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Controlled Markov processes: time discretization

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CONTENTS

0.1. INTRODUCTION AND SUMMARY

This part of the monograph is concerned with the approximation of controlled Markov processes with continuous-time parameter $\{t \mid t \geq 0\}$ by controlled Markov processes with discrete-time parameter $\{nh \mid n = 0,1,2,\ldots\}$, where h is a small step-size.

A Markov process is a stochastic process satisfying the so-called *'Markov property'*: "Given the actual state at a time-point t (present), the behaviour (evolution) of the process from t onward (future) is independent of the history up to time-point t (past)".

A controlled Markov process with continuous-time parameter can be described informally as follows. The state of a process (e.g. the number of customers in a service facility or the value of an investment fund) is observed continuously. A control prescribes at any time-point which decision rule has to be used. The decision rule in turn prescribes for any observed state the control variable (e.g. the service rate of the facility or the investment opportunity of the fund).
If at a time-point t the current decision rule is δ, then the evolution of the process during the time interval $[t,t+\Delta t]$, where Δt is small, is determined, approximately, by an *infinitesimal operator* A^{δ}.
In applications this operator will be given by means of infinitesimal characteristics depending on the actual state, say x, and the corresponding control variable $\delta(x)$ (e.g. the arrival and service rate at the facility (jump characteristics) or the profit rate (drift) and risk (diffusion) coefficient).
A cost-rate function L^{δ} yields costs incurred per unit of time.
Particular of interest for the above model is the so-called *continuous-time optimality equation (Bellman equation)*:

$$(1) \qquad \left\{ \begin{array}{ll} -\frac{d}{dt}\phi_t(x) = \inf\limits_{\delta\in\Delta} [L^{\delta} + A^{\delta}\phi_t](x) , & t \leq Z \\ \phi_Z(x) = 0 & \end{array} \right. , \quad x \in S ,$$

where Δ denotes a set of possible decision rules from which at each time-point a decision rule has to be chosen, and S is the state space.
The solution ϕ_t, provided it exists, is well-known to represent *'optimal (minimal) expected costs of controlling'* on $[t,Z]$. In general, however,

neither these costs nor a corresponding optimal control can be given explicitly.

The objective of this study is to present a general method of approximating continuous-time controlled Markov processes and related functions such as expected costs or optimal expected costs of controlling. Such a method is at least of theoretical interest, since continuous-time model formulations are often given via a limit of discrete-time descriptions. Especially, however, the computation by means of *recursive systems* and *inductive procedures* which are well-known for discrete-time structures makes time-discretization attractive from a *computational* point of view. For instance, optimal expected costs as well as a corresponding control for a discrete-time Markov process can be obtained recursively by using dynamic programming (see relation (2)).
In view of the computational aspect we focus on an approach which not only yields the convergence of discrete-time approximations as the step size h tends to 0, but which also provides *rates of convergence* or *bounds* and which is applicable to a wide *class of discretization-methods* possibly including *numerical* procedures. In this respect we emphasize, however, that this monograph must be seen just as a first step in this direction and that it is not concerned with obtaining 'good' numerical procedures.
Further analysis and developments in this respect would certainly be valuable. The discretizations which will be given explicitly in this monograph as well as the convergence rates and bounds obtained are not in the first place of computational interest. They illustrate, however, the application of our method which, to the best of our knowledge, forms a new approach to obtaining approximations for controlled stochastic models.

First, in chapter I, we analyze the approximation method for uncontrolled and (time-) homogeneous Markov processes as direct application of a well-known approximation theorem ad pted from numerical analysis. The approximation concerns expectations induced by transition probabilities (see (2.6) of chapter I) and, as an implication, also the probability law of the process. Next, in chapter II, we proceed along the same lines for controlled and (time-) inhomogeneous Markov processes. For fixed control we study the approximation of expectations induced by *transition probabilities* again (see (2.3.1) of chapter II) as well as of the *finite horizon cost function* (see (2.3.2) of chapter II).

Furthermore , attention is paid to the approximation of *optimal cost functions* as given by (1) and additionally, in jump- and diffusion-type applications to constructing *'nearly-(ε-) optimal' (discrete-time) controls*.

Therefore, we consider a discrete-time controlled Markov process at time-points $\{nh|n = 0,1,2,\ldots\}$, where h is a small step-size. (The setting for uncontrolled processes is included by neglecting the control characteristics). Such a process is determined as follows.
At each time-point nh the state of the process is observed. A control prescribes at any time-point a decision rule, which in its turn prescribes for any observed state a decision (action) to be chosen. If at time-point nh the actual state is x and decision rule δ is used, then the state at $nh + h$ is determined by the *one-step transition probability*:
$P_h^\delta(x;.)$. Further, a one-step cost $hL^\delta(x)$ is incurred.

In view of the approximation analysis, we introduce an operator T_h^δ on functions $f : S \to \mathbb{R}$, where S denotes the state space, by defining

$$T_h^\delta f(x) = \int f(y)\, P_h^\delta(x;dy) \qquad , \quad x \in S$$

(denoting the expectation of f induced by $P_h^\delta(x;.)$).
Then the inductive structure leads, for instance, to the possibility of recursively solving the *discrete-time optimality equation (dynamic programming equation)*:

$$\text{(2)} \qquad \begin{cases} \phi_{jh}^h(x) = \inf_{\delta\in\Delta} [hL^\delta + T_h^\delta(\phi_{jh+h}^h)](x) & , \quad j < \ell \\ \phi_{\ell h}^h(x) = 0 \end{cases} \qquad , \; x \in S \; .$$

The function ϕ_{jh}^h represents the optimal expected costs on $[jh, \ell h]$.
By subtracting ϕ_{jh+h}^h from both sides, and writing

$$A_h^\delta = [T_h^\delta - I]h^{-1},$$

(A_h^δ will be called a *one-step generator*), one easily derives

$$\text{(3)} \qquad [\phi_{jh}^h - \phi_{jh+h}^h](x) = \inf_{\delta\in\Delta} h[L^\delta + A_h^\delta(\phi_{jh+h}^h)](x) \qquad , \quad j < \ell \; , \; x \in S,$$

which can be seen as discrete-time analogue of the relation (1).
(Note that the right hand sides of (1) and (3) are non-linear in $\emptyset$ and $\emptyset^h$).

Intuitively, one might expect that as h tends to 0, convergence, in some appropriate norm, of the one-step generators A_h^δ to the infinitesimal operator A^δ for all decision rules δ, implies convergence also of the corresponding discrete-time processes and related functions, such as the convergence of $\emptyset^h$ to $\emptyset$. Actually, results of this type are well-known in the literature.
Without being exhaustive, we like to mention Skohorod (1958), Trotter (1958), Kurtz (1969) as well as Kushner and Yu (1973), (1974) for uncontrolled processes, and Whitt (1975), Kushner (1977), (1978), Kakumanu (1977), Nisio (1978), Gihman and Skorohod (1979), Van Der Duyn Schouten (1979), Hordijk and Van Der Duyn Schouten (1980), (1983a), (1983b), (1983c), Bensoussan and Robin (1983) as well as Christopeit (1983) for controlled processes.
All these references, with exception of Kakumanu (1977) for a specific model, are only concerned with the convergence of discrete-time approximations as the step size h tends to 0. With exception of Kushner (1977) and Haussmann (1980) for specific examples, rates of convergence or bounds are not provided. Moreover, the approaches used are quite different and several of them are especially developed for specific models or discretizations and require a detailed study of stochastic processes.

The approximation method developed in this monograph is of a unifying form and makes use of deterministic respresentations, more precisely, of deterministic *time-evolution equations* based on the 'Markov property'. For *uncontrolled* and time-homogeneous Markov processes these equations are the well-known *time-differential equations* (cf. Dynkin (1965)):

$$(4) \qquad \frac{d}{dt} T_t f = A(T_t f) \qquad , \quad t \geq 0 \quad , \quad T_0 f = f \ ,$$

where A is a *linear* operator on a domain of functions D_A and f denotes an initial function.
For *controlled* and time-inhomogeneous Markov processes it is more convenient to present these equations as *time-difference equations*

$$(5) \qquad U_t - U_{t+h} = hA_t(U_{t+h}) + R_t(h) \ , \ t+h \leq Z, \ \ U_Z = u,$$

where A_t is a *non-linear* operator on some domain of functions D_A again, u is a fixed terminal function at time-point Z, and $R_t(h)$ is a term which in a particular application has to satisfy:

(6) $$R_t(h)h^{-1} \to 0 \quad \text{(in some appropriate norm) as } h \to 0.$$

The literature on numerical analysis presents a well-known approximation theorem, known as *Lax-Richtmeyer theorem*, which deals with the convergence of finite difference-methods for initial value problems as given by (4). Consequently, application of this theorem enables us to study the discrete-time approximation for uncontrolled and homogeneous Markov processes. This is done in chapter I.

In order to deal with the convergence of non-linear and time-inhomogeneous difference-methods for the backwards time-difference equation (5), we present, as a slight extension of the Lax-Richtmeyer theorem, an *approximation lemma*. Although, also results of such a type are well-known in the literature on numerical analysis, we prefer to present a somewhat different form, which is particularly suitable for our purposes. Application of the approximation lemma is possible for: expectations induced by *transition probabilities*, finite horizon *cost functions*, and finite horizon *optimal cost functions*. This is shown in chapter II.

The approximation theorem as well as the approximation lemma concern approximations with respect to some appropriate norm and directly show how to obtain orders of convergence in that norm. As a result, application of the approximation theorem or lemma yields the following differences with the results given by the references mentioned above:

(i) Appropriate norms can be used.

(ii) Orders of convergence or bounds can be obtained.

By choosing appropriate norms we are able to study the approximation also for unbounded functions as well as to deal with unbounded charactericstics such as cost rates or infinitesimal characteristics.

Orders of convergence can be used to conclude that the convergence of discrete-time approximations for fixed control is uniform in a class of controls, or to show that ε- optimal controls can be constructed by using discrete-time dynamic programming.

More detailed comparisons with results of references as well as brief discussions on related literature can be found in the chapters I and II.

This monograph pays much attention to applying the method of time-discretization to uncontrolled and especially to controlled Markov processes of *jump* and *diffusion* type.
The discrete-time approximations given for these applications are quite natural and of a simple form. It may be remarked that the approximation lemma allows just as well more advanced difference-methods adopted from the literature on numerical analysis,which yield much better orders of convergence and computational results. Since, however, the application of such methods would require further analysis, they are not considered in this monograph. Nevertheless, as stated earlier, further investigation on computational aspects would be useful.

Since each of the chapters I and II contain a detailed introduction and summary itself, we only give a brief outline of the scope here.
First of all, we will conclude this introduction by presenting the necessary material on probability theory,such as the definitions of: stochastic processes, transition probabilities, Markov processes and weak convergence. Further,this introduction includes a list of (notational) conventions.
Next,chapter I studies time-discretization for uncontrolled and (time-) homogeneous Markov processes. First,time-discretization is analyzed in a general framework and yields as main approximation results of this chapter: theorem 4.3.1 and 4.3.7. Thereafter,application of these results is shown for Markov jump processes with bounded jump rates, an infinite server queue and solutions of stochastic differential equations (diffusions).
Chapter II examines the method of time-discretization for controlled Markov processes,more or less parallel to chapter I.
First of all,a formal description of continuous-time controlled Markov processes is given. There are three functions of interest to be approximated. In a general framework, again, the approximation of these functions is shown by the main approximation results of this chapter: theorems 6.3.2, 6.4.2 and 6.5.2 respectively. Next, special attention is paid to the applications: Controlled Markov jump processes and controlled stochastic differential equations (diffusions). As a special result of time-discretization, also the construction of ε-optimal controls will be investigated. In addition, at the end of this chapter a brief discussion on related literature will be given.
Finally, the Appendix contains auxiliary material on weak convergence of Markov processes on so-called D-spaces. A list of references as well as a list of symbols are included.

0.2. PROBABILITY CONCEPTS ; NOTATION

This section only collects some basic concepts and notation of probability theory which are essential for the sequel. For a more extensive introduction of probability theory we especially refer to Breiman (1968), Feller (1970) or Gihman and Skorohod (1969). In particular with respect to detailed studies of stochastic processes the books of Billingsley (1968), Dynkin (1965) and Gihman and Skorohod (1974), (1975), (1979) are also recommended. The definitions given below are adapted from Gihman and Skorohod (1974). The end of a definition or notation will be indicated by the symbol □.

DEFINITION 0.1. Let Ω be a set with σ - algebra Σ.
A probability measure $\mathbb{P}$ on Ω is a σ - additive non-negative measure such that $\mathbb{P}(\Omega) = 1$. The 3 - tuple $(\Omega,\Sigma,\mathbb{P})$ is called a probability space.

A random element X on a metric space S with Borel-field β is a measurable function from some probability space $(\Omega,\Sigma,\mathbb{P})$ into S. The probability measure $\mathbb{P}_X$ is defined by $\mathbb{P}_X(B) = \mathbb{P}(\{\omega | X(\omega)\in B\})$ for all $B\in\beta$.

Let $(\Omega,\Sigma,\mathbb{P})$ be a probability space. S a metric space and θ some parameter set. A function X with domain $\theta\times\Omega$ such that $X(t,.)$ is a random element on S for each $t\in\theta$ is called *a random function* or *stochastic process*. We call θ its domain of definiton and S its range or state space.
$X(.,\omega)$ for fixed $\omega\in\Omega$ is called a sample path. □

A random function X will be denoted by $\{X_t | t\in\theta\}$ or $(X_t)_{t\in\theta}$, where X_t indicates random element $X(t,.)$, $t\in\theta$.
In the sequel the domain of definition θ will always be given by either
(i) $\theta = \{t\in\mathbb{R} \, | t\geq 0\}$, or
(ii) $\theta = \{nh \; | n=0,1,2,...\}$ for some $h > 0$.
For case (ii) we also let X be denoted by $\{X_{nh} | n\in\mathbb{N}\}$ or $(X_{nh})_{n\in\mathbb{N}}$.
In the rest of this introduction let S be a metric space with Borel-field β.

DEFINITION 0.2. *A transition probability* from S_1 into S_2, where S_1 and S_2 are metric spaces with Borel-fields β_1 and β_2 respectively, is a mapping P from $S_1 \times \beta_2$ into $[0,1]$ such that
(i) $P(x;B)$ is β_1 - measurable in x for any $B\in\beta_2$, and
(ii) $P(x;.)$ is a probability measure on S_2 for any $x\in S_1$. □

DEFINITION 0.3. A collection of transition probabilities from S into S: $\{P_{s,t} | s,t \in \theta,\ s<t\}$ satisfies the *Chapman-Kolmogorov equation** if for all $t_1, t_2, t_3 \in \theta$ with $t_1<t_2<t_3$, $x \in S$ and $B \in \beta$ it satisfies

$$(0.1) \qquad P_{t_1,t_3}(x;B) = \int_S P_{t_2,t_3}(y;B)\, P_{t_1,t_2}(x;dy). \qquad \square$$

The definition of a Markov process is often given by using conditional probabilities; see for instance Dynkin (1965) p.77/78 or Gihman and Skorohod (1974) p.160. On p.162 and 163 of Gihman and Skorohod (1974), however, it is shown that the following definition can also be given.

DEFINITION 0.4. Let $\{P_{s,t} | s,t \in \theta, s<t\}$ be a collection of transition probabilities from S into S which satisfies the Chapman-Kolmogorov equation.

Then $\{X_t | t \in \theta\}$ is a *Markov process with transition probabilities* $\{P_{s,t} | s,t \in \theta, s<t\}$ if for any $n = 1,2,\ldots$, $0 \le t_1 < t_2 < \ldots < t_n$ with $t_i \in \theta$, $i = 1\ldots,n$ and $B_1, B_2, \ldots, B_n \in \beta$ it satisfies

$$(0.2) \qquad \begin{cases} \mathbb{P}(X_{t_1} \in B_1, X_{t_2} \in B_2, \ldots, X_{t_n} \in B_n) = \\ \int_{B_1} P_{X_{t_1}}(dx_1) [\int_{B_2} P_{t_1,t_2}(x_1;dx_2) [\ldots [\int_{B_n} P_{t_{n-1},t_n}(x_{n-1};dx_n)]]] \end{cases} \qquad \square$$

In accordance with (0.2), also let the set of transition probabilities include for any $t \ge 0$ the identity function $P_{t,t}$, i.e.; $P_{t,t}(x;B) = 1$ if $x \in B$ and 0 otherwise. In the sequel we always represent the transition probabilities by $\{P_{s,t} | s,t \in \theta\}$.

REMARK 0.5. It is shown by theorem 6 on p. 162 of Gihman and Skorohod (1974) that a Markov process satisfies the so-called 'absence of after - effect'. which informally says:
" The behaviour of the process for $s > t$ (future) only depends on the state of the process at $s = t$ (present) and not on its history for $s < t$ (past)."
The absence of after-effect is also known as '*Markov property*'. It is this Markov property or, more precisely, the Chapman-Kolmogorov equation on which the approximation analysis in the sequel will be based. $\square$

*In that case we also call the collection a <u>semigroup</u> of transition probabilities.

REMARK 0.6. A process $(X_t)_{t\in\theta}$ can be Markov with different collections of transition probabilities, say $\{P^1_{s,t}|s,t\in\theta\}$ and $\{P^2_{s,t}|s,t\in\theta\}$. By using (0.2) however it can be shown that, (also see p. 29, 160 and 161 of Gihman and Skorohod(1974) for any $s,t\in\theta$ and $B\in\beta$:
$\mathbb{P}\,(X_s\in\{x|P^1_{s,t}(x;B)\neq P^2_{s,t}(x;B)\})=0.$
Therefore, we let definition 0.4 also include transition probabilities.

□

DEFINITION 0.7. A Markov process $(X_t)_{t\in\theta}$ with transition probabilities $\{P_{s,t}|s,t\in\theta\}$ is called *homogeneous* if for some collection $\{P_t \mid t\in\theta\}$ of transition probabilities and all $s,t\in\theta$, $x\in S$ and $B\in\beta$;
$P_{s,s+t}(x;B) = P_t(x;B)$. □

DEFINITION 0.8. Let $\mathbb{P}$ be a probability measure on S and let f be a real-valued measurable and $\mathbb{P}$-integrable function. Then the *expectation* of f induced by $\mathbb{P}$ is given by $\mathbb{E}_{\mathbb{P}}f$ where

(0.3) $$\mathbb{E}_{\mathbb{P}}f = \int_S f\,d\mathbb{P}.$$

If it is clear which probability measure is considered, then we write $\mathbb{E}\,f(X)=\mathbb{E}_{\mathbb{P}_X} f.$ □

NOTATION 0.9.
$C(S) = \{f:S\to\mathbb{R} \mid f \text{ bounded and continuous }\}.$
$C^u(S)= \{f:S\to\mathbb{R} \mid f \text{ bounded and uniformly continuous }\}.$

DEFINITION 0.10. Let $h_0>0$. A collection $\{\mathbb{P}^h \mid h\in(0,h_0]\}$ of probability measures on S *converges weakly* to a probability measure $\mathbb{P}$ on S as $h\to 0$ if

(0.4) $$\lim_{h\to 0}\int_S f\,d\mathbb{P}^h = \int_S f\,d\mathbb{P} \text{ for all } f\in C(S).$$

NOTATION: $\mathbb{P}^h\Rightarrow\mathbb{P}$. □
For a collection of random elements $\{X^h|h\in(0,h_0]\}$ on S and X a random element on S write $X^h\Rightarrow X$ if $\mathbb{P}_{X^h}\Rightarrow\mathbb{P}_X$.

The *portmanteau theorem*, see p.11/12 of Billingsley(1968), gives several equivalent conditions for weak convergence.
Let us conclude this introduction with several notational conventions.

CONVENTIONS 0.11.

(0.5) An integral sign: $\int$ without subscript indicates an integral with domain of integration the whole metric space S.

(0.6) The symbol h always denotes a real-valued positive step-size. $h \to 0$ indicates: as h tends to 0 from above. $h \le h_0$ denotes: $h \in (0,h_0]$.

(0.7) The symbol Z always denotes a real-valued positive time-point. In the sequel we focus on time intervals $[0,Z]$ where Z is fixed but arbitrarily chosen. This fact will not be mentioned repeatedly. $t \le Z$ indicates for all $t \in [0,Z]$.

(0.8) For $x \ge 0$ let $[x]$ denote the 'entier' of x;i.e.; the number $n \in \{0,1,2,...\}$ such that $x-1 < n \le x$.

(0.9) For a collection $\{x^h | h \in (0,h_0]\} \subset \mathbb{R}$ and $p > 0$ we say that x^h is convergent of order $O(h^p)$ if $|x^h| \le Kh^p$ for all $h \le h_0$ and some constant K.

(0.10) Let S be a metric space with metric d_S and $x \in S$, $x^h \in S$ for all $h \le h_0$. Then $x^h \to x$ denotes that $d_S(x^h,x) \to 0$ as $h \to 0$.

(0.11) For $B \subset \beta$ define $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ otherwise.

(0.12) I always denotes an identity operator.

(0.13) The function $\bar{0} : S \to \mathbb{R}$ is given by: $\bar{0}(x) = 0$ for all $x \in S$.

(0.14) $\mathbb{N} = \{0,1,2,...\}$.

(0.15) The symbol := or =: indicates a defining relation.

(0.16) An equality sign = between two random elements on a same probability space denotes equality with probability one.

(0.17) For $f : \mathbb{R} \to \mathbb{R}$ a measurable function let f' resp. f'' resp. f''' denote the first resp. second resp. third derivative, provided it exists.

(0.18) The notation . ... , which in this monograph will be frequently called a collection, always indicates a family of elements parametrized by parameters ranging through a parameterset given by ... . Consequently, in this mongraph the word 'collection' must be interpreted as 'family' and not as 'set'.
Furthermore, with respect to the concepts of boundedness, continuity and differentiability of a family one can also interpret a family as a function.

(0.19) Two processes $\{X_t^1 | t \geq 0\}$ and $\{X_t^2 | t \geq 0\}$ on $D[0,\infty)$, where $D[0,\infty)$ is defined in the Appendix A, are called equal if they have the same probability law on $D[0,\infty)$.
Let $h > 0$. Two processes $\{X_{nh}^1 | n = 0,1,2...\}$ and $\{X_{nh}^2 | n = 0,1,2,...\}$ are said to be equal if for any $B \in \beta^\infty$, where β^∞ is the infinite product-σ-field of β, it holds that
$\mathbb{P}((X_{nh}^1)_{n\in\mathbb{N}} \in B) = \mathbb{P}((X_{nh}^2)_{n\in\mathbb{N}} \in B)$.

(0.20) The two chapters are numbered by a Roman capital. Each of them is subdivided in sections numbered by arithmetics and subsections numbered by an additional arithmetic. For instance subsection 5.3. Each section or subsection has its own numbering for definitions, notations, lemmas, propositions, theorems and remarks as well as separately a numbering between brackets for expressions, relations, formulas etc. For instance, lemma 5.3.2 and expression (5.3.2) of Chapter I.
Reference to a numbered statement in the same chapter is direct, but for a numbered statement in the other chapter or appendix we add the corresponding capital I or II, or letter A.
For instance lemma I.5.3.2 and expression I.(5.3.2) if referred to from chapter II.

(0.21) The end of a definition and notation, as said before, as well as that of an assumption, remark and proof will be indicated by the symbol □.

CHAPTER I

MARKOV PROCESSES; TIME-DISCRETIZATION

1. INTRODUCTION AND SUMMARY

This chapter is concerned with the *approximation* of *continuous-time homogeneous Markov processes* by means of *discrete-time Markov processes*. More precisely, given a process with continuous-time parameter $\{t \mid t \geq 0\}$ we consider for h sufficiently small processes with time parameter $\{nh \mid n = 0,1,2...\}$ and investigate convergence as h tends to 0.

Methods of time-discretization for stochastic processes are well-known in the literature. As references we like to mention Skorohod (1957), (1958), Trotter (1958), Kurtz (1969),(1970),(1975), Kushner and Yu (1973), (1974), Whitt (1975), Kushner (1977), Gihman and Skorohod (1979), Hordijk and Van Der Duyn Schouten (1983a).

The results of these references concern weak convergence of probability laws and do *not* provide *rates* of *convergence* or *bounds*. Their proofs of convergence are based on showing convergence of the *one-step generators* of the discrete-time processes to the *infinitesimal operator* of the continuous-time process. These proofs can be subdivided in those of probabilistic and those of analytic type.

The proofs of probabilistic type use probabilistic arguments such as: Relative compactness of the approximations (Kushner and Yu (1974a),(1974b)), weak convergence of embedded processes for jump type processes (Whitt (1975), Hordijk and Van Der Duyn Schouten (1983a)), or approximation of stochastic differential equations (Kushner (1977), Gihman and Skorohod (1979)).

The proofs of analytic type among the above mentioned references are based on a *semigroup* description for expectations induced by *transition probabilities* (Skorohod (1957), (1958), Trotter (1958) and Kurtz (1969), (1970), (1975)).

In this chapter, we will also make use of the semigroup and thus analytic approach. In contrast with Skorohod, Trotter and Kurtz, we will not investigate the convergence of resolvents of discrete-time semigroups and thus conclude convergence of the semigroups.
However, we will apply a well-known approximation theorem from which the convergence of the semigroups can be directly concluded.

This theorem, known as the *Lax-Richtmeyer theorem*, is adapted from the literature on numerical analysis. It concerns the convergence of *difference-methods* for time-evolution equations with a given initial value or more precisely for so-called *properly-posed initial value problems.*

Since the semigroup induced by a continuous-time Markov process corresponds to such a problem, the Lax-Richtmeyer theorem can be applied in order to study its approximation by means of discrete-time Markov processes. Moreover, besides its simple form it has the advantage above the weak convergence results mentioned above, that it provides *rates of convergence* with respect to some appropriate norm.

According to the Lax-Richtmeyer theory the so-called concepts of *consistency* and *stability* appear to be essential. Intuitively these concepts have the following interpretation.
Consistency states the *convergence* of the one-step generators to the infinitesimal operator as the step size h tends to 0.
Stability requires *boundedness* of the discrete-time semigroups on a finite time-interval uniformly in all step-sizes h and with respect to some norm.
The Lax-Richtmeyer theorem guarantees that consistency together with stability implies convergence of the difference-method. In addition, an order of convergence may be concluded from an order of consistency.

Application of the Lax-Richtmeyer theorem to a difference-method induced by discrete-time Markov processes enables us to study:
(i) Convergence as well as reates of convergence for
expectations induced by transition probabilities,
(ii) weak convergence of the *transition probabilities*, and
(iii) weak convergence of the *stochastic processes* on $D[0,\infty)$.
The possibility of choosing appropriate norms allows us to deal with
expectations for *unbounded functions*, and
unbounded infinitesimal characteristics.

After presenting convergence results in general form this chapter studies the method of time-discretization for three applications:

Jump-processes with bounded jump rates (subsection 5.1),

an infinite server queue as special example of a jump process with unbounded jump rates (subsection 5.2), and

solutions of stochastic differential equations or shortly *diffusion* processes (subsection 5.3).

For each of these applications we explicitly present one discrete-time construction. Especially for the jump and diffusion process these constructions can be seen as natural stochastic approximations. Further, they illustrate how to obtain rates of convergence as well as how to choose appropriate norms. It may be noted, however, that any other difference-method can be applied just as well. From a numerical point of view this may be better.

In the general setting as well as in these three applications the Markov processes are assumed to be homogeneous in time. Analogous results for time-inhomogeneous Markov processes however, can be obtained directly from chapter II by considering constant controls.

The organization of the chapter is as follows. First, in section 2, we present the semigroup of operators induced by transition probabilities of continuous-time Markov processes and we show that it corresponds to a so-called properly-posed initial value problem.

Section 3 concerns the discrete-time approximation of this problem. First, the general concepts of a difference-method, consistency, and stability are introduced and next, the Lax-Richtmeyer theorem is presented.

Further, for direct application later on, the end of section 3 contains a specific lemma from which consistency as well as an order of convergence can be concluded.

The results of sections 2 and 3 are applied in section 4 to difference methods for continuous-time Markov processes which are induced by discrete-time Markov processes. More precisely, in a general setting we present sufficient conditions for convergence of expectations, transition probabilities and processes.

The proofs given in this section make use of weak convergence results, which are collected in the Appendix, for processes.

Finally, section 5 contains the three applications for which the method of time-discretization is developed.

2. TRANSITION PROBABILITIES AND SEMIGROUPS

This section concerns a representation for transition probabilities of homogeneous Markov processes. Therefore, in analogy with Dynkin (1965) it is shown that these probabilities induce a *semigroup* of *linear* and *bounded operators*. (lemma 2.6). The semigroup property results from the essential Markov property or more precisely the Chapman-Kolmogorov equation.
Furthermore, this semigroup appears to be the unique solution of a particular *time-evolution equation* given by a so-called *initial value problem*. (lemmas 2.10 and 2.11). Combining these results with some additional properties yields (theorem 2.12):

> The semigroup induced by the transition probabilities corresponds to a so-called properly-posed initial value problem.

These results are directly adapted from Dynkin (1965). In this section nevertheless, the semigroup need not to be contractive as in Dynkin, but is only assumed to be bounded for some appropriate norm. This fact will appear to be useful for the applications given in section 5.

<u>DEFINITION 2.1.</u> Let S be a metric space with Borel-field β, then $\{P_t | t \geq 0\}$ with $P_t : S x \beta \to \mathbb{R}$, $t \geq 0$, is called a semigroup of transition probabilities on S, if

(i) for all $x \in S$ and $B \in \beta$
$P_0(x;B) = 1_B(x)$,

(ii) for all $x \in S$ and $t \geq 0$
$P_t(x;.)$ is a probability measure on S,

(iii) for all $B \in \beta$ and $t \geq 0$
$P_t(x;B)$ is β-measurable in x, and

(iv) for all $x \in S$, $t_1 \geq 0$, $t_2 \geq 0$ and $B \in \beta$

$$P_{t_1+t_2}(x;B) = \int P_{t_2}(z;B)\ P_{t_1}(x;dz). \tag{2.1}$$

Relation (2.1) is called the Chapman-Kolmogorov equation (cf. relation (0.1) of the introduction).
For the rest of this section let S be a complete metric space with Borel-field β and consider a semigroup of transition probabilities $\{P_t | t \geq 0\}$.

DEFINITION 2.2. A real-valued function $\mu: S \to \mathbb{R}$ is called a *bounding function* on S if for some $\delta_0 > 0$: $\mu(x) \geq \delta_0$ for all $x \in S$. □

NOTATION 2.3. With μ a bounding function on S define

$$(2.2) \qquad B^{\mu} = \{f: S \to \mathbb{R} \mid f \text{ measurable and } \sup_{x \in S} \mu(x)^{-1} |f(x)| < \infty\},$$

and for $f \in B^{\mu}$ write

$$(2.3) \qquad \| f \|_{\mu} = \| f(.) \|_{\mu} = \sup_{x \in S} \mu(x)^{-1} |f(x)|.$$

If $\mu(x) = 1$ for all $x \in S$ we also write

$$(2.4) \qquad \| f \|_{\mu} = \| f \|_{\infty}.$$ □

LEMMA 2.4. *Let μ be a bounding function on S. Then $\| . \|_{\mu}$ induces a norm on B^{μ} and B^{μ} is a Banach space with respect to this norm.*
The norm is called μ-norm.

PROOF. Immediately from the completeness of S. □

If some bounding function μ is under consideration,then we assume, unless explicitly stated otherwise, that any subset of B^{μ} is endowed with the μ-norm $\| . \|_{\mu}$ and the metric induced by it. Especially in the approximation analysis later on,this natural convention must be kept in mind.
Further, for any bounding function μ it may be noted that
$B = \{f : S \to \mathbb{R} \mid f \text{ measurable and } \sup_{x \in S} |f(x)| < \infty \} \subset B^{\mu}$. □

In the rest of this subsection let $Z>0$ and bounding function μ be fixed and suppose that the following assumption is satisfied.

ASSUMPTION 2.5. For the semigroup of transition probabilities $\{P_t \mid t \geq 0\}$, the bounding function μ and constant M:

$$(2.5) \qquad \| \int \mu(y) P_t(.;dy) \|_{\mu} \leq M \qquad , t \leq Z.$$ □

Definition 2.1 together with assumption 2.5 enables us to define for any $t \leq Z$ a linear operator $T_t : B^{\mu} \to B^{\mu}$ by

$$(2.6) \qquad T_t f(x) = \int f(y) \, P_t(x;dy) \qquad , x \in S.$$ □

LEMMA 2.6. *The collection of operators* $\{T_t | t \le Z\}$ *is a semigroup of bounded linear operators on* B^μ *satisfying for any* $f \in B^\mu$:

(2.7) $T_0 f = f$ *and* $T_t f \in B^\mu$, $t \le Z$,

(2.8) $T_{t_1+t_2} f = T_{t_1}(T_{t_2} f)$, $t_1 \le t_2 \le Z$,

(2.9) $\| T_t f \|_\mu \le M \| f \|_\mu$, $t \le Z$.

PROOF. (2.7): The measurability of $T_t f$ follows directly from the measurability of f and condition (iii) of definition 2.1 . The fact that $T_t f$ is μ-bounded follows from (2.5) through

$$(*) \qquad | \int f(y) P_t(x;dy) | \le \|f\|_\mu \int \mu(y) P_t(\cdot;dy) \le \| f \|_\mu M \mu(x) .$$

(2.8): Since f B^μ one can show as in theorem 11.20 of Rudin (1964) that there exists a sequence of simple functions $\{f^n\}_{n=1}^\infty$ such that $f^n(x) \to f(x)$ for all x S and $\| f^n \|_\mu \le \| f \|_\mu$ for any n. According to the Chapman-Kolmogorov relation we have that

$$\int f^n(y) P_{t_1+t_2}(x;dy) = \int [\int f^n(y) P_{t_2}(z;dy)] P_{t_1}(x;dz),$$

for any $n \in \mathbb{N}$, $t_1 \ge 0$, $t_2 \ge 0$ and $x \in S$. By letting n tend to ∞, and using the dominated convergence theorem for both sides of the last expression the proof will be completed.
(2.9): This is shown by (*). □

DEFINITION 2.7. A sequence of functions $\{f^n\}_{n=1}^\infty \subset B^\mu$ *converges in* μ*-norm* to $f \in B^\mu$ if $\|f^n - f\|_\mu \to 0$ as $n \to \infty$, notation: $f = \mu - \lim_{n\to\infty} f^n$.

A collection of functions $\{f_t | t \in [0,Z]\} \subset B^\mu$ is called:
μ - *bounded* if: $\|f_t\|_\mu \le M$ for some constant M and all $t \le Z$,
μ - *continuous* on [0,Z] if for all $t \le Z$:

$$\|f_{t+s} - f_t\|_\mu \to 0 \text{ as } s \to 0 ,$$

μ - *differentiable* on [0,Z] if for some collection $\{g_t | t \in [0,Z]\}$ and all $t \le Z$:

$$\|[f_{t+s} - f_t] s^{-1} - g_t\|_\mu \to 0 \quad \text{as } s \to 0 \text{ ,notation :}$$

$$\frac{d}{dt} f_t = g_t \quad , \quad t \le Z.$$

(Here we assume that $t+s \le Z$ and for the endpoints 0 and Z we only consider the limits as $s \downarrow 0$ resp. $s \uparrow 0$) □

NOTATION 2.8. Let us write

$$(2.10)\quad \begin{cases} B_0^{\mu} = \{f\in B^{\mu} \mid \mu\text{-}\lim_{h\to 0} T_h f = f\} \\ D_A^{\mu} = \{f\in B^{\mu} \mid \mu\text{-}\lim_{h\to 0} [T_h f - f]h^{-1} \text{ exists }\}. \end{cases}$$ □

DEFINITION 2.9. The *infinitesimal operator* of the semigroup $\{T_t \mid t\le Z\}$ is a linear operator $A : D_A^{\mu} \to B_0^{\mu}$ defined by

$$(2.11)\qquad Af = \mu\text{-}\lim_{h\to 0} [T_h f - f]h^{-1}, \qquad f\in D_A^{\mu}.$$ □

Note that the domain of the operator $A : D_A^{\mu}$ depends on μ, whereas the operator A itself does not. The next lemma presents results similar to Dynkin (1965) p.22/23. Particularly, it shows that the semigroup satisfies a so-called *time-evolution equation.*

LEMMA 2.10.

(i) B_0^{μ} *is a Banch space,*
$Af\in B_0^{\mu}$ *for any* $f\in D_A^{\mu}$, *and*
$T_t f$ *is* μ *-continuous on* $[0,Z]$ *for any* $f\in B_0^{\mu}$.

(ii) *The* μ*-closure of* D_A^{μ} *coincides with* B_0^{μ}.

(iii) *For* $f\in D_A^{\mu}$ *it holds that* $T_t f\in D_A^{\mu}$ *for all* $t\le Z$, *the collection* $\{T_t f \mid t\le Z\}$ *is* μ*- differentiable on* $[0,Z]$, *and*

$$(2.12)\qquad \frac{d}{dt}T_t f = A(T_t f) = T_t(Af) \quad, t\le Z.$$

PROOF. The first statement only requires to show : B_0^{μ} is closed. Therefore, let $f = \mu\text{-}\lim f_n$ with $\{f_n \mid n\in \mathbb{N}\}\subset B_0^{\mu}$ and take $\varepsilon>0$. Then,

$$\begin{aligned} \|T_h f - f\|_{\mu} &\le \|T_h f - T_h f_n\|_{\mu} + \|T_h f_n - f_n\|_{\mu} + \|f_n - f\|_{\mu} \\ &\le (M+1)\|f_n - f\|_{\mu} + \|T_h f_n - f_n\|_{\mu}. \end{aligned}$$

First fix n such that $\|f_n - f\|_{\mu} \le \varepsilon[2(M+1)]^{-1}$, next choose δ such that for $h<\delta$: $\|T_h f_n - f_n\|_{\mu} < \frac{1}{2}\varepsilon$. Then the above inequality yields $\|T_h f - f\|_{\mu} < \varepsilon$. Hence, B_0^{μ} is closed. Consequently, relation (2.11) implies $Af\in B_0^{\mu}$ for $f\in D_A^{\mu}$. Finally, let $f\in B_0^{\mu}$. Then with $t\in[0,Z)$, $h>0$ such that $t+h\le Z$:

$$\|T_{t+h} f - T_t f\|_{\mu} \le \|T_t(T_h - f)\|_{\mu} \le M\,\|T_h f - f\|_{\mu},$$

and for $t\in(0,Z]$, $h>0$ such that $t-h>0$:

$$\|T_{t-h}f - T_tf\|_\mu \leq \|T_{t-h}(f-T_hf)\|_\mu \leq M\ \|T_hf - f\|_\mu .$$

Hence, by letting h tend to 0 and using that $f\in B_0^\mu$ we find that T_tf is μ-continuous on $[0,Z]$.

(ii) Obviously, $D_A^\mu \in B_0^\mu$. Let $f\in B_0^\mu$ and define $g_a(x) = \int_0^a T_sf(x)ds$. Then,

$$T_hg_a(x) = \int_h^{a+h} T_sf(x)ds = \int_a^{a+h} T_sf(x)ds - \int_0^h T_sf(x)ds + g_a(x) .$$

By virtue of the μ-continuity of T_sf in s this inequality yields:
$\mu\text{-}\lim_{h\to 0} [T_hg_a - g_a]h^{-1} = T_af - f$, and thus $g_a\in D_A^\mu$.
The proof of (ii) is concluded with $f = \mu\text{-}\lim_{h\to 0} g_h$.

(iii) Let $f\in D_A^\mu$. Then for $t\in[0,Z)$, $h>0$ such that $t+h\leq Z$, we have

$$\|[T_{t+h}f - T_tf]h^{-1} - T_tAf\|_\mu =$$

$$\|T_t([T_hf - f]h^{-1} - Af)\|_\mu \leq M\ \|[T_hf - f]h^{-1} - Af\|_\mu .$$

And for $t\in(0,Z]$, $h>0$ and $t-h\geq 0$:

$$\|[T_{t-h}f - T_tf]\ h^{-1} - T_tAf\|_\mu = \|T_{t-h}([f - T_hf]h^{-1} - Af)\|_\mu +$$

$$\|T_{t-h}(Af - T_hAf)\|_\mu \leq M\ \|[f - T_hf]h^{-1} - Af\|_\mu + M\ \|Af - T_hAf\|_\mu .$$

Hence, by letting h tend to 0 in the above inequalities and using that $f\in D_A^\mu$ and $Af\in B_0^\mu$, as shown under (i), we may conclude :

$$T_tf \text{ is } \mu\text{-differentiable on } [0,Z] \text{ and } : \frac{d}{dt}T_tf = T_t(Af) \quad , t\leq Z .$$

However, since also

$$[T_{t+h}f - T_tf]h^{-1} = [T_h(T_tf) - T_tf]h^{-1} ,$$

the μ-differentiability implies for $f\in D_A^\mu$: $T_tf\in D_A^\mu$ and

$$\frac{d}{dt}T_tf = T_t(Af) = A(T_tf) \quad , \quad t\leq Z .$$

□

LEMMA 2.11. *Let* A *be the operator on* D_A^μ *as defined by* (2.11). *Then for any* $f \in D_A^\mu$ *there exists a unique* μ *-bounded and* μ *-differentiable collection* $\{u_t \mid t \le Z\} \subset D_A^\mu$ *satisfying*

$$(2.13) \qquad \frac{d}{dt} u_t = A u_t \quad , t \le Z \ , u_0 = f$$

PROOF. According to (iii) of lemma 2.10 and relation (2.9) the collection $\{T_t f \mid t \in [0,Z]\}$ is such a solution. To prove the uniqueness let $\{u_t^1 \mid t \le Z\}$ and $\{u_t^2 \mid t \le Z\}$ be μ-bounded solutions of (2.13). Define for $t \le Z$: $g_t = u_t^1 - u_t^2$. Then $\{g_t \mid t \le Z\} \subset D_A^\mu$ is μ-bounded and satisfies

$$\frac{d}{dt} g_t = A g_t \ , t \le Z \quad , g_0 = \bar{0}.$$

Next, since g_t is μ-bounded uniformly in $t \le Z$ and $g_t \in B^\mu$ the expression $T_{s-t} g_t$ is well-defined by (2.6) if $s \ge t$. Let $s \in (0,Z]$, $h > 0$ and $t \le s-h$, then

$$\| [T_{s-(t+h)} g_{(t+h)} - T_{s-t} g_t] h^{-1} - T_{s-t}[-A g_t + \frac{d}{dt} g_t] \|_\mu =$$

$$\|T_{s-t-h}([g_t - T_h g_t] h^{-1} + A g_t)\|_\mu + \|T_{s-t-h} A g_t - T_{s-t} A g_t\|_\mu +$$

$$\|T_{s-t-h}([g_{t+h} - g_t] h^{-1} - \frac{d}{dt} g_t)\|_\mu + \|T_{s-t-h} \frac{d}{dt} g_t - T_{s-t} \frac{d}{dt} g_t\|_\mu \ .$$

Since $g_t \in D_A^\mu$ it holds that $A g_t \in B_0^\mu$ and hence $\frac{d}{dt} g_t \in B^\mu$. Together with the differentiability of g_t this implies that all four terms in the right-hand side converge to 0 as h tends to 0.

$$\text{Hence} \qquad \frac{d^+}{dt} T_{s-t} g_t = T_{s-t}[-A g_t + \frac{d}{dt} g_t] = \bar{0} \quad , \quad t \in [0,s).$$

$$\text{Similarly} \qquad \frac{d^-}{dt} T_{s-t} g_t = T_{s-t}[-A g_t + \frac{d}{dt} g_t] = \bar{0} \quad , \quad t \in (0,s].$$

$$\text{Consequently,} \qquad T_s g_0(x) - T_0 g_s(x) = g_s(x) = 0 \quad , \quad x \in S \ , \ s \in [0,Z] \ .$$ □

Relation (2.13) is called an initial value problem or a time-evolution equation. The collection $\{u_t \mid t \in [0,Z]\}$ is called its solution. Combination of lemma 2.10 and lemma 2.11 yields as final result:

THEOREM 2.12. *The domain* D_A^μ *is dense in* B_0^μ *and for any* $f \in D_A^\mu$ *the collection* $\{T_t f \mid t \in [0,Z]\}$ *is the unique* μ*-bounded solution within* D_A^μ *of the initial value problem* (2.13). □

3. LAX-RICHTMEYER THEORY

This section studies the convergence of difference-methods for initial value problems given by (2.13). Lax and Richtmeyer (1956) presented an essential theorem from which convergence can be concluded. This theorem is well-known in the literature on numerical analysis. The Lax-Richtmeyer theory as presented below is directly adapted from Meis and Marcowitz (1981).

