SINGULAR PERTURBATIONS OF HYPERBOLIC TYPE

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CONTENTS

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INTRODUCTION

This work deals with initial value problems and initial-boundary value problems in $\mathbb{R}^2$ which are governed by a differential equation of the type

(1) $\qquad L^{(\varepsilon)}[u] \equiv \varepsilon L_2[u] + L_1[u] = f(x,t)$

where ε is a small positive parameter (i.e. $0 < \varepsilon \ll 1$), the operator L_2 is a linear hyperbolic differential operator of the second order given by

(2) $\qquad L_2[u] = \dfrac{\partial^2 u}{\partial t^2} + \{c_1(x,t) + c_2(x,t)\}\dfrac{\partial^2 u}{\partial x \partial t} + c_1(x,t)c_2(x,t)\dfrac{\partial^2 u}{\partial x^2}$

(with $c_1 < c_2$) and the operator L_1 is a nonvanishing first order differential operator which is either linear

(3) $\qquad L_1[u] = a(x,t)\dfrac{\partial u}{\partial t} + b(x,t)\dfrac{\partial u}{\partial x} + d(x,t)u$

or quasilinear

(4) $\qquad L_1[u] = a(x,t,u)\dfrac{\partial u}{\partial t} + b(x,t,u)\dfrac{\partial u}{\partial x} + d(x,t,u).$

The equation (1) describes the propagation of waves and occurs in many problems in mathematical physics. One class of physical problems which lead to the equation (1) concerns certain phenomena which are governed in first approximation by a quasilinear conservation law of the type

$\qquad \dfrac{\partial u}{\partial t} + \phi(u)\dfrac{\partial u}{\partial x} = 0.$

If higher order effects are taken into account there results a nonlinear mathematical model which yields after linearization an equation of the type (1). Phenomena which can thus be described are for instance traffic flow, the flow of water in long rivers and glacier flow. Also certain chemical exchange processes and the process of sedimentation in rivers are governed by a conservation law of the above mentioned type. For details we refer to WHITHAM [56, chapter 3] where also an extensive bibliography can be found.

Other physical problems which lead to the equation (1) are so-called

overdamped vibration problems. Problems of this kind are governed by an equation of the type

(5) $$L_2[u] = \mu\{f(x,t) - L_1[u]\}$$

where μ is some large positive constant (i.e. $\mu \gg 1$) and L_2 and L_1 are given by (2) and (3). Putting $\mu = \varepsilon^{-1}$ in (5) we obtain the equation (1). Physically the large constant μ corresponds to a strong suppression of the vibration which may be caused for instance by a heavy damping (mechanical problems) or by a highly absorbing surrounding medium (radiation problems). An example of an overdamped vibration is given by the propagation of electrical signals along a conducting wire of large resistance. In this case the mathematical description leads to the well-known telegraph equation with a small factor $\varepsilon = \mu^{-1}$ in front of the wave operator (see SMITH & PALMER [46]). Other overdamped vibration problems are for instance the motion of a vibrating string imbedded in a highly viscous medium and the propagation of radiation through a highly absorbing medium.

Since problems which are governed by an equation of the type (1) belong to the class of so-called *singular perturbation problems* we shall give a brief description of the concept of singular perturbation problem. This description will be followed by a discussion of some techniques which are characteristic of singular perturbation theory.

Let Ω be some domain in $\mathbb{R}^n$ with boundary ∂D and let $\vec{x} = (x_1, x_2, \ldots, x_n)$ denote a point in $\mathbb{R}^n$. We consider the boundary value problem

(6) $$\begin{cases} L^{(\varepsilon)}[u] \equiv \varepsilon L_2[u] + L_1[u] = f(\vec{x}), & \vec{x} \in \Omega \\ \\ B_i[u] = g_i(\vec{x}) & \text{along } \partial D \ (i = 1,2,\ldots,k) \end{cases}$$

where f and g_i $(i = 1,2,\ldots,k)$ are given functions and the $B_i[u]$ $(i = 1,2,\ldots,k)$ are expressions involving u and its derivatives; the constant k is a positive integer depending on the type of boundary value problem. L_2 and L_1 are differential operators of arbitrary type (either ordinary differential operators or partial differential operators of elliptic, parabolic or hyperbolic type) and of arbitrary order; these operators

may be linear or nonlinear. The constant ε is a small positive parameter which appears only as a factor in front of the operator L_2.

One can think of the left-hand side of the differential equation (6) as consisting of a term $L_1[u]$ together with a small perturbation $\varepsilon L_2[u]$. For this reason problems of the type (6) are called *perturbation problems*; the term $L_1[u]$ is referred to as the *unperturbed ε-independent part* of the operator $L^{(\varepsilon)}$ and the term $\varepsilon L_2[u]$ is called the *perturbation*.

One distinguishes between *regular* perturbation problems and *singular* perturbation problems. A class of singular perturbation problems is characterized by the fact that the order of L_2 is larger than the order of L_1; in case the order of L_2 is lower than or equal to that of L_1 we speak of regular perturbation problems (for an analysis of the concepts regular perturbation versus singular perturbation we refer to ECKHAUS [10]). From now on we shall assume that (6) is a singular perturbation problem in which the order of L_2 is larger than that of L_1.

A solution u of problem (6) is in general difficult to obtain explicitly. In practice, however, it often occurs that one is content with an approximation $\tilde{u}$ of u which is valid for small values of ε. In that case the problem reduces to the determination of a function $\tilde{u}$ which satisfies·

(7) $\lim\limits_{\varepsilon\downarrow 0} \|u - \tilde{u}\|_{\bar{\Omega}} = 0.$

Here $\|\cdot\|_{\bar{\Omega}}$ denotes some norm which has been given in advance. We may take for instance the sup-norm

$$\|u - \tilde{u}\|_{\bar{\Omega}} = \sup_{\bar{\Omega}} |u - \tilde{u}|$$

or the L_2-norm

$$\|u - \tilde{u}\|_{\bar{\Omega}} = [\int_{\bar{\Omega}} |u - \tilde{u}|^2 d\vec{x}]^{\frac{1}{2}}$$

where $\bar{\Omega}$ denotes the closure of Ω; also norms involving derivatives are possible. The construction of the approximation $\tilde{u}$ is usually carried out in two steps. First one constructs a so-called *formal approximation* $\tilde{u}$ which satisfies the differential equation and the boundary conditions (6) up to

some order of ε. In a second step one proves the correctness of the formal approximation by showing that (7) holds.

We shall now discuss both steps in some more detail, where it will be assumed for simplicity that the operators L_1 and L_2, as well as the expressions B_i (i = 1,2,...,k), are linear. However, much of what will be said applies also to nonlinear problems.

One always starts the construction of the formal approximation by solving the *reduced equation*

$$L_1[w_0] = f(\vec{x}), \qquad \vec{x} \in \Omega$$

which follows from the equation (6) by putting $\varepsilon = 0$. The solution w_0 is chosen such that it satisfies as many of the boundary conditions (6) as possible (the reduced equation together with the selected boundary conditions is also referred to as the *reduced problem*). If w_0 is sufficiently smooth it satisfies the differential equation (6) up to order ε, namely

$$L^{(\varepsilon)}[w_0] = f(\vec{x}) + \varepsilon L_2[w_0] = f(\vec{x}) + O(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly valid in subsets of Ω. However, since the order of L_2 is larger than that of L_1 the function w_0 generally does not satisfy all boundary conditions (6), so w_0 is not a formal approximation of u.

In order to cope with the remaining boundary conditions (6) a so-called *boundary layer function* v_0 is added to w_0 which, for ε approaching zero, tends asymptotically to zero everywhere in Ω with the exception of a small neighborhood of the boundary $\partial\Omega$ where it contributes to w_0 in such a way that the remaining boundary conditions are fulfilled; the appearance of boundary layers is characteristic of singular perturbation problems. Usually a boundary layer function can be constructed by stretching the coordinate, say x_n, which measures the distance of a point $\vec{x}$ to the boundary (if necessary, one introduces local coordinates in a neighborhood of the boundary). The following type of boundary layer function occurs frequently in singular perturbation problems:

$$v_0 = v_0(x_1,x_2,\ldots,x_{n-1},\frac{x_n}{\varepsilon^p}) = \phi(x_1,\ldots,x_{n-1})\exp(-\frac{x_n}{\varepsilon^p}), \qquad x_n \geq 0$$

where p is some positive constant.

Once a boundary layer function v_0 has been determined a formal approxima-
tion $\tilde{u}$ of u (up to order ε) can be written down, namely

$$\tilde{u} = w_0 + v_0.$$

Formal approximations of higher order can be obtained by taking more terms
into account. For instance, the function

(8)
$$\tilde{u} = \sum_{k=0}^{n} \varepsilon^k w_k + \sum_{k=0}^{n} \varepsilon^k v_k$$

with an appropriate choice for the functions w_k ($k = 0,1,\ldots,n$) and the
boundary layer functions v_k ($k = 0,1,\ldots,n$) may give a formal approximation
of u up to order ε^{n+1}.

As an illustration we shall now construct a formal approximation for the
solution of a simple initial value problem in one independent variable,
namely

(9)
$$\begin{cases} L^{(\varepsilon)}[u] \equiv \varepsilon\dfrac{d^2u}{dt^2} + \dfrac{du}{dt} + u = f(t), & 0 < t < \infty \\ u(0;\varepsilon) = C_1, \quad u_t(0;\varepsilon) = C_2 \end{cases}$$

with f some C^{∞}-function. This problem can of course easily be solved expli-
citly, but we give this example in this introduction merely to illustrate
the construction of a formal approximation. Stretching the coordinate t by
introducing

$$\tau = \frac{t}{\varepsilon}$$

we can write the differential operator (9) as

$$L^{(\varepsilon)}[u] \equiv \varepsilon^{-1}(\frac{d^2u}{d\tau^2} + \frac{du}{d\tau}) + u.$$

Let $\tilde{u}$ be defined by (8) where the functions w_k depend on t and the boundary
layer functions v_k depend on τ. Then we obtain

x

$$L^{(\varepsilon)}[\tilde{u}] = L^{(\varepsilon)}[\sum_{k=0}^{n} \varepsilon^k w_k] + L^{(\varepsilon)}[\sum_{k=0}^{n} \varepsilon^k v_k]$$

$$= \frac{dw_0}{dt} + w_0 + \varepsilon^{-1}(\frac{d^2 v_0}{d\tau^2} + \frac{dv_0}{d\tau})$$

(10)

$$+ \sum_{k=1}^{n} \varepsilon^k (\frac{dw_k}{dt} + w_k + \frac{d^2 w_{k-1}}{dt^2})$$

$$+ \sum_{k=1}^{n} \varepsilon^{k-1} (\frac{d^2 v_k}{d\tau^2} + \frac{dv_k}{d\tau} + v_{k-1})$$

$$+ \varepsilon^{n+1} \frac{d^2 w_n}{dt^2} + \varepsilon^n v_n$$

and also

(11)
$$\begin{cases} \tilde{u}(0;\varepsilon) = w_0(0) + \sum_{k=1}^{n} \varepsilon^k \{w_k(0) + \varepsilon^{-1} v_{k-1}(0)\} + \varepsilon^n v_n(0) \\ \\ \tilde{u}_t(0;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \{\frac{dw_k}{dt}(0) + \varepsilon^{-1} \frac{dv_k}{d\tau}(0)\}. \end{cases}$$

The functions w_k $(k = 0,1,\ldots,n)$ and v_k $(k = 0,1,\ldots,n)$ can now be obtained by solving the following set of initial value problems

(12)
$$\begin{cases} \frac{dw_0}{dt} + w_0 = f(t), \qquad 0 < t < \infty \\ \\ w_0(0) = C_1 \end{cases}$$

(13)
$$\begin{cases} \frac{d^2 v_0}{d\tau^2} + \frac{dv_0}{d\tau} = 0, \qquad 0 < \tau < \infty \\ \\ \frac{dv_0}{d\tau}(0) = \varepsilon\{C_2 - \frac{dw_0}{dt}(0)\} \\ \\ \lim_{\tau \to \infty} v_0(\tau) = 0 \end{cases}$$

and for $k = 1,2,\ldots,n$

$$(14) \quad \begin{cases} \dfrac{dw_k}{dt} + w_k = - \dfrac{d^2 w_{k-1}}{dt^2}, & 0 < t < \infty \\[3mm] w_k(0) = -\varepsilon^{-1} v_{k-1}(0) \end{cases}$$

$$(15) \quad \begin{cases} \dfrac{d^2 v_k}{d\tau^2} + \dfrac{dv_k}{d\tau} = -v_{k-1} \\[3mm] \dfrac{dv_k}{d\tau}(0) = -\varepsilon \dfrac{dw_k}{dt}(0) \\[3mm] \lim_{\tau \to \infty} v_k(\tau) = 0. \end{cases}$$

It is easily verified that the functions v_k are of the type

$$v_k(\tfrac{t}{\varepsilon}) = \varepsilon P_k(\tfrac{t}{\varepsilon}) \exp(-\tfrac{t}{\varepsilon})$$

with $P_k(t)$ some polynomial in t; these functions are typical boundary layer functions. Moreover it follows from (10), (11), (12), (13), (14) and (15) that

$$\begin{cases} L^{(\varepsilon)}[\tilde{u}] = f(t) + O(\varepsilon^{n+1}), & 0 < t < \infty \\[3mm] \tilde{u}(0;\varepsilon) = C_1 + O(\varepsilon^{n+1}) \\[3mm] \tilde{u}_t(0;\varepsilon) = C_2 \end{cases}$$

so $\tilde{u}$ is a formal approximation (up to order ε^{n+1}) for the solution u of (9).

For an exposition of formal constructive methods we refer to COLE [6] and NAYFEH [32]. FRIEDRICHS [14] has written a survey article on the occurrence of boundary layers in various branches of mathematical physics. A generalization of the concept of formal approximation can be found in ECKHAUS [11].

After a formal approximation up to some order of ε, say ε^{n+1}, has been constructed its correctness (that is, the validity of (7)) remains to be

shown. This can be done by introducing the so-called *remainder term* z defined by

(16) $z = u - \tilde{u}$

and proving that

(17) $\lim_{\varepsilon \downarrow 0} \|z\|_{\bar{\Omega}} = 0.$

Since $\tilde{u}$ is a formal approximation of u up to order ε^{n+1} it follows that z satisfies a boundary value problem of the type

(18)
$$\begin{cases} L^{(\varepsilon)}[z] = L^{(\varepsilon)}[u] - L^{(\varepsilon)}[\tilde{u}] = O(\varepsilon^{n+1}), & \vec{x} \in \Omega \\ B_i[z] = O(\varepsilon^{n+1}) \text{ along } \partial\Omega \ (i = 1,2,\dots,k) \end{cases}$$

so in order to prove (17) it suffices to derive an a *priori estimate* for the solution z of (18) which estimates $\|z\|_{\bar{\Omega}}$ in terms of the right-hand side function and the boundary functions. Once an a priori estimate has been established, say for instance

$$\|z\|_{\bar{\Omega}} = O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0$$

we obtain from (8) and (16)

(19) $\left\| u - \sum_{k=0}^{n} \varepsilon^k (w_k + v_k) \right\|_{\bar{\Omega}} = O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0$

from which the correctness of the formal approximation follows. The series in the left-hand side of (19) is called an *asymptotic expansion* of u into powers of ε.

REMARK. Not every formal approximation is a correct approximation. Examples of formal approximations which are not approximations can be found for instance in ECKHAUS [11] and will also be given in this thesis.

A priori estimates for the solution z of (18) can be obtained by various methods, depending on the nature of the operators L_1 and L_2. For problems with a singular perturbation of elliptic type (that is, for problems

with an elliptic operator L_2) a priori estimates have been established amongst others by BESJES [2], ECKHAUS & DE JAGER [12], VAN HARTEN [23], HUET [24[*]], LIONS [28] and VISIK & LYUSTERNIK [49], whereas a priori estimates for problems with a singular perturbation of parabolic type can be found for instance in BESJES [2] and LIONS [28]. In this thesis we shall be concerned primarily with singular perturbation problems in $\mathbb{R}^2$ which are governed by an equation of the type (1) in which the operator L_2 is a linear hyperbolic differential operator of the second order. A priori estimates for solutions of these kind of problems can be obtained by applying the so-called *energy integral method*. This method can briefly be described as follows:

consider the initial-boundary value problem

$$(20) \quad \begin{cases} L_2[u] + L_1[u] = f(x,t) & \text{in } \Omega \subset \mathbb{R}^2 \\ \\ u \text{ or } u \text{ and its first normal derivative prescribed along } \partial\Omega \end{cases}$$

where the operators L_2 and L_1 are given by (2) and (3) (in order to fix the idea one may think of L_2 as the wave operator and take for Ω the strip $0 < x < 1$, $0 < t < \infty$; in that case u should be prescribed along $x = 0$, $t \geq 0$ and $x = 1$, $t \geq 0$, whereas u and u_t should be prescribed along $t = 0$, $0 \leq x \leq 1$). Note that (20) is not a singular perturbation problem. We multiply the differential equation by a linear combination of $u(x,t)$ and its first derivatives and integrate the resulting formula over a suitably chosen region $S \subset \bar{\Omega}$, which usually contains a part of the boundary $\partial\Omega$. This yields

$$\iint\limits_S (\alpha u_t + \beta u_x + \gamma u)(L_2[u] + L_1[u])\,dxdt = \iint\limits_S (\alpha u_t + \beta u_x + \gamma u)f\,dxdt.$$

Selecting the coefficients $\alpha = \alpha(x,t)$, $\beta = \beta(x,t)$ and $\gamma = \gamma(x,t)$ in an appropriate way, applying GREEN's theorem in the plane to the left-hand side integral and taking into account the prescribed initial and boundary conditions we obtain certain integrals of positive definite quadratic forms in u, u_x and u_t, usually referred to as *energy integrals*. A careful manipulation of these energy integrals leads then to an a priori estimate for the solution u of (20).

The energy integral method has been known for a long time as a powerful tool

for obtaining a priori estimates and uniqueness proofs for solutions of initial value problems, not only in $\mathbb{R}^2$ but also in $\mathbb{R}^n$ (see for instance HADAMARD [22], RUBINOWICZ [42], WEBER [53], ZAREMBA [57] and the review in the article by FRIEDRICHS [13]). In its simplest form, with the choice

$$(21) \qquad \alpha \equiv 1, \ \beta \equiv \gamma \equiv 0$$

the method has been applied to CAUCHY's problem for linear hyperbolic equations of the second order (see FRIEDRICHS & LEWY [15]); a less trivial choice of α, β and γ is required for a proof of the uniqueness of the solution of TRICOMI's problem (cf. PROTTER [40]) or FRANKL's problem (cf. MORAWETZ [31]). Although the energy integral method is usually applied to hyperbolic equations it has also been used for elliptic equations (see GARABEDIAN [16, p.426]).

If the energy integral method described above is applied to singularly perturbed initial value problems which are governed by an equation of the type (1), (2), (3) a slight complication is formed by the presence of the small factor ε in front of the hyperbolic operator L_2 since it turns out that due to this factor the choice (21) for α, β and γ does no longer lead to useful estimates (see DE JAGER [26]). In case the coefficient $d(x,t)$ in (3) is identically equal to zero this difficulty can be overcome by choosing α and β such that

$$(22) \qquad \alpha(x,t) \equiv a(x,t), \quad \beta(x,t) \equiv b(x,t)$$

(where $a(x,t)$ and $b(x,t)$ are the coefficients occurring in L_1).
With this choice it is possible to derive by means of the energy integral method a priori estimates for solutions of initial value problems with a singular perturbation of hyperbolic type, provided that the coefficients a, b, c_1 and c_2 in L_1 and L_2 satisfy a so-called "timelike" condition (see DE JAGER [26]).

A generalized version of the energy integral method described above has been applied by DZAVADOV [7] to singularly perturbed initial value problems in $\mathbb{R}^n$. In that case the differential equation is multiplied by $L_1[u]$ and integrated over a suitably chosen region. For a problem in $\mathbb{R}^2$ multiplication by $L_1[u]$ comes down to the choice (22), provided that $d(x,t) \equiv 0$.

An application of the energy integral method to singularly perturbed initial-boundary value problems in $\mathbb{R}^n$ with a quasilinear first order operator L_1 of the type

$$L_1[u] = a(\vec{x},t)\frac{\partial u}{\partial t} + \sum_{k=1}^{n} b_k(\vec{x},t)\frac{\partial u}{\partial x_k} + c(\vec{x},t)|u|^\rho u, \qquad \rho > 0$$

can be found in GENET & MADAUNE [17].
A different form of the energy integral method, based on a multiplication of the equation by a second derivative of u, has been used by BLONDEL [3].

There exists a large amount of literature on singular perturbation problems. Singular perturbations of elliptic type have been studied for instance by BESJES [2], ECKHAUS [9], ECKHAUS & DE JAGER [12], GRASMAN [20], DE GROEN [21], VAN HARTEN [23], HUET [24*], DE JAGER [25], LIONS [28], O'MALLEY [37] and VISIK & LYUSTERNIK [49]; for singular perturbations of parabolic type we refer to BESJES [2] and LIONS [28]. Extensive bibliographies can be found in LIONS [28], NAYFEH [32] and O'MALLEY [37], [38]. Considering this large amount of literature the number of articles that deal with singular perturbations of hyperbolic type is relatively small. BLONDEL [3], [4], DZAVADOV [7], DE JAGER [26], SMITH [44], SMITH & PALMER [46] and WEINSTEIN & SMITH [55] have studied singularly perturbed initial value problems for linear hyperbolic differential equations, whereas singularly perturbed initial-boundary value problems have been investigated by DZAVADOV [8], GENET & PUPION [18] and WEINSTEIN & SMITH [55] (linear problems) and by GENET & MADAUNE [17] and MADAUNE [30] (nonlinear problems). Singular perturbations of hyperbolic type are also discussed in the books by COLE [6], LIONS [28], ROSEAU [41] and WHITHAM [56].

We shall end this introduction with a survey of the contents of this thesis.

The first two chapters of the thesis are devoted to singularly perturbed initial value problems for ordinary differential equations of the second order, because problems of this kind can be considered as special cases of similar problems for hyperbolic equations and in our opinion the methods that will be used later on in the analysis of singularly perturbed initial value problems for hyperbolic differential equations can best be understood by applying these methods first to relatively simple problems for ordinary differential equations.

In chapter I we deal with linear problems of the type

$$(23) \quad \begin{cases} \varepsilon\dfrac{d^2u}{dt^2} + a(t)\dfrac{du}{dt} + b(t)u = f(t), \quad 0 < t < \infty \\[2mm] u(0;\varepsilon) = c_1, \quad u_t(0;\varepsilon) = c_2 \end{cases}$$

where the coefficient $a(t)$ is strictly positive.

Applying the energy integral method we establish pointwise a priori esti-
mates for the solution of the initial value problem and its first deriva-
tive, which are uniformly valid in any bounded segment $0 \le t \le T$ for ε suf-
ficiently small. Next we construct a formal approximation for the solution
and use the a priori estimates to prove its correctness. It is shown that
the difference between the solution and the formal approximation and also
the difference between their first derivatives tend pointwise to zero as ε
approaches zero, uniformly in any bounded segment $0 \le t \le T$. We investigate
also problems of the type (23) with a strictly negative coefficient $a(t)$.
It is shown that in this case the solution tends pointwise to infinity as
ε approaches zero.

In chapter II we deal with nonlinear problems of the type

$$\begin{cases} \varepsilon\dfrac{d^2u}{dt^2} + a(t,u)\dfrac{du}{dt} + b(t,u) = 0, \quad 0 < t < \infty \\[2mm] u(0;\varepsilon) = c_1, \quad u_t(0;\varepsilon) = c_2 \end{cases}$$

where the coefficient $a(t,u)$ is strictly positive.

We construct a formal approximation and formulate an estimation theorem
from which the existence of a solution near the formal approximation fol-
lows; the difference between this solution and the formal approximation
and also the difference between their first derivatives are shown to con-
verge pointwise to zero for ε tending to zero, uniformly in any bounded
segment $0 \le t \le T$ where the solution w_0 of the reduced problem is regular.
The estimation theorem is a modified version of a similar theorem by
VAN HARTEN [23] and its proof is based on a contraction mapping principle
for operators in a Banach space.

In the two chapters that follow we extend the results of the chapters I
and II to singularly perturbed initial value problems for hyperbolic dif-
ferential equations. In chapter III we discuss linear problems of the type

$$\begin{cases} \varepsilon L_2[u] + L_1[u] = f(x,t), & -\infty < x < \infty, \ 0 < t < \infty \\[2mm] u(x,0;\varepsilon) = g(x), & -\infty < x < \infty \\[2mm] u_t(x,0;\varepsilon) = h(x), & -\infty < x < \infty \end{cases}$$

where the operators L_2 and L_1 are given by (2) and (3). The coefficient $a(x,t)$ is assumed to be strictly positive. An energy integral method is employed which yields pointwise a priori estimates for the solution of the initial value problem and its first derivatives, provided that the coefficients satisfy the so-called "timelike" condition

$$c_1(x,t) < \frac{b(x,t)}{a(x,t)} < c_2(x,t) \qquad \text{for } -\infty < x < \infty, \ 0 \le t < \infty.$$

These a priori estimates are, for ε sufficiently small, uniformly valid in any compact subset of $-\infty < x < \infty$, $0 \le t < \infty$. The energy integral method that is employed is an extension of a similar method developed by DE JAGER [26] and is basically different from the methods used by other authors (cf. [3], [4], [7], [44], [46] and [55]). Next we construct a formal approximation for the solution and use the a priori estimates to show that the difference between the solution and the formal approximation and also the difference between their first derivatives tend pointwise to zero as ε approaches zero, uniformly in any compact subset of $-\infty < x < \infty$, $0 \le t < \infty$. In chapter IV we investigate nonlinear problems of the type

$$\begin{cases} \varepsilon\{\dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2}\} + L_1[u] = 0, & -\infty < x < \infty, \ 0 < t < \infty \\[2mm] u(x,0;\varepsilon) = f(x), & -\infty < x < \infty \\[2mm] u_t(x,0;\varepsilon) = g(x), & -\infty < x < \infty \end{cases}$$

where the operator L_1 is given by (4) and the functions $c(x,t)$ and $a(x,t,u)$ are strictly positive. It is shown that a formal approximation can be constructed if the solution w_0 of the quasilinear reduced problem is sufficiently smooth. The existence of a solution of the full equation near this formal approximation is established by applying the estimation theorem of chapter II; the difference between this solution and the formal approximation and also the difference between their first derivatives are shown to converge pointwise to zero for ε tending to zero, uniformly in compact subsets

of $-\infty < x < \infty$, $0 \le t < \infty$ where w_0 is regular. As in chapter III the results
are obtained under the condition that the coefficients satisfy a "timelike"
condition. Here this condition reads

$$-c(x,t) < \frac{b(x,t,u)}{a(x,t,u)} < c(x,t) \qquad \text{for } -\infty < x,u < \infty, \ 0 \le t < \infty.$$

Generally, however, the solution w_0 of the quasilinear reduced problem does
not exist in the classical sense for all $t \ge 0$. Hence we are led to the
concept of "generalized solution of the reduced problem". A discussion of
generalized solutions and their relevance to singular perturbations of hyper-
bolic type is presented.

The last two chapters of this thesis are devoted to singularly per-
turbed initial-boundary value problems for hyperbolic differential equations.
In chapter V we consider the following problem in a strip:

$$\begin{cases} \varepsilon\{\dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2}\} + L_1[u] = f(x,t), & 0 < x < \ell, \ 0 < t < \infty \\[2mm] u(x,0;\varepsilon) = g(x), & 0 \le x \le \ell \\[2mm] u_t(x,0;\varepsilon) = h(x), & 0 \le x \le \ell \\[2mm] u(0,t;\varepsilon) = k_1(t), & 0 \le t < \infty \\[2mm] u(\ell,t;\varepsilon) = k_2(t), & 0 \le t < \infty \end{cases}$$

where the operator L_1 is given by (3), the coefficients $c(x,t)$ and $a(x,t)$
are strictly positive and the boundary functions satisfy

$$g(0) = k_1(0), \qquad g(\ell) = k_2(0).$$

Applying a modified version of the energy integral method developed in chap-
ter III we derive pointwise a priori estimates for the solution of this
problem. We give also pointwise estimates for the first derivatives of the
solution. The estimates are, for ε sufficiently small, uniformly valid in
any compact subset of $0 \le x \le \ell$, $0 \le t < \infty$.
Under the additional condition $b(x,t) > 0$ we construct a formal approxima-
tion for the solution. The correctness of the approximation is established
by applying the a priori estimates; we show that the difference between the

solution and the formal approximation tends pointwise to zero as ε approaches zero, uniformly in any compact subset of $0 \le x \le \ell$, $0 \le t < \infty$. Here, too, the results are obtained under the restriction that the coefficients satisfy the "timelike" condition.

$$-c(x,t) < \frac{b(x,t)}{a(x,t)} < c(x,t) \qquad \text{for } 0 \le x \le \ell, \ 0 \le t < \infty.$$

In chapter VI we investigate boundary value problems of the type

$$\begin{cases} \varepsilon \frac{\partial^2 u}{\partial x \partial y} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y), \qquad (x,y) \in D \\[2mm] u(x,0;\varepsilon) = g(x), \qquad 0 \le x < \infty \\[2mm] u(\phi(y),y;\varepsilon) = h(y), \qquad 0 \le y < \infty \end{cases}$$

where the region D is given by

$$D = \{(x,y) \mid \phi(y) < x < \infty, \ 0 < y < \infty\}$$

with $\phi(y)$ some non-decreasing function defined for $0 \le y < \infty$ and satisfying $\phi(0) = 0$. The coefficients $a(x,y)$ and $b(x,y)$ are strictly positive and the boundary functions satisfy

$$g(0) = h(0).$$

Note that in this problem the part of the boundary given by $y = 0$, $0 \le x < \infty$ coincides with a characteristic of the second order differential operator

$$\frac{\partial^2}{\partial x \partial y} \ .$$

Problems of this kind are sometimes referred to as Goursat problems. Applying a modified version of the energy integral method developed in chapter III we establish pointwise a priori estimates for the solution of this boundary value problem which are uniformly valid in any compact subset of $\bar{D}$ for ε sufficiently small. Next we construct a formal approximation for the solution and use the a priori estimates to show that the difference between the solution and the formal approximation converges pointwise to zero for ε tending to zero, uniformly in any compact subset of $\bar{D}$.

NOTATIONS

Numbers

$\mathbb{Z}$: integers.

$\mathbb{N}$: positive integers.

$\mathbb{R}$: real numbers.

$\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) \mid x_k \in \mathbb{R}; \ k = 1, 2, \ldots, n\}$ with $n \in \mathbb{N}$, $n \geq 2$.

$\mathbb{R}^+$: positive real numbers.

$\mathbb{R}^+$: nonnegative real numbers.

$\mathbb{C}$: complex numbers.

$\text{Re}\lambda$: real part of $\lambda \in \mathbb{C}$.

$\text{Im}\lambda$: imaginary part of $\lambda \in \mathbb{C}$.

$\arg\lambda$: argument of $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

$\vec{a} \cdot \vec{b}$: dot or scalar product of vectors $\vec{a}$ and $\vec{b}$.

$\vec{a} \times \vec{b}$: cross or vector product of vectors $\vec{a}$ and $\vec{b}$.

$\bar{D}$: closure of the open set $D \subset \mathbb{R}^2$.

∂D: boundary $\bar{D} \backslash D$ of the open set $D \subset \mathbb{R}^2$.

Functions

$u(t)$: real-valued function of the variable $t \in \mathbb{R}$.

$u(t;\varepsilon)$: real-valued function of the variable $t \in \mathbb{R}$, depending on the
   parameter $\varepsilon \in \mathbb{R}$.

$u' = u_t = \dfrac{du}{dt}$: first derivative of $u(t)$ or $u(t;\varepsilon)$.

$u'' = u_{tt} = \dfrac{d^2u}{dt^2}$: second derivative of $u(t)$ or $u(t;\varepsilon)$.

$u(x,t)$: real-valued function of the variables $x,t \in \mathbb{R}$.

$u(x,t;\varepsilon)$: real-valued function of the variables $x,t \in \mathbb{R}$, depending on
   the parameter $\varepsilon \in \mathbb{R}$.

$u_t = \dfrac{\partial u}{\partial t}$, $u_x = \dfrac{\partial u}{\partial x}$: first derivatives of $u(x,t)$ or $u(x,t;\varepsilon)$.

$u_{tt} = \dfrac{\partial^2 u}{\partial t^2}$, $u_{xt} = \dfrac{\partial^2 u}{\partial t \partial x}$, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$: second derivatives of $u(x,t)$ or $u(x,t;\varepsilon)$.

Order symbols

If $n \in \mathbb{N}$ and $u(x,t;\varepsilon)$ is a function defined in $\Omega \subset \mathbb{R}^2$ then
$u(x,t;\varepsilon) = O(\varepsilon^n)$, $\varepsilon \downarrow 0$ uniformly in Ω iff there exist positive constants
C and ε_0 such that $|u(x,t;\varepsilon)| \leq C\varepsilon^n$ for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

Function spaces

$C^0[0,T]$: space of real-valued, continuous functions defined in $[0,T] \subset \mathbb{R}$.

$C^k[0,T]$, $k \in \mathbb{N}$: space of real-valued, k times continuously differen-
tiable functions defined in $[0,T] \subset \mathbb{R}$.

$C^\infty[0,T] := \bigcap_{k=0}^{\infty} C^k[0,T]$.

$C^0(\Omega)$, $C^k(\Omega)$ and $C^\infty(\Omega)$ denote similar spaces of real-valued functions
defined in $\Omega \subset \mathbb{R}^2$.

Norms

For functions of the variable t defined in $0 \leq t \leq T$:

$$|u|_0 = \sup_{[0,T]} |u|, \qquad |u|_1^{(\varepsilon)} = \sup_{[0,T]} |u| + \varepsilon^{\frac{1}{2}} \sup_{[0,T]} |u_t|$$

$$\|u\|_{[0,T]} = \left[\int_0^T u^2 dt \right]^{\frac{1}{2}}, \qquad \|u\|_{1,[0,T]} = \int_0^T |u| dt$$

For functions of the variables x and t defined in a compact subset $\Omega \subset \mathbb{R}^2$:

$$|u|_0 = \sup_{\Omega} |u|, \qquad |u|_1^{(\varepsilon)} = \sup_{\Omega} |u| + \varepsilon^{\frac{3}{4}} \sup_{\Omega} |u_x| + \varepsilon^{\frac{3}{4}} \sup_{\Omega} |u_t|$$

$$\|u\|_{\Omega} = \left[\iint_{\Omega} u^2 dxdt \right]^{\frac{1}{2}}$$

Generic constants

Sometimes C and ε_0 denote generic constants of which the value may be
different on each appearance.

