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Dynamic feedback in nonlinear synthesis problems

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Preface

Since 1985, numerous authors have worked on the solution of nonlinear decoupling problems via dynamic state feedback. It is the purpose of this monograph to give an overview of the state of the art. Furthermore the results are applied to Hamiltonian control systems.

This monograph grew out of a research project performed at the Department of Applied Mathematics of the University of Twente. I gratefully acknowledge the support during this project of the two founding members of the Dutch Nonlinear Systems Group, Arjan van der Schaft and Henk Nijmeijer, who introduced me in the field of nonlinear control theory and who were always willing to lend me an ear. Furthermore, I thank Leo van der Wegen for many helpful discussions and for his contribution to the solution of the problems treated in part of Chapter 3 and in Chapter 4.

Eindhoven, December 1993,

Henri Huijberts.

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Chapter 1

Introduction

"In SCIENCE -in fact, in most things- it is usually best to begin at the beginning. *In* some *things, of course, it's better to begin at the* other *end. For instance, if you wanted to paint a dog green, it* might *be best to begin with the* tail, *as it doesn't bite at* that *end."*

Lewis Carroll, Sylvie and Bruno concluded.

The purpose of this monograph is to explain the role that is played by dynamic (and static) state feedback in the solution of synthesis problems for nonlinear control systems. In particular we pay attention to the disturbance decoupling problem, the model matching problem and the input-output decoupling problem.

In Section 1.1 a general introduction to some of the technical terms mentioned above is given. A basic tool that is used in this monograph, is *differential geometry.* In Section 1.2 we give an overview of concepts from differential geometry that are used in this monograph. In Section 1.3 a survey of the rest of the monograph is given. Here we pay special attention to the new contributions of this monograph.

1.1 General introduction

Nonlinear control system

A control system Σ is a part of the "real world" that is influenced by its environment via so called *inputs* and on its turn influences its environment via so called *outputs.* The inputs are divided in two groups. First, we have the inputs that we can control as we wish. These inputs are called *controls* and are denoted by u. Second, we have the inputs that we cannot control as we wish. These inputs are called the *disturbances* and are denoted by q. Also the outputs are divided in two groups. First, we have the outputs in whose behavior we are specifically interested. These outputs are called the *outputs-to-be-controlled* and are denoted by y. Second, we have the *measurements* that we can perform on the system. These outputs are denoted by η. Pictorally, this is given by Figure 1.1. In this monograph we are especially interested in *nonlinear control systems* (or briefly: nonlinear systems). These are control systems for which the interdependence of q, u, y, η is described by equations of

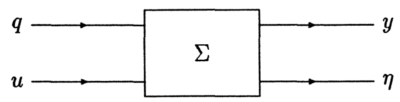

Figure 1.1: Control system

the form

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1,\cdots,x_n) + \sum_{i=1}^{m} g_{i1}(x_1,\cdots,x_n)u_i + \sum_{i=1}^{r} p_{i1}(x_1,\cdots,x_n)q_i \\
&\vdots \\
&\vdots \\
\dot{x}_n &= f_n(x_1,\cdots,x_n) + \sum_{i=1}^{m} g_{in}(x_1,\cdots,x_n)u_i + \sum_{i=1}^{r} p_{in}(x_1,\cdots,x_n)q_i \\
y_1 &= h_1(x_1,\cdots,x_n) \\
&\vdots \\
y_p &= h_p(x_1,\cdots,x_n) \\
\eta_1 &= k_1(x_1,\cdots,x_n) \\
&\vdots \\
\eta_s &= k_s(x_1,\cdots,x_n)
\end{aligned}
\tag{1.1}
$$

where f_i $(i = 1,\cdots,n)$, g_{ij} $(i = 1,\cdots,m; j = 1,\cdots,n)$, p_{ij} $(i = 1,\cdots,r; j = 1,\cdots,n)$, h_i $(i = 1,\cdots,p)$ and k_i $(i = 1,\cdots,s)$ belong to some function class and x_i $(i = 1,\cdots,n)$, u_i $(i = 1,\cdots,m)$ are time-dependent functions. The variables $x_1,\cdots,x_n$ are called the ***state variables*** of the system and $\dot{x}_i$ denotes the time-derivative of x_i, i.e., $\dot{x}_i = (d/dt)x_i$. The system is initialized by choosing $x_i(0) = x_{i0}$ $(i = 1,\cdots,n)$. Defining

$$
x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ \vdots \\ q_r \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_s \end{pmatrix},
$$

$$
f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \quad h(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_p(x) \end{pmatrix}, \quad k(x) = \begin{pmatrix} k_1(x) \\ \vdots \\ k_s(x) \end{pmatrix},
$$

$$
g(x) = \begin{pmatrix} g_{11}(x) & \cdots & g_{m1}(x) \\ \vdots & & \vdots \\ g_{1n}(x) & \cdots & g_{mn}(x) \end{pmatrix}, \quad p(x) = \begin{pmatrix} p_{11}(x) & \cdots & p_{r1}(x) \\ \vdots & & \vdots \\ p_{1n}(x) & \cdots & p_{rn}(x) \end{pmatrix},
$$

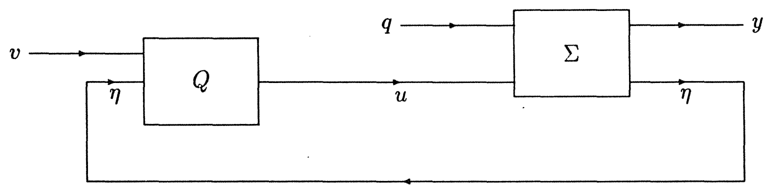

Figure 1.2: Feedback control system

we write (1.1) in the compact form

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ y & = & h(x) \\ \eta & = & k(x) \end{array} \right. \tag{1.2}$$

Static and dynamic state feedback

Consider a nonlinear control system of the form (1.2). In control theory, one is interested in the question how the controls u of the system should be chosen to assure that the outputs-to-be-controlled y meet some prespecified requirements. To make this more clear, we consider the example of a rigid robot arm. Typically, for such a system the controls are the torques that can be applied at the joints. If the aim is to let the endpoint of the robot arm follow a certain prespecified path, we take as outputs-to-be-controlled the (Cartesian) coordinates of the endpoint. With these sets of controls and outputs-to-be-controlled the behavior of the robot arm can be described via equations of the form (1.2) (see e.g. [5]). The question then is how to choose the controls in order that the endpoint follows the prespecified path, and remains close to it if disturbances are present.

In the control problems that we will encounter in this monograph, the solution to the problem comes down to changing the structure of the system (that is, the form of the equations (1.2)) by an appropriate choice of the controls u. Very often (and this monograph will be no exception) this is done via *feedback*. To make the idea of feedback clear, consider Figure 1.2. We see that the measurements η are fed back to Q, and, based on the value of η, Q "decides" by some kind of rule what the control u for the system Σ should be. The signal v in Figure 1.2 indicates a set of new controls that can be used to achieve further control objectives.

Above, we have been a bit vague about the form of Q. We will be more specific now. We will especially be interested in two types of feedback for a system (1.2). First, we assume throughout that the whole state vector of the system can be measured, i.e., in (1.2) we have $s = n$ and $\eta_1 = x_1, \cdots, \eta_n = x_n$. Therefore, in what follows we do not explicitly mention the measurements η any more and the outputs-to-be-controlled y will be simply referred to as the *outputs* y.

A *static state feedback* is a feedback of the form

$$u = \alpha(x) + \beta(x)v \tag{1.3}$$

where

$$\alpha(x) = \begin{pmatrix} \alpha_1(x) \\ \vdots \\ \alpha_m(x) \end{pmatrix}, \quad \beta(x) = \begin{pmatrix} \beta_{11}(x) & \cdots & \beta_{1m}(x) \\ \vdots & & \vdots \\ \beta_{m1}(x) & \cdots & \beta_{mm}(x) \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

with v denoting the new controls to the system. The adjective "static" indicates the fact that the feedback (1.3) decides what the value of u at a time-instance t should be *only* on the basis of what the value of x and v at that specific time-instance is. In this sense (1.3) could also be called a memoryless feedback. If we also incorporate some kind of memory in the feedback, we arrive at what is called a *dynamic state feedback*. This is a feedback of the form

$$\begin{cases} \dot{z} &= \alpha(x,z) + \beta(x,z)v \\ u &= \gamma(x,z) + \delta(x,z)v \end{cases} \tag{1.4}$$

where

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_\nu \end{pmatrix}, \quad \alpha(x,z) = \begin{pmatrix} \alpha_1(x,z) \\ \vdots \\ \alpha_\nu(x,z) \end{pmatrix}, \quad \beta(x,z) = \begin{pmatrix} \beta_{11}(x,z) & \cdots & \beta_{1m}(x,z) \\ \vdots & & \vdots \\ \beta_{\nu 1}(x,z) & \cdots & \beta_{\nu m}(x,z) \end{pmatrix}$$

$$\gamma(x,z) = \begin{pmatrix} \gamma_1(x,z) \\ \vdots \\ \gamma_m(x,z) \end{pmatrix}, \quad \delta(x,z) = \begin{pmatrix} \delta_{11}(x,z) & \cdots & \delta_{1m}(x,z) \\ \vdots & & \vdots \\ \delta_{m1}(x,z) & \cdots & \delta_{mm}(x,z) \end{pmatrix}$$

Here a part of $z_1, \cdots, z_\nu$ can be interpreted as the memory of the feedback.

Disturbance decoupling problem

A problem occurs when there are disturbances q entering the system that influence the outputs y. In this case the behavior of the outputs is unknown, since (by definition) the behavior of the disturbances is unknown. To tackle this problem, a first step in the controller design is to change the structure of the system (that is, the form of the equations (1.2)) via a static state feedback (1.3) or a dynamic state feedback (1.4) in such a way that the disturbances no longer influence the outputs.

The *disturbance decoupling problem* is thus defined as follows: given a system of the form (1.2), find (if possible) a static state feedback (1.3) or a dynamic state feedback (1.4) such that for the resulting system (1.2,1.3) (or (1.2,1.4)) the disturbances q no longer influence the outputs y.

Input-output decoupling problem

Consider a system (1.2), for which the number of controls equals the number of outputs (i.e., in (1.2) we have $p = m$). Moreover, assume that there are no disturbances entering the system, i.e., in (1.2) we have $r = 0$. The *input-output decoupling problem* for this system is defined as: find (if possible) a static state feedback (1.2) or a dynamic state feedback (1.4) for (1.2) such that for the resulting system (1.2,1.3) (or (1.2,1.4)), with controls $v_1, \cdots, v_m$ and outputs $y_1, \cdots, y_m$, the control v_i $(i = 1, \cdots, m)$ *does* influence the output y_i, but does *not* influence the other outputs $y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_m$.

Having designed a static (or dynamic) state feedack that solves the input-output decoupling problem, the control of the outputs of the system is greatly facilitated. Namely, if we want a certain output to behave in a certain way, it suffices to manipulate the control v_i, whilst we do not have to worry that any changes in the control v_i have a negative influence on the behavior of the other outputs $y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_m$.

Model matching problem
The *input-output behavior* of a system of the form (1.2) without disturbances (i.e., $r = 0$) is the set of all possible input-output trajectories $(u(t), y(t))$ that the system can produce. Now consider a system of the form (1.2), that we will call the *plant* P. It may be that the input-output behavior of P has some undesirable properties and that we would like to achieve some other (nicer) structure, say the input-output behavior of some fictitious system of the form (1.2), that we call the model M. The question if this is possible and, if the answer is positive, how this could be achieved is known as the *model matching problem.* It will become clear that the model matching is related to the disturbance decoupling problem.

1.2 Overview of differential geometry

In this section an overview is given of concepts from differential geometry that are used in this monograph. Standard references on differential geometry are [1],[2],[8],[16],[42],[92] (see also [61],[81]). An easily accessible account on differential geometry is [18]. Some concepts in this overview are taken from [61],[81],[42], [92].

Consider the space $I\!R^N$ with *Euclidean norm* $\|\cdot\|$ defined by $\|x\| = (\sum_{i=1}^N x_i^2)^{\frac{1}{2}}$. A subset $U \subset I\!R^N$ is called an *open subset* of $I\!R^N$ if for every $\bar{x} \in U$ there exists an $\epsilon > 0$ such that $\{x \in I\!R^N \mid \|x - \bar{x}\| < \epsilon\} \subset U$. For $x \in I\!R^N$, an open subset $U \subset I\!R^N$ containing x is called a *neighborhood* of x. Let U be a (not necessarily open) subset of $I\!R^N$. A subset $\bar{U} \subset U$ is called a *relatively open subset* (or briefly: open subset, when no confusion arises) of U if there exists an open subset $\tilde{U}$ of $I\!R^N$ such that $\tilde{U} \cap U = \bar{U}$. Note that a relatively open subset need not be an open subset of $I\!R^N$. For example, take $N = 2$, $U = \{x \in I\!R^2 \mid x_2 = 0\}$, $\bar{U} = \{x \in I\!R^2 \mid -1 < x_1 < 1, x_2 = 0\}$. For $\bar{x} \in U$, a relatively open subset $\bar{U}$ of U containing $\bar{x}$ is called a *neighborhood* of $\bar{x}$ in U.

Let U, V be subsets of $I\!R^N$ and consider a mapping $\Phi : U \rightarrow V$. Φ is called *continuous* if the inverse image of every open subset of V is an open subset of U. If U is open, Φ is called *smooth* if it has continuous partial derivatives of all orders. If U is not open, Φ is called *smooth* if for every $\bar{x} \in U$ there exists a neighborhood $\bar{U}$ of $\bar{x}$ in $I\!R^N$ and a smooth mapping $\Psi : \bar{U} \rightarrow V$ such that Ψ equals Φ on $U \cap \bar{U}$. The set of smooth real-valued functions on U is denoted by $C^\infty(U)$. Φ is called a *diffeomorphism* if $\Phi^{-1} : V \rightarrow U$ exists and both Φ and Φ^{-1} are smooth.

A subset $M \subset I\!R^N$ is called a *manifold* of dimension n if there exist an index set I, relatively open subsets U_i $(i \in I)$ of M, open subsets V_i $(i \in I)$ of $I\!R^n$ and diffeomorphisms $\phi_i : U_i \rightarrow V_i$ $(i \in I)$, such that $\bigcup_{i \in I} U_i = M$, and, whenever $U_i \cap U_j \neq \emptyset$ $(i, j \in I)$, the mapping $\phi_{ji} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ defined by $\phi_{ji}(y) = \phi_j \circ \phi_i^{-1}(y)$ is a diffeomorphism. For every $i \in I$, the pair (U_i, ϕ_i) is called a *local coordinate chart* on M, and the collection $\mathcal{A} = \{(U_i, \phi_i) \mid i \in I\}$ is called an *atlas* on M. A subset N of M is called a *submanifold* of

M if N itself is a manifold.

For $x \in \mathbb{R}^n$, the *tangent space* $T_x\mathbb{R}^n$ at x is the set of tangent vectors to $\mathbb{R}^n$ at x (and so, $T_x\mathbb{R}^n$ is a copy of $\mathbb{R}^n$). The natural basis of $T_x\mathbb{R}^n$ is denoted by $\{\frac{\partial}{\partial x_1}|_x, \cdots, \frac{\partial}{\partial x_n}|_x\}$. Consider a manifold M of dimension n and let $m \in M$. Let (U, ϕ) be a local coordinate chart around m. For $\tau \in T_{\phi(m)}\mathbb{R}^n$, define $\phi^*(m)\tau := \frac{\partial \phi^{-1}}{\partial x}(\phi(m))\tau$. The *tangent space* T_mM of M at m is defined as

$$T_mM = \text{span}_{\mathbb{R}}\{\phi^*(m)\frac{\partial}{\partial x_k}|_{\phi(m)} \mid k = 1, \cdots, n\} \tag{1.5}$$

The elements of T_mM are called *tangent vectors* at m. Since ϕ is a diffeomorphism, the mapping $\phi^*(m) : T_{\phi(m)}\mathbb{R}^n \to T_mM$ is a linear isomorphism. Hence the vectors $\frac{\partial}{\partial \phi_k}|_m :=$ $\phi^*(m)\frac{\partial}{\partial x_k}|_{\phi(m)}$ $(k = 1, \cdots, n)$ form a basis of T_mM. The set $TM = \{(m, \tau) \mid m \in M, \tau \in T_xM\}$ is called the *tangent bundle* of M.

Remark 1.2.1 It can be shown that the above definition of T_mM coincides with the intuitive notion of a tangent space to a surface in $\mathbb{R}^N$. Moreover, it can be shown that the definition of T_mM is independent of the local coordinate chart around m that is chosen (see e.g. [42]). ■

A *vector field* on M is a mapping τ that assigns to each $m \in M$ a tangent vector $\tau(m) \in T_mM$. τ is called a *smooth* vector field if for each $m \in M$ there exists a local coordinate chart (U, ϕ) around m and functions $\tau_1, \cdots, \tau_n \in C^\infty(U)$ such that for all $\bar{m} \in U$ we have $\tau(\bar{m}) = \sum_{i=1}^n \tau_i(\bar{m})\frac{\partial}{\partial \phi_i}|_{\bar{m}}$. The set of smooth vector fields on M is denoted by $V(M)$.

For $k = 1, \cdots, n$, let r_k denote the natural *coordinate functions* on $\mathbb{R}^n$, i.e., $r_k(a_1, \cdots, a_n) = a_k$. The functions $x_i = r_i \circ \phi$ $(i = 1, \cdots, n)$ are called *local coordinate functions* and the values $x_1(m), \cdots, x_n(m)$ of a point $m \in U$ are called the *local coordinates* of m. Let $F : M \to \mathbb{R}$ be a map. Then F yields a function $\hat{F} : \phi(U) \to \mathbb{R}$ defined as $\hat{F} = F \circ \phi^{-1}$, that is, for $m \in U$: $F(m) = \hat{F}(x_1(m), \cdots, x_n(m))$. The function $\hat{F}$ is called the local representative of F. It is customary to use the same letter "F" for F defined on M and for $\hat{F}$, its expression in local coordinates. Hence instead of $\hat{F}(x_1(m), \cdots, x_n(m))$ we write $F(x_1(m), \cdots, x_n(m))$. Furthermore, usually we delete the dependence on m, and write $F(x_1, \cdots, x_n)$, where $(x_1, \cdots, x_n)$ are the local coordinates of some (unspecified) point $m \in U$. Consider a smooth vector field τ on M, on U given by $\tau(m) = \sum_{i=1}^n \tau_i(m)\frac{\partial}{\partial \phi_i}|_m$. The expression of τ in local coordinates is denoted by $\tau(x_1, \cdots, x_n) = \sum_{i=1}^n \tau_i(x_1, \cdots, x_n)\frac{\partial}{\partial x_i}|_x$ or, briefly, $\tau(x) = \sum_{i=1}^n \tau_i(x)\frac{\partial}{\partial x_i}|_x$, where $\tau_i(x_1, \cdots, x_n)$ is the expression of τ_i in local coordinates. A vector field $\tau \in V(M)$ given in local coordinates, is often identified with the n-dimensional column vector $\text{col}(\tau_1(x), \cdots, \tau_n(x))$.

A *smooth curve* σ on M is a smooth mapping $\sigma : (a, b) \to M$, where (a, b) is an open interval of $\mathbb{R}$. For $t \in (a, b)$, let (U, ϕ) be a local coordinate chart around $\sigma(t) \in M$. Define $\dot{\sigma}(t) \in T_{\sigma(t)}M$ by the conventional limit (in local coordinates)

$$\dot{\sigma}(t) = \lim_{h \to 0} \frac{\sigma(t + h) - \sigma(t)}{h} \tag{1.6}$$

σ is called an *integral curve* of a given vector field f on M if $\dot{\sigma}(t) = f(\sigma(t))$ for all $t \in (a,b)$. In local coordinates $x_1, \cdots, x_n$ this just means that $\sigma(t) = (\sigma_1(t), \cdots, \sigma_n(t))$ is a solution of the set of differential equations

$$\left\{ \begin{array}{rcl} \dot{\sigma}_1(t) & = & f_1(\sigma_1(t), \cdots, \sigma_n(t)) \\ & \vdots & \\ \dot{\sigma}_n(t) & = & f_n(\sigma_1(t), \cdots, \sigma_n(t)) \end{array} \right. \qquad (t \in (a,b)) \tag{1.7}$$

where f is identified with the column vector $\mathrm{col}(f_1, \cdots, f_n)$. So, to a vector field f given in local coordinates we associate in a one-to-one way the set of differential equations

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & f_1(x_1, \cdots, x_n) \\ & \vdots & \\ \dot{x}_n(t) & = & f_n(x_1, \cdots, x_n) \end{array} \right. \tag{1.8}$$

also abbreviated as $\dot{x} = f(x)$.

For any two $\sigma, \tau \in V(M)$, in local coordinates given by $\sigma(x) = \mathrm{col}(\sigma_1(x), \cdots, \sigma_n(x))$, $\tau(x) = \mathrm{col}(\tau_1(x), \cdots, \tau_n(x))$ the *Lie bracket* $[\sigma, \tau]$ is defined by (in local coordinates):

$$[\sigma, \tau] = \begin{pmatrix} \frac{\partial \tau_1}{\partial x_1} & \cdots & \frac{\partial \tau_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \tau_n}{\partial x_1} & \cdots & \frac{\partial \tau_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} - \begin{pmatrix} \frac{\partial \sigma_1}{\partial x_1} & \cdots & \frac{\partial \sigma_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \sigma_n}{\partial x_1} & \cdots & \frac{\partial \sigma_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_n \end{pmatrix} = \frac{\partial \tau}{\partial x}\sigma - \frac{\partial \sigma}{\partial x}\tau \tag{1.9}$$

If $\sigma, \tau \in V(M)$, then also $[\sigma, \tau] \in V(M)$. $V(M)$ endowed with the Lie bracket $[\cdot, \cdot]$ has a special algebraic structure:

Definition 1.2.2 Lie algebra
A pair $(V, [\cdot, \cdot])$ is called a Lie-algebra (over $\mathbb{R}$) if V is a vector space over $\mathbb{R}$ with a product $[\cdot, \cdot] : V \times V \to V$ that has the following properties for all $v, w, z \in V$:

$$[a_1 v + a_2 w, z] = a_1[v, z] + a_2[w, z] \quad \forall a_1, a_2 \in \mathbb{R} \quad (\textit{bilinearity over } \mathbb{R}) \tag{1.10}$$

$$[v, w] = -[w, v] \quad (\textit{skew}-\textit{symmetry}) \tag{1.11}$$

$$[v, [w, z]] + [z, [v, w]] + [w, [z, v]] = 0 \quad (\textit{Jacobi}-\textit{identity}) \tag{1.12}$$

■

It can be shown that $(V(M), [\cdot, \cdot])$ (with $[\cdot, \cdot]$ the Lie bracket defined in (1.9)) is a Lie-algebra (over $\mathbb{R}$).

The *cotangent space* T_m^*M of M at m is defined as the dual space of T_mM, i.e., the space of linear mappings $\omega : T_mM \to \mathbb{R}$. The elements of T_m^*M are called *cotangent vectors* at m. If $\omega \in T_m^*M$, then the value of ω at $\tau \in T_mM$ is denoted by $<\omega, \tau>$ (or $\omega(\tau)$). Let $\tau_1, \cdots, \tau_n$ be a basis for T_mM. Then the unique basis $\omega_1, \cdots, \omega_n$ of T_m^*M which satisfies $<\omega_i, \tau_j> = \delta_{ij}$, is called the *dual basis* of T_m^*M with respect to $\tau_1, \cdots, \tau_n$. Given a coordinate chart (U,ϕ) around m, the dual basis with respect to $\frac{\partial}{\partial\phi_1}|_m,\cdots,\frac{\partial}{\partial\phi_n}|_m$ is denoted by $d\phi_1|_m,\cdots,d\phi_n|_m$. The set $T^*M=\{(m,\omega)\mid m\in M,\omega\in T^*_mM\}$ is called the *cotangent bundle* of M. A *covector field* (or *one-form*) ω on M is a mapping that to each $m\in M$ assigns a cotangent vector $\omega(m)\in T^*_mM$. ω is a *smooth* covector field if for each $m\in M$ there exists a local coordinate chart (U,ϕ) around m and functions $\omega_1,\cdots,\omega_n\in C^\infty(U)$ such that for all $\bar{m}\in U$ we have $\omega(\bar{m})=\sum_{i=1}^n\omega_i(\bar{m})d\phi_i|_{\bar{m}}$. The set of smooth covector fields on M is denoted by $V^*(M)$. Let $\omega\in V^*(M)$. The expression of ω in local coordinates is denoted by $\omega(x)=\sum_{i=1}^n\omega_i(x)dx_i|_x$, where $\omega_i(x)$ is the expression of ω_i in local coordinates. A covector field $\omega\in V^*(M)$ given in local coordinates, is often identified with the n-dimensional row-vector $(\omega_1(x)\ \cdots\ \omega_n(x))$. For $\tau\in V(M)$, $\omega\in V^*(M)$, the smooth function $<\omega,\tau>$: $M\to I\!R$ is defined by $<\omega,\tau>(m)=<\omega(m),\tau(m)>$. Note that in local coordinates $<\omega,\tau>$ is just the product of the row vector ω and the column vector τ. With every $F\in C^\infty(M)$ we can associate a covector field dF by defining (in local coordinates) $dF(x)=\sum_{i=1}^n\frac{\partial F}{\partial x_i}(x)dx_i|_x$.

Definition 1.2.3 Lie derivative
Let $\tau\in V(M)$. Then the following Lie-derivatives may be associated to τ (the definitions are given in local coordinates):

(i) $\mathcal{L}_\tau : C^\infty(M)\to C^\infty(M)$:

$$\mathcal{L}_\tau F=\sum_{i=1}^n\frac{\partial F}{\partial x_i}\tau_i \quad (\forall F\in C^\infty(M)) \tag{1.13}$$

(ii) $\mathcal{L}_\tau : V(M)\to V(M)$:

$$\mathcal{L}_\tau\sigma=[\tau,\sigma] \quad (\forall\sigma\in V(M)) \tag{1.14}$$

(iii) $\mathcal{L}_\tau : V^*(M)\to V^*(M)$:

$$\mathcal{L}_\tau\omega=(\frac{\partial\omega^T}{\partial x}\tau)^T+\omega\frac{\partial\tau}{\partial x} \quad (\forall\omega\in V^*(M)) \tag{1.15}$$

where "T" denotes the transpose. ■

The three types of Lie derivatives are related by the *Leibniz formula*:

$$\mathcal{L}_\tau<\omega,\sigma>=<\mathcal{L}_\tau\omega,\sigma>+<\omega,\mathcal{L}_\tau\sigma> \quad (\forall\sigma,\tau\in V(M),\omega\in V^*(M)) \tag{1.16}$$

A *distribution* Δ on M is a mapping that assigns to each $m\in M$ a linear subspace of T_mM. Δ is called a *smooth distribution* if for each $\bar{m}\in M$ there exist a neighborhood $U\subset M$ of $\bar{m}$ and a set of smooth vector fields τ_i, $i\in I$, with I some (possibly infinite) index set, such that $\Delta(m)=\mathrm{span}_{I\!R}\{\tau_i(m)\mid i\in I\}$ for every $m\in U$. In the sequel distribution will ***always*** mean smooth distribution. If $\{\tau_i\mid i\in I\}$ is a set of smooth vector fields on M, then their span, denoted by span $\{\tau_i\mid i\in I\}$ is the distribution defined by

$$\mathrm{span}\,\{\tau_i\mid i\in I\}:m\to\mathrm{span}_{I\!R}\{\tau_i(m)\mid i\in I\}\quad(m\in M) \tag{1.17}$$

The *sum* and *intersection* of two distributions Δ_1 and Δ_2 are defined as:

$$\Delta_1 + \Delta_2 : m \rightarrow \Delta_1(m) + \Delta_2(m) \quad (m \in M) \tag{1.18}$$

$$\Delta_1 \cap \Delta_2 : m \rightarrow \Delta_1(m) \cap \Delta_2(m) \quad (m \in M) \tag{1.19}$$

Note that the sum of two smooth distributions is a smooth distribution again, but that the intersection need not be smooth. A vector field τ on M is said to belong to a distribution Δ, denoted by $\tau \in \Delta$, if for every $m \in M$ we have $\tau(m) \in \Delta(m)$. A distribution Δ_1 is said to be contained in a distribution Δ_2, denoted by $\Delta_1 \subset \Delta_2$, if every vector field belonging to Δ_1 also belongs to Δ_2. The *dimension* of a distribution Δ at $m \in M$ is the dimension of the linear subspace $\Delta(m)$. A distribution is called *constant dimensional* if the dimension of $\Delta(m)$ does not depend on the point $m \in M$. If Δ is a distribution of constant dimension, say k, then for every $\bar{m} \in M$ there exist a neighborhood $U \subset M$ of $\bar{m}$ and vector fields $\tau_1, \cdots, \tau_k \in V(M)$ such that for every $m \in U$: $\Delta(m) = \operatorname{span}_{\mathbb{R}}\{\tau_1(m), \cdots, \tau_k(m)\}$. A distribution Δ is called *involutive* if $[\sigma, \tau] \in \Delta$ whenever $\sigma, \tau \in \Delta$. If Δ is not involutive, there always exists a smallest involutive distribution containing Δ. This distribution is called the *involutive closure* of Δ and is denoted by $\bar{\Delta}$.

A submanifold N of M is an *integral manifold* of a distribution Δ on M if $T_mN = \Delta(m)$ for every $m \in N$. A k-dimensional distribution Δ on M is *integrable* if around any $m \in M$ there exists a coordinate chart (U, ϕ) such that

$$\Delta(m) = \operatorname{span}\left\{\frac{\partial}{\partial \phi_1}|_m, \cdots, \frac{\partial}{\partial \phi_k}|_m\right\} \tag{1.20}$$

Δ as in (1.20) is called a *flat distribution* in the local coordinates ϕ. Note that the integral curve of a vector field $f \neq 0$ we encountered before is an integral manifold of the one-dimensional distribution span $\{f\}$.

Theorem 1.2.4 Frobenius theorem *(local version)*
A constant dimensional distribution Δ is integrable if and only if it is involutive. ∎

The Frobenius theorem states that for a constant dimensional distribution Δ involutivity is equivalent to the existence, for every $m \in M$, of a coordinate chart (U, ϕ) around m, with $\phi(m) = 0$, $\phi(U) = (-\epsilon, \epsilon) \times \cdots \times (-\epsilon, \epsilon)$ $(\epsilon > 0)$, such that for every $a_{k+1}, \cdots, a_n$, smaller in absolute value than ϵ, the submanifold

$$\{\bar{m} \in U \mid \phi_{k+1}(\bar{m}) = a_{k+1}, \cdots, \phi_n(\bar{m}) = a_n\} \tag{1.21}$$

is an integral manifold of Δ. Moreover every integral manifold is locally of this form.

There also exists a global version of the Frobenius theorem. For details, we refer to [92].

A *codistribution* Ω on M is a mapping that assigns to each $m \in M$ a linear subspace of T_m^*M. Ω is called smooth if for each $\bar{m} \in M$ there exist a neighborhood $U \subset M$ of $\bar{m}$ and a set of smooth covector fields ω_i, $i \in I$, with I some (possibly infinite) index set, such that $\Omega(m) = \operatorname{span}_{\mathbb{R}}\{\omega_i(m) \mid i \in I\}$ for every $m \in U$. If Δ is a distribution on M, then its *annihilator*, denoted by $\operatorname{ann} \Delta$, is the codistribution defined by

$$\operatorname{ann} \Delta(m) = \operatorname{span}_{\mathbb{R}}\{\omega(m) \mid \omega \text{ is a covector field s.t. } <\omega, \tau>= 0, \forall \tau \in \Delta\} \tag{1.22}$$

If Ω is a codistribution on M, then its *kernel*, denoted by $\operatorname{Ker}\Omega$, is the distribution defined by

$$\operatorname{Ker}\Omega(m) = \operatorname{span}_{\mathbb{R}}\{\tau(m) \mid \tau \text{ is a vector field s.t. } <\omega,\tau>=0, \forall\omega\in\Omega\} \tag{1.23}$$

In general the annihilator and kernel need not be smooth. However, if Δ and Ω are smooth and constant dimensional, then so are $\operatorname{ann}\Delta$ and $\operatorname{Ker}\Omega$, while $\operatorname{Ker}\operatorname{ann}\Delta = \Delta$, $\operatorname{ann}\operatorname{Ker}\Omega = \Omega$. (see e.g [81]).

Remark 1.2.5 In the sequel we use the following shorthand notation

(i) If $F_1, \cdots, F_p \in C^\infty(M)$, the codistribution $\operatorname{span}\{dF_1, \cdots, dF_p\}$ will be denoted by $\operatorname{span}\{dF\}$, while $\operatorname{Ker} dF$ denotes $\operatorname{Ker}\operatorname{span}\{dF_1, \cdots, dF_p\}$.

(ii) If Δ is a distribution and τ a vector field on M, then $[\tau, \Delta]$ denotes the distribution $\operatorname{span}\{[\tau,\sigma] \mid \sigma \in \Delta\}$.

(iii) $\frac{\partial}{\partial x_i}$ denotes $\frac{\partial}{\partial x_i}|_x$ and dx_i denotes $dx_i|_x$. ■

1.3 Organization of the monograph

The monograph is organized as follows.

Chapter 2 The theory on the linear and nonlinear disturbance decoupling problem via (regular) static state feedback (DDP) is recapitulated, based on [110] for the linear case and [61],[81] for the nonlinear case. The solution in the linear case is obtained by introducing the concept of a (controlled) invariant subspace. In the nonlinear case, the solution is obtained by introducing the concept of a locally (controlled) invariant distribution, a nonlinear generalization of a (controlled) invariant subspace. The solutions in the linear and nonlinear case show clear analogies. However, there are also some differences, which are indicated at the end of the chapter.

Chapter 3 We first repeat the theory on the strong input-output decoupling problem via regular static feedback (SIODP), following [81],[61]. For systems for which the SIODP is solvable, a local normal form is obtained. This new result was derived by Huijberts and Van der Schaft in [59] (see also [69]).

For the treatment of the strong input-output decoupling problem via regular dynamic state feedback (SDIODP) an algebraic theory for nonlinear control systems is reviewed. The presentation closely follows [30] (see also [35],[36]). A convenient way of performing the necessary calculations in this algebraic theory is provided by Singh's algorithm ([90]). The version of Singh's algorithm presented here is from [30]. Based on the algebraic theory a definition of a regular dynamic state feedback is given ([72],[30]). Singh's algorithm applied to a nonlinear control system provides a special sort of regular dynamic state feedback, that will be called a Singh compensator. The definition of a Singh compensator can be found in a series of papers by Huijberts, Nijmeijer and Van der Wegen [51],[52],[53],[54] (see also [91]). In general, there exists more than one Singh compensator for a nonlinear control system. It is shown that every Singh compensator has the same dimension. This result is from [54]. Moreover, it is shown that a nonlinear control system equipped with one specific Singh compensator is state space equivalent to the same system equipped with another Singh compensator. This result is new. A first step in proving this result was taken in [107].

A solution of the SDIODP is given, using the algebraic theory introduced before. This solution can be found in [30],[61],[71],[78],[81]. It is shown that the SDIODP is solvable if and only if it is solvable via a Singh compensator. Moreover, it is shown that a Singh compensator is a regular dynamic state feedback of *minimal* dimension solving the SDIODP. This means that the dimension of *every* regular dynamic state feedback that solves the SDIODP is greater than or equal to the dimension of a Singh compensator. This last result is new and is taken from a paper by Huijberts, Nijmeijer and Van der Wegen [54].

Chapter 4 The new problem of disturbance decoupling via regular dynamic state feedback (DDDP) is formulated and solved. It turns out that the problem is solvable if and only if it is solvable via a Singh compensator. Geometric conditions for solvability that are reminiscent of the conditions for solvability of the DDP (see Chapter 2) are also given. Moreover, algebraic conditions for solvability in terms of the algebraic structure at infinity ([71]) are given. Analogously to the DDDP, the disturbance decoupling problem via regular dynamic state feedback and disturbance measurements (DDDPdm) is formulated and solved, and moreover geometric and algebraic conditions for solvability of the DDDPdm are given. This chapter is based on a series of papers by Huijberts, Nijmeijer and Van der Wegen [51],[52],[53].

Chapter 5 In Chapter 2 the notion of a (controlled) invariant subspace for a linear system was generalized to nonlinear systems via the notion of a locally (controlled) invariant distribution. In this chapter another generalization, that of a locally (controlled) invariant submanifold, is introduced. This leads to the introduction of the clamped dynamics and the clamped dynamics algorithm for a nonlinear control system. These notions were first identified, in the single-input single-output case, in [11],[67]. For the multi-input multi-output case they were further elaborated in [12],[64],[101],[104]. A new result is given that establishes a connection between the clamped dynamics of a nonlinear control system and the clamped dynamics of this system together with a dynamic state feedback. This result was first derived by Huijberts in [50]. Also a result from [102], that gives a connection between the clamped dynamics algorithm and Singh's algorithm, is stated.

Chapter 6 The disturbance decoupling problem via nonregular dynamic state feedback (nDDDP) and the disturbance decoupling problem via nonregular dynamic state feedback and disturbance measurements (nDDDPdm) are formulated. For both problems an algorithm for solving the problems is given. These algorithms are based on Singh's algorithm and the clamped dynamics algorithm from Chapters 3 and 5, respectively. The results in the chapter are taken from a paper by Huijberts [50].

Chapter 7 The nonlinear model matching problem (MMP) is formulated. Sufficient conditions for solvability of the MMP from [31],[73] are given. It is shown that the solvability of the MMP is equivalent to the solvability of an associated disturbance decoupling via nonregular dynamic state feedback and disturbance measurements. This result can be found in a paper by Huijberts [50] (see also [31]) and forms a nonlinear generalization of results obtained in [76],[33] in the linear case. Based on the theory developed in Chapter 6 and this result necessary and sufficient conditions for solvability of the MMP are given.

Another new set of necessary and sufficient conditions for the solution of the MMP is given in case the SIODP is solvable for the model. It is also shown that under generic conditions on the plant the MMP is solvable around equilibrium points for the plant and the model if and only if it is solvable for the linearizations of the plant and the model around these equilibrium points. As an important intermediate step, it is shown that, under generic conditions, the SDIODP is solvable for a system around an equilibrium point if and only if the SDIODP is solvable for the linearization of this system around the equilibrium point. These results are from a paper by Huijberts and Nijmeijer [55]. The theory is illustrated via two examples from a paper by Huijberts [48].

Chapter 8 A special class of nonlinear control systems, the Hamiltonian control systems, is introduced. The concept of a Hamiltonian control system was introduced in [9] and further elaborated in e.g. [98]. The exposition closely follows Chapter 12 of [81]. The structure of Hamiltonian systems for which the SIODP is solvable is investigated. It is shown that the clamped dynamics of a Hamiltonian system for which the SIODP is solvable constitute a Hamiltonian system without inputs. This result is taken from [100],[103] (see also [81]) and is based on Dirac's theory of constrained Hamiltonian systems [32]. The result is used to give conditions for the solvability of the SIODP with stability for Hamiltonian systems. These conditions are taken from a paper by Huijberts and Van der Schaft [59]. Also the question to what extent a strongly input-output decoupled Hamiltonian still has a Hamiltonian structure is investigated. Up till now a complete answer to this question is not known, but in some particular cases the solution is obtained. Based on this a conjecture is stated. The conjecture and the treatment of the special cases are based on unpublished joint work by Huijberts and Van der Schaft. If the conjecture holds true, it may provide important structural information in the solution of synthesis problems for Hamiltonian control systems. This is illustrated via the model matching problem with prescribed tracking error for Hamiltonian control systems. This illustration is taken from a paper by Huijberts [47].

Chapter 2

Controlled invariance and disturbance decoupling via static state feedback

In this chapter we treat the disturbance decoupling problem via (regular) static state feedback for linear and nonlinear systems. Instrumental in the solution of the problem in the linear as well as in the nonlinear case is an appropriate notion of (controlled) invariance. This notion is introduced. In Section 2.1 we first treat the linear case, while in Section 2.2 the nonlinear case is discussed. Moreover in this section we discuss the analogies and differences between the solutions in the linear and nonlinear case.

For linear systems the notion of controlled invariance and its use in the solution of the disturbance decoupling problem dates back to the end of the sixties, see [6],[111]. A detailed account of the use of controlled invariance in linear synthesis problems is given in [110]. When no specific references are given, the results in Section 2.1 can be found in [110] (and the references therein). The nonlinear generalization of the notion of controlled invariance together with its applicability in various nonlinear synthesis problems (amongst which the disturbance decoupling problem) has been initiated in the beginning of the eighties in [45],[63]. When no specific references are given, the results in Section 2.2 can be found in [61],[81] and the references therein.

2.1 The linear case

Consider a linear time-invariant system of the form

$$\begin{cases} \dot{x} &= Ax + Bu + Eq \\ y &= Cx \end{cases} \tag{2.1}$$

with *states* $x \in \mathbb{R}^n$, *controls* $u \in \mathbb{R}^m$, *disturbances* $q \in \mathbb{R}^r$, *outputs* $y \in \mathbb{R}^p$ and A, B, C, E matrices of appropriate dimensions with constant coefficients.

For a given initial condition $x(0) = x_0 \in \mathbb{R}^n$, given control functions $\bar{u}(t)$ $(t \geq 0)$ and given disturbance functions $\bar{q}(t)$ $(t \geq 0)$ the outputs $y(t, x_0, \bar{u}, \bar{q})$ of (2.1) at time $t \geq 0$ are given by the well known variation of constants formula

$$y(t, x_0, \bar{u}, \bar{q}) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}B\bar{u}(\tau)d\tau + \int_0^t Ce^{A(t-\tau)}E\bar{q}(\tau)d\tau \tag{2.2}$$

Remark 2.1.1 *(i)* Throughout we assume that the controls $u(\cdot)$ belong to the class of piecewise continuous functions and that the disturbances $q(\cdot)$ belong to a sufficiently large class of functions (for example $\mathcal{L}^1_{\mathrm{loc}}$) such that the expression (2.2) is well-defined.

(ii) By the time-invariance of the system (2.1), we may take, without loss of generality, the initial time equal to zero. ■

In the disturbance decoupling problem via *static state feedback* for the system (2.1) we look for a static state feedback

$$u = Fx + Gv \tag{2.3}$$

where F, G are matrices of appropriate dimensions with constant coefficients and $v \in \mathbb{R}^m$ denotes the new controls to the system, such that after application of this feedback the outputs are independent of the disturbances, i.e., the last term in (2.2) vanishes for A replaced by $A + BF$. We give a formal definition:

Definition 2.1.2 Disturbance decoupling problem via static state feedback (DDP)
Given a system (2.1), under what conditions does there exist a static state feedback (2.3) such that for the system (2.1,2.3) the outputs are independent of the disturbances? ■

To arrive at necessary and sufficient conditions for the solvability of the DDP we define the notion of a (controlled) invariant subspace for (2.1).

Definition 2.1.3 (Controlled) invariant subspace

(i) A subspace $\mathcal{V} \subset \mathbb{R}^n$ is said to be an invariant subspace for (2.1) if

$$A\mathcal{V} \subset \mathcal{V} \tag{2.4}$$

(ii) A subspace $\mathcal{V} \subset \mathbb{R}^n$ is said to be a controlled invariant subspace for (2.1) if there exists a linear map $F : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$(A + BF)\mathcal{V} \subset \mathcal{V} \tag{2.5}$$

■

Condition (2.5) can be verified by checking the solvability of a linear matrix equation. The following lemma gives an equivalent but easier verifiable characterization of controlled invariance.

Lemma 2.1.4 *A subspace $\mathcal{V} \subset \mathbb{R}^n$ is a controlled invariant subspace for (2.1) if and only if*

$$A\mathcal{V} \subset \mathcal{V} + \operatorname{Im} B \tag{2.6}$$

■

Given a controlled invariant subspace $\mathcal{V}$, the set of *friends of* $\mathcal{V}$, denoted by $\mathcal{F}(\mathcal{V})$, consists of all linear maps $F : \mathbb{R}^n \to \mathbb{R}^m$ satisfying (2.5). A map $F \in \mathcal{F}(\mathcal{V})$ is said to *render* $\mathcal{V}$ *invariant for (2.1).*

Let a subspace $\mathcal{K} \subset \mathbb{R}^n$ be given and let $I(A, B; \mathcal{K})$ denote the set of controlled invariant subspaces contained in $\mathcal{K}$. Suppose $\mathcal{V}_1, \mathcal{V}_2 \in I(A, B; \mathcal{K})$ and define $\mathcal{V}_1 + \mathcal{V}_2 := \{v = v_1 + v_2 \mid v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\}$. Then, using (2.6), it can easily be checked that also $\mathcal{V}_1 + \mathcal{V}_2 \in I(A, B; \mathcal{K})$. Furthermore, the subspace $\{0\}$ is trivially contained in $\mathcal{K}$ and satisfies (2.6). This implies in particular that $I(A, B; \mathcal{K})$ contains a unique maximal element $\mathcal{V}^*(A, B; \mathcal{K})$, meaning that $\mathcal{V}^*(A, B; \mathcal{K}) \in I(A, B; \mathcal{K})$ and for any $\mathcal{V} \in I(A, B; \mathcal{K})$: $\mathcal{V} \subset \mathcal{V}^*(A, B; \mathcal{K})$. $\mathcal{V}^*(A, B; \mathcal{K})$ can be constructed by means of the algorithm given below. In the algorithm we employ the following notation. Given a subspace $\mathcal{V} \subset \mathbb{R}^n$, we define $\mathcal{V}^\perp := \{z \in \mathbb{R}^n \mid z^T v = 0, \forall v \in \mathcal{V}\}$.

Algorithm 2.1.5 Controlled invariant subspace algorithm
Consider the system (2.1) and let a subspace $\mathcal{K} \subset \mathbb{R}^n$ be given. Define the subspaces $\mathcal{W}^\mu$ $(\mu = 0, 1, 2, \cdots)$ according to

$$\begin{aligned} \mathcal{W}^0 &= \mathcal{K}^\perp \\ \mathcal{W}^{\mu+1} &= \mathcal{W}^\mu + A^T(\mathcal{W}^\mu \cap \operatorname{Im} B^\perp); \; \mu = 0, 1, 2, \cdots \end{aligned} \tag{2.7}$$

■

Theorem 2.1.6 *Consider Algorithm 2.1.5. The subspaces* $\mathcal{W}^\mu$ $(\mu = 0, 1, 2, \cdots)$ *satisfy* $\mathcal{W}^{\mu+1} \supset \mathcal{W}^\mu$ *and for some* $k^* \leq \dim(\mathcal{K})$*:*

$$\mathcal{W}^{k^*} = \mathcal{W}^{k^*+1} = \cdots$$

and

$$\mathcal{V}^*(A, B; \mathcal{K}) = (\mathcal{W}^{k^*})^\perp$$

■

Now let us return to the DDP. Of crucial importance in the solution of this problem is the controlled invariant subspace $\mathcal{V}^*(A, B; \operatorname{Ker} C)$. When no confusion arises, we denote this subspace by $\mathcal{V}^*$ in the sequel. The solution of the DDP is given by the following theorem.

Theorem 2.1.7 *The DDP is solvable for (2.1) if and only if*

$$\operatorname{Im} E \subset \mathcal{V}^* \tag{2.8}$$

where $\mathcal{V}^* = \mathcal{V}^*(A, B; \operatorname{Ker} C)$*, the maximal controlled invariant subspace for (2.1) that is contained in* $\operatorname{Ker} C$*. Moreover, any static state feedback (2.3) with* $F \in \mathcal{F}(\mathcal{V}^*)$ *and an arbitrary* G *solves the DDP for (2.1).* ■

Remark 2.1.8 *(i)* Note that the choice of G is free in (2.3). One will often choose G to be an invertible matrix, reflecting the fact that after we have solved the DDP we still have m independent controls. Note that we can find F in (2.3) by solving a linear matrix equation.

(ii) One might wonder if by allowing for a more general type of feedback, e.g. a *dynamic* state feedback of the form

$$\left\{ \begin{array}{lcl} \dot{z} & = & Kz + Lx + Mv \\ u & = & Rz + Sx + Tv \end{array} \right. \tag{2.9}$$

with $z \in \mathbb{R}^\nu$ for some ν and $v \in \mathbb{R}^m$ denoting the new controls, we can render the outputs independent of the disturbances under less restrictive conditions than (2.8). The answer to this question is negative. In fact one can easily show that the disturbance decoupling problem is solvable via a feedback (2.3) if and only if it is solvable via a feedback (2.9) (cf. [7]). For nonlinear systems a similar result does not hold any more, as we will see in the following section. ∎

A problem closely related to the DDP is the disturbance decoupling problem via static state feedback and disturbance measurements (DDPdm). In this problem we assume that we can measure the disturbances entering the system. We use this extra information by adding a feedthrough term to the feedback (2.3), i.e., we allow for a control of the form

$$u = Fx + Gv + Hq \tag{2.10}$$

This leads to the following definition:

Definition 2.1.9 Disturbance decoupling problem via static state feedback and disturbance measurements (DDPdm)
Given a system (2.1), does there exist a static state feedback with disturbance feedthrough (2.10) such that for the system (2.1,2.10) the outputs are independent of the disturbances?

∎

Analogously to the proof of Theorem 2.1.7 we can prove:

Theorem 2.1.10 *The DDPdm is solvable for (2.1) if and only if*

$$\operatorname{Im} E \subset \mathcal{V}^* + \operatorname{Im} B \tag{2.11}$$

where $\mathcal{V}^* = \mathcal{V}^*(A, B; \operatorname{Ker} C)$, the maximal controlled invariant subspace for (2.1) that is contained in Ker C*. Moreover, any static static state feedback with disturbance feedthrough (2.10), with $F \in \mathcal{F}(\mathcal{V}^*)$, H satisfying $\operatorname{Im}(E + BH) \subset \mathcal{V}^*$ and an arbitrary G solves the DDPdm for (2.1).* ∎

2.2 The nonlinear case

Consider an analytic nonlinear time-invariant system of the form

$$\begin{cases} \dot{x} &= f(x) + g(x)u + p(x)q \\ y &= h(x) \end{cases} \tag{2.12}$$

where $x = (x_1, \cdots, x_n)^T$ are local coordinates for the state space manifold M, $u \in \mathbb{R}^m$, $q \in \mathbb{R}^r$, $y \in \mathbb{R}^p$, $g(x) = (g_1(x) \cdots g_m(x))$, $p(x) = (p_1(x) \cdots p_r(x))$, $h(x) = \mathrm{col}(h_1(x), \cdots, h_p(x))$, $f, g_1, \cdots, g_m, p_1, \cdots, p_r$ are analytic vector fields on M and $h_1, \cdots, h_p$ are analytic functions on M. Throughout we assume that the controls $u(\cdot)$ belong to the class of piecewise continuous functions and the disturbances $q(\cdot)$ belong to a sufficiently large class of functions such that the local solution of the state equation of (2.12) is well-defined. Define the distributions $\mathcal{G} := \mathrm{span}\{g_1, \cdots, g_m\}$, $\mathcal{P} := \mathrm{span}\{p_1, \cdots, p_r\}$.

In the disturbance decoupling problem via *regular static state feedback* for the system (2.12) we look for a regular static state feedback of the form:

$$u = \alpha(x) + \beta(x)v \tag{2.13}$$

where $\alpha : M \to \mathbb{R}^m$ and $\beta : M \to \mathbb{R}^{m \times m}$ are analytic mappings with $\beta(x)$ regular (i.e., $\mid \beta(x) \mid \neq 0$) for all $x \in M$ and where $v \in \mathbb{R}^m$ denotes the new controls, such that for the system (2.12,2.13) the outputs are independent of the disturbances. This means that for any initial state x_0, any control $\bar{v}(t)$ $(t \geq 0)$ and any pair of disturbances $\bar{q}_1(t)$,$\bar{q}_2(t)$ $(t \geq 0)$ the outputs of (2.12,2.13) satisfy $y(t, x_0, \bar{v}, \bar{q}_1) = y(t, x_0, \bar{v}, \bar{q}_2)$ $(t \geq 0)$. Formally, this gives the following definition.

Definition 2.2.1 Disturbance decoupling problem via regular static state feedback (DDP)

(i) Given a system (2.12), under what conditions does there exist a regular static state feedback (2.13) such that for the system (2.12,2.13) the outputs are independent of the disturbances?

(ii) We say that the DDP is *locally solvable* if for every point $x_0 \in M$ there exist a neighborhood $U \subset M$ of x_0 and an analytic regular static state feedback (2.13) defined on U, such that for the system (2.12,2.13) defined on U the outputs are independent of the disturbances. ∎

In this section we give, under some regularity assumptions, a local solution of the DDP. Instrumental in this solution is the notion of a (controlled) invariant distribution.

Definition 2.2.2 Invariant and (locally) controlled invariant distribution

(i) An analytic distribution Δ on M is said to be invariant for (2.12) if

$$\begin{aligned} &[f, \Delta] \subset \Delta \\ &[g_j, \Delta] \subset \Delta \quad (j = 1, \cdots, m) \end{aligned} \tag{2.14}$$

(ii) An analytic distribution Δ on M is said to be controlled invariant for (2.12) if there exists an analytic regular static state feedback (2.13) such that Δ is invariant for the feedback modified system (2.12,2.13), i.e.,

$$\begin{aligned} &[f+g\alpha,\Delta]\subset\Delta \\ &[\sum_{i=1}^{m} g_i\beta_{ij},\Delta]\subset\Delta \quad (j=1,\cdots,m) \end{aligned} \tag{2.15}$$

(iii) An analytic distribution Δ on M is said to be locally controlled invariant for (2.12) if for each $x\in M$ there exist a neighborhood $U\in M$ of x and a regular static state feedback (2.13) defined on U such that (2.15) is satisfied on U. ∎

Given a locally controlled invariant distribution Δ, the set of *friends of* Δ, denoted by $\mathcal{F}(\Delta)$, consists of all pairs (α,β) satisfying (2.15). A pair $(\alpha,\beta)\in\mathcal{F}(\Delta)$ is said to *render* Δ *invariant for (2.12)*.

The following lemma gives another very useful characterization of (locally) controlled invariance of involutive distributions:

Lemma 2.2.3 *Consider the analytic nonlinear system (2.12) and assume that the distribution $\mathcal{G}$ has constant dimension. Let Δ be an analytic distribution of constant dimension and assume that $\Delta\cap\mathcal{G}$ has constant dimension. Then the distribution Δ is controlled invariant only if*

$$\begin{aligned} [f,\Delta] &\subset \Delta+\mathcal{G} \\ [g_j,\Delta] &\subset \Delta+\mathcal{G} \quad (j=1,\cdots,m) \end{aligned} \tag{2.16}$$

If moreover Δ is involutive, (2.16) is also a sufficient condition for locally controlled invariance. ∎

Remark 2.2.4 *(i)* Notice that a locally controlled invariant distribution is in general not controlled invariant. The point is that the locally defined feedbacks of Definition 2.2.2.*(iii)* need not patch together into a globally defined analytic feedback which renders the distribution invariant (see [21]).

(ii) Given an involutive locally controlled invariant distribution Δ, a pair $(\alpha,\beta)\in\mathcal{F}(\Delta)$ can be found by solving a set of partial differential equations. This can be seen as follows. By the Frobenius theorem we may assume without loss of generality that locally $\Delta=\text{span}\,\{\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_r}\}$. By (2.15), α has to satisfy

$$[f+g\alpha,\frac{\partial}{\partial x_i}]\in\text{span}\,\{\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_r}\} \quad (i=1,\cdots,r) \tag{2.17}$$

(2.17) is equivalent to the following set of partial differential equations for α

$$\frac{\partial f_k}{\partial x_i}+\sum_{s=1}^{m}(\frac{\partial g_{ks}}{\partial x_i}\alpha_s+g_{ks}\frac{\partial\alpha_s}{\partial x_i})=0 \quad (k=r+1,\cdots,n;i=1,\cdots,r) \tag{2.18}$$

Analogously we can derive partial differential equations for β. ∎

We can check that if a distribution Δ satisfies (2.16), then also its involutive closure $\bar{\Delta}$ satisfies (2.16). Let Δ_1 and Δ_2 be analytic distributions in $\operatorname{Ker} dh$ satisfying (2.16). Then it is easily seen that also the analytic distribution $\Delta_1 + \Delta_2$ satisfies (2.16). Because the zero-distribution is trivially contained in $\operatorname{Ker} dh$ and satisfies (2.16) we have as a result:

Proposition 2.2.5 *There exists a unique involutive distribution in* $\operatorname{Ker} dh$ *that satisfies (2.16) and that contains all distributions in* $\operatorname{Ker} dh$ *satisfying (2.16). This distribution is denoted by* $\Delta^*(f, g; \operatorname{Ker} dh)$ *or, when no confusion arises, by* Δ^**.* ■

Under some regularity assumptions there exists an algorithm to calculate Δ^*. For this, consider the following algorithm.

Algorithm 2.2.6 Define the codistributions P^μ $(\mu = 0, 1, 2, \cdots)$ by

$$\begin{aligned} P^0 &= \operatorname{span}\{dh_1, \cdots, dh_p\} \\ P^{\mu+1} &= P^\mu + \mathcal{L}_f(P^\mu \cap \operatorname{ann} \mathcal{G}) + \sum_{j=1}^{m} \mathcal{L}_{g_j}(P^\mu \cap \operatorname{ann} \mathcal{G}) \quad \mu = 0, 1, 2, \cdots \end{aligned} \tag{2.19}$$

■

Then we have the following result:

Theorem 2.2.7 *If the codistributions* $\operatorname{ann} \mathcal{G}$*,* P^μ *and* $P^\mu \cap \operatorname{ann} \mathcal{G}$ *have constant dimension on* M*, then for some* $k^* \leq \dim(\operatorname{Ker} dh)$*:*

$$P^{k^*} = P^{k^*+1} = \cdots \tag{2.20}$$

and

$$\Delta^* = \operatorname{Ker} P^{k^*} \tag{2.21}$$

■

In the solution of the DDP that will be presented shortly, we have to calculate Δ^* and a pair $(\alpha, \beta) \in \mathcal{F}(\Delta^*)$. If we want to find $(\alpha, \beta) \in \mathcal{F}(\Delta^*)$ using (2.15), a set of partial differential equations has to be solved, as was shown in Remark 2.2.4.*(ii)*. This may become quite tedious. However, we show that it is not necessary to solve a set of partial differential equations. In fact, an effective way of determining the codistributions P^μ $(\mu = 0, 1, 2, \cdots)$ locally around a point $x_0 \in M$ is given, provided the regularity assumptions of Theorem 2.2.7 are satisfied. At the same time a pair $(\alpha, \beta) \in \mathcal{F}(\Delta^*)$ is obtained.

Algorithm 2.2.8 ([65]) **Krener's algorithm**
Consider the analytic nonlinear system (2.12). Let $x_0 \in M$ be given. Assume that the regularity assumptions of Theorem 2.2.7 are satisfied around x_0. Furthermore, assume that the disturbances $q \equiv 0$.

Step 0

Suppose that the dimension of $P^0 = \text{span}\,\{dh_1, \cdots, dh_p\}$ equals p_0 around x_0. Then after a possible permutation of the outputs we may define $\phi_0(x) := \text{col}(h_1(x), \cdots, h_{p_0}(x))$ such that $P^0 = \text{span}\,\{d\phi_0\}$ around x_0.