The problem considered is concerned with the approximation of semigroups which correspond to initial value problems by means of semigroups which are constructed from a difference-method. In order to conclude convergence two concepts appear to be essential:

consistency and *stability*.

The Lax-Richtmeyer theorem states that

consistency implies *convergence* if and only if there is *stability*.

In view of subsequent applications we also give a lemma which presents a sufficient condition for consistency and which concerns a convergence-order.

In this section we consider a Banach space B with norm $\|.\|$ and time-interval $[0,Z]$. For a collection $\{f_t \mid t\in[0,Z]\}\subset B$ the concepts strongly continuous as well as differentiable mean continuity respectively differentiability with respect to the time parameter in norm $\|.\|$.

<u>DEFINITION 3.1</u>. Let $D_A\subset B$, $A: D_A\to B$ a linear operator. Then we have a *properly-posed initial value problem* on D_A if:

(i) D_A is dense in B.

(ii) For any $c\in D_A$ exists a unique collection $\{U_t(c) \mid t\in[0,Z]\}$ in D_A which is strongly differentiable on $[0,Z]$ and satisfies:

$$(3.1)\qquad \frac{d}{dt}U_t(c) = AU_t(c)\ ,\quad t\le Z\quad ,\ U_0(c) = c.$$

(iii) For some constant M and any $c\in D_A$ the collection $\{U_t(c) \mid t\in[0,Z]\}$ given in (ii) satisfies: $\|U_t(c)\|\le M\|c\|$ for all $t\in[0,Z]$.

NOTATION: P(B,Z,A) denotes this properly-posed initial value problem. □

<u>LEMMA 3.2</u>. *Let* P(B,Z,A) *be a properly-posed initial value problem. Then there exists a unique collection* $\{E_t \mid t\in[0,Z]\}$ *of linear operators*

$E_t: B \to B$, $t \le Z$, *such that* $E_t(c) = U_t(c)$, *for all* $c \in D_A$ *and* $t \in [0,Z]$. *Further*

(3.2) $E_{t_1+t_2} = E_{t_1} E_{t_2}$, $t_1+t_2 \le Z$.

(3.3) $\|E_t c\| \le M \| c \|$, $t \le Z$, $c \in B$.

(3.4) $E_t c$ *is strongly continuous in* $t \in [0,Z]$ for all $c \in B$.

(3.5) $A\, E_t c$ *is strongly continuous in* $t \in [0,Z]$ *and*

$A\, E_t c = E_t A c$, $t \le Z$, *for all* $c \in D_A$.

PROOF See theorem 4.12 and theorem 5.5 of Meis and Marcowitz (1981). □

DEFINITION 3.3. Let P(B,Z,A) be a properly-posed initial value problem, $\{E_t \mid t \in [0,Z]\}$ its semigroup and $h_0 > 0$.

(i) A collection of linear operators $M_D = \{C_h \mid h \in (0,h_0]\}$ with for $h \le h_0$ $C_h: B \to B$, is called a *difference-method* if for some constant K: $\|C_h c\| \le K\|c\|$ for all $h \le h_0$ and $c \in B$.

(ii) M_D is called *consistent* for *c* where $c \in B$, if

$$
(3.6) \quad \begin{cases} \| [C_h - E_h] E_t c \| \, h^{-1} \\ \text{converges to 0 uniformly in } t \le Z, \text{ as } h \to 0. \end{cases}
$$

M_D is called *consistent* if for some subset D_C dense in B it is consistent for all $c \in D_C$.

(iii) M_D is called *stable* if for some constant K_C:

$$
(3.7) \quad \begin{cases} \| [C_h]^n c \| \le K_C \|c\| \\ \text{for all } h \le h_0 \text{ and n such that } nh \le Z \text{ and all } c \in B. \end{cases}
$$

(iii) M_D is called *convergent* for *c*, if for all $t \le Z$:

$$
(3.8) \quad \begin{cases} \| [C_h]^n \, c - E_t c \| \\ \text{with } |nh - t| \le h \\ \text{converges to 0 uniformly in } nh \le Z, t \le Z \text{ as } h \to 0. \end{cases}
$$

M_D is called *convergent* if it is convergent for all $c \in B$. □

We are now able to present the essential approximation theorem. The proof which will be given is adapted from Meis and Marcowitz (1981). It is included in order to show its simplicity and to illustrate the above concepts. Moreover, we use parts of the proof later on.

THEOREM 3.4. (LAX-RICHTMEYER)
A consistent difference method M_D *for* $P(B,Z,A)$ *is convergent if and only if* M_D *is stable* .

PROOF. We only give the proof of sufficiency (if part) since the necessity (only if part) will not be used in the sequel. A proof of the necessity however can be found in Meis and Marcowitz (1981) p. 62/63.

Let M_D be stable. According to the semigroup property (3.2) we have for any $h \le h_0$, $n \in \mathbb{N}$ such that $nh \le Z$ and $c \in B$:

$$[C_h]^n c - E_{nh} c = \sum_{k=0}^{n-1} [C_h]^k [C_h - E_h] E_{(n-1-k)h} c. \tag{3.9}$$

Hence, for $c \in D_C$ we conclude from (3.6), (3.7) and (3.9) that for any $\varepsilon > 0$ exists a $\delta_1 > 0$ such that for all $h \le \delta_1$ and $nh \le Z$:

$$\| [C_h]^n c - E_{nh} c \| \le n K_C h \ \varepsilon \le \varepsilon Z K_C. \tag{3.10}$$

By virtue of (3.4) there exists a $\delta_2 > 0$ such that if $|nh - t| < \delta_2$, then

$$\| E_{nh} c - E_t c \| \le \varepsilon. \tag{3.11}$$

Hence, if $c \in D_C$, $h < \delta_1$ and $|nh - t| < \delta_2$, then

$$\| [C_h]^n c - E_t c \| \le \varepsilon (Z K_C + 1). \tag{3.12}$$

Since ε is chosen arbitrarily, this proves the convergence of M_D on D_C.

Now let $c \in B$ and $\varepsilon > 0$. Then there exists a $\bar{c} \in D_C$ with $\| c - \bar{c} \| < \varepsilon$. Relation (3.3) and the stability condition (3.7) yield

$$\|[C_h]^n c - E_t c\| \leq \|[C_h]^n \bar{c} - E_t \bar{c}\| + \|[C_h]^n (c - \bar{c})\| + \|E_t(\bar{c} - c)\|$$
$$\leq \|[C_h]^n \bar{c} - E_t \bar{c}\| + (M + K_C)\, \|c - \bar{c}\|$$
$$\leq \varepsilon (ZK_C + 1) + \varepsilon\, (M + K_C),$$

if $|nh - t|$ is sufficiently small. The last inequality results from (3.12) with c replaced by $\bar{c}$ and $\|c - \bar{c}\| < \varepsilon$. Again, since ε is chosen arbitrarily this proves the convergence of M_D also on B. □

COROLLARY 3.5. *Let* M_D *be a stable difference-method for* P(B,Z,A) *which is consistent for* c *with* $c \in B$, *then it is convergent for* c.

PROOF. Immediately from (3.9) up to (3.12) □

REMARK 3.6. By using (3.9) it also follows that if M_D is stable, then the left-hand side of (3.10) is *convergent*, as h tends to 0, of an *order* equal to the order of convergence in (3.6) *(order of consistency)*. □

Although, the following lemma can be proven, analogously to p.50 of Meis and Marcowitz, by using the generalized mean value theorem, we prefer to prove it by an integral representation, since we also make use of this representation later on. Therefore, let P(B,Z,A) be a properly-posed initial value problem and $c \in D_A$. Then by virtue of the time-evolution equation(3.1), $E_t c = U_t(c)$ and the continuity of $AE_t c$ we can write

$$E_t c - c = \int_0^t AE_s c \, ds \quad , \quad t \leq Z, \tag{3.13}$$

where the integral stands for the Bochner-integral (see p.42/43 of Hille (1948)). Further, by using (3.13) together with the semigroup property (3.2) and the strong continuity of $A\, E_t c$ we find, that

$$\begin{cases} \|([E_h - I]h^{-1} - A)\, E_t c\| = \|h^{-1} \int_t^{t+h} (A\, E_s c - A\, E_t c)\, ds\|, \\ \text{which converges to 0 uniformly in } t+h \leq Z \text{ as } h \to 0. \end{cases} \tag{3.14}$$

Next, let us consider a difference - method $M_D = \{ C_h \,|\, h \in (0, h_0] \}$. Then, for any $h \leq h_0$ we define a linear operator $A_h : B \to B$ by

$$A_h = [C_h - I]h^{-1} \quad , \tag{3.15}$$

which we call a *one - step generator*.

LEMMA 3.7. *Let* $P(B,Z,A)$ *be a properly-posed initial value problem and* $M_D=\{C_h \mid h\in(0,h_0]\}$ *a difference-method . Then:*

(i) M_D *is consistent for* c *with* $c\in D_A^{\mu}$ *if*

$$(3.16)\qquad \begin{cases} \|(A_h - A)E_t c\| \\ \text{converges to } 0 \text{ uniformly in } t\in[0,Z] \text{ as } h\to 0 . \end{cases}$$

(ii) *Let* M_D *be stable ,* $c\in D_A^{\mu}$ *,* $p\leq 1$ *and suppose that*

$$(3.17)\qquad \begin{cases} \|(A_h - A)E_t c\| + \|([E_h - I]h^{-1} - A)E_t c\| \\ \text{is convergent of order } O(h^p) \text{ uniformly in } t\in[0,Z] . \end{cases}$$

Then , the expression

$$(3.18)\qquad \|[C_h]^n c - E_t c\|$$

with $n=[th^{-1}]$ *is convergent of order* $O(h^p)$ *uniformly in* $t\in[0,Z]$.

PROOF. (i) For all $h\leq h_0$ and $t+h\leq Z$ we find by writing $C_h = I+hA+h[A_h-A]$:

$$(3.19)\qquad \|(E_h - C_h)E_t c\| h^{-1} \leq \|([E_h - I]h^{-1} - A)E_t c\| + \|(A_h - A)E_t c\| .$$

By virtue of (3.14) and (3.16) the right-hand side of (3.19) converges to 0 uniformly in $t\leq Z$ as h tends to 0 , which proves the consistency for c .

(ii) First note that according to (3.17) and (3.19) the expression

$$(3.20)\qquad \|(E_h - C_h)E_t c\| h^{-1}$$

is convergent of order $O(h^p)$ uniformly in $t \leq Z$.
By using $\|A E_t c\| = \|E_t Ac\| \leq M \|Ac\| < \infty$, $t \leq Z$, we obtain from (3.13):

$$(3.21)\qquad \|E_{nh} c - E_t c\| \leq h\, M\, \|Ac\|$$

with $n = [th^{-1}]$. The proof is concluded by noting that $p\leq 1$ and using the above facts in the proof of the Lax-Richtmeyer theorem for fixed c . □

REMARK 3.8. Clearly, the restriction $p \leq 1$ can be relaxed to $p > 0$ if one replaces t by nh in expression (3.18). □

4. APPROXIMATIONS BY DISCRETE-TIME MARKOV PROCESSES

4.1. INTRODUCTION AND SUMMARY

First, recall from the introduction and definition 2.1 that a continuous-time homogeneous Markov process has a corresponding semigroup of transition probabilities. Now let us consider such a semigroup as well as the corresponding semigroup of linear operators defined by means of relation (2.6). Then theorem 2.12 together with definition 3.1 imply, as already stated in section 2, that the semigroup of linear operators corresponds to a properly-posed initial value problem.

Consequently, the Lax-Richtmeyer theory can be applied to difference-methods for continuous-time homogeneous Markov processes. Hence, the concepts of consistency and stability will be essential.
Consistency will be guaranteed by a so-called *consistency relation,* see (4.3.1), and stability by a so-called *stability relation*, see (4.3.2).

Approximations for the semigroups might be obtained by using any difference-method. However, we especially focus on difference-methods which are induced by *one-step transition probabilities* P^h for all step sizes h.

A one-step transition probability induces a semigroup of transition probabilities $\{P^h_{nh} \mid n = 0,1,2,\dots\}$ As direct application of sections 2 and 3, we first study the approximation of *expectations* $\int f(y)P_t(.;dy)$ by expectations $\int f(y)P^h_{nh}(.;dy)$. (Theorem 4.3.1)

This also yields the weak convergence of the *transition probabilities* $P^h_{nh}(.;.)$ to $P_t(.;.)$. (Theorem 4.3.4.)

Finally, let $\{X_t \mid t \geq 0\}$ be a Markov process with transition probabilities $\{P_t \mid t \geq 0\}$. Then by letting $\{P^h_{nh} \mid n = 0,1,2,\dots\}$ induce discrete-time Markov processes $\{X^h_{nh} \mid n = 0,1,2,\dots\}$ we can consider the weak convergence of the *discrete-time processes* $\{X^h_{nh} \mid n = 0,1,2,\dots\}$ to $\{X_t \mid t \geq 0\}$ on an appropriate space. (Theorem 4.3.7.)

4.2. MODEL

Continuous-time. Let S be a separable and complete metric space with Borel-field β and $\{P_t \mid t \geq 0\}$ be a semigroup of transition probabilities on S. Suppose that assumption 2.5 is satisfied.

Further, recall the definitions of $T_t f$, B_0^{μ} , D_A^{μ} and Af given by expression (2.6), (2.10), (2.10) and (2.11) resprectively.
Finally, remember that the collection $\{T_t \mid t\in[0,Z]\}$ is the unique semigroup corresponding to a properly-posed initial value problem $P(B_0^{\mu},Z,A)$.

<u>Discrete-time</u>. Let for some $h_0>0$: $\{P^h \mid h\in(0,h_0]\}$ be a collection of transition probabilities from S into S; hence for any $h\le h_0$:
$P^h(x;.)$ is a probability measure on S for any $x\in S$, and
$P^h(x;B)$ is β-measurable in $x\in S$ for any $B\in\beta$.
For any step-size h we call P^h a *one-step transition probability*.
Suppose that for any $h\le h_0$

(4.2.1) $\qquad \|\int \mu(y)\ P^h(.;dy)\|_{\mu} < \infty.$

Then we introduce a difference-method $M_D=\{C_h \mid h\in(0,h_0]\}$ on B^{μ} by defining for all $h\le h_0$, $f\in B^{\mu}$ and $x\in S$:

(4.2.2) $\qquad C_h f(x) = \int f(y)\ P^h(x;dy).$

Further , for any $h\le h_0$ we obtain a collection $\{P^h_{nh} \mid n=0,1,2,\ldots\}$ of transition probabilities from the recursive scheme

(4.2.3) $$\begin{cases} P^h_0(x;B) = 1_B(x) & \\ P^h_{nh}(x;B) = \int P^h(y;B)\ P^h_{(n-1)h}(x;dy) & ,n\ge 1 \end{cases} \quad x\in S\ ,\ B\in\beta.$$

Hence , from (4.2.1),(4.2.2) and (4.2.3) it follows that

(4.2.4) $\qquad [C_h]^n f(x) = \int f(y)\ P^h_{nh}(x;dy)$

for all $h\le h_0$, $n\in\mathbb{N}$, $f\in B^{\mu}$ and $x\in S$.
Finally , recall for $h\le h_0$ the expression (3.15) : $A_h=[C_h-I]h^{-1}$.

4.3 DISCRETE-TIME APPROXIMATIONS

By applying the Lax-Richtmeyer theory,presented in section 3, to the difference-method given above,we obtain the following main approximation result .

THEOREM 4.3.1. *Let* $f \in D_A^{\mu}$ *and suppose that*

$$(4.3.1)\qquad \begin{cases} \| [A_h - A] T_t f \|_{\mu} \\ \text{converges to } 0 \text{ uniformly in } t \le Z, \text{ as } h \to 0, \end{cases}$$

and

$$(4.3.2)\qquad \begin{cases} \| \int \mu(y)\, P^h(.;dy) \|_{\mu} \le (1 + h M_C) \\ \text{for all } h \le h_0 \text{ and some constant } M_C. \end{cases}$$

Then the expression

$$(4.3.3)\qquad \| \int f(y)\, P_{nh}^h(.;dy) - \int f(y)\, P_t(.;dy) \|_{\mu}$$

with $n = [th^{-1}]$, *converges to* 0 *uniformly in* $t \le Z$ *as* $h \to 0$.

PROOF. First, note that relation (4.3.2) guarantees relation (4.2.1). By virtue of lemma 3.7., relation (4.3.1) implies consistency. According to lemma 4.3.2 below, relation (4.3.2) implies stability. Hence, the Lax-Richtmeyer theorem 3.4 together with the expression (2.6) for $T_t f$ and (4.2.4) for $[C_h]^n f$ completes the proof. □

LEMMA 4.3.2. *Let relation* (4.3.2) *be satisfied, then for any* $Z > 0$ *the difference-method is stable on* $[0,Z]$.

PROOF.

Since $$| \int f(y)\, P^h(x;dy) | \le \int \mu(y) \| f \|_{\mu}\, P^h(x;dy),$$

we have : $$\| C_h f \|_{\mu} \le \| f \|_{\mu} (1 + hM_C).$$

Hence, by iterating this inequality n-times, where $nh \le Z$, we find

$$(4.3.4)\qquad \| [C_h]^n f \|_{\mu} \le \| f \|_{\mu} (1 + hM_C)^n \le \| f \|_{\mu}\, e^{ZM_C}$$

for all $h \le h_0$ and n such that $nh \le Z$, and $f \in B^{\mu}$. □

REMARKS 4.3.3

1. An order of convergence for expression (4.3.3) can be obtained by using lemma 3.7. □

2. Note that relation(4.3.2), implying stability, depends only on the one-step transition probabilities $\{P^h \mid h\in(0,h_0]\}$ and μ .
3. Since relation (4.3.1) resp. (4.3.2) imply consistency for f resp. stability, we will refer to these relations in the rest of this chapter as ; the *consistency relation* (4.3.1) resp. the *stability relation* (4.3.2). It may be noted that by virtue of (3.14), relation (4.3.1) is necessary for consistency whereas (4.3.2) is not necessary for stability.
4. If for all f in some set G relation (4.3.1) and relation (4.3.2) are satisfied, then it can be shown as in the proof of the Lax-Richtmeyer theorem that for any f within the μ-closure of G expression (4.3.3) converges to 0 as h tends to 0.

Next, let us study weak convergence of the transition probabilities and corresponding processes. For the definition and notation of weak convergence see definition 0.10 of the general introduction. For definitions, notation and analysis of weak convergence of processes on so-called D-spaces, we refer to the Appendix A.

First, in order to conclude weak convergence of the transition probabilities recall the notation $C^u(S)$ for the set of real-valued uniformly continuous and bounded functions, also see notation 0.9.

THEOREM 4.3.4. *Let* G *be a subset of* B^μ *with* μ*-closure containing* $C^u(S)$. *Suppose that the consistency relation* (4.3.1) *is satisfied for all* $f\in G$ *and let the stability relation* (4.3.2) *hold.*
Then, for any $t\le Z$, $x\in S$ *and with* $n = [th^{-1}]$:

$$(4.3.5) \qquad P^h_{nh}(x;.) \Rightarrow P_t(x;.) \quad , \quad as \ h\to 0.$$

PROOF. According to theorem 4.3.1 and statement 4 of remark 4.3.3 it holds that expression (4.3.3) converges to 0 as h tends to 0 for any $f\in C^u(S)$. This fact together with the portmanteau theorem, see Billingley (1968) p.11, completes the proof. □

Next, let us consider a continuous-time homogeneous Markov process $\{X_t \mid t\ge 0\}$ with collection of transition probabilities $\{P_t \mid t\ge 0\}$. Further, let $\{Z^h_0 \mid h\in(0,h_0]\}$ be a collection of random elements on S. First, let us focus on the existence and construction of discrete-time Markov processes.

LEMMA 4.3.5. *For any* $h \le h_0$ *there exists a unique discrete-time Markov process* $\{X^h_{nh} \mid n = 0,1,2\ldots\}$ *with transition probabilities* $\{P^h_{nh} \mid n = 0,1,2\ldots\}$ *and* $X^h_0 = Z^h_0$.

PROOF: According to the theorem of Ionescu Tulcea (see Neveu (1964) p. 145) there exists a unique random process $\{X^h_{nh} \mid n = 0,1,2\ldots\}$ such that for any $n \in \mathbb{N}$ and $B_0, B_1 \ldots B_{n+1} \in \beta$:

$$(4.3.6) \qquad \begin{cases} \mathbb{P}(X^h_0 \in B_0) = \mathbb{P}(Z^h_0 \in B_0) \\ \mathbb{P}(X^h_0 \in B_0, X^h_1 \in B_1, \ldots, X^h_{(n+1)h} \in B_{n+1}) = \\ \int_{B_0 \times B_1 \times \ldots \times B_n} P^h(x_n; B_{n+1})\, d\mathbb{P}(x_0, x_1, \ldots, x_n) \end{cases}$$

By construction (4.2.3) of the transition probabilities $\{P^h_{nh} \mid n = 0,1,2\ldots\}$ we have for all $j,n \in \mathbb{N}$, $x \in S$ and $B \in \beta$:

$$(4.3.7) \qquad P^h_{(n+j)h}(x;B) = \int P^h_{jh}(y;B)\, P^h_{nh}(x;dy).$$

Consequently, the collection $\{P^h_{nh} \mid n = 0,1,2\ldots\}$ satisfies the Chapman-Kolmogorov equation. By comparing the constructions (4.2.3) and (4.3.6) and using (4.3.7) we can conclude that $\{X^h_{nh} \mid n = 0,1,2\ldots\}$ is a homogeneous Markov process with transition probabilities $\{P^h_{nh} \mid n = 0,1,2\ldots\}$. □

Note that the system (4.3.6) gives a recursive construction of the process $\{X^h_{nh} \mid n = 0,1,2\ldots\}$.
In order to analyse weak convergence of the processes $\{X^h_{nh} \mid n = 0,1,2\ldots\}$ as h tends to 0 let us first give a slight extension ot theorem 4.3.4 which guarantees relation (3.5) of the Appendix.

LEMMA 4.3.6. *Let the hypotheses of theorem 4.3.4 be satisfied and suppose that*

$$(4.3.8) \qquad \int f(y)\, P_t(x;dy) \text{ is continuous in } x \text{ for any } t \le Z \text{ and } f \in C^u(S),$$

and

$$(4.3.9) \qquad \sup_{x \in Q} \mu(x) < \infty \qquad \text{for any compact set } Q \subset S.$$

Then for any $x \in S$ *and collection* $\{x^h \mid h \in (0,h_0]\}$ *with* $x^h \to x$ *as* $h \to 0$, *and* $t \le Z$, $n \in \mathbb{N}$ *with* $n = [th^{-1}]$, *we have*

(4.3.10) $\quad P^h_{nh}(x^h;.) \Rightarrow P_t(x;.) \qquad , \text{ as } h \to 0.$

PROOF. According to the hypotheses of theorem 4.3.4, theorem 4.3.1 and statement 4 of remark 4.3.3 it holds that expression (4.3.3) converges to 0, as h tends to 0, for any $f \in C^u(S)$. Hence, by using (4.3.9):

$$\sup_{x \in Q} \left| \int f(y) P^h_{nh}(x;\, dy) - \int f(y) P_t(x,dy) \right| \to 0 \quad , \text{ as } h \to 0,$$

for any Q compact and $f \in C^u(S)$. Consequently, by using (4.3.8)

$$\left| \int f(y) P^h_{nh}(x^h;dy) - \int f(y) P_t(x;dy) \right| \to 0 \quad , \text{ as } h \to 0,$$

for any $f \in C^u(S)$. Application of the portmanteau theorem, see Billingsley (1968) p.11, completes the proof. □

Define for all $h \in (0,h_0]$ a process $(\overline{X}^h_t)_{t \geq 0}$ on $D[0,\infty)$ by

(4.3.11) $\quad \overline{X}^h_t = X^h_{nh} \quad , \quad t \in [nh,nh+h) \quad , n \in \mathbb{N}$

THEOREM 4.3.7. *Let for each* $Z \geq 0$ *assumption* 2.5 *be satisfied, with* M *replaced by* M_Z, *as well as the hypotheses of theorem* 4.3.4 *and the conditions* (4.3.8) *and* (4.3.9). *Further, assume*

(4.3.12) $\quad X^h_0 \Rightarrow X_0$,

(4.3.13) $\quad \mathbb{P}\left((X_t)_{t \geq 0} \in D[0,\infty) \right) = 1$,

and

(4.3.14) *one of the following conditions holds:*

(i) *Condition* (ii) *of theorem* A.3.5 .

(ii) *Condition* (ii) *of theorem* A.3.6 .

(iii) *Conditions* (ii) *and* (iii) *of theorem* A.3.7.

Then,

(4.3.15) $\quad (\overline{X}^h_t)_{t \geq 0} \Rightarrow (X_t)_{t \geq 0}$ *on* $D[0,\infty)$ *as*, $h \to 0$.

Proof. Immediately from relations (4.3.12) and (4.3.13), lemma 4.3.6 and the theorem A.3.5 , or A.3.6 or A.3.7 corresponding to the condition in (4.3.14) which is satisfied . □

5. APPLICATIONS

5.1. JUMP PROCESSESS

5.1.1. CONTINUOUS TIME MODEL

A detailed introduction and study of *Markov jump processes* can be found in Breiman (1968) and Gihman and Skorohod (1969), (1975). Informally, a Markov jump process satisfies the following description.
Given that at time-point t the state of the process is x, then the state remains unchanged thereafter for an exponential time with parameter $q(x)$; hence, with probability $[1 - q(x)h] + O(h^2)$ the state will not change during $[t, t+h]$, where h is small, and with probability $q(x)h + O(h^2)$ a change of that state, called a *jump*, will occur during this interval. Given that a jump out of x occurs, then the state changes according to a transition probability $H(x;.)$; hence, $H(x;B)$ is the probability that the jump brings the state in set B.

To proceed formally let us consider the 3-tuple (S,q,H), where
S is a separable and complete metric space with Borel-field β,
$q : S \to \mathbb{R}$ is a measurable function, called *jump rate*, and
$H : S \times \beta \to [0,1]$ is a transition probability, called *jump measure*.
Throughout this subsection the following assumption is made.

ASSUMPTION 5.1.1.

(i) $H(x;\{x\}) = 0$ for any $x \in S$.

(ii) For some constant $Q < \infty$ and all $x \in S$: $0 \le q(x) \le Q$. □

THEOREM 5.1.2. *There exists a unique semigroup of transition probabilities* $\{P_t \mid t \ge 0\}$ *on* S *such that for all* $x \in S$, *and* $B \in \beta$:

$$(5.1.1) \qquad [P_h(x;B) - 1_B(x)]h^{-1} \to q(x)[H(x;B) - 1_B(x)],$$

as $h \to 0$, *uniformly in all* $x \in S$ *and* $B \in \beta$.

PROOF Write $a(x;B) = q(x)H(x;B)$ and $a(x) = q(x)$ for all $x \in S$ and $B \in \beta$. Then the conditions a) and b) on p.25 of Gihman and Skorohod (1975) are satisfied. Hence, the proof directly follows from theorem 5 on p.27 of this reference. □

REMARKS 5.1.3

1. Condition (i) of assumption 5.1.1 is not essential for theorem 5.1.2 nor the analysis of this subsection. This may be seen by using a transformation as given on p.312 of Gihman and Skorohod (1969).
2. Expressions for the transition probabilities can be found on p.335 of Breiman or p.364 of Gihman and Skorohod (1969). In this subsection however, we make use of another expression given in proposition 5.1.5 below, which is more convenient for our purposes. □

THEOREM 5.1.4. *Let* Z_0 *be a random element on* S. *Then there exists a unique homogeneous Markov process,* $(X_t)_{t\geq 0}$, *with transition probabilities* $\{P_t \mid t\geq 0\}$ *given by theorem* 5.1.2 *and such that*

(i) $X_0 = Z_0$, *and*

(ii) $\mathbb{P}((X_t)_{t\geq 0} \in D[0,\infty)) = 1$.

PROOF. For the existence and construction of such a process see theorem 4 on p.364 of Gihman and Skorohod (1969) or theorem 15.37 together with corollary 15.44 of Breiman (1968). Since the transition probabilities determine the finite-dimensional distributions, see relation (0.2), the uniqueness follows from theorem 14.5 of Billingsley (1968). □

The process $(X_t)_{t\geq 0}$ given by theorem 5.1.4 will be called a *Markov (or pure) jump process* (see Gihman and Skorohod (1969) p.312 or Breiman (1968) p.328) corresponding to (S,q,H).
In the rest of this subsection, consider the unique collection of transition probabilities $\{P_t \mid t\geq 0\}$ given by theorem 5.1.2.

PROPOSITION 5.1.5. *Let the transition probability* $\bar{H}$ *from* S *into* S *be defined by*

$$(5.1.2) \qquad \bar{H}(x;B) = [1 - \frac{q(x)}{Q}]\, 1_B(x) + \frac{q(x)}{Q} H(x;B) \quad , \; x\in S,\; B\in\beta,$$

where Q *is the constant given by assumption* 5.1.1. *Further, define for all* $n\in N$ *transition probabilities* $\bar{H}^n$ *from* S *into* S *by*

$$(5.1.3) \quad \begin{cases} \bar{H}^0(x;B) = 1_B(x) \\ \bar{H}^n(x;B) = \int \bar{H}^{(n-1)}(y;B)\, \bar{H}(x;dy) \quad , \; n = 1,2,\ldots \end{cases} \quad , \; x\in S,\; B\in\beta.$$

Then, for all $t \geq 0$, $x \in S$, $B \in \beta$:

$$(5.1.4) \qquad P_t(x;B) = \sum_{n=0}^{\infty} \frac{(tQ)^n}{n!} e^{-tQ} \bar{H}^n(x;B)$$

PROOF. One can verify that the convergence relation (5.1.1) holds uniformly in all $x \in S$ and $B \in \beta$. Hence, theorem 5.1.2 completes the proof. □

In order to apply the results of section 2, we consider a bounding funtion μ which satisfies for some constant $K < \infty$:

$$(5.1.5) \qquad \| \int \mu(y) H(.;dy) \|_{\mu} \leq K.$$

REMARK 5.1.6 Since $H(x;.)$ is a probability measure for any $x \in S$ it can be shown that $K \geq 1$. If $\mu(x) = 1$ for all $x \in S$ then we can take: $K = 1$. □

LEMMA 5.1.7. *Let* μ *satisfy* (5.1.5). *Then assumption* 2.5 *is satisfied.*

PROOF. From (5.1.5) it follows that

$$(5.1.6) \qquad \| \int \mu(y) \bar{H}(.;dy) \|_{\mu} \leq K.$$

Hence, by n-times iterating:

$$(5.1.7) \qquad \| \int \mu(y) \bar{H}^n(.;dy) \|_{\mu} \leq K^n.$$

Combination of (5.1.4) and (5.1.7) yields for $t \leq Z$:

$$(5.1.8) \qquad \| \int \mu(y) P_t(.;dy) \|_{\mu} \leq \exp(ZQ[K-1])$$ □

In view of lemma 5.1.7 the results of section 2 can be adapted with bounding function μ satisfying (5.1.5). Therefore, recall relations (2.6), (2.10) and (2.11) for $T_t f$, B_0^{μ}, D_A^{μ} and Af.

LEMMA 5.1.8. (i) $D_A^{\mu} = B_0^{\mu} = B^{\mu}$.
(ii) *For all* $g \in B^{\mu}$ *and* $x \in S$:

$$(5.1.9) \qquad Ag(x) = q(x) \int [g(y) - g(x)] H(x;dy).$$

PROOF. Let $g \in B^{\mu}$. According to expression (5.1.4) for P_h:

$$T_h g(x) = \sum_{n=0}^{\infty} \frac{(hQ)^n}{n!} e^{-hQ} [\int g(y)\ \bar{H}^n(x;dy)] \qquad , \tag{5.1.10}$$

Further, from (5.1.2) and (5.1.5)

$$Q \int g(y)\ \bar{H}(x;dy) = q(x) \int [g(y) - g(x)]\ H(x;dy) + Qg(x) \tag{5.1.11}$$

and

$$\| \int [g(y) - g(x)]\ H(.;dy)\|_{\mu} \leq (K+1)\|g\|_{\mu}. \tag{5.1.12}$$

By using (5.1.7), (5.1.10), (5.1.11) and (5.1.12) we find that

$$\| [T_h g - g](\cdot)\, h^{-1} - q(\cdot) \int [g(y) - g(\cdot)]\ H(.;dy)\|_{\mu} \tag{5.1.13}$$

converges to 0 as h tends to 0. This proves (i) and (ii) of the lemma . □

5.1.2. APPROXIMATIONS

Take $h_0 \leq Q^{-1}$ and let $\{P^h \mid h \in (0,h_0]\}$ be a collection of one-step transition probabilities defined by

$$P^h(x;B) = hq(x)\ H(x;B) + [1 - hq(x)]1_B(x) \tag{5.1.14}$$

for all $x \in S$, $B \in \beta$ and $h \leq h_0$. Let $\{C_h \mid h \in (0,h_0]\}$ be the corresponding difference-method defined by (4.2.2) and $A_h = [C_h - I]h^{-1}$. Then, for $g \in B^{\mu}$:

$$A_h g(x) = q(x) \int [g(y) - g(x)]\ H(x;dy) = Ag(x) \qquad , x \in S, h \leq h_0. \tag{5.1.15}$$

LEMMA 5.1.9. *The consistency relation* (4.3.1) *holds for any* $f \in B^{\mu}$.

PROOF. Immediately from (5.1.15). □

LEMMA 5.1.10 *The stability relation* (4.3.2) *is satisfied* .

PROOF. Relation (5.1.5), with $K \geq 1$, yields

$$\| \int \mu(y)\ P^h(.;dy)\|_{\mu} \leq (1 + hQ[K-1]). \tag{5.1.16}$$

□

Recall expression (4.2.3) for the collection $\{P^h_{nh} \mid n = 0,1,2,\ldots\}$.

THEOREM 5.1.11. *For any* $f \in B^{\mu}$ *the expression*

$$(5.1.17) \qquad \| \int f(y)\, P^h_{nh}(.;dy) - \int f(y)\, P_t(.;dy) \|_{\mu}$$

with $n = [th^{-1}]$ *is convergent of order* $O(h)$ *uniformly in* $t \le Z$.

PROOF. In view of theorem 4.3.1, lemma 5.1.9 and lemma 5.1.10, it remains to show that the order of convergence is $O(h)$. Therefore, we use relation (3.17) of lemma 3.7 with $E_t = T_t$ and $c = f$ together with expression (3.14).

Since $T_t f \in B^{\mu} = D^{\mu}_A$ for all $t \le Z$, expression (5.1.9) for A together with the boundedness of $q(\cdot)$ by Q and relation (5.1.5) yield

$$(5.1.18) \qquad \|AT_s f - AT_t f\|_{\mu} \le 2QK\|T_s f - T_t f\|_{\mu}.$$

By using (3.13), $AT_t f = T_t Af$ and $\|T_t g\|_{\mu} \le M\|g\|_{\mu}$ if $g \in B^{\mu}$ with $M = e^{ZQ[K-1]}$, we find with $0 \le t \le s \le t+h \le Z$:

$$(5.1.19) \qquad \|T_s f - T_t f\|_{\mu} \le (s-t)\, e^{ZQ(K-1)} \|Af\|_{\mu} \le h e^{ZQ(K-1)} 2QK\|f\|_{\mu}.$$

By combining (5.1.18) with (5.1.19) and using (3.14) we obtain

$$(5.1.20) \qquad \|([T_h f - I]h^{-1} - A)T_t f\|_{\mu} \le h4Q^2K^2e^{ZQ(K-1)} \| f \|_{\mu}.$$

The proof is completed by using (5.1.15) and (5.1.20) in (3.17) and applying lemma 3.7. □

REMARK 5.1.12. Note that (5.1.5) is satisfied with K = 1 if we take $\mu(x) = 1$ for all $x \in S$. In that case we have: $\|f\|_{\mu} = \|f\|_{\infty}$ and $B^{\mu} = \{f: S \to \mathbb{R} \mid f \text{ measurable and bounded }\} =: B$.
By using (5.1.15) and (5.1.20) with K = 1 in (3.19), it follows that

$$(5.1.21) \qquad \|(h^{-1}[C_h - T_h]T_t f)\|_{\infty} \le h4Q^2\|f\|_{\infty}.$$

Hence, from the proof of the Lax-Richtmeyer theorem with constant $K_C = 1$, relation (5.1.19) with $s = [th^{-1}]h$ and K = 1, and relation (5.1.21), we can conclude that for all $f \in B$ and $t \le Z$:

$$(5.1.22)\quad \begin{cases} \text{Expression (5.1.17),} \\ \text{with } \mu \text{ replaced by } \infty\text{, is bounded by:} \qquad h[4ZQ^2+2Q]\|f\|_\infty . \end{cases}$$ □

PROPOSITION 5.1.13. *For all* $x\in S$, $t\geq 0$ *and with* $n=[th^{-1}]$ *the weak convergence relation* (4.3.5) *for the transition probabilities is satisfied.*

PROOF. Since $C^u(S)\subset B^\mu$ this follows from theorem 4.3.4 together with lemma 5.1.9 and 5.1.10. □

Let us conclude this subsection by showing weak convergence of discrete-time processes. Therefore, consider a Markov jump process $(X_t)_{t\geq 0}$ given by theorem 5.1.4 and for any $h\leq h_0$ a discrete-time Markov process $\{X^h_{nh} \mid n=0,1,2\ldots\}$ as given by lemma 4.3.5.
Further, recall expression (4.3.11) for the process $(\bar{X}^h_t)_{t\geq 0}$.

LEMMA 5.1.14. *Suppose that*

(i) $q(x)$ *is continuous in* x, *and*

(ii) $H(x;.)$ *is weakly continuous in* x.

Then relation (4.3.8) *is satisfied for any* $Z>0$.

PROOF. Expression (5.1.2) for $\bar{H}$ together with (i) and (ii) imply

$$(5.1.23)\qquad \int f(y)\,\bar{H}(x;dy) \text{ is continuous in } x \text{ for any } f\in C^u(S).$$

Repeatedly applying (5.1.23) to expression (5.1.3) and making use of expression (5.1.4) completes the proof. □

THEOREM 5.1.15. *Let the conditions* (i) *and* (ii) *of lemma* 5.1.14 *be satisfied and assume*: $X^h_0\Rightarrow X_0$ *as* h *tends to* 0. *Then,*

$$(5.1.24)\qquad (\bar{X}^h_t)_{t\geq 0}\Rightarrow (X_t)_{t\geq 0} \quad \text{on } D[0,\infty) \text{ as } h\to 0.$$

PROOF. We will apply theorem 4.3.7. First of all, note that $Z>0$ has been chosen arbitrarily. Hence by taking $\mu(x)=1$ for all $x\in S$ in (5.1.5) and using the lemmas 5.1.9 and 5.1.10 as well as $C^u(S)\subset B^\mu$, we conclude that for any $Z>0$ assumption 2.5 as well as the hypotheses of theorem 4.3.4 are satisfied. Further, by virtue of lemma 5.1.14 relation (4.3.8) holds, and clearly relation (4.3.9) is valid with $\mu(x)=1$ for all $x\in S$.

Since also relation (4.3.12) and (4.3.13) are guaranteed, the proof will be completed by verifying relation (4.3.14).
Therefore, we focus on condition (ii) of theorem A.3.6.
According to expression (5.1.14) for P^h:

$$(5.1.25) \qquad P^h(x;S-\{x\}) \le hQ$$

for any $x \in S$ and $h \le h_0$. Consequently, for fixed $\delta > 0$:

$$(5.1.26) \qquad \sup_{nh \le \delta} \sup_{x \in S} P^h_{nh}(x;S-\{x\}) \le [\delta h^{-1}]hQ \le \delta Q.$$

This proves condition (ii) of theorem A.3.6 and thus completes the proof. □

5.2. M|M|∞ - QUEUE

5.2.1. CONTINUOUS-TIME MODEL

This subsection is concerned with a specific jump process $\{X_t | t \ge 0\}$ where X_t denotes the number of customers in a so-called *infinite-server-queue*. Informally, this process is described as follows.
According to a Poisson process, say with parameter λ, customers arrive at a service facility. Each customer present at the facility is being served with constant service-rate, say ν. The arrival and service process are assumed to be independent. Hence, the process satisfies the following exponential structure:
Given that at time-point t the number of customers present is i, then the waiting time up to the next arrival or service-completion, which we call a *jump*, has an exponential distribution with parameter $(\lambda+i\nu)$. Given that a jump occurs, then with probability $\lambda(\lambda+i\nu)^{-1}$ the state becomes $(i+1)$ and with probability $i\nu(\lambda+i\nu)^{-1}$ it becomes $(i-1)$.
Note that the above exponential description resembles that of subsection 5.1. In contrast, however, the jumprate $(\lambda+i\nu)$ is not bounded uniformly in i. Nevertheless, the study of (pure) jump processes given in Breiman (1968) includes the above process.

Hence, in this subsection we consider the 3-tuple $(\mathbb{N},\lambda,\nu)$ where $\mathbb{N}$, the set of natural numbers, denotes the state space, λ is a positive constant, the *arrival rate*, and ν is a positive constant, the *service rate* per customer . The following proposition presents transition probabilities which correspond to the above description .

