with I and J replaced by finite subsets. Set

$$g := \bigwedge_{j\in J} (y_j 1_E) \vee 1_{E\setminus G_j}).$$

Then g is usc by Corollaries 1.3(a,b), and $\bigcap_{j\in J} F_{2,j} = \{f : f \leq g\}$. Furthermore, if $f_1,\ f_2 \in US, f_1 \leq f_2$ and $\bigcap_{i\in I} F_{1,i}$ contains f_1, then also f_2. So (4.1) holds iff

$$g^\vee(B_i) < x_i \text{ for some } i \in I. \tag{4.2}$$

(iii) $\Rightarrow$ (i). Assume $\mathcal{B} \subset \mathcal{K}$. Assume further that $E = \bigcup_{j\in J} G_j$ (this is no restriction: if necessary, add a j with $G_j = E$ and $y_j = 1$ in (4.1)). Suppose that (4.1) holds and fix an i that realizes (4.2). Let $J_i \subset J$ be the collection of j such that $B_i \cap G_j \neq \emptyset$ and $y_j < x_i$. Then $B_i \subset \bigcup_{j\in J_i} G_j$. As B_i is qcompact, we have $B_i \subset \bigcup_{j\in J_\#} G_j$ for some finite $J_\# \subset J_i$. Defining $g_\#$ by reducing J to $J_\#$ in the definition of g we find

$$g_\#^\vee(B_i) \leq \bigvee_{j\in J_\#} g_\#^\vee(G_j) \leq \bigvee_{j\in J_\#} y_j < x_i,$$

so (4.1) already holds with $\{i\}$ instead of I and $J_\#$ instead of J. We have proved that US is qcompact.
(i) $\Rightarrow$ (iii). Conversely, if US is qcompact, consider (4.1) with only one $i, B_i = B \in \mathcal{B},\ x_i := 1,\ y_j := 0$ for $j \in J$. (4.1) is equivalent to (4.2), thus to

$$B \subset \bigcup_{j\in J} G_j. \tag{4.3}$$

Reduction to finite J, being possible as US is qcompact, is equivalent to the same reduction in (4.3). Hence B is qcompact, which proves $\mathcal{B} \subset \mathcal{K}$.
(iii) $\Leftrightarrow$ (ii). Repeat the proof of (iii) $\Leftrightarrow$ (i) with ind($\mathcal{F}$) instead of US, $\{0,1\}$-valued $f, x_i := 1,\ y_j := 0$. □

PROOF OF THEOREM 4.3 (a,b,c). It is sufficient to prove the statements for US, as SM is homeomorphic and $\mathcal{F}$ is a subspace. So suppose $g, h \in US$ and $g \neq h$. Then $g(t) \neq h(t)$ for some $t \in E$, say $g(t) < h(t)$. Let $g(t) < x < h(t)$. Then $g^\vee(G) < x$ for some open $G \ni t$, so the basic open set $\{f \in US : f^\vee(G) > x\}$ contains h, but not g. This proves US to be T_0.

If moreover, E is fragmentally $\mathcal{B}$, then we can find $\mathcal{B}$ with $t \in B \subset G$, and the basic open set $\{f \in US : f^\vee(B) < x\}$ contains g, but not h. This proves US to T_1.

Finally, if E is locally $\mathcal{B}$, then select $B \in \mathcal{B}$ with $t \in \text{int}B \subset B \subset G$. Then $\{f \in US : f^\vee(B) < x\}$ and $\{f \in US : f^\vee(\text{int}B) > x\}$ are disjoint neighborhoods of g and h, which proves US to be T_2. □

Before proving Theorem 4.3(d), we examine first the basic open sets in $\mathcal{F}$, intersections of finitely many sets in the subbase (3.8). If $\mathcal{B}$ is closed for finite unions, then they have the form:

$$\{F \in \mathcal{F} : F \cap B = \emptyset,\ F \cap G_j \neq \emptyset \text{ for } j = 1, 2, \ldots, n\} \tag{44}$$

with $n \in \mathbb{N}_0, B \in \mathcal{B}$ and $G_j \in \mathcal{G}$ for $j = 1, 2, \ldots, n$. In particular, we must know when a set as in (4.4) is not empty.

4.5. LEMMA. *The set in (4.4) is empty iff $G_j \subset \text{sat}B$ for some j.*

PROOF. The set in (4.4) is empty iff for each open $H \supset B(H := E\backslash F)$, there is a G_j with $H \supset G_j$. The latter holds if $G_j \subset \text{sat}B$ for some j. If, on the other hand, $G_j \subset \text{sat}B$ for no j, then there is for each j an open $H_j \supset B$ such that $H_j \not\supset G_j$. Consequently, $H := \bigcap_{j=1}^n H_j \supset B$ and $H \supset G_j$ for no j. □

PROOF OF THEOREM 4.3(d). It is no restriction to assume $\mathcal{B}$ closed for finite unions. For if $\mathcal{C}$ is the collection of finite unions in $\mathcal{B}$, then $\mathcal{C}^{\text{sat}}$ is the collection of finite unions in $\mathcal{B}^{\text{sat}}$, and if E is internally $\mathcal{C}^{\text{sat}}$, then E is also internally $\mathcal{B}^{\text{sat}}$. So let $\mathcal{B}$ be closed for finite unions and let $\mathcal{F}$ be T_2. Fix $t \in E$. Then $\emptyset$ and clos$t$ have disjoint neighborhoods in $\mathcal{F}$. By (4.4) their form is

$$\begin{aligned} U &= \{F \in \mathcal{F} : F \cap B_1 = \emptyset\}, \\ V &= \{F \in \mathcal{F} : F \cap B_2 = \emptyset,\ F \cap G_j \neq \emptyset \text{ for } j = 1, 2, \ldots, n\} \end{aligned}$$

with $t \in G_j$ for $j = 1, 2, \ldots, n$ (note that $G_j \cap \text{clos}\, t \neq \emptyset$ iff $t \in G_j$). By Lemma 4.5 we have $U \cap V = \emptyset$ iff $G_j \subset \text{sat}(B_1 \cup B_2)$ for some j. So there is a j with $t \in G_j \subset \text{sat}(B_1 \cup B_2)$, which proves E to be internally $\mathcal{B}^{\text{sat}}$. □

4.6. LITERATURE. Special cases of Theorem 4.1 occur in KLEIN & THOMPSON (1984). The sup narrow topology in $\mathcal{F}$ is more 'hereditary' in its properties (cf. Property 4.4(b)). This could explain why this topology received almost exclusive attention from set topologists. The fact that $\mathcal{F}$ is sup vaguely qcompact has been proved by many authors, e.g. CHOQUET (1948), FELL (1962), KLEIN & THOMPSON (1984), and MATHERON (1975) for E locally compact with countable base. Sup vague qcompactness of US has been proved by BUTTAZZO (1977) for E being locally qcompact with countable base and SALINETTI & WETS (1981) for $E = \mathbb{R}^d$. It also follows from Th.II.4.7 and Th.III.1.10 of GIERZ ET AL. (1980). The last part of Corollary 4.4(a) has been proved also by DIXMIER (1968) and GIERZ ET AL. (1980). The latter reference contains also an extension of the equivalence to non-Hausdorff spaces: If E is *sober* (cf. Section 9), then US and $\mathcal{F}$ are sup vaguely Hausdorff iff E is locally qcompact. For a new and very simple proof that US is sup vaguely compact, see HOLWERDA (1993b).

5. THINNING THE CONVERGENCE CRITERIA TO BASES

The object of the present section is to thin out characterization (3.3) of the sup $\mathcal{B}$ topology on SM (and thus also on US and $\mathcal{F}$) to equivalent characterizations with $\mathcal{B}$ and $\mathcal{G}$ replaced by subclasses $\mathcal{B}_0$ and $\mathcal{G}_0$. The $\mathcal{G}$ part is easy.

5.1. LEMMA. *Let $\mathcal{G}_0$ be a base of $\mathcal{G}$. If the evaluation*

$$SM \ni m \mapsto m(G) \in \mathbb{I}$$

is lsc for $G \in \mathcal{G}_0$, then also for $G \in \mathcal{G}$.

PROOF. If $G \in \mathcal{G}$, then $G = \bigcup_{j \in J} G_j$ for some collection $\{G_j\}_{j \in J} \subset \mathcal{G}_0$, so $m \mapsto m(G) = \bigvee_{j \in J} m(G_j)$ is lsc as supremum of lsc functions. □

The $\mathcal{B}$ part is more demanding, and will in fact be handled only in case $\mathcal{B} = \mathcal{K}$.

5.2. LEMMA. *Let E be locally $\mathcal{K}_0$ with $\mathcal{K}_0 \subset \mathcal{K}$. If*

$$SM \ni m \mapsto m(K) \in \mathbb{I}$$

is usc for $K \in \mathcal{K}_0$, then also for $K \in \mathcal{K}$.

PROOF. We may and will assume $\mathcal{K}_0$ to be closed for finite unions (cf. Remark 3.2(e1)). We will show that

$$m(K) = \bigwedge_{Q \in \mathcal{K}_0 : Q \supset K} m(Q) \text{ for } K \in \mathcal{K}, \tag{5.1}$$

from which it follows that $m \mapsto m(K)$ is usc as infimum of usc functions. Trivially we have $\leq$ instead of $=$ in (5.1). To prove the reverse inequality, let $m(K) < x$. We will show that $m(Q) \leq x$ for some $Q \in \mathcal{K}_0$ with $Q \supset K$. We have $K \subset G := (d^{\vee} m)^{\leftarrow}[0, x)$, which is open by Lemma 2.2 and Theorem 1.2. By Property 3.7(b) we have $G = \bigcup_{j \in J} \mathrm{int} K_j = \bigcup_{j \in J} K_j$ for some collection $\{K_j\}_{j \in J} \subset \mathcal{K}_0$. As $K \in \mathcal{K}$ and $K \subset G$, there is a finite subset $J_{\#}$ of J such that

$$K \subset \bigcup_{j \in J_{\#}} \mathrm{int} K_j \subset \bigcup_{j \in J_{\#}} K_j =: Q \subset G.$$

Now $Q \in \mathcal{K}_0, Q \supset K$ and $m(Q) \leq m(G) \leq x$. □

5.3. THEOREM. *If $\mathcal{G}_0$ is a base of $\mathcal{G}, \mathcal{K}_0 \subset \mathcal{K}$ and E is locally $\mathcal{K}_0$, then the sup vague (= sup $\mathcal{K}$) topology on SM is the smallest that makes the evaluations*

$$m \mapsto m(A) \text{usc for } A \in \mathcal{K}_0, \text{ lsc for } A \in \mathcal{G}_0.$$

PROOF. Combine Lemmas 5.1 and 5.2. □

5.4. LEMMA. *If $\mathcal{G}_0$ is a countable base of $\mathcal{G}$ and E is locally $\mathcal{K}$, then E is locally $\mathcal{K}_0$ for some countable $\mathcal{K}_0 \subset \mathcal{K}$.*

PROOF. For all $G_1, G_2 \in \mathcal{G}_0$ such that there is at least one $K \in \mathcal{K}$ with $G_1 \subset \mathrm{int} K \subset K \subset G_2$, select one $K(G_1, G_2) \in \mathcal{K}$. Set $\mathcal{K}_0 := \{K(G_1, G_2)\}$. □

5.5. THEOREM. *If E is locally qcompact, then SM, US or $\mathcal{F}$ are sup vaguely metrizable iff E has a countable base.*

5.6. REMARK. If E is locally qcompact, then SM, US and $\mathcal{F}$ are sup vaguely compact by Corollary 4.4(a), so metrizable iff they have a countable base.

PROOF OF THEOREM 5.5. By Remark 5.6 we must show that SM, US or $\mathcal{F}$ has a countable base iff E has one. If E has a countable base $\mathcal{G}_0$, then E is locally $\mathcal{K}_0$ for some countable $\mathcal{K}_0 \subset \mathcal{K}$ by Lemma 5.4, and a countable subbase of the sup vague topology on SM is given by the subbase in (3.4) with $B \in \mathcal{K}_0, G \in \mathcal{G}_0$ and $x \in \mathbb{I} \cap \mathbb{Q}$. So SM has a countable base, as does US and its subspace $\mathcal{F}$.

If $\mathcal{F}$ has a countable base $\mathcal{U}$ of the sup vague topolgy, then it has also a countable base $\mathcal{V}$ consisting of finite intersections of subbase sets as in (3.8) (select one such set between each pair of $U_1 \subset U_2$ in $\mathcal{U}$). Let τ be the coarser topology in $\mathcal{F}$ with subbase consisting of $\emptyset$ and $\{F \in \mathcal{F} : F \cap G \neq \emptyset\}$ for $G \in \mathcal{G}$. Then each $\{F \in \mathcal{F} : F \cap G \neq \emptyset\}$ is union of elements of $\mathcal{V}$, but does not have any $\{F \in \mathcal{F} : F \cap K = \emptyset\}$ as subset, since it does not contain $\emptyset \in \{F \in \mathcal{F} : F \cap K = \emptyset\}$. So $\mathcal{W}$ consisting of all elements of $\mathcal{V}$ that are finite intersections of sets $\{F \in \mathcal{F} : F \cap G \neq \emptyset\}$ is a countable base for τ. Let c be the map

$$E \ni t \mapsto \operatorname{clos} t \in \mathcal{F}.$$

Then $c(t) = c(u)$ iff no open set in E separates t and u. Identifying such points we make E a T_0 space, and c an injection. Furthermore, c is bicontinuous if $\mathcal{F}$ is provided with the topology τ:

$$\{t \in E : c(t) \cap G \neq \emptyset\} = G$$

for $G \in \mathcal{G}$. So E is a subspace of $(\mathcal{F}, \tau)$ after identification via c. As $(\mathcal{F}, \tau)$ has a countable base, E does. □

We now investigate how we can select the subclasses $\mathcal{K}_0$ of $\mathcal{K}$ as in Theorem 5.3 or Lemma 5.4 under more specific assumptions. Note that sat$K$ is qcompact if K is.

5.7. EXAMPLE. If E is locally compact (thus Hausdorff), then with a base $\mathcal{G}_0$ we can choose $\mathcal{K}_0 := \{\operatorname{clos} G : G \in \mathcal{G}_0\} \cap \mathcal{K}$.

5.8. EXAMPLE. If E is locally compact with countable base, then E is metrizable, say by d. Set for $t \in E$ and $r \in (0, \infty)$:

$$\begin{aligned} B(t,r) &:= \{u \in E : d(t,u) < r\}, \\ B(t,r+) &:= \{u \in E : d(t,u) \leq r\}. \end{aligned} \tag{5.2}$$

Let D be a countable dense subset of E. Then a countable base $\mathcal{G}_0$ and a countable $\mathcal{K}_0 \subset \mathcal{K}$ such that E is locally $\mathcal{K}_0$ are given by

$$\begin{aligned} \mathcal{G}_0 &:= \{B(t,r) : t \in D,\ r \in \mathbb{Q} \cap (0,\infty),\ B(t,r+) \in \mathcal{K}\}, \\ \mathcal{K}_0 &:= \{B(t,r+) : t \in D,\ r \in \mathbb{Q} \cap (0,\infty),\ B(t,r+) \in \mathcal{K}\}. \end{aligned} \tag{5.3}$$

Note that for fixed t we have $B(t, r+)$ compact for all sufficiently small r (not for all r: consider $E = (0,1)$ with the usual metric and topology). One can metrize the same topology in such a way that all $B(t, r+)$ are compact (cf. VAUGHAN (1937)). Our present choice of $\mathcal{K}_0$ with $\mathcal{G}_0$ does *not* follow the recipe of Example 5.7, as $\mathrm{clos}B(t,r) = B(t,r+)$ need not hold in general (consider $r = 1$ in a discrete E with $d(t,u) = 1$ for $t \neq u$).

5.9. LITERATURE. The combination of Theorem 5.3 with Example 5.8 for $E = \mathbb{R}^d$ has been proved by SALINETTI & WETS (1981). For a completely different approach (cf. lines following Example 10.2), see NORBERG (1984, 1986). The 'if' part of Theorem 5.5 has been proved by DIXMIER (1968) and MATHERON (1975).

6. EXAMPLES AND FURTHER PROPERTIES

The following examples exhibit some properties of the sup $\mathcal{K}$ and $\mathcal{F}$ topologies in $US(\mathbb{R})$.

6.1. EXAMPLES. $E = \mathbb{R}, n = 1, 2, \ldots, f_n \in US(\mathbb{R})$.
(a) $f_n := 1_{\{n\}}$. Then $f_n \to 0_{\mathbb{R}}$ sup $\mathcal{K}$, but (f_n) does not converge sup $\mathcal{F}$.
(b) $f_n(t) := \frac{1}{2} + \frac{1}{2} \cdot (-1)^n \cos n^{\frac{1}{2}} t$. Then $f_n \to 1_{\mathbb{R}}$ sup $\mathcal{K}$ and $\mathcal{F}$, whereas $(f_n(t))$ does not converge in $\mathbb{I}$ for any t.
(c) $f_n := 1_{\{1/n\}}$. Then $f_n \to 1_{\{0\}}$ sup $\mathcal{K}$ and $\mathcal{F}$, whereas $f_n \to 0_{\mathbb{R}}$ pointwise.
(d) $f_n := 1_{\{1/n\}}$ for even n, $1_{\{1-1/n\}}$ for odd n. Then (f_n) does not converge sup $\mathcal{K}$ of $\mathcal{F}$, whereas $f_n \to 0_{\mathbb{R}}$ pointwise.
(e) $f_n := 1_{(-\infty,1/n]} + 1_{[2/n,\infty)}$. Then $f_n \to 1_{\mathbb{R}}$ sup $\mathcal{K}$, sup $\mathcal{F}$ and pointwise.

We now show that in many instances monotone nets in SM and US converge.

6.2. THEOREM. *(a) If (m_n) is an increasing net in SM and $m(G) := \bigvee_n m_n(G)$ for $G \in \mathcal{G}$, then $m \in SM$ and $m_n \to m$ sup $\mathcal{B}$ for any bounding class $\mathcal{B}$.*
(b) If (f_n) is a decreasing net in US with pointwise infimum f, then $f \in US, f_n \to f$ sup $\mathcal{K}$ and $f_n^\vee(K) \to f^\vee(K)$ for $K \in \mathcal{K}$.

PROOF. (a) Obviously, SM is closed for arbitrary suprema by (2.2), and $\liminf_n m_n(G) = \lim_n m_n(G) = m(G)$ for $G \in \mathcal{G}$. Furthermore, $m_n(B) \leq m(B)$, so $\limsup_n m_n(B) \leq m(B)$ for $B \in \mathcal{B}$.
(b) US is closed for arbitrary infima (Corollary 1.3(b)), so $f \in US$. Let $K \in \mathcal{K}$. Since $f_n^\vee(K)$ is nonincreasing in n and $f_n^\vee(K) \geq f^\vee(K)$, we have $\lim_n f_n^\vee(K) \geq f^\vee(K)$. If $\lim_n f_n^\vee(K) > x > f^\vee(K)$ for some $x \in \mathbb{I}$, then the nonempty qcompact sets $K \cap f_n^\leftarrow[x,1]$ would decrease to the empty set $K \cap f^\leftarrow[x,1]$, which is impossible. So $f_n^\vee(K) \to f^\vee(K)$. Trivially, $\lim_n f_n^\vee(G) \geq f^\vee(G)$ for $G \in \mathcal{G}$, as $f_n^\vee(G) \geq f^\vee(G)$, so $f_n \to f$ sup $\mathcal{K}$. □

6.3. COROLLARY. If $K_n \downarrow K$ in $\mathcal{K} \cap \mathcal{F}$ and $m \in SM$, then $m(K_n) \downarrow m(K)$.

PROOF. Apply Theorem 6.2(b) to $f_n := 1_{K_n} d^\vee m$ with K_1 instead of the K in Theorem 6.2(b). □

Even for nonmonotone nets the convergence $f_n^\vee(K) \to f^\vee(K)$ in Theorem 6.2(b) is interesting.

6.4. THEOREM. *Let $f_n, f \in US$. Then the following statements are equivalent.*
(i) $f_n \to f$ *sup* $\mathcal{K}$ *and pointwise;*
(ii) $f_n^\vee(K) \to f^\vee(K)$ *for* $K \in \mathcal{K}$.

PROOF. (i) $\Rightarrow$ (ii). Let $K \in \mathcal{K}$. By 3.7 and Corollary 1.3(c) we have for some $t_K \in K$:

$$\limsup_n f_n^\vee(K) \le f^\vee(K) = f(t_K) = \lim_n f_n(t_K) \le \liminf_n f_n^\vee(K) \, .$$

(ii) $\Rightarrow$ (i). Choosing $K = \{t\}$ we obtain pointwise convergence. For $G \in \mathcal{G}$ we have

$$\begin{aligned} \liminf_n f_n^\vee(G) &= \liminf_n \textstyle\bigvee_{t\in G} f_n(t) \ge \bigvee_{t\in G} \liminf_n f_n(t) \\ &= \textstyle\bigvee_{t\in G} \lim_n f_n(t) = \bigvee_{t\in G} f(t) = f^\vee(G). \end{aligned}$$

Together with the hypothesis this implies $f_n \to f$ sup $\mathcal{K}$ by (3.7). □

6.5. REMARK. One can prove that $f_n \to f$ sup $\mathcal{K}$ and pointwise iff $f_n \to f$ locally uniformly in the semimetric $d(x,y) := (x-y)^+$ on $\mathbb{I}\!\uparrow$, i.e. iff $d(f(t), f_n(t)) \to 0$ uniformly on qcompact sets in E. See Section 8 for the definition of 'semimetric'.

6.6. LITERATURE. The results in Remark 6.5 and related relative compactness criteria have been obtained by SALINETTI & WETS (1979). DOLECKI ET AL. (1983) and BEER (1982).

7. HYPO TOPOLOGIES

Here is a diagram of one-to-one maps that we have found in Sections 1, 2 and 3. Horizontal arrows denote surjections, vertical arrows injections; $SM = SM(E)$ is the lattice of sup measures on $\mathcal{G}(E)$, $US = US(E)$ the lattice of usc functions on E, 'ind' the indicator map $\mathcal{F}(E) \ni F \mapsto 1_F$ and 'id' is the identity map. All maps are in fact lattice isomorphisms, since they are order preserving.

$$\begin{array}{ccccc} SM(E) & \underset{i^\vee}{\overset{d^\vee}{\rightleftarrows}} & US(E) & \overset{\text{hypo}}{\rightleftarrows} & \mathcal{F}(E \times \mathbb{I}'\!\uparrow) \\ & & \uparrow\text{ind} & & \downarrow\text{id} \\ & & \mathcal{F}(E) & & \mathcal{F}(E \times \mathbb{I}') \end{array}$$

In Sections 3 and 4 we considered topologies on the different spaces in relation with the maps $d^\vee, i^\vee$ and 'ind'. In the present section we will concentrate on relations with the maps 'hypo' and 'id'.

Set $E^* := E \times \mathbb{I}'\!\uparrow$. Each class $\mathcal{B}^*$ of subsets of E^* determines a sup $\mathcal{B}^*$ topology on $\mathcal{F}(E^*)$, by Definitions 3.1 and 3.3. We carry this over to a topology on $US(E)$ by 'hypo'.

7.1. DEFINITION. *The **hypo $\mathcal{B}^*$ topology** is the topology on $US(E)$ that makes $US(E)$ homeomorphic to $\mathcal{F}(E \times \mathbb{I}'\!\uparrow)$ with the sup $\mathcal{B}^*$ topology via* hypo.

Set $\mathcal{K}^*(E) := \mathcal{K}(E^*)$. Then the sup vague topology on $\mathcal{F}(E^*)$ is the sup $\mathcal{K}^*$ topology (cf. Remark 3.2(b)). Let us call the hypo $\mathcal{K}^*$ topology on $US(E)$ the *hypo vague topology.* The following is a very convenient property that justifies our preference for vague topologies. Note that there is no condition at all on the underlying topological space E.

7.2. THEOREM. *The sup vague and hypo vague topolgies on $US(E)$ coincide.*

PROOF. Recall that all elements of $\mathcal{F}(E^*)$ have the form hypo$f$ for some $f \in US(E)$, by Theorem 1.6. For $G \in \mathcal{G}(E), x \in [0,1)$ and $f : E \to \mathbb{I}$ we have

$$f^\vee(G) > x \Leftrightarrow \text{hypo}\, f \cap (G \times (x,1]) \neq \emptyset. \tag{7.1a}$$

For $K \in \mathcal{K}(E), x \in (0,1]$ and $f \in US(E)$ we have by Corollary 1.3(a)

$$f^\vee(K) < x \Leftrightarrow \text{hypo}\, f \cap (K \times [x,1]) = \emptyset. \tag{7.1b}$$

The $f \in US(E)$ satisfying the left-hand sides of (7.1) form a subbase of the sup vague topology on $US(E)$. Note that $G \times (x,1] \in \mathcal{G}^* := \mathcal{G}(E^*)$ and that $K \times [x,1] \in \mathcal{K}^*$ (cf. lines preceding Definition 1.1 with $A\uparrow$ instead of $A\downarrow$). Consequently, the sets hypo $f \in \mathcal{F}(E^*)$ satisfying the right-hand sides of (7.1) are open as subbase sets of the sup vague topology on $\mathcal{F}(E^*)$. We have shown that hypo$^\leftarrow$ is continuous.

The remainder of this proof serves to show that also 'hypo' is continuous. So we must show that

$$\{f \in US(E) : \text{hypo} f \cap G^* \neq \emptyset\} \text{ for } G^* \in \mathcal{G}^* \tag{7.2a}$$

and

$$\{f \in US(E) : \text{hypo} f \cap K^* = \emptyset\} \text{ for } K^* \in \mathcal{K}^* \tag{7.2b}$$

are open subsets of $US(E)$. By Lemma 5.1 applied to $\mathcal{F}(E^*)$ as subspace of $US(E^*) \simeq SM(E^*)$ we need to show (7.2a) to be open only for G^* varying through a base $\mathcal{G}_0^*$ of $\mathcal{G}^*$. Such a base are the open rectangles $G \times (x,1]$ as in (7.1a), and (7.1a) gives us what we need.

We now consider (7.2b). Let $\pi_1 : (t,x) \mapsto t$ be the projection in E^* onto the first component. Set for $n = 1, 2, \ldots$

$$K_n^* := \bigcup_{k=1}^{2^n} \left(\pi_1(K^* \cap (E \times (0, k2^{-n}])) \times [(k-1)2^{-n}, 1]\right).$$

Then $K_n^* \supset K^*$, so hypo $f \cap K^* = \emptyset$ if hypo $f \cap K_n^* = \emptyset$ for some n. Conversely, if hypo $f \cap K_n^* = \emptyset$ for all n, then there are $t_n \in \pi_1 K^*$ and $(t_n, x_n) \in K^*$ such that $x_n - f(t_n) \leq 2^{-n}$. Since $\pi_1 K^*$ and K^* are qcompact, we arrive after passing to subsequences at the situation $t_n \to t_0$ in $\pi_1 K^*$ and $(t_n, x_n) \to (t_0, x_0)$ in K^*. The latter convergence implies $x_n \to x_0$ in $\mathbb{I}'\uparrow$, i.e., $\liminf x_n \geq x_0$ in $\mathbb{I}'$. Since f is usc, it follows that

$$f(t_0) \geq \limsup f(t_n) \geq \limsup (x_n - 2^{-n}) \geq x_0,$$

while $(t_0, x_0) \in K^*$. So hypo $f \cap K^* \neq \emptyset$. We have proved

$\{f \in US(E) : \text{hypo}\, f \cap K^* = \emptyset\} = \bigcup_{n=1}^{\infty}\{f \in US(E) : \text{hypo}\, f \cap K_n^* = \emptyset\}$

$= \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n}\{f \in US(E) : f^\vee(\pi_1(K^* \cap (E \times (0, k2^{-n}])) < (k-1)2^{-n}\}.$

The right-hand side is a union of open sets, since $f^\vee$ is applied to qcompact sets. For the last observation, note that a continuous function π_1 is applied to a qcompact set, an intersection of the qcompact set K^* with the closed set $E \times (0, k2^{-n}]$ in E^*. □

We now consider the last (vertical) arrow 'id' in the diagram at the beginning of this section. From the next theorem it follows that it is a homeomorphism if the spaces on its both sides are provided with the sup vague topology.

7.3. THEOREM. (a) $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ *with the sup vague topology is a subspace of* $\mathcal{F}(E \times \mathbb{I}')$ *with the sup vague topology.*
(b) *If E is locally qcompact, then this subspace is closed.*

Before proving the theorem we introduce some convenient notation and a lemma. For $x \in \mathbb{I}'$, set $\uparrow x := [x, 1]$. For $C \subset \mathbb{I}'$, set $\uparrow C := \bigcup_{x \in C} \uparrow x$. For $A \subset E \times \mathbb{I}'$, set $\uparrow A := \bigcup_{(t,x)\in A}\{t\} \times \uparrow x$. Note that $\uparrow A \subset \text{sat} A$ in $E \times \mathbb{I}'\uparrow$, so that for $F^* \in \mathcal{F}(E \times \mathbb{I}'\uparrow)$ we have

$$F^* \cap A = \emptyset \;\Leftrightarrow\; F^* \cap \uparrow A = \emptyset \;\Leftrightarrow\; F^* \cap \text{sat} A = \emptyset \tag{7.3}$$

(cf. lines following Definition 1.7). In general, saturations of qcompact sets are qcompact. Consequently, $\uparrow K^* \in \mathcal{K}(E \times \mathbb{I}'\uparrow)$ if $K^* \in \mathcal{K}(E \times \mathbb{I}'\uparrow)$, but we can say more.

7.4. LEMMA. *If* $K^* \in \mathcal{K}(E \times \mathbb{I}'\uparrow)$, *then* $\uparrow K^* \in \mathcal{K}(E \times \mathbb{I}')$.

PROOF. Let $\mathcal{H}$ be the base of $\mathcal{G}(E \times \mathbb{I}')$ consisting of rectangles $H = G \times I$ with $G \in \mathcal{G}(E)$ and I an open interval in $\mathbb{I}'$. Then $\uparrow H = G \times \uparrow I \in \mathcal{G}(E \times \mathbb{I}'\uparrow)$. Suppose $\uparrow K^* \subset \bigcup_{j \in J} H_j$ with $H_j \in \mathcal{H}$. Then also $\uparrow K^* \subset \bigcup_{j\in J} \uparrow H_j$. Since $\uparrow K^* \in \mathcal{K}(E \times \mathbb{I}'\uparrow)$, there is a finite $J_\# \subset J$ such that $\uparrow K^* \subset \bigcup_{j \in J_\#} \uparrow H_j$. Let π_1 and π_2 be the projections on the first and second component of $E \times \mathbb{I}'$, and define $\downarrow H$ starting from $\downarrow x := (0, x]$ for $x \in \mathbb{I}'$. Then $\pi_1 \uparrow K^* = \pi_1 K^* \in \mathcal{K}(E)$, and $\pi_2 \uparrow K^*$ has the form $[x, 1]$, so belongs to $\mathcal{K}(\mathbb{I}')$. Consequently, $\pi_1 \uparrow K^* \times \pi_2 \uparrow K^*$ belongs to $\mathcal{K}(E \times \mathbb{I}')$ and so does

$$(\pi_1 \uparrow K^* \times \pi_2 \uparrow K^*) \setminus \bigcup_{j \in J_\#} \downarrow H_j = \uparrow K^* \setminus \bigcup_{j \in J_\#} H_j.$$

Consequently, a finite subcollection of $(H_j)_{j \in J}$ covers the right-hand side, so a finite subcollection covers $\uparrow K^*$. □

PROOF OF THEOREM 7.3. (a) First of all, the topology of $E \times \mathbb{I}'\uparrow$ is coarser than that of $E \times \mathbb{I}'$, so $\mathcal{F}(E \times \mathbb{I}'\uparrow) \subset \mathcal{F}(E \times \mathbb{I}')$. The subbase open sets of $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ are $\{F^* : F^* \cap G^* \neq \emptyset\}$ and $\{F^* : F^* \cap K^* = \emptyset\}$ for $G^* \in \mathcal{G}(E \times \mathbb{I}'\uparrow)$ and $K^* \in \mathcal{K}(E \times \mathbb{I}'\uparrow)$. Recalling that $\mathcal{G}(E \times \mathbb{I}'\uparrow) \subset \mathcal{G}(E \times \mathbb{I}')$ we identify

$\{F^* : F^* \cap G^* \neq \emptyset\}$ as the trace in $(\mathcal{F} \times \mathbb{I}'\uparrow)$ of a subbase open subset of $\mathcal{F}(E \times \mathbb{I}')$. By (7.3) we have

$$\{F^* : F^* \cap K^* = \emptyset\} = \{F^* : F^* \cap \uparrow K^* = \emptyset\} ,$$

and by Lemma 7.4 we can identify the right hand side as the trace in $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ of a subbase open subset of $\mathcal{F}(E \times \mathbb{I}')$. We have proved (a).
(b) For general $E, \mathcal{F}(E \times \mathbb{I}'\uparrow)$ is sup vaguley qcompact by Corollary 4.4(a). If E is locally qcompact, then so is $E \times \mathbb{I}'$, so $\mathcal{F}(E \times \mathbb{I}')$ is Hausdorff by Theorem 4.3(b). In this case the qcompact subset $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ is closed. □

7.5. LITERATURE. BUTTAZZO (1977, Prop. (1.12)) proved that $US(E)$ is sup vaguely homeomorphic to a subset of $\mathcal{F}(E \times \mathbb{I}')$. Theorem 7.2 is a consequence of a general representation theorem in GIERZ ET AL. (1980) that identifies certain continuous lattices as $\mathcal{F}(E^*)$ with E^* the set of primes of the lattice in question. For more and very general results on closed epigraphs, see HOLWERDA (1993).

8. NON-HAUSDORFF LOCALLY QCOMPACT SPACES

The next two sections can be skipped by readers who are not interested in non-Hausdorff spaces. In the present section all material is concentrated that may be relevant for readers who want to restrict their considerations of non-Hausdorffness to $E \times \mathbb{I}'\uparrow$ with E Hausdorff.

Let us consider the diagram at the beginning of Section 7. From the theorems in Sections 4 and 7 we know that things are particularly nice if E is locally qcompact, but not necessarily Hausdorff, and all spaces are endowed with the sup vague topology. Then all spaces are compact, and all arrows are homeomorphisms (into when vertical). If, in addition, E is Hausdorff, or more specially, metric, then Examples 5.7 and 5.8 indicate convenient choices of subcollections $\mathcal{K}_0$ of $\mathcal{K}$ for defining smaller subbases of the sup vague topologies on the spaces $SM(E), US(E)$ and $\mathcal{F}(E)$ (cf. Theorem 5.3).

In this section we explore what remains of this when E is not Hausdorff. This is useful, because we want to be able to consider also $E^* := E \times \mathbb{I}'\uparrow$, which is not Hausdorff, but is locally qcompact if E is. The following examples are instructive.

8.1. EXAMPLE. Let $E = \mathbb{R}\downarrow$ as in Example 1.9. Nonempty $A \subset \mathbb{R}\downarrow$ are qcompact iff sup $A \in A$. Thus $K_n := (-\infty, 0) \cup \{n\}$ is qcompact for $n = 1, 2$, but $K_1 \cap K_2 = (-\infty, 0)$ is not. We see that $\mathcal{K}(\mathbb{R}\downarrow)$ is not closed for finite intersections. Let G be open and nontrivial, so $G = (-\infty, x)$ for some $x \in \mathbb{R}$. Then G is relatively qcompact, i.e., contained in some qcompact set. There is even a smallest saturated qcompact set Q containing G, viz. $Q = (-\infty, x]$. We cannot obtain Q by taking closures as in Example 5.7, since clos$G = \mathbb{R}$. In fact, the only closed qcompact set is $\emptyset$.

8.2. EXAMPLE. Let $E = \mathbb{Q}\downarrow$ with the relative lower topology from $\mathbb{R}\downarrow$. Again, nonempty $A \subset \mathbb{Q}\downarrow$ are qcompact iff $\sup A \in A$. The generic open set is

$(-\infty, x) \cap \mathbb{Q}$ with $x \in \mathbb{R}$. Now $G := (-\infty, \pi) \cap \mathbb{Q}$ is relatively qcompact since $G \subset (-\infty, q] \cap \mathbb{Q}$ for $q > \pi, q \in \mathbb{Q}$. However, there is no smallest qcompact set containing G.

The first step to overcome these problems is considering *saturated* qcompact sets rather than qcompact sets. We write $\mathcal{Q} = \mathcal{Q}(E)$ for the collection of saturated qcompact sets, with generic element Q. It is immediate that sat$K \in \mathcal{Q}$ iff $K \in \mathcal{K}$.

We have

$$\mathcal{Q}(\mathbb{R}\downarrow) = \{\emptyset, (-\infty, x] : x \in \mathbb{R}\} \text{ and } \mathcal{Q}(\mathbb{Q}\downarrow) = \{\emptyset, (-\infty, q] \cap \mathbb{Q} : q \in \mathbb{Q}\}.$$

Note that in both cases $\mathcal{Q}$ is closed for finite intersections, but that only $\mathcal{Q}(\mathbb{R}\downarrow)$ is closed for arbitrary intersections. There are E for which $\mathcal{Q}(E)$ is not even closed for finite intersections (cf. Example 9.7(b)).

These observations lead us to the following regularity condition that we will impose on E.

8.3. DEFINITION. *A topological space is a Q_δ space if the collection $\mathcal{Q}$ of its saturated qcompact sets is closed for arbitrary intersections.*

Hausdorff spaces are Q_δ, and so are $\mathbb{R}\downarrow$ and $\mathbb{I}'\uparrow$, but $\mathbb{Q}\downarrow$ is not. If $A \subset E$ is relatively qcompact and E is Q_δ, then the intersection of all saturated qcompact sets containing A is the smallest such set. We will denote it by sqc$A$, the *saturated qcompactification* of A. For Hausdorff E we have sqc$A$ = clos$A$ for relatively compact subsets A. For non-Hausdorff spaces which are Q_δ, 'sqc' takes over the role of 'clos'. We now can generalize Example 5.7 to

8.4. EXAMPLE. If E is locally qcompact and Q_δ, then with a base $\mathcal{G}_0$ we can choose

$$\mathcal{K}_0 := \{\text{sqc}G : G \in \mathcal{G}_0,\ G \text{ relatively qcompact }\}$$

in Theorem 5.3.

It would be nice to generalize Example 5.8 as well to non-Hausdorff spaces. The only way to do this is by generalizing the notion of 'metric', since all metric spaces are Hausdorff. Here are some partial results.

8.5. DEFINITION. *A* ***semimetric*** *on E is a map $d : E \times E \to [0, \infty)$ such that $d(t,t) = 0$ for $t \in E$ and satisfying the* ***triangle inequality***

$$d(t, v) \leq d(t, u) + d(u, v) \text{ for } t,\ u,\ v \in E.$$

Note that we do not require $d(t,u) = d(u,t)$. We define the balls $B(t,r)$ and $B(t, r+)$ for semimetrics as in (5.2). As for metrics, one proves that the balls $B(t,r)$ form a base of a topology, by definition the topology *generated* or *semimetrized* by d. For example, $\mathbb{R}\downarrow$ (Example 8.1) is semimetrized by $d(t,u) := (u-t)^+$, and more generally, $(\mathbb{R}\downarrow)^n$ by $d(t,u) := \bigvee_{k=1}^n (u_k - t_k)^+$.

In general we have for semimetric E
(a) sat$t = \{u \in E : d(t,u) = 0\}$,
(b) clos$u = \{t \in E : d(t,u) = 0\}$,
(c) the net $(t_n)_n$ converges to t in E iff $d(t,t_n) \to 0$.
Note that (a) and (b) express the more general equivalence $u \in$ sat$t \Leftrightarrow t \in$ clos$u$.

8.6. THEOREM. *If E has a countable base, then E is semimetrizable.*

PROOF. Let $G_1, G_2, \ldots$ be a base for E. Define for $t, u \in E$

$$d(t,u) := \sum_{n=1}^{\infty} 1_{G_n}(t) 1_{G_n^c}(u) 2^{-n}.$$

One easily checks that d is a semimetric. We now show that d generates the same topology as $G_1, G_2, \ldots$. If $t \in G_n$, then $\{s : d(t,s) < 2^{-n}\} \subset G_n$, so G_n is d-open. On the other hand, with N such that $\sum_{n>N} 2^{-n} < \epsilon$ we find

$$t \in \bigcap_{n \le N : t \in G_n} G_n \subset \{u : d(t,u) < \epsilon\}. \qquad \square$$

Recall that the balls $B(t, r+)$ are defined as in (5.2) for semimetrics d. In general, the balls $B(t, r+)$ need not be closed. If E is locally compact (thus also Hausdorff) and is metrized by d, then for fixed t the (then closed) balls $B(t, r+)$ are compact for all small r (cf. Example 5.7). If E is locally qcompact and semimetrized by d, then the balls $B(t, r+)$ are saturated (as intersection of the open sets $B(t,s)$ for $s > r$), but not necessarily qcompact, even for small r. However, in the Q_δ space $\mathbb{R}\!\downarrow$ of Example 8.1 all balls $B(t, r+) = (-\infty, t+r]$ are qcompact.

8.7. DEFINITION. *If E is semimetrized by d, then d is said to be $\mathcal{Q}$* ***compatible****, and E is said to be semimetrized $\mathcal{Q}$* ***compatibly*** *by d if for each $t \in E$ we have $B(t, r+) \in \mathcal{Q}$ for all small r.*

8.8. COROLLARY. If E is locally qcompact and $\mathcal{Q}$ compatibly semimetrized by d, then $\mathcal{K}_0$ as in (5.3) can be substituted in Theorem 5.3.

8.9. EXAMPLES. (a) All metrics on locally compact spaces are $\mathcal{Q}$ compatible.
(b) The semimetric $d(t,u) = (u-t)^+$ on $\mathbb{R}\!\downarrow$ is $\mathcal{Q}$ compatible.
(c) The semimetric $d(t,u) = (u-t)^+$ on $\mathbb{Q}\!\downarrow$ is not $\mathcal{Q}$ compatible: $B(0, \pi+) = (-\infty, \pi) \cap \mathbb{Q}$ is not qcompact. However, there is another semimetric d' that generates the same topology and is $\mathcal{Q}$ compatible: $d'(t,u) := \varphi((u-t)^+)$, where $\varphi(0) := 0, \varphi(t) := 2^{-n}$ for $t \in [2^{-(n+1)}, 2^{-n}), n \in \mathbb{Z}$.

We do not know whether all semimetrizable locally qcompact spaces can be semimetrized $\mathcal{Q}$ compatibly. It is even hard to verify if specific spaces are $\mathcal{Q}$ compatibly semimetrizable, as for instance $(([-\infty, 0) \cup [1, \infty))\!\downarrow)^2$. However, in many specific cases it is easy to find $\mathcal{Q}$ compatible semimetrics, and the number of such cases is extended by

8.10. LEMMA. *If $E^{(1)}$ and $E^{(2)}$ are locally qcompact and $\mathcal{Q}$ compatibly semimetrizable, then so is $E := E^{(1)} \times E^{(2)}$.*

PROOF. First of all, E is locally $\mathcal{K}_0$ with $\mathcal{K}_0$ the qcompact rectangles, so E is locally qcompact. If $E^{(n)}$ is $\mathcal{Q}$ compatibly semimetrized by the semimetric $d^{(n)}$ for $n = 1, 2$, then E is by the semimetric $d(t, u) := \bigvee_{n=1}^{2} d^{(n)}(t^{(n)}, u^{(n)})$. □

8.11. COROLLARY. If E is locally qcompact and $\mathcal{Q}$ compatibly semimetrizable, then so are $E \times \mathbb{I}'\!\uparrow$ and $E \times \mathbb{I}'$. In particular, the conclusion holds true if E is locally compact and metrizable.

8.12. LITERATURE. The same notion of 'semimetric' occurs in NACHBIN (1965).

9. MORE ABOUT NON-HAUSDORFF SPACES

First we make a fundamental observation about the spaces in the diagram of lattice isomorphisms in the beginning of Section 7. If we are given a toplogical space $(E, \mathcal{G})$, then $US(E)$ depends only on this space via $\mathcal{G}$. More specifically, if $(E_1, \mathcal{G}_1)$ and $(E_2, \mathcal{G}_2)$ are two topological spaces such that $\mathcal{G}_1$ and $\mathcal{G}_2$ are lattice isomorphic, then $US(E_1)$ and $US(E_2)$ are lattice isomorphic. This is obvious in the diagram on the left side since $SM(E)$ is a space of functions on $\mathcal{G}$, and on the right side since $\mathcal{F}(E \times \mathbb{I}'\!\uparrow)$ depends on $(E, \mathcal{G})$ only via $\mathcal{G}$. If, moreover, the bounding class $\mathcal{B}$ in the sup $\mathcal{B}$ topology on $US(E)$ depends on $(E, \mathcal{G})$ only via $\mathcal{G}$ (which is the case for $\mathcal{B} = \mathcal{F}$ but not for $\mathcal{B} = \mathcal{K}$) then $US(E)$ as a topological space depends on $(E, \mathcal{G})$ only via $\mathcal{G}$.

This makes it useful to study which topological spaces E have lattice isomorphic topologies $\mathcal{G}$. First we must get rid of a trivial complication. If two points in E are not separated by any open set, then we can identify them without affecting the lattice of open sets. By identifying all nonseparated points we make E a T_0 space. Therefore we will often assume that E is T_0.