Step k+1

Assume that in steps $0, \cdots, k$ the functions $\phi_0(x), \cdots, \phi_k(x)$ have been defined such that $P^k = \text{span}\,\{d\phi_k\}$. Then calculate

$$\dot{\phi}_k = \frac{\partial \phi_k}{\partial x}(x)[f(x) + g(x)u] =: A_{k+1}(x) + B_{k+1}(x)u \tag{2.22}$$

Since $P^k \cap \text{ann}\,\mathcal{G}$ has constant dimension around x_0, the matrix $B_{k+1}(x)$ has constant rank, say r_{k+1}, around x_0. After a possible permutation of the entries of ϕ_k we may assume that the first r_{k+1} rows of B_{k+1} are linearly independent around x_0. Accordingly, write (2.22) as:

$$\begin{pmatrix} \dot{\tilde{\phi}}_k \\ \dot{\hat{\phi}}_k \end{pmatrix} = \begin{pmatrix} \tilde{A}_{k+1}(x) + \tilde{B}_{k+1}(x)u \\ \hat{A}_{k+1}(x) + \hat{B}_{k+1}(x)u \end{pmatrix} \tag{2.23}$$

where $\tilde{B}_{k+1}$ has full row rank r_{k+1} around x_0. Let $\tilde{B}^+_{k+1}(x)$ be a right inverse of $\tilde{B}_{k+1}(x)$. Furthermore, let $\bar{B}_{k+1}(x)$ be an $(m, m - r_{k+1})$-matrix of full column rank satisfying $\tilde{B}_{k+1}(x)\bar{B}_{k+1}(x) = 0$ (i.e., the columns of $\bar{B}_{k+1}(x)$ span $\text{Ker}\,\tilde{B}_{k+1}(x)$). Define

$$\alpha_{k+1}(x) = -\tilde{B}^+_{k+1}(x)\tilde{A}_{k+1}(x) \tag{2.24}$$

$$\beta_{k+1}(x) = \begin{pmatrix} \tilde{B}^+_{k+1}(x) & \bar{B}_{k+1}(x) \end{pmatrix} \tag{2.25}$$

Then obviously $\beta_{k+1}(x)$ is regular for all x around x_0. Setting $u = \alpha_{k+1}(x) + \beta_{k+1}(x)v$ we find from (2.23):

$$\begin{pmatrix} \dot{\tilde{\phi}}_k \\ \dot{\hat{\phi}}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{A}_{k+1}(x) - \hat{B}_{k+1}(x)\tilde{B}^+_{k+1}(x)\tilde{A}_{k+1}(x) \end{pmatrix} + \begin{pmatrix} I_{r_{k+1}} & 0 \\ \hat{B}_{k+1}(x)\tilde{B}^+_{k+1}(x) & 0 \end{pmatrix} v =: \tag{2.26}$$

$$\begin{pmatrix} 0 \\ \sigma_{k+1}(x) \end{pmatrix} + \begin{pmatrix} I_{r_{k+1}} & 0 \\ \tau_{k+1}(x) & 0 \end{pmatrix} v$$

Note that, since each row of $\hat{B}_{k+1}(x)$ is linearly dependent on the rows of $\tilde{B}_{k+1}(x)$, $\sigma_{k+1}(x)$ and $\tau_{k+1}(x)$ are independent of the choice of $\tilde{B}^+_{k+1}(x)$ (and $\bar{B}_{k+1}(x)$). Now it can be proved that

$$P^{k+1} = \text{span}\,\{d\phi_k, d\sigma_{k+1}, d\tau_{k+1}\} \tag{2.27}$$

Since by assumption P^{k+1} has fixed dimension, say p_{k+1}, around x_0, we can find a $(p_{k+1} - p_k)$-vector of functions $\bar{\phi}_{k+1}(x)$, consisting of suitably chosen entries of σ_{k+1} and τ_{k+1}, such that around x_0

$$P^{k+1} = \text{span}\,\{d\phi_k, d\bar{\phi}_{k+1}\} \tag{2.28}$$

Finally, define $\phi_{k+1}(x) := (\phi_k^T(x) \;\; \bar{\phi}^T_{k+1}(x))^T$. ■

This algorithm terminates after a finite number, say k^*, of steps. It can be shown that k^* is the same as the k^* defined in Theorem 2.2.7. Moreover, it can be shown that locally $(\alpha_{k^*}, \beta_{k^*}) \in \mathcal{F}(\Delta^*)$.

Now we return to the DDP. We first formulate a result that states under which conditions the disturbances do not influence the outputs for a system (2.12).

Proposition 2.2.9 *(i) Consider the analytic nonlinear system (2.12). The disturbances q do not influence the outputs y if and only if there exists a distribution $\Delta \subset \operatorname{Ker} dh$ on M that is invariant for (2.12) and satisfies*

$$\mathcal{P} \subset \Delta \tag{2.29}$$

(ii) The above condition is on its turn equivalent to the condition that for all $k \geq 0$ and any choice of vector fields $X_1, \cdots, X_k$ in the set $\{f, g_1, \cdots, g_m\}$ we have

$$\mathcal{L}_p \mathcal{L}_{X_1} \cdots \mathcal{L}_{X_k} h(x) = 0 \quad \text{for all } x \tag{2.30}$$

■

Using Proposition 2.2.9, we can prove the following result.

Theorem 2.2.10 *Consider the analytic nonlinear system (2.12). Assume that the distributions Δ^*,$\Delta^* \cap \mathcal{G}$ and $\mathcal{G}$ are constant dimensional. Then the DDP is locally solvable if and only if*

$$\mathcal{P} \subset \Delta^* \tag{2.31}$$

Moreover, if the DDP is locally solvable then any $(\alpha, \beta) \in \mathcal{F}(\Delta^*)$ solves the DDP for (2.12).

■

Remark 2.2.11 Note that if the regularity conditions of Theorem 2.2.7 are satisfied, the distributions $\mathcal{G}$,Δ^* and $\Delta^* \cap \mathcal{G}$ have constant dimension on M. ■

We have now established necessary and sufficient conditions for local solvability of the DDP under some extra regularity assumptions. If we want to solve the DDP for a given system (2.12) satisfying these assumptions we proceed by calculating Δ^* using Algorithm 2.2.8 and then checking if (2.31) holds. If (2.31) indeed holds, a regular static state feedback that locally solves the DDP for (2.12) is produced by Algorithm 2.2.8. The algorithm also gives intermediate checks concerning the solvability of the DDP. Namely, from Algorithm 2.2.6 it is clear that $P^0 \subset P^1 \subset \cdots \subset P^{k^*}$ and thus $\operatorname{Ker} P^0 \supset \operatorname{Ker} P^1 \supset \cdots \supset \Delta^*$. Hence at every step of the algorithm we may check if

$$\mathcal{P} \subset \operatorname{Ker} P^k \tag{2.32}$$

If (2.32) does not hold for some k, then clearly the DDP is not solvable and we may stop the algorithm.

As in the linear case we can also define the disturbance decoupling problem via regular static state feedback and disturbance measurements (DDPdm). Again we assume that we can measure the disturbances entering the system. We use this extra information in our feedback, i.e. we allow for a feedback of the form

$$u = \alpha(x) + \beta(x)v + \gamma(x)q \tag{2.33}$$

where $\alpha : M \to \mathbb{R}^m$, $\beta : M \to \mathbb{R}^{m\times m}$, $\gamma : M \to \mathbb{R}^{m\times r}$ are analytic mappings with $\beta(x)$ regular for all $x \in M$ and where $v \in \mathbb{R}^m$ denotes the new controls. This leads to the following definition:

Definition 2.2.12 Disturbance decoupling problem via regular static state feedback and disturbance measurements (DDPdm)

(i) Given an analytic nonlinear system (2.12), under what conditions does there exist a regular static state feedback with disturbance feedthrough (2.33) such that for the system (2.12,2.33) the outputs are independent of the disturbances?

(ii) We say that the DDP is *locally solvable* if for every point $x_0 \in M$ there exist a neighborhood $U \subset M$ of x_0 and a regular static state feedback with disturbance feedthrough (2.33) defined on U, such that for the system (2.12,2.33) defined on U the outputs are independent of the disturbances. ∎

Analogously to Theorem 2.2.10 we have:

Theorem 2.2.13 *Consider the analytic nonlinear system (2.12). Assume that the distributions Δ^*,$\Delta^* \cap \mathcal{G}$ and $\mathcal{G}$ are constant dimensional. Then the DDPdm is locally solvable if and only if*

$$\mathcal{P} \subset \Delta^* + \mathcal{G} \tag{2.34}$$

∎

The local solutions of the nonlinear DDP and DDPdm that we have established show clear analogies with the solution of the linear DDP and DDPdm we have found in Section 2.1. Indeed, Δ^* for a linear system is just $\mathcal{V}^*$, while the equations (2.16),(2.31),(2.34) form the nonlinear counterparts of equations (2.5),(2.8),(2.11) respectively. However, there are also some differences between the solutions in the linear and the nonlinear case. We mention three important ones.

(i) In the nonlinear case we have only established a local solution under some regularity assumptions (see e.g. Theorem 2.2.10 and Theorem 2.2.13), whereas in the linear case the solution holds globally and we do not have to impose any regularity assumptions.

(ii) In the solution of the linear DDP we do not have to impose any constraint on the matrix G in (2.3), whereas in the nonlinear case we require its nonlinear counterpart, $\beta(x)$ in (2.13), to be regular. This regularity is essential in the "only if" part of the proof of Lemma 2.2.3. The following simple example illustrates that by allowing nonregular static state feedback we may be able to render the outputs independent of the disturbances, while this is not possible via a regular static state feedback.

Example 2.2.14 ([50]) Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 u\\ \dot{x}_2 &= x_3 + q\\ \dot{x}_3 &= x_1 u\\ y_1 &= x_1\\ y_2 &= x_3\end{aligned}\tag{2.35}$$

on the manifold $M = \{x \in I\!R^3 \mid x_1 > 0, x_2 > 0\}$. A simple calculation shows that for this system $\Delta^* = \{0\}$, hence the DDP is not solvable for (2.35). However, we can render the outputs independent of the disturbances by applying the nonregular static state feedback $u = 0$.

■

Note that although we gain something by allowing nonregular static state feedback, also something is lost. For the system (2.35) it is clear that after we have rendered the outputs independent of the disturbance via the feedback $u = 0$, we have no possibility to control the system any more.

(iii) In the linear case the disturbance decoupling problem is solvable via dynamic state feedback if and only if it is solvable via static state feedback (see Remark 2.1.8). This equivalence does not hold any more in the nonlinear case, as is shown in the following example.

Example 2.2.15 ([51]) Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 u_1 & y_1 &= x_1\\ \dot{x}_2 &= x_5 & y_2 &= x_3\\ \dot{x}_3 &= x_2 + x_4 + x_4 u_1\\ \dot{x}_4 &= u_2\\ \dot{x}_5 &= x_1 u_1 + q\end{aligned}\tag{2.36}$$

on the manifold $M = \{x \in I\!R^5 \mid x_2 > 0, x_4 > 0\}$. One straightforwardly shows that for this system $\Delta^* = \{0\}$, hence the DDP is not solvable for (2.36). However, if we apply the dynamic state feedback

$$\begin{aligned}\dot{z} &= v_1\\ u_1 &= z\\ u_2 &= v_2\end{aligned}\tag{2.37}$$

where v_1, v_2 denote the new controls, we find for (2.36,2.37) that the maximal locally controlled invariant distribution contained in $\operatorname{Ker} dh$ is given by

$$\Delta_e^* = \operatorname{span}\{\frac{\partial}{\partial x_5}, x_2(1+z)\frac{\partial}{\partial x_2} + (z_2 - z x_4)\frac{\partial}{\partial x_4} - z(1+z)\frac{\partial}{\partial z}\}\tag{2.38}$$

Since $\mathcal{P} = \operatorname{span}\{\frac{\partial}{\partial x_5}\}$, we see that $\mathcal{P} \subset \Delta_e^*$ and hence the DDP is locally solvable for (2.36,2.37). This means that we can render the outputs independent of the disturbances by applying a well chosen dynamic state feedback to (2.36). ■

This difference leads to the formulation of the disturbance decoupling problem via regular or nonregular dynamic state feedback (DDDP or nDDDP) that wil be studied in Chapters 4 and 6 respectively.

Chapter 3

Strong input-output decoupling for nonlinear systems

In this chapter we consider analytic nonlinear systems of the form

$$\left\{ \begin{array}{lcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array} \right. \tag{3.1}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ are local coordinates for the state space manifold M, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $g(x) = (g_1(x) \cdots g_m(x))$, $h(x) = \mathrm{col}(h_1(x), \cdots, h_p(x))$, $f, g_1, \cdots, g_m$ are analytic vector fields on M, and $h_1, \cdots, h_p$ are analytic functions on M.

Besides disturbance decoupling, another typical design problem for nonlinear (and linear) systems is the problem of achieving input-output decoupling. Roughly speaking, the input-output decoupling problem may be defined as follows. Given a square (i.e., $p = m$) system (3.1), find (if possible) a feedback law for (3.1) such that each of the outputs is controlled by one and only one of the newly defined controls.

In this chapter we treat a strong version of the input-output decoupling problem. In Section 3.1 we first discuss the case of regular static state feedback. In Section 3.2 the case of regular dynamic state feedback is discussed. Here a solution is obtained via the so called Singh's algorithm.

3.1 Strong input-output decoupling via regular static state feedback

In subsection 3.1.1 we formulate and solve the strong input-output decoupling problem via regular static feedback (SIODP). In subsection 3.1.2 we derive a local normal form for strongly input-output decouplable systems.

3.1.1 Solution of the SIODP

When no specific references are given, the results in this subsection can be found in [81],[61].

Consider a square analytic nonlinear system of the form (3.1), that is, $p = m$. Before giving the definition of the (strong) input-output decoupling problem for such a system, we first define what we mean by a (strongly) input-output decoupled system.

Definition 3.1.1 Input-output decoupled system
The square nonlinear system (3.1) is said to be input-output decoupled if, after a possible permutation of the controls, the following properties hold

(i) For each $i \in \{1, \cdots, m\}$ the output y_i is not affected by the controls u_j $(j \neq i)$.

(ii) The output y_i is *affected by the control u_i $(i = 1, \cdots, m)$.* ∎

Using Proposition 2.2.9.*(ii)* we can prove the following results.

Proposition 3.1.2 *Consider the square nonlinear system (3.1). Then requirement (i) of Definition 3.1.1 holds only if for $i, j \in \{1, \cdots, m\}, j \neq i$*

$$\mathcal{L}_{g_j}\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_k} h_i(x) = 0 \quad \forall k \geq 0,\ X_1, \cdots, X_k \in \{f, g_1, \cdots, g_m\},\ x \in M \tag{3.2}$$

∎

Next requirement *(ii)* of Definition 3.1.1 is discussed. We derive sufficient conditions under which the requirement is satisfied. This leads to the definition of a strongly input-output decoupled nonlinear system.

Assume that (3.2) holds. For each $i \in \{1, \cdots, m\}$, define the nonnegative integer $\bar{r}_i$ as the smallest integer such that

$$\begin{cases} \mathcal{L}_{g_i}\mathcal{L}_f^k h_i(x) = 0 & \forall x \in M, k = 0, \cdots, \bar{r}_i - 2 \\ \mathcal{L}_{g_i}\mathcal{L}_f^{\bar{r}_i - 1} h_i(x) \neq 0 & \text{for some } x \in M \end{cases} \tag{3.3}$$

Define the open subset M_0 of M by

$$M_0 = \{x \in M \mid \mathcal{L}_{g_i}\mathcal{L}_f^{\bar{r}_i - 1} h_i(x) \neq 0, i \in \{1, \cdots, m\}\} \tag{3.4}$$

Then the time-derivatives of y_i $(i = 1, \cdots, m)$ satisfy

$$\begin{aligned} y_i^{(k)} &= \mathcal{L}_f^k h_i(x) \quad k = 0, \cdots, \bar{r}_i - 1 \\ y_i^{(\bar{r}_i)} &= \mathcal{L}_f^{\bar{r}_i} h_i(x) + \mathcal{L}_{g_i}\mathcal{L}_f^{\bar{r}_i - 1} h_i(x) u_i \end{aligned} \tag{3.5}$$

Hence if all $\bar{r}_i$ $(i = 1, \cdots, m)$ are finite and M_0 coincides with M, we see from (3.5) that the controls u_i affect the outputs y_i for all $x \in M$. Together with the fact that (3.2) holds, this implies that the system (3.1) is input-output decoupled. This leads to the following definition:

Definition 3.1.3 Strongly input-output decoupled system
The square nonlinear system (3.1) is said to be strongly input-output decoupled if (3.2) holds and if there exist finite nonnegative integers as defined in (3.3) such that the subset M_0 as defined by (3.4) coincides with M. ∎

Motivated by Definition 3.1.3 we define:

Definition 3.1.4 Strong input-output decoupling problem via regular static state feedback (SIODP)

(i) Given a square nonlinear system (3.1), under what conditions does there exist a regular static static state feedback

$$u = \alpha(x) + \beta(x)v \tag{3.6}$$

such that the compensated system (3.1,3.6) with controls $v \in I\!R^m$ is strongly input-output decoupled?

(ii) Given the square nonlinear system (3.1) and a point $x_0 \in M$, we say that the SIODP is locally solvable around x_0 if there exist a neighborhood $U \subset M$ of x_0 and a regular static state feedback (3.6) defined on U such that the compensated system (3.1,3.6) defined on U is strongly input-output decoupled. ■

To arrive at necessary and sufficient conditions for (local) solvability of the SIODP we define for each $i \in \{1, \cdots, m\}$ the nonnegative integer r_i as the smallest integer such that

$$\begin{cases} \left(\mathcal{L}_{g_1}\mathcal{L}_f^k h_i(x) \quad \cdots \quad \mathcal{L}_{g_m}\mathcal{L}_f^k h_i(x) \right) = 0 & \forall x \in M, k = 0, \cdots, r_i - 2 \\ \left(\mathcal{L}_{g_1}\mathcal{L}_f^{r_i-1} h_i(x) \quad \cdots \quad \mathcal{L}_{g_m}\mathcal{L}_f^{r_i-1} h_i(x) \right) \neq 0 & \text{for some } x \in M \end{cases} \tag{3.7}$$

We call $r_1, \cdots, r_m$ the *relative degrees* of the nonlinear system (3.1). If all relative degrees are finite we define the (m,m)-matrix $A(x)$ with entries

$$a_{ij}(x) = \mathcal{L}_{g_j}\mathcal{L}_f^{r_i-1} h_i(x) \quad (i = 1, \cdots, m; j = 1, \cdots, m) \tag{3.8}$$

The matrix $A(x)$ is called the *decoupling matrix* of (3.1) for reasons that will become clear in the following theorem.

Theorem 3.1.5 *Consider the square nonlinear system (3.1).*

(i) The SIODP is solvable for (3.1) if and only if all relative degrees are finite and

$$\operatorname{rank} A(x) = m \quad \text{for all } x \in M \tag{3.9}$$

Moreover, a regular static state feedback (3.6) solving the SIODP for (3.1) is given by

$$\begin{aligned} \alpha(x) &= -A^{-1}(x)b(x) \\ \beta(x) &= A^{-1}(x) \end{aligned} \tag{3.10}$$

where $b(x) = \operatorname{col}(\mathcal{L}_f^{r_1} h_1(x), \cdots, \mathcal{L}_f^{r_m} h_m(x))$.

(ii) Given a point $x_0 \in M$, the SIODP is locally solvable around x_0 if and only if all relative degrees are finite and

$$\operatorname{rank} A(x_0) = m \tag{3.11}$$

Moreover, a regular static state feedback that locally solves the SIODP for (3.1) is given by (3.10) defined on a properly chosen neighborhood $U \subset M$ of x_0. ■

Remark 3.1.6 The results in Theorem 3.1.5 form a nonlinear generalization of the results in [34]. ■

To explain the difference between an input-output decoupled system and a strongly input-output decoupled system, consider the following example from [81]:

Example 3.1.7 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned} \tag{3.12}$$

For this system we find the relative degree $r = 2$ and the decoupling matrix $A(x) = \left(\begin{array}{c} 3x_2^2 \end{array} \right)$. Hence by Theorem 3.1.5 this system is not (globally) strongly input-output decoupled, whereas it is (locally) strongly input-output decoupled around any $x_0 \in \{x \in \mathbb{R}^2 \mid x_2 \neq 0\}$. Furthermore,

$$y^{(4)} = 6u^3 + 12x_2u\dot{u} + 3x_2^2\ddot{u} \tag{3.13}$$

Hence the output is influenced by the control u for every $x \in \mathbb{R}^2$. This implies that the system is globally input-output decoupled. ■

The difference between the two notions appears because the controls affect the outputs via all repeated Lie-derivatives of the form $\mathcal{L}_{g_j}\mathcal{L}_{X_1}\cdots\mathcal{L}_{X_k}h_i(x)$ $(i, j = 1, \cdots, m; k \in \mathbb{N})$, where $X_1, \cdots, X_k \in \{f, g_1, \cdots, g_m\}$, whereas in the case of strong input-output decoupling we only consider repeated Lie-derivatives of the form $\mathcal{L}_{g_j}\mathcal{L}_f^k h_i(x)$ $(i, j = 1, \cdots, m; k \in \mathbb{N})$. The reason for studying the strong input-output decoupling problem rather than the input-output decoupling problem is that we can obtain a simple analytic test for solvability of the SIODP. To obtain necessary and sufficient conditions under which a system can be rendered input-output decoupled via regular static state feedback, we need to take recourse to geometric methods, as introduced in Chapter 2. For this, we refer to [81],[61].

3.1.2 A local normal form for strongly input-output decouplable systems

Theorem 3.1.5 gives a complete (local) solution of the SIODP. We now investigate the structure of the system (3.1) after we have (locally) solved the SIODP via a regular static state feedback (3.10). We restrict ourselves to the case that we can solve the SIODP globally. In case of local solvability around a point $x_0 \in M$ all results are the same when restricted to a properly chosen neighborhood $U \subset M$ of x_0. When no specific references are made, the results in this subsection can be found in [59] (see also [69]).

Lemma 3.1.8 *Consider the square nonlinear system (3.1). If all relative degrees are finite and (3.9) holds, the row vectors $\{d\mathcal{L}_f^k h_i(x) \mid i = 1, \cdots, m; k = 0, \cdots, r_i - 1\}$ are linearly independent for each $x \in M$.*

Proof See [61],[81]. ∎

Assume that the SIODP is solvable for (3.1). Define

$$\xi_{ij}(x) := \mathcal{L}_f^{j-1} h_i(x) \quad (i = 1, \cdots, m; j = 1, \cdots, r_i) \tag{3.14}$$

and let $\xi_i := \mathrm{col}(\xi_{i1}, \cdots, \xi_{ir_i})$, $\xi := (\xi_1^T \ \cdots \ \xi_m^T)^T$. Then the result of Lemma 3.1.8 implies that we can find an additional vector of functions $\xi_{m+1}(x) = \mathrm{col}(\xi_{m+1\,1}(x), \cdots, \xi_{m+1\,d}(x))$, that is analytic in x and where $d = n - \sum_{i=1}^m r_i$, such that $\{\{\xi_{ij} \mid 1 \le i \le m, 1 \le j \le r_i\}, \{\xi_{m+1\,j} \mid 1 \le j \le d\}\}$ forms a new set of local coordinates for M. Denote $\Phi(x) := (\xi^T \ \xi_{m+1}^T)^T(x)$.

Let us see what form the system (3.1,3.10) takes in these new coordinates. Obviously,

$$\dot{\xi}_{ij} = \xi_{ij+1} \quad (i = 1, \cdots, m; j = 1, \cdots, r_i - 1) \tag{3.15}$$

Furthermore, by the definition of the relative degrees and the decoupling matrix we have for $i = 1, \cdots, m$:

$$\dot{\xi}_{ir_i} = \mathcal{L}_f^{r_i} h_i(x) + A_{i*}(x) u = v_i \tag{3.16}$$

where $A_{i*}(x)$ denotes the i-th row of $A(x)$. Hence in the new coordinates the system (3.1,3.10) takes the form

$$\begin{aligned} \dot{\xi}_i &= A_i \xi_i + B_i v_i \quad (i = 1, \cdots, m) \\ \dot{\xi}_{m+1} &= \bar{f}(\xi, \xi_{m+1}) + \bar{g}(\xi, \xi_{m+1}) v \\ y_i &= \xi_{i1} \quad (i = 1, \cdots, m) \end{aligned} \tag{3.17}$$

where $\bar{f}(\xi, \xi_{m+1}) = \mathcal{L}_f \xi_{m+1} \circ \Phi^{-1}(\xi, \xi_{m+1}), \bar{g}(\xi, \xi_{m+1}) = (\mathcal{L}_{g_1} \xi_{m+1} \cdots \mathcal{L}_{g_m} \xi_{m+1}) \circ \Phi^{-1}(\xi, \xi_{m+1})$ and (A_i, B_i) are in Brunovsky canonical form.

Thus in the new coordinates the system (3.1,3.10) consists of m independent linear subsystems and a nonlinear subsystem whose dynamics are driven by the dynamics of the linear subsystems. Using the following lemma, we can give an interpretation of the coordinates ξ_{m+1} of the nonlinear subsystem.

Lemma 3.1.9 *Consider the square nonlinear system (3.1). If all relative degrees are finite and (3.9) holds, we have*

$$\Delta^* = \bigcap_{i=1}^{m} \bigcap_{k=0}^{r_i - 1} \mathrm{Ker}\, d\mathcal{L}_f^k h_i = \bigcap_{i=1}^{m} \bigcap_{k=1}^{r_i} \mathrm{Ker}\, d\xi_{ik} \tag{3.18}$$

Proof See [61],[81]. ∎

The result of Lemma 3.1.9 implies that we can choose ξ_{m+1} in such a way that in the new local coordinates we have $\Delta^* = \mathrm{span}\,\{\partial/\partial\xi_{m+1}\}$. An easy calculation shows that Δ^* is invariant for the dynamics (3.17). Hence the static state feedback (3.10) that solves the SIODP for (3.1) at the same time renders Δ^* invariant. This implies that for a system (2.12) for which the SIODP (with $q \equiv 0$) and the DDP are solvable, we can simultaneously solve the SIODP and the DDP via the regular static state feedback (3.10).

As we can see from (3.17), the dynamics of the nonlinear subsystem do not affect the outputs. It is natural to question whether we can choose ξ_{m+1} in such a way that these dynamics are not directly influenced by the controls, i.e., $\bar{g}_j(\xi_1, \cdots, \xi_{m+1}) = 0$ for $j = 1, \cdots, m$. Because $\beta(x)$ is assumed to be regular, this is the case if and only if ξ_{m+1} can be chosen in such a way that in the original system (before feedback) the dynamics of ξ_{m+1} are not influenced by the old controls, i.e.,

$$\mathcal{L}_{g_j}\xi_{m+1\,k} = 0 \text{ for all } j \in \{1, \cdots, m\}, k \in \{1, \cdots, d\} \tag{3.19}$$

or, equivalently, $\xi_{m+1\,k} \in \operatorname{ann}\mathcal{G}$ for all $k \in \{1, \cdots, d\}$. Define the codistribution F by:

$$F := \operatorname{span}\{d\mathcal{L}_f^k h_i \mid 1 \le i \le m, 0 \le k \le r_i - 1\} \tag{3.20}$$

Since ξ_{m+1} has to be such that $(\xi_1, \cdots, \xi_{m+1})$ form a local coordinate system, the following conditions on ξ_{m+1} have to be satisfied:

$$d\xi_{m+1\,k} \notin F \quad (k \in \{1, \cdots, d\}) \tag{3.21}$$

$$\dim \operatorname{span}\{d\xi_{m+1\,1}, \cdots, d\xi_{m+1\,d}\} = d \tag{3.22}$$

We first prove a technical lemma.

Lemma 3.1.10 *Consider the square nonlinear system (3.1). If all relative degrees are finite and if (3.9) holds, then*

$$\operatorname{ann}\mathcal{G} \cap F = \operatorname{span}\{d\mathcal{L}_f^k h_i \mid 1 \le i \le m, 0 \le k \le r_i - 2\}$$

Proof From the fact that $\mathcal{L}_{g_j}\mathcal{L}_f^k h_i = 0$ $(1 \le j \le m, 0 \le k \le r_i - 2)$, it follows that $\operatorname{span}\{d\mathcal{L}_f^k h_i \mid 1 \le i \le m, 0 \le k \le r_i - 2\} \subset \operatorname{ann}\mathcal{G} \cap F$. Assume that there exist functions $\alpha_1(x), \cdots, \alpha_m(x)$ that are not all equal to zero, such that $\sum_{i=1}^m \alpha_i d\mathcal{L}_f^{r_i-1}h_i \in \operatorname{ann}\mathcal{G}$. This implies that for all $j = 1, \cdots, m$ we have $\sum_{i=1}^m \alpha_i \mathcal{L}_{g_j}\mathcal{L}_f^{r_i-1}h_i = 0$, which means that $\operatorname{rank}A(x) < m$ for all $x \in M$, which gives a contradiction. ∎

Now we reach the final result.

Theorem 3.1.11 *Consider the square nonlinear system (3.1). Assume that all its relative degrees are finite and that (3.9) holds. Then there are analytic functions $\xi_{m+11}, \cdots, \xi_{m+1d} : M \to I\!R$, with $d = n - \sum_{i=1}^m r_i$, that satisfy (3.19,3.21,3.22) if and only if $\mathcal{G}$ is involutive.*

Proof *(necessity)* Assume that $\xi_{m+11}, \cdots, \xi_{m+1d}$ satisfying (3.19,3.21,3.22) exist. Consider $H := \operatorname{span}\{\{d\mathcal{L}_f^k h_i \mid 1 \le i \le m, 0 \le k \le r_i - 2\}, d\xi_{m+11}, \cdots, d\xi_{m+1d}\}$. From (3.19) and Lemma 3.1.9 it follows that $H \subset \operatorname{ann}\mathcal{G}$. Moreover, from Lemma 3.1.9 and (3.21,3.22) it follows that $\dim H = \sum_{i=1}^m (r_i - 1) + d = \sum_{i=1}^m (r_i - 1) + n - \sum_{i=1}^m r_i = n - m = \dim \operatorname{ann}\mathcal{G}$. Hence $H = \operatorname{ann}\mathcal{G}$. So $\operatorname{ann}\mathcal{G}$ is spanned by exact one forms and this implies that $\mathcal{G}$ is involutive.
(sufficiency) Involutivity of $\mathcal{G}$ implies that there are functions $z_1, \cdots, z_{n-m}$ such that $\operatorname{ann}\mathcal{G} = \operatorname{span}\{dz_1, \cdots, dz_{n-m}\}$. By Lemma's 3.1.8 and 3.1.10 we can take $(z_{d+1}, \cdots, z_{n-m}) = (\mathcal{L}_f^k h_i \mid 1 \le i \le m, 0 \le k \le r_i - 2)$. Then the functions $z_1, \cdots, z_d$ satisfy (3.19,3.21,3.22). Hence, taking $\xi_{m+1k} = z_k$ $(k = 1, \cdots, d)$, we establish our claim. ∎

We see that if $\mathcal{G}$ is involutive, then the system (3.1,3.10) has the structure shown in Figure 3.1 in local coordinates $\xi_1, \cdots, \xi_{m+1}$ as before.

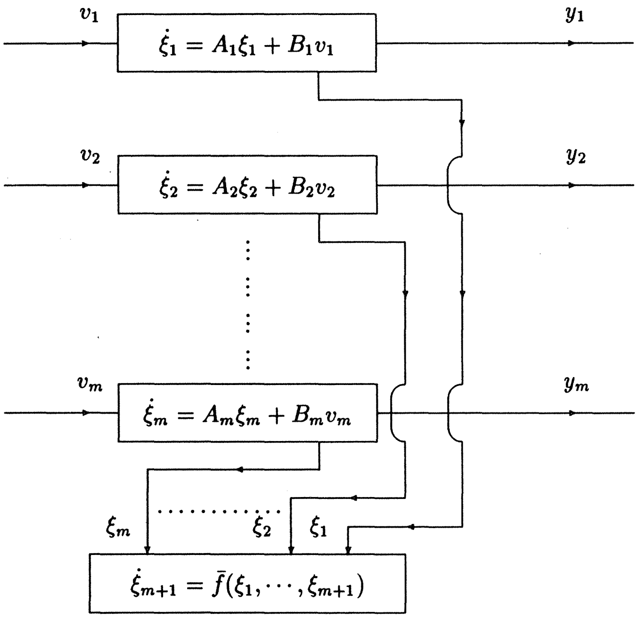

Figure 3.1: Local normal form for strongly input-output decouplable system

3.2 Strong input-output decoupling via regular dynamic state feedback

In Section 3.1 we have seen that a nonlinear system (3.1) can be strongly input-output decoupled via regular static state feedback if and only if it has finite relative degrees and its decoupling matrix has full rank for all $x \in M$. Also we have seen that if $\operatorname{rank} A(x_0) < m$ for some $x_0 \in M$, we may still be able to solve the SIODP locally around points x for which $\operatorname{rank} A(x) = m$. Another situation arises when $\operatorname{rank} A(x) < m$ for all $x \in M$. The following example illustrates this situation.

Example 3.2.1 Consider the following nonlinear system on $I\!R^5$:

$$\begin{array}{rcll} \dot{x}_1 & = & e^{x_2}u_1 & y_1 = x_1 \\ \dot{x}_2 & = & x_5 & y_2 = x_3 \\ \dot{x}_3 & = & x_2 + e^{x_4}u_1 & \\ \dot{x}_4 & = & u_2 & \\ \dot{x}_5 & = & e^{x_1}u_1 & \end{array} \tag{3.23}$$

For this system $r_1 = r_2 = 1$ and

$$A(x) = \begin{pmatrix} e^{x_2} & 0 \\ e^{x_4} & 0 \end{pmatrix} \tag{3.24}$$

Hence $\operatorname{rank} A(x) = 1 < 2 = m$ for all $x \in I\!R^5$ and so the SIODP cannot be solved. Consider the dynamic state feedback

$$\begin{array}{rcl} \dot{z} & = & \tilde{u}_1 \\ u_1 & = & e^z \\ u_2 & = & \tilde{u}_2 \end{array} \tag{3.25}$$

where $\tilde{u}_1, \tilde{u}_2$ denote the new controls. The relative degrees r_1^e, r_2^e of (3.23,3.25) are $r_1^e = r_2^e = 2$ and the decoupling matrix $A^e(x,z)$ of (3.23,3.25) is given by

$$A^e(x,z) = \begin{pmatrix} e^{(x_2+z)} & 0 \\ e^{(x_4+z)} & e^{(x_4+z)} \end{pmatrix} \tag{3.26}$$

Hence $\operatorname{rank} A^e(x,z) = 2 = m$ for all $(x,z) \in I\!R^5 \times I\!R$, which implies that we can solve the SIODP for (3.23,3.25). So although we cannot achieve strong input-output decoupling for (3.23) via static state feedback, we can achieve it via a well chosen dynamic state feedback.

■

The above example motivates the formulation of the strong input-output decoupling problem via regular dynamic state feedback (SDIODP). We study this problem in this section.

Before being able to formulate the SDIODP we of course have to define what we mean by a regular dynamic state feedback. One way of formulating this definition is via an algebraic theory for nonlinear control systems. A convenient way of performing the necessary calculations required in this algebraic theory is offered by Singh's algorithm. We introduce these tools in Subsection 3.2.1. The exposition is based on [30] (see also [35],[36]). In Subsection 3.2.2 we define what we mean by a regular dynamic state feedback, using the tools introduced in Subsection 3.2.1. Singh's algorithm applied to a nonlinear control system (3.1) provides us with a special sort of regular dynamic state feedback, which we call a Singh compensator. This compensator is also introduced in Subsection 3.2.2. In Subsection 3.2.3 we formulate and solve the SDIODP which was studied in e.g. [89],[91],[24],[71],[78],[30]. It will turn out that we can (locally) solve the SDIODP if and only if we can (locally) solve it via a Singh compensator. We also show that a Singh compensator is in fact a regular dynamic state feedback of minimal order solving the SDIODP. This means that the dimension of the state space of any regular dynamic state feedback that solves the SDIODP is greater than or equal to the dimension of the state space of a Singh compensator.

3.2.1 Algebraic tools

We first introduce an algebraic theory for nonlinear control systems. The material in this subsection is taken from [30].

Consider a (not necessarily square) analytic nonlinear system (3.1). Recall that a *meromorphic* function η is a function of the form $\eta = \pi/\theta$, where π and θ are analytic functions. Assume that the control functions $u(t)$ are n times continuously differentiable. Then define $u^{(0)} := u(t)$, $u^{(i+1)} := (d/dt)\,u^{(i)}(t)$. View $x, u, \cdots, u^{(n-1)}$ as variables and let $\mathcal{K}$ denote the field consisting of the set of rational functions of $(u, \cdots, u^{(n-1)})$ with coefficients that are meromorphic in x. Recall that given such a field, say in variables $\nu = (\nu_1, \cdots, \nu_j)$, we define $\partial/\partial\nu_i$ acting on a meromorphic function $\eta(\nu) = \pi(\nu)/\theta(\nu)$, where $\pi(\cdot)$ and $\theta(\cdot)$ are analytic, by the usual quotient rule of calculus,

$$\frac{\partial}{\partial \nu_i}\left(\frac{\pi(\nu)}{\theta(\nu)}\right) := \frac{\theta(\nu)\frac{\partial}{\partial \nu_i}\pi(\nu) - \pi(\nu)\frac{\partial}{\partial \nu_i}\theta(\nu)}{\theta^2(\nu)} \tag{3.27}$$

Then the differential of η is defined by

$$d\eta(\nu) := \sum_{i=1}^{j} \frac{\partial \eta(\nu)}{\partial \nu_i} d\nu_i \tag{3.28}$$

Returning to the system (3.1), we define in a natural way

$$\begin{aligned} \dot{y} &= \dot{y}(x,u) = \frac{\partial h}{\partial x}[f(x) + g(x)u] \\ y^{(k+1)} &= y^{(k+1)}(x, u, \cdots, u^{(k)}) \\ &= \frac{\partial y^{(k)}}{\partial x}[f(x) + g(x)u] + \sum_{i=0}^{k-1} \frac{\partial y^{(k)}}{\partial u^{(i)}} u^{(i+1)} \end{aligned} \tag{3.29}$$

Note that $\dot{y}, \cdots, y^{(n)}$ so defined have components in the field $\mathcal{K}$.

Let $\mathcal{E}$ denote the vector space (over $\mathcal{K}$) spanned by $\{dx, du, \cdots, du^{(n-1)}\}$. We introduce the chain of subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n$ of $\mathcal{E}$ by

$$\mathcal{E}_k := \operatorname{span}_{\mathcal{K}}\{dx, d\dot{y}, \cdots, dy^{(k)}\} \tag{3.30}$$

Note that in (3.30) the span is taken with respect to the field $\mathcal{K}$ and *not* with respect to the real numbers. Note also that we abuse notation slightly, because $\mathcal{E}_0 := \operatorname{span}\{dx, dy\}$, which is easily seen to be equal to span $\{dx\}$, since the outputs y only depend on x.

Definition 3.2.2 Rank of a nonlinear system
Consider the nonlinear system (3.1). Then the number

$$\rho^* := \dim_{\mathcal{K}} \mathcal{E}_n - \dim_{\mathcal{K}} \mathcal{E}_{n-1} \tag{3.31}$$

is called the rank of (3.1). ∎

It can be checked that $\rho^* \leq \min(m,p)$. The system (3.1) is said to be of ***full rank*** if $\rho^* = \min(m,p)$.

A convenient way for performing the necessary calculations required in the algebraic theory is provided by Singh's algorithm. Singh's algorithm was introduced in [91] for calculation of the left inverse of a nonlinear system. It is a generalization of the algorithm from [44], which was only applicable under some restrictive assumptions. The version of Singh's algorithm presented is from [30].

Algorithm 3.2.3 Singh's algorithm
Consider the analytic nonlinear control system (3.1).

Step 1

Calculate

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] =: a_1(x) + b_1(x)u \tag{3.32}$$

and define

$$\rho_1 = s_1 := \operatorname{rank} b_1(x) \tag{3.33}$$

where the rank is taken over the field of meromorphic functions of x. Permute, if necessary, the components of the output so that the first ρ_1 rows of $b_1(x)$ are linearly independent. Decompose y according to

$$\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix} \tag{3.34}$$

where $\dot{\tilde{y}}_1$ consists of the first ρ_1 rows of $\dot{y}$. Since the last rows of $b_1(x)$ are linearly dependent on the first ρ_1 rows, we can write

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u \\ \dot{\hat{y}}_1 &= \dot{\hat{y}}_1(x, \dot{\tilde{y}}_1) \end{aligned} \tag{3.35}$$

where the last equation is affine in $\dot{\tilde{y}}_1$. Finally, set $\tilde{B}_1(x) := \tilde{b}_1(x)$.

Step k+1

Suppose that in Steps 1 through k, $\dot{\tilde{y}}_1, \cdots, \tilde{y}_k^{(k)}, \hat{y}_k^{(k)}$ have been defined so that

$$\begin{aligned}
\dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u \\
&\vdots \\
\tilde{y}_k^{(k)} &= \tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k\}) \\
&\quad + \tilde{b}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1\})u \\
\hat{y}_k^{(k)} &= \hat{y}_k^{(k)}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\})
\end{aligned} \tag{3.36}$$

and so that they are rational functions of $\tilde{y}_i^{(j)}$ with coefficients in the field of meromorphic functions of x. Suppose also that the matrix $\tilde{B}_k := [\tilde{b}_1^T, \cdots, \tilde{b}_k^T]^T$ has full rank equal to ρ_k, where the rank is taken with respect to the field of rational functions of $\{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1\}$ with coefficients in the field of meromorphic functions of x. Then calculate

$$\hat{y}_k^{(k+1)} = \frac{\partial}{\partial x}\hat{y}_k^{(k)}[f(x) + g(x)u] + \sum_{i=1}^{k}\sum_{j=i}^{k}\frac{\partial \hat{y}_k^{(k)}}{\partial \tilde{y}_i^{(j)}}\tilde{y}_i^{(j+1)} \tag{3.37}$$

and write it as

$$\begin{aligned}
\hat{y}_k^{(k+1)} &= a_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k+1\}) \\
&\quad + b_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\})u
\end{aligned} \tag{3.38}$$

Define $B_{k+1} := [\tilde{B}_k^T, b_{k+1}^T]^T$, and $\rho_{k+1} := \operatorname{rank} B_{k+1}$ where the rank is taken with respect to the field of rational functions of $\{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k\}$ with coefficients in the field of meromorphic functions of x. Permute, if necessary, the components of $\hat{y}_k^{(k+1)}$ so that the first ρ_{k+1} rows of B_{k+1} are linearly independent. Decompose $\hat{y}_k^{(k+1)}$ as

$$\hat{y}_k^{(k+1)} = \begin{pmatrix} \tilde{y}_{k+1}^{(k+1)} \\ \\ \hat{y}_{k+1}^{(k+1)} \end{pmatrix} \tag{3.39}$$

where $\tilde{y}_{k+1}^{(k+1)}$ consists of the first $s_{k+1} := (\rho_{k+1} - \rho_k)$ rows. Since the last rows of B_{k+1} are linearly dependent on the first ρ_{k+1} rows, we can write

$$\begin{aligned}
\dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u \\
&\vdots \\
\tilde{y}_{k+1}^{(k+1)} &= \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k+1\}) \\
&\quad + \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\})u \\
\hat{y}_{k+1}^{(k+1)} &= \hat{y}_{k+1}^{(k+1)}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k+1, i \le j \le k+1\})
\end{aligned} \tag{3.40}$$

where once again everything is rational in $\tilde{y}_i^{(j)}$. Finally, set $\tilde{B}_{k+1} := [\tilde{B}_k^T, \tilde{b}_{k+1}^T]^T$. ■

We now have the following result (cf. [30]).

Theorem 3.2.4 *Consider the nonlinear system (3.1) and apply Singh's algorithm to it. Then for each* $1 \leq k \leq n$

(i) $\{dx, \{d\tilde{y}_i^{(j)} \mid 1 \leq i \leq k, i \leq j \leq k\}\}$ *is a basis for* $\mathcal{E}_k$.

(ii) $\dim_{\mathcal{K}} \mathcal{E}_k = n + \rho_1 + \cdots + \rho_k$ ∎

Corollary 3.2.5 *Consider the nonlinear system (3.1) and apply Singh's algorithm to it. Then* $\rho^* = \rho_n$. ∎

The rank of a nonlinear system can be given the following interpretation.

Proposition 3.2.6 *Consider the nonlinear system (3.1). Then the outputs of (3.1) satisfy* $(p-\rho^*)$ *independent differential equations. More specifically, it is possible to find an integer* $N \in \{1, \cdots, n\}$ *and a* $(p-\rho^*)$*-vector of functions* $\phi(\hat{y}_n, \cdots, \hat{y}_n^{(N-1)}, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq n, i \leq j \leq N\})$ *such that*

$$\hat{y}_n^{(N)} = \phi(\hat{y}_n, \cdots, \hat{y}_n^{(N-1)}, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq n, i \leq j \leq N\}) \tag{3.41}$$

Proof See Lemma 1 of [71] and Lemma 2.8 of [30]. ∎

Remark 3.2.7 The above proposition gives an analytic version of a result by Fliess ([35]) which holds in the differential algebraic setting. ∎

3.2.2 Regular dynamic state feedback and Singh compensator

When no specific references are given, the results in this subsection can be found in a series of papers [51],[52],[53],[54].

Using the theory developed in Subsection 3.2.1, we are able to define what we mean by a regular dynamic state feedback.

Definition 3.2.8 Rank of dynamic state feedback
Consider the nonlinear system (3.1), together with a dynamic state feedback

$$\begin{cases} \dot{z} &= \alpha(x,z) + \beta(x,z)v \\ u &= \gamma(x,z) + \delta(x,z)v \end{cases} \tag{3.42}$$

with $z \in I\!R^\nu$, $v \in I\!R^m$ denoting the new controls and $\alpha, \beta, \gamma, \delta$ analytic functions of x and z. Then the rank of (3.42) is defined as the rank of the system

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ \dot{z} &= \alpha(x,z) + \beta(x,z)v \\ u &= \gamma(x,z) + \delta(x,z)v \end{cases} \tag{3.43}$$

with controls v and outputs u. ∎

Definition 3.2.9 Regular dynamic state feedback
A dynamic state feedback (3.42) is said to be regular if it has full rank, i.e., the system (3.43) with controls v and outputs u has rank equal to m. ∎

Remark 3.2.10 The above definition of regular dynamic state feedback can be found in [72],[30]. ∎

To motivate Definition 3.2.9, we show that a regular dynamic state feedback has two essential properties that also a regular static state feedback has. These properties could be called an *input reproducibility property* and a *rank preserving property.*

We start with the input reproducibility property. Input reproducibility for nonlinear systems is usually called left-invertibility in the literature. Roughly speaking, the problem of left-invertibility is stated as follows. Given a nonlinear system (3.1) and an initial state x_0 can we, based on the knowledge of x_0 and the outputs $y(t) = y(t, x_0, u)$ on a (small) time-interval $[0, T)$ $(T > 0)$ reproduce the input $u(t)$ on this interval? For an overview of the theory of left- (and right-) invertibility of nonlinear systems we refer to [84],[82].

Consider the nonlinear system (3.43) where the feedback is a regular static state feedback, that is, we have $u = \alpha(x) + \beta(x)v$ where $\beta(x)$ is regular for all $x \in M$. Consider the system

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & f(\xi) + g(\xi)u \\ w & = & \beta^{-1}(\xi)(u - \alpha(\xi)) \end{array} \right. \tag{3.44}$$

Then it is easily verified that if we take $\xi(0) = x(0)$, we have that $w(t) = v(t)$ $(t \geq 0)$. Hence for the case of regular *static* state feedback the system (3.43) is left-invertible. The same result holds for regular *dynamic* state feedback, as follows from the following proposition (see e.g. [84],[82]).

Proposition 3.2.11 *Consider a square analytic nonlinear system (3.1). Then the system has full rank if and only if there exists an analytic subset $L \subset M \times \mathbb{R}^{mn}$ satisfying* int $L = \emptyset$, *such that for every $(x_0, u(0), \cdots, u^{(n-1)}(0)) \notin L$ there exists an open set $U \subset M \times \mathbb{R}^{mn}$ and an analytic system of the form*

$$\begin{array}{rcl} \dot{\xi} & = & a(\xi, y, \cdots, y^{(n)}) \\ w & = & b(\xi, y, \cdots, y^{(n)}) \end{array} \tag{3.45}$$

with $\xi \in \mathbb{R}^{\mu}$, and a point $\xi_0 \in \mathbb{R}^{\mu}$ with the property that $w(t) = u(t)$ $(t \in [0, T), T > 0)$ if we set $\xi(0) = \xi_0$. ∎

Remark 3.2.12 A system (3.45) that reproduces the inputs of a system (3.1) is called a left-inverse of (3.1). This left-inverse can be obtained via Singh's algorithm (cf. [64]). ∎

As for the rank preserving property, it is easily checked that the system (3.1,3.44) has the same rank as the system (3.1) (cf. [30]). A regular dynamic state feedback also has this property, as follows from the following proposition from [30].

Proposition 3.2.13 *Consider a nonlinear system (3.1) and a regular dynamic state feedback (3.42). Then the composed system (3.1,3.42) has the same rank as the nonlinear system (3.1).* ∎

Note that in general the converse of Proposition 3.2.13 does not hold. A simple counter example is provided by a system of rank zero. For such a system any dynamic state feedback (regular or not) will result in a composed system of rank zero.

Singh's algorithm applied to a (not necessarily square) nonlinear system (3.1) provides us with a special sort of regular dynamic state feedback, that we call a Singh compensator. For this we have to associate a notion of regularity with Singh's algorithm in the following way.

Definition 3.2.14 (Strongly) regular point
Consider the nonlinear system (3.1) and let a point $x_0 \in M$ be given.

(i) We call x_0 a regular point for (3.1) if for an appropriate application of Singh's algorithm to (3.1) there exists a point $(\tilde{y}_{i0}^{(j)} \mid 1 \le i \le n-1, i \le j \le n-1)$ such that

$$\mathrm{rank}_{\mathbb{R}} \tilde{B}_k(x_0, \{\tilde{y}_{i0}^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1\}) = \rho_k \quad (k = 1, \cdots, n) \tag{3.46}$$

(ii) We call x_0 a strongly regular point for (3.1) if for each application of Singh's algorithm to (3.1) there exists a point $(\tilde{y}_{i0}^{(j)} \mid 1 \le i \le n-1, i \le j \le n-1)$ such that (3.46) holds.

∎

The Singh compensator is obtained as follows (see also [91]). Consider the nonlinear system (3.1) and let $x_0 \in M$ be a strongly regular point for (3.1). Apply Singh's algorithm to (3.1). This yields at the n-th step:

$$\begin{aligned} \dot{\tilde{Y}}_n &= \tilde{A}_n(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-1, i \le j \le n\}) + \\ &\quad \tilde{B}_n(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-1, i \le j \le n-1\})u \\ \hat{y}_n^{(n)} &= \hat{y}_n^{(n)}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n, i \le j \le n\}) \end{aligned} \tag{3.47}$$

where $\tilde{Y}_n = (\tilde{y}_1^T \cdots \tilde{y}_n^{(n-1)^T})^T$ and

$$\tilde{A}_n(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-1, i \le j \le n\}) = \begin{pmatrix} \tilde{a}_1(x) \\ \tilde{a}_2(x, \dot{\tilde{y}}_1, \tilde{y}_1^{(2)}) \\ \vdots \\ \tilde{a}_n(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-1, i \le j \le n\}) \end{pmatrix}$$

Moreover we know that there exist a neighborhood $U \subset M$ of x_0, a point $(\tilde{y}_{i0}^{(j)} \mid 1 \le i \le n-1, i \le j \le n)$ and a neighborhood $\mathcal{Y}_0$ of this point such that $\tilde{B}_n$ has full row rank ρ^* on $U \times \mathcal{Y}_0$. After a possible permutation of the controls we may assume that the matrix

$\tilde{B}_{n1}$ consisting of the first ρ^* columns of $\tilde{B}_n$ is invertible on this neighborhood. Let $\tilde{B}_{n2}$ be such that $\tilde{B}_n = (\tilde{B}_{n1} \;\; \tilde{B}_{n2})$ on this neighborhood and define $u^1 = \mathrm{col}(u_1, \cdots, u_{\rho^*})$, $u^2 = \mathrm{col}(u_{\rho^*+1}, \cdots, u_m)$. Then on $U \times \mathcal{Y}_0$ (3.47) yields in particular:

$$u^1 = \tilde{B}_{n1}^{-1}[\dot{\tilde{Y}}_n - \tilde{A}_n] - \tilde{B}_{n1}^{-1}\tilde{B}_{n2}u^2 \tag{3.48}$$

For $i = 1, \cdots, \rho^*$, let γ_i be the lowest time-derivative and δ_i be the highest time-derivative of y_i appearing in (3.48). Then we can rewrite (3.48) as

$$\begin{aligned} u^1 = \;& \phi_1(x, \{y_i^{(j)} \mid 1 \le i \le \rho^*, \gamma_i \le j \le \delta_i - 1\}) \\ & + \sum_{i=1}^{\rho^*} \phi_{2i}(x, \{\, y_i^{(j)} \mid 1 \le i \le \rho^*, \gamma_i \le j \le \delta_i - 1\}) y_i^{(\delta_i)} \\ & + \sum_{i=\rho^*+1}^{m} \phi_{2i}(x, \{y_i^{(j)} \mid 1 \le i \le \rho^*, \gamma_i \le j \le \delta_i - 1\}) u_i \end{aligned} \tag{3.49}$$

for certain vectors of functions ϕ_1, ϕ_{2i} $(i = 1, \cdots, m)$.

Let z_i $(i = 1, \cdots, \rho^*)$ be a vector of dimension $\delta_i - \gamma_i$ and consider the system:

$$\begin{cases} \dot{z}_i &= A_i z_i + B_i v_i \quad (i = 1, \cdots, \rho^*) \\ u^1 &= \phi_1(x, z_1, \cdots, z_{\rho^*}) + \sum_{i=1}^{m} \phi_{2i}(x, z_1, \cdots, z_{\rho^*}) v_i \\ u_i &= v_i \quad (i = \rho^* + 1, \cdots, m) \end{cases} \tag{3.50}$$

where (A_i, B_i) $(i = 1, \cdots, \rho^*)$ are in Brunovsky canonical form. Then (3.50) is called a *Singh compensator* for (3.1) around x_0.

Remark 3.2.15 Note that the Singh compensator (3.50) in general will be a meromorphic feedback on an open subset $U \times \mathcal{Z} \subset M \times \mathbb{R}^\sigma$. However, we can always find a neighborhood $\tilde{U} \subset U$ of x_0 and a subset $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that (3.50) defined on $\tilde{U} \times \tilde{\mathcal{Z}}$ is analytic. ∎

Proposition 3.2.16 *Consider a nonlinear system (3.1) and let x_0 be a strongly regular point for (3.1). Then any Singh compensator for (3.1) around x_0 is a regular dynamic state feedback for (3.1).*

Proof See Appendix A. ∎

Let a strongly regular point $x_0 \in M$ for (3.1) be given. Then it is obvious that every different application of Singh's algorithm to (3.1) yields a different Singh compensator around x_0. The question arises if there exists a connection between these different Singh compensators. This is indeed the case. First we show that any Singh compensator has the same (intrinsically defined) dimension.

Assume that Singh's algorithm is applied to (3.1) in one specific way. Denote the dimension of the Singh compensator for (3.1) around x_0 obtained in this way by σ. Then obviously

$\sigma = \sum_{i=1}^{\rho^*}(\delta_i - \gamma_i)$. Next note that γ_i is the smallest $k \in \mathbb{N}$ for which y_i is an entry of $\tilde{y}_k$. Inspection of Singh's algorithm gives that the set $\{y_i \mid \gamma_i = k\}$ has s_k elements. Therefore,

$$\sum_{i=1}^{\rho^*} \gamma_i = \sum_{k=1}^{n} k s_k \tag{3.51}$$

Given the fact that $s_1 = \rho_1$ and $s_k = \rho_k - \rho_{k-1}$ $(k = 2, \cdots, n)$, $s_1, \cdots, s_n$ are defined intrinsically (i.e. independent of Singh's algorithm, see Theorem 3.2.4). Hence from (3.51) it follows that $\sum_{i=1}^{\rho^*} \gamma_i$ is independent of the specific way we apply Singh's algorithm. Next we show that the δ_i $(i = 1, \cdots, \rho^*)$ are also intrinsically defined. For this we need the following definition:

Definition 3.2.17 ([19]) **Essential vector**
Let V be a given vector space over a field $\mathcal{F}$. Let $\Lambda = \{\lambda_1, \cdots, \lambda_r\}$ be a family of vectors in V. Then λ_i is called an essential vector of Λ if

$$\nexists \alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_r \in \mathcal{F} : \; \lambda_i = \sum_{j \neq i} \alpha_j \lambda_j \tag{3.52}$$

■

Remark 3.2.18 The above definition means that an essential vector of Λ is linearly independent of all other vectors of Λ. This implies that every subset of Λ that forms a basis of span $\{\lambda_1, \cdots, \lambda_r\}$ necessarily contains the essential vectors of Λ. ■

Using simple arguments from linear algebra we can prove (see also [112]):

Lemma 3.2.19 *Let V be a given vector space over a field $\mathcal{F}$. Let $\Lambda = \{\lambda_1, \cdots, \lambda_r\}$ be a family of vectors in V. Let $s :=$ dim span $\{\lambda_1, \cdots, \lambda_r\}$ and assume that $\{\lambda_1, \cdots, \lambda_s\}$ is a set of linearly independent vectors. Then λ_i $(i = 1, \cdots, s)$ is an essential vector of Λ if and only if for all $j = s+1, \cdots, r$*

$$\lambda_j = \sum_{k=1}^{s} \alpha_{jk} \lambda_k \Rightarrow \alpha_{ji} = 0$$

■

Definition 3.2.20 ([39]) **Essential orders of a nonlinear system**
The essential order ϵ_i of the output y_i of (3.1) is defined as the smallest $k \in \{1, \cdots, n\}$ for which $dy_i^{(k)}$ is an essential vector of $\mathcal{E}_n$ (provided such a k exists). If for an output y_i such a k does not exist, we set $\epsilon_i = +\infty$. ■

Remark 3.2.21 For a definition of essential orders for a linear system and their use in the linear input-output decoupling problem, see [17],[23]. ■

Lemma 3.2.22 *Consider the nonlinear system (3.1) and apply Singh's algorithm to it. Recall that δ_i is the highest time-derivative of y_i appearing in (3.48). Then for $i = 1, \cdots, \rho^*$: $\delta_i = \min(n, \epsilon_i)$.*

Proof Assume that $\epsilon_i < +\infty$. By definition of the essential orders and Theorem 3.2.4 we have that $dy_i^{(k)}$ is not an essential vector of $\mathcal{E}_k$ for $k = 1, \cdots, \epsilon_i - 1$. This implies by Lemma 3.2.19 that $(\partial \hat{y}_{\epsilon_i - 1}^{(\epsilon_i-1)} / \partial y_i^{(\epsilon_i-1)}) \neq 0$ and hence $\delta_i \geq \epsilon_i$. Moreover, by the definition of the essential orders and Theorem 3.2.4 we have that $dy_i^{(\epsilon_i)}$ is an essential vector of $\mathcal{E}_k$ for $k = \epsilon_i, \cdots, n$. Again by Lemma 3.2.19 this implies that $(\partial \hat{y}_k^{(k)} / \partial y_i^{(r)}) = 0$ for $k = \epsilon_i, \cdots, n$ and $r = \epsilon_i, \cdots, n$. This means that $\delta_i \leq \epsilon_i$. Hence $\delta_i = \epsilon_i$. Now assume that $\epsilon_i = +\infty$. Then by Lemma 3.2.19 we have that $(\partial \hat{y}_k^{(k)} / \partial y_i^{(k)}) \neq 0$ for $k = 1, \cdots, n$. Hence $\delta_i = n$. Combining both results we find that $\delta_i = \min(n, \epsilon_i)$, which establishes our claim. ∎

Remark 3.2.23 Note that for determining the essential orders we do not have to use Definition 3.2.20. In fact, by the result of Lemma 3.2.22 we can again rely on Singh's algorithm. ∎

We have now proved:

Proposition 3.2.24 *Consider the nonlinear system (3.1) and let x_0 be a strongly regular point for (3.1). Then every Singh compensator for (3.1) around x_0 has dimension*

$$\sigma = \sum_{i=1}^{\rho^*} \epsilon_i - \sum_{k=1}^{n} k s_k \tag{3.53}$$

∎

The following proposition gives another connection between different Singh compensators (see also [107]).

Proposition 3.2.25 *Consider a nonlinear system (3.1). Let $x_0 \in M$ be a strongly regular point for (3.1). Consider a Singh compensator (3.50) around x_0 defined on $U \times \mathcal{Z} \subset M \times \mathbb{R}^\sigma$, where U is a neighborhood of x_0. Moreover, let*

$$\begin{cases} \dot{\bar{z}}_i &= A_i \bar{z}_i + B_i \bar{v}_i \quad (i = 1, \cdots, \rho^*) \\ \bar{u}^1 &= \bar{\phi}_1(x, \bar{z}_1, \cdots, \bar{z}_{\rho^*}) + \sum_{i=1}^{m} \bar{\phi}_{2i}(x, \bar{z}_1, \cdots, \bar{z}_{\rho^*}) \bar{v}_i \\ \bar{u}_i &= \bar{v}_i \quad (i = \rho^* + 1, \cdots, m) \end{cases} \tag{3.54}$$

be another Singh compensator around x_0 defined on $U \times \bar{\mathcal{Z}} \subset M \times \bar{\mathcal{Z}}$, and assume that in the construction of (3.54) the same permutation of controls is employed as in the construction of (3.50). Then there exist open subsets $\mathcal{V} \subset \mathcal{Z}$ and $\bar{\mathcal{V}} \subset \bar{\mathcal{Z}}$ and a diffeomorphism $\Psi : U \times \mathcal{V} \to U \times \bar{\mathcal{V}}$ that transforms the system (3.1,3.50) defined on $U \times \mathcal{V}$ into the system (3.1,3.54) defined on $U \times \bar{\mathcal{V}}$.

