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Stochastic processes and point processes of excursions

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1991 Mathematics Subject Classification: 60G55, 60J25, 60J55, 60J65.
ISBN 90 6196 438 5
NUGI-code: 811

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Printed in the Netherlands

Preface

In 1970 K. Itô published a paper which was titled: "Poisson point processes attached to Markov processes". In this paper he considers excursions from a given state a of a standard Markov process. He notes that a special role is played by the local time at a: labeled with the local time these excursions can be considered as the points of a Poisson point process. This tells us something about the randomness of the Markov process. In this paper, it is also remarked that the stochastic process can be reconstructed from the point process. In this book we will present a detailed discussion of excursions from the point of view of the theory of point processes and random measures. We also give the precise reconstruction of stochastic processes from point processes of excursions.

At this place I should like to thank Prof. dr. C.L. Scheffer for his help by the preparation of this book. I also should like to thank dr. J.G.M. Schoenmakers for his critical comments and dr. W.H. Penninx, who was able to solve all tex-problems.

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Chapter 1

Introduction

In his studies [35] and [36] of the sample paths of Brownian motion, Lévy developed the idea to decompose the time set $[0, \infty[$ in a part Z at which the process is in state 0 and intervals of time spent in $\mathbb{R}\backslash\{0\}$. Throughout the years this has proved to be a very fruitful idea. On one hand the study of the set of zeros Z led Lévy to the description of local time as an occupation density (Lévy used in [35] the term "mesure du voisinage". See for occupation densities the survey article of Geman and Horowitz [14], who discuss connections between the behaviour of a (non-random) real-valued Borel function and the behaviour of its occupation density. Local times for general Markov processes were introduced by Blumenthal and Getoor in [3]). On the other hand Lévy's study of the behaviour of Brownian motion on zero-free intervals was the starting point of excursion theory. Lévy's theory was extended in Itô-McKean [27], (2.9) and (2.10). See also Chung's article [6], in which elementary derivations are given of a number of Lévy's results. This research led to many deep theorems about the behaviour of the paths of diffusions, see for instance Williams [58] and Walsh's discussion of Williams' results in [53]. Another important application of excursion theory can be found in the construction of those strong Markov processes, which behave outside a fixed state (or more generally outside a set D) as a given Markov process X. In this area the works of Dynkin [9], [10] and Watanabe [54], [55] are important. For excursions from a subset S, see the works of Maisonneuve [38], [39] and Getoor [16]. Getoor gives also an application to invariant measures, see also Kaspi [30] and [31]. Unlike occupation densities, which are also useful in the study of non-random functions, excursion theory takes its use from the Markov character of the random process. To make clear the ideas behind excursion theory, let $X = (X_n)_{n\geq 0}$ be a homogeneous Markov chain with state space E. Let

$a \in E$ be a given state. Denote for $k = 1, 2, \ldots$ by ν_a^k the time at which the Markov chain X visits the state a for the k^{th} time: $\nu_a^k = \infty$ if there are less than k visits to a. Suppose that a is a recurrent state, i.e. $\mathbb{P}_a[\nu_a^1 < \infty] = 1$. Then the k^{th} excursion $V_k = (V_k(n))_{n \geq 0}$ from a of the Markov Chain X is defined as follows

$$V_k(n) = \begin{cases} X_{\nu_a^k + n} & \text{for } 0 \leq n < \nu_a^{k+1} - \nu_a^k \\ a & \text{for } n \geq \nu_a^{k+1} - \nu_a^k \end{cases}$$

Let $V_0 = (V_0(n))_{n \geq 0}$ be defined by

$$V_0(n) = \begin{cases} X_n & \text{for } n < \nu_a^1 \\ a & \text{for } n \geq \nu_a^1 \end{cases}$$

It follows from the strong Markov property that the sequence of excursions $(V_k)_{k \geq 1}$ is independent and identically distributed. It is clear that the process X can be reconstructed pathwise from the sequence $(V_k)_{k \geq 0}$.

For Markov processes with continuous time parameter the situation is more complicated. As an example take standard Brownian motion $B = (B_t)_{t \geq 0}$ and consider the excursions from state 0. Let Z be the set of zeros of B. The component intervals of $[0, \infty[\setminus Z$ are called excursion intervals. Since Z is a topological Cantor set of Lebesgue measure 0 (see Itô-McKean [27], problem 5, p.29), with probability 1 there is no first excursion interval. Let $I =]\alpha, \beta[$ be an excursion interval. The map $V_I : [0, \infty[\rightarrow\mapsto \mathbb{R}$ defined by

$$V_I(t) = \begin{cases} B_{\alpha+t} & \text{for } 0 \leq t < \beta - \alpha \\ 0 & \text{for } t \geq \beta - \alpha \end{cases}$$

is the excursion made by B from 0 corresponding to the excursion interval I; $\zeta = \beta - \alpha$ is called the length of the excursion. Put $\tau_I = \phi(\alpha)$, where ϕ is the local time of B at zero. Itô proved in [25] (see also Meyer [41]) that the random distribution of points (τ_I, V_I) in $[0, \infty[\times U, I$ running through the excursion intervals and U being the space of excursions from 0, is a Poisson point process on $[0, \infty[\times U$ whose intensity measure is the product of Lebesgue measure λ on $[0, \infty[$ and some σ-finite measure ν on U. This means that the number of points (τ_I, V_I) in a set $[u, v[\times U_0 \subset [0, \infty[\times U$ is Poisson distributed with expectation $(v - u)\nu(U_0)$ whilst the numbers of points (τ_I, V_I) in disjoint subsets of $[0, \infty[\times U$ are independent. Itô proved this result actually for excursions of a standard Markov process X from a regular point a, and he gave a characterization of the excursion law ν of a recurrent extension of X, i.e. a strong Markov process, which behaves as X until the first hitting of state a.

It is interesting to look at Itô's definition of a point process. Let $(S, \mathcal{S})$ be a measurable space. A point function $p :]0, \infty[\mapsto U$ is defined to be a map from a countable set $D_p \subset]0,\infty[$ into U. Meyer in [41] considers a point function p as a map defined for all points in $]0,\infty[$ by putting $p(x) = \partial$ for $x \in]0,\infty[\setminus D_p$ where ∂ is an extra point added to U. Let now $\prod$ be the space of all point functions: $]0,\infty[\mapsto U$. Denote for $p \in \prod$ and for $E \in \mathcal{B}(]0,\infty[) \otimes \mathcal{S}$ by $N(E,p)$ the number of the time points $t \in D_p$ for which $(t,p(t)) \in E$. The Borel σ-algebra $\mathcal{B}(\prod)$ on $\prod$ is defined as the σ-algebra generated by the sets $\{p \in \prod : N(E,p) = k\}$, $E \in \mathcal{B}(]0,\infty[) \otimes \mathcal{S}$, $k = 0,1,2,\ldots$ Itô defined a point process as a $(\prod, \mathcal{B}(\prod))$-valued random variable. For instance the point process of excursions from 0 of Brownian motion is the (stochastic) point function p defined by

$$D_p = \{\tau_I : I \text{ an excursion interval}\} \text{ and}$$
$$p(t) = V_I, t = \tau_I \in D_p.$$

This definition gives a clear picture of point processes such as they appear in excursion theory and that is presumably the reason why in studies about excursion theory this definition is always used, see for instance Watanabe [54] and Greenwood and Pitman [17]. Beside this definition of a point process as a stochastic point function, there exists a fairly general theory of point processes which views a point process as a discrete random measure. See Neveu [44], Jagers [29] and Krickeberg [34] for point processes on a locally compact space and Matthes, Kerstan and Mecke [40] for point processes on a complete, separable metric space. This measure-theoretical approach to excursions makes it possible to use some important results from this theory, such as e.g. the Palm-formula, which were up to now not used in the literature about excursion theory. An example of the use of the Palm-formula can be found in the construction of a Markov process from a Poisson point process of excursions. Itô only remarks in [25] that this can be done by reversing the procedure of deriving the excursion process from a Markov process. In Ikeda & Watanabe [22] Brownian motion is constructed from its excursion process using the general theory of stochastic processes (compensators and stochastic integrals). And in [2] Blumenthal gives a construction of which he claims that it is the construction Itô had in mind; this construction consists of a pathwise approximation of the Markov process. The most recent and complete work along these lines can be found in Salisbury [47] and [48]. The construction that we will give is based on an application of the Palm formula and on the so-called renewal property of a Poisson point process of excursions. This construction has in our opinion the advantage that it makes clear why the constructed process has the Markov property and it displays the role of local time in the construction. The same method can be used to write down a formula for the resolvent of the constructed process.

We continue with the definition of a point process as a discrete random measure. Let X be a topological space with Borel σ-algebra $\mathcal{B}(X)$. Roughly stated, a point process on X is a probability measure on the space of locally finite point measures on $(X, \mathcal{B}(X))$, or a random variable with values in the space of locally finite point measures on $(X, \mathcal{B}(X))$ by which we identify a random variable with its distribution. From now on we will use the word point process only in this sense. In excursion theory the topological space X is the (topological) product of the set of nonnegative reals $[0, \infty[$ with the usual topology and the space of excursions U endowed with the Skorohod topology. For example the point process of excursions from 0 of Brownian motion is the random measure $\sum\{\delta_{(\tau_I, V_I)} : I \text{ an excursion interval}\}$ where δ_x is the notation for the Dirac measure in x. Note that $X = [0, \infty[\times U$ is a polish space. The main reference on point processes on polish spaces is the book [40] of Matthes, Kerstan and Mecke. The theory which they develop depends essentially on a fixed metric d on X, chosen in advance, such that the metric topology coincides with the topology of X and (X, d) is a complete, separable metric space. A nonnegative Borel measure on X is locally finite if it is finite on the sets in $\mathcal{B}(X)$, which are bounded in the sense of the metric d. This theory is not directly applicable to excursion theory. The point measures which arises in excursion theory are finite on the sets $[a, b[\times[\zeta > \ell], \ell > 0$, (remember that ζ is the length of the excursion) and the most interesting cases are those where the set $[a, b[\times U$ has infinite mass. Note that the set $[\zeta > \ell]$ is dense in U. Thus it is not clear how to choose a metric d on X for which the set of locally finite measure contains this family of point measures. Instead of trying to find such a metric, it seems more natural to define local finiteness directly in terms of the sets $[a, b[\times[\zeta > \ell]$. More general, let $\mathcal{S}$ be a family of Borel subsets of X. A nonnegative Borel measure μ on X is called $\mathcal{S}$-finite if $\mu(A) < \infty$ for every $A \in \mathcal{S}$ and a point process P is an $\mathcal{S}$-finite point process if the probability measure P is concentrated on the space of $\mathcal{S}$-finite measures. The set of locally finite measures in the sense of Matthes, Kerstan and Mecke coincides then with the $\mathcal{S}$-finite measures, $\mathcal{S}$ being the family of all open balls with finite radius. Point processes on locally compact spaces are probability measures on the set of Radon measures, which is the same as the set of $\mathcal{S}$-finite measures with $\mathcal{S}$ consisting of the compact subsets.

So far we did not discuss a measurable structure on the set $\mathcal{M}^+(\mathcal{S})$ of $\mathcal{S}$-finite measures, which is of course necessary for the definition of probability measures on $\mathcal{M}^+(\mathcal{S})$. A σ-algebra on $\mathcal{M}^+(\mathcal{S})$ should at least measure the maps $\mu \in \mathcal{M}^+(\mathcal{S}) \mapsto \mu(A), A \in \mathcal{B}(X)$. In Matthes et al. [40] the σ-algebra on $\mathcal{M}^+(\mathcal{S})$ ($\mathcal{S}$ being the family of open balls of finite radius) is defined in an abstract way as the σ-algebra $\mathcal{A}$ generated by these maps. In the literature about point processes on locally compact spaces, on the other hand a σ-

algebra on the set of Radon measures is introduced in a topological way as the Borel σ-algebra $\mathcal{B}(\mathcal{M}^+)$ corresponding to the vague topology on $\mathcal{M}^+(\mathcal{S})$. It turns out that $\mathcal{B}(\mathcal{M}^+) = \mathcal{A}$ in this case, so we have a definition of $\mathcal{A}$ as a Borel σ-algebra corresponding to a nice topology on $\mathcal{M}^+(\mathcal{S})$, which makes it possible to use the apparatus of topological measure theory. In section (2.1) we will define a topology on the set $\mathcal{M}^+(\mathcal{S})$ of $\mathcal{S}$-finite measures on an arbitrary polish space X. Let $\mathcal{H}(\mathcal{S}) = \{f \in C_b(X) : \exists A \in \mathcal{S} : \operatorname{supp}(f) \subset A\}$ and let $\tau(\mathcal{S})$ be the topology $\sigma(\mathcal{M}^+(\mathcal{S}), \mathcal{H}(\mathcal{S}))$ of pointwise convergence on $\mathcal{H}(\mathcal{S})$. If $\mathcal{S}$ is a family of open subsets of X filtering to the right such that $\mathcal{S}$ covers X and such that $\mathcal{S}$ contains a countable, cofinal subset, then it will turn out that $(\mathcal{M}^+(\mathcal{S}), \tau(\mathcal{S}))$ is a Suslin space whilst the Borel σ-algebra on $\mathcal{M}^+(\mathcal{S})$ coincides with $\mathcal{A}$. At the end of the section we compare our results with the results of Harris in [19] and [20], who also defined a topology on some family of nonnegative Borel measures on a complete, separable metric space.

Section (2.2) contains standard results for $\mathcal{S}$-finite point processes, in particular the Palm-formula which is now a direct consequence of a general theorem on disintegrations of measures from topological measure theory. Further $\mathcal{S}$-finite Poisson point processes and Cox processes are discussed. Section (2.3) is devoted to the study of a special class of $\mathcal{S}$-finite Poisson point processes on $X = [0, \infty[\times U$, $\mathcal{S}$ being the family of subsets $I \times U_n$ of X where I is a bounded, open sub-interval of $[0, \infty[$ and $(U_n)_{n\geq 1}$ is a sequence of open subsets of U, increasing to U. It is clear that $\mathcal{S}$ is a filtering family of open subsets of X which covers X and has a countable, cofinal subsequence. Denote by $\mathcal{M}_1^\bullet(\mathcal{S})$ the set of $\mathcal{S}$-finite point measures μ for which $\mu(\{t\} \times U) \leq 1, t \geq 0$. An Itô-Poisson point process is a Poisson point process P on X with intensity measure $\lambda \otimes \nu$, λ denoting Lebesgue measure on $[0, \infty[$ and ν a σ-finite measure on U satisfying $\nu(U_n) < \infty, n \geq 1$. We choose the name Itô-Poisson point process, because the point process of excursions, as constructed by Itô, is of this type. Following Itô, the measure ν is called the characteristic measure of P. Further $P(\mathcal{M}_1^\bullet(\mathcal{S})) = 1$ for an Itô-Poisson point process P. The first important property of Itô-Poisson point processes is the renewal property which is treated here as a generalization of the property that a Poisson process is free from after-effects. The renewal property was already mentioned in Itô [25], but without a proof. We continue with Itô's characterization of Itô-Poisson point processes with a proof using "point process techniques". We end section (2.3) with a beautiful theorem of Greenwood and Pitman [17], which states that an Itô-Poisson point process P has an intrinsic time clock in the following sense: if $\mu \in \mathcal{M}_1^\bullet(\mathcal{S})$, denote by $\xi_{k1}(\mu), \xi_{k2}(\mu), \ldots$ the U_k-sequence of μ, i.e. $\operatorname{supp}(\mu) \cap ([0, \infty[\times U_k) = (\tau_{ki}(\mu), \xi_{ki}(\mu))_{i\geq 1}$ where the enumeration is such that the sequence $(\tau_{ki}(\mu))_{i\geq 1}$ is increasing in the order of $\mathbb{R}$. The sequence

$\xi_k = (\xi_{ki})_{i\geq 1}$ is an i.i.d. sequence on the probability space $(\mathcal{M}_1^\bullet(\mathcal{S}), P)$. The theorem of Greenwood and Pitman states that the time coordinates τ_{ki} can be reconstructed from the sequence $\xi_{k1}, \xi_{k2}, \dots$ if $\nu(U) = +\infty$. We give a complete proof of a slightly more general version of this theorem, which was formulated in [17] as a theorem on stochastic point functions.

In chapter 3 excursion theory is treated for Ray processes. We have chosen to treat excursion theory for Ray processes, since this class is in some sense the most general class of strong Markov processes, see Getoor [15] and Wiliams [59]. After a brief survey of Ray processes in section (3.1), we construct in section (3.2) the Itô-Poisson point process of excursions from a given state a of a Ray process Y. Since we want to include branchpoints in our discussion, we use a definition for excursions which differs a bit from Itô's definition, see also Rogers [45] who uses the same definition. We call excursion intervals the connected components of the complement in $[0,\infty[$ of the closed set of time points where the process hits or approaches the state a. Let $(r_k)_{k\geq 1}$ be a decreasing sequence of positive real numbers and let $U_k = \{u \in U : \zeta_u > r_k\}$. Denote by V_{kn} the n^{th} excursion of Y with length exceeding r_k. The strong Markov property implies that the sequence $(V_{kn})_{n\geq 1}$ is an independent, identically distributed sequence. Let $\tau_a = \inf\{t > 0 : Y_t = a \text{ or } Y_{t-} = a\}$. An application of the theorem of Greenwood and Pitman yields:

- If $\mathbb{P}_a[\tau_a = 0] = 1$ there exists and $\mathcal{S}$-finite Itô-Poisson point process N defined on $(\Omega, \mathcal{F}, \mathbb{P}_a)$ whose $[\zeta > \ell]$-subsequence is the sequence of excursions of Y of length greater than ℓ. The characteristic measure ν of N is the unique (modulo a multiplicative constant) measure on U of which the conditional distribution $\nu|_{U_j}$ is the probability distribution of $V_{j1}; \nu$ is a σ-finite measure with total mass $\nu(U) = +\infty$.

- In the remaining case where $\mathbb{P}_a[\tau_a = 0] = 0$ there exists an i.i.d. sequence $(\xi_n)_{n\geq 1}$ of U-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P}_a)$ whose $[\varsigma > \ell]$-subsequence of excursions of Y of length greater than ell.

Note that it was not necessary for this construction to introduce explicitly a local time at state a. Local time at state a will be discussed in section (3.3), in which we construct Markov processes from an $\mathcal{S}$-finite Itô-Poisson point process. The basic idea is the following. If $\mu \in \mathcal{M}_1^\bullet(\mathcal{S})$, then supp$(\mu)$ can be considered as a countable, ordered subset $(u_\sigma)_{\sigma\in J(\mu)}$ of U where $J(\mu)$ denotes the projection on $[0,\infty[$ of supp(μ) and where $u_\sigma = u$ iff $(\sigma, u) \in \sup(\mu)$. Note that $(u_\sigma)_{\sigma\in J(\mu)}$ is not necessarily a totally ordered subset of U. Let $L : U \to [0,\infty[$ be a given, measurable function on U. Define for $\sigma \in [0,\infty[$

$$B(\sigma,\mu) = \sum\{L(u_\tau) : \tau \in J(\mu) \cap [0,\sigma]\}$$

$$= \int \mu(d\tau du) 1_{[0,\sigma]}(\tau) L(u)$$

and

$$C(\mu) = U_{\sigma \in J(\mu)} [B(\sigma-,\mu), B(\sigma,\mu)[.$$

If $T = C(\mu)$ then denote by $\tilde{\mu}$ the concatenation of the functions $u_{\sigma|[0,L(u_\sigma)[}$, $\sigma \in J(\mu)$, that is

$$\tilde{\mu} : [0,\infty[\to E$$

$$\tilde{\mu}(s) = u_\sigma(s - B(\sigma-,\mu)) = \int \mu(d\tau dv)(v 1_{[0,L(v)[})(s - B(\tau-,\mu))$$

where $\sigma \in J(\mu)$ such that $s \in [B(\sigma-,\mu), B(\sigma,\mu)[$. In general we do not have that $[0,\infty[= C(\mu)$. If $B(\sigma,\mu)$ is strictly increasing as a function of σ, then $[0,\infty[$ is the disjoint union of $C(\mu)$ and the range of R of $B(.,\mu)$. Let now P be an $\mathcal{S}$-finite Itô-Poisson point process with charateristic measure ν. We want to construct Markov processes, so we have to assume that ν satisfies the properties of the characteristic measures which arose by the construction of the Itô-Poisson point processes of excursions in section (3.2). But it is not necessary to assume that $\nu(U) = +\infty$. In this context it is more natural to consider a family $(P_x)_{x\in E}$ of point processes, where P_x is the $\mathcal{S}$-finite Itô-Poisson point process P to which is added a first excursion corresponding to a start from x, taking in account the transition mechanism which is contained in the measure ν. For our construction we will follow the above described idea with the lifetime ζ in the role of L. Considered as a function of $\mu \in \mathcal{M}_1^\bullet(\mathcal{S})$, $B(\tau,\mu)$ is a random variable on the probability space $(\mathcal{M}_1^\bullet(\mathcal{S}), P_x)$. The Poisson-property of the point process P_x implies that the stochastic process $(B(\tau))_{\tau\geq 0}$ is a subordinator (i.e. the process $(B(\tau))_{\tau\geq 0}$ has nondecreasing càdlàg realizations and stationary independent increments). In our construction we add a linear term $\gamma\tau$ to $B(t)$, with γ a nonnegative real parameter, which gives us the general form of a subordinator with the same Lévy measure as $B(\tau)$. The simple Markov property for the constructed process is proved in theorem (2.3.6). In theorem (2.3.8) we give an expression for the resolvent and in theorem (2.3.9) the strong Markov property is proved under a weak extra condition. In theorem (2.3.10) we give an explicit formula for the Blumenthal-Getoor local time at state a. We end this section with an example of the construction of a stochastic process from a more general point process than an Itô-Poisson point process. This construction is based on a Cox process and leads to a strong Markov process which is killed exponentially in the local time at a.

In chapter 4 we give some applications of excursion theory. In the first two sections we derive explicit expressions for the characteristic measures of the Itô-Poisson point processes of excursions from 0 attached to standard Brownian motion and Brownian motion with constant drift. A natural problem is to describe all strong Markov processes which behave like a given Ray process Y until the first hitting or approach of a given state a. As far as we know the only complete solution for this problem is given in Itô and McKean [26] for the case of reflecting Brownian motion on $[0, \infty[$. In section (4.3.) we given an interpretation in terms of excursion theory of the parameters which appear in its description. In section (4.4) we will construct a model for random motion on an n-pod E_n, that is a tree with one single vertex 0 and with n leggs having infinite length. This is the most simple example of random motion on a graph. We want to define a process on E_n which is Markovian with stationary transition probabilities. We should also like to have the process to behave like standard Brownian motion restricted to a half line, when restricted to a single leg. Using the results for reflecting Brownian motion from section (4.3) we are able to characterize all strong Markov processes which satisfy this description. Frank and Durham present in [12] for the first time an intuitive description of such a process for the case n=3. They considered the case of continuous entering from 0 in a leg, which was chosen according to some given probability distribution. The difficulty which arises in the construction of this process is that the process, when starting from 0, will visit 0 infinitely many times in a finite time interval. It is therefore not possible to indicate the leg which is visited first starting from 0. The construction that we will give is based on section (3.3); our model allows also jumping in a leg, stickiness at 0 and killing with a rate proportional to local time at 0. In section (4.5) we show how theory of section (3.3) can be applied to the construction of certain Markov processes with Blumenthal uses in [2] and for the construction of which he refers to Meyer [42]. In this book only excursions from a single state a are treated. Recently this approach has been generalized by J.G.M. Schoenmakers to the description of excursions from a finite set of states.

Chapter 2

Point Processes

A point process is a random distribution of points in some space X. The case where X is the real line, more generally a locally compact, second countable Hausdorff space or a separable metric space, has been studied extensively. One always assumes that there is a family $\mathcal{S}$ of subsets of X, each of which can contain only a finite number of points. If X is locally compact then $\mathcal{S}$ consists of the compact subsets, if X has a metrical structure then $\mathcal{S}$ consists of the bounded subsets of X.

Mathematically the concept of a point process is formalized as follows. Let X be a topological space and let $\mathcal{S}$ be a family of open subsets of X. To a distribution Z of points in X we assign the point measure $\sum_{z\in Z}\delta_z$, where δ_z is the Dirac measure in z. The description with measures on X has greater flexibility than the description with subsets of X and is mathematically more convenient because of the richer structure of the linear topological nature of the space of measures. Moreover in the case of point processes with multiple points the approach via measures is more natural. So let $\mathcal{M}^+ = \mathcal{M}^+(\mathcal{S})$ be the set of all nonnegative Borel measures on X which are finite on the elements of $\mathcal{S}$. Denote by $\mathcal{A}$ the smallest σ-algebra on $\mathcal{M}^+$ which measures the maps

$$\mu \in \mathcal{M}^+ \mapsto \mu(A), \ A \in \mathcal{B}(X).$$

Let $\mathcal{M}^{\bullet\bullet} = M^{\bullet\bullet}(\mathcal{S})$ be the subset of $\mathcal{M}^+$ consisting of the point measures on X. An $\mathcal{S}-$finite point process on X is a probability measure on $(\mathcal{M}^+, \mathcal{A})$ which is concentrated on $\mathcal{M}^{\bullet\bullet}$, or an $\mathcal{M}^{\bullet\bullet}-$ valued random variable where we identify a random variable with its distribution. However, the measure-theoretic introduction of the σ-algebra $\mathcal{A}$ is not quite satisfactory. There are several reasons to prefer a definition of $\mathcal{A}$ as the Borel σ-algebra corresponding to some topological structure on $\mathcal{M}^+$: a topology on $\mathcal{M}^+$, which induces the corresponding narrow topology on the space of measures on $\mathcal{M}^+$, makes it possible to discuss weak convergence of point processes. Further, measurability properties of subsets of $\mathcal{M}^+$ (for example $\mathcal{M}^{\bullet\bullet}$) can be derived from topological properties and there is a powerful disintegration theorem for measures on topological spaces.

As an example, consider the set of nonnegative Borel measures on a locally compact, second countable Hausdorff space X. Then $\mathcal{S}$ is the set of all compact sets and $\mathcal{M}^+$ is the set of all Radon measures. Let $\mathcal{C}_K$ be the set of all real-valued, continuous functions on X with compact support. Endow $\mathcal{M}^+$ with the vague topology $\tau = \sigma(\mathcal{M}^+, \mathcal{C}_K)$ of pointwise convergence on the elements of $\mathcal{C}_K$: a net (μ_α) in $\mathcal{M}^+$ converges vaguely to $\mu \in \mathcal{M}$ iff $\mu_\alpha(f) \mapsto \mu(f)$ for each $f \in \mathcal{C}_K$, where $\mu(f)$ is the functional-analytic notation for the integral of f with respect to μ. The vague topology renders $\mathcal{M}^+$ a polish space, i.e. the vague topology on $\mathcal{M}^+$ is metrizable with a complete metric. The Borel σ-algebra $\mathcal{B}$ on $(\mathcal{M}^+, \tau)$ coincides with the σ-algebra $\mathcal{A}$ generated by the maps $\mu \in \mathcal{M}^+ \mapsto \mu(A)$, $A \in \mathcal{B}$. The basic result on weak convergence is Prohorov's theorem, which gives a characterization of the relative compact subsets of $(\mathcal{M}^+, \tau)$. The set of point measures $\mathcal{M}^{\bullet\bullet}$ is a vaguely closed subset of $\mathcal{M}^+$. See for proofs Bourbaki [5] and Krickeberg [34].

In the literature about point processes on complete, separable metric spaces (X, ρ) one studies always point processes which are finite on the family $\mathcal{S}$ of bounded Borel subsets of X In excursion theory we study point processes on a polish space U which are finite on some family $\mathcal{S}$ of open subsets. In this we cannot apply the theory of point processes on complete, separable metric spaces since it is not clear whether there exists a complete metric d for U such that $\mathcal{S}$ coincides with the family of d-bounded subsets of U. So, before we can study excursion theory we have to study the set $\mathcal{M}^+(\mathcal{S})$ of $\mathcal{S}$-finite Borel measures on a polish space U, $\mathcal{S}$ being some family of open subsets.

2.1 Topological spaces of Borel measures

We will first introduce some notations. Let X be a Suslin space with Borel σ-algebra $\mathcal{B}(X)$. A Suslin space is a Hausdorff topological space which is the image of a polish space under a continuous map, see Schwartz[49], p.96. The space of nonnegative, bounded Borel measures on X will be denoted by $\mathcal{M}_b^+(X)$. To have a sufficient amount of continuous real-valued functions on X, we will assume that X is a completely regular space, i.e. for each $x_0 \in X$ and each open neighbourhood U of x_0 there is a continuous function $f : X \mapsto [0,1]$ such that $f(x_0) = 1$ and f is identical zero on $X \backslash U$. The space of bounded continuous functions on X will be denoted by $\mathcal{C}_b(X)$.

To define a topology on the space of measures $\mathcal{M}_b^+(X)$, let $\mathcal{F}$ be a class of real-valued functions. Assume that each $f \in \mathcal{F}$ is $\mu-$integrable for each $\mu \in \mathcal{M}_b^+(X)$. Then we will denote by $\sigma(\mathcal{M}_b^+(X), \mathcal{F})$ the coarsest topology on $\mathcal{M}_b^+(X)$ for which the maps $\mu \in \mathcal{M}_b^+(X) \mapsto \mu(f), f \in \mathcal{F}$, are continuous. If we take $\mathcal{F} = \mathcal{C}_b(X))$ we get the narrow topology $\sigma(\mathcal{M}_b^+(X), \mathcal{C}_b(X))$ on $\mathcal{M}_b^+(X)$ which will be also denoted by $\tau_1(X)$. The topological space $(\mathcal{M}_b^+(X), \tau_1(X))$ is a Suslin space, see Bourbaki [5], p.6.

Let G be an open subset of X. Equipped with the relative topology, G is a completely regular Suslin space (Schwartz [49] theorem 3, p.96). If $f \in \mathcal{C}_b(X)$ then its restriction $f_{|G}$ to G is element of $\mathcal{C}_b(G)$. Define

$$\mathcal{H}(G) = \{f_{|G} : f \in \mathcal{C}_b(X), \mathrm{supp}(f) \subset G\}.$$

Denote by $\tau_2(G)$ the topology $\sigma(\mathcal{M}_b^+(G), \mathcal{H}(G))$. It is clear that $\tau_2(G) \subset \tau_1(G)$. Note that equality does not necessarily hold. Indeed if (x_n) is a sequence in G converging to a point x in the boundary of G, then the sequence of Dirac measures (δ_{x_n}) converges in the space $(\mathcal{M}_b^+(G), \tau_2(G))$ while it diverges in the space $(\mathcal{M}_b^+(G), \tau_1(G))$.

Proposition 2.1.1 *Let X be a completely regular Suslin space. If G is an open subset of X, then $(\mathcal{M}_b^+(G), \tau_2(G))$ is a Suslin space.*

Proof. Since $\tau_2(G) \subset \tau_1(G)$ and $(\mathcal{M}_b^+(G), \tau_1(G))$ is a Suslin space, it follows that $(\mathcal{M}_b^+(G), \tau_2(G))$ is the image of a polish space under a continuous map. So we only have to prove that $(\mathcal{M}_b^+(G), \tau_2(G))$ is a Hausdorff space. For this it is sufficient that $\mathcal{H}(G)$ separates the points of $\mathcal{M}_b^+(G)$.

So let O be an open subset of G and let $x \in O$. Since G is completely regular, there exists an open neighbourhood V of x whose closure $\bar{V}$ is contained in O and a continuous function $\phi_x : G \mapsto [0,1]$ such that $\phi_x(x) = 1$ and ϕ_x is zero outside V. It is clear that

$$\phi_x \in \mathcal{H}(G), x \in O,$$

$$\mathrm{supp}(\phi_x) \subset \bar{V} \subset O \subset G$$

and

$$\mathrm{and} 1_O = \bigvee_{x \in O} \phi_x.$$

Since any Suslin space is a Lindelöf space, the family $\{\phi_x; x \in O\}$ has a countable subfamily $(\phi_n)_{n \geq 1}$ with the same upper envelope 1_O, see Schwartz [49], pp. 103 and 104. Define for $n \geq 1$

$$g_n = \bigvee_{1 \leq i \leq n} \phi_i.$$

It follows that (g_n) is an sequence in $\mathcal{H}(G)$, increasing to 1_O. Hence for $\nu, \mu \in \mathcal{M}_b^+(G)$ we have

$$\forall f \in \mathcal{H}(G) : \nu(f) = \mu(f) \Longrightarrow \nu(O) = \mu(O)$$

for all open subsets $O \subset G$. By a monotone class argument we may conclude that $\mathcal{H}(G)$ separates the points of $\mathcal{M}_b^+(G)$.

Corollary 2.1.2 *Let X be a completely regular Suslin space. If G is an open subset of X, then $\mathcal{H}(G)$ separates the points of $\mathcal{M}^+(G)$.*

Remark 2.1.3 A Hausdorff topological space X is said to be a Lusin space if it is the image of a polish space under a continuous bijection, see Schwarz [49] p.94. It is clear that any Lusin space is a Suslin space. If we assume that X is itself a polish space, then $(\mathcal{M}_b^+(G), \tau_1(G))$ is also polish and it follows that $(\mathcal{M}_b^+(G), \tau_2(G))$ is a Lusin space.