CHAPTER I

INITIAL VALUE PROBLEMS FOR LINEAR
ORDINARY DIFFERENTIAL EQUATIONS

In this chapter we shall discuss the initial value problem (singular perturbation problem)

(1)
$$
\begin{cases}
\varepsilon \dfrac{d^2 u}{dt^2} + a(t;\varepsilon)\dfrac{du}{dt} + b(t;\varepsilon)u = f(t;\varepsilon), \qquad 0 < t < \infty \\[2ex]
u(0;\varepsilon) = C_1(\varepsilon), \quad u_t(0;\varepsilon) = C_2(\varepsilon)
\end{cases}
$$

where ε is a small positive parameter, i.e. $0 < \varepsilon \ll 1$.

The functions a, b and f depend on the variable t and on the parameter ε.

The initial data $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are constants depending on ε.

It will be assumed that the coefficient $a(t;\varepsilon)$ is either strictly positive

$$ a(t;\varepsilon) \geq a_0 > 0 \qquad \text{for } 0 \leq t \leq T $$

or strictly negative

$$ a(t;\varepsilon) \leq -a_0 < 0 \qquad \text{for } 0 \leq t \leq T $$

valid for each constant T > 0 provided that ε is sufficiently small, say $0 < \varepsilon \leq \varepsilon_0$ with ε_0 depending on T. Here a_0 is some positive constant depending on T and ε_0 but independent of ε.

There exists a large amount of literature on singularly perturbed initial value problems for ordinary differential equations. Most authors investigate these problems by writing them as a system of first order equations, see for instance HOPPENSTEADT [24], O'MALLEY [38, chapter 4], PONZO & WAX [39], VASIL'EVA [47], VASIL'EVA & VOLOSOV [48] and WASOW [52, chapter 10]. The problems considered by these authors are generally nonlinear.

Singularly perturbed initial value problems in the form of a single differ-
ential equation of higher order are less frequently encountered in litera-
ture. We mention here the work of GRADSTEIN [19], O'MALLEY [37], SMITH [45],
VOLOSOV [50], [51] and WEINSTEIN & SMITH [54].
In spite of this wide variety of literature we have thought it appropriate
to devote the first two chapters of this thesis to singularly perturbed
initial value problems for linear and nonlinear ordinary differential equa-
tions of order two. Our considerations are that initial value problems for
hyperbolic differential equations (which form the main subject of our in-
vestigations) can be solved essentially in the same way as initial value
problems for ordinary differential equations, so the methods that will be
used in the analysis of linear and nonlinear hyperbolic equations (chapters
III and IV) can best be understood by applying them first to relatively
simple problems which are governed by linear and nonlinear ordinary differ-
ential equations (chapters I and II).

In the sections 1 and 2 of this chapter we shall study problem (1)
with a strictly positive coefficient $a(t;\varepsilon)$. We shall derive a priori esti-
mates for the solution u of (1) (section 1) and construct an asymptotic ex-
pansion into powers of ε for this solution (section 2).
In section 3 we shall discuss problem (1) in case the coefficient $a(t;\varepsilon)$ is
strictly negative.

1. A priori estimates

We consider the initial value problem (1) under the following condi-
tions:

(2) for each positive constant T there exists a positive constant ε_0
(depending on T) such that

(i) $a(t;\varepsilon) \in C^1[0,T]$, $b(t;\varepsilon) \in C^1[0,T]$, $f(t;\varepsilon) \in C^0[0,T]$

valid for every value of ε satisfying $0 < \varepsilon \le \varepsilon_0$

(ii) the functions $a(t;\varepsilon)$, $b(t;\varepsilon)$ and their first derivatives are
uniformly bounded in $0 \le t \le T$ for $0 < \varepsilon \le \varepsilon_0$, i.e. there exist
positive constants A and B (depending on T and ε_0 but independ-
ent of ε) such that for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$

$$
(2) \quad
\begin{cases}
\begin{cases}
|a(t;\varepsilon)| + |\frac{da}{dt}(t;\varepsilon)| \le A \\[2mm]
|b(t;\varepsilon)| + |\frac{db}{dt}(t;\varepsilon)| \le B
\end{cases} \\[4mm]
\text{(iii)} \quad a(t;\varepsilon) \ge a_0 > 0
\end{cases}
$$

valid for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$, where a_0 is some constant depending on T and ε_0 but independent of ε.

For the moment it will be assumed that the initial conditions are homogeneous, so

$$
(3) \quad C_1(\varepsilon) = C_2(\varepsilon) = 0.
$$

In order to derive an estimate for the solution $u(t;\varepsilon)$ of (1) in an interval $0 \le t \le T$ we multiply the differential equation (1) by $2u$. This yields

$$
(4) \quad \frac{d}{dt}(au^2 + 2\varepsilon uu_t) + (2b - a_t)u^2 - 2\varepsilon u_t^2 = 2fu
$$

valid for $0 \le t \le T$ and $0 < \varepsilon \le \varepsilon_0$, where arguments have been omitted for simplicity.

Similarly, multiplication of (1) by $2au_t$ leads to

$$
(5) \quad \frac{d}{dt}(\varepsilon au_t^2) - \varepsilon a_t u_t^2 + 2a^2 u_t^2 + 2abuu_t = 2afu_t
$$

valid for $0 \le t \le T$ and $0 < \varepsilon \le \varepsilon_0$.

Adding (4) and (5) and using the inequalities

$$
\begin{cases}
-2a^2 u_t^2 - 2abuu_t + 2afu_t \le b^2 u^2 + f^2 \\[2mm]
2fu \le u^2 + f^2
\end{cases}
$$

we obtain

$$
(6) \quad \frac{d}{dt}(au^2 + 2\varepsilon uu_t + \varepsilon au_t^2) \le (1 + a_t + b^2 - 2b)u^2 + \varepsilon(2 + a_t)u_t^2 + 2f^2
$$

valid for $0 \le t \le T$ and $0 < \varepsilon \le \varepsilon_0$.

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From the positiveness of $a(t;\varepsilon)$ it follows that for $0 \leq t \leq T$ and ε sufficiently small, say $0 < \varepsilon \leq \varepsilon_0$ with ε_0 a generic constant

$$au^2 + 2\varepsilon uu_t + \varepsilon au_t^2 \geq a_0 u^2 + 2\varepsilon uu_t + \varepsilon a_0 u_t^2$$

$$\geq (a_0 - \varepsilon^{\frac{1}{2}})u^2 + \varepsilon(a_0 - \varepsilon^{\frac{1}{2}})u_t^2$$

$$\geq m(u^2 + \varepsilon u_t^2)$$

where the positive constant m (independent of ε) may be defined by

$$m = \frac{1}{2}a_0 .$$

So integrating (6) with respect to t and using this inequality as well as (3) we obtain

$$m(u^2 + \varepsilon u_t^2) \leq M \int_0^t (u^2 + \varepsilon u_t^2)d\tau + M\|f\|^2_{[0,T]}$$

or

$$(7) \qquad (u^2 + \varepsilon u_t^2) - \frac{M}{m}\int_0^t (u^2 + \varepsilon u_t^2)d\tau \leq \frac{M}{m}\|f\|^2_{[0,T]}$$

valid for $0 \leq t \leq T$ and $0 < \varepsilon \leq \varepsilon_0$, where the positive constant M (independent of ε) is given by

$$M = \max(2 + A, 1 + A + B^2 + 2B)$$

and $\|\cdot\|_{[0,T]}$ denotes the L_2-norm.

Using GRONWALL's lemma (see for instance RUBINSTEIN [43, p.26]) we obtain from (7)

$$u^2 + \varepsilon u_t^2 \leq \frac{M}{m}\|f\|^2_{[0,T]} \exp(\frac{Mt}{m})$$

$$\leq \frac{M}{m}\|f\|^2_{[0,T]} \exp(\frac{MT}{m})$$

and hence

$$(8) \quad \begin{cases} |u(t;\epsilon)| \leq C(T)\|f\|_{[0,T]} \\[2mm] |\frac{du}{dt}(t;\epsilon)| \leq C(T)\epsilon^{-\frac{1}{2}}\|f\|_{[0,T]} \end{cases}$$

valid for $0 \leq t \leq T$ and $0 < \epsilon \leq \epsilon_0$, where the positive constant $C(T)$, depending on T but independent of ϵ, is given by

$$C(T) = (\frac{M}{m})^{\frac{1}{2}} \exp(\frac{MT}{2m}).$$

Note that so far we did not use the differentiability of the coefficient $b(t;\epsilon)$ required in (2, i and ii).

Now that we have established estimates for the solution of the homogeneous initial value problem (1) it is not difficult to obtain estimates for the inhomogeneous problem. We put

$$(9) \quad u(t;\epsilon) = w(t;\epsilon) + p \exp\{-\int_0^t \frac{b(\tau;\epsilon)}{a(\tau;\epsilon)} \, d\tau\} - \epsilon q \exp\{-\frac{a(0;\epsilon)t}{\epsilon}\}$$

where the ϵ-dependent constants p and q are defined by

$$\begin{cases} p = \dfrac{a(0;\epsilon)\{a(0;\epsilon)C_1(\epsilon) + \epsilon C_2(\epsilon)\}}{a^2(0;\epsilon) - \epsilon b(0;\epsilon)}, \\[4mm] q = \dfrac{b(0;\epsilon)C_1(\epsilon) + a(0;\epsilon)C_2(\epsilon)}{a^2(0;\epsilon) - \epsilon b(0;\epsilon)}. \end{cases}$$

Substitution of (9) into (1) yields

$$\begin{cases} \epsilon\dfrac{d^2w}{dt^2} + a(t;\epsilon)\dfrac{dw}{dt} + b(t;\epsilon)w = F(t;\epsilon), \qquad 0 < t < \infty \\[4mm] w(0;\epsilon) = w_t(0;\epsilon) = 0 \end{cases}$$

with

$$F(t;\epsilon) = f(t;\epsilon) - \epsilon p \frac{d^2}{dt^2} \exp\{-\int_0^t \frac{b(\tau;\epsilon)}{a(\tau;\epsilon)} \, d\tau\}$$

$$- qa(0;\epsilon)\{a(t;\epsilon) - a(0;\epsilon)\}\exp\{-\frac{a(0;\epsilon)t}{\epsilon}\} +$$

$$+ \varepsilon q b(t;\varepsilon) \exp\{- \frac{a(0;\varepsilon)t}{\varepsilon}\}.$$

Using the positiveness of $a(t;\varepsilon)$ it is easily verified that there exists a generic constant $C(T)$, depending on T but independent of ε, such that

$$\|F\|_{[0,T]} \le C(T)\{\|f\|_{[0,T]} + \varepsilon|c_1(\varepsilon)| + \varepsilon^{3/2}|c_2(\varepsilon)|\}.$$

So applying the estimates (8) to the function w and using (9) as well as the positiveness of $a(t;\varepsilon)$ we obtain

$$\begin{cases} |u(t;\varepsilon)| \le C(T)\{\|f\|_{[0,T]} + |c_1(\varepsilon)| + \varepsilon|c_2(\varepsilon)|\} \\[2mm] |\frac{du}{dt}(t;\varepsilon)| \le C(T)\{\varepsilon^{-\frac{1}{2}}\|f\|_{[0,T]} + |c_1(\varepsilon)| + |c_2(\varepsilon)|\} \end{cases}$$

valid for $0 \le t \le T$ and $0 < \varepsilon \le \varepsilon_0$, with $C(T)$ a generic constant.

We summarize the results in

THEOREM 1. *Let* $u(t;\varepsilon)$ *be the solution of the initial value problem* (1) *under the conditions* (2). *Then we have in every interval* $0 \le t \le T$ *the estimates*

(10a)
$$\begin{cases} \sup_{[0,T]} |u(t;\varepsilon)| \le C(T)\{\|f\|_{[0,T]} + |c_1(\varepsilon)| + \varepsilon|c_2(\varepsilon)|\} \\[4mm] \end{cases}$$

(10b)
$$\sup_{[0,T]} |\frac{du}{dt}(t;\varepsilon)| \le C(T)\{\varepsilon^{-\frac{1}{2}}\|f\|_{[0,T]} + |c_1(\varepsilon)| + |c_2(\varepsilon)|\}$$

valid for all values of ε *satisfying* $0 < \varepsilon \le \varepsilon_0$, *where* ε_0 *is a constant depending on* T *and* $C(T)$ *is a positive constant depending on* T *but independent of* ε.

REMARKS.
1. The differentiability of the coefficient $b(t;\varepsilon)$ can be disposed of in case $c_1(\varepsilon) = c_2(\varepsilon) = 0$ (see formula (8)).
2. An estimate for the second derivative $\frac{d^2u}{dt^2}$ follows by substitution of (10a) and (10b) into the differential equation (1).

2. Asymptotic expansions

As an application of the a priori estimates that have been established in the preceding section we shall now construct an asymptotic expansion into powers of ε for the solution $u(t;\varepsilon)$ of the initial value problem (1). In order not to complicate the calculations we shall assume that the functions a, b and f as well as the initial data C_1 and C_2 are independent of the parameter ε. Furthermore we shall assume that a, b and f are C^{∞}-functions. So we consider the initial value problem

$$(11) \quad \begin{cases} \varepsilon\dfrac{d^2 u}{dt^2} + a(t)\dfrac{du}{dt} + b(t)u = f(t), \quad & 0 < t < \infty \\[2mm] u(0;\varepsilon) = C_1, \quad u_t(0;\varepsilon) = C_2 \end{cases}$$

with $0 < \varepsilon \ll 1$ and

$$(12) \quad \begin{cases} a(t) \in C^{\infty}(\overline{\mathbb{R}^+}), \quad b(t) \in C^{\infty}(\overline{\mathbb{R}^+}), \quad f(t) \in C^{\infty}(\overline{\mathbb{R}^+}) \\[3mm] a(t) > 0 \quad \text{for} \quad 0 \le t < \infty. \end{cases}$$

Expanding the functions $a(t)$ and $b(t)$ into a Taylor series we obtain for any $n \in \mathbb{N} \cup \{0\}$

$$\begin{cases} a(t) = \displaystyle\sum_{k=0}^{n} a_k t^k + a_{n+1}(t) t^{n+1} \\[4mm] b(t) = \displaystyle\sum_{k=0}^{n} b_k t^k + b_{n+1}(t) t^{n+1} \end{cases}$$

with $a_0 = a(0)$, $b_0 = b(0)$ etc.
We define the differential operator $L^{(\varepsilon)}$ by

$$L^{(\varepsilon)}[u] = \varepsilon\frac{d^2 u}{dt^2} + a(t)\frac{du}{dt} + b(t)u.$$

Putting $\tau = \dfrac{t}{\varepsilon}$ and expanding $L^{(\varepsilon)}$ into powers of ε we obtain

$$L^{(\varepsilon)}[u] = \varepsilon^{-1}\frac{d^2 u}{d\tau^2} + \varepsilon^{-1}\{ \sum_{k=0}^{n} a_k \varepsilon^k \tau^k + a_{n+1}(\varepsilon\tau)\varepsilon^{n+1}\tau^{n+1}\}\frac{du}{d\tau}$$

$$+ \sum_{k=0}^{n} b_k \varepsilon^k \tau^k u + b_{n+1}(\varepsilon\tau)\varepsilon^{n+1}\tau^{n+1} u.$$

We construct now a formal approximation $\tilde{u}(t;\varepsilon)$ of $u(t;\varepsilon)$ in the form

(13) $\qquad \tilde{u}(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^{k} w_{k}(t) + \sum_{k=0}^{n} \varepsilon^{k+1} v_{k}(\tau)$

where the functions w_{k} and v_{k} $(k = 0,1,\ldots,n)$ have to be determined. A calculation shows that

$$L^{(\varepsilon)}[\tilde{u}] = L^{(\varepsilon)}\left[\sum_{k=0}^{n} \varepsilon^{k} w_{k}\right] + L^{(\varepsilon)}\left[\sum_{k=0}^{n} \varepsilon^{k+1} v_{k}\right]$$

$$= \left\{a(t)\frac{dw_{0}}{dt} + b(t)w_{0}\right\} + \left\{\frac{d^{2}v_{0}}{d\tau^{2}} + a(0)\frac{dv_{0}}{d\tau}\right\}$$

(14) $$+ \sum_{k=1}^{n} \varepsilon^{k}\left\{a(t)\frac{dw_{k}}{dt} + b(t)w_{k} + \frac{d^{2}w_{k-1}}{dt^{2}}\right\}$$

$$+ \sum_{k=1}^{n} \varepsilon^{k}\left\{\frac{d^{2}v_{k}}{d\tau^{2}} + a(0)\frac{dv_{k}}{d\tau} + \sum_{j=1}^{k} a_{j}\tau^{j}\frac{dv_{k-j}}{d\tau}\right.$$

$$\left. + \sum_{j=0}^{k-1} b_{j}\tau^{j} v_{k-j-1}\right\} + R(t;\varepsilon)$$

where the function $R(t;\varepsilon)$ is defined by

$$R(t;\varepsilon) = \varepsilon^{n+1}\frac{d^{2}w_{n}}{dt^{2}} + \sum_{k=n+1}^{2n} \varepsilon^{k}\left(\sum_{j=k-n}^{n} a_{j}\tau^{j}\frac{dv_{k-j}}{d\tau}\right)$$

$$+ a_{n+1}(t)\varepsilon^{n+1}\tau^{n+1}\sum_{k=0}^{n} \varepsilon^{k}\frac{dv_{k}}{d\tau} + \sum_{k=n}^{2n} \varepsilon^{k+1}\left(\sum_{j=k-n}^{n} b_{j}\tau^{j} v_{k-j}\right)$$

$$+ b_{n+1}(t)\varepsilon^{n+1}\tau^{n+1}\sum_{k=0}^{n} \varepsilon^{k+1} v_{k}.$$

N.B. Empty sums, i.e. sums of which the upper limit of summation is less than the lower limit, are to be interpreted as zero.

Putting $t = 0$ in (13) we get

$$(15) \quad \begin{cases} \tilde{u}(0;\varepsilon) = w_0(0) + \sum_{k=1}^{n} \varepsilon^k \{w_k(0) + v_{k-1}(0)\} + \varepsilon^{n+1} v_n(0) \\[2mm] \tilde{u}_t(0;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \{\frac{dw_k}{dt}(0) + \frac{dv_k}{d\tau}(0)\}. \end{cases}$$

The formal approximation $\tilde{u}$ of the solution u of (11) is now obtained by solving the following set of initial value problems

$$(16) \quad \begin{cases} a(t)\frac{dw_0}{dt} + b(t)w_0 = f(t), \qquad 0 < t < \infty \\[3mm] w_0(0) = c_1 \end{cases}$$

$$(17) \quad \begin{cases} \frac{d^2v_0}{d\tau^2} + a(0)\frac{dv_0}{d\tau} = 0, \qquad 0 < \tau < \infty \\[3mm] \frac{dv_0}{d\tau}(0) = c_2 - \frac{dw_0}{dt}(0) \\[3mm] \lim_{\tau \to \infty} v_0(\tau) = 0 \end{cases}$$

and for $k = 1,2,\ldots,n$

$$(18) \quad \begin{cases} a(t)\frac{dw_k}{dt} + b(t)w_k = -\frac{d^2w_{k-1}}{dt^2}, \qquad 0 < t < \infty \\[3mm] w_k(0) = -v_{k-1}(0) \end{cases}$$

$$(19) \quad \begin{cases} \frac{d^2v_k}{d\tau^2} + a(0)\frac{dv_k}{d\tau} = -\sum_{j=1}^{k} a_j \tau^j \frac{dv_{k-j}}{d\tau} - \sum_{j=0}^{k-1} b_j \tau^j v_{k-j-1}, \qquad 0 < \tau < \infty \\[3mm] \frac{dv_k}{d\tau}(0) = -\frac{dw_k}{dt}(0) \\[3mm] \lim_{\tau \to \infty} v_k(\tau) = 0. \end{cases}$$

It is easily established that the functions $v_k(\tau)$, being of the form

$$(20) \qquad v_k(\tau) = P_k(\tau)\exp\{-a(0)\tau\}$$

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with $P_k(\tau)$ a polynomial in τ, are typical boundary layer functions and also that (14) and (15) can be written as

(21)
$$\begin{cases} L^{(\varepsilon)}[\tilde{u}] = f(t) + R(t;\varepsilon), & 0 < t < \infty \\ \\ \tilde{u}(0;\varepsilon) = C_1 + \varepsilon^{n+1} v_n(0) \\ \\ \tilde{u}_t(0;\varepsilon) = C_2. \end{cases}$$

Finally it is easily verified that for any nonnegative number T

$$R(t;\varepsilon) = 0(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0$$

uniformly valid in $0 \le t \le T$. Hence the function $\tilde{u}$ provides a formal approximation (up to order ε^{n+1}) for the solution u of problem (11).

Introducing the remainder term $z(t;\varepsilon)$ by

(22) $z(t;\varepsilon) = u(t;\varepsilon) - \tilde{u}(t;\varepsilon)$

it follows from (11) and (21) that

$$\begin{cases} L^{(\varepsilon)}[z] = -R(t;\varepsilon), & 0 < t < \infty \\ \\ z(0;\varepsilon) = -\varepsilon^{n+1} v_n(0) \\ \\ z_t(0;\varepsilon) = 0 \end{cases}$$

so we obtain from theorem 1

(23)
$$\begin{cases} z(t;\varepsilon) = 0(\varepsilon^{n+1}), & \varepsilon \downarrow 0 \\ \\ \dfrac{dz}{dt}(t;\varepsilon) = 0(\varepsilon^{n+\frac{1}{2}}), & \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in $0 \le t \le T$.
Since $\varepsilon^{n+1} v_n(\tau)$ is uniformly of order ε^{n+1} in $0 \le \tau < \infty$ it follows from (13), (22) and (23) that

$$\begin{cases} u(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(t) + \sum_{k=0}^{n-1} \varepsilon^{k+1} v_k(\tau) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0 \\[4mm] \dfrac{du}{dt}(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \dfrac{dw_k}{dt}(t) + \sum_{k=0}^{n} \varepsilon^k \dfrac{dv_k}{d\tau}(\tau) + O(\varepsilon^{n+\frac{1}{2}}), & \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in $0 \le t \le T$.

The accuracy in the expansion for $\dfrac{du}{dt}$ can be improved by applying the formula with n replaced by n+1 and using the fact that the terms

$$\varepsilon^{n+1} \frac{dw_{n+1}}{dt}(t) \quad \text{and} \quad \varepsilon^{n+1} \frac{dv_{n+1}}{d\tau}(\tau)$$

are uniformly of order ε^{n+1}. In this way we obtain

THEOREM 2. Let $u(t;\varepsilon)$ be the solution of the initial value problem (11) under the conditions (12). Then we have for any $n \in \mathbb{N} \cup \{0\}$

$$(24) \quad \begin{cases} u(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(t) + \sum_{k=0}^{n-1} \varepsilon^{k+1} v_k(\tau) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0 \\[4mm] \dfrac{du}{dt}(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \dfrac{dw_k}{dt}(t) + \sum_{k=0}^{n} \varepsilon^k \dfrac{dv_k}{d\tau}(\tau) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in every bounded interval $0 \le t \le T$, where $\tau = \dfrac{t}{\varepsilon}$ and the functions w_k and v_k (k = 0,1,...,n) follow successively from (16), (17), (18) and (19).

In case n = 0 empty sums are to be interpreted as zero.

COROLLARY 1. It follows from (24) that first approximations (n=0) for the solution u of (11) and its derivative are given by

$$\begin{cases} u(t;\varepsilon) = w_0(t) + O(\varepsilon), & \varepsilon \downarrow 0 \\[4mm] \dfrac{du}{dt}(t;\varepsilon) = \dfrac{dw_0}{dt}(t) + \{c_2 - \dfrac{dw_0}{dt}(0)\}\exp[-a(0)\dfrac{t}{\varepsilon}] + O(\varepsilon), & \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in every bounded interval $0 \le t \le T$, where w_0 is the solution of the so-called reduced differential equation (16) which follows from (11) by putting $\varepsilon = 0$.

COROLLARY 2. *It follows from (24) and (20) that for any* $n \in \mathbb{N} \cup \{0\}$

$$
\begin{cases}
u(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(t) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0 \\[2em]
\dfrac{du}{dt}(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \dfrac{dw_k}{dt}(t) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0
\end{cases}
$$

uniformly valid in every bounded interval $0 < \delta \le t \le T$, *where* δ *is an arbitrarily small positive constant independent of* ε.

REMARK. If the condition of infinite smoothness of the functions a, b and f is weakened, then the expansions (24) are valid for all $n \in \mathbb{N} \cup \{0\}$ with $n \le n_0$, the bound n_0 being determined by the differentiability of the functions a, b and f.

3. The case $a(t;\varepsilon) \le -a_0 < 0$

In this section we shall study problem (1) with a strictly negative coefficient $a(t;\varepsilon)$. In order not to complicate the calculations it will be assumed that the functions a, b and f are independent of ε; without loss of generality it will further be assumed that $C_1(\varepsilon) = C_2(\varepsilon) = 0$. Thus we consider the initial value problem

$$
(25) \qquad
\begin{cases}
\varepsilon \dfrac{d^2 u}{dt^2} + a(t)\dfrac{du}{dt} + b(t)u = f(t), & 0 < t < \infty \\[1.5em]
u(0;\varepsilon) = u_t(0;\varepsilon) = 0
\end{cases}
$$

with $0 < \varepsilon \ll 1$ and

$$
(26) \qquad
\begin{cases}
a(t) \in C^2(\overline{\mathbb{R}^+}), \quad b(t) \in C^1(\overline{\mathbb{R}^+}), \quad f(t) \in C^1(\overline{\mathbb{R}^+}) \\[1.5em]
\qquad\quad a(t) < 0 \quad \text{for} \quad 0 \le t < \infty .
\end{cases}
$$

Without loss of generality we may assume that there exists no interval $[0,t_0)$ with the property that $f(t) \equiv 0$ for $0 \le t < t_0$. Otherwise we could take the largest interval with this property, say $[0,t_0^*)$, and formulate a similar initial value problem for u in $t_0^* < t < \infty$ with homogeneous initial conditions at $t = t_0^*$, the solution in $[0,t_0^*)$ being given by $u(t;\varepsilon) \equiv 0$.

In the preceding section we have shown that the solution $u(t;\varepsilon)$ of pro-
blem (25) with a positive coefficient $a(t)$ satisfies

$$\lim_{\varepsilon \downarrow 0} u(t;\varepsilon) = w(t)$$

uniformly in $0 \leq t \leq T$, where w is the solution of the reduced problem

$$(27) \quad \begin{cases} a(t)\dfrac{dw}{dt} + b(t)w = f(t), & 0 < t < \infty \\[2ex] w(0) = 0. \end{cases}$$

It will turn out that the solution $u(t;\varepsilon)$ of problem (25) with a negative
coefficient $a(t)$ behaves quite differently. We shall show that for every
value of t with $t \geq \delta > 0$ (δ arbitrarily small but independent of ε) the
solution $u(t;\varepsilon)$ (and hence also the difference $u(t;\varepsilon) - w(t)$) increases ex-
ponentially as ε tends to zero.

In order to analyse the solution u of (25) we put

$$(28) \quad u(t;\varepsilon) = w(t) + U(t;\varepsilon)\exp\{-\frac{1}{\varepsilon}\int_0^t a(\tau)d\tau\}$$

where w is the solution of the reduced problem (27) which is given by

$$(29) \quad w(t) = \int_0^t \frac{f(\tau)}{a(\tau)} \exp\{-\int_\tau^t \frac{b(\xi)}{a(\xi)} d\xi\}d\tau.$$

Then we obtain for U the initial value problem

$$(30) \quad \begin{cases} \varepsilon\dfrac{d^2U}{dt^2} - a(t)\dfrac{dU}{dt} + \{b(t) - a'(t)\}U = \varepsilon\Psi_1(t)\exp\{\dfrac{1}{\varepsilon}\int_0^t a(\tau)d\tau\}, & 0 < t < \infty \\[2ex] U(0;\varepsilon) = 0, \quad U_t(0;\varepsilon) = -\dfrac{f(0)}{a(0)} \end{cases}$$

with

$$\Psi_1 = -\frac{d}{dt}\frac{f}{a} + \frac{bf}{a^2} + (\frac{d}{dt}\frac{b}{a} - \frac{b^2}{a^2})w,$$

the prime denoting differentiation with respect to t. Defining the function
$v(t;\varepsilon)$ by

14

$$v(t;\epsilon) = -\int_0^t \frac{f(\tau)}{a(\tau)} \exp[\int_\tau^t \{\frac{b(\xi) - a'(\xi)}{a(\xi)}\}d\xi + \frac{1}{\epsilon}\int_0^\tau a(\xi)d\xi]d\tau$$

(31)

$$= -\frac{1}{a(t)}\int_0^t f(\tau)\exp\{\int_\tau^t \frac{b(\xi)}{a(\xi)}d\xi + \frac{1}{\epsilon}\int_0^\tau a(\xi)d\xi\}d\tau$$

it is easily established that v satisfies the initial value problem

(32)

$$\begin{cases} \epsilon\frac{d^2v}{dt^2} - a(t)\frac{dv}{dt} + \{b(t) - a'(t)\}v = -\epsilon\Phi(t)v(t;\epsilon) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \epsilon\Psi_2(t)\ \exp\{\frac{1}{\epsilon}\int_0^t a(\tau)d\tau\} \\ v(0;\epsilon) = 0, \quad v_t(0;\epsilon) = -\frac{f(0)}{a(0)} \end{cases}$$

where

(33)

$$\begin{cases} \Phi = -\frac{d}{dt}\ (\frac{b-a'}{a}) \ - \ (\frac{b-a'}{a})^2 \\ \Psi_2 = -\frac{d}{dt}\frac{f}{a} + (\frac{a'-b}{a^2})f. \end{cases}$$

So putting

(34) $$U(t;\epsilon) = v(t;\epsilon) + z(t;\epsilon)$$

we obtain from (30) and (32)

(35)

$$\begin{cases} \epsilon\frac{d^2z}{dt^2} - a(t)\frac{dz}{dt} + \{b(t) - a'(t)\}z = F(t;\epsilon), \qquad 0 < t < \infty \\ z(0;\epsilon) = z_t(0;\epsilon) = 0 \end{cases}$$

with

(36)

$$\begin{cases} F(t;\epsilon) = \epsilon\Phi(t)v(t;\epsilon) + \epsilon\Psi(t)\exp\{\frac{1}{\epsilon}\int_0^t a(\tau)d\tau\} \\ \Psi = \Psi_1 - \Psi_2 = (\frac{2b-a'}{a^2})f + (\frac{d}{dt}\frac{b}{a} - \frac{b^2}{a^2})w. \end{cases}$$

We shall show that z(t;ε) is relatively small with respect to v(t;ε), or more precisely

(37) $\qquad \dfrac{z(t;\varepsilon)}{v(t;\varepsilon)} = O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$

uniformly in each interval $0 < \delta \le t \le T$, where δ is an arbitrarily small positive constant independent of ε.

Let us assume for the moment that $f(0) = \ell \neq 0$ (without loss of generality we may assume the constant ℓ to be positive). From the differentiability of f it follows that there exists an interval $[0,t_0]$ such that

(38) $\qquad \dfrac{1}{2}\ell \le f(t) \le \dfrac{3}{2}\ell \qquad \text{for } 0 \le t \le t_0.$

Using this inequality we obtain from (31) the following estimates for $v(t;\varepsilon)$ which are valid for sufficiently small values of ε:

(39) $\qquad \begin{cases} 0 \le v(t;\varepsilon) \le A\varepsilon & \text{for } 0 \le t \le T \\[2ex] v(t;\varepsilon) \ge B\varepsilon & \text{for } 0 < \delta \le t \le T. \end{cases}$

Here A and B are positive constants independent of ε, and δ is some arbitrarily small positive constant independent of ε. Furthermore it is easily established that the function F defined by (36), (29) and (33) satisfies

(40) $\qquad |F(t;\varepsilon)| \le C(T)\varepsilon\{v(t;\varepsilon) + |f(t;\varepsilon)|\exp(-\dfrac{a_0 t}{\varepsilon})$

$\qquad\qquad\qquad\qquad + \displaystyle\int_0^t |f(\tau;\varepsilon)|d\tau \, \exp(-\dfrac{a_0 t}{\varepsilon})\}$

valid for $0 \le t \le T$, where $C(T)$ is some generic constant depending on T but not on ε, and $a_0 = \min_{[0,T]} |a(t)|$.

From this inequality we obtain with the aid of (38) and (39) an estimate for the L_2-norm of F, namely

$\qquad \|F\|_{[0,T]} = O(\varepsilon^{3/2}), \qquad \varepsilon \downarrow 0$

so it follows from theorem 1 that the solution z of (35) satisfies

(41) $\qquad z(t;\varepsilon) = O(\varepsilon^{3/2}), \qquad \varepsilon \downarrow 0$

uniformly in $0 \le t \le T$.

Finally we obtain from (39) and (41)

$$\frac{z(t;\varepsilon)}{v(t;\varepsilon)} = O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$$

uniformly valid for $0 < \delta \le t \le T$.