Proof See Appendix B. ∎

3.2.3 Solution of the SDIODP

In this subsection a local solution of the SDIODP is given, and we show that a Singh compensator is a regular dynamic state feedback of minimal order that solves the SDIODP. We first formulate the SDIODP.

Definition 3.2.26 Strong input-output decoupling problem via regular dynamic state feedback (SDIODP)

(i) Given a square nonlinear system (3.1), under what conditions does there exist an analytic regular dynamic state feedback (3.42) for (3.1) such that the system (3.1,3.42) is strongly input-output decoupled?

(ii) Given a point $x_0 \in M$, we say that the SDIODP is locally solvable around x_0 if there exist a neighborhood $U \subset M$ of x_0 and an analytic regular dynamic state feedback (3.42) around x_0 for (3.1) such that the system (3.1,3.42) defined on $U \times \mathcal{Z}$ is strongly input-output decoupled. ■

Remark 3.2.27 Part *(i)* of Definition 3.2.26 implies that for any initial condition $(x(0), z(0)) \in M \times \mathbb{R}^\nu$ the system (3.1,3.42) is strongly input-output decoupled. Part *(ii)* of Definition 3.2.26 implies that for any initial condition $(x(0), z(0)) \in U \times \mathcal{Z}$ the system (3.1,3.42) restricted to $U \times \mathcal{Z}$ is strongly input-output decoupled. ■

Theorem 3.2.28 *Consider a square nonlinear system (3.1). Let x_0 be a strongly regular point for (3.1). Then the following statements hold.*

(i) The SDIODP is locally solvable around x_0 if and only if $\rho^* = m$.

(ii) If $\rho^* = m$, the SDIODP can be solved for (3.1) around x_0 via a Singh compensator around x_0.

(iii) Moreover, if $\rho^* = m$, any Singh compensator around x_0 is a regular dynamic state feedback of minimal order solving the SDIODP for (3.1) around x_0.

Proof

(i) See [30],[24],[71],[78].

(ii) (See also [91].) Assume that $\rho^* = m$. Then we know from Proposition 3.2.16 that any Singh compensator around x_0 is a regular dynamic state feedback for (3.1) around x_0. Apply a Singh compensator (3.50) around x_0 to (3.1). By construction of the Singh compensator there exist a neighborhood $U \subset M$ of x_0 and an open subset $\mathcal{Z} \subset \mathbb{R}^\sigma$ (with σ as defined in (3.53)) such that the outputs of (3.1,3.50) defined on $U \times \mathcal{Z}$ satisfy

$$\begin{aligned} y_i^{(k)} &= \text{function of } x \text{ and } z && (k = 0, \cdots, \gamma_i - 1) \\ y_i^{(k)} &= z_{ik} && (k = \gamma_i, \cdots, \epsilon_i - 1) \\ y_i^{(\epsilon_i)} &= v_i \end{aligned} \tag{3.55}$$

Hence the system (3.1,3.50) defined on $U \times \mathcal{Z}$ is strongly input-output decoupled.

■

For the proof of Theorem 3.2.28.*(iii)* (which is taken from [54]) we need some technical lemmas. Before stating these lemmas, we introduce some notation. Consider a (square) nonlinear system (3.1) together with a dynamic state feedback (3.42). For (3.1,3.42), let $\mathcal{K}^e$ denote the field consisting of the set of rational functions of $(v, \cdots, v^{(n+\nu-1)})$ with coefficients that are meromorphic in x and z. Let $\mathcal{E}_k^e := \operatorname{span}_{\mathcal{K}^e}\{dx, dz, d\dot{y}, \cdots, dy^{(k)}\}$ $(k = 1, \cdots, n+\nu-1)$. Furthermore, denote the relative degrees of (3.1,3.42) by $r_1^e, \cdots, r_m^e$ and its essential orders by $\epsilon_1^e, \cdots, \epsilon_m^e$.

Lemma 3.2.29 *Consider a square nonlinear system (3.1) of full rank. Then*

(i) $\epsilon_i, r_i < +\infty \quad (i = 1, \cdots, m)$

(ii) $\epsilon_i \geq r_i \quad (i = 1, \cdots, m)$

(iii) $\epsilon_i = r_i$ *for all* $i \in \{1, \cdots, m\}$ *if and only if the SIODP is locally solvable around a point* $x_0 \in M$.

Proof See [39]. ■

Lemma 3.2.30 *If for (3.1)* $dy_i^{(k)}$ *is not an essential vector of* $\mathcal{E}_n$*, then for (3.1,3.42)* $dy_i^{(k)}$ *is not an essential vector of* $\mathcal{E}_n^e$*. Hence* $\epsilon_i^e \geq \epsilon_i$ $(i = 1, \cdots, m)$. ■

Proof See Lemma 2 of [39]. ■

Proposition 3.2.31 *Consider the square nonlinear system (3.1). Let* x_0 *be a strongly regular point for (3.1) and consider a regular dynamic state feedback (3.42) of dimension* ν *that solves the SDIODP around* x_0*. Then* $\nu \geq \sigma$.

Proof Let $x_0 \in M$ be a strongly regular point for (3.1) and consider a regular dynamic state feedback (3.42) that solves the SDIODP around x_0. Then by Lemma 3.2.29 and Lemma 3.2.30 we have that $\epsilon_i^e = r_i^e \geq \epsilon_i$ $(i = 1, \cdots, m)$. By Lemma 3.1.8 this implies that the differentials $dy_i^{(j)}(x, z)$ $(i = 1, \cdots, m; j = 0, \cdots, r_i^e - 1)$ are linearly independent (over $\bar{\mathcal{K}}^e$, the subfield of $\mathcal{K}^e$ consisting of the meromorphic functions of x and z). By Theorem 3.2.4 and the proof of Proposition 3.2.24 we can find a reordering of the outputs of (3.1) and integers $\gamma_1, \cdots, \gamma_m$ satisfying $\sum_{i=1}^m \gamma_i = \sum_{i=1}^m \epsilon_i - \sigma$, such that for (3.1) the differentials $\{dx, \{dy_i^{(j)} \mid 1 \leq i \leq m, \gamma_i \leq j \leq \epsilon_i - 1\}\}$ are linearly independent over $\mathcal{K}$. Assume that for (3.1,3.42) these differentials are *not* linearly independent over $\bar{\mathcal{K}}^e$. This implies that we can find $r \in \{1, \cdots, m\}$, $s \in I\!N$ and a function $\phi_{rs}(x, \{u_i^{(j)} \mid i \neq r, 0 \leq j \leq s\}, \{u_r^{(j)} \mid 0 \leq j \leq s-1\})$ such that for (3.1,3.42)

$$u_r^{(s)} = \phi_{rs}(x, \{u_i^{(j)} \mid i \neq r, 0 \leq j \leq s\}, \{u_r^{(j)} \mid 0 \leq j \leq s-1\}) \tag{3.56}$$

By the proof of Lemma 3.3 in [30] this means that for all $k \geq s$ there exists a function $\phi_{rk}(x, \{u_i^{(j)} \mid i \neq r, 0 \leq j \leq k\}, \{u_r^{(j)} \mid 0 \leq j \leq s-1\})$ such that

$$u_r^{(k)} = \phi_{rk}(x, \{u_i^{(j)} \mid i \neq r, 0 \leq j \leq k\}, \{u_r^{(j)} \mid 0 \leq j \leq s-1\}) \tag{3.57}$$

This implies that if we apply Singh's algorithm to (3.43) we find that $\rho_{n+\nu} < m$. Hence by Definition 3.2.8 and Corollary 3.2.5, (3.42) is not a *regular* dynamic state feedback, which gives a contradiction. Hence for (3.1,3.42) the differentials $\{dx, \{dy_i^{(j)} \mid 1 \leq i \leq m, \gamma_i \leq j \leq \epsilon_i - 1\}\}$ are linearly independent over $\bar{\mathcal{K}}^e$. In particular this implies that

$$\operatorname{rank}_{\bar{\mathcal{K}}^e} \left(\frac{\partial y_i^{(j)}}{\partial z} \right)_{1 \leq i \leq m, \gamma_i \leq j \leq \epsilon_i - 1} = \sum_{i=1}^{m} (\epsilon_i - \gamma_i) = \sigma \tag{3.58}$$

Obviously,

$$\operatorname{rank}_{\bar{\mathcal{K}}^e} \left(\frac{\partial y_i^{(j)}}{\partial z} \right)_{1 \leq i \leq m, \gamma_i \leq j \leq \epsilon_i - 1} \leq \dim z \tag{3.59}$$

and hence $\nu \geq \sigma$, which establishes our claim. ■

Theorem 3.2.28.*(iii)* is now an immediate consequence of Proposition 3.2.31.

We conclude this section with an illustrative example

Example 3.2.32 Consider the following system on $\mathbb{R}^5$:

$$\begin{array}{rcll} \dot{x}_1 &=& x_2 u_1 & \quad y_1 = x_1 \\ \dot{x}_2 &=& x_5 & \quad y_2 = x_3 \\ \dot{x}_3 &=& x_2 + x_4 u_1 & \\ \dot{x}_4 &=& u_2 & \\ \dot{x}_5 &=& x_1 u_1 & \end{array} \tag{3.60}$$

Apply the first step of Singh's algorithm to this system. This yields:

$$\begin{array}{rcl} \dot{y}_1 &=& x_2 u_1 \\ \dot{y}_2 &=& x_2 + x_4 u_1 \end{array} \tag{3.61}$$

Hence $\rho_1 = 1$ and we can either choose $\tilde{y}_1 = y_1, \hat{y}_1 = y_2$ or we can choose $\tilde{y}_1 = y_2, \hat{y}_1 = y_1$. We first choose $\tilde{y}_1 = y_1, \hat{y}_1 = y_2$. Then

$$\dot{\hat{y}}_1 = \dot{\hat{y}}_1(x, \dot{\tilde{y}}_1) = x_2 + \frac{x_4}{x_2}\dot{\tilde{y}}_1 \tag{3.62}$$

Proceeding with the second step of Singh's algorithm we find

$$\hat{y}_1^{(2)} = x_5 + \frac{u_2}{x_2}\dot{\tilde{y}}_1 - \frac{x_4 x_5}{x_2^2}\dot{\tilde{y}}_1 + \frac{x_4}{x_2}\tilde{y}_1^{(2)} \tag{3.63}$$

Hence $\rho_2 = 2$ and we can take $\tilde{y}_2 = y_2$. Note that $\hat{y}_2$ does not appear. This implies that at the last step of Singh's algorithm applied in this way we obtain

$$\begin{array}{rcl} \dot{y}_1 &=& x_2 u_1 \\ \ddot{y}_2 &=& x_5 + \frac{u_2}{x_2}\dot{y}_1 - \frac{x_4 x_5}{x_2^2}\dot{y}_1 + \frac{x_4}{x_2}\ddot{y}_1 \end{array} \tag{3.64}$$

Hence $\gamma_1 = 1, \delta_1 = \epsilon_1 = 2, \gamma_2 = \delta_2 = \epsilon_2 = 2$. From (3.64) we see that all points $x_0 \in \{x \in I\!R^5 \mid x_2 \neq 0\}$ are regular points for (3.60). Moreover, from (3.64) we obtain the Singh compensator

$$\begin{aligned} \dot{z} &= v_1 \\ u_1 &= \tfrac{z}{x_2} \\ u_2 &= \tfrac{x_4 x_5}{x_2} - \tfrac{x_2 x_5}{z} - \tfrac{x_4}{z} v_1 + \tfrac{x_2}{z} v_2 \end{aligned} \tag{3.65}$$

We now choose $\tilde{y}_1 = y_2, \hat{y}_1 = y_1$. Then

$$\dot{\hat{y}}_1 = \dot{\hat{y}}_1(x, \dot{\tilde{y}}_1) = \frac{x_2}{x_4}(\dot{\tilde{y}}_1 - x_2) \tag{3.66}$$

Proceeding with the second step of Singh's algorithm we find

$$\hat{y}_1^{(2)} = \left(\tfrac{x_5}{x_4} - \tfrac{x_2 u_2}{x_4^2} \right)(\dot{\tilde{y}}_1 - x_2) + \frac{x_2}{x_4}(\tilde{y}_1^{(2)} - x_5) \tag{3.67}$$

We can now take $\tilde{y}_2 = y_1$. At the last step of Singh's algorithm applied in this way we obtain

$$\begin{aligned} \dot{y}_2 &= x_2 + x_4 u_1 \\ \ddot{y}_1 &= \left(\tfrac{x_5}{x_4} - \tfrac{x_2 u_2}{x_4^2} \right)(\dot{y}_2 - x_2) + \tfrac{x_2}{x_4}(\ddot{y}_2 - x_5) \end{aligned} \tag{3.68}$$

Hence $\gamma_1 = \delta_1 = \epsilon_1 = 2, \gamma_2 = 1, \delta_2 = \epsilon_2 = 2$. From (3.68) we see that also all points $x_0 \in \{x \in I\!R^5 \mid x_4 \neq 0\}$ are regular points for (3.60). Moreover, from (3.64) and (3.68) we see that all points $x_0 \in \{x \in I\!R^5 \mid x_2 \neq 0, x_4 \neq 0\}$ are strongly regular points for (3.60). (3.68) yields the Singh compensator

$$\begin{aligned} \dot{\bar{z}} &= v_1 \\ u_1 &= \frac{\bar{z} - x_2}{x_4} \\ u_2 &= \frac{x_4 x_5}{(x_2 - \bar{z})} - \frac{x_4 x_5}{x_2} + \frac{x_4^2}{x_2(x_2 - \bar{z})} v_1 - \frac{x_4}{(x_2 - \bar{z})} v_2 \end{aligned} \tag{3.69}$$

The diffeomorphism Ψ that transforms the system (3.60,3.65) into the system (3.60,3.69) is given by

$$(x, \bar{z}) = \Psi(x, z) = (x, \frac{x_4 z + x_2^2}{x_2}) \tag{3.70}$$

Chapter 4

Disturbance decoupling via regular dynamic state feedback

In this chapter we consider a nonlinear analytic system Σ of the form

$$\Sigma \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ y & = & h(x) \end{array} \right. \tag{4.1}$$

where $x = (x_1, \cdots, x_n)^T \in I\!R^n$ are local coordinates for the state space manifold M, $u \in I\!R^m$, $q \in I\!R^r$, $y \in I\!R^p$, $g(x) = (g_1(x) \cdots g_m(x))$, $p(x) = (p_1(x) \cdots p_r(x))$, $h(x) = \mathrm{col}(h_1(x), \cdots, h_p(x))$, $f, g_1, \cdots, g_m, p_1, \cdots, p_r$ are analytic vector fields on M and $h_1, \cdots, h_p$ are analytic functions on M. The system Σ with $q \equiv 0$ is denoted by Σ_0. Recall that the distributions $\mathcal{G}$ and $\mathcal{P}$ are defined by $\mathcal{G} := \mathrm{span}\,\{g_1, \cdots, g_m\}$ and $\mathcal{P} := \mathrm{span}\,\{p_1, \cdots, p_r\}$.

We have seen in Chapter 2 that if the DDP(dm) is not solvable for Σ, we may still be able to render the outputs independent of the disturbances by allowing dynamic state feedback. This gives rise to the formulation of the disturbance decoupling problem via regular dynamic state feedback (DDDP) and the disturbance decoupling problem via regular dynamic state feedback and disturbance measurements (DDDPdm).

In this chapter we formulate and solve the DDDP(dm). Instrumental in the solution of the DDDP is the Singh compensator that was introduced in Chapter 3. For the solution of the DDDPdm we need to modify the Singh compensator, yielding what we call a Singh compensator with disturbance feedthrough. We also translate the conditions obtained for the solvability of the DDDP(dm) into intrinsic algebraic and geometric conditions.

The results in this chapter can be found in a series of papers [51],[52],[53]. Another approach to the DDDP can be found in [83].

We first formulate the DDDP and the DDDPdm. We restrict ourselves to a local formulation and solution of these problems.

Definition 4.0.1 Disturbance decoupling problem via regular dynamic state feedback (DDDP)
Consider a nonlinear system Σ and let a point $x_0 \in M$ be given. The DDDP is said to be locally solvable around x_0 if there exist a regular analytic dynamic state feedback for Σ around x_0 of the form (3.42), denoted as R, such that the outputs of the composite system $\Sigma \circ R$ restricted to $U \times \mathcal{Z}$ are independent of the disturbances. ■

Definition 4.0.2 Disturbance decoupling problem via regular dynamic state feedback and disturbance measurements (DDDPdm)
Consider a nonlinear system Σ and let a point $x_0 \in M$ be given. The DDDPdm is said to be locally solvable around x_0 if there exists a dynamic state feedback for Σ of the form

$$Q \left\{ \begin{array}{rcl} \dot{z} &=& \alpha(x,q,z) + \beta(x,q,z)v \\ u &=& \gamma(x,q,z) + \delta(x,q,z)v \end{array} \right. \tag{4.2}$$

with $z \in \mathbb{R}^\nu$, such that for all $q \in \mathbb{R}^r$ (4.2) is a regular analytic dynamic state feedback for Σ around x_0, and the outputs of the composite system $\Sigma \circ Q$ restricted to $U \times \mathcal{Z}$ are independent of the disturbances. ∎

Remark 4.0.3 Note that if we require the dynamic state feedbacks R and Q to be static state feedback compensators (i.e. $\nu = 0$), we obtain the DDP and DDPdm that were studied in Chapter 2. ∎

Instrumental in the solution of the DDDP is the Singh compensator (3.50) that was introduced in Chapter 3. As the Singh compensator around a strongly regular point was introduced in Chapter 3 for systems without disturbances, we first have to define what is meant by a (strongly) regular point and a Singh compensator for Σ.

Definition 4.0.4 Let a point $x_0 \in M$ be given. We call x_0 a (strongly) regular point for Σ if x_0 is a (strongly) regular point for Σ_0 in the sense of Definition 3.2.14. ∎

Definition 4.0.5 Consider the analytic nonlinear system Σ. Let x_0 be a strongly regular point for Σ. Then any Singh compensator for Σ_0 around x_0 is also called a Singh compensator for Σ around x_0. ∎

We now give a solution of the DDDP. In the statement of the solution we employ the following notation. If we apply Singh's algorithm to Σ_0, the $\hat{y}_k^{(k)}$ $(k = 0, \cdots, n; \hat{y}_0 := y)$ can be viewed as functions on $M_e := M \times \mathbb{R}^{nm}$. By the same token, $\operatorname{Ker} d\hat{y}_k^{(k)}$ $(k = 0, \cdots, n)$ defines a distribution on M_e. Define the distributions $\mathcal{G}_e, \mathcal{P}_e$ on M_e by $\mathcal{G}_e := \mathcal{G} \times \{0\}, \mathcal{P}_e := \mathcal{P} \times \{0\}$.

Theorem 4.0.6 *Consider the nonlinear system Σ. Assume that x_0 is a strongly regular point for Σ.*

(i) The DDDP is locally solvable around x_0 if and only if it is solvable via a Singh compensator for Σ around x_0.

(ii) This condition is equivalent to the condition that for each application of Singh's algorithm to Σ_0 we have for $k = 0, \cdots, n-1$:

$$\mathcal{P}_e \subset \operatorname{Ker} d\hat{y}_k^{(k)} \tag{4.3}$$

Proof *(sufficiency)* Consider Σ and let x_0 be a strongly regular point for Σ. Assume that for each application of Singh's algorithm to Σ_0 (4.13) holds for $k = 0, \cdots, n-1$. Apply Singh's algorithm to Σ_0, yielding a reordering $\tilde{y}_1, \cdots, \tilde{y}_n, \hat{y}_n$ of the outputs. Note that without loss of generality we may assume that $\hat{y}_k = (\tilde{y}_{k+1}^T, \cdots, \tilde{y}_n^T, \hat{y}_n^T)^T$ for $k = 0, \cdots, n-1$. We now show that for Σ we have that for $k = 0, \cdots, n-1$

$$\begin{aligned} \tilde{y}_{k+1}^{(k+1)} &= \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k+1\}) + \\ &\quad \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\})u \\ \hat{y}_{k+1}^{(k+1)} &= \hat{a}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k+1\}) + \\ &\quad \hat{b}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k, i \le j \le k\})u \end{aligned} \tag{4.4}$$

i.e., the result of Singh's algorithm applied to Σ, where q is considered to be a parameter, is the same as the result of Singh's algorithm applied to Σ_0. For Σ we have

$$\begin{aligned} \dot{\tilde{y}}_1 &= \frac{\partial \tilde{y}_1}{\partial x}[f(x) + g(x)u + p(x)q] =: \tilde{a}_1(x) + \tilde{b}_1(x)u + \tilde{c}_1(x)q \\ \dot{\hat{y}}_1 &= \frac{\partial \hat{y}_1}{\partial x}[f(x) + g(x)u + p(x)q] =: \hat{a}_1(x) + \hat{b}_1(x)u + \hat{c}_1(x)q \end{aligned} \tag{4.5}$$

Since (4.3) holds for $k = 0$, we have that $\tilde{c}_1 \equiv 0$, $\hat{c}_1 \equiv 0$. Hence (4.4) holds for $k = 0$. Applying the above arguments repeatedly, we establish that (4.4) holds for $k = 0, \cdots, n-1$. Hence by (4.4) and Proposition 3.2.6 we have that

$$\begin{aligned} \dot{\tilde{Y}}_n &= \tilde{A}_n(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-1, i \le j \le n\}) \\ &\quad + \tilde{B}_n(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-1, i \le j \le n-1\})u \\ \hat{y}_n^{(n)} &= \phi(\hat{y}_n, \cdots, \hat{y}_n^{(n-1)}, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n, i \le j \le n\}) \end{aligned} \tag{4.6}$$

This implies that if we apply the Singh compensator to Σ, the outputs of the resulting system satisfy

$$\begin{aligned} \frac{\partial y_i^{(j)}}{\partial q} &= 0 \quad (1 \le i \le \rho^*, 0 \le j \le \delta_i - 1) \\ y_i^{(\delta_i)} &= v_i \quad (1 \le i \le \rho^*) \\ \frac{\partial y_i^{(j)}}{\partial q} &= 0 \quad (\rho^* + 1 \le i \le m, j \ge 0) \end{aligned} \tag{4.7}$$

Hence the Singh compensator locally solves the DDDP for Σ around x_0.

(necessity) Assume that the DDDP is locally solvable around x_0 via a regular dynamic state feedback R of the form (3.42). Apply Singh's algorithm to Σ_0, yielding a reordering $\tilde{y}_1, \cdots, \tilde{y}_n, \hat{y}_n$ of the outputs. Assume that (4.3) does not hold for all $k = 0, \cdots, n-1$. Let the integer τ be defined by $\tau := \min\{k \in \{0, \cdots, n-1\} \mid \mathcal{P}_e \not\subset \operatorname{Ker} d\hat{y}_k^{(k)}\}$. By the

sufficiency-part of this proof we then know that (4.4) holds for $k = 0, \cdots, \tau - 1$. Hence we have for $\Sigma \circ R$:

$$\begin{aligned}
\tilde{y}_{\tau+1}^{(\tau+1)} &= \tilde{a}_{\tau+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau, i \leq j \leq \tau+1\}) + \\
&\quad \tilde{b}_{\tau+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau, i \leq j \leq \tau\})(\gamma(x,z) + \delta(x,z)v) + \\
&\quad \tilde{c}_{\tau+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau, i \leq j \leq \tau\})q \\
\hat{y}_{\tau+1}^{(\tau+1)} &= \hat{a}_{\tau+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau, i \leq j \leq \tau+1\}) + \\
&\quad \hat{b}_{\tau+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau, i \leq j \leq \tau\})(\gamma(x,z) + \delta(x,z)v) + \\
&\quad \hat{c}_{\tau+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau, i \leq j \leq \tau\})q
\end{aligned} \tag{4.8}$$

where $(\tilde{c}_{\tau+1}^T \ \ \hat{c}_{\tau+1}^T)^T \not\equiv 0$. Denote $(\tilde{c}_{\tau+1}^T \ \ \hat{c}_{\tau+1}^T)^T =: C(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau, i \leq j \leq \tau\})$. Then the fact that R solves the DDDP for Σ implies that R must be such that $C(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau, i \leq j \leq \tau\}) = 0$ along all trajectories of $\Sigma \circ R$. Hence we can find $r \in \{1, \cdots, m\}$,$s \geq 0$ and a function $\phi_{rs}(x, \{u_i^{(j)} \mid i \neq r, 0 \leq j \leq s\}, \{u_r^{(j)} \mid 0 \leq j \leq s-1\})$ such that for $\Sigma \circ R$ we have

$$u_r^{(s)} = \phi_{rs}(x, \{u_i^{(j)} \mid i \neq r, 0 \leq j \leq s\}, \{u_r^{(j)} \mid 0 \leq j \leq s-1\}) \tag{4.9}$$

By the proof of Lemma 3.3 in [30] this means that for all $k \geq s$ there exist functions $\phi_{rk}(x, \{u_i^{(j)} \mid i \neq r, 0 \leq j \leq k\}, \{u_r^{(j)} \mid 0 \leq j \leq s-1\})$ such that

$$u_r^{(k)} = \phi_{rk}(k)(x, \{u_i^{(j)} \mid i \neq r, 0 \leq j \leq k\}, \{u_r^{(j)} \mid 0 \leq j \leq s-1\}) \tag{4.10}$$

which contradicts the fact that R is a *regular* dynamic state feedback. Hence (4.3) holds for all $k = 0, \cdots, n-1$. ∎

In a similar way we can obtain conditions for solvability of the DDDPdm. The role that is played by the Singh compensator in the solution of the DDDP is taken by the Singh compensator with disturbance feedthrough in the solution of the DDDPdm. Given a Singh compensator (3.50) for Σ around a strongly regular point x_0, we obtain a Singh compensator with disturbance feedthrough for Σ around x_0 by extending (3.50) in the following way. Note that (in the notation of Chapter 3) $\tilde{Y}_n = \tilde{Y}_n(x, \{y_i^{(j)} \mid 1 \leq i \leq \rho^*, \gamma_i \leq j \leq \delta_i - 1\})$. Define in the notation of Chapter 3

$$\begin{aligned}
&\phi_3(x, \{y_i^{(j)} \mid 1 \leq i \leq \rho^*, \gamma_i \leq j \leq \delta_i - 1\}) := \\
&\qquad -\tilde{B}_{n1}^{-1}(x, \{y_i^{(j)} \mid 1 \leq i \leq \rho^*, \gamma_i \leq j \leq \delta_i - 1\}) \cdot \\
&\qquad\qquad \frac{\partial \tilde{Y}_n}{\partial x}(x, \{y_i^{(j)} \mid 1 \leq i \leq \rho^*, \gamma_i \leq j \leq \delta_i - 1\})p(x)
\end{aligned} \tag{4.11}$$

and consider the following extension of (3.50)

$$\left\{ \begin{aligned}
\dot{z}_i &= A_i z_i + B_i v_i \quad (i = 1, \cdots, \rho^*) \\
u^1 &= \phi_1(x, z_1, \cdots, z_{\rho^*}) + \sum_{i=1}^{m} \phi_{2i}(x, z_1 \cdots, z_{\rho^*})v_i + \phi_3(x, z_1, \cdots, z_{\rho^*})q \\
u_i &= v_i \quad (i = \rho^* + 1, \cdots, m)
\end{aligned} \right. \tag{4.12}$$

Then (4.12) is called a *Singh compensator with disturbance feedthrough* for Σ around x_0. It can be shown that, like a Singh compensator (3.50), a Singh compensator with disturbance feedthrough (4.12) constitutes a regular dynamic state feedback for Σ around x_0 for every $q \in \mathbb{R}^r$. Without proof we state:

Theorem 4.0.7 *Consider a nonlinear system Σ and let $x_0 \in M$ be a strongly regular point for Σ. Then*

(i) The DDDPdm is locally solvable around x_0 if and only if it is solvable via a Singh compensator with disturbance feedthrough for Σ around x_0.

(ii) The above condition is equivalent to the condition that for each application of Singh's algorithm to x_0 we have for $k = 0, \cdots, n-1$:

$$\mathcal{P}_e \subset \operatorname{Ker} d\hat{y}_k^{(k)} + \mathcal{G}_e \tag{4.13}$$

■

The conditions in Theorems 4.0.6 and 4.0.7 can be given the following geometric interpretation (cf. [51]).

Proposition 4.0.8 *Consider a nonlinear system Σ and let $x_0 \in M$ be a strongly regular point for Σ. Let R be a Singh compensator for Σ around x_0. Denote the maximal locally controlled invariant distribution contained in* $\operatorname{Ker} dh$ *for $\Sigma_0 \circ R$ by Δ_e^*. Then*

$$\begin{aligned}\Delta_e^* &= \Big(\bigcap_{i=1}^{\rho^*}\bigcap_{k=0}^{\delta_i-1} \operatorname{Ker} dy_i^{(k)}\Big) \cap \Big(\bigcap_{i=\rho^*+1}^{p}\bigcap_{k=0}^{n-1} \operatorname{Ker} dy_i^{(k)}\Big) = \\ &\quad \Big(\bigcap_{i=1}^{\rho^*}\bigcap_{k=0}^{\gamma_i-1} \operatorname{Ker} dy_i^{(k)}\Big) \cap \operatorname{Ker} dz \cap \Big(\bigcap_{i=\rho^*+1}^{p}\bigcap_{k=0}^{n-1} \operatorname{Ker} dy_i^{(k)}\Big)\end{aligned} \tag{4.14}$$

Proof The first equality follows by applying Krener's algorithm to $\Sigma_0 \circ R$ and invoking Proposition 3.2.6. Moreover, by the construction of the Singh compensator we have that $y_i^{(\gamma_i+j-1)} = z_{ij}$ $(i = 1, \cdots, \rho^*; j = 1, \cdots, \delta_i - \gamma_i)$. This yields the second equality. ■

Using Theorem 2.2.10 we immediately obtain:

Proposition 4.0.9 *Consider a nonlinear system Σ and let $x_0 \in M$ be a strongly regular point for Σ around x_0. Let Q be a Singh compensator for Σ around x_0 and let Δ_e^* be as defined in Proposition 4.0.8. Then the DDDP is locally solvable around x_0 if and only if*

$$\mathcal{P}_e \subset \Delta_e^* \tag{4.15}$$

■

Obviously, Δ_e^* depends on how we apply Singh's algorithm to Σ_0, so Δ_e^* is by no means uniquely defined. However, the solvability of the DDDP does not depend on the way Singh's algorithm is performed (cf. Theorem 4.0.6). Hence for any distribution Δ of the form (4.14) generated by applying Singh's algorithm to Σ_0 we have $\mathcal{P}_e \subset \Delta$. Consequently, the distribution $\mathcal{P}_e$ is always contained in Δ_e^*. By construction we have that $\Delta_e^* \subset TM \times \{0\} \subset TM \times T\mathbb{R}^\sigma$ (with abuse of notation). However, the vector fields that span Δ_e^* may very well depend on z. Since the vector fields $p_1, \cdots, p_r$ only depend on x, they are contained in the (not necessarily locally controlled invariant) maximal distribution $\tilde{\Delta}_e^* \subset \Delta_e^*$ that contains the vector fields in Δ_e^* that only depend on x. The distribution $\tilde{\Delta}_e^*$ can be found via the following algorithm from [96].

Algorithm 4.0.10

Step 0

Define

$$\Delta_0 := \Delta_e^*$$

Step k

Define

$$\Delta_k := \{\tau \in \Delta_{k-1} \mid [\tau, \frac{\partial}{\partial z}] \in \Delta_{k-1} + \text{span}\{\frac{\partial}{\partial z}\}\}$$

■

Lemma 4.0.11 *Assume that the distributions Δ_k obtained in Algorithm 4.0.10 have constant dimension. Then for all k, $\Delta_k \supset \Delta_{k-1}$ and there is a k^* such that $\Delta_k = \Delta_{k^*}$ for all $k \geq k^*$.*

Proof See [96] ■

Assume that Algorithm 4.10 converges to Δ_{k^*}. Then the first n components of any vector field in Δ_{k^*} do not depend on z (cf. [96]). Moreover, we have that $\Delta_{k^*} \subset \Delta_e^*$ and hence the last σ components of any vector field in Δ_{k^*} equal zero. So the vector fields in Δ_{k^*} do not depend on z at all. By construction, Δ_{k^*} is the largest distribution in Δ_e^* having this property. Hence $\Delta_{k^*} = \tilde{\Delta}_e^*$.

Lemma 4.0.12 *$\tilde{\Delta}_e^*$ is uniquely defined, i.e., it is independent of the way we apply Singh's algorithm to Σ_0.*

Proof Assume that we have applied Singh's algorithm in two different ways, yielding Δ_{e1}^* and Δ_{e2}^*. Moreover, assume that by applying Algorithm 4.0.10 to these two distributions we obtain the distributions $\tilde{\Delta}_{e1}^*$ and $\tilde{\Delta}_{e2}^*$, where $\tilde{\Delta}_{e1}^* \neq \tilde{\Delta}_{e2}^*$. This implies that there are disturbance vector fields for which the DDDP is solvable by applying Singh's algorithm in one way, but not solvable by applying Singh's algorithm in the other way. This contradicts Theorem 4.5. Hence $\tilde{\Delta}_{e1}^*$ equals $\tilde{\Delta}_{e2}^*$. ■

Let $\tilde{\Delta}^*$be defined as the projection of $\tilde{\Delta}_e^*$ on TM. Then $\tilde{\Delta}^*$ is a well-defined distribution on M (cf. [96]). From the considerations above we immediately have the following theorem:

Theorem 4.0.13 *Consider the nonlinear system Σ and let $x_0 \in M$ be a strongly regular point for Σ. Assume that the distributions Δ_k obtained in Algorithm 4.0.10 have constant dimension. Then the DDDP is locally solvable around x_0 if and only if*

$$\mathcal{P} \subset \tilde{\Delta}^* \tag{4.16}$$

■

For the DDDPdm the reasoning is slightly different, although it follows the same lines. Here, Algorithm 4.0.10 is applied starting from the distribution $\Delta_0 = \Delta_{e\mathcal{G}}^* =: \Delta_e^* + \mathcal{G}_e$, resulting in the distribution $\tilde{\Delta}_{e\mathcal{G}}^* := \Delta_{k^*}$. Analogously to Lemma 4.0.12 it is then possible to prove:

Lemma 4.0.14 $\tilde{\Delta}_{e\mathcal{G}}^*$ *obtained in the way described above is uniquely defined, i.e., it is independent of the way we apply Singh's algorithm to Σ_0.* ■

Let $\tilde{\Delta}_{\mathcal{G}}^*$ be defined as the projection of the distribution $\tilde{\Delta}_{e\mathcal{G}}^*$ on TM.

Theorem 4.0.15 *Consider the nonlinear system Σ and let x_0 be a strongly regular point for Σ. Assume that the distributions Δ_k obtained in Algorithm 4.0.10, with $\Delta_0 = \Delta_{e\mathcal{G}}^*$, have constant dimension. Then the DDDPdm is locally solvable around x_0 if and only if*

$$\mathcal{P} \subset \tilde{\Delta}_{\mathcal{G}}^* \tag{4.17}$$

■

The conditions for solvability of the DDDP and the DDDPdm in Theorems 4.0.6,4.0.7 respectively, can also be translated into intrinsic (algebraic) conditions in the following way. Let Σ_q denote the system obtained from Σ by considering the disturbances q as an extra set of controls. For Σ_0, denote $\rho_{0k} = \dim \mathcal{E}_k - \dim \mathcal{E}_{k-1}$ $(k = 1, \cdots, n)$. Similarly, define ρ_{qk} $(k = 1, \cdots, n)$ for Σ_q.

Theorem 4.0.16 *Consider the nonlinear system Σ and let $x_0 \in M$ be a strongly regular point for Σ. Then the DDDPdm is locally solvable around x_0 if and only if for $k = 1, \cdots, n$*

$$\rho_{0k} = \rho_{qk} \tag{4.18}$$

Proof Apply Singh's algorithm to Σ_0, yielding a reordering $\tilde{y}_1, \cdots, \tilde{y}_n, \hat{y}_n$ of the outputs. Then for Σ_q we obtain

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u + \tilde{c}_1(x)q \\ \dot{\hat{y}}_1 &= \hat{a}_1(x) + \hat{b}_1(x)u + \hat{c}_1(x)q \end{aligned} \tag{4.19}$$

where $\tilde{a}_1, \tilde{b}_1, \hat{a}_1, \hat{b}_1$ are the same as in Singh's algorithm applied to Σ_0 and $\tilde{c}_1(x) = (\partial\tilde{y}_1/\partial x)p(x)$, $\hat{c}_1(x) = (\partial\hat{y}_1/\partial x)p(x)$. It can be shown that the fact that (4.13) holds for $k = 0$ is equivalent to the existence of a $\sigma_1(x)$ such that $\tilde{b}_1(x)\sigma_1(x) = \tilde{c}_1(x)$,$\hat{b}_1(x)\sigma_1(x) = \hat{c}_1(x)$. Hence the fact that (4.13) holds for $k = 0$ is equivalent to

$$\rho_{q1} = \text{rank}\begin{pmatrix} \tilde{b}_1 & \tilde{c}_1 \\ \hat{b}_1 & \hat{c}_1 \end{pmatrix} = \text{rank}\begin{pmatrix} \tilde{b}_1 & \tilde{b}_1\sigma_1 \\ \hat{b}_1 & \hat{b}_1\sigma_1 \end{pmatrix} = \text{rank}\begin{pmatrix} \tilde{b}_1 \\ \hat{b}_1 \end{pmatrix} = \rho_{01} \tag{4.20}$$

Let $\tilde{b}_1^+(x)$ be a right inverse of $\tilde{b}_1(x)$ and take $u = \tilde{b}_1^+(x)(\dot{\tilde{y}}_1 - \tilde{a}_1(x) - \tilde{c}_1(x)q)$ in (4.19). Then (4.19) yields in particular

$$\begin{aligned} \dot{\hat{y}}_1 = \; & \hat{a}_1(x) + \hat{b}_1(x)\tilde{b}_1^+(x)(\dot{\tilde{y}}_1 - \tilde{a}_1(x) - \tilde{b}_1(x)\sigma_1(x)q) + \hat{b}_1(x)\sigma_1(x)q = \\ & \hat{a}_1(x) + \hat{b}_1(x)\tilde{b}_1^+(x)(\dot{\tilde{y}}_1 - \tilde{a}_1(x)) \end{aligned} \tag{4.21}$$

Hence for Σ_q, $\dot{\hat{y}}_1$ is given by the same expression as for Σ_0. Applying the above arguments repeatedly, we can show that (4.13) holds for $k = 0, \cdots, n-1$ if and only if $\rho_{0k} = \rho_{qk}$ for $k = 1, \cdots, n$, which establishes our claim. ∎

Remark 4.0.17 In the terminology of [71], Theorem 4.0.16 states that the DDDPdm is locally solvable if and only if Σ_0 and Σ_q have the same *algebraic* structure at infinity. ∎

In order to obtain intrinsic conditions for the solvability of the DDDP, we follow an idea from [108]. Define an auxiliary system Σ_a by

$$\Sigma_a \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)w + p(x)q \\ \dot{w} & = & v \\ y & = & h(x) \end{array} \right. \tag{4.22}$$

where $w \in I\!\!R^m$, and $v \in I\!\!R^m$ denotes the controls of Σ_a. Let Σ_{a0} denote the system obtained from Σ_a by setting $q \equiv 0$ and let Σ_{aq} denote the system Σ_a where the disturbances q are considered to be an extra set of controls. Similarly to ρ_{0k}, ρ_{qk} for Σ, define ρ^a_{0k}, ρ^a_{qk} for Σ_a.

Theorem 4.0.18 *Consider the nonlinear system Σ. Let x_0 be a strongly regular point for Σ. Then the DDDP is locally solvable around x_0 if and only if for $k = 0, \cdots, n-1$*

$$\rho^a_{0k} = \rho^a_{qk} \tag{4.23}$$

Proof *(necessity)* Assume that the DDDP is locally solvable around x_0 via a regular dynamic state feedback

$$R \left\{ \begin{array}{rcl} \dot{z} & = & \alpha(x,z) + \beta(x,z)\tilde{u} \\ u & = & \gamma(x,z) + \delta(x,z)\tilde{u} \end{array} \right. \tag{4.24}$$

Consider the following dynamic state feedback for Σ_a:

$$R_a \left\{ \begin{array}{rcl} \dot{z}_1 & = & \alpha(x,z_1) + \beta(x,z_1)z_2 \\ \dot{z}_2 & = & \hat{u} \\ v & = & \sigma(x,w,z_1,z_2,q) \end{array} \right. \tag{4.25}$$

with

$$\sigma(x,w,z_1,z_2,q) = [\frac{\partial\gamma}{\partial x}(x,z_1) + \frac{\partial\delta}{\partial x}(x,z_1)z_2][f(x) + g(x)w + p(x)q] +$$

$$+[\frac{\partial\gamma}{\partial z_1}(x,z_1) + \frac{\partial\delta}{\partial z_1}(x,z_1)z_2][\alpha(x,z_1) + \beta(x,z_1)z_2] + \delta(x,z_1)\hat{u}$$

and $\hat{u}$ denoting the new controls. Then we find:

$$w = \int v dt = \gamma(x,z_1) + \delta(x,z_1)z_2$$

and thus R_a locally solves the DDDPdm for Σ_a, since R solves the DDDP for Σ. Furthermore we have as an (almost) immediate consequence of the fact that R is regular, that R_a is also regular. So the DDDPdm is locally solvable for Σ_a and hence by Theorem 4.0.16: (4.23) holds for $k = 1, \cdots, n$.
(sufficiency) Assume that (4.23) holds for $k = 1, \cdots, n$, i.e., the DDDPdm is locally solvable for Σ_a, say via a regular dynamic state feedback

$$Q_a \begin{cases} \dot{z} &= \alpha(x,w,z,q) + \beta(x,w,z,q)\tilde{v} \\ v &= \gamma(x,w,z,q) + \delta(x,w,z,q)\tilde{v} \end{cases} \tag{4.26}$$

with $\tilde{v}$ denoting the new controls. Then it is obvious that the dynamic state feedback

$$Q \begin{cases} \dot{z}_1 &= \gamma(x,z_1,z_2,q) + \delta(x,z_1,z_2,q)\tilde{v} \\ \dot{z}_2 &= \alpha(x,z_1,z_2,q) + \beta(x,z_1,z_2,q)\tilde{v} \\ u &= z_1 \end{cases} \tag{4.27}$$

locally solves the DDDPdm for Σ. Apply Singh's algorithm to Σ_0, yielding a reordering $\tilde{y}_1, \cdots, \tilde{y}_n, \hat{y}_n$ of the outputs. Employ the notation of the proof of Theorem 4.0.6. Then for Σ we have in particular:

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u + \tilde{c}_1(x)q \\ \dot{\hat{y}}_1 &= \hat{a}_1(x) + \hat{b}_1(x)u + \hat{c}_1(x)q \end{aligned} \tag{4.28}$$

Since Q locally solves the DDDPdm for Σ, the q-dependence in (4.22) must have vanished if we put u in (4.22) equal to the output of Q. This implies that actually $\tilde{c}_1 \equiv 0, \hat{c}_1 \equiv 0$, since the output of Q does not depend on q. It can be checked that this implies that (4.3) holds for $k = 0$. Applying the above arguments repeatedly, we can show that (4.3) holds for $k = 0, \cdots, n-1$. By Theorem 4.0.6 this implies that the DDDP is solvable around x_0.

■

Remark 4.0.19 Theorem 4.0.6 provides a complete solution of the DDDP, together with a regular dynamic state feedback that (locally) solves the problem. However, the Singh compensator is certainly not a regular dynamic state feedback of *minimal* dimension that solves the problem. This is illustrated by the following linear system:

$$\begin{aligned} \dot{x}_1 &= u_1 & y_1 &= x_1 \\ \dot{x}_2 &= x_3 + u_1 & y_2 &= x_2 \\ \dot{x}_3 &= x_4 + u_2 & & \\ \dot{x}_4 &= q & & \end{aligned} \tag{4.29}$$

It is easily seen that this system has full rank (equal to 2) and that a Singh compensator is given by

$$\begin{aligned} \dot{z} &= v_1 \\ u_1 &= z \\ u_2 &= -x_4 - v_1 + v_2 \end{aligned} \tag{4.30}$$

By Theorem 4.0.6 the Singh compensator (4.30) solves the DDDP for (4.29). However, as was already mentioned in Remark 2.1.8.*(ii)*, for a linear system the DDDP is solvable if and only if the DDP is (cf. [7]). Indeed, it can be checked that the regular *static* state feedback

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= -x_4 + v_2 \end{aligned} \tag{4.31}$$

solves the DDP for (4.29). The non-minimality of the Singh compensator (with respect to solving the DDDP) occurs since it produces a compensated system for which the rank of the decoupling matrix equals the rank of the original system, while this property does not have to hold for a regular feedback that renders outputs independent of the disturbances (the decoupling matrix of (4.29,4.31) has rank one, while the system (4.29) has rank 2).

As we have seen in Example 2.2.14, for nonlinear systems solvability of the DDDP does not necessarily imply that the DDP is solvable. Also, the example in this remark shows that *if* the DDP (and so also the DDDP) is solvable, the procedure described in this chapter only leads us to the conclusion that the DDDP is solvable and *not* to any conclusion about the solvability of the DDP. Therefore, the question arises if one can decide from a solution of the DDDP if possibly also the DDP is solvable, without having to invoke the theory presented in Chapter 2. If the answer to this question is positive, a second question is if a regular static state feedback solving the DDP can be derived from a Singh compensator. The answers to these questions are not available at the moment and the questions remain a topic for future research. It seems that a clue to the answer can be obtained by a thorough comparison of Krener's algorithm and Singh's algorithm. ■

We conclude this section with two examples.

Example 4.0.20 Consider the system (3.60) from Example 3.2.32. The maximal locally controlled invariant distribution contained in $\operatorname{Ker} dh$ for (3.60,3.65), denoted by Δ^*_{e1}, is given by

$$\Delta^*_{e1} = \operatorname{span}\{x_2 z\frac{\partial}{\partial x_2} - (x_2^2 - x_4 z)\frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\} \tag{4.32}$$

and we find

$$\tilde{\Delta}^*_1 = \operatorname{span}\{\frac{\partial}{\partial x_5}\} \tag{4.33}$$

For (3.60,3.69) the maximal locally controlled invariant distribution contained in $\operatorname{Ker} dh$, to be denoted by Δ^*_{e2}, is given by

$$\Delta^*_{e2} = \operatorname{span}\{x_2\frac{\partial}{\partial x_2} + x_4(\bar{z} - 2x_2)\frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\} \tag{4.34}$$

and we find

$$\tilde{\Delta}_2^* = \text{span}\,\{\frac{\partial}{\partial x_5}\} \tag{4.35}$$

Now consider the system Σ defined by

$$\begin{array}{rcll} \dot{x}_1 & = & x_2 u_1 & y_1 = x_1 \\ \dot{x}_2 & = & x_5 & y_2 = x_3 \\ \dot{x}_3 & = & x_2 + x_4 u_1 & \\ \dot{x}_4 & = & u_2 & \\ \dot{x}_5 & = & x_1 u_1 + q & \end{array} \tag{4.36}$$

Note that the system (3.60) is the same as Σ_0. Then by (4.35) and Theorem 4.0.13 the DDDP is locally solvable for Σ around all points $x_0 \in \{x \in I\!R^5 \mid x_2 \neq 0, x_4 \neq 0\}$. ∎

Example 4.0.21 Consider the voltage frequency controlled induction motor described in [26]. As state variables we take the projections of the stator current and flux vectors on a reference frame (α, β) which is fixed to the stator windings, and the angular position of the voltage input vector. As inputs we take the amplitude of the voltage input vector and the voltage supply frequency. The parameters R_s and R_r are the stator and rotor resistances, L_s and L_r are the stator and rotor self-inductances and M is the mutual inductance. The speed ω can be considered as a slowly varying parameter, owing to the large separation of time scales between the mechanical and the electromagnetic dynamics. In the sequel ω is assumed to be constant.

We define $\bar{x} = (x_1, \cdots, x_4)$ and $x = (\bar{x}, x_5)$ and we assume that a one-dimensional disturbance q influences the dynamics through the disturbance vector field $p(x) = (x_3\ x_4\ 0\ 0\ 0)^T$. Then the state equations are written as

$$\dot{x} = \begin{pmatrix} A\bar{x} \\ 0 \end{pmatrix} + \begin{pmatrix} g_1(x_5) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + p(x)q \tag{4.37}$$

where

$$A = \begin{pmatrix} -(\alpha+\beta) & -\omega & \frac{\beta}{L_s} & \frac{\omega}{\sigma L_s} \\ \omega & -(\alpha+\beta) & -\frac{\omega}{\sigma L_s} & \frac{\beta}{L_s} \\ -\alpha\sigma L_s & 0 & 0 & 0 \\ 0 & -\alpha\sigma L_s & 0 & 0 \end{pmatrix} \quad g_1(x_5) = \begin{pmatrix} \frac{\cos x_5}{\sigma L_s} \\ \frac{\sin x_5}{\sigma L_s} \\ \cos x_5 \\ \sin x_5 \end{pmatrix} \tag{4.38}$$

and $\alpha = \frac{R_s}{\sigma L_s}$, $\beta = \frac{R_r}{\sigma L_r}$, $\sigma = 1 - \frac{M^2}{L_s L_r}$.

Suitable outputs for the system are defined in terms of the stator flux Φ_s and the torque T_m. Hence, the following nonlinear output functions are used

$$\begin{array}{rcccl} h_1(x) & = & \Phi_s^2 & = & x_3^2 + x_4^2 \\ h_2(x) & = & T_m & = & x_2 x_3 - x_1 x_4 \end{array} \tag{4.39}$$

Applying Krener's algorithm, using REDUCE, we find that $\Delta^* = \{0\}$. Hence neither the DDPdm nor the DDP is solvable for (4.37,4.39). However, by applying Singh's algorithm to (4.37,4.39) we find that we can solve the DDDPdm by applying the following Singh compensator with disturbance feedthrough:

$$\begin{aligned} \dot{z} &= v_1 \\ u_1 &= \phi_1(x,z) \\ u_2 &= \phi_2(x,z,q,v_1,v_2) \end{aligned} \tag{4.40}$$

where

$$\phi_1(x,z) = \frac{2\alpha\sigma L_s(x_1x_3 + x_2x_4) + z}{2(x_3\cos x_5 + x_4 \sin x_5)} \tag{4.41}$$

and where $\phi_2(x,z,q,v_1,v_2)$ can be calculated from

$$\begin{aligned} &\phi_2(x,z,q,v_1,v_2) = \frac{1}{\mathcal{L}_{g_2}\mathcal{L}_f y_2 + \phi_1\mathcal{L}_{g_2}\mathcal{L}_{g_1}y_2}[v_2 - \frac{\partial\phi_1}{\partial z}\mathcal{L}_{g_1}y_2 \cdot v_1 - \\ &(\mathcal{L}_{f+\phi_1 g_1}\mathcal{L}_f y_2 + \mathcal{L}_{g_1}y_2\mathcal{L}_{f+\phi_1 g_1}\phi_1 + \phi_1\mathcal{L}_{f+\phi_1 g_1}\mathcal{L}_{g_1}y_2) - \\ &q(\mathcal{L}_p\mathcal{L}_f y_2 + \mathcal{L}_{g_1}y_2\mathcal{L}_p\phi_1 + \phi_1\mathcal{L}_p\mathcal{L}_{g_1}y_2)] \end{aligned} \tag{4.42}$$

Chapter 5

Controlled invariant submanifolds and clamped dynamics

In Chapter 2 we have seen that the notion of a (controlled) invariant subspace for a linear system can be conveniently generalized to nonlinear systems by introducing the notion of a (controlled) invariant distribution, at least for applications like the disturbance decoupling problem via regular feedback. In this chapter we present another generalization, namely that of a (controlled) invariant submanifold. The generalization of the maximal controlled invariant subspace contained in $\operatorname{Ker} C$ leads to the introduction of the clamped dynamics of a nonlinear system. This notion was first identified, in the single-input single-output case, in [11],[67]. For the multi-input multi-output case it was further elaborated in [12],[64],[101],[104].

Consider a linear system of the form

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \tag{5.1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and A, B, C matrices of appropriate dimensions with constant coefficients. Consider a subspace $\mathcal{V} \subset \mathbb{R}^n$. First set $u \equiv 0$ and assume that $A\mathcal{V} \subset \mathcal{V}$ (i.e., $\mathcal{V}$ is an invariant subspace for (5.1), cf. Chapter 2). It is readily checked that this requirement is equivalent to the requirement that the solutions of $\dot{x} = Ax$, $x(0) \in \mathcal{V}$, remain in $\mathcal{V}$ for all $t \geq 0$. Similarly, the requirement that $A\mathcal{V} \subset \mathcal{V} + \operatorname{Im} B$ (i.e., $\mathcal{V}$ is a controlled invariant subspace for (5.1), cf. Chapter 2) is equivalent to the existence of a feedback $u = Fx$ for (5.1) such that the solutions of $\dot{x} = (A + BF)x$, $x(0) \in \mathcal{V}$, remain in $\mathcal{V}$ for all $t \geq 0$.

Now consider a nonlinear system of the form

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases} \tag{5.2}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ are local coordinates for the state space manifold M, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $g(x) = (g_1(x) \cdots g_m(x))$, $h(x) = \operatorname{col}(h_1(x), \cdots, h_p(x))$, $f, g_1, \cdots, g_m$ are analytic vector fields on M and $h_1, \cdots, h_p$ are analytic functions on M. Recall that the distribution $\mathcal{G}$ is defined by $\mathcal{G} := \operatorname{span}\{g_1, \cdots, g_m\}$.

The above interpretation of (controlled) invariance for the linear system (5.1) leads to the notion of a (controlled) invariant submanifold for the nonlinear system (5.2).

Definition 5.0.1 (Controlled) invariant submanifold
Consider a nonlinear system (5.2) and let N be a submanifold of M.

(i) N is called invariant for (5.2) if

$$f(x) \in T_xN \quad \text{for all } x \in N \tag{5.3}$$

(ii) N is called (locally) controlled invariant for (5.2) if there exists (locally on N) a strict static state feedback $u = \alpha(x)$, $x \in N$, such that

$$f(x) + g(x)\alpha(x) \in T_xN \quad \text{for all } x \in N \tag{5.4}$$

i.e. N is invariant for $\dot{x} = f(x) + g(x)\alpha(x)$. ∎

Remark 5.0.2 (5.3) immediately implies that the solutions of $\dot{x} = f(x)$, $x(0) \in N$, remain in N for $t \in [0, T)$, with $T > 0$ small enough. ∎

We obtain the following generalization of Lemma 2.1.4 (see also Lemma 2.2.3).

Lemma 5.0.3 *Consider the nonlinear system (5.2) and a submanifold $N \subset M$. Assume that $\dim(T_xN + \mathcal{G}(x))$ is constant on N. Then N is locally controlled invariant for (5.2) if and only if*

$$f(x) \in T_xN + \mathcal{G}(x) \quad \text{for all } x \in N \tag{5.5}$$

∎

We proceed by giving a generalization of $\mathcal{V}^*$, the maximal controlled invariant subspace contained in $\operatorname{Ker} C$ for (5.1).

Definition 5.0.4 Output-nulling submanifold
A submanifold $N \subset M$ is called output-nulling if $N \subset h^{-1}(0)$, i.e. if the output value corresponding to states in N is zero. ∎

Algorithm 5.0.5 Clamped dynamics algorithm
Consider the system (5.2) and suppose $h(x_0) = 0$. Let $U \subset M$ be a neighborhood of x_0.

Step 0

Define $N_0 = h^{-1}(0) \cap U$.

Step k

Assume that N_{k-1} is a submanifold through x_0. Then define

$$N_k = \{x \in N_{k-1} \mid f(x) \in T_x N_{k-1} + \mathcal{G}(x)\}$$

∎

Proposition 5.0.6 *Consider a nonlinear system (5.2) and let $x_0 \in M$ be such that $h(x_0) = 0$. Assume that Algorithm 5.0.5 can be applied successfully to (5.2), i.e., we can find a neighborhood $U \subset M$ of x_0 such that at every step N_k, $k \geq 0$, is a submanifold through x_0. Then*

(i) $N_0 \supset N_1 \supset \cdots \supset N_{k-1} \supset N_k \supset \cdots$

(ii) There exists a $k^* \leq n$ such that $N_{k^*} = N_{k^*+1} = \cdots$

Furthermore, if we let N^* denote the maximal connected component of N_{k^*} containing x_0, we have

(iii) N^* satisfies (5.5).

(iv) For any output-nulling submanifold N satisfying (5.5) there exists some neighborhood $\tilde{U}$ of x_0 such that $N \cap \tilde{U} \subset N^*$.

Proof See e.g. [61],[81]. ∎

The result of Proposition 5.0.6 implies that N^* is the *maximal output-nulling submanifold through* x_0 with respect to property (5.5). If moreover $\dim(T_x N^* + \mathcal{G}(x))$ is constant for every $x \in N^*$ then it immediately follows from Lemma 5.0.3 that N^* is the *maximal locally controlled invariant output-nulling submanifold* around x_0.

Under some extra regularity assumptions we can give a more constructive version of the clamped dynamics algorithm. The version presented here is a modified version of the algorithm from [104] and is very much related to Krener's algorithm.

Algorithm 5.0.7 Clamped dynamics algorithm
Consider the system (5.2). Let $x_0 \in M$ be such that $h(x_0) = 0$ and assume that h has constant rank p_0 in a neighborhood of x_0 in $h^{-1}(0)$.

Step 0

Locally around x_0 the set $N_0 := h^{-1}(0)$ is an $(n - p_0)$-dimensional manifold through x_0. Permute the entries of h such that the first p_0 entries are independent on N_0 and denote $\phi_0 := \mathrm{col}(h_1, \cdots, h_{p_0})$, $\Phi_0 = \phi_0$.

Step 1

Calculate

$$\dot{\phi}_0 = \frac{\partial \phi_0}{\partial x}[f(x) + g(x)u] =: d_1(x) + e_1(x)u \tag{5.6}$$

Assume that $e_1(x)$ has constant rank r_1 on a neighborhood of x_0 in N_0. After a possible permutation of the entries of ϕ_0 we may assume that the first r_1 rows of e_1 are linearly independent on this neighborhood. Accordingly, write (5.6) as

$$\begin{pmatrix} \dot{\tilde{\phi}}_0 \\ \dot{\hat{\phi}}_0 \end{pmatrix} = \begin{pmatrix} \tilde{d}_1(x) + \tilde{e}_1(x)u \\ \hat{d}_1(x) + \hat{e}_1(x)u \end{pmatrix} \tag{5.7}$$

where $\tilde{e}_1(x)$ has full row rank r_1. Let $\tilde{e}_1^+(x)$ be a right inverse of $\tilde{e}_1(x)$ and set

$$u = -\tilde{e}_1^+(x)\tilde{d}_1(x) \tag{5.8}$$

Then (5.7) and (5.8) yield

$$\begin{pmatrix} \dot{\tilde{\phi}}_0 \\ \dot{\hat{\phi}}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{d}_1(x) - \hat{e}_1(x)\tilde{e}_1^+(x)\tilde{d}_1(x) \end{pmatrix} =: \begin{pmatrix} 0 \\ \bar{\phi}_1(x) \end{pmatrix} \tag{5.9}$$

Note that, since each row of $\hat{e}_1(x)$ is linearly dependent on the rows of $\tilde{e}_1(x)$, $\bar{\phi}_1(x)$ is independent of the choice of $\tilde{e}_1^+(x)$. Assume that $\bar{\phi}_1(x_0)$ has constant rank s_1 on a neighborhood of x_0 in N_0. Then locally around x_0, $N_1 := \{x \in N_0 \mid \bar{\phi}_1(x) = 0\}$ is an $(n-p_1)$-dimensional submanifold through x_0, with $p_1 := p_0 + s_1$. Permute the entries of $\bar{\phi}_1$ such that the first s_1 entries are independent on N_1, and denote $\phi_1 := \mathrm{col}(\bar{\phi}_{11}, \cdots, \bar{\phi}_{1s_1})$, $\Phi_1 := (\phi_0^T, \phi_1^T)^T$. Finally, set $\tilde{E}_1(x) := \tilde{e}_1(x)$, $\tilde{D}_1(x) = \tilde{d}_1(x)$, $\tilde{\Phi}_0 := \tilde{\phi}_0$.