We now start with an example. As in Example 8.2, let $\mathbb{Q}\!\downarrow$ be the rationals provided with the lower topology, the relative topology from $\mathbb{R}\!\downarrow$. Then its nontrivial open sets are given by $(-\infty, x) \cap \mathbb{Q}$ for $x \in \mathbb{R}$. We see that the topology of $\mathbb{Q}\!\downarrow$ is lattice isomorphic to that of $\mathbb{R}\!\downarrow$. Intuitively we may feel that $\mathbb{R}$ as a total space fits better in the topology than $\mathbb{Q}$. We now provide theoretical support for this feeling. At this point it is more convenient to regard a topology determined by the closed sets $\mathcal{F}$ rather than the open sets $\mathcal{G}$.

9.1. DEFINITION. *A set $F \in \mathcal{F}$ is called **prime** if $F \neq \emptyset$ and $F = F_1 \cup F_2$ with $F_1, F_2 \in \mathcal{F}$ implies $F = F_1$ or F_2.*

9.2. REMARK. From the definition it follows that clos$A$ $(A \subset E)$ is prime iff $A \neq \emptyset$ and $A \cap G_n \neq \emptyset$ for open G_n $(n = 1, 2)$ implies $A \cap G_1 \cap G_2 \neq \emptyset$. In particular singleton closures are prime. Moreover, in a Hausdorff space a prime closed set cannot contain two points, so the prime closed sets are just the singletons. The characterization in the first clause of this remark remains valid with G_1 and G_2 coming from a base $\mathcal{G}_0$ of $\mathcal{G}$.

9.3. EXAMPLE. In $\mathbb{Q}\downarrow$ the prime closed sets $\mathbb{Q}$ and $[x,\infty) \cap \mathbb{Q}$ with $x \in \mathbb{R}\backslash\mathbb{Q}$ are no singleton closures. In $\mathbb{R}\downarrow$ the total set $\mathbb{R}$ is prime closed and no singleton closure.

The observations in the example suggest us what to do. If F is a prime closed set that is not a singleton closure, then add a new point x to E that by definition is contained in each open set that intersects F, to obtain $F = \text{clos}x$. Formally one performs this by making a new topological space whose points are the primes in $\mathcal{F}(E)$. See Section 1 of HOFMANN & MISLOVE (1981), from which we borrow the following definition and result.

9.4. DEFINITION. *A topological space E is **sober** if it is T_0 and each prime closed set in E is a singleton closure.*

9.5. THEOREM. *For each space E there is a sober space* sob$E$, *unique up to homeomorphism, such that $\mathcal{G}(\text{sob}E)$ is lattice isomorphic to $\mathcal{G}(E)$.*

9.6. EXAMPLE. $\text{sob}\mathbb{Q}\downarrow \simeq \text{sob}\mathbb{R}\downarrow \simeq [-\infty,\infty)\downarrow$.

We call sob$E$ the *sobrification* of E. The term is not very suggestive, as sob$E$ is a kind of completion of E. It is the largest T_0 space with topology lattice isomorphic to $\mathcal{G}(E)$. We make a T_0 space E a topological subspace of sob$E$ by identifying points whose closure complements are mapped on each other by the lattice isomorphism between the topologies. We already noticed that $US(E)$ and $US(\text{sob}E)$ are lattice isomorphic, and homeomorphic with the sup weak topologies but not necessarily with the sup vague topologies.

It is hard to find examples of the latter, but HOFMANN & LAWSON (1978, §7) exhibit one in which every qcompact set in E has empty interior, whereas sob$E$ is locally qcompact. Consequently, $US(\text{sob}E)$ is sup vaguely Hausdorff by Theorem 4.3(c), whereas $US(E)$ is not, by Theorem 4.3(d).

In Examples 8.1 and 8.2 we observed that in general $\mathcal{K}$ is not closed for intersections, but that $\mathcal{Q} := \{\text{sat}K : K \in \mathcal{K}\}$ is closed for intersections in some of the cases where $\mathcal{K}$ fails to be so. We called E a Q_δ space if $\mathcal{Q}$ is closed for arbitrary intersections, and found that $\mathbb{R}\downarrow$ is Q_δ. The following list of examples is instructive.

9.7. EXAMPLES. (a) E is countable, the open sets are empty or cofinite. Then E is T_1 but not T_2, and not sober as the total set E is prime closed. All subsets are qcompact and saturated, so E is locally $\mathcal{Q}$ and Q_δ. The sobrification of E is obtained by adding a point ∞ to each nonempty open set.
(b) $E = \mathbb{N} \cup \{\infty_1, \infty_2\}$ with as open sets all subsets of $\mathbb{N}$ and all cofinite subsets of E that intersect $\{\infty_1, \infty_2\}$. Then E is T_1 (so all subsets are saturated) but not T_2; E is sober; $A \subset E$ is qcompact iff A is finite or A intersects $\{\infty_1, \infty_2\}$. E is locally $\mathcal{Q}$, but $\mathcal{Q}$ is not closed for finite intersections: consider $Q_n := \mathbb{N} \cup \{\infty_n\}$ for $n = 1, 2$. However, $\mathcal{Q}$ is closed for intersections of decreasing nets in $\mathcal{Q}$.

(c) E is Hausdorff, but not necessarily locally compact. Then E is sober and Q_δ.

From HOFMANN & MISLOVE (1981) and GIERZ & AL. (1980) we quote the following definition and results.

In a sober space, $\mathcal{Q}$ is closed for intersections of decreasing nets in $\mathcal{Q}$ (HOFMANN & MISLOVE (1981, Prop. 2.19)). Consequently, a sober space is Q_δ iff $\mathcal{Q}$ is closed for finite intersections. A space E is called *supersober* if the set of limit points of each ultrafilter on E is either empty or a singleton closure. A T_1 space which is not T_2, is not supersober. If E is supersober, then E is sober and Q_δ (GIERZ & AL. (1980, VII-1.11), NORBERG & VERVAAT (1989, Prop. 1.3)). If E is sober, Q_δ and locally $\mathcal{Q}$, then E is supersober (HOFMANN & MISLOVE (1981, Th.4.8)).

We will not prove or use these results here, but rather content ourselves with obtaining directly a collection of weaker results which serves our needs.

9.8. LEMMA. *Let E be locally $\mathcal{K}_0$ with $\mathcal{K}_0 \subset \mathcal{K}$ and such that $\mathcal{K}_0$ is closed for finite intersections.*
(a) *If $(t_\alpha)_\alpha$ is a convergent net in E and $\mathrm{Lim}t_\alpha$ its set of limits, then $\mathrm{Lim}t_\alpha$ is prime closed.*
(b) *If in addition E is sober, then E is Q_δ.*

PROOF. (a) In general, the set $\mathrm{Lim}t_\alpha$ is closed. Let G_1, G_2 be two open sets intersecting $\mathrm{Lim}t_\alpha$. We must prove that $G_1 \cap G_2 \cap \mathrm{Lim}t_\alpha \neq \emptyset$. Select $u_n \in G_n \cap \mathrm{Lim}t_\alpha$ and $K_n \in \mathcal{K}_0$ such that $u_n \in \mathrm{int}K_n \subset K_n \subset G_n$ for $n = 1, 2$. Since $u_n = \lim t_\alpha$, we have that $t_\alpha \in \mathrm{int}K_n \subset K_n$ for all sufficiently large α, so $t_\alpha \in K_1 \cap K_2$ for all sufficiently large α. Since $K_1 \cap K_2$ is qcompact, there is a $u \in K_1 \cap K_2$ with $u = \lim t_\alpha$. So $u \in K_1 \cap K_2 \cap \mathrm{Lim}t_\alpha \subset G_1 \cap G_2 \cap \mathrm{Lim}t_\alpha$.
(b) Let $Q_j \in \mathcal{Q}$ for $j \in J$ and set $Q := \bigcap_{j \in J} Q_j$. Then Q is saturated as intersection of saturated sets. It remains to show that Q is qcompact. Let (t_α) be a net in Q. Then (t_α) is a net in Q_j (for some fixed j) and Q_j is qcompact, so there is a convergent subnet with at least one of its limits in Q_j. Think (t_α) replaced by this convergent subnet. By (a) and the sobriety of E there is a $u \in E$ such that $\mathrm{Lim}t_\alpha = \mathrm{clos}u$. For all j we have that t_α is a convergent net in the qcompact set Q_j, so $Q_j \cap \mathrm{clos}\,u \neq \emptyset$. As Q_j is saturated, it follows that $u \in Q_j$ for all j, so $u \in Q$. We have proved that $Q \cap \mathrm{Lim}t_\alpha \neq \emptyset$, so Q is qcompact. □

We now turn to product spaces.

9.9. LEMMA. *Let $E := E^{(1)} \times E^{(2)}$ with the product topology. Then the prime closes sets in E are the rectangles with prime closed sides in $E^{(1)}$ and $E^{(2)}$.*

PROOF. Let π^n for $n = 1, 2$ be the projection in E on the nth component $E^{(n)}$. The key observations are that for open $G^{(1)}$ in $E^{(1)}$ and closed F in E we have

$$G^{(1)} \cap \pi^1 F \neq \emptyset \Leftrightarrow (G^{(1)} \times E^{(2)}) \cap F \neq \emptyset \,, \tag{9.1}$$

and that the open sets for testing primality of F as in Remark 9.2 may be the open rectangles

$$G^{(1)} \times G^{(2)} = (G^{(1)} \times E^{(2)}) \cap (E^{(1)} \times G^{(2)}). \tag{92}$$

Considering (9.1) for two open sets $G^{(1)}$ we see that $\text{clos}\pi^1 F$ (and similarly clos $\pi^2 F$) is prime, if F is prime. If $F^{(n)}$ is prime closed in $E^{(n)}$ for $n = 1, 2$, then $F^{(1)} \times F^{(2)}$ is in E, which one verifies by intersecting $F^{(1)} \times F^{(2)}$ with open rectangles as in (9.2).

It remains to prove that $F = \text{clos}\pi^1 F \times \text{clos}\pi^2 F =: F^{(1)} \times F^{(2)}$ for each prime closed in F in E. So suppose there is a $t \in (F^{(1)} \times F^{(2)}) \backslash F$. As F is closed, there is an open rectangle $G^{(1)} \times G^{(2)}$ containing t but not intersecting F, which contradicts the primality of F and $F \cap (G^{(1)} \times E^{(2)}) \neq \emptyset$, $F \cap (E^{(1)} \times G^{(2)}) \neq \emptyset$ (note that $\pi^1 t \in G^{(1)}$ and $\pi^1 t \in F^{(1)} = \text{clos}\pi^1 F$, so $G^{(1)} \cap \pi^1 F \neq \emptyset$). □

9.10. THEOREM. *Let $E = E^{(1)} \times E^{(2)}$ with the product topology.*
(a) *If $E^{(1)}$ and $E^{(2)}$ are locally qcompact, then so is E.*
(b) *If $E^{(1)}$ and $E^{(2)}$ are sober, then so is E.*
(c) *If $E^{(1)}$ and $E^{(2)}$ are sober, locally qcompact and Q_δ, then so is E.*

PROOF. (a) E is locally $\mathcal{K}_0$ with $\mathcal{K}_0$ the qcompact rectangles.
(b) Follows from Lemma 9.9 and $\text{clos}(t,u) = (\text{clos}\, t) \times (\text{clos}\, u)$.
(c) Let $\mathcal{K}_0$ be the qcompact rectangles in E. Then E is locally $\mathcal{K}_0$ and $\mathcal{K}_0$ is closed for finite (even arbitrary) intersections because $E^{(1)}$ and $E^{(2)}$ are Q_δ. By (b), E is sober. So E satisfies all assumptions of Lemma 9.8(b), which proves E to be Q_δ. □

9.11. COROLLARY. If E is sober, locally qcompact and Q_δ, then so are $E \times \mathbb{I}'\!\uparrow$ and $E \times \mathbb{I}'$.

9.12. LITERATURE. Most results of this section can be found in HOFMANN & MISLOVE (1981) and GIERZ ET AL. (1980).

10. OTHER CRITERIA FOR CONVERGENCE

Let E be locally qcompact and $\mathcal{Q}$ compatibly semimetrized by d, which is in particular the case if E is locally compact and metrized by d. Let the balls $B(t,r)$ and $B(t,r+)$ be defined by (5.2). If $f_n \to f$ sup $\mathcal{K}$ in US, then

$$\begin{aligned} f^\vee(B(t,r)) &\leq \liminf_n f_n^\vee(B(t,r)) \\ &\leq \limsup_n f_n^\vee(B(t,r+)) \leq f^\vee(B(t,r+)) \end{aligned} \tag{101}$$

for all $t \in E$ and $r > 0$ such that $B(t,r+)$ is qcompact (which is the case for all sufficiently small r, depending on t). If

$$f^\vee(B(t,r)) = f^\vee(B(t,r+)) \tag{102}$$

for some t and r, then

$$f^\vee(B(t,r)) = \lim_n f_n^\vee(B(t,r)). \tag{103}$$

As the function $r \mapsto f^\vee(B(t,r))$ is monotone, we have (10.2) for fixed t violated for at most countably many r. Consequently, if

$$\begin{array}{lcl} \mathcal{G}_f & := & \{B(t,r) : f^\vee(B(t,r)) = f^\vee(B(t,r+)),\ B(t,r+) \in \mathcal{K}\}, \\ \mathcal{K}_f & := & \{B(t,r+) : f^\vee(B(t,r)) = f^\vee(B(t,r+)),\ B(t,r+) \in \mathcal{K}\}, \end{array} \tag{10.4}$$

then $\mathcal{G}_f$ is a base of $\mathcal{G}$ and E is locally $\mathcal{K}_f$. So (10.1) restricted to balls in $\mathcal{G}_f$ or $\mathcal{K}_f$ (which is (10.3) with the same restriction) implies $f_n \to f \sup\mathcal{K}$ by Theorem 5.3. We conclude:

10.1. THEOREM. *We have $f_n \to f$ sup vaguely in US iff $f_n^\vee(B) \to f^\vee(B)$ for all $B \in \mathcal{G}_f$ or all $B \in \mathcal{K}_f$ defined in (10.4).*

10.2. EXAMPLE. $E = \mathbb{R}$. $f_n \to f$ sup vaguely in $US(\mathbb{R})$ iff $\lim_n f_n^\vee(B) = f^\vee(B)$ for all open bounded intervals B such that $f^\vee(B) = f^\vee(\mathrm{clos}B)$.

A unifying approach to some of the preceding results is based on semiseparating classes as considered by NORBERG (1984, 1986). First, let E be locally compact (thus Hausdorff) with countable base. A class $\mathcal{A}$ of subsets of E is called *separating* if for all open G and compact K with $G \supset K$ there is an $A \in \mathcal{A}$ such that $G \supset A \supset K$. A class $\mathcal{A}$ is *semiseparating* if the class of finite unions of elements in $\mathcal{A}$ is separating. Examples of semiseparating classes are $\mathcal{A} = \mathcal{G}_f$ and $\mathcal{A} = \mathcal{K}_f$. NORBERG (1986) related sup vague convergence of sup measures to the limiting behavior of their values on semiseparating classes. We refer to his work for the results, and confine ourselves to indicating some connections and a possible generalization to non-Hausdorff E.

A sup measure is *inner continuous* on $\mathcal{G}$ in the sense that

$$m(G_n)\uparrow m(G) \quad \text{if } G_n\uparrow G,\ G_n, G \in \mathcal{G}$$

(cf. (2.2)). An inner continuous set function $m : \mathcal{G} \to \mathbb{I}$ is a sup measure iff

$$m(G_1 \cup G_2) = m(G_1) \vee m(G_2) \quad \text{for } G_1, G_2 \in \mathcal{G}.$$

A sup measure is *outer continuous* on $\mathcal{K} \cap \mathcal{F}$, i.e., Corollary 6.3 holds true. This suffices for the case of Hausdorff E considered by NORBERG.

Generalization to the non-Hausdorff case is possible for locally qcompact sober E. In this case it is necessary to consider only semiseparating classes of saturated sets that separate open and saturated qcompact sets. The role of compact closure of relatively compact sets is taken over by the sqc operation in Example 8.4. The following lemma shows that sup measures are outer continuous on the saturated qcompact sets $\mathcal{Q}$.

10.3. LEMMA. *If E is locally qcompact and sober, m is a sup measure and $(Q_n)_n$ is a decreasing net in $\mathcal{Q}$ with intersection Q, then $m(Q_n)\downarrow m(Q)$.*

PROOF. Obviously, $\lim_n m(Q_n) \geq m(Q)$. By Theorem 2.5 and Corollary 1.3(c) there is a $t_n \in Q_n$ such that $d^\vee m(t_n) = m(Q_n)$. Since the Q_n's are qcompact, there is a convergent subnet $(t_{n'})$. By Lemma 9.8(a) the set of its limits is prime closed, so has the form $\operatorname{clos} u$ for a $u \in E$, as E is sober. Since the Q_n's are saturated and $Q_n \cap \operatorname{clos} u \neq \emptyset$, we have $u \in Q_n$ for all n, so $u \in Q$. As $u = \lim_{n'} t_{n'}$, we have

$$\begin{aligned} m(Q) &\geq d^\vee m(u) = \textstyle\bigwedge_{G \ni u} m(G) \geq \limsup_{n'} d^\vee m(t_{n'}) \\ &\geq \limsup_{n'} m(Q_{n'}) = \lim_n m(Q_n). \end{aligned}$$

This combined with the first observation proves the lemma. □

NORBERG (1984, 1986) assumed the sets in the semiseparating classes to be Borel measurable, which becomes necessary in the context of SM- or US-valued random variables.

10.4. LITERATURE. Theorem 10.1 has been proved also by SALINETTI & WETS (1981, 1986) and NORBERG (1986).

11. MEASURABILITY, RANDOM VARIABLES AND EXTREMAL PROCESSES

Let in general Bor E denote the Borel field of a topological space E, the σ-field generated by $\mathcal{G}(E)$. We begin with investigating Bor SM and Bor $\mathcal{F}$, where throughout this section SM and $\mathcal{F}$ are endowed with the sup vague topology. In general it is hard to characterize Bor SM further, but if SM has a countable base, then Bor SM is already generated by its subbase (3.4), as now each open set in SM is countable union of finite intersections of subbase elements. Now Bor SM can be characterized succinctly.

11.1. THEOREM. *If $\mathcal{G}(E)$ has a countable base, $\mathcal{G}_0$ is a base of $\mathcal{G}$ and E is locally $\mathcal{K}_0$ with $\mathcal{K}_0 \subset \mathcal{K}$, then Bor SM is the smallest σ-field that makes the evaluations $m \mapsto m(A)$ measurable for all $A \in \mathcal{G}_0$ or all $A \in \mathcal{K}_0$.*

PROOF. SM has a countable base by Theorem 5.5 and Remark 5.6. In the proofs of Lemmas 5.1 and 5.2 all J can be taken or made countable, which shows measurability of $A \mapsto m(A)$ for $A \in \mathcal{G}_0$ (or $\mathcal{K}_0$) to be equivalent to that for $A \in \mathcal{G}$ (or $\mathcal{K}$). Measurability for all $A \in \mathcal{G}$ or $\mathcal{K}$ implies measurability for all $A \in \mathcal{G} \cup \mathcal{K}$ by (3.1) with $G_n \downarrow A \in \mathcal{K}$ and Property 3.7(b) with J made countable.

11.2. DEFINITION. *An **extremal process** is an SM-valued random variable (rv). A **random usc function** is a US-valued rv. A **random closed set** is an $\mathcal{F}$-valued rv.*

11.3. COROLLARY. In the situation of Theorem 11.1 an extremal process is a mapping M from some probability space into SM such that $M(A)$ is an $\mathbb{I}$-valued rv for each $A \in \mathcal{G}_0$ or each $A \in \mathcal{K}_0$.

11.4. REMARK. Let $\mathcal{A}$ be the smallest σ-field in SM that makes all evaluations $m \mapsto m(\{t\}) = d^\vee m(t)$ measurable. Then $\mathcal{A} \subset$ Bor SM, but $\mathcal{A}$ is in general strictly smaller than Bor SM. To see this, set $E := [0,1]$ and let the rv ξ have a uniform distribution in E. Set $M_1 :\equiv 0, M_2 := 1_{\{\xi\}}$. Then M_1 and M_2 are extremal processes with different distributions on Bor SM: $M_1(E) = 0$ wp1, $M_2(E) = 1$ wp1, but equal distributions on $\mathcal{A}$: $M_1(\{t\}) = M_2(\{t\}) = 0$ wp1 for each $t \in E$.

11.5. THEOREM. *Let E, $\mathcal{G}_0$ and $\mathcal{K}_0$ be as in Theorem 11.1 with, moreover, $\mathcal{G}_0$ and $\mathcal{K}_0$ closed for finite intersections, and let M be an extremal process. Then the probability distribution of M is determined by the finite-dimensional distributions of $(M(G))_{G \in \mathcal{G}_0}$ or $(M(K))_{K \in \mathcal{K}_0}$.*

PROOF. The family of sets $\bigcap_{j \in J}\{m : m(G_j) \leq x_j\}$ for finite subcollections $(G_j)_{j \in J}$ of $\mathcal{G}_0$ generates Bor SM by Theorem 11.1, and is closed for finite intersections. Apply Theorem 10.3 of BILLINGSLEY (1979). The proof for $\mathcal{K}_0$ is similar. □

11.6. REMARK. If M is an extremal process, then $d^\vee M$ is a random usc function. If X is a random usc function, then $X^\vee$ is an extremal process.

Let M be an extremal process. So far we have seen that $M(A)$ is a rv in $\mathbb{I}$ for $A \in \mathcal{G} \cup \mathcal{K}$. Although $M(A)$ need not be a rv for all $A \subset E$, even not for all $A \in$ Bor E in case the σ-field in the underlying probability space does not contain all $\mathbb{P}$ nullsets, we can extend $\mathcal{G} \cup \mathcal{K}$ a bit further. Obviously, $M(\bigcup_{n=1}^\infty A_n) = \bigvee_{n=1}^\infty M(A_n)$ is a rv if each $M(A_n)$ is. So $M(A)$ is a rv for each $A \in (\mathcal{G} \cup \mathcal{K})^\sigma$, the family of countable unions of elements of $\mathcal{G} \cup \mathcal{K}$. If E is locally qcompact with countable base, then $\mathcal{G} \subset \mathcal{K}^\sigma$, so $(\mathcal{G} \cup \mathcal{K})^\sigma = \mathcal{K}^\sigma$. We have found

11.7. THEOREM. *If E is locally qcompact with countable base and M is an extremal process on E, then $M(A)$ is a rv for each $A \in \mathcal{K}^\sigma$ (the σ-qcompact sets), in particular for open A.*

11.8. REMARK. By (3.2) we have $M(A) = M(\text{sat}A)$ wp1 for all $A \in \mathcal{K}^\sigma$ simultaneously. So we do not lose anything by restricting $\mathcal{K}^\sigma$ to the saturated sets in $\mathcal{K}^\sigma$. As $\bigcap \text{sat} = \text{sat} \bigcap$ (cf. §1), we have $\{A \in \mathcal{K}^\sigma : A = \text{sat}A\} = \mathcal{Q}^\sigma$, the class of countable unions of saturated qcompact sets. In the next section it will be convenient to restrict $\mathcal{Q}^\sigma$ a bit further to

$$\mathcal{D} := \{A \in \mathcal{Q}^\sigma : A \subset Q \text{ for some } Q \in \mathcal{Q}\}. \tag{11.1}$$

We call $\mathcal{D}$ the *natural domain* of extremal processes.

We now turn to random closed sets (cf. Definition 11.2). They can be regarded as $\{0,1\}$-valued extremal processes or $\{0,1\}$-valued random usc functions. The previous theorems specialize to the following result.

11.9. THEOREM. *Let $\mathcal{G}$ have a countable base, $\mathcal{G}_0$ be a base of $\mathcal{G}$ and E be locally $\mathcal{K}_0$ with $\mathcal{K}_0 \subset \mathcal{K}$. Then the following holds.*
(a) Bor $\mathcal{F}$ *is the smallest σ-field that contains $\{F \in \mathcal{F} : F \cap A \neq \emptyset\}$ for all $A \in \mathcal{G}_0$ or all $A \in \mathcal{K}_0$.*
(b) *A random closed set is a mapping X from some probability space into $\mathcal{F}$*

such that $[X \cap A \neq \emptyset]$ is an event for all $A \in \mathcal{G}_0$ or all $A \in \mathcal{K}_0$.
(c) *If in addition $\mathcal{G}_0$ (or $\mathcal{K}_0$) is closed for finite unions, then the probability distribution of a random closed set X is determined by $T(A) := \mathbb{P}[X \cap A \neq \emptyset]$ for $A \in \mathcal{G}_0$ (or $\mathcal{K}_0$); T is called the* ***distribution function*** *of X.*
(d) *If X is a random closed set, then $[X \cap A \neq \emptyset]$ is an event for each $A \in \mathcal{K}^\sigma$.*

PROOF. (a,b,d) Straightforward from Theorem 11.1 and Corollary 11.3.
(c) In the first instance, Theorem 11.5 translates into the distribution of X being determined by the finite-dimensional distributions of

$$\left(1_{[X\cap A\neq\emptyset]}\right)_{A\in\mathcal{G}_0} \text{ or } \left(1_{[X\cap A\neq\emptyset]}\right)_{A\in\mathcal{K}_0}, \tag{112}$$

where $\mathcal{G}_0$ and $\mathcal{K}_0$ need not yet be closed for finite unions. In general, the finite-dimensional distributions of a collection of $\{0,1\}$-valued rv's $(\epsilon_j)_{j\in J}$ determine and are determined by $\mathbb{P}[\epsilon_i = 0 \text{ for } i \in I]$ for all finite $I \subset J$. So

$$\mathbb{P}[X \cap A = \emptyset] = 1 - \mathbb{P}[X \cap A \neq \emptyset]$$

with A varying through the finite unions in $\mathcal{G}_0$ (or $\mathcal{K}_0$) determines the finite-dimensional distributions of (11.2). The relevant direction of determination can be read from

$$\begin{aligned}&\mathbb{P}[\epsilon_i = 0 \text{ for } i \in K,\ \ \epsilon_i = 1 \text{ for } i \in L\backslash K]\\ &= \sum_{I:K\subset I\subset L} (-1)^{\#(I\backslash K)}\mathbb{P}[\epsilon_i = 0 \text{ for } i \in I]\end{aligned}$$

for finite K, L with $K \subset L \subset J$. □

11.10. REMARK. It is possible to characterize those $T : \mathcal{K} \to [0,1]$ such that T is the distribution function of a random closed set X. See MATHERON (1975, §2.2), SALINETTI & WETS (1986) and ROSS (1986) for Hausdorff E, and REVUZ (1955) and HONEYCUT (1971) for more general E.

11.11. LITERATURE. Random closed sets ($= \mathcal{F}(E)$-valued rv's) are the subject of the monograph by MATHERON (1975). They appear in the shape of 'measurable closed multifunctions' in the optimization literature (ROCKAFELLAR (1976), CASTAING & VALADIER (1977)). SALINETTI & WETS (1981) conciliate the two points of view. Random lower semicontinuous functions appear in the shape of 'normal integrands' in the optimization literature (ROCKAFELLAR (1976)). SALINETTI & WETS (1986) conciliate the two points of view. See also NORBERG (1984) for random closed sets and NORBERG (1986) for random usc functions.

12. CONVERGENCE IN DISTRIBUTION

As in the previous section, E is locally qcompact with countable base, and SM and $\mathcal{F}$ are provided with the sup vague topology. By Corollary 4.4(a) and Theorem 5.5, SM and $\mathcal{F}$ are metrizable and compact. So the general theory about convergence in distribution as treated in BILLINGSLEY (1968) applies immediately to extremal processes and random closed sets, with the pleasant circumstance that the collection of all probability distributions on BorSM or Bor$\mathcal{F}$ is narrowly (=weakly) compact, so we need not worry about tightness conditions. Since the distribution of an extremal process M is determined by that of $(M(G))_{G\in\mathcal{G}_0}$ with $\mathcal{G}_0$ a base of $\mathcal{G}$, we may expect that convergence in distribution of M_n to M in SM is determined by something like convergence in distribution of $(M_n(G))_{G\in\mathcal{G}_0}$ to $(M(G))_{G\in\mathcal{G}_0}$ in $\mathbb{I}^{\mathcal{G}_0}$. We are going to make this precise.

As in the classical theory of convergence in distribution, we must be careful with sets at which the limit M is discontinuous with positive probability. Recall the definition of the natural domain $\mathcal{D}$ of extremal processes in (11.1), the defintion of Q_δ in Definition 8.3 and the definition of sqc after Definition 8.3, that each Hausdorff space E is Q_δ and that in Hausdorff spaces the sqc operation is the same as taking closure for relatively compact sets.

12.1. Definition. *Let E be locally qcompact and Q_δ with countable base.*
(a) *Let M be an extremal process on E. A set $A \in \mathcal{D}$ is called a* ***continuity set*** *of M if $M(\text{int}A) = M(\text{sqc}A)$ wp1. The family of all continuity sets of M is denoted by $\mathcal{C}(M)$.*
(b) *A class $\mathcal{A} \subset \mathcal{Q}^\sigma$ is* ***probability determining*** *if the distributions of extremal processes M are determined by the finite-dimensional distributions of $(M(A))_{A\in\mathcal{A}}$.*
(c) *A class $\mathcal{A} \subset \mathcal{Q}^\sigma$ is* ***convergence determining*** *if for each two extremal processes M_1 and M_2 the class $\mathcal{A}\cap\mathcal{C}(M_1)\cap\mathcal{C}(M_2)$ is probability determining.*

For the next theorem, recall Definition 8.7 of $\mathcal{Q}$ compatible semimetric and note that all metrizable locally compact (Hausdorff) E are $\mathcal{Q}$ compatibly semimetrized by their metrics.

12.2. Theorem. *Let E be $\mathcal{Q}$ compatibly semimetrized by semimetric d and have a countable base. Let D be a dense subset of E. Then the classes of balls*

$$\mathcal{G}_0 := \{B(t,r) : t \in D,\ r > 0,\ B(t,r+) \in \mathcal{K}\},$$

$$\mathcal{K}_0 := \{B(t,r+) : t \in D,\ r > 0,\ B(t,r+) \in \mathcal{K}\}$$

both are convergence determining.

Proof. Since $\mathcal{G}_0$ is a base of $\mathcal{G}$ and E is locally $\mathcal{K}_0$ (cf. Example 5.8 and Corollary 8.8), $\mathcal{G}_0$ and $\mathcal{K}_0$ are probability determining by Theorem 11.5. It is obvious that $\mathcal{G}_0$ and $\mathcal{K}_0$ keep these properties if r is allowed to vary only through a dense subset of $(0,\infty)$ for each $t \in D$. So we are done if we prove that $B(t,r)$, $B(t,r+) \in \mathcal{C}(M)$ for all but countably many $r \in (0,\infty)$, where M is an extremal process.

Let $t \in D$. Then $r \mapsto M(B(t,r))$ is a nondecreasing left-continuous function, whereas $M(B(t,r+)) = \lim_{s\downarrow r} M(B(t,s))$ (wp1). In this situation we have $M(B(t,r)) = M(B(t,r+))$ wp1 iff $M(B(t,s)) \to_d M(B(t,r))$ in $\mathbb{I}$ as $s\downarrow r$. So it is sufficient to show that the map $r \mapsto$ law $M(B(t,r))$ has only countably many discontinuities. The countable collection of bounded continuous nondecreasing functions

$$\Phi := \{0 \vee (ax + b) \wedge 1 : \; a, b \in \mathbb{Q}, \; a > 0\}$$

determines convergence in distribution in $\mathbb{I}$:

$$X_n \to_d X \text{ in } \mathbb{I} \text{ iff } \mathbb{E}\varphi(X_n) \to \mathbb{E}\varphi(X) \text{ for } \varphi \in \Phi.$$

Furthermore, $r \to \mathbb{E}\varphi(M(B(t,r)))$ is nondecreasing for $\varphi \in \Phi$, as $r \mapsto \varphi\ (M(B(t,r)))$ is nondecreasing wp1. So there are only countably many r at which $r \mapsto \mathbb{E}\varphi(M(B(t,r)))$ is discontinuous for at least one $\varphi \in \Phi$. Only at these points $r \mapsto$ law $M(B(t,r))$ can be discontinuous, so only at these points we may have $M(B(t,r)) \neq M(B(t,r+))$ with positive probability. □

The next theorem clarifies the term 'continuity set' in Definition 12.1(a). Note that we can also speak about continuity sets of deterministic sup measures m, as they can be regarded as degenerate extremal processes. Consequently, $\mathcal{C}(m) = \{A \in \mathcal{D} : m(\text{int}A) = m(\text{sqc}A)\}$.

12.3. THEOREM. *Let E be locally qcompact and Q_δ with countable base.*
(a) *If $m_0 \in SM$ and $A \in \mathcal{C}(m_0)$, then the map $SM \ni m \mapsto m(A) \in \mathbb{I}$ is continuous at m_0.*
(b) *Let $\mathcal{A}$ be a convergence determining class and M_n, M be extremal processes. Then $M_n \to_d M$ in SM iff*

$$(M_n(A))_{A \in \mathcal{C}(M)} \to_d (M(A))_{A \in \mathcal{C}(M)} \text{ in } \mathbb{I}^{\mathcal{C}(M)} \tag{12.1}$$

(i.e., the finite-dimensional distributions of the left-hand side converge to those of the right-hand side).

PROOF. (a) Suppose $m_n \to m_0$ in SM. Note that $A \subset \text{sqc}A \in \mathcal{Q} \subset \mathcal{K}$. By (3.5) we have

$$\begin{aligned} m_0(A) &= m_0(\text{int}A) \le \liminf m_n(\text{int}A) \le \liminf m_n(A) \\ &\le \limsup m_n(A) \le \limsup m_n(\text{sqc}A) \le m_0(\text{sqc}A) = m_0(A). \end{aligned}$$

(b) If $M_n \to_d M_0$ in SM and $A_1, A_2, \ldots, A_k \in \mathcal{C}(M_0)$, then $SM \ni m \mapsto (m(A_i))_{i=1}^k \in \mathbb{I}^k$ is wp1 continuous at M_0, so $(M_n(A_i))_{i=1}^k \mapsto_d (M_0(A_i))_{i=1}^k$ in $\mathbb{I}^k$ by the Continuous Mapping Theorem (BILLINGSLEY (1968), §5). Conversely, if (12.1) holds, then each M_0 to which some subsequence of (M_n) converges in distribution must have the same finite-dimensional distributions as M for $A \in \mathcal{A} \cap \mathcal{C}(M) \cap \mathcal{C}(M_0)$, so $M_0 =_d M$. Since SM is compact, (M_n) is relatively compact for convergence in distribution, so $M_n \to_d M$ in SM. □

12.4. REMARK. One can prove that $A \in \mathcal{C}(m_0)$ is also necessary for continuity of $m \mapsto m(A)$ at m_0 in case $A \in \mathcal{D}$, $\text{int}\,\text{sqc}\,\text{int}A = \text{int}\,A$ and $\text{sqc}\,\text{int}\,\text{sqc}\,A = \text{sqc}\,A$.

Identifying random closed sets X with the associated $\{0,1\}$-valued extremal processes $M := 1_X^\vee$ we can translate Definition 12.1(a) into the following.

12.5. DEFINITION. *Let X be a random closed set in E with distribution function $T := \mathbb{P}[X \cap \cdot \neq \emptyset]$ considered on $\mathcal{G} \cup \mathcal{K}$. Then $A \in \mathcal{D}$ is called a continuity set of X if $T(\text{int}A) = T(\text{sqc}A)$, and $\mathcal{C}(X)$ is the class of all such A.*

12.6. THEOREM. *Let E be locally qcompact and Q_δ with countable base, and let $\mathcal{A}$ be a convergence determining class which is closed for finite unions. Let X_n, X be random closed sets in E with distribution functions T_n, T. Then $X_n \to_d X$ in $\mathcal{F}$ iff $T_n(A) \to T(A)$ for all $A \in \mathcal{A} \cap \mathcal{C}(X)$.*

PROOF. Similar to the proof of Theorem 12.3(b). Use Theorem 11.9(c) for the uniqueness of limit points of convergent subsequences and note that $\mathcal{C}(X)$ is closed for finite unions. For the latter, note that $\text{sqc}(A \cup B) = \text{sqc}A \cup \text{sqc}B$ and $\text{int}(A \cup B) \supset \text{int}A \cup \text{int}B$, so $(\text{sqc}(A \cup B)) \backslash \text{int}(A \cup B) \subset ((\text{sqc}A) \backslash \text{int}A) \cup ((\text{sqc}B) \backslash \text{int}B)$. □

12.7. APPLICATIONS. (a) If M_n, M are extremal processes on an interval $E \subset \mathbb{R}$, then $M_n \to_d M$ iff

$$(M_n(J_i))_{i=1}^k \to_d (M(J_i))_{i=1}^k \text{ in } \mathbb{I}^k$$

for each finite sequence $(J_i)_{i=1}^k$ of open intervals which are relatively compact in E and such that $M(J_i) = M(\text{clos}J_i)$ wp1 for $i = 1, 2, \ldots, k$.
(b) If X_n, X are random closed sets in $\mathbb{R}^d$ with distribution functions T_n, T, then $X_n \to_d X$ iff $T_n(A) \to T(A)$ for all finite unions A of blocks in $\mathbb{R}^d$ such that $T(\text{int}A) = T(\text{clos}A)$.

12.8. LITERATURE. Convergence in distribution for random closed sets is studied by SALINETTI & WETS (1981) for $E = \mathbb{R}^d$ and by NORBERG (1984). Convergence in distribution for random usc functions is studied by SALINETTI & WETS (1986) for $E = \mathbb{R}^d$ and NORBERG (1986). For convergence in probability, see SALINETTI, VERVAAT & WETS (1986). Convergence of probability measures on semi-lattices is studied by NORBERG (1989).

13. THE EXISTENCE THEOREM FOR EXTREMAL PROCESSES

As in the previous sections we assume that E is locally qcompact with countable base. We need the following lemma, which will be proved in Section 14 (cf. Remark 14.15(a)).

13.1 LEMMA. *Let J be countable. Then the mapping $US^J \ni (f_j)_{j \in J} \mapsto \bigwedge_{j \in J} f_j \in US$ is measurable, so $\bigwedge_{j \in J} X_j$ is a US-valued rv if all X_j are.*

Let M be an extremal process and $\mathcal{G}_0$ a base of $\mathcal{G}$. By Theorem 11.5 the probability distribution of M is determined by the distribution of the $\mathbb{I}^{\mathcal{G}_0}$-valued rv $(M(G))_{G \in \mathcal{G}_0}$. However, if we do not assume an extremal process to be given, but start only with an $\mathbb{I}^{\mathcal{G}_0}$-valued rv $(N(G))_{G \in \mathcal{G}_0}$, then it need not be true that there is an extremal process M such that $M(G) = N(G)$ wp1 for each $G \in \mathcal{G}_0$ (separately). Obviously, a necessary condition for the existence of such an M is

$$N(\bigcup_{j=1}^{\infty} G_j) = \bigvee_{j=1}^{\infty} N(G_j) \text{ wp1} \tag{13.1}$$

for each separate sequence $(G_j)_{j=1}^{\infty}$ in $\mathcal{G}_0$ with $\bigcup_{j=1}^{\infty} G_j \in \mathcal{G}_0$. The next theorem tells us that this condition is also sufficient.

13.2. THEOREM. (Existence Theorem for extremal processes). *Let E be locally qcompact and Q_δ with countable base, and let $\mathcal{G}_0$ be a base of $\mathcal{G}$ that does not contain $\emptyset$. Let $(N(G))_{G\in\mathcal{G}_0}$ be an $\mathbb{I}^{\mathcal{G}_0}$-valued rv such that (13.1) holds wp1 for each separate sequence $(G_j)_{j=1}^{\infty}$ in $\mathcal{G}_0$ with $\bigcup_{j=1}^{\infty} G_j \in \mathcal{G}_0$. Then there is an extremal process M such that $M(G) = N(G)$ wp1 for each $G \in \mathcal{G}_0$ separately.*

13.3. REMARKS. Note that in the theorem the exceptional event of probability 0 that (13.1) does not hold may depend on the sequence $(G_j)_{j=1}^{\infty}$. The stronger condition that (13.1) holds for all sequences $(G_j)_{j=1}^{\infty}$ simultaneously reduces Theorem 13.2 to a trivial consequence of the Extension Theorem 2.6. If $\mathcal{G}_0$ is countable, then it follows that $M = N$ wp1 on $\mathcal{G}_0$, so N is wp1 the restriction of the extremal process M (again by Theorem 2.6.). If $\mathcal{G}_0$ is uncountable (for instance if $\mathcal{G}_0 = \mathcal{G}$), this need not be true, as shows the following example.

13.4. EXAMPLE. $E = \mathbb{R}$, $\mathcal{G}_0 = \{\text{open intervals}\}$, ξ is a rv with a uniform distribution in (0,1), $N(G) := 1_{[\xi\in\partial G]}$ for $G \in \mathcal{G}_0$, where ∂G is the boundary of G. Then N is wp1 not the restriction to $\mathcal{G}_0$ of an extremal process, but $M \equiv 0$ makes the theorem work.

13.5. REMARK. The complication in Example 13.4 is avoided by assuming N to be monotone on $\mathcal{G}_0$. Then the conclusion of Theorem 13.2 can be strengthened to $M(G) = N(G)$ wp1 for all $G \in \mathcal{G}_0$ simultaneously.

PROOF OF THEOREM 13.2. Let $\mathcal{G}_1 \subset \mathcal{G}_0$ be a countable base of $\mathcal{G}$ consisting of relatively qcompact sets, and let

$$X := \bigwedge_{G\in\mathcal{G}_1} (N(G)1_G \vee 1_{G^c}). \tag{13.2}$$

Then X is a US-valued rv by Lemma 13.1, so $M := X^\vee$ is an extremal process. It is obvious that

$$M(G) \le N(G) \text{ wp1 for all } G \in \mathcal{G}_1. \tag{13.3}$$

Let $(G_k)_{k=1}^{\infty}$ be an enumeration of $\mathcal{G}_1$ and set

$$X_n := \bigwedge_{k=1}^{n} (N(G_k)1_{G_k} \vee 1_{G_k^c}).$$

We are going to prove

$$N(G_k) \le X_n^\vee(G_k) \text{ for } n \ge k. \tag{13.4}$$

Let Δ_n be the collection of atoms of the field generated by $G_1, G_2, \ldots, G_n$. Then X_n is constant at each $D \in \Delta_n$ with value $\bigwedge_{j\le n, G_j\supset D} N(G_j)$. Hence

$$X_n^\vee(G_k) = \bigvee_{D\in\Delta_n, D\subset G_k} \bigwedge_{j\le n, G_j\supset D} N(G_j) \text{ for } k \le n. \tag{135}$$

If (13.4) would not hold for some fixed $n \ge k$, then for each atom $D \subset G_k$ as in (13.5) there is a $G_{j(D)}$ with $j(D) \le n$ and $G_{j(D)} \supset D$ such that $N(G_{j(D)}) < N(G_k)$. Hence

$$N(G_k) > \bigvee_{D\subset G_k} N(G_{j(D)}) = N(\bigcup_{D\subset G_k} G_{j(D)}) \text{ wp1},$$

contradicting (13.1), since $\bigcup_{D\subset G_k} G_{j(D)} \supset G_k$. This proves (13.4). As $X_n{\downarrow}X$ pointwise, we have by (13.3), (13.4) and Theorems 6.2(b) and 6.4

$$N(G_k) = \lim_{n\to\infty} X_n^\vee(G_k) \le \lim_{n\to\infty} X_n^\vee(\text{sqc}G_k) = M(\text{sqc}G_k).$$

Combining this result with (13.3) we find

$$M(G) \le N(G) \le M(\text{sqc}G) \text{ wp1 for all } G \in \mathcal{G}_1. \tag{136}$$

Now take $G_0 \in \mathcal{G}_0$. Then there is a (countable) subcollection $\mathcal{G}_2$ of $\mathcal{G}_1$ such that $G_0 = \bigcup_{G\in\mathcal{G}_2} G = \bigcup_{G\in\mathcal{G}_2} \text{sqc}G$. By (13.1) and (13.6) we have wp1

$$M(G_0) = \bigvee_{G\in\mathcal{G}_2} M(G) \le \bigvee_{G\in\mathcal{G}_2} N(G) = N(G_0) \le \bigvee_{G\in\mathcal{G}_2} M(\text{sqc}G) = M(G_0),$$

so $M(G_0) = N(G_0)$ wp1 for each separate $G_0 \in \mathcal{G}_0$. □

13.6 LITERATURE. For existence theorems for random closed sets based on their probability distribution functions, see REVUZ (1955), MATHERON (1975), BERG ET AL. (1984, Th.4.6.18), SALINETTI & WETS (1986) and NORBERG (1989). Where in NORBERG (1984, 1987) theorems are claimed to generalize Theorem 13.2, it is ignored that 'wp1' in Theorem 13.2 refers to each separate sequence (G_j). The exceptional null event may vary with it. For existence theorems for probability measures on semi-lattices, see NORBERG (1990).

14. SEMICONTINUITY OF THE LATTICE OPERATIONS

In the present section we first return to the generality of a topological space E without further assumptions, and the sup $\mathcal{B}$ topologies on SM, US and $\mathcal{F}$. The spaces SM, US and $\mathcal{F}$ are lattices with as partial orders the pointwise order of functions on $\mathcal{G}$ for SM, the pointwise order of functions on E for US, and set inclusion for $\mathcal{F}$. The lattices SM and US are isomorphic via $d^\vee$ and $i^\vee$, and $F \mapsto 1_F$ maps $\mathcal{F}$ isomorphically onto a sublattice of US, in which $\wedge$ and $\vee$ give the same result as in US.