Let us now consider the case where $f(0) = 0$.
We assume that there exists an interval $(0,t_0]$ with the property that $f(t) \ne 0$ for $0 < t \le t_0$ (without loss of generality we may assume $f(t)$ to be positive for $0 < t \le t_0$). In addition it will be assumed that there exist positive constants C_1, C_2 and α such that

$$(42) \qquad c_1 t^\alpha \le f(t) \le c_2 t^\alpha \qquad \text{for } 0 \le t \le t_0$$

(note that necessarily $\alpha \ge 1$ because of the differentiability of f at $t = 0$). Using (42) we obtain from (31) the following estimates for $v(t;\varepsilon)$ which are valid for sufficiently small values of ε:

$$(43) \qquad \begin{cases} 0 \le v(t;\varepsilon) \le A\varepsilon^{\alpha+1} & \text{for } 0 \le t \le T \\[2mm] v(t;\varepsilon) \ge B\varepsilon^{\alpha+1} & \text{for } 0 < \delta \le t \le T. \end{cases}$$

Here A and B are positive constants independent of ε, and δ is some arbitrarily small positive constant independent of ε. We also obtain from (40), (42) and (43) the estimate

$$\|F\|_{[0,T]} = O(\varepsilon^{\alpha+3/2}), \qquad \varepsilon \downarrow 0$$

so theorem 1 yields

$$(44) \qquad z(t;\varepsilon) = O(\varepsilon^{\alpha+3/2}), \qquad \varepsilon \downarrow 0$$

uniformly valid in $0 \le t \le T$.
Finally it follows from (43) and (44) that

$$\frac{z(t;\varepsilon)}{v(t;\varepsilon)} = O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$$

uniformly valid for $0 < \delta \leq t \leq T$.

Using the formulae (28), (31), (34) and (37) we obtain

$$u(t;\varepsilon) = w(t) + \{1 + O(\varepsilon^{\frac{1}{2}})\}v(t;\varepsilon)\exp[-\frac{1}{\varepsilon}\int_0^t a(\tau)d\tau]$$

(45)

$$= w(t) - \frac{1 + O(\varepsilon^{\frac{1}{2}})}{a(t)}\int_0^t f(\tau)\exp\{\int_\tau^t \frac{b(\xi)}{a(\xi)}d\xi - \frac{1}{\varepsilon}\int_\tau^t a(\xi)d\xi\}d\tau$$

uniformly valid in every bounded interval $0 < \delta \leq t \leq T$.

REMARK. One easily verifies that the solution $w(t)$ of the reduced problem (27) provides a formal approximation for $u(t;\varepsilon)$ in case $f(0) = 0$. However, it follows from formula (45) that $w(t)$ does not approximate $u(t;\varepsilon)$ for small values of ε since the difference $u - w$ increases exponentially as ε tends to zero. Hence we conclude that a formal approximation for the solution u of problem (25) with a negative coefficient $a(t)$ is not necessarily a good approximation.

Finally we obtain from (45)

THEOREM 3. *Let* $u(t;\varepsilon)$ *be the solution of the initial value problem* (25) *under the conditions* (26) *and let the function* $f(t)$ *satisfy*

(46) $$c_1 t^\alpha \leq f(t) \leq c_2 t^\alpha$$

in some interval $0 \leq t \leq t_0$, *where the constants* c_1 *and* c_2 *are either both positive or both negative and the constant* α *is nonnegative. Then we have*

(47) $$u(t;\varepsilon) = -\frac{1 + O(\varepsilon^{\frac{1}{2}})}{a(t)}\int_0^t f(\tau)\exp\{\int_\tau^t \frac{b(\xi)}{a(\xi)}d\xi - \frac{1}{\varepsilon}\int_\tau^t a(\xi)d\xi\}d\tau, \quad \varepsilon \downarrow 0$$

uniformly valid in every bounded interval $0 < \delta \leq t \leq T$, *where* δ *is an arbitrarily small positive constant independent of* ε.

COROLLARY. *It follows from* (39), (43) *and* (45) *that the solution* $u(t;\varepsilon)$ *of* (25) *under the conditions* (26) *and* (46) *satisfies*

$$K_1 \varepsilon^{\alpha+1}\exp\{-\frac{1}{\varepsilon}\int_0^t a(\tau)d\tau\} \leq u(t;\varepsilon) \leq K_2\varepsilon^{\alpha+1}\exp\{-\frac{1}{\varepsilon}\int_0^t a(\tau)d\tau\}$$

valid for $0 < \delta \leq t \leq T$ and ε sufficiently small. Here K_1 and K_2 are ε-independent constants which are either both positive or both negative.

REMARK. Using a result of SMITH [45, p.245] (see also formula (49)) it is possible to replace the term $O(\varepsilon^{\frac{1}{2}})$ in the right-hand side of (47) by $O(\varepsilon)$.

4. Discussion

The a priori estimates for u and $\frac{du}{dt}$ that were established in theorem 1 are, under the conditions stated, the sharpest possible in the sense that the ε-dependent factors in front of the quantities $\| f \|_{[0,T]}$, $|c_1(\varepsilon)|$ and $|c_2(\varepsilon)|$ are the smallest possible. This can easily be checked by considering appropriate initial value problems. As an example we shall prove that the factor $\varepsilon^{-\frac{1}{2}}$ in the right-hand side of (10b) is optimal.

Let u be the solution of the initial value problem

(48)
$$\begin{cases} \varepsilon \frac{d^2 u}{dt^2} + \frac{du}{dt} = f(t;\varepsilon), & 0 < t < \infty \\ u(0;\varepsilon) = u_t(0;\varepsilon) = 0 \end{cases}$$

with

$$f(t;\varepsilon) = \exp\left(-\frac{t}{\varepsilon}\right).$$

It is easily verified that this solution satisfies

$$\frac{du}{dt}(t;\varepsilon) = \frac{t}{\varepsilon} \exp\left(-\frac{t}{\varepsilon}\right)$$

so it follows from the relations

$$\begin{cases} \| f \|_{[0,T]} = O(\varepsilon^{\frac{1}{2}}), & \varepsilon \downarrow 0 \\ \sup_{[0,T]} |\frac{du}{dt}(t;\varepsilon)| = O(1), & \varepsilon \downarrow 0 \end{cases}$$

that the factor $\varepsilon^{-\frac{1}{2}}$ in the right-hand side of (10b) is the smallest possible.

In a similar way the other factors in (10a) and (10b) can be shown to be optimal.

Alternative methods for obtaining a priori estimates have been given by O'MALLEY [37] and SMITH [45]: an estimate for the solution u of (1) (with a positive coefficient $a(t;\varepsilon)$) is derived by integrating the differential equation twice and then either solving the resulting Volterra equation by successive approximations (O'MALLEY) or applying GRONWALL's lemma (SMITH). The latter method yields the estimate

(49) $$\sup_{[0,T]} |u(t;\varepsilon)| \leq C(T)\{\|f\|_{1,[0,T]} + |C_1(\varepsilon)| + \varepsilon|C_2(\varepsilon)|\}$$

where

$$\|f\|_{1,[0,T]} = \int_0^T |f(\tau;\varepsilon)|d\tau.$$

This estimate may for certain problems (e.g. problem (48)) be sharper than (10a).

Estimates which are valid in the infinite interval $0 \leq t < \infty$ have been derived by WEINSTEIN & SMITH [54] for an equation of the type (1) and by HOPPENSTEADT [24] for a nonlinear system of first order equations.

The estimates obtained in section 1 for solutions of linear differential equations will turn out to play an important role in the investigations of nonlinear initial value problems. We shall deal with these problems in chapter II.

The energy integral method underlying the proof of theorem 1 can be generalized to yield a priori estimates for solutions of singularly perturbed initial value problems for hyperbolic differential equations and asymptotic expansions for these solutions can be obtained by a generalization of the expansion technique developed in section 2. This subject will be discussed in chapter III.

INITIAL VALUE PROBLEMS FOR NONLINEAR
ORDINARY DIFFERENTIAL EQUATIONS

In this chapter we shall extend the linear theory developed in chapter I to nonlinear initial value problems. In particular we shall investigate the quasilinear problem

(1)
$$\begin{cases} \varepsilon \dfrac{d^2 u}{dt^2} + a(t,u;\varepsilon)\dfrac{du}{dt} + b(t,u;\varepsilon) = f(t;\varepsilon), & 0 < t < \infty \\ u(0;\varepsilon) = u_t(0;\varepsilon) = 0 \end{cases}$$

where ε is a small positive parameter, i.e. $0 < \varepsilon \ll 1$.
Under appropriate conditions on the data we shall establish in section 1 the existence of a solution for problem (1), together with an a priori estimate for this solution.
In section 2 we shall construct a formal approximation $\tilde{u}$ for the solution u of (1). The correctness of this approximation will then be verified with the aid of the results of section 1.

1. <u>Existence and a priori estimates</u>

We shall start this section with the formulation of a theorem on nonlinear operators defined in some normed linear space. This theorem is a modified version of a theorem by VAN HARTEN [23, p.190] and its proof is based on a contraction mapping principle for operators in a Banach space.

Let us define the following spaces:

$$\begin{cases} R: \text{ a normed linear space with norm } \| \cdot \| \\ B: \text{ a Banach space with norm } | \cdot | \end{cases}$$

Let $\bar{\rho}$ be a certain positive number and let $\Omega(\bar{\rho}) \subset R$ denote the ball with radius $\bar{\rho}$, that is

$$\Omega(\bar{\rho}) = \{\omega \in R \mid \|\omega\| \leq \bar{\rho}\}.$$

We consider a nonlinear operator F from $\Omega(\bar{\rho})$ into B satisfying

(2) $F[\emptyset] = \emptyset$

where $\emptyset$ denotes the zero element in R or B.

It is assumed that F can be split up into a linear part L and a nonlinear part Ψ:

(3) $F[u] = L[u] + \Psi[u].$

The linear operator L can be extended to a map from R into B and this extension will also be denoted by L. In addition we assume that the following conditions for the operators L and Ψ are satisfied:

Condition a. L: $R \rightarrow B$ is bijective and the inverse operator $L^{-1}: B \rightarrow R$ is continuous, that is

$$\|L^{-1}[v]\| \leq \ell^{-1}|v| \qquad \text{for all } v \in B$$

where ℓ is some positive constant.

Condition b. Ψ: $\Omega(\bar{\rho}) \rightarrow B$ satisfies the Lipschitz condition

$$|\Psi[\omega_1] - \Psi[\omega_2]| \leq m(\rho)\|\omega_1 - \omega_2\|$$

valid for all $\omega_1, \omega_2 \in \Omega(\rho)$ and all $\rho \in [0,\bar{\rho}]$.

Here $m(\rho)$ is supposed to be continuous for $\rho \in [0,\bar{\rho}]$ and monotonically decreasing for decreasing ρ with

$$\lim_{\rho \downarrow 0} m(\rho) = 0.$$

Our problem is to determine an element $u \in \Omega(\bar{\rho})$ which satisfies

(4)   F[u] = f

for a given f ∈ B.
Putting u = L^{-1}[v] we obtain the following equivalence:

$$F[u] = f, \quad u \in \Omega(\bar{\rho}), \quad f \in B \leftrightarrow T[v] = v, \quad v \in L[\Omega(\bar{\rho})], \quad f \in B$$

where the nonlinear operator T: L[Ω(ρ̄)] → B is given by

$$T[v] = f - \Psi \circ L^{-1}[v].$$

After these preparations we can formulate

THEOREM 1. *Suppose that the nonlinear operator* F: Ω(ρ̄) → B *with* Ω(ρ̄) *a*
closed ball with radius ρ̄ *in a normed linear space* R *(norm* ‖·‖*) and* B *a*
Banach space (norm |·|*) satisfies the conditions* (2) *and* (3) *and the condi-*
tions a and b.
Let ρ_0 *be defined as the largest number* ρ ∈ [0,ρ̄] *for which*

$$m(\rho_0) \leq \frac{1}{2}\ell.$$

Then problem (4) *has for sufficiently small* f ∈ B, *viz.*

$$|f| \leq \frac{1}{2}\ell\rho_0,$$

a solution u *in the ball* Ω(ρ_0) *and this solution satisfies the estimate*

(5) $\|u\| \leq 2\ell^{-1}|f|.$

PROOF. Since problem (4) is equivalent to

$$T[v] = v, \quad v \in L[\Omega(\bar{\rho})], \quad f \in B$$

it can be considered as a search for a fixed point of an operator T from
L[Ω(ρ̄)] ⊂ B into B. The existence of such a fixed point can easily be esta-
blished. Following VAN HARTEN [23, p.191] we show that for $2\ell^{-1}|f| \leq \rho \leq \rho_0$

(i)   T is defined in the ball $B(\ell\rho) = \{v \in B \mid |v| \le \ell\rho\}$,

(ii)   T is a strict contraction on $B(\ell\rho)$,

(iii) $T[B(\ell\rho)] \subset B(\ell\rho)$.

Ad (i):   for any $v \in B(\ell\rho)$ we have

$$v = L \circ L^{-1}[v] \quad \text{and} \quad \|L^{-1}[v]\| \le \ell^{-1}|v| \le \rho \le \rho_0 \le \bar\rho$$

so it follows that $B(\ell\rho) \subset L[\Omega(\bar\rho)]$. Since T is defined in $L[\Omega(\bar\rho)]$ it is also defined in $B(\ell\rho)$.

Ad (ii):   for $v_1, v_2 \in B(\ell\rho)$ we have

$$|T[v_1] - T[v_2]| \le |\Psi \circ L^{-1}[v_1] - \Psi \circ L^{-1}[v_2]|$$

$$\le m(\rho)\|L^{-1}[v_1] - L^{-1}[v_2]\|$$

$$\le \ell^{-1}m(\rho)|v_1 - v_2|$$

$$\le \frac{1}{2}|v_1 - v_2|.$$

Ad (iii): for $v \in B(\ell\rho)$ we have

$$|T[v]| \le |f| + |\Psi \circ L^{-1}[v]|$$

$$\le \frac{1}{2}\ell\rho + m(\rho)\|L^{-1}[v]\|$$

$$\le \frac{1}{2}\ell\rho + m(\rho)\rho$$

$$\le \ell\rho.$$

It is well known that a strictly contractive map T from a sphere (in a Banach space) into itself has exactly one fixed point in that sphere. This leads to the unique existence of a solution $v \in B(\ell\rho)$. Due to the equivalence there exists also a solution $u \in \Omega(\rho)$ of problem (4); actually we have

$$\|u\| = \|L^{-1}[v]\| \le \ell^{-1}|v| \le \rho.$$

Note that uniqueness of u has not been established in the whole ball $\Omega(\rho)$, but only in $L^{-1}[B(\ell\rho)] \subset \Omega(\rho)$.

Finally, choosing $\rho = \rho_0$ we obtain the largest domain for which uniqueness can be guaranteed; choosing $\rho = 2\ell^{-1}|f|$ we find the estimate (5), which concludes the proof of theorem 1.

The proof given above is, apart from the definition of the operator T, identical to the proof of a similar theorem by VAN HARTEN [23, p.191]. The difference between VAN HARTEN's theorem and ours is formed by the definition of the operator F: in our case F maps a ball belonging to a normed linear space into a Banach space, whereas VAN HARTEN considers an operator F which maps a (linear subspace of a) Banach space into a normed linear space. Our formulation has the advantage that the theory is applicable to more general operators F, since F is allowed to be defined only in a ball instead of in a linear subspace. This fact will enable us for instance to consider quasilinear differential operators F with coefficients having singularities when regarded as functions of u.

Another advantage is lying in the fact that F is defined in a subset of a normed linear space instead of in a subset of a Banach space. This will become clear when we apply theorem 1 to hyperbolic differential equations containing a quasilinear first order part (see chapter IV, section 1).

After these preliminaries we turn our attention to the initial value problem (1). We impose the following conditions on the data:

(6)
$$\begin{cases} \text{there exists a positive constant } T_1 \text{ such that for each positive constant } T \text{ satisfying } T < T_1 \text{ there exists a positive constant } \varepsilon_1 \text{ (depending on T) such that} \\[2mm] \text{(i)} \quad a(t,0;\varepsilon) \in C^1[0,T], \quad b_u(t,0;\varepsilon) \in C^0[0,T], \quad f(t;\varepsilon) \in C^0[0,T] \\[2mm] \qquad \text{valid for every value of } \varepsilon \text{ satisfying } 0 < \varepsilon \leq \varepsilon_1 \\[2mm] \text{(ii)} \quad \text{the function } a(t,0;\varepsilon) \text{ and its first derivative } \frac{\partial a}{\partial t}(t,0;\varepsilon) \text{ as well} \\ \qquad \text{as the function } b_u(t,0;\varepsilon) \text{ are uniformly bounded in } 0 \leq t \leq T \text{ for} \\ \qquad 0 < \varepsilon \leq \varepsilon_1 \end{cases}$$

(6)
$$(iii) \quad a(t,0;\varepsilon) \geq a_0 > 0$$

valid for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_1]$, where a_0 is some constant depending on T and ε_1 but independent of ε

there exists a positive constant T_2 such that for each positive constant T satisfying $T < T_2$ there exist a positive constant U_0 and a positive constant ε_2 (depending on T and U_0) such that

(i) $\quad a(t,u;\varepsilon) \in C^0[0,T], \quad b(t,u;\varepsilon) \in C^0[0,T]$

valid for all $u \in [-U_0,U_0]$ and all $\varepsilon \in (0,\varepsilon_2]$

(7)

(ii) the derivative $a_u(t,u;\varepsilon)$ exists and is uniformly bounded in $[0,T] \times [-U_0,U_0]$ for $0 < \varepsilon \leq \varepsilon_2$

(iii) the derivative $b_u(t,u;\varepsilon)$ exists in $[0,T] \times [-U_0,U_0]$ for $0 < \varepsilon \leq \varepsilon_2$ and satisfies a Lipschitz condition with exponent $\alpha > 0$, viz.

$$|b_u(t,u;\varepsilon) - b_u(t,0;\varepsilon)| \leq C(T,U_0,\varepsilon_2)|u|^\alpha$$

valid for all $(t,u) \in [0,T] \times [-U_0,U_0]$ and all $\varepsilon \in (0,\varepsilon_2]$. Here $C(T,U_0,\varepsilon_2)$ is some positive constant depending on T, U_0 and ε_2 but independent of t, u or ε.

It follows from the conditions (6,i) and (7,i) that we may assume without loss of generality that

(8) $\quad\quad b(t,0;\varepsilon) \equiv 0 \quad$ for $0 \leq t < \infty$ and $0 < \varepsilon << 1$.

The existence of a solution for problem (1) can now be established by applying theorem 1. In terms of this theorem we introduce for any positive T with $T < T_0 = \min(T_1,T_2)$ and for any fixed value of ε with $0 < \varepsilon \leq \varepsilon_0 = \min(\varepsilon_1,\varepsilon_2)$ the Banach space

$$B = \{u \mid u \in C^0[0,T]\}$$

with norm

$$|u|_0 = \sup_{[0,T]} |u|$$

and the normed linear space

(9) $$R = \{u \mid u \in C^2[0,T], \quad u(0) = u_t(0) = 0\}$$

with norm

(10) $$|u|_1^{(\varepsilon)} = \sup_{[0,T]} |u| + \varepsilon^{\frac{1}{2}} \sup_{[0,T]} |u_t|.$$

Choosing $\bar{\rho} = U_0$ we define the operator $F: \Omega(\bar{\rho}) \to B$ by

$$F[u] = \varepsilon \frac{d^2 u}{dt^2} + a(t,u;\varepsilon)\frac{du}{dt} + b(t,u;\varepsilon)$$

which according to (8) satisfies the relation (2). For the operator $L: R \to B$ we take the linearization of F in the origin, thus

$$L[u] = \varepsilon \frac{d^2 u}{dt^2} + a(t,0;\varepsilon)\frac{du}{dt} + b_u(t,0;\varepsilon)u.$$

Then the operator $\Psi: \Omega(\bar{\rho}) \to B$ is given by

$$\Psi[u] = F[u] - L[u] = \{a(t,u;\varepsilon) - a(t,0;\varepsilon)\}\frac{du}{dt}$$

$$+ b(t,u;\varepsilon) - b_u(t,0;\varepsilon)u.$$

Using the conditions (6) we obtain from theorem 1 in chapter I (and from the first remark following that theorem)

$$|L^{-1}[v]|_1^{(\varepsilon)} \le M(T)|v|_0$$

valid for all $\varepsilon \in (0,\varepsilon_0]$ and all $v \in B$, where $M(T)$ is some positive constant depending on T but independent of ε, and ε_0 is some generic constant. So in terms of theorem 1 we have

$$\ell = \frac{1}{M(T)} .$$

It can easily be verified that the operator Ψ satisfies a Lipschitz condition. Using the conditions (7) we obtain for every $\rho \in [0,\bar{\rho}]$ and every

$v,w \in R$ with $|v|_1^{(\epsilon)} \le \rho$, $|w|_1^{(\epsilon)} \le \rho$ the following relation

$$\Psi[v] - \Psi[w] = \{a(t,v;\epsilon) - a(t,0;\epsilon)\}(v_t - w_t)$$

$$+ w_t\{a(t,v;\epsilon) - a(t,w;\epsilon)\}$$

$$+ b(t,v;\epsilon) - b(t,w;\epsilon) - b_u(t,0;\epsilon)(v-w)$$

$$= va_u(t,\theta_1 v;\epsilon)(v_t - w_t) + w_t a_u(t,w + \theta_2(v-w);\epsilon)(v-w)$$

$$+ \{b_u(t,w + \theta_3(v-w);\epsilon) - b_u(t,0;\epsilon)\}(v-w)$$

with $0 < \theta_i < 1$ $(i = 1,2,3)$. Hence it follows that

$$|\Psi[v] - \Psi[w]|_0 \le C(T)\{\rho \sup_{[0,T]} |v_t - w_t| + \max(\epsilon^{-\frac{1}{2}}\rho,\rho^\alpha) \sup_{[0,T]} |v - w|\}$$

$$\le C(T)\max(\epsilon^{-\frac{1}{2}}\rho,\rho^\alpha)|v - w|_1^{(\epsilon)}$$

valid for $0 < \epsilon \le \epsilon_0$. Here $C(T)$ is a positive constant which depends (apart from U_0 and ϵ_0) on T but not on ϵ.
So still using the terminology of theorem 1 we conclude that

$$m(\rho) = C(T)\max(\epsilon^{-\frac{1}{2}}\rho,\rho^\alpha).$$

N.B. The following simplifications are easily established:
    if a does not depend on u then $m(\rho) = C(T)\rho^\alpha$,
    if b is linear in u then $m(\rho) = C(T)\epsilon^{-\frac{1}{2}}\rho$.

Solving $m(\rho_0) = \frac{1}{2}\ell$ we obtain

$$\rho_0 = N(T)\epsilon^\lambda$$

with

$$(11) \qquad \lambda = \begin{cases} 0 & \text{if a does not depend on u} \\ \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

and $N(T)$ some positive constant depending on T but independent of ε.

Finally, applying theorem 1 we obtain

THEOREM 2. *Let the functions* $a(t,u;\varepsilon)$, $b(t,u;\varepsilon)$ *and* $f(t;\varepsilon)$ *satisfy the conditions* (6), (7) *and* (8), *and let* T *be some positive constant satisfying* $T < T_0 = \min(T_1,T_2)$. *Then there exist positive constants* ε_0, $M(T)$ *and* $N(T)$ *(depending on* T *but independent of* ε*) such that for* $0 \le t \le T < T_0$ *and* $0 < \varepsilon \le \varepsilon_0$ *problem* (1) *has a solution* $u \in R$*, provided that* f *is sufficiently small, viz.*

$$\sup_{[0,T]} |f| \le \frac{N(T)}{2M(T)} \varepsilon^\lambda .$$

Here the normed linear space R *is given by* (9) *and* (10)*, and the nonnegative constant* λ *is given by* (11).
The solution u *satisfies the estimate*

$$|u|_1^{(\varepsilon)} \le 2M(T) \sup_{[0,T]} |f| .$$

REMARKS.

1. Note that the coefficients $a(t,u;\varepsilon)$ and $b(t,u;\varepsilon)$ are allowed to have singularities in u. For instance, if

$$a(t,u;\varepsilon) = \frac{1}{1-u}$$

we choose the positive constant U_0 mentioned in (7) such that $U_0 < 1$. With this choice the conditions (6) and (7) for $a(t,u;\varepsilon)$ are fulfilled, as one easily verifies.

2. A similar theorem can be formulated for more general nonlinear initial value problems of the type

$$\begin{cases} \varepsilon \dfrac{d^2 u}{dt^2} + F(t,\vec{u};\varepsilon) = f(t;\varepsilon), & 0 < t < \infty \\ \\ u(0;\varepsilon) = u_t(0;\varepsilon) = 0 \end{cases}$$

(where $\vec{u}$ denotes the vector (u,u_t)) provided that suitable conditions are imposed on the nonlinear function F, namely

(i) $F(t,\vec{0};\varepsilon) \equiv 0$ for $0 \le t < \infty$ and $0 < \varepsilon \ll 1$,

(ii)   in order that theorem 1 of chapter I is applicable to the lineari-
       zed problem one must require that (6) holds, with $a(t,0;\varepsilon)$ and
       $b_u(t,0;\varepsilon)$ replaced by $F_{u_t}(t,\vec{0};\varepsilon)$ respectively $F_u(t,\vec{0};\varepsilon)$,

(iii) in order that the nonlinear part Ψ given by

$$\Psi[u] = F(t,\vec{u};\varepsilon) - F_{u_t}(t,\vec{0};\varepsilon)\frac{du}{dt} - F_u(t,\vec{0};\varepsilon)u$$

satisfies a Lipschitz condition it is sufficient to require that
there exist positive constants T, U_0 and ε_0 such that the functions
F, F_{uu}, F_{uu_t} and $F_{u_t u_t}$ are continuous and uniformly bounded in
$0 \le t \le T$, $|u| \le U_0$, $|u_t| \le U_0$ for $0 < \varepsilon \le \varepsilon_0$.

2. Asymptotic expansions

As an application of the a priori estimates obtained in the foregoing
section we shall construct an asymptotic expansion into powers of ε for the
solution u of the quasilinear initial value problem

(12)
$$\begin{cases} L^{(\varepsilon)}[u] \equiv \varepsilon\dfrac{d^2u}{dt^2} + a(t,u)\dfrac{du}{dt} + b(t,u) = 0, & 0 < t < \infty \\ \\ u(0;\varepsilon) = C_1, \quad u_t(0;\varepsilon) = C_2. \end{cases}$$

For simplicity it will be assumed that all data are C^∞-functions, that is

(13) $a(t,u) \in C^\infty(\overline{\mathbb{R}^+} \times \mathbb{R})$, $b(t,u) \in C^\infty(\overline{\mathbb{R}^+} \times \mathbb{R})$.

In addition we assume that

(14) $a(t,u) > 0$ for $0 \le t < \infty$ and $-\infty < u < \infty$.

However, we emphasize that the conditions (13) and (14) are not necessary
and can be weakened considerably (see the remarks following theorem 3 of
this section).

According to a standard procedure in singular perturbation theory a formal
approximation $\tilde{u}$ for the solution u of problem (12) can be constructed by

putting

(15) $\qquad \tilde{u}(t;\varepsilon) = w_0(t) + \varepsilon v_0(\tau) + \varepsilon w_1(t) + \varepsilon^2 v_1(\tau)$

where $\tau = \dfrac{t}{\varepsilon}$ and w_k and v_k $(k = 0,1)$ are functions which have to be determined. For the moment we shall assume that these functions have the following properties

(16) $\begin{cases} \text{there exists a positive number } T_0 \text{ (possibly } T_0 = \infty) \text{ such that for} \\ k = 0,1 \\[1em] \qquad w_k(t) \in C^\infty[0,T] \\[1em] \text{valid for every positive } T \text{ satisfying } T < T_0; \\[1em] \qquad v_k(\tau) \in C^\infty(\overline{\mathbb{R}^+}) \qquad \text{for } k = 0,1; \\[1em] \text{for any } n \in \mathbb{N} \cup \{0\} \text{ and any polynomial } P(\tau) \text{ the expression} \\[1em] \qquad P(\tau)\dfrac{d^n v_k}{d\tau^n} \qquad (k = 0,1) \qquad \cdot \\[1em] \text{is uniformly bounded in } 0 \leq \tau < \infty. \end{cases}$

We shall verify the validity of these properties later on, when the functions w_k and v_k have been determined explicitly.

Let T be some positive number satisfying $T < T_0$. Expanding the coefficient $a(t,\tilde{u})$ we obtain using (15) and (16)

$$a(t,\tilde{u}) = a(t,w_0 + \varepsilon v_0 + \varepsilon w_1 + \varepsilon^2 v_1)$$

$$= a(t,w_0) + \varepsilon(v_0 + w_1)\frac{\partial a}{\partial u}(t,w_0) + O(\varepsilon^2)$$

uniformly valid in $0 \leq t \leq T < T_0$.

Using

$$\frac{d\tilde{u}}{dt} = \frac{dw_0}{dt} + \frac{dv_0}{d\tau} + \varepsilon\frac{dw_1}{dt} + \varepsilon\frac{dv_1}{d\tau}$$

we get

$$a(t,\tilde{u})\frac{d\tilde{u}}{dt} = a(t,w_0)\frac{dw_0}{dt} + a(t,w_0)\frac{dv_0}{d\tau}$$

$$+ \varepsilon\{a(t,w_0)\frac{dw_1}{dt} + a(t,w_0)\frac{dv_1}{d\tau}\}$$

$$+ \varepsilon(w_1 + v_0)(\frac{dw_0}{dt} + \frac{dv_0}{d\tau})\frac{\partial a}{\partial u}(t,w_0)$$

$$+ O(\varepsilon^2)$$

uniformly valid in $0 \le t \le T < T_0$.

So putting $t = \varepsilon\tau$ and expanding into powers of ε we arrive at

$$a(t,\tilde{u})\frac{d\tilde{u}}{dt} = a(t,w_0)\frac{dw_0}{dt} + a(0,w_0(0))\frac{dv_0}{d\tau}$$

$$+ \varepsilon[\{\frac{\partial a}{\partial t}(0,w_0(0)) + \frac{\partial a}{\partial u}(0,w_0(0))\frac{dw_0}{dt}(0)\}\tau\frac{dv_0}{d\tau}$$

$$+ a(t,w_0)\frac{dw_1}{dt} + a(0,w_0(0))\frac{dv_1}{d\tau}$$

$$+ \frac{\partial a}{\partial u}(t,w_0)w_1\frac{dw_0}{dt} + \frac{\partial a}{\partial u}(0,w_0(0))w_1(0)\frac{dv_0}{d\tau}$$

$$+ \frac{\partial a}{\partial u}(0,w_0(0))\frac{dw_0}{dt}(0)v_0 + \frac{\partial a}{\partial u}(0,w_0(0))v_0\frac{dv_0}{d\tau}]$$

$$+ O(\varepsilon^2)$$

uniformly valid in $0 \le t \le T < T_0$.

N.B. Unless otherwise stated the argument of w_0, $\frac{dw_0}{dt}$, w_1, $\frac{dw_1}{dt}$ is t and the argument of v_0, $\frac{dv_0}{d\tau}$, v_1, $\frac{dv_1}{d\tau}$ is τ.

A similar expansion of the other terms in $L^{(\varepsilon)}[\tilde{u}]$ leads to the following expansion of the operator:

$$L^{(\varepsilon)}[\tilde{u}] = a(t,w_0)\frac{dw_0}{dt} + b(t,w_0) + \frac{d^2v_0}{d\tau^2} + a(0,w_0(0))\frac{dv_0}{d\tau}$$

$$(17) \qquad + \varepsilon[a(t,w_0)\frac{dw_1}{dt} + \tilde{b}(t,w_0)w_1 + \Phi_1(t)]$$

$$+ \varepsilon[\frac{d^2v_1}{d\tau^2} + a(0,w_0(0))\frac{dv_1}{d\tau} + \Psi_1(\tau)] + O(\varepsilon^2)$$

uniformly valid in $0 \le t \le T < T_0$, where

$$(18) \quad \begin{cases} \tilde{b}(t,w_0) = \frac{\partial a}{\partial u}(t,w_0)\frac{dw_0}{dt} + \frac{\partial b}{\partial u}(t,w_0) \\[2mm] \Phi_1(t) = \frac{d^2w_0}{dt^2} \\[2mm] \Psi_1(\tau) = \{\frac{\partial a}{\partial u}(0,w_0(0))\frac{dw_0}{dt}(0) + \frac{\partial b}{\partial u}(0,w_0(0))\}v_0 \\[2mm] \qquad + [\{\frac{\partial a}{\partial t}(0,w_0(0)) + \frac{\partial a}{\partial u}(0,w_0(0))\frac{dw_0}{dt}(0)\}\tau \\[2mm] \qquad\qquad + \frac{\partial a}{\partial u}(0,w_0(0))w_1(0)]\frac{dv_0}{d\tau} \\[2mm] \qquad + \frac{\partial a}{\partial u}(0,w_0(0))v_0\frac{\dot{dv_0}}{d\tau} . \end{cases}$$

Now we take for w_0 the solution of the nonlinear initial value problem

$$(19) \quad \begin{cases} a(t,w_0)\frac{dw_0}{dt} + b(t,w_0) = 0, \qquad t > 0 \\[2mm] w_0(0) = C_1 . \end{cases}$$

Note that this solution does not necessarily belong to the class $C^\infty(\overline{\mathbb{R}^+})$ but may have singularities for finite values of t. As an example consider the case where

$$a(t,w_0) \equiv 1, \quad b(t,w_0) = -w_0^2, \quad C_1 = 1 .$$

The solution is given by $w_0 = \frac{1}{1-t}$ and has a singularity at $t = 1$.

We shall denote by T_0 the smallest value of t for which the solution w_0 of (19) becomes singular ($T_0 = \infty$ if w_0 has no singularities). Then we have

$w_0 \in C^{\infty}[0,T]$ for each positive T with $T < T_0$.

The functions v_0, w_1 and v_1 can now be obtained by solving the following set of initial value problems

(20)
$$
\begin{cases}
\dfrac{d^2 v_0}{d\tau^2} + a(0,w_0(0))\dfrac{dv_0}{d\tau} = 0, & 0 < \tau < \infty \\[3ex]
\dfrac{dv_0}{d\tau}(0) = C_2 - \dfrac{dw_0}{dt}(0) \\[3ex]
\lim_{\tau \to \infty} v_0(\tau) = 0
\end{cases}
$$

(21)
$$
\begin{cases}
a(t,w_0)\dfrac{dw_1}{dt} + \tilde{b}(t,w_0)w_1 = -\Phi_1(t), & 0 < t < T_0 \\[3ex]
w_1(0) = -v_0(0)
\end{cases}
$$

(22)
$$
\begin{cases}
\dfrac{d^2 v_1}{d\tau^2} + a(0,w_0(0))\dfrac{dv_1}{d\tau} = -\Psi_1(\tau), & 0 < \tau < \infty \\[3ex]
\dfrac{dv_1}{d\tau}(0) = -\dfrac{dw_1}{dt}(0) \\[3ex]
\lim_{\tau \to \infty} v_1(\tau) = 0.
\end{cases}
$$

Note that only the first term w_0 of the expansion of $\tilde{u}$ follows from the solution of a nonlinear initial value problem; the remaining functions v_0, w_1 and v_1 satisfy linear equations which can easily be solved.

