CWI Tracts

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CWI Tract 1

Surfaces with canonical hyperplane sections

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PREFACE

This volume is a reprint of my thesis. Except for the correction of a few typing errors, there is only one change with respect to the original version. To chapter V, a new section has been added, in which we discuss how and to what extent the theory concerning the moduli and period map, as explained in V.§5 for the double covers of $\mathbb{P}^2$, can be applied to the two types of singular quartics in $\mathbb{P}^3$, containing two simple elliptic singularities.

I would like te express my gratitude to prof.dr. J.P. Murre for his help and encouragement during the preparation of my thesis, and also to dr. C.A.M. Peters, especially for his advices how to turn the rough material into presentable mathematics.

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Dick Epema

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CHAPTER V

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INTRODUCTION

The subject of this thesis is the study of *projective algebraic surfaces* X *with canonical hyperplane sections*, in particular of those, which are birational to an irrational ruled surface. We say that X has canonical hyperplane sections, if it can be embedded in some projective space $\mathbb{P}^g$ in such a way, that a general hyperplane section C is a smooth canonically embedded curve. Then C has genus g and X is of degree 2g - 2 . Examples are smooth K3 surfaces embedded with a complete linear system, which are classically known to be the only smooth ones, and quartic surfaces in $\mathbb{P}^3$ with isolated singularities.

We will show, that a surface with canonical hyperplane sections can only be

(i) a K3 surface,

(ii) a rational surface with a minimally elliptic singularity (see [L] for a definition; here we only prove, that it is a singularity of genus 1),

(iii) a ruled surface over a curve of genus $q \geqq 1$ with a singularity of genus $q + 1$, or

(iv) a ruled surface over an elliptic curve with two simple elliptic singularities.

Moreover, all of these surfaces may contain additional rational double points. An example of such a rational surface is a quartic in $\mathbb{P}^3$ with a triple point. The cones over canonically embedded curves show, that ruled surfaces occur.

Now if L is a complete system of hyperplane sections on a K3 surface and if $C \in L$ is a smooth, irreducible curve, then L cuts on C its canonical system, because the canonical class of the surface itself is zero (and a K3 is regular). The general principle of the surfaces (ii) - (iv) is, that their canonical class is contained in the non-rational singularities. This means that in those cases, if $\pi: X' \to X$ is the minimal resolution of the singularities of X , there exists a canonical divisor on X' with support contained in the exceptional set of π . This divisor turns out to be strictly negative.

Because, as we will presently see, K3 and rational surfaces have been dealt with elsewhere, we will mainly be concerned with irrational ruled surfaces.

When X is birationally equivalent to a K3 surface, we will find it to be the image of a minimal K3 surface X' under the map associated to a complete linear system without base points. These have been extensively studied in [S-D1]. Now there exist K3 surfaces,which carry complete base-point free systems consisting of hyperelliptic curves. These systems represent the surface, perhaps with some rational curves contracted to rational double points, as a double cover of a rational surface of degree $g-1$ in $\mathbb{P}^g$. In particular, if $g = 2$ one gets in this way double covers of $\mathbb{P}^2$ branched along a curve of degree 6. Guided by this phenomenon, the defining property of our surfaces is formulated in such a way as to include a certain hyperelliptic analogon of "canonical hyperplane sections". In chapter III and IV we will see that this "hyperelliptic case" fits in a natural way into our treatment.

As to rational surfaces, there is the beautiful paper of P. Du Val, "On rational surfaces whose prime sections are canonical curves" ([DV], 1933), which is written in the classical Italian style and which is therefore at first not quite comprehensible to the modern reader. It gives an explicit procedure to construct all (generic) rational surfaces X with canonical hyperplane sections in a fixed $\mathbb{P}^g$. At the end of the article a list is given describing the linear systems on $\mathbb{P}^2$ transforming it into the desired surfaces X for $2 \leqq g \leqq 6$.

Though our method of construction used in chapters III, IV is different from Du Val's, it has the same underlying main idea. This consists in constructing linear systems L'' on minimal (rational/ruled) surfaces X'', such that after blowing up all their base points to get X', the strict transform L' of L'' is disjoint from a fixed (anti-)canonical divisor W' on X'. As a consequence, this W' is blown down by $\phi_{L'}$ (to non-rational singularities), and we find $X = \phi_{L'}(X')$. However, Du Val makes use of the successive adjoint systems of L'', whereas we try to find a suitable minimal model X'' of X', on which L'' has a simple form.

Besides a hint at ruled surfaces in [DV], the only other traces of surfaces with canonical hyperplane sections in the classical literature I know of,are casual remarks of Enriques in [En], p.250 and in [Co], cap.VIII.§41, p.184 only pertaining to K3 and rational surfaces.

The reason to investigate surfaces with canonical hyperplane sections was a question of prof. J.P. Murre as to their nature, especially when they acquire singularities. Together with prof. A. Conte he came across surfaces of this kind when they were filling in the gaps and rewriting in modern language the paper of G. Fano ([F], 1938)) on threefolds W whose hyperplane sections are Enriques surfaces ([C-M]). It turns out that such a W carries a linear system M of surfaces with canonical hyperplane sections (see [C-M], lemma I.3.7). Fano concluded that the members of M are K3 surfaces, which is shown to be true under some assumptions, assuring W to be sufficiently general, in [C-M], lemma I.4.12. However, by the time Fano wrote this, Du Val's paper [DV] had appeared a few years ago, and there seems to be no justification for his conclusion without a study of the singularities of the surfaces in question.

We will now give a quick overview of the way this thesis is organized. To obtain a better overall picture of its contents the reader is referred to the introductions to each chapter, or, for more details, each section separately.

In ch. I we start with a precise definition of surfaces with canonical hyperplane sections and a discussion of the hyperelliptic case. Then we gather partly well-known generalities on these surfaces and give some examples. From ch. II on we only consider irrational ruled surfaces. As we mentioned above, the minimal resolution X' carries an anticanonical divisor W', and a fortiori a relatively minimal model X" of X' does. The main point of ch. II is to determine all possibilities for anticanonical divisors on minimal ruled surfaces and to describe the implications of this information for the non-rational singularities of X .

Chapter III deals with the "one non-rational singularity case" and is really the heart of the matter. After having formulated the way of construction (III.1.3), we give an effective method to find in principle all (families of) surfaces X with canonical hyperplane sections in a fixed $\mathbb{P}^g$. If X is birational to a ruled surface over a curve of genus $q \geqq 2$, we list them all in terms of certain invariants for $2 \leqq g \leqq 10$. For elliptic ruled surfaces we carry out this construction in detail, but only for $g = 2$, i.e. for the double covers of $\mathbb{P}^2$ branched along a sextic, and for $g = 3$, which gives us a complete classification of normal quartics in $\mathbb{P}^3$ with a singularity of genus 2. However, we do not describe these surfaces up to projective equivalence, because more complicated transformations than projective ones are involved to get suitable equations. Ch. IV deals in a way strictly analogous to that of ch. III with elliptic ruled surfaces with two simple elliptic singularities. Here we find equations for all normal quartics with two non-rational singularities, again up to isomorphism and not only up to projective equivalence. In the last section of this chapter we have a look at the moduli of the double covers of $\mathbb{P}^2$. Let E be a fixed elliptic curve and let $j = j(E)$ be its j-invariant. Let $\bar{M}_j$ be the compactified moduli variety of surfaces with canonical hyperplane sections of genus 2 containing two simple elliptic singularities, which are birational to $E \times \mathbb{P}^1$. Then we prove that $\bar{M}_j \cong \mathbb{P}^1$, if $j \neq 0,1728$.

Finally, in ch. V we take the groundfield k to be the field $\mathbb{C}$ of complex numbers and study the mixed Hodge structures on the cohomology groups $H^i(X_0,\mathbb{C})$, $X_0 \subset X'$ the open part obtained by leaving out on X' the exceptional divisors of *only* the non-rational singularities of X. We investigate the period map $\rho\colon \bar{M}_j \to H_j$, H_j the moduli variety of polarized mixed Hodge structures (MHS's) occurring as a certain sub-MHS of the MHS's on $H^2(X_0,\mathbb{C})$, with $X_0 \subset X' \xrightarrow{\pi} X$ as parametrized by $\bar{M}_j$. It turns out, that $H_j \cong \mathbb{P}^1$, and that ρ is a morphism of degree 4, if $j \neq 0,1728$. We conclude with some remarks on moduli and period maps in two other special cases. Maybe we should say one word here about the terminology in this chapter. We will use the notion of "moduli variety" (of surfaces, MHS's) in a naive sense, that is, it is simply meant to be a variety which parametrizes the objects represented by its points in a natural way.

The last few years surfaces with canonical hyperplane sections, or at least surfaces very much alike, have been studied from different points of view, amongst which that of surface singularity theory. Restricting ourselves to those papers which deal with these surfaces in a way similar to the way we do, let us first mention [C-M]. This paper contains some information on surfaces with canonical hyperplane sections concerning their singularities, birational type and invariants (see [C-M], I.§1, and lemma I.3.8, cor.I.4.6 and lemma I.4.12). These results, together with Du Val's paper [DV], were the origin of this thesis. In fact, they first consider surfaces X with isolated singularities with the following property: if $\pi\colon X' \to X$ is the minimal resolution, then the canonical class of X' is *numerically equivalent* to a divisor with support contained in the exceptional set of π. It is not difficult to show that this is a weaker condition.

In his C.R.-note [M], J.Y. Mérindol considers normal analytic surfaces with trivial dualizing sheaf, a property shared by our surfaces (I.cor.5.4.d). It was him,who drew my attention to the possibilities for anticanonical divisors on minimal ruled surfaces (II.prop.2.1), and consequently to the number and type of the occurring singularities on irrational ruled surfaces of the required type (II.cor. 3.3). These anticanonical divisors can also be found without proof in [K], lemma 2.18, but type a.2 stated there, the 2-section on an elliptic ruled surface, does not exist. Because of the central role played by these anticanonical divisors we include a proof here. The idea for the least trivial part of it came from Mérindol. This concerns the case of ruled surfaces over elliptic curves, the associated rank 2 locally free sheaves of which are indecomposable. By the way, we could have done without these because of our way of construction in ch. III, IV. Also I.lemma 5.2.a,b, cor.5.4.a,b,c and II.prop.3.1. and their proofs were inspired by him.

Independently, Y. Umezu has been working on the subject, though in [U] the starting point is also "trivial dualizing sheaf". However, she proofs that if one excludes Abelian surfaces (and assumes normality),one gets surfaces with canonical hyperplane sections ([U], thm. 2). Furthermore, she obtains many of the results contained in our chapters I, II. Moreover, from the same author I received a preprint, "Quartic surfaces of elliptic ruled type", in which normal quartics X in $\mathbb{P}^3$ birational to elliptic ruled surfaces are studied. Here she arrives at the same possibilities for the set $\mathrm{Sing}(X)$ of singularities as we do in III.thm. 3.4.b and IV.thm. 2.3.b, finds the same equations for the quartics with two simple elliptic singularities we have in the second of these theorems, and gives only sample equations for those with one singularity of genus 2.

In [N], I. Naruki investigates isolated singularities of quartic surfaces by means of their defining equations. Finally, in [Ep] a summary of the results of chapters I-IV were published.

All varieties in ch. I-IV are defined over an algebraically closed field k of $\mathrm{char}(k) \neq 2,3$, unless specified otherwise. In ch. V, $k = \mathbb{C}$.

LIST OF SYMBOLS AND NOTATIONS

$\mathbb{C}$ - field of complex numbers

$C \cdot D$, D^2 - intersection number resp. self-intersection of divisor(classes) C,D on a smooth surface

$|D|$, $|\mathcal{O}_X(D)|$ - the complete linear system (on a variety X) of which D is a member

deg D - degree of a divisor D on a curve

deg X - degree of a variety X in a certain embedding

Div(X) - group of divisors on a smooth variety X

$E/\langle\pm \mathrm{id}\rangle$ - the smooth elliptic curve E modulo its subgroup of automorphisms consisting of plus and minus the identity with respect to a chosen origin for the addition on E

$E(\lambda)$ - the smooth elliptic curve which is a double cover of $\mathbb{P}^1$ branched in $0,1,\lambda$ and ∞

g_d^r - a linear system of dimension r and degree d on a curve

$H^i(\mathcal{F})$ - the i-th cohomology group of the coherent sheaf $\mathcal{F}$ if the base space is understood

$h^i(\mathcal{F})$ - $\dim_k H^i(\mathcal{F})$

$j(E)$ - j-invariant of the smooth elliptic curve E

$j(\lambda)$ - $j(E(\lambda))$

$j(\tau)$ - j-invariant of the smooth elliptic curve defined over $\mathbb{C}$ which is isomorphic to the torus $\mathbb{C}/\langle 1,\tau\rangle$

K_X - canonical divisor(class) of a smooth variety X

$\sim$ - linear equivalence of divisors on a smooth variety, resp. of Weil divisors on a surface with isolated singularities; in the last chapter also (co-)homological equivalence

$\mathcal{N}_{Y/X}$ - normal sheaf of a nonsingular subvariety Y on a nonsingular variety X

$p_a(X)$ - arithmetic genus of a variety X

$p_g(X)$ - geometric genus of a smooth variety X resp. of a desingularization of X if X is singular

$\mathrm{Pic}(X)$	- group of Cartier divisors on a variety X modulo linear equivalence
$\mathbb{P}_X(\mathcal{E})$	- projective space bundle associated to a locally free coherent sheaf $\mathcal{E}$ on a variety X
$\mathbb{Q}$	- field of rational numbers
$q(X)$	- irregularity $h^1(\mathcal{O}_X)$ of a variety X
$R^i f_* \mathcal{F}$	- i-th direct image of a sheaf $\mathcal{F}$ relative to the morphism f
$S^m A$	- m-th symmetric power of a module A
$\mathrm{Sing}(X)$	- set of singular points of a variety X
$\mathrm{supp}(D)$, $\mathrm{supp}(\mathcal{F})$	- support of a divisor D resp. a sheaf $\mathcal{F}$
$\mathrm{Tr}_Y L$	- the trace on the subvariety Y of the linear system L , which is obtained by restricting L to Y
$\bar{\tau}$	- the complex conjugate of $\tau \in \mathbb{C}$
$\langle 1,\tau \rangle$, $\langle 2\pi i, 2\pi i\tau \rangle$	- the lattice in $\mathbb{C}$ generated over $\mathbb{Z}$ by 1 and τ (resp. by $2\pi i$ and $2\pi i\tau$)
ϕ_L	- the rational map from X to some projective space associated to the linear system L
$\chi(\mathcal{F})$	- the Euler characteristic of a coherent sheaf $\mathcal{F}$
ω_X^0	- the dualizing sheaf of a variety X
$\mathbb{Z}$	- the integers

CHAPTER I

PROPERTIES OF SURFACES WITH CANONICAL HYPERPLANE SECTIONS

1 DEFINITION

Let X be a projective algebraic surface.

DEFINITION 1.1. X is called a *surface with canonical hyperplane sections* if either:

(a) there exists an embedding $i: X \hookrightarrow \mathbb{P}^g$, $g \geqq 3$, such that a general hyperplane section C of $i(X)$ is a canonically embedded curve in that hyperplane, or

(b) X contains a g-dimensional linear system L , $g \geqq 2$, a general element of which is a smooth hyperelliptic curve C of genus g , such that $\mathrm{Tr}_C L = |K_C|$ and such that the corresponding rational map $h = \phi_L: X \to \mathbb{P}^g$ is a finite morphism of degree 2 onto a surface $\bar{X}$ in $\mathbb{P}^g$.

Already here we warn the reader that from §2 of this chapter on we will exclude from this definition a certain class of surfaces belonging to (b), which behave quite differently from the other ones; for this see the last part of §2 following the proof of prop. 2.1.

We note that in (a) of the definition, in which case we will simply write X instead of $i(X)$, a general hyperplane section C is canonically embedded in a $\mathbb{P}^{g-1}$, so $p_g(C) = g$. Furthermore, denoting the system of hyperplane sections in (a) also by L , (a) is equivalent to requiring that a general hyperplane section C of X is smooth and $\mathrm{Tr}_C L = |K_C|$. As such, (b) is the hyperelliptic analogon of (a). Also in (b), the curves of L will be called hyperplane sections. Because both in (a) and in (b), $\mathrm{Tr}_C L = |K_C|$ is complete, L is complete in both cases.

In this whole chapter from now on X will stand for a surface with canonical hyperplane sections. Let $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^g}(1)$ resp. $h^*\mathcal{O}_{\mathbb{P}^g}(1)$. Then

$\mathcal{O}_X(1) \otimes \mathcal{O}_C \cong \mathcal{O}_X(C) \otimes \mathcal{O}_C \cong \mathcal{O}_C(K_C)$.

The above definition implies that X has at most isolated singularities. Let $\pi: X' \to X$ be the minimal resolution of the singularities of X , and let $\pi^* C = C'$ be the inverse image of a general hyperplane section. Because π is an isomorphism in a neighbourhood of C , also on X' we have $\mathcal{O}_{X'}(C') \otimes \mathcal{O}_{C'} \cong \mathcal{O}_{C'}(K_{C'})$, so using the adjunction formula we get $\mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_{C'} \cong \mathcal{O}_{C'}$. If X is smooth, so X = X' , and if C is non-hyperelliptic we will show in prop. 3.1 that this indeed implies $K_X \sim 0$.

The reason why in this case only K3 surfaces appear and Abelian surfaces do not, lies in the fact that we want the hyperplane sections of X to be embedded by their complete canonical system. If however one embeds an Abelian surface X with a complete linear system L , then for any $C \in L$ one will find that $\mathrm{Tr}_C L$ is of codimension 2 (= the irregularity of X in this case) in, instead of coinciding with, the corresponding complete system on C .

2 SURFACES WITH HYPERELLIPTIC CANONICAL HYPERPLANE SECTIONS

In §3 we will determine, among other things, the invariants $h^i(\mathcal{O}_X)$, i = 1,2 , of surfaces X with canonical hyperplane sections. As the method employed there works only in case these are non-hyperelliptic, our aim in this section is to answer the following question:

(*) *Let* X *be a surface with hyperelliptic canonical hyperplane sections,* $\bar{X} = h(X) \subset \mathbb{P}^g$. *Because* $\mathcal{O}_{\mathbb{P}^g}(1) \otimes \mathcal{O}_{\bar{X}} = \mathcal{O}_{\bar{X}}(1)$ *is very ample and* $h: X \to \bar{X}$ *is finite,* $\mathcal{O}_X(1) = h^* \mathcal{O}_{\bar{X}}(1)$ *is ample* ([H], III.Ex. 5.7). *Does there exist an* $m \geqq 2$ *such that* $\mathcal{O}_X(m) = \mathcal{O}_X(mC)$ *is very ample and such that* $\phi_{|mC|}(X)$ *is a surface with (non-hyperelliptic) canonical hyperplane sections?*

It turns out that there are surfaces for which the answer is "no". For those surfaces, for which we prove in prop. 2.1 the answer to be "yes", the situation is as one would expect: if g = 2 , then we must take $m \geqq 3$, if $g \geqq 3$ then $m \geqq 2$ will do. (See prop. 2.1 and remark 2.1.1).

Before we can start proving this, we have to look a little closer at the surfaces $\bar{X}$ arising as the image of X in $\mathbb{P}^g$ in the hyperelliptic case. Because on X, $C^2 = \deg K_C = 2\,p_g(C) - 2 = 2g - 2$ and h is of degree 2, $\deg(\bar{X}) = g - 1$. Now surfaces of degree $g - 1$ in $\mathbb{P}^g$, not lying in a hyperplane, are completely classified and each of them must be one of the following (see [S-D1], thm. 1.10):

(i) $\mathbb{P}^2$;

(ii) the Veronese surface in $\mathbb{P}^5$, i.e. $\phi_2(\mathbb{P}^2)$, $\phi_2\colon \mathbb{P}^2 \to \mathbb{P}^5$ the Veronese embedding of $\mathbb{P}^2$ of degree 2;

(iii) the cones in $\mathbb{P}^g$ over a rational normal curve of degree $g - 1$ in $\mathbb{P}^{g-1}$, $g \geqq 3$;

(iv) a smooth rational scroll.

As to the surface mentioned under (iv) we use the following notation and facts, for the greatest part taken from [H], V.§2:

- $F_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, $n \geqq 0$, is a ruled surface over $\mathbb{P}^1$;
- $p\colon F_n \to \mathbb{P}^1$ is the natural projection;
- f is a fibre of p ;
- C_0 is the unique section of p with negative self-intersection if $n > 0$, or, if $n = 0$, a section with $C_0^2 = 0$. Then in all cases, $C_0^2 = -n$;
- $\operatorname{Pic} F_n \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by C_0 and f ; the intersection product of divisors on F_n is defined by $C_0^2 = -n$, $C_0 \cdot f = 1$, $f^2 = 0$;
- $K_{F_n} \sim -2C_0 - (n+2)f$;
- if $a,b \geqq 0$, then

$$H^0(\mathcal{O}_{F_n}(aC_0+bf)) = \bigoplus_{i=0}^{a} P_{b-in}\,, \tag{1}$$

P_ℓ being the vectorspace of homogeneous polynomials of degree ℓ in two variables if $\ell \geqq 0$, $P_\ell = (0)$ if $\ell < 0$; consequently, if $b \geqq an$, then

$$h^0(\mathcal{O}_{F_n}(aC_0+bf)) = \sum_{i=0}^{a} (b-in+1) \qquad (1');$$

$$- h^i(\mathcal{O}_{F_n}(-C_0+bf)) = 0 \ , \forall b \ , \ i = 0,1,2 \qquad (2)$$

(for $i = 0$ this is clear, for $i = 2$ use Serre duality and then, for $i = 1$, apply the Riemann-Roch theorem for surfaces);

- if $a \geqq 2$, $b \geqq 0$ then $2C_0$ is not a fixed part of $|aC_0+bf|$ on F_n iff

$$b \geqq (a-1)n \qquad (3)$$

- a surface of type (iv) is the image of an F_n, $n \geqq 0$, under the embedding associated to the system $|C_0+bf|$, with $b > n$. The fibres f are mapped to lines in $\mathbb{P}^g$. In this case we have the relation (cf. [H], V.cor. 2.19):

$$2b - n + 1 = g . \qquad (4)$$

Of course, in all four cases (i),...,(iv) a smooth hyperplane section of the surface, being a curve of degree $g - 1$ spanning a $\mathbb{P}^{g-1}$, is a smooth rational curve.

We now recall a few facts concerning double covers of surfaces. So let $g: Y \to Z$ be a double cover of a smooth surface Z branched along the reduced but not necessarily smooth curve B. Then:

- $g_*\mathcal{O}_Y \cong \mathcal{O}_Z \oplus \mathcal{O}_Z(-F)$ for some divisor (class) F on Z such that $B \sim 2F$;

- for every invertible sheaf L on Y,

$$h^i(Y,L) = h^i(Z,g_*L) \quad , i = 0,1,2 \tag{5}$$

- let $\mathcal{O}_Z(1)$ be a very ample invertible sheaf on Z ,and let $\mathcal{O}_Y(k) = \mathcal{O}_Y \otimes g^*\mathcal{O}_Z(k)$, $k \in \mathbb{Z}$. Then the projection formula gives

$$g_*\mathcal{O}_Y(k) \cong g_*\mathcal{O}_Y \otimes \mathcal{O}_Z(k) \quad , \tag{6}$$

and so (5), (6) and the formula for $g_*\mathcal{O}_Y$ give:

$$h^i(\mathcal{O}_Y(k)) = h^i(\mathcal{O}_Z(k)) + h^i(\mathcal{O}_Z(-F)\otimes\mathcal{O}_Z(k)) \text{ for } i = 0,1,2. \tag{7}$$

In prop. 2.1 we will answer question (*) posed in the beginning of this section, at least when $\bar{X}$ is not a cone (type (iii)). In that case we will content ourselves with two examples, the first of which we will come across later (see examples 4.3, 4).

PROPOSITION 2.1. *Let* X *be a surface with hyperelliptic canonical hyperplane sections* C *of genus* $g \geq 2$, *and let* $\bar{X} = h(X) \subset \mathbb{P}^g$. *Then:*

(a) *if* $g = 2$, *i.e. if* X *is a double cover of* $\bar{X} = \mathbb{P}^2$, $\phi_{|2C|} = \phi_2 \circ \phi_{|C|}$, $\phi_2\colon \mathbb{P}^2 \to \mathbb{P}^5$ *the Veronese embedding of* $\mathbb{P}^2$ *in* $\mathbb{P}^5$, *so* $\phi_{|2C|}$ *represents* X *as a double cover of the Veronese surface in* $\mathbb{P}^5$, *and* $\phi_{|3C|}$ *is an isomorphism,* $\phi_{|3C|}(X)$ *being a surface with (non-hyperelliptic) canonical hyperplane sections in* $\mathbb{P}^{10}$;

(b) *if* $g = 5$ *and* $\bar{X} = \phi_2(\mathbb{P}^2)$ *is the Veronese surface in* $\mathbb{P}^5$, *then via* $\phi_2^{-1} \circ \phi_{|C|}\colon X \to \mathbb{P}^2$, *the surface* X *belongs to the ones mentioned in* (a);

(c) *if* $g \geq 3$ *and* $\bar{X}$ *is a smooth rational scroll isomorphic to* F_n , $n \geq 0$, *let* $B \subset \bar{X}$ *be the branch curve of* $\phi_{|C|}\colon X \to \bar{X}$. *Let* $B \sim 2aC_0 + 2df$ *on* F_n , $a,d \geq 0$. *Then we can assume we are in one of the following situations:*

(i) $a = 0$, $d = g + 1$, *and* X *is isomorphic to a smooth minimal ruled surface over a curve* Γ *of genus* g , *so* $h^1(\mathcal{O}_X) = g$ *and* $h^2(\mathcal{O}_X) = 0$;

(ii) $a = 1$, $d = g + 1 + n - b$, *and* $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$;

(iii) $a = 2$, $d = n + 2$ *and* $0 \leq n \leq 4$. *In this case* $\phi_{|2C|}$ *maps* X *isomorphically onto a surface with (non-hyperelliptic) canonical hyperplane sections in* $\mathbb{P}^{4g-3}$;

(iv) $a = 3$, $d = 0$, $n = 0$, $g = 5$, *and* $X \cong \Gamma \times \mathbb{P}^1$, Γ *a smooth curve of genus* 2, *so* $h^1(\mathcal{O}_X) = 2$, *and* $h^2(\mathcal{O}_X) = 0$.

PROOF. (a) Let z_0, z_1, z_2 be homogeneous coordinates on $\mathbb{P}^2$, let $\ell \subset \mathbb{P}^2$ be a general line, $C = h^{-1}(\ell)$ and let $B \subset \mathbb{P}^2$ be the branch curve of $h: X \to \mathbb{P}^2$. Because $p_g(C) = g = 2$, we must have $B \cdot \ell = 2g + 2 = 6$, so B is a (reduced) curve of degree 6. Let $b(z) = 0$ be an equation of B, and let $h_i = h^*(z_i) \in H^0(X, \mathcal{O}_X(C))$, $i = 0,1,2$. The h_i form a basis of this space and $h: X \to \mathbb{P}^2$ is defined by

$$(z_0, z_1, z_2) = (h_0, h_1, h_2) .$$

Now in the notation for double covers preceding the proposition, $\mathcal{O}_Z(-F) \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ because $\deg(B) = 6$. Applying formula (7) with $k = 2,3$ we get $h^0(\mathcal{O}_X(2C)) = 6$ and $h^0(\mathcal{O}_X(3C)) = 11$.

So $h^0(\mathcal{O}_X(2C)) = h^0(\mathcal{O}_{\mathbb{P}^2}(2))$ which indeed implies that $\phi_{|2C|} = \phi_2 \circ \phi_{|C|}$.

As $H^0(X, \mathcal{O}_X(3C))$ contains all ten forms of degree 3 in the h_i and $\sqrt{b}(z)$ and these elements are independent, they form a basis, so $\phi_{|3C|}: X \to \mathbb{P}^{10}$ is given by:

$$(w_0, \ldots, w_{10}) = (h_0^3, h_0^2 h_1, \ldots, h_2^3, \sqrt{b}) ,$$

showing that $\phi_{|3C|}$ is an isomorphism.

Finally, to see that $\phi_{|3C|}(X)$ has canonical hyperplane sections, we have to show that if $C_3 \in |3C|$ is a smooth hyperplane section of this surface and if $M = \mathrm{Tr}_{C_3}|3C|$, $M = |K_{C_3}|$. Because $\phi_{|3C|}$ is an isomorphism, this will also imply that C_3 is non-hyperelliptic.

Now first of all, if $C_3' = \pi^{-1}(C_3) \cong C_3$ on the minimal resolution X', then $K_{X'} \cdot C_3' = 3K_{X'} \cdot C' = 0$ because $\mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_{C'} \cong \mathcal{O}_{C'}$, so using the adjunction formula and the fact that $(C')^2 = C^2 = 2$, we get $p_g(C_3) = p_a(C_3') = 1 + \frac{1}{2}C_3' \cdot (C_3' + K_{X'}) = 1 + \frac{1}{2} \cdot (3C')^2 = 10$.

Secondly, $\deg M = C_3^2 = 9\,C^2 = 18 = 2p_g(C_3) - 2 = \deg K_{C_3}$ and thirdly $\dim M = \dim|3C| - 1 = 9 = p_g(C_3) - 1$. Together this says that $M = |K_{C_3}|$.

(b) Is clear.

(c) Let now $\bar{X} = h(X) \cong F_n$, which we call F now and for which we use the notation introduced above, $n \geq 0$. Let $B \subset F$ be the (reduced) branch curve of $h: X \to \bar{X}$. As B is an even element in $\mathrm{Pic}(F)$,

$B \sim 2aC_0 + 2df$ for some $a,d \geq 0$.

Let $\Gamma \in |C_0+b\cdot f|$ be a smooth hyperplane section of $\bar{X}$. Because Γ is rational, $h^{-1}(\Gamma) = C$ is a double cover of Γ and $p_g(C) = g$, we must have $B\cdot\Gamma = 2g + 2$, so $(2aC_0+2df)\cdot(C_0+bf) = -2an + 2d + 2ab = 2g + 2$, i.e.,

$$a(b-n) + d = g + 1 . \tag{8}$$

Now if $a = 0$, $d = g + 1$, so $B \sim (2g+2)\cdot f$ and B is a curve consisting of $2g + 2$ different fibres on F . Then the double cover X of $\bar{X} \cong F$ is of course a smooth minimal ruled surface over a smooth curve Γ of genus g , Γ being isomorphic to the double cover of any section on F branched in the points of intersection with B . This gives (i).

If $a = 1$, then by (8), $d = g + n + 1 - b$. In this case $h_*\mathcal{O}_X \cong \mathcal{O}_F \oplus \mathcal{O}_F(-\frac{1}{2}B) \cong \mathcal{O}_F \oplus \mathcal{O}_F(-C_0-df)$, so by (5) $h^i(\mathcal{O}_X) = h^i(h_*\mathcal{O}_X) = h^i(\mathcal{O}_F) + h^i(\mathcal{O}_F(-C_0-df)) = 0$ for $i = 1,2$, using the fact that F is smooth, rational, so $h^i(\mathcal{O}_F) = 0$, $i = 1,2$ and formula (2). This is (ii).

Let now $a \geq 2$. Because B is reduced, $|2aC_0+2df|$ does not contain $2C_0$ as a fixed component, and in this case formula (3) reads

$$2d \geq (2a-1)n . \tag{9}$$

By (8) and (4), $d = g + 1 + a(n-b) = 2b - n + 2 + a(n-b) = (a-1)n + (2-a)b + 2$, which gives, because $a \geq 2$ and $b \geq n + 1$:

$$d \leq (a-1)n + (2-a)(n+1) + 2 = n - a + 4 . \tag{10}$$

Combining (9) and (10) we get

$$(2a-1)n \leq 2d \leq 2n - 2a + 8 ,$$

so $(2a-3)n \leq 8 - 2a$. This implies $8 - 2a \geq 0$, so $a \leq 4$ and also, if $a = 3$ or 4 then $n = 0$, and if $a = 2$ then $n \leq 4$.

Let's now examine these different cases.

If $a = 4$ and $n = 0$, by (8) $d = g + 1 - 4b$ and by (4) $2b = g - 1$, so $d = -g + 3 \geq 0$. This is only possible if $d = 0$, $g = 3$, so $b = 1$. But, as $n = 0$ and $b = 1$, $\bar{X} \cong \mathbb{P}^1 \times \mathbb{P}^1$, embedded in $\mathbb{P}^3$ by $|C_0+f|$, so

C_0 and f play the same rôle and we can interchange them to get $a = 0$, $d = 4$, so we can forget about this case.

If $a = 3$, $n = 0$, then $d = g + 1 - 3b$ and $2b = g - 1$, so $d = -\frac{1}{2}g + 2\frac{1}{2} \geqq 0$ and so $g = 3$ or 5. If $g = 3$, $d = 1$ and $b = 1$, so again we can interchange a and d to get $a = 1$ and $d = 3$. If $g = 5$ then $b = 2$, $d = 0$, so X is the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along $2a = 6$ different sections equivalent to C_0, so indeed $X \cong \Gamma \times \mathbb{P}^1$, $p_g(\Gamma) = 2$. This gives (iv).

Finally let $a = 2$. Then by (4) and (8) $d = n + 2$, so $B \sim 4C_0 + 2(n+2)\cdot f \sim -2K_F$.

To prove c(iii) we only have to show that

(α) for every $m \geqq 2$ the natural map $S^m H^0(X,\mathcal{O}_X(2)) \to H^0(X,\mathcal{O}_X(2m))$ is surjective, and

(β) $\dim|2C| = 4g - 3$ and if $C_2 \in |2C|$ is a smooth curve, then $\mathrm{Tr}_{C_2}|2C| = |K_{C_2}|$.

For then by (α), because $\phi_{|2mC|}$ is an isomorphism for $m >> 0$, $\phi_{|2C|}$ already is, and (β) says that $\phi_{|2C|}(X)$ is a surface in $\mathbb{P}^{4g-3}$ with canonical hyperplane sections.

To prove (α), let $m \geqq 2$. Writing for short $\mathcal{O}_F(aC_0+bf) = \mathcal{O}_F(a,b)$, we find, using (6), $h_*\mathcal{O}_X(2) \cong (\mathcal{O}_F \oplus \mathcal{O}_F(-\frac{1}{2}B)) \otimes \mathcal{O}_F(2) \cong \mathcal{O}_F(2,2b) \oplus \mathcal{O}_F(0,2b-n-2)$ and in the same way $h_*\mathcal{O}_X(2m) \cong \mathcal{O}_F(2m,2mb) \oplus \mathcal{O}_F(2m-2,2mb-n-2)$. Using these isomorphisms, the fact that $H^0(X,L) \cong H^0(F,h_*L)$ for a sheaf L on X and (1), we get a commutative diagram with the natural maps α_1 and α_2

$$\begin{tikzcd}
S^m H^0(X,\mathcal{O}_X(2)) \arrow[r, "\alpha_1"] \arrow[d, phantom, "\cong" sloped] & H^0(X,\mathcal{O}_X(2m)) \arrow[d, phantom, "\cong" sloped] \\
S^m((\bigoplus_{i=0}^{2} P_{2b-in}) \oplus P_{2b-n-2}) \arrow[r, "\alpha_2"] & ((\bigoplus_{j=0}^{2m} P_{2mb-jn}) \oplus (\bigoplus_{k=0}^{2m-2} P_{2mb-n-2-kn}))
\end{tikzcd}$$

Looking at the indices in the lower row, one sees that α_2 is surjective, so α_1 is, which proves (α).

As to (β), $\dim|2C| = h^0(\mathcal{O}_X(2)) - 1 = h^0(h_*\mathcal{O}_X(2)) - 1 = h^0(\mathcal{O}_F(2,2b)) + h^0(\mathcal{O}_F(0,2b-n-2)) - 1 = 8b - 4n + 1$ by (1), and using (8) with $a = 2$ and $d = n + 2$, this equals $4g - 3$.

Let $C_2 \in |2C|$ be a smooth curve, and let $M = \mathrm{Tr}_{C_2}|2C|$. Then dim M = dim|2C| - 1 = 4g - 4 , deg M = $(2C)^2$ = 4(2g-2) and with the same method as in (a) one computes $p_g(C_2)$ = 4g - 3 . As in (a), together this says M = $|K_{C_2}|$, which proves (β).

REMARK 2.1.1. We will see later, with the help of prop. 3.1, that for the surfaces mentioned in c(i), (ii), (iv) of the above proposition, the answer to (*) is "no", for they have the wrong invariants $h^i(\mathcal{O}_X)$, i = 1,2 .

REMARK 2.1.2. In fact, prop. 2.1.c is already contained in [DV], and only says in modern language what is stated there in the first part of "2. Two-sheeted surfaces", p. 3/4. Also Du Val excludes the surfaces of c(i), (ii) from his considerations and, according to me, overlooks the possibility of c(iv).

In [R1] , where double covers of the surfaces F_n which are K3 surfaces are studied, M. Reid arrives in the same context at the bound n ≦ 4 of prop. 2.1.c(iii).

As to surfaces with hyperelliptic canonical hyperplane sections, we will in the sequel only concern ourselves with those for which the answer to (*) is "yes".

*Therefore from now on "*X *is a surface with canonical hyperplane sections" will mean either that* X *satisfies def. 1.1.a or that* X *satisfies def. 1.1.b and the answer to* (*) *is "yes".*

REMARK 2.1.3. Of course this restriction we make in regard to the original definition means that of the surfaces which satisfy def. 1.1.b we only consider those which, via a multiple of the original linear system L , also satisfy def. 1.1.a.

This implies that in order to prove an abstract property of our surfaces, i.e. not depending on the embedding i or the map h , it is enough to show that the property holds for the surfaces of def. 1.1.a.