PROPOSITION 5.2.1. *Define for any* $i \in \mathbb{N}$, $B \subset \mathbb{N}$, *and* $t \geq 0$:

$$(5.2.1)\quad \begin{cases} P_t^{(0)}(i;B) = 1_B(x)\, e^{-(\lambda+i\nu)t} & \text{and for } n = 1,2,\ldots \\ P_t^{(n)}(i;B) = \int_0^t e^{-(\lambda+i\nu)s}\, [\lambda P_{t-s}^{(n-1)}(i+1;B) + i\nu P_{t-s}^{(n-1)}(i-1;B)]\, ds, & \end{cases}$$

and

$$(5.2.2)\quad P_t(i;B) = \sum_{n=0}^{\infty} P_t^{(n)}(i;B).$$

Then $\{P_t \mid t \geq 0\}$ *is a semigroup of transition probabilities.*

PROOF; One can show by induction on N that for any t,i,B the expression $\sum_{n=0}^{N} P_t^{(n)}(i;B)$ is dominated by 1 and monotone increasing in N.

Consequently, the right hand-side of (5.2.2) is well-defined. By using (5.2.1) and (5.2.2) one easily verifies the Chapman-Kolmogorov equation, see relation (2.1), and proves that P_t is a probability. □

PROPOSITION 5.2.2. *Let* Z_0 *be a random element on* S. *Then there exists a unique homogeneous Markov process* $(X_t)_{t \geq 0}$ *with transition probabilities* $\{P_t \mid t \geq 0\}$ *given by* (5.2.1) *and* (5.2.2), *such that*

(i) $X_0 = Z_0$

(ii) $\mathbb{P}\left((X_t)_{t \geq 0} \in D[0,\infty) \right) = 1$.

PROOF. The existence and construction of the process is shown by theorem 15.37 of Breiman, provided the condition of proposition 15.43 of this reference holds. This condition, however, is satisfied since $\sum_{n=0}^{\infty} (\lambda+i\nu+n\nu)^{-1} = \infty$ for any $i \in \mathbb{N}$. As in the proof of theorem 5.1.4, the uniqueness follows from theorem 14.5 of Billingsley (1968). □

In the rest of this subsection consider the semigroup $\{P_t \mid t \geq 0\}$ given by (5.2.1) and (5.2.2), and the Markov process $(X_t)_{t \geq 0}$ given by proposition 5.2.2. First, let us present some properties of the transition probabilities $\{P_t \mid t \geq 0\}$.

LEMMA 5.2.3. *Let* $h_0 > 0$. *Then for some constant* C *depending only on*

λ *and* ν, *we have for all* $h \le h_0$, $t \ge 0$:

$$(5.2.3) \qquad |h^{-1} P_h(i;i+1) - \lambda| \le hC \qquad , \quad i \ge 0,$$

$$(5.2.4) \qquad |h^{-1} P_h(i;i-1) - i\nu| \le i^2hC \qquad , \quad i \ge 1,$$

$$(5.2.5) \qquad |h^{-1} P_h(i;\ i-2,i-3,\ldots 0\ )| \le i^2hC \qquad , \quad i \ge 2,$$

$$(5.2.6) \qquad |P_t(i;i+n) \qquad | \le (\lambda t)^n\ (n!)^{-1} \qquad , \quad i,n \ge 0.$$

PROOF. Consider $i \in \mathbb{N}$ and write

(i) $\bar{p}_h(i;+1) = \int_o^h e^{-(\lambda+i\nu)s}\ \lambda\ e^{-(\lambda+i\nu+\nu)(h-s)}ds = \lambda e^{-h(\lambda+i\nu)}[e^{-h\nu}-1]\nu^{-1}$,

(ii) $\bar{p}_h(i;-1) = \int_o^h e^{-(\lambda+i\nu)s} i\nu\ e^{-(\lambda+i\nu-\nu)(h-s)}ds = ie^{-h(\lambda+i\nu)}[e^{h\nu}-1]$,

(iii) $\bar{p}_h(i;\ge +2) \le \int_o^h \lambda[\int_o^{h-s} \lambda dt\ ]\ ds = \frac{1}{2}\lambda^2h^2$,

(iv) $\bar{p}_h(i;\le -2) \le \int_o^h (i\nu+\nu)[\int_o^{h-s} (i\nu)dt\ ]\ ds \le (i+1)^2\nu^2h^2 \le 3i^2\nu^2h^2$, $\quad i \ge 1$.

(The interpretation of these expressions is the following: Given that $X_0 = i$ then all during $(0,h]$:
$\bar{p}_h(i;+1)$ is the probability of one arrival only,
$\bar{p}_h(i;-1)$ is the probability of one service-completion only,
$\bar{p}_h(i;\ge +2)$ dominates the probability of at least 2 arrivals, and
$\bar{p}_h(i;\le -2)$ dominates the probability of at least 2 service-completions.)

By using (5.2.2) it can be shown that

$$(5.2.7) \qquad |P_h(i;i+1) - \bar{p}_h(i;+1)| \le \bar{p}_h(i;\ge +2) \qquad , \quad i \ge 0,$$

$$(5.2.8) \qquad |P_h(i;i-1) - \bar{p}_h(i;-1)| \le \bar{p}_h(i;\le -2) \qquad , \quad i \ge 1.$$

Relation (5.2.7) together with (i) and (iii) yields (5.2.3).
Relation (5.2.8) together with (ii) and (iv) yields (5.2.4).
(5.2.5) is a direct consequence of (iv) and
(5.2.6) follows analoguously to (iii). □

DEFINITION 5.2.4. For any $p \in \mathbb{N}$ the bounding function μ_p is defined by $\mu_p(i) = (\lambda+i\nu)^p$. □

LEMMA 5.2.5. *For any* $p \in \mathbb{N}$ *there exists a constant* $M(p)$ *such that*

$$(5.2.9) \qquad \| \sum_{j=0}^{\infty} \mu_p(j)\; P_t(\cdot\,;j) \|_{\mu_p} \le M(p) \qquad , \qquad t \le Z.$$

PROOF. By virtue of (5.2.6), relation (5.2.9) can be verified with

$$(5.2.10) \qquad M(p) = 1 + \sum_{n=0}^{\infty} (\lambda Z)^n (n!)^{-1} (1 + n\tfrac{\nu}{\lambda})^p. \qquad \square$$

COROLLARY 5.2.6. *Assumption* 2.5 *is satisfied with* $\mu = \mu_p$ *for any* $p \in \mathbb{N}$. □

In view of corollary 5.2.6, the results of section 2 can be adapted with bounding function $\mu = \mu_{p+2}$ and $p \in \mathbb{N}$. Consider some fixed $p \in \mathbb{N}$. In what follows below the constants $M(p)$ and $M(p+1)$ are given by (5.2.10).

LEMMA 5.2.7 (i) $B^{\mu_p} \subset D_A^{\mu_{p+2}}$

(ii) *For all* $g \in B^{\mu_p}$ *and* $i \in \mathbb{N}$:

$$(5.2.11) \qquad Ag(i) = \lambda[g(i+1) - g(i)] + i\nu\; [g(i-1) - g(i)]$$

PROOF. Let $g \in B^{\mu_p}$ and write

$$(5.2.12) \qquad \begin{cases} T_h g(i) = P_h(i;i+1)g(i+1) + P_h(i;i-1)g(i-1) + \\ \sum_{j \ge i+2} P_h(i;j)g(j) + \sum_{j \le i-2} P_h(i;j)g(j) + (1 - \sum_{j \ne i} P_h(i;j))g(i). \end{cases}$$

By using (5.2.6) it follows that

$$(5.2.13) \qquad \sum_{j \ge i+2} P_h(i;j)g(j) \le$$
$$(\lambda^2 h^2) \sum_{n=0}^{\infty} \frac{(\lambda h)^n}{(n+2)!} \| g \|_{\mu_p} (\lambda + i\nu + 2\nu + n\nu)^p \le (\lambda^2 h^2) \| g \|_{\mu_p} (\lambda + i\nu + 2\nu)^p M(p).$$

With some calculation the relations (5.2.3), (5.2.4), (5.2.5), (5.2.12) and (5.2.13) together with the fact that $g \in B^{\mu_p}$ yield

$$(5.2.14) \qquad \| [T_h g - g](.)h^{-1} - (\lambda[g(.+1) - g(.)] + i\nu\, [g(.-1) - g(.)]) \|_{\mu_{p+2}}$$

is convergent of order $O(h)$ as h tends to 0. This completes the proof. □

5.2.2. APPROXIMATIONS

For $a,b\in\mathbb{R}$ we write $a\wedge b = \min(a,b)$.

Choose $h_0 \geq 0$ and let $\{P^h \mid h\in(0,h_0]\}$ be a collection of one-step transition probabilities on $\mathbb{N}$ defined by

$$(5.2.15)\qquad P^h(i;j) = \begin{cases} 1 - [h(\lambda+i\nu)\wedge 1] & , j = i \\ [h(\lambda+i\nu)\wedge 1]\frac{\lambda}{\lambda+i\nu} & , j = i+1 \\ [h(\lambda+i\nu)\wedge 1]\frac{i\nu}{\lambda+i\nu} & , j = i-1 \end{cases}$$

for all $i\in\mathbb{N}$. Let $\{C_h \mid h\in(0,h_0]\}$ be the corresponding difference-method defined by (4.2.2) and $A_h = [C_h - I]h^{-1}$. Then,

$$(5.2.16)\qquad \left\{\begin{array}{l} A_h g(i) = [1\wedge h^{-1}(\lambda+i\nu)^{-1}]. \\ (\lambda[g(i+1) - g(i)] + i\nu[g(i-1) - g(i)]) \end{array}\right. , \quad i\in\mathbb{N},\ g\in B^{\mu_p}.$$

Further, recall expression (4.2.3) for the collection $\{P^h_{nh} \mid n = 0,1,2...\}$.

LEMMA 5.2.8. *The consistency relation* (4.3.1) *holds with* $\mu=\mu_{p+2}$ *for any* $f\in B^{\mu_p}$.

PROOF. First consider some $g\in B^{\mu_p}$. Comparing (5.2.16) with (5.2.11) gives

$$(5.2.17)\qquad \left\{\begin{array}{l} [A_h g(i) - Ag(i)] = \{[(\lambda+i\nu)\wedge h^{-1}] - (\lambda+i\nu)\}. \\ (\frac{\lambda}{\lambda+i\nu}[g(i+1) - g(i)] + \frac{i\nu}{\lambda+i\nu}[g(i-1) - g(i)]). \end{array}\right.$$

Hence, for $(\lambda+i\nu) > h^{-1}$ and using $|g(j)| \leq (\lambda+j\nu)^p \| g\|_{\mu_p}$ for all j,we find

$$(5.2.18)\qquad \left\{\begin{array}{l} |A_h g(i) - Ag(i)|\,(\lambda+i\nu)^{-2} \leq [h^{-1}(\lambda+i\nu)^{-2} + (\lambda+i\nu)^{-1}]. \\ [(\lambda+i\nu+\nu)^p + 3(\lambda+i\nu)^p]\| g\|_{\mu_p} \leq 2h\, 4(1+\frac{\nu}{\lambda})^p(\lambda+i\nu)^p \| g\|_{\mu_p} . \end{array}\right.$$

Since also $|A_h g(i) - Ag(i)| = 0$ for $(\lambda+i\nu)\leq h^{-1}$, relation (5.2.18) yields

$$(5.2.19) \qquad \|(A_h-A)\,g\|_{\mu_{p+2}} \leq h\,8(1+\tfrac{\nu}{\lambda})^p \|g\|_{p} \;.$$

Next let $f\in B^{\mu_p}$. According to (5.2.9),

$$(5.2.20) \qquad T_t f \in B^{\mu_p} \text{ and } \|T_t f\|_{\mu_p} \leq M(p)\,\|f\|_{\mu_p} \quad , \; t\leq Z \;.$$

By combination of (5.2.19) and (5.2.20),

$$(5.2.21) \qquad \|(A_h-A)T_t f\|_{\mu_{p+2}} \leq h\,8(1+\tfrac{\nu}{\lambda})^p M(p)\,\|f\|_{\mu_p} \;, \; t\leq Z \;.$$

By letting h tend to 0 in (5.2.21) the proof is completed. □

<u>LEMMA 5.2.9.</u> *The stability relation* (4.3.2) *is satisfied with* $\mu = \mu_{p+2}$.

<u>PROOF</u>. From expression (5.2.15) for P^h :

$$(5.2.22) \qquad \|\sum \mu_{p+2}(j)\, P^h(i;j)\|_{\mu_{p+2}} \leq$$

$$1 + [h(\lambda+i\nu)\wedge 1]\left|-1 + \frac{\lambda}{\lambda+i\nu}\left(\frac{\lambda+i\nu+\nu}{\lambda+i\nu}\right)^{p+2} + \frac{i\nu}{\lambda+i\nu}\left(\frac{\lambda+i\nu-\nu}{\lambda+i\nu}\right)^{p+2}\right| \leq$$

$$1 + h\left|-(\lambda+i\nu) + \lambda(1+\tfrac{\nu}{\lambda})^{p+2} + i\nu(1-\tfrac{\nu}{\lambda+i\nu})^{p+2}\right| \leq 1 + h\,M_C \;.$$

□

<u>THEOREM 5.2.10</u>. *For any* $f\in B^{\mu_p}$ *the expression*

$$(5.2.23) \qquad \|\sum_{j\in\mathbb{N}} f(j)\, P^h_{nh}(\cdot;j) - \sum_{j\in\mathbb{N}} f(j)\, P_t(\cdot;j)\|_{\mu_{p+2}}$$

with $n = [th^{-1}]$ *is convergent of order* $O(h)$ *uniformly in* $t\leq Z$.

<u>PROOF</u>. By virtue of theorem 4.3.1, lemma 5.2.8 and lemma 5.2.9 it remains to verify the order of convergence O(h). Therefore, we will use relation (3.17) of lemma 3.7 with $E_t = T_t$ and $c = f$ where $f\in B^{\mu_p}$, together with expression (3.14).

First of all, note that relation (5.2.11) for A yields for any $n\in\mathbb{N}$ and $g\in B^{\mu_n}$,

$$(5.2.24) \qquad \|Ag\|_{\mu_{n+1}} \leq 4\,\|g\|_{\mu_n}.$$

Relations (5.2.24), (3.13) and (5.2.9) yield for $0\leq t\leq s\leq t+h\leq Z$:

$$(5.2.25)\qquad \|AT_s f - AT_t f\|_{\mu_{p+2}} \le 4\|T_s f - T_t f\|_{\mu_{p+1}} \le$$

$$4\int_t^s \|T_s Af\|_{\mu_{p+1}} ds \le 4(s-t)\, M(p+1)\|Af\|_{\mu_{p+1}} \le h\ 16\ M(p+1)\|f\|_{\mu_p}.$$

Applying (5.2.25) to (3.14) and combining (3.14) with (5.2.21) shows

$$(5.2.26)\qquad \begin{cases} \|(A_h - A)\ T_t f\|_{\mu_{p+2}} + \|([T_h - I]h^{-1} - A)\ T_t f\|_{\mu_{p+2}} \le \\ h\ \|f\|_{\mu_p}\ [8(1+\frac{\nu}{\lambda})^p\ M(p) + 16M(p+1)]. \end{cases}$$

Using (5.2.26) in (3.17) and applying lemma 3.7 completes the proof. □

PROPOSITION 5.2.11. *For all* $i \in \mathbb{N}$, $t \le 0$ *and with* $x = i$, $n = [th^{-1}]$ *the weak convergence relation* (4.3.5) *for the transition probabilities is satisfied.*

PROOF. Immediately from theorem 5.2.10 and the fact that $\{f : \mathbb{N} \to \mathbb{R} \mid f \text{ bounded}\} \subset B^{\mu_p}$ for $p \in \mathbb{N}$. □

Consider the Markov process $(X_t)_{t \ge 0}$ given by proposition 5.2.2, as well as for all $h \le h_0$ the discrete-time processes $\{X^h_{nh} \mid n = 0,1,2\ldots\}$ given by lemma 4.3.5. Further, recall expression (4.3.11) for $(\bar{X}^h_t)_{t \ge 0}$.

THEOREM 5.2.12. *Suppose that* $X^h_0 \Rightarrow X_0$ *as* $h \to 0$, *then*

$$(5.2.27)\qquad (\bar{X}^h_t)_{t \ge 0} \Rightarrow (X_t)_{t \ge 0} \quad \text{on } D[0,\infty) \quad \text{as } h \to 0.$$

PROOF. We will apply theorem 4.3.7. First note that $Z > 0$ and $p \in \mathbb{N}$ have been chosen arbitrarily. Hence, by taking $\mu = \mu_2$ and using the lemmas 5.2.5, 5.2.8 and 5.2.9 we conclude that assumption 2.5 as well as the hypotheses of theorem 4.3.4 are satisfied for all $Z > 0$. Further, clearly the conditions (4.3.8) and (4.3.9) hold. Since also the relations (4.3.12) and (4.3.13) are guaranteed, the proof is completed by showing (4.3.14). Therefore, we will verify the conditions (ii) and (iii) of theorem A.3.7.

For a compact set $C \subset \mathbb{N}$ with $Q = \max\ \{i \mid i \in C\}$ we find for all $h \le h_0$:

$$(5.2.28)\qquad \sup_{i\in C} P^h(i;\mathbb{N}-\{i\}) \le \sup_{i \in C} \{h(\lambda+i\nu) \wedge 1\} \le h(\lambda+Q\nu).$$

Consequently, for any $\delta > 0$ and all $h \le h_0$:

$$(5.2.29) \qquad \sup_{nh\le\delta} \ \sup_{i\in C} \ P^h_{nh}(i;\mathbb{N}-\{i\}) \ \le \ [\delta h^{-1}]h(\lambda+Q\nu) \le \ \delta(\lambda+Q\nu).$$

This proves condition (ii) of theorem A.3.7.

To proceed, first note that $P^h(i;j)$ does not exceed λh for $j = i+1$ and is equal to 0 for $j \ge i+2$. Consequently, for any $i\in\mathbb{N}$:

$$(5.2.30) \qquad \mathbb{P}\,(X^h_{nh} \ge i+\ell \text{ for some } nh \le Z \mid X^h_0 = i) \le$$

$$\binom{[Zh^{-1}]}{\ell}(\lambda h)^\ell \ \le \ \frac{[Zh^{-1}]^\ell}{\ell!}\ (\lambda h)^\ell \ \le \ \frac{(\lambda Z)^\ell}{\ell!}$$

where $\mathbb{P}\,(\cdot \mid X^h_0 = i)$ denotes the conditional probability given that $X^h_0 = i$.
From (5.2.30) we conclude for all $i \le Q$:

$$(5.2.31) \qquad \mathbb{P}\,(\bar{X}^h_t \le Q+\ell \text{ for all } t\in[0,Z] \mid X^h_0 = i) \ge 1 - \frac{(\lambda Z)^\ell}{\ell!},$$

which converges to 1 as $\ell \to \infty$ uniformly in $h \le h_0$.
Hence, for any compact set C, say with $Q = \max\{i \mid i\in C\}$, and any $\eta > 0$, one can find a compact set $\{0,1,\dots, Q+\ell\}$ such that the left-hand side of (5.2.31) is larger than $1-\eta$ uniformly in all $i\in C$ and $h \le h_o$. This guarantees condition (iii) of theorem A.3.7 and thus finalizes the proof. □

5.3. SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS (DIFFUSIONS)

5.3.1. CONTINUOUS-TIME MODEL

For an introduction, definitions, properties and techniques of *stochastic differential* and *integral equations* we refer to Arnold (1975) or Gihman and Skorohod (1972). In this subsection we frequently make use of the latter reference. Further, we note that the process which will be considered is also known as *diffusion (process)*, (see definition 2 on p.64 of Gihman and Skorohod (1972)).

With $(W_t)_{t\geq 0}$ denoting a Wiener process as defined on p.8 of Gihman and Skorohod (1972) and a,b: $\mathbb{R}\to\mathbb{R}$ measurable functions, consider the *stochastic differential equation:*

(5.3.1) $$d\eta_t = a(\eta_t)dt + b(\eta_t)\, dW_t.$$

A random function $(\eta_t)_{t\leq Z}$ is called a solution of (5.3.1) on $[0,Z]$ if it satisfies the conditions 1), 2) and 3) on p.33 of Gihman and Skorohod (1972). Particularly, with probability one the solution must satisfy the *stochastic integral equation:*

(5.3.2) $$\eta_t = \eta_0 + \int_0^t a(\eta_s)ds + \int_0^t b(\eta_s)dW_s \quad , \quad t\leq Z.$$

The first integral is a random element, which, with probability one, is equal to the ordinary Lebesque integral. The second integral is a random element, known as stochastic integral with respect to the Wiener measure, as defined on p.15 of Gihman and Skorohod (1972). The existence and uniqueness of a solution of (5.3.1) will be guaranteed by theorem 5.3.1 below, under a Lipschitz condition on the coefficients a and b.
First, let us formally present the 4-tuple $(\mathbb{R}, W, a, b)$, where
$\mathbb{R}$, the real line, is the state space, $(W_t)_{t\geq 0}$ is a Wiener process,
a is a measurable function from $\mathbb{R}$ into $\mathbb{R}$, called *drift function* and
b is a measurable function from $\mathbb{R}$ into $\mathbb{R}$, called *diffusion function*.
Further, it is useful to recall the conventions (0.16) and (0.17) of the introduction.

THEOREM 5.3.1. *Let the following assumptions be satisfied:*
(i) *There exists a constant* L *such that for all* $x,y \in \mathbb{R}$:

$$(5.3.3) \qquad |a(x) - a(y)| + |b(x) - b(y)| \leq L|x-y|.$$

(ii) η_0 *does not depend on* $(W_t)_{t \geq 0}$ *and* $\mathbb{E}\,[\eta_0]^2 < \infty$.
Then there exists a solution $(\eta_t)_{t \leq Z}$ of (5.3.1) *satisfying*:
(iii) *With probability one, the funtion* η_t *is continuous in* t.
(iv) $\mathbb{E}[\eta_t]^2 \leq C$ *for all* $t \leq Z$ *and some constant* C .

If $(\eta^1_t)_{t \leq Z}$ *and* $(\eta^2_t)_{t \leq Z}$ *are solutions of* (5.3.1) *satisfying* (iii) *and* (iv), *then:* $\mathbb{P}\,(\eta^1_t = \eta^2_t$ for all $t \leq Z) = 1$.

PROOF. With $K := \sqrt{2}\max\{L, |a(0)| + |b(0)|\}$ we obtain for all $x,y \in \mathbb{R}$:

$$(5.3.4) \qquad \begin{cases} |a(x) - a(y)| + |b(x) - b(y)| \leq K|x-y| \\ |a(x)|^2 + |b(x)|^2 \leq K^2(1+x^2) \; . \end{cases}$$

Hence, the proof is given by theorem 1 on p.40 of Gihman and Skorohod (1972).
□

Throughout this subsection let the Lipschitz relation (5.3.3) be satisfied, so that we can use relation (5.3.4). According to theorem 5.3.1 for any $x \in \mathbb{R}$ there exists a unique solution of (5.3.1) with $\eta_0 = x$. This solution is denoted by $(\eta_t(x))_{t \leq Z}$. Hence, we have

$$(5.3.5) \qquad \eta_t(x) = x + \int_0^t a(\eta_s(x))ds + \int_0^t b(\eta_s(x))dW_s \qquad , t \leq Z.$$

PROPOSITION 5.3.2. *We obtain a semigroup of transition probabilities* $\{P_t | t \leq Z\}$ *by defining for all* $t \leq Z$, $x \in \mathbb{R}$ *and* $B \in \beta$:

$$(5.3.6) \qquad P_t(x;B) = \mathbb{P}\,(\eta_t(x) \in B) \; .$$

PROOF. A direct consequence of theorem 1 on p.67 of Gihman and Skorohod (1972)
□

PROPOSITION 5.3.3. *Let* $(\eta_t)_{t\leq Z}$ *be a solution of* (5.3.1) *as given by theorem* 5.3.1. *Then* $(\eta_t)_{t\leq Z}$ *is a homogeneous Markov process with transition probabilities* $\{P_t \mid t\leq Z\}$ *defined by* (5.3.6).

PROOF. See theorem 1 on p.67 of Gihman and Skorohod (1972). □

The following two lemmas present results which will be used frequently without being mentioned explicitly. The first lemma concerns growth conditions with respect to the time-parameter. The second shows Hölder-type inequalities for integrals with random functions as integrands. In doing so, we assume that the measurability of the integrands as well as existence of the integrals, in (5.3.9) as ordinary integrals with probability one and in (5.3.10) as stochastic integrals, is guaranteed.

LEMMA 5.3.4. *For any* $m\in\mathbb{N}$ *there exists constants* $G(m)$, $L(m)$, *depending only on* m,Z *and* K *such that for all* $x\in\mathbb{R}$ *and* $t\leq Z$:

$$(5.3.7)\qquad \mathbb{E}\,|\eta_t(x)|^m \leq (1+|x|^m)\, e^{tG(m)}.$$

$$(5.3.8)\qquad \mathbb{E}\,|\eta_t(x)-x|^m \leq (1+|x|^m)\, t^{\frac{m}{2}}\, L(m)\, e^{tG(m)}.$$

PROOF. By using theorem 4 on p.48 of Gihman and Skorohod (1972) with $\eta_0 = x$ together with Schwartz' inequality. □

LEMMA 5.3.5. *Let* $(f_s)_{s\leq t}$ *be a random function. Then for any* $m\geq 1$:

$$(5.3.9)\qquad \mathbb{E}\,\Big|\int_0^t f_s\,ds\Big|^m \leq t^{m-1}\int_0^t \mathbb{E}\,|f_s|^m\,ds.$$

$$(5.3.10)\qquad \mathbb{E}\Big(\int_0^t f_s\,dW_s\Big)^{2m} \leq m(2m-1)^{m-1}\int_0^t \mathbb{E}\,(f_s)^{2m}\,ds.$$

PROOF. Relation (5.3.9) follows from using Hölder's inequality to the integral and next applying Fubini's theorem . Relation (5.3.10) is given by theorem 6 on p.26 of Gihman and Skorohod (1972) . □

In order to present a result on the dependence of initial data for solutions of stochastic differential equations, we need some further notation and a smoothness assumption on the functions a and b .

NOTATION 5.3.6. For any $m \in \mathbb{N}$ define the class of functions $C^{3;m}$ by

$$(5.3.11)\quad \begin{cases} C^{3;m} = \{f:\mathbb{R}\to\mathbb{R} \mid \text{for some constant } K_f, \text{ all } x\in S \text{ and } k=0,1,2,3: \\ \qquad \frac{d^k}{dx^k} f(x) \text{ exists and } |\frac{d^k}{dx^k} f(x)| \le K_f(1+|x|^m)\ \} \end{cases}$$ □

ASSUMPTION 5.3.7. For constant K and all $x\in S$ it holds that

$$(5.3.12)\quad \begin{cases} \frac{d^k}{dx^k} a(x) \text{ and } \frac{d^k}{dx^k} b(x) \text{ exist, are continuous in } x \text{ and} \\ |\frac{d^k}{dx^k} a(x)| + |\frac{d^k}{dx^k} b(x)| \le K, \quad k=1,2,3. \end{cases}$$ □

In the rest of this subsection let assumption 5.3.7 be satisfied. For $f\in C^{3;m}$ let K_f denote the constant given by (5.3.11).
Further, C or C_i, $i=1,2,\ldots$ always denotes a constant which depends only on m,Z and K.
The following proposition is an extension of theorem 1 on p.60 of Gihman and Skorohod (1972) in that it shows polynomial bounds.

PROPOSITION 5.3.8. *Let* $f\in C^{3;m}$ *and define for* $t>0$ *and all* $x\in S$:
$g_t(x) = \mathbb{E} f(\eta_t(x))$. *Then* $g_t\in C^{3;m}$ *and* $K_{g_t} \le CK_f$.

PROOF. Since $f\in C^{3;m}$, it follows from (5.3.11) and (5.3.7) that

$$(5.3.13)\qquad |\mathbb{E} f(\eta_t(x))| \le K_f \mathbb{E}(1+|\eta_t(x)|^m) \le 2K_f\, e^{tG(m)}(1+|x|^m).$$

Let us present the proof for $\frac{d^k}{dx^k}\mathbb{E} f(\eta_t(x))$ only for $k=1$. For $k=2,3$, it can be given analogously.
By virtue of theorem 1 on p.60 and its corollary on p.61 of Gihman and Skorohod (1972), the existence is shown by

$$(5.3.14)\quad \begin{cases} \frac{d}{dx}\mathbb{E} f(\eta_t(x)) = \mathbb{E}[f'(\eta_t(x))(\frac{d}{dx}\eta_t(x))], \text{ where} \\ (\frac{d}{dx}\eta_t(x))_{t\ge 0} \text{ is a random function satisfying:} \end{cases}$$

$$(5.3.15)\quad \begin{cases} \frac{d}{dx}\eta_t(x) = 1 + \int_0^t [a'(\eta_s(x))]\,(\frac{d}{dx}\eta_s(x))\,ds \\ \qquad\qquad + \int_0^t [b'(\eta_s(x))]\,(\frac{d}{dx}\eta_s(x))\,dW_s. \end{cases}$$

Since $|\frac{d}{dx} a(x)| + |\frac{d}{dx} b(x)| \leq K$ relation (5.3.15) yields

$$(5.3.16) \qquad \mathbb{E}\,(\tfrac{d}{dx}\eta_t(x))^2 \leq$$

$$3(1+tK^2 \int_0^t \mathbb{E}\,(\tfrac{d}{dx}\eta_s(x))^2 ds + K^2 \int_0^t \mathbb{E}\,(\tfrac{d}{dx}\eta_s(x))^2 ds).$$

Hence, the Gronwall-Bellman inequality, see for instance lemma 1 on p.41 of Gihman and Skorohod (1972), implies

$$(5.3.17) \qquad \mathbb{E}\,(\tfrac{d}{dx}\eta_t(x))^2 \leq 3e^{3(1+t)K^2 t}$$

Then, applying Schwartz' inequality to (5.3.14), using (5.3.17), $f'(y) \leq K_f(1+|y|^m)$ and (5.3.7) yield

$$(5.3.18) \qquad |\tfrac{d}{dx}\,\mathbb{E}\,f(\eta_t(x))| \leq \{\mathbb{E}\,[f'(\eta_t(x))]^2\}^{\frac{1}{2}}\,\{\mathbb{E}\,(\tfrac{d}{dx}\eta_t(x))^2\}^{\frac{1}{2}}$$

$$\leq K_f C(1+|x|^m).$$

□

Let us proceed by analyzing the results of section 2 for solutions of stochastic differential equations. Therefore, in the rest of this subsection let $\{P_t | t \leq Z\}$ be the semigroup of transition probabilities given by (5.3.6). Further, in this subsection let μ_m be given as follows:

<u>DEFINITION 5.3.9</u>. For any $m \in \mathbb{N}$ the bounding function μ_m is defined by $\mu_m(x) = (1+|x|^m)$ for all $x \in \mathbb{R}$. □

<u>LEMMA 5.3.10</u>. *Assumption* 2.5 *is satisfied with* $\mu = \mu_m$ *for any* $m \in \mathbb{N}$.

<u>PROOF</u>. By using the growth relation (5.3.7) □

Let $m \in \mathbb{N}$ be fixed. According to lemma 5.3.10 we can apply the results of section 2 with bounding function $\mu = \mu_{m+3}$.

<u>LEMMA 5.3.11</u>. (i) $C^{3;m} \subset D_A^{\mu_{m+3}}$

(ii) *For any* $g \in C^{3;m}$ *and* $x \in \mathbb{R}$:

$$(5.3.19) \qquad Ag(x) = a(x)\,\frac{d}{dx}\,g(x) + \frac{1}{2}\,b^2(x)\,\frac{d^2}{dx^2}\,g(x).$$

PROOF. Let $g \in C^{3;m}$. By Taylor expansion we find

$$(5.3.20)\quad \begin{cases} T_h g(x) - g(x) = \mathbf{E}\, g(\eta_h(x)) - g(x) = \\ \mathbf{E}\{[\frac{d}{dx} g(x)][\eta_h(x)-x] + \frac{1}{2}[\frac{d^2}{dx^2} g(x)][\eta_h(x)-x]^2 \\ + \frac{1}{6} g'''(x+\theta_{h,x}[\eta_h(x)-x])\ [\eta_h(x)-x]^3\}, \end{cases}$$

where $|\theta_{h,x}| \le 1$. The three terms of the right hand side will be considered under (i), (ii) and (iii) below.

(i) Since $\mathbf{E} \int_0^h b(\eta_s(x))dW_s = 0$ it follows from (5.3.5) that

$$(5.3.21)\quad |\mathbf{E}\{[\eta_h(x)-x] - ha(x)\}| = |\mathbf{E}\int_0^h [a(\eta_s(x)) - a(x)]ds| \le K\int_0^h \mathbf{E}\,|\eta_s(x)-x|\,ds \le h\sqrt{h}\ C_2(1+|x|).$$

(ii) Since $\mathbf{E}\,[\int_0^h b(\eta_s(x)dW_s]^2 = \mathbf{E}\int_0^h [b(\eta_s(x))]^2 ds =$

$$\mathbf{E}\int_0^h [b(\eta_s(x)) - b(x)][b(\eta_s(x)) + b(x)]ds + hb^2(x).$$

Schwartz' inequality implies

$$(5.3.22)\quad |\mathbf{E}[\int_0^h b(\eta_s(x))dW_s]^2 - hb^2(x)| \le [K^2\int_0^h \mathbf{E}\,(\eta_s(x)-x)^2 ds]^{\frac{1}{2}}[K^2\int_0^h \mathbf{E}\,(2+(\eta_s(x))^2)ds]^{\frac{1}{2}} \le h\sqrt{h}\ C_2(1+|x|^2).$$

By using (5.3.5), (5.3.22) and Schwartz' inequality we find

$$(5.3.23)\quad \begin{aligned} &|\mathbf{E}\{[\eta_h(x)-x]^2 - hb^2(x)\} \le h\sqrt{h}\ C_2(1+|x|^2) + \\ &|\mathbf{E}\,[\int_0^h a(\eta_s(x))ds]^2 + 2\ \mathbf{E}\,[\int_0^h a(\eta_s(x))ds]\ [\int_0^h b(\eta_s(x))dW_s]| \\ &\le h\sqrt{h}\ C_2(1+|x|^2) + hK^2\int_0^h \mathbf{E}\,(1+(\eta_s(x))^2)ds + \\ &2\{K^4 h\,[\int_0^h \mathbf{E}\,(1+(\eta_s(x))^2)ds]^2\}^{\frac{1}{2}} \le \\ &h\sqrt{h}\ C_3(1+|x|^2). \end{aligned}$$

(iii) Since $g \in C^{3;m}$ and $|\theta_{h,x}| \le 1$, we find

$$(5.3.24) \qquad |\mathbb{E}\, g'''(x+\theta_{h,x}[\eta_h(x)-x])|^2 \le$$

$$(K_g)^2\ \mathbb{E}\,(1+2|x|+|\eta_h(x)|)^{2m} \le C_4 (K_g)^2\ (1+|x|^{2m}) .$$

Schwartz' inequality and (5.3.24) imply that

$$(5.3.25) \qquad |\mathbb{E}[g'''\ (x+\theta_{h,x}(\eta_h(x)-x))][\eta_h(x)-x]^3| \le$$

$$C_5 K_g \{(1+|x|^{2m})\ h^3 (1+|x|^6)\}^{\frac{1}{2}} \le h\sqrt{h}\ C_6 K_g (1+|x|^{m+3}).$$

Finally, using that $g \in C^{3;m}$ and combining (5.3.20), (5.3.21), (5.3.23) and (5.3.25), one can show

$$(5.3.26) \quad \begin{cases} |[T_h g(x)-g(x)]h^{-1} - [a(x)\ \frac{d}{dx}\ g(x)+\frac{1}{2}b^2(x)\ \frac{d^2}{dx^2}\ g(x]| \le \\ \sqrt{h}\ C_7 K_g (1+|x|^{m+3}). \end{cases}$$

Relation (5.3.26) completes the proof if we let h tend to 0. □

5.3.2. APPROXIMATIONS

Take $h_0 > 0$ and let $\{P^h | h \in (0,h_0]\}$ be a collection of one-step transition probabilities defined by

$$(5.3.27) \qquad P^h(x;\{y\}) = \begin{cases} \frac{1}{2} & \text{for } y = x + a(x)h + b(x)\ \sqrt{h} \\ \frac{1}{2} & \text{for } y = x + a(x)h - b(x)\ \sqrt{h} \end{cases}$$

for all $x \in S$ and $h \le h_0$. Let $\{C_h | h \in (0,h_0]\}$ be the corresponding difference-method defined by (4.2.2) and $A_h = [C_h - I]h^{-1}$.
Further, recall expression (4.2.3) for $\{P^h_{nh} | n = 0,1,2...\}$.

LEMMA 5.3.12. *The consistency relation* (4.3.1) *holds with* $\mu = \mu_{m+3}$ *for any* $f \in C^{3;m}$.

PROOF. First consider $g \in C^{3;m}$. By Taylor expansion we obtain

$$(5.3.28) \qquad |A_h g(x) - [a(x) \frac{d}{dx} g(x) + \frac{1}{2} b^2(x) \frac{d^2}{dx^2} g(x)]| \le h^{-1}\{$$

$$|a^2(x)h^2 \frac{1}{2} \frac{d^2}{dx^2} g(x)| +$$

$$|\frac{1}{12} g'''(x+\theta^+_{h,x} [a(x)h+b(x)\sqrt{h}])[a(x)h+b(x)\sqrt{h}]^3 +$$

$$|\frac{1}{12} g'''(x+\theta^-_{h,x} [a(x)h-b(x)\sqrt{h}\})[a(x)h-b(x)\sqrt{h}]^3 \},$$

where $|\theta^+_{h,x}| \le 1$, $|\theta^-_{h,x}| \le 1$. Since $|a(x)| + |b(x)| \le K(1+|x|)$ and $g \in C^{3;m}$, it is easily shown that relation (5.3.28) and expression (5.3.19) for A yield

$$(5.3.29) \qquad \| (A_h - A)g \|_{\mu_{m+3}} \le \sqrt{h} C_1 K_g .$$

Next, consider $f \in C^{3;m}$. Then relation (5.3.29) together with proposition 5.3.8 guarantee for all $t \le Z$:

$$(5.3.30) \qquad \| (A_h - A) T_t f \|_{\mu_{m+3}} \le \sqrt{h} C_2 K_f .$$

By letting h tend to 0 in (5.3.30) the proof is concluded. □

<u>LEMMA 5.3.13</u>. *The stability relation* (4.3.2) *is satisfied with* $\mu = \mu_{m+3}$.

<u>PROOF</u>. Write $\bar{m} = m+3$. From expression (5.3.27) for P^h, Schwartz' inequality and $|a(x)| + |b(x)| \le K(1+|x|)$ it follows that

$$(5.3.31) \qquad | \int \mu_{\bar{m}}(y)\, P^h(x;dy) | =$$

$$| 1 + \frac{1}{2} [x+a(x)h + b(x)\sqrt{h}]^{\bar{m}} + \frac{1}{2} [x+a(x)h - b(x)\sqrt{h}]^{\bar{m}} | \le$$

$$1 + \{ \frac{1}{2} [x+a(x)h + b(x)\sqrt{h}]^{2\bar{m}} + \frac{1}{2} [x+a(x)h - b(x)\sqrt{h}]^{2\bar{m}} \}^{\frac{1}{2}} \le$$

$$(1 + |x|^{\bar{m}})(1 + hM_C)$$

for some constant M_C not depending on x,h. This proves (4.3.2) with $\mu = \mu_{m+3}$ □

<u>PROPOSITION 5.3.14</u>. *For any* $f \in C^{3;m}$ *the expression*

(5.3.32) $\| \int f(y)\, P^{h}_{nh}(\cdot;dy) - \int f(y)\, P_t(\cdot;dy)\|_{\mu_{m+3}}$

with $n = [th^{-1}]$ *is convergent of order* $O(\sqrt{h})$ *uniformly in* $t \leq Z$.

PROOF. In view of theorem 4.3.1, lemma 5.3.12 and lemma 5.3.13 it remains to show the order of convergence $O(\sqrt{h})$. This follows from lemma 3.7, relations (3.17), (5.3.26), (5.3.30) and proposition 5.3.8. □

In order to show weak convergence of the transition probabilities, we give the following lemmas. Although the result of lemma 5.3.15 is intuitively clear and the Weierstrass-theorem gives an analogous result for bounded intervals, we could not find a proof in the literature. Its proof is given in Appendix B.

LEMMA 5.3.15. $C^{3;0}$ *is dense in* $C^{u}(\mathbb{R})$ *in supremum - norm.* □

LEMMA 5.3.16. *Relation* (4.3.8) *is satisfied for any* $Z > 0$.

PROOF. It is shown by remark 1 on p.61 of Gihman and Skorohod (1972) that

(5.3.33) $\mathbb{E}\,[\eta_t(y+\Delta y) - \eta_t(y)]^2 \leq C[\Delta y]^2.$

Relation (5.3.33) together with Chebyshev's inequality implies that $\eta_t(x)$ is continuous in x in probability and thus in distribution.
The portmanteau theorem, see p.11 of Billingsley (1968), completes the proof.
□

PROPOSITION 5.3.17. *For any* $x \in S$ *and collection* $\{X^h \mid h \in (0,h_0]\}$ *with* $x^h \to x$ *as* $h \to 0$, *all* $t \leq Z$ *and with* $n = [th^{-1}]$ *the weak convergence relation* (4.3.10) *for the transition probabilities is satisfied.*

PROOF. First note that m was assumed to be fixed but arbitrarily chosen. Consider $m = 0$. Then lemma 5.3.12, 5.3.13 and 5.3.15 guarantee the hypotheses of theorem 4.3.4 with $\mu = \mu_3$ and $G = C^{3;0}$.
Consequently, since also lemma 5.3.16 implies (4.3.8) and clearly (4.3.9) is satisfied with $\mu(x) = (1+|x|^3)$, application of lemma 4.3.6 completes the proof. □

Let us conclude this subsection by showing weak convergence of

processes. Therefore, let $(X_t)_{t\geq 0}$ be the process such that for any $Z\geq 0$: $(X_t)_{t\leq Z} = (\eta_t)_{t\leq Z}$ where $(\eta_t)_{t\leq Z}$ is given by theorem 5.3.1.
Further, for any $h\leq h_0$ consider a collection $\{R^h_{nh}\,|\,n = 0,1,2,\ldots\}$ of identical and independent random elements with

$$\mathbb{P}\,(R^h_{nh} = \sqrt{h}) = \mathbb{P}\,(R^h_{nh} = -\sqrt{h}) = \frac{1}{2}\ , \quad n\in\mathbb{N}, \tag{5.3.34}$$

and let X^h_0 be an initial random element independent of $\{R^h_{nh}\,|\,n = 0,1,2\ldots\}$.
Next, for any $h\leq h_0$ define a discrete-time process $\{X^h_{nh}\,|\,n = 0,1,2\ldots\}$ by

$$X^h_{nh+h} = X^h_{nh} + a(X^h_{nh})h + b(X^h_{nh})\,R^h_{nh}\ , \quad n\in\mathbb{N}. \tag{5.3.35}$$

Then, it is not difficult to see that $\{X^h_{nh}\,|\,n = 0,1,2,\ldots\}$ is a homogeneous Markov process with transition probabilities
$\{P^h_{nh}\,|\,n = 0,1,2,\ldots\}$ defined by (4.3.2) with P^h given by (5.3.27).
Consequently, according to lemma 4.3.5 it equals the unique homogeneous Markov process constructed by (4.3.6).