It is not difficult to verify that the conditions (16) are satisfied. In particular, the functions v_0 and v_1 have the form

$$
\begin{cases}
v_0(\tau) = \dfrac{\dfrac{dw_0}{dt}(0) - C_2}{a(0,w_0(0))} \exp[-a(0,w_0(0))\tau] \\[4ex]
v_1(\tau) = P_1(\tau)\exp[-a(0,w_0(0))\tau] + P_2(\tau)\exp[-2a(0,w_0(0))\tau]
\end{cases}
$$

with $P_k(\tau)$ ($k = 1,2$) polynomials in τ. So we see that v_0 and v_1 are typical boundary layer functions.

Finally, we obtain from (15), (17), (19), (20), (21) and (22) the result

$$
\begin{cases}
L^{(\varepsilon)}[\tilde{u}] = R(t;\varepsilon), & 0 < t < T < T_0 \\[2ex]
\tilde{u}(0;\varepsilon) = C_1 + \varepsilon^2 v_1(0) \\[2ex]
\tilde{u}_t(0;\varepsilon) = C_2
\end{cases}
$$

where $R(t;\varepsilon)$ is some function satisfying

$$
R(t;\varepsilon) = O(\varepsilon^2), \qquad \varepsilon \downarrow 0
$$

uniformly in $0 \le t \le T < T_0$.

Hence the function $\tilde{u}(t;\varepsilon)$ defined by (15), (18), (19), (20), (21) and (22) provides a formal approximation (up to order ε^2) for the solution u of problem (12).

Higher order approximations can be obtained by considering

$$
(23) \qquad \tilde{u}(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(t) + \sum_{k=0}^{n} \varepsilon^{k+1} v_k(\tau), \qquad n \in \mathbb{N}, \ n \ge 2.
$$

In that case it can be shown that w_k and v_k ($k = 0,1$) are given by (18), (19), (20), (21) and (22), whereas w_k and v_k ($k \ge 2$) follow from the linear problems

$$
(24) \qquad
\begin{cases}
a(t,w_0)\dfrac{dw_k}{dt} + \tilde{b}(t,w_0)w_k = -\Phi_k(t), & 0 < t < T_0 \\[2ex]
w_k(0) = -v_{k-1}(0)
\end{cases}
$$

respectively

$$
(25) \qquad
\begin{cases}
\dfrac{d^2 v_k}{d\tau^2} + a(0,w_0(0))\dfrac{dv_k}{d\tau} = -\Psi_k(\tau), & 0 < \tau < \infty \\[2ex]
\dfrac{dv_k}{d\tau}(0) = -\dfrac{dw_k}{dt}(0) \\[2ex]
\lim_{\tau \to \infty} v_k(\tau) = 0,
\end{cases}
$$

the functions Φ_k and Ψ_k ($k \in \mathbb{N}$, $k \geq 2$) being completely determined by the functions $w_0, w_1, \ldots, w_{k-1}$ respectively the functions $w_0, w_1, \ldots, w_k$ and $v_0, v_1, \ldots, v_{k-1}$.

The functions $v_k(\tau)$ ($k \in \mathbb{N} \cup \{0\}$) are boundary layer functions of the type

$$(26) \qquad v_k(\tau) = \sum_{j=1}^{k+1} P_j^{(k)}(\tau) \exp[-ja(0, w_0(0))\tau]$$

with $P_j^{(k)}(\tau)$ a polynomial in τ.

It follows that

$$(27) \qquad \begin{cases} L^{(\varepsilon)}[\tilde{u}] = R(t;\varepsilon), & 0 < t < T < T_0 \\[2ex] \tilde{u}(0;\varepsilon) = C_1 + \varepsilon^{n+1} v_n(0) \\[2ex] \tilde{u}_t(0;\varepsilon) = C_2 \end{cases}$$

with

$$R(t;\varepsilon) = O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0.$$

uniformly valid in $0 \leq t \leq T < T_0$.

Next we shall prove that a formal approximation $\tilde{u}$ is indeed a good approximation for the solution u of (12). To this end we put

$$(28) \qquad z(t;\varepsilon) = u(t;\varepsilon) - \tilde{u}(t;\varepsilon).$$

Then we obtain from (12) and (27)

$$\begin{cases} \varepsilon \dfrac{d^2 z}{dt^2} + a(t, z+\tilde{u}) \dfrac{dz}{dt} + F(t,z;\varepsilon) = -R(t;\varepsilon), & 0 < t < T < T_0 \\[2ex] z(0;\varepsilon) = -\varepsilon^{n+1} v_n(0) \\[2ex] z_t(0;\varepsilon) = 0 \end{cases}$$

where

$$F(t,z;\varepsilon) = \{a(t, z+\tilde{u}) - a(t,\tilde{u})\} \dfrac{d\tilde{u}}{dt} + b(t, z+\tilde{u}) - b(t,\tilde{u}).$$

Defining a function $\tilde{z}(t;\varepsilon)$ by

(29) $\qquad \tilde{z}(t;\varepsilon) = z(t;\varepsilon) + \varepsilon^{n+1} v_n(0)$

we obtain

(30) $\qquad \begin{cases} \varepsilon \dfrac{d^2 \tilde{z}}{dt^2} + a^*(t,\tilde{z};\varepsilon)\dfrac{d\tilde{z}}{dt} + b^*(t,\tilde{z};\varepsilon) = -R^*(t;\varepsilon), & 0 < t < T < T_0 \\[2mm] \tilde{z}(0;\varepsilon) = \tilde{z}_t(0;\varepsilon) = 0 \end{cases}$

where

$$\begin{cases} a^*(t,\tilde{z};\varepsilon) = a(t,\tilde{z}+\tilde{u}-\varepsilon^{n+1} v_n(0)) \\[3mm] b^*(t,\tilde{z};\varepsilon) = \{a(t,\tilde{z}+\tilde{u}-\varepsilon^{n+1} v_n(0)) - a(t,\tilde{u}-\varepsilon^{n+1} v_n(0))\}\dfrac{d\tilde{u}}{dt} \\[3mm] \qquad\qquad + b(t,\tilde{z}+\tilde{u}-\varepsilon^{n+1} v_n(0)) - b(t,\tilde{u}-\varepsilon^{n+1} v_n(0)) \\[3mm] R^*(t;\varepsilon) = R(t;\varepsilon) + \{a(t,\tilde{u}-\varepsilon^{n+1} v_n(0)) - a(t,\tilde{u})\}\dfrac{d\tilde{u}}{dt} \\[3mm] \qquad\qquad + b(t,\tilde{u}-\varepsilon^{n+1} v_n(0)) - b(t,\tilde{u}). \end{cases}$$

It is easily established that the functions a^*, b^* and R^* satisfy the conditions (6), (7) and (8) of theorem 2 (with Lipschitz exponent $\alpha = 1$) and that

$$R^*(t;\varepsilon) = O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0$$

uniformly valid in $0 \leq t \leq T < T_0$.
So according to theorem 2 there exists for sufficiently small values of ε a solution $\tilde{z}$ of (30) which satisfies

(31) $\qquad |\tilde{z}|_1^{(\varepsilon)} \leq C(T)\varepsilon^{n+1}.$

Here $C(T)$ is some positive constant depending on T but independent of ε. Using (23), (28) and (29) we obtain from the estimate (31) the following theorem, which is a nonlinear generalization of theorem 2 of chapter I:

THEOREM 3. *Let the functions* $a(t,u)$ *and* $b(t,u)$ *satisfy the conditions* (13) *and* (14), *and let* w_0 *be the solution of the nonlinear initial value problem* (19). *Let* T_0 *denote the smallest value of* t *for which* w_0 *becomes singular* $(T_0 = \infty$ *if* w_0 *has no singularities).*

Then there exists for $0 \leq t \leq T < T_0$ *and for sufficiently small values of* ε *a solution* u *of problem* (12). *This solution and its derivative satisfy for any* $n \in \mathbb{N} \cup \{0\}$ *the relations*

(32)
$$
\begin{cases}
u(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(t) + \sum_{k=0}^{n-1} \varepsilon^{k+1} v_k(\tau) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0 \\[2ex]
\dfrac{du}{dt}(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \dfrac{dw_k}{dt}(t) + \sum_{k=0}^{n} \varepsilon^k \dfrac{dv_k}{d\tau}(\tau) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0
\end{cases}
$$

uniformly in every bounded interval $0 \leq t \leq T < T_0$; *the functions* w_k *and* v_k $(k = 0,1,\ldots,n)$ *follow successively from* (19), (20), (21), (22), (24) *and* (25) *where* τ *is defined by* $\tau = \dfrac{t}{\varepsilon}$.

In case $n = 0$ *empty sums are to be interpreted as zero.*

COROLLARY 1. *It follows from* (32) *that first approximations* $(n=0)$ *for the solution* u *of* (12) *and its derivative are given by*

$$
\begin{cases}
u(t;\varepsilon) = w_0(t) + O(\varepsilon), & \varepsilon \downarrow 0 \\[2ex]
\dfrac{du}{dt}(t;\varepsilon) = \dfrac{dw_0}{dt}(t) + \{C_2 - \dfrac{dw_0}{dt}(0)\}\exp[-a(0,w_0(0))\dfrac{t}{\varepsilon}] + O(\varepsilon), & \varepsilon \downarrow 0
\end{cases}
$$

uniformly valid in every bounded interval $0 \leq t \leq T < T_0$, *where* w_0 *is the solution of the reduced problem* (19).

COROLLARY 2. *It follows from* (32) *and* (26) *that for any* $n \in \mathbb{N} \cup \{0\}$

$$
\begin{cases}
u(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(t) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0 \\[2ex]
\dfrac{du}{dt}(t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \dfrac{dw_k}{dt}(t) + O(\varepsilon^{n+1}), & \varepsilon \downarrow 0
\end{cases}
$$

uniformly valid in every bounded interval $0 < \delta \leq t \leq T < T_0$, *where* δ *is an arbitrarily small positive constant independent of* ε.

REMARK 1. The statements of theorem 3 remain valid if the conditions (13) and (14) are replaced by the less stringent requirements

(33) $\quad\begin{cases} a(t,u) \in C^{\infty}([0,T_1] \times I), \quad b(t,u) \in C^{\infty}([0,T_1] \times I) \\[2mm] a(t,u) > 0 \quad \text{for} \quad 0 \le t \le T_1 \quad \text{and} \quad u \in I \end{cases}$

where T_1 is some positive constant with $T_1 < T_0$ and I is some open subset of $\mathbb{R}$ containing the range R_{w_0} of w_0:

$$I \supset R_{w_0} = \{y \in \mathbb{R} \mid \underset{t \in [0,T_1]}{\exists} \; y = w_0(t) \}.$$

In that case the relations (32) are uniformly valid in every bounded interval $0 \le t \le T$ with $T \le T_1$.

The conditions (33) have the advantage that they allow the functions $a(t,u)$ and $b(t,u)$ to have singularities in u, provided that these singularities do not belong to R_{w_0}. As an illustration we consider the nonlinear initial value problem

$$\begin{cases} \varepsilon \dfrac{d^2u}{dt^2} + \dfrac{1}{u}\dfrac{du}{dt} + u = 0, \qquad 0 < t < \infty \\[4mm] u(0;\varepsilon) = c_1 > 0, \quad u_t(0;\varepsilon) = c_2. \end{cases}$$

It is easily established that

$$w_0(t) = \frac{c_1}{1 + c_1 t} \qquad \text{for } 0 \le t < \infty$$

and also that

$$\frac{c_1}{1 + c_1 T_1} \le w_0(t) \le c_1$$

valid for $0 \le t \le T_1$, where T_1 is an arbitrarily large positive constant. Hence it follows that the conditions (33) can be fulfilled by choosing

$$I = \{u \in \mathbb{R} \mid \frac{c_1}{2(1+c_1 T_1)} < u < c_1 + 1\},$$

and we obtain

$$u(t;\varepsilon) = \frac{C_1}{1 + C_1 t} + O(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly valid in every bounded interval $0 \le t \le T$.

REMARK 2. The condition of infinite smoothness of the data can be weakened. In that case the expansions (32) are valid for all $n \in \mathbb{N} \cup \{0\}$ with $n \le n_0$, the bound n_0 being determined by the differentiability of the data.

REMARK 3. Since the solution u of an initial value problem which is governed by a differential equation of the type

$$\frac{du}{dt} = F(t,u)$$

(with u and F vector-valued functions) is uniquely determined if F is continuous with respect to t and locally lipschitzian with respect to u (PICARD-LINDELÖF theorem), we may conclude that the solution u of which the existence has been established in theorem 3 is unique. A similar remark can be made for the solution u mentioned in theorem 2.

CHAPTER III

INITIAL VALUE PROBLEMS FOR LINEAR
HYPERBOLIC DIFFERENTIAL EQUATIONS

In this chapter we shall investigate the initial value problem (singular perturbation problem)

$$(1) \quad \begin{cases} \varepsilon L_2[u] + L_1^{(\varepsilon)}[u] = f(x,t;\varepsilon), & (x,t) \in D \\[2mm] u(x,0;\varepsilon) = g(x;\varepsilon), & -\infty < x < \infty \\[2mm] u_t(x,0;\varepsilon) = h(x;\varepsilon), & -\infty < x < \infty \end{cases}$$

where ε is a small positive parameter, i.e. $0 < \varepsilon \ll 1$, the region D is the upper half-plane

$$D = \{(x,t) \mid -\infty < x < \infty, \ 0 < t < \infty\}$$

and the differential operators L_2 and $L_1^{(\varepsilon)}$ are defined by

$$(2) \quad L_2[u] = \frac{\partial^2 u}{\partial t^2} + \{c_1(x,t) + c_2(x,t)\}\frac{\partial^2 u}{\partial x \partial t} + c_1(x,t)c_2(x,t)\frac{\partial^2 u}{\partial x^2}$$

respectively

$$(3) \quad L_1^{(\varepsilon)}[u] = a(x,t;\varepsilon)\frac{\partial u}{\partial t} + b(x,t;\varepsilon)\frac{\partial u}{\partial x} + d(x,t;\varepsilon)u.$$

The functions a, b, d, f, g and h depend on the variables x and/or t and on the parameter ε; the functions c_1 and c_2 satisfy

$$c_1(x,t) < c_2(x,t) \quad \text{in } \bar{D},$$

which implies that L_2 is a hyperbolic operator.

Furthermore it will be assumed that the coefficient $a(x,t;\varepsilon)$ is strictly positive, i.e.

$$a(x,t;\varepsilon) \geq a_0 > 0 \qquad \text{in } \Omega$$

valid for each compact subset $\Omega \subset \bar{D}$ provided that ε is sufficiently small, say $0 < \varepsilon \leq \varepsilon_0$. Here a_0 is some positive constant depending on Ω and on ε_0 but independent of ε.

Compared to singular perturbations of elliptic type there exists relatively few literature on singular perturbations of hyperbolic type. BLONDEL [3], [4], DZAVADOV [7], DE JAGER [26], SMITH [44] and SMITH & PALMER [46] studied singularly perturbed initial value problems for linear hyperbolic differential equations, whereas singularly perturbed initial-boundary value problems were investigated by DZAVADOV [8] and GENET & PUPION [18] (linear problems) and by GENET & MADAUNE [17] and MADAUNE [30] (nonlinear problems). Singular perturbations of hyperbolic type are also discussed in the books by COLE [6, pp.129-140], ROSEAU [41, chapter 6] and WHITHAM [55, chapters 3 and 10]. The energy integral method that will be applied in this chapter to linear initial value problems of the type (1) is an extension of a similar method developed by DE JAGER [26] and differs basically from the methods used in [3], [4], [7], [44] and [46]. A brief discussion of these papers together with a comparison of the results will be given in section 4.

Section 1 of this chapter will be devoted to some preliminaries. In section 2 we shall derive a priori estimates for the solution u of (1) and in section 3 we shall construct an asymptotic expansion into powers of ε for this solution.

1. Preliminaries

We bring into mind that the characteristics of the operator L_2 defined by (2) follow from the equations

$$\frac{dx}{dt} = c_1(x,t) \qquad \text{and} \qquad \frac{dx}{dt} = c_2(x,t).$$

We shall assume that c_1 and c_2 are C^1-functions. Under this condition it is possible to draw at each point (x_0, t_0) two vectors, tangent to the

characteristics through (x_0, t_0) and pointing in the direction of increasing t (see figure 1). These vectors define the so-called characteristic directions at (x_0, t_0).
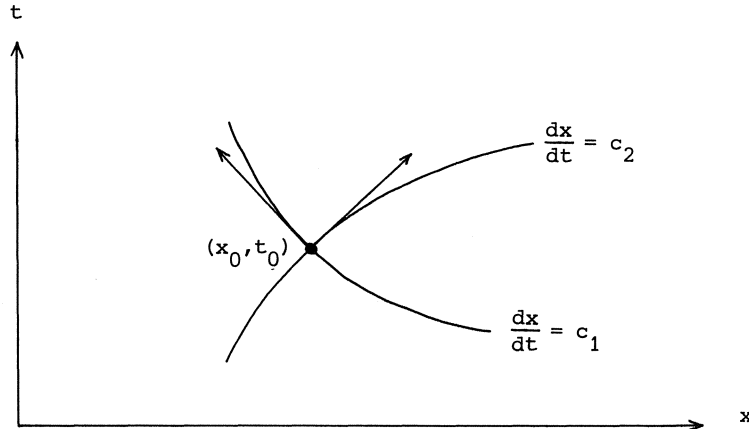

Figure 1: Characteristic directions

The characteristics of the operator $L_1^{(\varepsilon)}$ defined by (3) follow from the equation

$$\frac{dx}{dt} = \frac{b(x,t;\varepsilon)}{a(x,t;\varepsilon)} \; .$$

In order not to mix up the characteristics of L_2 and $L_1^{(\varepsilon)}$ we shall refer to the latter as subcharacteristics.

In the analysis of the initial value problem (1) the position of the subcharacteristics of the unperturbed operator $L_1^{(\varepsilon)}$ with respect to the characteristics of the perturbing operator L_2 plays an important role. Before entering into this subject we shall give a definition.

DEFINITION 1. Let $\phi(x, t)$ be a C^1-function satisfying $\phi_x^2 + \phi_t^2 > 0$. The curve $\phi(x,t) = 0$ has at one of its points (x_0, t_0) a spacelike (resp. timelike resp. characteristic) direction with respect to the characteristic directions connected with L_2 iff

$$(\phi_t + c_1\phi_x)(\phi_t + c_2\phi_x) > 0 \qquad \text{(resp. <0 resp. =0) at } (x_0,t_0).$$

The curve $\phi(x,t) = 0$ is said to be spacelike (resp. timelike resp. characteristic) if it has a spacelike (resp. timelike resp. characteristic) direction at each of its points.

This definition admits the following geometrical interpretation. Let the characteristic directions be described by the vectors $\vec{p} = \begin{pmatrix} c_1 \\ 1 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} c_2 \\ 1 \end{pmatrix}$. The direction of the tangent to the curve $\phi(x,t) = 0$ can be characterized by a vector $\vec{r} = \begin{pmatrix} \phi_t \\ -\phi_x \end{pmatrix}$. Evaluating cross products we obtain

$$(\vec{r} \times \vec{p}) \cdot (\vec{r} \times \vec{q}) = (\phi_t + c_1\phi_x)(\phi_t + c_2\phi_x),$$

the dot denoting the scalar product.

Hence it follows that the curve $\phi(x,t) = 0$ has a spacelike (resp. timelike) direction at (x_0,t_0) iff the vectors $\vec{p}$ and $\vec{q}$ are lying on the same side (resp. on different sides) of the tangent to the curve at (x_0,t_0) (see figure 2).
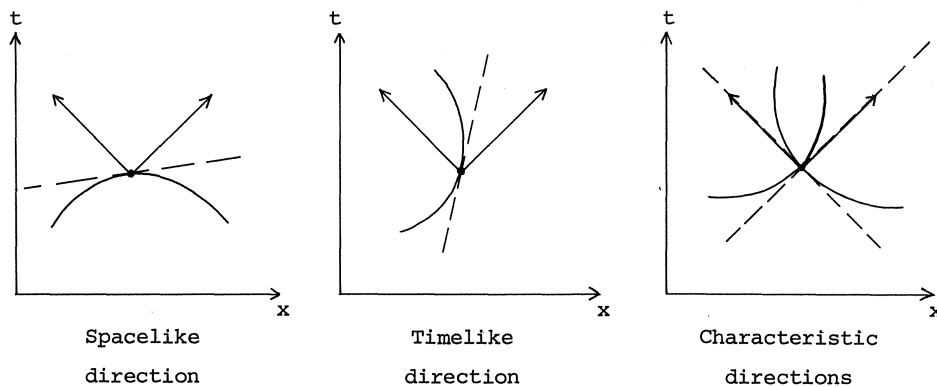

| Spacelike direction | Timelike direction | Characteristic directions |

Figure 2: Direction of a curve

With this knowledge we turn our attention to the following model problem (cf. COLE [6, p. 129])

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$$\begin{cases} \varepsilon\{\dfrac{\partial^2 u}{\partial t^2} + (c_1 + c_2)\dfrac{\partial^2 u}{\partial x \partial t} + c_1 c_2\dfrac{\partial^2 u}{\partial x^2}\} + a\dfrac{\partial u}{\partial t} + b\dfrac{\partial u}{\partial x} = 0, \qquad (x,t) \in D \\[2mm] u(x,0;\varepsilon) = g(x), \qquad -\infty < x < \infty \\[2mm] u_t(x,0;\varepsilon) = 0, \qquad -\infty < x < \infty \end{cases}$$

(4)

where a, b, c_1 and c_2 are constants with $a > 0$ and $c_1 < c_2$ and g is a C^∞-function.

We shall investigate under which conditions the solution $w(x,t)$ of the reduced problem

$$\begin{cases} a\dfrac{\partial w}{\partial t} + b\dfrac{\partial w}{\partial x} = 0, \qquad (x,t) \in D \\[2mm] w(x,0) = g(x), \qquad -\infty < x < \infty \end{cases}$$

which is given by

(5) $\qquad w(x,t) = g(x - \dfrac{b}{a} t)$

provides for small values of ε an approximation for the solution $u(x,t;\varepsilon)$ of (4).

We shall distinguish between problems with spacelike subcharacteristics and problems with timelike subcharacteristics.

1. The subcharacteristics are spacelike.

In this case, according to definition 1, the coefficients satisfy
$(b - ac_1)(b - ac_2) > 0$, that is

either $\quad \dfrac{b}{a} < c_1 \quad$ or $\quad \dfrac{b}{a} > c_2$.

A possible situation is sketched in figure 3. Note that the subcharacteristic through P lies outside the characteristic triangle PRS.
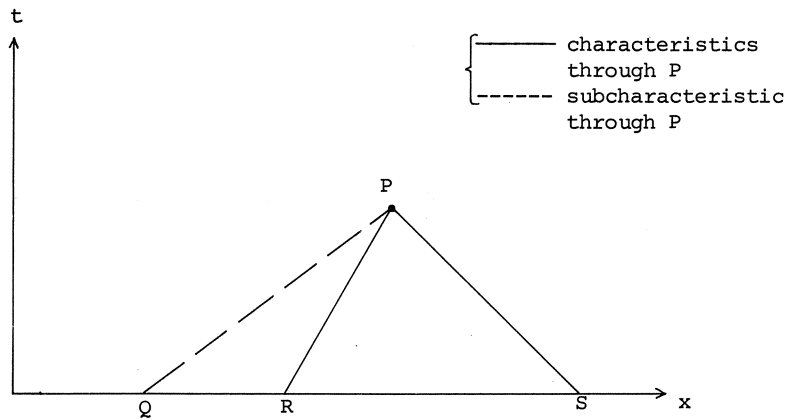
Figure 3: Spacelike subcharacteristics

From the theory of hyperbolic differential equations it is known (see for instance GARABEDIAN [16, p. 105]) that the value of the solution $u(x,t;\varepsilon)$ at the point P depends only on the values of $g(x)$ with x in the segment RS. On the other hand the value of $w(x,t)$ at the point P depends only on the value of g at the point Q (cf. (5)). Since the value of g at Q is independent of the values of $g(x)$ with x in RS (in fact we assumed that $g(x)$ belongs to $C^{\infty}(\mathbb{R})$, but not that $g(x)$ is analytic in $\mathbb{R}$) we expect that generally $w(x,t)$ does not approximate $u(x,t;\varepsilon)$ for small values of ε. This is confirmed by the following example.

EXAMPLE 1. Consider problem (4) with $g(x)$ some C^{∞}-function satisfying

$$\begin{cases} g(x) = x & \text{for} \quad x \le -1 \\ \\ g(x) = 0 & \text{for} \quad x \ge 0 \\ \\ -1 < g(x) < 0 & \text{for} \quad -1 < x < 0. \end{cases}$$

The coefficients are supposed to satisfy $a > 0$, $c_1 < c_2$ and $0 < c_2 < \dfrac{b}{a}$, which implies that the subcharacteristics are spacelike.
It follows from the theory of hyperbolic differential equations that

$$u(x,t;\varepsilon) \equiv 0 \quad \text{for} \quad 0 \le c_2 t \le x < \infty.$$

On the other hand the solution of the reduced problem satisfies

$$w(x,t) = g(x - \frac{b}{a} t) < 0 \quad \text{for} \quad 0 < x < \frac{b}{a} t < \infty$$

so the function $w(x,t)$ does not approximate $u(x,t;\varepsilon)$ in the region $0 < c_2 t < x < \frac{b}{a} t < \infty$ (see figure 4).
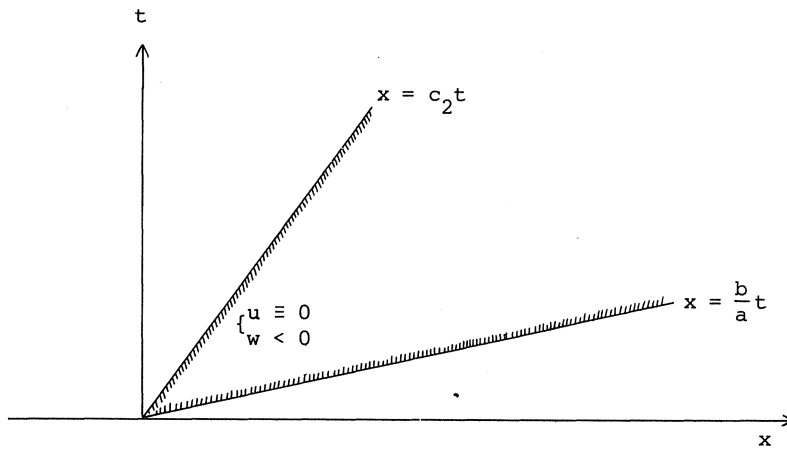

Figure 4

Hence we conclude that generally in the case of spacelike subcharacteristics the solution $w(x,t)$ of the reduced problem turns out to be a bad approxima- tion for the solution $u(x,t;\varepsilon)$ of (4). However, $w(x,t)$ does not *necessarily* provide a bad approximation. This is illustrated by

EXAMPLE 2. Consider problem (4) with $g(x) = x$ and $a > 0$, $c_1 < c_2$. No other conditions are imposed on the coefficients.
It is easily verified that the solution of this problem is given by

$$u(x,t;\varepsilon) = x - \frac{b}{a} t + \varepsilon \frac{b}{a^2} \{1 - \exp(-\frac{at}{\varepsilon})\}.$$

Since $w(x,t)$ is given by

$$w(x,t) = x - \frac{b}{a} t$$

it follows that

$$u(x,t;\varepsilon) = w(x,t) + 0(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly valid in $\bar{D}$, so in this example $w(x,t)$ approximates $u(x,t;\varepsilon)$ for small values of ε, whether the subcharacteristics are spacelike or not. Note that if in this example the condition $a > 0$ is replaced by $a < 0$ the solution u increases exponentially for all (x,t) with $t \geq \delta > 0$ if ε tends to zero, regardless whether the subcharacteristics are timelike or spacelike. In this case w does not approximate u for small values of ε.

2. The subcharacteristics are timelike.

In this case the coefficients satisfy $(b - ac_1)(b - ac_2) < 0$, that is

$$(6) \qquad c_1 < \frac{b}{a} < c_2.$$

A typical situation is sketched in figure 5. Note that the subcharacteristic through P lies inside the characteristic triangle PRS.
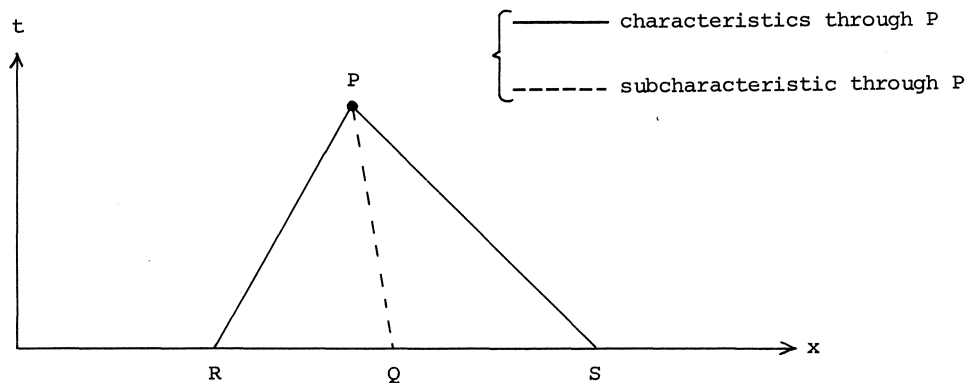

Figure 5: Timelike subcharacteristics

Using the same arguments as before (see text following figure 3) we have no reason to suspect $w(x,t)$ of being a bad approximation for $u(x,t;\varepsilon)$. However, it follows from example 2 that the timelike condition (6) alone does not

guarantee that w is a good approximation for u; it appears that we have to impose an additional condition and by the results of example 2 we get the suggestion to impose a > 0 as additional condition. Therefore we shall confine ourselves from now on to problems with timelike subcharacteristics and a positive coefficient a. It will be shown in the sections 2 and 3 that these restrictions are sufficient in order that the solution w of the reduced problem approximates the solution u for small values of ε, not only for problems of type (4) with constant coefficients but also for more general problems of type (1). For a discussion of initial value problems with timelike subcharacteristics and a negative coefficient a we refer to BLONDEL [3], [4] (see also section 4).

We shall end this section with a physical interpretation of the conditions

$$(7) \qquad a > 0, \ c_1 < \frac{b}{a} < c_2.$$

Introducing characteristic coordinates by

$$\begin{cases} r = x - c_1 t \\ s = x - c_2 t \end{cases}$$

we may write the differential equation

$$(8) \qquad \varepsilon \{ \frac{\partial^2 u}{\partial t^2} + (c_1 + c_2) \frac{\partial^2 u}{\partial x \partial t} + c_1 c_2 \frac{\partial^2 u}{\partial x^2} \} + a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = 0$$

(with constant coefficients and $c_1 < c_2$) in the form

$$(9) \qquad \varepsilon (c_2 - c_1)^2 \frac{\partial^2 u}{\partial r \partial s} - (b - ac_1) \frac{\partial u}{\partial r} - (b - ac_2) \frac{\partial u}{\partial s} = 0.$$

Let us consider a jump κ in the derivative $\frac{\partial u}{\partial s}$ across the characteristic $s = s_0$:

$$\kappa = \kappa(r) = \frac{\partial u}{\partial s}(r, s_0 + 0) - \frac{\partial u}{\partial s}(r, s_0 - 0).$$

We shall examine the propagation of κ along $s = s_0$.
Assuming that u itself is continuous across $s = s_0$ we can evaluate

equation (9) at $s = s_0+0$ and $s = s_0-0$ and form the difference to obtain

$$\varepsilon(c_2 - c_1)^2 \frac{\partial \kappa}{\partial r} = (b - ac_2)\kappa$$

and hence

$$\kappa = \kappa_0 \exp[\frac{b - ac_2}{\varepsilon(c_2 - c_1)^2}(r - r_0)].$$

Here $\kappa_0 = \kappa(r_0)$ is the jump across $s = s_0$ at the point (r_0, s_0). It follows that a jump across a characteristic propagates to infinity along that characteristic; in case $b - ac_2 < 0$ we have for $t \to \infty$ exponential decay, whereas for $b - ac_2 > 0$ we have exponential growth.

A similar discussion for jumps in $\frac{\partial u}{\partial r}$ across a characteristic $r = r_0$ shows that $b - ac_1 > 0$ implies exponential decay for $t \to \infty$, whereas $b - ac_1 < 0$ leads to exponential growth.

Hence we conclude that necessary and sufficient conditions for exponential decay are given by

$$\begin{cases} b - ac_1 > 0, \\ b - ac_2 < 0. \end{cases}$$

Together with the condition $c_1 < c_2$ these conditions are equivalent to

$$a > 0, \quad c_1 < \frac{b}{a} < c_2.$$

Thus we have shown that the conditions (7) are in fact necessary and sufficient conditions for exponential decay of jumps in the first derivatives. We may expect these conditions to be fulfilled for any problem describing some physical system.