We first investigate when the above partial orders are closed. Recall that a partial order $\le$ on a topological space T is closed if

$$\text{graph} \le := \{(x,y) \in T^2 : x \le y\}$$

is closed in T^2, or equivalently, if for all limits x and y of convergent nets (x_α) and (y_α) in T with $x_\alpha \le y_\alpha$ for all α we have $x \le y$. Note that the order in the subspace $\mathcal{F}$ is closed if the order in US is.

14.1 THEOREM. (a) *If E is locally $\mathcal{B}$, then the orders in SM, US and $\mathcal{F}$ are sup $\mathcal{B}$ closed.*
(b) *If the order in $\mathcal{F}$ is sup $\mathcal{B}$ closed, then E is internally $\mathcal{B}$.*

14.2 COROLLARY. If E is Hausdorff, then the orders in SM, US and $\mathcal{F}$ are sup $\mathcal{K}$ closed iff E is locally compact (cf. Property 3.7(d)).

PROOF OF THEOREM 14.1. (a) We give the proof for US. Let (h, g) be outside the graph of $\leq$, so $g(t) < h(t)$ for some $t \in E$. The construction in the proof of Theorem 4.3(c) gives the sides of an open rectangle around (h, g) that does not intersect graph $\leq$.
(b) It follows by Proposition I.2 of NACHBIN (1965) that $\mathcal{F}$ is Hausdorff. Apply Theorem 4.3(d). □

We now turn to the lattice operations. We write $\bigvee^{SM}$ and $\bigwedge^{SM}$ for the lattice operations in SM, and $\bigvee^{US}$ and $\bigwedge^{US}$ for the lattice operations in US. Note that $\bigvee^{SM}$ is the same as taking pointwise suprema of functions on $\mathcal{G}$, but that $\bigwedge^{SM}$ is more complicated:

$$\bigwedge_j^{SM} m_j = i^\vee d^\vee \bigwedge_j m_j = i^\vee \bigwedge_j d^\vee m_j. \tag{14.1}$$

PROOF. The first identity follows from Lemma 2.2(b) and Theorem 2.5(a). The second identity with $\leq$ instead follows from the monotonicity of $d^\vee$ and $i^\vee$ implying subsequently

$$\begin{aligned}
&\bigwedge_j m_j \leq m_k, \\
&d^\vee \bigwedge_j m_j \leq d^\vee m_k, \\
&d^\vee \bigwedge_j m_j \leq \bigwedge_j d^\vee m_j, \\
&i^\vee d^\vee \bigwedge_j m_j \leq i^\vee \bigwedge_j d^\vee m_j.
\end{aligned}$$

On the other hand we have $m_k = i^\vee d^\vee m_k \geq i^\vee \bigwedge_j d^\vee m_j$ with a sup measure on the right-hand side, so

$$\bigwedge_j^{SM} m_j \geq i^\vee \bigwedge_j d^\vee m_j.$$ □

Analogously, $\bigwedge^{US}$ is the same as taking pointwise infima of functions on E, but now $\bigvee^{US}$ is more complicated:

$$\bigvee_j^{US} f_j = d^\vee i^\vee \bigvee_j f_j = d^\vee \bigvee_j i^\vee f_j. \tag{14.2}$$

Consequently, we prefer considering $\bigvee$ in SM and $\bigwedge$ in SM (often without writing the upper indices).

We now want to investigate the topological properties of the lattice operations. The following concepts will be useful.

14.3. DEFINITION. (a) (cf. (3.3)). *The **upper topology** on SM is the topology with subbase consisting of $\{m : m(G) > x\}$ for $G \in \mathcal{G}$ and $x \in [0, 1)$. The $\mathcal{B}$ **lower topology** on SM is the topology with subbase consisting of*

$\{m : m(B) < x\}$ *for* $B \in \mathcal{B}$ *and* $x \in (0, 1]$.

The lower and upper topologies are defined on US *and* $\mathcal{F}$ *by declaring* $d^\vee$ *and 'ind' homeomorphisms (so have for subbases the corresponding halves of (3.7) and (3.8)). We write* $SM\!\uparrow$, $US\!\uparrow$ *and* $\mathcal{F}\!\uparrow$ *for the spaces with the upper topologies, and* $SM\!\downarrow$ *or* $SM\!\downarrow_\mathcal{B}$, *etc. for the spaces with the* $\mathcal{B}$ *lower topologies.*
(b) *Let* T *be a topological space. A mapping* $\varphi : T \to SM$, US *or* $\mathcal{F}$ *is called* ***lower semicontinuous (lsc)*** *if* $\varphi : T \to SM\!\uparrow$, $US\!\uparrow$ *or* $\mathcal{F}\!\uparrow$ *is continuous, and* φ *is called* $\mathcal{B}$ ***upper semicontinuous (usc)*** *if* $\varphi : T \to SM\!\downarrow_\mathcal{B}, US\!\downarrow_\mathcal{B}$ *or* $\mathcal{F}\!\downarrow_\mathcal{B}$ *is continuous. If* T *has the form* $T = (SM\!\uparrow)^J, (US\!\uparrow)^J$ *or* $(\mathcal{F}\!\uparrow)^J$ *with the product topology, then lsc functions on* T *are called* ***lower continuous***. *If* T *has the form* $T = (SM\!\downarrow_\mathcal{B})^J$, $(US\!\downarrow_\mathcal{B})^J$ *or* $(\mathcal{F}\!\downarrow_\mathcal{B})^J$ *with the product topology, then usc functions on* T *are called* $\mathcal{B}$ ***upper continuous***.

14.4. COROLLARY. In the situation

$$S \xrightarrow{\psi} T \xrightarrow{\varphi} SM,\ US \text{ or } \mathcal{F},$$

$\varphi \circ \psi$ is lsc ($\mathcal{B}$usc) if ψ is continuous and φ is lsc ($\mathcal{B}$usc), and $\varphi \circ \psi$ is lower ($\mathcal{B}$ upper) continuous for appropriate S and T if both ψ and φ are.

14.5. REMARK. If E is locally qcompact with countable base, then lsc and $\mathcal{B}$ usc functions are Borel measurable, by Theorem 11.1.

14.6. THEOREM. *Let* J *be an arbitrary index set. Then*

$$SM^J \ni (m_j)_{j \in J} \mapsto \bigvee_{j \in J} m_j \in SM \tag{14.3}$$

is lower continuous, and $\mathcal{B}$ *upper continuous (so* $\mathcal{B}$ *continuous) if* J *is finite.*

14.7. COROLLARY. The mapping $\mathcal{F}^J \ni (F_j)_{j \in J} \mapsto \operatorname{clos} \bigcup_{j \in J} F_j \in \mathcal{F}$ is lower continuous, and $\mathcal{B}$ upper continuous (so $\mathcal{B}$ continuous) if J is finite.

PROOF OF THEOREM 14.6. (a) Lower continuity follows from

$$\{(m_j) : \textstyle\bigvee_j m_j(G) > x\} \;=\; \textstyle\bigcup_k \{(m_j) : m_k(G) > x\} \in \mathcal{G}((SM\!\uparrow)^J).$$

It remains to prove that the mapping is $\mathcal{B}$ usc in case J is finite. If so, then $\bigvee_j^{US}$ corresponds to taking pointwise suprem of usc functions, and by Theorem 2.5(c) we have for $B \in \mathcal{B}$

$$\textstyle\bigvee_j^{SM} m_j(B) = \bigvee_{t \in B} \left[\bigvee_j d^\vee m_j\right](t) = \bigvee_j \bigvee_{t \in B} d^\vee m_j(t) = \bigvee_j m_j(B).$$

Consequently,

$$\{(m_j) : \textstyle\bigvee_j^{SM} m_j(B) < x\} = \textstyle\bigcap_k \{(m_j) : m_k(B) < x\} \in \mathcal{G}((SM\!\downarrow_\mathcal{B})^J). \ \square$$

14.8. REMARK. If J is infinite, then the mapping (14.3) need not be continuous. We exhibit this in $\mathcal{F}$ rather than SM. Let E be a separable metric space without isolated points, $(t_j)_{j=1}^\infty$ a dense sequence in E, $\mathcal{B} = \mathcal{K}$ and $F_{j,n} := \emptyset$ for $j < n$, $\{t_j\}$ for $j \geq n$. Then we have $F_{j,n} \to \emptyset =: F_j$ as $n \to \infty$, so that

$$\operatorname{clos} \bigcup_{j=1}^{\infty} F_{j,n} = \operatorname{clos}\{t_n, t_{n+1}, \ldots\} \to E,$$

whereas $\bigcup_{j=1}^{\infty} F_j = \emptyset$.

We now are going to study the semicontinuity of $\bigwedge$. We will restrict our attention to $\mathcal{B} = \mathcal{K}$, or equivalently, $\mathcal{B} = \mathcal{Q}$. The following assumption will be crucial. Note that it is equivalent to its restriction from $\mathcal{K}$ to $\mathcal{Q}$.

14.9. ASSUMPTION. *For $K \in \mathcal{K}$ and $G_1, G_2 \in \mathcal{G}$ such that $K \subset G_1 \cup G_2$ there are $K_1, K_2 \in \mathcal{K}$ such that $K_1 \subset G_1, K_2 \subset G_2$ and $K \subset K_1 \cup K_2$.*

14.10. LEMMA. *Sufficient conditions for Assumption 14.9 to hold are that E is locally qcompact or that E is Hausdorff. Assumption 14.9 does not hold if E is the one-point qcompactification of a space which is not locally qcompact.*

PROOF. See VERVAAT (1988a). □

14.11. THEOREM. *Let J be an arbitrary index set. Then*

$$\mathcal{F}^J \ni (F_j)_{j\in J} \mapsto \bigcap_{j\in J} F_j \in \mathcal{F} \tag{14.4}$$

is $\mathcal{K}$ upper continuous iff Assumption 14.9 holds. The mapping is not $\mathcal{K}$ continuous, even if J is finite.

PROOF. First an example showing that (14.4) is not $\mathcal{K}$ continuous if $\#J = 2$. Let $E := \mathbb{R}$, $F_{\pm n} := \{\pm 1/n\}$ for $n = 1, 2, \ldots$. Then $F_{jn} \to \{0\} =: F_j$ for $j = +, -$, whereas $F_{+n} \cap F_{-n} = \emptyset$, $F_+ \cap F_- = \{0\}$.

Necessity and sufficiency of Assumption 14.9 for the case $\#J = 2$ has been proved in VERVAAT (1988a). This implies already necessity of Assumption 14.9 for all larger J (take $F_j = E$ for all j but two). Sufficiency for finite J follows by induction. For infinite J, note that for qcompact K

$$\{(F_j)_{j\in J} : K \cap \bigcap_{j\in J} F_j = \emptyset\} = \bigcup_{\text{finite} J_\# \subset J} \{(F_j)_{j\in J} : K \cap \bigcap_{j\in J_\#} F_j = \emptyset\}.$$

The set on the right-hand side is open because of our previous result for finite J. □

14.12. THEOREM. *Let J be an arbitrary index set. Then*

$$US^J \ni (f_j)_{j\in J} \mapsto \bigwedge_{j\in J} f_j \in US \tag{14.5}$$

is $\mathcal{K}$ upper continuous if Assumption 14.9 holds. The mapping is not $\mathcal{K}$ continuous, even if J is finite.

PROOF. By Theorem 7.2 the spaces $US(E)$ and $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ are sup $\mathcal{K}$ homeomorphic. So Theorem 14.12 follows from Theorem 14.11 if we show that

Assumption 14.9 also holds for $E^* := E \times \mathbb{I}'\!\uparrow$. To this end, suppose that $F_1^*\cap, F_2^*\cap \in \mathcal{F}(E^*)$ and $K^* \in \mathcal{K}(E^*)$ and that $F_1^* F_2^* K^* = \emptyset$. Adopt the notations after Theorem 7.3. Then $F_1^* \cap F_2^* \cap \uparrow K^* = \emptyset$ by (7.3), and also $\uparrow K^* \in \mathcal{K}(E^*)$. Because F_1^* and F_2^* are hypographs, we have $\pi_1(F_1^* \cap \uparrow K^*) \cap \pi_1(F_2^* \cap \uparrow K^*) = \emptyset$. Now $\pi_1(\uparrow K^*)$ is qcompact since π_1 is continuous, and $\pi_1(F_n^* \cap \uparrow K^*)$ is closed in $\pi_1(\uparrow K^*)$ for $n = 1, 2$, since π_1 is a closed mapping when restricted to the qcompact domain $\pi_1 \uparrow K^* \times \pi_2 \uparrow K^*$. By Assumption 14.9 holding for E we can find K_1 and $K_2 \in \mathcal{K}(E)$ such that

$$K_1 \cap \pi_1(F_2^* \cap \uparrow K^*) = K_2 \cap \pi_1(F_1^* \cap \uparrow K^*) = \emptyset$$

and $\pi_1 \uparrow K^* \subset K_1 \cup K_2$. Then with $K_n^* := K_n \times \pi_2 \uparrow K^*$ we have $K_1^* \cap F_2^* = K_2^* \cap F_1^* = \emptyset$ and $K^* \subset K_1^* \cup K_2^*$. □

14.13. REMARKS. (a) If J is countable, then BorSM^J is the J-fold product σ-field of BorSM. So if $(M_j)_{j\in J}$ is a countable collection of extremal processes, then $\bigvee_{j\in J} M_j$ and $\bigwedge_{j\in J}^{SM} M_j$ are extremal processes. Considering the sup derivatives of M_j we obtain Lemma 13.1.
(b) If J is uncountable, then BorSM^J is strictly larger than the J-fold product σ-field of BorSM (cf. NELSON (1959), Theorem 2.1 and Corollary 2.1), so $\bigvee_j M_j$ and $\bigwedge_j^{SM} M_j$ need no longer be extremal processes if all M_j are. However, for each system of extremal processes there is a 'version' (i.e., another system of extremal processes with the same joint distributions for each finite subsystem of extremal processes) which is BorSM^J measurable. Its distribution over BorSM^J is unique if we require in addition that it is regular. All this is an immediate application of Theorem 1.1 of NELSON (1959).

14.14. LITERATURE. For special cases of Theorems 14.6 and 14.11, see BERGE (1963), KURATOWSKI (1968, §43) and MATHERON (1975). Assumption 14.9, Lemma 14.10 and Theorem 14.11 is the central topic of VERVAAT (1988a). Condition 14.9 already occurs in WILKER (1970).

15. CAPACITIES

15.1. DEFINITION. *A* ***precapacity*** *is a function* $c : \mathcal{K}(E) \to [0, \infty] =: \mathbb{J}$ *such that* $c(\emptyset) = 0$ *and* c *is increasing:* $c(K_1) \le c(K_2)$ *if* $K_1 \subset K_2$.

Examples of precapacities are obtained by restricting countably additive measures μ on Bor$E$ to $\mathcal{K}(E)$. In this case we have $c(K_1 \cup K_2) = c(K_1) + c(K_2)$ for disjoint $K_1, K_2 \in \mathcal{K}(E)$, or more generally,

$$c(K_1 \cup K_2) + c(K_1 \cap K_2) = c(K_1) + c(K_2) \text{ in case also } K_1 \cap K_2 \in \mathcal{K}(E). \tag{151}$$

Other examples of precapacities are the canonical extensions of sup measures m on $\mathcal{G}(E)$ restricted to $\mathcal{K}(E) : c(K) := \bigwedge_{G \supset K} m(G)$ (cf. Theorem 2.5(c)). In this case we have

$$c(K_1 \cup K_2) = c(K_1) \vee c(K_2). \tag{152}$$

Equivalently, if $f \in US(E)$, then $c(K) := f^{\vee}(K)$ defines a precapacity with the same properties. Finally, if $F \in \mathcal{F}(E)$, then the same procedure for $f = 1_F$ gives a $\{0,1\}$-valued precapacity c satisfying (15.2).

Precapacities can be extended to all subsets of E by

$$c(A) := \bigvee_{K \subset A} c(K) \text{ for } A \subset E. \tag{15.3}$$

In particular the extension to $\mathcal{G}(E)$ is important.

15.2. DEFINITION. *A precapacity is* ***upper semicontinuous (usc)***, *if*

$$c(K) = \bigwedge_{G \supset K} c(G) \text{ for } K \in \mathcal{K}(E). \tag{15.4}$$

Obviously, $c(K) = c(\mathrm{sat}K)$ for usc precapacities, so we can restrict their domain to the saturated qcompact sets $\mathcal{Q}(E)$. However, we now must *require* $c(K) = c(\mathrm{sat}K)$ for $K \in \mathcal{K}$ before applying (15.3). We then have $c(A) = c(\mathrm{sat}A)$ for $A \subset E$.

15.3. DEFINITION. *A* ***capacity*** *is an usc precapacity with domain restricted to* $\mathcal{Q}(E)$.

In the literature one sees often the following 'upper continuity' condition, which reads in a generalization to the non-Hausdorff case:

$$\begin{aligned} &c(Q_n)\downarrow c(\textstyle\bigcap Q_n) \text{ for all decreasing nets } (Q_n) \text{ in } \mathcal{Q}(E) \\ &\quad \text{with } \textstyle\bigcap Q_n \in \mathcal{Q}(E). \end{aligned} \tag{15.5}$$

15.4. THEOREM. (a) *If E is sober, then capacities c satisfy (15.5).*
(b) *If a precapacity c satisfies (15.5) and E is sober and locally qcompact (in particular if E is locally compact), then (15.4) holds, so c is a capacity.*

PROOF. (a) If $Q_n \downarrow Q := \bigcap Q_n$ in $\mathcal{Q}$, then $c(Q) \leq \lim c(Q_n)$ since $Q \subset Q_n$ for all n. Conversely, if $G \supset Q$ and $Q_n \downarrow Q$, then $Q_n \subset G$ for large n in case E is sober (HOFMANN & MISLOVE (1981)). Hence $\lim c(Q_n) \leq c(Q)$.
(b) Let $Q \in \mathcal{Q}$. By applying Property 3.7(b) we find for each instance of $Q \subset G$ a $Q' \in \mathcal{Q}(E)$ such that $Q \subset \mathrm{int}Q' \subset Q' \subset G$. We have $Q = \bigcap_{G \supset Q} G$ because Q is saturated. Selecting with each such G a Q' as above and applying (15.5) to the net of finite intersections of such Q' we find $c(Q) \geq \bigwedge_{G \supset Q} c(G)$. The reverse inequality is obvious. □

15.5. EXAMPLES. (a) Let $E = \mathbb{N} \cup \{\infty\}$ be the Appert-Varadarajan space, i.e., all sets $\{n\} \subset \mathbb{N}$ are open and a subset $G \subset E$ containing ∞ is open iff $\lim n^{-1} \#(G \cap \{1, \ldots, n\}) = 1$. Then E is Hausdorff and $\mathcal{K}$ consists of its finite subsets. If $c = \#$, then c is the extension of a finite precapacity on $\mathcal{K}$ to all subsets of E. Obviously, c satisfies (15.5). However, c is not usc since $c(G) = \infty$ for all G containing ∞.
(b) Here is an example of a precapacity c with different limits $\lim c(K_n)$ for

different decreasing sequences (K_n) with the same intersection. Let $E := \{(0,0)\} \cup (0,1]^2$ with the trace topology and trace distance d from $\mathbb{R}^2$. Set $V := \{0\} \times \mathbb{R}$ and $o := (0,0)$. Then

$$c(K) := \frac{d(V,K) + d(o,K)}{d(V,K)} \text{ for } K \in \mathcal{K}(E\backslash\{o\})$$

defines a capacity on $E\backslash\{o\}$. Let c' be the precapacity on E defined by $c'(K) := c(K\backslash\{o\})$. If (K_n) is a decreasing sequence of line segments starting at o, then $\lim c(K_n)$ depends on the slope of these segments.

For the moment, we return to precapacities. For sufficiently nice E, for instance metric, consider precapacities c which are restrictions to $\mathcal{K}(E)$ of Radon measures μ on Bor$E$ (i.e., μ is finite on $\mathcal{K}(E)$ and $\mu(A) = \bigvee_{K\subset A} \mu(K)$ for $A \in$ Bor$E$). The well-known vague topology on spaces of Radon measures (cf. BERG ET AL. (1984, §2.4)) suggests us the *vague topology* on spaces of precapacities, with subbase

$$\{c : c(G) > x\},\ \{c : c(K) < x\} \text{ for } G \in \mathcal{G}, K \in \mathcal{K} \text{ and } x \in \mathbb{J}. \tag{15.6}$$

Note that the trace topology on the space of the c arising from sup measures, usc functions or closed sets coincides with what we called the sup vague topologies on SM, US and $\mathcal{F}$. Similarly, the case of bounded measures μ on Bor E suggests us to extend the notion of *narrow (= weak) topology* to spaces of precapacities, with subbase

$$\{c : c(G) > x\},\ \{c : c(F) < x\} \text{ for } G \in \mathcal{G}, F \in \mathcal{F} \text{ and } x \in \mathbb{J}. \tag{15.7}$$

Again, the trace topology on the precapacities coming from SM, US or $\mathcal{F}$ corresponds to the sup narrow topology. We will study these topologies, in the case of the vague topology including the relations with spaces of Radon measures and spaces of sup measures. The latter aspect for the narrow topology is more complicated, and will be dealt with in another paper.

In the previous sections we have assumed that sup measures and usc functions have their values in $\mathbb{I} = [0,1]$. By obvious transformations we may replace $\mathbb{I}$ by any compact interval in $[-\infty,\infty]$, in particular by $\mathbb{J} = [0,\infty]$, the range of capacities. In the present section we will think $\mathbb{I}$ replaced by $\mathbb{J}$.

We now take the following point of view. We consider $\mathcal{Q}(E)$ as space on its own, with as points the saturated qcompact subsets Q of E, and want to regard (15.6) and (15.7) as special cases of sup topologies on $US(\mathcal{Q}(E))$. In particular, this implies that we provide $\mathcal{Q}(E)$ with a (non-Hausdorff) topology $\mathcal{G}^{\mathcal{Q}}$ with base $\mathcal{G}_0^{\mathcal{Q}}$ consisting of

$$\mathcal{Q}(G) = \{Q \in \mathcal{Q}(E) : Q \subset G\} \text{ for } G \in \mathcal{G}(E). \tag{15.8}$$

Then a precapacity c determines a sup measure $c^\vee$ on $\mathcal{G}^{\mathcal{Q}}$ by

$$c^\vee(G^{\mathcal{Q}}) = \bigvee_{Q\in G^{\mathcal{Q}}} c(Q) \text{ for } G^{\mathcal{Q}} \in \mathcal{G}^{\mathcal{Q}}.$$

For $G^{\mathcal{Q}} \in \mathcal{G}_0^{\mathcal{Q}}$ this specializes to

$c^\vee(\mathcal{Q}(G)) = c(G)$ as defined in (15.3).

Our new upper semicontinuity assumption about c is that $c \in US(\mathcal{Q}(E))$, so $c = d^\vee i^\vee c$, which may be written as (cf. proof of Theorem 2.6)

$$\begin{aligned} c(Q) &= \bigwedge\nolimits_{G^{\mathcal{Q}} \in \mathcal{G}^{\mathcal{Q}}: G^{\mathcal{Q}} \ni Q} c^\vee(G^{\mathcal{Q}}) = \bigwedge\nolimits_{G^{\mathcal{Q}} \in \mathcal{G}_0^{\mathcal{Q}}: G^{\mathcal{Q}} \ni Q} c^\vee(G^{\mathcal{Q}}) \\ &= \bigwedge\nolimits_{G \in \mathcal{G}(E): G \supset Q} c^\vee(\mathcal{Q}(G)) \\ &= \bigwedge\nolimits_{G \in \mathcal{G}(E): G \supset Q} c(G), \end{aligned}$$

which is (15.4). Consequently,

15.6. THEOREM. *A precapacity c is a capacity, i.e., (15.4) holds, iff $c \in US(\mathcal{Q}(E))$ and $c(\emptyset) = 0$.*

We write $CAP = CAP(E)$ for the family of all capacities on E (or rather $\mathcal{Q}(E)$). Recall that $CAP(E) = \{c \in US(\mathcal{Q}(E))\colon c(\emptyset) = 0\} =: US_0(\mathcal{Q}(E))$, where $\mathcal{Q}(E)$ is provided with the topology with base $\mathcal{G}_0^{\mathcal{Q}}$ consisting of the sets in (15.8). Let $\mathcal{B}$ be a class of subsets of E. Then

$$\mathcal{B}^{\mathcal{Q}} := \{\mathcal{Q}(B) : B \in \mathcal{B}\}$$

is a class of subsets of $\mathcal{Q}(E)$. In view of (15.6) and (15.7) we define the $\mathcal{B}$ topology on CAP as topology with subbase

$$\{c : c(G) > x\},\ \{c : c(B) < x\} \text{ for } G \in \mathcal{G},\ B \in \mathcal{B} \text{ and } x \in \mathbb{J}.$$

Since $c(B) = c(\text{sat}B) = c^\vee(\mathcal{Q}(B))$, we see by Lemma 5.1 that the $\mathcal{B}$ topology on $CAP(E)$ is the same as the sup $\mathcal{B}^{\mathcal{Q}}$ topology on $US_0(\mathcal{Q}(E))$.

15.7. THEOREM. *If E is locally $\mathcal{B}$ and $\mathcal{B}$ is closed for finite unions, then $\mathcal{Q}(E)$ is locally $\mathcal{B}^{\mathcal{Q}}$, and CAP is $\mathcal{B}^{\mathcal{Q}}$ Hausdorff.*

PROOF. The generic element of the base $\mathcal{G}_0^{\mathcal{Q}}$ is in (15.8). Let $Q_0 \in \mathcal{Q}(G)$, so $Q_0 \subset G$. For each $t \in Q_0$, select a $B(t) \in \mathcal{B}$ such that $t \in \text{int}B(t) \subset B(t) \subset G$. Then $Q_0 \subset \bigcup_{t \in Q_0} \text{int}B(t)$. Select a finite subset $Q_\#$ of Q_0 such that $Q_0 \subset \bigcup_{t \in Q_\#} \text{int}B(t)$, and set $B := \bigcup_{t \in Q_\#} B(t)$. Then $B \in \mathcal{B}$ and $Q_0 \subset \text{int}B \subset B \subset G$. So

$$Q_0 \in \mathcal{Q}(\text{int}B) \subset \mathcal{Q}(B) \subset \mathcal{Q}(G),$$

where $\mathcal{Q}(\text{int}B) \in \mathcal{G}_0^{\mathcal{Q}}$ and $\mathcal{Q}(B) \in \mathcal{B}^{\mathcal{Q}}$. So $\mathcal{Q}(E)$ is locally $\mathcal{B}^{\mathcal{Q}}$. Then $US(\mathcal{Q}(E))$ and also $US_0(\mathcal{Q}(E))$ is sup $\mathcal{B}$ Hausdorff by Theorem 4.3(c), so CAP is $\mathcal{B}$ Hausdorff. □

15.8. COROLLARY. If E is locally closed ($= T_3$, cf. Property 3.7(e)), then CAP is narrowly Hausdorff.

15.9. THEOREM. (a) *The space CAP is vaguely qcompact.*
(b) *If E is locally qcompact, then CAP is vaguely compact.*

PROOF. (a) Follows from Corollary 4.4(a), provided that $\mathcal{Q}^{\mathcal{Q}} \subset \mathcal{K}(\mathcal{Q}(E))$ (cf. Theorem 4.2.(iii)). To prove the latter, let $Q \in \mathcal{Q}(E)$ and consider $\mathcal{Q}(Q) = \{H \in \mathcal{Q} : H \subset Q\}$. Let $\mathcal{Q}(Q) \subset \bigcup_\alpha \mathcal{Q}(G_\alpha)$. Then $Q \in \mathcal{Q}(Q) \subset \bigcup_\alpha \mathcal{Q}(G_\alpha)$, so $Q \in \mathcal{Q}(G_\alpha)$ for some $\alpha =: \beta$. Then $Q \subset G_\beta$, so $\mathcal{Q}(Q) \subset \mathcal{Q}(G_\beta)$.
(b) Combine (a) and Theorem 15.7 for $\mathcal{B} = \mathcal{Q}$. □

15.10. LITERATURE. The present section complements and generalizes aspects of NORBERG (1986). For similar results on narrow convergence of capacities, see SALINETTI & WETS (1987) and VERVAAT (1988). DAL MASO (1980) has a similar approach to capacities based on topologies in spaces of increasing functions (DAL MASO (1979)). The topics of the present section have been developed further by NORBERG & VERVAAT (1989) and HOLWERDA & VERVAAT (1993). For simpler and direct proofs of the vague qcompactness of CAP, see HOLWERDA & VERVAAT (1993).

16. SUP AND RADON MEASURES AS SPECIAL CAPACITIES

In the beginning of the previous section we observed that restrictions of sup and Radon measures to $\mathcal{K}$ are precapacities with specific behavior for unions in $\mathcal{K}$ (cf. 15.2) and (15.1)). In the present section we are going to characterize the spaces of these restrictions as subspaces of the (pre)capacities. The presentation is self-contained for sup measures. The corresponding results for Radon measures demand much more theory and are quoted from the literature.

We start with some generalities about precapacities.

16.1. LEMMA. *If c is a precapacity and (G_n) is an increasing net in $\mathcal{G}$ with union $G := \bigcup_n G_n$, then $c(G_n)\uparrow c(G)$.*

PROOF. If $x < c(G) = \bigvee_{K\subset G} c(K)$, then there is a $K \subset G$ such that $x < c(K)$. Since K is qcompact, there is an n such that $K \subset G_n$, so $x < c(G_n)$. Hence $x < \lim c(G_n)$, which proves $c(G) \le \lim c(G_n)$. The reverse inequality is trivial. □

16.2. COROLLARY. Let c be a precapacity.
(a) If

$$c(G_1 \cup G_2) = c(G_1) \vee c(G_2) \text{ for } G_1, G_2 \in \mathcal{G}, \tag{161}$$

then $c(\bigcup_j G_j) = \bigvee_j c(G_j)$ for arbitrary collections in $\mathcal{G}$ (apply the lemma to the net of finite unions).
(b) If $c(G_1 \cup G_2) = c(G_1) + c(G_2)$ for disjoint G_1, G_2 in $\mathcal{G}$, then $c(\bigcup_j G_j) = \sum_j c(G_j)$ for arbitrary collections of disjoint sets in $\mathcal{G}$ (idem).

16.3. LEMMA. *If Assumption 14.9 holds and*

$$c(K_1 \cup K_2) = c(K_1) \vee c(K_2) \text{ for } K_1, K_2 \in \mathcal{K}, \tag{162}$$

then (16.1) holds.

PROOF. If $x < c(G_1 \cup G_2)$, then there is a qcompact $K \subset G_1 \cup G_2$ such that

$x < x(K)$. By Assumption 14.9 there are $K_1 \subset G_1$ and $K_2 \subset G_2$ such that $K \subset K_1 \cup K_2$. Hence

$$x < c(K) < c(K_1 \cup K_2) = c(K_1) \vee c(K_2) \leq c(G_1) \vee c(G_2).$$

We have proved $c(G_1 \cup G_2) \leq c(G_1) \vee c(G_2)$. The reverse inequality is trivial. □

16.4. THEOREM. *If Assumption 14.9 holds, then a capacity c is the restriction to $\mathcal{K}$ of the extension of a sup measure on $\mathcal{G}$ iff (16.2) holds.*

PROOF. By Lemma 16.3 and Corollary 16.2(a) we see that c is a sup measure on $\mathcal{G}$. Upper semicontinuity of c guarantees that $c(K) = \bigwedge_{G \supset K} c(G)$, in accordance with Theorem 2.5(c). □

16.5. COROLLARY. If Assumption 14.9 holds, then there is for each capacity c satisfying (16.2) a unique $f \in US$ such that $c(K) = f^{\vee}(K)$ for $K \in \mathcal{Q}$.

We now turn to Radon measures and henceforth assume that E is Hausdorff. There are two different definitions of Radon measures in the literature. Following BERG ET AL. (1984) we say that a countably additive measure μ on Bor E is *Radon* if μ is finite on $\mathcal{K}$ and $\mu(A) = \bigvee_{K \subset A} \mu(K)$ for $A \in \text{Bor}E$. Most other authors, starting with BOURBAKI (1965), require in addition that μ is *locally finite*: for each $t \in E$ there is an open $G \ni t$ such that $\mu(G) < \infty$. It is not hard to see that a Radon measure is locally finite iff its restriction to $\mathcal{K}$ is usc as a precapacity, so is a capacity.

Here is a list of plausible characterizations of finite additivity of precapacities on $\mathcal{K}$. Each line is implied by the next.

$$c(K_1 \cup K_2) = c(K_1) + c(K_2) \text{ if } K_1 \cap K_2 = \emptyset \; ; \tag{16.3a}$$

$$c(K_1 \cup K_2) \leq c(K_1) + c(K_2) \;\&\; (16.3a) \; ; \tag{16.3b}$$

$$c(K_1 \cup K_2) + c(K_1 \cap (K_2) = c(K_1) + (K_2); \tag{16.3c}$$

$$c(K_1) = c(K_1 \backslash K_2) + c(K_2) \text{ if } K_1 \supset K_2. \tag{16.3d}$$

16.6. THEOREM. *A precapacity c is the restriction of a Radon measure to $\mathcal{K}$ iff c is finite-valued on $\mathcal{K}$ and (16.3d) holds.*

PROOF. BERG ET AL. (1984, Th.2.1.4). □

16.7. THEOREM. *A capacity c is the restriction of a (necessarily locally finite) Radon measure to $\mathcal{K}$ iff c is finite-valued on $\mathcal{K}$ and (16.3b) holds.*

PROOF. BOURBAKI (1965, Th.IX.3.1 + Remark 1). □

16.8. EXAMPLE Let $E = \mathbb{R}$ and let $c([a, b]) := e^{b-a} - 1$ for compact intervals $[a, b]$. Extend c to finite disjoint unions of such intervals by (16.3a), and subsequently to all of $\mathcal{K}$ by (15.5). Then c is a capacity by Theorem 15.4(b). Furthermore, c satisfies (16.3a), but is not the restriction of a Radon measure, since it does not satisfy (16.3b).

16.9. LITERATURE. For related problems in partially ordered sets, see NORBERG (1989). For related problems in non-Hausdorff spaces, see NORBERG & VERVAAT (1989).

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On the Convergence of Probability Measures on Continuous Posets

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ABSTRACT. We study convergence in distribution of random variables in a countably based continuous poset L. The convergence is with respect to the Lawson topology on L. The main result is the following:

Let $\xi, \xi_1, \xi_2, \ldots$ be random variables in L. Then ξ_n converges in distribution to ξ if, and only if,

$$\limsup_n P\cap_i\{x_i \leq \xi_n\} \leq P\cap_i\{x_i \leq \xi\}$$

for finite collections $\{x_i\} \subseteq L$, and

$$\liminf_n P\{\xi_n \in \cap_i F_i\} \geq P\{\xi \in \cap_i F_i\}$$

for finite collections $\{F_i\}$ of Scott open filters on L.

We also derive some new existence results for probability measures on L. A lattice theoretical notion of tightness is introduced and related to the classical notion of tightness for random elements in a topological space.

Our results apply to random elements and various kinds of random sets in locally compact second countable sober spaces. They furthermore apply to semicontinuous processes and, more generally, random capacities on such a topological space.

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1. Introduction

Let L be a countably based continuous partially ordered set (poset for short). In a previous paper (Norberg (1989)), we have discussed the existence of random variables in L. Now our aim is to study their convergence in distribution, which by definition is equivalent to weak convergence of the induced probability measures.

On L there is a canonical topology, called the Lawson topology, which is completely regular, second countable and Hausdorff, hence Polish (i.e., admits a complete separable metric). The convergence in distribution is relative to this well behaved topology.

An L-valued mapping ξ, defined on some probability space, is a *random variable* in L if all events of the form $\{x \leq \xi\}$ are measurable. It is not hard to see that this is equivalent to require the event $\{\xi \in F\}$ measurable for $F \in \mathscr{L}$ – the collection of (Scott) open filters on L.

Let $\xi, \xi_1, \xi_2, \ldots$ be random variables in L. Our main result (Theorem 5.3) states that ξ_n converges in distribution to ξ, which we denote by $\xi_n \to_d \xi$, if and only if

(1.1) $$\limsup_n \mathrm{P}\cap_{i=1}^m \{x_i \leq \xi_n\} \leq \mathrm{P}\cap_{i=1}^m \{x_i \leq \xi\},\ m = 1,2,\ldots,\ x_1,\ldots,x_m \in L,$$

(1.2) $$\liminf_n \mathrm{P}\{\xi_n \in \cap_{i=1}^m F_i\} \geq \mathrm{P}\{\xi \in \cap_{i=1}^m F_i\},\ m = 1,2,\ldots,\ F_1,\ldots,F_m \in \mathscr{L}.$$

Let L be $\vee_f$-closed, i.e., closed for finite suprema. Then

$$x \leq z,\ y \leq z \Leftrightarrow x \vee y \leq z,$$

and condition (1.1) simplifies to

(1.3) $$\limsup_n \mathrm{P}\{x \leq \xi_n\} \leq \mathrm{P}\{x \leq \xi\},\ x \in L.$$

By definition every continuous poset is closed for directed suprema. Thus, if $A \subseteq L$ is non-empty, then the supremum of A, denoted $\vee A$, exists. Hence if $A \subseteq L$ has a lower bound, then A has a greatest lower bound, denoted $\wedge A$. It follows that $\mathscr{L}$ is closed for finite intersection ($\cap_f$-closed), and condition (1.2) reduces to

(1.4) $$\liminf_n \mathrm{P}\{\xi_n \in F\} \geq \mathrm{P}\{\xi \in F\},\ F \in \mathscr{L}.$$

Thus the characterization of convergence in distribution is particularly simple if L is $\vee_f$-closed.

The line $(-\infty,\infty]$ is a simple example of such an L. Another simple example is the non-negative reals $[0,\infty)$ with the reversed order. These two examples are also $\wedge_f$-closed. The union $\{1\}\times(0,1] \cup (0,1]\times\{1\}$ in its coordinatewise order is an example of a continuous poset which is $\vee_f$- but not $\wedge_f$-closed.

Other well-known (and important) examples of continuous posets are the collections of closed subsets of a locally compact second countable Hausdorff space and, more generally, all extended real-valued upper semicontinuous functions on such a space (cf. Gerritse (1985) and Vervaat (1988)). Also many of the collections of capacities discussed in Norberg (1986) and Norberg & Vervaat (1989) are continuous posets.

The reader who wants more examples of continuous posets is also referred to Lawson (1979), Gierz, Hofmann, Keimel, Mislove & Scott (1980) and Hofmann & Mislove (1981). These references contain all the general information on continuous posets that we need in this paper.

There is a well known simple characterization of convergence in distribution of real-valued random variables in terms of weak convergence of the associated distribution functions. A similar result can be proved if L is assumed both $\vee_f$- and $\wedge_f$-closed.

So assume this. Clearly any mapping of the form $P\{\cdot \le \xi\}$ is continuous from below in the sense that $P\{x_n \le \xi\} \downarrow P\{x \le \xi\}$ as $x_n \uparrow x$. There is no analogous notion of continuity from above, since L need not be closed for countable decreasing infima. It can, however, be shown that the set of all $x \in L$, for which there is some $F \in \mathscr{L}$ satisfying

$$\{\xi \in F\} \subseteq \{x \le \xi\},\ P\{\xi \in F\} = P\{x \le \xi\}$$

is dense in L. We could refer to such an x as a *continuity point* of $P\{\cdot \le \xi\}$, and we will see below that $\xi_n \to_d \xi$ if and only if

$$\lim_n P\{x \le \xi_n\} = P\{x \le \xi\} \tag{1.5}$$

for continuity points (Proposition 5.6).

By Zorn's lemma, any continuous poset has a maximal point, though there need of course not be a unique one. But if there is, then this point is the largest member of L and may in some cases act as the point of infinity (consider, e.g., the case $(-\infty,\infty]$ mentioned above). Assume now that L indeed has a largest member (a top), which we regard as the point of infinity and denote by τ.

A random variable ξ in L may be called *tight* if $\xi \ne \tau$ with probability one. It is easy to see that ξ is tight in this sense if, and only if, for each $\epsilon > 0$ there is some $F \in \mathscr{L}$ with $P\{\xi \in F\} \le \epsilon$. Similarly, a sequence $\xi_1, \xi_2, \ldots$ or collection (ξ_n) of random variables in L

may be called *tight*, if for each $\epsilon > 0$ we have

$$\sup_n \mathrm{P}\{\xi_n \in F\} \leq \epsilon$$

for some $F \in \mathscr{L}$. Some simple argumentation will show us that (ξ_n) is tight if, and only if,

$$\sup_n \mathrm{P}\{x \leq \xi_n\} \leq \epsilon$$

for all $\epsilon > 0$ and some $x \ll \tau$ (Proposition 5.8).

Let $\xi_1, \xi_2, \ldots$ be a sequence of tight random variables in L, which converges in distribution to some random variable ξ. If (ξ_n) is tight, so is ξ by (1.2). The converse follows by (1.1) (Proposition 5.9).

As remarked above our results apply to the collection of closed subsets of any locally compact second countable Hausdorff space. This is true also if we replace the Hausdorff separation assumption by the weaker condition that the space is sober. So, let S be such a space and let $\mathscr{F}$ denote its collection of closed subsets. Furthermore write $\mathscr{G}$ for its collection of open sets and $\mathscr{Q}$ for its collection of compact saturated sets. We will remind the reader of some basic topological notions soon. Let us only note at this point that a subset of a topological space is saturated if it coincides with the intersection of its open neigborhoods and that all subsets of a Hausdorff space are saturated.

A mapping φ from some probability space into $\mathscr{F}$ is called a *random closed set* in S, if $\{\varphi \cap G \neq \emptyset\}$ is a measurable event for all $G \in \mathscr{G}$. (Cf. Matheron (1975), who treats the Hausdorff case.) Clearly this holds if and only if $\{\varphi \subseteq F\}$ is measurable whenever $F \in \mathscr{F}$. Next note that $\mathscr{F}$ is a continuous poset relative to the exclusion order $\supseteq$. Thus a random closed set in S is nothing but a random variable in $\mathscr{F}$. Moreover, the Lawson topology on $\mathscr{F}$ coincides with Fell's (1962) "hit or miss" topology.

Let $\varphi, \varphi_1, \varphi_2, \ldots$ be random closed sets in S. Clearly (1.3) is equivalent to

$$\liminf_n \mathrm{P}\{\varphi_n \cap G \neq \emptyset\} \geq \mathrm{P}\{\varphi \cap G \neq \emptyset\},\ G \in \mathscr{G}, \tag{1.6}$$

and it follows directly from a result of Hofmann & Mislove (1981) that (1.4) holds if and only if

$$\limsup_n \mathrm{P}\{\varphi_n \cap Q \neq \emptyset\} \leq \mathrm{P}\{\varphi \cap Q \neq \emptyset\},\ Q \in \mathscr{Q}. \tag{1.7}$$

Thus, (1.6) and (1.7) together are necessary and sufficient for $\varphi_n \rightarrow_{\mathrm{d}} \varphi$.

A class $\mathscr{A}$ of subsets of S is said to be *separating* if, whenever $Q \subseteq G$, where $Q \in \mathscr{Q}$ and $G \in \mathscr{G}$, we have $Q \subseteq A \subseteq G$ for some $A \in \mathscr{A}$. Below it will be seen that $\varphi_n \rightarrow_{\mathrm{d}} \varphi$ if and only if

$$\lim_n \mathrm{P}\{\varphi_n \cap A \neq \emptyset\} = \mathrm{P}\{\varphi \cap A \neq \emptyset\} \tag{1.8}$$

for all sets A in some separating class of Borel sets in S (Proposition 6.1). This result was proved for the Hausdorff case in Norberg (1984).

The largest member of $\mathscr{F}$ with respect to the exclusion order is the empty set $\emptyset$. So φ is tight if and only if φ is non-empty with probability one. The result of Hofmann & Mislove (1981), which we just referred to, tells us that this holds if, and only if, for every $\epsilon > 0$ we have

$$\mathrm{P}\{\varphi \cap Q \neq \emptyset\} \geq 1 - \epsilon \tag{1.9}$$

for some $Q \in \mathscr{Q}$.

The topological space S can be endowed with a finer topology, which is completely regular, second countable and Hausdorff, in particular Polish. It is called the patch topology and may be characterized by the fact that $s_n \longrightarrow s$ in S if and only if $\{s_n\}^- \longrightarrow \{s\}^-$ in $\mathscr{F}$ with respect to Fell's topology (Proposition 6.2). It is easy to write down a characterization of convergence in distribution (with respect to the Polish patch topology) of random elements $\xi_1, \xi_2, \ldots$ to ξ. The following is obtained: $\xi_n \longrightarrow_{\mathrm{d}} \xi$ if and only if

$$\operatorname{liminf}_n \mathrm{P}\{\xi_n \in G\} \geq \mathrm{P}\{\xi \in G\},\ G \in \mathscr{G}, \tag{1.10}$$

$$\operatorname{limsup}_n \mathrm{P}\{\xi_n \in Q\} \leq \mathrm{P}\{\xi \in Q\},\ Q \in \mathscr{Q} \tag{1.11}$$

(Theorem 6.3). If S is Hausdorff, then $\mathscr{Q} \subseteq \mathscr{F}$ and (1.11) follows from (1.10). In this case the patch topology coincides with the original topology and nothing new is obtained.

Call a collection (ξ_n) of random elements in S *tight* if whenever $\epsilon > 0$ there is a compact saturated $Q \subseteq S$ with

$$\inf_n \mathrm{P}\{\xi_n \in Q\} \geq 1 - \epsilon.$$

A random element is *tight* if the corresponding singleton is. We must be a little careful when discussing tightness in the continuous poset L because we have several notions of it. We refer to the classical notion defined in this paragraph as *topological* tightness. The topology underlying it is always the Scott topology.