Definition 2.1.4 *Let $\mathcal{S}$ be a family of subsets of X. $\mathcal{S}$ is filtering to the right with respect to inclusion if*

$$\forall A, B \in \mathcal{S}, \exists C \in \mathcal{S} : A \subset C \text{ and } B \subset C.$$

Definition 2.1.5 *Let $\mathcal{S}$ be a family of subsets of X. A subfamily $\mathcal{D}$ of $\mathcal{S}$ is a cofinal subset if*

$$\forall A \in \mathcal{S}, \exists D \in \mathcal{D} : A \subset D.$$

Definition 2.1.6 *Let $\mathcal{S}$ be a family of measurable subsets of the measurable space X. A measure $\mu \in \mathcal{M}_b^+(X)$ is $\mathcal{S}$-finite if $\mu(A) < \infty$ for all $A \in \mathcal{S}$.*

Let $\mathcal{S}$ be a family of open subsets of X, which is filtering to the right with respect to inclusion. The space of $\mathcal{S}$–finite measures will be denoted by $\mathcal{M}^+ = \mathcal{M}^+(\mathcal{S})$. We continue with a precise description of the space $\mathcal{M}^+$. Define for all pairs $A, B \in \mathcal{S}$, $A \subset B$, the map π_{AB} by

$$\pi_{AB} : \mathcal{M}_b^+(B) \mapsto \mathcal{M}_b^+(A), \pi_{AB}(\mu) = {}_A\mu$$

where ${}_A\mu$ denotes the restriction of μ to A. Then

$$((\mathcal{M}_b^+(A), \tau_2(A)), \pi_{AB})_{A,B \in \mathcal{S}}$$

is a projective system of Suslin spaces. The projective limit

$$M = M(\mathcal{S}) = \lim_{\leftarrow} \pi_{AB}\mathcal{M}_b^+(B)$$

is the subspace of the productspace $\prod_{A \in \mathcal{S}} \mathcal{M}_b^+(A)$ whose elements $\mu = (\mu_A)$ satisfy the relation $\mu_A = \pi_{AB}(\mu_B)$ whenever $A \subset B$. If $\mu \in \mathcal{M}^+$, then

$({}_A\mu)_{A\in\mathcal{S}}$ is an element of M. To describe the relation between the spaces $\mathcal{M}^+$ and M, define the map ϕ by

$$\phi : \mu \in \mathcal{M}^+ \mapsto ({}_A\mu) \in M.$$

Proposition 2.1.7 *Let X be a completely regular Suslin space and $\mathcal{S}$ be a family of open subsets of X, which is filtering to the right. If $\mathcal{S}$ is a covering of X, then ϕ is a bijection of $\mathcal{M}^+$ onto M.*

Proof. Since $\mathcal{S}$ is filtering, $\bigcup_{A\in\mathcal{S}} \mathcal{B}(A)$ is a ring of subsets of X. To show that every $(\mu_A)_{A\in\mathcal{S}} \in M$ is ϕ–image of an element $\bar{\mu} \in \mathcal{M}_+$, we define the setfunction μ by

$$\mu : \bigcup_{A\in\mathcal{S}} \mathcal{B}(A) \mapsto \mathrm{I\!R},\ \mu(G) = \mu_A(G) \text{ if } G \in \mathcal{B}(A).$$

One easily verifies that μ is unambiguously defined. To see that μ is σ-additive, let $(G_n)_{n\geq 1}$ be a pairwise disjoint sequence in $\bigcup \mathcal{B}(A)$ with union also contained in $\bigcup \mathcal{B}(A)$. Then $\bigcup G_n \in \mathcal{B}(C)$ for some $C \in \mathcal{S}$. Hence every $G_n \in \mathcal{B}(C)$. It follows that

$$\mu(\bigcup G_n) = \mu_C(\bigcup G_n) = \sum \mu_C(G_n) = \sum \mu(G_n).$$

So μ is a finite, σ-additive measure on $(X, \bigcup \mathcal{B}(A))$. Being an open cover of a Lindelöf space, $\mathcal{S}$ has a countable subcover. It follows that $\mathcal{B}(X)$ is the σ-ring generated by the ring $\cup\mathcal{B}(A)$ and that a σ-additive measure μ on $\cup\mathcal{B}(A)$ has a unique extension to a σ-additive measure $\bar{\mu} \in \mathcal{M}^+$ on $\mathcal{B}(X)$, see Halmos [18], p. 54. It is clear that $\phi(\bar{\mu}) = (\mu_A)$ and that ϕ is one-to-one.

On M we define the projective topology, the coarsest topology which makes all the projections

$$\pi_B : M \mapsto \mathcal{M}_b^+(B), \pi_B((\mu_A)) = \mu_B$$

continuous and is therefore the trace on M of the product topology on $\prod_{A\in\mathcal{S}} \mathcal{M}_b^+(A)$. Assume that $\mathcal{S}$ covers X. Denote by $\tau = \tau(\mathcal{S})$ the coarsest topology on $\mathcal{M}^+$ which makes the bijection ϕ continuous. Define

$$\mathcal{H}(\mathcal{S}) = \{f \in \mathcal{C}_b(X) \mid \exists A \in \mathcal{S} : \operatorname{supp}(f) \subset A\}.$$

Then

$$\tau = \sigma(\mathcal{M}^+, \mathcal{H}(\mathcal{S})).$$

Let $\mathcal{B}$ be the Borel σ-algebra on $(\mathcal{M}^+, \tau)$. There are other natural definitions of σ-algebras on $\mathcal{M}^+$: define

$$\mathcal{A}_1 = \sigma[\mu \in \mathcal{M}^+ \mapsto \mu(B), B \in \mathcal{B}(X)]$$

and

$$\mathcal{A}_1 = \sigma[\mu \in \mathcal{M}^+ \mapsto \mu(f), f \in \mathcal{H}(\mathcal{S})].$$

Theorem 2.1.8 *Let X be a completely regular Suslin space and let $\mathcal{S}$ be a family of open subsets of X which*

- *is filtering to the right with respect to inclusion,*
- *covers X and*
- *has a countable cofinal subset.*

Then

(i) $(\mathcal{M}^+(\mathcal{S}), \tau(\mathcal{S}))$ *is a Suslin space and*

(ii) $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{B}(\mathcal{M}^+)$

Proof. Let $\mathcal{D}$ be a countable cofinal subset of $\mathcal{S}$. It is clear that $\mathcal{D}$ is also filtering to the right and covers X. Then $\mathcal{H}(\mathcal{D}) = \mathcal{H}(\mathcal{S})$ and $\mathcal{M}^+(\mathcal{D}) = \mathcal{M}^+(\mathcal{S})$, hence $\sigma(\mathcal{M}^+, \mathcal{H}(\mathcal{D})) = \sigma(\mathcal{M}^+, \mathcal{H}(\mathcal{S}))$. Being a projective limit of of a countable family of Suslin spaces, $M(\mathcal{D})$ is a Suslin space, see Schwartz [49], p. 111. Hence $(\mathcal{M}^+(\mathcal{D}), \tau(\mathcal{D}))$ is a Suslin space and (i) follows. To prove (ii) we first remark that the family of continuous maps

$$\mu \in \mathcal{M}^+ \mapsto \mu(f), f \in \mathcal{H}(\mathcal{S}),$$

separates the points of $\mathcal{M}^+$. Indeed, suppose that for $\mu, \nu \in \mathcal{M}^+$

$$\mu(f) = \nu(f)$$

for every $f \in \mathcal{H}(\mathcal{S})$. Let $A \in \mathcal{S}$. Every $\phi \in \mathcal{H}(A)$ is the restriction to A of a function $f \in \mathcal{H}(\mathcal{S})$. Hence

$${}_A\mu(\phi) = \mu(f) = \nu(f) = {}_A\nu(\phi).$$

So by corollary (2.1.2) we have that ${}_A\mu = {}_A\nu$. It follows that $\mu = \nu$ since A was arbitrarily chosen in $\mathcal{S}$. Since $\mathcal{M}^+$ is a Suslin space, there is a countable subfamily $(f_n)_{n\geq 1}$ of $\mathcal{H}(\mathcal{S})$ such that the points of $\mathcal{M}^+(\mathcal{S})$ are separated by the maps $\psi_n : \mu \in \mathcal{M}^+(\mathcal{S}) \mapsto \mu(f_n)$. By Fernique's lemma, the sequence

$(\psi_n)_{n\geq 1}$ generates $\mathcal{B}$, see Schwartz [49] p. 104, 105 and 108. It follows now that

$$\mathcal{B} \subset \mathcal{A}_2.$$

Let now $f \in \mathcal{H}(\mathcal{S})$ and let $\mathrm{supp}(f) \subset A, A \in \mathcal{S}$. Since f is continuous, there exists a sequence of $\mathcal{B}(X)-$stepfunctions, zero outside A and converging uniformly to f. Hence the map $\mu \in \mathcal{M}^+(\mathcal{S}) \mapsto \mu(f)$ is $\mathcal{A}_1-$measurable and it follows that

$$\mathcal{A}_2 \subset \mathcal{A}_1.$$

Let $A \in \mathcal{S}$ and let $O \subset A$ be open. As in the proof of proposition (2.1.1) we construct an increasing sequence $(f_n)_{n\geq 1}$ in $\mathcal{H}(\mathcal{S})$ with $\mathrm{supp}(f_n) \subset A$ and with supremum 1_O. It follows that the map $\mu \in \mathcal{M}^+(\mathcal{S}) \mapsto \mu(O)$ is $\mathcal{B}-$measurable. A monotone class argument yields the $\mathcal{B}-$measurability of the maps $\mu \in \mathcal{M}^+(\mathcal{S}) \mapsto \mu(G)$ for all $G \in \bigcup_{A\in\mathcal{S}} \mathcal{B}(A)$. Since $\mathcal{S}$ has a countable cofinal subset, every Borel set in X can be written as a countable union of elements of $\bigcup_{A\in\mathcal{S}} \mathcal{B}(A)$ hence

$$\mathcal{A}_1 \subset \mathcal{B}.$$

We may conclude that (ii) holds.

From now on we will assume that the space X is a polish space. Let d be a metric on X such that the metric topology is the topology of X and (X, d) is a complete metric space. Let $\mathcal{S}$ be a fixed family of open subsets of X satisfying the conditions of theorem (2.1.8). Define

$$\mathcal{S}' = \{G \in \mathcal{B}(X) : \exists A \in \mathcal{S} : G \subset A\}.$$

Remark 2.1.9 **(i)** The topological space $(\mathcal{M}^+, \tau)$ of $\mathcal{S}$-finite measures on X is a Lusin space, see Remark (2.1.3).

(ii) Even for polish spaces it need not be true that a filtering family of open subsets, which covers the space, has a countable cofinal subset. For example, let X be the space of all pairs of non-negative integers with the discrete topology. Then X is a polish space. A set A is a member of the family $\mathcal{S}$ iff for all except a finite number of integers m the set $\{n : (m, n) \in A\}$ is finite. The family $\mathcal{S}$ is filtering to the right and covers X. But $\mathcal{S}$ does not have a countable cofinal subset. Indeed, let $(A_k)_{k\geq 1}$ be a sequence of subsets of X contained in $\mathcal{S}$. For every $k \geq 1$ we can choose an element $x_k = (m, n) \in X$ such that $n \geq k$ and $x_k \notin A_k$. The set $B = \{x_k : k \geq 1\}$ is an element of $\mathcal{S}$ and there is no A_k such that $B \subset A_k$.

Theorem 2.1.10 *Let X be a polish space and let $\mathcal{S}$ be a family of open subsets of X satisfying the conditions of theorem (2.1.8). Let (μ_α) be a net in $\mathcal{M}^+$ and $\mu \in \mathcal{M}^+$. Then the following statements are equivalent:*

(i) $\mu_\alpha \to \mu$ *in* $(\mathcal{M}^+, \tau)$;

(ii) $\limsup \mu_\alpha(F) \le \mu(F)$ *for all closed* $F \in \mathcal{S}'$ *and* $\liminf \mu_\alpha(O) \ge \mu(O)$ *for all open* $O \in \mathcal{S}'$.

Proof. (i) $\Rightarrow$ (ii)
Let F be a closed subset of X, $F \subset A$ for some $A \in \mathcal{S}$ and let D be a countable dense subset of X. Define

$$I = \{(x,q) : x \in D, q \in \mathbb{Q}_+, B_x(q) \cap F = \emptyset\}$$

where

$$B_x = \{y : y \in X, d(x,y) \le q\}.$$

The set I is countable. For $i = (x,q) \in I$, the closed sets F and $A^c \cup B_x(q)$ are disjoint. Since X is a normal topological space, there are disjoint open sets U and V such that $F \subset U$ and $A^c \cup B_x(q) \subset V$. By Urysohn's lemma there is a continuous function $f_i : X \mapsto [0,1]$ which $\equiv 0$ on U^c and $\equiv 1$ on F. It is clear that $\mathrm{supp}(f_i) \subset U \cap V^c \subset A$, so $f_i \in \mathcal{H}(\mathcal{S})$. If $y \in F^c$, then there is an element $i = (x,q) \in I$ such that $y \in B_x(q)$, hence $f_i(y) = 0$. It follows that

$$1_F = \inf\{f_i : i \in I\}.$$

Define

$$g_n = \inf\{f_{i_1}, \dots, f_{i_n}\},\ n \ge 1,$$

where $(i_n)_{n\ge1}$ is an enumeration of I. It is clear that (g_n) is a decreasing sequence in $\mathcal{H}(\mathcal{S})$ converging pointwise to 1_F. So for each $n \ge 1$

$$\limsup_\alpha \mu_\alpha(F) \le \limsup_\alpha \mu_\alpha(g_n) = \mu(g_n),$$

hence

$$\limsup_\alpha \mu_\alpha(F) \le \mu(F).$$

Let O be an open subset of X, $O \subset A$ for some $A \in \mathcal{S}$. Let, as in proposition (2.1.1), (g_n) be an increasing sequence of bounded continuous functions such that $\mathrm{supp}(g_n) \subset A$ and $1_O = \sup g_n$. So for each $n \ge 1$

$$\liminf_\alpha \mu_\alpha(O) \ge \liminf_\alpha \mu_\alpha(g_n) = \mu(g_n),$$

hence

$$\liminf_\alpha \mu_\alpha(O) \ge \mu_\alpha(O).$$

This completes the proof of the implication (i) $\Rightarrow$ (ii).
(ii) $\Rightarrow$ (i)
Let f be a bounded nonnegative continuous function on X with $\mathrm{supp}(f) \subset A$ for some $A \in \mathcal{S}$. Define for $k \geq 1$ the functions $u_k,\ v_k : X \mapsto \mathbb{R}$ by

$$u_k = \sum_{i \geq 1} \frac{1}{k} 1_{[f < \frac{i}{k}] \cap A}$$

and

$$v_k = \frac{1}{k} 1_{\mathrm{supp}(f)} + \sum_{i \geq 1} \frac{1}{k} 1_{[f < \frac{i}{k}]},$$

the summations being finite summations since f is bounded. It is clear that $u_k \leq f \leq v_k$ for all $k \geq 1$ and that $u_k \uparrow f$ and $v_k \downarrow f$. Hence

$$\begin{aligned} \liminf_\alpha \mu_\alpha(u_k) &\geq \sum_{i \geq 1} \frac{1}{k} \liminf_\alpha \mu_\alpha([f < \frac{i}{k}] \cap A) \\ &\geq \sum_{i \geq 1} \frac{1}{k} \mu([f < \frac{i}{k}] \cap A) \\ &= \mu(u_k) \end{aligned}$$

and analogously

$$\limsup_\alpha \mu_\alpha(v_k) \leq \mu(v_k).$$

It follows that

$$\mu(u_k) \leq \liminf_\alpha \mu_\alpha(f) \leq \limsup_\alpha \mu_\alpha(f) \leq \mu(v_k.$$

Taking limits for $k \to \infty$ we get

$$\lim \mu_\alpha(f) = \mu(f)$$

which completes the proof of the implication (ii) $\Rightarrow$ (i).

Definition 2.1.11 *Let X and $\mathcal{S}$ be as in theorem (2.1.8). A measure $\mu \in \mathcal{M}^+$ is an $\mathcal{S}$-finite point measure if*

$$\forall G \in \mathcal{S}' : \mu(G) \in \mathbb{N}.$$

An $\mathcal{S}$-finite point measure is called simple if

$$\forall x \in X, \mu_x = \mu(\{x\}) \in \{0, 1\}.$$

The set of $\mathcal{S}$-finite point measures will be denoted by $\mathcal{M}^{\bullet\bullet} = M^{\bullet\bullet}(\mathcal{S})$ and the set of simple $\mathcal{S}$-finite point measures by $\mathcal{M}^{\bullet} = M^{\bullet}(\mathcal{S})$.

Proposition 2.1.12 *Let X be a polish space and let $\mathcal{S}$ be a family of open subsets of X satisfying the conditions of theorem (2.1.8). Then $\mathcal{M}^{\bullet\bullet}$ is a closed subset of $(\mathcal{M}^+, \tau)$.*

Proof. Let (μ_α) be a net in $\mathcal{M}^{\bullet\bullet}$ converging to $\mu \in \mathcal{M}^+$. Take $x \in \text{supp}(\mu)$ and let U be an open neighbourhood of x, $U \in \mathcal{S}'$. Then by proposition (2.1.10)

$$0 < \mu(U) \leq \liminf \mu_\alpha(U).$$

Since $\mu_\alpha(U) \in \mathbb{N}$, it follows that

$$\liminf \mu_\alpha(U) \geq 1.$$

Consider now a decreasing sequence $(U_n)_{n\geq 1}$ of open neighbourhoods of x in $\mathcal{S}'$ such that $U_n \downarrow \{x\}$ and $\bar{U}_{n+1} \subset U_n$ for every $n \geq 1$. From Urysohn's lemma follows the existence of a sequence (h_n) in $\mathcal{H}(\mathcal{S})$ such that for every $n \geq 1$

$$1_{\bar{U}_{n+1}} \leq h_n \leq 1_{U_n}.$$

Then

$$\begin{aligned} \mu(h_n) = \lim_\alpha \mu_\alpha(h_n) &\geq \limsup_\alpha \mu_\alpha(\bar{U}_{n+1}) \\ &\geq \liminf_\alpha \mu_\alpha(U_{n+2}) \geq 1 \end{aligned}$$

and

$$\mu_x = \lim_{n\to\infty} \mu(h_n) \geq 1.$$

It follows that $\text{supp}(\mu)$ is a discrete set and therefore for n sufficiently large

$$\mu_x = \mu(U_n) = \mu(\bar{U}_n).$$

Proposition (2.1.10) implies that

$$\begin{aligned} \mu(\bar{U}_n) \geq \limsup_\alpha \mu_\alpha(\bar{U}_n) &\geq \limsup_\alpha \mu_\alpha(U_n) \\ &\geq \liminf_\alpha \mu_\alpha(U_n) \geq \mu(U_n). \end{aligned}$$

Hence for n sufficiently large

$$\mu_x = \lim_\alpha \mu_\alpha(U_n) \in \mathbb{N},$$

and it follows that $\mu \in \mathcal{M}^{\bullet\bullet}$.

Let $U_1, \ldots, U_n$ be a finite sequence of open subsets of X such that $U_1, \ldots, U_n \in \mathcal{S}'$ and let $k_1, \ldots, k_n \in \mathbb{N}$. Define

$$V_{U_1,\ldots,U_n;k_1,\ldots,k_n} = \{\mu \in \mathcal{M}^{\bullet\bullet} : \mu(U_i) = \mu(\bar{U}_i) = k_i, i = 1, \ldots, n\}.$$

It follows from proposition (2.1.10) that the map

$$\mu \in \mathcal{M}^+ \mapsto \mu(G)$$

is lower semicontinuous (resp. upper semicontinuous) for each open (resp. closed) subset $G \subset X$ in $\mathcal{S}'$. Hence

$$V_{U_1,\ldots,U_n;k_1,\ldots,k_n} = \mathcal{M}^{\bullet\bullet} \cap \{\mu : \mu(U_i) > k_i - \frac{1}{2} \text{ and } \mu(\bar{U}_i) < k_i + \frac{1}{2}, 1 \leq i \leq n\}$$

is open in $\mathcal{M}^{\bullet\bullet}$. Let $\mathcal{U}$ be a countable base for the topology of X consisting of open subsets with closure in $\mathcal{S}'$ (see Appendix A1) and let $A_1, A_2, \ldots$ be an increasing, countable cofinal subfamily of $\mathcal{S}$. Define for $k.n \geq 1$

$$O_{k,n} = \cup V_{U_1,\ldots,U_n;1,\ldots,1}$$

where the union is taken over all finite sequences $U_1, \ldots, U_n$ in $\mathcal{U}$ whose elements are contained in A_k. It follows that $O_{k,n}$ is open in $\mathcal{M}^{\bullet\bullet}$ and that the set

$$\begin{aligned}\{\mu \in \mathcal{M}^{\bullet\bullet} : \mu(A_k) = n\} &= \\ &\{\mu \in \mathcal{M}^{\bullet\bullet} : \mu(A_k) > n-1\} \backslash \{\mu \in \mathcal{M}^{\bullet\bullet} : \mu(A_k) > n\}\end{aligned}$$

is a Borel subset of $\mathcal{M}^{\bullet\bullet}$. So

$$\mathcal{M}^{\bullet} = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{\mu \in \mathcal{M}^{\bullet\bullet} : \mu(A_k) = n\} \cap O_{k,n}$$

is also a Borel subset of $\mathcal{M}^{\bullet\bullet}$ and we have derived the following proposition:

Proposition 2.1.13 *Let X be a polish space and let $\mathcal{S}$ be a family of open subsets of X satisfying the conditions of theorem (2.1.8). Then $\mathcal{M}^{\bullet}$ is a Borel subset of $(\mathcal{M}^+, \tau)$.*

In chapter 3 we will be interested in a special class of point measures on a product space. Let X be the product $T \times U$ of the halfline $T = [0, \infty[$ with the usual topology and a polish space U. The space X with the product topology is a polish space. Let $(U_k)_{k \geq 1}$ be an increasing sequence of open subsets of U, $U_k \uparrow U$. Define

$$\mathcal{S} = \{I \times G : I \subset T \text{ open and bounded, } G \subset U \text{ open}$$

and $G \subset U_k$ for some $k \geq 1\}$.

This family $\mathcal{S}$ satisfies the conditions of theorem (2.1.8). Denote by $\mathcal{M}^+$ the set of $\mathcal{S}$-finite measures on $(X, \mathcal{B}(X))$ and by $\mathcal{M}_1^\bullet$ the set of simple $\mathcal{S}$-finite point measures μ satifying the condition

$$\forall t \in T : \mu(\{t\} \times U) \leq 1.$$

Proposition 2.1.14 *Let X be the product $T \times U$ of the halfline $T = [0, \infty[$ with the usual topology and a polish space U. Let*

$$\mathcal{S} = \{I \times G : I \subset T \textit{ open and bounded, } G \subset U \textit{ open}$$

$$\textit{and } G \subset U_k \textit{ for some } k \geq 1\}.$$

Then $\mathcal{M}_1^\bullet$ is a Borel subset of $(\mathcal{M}_+, \tau)$.

Proof. The proof is analogous to the proof of proposition (2.1.12) and is therefore omitted.

Remark 2.1.15 (i) If X is a locally compact, second countable Hausdorff space and $\mathcal{S}$ the family of compact subsets of X, then $\mathcal{M}^\bullet$ is a dense G_δ set in $\mathcal{M}^{\bullet\bullet}$.

(ii) Let (X, d) be a complete, separable metric space and let $\mathcal{S}$ be the family of all bounded open subsets of X. The family $\mathcal{S}$ satisfies the conditions of theorem (2.1.8): a countable cofinal subset of $\mathcal{S}$ is the sequence of open balls $(B_n(z))_{n \geq 1}$ with radius $n \in \mathbb{N}$ and center a fixed point $z \in X$. Matthes, Kerstan and Mecke define in [40], section (1.15) a metric ρ on $\mathcal{M}^{\bullet\bullet}$. It turns out that $(\mathcal{M}^{\bullet\bullet}, \rho)$ is a complete, separable metric space and the metric topology on $\mathcal{M}^{\bullet\bullet}$ coincides with the relative topology on $\mathcal{M}^{\bullet\bullet}$ as a subspace of $(\mathcal{M}^{\bullet\bullet}, \tau)$.

(iii) Let (X, d) be a complete, separable metric space and x_∞ be a fixed point of X. Harris defines in [19] a (nonnegative Borel) measure μ on X to be x_∞-finite if

(a) $\mu(X \backslash V) < \infty$ for each open set V containing X_∞ and

(b) $\mu(\{x_\infty\}) = 0$.

Let M be the class of x_∞-finite measures. Define for $t > 0$

$$E_t = \{x \in X : d(x, x_\infty) \geq \frac{1}{t}\}.$$

The sets E_t are closed and have disjoint boundaries. It is clear that

$$\mu \in M \Longleftrightarrow \forall t > 0 : \mu(E_t) < \infty.$$

Harris introduced in [19] a topology on M, which we describe now. Denote for $t > 0$ by L_t^* the Levy-Prohorov distance on $\mathcal{M}_b^+(E_t)$, that is a metric on $\mathcal{M}_b^+(E_t)$ such that $(\mathcal{M}_b^+(E_t), L_t^*)$ is a complete, separable metric space and the L_t^*-topology on $\mathcal{M}_b^+(E_t)$ is the narrow topology $\sigma(\mathcal{M}_b^+(E_t), C_b(E_t))$. If $\mu, \nu \in M$ put

$$L(\mu, \nu) = \int_0^\infty \frac{e^{-t} L_t(\mu, \nu)}{1 + L_t(\mu, \nu)} dt,$$

where $L_t(\mu, \nu)$ denotes the L_t^* distance between the restrictions of μ and ν to E_t. The integral converges and L is a metric for M such that (M, L) is a complete, separable metric space.

Consider now the polish space $X \backslash \{x_\infty\}$. Let $\mathcal{S}$ be the family of open subsets

$$A_t = \{x \in X \backslash \{x_\infty\} : d(x, x_\infty) > \frac{1}{t}\}, t > 0.$$

Then $\mathcal{S}$ satisfies the conditions of theorem (2.1.8). Denote by $\bar{\mu}$ the restriction of $\mu \in M$ to $X \backslash \{x_\infty\}$. Then $\bar{\mu}$ is a $\mathcal{S}$-finite measure on $X \backslash \{x_\infty\}$.

Proposition 2.1.16 *The map*

$$\chi : \mu \in M \mapsto \bar{\mu} \in \mathcal{M}^+$$

is a continuous bijection from (M, L) *on* $(\mathcal{M}^+, \tau)$.

Proof. It is clear that χ is a bijection. To see that χ is continuous, let $f \in \mathcal{H}$ and let (μ_n) be a sequence in M converging to μ, i.e. $\lim_{n\to\infty} L(\mu_n, \mu) = 0$. Then $\mathrm{supp}(f) \subset A_t$ for all t sufficiently small. It follows that there exists a $t > 0$ such that $\mathrm{supp}(f) \subset A_t$ and $\mu(\{x \in X : d(x, x_\infty) = \frac{1}{t}\}) = 0$. From Harris [19], theorem (2.2) we conclude that

$$\begin{aligned} \lim_{n\to\infty} \mu_n(f) &= \lim_{n\to\infty} {}_{A_t}\mu_n(f \mid_{A_t}) \\ &= {}_{A_t}\mu(f \mid_{A_t}) = \mu(f). \end{aligned}$$

So for each $f \in \mathcal{H}$ the map

$$\mu \in M \mapsto (\chi(\mu))(f)$$

is continuous, which implies the continuity of χ. If $\mu, \nu \in \mathcal{M}^+$ put

$$d(\mu, \nu) = L(\chi^{-1}(\mu), \chi^{-1}(\nu).$$

Then $(\mathcal{M}^+, d)$ is a complete, separable metric space. Let τ_d denote the d-topology on $\mathcal{M}^+$. From the foregoing proposition it follows that $\tau \subset \tau_d$.

Proposition 2.1.17 *Let (μ_n) be a sequence in $\mathcal{M}^+$ and $\mu \in \mathcal{M}^+$. Then*

$$\tau\text{-}\lim_{n\to\infty} \mu_n = \mu \Longleftrightarrow d\text{-}\lim_{n\to\infty} \mu_n = \mu.$$

Proof. The implication $(\Leftarrow)$ holds since $\tau \subset \tau_d$. So assume that

$$\tau\text{-}\lim_{n\to\infty} \mu_n = \mu.$$

If

$$\mu(\{x \in X\backslash\{x_\infty\} : d(x, x_\infty) = \frac{1}{t}\}) = 0,$$

then $\mu(\partial E_t) = 0$, where ∂E_t is the boundary of E_t. Identifying μ and $\chi(\mu)$, it follows that the restrictions of (μ_n) to E_t converge in $(\mathcal{M}_b^+(E_t), L_t^*)$ to μ, see Topsoe [50], p.40. From Harris [19], theorem 2.2 we may conclude that $d\text{-}\lim \mu_n = \mu$.

So for the topologies τ and τ_d we have:

$$(\mathcal{M}^+, \tau_d) \text{ is a polish space,}$$

$$\tau \subset \tau_d,$$

$$\mu_n \xrightarrow{\tau} \mu \Longleftrightarrow \mu_n \xrightarrow{\tau_d} \mu.$$

One cannot conclude from this that $\tau = \tau_d$. Take for instance (X, τ) as in example E of Kelley [33], p.77 and take for τ_d the discrete topology on X. It is clear that τ and τ_d satisfy the above conditions and that $\tau \neq \tau_d$.

2.2 Poisson point processes

Let X be a polish space and let $\mathcal{S}$ be a family of open subsets of X which is filtering to the right with respect to inclusion. Assume that $\mathcal{S}$ has a countable cofinal subset and that $\mathcal{S}$ covers X. Denote by $\mathcal{M}^+$ the Lusin space of nonegative Borel measures on X which are finite on $\mathcal{S}$. Let P be a probability measure on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+))$. For a finite sequence $B_1, \ldots, B_m$ in $\mathcal{B}(X)$ the finite-dimensional distribution $P_{B_1,\ldots,B_m}$ is defined as the image of P under the map

$$\mu \in \mathcal{M}^+ \mapsto [\mu(B_1), \ldots, \mu(B_m)] \in ([0, \infty])^m.$$

Note that it is a consequence of theorem (2.1.8) that probability measures on $\mathcal{M}^+$ with the same finite-dimensional distributions are identical. The Laplace transform $\hat{P}$ of P is defined by

$$\hat{P}(f) = \int_{\mathcal{M}^+} P(d\mu) \exp[-\int_X f(x)\mu(dx)]$$

where f runs through the cone $\mathcal{B}(X)_+$ of nonnegative, measurable functions. The momentgenerating functions of the finite-dimensional distributions $P_{B_1,\ldots,B_m}$ are determined by $\hat{P}$ as follows from

$$\int u_1^{x_1} \cdots u_m^{x_m} P_{B_1,\ldots,B_m}(dx_1 \cdots x_m) = \hat{P}(-\sum_1^m \ln(u_i) 1_{B_i}),$$

where $0 < u_i \leq 1$, $i = 1, \ldots, m$. So P is uniquely determined by its Laplace transform. The intensity measure $i = i_P$ of P is the Borel measure on X defined by

$$i(B) = \int_{\mathcal{M}^+} P(d\mu)\mu(B), B \in \mathcal{B}(X).$$

We say that P has $\mathcal{S}$-finite intensity if $i \in \mathcal{M}^+$. Denote by ρ the Campbell measure of P, that is the measure on $\mathcal{M}^+ \times X$ defined by

$$\int_{\mathcal{M}^+ \times X} F(\mu, x)\rho(d\mu, dx) = \int_{\mathcal{M}^+} P(d\mu) \int_X \mu(dx) F(\mu, x).$$

It is clear that ρ is a σ-finite measure if the intensity measure i_P is $\mathcal{S}$-finite. The projection $\rho(\mathcal{M}^+ \times .)$ of ρ on X is the intensity measure i_P. If the intensity measure i_P is $\mathcal{S}$-finite, then a general theorem on disintegrations of measures (see Bourbaki [5], section (2.7)) implies the existence of a measurable family of probability measures $(P_x)_{x\in X}$ on $\mathcal{M}^+$ such that

$$\begin{aligned} \int_{\mathcal{M}^+ \times X} F d\rho &= \int_{\mathcal{M}^+} P(d\mu) \int_X \mu(dx) F(\mu, x) \\ &= \int_X i(dx) \int_{\mathcal{M}^+} P_x(d\mu) F(\mu, x) \end{aligned}$$

for every measurable, nonnegative function $F : \mathcal{M}^+ \times X \mapsto \mathbb{R}$. This formula is called the Palm formula and the probability measures P_x on $\mathcal{M}^+$ are called the Palm measures of P. From the Palm formula it follows that P is completely determined by the intensity measure i_P and the Palm measures $(P_x)_{x\in X}$. A straightforward calculation yields a formula for the Laplace transforms of the Palm measures P_x. Let $f, g \in \mathcal{B}(X)_+$, then

$$\int i_P(dx)\hat{P}_x(f)g(x) = -\frac{d}{dt}\hat{P}(f + tg) \mid_{t=0} .$$

Definition 2.2.1 *A probability measure P on $\mathcal{M}^+$ will be called an $\mathcal{S}$-finite point process with phase space X (or an $\mathcal{S}$-finite point process on X or a point process if the phase space and the family $\mathcal{S}$ are clear from the context) if*

$$P(\mathcal{M}^{\bullet\bullet}(\mathcal{S})) = 1.$$

A point process P is simple if

$$P(\mathcal{M}^{\bullet}(\mathcal{S})) = 1.$$

As usual in probability theory, an $\mathcal{M}^+$-valued random variable N will also be called a (simple) point process if its distribution on $\mathcal{M}^+$ is so.