The *stochastic difference equation* (5.3.35) can be seen as the discrete-time analogue of the stochastic differential equation (5.3.1).
More precisely, the following theorem shows that the solutions of (5.3.35) weakly converge to that of (5.3.1) as h tends to 0.

<u>THEOREM 5.3.18</u>. *If* $X^h_0 \Rightarrow X_0$ *as* $h\to 0$ *and* $\sup_{h\leq h_0} \mathbb{E}\,|X^h_0|^3 < \infty$, *then*

$$(\bar{X}^h_t)_{t\geq 0} \Rightarrow (X_t)_{t\geq 0} \text{ on } D[0,\infty) \text{ as } h\to 0. \tag{5.3.36}$$

<u>PROOF</u>. We will apply theorem 4.3.7. First of all, let us repeat that $Z > 0$ and m are chosen arbitrarily. Hence, by taking $\mu = \mu_3$ and $G = C^{3;0}$, we conclude from lemmas 5.3.10, 5.3.12, 5.3.13 and 5.3.15 that assumption 2.5 as well as the hypotheses of theorem 4.3.4 are satisfied for all $Z > 0$.
Further, recall that lemma 5.3.15 implies (4.3.8) and $\mu = \mu_3$ satisfies (4.3.9). Relation (4.3.12) is guaranteed by assumption and relation (4.3.13) is implied by theorem 5.3.1 since $C[0,\infty) \subset D[0,\infty)$.
Consequently, theorem 4.3.7 completes the proof if we verify (4.3.14).
Therefore, we will focus on condition (ii) of theorem A3.5 and let

$(X^h_{nh}(y))_{n\in\mathbb{N}}$ denote the solution of (5.3.35) with $X^h_0 = y$. Then, for $n<\ell$,

$$(5.3.37) \qquad \mathbb{E}\,|X^h_{\ell h} - X^h_{nh}|^3 = \int \mathbb{E}\,|X^h_{\ell h}(y) - X^h_{nh}(y)|^3 \; \mathbb{P}_{X^h_0}(dy).$$

Consequently, the conditions of the theorem together with lemma 5.3.19 given below yields,

$$(5.3.38) \qquad \mathbb{E}\,|X^h_{\ell h} - X^h_{nh}|^3 \le C_Z(\ell h - nh)^{\frac{3}{2}}$$

uniformly for all $0 \le nh \le \ell h \le Z$, some constant C_Z and all $Z>0$. This verifies condition (ii) of theorem A.3.5. □

The following lemma, which was essential for the above proof, can be seen as the discrete-time analogue of the growth relation (5.3.8).

LEMMA 5.3.19. *For any* $Z>0$ *there exists a constant* M_Z *depending only on* Z *and* K *such that for all* $h\le h_0, y\in S$ *and* $0\le nh\le \ell h\le Z$:

$$(5.3.39) \qquad \mathbb{E}\,|X^h_{\ell h}(y) - X^h_{nh}(y)|^3 \le M_Z(1+|y|^3)(\ell h - nh)^{\frac{3}{2}}.$$

PROOF. Let $h\le h_0$ and $y\in S$.
Similarly to (5.3.31) we obtain for any $m\in\mathbb{N}$, and $jh\le Z$

$$(5.3.40) \qquad \mathbb{E}\,|X^h_{jh}(y)|^m \le (1+hM_C)\,\mathbb{E}|X^h_{(j-1)h}(y)|^m \le \dots \le e^{ZM_C}|y|^m.$$

To proceed, let us write: $\eta_j = \eta^h_{jh}(y)$ and $R_j = R^h_{jh}$, $j\in\mathbb{N}$. From (5.3.35):

$$(5.3.41) \qquad \mathbb{E}\,|\eta_\ell - \eta_n|^3 \le 8\;\mathbb{E}\Big|\sum_{j=n}^{\ell-1} a(\eta_j)h\Big|^3 + 8\,\mathbb{E}\Big|\sum_{j=n}^{\ell-1} b(\eta_j)\;R_j\Big|^3.$$

By using Hölder's inequality, $|a(x)| \le K(1+|x|)$ and (5.3.40) we find

$$(5.3.42) \qquad \mathbb{E}\,\Big|\sum_{j=n}^{\ell-1} a(\eta_j)h\Big|^3 \le$$

$$\mathbb{E}\,(\ell h - nh)^2 \sum_{j=n}^{\ell-1} |a(\eta_j)|^3 h \le (\ell h - nh)^3\; C\;(1+|y|^3)$$

In order to give a bound for the second term in the right-hand side of (5.3.41), first use Hölder's inequality to write

(5.3.43) $$\mathbb{E}\,\left|\sum_{j=n}^{\ell-1} b(\eta_j)\ R_j\right|^3 \le \left[\mathbb{E}\,(\sum_{j=n}^{\ell-1} b(\eta_j)\ R_j)^4\right]^{\frac{3}{4}}.$$

Next,we will proceed in analogy with p.385/386 of Gihman and Skorohod (1969) as follows. First,by using

$\mathbb{E}\,(R_j)^3 = \mathbb{E}(R_j) = 0$, R_j is independent of η_i for all $i < j$ and of R_i for all $i \le j$, and relation (5.3.35) we find

(5.3.44) $$\mathbb{E}\,[\sum_{j=n}^{\ell-1} b(\eta_j)\ R_j]^4 = \mathbb{E}\sum_{j=n}^{\ell-1}[b(\eta_j)\ R_j]^4 + 6\mathbb{E}\sum_{j=n}^{\ell-1}[\sum_{i=n}^{j-1} b(\eta_i)\ R_i]^2\,[b(\eta_i)]^2\,[R_i]^2$$

From Schwartz' inequality, $|b(x)| \le K(1+|x|)$ and $\mathbb{E}\,(R_j)^8 = h^4$:

(5.3.45) $$\mathbb{E}\sum_{j=n}^{\ell-1}[b(\eta_j)\ R_j]^4 \le \sum_{j=n}^{\ell-1}\{\ \mathbb{E}[b(\eta_j)]^8 h^4\}^{\frac{1}{2}} \le (\ell h - nh)hC_2(1+|y|^4).$$

Again noting that R_j is independent of η_i, $i < j$ and R_i, $i \le j$, using $\mathbb{E}\,(R_j)^2 = h$, and Schwartz' inequality

(5.3.46) $$\mathbb{E}\sum_{j=n}^{\ell-1}[\sum_{i=n}^{j-1} b(\eta_i)\ R_i]^2\,[b(\eta_i)]^2[R_j]^2 =$$
$$\mathbb{E}\sum_{j=n}^{\ell-1}([\sum_{i=n}^{j-1} b(\eta_i)\ R_i]^2\,[b(\eta_i)]^2])h \le$$
$$\{\sum_{j=n}^{\ell-1}\mathbb{E}\,[\sum_{i=n}^{j-1} b(\eta_i)\ R_i]^4 h\}^{\frac{1}{2}}\{\sum_{j=n}^{\ell-1}\mathbb{E}[b(\eta_i)]^4 h\}^{\frac{1}{2}}.$$

Since,according to (5.3.44),the expression $\mathbb{E}[\sum_{i=n}^{j-1} b(\eta_i)\ R_i]^4$ is increasing in j,the last term of (5.3.46) is bounded by

(5.3.47) $$\{(\ell h - nh)\ \mathbb{E}\,[\sum_{i=n}^{\ell-1} b(\eta_i)\ R_i]^4\}^{\frac{1}{2}}\{(\ell h - nh)\quad C_3(1+|y|^4)\}^{\frac{1}{2}}$$

Finally, from the relations (5.3.44) up to (5.3.47) one easily derives

(5.3.48) $$\mathbb{E}\,[\sum_{j=n}^{\ell-1} b(\eta_j)\ R_j]^4 \le (\ell h - nh)^2\ C_4(1+|y|^h).$$

The combination of (5.3.41), (5.3.42), (5.3.43) and (5.3.48) completes the proof. □

CHAPTER II

CONTROLLED MARKOV PROCESSES; TIME-DISCRETIZATION

1. INTRODUCTION AND SUMMARY

Parallel to the preceding chapter, the present chapter is concerned with applying the method of time-discretization to controlled Markov processes. More precisely, given a continuous-time Markov process with controlled generators, we study for h sufficiently small a discrete-time Markov process at time-points $\{nh|n = 0,1,2,\ldots\}$ with controlled one-step generators. The approximation of several functions associated with the continuous-time model is investigated by considering the corresponding functions for the discrete-time model.

In this chapter we study time-discretization for controlled processes from an *approximative* (computational) point of view. Another approach of applying time-discretization, which have appeared to be useful in the literature, is to transpose results of discrete-time models to a continuous-time model. Particularly, Van Der Duyn Schouten (1979) and Hordijk and Van Der Duyn Schouten (1980), (1983a), (1983b), (1983c) have developed this approach. Especially they have been succesful in transposing the *structure* of optimal policies from discrete-time models to a continuous-time model. They analyze the convergence of a time-discretization method in a general framework of so-called Markov decision drift processes. This framework allows generator as well as impulsive controls simultaneously and includes semi-Markov, Markov renewal as well as many other jump-type models with deterministic evolutions (drifts) between the jumps. However, their framework does not include diffusion type processes nor does their approximation approach yield rates of convergence or bounds. Actually, the impulsive control aspect makes it essentially more difficult to obtain bounds of approximation. It is not considered in this monograph.

Although the setting of this chapter is quite different from that of the Markov decision drift processes as introduced by Hordijk and Van Der Duyn Schouten (1980),there are strong relations and several of their techniques have been used.

As further references with respect to time-discretization of controlled stochastic processes,we like to mention without being exhaustive: Whitt (1975), Kushner (1977), (1978), Kakumanu (1977), Nisio (1978), Gihman and Skorohod (1979), Haussmann (1980) ,Bensoussan and Robin (1983) and Christopeit (1983). For each of these references a brief discussion and comparison with the current chapter is included in section 9. Below, only some global considerations are presented.

Nisio (1978), Gihman and Skorohod (1979), Bensoussan and Robin (1983) and Christopeit (1983) use methods of time-discretization which have in common that the one-step generators are induced by one-step transition probabilities of a continuous-time model but under a constant control variable during the interval of discretization. Nisio (1978) as well as Bensoussan and Robin (1983) are concerned with optimal control problems associated with time homogeneous Markov semigroups. Nisio (1978) uses such a method to show a unique semigroup representation for optimal stopping functions. Bensoussan and Robin (1983) just study the convergence of optimal cost functions related with continuous, stopping and impulsive control problems. Gihman and Skorohod (1979) focus on a fairly general framework of controlled stochastic processes.
Especially in view of their results on controlled Markov jump processes as well as controlled stochastic differential equations,their book has been a basic reference for our study.
Gihman and Skorohod (1979) apply time-discretization for several purposes, such as to show the sufficiency of step-controls in a general setting and to prove the optimality of a control for Markov jump processes.
Christopeit (1983) examines controlled diffusion processes and proves the optimality of a control.
Particularly,Kushner (1977),(1978) and Haussmann(1980) study for controlled diffusion processes discrete-time approximations which can possibly be obtained numerically and seem to allow for several modifications.
Kushner (1977), (1978) studies finite horizon-, impulsive- as well as

average cost control problems. Haussmann (1980) gives a special result for the finite horizon case.
Whitt (1975) and Kakumanu (1977) present results related with time-discretization for the infinite horizon cost case of controlled Markov jump processes. None of all the above mentioned references focuses on convergence rates or bounds. As far as we know , the literature does not provide such results nor a general approach to obtain them for controlled stochastic processes.

This chapter concentrates on a general framework of time-discretization for controlled Markov processes in order to approximate several functions of interest associated with the continuous-time model. The approximation analysis deals with rates of convergence or bounds with respect to some appropriate norm and is developed for a wide class of time-discretizations. Especially from a numerical point of view the latter fact might be useful, although numerical analysis is not included in this monograph. The approximation results are obtained by considering the discrete-time approximation of *time-difference equations*. The derivation of these equations follows from the Markov property.
Since non-linear and time-dependent operators has to be taken into account, we present an *approximation lemma*, to be seen as extension of the Lax-Richtmeyer theorem. The concepts *consistency* and *stability* are redefined in analogy with chapter I. Similarly to the Lax-Richtmeyer theorem,the approximation lemma states that consistency together with stability implies convergence. In addition, the *order of convergence* can be concluded from the *order of consistency*.
Since,however, in contrast with chapter I we consider time-difference in stead of time-differential equations, the consistency will not be implied by convergence of the discrete-time generators to the continuous-time generator but requires also sufficient *smoothness* with respect to the time parameter of the continuous-time function.
The approximation lemma yields the discrete-time approximation for

- *transition probabilities*,
- *finite horizon costs functions*, and
- *finite horizon optimal cost functions*

by verifying a corresponding and so-called

- *smoothness assumption* as well as a
- *(strong)-consistency-* and *(strong)-stability* relation.

The possibility of choosing an appropriate norm enables us to deal with *unbounded* functions and unbounded infinitesmal characteristics. The possibility of obtaining rates of convergence can be helpful for concluding convergence uniformly within a class of controls as well as for constructing a *nearly-optimal control*.

As in chapter I much attention is paid to applying the method of time-discretization to controlled processes of *jump* and *diffusion* type. The discretizations given are just the controlled analogues of those presented in chapter I and can be seen as natural approximations. More advanced discretizations will certainly be better from a computational point of view. Nevertheless, these 'naive' discretizations must just be seen as illustrations and as a first step to a more computational approach for controlled jump and diffusion models.

The scope of this chapter is as follows. Section 2 starts with the formal description of controlled Markov processes by means of introducing the concepts of a *control object* and an *admissible Markov control*. Thereafter, it presents the semigroup description for transition probabilities and the three types of functions for which the approximation analysis will be developed. Section 3 provides the time-difference equations for these functions. Next, in analogy with the continuous-time model also the discrete-time Markov process is given by means of introducing the concept of an h-control object. The admissibility of a Markov control for the discrete-time model is almost automatically fulfilled. Section 5 contains the approximation lemma as well as an additional lemma from which consistency and orders of convergence can be concluded. Section 6 applies the general approximation results to controlled Markov processes. First, the essential (strong)-consistency and (strong)-stability relations are collected in subsection 6.2. Next, in a general setting, the convergence of discrete-time approximations for each of the three types of functions is shown succesively in the subsections 6.3, 6.4 and 6.5.
Application of the approximation-method is shown for controlled Markov jump processes in section 7 and for controlled stochastic differentials equations (diffusion processes) in section 8. Besides the approximation of the three types of functions, both sections give, as special application, the construction of nearly - optimal controls. Finally, section 9 contains a discussion on related literature.

2. CONTINUOUS-TIME CONTROLLED MARKOV PROCESSES

2.1 DESCRIPTION AND DEFINITIONS

This chapter is concerned with continuous-time controlled Markov processes. Such a process satisfies the following informal description.

The state of a process is observed continuously. A *control* π prescribes at any time-point t a *decision rule* $\pi(t)$. Given that at a time-point the current decision rule is δ, then the evolution thereafter is infinitesimally determined by an *infinitesimal operator* A^{δ}.
The infinitesimal operator itself is determined by infinitesimal characteristics, such as jump rates, or drift and diffusion functions, which depend on the actual state and decision rule.
A decision rule, say δ, in turn prescribes for the actual (observed) state, say x, a *decision (controle variable)* $\delta(x)$ which has to be chosen. Usually, the infinitesimal characteristics depend on a decision rule through the decision.
Costs are taken into account by means of a *cost rate* function depending on the actual state and decision.

In order to give a formal presentation of continuous-time controlled Markov processes, we introduce the concept of a control object.

DEFINITION 2.1.1. A *control object* is a 7-tuple
$(S,\Gamma,\Delta,\mu,D_A,\{A^{\delta}|\delta\in\Delta\},L)$, where

(i) S is a separable complete metric space with Borel-field β.

(ii) Γ is a separable complete metric space with Borel-field $\beta(\Gamma)$.

(iii) Δ denotes a set of Borel-measurable functions $\delta:S \to \Gamma$.

(iv) μ denotes a bounding function, (cf. definition 2.2 of chapter I).

(v) D_A is a nonempty subset of B^{μ} (cf. notation 2.3 of chapter I).

(vi) For any $\delta\in\Delta$: A^{δ} is a linear operator from D_A to B^{μ}.

(vii) $L : S \times \Gamma \to \mathbb{R}$ is a Borel-measurable function.

Throughout this chapter the above characteristics are given the following interpretation:

(i) S is the *state space* of the process.

(ii) Γ denotes a set of *decisions*. At each time-point a decision has to be taken from Γ.

(iii) Δ presents a set of *decision rules* i.e.; if the current decision rule is $\delta \in \Delta$ and the actual state is x, then the decision $\delta(x) \in \Gamma$ is chosen.

(iv) μ is a bounding function and determines the class B^{μ}.

(v) D_A is a *domain* on which for all decision rules $\delta \in \Delta$ the operator A^{δ} is defined.

(vi) A^{δ} indicates an *infinitesimal operator* under decision rule $\delta \in \Delta$.

(vii) L represents a *cost-rate function* i.e.; if during $[t, t+\Delta t]$, where Δt is small, the state remains constant, say x, and the decision chosen is always γ, then the costs incurred are $\Delta t L(x,\gamma)$. □

<u>REMARKS 2.1.2.</u>

1. It is well-known that a domain D_A of infinitesimal operators is very important in view of the uniqueness of a corresponding process. For instance, uniqueness can not be guaranteed if D_A is too small. Therefore, it may be noted that, as for the applications in chapter I, also for the applications in this chapter, given in section 7 and 8, the μ-closure of D_A contains $C^u(S)$.

2. The approximation analysis which follows does not require further specification on the set of decision rules Δ. As examples consider:
   (i) $\Delta = \{\delta : S \to \Gamma \mid \delta \text{ measurable}\}$.
   (ii) For any $x \in S$ there exists a $\Gamma(x) \subset \Gamma$ and
   $\Delta = \{\delta : S \to \Gamma \mid \delta(x) \in \Gamma(x) \text{ for all } x \in S\}$.

   However, we only make the assumption (i) in parts of section 7 and 8. □

For the rest of this chapter, with exception of the applications given in section 7 and 8, consider a fixed control object. Although the definitions, notations and results which follow depend on the control object under consideration, this dependence will not be mentioned explicitly. This fact must be kept in mind throughout this chapter.

A continuous-time controlled Markov process is determined by its control object and a control. In the setting of this chapter we restrict ourselves to non-randomized Markov controls defined below.

<u>DEFINITION 2.1.3.</u> A non-randomized *Markov control* is a function $\pi : [0\,\infty) \to \Delta$. Let $\Pi(M)$ denote the set of all non-randomized Markov controls. □

Not for any non-randomized Markov control the existence and uniqueness of a continuous-time Markov process is guaranteed. Therefore, we introduce the notion of an admissible Markov control.

DEFINITION 2.1.4. A Markov control $\pi \in \Pi(M)$ is called *admissible* if the following two conditions are fulfilled:

(i) There exists a unique collection of transition probabilities $\{P^{\pi}_{s,t} \mid s,t \geq 0\}$ and this collection satisfies the Chapman-Kolmogorov equation (see definition 0.3)as well as for any $f \in D_A$ and $t \geq 0$:

$$(2.1.1) \qquad \| [\int f(y) P^{\pi}_{t,t+h}(\cdot;dy) - f(\cdot)]h^{-1} - A^{\pi(t)} f(\cdot)\|_{\mu} \to 0 \quad \text{as } h \to 0.$$

(ii) For some initial random element Z_0 there exists a unique Markov process $\{X^{\pi}_t \mid t \geq 0\}$ with transition probabilities $\{P^{\pi}_{s,t} \mid s,t \geq 0\}$ and such that: $X^{\pi}_0 = Z_0$ and $\mathbb{P}((X^{\pi}_t)_{t \geq 0} \in D[0,\infty)) = 1$. □

For $\pi \in \Pi(M)$ an admissible control the Markov process $\{X^{\pi}_t \mid t \geq 0\}$ is called a *continuous-time controlled Markov process*.

REMARK 2.1.5. In the above definition of an admissible control the uniqueness is just stated and does not necessarily result only from the additional requirements. For instance, the uniqueness of a semigroup of transition probabilities is not guaranteed only by (2.1.1). In fact, we need implicit conditions on the control object, especially with respect to the operators A^{δ} and the domain D_A, and on the control π which must guarantee the existence and uniqueness.
For the applications given in section 7 and 8 the admissibility of controls, i.e.; the existence and uniqueness of transition probabilities and a corresponding Markov process, will be verified. □

From now on let $Z > 0$ be fixed. We also call a control $\pi \in \Pi(M)$ admissible if the conditions of definition2.1.4 are satisfied only on the finite time interval $[0,Z]$.

DEFINITION 2.1.6. Let $\Pi(AB)$ be the set of *admissible* Markov controls such that any $\pi \in \Pi(AB)$ satisfies for some *bounding* constant M^{π}:

$$(2.1.2) \qquad \| \int \mu(y)\ P^{\pi}_{s,t}(\cdot\,;dy) \|_{\mu} \leq M^{\pi} \qquad , s,t \leq Z. \qquad \square$$

In the rest of this chapter let for $\pi \in \Pi(AB)$: $\{P^{\pi}_{s,t} \mid s,t \leq Z\}$ denote the unique collection of transition probabilities given by definition 2.1.4 and M^{π} the bounding constant given by (2.1.2).
Further, we make use of the following notational conventions:

NOTATION 2.1.7. If $\{f_t \mid t \in [0,Z]\}$ is a μ-bounded family in B^{μ}, such that for any $x \in S$: $f_t(x)$ is Lebesque integrable in $t \in [0,Z]$,
then for $t \leq Z$ we write :

$$(2.1.3) \quad \begin{cases} g_t = \int_t^Z f_s ds & \text{, if for all } x \in S: \\ g_t(x) = \int_t^Z f_s(x)ds, & \text{as Lebesque integral.} \end{cases}$$

In particular, conclude that $\{g_t \mid t \in [0,Z]\}$ is a μ-bounded subset of B^{μ}. $\square$

NOTATION 2.1.8. For $\delta \in \Delta$: $L^{\delta}(x) = L(x,\delta(x))$ $\quad ,x \in S$.
For $\pi \in \Pi(M), t \geq 0$ and $\pi(t) = \delta$: $\quad L^{\pi}_t(x) = L^{\delta}(x)$ $,x \in S$. $\square$

REMARK 2.1.9. In our setting we consider a cost-rate function $L^{\pi(t)}$ and an operator $A^{\pi(t)}$ which depend on the actual time parameter t only through the current decision rule $\pi(t)$. It may be clear to the reader, however, that the analysis of this monograph can simply be extended by allowing an additional and explicit time dependence as $L^{\pi(t)}_t$ and $A^{\pi(t)}_t$ under current decision rule $\pi(t)$ at time-point t. Such an extension is not included for notational convenience. $\square$

2.2 SEMIGROUP DESCRIPTION

Let $\pi\in\Pi(AB)$. Then for any $s,t\leq Z$ relation (2.1.2) justifies the definition of an operator: $T^{\pi}_{s,t} : B^{\mu}\to B^{\mu}$ given by

$$(2.2.1) \qquad T^{\pi}_{s,t}f(x) = \int f(y)\, P^{\pi}_{s,t}(x;dy) \qquad , \quad x\in S.$$

Moreover, since the transition probabilities satisfy the Chapman-Kolmogorov relation (see (0.1)),the proof of the following lemma can be given in analogy with that of lemma 2.6 of chapter I.

<u>LEMMA 2.1.1</u>. *The collection* $\{T^{\pi}_{s,t} \mid s,t\leq Z\}$ *is a semigroup of linear operators on* B^{μ} *such that for any* $f\in B^{\mu}$:

$$(2.2.2) \qquad T^{\pi}_{s,t}f\in B^{\mu} \quad ; \; T^{\pi}_{s,s}f = f \qquad , \quad s\leq t\leq Z.$$

$$(2.2.3) \qquad T^{\pi}_{s,t}f = T^{\pi}_{s,\theta}\,(T^{\pi}_{\theta,t}f) \qquad , \quad s\leq \theta\leq t\leq Z.$$

$$(2.2.4) \qquad \|T^{\pi}_{s,t}f\|_{\mu} \leq \|f\|_{\mu}M^{\pi} \qquad , \quad s\leq t\leq Z.$$

□

<u>REMARK 2.1.2</u>. Since the semigroup $\{T^{\pi}_{s,t} \mid s,t\geq 0\}$ is inhomogeneous in the time parameter,the results of section 2 chapter I can not be transferred directly. Particularly, several additional smoothness conditions with respect to the time parameter have to be made in order to show that the semigroup corresponds to unique solutions of time-differential equations or equivalently to a properly-posed initial value problem.
In this chapter, however, we prefer to deal with time-difference equations and we give smoothness conditions later on for direct application to the approximation analysis. □

2.3 FUNCTIONS OF INTEREST

This subsection presents the three types of functions for which the method of discrete-time approximation will be examined. In order to present these functions as well as for purposes in subsequent sections, some assumptions are included. The three types of functions and the corresponding assumptions are given below in §1, §2 and §3 respectively.

§1. EXPECTATION OF F ; FIXED CONTROL

Let $\pi \in \Pi(AB)$ and $f \in D_A$. For all $t \le Z$ and $x \in S$ consider the *expectation of f* induced by $P^{\pi}_{t,Z}(x;\cdot)$:

(2.3.1) $$T^{\pi}_{t,Z} f(x) = \int f(y)\, P^{\pi}_{t,Z}(x;dy).$$

<u>ASSUMPTION 2.3.1.</u>

(i) $\{T^{\pi}_{t,Z} f \mid t \in [0,Z]\} \subset D_A$.

(ii) $\{A^{\pi(t)}\, T^{\pi}_{t,Z} f \mid t \in [0,Z]\}$ is μ-bounded. □

§2. FINITE HORIZON COST FUNCTION ; FIXED CONTROL

Let $\pi \in \Pi(AB)$ and suppose that the following assumption holds.

<u>ASSUMPTION 2.3.2.</u> For any $t \le Z$ the function $T^{\pi}_{t,s}\, L^{\pi(s)}$ is μ-continuous and μ- bounded in $s \in [t,Z]$. □

The *finite horizon cost function* $V^{\pi}_t, t \le Z$, is defined by

(2.3.2) $$V^{\pi}_t = \int_t^Z T^{\pi}_{t,s}\, L^{\pi(s)}\, ds.$$

The value $V^{\pi}_t(x)$ represents the expected total costs from the time-point t up to Z given that the state at time-point t is x .

<u>ASSUMPTION 2.3.3.</u>

(i) $\{V^{\pi}_t \mid t \in [0,Z]\} \subset D_A$.

(ii) $\{A^{\pi(t)}\, V^{\pi}_t \mid t \in [0,Z]\}$ is μ-bounded. □

§3. FINITE HORIZON OPTIMAL COST FUNCTION

ASSUMPTION 2.3.4. There exists an operator $J : D_A \to B^{\mu}$ such that

$$(2.3.3) \qquad Jf(x) = \inf_{\delta \in \Delta} [L^{\delta}(x) + A^{\delta}f(x)]$$

for all $f \in D_A$ and $x \in S$. □

Note that this assumption requires that the right hand side of (2.3.3) exists and is finite for all $f \in D_A$ and $x \in S$. Suppose that assumption 2.3.4 is satisfied and consider:

ASSUMPTION 2.3.5. There exists a unique collection $\{\phi_t \mid t \in [0,Z]\} \subset D_A$ with $\{J(\phi_t) \mid t \in [0,Z]\}$ μ- bounded and satisfying the *finite horizon continuous-time optimality equation*:

$$(2.3.4) \qquad \begin{cases} \phi_t = \int_t^Z J(\phi_s)\, ds & , \ t \le Z. \\ \phi_Z = \bar{0}\,. \end{cases}$$
□

ϕ_t, $t \le Z$, is called a *finite horizon optimal cost function.*
Note that (2.3.4) requires that for any $x \in S$ the function $J(\phi_s)(x)$ is Lebesque integrable in s.

REMARK 2.3.6. It is well-known that for jump- and diffusion-type applications the value $\phi_t(x)$ can be interpreted as the *optimal ('minimal')* expected total costs from time-point t up to Z given that the state at time-point t is x; where for $t > 0$ the 'minimum' is taken over a wide class of Markov controls and for $t = 0$ also history dependent controls can be taken into account. (cf. Yushkevich (1980), Fleming and Rishel (1975)). □

REMARK 2.3.7. By combining (2.3.1) and (2.3.2) one can also consider a finite horizon cost function with a terminal cost function, say f, at time-point Z. Correspondingly, $\bar{0}$ can be replaced by f in (2.3.4). □

3. TIME-DIFFERENCE EQUATIONS

This chapter is concerned with the discrete-time approximation for each of the functions given by (2.3.1), (2.3.2) and (2.3.4).
In chapter I we dealt with discrete-time approximations for so-called initial value problems or equivalently for functions satisfying a time-differential equation.
In order to proceed in analogy with chapter I,the functions to be approximated will be presented by time-evolution equations. In the setting of this chapter,however, it is convenient to give such equations as time-difference in stead of time-differential equations.

The time-difference equations presented are direct consequences of the Markov- or equivalently semigroup property (2.2.3),respectively the integral representation (2.3.4). Moreover, we let the form of these equations correspond to time-difference equations which are given in section 4 for discrete-time controlled Markov processes. This can be seen by comparing (3.1.3) with (4.2.3), (3.2.3) with (4.2.6) and (3.3.2) with (4.2.9). Since also the structure of all these equations is one and the same, we are able to present one approximation lemma in section 5 which can be applied for each of the functions of interest.

Let $h>0$ and $t\leq Z$ such that $t+h\leq Z$.

3.1. TRANSITION PROBABILITIES

Let $\pi\in\Pi(AB)$, $f\in D_A$ and suppose that assumption 2.3.1 holds. According to the semigroup property (2.2.3) we can write

$$(3.1.1)\qquad T^{\pi}_{t,Z}f = T^{\pi}_{t,t+h}\,(T^{\pi}_{t+h,Z}f).$$

Hence, by defining

$$(3.1.2)\qquad R^{\pi}_{t}(T,f,h) = ([T^{\pi}_{t,t+h}-1] - hA^{\pi(t)})\; T^{\pi}_{t+h,Z}\,f\,.$$

relation (3.1.1) can be rewritten as

$$(3.1.3)\qquad T^{\pi}_{t,Z}\,f - T^{\pi}_{t+h,Z}\,f = hA^{\pi(t)}(T^{\pi}_{t+h,Z}\,f) + R^{\pi}_{t}(T,f,h).$$

3.2. FINITE HORIZON COST FUNCTION

Let $\pi \in \Pi(AB)$ and suppose that the assumptions 2.3.2 and 2.3.3 hold. By virtue of the semigroup property (2.2.3) and the fact that $T^{\pi}_{t,t+h}$ is a linear and bounded operator on B^{μ}, we can write

$$V^{\pi}_{t} = \int_{t}^{t+h} T^{\pi}_{t,s} \, L^{\pi(s)} ds + T^{\pi}_{t,t+h} \, (V^{\pi}_{t+h}). \tag{3.2.1}$$

Hence, by defining

$$\begin{aligned} R^{\pi}_{t}(V,h) = {} & \left(\int_{t}^{t+h} T^{\pi}_{t,s} \, L^{\pi(s)} ds - hL^{\pi(t)} \right) + \\ & \left([T^{\pi}_{t,t+h} - I] - hA^{\pi(t)} \right) V^{\pi}_{t+h} \quad , \end{aligned} \tag{3.2.2}$$

relation (3.2.1) can be rewritten as

$$V^{\pi}_{t} - V^{\pi}_{t+h} = h \, [L^{\pi(t)} + A^{\pi(t)} (V^{\pi}_{t+h})] + R^{\pi}_{t}(V,h). \tag{3.2.3}$$

3.3 FINITE HORIZON OPTIMAL COST FUNCTION

Suppose that the assumptions 2.3.4 and 2.3.5 hold. Define

$$R_{t}(\emptyset,h) = \int_{t}^{t+h} J(\emptyset_{s}) ds - hJ(\emptyset_{t+h}). \tag{3.3.1}$$

Then, the optimality equation (2.3.4) becomes

$$\emptyset_{t} - \emptyset_{t+h} = hJ(\emptyset_{t+h}) + R_{t}(\emptyset,h). \tag{3.3.2}$$

4. DISCRETE-TIME CONTROLLED MARKOV PROCESSES

4.1. DESCRIPTION AND DEFINITIONS.

Let $h > 0$. A discrete-time controlled Markov process at time-points $\{nh \mid n = 0,1,2,\ldots\}$ can be briefly described as follows.

At each of the time-points the state of a process is observed. A *control* π prescribes for any time-point nh a *decision rule* $\pi(nh)$.
If at a time-point nh the observed state is x and the decision rule is δ, then the state at time-point nh+h is determined according to a *one-step transition probability* $P_h^{\delta}(x;.)$.
Consequently, on a finite time interval $[0,Z]$ a discrete time controlled Markov process is completely determined by a finite number of decision rules and one-step transition probabilities.
Further, at each time-point the current decision rule ,say δ, prescribes for the observed state, say x, *decision* $\delta(x)$ which has to be chosen. If the observed state is x and decision a is chosen,then a *one-step cost* $hL(x,a)$ is incurred.

In contrast with a continuous-time controlled Markov process,the existence and uniqueness of a discrete-time controlled Markov process can be proven constructively. This will be shown below. First let us give the necessary notation and definitions.

DEFINITION 4.1.1. An h-*control object* is a 7-tuple $(S,\Gamma,\Delta,\mu,h,\{P_h^{\delta} \mid \delta\in\Delta\}, L)$, where

(i) S,Γ,Δ and μ are as defined in section 2.

(ii) $h > 0$ denotes the *step size* of the process i.e.; the distance between the equidistant time-points at which the process is generated.

(iii) For any $\delta\in\Delta : P_h^{\delta} : S\times\beta \to \mathbb{R}$ is a transition probability, and can be interpreted as the *one-step transition probability* under current decision rule δ.

(iv) $L : S\times\Gamma \to \mathbb{R}$ is the measurable function defined in section 2, but here we let hL represent a *one-step cost function*.

□

In the rest of this section,consider a fixed h-control object $(S,\Gamma,\Delta,\mu,h,\{P_h^{\delta} \mid \delta\in\Delta\}, L)$.

DEFINITION 4.1.2. A non-randomized h-*Markov control* is a sequence of decisions rules $(\pi(0), \pi(h), \pi(2h), \ldots, \pi(nh), \ldots)$ such that for all $n \in \mathbb{N} : \pi(nh) \in \Delta$.

Let $\Pi^h(M)$ denote the set of all non-randomized h-Markov controls.

The following lemma, which may be seen in contrast with definition 2.1.4, guarantees the existence and uniqueness of transition probabilities and a Markov process for any $\pi \in \Pi^h(M)$.

LEMMA 4.1.3. *Let* $\pi = (\pi(0), \pi(h), \pi(2h), \ldots, \pi(nh), \ldots) \in \Pi^h(M)$. *Then*:

(i) *There exists a unique collection of transition probabilities* $\{P^h_{j,n} \mid j \le n; j,n \in \mathbb{N}\}$ *such that for any* $j \in \mathbb{N}$, $x \in S$ *and* $B \in \beta$:

$$(4.1.1) \quad \begin{cases} P^h_{j,j}(x;B) = 1_B(x) & , \text{ and for } n \ge j: \\ P^h_{j,n+1}(x;B) = \int P_h^{\pi(nh)}(z;B)\, P^h_{j,n}(x;dz) \end{cases}$$

(ii) *For any random element* Z^h_0 *on* S *there exists a unique Markov process* $\{X^h_{nh} \mid n = 0,1,2,\ldots\}$ *with* $X^h_0 = Z^h_0$ *and transition probabilities* $\{P_{jh,nh} \mid j,n \in \mathbb{N}\}$ *where* $P_{jh,nh} = P^h_{j,n}$ *for all* $j \le n$; $j,n \in \mathbb{N}$.

PROOF.

(i) Directly by construction (4.1.1).

(ii) According to the theorem of Ionescu-Tulcea (see Neveu (1964) p.145) there exists a unique random process $\{X^h_{nh} \mid n = 0,1,2,\ldots\}$ such that for any $n \in \mathbb{N}$ and $B_0, B_1, \ldots, B_{n+1} \in \beta$:

$$(4.1.2) \quad \begin{cases} \mathbb{P}(X^h_0 \in B_0) = \mathbb{P}(Z^h_0 \in B_0) \quad , \text{ and} \\ \mathbb{P}(X^h_0 \in B_0, X^h_{1h} \in B_1, \ldots, X^h_{(n+1)h} \in B_{n+1}) = \\ \int_{B_0 \times B_1 \times \ldots \times B_n} P_h^{\pi(nh)}(x_n; B_{n+1})\, d\mathbb{P}(x_0, x_1, \ldots, x_n). \end{cases}$$

By construction (4.1.1) of the transition probabilities $\{P^h_{j,n} \mid j \le n; j,n \in \mathbb{N}\}$, we have for all $j \le \ell \le n$, $x \in S$ and $B \in \beta$:

$$(4.1.3) \qquad P^h_{j,n}(x;B) = \int P^h_{\ell,n}(z;B)\, P^h_{j,\ell}(x;dz) \; .$$

Consequently, the Chapman-Kolmogorov equation is satisfied. Hence, from (4.1.1), (4.1.2) and (4.1.3) we conclude that $\{X^h_{nh} | n = 0,1,2,\ldots\}$ is a Markov process with transition probabilities $\{P^h_{j,n} \mid j,n \in \mathbb{N}\}$ given by (4.1.1). □

REMARKS 4.1.4.

1. Note that the systems (4.1.1) and (4.1.2) give a recursive contruction of the transition probabilities and the Markov process.
2. In view of lemma 4.1.3 any control $\pi \in \Pi^h(M)$ may be called admissible for the h-control object.
3. In order to avoid too much notational complexity we have omitted to indicate the control as an index if we consider a discrete-time controlled Markov process. It will always be made clear in advance which control is under consideration.
4. In accordance with lemma 4.1.3, let for $\pi \in \Pi^h(M)$ the collection $\{P^h_{j,n} \mid j,n \in \mathbb{N}\}$, defined by (4.1.1), denote the transition probabilities of the process $\{X^h_{nh} | n = 0,1,2,\ldots\}$.
5. It may be remarked that the step-size h is indicated by a subindex for the collection $\{P^\delta_h \mid \delta \in \Delta\}$ given above as well as for the collections $\{T^\delta_h \mid \delta \in \Delta\}$ and $\{A^\delta_h \mid \delta \in \Delta\}$ defined below.
   For all other symbols, h will be indicated as a superindex. □

DEFINITION 4.1.5. Let $\delta \in \Delta$ and assume that

$$(4.1.4) \qquad \| \int \mu(y)\; P^\delta_h(\cdot;dy) \|_\mu < \infty.$$

Then, let $T^\delta_h : B^\mu \to B^\mu$ be defined by

$$(4.1.5) \qquad T^\delta_h f(x) = \int f(y)\; P^\delta_h(x;dy) \qquad , x \in S \;,\; f \in B^\mu.$$

Further, define the operator $A^\delta_h : B^\mu \to B^\mu$ by

$$(4.1.6) \qquad A^\delta_h = [T^\delta_h - I]\; h^{-1} \;,$$

which we call the *one-step generator* under decision rule δ. □

DEFINITION 4.1.6. Let $\Pi^h(AB)$ be the set of controls $\pi\in\Pi^h(M)$ such that any $\pi\in\Pi^h(AB)$ satisfies for some constant M^h:

$$(4.1.7) \qquad \| \int \mu(y)\, P^h_{j,n}(\cdot;dy)\|_\mu \le M^h \qquad , \quad j,n \le \ell. \qquad \square$$

Then for $\pi\in\Pi^h(AB)$ and $j,n\in\mathbb{N}$ we can define an operator $T^h_{j,n} : B^\mu \to B^\mu$ by

$$(4.1.8) \qquad T^h_{j,n} f(x) = \int f(y)\, P^h_{j,n}(x;dy) \qquad , \quad \forall x \in S,$$

and according to (4.1.8) we find

$$(4.1.9) \qquad \|T^h_{j,n} f\|_\mu \le M^h \|f\|_\mu$$

Further, since (4.1.7) is satisfied for any $j\in\mathbb{N}$ and with $n = j+1$, it follows from (4.1.1) that relation (4.1.4) is valid for any $\delta \in \{\pi(0), \pi(h), \pi(2h),\ldots, \pi(nh),\ldots\} \subset \Delta$. Hence, the notations (4.1.5) and (4.1.6) are justified if we consider $\pi\in\Pi^h(AB)$.

4.2. FUNCTIONS OF INTEREST.

This subsection presents the discrete-time analogues of the three types of functions, which for continuous-time processes are defined by (2.3.1), (2.3.2) and (2.3.4).
In view of the approximation analysis which will follow, we also present one-step recursion relations: see (4.2.2), (4.2.5) and (4.2.8). These relations show that the discrete-time functions can be computed recursively. In addition, the relations will be rewritten as time-difference equations in order to illustrate the correspondence with the time-difference equations (3.1.3), (3.2.3) and (3.3.2) for the continuous-time functions.

The three functions of interest and corresponding relations are given below in §1., §2 and §3 respectively. Let $\ell = [Zh^{-1}]$.