Let (ξ_n) be a family of random variables in L. Below we will see that (ξ_n) is topologically tight if and only if $\epsilon > 0$ implies the existence of finitely many $x_1, \ldots, x_m \in L$ satisfying

$$\inf_n P \cup_{i=1}^{m} \{x_i \le \xi\} \ge 1 - \epsilon.$$

We may of course take $m = 1$ here if L is $\wedge_f$-closed. The reader might want to compare this with our previously introduced lattice theoretical notion of tightness.

Let ξ be a random element in S and write $\varphi = \{\xi\}^-$ for its singleton closure. Not surprisingley ξ is tight if and only if φ is so in the lattice theoretical sense. The latter is obviously always true. Hence any random element in S is tight.

Assume the collection $\mathcal{Q}$ of compact saturated sets in S to be $\cap_f$-closed. Then the set

$$\left\{\{s\}^- : s \in S\right\} \cup \left\{\emptyset\right\}$$

is compact with respect to Fell's topology (Proposition 6.5). This fact allows the following conclusion for a collection (ξ_n) of random elements in S: Every subsequence of (ξ_n) has a further subsequence which is convergent in distribution if and only if (ξ_n) is tight (Theorem 6.7).

Note that the tightness is relative to the original topology, while the convergence in distribution is with respect to the Polish patch topology. So this result is not an extension of Prohorov's theorem (see, e.g., Billingsley (1968), p. 37) to a class of non-Hausdorff spaces. Instead it is an improvement of it when applied to a Polish space which arises as the patch space of a locally compact sober space (see Lawson (1989)). The advantage of our result over Prohorov's is that the collection $\mathcal{Q}$ of sets which are compact and saturated with respect to the original topology on S generally is, in a distinctive way, smaller than the collection $\mathcal{K}$ of subsets of S, which are compact with respect to the Polish patch topology.

We continue with a description of the contents of the various sections of this paper. In Section 2 we give the basic preliminaries on continuous posets and topology. The real development of our results begins in Section 3 when we introduce a convenient σ-field on L and discuss measurability.

Section 4 begins with a discusion of uniqueness of probability measures on L. For the reader's convenience we include here three existence theorems for random variables and one for random open filters in L. The latter are by definition random elements in $\mathcal{L}$, which is continuous under inclusion $\subseteq$. These four theorems have various assumptions on L and are proved in Norberg (1989). We conclude Section 4 by proving two new theorems. The first gives existence criteria for random variables in L, under assumptions on the latter that are not studied earlier, while the second discusses the existence of random open filters on L.

Section 5 contains the main result of this paper (i.e., Theorem 5.3), which gives necessary and sufficient conditions for convergence in distribution of random elements in L. The proof goes as follows: First we conclude by the Lawson duality (cf. Lawson (1979) or Hofmann & Mislove (1981)) that there is an equivalent theorem giving necessary and sufficient conditions for convergence in distribution of random open filters on L (Theorem 5.4). We then prove that any random open filter on L also is a random Scott open set in L. The latter are by definition random variables taking their values in Scott(L) – the collection of Scott open sets in L – which is continuous under inclusion. Next, we prove that a sequence of (random) open filters converge in $\mathscr{L}$, if and only if it converge in Scott(L). The final argument uses a characterization of convergence in distribution for random Scott open sets, which is proved earlier in Section 5.

After proving our main result we briefly discuss simple necessary and sufficient conditions for convergence in distribution, assuming L both $\vee_f$- and $\wedge_f$-closed. We then discuss tightness in the lattice theoretical sense. Finally in Section 5, we study convergence in distribution of rowwise infima $\wedge_j \xi_{nj}$, where the ξ_{nj}'s form a null array. It turns out that any limiting random variable must be infinitely divisible with respect to $\wedge$.

In Section 6 we discuss convergence in distribution of random elements in locally compact sober spaces. As already remarked, the convergence is relative to the Polish patch topology. The important results of this section are Theorem 6.2, which characterizis convergence in distribution and Theorem 6.7, which is our improvement of Prohorov's theorem.

2. Continuous posets and locally compact sober spaces

In this section we introduce some of the notation and terminology needed in this paper. Let L be a poset. For $x \in L$, we write $\uparrow x = \{y \in L: x \leq y\}$ and $\downarrow x = \{y \in L: y \leq x\}$. Say that a non-empty set $A \subseteq L$ is *directed* (*filtered*) if given $x,y \in A$, $\uparrow x \cap \uparrow y \cap A \neq \emptyset$ ($\downarrow x \cap \downarrow y \cap A \neq \emptyset$). A non-empty set $A \subseteq L$ is a *filter* if it is filtered and *upper* in the sense that $\uparrow x \subseteq A$ whenever $x \in A$.

Assume L *up-complete* in the sense that every directed subset D has a supremum $\bigvee D$ in L. Let $x,y \in L$. Then x is said to be *way below* y and we write $x \ll y$, if for every directed set $D \subseteq L$ with $y \leq \bigvee D$ we have $x \leq z$ for some $z \in D$.

Assume further that for each $x \in L$, $x = \bigvee\{y \in L: y \ll x\}$ and this set is directed. Then L is called *continuous*. A set $U \subseteq L$ is *Scott open* if *(i)* U is an upper set and *(ii)* $D \cap U \neq \emptyset$ whenever $D \subseteq L$ is directed with $\bigvee D \in U$. The collection of Scott open sets in L is a topology which we call the *Scott topology* and denote by Scott(L). Its subcollection of open filters is denoted $\mathscr{L}$ or OFilt(L). Note that $\mathscr{L}$ is an open base for Scott(L). Another open base for Scott(L) is formed by the sets $\{y \in L: x \ll y\}$, $x \in L$. Both $\mathscr{L}$ and Scott(L) are continuous posets relative to the inclusion order $\subseteq$.

The coarsest topology on L that contains Scott(L) (or $\mathscr{L}$) and all sets of the form $L\setminus\uparrow x$, where $x \in L$, is called the *Lawson* topology. This topology is completely regular, second countable and Hausdorff, hence Polish.

The above definitions and results can be found, e.g., in the papers Lawson (1979) and Hofmann & Mislove (1981) or the monograph Gierz, Hofmann, Keimel, Lawson, Mislove & Scott (1980).

Lawson (1979) shows that the mapping

$$x \longrightarrow \mathscr{F}_x = \{F \in \mathscr{L}: x \in F\}$$

is an order-isomorphism between L and OFilt($\mathscr{L}$). This is the object level of what is nowadays called the *Lawson duality*, and $\mathscr{L}$ is often called the *(Lawson) dual* of L.

Say that a set $Q \subseteq L$ is *separating*, if for each $x,y \in L$ with $x \ll y$, there exists some $z \in Q$ with $x \leq z \leq y$. This notion is very similar to the notion of a basis employed for continuous lattices in Gierz et al. (1980). Do not confuse it with our notion of a separating class of sets mentioned in the introduction. Norberg (1989) proves that the following four assertions are equivalent: *(i)* Scott(L) is second countable, *(ii)* L contains a countable separating subset, *(iii)* $\mathscr{L}$ contains a countable separating subset and *(iv)* Scott($\mathscr{L}$) is second countable. Moreover, a continuous topology has a countable separating subset if and only if it is second countable. Lawson (1988) calls L *countably based*, if L contains a countable separating subset.

Let S be a topological space. Call a closed set $F \subseteq S$ *irreducible* if, whenever $F \subseteq F_1 \cup F_2$ for any closed sets $F_1, F_2 \subseteq S$, then $F \subseteq F_1$ or $F \subseteq F_2$. Every singleton closure is irreducible and S is called *sober* if every non-empty irreducible closed set is the singleton closure of a unique $s \in S$. Hausdorff spaces are sober, but the converse is not true. Recall from the introduction that a set $A \subseteq S$ is *saturated* if it equals its *saturation* $\mathrm{sat}A = \cap\{G: A \subseteq G$ and $G \subseteq S$ is open$\}$. Clearly a set $K \subseteq S$ is compact if and only if its saturation is. We call S *locally compact* if whenever $s \in G \subseteq S$, where G is open, we have $s \in K^{\circ} \subseteq K \subseteq G$ for a compact set $K \subseteq S$. Note that we may take K saturated here.

The Scott topology on a continuous poset is sober and locally compact (Lawson (1979)). It is quite obvious that the topology $\mathscr{G}$ of any locally compact sober space S is continuous. Hofmann & Lawson (1978) shows the converse, i.e., a sober space is locally compact if its topology is continuous. Hofmann & Mislove (1981) shows that the collection $\mathscr{Q}$ of compact saturated subsets of S is anti-order isomorphic to OFilt($\mathscr{G}$). This fact is very important in random set theory.

We end this section by proving two lemmata, which will be needed below. The first is included at this point mainly for pedagogical reasons. Its proof is given in detail. The standard facts used are simple and can be found, e.g., in Lawson (1979)). The second lemma is an important part in the proof of Theorem 5.3. It has some independent value.

LEMMA 2.1. *Let L be a countably based continuous poset. Let $Q \subseteq L$ be Scott compact saturated. Then there are some $x_{ni} \in L$, where $n = 1,2,\ldots$ and $1 \le i \le k_n < \infty$, such that*

$$Q \subseteq \left[\cup_i \uparrow x_{ni}\right] \downarrow Q$$

as $n \to \infty$.

PROOF. Note that if $F \in \mathscr{L}$, then $(\uparrow x_n) \uparrow F$ for some $x_1, x_2, \ldots \in F$ with $x_{n+1} \ll x_n$. To see this, recall that $\mathscr{L}$ is continuous relative to $\subseteq$ and that $\mathscr{L}$ contains a countable separating subset $\mathscr{D}$. Then

$$F = \vee\{G \in \mathscr{L}: G \ll F\}$$

($= \cup\{G \in \mathscr{L}: G \ll F\}$). By the interpolation property (Lawson (1979), Proposition 1.6), if $G \ll F$, then $G \ll H$ and $H \ll F$ for some $H \in \mathscr{L}$. Choose $D \in \mathscr{D}$ such that $G \subseteq D \subseteq H$. Clearly $D \ll F$. We conclude

$$F = \cup\{G \in \mathscr{D}: G \ll F\}.$$

The set on the right is countable and directed. To see the latter, use the fact that the set $\{G \in \mathscr{L}: G \ll F\}$ is directed and repeat the argument already given. Thus there are $F_1, F_2, \ldots \in \mathscr{L}$ such that $F_n \ll F$ and $F_n \uparrow F$. Use the interpolation property to conclude that we may take $F_n \ll F_{n+1}$ here. (Up to now we have not used the fact that $\mathscr{L}$ is a collection of filters.) By Lawson (1979), Proposition 3.3, if $F_n \ll F_{n+1}$, then $F_n \subseteq \uparrow x_n \subseteq F_{n+1}$. By construction, it is clear that $(\uparrow x_n) \uparrow F$. Note that $\uparrow x_n \subseteq F_{n+1} \subseteq \uparrow x_{n+1}$. By Lawson (1979), Proposition 2.2, $x_{n+1} \ll x_n$. Our claim in the first sentence of the proof is thereby proved.

By Hofmann & Mislove (1981), Theorem 2.16,

$$\{U \in \text{Scott}(L): Q \subseteq U\} \in \text{OFilt}(\text{Scott}(L))$$

(i.e., is an open filter on Scott(L)). Recall that Scott(L) is continuous and, being second countable, contains a countable separating subset. By the already proved result,

$$(\uparrow U_n) \uparrow \{U \in \text{Scott}(L): Q \subseteq U\},$$

for some $U_1, U_2, \ldots \in \text{Scott}(L)$ with $Q \subseteq U_n$ and $U_{n+1} \ll U_n$. It is easy to see that $U_n \downarrow Q$.

We next prove that if $U \ll V$, where $U, V \in \text{Scott}(L)$, then $U \subseteq \cup_{i=1}^{m} \uparrow x_i \subseteq V$ for some $x_1, \ldots, x_m \in L$. From this fact, the lemma clearly follows. To see it, note first that $U \subseteq \cup_{i=1}^{m} F_i \subseteq V$ for some $F_1, \ldots, F_m \in \mathscr{L}$, since $\mathscr{L}$ contains a countable base for Scott(L). By the interpolation property, we may assume here that $U \ll \cup_{i=1}^{m} F_i$. For $1 \leq i \leq m$, choose $F_{i1}, F_{i2}, \ldots \in \mathscr{L}$ such that $F_{in} \ll F_i$ and $F_{in} \uparrow F_i$. Then $U \subseteq \cup_{i=1}^{m} F_{in}$ for some sufficiently large n.

By Lawson (1979), Proposition 3.3, we may choose $x_1, \ldots, x_m \in L$ such that $F_{in} \subseteq \uparrow x_i \subseteq F_i$. Hence

$$U \subseteq \cup_{i=1}^{m} \uparrow x_i \subseteq V. \qquad \square$$

LEMMA 2.2. *Let L be as in the above lemma. Then the Lawson topology on $\mathscr{L}$ = $\mathrm{OFilt}(L)$ is the trace of the Lawson topology on the larger* $\mathrm{Scott}(L)$.

PROOF. The Lawson topology on $\mathscr{L}$ is generated by the sets

$$\mathscr{F}_x = \{F \in \mathscr{L}\colon x \in F\} = \{U \in \mathrm{Scott}(L)\colon \uparrow x \subseteq U\} \cap \mathscr{L},$$

where $x \in L$, and the sets

$$\mathscr{L}\setminus\uparrow H = \{F \in \mathscr{L}\colon H \not\subseteq F\} = \{U \in \mathrm{Scott}(L)\colon H \not\subseteq U\} \cap \mathscr{L},$$

where $H \in \mathscr{L}$. Here

$$\{U \in \mathrm{Scott}(L)\colon \uparrow x \subseteq U\} \in \mathrm{OFilt}(\mathrm{Scott}(L))$$

since $\uparrow x$ is Scott compact saturated. So this set is open with respect to the Lawson topology on $\mathrm{Scott}(L)$. Clearly so is also

$$\{U \in \mathrm{Scott}(L)\colon H \not\subseteq U\} = \mathrm{Scott}(L)\setminus\uparrow H,$$

since $H \in \mathscr{L} \subseteq \mathrm{Scott}(L)$. Hence the Lawson topology on $\mathscr{L}$ is included in the trace of the Lawson topology on $\mathrm{Scott}(L)$.

To see the converse, first conclude by Hofmann & Mislove (1981), Theorem 2.16, that if $\mathscr{U} \in \mathrm{OFilt}(\mathrm{Scott}(L))$, then

$$\mathscr{U} = \{U \in \mathrm{Scott}(L)\colon Q \subseteq U\}$$

for some Scott compact saturated $Q \subseteq L$. Thus the Lawson topology on $\mathrm{Scott}(L)$ is the coarsest topology containing all sets of the form

$$\{U \in \mathrm{Scott}(L)\colon Q \subseteq U\},$$

for $Q \subseteq L$ Scott compact saturated, and all sets of the form

$$\{U \in \mathrm{Scott}(L)\colon V \not\subseteq U\},$$

where $V \in \mathrm{Scott}(L)$.

Now, if $V \in \mathrm{Scott}(L)$, then $V = \cup_n F_n$ for some $F_1, F_2, \ldots \in \mathscr{L}$, since $\mathscr{L}$ contains a countable base for $\mathrm{Scott}(L)$. Hence

$$\{U \in \mathrm{Scott}(L)\colon V \not\subseteq U\} = \cup_n\{U \in \mathrm{Scott}(L)\colon F_n \not\subseteq U\}.$$

Next, if $Q \subseteq U \in \mathrm{Scott}(L)$, where Q is Scott compact saturated, then $Q \subseteq \cup_{i=1}^m \uparrow x_i \subseteq U$ for some $x_1, \ldots, x_m \in L$ (cf. the proof of Lemma 2.1). The set $\cup_{i=1}^m \uparrow x_i$ is Scott compact saturated. Moreover, $\cup_{i=1}^m \uparrow x_i \subseteq U$ if, and only if, $x_1, \ldots, x_m \in U$. Thus the Lawson topology on $\mathrm{Scott}(L)$ is generated by all sets in the two families

$$\Big\{\{U \in \mathrm{Scott}(L)\colon F \nsubseteq U\}\colon F \in \mathscr{L}\Big\},$$
$$\Big\{\{U \in \mathrm{Scott}(L)\colon x \in U\}\colon x \in L\Big\}.$$

Hence the relativization of the Lawson topology on Scott(L) to $\mathscr{L}$ is included in the Lawson topology. □

3. Measurability

For the remaining part of this paper L is a countably based continuous poset. Its Scott topology, which is second countable, is denoted by Scott(L) and $\mathscr{L}$ denotes its collection of open filters. We let D and $\mathscr{D}$ be countable separating subsets of L and $\mathscr{L}$, resp. Write Σ (or $\Sigma(L)$) for the σ-field on L generated by the sets $\uparrow x$, $x \in L$.

Our first result gives several equivalent conditions for measurability. It says in particular that Σ is generated by $\mathscr{L}$ and also by the larger Scott(L).

PROPOSITION 3.1. *Let* $(\Omega, \mathscr{R})$ *be a measurable space and consider a mapping* ξ *from* Ω *into* L. *Then the following five conditions are equivalent:*

(i) ξ is measurable w.r.t. $\mathscr{R}$ and Σ,

(ii) $\{x \leq \xi\} \in \mathscr{R}$, $x \in L$,

(iii) $\{x \ll \xi\} \in \mathscr{R}$, $x \in L$,

(iv) $\{\xi \in F\} \in \mathscr{R}$, $F \in \mathscr{L}$,

(v) $\{\xi \in U\} \in \mathscr{R}$, $U \in$ Scott(L).

They imply

(vi) $\{\xi \leq x\} \in \mathscr{R}$, $x \in L$.

PROOF. Since Σ by definition is generated by all sets of the form $\uparrow x$, (i) and (ii) are equivalent. If $F \in \mathscr{L}$, then obviously $F = \bigcup_{x \in F} \uparrow x$. Moreover, if $x \in F \in \mathscr{L}$, then $\uparrow x \subseteq \uparrow y$ for some $y \in F \cap D$. Thus $F = \bigcup_{x \in F \cap D} \uparrow x$. Hence (ii) implies (iv). We have already remarked that $\mathscr{L}$ is an open base for Scott(L). Since the latter is second countable, (iv) implies (v), which in turn trivially implies (iii). It is an easy exercise to show that $\uparrow x = \bigcap_{y \ll x} \uparrow y = \bigcap_{y \ll x} \{z \in L : y \ll z\}$, and that the latter intersection can be thinned to a countable one, i.e., we may restrict y to D. So (iii) implies (ii).

Condition (vi) follows from (v), since $\downarrow x$ is Scott closed for all $x \in L$. □

We have not been able to prove that condition (vi) of Proposition 3.1 implies that ξ is measurable. It is true in many particular cases. We believe, however, that it is not a general fact.

4. Probability distributions

Let $(\Omega, \mathscr{R}, \mathrm{P})$ be a probability space. Then $\xi:\Omega \longrightarrow L$ is called a *random variable* in L if ξ is measurable w.r.t. $\mathscr{R}$ and Σ (cf. Proposition 3.1). In this case, the two mappings $x \longrightarrow \mathrm{P}\{x \leq \xi\}$ and $F \longrightarrow \mathrm{P}\{\xi \in F\}$ defined on L and $\mathscr{L}$, resp., have important interrelations.

PROPOSITION 4.1. *Let ξ be a random variable in L. Then*

(a) $\mathrm{P}\{x \leq \xi\} = \wedge_{y \ll x}\mathrm{P}\{y \leq \xi\} = \wedge_{y \ll x}\mathrm{P}\{y \ll \xi\} = \wedge_{x \in F \in \mathscr{L}}\mathrm{P}\{\xi \in F\},\ x \in L;$

(b) $\mathrm{P}\{\xi \in F\} = \vee_{G \ll F}\mathrm{P}\{\xi \in G\} = \vee_{x \in F}\mathrm{P}\{x \leq \xi\} = \vee_{x \in F}\mathrm{P}\{x \ll \xi\},\ F \in \mathscr{L}.$

PROOF. Let $F \in \mathscr{L}$. Then, since $\mathscr{L}$ is continuous and contains a countable separating subset $\mathscr{D}$, $F = \cup_{G \ll F} G$, where $G \in \mathscr{D}$. It is easy to extract $G_1, G_2, \ldots \in \mathscr{D}$ such that $G_n \uparrow F$. Next argue as in the proofs of Lemma 2.1 and Proposition 3.1 to conclude that there are $x_1, x_2, \ldots \in F$ satisfying $x_{n+1} \ll x_n$ and both $(\uparrow x_n) \uparrow F$ and $\{y \in L: x_n \ll y\} \uparrow F$. This shows (b) – (a) follows similarly. □

The *distribution* of a random variable ξ in L is by definition the induced probability measure $\mathrm{P}\xi^{-1}$ on Σ. *Equality in distribution* is denoted $=_\mathrm{d}$. Thus, for random variables ξ, η in L, $\xi =_\mathrm{d} \eta$ if, and only if, $\mathrm{P}\xi^{-1} = \mathrm{P}\eta^{-1}$ on Σ. Here, and in similar instances, the random variables need not be defined on the same probability space although we, for convenience, denote by P all underlying probability measures.

THEOREM 4.2. *Let ξ, η be random variables in L. Then the following statement is equivalent to $\xi =_\mathrm{d} \eta$:*

(i) $\mathrm{P}\{\xi \in U\} = \mathrm{P}\{\eta \in U\},\ U \in \mathrm{Scott}(L).$

Moreover, if L is $\vee_\mathrm{f}$- or $\wedge_\mathrm{f}$-closed, then also the following three statements are equivalent to $\xi =_\mathrm{d} \eta$:

(ii) $\mathrm{P}\{x \ll \xi\} = \mathrm{P}\{x \ll \eta\},\ x \in L,$

(iii) $\mathrm{P}\{x \leq \xi\} = \mathrm{P}\{x \leq \eta\},\ x \in L,$

(iv) $\mathrm{P}\{\xi \in F\} = \mathrm{P}\{\eta \in F\},\ F \in \mathscr{L}.$

PROOF. The first assertion of the theorem is obvious. Clearly $\xi =_{\mathrm{d}} \eta$ implies (ii), which, by Proposition 4.1, in turn implies both (iii) and (iv). This proposition also shows that (iii) and (iv) are equivalent. If L is $\wedge_{\mathrm{f}}$-closed, then $\mathscr{L} \cup \{\emptyset\}$ is $\cap_{\mathrm{f}}$-closed and $\xi =_{\mathrm{d}} \eta$ follows by a standard monotone class argument from (iv). If L is $\vee_{\mathrm{f}}$-closed, then $\xi =_{\mathrm{d}} \eta$ follows in a similar way from (iii). □

A part of the above theorem can be found in Norberg (1989), from which we now fetch three existence theorems for random variables in L. Let $\Lambda:L \longrightarrow \mathbb{R}$, and let $n = 1,2,...$ and $x,x_1,...,x_n \in L$. If L is $\vee_{\mathrm{f}}$-closed, we recursively define

$$\Lambda_n(x;x_1,...,x_n) = \Lambda_{n-1}(x;x_1,...,x_{n-1}) - \Lambda_{n-1}(x \vee x_n;x_1,...,x_{n-1}),$$

where

$$\Lambda_1(x;x_1) = \Lambda(x) - \Lambda(x \vee x_1).$$

The first existence theorem pressumes that L is both $\vee_{\mathrm{f}}$- and $\wedge_{\mathrm{f}}$-closed, i.e., a lattice.

THEOREM 4.3 (Norberg (1989)). *Assume L both $\vee_{\mathrm{f}}$- and $\wedge_{\mathrm{f}}$-closed and let $\Lambda:L \longrightarrow \mathbb{R}_+$. Then there is a random variable ξ in L satisfying*

$$\mathrm{P}\{x \le \xi\} = \Lambda(x),\ x \in L,$$

if, and only if, the following three conditions hold true:

(i) $$\Lambda_n(x;x_1,...,x_n) \ge 0,$$

(ii) $$\Lambda(x_n) \downarrow \Lambda(x) \text{ as } x_n \uparrow x,$$

(iii) $$\sup_{x \in L} \Lambda(x) = 1.$$

Before proceeding, let us look at the case when L only is $\vee_{\mathrm{f}}$-closed and $\Lambda:L \longrightarrow [0,1]$ is a function satisfying conditions (i) and (ii) of Theorem 4.3. It seems natural to add a bottom to L, i.e., form $L^{\perp} = L \cup \{\perp\}$, where $\perp$ is an artificial point satisfying $\perp \le x$ for all $x \in L^{\perp}$. It is trivial that $L^{\perp}$ is a countably based continuous poset, which is both $\vee_{\mathrm{f}}$- and $\wedge_{\mathrm{f}}$-closed. It is also trivial that the mapping $\tilde{\Lambda}:L^{\perp} \longrightarrow \mathbb{R}_+$, given by letting $\tilde{\Lambda}(x) = \Lambda(x)$ if $x \in L$ and $= 1$ if $x = \perp$, satisfies conditions (ii) and (iii) of Theorem 4.3.

Now note that

$$\tilde{\Lambda}_n(x;x_1,...,x_n) = \tilde{\Lambda}_{n-1}(x;x_1,...,x_{n-1})$$

if $x_n = \perp$ and that this iterated difference does not depend on the order in which we enumerate $x_1,...,x_n$. Thus we only need to check whether

$$\tilde{\Lambda}_n(\perp;x_1,...,x_n) \geq 0 \tag{4.1}$$

for $x_1,...,x_n \in L$. If (4.1) holds, we may conclude by Theorem 4.3 that there is a probability measure $\tilde{\lambda}$ on $L^{\perp}$ satisfying

$$\tilde{\lambda}(\uparrow x) = \tilde{\Lambda}(x),\ x \in L^{\perp}.$$

Write λ for the restriction of $\tilde{\lambda}$ to L. Clearly $\lambda(\uparrow x) = \Lambda(x)$ for $x \in L$. Moreover, (4.1) is of course necessary for the existence of λ.

Is λ a probability measure? In order to answer this question, we note that

$$\lambda(\cup_{i=1}^{n} \uparrow x_i) = \tilde{\lambda}(\cup_{i=1}^{n} \uparrow x_i) = 1 - \tilde{\Lambda}_n(\perp;x_1,...,x_n)$$

if $x_1,...,x_n \in L$. Hence λ is indeed a probability measure if

$$\inf \tilde{\Lambda}_n(\perp;x_1,...,x_n) = 0, \tag{4.2}$$

where the infimum is taken over $n = 1,2,...$ and $x_1,...,x_n \in L$.

Conversely, if λ is a probability measure on L, then λ is topologically tight by the discusion in the introduction and

$$\sup \lambda(\cup_{i=1}^{n} \uparrow x_i) = 1,$$

i.e., (4.2), follows by Proposition 6.4 below (please forgive us for relying on an unproven result). Thus (4.2) is a necessary and sufficient condition for $\lambda(L) = 1$.

It is quite natural to refer to any function satisfying condition (i) of Theorem 4.3 as a *completely monotone* function. This is the case also with functions satisfying condition (i) in Theorems 4.4, 4.6 and 4.7 below. Functions satisfying

$$\Lambda(x_n) \longrightarrow \Lambda(x) \text{ as } x_n \uparrow x$$

may be called *inner continuous.*

Whenever c is a real-valued function defined on an $\wedge$f-closed poset M and $y \in M$, we write $\Delta_y c$ for the mapping on M given by

$$\Delta_y c(x) = c(x) - c(x \wedge y).$$

Note that if L is $\vee$f-closed, then the collection $\{\uparrow x \colon x \in L\}$ is $\cap$f-closed and, writing $\Lambda(x) = c(\uparrow x)$,

$$\Lambda_1(x;y) = \Delta_{\uparrow y} c(\uparrow x).$$

The next existence theorem pressumes that L is $\wedge$f-closed and has a top. Then $\mathscr{L}$ is

$\cap_f$-closed and contains L.

THEOREM 4.4 (Norberg (1989)). *Suppose L is $\wedge_f$-closed and has a top, and let $\Phi: \mathscr{L} \longrightarrow \mathbb{R}_+$. Then there is a random variable ξ in L satisfying*

$$\mathrm{P}\{\xi \in F\} = \Phi(F),\ F \in \mathscr{L},$$

if, and only if, the following three conditions hold true:

(i) $$\Delta_{F_1} \ldots \Delta_{F_n} \Phi(F) \geq 0,$$

(ii) $$\Phi(F_n) \uparrow \Phi(F) \text{ as } F_n \uparrow F,$$

(iii) $$\Phi(L) = 1.$$

THEOREM 4.5 (Norberg (1989)). *A increasing function μ on* Scott(L) *extends to a unique probability measure on (L, Σ) if, and only if, the following three conditions hold true:*

(i) $$\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V),$$

(ii) $$\mu(U_n) \uparrow \mu(U) \text{ as } U_n \uparrow U,$$

(iii) $$\mu(\emptyset) = 0,\ \mu(L) = 1.$$

Functions satisfying (i) in Theorem 4.5 above are sometimes called *modular.* We have already remarked that the Scott topology is a locally compact, second countable and sober topology on L (cf. Lawson (1979)). Norberg (1989) shows that Theorem 4.5 above applies to all such topological spaces, provided that Scott(L) and Σ are replaced by the collections of open and Borel sets, resp.

The Lawson duality is of particular interest in the setting of Theorem 4.4, because if L is $\wedge_f$-closed and has a top, so is the case for its dual $\mathscr{L}$ and conversely. It yields the following existence theorem for random variables in $\mathscr{L}$, which quite naturally are called *random open filters* on L.

THEOREM 4.6 (Norberg (1989)). *Suppose L is $\wedge_f$-closed and has a top denoted by $\top$. Let $M: L \longrightarrow \mathbb{R}_+$. Then there is a random open filter φ on L satisfying*

$$\mathrm{P}\{x \in \varphi\} = M(x),\ x \in L,$$

if, and only if, the following three conditions hold true:

(i) $$\Delta_{x_1}\ldots\Delta_{x_n} M(x) \geq 0,$$

(ii) $$M(x_n) \uparrow M(x) \text{ as } x_n \uparrow x,$$

(iii) $$M(\top) = 1.$$

To this line of existence theorems we now add two new. The first considers the case when L only is assumed to be $\wedge_f$-closed, thus is a slight generalization of Theorem 4.4. Note that $\mathscr{L}_0 = \mathscr{L} \cup \{\emptyset\}$ is $\cap_f$-closed and that $L \in \mathscr{L}$.

THEOREM 4.7. *Assume L $\wedge_f$-closed and let $\Psi: \mathscr{L}_0 \longrightarrow \mathbb{R}_+$. Then there is a random variable ξ in L satisfying*

$$P\{\xi \in F\} = \Psi(F),\ F \in \mathscr{L},$$

if, and only if, the following three conditions hold true:

(i) $$\Delta_{F_1}\ldots\Delta_{F_n} \Psi(F) \geq 0,$$

(ii) $$\Psi(F_n) \uparrow \Psi(F) \text{ as } F_n \uparrow F,$$

(iii) $$\Psi(\emptyset) = 0,\ \Psi(L) = 1.$$

PROOF. Note first that $\mathscr{L}_0$ is a continuous poset with a second countable Scott topology, the latter because $\mathscr{D} \cup \{\emptyset\}$ is a separating subset of $\mathscr{L}_0$. By Theorem 4.6, there is a random open filter φ in $\mathscr{L}_0$ satisfying

$$P\{F \in \varphi\} = \Psi(F),\ F \in \mathscr{L}_0.$$

Let $\mathscr{F}$ be an open filter in $\mathscr{L}_0$. If $\emptyset \in \mathscr{F}$, then $\mathscr{F} = \mathscr{L}_0$, so assume $\emptyset \notin \mathscr{F}$. Then $\mathscr{F}$ must be an open filter in $\mathscr{L}$. By the Lawson duality, $\mathscr{F} = \mathscr{F}_x$ for some $x \in L$. Thus

$$\text{OFilt}(\mathscr{L}_0) = \text{OFilt}(\mathscr{L}) \cup \{\mathscr{L}_0\}.$$

Now note that

$$P\{\varphi = \mathscr{L}_0\} = P\{\emptyset \in \varphi\} = \Psi(\emptyset) = 0.$$

Hence $\varphi \in \text{OFilt}(\mathscr{L})$ a.s. Then $\varphi = \mathscr{F}_\xi$ a.s. for some random variable ξ in L satisfying

$$P\{\xi \in F\} = P\{F \in \varphi\} = \Psi(F),\ F \in \mathscr{L}. \qquad \square$$

REMARK 4.8. In the setting of Theorem 4.7 one in practise starts with a mapping $\Psi: \mathscr{L} \longrightarrow \mathbb{R}_+$, extends it to $\mathscr{L}_0$ by putting $\Psi(\emptyset) = 0$ and then checks whether the three conditions of the theorem are at hand.

We have already remarked that $\mathscr{L}$ is $\cap_f$-closed, if L is $\vee_f$-closed. Assume this and let $L^\perp = L \cup \{\perp\}$ as in the discusion following Theorem 4.3. Assume further that the mapping $N:L^\perp \longrightarrow \mathbb{R}_+$ satisfies

(4.3) $$\Delta_{x_1} \ldots \Delta_{x_n} N(x) \geq 0,$$

(4.4) $$N(x_n) \uparrow N(x) \text{ as } x_n \uparrow x,$$

(4.5) $$N(\perp) = 0,\ N(\top) = 1,$$

where $\top$ denotes the top of L, which exists by assumption. Then there is a random open filter φ in L satisfying

$$\mathbf{P}\{x \in \varphi\} = N(x),\ x \in L.$$

We leave the proof of this fact to the reader. Cf. Theorem 4.6. It is clear that conditions (4.3)-(4.5) also are necessary for the existence of φ.

The next result should be compared with Theorem 4.5.

THEOREM 4.9. *Let λ:Scott$(L) \longrightarrow \mathbb{R}_+$ be increasing. Then there is a random open filter φ on L satisfying*

$$\mathbf{P}\{\varphi \subseteq U\} = \lambda(U),\ U \in \text{Scott}(L)$$

if, and only if, the following three conditions hold true:

(i) $$\lambda(U \cup V) + \lambda(U \cap V) = \lambda(U) + \lambda(V),$$

(ii) $$\lambda(U_n) \downarrow \lambda(U) \text{ as } U_n \downarrow U = (\cap_n U_n)^\circ,$$

(iii) $$\lambda(\emptyset) = 0,\ \lambda(L) = 1.$$

PROOF. Write CoScott$(\mathscr{L})$ for the collection of Scott closed subsets of $\mathscr{L}$. For $U \in$ Scott(L), put

$$\mathscr{A}(U) = \{F \in \mathscr{L}\colon F \subseteq U\}.$$

It is not hard to see that $\mathscr{A}$ takes values in CoScott$(\mathscr{L})$ and that

$$\mathscr{A}(\cup_{k=1}^n U_k) = \cup_{k=1}^n \mathscr{A}(U_k),\ U_1, \ldots, U_n \in \text{Scott}(L),$$

$$\mathscr{A}(\wedge_\alpha U_\alpha) = \cap_\alpha \mathscr{A}(U_\alpha),\ \{U_\alpha\} \subseteq \text{Scott}(L).$$

(Note that $\wedge_\alpha U_\alpha = (\cap_\alpha U_\alpha)^\circ$.) Moreover,

$$U = \cup \mathscr{k}(U), \ U \in \mathrm{Scott}(L),$$

since $\mathscr{L}$ is a base for Scott(L). Cf. Lawson (1979), which furthermore shows that if $\mathscr{F} \in$ CoScott($\mathscr{L}$), then $\mathscr{F} = \mathscr{k}(U)$ for a unique $U \in$ Scott(L). Thus, $\mathscr{k}$ is an order-isomorphism between Scott(L) and CoScott($\mathscr{L}$).

For $\mathscr{U} \in$ Scott($\mathscr{L}$), put

$$\mu(\mathscr{U}) = 1 - \lambda(\mathscr{k}^{-1}(\mathscr{L}\setminus\mathscr{U})).$$

Then μ maps Scott($\mathscr{L}$) into [0,1]. Moreover, $\mu(\emptyset) = 1 - \lambda(L) = 0$ and $\mu(\mathscr{L}) = 1 - \lambda(\emptyset) = 1$. Next, let $\mathscr{U}_1, \mathscr{U}_2 \in$ Scott($\mathscr{L}$). If $\mathscr{U}_1 \subseteq \mathscr{U}_2$, then $\mathscr{L}\setminus\mathscr{U}_2 \subseteq \mathscr{L}\setminus\mathscr{U}_1$, which implies $\mathscr{k}^{-1}(\mathscr{L}\setminus\mathscr{U}_2) \subseteq \mathscr{k}^{-1}(\mathscr{L}\setminus\mathscr{U}_1)$ and $\mu(\mathscr{U}_1) \leq \mu(\mathscr{U}_2)$ follows. So μ is increasing. To see that μ is modular, let $U_i = \mathscr{k}^{-1}(\mathscr{L}\setminus\mathscr{U}_i)$, $i = 1,2$. Then

$$\mathscr{k}(U_1 \cup U_2) = \mathscr{k}(U_1) + \mathscr{k}(U_2) = \mathscr{L}\setminus(\mathscr{U}_1 \cap \mathscr{U}_2).$$

Hence

$$U_1 \cup U_2 = \mathscr{k}^{-1}(\mathscr{L}\setminus(\mathscr{U}_1 \cap \mathscr{U}_2)).$$

Similarly,

$$U_1 \cap U_2 = \mathscr{k}^{-1}(\mathscr{L}\setminus(\mathscr{U}_1 \cup \mathscr{U}_2)).$$

Hence

$$\begin{aligned}\mu(\mathscr{U}_1 \cup \mathscr{U}_2) + \mu(\mathscr{U}_1 \cap \mathscr{U}_2) &= 1 - \lambda(U_1 \cup U_2) + 1 - \lambda(U_1 \cap U_2) \\ &= 2 - \lambda(U_1) - \lambda(U_2) = \mu(\mathscr{U}_1) + \mu(\mathscr{U}_2).\end{aligned}$$

Finally, let $\mathscr{U}, \mathscr{U}_1, \mathscr{U}_2, \ldots \in$ Scott($\mathscr{L}$) satisfy $\mathscr{U}_n \uparrow \mathscr{U}$. Then $\mathscr{L}\setminus\mathscr{U}_n \downarrow \mathscr{L}\setminus\mathscr{U}$, which implies

$$U_n = \mathscr{k}^{-1}(\mathscr{L}\setminus\mathscr{U}_n) \downarrow \mathscr{k}^{-1}(\mathscr{L}\setminus\mathscr{U}) = U.$$

Hence $\lambda(U_n) \downarrow \lambda(U)$ and $\mu(\mathscr{U}_n) \uparrow \mu(\mathscr{U})$ follows.

By Theorem 4.5, μ extends to a probability measure on $\mathscr{L}$. Let φ be a random open filter on L with distribution μ. If $U \in$ Scott(L), then

$$\begin{aligned}\mathbf{P}\{\varphi \subseteq U\} &= \mathbf{P}\{\varphi \in \mathscr{k}(U)\} = \mu(\mathscr{k}(U)) \\ &= 1 - \mu(\mathscr{L}\setminus\mathscr{k}(U)) = \lambda(\mathscr{k}^{-1}(\mathscr{k}(U))) = \lambda(U).\end{aligned}$$

The necessity, which easily follows from the fact that

$$\{\varphi \in \mathscr{k}(U)\} = \{\varphi \subseteq U\}, \ U \in \mathrm{Scott}(L),$$

and the properties of $\mathscr{k}$, is left to the reader. □

5. Weak convergence results

In this section we shall discuss the convergence in distribution of random variables in L w.r.t. its Lawson topology. Recall from Section 2 that the Lawson topology is the coarsest topology that contains $\mathscr{L}$ and all sets of the form $L\backslash\uparrow x$, where $x \in L$. If L is $\wedge_f$-closed, then the mapping $(x,y) \longrightarrow x\wedge y$ from $L\times L$ into L is (Lawson) continuous. By Proposition 3.1, Σ is the Borel σ-field on L.

Recall that $\longrightarrow_d$ denotes *convergence in distribution*, i.e., for random variables $\xi,\xi_1,\xi_2,\ldots$ in L, $\xi_n \longrightarrow_d \xi$ if $P\xi_n^{-1}$ converges weakly to $P\xi^{-1}$. The latter holds by definition if $E[g(\xi_n)] \longrightarrow E[g(\xi)]$ whenever $g:L \longrightarrow \mathbb{R}$ is continuous and bounded (cf. Billingsley (1968)).

By the Portmanteu Theorem of Billingsley (1968) (or Ash (1972), Theorem 4.5.1), the following two conditions are implied by $\xi_n \longrightarrow_d \xi$:

(5.1) $$\limsup_n P\cap_{i=1}^m\{x_i \le \xi_n\} \le P\cap_i\{x_i \le \xi\},\ m = 1,2,\ldots,\ x_1,\ldots,x_m \in L;$$

(5.2) $$\liminf_n P\{\xi_n \in U\} \ge P\{\xi \in U\},\ U \in \mathrm{Scott}(L).$$

We begin with a preliminary result in which L is $\vee_f$-closed and has a bottom, i.e., is a *complete lattice.* This case is particularly simple, since the Lawson topology is compact (cf. Gierz et al. (1980)). Soon we will see that the following characterization of convergence in distribution is valid also if L only is assumed to be $\vee_f$-closed. See Remark 5.5 below.

PROPOSITION 5.1. *Suppose L is $\vee_f$-closed and has a bottom. Let $\xi,\xi_1,\xi_2,\ldots$ be random variables in L. Then $\xi_n \longrightarrow_d \xi$ if, and only if, the following two conditions hold true:*

(i) $$\limsup_n P\{x \le \xi_n\} \le P\{x \le \xi\},\ x \in L,$$

(ii) $$\liminf_n P\{\xi_n \in F\} \ge P\{\xi \in F\},\ F \in \mathscr{L}.$$

Moreover, condition (ii) *is a consequence of*

(iii) $$\liminf_n P\{x \ll \xi_n\} \ge P\{x \ll \xi\},\ x \in L.$$

PROOF. The necessity is already proved. We first show the last assertion of the theorem. Fix $F \in \mathscr{L}$ and let $\epsilon > 0$. By part (b) of Proposition 4.1,

$$P\{\xi \in F\} \le P\{x \ll \xi\} + \epsilon$$

for some $x \in L$. Now (ii) follows by a routine estimation.

Next assume (i) and (ii). By the compactness of the Lawson topology, (ξ_n) contains a convergent subsequence. Thus we may select $\xi_{n_1}, \xi_{n_2}, \ldots$ and a random variable η in L such that $\xi_{n_k} \to_{\mathrm{d}} \eta$. Suppose $x \in F \in \mathscr{L}$. Choose $y \in L$ and $G \in \mathscr{L}$ such that $x \in G \subseteq \uparrow y \subseteq F$. By (5.1) and (5.2),

$$\mathrm{P}\{x \le \eta\} \le \mathrm{P}\{\eta \in G\} \le \liminf_k \mathrm{P}\{\xi_{n_k} \in G\}$$
$$\le \limsup_n \mathrm{P}\{\xi_n \in \uparrow y\} \le \mathrm{P}\{\xi \in \uparrow y\} \le \mathrm{P}\{\xi \in F\}$$

and

$$\mathrm{P}\{x \le \xi\} \le \mathrm{P}\{\xi \in G\} \le \liminf_n \mathrm{P}\{\xi_n \in G\}$$
$$\le \limsup_k \mathrm{P}\{\xi_{n_k} \in \uparrow y\} \le \mathrm{P}\{\eta \in \uparrow y\} \le \mathrm{P}\{\eta \in F\}.$$

By Proposition 4.1,

$$\mathrm{P}\{x \le \xi\} = \mathrm{P}\{x \le \eta\},\ x \in L.$$

Thus $\eta =_{\mathrm{d}} \xi$ by Theorem 4.2, and we may conclude by Billingsley (1968), Theorem 2.3, that $\xi_n \to_{\mathrm{d}} \xi$. □

The Scott topology on L is a countably based continuous poset, satisfying the hypotheses of the foregoing result. Random variables in Scott(L) are quite naturally called *random Scott open sets.*

THEOREM 5.2. *Let $\gamma, \gamma_1, \gamma_2, \ldots$ be random Scott open sets in L. Then $\gamma_n \to_{\mathrm{d}} \gamma$ if, and only if, the following two conditions hold true:*

(i) $\limsup_n \mathrm{P}\cap_{i=1}^m \{F_i \subseteq \gamma_n\} \le \mathrm{P}\cap_{i=1}^m \{F_i \subseteq \gamma\},\ m = 1,2,\ldots,\ F_1,\ldots,F_m \in \mathscr{L},$

(ii) $\liminf_n \mathrm{P}\cap_{i=1}^m \{x_i \in \gamma_n\} \ge \mathrm{P}\cap_{i=1}^m \{x_i \in \gamma\},\ m = 1,2,\ldots,\ x_1,\ldots,x_m \in L.$

PROOF. Assume (i) and (ii). Let $U \in$ Scott(L). Then $U = \cup_i F_i$ for some $F_1, F_2, \ldots \in \mathscr{L}$. By (i), for each fixed m,

$$\limsup_n \mathrm{P}\{U \subseteq \gamma_n\} \le \limsup_n \mathrm{P}\{\cup_{i=1}^m F_i \subseteq \gamma_n\} = \limsup_n \mathrm{P}\cap_{i=1}^m \{F_i \subseteq \gamma_n\}$$
$$\le \mathrm{P}\cap_{i=1}^m \{F_i \subseteq \gamma\} = \mathrm{P}\{\cup_{i=1}^m F_i \subseteq \gamma\} \downarrow \mathrm{P}\{U \subseteq \gamma\}.$$

This shows condition (i) of Proposition 5.1.