Another way to obtain the conditions (7) by physical arguments is the following (see also ROSEAU [41, p. 198]). Let us look for solutions of the equation (8) which are periodic in the space variable and exponentially decaying for $t \to \infty$. To this end we set

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$$u = \exp(ikx - \lambda t)$$

and we require that

$$\begin{cases} k \in \mathbb{R}, & k > 0 \\ \\ \lambda \in \mathbb{C}, & \mathrm{Re}\ \lambda > 0. \end{cases}$$

Substitution into the differential equation yields the following relation between λ and k

$$f(\lambda) = \varepsilon(ikc_1 - \lambda)(ikc_2 - \lambda) + (ikb - a\lambda) = 0.$$

The roots λ of this equation will have a positive real part if and only if the quantity $\mathrm{arg} f(iy)$ decreases by 2π as y increases from $-\infty$ to $+\infty$. For y running through these values the point

$$f(iy) = -\varepsilon(kc_1 - y)(kc_2 - y) + i(kb - ay)$$

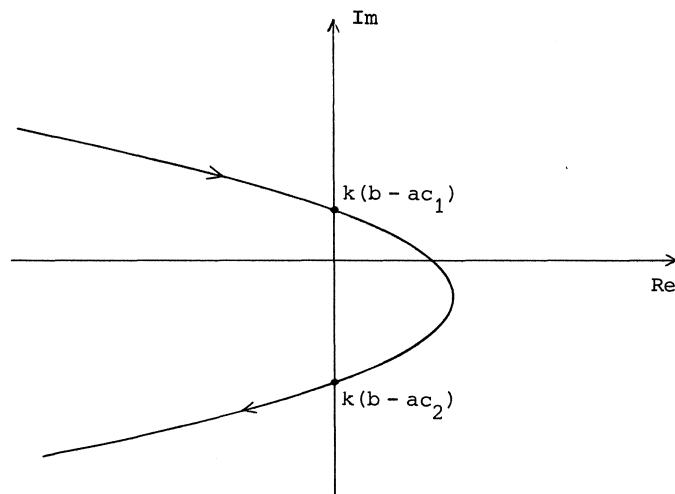describes a parabola in the complex plane (see figure 6).

Figure 6: Position of f(iy)

It is easily verified that we have exponential decay if and only if a > 0 (in which case the point f(iy) travels in the proper direction along the parabola) and the origin lies inside the parabola. By calculating the points of intersection with the imaginary axis we see that the latter condition is equivalent to

$$c_1 < \frac{b}{a} < c_2$$

indicating that the conditions (7) can also be considered as necessary and sufficient conditions for the equation (8) to admit exponentially decaying periodic solutions.

Now that we have established the physical relevancy of the conditions (7) we are going to derive in the sections 2 and 3 a priori estimates and asymptotic expansions for the solution of problem (1) under the conditions (7).

2. A priori estimates

We consider the initial value problem (1) under the following conditions:

(10) $\begin{cases} c_1(x,t) \in C^1(\bar{D}), \quad c_2(x,t) \in C^1(\bar{D}) \\[2ex] c_1(x,t) < c_2(x,t) \quad \text{in } \bar{D} \\[2ex] c_1(x,t) \text{ resp. } c_2(x,t) \text{ is uniformly bounded in any strip} \\ \{(x,t) \mid 0 \le x < \infty, \ 0 \le t \le T\} \text{ resp. } \{(x,t) \mid -\infty < x \le 0, \ 0 \le t \le T\} \\ \text{i.e.} \\[2ex] \begin{cases} |c_1(x,t)| \le M_1 \quad \text{for} \quad 0 \le x < \infty, \quad 0 \le t \le T \\[2ex] |c_2(x,t)| \le M_1 \quad \text{for} \quad -\infty < x \le 0, \quad 0 \le t \le T \end{cases} \\[2ex] \text{where } M_1 \text{ is some positive constant which may depend on T} \end{cases}$

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$$\left\{ \begin{array}{l} \text{for each compact subset } \Omega \subset \bar{D} \text{ there exists a positive constant } \varepsilon_1 \\ \text{(depending on } \Omega) \text{ such that} \end{array} \right.$$

(i) $a(x,t;\varepsilon) \in C^1(\Omega), \quad b(x,t;\varepsilon) \in C^1(\Omega)$

       $d(x,t;\varepsilon) \in C^0(\Omega), \quad f(x,t;\varepsilon) \in C^0(\Omega)$

       valid for every value of ε satisfying $0 < \varepsilon \leq \varepsilon_1$

(ii)   the functions $a(x,t;\varepsilon)$, $b(x,t;\varepsilon)$ and their first derivatives, as well as the function $d(x,t;\varepsilon)$, are uniformly bounded in Ω for $0 < \varepsilon \leq \varepsilon_1$, i.e. there exists a positive constant M_2 (depending on Ω and ε_1 but independent of ε) such that for all $(x,t) \in \Omega$ and all $\varepsilon \in (0,\varepsilon_1]$

(11)

$$\left\{ \begin{array}{c} |a| + \left|\frac{\partial a}{\partial x}\right| + \left|\frac{\partial a}{\partial t}\right| \leq M_2 \\[2mm] |b| + \left|\frac{\partial b}{\partial x}\right| + \left|\frac{\partial b}{\partial t}\right| \leq M_2 \\[2mm] |d| \leq M_2 \end{array} \right.$$

(iii) $a(x,t;\varepsilon) \geq a_0 > 0$

       valid for all $(x,t) \in \Omega$ and all $\varepsilon \in (0,\varepsilon_1]$, where a_0 is some constant depending on Ω and ε_1 but independent of ε

(iv) $\left\{ \begin{array}{l} b(x,t;\varepsilon) - a(x,t;\varepsilon)c_1(x,t) \geq p > 0 \\[2mm] a(x,t;\varepsilon)c_2(x,t) - b(x,t;\varepsilon) \geq p > 0 \end{array} \right.$

       valid for all $(x,t) \in \Omega$ and all $\varepsilon \in (0,\varepsilon_1]$, where p is some constant depending on Ω and ε_1 but independent of ε

(12)

$$\left\{ \begin{array}{l} \text{for each interval } [x_1,x_2] \text{ there exists a positive constant } \varepsilon_2 \text{ (depending on } x_1 \text{ and } x_2) \text{ such that} \\[3mm] \quad g(x;\varepsilon) \in C^1[x_1,x_2], \quad h(x;\varepsilon) \in C^0[x_1,x_2] \\[3mm] \text{valid for every value of } \varepsilon \text{ satisfying } 0 < \varepsilon \leq \varepsilon_2. \end{array} \right.$$

<u>REMARK</u>. The conditions (11,iv) imply that the subcharacteristics of $L_1^{(\varepsilon)}$ are, for ε sufficiently small, timelike with respect to the characteristic directions connected with L_2.

In order to derive a priori estimates for the solution $u(x,t;\varepsilon)$ of (1) in some compact subset $S \subset \bar{D}$ we enclose S by a compact subset $\Omega \subset \bar{D}$ in the following way (see figure 7):

we draw the line $t = T$ where $T \geq \sup_{(x,t)\in S} t$.

Next we draw a characteristic $\{(x,t) \mid x = \gamma_1(t)\}$ satisfying $\frac{dx}{dt} = c_1$ and lying to the right of S. This characteristic intersects the lines $t = 0$ resp. $t = T$ at the points B resp. C.

Likewise we draw a characteristic $\{(x,t) \mid x = \gamma_2(t)\}$ to the left of S, satisfying $\frac{dx}{dt} = c_2$ and intersecting $t = 0$ resp. $t = T$ at A resp. D. It is assumed that C lies to the right of D.

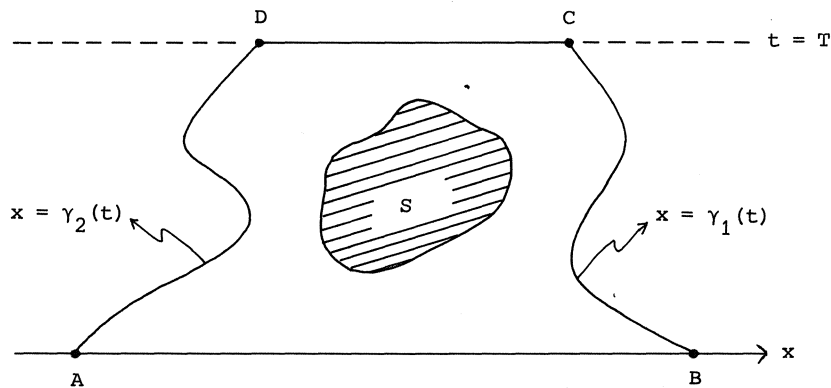Then Ω is defined as the closure of the region ABCD.

Figure 7: Construction of Ω

Note that the construction of Ω is possible because of the boundedness of the functions c_1 and c_2, which has been postulated in (10).

Now that Ω has been determined we multiply the differential equation (1) by $2u$ to obtain (omitting arguments)

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$$\frac{\partial}{\partial t}\{au^2 + 2\varepsilon uu_t\} + \frac{\partial}{\partial x}\{bu^2 + 2\varepsilon(c_1 + c_2)uu_t + 2\varepsilon c_1 c_2 uu_x\} =$$

(13)
$$u^2\{-2d + \frac{\partial a}{\partial t} + \frac{\partial b}{\partial x}\} + 2\varepsilon u_t^2 + 2\varepsilon(c_1 + c_2)u_t u_x + 2\varepsilon c_1 c_2 u_x^2$$

$$+ 2\varepsilon uu_t \frac{\partial(c_1 + c_2)}{\partial x} + 2\varepsilon uu_x \frac{\partial(c_1 c_2)}{\partial x} + 2fu$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$, the values of ε_1 and ε_2 depending on Ω (see the conditions (11,i) and (12)).

Multiplication of (1) by $2au_t$ yields

$$\frac{\partial}{\partial t}\{\varepsilon au_t^2 - \varepsilon ac_1 c_2 u_x^2\} + \frac{\partial}{\partial x}\{\varepsilon a(c_1 + c_2)u_t^2 + 2\varepsilon ac_1 c_2 u_t u_x\} =$$

(14)
$$\varepsilon u_t^2\{\frac{\partial a}{\partial t} + \frac{\partial a(c_1 + c_2)}{\partial x}\} + 2\varepsilon u_t u_x \frac{\partial(ac_1 c_2)}{\partial x} - \varepsilon u_x^2 \frac{\partial(ac_1 c_2)}{\partial t}$$

$$- 2aduu_t - 2a^2 u_t^2 - 2abu_t u_x + 2afu_t$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$.

Finally, multiplication of (1) by $2bu_x$ leads to

$$\frac{\partial}{\partial t}\{\varepsilon b(c_1 + c_2)u_x^2 + 2\varepsilon bu_t u_x\} - \frac{\partial}{\partial x}\{\varepsilon bu_t^2 - \varepsilon bc_1 c_2 u_x^2\} =$$

(15)
$$- \varepsilon u_t^2 \frac{\partial b}{\partial x} + 2\varepsilon u_t u_x \frac{\partial b}{\partial t} + \varepsilon u_x^2\{\frac{\partial b(c_1 + c_2)}{\partial t} + \frac{\partial(bc_1 c_2)}{\partial x}\}$$

$$- 2bduu_x - 2abu_t u_x - 2b^2 u_x^2 + 2bfu_x$$

likewise valid for $(x,t) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$.

Adding (13), (14) and (15) and using the inequalities

$$- 2aduu_t - 2bduu_x - 2a^2 u_t^2 - 4abu_t u_x - 2b^2 u_x^2 + 2fu + 2afu_t + 2bfu_x =$$

$$- 2(au_t + bu_x)^2 - 2du(au_t + bu_x) + 2f(au_t + bu_x) + 2fu \le$$

$$(1 + d^2)u^2 + 2f^2$$

and

$$2\varepsilon uu_t \frac{\partial(c_1+c_2)}{\partial x} + 2\varepsilon uu_x \frac{\partial(c_1c_2)}{\partial x} \leq$$

$$2\varepsilon u^2 + \varepsilon u_t^2 \{\frac{\partial(c_1+c_2)}{\partial x}\}^2 + \varepsilon u_x^2 \{\frac{\partial(c_1c_2)}{\partial x}\}^2 \leq$$

$$2u^2 + \varepsilon u_t^2 \{\frac{\partial(c_1+c_2)}{\partial x}\}^2 + \varepsilon u_x^2 \{\frac{\partial(c_1c_2)}{\partial x}\}^2$$

we obtain the relation

(16)
$$\frac{\partial}{\partial t}\mathcal{Q}_1 + \frac{\partial}{\partial x}\mathcal{Q}_2 \leq \mathcal{Q}_3 + 2f^2$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$, where the expressions $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ (quadratic in u, u_t and u_x) are defined by

(17)
$$\begin{cases} \mathcal{Q}_1 = au^2 + 2\varepsilon uu_t + \varepsilon au_t^2 + 2\varepsilon bu_t u_x + \varepsilon\{b(c_1 + c_2) - ac_1c_2\}u_x^2 \\ \\ \mathcal{Q}_2 = bu^2 + 2\varepsilon(c_1 + c_2)uu_t + 2\varepsilon c_1c_2 uu_x \\ \\ \qquad + \varepsilon\{a(c_1 + c_2) - b\}u_t^2 + 2\varepsilon ac_1c_2 u_t u_x + \varepsilon bc_1c_2 u_x^2 \\ \\ \mathcal{Q}_3 = \gamma_{11}u^2 + \varepsilon\gamma_{22}u_t^2 + 2\varepsilon\gamma_{23}u_t u_x + \varepsilon\gamma_{33}u_x^2 \end{cases}$$

with

$$\begin{cases} \gamma_{11} = 3 - 2d + d^2 + \frac{\partial a}{\partial t} + \frac{\partial b}{\partial x} \\ \\ \gamma_{22} = 2 + \frac{\partial a}{\partial t} - \frac{\partial b}{\partial x} + \frac{\partial a(c_1+c_2)}{\partial x} + \{\frac{\partial(c_1+c_2)}{\partial x}\}^2 \\ \\ \gamma_{23} = c_1 + c_2 + \frac{\partial b}{\partial t} + \frac{\partial(ac_1c_2)}{\partial x} \\ \\ \gamma_{33} = 2c_1c_2 + \frac{\partial b(c_1+c_2)}{\partial t} - \frac{\partial(ac_1c_2)}{\partial t} + \frac{\partial(bc_1c_2)}{\partial x} + \{\frac{\partial(c_1c_2)}{\partial x}\}^2. \end{cases}$$

Integrating (16) over Ω and using GREEN's theorem in the plane we obtain (see figure 7)

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(18)
$$\int_D^C \mathcal{Q}_1 dx + \int_B^C (\mathcal{Q}_2 - c_1\mathcal{Q}_1)dt + \int_A^D (c_2\mathcal{Q}_1 - \mathcal{Q}_2)dt \leq$$

$$\iint_\Omega \mathcal{Q}_3 dxdt + 2 \iint_\Omega f^2 dxdt + \int_A^B \mathcal{Q}_1 dx$$

valid for $0 < \varepsilon \leq \varepsilon_0$.

In order to obtain a priori estimates for u, u_x and u_t in Ω it is important that for ε sufficiently small the following crucial requirements are fulfilled:

(i)   the expression $\mathcal{Q}_1$, quadratic in u, u_t and u_x, is positive definite in Ω

(ii)   the expression $\mathcal{Q}_2 - c_1\mathcal{Q}_1$, quadratic in u and $u_t + c_1u_x$, is positive definite in Ω

(iii) the expression $c_2\mathcal{Q}_1 - \mathcal{Q}_2$, quadratic in u and $u_t + c_2u_x$, is positive definite in Ω.

It is easily verified from (17) that (i) is fulfilled iff

$$\begin{cases} a > 0 \qquad \text{in } \Omega \\[2mm] (b - ac_1)(ac_2 - b) > 0 \qquad \text{in } \Omega. \end{cases}$$

These conditions were supposed to be satisfied (see the conditions (11, iii and iv)).

REMARK. Note that these conditions are just the conditions (7) discussed at the end of section 1, which follow here in a natural way from the mathematical analysis.

Since we have for ε sufficiently small

$$b(c_1 + c_2) - ac_1c_2 = \frac{b^2 + (b - ac_1)(ac_2 - b)}{a} \geq \frac{b^2 + p^2}{a} > 0 \quad \text{in } \Omega$$

it follows that

$$Q_1 = au^2 + 2\varepsilon uu_t + \varepsilon au_t^2 + 2\varepsilon bu_t u_x + \varepsilon\{b(c_1 + c_2) - ac_1c_2\}u_x^2$$

$$\geq (a - \varepsilon^{\frac{1}{2}})u^2 + \varepsilon\{a - \frac{b^2 + \frac{1}{2}p^2}{b(c_1+c_2) - ac_1c_2} - \varepsilon^{\frac{1}{2}}\}u_t^2$$

$$(19) \qquad\qquad + \varepsilon(1 - \frac{b^2}{b^2 + \frac{1}{2}p^2})\{b(c_1 + c_2) - ac_1c_2\}u_x^2$$

$$\geq (a - \varepsilon^{\frac{1}{2}})u^2 + \varepsilon(\frac{1}{2}\frac{ap^2}{b^2 + p^2} - \varepsilon^{\frac{1}{2}})u_t^2 + \frac{1}{2}\varepsilon\frac{p^2}{a}u_x^2$$

$$\geq \frac{1}{2}a_0u^2 + \frac{1}{4}\varepsilon\frac{a_0p^2}{b^2 + p^2}u_t^2 + \frac{1}{2}\varepsilon\frac{p^2}{a}u_x^2 .$$

It is easily established from (17) that

$$Q_2 - c_1Q_1 = (b-ac_1)u^2 + 2\varepsilon c_2u(u_t + c_1u_x) + \varepsilon(ac_2-b)(u_t + c_1u_x)^2$$

so we find for ε sufficiently small

$$(20) \qquad Q_2 - c_1Q_1 \geq \frac{1}{2}pu^2 + \frac{1}{2}\varepsilon p(u_t + c_1u_x)^2 .$$

Similarly we obtain for ε sufficiently small

$$(21) \qquad c_2Q_1 - Q_2 \geq \frac{1}{2}pu^2 + \frac{1}{2}\varepsilon p(u_t + c_2u_x)^2 .$$

Hence, using (19), (20) and (21) we arrive at

$$(22) \quad \begin{cases} Q_1 \geq m(u^2 + \varepsilon u_t^2 + \varepsilon u_x^2) \\[2mm] Q_2 - c_1Q_1 \geq m\{u^2 + \varepsilon(u_t + c_1u_x)^2\} \\[2mm] c_2Q_1 - Q_2 \geq m\{u^2 + \varepsilon(u_t + c_2u_x)^2\} \end{cases}$$

valid for $(x,t) \in \Omega$ and ε sufficiently small, say $0 < \varepsilon \leq \varepsilon_0$, where ε_0 is some generic constant and the positive constant m, depending on Ω but independent of ε, is given by

$$m = \frac{1}{4}p \min(2, \frac{a_0p}{p^2 + M_2^2}, \frac{2p}{M_2}) .$$

It is easily established that there exists a constant $M \geq 2$, depending on Ω but independent of ε, such that

$$(23) \qquad |Q_1| + |Q_3| \leq M(u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

Using (22) and (23) we obtain from (18)

$$m \int_D^C (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)dx + m \int_B^C \{u^2 + \varepsilon(u_t + c_1 u_x)^2\}dt$$

$$(24) \qquad \qquad + m \int_A^D \{u^2 + \varepsilon(u_t + c_2 u_x)^2\}dt \leq$$

$$M \iint_\Omega (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)dxdt + MK_1^2(\Omega,\varepsilon)$$

valid for $0 < \varepsilon \leq \varepsilon_0$, where the positive constant $K_1(\Omega,\varepsilon)$, depending on Ω and ε, is defined by

$$K_1(\Omega,\varepsilon) = \|f\|_\Omega + \|g\|_I + \varepsilon^{\frac{1}{2}}\|g_x\|_I + \varepsilon^{\frac{1}{2}}\|h\|_I,$$

I denoting the segment $AB = [\gamma_2(0),\gamma_1(0)]$ (see figure 7).

One easily verifies that formula (24) with the same values of m and M is also valid for any similar region $\Omega' \subset \Omega$ with $A'B' \subset AB$, $D'C'//DC$ and $B'C'$ and $A'D'$ characteristic segments with directions c_1 resp. c_2 (see figure 8).
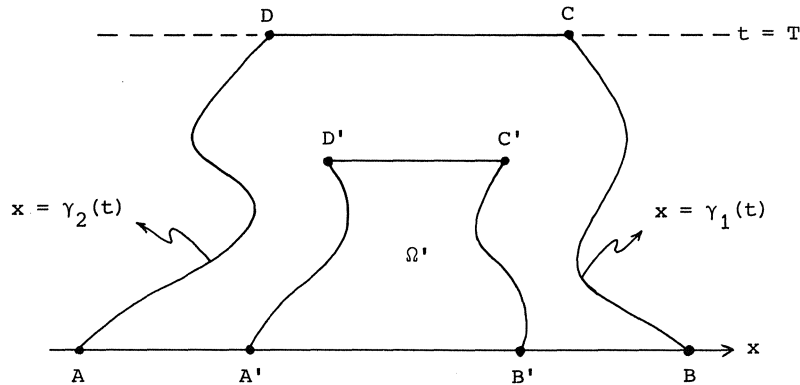

Figure 8: The region $\Omega' \subset \Omega$

So applying (24) to regions $\Omega' \subset \Omega$ with $A' = A$, $B' = B$ and $D'C'//DC$ we obtain

$$(25) \qquad \int_{\gamma_2(t)}^{\gamma_1(t)} (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)dx - \frac{M}{m} \int_0^t \int_{\gamma_2(\tau)}^{\gamma_1(\tau)} (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)dxd\tau \le \frac{M}{m}K_1^2(\Omega,\varepsilon)$$

valid for $0 \le t \le T$ and $0 < \varepsilon \le \varepsilon_0$.

Using GRONWALL's lemma we derive from (25) the estimate

$$(26) \qquad \int_{\gamma_2(t)}^{\gamma_1(t)} (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)dx \le C(\Omega)K_1^2(\Omega,\varepsilon)$$

valid for $0 \le t \le T$ and $0 < \varepsilon \le \varepsilon_0$, where the positive constant $C(\Omega)$, depending on Ω but independent of ε, is given by

$$C(\Omega) = \frac{M}{m} \exp(\frac{MT}{m}).$$

Applying (24) to a region Ω' inside Ω and using the estimate (26) we obtain

$$(27) \qquad \begin{cases} \displaystyle\int_{B'}^{C'} \{u^2 + \varepsilon(u_t + c_1 u_x)^2\}dt \le C(\Omega)K_1^2(\Omega,\varepsilon) \\[2em] \displaystyle\int_{A'}^{D'} \{u^2 + \varepsilon(u_t + c_2 u_x)^2\}dt \le C(\Omega)K_1^2(\Omega,\varepsilon) \end{cases}$$

valid for all characteristic segments $B'C'$ and $A'D'$ lying inside Ω (with directions c_1 resp. c_2) and all ε satisfying $0 < \varepsilon \le \varepsilon_0$. Here $C(\Omega)$ denotes some generic positive constant, depending on Ω but independent of ε.

Now that we have established estimates in L_2-norm it is not difficult to derive pointwise estimates for u and its first derivatives. Using (26) we obtain for every $(x,t) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$

$$|u^2(x,t;\varepsilon) - u^2(\gamma_2(t),t;\varepsilon)| =$$

$$\left|2\int_{\gamma_2(t)}^{x} u(\xi,t;\varepsilon)u_x(\xi,t;\varepsilon)d\xi\right| \leq$$

$$(28) \qquad 2\int_{\gamma_2(t)}^{\gamma_1(t)} |u(\xi,t;\varepsilon)u_x(\xi,t;\varepsilon)|d\xi \leq$$

$$2\left[\int_{\gamma_2(t)}^{\gamma_1(t)} u^2(\xi,t;\varepsilon)d\xi\right]^{\frac{1}{2}}\left[\int_{\gamma_2(t)}^{\gamma_1(t)} u_x^2(\xi,t;\varepsilon)d\xi\right]^{\frac{1}{2}} \leq$$

$$C(\Omega)\varepsilon^{-\frac{1}{2}}K_1^2(\Omega,\varepsilon)$$

or

$$u^2(\gamma_2(t),t;\varepsilon) \leq u^2(x,t;\varepsilon) + C(\Omega)\varepsilon^{-\frac{1}{2}}K_1^2(\Omega,\varepsilon).$$

Integrating this inequality with respect to x between the limits $\gamma_2(t)$ and $\gamma_1(t)$ we obtain

$$(29) \qquad \Theta_1 u^2(\gamma_2(t),t;\varepsilon) \leq C(\Omega)K_1^2(\Omega,\varepsilon) + \Theta_2 C(\Omega)\varepsilon^{-\frac{1}{2}}K_1^2(\Omega)$$

with

$$\begin{cases} \Theta_1 = \inf_{[0,T]} \{\gamma_1(t) - \gamma_2(t)\} > 0 \\ \\ \Theta_2 = \sup_{[0,T]} \{\gamma_1(t) - \gamma_2(t)\} > 0, \end{cases}$$

so we derive from (28) and (29) the pointwise estimate

$$|u(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-\frac{1}{4}}K_1(\Omega,\varepsilon)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$, where $C(\Omega)$ is a positive constant depending on Ω but independent of ε, and ε_0 is some generic constant.

In order to obtain pointwise estimates for $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial t}$ in Ω we proceed as follows (see figure 9):
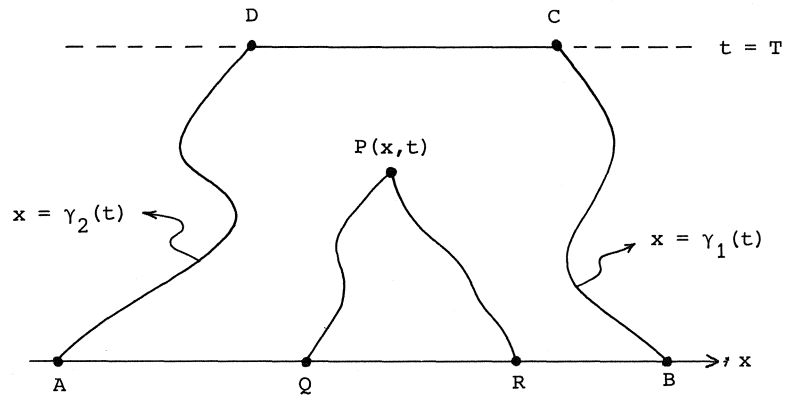
Figure 9

Let P(x,t) be an arbitrary point in Ω. We draw two characteristics through P, one with direction c_2 and intersecting the line $t = 0$ at Q, the other with direction c_1 and intersecting the initial line at R.

We multiply the differential equation (1) by $2(u_t + c_1 u_x)$ to obtain after some calculations

$$\varepsilon\left(\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x}\right)(u_t + c_1 u_x)^2 = -\frac{2(ac_2-\tilde{b})}{c_2 - c_1}(u_t + c_1 u_x)^2$$

(30)
$$-\frac{2(\tilde{b}-ac_1)}{c_2 - c_1}(u_t + c_2 u_x)(u_t + c_1 u_x)$$

$$-2du(u_t + c_1 u_x) + 2f(u_t + c_1 u_x)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$, where

$$\tilde{b} = b - \varepsilon\left(\frac{\partial c_1}{\partial t} + c_2\frac{\partial c_1}{\partial x}\right).$$

From the timelike condition (11,iv) we conclude that for ε sufficiently small, say $0 < \varepsilon \le \varepsilon_0$

$$a(x,t;\varepsilon)c_2(x,t) - \tilde{b}(x,t;\varepsilon) \geq \frac{1}{2}p \quad \text{in } \Omega.$$

Hence if follows from (30) that

$$\varepsilon\left(\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x}\right)(u_t + c_1 u_x)^2 \leq$$

$$-\frac{p}{c_2-c_1}(u_t + c_1 u_x)^2 - \frac{2}{c_2-c_1}(u_t + c_1 u_x)(\tilde{b}-ac_1)(u_t + c_2 u_x)$$

$$-\frac{2}{c_2-c_1}(u_t + c_1 u_x)\{d(c_2-c_1)u - (c_2-c_1)f\} \leq$$

$$\frac{1}{p(c_2-c_1)}\{(\tilde{b}-ac_1)(u_t + c_2 u_x) + d(c_2-c_1)u - (c_2-c_1)f\}^2 \leq$$

$$\frac{3(\tilde{b}-ac_1)^2}{p(c_2-c_1)}(u_t + c_2 u_x)^2 + \frac{3d^2(c_2-c_1)}{p}u^2 + \frac{3(c_2-c_1)}{p}f^2$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

Integrating this inequality with respect to t along the characteristic QP and using the second estimate in (27) we obtain

$$(31) \qquad |(u_t + c_1 u_x)(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-1}K_2(\Omega,\varepsilon)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$, where $C(\Omega)$ is a positive constant (depending on Ω but independent of ε) and the positive constant $K_2(\Omega,\varepsilon)$ (depending on both Ω and ε) is given by

$$K_2(\Omega,\varepsilon) = \sup_{\Omega} |f| + \|g\|_I + \varepsilon^{\frac{1}{2}}\sup_I |g_x| + \varepsilon^{\frac{1}{2}}\sup_I |h|.$$

Likewise, multiplying (1) by $2(u_t + c_1 u_x)$ and integrating along the characteristic RP we obtain

$$(32) \qquad |(u_t + c_2 u_x)(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-1}K_2(\Omega,\varepsilon)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

Finally, it follows from (31) and (32) that

$$\left\{ \begin{array}{l} |\frac{\partial u}{\partial x}(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-1}K_2(\Omega,\varepsilon) \\[4mm] |\frac{\partial u}{\partial t}(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-1}K_2(\Omega,\varepsilon) \end{array} \right.$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

We summarize the results in

THEOREM 1. *Let* $u(x,t;\varepsilon)$ *be the solution of the initial value problem* (1) *under the conditions* (10), (11) *and* (12), *and let the region* Ω *be given as in figure 7.*

Then we have for ε *sufficiently small, say* $0 < \varepsilon \leq \varepsilon_0$ *with* ε_0 *depending on* Ω, *the following estimates in* L_2-*norm*

$$\left\{ \begin{array}{l} \int\limits_{\gamma_2(t)}^{\gamma_1(t)} (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)dx \leq C(\Omega)K_1^2(\Omega,\varepsilon) \quad \text{valid for } 0 \leq t \leq T \\[8mm] \int\limits_{B'}^{C'} \{u^2 + \varepsilon(u_t + c_1 u_x)^2\}dt \leq C(\Omega)K_1^2(\Omega,\varepsilon) \\[8mm] \int\limits_{A'}^{D'} \{u^2 + \varepsilon(u_t + c_2 u_x)^2\}dt \leq C(\Omega)K_1^2(\Omega,\varepsilon) \end{array} \right.$$

where $B'C'$ *and* $A'D'$ *are arbitrary characteristic segments in* Ω *with directions* c_1 *resp.* c_2. *The positive constant* $C(\Omega)$ *depends on* Ω *but not on the parameter* ε. *The positive constant* $K_1(\Omega,\varepsilon)$, *depending on both* Ω *and* ε, *is given by*

$$K_1(\Omega,\varepsilon) = \|f\|_\Omega + \|g\|_I + \varepsilon^{\frac{1}{2}}\|g_x\|_I + \varepsilon^{\frac{1}{2}}\|h\|_I$$

with $I = \Omega \cap \{(x,t) \mid t = 0\}$.