§1. EXPECTATION OF F; FIXED CONTROL

Let $\pi\in\Pi^h(AB)$ and $f\in B^\mu(S)$. For any $j < \ell$ and $x\in S$ consider the *expectation of* f induced by the transition probability $P^h_{j,\ell}(x;.)$:

(4.2.1) $\quad T^h_{j,\ell} f(x) = \int f(y)\, P^h_{j,\ell}(x;dy).$

By virtue of (4.1.1) it can be shown analogously to lemma 2.6 of chapter I that for any $f \in B^{\mu}$ and $j<\ell$:

(4.2.2) $\quad T^h_{j,\ell} f = T^{\pi(jh)}_h (T^h_{j+1,\ell} f).$

Further, by using (4.1.6) and (4.2.2) we easily obtain

(4.2.3) $\quad T^h_{j,\ell} f - T^h_{j+1,\ell} f = hA^{\pi(jh)}_h (T^h_{j+1,\ell} f).$

§2. FINITE HORIZON COST FUNCTION; FIXED CONTROL

Let $\pi \in \Pi^h(AB)$ and suppose that the following assumption is satisfied:

<u>ASSUMPTION 4.2.1.</u> $\{L^{\pi(jh)} \mid j < \ell\} \subset B^{\mu}$. □

The *finite horizon cost function* V^h_j is defined by

(4.2.4) $\quad V^h_j = \sum_{n=j}^{\ell-1} T^h_{j,n} (L^{\pi(nh)})h \quad .$

Since assumption 4.2.1 is satisfied it follows from (4.2.2) and (4.2.4) that the collection of cost functions $\{V^h_j \mid j \le \ell\}$ satisfies the system:

(4.2.5) $$\begin{cases} V^h_\ell = \bar{0} \\ V^h_j = hL^{\pi(jh)} + T^{\pi(jh)}_h (V^h_{j+1}) \quad , \quad j < \ell. \end{cases}$$

Note that (4.2.5) can be solved recursively.
Further, by using (4.1.6) and (4.2.5) we easily find

(4.2.6) $\quad V^h_j - V^h_{j+1} = h[L^{\pi(jh)} + A^{\pi(jh)}_h (V^h_{j+1})].$

§3. FINITE HORIZON OPTIMAL COST FUNCTION

<u>NOTATION 4.2.2.</u> Consider a collection of functions $\{g^{\delta} \mid \delta \in \Delta\} \subset B^{\mu}$.
The function $g^0 : S \to \mathbb{R} \cup \{-\infty\}$ given by
$g^0(x) = \inf_{\delta\in\Delta} [g^{\delta}(x)]$, $x \in S$, will be denoted by: $\inf_{\delta\in\Delta} [g^{\delta}]$. □

ASSUMPTION 4.2.3. For some subset $F \subset B^{\mu}$, with $\bar{0} \in F$, and all $f \in F$:

(4.2.7) $$\inf_{\delta\in\Delta} [hL^{\delta} + T_h^{\delta} f] \in F.$$ □

Let assumption 4.2.3 be satisfied.
Then, there exists a unique collection $\{\emptyset_j^h \mid j \le \ell\} \subset F$
satisfying the *finite horizon discrete-time optimality equation* :

(4.2.8) $$\begin{cases} \emptyset_{\ell}^h = \bar{0} \\ \emptyset_j^h = \inf_{\delta\in\Delta} [hL^{\delta} + T_h^{\delta}(\emptyset_{j+1}^h)] \qquad , \; j < \ell. \end{cases}$$

Note that (4.2.8) can be solved recursively. Further, by using (4.1.6) and (4.2.8) we easily find

(4.2.9) $$\emptyset_j^h - \emptyset_{j+1}^h = \inf_{\delta\in\Delta} h[L^{\delta} + A_h^{\delta}(\emptyset_{j+1}^h)].$$

5. APPROXIMATION LEMMA

In analogy with section 3 of chapter I, this section is concerned with the convergence of time-difference methods and therefore contains an approximation lemma.
This lemma can be seen as a partial extension of the standard Lax-Richtmeyer theorem in that it allows *non-linear* and *time-inhomogeneous* methods. Such extensions are well-known in the literature on numerical analysis, see for instance Ansorge and Hass (1970). The approximation lemma presented in this section, however, is given in a form which is more suitable for our purposes. Therefore, it concerns *time-difference equations* in stead of properly-posed initial value problems. In view of the time-difference relation (3.1.3), (3.2.3) and (3.3.2), these equations are given as *backwards time-evolution equations.*

In analogy with chapter I we redefine the essential concepts of *consistency* and *stability* and the approximation lemma of this chapter also states that *consistency* together with *stability* implies *convergence.*

Especially for direct application to the time-difference equations (3.1.3), (3.2.3) and (3.3.2) we also include a more specific lemma. This lemma concerns consistency and orders of convergence.

In this section let B denote a Banach space endowed with norm $\|.\|$. Further, let $h_0 > 0$ be fixed and as before write $\ell = [Z h^{-1}]$.

DEFINITION 5.1. Let $\{E_j^h \mid jh \leq Z,\ j \in \mathbb{N},\ h \in (0,h_0]\}$ be a family of operators $E_j^h : B \to B$ and let $u \in B$. Then we have a *properly-posed time-difference problem* if there exists a family $\{U_t \mid t \in [0,Z]\} \subset B$ such that for all $h \leq h_0$:

$$(5.1) \qquad U_{jh} = E_j^h\,(U_{jh+h})\ , \qquad j < \ell\ , \qquad U_Z = u\ ,$$

and U_t is strongly continuous in $t \in [0,Z]$.

NOTATION: $P(B,Z,E,u)$ denotes this problem.
The collection $\{U_t \mid t \in [0,Z]\}$ is called the solution of $P(B,Z,E,u)$. □

Note that, since (5.1) must be satisfied for any $h \in (0,h_0]$ there can only exist a unique family $\{U_t \mid t \in [0,Z]\}$. Further, it is emphasized that the operators E_j^h can be *non-linear.*

DEFINITION 5.2. Let $P(B,Z,E,u)$ be a properly-posed time-difference problem and $\{U_t \mid t \le Z\}$ its solution.

(i) A family of operators $M_D = \{C_j^h \mid jh < Z,\ j \in \mathbb{N},\ h \in (0,h_0]\}$ with $C_j^h : B \to B$, $jh < Z$, is called a *difference-method* if for some constant $K : \|C_j^h c\| \le K\|c\|$ for all $jh < Z$ and $c \in B$.

(ii) M_D is called *consistent* if

$$(5.2)\qquad \begin{cases} \|C_j^h\ (U_{jh+h}) - E_j^h\ (U_{jh+h})\|\, h^{-1} \\ \text{convergences to 0 uniformly in } j < \ell \text{ as } h \to 0. \end{cases}$$

(iii) M_D is called *stable* if for constants K_j^h, with $j < \ell$ and $h \in (0,h_0]$, and for some constant K_C the following conditions are satisfied:

$$(5.3)\qquad \begin{cases} \|C_j^h(c_1) - C_j^h\ (c_2)\| \le K_j^h\ \|c_1 - c_2\| \\ \text{for all } c_1,\ c_2 \in B,\ j < \ell \text{ and } h \in (0,h_0], \end{cases}$$

and

$$(5.4)\qquad \begin{cases} \prod_{j=n}^{m}\ [K_j^h] \le K_C \\ \text{for all } n < m < \ell \text{ and } h \in (0,h_0]. \end{cases}$$

(iv) M_D is called *convergent* if by defining for all $h \le h_0$:

$$(5.5)\qquad U_\ell^h = u;\ U_j^h = C_j^h\ C_{j+1}^h \cdots\ C_{\ell-1}^h\ (U_\ell^h)\quad,\quad j < \ell,$$

we obtain

$$(5.6)\qquad \begin{cases} \|U_n^h - U_t\| \\ \text{with } |nh-t| < h \text{ converges to 0 uniformly in } t \le Z \text{ as } h \to 0. \end{cases}$$ □

These definitions enable us to present the main approximation lemma.

LEMMA 5.3. (APPROXIMATION LEMMA)
Let M_D *be a difference-method for* $P(B,Z,E,u)$.
Then M_D *is convergent if it is consistent and stable.*

PROOF. Let M_D be consistent for u and stable. For all $h \in (0,h_0]$ write

$$(5.7)\quad \begin{cases} \delta_j^h = U_j^h - U_{jh} & , \quad j \le \ell \ , \\ \varepsilon_j^h = C_j^h\,(U_{jh+h}) - E_j^h\,(U_{jh+h}) & , \quad j < \ell \ . \end{cases}$$

According to the relations (5.1) and (5.5) we have

$$(5.8)\qquad \delta_j^h = C_j^h\,(U_{j+1}^h) - C_j^h\,(U_{jh+h}) + \varepsilon_j^h .$$

Consequently, relation (5.3) directly implies

$$(5.9)\qquad \|\delta_j^h\| \le K_j^h \|\delta_{j+1}^h\| + \|\varepsilon_j^h\| .$$

By using this inequality for $j = n, n+1, \dots, \ell-1$ and applying the stability condition (5.4) we conclude that

$$(5.10)\qquad \|\delta_n^h\| \le K_C \|\delta_\ell^h\| + \|\varepsilon_n^h\| + K_C \sum_{j=n+1}^{\ell-1} \|\varepsilon_j^h\| .$$

Further, by virtue of the continuity condition of definition 5.1:

$$(5.11)\qquad \|\delta_\ell^h\| = \|U_Z - U_{\ell h}\| \to 0 \qquad , \text{ as } h \to 0.$$

And the consistency condition (5.2) yields that uniformly in $j < \ell$:

$$(5.12)\qquad \|\varepsilon_j^h\|\, h^{-1} \to 0 \qquad , \text{ as } h \to 0.$$

Combination of (5.10), (5.11) and (5.12) shows that uniformly in $n < \ell$:

$$(5.13)\qquad \|\delta_n^h\| \to 0 \qquad , \text{ as } h \to 0.$$

Finally, the proof is completed by writing with $|nh - t| \le h$.:

$$(5.14)\qquad \|U_n^h - U_t\| \le \|\delta_n\| + \|U_{nh} - U_t\| \qquad ,$$

and using (5.13) together with the continuity condition of definition 5.1.

□

In order to apply the above approximation lemma to time-difference problems corresponding to the equations (3.1.3), (3.2.3) and (3.3.2), let us consider the following assumption.

ASSUMPTION 5.4. Let $D \subset B$, $\{U_t \mid t \in [0,Z]\} \subset D$, $\{A_t \mid t \in [0,Z]\}$ a family of operators $A_t : D \to B$, $t \le Z$, and $\{R_t(h) \mid t+h \le Z,\ t \in [0,Z],\ h \in (0,h_0]\} \subset B$ such that for all $h \le h_0$:

$$(5.15) \qquad U_t - U_{t+h} = hA_t(U_{t+h}) + R_t(h) \qquad , \ t+h \le Z \ ,$$

and $A_t(U_{t+h})$ is $\|.\|$-bounded uniformly in $t+h \le Z$. □

Next, consider a difference-method $M_D = \{C_j^h \mid jh \le Z,\ j \in \mathbb{N},\ h \in (0,h_0]\}$ and define for all $jh < Z$ an operator $A_j^h : B \to B$ by

$$(5.16) \qquad A_j^h = [C_j^h - I]h^{-1}.$$

LEMMA 5.5. *Let assumption* 5.4 *be satisfied and suppose that*

$$(5.17) \qquad \begin{cases} \|R_t(h)\| h^{-1} \\ \textit{converges to } 0 \textit{ uniformly in } t \le Z \textit{ as } h \to 0. \end{cases}$$

Then $\{U_t \mid t \in [0,Z]\}$ *is the solution of a properly-posed time-difference problem* $P(B,Z,E,u)$ *with* $u = U_Z$ *and for all* $c \in B$, $jh+h \le Z$:

$$(5.18) \qquad E_j^h(c) = [I + hA_{jh}](c) + R_{jh}(h).$$

Further, we have:

(i) M_D *is consistent if*

$$(5.19) \qquad \begin{cases} \|A_j^h(U_{jh+h}) - A_{jh}(U_{jh+h})\| \\ \textit{converges to } 0 \textit{ uniformly in } jh+h \le Z \textit{ as } h \to 0. \end{cases}$$

(ii) *Let* M_D *be stable,* $p \le 1$ *and suppose that*

$$(5.20) \qquad \begin{cases} \|A_j^h(U_{jh+h}) - A_{jh}(U_{jh+h})\| + \|R_{jh}(h)\| h^{-1} \\ \textit{is convergent of order } O(h^p) \textit{ uniformly in } jh+h \le Z. \end{cases}$$

Then,

$$(5.21) \quad \begin{cases} \|U_n^h - U_t\| \ \tau_\tau \\ \textit{with } n=[th^{-1}] \textit{ is convergent of order } O(h^p) \textit{ uniformly in } t \le Z. \end{cases}$$

<u>PROOF</u>. The relations (5.15) and (5.18) directly imply (5.1) with $u = U_Z$. Further, since $A_t\ (U_{t+h})$ is $\|.\|$ bounded uniformly in $t+h \le Z$, we conclude from (5.15) and (5.17) that for some constant L and all $h \le h_0$:

$$(5.22) \qquad \|U_t - U_{t+h}\| \le hL.$$

Consequently, U_t is Lipschitz in t with respect to the norm $\|.\|$ which implies the strong continuity of U_t in t. This completes the conditions of definition 5.1 for P(B,Z,E,u).
We proceed by proving (i) and (ii).

(i) By using (5.16) and (5.18) we can write

$$(5.23) \qquad [C_j^h(U_{jh+h}) - E_j^h(U_{jh+h})]h^{-1} =$$

$$[A_j^h(U_{jh+h}) - A_{jh}(U_{jh+h})] - R_{jh}(h)h^{-1}$$

The relations(5.17), (5.19) and (5.23) directly yield the consistency.

(ii) Consider the proof of the approximation lemma 5.3 and note that $\varepsilon_j^h h^{-1}$ is given by expression (5.23). Hence, (5.20) implies that expression (5.12) is convergent of order $O(h^p)$. Since also (5.22) implies an order of convergence O(h) in (5.11), the relations (5.10), (5.11), (5.12) and together with the fact that $p \le 1$, yield an order of convergence $O(h^p)$ in (5.13).
Finally, this latter fact together with the Lipschitz relation (5.22) again, implies that the right-hand side of (5.14) is convergent of order $O(h^p)$. □

<u>REMARKS 5.6</u>. 1. Obviously, we could have given an extended definition of a stable difference-method more general than by (5.3) and (5.4), such that the if-part of the standard Lax-Richtmeyer theorem follows directly fromm lemma 5.3. For our purposes, however, (5.3) and (5.4) are sufficient and more convenient.
2. Similarly to remark I.3.8, the restriction $p \le 1$ can be relaxed to $p > 0$, if t is replaced by nh in (5.21) and if, in addition, $U_{\ell h}$-u is of order $O(h^p)$. □

6. DISCRETE-TIME APPROXIMATIONS

6.1. INTRODUCTION AND SUMMARY

The approximation analysis of section 5 will be applied in this section to conclude that the three types of continuous-time functions given by (2.3.1), (2.3.2) and (2.3.4) can be approximated by the corresponding discrete-time functions given by (4.2.1), (4.2.4) and (4.2.8) respectively.
In view of the time-difference equations presented in section 3 especially lemma 5.5 will be used. As a result, for each of the three types of continuous-time functions convergence of discrete-time approximations can be concluded if the following three conditions are fulfilled:

(i) The continuous-time function is sufficiently smooth with respect to the time-parameter so that it satisfies a so-called *smoothness-assumption* (see 6.3.1, 6.4.1 or 6.5.1).

(ii) The discrete-time generators converge to the infinitesimal operator of the continuous-time process as required by the *consistency* relation (6.2.1) or the *strong consistency* relation (6.2.2).

(iii) The discrete-time one-step transition probabilities satisfy the *stability* relation (6.2.3) or the *strong stability* relation (6.2.4).

By verifying these conditions the discrete-time approximation is shown for:

1. Transition probabilities (subsection 6.3).
2. Finite horizon cost functions (subsection 6.4).
3. Finite horizon optimal cost functions (subsection 6.5).

As an implication of 1 , also weak convergence of processes is considered.

Throughout this section we consider
$(S,\Gamma,\Delta,\mu,D_A,\{A^\delta|\delta\in\Delta\},L)$ as fixed control object and
$(S,\Gamma,\Delta,\mu,h\;,\{P_h^\delta|\delta\in\Delta\},L)$ as h-control object for all $h\in(0,h_0]$.

For $\pi\in\Pi(AB)$ let $\{P^\pi_{s,t}|s,t\geq 0]$ denote the transition probabilities of the continuous-time controlled Markov process under control π as given by definition 2.1.4. And for any $h\leq h_0$ let $\{P^h_{j,n}|j,n\in\mathbb{N}\}$ be the transition probabilities of the discrete-time controlled Markov process as given by lemma 4.1.3 under h-control $\pi^h = (\pi(0),\pi(h),\pi(2h),\ldots)$.

Further, recall (4.1.5) for T_h^δ and (4.1.6) for $A_h^\delta = [T_h^\delta - I]h^{-1}$.

Finally, we note that for each of the three types of continuous-time functions the so-called smoothness assumption 6.3.1, 6.4.1, or 6.5.1 includes all assumptions made with respect to that function in preceding sections. The so-called consistency and stability relations for this chapter are defined in the next subsection. Recall that $\ell=[Zh^{-1}]$.

6.2. CONSISTENCY AND STABILITY RELATIONS

In the definitions below consider a collection $\{U_t \mid t \leq Z\} \subset D_A$ and $\pi\in\Pi(AB)$. The collection will be specified by

$U_t = T^{\pi}_{t,Z} f$ with $\pi\in\Pi(AB)$ and $f\in D_A$ in subsection 6.3 ,

$U_t = V^{\pi}_t$ with $\pi\in\Pi(AB)$ in subsection 6.4 , and

$U_t = \phi_t$ in subsection 6.5.

Consistency relation:

$$
(6.2.1) \quad \begin{cases} \| (A_h^{\pi(jh)} - A^{\pi(jh)})\, U_{jh+h} \|_\mu \\ \text{converges to 0 uniformly in } jh+h \leq Z \text{ as } h \to 0. \end{cases}
$$

Strong consistency relation:

$$
(6.2.2) \quad \begin{cases} \sup\limits_{\delta\in\Delta} \| (A_h^{\delta} - A^{\delta})\, U_{jh} \|_\mu \\ \text{converges to 0 uniformly in } jh+h \leq Z \text{ as } h \to 0. \end{cases}
$$

Stability relation:

$$
(6.2.3) \quad \begin{cases} \| \int \mu(y) P_h^{\pi(jh)}(.\,;dy) \|_\mu \leq (1+hK^{\pi}) \\ \text{uniformly in } jh < Z,\ h \leq h_0 \text{ and for some constant } K^{\pi}. \end{cases}
$$

Strong stability relation:

$$
(6.2.4) \quad \begin{cases} \sup\limits_{\delta\in\Delta} \| \int \mu(y)\, P_h^{\delta}(.\,;dy) \|_\mu \leq (1+hK_\Delta) \\ \text{uniformly in } jh < Z,\ h \leq h_0 \text{ and for some constant } K_\Delta. \end{cases}
$$

REMARKS 6.2.1.

1. Obviously, the strong relation (6.2.2) implies (6.2.1) and the strong relation (6.2.4) implies (6.2.3) for any $\pi\in\Pi(AB)$.

2. Let $\pi\in\Pi(AB)$ and suppose that the stability relation (6.2.3) holds. Then from (4.1.1) and (6.2.3) it follows that for all $j\leq n\leq \ell$:

$$\|\int\mu(y)\ P^h_{j,n}(.;dy)\|_\mu \leq (1+hK^\pi)^{(n-j)} \leq e^{ZK^\pi}. \tag{6.2.5}$$

So that according to definition 4.1.6: $\pi^h = (\pi(0),\pi(h),\pi(2h),\dots)\in\Pi^h(AB)$. Consequently, the notations (4.1.5) for T^δ_h and (4.1.6) for A^δ_h are justified for any $\delta\in(\pi(0),\pi(h),\pi(2h),\dots)$. If the strong stability relation (6.2.4) holds, then these notations are allowed for any $\delta\in\Delta$. These facts will be used in subsection 6.3, 6.4 and 6.5.

□

6.3. TRANSITION PROBABILITIES AND PROCESS

Let $\pi\in\Pi(AB)$ be fixed, consider $f\in D_A$ and recall the expressions (2.3.1) for $T^\pi_{t.Z}f$, (3.1.2) for $R^\pi_t(T,f,h)$ as well as (4.2.1) for $T^h_{j,\ell}f$ with $\pi^h = (\pi(0),\pi(h),\pi(2h),\dots)$.

SMOOTHNESS ASSUMPTION 6.3.1.

(i) Assumption 2.3.1 is satisfied, and

(ii)

(6.3.1) $\begin{cases} \|R^\pi_t(T,f,h)\|_\mu h^{-1} \\ \text{converges to 0 uniformly in } t+h\leq Z \text{ as } h\to 0. \end{cases}$ □

THEOREM 6.3.2. *Suppose that with collection* $\{U_t\,|\,t\in[0,Z]\} = \{T^\pi_{t,Z}f\,|\,t\in[0,Z]\}$ *the following conditions are satisfied:*

(i) *The smoothness assumption* 6.3.1.

(ii) *The consistency relation* (6.2.1).

(iii) *The stability relation* (6.2.3).

Then,

(6.3.2) $\begin{cases} \|\int f(y)\ P^h_{n,\ell}(x;dy) - \int f(y)\ P^\pi_{t,Z}(x;dy)\ \|_\mu \\ \textit{with } n=[th^{-1}] \textit{ converges to } 0 \textit{ uniformly in } t\leq Z \textit{ as } h\to 0\ . \end{cases}$

PROOF. By virtue of assumption 2.3.1 and the time-difference equation (3.1.3), the assumption 5.4 is satisfied with

$$(6.3.3)\quad \begin{cases} B = B^{\mu} \;;\; D = D_A \;;\; U_t = T^{\pi}_{t,Z}f \;;\; A_t = A^{\pi(t)} \;; \\ R_t(h) = R^{\pi}_t(T,f,h) \;;\quad t \le Z. \end{cases}$$

Further, relation (6.3.1) implies (5.17).
Next, define a difference-method M_D by

$$(6.3.4)\qquad C^h_j\,(f) = T^{\pi(jh)}_h f \quad,\quad j < \ell,\ h\in(0,h_0].$$

Then, the one-step generator A^h_j defined by (5.16) equals the one-step generator $A^{\pi(jh)}_h$ as defined by (4.1.6) for all j,h.

Consequently, application of lemma 5.5 shows that conditions (i) and (ii) of the theorem imply consistency of M_D for the collection $\{U_t \mid t\in[0,Z]\} = \{T^{\pi}_{t,Z}f \mid t\in[0,Z]\}$.

Next let us examine the stability of M_D.
With (6.3.4) and the stability relation (6.2.3) we find

$$(6.3.5)\qquad \|C^h_j(f_1) - C^h_j(f_2)\|_{\mu} = \|T^{\pi(jh)}_h(f_1-f_2)\|_{\mu} \le (1+hK^{\pi})\,\|f_1-f_2\|_{\mu}$$

for all $f_1,f_2\in B^{\mu}$ and $j<\ell$. Hence, with $K^h_j = (1+hK^{\pi})$ and $K_C = e^{ZK^{\pi}}$ the relations (5.3) and (5.4) are satisfied which implies the stability. □

Further, from comparing (4.2.2) and (6.3.4) it follows that

$$(6.3.6)\qquad T^h_{j,\ell}f = U^h_j \quad \text{with } U^h_j \text{ defined by (5.5) with } u = f.$$

Finally, the approximation lemma 5.3 completes the proof if we recall the consistency and stability as well as the expressions (2.3.1) and (4.2.1) □

REMARK 6.3.3. An order of convergence in (6.3.2) can be obtained by using (ii) of lemma 5.5 together with orders of convergence in the consistency relation (6.2.1) and the 'smoothness' relation (6.3.1). □

The above theorem concerns expectations induced by transition probabilities. Particularly, it enables us to study the weak convergence of the transition probabilities themselves as well as of the underlying controlled processes. Therefore, as in chapter I, we refer to definition 0.10 for the notion of weak convergence and to the Appendix A for weak convergence on D-spaces. Further, recall that $C^u(S)$ is the set of real-valued uniformly continuous and bounded functions on S.

THEOREM 6.3.4. *Let* G *be a subset of* B^μ *with* μ*-closure containing* $C^u(S)$. *Suppose that the conditions of theorem* 6.3.2 *are satisfied for any* $f \in G$. *Then for all* $t \leq Z$, $x \in S$ *and with* $n = [th^{-1}]$, $\ell = [Zh^{-1}]$:

$$(6.3.7) \qquad P^h_{n,\ell}(x;.) \Rightarrow P^\pi_{t,Z}(x;.) \quad , \textit{as } h \to 0.$$

PROOF. First of all, conclude that (6.3.2) is satisfied for any $f \in G$. Further, for $f_1, f_2 \in B^\mu$ it follows from (2.1.2) that for $t \leq Z$:

$$(6.3.8) \qquad \| \int (f_1 - f_2)(y) \; P^\pi_{t,Z}(.;dy) \|_\mu \leq M^\pi \| f_1 - f_2 \|_\mu ,$$

and from (4.1.9) that for $nh \leq Z$:

$$(6.3.9) \qquad \| \int (f_1 - f_2)(y) \; P^h_{n,\ell}(.;dy) \|_\mu \leq M^h \| f_1 - f_2 \|_\mu .$$

Next, let $f \in C^u(S)$ and $\{f^n\}_{n=1}^\infty \subset B^\mu$ such that $\|f^n - f\| \to 0$ as $n \to \infty$.

Then, by using (6.3.2) for all f^n, together with (6.3.8) and (6.3.9) and by letting $n \to \infty$, one can show that (6.3.2) holds for f.
Finally, since convergence of functions within B^μ in μ-norm implies convergence of their values for any fixed $x \in S$, the proof is concluded from the portmanteau theorem. (cf. Billingsley (1968) p.11). □

Let $\{X^\pi_t \mid t \geq 0\}$ be the continuous-time controlled Markov process as given by definition 2.1.4 induced by control π and some $X^\pi_0 = Z_0$, and let for all $h \leq h_0$: $\{X^h_{nh} \mid n = 0,1,2,\ldots\}$ be the discrete-time controlled Markov process given by lemma 4.1.3 induced by $\pi = (\pi(0), \pi(h), \pi(2h), \ldots)$ and some $X^h_0 = Z^h_0$.
Further, define for all $h \leq h_0$ a process $(\bar{X}^h_t)_{t \geq 0}$ on $D[0,\infty)$ by

(6.3.10) $\bar{X}_t^h = X_{nh}^h$, $t \in [nh, nh+h)$, $n \in \mathbb{N}$.

THEOREM 6.3.5. *Let* G *be a subset of* B^μ *with* μ*-closure containing* $C^u(S)$. *Suppose that for each* $Z > 0$ *and* $f \in G$ *the conditions of theorem* 6.3.2 *as well as the following conditions are satisfied:*

(6.3.11) $X_0^h \Rightarrow X_0^\pi$, *as* $h \to 0$.

(6.3.12) $\int f(y)\, P_{t,Z}^\pi(x.dy)$ *is continuous in* $x \in S$.

(6.3.13) $\sup_{x \in Q} \mu(x) < \infty$ *for any compact set* $Q \subset S$.

(6.3.14) *One of the following holds:*
(i) *Condition* (ii) *of theorem* A.3.5.
(ii) *Condition* (ii) *of theorem* A.3.6.
(iii) *Conditions* (ii) *and* (iii) *of theorem* A.3.7.

Then,

(6.3.15) $(\bar{X}_t^h)_{t \geq 0} \Rightarrow (X_t^\pi)_{t \geq 0}$ *on* $D[0,\infty)$ *as* $h \to 0$.

PROOF. As in the proof of theorem 6.3.4, we can show that for any $f \in C^u(S)$ and $Z \in (0,\infty)$ relation (6.3.2) is satisfied.
Consequently, together with (6.3.13) we obtain for any $f \in C^u(S)$, $Z > 0$:

(6.3.16) $\sup_{x \in Q} \left| \int f(y)\, P_{n,\ell}^h(x;dy) - \int f(y)\, P_{t,Z}^\pi(x;dy) \right| \to 0$

as $h \to 0$ for any compact set $Q \subset S$, with $n = [th^{-1}]$, $\ell = [Zh^{-1}]$.
From (6.3.12), (6.3.16) and the portmanteau theorem we obtain for any $x \in S$ and collection $\{x^h \mid h \in (0,h_0]\}$ with $x^h \to x$:

(6.3.17) $P_{n,\ell}^h(x^h;.) \Rightarrow P_{t,Z}^\pi(x;.)$ as $h \to 0$,

which proves relation (3.5) of the appendix A. The proof is completed by theorem A.3.5, A.3.6 or A.3.7 corresponding to (6.3.14). □

6.4. FINITE HORIZON COST FUNCTION

Let $\pi\in\Pi(AB)$ be fixed and recall the expressions (2.3.2) for V_t^π, (3.2.2) for $R_t^\pi(V,h)$ and (4.2.4) for V_j^h with $\pi^h = (\pi(0),\pi(h),\pi(2h),\ldots)$.

SMOOTHNESS ASSUMPTION 6.4.1. The following conditions hold:

(i) Assumption 2.3.2,

(ii) assumption 2.3.3,

(iii) assumption 4.2.1, and

(iv)

$$(6.4.1)\quad \begin{cases} \|R_t^\pi(V,h)\|_\mu\, h^{-1} \\ \text{converges to 0 uniformly in } t+h\leq Z \text{ as } h\to 0. \end{cases}$$

□

THEOREM 6.4.2. *Suppose that with collection* $\{U_t\,|\,t\in[0,Z]\} = \{V_t^\pi\,|\,t\in[0,Z]\}$ *the following conditions are satisfied:*

(i) *The smoothness assumption* 6.4.1.

(ii) *The consistency relation* (6.2.1).

(iii) *The stability relation* (6.2.3).

Then,

$$(6.4.2)\quad \begin{cases} \|V_n^h - V_t^\pi\|_\mu \\ \textit{with } n = [th^{-1}] \;\; \textit{converges to } 0 \textit{ uniformly in } t\in[0,Z] \textit{ as } h\to 0. \end{cases}$$

□

PROOF. By virtue of the assumptions 2.3.2, 2.3.3 and 4.2.1 and the time-difference equation (3.2.3), the assumption 5.4 is guaranteed with

$$(6.4.3)\quad \begin{cases} B = B^\mu \quad ; \; D = D_A \; ; \; U_t = V_t^\pi \quad ; \\ A_t(U_t) = L^{\pi(t)} + A^{\pi(t)}(U_t) \quad ; \\ R_t(h) = R_t^\pi(V,h) \quad , \; t\leq Z. \end{cases}$$

Further, relation (6.4.1) guarantees (5.17).
Next, define a difference-method M_D by

$$(6.4.4)\qquad C_j^h(f) = hL^{\pi(jh)} + T_h^{\pi(jh)}(f) \qquad , \; j<\ell,\; h\in(0,h_0\,].$$

Then, the one-step generator A_j^h defined by (5.16) is given by

$$A_j^h(f) = L^{\pi(jh)} + A_h^{\pi(jh)}(f). \tag{6.4.5}$$

Combining (6.4.3) and (6.4.5) yields

$$A_j^h(f) - A_{jh}(f) = (A_h^{\pi(jh)} - A^{\pi(jh)})(f). \tag{6.4.6}$$

Consequently, application of lemma 5.5 shows that the conditions (i) and (ii) of the theorem imply consistency of M_D for the collection $\{U_t \mid t \in [0,Z]\} = \{V_t^\pi \mid t \in [0,Z]\}$.

Next, let us examine the stability of M_D.
From (6.4.4) and the stability relation (6.2.3) it follows that

$$\|C_j^h(f_1) - C_j^h(f_2)\|_\mu = \|T_h^{\pi(jh)}(f_1 - f_2)\|_\mu \le (1+hK^\pi)\ \|f_1 - f_2\|_\mu \tag{6.4.7}$$

for all $f_1, f_2 \in B^\mu$ and $j < \ell$. Hence, with $K_j^h = (1+hK^\pi)$ and $K_C = e^{ZK^\pi}$, the relations (5.3) and (5.4) are satisfied which proves the stability.

Further, from comparing (4.2.5) and (6.4.4) it follows that

$$V_j^h = U_j^h \qquad \text{with } U_j^h \text{ defined by (5.5) with } u = \bar{0}. \tag{6.4.8}$$

Finally, the approximation lemma 5.3 completes the proof if we recall the consistency and stability. □

REMARK 6.4.3.

1. An order of convergence in (6.4.2) can be obtained by using (ii) of lemma 5.5 together with orders of convergence in the consistency relation (6.2.1) and the 'smoothness' relation (6.4.1).
2. Recall that the discrete-time cost functions V_j^h can be computed by recursively solving (4.2.5).
3. By considering orders of convergence, one can sometimes conclude that the convergence in (6.4.2) is uniform in some class of controls, say $\Pi(U)$. As a result, we would obtain for any $x \in S$ and $t \le Z$:

$$\left|\inf_{\Pi(U)} V_n^h(x) - \inf_{\Pi(U)} V_t^\pi(x)\right| \to 0 \quad \text{as } h \to 0. \tag{6.4.9}$$

Hence, the 'optimal cost functions within $\Pi(U)$' could be approximated. □

6.5. FINITE HORIZON OPTIMAL COST FUNCTION

Recall the expressions (2.3.3) for the infimum-operator J, (2.3.4) for $\emptyset_t$, (3.3.1) for $R_t(\emptyset,h)$ and (4.2.8) for $\emptyset_j^h$.

SMOOTHNESS ASSUMPTION 6.5.1. The following conditions hold:

(i) Assumption 2.3.4,

(ii) assumption 2.3.5,

(iii) assumption 4.2.3, and

(iv)

$$\text{(6.5.1)} \quad \begin{cases} \|R_t(\emptyset,h)\|_\mu h^{-1} \\ \text{converges to 0 uniformly in } t+h \le Z \text{ as } h \to 0. \end{cases}$$

□

THEOREM 6.5.2. *Suppose that with collection* $\{U_t \mid t\in[0,Z]\} = \{\emptyset_t \mid t\in[0,Z]\}$ *the following conditions are satisfied:*

(i) *The smoothness assumption* 6.5.1.

(ii) *The strong consistency relation* (6.2.2).

(iii) *The strong stability relation* (6.2.4).

Then,

$$\text{(6.5.2)} \quad \begin{cases} \|\emptyset_n^h - \emptyset_t\|_\mu \\ \textit{with } n = [th^{-1}] \textit{ converges to 0 uniformly in } t \le Z \textit{ as } h \to 0. \end{cases}$$

PROOF. By virtue of the assumptions 2.3.4 and 2.3.5 as well as the time-difference equation (3.3.2), the assumption 5.4 holds with

$$\text{(6.5.3)} \quad \begin{cases} B = B^\mu \quad ; \quad D = D_A \quad ; \quad U_t = \emptyset_t ; \\ A_t(U_t) = J(U_t) = \inf\limits_{\delta\in\Delta} [L^\delta + A^\delta(U_t)] ; \\ R_t(h) = R_t(\emptyset,h) \qquad\qquad , \ t \le Z. \end{cases}$$

Further, relation (6.5.1) guarantees (5.17).
Next, define a difference-method M_D by

(6.5.4) $$C_j^h(f) = \inf_{\delta\in\Delta} [hL^\delta + T_h^\delta(f)] .$$

Then A_j^h, the one-step generator defined by (5.16), is given by

(6.5.5) $$A_j^h(f) = \inf_{\delta\in\Delta} [L^\delta + A_h^\delta(f)] .$$

Combining (6.5.3) and (6.5.5) yields

(6.5.6) $$\|A_j^h(f) - A_{jh}(f)\|_\mu \le$$

$$\sup_{x\in S} \mu(x)^{-1} [\sup_{\delta\in\Delta} | A_h^\delta(f)(x) - A^\delta(f)(x)|] =$$

$$\sup_{\delta\in\Delta} \sup_{x\in S} \mu(x)^{-1} | A_h^\delta(f)(x) - A^\delta(f)(x)| =$$

$$\sup_{\delta\in\Delta} \| A_h^\delta(f) - A^\delta(f) \|_\mu$$

Consequently, application of lemma 5.5 shows that the conditions (i) and (ii) imply consistency of M_D for the collection $\{U_t | t\in[0,Z]\} = \{\emptyset_t | t\in[0,Z]\}$.

Next, let us examine the stability of M_D.
First, in analogy with the steps of (6.5.6) it can be proven that

(6.5.7) $$\|C_j^h(f_1) - C_j^h(f_2)\|_\mu \le \sup_{\delta\in\Delta} \|T_h^\delta(f_1) - T_h^\delta(f_2)\|_\mu .$$

Next, the strong stability relation (6.2.4) and (6.5.7) imply

(6.5.8) $$\|C_j^h(f_1) - C_j^h(f_2)\|_\mu \le (1+hK_\Delta) \|f_1 - f_2\|_\mu$$

for all $f_1, f_2 \in B^\mu$ and $j < \ell$. Hence, with $K_j^h = (1+hK_\Delta)$ and $K_C = e^{ZK_\Delta}$, the relations (5.3) and (5.4) are satisfied which proves the stability.

Further, from comparing (4.2.8) and (6.5.4) it follows that

(6.5.9) $$\emptyset_j^h = U_j^h \qquad \text{with } U_j^h \text{ defined by (5.5) with } u = \bar{0}.$$

Finally, the approximation lemma 5.3 completes the proof if we recall the consistency and stability. □

REMARK 6.5.3.

1. An order of convergence in (6.5.2) can be obtained by using (ii) of lemma 5.5 together with orders of convergence in the strong consistency relation (6.2.2) and the 'smoothness' relation (6.5.1).

2. Recall that the discrete-time optimal cost functions $\emptyset_j^h$ can be computed by recursively solving (4.2.8).

3. By means of orders of convergence and relation (4.2.8), it is possible to give piecewise constant controls which are *'nearly-(ε-) optimal'* for the continuous-time model. This will be shown for the applications in sections 7 and 8.

4. The discrete-time approximations given in subsections 6.3 and 6.4 are all induced by one and the same fixed control $\pi\in\Pi(AB)$. However, let $\{\pi^h \mid h\in(0,h_0]\}$ be a collection of controls $\pi^h\in\Pi^h(AB)$. Then we can just as well consider discrete-time approximations induced by discrete-time controlled Markov processes under control π^h, say with corresponding transition probabilities $\{P^h_{j,n} \mid j,n\in\mathbb{N}\}$.

It is easily seen that all results of subsections (6.3) and (6.4) remain valid if we replace the consistency relation (6.2.1) and the stability relation (6.2.3) by

$$(6.5.10)\quad \begin{cases} \|(A_h^{\pi^h}(jh) - A^{\pi}(jh))\, U_{jh+h}\|_\mu \\ \text{converges to 0 uniformly in } jh+h\le Z \quad \text{as } h\to 0, \end{cases}$$

and

$$(6.5.11)\quad \begin{cases} \|\int \mu(y)\; P_h^{\pi^h}(jh)(.;dy)\|_\mu \le (1+hK^{\pi}) \\ \text{uniformly in } jh< Z,\ h\le h_0 \text{ and for some constant } K^{\pi}. \end{cases}$$

(Note that (6.2.2) does not imply the relation (6.5.10)).
Particularly, discrete-time controls π_0^h which correspond to the discrete-time optimality equations (4.2.8) are of interest.
If the controls $\{\pi_0^h\}$ contain a *limit control* π_0 for $h\to 0$ such that *the convergence relation* (6.4.2) *holds* and if in addition *the convergence relation* (6.5.2) *is satisfied, then* it can be shown that π_0 is an *optimal control* for the continuous-time model. □

7. CONTROLLED MARKOV JUMP PROCESSES

7.1. CONTINUOUS-TIME MODEL

This section concerns controlled Markov jump processes. For a general description of such a process one may consider the informal description given in subsection 5.1 of chapter I where the jump-characteristics q and H must be replaced by q_t^π and H_t^π with π representing a control.
As a specific example of a controlled Markov jump process consider a service facility with one server. Customers arrive according to a Poisson process with parameter λ. Each customer demands an amount of service which has an exponential distribution with mean 1.
The customers are served one at a time and in order of arrival. The service rate ν can be controlled within a finite interval $[\nu_1,\nu_2]$.
Consequently, if at epoch t the number of customers present is $i \geq 1$ and if during $[t,t+\Delta t]$ a constant service rate ν is used, then with probability $\lambda\Delta t + o(\Delta t)$ a new customer arrives and with probability $\nu\Delta t + o(\Delta t)$ a customer completes a service during $[t,t+\Delta t]$, where Δt is assumed to be small. Costs are incurred by a holding cost rate linear in the number of waiting customers plus a service cost rate linear in the controlled service rate.

To proceed formally, let us consider the control object $(S,\Gamma,\Delta,\mu,D_A, \{A^\delta|\delta\in\Delta\}, L)$ as well as
q a measurable real-valued function on S x Γ, called *jump rate*, and
H a transition probability from S x Γ to S, called *jump measure*;
such that for any $\delta\in\Delta$, $f\in D_A$ and $x\in S$:

$$(7.1.1) \qquad A^\delta f(x) = q(x,\delta(x)) \int [f(y) - f(x)] \, H(x,\delta(x);dy).$$

Specifications on the jump characteristics q and H, the bounding function μ, the domain D_A and cost-rate function L will follow.
First of all throughout this section the following assumption is made:

ASSUMPTION 7.1.1.

(i) $H(x,\gamma;\{x\}) = 0$ for any $(x,\gamma)\in S \times \Gamma$.

(ii) For some constant $Q<\infty$ and all $(x,\gamma)\in S \times \Gamma$: $0 \leq q(x,\gamma) \leq Q$. □

As in subsection 5.1 of chapter I condition (i) is not necessary for the analysis of this section, whereas condition (ii) is.
Particularly, assumption 7.1.1 enables us to show the existence and uniqueness of transition probabilities and a corresponding process for any measurable control. First consider:

NOTATION 7.1.2.

$$q^{\delta}(x) = q(x,\delta(x)), \quad H^{\delta}(x;.) = H(x,\delta(x);.) \qquad \text{for } x\in S,\ \delta\in\Delta.$$

$$q_t^{\pi} = q^{\delta}, \quad H_t^{\pi} = H^{\delta} \qquad \text{for } \pi\in\Pi(M),\ t\geq 0 \text{ and } \delta=\pi(t). \qquad \square$$

Then the following theorem is to be seen as an extension of theorem I 5.1.4.

THEOREM 7.1.3. *Let* $\pi\in\Pi(M)$ *such that* $\pi_t(x)$ *is measurable in* (t,x). *Then we have:*

(i) *There exists a unique semigroup of transition probabilities* $\{P^{\pi}_{s,t}|s,t\geq 0\}$ *such that for any* $t\geq 0$, $h>0$, $x\in S$ *and* $B\in\beta$:

$$(7.1.2)\quad \begin{cases} [P^{\pi}_{t,t+h}(x;B) - 1_B(x)]h^{-1} & \textit{is uniformly bounded} \\ \textit{and as } h\to 0 \textit{ converges to: } q_t^{\pi}(x)[H_t^{\pi}(x;B) - 1_B(x)]. \end{cases}$$

(ii) *For any random element* Z_0 *on* S *there exists a unique Markov process* $\{X_t^{\pi}|t\geq 0\}$ *with transition probabilities* $\{P^{\pi}_{s,t}|s,t\geq 0\}$ *such that* $X_0^{\pi}=Z_0$ *and* $\mathbb{P}((X_t^{\pi})_{t\geq 0}\in D[0,\infty)) = 1$.