3 INVARIANTS OF SURFACES WITH CANONICAL HYPERPLANE SECTIONS

In this section we will establish the first properties of surfaces with canonical hyperplane sections.

PROPOSITION 3.1. *Let* X *be a surface with canonical hyperplane sections of genus* $g \geqq 2$, *and let* $\pi: X' \to X$ *be the minimal resolution of its singularities. Then:*

(a) *if the hyperplane sections of* X *are non-hyperelliptic,* $\deg(X) = 2g - 2$;

(b) $h^1(\mathcal{O}_X) = 0$ *and* $h^2(\mathcal{O}_X) = 1$, *so* $p_a(X) = 1$;

(c) X *is normal, and if the hyperplane sections of* X *are non-hyperelliptic,* X *is projectively normal;*

(d) *the Kodairadimension* $\kappa(X')$ *equals* $-\infty$ *or* 0 . *In the second case* X' *is a minimal* K3 *surface.*

PROOF. (a) Of course, if X has non-hyperelliptic hyperplane sections, $\deg(X) = C^2 = \deg K_C = 2g - 2$.

(b) By remark 2.1.3 we can assume X to have non-hyperelliptic hyperplane sections. Consider the idealsheaf sequence of C on X , tensorized with $\mathcal{O}_X(n)$:

$$0 \to \mathcal{O}_X(n-1) \to \mathcal{O}_X(n) \to \mathcal{O}_C(n) \to 0 . \tag{1}$$

Because C is canonically embedded and hence projectively normal (see [S-D2], thm. 2.10), the natural map $H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_C(n))$ is surjective for all $n \geqq 0$, so the long exact cohomology sequence associated to (1) breaks up in two parts, the second being:

$$0 \to H^1(\mathcal{O}_X(n-1)) \to H^1(\mathcal{O}_X(n)) \to H^1(\mathcal{O}_C(n)) \to$$
$$\to H^2(\mathcal{O}_X(n-1)) \to H^2(\mathcal{O}_X(n)) \to 0 . \tag{2}$$

By [H], III.thm. 5.2.b, $H^i(\mathcal{O}_X(n)) = (0)$ for $i > 0$, $n >> 0$. Taking $i = 1$ and applying descending induction on n in (2) we find $H^1(\mathcal{O}_X(n)) = (0)$ for all $n \geqq 0$, in particular $h^1(\mathcal{O}_X) = 0$.

Furthermore, $H^1(\mathcal{O}_C(n)) = H^1(\mathcal{O}_C(nK_C)) = (0)$ for $n \geqq 2$, and $H^1(\mathcal{O}_C(1)) = H^1(\mathcal{O}_C(K_C)) \cong k$. Again applying descending induction on n , we see that $h^2(\mathcal{O}_X(n)) = 0$ if $n \geqq 1$, and finally, taking $n = 1$ in (2),

$h^2(\mathcal{O}_X) = h^1(\mathcal{O}_C(1)) = 1$.

(c) Let $n\colon \tilde{X} \to X$ be the normalization of X . Then we have the following exact sequence on X :

$$0 \to \mathcal{O}_X \to n_*\mathcal{O}_{\tilde{X}} \to \mathcal{F} \to 0 \ , \tag{3}$$

in which $\mathcal{F}$ is the (coherent) cokernel of the natural map $\mathcal{O}_X \to n_*\mathcal{O}_{\tilde{X}}$, and $\operatorname{supp}(\mathcal{F}) \subset \operatorname{Sing}(X)$. We know that X has isolated singularities, so let $\operatorname{Sing}(X) = \{x_1,\dots,x_r\}$. Denoting by $\mathcal{O}_i$ the local ring of x_i on X , the stalk $(n_*\mathcal{O}_{\tilde{X}})_{x_i} = \tilde{\mathcal{O}}_i$ is the normalization of $\mathcal{O}_i$ in the function field of X, $i = 1,\dots,r$.

Now on the one hand, because $H^0(\mathcal{O}_X) \cong H^0(n_*\mathcal{O}_{\tilde{X}}) \cong k$ and $H^1(\mathcal{O}_X) = (0)$ by (b), the long exact cohomology sequence of (3) gives $H^0(\mathcal{F}) = (0)$, on the other hand $H^0(\mathcal{F}) = \bigoplus_{i=1}^{r} (\tilde{\mathcal{O}}_i/\mathcal{O}_i)$, so $\tilde{\mathcal{O}}_i \cong \mathcal{O}_i$ for $i = 1,\dots,r$ and we conclude that X is normal.

Let us now assume that the hyperplane sections of X are non-hyperelliptic. To prove that X is projectively normal, by [H], II.Ex. 5.14.a it is enough to show that the natural map $H^0(\mathcal{O}_{\mathbb{P}^g}(n)) \to H^0(\mathcal{O}_X(n))$ is surjective for every $n \geqq 1$, which we know to be true for $n = 1$ (see §1, L is complete).

We now apply induction to n in the following commutative diagram, in which H is a general hyperplane in $\mathbb{P}^g$ and $C = X \cap H$:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}_X(n-1)) \arrow[r] & H^0(\mathcal{O}_X(n)) \arrow[r] & H^0(\mathcal{O}_C(n)) \arrow[r] & 0 \\
0 \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^g}(n-1)) \arrow[r] \arrow[u, "r_1"'] & H^0(\mathcal{O}_{\mathbb{P}^g}(n)) \arrow[r] \arrow[u, "r_2"'] & H^0(\mathcal{O}_H(n)) \arrow[r] \arrow[u, "r_3"'] & 0
\end{tikzcd} .$$

Here the top row is the first of the two parts in which the long exact cohomology sequence of (1) breaks up (see (a)), the bottom row is the analogous sequence for $H \subset \mathbb{P}^g$, exactness of which is a general property of projective spaces, and the r_i , $i = 1,2,3$ are the natural restriction maps.

Now r_3 is surjective because C is projectively normal, r_1 is surjective by induction, and so r_2 is surjective.

(d) Consider the ideal sheaf sequence of C' on X' , tensorized with $\mathcal{O}_{X'}(mK_{X'})$, $m > 0$. Notice that $\mathcal{O}_{X'}(mK_{X'}) \otimes \mathcal{O}_{C'} \cong \mathcal{O}_{C'}$.

$$0 \to \mathcal{O}_{X'}(-C'+mK_{X'}) \to \mathcal{O}_{X'}(mK_{X'}) \to \mathcal{O}_{C'} \to 0 .$$

Because $(-C'+mK_{X'})\cdot C' = -(C')^2 = 2 - 2g < 0$, and $|C'|$ does not have fixed components, $h^0(\mathcal{O}_{X'}(-C'+mK_{X'})) = 0$. Hence, the m-th plurigenus of X' , $P_m(X') = h^0(\mathcal{O}_{X'}(mK_{X'})) \leqq h^0(\mathcal{O}_{C'}) = 1$, and we conclude that $\kappa(X') = -\infty$ or 0 .

Now assume $\kappa(X') = 0$. Then X' is minimal. For if not, X' would contain an exceptional curve of the first kind E' . Let $m > 0$ be such that $|mK_{X'}| \neq \emptyset$, then $|mK_{X'}|$ consists of one divisor D' , and E' is a component of D' . Now $D'\cdot C' = mK_{X'}\cdot C' = 0$, so $E'\cdot C' = 0$, and E' is contracted to a point on X by π , contradicting the minimality of the resolution π .

Because X' is minimal and $\kappa(X') = 0$, there exists a smallest $m \geqq 1$, such that $mK_{X'} \sim 0$ (see [H], V.thm. 6.3). We will show that $m = 1$, so subsequently we only have to exclude the case of Abelian surfaces according to the classification of surfaces (see same reference).

If $m > 1$, let $f: Y \to X'$ be the m-fold cover of X' associated to $K_{X'}$. As $12K_{X'} \sim 0$ because X' is minimal of Kodaira dimension 0 , the only possible prime factors of m are 2 and 3, so, as we exclude these values for char(k) , f is étale. Because $\mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_{C'} \cong \mathcal{O}_{C'}$, the restriction of this cover to C' is trivial, $f^{-1}(C')$ consisting of m disjoint copies $C_1',\dots,C_m'$ of C' ; now on Y we have $(C_i')^2 = (C')^2 = 2g - 2$ and $C_i'\cdot C_j' = 0$, $i,j = 1,\dots,m$, $i \neq j$. But this contradicts the Hodge index theorem. So $m = 1$, and X' is a minimal K3 or Abelian surface.

If X' would be Abelian, X would be smooth. For if X would contain a singular point x , $\pi^{-1}(x)$ would, because X is normal by (c), consist of curves with negative self-intersection, but a minimal Abelian surface cannot contain such. But then $p_a(X) = -1$, which contradicts (b). So we end up with only minimal K3 surfaces.

4 EXAMPLES

In prop. 3.1.d in fact we proved that a surface X with canonical hyperplane sections is birationally equivalent to either a rational or ruled surface (when $\kappa(X') = -\infty$) or to a K3 surface. We will now give examples of these three types, both with non-hyperelliptic and hyperelliptic hyperplane sections.

EXAMPLES 4.1. Any smooth minimal K3 surface X, embedded by a complete system $|C|$ will do. For, as $K_X \sim 0$, the adjunction formula shows that $\mathcal{O}_X(C) \otimes \mathcal{O}_C \cong \mathcal{O}_C(K_C)$, and as $q(X) = h^1(\mathcal{O}_X) = 0$, $\mathrm{Tr}_C|C|$ on a general element of $|C|$ is complete.

Taking $g = 2$ for K3 surfaces one gets the double covers of $\mathbb{P}^2$ branched along a smooth sextic curve (or at least without too bad singularities). Then the 2-dimensional system of "hyperplane sections" is the system of the inverse images of lines in $\mathbb{P}^2$, which are curves of genus 2 and thus are hyperelliptic.

4.2. All surfaces of degree 4 in $\mathbb{P}^3$ with isolated singularities have plane curves of degree 4 as plane sections, and these are of course canonically embedded.

4.3. Let C be a non-hyperelliptic curve of genus $g \geqq 3$. Then the cone over the canonically embedded curve $\phi_{|K_C|}(C)$ is a surface of the required type. The corresponding surface X for a hyperelliptic C is a double cover of the cone over a rational normal curve of degree $g - 1$ in $\mathbb{P}^{g-1}$, which is the image of C under the canonical map. The branch curve consists of $2g + 2$ lines through the vertex of the cone, and a general "hyperplane section" of X is the double cover of a hyperplane section of the cone over the rational normal curve, branched over the points of intersection with the $2g + 2$ lines, and is of course isomorphic to C. One can show that for these surfaces the answer to question (*) of §2 is "yes" (for $g = 2$, see also III.thm. 2.2.(i),(iv)).

4.4. We will finally give an example of a rational surface with hyperelliptic canonical hyperplane sections. Let $Y = \mathbb{P}^2$ with homogeneous coordinates x_0, x_1, x_2, let C_3 be a smooth curve of degree 3 in $\mathbb{P}^2$, given by $f_3(x) = 0$, and assume that $P = (0,0,1)$ lies on C_3. Let L be the 3-dimensional system of curves of degree 5 with a triple point in P and 12 simple base points on C_3. These curves are clearly hyperelliptic. Let C_5, given by $f_5(x) = 0$, be an irreducible element of L with only singularity P. The rational map $\phi_L : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ is of degree 2 and is given by $(z_0, z_1, z_2, z_3) = (x_0^2 f_3, x_0 x_1 f_3, x_1^2 f_3, f_5)$, with z_i, $i = 0, \ldots, 3$, coordinates on $\mathbb{P}^3$, so the image $\phi_L(\mathbb{P}^2)$ is the quadric cone Q given by $z_1^2 = z_0 z_2$. The surface X with hyperelliptic canonical hyperplane sections of genus 3 is the double cover of Q, which can be obtained by first blowing up P and the other 12 base points on C_3 and then contracting the strict transform $\tilde{C}_3$, which is now exceptional. Note that on the blown-up surface, $\tilde{C}_3$ respresents the anticanonical class and is disjoint from the strict transform of a general curve of L, so indeed the "hyperplane sections" on X cut on each other their canonical class. One can show that in this case the answer to (*) of §2 again is "yes".

5 THE SINGULARITIES AND THE CANONICAL CLASS OF SURFACES WITH CANONICAL HYPERPLANE SECTIONS

We will now gather some information about the singularities that occur on surfaces with canonical hyperplane sections. First we recall some definitions and facts about normal surface singularities.

Let Y be a normal algebraic surface, $y \in Y$ a singularity and $\rho: Y' \to Y$ the minimal resolution of y.

DEFINITION 5.1. (a) The number $p_g(y) = \dim_k(R^1\rho_*\mathcal{O}_{Y'})_y$ is called *the (geometric) genus of* y.

(b) y is called *a rational singularity* if $p_g(y) = 0$.

(c) y is called *simple elliptic* if $\rho^{-1}(y)$ consists of one smooth elliptic curve. (cf. [S]).

We will use the following fact in lemma 5.2: if Z' is any (non-zero) positive divisor on Y', $\mathrm{supp}(Z') \subset \rho^{-1}(y)$, then $p_g(y) \geq p_a(Z')$. As we do not know a good reference for this we will include a proof here.

Let $U \subset Y$ be an affine neighbourhood of y not containing other singularities, let $V = \rho^{-1}(U)$ and let I be the idealsheaf of Z' on V. Because U is normal, $\rho_*\mathcal{O}_V = \mathcal{O}_U$. The dimension of the fibres of ρ is at most one, so $R^i\rho_*F = 0$ for $i > 1$, F a coherent sheaf on V. Because U is affine, $H^j(U,G) = (0)$ for $j > 0$, G a coherent sheaf on U. Consequently, in the Leray spectral sequence for a coherent sheaf F on V and for ρ we have $E_2^{pq} = H^p(U,R^q\rho_*F) = (0)$ if $p > 0$ or if $q > 1$, so only $E_2^{0,0}$ and $E_2^{0,1}$ are possibly non-zero and we have $H^0(V,F) \cong H^0(U,\rho_*F)$, $H^1(V,F) \cong H^0(U,R^1\rho_*F) = (R^1\rho_*F)_y$ and $H^2(V,F) = (0)$. Taking $F = \mathcal{O}_V$ we get $p_g(y) = h^1(\mathcal{O}_V)$ and taking $F = I$ we see that

$H^2(V,\mathcal{I}) = (0)$.

So the exact cohomology sequence associated to the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_V \to \mathcal{O}_{Z'} \to 0$ gives a surjection $H^1(\mathcal{O}_V) \to H^1(\mathcal{O}_{Z'})$ and thus $p_g(y) \geq h^1(\mathcal{O}_{Z'})$. On the other hand, $p_a(Z') = 1 - \chi(\mathcal{O}_{Z'}) = 1 - h^0(\mathcal{O}_{Z'}) + $ $+ h^1(\mathcal{O}_{Z'}) \leq h^1(\mathcal{O}_{Z'})$. Now combine these two inequalities.

We will now first prove the general lemma 5.2. In prop. 5.3 we show that our surfaces suffice the condition of this lemma.

<u>LEMMA 5.2.</u> *Let* Y *be a normal projective surface,* $\rho\colon Y' \to Y$ *the minimal resolution of the singularities of* Y *and suppose that* $\rho_* K_{Y'} \sim 0$ *as a Weil divisor on* Y . *Then:*

(a) $\dim|-K_{Y'}| = 0$ *; if* W' *is the unique anticanonical divisor on* Y' *, then* $\operatorname{supp}(W') = \rho^{-1}(\{y_1,\ldots,y_r\})$ *for certain singularities* $y_i \in Y$, $i = 1,\ldots,r$ *;*

(b) *a singularity* $y \in Y$ *is rational iff* $\rho^{-1}(y)$ *does not meet* supp(W') , *and then* y *is a rational double point;*

(c) *all singularities of* Y *are Gorenstein and even more is true: the dualizing sheaf of* Y , ω_Y^0 , *is isomorphic to* $\mathcal{O}_Y$.

<u>PROOF.</u> (a) Because $\rho_* K_{Y'} \sim 0$, there exists a canonical divisor $K_{Y'}$ on Y' with support in $\rho^{-1}(\operatorname{Sing}(Y))$. Let $K_{Y'} = \Sigma m_i F_i - \Sigma n_j G_j$ be the decomposition of this divisor in reduced, irreducible components with $m_i, n_j > 0$, $F_i \neq G_j$, for all i,j and let $F = \Sigma m_i F_i$, $G = \Sigma n_j G_j$.

Suppose $F \neq 0$. Because the intersection form on $\rho^{-1}(\operatorname{Sing}(Y))$ is negative definite, $F^2 < 0$, so there exists an i_0 such that $F \cdot F_{i_0} < 0$, say $i_0 = 1$. Also $F_1^2 < 0$ and $F_1 \cdot G \geq 0$, so $0 \leq p_a(F_1) =$ $= 1 + \frac{1}{2}F_1 \cdot (F_1 + K_{Y'}) = 1 + \frac{1}{2}F_1^2 + \frac{1}{2}F_1 \cdot F - \frac{1}{2}F_1 \cdot G$ is only possible if $F_1^2 = -1$ and $p_a(F_1) = 0$, i.e. F_1 is exceptional of the first kind. But this contradicts the minimality of ρ . So there exist positive anticanonical divisors, but because there is one with support in $\rho^{-1}(\operatorname{Sing}(Y))$ there can only be one in its linear equivalence class.

Let now $|-K_{Y'}| = \{W'\}$. For the second part of (a) we have to show that if $y \in \operatorname{Sing}(Y)$ and $\rho^{-1}(y)$ meets supp(W') , then $\rho^{-1}(y)$ does not contain curves which are not part of supp(W') . Assume the contrary. Then, $\rho^{-1}(y)$ being connected because Y is normal, we can assume that $\rho^{-1}(y)$ contains an irreducible curve E which is not part of supp(W') but intersects it, so $E \cdot W' > 0$. Hence $0 \leq p_a(E) = 1 + \frac{1}{2}E \cdot (E + K_{Y'}) =$

$= 1 + \frac{1}{2}E^2 - \frac{1}{2}E\cdot W'$ is only possible if $E^2 = -1$ and $p_a(E) = 0$, again contradicting the minimality of ρ .

(b) By (a), if $\rho^{-1}(y)$ meets $\mathrm{supp}(W')$, $\rho^{-1}(y)$ is a connected component of it. Let in that case W'_y be the part of W' supported on $\rho^{-1}(y)$. By the adjunction formula $p_a(W'_y) = 1$, so by the remark preceding this lemma y is not a rational singularity.

On the other hand, if $\rho^{-1}(y)$ does not meet $\mathrm{supp}(W')$, then the adjunction formula shows that $\rho^{-1}(y)$ consists of only smooth rational curves with self-intersection -2, so y must be a rational double point.

(c) A normal surface singularity is Gorenstein iff on a neighbourhood of the exceptional set in some resolution, the canonical divisor is linearly equivalent to a divisor with support in the exceptional set, which is here the case by (a).

Because all singularities are Gorenstein, the dualizing sheaf ω^0_Y is locally free, $\omega^0_Y \cong \mathcal{O}_Y(D)$ for some Cartier divisor D on Y . Let $U = Y\setminus \mathrm{Sing}(Y)$. By [R2], prop. 6 of the appendix to section 1, $\omega^0_Y|_U$ can be computed using differentials, and looking at $\rho^{-1}(U) \cong U$ on Y' we see that $\mathcal{O}_Y(D)|_U \cong \omega^0_Y|_U \cong \mathcal{O}_U$.

Because $\mathrm{codim}(Y\setminus U) \geqq 2$, this means that the Weil divisor associated to D under the natural map $\mathrm{Pic}(Y) \to C\ell(Y)$ from Cartier divisors modulo linear equivalence to Weil divisors modulo linear equivalence, is equivalent to zero. By [G-D], IV.cor. 21.6.10 the map $\mathrm{Pic}(Y) \to C\ell(Y)$ is injective because Y is normal. Hence $\omega^0_Y \cong \mathcal{O}_Y(D) \cong \mathcal{O}_Y$.

PROPOSITION 5.3. *Let* X *be a surface with canonical hyperplane sections and let* $\pi\colon X' \to X$ *be the minimal resolution of the singularities of* X . *Then* $\pi_* K_{X'} \sim 0$.

PROOF. By remark 2.1.3 we can assume that a general hyperplane section C of X is non-hyperelliptic. Let $C_m \in |m.C|$ be a smooth hypersurface section of degree m of X , $C'_m = \pi^{-1}(C_m)$, and let $L = \mathcal{O}_{X'}(C'_m) \otimes \mathcal{O}_{C'_m}$ be a locally free sheaf on C'_m , $m \geqq 1$. We will show by induction on m that $L \cong \mathcal{O}_{C'_m}(K_{C'_m})$, which for $m = 1$ is the hypothesis for X . In fact we will show that $h^0(L) = p_g(C'_m)$ and $\deg(L) = 2p_g(C'_m) - 2$.

Firstly, $p_g(C'_m) = p_a(C'_m) = 1 + \frac{1}{2}C'_m\cdot(C'_m+K_{X'}) = 1 + \frac{1}{2}(mC')^2 =$ $= 1 + \frac{1}{2}m^2\cdot(2g-2) = 1 + m^2(g-1)$, where $C' = \pi^{-1}(C)$ is the inverse image of a hyperplane section C of X of genus g .

Secondly, $\deg(L) = (C'_m)^2 = m^2\cdot(C')^2 = m^2(2g-2)$, which is what we want.

Thirdly, consider the ideal sheaf sequence of C' on X' tensorized with $\mathcal{O}_{X'}(m\cdot C')$, $m \geqq 1$:

$$0 \to \mathcal{O}_{X'}((m-1)C') \to \mathcal{O}_{X'}(m\cdot C') \to \mathcal{O}_{C'}(mK_{C'}) \to 0 \cdot$$

Because $C' \cong C$ is projectively normal in its canonical embedding, for every $m \geqq 1$ the corresponding sequence of global sections is exact, so $h^0(\mathcal{O}_{X'}(mC')) = h^0(\mathcal{O}_{C'}(mK_{C'})) + h^0(\mathcal{O}_{X'}((m-1)C')) = m(2g-2) + 1 - g + + h^0(\mathcal{O}_{X'}((m-1)C'))$. Now using induction on m we get $h^0(\mathcal{O}_{X'}(C'_m)) = h^0(\mathcal{O}_{X'}(mC')) = \Sigma_{i=2}^{m}(i(2g-2)+1-g) + h^0(\mathcal{O}_{X'}(C')) = (2g-2)(\frac{1}{2}m(m+1)-1) + + (m-1)(1-g) + (g+1) = m^2(g-1) + 2$. So $h^0(L) = h^0(\mathcal{O}_{X'}(C'_m)) - 1 = = m^2(g-1) + 1$, again the right value.

Because $\mathcal{O}_{X'}(C'_m) \otimes \mathcal{O}_{C'_m} \cong \mathcal{O}_{C'_m}$ the adjunction formula gives that the divisor $K_{X'}|_{C'_m} \sim 0$ on C'_m .

Let $D_0 = \pi_* K_{X'}$ as a Weil divisor on X . Of course also $D_0|_{C_m} \sim 0$ on C_m . We can now apply [Z], thm. 4, p. 570 or rather its consequence, stated on the same page, which says that if D_0 is a Weil divisor on a normal variety V , then the existence of non-negative divisors on C_m which are linearly equivalent to $D_0|_{C_m}$, for all $m \geqq m_0$, m_0 some positive integer, implies the existence of non-negative (Weil-) divisors on V which are linearly equivalent to D_0 (linear equivalence of Weil divisors which is possible because $\operatorname{codim}(\operatorname{Sing}(V)) \geqq 2$). In our case these conditions are satisfied, so there exists a divisor $D_1 \sim D_0$, $D_1 > 0$ on X and still $D_1|_{C_m} \sim 0$ on C_m . Because C_m is very ample on X , this says that $D_1 = 0$, and thus $\pi_* K_{X'} \sim 0$.

COROLLARY 5.4. *Let* X *be a surface with canonical hyperplane sections,* $\pi\colon X' \to X$ *its minimal resolution. Then:*

(a) X' *contains a unique positive anticanonical divisor* W' *;*

(b) *if* $x \in \operatorname{Sing}(X)$, *then either* x *is a rational double point and* $\pi^{-1}(x)$ *does not meet* $\operatorname{supp}(W')$, *or* x *is a non-rational singularity and* $\pi^{-1}(x)$ *is a connected component of* $\operatorname{supp}(W')$ *;*

(c) $\operatorname{supp}(W') = \cup\, \pi^{-1}(x)$, *the union being taken over the non-rational singularities* x *of* X *;*

(d) *all singularities of* X *are Gorenstein, and* $\omega_X^0 \cong \mathcal{O}_X$.

PROOF. By prop. 3.1.c X is normal. Now combine prop. 5.3 and lemma 5.2. (Of course in (b) $\pi^{-1}(x)$ is connected because X is normal).

In the sequel we will as in cor. 5.4.a denote the unique element in $|-K_{X'}|$ by W' .

COROLLARY 5.5. *Let* X *be a surface with canonical hyperplane sections and let* $\mathrm{Sing}(X) = \{x_1,\dots,x_r\}$. *Then:*

(a) *if* X *is a* K3 *surface,* X *can only contain rational double points as singularities;*

(b) *if* X *is rational,* $\sum_{i=1}^{r} p_g(x_i) = 1$;

(c) *if* X *is ruled over a curve of genus* q , $\sum_{i=1}^{r} p_g(x_i) = q + 1$.

PROOF. (a) By prop. 3.1.d, in this case X' is a minimal K3 surface, so $W' = 0$. Then use cor. 5.4.b.

(b),(c) These can both be obtained by computing dimensions in the following exact sequence, which one gets from the Leray spectral sequence for the sheaf $\mathcal{O}_{X'}$ and the morphism π . Note that $\pi_*\mathcal{O}_{X'} \cong \mathcal{O}_X$ because X is normal.

$$0 \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_{X'}) \to H^0(R^1\pi_*\mathcal{O}_{X'}) \to H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_{X'}) \to 0 \ .$$

REMARK 5.5.1. We will see what happens in case (c) later, see II.cor. 3.3.

CHAPTER II

RULED SURFACES WITH CANONICAL HYPERPLANE SECTIONS

In this chapter we study in more detail surfaces with canonical hyperplane sections, which are birationally equivalent to irrational ruled surfaces, in particular their non-rational singularities (cf. cor. 3.3), and also we make some preparations for the constructions in chapters III and IV.

1 PRELIMINARIES ON MINIMAL RULED SURFACES

We will first state the notation, conventions and facts we use for minimal smooth ruled surfaces, in which we follow [H], V.§2.

NOTATION 1.1. So let Y be such a surface, let $p: Y \to \Gamma$ be the natural projection onto the base curve and let $p_g(\Gamma) = q \geqq 1$. Then:

- $Y \cong \mathbb{P}_{\Gamma}(\mathcal{E})$, $\mathcal{E}$ a locally free sheaf of rank 2 on Γ ;

- we take $\mathcal{E}$ to be normalized, i.e. $H^0(\Gamma,\mathcal{E}) \neq 0$, but $H^0(\Gamma,\mathcal{E}\otimes\mathcal{L}) = (0)$ for every invertible sheaf $\mathcal{L}$ of negative degree on Γ (this normalized $\mathcal{E}$ need not be unique);

- sections of p on Y will, as abstract curves, often be identified with Γ ;

- let $\Lambda^2\mathcal{E} = \mathcal{O}_{\Gamma}(D)$, $D \in \mathrm{Div}(\Gamma)$, then there is an exact sequence

$$0 \to \mathcal{O}_{\Gamma} \to \mathcal{E} \to \mathcal{O}_{\Gamma}(D) \to 0 \; ; \qquad (1.1.1)$$

- the invariant e of Y is defined as $e = -\deg D$ (because $\deg D$

is equal to the minimum of the numbers S^2 , $S \subset Y$ a section, it is independent of the choice of normalization);

- let C_0 be a section of p such that $\mathcal{O}_Y(C_0) \cong \mathcal{O}_Y(1)$. Then $\mathcal{N}_{C_0/Y} = \mathcal{O}_\Gamma(D)$, so $C_0^2 = -e$ (the linear equivalence class of C_0 depends on the choice of the normalized $\mathcal{E}$; $\dim|C_0| > 0$ is only possible if $e \leqq 0$);

- if $\mathcal{E}$ is decomposable, $\mathcal{E} \cong \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$ with $e = -\deg D \geqq 0$; $\mathcal{E}$ is decomposable iff Y contains disjoint sections;

- if $\mathcal{E}$ is decomposable we will denote by C_1 a fixed section of p disjoint from C_0 . Then $\mathcal{N}_{C_1/Y} \cong \mathcal{O}_\Gamma(-D)$, so $C_1^2 = e$ (also the class of C_1 depends on $\mathcal{E}$; in general $\dim|C_1| > 0$);

- if $e > 0$, the normalized $\mathcal{E}$ is uniquely determined and so is the curve C_0, C_0 then being the unique section with negative self-intersection. If in this case $\mathcal{E}$ is decomposable, also the system $|C_1|$ is unique;

- if Y contains a section S with $S^2 < 2 - 2q$, then $Y \cong \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{N}_{S/Y})$, $e = -S^2 > 0$, and $S = C_0$; (1.1.2)

- $\mathrm{Pic}(Y) \cong \mathrm{Pic}(\Gamma) \oplus \mathbb{Z}$, generated by the fibres of p and C_0 ; we will denote a divisor (class) by $aC_0 + \Delta\cdot f$, $a \in \mathbb{Z}$, $\Delta \in \mathrm{Pic}(\Gamma)$;

- $K_Y \sim -2C_0 + (K_\Gamma + D)\cdot f$; (1.1.3)

- if $\mathcal{E}$ is decomposable, $C_1 \sim C_0 - D\cdot f$;

- if $a \geqq 0$, $h^0(Y,\mathcal{O}_Y(aC_0+\Delta\cdot f)) = h^0(\Gamma,S^a\mathcal{E}\otimes\mathcal{O}_\Gamma(\Delta))$ (1.1.4)

- by elm_P we understand the elementary transformation of Y centered at P , which is the composition of first blowing up P and then blowing down the strict transform of the fibre through P . The transformed surface is again a minimal smooth ruled surface over Γ .

Let now $q = 1$, so $\Gamma = E$ is an elliptic curve. Then:

- there exist exactly two ruled surfaces over E with indecomposable $\mathcal{E}$, one with $e = 0$ and one with $e = -1$, which we will denote by Y_0 resp. Y_{-1} if E is understood;
- for Y_0 the normalized $\mathcal{E}$ and the section C_0 are uniquely determined;
- on Y_{-1} there exists a one-dimensional algebraic family, parametrized by the points of E , of sections C_0 with $C_0^2 = 1$, each of which has its own normalized $\mathcal{E}$ and each of which is isolated in its linear equivalence class;
- if Y is a minimal ruled surface over E , Y must be one of the following:
  (i) if Y does not contain sections S with $S^2 < 1$, $Y = Y_{-1}$;
  (ii) if Y contains exactly one section S with $S^2 = 0$, $Y = Y_0$ and $S = C_0$;
  (iii) if Y contains exactly two sections S_i with $S_i^2 = 0$, $i = 0,1$, then $Y \cong \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(D))$, $e = -\deg D = 0$ and $D \not\sim 0$; either of the S_i may be chosen to be C_0 , the other C_1 ;
  (iv) if Y contains a pencil of sections S with $S^2 = 0$, then $Y \cong E \times \mathbb{P}^1$, the pencil $|S|$ being formed by the fibres of the projection $Y \to \mathbb{P}^1$; we then choose C_0 and C_1 to be two different elements of $|S|$;
  (v) if Y contains a section S with $S^2 < 0$, $Y \cong \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(D))$, with $\mathcal{O}_E(D) \cong \mathcal{N}_{S/Y}$, $S = C_0$ and $e = -\deg D = -S^2$.

In the following three propositions we will prove some facts about ruled surfaces which we need in the sequel.

PROPOSITION 1.2. *Let* E *be an elliptic curve, let* $D_1, D_2 \in \mathrm{Div}(E)$ *and assume* $e = -\deg D_1 = -\deg D_2 > 0$. *Let* $\mathcal{E}_i = \mathcal{O}_E \oplus \mathcal{O}_E(D_i)$ *and let* $X_i = \mathbb{P}_E(\mathcal{E}_i)$, $i = 1,2$. *Then the surfaces* X_1 *and* X_2 *are isomorphic.*

PROOF. Let $D_1 \sim \sum_{i=1}^{e} P_i$ and $D_2 \sim \sum_{j=1}^{e} Q_j$, P_i , $Q_j \in E$. Let $T: E \to E$ be a translation such that $T^*\mathcal{O}_E(D_2) \cong \mathcal{O}_E(D_1)$. Such a T exists, for the linear equivalence $D_2 + e(R-S) \sim \Sigma(Q_j+R-S) \sim \Sigma P_i \sim D_1$, i.e. $D_2 - D_1 \sim$ $\sim e(S-R)$ always has a solution $R = R_0$, $S = S_0$ because multiplication by $e \geqq 1$ of the Jacobian variety $J(E)$ of E onto itself is surjective. Then take T to be the translation defined by $T(R_0) = S_0$.

Now, as also $T^*\mathcal{O}_E \cong \mathcal{O}_E$, $T^*\mathcal{E}_2 \cong \mathcal{E}_1$ and so $X_1 \cong X_2$.

PROPOSITION 1.3. *Let* E *be an elliptic curve.*

(a) *If* $R \in Y_{-1}$, $p(R) = Q \in E$ *and* R *lies on a section* C_0 *with* $N_{C_0/Y_{-1}} \cong \mathcal{O}_E(Q)$, *then* elm_R *transforms* Y_{-1} *into* Y_0 .

(b) *Let* $Y = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-Q))$, $Q \in E$. *Then* $\dim|C_1| = 1$, $|C_1|$ *has a fixed point* R *on the fibre over* Q *and* elm_R *transforms* Y *into* $E \times \mathbb{P}^1$.

(c) *Let* $Y = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-Q_1-Q_2))$, $Q_1, Q_2 \in E$; *let* f_i *be the fibre over* Q_i , $i = 1,2$, *and let* F *be the subscheme of* Y *consisting of the two possibly coinciding fibres* f_i . *Then* $\dim|C_1| = 2$, $\dim \mathrm{Tr}_{f_i}|C_1| =$ $= \dim \mathrm{Tr}_F|C_1| = 1$, $i = 1,2$.

If $Q_1 \neq Q_2$, *let* $R_i \in f_i \smallsetminus C_0$, $i = 1,2$, *such that* $R_1 + R_2 \in \mathrm{Tr}_F|C_1|$; *if* $Q_1 = Q_2$, *let* $R_1 \in f_1 \smallsetminus C_0$ *and let* R_2 *be the direction in* R_1 *of the divisors in* $|C_1|$ *going through* R_1 . *Then* $\mathrm{elm}_{R_2} \circ \mathrm{elm}_{R_1}$ *transforms* Y *into* $E \times \mathbb{P}^1$.

PROOF. (a) Let $\tilde{Y} = \mathrm{elm}_P(Y_{-1})$, and let $\tilde{C}$ be the strict transform of C_0 . Then $\tilde{C}^2 = 0$ and because any section on Y_{-1} has self-intersection at least 1 any section on $\tilde{Y}$ has self-intersection at least 0. So, looking at the list of elliptic ruled surfaces above, we have to show that $\tilde{C}$ is the only section on $\tilde{Y}$ with self-intersection 0.

Assume $\tilde{D}$ is another, so we are in case (iii) or (iv) of the list above. Then the preimage D of $\tilde{D}$ on Y_{-1} must have been a section with $D^2 = 1$, going through R . Because sections with self-intersection 0 on the surfaces in cases (iii) and (iv) do not intersect each other, $\mathcal{O}_{\tilde{Y}}(\tilde{D}) \otimes \mathcal{O}_{\tilde{C}} \cong \mathcal{O}_{\tilde{C}}$, so $\mathcal{O}_{Y_{-1}}(D) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_E(Q)$. Let the linear equivalence class of D on Y_{-1} be $C_0 + \Delta\cdot f$, $\Delta \in \mathrm{Pic}(E)$. Then on the other hand we have $\mathcal{O}_{Y_{-1}}(D) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_{Y_{-1}}(C_0+\Delta\cdot f) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_E(Q+\Delta)$, so $\Delta \sim 0$ and $D \sim C_0$. But $\dim|C_0| = 0$, so $D = C_0$ which contradicts $\tilde{D} \neq \tilde{C}$.

(b) Using (1.1.4), $\dim|C_1| = h^0(\mathcal{O}_Y(C_1)) - 1 = h^0(\mathcal{O}_Y(C_0+Q\cdot f)) - 1 =$

$= h^0(\Gamma, \mathcal{E} \otimes \mathcal{O}_E(Q)) - 1 = h^0(\Gamma, \mathcal{O}_E(Q) \oplus \mathcal{O}_E) - 1 = 1$.

Consider the ideal sheaf sequence of the fibre f_0 over Q on Y , tensorized with $\mathcal{O}_Y(C_1)$:

$$0 \to \mathcal{O}_Y(C_0) \to \mathcal{O}_Y(C_1) \to \mathcal{O}_{f_0}(1) \to 0 .$$

$\mathrm{Dim}\ \mathrm{Tr}_{f_0}|C_1| = h^0(\mathcal{O}_Y(C_1)) - h^0((\mathcal{O}_Y(C_0)) - 1 = 0$, so indeed $|C_1|$ has a fixed point R on f_0 , which does not lie on C_0 , because $C_0 \cap C_1 = \emptyset$.

After performing elm_R the strict transform of $|C_1|$ is a one-dimensional system of sections with self-intersection 0, so checking the list above we see that we have obtained $E \times \mathbb{P}^1$.

(c) The assertions about the dimensions of the linear systems follow from considering exact sequences analogous to the one in (b).