We also have for ε *sufficiently small the following pointwise estimates*

$$
\begin{array}{ll}
(33a) & \sup\limits_{\Omega} |u(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-\frac{1}{4}}K_1(\Omega,\varepsilon) \\[4mm]
(33b) & \sup\limits_{\Omega} |\frac{\partial u}{\partial x}(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-1}K_2(\Omega,\varepsilon) \\[4mm]
(33c) & \sup\limits_{\Omega} |\frac{\partial u}{\partial t}(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-1}K_2(\Omega,\varepsilon)
\end{array}
$$

where again the positive constant $C(\Omega)$ *depends on* Ω *but not on* ε. *The posi-tive constant* $K_2(\Omega,\varepsilon)$, *depending on both* Ω *and* ε, *is given by*

$$K_2(\Omega,\varepsilon) = \sup_{\Omega} |f| + \|g\|_I + \varepsilon^{\frac{1}{2}} \sup_{I} |g_x| + \varepsilon^{\frac{1}{2}} \sup_{I} |h|.$$

3. Asymptotic expansions

As an application of the a priori estimates that we derived in the pre-ceding section we shall now construct an asymptotic expansion into powers of ε for the solution $u(x,t;\varepsilon)$ of the initial value problem

(34)
$$\begin{cases} \varepsilon L_2[u] + L_1[u] = f(x,t), & (x,t) \in D \\ u(x,0;\varepsilon) = g(x), & -\infty < x < \infty \\ u_t(x,0;\varepsilon) = h(x), & -\infty < x < \infty \end{cases}$$

where $0 < \varepsilon \ll 1$ and

$$\begin{cases} L_2[u] = \dfrac{\partial^2 u}{\partial t^2} + \{c_1(x,t) + c_2(x,t)\}\dfrac{\partial^2 u}{\partial x \partial t} + c_1(x,t)c_2(x,t)\dfrac{\partial^2 u}{\partial x^2} \\ L_1[u] = a(x,t)\dfrac{\partial u}{\partial t} + b(x,t)\dfrac{\partial u}{\partial x} + d(x,t)u. \end{cases}$$

In order not to complicate the calculations it will be assumed that all functions belong to the class C^{∞}. In addition we shall require

(35)
$$\begin{cases} c_1(x,t) < c_2(x,t) & \text{in } \bar{D} \\[6pt] c_1(x,t) \text{ uniformly bounded in any strip } \{(x,t) \mid 0 \le x < \infty, \ 0 \le t \le T\} \\ c_2(x,t) \text{ uniformly bounded in any strip } \{(x,t) \mid -\infty < x \le 0, \ 0 \le t \le T\} \\[6pt] a(x,t) > 0 & \text{in } \bar{D} \\[6pt] a(x,t)c_1(x,t) < b(x,t) < a(x,t)c_2(x,t) & \text{in } \bar{D}. \end{cases}$$

The construction of the formal approximation is similar to the construction given in chapter I, section 2 for ordinary differential equations. Expand-ing the coefficients into a Taylor series we obtain for any $n \in \mathbb{N} \cup \{0\}$

$$
\begin{cases}
a(x,t) = \sum_{k=0}^{n} a_k(x)t^k + a_{n+1}(x,t)t^{n+1} \\[2ex]
b(x,t) = \sum_{k=0}^{n} b_k(x)t^k + b_{n+1}(x,t)t^{n+1} \\[2ex]
c_1(x,t) + c_2(x,t) = \sum_{k=0}^{n} \tilde{c}_k(x)t^k + \tilde{c}_{n+1}(x,t)t^{n+1} \\[2ex]
c_1(x,t)c_2(x,t) = \sum_{k=0}^{n} c_k^*(x)t^k + c_{n+1}^*(x,t)t^{n+1} \\[2ex]
d(x,t) = \sum_{k=0}^{n} d_k(x)t^k + d_{n+1}(x,t)t^{n+1}
\end{cases}
$$

where $a_0(x) = a(x,0)$ etc.
We define the differential operator $L^{(\varepsilon)}$ by

$$
L^{(\varepsilon)} = \varepsilon L_2 + L_1.
$$

Putting $\tau = \dfrac{t}{\varepsilon}$ and expanding $L^{(\varepsilon)}$ into powers of ε we obtain

$$
L^{(\varepsilon)} = \varepsilon^{-1} \frac{\partial^2}{\partial \tau^2} + \left\{ \sum_{k=0}^{n} \tilde{c}_k(x)\varepsilon^k\tau^k + \tilde{c}_{n+1}(x,\varepsilon\tau)\varepsilon^{n+1}\tau^{n+1} \right\} \frac{\partial^2}{\partial x \partial \tau}
$$

$$
+ \varepsilon\left\{ \sum_{k=0}^{n} c_k^*(x)\varepsilon^k\tau^k + c_{n+1}^*(x,\varepsilon\tau)\varepsilon^{n+1}\tau^{n+1} \right\} \frac{\partial^2}{\partial x^2}
$$

$$
+ \varepsilon^{-1}\left\{ \sum_{k=0}^{n} a_k(x)\varepsilon^k\tau^k + a_{n+1}(x,\varepsilon\tau)\varepsilon^{n+1}\tau^{n+1} \right\} \frac{\partial}{\partial \tau}
$$

$$
+ \left\{ \sum_{k=0}^{n} b_k(x)\varepsilon^k\tau^k + b_{n+1}(x,\varepsilon\tau)\varepsilon^{n+1}\tau^{n+1} \right\} \frac{\partial}{\partial x}
$$

$$
+ \left\{ \sum_{k=0}^{n} d_k(x)\varepsilon^k\tau^k + d_{n+1}(x,\varepsilon\tau)\varepsilon^{n+1}\tau^{n+1} \right\}.
$$

We shall construct a formal approximation $\tilde{u}(x,t;\varepsilon)$ of $u(x,t;\varepsilon)$ in the form

$$
(36) \qquad \tilde{u}(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(x,t) + \sum_{k=0}^{n} \varepsilon^{k+1} v_k(x,\tau)
$$

where the functions w_k and v_k ($k = 0,1,\ldots,n$) have to be determined. A calculation shows that

$$L^{(\varepsilon)}[\tilde{u}] = L^{(\varepsilon)}\left[\sum_{k=0}^{n} \varepsilon^k w_k\right] + L^{(\varepsilon)}\left[\sum_{k=0}^{n} \varepsilon^{k+1} v_k\right]$$

$$= L_1[w_0] + \left\{\frac{\partial^2 v_0}{\partial \tau^2} + a(x,0)\frac{\partial v_0}{\partial \tau}\right\}$$

(37)
$$+ \sum_{k=1}^{n} \varepsilon^k (L_1[w_k] + L_2[w_{k-1}])$$

$$+ \sum_{k=1}^{n} \varepsilon^k \left\{\frac{\partial^2 v_k}{\partial \tau^2} + a(x,0)\frac{\partial v_k}{\partial \tau} + \Phi_k(x,\tau)\right\}$$

$$+ R(x,t;\varepsilon)$$

where the functions $\Phi_k(x,\tau)$ and $R(x,t;\varepsilon)$ are defined by

$$\Phi_k(x,\tau) = \sum_{j=0}^{k-1} \tilde{c}_j(x)\tau^j \frac{\partial^2 v_{k-j-1}}{\partial x \partial \tau} + \sum_{j=0}^{k-2} c_j^*(x)\tau^j \frac{\partial^2 v_{k-j-2}}{\partial x^2}$$

$$+ \sum_{j=1}^{k} a_j(x)\tau^j \frac{\partial v_{k-j}}{\partial \tau} + \sum_{j=0}^{k-1} b_j(x)\tau^j \frac{\partial v_{k-j-1}}{\partial x}$$

$$+ \sum_{j=0}^{k-1} d_j(x)\tau^j v_{k-j-1}$$

and

$$R(x,t;\varepsilon) = \varepsilon^{n+1} L_2[w_n] + \sum_{k=n}^{2n} \varepsilon^{k+1}\left\{\sum_{j=k-n}^{n} \tilde{c}_j(x)\tau^j \frac{\partial^2 v_{k-j}}{\partial x \partial \tau}\right\}$$

$$+ \varepsilon^{n+1} \sum_{j=0}^{n-1} c_j^*(x)\tau^j \frac{\partial^2 v_{n-j-1}}{\partial x^2}$$

$$+ \sum_{k=n}^{2n} \varepsilon^{k+2}\left\{\sum_{j=k-n}^{n} c_j^*(x)\tau^j \frac{\partial^2 v_{k-j}}{\partial x^2}\right\}$$

$$+ \tilde{c}_{n+1}(x,t)\varepsilon^{n+1}\tau^{n+1} \sum_{k=0}^{n} \varepsilon^{k+1} \frac{\partial^2 v_k}{\partial x \partial \tau} +$$

$$+ c^*_{n+1}(x,t)\varepsilon^{n+1}\tau^{n+1} \sum_{k=0}^{n} \varepsilon^{k+2} \frac{\partial^2 v_k}{\partial x^2}$$

$$+ \sum_{k=n+1}^{2n} \varepsilon^k \{ \sum_{j=k-n}^{n} a_j(x)\tau^j \frac{\partial v_{k-j}}{\partial \tau} \}$$

$$+ a_{n+1}(x,t)\varepsilon^{n+1}\tau^{n+1} \sum_{k=0}^{n} \varepsilon^k \frac{\partial v_k}{\partial \tau}$$

$$+ \sum_{k=n}^{2n} \varepsilon^{k+1} \{ \sum_{j=k-n}^{n} b_j(x)\tau^j \frac{\partial v_{k-j}}{\partial x} \}$$

$$+ b_{n+1}(x,t)\varepsilon^{n+1}\tau^{n+1} \sum_{k=0}^{n} \varepsilon^{k+1} \frac{\partial v_k}{\partial x}$$

$$+ \sum_{k=n}^{2n} \varepsilon^{k+1} \{ \sum_{j=k-n}^{n} d_j(x)\tau^j v_{k-j} \}$$

$$+ d_{n+1}(x,t)\varepsilon^{n+1}\tau^{n+1} \sum_{k=0}^{n} \varepsilon^{k+1} v_k.$$

N.B. Empty sums are to be interpreted as zero.

Putting $t = 0$ in (36) we get

$$(38) \quad \begin{cases} \tilde{u}(x,0;\varepsilon) = w_0(x,0) + \sum_{k=1}^{n} \varepsilon^k \{ w_k(x,0) + v_{k-1}(x,0) \} + \varepsilon^{n+1} v_n(x,0) \\ \tilde{u}_t(x,0;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \{ \frac{\partial w_k}{\partial t}(x,0) + \frac{\partial v_k}{\partial \tau}(x,0) \}. \end{cases}$$

The formal approximation $\tilde{u}$ of the solution u of (34) is now obtained by solving the following set of initial value problems

$$(39) \quad \begin{cases} a(x,t)\dfrac{\partial w_0}{\partial t} + b(x,t)\dfrac{\partial w_0}{\partial x} + d(x,t)w_0 = f(x,t), \quad (x,t) \in D \\ \\ w_0(x,0) = g(x), \quad -\infty < x < \infty \end{cases}$$

$$(40) \quad \begin{cases} \dfrac{\partial^2 v_0}{\partial \tau^2} + a(x,0)\dfrac{\partial v_0}{\partial \tau} = 0, \quad (x,\tau) \in D \\ \dfrac{\partial v_0}{\partial \tau}(x,0) = h(x) - \dfrac{\partial w_0}{\partial t}(x,0), \quad -\infty < x < \infty \\ \lim_{\tau \to \infty} v_0(x,\tau) = 0, \quad -\infty < x < \infty \end{cases}$$

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and for k = 1,2,...,n

$$
(41) \quad
\begin{cases}
a(x,t)\dfrac{\partial w_k}{\partial t} + b(x,t)\dfrac{\partial w_k}{\partial x} + d(x,t)w_k = -L_2[w_{k-1}], & (x,t) \in D \\[3mm]
w_k(x,0) = -v_{k-1}(x,0), & -\infty < x < \infty
\end{cases}
$$

$$
(42) \quad
\begin{cases}
\dfrac{\partial^2 v_k}{\partial \tau^2} + a(x,0)\dfrac{\partial v_k}{\partial \tau} = -\Phi_k(x,\tau), & (x,\tau) \in D \\[3mm]
\dfrac{\partial v_k}{\partial \tau}(x,0) = -\dfrac{\partial w_k}{\partial t}(x,0), & -\infty < x < \infty \\[3mm]
\lim_{\tau \to \infty} v_k(x,\tau) = 0, & -\infty < x < \infty.
\end{cases}
$$

It is easily established that the functions $v_k(x,\tau)$ are of the form

$$
(43) \qquad v_k(x,\tau) = P_k(\tau;x)\exp\{-a(x,0)\tau\}
$$

with $P_k(\tau;x)$ some polynomial in τ with coefficients depending on x; they are typical so-called boundary layer functions.
The formulae (37) and (38) can now be written as

$$
(44) \quad
\begin{cases}
L^{(\varepsilon)}[\tilde{u}] = f(x,t) + R(x,t;\varepsilon), & (x,t) \in D \\[3mm]
\tilde{u}(x,0;\varepsilon) = g(x) + \varepsilon^{n+1}v_n(x,0), & -\infty < x < \infty \\[3mm]
\tilde{u}_t(x,0;\varepsilon) = h(x), & -\infty < x < \infty.
\end{cases}
$$

One easily verifies that for every compact subset $\Omega \subset \bar{D}$

$$
(45) \qquad R(x,t;\varepsilon) = O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0
$$

uniformly valid in Ω. Hence the function $\tilde{u}$ provides a formal approximation (up to order ε^{n+1}) for the solution u of problem (34).

REMARK. Note that so far we did not use the timelike condition $ac_1 < b < ac_2$. Apparently a formal approximation can also be constructed if the subcharacteristics are spacelike. It is only in the proof of the correctness of the formal approximation (which follows presently) that the timelike condition becomes essential.

Introducing the remainder term $z(x,t;\varepsilon)$ by

(46) $z(x,t;\varepsilon) = u(x,t;\varepsilon) - \tilde{u}(x,t;\varepsilon)$

we obtain from (34) and (44)

$$\begin{cases} L^{(\varepsilon)}[z] = -R(x,t;\varepsilon), \qquad (x,t) \in D \\[2mm] z(x,0;\varepsilon) = -\varepsilon^{n+1} v_n(x,0), \qquad -\infty < x < \infty \\[2mm] z_t(x,0;\varepsilon) = 0, \qquad -\infty < x < \infty \end{cases}$$

so using (45) we get from theorem 1

$$\begin{cases} z(x,t;\varepsilon) = 0(\varepsilon^{n+\frac{3}{4}}), \qquad \varepsilon \downarrow 0 \\[3mm] z_x(x,t;\varepsilon) = 0(\varepsilon^n), \qquad \varepsilon \downarrow 0 \\[3mm] z_t(x,t;\varepsilon) = 0(\varepsilon^n), \qquad \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$.
Hence it follows from (36) and (46) that

$$\begin{cases} u(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(x,t) + \sum_{k=0}^{n-1} \varepsilon^{k+1} v_k(x,\tau) + 0(\varepsilon^{n+\frac{3}{4}}), \qquad \varepsilon \downarrow 0 \\[4mm] u_x(x,t;\varepsilon) = \sum_{k=0}^{n-1} \varepsilon^k \frac{\partial w_k}{\partial x}(x,t) + \sum_{k=0}^{n-2} \varepsilon^{k+1} \frac{\partial v_k}{\partial x}(x,\tau) + 0(\varepsilon^n), \qquad \varepsilon \downarrow 0 \\[4mm] u_t(x,t;\varepsilon) = \sum_{k=0}^{n-1} \varepsilon^k \frac{\partial w_k}{\partial t}(x,t) + \sum_{k=0}^{n-1} \varepsilon^k \frac{\partial v_k}{\partial \tau}(x,\tau) + 0(\varepsilon^n), \qquad \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$. Applying these estimates with
n replaced by n+1 we finally obtain

THEOREM 2. *Let* $u(x,t;\varepsilon)$ *be the solution of the initial value problem* (34),
where all data are C^{∞}*-functions which in addition satisfy the conditions*
(35). *Then we have for any* $n \in \mathbb{N} \cup \{0\}$

$$(47) \quad \begin{cases} u(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(x,t) + \sum_{k=0}^{n-1} \varepsilon^{k+1} v_k(x,\tau) + O(\varepsilon^{n+1}), \quad \varepsilon \downarrow 0 \\[2em] u_x(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \frac{\partial w_k}{\partial x}(x,t) + \sum_{k=0}^{n-1} \varepsilon^{k+1} \frac{\partial v_k}{\partial x}(x,\tau) + O(\varepsilon^{n+1}), \quad \varepsilon \downarrow 0 \\[2em] u_t(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \frac{\partial w_k}{\partial t}(x,t) + \sum_{k=0}^{n} \varepsilon^k \frac{\partial v_k}{\partial \tau}(x,\tau) + O(\varepsilon^{n+1}), \quad \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$, *where* $\tau = \dfrac{t}{\varepsilon}$ *and the functions* w_k *and* v_k *($k = 0,1,\ldots,n$) follow successively from* (39), (40), (41) *and* (42).

In case $n = 0$ *empty sums are to be interpreted as zero.*

COROLLARY 1. *It follows from* (47) *that first approximations* ($n = 0$) *for the solution* u *of* (34) *and its first derivatives are given by*

$$\begin{cases} u(x,t;\varepsilon) = w_0(x,t) + O(\varepsilon), \quad \varepsilon \downarrow 0 \\[2em] u_x(x,t;\varepsilon) = \frac{\partial w_0}{\partial x}(x,t) + O(\varepsilon), \quad \varepsilon \downarrow 0 \\[2em] u_t(x,t;\varepsilon) = \frac{\partial w_0}{\partial t}(x,t) + \{h(x) - \frac{\partial w_0}{\partial t}(x,0)\}\exp[-a(x,0)\frac{t}{\varepsilon}] + O(\varepsilon), \end{cases}$$
$$\varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$, *where* w_0 *is the solution of the reduced problem* (39).

COROLLARY 2. *It follows from* (47) *and* (43) *that for any* $n \in \mathbb{N} \cup \{0\}$

$$\begin{cases} u(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(x,t) + O(\varepsilon^{n+1}), \quad \varepsilon \downarrow 0 \\[2em] u_x(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \frac{\partial w_k}{\partial x}(x,t) + O(\varepsilon^{n+1}), \quad \varepsilon \downarrow 0 \\[2em] u_t(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k \frac{\partial w_k}{\partial t}(x,t) + O(\varepsilon^{n+1}), \quad \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in any compact subset $\Omega \subset \{(x,t) \mid -\infty < x < \infty, \; 0 < \delta \leq t < \infty\}$ *where* δ *is an arbitrarily small positive constant independent of* ε.

REMARK. If the condition of infinite smoothness of the data is weakened then the expansions (47) are valid for all $n \in \mathbb{N} \cup \{0\}$ with $n \leq n_0$, the bound n_0 being determined by the differentiability of the data.

4. Discussion

The a priori estimate (33a) that has been established in theorem 1 is, under the conditions stated, the sharpest possible in the sense that the factor $\varepsilon^{-\frac{1}{4}}$ in front of the constant K_1 is the smallest possible. This can easily be verified by considering the initial value problem.

$$
\begin{cases}
\varepsilon \left(\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} \right) + \dfrac{\partial u}{\partial t} = f(x,t;\varepsilon), & (x,t) \in D \\[2ex]
u(x,0;\varepsilon) = u_t(x,0;\varepsilon) = 0, & -\infty < x < \infty
\end{cases}
$$

with

$$
f(x,t;\varepsilon) = \left\{ \varepsilon + t + \left(1 - \dfrac{2x^2}{\varepsilon}\right) t^2 \right\} \exp\left(-\dfrac{x^2}{\varepsilon}\right).
$$

Its solution is given by

$$
u(x,t;\varepsilon) = \dfrac{1}{2} t^2 \exp\left(-\dfrac{x^2}{\varepsilon}\right).
$$

Defining the compact subset Ω by

$$
\Omega = \{(x,t) \mid t - 2 \leq x \leq 2 - t, \ 0 \leq t \leq 1\}
$$

we have

$$
\begin{cases}
\| f \|_\Omega = O(\varepsilon^{\frac{1}{4}}), & \varepsilon \downarrow 0 \\[2ex]
\sup_\Omega |u| = O(1), & \varepsilon \downarrow 0
\end{cases}
$$

so it follows that the factor $\varepsilon^{-\frac{1}{4}}$ in the right-hand side of (33a) is optimal.

The energy integral method presented in section 2 is an extension of a similar method developed by DE JAGER [26], which will now briefly be

discussed. DE JAGER considered the initial value problem

$$
\begin{cases}
\varepsilon\{\dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2}\} + a(x,t)\dfrac{\partial u}{\partial t} + b(x,t)\dfrac{\partial u}{\partial x} = f(x,t), \quad (x,t) \in D \\[2mm]
u(x,0;\varepsilon) = g(x), \qquad -\infty < x < \infty \\[2mm]
u_t(x,0;\varepsilon) = h(x), \qquad -\infty < x < \infty
\end{cases}
$$

under the conditions

$$
\begin{cases}
a(x,t) > 0 \quad \text{in } \bar{D}, \quad c(x,t) > 0 \quad \text{in } \bar{D} \\[2mm]
\dfrac{|b(x,t)|}{a(x,t)} < c(x,t) \quad \text{in } \bar{D}.
\end{cases}
$$

A formal approximation $\tilde{u}$ for the solution u of this problem can be constructed as in section 3. In order to prove the correctness of this approximation one needs an a priori estimate for the remainder term z which is a solution of the initial value problem

(48)
$$
\begin{cases}
\varepsilon\{\dfrac{\partial^2 z}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 z}{\partial x^2}\} + a(x,t)\dfrac{\partial z}{\partial t} + b(x,t)\dfrac{\partial z}{\partial x} = F(x,t;\varepsilon), \quad (x,t) \in D \\[2mm]
z(x,0;\varepsilon) = z_t(x,0;\varepsilon) = 0, \qquad -\infty < x < \infty
\end{cases}
$$

with $F(x,t;\varepsilon) = O(\varepsilon)$, $\varepsilon \downarrow 0$, uniformly valid in any compact subset $\Omega \subset \bar{D}$. Multiplying the differential equation by a special linear combination of z_t and z_x, namely $az_t + bz_x$ (which represents the derivative of z along a subcharacteristic) and integrating the resulting formula over a suitably chosen rectangle $\Omega = \{(x,t) \mid x_1 \le x \le x_2, \ 0 \le t \le T\}$ one obtains the pointwise estimate

(49)
$$
\sup_{\Omega} |z(x,t;\varepsilon)| \le C(\Omega)\varepsilon^{-\frac{1}{2}}\|F\|_{\Omega}
$$

from which the correctness of the formal approximation follows.

Compared to DE JAGER's method the extended energy integral method used in section 2 yields some obvious improvements. Firstly, by multiplying the differential equation (48) not only by $az_t + bz_x$ but also by z we obtain an estimate which is sharper than (49), namely (cf. theorem 1)

$$\sup_{\Omega} \ |z(x,t;\varepsilon)| \ \le \ C(\Omega)\varepsilon^{-\frac{1}{4}}\|F\|_{\Omega},$$

a result which has been shown to be optimal.

Secondly, by integrating over a region with characteristic boundaries instead of over a rectangle it is possible to obtain pointwise estimates for the derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial t}$, namely (cf. theorem 1)

$$\sup_{\Omega} \ |\frac{\partial z}{\partial x}| \ + \ \sup_{\Omega} \ |\frac{\partial z}{\partial t}| \ \le \ C(\Omega)\varepsilon^{-1}\sup_{\Omega} \ |F|.$$

Finally, the extended energy integral method enables us to deal with more general differential equations which compared to (48) contain an additional term $d(x,t)z$ in the left-hand side.

We shall end this section with a discussion of some other work in the same field.

The singularly perturbed initial value problem (1) has also been investigated by BLONDEL [3]. The following general result was established: if the solution u of (1) tends for $\varepsilon \downarrow 0$ pointwise to a continuously differentiable limit function w in a compact subset $\Omega \subset \bar{D}$ which is bounded by the line $t = 0$ and two characteristics (in case of timelike subcharacteristics) or by the line $t = 0$, a characteristic and a subcharacteristic (in case of spacelike subcharacteristics) (see figure 10) then $w \equiv w_0$ in Ω, where w_0 is the solution of the reduced problem.

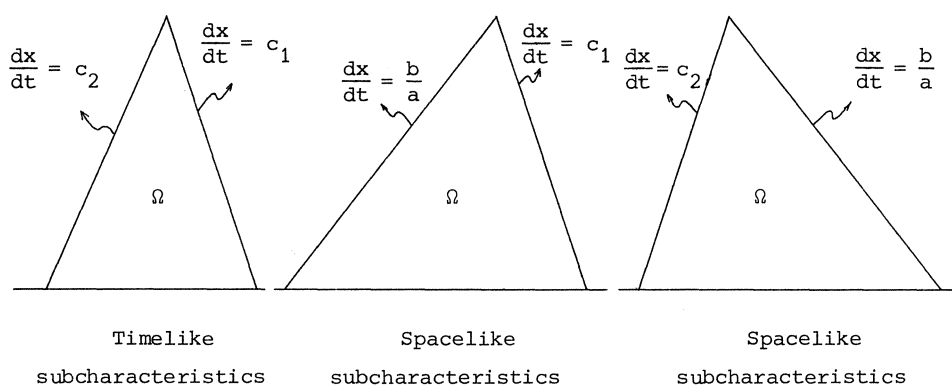|		
Timelike subcharacteristics	Spacelike subcharacteristics	Spacelike subcharacteristics

Figure 10: The region Ω

From this result BLONDEL concluded that the solution u of problem (1) with spacelike subcharacteristics cannot converge pointwise to a C^1-function in a region Ω as described above, unless all data are suitably chosen constants. Since example 2 at page 46 contradicts this conclusion a further investigation of problem (1) with spacelike subcharacteristics is justified.

For timelike subcharacteristics and a positive coefficient a BLONDEL showed that

$$u(x,t;\varepsilon) = w_0(x,t) + O(\varepsilon^{1/8}), \qquad \varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$, where w_0 denotes the solution of the reduced problem. Note that this result is considerably weaker than the result mentioned in corollary 1 at page 70.

It was also shown in [3] that the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ converge in L_2-norm to the corresponding derivatives $\frac{\partial w_0}{\partial x}$ and $\frac{\partial w_0}{\partial t}$ of the solution of the reduced problem as ε tends to zero. However, pointwise convergence of these derivatives remained an open question. This question seems to have been answered for the first time by SMITH & PALMER [46] and SMITH [44] for hyperbolic systems with constant coefficients (see further on). Our results (47) for hyperbolic equations with variable coefficients correspond with those obtained by SMITH and PALMER.

BLONDEL [3], [4] investigated also the behaviour of the solution u of (1) with timelike subcharacteristics and a negative coefficient a. It was shown that if the functions f, g and h in (1) are given by $f(x,t;\varepsilon) \equiv F(x)$, $g(x;\varepsilon) \equiv h(x;\varepsilon) \equiv 0$ with F a smooth positive-valued function independent of t and ε, then the solution u of (1) tends exponentially to infinity in the interior of $\bar{D}$ as ε approaches zero. A consequence of this result is the following: suppose that there exists for suitably chosen functions f, g and h a solution u of (1) which converges pointwise to a limit function as ε tends to zero. Then it suffices to replace f by f + F (with F independent of t and ε, positive and arbitrarily small) in order that the solution of the resulting initial value problem tends exponentially to infinity as ε approaches zero. In other words, the convergence of u is a phenomenon which is unstable with respect to the right-hand side function f.

We strongly believe that the solution u of problem (1) with a negative coefficient a tends exponentially to infinity in every point (x,t) with

$t \geq \delta > 0$ as ε tends to zero, regardless whether the subcharacteristics are spacelike or timelike (cf. chapter I, section 3); an exception has to be made for very special choices of the functions f, g and h (for instance $f \equiv g \equiv h \equiv 0$). However, so far we have not been able to prove this conjecture.

Singularly perturbed initial value problems in more than one space variable have been studied by BLONDEL [4] and DZAVADOV [7]. It was shown by these authors that the solution u and its first derivatives converge in L_2-norm to the solution w_0 of the reduced problem and the corresponding derivatives of w_0 as ε tends to zero.

SMITH [44] studied the singularly perturbed hyperbolic system

$$(50) \quad \begin{cases} \varepsilon(\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}) + A_1 u + B_1 v = f_1(x,t), & (x,t) \in D \\[2ex] \varepsilon(\dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x}) + A_2 u + B_2 v = f_2(x,t), & (x,t) \in D \end{cases}$$

with $u(x,0;\varepsilon)$ and $v(x,0;\varepsilon)$ prescribed for $-\infty < x < \infty$.
It was shown that if the constant coefficients A_1, B_1, A_2 and B_2 satisfy the conditions

$$(51) \qquad A_1 \geq 0, \quad B_2 \geq 0, \quad A_1 + B_2 > 0, \quad 0 \leq A_2 B_1 \leq A_1 B_2$$

then the solution u, v of (50) is for small values of ε approximately given by a solution of the reduced problem obtained from (50) by putting $\varepsilon = 0$.

A simplified version of (50) with $A_1 = B_1 = A_2 = B_2 = \dfrac{1}{2}$ and $f_1 = f_2 = f$ was investigated by SMITH & PALMER [46].

There is an overlap between our results and those of SMITH. One easily verifies that with the special choice

$$\begin{cases} A_1 = A_2 = \dfrac{1}{2}(a - b), \qquad B_1 = B_2 = \dfrac{1}{2}(a + b) \\[2ex] f_1(x,t) = f_2(x,t) = f(x,t) \\[2ex] u = \dfrac{\partial \phi}{\partial t} - \dfrac{\partial \phi}{\partial x}, \qquad v = \dfrac{\partial \phi}{\partial t} + \dfrac{\partial \phi}{\partial x} \end{cases}$$

(50) is equivalent to the second order equation

$$\varepsilon(\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2}) + a\frac{\partial\phi}{\partial t} + b\frac{\partial\phi}{\partial x} = f(x,t), \qquad (x,t) \in D$$

with $\phi(x,0;\varepsilon)$ and $\phi_t(x,0;\varepsilon)$ prescribed for $-\infty < x < \infty$.
In this case the conditions (51) take the form

$$a > 0, \quad |b| \leq a$$

so for constant coefficients our results correspond with those of SMITH.

Results obtained by means of a comparison technique have recently been published by WEINSTEIN & SMITH [55].

The a priori estimates obtained in theorem 1 will play an important role in the investigations of nonlinear initial value problems. We shall deal with these problems in chapter IV.
Moreover, the energy integral method underlying the proof of theorem 1 can be modified to yield a priori estimates for solutions of singularly perturbed initial-boundary value problems. The chapters V and VI will be devoted to this subject.

INITIAL VALUE PROBLEMS FOR NONLINEAR

HYPERBOLIC DIFFERENTIAL EQUATIONS

In this chapter we shall extend the linear theory developed in chapter III to nonlinear initial value problems. In particular we shall investigate the quasilinear problems

$$(1) \quad \begin{cases} \varepsilon L_2[u] + M_1^{(\varepsilon)}[u] = f(x,t;\varepsilon), & (x,t) \in D \\ \\ u(x,0;\varepsilon) = u_t(x,0;\varepsilon) = 0, & -\infty < x < \infty \end{cases}$$

and

$$(2) \quad \begin{cases} \varepsilon L_2[u] + M_1[u] = 0, & (x,t) \in D \\ \\ u(x,0;\varepsilon) = f(x), & -\infty < x < \infty \\ \\ u_t(x,0;\varepsilon) = g(x), & -\infty < x < \infty \end{cases}$$

where ε is a small positive parameter, i.e. $0 < \varepsilon \ll 1$, and the region D is the upper half-plane

$$D = \{(x,t) \mid -\infty < x < \infty, \; 0 < t < \infty\}.$$

Here the hyperbolic operator L_2 is defined by

$$(3) \qquad L_2[u] = \frac{\partial^2 u}{\partial t^2} - c^2(x,t)\frac{\partial^2 u}{\partial x^2}$$

with $c(x,t) > 0$ in $\bar{D}$, and the quasilinear operators $M_1^{(\varepsilon)}$ and M_1 are given by

(4a)
$$M_1^{(\varepsilon)}[u] = a(x,t,u;\varepsilon)\frac{\partial u}{\partial t} + b(x,t,u;\varepsilon)\frac{\partial u}{\partial x} + d(x,t,u;\varepsilon)$$

(4b)
$$M_1[u] = a(x,t,u)\frac{\partial u}{\partial t} + b(x,t,u)\frac{\partial u}{\partial x} + d(x,t,u).$$

Under appropriate conditions on the data we shall establish in section 1 the existence of a solution for problem (1); among other things it will be required that

(5)
$$\varepsilon L_2[0] + M_1^{(\varepsilon)}[0] = 0$$

and also that the right-hand side function $f(x,t;\varepsilon)$ is of sufficiently small order in ε. The existence proof is similar to the proof that we gave in section 1 of chapter II for ordinary differential equations: using a priori estimates for the solution of the linearized problem (which can be obtained from theorem 1 in chapter III) we can apply theorem 1 of chapter II which leads to the existence of a solution for problem (1), together with an estimate for this solution.

In section 2 we shall construct a formal approximation $\tilde{u}$ for the solution u of (2). The correctness of this approximation will then be verified by applying the results of section 1 to the remainder term.

Section 3 will be devoted to a further discussion of the construction of the formal approximation.

In section 4 we shall deal with two special cases of problem (2) in which the coefficients a and b are independent of u.

REMARK. It is possible to consider the problems (1) and (2) with a more general hyperbolic operator L_2 defined by

(6)
$$L_2[u] = \frac{\partial^2 u}{\partial t^2} + \{c_1(x,t) + c_2(x,t)\}\frac{\partial^2 u}{\partial x \partial t} + c_1(x,t)c_2(x,t)\frac{\partial^2 u}{\partial x^2}$$

where $c_1(x,t) < c_2(x,t)$ in $\bar{D}$ (cf. chapter III). Although we restrict ourselves for reasons of simplicity to the case where L_2 represents the wave operator, that is

$$-c_1(x,t) \equiv c_2(x,t) \equiv c(x,t) > 0 \qquad \text{in } \bar{D},$$

it is emphasized that more general problems with L_2 given by (6) can be

treated in a similar way.

1. Existence and a priori estimates

We consider the initial value problem (1) where the operators L_2 and $M_1^{(\epsilon)}$ are defined by (3) and (4a). In order to comply with the condition (5) (cf. theorem 1 of chapter II) we assume that

(7) $d(x,t,0;\epsilon) \equiv 0$ for $(x,t) \in \bar{D}$ and $0 < \epsilon \ll 1$.

This assumption can be made without loss of generality. Apart from (7) we impose the following conditions on the data:

(8)
$$
\begin{cases}
c(x,t) \in C^1(\bar{D}) \\[2ex]
c(x,t) > 0 \quad \text{in } \bar{D} \\[2ex]
c(x,t) \text{ is uniformly bounded in any strip} \\
\{(x,t) \mid -\infty < x < \infty, \ 0 \le t \le T\}
\end{cases}
$$

(9)
$$
\begin{cases}
\text{for each compact subset } \Omega \subset \bar{D} \text{ there exists a positive constant } \epsilon_1 \\
\text{(depending on } \Omega) \text{ such that} \\[2ex]
\text{(i)} \quad a(x,t,0;\epsilon) \in C^1(\Omega), \quad b(x,t,0;\epsilon) \in C^1(\Omega) \\
\qquad d_u(x,t,0;\epsilon) \in C^0(\Omega), \quad f(x,t;\epsilon) \in C^0(\Omega) \\[2ex]
\qquad \text{valid for every value of } \epsilon \text{ satisfying } 0 < \epsilon \le \epsilon_1 \\[2ex]
\text{(ii)} \quad \text{the functions } a(x,t,0;\epsilon), \ b(x,t,0;\epsilon) \text{ and their first derivatives} \\
\qquad \text{as well as the function } d_u(x,t,0;\epsilon) \text{ are uniformly bounded in } \Omega \\
\qquad \text{for } 0 < \epsilon \le \epsilon_1 \\[2ex]
\text{(iii)} \quad a(x,t,0;\epsilon) \ge a_0 > 0 \\[2ex]
\qquad \text{valid for all } (x,t) \in \Omega \text{ and all } \epsilon \in (0,\epsilon_1], \text{ where } a_0 \text{ is some} \\
\qquad \text{constant depending on } \Omega \text{ and } \epsilon_1 \text{ but independent of } \epsilon
\end{cases}
$$

(9)

(iv) $\begin{cases} a(x,t,0;\varepsilon)c(x,t) + b(x,t,0;\varepsilon) \geq p > 0 \\ a(x,t,0;\varepsilon)c(x,t) - b(x,t,0;\varepsilon) \geq p > 0 \end{cases}$

   valid for all $(x,t) \in \Omega$ and all $\varepsilon \in (0,\varepsilon_1]$, where p is some constant depending on Ω and ε_1 but independent of ε

(10)

for each compact subset $\Omega \subset \bar{D}$ there exist a positive constant U_0 and a positive constant ε_2 (depending on Ω and U_0) such that

(i) $a(x,t,u;\varepsilon) \in C^0(\Omega)$, $b(x,t,u;\varepsilon) \in C^0(\Omega)$, $d(x,t,u;\varepsilon) \in C^0(\Omega)$

   valid for all $u \in [-U_0,U_0]$ and all $\varepsilon \in (0,\varepsilon_2]$

(ii)   the derivatives $a_u(x,t,u;\varepsilon)$ and $b_u(x,t,u;\varepsilon)$ exist and are uniformly bounded in $\Omega \times [-U_0,U_0]$ for $0 < \varepsilon \leq \varepsilon_2$

(iii)   the derivative $d_u(x,t,u;\varepsilon)$ exists in $\Omega \times [-U_0,U_0]$ for $0 < \varepsilon \leq \varepsilon_2$ and satisfies a Lipschitz condition with exponent $\alpha > 0$, viz.

   $$|d_u(x,t,u;\varepsilon) - d_u(x,t,0;\varepsilon)| \leq C(\Omega,U_0,\varepsilon_2)|u|^\alpha$$

   valid for all $(x,t,u) \in \Omega \times [-U_0,U_0]$ and all $\varepsilon \in (0,\varepsilon_2]$. Here $C(\Omega,U_0,\varepsilon_2)$ is some positive constant depending on Ω, U_0 and ε_2 but independent of x, t, u or ε.