Definition 2.2.2 *A point process is said to be free from after-effects if for every finite sequence $B_1, \ldots, B_m$ in $\mathcal{B}(X)$ we have*

$$P_{B_1,\ldots,B_m} = P_{B_1} \otimes \cdots \otimes P_{B_m}$$

where $P_{B_1} \otimes \cdots \otimes P_{B_m}$ is the product of the measures P_{B_i} on $\bar{\mathbb{N}}$.

Definition 2.2.3 *A Poisson point process with intensity measure ν is a point process P which is free from after-effects and whose one-dimensional distributions $P_B, B \in \mathcal{B}(X)$, are Poisson distributions with parameter $\nu(B)$, i.e. if $\nu(B) < \infty$*

$$P_B(\{k\}) = \frac{[\nu(B)]^k}{k!} \exp(-\nu(B)), k \geq 0.$$

If $\nu(B) = \infty$ then $P_B = \delta_\infty$.

Proposition 2.2.4 *Let P be a Poisson point process with $\mathcal{S}$-finite intensity measure ν. Then the Laplace transform $\hat{P}$ and the Palm-measures P_x are given by*

$$\hat{P}(f) = \exp[-\int_X \nu(dx)(1 - e^{-f(x)})], f \in \mathcal{B}(X)_+$$

and

$$P_x = \tilde{\delta}_x * P,$$

where $\tilde{\delta}_x$ denotes the Dirac measure on $\mathcal{M}^+$ in the point δ_x.

Proof. Follows from standard calculations.

Proposition 2.2.5 *For every $\nu \in \mathcal{M}^+(\mathcal{S})$ there exists a unique $\mathcal{S}$-finite Poisson point process on X with intensity measure ν.*

Proof. Let $A \in \mathcal{S}$. The restriction $_A\nu$ of ν to A is a finite measure on $(A, \mathcal{B}(A))$. So there is a unique $\{A\}$-finite Poisson point process $_AP$ on A with intensity measure $_A\nu$, see Matthes et al. [40], section (1.7). The probability measure $_AP$ is defined on $(\mathcal{M}_b^+(A), \mathcal{B}(\mathcal{M}_b^+(A)))$, where $\mathcal{B}(\mathcal{M}_b^+(A))$ is the Borel σ-algebra on $(\mathcal{M}_b^+(A), \tau_2(A))$, see section (2.1) for the definition of the topology τ_2.
Let $A, B \in \mathcal{S}, A \subset B$ and let π_{AB} be the projection of $\mathcal{M}_b^+(B)$ on $\mathcal{M}_b^+(A)$ as defined in section (2.1). The image $\pi_{AB}(_BP)$ of $(_BP)$ is a probability measure on $(\mathcal{M}_b^+(A), \mathcal{B}(\mathcal{M}_b^+(A)))$. A straightforward calculation gives $\pi_{AB}(_BP)\hat{} = {_AP}\hat{}$. So $\pi_{AB}(_BP) ={_A P}$ and it follows that
$(\mathcal{M}_b^+(A), \mathcal{B}(\mathcal{M}_b^+(A)), {_A P}, \pi_{AB})$ is a projective system of probability spaces. Since $\mathcal{S}$ has a countable cofinal subset, it is a consequence of Bochner's theorem (see Bochner [4], p. 120) that there exists a projective limit P, which is a probability measure on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+)$. An easy calculation yields that P is the $\mathcal{S}$-finite Poisson point process on X with intensity measure ν.

Denote by P_ν the $\mathcal{S}$-finite point process on X with intensity measure $\nu \in \mathcal{M}^+$. Note that P_ν is a simple point process iff the intensity measure ν is a diffuse measure (i.e. $\nu(\{x\}) = 0$ for every $x \in X$). The family of point processes $\{P_\nu; \nu \in \mathcal{M}^+\}$ is a measurable family, i.e. for every $G \in \mathcal{B}(\mathcal{M}^+)$ the map $\nu \in \mathcal{M}^+ \mapsto P_\nu(G)$ is measurable. Let V be a probability measure on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+))$ and let Q be the probability measure on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+))$ defined by

$$Q = \int_{\mathcal{M}^+} V(d\nu) P_\nu .$$

It is clear that Q is a point process on X, which is simple iff V is concentrated on the diffuse measures in $\mathcal{M}^+$. Such a process is called a Cox process.

Proposition 2.2.6 *Let Q be a Cox process as defined above. The intensity measure i_Q, the Laplace transform $\hat{Q}$ and the Palm measures $(Q_x)_{x \in X}$ of Q are given by*

$$\begin{array}{lll} i_Q(B) & = i_V(B) & , B \in \mathcal{B}(X), \\ \hat{Q}(f) & = \hat{V}(1 - e^{-f}) & , f \in \mathcal{B}(X)_+, \\ Q_x & = \tilde{\delta} * \int V_x(d\nu) P_\nu & , x \in X, \end{array}$$

where $\tilde{\delta}_x$ denotes the Dirac measure on $\mathcal{M}^+$ in the point δ_x.

Proof. The formulas for i_Q and $\hat{Q}$ follow directly from the definitions. To prove the formula for Q_x, let $F : \mathcal{M}^+ \times X \mapsto \mathbb{R}$ be a measurable, nonnegative function. Then

$$\begin{aligned}
\int Q(d\mu)\int \mu(dx)F(\mu,x) &= \int V(d\nu)\int P_\nu(d\mu)\int \mu(dx)F(\mu,x)\\
&= \int V(d\nu)\int \nu(dx)\int(\tilde{\delta}*P_\nu)(d\mu)F(\mu,x)\\
&= \int i_V(dx)\int V_x(d\nu)\int P_\nu(d\mu)F(\mu+\delta_x,x)\\
&= \int i_Q(dx)\int(\tilde{\delta}*\int V_x(d\nu)P_\nu)(d\mu)F(\mu,x)
\end{aligned}$$

from which the result follows.

2.3 Itô-Poisson point processes

Let X be the product $T\times U$ of the halfline $T=[0,\infty[$ with the usual topology and a polish space U. The Borel σ-algebras on T and U will be denoted by $\mathcal{B}_T$ and $\mathcal{U}$. Endowed with the product topology X is a polish space and its Borel σ-algebra $\mathcal{B}(X)$ is identical to the product σ-algebra $\mathcal{B}_T\otimes\mathcal{U}$. Let $(U_k)_{k\geq 1}$ be an increasing sequence of open subsets of $\mathcal{U}$ which covers $\mathcal{U}$. Define

$$\begin{aligned}
\mathcal{S}=\{A:A=I\times G, I\subset T &\text{ open and bounded},\\
G\subset U &\text{open and } G\subset U_k \text{ for some } k\geq 1\}.
\end{aligned}$$

Then $\mathcal{S}$ is a family of open subsets of X which is filtering to the right with respect to inclusion, contains a countable cofinal subfamily and covers X. The topological space of $\mathcal{S}$-finite measures on X will be denoted by $(\mathcal{M}^+,\tau)$, see section (2.1). The Borel σ-algebra $\mathcal{G}$ on $\mathcal{M}^+$ coincides with the σ-algebra generated by the family of maps $\{p_A : A\in\mathcal{B}(X)\}$, where p_A is the map $p_A:\nu\in\mathcal{M}^+\mapsto\nu(A)$. The family $(\mathcal{G})_{t\geq 0}$ of sub-σ-algebras of $\mathcal{G}$ defined by

$$\mathcal{G}_t=\sigma(p_A, A\in\mathcal{B}(X), A\subset[0,t]\times U)$$

is a filtration on $(\mathcal{M}^+,\mathcal{G})$. A measurable map $\psi:\mathcal{M}^+\mapsto T$ is called $(\mathcal{G}_t)$-adapted if $[\psi\leq t]\in\mathcal{G}_t$ for every $t\in T$.

Definition 2.3.1 *Let U be a polish space and let $(U_k)_{k\geq 1}$ be a sequence of open subsets increasing to U. Let $T=[0,\infty[$. An Itô-Poisson point process on U is an $\mathcal{S}$-finite Poisson point process P with phase space $X=T\times U$ and intensity measure $\mu=\lambda\otimes\nu$ where λ denotes the Lebesgue measure on T and ν a nonnegative Borel measure on U, which is finite on the sequence (U_k). Following Itô [25], ν is also called the characteristic measure of the Itô-Poisson point process P.*

Remember from section (2.1) that $\mathcal{M}_1^\bullet$ is the (measurable) set of simple $\mathcal{S}$-finite point measures μ satisfying the condition

$$\forall t \in T; \mu(\{t\} \times U) \leq 1.$$

Proposition 2.3.2 *Let U be a polish space and P an $\mathcal{S}$-finite Itô-Poisson point process on U with characteristic measure ν. Then*

$$P(\mathcal{M}_1^\bullet) = 1$$

Proof. Since the intensity measure $\mu = \lambda \otimes \nu$ of P is diffuse, P is a simple point process on X. Define the maps $\pi_k, k \geq 1$, by

$$\pi_k : \mathcal{M}^\bullet \mapsto \mathcal{M}_T^{\bullet\bullet}, \pi_k(\mu) = [B \in \mathcal{B}_T \mapsto \mu(B \times U_k)],$$

where $\mathcal{M}_T^{\bullet\bullet}$ denotes the space of point measures on T which are finite on all bounded subintervals of T. The maps π_k are P-a.e. defined, measurable maps on $\mathcal{M}^+$. The measure $P_k = \pi_k(P)$ is the Poisson point process on T with intensity measure $i_k = \nu(U_k)\lambda$. Since the intensity measure i_k is diffuse, the point process P_k is a simple point process. It follows that

$$P(\pi_k^{-1}(\mathcal{M}_T^{\bullet\bullet})) = P(\mathcal{M}^\bullet) = 1$$

and

$$P(\mathcal{M}_1^\bullet) = P(\bigcap_{k\geq 1} \pi_k^{-1}(\mathcal{M}_T^\bullet)) = 1,$$

which completes the proof.

Let $\phi : \mathcal{M}^+ \mapsto T$ be a measurable map. Define the transformation $R_\phi : \mathcal{M}^+ \mapsto \mathcal{M}^+$ by

$$\int_{T\times U} R_\phi(\mu)(d\sigma, du) f(\sigma, u) =$$
$$\int_{T\times U} \mu(d\sigma, du) f(\sigma, u) 1_{[0,\phi(\mu)]}(\sigma), f \in \mathcal{B}(X)_+.$$

For $\sigma \in T$ we define the map t_σ by

$$t_\sigma : (\tau, v) \in]\sigma, \infty[\times U \mapsto (\tau - \sigma, v) \in X.$$

Finally we define the transformation $T_\phi : \mathcal{M}^+ \mapsto \mathcal{M}^+$ by

$$\begin{aligned}\int_{T\times U} T_\phi(\mu)(d\sigma, du) f(\sigma, u) &= \int_{T\times U} \mu(d\sigma, du) f \circ t_{\phi(\mu)}(\sigma, u) 1_{]\phi(\mu),\infty[}(\sigma) \\ &= \int_{T\times U} \mu(d\sigma, du) f(\sigma - \phi(\mu), u) 1_{]\phi(\mu),\infty[}(\sigma).\end{aligned}$$

We will write simply R_s and T_s if ϕ is the constant map $\mu \in \mathcal{M}^+ \mapsto s$.

Lemma 2.3.3 *Let $\phi : \mathcal{M}^+ \mapsto T$ be a measurable map. The above defined maps R_ϕ and T_ϕ are both measurable.*

Proof. Define for $k, n = 1, 2, \ldots$

$$A_{kn} = \{\mu \in \mathcal{M}^+; k2^{-n} \leq \phi(\mu) < (k+1)2^{-n}\}$$

and

$$\phi_n = \sum_k (k+1)2^{-n} 1_{A_{kn}}.$$

The sequence of measurable stepfunctions $(\phi_n)_{n\geq 1}$ is a strictly decreasing sequence, which converges pointwise to ϕ. It is clear that for every bounded continuous function $f : X \mapsto \mathbb{R}$ with support contained in some element of $\mathcal{S}$ and for every $\mu \in \mathcal{M}^+$

$$\lim_{n\to\infty} (T_{\phi_n}\mu)(f) = (T_\phi \mu)(f)$$

and

$$\lim_{n\to\infty} (R_{\phi_n}\mu)(f) = (R_\phi \mu)(f).$$

It follows that the sequences $(T_{\phi_n})_{n\geq 1}$ and $(R_{\phi_n})_{n\geq 1}$ converge pointwise to T_ϕ and R_ϕ. Let $A \in \mathcal{B}(X)$ and $\mu \in \mathcal{M}^+$. Since

$$(R_{\phi_n}\mu)(A) = \sum_k 1_{A_{kn}}(\mu)\mu(A \cap [0, (k+1)2^{-n}[\times U)$$

and

$$(T_{\phi_n}\mu)(A) = \sum_k 1_{A_{kn}}(\mu)\mu((t_{(k+1)2^{-n}})^{-1}(A)),$$

it is clear that the maps $\mu \in \mathcal{M}^+ \mapsto (R_{\phi_n}\mu)(A)$ and $\mu \in \mathcal{M}^+ \mapsto (T_{\phi_n}\mu)(A)$ are measurable maps which implies the measurability of the transformations R_{ϕ_n} and T_{ϕ_n}.

Theorem 2.3.4 *(Renewal property). Let $\phi : \mathcal{M}^+ \mapsto T$ be a measurable map and let P be an Itô-Poisson point process with characteristic measure ν. If ϕ is $(\mathcal{G}_t)$-adapted, then R_ϕ and T_ϕ are independent $\mathcal{M}^+$-valued random variables on $(\mathcal{M}^+, \mathcal{G}, P)$ and $T_\phi(P) = P$.*

Proof. Consider first the case that ϕ is a stepfunction, say

$$\phi = \sum_{i\geq 1} s_i 1_{A_i}, A_i \in \mathcal{G}_{s_i}.$$

Since P is free from after-effects

$$\begin{aligned}
&\int P(d\mu)\exp[-(R_\phi\mu)(f)-(T_\phi\mu)(g)] \\
&\quad=\sum_i \int P(d\mu)1_{A_i}(\mu)\exp[-(R_{s_i}\mu)(f)-(T_{s_i}\mu)(g)] \\
&\quad=\sum_i \int P(d\mu)1_{A_i}(\mu)\exp(-(R_{s_i}\mu)(f))\int P(d\mu)\exp(-(T_{s_i}\mu)(g))
\end{aligned}$$

for $f,g\in\mathcal{B}(X)_+$. Also for every $s\geq 0$ we have:

$$\begin{aligned}
\int P(d\mu)e^{-(T_s\mu)(g)} &= \exp[-\int_s^\infty d\sigma\int\nu(d\mu)(1-e^{-g(\sigma-s,\mu)})] \\
&= \exp[-\int_0^\infty d\sigma\int\nu(d\mu)(1-e^{-g(\sigma,\mu)})] \\
&= \hat{P}(g).
\end{aligned}$$

Hence

$$\int P(d\mu)\exp[-(R_\phi\mu)(f)-(T_\phi\mu)(g)]=\int P(d\mu)e^{-(R_\phi\mu)(f)}\hat{P}(g)$$

which completes the proof of the theorem for stepfunctions. The general case will follow by approximating ϕ from above by a sequence of stepfunctions as in the proof of lemma (2.3.3).

Remark 2.3.5 Without further assumptions, it is not possible to say more about R_ϕ. As an example, let P be an Itô-Poisson point process on X with $\mathcal{S}$-finite characteristic measure ν. Let $U_0\in\mathcal{S}$ be a subset of U such that $\nu(U_0)>0$. Define the map $\phi:\mathcal{M}_1^\bullet\mapsto T$ by

$$\phi(\mu)=\min\{t\in T:\mu(\{t\}\times U_0)=1\}.$$

Since

$$\{\mu\in\mathcal{M}_1^\bullet:\phi(\mu)\leq t\}=\{\mu\in\mathcal{M}_1^\bullet:\mu([0,t]\times U_0)\geq 1\},$$

μ is a $(\mathcal{G}_t)$-adapted map on $\mathcal{M}_1^\bullet$. For $f\in\mathcal{B}(X)_+$

$$\begin{aligned}
&(R_\phi(P))\hat{} \\
&\quad=\int P(d\mu)e^{-(R_\phi\mu)(f)} \\
&\quad=\int P(d\mu)\int\mu(d\tau d\nu)1_{[\phi(\mu)=\tau]}e^{-(R_\tau\mu)(f)}
\end{aligned}$$

$$= \int_0^\infty d\tau \int_U \nu(dv) \int P(d\mu) 1_{[\phi(\mu+\delta_{(\tau,v)})=\tau]} e^{-(R_\tau(\mu+\delta_{(\tau,v)}))(f)}$$
$$= \int_0^\infty d\tau \int_U \nu(dv) \int P(d\mu)\{1_{[\phi(\mu)=\tau]} + 1_{[\phi(\mu)>\tau]} 1_{U_0}(v)\} e^{-(R_\tau\mu)(f)-f(\tau,v)}$$
$$= \int_0^\infty d\tau \int_{U_0} \nu(dv) \int P(d\mu) 1_{[\mu([0,\tau]\times U_0)=0]} e^{-\mu(1_{[0,\tau]\times U\setminus U_0} f)-f(\tau,v)}$$
$$= \int_0^\infty d\tau \int_{U_0} \nu(dv) e^{-\tau\nu(U_0)-f(\tau,v)} \int P(d\mu) e^{-\mu(1_{[0,\tau]\times U\setminus U_0} f)}.$$

Let Q_τ be the image of the probability measure $\nu(. \mid U_0)$ under the map $u \in U \mapsto \delta_{(\tau,u)} \in \mathcal{M}^+$. It is clear that for $f \in \mathcal{B}(X)_+$

$$\hat{Q}_\tau(f) = \frac{1}{\nu(U_0)} \int_{U_0} \nu(dv) e^{-f(\tau,v)}.$$

Let S_τ be the image of the probability measure P under the map $\mu \in \mathcal{M}^+ \mapsto 1_{[0,\tau]\times U\setminus U_0}\mu \in \mathcal{M}^+$. It is easy to see that S_τ is the Poisson point process with intensity measure $1_{[0,\tau]\times U\setminus U_0} \times (\lambda \otimes \nu)$. It follows that

$$\begin{aligned} (R_\phi(P))\hat{}(f) &= \int_0^\infty d\tau \nu(U_0) e^{-\tau\nu(U_0)} \hat{Q}_\tau(f) \hat{S}_\tau(f) \\ &= \int_0^\infty d\tau \nu(U_0) e^{-\tau\nu(U_0)} (Q_\tau * S_\tau)\hat{}(f) \end{aligned}$$

and

$$R_\phi(P) = \int_0^\infty d\tau \nu(U_0) e^{-\tau\nu(U_0)} (Q_\tau * S_\tau).$$

Definition 2.3.6 *A measure $\mu \in \mathcal{M}^+$ is called recurrent if*

$$\forall t > 0, \forall k \geq 1 : \mu([t,\infty[\times U_k) > 0.$$

A point process P is called recurrent if $P(\mathcal{M}_\Gamma) = 1$ where $\mathcal{M}_\Gamma$ is the set of recurrent measures.

Note that an Itô-Poisson point process with characteristic measure ν is recurrent if $\nu(U_k) > 0$ for every $k \geq 1$.
Let $\mu \in \mathcal{M}_1^\bullet \cap \mathcal{M}_\Gamma$. For every $k \geq 1$ the support of the restriction ${}_k\mu$ of μ to $T \times U_k$ is a countable infinite set whose projection on T has finite intersections with bounded subintervals of T. So we can write

$$\operatorname{supp}({}_k\mu) = ((t_{ki}, u_{ki}))_{i\geq 1}$$

where $t_{ki} \in T$, $t_{ki} < t_{k,i+1}$ and $u_{ki} \in U_k$, $i \geq 1$. For $i, k = 1, 2, \dots$ define the maps τ_{ki}, ξ_{ki} and σ_{ki} on $\mathcal{M}_1^\bullet \cap \mathcal{M}_\Gamma$ by

$$\begin{aligned} \tau_{ki}(\mu) &= t_{ki}, \\ \xi_{ki}(\mu) &= u_{ki}, \\ \sigma_{ki}(\mu) &= \begin{cases} \tau_{k1}(\mu) & \text{if } i = 1 \\ \tau_{ki}(\mu) - \tau_{k,i-1}(\mu) & \text{if } i > 1 \end{cases} \end{aligned}$$

All these maps are measurable. Denote by σ_k and ξ_k the vectors $(\sigma_{k1}, \sigma_{k2}, \dots)$ and $(\xi_{k1}, \xi_{k2}, \dots)$, $k \geq 1$.

Theorem 2.3.7 *(Itô). Let U be a polish space and $(U_k)_{k\geq 1}$ be a sequence of open subsets increasing to U. Let P be an $\mathcal{S}$-finite, recurrent, simple point process on $X = T \times U$ and let ν be a measure on $(U, \mathcal{U})$ such that*

$$0 < \nu(U_k) < \infty, k \geq 1.$$

Then, P is the Itô-Poisson point process on U with characteristic measure ν iff for each $k \geq 1$

(i) *$(\xi_{ki})_{i\geq 1}$ is a sequence of iid variables with distribution*

$$P[\xi_{ki} \in A] = \frac{\nu(A \cap U_k)}{\nu(U_k)}, A \in \mathcal{U};$$

(ii) *$(\sigma_{ki})_{i\geq 1}$ is a sequence of iid variables with distribution*

$$P[\sigma_{ki} > t] = e^{-t\nu(U_k)}, t > 0;$$

(iii) *σ_k and ξ_k are independent vectors.*

Proof. Let P be the Itô-Poisson point process on U with characteristic measure ν. Then for $k \geq 1, \lambda > 0$ and $A \in \mathcal{U}$ we have

$$\begin{aligned} &\int P(d\mu)(e^{-\lambda\sigma_{k1}} 1_A(\xi_{k1}))(\mu) \\ &= \int P(d\mu) \int \mu(d\sigma du) 1_{\sigma_{k1}(\mu)\times(A\cap U_k)}(\sigma, u) e^{-\lambda\sigma} \\ &= \int_0^\infty d\sigma \int \nu(du) \int P(d\mu) 1_{\{\sigma_{k1}(\mu+\delta_{(\sigma,u)})\}\times(A\cap U_k)}(\sigma, u) e^{-\lambda\sigma} \\ &\quad \text{by an application of the Palm formula and proposition (2.2.4)} \\ &= \int_0^\infty d\sigma \int \nu(du) \int P(d\mu) 1_{A\cap U_k}(u) 1_{\{\sigma_{k1} > \sigma\}} e^{-\lambda\sigma} \\ &= \nu(A \cap U_k) \int_0^\infty d\sigma e^{-\sigma\nu(U_k) - \lambda\sigma} \\ &= \frac{\nu(A \cap U_k)}{\nu(U_k)} \frac{\nu(U_k)}{\lambda + \nu(U_k)}. \end{aligned}$$

Let now $k, n \geq 1, \lambda_1, \ldots, \lambda_n > 0$ and $A_1, \ldots, A_n \in \mathcal{U}$. Then

$$\int P(d\mu) \prod_{i=1}^{n} (e^{-\lambda_i \sigma_{ki}} 1_{A_i}(\xi_{ki}))(\mu)$$

$$= \int P(d\mu)(e^{-\lambda_1 \sigma_{k1}} 1_{A_1}(\xi_{k1}))(R_{\sigma_{k1}}\mu) \prod_{i=2}^{n} (e^{-\lambda_i \sigma_{k,i-1}} 1_{A_i}(\xi_{k,i-1}))(T_{\sigma_{k1}}\mu)$$

$$= \int P(d\mu)(e^{-\lambda_1 \sigma_{k1}} 1_{A_1}(\xi_{k1}))(\mu) \int P(d\mu) \prod_{i=1}^{n-1} (e^{-\lambda_{i+1} \sigma_{ki}} 1_{A_{i+1}}(\xi_{ki}))(\mu)$$

by an application of theorem (2.3.4)

$$= \prod_{i=1}^{n} \int P(d\mu)(e^{-\lambda_i \sigma_{k1}} 1_{A_i}(\xi_{k1}))(\mu)$$

by mathematical induction

$$= \prod_{i=1}^{n} \frac{\nu(A \cap U_k)}{\nu(U_k)} \frac{\nu(U_k)}{\lambda_i + \nu(U_k)}.$$

It follows that (i), (ii) and (iii) hold.
To prove the converse, let $f \in \mathcal{B}(X)_+$.

$$\begin{aligned}
\hat{P}(f) &= \int P(d\mu) e^{-\int f d\mu} \\
&= \lim_{k,t \to \infty} \int P(d\mu) e^{-\int_{[0,t] \times U_k} f d\mu} \\
&= \lim \{ \sum_{n=1}^{\infty} \int_{\{\tau_{kn} \leq t < \tau_{k,n+1}\}} P[\xi_{ki} \in du_i, \tau_{ki} \in dt_i, 1 \leq i \leq n] \\
&\qquad \exp(-\sum_{1}^{n} f(t_i, u_i)) + P[\tau_{k1} > t] \} \\
&= \lim \{ \sum_{n=1}^{\infty} e^{-\nu(U_k)t} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n \int_{U_k} \nu(du_1) \\
&\qquad \cdots \int_{U_k} \nu(du_n) \exp[-\sum_{1}^{n} f(t_i, u_i)] + e^{-\nu(U_k)t} \} \\
&= \lim \sum_{n=0}^{\infty} \frac{1}{n!} \{ \int_0^t d\sigma \int_{U_k} \nu(du) e^{-f(\sigma,u)} \}^n e^{-\nu(U_k)\tau} \\
&= \exp(-\int_0^{\infty} d\sigma \int \nu(du)(1 - e^{-f(\sigma,u)})).
\end{aligned}$$

Hence P is the Itô-Poisson process with characteristic measure ν.

Theorem 2.3.8 *(Greenwood & Pitman). Let N be an $\mathcal{M}^+$-valued random variable defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let N be an Itô-Poisson point process with characteristic measure ν. If $0 < \nu(U_n) < \infty, n \geq 1$, and $\nu(U) = \infty$ then*

$$\lim_{n\to\infty} \frac{1}{\nu(U_n)} N([0,t] \times U_n) = t$$

uniformly on bounded t-intervals $\mathbb{P}$-a.e..

Proof. Fix $t \geq 0$ and define for $n \geq 1$

$$\mathcal{F}_n = \sigma(\omega \in \Omega \mapsto N_\omega([0,t] \times U_k), k \geq n).$$

The random variables $N([0,t] \times U_k)$ are integrable with expecations given by

$$\mathbb{E}N([0,t] \times U_k) = t\nu(U_k).$$

A straightforward calculation yields

$$\mathbb{E}(N([0,t] \times U_n \mid \mathcal{F}_{n+1}) = \frac{\nu(U_n)}{\nu(U_{n+1}} N([0,t] \times U_{n+1}).$$

Therefore $(\frac{1}{\nu(U_n)} N([0,t] \times U_n), \mathcal{F}_n)_{n\geq 1}$ is a reversed martingale on $(\Omega, \mathcal{F}, \mathbb{P})$. It follows that $(\frac{1}{\nu(U_n)} N([0,t] \times U_n)$ converges a.s. and in L_1 to a limit which is a random variable measurable with respect to $\mathcal{F}_\infty = \bigcap_{n\geq 1} \mathcal{F}_n$. See Neveu [43]. Since for $\lambda > 0$ as $n \to \infty$

$$\exp(-\lambda \frac{1}{\nu(U_n)} N([0,t] \times U_n)) = \exp\{t\nu(U_n)(1 - e^{-\frac{\lambda}{\nu(U_n)}})\} \to e^{-\lambda t},$$

we may conclude

$$\lim_{n\to\infty} \frac{1}{\nu(U_n} N([0,t] \times U_n) = t \text{ a.s. and in } L^1.$$

The statement of the theorem now follows from a general lemma on the convergence of positive non-decreasing functions for which we refer to Appendix A3.

Let N be as in theorem (2.3.8). For $\omega \in \Omega$ such that $N_\omega \in \mathcal{M}_1^\bullet \cap \mathcal{M}_\Gamma$ we write $\tau_{ki}(\omega), \xi_{ki}(\omega)$ and $\sigma_{ki}(\omega)$ for $\tau_{ki}(N_\omega), \xi_{ki}(N_\omega)$ and $\sigma_{ki}(N_\omega)$, see the definitions preceeding theorem (2.3.7). Denote for $k > j$ by $T_{kji}(\omega)$ the index at which the i$^{\text{th}}$ point of type U_j in the vector $\xi_k(\omega) = \xi_k(N_\omega$. If N is an Itô-Poisson point process, then $\tau_{ki}(\omega), \xi_{ki}(\omega)$ and $\sigma_{ki}(\omega)$ are $\mathbb{P}$-a.e. defined random variables.

Corollary 2.3.9 *Under the assumptions of theorem (2.3.8) we have*

$$\lim_{k\to\infty} \frac{1}{\nu(U_k)} T_{kji}(\omega) = \tau_{ji}(\omega) \ \mathbb{P} - a.s..$$

Proof. By definition of $T_{kji}(\omega)$ we have

$$N_\omega([0,\tau_{ji}(\omega)] \times U_k) = T_{kji}(\omega).$$

The corollary implies that an Itô-Poisson point process N can be constructed from (U_k, ξ_k) whenever $\nu(U) = +\infty$; the times τ_{ki} at which the ξ_{ki} occur, are already determined by (U_k, ξ_k). We will put this in a more general framework.
Let $V_k = (V_{kj})_{j\geq 1}$ be a sequence of U_k-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, $k = 1, 2, \ldots$.

Definition 2.3.10 *The sequence $(U_k, V_k)_{k\geq 1}$ is called a nested array if*

(i) *V_k is an iid sequence, and*

(ii) *for $j < k$, V_j is the U_j-subsequence of V_k consisting of those terms which are in U_j.*

See Greenwood & Pitman [17].

From theorem (2.3.7) it is clear that $(U_k, \xi_k)_{k\geq 1}$ is an example of a nested array. If μ is a measure on $(U, \mathcal{U})$ and if $E \in \mathcal{U}$ is such that $0 < \mu(E) < \infty$, then $\mu \mid_E$ denotes the measure on U defined by

$$\mu \mid_E (A) = \frac{\mu(A \cap E)}{\mu(E)}, A \in \mathcal{U}.$$

Proposition 2.3.11 *If $(U_k, V_k)_{k\geq 1}$ is a nested array on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then there exists a unique measure ν on $(U, \mathcal{U})$ such that*

$$\nu(U_1) = 1 \ \text{and} \ \nu \mid_{U_j} = \nu_j,$$

where ν_j denotes the probability distribution of V_{j1}.

Proof. Let $j < k$. Define S_{kji} as the index at which the the i^{th} point of type U_j occurs in the sequence V_k. For $A \in \mathcal{U}$

$$\begin{aligned} \nu_j(A) &= \mathbb{P}(V_{j1} \in A) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(V_{ki} \in A, S_{kj1} = i) \\ &= \sum_{i=1}^{\infty} \{\mathbb{P}(V_{k1} \notin U_j\}^{i-1} \mathbb{P}(V_{k1} \in A \cap U_j) \\ &= \nu_k \mid_{U_j} (A). \end{aligned}$$

Substitution of $A = U_1$ yields

$$\nu_j(U_1) = \frac{\nu_k(U_1)}{\nu_k(U_j)}.$$

It follows that for $j < k$ and $A \in \mathcal{U}$

$$\frac{\nu_j(A \cap U_j)}{\nu_j(U_1)} = \frac{\nu_k(A \cap U_j)}{\nu_k(U_1)}.$$

Define

$$\mathcal{R} = \bigcup_j \{A \in \mathcal{U} : A \subset U_j\}.$$

The collection $\mathcal{R}$ is a ring of subsets of U, and the σ-ring generated bu $\mathcal{R}$ is $\mathcal{U}$. From the above it follows that we can define consistently a setfunction $\bar{\nu}$ on $\mathcal{R}$ by putting for $A \in \mathcal{R}, A \subset U_j, j \geq 1$,

$$\bar{\nu}(A) = \frac{\nu_j(A)}{\nu_j(U_1)}.$$

The setfunction $\bar{\nu}$ is a σ-finite measure on $\mathcal{R}$. Hence $\bar{\nu}$ has a unique extension ν to a measure on $(U, \mathcal{U})$, see Halmos [18]. From the construction it is clear that ν has the claimed properties. Define for $k \geq 1$

$$p_k = \mathbb{P}(V_{k1} \in U_1) = \frac{1}{\nu(U_k)}.$$

Then $(p_k)_{k \geq 1}$ is a decreasing sequence of positive real numbers:

$$\nu(U) = \infty \Leftrightarrow \lim_{k \to \infty} p_k = 0.$$

In the next theorem we associate an Itô-Poisson point process N to a given nested array (U_k, V_k) in such a way that V_k is the U_k-subsequence $(\xi_{ki})_{i \geq 1}$ of N.