Next, let $\mathscr{F}$ be an open filter on Scott(L). By Theorem 2.16 of Hofmann & Mislove (1981),

$$\mathscr{F} = \{U \in \text{Scott}(L): Q \subseteq U\}$$

for some Scott compact saturated $Q \subseteq L$. By Lemma 2.1, there are $x_{ni} \in L$, where $n = 1,2,\ldots$ and $1 \le i \le k_n < \infty$, such that

$$Q \subseteq \left[\cup_i \uparrow x_{ni}\right] \downarrow Q.$$

By (ii), for every fixed m,

$$\liminf_n \mathrm{P}\{\gamma_n \in \mathscr{F}\} = \liminf_n \mathrm{P}\{Q \subseteq \gamma_n\} \ge \liminf_n \mathrm{P}\{\cup_i \uparrow x_{mi} \subseteq \gamma_n\}$$
$$= \liminf_n \mathrm{P}\cap_i\{x_{mi} \in \gamma_n\} \ge \mathrm{P}\cap_i\{x_{mi} \in \gamma\} = \mathrm{P}\{\cup_i \uparrow x_{mi} \subseteq \gamma\} \uparrow \mathrm{P}\{Q \subseteq \gamma\} = \mathrm{P}\{\gamma \in \mathscr{F}\}.$$

This shows condition (ii) of Proposition 5.1 and $\gamma_n \to_{\mathrm{d}} \gamma$ follows.

To see the necessity, it is enough to note that

$$\{U \in \text{Scott}(L): \cup_{i=1}^m \uparrow x_i \subseteq U\}$$

is an open filter on Scott(L), since $\cup_{i=1}^m \uparrow x_i$ is Scott compact saturated, and that

$$\{U \in \text{Scott}(L): \cup_{i=1}^m F_i \subseteq U\}$$

is closed, since $\mathscr{L} \subseteq$ Scott(L). □

Here are our main convergence theorems for distributions of random variables in L and its Lawson dual $\mathscr{L}$. The first result replaces the preliminary Proposition 5.1.

THEOREM 5.3. *Let $\xi, \xi_1, \xi_2, \ldots$ be random variables in L. Then $\xi_n \to_{\mathrm{d}} \xi$ if, and only if, the following two conditions hold true:*

(i) $\limsup_n \mathrm{P}\cap_{i=1}^m\{x_i \le \xi_n\} \le \mathrm{P}\cap_{i=1}^m\{x_i \le \xi\}$, $m = 1,2,\ldots$, $x_1,\ldots,x_m \in L$,

(ii) $\liminf_n \mathrm{P}\{\xi_n \in \cap_{i=1}^m F_i\} \ge \mathrm{P}\{\xi \in \cap_{i=1}^m F_i\}$, $m = 1,2,\ldots$, $F_1,\ldots,F_m \in \mathscr{L}$.

Moreover, condition (ii) *holds if*

(iii) $\liminf_n \mathrm{P}\cap_{i=1}^m\{x_i \ll \xi_n\} \ge \mathrm{P}\cap_{i=1}^m\{x_i \ll \xi\}$, $m = 1,2,\ldots$, $x_1,\ldots,x_m \in L$.

THEOREM 5.4. *Let $\varphi, \varphi_1, \varphi_2, \ldots$ be random open filters in L, i.e., random variables in $\mathscr{L}$. Then $\varphi_n \to_{\mathrm{d}} \varphi$ if, and only if, the following two conditions hold true:*

(i) $\limsup_n \mathrm{P}\cap_{i=1}^m\{F_i \subseteq \varphi_n\} \le \mathrm{P}\cap_{i=1}^m\{F_i \subseteq \varphi\}$, $m = 1,2,\ldots$, $F_1,\ldots,F_m \in \mathscr{L}$,

(ii) $\liminf_n \mathrm{P}\cap_{i=1}^m\{x_i \in \varphi_n\} \ge \mathrm{P}\cap_{i=1}^m\{x_i \in \varphi\}$, $m = 1,2,\ldots$, $x_1,\ldots,x_m \in L$.

Moreover, condition (ii) *holds if*

(iii) $\liminf_n \mathrm{P}\cap_{i=1}^m\{F_i \ll \varphi_n\} \ge \mathrm{P}\cap_{i=1}^m\{F_i \ll \varphi\}$, $m = 1,2,\ldots$, $F_1,\ldots,F_m \in \mathscr{L}$.

PROOF OF THEOREMS 5.3 AND 5.4. Due to the Lawson duality, the two theorems are equivalent. To see this, recall from Section 2 that OFilt($\mathscr{L}$) = $\{\mathscr{F}_x: x \in L\}$ and note that

$$\{\varphi \in \mathscr{F}_x\} = \{x \in \varphi\},$$

for $x \in L$ and random open filters φ in L. Thus we only need to prove one of the theorems.

The proof of the fact that (i) and (ii) together is a necessary and sufficient condition for convergence is very easy in the setting of Theorem 5.4, since $\varphi, \varphi_1, \varphi_2, \ldots$ also are random variables in Scott(L) and $\varphi_n \to_d \varphi$ in $\mathscr{L}$ if, and only if, $\varphi_n \to_d \varphi$ in Scott(L) (cf. Lemma 2.2).

In the proof of the implication from (iii) to (ii) we work in the setting of Theorem 5.3. Fix $F_1, \ldots, F_m \in \mathscr{L}$. For $1 \leq i \leq m$, choose $x_{i1}, x_{i2}, \ldots \in L$ such that

$$\{y \in L: x_{in} \ll y\} \uparrow F_i$$

(cf. the proof of Proposition 4.1). Then, for each fixed m,

$$\begin{aligned} \liminf_n \mathrm{P}\{\xi_n \in \cap_{i=1}^m F_i\} &\geq \liminf_n \mathrm{P} \cap_{i=1}^m \{x_{im} \ll \xi_n\} \\ &\geq \mathrm{P} \cap_{i=1}^m \{x_{im} \ll \xi\} \uparrow \mathrm{P}\{\xi \in \cap_{i=1}^m F_i\}. \end{aligned}$$

Thus (ii) is a consequence of (iii). □

REMARK 5.5. If L is $\vee_f$-closed, then the two equivalences

$$x \leq z,\ y \leq z \Leftrightarrow x \vee y \leq z,$$

$$x \ll z,\ y \ll z \Leftrightarrow x \vee y \ll z$$

hold for $x, y, z \in L$, and, moreover, $\mathscr{L}$ is $\cap_f$-closed. Thus, in this case the conditions of Theorem 5.3 reduce to the corresponding conditions of Proposition 5.1.

If L only is $\wedge_f$-closed, then $\mathscr{L} \cup \{\emptyset\}$ is $\cap_f$-closed. In this case statement (ii) of Theorem 5.3 reduces to statement (ii) of Proposition 5.1.

There is always a need for simple necessary and sufficient conditions for convergence. We confine ourselves to the simple case when L is both $\vee_f$- and $\wedge_f$-closed.

PROPOSITION 5.6. *Assume L both $\vee_f$- and $\wedge_f$-closed and let $\xi, \xi_1, \xi_2, \ldots$ be random variables in L. Then, if $\xi_n \to_d \xi$, there are separating sets $A \subseteq L$ and $\mathscr{A} \subseteq \mathscr{L}$, such that A is $\vee_f$-closed and*

(i) $$\lim_n \mathrm{P}\{x \leq \xi_n\} = \mathrm{P}\{x \leq \xi\},\ x \in A,$$

while $\mathscr{A}$ *is* $\cap_f$*-closed and*

(ii) $$\lim_n \mathrm{P}\{\xi_n \in F\} = \mathrm{P}\{\xi \in F\},\ F \in \mathscr{A}.$$

Moreover, $\xi_n \to_d \xi$ *if there are separating sets* $A \subseteq L$ *and* $\mathscr{A} \subseteq \mathscr{L}$*, such that*

(iiia) $$\limsup_n \mathrm{P}\{x \leq \xi_n\} \leq \mathrm{P}\{x \leq \xi\},\ x \in A,$$

(iiib) $$\liminf_n \mathrm{P}\{\xi_n \in F\} \geq \mathrm{P}\{\xi \in F\},\ F \in \mathscr{A};$$

or if there is a separating $A \subseteq L$ *such that*

(iv) $$\lim_n \mathrm{P}\{x \leq \xi_n\} = \mathrm{P}\{x \leq \xi\},\ x \in A,$$

or if there is a separating $\mathscr{A} \subseteq \mathscr{L}$ *such that*

(v) $$\lim_n \mathrm{P}\{\xi_n \in F\} = \mathrm{P}\{\xi \in F\},\ F \in \mathscr{A}.$$

Our proof of the necessity of (i) and (ii) requires the following lemma, which tells us that the collection of all pairs $(x,F) \in L \times \mathscr{L}$, satisfying $F \subseteq \uparrow x$ and $\mathrm{P}\{\xi \in \uparrow x \backslash F\} = 0$, is sufficiently rich for our purposes.

LEMMA 5.7. *Let* ξ *be a random variable in L. Then, whenever* $x \in F \in \mathscr{L}$*, we have* $x \in H \subseteq \uparrow y \subseteq F$ *for some pair* $(y,H) \in L \times \mathscr{L}$ *with*

$$\mathrm{P}\{y \leq \xi\} = \mathrm{P}\{\xi \in H\}.$$

PROOF. Let $x(1) = x$ and choose $x(0) \in F$ such that $x(0) \ll x(1)$. Then choose recursively $x(k2^{-n})$ for $n = 1,2,\dots$ and $k = 1,2,\dots,2^n-1$, such that $x(k2^{-n}) \ll x(l2^{-m})$ if $k2^{-n} < l2^{-m}$. For $0 < t < 1$, put

$$y_t = \vee\{x(k2^{-n}) \colon k2^{-n} < t\}.$$

It is easily seen that

$$y_t = \vee_{s<t} y_s.$$

Hence the mapping

$$t \to \mathrm{P}\{y_t \leq \xi\},\ 0 < t < 1,$$

is decreasing and left continuous. Let t be a point of continuity. Then, by monotone convergence, writing $y = y_t$ and $H = \cup_{t<s} \uparrow y_s$,

$$\mathrm{P}\{y \leq \xi\} = \mathrm{P}\{\xi \in H\}.$$

Clearly $H \in \mathscr{L}$ and $x \in H \subsetneq \uparrow y \subsetneq F$. □

PROOF OF PROPOSITION 5.6. Suppose $\xi_n \to_d \xi$ and let $x,y \in L$, $x \ll y$. Then $y \in F \subsetneq \uparrow x$ for some $F \in \mathscr{L}$. By Lemma 5.7, there is a pair $(z,H) \in L \times \mathscr{L}$ with $y \in H \subsetneq \uparrow z \subsetneq F$ and $P\{\xi \in \uparrow z \backslash H\} = 0$. By Theorem 5.3,

$$P\{z \leq \xi_n\} \to P\{z \leq \xi\}.$$

Thus the set A of all $z \in L$ for which the above convergence holds is separating.

Similarly the reader may prove that the set $\mathscr{A}$ of all $F \in \mathscr{L}$ satisfying

$$P\{\xi_n \in F\} \to P\{\xi \in F\}$$

is separating, too. Now suppose $F,H \in \mathscr{A}$. Then $F \cap H \in \mathscr{L}$. To see that $F \cap H \in \mathscr{A}$, note first that

$$\text{liminf}_n P\{\xi_n \in F \cup H\} \geq P\{\xi \in F \cup H\},$$

since $F \cup H \in \text{Scott}(L)$, and then that

$$\begin{aligned} \text{limsup}_n P\{\xi_n \in F \cap H\} &= P\{\xi \in F\} + P\{\xi \in H\} - \text{liminf}_n P\{\xi_n \in F \cup H\} \\ &\leq P\{\xi \in F \cap H\} \leq \text{liminf}_n P\{\xi_n \in F \cap H\}. \end{aligned}$$

The proof of the fact that A is a $\vee_f$-closed is similar.

This shows the necessity of (i) and (ii). To see that (iii) is a sufficient condition for $\xi_n \to_d \xi$, choose for $x \in L$ some $x_1, x_2, \ldots \in A$ such that $x_n \ll x$ and $x_n \uparrow x$. Then

$$\text{limsup}_n P\{x \leq \xi_n\} \leq \text{limsup}_n P\{x_m \leq \xi_n\} \leq P\{x_m \leq \xi\} \downarrow P\{x \leq \xi\}.$$

Thus condition (i) of Theorem 5.3 follows from (iiia). Analogously the reader easily shows that (iiib) implies condition (ii) of Theorem 5.3. So (iii) is sufficient for $\xi_n \to_d \xi$.

To see that (iv) implies $\xi_n \to_d \xi$, let first $F \in \mathscr{L}$. Choose $x_m \in A$ such that $(\uparrow x_m) \uparrow F$. Now condition (ii) of Theorem 5.3 follows from

$$\text{liminf}_n P\{\xi_n \in F\} \geq \text{liminf}_n P\{\xi_n \in \uparrow x_m\} = P\{\xi \in \uparrow x_m\} \uparrow P\{\xi \in F\}.$$

Similarly the reader may show condition (i) of Theorem 5.3 and $\xi_n \to_d \xi$ follows. The remaining part of the proof is similar and, therefore, left to the reader. □

Assume, for now, that L has a top, which we denote by $\top$. In some applications it is important to be able to determine whether the limit of a sequence of distributions of random variables in L charges $\top$ or not.

Recall from the introduction that we call a collection (ξ_n) of random variables in L tight, if for all $\epsilon > 0$ there is some $F \in \mathscr{L}$ with

(5.3) $$\sup_n \mathrm{P}\{\xi_n \in F\} \leq \epsilon.$$

A random variable ξ in L is of course tight if the collection (ξ) is. There are $F_1, F_2, \ldots \in \mathscr{L}$ satisfying $F_n \downarrow \{\top\}$. So ξ is tight if, and only if, $\xi \neq \top$ a.s.

PROPOSITION 5.8. *Let L have a top $\top$ and consider a collection (ξ_n) of random variables in L. Then (ξ_n) is tight if, and only if, for every $\epsilon > 0$ there is some $x \in L$, $x \ll \top$, such that*

$$\sup_n \mathrm{P}\{x \leq \xi_n\} \leq \epsilon.$$

PROOF. The equivalence follows from the facts that every $F \in \mathscr{L}$ contains some $x \ll \top$, and that if $x \ll \top$, then some $F \in \mathscr{L}$ satisfies $F \subseteq \uparrow x$. □

PROPOSITION 5.9. *Let L have a top $\top$ and consider a sequence $\xi_1, \xi_2, \ldots$ of tight random variables in L. Let ξ be a random variable in L and assume $\xi_n \to_d \xi$. Then, if the sequence (ξ_n) is tight, so is ξ and vice versa.*

PROOF. As remarked in the introduction, the direct assertion follows at once from Theorem 5.3. So assume ξ tight. Fix $\epsilon > 0$. Choose $x_0 \in L$, $x_0 \ll \top$, such that $\mathrm{P}\{x_0 \leq \xi\} < \epsilon$. By Theorem 5.3, $\mathrm{P}\{x_0 \leq \xi_n\} \leq \epsilon$ for all $n > m$, say. Next, choose $x_i \in L$, $x_i \ll \infty$, such that $\mathrm{P}\{x_i \leq \xi_i\} \leq \epsilon$ for $1 \leq i \leq m$. Let $x \ll \top$ satisfy $x_i \leq x$ for all i. Then, clearly,

$$\sup_n \mathrm{P}\{x \leq \xi_n\} \leq \epsilon.$$ □

For the remaining part of this section, we assume L both $\vee_f$- and $\wedge_f$-closed. Then L has a top, which we still denote by $\top$. Consider a triangular array $(\xi_{nj},\, n \in \mathbb{N},\, 1 \leq j < \infty)$ of random variables in L. We assume the ξ_{nj}'s independent for each fixed n and that

$$\lim_n \sup_j \mathrm{P}\{\xi_{nj} \notin F\} = 0$$

for $F \in \mathscr{L}$, in which case we say that they form a *null-array*. Note that this holds if, and only if,

$$\lim_n \inf_j \mathrm{P}\{x \le \xi_{nj}\} = 1$$

for $x \in L$ with $x \ll \tau$. This is easy to see.

We are concerned with the possible convergence in distribution of the rowwise infima $\wedge_j \xi_{nj}$ of a null-array (ξ_{nj}).

THEOREM 5.10. *Assume L both $\vee_f$- and $\wedge_f$-closed. Let ξ be a random variable and let (ξ_{nj}) be a null-array of random variables in L. Then $\wedge_j \xi_{nj} \to_d \xi$ if, and only if,*

(ia) $$\liminf_n \Sigma_j \mathrm{P}\{x \le \xi_{nj}\}^c \ge -\log \mathrm{P}\{x \le \xi\},\ x \in L,$$

(ib) $$\limsup_n \Sigma_j \mathrm{P}\{\xi_{nj} \notin F\} \le -\log \mathrm{P}\{\xi \in F\},\ F \in \mathscr{L}.$$

Moreover, if $\wedge_j \xi_{nj} \to_d \xi$, then there are separating subsets $A \subseteq L$ and $\mathscr{A} \subseteq \mathscr{L}$, such that

(iia) $$\lim_n \Sigma_j \mathrm{P}\{x \le \xi_{nj}\}^c = -\log \mathrm{P}\{x \le \xi\},\ x \in A,$$

(iib) $$\lim_n \Sigma_j \mathrm{P}\{\xi_{nj} \notin F\} = -\log \mathrm{P}\{\xi \in F\},\ F \in \mathscr{A}.$$

Here A may be chosen $\vee_f$-closed and $\mathscr{A}$ $\cap_f$-closed.

Conversely, $\wedge_j \xi_{nj} \to_d \xi$ if there are separating sets $A \subseteq L$ and $\mathscr{A} \subseteq \mathscr{L}$ such that

(iiia) $$\liminf_n \Sigma_j \mathrm{P}\{x \le \xi_{nj}\}^c \ge -\log \mathrm{P}\{x \le \xi\},\ x \in A,$$

(iiib) $$\limsup_n \Sigma_j \mathrm{P}\{\xi_{nj} \notin F\} \le -\log \mathrm{P}\{\xi \in F\},\ F \in \mathscr{A},$$

or if

(iv) $$\Sigma_j \mathrm{P}\{\xi \le \xi_{nj}\}^c \to -\log \mathrm{P}\{x \le \xi\}$$

holds for all x in some separating subset of L, or if

(v) $$\Sigma_j \mathrm{P}\{\xi_{nj} \notin F\} \to -\log \mathrm{P}\{\xi \in F\}$$

holds for all F in some separating subset of $\mathscr{L}$.

PROOF. Write $\xi_n = \wedge_j \xi_{nj}$. We only prove that $\xi_n \to_d \xi$ if, and only if, both (ia) and (ib) hold. The remaining part of the proof is very similar to the proof of Proposition 5.6.

First note that trivial manipulations, using the well-known inequality

$$\alpha \le -\log(1-\alpha),$$

show us that condition (i) of Theorem 5.3 follows by (ia) and that (ib) follows by condition (ii) of Theorem 5.3.

Suppose condition (i) of Theorem 5.3. Fix $x \in L$ and let $x_m \ll x$. Let $\epsilon > 0$ be given and pick $\delta > 0$ such that

$$-\log(1-\alpha) \leq (1+\epsilon)\alpha$$

whenever $0 \leq \alpha < \delta$. Note that $x_m \ll \tau$. Hence

$$\sup_j \mathrm{P}\{x_m \leq \xi_{nj}\} < \delta$$

for sufficiently large n. We may now conclude that

$$-\log\mathrm{P}\{x_m \leq \xi\} \leq (1+\epsilon)\liminf_n \Sigma_j \mathrm{P}\{x_m \leq \xi_{nj}\}^c \leq (1+\epsilon)\liminf_n \Sigma_j \mathrm{P}\{x \leq \xi_{nj}\}^c,$$

and (ia) follows by letting $\epsilon \downarrow 0$ and then $x_m \uparrow x$.

Finally assume (ib). Fix $F \in \mathscr{L}$. Let $\epsilon > 0$ be given and pick $\delta > 0$ as in the above paragraph. Then

$$\sup_j \mathrm{P}\{\xi_{nj} \notin F\} < \delta$$

for $n \geq m$, say. Hence

$$\limsup_n -\log\mathrm{P}\{\xi_n \in F\} = \limsup_n \Sigma_j -\log\mathrm{P}\{\xi_{nj} \in F\}$$
$$\leq \limsup_n \Sigma_j (1+\epsilon)\mathrm{P}\{\xi_{nj} \notin F\} \leq -(1+\epsilon)\log\mathrm{P}\{\xi \in F\},$$

from which condition (ii) of Theorem 5.3 trivially follows. □

A random variable ξ in L is called *infinitely divisible* if, for $n = 1,2,\ldots$, we have $\xi =_d \wedge_j \xi_{nj}$ for some independent and identically distributed random variables $\xi_{n1},\ldots,\xi_{nn}$ in L.

If ξ is infinitely divisible, then, by Norberg (1989), $\xi \leq m$ a.s., where

$$m = \vee\{x \in L: \mathrm{P}\{x \leq \xi\} > 0\}.$$

Thus we may assume that $\xi_{nj} \leq m$ a.s. Let $F \in \mathscr{L}$. A simple argument shows that

$$\lim_n \mathrm{P}\{\xi_{n1} \notin F\} = 0$$

if, and only if, $\mathrm{P}\{\xi \in F\} > 0$, which clearly is equivalent to $m \in F$. We conclude that the ξ_{nj}'s form a null-array if $m = \tau$, or if we regard them as random variables in $M = \downarrow m$.

Next, let (ξ_{nj}) be a null-array of random variables in L and assume that $\wedge_j \xi_{nj} \to_d \xi$. Our aim is to show that ξ must be infinitely divisible. The following characterization of the infinitely divisible distributions is sufficient for our needs.

THEOREM 5.11 (Norberg (1989)). *Assume L both $\vee_f$- and $\wedge_f$-closed, and let ξ be a random variable in L. Define*

$$\mathscr{L}_\xi = \{F \in \mathscr{L}: \Phi_\xi(F) < \infty\},$$

where

$$\Psi_\xi(F) = -\log P\{\xi \in F\},\ F \in \mathscr{L}.$$

Then ξ is infinitely divisible if, and only if, $\mathscr{L}_\xi$ is $\cap_f$-closed and

$$\Delta_{F_1}\cdots\Delta_{F_n}\Psi_\xi(F) \le 0,\ n = 1,2,\ldots,\ F,F_1,\ldots,F_n \in \mathscr{L}_\xi.$$

We prove that $\mathscr{L}_\xi$ is $\cap_f$-closed. The proof of the second condition of Theorem 5.11 is similar but more cumbersumb. Let $F,H \in \mathscr{L}_\xi$. We know from Theorem 5.10 that

$$\Psi_n(F) = \Sigma_j P\{\xi_{nj} \notin F\} \longrightarrow \Psi_\xi(F)$$

if $F \in \mathscr{A}$, where $\mathscr{A} \subseteq \mathscr{L}$ is separating and $\cap_f$-closed. If $F,H \in \mathscr{A}$, we now get

$$\Psi_\xi(F \cap H) = \lim_n \Psi_n(F \cap H) \le \lim_n[\Psi_n(F) + \Psi_n(H)] = \Psi_\xi(F) + \Psi_\xi(H).$$

In the general case, choose $F_n,H_n \in \mathscr{A}$, such that $F_n \uparrow F$ and $H_n \uparrow H$. Then

$$\Psi_\xi(F \cap H) = \lim_n \Psi_\xi(F_n \cap H_n) \le \lim_n[\Psi_\xi(F_n) + \Psi_\xi(H_n)] = \Psi_\xi(F) + \Psi_\xi(H).$$

It follows that $\Psi_\xi(F \cap H) < \infty$, if $\Psi_\xi(F) < \infty$ and $\Psi_\xi(H) < \infty$, i.e., $\mathscr{L}_\xi$ is $\cap_f$-closed as claimed.

6. Random elements in locally compact sober spaces

In this section S is a locally compact, second countable and sober topological space. Write $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{Q}$ for the collections of closed, open and compact saturated sets in S.

Moreover write $\mathcal{S}$ for the Borel-σ-field on S, i.e., $\mathcal{S} = \sigma(\mathcal{G})$. Note that if $Q \in \mathcal{Q}$, then $Q = \cap_n G_n$ for some $G_1, G_2, \ldots \in \mathcal{G}$, and, if $G \in \mathcal{G}$, then $G = \cup_n Q_n$ for some $Q_1, Q_2, \ldots \in \mathcal{Q}$. Thus $\mathcal{S} = \sigma(\mathcal{Q}) = \sigma(\mathcal{Q} \cup \mathcal{G})$.

Note that $\mathcal{G}$ is continuous and, for $G_1, G_2 \in \mathcal{G}$, $G_1 \ll G_2$ if, and only if, $G_1 \subseteq Q \subseteq G_2$ for some $Q \in \mathcal{Q}$. Cf. Hofmann & Mislove (1981). The mapping which identifies an open set with its complement is of course an anti-order-isomorphism between $\mathcal{G}$ and $\mathcal{F}$. Thus $\mathcal{F}$ is continuous relative to the *exclusion* order $\supseteq$. Hofmann & Mislove (1981) also shows that

$$\text{OFilt}(\mathcal{G}) = \Big\{\{G \in \mathcal{G}: Q \subseteq G\}: Q \in \mathcal{Q}\Big\}. \tag{6.1}$$

Thus also $\mathcal{Q}$ is continuous relative to the exclusion order $\supseteq$. It follows by (6.1) and the Lawson duality that

$$\text{OFilt}(\mathcal{Q}) = \Big\{\{Q \in \mathcal{Q}: Q \subseteq G\}: G \in \mathcal{G}\Big\}. \tag{6.2}$$

The Scott topologies on $\mathcal{F}, \mathcal{G}$ and $\mathcal{Q}$ are second countable, since $\mathcal{G}$ is so.

The reader easily writes down characterization theorems for distributions of random variables in $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{Q}$, i.e., *random closed, open* and *compact saturated* sets in S (for the closed sets, see (1.6)–(1.7)). Let us instead note that some simple manipulation show that the Lawson topology on $\mathcal{F}$ is generated by the two families

$$\Big\{\{F \in \mathcal{F}: F \cap G \neq \emptyset\}: G \in \mathcal{G}\Big\},$$
$$\Big\{\{F \in \mathcal{F}: F \cap Q = \emptyset\}: Q \in \mathcal{Q}\Big\}.$$

Thus it coincides with Fell's (1962) topology.

Recall from the introduction that a class $\mathcal{A}$ of subsets of S is called separating, if whenever $Q \subseteq G$, where $Q \in \mathcal{Q}$ and $G \in \mathcal{G}$, we have $Q \subseteq A \subseteq G$ for some $A \in \mathcal{A}$.

PROPOSITION 6.1. *Let $\varphi, \varphi_1, \varphi_2, \ldots$ be random closed sets in S. Then $\varphi_n \to_d \varphi$ if, and only if,*

$$\mathrm{P}\{\varphi \cap A \neq \emptyset\} = \lim_n \mathrm{P}\{\varphi_n \cap A \neq \emptyset\} \tag{6.3}$$

for A in some separating class of subsets of S.

PROOF. Suppose $\varphi_n \to_d \varphi$. Whenever $Q \subseteq G$, where $Q \in \mathcal{Q}$ and $G \in \mathcal{G}$, there are $Q_1 \in \mathcal{Q}$ and $G_1 \in \mathcal{G}$ satisfying $Q \subseteq G_1 \subseteq Q_1 \subseteq G$ and $\mathrm{P}\{\varphi \cap G_1 \neq \emptyset\} = \mathrm{P}\{\varphi \cap Q_1 \neq \emptyset\}$. This follows by Lemma 5.7. Clearly

$$\mathrm{P}\{\varphi_n \cap G_1 \neq \emptyset\} \to \mathrm{P}\{\varphi \cap G_1 \neq \emptyset\}.$$

Thus the class of sets satisfying (6.3) is separating.

Conversely, assume that (6.3) holds for all $A \in \mathcal{A}$, where $\mathcal{A}$ is separating. We prove (1.6) and leave (1.7) to the reader. If $G \in \mathcal{G}$, choose $A_1, A_2, \ldots \in \mathcal{A}$ such that $A_n \uparrow G$. Then

$$\begin{aligned}\mathrm{P}\{\varphi \cap G \neq \emptyset\} &= \lim_m \mathrm{P}\{\varphi \cap A_m \neq \emptyset\} \\ &= \lim_m \lim_n \mathrm{P}\{\varphi_n \cap A_m \neq \emptyset\} \leq \liminf_n \mathrm{P}\{\varphi_n \cap G \neq \emptyset\}.\end{aligned}$$ □

A similar result may of course be proved for random compact saturated sets. This is left to the reader. We only note that a subcollection $\mathcal{A} \subseteq \mathcal{Q}$ is separationg as a subset of $\mathcal{Q}$ if, and only if, it is so as a class of sets in S.

A sequence $\xi_1, \xi_2, \ldots$ of random closed (resp. compact saturated) sets in S is tight if, and only if, for all $\epsilon > 0$, we have

$$\inf_n \mathrm{P}\{\xi_n \cap Q \neq \emptyset\} \geq 1 - \epsilon$$

for some compact and saturated $Q \subseteq S$ (resp.

$$\sup_n \mathrm{P}\{\xi_n \subseteq G\} \leq \epsilon$$

for some open $G \subseteq S$). To see this, use (6.1) (resp. (6.2)). For the closed sets, cf. (1.9).

An S-valued mapping on some probability space $(\Omega, \mathcal{R}, \mathrm{P})$ is called a *random element* in S if it is measurable w.r.t. $\mathcal{R}$ and $\mathcal{S}$.

The ***patch topology*** on S is by definition the coarsest topology containing $\mathcal{G}$ and all $S \backslash Q$, with $Q \in \mathcal{Q}$. It is completely regular, second countable and Hausdorff (Hofmann & Mislove (1981), Corollary 4.6), in particular Polish. Clearly $\mathcal{S}$ is the Borel-σ-field w.r.t. the patch topology. As an example of this construction we mention that the Lawson topology on L is the patch of the Scott topology.

We shall write down a simple convergence theorem for distributions of random variables in S. The convergence will be with respect to the patch topology. A part of the following proposition is needed.

PROPOSITION 6.2. *For $s, s_1, s_2, \ldots \in S$, the following three statements are equivalent:*

(i) $s_n \to s$ w.r.t. the patch topology;

(ii) $\mathrm{sat}\{s_n\} \to \mathrm{sat}\{s\}$ in $\mathcal{Q}$ w.r.t. the Lawson topology;

(iii) $\{s_n\}^- \to \{s\}^-$ in $\mathcal{F}$ w.r.t. Fell's topology.

PROOF. Let $A \subseteq S$ be saturated and fix $s \in S$. Then

$$s \in A \Leftrightarrow \mathrm{sat}\{s\} \subseteq A \Leftrightarrow \{s\}^- \cap A \neq \emptyset.$$

The proof is easy, thus omitted. In order to complete the proof, just note that the sets in $\mathcal{G}$ and $\mathcal{Q}$ are saturated. □

We next characterize convergence in distribution w.r.t. the patch topology of random variables in S. Denote by GSC_+ the class of non-negative functions g on S satisfying $\{g > x\} \in \mathcal{G}$ for all $x > 0$, and by QSC_+ those non-negative functions q on S that satisfy $\{q \geq x\} \in \mathcal{Q}$ for $x > 0$.

THEOREM 6.3. *Let $\xi, \xi_1, \xi_2, \ldots$ be random variables in S. Then the following two conditions are equivalent to $\xi_n \to_d \xi$:*

(ia) $\limsup_n \mathbf{P}\{\xi_n \in Q\} \leq \mathbf{P}\{\xi \in Q\}$, $Q \in \mathcal{Q}$,

(ib) $\liminf_n \mathbf{P}\{\xi_n \in G\} \geq \mathbf{P}\{\xi \in G\}$, $G \in \mathcal{G}$;

(iia) $\limsup_n \mathbf{E}[q(\xi_n)] \leq \mathbf{E}[q(\xi)]$, $q \in \mathrm{QSC}_+$,

(iib) $\liminf_n \mathbf{E}[g(\xi_n)] \geq \mathbf{E}[g(\xi)]$, $g \in \mathrm{GSC}_+$.

PROOF. It should be clear from what has been said before the theorem that $\xi_n \to_d \xi$ if, and only if, the two parts of (i) hold true. Next note that $1_G \in \mathrm{GSC}_+$ if $G \in \mathcal{G}$, while $1_Q \in \mathrm{QSC}_+$ if $Q \in \mathcal{Q}$. Thus (ii) implies (i). To see the implication from $\xi_n \to_d \xi$ to (ii), note first that the functions of GSC_+ are lower semicontinuous, while those of QSC_+ are upper semicontinuous w.r.t. the patch topology. This is trivial. Slightly less trivial is the fact that each $q \in \mathrm{QSC}_+$ is bounded. For $\{q \geq n\} \downarrow \{q = \infty\} = \emptyset$, so $\{q \geq n\} = \emptyset$ for some n. Now refer, e.g., to Ash (1972), Theorem 4.5.1. □

By definition, a collection (ξ_n) of random variables in S is tight if, for all $\epsilon > 0$, we have

(6.4) $$\inf_n \mathrm{P}\{\xi_n \in Q\} \geq 1 - \epsilon$$

for some $Q \in \mathcal{Q}$. Let us see what this means for a collection of random variables in a continuous poset.

Proposition 6.4. *Let (ξ_n) be a collection of random variables in L. Then (ξ_n) is (topologically) tight if, and only if, for every $\epsilon > 0$, there are some $x_1,\dots,x_m \in L$ such that*

$$\inf_n \mathrm{P} \cup_{i=1}^m \{x_i \leq \xi_n\} \geq 1 - \epsilon.$$

Proof. By Lemma 2.1, if $Q \subseteq L$ is Scott compact saturated, then $Q \subseteq \cup_{i=1}^m \uparrow x_i$ for some $x_1,\dots,x_m \in L$. This shows the only if part. To see the if part, just note that $\cup_{i=1}^m \uparrow x_i$ is Scott compact saturated. □

Clearly (6.4) is equivalent to

$$\inf_n \mathrm{P}\{\{\xi_n\}^- \cap Q \neq \emptyset\} \geq 1 - \epsilon$$

(cf. the proof of Proposition 6.2). So (ξ_n) is tight if, and only if, $(\{\xi_n\}^-)$ is so in the lattice theoretical sense. This is the idea behind what follows.

Proposition 6.5. *Assume $\mathcal{Q}$ $\cap_f$-closed. Let $s_1,s_2,\dots \in S$ and assume that $\{s_n\}^- \longrightarrow F \in \mathcal{F}$ w.r.t. Fell's topology. Then either $F = \emptyset$ or $F = \{s\}^-$ for some $s \in S$.*

Remark 6.6. The result says that the set

$$\mathcal{F}_i = \left\{\{s\}^- : s \in S\right\} \cup \left\{\emptyset\right\}$$

is closed and therefore compact w.r.t. Fell's topology. So this is a kind of a compactification of S.

Proof of Proposition 6.5. We prove that F is irreducibel, because the result then follows by sobriety. Suppose $F \subseteq H_1 \cup H_2$, where $H_i \subseteq S$ is closed. Choose $H_{im} \in \mathcal{F}$ and $Q_{im} \in \mathcal{Q}$ such that $H_i \subseteq S \backslash Q_{im} \subseteq H_{im} \downarrow H_i$. Fix m. Then $F \subseteq S\backslash Q_{1m} \cup S\backslash Q_{2m}$. Hence

$F \cap Q_{1m} \cap Q_{2m} = \emptyset$ and, since $Q_{1m} \cap Q_{2m} \in \mathcal{Q}$, $s_n \notin Q_{1m} \cap Q_{2m}$ for sufficiently large n. For such n, $s_n \in H_{1m} \cup H_{2m}$. Hence s_n is in H_{1m} or H_{2m} infinitely often. In the former case, $F \subseteq H_{1m}$ while $F \subseteq H_{2m}$ in the latter. Hence, $F \subseteq H_1 = \cap_m H_{1m}$ or $F \subseteq H_2 = \cap_m H_{2m}$, showing our claim that F is irreducibel. □

Here is finally our variant of Prohorov's theorem (cf. Billingsley (1968)). Note that the convergence in distribution is with respect to the patch topology, while the tightness is relative to the (coarser) original topology.

THEOREM 6.7. *Assume $\mathcal{Q}$ $\cap_f$-closed. Let (ξ_n) be a sequence of random variables in S. Then every subsequence of (ξ_n) has a further subsequence which converges in distribution to some random variable ξ, if, and only if, (ξ_n) is tight.*

PROOF. Suppose (ξ_n) is tight. Then so is $(\{\xi_n\}^-)$ in the lattice theoretical sense. By compactness, every subsequence of $(\{\xi_n\}^-)$ has a further subsequence $(\{\xi_{n_k}\}^-)$, which converges in distribution to some random closed set η in S. By tightness, $\eta \neq \emptyset$ a.s., so by Proposition 6.5, η is a singleton closure a.e. Thus there is a random element ξ in S satisfying $\eta = \{\xi\}^-$. By Proposition 6.2, $\xi_{n_k} \rightarrow_d \xi$.

Suppose, conversely, that (ξ_n) is not tight. Then so is the case for $(\{\xi_n\}^-)$, too. Thus there must be a subsequence $(\{\xi_{n_k}\}^-)$, which converges in distribution to some non-tight random closed set η. Then $\mathbf{P}\{\eta = \emptyset\} > 0$. This clearly rules out the possibility that (ξ_{n_k}) or some subsequence thereof converges in distribution. □

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Lattice-valued Semicontinuous functions

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Abstract: Three kinds of upper semicontinuity of functions $f: E \to L$ are studied and compared, where E is an arbitrary topological space and L is a complete lattice. Special attention is paid to continuous lattices L, in which setting the three types of semicontinuity coincide. The space of these semicontinuous functions is topologized. Major examples are the lattice of sup measures as considered by Vervaat, and the lattice of closed subsets of a locally quasicompact topological space.

Key words: upper semicontinuity, continuous lattice, lattice of closed subsets.

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0. Introduction

The concept of upper semi-continuous (usc) functions from a topological space E to R or $\bar{R} := [-\infty,\infty]$ is classical. Recently, interest has grown in the case where a more general space L replaces the range R or $\bar{R}$, with the complication that previously equivalent characterizations of upper semi-continuity split into different concepts.

Several results in this direction were obtained by Michel [1973], Penot & Théra [1979,1982] and Beer [1984]. Penot & Théra require L to be a preordered topological space, and in all their definitions the topology of L (besides that of E) plays a role. Their main motivation is application to optimization, with L a linear preordered topological space.

In the present paper it is assumed that L is a complete lattice, and one more definition of upper semi-continuity is given, suggested by Vervaat [1982,1985], for which no topology of L is needed. However, we are interested in the relation between the definitions of Michel, Penot & Théra and Vervaat, and here the topology of L comes in. As is to be expected, several conditions in the interplay between topology and lattice structure of L play a role in the results. This is what Sections 1-4 (and the Appendix) deal with.

In Section 5, the main aim of the present paper is presented: breaking the ground for a future analysis of random lattice-valued usc functions and lattice-valued extremal processes in the sense of Vervaat [1982] (cf. also Norberg [1985]). Here it will turn out why Vervaat's definition of semicontinuity is the most useful for us.

As a preparation for this future analysis, we study the following important example:

$$L = US(E,\bar{R}),$$

the space of usc functions from a topological space E to $\bar{R}$ (so in fact we study usc functions which take usc functions as their values!). A study of the structure of this space shows that it is a so-called continuous lattice. To continuous lattices a whole monograph, Gierz et al. [1980], is devoted. Several results on continuous lattices from the present paper can be found there as well. Nevertheless, we have tried to make this paper self-contained, because it is directed to researchers in probability (random closed sets, extremal processes) and optimization theory, for whom it is hard to find their way in the book, as we know by experience.

After a short introduction on continuous lattices, we prove that all forementioned notions of upper semicontinuity coincide if L is a continuous lattice (Section 6).

In Sections 7-10, we specialize to

$$L = \mathcal{F}(E \times L_2),$$

the space of all closed subsets of $E \times L_2$, where L_2 is another continuous lattice (Section 8), and to

$$L = US(E, L_2)$$

(Section 10), which brings us back to our starting point. Sections 7 and 9 do the preparing work.

The topology that we find on these spaces, starting from the lattice viewpoint, turns out to coincide with the sup vague topology as introduced by Fell ([1962]; see Section 7 for more references), under the assumption that E is locally quasicompact. In view of the future applications, we do not assume that E is Hausdorff.

List of notations

$\mathbb{N}$	$\{1,2,\ldots\}$
$\mathbb{N}_0$	$\{0,1,2,\ldots\}$
$\bar{\mathbb{R}}$	$[-\infty,\infty]$
A^c	complement of A
int A	interior of A
clos A	closure of A
$\mathcal{P}(E)$	$\{A : A \subset E\}$
$\mathcal{F}(E)$	$\{A : A \subset E, A$ closed$\}$
$\mathcal{G}(E)$	$\{A : A \subset E, A$ open$\}$
x_A	the function: $t \mapsto \begin{cases} x & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}$ $(x \in L, A \subset E)$
$\mathbf{0}$	lattice bottom
$\mathbf{1}$	lattice top
$\downarrow x$	$\{y \in L : y \leq x\}$
$\uparrow x$	$\{y \in L : y \geq x\}$
$\downarrow A$	$\bigcup_{x \in A} \downarrow x$

$\uparrow A$	$\bigcup_{x \in A} \uparrow x$
$x>y$	$x \geqslant y$ and $x \neq y$
$x<y$	$x \leqslant y$ and $x \neq y$
$x \| y$	$x \nleqslant y$ and $x \ngeqslant y$
$x \gg y$	if $B \subset L$ is filtered (see below) and $\inf B \leqslant y$, then there is a $z \in B$ with $z \leqslant x$
$\Downarrow x$	$\{y \in L: x \gg y\}$
$\Uparrow x$	$\{y \in L: y \gg x\}$
$\Downarrow A$	$\{y \in L: x \gg y$ for some $x \in A\}$
$E\uparrow$	the space E with the topology generated by the open sets $(\downarrow x)^c$
$E\Downarrow\uparrow$	the space E with the topology generated by the open sets $(\downarrow x)^c$ and $\Downarrow x$
$f(A)$	$\{f(t): t \in A\}$
$f^{\vee}(A)$	$\sup f(A)$
$\bigwedge A$	$\inf A$
$\bigvee A$	$\sup A$

Some definitions

quasicompact	finite open subcover property
compact	quasicompact + Hausdorff
$B \subset L$ is filtered	$B \cap \downarrow x \cap \downarrow y \neq \varnothing$ for all $x, y \in B$

1. Three types of semicontinuity

Let E be a topological space and L a complete lattice, provided with a topology. Nothing is required in advance about the relation between lattice structure and topology. We will formulate three notions of upper semicontinuity for functions $f: E \rightarrow L$, which all agree in the classical case $L=I$, I some closed interval in $\overline{\mathbb{R}}$. The first (and our main) one was suggested by Vervaat [1982].

Definition 1.1. *A function $f: E \to L$ is said to be **lattice upper semicontinuous (lat usc)** at $t \in E$, if $f(t) = \bigwedge \{f^{\vee}(G)$: G open, $t \in G\} =: f^{*}(t)$.*

Note that $f \leq f^{*}$ in general. As usual, f is said to be lat usc (on E) if it is at every $t \in E$. We can characterize this global semicontinuity using the notation

$$(1.1) \qquad f^{G}(t) := \begin{cases} f^{\vee}(G) & \text{if } t \in G \\ \mathbf{1} & \text{if } t \notin G \end{cases}$$

for every open $G \subset E$, where $\mathbf{1}$ is the top element of L (recall that L is complete). Obviously, f is lat usc iff $f = \bigwedge_{G} f^{G} =: f^{*}$.

The second definition is due to Penot & Théra [1979,1982], and involves a topological structure on L. Unfortunately, their research concentrates on lower rather than upper semicontinuity, so their results can be cited only after reversing the order.

Definition 1.2. *A function $f: E \to L$ is said to be **topologically upper semicontinuous (top usc)** at $t \in E$, if for every neighbourhood U of $f(t)$ there exists an open neighbourhood G of t such that $f(G) \subset \downarrow U$.*

If $\downarrow U$ is open for all open $U \subset L$, and τ is the topology on L, then τ_d: = $\{\downarrow U$: U open$\}$ defines a topology on L, the "decreasing topology generated by τ". In that case $f: E \to L$ is top usc iff $f: E \to (L, \tau_d)$ is continuous. This was observed by Penot & Théra [1982] and worked out by Beer [1984]; see the latter for more details.

For our last definition of upper semicontinuity, originally from Michel [1973], we need the notion of hypograph.

Definition 1.3A. *Set $L' := L \setminus \{\mathbf{0}\}$. The **hypograph** of a function $f: E \to L$ is the set*

$$\text{hypo } f := \{(t,x) \in E \times L' : x \leq f(t)\}.$$

The appearance of L' in this definition (in which it differs from Michel's) may be surprising, but its convenience will become clear later on. As a first hint, note that the bottom $\mathbf{0}_E$ of L^E: = $\{f: E \to L\}$ has the bottom $\varnothing$ of $\mathcal{P}(E \times L')$ as its hypograph.