PROOF. (i) The existence of a semigroup $\{P^{\pi}_{s,t}|s,t\geq 0\}$ satisfying (7.1.2) follows from defining, as in theorem 4 on p.364 of Gihman and Skorohod (1969), for any $s,t\geq 0$, $x\in S$, $B\in\beta$:

$$(7.1.3)\quad \begin{cases} P^{\pi}_{s,t}(x;B) = \sum_{n=0}^{\infty} P^{n}_{s,t}(x;B) \quad , \quad \text{where} \\ P^{0}_{s,t}(x;B) = 1_B(x)\ \exp\left(-\int_s^t q_u^{\pi}(x)du\right), \quad \text{and for } n\geq 1: \\ P^{n}_{s,t}(x;B) = \int_s^t \exp\left[\left(-\int_s^u q_t^{\pi}(x)dt\right)\right] q_u^{\pi}(x). \\ \qquad\qquad \left[\int P^{n-1}_{u,t}(y;B)\ H_u^{\pi}(x;dy)\right] du \quad . \end{cases}$$

If $\{P^{\pi}_{s,t} | s,t \geq 0\}$ is a collection satisfying (7.1.2),then it can be shown in analogy with p.347 - 353 and p.364 - 366 of Gihman and Skorohod (1969) that the collection is given by (7.1.3). (As only difference with the above reference we must use dominated convergence in (7.1.2) in stead of convergence uniformly in t).

(ii) The existence and construction of such a process is shown by theorem 4 on p. 364 of Gihman and Skorohod (1969). Since the transition probabilities determine the finite-dimensional distributions,the uniqueness follows from theorem 14.5 of Billingsley (1968). □

Before further investigating the admissibility of a control,let us first present an assumption on μ which is made throughout this section.

ASSUMPTION 7.1.4. For some constant $1 \leq K < \infty$:

$$(7.1.4) \qquad \sup_{\delta \in \Delta} \| \int \mu(y)\, H^{\delta}(.;dy) \|_{\mu} \leq K. \qquad \square$$

The control which will be given below obviously satisfies the measurability condition. Therefore, let $\{P^{\pi}_{s,t} | s,t \geq 0\}$ denote the transition probabilities given by (7.1.3). Recall (7.1.1) for A^{δ} and as before let Z be a finite time-point.
Below we always use a symbol C to indicate a constant depending only on Z,Q,K and a control π. Further, Δt always denotes a positive number, representing a length of time.

LEMMA 7.1.5. *Let* $\pi \in \Pi(M)$ *such that*

$$(7.1.5) \qquad \| q^{\pi}_{t+\Delta t} - q^{\pi}_{t} \|_{\infty} \leq \Delta t C \qquad , \; t+\Delta t \leq Z.$$

$$(7.1.6) \qquad \begin{cases} \| \int f(y) H^{\pi}_{t+\Delta t}(.;dy) - \int f(y) H^{\pi}_{t}(.;dy) \|_{\mu} \; \|f\|^{-1}_{\mu} \leq \Delta t C \\ \text{uniformly in } f \in B^{\mu} \qquad , \; t+\Delta t \leq Z. \end{cases}$$

Then for all $f \in B^{\mu}$ *and* $t+h \leq Z$:

$$(7.1.7) \qquad \| [\int f(y) P^{\pi}_{t,t+h}(.;dy) - f(.)] h^{-1} - A^{\pi(t)} f(\cdot) \|_{\mu} \leq hC \, \|f\|_{\mu} \; .$$

PROOF. First of all let us prove that for all $s \leq t \leq Z$, and $n \in \mathbb{N}$:

$$\| \int \mu(y)\, P^n_{s,t}(.;dy)\|_\mu \leq \frac{(t-s)^n}{n!}\,(QK)^n, \tag{7.1.8}$$

where $P^n_{s,t}$ is defined recursively by (7.1.3).
Clearly, for n = 0 relation (7.1.8) is satisfied. Let us proceed by induction on n. Let (7.1.8) be satisfied for n - 1. Then according to (7.1.3) and (7.1.4):

$$\begin{aligned} &\| \int \mu(y)\, P^n_{s,t}(.;dy)\|_\mu \leq \\ &\int_s^t Q\, \| \int [\int \mu(y)\, P^{n-1}_{u,t}\,(z;dy)]\, H^\pi_u(.;dz)\|_\mu\, du \leq \\ &\int_s^t Q\, \| \int \mu(y)\, P^{n-1}_{u,t}(.;dy)\|_\mu \cdot \| \int \mu(z)\, H^\pi_u(.;dz)\|_\mu\, du \leq \\ &\int_s^t (QK)^n\, \frac{(t-u)^{n-1}}{(n-1)!}\, du \leq \frac{(t-s)^n}{n!}\,(QK)^n . \end{aligned} \tag{7.1.9}$$

Next let $f \in B^\mu$, and let $t \leq t+h \leq Z$. As a direct consequence of (7.1.8):

$$\begin{aligned} &\| \sum_{n=2}^{\infty} \int f(y)\, P^n_{t,t+h}(.;dy)\|_\mu \leq \\ &\sum_{n=2}^{\infty} \frac{h^n}{n!}\,[QK]^n \|f\|_\mu \leq h^2 e^{hQK}\, \|f\|_\mu . \end{aligned} \tag{7.1.10}$$

Since q is bounded by Q we have

$$\| \exp\,(-\int_t^{t+\Delta t} q^\pi_u(\cdot)du) - 1_{\mathbb{R}}(\cdot)\,\|_\infty \leq \Delta t C\, . \tag{7.1.11}$$

Expression (7.1.3) for $P^1_{t,t+h}$ together with (7.1.6) and (7.1.11) gives after some calculation:

$$\| h^{-1} \int_t^{t+h} f(y)\, P^1_{t,t+h}(.;dy) - q^\pi_t(\cdot) \int f(y)\, H^\pi_t(.;dy)\|_\mu \leq hC\|f\|_\mu\, . \tag{7.1.12}$$

Further, by virtue of (7.1.5) one easily shows:

$$\| [\exp\,(-\int_t^{t+h} q^\pi_u(\cdot)du) - (1-hq^\pi_t(\cdot))]f(\cdot)\|_\mu \leq h^2 C\|f\|_\mu \tag{7.1.13}$$

Finally, by combination of expression (7.1. 3) for $P^\pi_{t,t+h}$ and the relations (7.1.10), (7.1.12) and (7.1.13) the proof is completed. □

The above lemma and its proof enable us to conclude admissibility of a control, as defined in definition 2.1.4, as well as boundedness, as given in definition 2.1.6.

THEOREM 7.1.6. *Let* $\pi \in \Pi(M)$ *which satisfies the relations* (7.1.5) *and* (7.1.6). *Then*,

(i) π *is an admissible control for the control object with* $D_A = B^{\mu}$ *and*

(ii) π *satisfies the boundedness relation* (2.1.2).

Consequently, $\pi \in \Pi(AB)$.

PROOF.

(i) Directly from definition 2.1.4, theorem 7.1.3 and lemma 7.1.5.

(ii) According to (7.1.3) and (7.1.8) relation (2.1.2) is satisfied with bounding constant $M^{\pi} = \exp(ZQK)$. □

REMARKS 7.1.7.

1. The Lipschitz relations (7.1.5) and (7.1.6) are quite strong. However, in view of proving the admissibility of a control as well as of obtaining approximation results later on, several relaxations are possible. For instance, let us only suppose that the left-hand side of (7.1.5) and (7.1.6) converge to 0 as $\Delta t \to 0$. Then it can be shown analogously to the proof of the above lemma that also the left hand side of (7.1.7) converges to 0 as $\Delta t \to 0$. And, as a result, it can be shown in analogy with the approximation analysis which follows that the theorems 7.2.3 and 7.2.5 are valid if the order of converge $O(h)$ is omitted.

An other relaxation which can be desirable for applications is to replace (7.1.5) and (7.1.6) by piecewise Lipschitz (or continuity) conditions on q and H with respect to the time-parameter. A specific relaxation of this type is made in §4 of subsection 7.2. In order not to complicate things too much, we have omitted to include such relaxations in generality.

2. The Lipschitz relations (7.1.5) and (7.1.6) are guaranteed in each of the following two cases:

(i) π is stationary, i.e.; for some $\delta \in \Delta$ and all $t \geq 0$: $\pi(t) = \delta$.

(ii) With d_{Γ} the metric on Γ it holds for some Lipschitz constant L:

$$(7.1.14)\quad \begin{cases} \|q(.;\gamma_2) - q(.;\gamma_1)\|_\infty \le d_\Gamma(\gamma_1,\gamma_2)\, L & ,\ \gamma_1,\gamma_2 \in \Gamma \\ \|\int f(y)\, H(.;\gamma_2;dy) - \int f(y)\, H(.,\gamma_1;dy)\|_\mu & \\ .\|f\|_\mu^{-1} \le d_\Gamma(\gamma_1,\gamma_2)\, L\ ,\ \text{for all } f\in B^\mu, & ,\ \gamma_1,\gamma_2 \in \Gamma \\ \|d_\Gamma(\pi_{t+\Delta t}(\cdot),\pi_t(\cdot))\|_\infty \le \Delta t\, L & ,\ t+\Delta t \le Z. \end{cases}$$

The Lipschitz conditions in (7.1.14) on q and H are quite natural in queueing models where the state variable denotes the number of customers present. The Lipschitz condition on π as given in (7.1.14) may not always be satisfied in realistic models.
In this respect, however, (also see remark 1) relaxations as piecewise Lipschitz conditions on π as given by (7.1.14) will be useful.

3. By using standard arguments it can be shown that relation (7.1.6) is satisfied for all $f\in B^\mu$ if and only if

$$(7.1.15)\qquad \|\int \mu(y)\,|H^\pi_{t+\Delta t} - H^\pi_t|(.;dy)\|_\mu \le \Delta tC \quad ,\ t+\Delta t \le Z\ ,$$

where the measure $|H^\pi_{t+\Delta t} - H^\pi_t|(x;.)$ denotes the *total variation* of $(H_{t+\Delta t} - H_t)(x;.)$. (see Neveu (1964) p.101 for a definition). □

In view of theorem 7.1.6, the notations and results of section 2 can be applied for any $\pi\in\Pi(M)$ satisfying (7.1.5) and (7.1.6) for all $f\in B^\mu$ and $t+\Delta t \le Z$. Especially, recall expression (2.2.1) for $T^\pi_{s,t}f$.
Then, by using the results of this subsection one easily verifies the following relations, which are given for application later on.
For all $f\in B^\mu$, $s,t\le Z$, $t+\Delta t\le Z$ and $\delta\in\Delta$:

$$(7.1.16)\qquad \|T^\pi_{s,t}f\|_\mu \le e^{ZQK}\|f\|_\mu \quad ,$$

$$(7.1.17)\qquad \|A^\delta f\|_\mu \le 2QK\|f\|_\mu \quad ,$$

$$(7.1.18)\qquad \|[T^\pi_{t,t+\Delta t} - I]\, f\,\|_\mu \le \Delta tC\|f\|_\mu \ ,$$

$$(7.1.19)\qquad \|([T^\pi_{t,t+\Delta t} - I](\Delta t)^{-1} - A^{\pi(t)})f\,\|_\mu \le \Delta tC\|f\|_\mu \ .$$

7.2. APPROXIMATIONS

Take $h_0 \le Q^{-1}$ and let for any $h \le h_0 : \{P_h^\delta | \delta \in \Delta\}$ be a collection of one-step transition probabilities defined by

$$(7.2.1) \qquad P_h^\delta(x;B) = hq^\delta(x)\, H^\delta(x;B) + [1-hq^\delta(x)]1_B(x)$$

for all $x \in S$, $B \in \beta$ and $\delta \in \Delta$. Obviously, relation (7.1.4) guarantees relation (4.1.4), which justifies the notation of the operators T_h^δ and A_h^δ as given by (4.1.5) respectively (4.1.6) for any $\delta \in \Delta$.
As a result, we obtain for any $f \in B^\mu$, $x \in S$ and $h \le h_0$:

$$(7.2.2) \qquad A_h^\delta f(x) = q^\delta(x) \int [f(y) - f(x)]\, H^\delta(x;dy) = A^\delta f(x)$$

LEMMA 7.2.1. *The strong consistency relation* (6.2.2) *holds for any collection* $\{U_t | t \in [0,Z]\} \subset B^\mu$.

PROOF. Immediately from (7.2.2). □

LEMMA 7.2.2. *The strong stability relation* (6.2.4) *is satisfied.*

PROOF. Relation (7.1.4), with $K \ge 1$, and (7.2.1) yield:

$$(7.2.3) \qquad \sup_{\delta \in \Delta} \| \int \mu(y)\, P_h^\delta(.;dy) \|_\mu \le (1+hQ[K-1]). \qquad \square$$

Lemma 7.2.2 implies, also see remark 6.2.1, that for any control $\pi \in \Pi(M)$ and $h \le h_0$: $\pi^h = (\pi(0), \pi(h), \pi(2h), \ldots) \in \Pi^h(AB)$, (see definition 4.1.6). As a result, below we will examine in correspondence to the subsections 6.3, 6.4 and 6.5 respectively, the discrete-time approximation for the continuous-time model given in subsection 7.1.1 of:

- transition probabilities in §1,
- finite horizon cost functions in §2, and
- finite horizon optimal cost functions in §3.

In addition, the construction of nearly-(ε-) optimal controls is studied in §4. The notation which will be used can be found in the (corresponding parts of) sections 2,3 and 4.

NOTE It is easily seen that lemma 7.2.1 also holds if in the right-hand side of relation (7.2.1) we add an arbitrary term of order $o(h)$, as h tends to 0. □

§1. TRANSITION PROBABILITIES

Let $\pi \in \Pi(M)$ which satisfies (7.1.5) and (7.1.6).

THEOREM 7.2.3. *For any* $f \in B^{\mu}$ *the convergence relation* (6.3.2) *is satisfied with order of convergence* $O(h)$.

PROOF. By virtue of theorem 6.3.2, lemma 7.2.1 and lemma 7.2.2, relation (6.3.2) is proven by verifying the smoothness assumption 6.3.1.

Since $D_A = B^{\mu}$, assumption 2.3.1 directly follows from (7.1.16) and (7.1.17).

The 'smoothness' relation (6.3.1) is satisfied since (7.1.16) and (7.1.19) yield:

$$(7.2.4) \qquad \|R_t^{\pi}(T,f,h)\|_{\mu} h^{-1} = \|(h^{-1}[T_{t,t+h}^{\pi} - I] - A^{\pi(t)})\, T_{t+h,Z}^{\pi} f\|_{\mu} \leq hCe^{ZQK}\|f\|_{\mu} .$$

Hence, the smoothness assumption 6.3.1 is guaranteed. Furthermore, lemma 5.5 together with the relations (7.2.2) and (7.2.4) shows the order of convergence $O(h)$. □

In analogy with subsection 5.1 of chapter I, also weak convergence of the transition probabilities can be concluded as well as, under additional continuity conditions on q and H, weak convergence of processes on D-spaces. However, such a result is included by much more general results shown by Hordijk and Van Der Duyn Schouten (1983a). See also Van Der Duyn Schouten (1979).

§2. FINITE HORIZON COST FUNCTION

Let $\pi \in \Pi(M)$ which satisfies (7.1.5) and (7.1.6).

ASSUMPTION 7.2.4. For constants L and C:

$$\|L^{\delta}\|_{\mu} \leq L \qquad \text{for all } \delta \in \Delta, \tag{7.2.4}$$

$$\|L^{\pi}_{t+\Delta t} - L^{\pi}_{t}\|_{\mu} \leq \Delta t C. \tag{7.2.5}$$

THEOREM 7.2.5. *Let assumption* 7.2.4 *be satisfied. Then the convergence relation* (6.4.2) *holds with order of convergence* O(h).

PROOF. By virtue of theorem 6.4.2, lemma 7.2.1 and 7.2.2, relation (6.4.2) is proven by verifying the smoothness assumption 6.4.1.

By using (7.1.16), (7.1.18), (7.2.4) and (7.2.5) we obtain with $\Delta s \geq 0$:

$$\|T^{\pi}_{t,s+\Delta s} L^{\pi}_{s+\Delta s} - T^{\pi}_{t,s} L^{\pi}_{s}\|_{\mu} \leq \tag{7.2.6}$$

$$\|T^{\pi}_{t,s+\Delta s} (L^{\pi}_{s+\Delta s} - L^{\pi}_{s})\|_{\mu} + \|T^{\pi}_{t,s} (T^{\pi}_{s,s+\Delta s} - I) L^{\pi}_{s}\|_{\mu} \leq \Delta s C.$$

Hence, $T^{\pi}_{t,s} L^{\pi}_{s}$ is μ-continuous in $s \in [t,Z]$. Since also (7.1.16) and (7.2.4) imply that $T^{\pi}_{t,s} L^{\pi}_{s}$ is μ-bounded uniformly in $0 \leq t \leq s \leq Z$, the assumption 2.3.2 is guaranteed. Consequently, $\{V^{\pi}_{t} | t \in [0,Z]\}$ is μ-bounded. Together with $D_A = B^{\mu}$ and (7.1.17) this also implies assumption 2.3.3. Clearly, assumption 4.2.1 is implied by (7.2.4).

Further, expression (3.2.2) for $R_t(V,h)$, relation (7.2.6) with $s = t$, the μ-boundedness of $\{V_t | t \in [0,Z]\}$ and relation (7.1.19) imply that

$$\|R^{\pi}_{t}(V,h)\|_{\mu} h^{-1} \leq \tag{7.2.7}$$

$$\|h^{-1} \int_{t}^{t+h} T^{\pi}_{t,s} L^{\pi}_{s} ds - L^{\pi}_{t}\|_{\mu} +$$

$$\|h^{-1}([T^{\pi}_{t,t+h} - I] - hA^{\pi(t)}) V^{\pi}_{t+h}\|_{\mu} \leq hC.$$

Consequently, the smoothness assumption 6.4.1 is satisfied. Furthermore, lemma 5.5 together with the relations (7.2.2) and (7.2.7) guarantees the order of convergence O(h). □

Note that analogous to (7.1.14) one can give Lipschitz conditions on L with respect to γ, and π with respect to t, which guarantee (7.2.5). For the case of a stationary control π and bounded cost-rates we present a special approximation result. Namely, a rate of convergence which is linear in the length Z of the time interval, for Z larger than 1.

APPLICATION 7.2.6.

Let π be a stationary control i.e.; $\pi(t) = \delta$ for all $t > 0$ and some $\delta \in \Delta$. Further, assume that for some constant L :

$$(7.2.8) \qquad \|L^{\delta}\|_{\infty} \le L \quad \text{for all } \delta \in \Delta.$$

Since $q_t^{\pi} = q^{\delta}$; $H_t^{\pi} = H^{\delta}$ for all $t \ge 0$, one easily shows that the transition probabilities $\{P^{\pi}_{s,t} \mid s,t \ge 0\}$ defined by (7.1.3) are time-homogeneous, i.e.: $P^{\pi}_{s_1,s_1+t} = P^{\pi}_{s_2,s_2+t}$ for all s_1, s_2 and t.

Write: $\qquad P_t^{\delta} = P^{\pi}_{0,0+t};\ T_t^{\delta} = T^{\pi}_{0,0+t};\ t \ge 0,$

and choose $\mu(x) = 1$, $x \in S$. Hence $\|f\|_{\mu} = \|f\|_{\infty}$ for $f \in B^{\mu}$, relation (7.1.4) holds with $K = 1$, and $\|T_t f\|_{\infty} \le \|f\|_{\infty}$, $t \ge 0$.
First of all V_t^{π} becomes

$$(7.2.9) \qquad V_t^{\pi} = \int_0^{Z-t} T_s^{\delta} L^{\delta}\, ds \quad ,$$

which implies immediately,

$$(7.2.10) \qquad \|V^{\pi}_{t+\Delta t} - V^{\pi}_t\|_{\infty} \le \int_{Z-t-\Delta t}^{Z-t} \|T_s^{\delta} L^{\delta}\|_{\infty}\, ds \le \Delta t L \quad ,$$

and together with (7.1.17) with K = 1 :

$$(7.2.11) \qquad \|A^{\delta}(V^{\pi}_{t+\Delta t} - V^{\pi}_t)\|_{\infty} \le 2Q\|V^{\pi}_{t+\Delta t} - V^{\pi}_t\|_{\infty} \le \Delta t 2Q L .$$

Further, by writing

$$(7.2.12) \qquad \|h^{-1}(V^{\pi}_{t-h} - V^{\pi}_t) - (L^{\delta} + A^{\delta}V^{\pi}_t)\|_{\infty} \le$$
$$\|h^{-1}\int_0^h T_s^{\delta} L^{\delta} ds - hL^{\delta}\|_{\infty} + \|(h^{-1}[T_h^{\delta} - I] - A^{\delta})V^{\pi}_t\|_{\infty} ,$$

the relations (7.1.18) and (7.1.19) imply that $\frac{d^-}{dt} V_t^{\pi} = (L^{\delta} + A^{\delta} V_t^{\pi})$.

Since also the relations (7.2.10) and (7.2.11) guarantee the continuity of V_t^π and $A^\delta V_t^\pi$ in t in supremum-norm and $V_Z^\pi = \bar{0}$, we can conclude (cf. 1.13 on p.38 of Dynkin (1965)):

$$(7.2.13) \qquad V_t^\pi = \int_t^Z (L^\delta + A^\delta V_s^\pi)\, ds.$$

Consequently, from (3.2.3) and (7.2.13):

$$(7.2.14) \qquad R_t^\pi(V,h) = \int_t^{t+h} (L^\delta + A^\delta V_s^\pi) ds - h(L^\delta + A^\delta V_{t+h}^\pi),$$

So that (7.2.11) and (7.2.14) yield:

$$(7.2.15) \qquad \|R_t^\pi(V,h)\|_\infty h^{-1} \le \sup_{s\in[t,t+h]} \|A^\delta(V_s^\pi - V_{t+h}^\pi)\|_\infty \le h2QL .$$

Hence, the 'smoothness' relation (6.4.2) is satisfied with order of convergence not depending on Z.

Next, note that the strong stability relation (6.2.4) is satisfied with $K_\Delta = 0$ and proceed analogously to the proof of theorem 6.5.2. Then, the stability condition holds with $K_C = 1$.
As a result, by using relations(5.23), (7.2.2), (7.2.10) and (7.2.15), and reconsidering the proof of the approximation lemma 5.3, one can conclude that

$$(7.2.16) \qquad \begin{cases} \|V_n^h - V_t^\pi\|_\infty \le h2QLZ + hLZ + hL \\ \text{with } n = [th^{-1}], \text{ uniformly in } t \le Z. \end{cases}$$

Since (7.2.16) holds uniformly in all stationary controls π, this approximation result can be useful in order to approximate 'optimal average cost functions'. □

§3. FINITE HORIZON OPTIMAL COST FUNCTION

In addition to assumptions 7.1.1 and 7.1.4 let be satisfied:

ASSUMPTION 7.2.7.

(i) Γ is compact.

(ii) $\Delta = \{\delta : S \to \Gamma \mid \delta \text{ measurable}\}$.

(iii) $q(x,\gamma)$ is continuous in $(x,\gamma) \in S \times \Gamma$.
$H(x,\gamma;.)$ is weakly continuous in $(x,\gamma) \in S \times \Gamma$.
$L(x,\gamma)$ is continuous in $(x,\gamma) \in S \times \Gamma$ and satisfies (7.2.4). □

This assumption enables us to prove the existence and the uniqueness of a solution of the optimality equation (2.3.4) with A^{δ} given by (7.1.1). Therefore,however, we first give some auxiliaries. First,recall expression (2.3.3) for the infimum operator J and let C^{μ} denote the subclass within B^{μ} of continuous functions. Then we have cf. lemma 1.4 on p.16 of Gihman and Skorohod (1979) :

LEMMA 7.2.8. $J = C^{\mu} \to C^{\mu}$

PROOF. Let $f \in C^{\mu}$. Then,according to (iii) of assumption 7.2.7,the function $g : S \times \Gamma \to \mathbb{R}$ defined by

$$g(x,\gamma) = L(x,\gamma) + q(x,\gamma) \int [f(y) - f(x)] H(x,\gamma;dy)$$

is continuous in $(x,\gamma) \in S \times \Gamma$. So that from (i) and (ii) of assumption 7.2.7.,

$$Jf(x) = \inf_{\delta \in \Delta} g(x,\delta(x)) = \min_{\gamma \in \Gamma} g(x,\gamma) = g(x,\gamma^{0})$$

for some $\gamma^{0} \in \Gamma$ and any fixed $x \in S$.
Next, let $x \in S$ and $\{x_n\} \subset S$ with $x_n \to x$ as $n \to \infty$. Then clearly

$$(7.2.17) \qquad Jf(x) = g(x,\gamma^{0}) = \lim_{n\to\infty} g(x_n,\gamma^{0}) \geq \limsup_{n\to\infty} Jf(x_n).$$

On the other hand, let $\{n_j\}$ and $\{\gamma_{n_j}\}$ be sequences such that $\liminf_{n\to\infty} Jf(x_n) = \lim_{j\to\infty} Jf(x_{n_j}) = \lim_{j\to\infty} g(x_{n_j},\gamma_{n_j})$, then the compactness of Γ implies the existence of $\gamma^* \in \Gamma$ such that for the metric d_Γ on Γ: $d_\Gamma(\gamma_{n_j}, \gamma^*) \to 0$ as $j \to \infty$. Hence, we obtain

$$(7.2.18) \qquad Jf(x) \le g(x,\gamma^*) = \lim_{j\to\infty} g(x_{n_j}, \gamma_{n_j}) = \lim_{n\to\infty}\inf Jf(x_n).$$

Combining (7.2.17) and (7.2.18) proves the continuity of $Jf(x)$ in $x \in S$. Further, according to expression (2.3.3) for Jf, (7.1.17) and (7.2.4):

$$(7.2.19) \qquad \|Jf\|_\mu \le L + 2\,QK\|f\|_\mu \ .$$

which completes the proof. □

PROPOSITION 7.2.9. *Assumption* 2.3.5 *holds i.e.; there exists a unique* μ-*bounded solution* $\{\emptyset_t | t \in [0,Z]\}$ *of the optimality equation* (2.3.4).

PROOF. Let $C^\mu[0,Z]$ be the set of functions $g:[0,Z] \to C^\mu$ which are μ-continuous in t. For $g \in C^\mu[0,Z]$ write $g_t = g(t)$, $t \le Z$. Let $g \in C^\mu[0,Z]$. Then,

$$(7.2.20) \qquad \|J(g_{t+\Delta t}) - J(g_t)\|_\mu \le \sup_{\delta\in\Delta} \|A^\delta(g_{t+\Delta t} - g_t)\|_\mu \le 2QK\|g_{t+\Delta t} - g_t\|_\mu$$

proves the μ-continuity in t of $J(g_t)$. Hence, by also taking lemma 7.2.8 into account we can define an operator
$B: C^\mu[0,Z] \to C^\mu[0,Z]$ by

$$(7.2.21) \qquad (Bg)_t = \int_t^Z J(g_s)ds \quad , t \le Z.$$

Then, for $g^1, g^2 \in C^\mu[0,Z]$ and any $t \le Z$:

$$(7.2.22) \qquad \sup_{t\le Z} \|(Bg^1)_t - (Bg^2)_t\|_\mu \le \sup_{t\le Z} \int_t^Z \sup_{\delta\in\Delta} \|A^\delta(g^1 - g^2)_s\|_\mu ds \le$$
$$\sup_{t\le Z} \int_t^Z 2QK\|(g^1 - g^2)_s\|_\mu ds \le (2QKZ) \sup_{t\le Z} \|(g^1 - g^2)_t\|_\mu.$$

Consequently, the operator B on $C^\mu[0,Z]$ is Lipschitz with respect to the norm $\|g\| = \sup_{t\le Z} \|g_t\|_\mu$ for $g \in C^\mu[0,Z]$.

The proof proceeds by the well-known method of successive approximation (also known as Picard-iteration), (cf. lemma 11.4 of Fleming and Rishel(1975)).

□

Further, according to (7.2.19):

$$(7.2.23)\qquad \|\phi_t\|_\mu \le \int_t^Z (L + 2QK\|\phi_s\|_\mu)\, ds$$

so that the Gronwall-Bellman inequality implies

$$(7.2.24)\qquad \|\phi_t\|_\mu \le ZL \exp(2QKZ) =: C_\phi.$$

Next consider the approximation-method given in subsection 6.5. Then the following theorem shows that the continuous-time optimal cost functions $\{\phi_t | t\in[0,Z]\}$ can be approximated by the discrete-time optimal cost functions $\{\phi_n^h | n = 0,1,2,\ldots\}$.

THEOREM 7.2.10. *The convergence relation* (6.5.2) *is satisfied with order of convergence* O(h).

PROOF. By virtue of theorem 6.5.2, lemma 7.2.1 and 7.2.2 relation (6.5.2) follows from verifying the smoothness assumption 6.5.1.
Lemma 7.2.8 guarantees assumption 2.3.4 with $D_A = C^\mu$ and proposition 7.2.9 shows assumption 2.3.5 also with $D_A = C^\mu$.
Further, lemma 7.2.8 together with the fact that $T_h^\delta = [hA_h^\delta + I] = [hA^\delta + I]$ yields assumption 4.2.3 with $F = C^\mu$.
To prove (6.5.1) first conclude from (7.2.19) and (7.2.24):

$$(7.2.25)\qquad \|\phi_{t+\Delta t} - \phi_t\|_\mu \le \left\|\int_t^{t+\Delta t} J(\phi_s)\, ds\right\|_\mu \le \Delta t(L+2QKC_\phi)$$

Then, from expression (3.3.1) for $R_t(\phi,h)$, the fact that $\|J(\phi_{t+\Delta t}) - J(\phi_t)\|_\mu \le 2QK\|\phi_{t+\Delta t} - \phi_t\|_\mu$ and relation (7.2.25):

$$(7.2.26)\qquad \|R_t(\phi,h)\|_\mu h^{-1} \le hC.$$

Hence, the 'smoothness' relation (6.5.1) is satisfied. Furthermore, lemma 5.5 together with the relations (7.2.2) and (7.2.26) guaranteesthe order of convergence O(h). □

REMARK 7.2.11.

1. Clearly, the constant C in (7.2.26) can be given explicitly by using (7.2.24) and (7.2.25). Together with the consistency shown by (7.2.2)

and the stability with constant K_C = exp ZQ(K-1), relation (7.2.26) in turn may yield a precise rate of convergence in (6.5.2).
E pecially this latter fact is of interest since the functions $\emptyset_j^h$ can be obtained recursively and provide a corresponding discrete-time optimal control by applying dynamic programming. (see lemma 7.2.12 below).
For h sufficiently small this discrete-time optimal control is an ε-optimal control for the continuous time model if applied to the discrete-time model.

2. The finite horizon optimality equation (2.3.4) for Markov jump processes has been studied by several authors. The existence and uniqueness of a solution is well-known. First of all, the case of a finite state and action space is analyzed by Miller (1968). Pliska (1975) considers a general state space and compact action set and requires somewhat stronger continuity conditions as given by assumption 7.2.7 as well as a convexity condition on the set of decision rules.

Gihman and Skorohod (1979) also deal with a general state space, assume a compact decision set and use the continuity conditions of assumption 7.2.7. All these references concern bounded cost rates i.e.; $\|L^\delta\|_\infty \leq C$ uniformly in $\delta \in \Delta$. Yushkevich (1980) extends the above models to general state and action spaces as well as unbounded cost rates.
Moreover, he relaxes the continuity conditions to measurability assumptions. The cost rates, however, are assumed to be non-negative and the cost functions to be finite for any initial state and admissible control. Proposition 7.2.9 partially extends his results in that μ-bounded functions can be dealt with. Moreover, its proof is constructive.

3. The 'optimality' of the solution of the optimality equation has been shown by the above mentioned references as well as by Rishel (1976) and Boel and Varaiya (1977).
Miller (1968) proves the optimality within the class of all piecewise-constant controls. Pliska (1975) considers all Markov controls. Rishel (1976), Boel and Varaiya (1977), Gihman and Skorohod (1979) as well as Yushkevich (1980) also include history dependent controls.

4. The existence of optimal Markov controls i.e.; with corresponding cost function satisfying the optimality equation is well-known for the case of bounded cost rates and under the continuity conditions on the

jump characteristics. See Pliska (1975) and Gihman and Skorohod (1979). Under the weaker conditions, Yushkevich (1980) shows the existence of ε-optimal Markov or optimal Markov controls in several specific situations (see theorem 6.2, 7.1, 8.1 of this reference).
We remark that in our setting, where the cost-rates are allowed to be unbounded, if bounded by the μ-norm, but where the continuity conditions on q and H are made, the existence of an optimal Markov control can be shown analogously to Pliska (1975) or Gihman and Skorohod (1979).
In the next paragraph we will concentrate on constructing ε-optimal Markov controls.

§4. CONSTRUCTION OF ε-OPTIMAL MARKOV CONTROLS.

With the method of time-discretization we can construct ε-optimal Markov controls as follows. First, by using dynamic programming we can obtain an h-Markov control which is optimal for the h-discrete-time model. This control is implemented in the continuous-time model as a control, say π, which is stationary on the intervals [nh, nh+h). Let V^{π} denote the corresponding cost function. Then, showing that the discrete-time and continuous-time cost function under that control are equal up to an order O(h) and using the approximation result of §3 also imply that V^{π} approximates the optimal cost function $\emptyset$ with order O(h).

As in §3, let the assumptions 7.1.1, 7.1.4 as well as 7.2.7 be satisfied. Recall the verification of assumption 4.2.3 with $F = C^{\mu}$ as well as the existence of a unique μ-bounded solution $\{\emptyset_t \mid t \in [0,Z]\} \subset C^{\mu}$ of (2.3.4) and $\{\emptyset_j^h \mid jh \le Z,\ j \in \mathbb{N}\} \subset C^{\mu}$ of (4.2.8).
Fix $h \le h_0$ and let $\ell = [Zh^{-1}]$.

<u>LEMMA 7.2.12</u>. *There exist* $\delta(0), \delta(1), \ldots, \delta(\ell-1) \in \Delta$ *such that*

$$\emptyset_j^h = \inf_{\delta \in \Delta} [hL^{\delta} + T_h^{\delta} \emptyset_{j+1}^h] = hL^{\delta(j)} + T_h^{\delta(j)} \emptyset_{j+1}^h \quad , \ j < \ell. \tag{7.2.27}$$

<u>PROOF</u>. Consider some $j < \ell$. Then first conclude from the continuity of $\emptyset_{j+1}^h(x)$ in x and of q, H and L in (x,γ) that the function g defined by

$$g(x,\gamma) = hL(x,\gamma) + hq(x,\gamma) \int \emptyset_{j+1}^h(y) H(x,\gamma;dy) + [1-hq(x,\gamma)] \emptyset_{j+1}^h(x) \quad ,$$

is continuous in $(x,\gamma)\in S\times\Gamma$. Next, recall the compactness of Γ. Then, lemma 1.4 on p.16 of Gihman and Skorohod (1979) guarantees the existence of a measurable function (selection) $\delta^0: S\to\Gamma$ such that

$$\inf_{\delta\in\Delta} g(x,\delta(x)) = \inf_{\gamma\in\Gamma} g(x,\gamma) = g(x,\delta^0(x)) \qquad , \quad x\in S.$$

Let $\delta(j) = \delta^0$, then (7.2.1) for $P_h^{\delta(j)}$ and (4.1.5) for $T_h^{\delta(j)}$ yield the proof. □

Let $\pi^h\in\Pi^h(M)$ such that $\pi^h = (\delta(0),\delta(1),\ldots,\delta(\ell-1),\delta(\ell-1),\delta(\ell-1),\ldots)$ with $\delta(i), i=1,\ldots,\ell-1$ given by lemma 7.2.12. According to lemma 7.2.2 we have: $\pi^h\in\Pi^h(AB)$.

From (4.2.5) and (7.2.27) it follows that

$$(7.2.28) \qquad V_j^h = \phi_j^h \qquad , \quad j\le\ell.$$

Next, let $\pi\in\Pi(M)$ be defined by

$$(7.2.29) \qquad \pi(t) = \pi^h(nh), \qquad t\in[nh,nh+h), \quad n\in\mathbb{N}.$$

According to theorem 7.1.3, there exists a unique semigroup $\{P^\pi_{s,t}\,|\,s,t\ge 0\}$ of transition probabilities given by (7.1.3) with

$$q_t^\pi = q^{\delta(n)} \;;\; H_t^\pi = H^{\delta(n)} \quad \text{if} \quad t\in[nh,nh+h), \quad t\le Z.$$

Consequently, by reconsidering the proof of lemma 7.1.5, but only with $t = nh, nh\le Z, n\in\mathbb{N}$, we conclude that (7.1.7) still holds for $t = nh, nh\le Z$, $n\in\mathbb{N}$. Hence, the relations (7.1.18) and (7.1.19) remain valid with $t = nh, \Delta t\le h$.

These facts imply that relation (7.2.6) is true for any $t\le Z$ but with $s = jh$, $j\in\mathbb{N}$ and $\Delta s\le h$. This shows that $T^\pi_{t,s}L^\pi_s$ is μ-continuous in $s\in[jh,jh+h)$, $j\in\mathbb{N}$, $jh\le Z$, so that V^π_t is well-defined by (2.3.2) for any $t\le Z$. Further, it is easily seen that (3.2.1), (3.2.2) and (3.2.3) remain valid. Moreover, analogously to (7.2.7) it is shown that

$$(7.2.30) \qquad \|R^\pi_{nh}(V,h)\|_\mu h^{-1} \le hC \quad , \quad nh\le Z, \; n\in\mathbb{N}.$$

THEOREM 7.2.13. *For all* $t \leq Z$ *and some constant* C:

(7.2.31) $\|V_t^\pi - \phi_t\|_\mu \leq hC.$

PROOF. By virtue of theorem 7.2.10 and equality (7.2.28) it suffices to prove for all $t \leq Z$, with $n = [th^{-1}]$ and C some constant:

(7.2.32) $\|V_n^h - V_t^\pi\|_\mu \leq hC$

Writing

(7.2.33) $V_{nh}^\pi - V_t^\pi = \int_{nh}^{t} T_{nh,s}^\pi L_s^\pi \, ds + [T_{nh,t}^\pi - I]V_t^\pi$

and using the relations (7.1.16) and (7.1.18) one easily concludes

(7.2.34) $\|V_{nh}^\pi - V_t^\pi\|_\mu \leq hC,$

so that the proof is completed by showing

(7.2.35) $\|V_n^h - V_{nh}^\pi\|_\mu \leq hC.$

From (3.2.3), (4.2.5) and $A^\delta = A_h^\delta = [T_h^\delta - I]h^{-1}$:

(7.2.36) $\begin{cases} V_j^h = hL^{\delta(j)} + T_h^{\delta(j)} V_{j+1}^h \\ V_{jh}^\pi = hL^{\delta(j)} + T_h^{\delta(j)} V_{jh+h}^\pi + R_{jh}^\pi(V,h). \end{cases}$

Write: $\delta_j = V_j^h - V_{jh}^\pi$. Then (7.2.3), (7.2.30) and (7.2.36) imply

(7.2.37) $\|\delta_j\|_\mu \leq (1+hC)\|\delta_{j+1}\|_\mu + h^2C$.

Iterating (7.2.37) for $j = n, n+1, \ldots, \ell-1$ and using (7.2.32) with $t = Z$, yields (7.2.35). □

8. CONTROLLED STOCHASTIC DIFFERENTIAL EQUATIONS (DIFFUSIONS)

8.1. CONTINUOUS TIME MODEL

This section is concerned with the solutions of controlled stochastic differential equations, also known as controlled diffusion processes. As example of such a process consider an investment fund. The owner (controller) of the fund can continuously control the fund by choosing an investment opportunity from an available set. An investement opportunity is characterized by a pair (γ_1,γ_2), where γ_1 denotes the rate of return (profit) per dollar invested and γ_2 is the value of risk given by the variance per dollar invested. Let the state variable of the process denote the value of the fund. Then, given that at time-point t the state is x and that during $[t,t+\Delta t]$ one and the same investment opportunity with associated pair (γ_1,γ_2) is chosen, the state at time $t+\Delta t$ is given by the random variable $X_{t+\Delta t}$ satisfying

$$X_{t+\Delta t} = x + x\gamma_1 \Delta t + x\sqrt{\gamma_2}\, W_{\Delta t} \quad ,$$

where $W_{\Delta t}$ is a stochastic increment, due to risk, which has a normal distribution with mean 0 and variance Δt. Costs are involved by means of a cost rate function depending on the actual value of the fund as well as the rate of return and value of risk.

To proceed formally, let $S = \mathbb{R}$, $\Gamma = \Gamma_1 \times \Gamma_2 \subset \mathbb{R}^2$ and for $\delta : \mathbb{R} \to \Gamma$ and $x \in S$ write $\delta(x) = (\delta_1(x),\delta_2(x))$. Then, in this section, we consider a control object $(\mathbb{R}, \Gamma,\Delta,\mu,D_A,\{A^\delta|\delta\in\Delta\},L)$ as well as
$a : \mathbb{R} \times \Gamma_1 \to \mathbb{R}$ a measurable function, called *drift function*, and
$b : \mathbb{R} \times \Gamma_2 \to \mathbb{R}$ a measurable function, called *diffusion function*,
such that for all $\delta\in\Delta$, $f \in D_A$ and $x \in S$:

$$A^\delta f(x) = a(x,\delta_1(x))\frac{d}{dx} f(x) + \frac{1}{2} b^2(x,\delta_2(x)) \frac{d^2}{dx^2} f(x). \tag{8.1.1}$$

As will be shown below, for a sufficiently smooth control $\pi\in\Pi(M)$ there exists a controlled Markov process on $[0,Z]$, denoted by $(\eta_t^\pi)_{t\leq Z}$, which corresponds to the above control object and is given by the *stochastic differential equation*:

(8.1.2) $\qquad d\eta_t^\pi = a_t^\pi(\eta_t)\ dt + b_t^\pi(\eta_t^\pi)\ dW_t \qquad , \quad t \le Z$

where a_t^π, b_t^π are given by notation 8.1.1 below. As in subsection 5.3 of chapter I, we refer to the books of Gihman and Skorohod (1972) or Arnold (1975) for the definitions, properties and techniques of stochastic differential and integral equations with respect to the Wiener measure $(W_t)_{t \ge 0}$. Particularly, detailed studies of controlled stochastic differential equations can be found in Fleming and Rishel (1975), Gihman and Skorohod (1979) and Krylov (1980) as well as many others.

Specifications on the functions a and b, the bounding function μ, the domain D_A and the cost-rate function L will follow. First, let us present some notation before analyzing the admissibility of a control.