Performing elm_{R_1} gives an elliptic ruled surface $\tilde{Y}$ with a section $\tilde{C}$, the image of C_0 , with $N_{\tilde{C}/\tilde{Y}} \cong \mathcal{O}_E(-Q_2)$, so again by the list of elliptic ruled surfaces above, $\tilde{Y} \cong \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-Q_2))$, and we are in the situation of (b) with R the image point of R_2 . But then the rest is clear.

PROPOSITION 1.4. *Let* $Y = \mathbb{P}_\Gamma(\mathcal{E})$ *be the minimal ruled surface over a smooth curve* Γ *of genus* $g \geq 1$ *associated to a locally free sheaf* $\mathcal{E}$ *on* Γ *of rank* 2 *which is a nontrivial extension of* $\mathcal{O}_\Gamma(-K_\Gamma)$ *by* $\mathcal{O}_\Gamma$. *Let* G *be an irreducible, reduced curve on* Y , $G \neq C_0$. *Then* $G \cap C_0 \neq \emptyset$.

PROOF. First we note that extensions of $\mathcal{O}_\Gamma(-K_\Gamma)$ by $\mathcal{O}_\Gamma$ are classified by $\mathrm{Ext}^1(\mathcal{O}_\Gamma(-K_\Gamma), \mathcal{O}_\Gamma) \cong \mathrm{Ext}^1(\mathcal{O}_\Gamma, \mathcal{O}_\Gamma(K_\Gamma)) \cong H^1(\Gamma, \mathcal{O}_\Gamma(K\ )) \cong k$, so up to a constant there exists only one such a nontrivial extension $\mathcal{E}$ and so for every curve Γ the surface Y is uniquely determined. Clearly $\mathcal{E}$ is normalized, and as $\Lambda^2\mathcal{E} = \mathcal{O}_\Gamma(-K_\Gamma)$, the invariant e of Y is equal to $-\deg(-K_\Gamma) = 2q - 2$. If $q > 1$, $e > 0$, and if $q = 1$, $Y = Y_0$, so indeed on these surfaces the section C_0 is uniquely determined.

Let us now assume that Y contains an irreducible, reduced curve G , $G \neq C_0$ and $G \cap C_0 = \emptyset$. Let P be any point on C_0 , let $p(P) = Q \in \Gamma$ and apply elm_P to Y . Let $\tilde{Y}$ be the transformed surface and $\tilde{C}_0$, $\tilde{G}$ the strict transforms on $\tilde{Y}$ of C_0 resp. G . As $N_{C_0/Y} \cong \mathcal{O}_\Gamma(-K_\Gamma)$, $N_{\tilde{C}_0/\tilde{Y}} \cong \mathcal{O}_\Gamma(-K_\Gamma-Q)$, so by (1.1.2), $\tilde{Y} = \mathbb{P}_\Gamma(\tilde{\mathcal{E}})$, with $\tilde{\mathcal{E}} = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-K_\Gamma-Q)$, and $\tilde{C}_0$ plays the role of C_0 on $\tilde{Y}$.

Because $P \in C_0$, on $\tilde{Y}$ we still have $\tilde{G} \cap \tilde{C}_0 = \emptyset$, so

$\mathcal{O}_{\tilde{Y}}(\tilde{G}) \otimes \mathcal{O}_{\tilde{C}_0} \cong \mathcal{O}_\Gamma$. On the other hand, if $\tilde{G} \sim a\tilde{C}_0 + \Delta\cdot\tilde{f}$, $\tilde{f}$ a fibre on $\tilde{Y}$, then $\mathcal{O}_{\tilde{Y}}(\tilde{G}) \otimes \mathcal{O}_{\tilde{C}_0} \cong \mathcal{O}_\Gamma(a(-K_\Gamma-Q)+\Delta)$, so $\Delta \sim a(K_\Gamma+Q)$ and $\tilde{G} \sim a(\tilde{C}_0+(K_\Gamma+Q)\cdot\tilde{f}) \sim a\tilde{C}_1$, $\tilde{C}_1$ a section on $\tilde{Y}$ disjoint from $\tilde{C}_0$.

Let $\tilde{f}_0$ be the fibre on $\tilde{Y}$ over Q , and let $S = \tilde{C}_0 \cap \tilde{f}_0$. Because elm_P contracts the fibre on Y over Q , $\tilde{G}$ intersects $\tilde{f}_0$ in only one point R , which has multiplicity a on $\tilde{G}$, so $aR \in \mathrm{Tr}_{\tilde{f}_0}|a\tilde{C}_1|$. Because $\tilde{G} \cap \tilde{C}_0 = \emptyset$ and $S \in \tilde{C}_0$, $R \neq S$.

We claim that the linear system $|\tilde{C}_1|$ has a fixed point T on $\tilde{f}_0$. For this, consider the ideal sheaf sequence of $\tilde{f}_0$ on $\tilde{Y}$, tensorized with $\mathcal{O}_{\tilde{Y}}(\tilde{C}_1)$:

$$0 \to \mathcal{O}_{\tilde{Y}}(\tilde{C}_0+K_\Gamma\cdot\tilde{f}) \to \mathcal{O}_{\tilde{Y}}(\tilde{C}_1) \to \mathcal{O}_{\tilde{f}_0}(1) \to 0 .$$

Using (1.1.4), $\dim \mathrm{Tr}_{\tilde{f}_0}|\tilde{C}_1| = h^0(\mathcal{O}_{\tilde{Y}}(\tilde{C}_1)) - h^0(\mathcal{O}_{\tilde{Y}}(\tilde{C}_0+K_\Gamma\cdot\tilde{f})) - 1 =$ $h^0(\Gamma,\tilde{\mathcal{E}}\otimes\mathcal{O}_\Gamma(K_\Gamma+Q)) - h^0(\Gamma,\tilde{\mathcal{E}}\otimes\mathcal{O}_\Gamma(K_\Gamma)) - 1 = h^0(\Gamma,\mathcal{O}_\Gamma(K_\Gamma+Q)\oplus\mathcal{O}_\Gamma) -$ $- h^0(\Gamma,\mathcal{O}_\Gamma(K_\Gamma)\oplus\mathcal{O}_\Gamma(-Q)) - 1 = q + 1 - q - 1 = 0$, which proves our claim. Because $\tilde{C}_0 \cap \tilde{C}_1 = \emptyset$, $T \neq S$.

Also $R \neq T$. For if not, elm_R , which is the inverse of elm_P , would give as strict transforms of $\tilde{C}_0$ and $\tilde{C}_1$, which then goes throug $R = T$, two disjoint sections on Y , contradicting the fact that $\mathcal{E}$ is indecomposable.

Finally one can show that, if $M_a = \mathrm{Tr}_{\tilde{f}_0}|a\tilde{C}_1|$, $a \geqq 1$, $\dim M_a = a - 1$ by the same method as above for $a = 1$. Intersecting $\tilde{f}_0$ with curves contained in $|a\tilde{C}_1|$ of the form $(a-i)\tilde{C}_0 + i\tilde{C}_1 + \Delta_i\cdot f$, $\Delta_i \in \mathrm{Div}(\Gamma)$, $\Delta_i > 0$, Δ_i not containing Q , $i = 0,1,2,\dots,a-2,a$, we see that M_a is spanned as a projective space by $aS,(a-1)S+T,\dots,2S+(a-2)T,aT$, but that $S + (a-1)T \notin M_a$. Indeed $\Delta_i \sim (a-i)(K_\Gamma+Q)$ is base-point free if $i \neq a-1$, but $\Delta_{a-1} \sim K_\Gamma + Q$ has Q as a fixed point, so a divisor in $|a\tilde{C}_1|$ intersecting $\tilde{f}_0$ in $S + (a-1)T$ would contain $\tilde{f}_0$ as a component. This implies that S and T are the only points $U \in \tilde{f}_0$, such that $aU \in M_a$. As $aR \in M_a$, this would imply $R = S$ or $R = T$, which we know not to be true, and indeed any curve on Y intersects C_0 .

2 ANTICANONICAL DIVISORS ON MINIMAL RULED SURFACES

Let X be a surface with canonical hyperplane sections, let $\pi: X' \to X$ be the minimal resolution of the singularities of X and let $\phi: X' \to X''$ be a relatively minimal model for X'. By I.cor. 5.4.a X' contains a unique (positive) anticanonical divisor W'. Because $\phi_* W' \in$ $\in |-K_{X''}|$, also on X'' the anticanonical system is nonempty, though in general we will have $\dim|-K_{X''}| > 0$.

We will now determine the possibilities for anticanonical divisors on minimal ruled surfaces.

PROPOSITION 2.1. *Let* $Y = \mathbb{P}_\Gamma(E)$ *be a minimal ruled surface over a smooth curve* Γ *of genus* $q \geq 1$, *and assume* E *to be normalized. If* $|-K_Y| \neq \emptyset$, *then:*

(a) *if* E *is indecomposable and* $q \geq 2$, E *is a non-trivial extension of* $\mathcal{O}_\Gamma(-K_\Gamma)$ *by* $\mathcal{O}_\Gamma$ *which is unique up to a constant, and* $\dim|-K_Y| = 0$; $|-K_Y| = \{2C_0\}$ *in this case;*

(b) *if* E *is decomposable and* $q \geq 2$, $E \cong \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$, $D \in \mathrm{Div}(\Gamma)$ *such that* $|-K_\Gamma-D| \neq \emptyset$. *Then* $\dim|-K_Y| = \dim|-K_\Gamma-D|$ *and any anticanonical divisor is of the form* $2C_0 + D_0\cdot f$, $D_0 \in |-K_\Gamma-D|$;

(c) *if* E *is indecomposable and* $q = 1$, E *is the, up to a constant unique, non-trivial extension of* $\mathcal{O}_\Gamma$ *by* $\mathcal{O}_\Gamma$, *so* $Y \cong Y_0$, $\dim|-K_Y| = 0$, *and* $|-K_Y| = \{2C_0\}$;

(d) *if* E *is decomposable and* $q = 1$, *there is no restriction on* E. *Let* $E = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$, $D \in \mathrm{Div}(\Gamma)$, $e = -\deg D \geq 0$. *Then:*

(i) *if* $e = 0$ *and* $D \sim 0$, $Y = E \times \mathbb{P}^1$ *and* $\dim|-K_Y| = 2$, *any anticanonical divisor consisting of two fibres of the projection* $Y \to \mathbb{P}^1$;

(ii) *if* $e = 0$ *and* $D \not\sim 0$, $\dim|-K_Y| = 0$, $|-K_Y| = \{C_0+C_1\}$;

(iii) *if* $e > 0$, $\dim|-K_Y| = e$ *and any anticanonical divisor is either of the form* $2C_0 + D_0\cdot f$, $D_0 \in |-D|$ *or the form* $C_0 + C_1'$, $C_1' \in |C_1|$ *a section.*

PROOF. (a), (b). Let $Y = \mathbb{P}_\Gamma(E)$, $q = p_g(\Gamma) \geq 2$ and $\Lambda^2 E = \mathcal{O}_\Gamma(D)$, $D \in \mathrm{Div}(\Gamma)$. Using (1.1.3,4) $\dim|-K_Y| = h^0(\mathcal{O}_Y(2C_0+(-K_\Gamma-D)\cdot f)) - 1 =$ $= h^0(\Gamma, S^2E\otimes\mathcal{O}_\Gamma(-K_\Gamma-D)) - 1$.

Assume $|-K_Y| \neq \emptyset$. Then $h^0(S^2E\otimes\mathcal{O}_\Gamma(-K_\Gamma-D)) \geq 1$. The sequence (1.1.1) induces the exact sequence $0 \to E \to S^2E \to \mathcal{O}_\Gamma(2D) \to 0$ (see [H], II.Ex. 5.16.c), which, tensorized with $\mathcal{O}_\Gamma(-K_\Gamma-D)$, gives:

$$0 \to \mathcal{E} \otimes \mathcal{O}_\Gamma(-K_\Gamma-D) \to S^2\mathcal{E} \otimes \mathcal{O}_\Gamma(-K_\Gamma-D) \to \mathcal{O}_\Gamma(D-K_\Gamma) \to 0 \ , \qquad (1)$$

so $h^0(S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-K_\Gamma-D)) \geqq 1$ induces either $h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-K_\Gamma-D)) > 0$ or $h^0(\mathcal{O}_\Gamma(D-K_\Gamma)) > 0$. Because $\mathcal{E}$ is normalized, $h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-K_\Gamma-D)) > 0$ implies $\deg(-K_\Gamma-D) = -2q + 2 + e \geqq 0$, and $h^0(\mathcal{O}_\Gamma(D-K_\Gamma)) > 0$ implies $\deg(D-K_\Gamma) =$ $= e - 2q + 2 \geqq 0$, so either $e \geqq 2q - 2$ or $e \leqq -2q + 2$.

Now if $e \leqq -2q + 2$, $e < 0$ because $q \geqq 2$, so $\mathcal{E}$ is indecomposable. By [H], V.Ex. 2.5 then also $e \geqq -q$, so this can only happen if $q = 2$ and $e = -\deg D = -2$. Then $\deg(-K_\Gamma-D) = -4$, so $h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-K_\Gamma-D)) = 0$ because $\mathcal{E}$ is normalized, so we must have $h^0(\mathcal{O}_\Gamma(D-K_\Gamma)) > 0$. As $\deg(D-K_\Gamma) = 0$ in this case, this implies $D \sim K_\Gamma$. We will show that this case does not occur.

So let $W \in |-K_Y| = |2C_0+(-2K_\Gamma)\cdot f|$. Because $h^0(\mathcal{O}_Y(C_0+(-2K_\Gamma)\cdot f) =$ $= h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-2K_\Gamma)) = 0$, C_0 is not a component of W . As $C_0\cdot W =$ $= C_0\cdot(2C_0-2K_\Gamma\cdot f) = -2e - 2\deg(K_\Gamma) = 4 - 4 = 0$, $W \cap C_0 = \emptyset$, which is impossible by prop. 1.4.

So if $|-K_Y| \neq \emptyset$, $e \geqq 2q - 2$. Then $\deg(D-K_\Gamma) = -e - 2q + 2 < 0$, so by (1), $h^0(S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-K_\Gamma-D)) = h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-K_\Gamma-D))$. Tensorizing (1.1.1) with $\mathcal{O}_\Gamma(-K_\Gamma-D)$ we get:

$$0 \to \mathcal{O}_\Gamma(-K_\Gamma-D) \to \mathcal{E}\otimes\mathcal{O}_\Gamma(-K_\Gamma-D) \to \mathcal{O}_\Gamma(-K_\Gamma) \to 0 \ ,$$

so $h^0(\mathcal{E}\otimes\mathcal{O}_\Gamma(-K_\Gamma-D)) = h^0(\mathcal{O}_\Gamma(-K_\Gamma-D))$.

Taking all this together we see that $\dim|-K_Y| = \dim|-K_\Gamma-D|$, so $2C_0$ is a fixed part of $|-K_Y|$. If $\mathcal{E}$ is decomposable, this gives (b). If $\mathcal{E}$ is indecomposable, by [H], V.thm. 2.12.b, $e \leqq 2q - 2$. As we also have $e \geqq 2q - 2$, we get $e = 2q - 2$. But then $\dim|-K_Y| = \dim|-K_\Gamma-D| \geqq$ $\geqq 0$ is only possible if $D \sim -K_\Gamma$. Then indeed $\mathcal{E}$ is a non-trivial extension of $\mathcal{O}_\Gamma(-K_\Gamma)$ by $\mathcal{O}_\Gamma$ which is unique up to a constant as we have seen in the beginning of the proof of prop. 1.4. Now $\dim|-K_Y| = 0$, and as $-K_Y \sim 2C_0$, $|-K_Y| = \{2C_0\}$.

(c) According to the list of elliptic ruled surfaces given in §1, we only have to consider $Y = Y_0$ or Y_{-1} .

Let's first take $Y = Y_0$. Then $\mathcal{E}$ is a non-trivial extension of $\mathcal{O}_\Gamma$ by $\mathcal{O}_\Gamma$, $\Lambda^2\mathcal{E} \cong \mathcal{O}_\Gamma$ so $D \sim 0$ and $-K_Y \sim 2C_0$. We have to show that $2C_0$ is the only anticanonical divisor on Y .

Suppose not, and let $W \in |-K_Y|$, $W \neq 2C_0$. Then because $h^0(\mathcal{O}_\Gamma(C_0)) =$

$h^0(\Gamma,\mathcal{E}) = 1$, C_0 is not a component of W . So, as $C_0 \cdot W = 4C_0^2 = 4e = 0$, $W \cap C_0 = \emptyset$, which is impossible by prop. 1.4, and indeed $|-K_Y| = \{2C_0\}$.

We will now finish the proof of (c) by showing that $|-K_{Y-1}| = \emptyset$. So let $Y = Y_{-1}$. In this case there is no unique normalized $\mathcal{E}$ associated to Y . We can take $\mathcal{E}$ to be the, up to a constant unique, non-trivial extension of $\mathcal{O}_\Gamma(Q)$ by $\mathcal{O}_\Gamma$ for some $Q \in \Gamma$. Now $\Lambda^2\mathcal{E} \cong \mathcal{O}_\Gamma(Q) \cong \mathcal{N}_{C_0/Y}$ and $-K_Y \sim 2C_0 - Q\cdot f$.

Suppose $W \in |-K_Y|$ · Because $\mathcal{E}$ is normalized, $h^0(\mathcal{O}_Y(-K_Y-C_0)) = h^0(\mathcal{O}_Y(C_0-Q\cdot f)) = h^0(\Gamma,\mathcal{E}\otimes\mathcal{O}_\Gamma(-Q)) = 0$, so C_0 is not a component of W . Also W does not contain fibres, for if so, because $\mathcal{O}_Y(-K_Y) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_\Gamma(Q)$, it would be the fibre over Q . But then $h^0(\mathcal{O}_Y(2C_0-2Q\cdot f)) > 0$, which is not true, for if we tensorize (1) in the proof of (a), (b), in which now $K_\Gamma \sim 0$ and $D = Q$, with $\mathcal{O}_\Gamma(-Q)$, we get $h^0(\mathcal{O}_Y(2C_0-2Q\cdot f)) = h^0(\Gamma,S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-2Q)) \leqq h^0(\Gamma,\mathcal{O}_\Gamma(-2Q)) + h^0(\Gamma,\mathcal{E}\otimes\mathcal{O}_\Gamma(-Q)) = 0$, again because $\mathcal{E}$ is normalized.

Because $W\cdot f = 2$, the only possibilities left for W are $W = S_0 + S_1$, the S_i two sections different from C_0 , or W is an irreducible reduced curve. In the first case, because $W\cdot C_0 = (2C_0-Q\cdot f)\cdot C_0 = -2e - 1 = 1$, $S_0 \neq S_1$, and one of the S_i must be disjoint from C_0 , contradicting the fact that $\mathcal{E}$ is indecomposable. So W is an irreducible, reduced curve, represented by p as a 2-fold cover of Γ . Because W dominates Γ , $p_g(W) \geqq p_g(\Gamma) = 1$; on the other hand, the adjunction formula gives $p_g(W) \leqq p_a(W) = p_a(-K_Y) = 1$, so $p_g(W) = 1$ and W is smooth, elliptic.

Because $\mathcal{O}_Y(-K_Y) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_\Gamma(Q)$, W intersects C_0 in $P = f_0 \cap C_0$, f_0 the fibre over Q . Now we perform elm_P . By prop. 1.3.a this transforms Y into Y_0 . Then the strict transform of W , again a smooth elliptic curve, would be an anticanonical divisor on Y_0 , but we already know this is impossible, so indeed $|-K_Y| = \emptyset$.

(d) If $Y = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma\oplus\mathcal{O}_\Gamma(D))$ with $p_g(\Gamma) = 1$, then $-K_Y \sim 2C_0 - D\cdot f \sim C_0 + C_1$ and $h^0(\mathcal{O}_Y(-K_Y)) = h^0(\Gamma,S^2\mathcal{E}\otimes\mathcal{O}_\Gamma(-D)) = h^0(\Gamma,\mathcal{O}_\Gamma(-D)\oplus\mathcal{O}_\Gamma\oplus\mathcal{O}_\Gamma(D))$, which gives the claimed values for $\dim|-K_Y|$. For $e = 0$ also the possiblilities for anticanonical divisors are clear.

If $e > 0$, $h^0(\mathcal{O}_Y(-K_Y-C_0)) = h^0(\mathcal{O}_Y(C_0-D\cdot f)) = h^0(\Gamma,\mathcal{E}\otimes\mathcal{O}_\Gamma(-D)) = h^0(\Gamma,\mathcal{O}_\Gamma(-D)\oplus\mathcal{O}_\Gamma) = e + 1 = h^0(\mathcal{O}_Y(-K_Y))$, so then C_0 is a fixed component of $|-K_Y|$, leaving as variable part the system $|C_1|$. Now any divisor of $|C_1|$ is either a section C_1' , or is of the form $C_0 + D_0\cdot f$,

$D_0 \in |-D|$, so indeed any anticanonical divisor is of the desired form.

3 SINGULARITIES ON RULED SURFACES WITH CANONICAL HYPERPLANE SECTIONS

Before we can draw conclusions from prop. 2.1 in regard to the type and number of the occurring singularities on ruled surfaces with canonical hyperplane sections, we have to look closer at the way in which the minimal resolution X' arises from its relatively minimal model X'' , which of course in general is not unique, i.e. we will describe the inverse of the morphism $\phi: X' \to X''$ introduced in §2. We put $W'' = \phi_* W' \in |-K_{X''}|$, the image of the unique anticanonical divisor W' on X' .

PROPOSITION 3.1. *Let* $X' = X'_s \xrightarrow{\phi_{s-1}} X'_{s-1} \to \dots \xrightarrow{\phi_1} X'_1 = X''$ *be the factorization of* ϕ *in such a way that* ϕ_1^{-1} *blows up the points on* X'' *in which* ϕ^{-1} *is not defined giving* X'_1 , ϕ_2^{-1} *blows up the points in which* $\phi^{-1} \circ \phi_1$ *is not defined etc., and let* $W'_t = (\phi_t \circ \dots \circ \phi_{s-1})_* (W') \in |-K_{X'_t}|$, $t = 1,2,\dots,s-1$. *Then:*

(a) *all points blown up by* ϕ_t^{-1} *on* X'_t *lie on* $\operatorname{supp}(W'_t)$, $t = 1,2,\dots,s-1$;

(b) *if* ϕ_t^{-1} *blows up the points* $P_{t,1},\dots,P_{t,i(t)} \in X'_t$, *which have multiplicity* $\mu_{t,i} \geqq 1$ *on* W'_t , *if* $\phi_t^{-1}(P_{t,i}) = E_{t+1,i}$ *and if* $\widetilde{W}'_{t+1}$ *is the strict transform of* W'_t *on* X'_{t+1} , *then:*

$$W'_{t+1} = \widetilde{W}'_{t+1} + \sum_{i=1}^{i(t)} (\mu_{t,i}-1)\cdot E_{t+1,i} \,, \tag{1}$$

$t = 1,2,\dots,s-1$, $i = 1,2,\dots,i(t)$; *so for every point* $P_{t,i}$ *with* $\mu_{t,i} \geqq 2$, *one new smooth rational component is introduced in the anticanonical divisor with multiplicity* $\mu_{t,i} - 1$;

(c) *the number of connected component of the anticanonical divisors* W' *on* X' *and* $W'' = W'_1$ *on* X'' *is the same.*

PROOF. (a) Let $E \subset X'_{t+1}$ be a smooth rational curve such that $\phi_t(E)$ is a point P on X'_t , $t \in \{1,2,\dots,s-1\}$. We have to show that $P \in \operatorname{supp}(W'_t)$.

Let $W'_{t+1} = m\cdot E + D'_{t+1}$, $m \geqq 0$, E not a component of D'_{t+1} . By the adjunction formula $E\cdot W'_{t+1} = -E\cdot K_{X'_{t+1}} = 2 + E^2 = 1$, so $E\cdot D'_{t+1} =$ $= 1 - m\cdot E^2 = 1 + m > 0$. Hence $P = \phi_t(E)$ is a point on $\operatorname{supp}((\phi_t)_* D'_{t+1}) =$

= $\mathrm{supp}(W_t')$.

(b) We may assume $t = 1$, $i(1) = 1$, so ϕ_1^{-1} blows up only one point $P_{1,1} = P$ with multiplicity $\mu_{1,1} = \mu$ on W_1' to a curve E on X_2' . Then using the general formula $K_{X_2'} \sim \phi_1^* K_{X_1'} + E$ we get $W_2' = \widetilde{W}_2' + (\mu-1)E$, which is the desired formula in this case.

(c) First of all, it is enough to show that W_1' and W_2' have the same number of connected components. Furthermore we may assume that ϕ_1^{-1} blows up only one point P to $E \subset X_2'$ and as we can deal with the connected components of W_1' separately we may as well assume that W_1' is connected.

Now if P lies on only one irreducible component of W_1' , $\widetilde{W}_2'$ is of course still connected, and as E intersects $\widetilde{W}_2'$, formula (1) shows that W_2' is, too.

If P lies on two or more irreducible components of W_1' , $\widetilde{W}_2'$ may be disconnected, but in this case the multiplicity μ of P on W_1' is at least 2, so E does occur in (1), and of course E intersects all connected components of $\widetilde{W}_2'$, so W_2' is connected.

DEFINITION 3.2. A resolution $\rho: Y' \to Y$ of a normal surface singularity $y \in Y$ is called *good*, if the exceptional divisor $E = \rho^{-1}(y)$ has the following properties:

(i) every irreducible component of E is smooth;

(ii) E has only normal crossings;

(iii) any two irreducible components of E intersect in at most one point.

COROLLARY 3.3. *Let* X *be a surface with canonical hyperplane sections, birationally equivalent to a ruled surface over a curve* Γ *of genus* $q \geq 1$, $\pi: X' \to X$ *its minimal resolution. Then:*

(a) *if* $q = 1$, *the non-rational singularities of* X *consist of either one singularity* x *with* $p_g(x) = 2$ *or two simple elliptic singularities* x_i *with* $\pi^{-1}(x_i) \cong \Gamma$, $i = 0,1$ *;*

(b) *if* $q \geq 2$, X *contains exactly one non-rational singularity* x *with* $p_g(x) = q + 1$ *;*

(c) *the minimal resolution* π *is good in regard to all singularities of* X *and in the first case of* (a) *and in* (b), $\pi^{-1}(x)$ *consists of one non-rational curve isomorphic to* Γ , *plus smooth rational curves, and does not contain cycles.*

PROOF. (a), (b). By I.cor. 5.4.b,c, the exceptional divisors on X' of the non-rational singularities of X are the connected components of $\mathrm{supp}(W')$. By prop. 3.1.c this number is the same as the number of connected components of W'' . By prop. 2.1 this number is one, except for the case when $q = 1$ and W'' consists of two disjoint sections on X'' . So besides this case, X contains only one non-rational singularity x , which has $p_g(x) = = q + 1$ by I.cor. 5.5.c. This gives the first part of (a) and (b).

In the remaining case, $q = 1$ and $W_1' = W'' = C_0 + C_1'$, C_0 and C_1' two disjoint sections on X'' . Using prop. 3.1.a,b we see that $W' = \tilde{C}_0 + \tilde{C}_1'$, $\tilde{C}_0$ and $\tilde{C}_1'$ the strict transforms on X' of C_0 resp. C_1' , so indeed we get two simple elliptic singularities, and of course $\tilde{C}_0$, $\tilde{C}_1' \cong \Gamma$.

(c) Using [A], prop. 1, it is an easy exercise to show that any resolution of a rational singularity is good, and as also of course the minimal resolution of a simple elliptic singularity is good, we only have to consider the singularity x with $p_g(x) = q + 1$.

In that case, by prop. 2.1, the reduced divisor W''_{red} on X'' suffices (i), (ii) and (iii) of the above definition, contains one curve isomorphic to Γ and for the rest only rational curves, and does not contain cycles. Now blowing up points on X'' in the way described in prop. 3.1 to get X' does not spoil any of these properties for $W'_{\mathrm{red}} = \pi^{-1}(x)$, which proves (c).

CHAPTER III

CONSTRUCTION OF RULED SURFACES WITH CANONICAL HYPERPLANE SECTIONS CONTAINING ONE NON-RATIONAL SINGULARITY

In this chapter we will carry out, at least in low-dimensional projective spaces, the construction of surfaces with canonical hyperplane sections birationally equivalent to a ruled surface over a curve of genus $q \geqq 1$, containing one singularity of genus $q + 1$. For $q \geqq 2$ this is the only possibility by II.cor. 3.3.b. The case $q = 1$ will turn out to be by far the most interesting. We will deal with this situation in §3; in particular there we give equations for all normal quartic surfaces, up to isomorphism, with a singularity of genus 2.

1 DESCRIPTION OF THE CONSTRUCTION

We summarize the situation in the following diagram:

$$\begin{tikzcd}
X' \arrow[r, "\pi"] \arrow[dd, "\phi"'] & X \arrow[r, hook, "i"] & \mathbb{P}^g \\
 & x \arrow[u, phantom, sloped, "\in"] & \\
X'' \arrow[r, "p"] & \Gamma &
\end{tikzcd}$$

diagram 1.1.

Here:

- X is the surface with canonical hyperplane sections of genus $g \geqq 3$, embedded in $\mathbb{P}^g$; in the hyperelliptic case, the upper row is to be replaced by $X' \overset{\pi}{\to} X \overset{h}{\to} \mathbb{P}^g$, $g \geqq 2$, and then $\bar{X} = h(X)$;

- $\pi: X' \to X$ is the minimal resolution of the singularities of X;

- $\phi: X' \to X''$ is a relatively minimal model of X' ;

- $p: X'' \to \Gamma$ is the natural projection of X'' onto its base curve Γ ; $p_g(\Gamma) = q \geqq 1$; $X'' \cong \mathbb{P}_\Gamma(E)$ with E normalized; $\Lambda^2 E = \mathcal{O}_\Gamma(D)$, $D \in \mathrm{Div}(\Gamma)$ and $e = -\deg D$;

- $x \in X$ is the only non-rational singularity of X , $p_g(x) = q + 1$.

Furthermore we use the following notation and facts:

- L is the g-dimensional linear system of hyperplane sections of X , $C \in L$ a general curve; in I.§1 we saw that L is complete;

- $L'' = \phi_* L'$ on X'' , $C'' \in L''$ a general curve; $\dim L'' = g$, and $\phi_{L''}(X'') = X$ or $\bar{X} \subset \mathbb{P}^g$ in the non-hyperelliptic resp. hyperelliptic case;

- $L'' \subset |aC_0+\Delta\cdot f|$ on X'' for some $a \in \mathbb{Z}$, $\Delta \in \mathrm{Pic}(\Gamma)$. Let its base points be P_i with multiplicity r_i on a general $C'' \in L''$, $i = 1,\dots,k$. Among these there may be infinitely near base points, i.e. not only base points of first order lying on X'' itself, but also points of second order, which are fixed directions in base points on X'' , etc.;

- L'' is complete with respect to the conditions imposed by its base points, because L' is complete;

- let $X' = X'_s \xrightarrow{\phi_{s-1}} X'_{s-1} \longrightarrow \dots \xrightarrow{\phi_1} X'_1 = X''$ be the factorization of ϕ described in II.prop. 3.1; ϕ_t blows up the base points of order t of L'' , which lie on $\mathrm{supp}(W'_t) \subset X'_t$ by II.prop. 3.1.a, W'_t the anticanonical divisor on X'_t arising from $\phi_* W' = W'' \in |-K_{X''}|$ with W' the unique anticanonical divisor on X' .

In addition to II.prop. 3.1, let us prove the following proposition concerning the way the base points P_i are situated on X'' .

PROPOSITION 1.2. (a) *A general* $C' \in L'$ *is disjoint from* W' .

(b) *A general* $C'' \in L''$ *has no variable intersections with the divisor* W'' *, i.e. for every* $t = 1,\dots,s$ *, the strict transform* C_t' *of* C'' *on* X_t' *has no variable intersections with* $\mathrm{supp}(W_t')$.

(c) *If* $P_i \in X_t'$ *is a base point of order* t *lying on only one component* D_t' *of* W_t' *, of which it is a smooth point and if moreover* D_t' *appears with multiplicity* 1 *in* W_t'*, then the only possible fixed direction of a general* C_t' *in* P_i *is the direction of* D_t' .

(d) *If* $P_i \in X_t'$ *is a base point of order* t *and if the multiplicity of* P_i *on* W_t' *is* $\mu_i \geqq 2$ *, then there must be base points infinitely near to* P_i *of order* j *, at least for every* $j \leqq \mu_i + t - 1$.

PROOF. (a) We have $\pi = \phi_{|C'|}$ or $h\circ\pi = \phi_{|C'|}$ in the non-hyperelliptic resp. hyperelliptic case, and h is a finite morphism. By I.cor. 5.4.c, π blows down W' , and so $\phi_{|C'|}$ does, which proves the assertion.

(b) If a general C_t' would have a variable intersection with $\mathrm{supp}(W_t')$, it would survive to give an intersection of C' with $\mathrm{supp}(W')$, contradicting (a).

(c) After blowing up P_i to the curve $E_i = \phi_t^{-1}(P_i) \subset X_{t+1}'$, E_i is no component of W_{t+1}' by II.prop. 3.1.b. Because base points on X_{t+1}' must lie on W_{t+1}' by II.prop. 3.1.a, the only possible base point on E_i , i.e. the only fixed direction in P_i , is the point $E_i \cap D_{t+1}'$, D_{t+1}' the strict transform of D_t' , which is the direction of D_t' .

(d) We can assume $P_i = P \in X''$, so $t = 1$, and let $\mu_i = \mu$. By II.prop. 3.1.b the exceptional curve $E = \phi_1^{-1}(P)$ appears with multiplicity $\mu - 1 \geqq 1$ in W' . The curves C_2' intersect E according to the directions of C'' in P , and by (b) the intersections of C_2' with E must be fixed to give base points on X_2' with multiplicity at least $\mu - 1$ on W_2' . Repeating this process we arrive at base points infinitely near to P at least on X_μ' , i.e. of order μ .

REMARK 1.2.1. We will use prop. 1.2.d only in case $P \in X''$ lies on a smooth component D'' of W'' which has multiplicity 2 in W'' . Then the direction of C'' in P must be fixed.

We are now able to give an outline of the program we will carry out to construct the surfaces X .

CONSTRUCTION 1.3. *To construct ruled surfaces* X *with canonical hyperplane sections with one non-rational singularity we have to*

(a) *take a minimal ruled surface* X'' *which has* $|-K_{X''}| \neq \emptyset$ *and fix a connected* $W'' \in |-K_{X''}|$ *(connected because we have only one non-rational singularity, cf. II. prop. 3.1.c);*

(b) *fix an* $a \geq 1$ *and a* $\Delta \in \mathrm{Pic}(\Gamma)$ *and find linear subsystems* $L'' \subset |aC_0+\Delta\cdot f|$ *which have their base points exactly on* W'' *(by which we mean the combination of the statements of II.prop. 3.1.a and prop. 1.2.b). Here, after constructing such an* L'' *, a general curve* C'' *of* L'' *may turn out to be hyperelliptic;*

(c) *in the non-hyperelliptic case (try to) determine* $\phi_{L''}(X'') = X \subset \mathbb{P}^g$ *, in the hyperelliptic case (try to) determine* $\phi_{L''}(X'') = h(X) = \bar{X} \subset \mathbb{P}^g$ *and the branch curve of* h *on* $\bar{X}$ *.*

Of course, explicitly blowing up the base points found in (b), *one finds the exceptional divisor* $\pi^{-1}(x) = \mathrm{supp}(W')$ *(cf. I.cor. 5.4.c) and possible exceptional divisors for rational singularities.*

In prop. 1.4 we show how we can replace X'' , if necessary, by a more suitable minimal model, on which the transform of the linear system L'' has a simpler form. As a consequence we get some relations between the numbers $a,e.q,g$ and r_i .

PROPOSITION 1.4. (a) $e \geqq 2q-2$ *and* $e = 2q-2$ *iff* $D \sim -K_\Gamma$. *So except for the case* $X'' = E \times \mathbb{P}^1$, $p_g(E) = 1$, *the curve* C_0 *is uniquely determined on* X'' *and* $2C_0$ *is a fixed part of* $|-K_{X''}|$.

(b) *After possibly replacing* X'' *by another relatively minimal model of* X' *we can assume* $P_i \notin C_0$, $i = 1,\dots,k$. *Then:*

(1) $r_i \leqq a-1$, $i = 1,\dots, k$ *;*

(2) E *is decomposable,* $E \cong \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$ *;*

(3) $\Delta \sim -aD$, *so* $L'' \subset |aC_1|$, $a \geqq 1$ *;*

(4) $\Sigma_{i=1}^k r_i = a(e-2q+2)$ *;*

(5) $g = 1 + \frac{1}{2}a^2e - \frac{1}{2}\Sigma_{i=1}^k r_i^2$ *;*

(6) $1 + a^2(q-1) + \frac{1}{2}a(e-2q+2) \leqq g \leqq 1 + \frac{1}{2}a^2e - \frac{1}{2}a(e-2q+2)$ *;*

(7) *as soon as base points are present (i.e. as soon as* $D \not\sim -K_\Gamma$ *), the number of conditions imposed by them is exactly one less than the expected number (i.e. they pose* $\frac{1}{2}\sum_{i=1}^{k} r_i(r_i+1) - 1$ *conditions).*

PROOF. (a) On X'' , $|-K_{X''}| \neq \emptyset$ and because X has only one non-rational singularity, $|-K_{X''}|$ contains connected divisors. Now checking the possibilities in II.prop. 2.1 gives (a).

(b) Assume L'' has base points on C_0 , so $C''\cdot C_0 > 0$. Let $P = P_i$ be such a point. We then apply elm_P to obtain another relatively minimal model $\tilde{X}''$. Denoting by $\tilde{C}''$ and $\tilde{C}_0$ the strict transforms of C'' resp. C_0 , we find $\tilde{C}''\cdot\tilde{C}_0 < C''\cdot C_0$, so repeating this, if necessary, a finite number of times, this intersection number becomes 0, which is what we want.