Theorem 2.3.12 *(Greenwood & Pitman). Let $(U_k, V_k)_{k \geq 1}$ be a nested array on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If*

$$\lim_{k \to \infty} p_k = 0,$$

then for $\mathbb{P}$-a.e ω the limits

$$t_{jn}(\omega) = \lim_{k \to \infty} p_k S_{kjn}(\omega)$$

exist for all $n, j = 1, 2, \dots$ where $S_{kjn}(\omega)$ is the index at which the n^{th} point of type U_j occurs in the sequence V_k. The sequence of limitpoints $(t_{nj}(\omega))_{n\geq 1}$ is a strictly increasing sequence of positive real numbers. The $\mathbb{P}$-a.e. defined random variable

$$\omega \in \Omega \mapsto \sum_{j=1}^{\infty} \sum_{n: V_{jn}(\omega) \notin U_{j-1}} \delta_{(t_{jn}(\omega), V_{jn}(\omega))}$$

is an Itô-Poisson point process on U with characteristic measure ν as defined in proposition (2.3.11).

Proof Define for $k > j, n \geq 1$,

$$D_{kj1} = S_{kj1},$$

$$D_{kj,n+1} = S_{kj,n+1} - S_{kjn},$$

and

$$D_{kj} = (D_{kjn})_{n\geq 1}.$$

Then for measurable sets $A_1, \dots, A_m \subset U_j$ and $d_1, \dots, d_m \in \mathbb{N}$:

$$\begin{aligned}
&\mathbb{P}(D_{kji} = d_i + 1, V_{ji} \in A_i, i = 1, \dots, m) \\
&= \{\mathbb{P}(V_{k1} \notin U_j)\}^{d_1 + \cdots d_m} \prod_{i=1}^{m} \mathbb{P}(V_{k1} \in A_i) \\
&= (1 - \nu_k(U_j))^{d_1 + \cdots d_m} \nu_k(A_1) \cdots \nu_k(A_m) \\
&= (1 - \frac{p_k}{p_j})^{d_1 + \cdots d_m} (\frac{p_k}{p_j})^m \nu_j(A_1) \cdots \nu_j(A_m)
\end{aligned}$$

since $\nu_j = \nu_k \mid_{U_j}$, see the proof of proposition (2.3.11). So D_{kj} and V_j are independent sequences of random variables. The random variables $(D_{kji})_{i\geq 1}$ form an iid sequence of random variables, geometrically distributed with expectation $\frac{p_j}{p_k}$. Define the filtration $(\mathcal{F}_j)_{j\geq 1}$ by

$$\mathcal{F}_j = \sigma(V_1, \dots, V_j).$$

Then for $l > k > j$ and $n \geq 1$ we have

$$\mathbb{E}(S_{ljn} \mid \mathcal{F}_k) = \mathbb{E}(\sum_{i=1}^{S_{kjn}} D_{lki} \mid \mathcal{F}_k) = \frac{p_k}{p_l} S_{kjn}.$$

So $(p_k S_{kjn}, \mathcal{F}_k)_{k\geq j}$ is a martingale on $(\Omega, \mathcal{F}, \mathbb{P})$, and it follows that outside a set of $\mathbb{P}$-measure 0 the sequence $(p_k S_{kjn})_{k\geq j}$ converges for all $n, j \geq 1$ to a finite limit

$$t_{jn}(\omega) = \lim_{k\to\infty} S_{kjn}(\omega).$$

It is clear that $0 \leq t_{j1}(\omega) \leq t_{j2}(\omega) \leq \ldots$ Since $p_k \to \infty$, we have for $i, j \geq 1$

$$\lim_{k\to\infty} \mathbb{E}(e^{-\lambda p_k D_{kji}}) = \frac{\frac{p_k}{p_j} e^{-\lambda p_k}}{1-(1-\frac{p_k}{p_j})e^{-\lambda p_k}} = \frac{\frac{1}{p_j}}{\frac{1}{p_j}+\lambda}.$$

It follows that the random variables $t_{j1}, t_{j,n+1} - t_{j,n}, n \geq 1$ are independent and exponentially distributed with expectation p_j. So $\mathbb{P}$-a.s. the sequence $(t_{jn}(\omega))_{n\geq 1}$ is strictly increasing. Theorem (2.3.7) implies that

$$\omega \in \Omega \mapsto \sum_{j=1}^{\infty} \sum_{n:V_{jn}(\omega)\notin U_{j-1}} \delta_{(t_{jn}(\omega), V_{jn}(\omega))}$$

is a Poisson point process on $T \times U$ with intensity measure $\lambda \otimes \nu$.

Let $N' : \Omega \mapsto \mathcal{M}^+$ be an Itô-Poisson point process with characteristic measure ν' such that the U_k-subsequence $(\xi'_{ki})_{i\geq 1}$ of N' is V_k. Using the notations and definitions of theorem (2.3.13) we get:

Corollary 2.3.13 *There exists a positive constant c such that for $j, n \geq 1$*

$$\nu' = c\nu \text{ and } \tau'_{jn} = \frac{1}{c} t_{jn}.$$

Proof. Since

$$\mathbb{P}(\xi'_{k1} \in U_1) = p_k \text{ but also } \mathbb{P}(\xi'_{k1} \in U_1) = \frac{\nu'(U_1)}{\nu'(U_k)},$$

it follows that

$$\nu'(U_k) = \frac{1}{p_k}\nu'(U_1).$$

Hence for $A \in \mathcal{U}, A \subset U_k, k \geq 1$

$$\nu'(A) = \nu'(U_k)\mathbb{P}(\xi'_{k1} \in A) = \frac{\nu'(U_1)}{p_k}\frac{\nu(A)}{\nu(U_k)} = \nu'(U_1)\nu(A).$$

So $\nu' = c\nu$ on the ring $\bigcup \mathcal{B}(U_k)$, where $c = \nu'(U_1)$. Finally for $j, n \geq 1$

$$\tau'_{jn} = \lim_{k\to\infty} \frac{1}{\nu'(U_k)} S'_{kjn} = \frac{1}{c} \lim_{k\to\infty} \frac{1}{p_k} S_{kjn} = \frac{1}{c} t_{jn}.$$

Corollary 2.3.14 *Let $(U_k, V_k)_{k\geq 1}$ be a nested array on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If*

$$\lim_{k\to\infty} p_k > 0,$$

then for $\mathbb{P}$*-a.e* ω *the limits*

$$\xi_n(\omega) = \lim_{k\to\infty} V_{kn}(\omega)$$

exist for every $n \geq 1$*. The sequence* $(\xi_n)_{n\geq 1}$ *is an iid sequence of random variables,*

$$\mathbb{P}(\xi_n \in A) = \frac{\nu(A)}{\nu(U)}, A \in \mathcal{U}.$$

Further the (U_k)*-subsequence of the sequence* $(\xi_n)_{n\geq 1}$ *is* $\mathbb{P}$*-a.s. equal to* $(V_{ki})_{i\geq 1}$*.*

Proof. Since $\lim_{k\to\infty} p_k > 0$, it follows from the proof of theorem (2.3.13) that $\lim_{k\to\infty} S_{kjn}(\omega)$ exists $\mathbb{P}$-a.es.. So for every $n \geq 1$, and for k sufficiently large, say $k > K_n$, the term S_{k1n} is constant. It follows that $(V_{k1}, \ldots, V_{kM_n})$ is constant for $k > K_n$, where $M_n = \lim_{k\to\infty} S_{k1n} \geq n$. Define for $n \geq 1$: $\xi_n = \lim_{k\to\infty} V_{kn}$. It is clear that the sequence $(\xi_n)_{n\geq 1}$ is an iid sequence of random variables. Let $A \in \mathcal{U}$, then

$$\mathbb{P}(\xi_n \in A) = \lim_{k\to\infty} \mathbb{P}(V_{kn} \in A) = \lim_{k\to\infty} \frac{\nu(A \cap U_k)}{\nu(U_k)} = \frac{\nu(A)}{\nu(U)}.$$

Finally, it is clear that the (U_k)-subsequence of the sequence $(\xi_n)_{n\geq 1}$ is $\mathbb{P}$-a.s. equal to $(V_{ki})_{i\geq 1}$.

Chapter 3

Excursion Theory

Let Y be a standard Markov process with state space S and let $a \in S$ be a given state, which is recurrent for Y. In [25] Itô defined the excursion process of Y with respect to $\mathbb{P}_a$ in the following way: let $S(t)$ be the inverse local time of Y at a. If t is such that $S(t-) < S(t)$, then the function u_t defined by

$$u_t(s) = \begin{cases} Y(S(t-)+s) & ,0 \leq s < S(t) - S(t-) \\ a & ,s \geq S(t) - S(t-) \end{cases}$$

is called the excursion of Y in $]S(t-), S(t)[$. Itô proved that the random distribution of the points $(t, u_t), t \in \{s : S(s-) < S(s)\}$, is a Poisson point process on $[0, \infty[\times U$, where U denotes the space of all excursions. In the first part of this chapter we will study the excursions of a Ray process Y in the maximal components of $[0, \infty[\setminus Z$, where $Z = \{t \in [0, \infty[: Y_t = a \text{ or } Y_{t-} = a\}$. The strong Markov property implies that the sequence of excursions in the intervals of length greater than a given positive real number r, is an iid sequence with respect to $\mathbb{P}_a$. Using the the theorem of Greenwood & Pitman (see section (2.3)) one can construct the Itô-Poisson point process of excursions without the explicit introduction of the local time; the characteristic measure ν is determined by the sub-Markov semigroup $(K_t)_{t\geq 0}$ defined by

$$K_t(x, dy) = \mathbb{P} - x[Y_t \in dy : Y_s, Y_{s-} \neq a \text{ for all } s \leq t]$$

and an entrance law for the semigroup (K_t). If the state a is regular for Y, the total mass of ν is ∞.

In the last part of the chapter we construct stochastic processes from Itô-Poisson point processes. Let P be an Itô-Poisson point process with characteristic measure ν determined by the semigroup (K_t) and an entrance law $(\eta_s)_{s>0}$ for the semigroup (K_t) such that $\eta(\{a\}) = 0$ for every $s > 0$ and

$$\int_U (1 - e^{-\zeta_u})\nu(du) < \infty.$$

The stochastic process Y constructed by concatenation of the excursions of P will turn out to have the simple Markov property: the proof of this property is based on an application of the renewal property of Itô-Poisson point processes and an application of the Palm formula. Of course, the assumptions about ν are necessary to get a Markov evolution of the process inside an excursion. We will give sufficient conditions for the process Y to be a Ray process. A simple calculation based on the Palm formula will give a formula for the resolvent of Y. Further we will give a formula for the Blumenthal-Getoor local time of Y at state a. Finally we will give an example of the construction of a Markov process from a Cox point process.

3.1 Ray processes

This section contains a summary of some important properties of Ray processes. For proofs we refer to the books of Getoor [15] and Williams [59]. Let E denote a compact metric space with Borel σ-algebra $\mathcal{E}$.

Definition 3.1.1 *Let $\mathcal{C}(E)$ be the space of continuous functions on E. A family $(R_\lambda)_{\lambda>0}$ of kernels on $(E, \mathcal{E})$ is called a Ray resolvent if*

(i) $\forall \lambda > 0 : \lambda R_\lambda 1 \leq 1$, *(sub-Markov property),*

(ii) $\forall \lambda, \mu > 0 : R_\lambda - R_\mu + (\lambda - \mu) R_\lambda R_\mu = 0$ *(resolvent equation),*

(iii) $\forall \lambda > 0 : R_\lambda(\mathcal{C}(E)) \subset \mathcal{C}(E)$,

(iv) $\bigcup_{\alpha \geq 0} CSM^\alpha$ *separates the points of E (Ray property),*

where CSM^α is the family of continuous α-supermedian functions relative to (R_λ).

There is a standard construction to change a Ray resolvent (R_λ) into a Markov Ray resolvent (i.e. $\lambda R_\lambda 1 = 1$ for every $\lambda > 0$) on a space E' which arises from E by adjoining an isolated point. So assume that $(R_\lambda)_{\lambda>0}$ is a Markov resolvent on E. The construction of the Ray process with resolvent (R_λ) goes via a Markov semigroup $(P_t)_{t\geq 0}$ whose existence and uniqueness is guaranteed by a theorem of Ray.

Theorem 3.1.2 *(Ray). Let (R_λ) be a Markov Ray resolvent on a compact metric space E. Then there exists a unique Markov semigroup $(P_t)_{t\geq 0}$ satisfying for $f \in \mathcal{C}(E)$ and $x \in E$*

(i) *$t \mapsto P_t f(x)$ is right continuous on $[0, \infty[$,*

(ii) *$\forall \lambda > 0; R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt$.*

Proof. See Williams [59], p.187.

We continue with a brief description of the construction of the canonical realization of the Markov process with transition semigroup (P_t). The sample space Ω will be the space $D_{[0,\infty[}(E)$ of càdlàg functions from $[0, \infty[$ to E. Let $Y = (Y_t)_{t\geq 0}$ be the coordinate process on Ω:

$$Y_t(\omega) = \omega(t) \text{ for } \omega \in \Omega, t \geq 0.$$

Let $\mathcal{F}_t^\circ$ be the σ-algebra $\sigma(Y_s, 0 \leq s \leq t)$ on Ω generated by the maps $Y_s, s \leq t$, and $\mathcal{F}^\circ = \sigma(Y_t, t \geq 0)$. For every probability measure μ on $(E, \mathcal{E})$, there exists a unique probability measure $\mathbb{P}_\mu$ on $(\Omega, \mathcal{F}^\circ)$ such that for $0 \leq t_1 \leq \ldots \leq t_n$

$$\begin{aligned}&\mathbb{P}_\mu(Y_0 \in dx_0, Y_{t_i} \in dx_i, i = 1, \ldots, n)\\ &\quad = \int \mu(dx) P_0(x, dx_0) P_{t_1}(x_0, dx_1) P_{t_2 - t_1}(x_1, dx_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n).\end{aligned}$$

Let $(\Omega, \mathcal{F}^\mu, \{\mathcal{F}_t^\mu\})$ be the usual $\mathbb{P}_\mu$-augmentation of $(\Omega, \mathcal{F}^\circ, \{\mathcal{F}_t^\circ\})$, where $\mathcal{F}^\mu$ is the $\mathbb{P}_\mu$-completion of $\mathcal{F}^\circ$ and $\mathcal{F}_t^\mu$ the σ-algebra generated by $\mathcal{F}_{t+}^\circ = \bigcap_{s>t} \mathcal{F}_s^\circ$ and the class of all $\mathbb{P}_\mu$-null sets in $\mathcal{F}^\mu$. Then

$$Y = (\Omega, \mathcal{F}^\mu, \{\mathcal{F}_t^\mu\}, (Y_t)_{t\geq 0}, \mathbb{P}_\mu)$$

is a strong Markov process, this means that for every $\{\mathcal{F}_t^\mu\}$-stopping time T and every bounded $\mathcal{F}^\mu$-measurable random variable η on Ω

$$\mathbb{E}[\eta \circ \theta_T 1_{[T<\infty]} \mid \mathcal{F}_T^\mu] = \mathbb{E}_{Y(T)}[\eta] 1_{[T<\infty]} \ \mathbb{P}_\mu\text{-a.s.},$$

see Getoor [15], p.24. Let D be the set of $x \in E$ such that $P_0(x, .) = \delta_x$. The set D is a Borel subset of E. Points in $B = E \backslash D$ are called branchpoints. For every μ we have

$$\mathbb{P}_\mu[Y_t \in D, \forall t \geq 0] = 1.$$

So the paths $t \in [0, \infty[\mapsto Y_t(\omega)$ are a.s. right continuous functions with values in D and left limits in E.

Theorem 3.1.3 *Let μ be a probability measure on E and let*

$$Y = (\Omega, \mathcal{F}^\mu, \{\mathcal{F}_t^\mu\}, (Y_t)_{t\geq 0}, \mathbb{P}_\mu)$$

be as above. Let (τ_n) be an increasing sequence of $\{\mathcal{F}_t^\mu\}$- stopping times. Let $\tau = \sup \tau_n$ and $\Lambda = [\tau < \infty$ and $\forall n : \tau_n < \tau]$. If f is a bounded universally measurable function on E, then $\mathbb{P}_\mu$-a.s. we have

$$\mathbb{E}_\mu[f \circ Y_\tau 1_{[\tau<\infty]} \mid \bigvee_{n\geq 1} \mathcal{F}^\mu_{\tau_n}] = f \circ Y_\tau 1_{[\tau<\infty]} 1_{\Lambda^c} + P_0 f(Y_{\tau-}) 1_\Lambda .$$

Proof. See Getoor [15], p.25.

Definition 3.1.4 *Let (R_λ) be a Markov Ray resolvent on the compact metric space E. The canonical realization of the Ray process associated with the resolvent (R_λ) is the collection of quintuples*

$$Y = (\Omega, \mathcal{F}^\mu, \{\mathcal{F}_t^\mu\}, (Y_t)_{t\geq 0}, \mathbb{P}_\mu)$$

where μ runs through the set of probability measures on $(E, \mathcal{E})$.

We will need the following notations

$$\mathcal{F} = \bigcap_\mu \mathcal{F}^\mu \text{ and } \mathcal{F}_t = \bigcap_\mu \mathcal{F}_t^\mu .$$

Note that $Y = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, (Y_t)_{t\geq 0}, \mathbb{P}_\mu)$ has also the strong Markov property.

3.2 Point processes of excursions

Let Y be the canonical realization of the Ray process with Ray resolvent $(R_\lambda)_{\lambda>0}$ on the compact metric state space E. We will use the notations introduced in section (3.1). Let a be a given state. The polish space of càdlàg functions $f : [0, \infty[\mapsto E$ will be denoted by U, see appendix A2. The Borel σ-algebra $\mathcal{U}$ on U coincides with the σ-algebra generated by the coordinate evaluations. For $f \in U$, let $Z(f)$be the closed set of points at which f approaches or hits the state a:

$$Z(f) = \{t \in [0, \infty[: f(t) = a \text{ or } f(t-) = a\}.$$

The connected components of $[0, \infty[\backslash Z(f)$ are called excursion intervals from a of f. Let $I =]D, T[$ be an excursion interval of f. The map $V_I(f) : [0, \infty[\mapsto E$ defined by

$$V_I(f)(t) = \begin{cases} f(D+t) & \text{if } 0 \leq t \leq T - D \\ a & \text{if } t \geq T - D \end{cases}$$

is called the excursion of the function f from point a on the excursion interval I. If it is clear from which point the excursions are considered, we will speak simply of excursions and excursion intervals of f. The map $\zeta : f \in U \mapsto \zeta_f \in [0, \infty[$ defined by

$$\zeta_f = \inf(Z(f)\backslash\{0\})$$

is a lower semi-continuous function, see appendix A2. If $\inf(Z(f)\backslash\{0\}) > 0$, then ζ_f is the first time after zero at which f hits or approaches the state a. The length (or the duration) of the excursion $V_I(f)$ is defined by $\zeta(V_I(f) = T - D$. Let r be a positive real number. We will now study the excursions of length greater than r of the realizations of the Ray process Y. Denote by $(]D_n(\omega), T_n(\omega)[)_{n\geq 1}$ the sequence of all excursion intervals of $Y_.(\omega)$ with length exceeding r, enumerated in such a way that $D_n(\omega) < D_{n+1}(\omega), n \geq 1$. This sequence is at most countable, and it is also possible that there are only finitely many excursion intervals of lenght exceeding r. The excursion corresponding to the excursion interval $]D_n(\omega), T_n(\omega)[$ will be denoted by $V_n(\omega)$. Note that $V_n : \omega \in \Omega \mapsto V_n(\omega) \in U$ is a partially defined, U-valued random variable. Define $T_n(\omega) = +\infty$ if there are less than n excursion intervals of length exceeding r. Note that the mappings T_n are $(\mathcal{F}_t)$-stopping times. With the above introduced definitions and notations we have the following theorem.

Theorem 3.2.1 *Let Y be a Ray process on $E, a \in E$ and $r > 0$.*

(i) *The sequence of excursion intervals from a of lenght greater than r is $\mathbb{P}_a$-a.s. an infinite sequence if and only if $\mathbb{P}_a[T_1 < \infty] = 1$.*

(ii) *If $\mathbb{P}_a[T_1 < \infty] = 1$, then the sequence $(V_n)_{n\geq 1}$ of excursions from a of length greater than r is an iid sequence of U-valued random variables.*

Proof. We start with the construction of a sequence $(\tau_n)_{n\geq 1}$ of $(\mathcal{F}_t^\circ)$-stopping times which increase to T_1. Let $(\delta_n)_{n\geq 1}$ be a sequence of positive real numbers, strictly decreasing to zero: $\delta_n \downarrow 0 (n \to \infty)$. Define for $n \geq 1$:

$$\tau_n = \inf\{t > r : Y_t \in B_a(\delta_n) \text{ and } Y_s, Y_{s-} \neq a, s \in [t - r, t[\}$$

where $B_a(\delta_n)$ is the open ball in E with center a and radius δ_n. Define

$$\tau = \sup \tau_n$$

and

$$\Lambda = [\tau < \infty; \forall n \geq 1 : \tau_n < \tau].$$

Let $\omega \in \Omega$ and let t be such that $Y_t(\omega)$ or $Y_{t-}(\omega) = a$ and $Y_s(\omega), Y_{s-}(\omega) \neq a$ for all $s \in [t - r, t[$. If $Y_t(\omega) = a$, then $\tau_n \leq t$ for all $n \geq 1$. If $Y_{t-}(\omega) = a$, then for every $n \geq 1$ there exists a positive real number ϵ_n such that $Y_r(\omega) \in B_a(\delta_n)$ for all $r \in]t - \epsilon_n, t[$. Since $Y_{(t-r)-}(\omega) \neq a$ there exists an $\eta > 0$ such that $Y_s(\omega), Y_{s-}(\omega) \neq a$ for all $s \in [t - r - \eta, t - r]$. Hence $\tau_n(\omega) \leq t - \min(\epsilon_n, \eta) \leq t$. Consequently in both cases we have $\forall n \geq 1$: $\tau_n(\omega) \leq t$. It follows that

$$\tau(\omega) \leq T_1(\omega).$$

To prove the converse inequality we consider first the case that $\omega \in \Lambda$. Since $Y_{\tau_n}(\omega) \in \overline{B_a(\delta_n)}$, it follows that $Y_{\tau-}(\omega) = a$. Further, it is clear that

$$\forall s \in [\tau(\omega) - r, \tau(\omega)[: Y_s(\omega), Y_{s-}(\omega), \neq a.$$

Hence $\tau(\omega)$ is the right-hand endpoint of an excursion interval of length greater than r and it follows that $T_1(\omega) \leq \tau(\omega)$ for $\omega \in \Lambda$.
Suppose now that $\omega \notin \Lambda$ and $\tau(\omega) < \infty$. For k sufficiently large, $\tau_k(\omega) = \tau(\omega)$. This implies that $Y_{\tau(\omega)}(\omega) = a$ and

$$\forall s \in [\tau(\omega) - r, \tau(\omega)[: Y_s(\omega), Y_{s-}(\omega) \neq a.$$

It follows that for every $\omega \in \Omega$

$$T_1(\omega) \leq \tau(\omega).$$

Hence

$$T_1 \equiv \tau.$$

To prove (i) let $A_n = [T_n < \infty]$ be the event that there are at least n excursions of length greater than r and let A be the event that there is an infinite sequence of excursions of length greater than r.

$$\begin{aligned}
\mathbb{P}_a(A_{n+1}) &= \mathbb{E}_a(1_{[T_1<\infty]} 1_{A_n} \circ \theta_{T_1}) \\
&= \mathbb{E}_a(1_{[T_1<\infty]} \mathbb{P}_{Y(T_1)}(A_n)) \text{ (strong Markov property)} \\
&= \mathbb{E}_a(\mathbb{E}_a(1_{[T_1<\infty]} \mathbb{P}_{Y(T_1)}(A_n) \mid \bigvee \mathcal{F}_{\tau_n})) \\
&= \mathbb{E}_a(1_{[T_1<\infty]} \mathbb{P}_{Y(T_1)}(A_n) 1_{\Lambda^c} + 1_\Lambda \int P_0(Y_{T_1-}, dy) \mathbb{P}_y(A_n))
\end{aligned}$$

by an application of theorem (3.1.3).
It is clear that $Y_{T_1} = a$ on $[T_1 < \infty] \cap \Lambda^c$ and $Y_{T_1-} = a$ on Λ. Hence

$$\mathbb{P}_a(A_{n+1}) = \mathbb{E}(1_{T_1<\infty]} \mathbb{P}_a(A_n)) = \mathbb{P}_a(A_1)\mathbb{P}_a(A_n)$$

and by mathematical induction

$$\mathbb{P}_a(A_n) = (\mathbb{P}_a[T_1 < \infty])^n.$$

It follows that

$$\mathbb{P}_a(A) = \lim_{n\to\infty} (\mathbb{P}_a[T_1 < \infty])^n = 1 \Longleftrightarrow \mathbb{P}_a[T_1 < \infty] = 1.$$

This completes the proof of (i).
To prove (ii) let $n \geq 2$ and let $(f_i)_{1\leq i\leq n}$ be a finite sequence of bounded measurable functions on U. An application of the strong Markov property yields

$$\begin{aligned}
\mathbb{E}_a[\prod_{i=1}^n f_i(V_i)] &= \mathbb{E}_a[f_1(V_1)\prod_{i=2}^n f_i(V_{i-1}\circ\theta_{T_1})] \\
&= \mathbb{E}_a[f_1(V_1)\mathbb{E}_{Y(T_1)}(\prod_{i=2}^n f_i(V_{i-1}))] \\
&= (*)
\end{aligned}$$

Consider first the case that a is a branch point. Then

$$\mathbb{P}_a(\Lambda^c) \leq \mathbb{P}_a[Y_\tau = a] = 0.$$

Hence $Y_{\tau-} = a(\mathbb{P}_a\text{-a.s.})$. So the sequence (τ_n) foretells $\tau = T_1$ and T_1 is predictable. It follows that

$$\mathcal{F}_{T_1-} = \bigvee \mathcal{F}_{\tau_n}.$$

An application of Galmarino's test (see Dellacherie & Meyer [7], p.149) yields the $\mathcal{F}_{T_1-}$-measurability of $f_1(V_1)$. So

$$\begin{aligned}
(*) &= \mathbb{E}_a[f_1(V_1)\mathbb{E}_a(\mathbb{E}_{Y(\tau)}(\prod_{i=2}^n f_i(V_{i-1}) \mid \bigvee \mathcal{F}_{\tau_n})] \\
&= \mathbb{E}_a[f_1(V_1)\int P_0(Y_{\tau-}, dy)\mathbb{E}_y(\prod_{i=2}^n f_i(V_{i-1}))] \\
&= \mathbb{E}_a[f_1(V_1)\int P_0(a, dy)\mathbb{E}_y(\prod_{i=2}^n f_i(V_{i-1}))] \\
&= \mathbb{E}_a[f_1(V_1)]\mathbb{E}_a[\prod_{i=2}^n f_i(V_{i-1})].
\end{aligned}$$

If a is not a branch point, then by theorem (3.1.3)

$$\mathbb{P}_a[Y(T_1) = a] = \mathbb{E}_a[1_{\{a\}}\circ Y_{T_1}1_{\Lambda^c} + \mathbb{E}_a[1_\Lambda \int P_0(Y_{T_1-}, dy)1_{\{a\}}(y)] = 1.$$

So in both cases we have

$$\mathbb{E}_a[\prod_{i=1}^{n} f_i(V_i)] = \mathbb{E}_a[f_1(V_1)]\mathbb{E}_a[\prod_{i=2}^{n} f_i(V_{i-1})].$$

An induction argument completes the proof of (ii).

We introduce some more notation and definitions. Define the maps

$$\begin{aligned}
\rho_a &: \Omega \mapsto [0,\infty], \rho_a(\omega) = \sup\{t \geq 0 : Y_s(\omega) = a \text{ for every } s \leq t\}, \\
\sigma_a &: \Omega \mapsto [0,\infty], \sigma_a(\omega) = \inf\{t \geq 0 : Y_t(\omega) = a \text{ or } Y_{t-}(\omega) = a\}, \\
\tau_a &: \Omega \mapsto [0,\infty], \tau_a(\omega) = \inf\{t > 0 : Y_t(\omega) = a \text{ or } Y_{t-}(\omega) = a\}.
\end{aligned}$$

The maps σ_a and τ_a are stopping times.

Definition 3.2.2 *The state $a \in E$ is called a holding point for the Ray process Y if*

$$\mathbb{P}_a[\rho_a > 0] > 0.$$

Define for $t \geq 0$ the kernel K_t on $(E, \mathcal{E})$ by

$$K_t(x, A) = \mathbb{P}_x[Y_t \in A, \sigma_a > T], x \in E, A \in \mathcal{E}.$$

The family $(K_t)_{t\geq 0}$ is a sub-Markov semigroup on $(E, \mathcal{E})$. Define for $s \geq r$ the measure e_s on $(E, \mathcal{E})$ by

$$e_s(A) = \mathbb{P}_a[Y_{D_1+s} \in A, D_1 + s < T_1 < \infty], A \in \mathcal{E}.$$

Let μ be the $\mathbb{P}_a$-distribution of V_1; μ is a finite measure on $(U, \mathcal{U})$. With the above introduced notations and definitions we have the following lemma.

Lemma 3.2.3 *Let r be a given positive real number.*

(i) $\forall s \geq r, \forall t \geq 0 : e_s K_t = e_{s+t}$.

(ii) *For $0 \leq t_1 \leq \ldots \leq t_n$ and $x_1, \ldots, x_n \in E\backslash\{a\}$*

$$\begin{aligned}
&\mu[u(r + t_i) \in dx_i, i = 1, \ldots, n] \\
&\quad = e_{r+t_1}(dx_1)K_{t_2-t_1}(x_1, dx_2) \cdots K_{t_n-t_{n-1}}(x_{n-1}, dx_n).
\end{aligned}$$

(iii) $\mathbb{P}_a[\tau_a = 0] = 0$ *or* 1.

Proof. Note that $D_1 + s$ is for $s \geq r$ a stopping time. Let $A \in \mathcal{E}, s \geq r$ and $t \geq 0$, then

$$\begin{aligned} e_s K_t(A) &= \int \mathbb{P}_a[Y_{D_1+s} \in dx, D_1 + s < T_1 < \infty]\mathbb{P}_x[Y_t \in A, \sigma_a > t] \\ &= \mathbb{E}_a(1_{[D_1+s<T_1<\infty]}\mathbb{E}_{Y_{D_1+s}}[1_A(Y_t); \sigma_a > t]) \\ &= \mathbb{E}_a(1_{[D_1+s<T_1<\infty]}1_{[\sigma_a \circ \theta_{D_1+s} > t]}1_A(Y_{D_1+s+t})] \\ &= e_{s+t}(A). \end{aligned}$$

This completes the proof of the first part of the lemma.
Let $0 \leq t_1 \leq \ldots \leq t_n$ and $x_1, \ldots, x_n \in E\backslash\{a\}$,

$$\begin{aligned} &\mu[u(r + t_i) \in dx_i, i = 1, \ldots, n] \\ &\quad = \mathbb{P}_a[D_1 + r < T_1 < \infty, \sigma_a \circ \theta_{D_1+r} > t_n, Y_{t_i} \circ \theta_{D_1+r} \in dx_i, i = 1, \ldots, n] \\ &\quad = \int e_r(dx)\mathbb{P}_x[Y_{t_i} \in dx_i, i = 1, \ldots, n, \sigma_a > t_n]. \end{aligned}$$

by an application of the strong Markov property on stopping time $D_1 + r$.
A repeated application of the simple Markov property yields

$$\begin{aligned} &\mathbb{P}_x[Y_{t_i} \in dx_i, i = 1, \ldots, n, \sigma_a > t_n] \\ &\quad = K_{t_1}(x, dx_1)K_{t_2-t_1}(x_1, dx_2) \cdots K_{t_n-t_{n-1}}(x_{n-1}, dx_n). \end{aligned}$$

An application of (i) completes the proof of (ii).
The third statement is a consequence of Blumenthal's 0-1 law, see Williams [59], p.126.