Definition 1.3B. *A function* $f: E \to L$ *is said to be* ***hypo-closed*** *if* hypo f *is closed in* $E \times L'$, *where* L' *is provided with the restriction topology of* L.

To lay the connection with Penot & Théra's work, note that their hypograph is closed as subset of $E \times L$ iff ours is as subset of $E \times L'$, provided that $\{0\}$ is closed in L.

Finally, we denote by US_l, US_t and HC the spaces of all lat usc, top usc and hypo-closed functions, respectively.

2. Lattice properties of the function spaces

First we study closedness for taking arbitrary (pointwise) infima.

Theorem 2.1. *US_l and HC are closed for arbitrary infima.*

Proof. Consider $\{f_i: i \in I\}$ for some index set I, and set $f := \bigwedge_{i \in I} f_i$.
(i) Suppose $\{f_i: i \in I\} \subset US_l$, and let $t \in E$. For each i, $f_i(t) \geq f_i^*(t) \geq f^*(t)$. Taking the infimum over all i gives $f(t) \geq f^*(t)$.
(ii) Notice that hypo $f = \bigcap_{i \in I}$ hypo f_i.

□

For US_t, a condition is needed, even for finite infima. Corollary 1.9 of Penot & Théra [1982] can be sharpened slightly.

Lemma 2.2. *If $\wedge: L \times L \to L$ is top usc, then US_t is closed for finite infima.*

Proof. Let $f_1, f_2 \in US_t$, $f := f_1 \wedge f_2$, $t \in E$, U a neighbourhood of $f(t)$. Choose neighbourhoods U_i of $f_i(t)$ such that $x_1 \wedge x_2 \in \downarrow U$ for all $x_1 \in U_1$, $x_2 \in U_2$. There are open neighbourhoods G_1, G_2 of t such that $f_i(G_i) \subset \downarrow U_i$ for $i=1,2$. Let $G := G_1 \cap G_2$. Then $f(G) \subset \downarrow U$.

□

The infinite case is a sharpening, in both conditions, of Corollary 1.13 in Penot & Théra [1982]. Recall that a set $B \subset L$ is filtered if every pair of elements of B has a common lower bound in B.

Theorem 2.3. *If* $\wedge: L \times L \to L$ *is top usc and*

(2.1) $\inf B \in \text{clos } B$ *for each nonempty filtered* $B \subset L$,

then US_t *is closed for arbitrary infima.*

Proof. Let $\{f_i: i \in I\} \subset US_t$, $f := \bigwedge_{i\in I} f_i$. Let $t \in E$, and U be a neighbourhood of $f(t)$. Set $f_J := \bigwedge_{i\in J} f_i$ for every finite $J \subset I$ and $B := \{f_J(t): J \subset I \text{ finite}\}$. Then B is a filtered set in L and $\inf B = f(t)$, so $f(t) \in \text{clos } B$. It follows that there is a finite $J \subset I$ such that $f_J(t) \in U$. As $f_J \in US_t$ (by Lemma 2.2), we can find an open neighbourhood G of t such that $f_J(G) \subset \downarrow U$. So $f(G) \subset \downarrow U$.

□

In taking suprema, we must be more careful. About infinite suprema, nothing can be expected, as shows the classical case $L = \overline{\mathsf{R}}$. But even finite suprema need not inherit semicontinuity from their superands. The following example is instructive.

Example 2.4. Let $E = \{0\} \cup \{\frac{1}{n}: n \in \mathsf{N}\}$, with the restriction topology of R, and let $L \subset \mathsf{R}^2$ be the set $L := \{(0,0),(1,0),(1,1)\} \cup \{(\frac{1}{n},1): n \in \mathsf{N}\}$, with the restriction topology of R^2 and componentwise ordering. Define $f, g: E \to L$ by $f(0) = (0,0)$, $f(\frac{1}{n}) = (\frac{1}{n},1)$ for all n, $g(t) = (1,0)$ for all $t \in E$. Then $(f \vee g)(0) = (1,0)$, $(f \vee g)(t) = (1,1)$ if $t>0$. Now f and g belong to US_l and HC, but $f \vee g$ belongs to neither of these. Unfortunately, $f \notin US_t$, so the example says nothing about this class.

□

We will restrict our attention now to US_t and US_l, as for HC no nice results seem to exist. For US_t, Penot & Théra [1982, Corollary 1.9] give a result, which we again sharpen slightly.

Theorem 2.5. *If* $\downarrow U$ *is open for all open* U *and if the mapping* $(x,y) \mapsto x \vee y$ *is top usc at every point of* $\{(x,x): x \in L\}$, *then* US_t *is closed for finite suprema.*

Proof. Let $f_1, f_2 \in US_t$ and $f := f_1 \vee f_2$. Let $t \in E$ and $U \subset L$ be a neighbourhood of $f(t)$. As $f(t) = f(t) \vee f(t)$, we can find open neighbourhoods V_1 and V_2 of $f(t)$ such that $x \vee y \in \downarrow U$ for all $x \in V_1, y \in V_2$. Set $U_1 := V_1 \cap V_2$. Now $\downarrow U_1$ is an open neighbourhood of $f(t)$, hence also of $f_i(t)$ $(i=1,2)$. So we can find open neighbourhoods G_i of t such that $f_i(G_i) \subset \downarrow U_1$. Set $G := G_1 \cap G_2$. For $s \in G$, we have $f_1(s), f_2(s) \in \downarrow U_1$, so $f(s) = f_1(s) \vee f_2(s) \in \downarrow U$.

□

Replacing the conditions of Theorem 2.5 by one sufficient condition we find:

Corollary 2.6. *If the mapping* $(x,y) \mapsto x \vee y$ *is top usc at every point of* $\{(x,y) \in L \times L : x \leq y\}$, *then* US_t *is closed for finite suprema.*

In order to deal with US_l, we adopt Definition 0.4.1 from Gierz et al. [1980].

Definition 2.7. *A lattice* L *is called* ***join-continuous*** *if for every* $x \in L$ *and every filtered set* $B \subset L$ *the following holds:*

$$x \vee (\textstyle\bigwedge_{y \in B} y) = \bigwedge_{y \in B}(x \vee y).$$

Theorem 2.8. *If* L *is join-continuous, then* US_l *is closed for finite suprema.*

Proof. Let $f_1, f_2 \in US_l$, $f := f_1 \vee f_2$, and $t \in E$. Observe that $\{f_i^{\vee}(G) : G \text{ open}, t \in G\}$ is a filtered set in L. The following is straightforward (with G, G_1, G_2 varying through the open neighbourhoods of t):

$$f(t) = f_1(t) \vee f_2(t) = (\textstyle\bigwedge_{G_1} f_1^{\vee}(G_1)) \vee (\bigwedge_{G_2} f_2^{\vee}(G_2)) =$$

$$= \textstyle\bigwedge_{G_2}((\bigwedge_{G_1} f_1^{\vee}(G_1) \vee f_2^{\vee}(G_2)) = \bigwedge_{G_2}\bigwedge_{G_1}(f_1^{\vee}(G_1) \vee f_2^{\vee}(G_2)) \geq$$

$$\geq \textstyle\bigwedge_{G_2}\bigwedge_{G_1} f^{\vee}(G_1 \cap G_2) = \bigwedge_G f^{\vee}(G).$$

□

3. Comparison of the function spaces for two-valued functions

Throughout this section, we will consider functions $f: E \to L$ which take at most two different values. For a function space S as defined in Section 1, we set ${}_2S$ for the subspace of those one-or-two-valued functions. Our final result will be: ${}_2HC \subset {}_2US_l \subset {}_2US_t$, where in general no inclusion can be reversed. Conditions under which they can will also be given.

To begin with, note that all one-valued (=constant) functions belong to these classes.

As US_l is our main class of interest, we first characterize the (nontrivial) elements of ${}_2US_l$.

Lemma 3.1. *Let $f: E \to L$ be two-valued:*

(3.1) $f = x_A \vee y_{A^c}$ $(x \neq y,\ A, A^c \neq \varnothing)$.

Then $f \in {}_2US_l$ iff one of the following three conditions holds: (i) $x > y$ *and A is closed;* (ii) $x < y$ *and A is open;* (iii) $x \| y$ *and A is clopen.*

Proof. A quick argument shows that the theorem can be restated in the following way:
(a) if $x \not\geq y$, then: A is open iff f is lat usc at each point of A;
(b) if $x \not\leq y$, then: A is closed iff f is lat usc at each point of A^c.
We only prove (a). So let $x \not\geq y$. Then A is open $\Longleftrightarrow$ every $t \in A$ has an open neighbourhood $G \subset A$ $\Longleftrightarrow$ every $t \in A$ has an open neighbourhood G with $f^{\vee}(G) = x$ $\Longleftrightarrow$ for every $t \in A$, $f^{*}(t) \leq x$ $\Longleftrightarrow$ f is lat usc at every point of A.

□

An easy consequence of this theorem is the following.

Corollary 3.2. *Let f be two-valued. Then $f \in {}_2US_l$ iff $f^{\leftarrow}(\uparrow x)$ is closed for each $x \in L$.*

For ${}_2HC$ and ${}_2US_t$ we cannot give such explicit characterizations as for ${}_2US_l$ in Lemma 3.1, so we concentrate on comparing both of them with ${}_2US_l$. We first compare ${}_2US_l$ and ${}_2HC$.

Lemma 3.3. ${}_2HC \subset {}_2US_l$.

Proof. Let $x \neq \mathbf{0}$. Observe that

$$\text{(3.2)} \qquad \text{hypo}\, f \cap (E \times \{x\}) = f^{\leftarrow}(\uparrow x) \times \{x\}.$$

If hypo f is closed in $E \times L'$, then both sides of (3.2) are closed in $E \times \{x\}$, so $f^{\leftarrow}(\uparrow x)$ is closed in E. Apply Corollary 3.2.

□

Example 3.4. ${}_2US_l \not\subset {}_2HC$. Let L have the trivial topology $\{\emptyset, L\}$ and consist of at least three points: $\mathbf{0}, x, \mathbf{1}$. Then the function x_E belongs to ${}_2US_l$ but not to ${}_2HC$.

□

For a condition under which ${}_2US_l \subset {}_2HC$, we need the following definition.

Definition 3.5. *L has closed lower (upper) point shadows if $\downarrow x$ ($\uparrow x$) is closed for every $x \in L$.*

Remark. Penot & Théra [1982] call the order on L semi-closed if L has closed lower and upper point shadows. From Nachbin [1965, Prop. 1.2] we quote: if the order on L is closed (i.e., its graph is closed in L^2), then L has closed lower and upper point shadows. However, the order being closed forces L to be Hausdorff (cf. Nachbin), which is too strong for our purposes.

Lemma 3.6. *If L has closed lower point shadows, then ${}_2US_l \subset {}_2HC$.*

Proof. Note that $L' = L \backslash \{\mathbf{0}\}$ has the restriction topology of L, so $(\downarrow x)' := (\downarrow x) \backslash \{\mathbf{0}\}$ is closed in L' iff $\downarrow x$ is in L. Write f as in (3.1), and observe that

$$\text{(3.3)} \qquad \text{hypo}\, f = (A \times (\downarrow x)') \cup (A^c \times (\downarrow y)').$$

We consider the three cases in Lemma 3.1. (i) Suppose $x > y$ and A closed. Then hypo $f = (A \times (\downarrow x)') \cup (E \times (\downarrow y)')$, which is closed in $E \times L'$. (ii) The case $x < y$ and A open is similar. (iii) The case $x \| y$ and A clopen follows immediately from (3.3) as it stands.

□

Having finished comparing ${}_2US_l$ with ${}_2HC$, we now turn to ${}_2US_t$.

Lemma 3.7. ${}_2US_l \subset {}_2US_t$.

Proof. Let f be as in (3.1), let $t \in A$ (say) and U be a neighbourhood of $f(t)=x$. We must find an open neighbourhood G of t such that $f(G) \subset \downarrow U$. If A is open, take $G=A$. If A is not open, then by Lemma 3.1 $x>y$, so any G will do.

□

Example 3.8. ${}_2US_t \not\subset {}_2US_l$. Let $L\uparrow$ denote the complete lattice L, provided with the **upper topology** ,i.e., the nonempty closed subbase sets are $\downarrow x$ (so it is the coarsest topology with closed lower point shadows). As all nonempty open sets in $L\uparrow$ are increasing, their lower shadows coincide with L, so every $f: E \to L$ is top usc. But the definition of US_l is topology-free for L, so there are many non-lat usc functions, for example $f = \mathbf{1}_A$ for every nonclosed $A \subset E$.

□

Lemma 3.9. *If L has closed upper point shadows, then* ${}_2US_t \subset {}_2US_l$.

Proof. Let f be as in (3.1). In view of Lemma 3.1, it suffices to prove: (a) if $x \not\geq y$, then A is open; (b) if $x \not\leq y$, then A is closed. We only show (a). So let $t \in A$. Then $f(t)=x$. As $\uparrow y$ is closed and $x \notin \uparrow y$, there is a neighbourhood U of x with $U \cap \uparrow y = \emptyset$. Choose an open neighbourhood G of t such that $f(G) \subset \downarrow U$. As $y \notin \downarrow U$, $f(G) = \{x\}$, so $G \subset A$.

□

We collect our results:

Theorem 3.10. (i) ${}_2HC \subset {}_2US_l \subset {}_2US_t$.
(ii) *If L has closed lower point shadows, then* ${}_2HC = {}_2US_l$.
(iii) *If L has closed upper point shadows, then* ${}_2US_l = {}_2US_t$.

4. Comparison of the full function spaces

In this section we study relations between the function spaces US_l, US_t and HC. In general, these spaces are different, and, in contrast to the two-valued case, no general inclusions hold. Examples of functions belonging to exactly one or two of these spaces are given in the Appendix. Here we give conditions on L under which some inclusions hold. First we handle the relations between US_l and US_t.

Theorem 4.1. *Suppose*

(4.1) $\quad \inf B \in \operatorname{clos} B$ *for each nonempty filtered* $B \subset L$.

Then $US_l \subset US_t$.

Proof. Let $t \in E$, f lat usc at t. Let U be a neighbourhood of $f(t)$. Define $B := \{f^{\vee}(G)\colon G \text{ open}, t \in G\}$. As B is filtered and $\inf B = f(t)$, it follows that $f(t) \in \operatorname{clos} B$. Thus $U \cap B \neq \varnothing$, so there exists an open neighbourhood G of t such that $f^{\vee}(G) \in U$. Hence $f(G) \subset \downarrow U$.

□

Theorem 4.2. *Suppose*

(4.2) $\quad x = \inf \{\sup U\colon U \text{ open}, x \in U\}$ *for all* $x \in L$.

Then $US_t \subset US_l$.

Proof. Let $t \in E$, f top usc at t. For every neighbourhood U of $f(t)$ there is an open neighbourhood G of t such that $f(G) \subset \downarrow U$, whence $f^{\bullet}(t) \leqslant f^{\vee}(G) \leqslant \sup \downarrow U = \sup U$. So $f^{\bullet}(t) \leqslant \inf \{\sup U\colon U \text{ open}, f(t) \in U\} = f(t)$.

□

Remark. Another way of stating (4.2) is: the identity mapping : $L \to L$ is lat usc.

Next we turn to the relations between US_t and HC, which were found by Penot & Théra [1982, Prop. 1.3].

Proposition 4.3. (a) *If the order of* L *is closed, then* $US_t \subset HC$.

(b) *Let* {**0**} *be closed in* L. *If* $(\downarrow U)^c$ *is quasicompact for every nonempty open* $U \subset L$, *then* $HC \subset US_l$.

We had to add the condition on {**0**} because of the slight difference between our definitions and those of Michel and Penot & Théra (cf. Section 1). In (a), we do not need to add this condition, as it is guaranteed by the closed order of L, implying Hausdorffness (cf. Section 3).

What remains are the relations between US_l and HC. For $HC \subset US_l$, it seems that no essentially shorter way can be found than tying Theorem 4.2 and Proposition 4.3(b) together. The converse however is handled in a rather different way, using the results of the previous section on two-valued functions. The crucial step is the following:

Lemma 4.4. *US_l is the smallest function space which both contains $_2US_l$ and is closed for arbitrary infima.*

Proof. Let Σ be the smallest function space as indicated. By Theorem 2.1, $\Sigma \subset US_l$. It remains to prove that $US_l \subset \Sigma$. So let $f \in US_l$. For every open G in E, define f^G as in (1.1). Note that every f^G belongs to Σ: if $f^\vee(G) = \mathbf{1}$, then f^G is constant, and if $f^\vee(G) < \mathbf{1}$, then $f^G \in {}_2US_l$ by Lemma 3.1. As f is lat usc, $f = \bigwedge_G f^G$. So $f \in \Sigma$.

□

Combining Theorem 2.1, Lemma 3.6 and Lemma 4.4, we find:

Theorem 4.5. *If L has closed lower point shadows, then $US_l \subset HC$.*

5. Sup measures

The present section on supremum measures (sup measures) gives a motivation for our definition of upper semicontinuity, as the concept of lattice upper semicontinuity fits in a natural way in the theory of sup measures, introduced by Vervaat [1982]. On the other hand, this section explains our interest in continuous lattices, as this concept helps us to generalize Vervaat's results. Continuous lattices will be the main topic of the remaining sections.

Let E be a topological space, again, and $\mathcal{G}$ the collection of its open sets. Vervaat introduced the concept of $\bar{\mathbb{R}}$-valued sup measures $m: \mathcal{G} \to \bar{\mathbb{R}}$. We generalize some of his results to L-valued sup measures $m: \mathcal{G} \to L$, where L is a complete lattice (without a topology). In the beginning, this causes only few changes.

Definition 5.1. (a) *For every function $f: E \to L$, its **sup integral** $f^\vee: \mathcal{G} \to L$ is defined by*

$$f^\vee(G) := \sup \{f(t): t \in G\}.$$

(b) *For every function $m: \mathcal{G} \to L$, its **sup derivative** $d^\vee m: E \to L$ is defined by*

$$d^\vee m(t) := \inf \{m(G): t \in G\}.$$

In the following lemma the only nontrivial part is (c), which, however, follows immediately from (a) and (b).

Lemma 5.2. (a) *For every $m: \mathcal{G} \to L$, $m \geq (d^\vee m)^\vee$.*
(b) *For every $f: E \to L$, $f \leq d^\vee f^\vee$.*
(c) *For every $m: \mathcal{G} \to L$, $d^\vee m$ is lat usc.*
(d) *For every $f: E \to L$, $d^\vee f^\vee$ is the smallest lat usc majorant of f.*

The way to generalize sup measures is self-evident.

Definition 5.3. *A function $m: \mathcal{G} \to L$ is called a **sup measure** if*

$$m(\textstyle\bigcup_{j \in J} G_j) = \bigvee_{j \in J} m(G_j)$$

for every collection $\{G_j: j \in J\} \subset \mathcal{G}$.

Of course, every sup integral $f^\vee$ is a sup measure. In Vervaat [1982] it is proved that for $L = \bar{\mathbb{R}}$ every sup measure can be written as a sup integral, by showing the following: $m: \mathcal{G} \to \bar{\mathbb{R}}$ is a sup measure iff $m = (d^\vee m)^\vee$. Trying to generalize this result, we encounter the first problem. The relevant inequality in Theorem 2.5 of Vervaat [1982] is proved by showing that all *strict* majorants of one side majorize the other side, so the other side is majorized by the infimum of all strict majorants of the first side as $L = \bar{\mathbb{R}}$. So what we need for our general complete lattice L is some strict inequality relation with the property that every lattice element equals the infimum of all its strict majorants.

Studying in detail the concrete inequality that is to be proved, tells us that we need the following.

Definition 5.4. *In a complete lattice L, x is said to be **way above** y, in symbols $x \gg y$, if for each filtered set B with* inf $B \leq y$, *there is a $z \in B$ such that $z \leq x$. We write*

$$\Downarrow x := \{y \in L : x \gg y\}, \ \Uparrow x := \{y \in L : y \gg x\}.$$

Caution. The definition of "way above" is strongly asymmetric. In general, the corresponding "way below" relation $y \ll x$, based on directed sets (i.e., the reverse order analogue of filtered sets) D with sup $D \geq x$ need not coincide with the above relation!

These concepts, and much of the further terminology in this section, are borrowed from Gierz et al. [1980], who, however, develop their theory concentrating on the way below relation, which is the wrong choice for our purposes. For that reason, we will sometimes quote explicitly some of their results in our setting. Whenever we want to discriminate between the two, we insert the affix "lower" to the notions in the context of Gierz et al., and "upper" in ours.

We already discussed the need of the following.

Definition 5.5. *A complete lattice L is called **(upper) continuous** if $x =$* inf *$\Uparrow x$ for all $x \in L$.*

Usually, in whatever context, the word "continuity" is defined in terms of a topology. Notice that the notion of lattice continuity, however, does *not* involve a topology on the lattice. For the choice of this terminology, see Scott [1972].

Finally, we can give the analogue of Vervaat's Theorem 2.5 [1982].

Theorem 5.6. *Let L be a continuous lattice and $m : \mathcal{G} \to L$.*
(a) *If m is a sup measure, then $\bigvee_{t \in A} d^\vee m(t) = \bigwedge_{G: A \subset G} m(G)$ for all $A \subset E$.*
(b) *m is a sup measure iff $m = (d^\vee m)^\vee$.*

Proof. (a) Obviously $\bigvee_{t \in A} d^\vee m(t) \leq \bigwedge_{G: A \subset G} m(G)$. We will prove: $\Uparrow \bigvee_{t \in A} d^\vee m(t) \subset \uparrow \bigwedge_{G: A \subset G} m(G)$, which gives the converse inequality by lattice continuity. So let $x \gg \bigvee_{t \in A} d^\vee m(t)$, and set $B_t := \{m(G) : t \in G,\ G \text{ open}\}$ for every $t \in A$. Then B_t is a filtered set and $x \gg d^\vee m(t) = \inf B_t$, so there is an open neighbourhood G_t of t with $m(G_t) \leq x$. Set $G_0 := \bigcup_{t \in A} G_t$; then $A \subset G_0$ and $m(G_0) = \bigvee_{t \in A} m(G_t) \leq x$. So $x \geq \bigwedge_{G: A \subset G} m(G)$.

(b) The "if" part is trivial, the "only if" part is a special case of (a) for open A.

□

Corollary 5.7. *Let L be a continuous lattice and let $m: \mathcal{G} \to L$ be a sup measure and $\Sigma := \{f: E \to L: f^{\vee} = m\}$.*

(a) $d^{\vee}m \in \Sigma$;

(b) $d^{\vee}m = \sup \Sigma$;

(c) *if $f \in \Sigma$ is lat usc, then $f = d^{\vee}m$.*

Proof. (a) follows from Theorem 5.6, (b) from Lemma 5.2(b), while (c) is straightforward.

□

6. Continuous lattices and the Lawson topology

In Section 5 we introduced the notions of way above relation and continuous lattice. Here we give some elementary properties. Most of them can be found in dual form in Gierz et al. [1980].

The first proposition deals with the way above relation, and L is not assumed to be a continuous lattice. Its proof follows immediately from the definitions.

Proposition 6.1. *For all $x, y, z, w \in L$ we have:*

(a) *If $x \gg y$, then $x \geq y$;*

(b) *If $w \geq x \gg y \geq z$, then $w \gg z$;*

(c) *If $x \gg z$ and $y \gg z$, then $x \wedge y \gg z$;*

(d) *$x = \inf \Uparrow x$ iff $x \not\geq y$ implies the existence of a $z \gg x$ with $z \not\geq y$.*

The following lemma shows that in a continuous lattice the way above relation satisfies some interpolation property.

Lemma 6.2. *Let L be a continuous lattice. If $x \gg z$, then there is a y with $x \gg y \gg z$.*

Proof. Let $B := \{u \in L : u \gg y \gg z \text{ for some } y \in L\}$. We must show that $x \in B$. By Proposition 6.1(b,c), B is an increasing filtered set. Of course, $z^* := \inf B \geq z$. Now suppose $z^* \not\leq z$. Applying Proposition 6.1(d) twice, we find a $y \gg z$ with $y \not\geq z^*$ and a $u \gg y$ with $u \not\geq z^*$. Thus $u \in B$, so $u \geq \inf B = z^*$, a contradiction. So $z^* = z$.
It follows that $x \gg \inf B$, and as B is filtered, there is a $w \in B$ such that $x \geq w$. So $x \in \uparrow B = B$.

□

Corollary 6.3. *Let L be a continuous lattice. If $x \gg y$, then for every filtered set $B \subset L$ with $\inf B \leq y$, there is a $z \in B$ with $x \gg z$.*

Proof. By Lemma 6.2, there is a $w \in L$ such that $x \gg w \gg y$. From $w \gg y$ follows the existence of a $z \in B$ with $z \leq w$, so $x \gg z$.

□

The goal of this section is to study the results of the preceding sections on US_l, US_t and HC. Most of these require a topology on L, except one, which is the topic of the next theorem.

Theorem 6.4. *If L is a continuous lattice, then US_l is closed for finite suprema.*

Proof. By Theorem 2.8, it suffices to show that lattice continuity implies lattice join-continuity. So let $x \in L$, $B \subset L$ a filtered set; we must show

(6.1) $$x \vee \inf B \geq \inf\{x \vee y : y \in B\} =: w$$

(note that the reverse inequality is trivial).
Suppose (6.1) does not hold. By Proposition 6.1(d), there is a $z \gg x \vee \inf B$ with $z \not\geq w$. As $\inf B \leq x \vee \inf B$, there is a $y \in B$ with $y \leq z$. So $z \geq x \vee y \geq w$, a contradiction. Thus (6.1) is proved.

□

The time has come to provide our continuous lattice with a topology. It is the upper version of what Gierz et al. [1980] call the Lawson topology.

Definition 6.5. *Let L be a continuous lattice. The **(upper) Lawson topology** is defined by its subbase elements $\Downarrow x$ and $(\downarrow x)^c$, for all $x \in L$. Notation for this topological space: $L \Downarrow \uparrow$.*

To get an idea, note that $\bar{R}⇓↑$ is $\bar{R}$ with the usual topology, as both $(\downarrow x)^c = (x,\infty]$ and $⇓x = [-\infty,x)$ are open, the latter by Theorem 6.6. For general L, a generic base element is given by $⇓x \cap \bigcap_{k=1}^{n} (\downarrow x_k)^c$, for some $n \in N_0$, $x, x_1, x_2, \ldots, x_n \in L$ (note that $⇓1 = L$).

Theorem 6.6. *If L is a continuous lattice, then $L⇓↑$ has closed order. In particular, it is a Hausdorff space and has closed point shadows.*

Proof. That closed order implies the last two statements, was observed in previous sections, so only the closed order remains to be proved. Let $\Gamma := \{(x,y): x \leq y\}$ be the graph of the order and let $(x,y) \notin \Gamma$, so $y \ngeq x$. By Proposition 6.1(d), there is a $z \gg y$ with $z \ngeq x$. Now $(\downarrow z)^c \times ⇓z$ is a neighbourhood of (x,y), disjoint from Γ.

□

The structure of $L⇓↑$ has one more nice feature:

Theorem 6.7. *If L is a continuous lattice, then $L⇓↑$ is a compact space.*

Proof. We use Alexander's lemma: it suffices to select a finite subcover from each open subbase cover of L. (i) Let $(\downarrow y)^c \cup \bigcup_{i \in I} ⇓x_i = L$ for some index set I (at most one $(\downarrow y)^c$ suffices, as $\bigcup_j (\downarrow y_j)^c = (\downarrow \bigwedge_j y_j)^c$). As $y \in L$ and $y \notin (\downarrow y)^c$, there is an $i \in I$ such that $y \in ⇓x_i$, so $\downarrow y \subset ⇓x_i$, hence $\{⇓x_i, (\downarrow y)^c\}$ is a finite subcover of L. (ii) The cases, where the cover consists of only one type of subbase elements, are trivial.

□

We continue our research on the classes US_l, US_t and HC.

Theorem 6.8. *Let L be a continuous lattice with the Lawson topology. Then $US_t = US_l$.*

Proof. A direct proof is not difficult and uses Proposition 6.1(d) and Lemma 6.2. But the results of Section 4 allow a shorter proof.

(a) For $US_l \subset US_t$, we use Theorem 4.1, checking (4.1). So let $B \subset L$ be filtered, $x := \inf B$ and let U be a neighbourhood of x. We may assume that $U = ⇓y \cap \bigcap_{k=1}^{n} (\downarrow y_k)^c$ for some $n \in N_0$, $y, y_1, \ldots, y_n \in L$. As $y \gg x$, Corollary 6.3 gives a $z \in B$ with $y \gg z$. From $z \geq x$ and $x \nleq y_k$, we see that $z \nleq y_k$ for every k. So $z \in B \cap U$.

(b) For $US_t \subset US_l$, we use Theorem 4.2, checking (4.2):

inf {sup U: U open, $x \in U$} $\leq$ inf {sup $\Downarrow z$: $z \gg x$} $\leq$ inf {z: $z \gg x$} = inf $\Uparrow x$ = x
(the reverse inequality holds trivially).

□

The following lemma, needed to prove the last result of this section, goes back to the remark following Definition 1.2.

Lemma 6.9. *In a continuous lattice with the Lawson topology, $\downarrow U$ is open for every open U.*

Proof. We may assume that U is an open base set, i.e., $U = \Downarrow x \cap \bigcap_{j=1}^{n} (\downarrow x_j)^c$ for some $n \in \mathbb{N}_0$, $x, x_1, \ldots, x_n \in L$. Let $z \in \downarrow U$, so there is a $u \in U$ with $z \leq u$. As $x \gg u$, there is a y with $x \gg y \gg u$ (Lemma 6.2). Note that $y \in U$, so $z \in \Downarrow y \subset \downarrow U$.

□

Theorem 6.10. *Let L be a continuous lattice with the Lawson topology. Then $US_t = HC$.*

Proof. We check the conditions in Proposition 4.3.
(a) Already done in Theorem 6.6.
(b) First note that {0} is closed in L. Now let $U \subset L$ be open. By Theorem 6.7 and Lemma 6.9, $(\downarrow U)^c$ is a closed subset of a quasicompact space, hence itself quasicompact.

□

7. The lattice of closed subsets of a topological space

Let D be a **locally quasicompact** space, i.e., each point in D has a neighbourhood base of quasicompact sets. In absence of Hausdorffness, it is not sufficient that each point has some quasicompact neighbourhood. Even a quasicompact space need not be locally quasicompact.

In this section $L = \mathcal{F}(D)$ is the (complete) lattice of closed subsets of D. The order in $\mathcal{F}(D)$ is set inclusion, the infimum corresponds with the intersection, whereas the supremum corresponds with the closure of the union.

One can prove that $\mathcal{F}(D)$ is a continuous lattice iff it is lattice isomorphic to $\mathcal{F}(D')$ for some locally quasicompact topological space D' (cf. Gierz et al. [1980], Hofmann & Mislove [1981]). It is not necessary that D itself is locally quasicompact, although examples are hard to find. Hofmann & Lawson [1978, p. 304] construct one by the axiom of choice.

In Gierz et al. [1980] the lattice $L = \mathcal{G}(D)$ of open subsets of D is studied. Of course, properties of $\mathcal{F}(D)$ are dual to those of $\mathcal{G}(D)$. Nevertheless it is instructive to give some proofs, in order to get used to the specific meaning of the strict inequality relation $\gg$ in this type of lattice.

The first lemma is the dual of III.1.13 in Gierz et al. [1980].

Lemma 7.1. *Let D be a locally quasicompact space. For $F_1, F_2 \in \mathcal{F}(D)$, we have $F_1 \gg F_2$ iff there is a quasicompact $K \subset D$ such that $F_2 \subset K^c \subset F_1$.*

Proof. For the "if" part, let $\mathcal{B}$ be a filtered subset of $\mathcal{F}(D)$ and $B_0 := \bigcap_{B \in \mathcal{B}} B = \inf \mathcal{B} \subset F_2$. Then $K \subset F_2^c \subset B_0^c = \bigcup_{B \in \mathcal{B}} B^c$, an open cover of K, so there is a finite $\mathcal{B}_0 \subset \mathcal{B}$ such that $K \subset \bigcup_{B \in \mathcal{B}_0} B^c = (\bigcap_{B \in \mathcal{B}_0} B)^c$. As $\mathcal{B}$ is filtered, there is a $B_1 \in \mathcal{B}$ with $B_1 \subset \bigcap_{B \in \mathcal{B}_0} B \subset K^c \subset F_1$, which proves that $F_1 \gg F_2$.
The "only if" part uses the local quasicompactness of D. Suppose $F_1 \gg F_2$. As F_2^c is open, there is for every $t \in F_2^c$ a quasicompact K_t with $t \in \text{int } K_t \subset K_t \subset F_2^c$. Now $F_2^c = \bigcup_{t \in F_2^c} \text{int } K_t$, so $F_2 = \bigcap_{t \in F_2^c} (\text{int } K_t)^c$ is the infimum of the filtered set $\{\bigcap_{t \in A} (\text{int } K_t)^c : A \subset F_2^c \text{ finite}\}$. As $F_1 \gg F_2$, there is a finite $A \subset F_2^c$ such that $(\bigcup_{t \in A} \text{int } K_t)^c = \bigcap_{t \in A} (\text{int } K_t)^c \subset F_1$. It follows that $K := \bigcup_{t \in A} K_t$ satisfies $F_2 \subset K^c \subset F_1$.

□

The following, important, result is an easy consequence now.

Theorem 7.2. *If D is locally quasicompact, then $\mathcal{F}(D)$ is a continuous lattice.*

Proof. Let $F \in \mathcal{F}(D)$ and set $F_1 := \inf \Uparrow F = \bigcap \{H \in \mathcal{F}(D) : F \subset H \text{ and } F \subset K^c \subset H$ for some quasicompact $K\}$. Of course, $F \subset F_1$. For $F_1 \subset F$, let $t \in F^c$. As D is locally quasicompact, there is a quasicompact K such that $t \in \text{int } K \subset K \subset F^c$. Set $H := (\text{int } K)^c$, then $F \subset K^c \subset H$ and $t \notin H$, so $t \notin F_1$.

□

We are going to topologize $\mathcal{F}(D)$ by means of the Lawson topology. The resulting topology turns out to be well-known.

Theorem 7.3. *If D is locally quasicompact, then the Lawson topology on $\mathcal{F}(D)$ coincides with the so-called* ***sup vague topology****, generated by the open subbase sets*

(7.1) $\{F \in \mathcal{F}(D)\colon F \cap G \neq \varnothing\}$, *for G open*

and

(7.2) $\{F \in \mathcal{F}(D)\colon F \cap K = \varnothing\}$, *for K quasicompact.*

Proof. As the Lawson topology on $\mathcal{F}(D)$ is generated by its subbase elements $(\downarrow F)^c$ and $\Downarrow F$ for all $F \in \mathcal{F}(D)$, it suffices to show:
(a) the sets $(\downarrow F)^c$ generate the same topology as the sets (7.1);
(b) the sets $\Downarrow F$ generate the same topology as the sets (7.2).
(a) For $F_0 \in \mathcal{F}(D)$, $(\downarrow F_0)^c = \{F\colon F \notin \downarrow F_0\} = \{F\colon F \not\subset F_0\} = \{F\colon F \cap F_0^c \neq \varnothing\}$.
(b) Let $F_0 \in \mathcal{F}(D)$. Then
$\Downarrow F_0 = \{F\colon F_0 \gg F\} = \{F\colon F \subset K^c \subset F_0$ for some quasicompact $K\}$
$= \{F\colon F \cap K = \varnothing,\ F_0 \cup K = D$ for some quasicompact $K\}$
$= \bigcup_{K\colon F_0 \cup K = D} \{F\colon F \cap K = \varnothing\}$, which is open in the topology generated by the sets in (7.2).
Conversely, we will prove that $\{F\colon F \cap K = \varnothing\} = \bigcup_{F\colon F \cap K = \varnothing} \Downarrow F$. For the $\subset$ part (the other part is obvious), let $F_0 \cap K = \varnothing$. As F_0^c is open and $K \subset F_0^c$, we can find for every $t \in K$ a quasicompact K_t with $t \in \text{int}\, K_t \subset K_t \subset F_0^c$. Now K is quasicompact and $K \subset \bigcup_{t \in K} \text{int}\, K_t$, so there is a finite $K_0 \subset K$ with $K \subset \bigcup_{t \in K_0} \text{int}\, K_t$. Set $K_1 := \bigcup_{t \in K_0} K_t$ and $F := (\bigcup_{t \in K_0} \text{int}\, K_t)^c$, then $F \in \mathcal{F}(D)$, $F \cap K = \varnothing$, $F^c \subset K_1$, K_1 is quasicompact and $K_1 \subset F_0^c$, so $F_0 \subset K_1^c \subset F$, which proves that $F \gg F_0$, so $F_0 \in \Downarrow F$.

□

The sup vague topology on closed sets was studied first by Fell [1962]. For Hausdorff D, many authors worked on it, as Matheron [1975], Salinetti & Wets [1985+] and Norberg [1985]. For non-Hausdorff D see the revision of Vervaat [1982]. Note that for compact metric D, the sup vague topology coincides with the well-known Hausdorff topology.

8. A topology on US_l

Let E be a locally quasicompact space and L a complete lattice. In this section we want to topologize the space US_l of all lat usc functions $f: E \to L$. To this end, we use the mapping hypo (Definition 1.3A), which assigns to every $f \in US_l$ a subset of $E \times L'$ ($L' := L \backslash \{0\}$).

If $L = \bar{\mathbb{R}}$, hypo is a lattice isomorphism between US_l and a subspace of $\mathcal{P}(E \times \bar{\mathbb{R}}')$, by which we mean: a bijection which takes arbitrary infima and finite suprema into arbitrary intersections and finite unions, respectively. A particularly nice feature of this isomorphism is that its image hypo(US_l) is closed under arbitrary intersections and finite unions, so it defines a cotopology, i.e., the closed sets of a topology. It turns out that we can identify hypo(US_l) with $\mathcal{F}(E \times \bar{\mathbb{R}}' \uparrow)$, where $\uparrow$ denotes the upper topology as defined in Example 3.8. Thus one can topologize US_l with the induced sup vague (Lawson) topology on $\mathcal{F}(E \times \bar{\mathbb{R}}' \uparrow)$, introduced in Section 7. More about this can be found in Vervaat [1982].

Our goal is to generalize the above for arbitrary, complete lattices L. If L is not totally ordered, problems arise: in general, hypo $f \cup$ hypo g is a larger set than hypo $(f \vee g)$. For example, if $L = \bar{\mathbb{R}}^2$, $f = (0,1)_E$, $g = (1,0)_E$, then $f \vee g = (1,1)_E$, so hypo $(f \vee g) = E \times \downarrow'(1,1)$, (where, for a moment, $\downarrow' x$ denotes $\downarrow x \cap L'$), whereas hypo $f \cup$ hypo $g = E \times [\downarrow'(0,1) \cup \downarrow'(1,0)]$, which is not the hypograph of any function, as $\downarrow'(0,1) \cup \downarrow'(1,0)$ can not be written as $\downarrow' x$ for any $x \in \bar{\mathbb{R}}^2$.

This example suggests us to restrict the space $\mathcal{F}(E \times L' \uparrow)$ together with its topology to the subspace $\mathcal{F}_0$ of those $F \in \mathcal{F}(E \times L' \uparrow)$ which satisfy

(8.1) for all $t \in E$ there exists a $y \in L$ such that $\{x \in L' : (t,x) \in F\} = \downarrow' y$.

Note that in (8.1), y is an element of L, rather than L', for we want $\varnothing$ to be a member of $\mathcal{F}_0$.

It is easily verified that $\mathcal{F}_0$ is closed for arbitrary infima, from which it follows that $\mathcal{F}_0$ is a complete lattice (see, for example, 0.2.2 in Gierz et al. [1980]). However, the $\mathcal{F}_0$-supremum of two elements F_1, F_2 of $\mathcal{F}_0$ is no longer the union of both sets, but rather:

$$\text{hypo}\,[(\text{hypo}^{\leftarrow} F_1) \vee (\text{hypo}^{\leftarrow} F_2)].$$

Summarizing:

Theorem 8.1. *The mapping* hypo: $US_l \to \mathcal{F}_0$ *defines a lattice isomorphism, i.e., it is a bijection preserving infima and suprema.*

As the isomorphism hypo carries over the lattice structure to US_l, the question arises whether US_l is a continuous lattice. To answer this question, we must know something about the way above relation in US_l. Here we only derive what is directly needed; at the end of this section, a complete description of the way above relation is given.

As infima in the sublattice $\mathcal{F}_0$ coincide with infima in the mother lattice $\mathcal{F} = \mathcal{F}(E \times L' \uparrow)$, we see that the way above relation in $\mathcal{F}_0$ is the restriction of the way above relation in $\mathcal{F}$; in other words, for $F,G \in \mathcal{F}_0$: $F \gg G$ in $\mathcal{F}_0$ iff $F \gg G$ in $\mathcal{F}$. In the sequel, we work in US_l rather than in $\mathcal{F}_0$, because this gives more insight in US_l, which is after all the space we are studying. The first lemma is the reverse order analogue of II.4.20(ii) in Gierz et al. [1980], though in a quite different language.

Lemma 8.2. *Let E be locally quasicompact and L a continuous lattice. Then for every* $f \in US_l$,

$$f = \inf \{a_{\text{int } K} \vee \mathbf{1}_{(\text{int } K)^c} : a \in L,\ K \subset E \text{ is quasicompact and } a \gg f^\vee(K)\}.$$

Proof. Let $\mathcal{B}$ be the collection on the right hand side. Of course, $\inf \mathcal{B} \geq f$. For the converse, let $t \in E$. We must show that $(\inf \mathcal{B})(t) \leq f(t)$. Since L is a continuous lattice, it suffices to show that

(8.2) for all $x \gg f(t)$ there exists a $b \in \mathcal{B}$ such that $b(t) \leq x$.

So let $x \gg f(t)$. As $f \in US_l$, there is an open neighbourhood G of t with $x \gg f^\vee(G)$ (Corollary 6.3). As E is locally quasicompact, there is a quasicompact set $K \subset G$ with $t \in \text{int } K$. Set $b := x_{\text{int } K} \vee \mathbf{1}_{(\text{int } K)^c}$.

□

The following lemma states that the collection $\mathcal{B}$ in the proof of Lemma 8.2 is actually part of $\Uparrow f$:

Lemma 8.3. *Let E be locally quasicompact and L a continuous lattice. If* $a \in L, K \subset E$ *is quasicompact and* $a \gg f^\vee(K)$, *then* $a_{\text{int } K} \vee \mathbf{1}_{(\text{int } K)^c} \gg f$.

Proof. Let $\mathcal{B} \subset US_l$ be a filtered set with $\inf \mathcal{B} \leq f$. We must find a $b \in \mathcal{B}$ with $b \leq a_{\text{int } K} \vee \mathbf{1}_{(\text{int } K)^c}$. Let $t \in K$. Note that $(\inf \mathcal{B})(t) \leq f(t) \leq f^\vee(K)$, $\{b(t): b \in \mathcal{B}\}$ is filtered, and $a \gg f^\vee(K)$, so we find by Corollary 6.3 a $b_t \in \mathcal{B}$ such that $a \gg b_t(t)$. As $b_t \in US_l$, there is an open neighbourhood G_t of t such that $b_t^\vee(G_t) \leq a$. Now $K \subset \bigcup_{t \in K} G_t$, so there is a finite subset J of K such that $K \subset \bigcup_{t \in J} G_t$. As $\mathcal{B}$ is filtered, there is a $b \in \mathcal{B}$ with $b \leq \bigwedge_{t \in J} b_t$. This b satisfies the requirements, since $b^\vee(\text{int } K) \leq b^\vee(K) \leq \bigvee_{t \in J} b_t^\vee(G_t) \leq a$.

□

Combining the two preceding lemmas, we conclude:

Theorem 8.4. *If E is locally quasicompact and L is a continuous lattice, then US_l is a continuous lattice.*

Next, we want to provide US_l with the sup vague (Lawson) topology, by translating the corresponding topology on $\mathcal{F}_0$. As we have seen, the way above relation $\gg$ on $\mathcal{F}_0$ is just the restriction of $\gg$ to $\mathcal{F}$. Of course, the same holds for the $\leq$ relation. As the Lawson ($\Downarrow\uparrow$)−topology is defined in terms of these two relations, we immediately conclude: the Lawson topology on $\mathcal{F}_0$ is the restriction of the Lawson topology to $\mathcal{F}$. So we see, by Theorems 6.6 and 6.7:

Theorem 8.5. *If E is locally quasicompact and L is a continuous lattice, then $US_l \Downarrow\uparrow$ is a compact Hausdorff space, which can, via $\mathcal{F}_0$, be considered as a closed subspace of $\mathcal{F}(E \times L'\uparrow)$.*

Corollary 8.6. *$\mathcal{F}_0$ is closed in $\mathcal{F}$.*

As promised earlier, we finally completely characterize the way above relation in US_l.

Proposition 8.7. *Let E be locally quasicompact and L be a continuous lattice. Then for $f, g \in US_l$, $g \gg f$ if and only if there exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in L$ and quasicompact sets $K_1, \ldots, K_n$ in E such that $a_i \gg f^\vee(K_i)$ for all i and $g \geq \bigwedge_i (a_{i_{\text{int } K_i}} \vee \mathbf{1}_{(\text{int } K_i)^c})$.*

Proof. The "if" part: by Lemma 8.3, $a_{i_{\text{int } K_i}} \vee \mathbf{1}_{(\text{int } K_i)^c} \gg f$ for every i, so by Proposition 6.1(c), $g \geq \bigwedge_i (a_{i_{\text{int } K_i}} \vee \mathbf{1}_{(\text{int } K_i)^c}) \gg f$.