NOTATION 8.1.1.

$$a^\delta(x) = a(x,\delta_1(x)); b^\delta(x) = b(x,\delta_2(x)) \qquad \text{for all } \delta\in\Delta,\ x\in S.$$

$$a_t^\pi = a^\delta\ ;\ b_t^\pi = b^\delta \qquad \text{for } \pi\in\Pi(M),\ t \ge 0 \text{ and } \delta = \pi(t). \qquad \square$$

THEOREM 8.1.2. *Let* $\pi\in\Pi(M)$ *be such that the following conditions hold:*

(i) *For some constant* L, *all* $x,y\in\mathbb{R}$ *and* $t+\Delta t \le Z$:

(8.1.3) $\qquad |a_{t+\Delta t}^\pi(x) - a_t^\pi(y)| + |b_{t+\Delta t}^\pi(x) - b_t^\pi(y)| \le L(|x-y| + \Delta t).$

(ii) η_0^π *does not depend on* $(W_t)_{t\ge 0}$ *and* $\mathbb{E}\,[\eta_0^\pi]^2 < \infty$.

Then there exists a unique solution* $(\eta_t^\pi)_{t \le Z}$ *of* (8.1.2) *satisfying:*

(iii) *With probability one the function* η_t^π *is continuous in* t.

(iv) $\mathbb{E}\,[\eta_t^\pi]^2 \le C$ *for all* $t \le Z$ *and some constant* C.

* Here , the uniqueness holds in the sense of random elements on D[0,Z], but also in the stronger sense as given by theorem 5.3.1 in Chapter I .

PROOF. By defining $K := \max\{L, LZ + |a_o^\pi(o)| + |b_o^\pi(o)|\}$ one easily shows

$$(8.1.4)\quad \begin{cases} |a_t^\pi(x) - a_t^\pi(y)| + |b_t^\pi(x) - b_t^\pi(y)| \le K|x-y| \\ |a_t^\pi(x)| + |b_t^\pi(x)| \le K(1+|x|) \end{cases}$$

for all $x,y \subset \mathbb{R}$ and $t \le Z$. Further, recall that a_t^π and b_t^π are measurable in (t,x). Hence, the proof is given by theorem 1 on p.40 of Gihman and Skorohod (1972). □

In the rest of this subsection 8.1, let π satisfy (8.1.3) for some $Z > 0$. The Lipschitz and growth relation (8.1.4) will be used without explicitly mentioning it.
Particularly, after using a time-shift over time-s it directly follows from theorem 8.1.2 that for any $s,t \le Z$ and $x \in S$ there exists a unique solution $(\eta_{s,t}^\pi(x))_{s \le t \le Z}$ such that for all $s \le t \le Z$:

$$(8.1.5)\qquad \eta_{s,t}^\pi(x) = x + \int_s^t a_u^\pi(\eta_{s,u}^\pi(x))du + \int_s^t b_u^\pi(\eta_{s,u}^\pi(x))dW_u$$

These solutions enable us to present the following proposition.
Its proof is given by theorem 1 on p.67 of Gihman and Skorohod (1972).

PROPOSITION 8.1.3.

(i) *There exists a unique semigroup of transition probabilities* $\{P_{s,t}^\pi \mid s,t \le Z\}$ *such that for all* $s,t \le Z$ *and* $B \in \beta$:

$$(8.1.6)\qquad P_{s,t}^\pi(x;B) = \mathbb{P}(\eta_{s,t}^\pi(x) \in B) \qquad , x \in S.$$

(ii) *The unique solution of* (8.1.2) *given by theorem* 8.1.2 *is a Markov process with transition probabilities* $\{P_{s,t}^\pi | s,t \le Z\}$. □

Further, after using a time shift over -s again, the following lemma can be shown by applying theorem 4 on p.48 of Gihman and Skorohod (1972) together with Schwartz' inequality.

LEMMA 8.1.4. *For any* $m \in \mathbb{N}$ *there exist constants* $G(m)$ *and* $L(m)$ *depending only on* m, Z *and* K *such that for all* $0 \le s \le t \le Z$ *and* $x \in \mathbb{R}$:

$$(8.1.7) \qquad \mathbb{E}\,|\eta^{\pi}_{s,t}(x)|^m \leq (1+|x|^m)\; e^{(t-s)G(m)}$$

$$(8.1.8) \qquad \mathbb{E}\,|\eta^{\pi}_{s,t}(x)-x|^m \leq (1+|x|^m)(t-s)^{\frac{m}{2}}\, L(m)\; e^{(t-s)G(m)} \qquad \square$$

To proceed, let $\{P^{\pi}_{s,t} \mid s,t \leq Z\}$ be the transition probabilities given by (8.1.6) and fix $m \in \mathbb{N}$. The symbol C always denotes a constant depending only on m,Z,K and π.
In addition we recall the notation I.5.3.6 for $C^{3;m}$ as well as the constant K_f if $f \in G^{3;m}$. Further, let μ_m denote the bounding function given by definition I.5.3.9.

LEMMA 8.1.5. *For any* $f \in C^{3;m}$ *and* $t+h \leq Z$:

$$(8.1.9) \qquad \| [\int f(y) P^{\pi}_{t,t+h}(.;dy) - f(\cdot)]h^{-1} - A^{\pi(t)} f(\cdot) \|_{\mu_{m+3}} \leq \sqrt{h}\; CK_f$$

PROOF. Consider a fixed $f \in C^{3;m}$ and $t+h \leq Z$. Write for $s \geq 0$:
$\eta_s(x) = \eta^{\pi}_{t,t+s}(x)$, $a_s(\cdot) = a^{\pi}_{t+s}(\cdot)$, $b_s(\cdot) = b^{\pi}_{t+s}(\cdot)$ and $T_s f(x) = \mathbb{E}\, f(\eta_s(x))$.

Then by using the integral equation (8.1.5) as well as the growth and Lipschitz relation (8.1.7) and (8.1.8) respectively the proof can be given almost analogously to that of lemma I.5.3.11.
As only difference the time dependence of the functions a_s, b_s has to be taken into account. However, by using the Lipschitz relation (8.1.3), especially with respect to the time parameter, one easily verifies that this time dependence givesrise to extra terms of the form h^2C on the right-hand sides of I(5.3.21), (5.3.22), (5.3.23) and (5.3.26). $\square$

THEOREM 8.1.6. *Let* $\pi \in \Pi(M)$ *such that relation* (8.1.3) *is satisfied. Consider the control object with* $\mu = \mu_{m+3}$ *and* $D_A = C^{3;m}$. *Then,*
(i) π *is an admissible control for the control object, and*
(ii) π *satisfies the boundedness relation* (2.1.2).
Consequently, $\pi \in \Pi(AB)$.

PROOF. (i) Directly from definition 2.1.4, theorem 8.1.3 and lemma 8.1.5.
(ii) By using the growth relation (8.1.7). $\square$

REMARKS 8.1.7.

1. The Lipschitz relation (8.1.3) is quite strong. However, as in remark 7.1.7, it may be noted that, in view of proving the admissibility of a control as well as of obtaining approximation results later on, some relaxations, especially with respect to the Lipschitz condition the time-parameter, can possibly be made.
Particularly, relaxations to piecewise Lipschitz conditions are very useful for applications. A specific relaxation of this type will be made in §4 of subsection 8.2.
However, we have not dealt with more relaxations and prefer to give strong conditions in order to avoid too much complexity as well as to show how to obtain rates of convergence.

2. The Lipschitz relation (8.1.3) is satisfied if the functions a and b are Lipschitz i.e.; for some constant G, all $x,y \in \mathbb{R}$, $\gamma_1,\gamma_2 \in \Gamma_1, \eta_1,\eta_2 \in \Gamma_2$:

$$(8.1.10) \quad \begin{cases} |a(x,\gamma_1) - a(y,\gamma_2)| + |b(x,\eta_1) - b(y,\eta_2)| \leq \\ G(|x-y| + |\gamma_1 - \gamma_2| + |\eta_1 - \eta_2|) \end{cases}$$

and if in addition one of the following two conditions hold:

(i) π is stationary with a Lipschitz decision rule i.e.; for some $\delta \in \Delta$ and all $t \geq 0$: $\pi(t) = \delta$ and for all $x,y \in \mathbb{R}$ and some constant G :

$$(8.1.11) \quad |\delta_1(x) - \delta_1(y)| + |\delta_2(x) - \delta_2(y)| \leq G|x-y|.$$

(ii) The control π satisfies for all $t \leq Z$ and with $\delta = \pi(t)$ the Lipschitz relation (8.1.11) (with constant G uniformly in all $t \leq Z$) as well as with $d_\Gamma((\gamma_1,\eta_1),(\gamma_2,\eta_2)) := |\gamma_1-\gamma_2| + |\eta_1-\eta_2|$:

$$(8.1.12) \quad \|d_\Gamma(\pi_{t+\Delta t}(.),\pi_t(.))\|_\infty \leq \Delta t C .$$

(Note that (i) is included by (ii)).
Especially the conditions (8.1.11) and (8.1.12) will not always be satisfied in realistic models. As in 1 of this remark, however, extensions to piecewise Lipschitz conditions seems to be worthwhile in this respect. □

In view of theorem 8.1.6, the notations and results of section 2 can be applied with $\mu = \mu_{m+3}$ and $D_A = C^{3;m}$. Recall expression (2.2.1) for $T^{\pi}_{s,t}f$.
Then from relation (2.2.4), expression (8.1.1) for A^{δ}, the growth condition of (8.1.4) and the inequality (8.1.9) one easily verifies for all $f \in C^{3;m}$, $s,t \le Z$, $t+\Delta t \le Z$:

$$(8.1.13) \qquad \|T^{\pi}_{s,t}\, f\|_{\mu_{m+3}} \le M^{\pi}\|f\|_{\mu_{m+3}} ,$$

$$(8.1.14) \qquad \|A^{\pi(t)} f\|_{\mu_{m+3}} \le CK_f ,$$

$$(8.1.15) \qquad \|[T^{\pi}_{t,t+\Delta t} - I]f\|_{\mu_{m+3}} \le \Delta t CK_f ,$$

$$(8.1.16) \qquad \|([T^{\pi}_{t,t+\Delta t} - I](\Delta t)^{-1} - A^{\pi(t)})f\|_{\mu_{m+3}} \le \sqrt{\Delta t}\, CK_f .$$

Let us conclude this subsection by presenting, analogously to proposition I.5.3.8, a result on the differentiability of $T^{\pi}_{s,t}f(x)$ with respect to its initial data x. Therefore, however, as in chapter I we make the following additional assumption on a^{π} and b^{π}.

<u>ASSUMPTION 8.1.8</u>. For constant K and all $x \in S$, $t \le Z$ it holds that

$$(8.1.17) \quad \begin{cases} \frac{\partial^k}{\partial x^k} a^{\pi}_t(x) \text{ and } \frac{\partial^k}{\partial x^k} b^{\pi}_t(x) \text{ exist, are continuous in } (t,x) \text{ and} \\ |\frac{\partial^k}{\partial x^k} a^{\pi}_t(x)| + |\frac{\partial^k}{\partial x^k} b^{\pi}_t(x)| \le K , \quad k = 1,2,3. \end{cases}$$ □

<u>PROPOSITION 8.1.9</u>. *Let* $f \in C^{3;m}$ *and define for* $0 \le s \le t \le Z$ *and all* $x \in S$: $g_{s,t}(x) = T^{\pi}_{s,t}f(x)$. *Then* $g_{s,t} \in C^{3;m}$ *and* $K_{g_{s,t}} \le CK_f$.

<u>PROOF</u>. By using the same notation as given in the proof of lemma 8.1.5 and noting that the growth relation in (8.1.4) as well as the boundedness condition in (8.1.17) hold uniformly in $t \le Z$, the proof can be given analogously to that of proposition I.5.3.8. □

<u>NOTATION 8.1.10.</u> Let $C^{3;m}[0,Z]$ be the set of collections $\{g_t \mid t \in [0,Z]\}$ such that for some constant, denoted by $K_g[0,Z]$, and all t: $g_t \in C^{3;m}$ and $K_{g_t} \le K_g[0,Z]$. □

8.2. APPROXIMATIONS

Take $h_0 > 0$ and let for any $h \le h_0$: $\{P_h^\delta | \delta \in \Delta\}$ be a collection of transition probabilities defined by

$$(8.2.1) \qquad P_h^\delta(x;\{y\}) = \begin{cases} \frac{1}{2} & \text{for } y = x + a^\delta(x)h + b^\delta(x)\sqrt{h} \\ \frac{1}{2} & \text{for } y = x + a^\delta(x)h - b^\delta(x)\sqrt{h} \end{cases}$$

for all $x \in S$ and $\delta \in \Delta$.
Below we examine the discrete-time approximation of

- transition probabilities in §1, and
- finite horizon cost functions in §2.

Therefore, let $\pi \in \Pi(M)$ be a fixed control and suppose that the *Lipschitz relation* (8.1.3) as well as *assumption* 8.1.8 are satisfied. Then, for any $h \le h_0$ we consider the h-control object with $\mu = \mu_{m+3}$ and P_h^δ given by (8.2.1) as well as the h-Markov control : $\pi^h = (\pi(0), \pi(h), \pi(2h), \ldots)$.

According to lemma 8.2.2 below, relation (4.1.4), which justifies the use of the operators T_h^δ and A_h^δ given by (4.1.5) and (4.1.6), is guaranteed for any $\delta \in (\pi(0), \pi(h), \pi(2h), \ldots, \pi(\ell h))$, $\ell = [Zh^{-1}]$. First, recall that $|a_t^\pi(x)| + |b_t^\pi(x)| \le K(1 + |x|)$, uniformly in $t \le Z$. Then, by using this growth relation, the following two lemmas can be shown analogously to the proof of lemma I.5.3.12 and lemma I.5.3.13 respectively. Let $\ell = [Zh^{-1}]$.

<u>LEMMA 8.2.1</u>. *The consistency relation* (6.2.1) *with* $\mu = \mu_{m+3}$ *is satisfied for any collection* $\{U_t | t \in [0,Z]\} \in C^{3;m}[0,Z]$ *and such that for all* $h \le h_0$, $j < \ell$:

$$(8.2.2) \qquad \| (A_h^{\pi(jh)} - A^{\pi(jh)}) \, U_{jh+h} \|_{\mu_{m+3}} \le \sqrt{h} C K_U [0,Z]. \qquad \square$$

<u>LEMMA 8.2.2</u>. *The stability relation* (6.2.3) *is satisfied with* $\mu = \mu_{m+3}$. □

Note that (also see remark 6.2.1) lemma 8.2.2 implies:
$\pi^h = (\pi(0), \pi(h), \pi(2h), \ldots) \in \Pi^h(AB)$.

§1. TRANSITION PROBABILITIES

<u>THEOREM 8.2.3</u>. *For any* $f \in C^{3;m}$ *the convergence relation* (6.3.2) *is satisfied with order of convergence* $O(\sqrt{h})$.

<u>PROOF</u>. Let $f \in C^{3;m}$ and first conclude from proposition 8.1.9 that

$$(8.2.3) \qquad \{T^{\pi}_{t,Z} f \mid t \in [0,Z]\} \in C^{3;m}[0,Z] \text{ and } K_{T^{\pi}_{t,Z} f} \leq CK_f, \; t \leq Z.$$

Consequently, by virtue of theorem 6.3.2, lemma 8.2.1 and lemma 8.2.2, relation (6.3.2) is proven by verifying the smoothness assumption 6.3.1.

Condition (i) of assumption 2.3.1 with $D_A = C^{3;m}$ directly follows from (8.2.3). Condition (ii) of assumption 2.3.1 with $\mu = \mu_{m+3}$ is implied by (8.1.14) together with (8.2.3) again.

The 'smoothness' relation (6.3.1) is satisfied since (8.1.16) and (8.2.3) yield:

$$(8.2.4) \qquad \|R^{\pi}_t(T,f,h)\|_{\mu_{m+3}} h^{-1} =$$
$$\|(h^{-1}[T^{\pi}_{t,t+h} - I] - A^{\pi}(t))T^{\pi}_{t+h,Z} f\|_{\mu_{m+3}} \leq \sqrt{h}\, CK_f \; .$$

Hence, the smoothness assumption 6.3.1 is guaranteed. Furthermore, lemma 5.5 together with the relations (8.2.2) and (8.2.4) implies the order of convergence $O(\sqrt{h})$. □

In analogy with subsection 5.3 of chapter I, also weak convergence of the transition probabilities and discrete-time processes can be concluded by making use of lemma I.5.3.15 and an analogue of lemma I.5.3.19.

§2. FINITE HORIZON COST FUNCTION

<u>ASSUMPTION 8.2.4</u>.

$$(8.2.5) \qquad \{L^{\pi}_t \mid t \in [0,Z]\} \in C^{3;m}[0,Z] \; .$$

$$(8.2.6) \qquad \|L^{\pi}_{t+\Delta t} - L^{\pi}_t\|_{\mu_m} \leq \Delta t C \quad , \quad t+\Delta t \leq Z.$$

(8.2.7) $\frac{\partial^k}{\partial x^k} L_t^\pi(x)$ is continuous in (t,x) for $k = 1,2,3$. □

More specific conditions on L^π and π guaranteeing assumption 8.2.4 will be given in remark 8.2.6 below. First, let us present the approximation result.

THEOREM 8.2.5. *Let assumption* 8.2.4 *be satisfied. Then the convergence relation* (6.4.2) *holds with order of convergence* $O(\sqrt{h})$.

PROOF. First let us verify the smoothness assumption 6.4.1.
By using (8.1.13), (8.1.15), (8.2.5), (8.2.6) and the fact that $\|g\|_{\mu_{m+3}} \le 2\|g\|_{\mu_m}$ if $g \in B^{\mu_m}$, we conclude for all $o \le t \le s \le s+\Delta s \le Z$:

$$(8.2.8) \qquad \|T^\pi_{t,s+\Delta s} L^\pi_{s+\Delta s} - T^\pi_{t,s} L^\pi_s\|_{\mu_{m+3}} \le$$

$$\|T^\pi_{t,s+\Delta s}(L^\pi_{s+\Delta s} - L^\pi_s)\|_{\mu_{m+3}} + \|T^\pi_{t,s}(T^\pi_{s,s+\Delta s} - I)L^\pi_s\|_{\mu_{m+3}} \le \Delta s C.$$

Hence, $T^\pi_{t,s} L^\pi_s$ is μ_{m+3}-continuous in $s \in [t,Z]$ for any $t \le Z$.
Further, relation (8.2.5) implies the μ_m-and hence μ_{m+3}-boundedness of $\{L^\pi_t | t \in [0,Z]\}$. So that together with (8.1.13) we also conclude the μ_{m+3}-boundedness of $\{T^\pi_{t,s} L^\pi_s | s,t \in [0,Z]\}$, which completes the verification of assumption 2.3.2.
Next, we will show:

$$(8.2.9) \qquad \{V^\pi_t | t \in [0,Z]\} \in C^{3;m}[0,Z].$$

Therefore, first of all by using proposition 8.1.9 and the mean value theorem, conclude that for all $f \in C^{3;m}$, $x \in S$, $0 \le t \le s \le Z$ and with $|\Delta x| \le 1$:

$$(8.2.10) \qquad |(\frac{\partial^k}{\partial y^k} T^\pi_{t,s} f(y))_{y = x+\Delta x} - (\frac{\partial^k}{\partial y^k} T^\pi_{t,s} f(y))_{y=x}| \; |\Delta x|^{-1} \le (1+|x|^m) C K_f$$

where C depends only on Z, K and m. Relation (8.2.10) together with (8.2.5), which implies the boundedness of $K_{L^\pi_t}$ uniformly in $t \le Z$, and Lebesque's dominated convergence theorem will imply for $K = 1,2,3$, $x \in S$ and $t \le Z$:

$$(8.2.11) \qquad (\frac{\partial^k}{\partial x^k} V^\pi_t(x)) = \int_t^Z (\frac{\partial^k}{\partial x^k} T^\pi_{t,s} L^\pi_s(x)) \, ds$$

if in addition the integrals in the right hand side of (8.2.11) are well-defined as Lebesque-integral. However, by using the continuity in (t,x) of $a_t^\pi(x)$, $b_t^\pi(x)$ and $L_t^\pi(x)$, as resulting from (8.1.3) and (8.2.7), the continuity of the integrands in s for fixed t and x can be shown analogously to p.61/62 of Gihman and Skorohod (1972). Consequently, (8.2.11) holds for k = 1,2,3 all $t\in Z$ and $x\in S$.

Finally, (8.2.3), (8.2.5) and (8.2.11) yield (8.2.9).

This proves condition (i) of assumption 2.3.3 with $D_A = C^{3;m}$,
and together with (8.1.14) also condition (ii) of assumption 2.3.3 is shown.

Further, expression (3.2.2) for $R_t^\pi(V,h)$, relation (8.2.8) with $s=t$, (8.2.9) and (8.1.16) imply that

$$(8.2.12)\qquad \|R_t^\pi(V,h)\|_{\mu_{m+3}}\, h^{-1} \le \|h^{-1}\int_t^{t+h} T_{t,s}^\pi L_s^\pi\, ds - L_t^\pi\|_{\mu_{m+3}} + \|h^{-1}\,([T_{t,t+h}^\pi - I] - hA^{\pi(t)})V_{t+h}^\pi\|_{\mu_{m+3}} \le \sqrt{h}C.$$

Consequently, the smoothness assumption 6.4.1 is satisfied.

As a result, by virtue of theorem 6.4.2, lemma 8.2.1 together with (8.2.9) and lemma 8.2.2, the convergence relation (6.4.2) holds. Furthermore, lemma 5.5 together with (8.2.2) and (8.2.12) shows the order of convergence $O(\sqrt{h})$.

□

REMARK 8.2.6.

1. It is not difficult to verify that assumption 8.2.4 is satisfied if the following conditions are guaranteed:

(i) π satisfies (8.1.12) as well as for any $t \le Z$ and with $\delta = \pi(t)$ relation (8.1.11) with constant G uniformly in all $t \le Z$.

(ii) For all $x\in S$, (γ_1,η_1), $(\gamma_2,\eta_2)\in\Gamma$.

$$(8.2.13)\qquad |L(x,\gamma_1,\eta_1) - L(x,\gamma_2,\eta_2)| \le C(|\gamma_1-\gamma_2| + |\eta_1-\eta_2|)\;[1+|x|^m]\;.$$

(iii) For k = 1,2,3 , i = 1,2,3 :

$$(8.2.14)\quad \begin{cases} \dfrac{\partial^k}{\partial x_i^k} L(x_1,x_2,x_3) \text{ exists, is continuous in } (x_1,x_2,x_3) \text{ and} \\ \left|\dfrac{\partial^k}{\partial x_i^k} L(x_1,x_2,x_3)\right| \le C\,[1+|x_1|^m + |x_2|^m + |x_3|^m],\ (x_1,x_2,x_3)\in\mathbb{R}\times\Gamma_1\times\Gamma_2. \end{cases}$$

2. Note that the convergence in (6.4.2) is concluded uniformly within the class of controls for which the Lipschitz constant L in (8.1.3) and C in (8.2.6) as well as the constant $K_L \pi[0,Z]$ corresponding to (8.2.5) are uniformly bounded. □

§ 3. FINITE HORIZON OPTIMAL COST FUNCTION

In stead of assuming a fixed control and corresponding smoothness conditions as in §1 and §2 , in this paragraph the following assumptions are made:

ASSUMPTION 8.2.7.

(i) $\Gamma_1 \subset \mathbb{R}$ and $\Gamma_2 \subset \mathbb{R}$ are compact.

(ii) $\Delta = \{\delta : \mathbb{R} \to \Gamma \mid \delta \text{ measurable}\}$.

(iii) The function $L : \mathbb{R} \times \Gamma_1 \times \Gamma_2 \to \mathbb{R}$ satisfies (8.2.14).

(iv) The functions $a : \mathbb{R} \times \Gamma_1 \to \mathbb{R}$ and $b : \mathbb{R} \times \Gamma_2 \to \mathbb{R}$ are continuous and satisfy for some constant G, all $x,y \in \mathbb{R}$ and $(\gamma_1,\gamma_2) \in \Gamma$ the Lipschitz condition:

(8.2.15) $$|a(x,\gamma_1) - a(y,\gamma_1)| + |b(x,\gamma_2) - b(y,\gamma_2)| \le G|x-y|.$$ □

ASSUMPTION 8.2.8. There exists a unique collection $\{\phi_t \mid t \in [0,Z]\}$ satisfying (2.3.4) such that

(8.2.16) $$\{\phi_t \mid t \in [0,Z]\} \in C^{3;m}[0,Z] ,$$

as well as for all $o \le s \le s+\Delta s \le Z$:

(8.2.17) $$\left\|\frac{\partial}{\partial x} \phi_{s+\Delta s} - \frac{\partial}{\partial x} \phi_s\right\|_{\mu_m} + \left\|\frac{\partial^2}{\partial x^2} \phi_{s+\Delta s} - \frac{\partial^2}{\partial x^2} \phi_s\right\|_{\mu_m} \le \sqrt{\Delta s}\, C.$$ □

REMARK 8.2.9. We note that assumption 8.2.8 will not be satisfied in general. References and more detailed statements in this respect are given in remark 8.2.13 below. Further, assumption 8.2.8 is only likely to be satisfied under stronger conditions on a and b, such as sufficient differentiability and strictly positiveness of b. In our approach,we let assumption 8.2.7 contain conditions only for the approximation analysis, whereas assumption 8.2.8 guarantees the existence of the optimal cost function. □

In this §3, we will also consider for any $h \leq h_0$ an h-control object with $\mu = \mu_{m+3}$ and P_h^δ given by (8.2.1). According to lemma 8.2.11 below, relation (4.1.4) is guaranteed and hence, the notations T_h^δ and A_h^δ as given by (4.1.5) and (4.1.6) are justified for all $\delta \in \Delta$. From (8.2.15) one easily concludes for some constant K and all $x \in S$:

$$(8.2.18) \qquad \sup_{\delta \in \Delta} (|a^\delta(x)| + |b^\delta(x)|) \leq K(1+|x|).$$

As a result, by using the growth relation (8.2.18), the following two lemmas can also (cf. lemma 8.2.1 and 8.2.2) be shown analogously to the proofs of lemma I.5.3.12 and lemma I.5.3.13 respectively.

LEMMA 8.2.10. *The strong consistency relation* (6.2.2) *holds with* $\mu = \mu_{m+3}$ *for any collection* $\{U_t \mid t \in [0,Z]\} \in C^{3;m}[0,Z]$ *and such that for all* $h \leq h_0$, $jh+h \leq Z$:

$$(8.2.19) \qquad \sup_{\delta \in \Delta} \|(A_h^\delta - A^\delta)\, U_{jh+h}\|_{\mu_{m+3}} \leq \sqrt{h}\, CK_U[0,Z]. \qquad \square$$

LEMMA 8.2.11. *The strong stability relation* (6.2.4) *holds with* $\mu = \mu_{m+3}$. □

THEOREM 8.2.12. *The convergence relation* (6.5.2) *is satisfied with order of convergence* $O(\sqrt{h})$.

PROOF. By virtue of theorem 6.5.2, lemma 8.2.10 together with (8.2.16), and lemma 8.2.11, the relation (6.5.2) is shown by verifying the smoothness assumption 6.5.1.

Therefore, from the continuity of L, a and b one easily concludes that for any $f \in C^{3;m}$ and $f^h \in C^{\mu_m}$ the functions g and g^h on $\mathbb{R} \times \Gamma_1 \times \Gamma_2$, defined by

$$(8.2.20) \quad \begin{cases} g(x,\gamma_1,\gamma_2) = L(x,\gamma_1,\gamma_2) + [a(x,\gamma_1)\frac{d}{dx} f(x) + \frac{1}{2} b^2(x,\gamma_2)\frac{d^2}{dx^2} f(x)] \\ g^h(x,\gamma_1,\gamma_2) = hL(x,\gamma_1,\gamma_2) + [\frac{1}{2} f^h(x + a(x,\gamma_1)h + b(x,\gamma_2)\sqrt{h}) + \\ \qquad\qquad \frac{1}{2} f^h(x + a(x,\gamma_1)h - b(x,\gamma_2)\sqrt{h})], \end{cases}$$

are continuous in $(x,\gamma_1,\gamma_2) \in \mathbb{R} \times \Gamma_1 \times \Gamma_2$. Next, recall expression (2.3.3) for the infimum operator J with A^δ given by (8.1.1). Further, recall the conditions (i) and (ii) of assumption 8.2.7 as well as the growth relation of (8.2.14) and the growth relation (8.2.18). Then it can be shown analogously to the proof of lemma 7.2.8 that $J : C^{3;m} \to C^{\mu_m}$. This guarantees assumption 2.3.4 with $D_A = C^{3;m}$ as well as assumption 4.2.3 with $F = C^{\mu_m}$.

Furthermore, from (8.2.14) and (8.2.18) it is seen directly that

$$(8.2.21) \qquad \|Jf\|_{\mu_m} \le C(1 + K_f) \qquad \text{for } f \in C^{3;m},$$

so that together with (8.2.16) we can conclude that the collection $\{J\emptyset_t \mid t \in [0,Z]\}$ is μ_m- and hence μ_{m+3}-bounded. Together with assumption 8.2.8, this also proves assumption 2.3.5. Finally, from (8.2.17) and (8.2.18) it follows that

$$(8.2.22) \qquad \|J(\emptyset_{t+\Delta t}) - J(\emptyset_t)\|_{\mu_{m+3}} \le \sup_{\delta \in \Delta} \|A^{\delta}(\emptyset_{t+\Delta t} - \emptyset_t)\|_{\mu_{m+3}} \le \sqrt{\Delta t}\, C$$

Consequently, expression (3.3.1) for $R_t(\emptyset,h)$ and (8.2.22) imply

$$(8.2.23) \qquad \|R_t(\emptyset,h)\|_{\mu_{m+3}} h^{-1} \le \sqrt{h}\, C.$$

Hence, the 'smoothness' relation (6.5.1) is satisfied. Furthermore, lemma 5.5 together with the relations (8.2.19) and (8.2.23) guarantees the order of convergence $O(\sqrt{h})$. □

REMARKS 8.2.13.

1. The problem of the existence and the uniqueness of a solution of the optimality equation (2.3.4) for diffusion processes, as is required by assumption 8.2.8, is well-known in the literature.

Especially, the case of an uncontrolled and non-degenerate diffusion coefficient has been frequently studied. Existence results can be obtained by using results for ordinary differential equations together with a method of successive approximation, as shown for instance by Fleming and Rishel (1975). From an algorithmic and computational point of view, the method of policy improvement (iteration), as used by Puterman (1977), (1978), can be very valuable.

For linear systems, i.e. with coefficients linear in the control variable, it is sometimes possible to solve the optimality equation analytically, as shown, for instance, on p. 187 - 200 of Gihman and Skorohod (1979).

Krylov (1980), however, studies the Bellman equation, in general, for diffusion processes with controlled drift as well as diffusion coefficients. Under smoothness and growth conditions on the diffusion characteristics as well as the cost rate function (see p.130, 165 and 173 of Krylov) of the type as given by (8.1.17) and (8.2.14) up to second order derivatives as well as non-degeneracy of the diffusion coefficient, he proves (see chapter 4) the existence of a solution of the Bellman equation.
In addition, it is also shown that the first and second derivatives of a solution with respect to the state variable are polynomial bounded, such as required by (8.2.16). In analogy with his results, we trust that also the conditions for the third derivative can be verified. We like to note that his proofs are based on using stochastic differential equations for the (mean square) derivatives of solutions of differential equations with respect to the initial data, and using Lipschitz and growth conditions, uniform in all controls, of the coefficients.

2. It is well-known, see for instance p.159 of Fleming and Rishel (1975), p.174-180 of Gihman and Skorohod or Krylov (1980), that the solutions of the optimality equation of diffusion type present optimal cost of control within a wide class of controls, including history dependent controls. □

§4. CONSTRUCTION OF ε-OPTIMAL PIECEWISE CONSTANT CONTROLS

As in section 7, it will be worthwhile, in view of the optimality property stated in 2 of remark 8.2.13, to study the possibility of constructing (simple) controls such that the cost function of the corresponding continuous-time controlled process is close to the optimality function. If there exists a discrete-time control which is Lipschitz with respect to the state variable and which is optimal (nearly optimal) for an h-discrete-time controlled process, with h sufficiently small, then it can be shown analogously to §4 of subsection 7.2. that this control is also nearly optimal for the continuous-time model. Unfortunately, however, in general such a Lipschitz property for optimal or nearly optimal

controls is not guaranteed or may be difficult to prove.
In order to avoid Lipschitz requirements on the controls, we will use discrete-time controlled stochastic processes of which the one-step transition probabilities themselves are induced by *stochastic differential equations* under *constant control variable.*
A discrete-time controlled process so constructed is just a special case of a continuous-time controlled stochastic differential equation.
As a result, showing that the corresponding optimal cost functions of the discrete-time construction and the continuous-time model are close and recursively determining an optimal (or nearly-optimal) control for the discrete-time case, yields a nearly-optimal control.

In this paragraph, let the assumptions 8.2.7 and 8.2.8 also be satisfied. Further, for $\gamma = (\gamma_1,\gamma_2) \in \Gamma$ we will also use γ as a superindex to indicate a constant decision rule $\bar{\gamma} \in \Delta$ with $\bar{\gamma}(x) = \gamma$ for all $x \in S$ as well as a constant Markov control $\bar{\bar{\gamma}}$ with $\bar{\bar{\gamma}}(t) = \bar{\gamma}$ for all $t \geq 0$.
It will always be clear which of these two is used.

Let $h > 0$ be fixed and note that the Lipschitz relation (8.2.15) implies the Lipschitz relation (8.1.3) for the constant control $\pi = \bar{\bar{\gamma}}$, where $\bar{\bar{\gamma}}(t,x) = \gamma$ for all $x \in S$, $t \geq 0$, $\gamma \in \Gamma$. Consequently, according to theorem 8.1.2, there exists for any γ and $x \in S$ a unique (homogeneous) Markov process $(\eta_t^\gamma(x))_{t \leq h}$ satisfying:

$$(8.2.24) \qquad \eta_t^\gamma(x) = x + \int_0^t a^\gamma(\eta_s^\gamma(x))\, ds + \int_0^t b^\gamma(\eta_s^\gamma(x))\, dW_s \quad , \quad t \leq h.$$

Further, let $\{T_t^\gamma \mid t \in [0,Z]\}$ be the corresponding semigroup of operators defined on $B^{\mu_{m+3}}$ by

$$(8.2.25) \qquad T_t^\gamma f(x) = \mathbb{E}\, f(\eta_t^\gamma(x)) \quad , \quad x \in S .$$

Then, according to (8.1.7), (8.1.15) and (8.1.16) we obtain for all $n \in \mathbb{N}, f \in C^{3;m}$:

$$(8.2.26) \qquad \mathbb{E}\, |\eta_t^\gamma(x)|^n \leq C(1 + |x|^n),$$

$$(8.2.27) \qquad \| [T_{\Delta t}^\gamma - I] f \|_{\mu_{m+3}} \leq \Delta t\, C\, K_f \, ,$$

$$(8.2.28) \qquad \| ([T_{\Delta t}^\gamma - I](\Delta t)^{-1} - A^\gamma) f \|_{\mu_{m+3}} \leq \sqrt{\Delta t}\, C\, K_f \, ,$$

where, by virtue of (8.2.15) and (8.2.18), the constant C holds uniformly in $\gamma\in\Gamma$.

To proceed first conclude from (8.2.14) and the compactness of Γ that $L^\gamma\in C^{3;m}$. Consequently, by using (8.2.27) it is easily seen that $\{T_t^\gamma L^\gamma | t\in[0,h]\}$ is μ_{m+3}-continuous as well as μ_{m+3}-bounded in $t\in[0,h]$. As a result, the function $T_s^\gamma L^\gamma$ can be integrated.

Hence, by virtue of lemma 8.2.14 below we are able to define an operator G^h on $C^{\mu_{m+3}}$ by

$$(8.2.29)\qquad G^h f(x) = \inf_{\gamma\in\Gamma} [\int_0^h T_s^\gamma L^\gamma(x)ds + T_h^\gamma f(x)].$$

We obtain a collection $\{\phi_j^h | jh\le Z\} \subset C^{\mu_m}$ by recursively solving the *discrete-time optimality equation:*

$$(8.2.30)\qquad \begin{cases} \phi_\ell^h(x) = \inf_{\gamma\in\Gamma} [\int_0^{Z-\ell h} T_s^\gamma L^\gamma(x)ds] & , x\in S,\ \ell = [Zh^{-1}]. \\ \phi_j^h = G^h(\phi_{j+1}^h) & , j<\ell. \end{cases}$$

LEMMA 8.2.14. $G^h : C^{\mu_m} \to C^{\mu_m}$ *and* $\phi_\ell^h\in C^{\mu_m}$.

PROOF. Since the functions a and b are uniformly continuous in (x,γ) on the compact set $[-N,N] \times \Gamma$ for any $N>0$ it follows from theorem 2 on p.52 of Gihman and Skorohod (1972) that for $(x^n,\gamma^n) \to (x,\gamma)$:

$$(8.2.31)\qquad \sup_{t\le h} \mathbb{E}\, |\eta_t^{\gamma^n}(x^n) - \eta_t^\gamma(x)|^2 \to 0.$$

Further, note that relation (8.2.14) together with the compactness of Γ implies that $|L^\gamma(x) - L^\gamma(y)| \le |x-y|\ C(1 + |x|^m + |y|^m)$ as well as that relation (8.2.13) holds. Hence, we can write

$$(8.2.32)\qquad |T_t^{\gamma^n} L^{\gamma^n}(x^n) - T_t^\gamma L\ (x)| \le$$

$$|T_t^{\gamma^n} L^{\gamma^n}(x^n) - T_t^\gamma L^{\gamma^n}(x)| + |T_t^\gamma(L^{\gamma^n} - L^\gamma)(x)| \le$$

$$\mathbb{E}\, C|\eta_t^{\gamma^n}(x^n) - \eta_t^\gamma(x)|[1 + |\eta_t^{\gamma^n}(x^n)|^m + |\eta_t^\gamma(x)|^m] +$$

$$\mathbb{E}\, C\ (|\gamma_1^n - \gamma_1| + |\gamma_2^n - \gamma_2|)\ \ [1 + |\eta_t^\gamma(x)|^m].$$

Finally, applying Schwartz' inequality to the first term of the last sum, letting $(x^n,\gamma^n) \to (x,\gamma)$ and using (8.2.26), (8.2.31) and (8.2.32) shows:

(8.2.33) $\quad T_t^\gamma L^\gamma(x)$ is continuous in (x,γ) uniformly in $t \le h$.

Furthermore, for any $f \in C^{\mu_m}$ relation (8.2.31), implies by using the standard step of applying Chebyshev's inequality: $\eta_h^\gamma(x)$ is weakly continuous in (x,γ). Further, since $f \in C^{\mu_m}$ and $\mathbb{E}\mu_m(\eta_h^\gamma(x)) \le C(1+|x|^m)$ uniformly in γ (see (8.2.26) this also implies that
$T_h^\gamma f(x) = \mathbb{E} f(\eta_h^\gamma(x))$ is continuous in (x,γ) for any $f \in C^{\mu_m}$.

Consequently, for $f \in C^{\mu_m}$ the function $g^h : \mathbb{R} \times \Gamma \to \mathbb{R}$, defined by

$$(8.2.34) \qquad g^h(x,\gamma) = \int_0^h T_s^\gamma L^\gamma(x)ds + T_h^\gamma f(x),$$

is continuous in (x,γ). From this continuity it follows analogously to the proof of lemma 7.2.8 that $G^h f(x)$ is continuous in x.
Since also (8.2.14) and the compactness of Γ implies the μ_m-boundedness of L^γ, we obtain from (8.2.26) and (8.2.34): $G^h : C^{\mu_m} \to C^{\mu_m}$.
Further, from the definition of ϕ_ℓ^h we obtain analogously: $\phi_\ell^h \in C^{\mu_m}$. □

<u>LEMMA 8.2.15</u>. *There exist* $\delta_0,\delta_1,\ldots,\delta_\ell \in \Delta$ *such that for any* $j \le \ell$, $x \in S$ *and with* $\gamma = \delta_j(x)$:

$$(8.2.35) \quad \begin{cases} \phi_j^h(x) = \int_0^{Z-\ell h} T_s^\gamma L^\gamma(x)ds & \text{if } j = \ell, \\ \phi_j^h(x) = \int_0^h T_s^\gamma L^\gamma(x)ds + T_h^\gamma \phi_{j+1}^h(x) & \text{if } j < \ell. \end{cases}$$

<u>PROOF</u>. First conclude from the recursive scheme (8.2.30) and lemma 8.2.14 that $\{\phi_j^h \mid j \le \ell\} \subset C^{\mu_m}$. Then, the proof directly follows from the continuity in (x,γ) of the right-hand sides in (8.2.35), shown by the proof of lemma 8.2.14, together with the compactness of Γ and lemma 1.4 on p.16 of Gihman and Skorohod (1979). □

Let π represent the control which at time-point jh changes its value according to the decision rule δ_j given by lemma 8.2.15 and the current state and thereafter remains unchanged up to the next time-point jh+h, $j = 0,1,\ldots,\ell$.

Then, from the stochastic integral equation (8.2.24) together with lemma 8.2.15, it can be concluded that the optimal cost functions $\{\phi^h | jh \le Z\}$ represents cost functions $\{V^\pi_{jh} \mid jh \le Z\}$ corresponding to a controlled stochastic integral equation under the piecewise constant and almost-Markov control π.

(A precise formulation would require history dependent controls, which here we avoid). In view of the above, however, we write: $V^\pi_{jh} = \phi^h_j$, $j \le \ell$ and will show that π is (in the above mentioned informal sense) a nearly-optimal control.

<u>THEOREM 8.2.16</u>. *For some constant* C *all* $t \le Z$ *and with* $n = [th^{-1}]$.

$$\|V^\pi_{nh} - \phi_t\|_{\mu_{m+3}} \le \sqrt{h}C. \tag{8.2.36}$$

<u>PROOF</u>. First of all from (2.3.4) and (8.2.22) we easily conclude that

$$\|\phi_{nh} - \phi_t\|_{\mu_{m+3}} \le \sqrt{h}C. \tag{8.2.37}$$

Next, according to the systems (2.3.4) and (8.2.30), the fact that Δ contains all constant decision rules and that infima are taken pointwise:

$$\begin{cases} \phi^h_j = \inf_{\gamma\in\Gamma} [hL^\gamma + T^\gamma_h \phi^h_{j+1} + (\int_0^h T^\gamma_h L^\gamma ds - hL^\gamma)] , \\ \phi_{jh} = \inf_{\gamma\in\Gamma} [hL^\gamma + (hA^\gamma + I)\, \phi_{jh+h} + R_{jh}(\phi,h) . \end{cases} \tag{8.2.38}$$

Write $\delta_j = \phi^h_j - \phi_{jh}$. Then from (8.2.38):

$$\begin{aligned} \|\delta_j\|_{\mu_{m+3}} \le \\ &\sup_{\gamma\in\Gamma} \| T^\gamma_h(\delta_{j+1})\|_{\mu_{m+3}} + \sup_{\gamma\in\Gamma} \| (\int_0^h T^\gamma_s L^\gamma ds - hL^\gamma)\|_{\mu_{m+3}} + \\ &\sup_{\gamma\in\Gamma} \|([T^\gamma_h - I] - hA^\gamma)\, \phi_{jh+h}\|_{\mu_{m+3}} + \|R_{jh}(\phi,h)\|_{\mu_{m+3}} . \end{aligned} \tag{8.2.39}$$

First, conclude from (8.2.14) that $L^\gamma \in C^{3;m}$ and $\sup_{\gamma\in\Gamma} K_{L^\gamma} < \infty$.