Let X be the product $T \times U$ of the halfline $T = [0, \infty[$ with the usual topology and the polish space U of càdlàg functions $f : [0, \infty[\mapsto E$. Let $a \in E$ and let $\mathcal{S}$ be the family of open subsets of X defined by

$$\mathcal{S} = \{A \subset X : A = I \times [\zeta > t], I \subset T \text{ open and bounded, } t > 0\},$$

where ζ is the first time after zero at which f hits or approaches a. The elements of $U_\infty = \{u \in U : \zeta_u > 0\}$ will be called excursions. Let Y, σ_a and K_t be defined as above and let $\bar{Y}$ be the process defined by

$$\bar{Y}_t = \begin{cases} Y_t & \text{if } t < \sigma_a \\ a & \text{if } t \geq \sigma_a. \end{cases}$$

Denote by $\alpha_x, x \in E$, the $\mathbb{P}_x$-distribution of $\bar{Y}$. Then α_x is a probability measure on U and the finite-dimensional distributions of α_x are given by

$$\begin{aligned} &\alpha_x[u(t_i) \in dx_i, i = 1, \ldots, n] \\ &\quad = K_{t_1}(x, dx_1)K_{t_2-t_1}(x_1, dx_2) \cdots K_{t_n-t_{n-1}}(x_{n-1}, dx_n), \end{aligned}$$

where $0 < t_1 < \ldots < t_n$ and $x_1, \ldots, x_n \in E\backslash\{a\}$. The shift operator on U will also be denoted by θ_t, so for $t \geq 0$ and $u \in U$, $\theta_t u$ is the element of U defined by

$$(\theta_t u)(s) = u(s+t), s \geq 0.$$

With the above introduced notations and definitions we have the following theorem.

Theorem 3.2.4 *Let Y be a Ray process and $a \in E$. Assume that a is not a holding point and*

$$\exists r > 0 : \mathbb{P}_a[T_1^r < \infty] = 1$$

where T_1^r denotes the endpoint of the first excursion of length greater than r. Then there are two cases possible.

A. $\mathbb{P}_a[\tau_a = 0] = 0$.
In this case there exists an iid sequence $(\xi_n)_{n\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P}_a)$ of U_∞-valued random variables whose $[\zeta > l]$-subsequence is the sequence of excursions of Y of length exceeding l.

B. $\mathbb{P}_a[\tau_a = 0] = 1$.
In this case there exists an $\mathcal{S}$-finite Itô-Poisson point process N defined on $(\Omega, \mathcal{F}, \mathbb{P}_a)$ whose $[\zeta > l]$-subsequence is the sequence of excursions of Y of length exceeding l. The characteristic measure ν of N is a σ-finite measure on U_∞ having the following properties:

(i) *ν is concentrated on*

$$\{u \in U_\infty : \forall s \geq \zeta_u : u(s) = a\},$$

(ii) *for each $f \in b\mathcal{F}_\infty$, $t > 0$ and $\Lambda \in \sigma(u \in U_\infty \mapsto u(r), r \leq t)$ we have*

$$\int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\nu(du) = \int_{\Lambda\cap[\zeta>t]} \alpha_{u(t)}(f)\nu(du),$$

(iii) $\forall t > 0 : \nu([\zeta > t]) < \infty$.

Proof. Let $(r_k)_{k\geq 1}$ be a strictly decreasing sequence of positive real numbers, such that $\lim_{k\to\infty} r_k = 0$ and $\mathbb{P}_a[T_{11} < \infty] = 1$, where T_{11} is the endpoint of the first excursion of length exceeding r_1. For $k, n = 1, 2\ldots$ denote by $]D_{kn}(\omega), T_{kn}(\omega)[$ resp. $V_{kn}(\omega)$ the n^{th} excursion interval of the realization $Y_.(\omega)$ with length exceeding r_k. Since $T_{k1} \leq T_{11}$, it follows that

$\mathbb{P}_a[T_{k1} < \infty] = 1$, and the sequence $(V_{kn})_{n \geq 1}$ of excursions of length exceeding r_k is an iid sequence of U-valued random variables, see theorem (3.2.1). Define for $k = 1, 2, \ldots$

$$\begin{aligned} U_k &= \{u \in U : \zeta_u > r_k\}, \\ V_k &= (V_{k1}, V_{k2}, \ldots), \\ \nu_k &= V_{k1}(\mathbb{P}_a) \text{ and} \\ p_k &= \nu_k(U_1). \end{aligned}$$

Then $(U_k, V_k)_{k \geq 1}$ is a nested array on $(\Omega, \mathcal{F}, \mathbb{P}_a)$, see section (2.3) for the definition of a nested array.
Suppose that we are in case A. Since

$$\lim_{k \to \infty} 1_{U_1}(V_{k1}(\omega)) = 1_{[\tau_a > r_1]}(\omega) \text{ on } [\tau_a > 0],$$

it follows that

$$\lim_{k \to \infty} p_k = \lim_{k \to \infty} \mathbb{P}_a[V_{k1} \in U_1] = \mathbb{P}_a[\tau_a > r_1] > 0.$$

By corollary (2.3.15) there exists an iid sequence $(\xi_n)_{n \geq 1}$ of random varaibles whose U_k-subsequence is $\mathbb{P}_a$-a.s. equal to $(V_{kn})_{n \geq 1}$.
Suppose that we are in case B. Since a is not a holding point we have

$$\mathbb{P}_a(\{\omega : \exists K_\omega, \forall k \geq K_\omega : V_{k1}(\omega) \notin U_1\}) = 1.$$

Hence in case B

$$\lim_{k \to \infty} p_k = \lim_{k \to \infty} \mathbb{P}_a[V_{k1} \in U_1] = 0.$$

It follows from theorem (2.3.12) that $\mathbb{P}_a$-a.s. for $n, j = 1, 2, \ldots$ the limits

$$t_{jn} = \lim_{k \to \infty} p_k S_{kjn}$$

exist, where $S_{kjn}, k > j$ denotes the index of the n^{th} excursion of length greater than r_j in the sequence $(V_{kn})_{n \geq 1}$. The $\mathbb{P}_a$-a.s. defined random variable

$$N : \omega \in \Omega \mapsto \sum_{j=1}^{\infty} \sum_{n : V_{jn}(\omega) \notin U_{j-1}} \delta_{(t_{jn}(\omega), V_{jn}(\omega))}$$

is an $\mathcal{S}$-finite Itô-Poisson point process on U_∞ with characteristic measure ν determined by

$$\nu(U_1) = 1 \text{ and } \nu_{|U_j} = \nu_j.$$

Further the U_k-subsequence of N is the sequence $(V_{kn})_{n\geq 1}$ which implies that for every $l > 0$ the $[\zeta > l]$-subsequence of N is the sequence of excursions of Y of length exceeding l. If $(r'_k)_{k\geq 1}$ is another strictly decreasing sequence of positive real numbers with the same properties as the sequence $(r_k)_{k\geq 1}$, then the Itô-Poisson point process N' constructed as above starting from the sequence (r'_k) has a characteristic measure ν' wich differs only by a multiplicative constant from ν, see corollary (2.3.14). To prove property (ii), fix $t > 0$ and choose k such that $r_k \leq t$. Let for $0 \leq t_1 < \ldots t_l \leq t$, $0 \leq s_1 < \ldots s_n$, $A_1, \ldots, A_l \in \mathcal{E}$ and $f_1, \ldots, f_n \in b\mathcal{E}$

$$\Lambda = [u(t_i) \in A_i, i = 1, \ldots, l]$$

and for $u \in U$

$$f(u) = \prod_{j=1}^{n} f_j(u(s_j)).$$

Then by definition of ν

$$\begin{aligned}
&\int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\nu(du) \\
&= \frac{1}{\nu_k(U_1)} \int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\nu_k(du) \\
&= \frac{1}{\mathbb{P}_a[T_{k1} > D_{k1} + r_1]} \\
&\quad \times \mathbb{E}_a[\prod_i 1_{A_i}(V_{k1}(t_i)) \times 1_{[\zeta(V_{k1})>t]} \times \prod_j f_j(V_{k1}(s_j + t))] \\
&= \frac{1}{\mathbb{P}_a[T_{k1} > D_{k1} + r_1]} \\
&\quad \times \mathbb{E}_a[\prod_i 1_{A_i}(Y(D_{k1} + t_i)) \times 1_{[D_{k1}+t<T_{k1}]} \times \prod_j f_j(\bar{Y}(s_j)) \circ \theta_{D_{k1}+t}] \\
&= \frac{1}{\mathbb{P}_a[T_{k1} > D_{k1} + r_1]} \\
&\quad \times \mathbb{E}_a[\prod_i 1_{A_i}(Y(D_{k1} + t_i)) \times 1_{[D_{k1}+t<T_{k1}]} \times \alpha_{Y(D_{k1}+t)}(f)] \\
&\quad \text{(by an application of the Strong Markov property)} \\
&= \int_{\Lambda\cap[\zeta>t]} \alpha_{u(t)}(f)\nu(du).
\end{aligned}$$

A standard monotone class argument completes the proof of (ii).
For the proof of (i), first note that the set

$$W = \{u \in U_\infty : \forall s \geq \zeta_u : u(s) = a\}$$

is a measurable subset of U_∞. It is clear that

$$\forall x \in E : \alpha_x(1_W) = 0.$$

So (ii) implies

$$\int_{[\zeta>t]} 1_W(\theta_t u)\nu(du) = \int_{[\zeta>t]} \alpha_{u(t)}(1_W)\nu(du) = 0.$$

On the other hand

$$\zeta_u > t \Longrightarrow 1_W(\theta_t u) = 1_W(u).$$

Hence

$$\begin{aligned}\int_{U_\infty} 1_W(u)\nu(du) &= \lim_{t\downarrow 0}\int_{[\zeta>t]} 1_W(u)\nu(du)\\ &= \lim_{t\downarrow 0}\int_{[\zeta>t]} 1_W(\theta_t u)\nu(du) = 0,\end{aligned}$$

which completes the proof of (i).
Finally,

$$\forall t > 0 : \nu([\zeta > t]) = \eta_t(E) < \infty,$$

which implies (iii).

Definition 3.2.5 *A family of finite measures $(\epsilon_s)_{s>0}$ on $(E,\mathcal{E})$ will be called an entrance law for the (sub-Markov) semigroup $(K_t)_{t\geq 0}$ whenever*

$$\forall s, t \geq 0 : \epsilon_s K_t = \epsilon_{s+t}.$$

Theorem 3.2.6 *There is a one-to-one correspondence between σ-finite measures m on $(U_\infty,\mathcal{U}_\infty)$ satisfying properties (i), (ii) and (iii) of theorem (3.2.4) and entrance laws $(\epsilon_s)_{s>0}$ for the semigroup (K_t) satisfying $\epsilon_s(a) = 0$ for every $s > 0$.*

Proof. Let us first assume that m is a σ-finite measure on $(U_\infty,\mathcal{U}_\infty)$ satisfying properties (i), (ii) and (iii) of theorem (3.2.4). Define for $s > 0$ and $A \in \mathcal{E}$

$$\epsilon_s(A) = m(\{u \in U_\infty : u(s) \in A, \zeta_u > s\}).$$

Then $(\epsilon_s)_{s>0}$ is a family of finite measures on $(E,\mathcal{E})$ satisfying $\epsilon_s(a) = 0$ for every $s > 0$. For $A \in \mathcal{E}$, denoting $A\backslash\{a\}$ by A',

$$\epsilon_s K_t(A) = \int m(u(s) \in dx, \zeta_u > s)\mathbb{E}_x[1_{A'}(Y_t); \sigma_a > t]$$

$$\begin{aligned}
&= \int_{[\zeta_u > s]} m(du)\mathbb{E}_{u(s)}[1_{A'}(Y_t)] \\
&= \int_{[\zeta_u > s]} m(du)\alpha_{u(s)}[1_{A'}(u(t))] \\
&= \int_{[\zeta_u > s]} m(du)1_{A'}(u(s+t)) \text{ (by property (ii))} \\
&= \int_{[\zeta_u > s+t]} m(du)1_{A'}(u(s+t)) \text{ (by property (i))} \\
&= \epsilon_{s+t}(A') = \epsilon_{s+t}(A).
\end{aligned}$$

So $(\epsilon_s)_{s>0}$ is an entrance law, which proves the first half of the theorem. To prove the second half of the theorem, let $(\epsilon_s)_{s>0}$ be an entrance law for the semigroup (K_t) satisfying $\epsilon_s(a) = 0$ for every $s > 0$. Define for $t > 0$

$$\mathcal{G}_t = \{B \subset U_\infty : \exists A \in \mathcal{U}_\infty : B = [\zeta > t] \cap \theta_t^{-1}(A)\}.$$

The sets

$$\{u \in U_\infty : \zeta_u > t, u(t + s_i) \in F_i, i = 1, \dots, n\}$$

where $n \geq 1, 0 \leq s_1 \leq \dots \leq s_n$ and $F_1, \dots, F_n \in \mathcal{E}$, generate the σ-algebra $\mathcal{F}_t$. If $r < t$, then $\mathcal{U}_\infty \supset \mathcal{G}_r \supset \mathcal{G}_t$. Indeed, let $B \in \mathcal{G}_t$, say $B = [\zeta > t] \cap \theta_t^{-1}(A)$ with $A \in \mathcal{U}_\infty$. Then

$$B = [\zeta > r] \cap \theta_r^{-1}([\zeta > t - r]) \cap \theta_{t-r}^{-1}(A) \in \mathcal{G}_r.$$

Note that $\bigcup_{t>0} \mathcal{G}_t$ is a ring generating $\mathcal{U}_\infty$. The setfunction μ_t defined on $\mathcal{G}_t$ by

$$\mu_t([\zeta > t] \cap \theta_t^{-1}(A)) = \int \epsilon_t(dx)\alpha_x(A), A \in \mathcal{U}_\infty,$$

is a finite measure on the ring $\mathcal{G}_t$, whilst for $r < t$

$$\begin{aligned}
\mu_r([\zeta > t] \cap \theta_t^{-1}(A)) &= \int \epsilon_r(dx)\alpha_x[\zeta > t - r, 1_A \circ \theta_{t-r}] \\
&= \int \epsilon_r(dx)\mathbb{E}_x[1_{[\sigma > t-r]} \times 1_A \circ Y(\theta_{t-r})] \\
&= \int \epsilon_r(dx) \int K_{t-r}(x, dy)\alpha_y(A) \\
&\quad \text{(by the simple Markov property)} \\
&= \int \epsilon_t(dy)\alpha_y(A) \\
&= \mu_t([\zeta > t] \cap \theta_t^{-1}(A)).
\end{aligned}$$

It follows that we can define a setfunction μ on $\bigcup_{t>0} \mathcal{G}_t$ by putting

$$\mu(B) = \mu_t(B) \text{ if } B \in \mathcal{G}_t.$$

Since the setfunction μ is a σ-finite measure on the ring $\bigcup_{t>0} \mathcal{G}_t$, it has a unique extension to a σ-finite on $(U_\infty, \mathcal{U}_\infty)$. From standard monotone class arguments it follows that for $f \in b\mathcal{U}_\infty$ and $t > 0$

$$\int_{[\zeta > t]} \mu(du) f(\theta_t u) = \int \epsilon_t(dx) \alpha_x(f).$$

Fix $t > 0$ and define for $0 \leq r_1 < \cdots < r_l \leq t$ and $A_1, \ldots A_l \in \mathcal{E}$

$$\Lambda = [u(r_i) \in A_i, i = 1, \ldots, l].$$

For $0 < r < r_1$ and $f \in b\mathcal{U}_\infty$

$$\int_{\Lambda \cap [\zeta > t]} \mu(du) f(\theta_t u)$$

$$= \int \epsilon_r(dx) \int \alpha_x(du) \prod_{i=1}^{l} 1_{[u(r_i - r) \in A_i]}(u) \times 1_{[\zeta > t-r]}(u) f(\theta_{t-r} u)$$

$$= \int \epsilon_r(dx) \int \alpha_x(du) \prod_{i=1}^{l} 1_{[u(r_i - r) \in A_i]}(u) \times 1_{[\zeta > t-r]}(u) \alpha_{u(t-r)}(f)$$

by an application of the simple Markov property

$$= \int_{[\zeta > r]} \mu(du) \prod_{i=1}^{l} 1_{[u(r_i + r) \in A_i]}(\theta_r u) \times 1_{[\zeta > t-r]}(\theta_r u) \alpha_{\theta_r u(t-r)}(f)$$

$$= \int_{\Lambda \cap [\zeta > t]} \mu(du) \alpha_{u(t)}(f).$$

This proves the formula

$$\int_{\Lambda \cap [\zeta > t]} f(\theta_t u) \nu(du) = \int_{\Lambda \cap [\zeta > t]} \alpha_{u(t)}(f) \nu(du)$$

for elementary sets Λ. From a standard monotone class argument it follows that this formula is true for all $\Lambda \in \sigma(u \in U_\infty \mapsto u(r), r \leq t)$. As in the proof of theorem (3.2.4) it now follows that ν is concentrated on the set

$$\{u \in U_\infty : \forall s \geq \zeta_u : u(s) = a\}.$$

From the definition of ν it is clear that

$$\forall t > 0 : \nu([\zeta > t]) < \infty,$$

which completes the proof.

Remark 3.2.7 **(i)** In theorem (3.2.4) the excursions of Y are considered on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_a)$. On a probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$ we have to add to the Itô-Poisson point process (or to the iid sequence $(\xi_n)_{n\geq 1}$) a first excursion describing the process up to time σ_a. Let $x \in E, x \neq a$. The map $W : \Omega \mapsto U$ defined by

$$(W\omega)(t) = \begin{cases} Y_t(\omega) & \text{for } t < \sigma_a(\omega) \\ a & \text{for } t \geq \sigma_a(\omega) \end{cases}$$

is a U_∞-valued random variable, describing the process Y up to time σ_a. As in the proof of theorem (3.2.4), let $(r_k)_{k\geq 1}$ be a strictly decreasing sequence of positive real numbers, $r_k \downarrow 0$ as $k \to \infty$. Denote for $k, n = 1, 2, \ldots$ by $V^x_{kn}(\omega)$ the n^{th} excursion from a with length exceeding r_k of the realization $Y(\theta_{\sigma_a}\omega)$. So $V^x_{kn}(\omega) = V_{kn}(\theta_a\omega)$. Define the vector $V^x_k = (V^x_{k1}, V^x_{k2}, \ldots)$. As in lemma (3.2.3) we can prove that for every $x \in E\backslash\{a\}$ the sequence $(V^x_k, U_k)_{k\geq 1}$ is a nested array on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ and

$$\begin{aligned} &\mathbb{P}_x[W \in B, V^x_{ki} \in A_i, i = 1, \ldots, n] \\ &\quad = \mathbb{P}_x[W \in B]\mathbb{P}_a[V_{ki} \in A_i, i = 1, \ldots, n] \end{aligned}$$

for every $n \geq 1, B, A_1, \ldots, A_n \in \mathcal{U}$. When $\mathbb{P}_a[\tau_a = 0] = 1$, we define the point processes N^x and Q^x by

$$N^x : \omega \in \Omega \mapsto N(\theta_{\sigma_a}\omega),$$

N as in theorem (3.2.4), and

$$Q^x : \omega \in \Omega \mapsto \delta_{(0, W(\omega))}.$$

It follows that the point processes N^x and Q^x defined on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ are independent and as in theorem (3.2.4) the point process N^x is an Itô-Poisson point process with the same characteristic measure ν as N.

(ii) If state a is a holding point for Y, then there exists an iid sequence $(\xi_n)_{n\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P}_a)$ of U_∞-valued random variables whose $[\zeta > l]$-subsequence is the sequence of excursions of Y of length greater than l. Between two consecutive excursions the process Y remains in the state a. These time intervals are exponentially distributed.

3.3 Construction from point processes

Let (E, ρ) be a compact metric space with Borel σ-algebra $\mathcal{E}$ and $a \in E$ some given point of E. The space of càdlàg functions defined on $T = [0, \infty[$ with values in E will be denoted by U. Endowed with the Skorohod topology, U is a polish space. The Borel σ-algebra on U will be denoted by $\mathcal{U}$; this σ-algebra is also generated by the coordinate evaluations on U. The map $\zeta : U \mapsto [0,\infty]$ is defined by

$$\zeta(u) = \zeta_u = \inf\{t > 0 : u(t) = a \text{ or } u(t-) = a\}.$$

For $u \in U_\infty = [\zeta > 0]$ the number ζ_u is the first "time" after zero that u hits or approaches a. The map ζ is lower semi-continuous.
On the space $(E,\mathcal{E})$ there will be given a Markov semigroup of kernels $(\bar{P}_t)_{t\geq 0}$ such that for every $x \in E$ there exists a probability measure α_x on $(U,\mathcal{U})$ which is concentrated on the set $\{u \in U : \forall t \geq \zeta_u : u(t) = a\}$ and which has finite-dimensional distributions given by

$$\begin{aligned}&\alpha_x[u(t_i \in dx_i, i = 1,\ldots,n] \\ &\quad = \bar{P}_{t_1}(x,dx_1)\bar{P}_{t_2-t_1}(x_1,dx_2)\cdots\bar{P}_{t_n-t_{n-1}}(x_{n-1},dx_n)\end{aligned}$$

where $0 \leq t_1 \leq \cdots \leq t_n$ and $x_1,\ldots,x_n \in E$. For $t \geq 0$ the kernel K_t on $(E,\mathcal{E})$ is defined by

$$K_t(x,dy) = \alpha_x[u(t) \in dy, \zeta_u > t].$$

The family $(K_t)_{t\geq 0}$ is a sub-Markov semigroup of kernels on $(E,\mathcal{E})$. On $(E,\mathcal{E})$ there will also be given a family of finite measures $(\eta_s)_{s>0}$ which is an entrance law for the semigroup $(K_t)_{t\geq 0}$ with $\eta_s(\{a\}) = 0$ for every $s > 0$. By theorem (3.2.6) there is a unique measure ν on $(U_\infty,\mathcal{U}_\infty)$ having the three properties

(i) ν is concentrated on $\{u \in U_\infty : \forall s \geq \zeta_u : u(s) = a\}$,

(ii) $\forall f \in b\mathcal{U}_\infty, \forall t > 0, \forall\Lambda \in \sigma(u \in U_\infty \mapsto u(r), r \leq t)$:

$$\int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\nu(du) = \int_{\Lambda\cap[\zeta>t]} \alpha_{u(t)}(f)\nu(du),$$

(iii) $\forall t > 0 : \nu([\zeta > t]) < \infty$.

It follows that

$$\forall s > 0 : \eta_s(dx) = \nu([\zeta_u > s, u(s) \in dx]).$$

We will always consider ν as a measure on U by putting $\nu(U\backslash U_\infty) = 0$. The product topological space $T \times U$, where T is equipped with the usual topology will be denoted by X and $\mathcal{M}^+$ is the space of nonnegative Borel measures on $(X, \mathcal{B}(X))$ which are finite on the family $\mathcal{S}$ consisting of the subsets $I \times [\zeta > t]$ where I runs through the bounded subintervals of T and $t > 0$. The $\mathcal{S}$-finite Itô-Poisson point process on U with characteristic measure ν will be denoted by P. (The existence and unicity of P follows from proposition (2.2.5).) By proposition (2.3.2) $P(\mathcal{M}_1^\bullet) = 1$, where $\mathcal{M}_1^\bullet$ is the set of $\mathcal{S}$-finite point measures μ with $\mu(\{t\} \times U) \leq 1$ for every $t \geq 0$. Measures $\mu \in \mathcal{M}^\bullet$ have a countable support. For $\mu \in \mathcal{M}_1^\bullet$ the projection $J(\mu)$ of $\text{supp}(\mu)$ on T is an ordered subset. If $\sigma \in J(\mu)$, there is a $u \in U$ such that $(\sigma, u) \in \text{supp}(\mu)$. We will write u_σ for u. Let $L : U \mapsto [0, \infty[$ be a given, measurable function. Define for $\sigma \in T$ and $\mu \in \mathcal{M}_1^\bullet$

$$\begin{aligned} B(\sigma, \mu) &= \sum \{L(u_\tau) : \tau \in J(\mu) \text{ and } \tau < \sigma\} \\ &= \int \mu(d\tau d\nu) 1_{[o,\sigma]}(\tau) L(\nu) \end{aligned}$$

and

$$C(\mu) = \bigcup_{\sigma \in J(\mu)} [B(\sigma-, \mu), B(\sigma, \mu)[.$$

If $T = C(\mu)$ then denote by

$$\tilde{\mu} : T \mapsto E$$

the concatenation of the functions $u_{\tau|[0,L(u_\tau)[}, \tau \in J(\mu)$, that is

$$\tilde{\mu}(s) = u_\sigma(s - B(\sigma-, \mu)) = \int \mu(d\tau dv)(v \times 1_{[0,L(v)[})(s - B(\tau-, \mu))$$

with

$$\sigma \in J(\mu) \text{ such that } s \in [B(\sigma-, \mu), B(\sigma, \mu)[.$$

The function u_σ is called the excursion straddling s. So, if Q is an $\mathcal{S}$-finite point process with phase space X such that

$$Q(\{\mu \in \mathcal{M}_1^\bullet : T = C(\mu)\}) = 1,$$

then for $s \geq 0$ the maps

$$Y_s : \mu \in \{\mu \in \mathcal{M}_1^\bullet : T = C(\mu)\} \mapsto \tilde{\mu}(s)$$

are Q-a.e. defined random variables on the probability space $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+), Q)$.

In this section we want to consider a construction of this kind for the Itô-Poisson point process P. With an extra assumption about the characteristic measure ν of P it will turn out that the process $Y = (Y_t)_{t \geq 0}$ is a

Markov process. So it is natural to consider a family of probability measures $(P_x)_{x\in E}$ on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+))$, where P_x will be the distribution of the process Y starting in x. To this end we add a first point to P corresponding to a start from x taking in account the given transition probabilities $(\bar{P}_t)$. Consider the map

$$u \in U \mapsto \delta_{(0,u)} \in \mathcal{M}^+.$$

For $x \neq a$, Q_x denotes the image of the probability measure α_x under this map. Then Q_x is a point process with phase space X; its intensity measure i_{Q_x}, its Palm measures $(Q_x)_{(\tau,v)}$ and its Laplace transform $\hat{Q}_x$ are given by

$$\begin{array}{lcll} i_{Q_x} & = & \delta_0 \otimes \alpha_x, & \\ (Q_x)_{(\tau,v)} & = & \delta_{\delta_{(\tau,v)}}, & (\tau, v) \in X \\ \hat{Q}_x(f) & = & \int \alpha_x(du)e^{-f(0,u)}, & f \in \mathcal{B}(X)_+. \end{array}$$

Define the family of point processes $(P_x)_{x\in E}$ by

$$P_x = \begin{cases} Q_x * P & if x \in E\backslash\{a\} \\ P & if x = a. \end{cases}$$

Some important properties of the point processes P_x are collected in the following lemma, whose straightforward proof is deleted.

Lemma 3.3.1 *For $x \neq a$, the intensity measure i_{P_x}, the Palm measures $(P_x)_{(\tau,v)}$ and the Laplace transform $\hat{P}_x$ of the point process P_x are given by*

$$\begin{array}{lcl} i_{P_x} & = & i_{Q_x} + i_P, \\ (P_x)_{(\tau,v)} & = & \begin{cases} P_x * \delta_{\delta_{(\tau,v)}} & for\ v \in U\ and\ \tau > 0 \\ \delta_{\delta_{(0,v)}} * P & for\ v \in U\ and\ \tau = 0, \end{cases} \\ \hat{P}_x(f) & = & \int \alpha_x(du)e^{-f(0,u)} \exp[-\int(1-e^{-f})d\lambda \otimes \nu], f \in \mathcal{B}(X)_+. \end{array}$$

Further

$$P_x(\mathcal{M}_1^\bullet) = 1.$$

Let $\Omega = \mathcal{M}_1^\bullet$ and $\mathcal{F}$ the trace of $\mathcal{B}(\mathcal{M}^+)$ on Ω. Our basic family of probability spaces will be $(\Omega\mathcal{F}, P_x), x \in E$. Define for $\omega \in \Omega$ and $\tau \geq 0$

$$A(\tau, \omega) = \int_X \omega(d\sigma du)1_{]o,\tau]}(\sigma)\zeta_u.$$

The random variable $A(\tau)$ is the sum of the excursions up to and including time τ, leaving out the excursion at time 0. As a function of τ, $A(\tau, \omega)$ is a non-decreasing càdlàg function on $[0, \infty[$ for every $\omega \in \Omega$. The Laplace transform of $A(\tau)$ is given by

$$\int_\Omega e^{-\lambda A(\tau)}dP_x = \exp[-\tau \int_U (1 - e^{\lambda\zeta_u})\nu(du)], \lambda > 0, x \in E.$$

From now on we will assume that

$$\int_U (1 - e^{-\zeta_u})\nu(du) < \infty.$$

Then $A(\tau)$ is P_x-a.s. finite for every $\in E$. Note that the family of random variables $(A(\tau))_{\tau \geq 0}$ is a subordinator whose Lévy measure is the ζ-image $\zeta(\nu)$ of the measure ν. See for subordinators Itô [24]. Addition to $A(\tau)$ of a linear term $\gamma\tau$, $\gamma \geq 0$, gives the general form of a subordinator with Levy measure $\zeta(\nu)$. Define for $\tau \geq 0$ and $\omega \in \Omega$

$$\sigma_a(\omega) = \int \omega(d\sigma du) 1_{\{0\}}(\sigma)\zeta_u$$

and

$$B(\tau, \omega) = \sigma_a(\omega) + A(\tau, \omega) + \gamma\tau.$$

It follows from a straightforward calculation that the Laplace transform of the random variable $B(\tau)$ is given by

$$\int_\Omega e^{-\lambda B(\tau)} dP_x = \int e^{-\lambda\sigma_a} dP_x \times \exp[-\tau(\lambda\gamma + \int_U (1 - e^{\lambda\zeta_u})\nu(du))].$$

For $\omega \in \Omega$, denote by $R(\omega)$ the range of $B(.,\omega)$;

$$R(\omega) = \{s \in [0, \infty[: \exists\tau : s = B(\tau, \omega)\}$$

and let $\phi(.,\omega)$ be the right continuous inverse of $B(.,\omega)$:

$$\phi(s, \omega) = \inf\{\tau : B(\tau, \omega) > s\}, s \geq 0.$$

It follows from the definition of ϕ that

$$\forall s \geq 0 : B(\phi(s, \omega)-, \omega) \leq s \leq B(\phi(s, \omega), \omega)$$

with $B(0-, \omega) = 0$. Let $J(\omega)$ be the projection of the support of the measure ω on T:

$$J(\omega) = \{\sigma \in T : \omega(\{\sigma\} \times U) = 1\}.$$

Note that $J(\omega)$ is P_x-a.s. a discrete subset of T if $\nu(U) < \infty$, and a countable, dense subset of T if $\nu(U) = \infty$. Define for $\omega \in \Omega$

$$C(\omega) = \bigcup_{\sigma \in J(\omega)} [B(\sigma-, \omega), B(\sigma, \omega)[.$$

With the above introduced definitions and notations we have the following lemma.

Lemma 3.3.2 *Let $x \in E$. Then P_x-a.s.*

$$T = \begin{cases} R(\omega) + C(\omega) & \text{if } \nu(U) = \infty \quad \text{or} \quad \gamma > 0 \\ C(\omega) & \text{if } 0 < \nu(U) < \infty \quad \text{and} \quad \gamma = 0 \end{cases}$$

Proof. Assume that $\nu(U) = \infty$ or $\gamma > 0$. The function $B(.,\omega)$ is P_x-a.s. strictly increasing, $B(\tau,\omega) \uparrow \infty$ as $\tau \to \infty$. The assertion in the lemma follows from appendix (A3.3). We continue with the case $0 < \nu(U) < \infty$ and $\gamma = 0$. In this case $J(\omega)$ is P_x-a.s. a discrete set, which can be written as $J(\omega) = (\sigma_n(\omega))_{n \geq 1}$ with $\sigma_1(\omega) < \sigma_2(\omega) < \ldots$ Further $\sigma_n(\omega) \to \infty$ as $n \to \infty$. Since $\gamma = 0$, $B(\sigma_n(\omega)-,\omega) = B(\sigma_{n-1}(\omega),\omega)$ and the assertion of the lemma follows.