For the "only if" part, let $g \gg f$. We want to use Lemma 8.2, but the collection $\mathcal{B}$ on its right hand side need not be filtered. Therefore we study the collection $\mathcal{B}_0$ of all infima of finitely many elements of $\mathcal{B}$. Of course, $\mathcal{B}_0$ is filtered, and, by Lemma 8.2, $\inf \mathcal{B}_0 \leq \inf \mathcal{B} = f$, so there is a $b \in \mathcal{B}_0$ with $b \leq g$, which is exactly what we need.

□

Corollary 8.8. *Let E be locally quasicompact, L be a continuous lattice and $g \in US_l$. Then $\Downarrow g$ is nonempty iff g satisfies the following two conditions:*
(i) $\inf g(E) \gg \mathbf{0}$;
(ii) $\{t: g(t) \neq \mathbf{1}\}$ *is contained in the interior of a quasicompact set.*

Proof. Note that $\Downarrow g \neq \varnothing$ iff $g \gg \mathbf{0}_E$. For the "only if" part, suppose $g \gg \mathbf{0}_E$. By Proposition 8.7 there are $a_1, \ldots, a_n \in L$, all $\gg \mathbf{0}$, and quasicompact sets $K_1, \ldots, K_n$ in E such that $g \geq \bigwedge_i (a_{i_{\mathrm{int}\, K_i}} \vee \mathbf{1}_{(\mathrm{int}\, K_i)^c})$. Now $\inf g(E) \geq \bigwedge_i a_i \gg \mathbf{0}$ by Proposition 6.1(c), and $\{t: g(t) \neq \mathbf{1}\} \subset \bigcup_i \mathrm{int}\, K_i \subset \mathrm{int} \bigcup_i K_i$.
For the "if" part, let $\inf g(E) \gg \mathbf{0}$ and $\{t: g(t) \neq \mathbf{1}\} \subset \mathrm{int}\, K$ for some quasicompact K. Then $g \geq (\inf g(E))_{\mathrm{int}\, K} \vee \mathbf{1}_{(\mathrm{int}\, K)^c}$, so $g \gg \mathbf{0}$ by Lemma 8.3.

□

9. Quasicompact subsets of $L' \uparrow$

In Section 8 we have topologized US_l via the sup vague topology on $\mathcal{F}(E \times L' \uparrow)$. In Section 7 we have found a subbase for the sup vague topology on $\mathcal{F}(D)$ (Theorem 7.3). The goal of Section 10 is, by putting $D := E \times L' \uparrow$, to translate the subbase elements in terms of US_l. For translating (7.2), we need a characterization of quasicompactness in $L' \uparrow$. That is what this section deals with.

Theorem 9.1. *If $A \subset L' \uparrow$ is nonempty, then A is quasicompact iff* $\inf B \in \uparrow A$ *for each filtered $B \subset \uparrow A$.*

Proof. By Alexander's lemma, A is quasicompact iff each subbase cover $\{(\downarrow x_i)^c : i \in I\}$ of A contains a finite subcover. Note that all subbase elements are increasing sets. As $A \subset U$ iff $\uparrow A \subset U$ for increasing U and arbitrary A, we may assume that A is increasing.

For the "if" part, take a cover as above and set $B := \{\bigwedge_{i\in J} x_i : J \subset I \text{ finite}\}$. B is filtered, and $A \subset \bigcup_{i\in I} (\downarrow x_i)^c = (\downarrow \inf B)^c$. So $B \not\subset \uparrow A$, i.e., there is a finite $J \subset I$ with $\bigwedge_{i\in J} x_i \notin A$. As A is increasing, we see that $A \cap (\downarrow \bigwedge_{i\in J} x_i) = \varnothing$, hence $A \subset (\downarrow \bigwedge_{i\in J} x_i)^c = \bigcup_{i\in J} (\downarrow x_i)^c$, so A is quasicompact.
For the "only if" part, suppose that A is quasicompact, $B \subset A$ is filtered and $\inf B \notin A$. Then, since A is increasing, $A \subset (\downarrow \inf B)^c = \bigcup_{x\in B} (\downarrow x)^c$, an open cover of A, so there is a finite $B_0 \subset B$ such that $A \subset \bigcup_{x\in B_0} (\downarrow x)^c = (\downarrow \inf B_0)^c$. But as B is filtered, B contains a $b \leq \inf B_0$, so $b \notin A$, a contradiction.

□

It turns out to be very hard to give a more specific characterization of quasicompactness in $L' \uparrow$. However, if L is a continuous lattice, there is an important subclass of quasicompact sets:

Proposition 9.2. *If L is a continuous lattice, then all sets of the type $(\Downarrow x)^c$ $(x \in L')$ are quasicompact in $L' \uparrow$, and so are their intersections.*

Proof. Of course, $\uparrow(\Downarrow x)^c = (\Downarrow x)^c$. Let $B \subset (\Downarrow x)^c$ be filtered and set $y := \inf B$. If $y \notin (\Downarrow x)^c$, then $x \gg y$, so by Corollary 6.3 there is a $z \in B$ such that $x \gg z$. So $B \not\subset (\Downarrow x)^c$, a contradiction. Hence $y \in (\Downarrow x)^c$, and Theorem 9.1 applies. For the second statement, notice that it follows easily from Theorem 9.1 that the intersection of arbitrarily many *increasing* quasicompact sets is quasicompact.

□

From this proposition it will be an easy consequence (Corollary 9.5) that $L' \uparrow$ is a locally quasicompact space if L is a continuous lattice. However, we present this result in a somewhat more general context, as this will be useful in Section 10. The following definition is taken from the forthcoming revision of Vervaat [1982].

Definition 9.3. *Let D be a topological space and $\mathcal{B} \subset \mathcal{P}(D)$. D is called **locally** $\mathcal{B}$ if for every $t \in D$ and open neighbourhood G of t there is a $B \in \mathcal{B}$ such that $t \in \operatorname{int} B \subset B \subset G$.*

The same definition turns up already in a paper by Ceder [1961], who calls $\mathcal{B}$ a "quasibase" for the topology on D if D is locally $\mathcal{B}$. We prefer Vervaat's terminology, in view of our applications (cf. Corollary 9.5).

Lemma 9.4. *If L is a continuous lattice, then* $L'\uparrow$ *is locally* $\{\bigcap_{k=1}^{n}(\Downarrow x_k)^c: n\in\mathbb{N}, x_1,\ldots,x_n\in L'\}$.

Proof. Let $y\in L'$ and $y\in U$, U open. We may assume that U is a base set, i.e., $U=\bigcap_{i=1}^{m}(\downarrow z_i)^c$. For every i, $z_i\not\geq y$, so by Proposition 6.1(d) there are $x_i\gg z_i$ with $x_i\not\geq y$. Set $K:=\bigcap_{i=1}^{m}(\Downarrow x_i)^c$. As $(\Downarrow x_i)^c\subset(\Downarrow z_i)^c$ for each i, it follows that $K\subset U$. Finally, since $\bigcap_{i=1}^{m}(\downarrow x_i)^c$ is an open subset of K, we conclude that $y\in\bigcap_{i=1}^{m}(\downarrow x_i)^c\subset \text{int } K$.

□

Notice that we reobtain the definition of local quasicompactness by taking in Definition 9.3 for $\mathcal{B}$ the collection of all quasicompact subsets of $L'\uparrow$, or even a collection of some quasicompact subsets. This leads to the announced result:

Corollary 9.5. *If L is a continuous lattice, then* $L'\uparrow$ *is locally quasicompact.*

Proof. Combine Proposition 9.2 and Lemma 9.4.

□

10. Back to the topology on US_l

The first lemma in this section is an easy consequence of Definition 9.3.

Lemma 10.1. *Let* D_1,D_2 *be topological spaces which are locally* $\mathcal{B}_1,\mathcal{B}_2$, *respectively. Then* $D_1\times D_2$ *is locally* $\mathcal{B}_1\times\mathcal{B}_2(:=\{B_1\times B_2: B_1\in\mathcal{B}_1, B_2\in\mathcal{B}_2\}$, *provided with the product topology).*

Corollary 10.2. *If E is locally quasicompact and L is a continuous lattice, then* $E\times L'\uparrow$ *is locally quasicompact.*

Proof. Combine Corollary 9.5 and Lemma 10.1.

□

So, as a first step to realize the final goal sketched in the introduction of the preceding section, we apply Theorem 7.3: the sup vague topology on $\mathcal{F}(E \times L' \uparrow)$ is generated by the open subbase sets

(10.1) $\{F \subset E \times L' \uparrow : F \cap G \neq \varnothing\}$ for all open $G \subset E \times L' \uparrow$

and

(10.2) $\{F \subset E \times L' \uparrow : F \cap K = \varnothing\}$ for all quasicompact $K \subset E \times L' \uparrow$.

We want to thin out this untractable subbase by selecting sufficient subclasses of open sets and quasicompact sets in $E \times L' \uparrow$. Much of the following was done in Vervaat [1982]. However, his results are in terms of sup measures, and only deal with $L = \bar{\mathsf{R}}$, which was sometimes essentially made use of. We generalize his results and translate them into terms of closed sets. Here is the result:

Theorem 10.3. *If E is a locally quasicompact space and L is a continuous lattice, then the sup vague topology on $\mathcal{F}(E \times L' \uparrow)$ is generated by the open subbase sets*

(10.3) $\{F \subset E \times L' \uparrow : F \cap (G \times (\downarrow X)^c) \neq \varnothing\}$, *where G runs through a base of open sets in E and $X \subset L'$ is finite,*
and

(10.4) $\{F \subset E \times L' \uparrow : F \cap (K \times (\Downarrow X)^c) = \varnothing\}$, *where K runs through the quasicompact subsets of E and $X \subset L'$ is finite.*

Proof. (i) Starting with the open sets, we see that we can thin out (10.1) to open *base* sets $G \subset E \times L' \uparrow$, since $\{F : F \cap (\bigcup_{i \in I} G_i) \neq \varnothing\} = \bigcup_{i \in I} \{F : F \cap G_i \neq \varnothing\}$. This results in (10.3).
(ii) In order to derive (10.4), we set $\mathcal{B} := \mathcal{K}_E \times \mathcal{Q}$, where $\mathcal{K}_E$ denotes the class of all quasicompact subsets of E, and $\mathcal{Q}$ is the collection defined in Lemma 9.4. Furthermore we set $\mathcal{B}_1 := \{\bigcup_{k=1}^{n} B_k : n \in \mathsf{N},\ B_1,\dots,B_n \in \mathcal{B}\}$. It suffices to thin out (10.2) to $\mathcal{B}_1$, since $\{F : F \cap \bigcup_k B_k = \varnothing\} = \bigcap_k \{F : F \cap B_k = \varnothing\}$. To this end, it suffices to prove that

(10.5) $\{F : F \cap K = \varnothing\} = \bigcup_{B \in \mathcal{B}_1 :\, K \subset B} \{F : F \cap B = \varnothing\}$

for every quasicompact K in $E \times L' \uparrow$.
The only nontrivial inclusion in (10.5) is the $\subset$ part. Therefore, fix an F with $F \cap K = \varnothing$, i.e., $K \subset F^c$. By Lemmas 9.4 and 10.1, $E \times L' \uparrow$ is locally $\mathcal{B}$, so for every $t \in K$ there is a $B_t \in \mathcal{B}$ such that $t \in \operatorname{int} B_t \subset B_t \subset F^c$. Now K is quasicompact, so from the open cover $\{\operatorname{int} B_t : t \in K\}$ we can select a finite subcover $\{\operatorname{int} B_t : t \in J\}$ of K, for some finite $J \subset K$. Set $B := \bigcup_{t \in J} B_t$. Then $B \in \mathcal{B}_1$, $K \subset B$ and $F \cap B = \varnothing$, which

proves (10.5).

□

Finally, we translate to US_l:

Theorem 10.4. *If E is a locally quasicompact space and L is a continuous lattice, then the sup vague topology on US_l is generated by the open subbase sets*

$$\{f : f(G) \not\subset \downarrow X\}$$

and

$$\{f : f(K) \subset \Downarrow X\},$$

where G runs through a base of open sets in E, K runs through the quasicompact subsets of E and $X \subset L'$ is finite.

Proof. Apply the isomorphism hypo to Theorem 10.3. The only nontrivial observation is that both $(\downarrow X)^c$ and $(\Downarrow X)^c$ are increasing sets.

□

11. Appendix

In this appendix we give examples of functions belonging to exactly one or two of the three function spaces studied in Section 4.

Example 1 $US_l \setminus (US_t \cup HC)$:
$E = (0,1)$, usual topology;
$L = \{0\} \cup (1,2]$, lower topology (i.e., the reverse order analogue of the upper topology, cf. Example 3.8);
$f(0) = 0$, $f(t) = 1+t$ if $t>0$.

Example 2 $US_t \setminus (US_l \cup HC)$:
see Example 3.8.

Example 3 $HC \setminus (US_t \cup US_l)$:
$E = (0,1)$, usual topology;
$L = \{(x, 1-x) \in \mathsf{R}^2 : x \in (0,1)\}$;
$f(t) = 1-t$.

Example 4 $(US_l \cup US_t) \setminus HC$:
see Example 3.4.

Example 5 $(US_l \cup HC) \setminus US_t$:
$E = \mathsf{R}$, usual topology;
$L = \overline{\mathsf{R}}$, discrete topology;
f = identity mapping.

Example 6 $(US_t \cup HC) \setminus US_l$:
$E = \mathsf{R}$, usual topology;
$L = LS(\mathsf{R}, [-1,0])$ (lower semicontinuous functions from R to $[-1,0]$), with the topology induced by the sup vague topology on US via the identification $L = -US(\mathsf{R},[0,1])$;
$f(t) = -1_{\{t\}}$.

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Appendix: notes and complements

A General

Many results can be found also in Gierz et al. (1980), as the following list shows. Mislove (1982) is recommended for a first introduction to continuous lattices.

The present paper	*Gierz et al. (1980)*
Theorem 6.6	p. 155 above Definition III.2.10
Theorem 6.7	Theorem III.1.9 (more generally for complete lattices)
Lemma 6.9	Proposition III.2.1(a)
Lemma 7.1	dual of I.1.4 (cf. Exercise III.1.13)
Theorem 7.2	dual of Example I.1.7
Theorem 7.3	Exercice III.1.13

For a further study on versions of semicontinuity, see

- H. Holwerda (1993): Variations on lower semicontinuity. In: H. Holwerda: *Topology and Order.* Ph.D. Thesis, Cath. Univ. Nijmegen.

B Corrections

In the proof of Theorem 6.7 it is not true that "one $(\downarrow y)^c$ suffices". See Theorem III.1.9 in Gierz et al. (1980).

It is easier and more natural to replace (7.2) by its (equivalent) restriction to saturated qcompact sets, and to restrict attention to such sets throughout Section 9.

It is harmful and not necessary to replace L by $L' := L\backslash\{\mathbf{0}\}$ in Section 9. Proposition 9.2 is false for L' but correct for L.

Example 3 in Section 11 is false and must be replaced by the following, due to Henk Holwerda.

$E = [0, 1]$;

$L = ([0, 1] \times \{0\}) \cup (\{0, 1\} \times \{1\})$ with the induced topology and order of $[0, 1]^2$;

$f(0) = (0, 1)$, $f(t) = (t, 0)$ if $t \neq 0$.

C Typos and minor corrections

List of notations:	$f^{\nu}(A) \rightarrow f^{\vee}(A)$
Some definitions:	$B \neq \phi$ must be required in addition
line above Lemma 7.1:	III.1.13 → I.1.4
Section 8, 3rd paragraph, line 2:	larger → smaller
Lemma 8.3:	local qcompactness is not needed
Reference [6]:	1979 → 1981

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Spaces with Vaguely upper Semicontinuous Intersection[1]

Wim Vervaat[2]

Spaces are not assumed to be Hausdorff. The intersection operation in the hyperspace of closed sets is shown to be upper semicontinuous with respect to the vague topology iff covers of saturated quasicompact (qcompact) sets by open sets can be imitated by a similar cover by smaller saturated qcompact sets. A sufficient condition for this is that each saturated qcompact set is part of some locally qcompact set. Results are demonstrated on the example of one-point qcompactifications of spaces whose qcompact sets are closed.

0. Introduction

Let E be a topological space, not necessarily Hausdorff. Quasicompactness (qcompactness) refers to the finite-open-subcover property; the combination 'qcompact' and 'Hausdorff' is referred to as 'compact'. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}$ denote the families of closed, open and qcompact sets in E. Capitals F, G and K with or without index denote generic elements of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}$. The ***vague*** or ***Fell topology*** (Fell (1962), Matheron (1975), Norberg (1986), Vervaat (1988)) in $\mathcal{F}$ is generated by the subbase consisting of

$$\{F \in \mathcal{F}: FK = \phi\} \qquad \text{for } K \in \mathcal{K}, \tag{1a}$$
$$\{F \in \mathcal{F}: FG \neq \phi\} \qquad \text{for } G \in \mathcal{G}. \tag{1b}$$

It is well-known that $\mathcal{F}$ with the vague topology is qcompact, and is Hausdorff iff E is locally qcompact in case E is Hausdorff (Vervaat (1988)), or more generally, in case E is sober, i.e., T_0 and not a proper subspace of a larger T_0 space E_1 with topology lattice-isomorphic to that of E (Gierz et al. (1980), Hofmann & Mislove (1981)). One easily checks that the union

$$\mathcal{F} \times \mathcal{F} \ni (F_1, F_2) \mapsto F_1 \cup F_2 \in \mathcal{F}$$

[1] Revision of Report 88-30 of the Faculty of technical Mathematics and Informatics, Delft University of Technology, whose hospitality is gratefully acknowledged.

[2] Wim Vervaat died early 1994. His last work address was Université Claude Bernard Lyon 1.

is vaguely continuous (Vervaat (1988)). In contrast to this the intersection

$$\mathcal{F} \times \mathcal{F} \ni (F_1, F_2) \mapsto F_1 \cap F_2 \in \mathcal{F} \tag{2}$$

is almost never continuous, but in many instances it turns out to be vaguely upper semicontinuous (usc), by which we mean that the mapping is continuous when $\mathcal{F}$ on the range side is provided with the upper vague topology, with subbase consisting only of the sets in (1a). In this note we will investigate when the intersection is vaguely usc.

1. Saturated sets

For $A \subset E$ we write $\operatorname{sat} A := \bigcap\{G : G \supset A\}$. We call $\operatorname{sat} A$ the *saturation* of A, and say that A is *saturated* if $A = \operatorname{sat} A$. Note that E is T_1 iff all singletons are closed iff all subsets of E are saturated. One easily verifies that

$$t \in \operatorname{sat} A \quad \text{iff} \quad \operatorname{clos}\{t\} \cap \operatorname{sat} A \neq \phi,$$

so that $FA \neq \phi$ iff $F \cap \operatorname{sat} A \neq \phi$ in case F is closed. Consequently,

$$\{F \in \mathcal{F} : FK = \phi\} = \{F \in \mathcal{F} : F \cap \operatorname{sat} K = \phi\}.$$

We therefore may thin out the subbase of the (upper) vague topology in $\mathcal{F}$ by restricting K in (1a) to $\mathcal{Q}$, the family of saturated qcompact subsets of E. Generic elements of $\mathcal{Q}$ are denoted by Q, with or without index.

2. Qcompact imitations of open covers

In this section we show that the intersection is vaguely usc iff each cover of a $Q \in \mathcal{Q}$ by two open sets can be imitated by a smaller cover by two saturated qcompact sets (cf. also Vervaat (1988)).

Theorem 1. *The intersection is vaguely usc iff for each instance of $Q \subset G_1 \cup G_2$ there are $Q_1, Q_2 \in \mathcal{Q}$ such that $Q_1 \subset G_1$, $Q_2 \subset G_2$ and $Q \subset Q_1 \cup Q_2$.*

Proof. The intersection is vaguely usc iff for each $Q \in \mathcal{Q}$

$$U := \{(F_1, F_2) \in \mathcal{F}^2 : F_1 F_2 Q = \phi\}$$

is open in $\mathcal{F}^2$. Openness of U is equivalent to the existence for each $(F_{10}, F_{20}) \in U$ of Q_1, Q_2 such that

$$F_{10} Q_1 = F_{20} Q_2 = \phi, \tag{a}$$
$$F_1 Q_1 = F_2 Q_2 = \phi \quad \text{implies} \quad F_1 F_2 Q = \phi. \tag{b}$$

Because Q is saturated, implication (b) holds iff $Q \subset Q_1 \cup Q_2$ (to see this, consider $F_1 \cup \operatorname{clos}\{x\}$ and $F_2 \cup \operatorname{clos}\{x\}$ for $x \in Q \backslash (Q_1 \cup Q_2)$). Consequently, openness of U is equivalent to the condition in the theorem holding with $G_1 = F_{10}^c$, $G_2 = F_{20}^c$. □

Remarks. (a) Vague upper semicontinuity of the intersection was defined as continuity of the mapping in (2) with $\mathcal{F}$ on the range side provided with the upper vague topology. From the proof of Theorem 1 it is obvious that it does not make difference whether the spaces $\mathcal{F}$ on the domain side are provided with the vague topology or only with the upper vague topology.
(b) The Lawson and Scott topologies with reversed order of Gierz et al. (1980) coincide with the vague and upper vague topologies in case E is locally qcompact, and are finer in general (cf. proof of Theorem 7.3 in Gerritse (1985)). For locally qcompact E, upper semicontinuity of the intersection follows from Corollary II.4.13 in Gierz et al. (1980).
(c) The condition in Theorem 1 plays a prominent role in Norberg & Vervaat (1989), which triggered the research leading to the present paper. It is also a central condition (Condition (D)) in Wilker (1970). That's why this condition is referred to as 'Wilker' in recent work ($\geq$1993) of Holwerda and Vervaat.

3. Sufficient conditions

Recall that E is locally qcompact if the topology has a base of qcompact sets, or equivalently, if for each instance of $t \in G$ there are $G_1 \in \mathcal{G}$ and $Q_1 \in \mathcal{Q}$ such that $t \in G_1 \subset Q_1 \subset G$. It is well-known that for Hausdorff E this is equivalent to each point of E having a qcompact neighborhood, but not so for non-Hausdorff E. In particular a qcompact E need not be locally qcompact.

Here is a sufficient condition for vaguely upper semicontinuous intersection. It can be regarded as local qcompactness around each qcompact set.

Theorem 2. *If for each $Q \in \mathcal{Q}$ there is an $A \supset Q$ such that A is locally qcompact in the relative topology, then the equivalent statements in Theorem 1 hold.*

Proof. We prove the second statement in Theorem 1. So let $Q \subset G_1 \cup G_2$. First suppose that E is locally qcompact, so we may choose $A = E$. Select for each $t \in Q$ a $Q(t) \in \mathcal{Q}$ such that $t \in \operatorname{int} Q(t)$ and $Q(t) \subset G_1$ if $t \in G_1$ and $Q(t) \subset G_2$ if $t \in G_2$. Then there is a finite $J \subset Q$ such that already $Q \subset \bigcup_{t \in J} \operatorname{int} Q(t)$. Now set $Q_1 := \bigcup_{t \in JG_1} Q(t)$ and $Q_2 := \bigcup_{t \in J{G_1}^c} Q(t)$. Then $Q_1 \subset G_1$, $Q_2 \subset G_2$ and $Q \subset Q_1 \cup Q_2$.

Next suppose that $A \supset Q$ is locally qcompact. The previous result applies to $Q \subset G_1 A \cup G_2 A$ considered in A with the relative topology, to obtain Q_1 and Q_2 such that $Q_1 \subset G_1 A$, $Q_2 \subset G_2 A$ and $Q \subset Q_1 \cup Q_2$. □

Remark. The condition in Theorem 2 holds with maximal A: $A = E$ in case E is locally qcompact. If $\mathcal{Q}$ is closed for finite intersections, then the condition in Theorem 2 holds iff it holds with minimal A: $A = Q$, i.e., each $Q \in \mathcal{Q}$ is locally qcompact in the relative topology. The latter holds for instance if E is Hausdorff or if all qcompact sets are finite. In either case E need not be locally qcompact.

Under additional assumptions the condition in Theorem 2 is also necessary.

Theorem 3. *Let E be such that $\mathcal{K} \subset \mathcal{F}$ (so E is T_1 and $\mathcal{Q} = \mathcal{K}$). Then the equivalent conditions of Theorem 1 hold iff each $K \in \mathcal{K}$ is locally qcompact in the relative topology.*

Proof. The 'if' part follows from Theorem 2. For the 'only if' part, let $t \in GK$. We have $K \subset G \cup \{t\}^c$, with $\{t\}^c$ open because E is T_1. By Theorem 1 there are K_1, K_2 such that $K_1 \subset G$, $K_2 \subset \{t\}^c$ and $K \subset K_1 \cup K_2$. Then we have $t \in {K_2}^c K \subset K_1 K \subset GK$, with ${K_2}^c$ open because $\mathcal{K} \subset \mathcal{F}$. □

4. Application to one-point qcompactifications

Let E be a space such that $\mathcal{K} \subset \mathcal{F}$ (which is the case if E is Hausdorff). Let $E' := E \cup \{\infty\}$ its one-point qcompactification with open sets $G' = G$ or $K^c \cup \{\infty\}$ (subsets and families of subsets of E' will be marked by primes throughout). Also in this generality E' is qcompact and T_1, and E is a subspace of E'.

In this section we want to investigate which E' have vaguely usc intersection. The notions of *k-extension* and *k-space* (Kelley (1975) and Brown (1968)) will be useful here. Let E be a space such that $\mathcal{K}$ is closed for arbitrary intersections (which is the case if $\mathcal{K} \subset \mathcal{F}$). Call $A \subset E$ *k-closed* if $AK \in \mathcal{K}$ for all $K \in \mathcal{K}$, let $\mathcal{F}_k$ be the family of k-closed sets, and let $\mathcal{G}_k$ be the family of their complements, the *k-open* sets. If $\mathcal{K} \subset \mathcal{F}$, then $\mathcal{G}_k \supset \mathcal{G}$, and $\mathcal{G}_k$ is a topology in E, which is called the *k-extension* of $\mathcal{G}$ (Kelley (1975), Problem 6.K). It is the finest topology in E with the same qcompact sets as $\mathcal{G}$. If $\mathcal{G}_k = \mathcal{G}$, then E is called a *k-space.* This is the case if $\mathcal{K} \subset \mathcal{F}$ and E is first countable or locally qcompact (Kelley (1975), Theorem 7.17, stated for Hausdorff E, but the proof applies verbatim in the present generality).

Lemma. *Let E be such that $\mathcal{K} \subset \mathcal{F}$ and let E' be its one-point qcompactification. Then K' is qcompact in E' iff it is qcompact in E or has the form $K' = A \cup \{\infty\}$ with $A \in \mathcal{F}_k$.*

Proof. Since E is a subspace of E', subsets of E are qcompact in E' iff they are in E.

It remains to consider $K' = A \cup \{\infty\}$. If $A \cup \{\infty\} \in \mathcal{K}'$, then we have $F' \cap (A \cup \{\infty\}) \in \mathcal{K}'$ for each $F' \in \mathcal{F}'$, in particular for $F' = K \in \mathcal{K}$. Hence $AK \in \mathcal{K}'$, so $AK \in \mathcal{K}$ for $K \in \mathcal{K}$. It follows that $A \in \mathcal{F}_k$.

Conversely, let $A \in \mathcal{F}_k$, and suppose that $(A \cup \{\infty\}) \cap \bigcap_j F'_j = \phi$, with $F'_j \in \mathcal{F}'$. Then at least one F'_j, F'_0 say, avoids ∞, so belongs to $\mathcal{K}$. Hence $AF'_0 \in \mathcal{K}$ and

$$(A \cup \{\infty\}) \cap \bigcap_j F'_j = AF'_0 \cap \bigcap_j (F'_j \backslash \{\infty\}) = \phi,$$

where $F'_j \backslash \{\infty\} \in \mathcal{F}$. Consequently, AF'_0 has empty intersection with the intersection of finitely many F'_j. This proves $A \cup \{\infty\} \in \mathcal{K}'$. □

Corollary. In the situation of the lemma we have $\mathcal{F}' \subset \mathcal{K}'$, and $\mathcal{F}' = \mathcal{K}'$ iff E is a k-space.

Theorem 4. *Let E be a k-space such that $\mathcal{K} \subset \mathcal{F}$, and let E' be its one-point qcompactification. Then E' satisfies the equivalent conditions of Theorem 1 iff E is locally qcompact.*

Remark. For E such that $\mathcal{K} \subset \mathcal{F}$ (not necessarily a k-space) we have that E' is locally qcompact iff E is locally qcompact, and in this case both E and E' satisfy the equivalent conditions of Theorem 1, by Theorem 2 and the subsequent remark.

Proof of Theorem 4. In view of the remark, it remains to prove the 'only if' part. Now $\mathcal{K}' = \mathcal{F}'$ by the lemma and its corollary, so each $K' \in \mathcal{K}'$ is locally qcompact by Theorem 3. In particular, E' is. □

5. Extensions of the results

5.1. Let J be an index set and consider the intersection

$$\mathcal{F}^J \ni (F_j)_{j \in J} \mapsto \bigcap_{j \in J} F_j \in \mathcal{F}.$$

In this paper we have characterized those E for which the intersection is vaguely upper semicontinuous in case $\#J = 2$. By iteration we see that this implies vaguely usc intersection for all finite J, and then for arbitrary J (cf. Vervaat (1988, §14)).

5.2. It is immediate that the condition in Theorem 1 is equivalent to the analogous statement with covers by arbitrarily many open sets rather than two.

5.3. For functions $f : E \to [0,1]$, define the *hypograph*, hypo f, by

$$\operatorname{hypo} f := \{(t,x) \in E \times (0,1] : 0 < x \leq f(t)\}.$$

Let $\mathrm{US}(E)$ denote the space of usc functions $E \to [0,1]$. Let $(0,1]\uparrow$ denote the interval $(0,1]$ provided with the *upper topology,* whose nontrivial open sets are $(x,1]$. The *sup vague topology* in $\mathrm{US}(E)$ can be defined by declaring the bijection

$$\operatorname{hypo} : \mathrm{US}(E) \to \mathcal{F}(E \times (0,1]\uparrow)$$

a homeomorphism (Vervaat (1988)). Now the results of the previous sections translate into a characterization of $\wedge$ (pointwise infimum) in $\mathrm{US}(E)$ being sup vaguely usc, with the conditions on E now to be applied to $E \times (0,1]\uparrow$.

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Capacities on non-Hausdorff spaces[1]

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Abstract
Capacities of various types and Radon measures are generalized to a non-Hausdorff setting. This involves changes in the definitions (having no effect inside the Hausdorff context) and selection of new topological regularity conditions. Attention is paid to classical topics theory, and also to topologies on spaces on capacities.

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0. Introduction

Let E be a topological space. If E is Hausdorff, then the answers to the following are known. Let capacities be increasing outer continuous functions c on the compact subsets of E such that $c(\emptyset) = 0$. When is c the restriction of a Radon measure? When can c be extended to a Choquet capacity? It is the first object of this paper to answer the same questions for broad classes of non-Hausdorff spaces (Sections 3 and 4).

After this, the (initial) domain of the capacities (the compacts in case E is Hausdorff) is topologized in such a way that capacities can be interpreted as semicontinuous functions on this domain (Section 5). Consequently, conditions are known under which the capacities form a continuous lattice in the sense

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of GIERZ et al. (1980), but a surprising (and unusual) additional result is that they are also a continuous lattice with the order reversed (Section 6).

The non-Hausdorff setting requires its own regularity conditions, which are presented in Section 1. Capacities are introduced in Section 2. Compact Hausdorff topologies on the set of all capacities are defined in various ways in Section 7. Under mild conditions all turn out to be the same.

Non-Hausdorff spaces are the natural setting for spaces of closed sets and upper semicontinuous functions (cf. VERVAAT (1988a)). That was our major reason for writing this article. Moreover, we speculate that a theory of continuous characters of topological semigroups can be developed by allowing the topology to be non-Hausdorff. Such a theory could not be developed so far, in sharp contrast with the situation for topological groups.

We conclude this introduction with some general remarks on the notations: $\mathbb{R}$ denotes the real line and $\mathbb{R}_+$ is its non-negative part. We write $\bar{\mathbb{R}}_+$ for $[0,\infty]$. The natural numbers are denoted $\mathbb{N}$ and $\mathbb{Q}$ is the rationals. Let $A \subseteq E$, where E is a topological space. We then write A^- for its closure and A° for its interior.

1. Regularity conditions for non-Hausdorff spaces

Let E be a topological space. We write $\mathcal{G}$ and $\mathcal{F}$ for the collections of open and closed sets in E. Nonscript capitals G and F, with or without index, always denote open and closed subsets of E. We assume E to be T_0, i.e., for each pair of distinct points $s, t \in E$, there is an open set containing only one of them. Equivalently, $\{s\}^- = \{t\}^-$ implies $s = t$. If, more specifically, there are open sets containing s but not t and t but not s, then E is called T_1. This is equivalent to $\{s\}^- = \{s\}$ for all $s \in E$.

For $A \subseteq E$ we define the *saturation* of $A, \operatorname{sat}(A)$, by

$$\operatorname{sat}(A) = \bigcap_{G \supseteq A} G.$$

If $A = \operatorname{sat}(A)$, then A is called *saturated.* Clearly,

$$\operatorname{sat}(A) \cap F = \emptyset \Leftrightarrow A \cap F = \emptyset$$

It is a simple exercise to show that $s \in \operatorname{sat}(A)$ iff $\{s\}^- \cap A \neq \emptyset$. It follows that E is T_1 iff all subsets of E are saturated.

In what follows it is helpful to keep the following examples in mind. Let $\mathbb{R}{\downarrow}$ denote the reals with the *lower topology,* whose nontrivial open sets are the intervals $(-\infty, x)$ for $x \in \mathbb{R}$. Let $\mathbb{Q}{\downarrow}$ denote the rationals with the relative topology of $\mathbb{R}{\downarrow}$. Observe that $\mathcal{G}(\mathbb{R}{\downarrow})$ and $\mathcal{G}(\mathbb{Q}{\downarrow})$ are isomorphic as lattices in an obvious way. So non-Hausdorff spaces with the same (i.e., lattice isomorphic) topology need not be homeomorphic. Intuitively we feel that $\mathbb{R}{\downarrow}$ is a more complete T_0 space than $\mathbb{Q}{\downarrow}$ with the same topology. The following definitions and properties settle this.

A nonempty $F \subseteq E$ is said to be *irreducible* if for each instance of $F \subseteq F_1 \cap F_2$ we have $F \subseteq F_1$ or $F \subseteq F_2$. Obviously each singleton closure is irreducible. If, conversely, all irreducible closed sets are singleton closures, then E is called *sober.*

THEOREM 1.1 (HOFMAN & MISLOVE (1981)). *For each T_0 space E there is a sober T_0 space* $\text{sob}(E)$*, unique up to homeomorphism, in which E can be embedded as a subspace. The space* $\text{sob}(E)$ *is the largest T_0 space with the same topology as E.*

We call $\text{sob}(E)$ the *sobrification* of E. Obviously $\mathbb{Q}{\downarrow}$ is not sober, since the irreducible closed set $[x, \infty) \cap \mathbb{Q}$ is no singleton closure for $x \in \mathbb{R} \backslash \mathbb{Q}$. Even $\mathbb{R}{\downarrow}$ is not sober, because $\mathbb{R}$ itself is irreducible and not a singleton closure. In fact, both $\text{sob}(\mathbb{Q}{\downarrow})$ and $\text{sob}(\mathbb{R}{\downarrow})$ are homeomorphic to $[-\infty, \infty){\downarrow}$.

Hausdorff spaces are sober, and there are nonsober T_1 spaces (cf. GIERZ et al. (1980, page 79)).

We now consider quasicompact (qcompact) subsets of E, where *quasicompactness* is defined by the finite-open-subcover property, without resort to any further separation property, i.e., $Q \subseteq E$ is qcompact iff every open cover of Q has a finite subcover. The combination qcompactness and Hausdorff is referred to as *compact.*

In both $\mathbb{Q}{\downarrow}$ and $\mathbb{R}{\downarrow}$, a nonempty subset is qcompact iff it contains its supremum. From this it follows that the intersection of two qcompact sets need not be qcompact: Take $Q_1 = (-\infty, \pi]$ and $Q_2 = (-\infty, 4] \cap \mathbb{Q}$. Then both Q_1 and Q_2 are qcompact subsets of $\mathbb{R}{\downarrow}$, but $Q_1 \cap Q_2 = (-\infty, \pi] \cap \mathbb{Q}$ is not.

Things become better if we restrict our attention to the collection $\mathcal{Q}$ of saturated qcompact sets in E. Nonscript capitals Q, with or without index, always denote elements of $\mathcal{Q}$. Note that both $\mathcal{Q}(\mathbb{Q}{\downarrow})$ and $\mathcal{Q}(\mathbb{R}{\downarrow})$ are closed for finite intersections, and $\mathcal{Q}(\mathbb{R}{\downarrow})$ even for arbitrary nonempty intersections.

If E is Hausdorff, then $\mathcal{Q}$ coincides with the collection $\mathcal{K}$ of compact subsets of E.

We say that E is Q_δ if $\mathcal{Q}$ is closed for arbitrary nonempty intersections.

Henceforth we will consider mainly sober spaces, because of Theorem 1.2 below. A collection $(A_\alpha)_\alpha$ of sets is said to be a *decreasing net* if for each pair α_1, α_2 there is an α_3 such that $A_{\alpha_3} \subseteq A_{\alpha_1} \cap A_{\alpha_2}$.

THEOREM 1.2 (HOFMANN & MISLOVE (1981)). *Let E be sober and take a decreasing net $(Q_\alpha)_\alpha$ of sets in $\mathcal{Q}$. Then the following holds:*

(a) $\cap_\alpha Q_\alpha \in \mathcal{Q}$,
(b) If $\cap_\alpha Q_\alpha \subseteq G$, then $Q_\alpha \subseteq G$ for some α

From (a) it follows that a sober E is Q_δ iff $\mathcal{Q}$ is closed for finite intersections. If E is not sober, (b) may fail. Consider $\mathbb{R}{\downarrow}, Q_\alpha = (-\infty, \alpha]$ for $\alpha \in \mathbb{R}, G = \emptyset$, and recall that $\text{sob}(\mathbb{R}{\downarrow}) = [-\infty, \infty){\downarrow}$.

We say that E is locally qcompact if E has a base of qcompact sets, i.e., if $s \in G$ implies the existence of $Q \subseteq E$ such that $s \in Q^\circ \subseteq Q \subseteq G$. For the case that E is Hausdorff it is known that E is locally compact iff each $s \in E$ has a compact neighborhood, in particular if E is compact. The analogue for non-Hausdorff E is false. In particular the usual one-point qcompactification of a Hausdorff E is not Hausdorff and not locally qcompact if E is not locally compact.

We say that E is WILKER or has *upper semicontinuous (usc) intersection* if for each instance of $Q \subseteq G_1 \cup G_2$ there are Q_{-1}, Q_2 such that $Q_1 \subseteq G_1, Q_2 \subseteq G_2$ and $Q \subseteq Q_1 \cup Q_{-2}$ (cf. WILKER (1970)). The latter property turns out to be equivalent to the mapping

$$(F_1, F_2) \mapsto F_1 \cap F_2$$

from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$ being usc when $\mathcal{F}$ is provided with Fell's topology (cf. FELL (1962)), or the sup vague topology (cf. VERVAAT (1988b)), which explains our name of it.

Recall that E is T_0. In the remainder of the paper one or more of the following four hypotheses will be assumed:

- E is sober,
- E is Q_δ,
- E is locally qcompact,
- E has usc intersection.

Here is a little more background for these regularity assumptions.

A space is called ***supersober*** if the set limit points of each ultrafilter is either empty or a singleton closure (HOFMAN & MISLOVE (1980, Def. 1.2)). A T_1 space which is not Hausdorff is not supersober. If E is supersober, then E is sober (GIERZ et al. (1980, VII-1.11)) and Q_δ (see Proposition 1.3 below). If E is sober, Q_δ and locally qcompact, then E is supersober (HOFMANN & MISLOVE (1980, Th. 4.8)).

Sufficient conditions for the usc intersection property have been obtained in VERVAAT (1988b). Here are three: E is Hausdorff, E is locally qcompact, or each $Q \subseteq E$ is locally qcompact in the relative topology of E.

For completeness we enclose there following result, whose proof we could not find in the literature.

PROPOSITION 1.3 (VAN ROOIJ (1984)). *If E is supersober, then E is Q_δ.*

PROOF. Let $(Q_\alpha)_\alpha$ be a collection in $\mathcal{Q}$. We must prove that $R = \bigcap_\alpha Q_\alpha \in \mathcal{Q}$. As R is saturated, it suffices to prove that R is qcompact. To this end, let $(F_\beta)_\beta$ be a collection of closed sets such that R has nonempty intersections with all finite subcollections. We must show that $R \cap (\bigcap_\beta F_\beta) \neq \emptyset$. Extend $(R \cap F_\beta)_\beta$ to an ultrafilter $\mathcal{X}$ in E and set $Y = \bigcap_{X \in \mathcal{X}} X^-$. Note that $Y \subseteq \bigcap_\beta F_\beta$. We will show that $R \cap Y \neq \emptyset$.

As $Q_\alpha \supseteq R, Q_\alpha$ intersects every member of $\mathcal{X}$. As Q_α is compact, $Q_\alpha \cap Y \neq \emptyset$. In particular, $Y \neq \emptyset$, so $Y = \{x\}^-$ for some x, since E is supersober. We have $Q_\alpha \cap \{x\}^- \neq \emptyset$, so $x \in \mathrm{sat}(Q_\alpha) = Q_\alpha$. It follows that $x \in R$, so $R \cap Y \neq \emptyset$. □

2. CAPACITIES ON $\mathcal{Q}$

An increasing mapping $c : \mathcal{Q} \longrightarrow \bar{\mathbb{R}}_+$ is said to be *outer* (or *right*) *continuous* if $c(Q) < x \in \mathbb{R}_+$ implies the existence of a $G \supseteq Q$ such that $c(Q_1) < x$ whenever $Q_1 \subseteq G$. If addition $c(\emptyset) = 0$, then c is called a *capacity*, or longer, a *capacity on $\mathcal{Q}$*.

Capacities on $\mathcal{Q}$ extend to the powerset of E in two steps:

$$c(K) := c(\mathrm{sat}K)\ forK \in \mathcal{K} \tag{2.1a}$$

$$c(A) = \sup_{K \subseteq A} c(K). \tag{2.1b}$$

As a consequence the outer continuity extends from $\mathcal{Q}$ to $\mathcal{K}$, and $c(\mathrm{sat}A) = c(A)$ for $A \subset E$. The extension to $\mathcal{G}$ is of particular importance. It is *inner* (or *left*) *continuous* in the following sense: Whenever $c(G) > x$, there is a $Q \subseteq G$ such that $c(G_1) > x$ for all $G_1 \supseteq Q$. The following is an obvious consequence of outer continuity.

PROPOSITION 2.1. *Let c be a capacity on $\mathcal{Q}$. Then*

$$c(Q) = \inf_{G \supseteq Q} c(G).$$

The above notion of capacities coincides with VERVAAT's (1988a). Two other notions occur in the present paper, Choquet capacities (defined on the powerset of E) in Section 4, and capacities on $\mathcal{G}$ in Section 5.

In the literature one often sees the following 'upper continuity' condition for capacities on $\mathcal{Q}$, which reads in a generalization to sober E:

$$\inf_\alpha c(Q_\alpha) = c(\bigcap_\alpha Q_\alpha) \tag{2.2}$$

for all decreasing nets $(Q_\alpha)_\alpha)$ in $\mathcal{Q}$ (recall that $\mathcal{Q}$ is closed for intersections of decreasing nets if E is sober, cf. Theorem 1.2). Functions satisfying (2.2) are said to *preserve filtered intersections* (in $\mathcal{Q}$).

PROPOSITION 2.2. *Let E be sober and let $c : \mathcal{Q} \longrightarrow \bar{\mathbb{R}}_+$ be increasing. Then the following holds:*

(a) *If c is outer continuous, then c preserves filtered intersections.*
(b) *Let E be locally qcompact. If c preserves filtered intersections, then c is outer continuous.*

PROOF. Part (a) is a simple consequence of Proposition 2.1 and Theorem 1.2, while part (b) follows from the fact that for each $Q \subseteq E$ there is a decreasing net $(Q_\alpha)_\alpha$ with intersection $\bigcap_\alpha Q_\alpha = Q$ and $Q \subseteq Q_\alpha^\circ$ for all α. For details, we refer to VERVAAT (1988a), Theorem 15.4. □

Without local qcompactness, (b) need not be true. It fails for $c = \#$ (cardinality) in case $\mathcal{Q}$ happens to consist of all finite subsets of E and all nonempty open sets are infinite. On the other hand, if E is locally qcompact, sober and second countable, then c preserves filtered intersections if $c(Q_n) \downarrow c(Q)$ whenever $Q_n \downarrow Q$. Here are some additional properties that capacities will be assumed to possess in the next sections.

Let $\mathcal{L}$ be a lattice of subsets of E (i.e. closed for finite intersections and unions). A mapping $c : \mathcal{L} \longrightarrow \bar{\mathbb{R}}_+$ is called *modular* if

$$c(A_1 \cup A_2) + c(A_1 \cap A_2 = c(A_1) + c(A_2)$$

for $A_1, A_2 \in \mathcal{L}$. It is called *submodular* (or *strong subadditive*) if the equality sign above is replaced by $\leq$ and *supermodular* if it is replaced by $\geq$.