(b1) If P_j is a base point infinitely near to the base point $P_i \in X''$, then $r_j \leqq r_i$, so it is enough to show $r = r_i < a$ for a $P = P_i \in X''$. Moreover $P \notin C_0$.

Let f_0 be the fibre on X'' through P . As $C'' \in |aC_0+\Delta\cdot f|$, $r \leqq C''\cdot f_0 = a$. If $r = a$, after blowing up only P , the strict transforms $\tilde{C}''$ and $\tilde{f}_0$ of C'' resp. f_0 are disjoint and $\tilde{f}_0^2 = -1$. But then, if f_0' is the strict transform of f_0 on X' , also $(f_0')^2 = -1$, so f_0' is an exceptional curve of the first kind on X' , and C' and f_0' are disjoint so $\pi(f_0')$ is a point on X (which must be x because $\phi^{-1}(C_0) \cap f_0' \neq \emptyset$). But this contradicts the minimality of the resolution π and we conclude $r < a$.

(b2) By II.prop. 2.1 we can only have an indecomposable E if E is a non-trivial extension of $\mathcal{O}_\Gamma(-K_\Gamma)$ by $\mathcal{O}_\Gamma$. Then $W'' = 2C_0$ and as we assume there are no base points on C_0 , there are no base points at all. As a consequence, a general curve $C'' \in L''$ would be disjoint from C_0 . By II.prop. 1.4 this is impossible, so we can forget about this case.

(b3) Again, as there are no base points on C_0 , $C_0 \cap C'' = \emptyset$. So $\mathcal{O}_\Gamma \cong \mathcal{O}_{X''}(C'') \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_{X''}(aC_0+\Delta\cdot f) \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_\Gamma(aD+\Delta)$. Hence $\Delta \sim - aD$, so $C'' \sim a(C_0-D\cdot f) \sim aC_1$ and $L'' \subset |aC_1|$.

(b4) We will show that $\Sigma r_i = C''\cdot W''$. Assuming this for a moment, we get, using II.1.1.3, $\Sigma r_i = aC_1\cdot(2C_0-(K_\Gamma+D)\cdot f) = a(e-2q+2)$ which is (b4).

To prove $\Sigma r_i = C''\cdot W''$, because $C'\cdot W' = C_s'\cdot W_s' = 0$ by prop. 1.2.a, it is enough to show that $C_t'\cdot W_t' = C_{t+1}'\cdot W_{t+1}' + \Sigma^t r_i$, with C_t' the strict transform of C'' on X_t' and Σ^t denoting the sum over the base points of order t , lying on X_t' ; $t = 1,\dots,s-1$. To show this, we can assume $t = 1$.

Now formula (1) of II.prop. 3.1.b says that $W_2' = \overset{*}{\phi_1}W_1' - \Sigma^1 E_i$. Also $C_2' = \overset{*}{\phi_1}C_1' - \Sigma^1 r_i E_i$, so $C_2'\cdot W_2' = C_1'\cdot W_1' - \Sigma^1 r_i$ as asserted.

(b5) On the one hand $(C')^2 = 2g - 2$, on the other $(C')^2 = (C'')^2 - \sum_{i=1}^{k} r_i^2 = (aC_1)^2 - \sum_{i=1}^{k} r_i^2 = a^2e - \sum_{i=1}^{k} r_i^2$. Combine these two.

(b6) First, $\Sigma r_i^2 \geqq \Sigma r_i = a(e-2q+2)$ by (b4). Combining this with (b5) gives the upper bound. Second, Σr_i^2 attains its maximum value when as may as possible r_i's are equal to the maximum value $a-1$. Though Σr_i may not be divisible by $a - 1$, we still have the estimate $\Sigma r_i^2 \leqq (\Sigma r_i)\cdot(a-1)^{-1}\cdot(a-1)^2 = a(e-2q+2)(a-1)$ by (b4). Together with (b5) this gives the lower bound.

(b7) As $a \geqq 1$, by (b4) there are no base points iff $e = 2q - 2$ and by (a) this is the case iff $D \sim -K_\Gamma$.

Now assume there are base points. If the conditions imposed by them would be independent, we would have $\dim L'' = \dim|aC_1| - \frac{1}{2}\Sigma r_i(r_i+1) = h^0(\mathcal{O}_{X''}(aC_1)) - 1 - \frac{1}{2}\Sigma r_i^2 - \frac{1}{2}\Sigma r_i$. Computing the first term using II. 1.1.4 with $\mathcal{E} = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(D)$ and $\Delta = -aD$ because $C_1 \sim C_0 - D\cdot f$, and using (b4) for the last, we find this to be equal to $\frac{1}{2}a^2e - \frac{1}{2}\Sigma r_i^2$. However, $\dim L'' = g$ and by (b5) this is 1 bigger than calculated above, which we had to prove.

From now on we will assume the condition of prop. 1.4.b to be satisfied. We will use the bounds in (b6) of that proposition to determine, for fixed values of g , the possibilities for q , a and e .

The simplified form for the linear system L'' we found in prop. 1.4.b gives us the opportunity to prove the following proposition. Here, with respect to the occurrence of rational singularities on X , we only need to know in the sequel what happens when $a = 2$.

PROPOSITION 1.5. (a) *The number* e *is equal to minus the self-intersection of the only non-rational curve in the exceptional divisor* $\pi^{-1}(x)$.

(b) *In the non-hyperelliptic case, the number* a *is equal to the degree of embedding of a general fibre of* X'' *in* $\mathbb{P}^g$ *by* $\phi_{L''}$.

(c) *When* $a = 2$, X *contains at most ordinary double points (= singularities of type* A_1*) as rational singularities.*

PROOF. (a) Let C_0' be the strict transform of C_0 on X' , $C_0' \cong \Gamma$ is the only non-rational curve in $\pi^{-1}(x)$. Because no points are blown up on C_0 , $C_0' = \phi^{-1}(C_0)$, and $\mathcal{N}_{C_0'/X'} \cong \mathcal{N}_{C_0/X''} \cong \mathcal{O}_\Gamma(D)$, so $(C_0')^2 = C_0^2 = -e$.

(b) Follows from $C''\cdot f = (aC_1)\cdot f = a$.

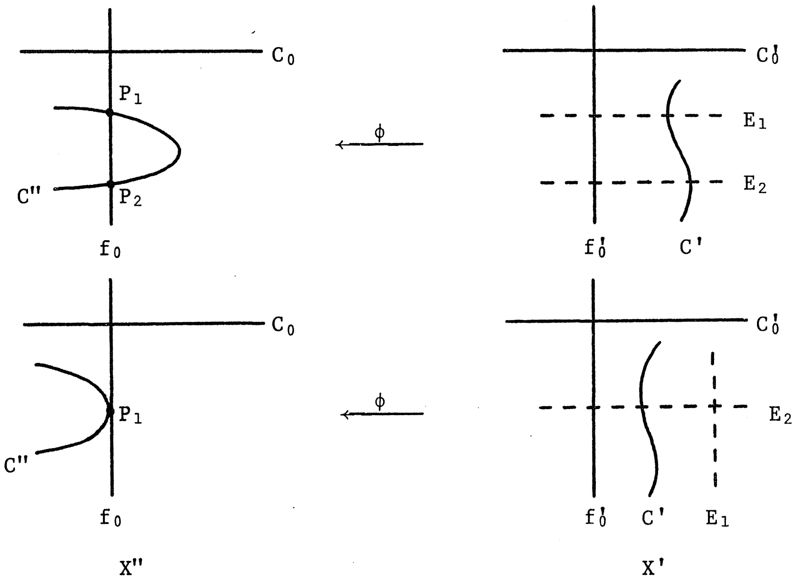
(c) To find rational singularities on X is the same as to find smooth rational curves E on X' with $E \not\subset \mathrm{supp}(W')$, $E \cap C' = \emptyset$ for general

$C' \in L'$ and $E^2 \leqq -2$. We claim that such an E must necessarily be exceptional for ϕ .

For, the only rational curves on X'' are fibres f , and if $f \not\subset \mathrm{supp}(W'')$, no points are blown up on it, so the strict transform $\phi^{-1}(f)$ still has $(\phi^{-1}(f))^2 = 0$, if $f \subset \mathrm{supp}(W'')$, its strict transform is contained in $\mathrm{supp}(W')$, which proves the claim.

So let us now assume that E arises from blowing up points on the fibre $f_0 \subset \mathrm{supp}(W'')$. Because $a = 2$, all base points of L'' are simple by prop. 1.4.b1 and so either a general C'' has two different simple base points on f_0 or is tangent to f_0 in one point, in both cases with the necessary higher order base points. If the multiplicity of f_0 in W'' is at least 2, one can, by blowing up the base points, assure oneself of the fact that no E of the required type arises. So f_0 must have multiplicity 1 in W'' .

But then, examining the two possible configurations of base points as sketched below, we prove our assertion.



Here E_i is the exceptional divisor arising from blowing up P_i , $i = 1,2$, P_2 in the second case being the direction of f_0 in P_1 , and f_0' is the strict transform of f_0 on X' . In the second case, $E_1^2 = -2$, and E_1 is contracted to an A_1-singularity by π .

REMARK 1.5.1. Because of prop. 1.5.a we must have $e > 0$, so we can forget about the only case where the curve C_0 is not uniquely determined on X'' described in prop. 1.4.a, which has $e = 0$. Hence using II.prop. 2.1, we know that $W'' = 2C_0$ + fibres, C_0 the unique section on X'' with negative self-intersection.

2 CONES AND RULED SURFACES OVER CURVES OF GENUS AT LEAST 2

In this section we prove two theorems, concerning the cases $a = 1$, $q \geqq 1$ and $q \geqq 2$. We will show that if $a = 1$, i.e. if the fibres of X'' become lines on X , then X is a cone over a canonically embedded curve. (see I.ex. 4.3).

If $q \geqq 2$, the lower bound of prop. 1.4.b6 will prove to be very restrictive; except for cones over canonically embedded curves or higher Veronese embeddings of these, there are very few possibilities in low-dimensional projective spaces. Up to $\mathbb{P}^{10}$, a limit which is chosen arbitrarily, we will list them all in terms of their numbers q,a,e . To find all possibilities of these surfaces in higher projective spaces, one can follow the same procedure as in the proof of thm. 2.2 below.

THEOREM 2.1. *If* $a = 1$, *i.e. if the fibres of* X'' *are transformed into straight lines on* X , $X' = X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-K_\Gamma))$, $g = q \geqq 2$, *and* $L'' = |C_1|$.

If Γ *is not hyperelliptic,* X *is the cone over the canonically embedded curve* $\phi_{|K_\Gamma|}(\Gamma)$.

If Γ *is hyperelliptic,* $\bar{X}$ *is the cone over the rational normal curve* $\phi_{|K_\Gamma|}(\Gamma)$ *of degree* $q - 1$ *in* $\mathbb{P}^{q-1}$, *and* X *is the double cover of* $\bar{X}$ *branched along* $2q + 2$ *different lines on* $\bar{X}$ *through the vertex.*

PROOF. The proof is a combination of the statements of prop. 1.4. By (b1) there are no base points, so $X' = X''$, and $\Sigma r_i = 0$. Then by (b3), $L'' = |C_1|$, so $g = p_g(C'') = p_g(C_1) = q$, and by (b4), $e = 2q - 2$. Hence prop. 1.4.a implies $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-K_\Gamma))$.

From all this it follows that $\phi_{L''}(X'')$ is the cone over $\phi_{L''}(C_1)$, which is nothing else but $\phi_{|K_\Gamma|}(\Gamma)$, and this yields the description of the surfaces in the theorem.

THEOREM 2.2. *Let* X *be a surface with canonical hyperplane sections of genus* g *, birational to* $\Gamma \times \mathbb{P}^1$ *,* Γ *a curve of genus* $q \geqq 2$ *, which is not a cone. Then* $g \geqq 5$ *, and for* $5 \leqq g \leqq 10$ *there are the following possibilities:*

(i) $g = 5 : q = a = e = 2$ *; then* $X' = X'' = \mathbb{P}(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-K_\Gamma))$ *and* $L'' = |2C_1|$ *, which is a system of hyperelliptic curves;* $\bar{X}$ *is the Veronese surface in* $\mathbb{P}^5$ *, and* X *is the double cover of* $\bar{X}$ *branched along six smooth conics on* $\bar{X}$ *going through one point;*

(ii) $g = 6,7,8 : q = a = 2 , e = g - 3$ *;*

(iii) $g = 9 : q = a = 2 , e = 6$ *, or* $q = 3 , a = 2 , e = 4$.

In the second case, $X' = X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-K_\Gamma))$ *and* $L'' = |2C_1|$ *; if* Γ *is not hyperelliptic,* X *is the double Veronese embedding of the cone* $\phi_{|C_1|}(X'')$ *, if* Γ *is hyperelliptic,* $X \cong \phi_{L''}(X'')$ *;*

(iv) $g = 10 : q = a = 2 , e = 7$ *or* $q = 3 , a = 2 , e = 5$ *or*
$q = 2 , a = 3 , e = 2$.

In the last case, $X' = X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-K_\Gamma))$ *and* $L'' = |3C_1|$ *, which is a system of non-hyperelliptic curves, and* $X \cong \phi_{L''}(X'')$.

PROOF. As to the possible values for q,a and e , by prop. 1.4.b6 we certainly have $1 + a^2(q-1) \leqq g$. Because $q,a \geqq 2$ this implies $g \geqq 5$, and if $g \leqq 8$ we must have $(q,a) = (2,2)$. Taking $(q,a) = (2,2)$ in (b6) we find $e = g - 3$ and this gives us the values in (i), (ii) and the first of (iii) and (iv). When $g = 9$ or 10 one can easily find the remaining possibilities for q,a,e with the help of (b6).

In all three special cases $(g,q,a,e) = (5,2,2,2)$, $(9,3,2,4)$ and $(10,2,3,2)$, $e = 2g - 2$, so by prop. 1.4.a $X'' = \mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-K_\Gamma))$, and by prop. 1.4.b7 there are no base points, so $X' = X''$ and $L'' = |aC_1|$.

Let us first consider the first and last of these three cases together. By thm. 2.1, taking the system $L'' = |C_1|$ on $\mathbb{P}_\Gamma(\mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(-K_\Gamma))$, $p_g(\Gamma) = 2$, gives us X as the double cover of $\mathbb{P}^2$ branched along six lines through a point. But this together with I.prop. 2.1.a yields the descriptions in (i) and (iv).

Finally, for $(g,q,a,e) = (9,3,2,4)$, if Γ is non-hyperelliptic, there is nothing to prove, so let us assume Γ to be hyperelliptic. Similarly as above for $q = 2$, the morphism associated to $L'' = |C_1|$ on X'' or rather to $|C|$ on X represents X as a double cover of a quadric cone $K \subset \mathbb{P}^3$ branched along a curve B consisting of 8 different lines through the vertex. Let z_0,z_1,z_2,z_3 be homogeneous coordinates on $\mathbb{P}^3$, assume K to be defined by $z_2^2 = z_0z_1$ and B by $b(z_0,z_1,z_2) = 0$, $\deg b(z) = 4$. Now $H^0(X,\mathcal{O}_X(2C)) \cong H^0(X'',\mathcal{O}_{X''}(2C_1))$ is of dimension 10, and contains all 10 forms z_iz_j, $i,j = 0,\ldots,3$, but because of the relation $z_2^2 = z_0z_1$ they span a subspace of codimension 1. However we can take $\sqrt{b}(z)$ as a tenth basiselement and in the same way as in the proof of I.prop. 2.1.a this shows that $\phi_{|2C|}$ is an isomorphism, so $X \cong \phi_{|2C|}(X) = \phi_{|2C_1|}(X'')$.

REMARK 2.2.1. As to possible rational singularities on the surfaces described in thm. 2.2, these only may occur if $q = a = 2$, $e = g - 3$, $g \geqq 6$, or if $q = 3$, $a = 2$, $e = g - 5$, $g \geqq 10$. (Also for $g > 10$ these values of q,a and e fit in the equations of prop. 1.4.b). Then, according to prop. 1.5.c, we find that there can be at most $g - 5$ resp. $g - 9$ (i.e. the maximal number of fibres in W'', which is equal to $e - 2q + 2$) A_1-singularities.

REMARK 2.2.2. In thm. 2.2 we do not assure the existence of the surfaces corresponding to the different values of g,q,a,e found there, but though we feel sure that for every set of values there exists at least one family of surfaces, we merely say which are at most possible, for so far we only have necessary conditions for their existence. Of course the surfaces described in thm. 2.2 in (i) and the last ones in (iii) and (iv) indeed occur. As to the others, we will content ourselves here with the following example, which describes a surface in $\mathbb{P}^6$ with canonical hyperplane sections which is not a cone or the Veronese embedding of a cone and which is constructed with the same method we will employ in §3 of this chapter for elliptic ruled surfaces.

EXAMPLE 2.2.3. We will give an example of a surface $X \subset \mathbb{P}^6$ with canonical hyperplane sections corresponding to the case $(g,q,a,e) = (6,2,2,3)$ in thm. 2.2(ii).

Let Γ be a curve of genus 2, and embed it in $\mathbb{P}^4$ with the complete system $|2K_\Gamma+2P|$, $P \in \Gamma$. Let $P_1,P_2 \in \Gamma$, such that $P_1 + P_2 \in |K_\Gamma|$, $P_1 \neq P_2$, $P_i \neq P$, $i = 1,2$. Let $y_0,\ldots,y_4$ be homogeneous coordinates on $\mathbb{P}^4$; we may assume them to be chosen such that, if H_i is the hyperplane defined by $y_i = 0$, then:

$$\begin{aligned}
\Gamma\cdot H_0 &= 2P_1 + 2P_2 + 2P \\
\Gamma\cdot H_1 &= P_1 + P_2 + 3P + Q \\
\Gamma\cdot H_2 &= 4P + 2Q \\
\Gamma\cdot H_3 &= P_1 + P + \sum_{i=1}^{4} S_i \\
\Gamma\cdot H_4 &= \sum_{j=1}^{6} T_j \ ,
\end{aligned}$$

with $Q, S_i, T_j \in \Gamma$, Q such that $P + Q \in |K_\Gamma|$, and $\{P, Q, P_1, P_2, S_1, \ldots, S_4, T_1, \ldots, T_6\}$ a set of fourteen different points, except for the possibility $P = Q$. Let x_0, x_1 be coordinates on $\mathbb{P}^1$, and let C_0 be the curve defined by $x_0 = 0$ on $\Gamma \times \mathbb{P}^1$.

Now X is the image of $\Gamma \times \mathbb{P}^1$ under the birational map $\psi\colon \Gamma \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^6$ defined by:

$$(z_0, \ldots, z_6) = (x_0^2 y_0, x_0^2 y_1, x_0^2 y_2, x_0^2 y_3, x_0 x_1 y_0, x_0 x_1 y_1, x_0^2 y_4 - x_1^2 y_0), \quad (1)$$

which has as its inverse

$$\begin{aligned}
&(x_0, x_1) \times (y_0, \ldots, y_4) = (z_0, z_4) \times \\
&\times (z_0^2, z_0 z_1, z_0 z_2, z_0 z_3, z_0 z_6 + z_4^2) \ . \qquad (2)
\end{aligned}$$

This shows that ψ induces a biregular correspondece of the open pieces $x_0 y_0 \neq 0$ resp. $z_0 \neq 0$, and $\psi(C_0) = (0, \ldots, 0, 1)$, which is the non-rational singularity of X .

To see that indeed X has canonical hyperplane sections consider the curve $C \subset X$ defined by

$$z_6 = x_0^2 y_4 - x_1^2 y_0 = 0 \ . \qquad (3)$$

Equation (3) represents C as a double cover of Γ, because $(y_0, y_4) \neq$ $\neq (0,0)$ on Γ . Let $\tau\colon C \to \Gamma$ be the morphism of degree 2; τ is branched over the six points T_i , $i = 1, \ldots, 6$, defined by $y_4 = 0$ on Γ , because for fixed y_0, y_4 , (3) has only a double root if $y_0 = 0$ or $y_4 = 0$, but over the points with $y_0 = 0$ the curve defined by (3) on $\Gamma \times \mathbb{P}^1$ has a double point, for there y_0 has a double zero, so these points do not count as branch points.

By (3), on C we have $x_1 = x_0 \sqrt{\frac{y_4}{y_0}}$; substituting this in (1) and dividing all coordinates by x_0^2 , we find that C is embedded in the hyperplane $H \subset \mathbb{P}^6$ given by $z_6 = 0$ in the following way:

$$(z_0,\ldots,z_5) = (y_0,y_1,y_2,y_3,y_0\sqrt{\frac{y_4}{y_0}},y_1\sqrt{\frac{y_4}{y_0}}) \ ,$$

with the y_i viewed as functions on C via the morphism $\tau\colon C \to \Gamma$.

Let R be the branch divisor on C. Then R is defined on C by $\sqrt{y_4} = 0$, and $K_C \sim \tau^* K_\Gamma + R$. Now one can check that the divisors D_i , cut on C by $z_i = 0$, all contain $\tau^*(P)$, and that $D_i - \tau^*(P) \sim$ $\sim K_C$, $i = 0,\ldots,5$. For instance,

$$\begin{aligned} D_5 &= C\cdot(z_5{=}0) = \operatorname{div}(y_1) + \operatorname{div}(\sqrt{y_4}) - \operatorname{div}(\sqrt{y_0}) = \\ &= \tau^*(P_1+P_2+3P+Q) + R - \tau^*(P_1+P_2+P) = \\ &= \tau^*(P+Q) + R + \tau^*(P) \sim \tau^* K_\Gamma + R + \tau^*(P) \ . \end{aligned}$$

3 ELLIPTIC RULED SURFACES WITH ONE NON-RATIONAL SINGULARITY

We will now treat the elliptic ruled surfaces, and for these we restrict ourselves to $g = 2,3$. So let now $\Gamma = E$ be an elliptic curve, $q = 1$, and let $Y = E \times \mathbb{P}^1$.

First we describe in prop. 3.1 the occurring surfaces X" together with the linear systems L" (cf. 1.3.a,b). Blowing up the base points gives us X' together with the shape of the exceptional divisor $\pi^{-1}(x) =$ $= \operatorname{supp}(W')$. There also we transform X" by elementary transformations into Y . Finally we find X in thm. 3.4, at least in the non-hyperelliptic case, as the image of Y under the birational map associated to the strict transform of L" on Y (cf. 1.3.c).

It turns out that the only occurring surfaces X" are of the form

- $X_1 = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-Q_1))$ (e=1) , and
- $X_2 = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-Q_1-Q_2))$ (e=2) , $Q_1,Q_2 \in E$,

and by II.prop. 1.2 we can assume the point Q_1 resp. the linear equivalence class of $Q_1 + Q_2$ to be fixed.

Let f_i be in both cases the fibre on X'' over Q_i , $i = 1,2$, and let $R = R_1 \in X_1$ and $R_1, R_2 \in X_2$ be points on f_1 resp. on f_1 and f_2 as in II.prop. 1.3.b,c. On X_1 , R_1 is fixed, on X_2 we will choose $R_1 \in f_1 \setminus C_0$ in a suitable way and then R_2 is determined. Using the same proposition, let

- $\varepsilon = \mathrm{elm}_{R_1} : X_1 \dashrightarrow Y$ resp. $\varepsilon = \mathrm{elm}_{R_2} \circ \mathrm{elm}_{R_1} : X_2 \dashrightarrow Y$; then we

denote by

- L_Y the strict transform via ε of L'' on Y , $C_Y \in L_Y$ a general curve,
- C_0 the strict transform of C_0 on Y , and by
- g_i the fibre on Y over Q_i , $i = 1,2$.
  (but a general fibre still by f).

In prop. 3.1 we give in each case the dual graph of the reduced divisor $W = W'_{red}$, but at that moment we cannot yet talk about it as the dual graph of the exceptional divisor $\pi^{-1}(x)$ of the only non-rational singularity, for there is still nothing that assures the existence of a corresponding surface X . However, the proof of thm. 3.4 will establish this existence for each of the cases below, so in fact we describe exceptional divisors $\pi^{-1}(x)$.

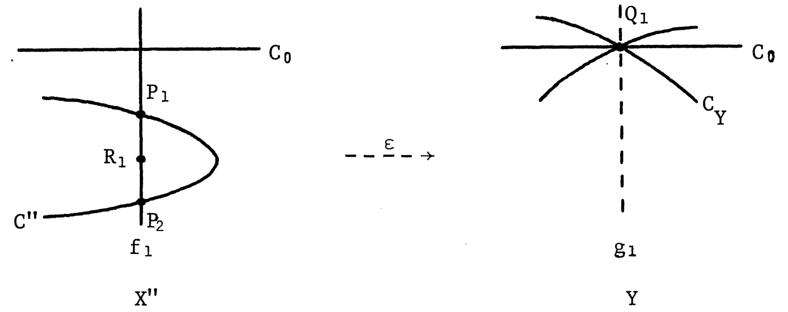
We use the following dual graph notation:

- $\overset{o}{-e}$ denotes a smooth elliptic curve with self-intersection $-e$;
- $*$ stands for a smooth rational curve with self-intersection -2 ;
- a line segment connecting two $*$'s or an o and an $*$ means that the corresponding curves intersect transversally in one point.

PROPOSITION 3.1. (a) *If* $g = 2$, *then* $a = 2$, $e = 1$, $r_1 = r_2 = 1$, $X'' = X_1$ *and* L'' *has two different simple base points* P_1, P_2 *on* f_1 , $P_i \neq R_1$, $i = 1,2$.

On Y , $L_Y \subset |2C_0+2Q_1\cdot f|$ *has one ordinary double point in* $Q_1 \in C_0$ *with both directions fixed and not equal to the direction of* C_0 *or* g_1 .

The dual graph of W *is* $\underset{-1}{\text{o}}\text{———}*$.

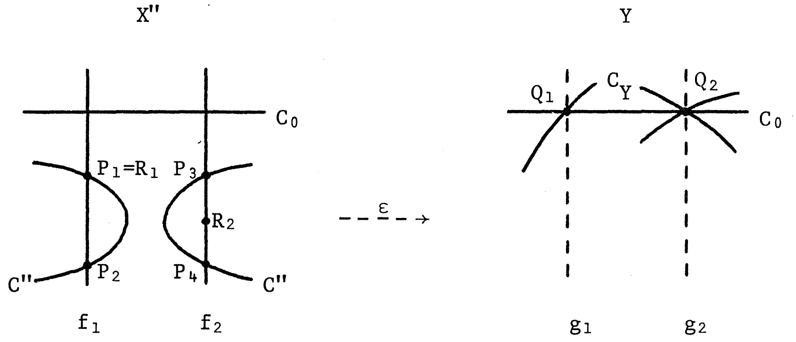


(b) *If* $g = 3$, *either* $(a,e) = (2,2)$ *or* $(3,1)$. *Then:*

(i) *if* $a = e = 2$, $r_1 = \dots = r_4 = 1$, $X'' = X_2$, *and assuming that a general* $C'' \in L''$ *is not hyperelliptic,* L'' *resp.* $L_Y \subset |2C_0+(Q_1+2Q_2)\cdot f|$ *on* Y *have one of the following configurations of base points:*

(i1)

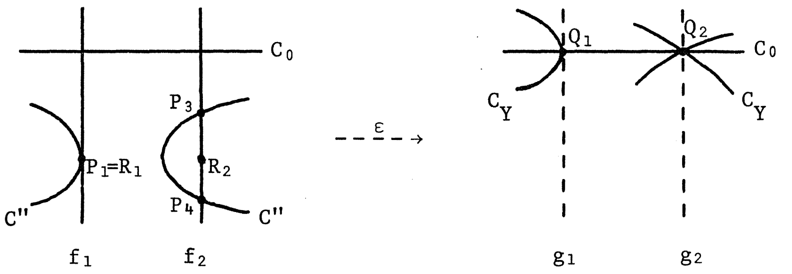
$Q_1 \neq Q_2$



dual graph of W: $* \text{———} \underset{-2}{o} \text{———} *$;

(i2)

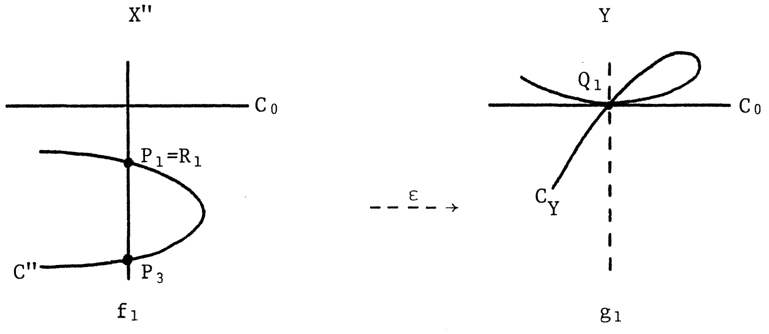
$Q_1 \neq Q_2$

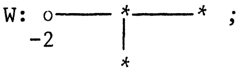


P_2 = *direction of* f_1 *in* P_1 ;

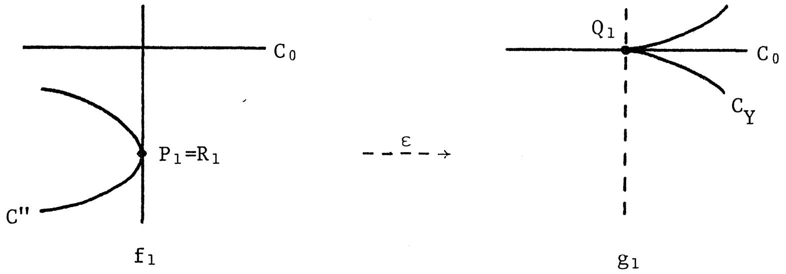
dual graph of W: $* \text{———} \underset{-2}{o} \text{———} *$;

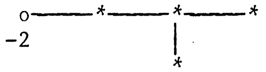
(i3)
$Q_1 = Q_2$.



P_2,P_4 = *fixed direction in* P_1 *resp.* P_3 ;
R_2 = *direction in* P_1 , $R_2 \neq P_2$;
dual graph of W:  ;

(i4)
$Q_1 = Q_2$

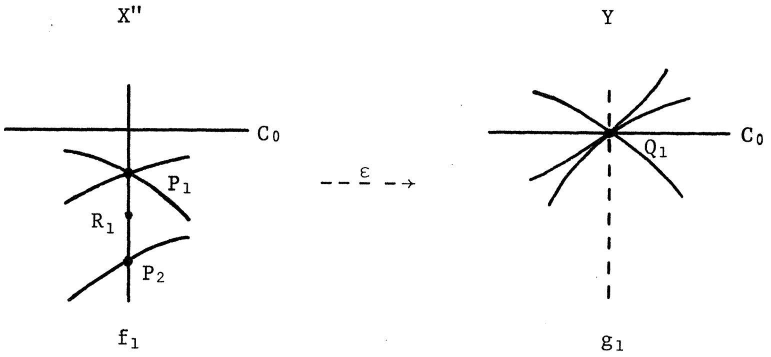


P_2 = *direction of* f_1 *in* P_1 ;
P_i *is a base point of order* i , P_i = *fixed direction in* P_{i-1} , i = 3,4 ;
R_2 = *direction in* P_1 ;
dual graph of W:  ;

(ii) *and if* $a = 3$, $e = 1$, $r_1 = 2$, $r_2 = 1$, $X'' = X_1$, *and assuming the curves of* L'' *to be non-hyperelliptic,* L'' *has two different base points* P_1 *and* P_2 *on* f_1 *of multiplicity* 2 *resp.* 1, $P_i \neq R_1$, $i = 1,2$.

On Y , $L_Y \subset |3C_0+3Q_1 \cdot f|$ *has one triple point in* $Q_1 \in C_0$ *consisting of a tacnode and a simple branch, both with fixed direction not equal to the direction of* C_0 *or* g_1 .

The dual graph of W *is* $\underset{-1}{\circ}\!-\!\!-\!\!-\!\!-\!*$.



PROOF. (a) In prop. 1.4, if $g = 2$, the only solution of (b6) is $a = 2$, $e = 1$, so $X'' = X_1$ and $L'' \subset |2C_1|$, and then (b4,5) give $\Sigma r_i = \Sigma r_i^2 = 2$, so $r_1 = r_2 = 1$.

On X_1, $W'' = 2C_0 + f_1$, and L'' has two simple base points P_1,P_2 on f_1 with possibly P_2 the direction of f_1 in P_1 (cf. prop. 1.2.c). Let us show that the configuration of the points P_1,P_2,R_1 must be as stated in the proposition.

First, $R_1 \neq P_i$, $i = 1,2$, for otherwise, because $C''\cdot C_1 = 2C_1^2 =$ $= 2e = 2$, and $\dim \mathrm{Tr}_{C''}|C_1| = \dim|C_1| = 1$ (use II. 1.1.4), we would have $\mathrm{Tr}_{C''}|C_1| = R_1 + g_1^1$, contradicting C'' being non-rational.

Second, if P_2 would be the direction of f_1 in P_1, $2P_1 \in \mathrm{Tr}_{f_1}|2C_1|$. However, $\dim \mathrm{Tr}_{f_1}|2C_1| = h^0(\mathcal{O}_{X_1}(2C_1)) - h^0(\mathcal{O}_{X_1}(2C_1-f_1)) - 1 = 1$ (again II. 1.1.4), so $\mathrm{Tr}_{f_1}|2C_1|$ contains exactly two divisors of the form $2S$, $S \in f_1$. Clearly, $S = C_0 \cap f_1$ or $S = R_1$. But $P_1 \notin C_0$ and $P_1 \neq R_1$, so we get a contradiction, and we are left with the desired configuration on X_1 .

To determine L_Y , apply $\varepsilon = \mathrm{elm}_{R_1}$ to X_1, and to get the dual graph of W , blow up P_1 and P_2 on X_1 .

(b) If $g = 3$, the formula of prop. 1.4.b6 has solutions $(a,e) =$ $= (2,2)$, $(3,1)$ and $(4,1)$.

If $(a,e) = (4,1)$, prop. 1.4.b4,5 give $\Sigma r_i = 4$ and $\Sigma r_i^2 = 12$, which is impossible. Let us examine the other two cases.

(i) If $a = e = 2$, and so $X'' = X_2$ and $L'' \subset |2C_1|$, the same formulas yield $\Sigma r_i = \Sigma r_i^2 = 4$, so $r_1 = \ldots = r_4 = 1$ and L'' has four simple base points $P_1,\ldots,P_4$ on $W'' = 2C_0 + f_1 + f_2$.

We now choose $R_1 = P_1 \in f_1$, then R_2 is fixed. Let M be the one-dimensional subsystem of $|C_1|$ passing through R_1 (and R_2). Then also $\dim \mathrm{Tr}_{C''}M = 1$, and if $R_2 = P_i$ for some $i \in \{2,3,4\}$, because $C''\cdot C_1 = $ $= 2C_1^2 = 2e = 4$, we find $\mathrm{Tr}_{C''}M = R_1 + R_2 + g_2^1$, and so C'' would be hyperelliptic.

We conclude that if we assume a general $C'' \in L''$ to be non-hyperelliptic, $R_2 \neq P_i$, $i = 2,3,4$. Using this, one finds with the help of prop. 1.2.c,d five possible configurations for the P_i , R_j , three with $Q_1 \neq Q_2$ and two with $Q_1 = Q_2$, four of which are sketched in the proposition. The only case not occurring is where $Q_1 \neq Q_2$ and C'' is tangent to both f_1 and f_2 , say in P_1 resp. P_3 with P_2 and P_4 the directions of f_1 and f_2 in resp. $P_1 = R_1$ and P_3 . As we shall presently show this case does not occur, because then we would necessarily have $R_2 = P_3$, which is not allowed as we saw above.

To prove this, let $M_0 = \mathrm{Tr}_F|2C_1|$, $F = f_1 + f_2$, $M_1 = \mathrm{Tr}_{f_1}|2C_1|$, N the subsystem of $|2C_1|$ of curves tangent to f_1 in P_1 , and $N_2 = \mathrm{Tr}_{f_2}N$. Now $\dim|2C_1| = 6$ (use II. 1.1.4), and with the help of ideal sheaf sequences as in II.prop. 1.3.b one finds $\dim M_0 = 3$ and $\dim M_1 = 2$, so M_1 is complete. This implies $\dim N_2 = \dim M_0 -$ $- \dim M_1 = 1$, and as $\deg(N_2) = 2$, N_2 contains exactly two divisors of the form $2S$, $S \in f_2$. Any curve of the form $2C_0 + f_1 + \Delta\cdot f$, $\Delta \sim Q_1 + 2Q_2$, is contained in N , so intersecting with f_2 we find $S = C_0 \cap f_2$ for one of them. Also, if $C_1^* \in M$, $2C_1^* \in N$, and as C_1^* passes through R_2 , we get $2R_2 \in N_2$, so $S = R_2$ is the other.

Now, if C'' is tangent to both f_1 and f_2 in P_1 resp. P_3 , we have $2P_3 \in N_2$, and as $P_3 \notin C_0$, this implies $P_3 = R_2$ as asserted.

The statements about L_Y and the dual graph of W follow in the same way as in (a).

(ii) If $a = 3$, $e = 1$, so $X'' = X_1$ and $L'' \subset |3C_1|$, the formulas give $\Sigma r_i = 3$ and $\Sigma r_i^2 = 5$, so $r_1 = 2$, $r_2 = 1$, L'' having a double base point P_1 and a simple one P_2 .

Again, R_1 cannot be equal to a base point, for if it were, because $C''\cdot C_1 = 3$ and $\dim \mathrm{Tr}_{C''}|C_1| = \dim|C_1| = 1$, the variable part of $\mathrm{Tr}_{C''}|C_1|$ would be a one-dimensional system on C'' of degree at most 2, implying C'' to be rational or hyperelliptic, which we do not want.