Let $\omega \in \Omega$ and $t \geq 0$. Let $t \in C(\omega)$. If there is some $\tau \geq 0$ such that $t \in [B(\tau-,\omega), B(\tau,\omega)[$, and if u is such that $\omega_{(\tau,u)} = 1$, then u is the excursion (in ω) straddling t. Note that $\tau = \phi(t,\omega)$ and that $\zeta_u = B(\phi(t,\omega),\omega) - B(\phi(t,\omega)-,\omega)$. With $\omega \in \Omega$ we associate a function $\tilde{\omega} : T \mapsto E$ defined by

$$\tilde{\omega}(t) = \begin{cases} u(t - B(\phi(t,\omega)-,\omega)) & \text{if } t \in C(\omega) \\ a & \text{if } t \neq C(\omega) \end{cases}, t \in T,$$

where u is the excursion straddling t. Note that for $t \in C(\omega)$

$$\tilde{\omega}(t) = \int \omega(d\sigma du)(u \times 1_{[0,\zeta_u[})(t - B(\sigma-,\omega))$$

and

$$1_{C(\omega)}(t) = \int \omega(d\sigma du) 1_{[0,\zeta_u[}(t - B(\sigma-,\omega)).$$

It follows that the map

$$\omega \in (\Omega, \mathcal{F}) \mapsto \tilde{\omega} \in (E^T, \mathcal{E}^T)$$

is measurable. Denote the coordinate evaluations on E^T by Y_t, $t \geq 0$, i.e.

$$Y_t : f \in E^T \mapsto f(t) \in E,$$

and the image of the probability measure P_x under the map $\omega \mapsto \tilde{\omega}$ by $\mathbb{P}_x$. Then, $Y = \{Y_t : t \geq 0\}$ is an E-valued stochastic process on the probability space $(E^T, \mathcal{E}^T, \mathbb{P}_x)$. We continue with the calculation of the finite-dimensional distributions of the process Y. From the definition of the measure ν it follows that for $s, l > 0$

$$\nu([u(s) \in dx, \zeta_u > s + l]) = \eta_s(dx)\bar{P}_l(x, E\backslash\{a\}).$$

Writing β_x for the α_x-distribution of ζ,

$$\beta_x(dl) = \alpha_x[\zeta \in dl] = d\bar{P}_l(x, \{a\}),$$

we get for $s, l > 0$

$$\nu([u(s) \in dx, \zeta_u - s \in dl]) = \eta_s(dx)\beta_x(dl).$$

Proposition 3.3.3 *Let $f \in b\mathcal{E}$ such that $f(a) = 0$.*

A. $x \neq a$.

$$\mathbb{E}_x[f(Y_t)] = \begin{cases} \int K_t(x, dy)f(y) + \int_0^t \beta_x(dl)\mathbb{E}_a[f(Y_{t-l})] & \textit{if } t > 0 \\ \int K_0(x, dy)f(y) + \beta_x(\{0\})\mathbb{E}_a[f(Y_0)] & \textit{if } t = 0 \end{cases}$$

B. $x = a$.

For $t > 0$:

$$\mathbb{E}_a[f(Y_t)] = \int P(d\omega) \int_0^t d\phi(q, \omega) \int \eta_{t-q}(dy)f(y)$$

For $t = 0$ we have to consider two cases:

(i) $\nu(U) = \infty$ *or* $\gamma > 0$.

$$\mathbb{E}_a[f(Y_0)] =; 0$$

(ii) $0 < \nu(U) < \infty$ *and* $\gamma = 0$.

$$\mathbb{E}_a[f(Y_0)] = \frac{1}{\nu(U)} \int \nu(du)f[u(0)].$$

Proof. The proof is based on an application of the Palm formula, see section (2.2). We start with case B. Let $t > 0$.

$$\begin{aligned} &\mathbb{E}_a[f(Y_t)] \\ &= \int P(d\omega)1_{C(\omega)}(t) \int \omega(d\sigma du)(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-, \omega)) \\ &= \int P(d\omega) \int \omega(d\sigma du)(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-, \omega)) \\ &= \int_0^\infty d\sigma \int \nu(du) \int P(d\omega)(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-, \omega + \delta_{(\sigma,u)})) \\ &= \int P(d\omega) \int_0^\infty d\sigma \int \eta_{t-B(\sigma-,\omega)}(dy)f(y) \\ &= \int P(d\omega) \int_0^\infty d\phi(q, \omega) \int \eta_{t-q}(dy)f(y) \end{aligned}$$

where in the last step we have used an integration formula for right continuous inverses, see appendix A3.
Let $t = 0$. If $0 < \nu(U) < \infty$ and $\gamma = 0$, then the formula for $\mathbb{E}_a[f(Y_0)]$ follows from theorem (2.3.7). If $\nu(U) = \infty$ or $\gamma > 0$, then

$$\begin{aligned} P[0 \in C(\omega)] &= P[B(0,\omega) > 0] \\ &= P[\omega(\{0\} \times U] = 0 \end{aligned}$$

since $\lambda \otimes \nu(\{0\} \times U) = 0$. It follows that $\mathbb{P}_a[Y_0 = a] = 1$, hence $\mathbb{E}_a[f(Y_0)] = 0$.
We continue with case A. Let $t > 0$.

$$\begin{aligned} &\mathbb{E}_x[f(Y_t)] \\ &= \int Q_x * P(d\omega) \int \omega(d\sigma du)(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-,\omega)) \\ &= \int Q_x(d\omega') \int \omega'(d\sigma du) 1_{[0]}(\sigma)(f \circ u \times 1_{[0,\zeta_u[})(t) \\ &\quad + \int Q_x(d\omega') \int P(d\omega) \int \omega(d\sigma du) \\ &\quad 1_{]0,\infty[}(\sigma)(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-,\omega+\omega')) \\ &= \int \alpha_x(du) 1_{[\zeta_u > t]} f(u(t)) \\ &\quad + \int \alpha_x(dv) \int P(d\omega) \int \omega(d\sigma du)(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-,\omega) - \zeta_v) \\ &= \int K_t(x,dy) f(y) + \int_0^t \beta_x(dl) \mathbb{E}_a[f(Y_{t-l})]. \end{aligned}$$

For $t = 0$ we only have to note that

$$\begin{aligned} &\int \alpha_x(dv) \int P(d\omega) \int \omega(d\sigma du)(f \circ u \times 1_{[0,\zeta_u[})(0 - B(\sigma-,\omega) - \zeta_v) \\ &= \alpha_x[\zeta_v = 0] \times \mathbb{E}_a[f(Y_0)] \\ &= \beta(\{0\}) \times \mathbb{E}_a[f(Y_0)]. \end{aligned}$$

Define the measure Φ on $[0,\infty[$ by

$$\Phi(dq) = \int_\Omega P(d\omega) d\phi(q,\omega).$$

Define the kernels $(S_t)_{t>0}$ on $(E,\mathcal{E})$ by:

$$\begin{aligned} S_t(x,dy) &= \begin{cases} (\Phi * \eta)_t(dy) & \text{for } x = a, y \neq a \\ K_t(x,dy) + \int_0^t \beta_x(dl)(\Phi * \eta)_{t-l}(dy) & \text{for } x, y \neq a \end{cases} \\ S_t(x,\{a\}) &= 1 - S_t(x, E\backslash\{a\}). \end{aligned}$$

Define the kernel S_0 on $(E,\mathcal{E})$ by:

(i) If $\nu(U)=\infty$ or $\gamma>0$ then

$$S_0(x,dy)=\begin{cases} 0 & \text{for } x=a, y\neq a \\ K_0(x,dy) & \text{for } x,y\neq a, \end{cases}$$

(ii) If $0<\nu(U)<\infty$ and $\gamma=0$ then

$$S_0(x,dy)=\begin{cases} \frac{1}{\nu(U)}\nu[u(0)\in dy,\zeta_u>0] & \text{for } x=a, y\neq a \\ K_0(x,dy)+\frac{\beta_x(\{0\})}{\nu(U)}\nu[u(0(\in dy,\zeta_u>0] & \text{for } x,y\neq a. \end{cases}$$

In both cases (i) and (ii)

$$S_0(x,\{a\})=1-S_0(x,E\backslash\{a\}).$$

The kernels $(S_t)_{t\geq 0}$ are a family of Markov kernels om $(E,\mathcal{E})$. The statement of proposition (3.3.3) can be written as

$$\mathbb{E}_x[f(Y_t)]=S_tf(x), t\geq 0, f\in b\mathcal{E}.$$

Define for $t\geq 0$ the map $\psi_t:\Omega\mapsto\Omega$ by

$$\psi_t(\omega)=\begin{cases} T_{\phi(t)}(\omega) & \text{if } t\in R(\omega) \\ \delta_{(0,\theta_{t-B(\phi(t,\omega)-,\omega)}u)}+T_{\phi(t)}(\omega) & \text{if } t\notin R(\omega) \end{cases}$$

where u is the excursion straddling t and where T_ϕ is defined as in section (2.3). The meaning of the map ψ_t is explained in the following lemma.

Lemma 3.3.4 *For $s,t\geq 0$ and $\omega\in\Omega$ we have*

$$\phi(s+t,\omega)=\phi(t,\omega)+\phi(s,\psi_t\omega)$$

and

$$Y_s[(\psi_t\omega)]=Y_{s+t}(\tilde{\omega}).$$

Proof. First note that $\sigma_a(\psi_t\omega)=B(\phi(t,\omega),\omega)-t$. Indeed, if $t\in R(\omega)$ then

$$\begin{aligned}\sigma_a(\psi_t\omega) &= \sigma_a(T_{\phi(t)}\omega) \\ &= \int(T_{\phi(t)}\omega)(d\sigma du)1_{\{0\}}(\sigma)\zeta_u \\ &= \int\omega(d\sigma du)1_{]\phi(t,\omega),\infty[}(\sigma)1_{\{0\}}(\sigma-\phi(t,\omega))\zeta_u \\ &= 0\end{aligned}$$

and the result follows, as $B(\phi(t,\omega),\omega) = t$ for $t \in R(\omega)$.
If $t \notin R(\omega)$ then

$$\begin{aligned}
\sigma_a(\psi_t\omega) &= \sigma_a(\delta_{(0,\theta_{t-B(\phi(t,\omega)-,\omega)}u)}) + \sigma_a(T_{\phi(t)}\omega) \\
&= \zeta_u - (t - B(\phi(t,\omega)-,\omega)) \\
&= B(\phi(t,\omega)-,\omega) - t
\end{aligned}$$

since u is the excursion straddling t.
We continue with the calculation of $B(\tau,\psi_t\omega)$.

$$\begin{aligned}
B(\tau,\psi_t\omega) &= \sigma_a(\psi_t\omega) + A(\tau,\psi_t\omega) + \gamma\tau \\
&= B(\phi(t,\omega),\omega) - t + \int \omega(d\sigma du) 1_{]\phi(t,\omega),\phi(t,\omega)+\tau]}(\sigma)\zeta_u + \gamma\tau \\
&= B(\phi(t,\omega),\omega) - t + B(\phi(t,\omega)+\tau,\omega) - B(\phi(t,\omega),\omega) \\
&= B(\phi(t,\omega)+\tau,\omega) - t. \ (*)
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(s,\psi_t\omega) &= \inf\{\tau : B(\tau,\psi_t\omega) > s\} \\
&= \inf\{\tau : B(\phi(t,\omega)+\tau,\omega) + \tau > s + t\} \text{by formula (*)} \\
&= -\phi(t,\omega) + \phi(s+t,\omega),
\end{aligned}$$

which proves the first part of the lemma.
For the second part, suppose first that $s \in R(\psi_t\omega)$. Then, for some $\tau \geq 0$

$$B(\tau,\psi_t\omega) = s,$$

hence by formula (*)

$$B(\phi(t,\omega)+\tau,\omega) = t + B(\tau,\psi_t\omega) = t + s,$$

and it follows that $s + t \in R(\omega)$. So

$$Y_s((\psi_t\omega)\tilde{}) = Y_{s+t}(\tilde{\omega}) = a$$

by definition of Y.
Suppose now that $s \notin R(\psi_t\omega)$. Let

$$s < \sigma_a(\psi_t\omega) = B(\phi(t,\omega),\omega) - t.$$

Then $\phi(s+t,\omega) = \phi(t,\omega)$ and there is one excursion, say u, (in ω) straddling both t and $s+t$, so

$$\begin{aligned}
Y_s((\psi_t\omega)\tilde{}) &= u(s+t - B(\phi(t,\omega)-,\omega)) \\
&= u(s+t - B(\phi(t+s,\omega)-,\omega)) \\
&= Y_{s+t}(\tilde{\omega}).
\end{aligned}$$

For

$$s > \sigma_a(\psi_t\omega)$$

we have $B(0, \psi_t\omega) < s$, so $\phi(s, \psi_t\omega) > 0$. It follows that

$$\begin{aligned}
(\psi_t\omega)_{(\phi(s,\psi_t\omega),u)} &= (T_{\phi(t)}\omega)_{(\phi(s+t,\omega)-\phi(t,\omega),u)} \\
&= \omega_{(\phi(s+t,\omega),u)}
\end{aligned}$$

and the excursion straddling s in $\psi_t\omega$ is the same as the excursion straddling $s+t$ in ω. From

$$\begin{aligned}
B(\phi(s,\psi_t\omega)-,\psi_t\omega) &= \lim_{\epsilon\downarrow 0} B(\phi(s,\psi_t\omega)-\epsilon,\psi_t\omega) \\
&= \lim_{\epsilon\downarrow 0} B(\phi(s+t,\omega)-\phi(t,\omega)-\epsilon,\psi_t\omega) \\
&= \lim_{\epsilon\downarrow 0} B(\phi(s+t,\omega)-\epsilon,\omega)-t \text{ by formula (*)} \\
&= B(\phi(s+t,\omega)-,\omega)-t,
\end{aligned}$$

it follows that

$$\begin{aligned}
Y_s((\psi_t\omega)\tilde{}) &= u(s-B(\phi(s,\psi_t\omega)-,\psi_t\omega)) \\
&= u(s+t-B(\phi(t+s,\omega)-,\omega)) \\
&= Y_{s+t}(\tilde{\omega}),
\end{aligned}$$

where u is the excursion straddling s in $\psi_t\omega$.

Theorem 3.3.5 *Let $n \geq 2$, $f_1, \ldots, f_n \in b\mathcal{E}$, $0 \leq t_1 \cdots \leq t_n$ and $x \in E$ then*

$$\begin{aligned}
&\mathbb{E}_x[\prod_{i=1}^n f_i(Y_{t_i})] \\
&\quad = \int S_{t_1}(x, dy_1) \int S_{t_2-t_1}(y_1, dy_2) \cdots \int S_{t_n-t_{n-1}}(y_{n-1}, dy_n) \prod_{i=1}^n f_i(y_i).
\end{aligned}$$

Proof. We will only consider the case $x \neq a$ and $t_1 > 0$. The proof for the remaining cases is analogous and is therefore deleted. Let $\bar{f}_1$ be defined on E by

$$\bar{f}_1(x) = f_1(x) - f(a).$$

Then

$$\mathbb{E}_x[\prod_{i=1}^n f_i(Y_{t_i})] = \mathbb{E}_x[\bar{f}_1(Y_{t_1}) \prod_{i=2}^n f_i(Y_{t_i})] + f_1(a)\mathbb{E}_x[\prod_{i=2}^n f_i(Y_{t_i})].$$

Since $\bar{f}_1(a) = 0$, we have

$$
\begin{aligned}
&\mathbb{E}_x[\bar{f}_1(Y_{t_1})\prod_{i=2}^{n} f_i(Y_{t_i})] \\
&= \mathbb{E}_x[1_C(t_1)\bar{f}_1(Y_{t_1})\prod_{i=2}^{n} f_i(Y_{t_i})] \\
&= \int P_x(d\omega)\int \omega(d\sigma du)(\bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1 - B(\sigma-,\omega)) \\
&\quad \prod_{i=2}^{n} f_i(Y_{t_i-t_1})((\psi_{t_1}\omega\,\mathfrak{f}) \\
&= \int Q_x(d\omega')\int P(d\omega)\int(\omega+\omega')(d\sigma du) \\
&\quad (\bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1) - B(\sigma-,\omega+\omega'))\prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\psi_{t_1}(\omega+\omega'\,\mathfrak{f}) \\
&= \int Q_x(d\omega')\int P(d\omega)\int \omega'(d\sigma du)\cdots \\
&\quad +\int Q_x(d\omega')\int P(d\omega)\int \omega(d\sigma du)\cdots \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}
$$

We first calculate I.

$$
\begin{aligned}
&\int Q_x(d\omega')\int \omega'(d\sigma du)(\bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1) \\
&\qquad \int P(d\omega)\prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\psi_{t_1}(\omega+\omega'\,\mathfrak{f}) \\
&\quad = \int \alpha_x(du)(\bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1) \\
&\qquad \int P(d\omega)\prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\psi_{t_1}(\omega+\delta_{(0,u)})\,\mathfrak{f}).
\end{aligned}
$$

From

$$\zeta_u > t_1 \Longrightarrow \forall \tau \geq 0 : B(\tau,\omega+\delta_{(0,u)}) = B(\tau,\omega)+\zeta_u > t_1$$

it follows that

$$\phi(t_1,\omega+\delta_{(0,u)}) = 0 \text{ and } t_1 \notin R(\omega+\delta_{(0,u)}).$$

Hence P-a.s.

$$\psi_{t_1}(\omega + \delta_{(0,u)}) = \delta_{(0,\theta_{t_1}u)} + T_0(\omega + \delta_{(0,u)}) = \delta_{(0,\theta_{t_1}u)} + \omega.$$

So

$$\begin{aligned}
\mathrm{I} &= \int \alpha_x(du)(\bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1) \int P(d\omega) \prod_{i=2}^{n} f_i(Y_{t_i - t_1})((\delta_{(0,\theta_{t_1}u)} + \omega)) \\
&= \int K_{t_1}(x, dy)\bar{f}_1(y) \int \alpha_y(du) \int P(d\omega) \prod_{i=2}^{n} f_i(Y_{t_i - t_1})((\delta_{(0,u)} + \omega)) \\
&= \int K_{t_1}(x, dy)\bar{f}_1(y)\mathbb{E}_y[\prod_{i=2}^{n} f_i(Y_{t_i - t_1})];
\end{aligned}$$

We continue with the calculation of II.

$$\begin{aligned}
& \int \alpha_x(dv) \int P(d\omega) \int \omega(d\sigma du) \\
& \qquad (\bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1 - \zeta_v - B(\sigma-, \omega)) \\
& \qquad \times \prod_{i=2}^{n} f_i(Y_{t_i - t_1})((\psi_{t_1}(\delta_{(0,v)} + \omega))) \\
= & \int \alpha_x(dv) \int_0^\infty d\sigma \int \nu(du) \int P(d\omega) \\
& \qquad (\bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1 - \zeta_v - B(\sigma-, \omega + \delta_{(\sigma,u)})) \\
& \qquad \times \prod_{i=2}^{n} f_i(Y_{t_i - t_1})((\psi_{t_1}(\delta_{(0,v)} + \omega + \delta_{(\sigma,u)})))
\end{aligned}$$

by an application of the Palm formula, see section (2.2).
Since

$$\begin{aligned}
t_1 - \zeta_v - B(\sigma-, \omega) &\implies B(\sigma-, \omega + \delta_{(0,v)} + \delta_{(\sigma,u)}) > t_1 \\
&\implies \phi(t_1, \omega + \delta_{(0,v)} + \delta_{(\sigma,u)}) \le \sigma
\end{aligned}$$

and

$$\begin{aligned}
0 \le t_1 - \zeta_v - B(\sigma-, \omega) &\implies B(\sigma-, \omega + \delta_{(0,v)} + \delta_{(\sigma,u)}) \le t_1 \\
&\implies \phi(t_1, \omega + \delta_{(0,v)} + \delta_{(\sigma,u)}) \ge \sigma,
\end{aligned}$$

it follows that

$$\phi(t_1, \omega + \delta_{(0,v)} + \delta_{(\sigma,u)}) = \sigma,$$

and

$$t_1 \notin R(\omega + \delta_{(0,v)} + \delta_{(\sigma,u)}),$$

hence P-a.s.

$$\begin{aligned} \psi_{t_1}(\delta_{(0,v)} + \omega + \delta_{(\sigma,u)}) &= \delta_{(0,\bar{u})} + T_\sigma(\delta_{(0,v)} + \omega + \delta_{(\sigma,u)}) \\ &= \delta_{(0,\bar{u})} + T_\sigma(\omega), \end{aligned}$$

where $\bar{u} = \theta_{t_1 - B(\sigma-,\omega)-\zeta_v} u$. So

$$\begin{aligned} &\mathrm{II} \\ &= \int \alpha_x(dv) \int_0^\infty d\sigma \int \nu(du) \int P(d\omega) \\ &\quad (\bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1 - \zeta_v - B(\sigma-,\omega + \delta_{(\sigma,u)})) \\ &\quad \times \prod_{i=2}^n f_i(Y_{t_i - t_1})(\delta_{(0,\bar{u})} + T_\sigma(\omega)) \\ &= \int \alpha_x(dv) \int_0^\infty d\sigma \int \nu(du) \int P(d\omega) \\ &\quad \bar{f}_1 \circ u \times 1_{[0,\zeta_u[})(t_1 - \zeta_v - B(\sigma-,\omega + \delta_{(\sigma,u)}) \\ &\quad \int P(d\omega') \prod_{i=2}^n f_i(Y_{t_i - t_1})(\delta_{(0,\bar{u})} + \omega') \\ &\quad \text{by an application of the renewal property} \\ &= \int \alpha_x(dv) \int (\Phi * \eta)_{t_1 - \zeta_v}(dy) \bar{f}_1(y) \int \alpha_y(du) \\ &\quad \int P(d\omega') \prod_{i=2}^n f_i(Y_{t_i - t_1})(\delta_{(0,\bar{u})} + \omega') \\ &= \int_0^1 \beta_x(dl) \int (\Phi * \eta)_{t_1 - l}(dy) \bar{f}_1(y) \mathbb{E}_y[\prod_{i=2}^n f_i(Y_{t_i - t_1})]. \end{aligned}$$

It follows that

$$\begin{aligned} &\mathbb{E}_x[\prod_{i=1}^n f_i(Y_{t_i})] \\ &= \int_{E\setminus\{a\}} S_{t_1}(x,dy)\bar{f}_1(y)\mathbb{E}_y[\prod_{i=2}^n f_i(Y_{t_i - t_1})] + f_1(a)\mathbb{E}_a[\prod_{i=2}^n f_i(Y_{t_i - t_1})] \\ &= \int_E S_{t_1}(x,dy) f_1(y) \mathbb{E}_y[\prod_{i=2}^n f_i(Y_{t_i - t_1})] \end{aligned}$$

An induction argument completes the proof of the theorem.

As a consequence of theorem (3.3.5), $(S_t)_{t\geq 0}$ is a Markov semigroup on $(E,\mathcal{E})$ and Y is the canonical representation of the Markov process with this transition semigroup. Let $(V_\lambda)_{\lambda>0}$ be the resolvent of the semigroup $(S_t)_{t\geq 0}$, i.e. the operator on $b\mathcal{E}$ defined by

$$V_\lambda f(x) = \int_0^\infty e^{-\lambda t} S_t f(x) dt = \mathbb{E}_x \int_0^\infty e^{-\lambda t} f(Y_t) dt, x \in E.$$

Denote the resolvent of the semigroup $(K_t)_{t\geq 0}$ by $(G_\lambda)_{\lambda>0}$. Define for $\lambda > 0$ and $f \in b\mathcal{E}$

$$\hat{\eta}_\lambda(f) = \int_0^\infty dt e^{-\lambda t} \int_E \eta_t(dx) f(x).$$

This integral converges because of the assumption

$$\int (1 - e^{-\zeta_u})\nu(du) = \hat{\eta}_1(1) < \infty.$$

Define for $\lambda > 0$ and $x \in E$

$$z_\lambda(x) = \int \alpha_x(du) e^{-\lambda\zeta_u}.$$

In the next lemma we prove some relations between $(G_\lambda), (\hat{\eta}_\lambda)$ and z_λ.

Lemma 3.3.6 *Let $\lambda, \mu > 0$ and $f \in b\mathcal{E}$, then*

(i) $(\mu - \lambda)\hat{\eta}_\lambda(G_\mu f) = \hat{\eta}_\lambda(f) - \hat{\eta}_\mu(f)$,

(ii) $z_\lambda = 1 - \lambda G_\lambda 1$,

(iii) $(\mu - \lambda)\hat{\eta}_\mu(z_\lambda) = \mu\hat{\eta}_\mu(1) - \lambda\hat{\eta}_\lambda(1)$.

Proof.
(i)

$$\begin{aligned}
(\mu - \lambda)\hat{\eta}_\lambda(G_\mu f) &= (\mu - \lambda)\int_0^\infty dt e^{-\lambda t} \int \eta_t(dx) \int_0^\infty e^{-\mu s} K_s f(x) ds \\
&= (\mu - \lambda)\int_0^\infty dt e^{-\lambda t} \int_0^\infty e^{-\mu s} \eta_{s+t}(f) ds \\
&= \int_0^\infty ds e^{-\mu s} \eta_s(f) \int_0^s (\mu - \lambda) e^{-(\mu-\lambda)t} dt \\
&= \hat{\eta}_\lambda(f) - \hat{\eta}_\mu(f).
\end{aligned}$$

(ii)

$$\begin{aligned} z_\lambda(x) &= \int \alpha_x(du)e^{-\lambda\zeta_u} \\ &= \int \alpha_x(du)\int_0^\infty 1_{]\zeta_u,\infty[}(s)\lambda e^{-\lambda s}ds \\ &= \int_0^\infty \lambda e^{-\lambda s}(1-K_s1(x))ds \\ &= 1-\lambda G_\lambda 1(x). \end{aligned}$$

(iii)

$$\begin{aligned} (\mu-\lambda)\hat{\eta}_\mu(z_\lambda) &= (\mu-\lambda)\hat{\eta}_\mu(1-\lambda G_\lambda 1) \text{ (by (iii))} \\ &= (\mu-\lambda)\hat{\eta}_\mu(1)+\lambda(\hat{\eta}_\mu(1)-\hat{\eta}_\lambda(1)) \text{ (by (i))} \\ &= \mu\hat{\eta}_\mu(1)-\lambda\hat{\eta}_\lambda(1). \end{aligned}$$

We continue with a theorem which gives an expression for the resolvent $(V_\lambda)_{\lambda>0}$.

Theorem 3.3.7 *Let $\lambda > 0$ and $f \in b\mathcal{E}$. Then*

$$V_\lambda f(x) = G_\lambda f(x) + z_\lambda(x)V_\lambda f(a)$$

where

$$V_\lambda f(a) = \frac{\hat{\eta}_\lambda(f)+\gamma f(a)}{\lambda\gamma+\lambda\hat{\eta}_\lambda(1)}.$$

Proof. Let $x \in E\backslash\{a\}$. Let $f \in b\mathcal{E}$ and $\bar{f}(.) = f(.) - f(a)$.

$$\begin{aligned} &V_\lambda f(x) \\ &= \int P_x(d\omega)\int \omega(d\sigma du)\int_0^\infty e^{-\lambda t}(f\circ u\times 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))dt \\ &\quad +\int P_x(d\omega)\int_0^\infty 1_{C^c(\omega)}(t)e^{-\lambda t}f(a)dt \\ &= \int P_x(d\omega)\int \omega(d\sigma du)\int_0^\infty e^{-\lambda t}(\bar{f}\circ u\times 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))dt \\ &\quad +\frac{1}{\lambda}f(a) \\ &= \int \alpha_x(dv)\int_0^{\zeta_v} e^{-\lambda t}\bar{f}(v(t))dt + \int \alpha_x(dv)\int_{\zeta_v}^\infty e^{-\lambda t}\mathbb{E}_a[\bar{f}(Y_{t-\zeta_v})]dt \\ &\quad +\frac{1}{\lambda}f(a) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-\lambda t} K_t \bar{f}(x)dt + \int \alpha_x(dv) e^{-\lambda\zeta_v} V_\lambda \bar{f}(a) + \frac{1}{\lambda} f(a) \\
&= \int 0^\infty e^{-\lambda t} K_t f(x)dt + (1 - \lambda \int_0^\infty e^{-\lambda t} K_t 1(x)dt) \times V_\lambda f(a)
\end{aligned}
$$

We continue with a calculation of $V_\lambda f(a)$:

$$
\begin{aligned}
&V_\lambda f(a) \\
&= \int P(d\omega) \int \omega(d\sigma du) \int_0^\infty e^{-\lambda t} (\bar{f} \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-,\omega))dt + \frac{1}{\lambda} f(a) \\
&= \int_0^\infty d\sigma \int \nu(du) \int P(d\omega) e^{-\lambda B(\sigma-,\omega)} \int_0^{\zeta_u} e^{-\lambda t} \bar{f}(u(t))dt + \frac{1}{\lambda} f(a) \\
&= \int_0^\infty d\sigma \int \nu(du) \exp[-\sigma(\lambda\gamma + \int (1 - e^{-\lambda\zeta_w})\nu(dw))] \\
&\quad \times \int_0^\infty e^{-\lambda t} \bar{f}(u(t)) 1_{[\zeta_u > t]} dt + \frac{1}{\lambda} f(a) \\
&= \frac{1}{\lambda\gamma + \int (1 - e^{-\lambda\zeta_w})\nu(dw)} \int_0^\infty e^{-\lambda t} \int \nu(du) \bar{f}(u(t)) 1_{[\zeta_u > t]} dt + \frac{1}{\lambda} f(a).
\end{aligned}
$$

Note that

$$
\int (1 - e^{-\lambda\zeta_w})\nu(dw) = \lambda \hat{\eta}_\lambda(1)
$$

and

$$
\int_0^\infty e^{-\lambda t} \int \nu(du) \bar{f}(u(t)) 1_{[\zeta_u > t]} dt = \hat{\eta}_\lambda(\bar{f}).
$$

Hence

$$
V_\lambda f(a) = \frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\lambda\gamma + \lambda\hat{\eta}_\lambda(1)}.
$$

The rest of the theorem follows from the following observation

$$
\begin{aligned}
1 - \lambda \int_0^\infty e^{-\lambda t} K_t 1(x)dt &= \int_0^\infty \lambda e^{-\lambda t}(1 - K_t 1(x))dt \\
&= \int \alpha_x(du) \int_0^\infty \lambda e^{-\lambda t} 1_{[\zeta_u \le t]} dt \\
&= \int \alpha_x(du) e^{-\lambda\zeta_u}.
\end{aligned}
$$

The next theorem states that the resolvent (V_λ) inherits the Ray property of the resolvent (G_λ) if the following extra condition

$$
\forall x \neq a : z_\lambda(x) < 1
$$

is satisfied.

Theorem 3.3.8 *If $z_\lambda(x) < 1$ for every $x \neq a$ and if (G_λ) is a Ray resolvent, then the same is true for (V_λ). In this case the process Y has càdlàg paths $\mathbb{P}_x$-a.s. and is therefore a strong Markov process.*

Proof. If (G_λ) is a Ray resolvent, then the same reasoning as in Rogers [ ] can be used to prove the Ray property for the resolvent (V_λ). By construction, Y is the canonical representation of the Markov process corresponding to (V_λ). So $\mathbb{P}_x$-a.s. the limits

$$X_t = \lim_{q \in \mathbb{Q} \downarrow\downarrow t} Y_q$$

exist for all $t \geq 0$. The process $X = (X_t)_{t \geq 0}$ is a càdlàg version of the process Y which has the strong Markov property, see Williams [ ], ch.III. So it is sufficient to show that the processes X and Y are $\mathbb{P}_x$-equal. It is clear that

$$\forall t \in C(\omega) : X_t(\omega) = Y_t(\omega).$$

So there is nothing to prove in the case $\gamma = 0$ and $0 < \nu(U) < \infty$, since in this case $T = C(\omega)$, see lemma (3.3.2). So let $\gamma > 0$ or $\nu(U) < \infty$. Suppose that $t \in R(\omega)$, say $t = B(r, \omega)$.
If $\nu(U) < \infty$, then there is a first interval $[B(\sigma-, \omega), B(\sigma, \omega)[$ following t, i.e.

$$t < B(\sigma-, \omega) \text{ and } [t, B(\sigma-, \omega)[\subset R(\omega),$$

and it is clear that $X_t(\omega) = a = Y_t(\omega)$.
If $\nu(U) = \infty$, there is no first interval $[B(\sigma-, \omega), B(\sigma, \omega)[$ following t, since this would imply that

$$\exists \sigma > r : \omega([r, \sigma[\times U) = 0$$

which is impossible. So for every $\epsilon > 0$, the interval $[t, t+\epsilon]$ contains an excursion interval. Since we can choose in each interval a rational number q so that $Y_q(\omega)$ is in a arbitrary small neighbourhood of a, it follows that $X_t(\omega) = a = Y_t(\omega)$.

In the next theorem we give an explicit formula for the Blumenthal-Getoor local time for Y at a, see Blumenthal&Getoor [ ]. Consider the map $\phi(t)$ defined on $\mathcal{M}_1^\bullet$ as a $\mathbb{P}_x$-a.s. defined map on the sample space E^T of the process Y, which is possible since the map

$$\omega \in \mathcal{M}_1^\bullet \mapsto \tilde{\omega} \in E^T$$

is an injection.