In order to establish the existence of a solution for problem (1) we consider a compact subset $\Omega \subset \bar{D}$ which is bounded by the lines $t = 0$ and $t = T$ ($T > 0$) and by two characteristics of the operator L_2, one with a negative slope, the other with a positive slope (see figure 1).
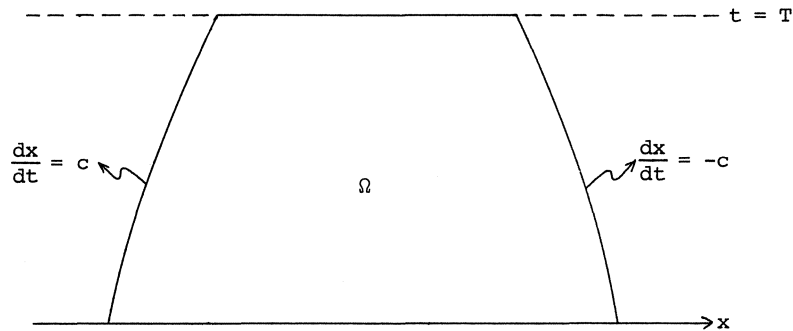
Figure 1

The existence proof will be based on theorem 1 of chapter II. In terms of this theorem we introduce for any fixed value of ε with

$$0 < \varepsilon \le \varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$$ the Banach space

$$B = \{u \mid u \in C^0(\Omega)\}$$

with norm

$$|u|_0 = \sup_{\Omega} |u|$$

and the normed linear space

(11) $$R = \{u \mid u \in C^1(\Omega), \; L_2[u] \in C^0(\Omega), \; u(x,0) = u_t(x,0) = 0\}$$

with norm

(12) $$|u|_1^{(\varepsilon)} = \sup_{\Omega} |u| + \varepsilon^{\frac{3}{4}} \sup_{\Omega} |\frac{\partial u}{\partial x}| + \varepsilon^{\frac{3}{4}} \sup_{\Omega} |\frac{\partial u}{\partial t}|.$$

We state here explicitly that the expression $L_2[u]$ in (11) may contain weak second derivatives.

Let $\mathcal{B}(\rho)$ be a ball in R with radius ρ, that is

$$\mathcal{B}(\rho) = \{u \in R \mid |u|_1^{(\varepsilon)} \le \rho\}.$$

Choosing $\bar{\rho} = U_0$ we define the operators $F: \mathcal{B}(\bar{\rho}) \to B$, $L: R \to B$ and $\Psi: \mathcal{B}(\bar{\rho}) \to B$ by

$$
\begin{cases}
F[u] = \varepsilon L_2[u] + M_1^{(\varepsilon)}[u] \\[2ex]
L[u] = \varepsilon L_2[u] + a(x,t,0;\varepsilon)\dfrac{\partial u}{\partial t} + b(x,t,0;\varepsilon)\dfrac{\partial u}{\partial x} + d_u(x,t,0;\varepsilon)u \\[2ex]
\Psi[u] = \{a(x,t,u;\varepsilon)-a(x,t,0;\varepsilon)\}\dfrac{\partial u}{\partial t} + \{b(x,t,u;\varepsilon)-b(x,t,0;\varepsilon)\}\dfrac{\partial u}{\partial x} \\[2ex]
\hphantom{\Psi[u] = } + d(x,t,u;\varepsilon) - d_u(x,t,0;\varepsilon)u.
\end{cases}
$$

Using the conditions (8) and (9) we obtain from theorem 1 in chapter III

$$
|L^{-1}[v]|_1^{(\varepsilon)} \le M(\Omega)\varepsilon^{-\frac{1}{4}}|v|_0
$$

valid for all $\varepsilon \in (0,\varepsilon_0]$ and all $v \in B$, where $M(\Omega)$ is some positive constant depending on Ω but independent of ε, and ε_0 is some generic constant. So in terms of theorem 1 in chapter II we have

$$
\ell = \frac{\varepsilon^{\frac{1}{4}}}{M(\Omega)} .
$$

It can easily be verified that the operator Ψ satisfies a Lipschitz condition. Using the conditions (10) we have for every $\rho \in [0,\bar{\rho}]$ and every $u,v \in R$ with $|u|_1^{(\varepsilon)} \le \rho$, $|v|_1^{(\varepsilon)} \le \rho$ the following relation

$$
\Psi[u] - \Psi[v] =
$$

$$
\{a(x,t,u;\varepsilon)-a(x,t,0;\varepsilon)\}(u_t-v_t) + v_t\{a(x,t,u;\varepsilon)-a(x,t,v;\varepsilon)\}
$$

$$
+ \{b(x,t,u;\varepsilon)-b(x,t,0;\varepsilon)\}(u_x-v_x) + v_x\{b(x,t,u;\varepsilon)-b(x,t,v;\varepsilon)\}
$$

$$
+ d(x,t,u;\varepsilon) - d(x,t,v;\varepsilon) - d_u(x,t,0;\varepsilon)(u-v) =
$$

$$
ua_u(x,t,\Theta_1 u;\varepsilon)(u_t-v_t) + v_t a_u(x,t,v+\Theta_2(u-v);\varepsilon)(u-v)
$$

$$
+ ub_u(x,t,\Theta_3 u;\varepsilon)(u_x-v_x) + v_x b_u(x,t,v+\Theta_4(u-v);\varepsilon)(u-v)
$$

$$
+ \{d_u(x,t,v+\Theta_5(u-v);\varepsilon)-d_u(x,t,0;\varepsilon)\}(u-v)
$$

with $0 < \Theta_i < 1$ $(i = 1,2,\ldots,5)$. Hence we obtain, again using (10)

$$|\Psi[u] - \Psi[v]|_0 \leq A(\Omega)\varepsilon^{-\frac{3}{4}}\rho\{\varepsilon^{\frac{3}{4}}\sup_{\Omega}|u_x-v_x| + \varepsilon^{\frac{3}{4}}\sup_{\Omega}|u_t-v_t|\}$$

$$+ B(\Omega)\max(\varepsilon^{-\frac{3}{4}}\rho,\rho^{\alpha})\sup_{\Omega}|u-v|$$

$$\leq C(\Omega)\max(\varepsilon^{-\frac{3}{4}}\rho,\rho^{\alpha})|u-v|_1 \qquad \overset{(\varepsilon)}{}$$

valid for $0 < \varepsilon \leq \varepsilon_0$. Here $A(\Omega)$, $B(\Omega)$ and $C(\Omega)$ are positive constants which depend (apart from U_0 and ε_0) on the region Ω. They do not, however, depend on ε.

Using the terminology of theorem 1 in chapter II we conclude from the fore-going that

$$m(\rho) = C(\Omega)\max(\varepsilon^{-\frac{3}{4}}\rho,\rho^{\alpha}).$$

N.B. The following simplifications are easily established:

if a and b do not depend on u then $m(\rho) = C(\Omega)\rho^{\alpha}$;

if d is linear in u then $m(\rho) = C(\Omega)\varepsilon^{-\frac{3}{4}}\rho$.

Solving $m(\rho_0) = \frac{1}{2}\ell$ we obtain

$$\rho_0 = N(\Omega)\varepsilon^{\lambda}$$

with

(13) $$\lambda = \begin{cases} 1 & \text{if d is linear in u} \\ \dfrac{1}{4\alpha} & \text{if a and b do not depend on u} \\ \max(1,\dfrac{1}{4\alpha}) & \text{otherwise} \end{cases}$$

and $N(\Omega)$ some positive constant depending on Ω but independent of ε. Finally, applying theorem 1 of chapter II we obtain

THEOREM 1. *Let Ω be some compact subset of $\bar{D}$ as shown in figure 1. Then there exist positive constants ε_0, $M(\Omega)$ and $N(\Omega)$ (depending on Ω but in-dependent of ε) such that for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$ problem (1) under*

the conditions (7), (8), (9) *and* (10) *has a solution* u ∈ R, *provided that*
f *is sufficiently small, viz.*

(14) $\sup_{\Omega} |f| \leq \frac{N(\Omega)}{2M(\Omega)} \varepsilon^{\lambda+\frac{1}{4}}$.

Here the normed linear space R *is given by* (11) *and* (12), *and the positive*
constant λ *is given by* (13).
The solution u *satisfies the estimate*

$$|u|_1^{(\varepsilon)} \leq 2M(\Omega)\varepsilon^{-\frac{1}{4}}\sup_{\Omega} |f|.$$

REMARKS.

1. Since theorem 1 of chapter II applies to nonlinear operators F mapping
   a (ball belonging to a) normed linear space into a Banach space it was
   sufficient to verify that the linear space R defined by (11) and (12)
   is a normed linear space; we did not have to prove the linear space R
   (supplied with some suitable norm) to be a Banach space. Such a proof
   would have been necessary if we had applied the theory developed by
   VAN HARTEN [23, chapter 7] for nonlinear operators F mapping a (linear
   subspace of a) Banach space into a normed linear space.

2. Note that in theorem 1 the functions $a(x,t,u;\varepsilon)$, $b(x,t,u;\varepsilon)$ and
   $d(x,t,u;\varepsilon)$ are allowed to have singularities in u (cf. chapter II).

3. A similar theorem can be formulated for more general nonlinear initial
   value problems of the type

$$\begin{cases} \varepsilon L_2[u] + F(x,t,\vec{u};\varepsilon) = f(x,t;\varepsilon), & (x,t) \in D \\[2mm] u(x,0;\varepsilon) = u_t(x,0;\varepsilon) = 0, & -\infty < x < \infty \end{cases}$$

(where $\vec{u}$ denotes the vector (u,u_t,u_x)) provided that suitable conditions
are imposed on the nonlinear function F, namely

(i) $F(x,t,\vec{0};\varepsilon) \equiv 0$ for $(x,t) \in \bar{D}$ and $0 < \varepsilon \ll 1$,

(ii)   in order that theorem 1 of chapter III is applicable to the
       linearized problem one must require that (9) holds, with
       $a(x,t,0;\varepsilon)$, $b(x,t,0;\varepsilon)$ and $d_u(x,t,0;\varepsilon)$ replaced by $F_{u_t}(x,t,\vec{0};\varepsilon)$,

$$F_{u_x}(x,t,\vec{0};\epsilon) \quad \text{resp.} \quad F_u(x,t,\vec{0};\epsilon),$$

(iii) in order that the nonlinear part Ψ given by

$$\Psi(x,t,\vec{u};\epsilon) = F(x,t,\vec{u};\epsilon) - F_{u_t}(x,t,\vec{0};\epsilon)\frac{\partial u}{\partial t} - F_{u_x}(x,t,\vec{0};\epsilon)\frac{\partial u}{\partial x}$$

$$- F_u(x,t,\vec{0};\epsilon)u$$

satisfies a Lipschitz condition it is sufficient to require that there exist for every compact subset $\Omega \subset \bar{D}$ positive constants U_0 and ϵ_0 (depending on Ω) such that the function F and its second derivatives with respect to u, u_t and u_x are continuous and uniformly bounded for $(x,t) \in \Omega$, $|u| \leq U_0$, $|u_t| \leq U_0$, $|u_x| \leq U_0$ and $0 < \epsilon \leq \epsilon_0$.

2. Asymptotic expansions

As an application of the a priori estimates obtained in the foregoing section we shall construct an asymptotic expansion into powers of ϵ for the solution u of the quasilinear initial value problem

(15)
$$\begin{cases} L^{(\epsilon)}[u] \equiv \epsilon L_2[u] + M_1[u] = 0, & (x,t) \in D \\ u(x,0;\epsilon) = f(x), & -\infty < x < \infty \\ u_t(x,0;\epsilon) = g(x), & -\infty < x < \infty \end{cases}$$

where the operators L_2 and M_1 are defined by (3) and (4b).
For simplicity it will be assumed that all data are C^∞-functions, that is

(16)
$$\begin{cases} a(x,t,u) \in C^\infty(\bar{D} \times \mathbb{R}), & b(x,t,u) \in C^\infty(\bar{D} \times \mathbb{R}), \\ c(x,t) \in C^\infty(\bar{D}), & d(x,t,u) \in C^\infty(\bar{D} \times \mathbb{R}), \\ f(x) \in C^\infty(\mathbb{R}), & g(x) \in C^\infty(\mathbb{R}). \end{cases}$$

In addition we require that

$$
(17) \quad
\begin{cases}
c(x,t) > 0 \qquad \text{in } \bar{D} \\[2ex]
c(x,t) \text{ is uniformly bounded in any strip} \\
\{(x,t) \mid -\infty < x < \infty, \ 0 \le t \le T\} \\[2ex]
a(x,t,u) > 0 \qquad \text{in } \bar{D} \times \mathbb{R} \\[2ex]
|b(x,t,u)| < a(x,t,u)c(x,t) \qquad \text{in } \bar{D} \times \mathbb{R}.
\end{cases}
$$

A formal approximation $\tilde{u}$ for the solution u of problem (15) can be constructed as follows: we put

$$
(18) \qquad \tilde{u}(x,t;\varepsilon) = w_0(x,t) + \varepsilon v_0(x,\tau) + \varepsilon w_1(x,t) + \varepsilon^2 v_1(x,\tau)
$$

where $\tau = \dfrac{t}{\varepsilon}$ and w_k and v_k $(k = 0,1)$ are functions which have to be determined. For the moment we shall assume that the functions w_k and v_k $(k = 0,1)$ have the following properties

$$
(19) \quad
\begin{cases}
w_k(x,t) \in C^\infty(\bar{D}), \qquad v_k(x,\tau) \in C^\infty(\bar{D}) \\[2ex]
\text{for any } n,m \in \mathbb{N} \cup \{0\} \text{ and any polynomial } P(\tau;x) \text{ in } \tau \text{ with} \\
\text{coefficients depending continuously on } x \text{ the expression} \\[2ex]
P(\tau;x)\dfrac{\partial^{n+m} v_k}{\partial\tau^n \partial x^m} \\[2ex]
\text{is uniformly bounded in every semi-infinite strip} \\
x_1 \le x \le x_2, \ 0 \le \tau < \infty.
\end{cases}
$$

We shall verify the validity of these properties later on, when the functions w_k and v_k have been determined explicitly.

Expanding the coefficient $a(x,t,\tilde{u})$ we obtain

$$
a(x,t,\tilde{u}) = a(x,t,w_0 + \varepsilon v_0 + \varepsilon w_1 + \varepsilon^2 v_1)
$$

$$
= a(x,t,w_0) + \varepsilon(v_0 + w_1)\frac{\partial a}{\partial u}(x,t,w_0) + O(\varepsilon^2)
$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$.

Using

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial w_0}{\partial t} + \frac{\partial v_0}{\partial \tau} + \varepsilon \frac{\partial w_1}{\partial t} + \varepsilon \frac{\partial v_1}{\partial \tau}$$

we obtain

$$a(x,t,\tilde{u})\frac{\partial \tilde{u}}{\partial t} = a(x,t,w_0)\frac{\partial w_0}{\partial t} + a(x,t,w_0)\frac{\partial v_0}{\partial \tau}$$

$$+ \varepsilon[a(x,t,w_0)\frac{\partial w_1}{\partial t} + a(x,t,w_0)\frac{\partial v_1}{\partial \tau}$$

$$+ (w_1 + v_0)(\frac{\partial w_0}{\partial t} + \frac{\partial v_0}{\partial \tau})\frac{\partial a}{\partial u}(x,t,w_0)]$$

$$+ 0(\varepsilon^2)$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$.

So putting $t = \varepsilon\tau$ and expanding into powers of ε we arrive at

$$a(x,t,\tilde{u})\frac{\partial \tilde{u}}{\partial t} =$$

$$a(x,t,w_0)\frac{\partial w_0}{\partial t} + a(x,0,w_0(x,0))\frac{\partial v_0}{\partial \tau}$$

$$+ \varepsilon[\{\frac{\partial a}{\partial t}(x,0,w_0(x,0)) + \frac{\partial a}{\partial u}(x,0,w_0(x,0))\frac{\partial w_0}{\partial t}(x,0)\}\tau\frac{\partial v_0}{\partial \tau}$$

$$+ a(x,t,w_0)\frac{\partial w_1}{\partial t} + a(x,0,w_0(x,0))\frac{\partial v_1}{\partial \tau}$$

$$+ \frac{\partial a}{\partial u}(x,t,w_0)w_1\frac{\partial w_0}{\partial t} + \frac{\partial a}{\partial u}(x,0,w_0(x,0))w_1(x,0)\frac{\partial v_0}{\partial \tau}$$

$$+ \frac{\partial a}{\partial u}(x,0,w_0(x,0))\frac{\partial w_0}{\partial t}(x,0)v_0 + \frac{\partial a}{\partial u}(x,0,w_0(x,0))v_0\frac{\partial v_0}{\partial \tau}]$$

$$+ 0(\varepsilon^2)$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$.

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N.B. Unless otherwise stated the arguments are x,t or x,τ. So w_0 stands for $w_0(x,t)$, v_0 stands for $v_0(x,\tau)$ etc.

A similar expansion of the other terms in $L^{(\varepsilon)}[\tilde{u}]$ leads to the following expansion of the operator:

$$
\begin{aligned}
L^{(\varepsilon)}[\tilde{u}] = {} & a(x,t,w_0)\frac{\partial w_0}{\partial t} + b(x,t,w_0)\frac{\partial w_0}{\partial x} + d(x,t,w_0) \\
& + \frac{\partial^2 v_0}{\partial \tau^2} + a(x,0,w_0(x,0))\frac{\partial v_0}{\partial \tau} \\
& + \varepsilon[a(x,t,w_0)\frac{\partial w_1}{\partial t} + b(x,t,w_0)\frac{\partial w_1}{\partial x} + \tilde{d}(x,t,w_0)w_1 + \Phi_1(x,t)] \\
& + \varepsilon[\frac{\partial^2 v_1}{\partial \tau^2} + a(x,0,w_0(x,0))\frac{\partial v_1}{\partial \tau} + \Psi_1(x,\tau)] + O(\varepsilon^2)
\end{aligned}
$$

(20)

uniformly valid in each compact subset $\Omega \subset \bar{D}$, where

(21)

$$
\begin{cases}
\tilde{d}(x,t,w_0) = \dfrac{\partial a}{\partial u}(x,t,w_0)\dfrac{\partial w_0}{\partial t} + \dfrac{\partial b}{\partial u}(x,t,w_0)\dfrac{\partial w_0}{\partial x} + \dfrac{\partial d}{\partial u}(x,t,w_0) \\[2ex]
\Phi_1(x,t) = \dfrac{\partial^2 w_0}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 w_0}{\partial x^2} \\[2ex]
\Psi_1(x,\tau) = \{\dfrac{\partial a}{\partial u}(x,0,w_0(x,0))\dfrac{\partial w_0}{\partial t}(x,0) + \dfrac{\partial b}{\partial u}(x,0,w_0(x,0))\dfrac{\partial w_0}{\partial x}(x,0) \\[2ex]
\qquad\qquad\qquad\qquad + \dfrac{\partial d}{\partial u}(x,0,w_0(x,0))\}v_0 \\[2ex]
\quad + [\{\dfrac{\partial a}{\partial t}(x,0,w_0(x,0)) + \dfrac{\partial a}{\partial u}(x,0,w_0(x,0))\dfrac{\partial w_0}{\partial t}(x,0)\}\tau \\[2ex]
\qquad\qquad\qquad + \dfrac{\partial a}{\partial u}(x,0,w_0(x,0))w_1(x,0)]\dfrac{\partial v_0}{\partial \tau} \\[2ex]
\quad + b(x,0,w_0(x,0))\dfrac{\partial v_0}{\partial x} + \dfrac{\partial a}{\partial u}(x,0,w_0(x,0))v_0\dfrac{\partial v_0}{\partial \tau}.
\end{cases}
$$

The functions w_k and v_k ($k = 0,1$) can now be obtained by solving the following set of initial value problems:

$$(22) \quad \begin{cases} a(x,t,w_0)\dfrac{\partial w_0}{\partial t} + b(x,t,w_0)\dfrac{\partial w_0}{\partial x} + d(x,t,w_0) = 0, & (x,t) \in D \\[2ex] w_0(x,0) = f(x), & -\infty < x < \infty \end{cases}$$

$$(23) \quad \begin{cases} \dfrac{\partial^2 v_0}{\partial \tau^2} + a(x,0,w_0(x,0))\dfrac{\partial v_0}{\partial \tau} = 0, & (x,\tau) \in D \\[2ex] \dfrac{\partial v_0}{\partial \tau}(x,0) = g(x) - \dfrac{\partial w_0}{\partial t}(x,0), & -\infty < x < \infty \\[2ex] \lim_{\tau \to \infty} v_0(x,\tau) = 0, & -\infty < x < \infty \end{cases}$$

$$(24) \quad \begin{cases} a(x,t,w_0)\dfrac{\partial w_1}{\partial t} + b(x,t,w_0)\dfrac{\partial w_1}{\partial x} + \tilde{d}(x,t,w_0)w_1 = -\Phi_1(x,t), & (x,t) \in D \\[2ex] w_1(x,0) = -v_0(x,0), & -\infty < x < \infty \end{cases}$$

$$(25) \quad \begin{cases} \dfrac{\partial^2 v_1}{\partial \tau^2} + a(x,0,w_0(x,0))\dfrac{\partial v_1}{\partial \tau} = -\Psi_1(x,\tau), & (x,\tau) \in D \\[2ex] \dfrac{\partial v_1}{\partial \tau}(x,0) = -\dfrac{\partial w_1}{\partial t}(x,0), & -\infty < x < \infty \\[2ex] \lim_{\tau \to \infty} v_1(x,\tau) = 0, & -\infty < x < \infty. \end{cases}$$

Note that only the first approximation w_0 follows from the solution of a nonlinear problem; the remaining functions v_0, w_1 and v_1 satisfy linear equations which can easily be solved.

In order to verify the properties (19) we make the crucial assumption

$$(26) \qquad w_0(x,t) \in C^\infty(\bar{D})$$

which is only true in case the reduced quasilinear equation (22) is of a very special kind. The regularity of $w_0(x,t)$ for general quasilinear first order differential equations will be discussed in section 3.

It follows from (26) that the conditions (19) are satisfied. In particular,

the functions v_0 and v_1 have the form

$$
\begin{cases}
v_0(x,\tau) = \dfrac{\dfrac{\partial w_0}{\partial t}(x,0) - g(x)}{a(x,0,w_0(x,0))} \exp[-a(x,0,w_0(x,0))\tau] \\[4mm]
v_1(x,\tau) = P_1(\tau;x)\exp[-a(x,0,w_0(x,0))\tau] \\[4mm]
\qquad\qquad + P_2(\tau;x)\exp[-2a(x,0,w_0(x,0))\tau]
\end{cases}
$$

with $P_k(\tau;x)$ $(k = 1,2)$ polynomials in τ with coefficients depending on x. So we see that v_0 and v_1 are typical boundary layer functions.

Finally, we obtain from (18), (20), (22), (23), (24) and (25) the result

$$
\begin{cases}
L^{(\varepsilon)}[\tilde{u}] = R(x,t;\varepsilon), \qquad (x,t) \in D \\[3mm]
\tilde{u}(x,0;\varepsilon) = f(x) + \varepsilon^2 v_1(x,0), \qquad -\infty < x < \infty \\[3mm]
\tilde{u}_t(x,0;\varepsilon) = g(x), \qquad -\infty < x < \infty
\end{cases}
$$

where $R(x,t;\varepsilon)$ is some C^∞-function satisfying

$$
R(x,t;\varepsilon) = 0(\varepsilon^2), \qquad \varepsilon \downarrow 0
$$

uniformly in any compact subset $\Omega \subset \bar{D}$.

Hence the function $\tilde{u}(x,t;\varepsilon)$ defined by (18), (21), (22), (23), (24) and (25) provides a formal approximation (up to order ε^2) for the solution u of problem (15).

Higher order approximations can be obtained by considering

$$
(27) \qquad \tilde{u}(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(x,t) + \sum_{k=0}^{n} \varepsilon^{k+1} v_k(x,\tau), \qquad n \in \mathbb{N}, \ n \geq 2.
$$

In that case w_k and v_k $(k = 0,1)$ are given by (21), (22), (23), (24) and (25), whereas w_k and v_k $(k \geq 2)$ follow from the linear problems

$$(28) \quad \begin{cases} a(x,t,w_0)\dfrac{\partial w_k}{\partial t} + b(x,t,w_0)\dfrac{\partial w_k}{\partial x} + \tilde{d}(x,t,w_0)w_k = -\Phi_k(x,t), \quad (x,t) \in D \\[2mm] w_k(x,0) = -v_{k-1}(x,0), \qquad -\infty < x < \infty \end{cases}$$

respectively

$$(29) \quad \begin{cases} \dfrac{\partial^2 v_k}{\partial \tau^2} + a(x,0,w_0(x,0))\dfrac{\partial v_k}{\partial \tau} = -\Psi_k(x,\tau), \qquad (x,\tau) \in D \\[2mm] \dfrac{\partial v_k}{\partial \tau}(x,0) = -\dfrac{\partial w_k}{\partial t}(x,0), \qquad -\infty < x < \infty \\[4mm] \lim_{\tau \to \infty} v_k(x,\tau) = 0, \qquad -\infty < x < \infty \end{cases}$$

the functions Φ_k and Ψ_k ($k \in \mathbb{N}$, $k \geq 2$) being completely determined by the functions $w_0, w_1, \ldots, w_{k-1}$ respectively the functions $w_0, w_1, \ldots, w_k$ and $v_0, v_1, \ldots, v_{k-1}$. The functions $v_k(x,\tau)$ ($k \in \mathbb{N} \cup \{0\}$) are boundary layer functions of the type

$$(30) \quad v_k(x,\tau) = \sum_{j=1}^{k+1} P_j^{(k)}(\tau;x)\exp[-ja(x,0,w_0(x,0))\tau]$$

with $P_j^{(k)}(\tau;x)$ a polynomial in τ with coefficients depending on x.

It follows that

$$(31) \quad \begin{cases} L^{(\varepsilon)}[\tilde{u}] = R(x,t;\varepsilon), \qquad (x,t) \in D \\[2mm] \tilde{u}(x,0;\varepsilon) = f(x) + \varepsilon^{n+1}v_n(x,0), \qquad -\infty < x < \infty \\[2mm] \tilde{u}_t(x,0;\varepsilon) = g(x), \qquad -\infty < x < \infty \end{cases}$$

with $R(x,t;\varepsilon)$ some C^∞-function satisfying

$$R(x,t;\varepsilon) = O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0$$

uniformly in any compact subset $\Omega \subset \bar{D}$.

Before entering into a discussion of the condition (26) we shall prove that a formal approximation $\tilde{u}$ is indeed a good approximation for the solution u of (15). To this end we put

(32) $z(x,t;\varepsilon) = u(x,t;\varepsilon) - \tilde{u}(x,t;\varepsilon).$

Then we obtain from (15) and (31)

$$
\begin{cases}
\varepsilon\{\dfrac{\partial^2 z}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 z}{\partial x^2}\} + a(x,t,z+\tilde{u})\dfrac{\partial z}{\partial t} + b(x,t,z+\tilde{u})\dfrac{\partial z}{\partial x} \\[2ex]
\qquad\qquad + \{a(x,t,z+\tilde{u}) - a(x,t,\tilde{u})\}\dfrac{\partial \tilde{u}}{\partial t} \\[2ex]
\qquad\qquad + \{b(x,t,z+\tilde{u}) - b(x,t,\tilde{u})\}\dfrac{\partial \tilde{u}}{\partial x} \\[2ex]
\qquad\qquad + d(x,t,z+\tilde{u}) - d(x,t,\tilde{u}) = -R(x,t;\varepsilon), (x,t)\in D \\[2ex]
z(x,0;\varepsilon) = -\varepsilon^{n+1}v_n(x,0), \qquad -\infty < x < \infty \\[2ex]
z_t(x,0;\varepsilon) = 0, \qquad -\infty < x < \infty.
\end{cases}
$$

Defining a function $\tilde{z}(x,t;\varepsilon)$ by

(33) $\tilde{z}(x,t;\varepsilon) = z(x,t;\varepsilon) + \varepsilon^{n+1}v_n(x,0)$

we get

(34)
$$
\begin{cases}
\varepsilon\{\dfrac{\partial^2 \tilde{z}}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 \tilde{z}}{\partial x^2}\} + a^*(x,t,\tilde{z};\varepsilon)\dfrac{\partial \tilde{z}}{\partial t} + b^*(x,t,\tilde{z};\varepsilon)\dfrac{\partial \tilde{z}}{\partial x} \\[2ex]
\qquad\qquad + d^*(x,t,\tilde{z};\varepsilon) = -R^*(x,t;\varepsilon), \qquad (x,t) \in D \\[2ex]
\tilde{z}(x,0;\varepsilon) = \tilde{z}_t(x,0;\varepsilon) = 0, \qquad -\infty < x < \infty
\end{cases}
$$

where

$$a^*(x,t,\tilde{z};\varepsilon) = a(x,t,\tilde{z}+\tilde{u}-\varepsilon^{n+1}v_n(x,0))$$

$$b^*(x,t,\tilde{z};\varepsilon) = b(x,t,\tilde{z}+\tilde{u}-\varepsilon^{n+1}v_n(x,0))$$

$$d^*(x,t,\tilde{z};\varepsilon) = \{a(x,t,\tilde{z}+\tilde{u}-\varepsilon^{n+1}v_n(x,0)) - a(x,t,\tilde{u}-\varepsilon^{n+1}v_n(x,0))\}\frac{\partial\tilde{u}}{\partial t}$$

$$+ \{b(x,t,\tilde{z}+\tilde{u}-\varepsilon^{n+1}v_n(x,0)) - b(x,t,\tilde{u}-\varepsilon^{n+1}v_n(x,0))\}\frac{\partial\tilde{u}}{\partial x}$$

$$+ d(x,t,\tilde{z}+\tilde{u}-\varepsilon^{n+1}v_n(x,0)) - d(x,t,\tilde{u}-\varepsilon^{n+1}v_n(x,0))$$

$$- \varepsilon^{n+1}\{b(x,t,\tilde{z}+\tilde{u}-\varepsilon^{n+1}v_n(x,0)) - b(x,t,\tilde{u}-\varepsilon^{n+1}v_n(x,0))\}\frac{d}{dx}v_n(x,0)$$

(35)

$$R^*(x,t;\varepsilon) = R(x,t;\varepsilon)$$

$$- \varepsilon^{n+1}b(x,t,\tilde{u}-\varepsilon^{n+1}v_n(x,0))\frac{d}{dx}v_n(x,0)$$

$$+ \varepsilon^{n+2}c^2(x,t)\frac{d^2}{dx^2}v_n(x,0)$$

$$+ \{a(x,t,\tilde{u}-\varepsilon^{n+1}v_n(x,0)) - a(x,t,\tilde{u})\}\frac{\partial\tilde{u}}{\partial t}$$

$$+ \{b(x,t,\tilde{u}-\varepsilon^{n+1}v_n(x,0)) - b(x,t,\tilde{u})\}\frac{\partial\tilde{u}}{\partial x}$$

$$+ d(x,t,\tilde{u}-\varepsilon^{n+1}v_n(x,0)) - d(x,t,\tilde{u}).$$

It is easily established that the functions a^*, b^*, d^* and R^* satisfy the conditions (7), (9) and (10) of theorem 1 (with Lipschitz exponent $\alpha = 1$) and that

$$R^*(x,t;\varepsilon) = O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$.

So taking for Ω some characteristic region as shown in figure 1 we conclude from theorem 1 that there exists for sufficiently small values of ε a solution $\tilde{z}$ of (34), provided that (cf. (14))

(36) $\qquad n + 1 > \lambda + \dfrac{1}{4}$

where the constant λ, according to (13), is given by

$$\lambda = \begin{cases} \dfrac{1}{4} & \text{if a and b do not depend on u} \\[2ex] 1 & \text{otherwise.} \end{cases}$$

The condition (36) can be simplified to yield $n \geq 0$ (if a and b are independent of u) or $n \geq 1$ (otherwise).

According to theorem 1 the solution $\tilde{z}$ satisfies the estimate

(37) $\qquad |\tilde{z}|_1^{(\varepsilon)} \leq C(\Omega)\varepsilon^{n+\frac{3}{4}}$

where $C(\Omega)$ is some positive constant depending on Ω but independent of ε. Using (27), (32) and (33) we obtain from (37) the following theorem which is a nonlinear generalization of theorem 2 of chapter III:

THEOREM 2. *If the solution* $w_0(x,t)$ *of the nonlinear problem* (22) *belongs to the class* $C^\infty(\bar{D})$ *then there exists for sufficiently small values of* ε *a solution u of problem* (15) *under the conditions* (16) *and* (17).
This solution and its first derivatives satisfy for any $n \in \mathbb{N} \cup \{0\}$ *the relations*

(38) $\left\{ \begin{array}{l} u(x,t;\varepsilon) = \displaystyle\sum_{k=0}^{n} \varepsilon^k w_k(x,t) + \sum_{k=0}^{n-1} \varepsilon^{k+1} v_k(x,\tau) + O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0 \\[3ex] \dfrac{\partial u}{\partial x}(x,t;\varepsilon) = \displaystyle\sum_{k=0}^{n} \varepsilon^k \dfrac{\partial w_k}{\partial x}(x,t) + \sum_{k=0}^{n-1} \varepsilon^{k+1} \dfrac{\partial v_k}{\partial x}(x,\tau) + O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0 \\[3ex] \dfrac{\partial u}{\partial t}(x,t;\varepsilon) = \displaystyle\sum_{k=0}^{n} \varepsilon^k \dfrac{\partial w_k}{\partial t}(x,t) + \sum_{k=0}^{n} \varepsilon^k \dfrac{\partial v_k}{\partial \tau}(x,\tau) + O(\varepsilon^{n+1}), \qquad \varepsilon \downarrow 0 \end{array} \right.$

uniformly in any compact subset $\Omega \subset \bar{D}$; *the functions* w_k *and* v_k
$(k = 0,1,\ldots,n)$ *follow successively from* (22), (23), (24), (25), (28) *and*
(29) *where* τ *is defined by* $\tau = \dfrac{t}{\varepsilon}$.
In case $n = 0$ *empty sums are to be interpreted as zero.*

COROLLARY 1. *It follows from the proof of theorem 2 that in case*
$w_0(x,t) \in C^\infty(\Omega)$, *with* Ω *some characteristic region as shown in figure 1, the results of theorem 2 remain valid for any compact subset* $\Omega' \subset \Omega$.