Step k

Suppose that at step $0, 1, \cdots, k-1$ we have defined $\phi_0, \cdots, \phi_{k-1}$, $\Phi_\ell := (\phi_0^T, \cdots, \phi_\ell^T)^T$ and $\tilde{\Phi}_0, \cdots, \tilde{\Phi}_{k-2}$ such that $N_\ell := \{x \mid \Phi_\ell(x) = 0\}$ $(\ell = 0, \cdots, k-1)$ is a submanifold through x_0 and

$$\dot{\tilde{\Phi}}_{k-2} = \tilde{D}_{k-1}(x) + \tilde{E}_{k-1}(x)u \tag{5.10}$$

where $\tilde{E}_{k-1}(x)$ has full row rank r_{k-1} on a neighborhood of x_0 in N_{k-2}. Calculate

$$\dot{\phi}_{k-1} := \frac{\partial \phi_{k-1}}{\partial x}[f(x) + g(x)u] =: d_k(x) + e_k(x)u \tag{5.11}$$

and set $E_k := (\tilde{E}_{k-1}^T \;\; e_k^T)^T$. Assume that E_k has constant rank r_k on a neighborhood of x_0 in N_{k-1}. After a possible permutation of the entries of ϕ_{k-1} we may assume that the first r_k rows of E_k are linearly independent on this neighborhood. This also implies that the first $r_k - r_{k-1}$ rows of $e_k(x)$ are linearly independent on this neighborhood. Accordingly, write (5.11) as

$$\begin{pmatrix} \dot{\tilde{\phi}}_{k-1} \\ \dot{\hat{\phi}}_{k-1} \end{pmatrix} = \begin{pmatrix} \tilde{d}_k(x) + \tilde{e}_k(x)u \\ \hat{d}_k(x) + \hat{e}_k(x)u \end{pmatrix} \tag{5.12}$$

where $\hat{e}_k(x)$ has full row rank $r_k - r_{k-1}$. Letting $\tilde{\Phi}_{k-1} := (\tilde{\Phi}_{k-2}^T, \hat{\phi}_{k-1}^T)^T$, $\tilde{D}_k := (\tilde{D}_{k-1}^T \; \hat{d}_k^T)^T$, $\tilde{E}_k := (\tilde{E}_{k-1}^T \; \hat{e}_k^T)^T$, we can rewrite (5.10),(5.12) as

$$\begin{aligned} \dot{\tilde{\Phi}}_{k-1} &= \tilde{D}_k(x) + \tilde{E}_k(x)u \\ \dot{\hat{\phi}}_{k-1} &= \hat{d}_k(x) + \hat{e}_k(x)u \end{aligned} \tag{5.13}$$

where $\tilde{E}_k(x)$ has full row rank r_k on a neighborhood of x_0 in N_{k-1}. Let $\tilde{E}_k^+(x)$ be right inverse of $\tilde{E}_k(x)$ and set

$$u = -\tilde{E}_k^+(x)\tilde{D}_k(x) \tag{5.14}$$

Then (5.13) and (5.14) yield

$$\begin{aligned} \dot{\tilde{\Phi}}_{k-1} &= 0 \\ \dot{\hat{\phi}}_{k-1} &= \hat{d}_k(x) - \hat{e}_k(x)\tilde{E}_k^+(x)\tilde{D}_k(x) =: \bar{\phi}_k(x) \end{aligned} \tag{5.15}$$

Note that, since each row of $\hat{e}_k(x)$ is linearly dependent on the rows of $\tilde{E}_k(x)$, $\bar{\phi}_k(x)$ is independent of the choice of $\tilde{E}_k^+(x)$. Assume that $\bar{\phi}_k(x_0) = 0$ and that $\bar{\phi}_k(x)$ has constant rank s_k on a neighborhood of x_0 in N_{k-1}. Then locally around x_0, $N_k := \{x \in N_{k-1} \mid \bar{\phi}_k(x) = 0\}$ is an $(n - p_k)$-dimensional submanifold through x_0, with $p_k := p_{k-1} + s_k$. Permute the entries of $\bar{\phi}_k$ such that the first s_k entries are independent on N_k, and denote $\phi_k := \mathrm{col}(\bar{\phi}_{k1}, \cdots, \bar{\phi}_{ks_k})$, $\Phi_k := (\Phi_{k-1}^T, \phi_k^T)^T$. ■

Remark 5.0.8 Note that Algorithm 5.0.7 is close to Krener's algorithm 2.2.8. The main difference is that in Algorithm 5.0.7 we do not have to compute the matrices $\beta_k(x)$, nor $\tau_k(x)$. Moreover, the constant rank assumptions in Algorithm 5.0.7 need only be made on neighborhoods of x_0 contained in the submanifolds N_k. ■

Definition 5.0.9 Regular point
We call $x_0 \in M$ a regular point for Algorithm 5.0.7 if this algorithm can be applied successfully to (5.2), i.e., at every step of the algorithm both constant rank assumptions are satisfied and $\bar{\phi}_k(x_0) = 0$ for all k. ■

Suppose that x_0 is a regular point for Algorithm 5.0.7. Then it is straightforward to show that the submanifolds N_k as produced by Algorithm 5.0.7 locally equal the *intrinsically defined* submanifolds N_k of Algorithm 5.0.5. Hence, the N_k of Algorithm 5.0.7 do not depend on the particular choice of the right inverse of $\tilde{E}_k$, nor on the selection of the independent entries of $\bar{\phi}_k$. Moreover, since the E_k have constant rank around x_0, $\dim(T_xN_k + \mathcal{G}(x))$ is constant on a neighborhood of x_0 in N_k ($k \geq 0$). This implies by Lemma 5.0.3 that $N^* = N_{k^*}$ as produced by Algorithm 5.0.7 is the maximal locally controlled invariant output-nulling submanifold around x_0.

N^* is called the *clamped dynamics manifold* of (5.2) around x_0. The feedbacks $u = \alpha^*(x)$, $x \in N^*$, which render N^* invariant, are given by the solutions of

$$\tilde{D}_{k^*}(x) + \tilde{E}_{k^*}(x)\alpha^*(x) = 0 \tag{5.16}$$

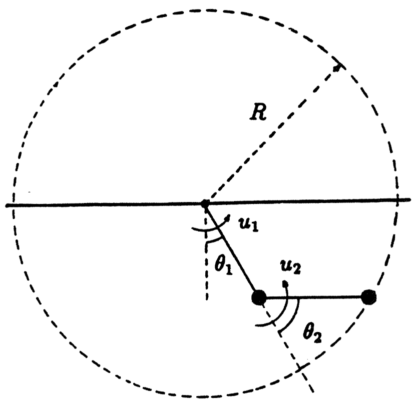

Figure 5.1: Constrained two-link robot arm

Denote $\tilde{m} := m - r_{k^*}$. Let $\tilde{E}_{k^*}^{+}(x)$ be a right inverse of $\tilde{E}_{k^*}(x)$ and define $\bar{\alpha}(x) := -\tilde{E}_{k^*}^{+}(x)\tilde{D}_{k^*}(x)$. Furthermore, let $\bar{\beta}(x)$ be an $(m, \tilde{m})$-matrix of full column rank satisfying $\tilde{E}_{k^*}(x)\bar{\beta}(x) = 0$. Then the full solution set of (5.16) is given as

$$\alpha^*(x) = \bar{\alpha}(x) + \bar{\beta}(x)v \tag{5.17}$$

where $v = \mathrm{col}(v_1, \cdots, v_{\tilde{m}}) \in I\!R^{\tilde{m}}$ can be chosen arbitrarily. Alternatively, we can interpret v as new controls and (5.17) as a nonregular static state feedback applied to (5.2). The system (5.2,5.17) restricted to N^* is then called the *clamped dynamics* of (5.2), i.e., all motions of (5.2) compatible with the constraints $h(x) = 0$.

We illustrate the theory developed so far with an example.

Example 5.0.10 Consider a frictionless, rigid two-link robot arm with control torques u_1 and u_2 applied at the joints. For simplicity we assume that both links have unit length and unit mass, where the mass is concentrated at the end of each link.

The dynamics of this manipulator are described by (see e.g. [40])

$$\frac{d}{dt}\begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ -M^{-1}(\theta)C(\theta,\dot{\theta}) - M^{-1}(\theta)k(\theta) \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(\theta) \end{pmatrix} u \tag{5.18}$$

where $\theta = \mathrm{col}(\theta_1, \theta_2)$, $u = \mathrm{col}(u_1, u_2)$ and

$$
\begin{aligned}
M(\theta) &= \begin{pmatrix} 3 + 2\cos\theta_2 & 1 + \cos\theta_2 \\ 1 + \cos\theta_2 & 1 \end{pmatrix} \\
C(\theta, \dot{\theta}) &= \begin{pmatrix} -\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2 \\ \dot{\theta}_1^2 \sin\theta_2 \end{pmatrix} \\
k(\theta) &= \begin{pmatrix} -2g\sin\theta_1 - g\sin(\theta_1 + \theta_2) \\ -g\sin(\theta_1 + \theta_2) \end{pmatrix} \\
g &= \text{constant of gravity}
\end{aligned}
$$

The matrix $M(\theta)$ is called the *inertia matrix* of the robot arm. The term $C(\theta, \dot{\theta})$ represents the *centripetal and Coriolis forces*, while the term $k(\theta)$ represents the *gravitational forces.* Assume that we want to control the robot arm in such a way that the endpoint remains on a circle with radius R, $1 < R < 2$, and with center at the base of the robot arm. The distance of the endpoint to the base of the robot arm is equal to $\sqrt{2 + 2\cos\theta_2}$. Hence if we define the output of (5.18) as

$$
y = 2 + 2\cos\theta_2 - R^2 \tag{5.19}
$$

the possible movements along a circle with radius R are given by the clamped dynamics of (5.18,5.19). Applying Algorithm 5.0.7 to (5.18,5.19) yields

$$
N^* = \{(\theta, \dot{\theta}) \mid \cos\theta_2 = \frac{R^2 - 2}{2}, \dot{\theta}_2 = 0\} \tag{5.20}
$$

and (on N^*)

$$
\begin{aligned}
&\tfrac{2\sqrt{4-R^2}(R^3+R)}{(R^4-4R^2-4)}\dot{\theta}_1^2 - \tfrac{g\sqrt{4-R^2}(R^3+2R)}{(R^4-4R^2-4)}\cos\theta_1 - g\sin\theta_1 + \\
&\qquad \tfrac{2R^2}{(R^4-4R^2-4)}\alpha_1^*(\theta, \dot{\theta}) - \tfrac{4(R^2+1)}{(R^4-4R^2-4)}\alpha_2^*(\theta, \dot{\theta}) = 0
\end{aligned} \tag{5.21}
$$

Hence

$$
\begin{aligned}
&\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_1^*(\theta, \dot{\theta}) \\ \alpha_2^*(\theta, \dot{\theta}) \end{pmatrix} = \\
&\begin{pmatrix} 0 \\ 2R\dot{\theta}_1^2\sqrt{4 - R^2} - \tfrac{g\sqrt{4-R^2}(R^3+2R)}{4(R^2+1)}\cos\theta_1 - \tfrac{g(R^4-4R^2-4)}{4(R^2+1)}\sin\theta_1 \end{pmatrix} + \begin{pmatrix} 1 \\ \tfrac{R^2}{2(R^2+1)} \end{pmatrix} v
\end{aligned} \tag{5.22}
$$

Note that on N^* the quantities θ_2 and $\dot{\theta}_2$ are constant. Hence we can take θ_1 and $\dot{\theta}_1$ as coordinates for the clamped dynamics. The clamped dynamics then are given by

$$
\frac{d}{dt}\begin{pmatrix} \theta_1 \\ \dot{\theta}_1 \end{pmatrix} = \begin{pmatrix} \dot{\theta}_1 \\ \tfrac{gR\sqrt{4-R^2}}{2(R^2+1)}\cos\theta_1 + \tfrac{g(R^2+2)}{2(R^2+1)}\sin\theta_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \tfrac{1}{R^2+1} \end{pmatrix} v \tag{5.23}
$$

Having designed a control (5.22) such that the endpoint of the robot arm remains on a circle with radius R, one might question if the motion of the endpoint along the circle can be controlled. The answer to this question is positive. Assume that a reference trajectory $\theta_{1r}(t)$ ($t \geq 0$) is given. Then it is easy to check that by choosing $\theta_1(0) = \theta_{1r}(0)$, $\dot{\theta}_1(0) = \dot{\theta}_{1r}(0)$ and

$$v(t) = (R^2+1)\dot{\theta}_{1r}(t) - \frac{gR}{2}\sqrt{4-R^2}\cos\theta_{1r}(t) - \frac{g}{2}(R^2+2)\sin\theta_{1r}(t) \tag{5.24}$$

in (5.23), we have $\theta_1(t) = \theta_{1r}(t)$ for $t \geq 0$. ∎

We conclude this chapter with two propositions that will prove their usefulness in the following chapter. The first result, that is taken from [50], gives a connection between the clamped dynamics manifold af a system (5.2) and the clamped dynamics manifold of a system (5.2) with a dynamic state feedback (3.42).

Proposition 5.0.11 *Consider a nonlinear system (5.2) and let x_0 be a regular point for Algorithm 5.0.7 applied to (5.2). Let the clamped dynamics manifold N^* of (5.2) around x_0 be given by $N^* = \{x \mid \Phi_{k^*}(x) = 0\}$. Furthermore, consider a dynamic state feedback (3.42) for (5.2) and let (x_0, z_0) be a regular point for Algorithm 5.0.7 applied to (5.2,3.42). Then there exists a vector of functions $\Psi(x,z)$ such that M^*, the clamped dynamics manifold of (5.2,3.42) around (x_0,z_0), is given by $M^* = \{(x,z) \mid \Phi_{k^*}(x) = 0, \Psi(x,z) = 0\}$.*

Proof See Appendix C. ∎

The following result was first derived in [102] (see also [81]). It gives a connection between Singh's algorithm and Algorithm 5.0.7. For this, consider the following augmented system obtained from (5.2):

$$\begin{array}{rcll} \dot{x} & = & f(x) + g(x)u & \\ \dot{w}_{i0} & = & w_{i1} & \\ & \vdots & & (i = 1,\cdots,p) \\ \dot{w}_{in-1} & = & v_i & \\ \bar{y}_i & = & h_i(x) - w_{i0} & (i = 1,\cdots,p) \end{array} \tag{5.25}$$

with controls $(u_1,\cdots,u_m,v_1,\cdots,v_p)$ and outputs $(\bar{y}_1,\cdots,\bar{y}_p)$.

Let a point $x_0 \in M$ be given. Assume that there exists a $w^0 := (w^0_{ij} \mid 1 \leq i \leq p, 0 \leq j \leq n-1)$ such that (x_0, w^0) is a regular point for Algorithm 5.0.7 applied to (5.25). Then obviously we have that $w^0_{*0} = h(x_0)$ $(i = 1,\cdots,m)$. Applying the first step of Algorithm 5.0.7 to (5.25), we find

$$\dot{\phi}_0 = d_1(x,w) + e_1(x)u = a_1(x) - w_{*1} + b_1(x)u \tag{5.26}$$

where a_1, b_1 are obtained by applying the first step of Singh's algorithm to (5.2). Since (x_0, w^0) is a regular point for Algorithm 5.0.7 applied to (5.25), the matrix $b_1(x)$ has constant rank r_1 on a neighborhood of x_0. After a possible permutation of the entries of ϕ_0 we may assume that the first r_1 rows of $b_1(x)$ are independent on this neighborhood. Accordingly, write (5.26) as

$$\begin{pmatrix} \dot{\tilde{\phi}}_0 \\ \dot{\hat{\phi}}_0 \end{pmatrix} = \begin{pmatrix} \tilde{a}_1(x) - \tilde{w}^1_{*1} + \tilde{b}_1(x)u \\ \hat{a}_1(x) - \hat{w}^1_{*1} + \hat{b}_1(x)u \end{pmatrix} \tag{5.27}$$

where $\tilde{b}_1(x)$ has full row rank r_1. Let $\tilde{b}_1^+(x)$ be a right inverse of $\tilde{b}_1(x)$ and set

$$u = -\tilde{b}_1^+(x)(\tilde{a}_1(x) - \tilde{w}_{*1}^1) \tag{5.28}$$

Then (5.27) and (5.28) yield

$$\begin{pmatrix} \dot{\tilde{\phi}}_0 \\ \dot{\hat{\phi}}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{a}_1(x) - \hat{w}_{*1}^1 - \hat{b}_1(x)\tilde{b}_1^+(x)(\tilde{a}_1(x) - \tilde{w}_{*1}^1) \end{pmatrix} =: \begin{pmatrix} 0 \\ \bar{\phi}_1(x,w) \end{pmatrix} \tag{5.29}$$

Note that by its structure, $\bar{\phi}_1(x,w)$ has full rank $s_1 = p - r_1$. Hence we have $\phi_1(x,w) = \bar{\phi}_1(x,w)$. Furthermore, if we compare (5.29) with the final result of the first step of Singh's algorithm applied to (5.2), we find that

$$\phi_1(x,w) = \dot{\hat{y}}_1(x, \dot{\tilde{y}}_1) \mid_{\dot{\tilde{y}}_1 = \tilde{w}_{*1}^1} - \hat{w}_{*1}^1 \tag{5.30}$$

Hence at the second step of Algorithm 5.0.7 applied to (5.25) we have

$$\dot{\phi}_1 = d_2(x,w) + e_2(x,w)u = a_2(x, \tilde{w}_{*1}^1, \tilde{w}_{*2}^1) - \hat{w}_{*2}^1 + b_2(x, \tilde{w}_{*1}^1)u \tag{5.31}$$

where a_2, b_2 are obtained by applying the second step of Singh's algorithm to (5.2). If we proceed by applying the above arguments repeatedly, we obtain the following result.

Proposition 5.0.12 *Consider the nonlinear system (5.2) and the augmented system (5.25) obtained from (5.2). Let a point $x_0 \in M$ be given. Then there exists the following connection between Singh's algorithm applied to (5.2) and Algorithm 5.0.7 applied to (5.25).*

(i) There exists a $w^0 = (w_{ij}^0 \mid 1 \leq i \leq p, 0 \leq j \leq n-1)$ such that (x_0, w^0) is a regular point for Algorithm 5.0.7 applied to (5.25) if and only if x_0 is a regular point for Singh's algorithm applied to (5.2).

(ii) If there exists a w^0 such that (x_0, w^0) is a regular point for Algorithm 5.0.7 applied to (5.25), then $r_i = \rho_i - \rho_{i-1}$ $(i = 1, \cdots, n)$, with $\rho_0 := 0$ and $\rho_1, \cdots, \rho_n$ as defined in Singh's algorithm.

(iii) Assume that Algorithm 5.0.7 is applied to (5.25) around a regular point (x_0, w^0). Then there is an application of Singh's algorithm to (5.2) such that the result of this application of Singh's algorithm to (5.2) can be obtained from the result of Algorithm 5.0.7 applied to (5.25). Conversely, assume that $x_0 \in M$ is a regular point for Singh's algorithm applied to (5.2). Then there is an application of Algorithm 5.0.7 to (5.25) around a regular point (x_0, w^0) such that the result of this application of Algorithm 5.0.7 to (5.25) can be obtained from the result of Singh's algorithm applied to (5.2). More specifically, assume that Algorithm 5.0.7 applied to (5.25) around a regular point (x_0, w^0) yields for $k = 1, \cdots, n$:

$$\begin{aligned} \dot{\phi}_{k-1} &= d_k(x, \{\tilde{w}_{*j}^i \mid 1 \leq i \leq k-1, i \leq j \leq k\}, \hat{w}_{*k}^{k-1}) \\ &\quad + e_k(x, \{\tilde{w}_{*j}^i \mid 1 \leq i \leq k-1, i \leq j \leq k-1\})u \end{aligned} \tag{5.32}$$

where $(\tilde{w}_{*0}^1, \cdots, \tilde{w}_{*0}^n, \hat{w}_{*0}^n)$ is a reordering of the components of w_{*0}, $(\tilde{w}_{*j}^1, \cdots, \tilde{w}_{*j}^n, \hat{w}_{*j}^n)$ is a corresponding reordering of the components of w_{*j} and $\hat{w}_{*k}^{k-1} = (\tilde{w}_{*k}^k, \cdots, \tilde{w}_{*k}^n, \hat{w}_{*k}^n)$. Furthermore, assume that Singh's algorithm applied to (5.2) yields for $k = 1, \cdots, n$

$$\begin{aligned} \hat{y}_{k-1}^{(k)} &= a_k(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq k-1, i \leq j \leq k\}) + \\ &\quad b_k(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq k-1, i \leq j \leq k-1\})u \end{aligned} \tag{5.33}$$

where $\hat{y}_k := (\tilde{y}_{k+1}, \cdots, \tilde{y}_n, \hat{y}_n)$. *Then the following connection between* d_k, e_k, a_k, b_k *exists:*

$$\begin{aligned}
&a_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k\}) = \\
&\qquad d_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k\}, 0) \\
&b_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1\}) = \\
&\qquad e_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1\})
\end{aligned} \tag{5.34}$$

$$\begin{aligned}
&d_k(x, \{\tilde{w}_{*j}^i \mid 1 \le i \le k-1, i \le j \le k\}, \hat{w}_{*k}^{k-1}) = \\
&\qquad a_k(x, \{\tilde{w}_{*j}^i \mid 1 \le i \le k-1, i \le j \le k\}) - \hat{w}_{*k}^{k-1} \\
&e_k(x, \{\tilde{w}_{*j}^i \mid 1 \le i \le k-1, i \le j \le k-1\}) = \\
&\qquad b_k(x, \{\tilde{w}_{*j}^i \mid 1 \le i \le k-1, i \le j \le k-1\})
\end{aligned} \tag{5.35}$$

■

Chapter 6

Disturbance decoupling via nonregular dynamic state feedback

"Only the poet or the saint can water an asphalt pavement in the confident anticipation that lilies will reward his labor."

W. Somerset Maugham, The moon and sixpence.

In this chapter we consider a nonlinear analytic system Σ of the form

$$\Sigma \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ y & = & h(x) \end{array} \right. \tag{6.1}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ are local coordinates for the state space manifold M, $u \in \mathbb{R}^m$, $q \in \mathbb{R}^r$, $y \in \mathbb{R}^p$, $g(x) = (g_1(x) \cdots g_m(x))$, $p(x) = (p_1(x) \cdots p_r(x))$, $h(x) = \mathrm{col}(h_1(x), \cdots, h_p(x))$, $f, g_1, \cdots, g_m, p_1, \cdots, p_r$ are analytic vector fields on M and $h_1, \cdots, h_p$ are analytic functions on M. The system Σ with $q \equiv 0$ is denoted by Σ_0. Recall that the distributions $\mathcal{G}$ and $\mathcal{P}$ are defined by $\mathcal{G} := \mathrm{span}\,\{g_1, \cdots, g_m\}$ and $\mathcal{P} := \mathrm{span}\,\{p_1, \cdots, p_r\}$.

We have seen in Example 2.2.14 that if the DDP(dm) is not solvable for Σ, we may still be able to render the outputs independent of the disturbances by allowing nonregular (dynamic or static) state feedback. This gives rise to the formulation of the disturbance decoupling problem via nonregular dynamic state feedback (nDDDP) and the disturbance decoupling problem via nonregular dynamic state feedback and disturbance measurements (nDDDPdm). In Section 6.1 we formulate both problems and we discuss the ideas behind the algorithm that is presented in Section 6.2 to solve the nDDDP(dm).

When no specific references are given, the results in this chapter may be found in [50].

6.1 Formulation of the problem and ideas behind the algorithm

We first formulate the nDDDP and the nDDDPdm. As for the DDDP and the DDDPdm in Chapter 4, we restrict ourselves to a local formulation and solution of both problems.

Definition 6.1.1 Disturbance decoupling problem via nonregular dynamic state feedback (nDDDP)

Consider the system Σ and let a point $x_0 \in M$ be given. Then the nDDDP is said to be solvable around x_0 if there exist a dynamic state feedback R of the form

$$R \left\{ \begin{array}{rcl} \dot{z} & = & \alpha(x,z) + \beta(x,z)v \\ u & = & \gamma(x,z) + \delta(x,z)v \end{array} \right. \tag{6.2}$$

with $z \in I\!R^{\nu}$, $v \in I\!R^{m}$ denoting the new controls and $\alpha, \beta, \gamma, \delta$ analytic functions of x and z, a neighborhood $U \subset M$ of x_0, an open subset $\mathcal{Z} \subset I\!R^{\nu}$ and a map $F : U \to \mathcal{Z}$ with the property that for all $\bar{x} \in U$ the outputs of $\Sigma \circ R$ restricted $U \times \mathcal{Z}$ and initialized at $(\bar{x}, F(\bar{x}))$ are independent of q. ∎

Definition 6.1.2 Disturbance decoupling problem via nonregular dynamic state feedback and disturbance measurements (nDDDPdm)
Consider the system Σ and let a point $x_0 \in M$ be given. Then the nDDDPdm is said to be solvable around x_0 if there exist a dynamic state feedback with disturbance feedthrough Q of the form

$$Q \left\{ \begin{array}{rcl} \dot{z} & = & \alpha(x,q,z) + \beta(x,q,z)v \\ u & = & \gamma(x,q,z) + \delta(x,q,z)v \end{array} \right. \tag{6.3}$$

with $z \in I\!R^{\nu}$, $v \in I\!R^{m}$ denoting the new controls and $\alpha, \beta, \gamma, \delta$ analytic functions of x,q and z, a neighborhood $U \subset M$ of x_0, an open subset $\mathcal{Z} \subset I\!R^{\nu}$ and a map $F : U \to \mathcal{Z}$ with the property that for all $\bar{x} \in U$ the outputs of $\Sigma \circ Q$ restricted to $U \times \mathcal{Z}$ and initialized at $(\bar{x}, F(\bar{x}))$ are independent of q. ∎

In Section 6.2 we give an algorithm to solve the nDDDP(dm). To get a better insight in how the algorithm works, we first discuss the ideas behind the algorithm. Consider a system Σ with disturbances, of the form (6.1), and assume that we wish to solve the nDDDP around a strongly regular point $x_0 \in M$.

We start by applying Singh's algorithm to Σ_0. Recall that the distribution $\mathcal{P}_e$ on $M_e := M \times I\!R^{nm}$ is defined by $\mathcal{P}_e := \mathcal{P}_e \times \{0\}$ (see Chapter 4). If for $k = 0, \cdots, n-1$ we have that $\mathcal{P}_e \subset \operatorname{Ker} d\hat{y}_k^{(k)}$, we know by Theorem 4.0.6 that we can solve the DDDP for Σ around x_0 and hence the nDDDP is solvable for Σ around x_0.

Assume that for some $k \in \{0, 1, \cdots, n-1\}$ we have that $\mathcal{P}_e \not\subset \operatorname{Ker} d\hat{y}_k^{(k)}$ and let $\tau_1 := \min\{k \in I\!N \mid \mathcal{P}_e \not\subset \operatorname{Ker} d\hat{y}_k^{(k)}\}$. Then for Σ

$$\begin{array}{rcl} \hat{y}_{\tau_1}^{(\tau_1+1)} & = & \dfrac{\partial \hat{y}_{\tau_1}^{(\tau_1)}}{\partial x}[f(x) + g(x)u + p(x)q] + \sum\limits_{i=1}^{\tau_1} \sum\limits_{j=i}^{\tau_1} \dfrac{\partial \hat{y}_{\tau_1}^{(\tau_1)}}{\partial \tilde{y}_i^{(j)}} \tilde{y}_i^{(j+1)} =: \\ & & a_{\tau_1+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le \tau_1 + 1, i \le j \le \tau_1\}) + \\ & & b_{\tau_1+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le \tau_1, i \le j \le \tau_1\})u + \\ & & D^1(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le \tau_1, i \le j \le \tau_1\})q \end{array} \tag{6.4}$$

where $D^1 \not\equiv 0$. Hence there is at least one entry of $\hat{y}_{\tau_1}$ that is affected by q. Moreover, using arguments from Section 3.1, it can be shown that this q-dependence is structural.

This means that at points $(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau_1, i \leq j \leq \tau_1\})$ where $D^1(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \tau_1, i \leq j \leq \tau_1\}) \neq 0$, $\hat{y}_{\tau_1}$ is affected by q. Hence, in order to render the outputs independent of the disturbances, we have to assure that $D^1(x(t), \{\tilde{y}_i^{(j)}(t) \mid 1 \leq i \leq \tau_1, i \leq j \leq \tau_1\}) = 0$ for all $t \geq 0$. Therefore we proceed by starting again with applying Singh's algorithm to Σ_0 under the constraint that D^1 is zero for all $t \geq 0$. Note that what is proposed here is in fact a combination of applying Singh's algorithm (with respect to the outputs $y = h(x)$) to Σ_0 and applying the clamped dynamics algorithm (with respect to the extra "output" D^1). Using Proposition 5.0.12, this combination can be incorporated in one algorithm. Namely, let $\hat{D}^1$ be a vector of functions consisting of the non-zero entries of D^1 and consider the following augmented system obtained from Σ:

$$\Sigma_a^2 \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ \dot{w}_{i0} & = & w_{i2} \\ & \vdots & \qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \cdots, p) \\ \dot{w}_{i\nu_2 - 1} & = & v_i \\ y^2 & = & \begin{pmatrix} h(x) - w_{*0} \\ \\ \hat{D}^1(x, \{\tilde{w}_{ij} \mid 1 \leq i \leq \tau_1, i \leq j \leq \tau_1\}) \end{pmatrix} \end{array} \right. \tag{6.5}$$

where $\nu_2 = n + \tau_1$ and $w_{*0} = \mathrm{col}(w_{10}, \cdots, w_{p0})$. Denote $w^2 := (w_{ij} \mid 1 \leq i \leq p, 0 \leq j \leq \nu_2 - 1)$. As for Σ, let Σ_{a0}^2 denote the system obtained from Σ_a^2 by setting $q \equiv 0$. Then from Proposition 5.0.12 it is clear that the combination of Singh's algorithm and the clamped dynamics algorithm described before, is equivalent to applying the clamped dynamics algorithm to Σ_{a0}^2, where w_{ij} takes the role of $y_i^{(j)}$. While applying the clamped dynamics algorithm, we again have to check at every step if a structural dependence of the outputs of Σ on the disturbances appears. For this, denote the submanifold obtained at the k-th step of applying the clamped dynamics algorithm to Σ_{a0}^2 by N_k^2 and the vector of functions obtained at the k-th step of the clamped dynamics algorithm applied to Σ_{a0}^2 by $\phi_k^2(x, w^2)$. Then it can be shown that a structural q-dependence does not appear if and only if for all $k \in \{0, \cdots, \nu_2 - 1\}$ and all $(x, w^2) \in N_k^2$

$$\mathcal{P}_e(x) \subset T_{(x,w^2)} N_k^2 \tag{6.6}$$

If (6.6) indeed holds, we can solve the nDDDP via a dynamic state feedback that is derived in the following section. If (6.6) does not hold, we define $\tau_2 := \min\{k \in I\!N \mid \exists (x, w^2) \in N_k^2$ such that $\mathcal{P}_e(x) \not\subset T_{(x,w^2)} N_k^2\}$ and $D^2(x, w^2) := (\partial \phi_{\tau_2}^2 / \partial x) p(x)$. D^2 characterizes the structural dependence of the outputs of Σ_a^2 on the disturbances. We then proceed by defining a new augmented system in the same way as described above, and applying the clamped dynamics algorithm to this system. If at a certain step σ^* we have that for all $k \in I\!N$ and for all $(x, w^{\sigma^*}) \in N_k^{\sigma^*}$: $\mathcal{P}_e(x) \subset T_{(x,w^{\sigma^*})} N_k^{\sigma^*}$, we can construct a dynamic state feedback for Σ that locally solves the nDDDP around x_0. The construction of this dynamic state feedback is given in the following section.

While going through the procedure described above, two intermediate checks should be applied. First, it may occur at a certain step that the submanifold on which the constraints are satisfied, is empty. In this case the nDDDP is not solvable and we can stop the procedure. Second, assume that at a certain step the set of constraint functions is given by $\Psi(x, w^\sigma)$. Then to guarantee solvability of the nDDDP around x_0, necessarily there must be a neighborhood $U \subset M$ of x_0 such that for every $\bar{x} \in U$ there is a $\bar{w}^\sigma$ satisfying $\Psi(\bar{x}, \bar{w}^\sigma) = 0$. If this is not the case, the nDDDP is not locally solvable and we can stop the procedure.

If we want to solve the nDDDPdm instead of the nDDDP, the procedure described above follows the same lines, with a few minor differences that are clarified in the following section.

6.2 Solution of the problem

Based on the considerations in Section 6.1, we formulate the following algorithm for solving the nDDDP.

Algorithm 6.2.1 Solving the nDDDP
Consider the system Σ and let a point $x_0 \in M$ be given.

Step 0

Define the augmented system Σ_a^1 obtained from Σ by

$$\Sigma_a^1 \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ \dot{w}_{i0} & = & w_{i1} \\ & \vdots & \qquad\qquad\qquad\qquad (i = 1, \cdots, p) \\ \dot{w}_{i\nu_1 - 1} & = & v_i \\ y^1 & = & h(x) - w_{*0} \end{array} \right. \tag{6.7}$$

where $\nu_1 = n$ and $w_{*0} := \mathrm{col}(w_{10}, \cdots, w_{p0})$. Denote $w^1 := (w_{ij} \mid 1 \le i \le p, 0 \le j \le \nu_1 - 1)$.

Step σ+1

Let $\nu_1, \cdots, \nu_\sigma$ and $\Sigma_a^1, \cdots, \Sigma_a^\sigma$ be defined. Let Σ_{a0}^σ denote the system obtained from Σ_a^σ by setting $q \equiv 0$. Assume that there is a point $\bar{w}^\sigma$ such that $(x_0, \bar{w}^\sigma)$ is a regular point for Algorithm 5.0.7 applied to Σ_a^σ. Apply Algorithm 5.0.7 to Σ_a^σ. Employ the notation of Algorithm 5.0.7 with a superscript σ. Let τ_σ be the smallest integer satisfying

$$\frac{\partial \phi_{\tau_\sigma}^\sigma}{\partial x} p(x) \not\equiv 0 \tag{6.8}$$

and define $D^\sigma(x, w^\sigma) := (\partial \phi_{\tau_\sigma}^\sigma / \partial x) p(x)$. If it turns out that $\tau_\sigma \ge \nu_\sigma$, set $\tau_\sigma := \nu_\sigma$.

Define the subset $p(N_{\tau_\sigma}^\sigma) \subset M$ by

$$p(N_{\tau_\sigma}^\sigma) := \{x \in M \mid \exists w^\sigma : \ (x, w^\sigma) \in N_{\tau_\sigma}^\sigma\} \tag{6.9}$$

Distinguish the following cases (which should be checked sequentially).

1. $N_{\tau_\sigma}^\sigma = \emptyset$. In this case we stop.

2. There is no neighborhood $\bar{U} \subset M$ of x_0 such that $p(N_{\tau_\sigma}^\sigma) \cap \bar{U} = \bar{U}$. In this case we stop.

3. $\tau_\sigma = \nu_\sigma$. In this case we stop, defining $N^* := N_{\tau_\sigma}^\sigma$.

4. If none of the cases 1,2,3 hold, define

$$D^\sigma(x, w^\sigma) := \frac{\partial \phi^\sigma_{\tau_\sigma}}{\partial x} p(x) \tag{6.10}$$

and $\nu_{\sigma+1} := \nu_\sigma + \tau_\sigma$. Let $\hat{D}^\sigma$ be the vector of functions consisting of the non-zero entries of D^σ. Then define the system

$$\Sigma_a^{\sigma+1} \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ \dot{w}_{i0} & = & w_{i1} \\ & \vdots & \qquad\qquad\qquad\qquad (i = 1, \cdots, p) \\ \dot{w}_{i\nu_{\sigma+1}-1} & = & v_i \\ y^{\sigma+1} & = & \begin{pmatrix} \Phi^\sigma_{\tau_\sigma} \\ \\ \hat{D}^\sigma \end{pmatrix} \end{array} \right. \tag{6.11}$$

and denote $w^{\sigma+1} := (w_{ij} \mid 1 \leq i \leq p, 0 \leq j \leq \nu_{\sigma+1} - 1)$. ■

We associate the following notion of regularity with Algorithm 6.2.1.

Definition 6.2.2 *Consider a nonlinear system Σ and let a point $x_0 \in M$ be given. We call x_0 a regular point for Algorithm 6.2.1 if the algorithm can be applied successfully to Σ around x_0, i.e., at each step σ of the algorithm there is a $\bar{w}^\sigma$ such that $(x_0, \bar{w}^\sigma)$ is a regular point for Algorithm 5.0.7 applied to Σ^σ_{a0}.* ■

It can be shown that if x_0 is a regular point for Algorithm 6.2.1 applied to Σ, the algorithm applied to Σ around x_0 terminates in a finite number of steps. The following theorem gives a solution of the nDDDP:

Theorem 6.2.3 *Consider a nonlinear system Σ and let $x_0 \in M$ be a regular point for Algorithm 6.2.1 applied to Σ. Then the nDDDP is solvable around Σ if and only if Algorithm 6.2.1 applied to Σ terminates because of case 3.*

In the proof of Theorem 6.2.3 we need the following lemma:

Lemma 6.2.4 *Consider a nonlinear system Σ and let $x_0 \in M$ be a regular point for Algorithm 6.2.1 applied to Σ. Define the system $\bar{\Sigma}^1_a$ by*

$$\bar{\Sigma}^1_a \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ \dot{w}_{i0} & = & w_{i1} \\ & \vdots & \qquad\qquad\qquad\qquad (i = 1, \cdots, p) \\ \dot{w}_{i\nu_{\sigma^*}-1} & = & v_i \\ y^1 & = & h(x) - w_{*0} \end{array} \right. \tag{6.12}$$

where $(u, v_1, \cdots, v_p)$ are the controls and y^1 the outputs. For $\sigma = 2, \cdots, \sigma^*$, let the systems $\bar{\Sigma}^\sigma_a$ be defined as $\bar{\Sigma}^1_a$ with y^1 replaced by $y^\sigma = (\Phi^{\sigma-1^T}_{\tau_{\sigma-1}} \;\; \hat{D}^{\sigma^T})^T$. As before, let $\bar{\Sigma}^\sigma_{a0}$ denote the *system obtained from $\bar{\Sigma}_a^\sigma$ by setting $q \equiv 0$. Denote $w := (w_{ij} \mid 1 \le i \le p, 0 \le j \le \nu_{\sigma^*} - 1)$. Consider a nonregular dynamic state feedback R for Σ of the form*

$$R \left\{ \begin{array}{rcl} \dot{z} & = & \alpha(x,z) \\ u & = & \gamma(x,z) \end{array} \right. \tag{6.13}$$

with $z \in I\!\!R^\nu$. Denote the clamped dynamics manifold of $\bar{\Sigma}_{a0}^\sigma$ by $\bar{N}^\sigma$ and that of $\bar{\Sigma}_{a0}^\sigma$ by $\bar{M}^\sigma$. Denote the distribution spanned by the disturbance vector fields of $\bar{\Sigma}_a^\sigma$ by $\mathcal{P}_e$. Assume that R solves the nDDDP for Σ. Then

$$\bar{M}^1 = \cdots = \bar{M}^{\sigma^*} \tag{6.14}$$

and

$$\forall\ (x,w,z) \in \bar{M}^1 :\ \mathcal{P}_e(x) \subset T_{(x,w,z)}\bar{M}^1 \tag{6.15}$$

Proof By the structure of $\bar{\Sigma}_a^1$ it is clear that R also solves the nDDDP for $\bar{\Sigma}_a^1$. By Theorem 2.1.10 this means that $\mathcal{P}_e \subset \Delta_e^*$, where Δ_e^* is the maximal locally controlled invariant distribution contained in $\operatorname{Ker} dy^1$ for $\bar{\Sigma}_a^1 \circ R$. It is straightforwardly checked that $\Delta_e^* = \bigcap_{k=1}^{\nu_{\sigma^*}-1} \operatorname{Ker} dy^{1^{(k)}}$, where $y^{1^{(k)}}$ denotes the k-th time-derivative of y^1. Furthermore, we can check that $\bar{M}^1 = \{(x,w,z) \mid y^{1^{(k)}}(x,w,z) = 0, 0 \le k \le \nu_{\sigma^*} - 1\}$. Hence $\bar{M}^1$ is an integral submanifold of Δ_e^*. This establishes (6.15). Next note that the systems $\bar{\Sigma}_a^\sigma$ only differ from the systems Σ_a^σ in that the number of w_{ij}'s is larger. Hence on $\bar{N}^\sigma$ we have that $\Phi_{\tau_1}^1(x,w) = 0$. By Proposition 5.0.11 this implies that we can find a vector of functions $\Psi^1(x,w,z)$ such that $\bar{M}^1$ can alternatively be written as $\bar{M}^1 = \{(x,w,z) \mid \Phi_{\tau_1}^1(x,w) = 0, \Psi^1(x,w,z) = 0\}$. By definition of τ_1 we have that $(\partial\Phi_{\tau_1}^1/\partial x)p(x) \not\equiv 0$. Hence, since (6.15) holds, we must also have that $\hat{D}^1(x,w) = 0$ on $\bar{M}^1$. Together with the fact that $\Phi_{\tau_1}^1(x,w) = 0$ on $\bar{M}^1$ this implies that $\bar{M}^2 = \bar{M}^1$. Using the above arguments and an induction argument, we establish (6.14). ∎

Proof of Theorem 6.2.3
(sufficiency) Assume that Algorithm 6.2.1 applied to Σ terminates because of case 3. Then obviously N^* is a locally controlled invariant output-nulling submanifold for $\Sigma_{a0}^{\sigma^*}$ and all vector fields in $\mathcal{P}_e$ are tangent to N^*. Let $\bar{\Sigma}_a^1$ be defined as in Lemma 6.2.4. Then N^* is also a locally controlled invariant output-nulling submanifold for $\bar{\Sigma}_{a0}^1$. Since the algorithm does not terminate because of case 2, there exists a neighborhood $U \subset M$ of x_0 such that $p(N^*) \cap U = U$. This means that there exists a mapping $F : U \to I\!\!R^{p\nu_{\sigma^*}}$ such that for all $\bar{x} \in U$ we have that $(\bar{x}, F(\bar{x})) \in N^*$. Recall from Lemma 6.2.4 that w is defined as $w := (w_{ij} \mid 1 \le i \le p, 0 \le j \le \nu_{\sigma^*} - 1)$. Let $\begin{pmatrix} u \\ v \end{pmatrix} = \alpha^*(x,w)$ be a control that renders N^* invariant for $\bar{\Sigma}_{a0}^1$. Consider the system $\bar{\Sigma}_{a0}^1$ with $\begin{pmatrix} u \\ v \end{pmatrix} = \alpha^*(x,w)$, restricted to N^*. By the above we have for this system that $\mathcal{P}_e \subset \bigcap_{k=0}^{\nu_{\sigma^*}-1} \operatorname{Ker} dh^{(k)}(x,w)$, where $h^{(k)}$ denotes the k-th time-derivative of h, and hence the disturbances do not influence $y = h(x)$ for $\bar{\Sigma}_a^0$ with $\begin{pmatrix} u \\ v \end{pmatrix} = \alpha^*(x,w)$. Let z_i $(i = 1, \cdots, p)$ be a vector of dimension ν_{σ^*} and consider the dynamic state feedback

$$R \left\{ \begin{array}{rcl} \dot{z}_i & = & Az_i + bv_i \quad (i = 1, \cdots, p) \\ \begin{pmatrix} u \\ v \end{pmatrix} & = & \alpha^*(x,z) \end{array} \right. \tag{6.16}$$

with (A, b) in Brunovsky canonical form, initialized at $z(0) = F(x(0))$ for $x(0) \in U$. Then from the above considerations it follows that R solves the nDDDP around x_0.

(necessity) Assume that the nDDDP is locally solvable around x_0 via a dynamic state feedback of the form (6.13). We first show that Algorithm 6.2.1 applied to Σ cannot terminate because of case 1. For $\sigma = 1, \cdots, \sigma^*$, let $\bar{\Sigma}_a^\sigma$ be defined as in Lemma 6.2.4. Furthermore, let M^σ denote the clamped dynamics manifold of $\Sigma_{a0}^\sigma \circ R$ ($\sigma = 1, \cdots, \sigma^*$). Recall that the clamped dynamics manifold of $\bar{\Sigma}_{a0}^\sigma \circ R$ ($\sigma = 1, \cdots, \sigma^*$) is denoted by $\bar{M}^\sigma$ and that by Lemma 6.2.4 and the structure of $\bar{\Sigma}_a^1$ we have that $\bar{M}^1 = \cdots = \bar{M}^{\sigma^*} = \{(x, w, z) \mid \bar{\Phi}(x, z) - w = 0\}$ for some vector of functions $\bar{\Phi}(x, z)$. Noting that Σ_a^σ only differs from $\bar{\Sigma}_a^\sigma$ in that for $\bar{\Sigma}_a^\sigma$ the number of w_{ij}'s is greater, we see that for $\sigma = 1, \cdots, \sigma^*$ there exist vectors of functions $\Phi^\sigma(x, z)$ such that $M^\sigma = \{(x, w^\sigma, z) \mid \Phi^\sigma(x, z) - w^\sigma = 0\}$. For $\sigma = 1, \cdots, \sigma^*$ consider the sets $\tilde{M}^\sigma := \{(x, w^\sigma) \mid \exists z \text{ such that } (x, w^\sigma, z) \in M^\sigma\}$. By the form of M^σ we see that $\tilde{M}^\sigma \neq \emptyset$. By Proposition 5.0.11 we know that $\Phi_{\tau_\sigma}^\sigma(x, w^\sigma) = 0$ on M^σ and hence any $(x, w^\sigma) \in \tilde{M}^\sigma$ necessarily satisfies $\Phi_{\tau_\sigma}^\sigma(x, w^\sigma) = 0$. This implies that $\tilde{M}^\sigma \subset \{(x, w^\sigma) \mid \Phi_{\tau_\sigma}^\sigma(x, w^\sigma) = 0\} = N_{\tau_\sigma}^\sigma$. Since $\tilde{M}^\sigma \neq \emptyset$, this means that also $N_{\tau_\sigma}^\sigma \neq \emptyset$ ($\sigma = 1, \cdots, \sigma^*$) and hence the algorithm does not terminate because of case 1.

Next, assume that the algorithm terminates because of case 2. This means that there is no neighborhood $U \subset M$ of x_0 such that $p(N_{\tau_{\sigma^*}}^{\sigma^*}) \cap U = U$. Hence for any neighborhood $\bar{U} \subset M$ of x_0 there is an $\bar{x} \in \bar{U}$ such that $\Phi_{\tau_{\sigma^*}}^{\sigma^*}(\bar{x}, w^{\sigma^*}) \neq 0$ for all w^{σ^*}. Since by Proposition 5.0.11 any $(x, w^{\sigma^*}, z) \in M^{\sigma^*}$ has to satisfy $\Phi_{\tau_{\sigma^*}}^{\sigma^*}(x, w^{\sigma^*}) = 0$, this implies that $(\bar{x}, w^{\sigma^*}, z) \notin M^{\sigma^*}$ for all (w^{σ^*}, z). However, this contradicts the fact that there exists a vector of functions $\Phi^{\sigma^*}(x, z)$ such that $M^{\sigma^*} = \{(x, w^{\sigma^*}, z) \mid \Phi^{\sigma^*}(x, z) - w^{\sigma^*} = 0\}$. Hence the algorithm cannot terminate because of case 2.

By the above we conclude that if the nDDDP is locally solvable for Σ around x_0, Algorithm 6.2.1 applied to Σ can only terminate because of case 3. ∎

We proceed by giving a solution of the nDDDPdm. For this, consider the following algorithm:

Algorithm 6.2.5 Solving the nDDDPdm
Consider the system Σ and let a point $x_0 \in M$ be given.

Step 0

Define the augmented system Σ_a^1 obtained from Σ by

$$\Sigma_a^1 \begin{cases} \dot{x} &= f(x) + g(x)u + p(x)q \\ \dot{w}_{i0} &= w_{i1} \\ &\vdots \qquad\qquad\qquad\qquad (i = 1, \cdots, p) \\ \dot{w}_{i\nu_1 - 1} &= v_i \\ y^1 &= h(x) - w_{*0} \end{cases} \tag{6.17}$$

where $\nu_1 = n$ and $w_{*0} := \mathrm{col}(w_{10}, \cdots, w_{p0})$. Denote $w^1 := (w_{ij} \mid 1 \le i \le p, 0 \le j \le \nu_1 - 1)$.

Step σ+1

Let $\nu_1, \cdots, \nu_\sigma$ and $\Sigma_a^1, \cdots, \Sigma_a^\sigma$ be defined. Let Σ_{a0}^σ denote the system obtained from Σ_a^σ by setting $q \equiv 0$. Assume that there is a point $\bar{w}^\sigma$ such that $(x_0, \bar{w}^\sigma)$ is a regular point for Algorithm 5.0.7 applied to Σ_a^σ. Apply Algorithm 5.0.7 to Σ_a^σ. Employ the notation of Algorithm 5.0.7 with a superscript σ. Let τ_σ be the smallest integer satisfying

$$(r^\sigma_{\tau_\sigma+1} - r^\sigma_{\tau_\sigma}) = \operatorname{rank}\begin{pmatrix} \frac{\partial \phi^\sigma_{\tau_\sigma}}{\partial x} g(x) \end{pmatrix} < \operatorname{rank}\begin{pmatrix} \frac{\partial \phi^\sigma_{\tau_\sigma}}{\partial x} g(x) & \frac{\partial \phi^\sigma_{\tau_\sigma}}{\partial x} p(x) \end{pmatrix} \tag{6.18}$$

If it turns out that $\tau_\sigma \geq \nu_\sigma$, set $\tau_\sigma := \nu_\sigma$.

Define the subset $p(N^\sigma_{\tau_\sigma}) \subset M$ by

$$p(N^\sigma_{\tau_\sigma}) := \{x \in M \mid \exists w^\sigma : (x, w^\sigma) \in N^\sigma_{\tau_\sigma}\} \tag{6.19}$$

Distinguish the following cases (which should be checked sequentially).

1. $N^\sigma_{\tau_\sigma} = \emptyset$. In this case we stop.

2. There is no neighborhood $\bar{U} \subset M$ of x_0 such that $p(N^\sigma_{\tau_\sigma}) \cap \bar{U} = \bar{U}$. In this case we stop.

3. $\tau_\sigma = \nu_\sigma$. In this case we stop, defining $N^* := N^\sigma_{\tau_\sigma}$.

4. If none of the cases 1,2,3 holds, define

$$\begin{aligned} \tilde{f}^\sigma(x, w^\sigma) &:= \frac{\partial \tilde{\phi}^\sigma_{\tau_\sigma}}{\partial x} p(x) \\ \hat{f}^\sigma(x, w^\sigma) &:= \frac{\partial \hat{\phi}^\sigma_{\tau_\sigma}}{\partial x} p(x) \end{aligned} \tag{6.20}$$

Let $\tilde{e}^{\sigma+}_{\tau_\sigma+1}(x, w^\sigma)$ be a right inverse of $\tilde{e}^\sigma_{\tau_\sigma+1}(x, w^\sigma)$. Then define

$$D^\sigma(x, w^\sigma) := \hat{f}^\sigma(x, w^\sigma) - \hat{e}^\sigma_{\tau_\sigma+1}(x, w^\sigma)\tilde{e}^{\sigma+}_{\tau_\sigma+1}(x, w^\sigma)\tilde{f}(x, w^\sigma) \tag{6.21}$$

and $\nu_{\sigma+1} := \nu_\sigma + \tau_\sigma$. Let $\hat{D}^\sigma$ be the vector of functions consisting of the non-zero entries of D^σ. Then define the system

$$\Sigma_a^{\sigma+1} \left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g(x)u + p(x)q \\ \dot{w}_{i0} &=& w_{i1} \\ &\vdots& \qquad\qquad\qquad (i = 1, \cdots, p) \\ \dot{w}_{i\nu_{\sigma+1}-1} &=& v_i \\ y^{\sigma+1} &=& \begin{pmatrix} \Phi^\sigma_{\tau_\sigma} \\ \hat{D}^\sigma \end{pmatrix} \end{array} \right. \tag{6.22}$$

and denote $w^{\sigma+1} := (w_{ij} \mid 1 \leq i \leq p, 0 \leq j \leq \nu_{\sigma+1} - 1)$. ■

The following notion of regularity is associated with Algorithm 6.2.5.

Definition 6.2.6 *Consider a nonlinear system Σ and let a point $x_0 \in M$ be given. We call x_0 a regular point for Algorithm 6.2.5 if at each step σ of the algorithm there is a $\bar{w}^\sigma$ such that $(x_0, \bar{w}^\sigma)$ is a regular point for Algorithm 5.0.7 applied to Σ^σ_{a0}.* ■

Without a proof we state:

Theorem 6.2.7 *Consider the nonlinear system Σ and let $x_0 \in M$ be a regular point for Algorithm 6.2.5 applied to Σ. Then the nDDDPdm is locally solvable around x_0 if and only if Algorithm 6.2.5 applied to Σ terminates because of case 3.* ∎

Remark 6.2.8 As stated before, the proof of Theorem 6.2.7 essentially follows the same lines as the proof of Theorem 6.2.3. The main difference is the construction of the dynamic state feedback solving the nDDDPdm. In this case the construction proceeds as follows (compare with (6.16) and the construction of the Singh compensator with disturbance feedthrough in Chapter 4). Assume that Algorithm 6.2.5 applied to Σ terminates because of case 3. Then there exist a neighborhood $U \subset M$ of x_0 and a mapping $F : U \to \mathbb{R}^{p\nu_{\sigma^*}}$ such that for all $\bar{x} \in U$ we have that $(\bar{x}, F(\bar{x})) \in N^*$. Let $\begin{pmatrix} u \\ v \end{pmatrix} = \alpha^*(x, w)$ be a control law that renders N^* invariant for $\bar{\Sigma}^1_{a0}$. Furthermore, define

$$\tilde{F}^{\sigma^*}(x, w) := \frac{\partial \Phi^{\sigma^*-1}_{\tau_{\sigma^*-1}}}{\partial x} p(x) \tag{6.23}$$

Recall that (locally) $\tilde{E}^{\sigma^*}_{\tau_{\sigma^*}}(x, w) := (\partial \Phi^{\sigma^*-1}_{\tau_{\sigma^*-1}} / \partial x) g(x)$ has full row rank. Let (locally) $\tilde{E}^{\sigma^*+}_{\tau_{\sigma^*}}(x, w)$ be a right inverse of $\tilde{E}^{\sigma^*}_{\tau_{\sigma^*}}(x, w)$ and define

$$\epsilon(x, w) := -\tilde{E}^{\sigma^*+}_{\tau_{\sigma^*}}(x, w) F^{\sigma^*}(x, w) \tag{6.24}$$

Let z_i $(i = 1, \cdots, p)$ be a vector of dimension ν_{σ^*} and consider the nonregular dynamic state feedback

$$Q \left\{ \begin{array}{rcl} \dot{z}_i & = & A z_i + b v_i \quad (i = 1, \cdots, p) \\ \begin{pmatrix} u \\ v \end{pmatrix} & = & \alpha^*(x, z) + \epsilon(x, z) q \end{array} \right. \tag{6.25}$$

with (A, b) in Brunovsky canonical form, initialized at $z(0) = F(x(0))$ for any $x(0) \in U$. Then Q solves the nDDDPdm around x_0. ∎

The form of Algorithms 6.2.1 and 6.2.5 and the construction of the nonregular dynamic state feedback solving the nDDDP and the nDDDPdm have been chosen as presented here to make the algorithms and the proof of Theorem 6.2.3 as transparent as possible. However, the bookkeeping while applying the algorithms may become quite troublesome. Moreover, the dynamic state feedbacks that are proposed in the proof of Theorem 6.2.3 and in Remark 6.2.8 for solving the nDDDP, nDDDPdm respectively, may have unnecessarily high dimension and it is not guaranteed that they have maximal achievable rank. Further research is required on the problem of obtaining dynamic state feedbacks of lower dimension and maximal achievable rank that solve the nDDDP(dm). For (relatively) simple examples the problems mentioned above can be circumvented to a great extent by using ***ad hoc*** arguments in the vein of the algorithm. This will be illustrated by the following example.

Example 6.2.9 Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_2u_1 + x_4 & y_1 = x_1 \\
\dot{x}_2 &= x_3 + q & y_2 = x_3 \\
\dot{x}_3 &= x_1u_1 + x_4 & y_3 = x_5 \\
\dot{x}_4 &= x_5 & y_4 = x_7 \\
\dot{x}_5 &= x_9 + u_2 \\
\dot{x}_6 &= x_8u_3 \\
\dot{x}_7 &= x_8 + x_9 + x_6x_9 + x_6u_2 \\
\dot{x}_8 &= u_3 \\
\dot{x}_9 &= x_{10} \\
\dot{x}_{10} &= q
\end{aligned}
\tag{6.26}
$$

for which we want to solve the nDDDP around points in the set $\{x \in I\!R^{10} \mid x_1 \neq 0, x_2 \neq 0, x_6 \neq 0, x_8 \neq 0\}$. We first restrict our attention to y_1 and y_2. Then:

$$
\begin{aligned}
\dot{y}_1 &= x_2u_1 + x_4 \Rightarrow u_1 = \frac{1}{x_2}(\dot{y}_1 - x_4) \\
\dot{y}_2 &= x_1u_1 + x_4 = \frac{x_1}{x_2}(\dot{y}_1 - x_4) + x_4 \\
\ddot{y}_2 &= \frac{\dot{y}_1}{x_2}(\dot{y}_1 - x_4) - \frac{x_1(x_3+q)}{x_2^2}(\dot{y}_1 - x_4) + \frac{x_1}{x_2}(\ddot{y}_1 - x_5) + x_5
\end{aligned}
\tag{6.27}
$$

Hence $\ddot{y}_2$ depends on q. This means that we should guarantee that

$$
\frac{\partial \ddot{y}_2}{\partial q} = -\frac{x_1}{x_2^2}(\dot{y}_1 - x_4) = 0 \tag{6.28}
$$

Since we are working around points for which $x_1 \neq 0, x_2 \neq 0$, this implies that

$$
\frac{1}{x_2}(\dot{y}_1 - x_4) = u_1 = 0 \tag{6.29}
$$

Having chosen $u_1 = 0$, it follows from the structure of the system that y_1 and y_2 can be made independent of the disturbance if and only if y_3 and y_4 can be made independent of the disturbance. Restricting our attention to y_3 and y_4, we find

$$
\begin{aligned}
\dot{y}_3 &= x_9 + u_2 \Rightarrow u_2 = \dot{y}_3 - x_9 \\
\dot{y}_4 &= x_8 + x_9 + x_6x_9 + x_6u_2 = x_8 + x_9 + x_6\dot{y}_3 \\
\ddot{y}_4 &= u_3 + x_{10} + x_8\dot{y}_3u_3 + x_6\ddot{y}_3 \Rightarrow u_3 = \frac{1}{1 + x_8\dot{y}_3}(\ddot{y}_4 - x_{10} - x_6\ddot{y}_3)
\end{aligned}
\tag{6.30}
$$

Hence the nDDDP is solved via the nonregular dynamic state feedback

$$
\begin{aligned}
\dot{z} &= v_1 \\
u_1 &= 0 \\
u_2 &= z - x_9 \\
u_3 &= \frac{1}{1 + x_8z}(v_2 - x_{10} - x_6v_1)
\end{aligned}
\tag{6.31}
$$

The rank of the dynamic state feedback (6.31) equals two. This is the maximal achievable rank since it is necessary that $u_1 = 0$. Note that the outputs can also be rendered independent of the disturbance via the following nonregular *static* state feedback of rank one:

$$\begin{aligned} u_1 &= 0 \\ u_2 &= x_9 \\ u_3 &= -x_{10} + v \end{aligned} \tag{6.32}$$

■

Chapter 7

The nonlinear model matching problem

The model matching problem is a synthesis problem that has received much attention over the last two decades (see [74],[75],[76],[33],[66],[49] for the linear case and [60],[31],[27],[28], [10],[47],[48],[50],[55],[73] for the nonlinear case). Roughly, the model matching problem may be stated as follows: given a (linear or nonlinear) system, called the plant P, and another (linear or nonlinear) system, called the model M, under what conditions is it possible to construct a dynamic static state feedback Q for P such that the input-output behavior of the compensated system $P \circ Q$ is the same as the input-output behavior of M?

For linear systems the model matching problem was first solved in [74], via Silverman's structure algorithm ([88]). A solution in geometric terms was obtained in [75]. In [76],[33] it was shown that the solvability of the linear model matching problem is equivalent to the solvability of an associated disturbance decoupling problem with disturbance measuremements.