Theorem 3.3.9 *Under the same assumptions as in theorem (3.3.7), the Blumenthal-Getoor local time $L = (L_t)_{t\geq 0}$ at the state a of the process Y is given by*

$$L_t = [\gamma + \int (1 - e^{-\zeta_u})\nu(du)] \times \phi(t).$$

Proof. Let σ_a be the first time that Y hits or approaches the state a.

$$\begin{aligned} \mathbb{E}_x(e^{-\sigma_a}) &= \int P_x(d\omega)e^{-B(0,\omega)} \\ &= \int Q_x(d\omega')\int P_x(d\omega)e^{-B(0,\omega+\omega')} \\ &= \int Q_x(d\omega')e^{-B(0,\omega')}. \end{aligned}$$

Note that

$$B(0,\omega+\omega') = B(0,\omega')\text{for } P \text{ almost every } \omega$$

and

$$B(\tau,\omega') = B(0,\omega') \text{ for } Q_x \text{ almost every } \omega'.$$

So for $t > B(0,\omega')$

$$\begin{aligned} \phi(t,\omega+\omega') &= \inf\{\tau : B(\tau,\omega+\omega') > t\} \\ &= \inf\{\tau : B(\tau,\omega) + B(0,\omega') > t\} \\ &= \phi(t - B(0,\omega'),\omega). \end{aligned}$$

It follows that

$$\begin{aligned} &\int P_x(d\omega)\int_0^\infty e^{-t}d\phi(t,\omega) \\ &= \int P_x(d\omega)\int_0^\infty e^{-t}1_{[B(0,\omega),\infty[}(t)d\phi(t,\omega) \\ &= \int Q_x(d\omega')\int P(d\omega)\int_0^\infty e^{-t}1_{[B(0,\omega+\omega'),\infty[}(t)d\phi(t,\omega+\omega') \\ &= \int Q_x(d\omega')\int P(d\omega)\int_0^\infty e^{-t}1_{[B(0,\omega'),\infty[}(t)d\phi(t - B(0,\omega'),\omega) \\ &= \int Q_x(d\omega')e^{-B(0,\omega')} \times \int P(d\omega)\int_0^\infty e^{-t}d\phi(t,\omega). \end{aligned}$$

Since

$$\begin{aligned} \int P(d\omega)\int_0^\infty e^{-t}d\phi(t,\omega) &= \int_0^\infty dt\int P(d\omega)e^{-B(t,\omega)} \\ &= \frac{1}{\gamma + \int(1 - e^{-\zeta_u})\nu(du)} \end{aligned}$$

we get

$$\mathbb{E}_a(e^{-\sigma_a}) = [\gamma + \int (1 - e^{-\zeta_u})\nu(du)] \int P_x(d\omega) \int_0^\infty e^{-t} d\phi(t, \omega).$$

Since the Itô-Poisson point process of excursions from a can be reconstructed from Y, it follows that $\phi(t)$ is measurable with respect to the σ-algebra $\sigma(Y_t : t \geq 0)$ generated by the process Y. An application of Galmarino's test, see Dellacherie&Meyer [ ], yields that the process $(\phi(t))_{t\geq 0}$ is adapted to the filtration of the process Y. Finally, it follows from lemma (3.3.4) that the functional $(\phi(t))_{t\geq 0}$ is additive, which completes the proof of the theorem.

We conclude this section by a short description of the process Y^δ, which is the process Y, constructed as above from the family of point processes (P_x), with killing on state a at a rate δ proportional to the local time. This is also an example of the construction of a stochastic process from a more general point process. Let for $s \geq 0$ the point process P^s be defined as the image of P under the map

$$\omega \in \mathcal{M}^+ \mapsto 1_{[0,s]\times U}\omega \in \mathcal{M}^+.$$

It is clear that P^s is a Poisson point process on X with intensity measure $1_{[0,s]}\lambda \otimes \nu$ where ν is the characteristic measure of P. Let for $\delta > 0$ the point process S^δ be the Cox process on X defined by

$$S^\delta = \int_0^\infty \delta e^{-\delta s} P^s ds,$$

and let for $x \in E$ the family of point processes S_x^δ be defined by

$$S_x^\delta = \begin{cases} Q_x * S^\delta & \text{for } x \neq a \\ S^\delta & \text{for } x = a. \end{cases}$$

where the point process Q_x is defined as in the beginning of this section. Define the map

$$\kappa : \omega \in \mathcal{M}^+ \mapsto \inf\{\tau : \omega([\tau, \infty[\times U) = 0\} \in [0, \infty]$$

and define

$$B(\tau, \omega) = \begin{cases} \int \omega(d\sigma du) 1_{[0,\tau]}(\sigma)\zeta_u + \gamma\tau & \text{for } \tau \leq \phi(\omega) \\ B(\phi(\omega), \omega) & \text{for } \tau > \phi(\omega) \end{cases}$$

where γ is a positive constant. The process Y^δ is now constructed from the family of point processes (S_x^δ) in the following way. Until time $B(\infty, \omega)$ the construction is the same as for the process Y associated with the family (P_x). At time $B(\infty,\omega)$ the process Y is killed. It can be shown that Y^δ has the simple Markov property. We will only give an expression for the resolvent of the process Y^δ. Let $(V_\lambda^\delta)_{\lambda>0}$ be the resolvent of the process Y^δ: for $x \in E, \lambda > 0$ and $f \in b\mathcal{E}$

$$V_\lambda^\delta f(x) = \mathbb{E}_x \int_0^\infty e^{-\lambda t} f(Y_t^\delta)dt.$$

Theorem 3.3.10 *Let $x \in E, \lambda > 0$ and $f \in b\mathcal{E}$. Then*

$$V_\lambda^\delta f(x) = G_\lambda f(x) + z_\lambda(x) V_\lambda^\delta f(a)$$

where

$$V_\lambda^\delta f(a) = \frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}.$$

Proof. We only calculate $V_\lambda^\delta f(a)$. The rest of the proof is analogous to the proof of theorem (3.3.7). Suppose first that $f(a) = 0$.

$$\begin{aligned}
&V_\lambda^\delta f(a) \\
&= \int S^\delta(d\omega) \int \omega(d\sigma du) \int_0^\infty e^{-\lambda t}(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-,\omega))dt \\
&= \int_0^\infty ds\delta e^{-\delta s} \int P(d\omega) \int \omega(d\sigma du) 1_{[0,s]}(\sigma) \\
&\quad \int_0^\infty dt e^{-\lambda t}(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-,\omega)) \\
&= \int_0^\infty ds\delta e^{-\delta s} \int_0^s d\sigma \int \nu(du) \int P(d\omega) \\
&\quad \int_0^\infty dt e^{-\lambda t}(f \circ u \times 1_{[0,\zeta_u[})(t - B(\sigma-,\omega)) \\
&= \int_0^\infty d\sigma \int \nu(du) \int P(d\omega) e^{-\delta\sigma - \lambda B(\sigma-,\omega)} \int_0^{\zeta_u} dt e^{-\lambda t} f(u(t)) \\
&= \frac{1}{\delta + \lambda\gamma + \int(1 - e^{-\lambda\zeta_w})\nu(dw)} \int_0^\infty dt e^{-\lambda t} \int \nu(du) f(u(t)) 1_{[\zeta_u > t]} \\
&= \frac{\hat{\eta}_\lambda(f)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}.
\end{aligned}$$

Further

$$V_\lambda^\delta f(a) = \int S^\delta(d\omega) \int_0^{B(\infty,\omega)} dt e^{-\lambda t}$$

$$
\begin{aligned}
&= \int_0^\infty ds\delta e^{-\delta s} \int P(d\omega) \int_0^{B(s,\omega)} dt e^{-\lambda t} \\
&= \frac{1}{\lambda} \int_0^\infty ds\delta e^{-\delta s}[1 - e^{-s(\lambda\gamma + \lambda\hat{\eta}_\lambda(1))}] \\
&= \frac{\gamma + \hat{\eta}_\lambda(1)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}.
\end{aligned}
$$

So for $f \in b\mathcal{E}$

$$
\begin{aligned}
V_\lambda^\delta f(a) &= [V_\lambda^\delta(f - f(a) \times 1)](a) + f(a)V_\lambda^\delta 1(a) \\
&= \frac{\hat{\eta}_\lambda(f - f(a) \times 1) + (\gamma + \hat{\eta}_\lambda(1))f(a)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)} \\
&= \frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}.
\end{aligned}
$$

Chapter 4

Applications

In chapter 3 we saw how to construct for a Ray process Y the Itô-Poisson point process of excursions from a recurrent state a, which is not a holding point and for which $\mathbb{P}_a[\tau_a = 0] = 1$ where τ_a is the infimum of the times $t > 0$ at which Y hits or approaches state a. In the first section of this chapter we will show how one can get an explicit formula for the characteristic measure of the Itô-Poisson point process of excursions from zero for standard Brownian motion using the elementary calculations of the distribution of Brownian excursions in Chung [6]. By means of adjunction of a Radon-Nikodym factor we get from this result an explicit formula for the characteristic measure of the Itô-Poisson point process of excursions from zero for standard Brownian motion with constant drift. This will be done in section (4.2). A well-known problem which can be treated with excursion theory is to describe all strong Markov processes which behave like a given strong Markov process outside a given state a (or more generally outside some given subset of states D). We will consider the problem to describe all Ray processes on $[0, \infty[$ which behave otside zero like Brownian motion. This problem was first treated by Feller [11] using theory of differential equations. Feller's solution was that the infinitesimal generator of such a process is the differential operator

$$\mathcal{G} = \frac{1}{2}\frac{d^2}{dx^2}$$

with domain

$$D = C_2([0, \infty[) \cap \{u : p_1 u(0) - p_2 u'(0) + p_3 u''(0) = \int p_4(dx)[u(x) - u(0)]\}$$

where p_1, p_2 and p_3 are nonnegative real numbers and p_4 a σ-finite measure on $]0,\infty[$ such that

$$p_1 + p_2 + p_3 + \int p_4(dx)(1 - e^{-x}) = 1.$$

Itô and McKean constructed in [26] the sample paths of these processes, which they called Feller's Brownian motions, from the reflecting Brownian motion and its local time and (independent) exponential holding times and differential processes. Rogers derived in [45] Feller's result using resolvent identities. We will give in section (4.3) an interpretation of the parameters p_1, p_2, p_3 and the measure p_4 by means of excursion theory. In section (4.4) we will use these results to construct a model for a random motion on an n-pod E_n, that is a tree with one single vertex 0 and with n legs having infinite length. This model can be used to describe the movement of nutrients in the root system of a plant, also there is a possible application to the description of the spread of pollutants in a stream system and to the analysis and desgn of circulatory systems, see Frank and Durham [12]. We will construct all strong Markov processes which behave like standard Brownian motion restricted to a halfline, when restricted to a single leg.
In the last section we will construct a Markov process on $[0,\infty[$ which behaves as follows: starting at a point $x \in]0,\infty[$ it evolves like a given strong Markov process until reaching 0 where it waits a length of time having an exponential distribution with parameter α after which time it jumps independently to a new position in $]0,\infty[$ according to a given probability measure η and then proceeds as before.

4.1 Point processes attached to Brownian motion

In this section we will apply the results of section (3.2) to Brownian motion. In particular, we will derive an explicit formula for the characteristic measure ν of the Itô-Poisson point process of excursions from zero. The derivation is based on theorem (3.2.4) and on the following elementary calculations of the distribution of Brownian excursions in Chung [6].
Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$. Let $r > 0$ and let V_1 be the first excursion from zero with length greater than r. Following Chung [6], we introduce

the following notations:

$$\begin{aligned}\beta(r) &= \inf\{t > r : B_t = 0\},\\ \gamma(r) &= \sup\{t < r : B_t = 0\},\\ L(r) &= \beta(r) - \gamma(r).\end{aligned}$$

So $]\gamma(r), \beta(r)[$ is the excursion interval containing r. As $\mathbb{P}_0[B_r = 0] = 0$ and B has continuous realizations, $L(r) > 0, \mathbb{P}_0$-a.s.. Denote as in section (3.2), the first excursion interval of length greater than r by $]D_1, T_1[$. Let $n \geq 1$, $0 < t_1 < \ldots < t_n$ and $x_1, \ldots, x_n \in \mathbb{R}_+$:

$$\begin{aligned}&\mathbb{P}_0[V_1(t_i) \in dx_i, i = 1, \ldots, n]\\ &\quad= \mathbb{P}_0[V_1(t_i) \in dx_i, i = 1, \ldots, n, L(r) \leq r]\\ &\qquad+\mathbb{P}_0[V_1(t_i) \in dx_i, i = 1, \ldots, n, L(r) > r]\\ &\quad= \mathrm{I} + \mathrm{II}.\end{aligned}$$

It is clear that

$$[L(r) \leq r] = [D_1 \geq \beta(r)]$$

and that

$$D_1 = \beta(r) + D_1 \circ \theta_{\beta(r)} \text{ on } [D_1 \geq \beta(r)].$$

It follows that

$$\begin{aligned}\mathrm{I} &= \mathbb{P}_0[V_1(t_i) \circ \theta_{\beta(r)} \in dx_i, i = 1, \ldots, n, \beta(r) - \gamma(r) \leq r]\\ &= \mathbb{P}_0[\beta(r) - \gamma(r) \leq r]\mathbb{P}_0[V_1(t_i) \in dx_i, i = 1, \ldots, n]\end{aligned}$$

by an application of the strong Markov property on stopping time $\beta(r)$. It is also clear that

$$[L(r) > r] = [T_1 = \beta(r), D_1 = \gamma(r)].$$

It follows that

$$\mathrm{II} = \mathbb{P}_0[B_{\gamma(r)+t_i} \in dx_i, i = 1, \ldots, n, L(r) > \max(t_n, r)].$$

Hence

$$\begin{aligned}&\mathbb{P}_0[V_1(t_i) \in dx_i, i = 1, \ldots, n]\\ &\quad= \frac{\mathbb{P}_0[B_{\gamma(r)+t_i} \in dx_i, i = 1, \ldots, n, L(r) > \max(t_n, r)]}{\mathbb{P}_0[L(r) > r]}.\end{aligned}$$

A simple calculation using Chung [6], formula (2.20) results in

$$\mathbb{P}_0[L(r) \geq r] = \frac{2}{\pi}$$

and the same reasoning as in Chung [6], theorem 6 yields for $l > \max(t_n, r)$

$$\begin{aligned} & \mathbb{P}_0[\gamma(r) \in ds, B(\gamma(r)+t_i) \in dx_i, i=1,\dots,n, L(r) \in dl] \\ = \; & p(s;0,0)g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\cdots \\ & q(t_n-t_{n-1};x_{n-1},x_n)g(l-t_n;0,x_n)ds dx_1\cdots dx_n dl \end{aligned}$$

where

$$\begin{aligned} p(t;x,y) &= \exp[-\tfrac{(x-y)^2}{2t}], \\ g(t;0,y) &= \sqrt{\tfrac{1}{2\pi t}}\tfrac{|y|}{t}\exp[-\tfrac{y^2}{2t}], \\ q(t;x,y) &= p(t;x,y) - p(t;x,-y). \end{aligned}$$

The probabilistic interpretations of p, q and g are as follows

$$\begin{aligned} \mathbb{P}_x[B(t) \in dy] &= p(t;x,y)dy, \\ \mathbb{P}_0[\sigma_y \in dt] &= g(t;0,y)dt, \\ \mathbb{P}_x[B(t) \in dy, \sigma_0 > t] &= q(t;x,y)dy. \end{aligned}$$

for $t > 0$ and $xy > 0$. It follows that for $t_n > r$

$$\begin{aligned} & \mathbb{P}_0[B(\gamma(r)+t_i) \in dx_i, i=1,\dots,n, L(r) > r] \\ = \; & \sqrt{\frac{2r}{\pi}}g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\cdots q(t_n-t_{n-1};x_{n-1},x_n)dx_1\cdots dx_n. \end{aligned}$$

hence

$$\begin{aligned} & \mathbb{P}_0[V_1(t_i) \in dx_i, i=1,\dots,n] \\ = \; & \sqrt{\frac{2r}{\pi}}g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\cdots q(t_n-t_{n-1};x_{n-1},x_n)dx_1\cdots dx_n. \end{aligned}$$

This formula enables us to calculate some important quantities.

$$\begin{aligned} \mathbb{P}_0[T_1 < \infty] &= \int \mathbb{P}_0[V_1(r) \in dx] \\ &= 2\sqrt{2\pi r}\int_0^\infty g(r;0,x)dx = 1 \end{aligned}$$

and

$$\mathbb{P}_0[T_1 - D_1 > r+s] = \sqrt{\frac{r}{r+s}}.$$

Let $(r_k)_{k\geq 1}$ be a strictly decreasing sequence of positive real numbers, such that $\lim_{k\to\infty} r_k = 0$. Let $(U_k, V_k)_{k\geq 1}$ be defined as in the proof of theorem (3.2.4), i.e. V_k is the sequence of excursions from 0 of length greater than

r_k and U_k is the set of functions $u \in U = D_{[0,\infty[}(\mathbb{R})$ for which $\zeta_u > r_k$. Then

$$lim_{k\to\infty} p_k = lim_{k\to\infty} \mathbb{P}_0(V_{k1} \in U_1) = lim_{k\to\infty} \sqrt{\frac{r_k}{r_k + r_1}} = 0,$$

so there is an Itô-Poisson point process N on U whose $[\zeta > l]$-subsequence is the sequence of excursions of B of length greater than l. The characteristic measure ν of N is given by

$$\begin{aligned} &\nu[u(t_i) \in dx_i, i = 1, \ldots, n] \\ &= \sqrt{\frac{\pi r_1}{2}} g(t_1; 0, x_1) q(t_2 - t_1; x_1, x_2) \cdots q(t_n - t_{n-1}; x_{n-1}, x_n) dx_1 \cdots dx_n \end{aligned}$$

where $0 < t_1 < \ldots < t_n$ and $x_1, \ldots, x_n > 0$. Taking $r_1 = \frac{8}{\pi}$ we get the same normalization of ν as in Ikeda & Watanabe [22]. With our notations it is more convenient to take $r_1 = \frac{2}{\pi}$.

4.2 Brownian motion with drift

With the results of section (4.1) for standard Brownian motion, it is not very difficult to write down a formula for the characteristic measure of the Itô-Poisson point process of excursions from zero of Brownian motion with constant drift. The passage from Brownian densities to densities of Brownian motion with drift is done by adjunction of a Radon-Nikodym factor, see for instance Imhof & Kummerling [23]. Let $Y = (Y_t)_{t\geq 0}$ be Brownian motion with constant drift δ, i.e.

$$Y(t) = \delta t + B(t), t \geq 0.$$

A straightforward calculation yields

$$\begin{aligned} &\mathbb{P}_x[Y(t_i) \in dy_i, i = 1, \ldots, n] \\ &= \exp[\delta(y_n - x) - \frac{1}{2}\delta^2 t_n] \mathbb{P}_x[B(t_i) \in dy_i, i = 1, \ldots, n], \end{aligned}$$

for $0 \leq t_1 \leq \cdots \leq t_n$ and $y_1, \ldots, y_n \in \mathbb{R}$. It follows that the Radon-Nikodym derivatives $\bar{p}(t; x, y), \bar{g}(t; 0, y)$ and $\bar{q}(t; x, y)$ of the measures $\mathbb{P}_x[Y(t) \in dy], \mathbb{P}_0[\sigma \in dt]$ and $\mathbb{P}_x[Y(t) \in dy, \sigma_0 > 0]$ with respect to the Lebesgue measure are given by

$$\begin{aligned} \bar{p}(t; x, y) &= \exp[\delta(y - x) - \tfrac{1}{2}\delta^2 t] p(t; x, y), \\ \bar{g}(t; 0, y) &= \exp[\delta y - \tfrac{1}{2}\delta^2 t] g(t; 0, y), \\ \bar{q}(t; x, y) &= \exp[\delta(y - x) - \tfrac{1}{2}\delta^2 t] q(t; x, y), \end{aligned}$$

for $t > 0$ and $xy > 0$. Let $]\bar{\gamma}(r), \bar{\beta}(t)[$ be the excursion interval of Y containing r with length $\bar{L}(r) = \bar{\beta}(r) - \bar{\gamma}(r)$ and let $]\bar{D}_1, \bar{T}_1[$ resp. $\bar{V}_1$ be the first excursion interval resp. the first excursion of the process Y with length greater than r, then a simple calculation yields

$$\begin{aligned} &\mathbb{P}_0[Y(\bar{\gamma}(r) + t_i) \in dy_i, i = 1, \ldots, n, \bar{L}(r) > t_n] \\ &= \{\int_0^r \sqrt{\frac{1}{2\pi s}} \exp(-\frac{1}{2}\delta^2 s) ds\} \\ &\quad \times \bar{g}(t_1; 0, y_1) \times \prod_{i=1}^{n} \bar{q}(t_{i+1} - t_i; y_i, y_{i+1}) dy_1 \cdots dy_n \end{aligned}$$

for $0 < t_1 < \cdots < t_n$, $y_1, \ldots, y_n > 0$ and $t_n > r$. It follows that

$$\mathbb{P}_0[\bar{L}(r) > r] = \int_0^r \sqrt{\frac{1}{2\pi s}} \exp(-\frac{1}{2}\delta^2 s) ds \int_{-\infty}^{\infty} \bar{g}(r; 0, x) dx$$

and

$$\begin{aligned} &\mathbb{P}_0[\bar{V}_1(t_i), i = 1, \ldots, n] \\ &= \frac{1}{\int_{-\infty}^{\infty} \bar{g}(r; 0, x) dx} \bar{g}(t_1; 0, y_1) \times \prod_{i=1}^{n} \bar{q}(t_{i+1} - t_i; y_i, y_{i+1}) dy_1 \cdots dy_n. \end{aligned}$$

So

$$\mathbb{P}_0[\bar{T}_1 < \infty] = 1$$

and

$$\mathbb{P}_0[\bar{T}_1 - \bar{D}_1 > r + s] = \frac{\int_{-\infty}^{\infty} \bar{g}(r + s; 0, x) dx}{\int_{-\infty}^{\infty} \bar{g}(r; 0, x) dx}.$$

Since

$$\int_{-\infty}^{\infty} \bar{g}(r; 0, x) dx = \frac{1}{\sqrt{r}} \exp(-\frac{1}{2}\delta^2 r) \mathbb{E}_0(| B(1) | e^{\delta B(1)\sqrt{r}})$$

we get

$$\begin{aligned} &\lim_{r \downarrow 0} \frac{\int_{-\infty}^{\infty} \bar{g}(r + s; 0, x) dx}{\int_{-\infty}^{\infty} \bar{g}(r; 0, x) dx} \\ &= \lim_{r \downarrow 0} \sqrt{\frac{r}{r + s}} \exp(-\frac{1}{2}\delta^2 r) \frac{\mathbb{E}_0(| B(1) | \exp(\delta B(1)\sqrt{r + s})}{\mathbb{E}_0(| B(1) | \exp(\delta B(1)\sqrt{r})} = 0. \end{aligned}$$

So in the same way as for standard Brownian motion it follows that there exists an Itô-Poisson point proces $\bar{N}$ on U whose $[\zeta > l]$-subsequence is the sequence of excursions of Y of length greater than l. An appropiate choice for r_1 yields the following formula for the characteristic measure $\bar{\nu}$ of $\bar{N}$:

$$\bar{\nu}[u(t_i) \in dy_i, i = 1, \ldots, n] = \bar{g}(t_1; 0, y_1) \times \prod_{i=1}^{n} \bar{q}(t_{i+1} - t_i; y_i, y_{i+1}) dy_1 \cdots dy_n$$

for $0 < t_1 < \cdots < t_n$ and $y_1, \ldots, y_n > 0$.

4.3 Feller's Brownian motions

Let $B = (B_t)_{t \geq 0}$ be a standard one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$. The process $Y = (Y_t)_{t \geq 0}$ defined by $Y_t = \mid B_t \mid$ is called reflecting Brownian motion. Let $r > 0$ be given and let V^Y(resp. V^B) be the first excursion from zero of the process Y (resp. B) with length greater than r. Then, for $n \geq 1$, $0 < t_1 < \cdots < t_n$ and $x_1, \ldots, x_n \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\mathbb{P}_0[V^Y(t_i) \in dx_i, i = 1, \ldots, n] \\
&= \mathbb{P}_0[V^B(t_i) \in dx_i, i = 1, \ldots, n] + \mathbb{P}_0[-V^B(t_i) \in dx_i, i = 1, \ldots, n] \\
&= 2\sqrt{\frac{\pi r}{2}} g(t_1; 0, x_1) \times \prod_{i=1}^{n} q(t_{i+1} - t_i; x_i, x_{i+1}) dx_1 \cdots dx_n
\end{aligned}$$

since $g(t; 0, x) = g(t; 0, -x)$ and $q(t; x, y) = q(t; -x, -y)$, see section (4.1). It is now clear that the characteristic measure ν of the Itô-Poisson point process of excursions from zero of reflecting Brownian motion is given by

$$\nu[u(t_i) \in dx_i, i = 1, \ldots, n] = g(t_1; 0, x_1) \times \prod_{i=1}^{n} q(t_{i+1} - t_i; x_i, x_{i+1}) dx_1 \cdots dx_n,$$

$0 < t_1 < \cdots < t_n$ and $x_1, \ldots, x_n \in \mathbb{R}_+$. The characteristic measures of the Itô-Poisson point process of excursions from zero of Brownian and reflecting Brownian motion corresponds to the same semigroup $(K_t)_{t \geq 0}$ which is defined by

$$K_t(x, dy) = q(t; x, y) dy.$$

The entrance laws $(\eta_s)_{s>0}$ are given by

$$\eta_s(dy) = g(s; 0, y) dy \text{ for Brownian motion}$$

and by

$$\eta_s(dy) = 1_{]0,\infty[}(y) g(s; 0, y) dy \text{ for reflecting Brownian motion.}$$

Each strong Markov process Y which behaves like Brownian motion until the first hitting τ_0^Y of 0, i.e.

$$\mathbb{P}_x[Y(t_i) \in dy_i, i = 1, \dots, n, \tau_0^Y > t] = \mathbb{P}_x[B(t_i) \in dy_i, i = 1, \dots, n, \tau_0 > t],$$

where $0 \leq t_1 < \cdots < t_n \leq t$, has a characteristic measure (for the Itô-Poisson point process of excursions from zero), which correspond to the same semigroup (K_t). It is clear from the construction of these processes from Itô-Poisson point processes that the converse also holds. A problem extensively studied is to describe all the Ray processes on $[0, \infty[$, which behave like Brownian motion until the first hitting or approach of 0, see for example Feller [11], Itô-McKean [26] and Rogers [45]. It follows from an application of the strong Markov property that the resolvent $(V_\lambda)_{\lambda > 0}$ of such a process satisfies the following formula for $f \in C_0([0, \infty[)$ and $x \geq 0$:

$$V_\lambda f(x) = G_\lambda f(x) + e^{-x\sqrt{2\lambda}} V_\lambda f(0)$$

where

$$G_\lambda f(x) = \int_0^\infty e^{-\lambda t} K_t f(x) dt = \mathbb{E}_x \int_0^{\tau_0} e^{-\lambda t} f(B_t) dt.$$

Rogers gives in [45] the following characterization of $V_\lambda f(0)$:

$$\exists p_1, p_2, p_3 \geq 0, \exists p_4 \in \mathcal{M}_+(]0, \infty[)$$

such that

$$\int_{]0,\infty[} p_4(dx)(1 - e^{-x}) < \infty$$

and such that

$$V_\lambda f(0) = \frac{2p_2 \int_0^\infty e^{-x\sqrt{2\lambda}} f(x) dx + p_3 f(0) + \int_{]0,\infty[} p_4(dx) G_\lambda f(x)}{p_1 + p_2\sqrt{2\lambda} + \lambda p_3 + \int_{]0,\infty[} p_4(dx)(1 - e^{-x\sqrt{2\lambda}})}.$$

Actually we should have considered these processes on the one-point - compactification $[0, \infty]$ of $[0, \infty[$, the point ∞ playing the role of a cemetery, where the process is sent to when killed. We have left this out to avoid annoying technicalities. Rogers' derivation of this characterization is based on the resolvent equation. He remarks that the parameters p_1, p_2, p_3 and p_4 have natural interpretations in excursion theory. To see this, let for $s > 0$ and $n \geq 0$ the measure ϵ_{xs} on $[0, \infty[$ be defined by

$$\epsilon_{xs}(dy) = \begin{cases} q(s; x, y) dy & \text{if } x > 0 \\ g(s; 0, y) dy & \text{if } x = 0. \end{cases}$$

The families $(\epsilon_{xs})_{s>0}$, $x \geq 0$, of finite measures on $[0,\infty[$ are entrance laws for the semigroup (K_t). According to theorem (3.2.6), the semigroup (K_t) and the entrance law $(\epsilon_{xs})_{s>0}$ determine a unique σ-finite measure ν_x on the excursion space $(U_\infty, \mathcal{U}_\infty)$ satisfying property (i), (ii) and (iii) of theorem (3.2.4) such that

$$\epsilon_{xs}(dy) = \nu_x([\zeta_u > s, u(s) \in dy]).$$

It is clear that ν_0 is the characteristic measure of the Itô-Poisson point process of excursions from zero of standard Brownian motion. For $x > 0$ the measure ν_x is identical to the distribution α_x on $(U_\infty, \mathcal{U}_\infty)$ of standard Brownian started from x, which is absorbed in state 0. Let p be a nonnegative measure on $[0,\infty[$ such that

$$\forall x > 0 : p([x,\infty[).$$

Define the measure ν on $(U_\infty, \mathcal{U}_\infty)$ by

$$\nu = \int_{[0,\infty[} p(dx)\nu_x.$$

Then the family $(\eta_s)_{s>0}$ defined by

$$\eta_s(dy) = \nu([\zeta_u > s, u(s) \in dy]) = \int_{[0,\infty[} p(dx)\epsilon_{xs}(dy)$$

is an entrance law for the semigroup (K_t). For $\lambda > 0$ and bounded, measurable functions f on $[0,\infty[$ we have

$$\begin{aligned} \hat{\eta}_\lambda(f) &= \int_0^\infty ds e^{-\lambda s} \int \eta_s(dy) f(y) \\ &= p(\{0\}) \int_0^\infty f(y) e^{-y\sqrt{2\lambda}} dy + \int_{[0,\infty[} p(dx) G_\lambda f(x). \end{aligned}$$

It follows that

$$\begin{aligned} \int_U (1 - e^{-\zeta_u})\nu(du) &= \hat{\eta}_1(1) \\ &= \frac{1}{\sqrt{2}} p(\{0\}) + \int_{[0,\infty[} p(dx)(1 - e^{-x\sqrt{2}}). \end{aligned}$$

Let P be the Itô-Poisson point process on $[0,\infty[$ with characteristic measure ν. As in section (3.3) we assume that

$$\int_U (1 - e^{-\zeta_u})\nu(du) < \infty$$

to guarantee that the sum $A(\tau)$ of the lengths of the excursions up to time τ is finite. This condition is equivalent to the following condition on the measure p:

$$\int_{[0,\infty[} p(dx)(1 - e^{-x\sqrt{2}}) < \infty.$$

Let $\gamma \geq 0$ and define as in section (3.3)

$$B(\tau, \omega) = \sigma_0(\omega) + A(\tau, \omega) + \gamma\tau.$$

Finally, let $\delta \geq 0$ be the killing-rate in local time at state 0. If $(V_\lambda)_{\lambda>0}$ is the resolvent of the strong Markov process attached to P then by theorem (3.3.10) we have

$$V_\lambda f(x) = \begin{cases} G_\lambda f(x) + V_\lambda f(0) e^{-x\sqrt{2\lambda}} & \text{for } x \neq 0 \\ \frac{p(\{0\}) \int_0^\infty f(y) e^{-y\sqrt{2\lambda}} dy + \int_{[0,\infty[} p(dx) G_\lambda f(x) + \gamma f(0)}{\delta + \lambda\gamma + \frac{1}{2} p(\{0\})\sqrt{2\lambda} + \int_{[0,\infty[} p(dx)(1 - e^{-x\sqrt{2\lambda}})} & \text{for } x = 0. \end{cases}$$

It follows that p_1 is the killing-rate in local time at state 0. The parameter p_3 corresponds to γ, which is a measure for the stickiness at state 0. Further, it is easy to see that ν_x is concentrated on the set of excursions $\{u \in U_\infty : u(0) = x\}$, so $p_4(dx)$ is the rate in local time at 0 by which there appear excursions starting at x. The parameter $2p_2$ is the rate in local time at 0 by which the process exits 0 continuously.