But this excludes the possibility of P_2 being the direction of f_1 in P_1 . For if this were the case, $3P_1 \in \mathrm{Tr}_{f_1}|3C_1|$. Using the ideal sheaf sequence of f_1 on X_1 one finds $\dim \mathrm{Tr}_{f_1}|3C_1| = 2$. Let $S = C_0 \cap f_1$. Intersecting f_1 with divisors of the form $3C_0$ + fibres, $2C_0 + C_1$ + fibres and $3C_1$, one finds $3S$, $2S + R_1$, $3R_1 \in \mathrm{Tr}_{f_1}|3C_1|$. This implies that S and R_1 are the only points $T \in f_1$ such that $3T \in \mathrm{Tr}_{f_1}|3C_1|$. So, as $P_1 \notin C_0$, this implies $P_1 = R_1$ which is not allowed. We conclude that only the configuration stated in the proposition is possible.

The rest is clear.

REMARK 3.1.1. As to additional rational singularities, looking at the figures in the above proposition and using prop. 1.5.c and its proof, one sees that only the surfaces corresponding to (b.i2) will contain one ordinary double point. See further (3.4.2).

We will now construct the surfaces X corresponding to each of the cases described in prop. 3.1. In particular, we get all possibilities for normal quartic surfaces in $\mathbb{P}^3$ with a singularity of genus 2. Instead of using the surfaces X'' with their linear systems L'' , we will work with $Y = E \times \mathbb{P}^1$ and the transformed systems L_Y .

Let us for a moment assume we are in the non-hyperelliptic case. We will then find X as the image of Y under the birational map associated to L_Y . Let L_Y be part of the complete system $|aC_0+\Delta\cdot f|$ on Y , $a \in \mathbb{Z}$, $a \geq 2$ and $\Delta \in \mathrm{Pic}(E)$, $b = \deg \Delta \geq 3$. Now consider the following diagram:

$$\begin{tikzcd}
Y = E \times \mathbb{P}^1 \arrow[r, "j"] \arrow[d, phantom] & \mathbb{P}^N \arrow[d, "r"] \arrow[dr, dashed, "P"] & \\
X'' \arrow[u, "\varepsilon"] \arrow[r, "k"] & \mathbb{P}^n \arrow[r, dashed, "q"] & \mathbb{P}^g \\
& X' \arrow[ul, "\phi"] \arrow[r, "\pi"] & X \arrow[u, "\phi_L = i \text{ (or h)}"']
\end{tikzcd}$$

diagram 3.2.

Here ε, ϕ, π and i are the same maps as before; the other maps are:

- j is the embedding corresponding to the complete system $|aC_0+\Delta\cdot f|$ on Y ;

- p is the projection of $j(Y) \subset \mathbb{P}^N$ according to the base points of L_Y , so $(p\circ j)(Y) = X$; we set $\psi = p\circ j = \phi_{L_Y}$;

- k is the morphism associated to the complete system $|aC_1|$ of which L'' is part;

- q is the projection of $k(X'') \subset \mathbb{P}^n$ according to the base points of L'' , so $(q\circ k)(X'') = X$;

- the strict transform via ε of $|aC_1|$ is a subsystem of $|aC_0+\Delta\cdot f|$ on Y ; r is the projection of $j(Y) \subset \mathbb{P}^N$ according to this inclusion of linear systems.

Let $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))$ and $H^0(E,\mathcal{O}_E(\Delta))$ have basis $\{x_0,x_1\}$ resp. $\{y_1,\ldots,y_b\}$. Then viewing the x_i and y_j as forms on Y via the projections onto $\mathbb{P}^1$ and E , $H^0(Y,\mathcal{O}_Y(aC_0+\Delta\cdot f))$, which has dimension $(a+1)\cdot b$, is spanned by all products of the form $x_0^{i_0}x_1^{i_1}y_j$, $i_0+i_1 = a$, $j = 1,\ldots,b$. In thm. 3.4 we will make a special choice of the bases $\{x\}$ and $\{y\}$, suitable with respect to the base points on Y . Now choosing these bases induces a linear transformation of the x_i's and y_j's and on $\mathbb{P}^N$ this amounts to a projective transformation. In general, however, this induces a non-linear transformation on $\mathbb{P}^n$ and a fortiori on $\mathbb{P}^g$. So in fact, we do not classify the surfaces X up to projective equivalence, but up to isomorphism, using more complicated transformations than projective ones on $\mathbb{P}^g$, which induce isomorphisms on X .

In the hyperelliptic case one also has a diagram as (3.2) above, but then $\phi_L = h$ and j need not be an embedding ($b=2$ is possible). Though then all surfaces which might appear as image $\bar{X} = h(X)$ (see the list in the beginning of I.§2) are projectively determined, still we will not find $\bar{X}$ together with the branch curve of h on it up to projective equivalence. For instance, it is easy to find an example of an isomorphism on a double cover X of $\mathbb{P}^2$ as in thm. 3.4.a below which induces a quadratic Cremona transformation on $\mathbb{P}^2$.

PROPOSITION 3.3. *Let* E *be a smooth elliptic curve over an algebraically closed field* k , $\mathrm{char}(k) \neq 2$. *Let* $Q_1,Q_2 \in E$ *and let* E *be embedded in* $\mathbb{P}^2$ *by the complete linear system* $|Q_1+2Q_2|$. *Then we can choose coordinates* y_0,y_1,y_2 *on* $\mathbb{P}^2$ *such that* E *is given by*

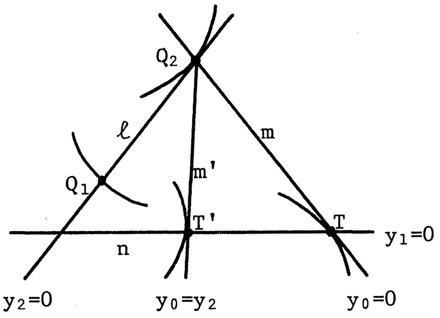
$$h(y) = y_0^3 - y_1^2y_2 - (1+\lambda)y_0^2y_2 + \lambda y_0y_2^2 +$$
$$+ \mu y_0^2y_1 - \mu y_0y_1y_2 = 0 ,$$

that $Q_2 = (0,1,0)$, *and that* $y_2 = 0$ *cuts on* E *the divisor* $Q_1 + 2Q_2$. *Furthermore,* $Q_1 = Q_2$ *iff* $\mu = 0$.

PROOF. Assume E to be embedded in $\mathbf{P}^2$ by $|Q_1+2Q_2|$. Let ℓ be the tangent line to E in Q_2 , let $m \neq \ell$ be a line through Q_2 tangent to E in some point T and let $n \neq m$ be a line through T intersecting E in another point T' such that the line m' through Q_2 and T' is tangent to E in T' .

We can assume $Q_2 \notin n$. If $Q_1 = Q_2$ this is clear, if $Q_1 \neq Q_2$ this follows from the facts that through Q_2 there pass four different tangent lines to E not equal to ℓ and that E is of degree 3.

Now choose coordinates y_i , $i = 0,1,2$, such that ℓ,m,n are defined by $y_2 = 0$, $y_0 = 0$ resp. $y_1 = 0$. Then $T' = (\alpha,0,\beta)$, $\alpha,\beta \in k$, $\alpha,\beta \neq 0$ and multiplying y_0 with a suitable scalar, we can assume $T' = (1,0,1)$.



Let $h(y) = 0$ be the equation of E . Because $Q_2 = (0,1,0)$ and $T = (0,0,1)$ lie on E , y_1^3 and y_2^3 do not appear in h , and because ℓ and m are tangent to E in Q_2 resp. T , $y_0y_1^2$ and $y_1y_2^2$ do not. So

$$h(y) = \alpha_0y_0^3 + \alpha_1y_1^2y_2 + \alpha_2y_0^2y_2 + \alpha_3y_0y_2^2 +$$
$$+ \alpha_4y_0^2y_1 + \alpha_5y_0y_1y_2 = 0 .$$

Because $(1,0,0) \notin E$, $\alpha_0 \neq 0$, and because E is smooth in $(0,1,0)$

and $(0,0,1)$, $\alpha_1,\alpha_3 \neq 0$. So dividing h by α_0 and replacing y_1 by a suitable scalar multiple we can assume $\alpha_0 = 1$, $\alpha_1 = -1$. Because $m: y_0 = y_2$ is tangent to E in T' ,

$$(1+\alpha_2+\alpha_3)y_0^3 + (\alpha_4+\alpha_5)y_0^2y_1 - y_0y_1^2 = 0$$

has $y_1 = 0$ as a double root, so $\alpha_2 = -1 - \alpha_3$ and $\alpha_5 = -\alpha_4$. Now put $\alpha_3 = \lambda$ and $\alpha_4 = \mu$ to get the desired equation.

The rest is clear.

REMARK 3.3.1. One can show that the condition for the curve defined by the equation $h(y) = 0$ above to be smooth is $\lambda(\lambda-1)((\mu^2-4)^2+16\lambda\mu^2) \neq 0$.

REMARK 3.3.2. If $\mu = 0$, $h(y) = 0$ is the familiar equation of an elliptic curve which is a double cover of $\mathbb{P}^1$ branched in $0,1,\lambda$ and ∞ , so $E \cong E(\lambda)$.

In thm. 3.4 we assume to be given coordinates x_0,x_1 on $\mathbb{P}^1$ such that C_0 on $Y = E \times \mathbb{P}^1$ is defined by $x_0 = 0$.

THEOREM 3.4. *Let* X *be a surface with canonical hyperplane sections of genus* g *, birationally equivalent to a ruled surface over an elliptic curve* E *with one non-rational singularity* x *. Let* $\pi: X' \to X$ *be the minimal resolution. Then:*

(a) *if* $g = 2$, X *is isomorphic to the double cover of* $\mathbb{P}^2$ *, branched along the three conics with fourfold contact in* $(0,0,1)$ *, given by*

$$z_1^2 = z_0z_2$$

$$z_1^2 = z_0z_2 - z_0^2$$

$$z_1^2 = z_0z_2 - \lambda z_0^2 \text{ , } \textit{with } \lambda \neq 0,1 \text{ .}$$

Here $E \cong E(\lambda)$, x *lies over the point* $(0,0,1)$, $\mathrm{Sing}(X) = \{x\}$ *, and* $\pi^{-1}(x)$ *has dual graph* $\underset{-1}{\circ}\text{———}*$ *;*

(b) *if* $g = 3$ *, and if we assume the hyperplane sections of* X *to be non-hyperelliptic, either:*

(i) X *is isomorphic to a surface in* $\mathbb{P}^3$ *given by*

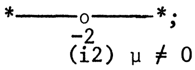
$$H(z) = z_0^3 z_1 - (z_1 z_3 - \alpha z_0 z_2 - z_2^2)^2 - (1+\lambda) z_0^2 z_1^2 + \lambda z_0 z_1^3 +$$

$$+ \mu z_0^2 (z_1 z_3 - \alpha z_0 z_2 - z_2^2) - \mu z_0 z_1 (z_1 z_3 - \alpha z_0 z_2 - z_2^2) = 0 \ ,$$

with $\lambda(\lambda-1)((\mu^2-4)^2+16\lambda\mu^2) \neq 0$, *and if* $\mu \neq 0$ *then* $\alpha^2 \neq 4\mu$;

Here $x = (0,0,0,1)$ *and if*

(i1) $\mu,\alpha \neq 0$, $\mathrm{Sing}(X) = \{x\}$ *and* $\pi^{-1}(x)$ *has dual graph*



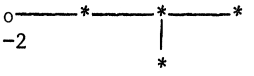
;

(i2) $\mu \neq 0$, $\alpha = 0$, $\mathrm{Sing}(X) = \{x,(-\mu,0,0,1)\}$, *with* $(-\mu,0,0,0,1)$ *an* A_1*-singularity and* $\pi^{-1}(x)$ *has dual graph as in* (i1) *;*

(i3) $\mu = 0$, $\alpha \neq 0$, $\mathrm{Sing}(X) = \{x\}$ *and* $\pi^{-1}(x)$ *has dual graph*

;

(i4) $\mu = \alpha = 0$, $\mathrm{Sing}(X) = \{x\}$ *and* $\pi^{-1}(x)$ *has dual graph*



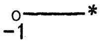
;

or:

(ii) X *is isomorphic to a surface in* $\mathbb{P}^3$ *given by*

$$H(z) = z_0^2 z_3^2 + (4z_1^3 + 6z_0 z_1 z_2) \cdot z_3 - 4z_0 z_2^3 - 3z_1^2 z_2^2 +$$

$$+ (1+\lambda)(2z_0 z_2 + 2z_1^2)^2 - \lambda(2z_0 z_2 + 2z_1^2) \cdot 2z_0^2 = 0 \ , \textit{ with } \lambda \neq 0,1 \ .$$

Here $E \cong E(\lambda)$, $x = (0,0,0,1)$, $\mathrm{Sing}(X) = \{x\}$, *and* $\pi^{-1}(x)$ *has dual graph* 

.

<u>PROOF.</u> (a) We will describe the rational map $\psi = \phi_{L_Y} : Y \dashrightarrow \mathbb{P}^2$ with L_Y as in prop. 3.1.a.

Let $y_0, y_1 \in H^0(E, \mathcal{O}_E(2Q_1))$ be a basis such that via $\phi_{|2Q_1|} : E \to \mathbb{P}^1$, Q_1 is the point $(y_0,y_1) = (0,1)$. Let $(y_0,y_1) = (1,0)$, $(1,1)$ and $(1,\lambda)$ be the other branch points of $\phi_{|2Q_1|}$. Now x_0 and y_0 have a zero of order 1 resp. 2 in $Q_1 \in Y$.

On E there exists a local parameter $y \in \mathcal{O}_{E,Q_1}$ in Q_1 , such that $y^2 \equiv y_0/y_1 \bmod m_{E,Q_1}^3$, $\mathcal{O}_{E,Q_1}$ and m_{E,Q_1} denoting the local ring of E in Q_1 and its maximal ideal. Let $\mathcal{O}$ be the local ring of Y in $Q_1 \in C_0$ and m its maximal ideal. Then $x = x_0/x_1$ and y are generators of m .

Because $L_Y \subset |2C_0+2Q_1\cdot f|$, the space of functions belonging to L_Y , $H^0(L_Y)$, is a subspace of $H^0(\mathcal{O}_Y(2C_0+2Q_1\cdot f))$, which is spanned by the six forms $x_0^{i_0}x_1^{i_1}y_j$, $i_0 + i_1 = 2$, $j = 0,1$. Let

$$G = \alpha_0x_0^2y_0 + \alpha_1x_0x_1y_0 + \alpha_2x_1^2y_0 + \alpha_3x_0^2y_1 +$$
$$+ \alpha_4x_0x_1y_1 + \alpha_5x_1^2y_1 \in H^0(L_Y) .$$

Because $G(Q_1) = 0$, $\alpha_5 = 0$, and because G must have a double point in Q_1 , $\alpha_4 = 0$. Writing the remaining form locally around Q_1 with $x_1 = y_1 = 1$, $x_0 = x$ and $y_0 \equiv y^2 \bmod m^3$, we get:

$$G \equiv \alpha_0x^2y^2 + \alpha_1xy^2 + \alpha_2y^2 + \alpha_3x^2 \mod m^3 .$$

As we must have an ordinary double point in Q_1 with fixed directions not equal to the direction of C_0 and the fibre g_1 , $\alpha_2,\alpha_3 \neq 0$ and the ratio α_2/α_3 is fixed. Multiplying x_0 with a suitable constant, we can assume $\alpha_2 = \alpha_3$, and so $H^0(L_Y)$ is spanned by $x_0^2y_0$, $x_0x_1y_0$ and $x_1^2y_0 + x_0^2y_1$.

Now ψ: $E \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ is given by:

$$(z_0,z_1,z_2) = (x_0^2y_0,x_0x_1y_0,x_1^2y_0+x_0^2y_1) . \qquad (1)$$

This shows that ψ factorizes through $\phi_{|2Q_1|} \times \mathrm{id} : E \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, which is branched along the four fibres lying over the branch points on E of $\phi_{|2Q_1|}$.

We can write ψ locally around $Q_1 \bmod m^3$, as

$$(z_0,z_1,z_2) = (x^2y^2,xy^2,y^2+x^2) . \qquad (2)$$

To blow up Q_1 , we put $y = xt$ in (2), and because Q_1 is a double point we then divide by x^2 , $x = 0$ giving the exceptional divisor E_1 arising from Q_1 , so then ψ is:

$$(z_0,z_1,z_2) = (x^2t^2,xt^2,t^2+1) . \qquad (3)$$

Now by diagram 3.2, $\psi = h\circ\pi\circ\phi^{-1}\circ\varepsilon^{-1}$. Indeed:

- by (1), ψ is only not defined in $x_0 = y_0 = 0$, i.e., in Q_1 , so Q_1 is blown up to a curve E_1 , and taking $y_0 = 0$ in (1) one sees that g_1 is contracted. Together this is $\varepsilon^{-1} = \mathrm{elm}_{Q_1} : E \times \mathbb{P}^1 \dashrightarrow X_1$; $E_1 = f_1$ on X_1 and g_1 is contracted to R_1 on f_1 . The fixed directions in Q_1 are now the points $P_1, P_2 \in f_1$. (see prop. 3.1.a).

- by (3), $\psi \circ \varepsilon$ is not defined in the points $(x,t) = (0, \pm\sqrt{-1})$, which are P_1 and P_2 , and so these points are blown up. This is $\phi^{-1} : X_1 \dashrightarrow X'$;

- taking $x_0 = 0$ in (1) and $x = 0$ in (3) one sees that C_0 resp. f_1 are contracted, both being mapped to the point $(0,0,1)$. These contractions together are $\pi : X' \to X$;

- let us finally consider h . The map $\phi_{|2Q_1|} \times \mathrm{id} : E \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ was branched along the fibres over $0, 1, \lambda$ and ∞ . Now the fibre over $\infty = (0,1)$ is g_1 , and is contracted to a point, so h is branched along the other three. Taking $y_1 = 0$, $y_1 = y_0$ and $y_1 = \lambda y_0$ in (1), we see that these three fibres are mapped to the conics with equations as announced in the theorem, and these have fourfold contact in $h(x) = (0,0,1)$. Now the rest is clear.

(b)(i) Here we describe the map ψ for L_Y as in prop. 3.1.b(i).

Let $y_0, y_1, y_2 \in H^0(E, \mathcal{O}_E(Q_1 + 2Q_2))$ be a basis such that E embedded by $|Q_1 + 2Q_2|$ has an equation $h(y) = 0$ as in prop. 3.3. Let us first assume $\mu \neq 0$, so $Q_1 \neq Q_2$ and $Q_1 = (-\mu, 1, 0)$. Let $\mathcal{O}_i$, m_i be the local ring with its maximal ideal of Q_i on Y , $i = 1,2$. Locally around Q_1 and Q_2 we take $x_1 = y_1 = 1$. Now m_1 is generated by $x = x_0$ and $t = y_2$, m_2 has generators $x = x_0$ and $u = y_0$ and $h(y) = 0$ gives the relation $y_2 \equiv \mu u^2 \bmod m_2^3$ in $\mathcal{O}_2$.

Now $H^0(L_Y) \subset H^0(\mathcal{O}_Y(2C_0 + (Q_1 + 2Q_2) \cdot f))$, which is spanned by the nine forms $x_0^{i_0} x_1^{i_1} y_j$, $i_0 + i_1 = 2$, $j = 0,1,2$. Of these, only $x_1^2 y_1$ is non-zero in Q_2 , and of the other eight, only $x_1^2 y_0$ is non-zero in Q_1 , so we can forget about these two. Of the remaining seven, $x_0 x_1 y_1$ is the only one with a zero of order 1 in Q_2 , where we want a double point, so also this form can be thrown away.

Consequently, any $G \in H^0(L_Y)$ can be written as

$$G = \alpha_0 x_0^2 y_0 + \alpha_1 x_0^2 y_1 + \alpha_2 x_0^2 y_2 + \alpha_3 x_0 x_1 y_0 + $$
$$+ \alpha_4 x_0 x_1 y_2 + \alpha_5 x_1^2 y_2 .$$

Writing G locally around Q_1 resp. Q_2 we get

$$G \equiv G_1 = -\alpha_0\mu x^2 + \alpha_1 x^2 + \alpha_2 x^2 t - \alpha_3\mu x +$$
$$+ \alpha_4 xt + \alpha_5 t \mod \mathfrak{m}_1^2 \text{ , and}$$
$$G \equiv G_2 = \alpha_0 x^2 u + \alpha_1 x^2 + \alpha_2\mu x^2 u^2 + \alpha_3 xu +$$
$$+ \alpha_4\mu xu^2 + \alpha_5\mu u^2 \mod \mathfrak{m}_2^3 .$$

In Q_2 we must have a double point with two different fixed directions, not equal to the direction of C_0 or g_2, so the quadratic part of G_2, $\alpha_1 x^2 + \alpha_3 xu + \alpha_5\mu u^2$, is a multiple of a fixed quadratic form

$$\beta_1 x^2 + \beta_3 xu + \beta_5\mu u^2 . \tag{4}$$

Now $\beta_1, \beta_5 \neq 0$, for otherwise G_2 is divisible by either u or x, and g_2 or C_0 would be a fixed component of L_Y. Dividing (4) by β_5 and replacing $x = x_0$ by a suitable multiple, we can assume $\beta_1 = \beta_5 = 1$. Then call $\beta_3 = \alpha$. As we must have two different directions in Q_2, (4), which is now $x^2 + \alpha xu + \mu u^2$, has two different roots, so $\alpha^2 \neq 4\mu$.

In Q_1 we must have a simple point with fixed direction. The linear part of G_1 is $-\alpha_3\mu x + \alpha_5 t = \alpha_5(-\alpha\mu x+t)$ by the arrangements above. So we already have a fixed direction in Q_1, never equal to the direction of C_0, and equal to the direction of g_1 if $\alpha = 0$ (case (i2)).

Summarizing, $H^0(L_Y)$ has basis $\{x_0^2 y_0, x_0^2 y_2, x_0 x_1 y_2, x_0^2 y_1 + \alpha x_0 x_1 y_0 + x_1^2 y_2\}$ if $\mu \neq 0$.

One can check that if $\mu = 0$ these four forms still satisfy the conditions of the base points,so then $H^0(L_Y)$ has the same basis. Then we are in case (i3) of prop. 3.1.b if $\alpha \neq 0$, in case (i4) if $\alpha = 0$.

The rational map $\psi = \phi_{L_Y} : E \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ is now given by

$$(z_0, z_1, z_2, z_3) =$$
$$= (x_0^2 y_0, x_0^2 y_2, x_0 x_1 y_2, x_0^2 y_1 + \alpha x_0 x_1 y_0 + x_1^2 y_2) , \tag{5}$$

and its inverse ψ^{-1} by

$$(x_0, x_1) \times (y_0, y_1, y_2) = (z_1, z_2) \times (z_0 z_1, z_1 z_3 - \alpha z_0 z_2 - z_2^2, z_1^2) . \tag{6}$$

This shows ψ to be birational and also $x = \psi(C_0) = (0,0,0,1)$.

We find the equation of $\psi(Y)$ by substituting the y_i as found in (6) in the equation $h(y) = 0$ of E in $\mathbb{P}^2$ (see prop. 3.3). This gives us an equation of degree 6 in the z_i , but it can be divided by z_1^2 , leaving us with the equation promised in the theorem. This indeed must be the equation of $\psi(Y)$, for, as ψ is birational, $\psi(Y)$ is a non-rational surface, so of degree at least 4 in $\mathbb{P}^3$.

Let us determine the singularities of this surface. By (5) and (6), ψ and ψ^{-1} are biregular on $x_0^2y_2 \neq 0$ resp. $z_1 \neq 0$, so $\mathrm{Sing}(\psi(Y)) \subset \subset \mathrm{Sing}(C)$, C being the plane section $z_1 = 0$; C has equation

$$z_2(\alpha z_0+z_2)(z_2^2+\alpha z_0z_2+\mu z_0^2) = 0 . \qquad (7)$$

If $\mu,\alpha \neq 0$, because $\alpha^2 \neq 4\mu$ (7) defines four different lines passing through $(0,0,0,1)$, so $\mathrm{Sing}(\psi(Y)) = \{x\}$.

If $\mu \neq 0$, $\alpha = 0$, (7) can be written as

$$z_2^2(z_2^2+\mu z_0^2) = 0 ,$$

so all singularities of $\psi(Y)$ lie on the line $z_1 = z_2 = 0$. Now one can compute that

$$\partial H/\partial z_1(\gamma,0,0,\delta) = \gamma^2(\gamma+\mu\delta) ,$$

which is 0 if $\gamma = 0$ giving x , or if $\gamma = -\mu$, $\delta = 1$. Writing H locally around $(-\mu,0,0,1)$ with $z_3 = 1$ and $z = z_0 + \mu$ we get:

$$H(z-\mu,z_1,z_2,1) = -(1+\lambda\mu^2)z_1^2 + \mu^2zz_1 - $$
$$- \mu^3z_2^2 + \text{terms of degree at least 3,}$$

so $(-\mu,0,0,1)$ is of type A_1 .

If $\mu = 0$, $\alpha \neq 0$, (7) gives two lines, both counted twice: $z_2 = 0$ and $\alpha z_0 + z_2 = 0$. In this case

$$\partial H/\partial z_1(\gamma,0,0,\delta) = \gamma^3 \text{ and} \qquad (8)$$
$$\partial H/\partial z_1(1,0,-\alpha,\delta) = 1 ,$$

so we get only x .

If $\mu = \alpha = 0$, (7) gives the line $z_2 = 0$ with multiplicity 4. Also here (8) holds, so again we get only x .

We conclude that $\psi(Y)$ is a quartic surface, birational to the elliptic ruled surface Y , with isolated singularities, containing only one non-rational singularity. But then only the situations of prop. 3.1.b(i) can occur, so $\psi(Y)$ must necessarily be the surface X corresponding to those situations, and $\pi^{-1}(x) = W$.

(ii) Let E embedded in $\mathbb{P}^2$ by $|3Q_1|$ have an equation as in prop. 3.3 with $\mu = 0$; $Q_1 = (0,1,0)$ with inflexional tangent $y_2 = 0$. Let $\mathcal{O},\mathfrak{m}$ be the local ring of Q_1 on Y with its maximal ideal, $\mathfrak{m}$ generated by $x = x_0/x_1$ and $y = y_0/y_1$. The equation of E gives, taking $y_1 = 1$ locally in Q_1 , $y_2 \equiv y^3 \mod \mathfrak{m}^4$.

In this case, $H^0(L_Y) \subset H^0(\mathcal{O}_Y(3C_0+3Q_1 \cdot f))$, which has as a basis the twelve forms $x_0^{i_0} x_1^{i_1} y_j$, $i_0 + i_1 = 3$, $j = 0,1,2$. In $Q_1 \in Y$ we must have a triple point. Now $x_0x_1^2y_0, x_1^3y_0, x_0^2x_1y_1, x_0x_1^2y_1$ and $x_1^3y_1$ have a zero of order less than 3 in Q_1 , and as their linear and quadratic part never cancel in a linear combination we can forget about them. So any $G \in H^0(L_Y)$ can be written as:

$$G = \alpha_0x_0^3y_0 + \alpha_1x_0^2x_1y_0 + \alpha_2x_0^3y_1 + \alpha_3x_0^3y_2 + \alpha_4x_0^2x_1y_2 + $$
$$+ \alpha_5x_0x_1^2y_2 + \alpha_6x_1^3y_2 .$$

Writing this locally around Q_1 with $x_1 = y_1 = 1$, $x_0 = x$, $y_0 = y$ and $y_2 \equiv y^3 \mod \mathfrak{m}^4$, this is:

$$G \equiv g = \alpha_0x^3y + \alpha_1x^2y + \alpha_2x^3 + \alpha_3x^3y^3 + \alpha_4x^2y_3 + $$
$$+ \alpha_5xy^3 + \alpha_6y^3 \mod \mathfrak{m}^4 . \tag{9}$$

Any such g has a triple point in Q_1 , so we only have to take care of the tacnode and the simple branch with fixed direction. To this end we blow up Q_1 . We put $y = xt$ in (9) and divide by x^3 to get:

$$g' = \alpha_0xt + \alpha_1t + \alpha_2 + \alpha_3x^3t^3 + \alpha_4x^2t^3 + \alpha_5xt^3 + \alpha_6t^3 . \tag{10}$$

The exceptional divisor E_1 arising from Q_1 is defined by $x = 0$, and t is an affine coordinate on E_1 . On E_1 we must have a double point

P_1 corresponding to the tacnode and a simple point P_2 corresponding to the simple branch. As both directions are not equal to either the direction of C_0 or of g_1, P_1 and P_2 are given in (10) by $(x,t) = (0,t_i)$, $t_i \neq 0,\infty$, $i = 1,2$. By multiplying $x = x_0$ by a suitable constant we can assume $t_1 = 1$, i.e. the direction of the tacnode is $y = x$ on Y. This implies:

$$g'(0,1) = \alpha_1 + \alpha_2 + \alpha_6 = 0 ,$$

$$\frac{\partial g'}{\partial x}(0,1) = \alpha_0 + \alpha_5 = 0 , \quad \text{and}$$

$$\frac{\partial g'}{\partial t}(0,1) = \alpha_1 + 3\alpha_6 = 0 , \quad \text{so}$$

$$g' = \alpha_0(xt-xt^3) + \alpha_3x^3t^3 + \alpha_4x^2t^3 + \alpha_6(-3t+2+t^3) . \tag{11}$$

Intersecting the resulting curves with E_1, i.e. taking $x = 0$ in (11), we get $(t^3-3t+2) = (t-1)^2(t+2) = 0$, so P_2 is now $(x,t) = (0,-2)$, and the direction of the simple branch in Q_1 on Y is given by $y = -2x$.

We conclude that $H^0(L_Y)$ has basis $\{x_0^3y_2, x_0^2x_1y_2, x_0^3y_0-x_0x_1^2y_2, -3x_0^2x_1y_0+2x_0^3y_1+x_1^3y_2\}$. Consequently, the birational map $\psi = \phi_{L_Y}$: $E \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ and its inverse are defined by

$$(z_0,z_1,z_2,z_3) =$$

$$= (x_0^3y_2, x_0^2x_1y_2, x_0^3y_0-x_0x_1^2y_2, -3x_0^2x_1y_0+2x_0^3y_1+x_1^3y_2) , \quad \text{and} \tag{12}$$

$$(x_0,x_1) \times (y_0,y_1,y_2) = (z_0,z_1) \times$$

$$\times (2z_0^2z_2+2z_0z_1^2, z_0^2z_3+3z_0z_1z_2+2z_1^3, 2z_0^3) . \tag{13}$$

Indeed $x = \psi(C_0) = (0,0,0,1)$, take $x_0 = 0$ in (12).

To get the equation $H(z) = 0$ of $\psi(Y)$, insert the expressions for the y_i of (13) in the equation $h(y) = 0$ of prop. 3.3 with $\mu = 0$, and divide the result by $2z_0^5$.

This time, ψ is biregular outside of $x_0y_2 = 0$ resp. $z_0 = 0$, so $\mathrm{Sing}(\psi(Y)) \subset \mathrm{Sing}(C)$, C the plane section $z_0 = 0$ with equation

$$z_1^2(4z_1z_3-3z_2^2+(1+\lambda)\cdot 4z_1^2) = 0 . \tag{14}$$

So C is a smooth conic with its tangent line $z_1 = 0$ counted twice, hence any singularity of $\psi(Y)$ lies on the line $z_0 = z_1 = 0$. However,

$$\partial H/\partial z_0(0,0,\gamma,\delta) = -4\gamma^3 ,$$

and this implies $\mathrm{Sing}(\psi(Y)) = \{x\}$.

Now the proof can be concluded in the same way as in (i) above.

REMARK 3.4.1. The equations in thm. 3.4 show that if $g = 2$ or if $g = 3$ and X is of type (ii), there exists up to isomorphism exactly one surface X for each elliptic curve. If $g = 3$ and X is of type (i), things are more complicated.

REMARK 3.4.2. In prop. 3.1.b(i) we excluded only one possible configuration of the base points, namely when $Q_1 \neq Q_2$ and $C'' \in L''$ is tangent to both f_1 and f_2 , because then we would get a system of hyperelliptic curves. By the same arguments as in (3.1.1), a surface X corresponding to this case would contain two ordinary double points, and indeed one can show them to exist. They turn out to be double covers of a quadric in $\mathbb{P}^3$ with vertex V , branched along four smooth hyperplane sections not going through V and having one point P in common, the singularity $x \in X$ lying over P and the rational double points both over V .

On the other hand, thm. 3.4.b(i) shows that all possible configurations, stated in prop. 3.1.b(i), give a non-hyperelliptic system.

Hence we conclude that degeneration of the surfaces of thm. 3.4.b(i) to a double quadric, which is then a cone, coincides with degeneration to a surface with two A_1-points.

CHAPTER IV

ELLIPTIC RULED SURFACES WITH TWO SIMPLE ELLIPTIC SINGULARITIES

In this chapter we will look into the one remaining case, corresponding to the second possibility of II.cor. 3.3.a. So now X is a surface with canonical hyperplane sections birationally equivalent to a ruled surface over an elliptic curve E and contains two simple elliptic singularities.

Our aim is to prove a theorem analogous to III.thm. 3.4 describing these surfaces when the genus g of their hyperplane sections equals 2 or 3. Because we will use the same method for this as in the preceding chapter, much of III.§1,3 applies here as well with only minor changes.

We conclude this chapter with a theorem about the moduli of these surfaces if $g = 2$, to be used in ch. V.

1 ADJUSTMENT OF THE CONSTRUCTION OF CHAPTER III

Here we indicate what can be taken over from III.§1 with the necessary modifications.

- III.diagram 1.1 is still valid if we write $\Gamma = E$, E a smooth elliptic curve, $q = 1$, and if we replace x by the two simple elliptic singularities x_0, x_1.

For the rest, everything up to III.prop. 1.2 still holds.

- In III.prop. 1.2 we omit (d).

- In III.construction 1.3 we have to take in (a) $W'' \in |-K_{X''}|$ to be a sum of two disjoint sections: $W'' = C_0 + C_1$. (cf. II.prop. 2.1.d).

Moreover, in the last sentence we can forget about the statement concerning $\pi^{-1}(x)$.

- Taking $q = 1$, III.prop. 1.4 still holds with the same proof if we leave out in (a) the equivalence $e = 2q - 2 = 0$ iff $D \sim -K_E$ because of the possibility of II.prop. 2.1.d with $e = 0$, $D \not\sim 0$.

Note that now, because all base points must lie on W" , they all lie on C_1 . Because of II.prop. 3.1.b(1), in every stage of blowing up points in passing from X" to X' , the anticanonical divisor arising from W" consists of the strict transforms of C_0 and C_1 .

Denoting by C_i' the strict transforms of these curves on X' , let $x_i = \pi(C_i')$, $i = 0,1$.

- Of III.prop. 1.5 we skip (c), leave (b) as it is, and replace (a) by

(PROPOSITION 1.5. a)' $(C_0')^2 = -e$ *and* $(C_1')^2 = e - k$, *where* $C_i' = \pi^{-1}(x_i)$ *is the exceptional divisor of* x_i *in the minimal resolution and* k *is the number of base points of* L" .

PROOF. As to C_0' , see the proof of III.prop. 1.5.a. As to C_1' , we saw above that all base points of L" lie on C_1 . So on C_1 (or on strict transforms of it) k smooth points are blown up. But then the assertion follows from the fact that $C_1^2 = e$.

- As in III.remark 1.5.1 C_0 is uniquely determined, as we must now be in the last case of II.prop. 2.1.d. The curve C_1 is a fixed section disjoint from C_0 .

2 CONSTRUCTION OF ELLIPTIC RULED SURFACES WITH CANONICAL HYPERPLANE SECTIONS CONTAINING TWO SIMPLE ELLIPTIC SINGULARITIES

This section is the analogon of III.§3.

Let us introduce the following notation:

- $Q = 0 \in E$ is a fixed point of E , assumed to be the zero with respect to addition on the abelian group E ;

- $\theta = \phi_{|2Q|}: E \to \mathbb{P}^1$. Let y_0, y_1 be coordinates on $\mathbb{P}^1$ such that $\theta(Q) = (1,0) = \infty$, and such that the other branch points of θ lie over $(y_0, y_1) = (0,1)$, $(1,1)$ and $(\lambda, 1)$;

- $\sigma: E \to E$ is the isomorphism interchanging the sheets of θ . Because we chose $Q = 0$, $\sigma = -\mathrm{id}$ on E , and so $\theta(E) = E/\langle \pm \mathrm{id} \rangle$;

- $N = \theta^* |\mathcal{O}_{\mathbb{P}^1}(2)| \subset |4Q|$;

- $X_i = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-iQ))$, $i = 1,2$. Let f_Q be on both surfaces the fibre over Q ;

- $R_i = f_Q \cap C_1$ on X_i , $i = 1,2$, and let $\varepsilon = \mathrm{elm}_{R_1}: X_1 \dashrightarrow Y$ resp. $\varepsilon = \mathrm{elm}_{R_1} \circ \mathrm{elm}_{R_2}: X_2 \dashrightarrow Y$; then we denote by

- C_i the strict transform of C_i on Y , $i = 0,1$, and by

- g_Q the fibre on Y over Q ,

L_Y, C_Y having the same meaning as in III.§3.

PROPOSITION 2.1. *Let* E *be a smooth elliptic curve over an algebraically closed field* k , $\mathrm{char}(k) \neq 2$.