In Subsection 7.1.1 we first formulate the model matching problem and we give the sufficient conditions for solvability of the problem that were obtained in [31],[73]. Also for nonlinear systems it can be shown that the solvability of the model matching problem is equivalent to the solvability of an associated disturbance decoupling problem with disturbance measurements. More specifically, the solvability of the nonlinear model matching problem is equivalent to the solvability of an associated nDDDPdm ([50],[31]). In Subsection 7.1.2 this equivalence is proved. Based on this equivalence, we also give necessary and sufficient conditions for solvability of the nonlinear model matching problem in this subsection.

Another set of necessary and sufficient conditions for local solvability of the nonlinear model matching problem was obtained in [55], under the conditions that both the plant and the model are square systems and the decoupling matrix of the model has full rank. In [55] a connection was made between the solvability of the model matching problem for the nonlinear plant and model and the solvability of the linear model matching problem for the linearizations of the plant and the model around an equilibrium point. This approach to the solution of the nonlinear model matching problem is treated in Section 7.2, together with some examples, that were taken from [31],[27],[55],[48].

7.1 Model matching and nonregular dynamic disturbance decoupling

7.1.1 Formulation of the problem and sufficient conditions for solvability

Consider a nonlinear analytic system, called the plant P, described by equations of the form

$$P\left\{\begin{array}{rcl}\dot{x} & = & f(x)+g(x)u \\ y & = & h(x)\end{array}\right. \tag{7.1}$$

with $x=(x_1,\cdots,x_n)^T \in I\!R^n$ local coordinates for the state space manifold $\mathcal{X}$, $u \in I\!R^m$ denoting the controls, $y \in I\!R^p$ denoting the outputs, $g(x)=(g_1(x)\cdots g_m(x))$, $h(x)=\mathrm{col}(h_1(x),\cdots,h_p(x))$, $f,g_1,\cdots,g_m$ analytic vector fields on $\mathcal{X}$, and $h_1,\cdots,h_p$ analytic functions on $\mathcal{X}$.

Moreover, consider another nonlinear analytic system, called the model M, described by equations of the form

$$M\left\{\begin{array}{rcl}\dot{\bar{x}} & = & \bar{f}(\bar{x})+\bar{g}(\bar{x})\bar{u} \\ \bar{y} & = & \bar{h}(\bar{x})\end{array}\right. \tag{7.2}$$

with $\bar{x}=(\bar{x}_1,\cdots,\bar{x}_{\bar{n}})^T \in I\!R^{\bar{n}}$ local coordinates for the state space manifold $\bar{\mathcal{X}}$, $\bar{u} \in I\!R^{\bar{m}}$ denoting the controls, $\bar{y} \in I\!R^p$ denoting the outputs, $\bar{g}(\bar{x})=(\bar{g}_1(\bar{x})\cdots\bar{g}_{\bar{m}}(\bar{x}))$, $\bar{h}(\bar{x})=\mathrm{col}(\bar{h}_1(\bar{x}),\cdots,\bar{h}_p(\bar{x}))$, $\bar{f},\bar{g}_1,\cdots,\bar{g}_m$ analytic vector fields on $\bar{\mathcal{X}}$, and $\bar{h}_1,\cdots,\bar{h}_p$ analytic functions on $\bar{\mathcal{X}}$. Moreover, assume that $\bar{m}\leq m$. Note that M has the same number of outputs as P.

For linear systems, the input-output behavior may be described via the transfer matrix of the system. The model matching problem then comes down to finding a dynamic state feedback Q for P, such that the transfer matrices of $P\circ Q$ and M coincide. For nonlinear systems, the input-output behavior may be described in terms of Volterra series expansions, that is, the output $y(t)=\mathrm{col}(y_1(t),\cdots,y_p(t))$ of a nonlinear system of the form (7.1) has a Volterra series expansion of the form (cf. [61],[81])

$$\begin{aligned} y(t)= \; & w_0(t,x_0)+\sum_{i=1}^{m}\int_0^t w_i(t,\tau_1,x_0)u_i(\tau_1)d\tau_1+ \\ & \sum_{i_1,i_2=1}^{m}\int_0^t\int_0^{\tau_1} w_{i_1i_2}(t,\tau_1,\tau_2,x_0)u_{i_1}(\tau_1)u_{i_2}(\tau_2)d\tau_1d\tau_2+\cdots \end{aligned} \tag{7.3}$$

where $x_0\in\mathcal{X}$ is the initial state at time $t=0$. The function $w_{j_1\cdots j_i}(t,\tau_1,\cdots,\tau_i,x_0)$ is called the $(j_1,\cdots,j_i)$-th Volterra kernel of i-th order for (7.1). The nonlinear model matching problem then comes down to finding (if possible) a dynamic state feedback Q for P such that all Volterra kernels of i-th order ($i\geq 1$) of $P\circ Q$ and M coincide. We make this more explicit below.

Let $\bar{w}_{j_1\cdots j_i}(t,\tau_1,\cdots,\tau_i,\bar{x}_0)$ denote the $(j_1,\cdots,j_i)$-th Volterra kernel of the model M and similarly $w^e_{j_1\cdots j_i}(t,\tau_1,\cdots,\tau_i,(x_0,z_0))$ the $(j_1,\cdots,j_i)$-th Volterra kernel of the compensated plant $P\circ Q$. Since $\bar{w}_{j_1\cdots j_i}$ depends on the initial state $\bar{x}_0$ of M and $w^e_{j_1\cdots j_i}$ on the initial state (x_0,z_0) of $P\circ Q$, when imposing the coincidence between these kernels one must specify how $\bar{x}_0$ and (x_0, z_0) are to be chosen. Depending on this choice, one may formulate different matching problems. The most common definition of the nonlinear model matching problem, which is taken from [31], is given below.

Definition 7.1.1 Nonlinear model matching problem (MMP)
Given a plant P of the form (7.1), a model M of the form (7.2) and a point $(x_0, \bar{x}_0) \in \mathcal{X} \times \bar{\mathcal{X}}$, the MMP is said to be locally solvable around $(x_0, \bar{x}_0)$ if there exist neighborhoods $U \subset \mathcal{X}$ of x_0 and $\bar{U} \subset \bar{\mathcal{X}}$ of $\bar{x}_0$, an integer ν, an open subset $\mathcal{Z} \subset \mathbb{R}^\nu$, a dynamic state feedback Q of the form

$$Q \left\{ \begin{array}{lcl} \dot{z} & = & \alpha(x,z) + \beta(x,z)v \\ u & = & \gamma(x,z) + \delta(x,z)v \end{array} \right. \tag{7.4}$$

with $z \in \mathbb{R}^\nu$, $v \in \mathbb{R}^{\bar{m}}$ denoting the new controls, and $\alpha, \beta, \gamma, \delta$ analytic functions defined on $U \times \mathcal{Z}$, and a map $F : U \times \bar{U} \to \mathcal{Z}$ with the property that

$$w^e_{j_1 \cdots j_i}(t, \tau_1, \cdots, \tau_i, (x, F(x, \bar{x}))) = \bar{w}_{j_1 \cdots j_i}(t, \tau_1, \cdots, \tau_i, \bar{x}) \tag{7.5}$$

for all $i \geq 1$, for all $1 \leq j_i \leq m$ and for all $(x, \bar{x}) \in U \times \bar{U}$. ∎

Remark 7.1.2 Note that after we have solved the MMP for (M, P), it will in general not be the case that the output trajectories of $P \circ Q$ and M are the same. This is due to the fact that we only require the i-th order Volterra kernels ($i \geq 1$) of $P \circ Q$ and M to coincide. If we also require the 0-th order Volterra kernels to coincide, we do have that the output trajectories of $P \circ Q$ and M are the same. This problem is known as the *strong* model matching problem and was studied in e.g. [27],[28] (see also [47]). ∎

Define an *extended system* E associated with M and P:

$$E \left\{ \begin{array}{lcl} \dot{x}^E & = & f^E(x^E) + g^E(x^E)u^E + p^E(x^E)q^E \\ y^E & = & h^E(x^E) \end{array} \right. \tag{7.6}$$

with $x^E = (x^T \;\; \bar{x}^T)^T$, controls $u^E \in \mathbb{R}^m$, disturbances $q^E \in \mathbb{R}^m$, outputs $y^E \in \mathbb{R}^p$ and

$$f^E(x^E) = \begin{pmatrix} f(x) \\ \bar{f}(\bar{x}) \end{pmatrix}, \; g^E(x^E) = \begin{pmatrix} g(x) \\ 0 \end{pmatrix}$$

$$p^E(x^E) = \begin{pmatrix} 0 \\ \bar{g}(\bar{x}) \end{pmatrix}, \;\; h^E(x^E) = h(x) - \bar{h}(\bar{x})$$

We have the following sufficient conditions for solvability of the nonlinear MMP.

Proposition 7.1.3 *([31]) Consider a nonlinear plant P and a nonlinar model M. Let $x_0 \in \mathcal{X}$, $\bar{x}_0 \in \bar{\mathcal{X}}$ be given. For the extended system E, let $\mathcal{G}^E :=$ span $\{g^E_1, \cdots, g^E_m\}$ and $\mathcal{P}^E :=$ span $\{p^E_1, \cdots, p^E_m\}$. Let Δ^{E^*} denote the maximal locally controlled invariant distribution contained in* Ker dh^E *for E. Assume that locally around $(x_0, \bar{x}_0)$, Δ^{E^*}, $\Delta^{E^*} \cap \mathcal{G}^E$ and $\mathcal{G}^E$ have constant dimension. Then the MMP is locally solvable around $(x_0, \bar{x}_0)$ if*

$$\mathcal{P}^E \subset \Delta^{E^*} + \mathcal{G}^E \tag{7.7}$$

∎

Proposition 7.1.4 *([73]) Consider a nonlinear plant P and a nonlinear model M. Let $x_0 \in \mathcal{X}$, $\bar{x}_0 \in \bar{\mathcal{X}}$ be given. Assume that $(x_0, \bar{x}_0)$ is a strongly regular point for the extended system E. Then the MMP is locally solvable around $(x_0, \bar{x}_0)$ if for $k = 1, \cdots, n + \bar{n}$*

$$\rho^E_{0k} = \rho^E_{q^E k} \tag{7.8}$$

■

Remark 7.1.5 Comparing the results of Propositions 7.1.3 and 7.1.4 with the results of Theorems 2.2.13 and 4.0.16 respectively, the idea arises that the solvability of the MMP is connected with the solvability of a disturbance decoupling problem with disturbance measurements. In the following subsection it is shown that this is indeed the case (see also Remark 7.1.11). ■

The following examples show that conditions (7.7) and (7.8) are not necessary conditions for solvability of the MMP.

Example 7.1.6 ([31],[27]) Consider the following nonlinear plant P:

$$\begin{aligned} \dot{x}_1 &= x_3 u_1 \qquad & y_1 &= x_2 - x_3 \\ \dot{x}_2 &= x_4 + u_2 & y_2 &= x_1 \\ \dot{x}_3 &= u_1 + u_2 \\ \dot{x}_4 &= u_2 \end{aligned} \tag{7.9}$$

Let the linear model M be given by

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 \qquad & \bar{y}_1 &= \bar{x}_1 \\ \dot{\bar{x}}_2 &= \bar{u}_1 & \bar{y}_2 &= \bar{x}_3 \\ \dot{\bar{x}}_3 &= \bar{x}_4 \\ \dot{\bar{x}}_4 &= \bar{u}_2 \end{aligned} \tag{7.10}$$

Consider the following dynamic state feedback Q for P:

$$\begin{aligned} \dot{z} &= -\frac{z^2}{(x_3+z)} - \frac{z}{(x_3+z)} v_1 + \frac{1}{(x_3+z)} v_2 \\ u_1 &= z \\ u_2 &= -\frac{z^2}{(x_3+z)} + \frac{x_3}{(x_3+z)} v_1 + \frac{1}{(x_3+z)} v_2 \end{aligned} \tag{7.11}$$

Then it can be checked that for $P \circ Q$ we have

$$\begin{aligned} \ddot{y}_1 &= v_1 \\ \ddot{y}_2 &= v_2 \end{aligned} \tag{7.12}$$

So $P \circ Q$ and M have the same input-output behavior and hence the MMP is solvable. For the extended system E we find

$$\begin{aligned} \mathcal{P}^E &= \operatorname{span}\{\frac{\partial}{\partial \bar{x}_2}, \frac{\partial}{\partial \bar{x}_4}\} \\ \mathcal{G}^E &= \operatorname{span}\{x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\} \\ \Delta^{E^*} &= \operatorname{span}\{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial \bar{x}_3}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial \bar{x}_1}, \frac{\partial}{\partial x_4} + \frac{\partial}{\partial \bar{x}_2}\} \end{aligned} \tag{7.13}$$

and hence (7.7) does not hold, i.e., (7.7) is not a necessary condition for solvability of the MMP. ■

Example 7.1.7 ([50]) Consider the nonlinear plant P:

$$\begin{array}{rclcl} \dot{x}_1 & = & x_2 + x_2 u_2 & \quad & y_1 = x_1 \\ \dot{x}_2 & = & u_1 & & y_2 = x_2 \\ \dot{x}_3 & = & u_2 & & y_3 = x_3 \end{array} \tag{7.14}$$

Let the model M be given by

$$\begin{array}{rclcl} \dot{\bar{x}}_1 & = & \bar{x}_2 & \quad & \bar{y}_1 = \bar{x}_1 \\ \dot{\bar{x}}_2 & = & \bar{x}_3 + \bar{u}_1 & & \bar{y}_2 = \bar{x}_2 \\ \dot{\bar{x}}_3 & = & \bar{u}_2 & & \bar{y}_3 = \bar{x}_4 \\ \dot{\bar{x}}_4 & = & -\bar{x}_4 & & \end{array} \tag{7.15}$$

Consider the following dynamic state feedback Q for P:

$$\begin{array}{rcl} \dot{z} & = & v_2 \\ u_1 & = & z + v_1 \\ u_2 & = & 0 \end{array} \tag{7.16}$$

Then it is easily seen that Q solves the MMP for (M, P). For the extended system E we find

$$\begin{array}{rclcrcl} \rho^E_{01} & = & 2 & \quad & \rho^E_{q^E 1} & = & 2 \\ \rho^E_{02} & = & 2 & & \rho^E_{q^E 2} & = & 3 \end{array} \tag{7.17}$$

Hence (7.8) is not a necessary condition for solvability of the MMP. ■

7.1.2 Necessary and sufficient conditions for solvability of the nonlinear model matching problem

The following theorem states that the solvability of the nonlinear model matching problem is equivalent to the solvability of an associated disturbance decoupling problem with disturbance measurements.

Theorem 7.1.8 *Consider a nonlinear plant P and a nonlinear model M. Let $x_0 \in \mathcal{X}$, $\bar{x}_0 \in \bar{\mathcal{X}}$ be given. Then the MMP is locally solvable around $(x_0, \bar{x}_0)$ if and only if the nDDDPdm for E is locally solvable around $(x_0, \bar{x}_0)$.*

Proof Note that we can alternatively define the MMP as follows: the nonlinear MMP is said to be locally solvable around $(x_0, \bar{x}_0)$ if there exist neighborhoods $U \subset \mathcal{X}$ of x_0 and $\bar{U} \subset \bar{\mathcal{X}}$ of $\bar{x}_0$, a dynamic state feedback Q of the form (7.4) for P and a map $F : U \times \bar{U} \to \mathcal{Z}$ such that, when we put $v \equiv \bar{u}$ in (7.4), we have that

$$y(x, F(x, \bar{x}), t) - \bar{y}(\bar{x}, t) \tag{7.18}$$

is independent of $\bar{u}$ for all $(x, \bar{x}) \in U \times \bar{U}$. This immediately gives the necessity-part of the proof. For the sufficiency, note that a dynamic state feedback Q^E solving the nDDDPdm for E around $(x_0, \bar{x}_0)$ also almost immmediately provides us a dynamic state feedback Q that solves the MMP around $(x_0, \bar{x}_0)$. The only problem that arises is the fact that Q^E depends on the state variables $\bar{x}$ of M. This problem is circumvented by adding a copy of M to Q^E. More specifically, assume that the nDDDPdm for E is locally solvable around $(x_0, \bar{x}_0)$ via a dynamic state feedback

$$Q^E \left\{ \begin{array}{rcl} \dot{z} & = & \alpha_1(x^E, z) + \alpha_2(x^E, z)q^E \\ u^E & = & \gamma_1(x^E, z) + \gamma_2(x^E, z)q^E \end{array} \right. \tag{7.19}$$

with $z \in I\!R^\nu$, initialized at $\hat{z} = F^E(\hat{x}^E)$ for any $\hat{x}^E \in U^E \subset \mathcal{X} \times \bar{\mathcal{X}}$, where U^E is a neighborhood of $(x_0, \bar{x}_0)$. Then consider the following dynamic state feedback Q for P:

$$Q \left\{ \begin{array}{rcl} \dot{z} & = & \alpha_1(x, w, z) + \alpha_2(x, w, z)v \\ \dot{w} & = & \bar{f}(w) + \bar{g}(w)v \\ u & = & \gamma_1(x, w, z) + \gamma_2(x, w, z)v \end{array} \right. \tag{7.20}$$

initialized at $(\hat{z}, \hat{w}) = (F^E(\hat{x}, \hat{\bar{x}}), \hat{\bar{x}})$ for any $(\hat{x}, \hat{\bar{x}}) \in U^E$. Then with the above alternative definition of the MMP it is easily seen that Q locally solves the MMP around $(x_0, \bar{x}_0)$. ■

Remark 7.1.9 Theorem 7.1.8 is taken from [50]. The sufficiency-part of the result can also be found in [31]. ■

Combining the results of Theorem 7.1.8 and Chapter 6, we obtain the following necessary and sufficient condition for the solvability of the MMP (cf. [50]).

Theorem 7.1.10 *Consider a nonlinear plant P and a nonlinear model M. Let $x_0 \in \mathcal{X}$, $\bar{x}_0 \in \bar{\mathcal{X}}$ be given. Assume that $(x_0, \bar{x}_0)$ is a regular point for Algorithm 6.2.5 applied to E. Then the MMP is locally solvable around $(x_0, \bar{x}_0)$ if and only if Algorithm 6.2.5 applied to E terminates because of case 3.* ■

Remark 7.1.11 From the result of Theorem 7.1.8 and the theory developed in Chapters 2 and 4 it immediately becomes clear why the conditions of Propositions 7.1.3 and 7.1.4 are sufficient conditions for the solvability of the MMP. Namely, Proposition 7.1.3 states that the DDPdm is solvable for E (compare with Theorem 2.2.13) and Proposition 7.1.4 states that the DDDPdm is solvable for E (compare with Theorem 4.0.16). In fact, the result of Proposition 7.1.3 was obtained in [31] by associating the MMP for (M, P) with a (regular) DDPdm for E. Note however that the result of Proposition 7.1.4 in [73] was *not* obtained by associating the MMP with a disturbance decoupling problem with disturbance measurements and that it was obtained independently of [51],[52]. ■

7.2 Nonlinear model matching and linearization

In this section we consider a square plant P of the form (7.1) and a square model M of the form (7.2). We first derive necessary and sufficient conditions for solvability of the MMP around a point $(x_0, \bar{x}_0) \in \mathcal{X} \times \bar{\mathcal{X}}$ under the condition that the decoupling matrix $\bar{A}(\bar{x})$ of M has full rank for $\bar{x} = \bar{x}_0$. Although this last assumption is certainly restrictive, it can be argued that in practical circumstances it is often desirable.

Next we investigate the connection between the solvability of the MMP for (M, P) around an equilibrium point $(x_0, \bar{x}_0)$ and the solvability of the associated linear MMP for the linearizations LP, LM of P and M around x_0 and $\bar{x}_0$ respectively, under the condition that the decoupling matrix $\bar{A}(\bar{x})$ of M has full rank for $\bar{x} = \bar{x}_0$. It is shown that under generic conditions -a mathematical phrasing of almost always- the nonlinear MMP is solvable around an equilibrium point $(x_0, \bar{x}_0)$, if and only if the corresponding linear MMP is solvable for (LM, LP). In our opinion this has important practical implications, in that in engineering practice one often studies a specific control problem by addressing the problem on the linearization around a given working point. The result that is given here may be viewed as an *a posteriori* justification of this methodology. In this way the result fits in the philosophy developed in [40], where the relation between the SIODP for a nonlinear system and its linearization around a working point is investigated (see also [85],[106],[56],[57],[58], where similar relations for several synthesis problems were obtained). When no specific references are given, the results in this section can be found in [55].

Consider a square nonlinear plant P of the form (7.1) and a square nonlinear model M of the form (7.2). Let $r_1, \cdots, r_m$ be the relative degrees of P and let $\epsilon_1, \cdots, \epsilon_m$ be its essential orders. Let $\bar{r}_1, \cdots, \bar{r}_m$ be the relative degrees of M. Furthermore, given a dynamic state feedback Q of the form (7.4) for P, the relative degrees of $P \circ Q$ are denoted by $r_1^e, \cdots, r_m^e$.

In what follows, we need the following results:

Lemma 7.2.1 *Consider a square nonlinear plant and a square nonlinear model M. Let $x_0 \in \mathcal{X}, \bar{x}_0 \in \bar{\mathcal{X}}$ be given. Assume that the MMP is locally solvable around $(x_0, \bar{x}_0)$ via a dynamic state feedback Q. Then*

(i) $r_i^e = \bar{r}_i$ $(i = 1, \cdots, m)$

(ii) If moreover the decoupling matrix $\bar{A}(\bar{x})$ of M has full rank for $\bar{x} = \bar{x}_0$, there exists a $z_0 \in \mathbb{R}^\nu$ such that the decoupling matrix $A^e(x, z)$ of $P \circ Q$ has full rank for $(x, z) = (x_0, z_0)$.

Proof Let $P \circ Q$ be given by

$$\left\{ \begin{array}{lcl} \dot{\tilde{x}} & = & \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})v \\ \tilde{y} & = & \tilde{h}(\tilde{x}) = h(x) \end{array} \right. \tag{7.21}$$

where $\tilde{x} \in \mathcal{X} \times \mathbb{R}^\nu$. The fact that Q locally solves the MMP around $(x_0, \bar{x}_0)$, implies that it locally solves the nDDDPdm for E around $(x_0, \bar{x}_0)$ when we take $v = q^E$ (see Theorem 7.1.8). By Proposition 2.2.9 and the structure of $P \circ Q$ this implies that there exist neighborhoods $U \subset \mathcal{X}$ and $\bar{U} \subset \bar{\mathcal{X}}$ of $x_0, \bar{x}_0$ respectively, and an open subset $\mathcal{Z} \subset \mathbb{R}^\nu$ so that for any $\bar{x}_* \in \bar{U}$ there is an $\tilde{x}_* \in U \times \mathcal{Z}$ such that for any q^E and any $k \in \mathbb{N}$: $\mathcal{L}_{\tilde{g}}\mathcal{L}_{\tilde{f}}^k\tilde{h}(\tilde{x}) = \mathcal{L}_{\bar{g}}\mathcal{L}_{\bar{f}}^k\bar{h}(\bar{x})$ along the trajectories of $E \circ Q$ starting at $(\tilde{x}_*, \bar{x}_*)$, that are contained in $U \times \bar{U} \times \mathcal{Z}$. This implies in particular that $r_i^e = \bar{r}_i$ $(i = 1, \cdots, m)$ and that there is a $z_0 \in \mathcal{Z}$ for which $A^e(x_0, z_0)$ has full rank. Hence we have established *(i)* and *(ii)*.

■

Lemma 7.2.2 *Consider a square plant P and a square model M. Let $x_0 \in \mathcal{X}$ and $\bar{x}_0 \in \bar{\mathcal{X}}$ be given. Then, if the decoupling matrix $A(x)$ of P has full rank for $x = x_0$, the MMP is locally solvable around $(x_0, \bar{x}_0)$ if and only if $r_i \leq \bar{r}_i$ $(i = 1, \cdots, m)$.*

Proof See [31],[47]. ∎

Lemma's 7.2.1 and 7.2.2 lead to:

Proposition 7.2.3 *Consider a square nonlinear plant P and a square nonlinear model M. Let $x_0 \in \mathcal{X}$ and $\bar{x}_0 \in \bar{\mathcal{X}}$ be given. Assume that the decoupling matrix $\bar{A}(\bar{x})$ of M has full rank for $\bar{x} = \bar{x}_0$. Then the MMP is locally solvable around $(x_0, \bar{x}_0)$ if and only if there is a dynamic state feedback Q that locally solves the SDIODP around x_0, with the property that $r_i^e \leq \bar{r}_i$ $(i = 1, \cdots, m)$.*

Proof *(sufficiency)* Assume that there is a dynamic state feedback Q with the above properties. Then by Lemma 7.2.2 the MMP is solvable for $(M, P \circ Q)$ and hence the MMP is also solvable for (M, P).
(necessity) Follows immediately from Lemma 7.2.1. ∎

Proposition 7.2.4 *Consider a square nonlinear plant P and a square nonlinear model M. Let $x_0 \in \mathcal{X}$, $\bar{x}_0 \in \bar{\mathcal{X}}$ be given. Assume that the decoupling matrix $\bar{A}(\bar{x})$ of M has full rank for $\bar{x} = \bar{x}_0$. Then the MMP is locally solvable around $(x_0, \bar{x}_0)$ if and only if $\epsilon_i \leq \bar{r}_i$ $(i = 1, \cdots, m)$.*

Proof By Proposition 7.11 the solvability of the MMP is equivalent to the existence of a dynamic state feedback Q that solves the SDIODP for P around x_0, with the property that $r_i^e \leq \bar{r}_i$ $(i = 1, \cdots, m)$. By the construction of the Singh compensator in Subsection 3.2.2 and Theorem 3.2.28 we know that there is a dynamic state feedback Q that solves the SDIODP for P with the property that $\epsilon_i = r_i^e$ $(i = 1, \cdots, m)$. Moreover we know by Lemma 3.2.29 that any dynamic state feedback Q solving the SDIODP for P has the property that $r_i^e \geq \epsilon_i$ $(i = 1, \cdots, m)$. These facts establish our claim. ∎

Using Proposition 7.2.4, a connection will be made between the solvability of the MMP for nonlinear M and P and the solvability of the MMP for their linearizations around an equilibrium point.

Let $x_0 \in \mathcal{X}, \bar{x}_0 \in \bar{\mathcal{X}}$ be equilibrium points for P and M respectively, i.e., $f(x_0) = 0$ and $\bar{f}(\bar{x}_0) = 0$. Assume (without loss of generality) that $h(x_0) = \bar{h}(x_0) = 0$. Then the linearizations LP and LM of P and M around x_0 and $\bar{x}_0$ are given by

$$LP \left\{ \begin{array}{rcl} \dot{x}^\ell & = & Fx^\ell + Gu \\ y^\ell & = & Hx^\ell \end{array} \right. \tag{7.22}$$

$$LM \left\{ \begin{array}{rcl} \dot{\bar{x}}^\ell & = & \bar{F}\bar{x}^\ell + \bar{G}\bar{u} \\ \bar{y}^\ell & = & \bar{H}\bar{x}^\ell \end{array} \right. \tag{7.23}$$

where

$$F = \left(\frac{\partial f}{\partial x} \right)(x_0), \quad G = g(x_0), \quad H = \left(\frac{\partial h}{\partial x} \right)(x_0)$$

$$\bar{F} = \left(\frac{\partial \bar{f}}{\partial \bar{x}} \right)(\bar{x}_0), \quad \bar{G} = \bar{g}(\bar{x}_0), \quad \bar{H} = \left(\frac{\partial \bar{h}}{\partial \bar{x}} \right)(\bar{x}_0)$$

Remark 7.2.5 Strictly speaking, the equilibria of a nonlinear system (7.1) are all (x_0, u_0) satisfying $f(x_0)+g(x_0)u_0 = 0$, whereas above we restrict ourselves to equilibria of the form $(x_0, 0)$, i.e., we let $u_0 = 0$. If we do not restrict ourselves to the case that $u_0 = 0$, the set of equilibrium points obviously gets larger. Note however that an equilibrium of the form (x_0, u_0) where $u_0 \neq 0$ can be transformed into an equilibrium of the form $(x_0, 0)$ by applying a preliminary control $u = u_0 + v$, where $v \in \mathbb{R}^m$ denotes the new controls. ∎

In the sequel the following assumption is made (compare with Definition 3.2.14):

Assumption 7.2.6 For every application of Singh's algorithm to P we have for $k = 1, \cdots, n$:

$$\operatorname{rank}\tilde{B}_k(x_0, 0, \cdots, 0) = \rho_k \tag{7.24}$$

∎

Remark 7.2.7 If we compare Assumption 7.2.6 with Definition 3.2.14, we see that the assumption implies that x_0 is a strongly regular point for P, with $\{\tilde{y}_{i0}^{(j)} \mid 1 \leq i \leq n-1, i \leq j \leq n-1\} = (0, \cdots, 0)$. ∎

A strongly regular equilibrium point $x_0 \in \mathcal{X}$ that satisfies Assumption 7.2.6 has the folowing property:

Lemma 7.2.8 *Consider a square nonlinear plant P of full rank. Let x_0 be a strongly regular equilibrium point satisfying Assumption 7.2.6. Let Q be a Singh compensator for P around x_0. Then $(x, z) = (x_0, 0)$ is an equilibrium point for $P \circ Q$.*

Proof By the form (3.50) of Q we see that an equilibrium point (x, z) of $P \circ Q$ has to satisfy $z = 0$. Applying Singh's algorithm to P in the way that leads to the Singh compensator Q yields

$$\tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq k-1, i \leq j \leq k\}) = \frac{\partial \hat{y}_k^{(k-1)}}{\partial x} f(x) + \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} \frac{\partial \hat{y}_k^{(k-1)}}{\partial \tilde{y}_i^{(j)}} \tilde{y}_i^{(j+1)} \tag{7.25}$$

Since $f(x_0) = 0$, (7.25) yields for $k = 1, \cdots, n$:

$$\tilde{a}_k(x_0, 0, \cdots, 0) = 0 \tag{7.26}$$

By (3.48) we have that for $P \circ Q$:

$$\begin{aligned} &u(x, z_1, \cdots, z_m, v_1, \cdots, v_m) = \\ &\qquad \tilde{B}_n^{-1}(x, z_1, \cdots, z_m)[z_{*1} - \tilde{A}_n(x, z_1, \cdots, z_m, v_1, \cdots, v_m)] \end{aligned} \tag{7.27}$$

Then (7.26) and (7.27) imply that for $P \circ Q$

$$f(x_0) + g(x_0, 0, \cdots, 0)u(x_0, 0, \cdots, 0) = 0 \tag{7.28}$$

Hence $(x, z) = (x_0, 0)$ is an equilibrium point for $P \circ Q$. ∎

Lemma 7.2.9 *Consider a square nonlinear plant P. Let $x_0 \in \mathcal{X}$ be a strongly regular equilibrium point for P that satisfies Assumption 7.2.6. Consider the linearization of P around x_0, to be denoted by LP. Then:*

(i) The rank of P is equal to the rank of LP.

(ii) Denote the essential orders of LP by ϵ_i^ℓ $(i = 1, \cdots, m)$. Then $\epsilon_i = \epsilon_i^\ell$ $(i = 1, \cdots, m)$.

Proof *(i)* Recall that for $k = 1, \cdots, n$

$$\mathcal{E}_k = \text{span}_{\mathcal{K}}\{dx, d\dot{y}, \cdots, dy^{(k)}\} \tag{7.29}$$

For $k = 1, \cdots, n$, introduce the matrices (cf. [77]):

$$J_k(x, u, \cdots, u^{(k-1)}) = \begin{pmatrix} \frac{\partial \dot{y}}{\partial u} & 0 & 0 & \cdots & 0 \\ \frac{\partial y^{(2)}}{\partial u} & \frac{\partial y^{(2)}}{\partial \dot{u}} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \frac{\partial y^{(k)}}{\partial u} & \frac{\partial y^{(k)}}{\partial \dot{u}} & \frac{\partial y^{(k)}}{\partial u^{(2)}} & \cdots & \frac{\partial y^{(k)}}{\partial u^{(k-1)}} \end{pmatrix} \tag{7.30}$$

Then from Theorem 3.2.4 it follows that the rank ρ^* of P is given by (see [30])

$$\rho^* = \text{rank}_{\mathcal{K}} J_n - \text{rank}_{\mathcal{K}} J_{n-1} \tag{7.31}$$

Since x_0 is a strongly regular point that satisfies Assumption 7.2.6, it follows from (B.18) that

$$\rho^* = \text{rank}_{\boldsymbol{R}} J_n(x_0, 0, \cdots, 0) - \text{rank}_{\boldsymbol{R}} J_{n-1}(x_0, 0, \cdots, 0) \tag{7.32}$$

Analogously to (7.30), define matrices J_k^ℓ $(k = 1, \cdots, n)$ for the linearized plant LP. It is easily checked that

$$J_k^\ell = \begin{pmatrix} HG & 0 & 0 & \cdots & 0 \\ HFG & HG & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ HF^{k-1}G & HF^{k-2}G & \cdots & \cdots & HG \end{pmatrix} \tag{7.33}$$

We show that $J_k(x_0, 0, \cdots, 0) = J_k^\ell$ $(k = 1, \cdots, n)$, i.e.,

$$\frac{\partial y^{(k)}}{\partial u^{(\ell)}}(x_0, 0, \cdots, 0) = HF^{k-\ell-1}G \quad (k = 1, \cdots, n; \ell = 0, \cdots, k-1) \tag{7.34}$$

First it is shown that for $k = 1, 2, \cdots$

$$\frac{\partial y^{(k)}}{\partial x}(x_0, 0, \cdots, 0) = HF^k \tag{7.35}$$

For $k = 1$ we have

$$\begin{aligned} &\frac{\partial \dot{y}}{\partial x}(x_0, 0, \cdots, 0) = \frac{\partial}{\partial x}(\frac{\partial y}{\partial x}[f(x) + g(x)u])_{(x_0,0,\cdots,0)} = \\ &\frac{\partial y}{\partial x}(x_0)\frac{\partial f}{\partial x}(x_0) = HF \end{aligned} \tag{7.36}$$

Hence (7.35) holds for $k = 1$. Assume that (7.35) holds for $k = 1, \cdots, \ell - 1$. Then

$$\begin{aligned} &\frac{\partial y^{(\ell)}}{\partial x}(x_0, 0, \cdots, 0) = \\ &\qquad \frac{\partial}{\partial x}(\frac{\partial y^{(\ell-1)}}{\partial x}[f(x) + g(x)u] + \sum_{r=0}^{\ell-2} \frac{\partial y^{(\ell-1)}}{\partial u^{(r)}} u^{(r+1)})_{(x_0,0,\cdots,0)} = \\ &\qquad \frac{\partial y^{(\ell-1)}}{\partial x}(x_0, 0, \cdots, 0)\frac{\partial f}{\partial x}(x_0) = HF^{k-1}F = HF^k \end{aligned} \tag{7.37}$$

Hence (7.35) holds for $k = 1, 2, \cdots$. Next we show that for $k = 1, \cdots, n$:

$$\frac{\partial y^{(k)}}{\partial u}(x_0, 0, \cdots, 0) = HF^{k-1}G \tag{7.38}$$

For $k = 1$ we have:

$$\begin{aligned} &\frac{\partial \dot{y}}{\partial u}(x_0, 0, \cdots, 0) = \frac{\partial}{\partial u}(\frac{\partial y}{\partial x}[f(x) + g(x)u])_{(x_0,0,\cdots,0)} = \\ &\frac{\partial y}{\partial x}(x_0)g(x_0) = HG \end{aligned} \tag{7.39}$$

Hence (7.38) holds for $k = 1$. Assume that (7.38) holds for $k = 0, \cdots, \ell - 1$. Then, using (7.35), we have:

$$\begin{aligned} &\frac{\partial y^{(\ell)}}{\partial u}(x_0, 0, \cdots, 0) = \frac{\partial}{\partial u}(\frac{\partial y^{(\ell-1)}}{\partial x}[f(x) + g(x)u] + \sum_{r=0}^{\ell-2} \frac{\partial y^{(\ell-1)}}{\partial u^{(r)}} u^{(r+1)})_{(x_0,0,\cdots,0)} = \\ &\frac{\partial y^{(\ell-1)}}{\partial x}(x_0, 0, \cdots, 0)g(x_0) = HF^{\ell-1}G \end{aligned} \tag{7.40}$$

Let $k \in \{1, 2, \cdots, n\}$, $\ell \in \{1, \cdots, k-1\}$. Then:

$$\begin{aligned} &\frac{\partial y^{(k)}}{\partial u^{(\ell)}}(x_0, 0, \cdots, 0) = \\ &\qquad \frac{\partial}{\partial u^{(\ell)}}(\frac{\partial y^{(k-1)}}{\partial x}[f(x) + g(x)u] + \sum_{r=0}^{k-2} \frac{\partial y^{(k-1)}}{\partial u^{(r)}} u^{(r+1)})_{(x_0,0,\cdots,0)} = \\ &\qquad \frac{\partial y^{(k-1)}}{\partial u^{(\ell-1)}}(x_0, 0, \cdots, 0) = \cdots = \frac{\partial y^{(k-\ell)}}{\partial u}(x_0, 0, \cdots, 0) = HF^{k-\ell-1}G \end{aligned} \tag{7.41}$$

(7.40) and (7.41) establish that (7.35) holds for $k = 1, 2, \cdots, n$, $\ell = 0, \cdots, k-1$ and hence for $k = 1, \cdots, n$ we have:

$$J_k(x_0, 0, \cdots, 0) = J_k^\ell \tag{7.42}$$

Then from (7.32) and (7.42) it follows that the rank of P is equal to the rank of LP.

(ii) Recall that for $i = 1, \cdots, m$ the essential order ϵ_i is defined as the smallest $k \in I\!N$ for which $dy_i^{(k)}$ is an essential vector of $\mathcal{E}_n$. It is readily shown that we can equivalently define ϵ_i as the smallest $k \in I\!N$ for which the row vector

$$\begin{pmatrix} \frac{\partial y_i^{(k)}}{\partial u} & \cdots & \frac{\partial y_i^{(k)}}{\partial u^{(k-1)}} & 0 & \cdots & 0 \end{pmatrix} \tag{7.43}$$

is an essential vector of the family of row vectors constituted by the row vectors of J_n. For $i = 1, \cdots, m$, the essential order ϵ_i^ℓ of LP is then defined as the smallest $k \in I\!N$ for which the row vector

$$\begin{pmatrix} H_i F^{k-1} G & \cdots & H_i G & 0 & \cdots & 0 \end{pmatrix} \tag{7.44}$$

is an essential vector of the family of row vectors constituted by the row vectors of J_n^ℓ. (N.B.: This is exactly the definition of the essential orders for a linear system that was given in [17].) Since (7.42) holds, the row vector (7.43) is an essential vector of the family of vectors constituted by the row vectors of J_n if and only if the row vector (7.44) is an essential vector of the family of vectors constituted by the row vectors of J_n^ℓ. Hence $\epsilon_i = \epsilon_i^\ell$ $(i = 1, \cdots, m)$. ∎

As an immediate consequence of Lemma 7.2.9 we have:

Corollary 7.2.10 *Consider a square nonlinear plant P. Let $x_0 \in \mathcal{X}$ be a strongly regular equilibrium point satisfying Assumption 7.2.6 for P. Denote the linearization of P around x_0 by LP. Then the SDIODP is solvable for P around x_0 if and only if it is solvable for LP.* ∎

Remark 7.2.11 The result of Corollary 7.2.10 can be found in [55]. (see also [56]). It generalizes a result of [40], where a similar result was obtained for the SIODP. ∎

We now come to the statement of our main result.

Theorem 7.2.12 *Consider a square nonlinear plant P and a square nonlinear model M. Let $x_0 \in \mathcal{X}, \bar{x}_0 \in \bar{\mathcal{X}}$ be equilibrium points for P and M respectively. Assume that x_0 is a strongly regular equilibrium point for P satisfying Assumption 7.2.6. Furthermore, assume that the decoupling matrix $\bar{A}(\bar{x})$ of M has full rank for $\bar{x} = \bar{x}_0$. Denote the linearizations of P and M around x_0 and $\bar{x}_0$ by LP,LM respectively. Then the MMP is locally solvable around $(x_0, \bar{x}_0)$ if and only if the MMP is solvable for (LM, LP).*

Proof By Proposition 7.2.4, solvability of the MMP for (M, P) around $(x_0, \bar{x}_0)$ is equivalent to $\epsilon_i \leq \bar{r}_i$ $(i = 1, \cdots, m)$. By Lemma's 7.2.9 and 3.2.29, this is equivalent to $\epsilon_i^\ell \leq \bar{r}_i^\ell$ $(i = 1, \cdots, m)$. By Lemma 7.2.2.*(i)* this is equivalent to the solvability of the MMP for (LM, LP). ∎

Example 7.2.13 Consider the plant and model that were introduced in Example 7.1.6. The set of equilibrium points of the form $(x_0, 0)$ satisfying $h(x_0) = 0$ is given by $\mathcal{E}_P = \{(x_0, 0) \mid x_{10} = x_{40} = 0, x_{20} = x_{30}\}$. Applying Singh's algorithm to P yields

$$\begin{aligned} \dot{y}_1 &= x_4 - u_1 \\ \dot{y}_2 &= x_3 u_1 = x_3(x_4 - \dot{y}_1) \\ \ddot{y}_2 &= (x_3 + x_4 - \dot{y}_1)u_2 + (x_4 - \dot{y}_1)^2 - x_3\ddot{y}_1 \end{aligned} \tag{7.45}$$

Hence the set of strongly regular equilibrium points satisfying Assumption 7.2.6 and $h(x_0) = 0$ is given by $\tilde{\mathcal{E}}_P = \{(x_0, 0) \in \mathcal{E}_P \mid x_{30} \neq 0\}$. The linearization of P around an $(x_0, 0) \in \tilde{\mathcal{E}}_P$ is given by

$$\begin{aligned} \dot{x}_1^\ell &= x_{30} u_1 & \qquad y_1^\ell &= x_2^\ell - x_3^\ell \\ \dot{x}_2^\ell &= x_4^\ell + u_2 & y_2^\ell &= x_1^\ell \\ \dot{x}_3^\ell &= u_1 + u_2 \\ \dot{x}_4^\ell &= u_2 \end{aligned} \tag{7.46}$$

Denoting by e_i $(i = 1, \cdots, 8)$ the i-th basis vector of the standard basis of $\mathbb{R}^8$, we find for the extended system E associated with LP and LM $(= M)$:

$$\begin{aligned} \operatorname{Im} B^E &= \operatorname{span}\{x_{30}e_1 + e_3, e_2 + e_3 + e_4\} \\ \operatorname{Im} E^E &= \operatorname{span}\{e_6, e_8\} \\ \mathcal{V}^{E^*} &= \operatorname{span}\{e_1 + e_7, e_2 + e_5, e_3 - e_5, e_4 + e_6, e_4 + x_{30}e_8\} \end{aligned} \tag{7.47}$$

Then it can be checked that for E we have $\operatorname{Im} E^E \subset \mathcal{V}^{E^*} + \operatorname{Im} B^E$ and hence the DDPdm is solvable for E. From the theory on the linear MMP (cf. [75]) it then follows that the MMP is solvable for (LM, LP). Hence by Theorem 7.2.12 the MMP for (M, P) is locally solvable around any $(x_0, \bar{x}_0) \in \tilde{\mathcal{E}}_P \times \mathbb{R}^4$.

A dynamic state feedback solving the MMP for (M, P) is obtained as follows. Let $x_0 \in \tilde{\mathcal{E}}_P$. From (7.45) we obtain the following Singh compensator around x_0 for P:

$$\begin{aligned} \dot{z} &= v_1 \\ u_1 &= x_4 - z \\ u_2 &= -\frac{(x_4 - z)^2}{x_3 + x_4 - z} + \frac{x_3}{x_3 + x_4 - z}v_1 + \frac{1}{x_3 + x_4 - z}v_2 \end{aligned} \tag{7.48}$$

Applying (7.48) to P around x_0 we find:

$$\begin{aligned} \ddot{y}_1 &= v_1 \\ \ddot{y}_2 &= v_2 \end{aligned} \tag{7.49}$$

Hence (7.48) locally solves the MMP for (M, P) around any $(x_0, \bar{x}_0) \in \tilde{\mathcal{E}}_P \times \mathbb{R}^4$. Note that the dynamic state feedback (7.48) is different from the one given in Example 7.1.6. However, applying the (local) state space diffeomorphism $\Psi(x, z) = (x, x_4 - z)$ to $P \circ Q$, we obtain the dynamic state feedback given in Example 7.1.6. ∎

Theorem 7.2.12 holds provided that Assumption 7.2.6 is satisfied. We now show that this assumption is *generically* satisfied. Before doing this, we first indicate what is meant by a generic property. Here we follow [95].

An analytic nonlinear control system of the form (7.1) can alternatively be interpreted as an analytic mapping $\Phi : \mathcal{X} \to \mathbb{R}^{n+nm+p}$, where $\Phi = (f^T\ g_1^T \cdots g_m^T\ h_1 \cdots h_p)^T$. Denote the set of all analytic mappings from $\mathcal{X}$ to $\mathbb{R}^{n+nm+p}$ by X_{nmp}. Then X_{nmp} can be endowed with a topology, e.g. the C^∞-Whitney topology. A property π of the elements of X_{nmp} is then said to be a *generic property* if the set of elements of X_{nmp} satisfying π forms an open and dense subset of X_{nmp} (with respect to the Whitney-topology).

Now consider Assumption 7.2.6 in more detail. As was already mentioned in Remark 7.2.7, Assumption 7.2.6 implies that x_0 is a strongly regular point for P, with $\{\tilde{y}_{i0}^{(j)} \mid 1 \le i \le n-1, i \le j \le n-1\} = (0, \cdots, 0)$ in (3.46). By Theorem 3.2.28 and Proposition 7.2.3 a necessary condition for solvability of the MMP for P and M around $(x_0, \bar{x}_0)$ is already that for $k = 1, \cdots, n$ and for *some* $\{\tilde{y}_{i0}^{(j)} \mid 1 \le i \le n-1, i \le j \le n-1\}$:

$$\operatorname{rank}\tilde{B}_k(x_0, \{\tilde{y}_{i0}^{(j)} \mid 1 \le i \le n-1, i \le j \le n-1\}) = \rho_k \tag{7.50}$$

So the *crucial* part of the assumption is that we can take $\{\tilde{y}_{i0}^{(j)} \mid 1 \le i \le n-1, i \le j \le n-1\} = (0, \cdots, 0)$. By analyticity of the data this crucial part is satisfied for an open and dense subset (in the induced C^∞-Whitney topology) of the set of systems in X_{nmp} that have x_0 as a strongly regular equilibrium point. Hence Assumption 7.2.6 is generically satisfied.

As remarked before, Theorem 7.2.12 forms an *a posteriori* justification of a methodology from engineering practice, where often a nonlinear synthesis problem is tackled by linearizing the system around a specific working point and then solving the problem for the linearization. Of course it still remains questionable if the control that was designed to solve the synthesis problem for the linearization of the system also is a good approximation for a control that solves synthesis problem for the original nonlinear system. In the following two examples from [48] this question is investigated for the nonlinear MMP.

Example 7.2.14 In this example we take as the plant P the robot arm that was described in Example 5.0.10. The relative degrees of P are $r_1 = r_2 = 2$. Moreover, it can be checked that the decoupling matrix of P is $M(\theta)$, which is an invertible matrix for all θ. Then by Lemma 7.2.2 the MMP is solvable for any model M with relative degrees $\bar{r}_1, \bar{r}_2 \ge 2$. In particular we can take M to be a linear system with transfer matrix

$$\bar{G}(s) = \begin{pmatrix} \frac{1}{(s+3)^2} & 0 \\ 0 & \frac{1}{(s+3)^2} \end{pmatrix} \tag{7.51}$$

It follows immediately that the model has relative degrees $\bar{r}_1 = \bar{r}_2 = 2$ and that its decoupling matrix has full rank. Consider the equilibrium point $(\theta, \dot{\theta}) = (0,0)$ for P. Since $M(\theta)$ is invertible for all θ, this equilibrium point satisfies Assumption 7.2.6. A static state feedback that solves the MMP for (LM, LP) (note: $LM = M$) is given by

$$u = F \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} + Gv \tag{7.52}$$

where

$$F = \begin{pmatrix} -15 & -30 & 42 & -12 \\ -8 & -12 & 21 & -6 \end{pmatrix}, \quad G = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

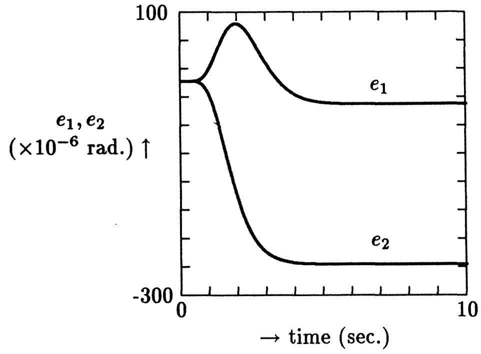

Figure 7.1: Double pendulum with control (7.52) and $\bar{u}_1 = 1$

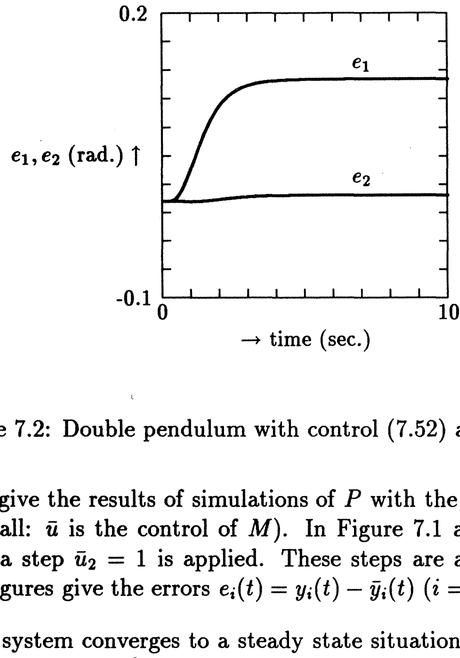

Figure 7.2: Double pendulum with control (7.52) and $\bar{u}_2 = 1$

Figures 7.1 and 7.2 give the results of simulations of P with the control (7.52), where we have set $v = \bar{u}$ (recall: $\bar{u}$ is the control of M). In Figure 7.1 a step $\bar{u}_1 = 1$ is applied, while in Figure 7.2 a step $\bar{u}_2 = 1$ is applied. These steps are applied during the whole time-interval. The figures give the errors $e_i(t) = y_i(t) - \bar{y}_i(t)$ $(i = 1, 2)$.

In the first case the system converges to a steady state situation with steady state errors $e_1 = -32 \cdot 10^{-6}$, $e_2 = -258 \cdot 10^{-6}$. The steady state error e_1 is 0.02% of the steady state value of $\bar{y}_1$. In the second case the system converges to a steady state situation with steady state errors $e_1 = 0.1290$, $e_2 = 0.0067$. The steady state error e_2 is 6.1% of the steady state value of $\bar{y}_2$. Hence in the first case the linear control (7.52) behaves reasonably well, while in the second case the results are less satisfying.

The fact that we end up with a steady state error suggests that the behavior of the system can be improved by extending the control (7.52) with a copy of the model and then applying an extra PI-action. Figure 7.3 shows that this indeed works. Here we have again applied a step $\bar{u}_2 = 1$, while to u_1 in (7.52) we have added a term $-12.8e_1(t) - 10\int_0^t e_1(\tau)d\tau$ and to u_2 in (7.52) a term $-10\int_0^t e_2(\tau)d\tau$ is added. We now end up with a steady state error $e_1 = 0.0014$ (1% of the steady state error in Figure 7.2) and $e_2 = 242 \cdot 10^{-6}$ (0.22% of the

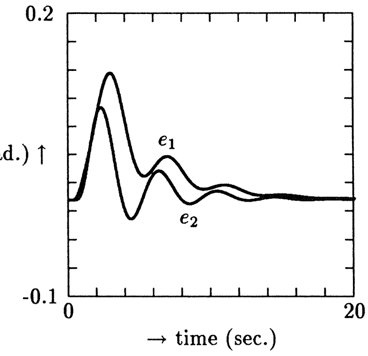

Figure 7.3: Double pendulum with control (7.52), PI-action and $\bar{u}_2 = 1$

steady state value of $\bar{y}_2$).

Further simulations (of which no figures are included) show that the stability of the closed loop system is maintained for step controls up to $(\bar{u}_1, \bar{u}_2) = (8, 0)$, $(\bar{u}_1, \bar{u}_2) = (0, 3)$, $(\bar{u}_1, \bar{u}_2) = (2, 2)$. In the first two cases (i.e., if we set one of the controls equal to zero) the maximum errors and steady state errors are proportional to the magnitude of the control.

Example 7.2.15 In this example the plant P is a two link robot arm moving in a vertical plane, as was described in e.g. [25]. The first link is actuated through a direct drive motor. The rotation of the first link with respect to a line perpendicular to the base is indicated by q_1. The second joint shows a significant elasticity. This elasticity is modelled by associating two variables to the second joint: q_2, the position of the second actuator with respect to the first link and q_3, the position of the second link with respect to the first link. The motor (q_2) is then coupled to the joint (q_3) by means of a transmission with transmission ratio $NT > 1$ and a torsional spring with spring constant K (see Figure 7.4).

In this example the links are of unit length, the motor inertia equals 0.001 and we assume the masses to be equal to one and to be concentrated at the joints (the motors) and at the tip (a load). Furthermore, we set $K = 1000$, $NT = 100$ and the constant of gravity $g = 10$ (note that we disregard the dimensions of the constants). The Euler-Lagrange equations for this system are:

$$M(q)\ddot{q} + N(q, \dot{q}) = u_E \tag{7.53}$$

where $q = \mathrm{col}(q_1, q_2, q_3)$, $u_E = \mathrm{col}(u_1, u_2, 0)$ and

$$M(q) = \begin{pmatrix} 3.002 + 2\cos q_3 & 0.001 & 1 + \cos q_3 \\ 0.001 & 0.001 & 0 \\ 1 + \cos q_3 & 0 & 1 \end{pmatrix}$$

$$N(q, \dot{q}) = \begin{pmatrix} -(2\dot{q}_1\dot{q}_3 + \dot{q}_3^2)\sin q_3 + 20\sin q_1 + 10\sin(q_1 + q_3) \\ \\ -\frac{K}{NT}(q_3 - \frac{q_2}{NT}) \\ \\ \dot{q}_1^2 \sin q_3 + K(q_3 - \frac{q_3}{NT}) + 10\sin(q_1 + q_3) \end{pmatrix}$$

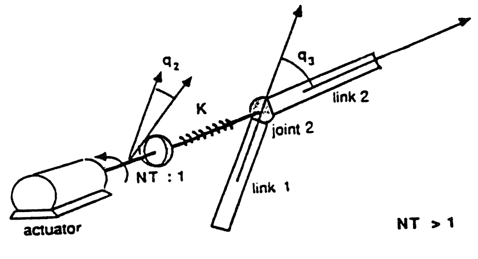

Figure 7.4: Definition of variables and parameters for elastic joint

As outputs for P we take $y_1 = q_1$, $y_2 = q_3$. The relative degrees of P are $r_1 = r_2 = 2$ and the essential orders are $\epsilon_1 = \epsilon_2 = 4$. We consider the equilibrium point $(q, \dot{q}) = (0, 0)$. Then it can be checked that this equilibrium point satisfies Asssumption 7.2.6 (see [25] for details).

The model M will be a linear system with transfer matrix

$$\bar{G}(s) = \begin{pmatrix} \dfrac{1}{(s+3)^4} & 0 \\ 0 & \dfrac{1}{(s+3)^4} \end{pmatrix} \tag{7.54}$$

It follows immediately that M has relative degrees $\bar{r}_1 = \bar{r}_2 = 4$ and that its decoupling matrix has full rank. A dynamic state feedback that solves the MMP for (LM, LP) is given by:

$$\begin{aligned} \dot{z} &= Kz + L\begin{pmatrix} q \\ \dot{q} \end{pmatrix} + Nv \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= R\begin{pmatrix} q \\ \dot{q} \\ z \end{pmatrix} + Sv \end{aligned} \tag{7.55}$$

where $z = \mathrm{col}(z_1, z_2)$ and

$$K = \begin{pmatrix} 0 & 1 \\ -54 & -12 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -81 & 0 & 0 & -108 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 9.0278 & 20.956 & -2106.5641 & -0.0096 & -0.012 & 1.2012 & -0.2 & 0 \\ -0.9722 & 1.056 & -106.5641 & -0.0096 & -0.012 & 1.2012 & -0.2 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 0.0002 & 0.0001 \\ 0.0002 & 0.0001 \end{pmatrix}$$

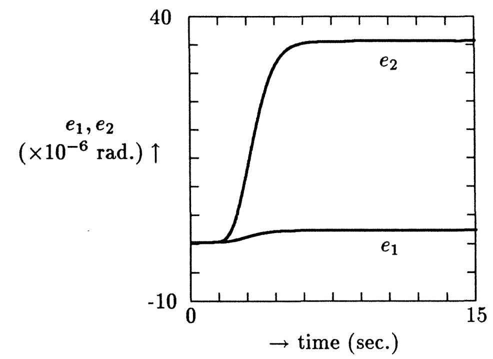

Figure 7.5: Robot arm with control (7.55) and $\bar{u}_1 = 1$

Figures 7.5 and 7.6 give the results of simulations of P with the dynamic state feedback (7.55) (where we have set $v = \bar{u}$). In Figure 7.5 a step $\bar{u}_1 = 1$ is applied, while in Figure 7.6 a step $\bar{u}_2 = 1$ is applied. The steps are applied during the whole time-interval. The figures give the errors $e_i(t) = y_i(t) - \bar{y}_i(t)$ $(i = 1, 2)$.

In the first case the system converges to a steady state situation with steady state errors $e_1 = 2 \cdot 10^{-6}$, $e_2 = 35 \cdot 10^{-6}$. The steady state error e_1 is 0.02% of the steady state value of $\bar{y}_1$. In the second case we end up with fluctuations around $e_1 = -2 \cdot 10^{-6}$, $e_2 = -102 \cdot 10^{-6}$. The value $e_2 = -102 \cdot 10^{-6}$ is 0.82% of the steady state value of $\bar{y}_2$.

Further simulations show that the stability of the closed loop is maintained for step controls up to $(\bar{u}_1, \bar{u}_2) = (12, 0)$, $(\bar{u}_1, \bar{u}_2) = (0, 9)$, $(\bar{u}_1, \bar{u}_2) = (5, 5)$.

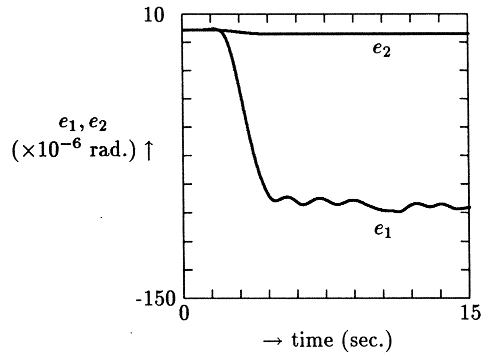

Figure 7.6: Robot arm with control (7.55) and $\bar{u}_2 = 1$

■

Chapter 8

Structure of strongly input-output decouplable Hamiltonian control systems

A special class of nonlinear control systems is given by the Hamiltonian control systems. Roughly speaking, these are nonlinear control systems for which the dynamics can be described via the Hamiltonian equations of motion. The concept of a Hamiltonian control system was introduced in [9] and further elaborated in e.g. [98].

In Subsection 8.1.1 we motivate the definition of a Hamiltonian control system via the Euler-Lagrangian and Hamiltonian equations of motion. In Subsection 8.1.2 a coordinate-free definiton of a Hamiltonian control system is given. In Section 8.2 we investigate the structure of strongly input-output decouplable Hamiltonian systems and these results are applied to the strong input-output decoupling problem with stability for Hamiltonian systems and to the model matching problem for Hamiltonian systems.

8.1 Definition of a Hamiltonian control system

In this section we closely follow Chapter 12 of [81] (see also [98],[20],[105]). An extensive geometrical treatment of Hamiltonian systems can be found in [1],[4].