4.4 Brownian motion on an n-pod

In this example we will construct Markov processes on an n-pod E_n. As a set E_n is defined by

$$E_n =]0, \infty[\times \{1, \ldots, n\} \cup \{0\}.$$

Let

$$d_n : E_n \times E_n \mapsto \mathrm{I\!R}$$

be defined by

$$d_n[(x,i),(y,j)] = \begin{cases} x + y & \text{for } i \neq j \\ \mid x - y \mid & \text{for } i = j, \end{cases}$$

$$d_n[0,(x,i)] = x,$$

$$d_n[0,0] = 0.$$

The function d_n is a metric on E_n and the topological space (E_n, d_n) is a locally compact, second countable Hausdorff space. The topological space E_n is called an n-pod. Denote by $\mathcal{E}_n$ the Borel σ-algebra on E_n and by $A_k, k = 1, \ldots, n$, the subset

$$A_k = \{e \in E_n : e = 0 \text{ or } \exists x > 0 : e = (x, k)\}$$

of E_n,which is called the k^{th} axis of E_n. It is clear that E_2 is homeomorphic to the real line $\mathbb{R}$ and that A_i is homeomorphic to $[0, \infty[$. We want to consider strong Markov processes Y on E_n, which behave like Brownian motion until the first hitting or approach σ_0 of state 0, i.e.

$$\mathbb{P}_{(x,i)}[Y_t \in (y, y + dy) \times \{j\}, \sigma_0 > t] = \begin{cases} q(t; x, y)\lambda_i(dy) & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$$

where λ_i is the image of the Lebesgue measure on $[0, \infty[$ under the map $\phi_i : [0, \infty[\mapsto A_i$ defined by

$$\phi_i(x) = \begin{cases} (x, i) & \text{for } x > 0 \\ 0 & \text{for } x > 0 \end{cases}$$

and where $q(t; x, y)$ is defined as in section (4.1). Define for $t > 0$ the kernel P_t^n on $(E_n, \mathcal{E}_n)$ by

$$P_t^n((x,i), F) = \int_0^\infty 1_F((y,i))q(t;x,y)\lambda_i(dy) + 1_F(0)\{1 - \int_0^\infty q(t;x,y)\lambda_i(dy)\}$$

$$P_t^n(0, F) = 1_F.$$

The family of kernels $(P_t^n)_{t\geq 0}$, where P_0^n is the identity kernel on $(E_n, \mathcal{E}_n)$, is a Markov semigroup of kernels which corresponds to a Feller-Dynkin semigroup on $C_0(E_n)$, see Williams [59]. Let $\alpha_e, e \in E_n$, be the measure on the function space $U^{(n)} = D_{E_n}([0, \infty[)$ whose finite-dimensional distributions are given by

$$\alpha_e[u(t_i \in de_i, i = 1, \ldots, m] = P_{t_1}^n(e, de_1) \prod_{i=1}^{m-1} P_{t_{i+1}-t_i}^n(e_i, de_{i+1})$$

where $m \geq 1$, $0 \leq t_1 \leq \cdots \leq t_m$ and $e_1, \ldots e_m \in E_n$. The measure α_e is concentrated on the set

$$\{u \in U^{(n)} : \forall t \geq \zeta_u : u(t) = 0\}$$

where ζ_u is the lifetime

$$\zeta_u = \inf\{t > 0 : u(t) = 0 \text{ or } u(t-) = 0\}.$$

Let $(K_t^n)_{t\geq 0}$ be the sub-Markov semigroup of kernels on $(E_n, \mathcal{E}_n)$ defined by

$$\begin{aligned} K_t^n(e,dy) &= \alpha_e[u(t) \in dy, \zeta_u > t] \\ &= \begin{cases} q(t;x,y)\lambda_i(dy) & \text{for } e=(x,i) \\ 0 & \text{for } e=0. \end{cases} \end{aligned}$$

Let Y be a strong Markov process on E_n, which behaves like Brownian motion until the first hitting or approach of 0, and let ν be the characteristic measure of the Itô-Poisson point process of excursions from zero of the process Y. The measure ν is determined by an entrance law $(\eta_s)_{s>0}$ for the semigroup (K_t^n) with $\forall s > 0 : \eta_s(\{0\}) = 0$. We have

$$\int \eta_s(de)K_t^n(e,dy) = \sum_{i=1}^n \int_0^\infty \eta_s^i(dx)q(t;x,y)\lambda_i(dy) = \sum_{i=1}^n 1_{A_i}(y)\eta_{s+t}(dy)$$

where $\eta_s^i = \phi_i^{-1}[1_{A_i}\eta_s]$ is the ϕ_i^{-1}-image of the restriction of η_s to the axis A_i. It follows that for $i = 1, \ldots, n$

$$(\int_0^\infty \eta_s^i(dy)q(t;x,y))dy = \eta_s^i(dy).$$

As in section (4.3) there exist measures $p^{(i)}, i = 1, \ldots, n$ on $[0, \infty[$ such that

$$\int_0^\infty p^{(i)}(dx)(1-e^{-x}) < \infty$$

and

$$\eta_s^i(dy) = \int_{[0,\infty[} p^{(i)}(dx)\epsilon_{xs}(du),$$

where ϵ_{xs} is defined as in section (4.3). It follows that the resolvent $(V_\lambda)_{\lambda>0}$ of the strong Markov process Y is given by

$$V_\lambda f(x,i) = \int_0^\infty G_\lambda(x,dy)f(y,i) + V_\lambda f(0)e^{-x\sqrt{2\lambda}}$$

with

$$\begin{aligned} &V_\lambda f(0) \\ &= \frac{\sum_{i=1}^n p^{(i)}(0)\int_0^\infty f(y,i)e^{-y\sqrt{2\lambda}}dy}{\delta + \lambda\gamma + \frac{1}{2}\sum_{i=1}^n p^{(i)}(0) + \sum_{i=1}^n \int_{]0,\infty[} p^{(i)}(dx)(1-e^{-x\sqrt{2\lambda}})} \\ &\quad + \frac{\sum_{i=1}^n \int_{]0,\infty[} p^{(i)}(dx)\int_0^\infty G_\lambda(x,dy)f(y,i) + \gamma f(0)}{\delta + \lambda\gamma + \frac{1}{2}\sum_{i=1}^n p^{(i)}(0) + \sum_{i=1}^n \int_{]0,\infty[} p^{(i)}(dx)(1-e^{-x\sqrt{2\lambda}})} \end{aligned}$$

where γ and δ are nonnegative constants and $f \in b\mathcal{E}_n$. As a special case take $n = 2, \delta = \gamma = p^{(1)}(]0,\infty[) = p^{(2)}(]0,\infty[) = 0$ and $\alpha + \beta > 0$, where $\alpha = p^{(1)}(0)$ and $\beta = p^{(2)}(0)$. Then we get

$$V_\lambda f(0) = \sqrt{\frac{2}{\lambda}}\frac{1}{\alpha+\beta}\{\alpha \int_0^\infty f(y,1)e^{-y\sqrt{2\lambda}}dy + \beta \int_0^\infty f(y,2)e^{-y\sqrt{2\lambda}}dy\}.$$

Mapping E_2 on $\mathbb{R}$ by the map ψ:

$$\begin{aligned} \psi(y,1) &= y \\ \psi(y,2) &= -y \\ \psi(0) &= 0 \end{aligned}$$

and writing f for the composed map $f \circ \psi$, we get after some straightforward calculation that

$$V_\lambda f(0) = \sqrt{\frac{2}{\lambda}}\frac{1}{\alpha+\beta}\{\alpha \int_0^\infty f(y)e^{-y\sqrt{2\lambda}}dy + \beta \int_0^\infty f(-y)e^{-y\sqrt{2\lambda}}dy\}$$

and

$$V_\lambda f(x) = \int_0^\infty \frac{1}{\sqrt{2\lambda}}[e^{-\sqrt{2\lambda}|x-y|} + \mathrm{sign}(y)\frac{\alpha-\beta}{\alpha+\beta}e^{-\sqrt{2\lambda}(|x|+|y|)}]f(y)dy$$

for all $x \in \mathbb{R}\backslash\{0\}$, where

$$\mathrm{sign}(y) = \begin{cases} -1 & \text{for } y < 0 \\ 0 & \text{for } y = 0 \\ 1 & \text{for } y > 0 \end{cases}.$$

This is the resolvent of skew Brownian motion, see Itô and McKean [27]. The numbers $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$ may be interpreted as the probabilities for an excursion on the right- resp. the left hand side of 0, see Harrison and Shepp [21]. In [12] Frank and Durham give an intuitive description of symmetric Brownian motion on a 3-pod, which corresponds to the case

$$\delta = \gamma = 0 \text{ and } p^{(1)} = p^{(2)} = p^{(3)} = \frac{1}{3}\delta_0.$$

We end this section by remarking that a similar construction is possible for processes which behave outside zero like Brownian motion with constant drift.

4.5 Blumenthal's construction

In [2] Blumenthal constructs for a given characteristic measure and entrance law the extension of the original process whose entrance law is the given one, claiming that this construction is the one that Itô was referring to in [25]. Let E be a compact metric space and let $a \in E$ be a fixed point. Let $X = (X_t)_{t \geq 0}$ be a Ray process with state space E. Blumenthal considers the case $E = [0, \infty[$, $a = 0$ and X is a standard Markov process. By a standard construction for Markov processes, X can be considered as a Ray process on the one-point compactification $[0, \infty]$, see Getoor [15]. Denote as in section (3.2) by α_x the distribution on $U = D_E([0, \infty[)$ of the process X starting at x and absorbed in a after the first hitting or approach σ_a of the point a and by $(K_t)_{t \geq 0}$ the sub-Markov semigroup of kernels on $(E, \mathcal{E})$ defined by

$$K_t(x, dy) = \alpha_x[u(t) \in dy, \zeta_u > t].$$

Blumenthal's construction is based on an approximation with Markov processes of the following type. Starting at a point $x \in E \backslash \{a\}$ the process evolves according to the transition probabilities of the process X until reaching the state a where it waits a length of time having an exponential distribution with mean $\alpha > 0$ after which time it jumps independently to a new position in $E \backslash \{a\}$ according to a given probability distribution η and so on. The measure η is called the jumping in measure and α is called the holding parameter. For the existence of such a Markov process Blumenthal refers to Meyer [41]. A simple calculation using the strong Markov property yields for the resolvent $(U_\lambda)_{\lambda > 0}$ of this process the formula

$$U_\lambda f(x) = G_\lambda f(x) + \mathbb{E}_a(e^{-\lambda \sigma_a}) \frac{\frac{1}{\alpha} f(a) + \int \eta(dy) G_\lambda f(y)}{\frac{1}{\alpha}\lambda + \int \eta(dy) \mathbb{E}_y(1 - e^{-\lambda \sigma_a})}$$

where

$$G_\lambda f(x) = \int_0^\infty e^{-\lambda t} K_t f(x) dt \text{ and } f \in b\mathcal{E}.$$

It is not difficult to construct this process with the methods of section (3.3). Define the family of finite measures $(\eta_s)_{s>0}$ on $(E, \mathcal{E})$ by

$$\eta_s(dy) = \int \eta(dx) K_s(x, dy).$$

The family $(\eta_s)_{s>0}$ is an entrance law for the semigroup (K_t) satisfying the property $\forall s > 0 : \eta(\{a\}) = 0$. Let ν be the σ-finite measure on $(U, \mathcal{U})$ corresponding to the entrance law (η_s) and the semigroup (K_t), see theorem (3.2.6), and let P be the Itô-Poisson point process on U with characteristic measure ν. Let Y be the Markov process attached to P as in section (3.3). Then by theorem (3.3.10) the resolvent $(V_\lambda)_{\lambda>0}$ of Y is given by

$$V_\lambda f(x) = G_\lambda f(x) + \mathbb{E}_a(e^{-\lambda\sigma_a})\frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\lambda\gamma + \lambda\hat{\eta}_\lambda(1)}, f \in b\mathcal{E}.$$

Since

$$\hat{\eta}_\lambda(f) = \int_0^\infty dt e^{-\lambda t}\int \eta(dx)K_t f(x) = \int \eta(dx)G_\lambda f(x)$$

and, for $x \neq a$,

$$\begin{aligned}
\lambda G_\lambda 1(x) &= \int_0^\infty \lambda e^{-\lambda t}K_t 1(x)dt \\
&= \int_0^\infty \lambda e^{-\lambda t}\mathbb{E}_x(1_{[\sigma_a>t]})dt \\
&= \mathbb{E}_x\int_0^{\sigma_a} \lambda e^{-\lambda t}dt \\
&= \mathbb{E}_x(1-e^{-\sigma_a}),
\end{aligned}$$

the process Y with $\gamma = \frac{1}{\alpha}$ is the above described Markov process. The strong Markov property for Y follows from the assumptions in Blumenthal [2] about the resolvent $(G_\lambda)_{\lambda>0}$.

Appendix A

A.1 The existence of an $\mathcal{S}$-finite base for the topology

Let X be a polish space and let $\mathcal{S}$ be a collection of open subsets of X. Denote by $\mathcal{S}'$ the family of all Borel subsets of X contained in some element of $\mathcal{S}$.

Proposition A.1.1 *If $\mathcal{S}$ covers X, then there exists a countable base for the topology consisting entirely of open subsets with closure in $\mathcal{S}'$.*

Proof. Let D be a dense subset of X and let d be a metric on X compatible with the topology of X. For each $x \in D$ there is an $A \in \mathcal{S}$ containing x. Let

$$\delta = d(x, A^c) = \inf\{d(x,y) : y \neq A\}.$$

Then $\delta > 0$, since A^c is closed and $x \notin A^c$. Let $r \in (0, \delta)$. The closure $\bar{B}_x(r)$ of the ball with center x and radius r is contained in A:

$$\begin{aligned} y \in \bar{B}_x(r) &\implies d(x,y) \leq r \\ &\implies d(y, A^c) \geq d(x, A^c) - d(x,y) \geq \delta - r > 0 \\ &\implies y \notin \overline{A^c} \\ &\implies y \in A. \end{aligned}$$

So $\bar{B}_x(r) \in \mathcal{S}'$. Define for $x \in D$

$$I_x = \{q \in \mathbb{Q} : \bar{B}_x(q) \in \mathcal{S}'\}.$$

Claim: $\mathcal{U} = \{B_x(q) : x \in D, q \in I_x\}$ is a countable base for the topology of X. To see this, let $O \subset X$ be open. It is clear that

$$O \supset \bigcup\{B_x(q) : B_x(q) \in \mathcal{U}, B_x(q) \subset O\}.$$

Let $y \in O$. Choose $A \in \mathcal{S}$ such that $y \in A$. Then

$$\epsilon = \min(d(y, O^c), d(y, A^c)) > 0.$$

For $x \in D \cap B_y(\frac{1}{4}\epsilon)$ and $q \in \mathbb{Q} \cap]\frac{1}{4}\epsilon, \frac{3}{4}\epsilon[$ we have

$$d(x, A^c) \geq d(y, A^c) - d(x, y) \geq \epsilon - \frac{1}{4}\epsilon > q.$$

So $q \in I_x$ and $y \in B_x(q)$. Hence

$$O \subset \bigcup \{B_x(q) : B_x(q) \in \mathcal{U}, B_x(q) \subset O\}.$$

It follows that each open set can be covered by elements of $\mathcal{U}$.

A.2 The Skorohod topology

Let (X, ρ) be separable metric space. Let for $t_0, T \in \mathbb{R}, t_0 < T$, the space of functions $u : [t_0, T] \mapsto X$ which are right continuous on $[t_0, T[$ and have left limits on $]t_0, T]$ be denoted by $D_X([t_0, T])$. Let $\Lambda([t_0, T])$ be the class of strictly increasing, continuous maps $\lambda : [t_0, T] \mapsto [t_0, T]$, such that $\lambda(t_0) = t_0$ and $\lambda(T) = T$. For $u, v \in D_X([t_0, T])$, define $d_1(u, v)$ to be the infimum of those positive real numbers ϵ for which there exists a map $\lambda \in \Lambda([t_0, T])$ such that

$$\sup\{\mid \lambda(t) - t \mid : t \in [t_0, T]\} \leq \epsilon$$

and

$$\sup\{\rho(u(t)), v \circ \lambda(t)) : t \in [t_0, T]\} \leq \epsilon.$$

The fuction d_1 is a metric on $D_X([t_0, T])$. The topology on $D_X([t_0, T])$ induced by d_1 is called Skorohod's J_1 topology. Equipped with the J_1 topology, $D_X([t_0, T])$ is a polish space, see Billingsley [1]. Let U be the space of càdlàg functions of $[0, \infty[$ in X. There are several papers about the extension of the J_1 topology to U, see among others Lindvall [37] and Whitt [57]. We will summarize the theory of Whitt [57]. Let for $0 < b < c$, $r_{bc} : U \mapsto D_X([b, c])$ be the restriction to $[b, c]$ defined by $(r_{bc}x)(t) = x(t), t \in [b, c]$. For any $x, y \in U$, let d be defined by

$$d(x, y) = \int_0^\infty dte^{-t} \min[d_{0t}(r_{0t}x, r_{0t}y), 1]$$

where d_{0t} is the metric on $D_X([0, t])$ as defined above. The function d is a metric on U. The topology induced by d is called the Skorohod topology on U. Note that a sequence $(x_n) \subset U$ converges to $x \in U$ iff $d_{0t}(r_{0t}x_n, d_{0t}x) \to 0$ for almost all t. The basic properties of the Skorohod topology are:

(i) U equipped with the Skorohod topology is a polish space,

(ii) the Borel σ-algebra on U coincides with the σ-algebra generated by the coordinate evaluations,

(iii) if $(P_n)_{n\geq 1}, P$ are probability measures on U, then

$$\begin{aligned} P_n \to P \Longleftrightarrow & \\ & \exists \text{ a sequence } [s_k, t_k]_{k\geq 1} : \bigcup_{k=1}^{\infty} [s_k, t_k] = [0, \infty[\\ & \text{and } \forall k : r_{s_k t_k}(P_n) \Longrightarrow r_{s_k t_k}(P) \text{ on } D_X([s_k, t_k]). \end{aligned}$$

Fix $a \in X$ and define the map $\zeta : u \in U \mapsto \zeta_u \in [0, \infty]$ by

$$\zeta_u = \inf\{t > 0 : u(t) = a \text{ or } u(t-) = a\}.$$

Lemma A.2.1 *The map ζ is lower semi-continuous.*

Proof. It is sufficient to show that the sets $\{u \in U : \zeta_u \leq k\}, k > 0$, are closed sets. So let $k > 0$ be fixed and let (u_n) be a sequence in $\{u \in U : \zeta_u \leq k\}$ converging to u. Let $\epsilon > 0$. If the restrictions of the u_n to $[0, k]$ converge in $D_X([0, k])$ to the restriction of u to $[0, k]$, there exists for every n sufficiently large a function $\lambda \in \Lambda([0, k])$ such that

$$\sup\{\mid \lambda(t) - t \mid : t \in [0, k]\} \leq \epsilon$$

and

$$\sup\{\rho(u_n(t)), u \circ \lambda(t)) : t \in [0, k]\} \leq \epsilon.$$

So

$$\rho(u \circ \lambda(t), a) \leq \rho(u_n(t)), u \circ \lambda(t)) + \rho(u_n(t)), a) \leq 2\epsilon$$

for some $t \in [0, k]$. It follows that

$$\forall \epsilon > 0, \exists s \in [0, k] : \rho(u(s), a) < \epsilon$$

and this implies that $\zeta_u \leq k$. If the restrictions of u_n to $[0, k]$ do not converge in $D_X([0, k])$ to the restriction of u, then there exists a sequence (k_m) decreasing to k, such that

$$\forall m \geq 1 : \lim_{n\to\infty} r_{0k-m} u_n = r_{0k-m} u.$$

As above we may conclude that $\forall m \geq 1 : \zeta_u \leq k_m$ and it follows that $\zeta_u \leq k$.

A.3 Some results on real functions

A.3.1 Result 1

Lemma A.3.1 *(Greenwood & Pitman). For each $n > 0$ let $f_n(t)$ be a positive, nondecreasing function of $t \in [0,\infty[$ and let $S \subset [0,\infty[$. If*

$$\forall s \in S : \lim_{n\to\infty} f_n(s) = f(s) \textit{ exists,}$$

and

$$\{f(s) : s \in S\} \textit{ is dense in } [0,\infty[,$$

then there is a continuous, nondecreasing function f defined on $[0,a[$, where $a = \sup S$, such that uniformly on bounded sub-intervals of $[0,a[$

$$\lim_{n\to\infty} f_n(t) = f(t).$$

Proof. For every $n > 0$ and $s \in S$ we have $0 \le f_n(0) \le f_n(s)$. So

$$0 \le \underline{\lim} f_n(0) \le \overline{\lim} f_n(0) \le \inf\{f(s) : s \in S\} = 0$$

and

$$\lim f_n(0) = 0.$$

Let $x \in]0,a[$. If $S \cap [0,x] = \emptyset$, then $\lim f_n(x) = 0$. In the remaining case we have

$$\sup\{f(s) : s \in S \cap [0,x]\} \le \underline{\lim} f_n(x) \le \overline{\lim} f_n(x) \le \inf\{f(s) : s \in S \cap [x,\infty[\}.$$

Since $\{f(s) : s \in S\}$ is dense in $[0,\infty[$, $\lim f_n(x)$ exists. Define the function $f : [0,a[\mapsto [0,\infty[$ as the pointwise limit of the sequence functions (f_n).It is clear that f is a nondecreasing, continuous function on $[0,a[$. If the convergence of the sequence (f_n) is not uniform on bounded sub-intervals of $[0,a[$, then there exists an $M < a$ and an $\epsilon > 0$ such that

$$\forall n \in \mathbb{N}, \exists t_n \in [0,M]; | f_n(t_n) - f(t_n) | > \epsilon.$$

Let $(t_{n'})$ be a convergent subsequence of (t_n), $t_\infty = \lim t_{n'}$. Choose $x_1, x_2 \in [0,a[$ such that

$$x_1 < t_\infty < x_2 \text{ and } f(x_2) - f(x_1) < \frac{1}{4}\epsilon.$$

If $t_\infty = 0$, take $x_1 = 0$. Then for n' sufficiently large

$$f_{n'}(x_1) - f(t_{n'}) < f_{n'}(t_{n'}) - f(t_{n'}) < f_{n'}(x_2) - f(t_{n'}),$$

and it follows by letting $n' \to \infty$ that

$$-\frac{1}{4}\epsilon \le \overline{\lim}[f_{n'}(t_{n'}) - f(t_{n'})] \le \frac{1}{4}\epsilon$$

which is a contradiction. So the convergence of the sequence (f_n) is uniform on bounded sub-intervals of $[0, a[$.

A.3.2 Result 2

Let A be a function on $[0, \infty[$ which is nonnegative, nondecreasing and right continuous. Denote by λ the Lebesgue measure on $[0, \infty[$, and let ϕ be the distribution function of the measure $\nu = A(\lambda)$ on $[0, \infty[$. Then for $t \ge 0$

$$\begin{aligned} \phi(t) &= \int 1_{[0,t]} dA(\lambda) \\ &= \lambda(\{x : A(x) \le t\}) \\ &= \sup\{x : A(x) \le t\}). \end{aligned}$$

The function ϕ is nonnegative, possibly infinite valued, nondecreasing and right continuous.

$$\begin{aligned} \phi(t) \le y &\implies y \text{ is an upperbound of } \{x : A(x) \le t\} \\ &\implies \forall \epsilon > 0 : A(y + \epsilon) > t \\ &\implies A(y) = A(y+) \ge t \\ &\implies A(y) \text{ is an upperbound of } \{t : \phi(t) \le y\}. \end{aligned}$$

On the other hand

$$\begin{aligned} u \text{ is an upperbound of } \{t : \phi(t) \le y\} &\implies \forall \epsilon > 0 : \phi(u + \epsilon) > y \\ &\implies \forall \epsilon > 0 : A(y) \le u + \epsilon. \end{aligned}$$

So

$$A(y) = \sup\{t : \phi(t) \le y\}$$

is the distribution function of the measure $\phi(\lambda)$. We call ϕ the right continuous inverse of A. We have shown that A is the right continuous inverse of ϕ. Let $F \in L^1(\nu)$, then

$$\int F \circ A(x)\lambda(dx) = \int F dA(\lambda) = \int F(y) d\phi(y).$$

A.3.3 Result 3

Let f be a nondecreasing, right continuous function on $[0,\infty[$, such that $f(0)=0$ and $\lim_{n\to\infty} f(x) = +\infty$. Define

$$J = \{t \in]0,\infty[: f(t-) < f(t)\}$$

and

$$R = \text{range}(f) = \{s \in [0,\infty[: \exists t \geq 0 : s = f(t)\}.$$

Lemma A.3.2 *If f is strictly increasing, the*

$$[0,\infty[= R + \sum_{t\in J} [f(t-), f(t)[$$

where the union is a union of disjoint intervals.

Proof. Let $t \in R$, say $t = f(r)$. Assume that there is an $s \in J$ such that $t \in [f(s-), f(s)[$. Then

$$f(s-) \leq f(r) < f(s).$$

It follows that $r < s$, so $f(r) = f(s-)$. This can only be the case when f is constant on $[r,s[$. This is impossible since f is strictly increasing. So

$$R \cap \sum_{t\in J} [f(t-), f(t)[= \emptyset.$$

Let $t \in [0,\infty[\backslash R$. Then $\forall s \in [0,\infty[: f(s) < t$ or $f(s) > t$. Define

$$u = \inf\{s : f(s) > t\} = \sup\{s : f(s) < t\}.$$

Then $f(u) \geq t$, so $f(u) > t$ and $f(u-) \leq t$. It follows that

$$t \in [f(u-), f(u)[\subset \bigcup_{t\in J} [f(t-), f(t)[,$$

which completes the proof of the lemma.

References

[1] Billingsley, P.: Convergence of Probability Measures. Wiley, New York 1968.

[2] Blumenthal, R.M.: On Construction of Markov Processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 63 (1983) 433-444.

[3] Blumenthal, R.M. and Getoor, R.K.: Markov Processes and Potential Theory. Academic Press, New York 1968.

[4] Bochner, S.: Harmonic Analysis and the Theory of Probability. Univ. of California Press, Berkeley, California, 1955.

[5] Bourbaki, N.: Eléments de mathématique. Intégration, chapitre IX, 2nd edition, Act. Sci. et Industr. 1343, Hermann, Paris 1969.

[6] Chung, K.L.: Excursions in Brownian Motion. Ark. Math. 14 (1976) 155-177.

[7] Dellacherie, C. and Meyer, P.A.: Probabilities and Potential. North Holland, Amsterdam 1978.

[8] Dynkin, E.B.: Markov Processes I. Springer Verlag, Heidelberg 1965.

[9] Dynkin, E.B.: On extensions of a Markov Process. Theor. Probability 13 (1968) 672-676.

[10] Dynkin, E.B.: Wanderings of a Markov Process. Theor. Probability 16 (1971) 401-428.

[11] Feller, W.: Generalised second order differential operators and their lateral condititons. Illinois J. of Math. 1 (1957) 459-504.

[12] Frank, H.F. and Durham, S.: Random motion on binary trees. J. Appl. Prob. 21 (1984) 58-69.

[13] Fristedt, B. and Taylor, S.J.: Constructions of Local Time for a Markov Process. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 62 (1983) 73-112.

[14] Geman, D. and Horowitz, J.: Occupation Densitites. Ann. Probability 8 (1980) 1-67.

[15] Getoor, R.K.: Markov Processes: Ray Processes and Right Processes. Springer Verlag, Heidelberg 1975.

[16] Getoor, R.K.: Excursions of a Markov process. Ann. of Probability 7 (1979) 244-266.

[17] Greenwood, P. and Pitman, J.: Construction of Local Time and Poisson point processes from nested arrays. J. London Math. Soc. (2) 22 (1980) 182-192.

[18] Halmos, P.R.: Measure Theory. Springer Verlag, Heidelberg 1974.

[19] Harris, T.E.: Counting Measures, Monotone Random Set Functions. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 10 (1968) 102-119.

[20] Harris, T.E.: Random Measures and Motions of Point Processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 18 (1971) 85-115.

[21] Harrison, J.M. and Shepp, L.A.: On skew Brownian Motion. Ann. Prob. 9 (1981) 309-313.

[22 Ikeda, N. and Watanabe, S.: Stochastic differential equations and diffusion processes. North Holland/Kodansha, Amsterdam 1981.

[23] Imhof, J.P. and Kummerling, P.: Operational derivation of some Brownian results. Intern. Statist. Review 54 (1986) 327-341.

[24] Itô, K.: Stochastic processes. Aarhus Universitet Lecture Notes Series No. 16 1969.

[25] Itô, K.: Poisson point processes attached to Markov processes. Proceedings of the sixth Berkeley Symposium on Mathematical Statistics and Probability. 19 255-329. University of California Press, 1970.

[26] Itô, K. and McKean, H.P.: Brownian motions on a half line. Illinois J. Math. 7 (1963) 181-231.

[27] Itô, K. and McKean, H.P.: Diffusion Processes and their Sample Paths. Springer Verlag, Heidelberg 1974.

[28] Jagers, P.: On Palm Probabilities. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 26 (1973) 17-32.

[29] Jagers, P.: Random measures and point processes. In: Advances in Probability, Vol. 3 (P. Ney and S. Port ed.) 179-239, M. Dekker, New York 1974.

[30] Kaspi, H.: Excursions of Markov Processes: An Approach Via Markov Additive Processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 64 (1983) 251-268.

[31] Kaspi, H.: On In variant Measures and Dual Excursions of Markov Processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 66 (1984) 185-204.

[32] Kato, T.: Perturbation Theory for Linear OPerators. Springer Verlag, Heidelberg 1966.

[33] Kelley, J.L.: General Topology. Van Nostrand, New York 1970.

[34] Krickeberg, K.: Fundamentos del analisis estadistico de procesos puntuales. Mimeographed notes of the university of Santiago, Chili, 1973.

[35] Lévy, P.: Sur certain processus stochastiques homogènes. Comp. Math. 7 (1939) 283-339.

[36] Lévy, P.: Processus stochastiques et mouvement Brownien. Paris 1948.

[37] Lindvall, T.: Weak Convergence of Probability Measures and Random Functions in the Function Space $D[0,\infty)$. J. Appl. Prob. 10 (1973) 109-121.

[38] Maisonneuve, B.: Systèmes régénératifs. Astérisque no 15. Soc. Math. de France, 1974.

[39] Maisonneuve, B.: Exit systems. Ann Probability 3 (1975) 399-411.

[40] Matthes, K., Kerstan, J. and Mecke, J.: Infinitely Divisible Point Processes. Wiley, New York 1978.

[41] Meyer, P.A.: Processus de Poisson ponctuels, d'après K. Itô. Sem. de Prob. V. Lecture Notes in Math 191, Springer Verlag, Heidelberg 1971.

[42] Meyer, P.A.: Renaissance, recollements, mélanges, ralentissement de processus de Markov. Ann. Inst. Fourier 25 (1975) 465-497.

[43] Neveu, J.: Discrete-parameter Martingales. North-Holland, Amsterdam 1975.

[44] Neveu, J.: Processus ponctuels. Ecole d'Eté de Prob. de Saint-Flour VI-1976. Lecture Notes in Math. 598, 250-445. Springer Verlag, Heidelberg.

[45] Rogers, L.C.G.: Itô Excursion Theory Via Resolvents. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 63 (1983) 237-255.

[46] Rogers, L.C.G.: Addendum to "Itô Excursion Theory Via Resolvents." Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 67 (1984) 473-476.

[47] Salisbury, T.S.: Construction of Right Processes from Excursions. Prob. Th. Rel. Fields 73 (1986) 351-367.

[48] Salisbury, T.S.: On the Itô Excursion Process. Prob. Th. Rel. Fields 73 (1986) 319-350.

[49] Schwartz, L.: Radon Measures on arbitrary Topological Spaces and Cylindrical Measures. Oxford University Press, Oxford 1973.

[50] Topsoe, F.: Topology and Measure. Springer Verlag, Heidelberg, 1970.

[51] Trooter, H.F.: Approximation of Semi-groups of Operators. Pac. J. Math. 8 (1958) 887-919.

[52] Walsh, J.: A diffusion with a discontinuous local time. Astérisque no 52-53, 37-47, Soc. Math. de France, 1978.

[53] Walsh, J.: Excursions and local time. Astérisque no 52-53, 159-192, Soc. Math. de France, 1978.

[54] Watanabe, S.: Poisson point processes of Brownian excursions and its applications to diffusion processes. Proc. Symp. in Pure Math. AMS, 31 (1977) 153-164.

[55] Watanabe, S.: Point processes and Martingales. Stochastic Analysis (A. Friedman and M. Pinsky eds.), 315-326, Academic Press, 1978.

[56] Watanabe, S.: A limit theorem for sums of i.i.d. random variables with slowly varying tail probability. Multivariate Analysis-V (P.R. Krisniah ed.), 249-261. North Holland, Amsterdam 1980.

[57] Whitt, W.: Some useful Functions for Functional Limit Theorems. Math. of Op. Res. (5) 1980 No. 1.

[58] Williams, D.: Path decomposition and continuity of local time for one-dimensional diffusions, I. Proc. London Math. Soc. (3) 28 (1974) 739-768.

[59] Williams, D.: Diffusions, Markov Processes and Martingales, Volume 1: Foundations. Wiley, New York 1979.

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13 H.A. Lauwerier. *Asymptotic expansions.* 1966, out of print; replaced by MCT 54.

14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

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21 E. Wattel. *The compactness operator in set theory and topology.* 1968.

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23 T.J. Dekker, W. Hoffmann. *ALGOL 60 procedures in numerical algebra, part 2.* 1968.

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26 European Meeting 1968. *Selected statistical papers, part I.* 1968.

27 European Meeting 1968. *Selected statistical papers, part II.* 1968.

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38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

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47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

52 P.C. Baayen (ed.). *Topological structures.* 1974.

53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

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