Further, recall: $\{\phi_t | t \in [0,Z]\} \in C^{3;m}\,[0,Z]$.

By using these facts we conclude from (8.2.23), (8.2.27), (8.2.28) and (8.2.39):

$$(8.2.40) \qquad \|\delta_j\|_{\mu_{m+3}} \le \sup_{\gamma\in\Gamma} \|T_h^\gamma(\delta_{j+1})\|_{\mu_{m+3}} + h\sqrt{h}C$$

Finally, by using lemma 8.2.17 given below, relation (8.2.35) for ϕ^n_0 and (8.2.37) for t = Z, we obtain by iterating (8.2.40) for $j = n, n+1, \ldots, \beta - 1$:

$$(8.2.41) \qquad \|\delta_j\|_{\mu_{m+3}} \le \sqrt{h}\,C$$

Combination of (8.2.37) and (8.2.41) completes the proof. □

LEMMA 8.2.17. *For any* $n\in\mathbb{N}$ *and all* $\gamma\in\Gamma$:

$$(8.2.42) \qquad \|T_h^\gamma \mu_n\|_{\mu_n} \le (1+hC)$$

for some constant C *not depending on* h *and* γ.

PROOF. Since Schwartz' inequality can be used if n is uneven it suffices to give the proof for n is even. First, we write

$$(8.2.43) \qquad \begin{aligned} &\mathbb{E}\,[\eta_h^\gamma(x)]^n = \mathbb{E}\,[(\eta_h^\gamma(x) - x) + x]^n \le \\ &|x|^n + \sum_{i=1,\ldots,n} \binom{n}{i} |x|^{n-i} \,|\mathbb{E}\,(\eta_h^\gamma(x) - x)^i|. \end{aligned}$$

Relation (8.1.8) directly implies (also noting that $h \le h_0$):

$$(8.2.44) \qquad \sum_{i=2,\ldots,n} \binom{n}{i} |x|^{n-i} \,|\mathbb{E}\,(\eta_h^\gamma(x) - x)^i| \le hC(1+|x|^n).$$

Next, recalling that a^γ is Lipschitz in x, uniformly in γ, we obtain analogously to relation (5.3.21) of chapter I:

$$(8.2.45) \qquad |\mathbb{E}\,\{[\eta_h^\gamma(x) - x] - h\,a^\gamma(x)\}| \le h\sqrt{h}\;C(1+|x|)\ ,$$

so that together with the growth relation for a^γ (see (8.2.18)):

$$(8.2.46) \qquad |\mathbb{E}\,[\eta_h^\gamma - x]| \le h\,C\,(1+|x|).$$

Combining (8.2.43), (8.2.44) and (8.2.46) yields

$$(8.2.47) \qquad T_h^\gamma\mu_n(x) = 1 + \mathbb{E}\,[\eta_h^\gamma(x)]^n \le (1+|x|^n) + h\,C\,(1+|x|^n).$$

□

<u>REMARKS 8.2.18.</u>

1. Note that lemma 8.2.17 is a slight extension of the growth relation (8.1.8).

2. In view of the discrete-time optimality equation (8.2.30), we note that for any fixed γ the term between brackets on the right-hand side of (8.2.29) can be given explicitly by using formula for the transition probabilities as shown on p.93-95 of Gihman and Skorohod (1972).

3. The existence of optimal and ε-optimal Markov controls as well as construction of ε-optimal Markov controls has been shown in chapter 5 of Krylov (1980) under a uniform Lipschitz condition on the coefficients in all admissible controls (see p.214 of Krylov (1980)).
In this respect compare the Lipschitz relation (8.2.15). Krylov's construction of ε-optimal controls follows from continuity arguments and choosing dense subsets of the decision set (cf. lemma 1.4.9 on p.28 of Krylov(1980)) and consequently, does not show a recursive construction such as given by lemma 8.2.15 above. □

9. RELATED LITERATURE

The topic of stochastic control has obtained a fast growing attention in the literature during the last decade,so that it is impossible to give a reasonable complete survey on this field.
In this respect,we only like to mention the survey paper of Fleming (1969), as well as the list of references included by Fleming and Rishel (1975), Kushner (1977), Gihman and Skorohod (1979) and Bensoussan (1983).
Also methods of time-discretization have been frequently applied in the literature for all types of 'ad hoc' applications with considerable success and seem to become an important tool for analyzing controlled stochastic processes (see for instance Mitchell (1973) and Doshi (1978)).

Therefore,we restrict ourselves to a discussion on the literature which is most closely related to the approximation analysis of this monograph. First in §1 we discuss the so-called Markov decision drift processes introduced by Hordijk and Van Der Duyn Schouten (1980). Their model receives special attention since it includes continuous as well as impulsive controls simultaneously and also because the approximation analysis of this monograph may yield some generalizations of their model such as the inclusion of diffusion processes as drifts. Next, in § 2 we focus on the work of Kushner (1977), (1978) and Haussmann (1980),which especially seem to be of interest also from a computational point of view, as well as the work of Nisio (1978), Bensoussan and Robin (1983), Gihman and Skorohod (1979), Christopeit (1983), Whitt (1975) and Kakumanu (1977).

In addition, § 3 and § 4 contain only a small survey of related literature on controlled Markov jump processes and controlled diffusion processes respectively. We note that these surveys are far from complete.

§ 1. MARKOV DECISION DRIFT PROCESSES; TIME-DISCRETIZATION

Van Der Duyn Schouten (1979) and Hordijk and Van Der Duyn Schouten (1980), (1983a), (1983b), (1983c) have introduced Markov decision drift processes. An informal description of such a process is the following.
A process is observed continuously. The process is assumed to be a jump process with deterministic drifts between the jumps. The jumps are induced by a controlled Markov jump process (for example, in a maintenance replacement model shocks of damage occur according to a Poisson process with controllable shock rate) as well as by possible impulsive controls

(for example,an immediate replacement if a certain level of damage is exceeded). The deterministic drift itself (such as a continuous increase of damage) is not influenced by the controls.
A policy pescribes at each time-point t a decision to be taken based on the history up to that time-point; it is either a 'continuous control variable' influencing the jump-characteristics of the Markov jump process or an 'impulsive control' which has an impulsive effect on the process. Costs per unit of time are incurred depending on the actual state and 'continuous control' as long as no impulsive control takes place as well as lump costs depending on the actual state and the 'impulsive control'.

The above description holds for many applications as queueing, replacement and inventory models,especially since the process is allowed to evolve between jumps and since impulsive controls are taken into account.

For the above framework,Van Der Duyn Schouten (1979) and Hordijk and Van Der Duyn Schouten (1980), (1983a), (1983b), (1983c) develop a method of time-discretization in order to obtain structural results for optimal controls in the continuous-time model. Therefore, a sequence of discrete-time Markov decision chains is constructed. In a discrete-time Markov decision chain the difference in their impact of continuous and impulsive controls vanishes. By making use of the dynamic programming method for discrete-time optimality equations together with weak convergence results for the approximating sequence of processes they are able to conclude the optimality of a limit control within a wide class of 'regular' policies.
Further,under several conditions they are able to deal with unbounded cost rates and it is worth noting that their analysis does not require solutions of the continuous-time optimality equations. The optimality of structured policies is shown, e.g. bang - bang type resp. monotone type resp. (s,S)-type in maintenance replacement resp. $M|M|1$-queueing resp. inventory models.

The discretization presented in section 7 for controlled Markov jump processes is equal to their discretization in case of absence of impulsive control. Therefore,in that case the analysis of this chapter can be applied in order to obtain approximations for the optimal cost functions. Furthermore,especially the generalization of a deterministic drift to a controlled stochastic process between jumps, is an important topic for further investigation where the results and techniques of Hordijk and Van Der Duyn Schouten can be combined with the analysis of this chapter.

§2. FURTHER GENERAL TIME-DISCRETIZATION METHODS.

Kushner (1977) develops probability methods for approximations in stochastic control, particularly controlled diffusion processes, and first or second order differential equations. His method is based on a combined discretization of time-and state variable and is induced by a numerical procedure, although a modification is required in order to guarantee discretizations associated with one-step transition probabilities. His type of discretization resembles that presented in section 8 for controlled diffusion processes. If the drift coefficient is always 0 then these discretizations are exactly the same.

The way of showing convergence, however, is somewhat different. Kushner only needs to verify the tightness of the approximating processes. Using a weakly convergent subsequence the existence of a limit process is proved. By showing the uniqueness of a limit process (cf. p.99) and by using continuity properties on D-spaces (cf. p.100/101), Kushner (1977) proves the convergence of the approximating processes as well as of the corresponding cost functionals.
The tightness is proven by showing that the fourth moments of the increments over time Δt for all discrete-time processes are uniformly bounded by $C(\Delta t)^2$ (cf. p.96/97 of Kushner (1977)).In this respect we note that lemma I.5.3.19 presents a similar type of result. Since, however, there we required a polynomial bound with respect to initial data and we allowed unbounded coefficients, we had to give a somewhat modified proof.

Kushner applies his method to a variety of interesting problems associated with diffusion processes as optimal stopping, impulsive control and reflection problems. He studies the approximation of the cost functionals (some specific numerical results are included) as well as the existence and optimality of a limit control.

Again, it is worth noting that the approach does not require analysis of continuous-time optimality equations. Further, we note that he assumes diffusion coefficients to be uncontrolled and the cost functions involved to be bounded.

The same approximation method is applied to the average cost case in Kushner (1978) and, under relaxations of Lipschitz conditions and for the finite horizon function again, by Haussmann. Haussmann shows that a limit control satisfies a stochastic maximum principle.

Gihman and Skorohod (1979), Nisio (1978) as well as Bensoussan and Robin (1983) use the same type of time-discretization. As one-step transition probabilities they take $P^{h}_{nh,nh+h} = P^{\gamma}_{nh,nh+h}$ where $P^{\gamma}_{nh,nh+h}$ denotes the transition probability of the continuous-time process under constant control variable γ during $[nh,nh+h)$.

In a general setting, Gihman and Skorohod (1979) consider controlled stochastic processes associated with a control object (different from our definition) and so-called generalized controls inducing a random process of control. For a fixed sample path of the control process the probability law of the state process is determined in a non-anticipative way and similarly for a fixed sample path of the state process the probability law of the control process is determined in a non-anticipative way. By using time-discretization, they show in this general setting the sufficiency of step-(that means with piecewise constant control variables) feedback controls with respect to optimality for finite horizon cost functions depending on the entire history of state and control process.
For a controlled Markov jump process these results yield the optimality of a solution of the optimality equation and of a corresponding control. Further, for controlled stochastic differential equations driven by a process of independent increments, which includes a Wiener process and/or Poisson processes, it is shown that the discretization induced by piecewise constant controlling the differential equations yields nearly-optimal controls. The discretization given in §4 of section 8 is of the same form. The results of Gihman and Skorohod require Lipschitz conditions on the drift and diffusion coefficient in the state variable uniformly in all controls, whereas in § 4 of section 8 we only require the

Lipschitz condition uniformly in any fixed control-variable (decision). Further, their results concern bounded cost functions.

Nisio (1978) shows that the optimal cost functions associated with optimal stopping problems for time-homogeneous Markov processes can be represented by a unique non-linear semigroup which is monotone; contractive and strongly continuous. In addition, he shows that the semigroup is the unique solution of an optimality equation. His results assume bounded and Lipschitz cost functions and are derived by controlling the model at step-sizes 2^{-i} and showing monotony properties of the corresponding cost functions.

Bensoussan and Robin (1983) use a same type of method as Nisio but also apply it to continuous and impulsive control problems associated with homogeneous Markov semigroups. Analogously to Nisio, the convergence of discrete-time optimal cost functions is shown for step-sizes 2^{-i} and by using monotony arguments. The boundedness of the cost functions is assumed but relaxations of Lipschitz conditions are made. For the continuous control problem the set of possible controls is assumed to be finite. Probabilistic interpretations of the convergence results are included.

Christopeit(1983) studies a stochastic differential equation with controlled bounded drift function and uncontrolled diffusion coefficient. As discretization, stochastic difference-equations with step-size 2^{-i} are considered. The one-step stochastic difference is taken linear in the step-size and drift function at the discrete-time point plus an addition of a Wiener increment. By showing tightness, convergence results are obtained and the optimality of a limit control is shown for the finite horizon cost case The existence of the limit control which has to satisfy a certain convergence condition is assumed a priori. The drift function is Lipschitz in all variables. The cost function is allowed to be exponentially bounded and is only required to be continuous.

Whitt (1975) studies in a general setting the convergence of a sequence of Markov renewal programs and related functions. As a particular application for exponential processes and countable state space, an approximating sequence of discrete-time Markov programs is studied.

Convergence results are obtained under (pointwise) convergence conditions of the one-step to the infinitesimal jump characteristics. (The time-discretization given in section 7 for Markov jump processes and with countable state space obviously satisfies that condition).

Kakumanu (1977) treats controlled Markov jump processes with countable state space and finite decision set, bounded jump rates and bounded cost-rate function. For the discounted as well as the average cost case he shows that for any control the cost functions and hence also the optimal cost functions are exactly equal, up to a factor, for the continuous- and discrete-time model.

§3. CONTROLLED MARKOV JUMP PROCESSES

A study of these processes can be found in Miller (1968), Pliska (1975), Davis (1976), Boel and Varaiya (1977), Kakumanu (1977), Gihman and Skorohod (1979) and Yushkevich (1980). See remark 7.2.11 for specific remarks on these references in view of optimality equations and optimal controls.

In more general settings controlled jump-type processes are examined by Stone (1973), Whitt (1975), Rishel(1976), Van Der Duyn Schouten (1979) and Hordijk and Van Der Duyn Schouten (1980), (1983a), (1983b), (1983c).

Applications of controlled jump-type models can especially be found in queueing models, storage models etc. See for instance Mitchell (1973), Doshi (1978), Van Der Duyn Schouten (1979) and Hordijk and Van Der Duyn Schouten (1983c).

§4. CONTROLLED DIFFUSION PROCESSES

A general study of these processes can be found in the books of Mandl (1968), Fleming and Rishel (1975), Kushner (1977), Gihman and Skorohod (1979), Krylov (1980) and Bensoussan (1983) as well as the theses and related papers of Pliska (1972), (1973) and Puterman (1972), (1974).

Particularly,the work of Mandl (1968), Pliska (1972), (1973) and Puterman (1972), (1977), (1978) are concerned with the continuous control of diffusion processes on bounded subsets of the state space. They present many existence and uniqueness results with respect to the optimality equations in the stationary case (Mandl, Pliska and Puterman) as well as the finite horizon case (Puterman), on the optimality of solutions of these equations and especially on proving the existence of optimal and ε-optimal Markov or piecewise constant controls.

Applications of controlled diffusion processes have been given,for instance,for dam, queueing, investment and particularly cash-balance models. See Bather (1968), Pliska (1972), Puterman (1972), Constantinides (1974), Constantinides and Richard (1978), Harrison and Taylor (1978), and Harrison and Taksar (1981).

APPENDIX

A WEAK CONVERGENCE OF MARKOV PROCESSES ON $D[0,\infty)$

1. INTRODUCTION AND DEFINITIONS

This appendix provides three theorems in order to conclude weak convergence of discrete-time Markov processes to a continuous-time Markov process on a special sample path space $D[0,\infty)$. Especially these theorems are developed for application to jump- and diffusion-type Markov processes and have been applied in subsections 5.1,5.2 and 5.3 of chapter I.
First,it is shown that weak convergence of transition probabilities also implies weak convergence of the finite-dimensional distributions of the processes (see proposition 3.4). Next,each of the three theorems presents conditions which guarantee 'weak compactness' (tightness) of the discrete-time processes. We note that these conditions are essentially based on results of Chentsov (1956) and Skorohod (1957), (1958).

D-spaces concern functions on a continuous-time parameter which at each time-point has a left-and right-hand limit and are right continuous. Analysis of D-spaces is initiated by Skorohod (1956). Studies of weak convergence of stochastic processes on D-spaces, that means with sample paths contained in a D-space,can also be found in Billingsley (1968), Lindvall (1973), Gihman and Skorohod (1974),and Whitt (1980).
Sample paths of jump-processes, as given in subsections I.5.1 and I.5.2, are contained in D-spaces. Sample paths of diffusion processes, as given in subsection I.5.3, are elements of the subclass of continuous functions.

Without restriction of generality,let the state space S be given by $\mathbb{R}$. For a function $x:[0,Z]\to\mathbb{R}$ with $Z>0$ and $t\in[0,Z]$ write $x(t-)=\lim_{s\uparrow t} x(s)$ and $x(t+)=\lim_{s\downarrow t} x(s)$, with the convention $x(0-)=x(0)$, $x(Z+)=x(Z)$.
First let us present the necessary notations:
For $a,b\in\mathbb{R}$ write: $a\wedge b=\min(a,b)$; $a\vee b=\max(a,b)$.

$$(1.1)\quad \begin{cases} D[0,Z]=\{x:[0,Z]\to\mathbb{R}\ |x(t+) \text{ exists and } x(t+)=x(t) \text{ for all } t\in[0,Z) \\ \qquad\qquad x(t-) \text{ exists for all } t\in(0,Z] \text{ and } x(Z-)=x(Z)\} \\ D[0,\infty)=\{x:[0,\infty)\to\mathbb{R}\ |x(t-),\ x(t+) \text{ exist for all } t\in[0,\infty) \\ \qquad\qquad x(t)=x(t+) \text{ for all } t\in[0,\infty)\}. \end{cases}$$

(1.2) For $x \in D[0,\infty)$ and $Z \in [0,\infty)$ the Skorohod modules on $[0,Z]$ is given by

$$\Delta_c[0,Z](x) = \sup_{0 \le t_2 - c \le t_1 \le t_2 \le t_3 \le t_2 + c \le Z} |x(t_1)-x(t_2)| \wedge |x(t_2)-x(t_3)|$$

(1.3) For $x,y \in D[0,Z]$, $Z \in [0,\infty)$ define

$$\begin{cases} d_Z(x,y) = \inf_{\Lambda_Z} \sup_{t \in [0,Z]} \{|x(t) - y(\lambda(t))| \vee |t-\lambda(t)|\}, \\ \text{where} \quad \Lambda_Z = \{\lambda : [0,Z] \to [0,Z] \,|\, \lambda(t) \text{ strictly increasing} \\ \qquad\qquad \text{and continuous in } t,\ \lambda(0) = 0,\ \lambda(Z) = Z\}. \end{cases}$$

(1.4) For $Z \in [0,\infty)$ define the mapping
$\Gamma_Z : D[0,\infty) \to D[0,Z]$ such that for $x \in D[0,\infty)$:
$\Gamma_Z(x) = y \in D[0,Z]$ with $y(t) = x(t)$ for all $t \in [0,Z)$, $y(Z)=x(Z-)$.

(1.5) For $x,y \in D[0,\infty)$ define

$$d_\infty(x,y) = \int_0^\infty e^{-s} [d_s(\Gamma_s(x),\Gamma_s(y)) \wedge 1]\, ds.$$

The literature especially studies $D[0,1]$. However, the following holds:

<u>LEMMA 1.1.</u>
$D[0,Z]$ *is a separable metric space with metric* d_Z *for all* $Z > 0$.
$D[0,\infty)$ *is a separable metric space with metric* d_∞.

<u>PROOF</u>. For Z = 1 especially see Skorohod (1956) or Billingsley (1968).
For $Z \le \infty$ see theorem 2.5 of Whitt (1980). □

<u>REMARKS 1.2.</u>

1. The metric d_1 was introduced by Skorohod (1956) and the corresponding topology is known as Skorohod's J_1-topology.

2. D-spaces with the above metrics are not complete. However, they are metrizable as complete metric spaces by means of metrics which are equivalent to the above metrics, see theorem VII of Kolmogorov (1956), theorem 14.2 of Billingsley (1968), theorem 1 of Lindvall (1973) and theorem 2.6 of Whitt (1978). □

2. WEAK CONVERGENCE ON $D[0,\infty)$

In what follows below, read $[0,Z] = [0,\infty)$ if $Z = \infty$.
Recall the definition of a stochastic process X given by definition 0.1 with $\theta = [0,Z]$, $S = \mathbb{R}$ and $Z \in [0,\infty) \cup \{\infty\}$. Then X is called a random element on $D[0,Z]$ if $\mathbb{P}(\{\omega | X(\cdot,\omega) \in D[0,Z]\}) = 1$.
Notation : $X \in R(D[0,Z])$.
In this monograph we always assume separable stochastic processes (cf. Gihman and Skorohod (1974) p.164).

For $X \in R(D[0,\infty))$ with probability measure $\mathbb{P}_X$ denote

$$T_X = \{t \,|\, \mathbb{P}(\{\omega | X(t+,\omega) = x(t-,\omega)\}) = 1\},$$

then it follows analogously to p.124 of Billingsley (1968) that

$$N_X = [0,\infty) - T_X \text{ is a countable subset of } \mathbb{R}.$$

Recall the concept of weak convergence and the notation $X^h \Rightarrow X$ as given by definition 0.10 with $S = D[0,Z]$, $Z \in [0,\infty) \cup \{\infty\}$, (continuity of functions has to be considered with respect to the metrics given by (1.3) for $Z < \infty$ and (1.5) if $Z = \infty$). □

LEMMA 2.1. *Let* $X^h \in R(D[0,\infty))$, $0 < h \le h_0$ *and* $X \in R(D[0,\infty))$. *Then*

$$(2.1) \quad \begin{cases} X^h \Rightarrow X \text{ on } D[0,\infty) \textit{ if and only if} \\ \Gamma_Z(X^h) \Rightarrow \Gamma_Z(X) \textit{ on } D[0,Z] \textit{ for all } Z \in T_X. \end{cases}$$

PROOF. See theorem 3 of Lindvall (1973) or theorem 2.8 of Whitt (1980). □

NOTATION 2.2. For $x \in D[0,\infty)$, $k \in \mathbb{N}$ and $0 \le t_1 < t_2 < \ldots < t_k$ write
$\pi_{t_1,t_2,\ldots,t_k}(x) = (x(t_i))_{i=1}^k \in \mathbb{R}^k$.

LEMMA 2.3. *Let* $X^h \in R(D[0,\infty))$, $0 < h \le h_0$ *and* $X \in R(D[0,\infty))$ *and suppose that*

$$(2.2) \quad \begin{cases} \textit{for any } k \in \mathbb{N} \textit{ and } 0 \le t_1 < t_2 < \ldots < t_k \textit{ with } t_i \in T_X \textit{ for all } i \in k: \\ \pi_{t_1,t_2,\ldots,t_k}(X^h) \Rightarrow \pi_{t_1,t_2,\ldots,t_k}(X) \text{ on } \mathbb{R}^k, \end{cases}$$

and

$$(2.3)\quad \begin{cases} \textit{for any } Z\in T_X \textit{ and any } \varepsilon>0: \\ \lim_{c\downarrow 0} \sup_{h\in(0,h_0]} \mathbb{P}\,(\Delta_c[0,Z](X^h)>\varepsilon) = 0. \end{cases}$$

Then: $X^h \Rightarrow X$ *on* $D[0,\infty)$.

PROOF. Consider $Z\in T_X$. Since d_Z is a natural extension of d_1, it results analogously to the proof of theorem 3.2.1 on p.283 of Skorohod (1956) that $\Gamma_Z(X^h) \Rightarrow \Gamma_Z(X)$ on $D[0,Z]$.
Application of lemma 2.1 completes the proof. □

REMARK 2.4. Condition (2.2) requires weak convergence of the so-called finite dimensional distributions. Condition (2.3) together with (2.2) implies that the collection of processes is weakly compact (tight).
In the next section we will consider Markov processes and present specific conditions on their transition probabilities which guarantee (2.2) and (2.3). □

3. WEAK CONVERGENCE OF MARKOV PROCESSES

In view of the approximation analysis, this section analyzes the conditions of lemma 2.3 with the following stochastic processes:

(i) $X = (X_t)_{t\geq 0} \in R(D[0,\infty))$ a Markov process with $\{P_{s,t} \mid s,t\geq 0\}$ its transition probabilities.

(ii) For some $h_0>0$ and any $h\leq h_0$:
$X^h = (X^h_t)_{t\geq 0}$ a Markov process with $X^h_t = X^h_{nh}$, $n = [th^{-1}]$, $t\geq 0$ and $\{P^h_{jh,mh} \mid j,m\in\mathbb{N}\}$ its transition probabilities at $\{nh \mid n=0,1,2,\ldots\}$.

Clearly, for any $h\in(0,h_0]$: $X^h\in R(D[0,\infty))$ and by denoting $P^h_{s,t} = P^h_{jh,mh}$ with $j = [sh^{-1}]$ and $m=[th^{-1}]$, we obtain the corresponding transition probabilities for all $s,t\geq 0$.

In order to analyze weak convergence of finite-dimensional distributions given by (2.2), we first present two lemmas. Therefore let S,S_1,S_2 denote separable metric spaces with Borel-field β,β_1 and β_2 respectively.

<u>LEMMA 3.1.</u>

Let X, X^h *for all* $h \in (0,h_0]$ *be random elements on* S.
and f, f^h *for all* $h \in (0,h_0]$ *real-valued bounded and measurable functions on* S.
If

$$X^h \Rightarrow X$$

and

$$\text{for } \mathbb{P}_X \text{ - almost all } x\text{: } f^h(x^h) \to f(x) \text{ as } x^h \to x ,$$

then

$$\mathbb{E}\, f(X^h) \Rightarrow \mathbb{E}\, f(X).$$

<u>PROOF</u>. See Hordijk and Van Der Duyn Schouten (1980) theorem 6.11. □

<u>DEFINITION 3.2.</u>

Let $Z = (Z_1, Z_2)$ be a random element on $S_1 \times S_2$
and Q a transition probability from S_1 into S_2.
Suppose that for any $B_1 \in \beta_1$ and $B_2 \in \beta_2$ it holds that

$$\mathbb{P}\,(Z_1 \in B_1, Z_2 \in B_2) = \int Q(z_1;B_2)\, I_{B_1}(z_1)\, d\mathbb{P}_{Z_1}(z_1). \tag{3.1}$$

Then Z_2 is said to be induced by $[Z_1,Q]$. □

<u>LEMMA 3.3.</u>

Let $Z = (Z_1, Z_2)$ *and* $Z^h = (Z_1^h, Z_2^h)$ *for all* $h \in (0,h_0]$ *be random elements on* $S_1 \times S_2$ *and* Q *and* Q^h *for all* $h \in (0,h_0]$ *transition probabilities such that*

Z_2 *is induced by* $[Z_1,Q]$, *and*
Z_2^h *is induced by* $[Z_1^h,Q^h]$ *for all* $h \in (0,h_0]$.

If

$$Z_1^h \Rightarrow Z_1 \text{ on } S_1 \quad \text{, and for } \mathbb{P}_{Z_1} \text{ - almost all } z_1\text{:} \tag{3.2}$$

$$Q^h(z_1^h;.) \Rightarrow Q(z_1;.) \quad \text{as } z_1^h \to z_1 , \tag{3.3}$$

then: $(Z_1^h, Z_2^h) \Rightarrow (Z_1, Z_2)$ on $S_1 \times S_2$.

<u>PROOF</u>. According to theorem 3.1 of Billingsley (1968), it suffices to show:

(i) $P_{(Z_1^h,Z_2^h)}(B_1 \times B_2) \to P_{(Z_1,Z_2)}(B_1 \times B_2)$ for all B_1, B_2 with

(ii) $\quad P_{(Z_1,Z_2)}(\partial B_1 \times S_2) = P_{(Z_1,Z_2)}(S_1 \times \partial B_2) = 0.$

Let (ii) be satisfied for $B_1 \times B_2$. Then, by expression (3.1):
$\mathbb{P}_Z(\partial B_1) = 0$ and $Q(z_1;\partial B_2) = 0$ for $z_1 : \mathbb{P}_{Z_1}$ -almost sure.

Hence, (3.3) together with the portmanteau theorem, see Billingsley (1968) p.19, implies that for $\mathbb{P}_{Z_1}$ -almost all z_1:

(iii) $\quad Q^h(z_1^h;B_2)\, 1_{B_1}(z_1^h) \to Q(z_1;B_2)\, 1_{B_1}(z_1) \quad$ as $z_1^h \to z_1$.

Consider expression (3.1). Then the application of lemma 3.1, condition (3.2) as well as condition (iii) yield condition (i). □

The above lemma enables us to conclude convergence of the finite-dimensional distributions from that of the transition probabilities as follows.

<u>PROPOSITION 3.4</u>. *Suppose that*

(3.4) $\qquad X_0^h \Rightarrow X_0 \qquad ,$

and

(3.5) $\qquad \begin{cases} P_{s,t}^h(x^h;.) \Rightarrow P_{s,t}(x;.) \\ \text{for all } x^h \to x,\ x \in \mathbb{R} \text{ and } s,t \geq 0. \end{cases}$

Then the weak convergence condition (2.2) *is satisfied.*

<u>PROOF</u>. Let $0 \leq t_1 < t_2 < \ldots < t_k$. From (0.2) conclude that

$X_{t_1}^h$ is induced by $[X_0^h, P_{0,t_1}^h]$, and

X_{t_1} is induced by $[X_0, P_{0,t_1}]$.

Hence, lemma 3.3 together with (3.4) and (3.5) yields $(X_0^h, X_{t_1}^h) \Rightarrow (X_0, X_{t_1})$ which implies $X_{t_1}^h \Rightarrow X_{t_1}$.

The proof proceeds by induction on $\ell < k$ as follows. Define random elements Y^h and Y on $\mathbb{R}^\ell$ by: $Y^h = \pi_{t_1,\ldots,t_\ell}(X^h)$ and $Y = \pi_{t_1,\ldots,t_\ell}(Y)$. Then again according to (0.2):

$$\text{(i)} \begin{cases} X^h_{t_{\ell+1}} \text{ is induced by } [Y^h, \bar{P}^h_{t_\ell, t_{\ell+1}}], \text{ and} \\ X_{t_{\ell+1}} \text{ is induced by } [Y, \bar{P}_{t_\ell, t_{\ell+1}}], \end{cases}$$

where

$$\text{(ii)} \begin{cases} \bar{P}^h_{s,t}((x_j)_{j=1}^{\ell};.) = P^h_{s,t}(x_\ell;.) \\ \bar{P}_{s,t}((x_j)_{j=1}^{\ell};.) = P_{s,t}(x_\ell;.). \end{cases}$$

Since for $y^h = (x^h_j)_{j=1}^{\ell} \in \mathbb{R}^\ell$ and $y = (x_j)_{j=1}^{\ell} \in \mathbb{R}^\ell$: $y^h \to y$ on $\mathbb{R}$ implies that $x^h_\ell \to x_\ell$, we conclude from conditions (3.5), (i) and (ii): condition (3.3) is satisfied with

$S_1 = \mathbb{R}^\ell$, $Z^h_1 = Y^h$, $Z_1 = Y$, $S_2 = \mathbb{R}$, $Q^h = \bar{P}^h_{t_\ell, t_{\ell+1}}$ and $Q = \bar{P}_{t_\ell, t_{\ell+1}}$.

Hence with an induction hypothesis $Y^h_\ell \Rightarrow Y_\ell$ we obtain from lemma 3.3: $(Y^h, X^h_{t_{\ell+1}}) \Rightarrow (Y, X_{t_{\ell+1}})$ on $\mathbb{R}^{\ell+1}$, or equivalently

$\pi_{t_1,\ldots,t_{\ell+1}}(X^h) \Rightarrow \pi_{t_1,\ldots,t_{\ell+1}}(X)$. □

The following theorems are given such that in section 5 of chapter I they can directly be applied to discrete-time approximations for solutions of stochastic differential equations, respectively jump processes with bounded respectively unbounded jump rates.

THEOREM 3.5. *Suppose that*

(i) *relations* (3.4) *and* (3.5) *are satisfied, and*

(ii) *for some* $\alpha > 0$, $\gamma > 0$ *and any* $Z > 0$ *exists a constant* K_Z *such that for all* $0 \le nh \le \ell h \le Z$ *and all* $h \in (0, h_0]$:

$$\mathbb{E}\,|X^h_{\ell h} - X^h_{nh}|^\gamma \le K_Z |\ell h - nh|^{1+\alpha}. \tag{3.6}$$

Then $X^h \Rightarrow X$ *on* $D[0,\infty)$.

PROOF. According to proposition 3.4, condition (i) implies (2.2). Let $Z \in T_X$. Then, since d_Z is the natural extension, it is justified to apply theorem 3 on p.341 of Gihman and Skorohod (1974). As a result, relation (2.1) is verified by showing:

$$(3.7)\qquad \mathbb{E}\,(|X^h_{t_1} - X^h_{t_2}|\ |X^h_{t_2} - X^h_{t_3}|)^{\frac{1}{2}\gamma} \le H(t_3 - t_1)^{1+\alpha}$$

for all $h\in(0,h_0]$ $0\le t_1\le t_2\le t_3\le Z$ and with constants $H,\alpha,\beta>0$. Since X^h_t is constant for $t\in[nh,nh+h)$, $n\in\mathbb{N}$, the left hand side of (3.7) is equal to 0 if $\frac{1}{2}h > \max(|t_1-t_2|, |t_2-t_3|)$. For $\frac{1}{2}h \le \max(|t_1-t_2|, |t_2-t_3|)$ we can bound it by using Schwartz' inequality by

$$(3.8)\qquad (\mathbb{E}\,|X^h_{t_1} - X^h_{t_2}|^\gamma \cdot \mathbb{E}\,|X^h_{t_2} - X^h_{t_3}|^\gamma)^{\frac{1}{2}} \le$$

$$\sup_{|\ell h-nh|\le(t_3-t_1)+h} \{\mathbb{E}\,|X^h_{\ell h} - X^h_{nh}|^\gamma\} \le 3K_Z(t_3-t_1)^{1+\alpha},$$

where the last inequality follows from (3.6) and $h\le 2(t_3-t_1)$. Consequently, for all $h\in(0,h_0]$ relation (3.7) is satisfied. □

For $x\in\mathbb{R}$ and $\varepsilon\ge 0$ write: $V_\varepsilon(x) = \{y\in\mathbb{R}|\ |y-x| > \varepsilon\}$

<u>THEOREM 3.6.</u> *Suppose that*

(i) *relations* (3.4) *and* (3.5) *hold, and*

(ii) *for any* $\varepsilon>0$ *there exists a constant* $Q(\varepsilon)$ *such that for all* $\delta>0$ *and* $h\in(0,h_0]$:

$$(3.9)\qquad \sup_{x\in\mathbb{R}}\ \sup_{|\ell h-nh|<\delta} P^h_{\ell h,nh}(x;V_\varepsilon(x)) \le \delta\,Q(\varepsilon)\ .$$

Then $X^h \Rightarrow X$ *on* $D[0,\infty)$.

<u>PROOF</u>. According to proposition 3.4, condition (i) implies (2.2). Let $Z\in T_X$. Recall relation (1.2) for the Skorohod modulus on $[0,Z]$. Then, since X^h_t is constant for $t\in[nh,nh+h)$, $n\in\mathbb{N}$, it follows that

$$(3.10)\qquad \begin{cases} \mathbb{P}\,(\Delta_c[0,Z](X^h)>0) = 0 \quad \text{if } c<\frac{1}{2}h \text{, and} \\ \mathbb{P}\,(\Delta_c[0,Z](X^h)\ge\varepsilon) \le \sup\limits_{0\le n_2h-c-h\le n_1h\le n_2h\le n_3h\le n_2h+c+h} \\ \mathbb{P}\,(|X^h_{n_1h}-X^h_{n_2h}| \wedge |X^h_{n_2h}-X^h_{n_3h}| \ge \varepsilon) \qquad \text{else}. \end{cases}$$

By applying lemma 3 on p.431 of Gihman and Skorohod (1974) with $\alpha = 3cQ(\varepsilon/4)$ and choosing c such that $3cQ(\varepsilon/4)\le\frac{1}{2}$, we conclude from (3.9) and (3.10):

(3.11) $\mathbb{P}\,(\Delta_c[0,Z](X^h) \geq \varepsilon) \leq 6cQ(\varepsilon/4)$.

Since this holds for any fixed $\varepsilon > 0$ uniformly in all $h \in (0,h_0]$, relation (2.3) follows from letting c tend to 0. Hence, lemma 2.3 completes the proof. □

The notation $\mathbb{P}\,(\cdot|\cdot)$ used below indicates a conditional probability which is assumed to be regular. For definitions of these probability concepts we refer to any standard book on probability theory.

THEOREM 3.7. *Suppose that*

(i) *relations* (3.4) *and* (3.5) *are satisfied,*

(ii) *for any* $\varepsilon > 0$ *and any compact set* K *there exists a constant* $Q(\varepsilon,K)$ *such that for all* $\delta > 0$ *and all* $h \in (0,h_0]$:

(3.12) $$\sup_{x\in K}\ \sup_{|\ell h - nh|<\delta}\ P^h_{\ell h,nh}(x;V_\varepsilon(x)) \leq \delta\, Q(\varepsilon,K),$$

(iii) *for any* $\eta > 0$ *any* $Z < \infty$ *and compact set* K_0 *there exists a compact set* $K = K(\eta,Z,K_0)$ *such that for all* $h \in (0,h_0]$:

(3.13) $$\inf_{x\in K_0}\ \mathbb{P}\,(X^h_t \in K \text{ for all } t \in [0,Z]\,|\,X^h_0 = x) > 1-\eta \quad .$$

Then $X^h \Rightarrow X$ *on* $D[0,\infty)$.

PROOF. According to proposition 3.4 condition (i) implies (2.2). Let $Z \in T_X$. Choose $\eta > 0$ arbitrarily. Since $X^h_0 \Rightarrow X_0$, it follows from theorem 2 on p.377 of Gihman and Skorohod (1974) that for some compact set K_0 and all $h \in (0,h_0]$:

(i) $\mathbb{P}\,(X^h_0 \in K_0) > 1-\eta$

Then, by conditioning on the initial state X^h_0 and using (i) together with (3.13) we also obtain for some compact set K depending on η,Z and K_0:

(ii) $\mathbb{P}\,(X^h_t \in K \text{ for all } t \in [0,Z]) > 1-2\eta$.

Next, let us recall the Skorohod modulus given by (2.1). Then by (ii),

$$(3.14)\quad \begin{cases} \mathbb{P}\,(\Delta_c[0,Z](X^h) \geq 4\varepsilon) \leq 2\eta + \\ \\ \mathbb{P}\,(\Delta_c[0,Z](X^h) \geq 4\varepsilon \text{ and } X_t^h \in K \text{ for all } t\in[0,Z]). \end{cases}$$

The last term can be bounded analogously to (3.10) but with extra requirements that: $X_t^h \in K$ for all $t\in[0,Z]$.
Then, in analogy with the proof of lemma 2 on p.420 and lemma 3 on p.431 of Gihman and Skorohod (1974), it can be shown that, after choosing c such that $3c\,Q\,(\frac{\varepsilon}{4}, K) \leq \frac{1}{2}$, the relation (3.10) with $X_t^h \in K$ for all $t\in[0,Z]$ and the relation (3.12) imply

$$(3.15)\qquad \mathbb{P}\,(\Delta_c[0,Z](X^h) \geq 4\varepsilon \text{ and } X_t^h \in K \text{ for all } t\in[0,Z]) \leq 6\,c\,Q\,(\varepsilon,K).$$

Consequently, for any fixed $\eta > 0$ the combination of (3.14) and (3.15) yields

$$(3.16)\qquad \lim_{c\downarrow 0}\ \sup_{h\in(0,h_0]}\ \mathbb{P}\,(\Delta_c[0,Z](X^h) \geq 4\varepsilon) \leq 2\eta.$$

Since ε and η are chosen arbitrarily, relation (3.16) proves (2.3).
Lemma 2.3 completes the proof. □

B PROOF OF LEMMA I.5.3.15.

Let $f\in C^u(\mathbb{R})$ and first consider $f(\cdot)$ on $[0,\infty)$. Let $\varepsilon > 0$.
Then there exists a $\delta_\varepsilon > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta_\varepsilon$.
Define with $t_0 = 0$ a sequence $\{t_i\}_{i=0}^{\infty}$ such that
$|f(x) - f(t_i)| < \varepsilon$ for all $x\in[t_i,t_{i+1})$ and $|f(t_{i+1}) - f(t_i)| = \varepsilon$. If
$|f(x) - f(t_i)| < \varepsilon$ for all $x \geq t_i$ then define $t_{i+1} := \infty$. Hence,
$|t_{i+1} - t_i| \geq \delta_\varepsilon > 0$ for $i = 0,1,2,\ldots$
Next define a function $g_\varepsilon^+ : [0,\infty) \to \mathbb{R}$ by

$$g_\varepsilon^+(x) = \begin{cases} f(t_i) + [f(t_{i+1}) - f(t_i)]\Big[-20\,\Big(\dfrac{x-t_i}{t_{i+1}-t_i}\Big)^7 + 70\,\Big(\dfrac{x-t_i}{t_{i+1}-t_i}\Big)^6 \\ \qquad - 84\,\Big(\dfrac{x-t_i}{t_{i+1}-t_i}\Big)^5 + 35\,\Big(\dfrac{x-t_1}{t_{i+1}-t_i}\Big)^4\Big], & \text{if } t_i \leq x < t_{i+1} < \infty, \\ \\ f(t_i) & \text{if } t_i \leq x, t_{i+1} = \infty. \end{cases}$$

Then, one easily verifies the following properties with $k=1,2,3$:

(i) $\frac{d^k}{dx^k} g_\varepsilon^+(x)$ exists and is equal to 0 for $x = t_i$ with $t_i < \infty$,

(ii) $\frac{d^k}{dx^k} g_\varepsilon^+(x) \leq \varepsilon\ 2.10^3\ [(\delta_\varepsilon)^{-3} \wedge 1]$, $x \geq 0$, and

(iii) $|g_\varepsilon^+(x) - f(x)| \leq \varepsilon .\ 211$, $x \geq 0$

Analogously, one can find a function $g_\varepsilon^- : (-\infty, 0] \to \mathbb{R}$. Combining g_ε^+ and g_ε^- and noting that ε is chosen arbitrarily completes the proof. □

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