If E is Q_δ, then $\mathcal{Q}$ is a lattice. The following is a simple exercise.

PROPOSITION 2.1. *If E is Q_δ and c is a supermodular capacity, then its extension to the powerset of E is supermodular too.*

Submodularity does not extend so easily. The best results are obtained in the context of capacitability in Section 4, but the following more modest results will be used in Section 3, which we wish to keep independent of Section 4.

Our first result is Lemma 9.9 of CHOQUET (1969) in a larger generality. Choquet's proof uses normality. It is easier to resort to the usc intersection property instead, as do DELLACHERIE & MEYER (1978) in their proof of Choquet's lemma.

PROPOSITION 2.4. *Let E be a Q_δ space with usc intersection, and let c be a submodular capacity on $\mathcal{Q}$. Then its extension to $\mathcal{G}$ is submodular.*

PROOF. Argue as in the last paragraph of DELLACHERIE & MEYER'S (1978) proof of their Theorem III.42, but replace $\mathcal{K}$ by $\mathcal{Q}$. □

PROPOSITION 2.5. *Let E be Q_δ with usc intersection, and let c be a modular capacity on $\mathcal{Q}$. Write*

$$\mathcal{H} = \{A \subseteq E : c(A) = \inf_{G \supseteq A} c(G)\},$$
$$\mathcal{H}_f = \{A \in \mathcal{H} : c(A) < \infty\}.$$

Then

$$c(A_1 \cup A_2) + c(A_1 \cap A_2) = c(A_1) + c(A_2) \tag{2.3}$$

for $A_1, A_2 \in \mathcal{H}$ and $\mathcal{H}_f$ is a lattice, so c is modular on $\mathcal{H}_f$.

PROOF. By Proposition 2.4, we know that c is submodular on $\mathcal{G}$. This implies that c is submodular on $\mathcal{H}$, and (2.3) follows by Proposition 2.3.

It remains to prove that $\mathcal{H}_f$ is a lattice. Fix $A_1, A_2 \in \mathcal{H}_f$, let $\epsilon > 0$ and choose $G_i \supseteq A_{-i}$ such that $c(G_i) < c(A_i) + \epsilon$, for $i = 1, 2$. Then

$$\begin{aligned} c(A_{-1} \cup A_2) + c(A_1 \cap A_2) + 2\epsilon &= c(A_1) + c(A_2) + 2\epsilon \\ > c(G_1) + c(G_2) &= c(G_1 \cup G_2) + c(G_1 \cap G_2). \end{aligned}$$

Hence

$$0 \leq c(G_1 \cup G_2) - c(A_1 \cup A_2) + c(G_1 \cap G_2) - c)A_1 \cap A_2) < 2\epsilon.$$

So

$$0 \leq c(G_1 \cup G_{-2}) - c(A_1 \cup A_2) < 2\epsilon$$

and

$$0 \leq c(G_1 \cap G_{-2}) - c(A_1 \cap A_2) < 2\epsilon,$$

showing that $A_1 \cup A_2, A_1 \cap A_2 \in \mathcal{H}_f$. □

3. Radon measures

In the present section we want to characterize those capacities which are restrictions to $\mathcal{Q}$ of Radon measures on the Borel-σ-field $\mathcal{B}$ of E. However, we must adapt the definition of Borel-σ-field and Radon measure to the larger generality of non-Hausdorff spaces.

We define the *Borel-σ-field* $\mathcal{B}$ as the σ-field generated by $\mathcal{G}$ and $\mathcal{Q}$. If $\mathcal{Q} \subseteq \mathcal{F}$ (in particular if E is Hausdorff) or $\mathcal{G}$ is second countable, then $\mathcal{B}$ is already generated by $\mathcal{G}$ alone, and we return to the usual definition of Borel-σ-field. Note that unsaturated qcompact sets need not be Borel measurable (consider $\mathbb{R}\downarrow$ in Section 1).

Let $\mathfrak{c}$ consist of the finite unions of sets of the form $Q \backslash G$. Note that $\mathfrak{c}$ consists of qcompact sets, not necessarily saturated. We say that a countably additive measure μ on $(E, \mathcal{B})$ is a *Radon measure* if μ is finite-valued on $\mathcal{Q}$ and $\mathfrak{c}$-*inner regular*, i.e.,

$$\mu(B) = \sup\{\mu(C) : \; B \supseteq C \in \mathfrak{c}\} \tag{3.1}$$

for $B \in \mathcal{B}$.

If E is T_1, then all sets are saturated and $\mathfrak{c} = \mathcal{Q}$. In this case our notion of Radon measure coincides with more classical versions as in Berg et al. (1984).

The following is known for locally compact Hausdorff E (cf. Bourbaki (1965), Theorem 3.1 and Remark 1). See also the discussion on p. 62-63 of Berg et al. (1984).

Theorem 3.1. *Let E be Hausdorff and locally compact. A capacity c is the restriction of a unique Radon measure to $\mathcal{Q}$ iff c is finite-valued on $\mathcal{Q}$,*

$$c(Q_1 \cup Q_2) \leq c(Q_1) + c(Q_2) \tag{3.2a}$$

for all $Q_{-1}, Q_2 \supseteq E$, *and, in addition,*

$$c(Q_1 \cup Q_2) = c(Q_1) + c(Q_2) \tag{3.2b}$$

if $Q_1 \cap Q_2 = \emptyset$.

If c extends to a Radon measure on $\mathcal{B}$ and $\mathcal{Q}$ is a lattice, then c is modular on $\mathcal{Q}$:

$$c(Q_1 \cup Q_2) + c(Q_1 \cap Q_2) = c(Q_1) + c(Q_2) \tag{3.3}$$

for $Q_1, Q_2 \subseteq E$. On the other hand, (3.3) implies (3.2) , so Theorem 3.1 also holds true with (3.2) replaced by (3.3).

We cannot expect that Theorem 3.1 holds as it stands for non-Hausdorff E because of the following example: If G is nonempty open subset of $\mathbb{R}_+ \downarrow$, then $G = [0, x)$ for some x, $0 < x \leq \infty$. Hence every nonempty saturated subset of $\mathbb{R}_+ \downarrow$ contains the point 0, and we see that if both $Q_1, Q_2 \subseteq \mathbb{R}_+ \downarrow$ are nonempty, then $Q_1 \cap Q_2 \neq \emptyset$. Thus, for $\mathbb{R}_+ \downarrow$ condition (3.2b) does not tell us anything about c. Neither does (3.3), but for $(\mathbb{R}_+ \downarrow)^2$ (3.3) becomes a restrictive and meaningful hypothesis.

We will generalize Theorem 3.1 with (3.3) instead of (3.2) to sober Q_δ spaces with usc intersection ($[0, \infty)\downarrow$ is such space). Our proof is based on the following two general results.

PROPOSITION 3.2. *Let $\mathcal{L}$ be a lattice of subsets of E containing $\emptyset$. Let $c : \mathcal{L} \longrightarrow \mathbb{R}_+$ be increasing and modular with $c(\emptyset) = 0$. Then c extends to a unique finitely additive finite measure μ on the ring $\mathcal{R}$ generated by $\mathcal{L}$.*

PROOF. There is a unique real-valued additive set function μ on $\mathcal{R}$ that extends c. This follows by a result of SMILEY (1944) and and PETTIS (1951) (see also KISYŃSKI (1968), LIPECKI (1971) and TOPSØE (1978, Corollary to Lemma 8.1)). The ring $\mathcal{R}$ consists of all finite disjoint unions of sets of the form $L \backslash L'$, and

$$\mu(L \backslash L') = c(L) - c(L \cap L') \geq 0,$$

since c is increasing. Hence μ is nonnegative. □

We say that a class $\mathfrak{c}$ of subsets of E is *compact* if $C_n \in \mathfrak{c}$ for $n = 1, 2, \ldots$ and $\bigcap_n C_n = \emptyset$ imply $\bigcap_{n=1}^m C_n = \emptyset$ for some $m \in \mathbb{N}$.

PROPOSITION 3.3 NEVEU (1965), Proposition I-6-2 and Exercise I-6-1). *Let $\mathcal{A}$ be a field of subsets of E and μ: $\mathcal{A} \longrightarrow \mathbb{R}_+$ a finitely additive finite measure. If $\mathfrak{c} \subseteq \mathcal{A}$ is a compact class and μ is $\mathfrak{c}$-inner regular, then μ is countably additive on $\mathcal{A}$ and can be extended to a unique $\mathfrak{c}$-inner regular measure on the σ-field generated by $\mathcal{A}$.*

We now apply these results to our case interest. It is convenient to fix a modular finite-valued capacity c on $\mathcal{Q}$. We write

$$\mathcal{G}_f = \{G \in \mathcal{G} : c(G) < \infty\}.$$

By Proposition 2.1 there is for each $Q \subseteq E$ a $G \in \mathcal{G}_f$ with $G \supseteq Q$, so $\bigcup_{G \in \mathcal{G}_f} G = E$.

LEMMA 3.4. *If E is sober and Q_δ, then the collection $\mathfrak{c}_\circ$ of all sets of the form $Q \backslash G$ is a compact class, as well as $\mathfrak{c}$ consisting of all finite unions from $\mathfrak{c}_\circ$.*

PROOF. Suppose $\bigcap_n (Q_n \backslash G_n) = \emptyset$. Then $\bigcap_n Q_n \subseteq \bigcup_n G_n$. Now $\bigcap_n Q_n \in \mathcal{Q}$ because E is Q_δ, so there is an $m \in \mathbb{N}$ such that $\bigcap_n Q_n \subseteq \bigcup_{n=1}^m G_n$. By sobriety of E there is an $l \in \mathbb{N}$ such that $\bigcap_{n=1}^l Q_n \subseteq \bigcup_{n=1}^m G_n$ (cf. Theorem 1.2 (b)), so $\bigcap_{n=1}^k (Q_n \backslash G_n) = \emptyset$ if $k \geq \max(l, m)$.

This proves that $\mathfrak{c}_\circ$ is a compact class. Now $\mathfrak{c}_\circ$ is closed for arbitrary intersections, because $\mathcal{Q}$ is so. By Lemma I-6-1 of Neveu (1965) or Theorem III.4 of DELLACHERIE & MEYER (1978), then also $\mathfrak{c}$ is a compact class. □

Let $\mathcal{L}$ be the lattice generated by $\mathcal{Q} \cup \mathcal{G}_f$ and write $\mathcal{R}$ for the ring generated by $\mathcal{L}$. Note that $\mathfrak{c}_\circ \subseteq \mathcal{R} : Q \backslash G = Q \backslash (G \cap G_1)$ for $Q \subseteq G_1 \in \mathcal{G}_f$. So $\mathfrak{c} \subseteq \mathcal{R}$ as well.

LEMMA 3.5. *Let E be Q_δ with usc intersection. There is a unique finitely additive finite measure μ on $\mathcal{R}$, such that $\mu = c$ on $\mathcal{L}$.*

PROOF. By Propositions 2.5 and 2.1, c is modular on $\mathcal{L} \subseteq \mathcal{H}_f$. The lemma follows by Proposition 3.2. □

In general we do not have $c = \mu$ on $\mathcal{R}$ (consider $\mathbb{R}\!\downarrow$ in Section 1). The next result explores to what extent $c = \mu$ can hold for Radon measures μ.

PROPOSITION 3.6. *Let μ be a Radon measure on $(E, \mathcal{B})$. if $\mu = c$ on $\mathcal{Q}$, then $\mu(B) = c(B)$ for all saturated $B \in \mathcal{B}$.*

PROOF. Note first that c is $\mathcal{Q}$-inner regular, while μ is $\mathfrak{c}$-inner regular. Since $\mathcal{Q} \subseteq \mathfrak{c}$, we have $c \leq \mu$ on $\mathcal{B}$.

Now let $B \in \mathcal{B}$ be saturated, and take $x < \mu(B)$. Then $x < \mu(C)$ for some $C \in \mathfrak{c}$, $C \subseteq B$. Since C is qcompact, so is $\text{sat}(C)$, i.e., $\text{sat}(C) \in \mathcal{Q}$. We also have $\text{sat}(C) \subseteq \text{sat}(B) = B$, so

$$x < \mu(C) \leq \mu(\text{sat}(C)) = c(\text{sat}(C)) \leq c(\text{sat}(B)) = c(B).$$

Thus $\mu(B) \leq c(B)$ and equality follows. □

Here is the main result of this section.

THEOREM 3.7. *Let E be sober and Q_δ with usc intersection, and let c be a finite-valued modular capacity on $\mathcal{Q}$. Then there exists a unique Radon measure μ on $\mathcal{B}$ such that $\mu = c$ on $\mathcal{Q}$ (and hence on all saturated Borel sets).*

PROOF. In the first instance we restrict our attention to the case $c(E) < \infty$. Then $\mathcal{G}_f = \mathcal{G}$ and $E \in \mathcal{R}$, so $\mathcal{R}$ is the field generated by $\mathcal{Q} \cup \mathcal{G}$. By Lemma 3.5, c in $\mathcal{L}$ extends to a unique finitely additive measure μ on $\mathcal{R}$. By Theorem 3.3 and Lemma 3.4 it only remains to check that μ is $\mathfrak{c}$-inner regular. Because $\mathcal{R}$ consists of finite unions of sets $L_1 \backslash L_2$ with $L_1, L_2 \in \mathcal{L}$, it is sufficient to check that μ is $\mathfrak{c}_\circ$-inner regular ($\mathfrak{c}_\circ$ as in Lemma 3.4) on the latter sets:

$$\mu(L_1 \backslash L_2) = \sup\{\mu(Q \backslash G) : Q \backslash G \subseteq L_1 \backslash L_2\}. \tag{3.4}$$

We may assume $L_2 \subset L_1$. The inequality $\geq$ in (3.4) is obvious, but $\leq$ needs a proof.

Let $\epsilon > 0$. Note that $L_i \in \mathcal{H}_f$ (cf. Propositions 2.5 and 2.1). Hence we may select $Q \subseteq L_1$ such that $c(Q) > c(L_1) - \epsilon$, and $G \supseteq L_2$ such that $c(G) < c(L_2) + \epsilon$. Then $Q \backslash G \subseteq L_1$ and

$$\begin{aligned} \mu(Q \backslash G) = \ & c(Q) - c(Q \cap G) \geq c(Q) - c(G) \\ & > c(G_1 \cap S_1) - c(G_2 \cap S_2) - 2\epsilon = \mu(L_1 \backslash L_2) - 2\epsilon. \end{aligned}$$

This proves (3.4) and completes the proof of the theorem in case $c(E) < \infty$.

We now drop this assumption. The conclusion of the theorem holds for each $G \in \mathcal{G}_f$, considered as space on its own, and the resulting μ_G on the Borel sets of G for $G \in \mathcal{G}_f$ are consistent in the sense that $\mu_{G_1 \cap G_2} = m\mu G_2$ on the Borel sets of $G_1 \cap G_2$. We now define μ on $\mathcal{B}$ by

$$\mu(B) = \sup_{G \in \mathcal{G}_f} \mu_G(G \cap B) \tag{3.5}$$

for $B \in \mathcal{B}$. Then $\mu = \mu_G$ on the Borel sets of G if $G \in \mathcal{G}_f$. It is routine to show that μ is countably additive, hence a measure on $(E, \mathcal{B})$.

We now prove that μ is $\mathfrak{c}$-inner regular. It suffices to prove that $\mu(B) \leq \sup_{C \subseteq B} \mu(C)$. If $x < \mu(B)$, then there is a $G \in \mathcal{G}_f$ such that $x < \mu(B \cap G)$, so there is a $C \in \mathfrak{c}$, $C \subseteq B \cap G$ such that $x < \mu(C)$, which proves the desired inequality.

As a Radon measure, μ is determined by its values on $\mathfrak{c}$, whence by the μ_G for $G \in \mathcal{G}_f$. So μ is unique. □

Note that outside the context of Radon measures, μ need not be unique (cf. SCHWARTZ (1973), pp. 44-45).

Recently NORBERG (1989) has shown that if $c : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ is increasing, finite and modular on

$$\{G \in \mathcal{G} : G \subseteq Q \text{ for some } Q\}$$

and such that $c(\emptyset) = 0$, then c extends to a unique measure on the σ-field generated by $\mathcal{G}$, provided that E is locally qcompact, sober and second countable, and that c is inner continuous. So our Theorem 3.7 both complements and extends Norberg's result.

4. CHOQUET CAPACITIES AND CAPACITABILITY

In this section E is assumed sober and $\mathcal{Q}_\delta$. By a *Choquet capacity* we mean an extended real-valued mapping e, defined at all subsets of E, such that

$$(i) \qquad e(A) \leq e(B) \text{ for } A \subseteq B \subseteq E,$$

$$(ii) \qquad e(A_n) \uparrow e(A) \text{ for } A, A_1, A_{-2}, \ldots \subseteq E \text{ with } A_n \uparrow A,$$

$$(iii) \qquad e(Q_n) \downarrow e(Q) \text{ as } Q_n \downarrow Q.$$

A set $A \subseteq E$ is called *capacitable* if

$$e(A) = \sup_{Q \subseteq A} e(Q).$$

Choquet's theorem on capacitability says that every $\mathcal{Q}$-analytic subset of E is capacitable (cf. DELLACHERIE & MEYER (1978), Theorem III.28 and Definition III.7). There it is also shown that the collection $\mathfrak{a}(\mathcal{Q})$ of all $\mathcal{Q}$-analytic sets is closed for countable unions and intersections, contains $\mathcal{Q}$ (Theorem III.8), that $E \in \mathfrak{a}(\mathcal{Q})$ iff E is a countable union of sets in $\mathcal{Q}$ (the remark after Definition III.7), and that $\mathfrak{a}(\mathcal{Q})$ contains the σ-field $\sigma(\mathcal{Q})$ generated by $\mathcal{Q}$ iff all sets of the form $E \backslash Q$ are $\mathcal{Q}$-analytic (Theorem III.12).

Let $e : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ be increasing and submodular. assume that $e(G_n) \uparrow e(G)$ whenever $G_n \uparrow G$. For $A \subseteq E$, we define

$$e^*(A) = \inf_{G \supseteq A} e^*(G).$$

Then e^* satisfies conditions (i) and (ii) above (cf. DELLACHERIE & MEYER (1978, Theorem II.32)). Consequently, e^* is a Choquet capacity iff (iii) holds, i.e., $e^*(Q_n) \downarrow e^*(Q)$ whenever $Q_n \downarrow Q$. Inner continuity of e is a sufficient condition for this, as shows the following proposition.

PROPOSITION 4.1. *Let E be a sober $\mathcal{Q}_\delta$ space, and let $e : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ be increasing, submodular and inner continuous. Then e^* is a submodular Choquet capacity.*

PROOF. DELLACHERIE & MEYER (1978) prove this result for Hausdorff E. Their proof applies with obvious changes. □

Also the next result is known in the Hausdorff case (cf. DELLACHERIE & MEYER (1978), Theorem III.42). It is a corollary to Proposition 4.1.

THEOREM 4.2. *Let E be a sober $\mathcal{Q}_\delta$ space with usc intersection, and let c be a submodular capacity on $\mathcal{Q}$. For $A \subseteq E$, let*

$$\tilde{c}(A) = \inf_{G \supseteq A} c(G).$$

Then $\tilde{c}$ is a submodular Choquet capacity relative to $\mathcal{Q}$. Hence $c(A) = \tilde{c}(A)$ for all $\mathcal{Q}$-analytic sets $A \subseteq E$. If c is modular on $\mathcal{Q}$, then so is $\tilde{c}$ on $\mathfrak{a}(\mathcal{Q})$.

PROOF. Every assertion except the last follows as in DELLACHERIE & MEYER (1978). If c is modular on $\mathcal{Q}$, then, by Proposition 2.3, its extension (2.1) is supermodular on the powerset of E, in particular on $\mathfrak{a}(\mathcal{Q})$, where it coincides with the submodular function $\tilde{c}$. □

We now apply Theorem 4.2 to random set theory. Let ξ be a random closed set in E (which of course is assumed to be sober and Q_δ with usc intersection). By this we mean that ξ is an $\mathcal{F}$-valued function on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$ such that

$$\{\xi \cap Q \neq \emptyset\} \in \mathcal{A}.$$

Let

$$T(Q) = \mathbf{P}\{\xi \cap Q \neq \emptyset\}.$$

It is not hard to see that T is a submodular capacity, and that its extension to $\mathcal{G}$ is

$$T(G) = \mathbf{P}\{\xi \cap G \neq \emptyset\}.$$

Let $B \in \mathfrak{a}(\mathcal{Q})$. By Theprem 4.2,

$$\sup_{Q \subseteq B} T(Q) = \inf_{G \supseteq B} T(G).$$

It follows that the event $\{\xi \cap B \neq \emptyset\}$ belongs to the completion of $\mathcal{A}$ w.r.t. $\mathbf{P}$, as can be seen by arguments on page 30 of MATHERON (1975).

An analogous result holds for the random usc functions discussed by VERVAAT (1988a). NORBERG (1986) treats the case when E is locally compact, second countable and Hausdorff.

5. SEMICONTINUITY

Clearly the collection

$$\{\{Q \in \mathcal{Q} : Q \subseteq G\} : G \in \mathcal{G}\}$$

is closed for finite intersections, so it may serve as a vase for a topology on $\mathcal{Q}$. Note that all sets $\mathcal{U}$ in this base are *decreasing* in the sense that

$$Q_1 \subseteq Q \in \mathcal{U} \Longrightarrow Q_1 \in \mathcal{U}.$$

Consequently, all open subset of $\mathcal{Q}$ are decreasing as well.

A mapping $c : \mathcal{Q} \longrightarrow \bar{\mathbb{R}}_+$ is usc with respect to the topology just introduced if the set $\{Q \in \mathcal{Q} : c(Q) < x\}$ is open for all $x \in \bar{\mathbb{R}}_+$. In particular, these sets are decreasing, so any usc c must be increasing. The remaining part of the next result is easy.

PROPOSITION 5.1 (VERVAAT (1988a), Theorem 15.6). *Let $c : \mathcal{Q} \longrightarrow \bar{\mathbb{R}}_+$. Then c is usc iff c is outer continuous (and hence increasing). In particular, when $c(\emptyset) = 0$, c is usc iff c is a capacity.*

Here it is appropriate to recall Proposition 2.2, which gives

COROLLARY 5.2. *Let E be sober and fix $c : \mathcal{Q} \longrightarrow \bar{\mathbb{R}}_+$. If c is usc, then c is increasing and preserves filtered intersections. The converse holds true is E is locally qcompact.*

So far, capacities (not Choquet capacities) have been defined in the first instance on $\mathcal{Q}$, with extensions afterwards to all subsets of E, in particular to $\mathcal{G}$:

$$c(G) = \sup_{Q \subseteq G} c(Q).$$

Hence forth we call such c *capacities on* $\mathcal{Q}$, and the induced functions on $\mathcal{G}$ their *extensions to* $\mathcal{G}$.

We now will introduce a topology on $\mathcal{G}$ and define *capacities on* $\mathcal{G}$ to be lsc functions c on $\mathcal{G}$ mapping $\emptyset$ on 0. These capacities turn out to be increasing, so they can be extended to all subsets of E, in particular to $\mathcal{Q}$:

$$c(Q) = \inf_{G \supseteq Q} c(G).$$

We call the induced function on $\mathcal{Q}$ the *extension* of c to $\mathcal{Q}$.

Directed sets in $\mathcal{G}$ play a central role in the definitions. A *directed* set in $\mathcal{G}$ is a parametrized collection $(G_i)_i \subseteq \mathcal{G}$, such that for each i_1, i_2 there is an i_3 with $G_{i_1} \cup G_{i_2} \subseteq G_{i_3}$. In other words, a directed set is nothing but an increasing net.

Here is the topology on $\mathcal{G}$. It is called the *Scott topology* (cf. SCOTT (1972), GIERZ et al. (1980) and HOFMANN & MISLOVE (1980)). A subclass $\mathcal{U} \subset \mathcal{G}$ belongs to this topology (or is *Scott open*) iff $\mathcal{U}$ is increasing:

$$G_1 \supseteq G \in \mathcal{U} \Longrightarrow G_1 \in \mathcal{U},$$

and directed sets in $\mathcal{G}$ cannot penetrate $\mathcal{U}$ only by their union:

$$(G_i)_i \text{ directed, } \bigcup_i G_i \in \mathcal{U} \Longrightarrow G_i \in \mathcal{U} \text{ for some } i.$$

A function $c : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ is said to *preserve directed unions* if

$$c(\bigcap_i G_i) = \sup_i c(G_i)$$

for all directed sets $(G_i)_i$. One can characterize the Scott topology as the coarsest for which all such c are continuous.

A mapping $c : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ is lsc iff the sets $\{G \in \mathcal{G} : c(G) > x\}$ are Scott open for $x \in \mathbb{R}_+$. In particular these sets are increasing, so such a c is increasing. An increasing $c : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ preserves directed unions iff $c(\bigcap_i G_i) > x$ with $(G_i)_i$ directed implies $c(G_i) > x$ for some i. It is clear that this holds iff c is lsc. Hence

PROPOSITION 5.3. *Let $c : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$. Then c is lsc iff c preserves directed unions.*

Recall that $c : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ is a capacity on $\mathcal{G}$ iff c is lsc and $c(\emptyset) = 0$. here is the connection with capacities on $\mathcal{Q}$.

THEOREM 5.4. *The following four propositions are true:*

a1 *If $c : \mathcal{Q} \longrightarrow \bar{\mathbb{R}}_+$ is increasing and $c(\emptyset) = 0$, then it extension to $\mathcal{G}$ is a capacity on $\mathcal{G}$.*

a2 *The extension of the latter capacity on $\mathcal{G}$ to $\mathcal{Q}$ equals c in case c was already a capacity on $\mathcal{Q}$.*

b1 *If $c : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ is increasing and $c(\emptyset) = 0$, then its extension to $\mathcal{Q}$ is a capacity on $\mathcal{Q}$*

b2 *The extension of the latter capacity on $\mathcal{Q}$ to $\mathcal{G}$ equals c in case c was already a capacity on $\mathcal{G}$ and E is locally qcompact.*

PROOF.
a1 We only need to show that the extension of c to $\mathcal{G}$ preserves directed unions. If $(G_i)_i$ is directed in $\mathcal{G}$ and $G = \bigcap_i G_i$, then $c(G) \geq \sup_i c(G_i)$ because c is increasing. For each $Q \subseteq G$ we have $Q \subseteq G_i$ for some i, so $c(Q) \leq \sup_i c(G_i)$. Hence $c(G) \leq \sup_i c(G_i)$.

a2 This is Proposition 2.1 once again.

b1 We only need to show that if $c(Q) < x$, then $\sup_{Q \subseteq G} c(Q) < x$ for some $G \supseteq Q$. But this is obvious from the definition of $c(Q)$.

b2 Let c be a capacity on $\mathcal{G}$ and write d for its extension to $\mathcal{Q}$. Let further e be the extension of d to $\mathcal{G}$. We shall prove that $c = e$. Fix $G \subseteq E$. If $Q \subseteq G$, then $d(Q) \leq c(G)$. Hence

$$e(G) = \sup_{Q \subseteq G} d(Q) \leq c(G).$$

Now let $x < c(G)$. Whenever $s \in G$, there is by local qcompactness a $Q \subseteq G$ with $s \in Q^\circ$. Thus we may choose an increasing net $(Q_\alpha)_\alpha$ such that $Q_\alpha \subseteq G$ for all α and $\bigcup_\alpha Q_\alpha^\circ = G$. But the $x < c(Q_\alpha^\circ)$ for some α. If $Q_\alpha \subseteq G'$, then $c(Q_\alpha^\circ) \leq c(G')$. Hence $c(Q_\alpha^\circ) \leq d(Q_\alpha)$. But $d(Q_\alpha) \leq e(G)$, so we have $x < e(G)$, showing that $c(G) \leq e(G)$. Hence $c(G) = e(G)$. □

In particular this result tells us that we do not need discriminate between capacities on $\mathcal{Q}$ and $\mathcal{G}$ in case E is locally qcompact.

The condition of local qcompactness cannot be omitted in b2 of the theorem, as shows the following example. Let E be the reals with the right half-open topology (Ex. 51 in STEEN & SEEBACH (1978)), having as base all intervals

$[a, b)$ for $a < b$. Then E is Hausdorff, so $\mathcal{Q}(E) = \mathcal{K}(E)$, and all compact sets are countable. Moreover, the Borel-σ-field of E coincides with the classical Borel-σ-field of $\mathbb{R}$. Let c be Lebesgue measure. We must show that c is lsc, i.e., that $c(\bigcup_i G_i) = \sup_i c(G_i)$ for all directed families $(G_i)_i$ of open sets. For classical open sets this follows by Theorem 2.1.5(i) in Berg et al. (1984). The result then extends to open sets in the new setting by arguments as in point 7 of Ex. 51 in STEEN & SEEBACH (1978). So c is a capacity on $\mathcal{G}$. Since all compact sets are countable, we have $d = 0$ on $\mathcal{K}$ and $e = 0$ on $\mathcal{G}$ in the terminology of the proof of Theorem 5.4.b2.

We have discussed functions $c : \mathcal{G} \longrightarrow \bar{\mathbb{R}}_+$ that preserve directed unions. A stronger condition is that c preserves all unions. Such c's are called *sup measures* in VERVAAT (1988a). There it is shown that each sup measure is of the form

$$c(G) = \sup_{s \in G} g(s),$$

where g is usc with range $\bar{\mathbb{R}}_+$, and given by

$$g(s) = \inf_{G \ni s} g(G)$$

for $s \in E$.

If c is a capacity on $\mathcal{G}$, i.e., already known to preserve directed unions, then c preserves all unions iff

$$c(G_1 \cap G_2) = c(G_1) \vee c(G_2).$$

6. Continuous lattices of capacities and related topologies

In the previous section we equipped $\mathcal{Q}$ with the topology generated by all sets of the form $\{Q : Q \subseteq G\}$, and saw that an $\bar{\mathbb{R}}_+$-valued function on $\mathcal{Q}$ is a capacity iff it is usc and maps $\emptyset$ to 0.

Now we take E locally qcompact sober. Then $\mathcal{Q}$ is locally qcompact, too. In fact $\mathcal{Q}$ is a continuous semi-lattice under reverse inclusion and its topology is the Scott topology (cf. HOFMANN & MISLOVE (1980) and GIERZ et al. (1980) Theorem 8.4 of GERRITSE (1985) or Theorem II.2.8 of GIERZ et al. (1980) now tell us that the collection of all usc functions from $\mathcal{Q}$ to $\bar{\mathbb{R}}_+$ is an upper continuous lattice, i.e., a continuous lattice under the reverse pointwise order. By Theorem I.2.7.(ii) in GIERZ et al. (1980) the sublattice of all capacities on $\mathcal{Q}$ is continuous as well.

In Section 5 we equipped also $\mathcal{G}$ with its Scott topology. It is well known that $\mathcal{G}$ is locally qcompact in this topology if $\mathcal{G}$ is a continuous lattice, which is the case iff sob(E) is locally qcompact. A dual form of Gerritse's Theorem 8.4 now tells us that the collection of all lsc functions from $\mathcal{G}$ into $\bar{\mathbb{R}}_+$ in this case is a continuous lattice under the pointwise ordering. It is obvious that its sublattice of capacities is continuous too.

So, for E locally qcompact and sober, the collection of all capacities on E is a continuous lattice under both the pointwise and the reverse pointwise ordering. (Recall that we may identify capacities on $\mathcal{Q}$ and capacities on $\mathcal{G}$ in this case.)

Next we discuss various ways of defining a compact Hausdorff topology on sets of capacities.

Every continuous lattice can be endowed with a canonical compact Hausdorff topology which is called the *Lawson topology* (cf. GIERZ et al. (1980) or HOFMANN & MISLOVE (1980)). This is the coarsest topology containing the Scott topology and the complements of all Scott qcompact saturated sets. Moreover, a result of LAWSON (1973) shows that this is the only possible compact Hausdorff topology on a continuous lattice in case lattice operations are continuous.

Write $\mathbf{C}_{\mathcal{G}}$ and $\mathbf{C}_{\mathcal{Q}}$ for the collections of capacities on $\mathcal{G}$ and $\mathcal{Q}$, resp. We just saw that $\mathbf{C}_{\mathcal{G}}$ is a continuous lattice if $\mathcal{G}$ is continuous or, equivalently, sob(E) is locally qcompact, and that $\mathbb{C}_{\mathcal{Q}}$ is a continuous lattice under the reverse order (or upper continuous) if E is locally qcompact and sober. In the latter case we need not distinguish between capacities on $\mathcal{G}$ and $\mathcal{Q}$ and we will write $\mathbf{C}$ for either of $\mathbf{C}_{\mathcal{G}}$ and $\mathbf{C}_{\mathcal{Q}}$.

Now the Lawson topology on $\mathbf{C}$ can be introduced in two different ways depending on whether we regard $\mathbb{C}$ as a continuous lattice in the natural pointwise ordering or its reverse. Note however that both methods must yield the same topology.

Let us turn to more explicit characterizations. The members of $\mathbf{C}$ are precisely the usc functions on $\mathcal{Q}$ into $\bar{\mathbb{R}}_+$ that map $\emptyset$ on 0. VERVAAT (1988a) introduces a topology for usc functions which he calls the *sup vague topology*, and shows it to be qcompact, and moreover Hausdorff if $\mathcal{Q}$ is locally qcompact, the case we are considering. Consequently, $\mathbf{C}$ is compact in the relative sup vague topology, which then must coincide with the Lawson topology.

By analogy with measure theory in locally compact spaces VERVAAT (1988a) introduces a topology on $\mathbf{C}$, which may be called the *vague*, since it is generated by all sets of the form $\{c : c(Q) < x\}$ and $\{c : c(G) > x\}$ for Q, G and $x \in \bar{\mathbb{R}}_+$. Note that a sequence (c_n) converges in this topology (or "vaguely") iff $\limsup_n c_n(Q) \le c(Q)$ and $\limsup_n c_n(G) \ge c(G)$. VERVAAT (1988a) shows that the vague topology on $\mathbf{C}$ coincides with the sup vague in the previous paragraph, so in the end with the Lawson topology.

NORBERG (1986) introduces the vague topology for capacities in case E is locally compact and proves that it is compact Hausdorff. His proof applies here too, provided the class $\mathcal{K}$ of compact sets in E is replaced by its counterpart $\mathcal{Q}$ in the non-Hausdorff setting.

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A note on Fell- and Epicompactness[1]

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The present note contains a lattice-theoretic proof of two well-known compactness results, for the Fell topology on the hyperspace of closed subsets of, and the epitopology on the space of all extended-real-valued lower semicontinuous maps on a given topological space.

1 Introduction

Let X be an arbitrary topological space and $\mathcal{F}(X)$ its hyperspace of all closed subsets, endowed with the *Fell topology* (Fell (1962)) with subbasic open sets $\{F \in \mathcal{F}(X) : F \cap K = \phi\}$ for $K \subseteq X$ compact and $\{F \in \mathcal{F}(X) : F \cap G \neq \phi\}$ for $G \subseteq X$ open. It has been proved by many authors that $\mathcal{F}(X)$ is Fell compact (but not necessarily Hausdorff), whatever is the underlying space X (see besides Fell (1962), e.g., Flachsmeyer (1964) and the monographs by Matheron (1975) and Attouch (1984)). The last author also proves compactness of the closely related space $\mathrm{LSC}(X)$ of all lower semicontinuous (lsc) maps from X to $\overline{\mathsf{R}} := [-\infty, \infty]$, equipped with the so-called *epitopology*. This epitopology arises naturally as the relative Fell topology from $\mathcal{F}(X \times \mathsf{R})$ if lsc functions on X are identified with their closed epigraphs in $X \times \mathsf{R}$. Compactness of $\mathrm{LSC}(X)$ can thus be proved via closedness in $\mathcal{F}(X \times \mathsf{R})$, which holds for locally compact (but not necessarily Hausdorff) X (cf. Attouch (1984) for Hausdorff X and Vervaat (1988) for the general case). A more direct approach to the epitopology occurs in the latter paper Vervaat (1988), who calls it the *inf vague topology* and characterizes it by having subbasic open sets $\{f \in \mathrm{LSC}(X) : \inf f(K) > c\}$ and $\{f \in \mathrm{LSC}(X) : \inf f(G) < c\}$ for $K \subseteq X$ compact and $G \subseteq X$ open, respectively, and $c \in \overline{\mathsf{R}}$ (here $\inf A$ denotes the infimum of A for $A \subseteq \overline{\mathsf{R}}$). This characterization gives rise to an alternative, direct compactness proof for $\mathrm{LSC}(X)$, without any restriction on the underlying space X (Vervaat (1988); cf. also the non-standard proof in Norberg (1990)).

The aim of the present note is to provide a lattice-theoretic interpretation of both the Flachsmeyer-Matheron-Attouch proof of Fell compactness of $\mathcal{F}(X)$ and Vervaat's proof of epicompactness of $\mathrm{LSC}(X)$. Obviously, both $\mathcal{F}(X)$ and

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LSC(X) carry a natural partial order, the (reverse) inclusion and pointwise order, respectively. By exhibiting their role in the cited proofs we show that these proofs in fact correspond to special instances of a well-known compactness result in lattice theory. For the special case of a locally compact underlying space X, the connection with lattice theory was made already by Gierz et al. (1980) (see Exc. III.1.13 and Thm. II.4.7; cf. also Gerritse (1985), Thms. 8.4 and 10.4).

2 Preliminaries on complete lattices

We briefly review the relevant lattice-theoretic notions here. For more information we refer to the monograph Gierz et al. (1980) on so-called continuous lattices, from which most of these notions are taken.

Let L be a *complete lattice*, i.e., L is a set equipped with a partial order $\leq$ such that every subset A of L has an infimum $\inf A$ and (hence also) a supremum $\sup A$. In particular, L has a bottom $\mathbf{0}$ and a top $\mathbf{1}$, which also appear as sup and inf, respectively, of the empty set ϕ.

For $x \in L$ and $A \subseteq L$ we write $\uparrow x := \{y \in L : x \leq y\}$ and $\uparrow A := \{y \in L : x \leq y$ for some $x \in A\}$, and we say that A is *increasing* if $A = \uparrow A$; the sets $\downarrow x$ and $\downarrow A$, and the notion of a *decreasing* set are defined dually.

Finally, a subset $D \subseteq L$ is called *directed* (*filtered*) if every finite subset of D has an upper (lower, respectively) bound in D. In particular, this must hold for $\phi \subseteq D$, so D cannot be empty.

We come to the *Lawson topology* on L, which typically stems from the theory of continuous lattices. It has subbasic open sets of two types, increasing and decreasing respectively. The open sets of the first type are those increasing subsets U of L for which

$$\sup D \in U \text{ implies } D \cap U \neq \phi \text{ for all directed sets } D \subseteq L.$$

The second class of subbasic open sets is $\{(\uparrow x)^c : x \in L\}$. The Lawson topology $\lambda(L)$ on L is the topology generated by these two types of sets.

Here is the fundamental compactness result that we are going to use. It has a completely elementary proof, based on Alexander's subbase lemma.

THEOREM (Gierz et al. (1980), Thm. III.1.9). *Each complete lattice is compact in its Lawson topology.*

For the rest we note that the subbasic Lawson open sets of the first type themselves also constitute a topology on L, the so-called *Scott topology* $\sigma(L)$. On the other hand, the topology generated by the sets $(\uparrow x)^c$ for $x \in L$ is called the *lower topology* and denoted by $\omega(L)$. For convenience we also introduce the *dual Scott topology* $\widetilde{\sigma}(L)$ defined with the help of infima of filtered sets, the *upper topology* $v(L)$ generated by the sets $(\downarrow x)^c$ for $x \in L$, and their common refinement $\widetilde{\lambda}(L)$, the *dual Lawson topology*. Of course, L is also $\widetilde{\lambda}$-compact.

3. $\mathcal{F}(X)$ and LSC(X) as complete lattices

Let X be an arbitrary topological space, and consider $\mathcal{F}(X)$ and LSC(X).

Obviously, $\mathcal{F}(X)$ is a complete lattice with respect to the inclusion order (i.e., $F_1 \leq F_2$ iff $F_1 \subseteq F_2$), in which the infimum operation means 'intersection', and the supremum 'closure of the union'. Likewise, $\mathrm{LSC}(X)$ with the pointwise order (i.e., $f_1 \leq f_2$ iff $f_1(x) \leq f_2(x)$ for all $x \in X$) is a complete lattice in which suprema can be taken pointwise; the infimum of a subfamily of $\mathrm{LSC}(X)$ is the largest lsc function smaller than or equal to its pointwise infimum. It remains to point out the relation between the topology and the lattice structure of $\mathcal{F}(X)$ and $\mathrm{LSC}(X)$, respectively. Here are the results.

PROPOSITION 1. *The Fell topology on $\mathcal{F}(X)$ is coarser than the dual Lawson topology.*

COROLLARY. *$\mathcal{F}(X)$ is Fell compact.*

PROOF OF PROPOSITION 1. Let L denote $\mathcal{F}(X)$ with the inclusion order.
First, let $K \subseteq X$ be compact and $U := \{F \in \mathcal{F}(X) \colon F \cap K = \phi\}$. We show that $U \in \widetilde{\sigma}(L)$. It is clear that U is decreasing. So let $D \subseteq L$ be filtered and suppose $\inf D \in U$, i.e., $(\bigcap_{F \in D} F) \cap K = \phi$. By compactness of K we have $(\bigcap_{i=1}^{n} F_i) \cap K = \phi$ for an $n \in \mathsf{N}$ and certain $F_1, \ldots, F_n \in D$. Since D is filtered, there is an $F_0 \in D$ such that $F_0 \subseteq F_i$ for $i = 1, \ldots, n$. Apparently, $F_0 \cap K = \phi$, i.e., $F_0 \in U$, so $D \cap U \neq \phi$.
On the other hand, for open $G \subseteq X$ we have $H := G^c \in L$ and $\{F \in \mathcal{F}(X) \colon F \cap G \neq \phi\} = \{F \in \mathcal{F}(X) \colon F \not\subseteq H\} = (\downarrow H)^c \in \upsilon(L)$. □

Using Vervaat's characterization of the epitopology we can give a completely similar proof for the case of $\mathrm{LSC}(X)$.

PROPOSITION 2. *The epitopology on $\mathrm{LSC}(X)$ is coarser than the Lawson topology.*

COROLLARY. *$\mathrm{LSC}(X)$ is epicompact.*

PROOF OF PROPOSITION 2. Let L now be $\mathrm{LSC}(X)$ with the pointwise order.
Firstly, let $K \subseteq X$ be compact, $c \in \overline{\mathsf{R}}$ and $U := \{f \in \mathrm{LSC}(X) \colon \inf f(K) > c\}$. We prove $U \in \sigma(L)$. Clearly, U is increasing. Now, let $D \subseteq L$ be directed and suppose $g := \sup D \in U$. Then for $x \in K$ we have $g(x) = \sup_{f \in D} f(x) > c$, so there exists an $f_x \in D$ with $f_x(x) > c$. Since f_x is lsc, x has an open neighbourhood G_x such that $f_x(G_x) \subseteq (c, \infty]$. Compactness of K then implies the existence of a finite number of points $x_1, \ldots, x_n$ such that $K \subseteq \bigcup_{i=1}^{n} G_{x_i}$. As D is directed, there is an $f_0 \in D$ such that $f_0 \geq f_{x_i}$ for $i = 1, \ldots, n$. It follows that $f_0(K) \subseteq (c, \infty]$, hence also $\inf f_0(K) > c$, since f_0 as an lsc function attains its minimum on the compact set K. We conclude that $f_0 \in U$, so $D \cap U \neq \phi$.
Secondly, let $G \subseteq X$ be open, $c \in \overline{\mathsf{R}}$ and $V := \{f \in \mathrm{LSC}(X) \colon \inf f(G) < c\}$. Now define $h : X \to \overline{\mathsf{R}}$ by

$$h(x) := \begin{cases} c & \text{if } x \in G, \\ -\infty & \text{if } x \notin G. \end{cases}$$

As G is open, h is lsc, i.e., $h \in L$. Now $V = \{f \in \mathrm{LSC}(X) : f(x) < c \text{ for some } x \in G\} = \{f \in \mathrm{LSC}(X) : f(x) < h(x) \text{ for some } x \in X\} = (\uparrow h)^c \in \omega(L)$. □

On closer inspection the compactness proofs for $\mathcal{F}(X)$ in Flachsmeyer (1964), Matheron (1975) and Attouch (1984) and that for LSC(X) in Vervaat (1988) turn out to reflect the structure of the lattice-theoretic proof of Gierz et al. (1980), Thm. III.1.9 (cited above). Quite different, however, is the original proof of Fell (1962) for $\mathcal{F}(X)$, in terms of universal nets.

Following Vervaat (1988) we could also have proved the results for $\mathcal{F}(X)$ and LSC(X) at once by identifying $\mathcal{F}(X)$ with the space of all lower(!) semicontinuous $\{0,1\}$-valued maps (via the characteristic functions of the *complements* of closed sets) and replacing the range $\overline{\mathsf{R}}$ in LSC(X) by an arbitrary compact subset of $\overline{\mathsf{R}}$ (or, even more generally, another continuous lattice, as in Gierz et al. (1980) and Gerritse (1985)).

If the underlying space X is locally compact (in the strong sense of Fell (1962) in case X fails to be Hausdorff), then the Fell and the epitopology on $\mathcal{F}(X)$ (LSC(X), respectively) are even known to *coincide* with the (dual) Lawson topology (cf. the references at the end of the introduction). Moreover, $\mathcal{F}(X)$ and LSC(X) are (reverse-order) continuous lattices in this case and the respective topologies are all Hausdorff.

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