8.1.1 Motivation of the definition

Consider a mechanical system with n degrees of freedom, locally represented by n generalized configuration coordinates $q = (q_1, \cdots, q_n)$. Assume that the system is conservative, that is, there are no dissipative forces present in the system. Let $T(q, \dot{q})$ denote the ***kinetic energy function*** of the system, and let $V(q)$ denote its *potential energy function.* Assume that the first m ($m \leq n$) configuration coordinates can be controlled via control variables (generalized forces) u_i ($i = 1, \cdots, m$). Define the *Lagrangian function* $L(q, \dot{q})$ of the system by

$$L(q, \dot{q}) := T(q, \dot{q}) - V(q) \tag{8.1}$$

Then the equations of motion of the system are given by the well-known *Euler-Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \begin{cases} u_i & (i = 1, \cdots, m) \\ 0 & i = m+1, \cdots, n \end{cases} \tag{8.2}$$

(8.2) is called a *Lagrangian control system.* Note that (8.2) does not constitute a system in state space form. However, for most mechanical systems the kinetic energy is of the form

$$T(q,\dot q) = \frac{1}{2}\dot q^T M(q)\dot q \tag{8.3}$$

where $M(q)$ is a positive definite matrix for all q. Let $C(q,\dot q)$ be an n-vector with entries

$$c_i(q,\dot q) = \sum_{k,s=1}^{n} \frac{\partial m_{is}}{\partial q_k}(q)\dot q_k \dot q_s - \frac{1}{2}\sum_{r,s=1}^{n} \frac{\partial m_{rs}}{\partial q_i}(q)\dot q_r \dot q_s \quad (i = 1,\cdots,n) \tag{8.4}$$

where m_{ij} denotes the (i,j)-th entry of M, and let

$$\begin{aligned} k(q) &= \frac{\partial V}{\partial q}(q) \\ B &= \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} \end{aligned} \tag{8.5}$$

Then (8.2) specializes to

$$M(q)\ddot q + C(q,\dot q) + k(q) = Bu \tag{8.6}$$

and we obtain the $2n$-dimensional state space system

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot q \end{pmatrix} = \begin{pmatrix} \dot q \\ -M^{-1}(q)C(q,\dot q) - M^{-1}(q)k(q) \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u \tag{8.7}$$

We now pass on to the Hamiltonian formulation. For the Lagrangian control system (8.2), assume that for all $(q,\dot q)$

$$\det\left(\frac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\right)_{1\le i,j\le n} \neq 0 \tag{8.8}$$

Define the *generalized momenta*

$$p_i = \frac{\partial L}{\partial \dot q_i}(q,\dot q) \quad (i = 1,\cdots,n) \tag{8.9}$$

Since (8.8) holds, we have that locally $(p_1,\cdots,p_n)$ is a set of independent functions. Define the *Hamiltonian function* $H(q,p)$ as the Legendre transform of $L(q,\dot q)$ with respect to $\dot q$, i.e.,

$$H(q,p) = \sum_{i=1}^{n} p_i \dot q_i - L(q,\dot q) \tag{8.10}$$

where $\dot q$ satisfies (8.9). Then the Euler-Lagrange equations of motion (8.2) transform into the *Hamiltonian* equations of motion

$$\begin{aligned} \dot q_i &= \frac{\partial H}{\partial p_i}(q,p) && (i = 1,\cdots,n) \\ \dot p_i &= -\frac{\partial H}{\partial q_i}(q,p) + u_i && (i = 1,\cdots,m) \\ \dot p_i &= -\frac{\partial H}{\partial q_i}(q,p) && (i = m+1,\cdots,n) \end{aligned} \tag{8.11}$$

We call (8.11) a *Hamiltonian control system.* A main advantage of (8.11) in comparison with (8.2) is that (8.11) immediately constitutes a control system in state space form, with state space variables (q,p). Moreover, if the kinetic energy function $T(q,\dot{q})$ is of the form (8.3), the Hamiltonian function $H(q,p)$ equals the total (internal) energy of the system. Namely, in this case (8.9) specializes to

$$p = M(q)\dot{q} \tag{8.12}$$

and hence in the new coordinates (q,p) the kinetic energy is given by

$$T(q,p) = \frac{1}{2}p^T M^{-1}(q)p \tag{8.13}$$

Then (8.10),(8.12),(8.13) yield

$$\begin{aligned} H(q,p) = \; & \sum_{i=1}^{n} p_i\dot{q}_i - L(q,\dot{q}) = \\ & p^T M^{-1}(q)p - (\tfrac{1}{2}p^T M^{-1}(q)p - V(q)) = \\ & \tfrac{1}{2}p^T M^{-1}(q)p + V(q) \end{aligned} \tag{8.14}$$

which establishes our claim.

(8.11) is not the most general form of Hamiltonian control systems we wish to study. In fact, what we want to study is a class of systems that is invariant w.r.t. a certain class of state space transformations. More specifically, consider a system of the form

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q,p) - \sum_{j=1}^{m} u_j\frac{\partial C_j}{\partial p_i}(q,p) \\ & \qquad\qquad\qquad\qquad\qquad (i = 1,\cdots,n) \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q,p) + \sum_{j=1}^{m} u_j\frac{\partial C_j}{\partial q_i}(q,p) \end{aligned} \tag{8.15}$$

Note that (8.11) is of the form (8.15) (with $C_j(q,p) = q_j$ for $j = 1,\cdots,m$). For any two functions $F(q,p), G(q,p)$, define

$$\{F,G\} = \sum_{i=1}^{n}(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}) \tag{8.16}$$

and consider state space transformations $(Q(q,p),P(q,p)) = (Q_1(q,p),\cdots,Q_n(q,p),$ $P_1(q,p),\cdots,P_n(q,p))$ that satisfy $\{Q_i,Q_j\} = \{P_i,P_j\} = 0$, $\{P_i,Q_j\} = \delta_{ij}$ $(i,j = 1,\cdots,n)$. Such state space transformations are called *canonical transformations.* It can be shown that the system (8.15), after we have applied a canonical transformation is again of the form (8.15) (with possibly different $H(Q,P), C_1(Q,P),\cdots,C_m(Q,P)$). Now if we equip a system of the form (8.15) with the natural outputs:

$$y_j = C_j(q,p) \quad (j = 1,\cdots,m) \tag{8.17}$$

we obtain a square input-output system. Any system of the form (8.15,8.17) is called an *affine Hamiltonian input-output system* (or briefly Hamiltonian system).

There are some good reasons to choose the outputs as in (8.17). We mention two of them (also see the discussion in [98],[20],[81]). First of all, with this choice of outputs we obtain from (8.15) the *energy balance*:

$$\dot{H} = \sum_{j=1}^{m} u_j\dot{y}_j \tag{8.18}$$

Hence in the case (8.11), where $y_j = q_j$, $\dot{H}$ equals the instantaneous external work performed on the system. Second, with this choice of outputs the external "forces" $u_1, \cdots, u_m$ influence the system via the external channels corresponding to the outputs $C_1, \cdots, C_m$, which are the "displacements" caused by these excitations along the same line of action. If we interpret the outputs as measurements performed on the system, the last point implies that we are dealing with a mechanical system with collocated actuators and sensors.

8.1.2 Coordinate-free definition of a Hamiltonian system

Definition 8.1.1 Poisson structure, Poisson bracket, Poisson manifold
Let M be a manifold and let $C^\infty(M)$ be the set of smooth real-valued functions on M. A Poisson structure on M is a bilinear map from $C^\infty(M) \times C^\infty(M)$ into $C^\infty(M)$, called the Poisson bracket and denoted as

$$(F,G) \mapsto \{F,G\} \quad F,G \in C^\infty(M) \tag{8.19}$$

which satisfies the following properties for any $F,G,H \in C^\infty(M)$

$$\{F,G\} = -\{G,F\} \quad (skew-symmetry) \tag{8.20}$$

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0 \quad (Jacobi-identity) \tag{8.21}$$

$$\{F,GH\} = \{F,G\}H + G\{F,H\} \quad (Leibniz\ rule) \tag{8.22}$$

M together with a Poisson structure is called a Poisson manifold. ∎

Lemma 8.1.2 *Let M be a Poisson manifold with local coordinates $x_1, \cdots, x_r$.*

(i) There exist locally smooth functions $w_{ij}(x)$ $(1 \le i \le r, 1 \le j \le r)$ such that the Poisson bracket is locally given as

$$\{F,G\}(x) = \sum_{i,j=1}^{r} w_{ij}(x) \frac{\partial F}{\partial x_i}(x) \frac{\partial G}{\partial x_j}(x) \tag{8.23}$$

(ii) The functions $w_{ij}(x)$ in (i) are determined by

$$w_{ij}(x) = \{x_i, x_j\}(x) \quad (1 \le i \le r, 1 \le j \le r) \tag{8.24}$$

and satisfy

$$w_{ij}(x) = -w_{ji}(x) \quad (1 \le i \le r, 1 \le j \le r) \tag{8.25}$$

$$\sum_{\ell=1}^{r} (w_{\ell j} \frac{\partial w_{ik}}{\partial x_\ell} + w_{\ell i} \frac{\partial w_{kj}}{\partial x_\ell} + w_{\ell k} \frac{\partial w_{ji}}{\partial x_\ell}) = 0 \quad (1 \le i,j,k \le r) \tag{8.26}$$

Proof See e.g. [81]. ∎

Remark 8.1.3 (8.25) reflects the skew-symmetry of the Poisson-bracket, while (8.26) reflects the fact that the Poisson-bracket satisfies the Jacobi-identity. ∎

From Lemma 8.1.2 we conclude that locally any Poisson bracket is determined by a skew-symmetric (r,r)-matrix $W(x)$ of which the entries $w_{ij}(x)$ $(1 \leq i,j \leq r)$ satisfy (8.26). The matrix $W(x)$ is called the *structure matrix* of the Poisson structure. The *rank* of the Poisson bracket in every $x \in M$ is defined as the rank of $W(x)$. Since $W(x)$ is skew-symmetric, necessarily its rank is even. A Poisson bracket is said to be non-degenerate if $\text{rank}W(x) = \dim M$ for every $x \in M$. In particular for a non-degenerate Poisson bracket we have for any $x \in M$

$$\text{rank}W(x) = \dim M = 2n \text{ for some } n \tag{8.27}$$

Example 8.1.4 Let $M = I\!R^{2n}$ with coordinates $(x_1,\cdots,x_{2n}) =: (q_1,\cdots,q_n,p_1,\cdots,p_n)$. Define the Poisson bracket of two functions $F(q,p)$,$G(q,p)$ as

$$\{F,G\}(q,p) = \sum_{i=1}^{n}(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i})(q,p) \tag{8.28}$$

It is easily checked that for this Poisson bracket we have $\{q_i,q_j\} = \{p_i,p_j\} = 0$ and $\{p_i,q_j\} = \delta_{ij}$, Hence the structure matrix of this bracket is given by

$$W(x) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{8.29}$$

and we conclude that the bracket is a non-degenerate Poisson-bracket. The bracket (8.28) is called the *standard* Poisson bracket on $I\!R^{2n}$. ∎

The following theorem states that locally every non-degenerate Poisson bracket is as the standard Poisson bracket defined by (8.29) (cf. [1],[4]).

Theorem 8.1.5 *(Darboux) Let M be a $2n$-dimensional manifold with non-degenerate Poisson-bracket $\{\cdot,\cdot\}$. Then locally around any $x_0 \in M$ we can find coordinates $(q,p) = (q_1,\cdots,q_n,\, p_1,\cdots,p_n)$, called* canonical coordinates, *such that*

$$\{F,G\}(q,p) = \sum_{i=1}^{n}(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i})(q,p) \tag{8.30}$$

∎

Consider a Poisson manifold M. For a given $F \in C^\infty(M)$ and arbitrary $x \in M$, define the mapping $X_F(x) : C^\infty(M) \to I\!R$ as

$$X_F(x)G := \{F,G\}(x) \tag{8.31}$$

By the fact that the Poisson bracket satisfies the Jacobi-identity and the Leibniz rule, it follows that $X_F(x) \in T_xM$ for any $x \in M$. Hence for any $F \in C^\infty(M)$ we obtain a smooth vector field X_F on M satisfying

$$X_F(G) = \{F,G\} \text{ for any } G \in C^\infty(M) \tag{8.32}$$

X_F is called the *Hamiltonian vector field* corresponding to the Hamiltonian function F and the Poisson bracket $\{\cdot,\cdot\}$ on M. We have the following connection between the Lie bracket of two Hamiltonian vector fields and the Poisson bracket of their Hamiltonians:

Lemma 8.1.6 *For any $F, G \in C^\infty(M)$ we have*

$$[X_F, X_G] = X_{\{F,G\}} \tag{8.33}$$

Proof See e.g. [81]. ∎

With any non-degenerate Poisson structure, we can associate another, dual, geometric object. Namely, let $\{\cdot,\cdot\}$ be the non-degenerate Poisson bracket on M, locally given by the skew-symmetric structure matrix $W(x)$. Define the bilinear map

$$\omega_x : T_xM \times T_xM \to I\!R \tag{8.34}$$

by setting

$$\omega_x(X_F(x), X_G(x)) = \{F, G\}(x) \tag{8.35}$$

By linearity and the fact that we can choose $2n$ functions $F_1, \cdots, F_{2n}$ such that $X_{F_1}, \cdots, X_{F_{2n}}$ are independent, (8.35) completely defines ω_x as a bilinear map (8.34). By (8.23), we have in local coordinates

$$\{F, G\}(x) = dF(x)W(x)(dG(x))^T \tag{8.36}$$

where

$$dF = \left(\frac{\partial F}{\partial x_1} \quad \cdots \quad \frac{\partial F}{\partial x_{2n}} \right) \tag{8.37}$$

Moreover, by (8.34) the Hamiltonian vector field X_F is represented in local coordinates by

$$X_F(x) = (dF \cdot W(x))^T \tag{8.38}$$

Hence

$$\begin{aligned} \{F, G\}(x) = \; & (X_F(x))^T W^{-1}(x) W(x) W^{-T}(x) X_G(x) = \\ & (X_F(x))^T W^{-T}(x) X_G(x) \end{aligned} \tag{8.39}$$

Then it follows that ω_x has the matrix representation

$$[\omega_x] = W^{-T}(x) = -W^{-1}(x) \tag{8.40}$$

In canonical coordinates $x = (q, p)$, ω_x equals the constant matrix $W(x)$ in (8.29). By letting x vary we obtain a so-called differential two-form ω, which is called a ***symplectic form*** on M. Furthermore, note that by (8.35) and $\{F, G\} = -\{G, F\} = -X_G(F) = -dF(X_G)$, the Hamiltonian vector field X_H corresponding to the Hamiltonian H is uniquely determined by the relation

$$\omega_x(X_H(x), Z) = -dH(x)(Z) \quad \text{for any } Z \in T_xM, x \in M \tag{8.41}$$

The manifold M endowed with the symplectic form ω is called a *symplectic manifold.*

We are now able to give a coordinate free definition of a Hamiltonian control system (8.15).

Definition 8.1.7 Hamiltonian system
Let M be a manifold with non-degenerate Poisson bracket. Let $H, C_1, \cdots, C_m \in C^\infty(M)$. Then

$$\begin{cases} \dot{x} &= X_H(x) - \sum_{j=1}^{m} X_{C_j}(x)u_j \\ \\ y_j &= C_j(x) \quad (j = 1, \cdots, m) \end{cases} \tag{8.42}$$

is called an affine Hamiltonian input-output system, or briefly, Hamiltonian system. ∎

By (8.32) and Theorem 8.1.5 we have in local canonical coordinates $x = (q, p)$:

$$X_F = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}\right) \tag{8.43}$$

Hence we immediately obtain:

Corollary 8.1.8 *Consider a Hamiltonian system (8.42) on M. Then around any $x_0 \in M$ there exist canonical coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ for M such that (8.42) takes the form (8.15).* ∎

Remark 8.1.9 If the Poisson bracket in Definition 8.1.7 is degenerate, the system (8.42) is called a *Poisson system* (see [86]). ∎

A special class of Hamiltonian systems that is of special interest for applications, are the *simple* Hamiltonian systems. Roughly, these are Hamiltonian systems with a Hamiltonian function (internal energy) H of the form (8.14) and with observation functions $C_1, \cdots, C_m$ that only depend on the configuration coordinates q. We give a formal definition of this class of Hamiltonian systems, following [98] (see [1] for the definition of a simple Hamiltonian system without controls). For this we need the following. Let Q be an n-dimensional manifold, denoting the configuration space, and let T^*Q be its cotangent bundle, denoting the phase space or state space. On T^*Q there is a naturally defined Poisson bracket, which is defined in local coordinates as follows. Let $q_0 \in Q$ and let $q = (q_1, \cdots, q_n)$ be local coordinates for Q around q_0. Then there exist natural coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ for T^*Q (cf. Section 1.2). Now, let $F, G \in C^\infty(T^*Q)$. Then their Poisson bracket is defined as

$$\{F, G\}(q, p) = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right)(q, p) \tag{8.44}$$

It can be shown (cf. [81]) that this Poisson bracket is well-defined, i.e., it does not depend on the particular choice of natural coordinates for T^*Q. In particular we see that natural coordinates for T^*Q are canonical coordinates for the Poisson bracket (8.44) on T^*Q.

Definition 8.1.10 Simple Hamiltonian system
A simple Hamiltonian system on T^*Q is a Hamiltonian system (8.42) where $H, C_1, \cdots, C_m$ are of the form (in natural coordinates (q,p) for T^*Q)

$$H(q,p) = \frac{1}{2}p^T G(q)p + V(q) \tag{8.45}$$

with $G(q)$ a positive definite (n,n)-matrix for every q, and

$$C_j(q,p) = C_j(q) \quad (j = 1, \cdots, m) \tag{8.46}$$

The expression $\frac{1}{2}p^T G(q)p$ is called the kinetic energy and $V(q)$ the potential energy. ■

8.2 Structural properties of strongly input-output decouplable Hamiltonian systems

8.2.1 Clamped dynamics of strongly input-output decouplable Hamiltonian systems

When no specific references are given, the results in this subsection can be found in [81].

Consider a $2n$-dimensional manifold M with a non-degenerate Poisson bracket $\{\cdot,\cdot\}$. Let an analytic Hamiltonian system on M be given by

$$\begin{cases} \dot{x} = X_H(x) - \sum_{j=1}^{m} X_{C_j}(x)u_j \\ y_j = C_j(x) \quad (j = 1, \cdots, m) \end{cases} \tag{8.47}$$

where $x = (x_1, \cdots, x_{2n})^T \in \mathbb{R}^{2n}$ are local coordinates for M and $H, C_1, \cdots, C_m$ are analytic functions on M. Assume that the SIODP is solvable for (8.47), i.e., the relative degrees $r_1, \cdots, r_m$ of (8.47) are finite and the decoupling matrix $A(x)$ of (8.47) has full rank m for all $x \in M$ (cf. Theorem 3.1.5). For any $F, G \in C^\infty(M)$ we define inductively:

$$\begin{aligned} \mathrm{ad}_F^0 G &:= G \\ \mathrm{ad}_F^k G &:= \{F, \mathrm{ad}_F^{k-1} G\} \quad (k = 1, 2, \cdots) \end{aligned} \tag{8.48}$$

By (8.32) we have that for any $k = 0, 1, \cdots$

$$\mathcal{L}_{X_H}^k C_i = \mathrm{ad}_H^k C_i \tag{8.49}$$

and

$$\mathcal{L}_{X_{C_j}} \mathcal{L}_{X_H}^k C_i = \{C_j, \mathrm{ad}_H^k C_i\} \tag{8.50}$$

Hence the relative degree r_i $(i = 1, \cdots, m)$ of (8.47) is defined as the smallest integer such that (see also (3.68)):

$$\begin{cases} \left(\{C_1, \mathrm{ad}_H^k C_i\} \ \cdots \ \{C_m, \mathrm{ad}_H^k C_i\} \right)(x) = 0 \quad (\forall x \in M, k = 0, \cdots, r_i - 2) \\ \left(\{C_1, \mathrm{ad}_H^{r_i-1} C_i\} \ \cdots \ \{C_m, \mathrm{ad}_H^{r_i-1} C_i\} \right)(x) \neq 0 \quad (\text{for some } x \in M) \end{cases} \tag{8.51}$$

and the decoupling matrix $A(x)$ has the entries

$$a_{ij}(x) = -\{C_j, \mathrm{ad}_H^{r_i-1}C_i\}(x) = \{\mathrm{ad}_H^{r_i-1}C_i, C_j\}(x) \tag{8.52}$$

(8.51) and (8.52) imply that for the computation of $r_1, \cdots, r_m$ and $A(x)$ we do not have to go through the equations of motion (8.47); the knowledge of $H, C_1, \cdots, C_m$ suffices.

For the system (8.47) we make the following assumption:

Assumption 8.2.1 The distribution $\mathcal{G} := \mathrm{span}\,\{X_{C_1}, \cdots, X_{C_m}\}$ is involutive. ■

Remark 8.2.2 If all relative degrees of (8.47) are greater than one, Assumption 8.2.1 is automatically satisfied. Namely, in this case we have for all $i, j = 1, \cdots, m$: $\{C_i, C_j\} = 0$ and hence by Lemma 8.1.6: $[X_{C_i}, X_{C_j}] = X_{\{C_i,C_j\}} = 0$. ■

Analogously to (3.14), define $\xi_{ij}(x) := \mathrm{ad}_H^{j-1}C_i(x)$ $(i = 1, \cdots, m; j = 1, \cdots, r_i)$. Since Assumption 8.2.1 is satisfied and the SIODP is solvable for the system, we can find an additional set of functions $\bar{x}_1(x), \cdots, \bar{x}_d(x)$ $(d = 2n - \sum_{i=1}^m r_i)$ that satisfy $\mathcal{L}_{X_{C_j}}\bar{x}_i = \{C_j, \bar{x}_i\} = 0$ such that $\{\{\xi_{ij} \mid 1 \le i \le m, 1 \le j \le r_i\}, \bar{x}_1, \cdots, \bar{x}_d\}$ forms a new set of local coordinates for M (cf. Theorem 3.1.11). Define $\xi_i := \mathrm{col}(\xi_{ij} \mid 1 \le j \le r_i)$ $(i = 1, \cdots, m)$, $\xi := (\xi_1^T \cdots \xi_m^T)^T$, $\bar{x} := \mathrm{col}(\bar{x}_1, \cdots, \bar{x}_d)$ and $\Phi(x) := (\xi^T(x)\ \bar{x}^T(x))^T$. For a function F in the coordinates x, we denote for convenience:

$$F(\xi, \bar{x}) = F \circ \Phi^{-1}(\xi, \bar{x}) \tag{8.53}$$

Then in the new coordinates $(\xi, \bar{x})$ the system (8.47) has the form (compare with (3.17)):

$$\begin{array}{rcll} \dot{\xi}_{i1} & = & \xi_{i2} & \\ & \vdots & & (i = 1, \cdots, m) \\ \dot{\xi}_{ir_i-1} & = & \xi_{ir_i} & \\ \dot{\xi}_{r_i} & = & \mathrm{ad}_H^{r_i}C_i(\xi, \bar{x}) + A_{i*}(\xi, \bar{x})u & \\ \dot{\bar{x}}_i & = & \{H, \bar{x}_i\}(\xi, \bar{x}) & (i = 1, \cdots, d) \\ y_j & = & \xi_{j1} & (j = 1, \cdots, m) \end{array} \tag{8.54}$$

We investigate the structure of the clamped dynamics of (8.54). Since the decoupling matrix of (8.54) has full rank for all $(\xi_1, \cdots, \xi_m, \bar{x})$, it is easily seen that the following result holds (see e.g. [81],[61]).

Proposition 8.2.3 *The clamped dynamics manifold N^* of (8.54) is given by*

$$N^* = \{(\xi_1, \cdots, \xi_m, \bar{x}) \mid \xi_1 = 0, \cdots, \xi_m = 0\} \tag{8.55}$$

Moreover, the clamped dynamics of (8.54) is given by

$$\dot{\bar{x}}_i = \{H, \bar{x}_i\}(0, \bar{x}) \quad (i = 1, \cdots, d) \tag{8.56}$$

■

The clamped dynamics manifold N^* of (8.54) and the clamped dynamics (8.56) have a special structure. To describe this structure, we need the following definition.

Definition 8.2.4 Symplectic submanifold
Consider a manifold M with symplectic form ω. Let N be a submanifold of M. Restrict the bilinear map $\omega_x : T_xM \times T_xM \to I\!R$, $x \in M$, to a bilinear map $\bar{\omega}_x : T_xN \times T_xN \to I\!R$, $x \in N$, i.e.,

$$\bar{\omega}_x(X,Y) := \omega_x(X,Y) \quad (X, Y \in T_xN,\ x \in N) \tag{8.57}$$

Then N is called a symplectic submanifold of M if $\bar{\omega}_x$ is a non-degenerate bilinear map for every $x \in N$, i.e., if the rank of a matrix representation of $\bar{\omega}_x$ equals $\dim N$ for every $x \in N$. ∎

Remark 8.2.5 Note that in particular a symplectic submanifold is even-dimensional. ∎

Proposition 8.2.6 *Consider a manifold M with symplectic form ω. Let a submanifold N of M be given by*

$$N = \{x \mid \phi_1(x) = \cdots = \phi_k(x) = 0\}$$

for functions $\phi_1, \cdots, \phi_k$. Then N is a symplectic submanifold of M if and only if for any $i \in \{1, \cdots, k\}$ there exists a $j \in \{1, \cdots, k\}$ such that $\{\phi_i, \phi_j\} \neq 0$.

Proof See e.g. [22]. ∎

We now prove that N^* is a symplectic submanifold of M. For this, we need the following lemma.

Lemma 8.2.7 *For all $k, \ell \in I\!N$ we have*

$$\{\mathrm{ad}_H^k C_j, \mathrm{ad}_H^\ell C_i\} = \sum_{r=0}^{\ell} \binom{\ell}{r} (-1)^{r+1} \mathrm{ad}_H^{\ell-r} \{C_i, \mathrm{ad}_H^{k+r} C_j\} \tag{8.58}$$

Proof By induction, using the Jacobi-identity. ∎

From Lemma 8.2.7 we obtain in particular:

Corollary 8.2.8 *For all $i, j = 1, \cdots, m$ and for all $\ell = 0, \cdots, r_j - 1$ we have*

$$\{\mathrm{ad}_H^{r_j-1-\ell} C_j, \mathrm{ad}_H^\ell C_i\} = (-1)^{\ell+1} \{C_i, \mathrm{ad}_H^{r_j-1} C_j\} \tag{8.59}$$

Proof Follows immediately from (8.58) and the definition of the relative degrees. ∎

Since the decoupling matrix $A(x)$ of (8.47) has full rank for all $x \in M$, we have that for any $j \in \{1, \cdots, m\}$ there exists an $i \in \{1, \cdots, m\}$ such that $\{C_i, \mathrm{ad}_H^{r_j-1} C_j\} \neq 0$. By Corollary 8.2.8 this implies that for any function $\mathrm{ad}_H^k C_j$ $(j = 1, \cdots, m; k = 0, \cdots, r_j - 1)$ there exist $i \in \{1, \cdots, m\}$, $\ell \in \{0, \cdots, r_i - 1\}$ such that $\{\mathrm{ad}_H^\ell C_i, \mathrm{ad}_H^k C_j\} \neq 0$. Hence by (8.55) and Proposition 8.2.6 N^* is a symplectic submanifold of M.

We now show that the clamped dynamics (8.56) of (8.54) is in fact a Hamiltonian system on N^*. For this, we define a Poisson bracket $\{\cdot, \cdot\}_{N^*}$ on N^* in the following way. Let $s := \sum_{i=1}^m r_i$ and denote the functions $C_i(x), \cdots, \mathrm{ad}_H^{r_i-1} C_i(x)$ $(i = 1, \cdots, m)$ as $\psi_j(x)$ $(j = 1, \cdots, s)$. Form the (s, s) skew-symmetric matrix $D(x)$ with entries

$$d_{ij}(x) = \{\psi_i, \psi_j\}(x) \quad (i, j = 1, \cdots, s) \tag{8.60}$$

It can be shown that $D(x)$ is an invertible matrix for all $x \in M$ (cf. [81]). Let the entries of $D^{-1}(x)$ be denoted by $d^{ij}(x)$. For any $F, G \in C^\infty(N^*)$, define

$$\{F, G\}_{N^*}(x) = \{F, G\}(x) - \sum_{i,j=1}^{s} \{F, \psi_i\}(x) d^{ij}(x) \{\psi_j, G\}(x) \quad (x \in N^*) \tag{8.61}$$

where the right-hand side of (8.61) is computed for any smooth extensions of F and G to a neighborhood of x in M. Then we have the following result.

Proposition 8.2.9 *Let ω denote the symplectic form on M associated with the Poisson bracket $\{\cdot, \cdot\}$. Let $\bar{\omega}$ denote the restriction of ω to N^* (see (8.57)). For any $F \in C^\infty(N^*)$, define the vector field $\bar{X}_F$ on N^* by setting (see (8.41))*

$$\bar{\omega}_x(\bar{X}_F, Z) = -dF(x)(Z) \quad \text{for any } Z \in T_x N^*, x \in N^* \tag{8.62}$$

Then:

(i) $\{\cdot, \cdot\}_{N^*}$ is a non-degenerate Poisson bracket on N^*, called the Dirac bracket, *and for any $F \in C^\infty(N^*)$ the vector field $\bar{X}_F$ is the Hamiltonian vector field on N^* with respect to F and the Poisson bracket $\{\cdot, \cdot\}_{N^*}$, i.e., for any $G \in C^\infty(N^*)$*

$$\bar{X}_F(G) = \{F, G\}_{N^*} \; (= \bar{\omega}(\bar{X}_F, \bar{X}_G)) \tag{8.63}$$

(ii) Define the (s, d)-matrix $U(x)$ with entries

$$u_{ij}(x) = \{\psi_i, \bar{x}_j\}(x) \quad (i = 1, \cdots, s; j = 1, \cdots, d) \tag{8.64}$$

and the (d, d)-matrix $S(x)$ with entries

$$s_{ij}(x) = \{\bar{x}_i, \bar{x}_j\}(x) \quad (i = 1, \cdots, d; j = 1, \cdots, d) \tag{8.65}$$

Then the structure matrix $\bar{W}(x)$ of $\{\cdot, \cdot\}_{N^*}$ is given by

$$\bar{W}(x) = S(x) + U^T(x) D^{-1}(x) U(x) \tag{8.66}$$

Proof See [81]. ■

Now let us return to the clamped dynamics (8.56) of (8.54) evolving on the symplectic submanifold N^*. We first introduce some notation. Recall that $\xi_{ij} := \mathrm{ad}_H^{j-1} C_i$ ($i = 1, \cdots, m; j = 1, \cdots, r_i$), $\xi_i = \mathrm{col}(\xi_{i1}, \cdots, \xi_{ir_i})$ ($i = 1, \cdots, m$) and $\xi = (\xi_1^T \cdots \xi_m^T)^T$. Furthermore, recall that $\bar{x}_1, \cdots, \bar{x}_d$ ($d = 2n - \sum_{i=1}^m r_i$) are chosen in such a way that $(\xi, \bar{x})$ forms a local coordinate system for M and $\{\xi_{j1}, \bar{x}_i\} = 0$ ($i = 1, \cdots, d; j = 1, \cdots, m$). Denote

$$
\begin{aligned}
\partial_\xi F &= \begin{pmatrix} \frac{\partial F}{\partial \xi_{11}} & \cdots & \frac{\partial F}{\partial \xi_{1r_1}} & \cdots & \frac{\partial F}{\partial \xi_{m1}} & \cdots & \frac{\partial F}{\partial \xi_{mr_m}} \end{pmatrix} \\
\partial_{\bar{x}} F &= \begin{pmatrix} \frac{\partial F}{\partial \bar{x}_1} & \cdots & \frac{\partial F}{\partial \bar{x}_d} \end{pmatrix}
\end{aligned}
\tag{8.67}
$$

Then, given another function $G \in C^\infty(M)$, in the new coordinates $(\xi, \bar{x})$ we have

$$
\{F, G\}(\xi, \bar{x}) =
$$

$$
\begin{pmatrix} \partial_\xi F(\xi, \bar{x}) & \partial_{\bar{x}} F(\xi, \bar{x}) \end{pmatrix}
\begin{pmatrix} D(\xi, \bar{x}) & U(\xi, \bar{x}) \\ -U^T(\xi, \bar{x}) & S(\xi, \bar{x}) \end{pmatrix}
\begin{pmatrix} (\partial_\xi G(\xi, \bar{x}))^T \\ (\partial_{\bar{x}} G(\xi, \bar{x}))^T \end{pmatrix}
\tag{8.68}
$$

Furthermore, denote

$$
\begin{aligned}
\{F, \xi\} &= \begin{pmatrix} \{F, \xi_{11}\} & \cdots & \{F, \xi_{1r_1}\} & \cdots & \{F, \xi_{m1}\} & \cdots & \{F, \xi_{mr_m}\} \end{pmatrix} \\
\{F, \bar{x}\} &= \begin{pmatrix} \{F, \bar{x}_1\} & \cdots & \{F, \bar{x}_d\} \end{pmatrix}
\end{aligned}
\tag{8.69}
$$

Then from (8.68) it follows that

$$
\begin{aligned}
\{F, \xi\}(\xi, \bar{x}) &= \partial_\xi F(\xi, \bar{x}) D(\xi, \bar{x}) - \partial_{\bar{x}} F(\xi, \bar{x}) U^T(\xi, \bar{x}) \\
\{F, \bar{x}\}(\xi, \bar{x}) &= \partial_\xi F(\xi, \bar{x}) U(\xi, \bar{x}) + \partial_{\bar{x}} F(\xi, \bar{x}) S(\xi, \bar{x})
\end{aligned}
\tag{8.70}
$$

Finally, for a function $F \in C^\infty(M)$, let $\bar{F}$ denote its restriction to N^*, i.e., $\bar{F}(\bar{x}) = F(0, \bar{x})$.

From (8.56) and (5.17) it follows that the (unique) control $u = \alpha^*(\xi, \bar{x})$ that renders N^* as an invariant submanifold is given by

$$
\alpha^*(\xi, \bar{x}) = -A^{-1}(\xi, \bar{x}) b(\xi, \bar{x})
\tag{8.71}
$$

where

$$
\begin{aligned}
b(\xi, \bar{x}) = &\begin{pmatrix} \mathrm{ad}_H^{r_1} C_1 & \cdots & \mathrm{ad}_H^{r_m} C_m \end{pmatrix}^T (\xi, \bar{x}) = \\
&\begin{pmatrix} \{H, \xi_{1r_1}\} & \cdots & \{H, \xi_{mr_m}\} \end{pmatrix}^T (\xi, \bar{x})
\end{aligned}
\tag{8.72}
$$

Define the function $H^a(\xi, \bar{x})$ by

$$
H^a(\xi, \bar{x}) = H(\xi, \bar{x}) - \sum_{i=1}^m \alpha_i^*(\xi, \bar{x}) \xi_{i1}
\tag{8.73}
$$

Since $\xi_{i1} = 0$ on N^* $(i = 1, \cdots, m)$ we have on N^*:

$$\begin{aligned}\{H, \xi\}(0, \bar{x}) - \sum_{i=1}^{m} \alpha_i^*(0, \bar{x})\{\xi_{i1}, \xi\}(0, \bar{x}) = \\ \{H - \sum_{i=1}^{m} \alpha_i^* \xi_{i1}, \xi\}(0, \bar{x}) = \{H^a, \xi\}(0, \bar{x})\end{aligned} \tag{8.74}$$

Hence on N^* we have

$$0 = \dot{\xi}^T = \{H, \xi\}(0, \bar{x}) - \sum_{i=1}^{m} \alpha_i^*(0, \bar{x})\{\xi_{i1}, \xi\}(0, \bar{x}) = \{H^a, \xi\}(0, \bar{x}) \tag{8.75}$$

By (8.70) this implies that

$$\partial_\xi H^a(0, \bar{x}) D(0, \bar{x}) - \partial_{\bar{x}} H^a(0, \bar{x}) U^T(0, \bar{x}) = 0 \tag{8.76}$$

and hence

$$\partial_\xi H^a(0, \bar{x}) = \partial_{\bar{x}} H^a(0, \bar{x}) U^T(0, \bar{x}) D^{-1}(0, \bar{x}) \tag{8.77}$$

Now note that $\bar{x}$ is constructed in such a way that $\{\xi_{i1}, \bar{x}_j\} = 0$ $(i = 1, \cdots, m; j = 1, \cdots, d)$. Hence on N^* we have

$$\{H, \bar{x}\}(0, \bar{x}) = \{H - \sum_{i=1}^{m} \alpha_i^* \xi_{i1}, \bar{x}\}(0, \bar{x}) = \{H^a, \bar{x}\}(0, \bar{x}) \tag{8.78}$$

Then by (8.56),(8.70),(8.77) the clamped dynamics of (8.54) are given by

$$\begin{aligned}\dot{\bar{x}}^T = \;& \{H, \bar{x}\}(0, \bar{x}) = \{H^a, \bar{x}\}(0, \bar{x}) = \\ & \partial_\xi H^a(0, \bar{x}) V(0, \bar{x}) + \partial_{\bar{x}} H^a(0, \bar{x}) S(0, \bar{x}) = \\ & \partial_{\bar{x}} H^a(0, \bar{x})(S(0, \bar{x}) + V^T(0, \bar{x}) D^{-1}(0, \bar{x}) V(0, \bar{x}) = \\ & \partial_{\bar{x}} \bar{H}^a(\bar{x}) \bar{W}(0, \bar{x}) = \partial_{\bar{x}} \bar{H}(\bar{x}) \bar{W}(0, \bar{x}) = \{\bar{H}, \bar{x}\}_{N^*}(0, \bar{x})\end{aligned} \tag{8.79}$$

This means that we have established:

Theorem 8.2.10 *Consider the Hamiltonian system (8.47) on M. Assume that* $\dim(N^*) > 0$ *and that* $\operatorname{rank} A(x) = m$ *for all $x \in M$. Then the clamped dynamics (8.56) is the Hamiltonian vector field on N^* with respect to the Dirac bracket (8.61) and Hamiltonian $\bar{H}$, i.e., the clamped dynamics are given as*

$$\dot{\bar{x}} = (\{\bar{H}, \bar{x}\}_{N^*})^T(0, \bar{x}) \tag{8.80}$$

where $\bar{H}(\bar{x}) = H(0, \bar{x})$. ■

Remark 8.2.11 The result of Theorem 8.2.10 can be found in [100],[103], where it was proved in a slightly different way. The Dirac bracket was introduced in [32] for the study of constrained Hamiltonian systems. See [70] for contributions on the control of constrained Hamiltonian systems. ■

8.2.2 Strong input-output decoupling with stability for Hamiltonian systems

When no specific references are given, the results in this subsection can be found in [59]

In this subsection we apply the result of Theorem 8.2.10 to the strong input-output decoupling problem with stability for the Hamiltonian system (8.47). We first give a definition of the problem. For this, we need the following definitions (cf. [43]):

Definition 8.2.12 Locally stable equilibrium point, locally asymptotically stable equilibrium point
Consider

$$\dot{x} = f(x) \tag{8.81}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ are local coordinates for a manifold M and f is an analytic vector field on M. Let $x_0 \in M$ be an equilibrium point of (8.81), i.e., $f(x_0) = 0$. Then

(i) x_0 is said to be locally stable if for any neighborhood $U \subset M$ of x_0 there exists a neighborhood $\bar{U} \subset M$ of x_0 such that for any $\bar{x} \in \bar{U}$, the solution of (8.81) with $x(0) = \bar{x}$ remains in U for all $t \geq 0$.

(ii) x_0 is said to be locally asymptotically stable if x_0 is locally stable and there exists a neighborhood $U_0 \subset M$ of x_0 such that for any $\bar{x} \in U_0$ the solution of (8.81) with $x(0) = \bar{x}$ converges to x_0 as $t \to \infty$.

■

Consider the Hamiltonian system (8.47). Let $x_0 \in M$ be an equilibrium point of (8.47), i.e., $X_H(x_0) = 0$. Then we say that the SDIODP with (asymptotic) stability for (8.47) is locally solvable around x_0, if we can find a regular static state feedback

$$u = \alpha(x) + \beta(x)v \tag{8.82}$$

with $\alpha(x_0) = 0$ for (8.47) such that (8.47,8.82) is strongly input-output decoupled around x_0 and x_0 is a locally (asymptotically) stable equilibrium point for (8.47,8.82). Note that the requirement $\alpha(x_0) = 0$ guarantees that x_0 is an equilibrium point for (8.47,8.82).

Assume that all relative degrees of (8.47) are finite and that its decoupling matrix $A(x)$ has full rank for $x = x_0$. Furthermore, assume that Assumption 8.2.1 is satisfied for (8.47). Then locally around x_0, (8.47) takes the form (8.54). Consider the following class of feedbacks for (8.54) (denoted by L):

$$u = A^{-1}(\xi, \bar{x})(\gamma(\xi) - b(\xi, \bar{x})) + A^{-1}(\xi, \bar{x})v \tag{8.83}$$

with $b(\xi, \bar{x})$ as defined in (8.72) and $\gamma(\xi)$ an m-vector of functions with entries

$$\gamma_i(\xi) = \sum_{j=1}^{r_i} \gamma_{ij}\xi_{ij} \quad (i = 1, \cdots, m) \tag{8.84}$$

Then (8.54,8.83) has the form

$$
\begin{aligned}
\Sigma_i: \quad \dot{\xi}_i &= A_i\xi_i + B_i v_i \qquad (i = 1, \cdots, m) \\
\tilde{\Sigma}: \quad \dot{\bar{x}} &= (\{H, \bar{x}\})^T(\xi, \bar{x}) \\
y_j &= \xi_{j1} \qquad (j = 1, \cdots, m)
\end{aligned}
\tag{8.85}
$$

where

$$
A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & 1 \\ \gamma_{i1} & \gamma_{i2} & \cdots & \cdots & \gamma_{ir_i} \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
$$

Obviously, (8.85) constitutes a strongly input-output decoupled system. Moreover, by a proper choice of the constants γ_{ij} $(i = 1, \cdots, m; j = 1, \cdots, r_i)$ the linear subsystems Σ_i $(i = 1, \cdots, m)$ can be made asymptotically stable. Denote the class of feedbacks from L that render the linear subsystems Σ_i asymptotically stable by L_s. Then we see that if we apply a feedback from the class L_s to (8.54) and if $N^* = \{0\}$, we arrive at an asymptotically stable system (8.85) (where the nonlinear subsystem $\tilde{\Sigma}$ is not present). The question arises under which conditions a feedback from the class L_s locally stabilizes the total system (8.85) if $N^* \neq \{0\}$. For this we consider the clamped dynamics (8.80) of (8.54). Assume that $\bar{x} = 0$ is an equilibrium point of (8.80) or, equivalently, $(\xi, \bar{x}) = (0, 0)$ is an equilibrium point of (8.85).

Proposition 8.2.13 *Consider the system (8.85). Then*

(i) The equilibrium point $(\xi, \bar{x}) = (0, 0)$ cannot be a locally asymptotically stable equilibrium point of (8.85).

(ii) The equilibrium point $(\xi, \bar{x}) = (0, 0)$ is a locally stable equilibrium point of (8.85) if and only if $\bar{x} = 0$ is a locally stable equilibrium point of (8.80). ∎

Remark 8.2.14 It should be noted that Proposition 8.2.13 still holds if we choose a decoupling feedback (8.83) where $\gamma_i(\xi)$ is a function of ξ_i that is not of the form (8.84). ∎

In the proof of Proposition 8.2.13 we use the following result from centre manifold theory (see [15]).

Proposition 8.2.15 *Consider the system*

$$
\begin{aligned}
\dot{x} &= Ax + f(x, z) \\
\dot{z} &= Bz + g(x, z)
\end{aligned}
\tag{8.86}
$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ and A and B are constant matrices such that A has all its eigenvalues in the open left half plane and B has all its eigenvalues on the imaginary axis. The functions f and g are C^2 with $f(0,0) = 0$, $g(0,0) = 0$, $Df(0,0) = 0$ and $Dg(0,0) = 0$ (here Df is the Jacobian matrix of f). Then *(i) The equation $Ax + f(x,z) = 0$ has a solution*

$$x = h(z), \quad | x |< \delta \tag{8.87}$$

where h is C^2 and $\delta > 0$. Moreover, the manifold $\{(x,z) \mid x = h(z)\}$, called a centre manifold, *is an invariant manifold for the system (8.86) (see Definition 5.0.1). A solution $(x(t), z(t))$ $(t \geq 0)$ of (8.86) starting at a point $(x_0, z_0) = (h(z_0), z_0)$ on the centre manifold is given by $(x(t), z(t)) = (h(\eta(t)), \eta(t))$, where $\eta(t)$ is a solution of the differential equation*

$$\dot{\eta} = B\eta + g(h(\eta), \eta) \tag{8.88}$$

with initial condition $\eta(0) = z_0$.

(ii) Assume that the equilibrium point of (8.88) is locally stable, locally asymptotically stable, unstable respectively. Then the equilibrium point of (8.86) is locally stable, locally asymptotically stable, unstable respectively.

■

Remark 8.2.16 The use of centre manifold theory in nonlinear control theory was introduced in [3] (see also [68]). ■

Proof of Proposition 8.2.13.

(i) The dynamics (8.80) are governed by a Hamiltonian vector field (cf. Theorem 8.2.10) and hence, by Liouville's theorem (cf. [4],[1]), $\bar{x} = 0$ can only be a locally stable equilibrium point of (8.80) and not a locally asymptotically stable one. Hence $(\xi, \bar{x}) = (0,0)$ is not a locally asymptotically stable equilibrium point of (8.85).

(ii) *(necessity)* By the proof of *(i)* we see that $(\xi, \bar{x}) = (0,0)$ can at most be a locally stable equilibrium point of (8.85) and that a necessary condition for stability is that $\bar{x} = 0$ is a stable equilibrium point for (8.80).

(sufficiency) Assume that $\bar{x} = 0$ is a stable equilibrium point of (8.80). Set $v = 0$ in (8.85). Then, taking Taylor series of (8.85) around $(\xi, \bar{x}) = (0,0)$, we obtain a system of the form

$$\begin{aligned} \dot{\xi} &= A\xi \\ \dot{\bar{x}} &= K\bar{x} + L\xi + g(\xi, \bar{x}) \end{aligned} \tag{8.89}$$

where the matrix A has all its eigenvalues in the open left half plane, $g(0,0) = 0$, $Dg(0,0) = 0$, and

$$\begin{aligned} K &= \partial_{\bar{x}}(\{H, \bar{x}\})^T(0,0) = \partial_{\bar{x}}(\{\bar{H}, \bar{x}\}_{N^*})^T(0,0) \\ L &= \partial_{\xi}(\{H, \bar{x}\})^T(0,0) \end{aligned} \tag{8.90}$$

Since $\bar{x} = 0$ is a locally stable equilibrium point of (8.80), the matrix K has all its eigenvalues in the closed left half plane. Moreover, K is the state matrice of a linear

Hamiltonian system and therefore it has an eigenvalue λ if and only if it also has an eigenvalue $-\lambda$ (cf. [1]). Hence K has all its eigenvalues on the imaginary axis. Therefore, the matrices A and K have distinct eigenvalues. Hence there exists a matrix P such that $PA - KP = -L$ (cf. [38]). Defining $\tilde{x} = P\xi + \bar{x}$, we obtain from (8.89):

$$\begin{aligned} \dot{\xi} &= A\xi \\ \dot{\tilde{x}} &= K\tilde{x} + \tilde{g}(\xi, \tilde{x}) \end{aligned} \tag{8.91}$$

where $\tilde{g}(\xi, \tilde{x}) = g(\xi, \tilde{x} - P\xi)$. Then (8.91) is of the same form as (8.86). Obviously, in this case the centre manifold is given by $\xi = 0$ and in the coordinates $(\xi, \tilde{x})$ the flow on the centre manifold is governed by

$$\dot{\tilde{x}} = \dot{\bar{x}} = (\{H, \bar{x}\})^T(0, \tilde{x}) \tag{8.92}$$

Since by assumption $\tilde{x} = 0$ is a locally stable equilibrium point of (8.92), we have by Proposition 8.2.15 that $(\xi, \tilde{x}) = (0, 0)$ is a locally stable equilibrium point of (8.91). By the definition of $\tilde{x}$ this immediately implies that $(\xi, \bar{x}) = (0, 0)$ is a locally stable equilibrium point of (8.85), which establishes our claim. ∎

Using Proposition 8.2.13 we can give the following sufficient condition for local stability of (8.85):

Proposition 8.2.17 *Consider the system (8.85). Recall that the function $\bar{H}$ is defined by $\bar{H}(\bar{x}) = H(0, \bar{x})$. Then $(\xi, \bar{x}) = (0, 0)$ is a locally stable (not locally asymptotically stable) equilibrium point of (8.85) if $\bar{H}$ has an isolated local minimum at $\bar{x} = 0$.*

Proof By (8.80) we have on N^*:

$$\dot{\bar{H}} = \partial_{\bar{x}}\bar{H} \cdot \dot{\bar{x}} = \partial_{\bar{x}}\bar{H}(\bar{x})\bar{W}^T(0, \bar{x})(\partial_{\bar{x}}\bar{H}(\bar{x}))^T = 0 \tag{8.93}$$

where the last equality follows from the skew-symmetry of $\bar{W}(\xi, \bar{x})$. Hence, if $\bar{H}$ has an isolated local minimum at $\bar{x} = 0$, then $\bar{H}$ is a Lyapunov function, which implies that $\bar{x} = 0$ is a locally stable equilibrium point of (8.80) (cf. [43]). Then, by Proposition 8.2.13, $(\xi, \bar{x}) = (0, 0)$ is a locally stable equilibrium point of (8.85). ∎

For the stability of a Hamiltonian system it is not necessary that its Hamiltonian function has an isolated local minimum at the stable equilibrium point, as follows from the following example from [1].

Example 8.2.18 Consider two harmonic oscillators, of which one is running backwards in time. The dynamics of this system are described by the following equations on $\mathbb{R}^4$:

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q, p) \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q, p) \end{aligned}\right. \quad (i = 1, 2) \tag{8.94}$$

where $H(q,p) = \frac{1}{2}(q_1^2 + p_1^2) - \frac{1}{2}(q_2^2 + p_2^2)$. (8.94) constitutes a linear Hamiltonian system with state matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{8.95}$$

It is easily checked that A has eigenvalues $-i$,i, where the algebraic as well as the geometric multiplicity of both eigenvalues equals two. Hence (cf. [43]) $(q,p) = (0,0)$ is a stable equilibrium point of (8.94). However, $(q,p) = (0,0)$ is a saddle point of H. ■

By Example 8.2.18 we see that the converse of Proposition 8.2.17 is not true in general. However, for *simple* Hamiltonian systems (see Definition 8.1.10) we can make a statement about the converse of Proposition 8.2.17. First consider a simple Hamiltonian systems without controls. Then we have the following result.

Proposition 8.2.19 *Consider a Hamiltonian function H belonging to a simple Hamiltonian system without controls. Assume that the origin is an equilibrium point of $\dot{x} = X_H(x)$, i.e. $X_H(0) = 0$. For a given kinetic energy function, consider the subset of $C^\infty(Q)$ consisting of all potential energy functions $V(q)$ for which the origin is a locally stable equilibrium point of $\dot{x} = X_H(x)$. Then for all $V(q)$ in an open and dense subset of this set (endowed with the C^∞-Whitney topology, cf. [1]), H has an isolated local minimum at the origin.*

Proof Let $H = \frac{1}{2}p^T G(q)p + V(q)$, where $G(q)$ is positive definite for all q. Define the matrices P and R by

$$\begin{aligned} P &= G(0) \\ R &= \frac{\partial^2 V}{\partial q^2}(0,0) \end{aligned} \tag{8.96}$$

The linearization of $\dot{x} = X_H(x)$ $(x = (q^T, p^T)^T)$ around $(q,p) = (0,0)$ is given by

$$\begin{pmatrix} \dot{q}^\ell \\ \dot{p}^\ell \end{pmatrix} = \begin{pmatrix} 0 & P \\ -R & 0 \end{pmatrix} \begin{pmatrix} q^\ell \\ p^\ell \end{pmatrix} \tag{8.97}$$

Let the eigenvalues of P be given by $p^1, \cdots, p^n$ and those of R by $r^1, \cdots, r^n$. Then it can be checked that the characteristic polynomial $p(\lambda)$ of the state matrix in (8.97) is given by $p(\lambda) = \prod_{i=1}^n (\lambda^2 + p^i r^i)$. the fact that $(q,p) = (0,0)$ is a stable equilibrium point of $\dot{x} = X_H(x)$ implies that the state matrix in (8.97) cannot have eigenvalues in the open right half plane (cf. [43]). Hence we have $p^i r^i \geq 0$ $(i = 1, \cdots, n)$. By the fact that P is positive definite, this implies that $r^i \geq 0$ $(i = 1, \cdots, n)$. Now the set of $V \in C^\infty(Q)$ satisfying $r^i > 0$ $(i = 1, \cdots, n)$ forms an open and dense subset of the set of $V \in C^\infty(Q)$ with the property that $r_i \geq 0$ $(i = 1, \cdots, n)$. Moreover, for all $V \in C^\infty(Q)$ with the property that $r^i > 0$ $(i = 1, \cdots, n)$ we have that the matrix

$$\frac{\partial^2 H}{\partial x^2}(0,0) = \begin{pmatrix} R & 0 \\ 0 & P \end{pmatrix} \tag{8.98}$$

is positive definite, which means that $(q,p) = (0,0)$ is an isolated local minimum for H. ■

Now consider a simple Hamiltonian system (8.47) on T^*Q and assume that its decoupling matrix $A(x)$ has full rank for all $x \in T^*Q$. By (8.23) and (8.52) we can write

$$A = -\begin{pmatrix} dC_1 \\ \vdots \\ dC_m \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ & \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \mathrm{dad}_H^{r_1-1}C_1 \\ \vdots \\ \mathrm{dad}_H^{r_m-1}C_m \end{pmatrix} \tag{8.99}$$

and hence the fact that $A(x)$ has full rank for all $x \in M$ implies that the codistribution $\mathrm{span}\,\{dC_1, \cdots, dC_m\}$ has constant dimension m. This means that we can find natural coordinates (q,p) for T^*Q such that $C_i(q) = q_i$ $(i = 1, \cdots, m)$, cf. [1]. It can be checked that for $i, j = 1, \cdots, m$

$$\{q_j, \mathrm{ad}_H q_i\} = g_{ij}(q) \tag{8.100}$$

where $g_{ij}(q)$ denotes the (i,j)-th entry of the matrix $G(q)$. Since $G(q)$ is positive definite for all $q \in Q$, we have that $g_{ii}(q) > 0$ for all $q \in Q$ $(i = 1, \cdots, m)$. Hence all relative degrees of (8.47) equal two and we have $\xi = (q_1, \cdots, q_m, \dot{q}_1, \cdots, \dot{q}_m)$. Since for all $i = 1, \cdots, m$ and all $j = m+1, \cdots, n$ we have $\{q_i, q_j\} = \{q_i, p_j\} = 0$, we can take $\bar{x} = (q_{m+1}, \cdots, q_n, p_{m+1}, \cdots, p_n)$. Denote $\tilde{q} = (q_1, \cdots, q_m)$, $\tilde{p} = (p_1, \cdots, p_m)$, $\bar{q} = (q_{m+1}, \cdots, q_n)$, $\bar{p} = (p_{m+1}, \cdots, p_n)$. Partition the matrix $G(q)$ according to

$$G(q) = \begin{pmatrix} G_{11}(q) & G_{12}(q) \\ G_{12}^T(q) & G_{22}(q) \end{pmatrix} \tag{8.101}$$

where $G_{11}(q)$ is an (m,m)-matrix and $G_{22}(q)$ is an $(n-m, n-m)$-matrix. In the new coordinates $(\xi, \bar{x})$ we obtain

$$H(\xi, \bar{x}) = \frac{1}{2}\dot{\tilde{q}}^T G_1(\tilde{q}, \bar{q})\dot{\tilde{q}} + \frac{1}{2}\bar{p}^T G_2(\tilde{q}, \bar{q})\bar{p} + V(\tilde{q}, \bar{q}) \tag{8.102}$$

where $G_1(\tilde{q}, \bar{q}) = G_{11}^{-1}(\tilde{q}, \bar{q})$ and $G_2(\tilde{q}, \bar{q}) = G_{22}(\tilde{q}, \bar{q}) - G_{12}^T(\tilde{q}, \bar{q})G_{11}^{-1}(q)G_{12}(\tilde{q}, \bar{q})$. It is a standard result from linear algebra (cf. [38]) that $G_2(\tilde{q}, \bar{q})$ is positive definite for all $(\tilde{q}, \bar{q})$. For (8.47) we obtain (cf. [81]):

$$D(q,p) = \begin{pmatrix} 0 & -G_{11}(q) \\ G_{11}(q) & F(q,p) \end{pmatrix} \tag{8.103}$$

where F is the (m,m)-matrix with (i,j)-th entry

$$f_{ij}(q,p) = \sum_{k,\ell=1}^{n} \left(g_{i\ell}\frac{\partial g_{jk}}{\partial q_\ell} - g_{j\ell}\frac{\partial g_{ik}}{\partial q_\ell}\right)(q)p_k \tag{8.104}$$

Moreover, $U(q,p)$ (see Proposition 8.2.9) has the form

$$U(q,p) = \begin{pmatrix} 0_{m,2(n-m)} \\ \\ \tilde{U}(q,p) \end{pmatrix} \tag{8.105}$$

Then it follows that

$$U^T(q,p)D^{-1}(q,p)U(q,p) = 0 \tag{8.106}$$

and hence the structure matrix $\bar{W}(q,p)$ of $\{\cdot,\cdot\}_{N^*}$ is given by (see Proposition 8.2.9)

$$\bar{W}(q,p) = S(q,p) = \begin{pmatrix} 0 & -I_{n-m} \\ I_{n-m} & 0 \end{pmatrix} \tag{8.107}$$

From (8.102) we obtain

$$\bar{H}(\bar{x}) = H(0, \bar{x}) = \frac{1}{2}\bar{p}^T G_2(0, \bar{q})\bar{p} + V(0, \bar{q}) \tag{8.108}$$

Hence by Theorem 8.22 the clamped dynamics of the simple Hamiltonian system (8.47) are given by

$$\begin{aligned} \dot{\bar{q}} &= \frac{\partial \bar{H}}{\partial \bar{p}}(\bar{q}, \bar{p}) \\ \dot{\bar{p}} &= -\frac{\partial \bar{H}}{\partial \bar{q}}(\bar{q}, \bar{p}) \end{aligned} \tag{8.109}$$

and we see that (8.109) constitutes a simple Hamiltonian system (without controls).

We have now obtained the following converse of Proposition 8.2.17:

Proposition 8.2.20 *Consider a* simple *Hamiltonian system (8.47). Assume that the SIODP is solvable for (8.47). Let $\bar{H}$ be defined as the restriction of the Hamiltonian H to N^* and assume that $\bar{x} = 0$ is an equilibrium point for the clamped dynamics (8.109). Then $\bar{H}$ has an isolated local minimum at the origin for all $V(q)$ in an open and dense subset of the set of functions in $C^\infty(Q)$ for which there exists a decoupling feedback from the class L_s that locally stabilizes $(\xi, \bar{x}) = (0, 0)$.* ■

Combining the results of Propositions 8.2.13,8.2.17 and 8.2.20 we have:

Theorem 8.2.21 *Consider the Hamiltonian system (8.47). Assume that the SIODP is solvable for (8.47) and that the system satisfies Assumption 8.2.1. Assume that $\bar{x} = 0$ is an equilibrium point for the clamped dynamics (8.56). Then:*

(i) Application of a decoupling feedback from the class L_s cannot result in a locally asymptotically stable equilibrium point $(\xi, \bar{x}) = (0, 0)$.

(ii) Any decoupling feedback from the class L_s renders $(\xi, \bar{x}) = (0, 0)$ a locally stable equilibrium point if and only if $\bar{x} = 0$ is a locally stable equilibrium point for the clamped dynamics (8.56).

(iii) If $\bar{H}$ has an isolated local minimum for $\bar{x} = 0$, then $\bar{x} = 0$ is a locally stable equilibrium point for (8.56).

(iv) If, moreover, (8.47) is a simple Hamiltonian system, Then $\bar{H}$ has an isolated local minimum at the origin for all $V(q)$ in an open and dense subset of the set of functions in $C^\infty(Q)$ for which there exists a decoupling feedback from the class L_s that locally stabilizes $(\xi, \bar{x}) = (0, 0)$. ■

Remark 8.2.22 Theorem 8.2.21 can be easily extended to the following generalized version of system equations (8.47). Assume that the control vector fields in (8.47) are not necessarily given by X_{C_j}, but instead take the more general form

$$X_{P_j \circ (C_1, \cdots, C_m)} \quad (j = 1, \cdots, m) \tag{8.110}$$

where the mapping $P = (P_1, \cdots, P_m) : \mathbb{R}^m \to \mathbb{R}^m$ is assumed to be a diffeomorphism. (For example, this may happen in the case of robot manipulators. Generally, the input torques correspond to the joint coordinates, but the outputs may be given in task space coordinates. In the case of an equal number of controls and outputs the joint cordinates are usually related to the task coordinates by a transformation which is invertible except for some singular points.) Then the relative degrees r_i $(i = 1, \cdots, m)$ are the same as if the control vector fields would equal X_{C_j} $(j = 1, \cdots, m)$. Moreover, the decoupling matrix in this generalized case is given as

$$-\begin{pmatrix} \{C_1, \mathrm{ad}_H^{r_1-1} C_1\} & \cdots & \{C_m, \mathrm{ad}_H^{r_1-1} C_1\} \\ \vdots & & \vdots \\ \{C_1, \mathrm{ad}_H^{r_m-1} C_m\} & \cdots & \{C_m, \mathrm{ad}_H^{r_m-1} C_m\} \end{pmatrix} \begin{pmatrix} \frac{\partial P_1}{\partial y_1} & \cdots & \frac{\partial P_m}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial P_1}{\partial y_m} & \cdots & \frac{\partial P_m}{\partial y_m} \end{pmatrix} \tag{8.111}$$

and hence, since P is a diffeomorphism, the decoupling matix has full rank if and only if the decoupling matrix for the system with control vector fields X_{C_j} has full rank. A decoupling feedback from the class L brings the system into the same form (8.54), and so Theorem 8.2.21 also applies to this more general case. ∎

We conclude this subsection with an example from robotics.