(a) *Assume* $E \cong E(\lambda)$, *let* $Q \in E$ *and let* E *be embedded in* $\mathbb{P}^3$ *by the complete linear system* $|4Q|$. *Then we can choose coordinates* y_0, y_1, y_2, y_3 *on* $\mathbb{P}^3$ *such that* E *is given by*

$$(1) \qquad y_0^2 = y_1 y_3 \quad \text{and}$$

$$(2) \qquad y_2^2 = y_0 y_3 - y_0^2 - \lambda y_1 y_3 + \lambda y_0 y_1 \text{ , with } \lambda \neq 0,1 \text{ ,}$$

that $Q = (0,0,0,1)$, *and that* $y_1 = 0$ *cuts on* E *the divisor* $4Q$. *Furthermore, then* y_0, y_1, y_2 *have a zero in* Q *of order* 2,4 *resp.* 1.

(b) *Let* $Q, P_1, P_2 \in E$, $Q \neq P_i$, $i = 1,2$, *such that* $3Q \sim 2P_1 + P_2$, *and let* E *be embedded in* $\mathbb{P}^2$ *by the complete linear system* $|3Q|$. *Then we can choose coordinates* y_0, y_1, y_2 *on* $\mathbb{P}^2$ *such that* E *is given by*

$$h(y) = y_0^3 + y_1^2 y_2 + y_1 y_2^2 + \lambda y_0^2 y_2 + \mu y_0 y_1 y_2 = 0 \text{ ,}$$

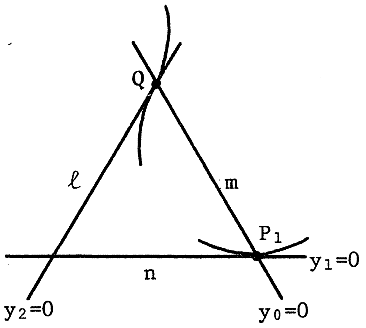
that $Q = (0,1,0)$, *that* $P_1 = (0,0,1)$, *and that* $y_2 = 0$ *and* $y_1 = 0$ *cut on* E *the divisors* $3Q$ *resp.* $2P_1 + P_2$. *Furthermore,* $P_1 = P_2$ *iff* $\lambda = 0$.

PROOF. (a) Let $\tilde{E}$ be the curve defined by (1) and (2) above, and project it from $(0,0,0,1) \in \tilde{E}$ into $\mathbb{P}^2(y_0,y_1,y_2)$. To find the image, substitute $y_3 = y_0^2/y_1$ ((1)) into (2). This gives the familiar equation of $E(\lambda)$, so $\tilde{E} \cong E$.

If $(0,0,0,1) = R \in E$, let $T: E \to E$ be the translation defined by $T(Q) = R$. Then $\phi_{|4R|} \circ T = \phi_{|4Q|}$ and because $y_1 = 0$ cuts on E four times the point $(0,0,0,1)$, $\phi_{|4Q|}(E)$ is indeed given by the desired equations with $Q = (0,0,0,1)$. The rest is clear.

(b) Assume E to be embedded in $\mathbb{P}^2$ by $|3Q|$. Let ℓ be the inflexional tangent line to E in Q , let m be the line connecting Q and P_1 and let n be the tangent line to E in P_1 .

Choose y_0,y_1,y_2 in a such a way that ℓ,m and n are defined by $y_2 = 0$, $y_0 = 0$ resp. $y_1 = 0$.



Because of these choices E has an equation

$$h(y) = \alpha_0 y_0^3 + \alpha_1 y_1^2 y_2 + \alpha_2 y_1 y_2^2 + \alpha_3 y_0^2 y_2 + \alpha_4 y_0 y_1 y_2 = 0 ,$$

with $\alpha_0,\alpha_1,\alpha_2 \neq 0$. Now divide by α_0 and multiply y_1 and y_2 with a suitable scalar to get $\alpha_0 = \alpha_1 = \alpha_2 = 1$. The rest is clear.

NOTATION 2.1.1. Let $\Delta_\alpha \in |4Q|$ be the divisor cut on $E \subset \mathbb{P}^3$ as in prop. 2.1.a by the plane $\sum_{i=0}^{3} \alpha_i y_i = 0$.

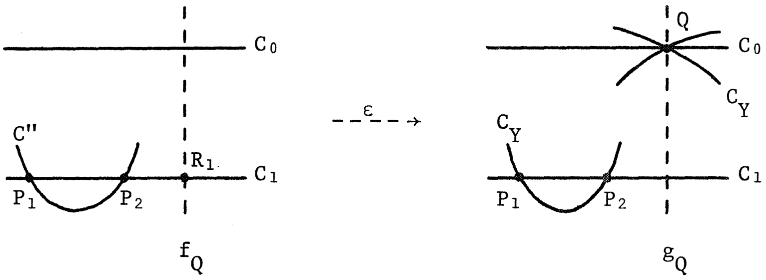
REMARK 2.1.2. In prop. 2.1.a, $y_0 = 0$ cuts on E the divisor $2Q + 2Q'$, $Q' = (0,1,0,0)$ (so $2Q' \in |2Q|$), $y_1 = 0$ the divisor 4Q and $y_3 = 0$ the divisor 4Q'. This means that $N = \{\Delta_\alpha/\alpha_2=0\}$, $N \subset |4Q|$ the earlier defined two-dimensional subsystem composite with $|2Q|$.

REMARK 2.1.3. For the equation in prop. 2.1.b there exists a polynomial in the coefficients, say $P(\lambda,\mu)$, such that it defines a smooth elliptic curve iff $P(\lambda,\mu) \neq 0$.

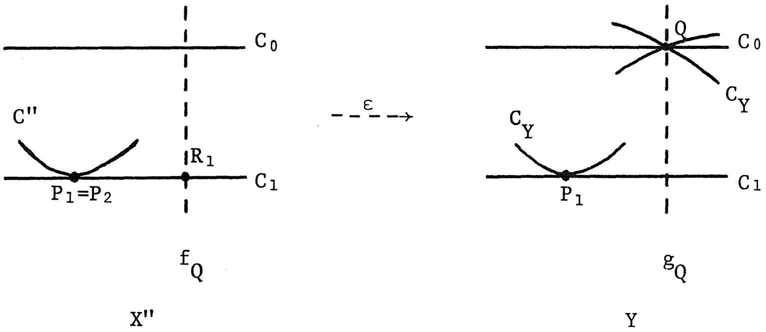
In prop. 2.2, which replaces III.prop. 3.1, we employ the usual terminology for rational double points of surfaces: a point is called of type A_ℓ if the dual graph of its exceptional divisor in the minimal resolution is $\underbrace{*\!-\!-\!-\!* \!-\!-\!- \cdots -\!-\!-\!* \!-\!-\!-\!*}_{\ell x}$, $\ell \geqq 1$ (dual graph notation of III.§3.).

PROPOSITION 2.2. (a) *If* $g = 2$, *then* $a = 2$, $e = 1$, $r_1 = r_2 = 1$, $X'' = X_1$ *and* L'' *resp.* $L_Y \subset |2C_0+2Q\cdot f|$ *have one of the following two configurations of base points:*

(a1)



(a2)
$1 \times A_1$

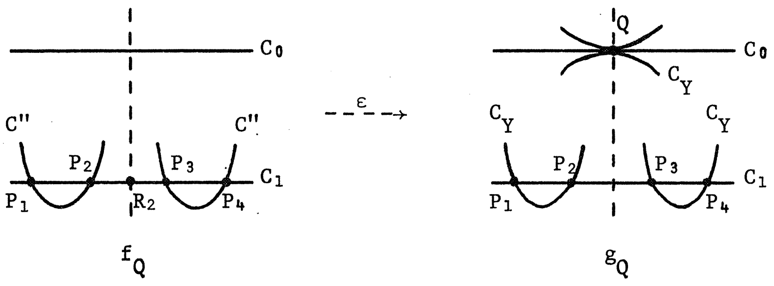


(b) *If* $g = 3$, *either* $(a,e) = (2,2)$ *or* $(3,1)$. *Then:*

(i) *if* $a = e = 2$, $r_1 = \ldots = r_4 = 1$, $X'' = X_2$, *and* $L'' \subset |2C_1| = |2C_0+4Q\cdot f|$. *Let* E *be given as in prop. 2.1.a, let* $\Delta_\alpha = \sum_{i=1}^{4} P_i$ *be the divisor of base points on* C_1 *and assume a general* C'' *to be non-hyperelliptic. Then* $\Delta_\alpha \not\subset N$.

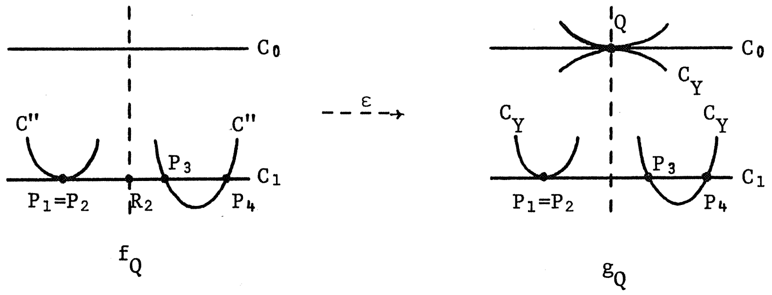
If Δ_α *does not contain* Q , L'' *and* $L_Y \subset |2C_0+4Q\cdot f|$ *have one of the following configurations of base points:*

(i1)



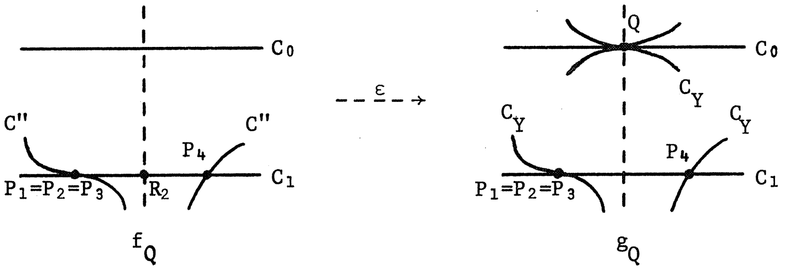
(i2)

1 × A_1



(i3)

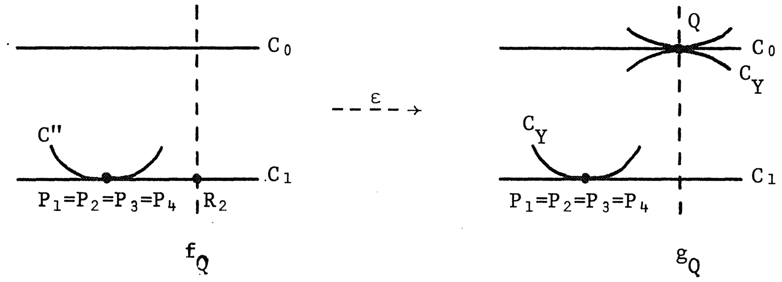
1 × A_2



X''　　　　Y

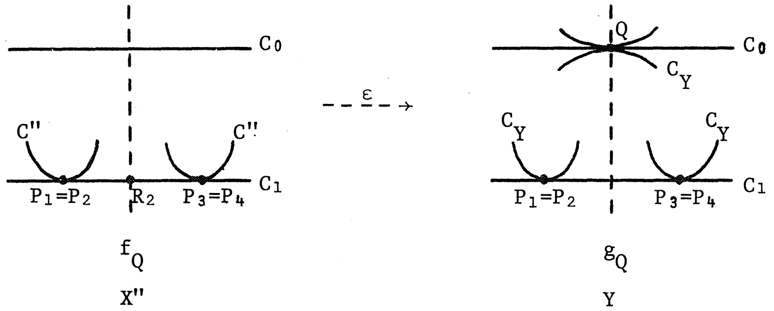
(i4)

1 × A_3



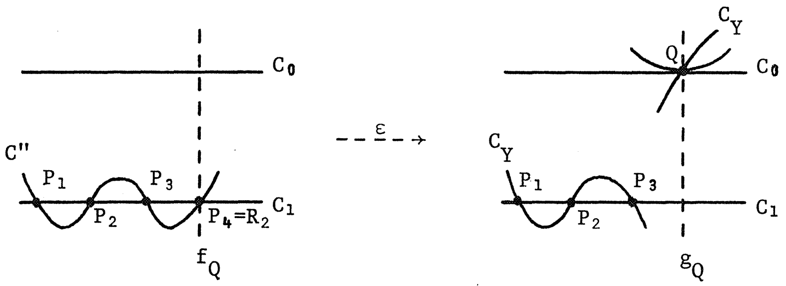
(i5)

2 × A_1



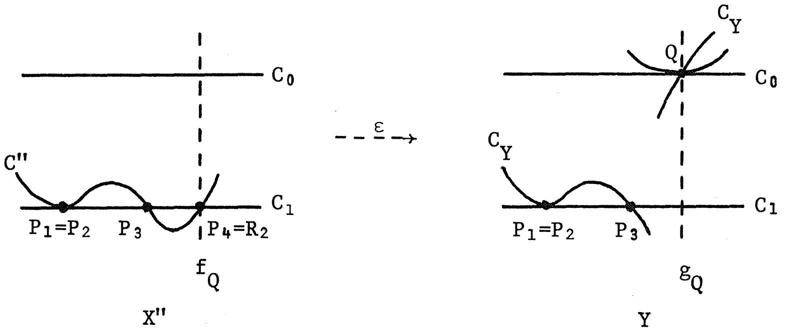
If Δ_α *contains* Q ; $L_Y \subset |2C_0+3Q\cdot f|$, *and* L'' *and* L_Y *have one of these configurations of base points:*

(i6)

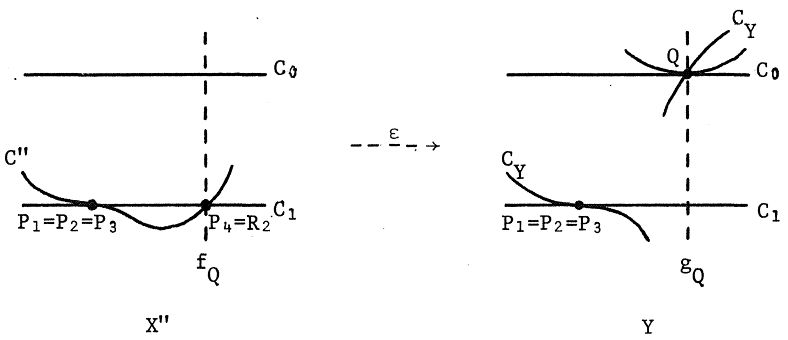


(i7)

1 × A_1

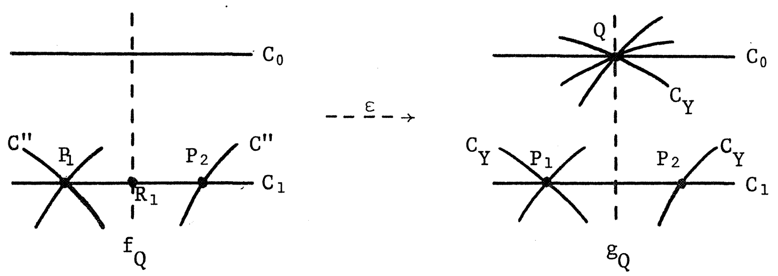


(i8)
$1 \times A_2$

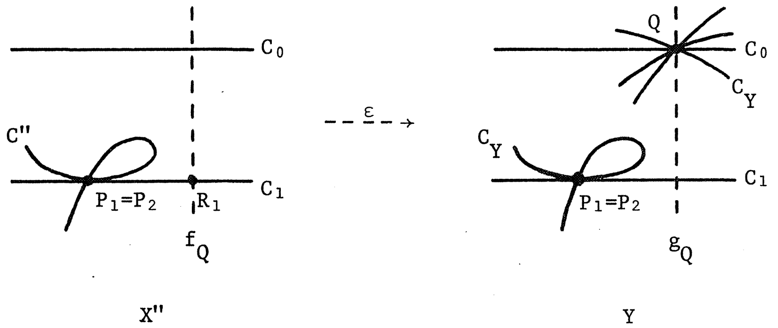


(ii) *and if* $a = 3$, $e = 1$, $r_1 = 2$, $r_2 = 1$, $X'' = X_1$, *and if we assume a general* $C'' \in L''$ *to be non-hyperelliptic,* L'' *resp.* $L_Y \subset |3C_0+3Q\cdot f|$ *have one of the following configurations of base points:*

(ii1)



(ii2)



Moreover, in each case of (a) *and* (b) *above the set of rational singularities, if present, of a corresponding surface* X *is indicated.*

<u>PROOF.</u> The values for a,e and the r_i are the same as in III.prop. 3.1 because III.prop. 1.4 still holds.

(a) It is clear that (a1,2) are the only possibilities once we know that $R_1 \neq P_i$, $i = 1,2$. But this is so for the same reason as in the proof of III.prop. 3.1.a.

To find additional rational singularities, note that the only rational curves with negative self-intersection on X' will be the strict transforms of the fibre(s) on which P_1, P_2 lie and the exceptional divisors arising from blowing up P_1, P_2 .

Blowing up P_1, P_2 one finds that in (a1) all these curves have self-intersection -1 , so no rational singularities, but in (a2) we get, with E_i the curve arising from blowing up P_i , $i = 1,2$:

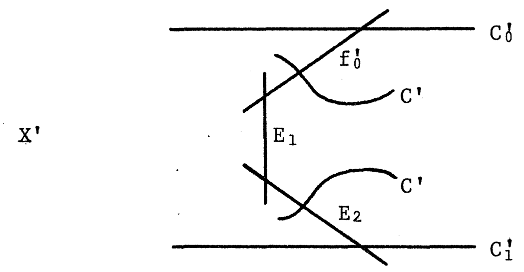


figure 2.2.1.

with $E_1^2 = -2$ and E_1 disjoint from a general C' , so if a corresponding surface X exists, $\pi(E_1)$ will be an A_1-singularity on it.

(b) (i) Let us first prove that $\Delta_\alpha \notin N$.

Assuming the contrary, let $M \subset |C_1|$ be the subsystem of curves passing through the base point P_1 of L'' , $\dim|C_1| = 2$, so $\dim M = 1$. Because $N_{C_1/X''} \cong \mathcal{O}_E(2Q)$, M passes through another fixed point $S \in C_1$ and $Tr_{C_1} M = P_1 + S \in |2Q|$.

But now, because Δ_α contains P_1 and $\Delta_\alpha \in N$, Δ_α also contains S , $S = P_i$ for some $i \in \{2,3,4\}$, and this implies that $Tr_{C''} M =$ $= P_1 + S + g_2^1$, for $C''.C_1 = 2C_1^2 = 4$, and this contradicts our assumption that C'' is non-hyperelliptic.

Now if Δ_α does not contain Q , applying ε it is easy to see that $L_Y \subset |2C_0+4Q\cdot f|$ and one finds configurations (i1)-(i5).

If Δ_α contains Q , it is necessarily with multiplicity 1 because $\Delta_\alpha \notin N$. Then in applying ε one has to blow up once $Q = R_2 \in C_1$, and so one finds $L_Y \subset |2C_0+3Q\cdot f|$, (i6)-(i8) giving no problems.

As to possible rational singularities, drawing pictures as in the proof of (a), it is easy to see that the coincidence of ℓ base points gives rise on X' to the exceptional divisor of an $A_{\ell-1}$-singularity disjoint from a general C' .

(ii) Here (ii1,2) are the only possibilities to be written down, because for the same reason as in the proof of III.prop. 3.1.b(ii), $R_1 \neq P_1,P_2$.

Blowing up P_1 and P_2 one will find that both in (ii1) and in (ii2) on X' all rational curves with negative self-intersection are intersected by a general C' , so no rational singularities occur.

At this point we could copy the discussion following III.remark 3.1.1 up to III.prop. 3.3; again not only projective transformations are involved to get the equations below. In this case we classify all normal quartic surfaces in $\mathbb{P}^3$ with two simple elliptic singularities up to isomorphism.

In the theorem below we choose coordinates x_0,x_1 on $\mathbb{P}^1$ such that on $Y = E \times \mathbb{P}^1$ the curve C_i is given by $x_i = 0$, $i = 0,1$. Moreover, we recall that a simple elliptic surface singularity $z \in Z$ is called of type $\tilde{E}_i$, if in the minimal resolution $\rho\colon Z' \to Z$ the smooth elliptic curve $\rho^{-1}(z)$ has self-intersection $i - 9$, $i = 7,8$.

<u>THEOREM 2.3.</u> *Let* X *be a surface with canonical hyperplane sections of genus* g , *birationally equivalent to a ruled surface over an elliptic curve* E , *with two simple elliptic singularities* x_0,x_1 . *Then:*

(a) *if* $g = 2$, X *is isomorphic to the double cover of* $\mathbb{P}^2$, *branched along the three conics given by*

$$z_1z_2 + \alpha z_0^2 = 0$$

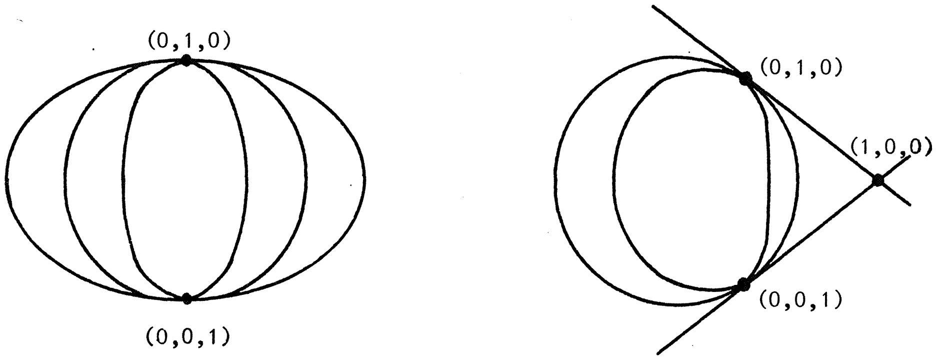
$$z_1z_2 + (\alpha-1)z_0^2 = 0$$

$$z_1z_2 + (\alpha-\lambda)z_0^2 = 0 \text{ , } \textit{with } \lambda \neq 0,1 \text{ ,}$$

which are tangent to each other in (0,1,0) *and* (0,0,1) . *Here* $E \cong E(\lambda)$, x_0 *and* x_1 *lie over* (0,1,0) *resp.* (0,0,1) , *both are of type* $\tilde{E}_8$, *and*

(a1) *if* $\alpha \neq 0,1,\lambda$, $\mathrm{Sing}(X) = \{x_0,x_1\}$,

(a2) *if* $\alpha = 0,1$ *or* λ , *one of the three conics consists of the tangent lines to the other two in* (0,1,0) *and* (0,0,1) , *which intersect in* (1,0,0) . *Then* $\mathrm{Sing}(X) = \{x_0,x_1,y\}$ *with* y *an* A_1*-singularity lying over* (1,0,0) .



$\alpha \neq 0,1,\lambda$ $\qquad$ $\alpha = 0,1$ or λ

branch curve of $h: X \to \mathbf{P}^2$.

(b) *if* $g = 3$, *and if we assume the hyperplane sections of* X *to be non-hyperelliptic, either:*

(i) X *is isomorphic to a surface in* $\mathbf{P}^3$ *given by*

$$H(z) = (z_2z_3-\alpha_0z_0z_1-\alpha_1z_1^2-\alpha_3z_0^2)^2 -$$
$$- \alpha_2^2(z_0^3z_1-z_0^2z_1^2-\lambda z_0^2z_1^2+\lambda z_0z_1^3) = 0 \text{ , } \textit{with}$$
$$\alpha_2 \neq 0 \text{ , } \lambda \neq 0,1 \text{ .}$$

Here $E \cong E(\lambda)$, $x_0 = (0,0,1,0)$, $x_1 = (0,0,0,1)$ *and both are of type* $\tilde{E}_7$.

Let $F(z_0,z_1) = H(z_0,z_1,0,0)$. *Then* $\mathrm{Sing}(X) = \{x_0,x_1\} \cup \cup \{(\zeta_0,\zeta_1,0,0)/F = 0$ *has a zero of order at least* 2 *in* $(\zeta_0,\zeta_1)\}$. *Moreover, if the order of zero of* F *in* (ζ_0,ζ_1) *is* ℓ , $(\zeta_0,\zeta_1,0,0)$ *is of type* $A_{\ell-1}$, *and so the set of rational singularities of* X *is* $\{A_1\}$, $\{2\times A_1\}$, $\{A_2\}$ *or* $\{A_3\}$;

or:

(ii) X *is isomorphic to a surface in* $\mathbf{P}^3$ *given by*

$$H(z) = z_0^3z_3 + z_1z_2^3 + z_1^2z_3^2 + \lambda z_0^2z_2^2 + \mu z_0z_1z_2z_3 = 0 \text{ ,}$$

with $P(\lambda,\mu) \neq 0$.

Here $x_0 = (0,0,0,1)$, $x_1 = (0,1,0,0)$, *both are of type* $\tilde{E}_8$ *and* $\mathrm{Sing}(X) = \{x_0,x_1\}$.

PROOF. (a) As in the proof of III.thm. 3.4 we have to describe $\psi = \phi_{L_Y}$, now with L_Y as in prop. 2.2.a.

Assume that $P_1 + P_2 \in |2Q|$ is cut on E via θ by $y_0' = y_0 - \alpha y_1 = 0$, $\alpha \in k$. Now any $G \in H^0(L_Y) \subset H^0(\mathcal{O}_Y(2C_0+2Q\cdot f))$ can be written as

$$G = \alpha_0 x_0^2 y_0' + \alpha_1 x_0^2 y_1 + \alpha_2 x_0 x_1 y_0' + \alpha_3 x_0 x_1 y_1 + \\ + \alpha_4 x_1^2 y_0' + \alpha_5 x_1^2 y_1 \ . \tag{1}$$

Because G must have a double point in $Q \in C_0$, $\alpha_2 = \alpha_4 = 0$, and because G has to cut $P_1 + P_2$ on C_1 , taking $x_0 = 1$, $x_1 = 0$ we find $\alpha_1 = 0$, and so a basis of $H^0(L_Y)$ is formed by $x_0x_1y_1, x_1^2y_1$ and $x_0^2(y_0-\alpha y_1)$. Consequently, $\psi\colon E \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ is defined by

$$(z_0,z_1,z_2) = (x_0x_1y_1, x_1^2y_1, x_0^2(y_0-\alpha y_1)) \ . \tag{2}$$

Omitting the little verifications as in the proof of III.thm. 3.4.a that this map is indeed as expected in prop. 2.2.a, let us find the branch curve of $h\colon X \to \mathbb{P}^2$.

By (2), ψ factorizes through $\theta \times \mathrm{id}\colon E \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, which is branched along the fibres over $(y_0,y_1) = (1,0)$, $(0,1)$, $(1,1)$ and $(\lambda,1)$, and using (2) we see that the first of these, which is the fibre over Q , is contracted. So, at least if $P_1 \neq P_2$ and hence these points do not lie on any of the three remaining fibres along which $\theta \times \mathrm{id}$ is branched, the branch curve of h consists of the images of these three fibres, the equations of which can be found by taking $y_0 = 0$, $y_0 = y_1$ and $y_0 = \lambda y_1$ in (2).

If $P_1 = P_2$, they lie on one of the three remaining fibres and $\alpha = 0,1$ or λ . Let us assume $\alpha = 0$, so $P_1 = P_2$ lies on $y_0 = 0$, things being analogous in the other two cases. Of course the images of $y_0 = y_1$ and $y_0 = \lambda y_1$ still belong to the branch curve of h , giving the second and third conic in thm. 2.3.a. Now consider figure 2.2.1. We will show that the branch curve of h is completed by the images of f_0' , the strict transform on X' of f_0 , the fibre through P_1 on X'' , and of E_2 .

To this end, let $x_0 = y_1 = 1$ locally at P_1, let $x_1 = x$ and let $y_0 \equiv y^2 \bmod \mathfrak{m}^3$, y a local parameter of E in P_1, $\mathfrak{m}$ the maximal ideal in the local ring of P_1 on E. Then ψ can be written mod $\mathfrak{m}^3$ as:

$$(z) = (x,x^2,y^2) . \qquad (3)$$

To blow up P_1, put $x = yt$, t a coordinate on E_1, in (3) and divide by the local equation $y = 0$ of E_1 to get

$$(z) = (t,yt^2,y) . \qquad (4)$$

Now P_2 is the point $(y,t) = (0,0)$. To blow it up, put $y = tu$, u a coordinate on E_2, in (4) and divide by the local equation $t = 0$ of E_2 to get

$$(z) = (1,t^2u,u) . \qquad (5)$$

Now by (3), f_0 (or f_0') given by $y = 0$ lies doubly over the line $z_2 = 0$, and by (5), E_2 given by $t = 0$ lies doubly over $z_1 = 0$ and so we find the conic $z_1z_2 = 0$ to form part of the branch curve too.

As to the singularities of X, by (2), $x_0 = \psi(C_0) = (0,1,0)$, $x_1 = \psi(C_1) = (0,0,1)$ and they are of type $\widetilde{E}_8$ by (prop. 1.5.a)'. Moreover, if $\alpha \neq 0,1,\lambda$, $(0,1,0)$ and $(0,0,1)$ are the only singularities of the branch curve and this gives (a1). If $\alpha = 0,1$ or λ, the branch curve has one more singularity, $(1,0,0)$, which is the intersection of two lines and so is an ordinary double point, and as a consequence then X has an A_1-point lying over it ((a2)). Indeed, by (4), E_1 given by $y = 0$ is mapped to $(1,0,0)$.

(b) (i) Referring to prop. 2.2.b(i), let us first assume that Δ_α does not contain Q. Then any $G \in H^0(L_Y) \subset H^0(\mathcal{O}_Y(2C_0+4Q\cdot f))$ is of the form

$$G = \sum_{i=0}^{3} \beta_i x_0^2 y_i + \sum_{j=0}^{3} \gamma_j x_0 x_1 y_j + \sum_{k=0}^{3} \delta_k x_1^2 y_k . \qquad (6)$$

Such a G has to cut on C_1 the divisor Δ_α, so taking $x_1 = 0$ we find that $\Sigma\beta_i x_0^2 y_i$ must be a multiple of the fixed form $\Sigma\alpha_i x_0^2 y_i$ and indeed this form is an element of $H^0(L_Y)$ because the factor x_0^2 takes care of the tacnode in $Q \in C_0$.

To satisy the conditions at Q , because $G(Q) = 0$, $\delta_3 = 0$, because $x_0x_1y_3$ and $x_1^2y_2$ are the only forms with a zero of order 1 at Q and their linear parts never cancel, $\gamma_3 = \delta_2 = 0$, and because $x_0x_1y_2$ and $x_1^2y_0$ are the only forms with a zero of order 2 (forgetting about the $x_0^2y_i$ for a moment) and a linear combination never has a tacnode at $Q \in C_0$, $\gamma_2 = \delta_0 = 0$.

We are now left with $x_0x_1y_0$, $x_0x_1y_1$ and $x_1^2y_1$. The second and third have a zero of order 5 resp. 4 in Q , so automatically satisfy the conditions of a tacnode, and the first does also, having a zero of order 3 and having C_0 as a direction in Q . We conclude that $\psi\colon E \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ and its inverse are defined by

$$(z_0,z_1,z_2,z_3) = (x_0x_1y_0, x_0x_1y_1, x_1^2y_1, \Sigma\alpha_i x_0^2y_i) \ , \tag{7}$$

and

$$(x_0,x_1) \times (y_0,y_1,y_2) = (z_1,z_2) \times$$
$$\times\ (\alpha_2z_0z_1, \alpha_2z_1^2, z_2z_3-\alpha_0z_0z_1-\alpha_1z_1^2-\alpha_3z_0^2) \ , \tag{8}$$

with $\alpha_2 \neq 0$ because $\Delta_\alpha \notin N$ and $\alpha_3 \neq 0$ because Δ_α does not contain Q .

Now if Δ_α contains Q , ψ and ψ^{-1} are still given by (7) and (8) with $\alpha_3 = 0$. For then, thinking of y_0,y_1,y_2 as being divided by a local parameter of E at Q , the subspace $H^0(\mathcal{O}_Y(2C_0+3Q\cdot f)) \subset H^0(\mathcal{O}_Y(2C_0+4Q\cdot f))$ is just spanned by the forms $x_0^{i_0}x_1^{i_1}y_j$, $i_0 + i_1 = 2$, $j = 0,1,2$, and one easily shows that in this case $H^0(L_Y)$ has exactly the same forms as in (7) as a basis.

To get the equation of $\psi(Y)$, substitute in prop. 2.1.a $y_3 = y_0^2/y_1$ ((1)) in (2) to get an equation of degree 3 in y_0,y_1,y_2 , replace these by the forms in the z_i found in (8) and divide by $\alpha_2z_1^2$.

Finally, let us have a look at $\mathrm{Sing}(\psi(Y))$. Writing $H(z) = $ $= H_2^2 - \alpha_2^2H_4 = 0$, we get

$$\partial H/\partial z_2 = 2z_3H_2 \quad \text{and} \tag{9}$$

$$\partial H/\partial z_3 = 2z_2H_2 \ , \tag{9'}$$

so if $x \in \mathrm{Sing}(\psi(Y))$, either $z_2(x) = z_3(x) = 0$ or $H_2(x) = 0$. If

$H_2(x) = 0$, $\partial H/\partial z_0(x) = \partial H/\partial z_1(x) = 0$ implies $z_0 = z_1 = 0$ because $H_4 = 0$ defines a reduced algebraic set in $\mathbb{P}^1(z_0,z_1)$, so we only find $x = x_0$ or x_1 . We conclude that anyway $\mathrm{Sing}(\psi(Y))$ is a finite set. Moreover, because $x_i = \psi(C_i)$, $i = 0,1$, x_0 and x_1 are non-rational singularities, so by II.cor. 3.3.a they have to be simple elliptic. But then, because the fibres of Y are embedded with degree 2 in $\mathbb{P}^3$, the surfaces $X = \psi(Y)$ necessarily have to fit the descriptions of prop. 2.2.b(i). On the one hand, with the help of (prop. 1.5.a)' this implies that the x_i are of type $\tilde{E}_7$. On the other, one easily computes that with E given as in prop. 2.1.a, finding $\Delta_\alpha = E \cap (\Sigma\alpha_i y_i=0)$ is solving $F = 0$, so $F = 0$ has a zero of order $\ell \geqq 2$ iff Δ_α contains a point P with multiplicity $\ell \geqq 2$ and by prop. 2.2.b(i) this is so iff X contains an $A_{\ell-1}$- rational singularity.

(b)(ii) Assume E to be given by $h(y) = 0$ as in prop. 2.1.b. Then with the same method as in foregoing cases one finds ψ and ψ^{-1} to be given by

$$(z_0,z_1,z_2,z_3) = (x_0^2x_1y_0,x_0^3y_1,x_0x_1^2y_2,x_1^3y_2) \ , \tag{10}$$

and

$$(x_0,x_1) \times (y_0,y_1,y_2) = (z_2,z_3) \times (z_0z_2z_3,z_1z_3^2,z_2^3) \ . \tag{11}$$

Now $h(z_0z_2z_3,z_1z_3^2,z_2^3) = 0$ can be divided by $z_2^3z_3^2$ to give the desired equation.

Let us find $\mathrm{Sing}(\psi(Y))$. By (10) and (11), ψ and ψ^{-1} are biregular on $x_0x_1y_2 \neq 0$ resp. $z_2z_3 \neq 0$, so $\mathrm{Sing}(\psi(Y))$ is contained in the set of singularities of the plane sections C_i defined by $z_i = 0$, $i = 2,3$, with equations

$$C_2\colon\ z_3(z_0^3+z_1^2z_3) = 0 \tag{12}$$

$$C_3\colon\ z_2^2(z_1z_2+\lambda z_0^2) = 0 \ . \tag{13}$$

This shows that C_2 is a cuspidal cubic with its inflexional tangent, so $\mathrm{Sing}(C_2)$ consists of its cusp, which is the point $(0,0,0,1) = x_0 = \psi(C_0)$ and its flex, which is $(0,1,0,0) = x_1 = \psi(C_1)$.

By (13), C_3 consists either of a smooth conic with its tangent line $z_2 = 0$ counted twice, or of two lines, $z_2 = 0$ counted three times. In both cases, $\mathrm{Sing}(C_3) = \{z_2=z_3=0\}$. However,

$$\partial H/\partial z_3(\gamma,\delta,0,0) = \gamma^3 ,$$

so for a point of $\mathrm{Sing}(C_3)$ to be singular on X, it must be x_1. We conclude that $\mathrm{Sing}(\psi(Y)) = \{x_0,x_1\}$. But then the proof of b(ii) can be finished in a similar but much less complicated way as the proof of b(i).

REMARK 2.3.1. In the same way as in ch. III for the surfaces of III.thm. 3.4.b(i) we want to avoid the impression that the coefficients appearing in the equations above form a set of moduli for the surfaces involved, though for $g = 2$ we are near (see §3).

REMARK 2.3.2. In prop. 2.2.b(i) we saw that if $C'' \in L''$ is non-hyperelliptic, $\Delta_\alpha \notin N$ (i.e. $\alpha_2 \neq 0$). However, if $\alpha_2 \neq 0$, $H(z) = 0$ in thm. 2.3.b(i) defines a proper quartic, so in fact X has non-hyperelliptic hyperplane sections iff $\Delta_\alpha \notin N$. Moreover, if $\alpha_2 = 0$, the same equation exhibits X as a double quadric.

3 MODULI OF THE DOUBLE COVERS OF $\mathbb{P}^2$

DEFINITION 3.1. We define M_j to be the *moduli variety* of surfaces with canonical hyperplane sections of genus 2, birational to $E \times \mathbb{P}^1$, E an elliptic curve with $j(E) = j$, and containing two simple elliptic singularities.

THEOREM 3.2. *If* $j \neq 0,1728$, $M_j \cong \mathbb{A}^1_k$.

PROOF. The variety M_j parametrizes the surfaces X of thm. 2.3.a for a fixed λ with $j = j(\lambda) \neq 0,1728$. Because two of these surfaces X are isomorphic iff their minimal resolutions X' are, let us look at the latter.

The description in prop. 2.2.a shows that such an X' is obtained by blowing up two points $P_1,P_2 \in C_1 \subset X_1$ defined by $y_0 - \alpha y_1 = 0$, $P_1 + P_2 \in |2Q|$, $P_i \neq Q$, $i = 1,2$. We now associate to $X' = X'_\alpha$ the point $\alpha = \theta(P_i) \in \mathbb{A}^1 = \mathbb{P}^1(y_0,y_1)\smallsetminus\{(1,0)\}$. The only thing left to do is to show that if $X'_{\alpha'}$ is gotten by blowing up $P_1',P_2' \in C_1$ on X_1, $P_1' + P_2'$ defined by $y_0 - \alpha' y_1 = 0$, $X'_\alpha \cong X'_{\alpha'}$ implies $\alpha = \alpha'$.

So let $f\colon X'_\alpha \to X'_{\alpha'}$ be an isomorphism. Then $f(C_0') = C_0'$ or C_1', so in either way f induces an automorphism $\tilde{f}\colon E \tilde{\to} E$. Because $\mathcal{N}_{C_i/X'_\alpha} \cong \mathcal{N}_{C'_i/X'_{\alpha'}} \cong \mathcal{O}_E(-Q)$ for $i = 0,1$, $\tilde{f}(Q) = Q = 0 \in E$. But then by [H],IV. 4.7, $\tilde{f} = \pm\mathrm{id}$. Also, assuming for a moment $P_1 \neq P_2$ (for $P_1 = P_2$ the argument is similar), because the only fibres on the surfaces X' consisting of two rational curves are those over the points to be blown up on X_1 , $\tilde{f}(\{P_1,P_2\}) = \{P'_1,P'_2\}$. Combining these two facts and reminding that P_1 and P_2 (resp. P'_1 and P'_2) are each others inverse on E , we find $P_1 + P_2 = P'_1 + P'_2 \in |2Q|$, and so $\alpha = \alpha'$.

The proof of thm. 3.2 shows that to every divisor $P_1 + P_2 \in |2Q|$, $P_i \neq Q$, i.e. to every $\alpha = \theta(P_i) \in \mathbb{P}^1$, $\alpha \neq \theta(Q) = \infty$, there corresponds up to isomorphism exactly one surface X with minimal resolution X' $(j\neq 0,1728)$. However, of course for $\alpha = \theta(Q)$, $Q = P_1 = P_2$, the surface X' still exists: blow up $Q \in C_1$ twice on X_1 . (The linear system L" on X_1 for this situation is composite with the pencil $|C_1| = |C_0+Q\cdot f|$, so we do not get an X). This gives us the opportunity to compactify M_j in a natural way to $\mathbb{P}^1$. To this end, let us make the following

DEFINITION 3.3. We define $\bar{M}_j$ to be the *moduli variety* of surfaces which are isomorphic to $X_1 = \mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{O}_E(-Q))$, E an elliptic curve with $j(E) = j$, $Q \in E$ a fixed point, blown up in two points P_1 , P_2 lying on a fixed section C_1 , $P_1 + P_2 \in |2Q|$.

COROLLARY 3.4. *If* $j \neq 0,1728$, $\bar{M}_j \cong \mathbb{P}^1_k$.

PROOF. Same as proof of thm. 3.2.

We can summarize this section in the following diagram:

$$\begin{tikzcd}
(E\setminus Q) \arrow[r, "\theta"] \arrow[d, hook] & M_j \cong \mathbb{A}^1 = \mathbb{P}^1\setminus\{\infty\} \arrow[d, hook] \\
E \arrow[r, "\theta"] & \bar{M}_j \cong \mathbb{P}^1
\end{tikzcd} \qquad \infty = \theta(Q) = (1,0) \ , \qquad \text{diagram 3.5.}$$

and so we can identify $\bar{M}_j$ with $\theta(E) = E/\langle\pm\mathrm{id}\rangle$, the points $0,1,\lambda,\infty \in \bar{M}_j$ corresponding to surfaces X' containing one special fibre consisting of three rational curves instead of two special fibres consisting of two, three of which $(0,1,\lambda)$ give an X with an A_1-double point, the fourth (∞) being the point of compactification not giving an X .

CHAPTER V

MIXED HODGE STRUCTURES ASSOCIATED TO RULED SURFACES WITH CANONICAL HYPERPLANE SECTIONS

In this chapter we study mixed Hodge structures (MHS's) associated to ruled surfaces with canonical hyperplane sections, so now we take the groundfield k to be the field of complex numbers $\mathbb{C}$. In fact we will be mainly interested in the MHS on the cohomologygroup $H^2(X_0,\mathbb{C})$, where $X_0 = X'\setminus\{\text{exceptional divisors of the non-rational singularities of } X\}$, X' being the minimal resolution of a surface X with canonical hyperplane sections birational to an irrational ruled surface, which we described in detail in chapters III and IV.

It turns out that this MHS on $H^2(X_0,\mathbb{C})$ is far more interesting in case X is birational to $E \times \mathbb{P}^1$, E an elliptic curve, and contains two simple elliptic singularities (see ch. IV) than when X only contains one non-rational singular point (ch. III), the reason being that in the first case the unique anti-canonical divisor $-K_{X'}$ on X' is reduced, so then X_0 carries a holomorphic 2-form with logarithmic poles on the two exceptional divisors, whereas in the second case it is non-reduced (see §2).

Before describing the MHS's of X_0 in §2, we give an outline of the MHS on the cohomology groups of an open surface in general (§1). In §3 we gather some information on extensions of MHS's to be applied in §5. Then in §4 we study more closely the MHS on $H^2(X_0,\mathbb{C})$ for X_0 as in ch. IV. Replacing $H^2(X_0,\mathbb{C})$ by a subspace defined in a way analogous to the way one defines primitive cohomology on a smooth proper surface, we derive an exact sequence (i.e. an extension) of polarized MHS's. In §5 we investigate this extension for the double covers of $\mathbb{P}^2$, and give a description of a period map for these surfaces. We conclude in §6 with some remarks on these matters for the two types of quartics in $\mathbb{P}^3$ with two simple elliptic singularities.

1 THE MIXED HODGE STRUCTURES OF A SMOOTH OPEN SURFACE

Here we will describe in as short a way as possible for our purposes the MHS on the cohomology groups of an open surface along the lines set out in [G-S],§5, sometimes using a little [D],§3, without explicitly referring to these papers.

As to MHS's, we use the standard notation contained in

DEFINITION 1.1. A *mixed Hodge structure* (MHS) H is a triple $(H_{\mathbb{Z}},\{W_k\},\{F^p\})$ with:

(i) $H_{\mathbb{Z}}$ a finitely generated Abelian group;

(ii) $\{W_k\}$ a finite increasing filtration of $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, the weight filtration (the numbers k such that $W_k/W_{k-1} \neq (0)$ are called the weights);

(iii) $\{F^p\}$ a finite decreasing filtration of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, the Hodge filtration,

such that $Gr_k^W H = W_k/W_{k-1}$ carries a Hodge structure (H.S.) of weight k with Hodge filtration induced by $\{F^p\}$:

$$F^p Gr_k^W H_{\mathbb{C}} = F^p \cap (W_k \otimes \mathbb{C})/F^p \cap (W_{k-1} \otimes \mathbb{C}) .$$

The numbers $h^{pq} = \dim_{\mathbb{C}}(W_k \otimes \mathbb{C}/W_{k-1} \otimes \mathbb{C})^{pq}$ are called the *Hodge numbers* of H $(k=p+q)$.

(1.2) Let us introduce the following notation:

- $\bar{Y}$ is a smooth projective complex surface;

- D is a reduced divisor on $\bar{Y}$, having only smooth components and normal crossings; let $D = \bigcup_j D_j$ be the decomposition in irreducible components;

- let $\bigcup_{k \neq \ell} (D_k \cap D_\ell) = \bigcup_r \{P_r\}$ be the union of points of intersection of the components of D . We assume to be chosen a fixed ordering of the D_j , so to each P_r there corresponds an ordered pair (k,ℓ) , $k < \ell$, such that $P_r \in D_k \cap D_\ell$;

- let $D^{[s]}$ be the disjoint union of the s-fold intersections of the components D_j of D , so $D^{[0]} = \bar{Y}$, $D^{[1]} = \coprod_j D_j$ and $D^{[2]} = \bigcup_r \{P_r\}$;

- let $Y = \bar{Y} \setminus D$ and let $j: Y \to \bar{Y}$ be the embedding.

Our aim is to describe the MHS's $H^i(Y)$, $i = 0,1,\dots,4$, denoting by $H^i(V)$ for a smooth quasi-projective complex variety V the MHS on the vector space $H^i(V,\mathbb{C})$ with lattice $H^i(V,\mathbb{Z})$.

Of course also in this case $H^i(Y,\mathbb{C}) \cong H^i(A^*(Y))$, $A^*(V)$ the complex of global complex-valued C^∞-forms on a smooth complex manifold V .

(1.3) Let $A^*(Y,\log D)$ be the "C^∞-log complex" which is a subcomplex of $A^*(Y)$; $A^i(Y,\log D) = \{\omega \in A^i(Y)/\omega$ extends to an i-form on $\bar{Y}$ with logarithmic poles along $D\}$. We say that ω has logarithmic poles along D if

(i) locally near a smooth point $P \in D$, where D is defined by $z_1 = 0$, ω can be written as $\omega = \eta \wedge \frac{dz_1}{z_1}$, η regular on $\bar{Y}$ near P , and if

(ii) locally near $P \in D_k \cap D_\ell$ where D is defined by $z_1 \cdot z_2 = 0$, ω can be written as $\omega = \eta \wedge \frac{dz_j}{z_j}$, $j = 1,2$ or as $\omega = \eta \wedge \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$, with η regular on $\bar{Y}$ near P .

Analogously one can define the analytic *sheaves* $\Omega^i_{\bar{Y}}(\log D)$ of holomorphic i-forms on $\bar{Y}$ with logarithmic poles along D , $i = 0,1,2$. Of course, then $\Omega^0_{\bar{Y}}(\log D) \cong \mathcal{O}_{\bar{Y}}$ and $\Omega^2_{\bar{Y}}(\log D) \cong \mathcal{O}_{\bar{Y}}(K_{\bar{Y}}+D)$.

(1.4) Now the most important step on the way to the desired MHS's is, that one can prove that the inclusion of complexes $A^*(Y,\log D) \hookrightarrow A^*(Y)$ induces an isomorphism $H^i(A^*(Y,\log D)) \xrightarrow{\sim} H^i(A^*(Y))$, $i = 0,1,\dots,4$ on cohomology, and so $H^i(Y,\mathbb{C}) \cong H^i(A^*(Y,\log D))$. As a consequence, every cohomology class in $H^i(Y,\mathbb{C})$ can be represented by a global differential form ω on Y which extends on $\bar{Y}$ to a form with logarithmic poles along D .

(1.5) If $\omega \in A^*(Y,\log D)$ and ω involves everywhere at most one dz_i/z_i , i.e. if ω can be written around any point $P \in D_j = (z_j=0)$ as $\omega = \eta \wedge \frac{dz_j}{z_j}$ for every j , with η regular, the residue map $R: A^*(Y,\log D) \to A^{*-1}(D^{[1]})$ is defined as $R\omega = (\mathrm{res}_{D_1}\omega,\dots,\mathrm{res}_{D_j}\omega,\dots)$, with

$$\operatorname{res}_{D_j}\omega = \operatorname{res}_{D_j}(\eta\wedge\frac{dz_j}{z_j}) = \eta/D_j \in A^{*-1}(D_j) .$$

It is a fact that R carries over to cohomology to give a residue map

$$R: H^i(Y,\mathbb{C}) \to H^{i-1}(D^{[1]},\mathbb{C}) .$$

(1.6) Let us define the following Gysin maps, which are induced by inclusions and are in fact already defined over $\mathbb{Z}$:

$$\begin{aligned} d_1&: H^0(\cup\{P_r\},\mathbb{Q}) \to H^2(\amalg D_j,\mathbb{Q}) \\ d_2&: H^0(\amalg D_j,\mathbb{Q}) \to H^2(\bar{Y},\mathbb{Q}) \\ d_3&: H^1(\amalg D_j,\mathbb{Q}) \to H^3(\bar{Y},\mathbb{Q}) \\ d_4&: H^2(\amalg D_j,\mathbb{Q}) \to H^4(\bar{Y},\mathbb{Q}) . \end{aligned}$$

To be more precise for d_1 , if $P \in D_k \cap D_\ell$, $k < \ell$, then on the $H^0(P)$-part of $H^0(\cup\{P_r\},\mathbb{Q})$, the map d_1 is as follows:

$$\begin{array}{ccc} H^0(P,\mathbb{Q}) & \to & ..\oplus H^2(D_k,\mathbb{Q}) \oplus \ldots \oplus H^2(D_\ell,\mathbb{Q}) \oplus \ldots \\ 1 & \to & (\ldots\ldots,1,\ldots\ldots\ldots\ldots\ldots,-1,\ldots\ldots\ldots) . \end{array}$$

<u>REMARK 1.6.1.</u> From this description it follows that d_1 is injective if D does not contain cycles.

(1.7) Finally, let us write down the resulting Hodge and weight filtration on $H^i(Y,\mathbb{C})$, $i = 0,1,\ldots,4$. As $H^0(Y,\mathbb{C}) = \mathbb{C}$ and $H^4(Y,\mathbb{C}) = (0)$ because for the MHS $H^4(Y)$ the equalities $H^4(Y,\mathbb{Q}) = W_4 = \operatorname{coker}(d_4)$ hold and d_4 is surjective (of course we assume $D \neq 0$) , the only interesting cases are $i = 1,2,3$.

Let $i = 1$. The weight filtration on $H^1(Y,\mathbb{Q})$ is

$$(1.7.1)\quad \left\{ \begin{array}{l} 0 \subset \underset{W_1}{\underset{\parallel}{H^1(\bar{Y},\mathbb{Q})}} \subset \underset{W_2}{\underset{\parallel}{H^1(Y,\mathbb{Q})}} \quad \text{(inclusion via } j^*\text{) , with} \\ \\ W_2/W_1 \cong \ker(H^0(\amalg D_j,\mathbb{Q}) \xrightarrow{d_2} H^2(\bar{Y},\mathbb{Q})) . \end{array} \right.$$

<u>REMARK 1.7.2.</u> Note that $W_1 = W_2$ if all D_j are exceptional for the same morphism.

As to the Hodge filtration ,

$$(1.7.3)\qquad \begin{cases} H^1(Y,\mathbb{C}) = F^0 \supset F^1 \supset F^2 = (0) \text{ , with} \\ F^0/F^1 \cong H^1(\bar{Y},\mathcal{O}_{\bar{Y}}) \text{ and } F^1 \cong H^0(\bar{Y},\Omega^1_{\bar{Y}}(\log D)) \text{ .} \end{cases}$$

Let $i = 2$. Now the weight filtration has length 3: $0 \subset W_2 \subset W_3 \subset$ $\subset W_4 = H^2(Y,\mathbb{Q})$, with

$$(1.7.4)\qquad \begin{cases} W_2 \cong \operatorname{coker}(d_2) = \operatorname{im}(j^*:H^2(\bar{Y},\mathbb{Q}) \to H^2(Y,\mathbb{Q})) \\ W_3/W_2 \cong \ker(d_3) = \ker(H^1(\amalg D_j,\mathbb{Q}) \to H^3(\bar{Y},\mathbb{Q})) \\ W_4/W_3 \cong \ker(d_1) = \ker(H^0(\cup\{P_r\},\mathbb{Q}) \to H^2(\amalg D_j,\mathbb{Q})) \text{ .} \end{cases}$$

REMARK 1.7.5. Note that $W_4 = W_3$ in case $\cup\{P_r\} = \emptyset$, or more generally, in case D does not contain cycles (cf. (1.6.1)).

REMARK 1.7.6. As a special instance of a general fact, the map $W_3 \otimes \mathbb{C} \to W_3 \otimes \mathbb{C}/W_2 \otimes \mathbb{C}$ coming from the inclusion $W_2 \subset W_3$ is induced by the residue map R (cf. (1.5)).

The Hodge filtration is the following:

$$(1.7.7)\qquad \begin{cases} H^2(Y,\mathbb{C}) = F^0 \supset F^1 \supset F^2 \supset F^3 = (0) \text{ with} \\ F^0/F^1 \cong H^2(\bar{Y},\mathcal{O}_{\bar{Y}}) \text{ , } F^1/F^2 \cong H^1(\bar{Y},\Omega^1_{\bar{Y}}(\log D)) \text{ ,} \\ \text{and } F^2 \cong H^0(\bar{Y},\mathcal{O}_{\bar{Y}}(K_{\bar{Y}}+D)) \text{ .} \end{cases}$$

Let $i = 3$. Then the weight filtration is $0 \subset W_3 \subset W_4 = H^3(Y,\mathbb{Q})$ with

$$(1.7.8)\qquad W_3 \cong \operatorname{coker}(d_3) = H^3(\bar{Y},\mathbb{Q})/d_3(H^1(\amalg D_j,\mathbb{Q})) \text{ , and}$$

$$(1.7.9)\qquad W_4/W_3 \cong \ker(d_4)/\operatorname{im}(d_1) = \\ = \text{homology of } (H^0(\cup\{P_r\},\mathbb{Q}) \xrightarrow{d_1} H^2(\amalg D_j,\mathbb{Q}) \xrightarrow{d_4} H^4(\bar{Y},\mathbb{Q})) \text{ .}$$

For the Hodge filtration we have

$$(1.7.10)\qquad \begin{cases} H^3(Y,\mathbb{C}) = F^1 \supset F^2 \supset F^3 = (0) \text{ with} \\ F^1/F^2 \cong H^2(\bar{Y},\Omega^1_{\bar{Y}}(\log D)) \text{ and} \\ F^2 \cong H^1(\bar{Y},\mathcal{O}_{\bar{Y}}(K_{\bar{Y}}+D)) \text{ .} \end{cases}$$

REMARK 1.7.11. In fact, taking into account that W_k/W_{k-1} is of weight k, it would be more precise to add in each case above a suitable factor of $2\pi i$ in the formulas for W_k/W_{k-1}. For instance, if $i = 2$, we had better write $W_3/W_2 \cong \frac{1}{2\pi i} \ker(d_3)$.

2 COMPUTATION OF MIXED HODGE STRUCTURES ASSOCIATED TO RULED SURFACES WITH CANONICAL HYPERPLANE SECTIONS

In this section we actually compute the MHS on the cohomology groups $H^i(X_0,\mathbb{C})$, X_0 a smooth open surface derived from a ruled surface with canonical hyperplane sections, to be defined in a minute.

In the same way as in ch. III, IV we will be able to prove certain facts about these MHS's in general for each of these two types of surfaces, the surfaces described in ch. IV turning out to be far more interesting in this respect than those of ch. III, but from a certain point on one has to examine different cases separately, which we will only do, at least in detail, for the double covers of $\mathbb{P}^2$ of IV.thm. 2.3.a.

Now let X be a surface with canonical hyperplane sections, birational to $\Gamma \times \mathbb{P}^1$, Γ a smooth curve of genus $q \geq 1$, and let $\pi: X' \to X$ be its minimal resolution. For these surfaces we will employ the notation as introduced in former chapters. By II.cor. 3.3 we know that either X contains one non-rational singularity x (see ch. III) or two simple elliptic ones x_0,x_1 (ch. IV), in which case $q = 1$, and we will call X (and X_0, see definition below) of type III resp. IV accordingly. We now define

- $D = \pi^{-1}(x)$ resp. $D = \pi^{-1}(x_0) + \pi^{-1}(x\ )$ to be the divisor on X' consisting of the exceptional divisors of the non-rational singularities,

- $X_0 = X' \setminus D$, and $j: X_0 \to X'$ to be the embedding.

If X is of type III, $D = C'_0$ + smooth rational curves, $C'_0 \cong \Gamma$, and if X is of type IV, $D = C'_0 + C'_1$, the $C'_i \cong \Gamma$ two disjoint sections (Γ elliptic), and in either case D has only normal crossings and does not contain cycles (II.cor. 3.3.c).

The reason why we study this X_0 rather than $X \setminus \mathrm{Sing}(X)$ is the following. As we will see, $H^2(X_0)$ with X_0 of type IV is the most interesting MHS we will encounter. Now taking all surfaces X of type IV, birational to $E \times \mathbb{P}^1$ with the elliptic curve E fixed, and with the same set of numbers $\{g,a,e,r_1,\dots,r_k\}$, we will find that $H^2(X_0,\mathbb{C})$ has constant rank, a certain extension in which $H^2(X_0,\mathbb{C})$ fits becoming more trivial when X acquires more additional rational singularities, whereas the rank of $H^2(X\setminus\mathrm{Sing}(X),\mathbb{C})$ decreases, the more rational singular points X contains.

DEFINITION 2.1. We denote by $\mathbb{Z}(k)$ the *one-dimensional Hodge structure* H *of weight* $-2k$, with $H_{\mathbb{Z}} = (2\pi i)^k\cdot\mathbb{Z} \subset \mathbb{C} = H_{\mathbb{C}} = H^{-k,-k}$.

As usual, we write $H^1(E)(-1)$ for the Hodgestructure H on the first cohomology of an elliptic curve E with weight shifted from 1 to 3, which has $H_{\mathbb{Z}} = H^1(E,\mathbb{Z}) \otimes \mathbb{Z}(-1) = \frac{1}{2\pi i}\cdot H^1(E,\mathbb{Z}) \subset H^1(E,\mathbb{C}) = H_{\mathbb{C}}$.

THEOREM 2.2. *Let* X *be a surface with canonical hyperplane sections birational to* $\Gamma \times \mathbb{P}^1$, Γ *a smooth curve of genus* $q \geq 1$, *and let* $\pi: X' \to X$ *be the minimal resolution. Let* X_0 *be defined as above. Then:*

(a) $H^1(X_0) \cong H^1(\Gamma)$;

(b) *if* X *is of type III,* $H^2(X_0) \cong j^*H^2(X')$ *is a pure H.S. of weight* 2 *and type* (1,1);

if X *is of type IV and so* $\Gamma = E$ *is an elliptic curve, there exists an exact sequence of MHS's*

$$0 \to j^*H^2(X') \to H^2(X_0) \to H^1(E)(-1) \to 0 ,$$

and the Hodge numbers of $H^2(X_0)$ *are* $h^{0,2} = h^{2,0} = 0$, $h^{1,1} = h^{1,1}(H^2(X')) - 2 > 0$, $h^{1,2} = h^{2,1} = 1$ *and* $h^{2,2} = 0$;

(c) *if* X *is of type III,* $H^3(X_0) = (0)$;

if X *is of type IV,* $H^3(X_0) \cong \mathbb{Z}(-2)$.

PROOF. (a) By (1.7.2) with $\bar{Y} = X'$ and $Y = X_0$, we get $W_1 = W_2$ for the weight filtration of $H^1(X_0,\mathbb{Q})$ because we only leave out curves exceptional for π, so by (1.7.1) $H^1(X_0,\mathbb{Q}) = W_1 = H^1(X',\mathbb{Q})$ and so $H^1(X_0) \cong H^1(X')$, which is of course isomorphic to $H^1(\Gamma)$ because X' is smooth, ruled over Γ.

(b) Referring to (1.7.4), let $0 \subset W_2 \subset W_3 \subset W_4 = H^2(X_0,\mathbb{Q})$ be the weight filtration. Because both when X_0 is of type III and of type IV, D does not contain cycles, $W_3 = W_4$ by (1.7.5).

In case X is of type III, D consists of $C_0' \cong \Gamma$ and smooth rational curves, so $H^1(\amalg D_j, \mathbb{Q}) \cong H^1(\Gamma, \mathbb{Q})$. Also, because X' is smooth and ruled over Γ, $H^3(X', \mathbb{Q}) \cong H^1(\Gamma, \mathbb{Q})$, and now, as $d_3 : H^1(\amalg D_j, \mathbb{Q}) \to$ $\to H^3(X', \mathbb{Q})$ is a Gysin map, d_3 is an isomorphism, and we get $W_2 = W_3$ because $W_3/W_2 = \ker(d_3) = (0)$. So in this case we end up with $H^2(X_0, \mathbb{Q}) =$ $= W_2 = j^* H^2(X', \mathbb{Q})$, hence $H^2(X_0)$ is of type $(1,1)$ because $H^2(X')$ is, X' being a ruled surface.

If X is of type IV, $D = C_0' + C_1'$, the $C_i' \cong E$ two disjoint sections, so $H^1(\amalg D_j, \mathbb{Q}) \cong H^1(C_0', \mathbb{Q}) \oplus H^1(C_1', \mathbb{Q})$, and in the same way as above $d_3 : H^1(C_0', \mathbb{Q}) \oplus H^1(C_1', \mathbb{Q}) \to H^3(X', \mathbb{Q}) \cong H^1(E, \mathbb{Q})$ is an isomorphism on both factors, so $W_3/W_2 = \ker(d_3)$ is, for instance by projection onto $H^1(C_1', \mathbb{Q})$, isomorphic to $H^1(E, \mathbb{Q})$. Now the filtration $0 \subset W_2 \subset W_3 = H^2(X_0, \mathbb{Q})$ produces the desired exact sequence, keeping in mind that W_3/W_2 has weight 3 so $H^1(E)$ appears tensorized with $\mathbb{Z}(-1)$.

As to the Hodgenumbers, the h^{pq} of $H^2(X_0)$ with $p + q = 2$ are those of $j^* H^2(X') \cong H^2(X')/d_2(H^0(C_0' \cup C_1'))$, so because X' is ruled, $h^{0,2} = h^{2,0} = 0$, and because C_0' and C_1' are exceptional for π, so independent in cohomology, $h^{1,1} = h^{1,1}(H^2(X')) - 2$, which is positive because we still have the class of a hyperplane section left. The numbers $h^{1,2} = h^{2,1}$ are those of the graded part of weight 3, $H^1(E)(-1)$, and so equal 1, and $h^{2,2} = 0$ because $W_4 = W_3$.

(c) Let $0 \subset W_3 \subset W_4 = H^3(X_0, \mathbb{Q})$ be the weight filtration, In (b) we saw that d_3 is surjective, so by (1.7.8) $W_3 = (0)$.

If X is of type III, consider the sequence (1.7.9) in our situation with $\bar{Y} = X'$. Then, because of II.cor. 3.3.c, $\#(\cup\{P_r\}) =$ (number of irreducible components D_j of D) $- 1$, and (1.6.1) says that d_1 is injective. As d_4, being a Gysin map, is surjective and $\dim H^4(X', \mathbb{Q}) = 1$, the alternating sum of dimensions in (1.7.9) is 0, so $W_4 = W_3 = (0)$, proving the assertion.

If X is of type IV, there are no P_r in (1.7.9), so then $W_4 =$ $= W_4/W_3 = \ker(H^2(C_0' \cup C_1', \mathbb{Q}) \to H^4(X', \mathbb{Q}))$, which implies that $\dim H^3(X_0, \mathbb{C}) =$ $= 1$, and $H^3(X_0)$ is pure of weight 4.

<u>REMARK 2.2.1.</u> Thm. 2.2.b indeed shows what we asserted in the beginning of this section, namely that $\dim H^2(X_0, \mathbb{C})$ is constant in a family of surfaces of type IV. For taking a set of numbers $\{g, a, e, r_1, \ldots, r_k\}$ means that we have to blow up k points to get X' from a minimal model X'', so $h^{1,1}(H^2(X')) = 2 + k$. Then $\dim H^2(X_0, \mathbb{C}) = \dim j^* H^2(X', \mathbb{C}) +$

$+ 2 = h^{1,1} + 2 = k + 2$.

REMARK 2.2.2. In thm. 2.2.c we found $H^3(X_0,\mathbb{C}) \cong \mathbb{C}$ for X_0 of type IV. Looking at the Hodge filtration in (1.7.10), we find $F^2 \cong H^1(X',\mathcal{O}_{X'})$, because $D = C_0' + C_1'$ and $K_{X'} \sim - C_0' - C_1'$ on X' . Because X' is ruled over an elliptic curve E , $H^1(X',\mathcal{O}_{X'}) \cong \mathbb{C}$, and so $H^3(X_0,\mathbb{C}) = F^2$. As a consequence, the fact that $H^3(X_0,\mathbb{C}) \neq (0)$ is apparently due to the fact that $K_{X'} + D \sim 0$, whereas in the case of surfaces of type III, though supp W' = supp D $(\{W'\}=|-K_{X'}|)$, $D - W' \not\sim 0$ because W' is not reduced. We will see in the sequel that exactly this fact causes the MHS $H^2(X_0)$ for a surface of type IV to be not a pure H.S.

3 EXTENSIONS OF MIXED HODGE STRUCTURES

In this section we gather some information about extensions of MHS's necessary for §5.

Let A and B be fixed MHS's. An extension of B by A is a short exact sequence

$$(3.1) \qquad 0 \to A \xrightarrow{i} H \xrightarrow{p} B \to 0$$

of MHS's. The question is, given A and B , which MHS's H can be placed in the middle? Now different exact sequences like (3.1) may define extensions which are in some sense the same. One equivalence relation, used by J. Carlson in [Ca], is congruence: the extension of (3.1) is said to be congruent with $0 \to A \xrightarrow{i'} H' \xrightarrow{p'} B \to 0$ if there exists a commutative diagram (with all maps morphisms of MHS's):

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"'] & H \arrow[r, "p"] \arrow[d, "\eta"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & A \arrow[r, "i'"] & H' \arrow[r, "p'"] & B \arrow[r] & 0
\end{tikzcd}$$

diagram 3.2.

in which η is an isomorphism and α and β are each the identity.

However, in the one special case we will deal with, we want to know the MHS's H in the middle up to isomorphism and then α and β in (3.2) are only required to be isomorphisms. Still, we first discuss extensions up to congruence as an intermediate step, because it will prove favourable for our descriptions in §5.

We now take:

- A a pure H.S. of weight 2 and type (1,1): $A_{\mathbb{C}} = A^{1,1}$, $\dim A_{\mathbb{C}} = a$, and assume $A_{\mathbb{Z}}$ to be torison free;
- B a pure H.S. of weight 3, only containing types (1,2) and (2,1): $B_{\mathbb{C}} = B^{1,2} \oplus B^{2,1}$, $\dim B_{\mathbb{C}} = 2$. Also assuming $B_{\mathbb{Z}}$ to be torsion free, let $\{f_1,f_2\}$ be a basis of $B_{\mathbb{Z}}$ and let $B^{2,1} = \mathbb{C}\cdot(f_1+\tau\cdot f_2)$, $\tau \in \mathbb{C}\setminus\mathbb{R}$, then $B^{1,2} = \mathbb{C}\cdot(f_1+\bar{\tau}f_2)$.

PROPOSITION 3.3. *The set* $\mathrm{Ext}(B,A)$ *of congruence classes of extensions of* B *by* A *with* A *and* B *as defined above can be put in bijective correspondence with the group* $(\mathbb{C}/\langle 1,\tau\rangle)^a$, *zero giving the trivial extension.*

PROOF. Let H be an extension of B by A. The weight filtration on $H_{\mathbb{Q}}$ must necessarily be $W_1 = (0)$, $W_2 = i(A_{\mathbb{Q}})$, $W_3 = H_{\mathbb{Q}}$, and the Hodge filtration must satisfy $F^1 = H_{\mathbb{C}}$, $F^2 \cap i(A_{\mathbb{C}}) = (0)$, $p(F^2) = B^{2,1}$, so $\dim F^2 = 1$, and $F^3 = (0)$. As a consequence, H is completely determined by the position of the line F^2.

Let $\{e_1,\ldots,e_a,F_1,F_2\}$ be a basis of $H_{\mathbb{Z}}$ such that $\{e_1,\ldots,e_a\}$ is a basis of $A_{\mathbb{Z}}$ and $p(F_i) = f_i$, $i = 1,2$. Let $\omega \in F^2$ be the vector satisfying $p(\omega) = f_1 + \tau f_2$. Now the position of F^2, and hence H, is determined by the coefficients $\zeta_i \in \mathbb{C}$ in $\omega = \sum_{i=1}^{a} \zeta_i\cdot e_i + F_1 + \tau F_2$. However, the F_j are only determined up to integral linear combinations of the e_i, and so the ζ_i are determined up to integral multiples of 1 and τ. Now assigning to H the a-tuple $(\zeta_1,\ldots,\zeta_a)$, all ζ_i viewed $\mathrm{mod}(1,\tau)$, gives the correspondence announced in the proposition, and clearly, if $\zeta_1 = \ldots = \zeta_a \equiv 0(1,\tau)$, $H \cong A \oplus B$ as a MHS.

REMARK 3.3.1. In fact this proof of prop. 3.3 is a simplified version of the proof of [Ca], prop. 2 for our special case. Indeed one can show that the assertion of [Ca], prop. 2 to the effect that $\mathrm{Ext}(B,A) \cong J^0\mathrm{Hom}(B,A)$ with our A and B is equivalent to the statement of prop. 3.3.

REMARK 3.3.2. Because a polarization on a MHS is defined via its graded pieces, prop. 3.3 is still valid if A and B carry polarizations and if we require H to be a polarized MHS.

DEFINITION 3.4. We define A_1 to be *the Hodge structure* $\mathbb{Z}(-1)$ *endowed with the polarization* $(\cdot,\cdot)$ given by $(e,e) = -2$ for $e \in A_{1,\mathbb{Z}}$ a generator.

DEFINITION 3.5. Let E be an elliptic curve, $j = j(E)$.

(a) We denote by Ext_j the *group of congruence classes of extensions* of $H^1(E)(-1)$ by A_1 .

(b) We define H_j to be the *moduli variety* of polarized MHS's which are an extension of $H^1(E)(-1)$ by A_1 .

We will now study Ext_j and H_j . First we have to introduce some notation. Let E be an elliptic curve. Then:

- $E \cong \mathbb{C}/<1,\tau>$ for some $\tau \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$; by $z \in \mathbb{C}$ we denote a complex variable or its class $\mathrm{mod}(1,\tau)$. Let $Q = 0 \in E$ be defined by $z = 0$;

- if $E \cong E(\lambda)$, let $\theta = \phi_{|2Q|} : E \to \mathbb{P}^1$ be the morphism branched in $0,1,\lambda$ and ∞ as in the beginning of IV.§2, $\theta(Q) = \infty$. Then the branch points of θ on E are the 2-torsion points, so $\theta^{-1}(\{0,1,\lambda\}) = \{z=\frac{1}{2},\frac{1}{2}\tau,\frac{1}{2}(1+\tau)\}$;

- let $\{\gamma_1,\gamma_2\}$ be a basis of $H_1(E,\mathbb{Z})$ as in figure 3.6, and let $\{c_1,c_2\} \subset H^1(E,\mathbb{Z})$ be the dual basis. Then $dz = c_1 + \tau c_2 \in H^0(E,\Omega^1_E)$. Now if $f_j = \frac{1}{2\pi i} c_j$, $j = 1,2$, then $\{f_1,f_2\}$ is a basis of $(H^1(E)(-1))_{\mathbb{Z}}$, and $dz = 2\pi i(f_1+\tau f_2)$.

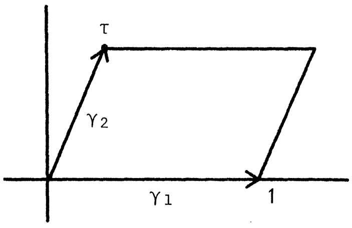


figure 3.6.

In cor. 3.7 the factor $2\pi i$, which of course does not matter for the isomorphisms, comes in because of the same factor $2\pi i$ in dz above. The reason why, which is not much more than a matter of choice, will become clear in §5.

<u>COROLLARY 3.7.</u> $\mathrm{Ext}_j \cong \mathbb{C}/\langle 2\pi i, 2\pi i\tau\rangle \cong E$, $j = j(\tau) = j(E)$.

Finally, denoting by $\mathrm{Aut}(H)$ the group of automorphisms of a (polarized) MHS H , which always contains plus and minus the identity ($\pm$id), we can state

<u>THEOREM 3.8.</u> *Let* $j \neq 0,1728$. *Then:*

(a) $H_j \cong \mathbb{P}^1_{\mathbb{C}}$. *More precisely, there exists a natural map* $\mathrm{Ext}_j \to H_j$ *sending a congruence class to its isomorphism class which can be identified with* $\theta \colon E \to \mathbb{P}^1 = E/\langle\pm\mathrm{id}\rangle$. *In this identification,* $\theta(0) = \infty$ *corresponds to the trivial extension;*

(b) *with the identification of* (a), *let* $\beta \in H_j = \mathbb{P}^1$, *and let* $H(\beta)$ *be a MHS in the isomorphism class corresponding to* β . *Then if*

(i) $\beta \neq 0,1,\lambda,\infty$, $\mathrm{Aut}(H(\beta)) = \{\pm\mathrm{id}\}$, *and if*

(ii) $\beta = 0,1,\lambda$ *or* ∞ , $\#\mathrm{Aut}(H(\beta)) = 4$.

<u>PROOF.</u> Let $E \cong \mathbb{C}/\langle 1,\tau\rangle$, $j = j(\tau)$ and let H,H' represent elements of H_j . Let $\{e,F_1,F_2\}$ and $\{e'_1,F'_1,F'_2\}$ be bases of $H_{\mathbb{Z}}$ resp. $H'_{\mathbb{Z}}$ with the same properties as in the proof of prop. 3.3. By the proof of the same proposition, H and H' can only differ in the position of their Hodge subspace F^2 . Let $\omega = \zeta\cdot e + F_1 + \tau F_2$ and $\omega' = \zeta'\cdot e' + F'_1 + \tau\cdot F'_2$ be basiselements of F^2_H resp. $F^2_{H'}$, so H and H' correspond to $\zeta,\zeta' \in$ $\in \mathrm{Ext}_j = \mathbb{C}/\langle 1,\tau\rangle$. Let us now find out when they are isomorphic.