Example 8.2.23 Consider a robot arm, consisting of two unit masses attached to massless links of length one, where the first link is flexible. To model this flexibility we divide the first link in two auxiliary links, one of length ϵ $(0 < \epsilon < 1)$ and the other of length $1 - \epsilon$, connected by a torsional spring with spring constant k (see Figure 8.1). We apply a torque u_1 on the auxiliary link, hence in the Hamiltonian formulation the first control vector field will be X_{D_1}, where $D_1(q) = q_1$. On the second link we apply a torque u_2. Because of the flexibility of the first link, we assume that the motor exerting this torque is situated in the base of the robot arm and that the torque is transfered by means of a chain belt. Since this introduces reaction forces (see [5]), in the Hamiltonian formulation the second control vector field will be X_{D_2}, where $D_2(q) = q_1 + q_3$. To guarantee positive definiteness of the kinetic energy matrix we introduce an auxiliary mass δ $(\ll 1)$ located at the joint between the first and second auxiliary link.

We are now interested in the stability properties of the equilibrium point $(q_1, q_2, q_3) = (\frac{\pi}{2}, 0, 0)$ after we have applied a decoupling feedback from the class L_s. This means that we are interested in the outputs $C_1(q) = q_1 - \frac{\pi}{2}$, $C_2(q) = q_3$ rather than D_1 and D_2. However, there exists a diffeomorphism $P : \mathbb{R}^2 \to \mathbb{R}^2$ such that $P(C_1, C_2) = (D_1, D_2)$. Hence, by Remark 8.2.22, the relative degrees and the clas L_s will be the same with both choices of outputs. Calculation yields

$$\begin{aligned} N^* = \ & \{(q,p) \mid q_1 = \tfrac{\pi}{2}, q_3 = 0, \\ & p_1 = \left(\frac{(3-2\epsilon)\epsilon \cos q_2}{2\epsilon^2 - 6\epsilon + 5} + 1 \right) p_2, p_3 = \frac{2-\epsilon}{2\epsilon^2 - 6\epsilon + 5} p_2\} \end{aligned} \tag{8.112}$$

and

$$\begin{aligned} & \bar{H}(q_2, p_2) = \\ & \frac{1}{2(2\epsilon^2 + 6\epsilon + 5)} p_2^2 + 2\epsilon g + (3 - 2\epsilon) g \cos q_2 + \frac{k q_2^2}{2} \end{aligned} \tag{8.113}$$

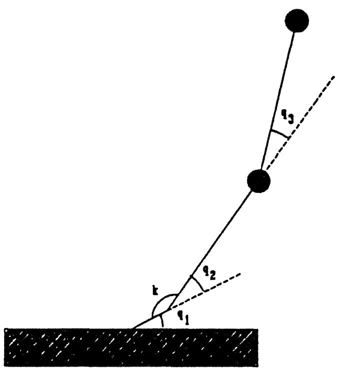

Figure 8.1: Robot arm with flexible link

where we have put $\delta = 0$. Hence the clamped dynamics of the system are given by

$$\begin{aligned} \dot{q}_2 &= \frac{\partial \bar{H}}{\partial p_2}(q_2, p_2) \\ \dot{p}_2 &= -\frac{\partial \bar{H}}{\partial q_2}(q_2, p_2) \end{aligned} \tag{8.114}$$

It is obvious that $(q_2, p_2) = (0, 0)$ is an equilibrium point of (8.114). In this point we have:

$$\frac{\partial^2 \bar{H}}{\partial (q_2, p_2)}(0,0) = \begin{pmatrix} k - (3-2\epsilon)g & 0 \\ 0 & \dfrac{1}{2\epsilon^2 + 6\epsilon + 5} \end{pmatrix} \tag{8.115}$$

Hence $\bar{H}$ has an isolated local minimum at $(q_2, p_2) = (0, 0)$ if $k > (3 - 2\epsilon)g$, i.e., if the flexibility is not too big. Thus $(q_2, p_2) = (0, 0)$ is a stable equilibrium point of the dynamics (8.114). Furthermore, it is easy to check that $(q_2, p_2) = (0, 0)$ is an unstable equilibrium point of (8.114) if $k < (3 - 2\epsilon)g$. Further, it may be shown that if $k = (3 - 2\epsilon)g$, $(q_2, p_2) = (0, 0)$ is still a stable equilibrium point of (8.114). Summarizing, $(q_1, q_2, q_3) = (\frac{\pi}{2}, 0, 0)$ can be made a stable equilibrium point by application of a decoupling feedback from the class L_s if and only if $k \geq (3 - 2\epsilon)g$. ■

8.2.3 Structure of strongly input-output decoupled Hamiltonian systems

When no specific references are made, the results in this subsection are based on unpublished joint work with A.J. van der Schaft.

In Subsection 8.2.1 we have seen that the clamped dynamics (8.56) of a strongly input-output decouplable Hamiltonian system (8.47) is again a Hamiltonian system (without controls) with respect to the Dirac bracket $\{\cdot, \cdot\}_{N^*}$ on N^*. In general there will not exist a decoupling feedback for (8.47) such that the resulting (strongly input-output decoupled) system is again a Hamiltonian system. In fact, the class of Hamiltonian systems satisfying this requirement is quite restrictive. For example, it was shown in [80] (see also [79]) that a necessary condition for the existence of a decoupling feedback for a Hamiltonian system that preserves the Hamiltonian structure, is that its decoupling matrix is a diagonal matrix. In [93] it was shown that if we choose a decoupling feedback from the class L with $\gamma_{ij} = 0$ $(i = 1, \cdots, m; j = 0, \cdots, r_i - 1)$ (i.e., the matrices (A_i, B_i) $(i = 1, \cdots, m)$ are in Brunovsky canonical form), the linear subsystems Σ_i $(i = 1, \cdots, m)$ in (8.85) are Hamiltonian with respect to some symplectic form $\{\cdot,\cdot\}_i$, provided that r_i is even.

In this subsection we consider the nonlinear subsystem

$$\tilde{\Sigma}: \quad \dot{\bar{x}} = (\{H, \bar{x}\})^T(\xi, \bar{x}) \tag{8.116}$$

in (8.85). The vector field $(\{H, \bar{x}\})^T(\xi, \bar{x})$ is considered as a vector field on $\mathbb{R}^d$, where $d = 2n - \sum_{i=1}^m r_i$, depending on the variable $\bar{x}$ and parametrized by ξ. Therefore, denote

$$f_\xi(\bar{x}) := (\{H, \bar{x}\})^T(\xi, \bar{x}) \tag{8.117}$$

For two smooth functions F_ξ, G_ξ of the variable $\bar{x}$ that are parametrized by ξ we define the parametrized Poisson bracket $\{\cdot,\cdot\}_\xi$ by:

$$\begin{aligned} \{F_\xi, G_\xi\}_\xi(\bar{x}) := \; & \partial_{\bar{x}} F_\xi(\bar{x})(S(\xi, \bar{x}) + U^T(\xi, \bar{x}) D^{-1}(\xi, \bar{x}) U(\xi, \bar{x}))(\partial_{\bar{x}} G_\xi(\bar{x}))^T = \\ & \partial_{\bar{x}} F_\xi(\bar{x}) \bar{W}(\xi, \bar{x})(\partial_{\bar{x}} G_\xi(\bar{x}))^T \end{aligned} \tag{8.118}$$

It follows as in Proposition 8.2.9 that this is *indeed* a Poisson bracket for every ξ.

We ask ourselves the question if, for every $\xi \in \mathbb{R}^s$, (recall that $s = \sum_{i=1}^m r_i$) the dynamics (8.116), parametrized by ξ, is Hamiltonian with respect to $\{\cdot,\cdot\}_\xi$, i.e., if there exists a function $\bar{H}_\xi(\bar{x})$, parametrized by ξ, such that for every $\xi \in \mathbb{R}^s$ we have

$$f_\xi(\bar{x}) = (\{\bar{H}_\xi, \bar{x}\}_\xi(\bar{x}))^T \tag{8.119}$$

This is motivated by the discussion in Subsection 8.2.1 since for $\xi = 0$ the Poisson bracket (8.117) is just the Dirac bracket on N^*.

A positive answer to this question may give important structural information in the solution of several synthesis problems for Hamiltonian systems. In the next subsection this is illustrated for the model matching problem with prescribed tracking error for Hamiltonian systems. Also it is to be expected that for the solution of the nonlinear regulator problem (see e.g. [37],[14],[46]) for Hamiltonian systems this result will be of help.

Unfortunately, up till now the result in its full generality has neither been proved, nor a counter example has been found. Therefore, instead of giving a proof or a counter example, we formulate a conjecture and treat some special cases on which the conjecture is based.

Conjecture 8.2.24 *There exists a function $\bar{H}_\xi(\bar{x})$, such that for every $\xi \in \mathbb{R}^s$ (8.119) is satisfied.* ■

We first consider the special cases of a single-input single-output Hamiltonian system with relative degree $r = 2$ and relative degree $r = 4$. Note that by Corollary 8.2.8 the relative degree of a single-input single-output Hamiltonian system is necessarily even. Hence the cases $r = 1$, $r = 3$ do not occur. In the exposition we use the following result from [13], which generalizes Theorem 3.1.11.

Theorem 8.2.25 *Consider the analytic single-input single-output system*

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases} \tag{8.120}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ are local coordinates for the state space manifold M, $u \in \mathbb{R}$ and $y \in \mathbb{R}$. Let r denote the relative degree of (8.120). Let $x_0 \in M$ be such that $\mathcal{L}_g \mathcal{L}_f^{r-1} h(x_0) \neq 0$. Denote $\xi_j = \mathcal{L}_f^{j-1} h$ $(j = 1, \cdots, r)$. Define inductively: $\mathrm{ad}_f^0 g = g$, $\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g]$ $(k = 1, 2, \cdots)$. Let $k \in \{0, 1, \cdots, r-1\}$. Then locally around x_0 there exist functions $z_1, \cdots, z_{n-r} : M \to \mathbb{R}$ such that $(\xi_1, \cdots, \xi_r, z_1, \cdots, z_{n-r})$ forms a local coordinate system for M around x_0 and

$$\mathcal{L}_{\mathrm{ad}_f^i g} z_j(x) = 0 \quad (1 \leq j \leq n-r, 0 \leq i \leq r-k) \tag{8.121}$$

if and only if the distribution

$$\Gamma_k = \mathrm{span}\,\{g, \mathrm{ad}_f g, \cdots, \mathrm{ad}_f^{r-k} g\} \tag{8.122}$$

has constant dimension $r - k + 1$ and is involutive, in a neighborhood of x_0. ∎

Using Theorem 8.2.25 we obtain:

Corollary 8.2.26 *Let M be an even-dimensional manifold with non-degenerate Poisson bracket $\{\cdot,\cdot\}$. Consider an analytic Hamiltonian control system on M of the form*

$$\begin{cases} \dot{x} &= X_H(x) - X_C(x)u \\ y &= C(x) \end{cases} \tag{8.123}$$

where $x = (x_1, \cdots, x_{2n})^T$ are local coordinates for M, $u \in \mathbb{R}$ and $y \in \mathbb{R}$. Let r denote the relative degree of (8.123). Let $x_0 \in M$ be such that $\{C, \mathrm{ad}_H^{r-1} C\}(x_0) \neq 0$. Denote $\xi_j := \mathrm{ad}_H^{j-1} C$ $(j = 1, \cdots, r)$. Recall that r is even, and define $\ell := (r/2)$. Then locally around x_0 there exist functions $\bar{x}_1, \cdots, \bar{x}_{2n-r} : M \to \mathbb{R}$ such that $(\xi_1, \cdots, \xi_r, \bar{x}_1, \cdots, \bar{x}_{2n-r})$ is a local coordinate system for M around x_0 and

$$\{\xi_j, \bar{x}_i\} = 0 \quad (j = 1, \cdots, \ell; i = 1, \cdots, 2n-r) \tag{8.124}$$

Proof By Lemma 8.2.7 and the definition of the relative degree we have that for all $i, j = 1, \cdots, \ell$:

$$\{\mathrm{ad}_H^{i-1} C, \mathrm{ad}_H^{j-1} C\} = 0 \tag{8.125}$$

Hence by a similar argument as was used in Remark 8.2.2 we have that the distribution

$$\Gamma_\ell = \mathrm{span}\,\{X_C, \cdots, X_{\mathrm{ad}_H^{\ell-1} C}\} \tag{8.126}$$

is involutive. The result now follows immediately from Theorem 8.2.25. ∎

SISO-system, relative degree r=2

Consider a SISO Hamiltonian system (8.123) with relative degree $r = 2$. Define ξ_1, ξ_2 as in Corollary 8.2.26. For convenience, assume that M has dimension four. The arguments that follow can be extended to higher dimensional systems. By Corollary 8.2.26 we can find functions $\bar{x}_1, \bar{x}_2$ such that $(\xi_1, \xi_2, \bar{x}_1, \bar{x}_2)$ forms a local coordinate system for M and $\{\xi_1, \bar{x}_1\} = \{\xi_1, \bar{x}_2\} = 0$. As in Proposition 8.2.9, form the matrices

$$
\begin{aligned}
D(\xi,\bar{x}) &= \begin{pmatrix} \{\xi_1,\xi_1\} & \{\xi_1,\xi_2\} \\ \{\xi_2,\xi_1\} & \{\xi_2,\xi_2\} \end{pmatrix}(\xi,\bar{x}) =: \begin{pmatrix} 0 & d_{12} \\ -d_{12} & 0 \end{pmatrix}(\xi,\bar{x}) \\
U(\xi,\bar{x}) &= \begin{pmatrix} \{\xi_1,\bar{x}_1\} & \{\xi_1,\bar{x}_2\} \\ \{\xi_2,\bar{x}_1\} & \{\xi_2,\bar{x}_2\} \end{pmatrix}(\xi,\bar{x}) =: \begin{pmatrix} 0 & 0 \\ u_{21} & u_{22} \end{pmatrix}(\xi,\bar{x}) \\
S(\xi,\bar{x}) &= \begin{pmatrix} \{\bar{x}_1,\bar{x}_1\} & \{\bar{x}_1,\bar{x}_2\} \\ \{\bar{x}_2,\bar{x}_1\} & \{\bar{x}_2,\bar{x}_2\} \end{pmatrix}(\xi,\bar{x}) =: \begin{pmatrix} 0 & s_{12} \\ -s_{12} & 0 \end{pmatrix}(\xi,\bar{x})
\end{aligned} \tag{8.127}
$$

Then

$$\bar{W}(\xi,\bar{x}) = S(\xi,\bar{x}) + U^T(\xi,\bar{x})D^{-1}(\xi,\bar{x})U(\xi,\bar{x}) = S(\xi,\bar{x}) \tag{8.128}$$

In particular, (8.128) implies that $S(\xi,\bar{x})$ is invertible. Using similar arguments as are used in the proof of Theorem 8.1.5 (see [1]), it can be shown that we can choose $\bar{x}_1, \bar{x}_2$ in such a way that $\{\bar{x}_1, \bar{x}_2\} = -1$, i.e.,

$$\bar{W}(\xi,\bar{x}) = S(\xi,\bar{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{8.129}$$

From the Jacobi-identity (8.21) we obtain:

$$
\begin{aligned}
0 =\ & \{\xi_1,\{\xi_2,\bar{x}_1\}\} + \{\bar{x}_1,\{\xi_1,\xi_2\}\} + \{\xi_2,\{\bar{x}_1,\xi_1\}\} = \\
& \{\xi_1,u_{21}\} + \{\bar{x}_1,d_{12}\} + 0 = \frac{\partial u_{21}}{\partial \xi_2}d_{12} - \frac{\partial d_{12}}{\partial \xi_2}u_{21} - \frac{\partial d_{12}}{\partial \bar{x}_2} \Rightarrow \\
\frac{\partial d_{12}}{\partial \bar{x}_2} =\ & \frac{\partial u_{21}}{\partial \xi_2}d_{12} - \frac{\partial d_{12}}{\partial \xi_2}u_{21}
\end{aligned} \tag{8.130}
$$

$$
\begin{aligned}
0 =\ & \{\xi_1,\{\xi_2,\bar{x}_2\}\} + \{\bar{x}_2,\{\xi_1,\xi_2\}\} + \{\xi_2,\{\bar{x}_2,\xi_1\}\} = \\
& \{\xi,u_{22}\} + \{\bar{x}_2,d_{12}\} + 0 = \\
& \frac{\partial u_{22}}{\partial \xi_2}d_{12} - \frac{\partial d_{12}}{\partial \xi_2}u_{22} + \frac{\partial d_{12}}{\partial \bar{x}_1} \Rightarrow \\
\frac{\partial d_{12}}{\partial \bar{x}_1} =\ & -\frac{\partial u_{22}}{\partial \xi_2}d_{12} + \frac{\partial d_{12}}{\partial \xi_2}u_{22}
\end{aligned} \tag{8.131}
$$

$$
\begin{aligned}
0 =\ & \{\xi_2,\{\bar{x}_1,\bar{x}_2\}\} + \{\bar{x}_2,\{\xi_2,\bar{x}_1\}\} + \{\bar{x}_1,\{\bar{x}_2,\xi_2\}\} = \\
& 0 + \{\bar{x}_2,u_{21}\} - \{\bar{x}_1,u_{22}\} = \\
& -\frac{\partial u_{21}}{\partial \xi_2}u_{22} + \frac{\partial u_{21}}{\partial \bar{x}_1} + \frac{\partial u_{22}}{\partial \xi_2}u_{21} + \frac{\partial u_{22}}{\partial \bar{x}_2} \Rightarrow \\
\frac{\partial u_{21}}{\partial \bar{x}_1} + & \frac{\partial u_{22}}{\partial \bar{x}_2} = \frac{\partial u_{21}}{\partial \xi_2}u_{22} - \frac{\partial u_{22}}{\partial \xi_2}u_{21}
\end{aligned} \tag{8.132}
$$

Furthermore, we have

$$\xi_2 = \dot{\xi}_1 = \{H, \xi_1\} = -\frac{\partial H}{\partial \xi_2} d_{12} \Rightarrow \frac{\partial H}{\partial \xi_2} = -\frac{1}{d_{12}} \xi_2 \tag{8.133}$$

Using (8.133) we obtain:

$$\begin{aligned} f_\xi(\bar{x}) = \begin{pmatrix} \{H, \bar{x}_1\} \\ \\ \{H, \bar{x}_2\} \end{pmatrix} (\xi, \bar{x}) = & \begin{pmatrix} \frac{\partial H}{\partial \xi_2} u_{21} + \frac{\partial H}{\partial \bar{x}_2} \\ \\ \frac{\partial H}{\partial \xi_2} u_{22} - \frac{\partial H}{\partial \bar{x}_1} \end{pmatrix} (\xi, \bar{x}) = \\ & \begin{pmatrix} -\frac{u_{21}}{d_{12}} \xi_2 + \frac{\partial H}{\partial \bar{x}_2} \\ \\ -\frac{u_{22}}{d_{12}} \xi_2 - \frac{\partial H}{\partial \bar{x}_1} \end{pmatrix} (\xi, \bar{x}) \end{aligned} \tag{8.134}$$

We are now interested in the question if there exists a function $\bar{H}_\xi(\bar{x})$, parametrized by ξ, such that for all $\xi \in I\!R^2$:

$$f_\xi(\bar{x}) = (\{\bar{H}_\xi, \bar{x}\}_\xi)^T(\bar{x}) = \begin{pmatrix} \frac{\partial \bar{H}_\xi}{\partial \bar{x}_2} \\ \\ -\frac{\partial \bar{H}_\xi}{\partial \bar{x}_1} \end{pmatrix} (\bar{x}) \tag{8.135}$$

A necessary and sufficient condition for existence of an $\bar{H}_\xi(\bar{x})$ satisfying (8.135) is (cf. [87])

$$\frac{\partial f_{\xi 1}}{\partial \bar{x}_1} + \frac{\partial f_{\xi 2}}{\partial \bar{x}_2} = 0 \tag{8.136}$$

We now show that (8.136) indeed holds. By (8.134) we have

$$\begin{aligned} \frac{\partial f_{\xi 1}}{\partial \bar{x}_1} + \frac{\partial f_{\xi 2}}{\partial \bar{x}_2} = & \frac{\partial}{\partial \bar{x}_1}\left(-\frac{u_{21}}{d_{12}}\xi_2 + \frac{\partial H}{\partial \bar{x}_2}\right) + \frac{\partial}{\partial \bar{x}_2}\left(-\frac{u_{22}}{d_{12}}\xi_2 - \frac{\partial H}{\partial \bar{x}_1}\right) = \\ & \xi_2\left(-\frac{1}{d_{12}}\frac{\partial u_{21}}{\partial \bar{x}_1} + \frac{u_{21}}{d_{12}^2}\frac{\partial d_{12}}{\partial \bar{x}_1} - \frac{1}{d_{12}}\frac{\partial u_{22}}{\partial \bar{x}_2} + \frac{u_{22}}{d_{12}^2}\frac{\partial d_{12}}{\partial \bar{x}_2}\right) \end{aligned} \tag{8.137}$$

Substituting (8.130),(8.131),(8.132) in (8.137) yields

$$\begin{aligned} \frac{\partial f_{\xi 1}}{\partial \bar{x}_1} + \frac{\partial f_{\xi 2}}{\partial \bar{x}_2} = & \xi_2\left(-\frac{1}{d_{12}}\left(\frac{\partial u_{21}}{\partial \xi_2}u_{22} - \frac{\partial u_{22}}{\partial \xi_2}u_{21}\right) + \frac{u_{21}}{d_{12}^2}\left(-\frac{\partial u_{22}}{\partial \xi_2}d_{12} + \frac{\partial d_{12}}{\partial \xi_2}u_{22}\right) + \right. \\ & \left. \frac{u_{22}}{d_{12}^2}\left(\frac{\partial u_{22}}{\partial \xi_2}d_{12} - \frac{\partial d_{12}}{\partial \xi_2}u_{21}\right)\right) = 0 \end{aligned} \tag{8.138}$$

So (8.136) holds and hence Conjecture 8.2.24 is true in the case of a single-input single-output system with relative degree $r = 2$.

SISO-system, relative degree r=4

Consider a SISO Hamiltonian system (8.123) with relative degree $r = 4$. Define $\xi_1, \xi_2, \xi_3, \xi_4$ as in Corollary 8.2.26. For convenience, assume that M has dimension six. By Corollary 8.2.26 we can find functions $\bar{x}_2, \bar{x}_2$ such that $(\xi, \bar{x})$ forms a local coordinate system for

M and $\{\xi_1, \bar{x}_1\} = \{\xi_1, \bar{x}_2\} = \{\xi_2, \bar{x}_1\} = \{\xi_2, \bar{x}_2\} = 0$. By Lemma 8.2.7 we have that $\{\xi_2, \xi_3\} = -\{\xi_1, \xi_4\}$. As in Proposition 8.2.9, form the matrices

$$
\begin{aligned}
D(\xi, \bar{x}) &:= \begin{pmatrix} 0 & 0 & 0 & d_{14} \\ 0 & 0 & -d_{14} & d_{24} \\ 0 & d_{14} & 0 & d_{34} \\ -d_{14} & -d_{24} & -d_{34} & 0 \end{pmatrix} (\xi, \bar{x}) \\
U(\xi, \bar{x}) &:= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ u_{31} & u_{32} \\ u_{41} & u_{42} \end{pmatrix} (\xi, \bar{x}) \\
S(\xi, \bar{x}) &:= \begin{pmatrix} 0 & s_{12} \\ -s_{12} & 0 \end{pmatrix} (\xi, \bar{x})
\end{aligned}
\tag{8.139}
$$

Then we have again $\bar{W}(\xi, \bar{x}) = S(\xi, \bar{x})$ and hence we can again choose $\bar{x}_1, \bar{x}_2$ in such a way that

$$\bar{W}(\xi, \bar{x}) = S(\xi, \bar{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{8.140}$$

From the Jacobi-identity we obtain the following equalities:

$$\frac{\partial d_{14}}{\partial \bar{x}_1} = \frac{\partial d_{14}}{\partial \xi_3} u_{32} - \frac{\partial u_{32}}{\partial \xi_3} d_{14} \tag{8.141}$$

$$\frac{\partial d_{14}}{\partial \bar{x}_2} = -\frac{\partial d_{14}}{\partial \xi_3} u_{31} + \frac{\partial u_{31}}{\partial \xi_3} d_{14} \tag{8.142}$$

$$\frac{\partial d_{24}}{\partial \bar{x}_1} = \frac{\partial d_{24}}{\partial \xi_3} u_{32} + \frac{\partial d_{24}}{\partial \xi_4} u_{42} + \frac{\partial u_{42}}{\partial \xi_3} d_{14} - \frac{\partial u_{42}}{\partial \xi_4} d_{24} \tag{8.143}$$

$$\frac{\partial d_{24}}{\partial \bar{x}_2} = -\frac{\partial d_{24}}{\partial \xi_3} u_{31} - \frac{\partial d_{24}}{\partial \xi_4} u_{41} - \frac{\partial u_{41}}{\partial \xi_3} d_{14} + \frac{\partial u_{41}}{\partial \xi_4} d_{24} \tag{8.144}$$

$$\frac{\partial u_{31}}{\partial \bar{x}_1} + \frac{\partial u_{32}}{\partial \bar{x}_2} = \frac{\partial u_{31}}{\partial \xi_3} u_{32} - \frac{\partial u_{32}}{\partial \xi_3} u_{31} \tag{8.145}$$

$$\frac{\partial u_{41}}{\partial \bar{x}_1} + \frac{\partial u_{42}}{\partial \bar{x}_t} = \frac{\partial u_{41}}{\partial \xi_3} u_{32} + \frac{\partial u_{41}}{\partial \xi_4} u_{42} - \frac{\partial u_{42}}{\partial \xi_3} u_{31} - \frac{\partial u_{42}}{\partial \xi_4} u_{41} \tag{8.146}$$

$$\frac{\partial d_{24}}{\partial \xi_4} = -\frac{\partial d_{14}}{\partial \xi_3} \tag{8.147}$$

$$\frac{\partial u_{41}}{\partial \xi_4} = \frac{\partial u_{31}}{\partial \xi_3} \tag{8.148}$$

$$\frac{\partial u_{42}}{\partial \xi_4} = \frac{\partial u_{32}}{\partial \xi_3} \tag{8.149}$$

Furthermore, we have

$$\xi_2 = \dot{\xi}_1 = \{H, \xi_1\} = -\frac{\partial H}{\partial \xi_4} d_{14} \Rightarrow \frac{\partial H}{\partial \xi_4} = -\frac{1}{d_{14}} \xi_2 \tag{8.150}$$

$$\xi_3 = \dot{\xi}_2 = \{H, \xi_2\} = \frac{\partial H}{\partial \xi_3} d_{14} - \frac{\partial H}{\partial \xi_4} d_{24} \Rightarrow \frac{\partial H}{\partial \xi_3} = -\frac{d_{24}}{d_{14}^2}\xi_2 + \frac{1}{d_{14}}\xi_3 \tag{8.151}$$

Using (8.150),(8.151) we obtain:

$$\begin{aligned} f_\xi(\bar{x}) = & \begin{pmatrix} \{H, \bar{x}_1\} \\ \\ \{H, \bar{x}_2\} \end{pmatrix} (\xi, \bar{x}) = -\xi_2 \begin{pmatrix} \frac{u_{31}d_{24} + u_{41}d_{14}}{d_{14}^2} \\ \\ \frac{u_{32}d_{24} + u_{42}d_{14}}{d_{14}^2} \end{pmatrix} (\xi, \bar{x}) + \\ & \xi_3 \begin{pmatrix} \frac{u_{31}}{d_{14}} \\ \\ \frac{u_{32}}{d_{14}} \end{pmatrix} (\xi, \bar{x}) + \begin{pmatrix} \frac{\partial H}{\partial \bar{x}_2} \\ \\ -\frac{\partial H}{\partial \bar{x}_1} \end{pmatrix} (\xi, \bar{x}) =: \\ & -\xi_2 \tilde{f}_\xi^1(\bar{x}) + \xi_3 \tilde{f}_\xi^2(\bar{x}) + \tilde{f}_\xi^3(\bar{x}) \end{aligned} \tag{8.152}$$

Analogously to the case that $r = 2$, we have to check if

$$\frac{\partial f_{\xi 1}}{\partial \bar{x}_1} + \frac{\partial f_{\xi 2}}{\partial \bar{x}_2} = 0 \tag{8.153}$$

Obviously, this is the case if for $i = 1, 2, 3$:

$$\frac{\partial \tilde{f}_{\xi 1}^i}{\partial \bar{x}_1} + \frac{\partial \tilde{f}_{\xi 2}^i}{\partial \bar{x}_2} = 0 \tag{8.154}$$

We see that (8.154) holds for $i = 3$. For $i = 2$ we have:

$$\begin{aligned} \frac{\partial \tilde{f}_{\xi 1}^2}{\partial \bar{x}_1} + \frac{\partial \tilde{f}_{\xi 2}^2}{\partial \bar{x}_2} = & \frac{\partial}{\partial \bar{x}_1}\left(\frac{u_{31}}{d_{14}}\right) + \frac{\partial}{\partial \bar{x}_2}\left(\frac{u_{32}}{d_{14}}\right) = \\ & \frac{1}{d_{14}}\left(\frac{\partial u_{31}}{\partial \bar{x}_1} + \frac{\partial u_{32}}{\partial \bar{x}_2}\right) - \frac{1}{d_{14}^2}\left(u_{31}\frac{\partial d_{14}}{\partial \bar{x}_1} + u_{32}\frac{\partial d_{14}}{\partial \bar{x}_2}\right) = 0 \end{aligned} \tag{8.155}$$

where the last equality follows by substituting (8.141),(8.142),(8.145). For $i = 1$ we have:

$$\begin{aligned} \frac{\partial \tilde{f}_{\xi 1}^1}{\partial \bar{x}_1} + \frac{\partial \tilde{f}_{\xi 2}^1}{\partial \bar{x}_2} = & \frac{\partial}{\partial \bar{x}_1}\left(\frac{u_{31}d_{24} + u_{41}d_{14}}{d_{14}^2}\right) + \frac{\partial}{\partial \bar{x}_2}\left(\frac{u_{32}d_{24} + u_{42}d_{14}}{d_{14}^2}\right) = \\ & \frac{1}{d_{14}^2}\left(\frac{\partial}{\partial \bar{x}_1}(u_{31}d_{24} + u_{41}d_{14}) + \frac{\partial}{\partial \bar{x}_2}(u_{32}d_{24} + u_{42}d_{14})\right) - \\ & \frac{2}{d_{14}^3}\left(\frac{\partial d_{14}}{\partial \bar{x}_1}(u_{31}d_{24} + u_{41}d_{14}) + \frac{\partial d_{14}}{\partial \bar{x}_2}(u_{32}d_{24} + u_{42}d_{14})\right) \end{aligned} \tag{8.156}$$

Using (8.141),(8.142) we obtain:

$$\begin{aligned} & \frac{\partial d_{14}}{\partial \bar{x}_1}(u_{31}d_{24} + u_{41}d_{14}) + \frac{\partial d_{14}}{\partial \bar{x}_2}(u_{32}d_{24} + u_{42}d_{14}) = \\ & \frac{\partial u_{31}}{\partial \xi_3}(u_{32}d_{24} + u_{42}d_{14})d_{14} - \frac{\partial u_{32}}{\partial \xi_3}(u_{31}d_{24} + u_{41}d_{14})d_{14} - \frac{\partial d_{14}}{\partial \xi_3}(u_{31}u_{42} - u_{32}u_{41}) \end{aligned} \tag{8.157}$$

Furthermore, using equations (8.141),$\cdots$,(8.149) and exercising some patience, we obtain:

$$\begin{aligned}&\frac{\partial}{\partial\bar{x}_1}(u_{31}d_{24}+u_{41}d_{14})+\frac{\partial}{\partial\bar{x}_2}(u_{32}d_{24}+u_{42}d_{14})=\\&2\frac{\partial u_{31}}{\partial\xi_3}(u_{32}d_{24}+u_{42}d_{14})-2\frac{\partial u_{32}}{\partial\xi_3}(u_{31}d_{24}+u_{41}d_{14})-2\frac{\partial d_{14}}{\partial\xi_3}(u_{31}u_{42}-u_{32}u_{41})\end{aligned}\tag{8.158}$$

From (8.156),(8.157),(8.158) we conclude that (8.154) holds for $i=1$. Hence Conjecture 8.2.24 is true in the case of a single-input single-output system with relative degree $r=4$.

In the two cases considered so far, we have established that a function $\bar{H}_\xi(\bar{x})$ satisfying (8.119) exists, without actually constructing $\bar{H}_\xi$. In the following case, that of a multi-input multi-output *simple* Hamiltonian system, we will give $\bar{H}_\xi(\bar{x})$ explicitly.

Simple Hamiltonian system

Consider a simple Hamiltonian system (8.123) with the Hamiltonian function

$$H(q,p)=\frac{1}{2}p^TG(q)p+V(q)\tag{8.159}$$

where $G(q)$ is positive definite for all q. As we have seen in Subsection 8.2.2, we can take $\xi=(\tilde{q},\dot{\tilde{q}})$ and $\bar{x}=(\bar{q},\bar{p})$, where $\tilde{q}=(q_1,\cdots,q_m)$, $\bar{q}=(q_{m+1},\cdots,q_n)$, $\bar{p}=(p_{m+1},\cdots,p_n)$. Partition $G(q)$ as in (8.101). Then

$$\dot{\tilde{q}}=G_{11}\tilde{p}+G_{12}\bar{p}\Rightarrow\tilde{p}=G_{11}^{-1}(\dot{\tilde{q}}-G_{12}\bar{p})\tag{8.160}$$

Then in the new coordinates $(\xi,\bar{x})$ we have

$$\dot{\bar{q}}=G_{12}^TG_{11}^{-1}\dot{\tilde{q}}+(G_{22}-G_{12}^TG_{11}^{-1}G_{12})\bar{p}\tag{8.161}$$

and for $i=m+1,\cdots,n$

$$\begin{aligned}\dot{p}_i=\;&-(\tfrac{1}{2}\tilde{p}^T\frac{\partial G_{11}}{\partial q_i}\tilde{p}+\tilde{p}^T\frac{\partial G_{12}}{\partial q_i}\bar{p}+\tfrac{1}{2}\bar{p}^T\frac{\partial G_{22}}{\partial q_i}\bar{p}+\frac{\partial V}{\partial q_i})=\\&-(\tfrac{1}{2}(\dot{\tilde{q}}^T-\bar{p}^TG_{12}^T)G_{11}^{-1}\frac{\partial G_{11}}{\partial q_i}G_{11}^{-1}(\dot{\tilde{q}}-G_{12}\bar{p})+(\dot{\tilde{q}}^T-\bar{p}^TG_{12}^T)G_{11}^{-1}\frac{\partial G_{12}}{\partial q_i}\bar{p}+\\&\tfrac{1}{2}\bar{p}^T\frac{\partial G_{22}}{\partial q_i}\bar{p}+\frac{\partial V}{\partial q_i})=\\&\tfrac{1}{2}(\dot{\tilde{q}}^T-\bar{p}^TG_{12}^T)\frac{\partial G_{11}^{-1}}{\partial q_i}(\dot{\tilde{q}}-G_{12}\bar{p})-(\dot{\tilde{q}}^T-\bar{p}^TG_{12}^T)G_{11}^{-1}\frac{\partial G_{12}}{\partial q_i}\bar{p}-\\&\tfrac{1}{2}\bar{p}^T\frac{\partial G_{22}}{\partial q_i}\bar{p}-\frac{\partial V}{\partial q_i}\end{aligned}\tag{8.162}$$

Consider the function

$$\begin{aligned}\bar{H}_\xi(\bar{x})=\;&-\tfrac{1}{2}\dot{\tilde{q}}^TG_{11}^{-1}(\tilde{q},\bar{q})\dot{\tilde{q}}+\tfrac{1}{2}\bar{p}^T(G_{22}(\tilde{q},\bar{q})-G_{12}^T(\tilde{q},\bar{q})G_{11}^{-1}(\tilde{q},\bar{q})G_{12}(\tilde{q},\bar{q}))\bar{p}+\\&\bar{p}^TG_{12}^T(\tilde{q},\bar{q})G_{11}^{-1}(\tilde{q},\bar{q})\dot{\tilde{q}}+V(\tilde{q},\bar{q})\end{aligned}\tag{8.163}$$

Then we have

$$\frac{\partial\bar{H}_\xi}{\partial\bar{p}}=G_{12}^TG_{11}^{-1}\dot{\tilde{q}}+(G_{22}-G_{12}^TG_{11}^{-1}G_{12})\bar{p}\tag{8.164}$$

and for $i = m+1, \cdots, n$:

$$
\begin{aligned}
\frac{\partial \bar{H}_\xi}{\partial q_i} = \; & -\frac{1}{2}\dot{\bar{q}}^T \frac{\partial G_{11}^{-1}}{\partial q_i}\dot{\bar{q}} + \frac{1}{2}\bar{p}^T \frac{\partial G_{22}}{\partial q_i}\bar{p} - \\
& \frac{1}{2}\bar{p}^T \Big(\frac{\partial G_{12}^T}{\partial q_i} G_{11}^{-1} G_{12} + G_{12}^T \frac{\partial G_{11}^{-1}}{\partial q_i} G_{12} + G_{12}^T G_{11}^{-1} \frac{\partial G_{12}}{\partial q_i}\Big)\bar{p} + \\
& \bar{p}^T \Big(\frac{\partial G_{12}^T}{\partial q_i} G_{11}^{-1} + G_{12}^T \frac{\partial G_{11}^{-1}}{\partial q_i}\Big)\dot{\bar{q}} + \frac{\partial V}{\partial q_i} = \\
& -\frac{1}{2}\Big(\dot{\bar{q}}^T \frac{\partial G_{11}^{-1}}{\partial q_i}\dot{\bar{q}} + \bar{p}^T G_{12}^T \frac{G_{11}^{-1}}{\partial q_i} G_{12}\bar{p} - \bar{p}^T G_{12}^T \frac{\partial G_{11}^{-1}}{\partial q_i}\dot{\bar{q}} - \dot{\bar{q}}^T \frac{\partial G_{11}^{-1}}{\partial q_i} G_{12}\bar{p}\Big) - \qquad (8.165) \\
& \bar{p}^T G_{12}^T G_{11}^{-1} \frac{\partial G_{12}}{\partial q_i}\bar{p} + \dot{\bar{q}}^T G_{11}^{-1} \frac{\partial G_{12}^T}{\partial q_i}\bar{p} + \frac{1}{2}\bar{p}^T \frac{\partial G_{22}}{\partial q_i}\bar{p} + \frac{\partial V}{\partial q_i} = \\
& -\frac{1}{2}(\dot{\bar{q}}^T - \bar{p}^T G_{12}^T) \frac{\partial G_{11}^{-1}}{\partial q_i}(\dot{\bar{q}} - G_{12}\bar{p}) + (\dot{\bar{q}}^T - \bar{p}^T G_{12}^T) G_{11}^{-1} \frac{\partial G_{12}}{\partial q_i}\bar{p} + \\
& \frac{1}{2}\bar{p}^T \frac{\partial G_{22}}{\partial q_i}\bar{p} + \frac{\partial V}{\partial q_i}
\end{aligned}
$$

From (8.161),(8.162),(8.164) and (8.165) we conclude that $\bar{H}_\xi$ satisfies (8.119) and hence Conjecture 8.2.24 holds true in this case.

We have now shown Conjecture 8.2.24 to be true in three special cases. It seems that for SISO Hamiltonian systems the approach taken here can be generalized to the cases that the relative degree is $r = 2k$ ($k = 3, 4, \cdots$). For MIMO Hamiltonian systems one should be more careful, as will be made clear below.

In all three cases that were considered, a basic step in the exposition is that $\bar{x}$ can be chosen in such a way that $\{\xi_i, \bar{x}\} = 0$ for at least half of the ξ_i's. This fact enables us to establish that $S(\xi, \bar{x}) = \bar{W}(\xi, \bar{x})$ and to choose $\bar{x}$ in such a way that (8.132) holds. A basic requirement here is of course that the relative degree r is even. For MIMO strongly input-output decouplable Hamiltonian systems the relative degrees need not be even, as follows from the following example:

Example 8.2.27 Consider the following Hamiltonian system on $I\!R^2$:

$$
\begin{aligned}
\dot{q} &= -u_2 \\
\dot{p} &= u_1 \\
y_1 &= q \\
y_2 &= p
\end{aligned}
\qquad (8.166)
$$

For this system we have the relative degrees $r_1 = r_2 = 1$ and the decoupling matrix

$$
A(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

Hence the system is strongly input-output decouplable and both relative degrees are odd.

■

We next show that a strongly input-output decouplable Hamiltonian system always has an even number of odd relative degrees. For this, we need the following lemma:

Lemma 8.2.28 *Consider a Hamiltonian system (8.51). Let $r_1, \cdots, r_m$ denote the relative degrees of (8.51). Then:*

(i) $\{C_j, \mathrm{ad}_H^{r_i-1} C_i\} = (-1)^{r_i}\{C_i, \mathrm{ad}_H^{r_i-1} C_j\}$

(ii) If $\{C_j, \mathrm{ad}_H^{r_i-1} C_i\} \not\equiv 0$, *then* $r_j \leq r_i$.

Proof *(i)* follows immediately from Lemma 8.2.7 and *(ii)* follows immediately from *(i)*. ■

Proposition 8.2.29 *Consider a strongly input-output decouplable Hamiltonian system (8.51). Assume that its outputs are arranged in such a way that $r_1 \leq \cdots \leq r_m$. Define integers $k_1, \cdots, k_N, w_1, \cdots, w_N$ in the following way: $k_1 < \cdots < k_N$ are the values taken by the relative degrees, and w_i $(i = 1, \cdots, N)$ is the number of relative degrees taking value k_i. Then: w_i is even if k_i is odd $(i = 1, \cdots, N)$.*

Proof By Lemma 8.2.28 it follows that the decoupling matrix $A(x)$ of (8.51) has the following form:

- $A(x)$ has N diagonal blocks $D_i(x)$, where $D_i(x)$ is a (w_i, w_i)-matrix.
- The elements of $A(x)$ above the diagonal blocks equal zero.
- The elements of $A(x)$ below the diagonal blocks may be non-zero.

Hence, since $A(x)$ has full rank, each of the diagonal blocks $D_i(x)$ $(i = 1, \cdots, N)$ must have full rank. Assume that k_i is odd. Then, by Lemma 8.2.28.*(i)*, $D_i(x)$ is skew-symmetric. Hence $D_i(x)$ is a skew-symmetric matrix of full rank. This implies that $D_i(x)$ is even-dimensional (cf. [1]). Hence w_i is even. ■

Remark 8.2.30 *(i)* If the Hamiltonian system (8.51) is not strongly input-output decouplable, Proposition 8.2.29 does not need to hold. A counter example is provided by a Hamiltonian system on $T^*\mathbb{R}^2$ (with standard Poisson bracket), with $H(q,p) = p_2 q_1 + p_1^2 \exp(q_1 + q_2)$, $C_1(q,p) = q_1$, $C_2(q,p) = q_2$.

(ii) Proposition 8.2.29 is taken from [47]. Together with the remark above, it answers a question posed by Grizzle and Nijmeijer ([41]). ■

We will now derive an extension of Corollary 8.2.26 to multi-input multi-output Hamiltonian systems for which all relative degrees are even. For this, we need some concepts from the theory of function groups, taken from [97],[109].

Definition 8.2.31 Function space, function group
Consider an even-dimensional manifold M with non-degenerate Poisson bracket $\{\cdot,\cdot\}$. A collection $\mathcal{F}$ of smooth functions from M to $\mathbb{R}$ is called a function space, if

(i) $\mathcal{F}$ is a linear subspace (over $\mathbb{R}$) of $C^\infty(M)$.

(ii) If $F_1, \cdots, F_s \in \mathcal{F}$ and $G : \mathbb{R}^s \to \mathbb{R}$ is a smooth function, then $G(F_1, \cdots, F_s) \in \mathcal{F}$.

Furthermore, we call $\mathcal{F}$ a function group if also

(iii) $\mathcal{F}$ is closed under Poisson bracket, i.e., if $F_1, F_2 \in \mathcal{F}$, then $\{F_1, F_2\} \in \mathcal{F}$. ∎

Note that by *(ii)* a non-empty function space always contains $\mathbb{R}$, the constant functions on M. Given some functions $F_1, \cdots, F_k$ on M we denote by span $\{F_1, \cdots, F_k\}$ the smallest function space containing these functions. The *sum* $\mathcal{F}^1 + \mathcal{F}^2$ of two function spaces is the smallest function space containing $\mathcal{F}^1$ as well as $\mathcal{F}^2$. Furthermore we define

$$\mathcal{F}^\perp = \{G \in C^\infty(M) \mid \{G, F\} = 0, \forall F \in \mathcal{F}\} \tag{8.167}$$

Using the Jacobi-identity, it can be shown that $\mathcal{F}^\perp$ is a function group, called the ***polar group***.

Lemma 8.2.32 *Let $\mathcal{F}$ be a function group. Define the codistribution $d\mathcal{F}$ on M by $d\mathcal{F}(x) = \{dF(x) \mid F \in \mathcal{F}\}$ $(x \in M)$ and similarly define the codistribution $d(\mathcal{F} \cap \mathcal{F}^\perp)$. Assume that $\dim d\mathcal{F} = k$ and $\dim d(\mathcal{F} \cap \mathcal{F}^\perp) = r$. Then locally there exist canonical coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ for M such that*

$$\mathcal{F} = \text{span}\,\{q_1, \cdots, q_\ell, p_1, \cdots, p_\ell, p_{\ell+1}, \cdots, p_{\ell+r}\} \tag{8.168}$$

with $2\ell + r = k$.

Proof See [109]. ∎

Corollary 8.2.33 *Consider a function group $\mathcal{F} = \text{span}\,\{F_1, \cdots, F_k\}$, where $F_1, \cdots, F_k$ are independent functions. Assume that $\dim d(\mathcal{F} \cap \mathcal{F}^\perp)$ is constant. Then locally there exist $2n - k$ independent functions $G_1, \cdots, G_{2n-k}$ such that $\mathcal{F}^\perp = \text{span}\,\{G_1, \cdots, G_{2n-k}\}$.*

Proof By Lemma 8.2.32 we may assume without loss of generality that $\mathcal{F} = \text{span}\,\{q_1, \cdots, q_\ell, p_1, \cdots, p_\ell, p_{\ell+1}, \cdots, p_{\ell+r}\}$, where $2\ell + r = k$. Obviously the $2n - k$ functions $p_{\ell+1}, \cdots, p_n, q_{\ell+r+1}, \cdots, q_n$ span $\mathcal{F}^\perp$, which establishes our claim. ∎

Proposition 8.2.34 *Let M be an even-dimensional manifold with non-degenerate Poisson bracket $\{\cdot, \cdot\}$. Consider an analytic Hamiltonian system (8.47) on M. Let $x_0 \in M$ be such that the decoupling matrix $A(x)$ of (8.47) has full rank for $x = x_0$. Assume that the relative degrees $r_1, \cdots, r_m$ of (8.47) are even and define $\ell_i := (r_i/2)$ $(i = 1, \cdots, m)$. Denote $\xi_{ij} = \text{ad}_H^{(j-1)} C_i$ $(i = 1, \cdots, m; j = 1, \cdots, r_i)$ and let $d = 2n - \sum_{i=1}^m r_i$. Then there exist functions $\bar{x}_1, \cdots, \bar{x}_d : M \to \mathbb{R}$ such that $((\xi_{ij} \mid 1 \le i \le m, 1 \le j \le r_i), \bar{x}_1, \cdots, \bar{x}_d)$ is a local coordinate system for M around x_0 and*

$$\{\xi_{ij}, \bar{x}_k\} = 0 \quad (i = 1, \cdots, m; j = 1, \cdots, \ell_i; k = 1, \cdots, d) \tag{8.169}$$

Proof Define the function spaces $\mathcal{F} = \text{span}\,\{\xi_{ij} \mid 1 \le i \le m, 1 \le j \le r_i\}$, $\mathcal{C} = \text{span}\,\{\xi_{ij} \mid 1 \le i \le m, 1 \le j \le \ell_i\}$. By Lemma 8.2.7 and the definition of the relative degrees it follows that $\{\xi_{ij}, \xi_{ks}\} = 0$ $(i, k = 1, \cdots, m; j = 1, \cdots, \ell_i; s = 1, \cdots, \ell_k)$ and for all ξ_{ij} $(i = 1, \cdots, m; j = 1, \cdots, \ell_i)$ there exist $k \in \{1, \cdots, m\}$, $s \in \{\ell_k + 1, \cdots, r_k\}$ such that $\{\xi_{ij}, \xi_{ks}\} \ne 0$. Hence $\mathcal{C}$ is a function group satisfying $\mathcal{C} \subset \mathcal{C}^\perp$ and

$$\mathcal{C}^\perp \cap \mathcal{F} = \mathcal{C} \tag{8.170}$$

Since the functions ξ_{ij} $(i = 1, \cdots, m; j = 1, \cdots, \ell_i)$ are independent around x_0 (cf. Lemma 3.1.8), by Corollary 8.2.33 there exist smooth functions $\bar{x}_1, \cdots, \bar{x}_d : M \to \mathbb{R}$ such that (around x_0):

$$\mathcal{C}^\perp = \text{span}\,\{\{\xi_{ij} \mid 1 \le i \le m, 1 \le j \le \ell_i\}, \bar{x}_1, \cdots, \bar{x}_d\} \tag{8.171}$$

Obviously, the functions $\bar{x}_1, \cdots, \bar{x}_d$ satisfy (8.169) and they are independent of ξ_{ij} $(i = 1, \cdots, m; j = 1, \cdots, \ell_i)$. Assume that the functions $\bar{x}_1, \cdots, \bar{x}_d, \xi_{ij}$ $(i = 1, \cdots, m; j = \ell_i + 1, \cdots, r_i)$ are not independent. This implies that there exists a $k \in \{1, \cdots, d\}$ and locally a function $\phi(\{\bar{x}_i \mid i \ne k\}, \{\xi_{ij} \mid 1 \le i \le m, \ell_i + 1, \cdots, r_i\})$ satisfying $(\partial\phi/\partial\xi_{js}) \ne 0$ for some $j \in \{1, \cdots, m\}$, $s \in \{\ell_j + 1, \cdots, r_j\}$ and

$$\bar{x}_k = \phi(\{\bar{x}_i \mid i \ne k\}, \{\xi_{ij} \mid 1 \le i \le m, \ell_i + 1 \le j \le r_i\}) \tag{8.172}$$

Then for $i = 1, \cdots, m$ we have

$$\begin{aligned} 0 = \; & \{\bar{x}_k, \xi_{i1}\} = \sum_{j=1}^{d} \frac{\partial\phi}{\partial\bar{x}_j}\{\bar{x}_j, \xi_{i1}\} + \sum_{s=1}^{m} \sum_{j=\ell_s+1}^{r_s} \frac{\partial\phi}{\partial\xi_{sj}}\{\xi_{sj}, \xi_{i1}\} = \\ & \sum_{s=1}^{m} \frac{\partial\phi}{\partial\xi_{sr_s}}\{\xi_{sr_s}, \xi_{i1}\} = A_{i*} \begin{pmatrix} \dfrac{\partial\phi}{\partial\xi_{1r_1}} \\ \vdots \\ \dfrac{\partial\phi}{\partial\xi_{mr_m}} \end{pmatrix} \end{aligned} \tag{8.173}$$

Since $A(x)$ has full rank around x_0, this implies that $(\partial\phi/\partial\xi_{ir_i}) = 0$ $(i = 1, \cdots, m)$. Repeating this argument, we find that $(\partial\phi/\partial\xi_{ij}) = 0$ $(i = 1, \cdots, m; j = \ell_i + 1, \cdots, r_i)$, which contradicts the form of ϕ. Hence the functions $((\xi_{ij} \mid 1 \le i \le m, 1 \le j \le r_i), \bar{x}_1, \cdots, \bar{x}_d)$ form a local coordinate system around x_0 and $\bar{x}_1, \cdots, \bar{x}_d$ satisfy (8.169), which establishes our claim. ∎

With the coordinates $((\xi_{ij} \mid 1 \le i \le m, 1 \le j \le r_i), \bar{x}_1, \cdots, \bar{x}_d)$ as in Proposition 8.2.34, the matrices D, U in Proposition 8.2.9 have the form (with $\ell = \sum_{i=1}^m \ell_i$):

$$\begin{aligned} D(\xi, \bar{x}) &= \begin{pmatrix} 0_{\ell\ell} & D_1 \\ -D_1^T & D_2 \end{pmatrix}(\xi, \bar{x}) \\ U(\xi, \bar{x}) &= \begin{pmatrix} 0_{\ell d} \\ \bar{U} \end{pmatrix}(\xi, \bar{x}) \end{aligned} \tag{8.174}$$

Hence we find again:

$$\bar{W}(\xi, \bar{x}) = S(\xi, \bar{x}) + U^T(\xi, \bar{x})D^{-1}(\xi, \bar{x})U(\xi, \bar{x}) = S(\xi, \bar{x}) \tag{8.175}$$

and we can again choose $\bar{x}_1, \cdots, \bar{x}_d$ in such a way that

$$S(\xi, \bar{x}) = \begin{pmatrix} 0 & -I_{d/2} \\ I_{d/2} & 0 \end{pmatrix} \tag{8.176}$$

This gives good reason to believe that with the approach taken for the case of single-input single-output systems one may also be able to establish that Conjecture 8.2.24 holds for multi-input multi-output Hamiltonian systems for which all relative degrees are even.

If some of the relative degrees are odd, however, it may not be possible to choose $\bar{x}$ in such a way that $\{\xi_{ij}, \bar{x}\} = 0$ for at least half of the ξ_{ij}'s. A consequence of this is that in this case we do not have that $U^T(\xi, \bar{x})D^{-1}(\xi, \bar{x})U(\xi, \bar{x}) = 0$ and hence $\bar{W}(\xi, \bar{x}) \neq S(\xi, \bar{x})$. The treatment of this case remains a topic for future research.

8.2.4 Model matching with prescribed tracking error for Hamiltonian systems

Consider a manifold $\mathcal{X}$ with non-degenerate Poisson bracket $\{\cdot,\cdot\}_P$. Let P be a Hamiltonian system on $\mathcal{X}$ of the form

$$P \left\{ \begin{array}{rcl} \dot{x} & = & X_H(x) - \sum\limits_{j=1}^{m} X_{C_j}(x)u_j \\ \\ y_j & = & C_j(x) \quad (j = 1, \cdots, m) \end{array} \right. \tag{8.177}$$

where $x = (x_1, \cdots, x_{2n})^T$ are local coordinates for $\mathcal{X}$ and $H, C_1, \cdots, C_m$ are analytic functions on $\mathcal{X}$. Assume that the distribution span$\{X_{C_1}, \cdots, X_{C_m}\}$ is involutive and that the relative degrees $r_1, \cdots, r_m$ of P are finite. Furthermore, assume that the decoupling matrix $A(x)$ of P has full rank for all $x \in \mathcal{X}$.

Let $\bar{\mathcal{X}}$ be a manifold with non-degenerate Poisson bracket $\{\cdot,\cdot\}_M$. Let M be a Hamiltonian system on $\bar{\mathcal{X}}$ of the form

$$M \left\{ \begin{array}{rcl} \dot{\bar{x}} & = & X_{\bar{H}}(\bar{x}) - \sum\limits_{j=1}^{m} X_{\bar{C}_j}(\bar{x})\bar{u}_j \\ \\ \bar{y}_j & = & \bar{C}_j(\bar{x}) \quad (j = 1, \cdots, m) \end{array} \right. \tag{8.178}$$

where $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_{2\bar{n}})^T$ are local coordinates for $\bar{\mathcal{X}}$ and $\bar{H}, \bar{C}_1, \cdots, \bar{C}_m$ are analytic functions on $\bar{\mathcal{X}}$. Denote the relative degrees of M by $\bar{r}_1, \cdots, \bar{r}_m$.

Let $\bar{x}_0 \in \bar{\mathcal{X}}$ be given. Since $A(x)$ has full rank for all $x \in \mathcal{X}$, it follows from Lemma 7.2.2 that for every $x_0 \in \mathcal{X}$ the MMP is locally solvable for P and M around $(x_0, \bar{x}_0)$ if and only if $r_i \leq \bar{r}_i$ $(i = 1, \cdots, m)$. Consider a dynamic state feedback Q of the form (7.4) that solves the MMP for (M, P) around $(x_0, \bar{x}_0)$. As was already noted in Remark 7.1.2, in general it will not be the case that, if we set $v = \bar{u}$ in (7.4), the output trajectories of $P \circ Q$ and M are the same. This is due to the fact that to achieve this, the Volterra kernels of *all* orders of $P \circ Q$ and M have to coincide, while in the formulation of the model matching problem it is only required that all i-th order Volterra kernels for $i \geq 1$ coincide. If we want *all* input-output trajectories of $P \circ Q$ and M to be the same the problem under consideration is known as the *strong* model matching problem (see [27],[28]). If strong model matching is not possible, one may be interested in constructing a feedback Q for the plant that solves the MMP for (M, P), with the property that the absolute value of the difference between the outputs of $P \circ Q$ and M decays exponentially. This problem is known as the model matching problem with prescribed tracking error (see [10]). In the special case considered in this subsection, we construct a dynamic state feedback Q for P that has the following properties:

(i) For every $\bar{x}_0 \in \bar{\mathcal{X}}$ there exists an $x_0 \in \mathcal{X}$ and a $z_0 \in I\!R^\nu$ (the state space of Q) such that for every $\bar{u}(t)$ $(t \geq 0)$:

$$y(t, x_0, z_0, \bar{u}) = \bar{y}(t, \bar{x}_0, \bar{u}) \quad (\forall t \geq 0) \tag{8.179}$$

where $y(t, x_0, z_0, \bar{u})$ is the output of $P \circ Q$ with control $\bar{u}$, initialized at $(x(0), z(0)) = (x_0, z_0)$ and $\bar{y}(t, \bar{x}_0, \bar{u})$ is the output of M with control $\bar{u}$, initialized at $\bar{x}(0) = \bar{x}_0$.

(ii) For every $x_0 \in \mathcal{X}$ and every $\bar{x}_0 \in \bar{\mathcal{X}}$ there exists a $z_0 \in I\!R^\nu$ and $\sigma, \tau > 0$ such that for every $\bar{u}(t)$ $(t \geq 0)$:

$$\|y(t, x_0, z_0, \bar{u}) - \bar{y}(t, \bar{x}_0, \bar{u})\|_m < \tau e^{-\sigma t} \quad (\forall t \geq 0) \tag{8.180}$$

where $\| \cdot \|_m$ denotes the Euclidean norm on $I\!R^m$.

Property *(i)* means that the strong model matching problem is solvable for (M, P) at $(x_0, \bar{x}_0)$. Property *(ii)* means that the model matching problem with prescribed tracking error is solvable for (M, P). The results on the model matching problem with prescribed tracking error presented in this subsection are taken from [47]. They are a generalization to multi-input multi-output systems of the results from [10] for single-input single-output systems.