To this end, let $\eta \colon H \to H'$ be an isomorphism. Because $\eta(A_{\mathbb{Z}}) = A_{\mathbb{Z}} \subset H'_{\mathbb{Z}}$, $\eta(e) = \pm e'$. Furthermore, η induces an isomorphism $\bar\eta$ on the graded part of weight 3 $H^1(E)(-1)$, and because $j \neq 0,1728$, $\bar\eta = \pm\mathrm{id}$ (see [H],IV.cor. 4.7), i.e. $\eta(F_i) = \pm F'_i + a_i\cdot e'$, $a_i \in \mathbb{Z}$, $i = 1,2$. Finally $\eta(F^2_H) = F^2_{H'}$, so

$$\pm\zeta\cdot e' + (\pm(F'_1+\tau F'_2)+(a_1+a_2\tau)e') =$$

$$= c\cdot(\zeta'\cdot e'+F'_1+\tau F'_2) \ , \ c \in \mathbb{C} \ , \ c \neq 0 \ .$$

This implies that either $c = 1$ and $\pm\zeta + a_1 + a_2\tau = \zeta'$ or $c = -1$ and $\pm\zeta + a_1 + a_2\tau = -\zeta'$, and we conclude that $H \cong H'$ iff $\zeta \equiv \pm\zeta' \bmod (1,\tau)$, which proves (a).

(b) Let $\beta \in H_j$, let $\theta^{-1}(\beta) = \{\zeta,-\zeta\}$ and let $\eta \colon H(\beta) \to H(\beta)$ be an automorphism. Using the proof of (a) with $H = H' = H(\beta)$ etc. we find that if

$\zeta \not\equiv -\zeta \bmod(1,\tau)$, i.e. if $\beta \neq 0,1,\lambda,\infty$, either $\eta(e) = e$ and $\bar{\eta} = \mathrm{id}$ or $\eta(e) = -e$ and $\bar{\eta} = -\mathrm{id}$ which proves (i) and if $\zeta \equiv -\zeta \bmod(1,\tau)$, i.e. if $\beta = 0,1,\lambda$ or ∞ , we get any of the four combinations of $\eta(e) = \pm e$ and $\bar{\eta} = \pm \mathrm{id}$ proving (ii).

4 THE MIXED HODGE STRUCTURE ON $H^2(X_0,\mathbb{C})$ FOR X_0 OF TYPE IV

In this section we will try to understand the meaning of the exact sequence

$$(4.1) \qquad 0 \to j^*H^2(X') \xrightarrow{i} H^2(X_0) \xrightarrow{p} H^1(E)(-1) \to 0$$

of thm. 2.2.b, in particular the extension class involved, the leading motive being to recover the surface X from $H^2(X_0)$. Here X is a surface with canonical hyperplane sections birational to $E \times \mathbb{P}^1$, E an elliptic curve, containing two simple elliptic singularities x_0,x_1, $\pi: X' \to X$ its minimal resolution, $\pi^{-1}(x_i) = C'_i$, $i = 0,1$ and $X_0 = X' \setminus (C'_0 \cup C'_1)$, $j: X_0 \to X'$ the embedding. In fact we will later on replace (4.1) by an exact sequence which has a certain polarized sub-MHS of $H^2(X_0)$ in the middle, see prop. 4.5.

We will first state explicitly what we know about $H^2(X_0)$.

PROPOSITION 4.2. (a) *The weight filtration on* $H^2(X_0,\mathbb{Q})$ *is given by* $0 \subset W_2 \subset W_3 = H^2(X_0,\mathbb{Q})$ *with* $W_2 =$ *image of* $(j^*: H^2(X',\mathbb{Q}) \to H^2(X_0,\mathbb{Q}))$ *and* $W_3/W_2 \cong (H^1(E)(-1))_{\mathbb{Q}}$.

(b) *The Hodgefiltration of* $H^2(X_0)$ *is* $H^2(X_0,\mathbb{C}) = F^1 \supset F^2 \cong \cong H^0(X',\mathcal{O}_{X'}) \supset F^3 = (0)$, *and* $F^1/F^2 \cong H^1(X',\Omega^1_{X'}(\log C'_0+C'_1))$.

PROOF. (a) Read the proof of thm. 2.2.b.

(b) Use (1.7.7) with $\bar{Y} = X'$ and $D = C'_0 + C'_1$. Then $H^2(X',\mathcal{O}_{X'}) = = (0)$ because X' is ruled, and $H^0(X',\mathcal{O}_{X'}(K_{X'}+D)) \cong H^0(X',\mathcal{O}_{X'})$ because $D \sim - K_{X'}$.

REMARK 4.2.1. In the above proposition $H^0(X',\mathcal{O}_{X'})$ should be thought of as $H^0(X',\Omega^2_{X'}(C'_0+C'_1))$, or rather as $\mathbb{C}\cdot\omega$, ω the up to a constant unique holomorphic 2-form on X' with poles of order one on C'_0 and C'_1 .

REMARK 4.2.2. The MHS $H^2(X_0)$ is completely determined by the position of the complex line $\mathbb{C}\cdot\omega = F^2$ relative to W_2 and its image on the quotient W_3/W_2 .

In (4.1) the inclusion i is derived from restricting cohomology classes of X' to X_0 . Prop. 4.3 gives information about the map p . Note that we can talk about the residue on C_0' and C_1' of an element of $H^2(X_0,\mathbb{C})$, because by (1.4) every cohomology class in $H^2(X_0,\mathbb{C})$ can be represented by a differential form with logarithmic poles on C_0' and C_1' , and because by (1.5) the residues of such a form only depend on its class.

PROPOSITION 4.3. *Identifying* E *and* C_1' , *the map* $p_{\mathbb{C}}$: $H^2(X_0,\mathbb{C}) \to H^1(E,\mathbb{C})$ *induced by* p *in (4.1) can be identified with the residue map* $\mathrm{res}_{C_1'}$: $H^2(X_0,\mathbb{C}) \to H^1(C_1',\mathbb{C})$.

PROOF. As we remarked in (1.7.6), the map $p_{\mathbb{C}}$ is induced by the residue map R: $H^2(X_0,\mathbb{C}) \to \ker(H^1(C_0',\mathbb{C})\oplus H^1(C_1',\mathbb{C})\xrightarrow{d_3}H^3(X',\mathbb{C}))$. In the proof of thm. 2.2.b we saw that this $\ker(d_3)$ projects isomorphically onto $H^1(C_1',\mathbb{C})$. Together this proves the proposition.

Prop. 4.4. shows explicitly in this case that $H^2(X_0,\mathbb{C})$ maps to (and even onto because $p_{\mathbb{C}}$ is surjective) $\ker(d_3: H^1(C_0',\mathbb{C})\oplus H^1(C_1',\mathbb{C}) \to H^3(X',\mathbb{C}))$, d_3 being the same isomorphism, identifying C_0' and C_1' with E , on both factors.

PROPOSITION 4.4. *For every* $\omega \in H^2(X_0,\mathbb{C})$, $\mathrm{res}_{C_0'}\omega = - \mathrm{res}_{C_1'}\omega$.

PROOF. Let $q: X' \to E$ be the projection of the ruled surface X' onto its base curve. There exists a non-empty Zariski-open subset $\tilde{E} \subset E$ such that $\tilde{X} = q^{-1}(\tilde{E}) \cong \tilde{E} \times \mathbb{P}^1$. Let z_0,z_1 be coordinates on $\mathbb{P}^1$ such that on $\tilde{X}$ the curve C_i' is defined by $z_i = 0$, $i = 0,1$. Let $\omega \in H^2(X_0,\mathbb{C})$ and assume it has a logarithmic pole, say on C_0' , so $\omega = \eta \wedge \frac{dz_0}{z_0}$. But then, because $z_1 = \frac{1}{z_0}$, $\omega = - \eta\wedge\frac{dz_1}{z_1}$, and so $\mathrm{res}_{C_0'}\omega = \eta = - \mathrm{res}_{C_1'}\omega$.

We will now replace $H^2(X_0)$ by a certain sub-MHS, which is polarized if we allow polarizations to have values in $\mathbb{Q}$ on the integral lattice (however cf. remark 4.5.1),and study a corresponding exact sequence with its own extension class instead of (4.1). To this end, let

- $C' = \pi^* C$ be the inverse image on X' of a general hyperplane section of X, denoting by $k\colon C' \hookrightarrow X_0, X'$ both inclusions;

- $(C_0', C_1', C')^\perp \subset H^2(X', \mathbb{Z})$ be the orthogonal complement of the subgroup generated by C_0', C_1' and C' relative to the intersection form on X', and, identifying $j^* H^2(X', \mathbb{Z})$ with $H^2(X', \mathbb{Z})/(\mathbb{Z}\cdot C_0' \oplus \mathbb{Z}\cdot C_1')$, let

- $A_{\mathbb{Z}} = \dfrac{\ker(k^*\colon H^2(X',\mathbb{Z}) \to H^2(C',\mathbb{Z}))}{(\mathbb{Z}\cdot C_0' \oplus \mathbb{Z}\cdot C_1')}$ be its subgroup (note that because $C_i' \cdot C' = 0$, $i = 0,1$, the denominator is a subgroup of the numerator).

Now the composition $(C_0', C_1', C')^\perp \hookrightarrow H^2(X', \mathbb{Z}) \to H^2(X', \mathbb{Z})/(\mathbb{Z}\cdot C_0' \oplus \mathbb{Z}\cdot C_1')$ is injective, for if $D \in (C_0', C_1', C')^\perp$ and $D \sim a_0 C_0' + a_1 C_1'$, $(D\cdot C_i')/(C_i')^2 = = a_i = 0$, $i = 0,1$, so $D \sim 0$, and we can identify $(C_0', C_1', C')^\perp$ with $j^*((C_0', C_1', C')^\perp) \subset H^2(X_0, \mathbb{Z})$. By definition we even have $(C_0', C_1', C')^\perp \subset A_{\mathbb{Z}}$.

PROPOSITION 4.5. *Assume that* $k^*\colon H^2(X', \mathbb{Z}) \to H^2(C', \mathbb{Z})$ *is surjective. Then there exists an exact sequence*

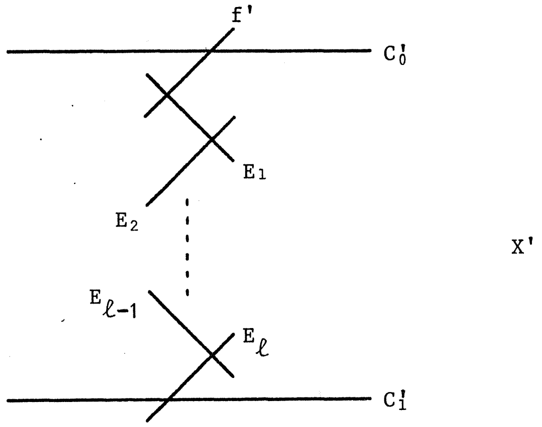
$$(4.5.1) \qquad 0 \to A \xrightarrow{i} H \xrightarrow{p} B \to 0$$

of polarized MHS's, with $A_{\mathbb{Z}}$ *as defined above,* $H_{\mathbb{Z}} = = \ker(k^*\colon H^2(X_0, \mathbb{Z}) \to H^2(C', \mathbb{Z}))$ *and* $B = H^1(E)(-1)$, *with the MHS's* A *and* H *induced by* $j^* H^2(X')$ *resp.* $H^2(X_0)$ *and with* i *and* p *the restrictions of the same maps as before. Moreover,* $A_{\mathbb{Z}}$ *is free, the polarization on* A *(with values in* $\mathbb{Q}$ *) is induced by the inclusion* $(C_0', C_1', C')^\perp \subset A_{\mathbb{Z}}$, *the weight filtration of* H *is given by* (4.5.1) *and the Hodge filtration is* $H_{\mathbb{C}} = F^1 \supset F^2 \supset F^3 = (0)$, *with* $F^2 = \mathbb{C}\cdot\omega$, ω *a holomorphic 2-form on* X_0 *with logarithmic poles on* C_0' *and* C_1'.

PROOF. Let $0 \to j^* H^2(X', \mathbb{Z}) \xrightarrow{i} H^2(X_0, \mathbb{Z}) \xrightarrow{p} (H^1(E)(-1))_{\mathbb{Z}} \to 0$ be the exact sequence over $\mathbb{Z}$ induced by (4.1). Let $F_1, F_2 \in H^2(X_0, \mathbb{Z})$, such that $p(F_i) = f_i \in B_{\mathbb{Z}} = (H^1(E)(-1))_{\mathbb{Z}}$, $\{f_1, f_2\}$ a basis of $B_{\mathbb{Z}}$. Because k^* is surjective, there exists a $G \in j^* H^2(X', \mathbb{Z})$, such that $k^*(G)$ is a generator of $H^2(C', \mathbb{Z})$. Now modifying the F_i if necessary with an integral multiple of G, we can assume $F_i \in \ker(k^*) \subset H^2(X_0, \mathbb{Z})$, i.e., $F_i \in H_{\mathbb{Z}}$, $H_{\mathbb{Z}}$ as defined in the proposition. Moreover, because $B_{\mathbb{Z}}$ is free, we can write $H^2(X_0, \mathbb{Z}) = j^* H^2(X', \mathbb{Z}) \oplus \mathbb{Z}\cdot F_1 \oplus \mathbb{Z}\cdot F_2$, and hence taking

$\ker(k^*)$ both in $H^2(X_0,\mathbb{Z})$ and in $H^2(X',\mathbb{Z})$ leaves us with the exact sequence over $\mathbb{Z}$ which induces (4.5.1).

The fact that $A_{\mathbb{Z}}$ is free follows from the same for $H^2(X',\mathbb{Z})/(\mathbb{Z}\cdot C_0'\oplus\mathbb{Z}\cdot C_1')$, which can be seen as follows. Let $P_1,\ldots,P_k$ be the points to be blown up to curves $E_i \subset X'$, $i = 1,\ldots,k$ to get X' from X'' . Then $\{C_0',f,E_1,\ldots,E_k\}$ is a basis of $H^2(X',\mathbb{Z})$. Now assume that $P_2,\ldots,P_{\ell}$ ($\ell\leq k$, possibly none) are infinitely near $P_1 \in C_1 \subset X''$. Then, because in every stage only a point on the strict transform of C_1 can be blown up, on X' the fibre over P_1 looks like



with $E_1^2 = \ldots = E_{\ell-1}^2 = -2$, $E_{\ell}^2 = (f')^2 = -1$. Using intersection numbers one easily finds $C_1' \sim C_0' + e\cdot f - \sum_{i=1}^{\ell} i\cdot E_i - \sum_{j=\ell+1}^{k} n_j\cdot E_j$, and so $\{C_0',C_1',f,E_2,\ldots,E_k\}$ is also a basis for $H^2(X',\mathbb{Z})$, proving freeness.

But now $(C_0',C_1',C')^{\perp} \subset A_{\mathbb{Z}}$ is an inclusion of free lattices of the same rank, so we can extend the integral, symmetric bilinear form on $(C_0',C_1',C')^{\perp}$, which is negative definite because $(C')^2 = 2g - 2 > 0$, over $\mathbb{Q}$ to the whole of $A_{\mathbb{Z}}$.

Finally, the two filtrations of H are induced by those of $H^2(X_0)$, so the assertion about the weight filtration is clear, and as to the Hodge filtration, we certainly have $F^1 = H_{\mathbb{C}}$ and $F^3 = (0)$, but because $\int_{C'}\omega = 0$, ω the up to a constant unique holomorphic 2-form on X_0 with logarithmic poles on C_0' and C_1' , the line $\mathbb{C}\cdot\omega \subset H^2(X_0,\mathbb{C})$ survives in $H_{\mathbb{C}}$ to give F^2 .

<u>REMARK 4.5.2.</u> In all cases for $g = 2,3$, $k^*: H^2(X',\mathbb{Z}) \to H^2(C',\mathbb{Z})$ is surjective (cf. IV.prop. 2.2.).

REMARK 4.5.3. The polarization on $A_{\mathbb{Z}}$ is now given by the following rule: if $\bar{D},\bar{D}' \in A_{\mathbb{Z}}$ are represented by $D,D' \in \ker(k^*)$, let $D\cdot C_i' = a_i$, $D'\cdot C_i' = a_i'$. If moreover, $(C_i')^2 = -e_i$, $e_0 = e$, $e_1 = k - e$, then $(\bar{D}\cdot\bar{D}') = (D+\frac{a_0}{e_0}C_0'+\frac{a_1}{e_1}C_1')\cdot(D'+\frac{a_0'}{e_0}C_0'+\frac{a_1'}{e_1}C_1') \in \mathbb{Q}$. This shows that if $e_0 = e_1 = 1$, the polarization takes values in $\mathbb{Z}$, and even $(C_0',C_1',C')^{\perp} = A_{\mathbb{Z}}$.

REMARK 4.5.4. In every case if $g = 2,3$, the polarization on $A_{\mathbb{Z}}$ has values in $\mathbb{Z}$. If $g = 2$ or $g = 3$, $a = 3$, $e = 1$, then $e_0 = e_1 = 1$, so so this follows from (4.5.3). If $g = 3$, $a = e = 2$, one can compute it directly. In this case, $(C_0',C_1',C')^{\perp}$ is of index 2 in $A_{\mathbb{Z}}$.

Now the question is, given the polarized MHS H, i.e. given an exact sequence as (4.5.1), *does it determine the surface* X, and maybe in the first place, *is there an* X *to every such an* H ? Obviously, one finds back the elliptic curve E from the quotient polarized H.S. B. The rest of the information contained in (4.5.1) is the extension class. In order to deal with this, let $F_1,F_2 \in H_{\mathbb{Z}}$, such that $p(F_i) = f_i$, with $\{f_1,f_2\}$ a basis of $B_{\mathbb{Z}}$ as defined in §3 preceding cor. 3.7. Now fix $\omega_0 \in F^2 \subset H_{\mathbb{C}}$, such that $p(\omega_0) = \mathrm{res}_{C_1'}\omega_0 = dz = 2\pi i(f_1+\tau f_2)$, i.e. if C_1' is locally defined by $w = 0$, $\omega_0 = dz \wedge \frac{dw}{w}$. If $e_1,\ldots,e_a$ is a basis of $A_{\mathbb{Z}}$, $\omega_0 = \Sigma\zeta_i e_i + 2\pi i(F_1+\tau F_2)$ and according to prop. 3.3 the ζ_i determine the extension. In the only case $(g=2)$ we treat in §5, we will trace the geometric meaning of the ζ_i (then there is only one $\zeta = \zeta_1$), and we will find that not the extension class but the isomorphism class of H is important.

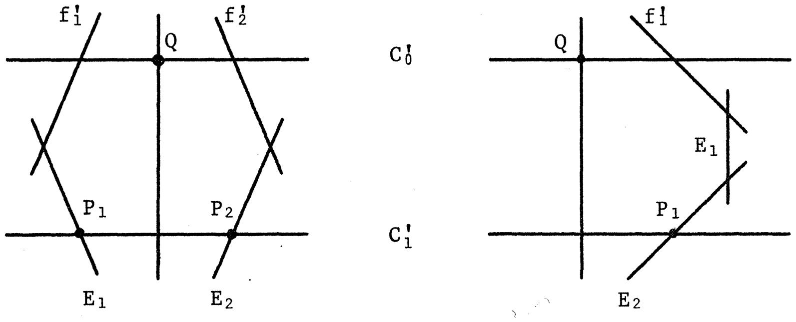
5 THE PERIOD MAP FOR THE DOUBLE COVERS OF $\mathbb{P}^2$

Let us now turn our attention to the surfaces X of IV.thm. 2.3.a, the double covers of $\mathbb{P}^2$. In view of the last part of the preceding section we assume throughout this section the base curve E to be fixed.

From IV.prop. 2.2.a we get this description of the minimal resolution X' of X: X' arises from $X_1 = \mathbb{P}_E(\mathcal{O}_E\oplus\mathcal{O}_E(-Q))$ by blowing up two points $P_1,P_2 \in C_1$ to curves E_1,E_2, and if $P_i = z_i$ on E, $z_1 + z_2 = 0$ because $P_1 + P_2 \in |2Q|$. Moreover, $P_i \neq Q$, $i = 1,2$. By IV.thm. 3.2, if

$j(E) \neq 0,1728$, the moduli variety M_j of the X is isomorphic to $\mathbb{C}$ by associating to X the point $\alpha = \theta(P_i) \in \mathbb{C}$, and we completed M_j to $\bar{M}_j \cong \mathbb{P}^1_{\mathbb{C}}$ by adding the point $\infty = \theta(Q)$ (IV.cor. 3.4), $\bar{M}_j$ parametrizing the X' , or, perhaps better to say inthis context, the X_0 .

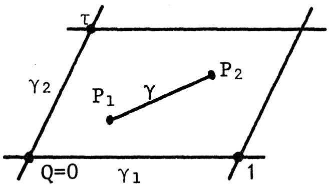
According to whether $P_1 \neq P_2$ or $P_1 = P_2$ we get the following picture of X' :



$P_1 \neq P_2$; $E_1^2 = E_2^2 = -1$. $P_1 = P_2$; $E_1^2 = -2$, $E_2^2 = -1$.

(a)   figure 5.1.   (b)

Let $\tilde{E}_i = E_i$, $i = 1,2$ in (a) resp. $\tilde{E}_1 = E_1 + E_2$, $\tilde{E}_2 = E_2$ in (b) and let f_i' denote the strict transforms on X' of the fibres f_i over P_i on X_1 . Then $\{C_0', f', \tilde{E}_1, \tilde{E}_2\}$, f' a general fibre on X', is a basis of $H^2(X',\mathbb{Z})$, because, writing down the intersection maxtrix one finds that it has determinant -1 . Now if $P_1 \neq P_2$, let us make the following construction.



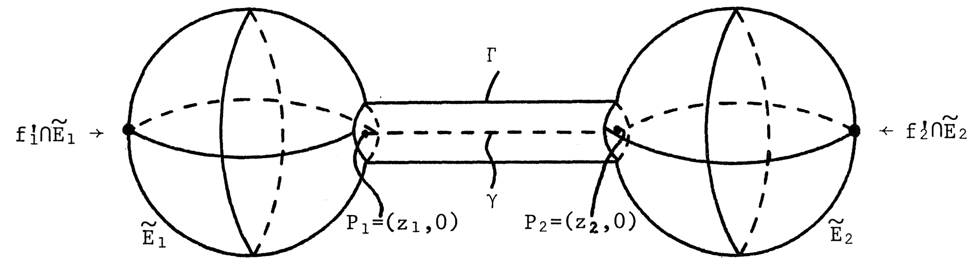
For a suitable union F of fibres on X' over points different from Q,P_1,P_2 , $X' \setminus (C_0' \cup f_1' \cup f_2' \cup F) \cong \tilde{E} \times \mathbb{C}(z,w)$, $\tilde{E} \subset E$ a Zariski-open set containing Q,P_1,P_2 . Let γ be the linear chain joining P_1 and P_2 on E .

Consider the subset

$$\Delta = \{(z_1,w)/|w|\geq\varepsilon\} \cup \{(z_2,w)/|w|\geq\varepsilon\} \cup$$

$$\cup \{(z,w)/z\in\gamma, w=\varepsilon\cdot e^{i\phi}, 0\leq\phi\leq 2\pi\} \cup \{f_1'\cap\tilde{E}_1\} \cup \{f_2'\cap\tilde{E}_2\} \ , \ \varepsilon > 0 \ ,$$

of X' , which looks like this:



the cycle Δ , if $P_1 \neq P_2$

and call Γ the tubular neighbourhood of γ in Δ . This Δ is homeomorphic to S^2 , so can be given an orientation to represent a cycle on X' (and even on X_0). Writing $\Delta \sim aC'_0 + b\cdot f' + c_1\tilde{E}_1 + c_2\tilde{E}_2$ (the right hand side viewed as a homology class) and intersecting with C'_0 and f' we find $a = b = 0$. Intersecting with $\tilde{E}_i$ we get $c_i = \pm 1$, and finally because $\Delta\cdot C'_1 = 0$, $c_1 = -c_2$, so in Δ one of the $\tilde{E}_i$, say $\tilde{E}_1$, has its analytic orientation, and then the other, $\tilde{E}_2$, its anti-analytic orientation. So $\Delta \sim \tilde{E}_1 - \tilde{E}_2$ as a homology class.

Also in case $P_1 = P_2$ denoting by Δ the homology class carried by the curve $E_1 = \tilde{E}_1 - \tilde{E}_2$, let

$-e$ be the integral cohomology class in $j^*H^2(X',\mathbb{Z}) \subset H^2(X_0,\mathbb{Z})$ which is intersecting with Δ , i.e. $e = j^*(\tilde{E}_1-\tilde{E}_2)$.

PROPOSITION 5.2. *In this case* $A \cong A_1$ *and* $A_{\mathbb{Z}}$ *is generated by* e .

PROOF. By (4.5.3) $(C'_0,C'_1,C')^{\perp} = A_{\mathbb{Z}}$. Using intersection numbers and the basis $\{C'_0,f',\tilde{E}_1,\tilde{E}_2\}$ of $H^2(X',\mathbb{Z})$ first one finds $C'_1 \sim C'_0 + f' - \tilde{E}_1 - \tilde{E}_2$ and $C' \sim 2C'_0 + 2f' - \tilde{E}_1 - \tilde{E}_2$ and with the help of this $(C'_0,C'_1,C')^{\perp} = \mathbb{Z}\cdot(\tilde{E}_1-\tilde{E}_2)$, so $A_{\mathbb{Z}} = \mathbb{Z}\cdot e$. Moreover, $(e,e) = (\tilde{E}_1-\tilde{E}_2)^2 = -2$, hence $A \cong A_1$.

PROPOSITION 5.3. *Let in the diagram*

$$\begin{tikzcd}
E \arrow[r, ".2"] \arrow[d, "\theta"'] & E \arrow[d, "\theta"] \\
\mathbb{P}^1(y_0,y_1) \arrow[r, dashed, "\rho"] & \mathbb{P}^1(y_0',y_1')
\end{tikzcd}$$

both maps θ *and the upper arrow which is multiplication by* 2 *on* $E \cong E(\lambda)$ *be given. Then the (unique) map making the square commutative is the morphism* ρ *given by*

$$y' = \rho(y) = \frac{y^4-2\lambda y^2+\lambda^2}{4(y^3-(1+\lambda)y^2+\lambda y)} \;,\; y = y_0/y_1 \;,\; y' = y_0'/y_1' \;.$$

PROOF. First of all, for $R \in \mathbb{P}^1$, $\theta^{-1}(R) = \{P,-P\}$, and as $\theta(2P) = = \theta(-2P)$, ρ can at least settheoretically be well defined in a unique way.

Embed E in $\mathbb{P}^2$ by $|3Q|$ to have equation $y_1y_2^2 = y_0^3 - y_0^2y_1 - - \lambda y_0^2y_1 + \lambda y_0y_1^2 = 0$, $Q = (0,0,1)$. Then $\theta\colon E \to \mathbb{P}^1(y_0,y_1)$ is projection onto the line $y_2 = 0$. Now the law of addition on a plane cubic says that $y' = y_0'/y_1' = \theta(2P) = \theta(-2P)$ equals the ratio y_0/y_1 evaluated in the third point of intersection of E with its tangent line in P , and an easy computation gives the desired formula.

Now we come to the description of the period map from $\bar{M}_j$, the surfaces X' , to H_j , the mixed Hodge structures occurring as $H^2(X_0)$, $X_0 \subset X'$. In short we prove the following. If $H \in H_j$, the polarization on the graded part of weight 2 A enables us to find the generators $\pm e \in A_{\mathbb{Z}}$, these being the only $v \in A_{\mathbb{Z}}$ with $(v,v) = -2$. Then, writing $\omega_0 \in F^2$ with respect to the basis $\{e,F_1,F_2\}$, F_1,F_2 as before, $\omega_0 = \zeta\cdot e + 2\pi i(F_1+\tau F_2)$, we find $-(2\zeta \bmod(2\pi i,2\pi i\tau))$. It turns out that $2\pi i(z_2-z_1) \equiv - 2\zeta(2\pi i,2\pi i\tau)$, $z_i = P_i$, $i = 1,2$, the points blown up on X_1 . Because also $z_1 + z_2 \equiv 0(1,\tau)$, to find the divisor $P_1 + P_2 \in |2Q|$, one has to solve

$$z_1 + z_2 \equiv 0 \mod(1,\tau) \text{ and}$$

$$z_1 - z_2 \equiv \frac{1}{\pi i}\,\zeta \mod(1,\tau) \;,$$

which in general has four solutions, so the period map will be of degree 4.

THEOREM 5.4. *Let* $E \cong E(\lambda)$ *with* $j(\lambda) \neq 0,1728$. *Then, with the identifications made in IV. 3.5, cor. 3.7 and thm. 3.8, there exists a commutative diagram*

$$\begin{tikzcd}
\alpha \arrow[d, phantom, sloped, "\in"] & E \arrow[r, ".2"] \arrow[d, "\theta"'] & \mathrm{Ext}_j \arrow[r, phantom, "\cong"] \arrow[d] & E \arrow[d, "\theta"] \\
E/\langle\pm id\rangle = \mathbb{P}^1_{\mathbb{C}} \arrow[r, phantom, "\cong"] \arrow[d, phantom, sloped, "\supset"] & \bar{M}_j \arrow[r, "\rho"] \arrow[d, phantom, sloped, "\supset"] & H_j \arrow[r, phantom, "\cong"] \arrow[d, phantom, sloped, "\ni"] & \mathbb{P}^1_{\mathbb{C}} = E/\langle\pm id\rangle \\
\mathbb{C} \arrow[r, phantom, "\cong"] & M_j & \beta &
\end{tikzcd}$$

in which the period map $\rho\colon \bar{M}_j \to H_j$ *is a morphism of degree 4, given by*

$$\beta = \rho(\alpha) = \frac{(\alpha^2-\lambda)^2}{4(\alpha^3-(1+\lambda)\alpha^2+\lambda\alpha)} .$$

Moreover,

(i) $\rho(\{0,1,\lambda,\infty\}) = \infty$, $\infty \in H_j$ *the point representing the trivial extension,* $0,1,\lambda \in M_j$ *the points representing the only surfaces which contain an ordinary double point and* $\infty \in \bar{M}_j$ *the point of compactification, and*

(ii) ρ *is branched over* $0,1,\lambda \in H_j$, $\rho^{-1}(\beta)$ *consisting of two points with ramification index* 2 *for* $\beta = 0,1,\lambda$.

PROOF. Let $\alpha \in \bar{M}_j$ be represented by X'_α , $\theta^{-1}(\alpha) = \{P_1,P_2\}$, $P_i = z_i$ the points to be blown up on $C_1 \subset X_1$ to get X'_α , $z_1 + z_2 = 0$. Let $H = H^2(X_0)$, $X_0 = X'_\alpha \setminus (C_0^1 \cup C_1^1)$, H represents the image $\beta \in H_j$ of α . Let $e \in A_{\mathbb{Z}}$ be as in prop. 5.2, and let $\omega_0 = \zeta \cdot e + 2\pi i(F_1+\tau F_2)$, so this choice of e fixes a congruence class $\zeta \in \theta^{-1}(\beta)$. We will now compute $\int_\Delta \omega_0$ in two ways.

On the one hand, if $P_1 = P_2$, $\int_\Delta \omega_0 = \int_{E_1} \omega_0 = 0$, because E_1 is analytic and ω_0 is holomorphic, if $P_1 \neq P_2$, $\int_\Delta \omega_0 = \int_\Gamma \omega_0$ because the parts of the $\tilde{E}_i$ appearing in Δ are defined by analytic coordinates, and $\int_\Gamma \omega_0 = \int_\Gamma dz \wedge \frac{dw}{w} = 2\pi i \int_\gamma dz = 2\pi i(z_2-z_1)$, so in both cases $\int_\Delta \omega_0 = 2\pi i(z_2-z_1) = -4\pi i z_1$, because $z_1 + z_2 = 0$.

On the other hand, $\int_\Delta \omega_0 = \int_\Delta \zeta \cdot e + 2\pi i(F_1+\tau F_2)$. Because $F_i \in H^2(X_0,\mathbb{Z})$,

$\int_\Delta F_i \in \mathbb{Z}$, and $\int_\Delta e = (e,e) = -2$, so $\int_\Delta \omega_0 \equiv -2\zeta(2\pi i, 2\pi i\tau)$. We conclude that $-2\zeta \equiv -4\pi i z_1(2\pi i, 2\pi i\tau)$, so $\zeta \equiv 2\pi i z_1(\pi i, \pi i\tau)$ (note that we view $z_1 \in E = \mathbb{C}/\langle 1,\tau\rangle$ but $\zeta \in \mathrm{Ext}_j = \mathbb{C}/\langle 2\pi i, 2\pi i\tau\rangle$), which shows that we get a map $E \to \mathrm{Ext}_j$ which is multiplication by 2. Now the diagram in the theorem can be completed by ρ as in prop. 5.3, which must necessarily be the period map.

As to (i) and (ii), these are a consequence of the formula for ρ and of identifications made before.

6 SOME REMARKS ON MODULI AND PERIOD MAPS FOR TWO TYPES OF SINGULAR QUARTICS IN $\mathbb{P}^3$

In this final section we will discuss what we know about the moduli and the period map for the two kinds of quartics described in IV.thm. 2.3.b. First of all, we think, that any family of elliptic ruled surfaces with canonical hyperplane sections containing two simple elliptic singularities can be dealt with in more or less the same way as we did in §5 with the double covers of $\mathbb{P}^2$. However, things can become quite complicated. An indication of this is already given by the surfaces of IV.thm. 2.3.b(i). In that case, both the moduli variety M of the surfaces X (birational to $E \times \mathbb{P}^1$, the elliptic curve E fixed), and the moduli variety H of the associated polarized sub-MHS's of $H^2(X_0)$ are, if they exist, quotients of the Abelian variety $E \times E \times E$ (for M this follows from IV.prop. 2.2.b(i): X is determined by choosing $P_1, P_2, P_3 \in E$; for H this is a consequence of prop. 3.3 and a dimension count), and seem difficult to determine. The best we can do at the moment, is to make a guess at the degree of the period map (16?), if it exists, by writing down congruences like those at the bottom of page 97, but that does not get us very far.

As to the other case, the surfaces of IV.thm. 2.3.b(ii), for these we would have written a section like the preceding one, which is mainly due to the fact, that also here only two points have to be blown up to get X' from X'' , so the moduli varieties in question are again one-dimensional. Retaining as much as possible the same notation, here are the facts.

Assume $j \neq 0,1728$. Then both $\bar{M}_j$ and H_j are isomorphic to $\mathbb{P}^1$. As to $\bar{M}_j$, referring to IV.prop. 2.2.b(ii), it is clear, that the choice of $P_1 \in E$ determines X , or rather X' , and it turns out, that the only other choice for the double base point, giving an isomorphic X , is the inverse $\theta(P_1)$ with respect to the addition on E. So $\bar{M}_j \tilde{=} \mathbb{P}^1 = E/<\pm id>$. Furthermore, one can show, that choosing $P_1 = Q$ does not give an X, but choosing $P_2 = Q$, $P_1 \neq Q$ (but $2P_1 \sim 2Q$, or, what is the same, $\theta(P_1) = P_1$, so three possibilities), give an X with hyperelliptic canonical hyperplane sections, which is then a double cover of a *smooth* quadric in $\mathbb{P}^3$, branched along two smooth rational normal curves of degree 3, which intersect in two points. This means, that also here, $M_j \tilde{=} \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$, but now the points $0,1,\lambda \in M_j$ correspond to the hyperelliptic cases, whereas in the case of §5 these points corresponded to the surfaces with an A_1-singularity.

As to H_j, the only difference with the case of the double covers of $\mathbb{P}^2$ is, that the polarization on A is now defined by $(e,e) = -4$ for $e \in A_{\mathbb{Z}}$ a generator, because the determinant of the intersection matrix of C_0', C_1', C' now equals 4. However, this does not matter for the isomorphism $H_j \tilde{=} \mathbb{P}^1$.

Let's now turn our attention to the period map, which we call $\tilde{\rho}$ this time, $\tilde{\rho}: \bar{M}_j \to H_j$. Identifying P_i with z_i on $E = \mathbb{C}/<1,\tau>$, $i = 1,2$, it turns out, that we have to solve the following two congruences, instead of those at the bottom of page 97:

$$\begin{aligned} 2z_1 + z_2 &\equiv 0(1,\tau) \\ -2z_1 + z_2 &\equiv -\frac{2}{\pi i}\,\zeta(1,\tau)\ , \end{aligned} \qquad (*)$$

the first of which is a consequnce of the fact, that $2P_1 + P_2 \sim 3Q$, the second of computing $\int_\Delta \omega_0$ in two ways, as we did in §5. Fot this cycle Δ, which lies in X_0, we can make in this case a similar, but more complicated, construction as in §5. Now (*) has 16 solutions, and so $\tilde{\rho}$ will be of degree 16. Indeed, writing (*) in another way, we find $\zeta \equiv 2\pi i z_1(\pi i/2, \pi i\tau/2)$, and this shows that we get a commutative diagram like the one of thm.5.4, but for the upper row, which is replaced by multiplication by 4 on E, so $\tilde{\rho}$ looks like $\rho o \rho$. Using this, we could now fomulate a theorem like thm.5.4 for these surfaces, determine the branch points of $\tilde{\rho}$ and trace their geometric meaning etc. We will not pursue this further, except for the following two remarks.

First, the three surfaces X with hyperelliptic canonical hyperplane sections ($0,1,\lambda \in M_j$) belong to those having trivial associated MHS (i.e.

map to $\infty \in H_j$), but they are not the only ones; second, for none of the surfaces X with $P_1 = P_2$, the associated MHS is trivial. (Of course, in both remarks we disregard the case $P_1 = P_2 = Q$ not giving an X , i.e. $\infty \in \bar{M}_j$).

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