As in (7.6), form the extended system E associated with M and P:

$$E \left\{ \begin{array}{rcl} \dot{x} &=& X_H(x) - \sum\limits_{j=1}^{m} X_{C_j}(x) u_j \\ \dot{\bar{x}} &=& X_{\bar{H}}(\bar{x}) - \sum\limits_{j=1}^{m} X_{\bar{C}_j}(\bar{x}) q_j^E \\ y_j^E &=& C_j(x) - \bar{C}_j(\bar{x}) \quad (j = 1, \cdots, m) \end{array} \right. \tag{8.181}$$

The dynamics of E evolve on the manifold $\mathcal{X}^E = \mathcal{X} \times \bar{\mathcal{X}}$. A non-degenerate Poisson bracket $\{\cdot, \cdot\}_E$ on $\mathcal{X}^E$ is defined by

$$\{F^E, G^E\}_E = \sum_{i,j=1}^{2n} \frac{\partial F^E}{\partial x_i} \frac{\partial G^E}{\partial x_j} \{x_i, x_j\}_P + \sum_{i,j=1}^{2\bar{n}} \frac{\partial F^E}{\partial \bar{x}_i} \frac{\partial G^E}{\partial \bar{x}_j} \{\bar{x}_i, \bar{x}_j\}_M \tag{8.182}$$

for every $F^E, G^E \in C^\infty(\mathcal{X}^E)$. Defining $x^E = (x^T, \bar{x}^T)^T$, $H^E(x^E) = H(x) + H(\bar{x})$, (8.181) can be written as

$$E \left\{ \begin{array}{rcl} \dot{x}^E &=& X_{H^E}^E - \sum\limits_{j=1}^{m} X_{C_j}^E(x^E) u_j - \sum\limits_{j=1}^{m} X_{\bar{C}_j}^E(x^E) q_j^E \\ y_j^E &=& C_j(x) - \bar{C}_j(\bar{x}) \quad (j = 1, \cdots, m) \end{array} \right. \tag{8.183}$$

where $X_{H^E}^E, X_{C_1}^E, \cdots, X_{C_m}^E, X_{\bar{C}_1}^E, \cdots, X_{\bar{C}_m}^E$ are the Hamiltonian vector fields corresponding to the Hamiltonian functions $H^E, C_1, \cdots, C_m, \bar{C}_1, \cdots, \bar{C}_m$ and the Poisson bracket $\{\cdot, \cdot\}_E$ on $\mathcal{X}^E$. Note that (8.183) does not constitute a Hamiltonian control system, since the Hamiltonian functions corresponding to the input vector fields are not the same as the output functions.

As before, define $\xi_{ij} = \mathrm{ad}_H^{j-1} C_i$ $(i = 1, \cdots, m; j = 1, \cdots, r_i)$, where "$\mathrm{ad}_H$" is with respect to the Poisson bracket $\{\cdot,\cdot\}_P$ on $\mathcal{X}$. Let $d = 2n - \sum_{i=1}^m r_i$ and construct functions $\eta_1(x), \cdots, \eta_d(x)$ such that $((\xi_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq r_i), \eta_1, \cdots, \eta_d)$ forms a local coordinate system for $\mathcal{X}$ and $\{C_j, \eta_i\}_P = 0$ $(i = 1, \cdots, d; j = 1, \cdots, m)$. Let $\zeta_{ij} := \xi_{ij} - \mathrm{ad}_{\bar{H}}^{j-1} \bar{C}_i$ $(i = 1, \cdots, m; j = 1, \cdots, r_i)$, where "$\mathrm{ad}_{\bar{H}}$" is with respect to the Poisson bracket $\{\cdot,\cdot\}_M$, and denote $\zeta_i = \mathrm{col}(\zeta_{i1}, \cdots, \zeta_{ir_i})$, $\zeta = (\zeta_1^T \ \cdots \ \zeta_m^T)^T$, $\eta = \mathrm{col}(\eta_1, \cdots, \eta_d)$. Then obviously $(\zeta, \eta, \bar{x})$ forms a new local coordinate system for $\mathcal{X}^E$. Denote $\Psi(x, \bar{x}) = (\zeta^T(x, \bar{x}), \eta^T(x, \bar{x}), \bar{x}^T)^T$. Given a function $F^E \in C^\infty(\mathcal{X}^E)$ in the coordinates $(x, \bar{x})$, its expression in the new coordinates $(\zeta, \eta, \bar{x})$, $F^E \circ \Psi^{-1}(\zeta, \eta, \bar{x})$, is denoted by $F^E(\zeta, \eta, \bar{x})$. Let $\bar{B}(\bar{x})$ be the (m, m)-matrix with entries $\bar{b}_{ij}(\bar{x}) = \{\mathrm{ad}_{\bar{H}}^{r_i - 1} \bar{C}_i, \bar{C}_j\}_M(\bar{x})$ $(i, j = 1, \cdots, m)$.

Assume that the MMP is solvable for M and P around $(x_0, \bar{x}_0)$, i.e., $r_i \leq \bar{r}_i$ $(i = 1, \cdots, m)$. Then in the new coordinates E has the form:

$$\begin{aligned}
\dot{\zeta}_{i1} &= \zeta_{i2} \\
&\ \vdots && \qquad (i = 1, \cdots, m) \\
\dot{\zeta}_{ir_i - 1} &= \zeta_{ir_i} \\
\dot{\zeta}_{ir_i} &= \mathrm{ad}_H^{r_i} C_i(\zeta, \eta, \bar{x}) + A_{i*}(\zeta, \eta, \bar{x}) u - \\
&\qquad \mathrm{ad}_{\bar{H}}^{r_i} \bar{C}_i(\bar{x}) - \bar{B}_{i*}(\bar{x}) q^E \\
\dot{\eta}_i &= \{H, \eta_i\}_P(\zeta, \eta, \bar{x}) && \qquad (i = 1, \cdots, d) \\
\dot{\bar{x}} &= X_{\bar{H}}(\bar{x}) - \sum_{j=1}^m X_{\bar{C}_j}(\bar{x}) q_j^E \\
y_j^E &= \zeta_{j1} && \qquad (j = 1, \cdots, m)
\end{aligned} \tag{8.184}$$

Let $d(\zeta, \eta, \bar{x}) = \mathrm{col}(\mathrm{ad}_H^{r_1} C_1, \cdots, \mathrm{ad}_H^{r_m} C_m)(\zeta, \eta, \bar{x})$, $e(\bar{x}) = \mathrm{col}(\mathrm{ad}_{\bar{H}}^{r_1} \bar{C}_1, \cdots, \mathrm{ad}_{\bar{H}}^{r_m} \bar{C}_m)(\bar{x})$ and let $\gamma(\zeta)$ be an m-vector with entries $\gamma_i(\zeta) = \sum_{j=1}^{r_i} \gamma_{ij} \zeta_{ij}$. Define

$$u = A^{-1}(\zeta, \eta, \bar{x})(\gamma(\zeta) - d(\zeta, \eta, \bar{x}) + e(\bar{x}) + \bar{B}(\bar{x}) q^E) \tag{8.185}$$

Then (8.184,8.185) becomes:

$$\begin{aligned}
\dot{\zeta}_{i1} &= \zeta_{i2} \\
&\ \vdots && \qquad (i = 1, \cdots, m) \\
\dot{\zeta}_{ir_i - 1} &= \zeta_{ir_i} \\
\dot{\zeta}_{ir_i} &= \gamma_{ir_i} \zeta_{ir_i} + \cdots + \gamma_{i1} \zeta_{i1} \\
\dot{\eta}_i &= \{H, \eta_i\}_P(\zeta, \eta, \bar{x}) && \qquad (i = 1, \cdots, d) \\
\dot{\bar{x}} &= X_{\bar{H}}(\bar{x}) - \sum_{j=1}^m X_{\bar{C}_j}(\bar{x}) q_j^E \\
y_j^E &= \zeta_{j1} && \qquad (j = 1, \cdots, m)
\end{aligned} \tag{8.186}$$

Hence the feedback (8.185) solves the nDDDPdm for E. It is clear that if we choose $\zeta_{ij}(0) = 0$ $(i = 1, \cdots, m; j = 1, \cdots, r_i)$, then $y^E(t) = 0$ $(\forall t \geq 0)$. By the definition of the ζ_{ij}, $\zeta_{ij} = 0$ $(i = 1,\cdots,m, j = 1,\cdots,r_i)$ is equivalent to $\xi_{ij}(x) = \mathrm{ad}_{\bar{H}}^{j-1}\bar{C}_i(\bar{x})$ $(i = 1,\cdots,m; j = 1,\cdots,r_i)$. Since the functions $\xi_{ij}(x)$ are independent (cf. Lemma 3.1.8), we have that for every $\bar{x}_0 \in \bar{\mathcal{X}}$ there exists an $x_0 \in \mathcal{X}$ such that $\xi_{ij}(x_0) = \mathrm{ad}_{\bar{H}}^{j-1}\bar{C}_i(\bar{x}_0)$ $(i = 1,\cdots,m; j = 1,\cdots,r_i)$. The outputs of (8.186) satisfy

$$y_j^{E(r_j)} = \gamma_{jr_j} y_j^{E(r_j-1)} + \cdots + \gamma_{j1} y_j^E \quad (j = 1,\cdots,m) \tag{8.187}$$

This means that for every initial condition $(\zeta_0, \eta_0, \bar{x}_0)$ for (8.186) there exist $\sigma, \tau > 0$ and γ_{ij} $(i = 1,\cdots,m; j = 1,\cdots,r_i)$ such that

$$\|y^E(t,\zeta_0,\eta_0,\bar{x}_0)\|_m \le \tau e^{-\sigma t} \quad (\forall t \ge 0) \tag{8.188}$$

By the proof of Theorem 7.1.8 and the above considerations, it follows that the dynamic state feedback Q defined by

$$Q \begin{cases} \dot{z} &= X_{\bar{H}}(z) - \sum\limits_{j=1}^{m} X_{\bar{C}_j}(z) v_j \\ u &= A^{-1}(\zeta,\eta,z)(\gamma(\zeta) - d(\zeta,\eta,z) + e(z) + \bar{B}(z)v) \end{cases} \tag{8.189}$$

with $z(0) = \bar{x}_0$ solves the MMP around every $(x_0, \bar{x}_0) \in \mathcal{X} \times \bar{\mathcal{X}}$. Moreover, Q satisfies property *(i)* whenever $(x_0, \bar{x}_0)$ are such that $\xi_{ij}(x_0) = \mathrm{ad}_{\bar{H}}^{j-1}\bar{C}_i(\bar{x}_0)$ $(i = 1,\cdots,m; j = 1,\cdots,r_i)$ and Q satisfies property *(ii)*, with an appropriate choice of γ_{ij} $(i = 1,\cdots,m; j = 1,\cdots,r_i)$.

We proceed by investigating the structure of the limiting behavior (the behavior as $t \to \infty$) of the system $P \circ Q$. Note that the dynamics of Q are an exact copy of the dynamics of M (i.e., the same dynamics *and* the same initialization of the dynamics). Moreover, u in (8.189) is constructed in such a way that $C_i^{(k)}(x) = \bar{C}_i^{(k)}(z)$ $(i = 1,\cdots,m; k = 0,1,\cdots)$ as $t \to \infty$ (where $C_i^{(k)}, \bar{C}_i^{(k)}$ denote the k-th time-derivative of $C_i,\bar{C}_i$ respectively). Therefore, investigation of the structure of the limiting behavior of $P \circ Q$ is equivalent to investigation of the clamped dynamics of E. From (8.186) we see that the clamped dynamics manifold N_E^* of E is given by

$$\begin{aligned} N_E^* = \; & \{(x,\bar{x}) \in \mathcal{X}^E \mid \zeta_{ij}(x,\bar{x}) = 0, 1 \le i \le m, 1 \le j \le r_i\} = \\ & \{(x,\bar{x}) \in \mathcal{X}^E \mid \xi_{ij}(x) = \mathrm{ad}_{\bar{H}}^{j-1}\bar{C}_i(\bar{x}), 1 \le i \le m, 1 \le j \le r_i\} \end{aligned} \tag{8.190}$$

and the clamped dynamics of E are given by

$$\begin{cases} \dot{\eta}_i &= \{H,\eta_i\}_P(0,\eta,\bar{x}) \quad (i = 1,\cdots,d) \\ \dot{\bar{x}} &= X_{\bar{H}}(\bar{x}) - \sum\limits_{j=1}^{m} X_{\bar{C}_j}(\bar{x}) q_j^E \end{cases} \tag{8.191}$$

Hence the clamped dynamics of E consist of the dynamics of M and a nonlinear system whose dynamics are driven by the dynamics of M. We consider these dynamics in more detail. Recall that we have defined $\Psi(x,\bar{x}) = (\zeta^T(x,\bar{x}), \eta^T(x), \bar{x})^T$. Define also $\Phi(x) = (\xi^T(x), \eta^T(x))^T$ and $\pi(\bar{x}) = \mathrm{col}(\mathrm{ad}_{\bar{H}}^{j-1}\bar{C}_i(\bar{x}) \mid 1 \le i \le m, 1 \le j \le r_i)$. Note that $\zeta(x,\bar{x}) = \xi(x) - \pi(\bar{x})$. Hence we have:

$$\Psi^{-1}(0,\eta,\bar{x}) = \begin{pmatrix} \Phi^{-1}(\pi(\bar{x}),\eta) \\ \bar{x} \end{pmatrix} \tag{8.192}$$

Therefore,

$$\dot{\eta} = \{H, \eta\}_P^T \circ \Phi^{-1}(\pi(\bar{x}), \eta) \tag{8.193}$$

Let $s = \sum_{i=1}^m r_i$ and interpret the vector field governing the dynamics (8.193) as a vector field parametrized by the parameter $\pi \in I\!R^s$, i.e., define the parametrized vector field $f_\pi(\eta)$ by

$$f_\pi(\eta) = \{H, \eta\}_P^T \circ \Phi^{-1}(\pi, \eta) \tag{8.194}$$

Now if Conjecture 8.2.24 is true, then for every $\pi \in I\!R^s$ there exists a function $\bar{H}_\pi(\eta)$ and a non-degenerate Poisson bracket $\{\cdot,\cdot\}_\pi$ such that

$$f_\pi(\eta) = \{\bar{H}_\pi, \eta\}_\pi^T(\eta) \tag{8.195}$$

and hence for every $\pi \in I\!R^s$ we have that f_π is a Hamiltonian vector field.

This has the following consequence. It is desirable that the closed loop system $P \circ Q$ has some internal stability properties. In particular, we would like the system to be bounded input-bounded state stable (BIBS-stable). Roughly, this means that whenever a bounded control is applied to the system, the state variables of the system will also remain bounded. As was already indicated in [10], for BIBS-stability of the closed loop system we must have that in the dynamics (8.193) $\eta(t)$ is bounded ($t \geq 0$) when $\pi(\bar{x}(t))$ ($t \geq 0$) is bounded. If for every $\pi \in I\!R^s$ the parametrized vector field f_π is a Hamiltonian vector field, the stability analysis of the dynamics (8.193) becomes quite delicate. The reason for this is that, by Liouville's theorem (cf. [1]), a Hamiltonian vector field cannot have exponentially equilibrium points (see also the proof of Proposition 8.2.13). This causes the appearance of so called peaking phenomena, which make the stability analysis very difficult and delicate (see e.g. [13] and [94]). Further research on this topic is required.

Chapter 9

Conclusions

In linear systems theory, the geometric approach has proved to be a powerful tool for solving synthesis problems. Motivated by this success, various authors have tried to generalize the concepts form the linear geometric theory to the nonlinear context using differential geometric tools. To a great extent, these attempts have been very successful. An example of this is provided by the nonlinear disturbance decoupling problem via regular static state feedback. This problem was reviewed in Chapter 2. Another example is the nonlinear input-output decoupling problem that was recapitulated in Chapter 3.

The problems that one encounters in generalizing concepts from the linear geometric theory to the nonlinear context are (at least) threefold:

(i) Most of the generalizations of solutions to synthesis problems from the linear to the nonlinear context only hold if certain regularity assumptions are satisfied. Moreover, most of these generalizations are only local solutions, whereas in the linear case the solutions hold globally.

(ii) Many of the concepts from the linear geometric theory do not give rise to a single nonlinear generalization. For instance, the concept of a controlled invariant subspace admits at least two nonlinear generalizations: it can be generalized to a controlled invariant distribution as well as to a controlled invariant submanifold. Another example of this phenomenon is provided by the notion of zero-dynamics (or clamped dynamics) (see [64], where three different nonlinear generalizations of the notion of zero-dynamics for linear systems are given). It strongly depends on the application one has in mind what generalization is most suitable.

(iii) It is no longer possible to make use of ideas from the linear geometric theory when typically nonlinear phenomena start to play a role. For instance, for linear systems the solvability conditions for the distrurbance decoupling problem via regular static state feedback (DDP), regular dynamic state feedback (DDDP) and nonregular dynamic state feedback (nDDDP) respectively, are all the same. For nonlinear systems this no longer holds true, as was shown in Examples 2.2.14 and 2.2.15.

The fact that this last problem arises calls for the development of a new (typically nonlinear) theory. In this monograph such a theory was developed for the disturbance decoupling problem via dynamic state feedback and the model matching problem. Moreover, new results on the strong input-output decouplng problem via regular static and dynamic state feedback were obtained.

A local solution of the disturbance decoupling problem via regular dynamic state feedback has been provided. The solution is based on an algebraic theory for nonlinear control systems that was developed in [35],[30],[36]. A central role in the solution of the DDDP is played by Singh's algorithm and a special regular dynamic state feedback derived from this algorithm, the Singh compensator. Some special properties of this Singh compensator have also been established.

The disturbance decoupling problem via nonregular dynamic state feedback has been locally solved by combining the algebraic theory mentioned above with the theory of controlled invariant submanifolds that was developed in [11],[67],[12],[64],[101],[104].

Another nonlinear synthesis problem for which a local solution has been given, is the nonlinear model matching problem (MMP). This solution has been obtained by showing that the solvability of the MMP is equivalent to the solvability of an associated disturbance decoupling problem via nonregular dynamic state feedback and disturbance measurements. For a square nonlinear plant and a square nonlinear model that is strongly input-output decouplable via regular static state feedback it has moreover been shown that, under generic assumptions, the MMP is solvable around an equilibrium point if and only if it is solvable for the linearizations of plant and model around this equilibrium point.

The theory on the input-output decoupling problem and the model matching problem has been applied to a special class of nonlinear control systems, the Hamiltonian systems. This class of control systems is of particular interest for applications. The problem of achieving strong input-output decoupling with stability for Hamiltonian systems has been investigated. Moreover a conjecture on the (Hamiltonian) structure of strongly input-output decouplable Hamiltonian systems has been formulated and proved to be true in some specific cases. If the conjecture holds true, it may provide important structural information in the solution of synthesis problems for Hamiltonian systems. This has been illustrated via the model matching problem with prescribed tracking error for Hamiltonian systems.

This monograph leaves open several interesting questions for further research. Some of them are listed below.

(i) A problem that still remains to be solved is the disturbance decoupling problem via nonregular *static* state feedback.

(ii) Can the results on the DDP and the DDDP with stability that were obtained in [107] be extended to the nDDP and the nDDDP with stability?

(iii) For the DDDP intrinsic (i.e., algorithm-independent) algebraic and geometric conditions for the solvability have been given, while the solution for the nDDDP has been given via an algorithm. Can we also come up with intrinsic conditions for the solvability of the nDDDP?

(iv) Is it possible to decide about the solvability of the DDP or the nDDP on the basis of the solvability of the DDDP, nDDDP respectively?

(v) What can be said about the internal stability of the compensated plant after we have solved the nonlinear MMP? Up till now this problem has only been addressed in [10] in case the plant is a SISO-system and in [47] in case the plant is strongly input-output decouplable via static state feedback. The problem consists in the fact that

one may very well have to introduce unstable unobservable modes in the closed loop, even if one starts from an internally stable plant and an internally stable model. To solve this problem, further investigation of the structure of a model matching configuration is needed, especially concerning the "fixed" and "free" modes of such a configuration. For linear systems this investigation has already been performed in [75]. For nonlinear systems the problem is undoubtedly much more difficult to solve. So far, only results about fixed modes in the solution of the input-output decoupling problem have been obtained in [62]. It is not clear if a similar analysis is applicable for the nonlinear MMP.

(vi) In Section 7.2 a result has been obtained that connects the solvability of the model matching problem around an equilibrium point with the solvability of this problem for the linearizations of plant and model around this equilibrium point. It is still an open question if a (linear) dynamic state feedback that solves the MMP for linearizations of plant and model is a good approximate solution of the MMP for the original (nonlinear) plant and model. Of course one should specify here what is meant by "good". It seems reasonable to call a dynamic state feedback that solves the MMP for the linearized plant and model "good" if it is a first order approximation of a dynamic state feedback that solves the MMP for the original (nonlinear) plant and model.

(vii) In Subsection 8.2.2 a result has been obtained on the problem of strong input-output decoupling with stability via regular static state feedback for Hamiltonian systems. One important conclusion is that with a static state feedback from the class L_s one can at most achieve stability (and not asymptotic stabiity) if $N^* \neq \{0\}$, where N^* is the maximal locally controlled invariant output-nulling submanifold for the system under consideration. However, it is easy to come up with examples where strong input-output decoupling with *asymptotic* stability can be achieved if $N^* \neq \{0\}$. Obviously, in this case the feedback that does the job will be outside the class L_s. It is still an open question under what conditions the strong input-output decoupling problem with asymptotic stability for Hamiltonian systems is solvable.

(vii) Hamiltonian systems are *conservative* systems, that is, systems which preserve energy. From a practical point of view it can be argued that in this sense Hamiltonian system are an idealization, since in practical circumstances there will always be some friction present in the system. It would be interesting to know if for a Hamiltonian control system to which some friction is added one can achieve strong input-output decoupling with asymptotic stability if a feedback from the class L_s is applied.

(viii) Can Conjecture 8.2.24, which concerns the Hamiltonian structure of strongly input-output decouplable Hamiltonian systems, be proved in its full generality? If not, can a counter example be found?

(ix) In general a feedback that solves a synthesis problem for a Hamiltonian system will not preserve the Hamiltonian structure of the system. It would be interesting to know when it does preserve the Hamiltonian structure. Till now, in the nonlinear context only results are known for the strong input-output decoupling problem via regular static state feedback (see [80]). A problem that one encounters in answering this question, is that a feedback that preserves the Hamiltonian structure should necessarily be an *output* feedback (see e.g. [98]). A theory on output feedback for nonlinear control systems is at the moment hardly available.

Finally it should be noted that in this monograph the attention was restricted to *analytic* nonlinear control systems. The theory that was developed can be extended to *smooth* nonlinear control systems. However, in extending the theory to smooth nonlinear control systems, one should be very careful with the notions of regularity one employs.

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Appendix A

Proof of Proposition 3.2.16

Proposition 3.2.16 *Consider a nonlinear system (3.1) and let x_0 be a strongly regular point for (3.1). Then any Singh compensator for (3.1) around x_0 is a regular dynamic state feedback for (3.1).*

Proof The proof exploits Proposition 3.2.11, i.e., we show that for the system (3.1,3.50) with controls v and outputs u we can reconstruct v given the knowledge of u. Obviously, by the structure of the Singh compensator (3.50), we can immediately reconstruct $v_{\rho^*+1}, \cdots, v_m$ given the knowledge of $u_{\rho^*+1}, \cdots, u_m$. Note that $\tilde{A}_n$,$\tilde{B}_n$ in (3.47) have the following structure:

$$\tilde{A}_n(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-1, i \le j \le n\}) =$$
$$\begin{pmatrix} \tilde{a}_1(x) \\ \tilde{a}_2(x, \{y_i^{(j)} \mid 1 \le i \le \rho_1, \gamma_i \le j \le \min(2, \delta_i)\}) \\ \vdots \\ \tilde{a}_n(x, \{y_i^{(j)} \mid 1 \le i \le \rho_{n-1}, \gamma_i \le j \le \min(n, \delta_i)\}) \end{pmatrix} \tag{A.1}$$
$$\tilde{B}_n(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-1, i \le j \le n\}) =$$
$$\begin{pmatrix} \tilde{b}_1(x) \\ \tilde{b}_2(x, \{y_i^{(j)} \mid 1 \le i \le \rho_1, \gamma_i \le j \le \min(1, \delta_i - 1)\}) \\ \vdots \\ \tilde{b}_n(x, \{y_i^{(j)} \mid 1 \le i \le \rho_{n-1}, \gamma_i \le j \le \min(n-1, \delta_i - 1)\}) \end{pmatrix}$$

where $\tilde{b}_k$ ($k = 1, \cdots, n$) has full row rank $\rho_k - \rho_{k-1}$ (putting $\rho_0 := 0$). Let $\tilde{a}_{kj}, \tilde{b}_{kj}$ denote the j-th row of $\tilde{a}_k$,$\tilde{b}_k$ respectively. By construction of the Singh compensator we have in particular for $r = 1, \cdots, \rho_1$:

$$\begin{aligned} z_{r1} &= \tilde{a}_{1r}(x) + \tilde{b}_{1r}(x)u \quad \text{if } \delta_r > 1 \\ v_r &= \tilde{a}_{1r}(x) + \tilde{b}_{1r}(x)u \quad \text{if } \delta_r = 1 \end{aligned} \tag{A.2}$$

Taking successive time-derivatives of the first equation in (A.2) we obtain

$$\begin{aligned} z_{rs} &= \phi_{sr}(x, u, \cdots, u^{(s-2)}) + \psi_{sr}(x, u, \cdots, u^{(s-2)})u^{(s-1)} \quad (s = 2, \cdots, \delta_r - 1) \\ v_r &= \phi_{\delta_r r}(x, u, \cdots, u^{(\delta_r-2)}) + \psi_{\delta_r r}(x, u, \cdots, u^{(\delta_r-2)})u^{(\delta_r-1)} \end{aligned} \tag{A.3}$$

Hence, given u we can reconstruct $v_1, \cdots, v_{\rho_1}$. Let $\hat{z}_2 := \{z_{ij} \mid 1 \leq i\rho_1, 1 \leq j \leq \delta_i - 1\}$. Then by construction of the Singh compensator (3.50) we have for $r = \rho_1 + 1, \cdots, \rho_2$:

$$\begin{aligned} z_{r1} &= \tilde{a}_{2r-\rho_1}(x, \hat{z}_2, \{v_i \mid 1 \leq i \leq \rho_1\}) + \tilde{b}_{2r-\rho_1}(x, \hat{z}_2)u \quad \text{if } \delta_r > 2 \\ v_r &= \tilde{a}_{2r-\rho_1}(x, \hat{z}_2, \{v_i \mid 1 \leq i \leq \rho_1\}) + \tilde{b}_{2r-\rho_1}(x, \hat{z}_2)u \quad \text{if } \delta_r = 2 \end{aligned} \tag{A.4}$$

Taking successive time-derivatives of the first equation in (A.4) and substituting (A.2),(A.3), we see that given u we can reconstruct $v_{\rho_1+1}, \cdots, v_{\rho_2}$. Applying the above arguments repeatedly, we can prove that given u we can reconstruct $v_1, \cdots, v_m$, which establishes our claim. ∎

Appendix B

Proof of Proposition 3.2.25

Proposition 3.2.25 *Consider a nonlinear system (3.1). Let $x_0 \in M$ be a strongly regular point for (3.1). Consider a Singh compensator (3.50) around x_0 defined on $U \times \mathcal{Z} \subset M \times I\!R^{\sigma}$, where U is a neighborhood of x_0. Moreover, let*

$$\left\{\begin{array}{rcl} \dot{\bar{z}}_i &=& A_i\bar{z}_i + B_i\bar{v}_i \quad (i = 1, \cdots, \rho^*) \\ \bar{u}^1 &=& \bar{\phi}_1(x, \bar{z}_1, \cdots, \bar{z}_{\rho^*}) + \sum_{i=1}^{m} \bar{\phi}_{2i}(x, \bar{z}_1, \cdots, \bar{z}_{\rho^*})\bar{v}_i \\ \bar{u}_i &=& \bar{v}_i \quad (i = \rho^* + 1, \cdots, m) \end{array}\right. \tag{B.1}$$

be another Singh compensator around x_0 defined on $U \times \bar{\mathcal{Z}} \subset M \times \bar{\mathcal{Z}}$, and assume that in the construction of (B.1) the same permutation of controls is employed as in the construction of (3.50). Then there exist open subsets $\mathcal{V} \subset \mathcal{Z}$ and $\bar{\mathcal{V}} \subset \bar{\mathcal{Z}}$ and a diffeomorphism $\Psi : U \times \mathcal{V} \to U \times \bar{\mathcal{V}}$ that transforms the system (3.1,3.50) defined on $U \times \mathcal{V}$ into the system (3.1,B.1) defined on $U \times \bar{\mathcal{V}}$.

Proof Let $x_0 \in M$ be a strongly regular point for (3.1). Apply Singh's algorithm, leading to the Singh compensator (3.50). This yields a reordering $\tilde{y}_1, \cdots, \tilde{y}_n, \hat{y}_n$ of the outputs of (3.1) such that for $k = 1, \cdots, n$

$$\begin{array}{rcl} \tilde{y}_k^{(k)} &=& \tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k\}) + \\ && \tilde{b}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1\})u \end{array} \tag{B.2}$$

where $\tilde{b}_k$ has full row rank over $\mathcal{K}$. Apply Singh's algorithm in another way, yielding the Singh compensator (B.1). This yields a reordering $\bar{y}_1, \cdots, \bar{y}_n, \hat{\bar{y}}_n$ of the outputs of (3.1) such that for $k = 1, \cdots, n$

$$\begin{array}{rcl} \bar{y}_k^{(k)} &=& \bar{a}_k(x, \{\bar{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k\}) + \\ && \bar{b}_k(x, \{\bar{y}_i^{(j)} \mid 1 \le i \le k-1, i \le j \le k-1\})u \end{array} \tag{B.3}$$

where $\bar{b}_k$ has full row rank over $\mathcal{K}$.

Consider $(B.2)$ for $k = 1$ and define $\tilde{\phi}_{11}(x, u) := \tilde{a}_1(x) + \tilde{b}_1(x)u$. Taking time-derivatives gives for $\ell = 2, \cdots, n$:

$$\begin{aligned} \tilde{y}_1^{(\ell)} = \; & \tilde{\phi}_{1\ell}(x, u, \cdots, u^{(\ell-1)}) = \\ & \frac{\partial \tilde{\phi}_{1\ell-1}}{\partial x}[f(x) + g(x)u] + \sum_{r=0}^{\ell-2} \frac{\partial \tilde{\phi}_{1\ell-1}}{\partial u^{(r)}} u^{(r+1)} \end{aligned} \tag{B.4}$$

Note that we have for $\ell = 1, \cdots, n$:

$$\tilde{\phi}_{1\ell}(x, u, \cdots, u^{(\ell-1)}) = \tilde{\phi}'_{1\ell}(x, u, \cdots, u^{(\ell-2)}) + \tilde{b}_1(x) u^{(\ell-1)} \tag{B.5}$$

Consider now (B.2) for $k = 2$ and define

$$\tilde{\phi}_{22}(x, u, \dot{u}) = \tilde{a}_2(x, \tilde{\phi}_{11}(x, u), \tilde{\phi}_{12}(x, u, \dot{u})) + \tilde{b}_2(x, \tilde{\phi}_{11}(x, u))u \tag{B.6}$$

Taking time-derivatives of (B.6), we obtain analogously to (B.4) the functions $\tilde{\phi}_{2\ell}(x, u, \cdots, u^{(\ell-1)})$ $(\ell = 3, \cdots, n)$. Note that for $\ell = 2, \cdots, n$ we have:

$$\begin{aligned} & \tilde{\phi}_{2\ell}(x, u, \cdots, u^{(\ell-1)}) = \\ & \qquad \tilde{\phi}'_{2\ell}(x, u, \cdots, u^{(\ell-3)}, \tilde{\phi}_{11}(x, u), \cdots, \tilde{\phi}_{1\ell}(x, u, \cdots, u^{(\ell-1)})) + \\ & \qquad \tilde{b}_2(x, \tilde{\phi}_{11}(x, u)) u^{(\ell-1)} \end{aligned} \tag{B.7}$$

Repeating this procedure yields functions $\tilde{\phi}_{k\ell}(x, u, \cdots, u^{(\ell-1)})$ $(1 \le k \le n, k \le \ell \le n)$ of the form

$$\begin{aligned} & \tilde{\phi}_{k\ell}(x, u, \cdots, u^{(\ell-1)}) = \\ & \qquad \tilde{\phi}'_{k\ell}(x, u, \cdots, u^{(\ell-k-1)}, \{\tilde{\phi}_{ij}(x, u, \cdots, u^{(j-1)} \mid 1 \le i \le k-1, i \le j \le \ell\}) + \\ & \qquad \tilde{b}_k(x, \{\tilde{\phi}_{ij}(x, u, \cdots, u^{(j-1)}) \mid 1 \le i \le k-1, i \le j \le k-1\}) u^{(\ell-k)} \end{aligned} \tag{B.8}$$

Let $\tau = \sum_{i=1}^{n} (n+1-i)\rho_i$ and define the mapping $\tilde{\Phi} : M \times I\!R^{nm} \to M \times I\!R^{\tau}$ by

$$\tilde{\Phi}(x, u, \cdots, u^{(n-1)}) = \begin{pmatrix} x \\ \tilde{\phi}_{k\ell}(x, u, \cdots, u^{(\ell-1)}) \end{pmatrix}_{(1 \le k \le n, \ell \le j \le n)} \tag{B.9}$$

Consider a point $(x_0, \tilde{y}_0) := (x_0, \{\tilde{y}_{i0}^{(j)} \mid 1 \le i \le n, i \le j \le n\}) \in M \times I\!R^{\tau}$ that satisfies

$$\text{rank}_{I\!R} \tilde{B}_k(x_0, \tilde{y}_0) = \rho_k \quad (k = 1, \cdots, n) \tag{B.10}$$

We show that there exists an $(x_0, u_0, \cdots, u_0^{(n-1)}) \in M \times I\!R^{nm}$, denoted by $\tilde{\Phi}^{-1}(x_0, \tilde{y}_0)$, such that

$$\tilde{\Phi}(x_0, u_0, \cdots, u_0^{(n-1)}) = \begin{pmatrix} x_0 \\ \tilde{y}_0 \end{pmatrix} \tag{B.11}$$

Define for $\ell = 1, \cdots, n$:

$$\begin{aligned} & \tilde{B}'_\ell(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-\ell, i \le j \le n-\ell\}) = \\ & \qquad \begin{pmatrix} \tilde{b}_1(x) \\ \vdots \\ \tilde{b}_{n+1-\ell}(x, \{\tilde{y}_i^{(j)} \mid 1 \le i \le n-\ell, i \le j \le n-\ell\}) \end{pmatrix} \end{aligned} \tag{B.12}$$

By (B.10) we have that $(u_0, \cdots, u_0^{(n-1)})$ has to satisfy

$$\begin{pmatrix} \tilde{y}_{10}^{(\ell)} \\ \tilde{y}_{20}^{(\ell+1)} \\ \vdots \\ \tilde{y}_{n+1-\ell 0}^{(n)} \end{pmatrix} = \begin{pmatrix} \tilde{\phi}'_{1\ell}(x_0, u_0, \cdots, u_0^{(\ell-2)}) \\ \tilde{\phi}'_{2\ell+1}(x_0, u_0, \cdots, u_0^{(\ell-2)}, \dot{\tilde{y}}_{10}, \cdots, \tilde{y}_{10}^{(\ell+1)}) \\ \vdots \\ \tilde{\phi}'_{n+1-\ell n}(x_0, u_0, \cdots, u_0^{(\ell-2)}, \{\tilde{y}_{i0}^{(j)} \mid 1 \leq i \leq n-\ell, i \leq j \leq n\}) \end{pmatrix} + \tilde{B}'_\ell(x_0, \{\tilde{y}_{i0}^{(j)} \mid 1 \leq i \leq n-\ell, i \leq j \leq n-\ell\})u_0^{(\ell-1)} \tag{B.13}$$

for $\ell = 1, \cdots, n$. Note that by (B.10) the matrices $\tilde{B}_\ell(x_0, \{\tilde{y}_{i0}^{(j)} \mid 1 \leq i \leq n-\ell, i \leq j \leq n-\ell\})$ have full row rank over $\mathbb{R}$ $(\ell = 1, \cdots, n)$. This means that from (B.13) $(u_0, \cdots, u_0^{(n-1)})$ can be solved in the following way. First consider (B.13) for $\ell = 1$. Then the first term on the right hand side of (B.13) does not depend on $(u_0, \cdots, u_0^{(n-1)})$. Since $\tilde{B}'_1(x_0, \{\tilde{y}_{i0}^{(j)} \mid 1 \leq i \leq n-1, i \leq j \leq n-1\})$ has full row rank, we can solve u_0 from (B.13) for $\ell = 1$. Next consider (B.13) for $\ell = 2$. The first term on the right hand side of (B.13) now does depend on u_0, but not on $(\dot{u}_0, \cdots, u^{(n-1)})$. Since $\tilde{B}'_2(x_0, \{\tilde{y}_{i0}^{(j)} \mid 1 \leq i \leq n-2, i \leq j \leq n-2\})$ has full row rank, $\dot{u}_0$ can be solved form (B.13) for $\ell = 2$, by first substituting the value of u_0 that was found from (B.13) for $\ell = 1$. Repeating this procedure, we solve $(u_0, \cdots, u_0^{(n-1)})$ from (B.13). Note however, that in general $(u_0, \cdots, u_0^{(n-1)})$ satisfying (B.13) is not uniquely determined by (B.13).

If we consider the application of Singh's algorithm leading to (B.3), we can analogously define a mapping $\bar{\Phi} : M \times \mathbb{R}^{nm} \to M \times \mathbb{R}^{r}$.

Define the subsets $(M_0 \times \tilde{\mathcal{Y}}_0)$, $(M_0 \times \bar{\mathcal{Y}}_0)$ of $M \times \mathbb{R}^r$ by

$$\begin{aligned} (M_0 \times \tilde{\mathcal{Y}}_0) &= \{(x_0, \tilde{y}_0) \in M \times \mathbb{R}^r \mid \operatorname{rank}_{\mathbb{R}} \tilde{B}_k(x_0, \tilde{y}_0) = \rho_k, k = 1, \cdots, n\} \\ (M_0 \times \bar{\mathcal{Y}}_0) &= \{(x_0, \bar{y}_0) \in M \times \mathbb{R}^r \mid \operatorname{rank}_{\mathbb{R}} \bar{B}_k(x_0, \bar{y}_0) = \rho_k, k = 1, \cdots, n\} \end{aligned} \tag{B.14}$$

Obviously, $(x_0, \tilde{y}_0) \in (M_0 \times \tilde{\mathcal{Y}}_0)$ for some $\tilde{y}_0 \in \mathbb{R}^r$ if and only if x_0 is a regular point for Singh's algorithm. This justifies the definition of

$$\begin{aligned} (M_0 \times \tilde{\mathcal{Y}}_0)_{\text{s.r.}} &= \{(x_0, \tilde{y}_0) \in (M_0 \times \tilde{\mathcal{Y}}_0) \mid x_0 \text{ is strongly regular}\} \\ (M_0 \times \bar{\mathcal{Y}}_0)_{\text{s.r.}} &= \{(x_0, \bar{y}_0) \in (M_0 \times \bar{\mathcal{Y}}_0) \mid x_0 \text{ is strongly regular}\} \end{aligned} \tag{B.15}$$

Consider the subsets $\tilde{\mathcal{N}}, \bar{\mathcal{N}} \subset M \times \mathbb{R}^{nm}$ defined by

$$\begin{aligned} \tilde{\mathcal{N}} &= \tilde{\Phi}^{-1}((M_0 \times \tilde{\mathcal{Y}}_0)_{\text{s.r.}}) \\ \bar{\mathcal{N}} &= \bar{\Phi}^{-1}((M_0 \times \bar{\mathcal{Y}}_0)_{\text{s.r.}}) \end{aligned} \tag{B.16}$$

Introduce the notation

$$\mathcal{E}_k(x_0, u_0, \cdots, u_0^{(k-1)}) = \operatorname{span}_{\mathbb{R}}\{dx, d\dot{y}(x_0), \cdots, dy^{(k)}(x_0, u_0, \cdots, u_0^{(k-1)})\} \tag{B.17}$$

For $(x_0, u_0, \cdots, u_0^{(n-1)}) \in \tilde{\mathcal{N}}$ it straightforwardly follows from (B.2) (see the proof of Theorem 2.3 in [30]) that

$$\begin{aligned} \dim_{\mathbb{R}} \mathcal{E}_k(x_0, u_0, \cdots, u_0^{(n-1)}) = \;& n + \sum_{\ell=1}^{k} \operatorname{rank}_{\mathbb{R}} \tilde{B}_\ell \circ \tilde{\Phi}(x_0, u_0, \cdots, u_0^{(n-1)}) = \\ & n + \sum_{\ell=1}^{k} \operatorname{rank}_{\mathbb{R}} \bar{B}_\ell \circ \bar{\Phi}(x_0, u_0, \cdots, u_0^{(n-1)}) \end{aligned} \tag{B.18}$$

Hence we have for $k = 1, \cdots, n$:

$$\operatorname{rank}_{\mathbb{R}} \tilde{B}_k \circ \tilde{\Phi}(x_0, u_0, \cdots, u_0^{(n-1)}) = \operatorname{rank}_{\mathbb{R}} \bar{B}_k \circ \bar{\Phi}(x_0, u_0, \cdots, u_0^{(n-1)}) \tag{B.19}$$

This implies that $(x_0, u_0, \cdots, u_0^{(n-1)}) \in \bar{\mathcal{N}}$ and so $\tilde{\mathcal{N}} \subset \bar{\mathcal{N}}$. Similarly, it can be shown that $\bar{\mathcal{N}} \subset \tilde{\mathcal{N}}$ and hence we have established that $\tilde{\mathcal{N}} = \bar{\mathcal{N}}$.

For convenience, denote $\mathcal{N} = \tilde{\mathcal{N}} = \bar{\mathcal{N}}$. Let $d = nm - \sum_{i=1}^{n} \rho_i$ and define $\tilde{\mathcal{S}} = (M_0 \times \tilde{\mathcal{Y}}_0)_{\text{s.r.}} \times \mathbb{R}^d$, $\bar{\mathcal{S}} = (M_0 \times \bar{\mathcal{Y}}_0)_{\text{s.r.}} \times \mathbb{R}^d$. Permute the entries of u in such a way that for $k = 1, \cdots, n$ the matrix $\tilde{B}_{k1}$ consisting of the first ρ_k columns of $\tilde{B}_k$ is invertible for every $(x_0, \tilde{y}_0) \in (M_0 \times \tilde{\mathcal{Y}}_0)_{\text{s.r.}}$. Note that by (B.19) this implies that for $k = 1, \cdots, n$ also the matrix $\bar{B}_{k1}$ consisting of the first ρ_k columns of $\bar{B}_k$ is invertible for every $(x_0, \bar{y}_0) \in (M \times \bar{\mathcal{Y}}_0)_{\text{s.r.}}$. For $k = 0, \cdots, n-1$, let $u_{01}^{(k)}$ consist of the first ρ_{k+1} entries of $u_0^{(k)}$ and $u_{02}^{(k)}$ of the remaining $m - \rho_{k+1}$ entries of $u_0^{(k)}$. Define the mappings $\tilde{\Phi}_e : \mathcal{N} \to \tilde{\mathcal{S}}$, $\bar{\Phi}_e : \mathcal{N} \to \bar{\mathcal{S}}$ by

$$\begin{aligned} \tilde{\Phi}_e(x_0, u_0, \cdots, u_0^{(n-1)}) &= \begin{pmatrix} \tilde{\Phi}(x_0, u_0, \cdots, u_0^{(-1)}) \\ u_{02} \\ \vdots \\ u_{02}^{(n-1)} \end{pmatrix} \\ \bar{\Phi}_e(x_0, u_0, \cdots, u_0^{(n-1)}) &= \begin{pmatrix} \bar{\Phi}(x_0, u_0, \cdots, u_0^{(n-1)}) \\ u_{02} \\ \vdots \\ u_{02}^{(n-1)} \end{pmatrix} \end{aligned} \tag{B.20}$$

Let $(x_0, u_0, \cdots, u_0^{(n-1)}) \in \mathcal{N}$. By analyticity of the data, $\tilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$ are open and dense subsets of $M \times \mathbb{R}^\tau \times \mathbb{R}^d$. This implies that there exist neighborhoods $\tilde{U}, \bar{U} \subset M \times \mathbb{R}^\tau \times \mathbb{R}^d$ of $(x_0, \tilde{y}_0, u_{02}, \cdots, u_{02}^{(n-1)}) = \tilde{\Phi}_e(x_0, u_0, \cdots, u_0^{(n-1)})$ and $(x_0, \bar{y}_0, u_{02}, \cdots, u_{02}^{(n-1)}) = \bar{\Phi}_e(x_0, u_0, \cdots, u_0^{(n-1)})$ respectively, such that $\tilde{U} \subset \tilde{\mathcal{S}}$, $\bar{U} \subset \bar{\mathcal{S}}$. It is readily checked that $\tilde{\Phi}_e$ and $\bar{\Phi}_e$ are one-to-one and onto and that $\tilde{\Phi}_e$ and $\bar{\Phi}_e$ are meromorphic functions. This implies that there exist neighborhoods $U_0 \subset \mathcal{N}$ of $(x_0, u_0, \cdots, u_0^{(n-1)})$, $\tilde{U}_0 \subset \tilde{U}$ of $(x_0, \tilde{y}_0, u_{02}, \cdots, u_{02}^{(n-1)})$ and $\bar{U}_0 \subset \bar{U}$ of $(x_0, \tilde{y}_0, u_{02}, \cdots, u_{02}^{(n-1)})$ such that $\tilde{\Phi}_e : U_0 \to \tilde{U}_0$, $\bar{\Phi}_e : U_0 \to \bar{U}_0$ are diffeomorphisms. Define the diffeomorphism $\Phi : \tilde{U}_0 \to \bar{U}_0$ by $\Phi = \bar{\Phi}_e \circ \tilde{\Phi}_e^{-1}$.

By the construction of $\tilde{\Phi}$ respectively $\bar{\Phi}$ (see the proof of Proposition 3.2.24) there exist $\tilde{\gamma}_1, \cdots, \tilde{\gamma}_{\rho^*}, \bar{\gamma}_1, \cdots, \bar{\gamma}_{\rho^*}, \delta_1, \cdots, \delta_{\rho^*}$ such that (after a possible permutation of the entries of

$\tilde{\Phi}, \bar{\Phi}$ respectively):

$$
\begin{aligned}
\tilde{\Phi}(x_0, u_0, \cdots, u_0^{(n-1)}) &= \begin{pmatrix} x \\ \tilde{y}_\gamma \\ \tilde{y}_\delta \\ \tilde{y}_r \end{pmatrix} (x_0, u_0, \cdots, u_0^{(n-1)}) \\
\bar{\Phi}(x_0, u_0, \cdots, u_0^{(n-1)}) &= \begin{pmatrix} x \\ \bar{y}_\gamma \\ \bar{y}_\delta \\ \bar{y}_r \end{pmatrix} (x_0, u_0, \cdots, u_0^{(n-1)})
\end{aligned}
\tag{B.21}
$$

where

$$
\begin{aligned}
\tilde{y}_\gamma &= (y_i^{(j)} \mid 1 \leq i \rho^*, \tilde{\gamma}_i \leq j \leq \delta_i - 1) \\
\bar{y}_\gamma &= (y_i^{(j)} \mid 1 \leq i \leq \rho^*, \bar{\gamma}_i \leq j \leq \delta_i - 1) \\
\tilde{y}_\delta &= (y_i^{(\delta_i)} \mid 1 \leq i \leq \rho^*) \\
\bar{y}_\delta &= (y_i^{(\delta_i)} \mid 1 \leq i \leq \rho^*) \\
\tilde{y}_r &= (y_i^{(j)} \mid 1 \leq i \leq \rho^*, \delta_i + 1 \leq j \leq n) \\
\bar{y}_r &= (y_i^{(j)} \mid 1 \leq i \leq \rho^*, \delta_i + 1 \leq j \leq n)
\end{aligned}
$$

Note that by Lemma 3.2.22 and Proposition 3.2.24 we have $\dim \tilde{y}_\gamma = \dim \bar{y}_\gamma = \sigma$. Denote $u_{02}^J = (u_{02}, \cdots, u_{02}^{(n-1)})$. the sets $(x, \tilde{y}_\gamma, \tilde{y}_\delta, \tilde{y}_r, u_{02}^J)$, $(x, \bar{y}_\gamma, \bar{y}_\delta, \bar{y}_r, u_{02}^J)$ form a set of local coordinates for $M \times I\!R^\tau \times I\!R^d$, connected by the diffeomorphism Φ. Note that the entries of $(\tilde{y}_\delta, \tilde{y}_r)$ and $(\bar{y}_\delta, \bar{y}_r)$ are the same. Moreover, note that the rows of $d\tilde{y}_\delta$ and $d\tilde{y}_r$ are essential in $\mathcal{E}_n$. By Theorem 3.2.4 and Lemma 3.2.19 this means that

$$
\operatorname{span}_{\mathcal{K}}\{d\bar{y}_\gamma\} \subset \operatorname{span}_{\mathcal{K}}\{dx, d\tilde{y}_\gamma\} \tag{B.22}
$$

These facts imply that Φ has the form

$$
\Phi(x, \tilde{y}_\gamma, \tilde{y}_\delta, \tilde{y}_r, u_{02}^J) = \begin{pmatrix} x \\ \Phi_\gamma(x, \tilde{y}_\gamma) \\ \tilde{y}_\delta \\ \tilde{y}_r \\ u_{02}^J \end{pmatrix} \tag{B.23}
$$

Consider the mapping $\Psi : M \times I\!R^\sigma \to M \times I\!R^\sigma$ defined by

$$
\Psi(x, \tilde{y}_\gamma) = \begin{pmatrix} x \\ \Phi_\gamma(x, \tilde{y}_\gamma) \end{pmatrix} \tag{B.24}
$$

Since Φ is a local diffeomorphism, it follows that there is a neighborhood $U \subset M$ of x_0 and open subsets $\mathcal{V}, \bar{\mathcal{V}} \subset I\!R^\sigma$ such that $\Psi : U \times \mathcal{V} \to U \times \bar{\mathcal{V}}$ is a diffeomorphism.

Obviously,

$$
\dot{\Phi}_\gamma = \bar{A}\Phi_\gamma(x, \tilde{y}_\gamma) + \bar{B}\bar{y}_\delta \tag{B.25}
$$

with $\bar{A} = \text{block diag}(\bar{A}_1, \cdots, \bar{A}_{\rho^*})$, $\bar{B} = \text{block diag}(\bar{B}_1, \cdots, \bar{B}_{\rho^*})$, and $\bar{A}_1, \cdots, \bar{A}_{\rho^*}, \bar{B}_1, \cdots, \bar{B}_{\rho^*}$ as defined in (B.1). For $(x, u, \cdots, u^{(n-1)}) \in M \times I\!R^{nm}$, define $\pi_u(x, u, \cdots, u^{(n-1)}) = u$. Comparing (3.48) and the construction of $\tilde{\Phi}_e^{-1}$, $\bar{\Phi}_e^{-1}$, it follows that

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}(x, \tilde{y}_\gamma, \tilde{y}_\delta, u_{02}) = \pi_u \tilde{\Phi}_e^{-1}(x, \tilde{y}_\gamma, \tilde{y}_\delta, \tilde{y}_r, u_{02}^J) \tag{B.26}$$

$$\begin{pmatrix} \bar{u}^1 \\ \bar{u}^2 \end{pmatrix}(x, \bar{y}_\gamma, \bar{y}_\delta, u_{02}) = \pi_u \bar{\Phi}_e^{-1}(x, \bar{y}_\gamma, \bar{y}_\delta, \bar{y}_r, u_{02}^J) \tag{B.27}$$

From the definition of Φ we then have

$$\begin{aligned} \begin{pmatrix} \bar{u}^1 \\ \bar{u}^2 \end{pmatrix}(x, \bar{y}_\gamma, \bar{y}_\delta, u_{02}) = {} & \pi_u \bar{\Phi}_e^{-1} \circ \Phi(x, \tilde{y}_\gamma, \tilde{y}_\delta, \tilde{y}_r, u_{02}^J) = \\ & \pi_u \tilde{\Phi}_e^{-1}(x, \tilde{y}_\gamma, \tilde{y}_\delta, \tilde{y}_r, u_{02}^J) = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}(x, \tilde{y}_\gamma, \tilde{y}_\delta, u_{02}) \end{aligned} \tag{B.28}$$

(B.25) and (B.28) establish that Ψ transforms the system (3.1,3.50) defined on $U \times \mathcal{V}$ into the system (3.1,B.1) on $U \times \bar{\mathcal{V}}$. ■

Appendix C

Proof of Proposition 5.0.11

Proposition 5.0.11 *Consider a nonlinear system (5.2) and let x_0 be a regular point for Algorithm 5.0.7 applied to (5.2). Let the clamped dynamics manifold N^* of (5.2) around x_0 be given by $N^* = \{x \mid \Phi_{k^*}(x) = 0\}$. Furthermore, consider a dynamic state feedback (3.42) for (5.2) and let (x_0, z_0) be a regular point for Algorithm 5.0.7 applied to (5.2,3.42). Then there exists a vector of functions $\Psi(x,z)$ such that M^*, the clamped dynamics manifold of (5.2,3.42) around (x_0, z_0), is given by $M^* = \{(x,z) \mid \Phi_{k^*}(x) = 0, \Psi(x,z) = 0\}$.*

Proof Denote the submanifolds obtained while applying the clamped dynamics algorithm to (5.2) and (5.2,3.42) by N_k $(= \{x \mid \Phi_k(x) = 0\})$, M_k respectively. We will prove by induction that we can find vectors of functions $\Psi_k(x,z)$ such that $M_k = \{(x,z) \mid \xi_k(x,z) = 0\}$, with $\xi_k(x,z) = \begin{pmatrix} \Phi_k(x) \\ \Psi_k(x,z) \end{pmatrix}$. Obviously, we have $N_0 = \{x \mid \Phi_0(x) = 0\}$, $M_0 = \{(x,z) \mid \xi_0(x,z) = 0\}$, with $\Phi_0(x) = \xi_0(x,z) = h(x)$. Hence our claim holds for $k = 0$. Now apply the first step of the clamped dynamics algorithm to (5.2), yielding matrices $A_1(x), B_1(x)$, a vector of functions $\Phi_1(x)$ and $N_1 = \{x \mid \Phi_1(x) = 0\}$. Applying the first step of the clamped dynamics algorithm to (5.2,3.42) yields:

$$\begin{aligned} \dot{\xi}_0(x,z) &= \frac{\partial \phi_0}{\partial x}(x)[f(x) + g(x)(\gamma(x,z) + \delta(x,z)v] = \\ & \quad A_1(x) + B_1(x)\gamma(x,z) + B_1(x)\delta(x,z)v \end{aligned} \tag{C.1}$$

Since (x_0, z_0) is a regular point for the clamped dynamics algorithm applied to (5.2,3.42), $B_1(x)\delta(x,z)$ has full rank $\tilde{r}_1$ in a neighborhood of (x_0, z_0) in M_0. It is clear that $\tilde{r}_1 \leq r_1 = \text{rank} B_1(x)$. Moreover the rows of $\hat{B}_1(x)\delta(x,z)$ are linearly dependent on the rows of $\tilde{B}_1(x)\delta(x,z)$, since the rows of $\hat{B}_1(x)$ are linearly dependent on the rows of $\tilde{B}_1(x)$. Thus we can permute the entries of $\xi_0(x,z)$ in such a way that the first $\tilde{r}_1$ rows of $B_1(x)\delta(x,z)$ are linearly independent and the last $(p_0 - r_1)$ entries of $\xi_0(x,z)$ consist of $\hat{\phi}_0(x)$, i.e. we can write (C.1) as

$$\begin{pmatrix} \dot{\tilde{\sigma}}_0(x,z) \\ \dot{\hat{\sigma}}_0(x,z) \\ \dot{\hat{\phi}}_0(x,z) \end{pmatrix} = \begin{pmatrix} \tilde{A}_{\sigma 1}(x) + \tilde{B}_{\sigma 1}(x)\gamma(x,z) \\ \hat{A}_{\sigma 1}(x) + \hat{B}_{\sigma 1}(x)\gamma(x,z) \\ \hat{A}_1(x) + \hat{B}_1(x)\gamma(x,z) \end{pmatrix} + \begin{pmatrix} \tilde{B}_{\sigma 1}(x)\delta(x,z) \\ \hat{B}_{\sigma 1}(x)\delta(x,z) \\ \hat{B}_1(x)\delta(x,z) \end{pmatrix} v \tag{C.2}$$

where $(\tilde{\sigma}_0^T, \hat{\sigma}_0^T)^T$ consists of the entries of $\tilde{\phi}_0$ and $\tilde{B}_{\sigma 1}(x)\delta(x,z)$ has full row rank $\tilde{r}_1$ in a neighborhood of (x_0, z_0) in M_0. Observe that $\begin{pmatrix} \tilde{B}_{\sigma 1}(x) \\ \hat{B}_{\sigma 1}(x) \end{pmatrix}$ has full row rank r_1 in a neighborhood of (x_0, z_0) in M_0. Thus, we can rewrite $\dot{\bar{\phi}}_0(x,z)$ in (C.2) as

$$\dot{\bar{\phi}}_0(x,z) = \hat{A}_1(x) + \hat{B}_1(x) \begin{pmatrix} \tilde{B}_{\sigma 1}(x) \\ \hat{B}_{\sigma 1}(x) \end{pmatrix}^+ [\begin{pmatrix} \dot{\bar{\sigma}}_0(x,z) \\ \dot{\sigma}(x,z) \end{pmatrix} - \begin{pmatrix} \tilde{A}_{\sigma 1}(x) \\ \hat{A}_{\sigma 1}(x) \end{pmatrix}] \tag{C.3}$$

where $\begin{pmatrix} \tilde{B}_{\sigma 1} \\ \hat{B}_{\sigma 1} \end{pmatrix}^+$ is a right inverse of $\begin{pmatrix} \tilde{B}_{\sigma 1} \\ \hat{B}_{\sigma 1} \end{pmatrix}$.

Let $(\tilde{B}_{\sigma 1}(x)\delta(x,z))^+$ be a right inverse of $\tilde{B}_{\sigma 1}(x)\delta(x,z)$. Let $v = -(\tilde{B}_{\sigma 1}(x)\delta(x,z))^+\tilde{A}_{\sigma 1}(x)$. Then we find from (C.2):

$$\begin{pmatrix} \dot{\bar{\sigma}}_0(x,z) \\ \dot{\bar{\sigma}}_0(x,z) \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{A}_{\sigma 1}(x) - \hat{B}_{\sigma 1}(x)(\tilde{B}_{\sigma 1}(x)\delta(x,z))^+\tilde{A}_{\sigma 1}(x) \end{pmatrix} =: \begin{pmatrix} 0 \\ \bar{\sigma}_1(x,z) \end{pmatrix} \tag{C.4}$$

Then M_1 is given as $M_1 = \{(x,z) \mid \Phi_0(x) = 0, \dot{\bar{\phi}}_0(x,z) = 0, \bar{\sigma}_1(x,z) = 0\}$. Since $\dot{\bar{\sigma}}_0(x,z) = 0$ on M_1 and (obviously) $\begin{pmatrix} \tilde{B}_{\sigma 1}(x) \\ \hat{B}_{\sigma 1}(x) \end{pmatrix}^+ \begin{pmatrix} \tilde{A}_{\sigma 1}(x) \\ \hat{A}_{\sigma 1}(x) \end{pmatrix} = \tilde{B}_1^+(x)\tilde{A}_1(x)$, we find from (C.3) that on M_1

$$\dot{\bar{\phi}}_0(x,z) = \hat{A}_1(x) - \hat{B}_1(x)\tilde{B}_1^+(x)\tilde{A}_1(x) = \bar{\phi}_1(x) \tag{C.5}$$

Thus, $M_1 = \{(x,z) \mid \Phi_0(x) = 0, \bar{\phi}_1(x) = 0, \bar{\sigma}_1(x,z) = 0\}$. Assume that $(\bar{\phi}_1^T(x), \bar{\sigma}_1^T(x,z))^T$ has constant rank $\tilde{s}_1$ in a neighborhood of (x_0, z_0) in M_1. Obviously, $\tilde{s}_1 > s_1$. Since $\bar{\phi}_1(x)$ has constant rank s_1 in a neighborhood of (x_0, z_0) in M_1, we can permute the entries of $\bar{\phi}_1(x)$ and $\bar{\sigma}_1(x,z)$ such that $(\bar{\phi}_{11}(x), \cdots, \bar{\phi}_{1s_1}(x), \bar{\sigma}_{11}(x,z), \cdots, \bar{\sigma}_{1\tilde{s}_1 - s_1}(x,z))$ are independent on M_1. Define $\Phi_1(x) = (\Phi_0^T(x), \bar{\phi}_{11}(x), \cdots, \bar{\phi}_{1s_1}(x))^T$, $\Psi_1(x,z) = (\bar{\sigma}_{11}(x,z), \cdots, \bar{\sigma}_{1\tilde{s}_1 - s_1}(x,z))^T$. Then we find $M_1 = \{(x,z) \mid \Phi_1(x) = 0, \Psi_1(x,z) = 0\}$. Hence our claim also holds for $k = 1$. Using similar arguments as above, we can prove that our claim holds for $k = 0, \cdots, k^*$, which completes the proof. ∎

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