COROLLARY 2. *It follows from* (38) *that first approximations* (n = 0) *for the solution* u *of* (15) *and its first derivatives are given by*

$$
\left\{
\begin{aligned}
u(x,t;\varepsilon) &= w_0(x,t) + O(\varepsilon), && \varepsilon \downarrow 0 \\[2mm]
u_x(x,t;\varepsilon) &= \frac{\partial w_0}{\partial x}(x,t) + O(\varepsilon), && \varepsilon \downarrow 0 \\[2mm]
u_t(x,t;\varepsilon) &= \frac{\partial w_0}{\partial t}(x,t) + \{g(x) - \frac{\partial w_0}{\partial t}(x,0)\}\exp[-a(x,0,w_0(x,0))\frac{t}{\varepsilon}] + O(\varepsilon), \\
&&& \varepsilon \downarrow 0
\end{aligned}
\right.
$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$, *where* w_0 *is the* C^∞*-solution of the reduced problem* (22).

COROLLARY 3. *It follows from* (38) *and* (30) *that for any* $n \in \mathbb{N} \cup \{0\}$

$$
\left\{
\begin{aligned}
u(x,t;\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k w_k(x,t) + O(\varepsilon^{n+1}), && \varepsilon \downarrow 0 \\[2mm]
u_x(x,t;\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \frac{\partial w_k}{\partial x}(x,t) + O(\varepsilon^{n+1}), && \varepsilon \downarrow 0 \\[2mm]
u_t(x,t;\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \frac{\partial w_k}{\partial t}(x,t) + O(\varepsilon^{n+1}), && \varepsilon \downarrow 0
\end{aligned}
\right.
$$

uniformly valid in any compact subset $\Omega \subset \{(x,t) \mid -\infty < x < \infty, \ 0 < \delta \le t < \infty\}$ *where* δ *is an arbitrarily small positive constant independent of* ε.

REMARKS.

1. It is not necessary that the conditions (16) and (17) hold for all $u \in \mathbb{R}$. It is sufficient if these conditions are fulfilled for $u \in I$ with I some open subset of $\mathbb{R}$ containing the range R_{w_0} of w_0. This implies that the functions $a(x,t,u)$, $b(x,t,u)$ and $d(x,t,u)$ are allowed to have singularities in u, provided that these singularities are not in R_{w_0} (cf. the first remark following theorem 3 in chapter II).

2. The condition of infinite smoothness of the data (and hence also the condition that w_0 belongs to $C^\infty(\bar{D})$) can be weakened. In that case the expansions (38) are valid for all $n \in \mathbb{N} \cup \{0\}$ with $n \le n_0$, the bound n_0 being determined by the differentiability of the data (see also subsection 4.2).

3. Generalized solutions of the reduced problem

In the preceding section we saw that for the construction of a first approximation (n = 0 in theorem 2) it is necessary that the solution w_0 of the reduced problem belongs at least to the class $C^2(\bar{D})$. However, in non-linear problems of the type (15) where the coefficients a and b depend on u this necessary condition is often violated, even if the data are C^∞-functions. This is caused by the possible multivaluedness of w_0, a phenomenon which will now briefly be discussed. For more details we refer to LAX [27].

We look for a solution u of the differential equation

$$(39) \qquad \frac{\partial u}{\partial t} + \Phi(u)\frac{\partial u}{\partial x} = 0, \qquad (x,t) \in D$$

which also satisfies the initial condition

$$(40) \qquad u(x,0) = f(x), \qquad -\infty < x < \infty.$$

We require that

$$\Phi(u) \in C^1(\mathbb{R}), \qquad f(x) \in C^1(\mathbb{R}).$$

It follows from the differential equation (39) that u is constant along characteristics x = x(t) satisfying

$$(41) \qquad \frac{dx}{dt} = \Phi(u).$$

Using the constancy of u we conclude from (41) that the characteristics are straight lines. Geometrically the solution u of (39), (40) can now be constructed by drawing through each point $(x_0,0)$ a straight line with slope $\frac{dt}{dx} = \frac{1}{\Phi(f(x_0))}$. Along this line u has the constant value $f(x_0)$.

One easily verifies that for sufficiently small values of t an analytical form of the solution is implicitly given by

$$u - f(x - t\Phi(u)) = 0$$

since we obtain from this formula

$$(42) \quad \begin{cases} \dfrac{\partial u}{\partial t} = - \dfrac{\Phi(u)\,f'(x - t\Phi(u))}{1 + t\Phi'(u)\,f'(x - t\Phi(u))} \\[4mm] \dfrac{\partial u}{\partial x} = \dfrac{f'(x - t\Phi(u))}{1 + t\Phi'(u)\,f'(x - t\Phi(u))} \end{cases}$$

the prime denoting differentiation with respect to the argument.

Let us now assume that Φ is monotonically increasing, i.e.

$$\Phi'(u) > 0 \qquad \text{for } u \in \mathbb{R}.$$

If f is a monotonically increasing function then it follows from (42) that $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$ remain bounded for all $t > 0$. On the other hand, if f is a decreasing function then $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$ tend to infinity as the denominators in (42) approach zero.

This is also illustrated by the geometrical construction mentioned before: for increasing f the characteristics issuing from $t = 0$ diverge in the direction of positive t and u remains single-valued for all $t > 0$; for decreasing f the characteristics intersect after some time $t = t_B$ and u becomes multivalued at the points of intersection (see figure 2).
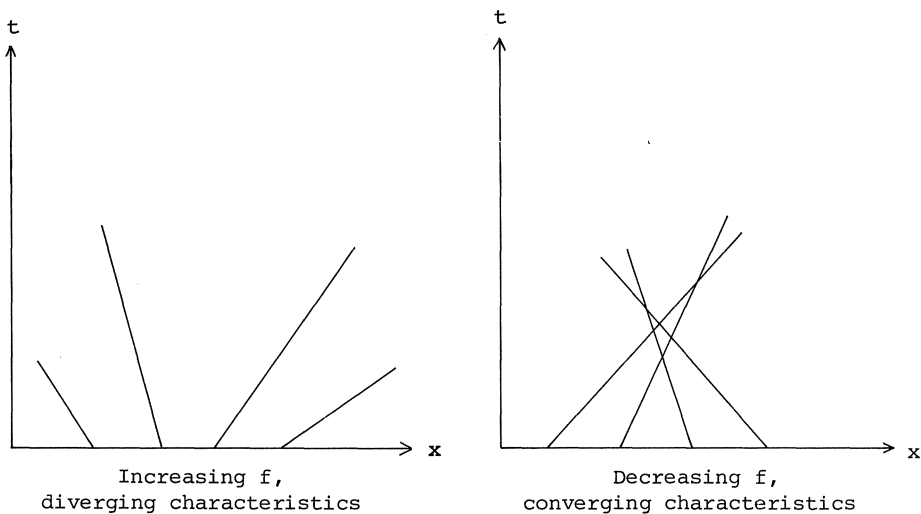

Increasing f,
diverging characteristics

Decreasing f,
converging characteristics

Figure 2

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We can think of u as a wave with propagation velocity $\Phi(u)$, higher values
of u travelling faster than lower ones. If the propagation velocity is a
decreasing function of x (which happens for instance if $\Phi'(u) > 0$, $f'(x) < 0$)
the wave will break after some time $t = t_B$ to give a multivalued function
$u(x,t)$ (see figure 3). It follows that for $t \geq t_B$ the derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$
may become infinite.

u

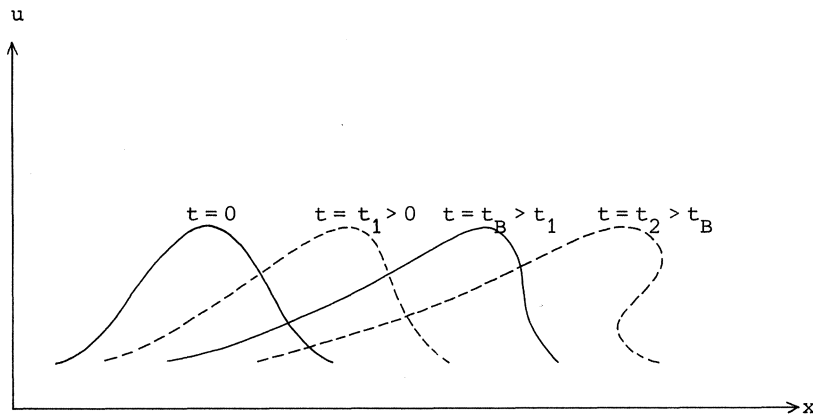t = 0 $t = t_1 > 0$ $t = t_B > t_1$ $t = t_2 > t_B$

x

Figure 3: u as function of x (t fixed)

Both from the analytical and the geometrical point of view it will be clear
that if the function f is not increasing then no continuous function $u(x,t)$
exists which solves (39), (40) for all $t \geq 0$ in the classical sense. It
seems therefore natural to introduce generalized solutions of (39).

DEFINITION 1. A piecewise continuous function $u(x,t)$, continuously differ-
entiable at its points of continuity, is a generalized (distributional, weak)
solution of (39) iff

(i)   u satisfies (39) in the classical sense at all points of continuity
      and

(ii) the jump condition

(43) $(u_\ell - u_r)\frac{dx}{dt} = \int_{u_r}^{u_\ell} \Phi(\xi)\,d\xi$

      holds at all lines of discontinuity $x = x(t)$, where u_ℓ and u_r are
      defined by

$$\begin{cases} u_\ell = u_\ell(x,t) = \lim_{y \uparrow x} u(y,t) \\ \\ u_r = u_r(x,t) = \lim_{y \downarrow x} u(y,t). \end{cases}$$

A generalized solution of (39), (40) with f a piecewise continuous function is not necessarily uniquely determined. However, if we restrict ourselves to generalized solutions which in addition satisfy

(44) $u_\ell > u_r$

at every point of discontinuity (x,t) then it can be shown that there exists a uniquely determined (single-valued) function u(x,t) which solves (39), (40) for all $t \geq 0$ in generalized sense. For a discussion of the condition (44) we refer to LAX [27].

The concept of generalized solution can be extended to include bounded measurable functions.

DEFINITION 2. A bounded, measurable function u(x,t) is a generalized (distributional, weak) solution of (39) iff

$$\iint_D \{\phi_t u + \phi_x \int_0^u \Phi(\xi)d\xi\}dxdt = 0$$

for all test functions $\phi \in C^1(D)$ with compact support in D.

It can be shown that for piecewise continuous functions the definitions 1 and 2 are equivalent (cf. LAX [27]).

As before, a generalized solution u of (39), (40) with f a bounded, measurable function is not necessarily uniquely determined. However, also in this case existence and uniqueness can be guaranteed if certain additional conditions are imposed on u (see OLEINIK [33], [35]).

As an illustration we shall give an example of a generalized solution in the sense of definition 1.

EXAMPLE 1.

(45)
$$\begin{cases} \dfrac{\partial u}{\partial t} + \arctan u \, \dfrac{\partial u}{\partial x} = 0, \qquad (x,t) \in D \\[3mm] u(x,0) = -x, \qquad -\infty < x < \infty. \end{cases}$$

The solution of this problem is implicitly given by

$$F(x,t,u) \equiv u - t \arctan u + x = 0$$

and it is easily verified that the lowest value of t for which

$$F_u(x,t,u) = 0$$

is given by $t = 1$ (with $x = u = 0$), so breaking takes place at $t_B = 1$. This can also be checked graphically. The characteristic through $(x_0,0)$ is for $t > 0$ given by

(46)
$$\arctan x_0 = \frac{x_0 - x}{t} .$$

If we consider (46) as an equation in x_0 with x and t given then it easily follows from the graphs of $\arctan x_0$ and $\dfrac{x_0 - x}{t}$ that breaking occurs at $(x,t) = (0,1)$ (see figure 4). For $t > 1$ and suitably chosen x there are several characteristics passing through (x,t).
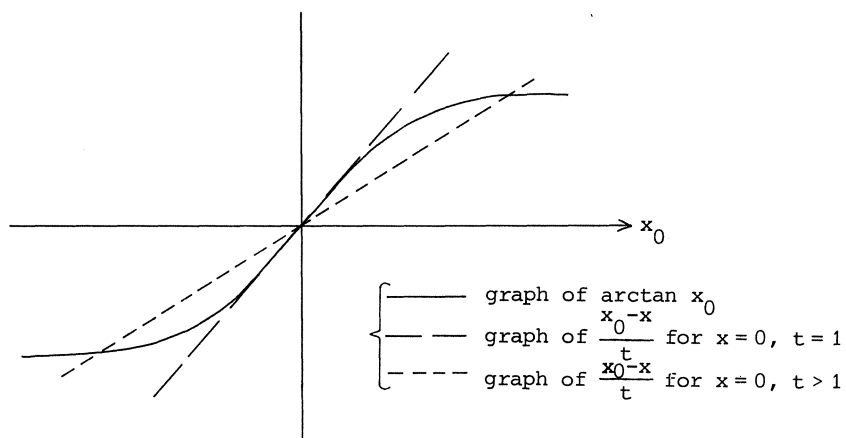

graph of $\arctan x_0$

graph of $\dfrac{x_0 - x}{t}$ for $x = 0$, $t = 1$

graph of $\dfrac{x_0 - x}{t}$ for $x = 0$, $t > 1$

Figure 4

The region in the (x,t)-plane where u(x,t) is multivalued is bounded by the envelope

(47) $\begin{cases} x = \xi - (1 + \xi^2) \arctan \xi \\ \\ t = 1 + \xi^2 \end{cases}$ ($\xi \in \mathbb{R}$)

(cf. WHITHAM [56, p.23]). We have sketched the situation in figure 5.

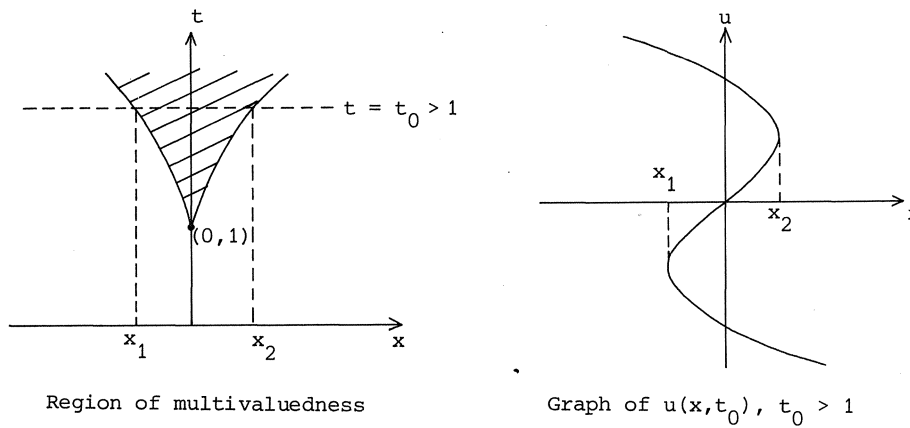Region of multivaluedness   Graph of $u(x,t_0)$, $t_0 > 1$

Figure 5

In order to obtain from this multivalued function the generalized solution of (45) we have to insert a discontinuity, often referred to as a shock. The line of discontinuity must pass through the point (0,1) and lie in the shaded region (see figure 5). In addition the shock position must be such that the jump condition (43) holds. It is easily verified that the line $\{(x,t) \mid x = 0, t \geq 1\}$ satisfies all requirements because of the uneven character of the problem. The generalized piecewise continuous solution is sketched in figure 6.

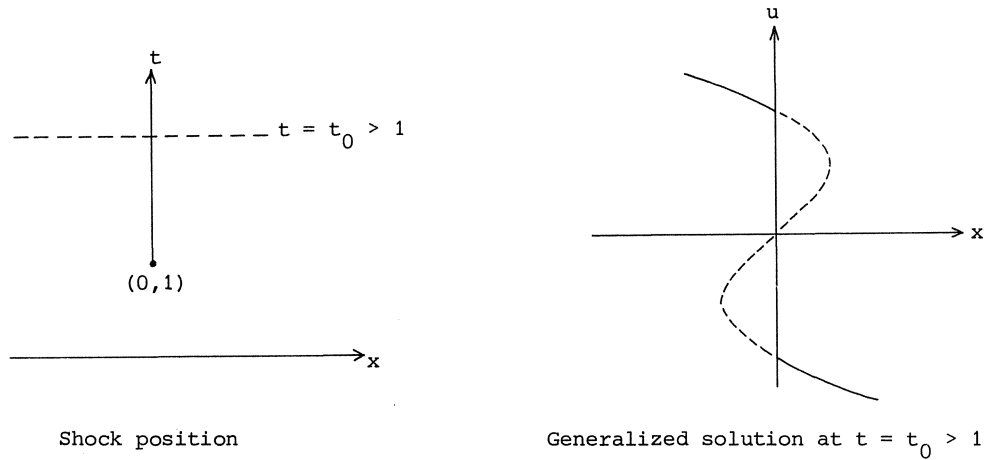Shock position   Generalized solution at $t = t_0 > 1$

Figure 6

From this example one might get the impression that the line of discontinuity necessarily coincides with the characteristic through the breaking point. However, this is not true, as an analysis of the following problem shows:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \arctan u \, \dfrac{\partial u}{\partial x} = 0, & (x,t) \in D \\[2ex]
u(x,0) = \begin{cases} -x - \dfrac{1}{3}x^3, & x \geq 0 \\[2ex] -x, & x < 0. \end{cases}
\end{cases}
$$

The breaking point is again given by $(0,1)$ but one may verify that the characteristic $x = 0$ does not satisfy the jump condition (43), so it cannot coincide with the line of discontinuity.

Finally, it should be remarked that the concept of generalized solutions can be extended to more general initial value problems of the type

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \Phi(x,t,u)\dfrac{\partial u}{\partial x} + \Psi(x,t,u) = 0, & (x,t) \in D \\[2ex]
u(x,0) = f(x), & -\infty < x < \infty.
\end{cases}
$$

Also the restriction that Φ be monotonic in u can be disposed of (see OLEINIK [36]).

Let us now return to singular perturbation problems and see what the introduction of generalized solutions leads to. For singular perturbations of parabolic type it is known (OLEINIK [33], [34]) that under appropriate conditions on the data the solution u of

$$(48) \quad \begin{cases} \varepsilon\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} + \Phi(x,t,u)\dfrac{\partial u}{\partial x} + \Psi(x,t,u), & (x,t) \in D \\[2mm] u(x,0;\varepsilon) = f(x), & -\infty < x < \infty \end{cases}$$

converges for $\varepsilon \downarrow 0$ in some integral norm to the generalized solution w of

$$\begin{cases} \dfrac{\partial w}{\partial t} + \Phi(x,t,w)\dfrac{\partial w}{\partial x} + \Psi(x,t,w) = 0, & (x,t) \in D \\[2mm] w(x,0) = f(x), & -\infty < x < \infty. \end{cases}$$

More strongly, if $\Phi(x,t,u) = u$ and $\Psi(x,t,u) \equiv 0$ we have BURGERS' equation and problem (48) can be solved explicitly (see BURGERS [5] and WHITHAM [56, chapter 4]). It turns out that u converges pointwise to w for $\varepsilon \downarrow 0$ with the exception of arbitrarily small neighborhoods of the lines of discontinuity where a smooth transition takes place (see figure 7).
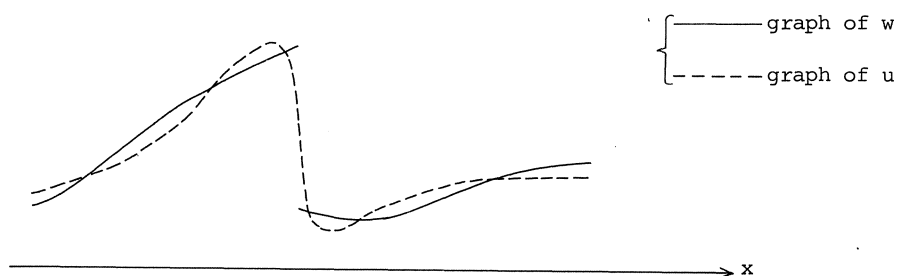

Figure 7: Convergence of u for fixed t

We strongly believe that similar results hold for problems with a singular perturbation of hyperbolic type. A possible strategy to solve this kind of problems will now be illustrated by the following example

$$\begin{cases} L^{(\varepsilon)}[u] \equiv \varepsilon(\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}) + \frac{\partial u}{\partial t} + \arctan u \frac{\partial u}{\partial x} = 0, \qquad (x,t) \in D \\ u(x,0;\varepsilon) = -x, \qquad -\infty < x < \infty \\ u_t(x,0;\varepsilon) = 0, \qquad -\infty < x < \infty \end{cases}$$

where c is some constant satisfying

$$(49) \qquad c \geq \frac{\pi}{2} .$$

This inequality implies that the timelike condition is fulfilled.

From example 1 we know the shock position and the region of multivaluedness for the solution of the reduced problem. We draw through the breaking point $(0,1)$ two characteristics satisfying $\frac{dx}{dt} = \pm c$. Using the timelike condition (49) it can be shown that these characteristics lie outside the region of multivaluedness (see (47) and figure 8).
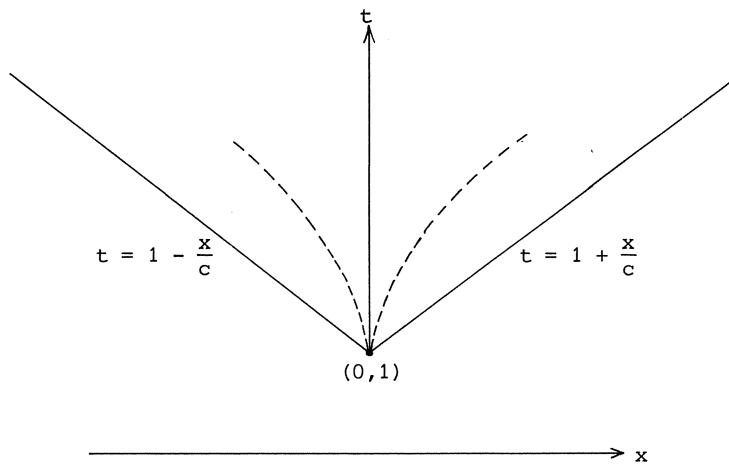

Figure 8

Since the generalized solution of the reduced problem is infinitely smooth in the region

$$D^* = \{(x,t) \mid 0 \le t < 1 + \frac{x}{c} \vee 0 \le t < 1 - \frac{x}{c}\}$$

we have the approximation

$$u(x,t;\varepsilon) = w(x,t) + O(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset D^*$ (see corollary 1 of theorem 2).

Approximations for u in the remaining region $D\backslash D^*$ have not been established yet. Since it follows from (43) that shocks originating from the reduced equation necessarily take place along timelike curves in the (x,t)-plane a promising strategy for future investigations seems to be the following (see also chapter VI):

1. Define for an arbitrarily small positive number δ (independent of ε) the region D_δ by

$$D_\delta = \{(x,t) \mid x > 0 \wedge t > 1 + \frac{x}{c} - \delta\}$$

and consider the Goursat problem $L^{(\varepsilon)}[u] = 0$ in D_δ, *assuming the values of u along the boundary to be known.*

2. Construct a formal approximation $\tilde{u}$ for u in $\bar{D}_\delta$, consisting of the generalized solution w augmented with a boundary layer function at $x = 0$.

3. Prove that $\lim_{\varepsilon \downarrow 0} \{u(x,t;\varepsilon) - \tilde{u}(x,t;\varepsilon)\} = 0$ in $\bar{D}_\delta$, using the fact that

$$u(x,1 + \frac{x}{c} - \delta;\varepsilon) - w(x,1 + \frac{x}{c} - \delta) = O(\varepsilon), \qquad \varepsilon \downarrow 0 \qquad \text{for } x \ge 0$$

(cf. chapter VI).

Since the unknown function $u(0,t;\varepsilon)$, $t \ge 1$ occurs only in the boundary layer function it follows that

$$\lim_{\varepsilon \downarrow 0} u(x,t;\varepsilon) = w(x,t) \quad \text{in} \quad \{(x,t) \mid x \ge \delta \wedge t \ge 1 + \frac{x}{c} - \delta\}.$$

In a similar way one obtains an estimate in $\{(x,t) \mid x \le -\delta \wedge t \ge 1 - \frac{x}{c} - \delta\}$.

The above mentioned procedure can be carried out if one knows a priori that both $u(x,t;\varepsilon)$ and its tangential derivative along the shock remain bounded

as ε tends to zero. Finally, approximations in regions containing a part of
the line along which the shock takes place may be constructed by adding to
the generalized solution a free boundary layer giving a smooth transition
(depending on ε) from values of $w(x,t)$ with $x < 0$ to values of $w(x,t)$ with
$x > 0$.

In the next section we shall discuss two nonlinear initial value prob-
lems of the type (15) in which the coefficients a and b are independent of
u.

4. Problems with a and b independent of u

4.1. The case $d(x,t,u) = d(x,t)\sin u$

We consider the quasilinear initial value problem

$$(50) \quad \begin{cases} \varepsilon\{\dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2}\} + a(x,t)\dfrac{\partial u}{\partial t} + b(x,t)\dfrac{\partial u}{\partial x} + d(x,t)\sin u = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in D \\ \\ u(x,0;\varepsilon) = f(x), \qquad -\infty < x < \infty \\ \\ u_t(x,0;\varepsilon) = g(x), \qquad -\infty < x < \infty \end{cases}$$

where ε is a small positive parameter, i.e. $0 < \varepsilon \ll 1$.
The following conditions are imposed on the data

$$(51) \quad \begin{cases} c(x,t) \in C^\infty(\bar{D}), \quad d(x,t) \in C^\infty(\bar{D}) \\ \\ f(x) \in C^\infty(\mathbb{R}), \quad g(x) \in C^\infty(\mathbb{R}) \\ \\ c(x,t) > 0 \qquad \text{in } \bar{D} \\ \\ c(x,t) \text{ is uniformly bounded in any strip} \\ \{(x,t) \mid -\infty < x < \infty, \ 0 \le t \le T\} \end{cases}$$

and

$$\begin{cases} a(x,t) \in C^\infty(\bar{D}), \quad b(x,t) \in C^\infty(\bar{D}) \\ \\ a(x,t) > 0 \quad \text{in } \bar{D} \\ \\ |b(x,t)| < a(x,t)c(x,t) \quad \text{in } \bar{D}. \end{cases}$$

We ask for an approximation of the solution u of (50), valid for small values of ε.

It has been shown in section 2 that the reduced problem

(52) $$\begin{cases} a(x,t)\dfrac{\partial w}{\partial t} + b(x,t)\dfrac{\partial w}{\partial x} + d(x,t)\sin w = 0, \quad (x,t) \in D \\ \\ w(x,0) = f(x), \quad -\infty < x < \infty \end{cases}$$

holds a key position in the construction of an approximation for the solution u of (50); it is in fact the smoothness of the solution w of (52) which determines the highest order of accuracy we can obtain.

Since an obvious change of coordinates transforms problem (52) into an equivalent problem of the type

(53) $$\begin{cases} \dfrac{\partial w}{\partial t} + d(x,t)\sin w = 0, \quad (x,t) \in D \\ \\ w(x,0) = f(x), \quad -\infty < x < \infty \end{cases}$$

it follows that instead of (50) we may consider without loss of generality the simpler problem

(54) $$\begin{cases} \varepsilon\{\dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2}\} + \dfrac{\partial u}{\partial t} + d(x,t)\sin u = 0, \quad (x,t) \in D \\ \\ u(x,0;\varepsilon) = f(x), \quad -\infty < x < \infty \\ \\ u_t(x,0;\varepsilon) = g(x), \quad -\infty < x < \infty \end{cases}$$

for as far as the construction of an approximation for u is concerned there is qualitatively no difference between the problems (50) and (54).

An approximation for the solution u of (54) can easily be given in explicit form. We find for the solution w of the reduced problem (53)

$$(55) \quad \begin{cases} w(x,t) = 2 \arctan[\tan\{\tfrac{1}{2}f(x)\}\exp\{\phi(x,t)\}] + 2k\pi \\ \qquad\qquad\qquad\qquad\qquad \text{if } (x,t) \in A_k, \; k \in \mathbb{Z} \\[2ex] w(x,t) = (2k + 1)\pi \qquad \text{if } (x,t) \in B_k, \; k \in \mathbb{Z} \end{cases}$$

where

$$(56) \quad \begin{cases} A_k = \{(x,t) \mid (2k - 1)\pi < f(x) < (2k + 1)\pi, \; 0 \le t < \infty\}, \quad k \in \mathbb{Z} \\[2ex] B_k = \{(x,t) \mid f(x) = (2k + 1)\pi, \; 0 \le t < \infty\}, \quad k \in \mathbb{Z} \\[2ex] \phi(x,t) = - \int_0^t d(x,\tau)d\tau. \end{cases}$$

Since $w(x,t) \in C^\infty(\bar{D})$ we obtain from corollary 2 of theorem 2

THEOREM 3. *For sufficiently small values of* ε *there exists a solution* u *of problem* (54) *under the conditions* (51). *This solution satisfies*

$$u(x,t;\varepsilon) = w(x,t) + O(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly in each compact subset $\Omega \subset \bar{D}$, *where* $w(x,t)$ *is given by* (55) *and* (56).

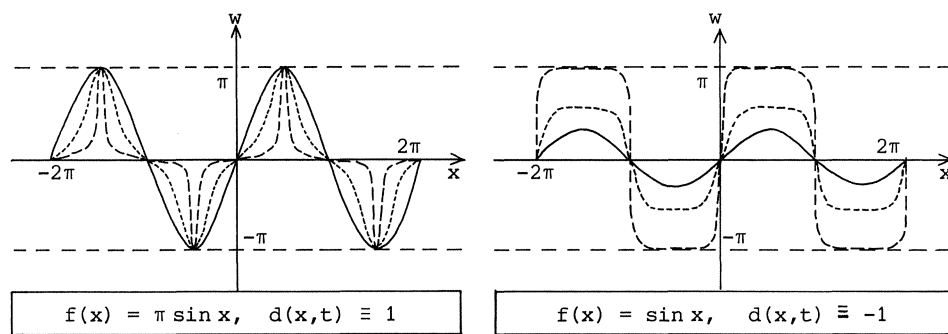A sketch of w as function of x for different values of t is given in figure 9.

| f(x) = π sin x, d(x,t) ≡ 1 | f(x) = sin x, d(x,t) ≡ -1 |

Figure 9: w as function of x for
$$\begin{cases} t = 0 \qquad \underline{\qquad\quad} \\ t = t_1 > 0 \qquad \text{-------} \\ t = t_2 \gg t_1 \qquad \text{-- -- --} \end{cases}$$

Note that in case $d(x,t) \equiv 1$

$$\lim_{t \to \infty} w(x,t) = 2k\pi \quad \text{if} \quad (2k - 1)\pi < f(x) < (2k + 1)\pi, \quad k \in \mathbb{Z}$$

and in case $d(x,t) \equiv -1$

$$\lim_{t \to \infty} w(x,t) = \begin{cases} (2k + 1)\pi & \text{if} \quad 2k\pi < f(x) < (2k + 1)\pi, \quad k \in \mathbb{Z} \\ \\ (2k - 1)\pi & \text{if} \quad (2k - 1)\pi < f(x) < 2k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

4.2. The case $d(x,t,u) = d_1(x,t)u + d_2(x,t)u|u|^{\nu}$

We consider the quasilinear initial value problem

$$(57) \quad \begin{cases} \varepsilon\{\dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2}\} + a(x,t)\dfrac{\partial u}{\partial t} + b(x,t)\dfrac{\partial u}{\partial x} \\ \qquad\qquad\qquad + d_1(x,t)u + d_2(x,t)u|u|^{\nu} = 0, \qquad (x,t) \in D \\ u(x,0;\varepsilon) = f(x), \qquad -\infty < x < \infty \\ u_t(x,0;\varepsilon) = g(x), \qquad -\infty < x < \infty \end{cases}$$

where ε is a small positive parameter, i.e. $0 < \varepsilon \ll 1$, and ν is a positive number.

The following conditions are imposed on the data

$$(58) \quad \begin{cases} c(x,t) \in C^{\infty}(\bar{D}), \quad f(x) \in C^{\infty}(\mathbb{R}), \quad g(x) \in C^{\infty}(\mathbb{R}) \\ \\ c(x,t) > 0 \qquad \text{in } \bar{D} \\ \\ c(x,t) \text{ is uniformly bounded in any strip} \\ \{(x,t) \mid -\infty < x < \infty, \ 0 \le t \le T\} \end{cases}$$

and

$$\begin{cases} a(x,t) \in C^\infty(\bar{D}), \quad b(x,t) \in C^\infty(\bar{D}) \\[2mm] d_1(x,t) \in C^\infty(\bar{D}), \quad d_2(x,t) \in C^\infty(\bar{D}) \\[2mm] a(x,t) > 0 \quad \text{in } \bar{D}, \quad d_2(x,t) > 0 \quad \text{in } \bar{D} \\[2mm] |b(x,t)| < a(x,t)c(x,t) \qquad \text{in } \bar{D}. \end{cases}$$

We ask for an approximation of the solution u of (57), valid for small values of ε. Note that in this problem the function d(x,t,u) does not belong to the class $C^\infty(\bar{D} \times \mathbb{R})$, except when ν is even.

In order not to complicate the calculations we shall restrict ourselves to the following special form of problem (57):

(59)
$$\begin{cases} L^{(\varepsilon)}[u] \equiv \varepsilon\{\frac{\partial^2 u}{\partial t^2} - c^2(x,t)\frac{\partial^2 u}{\partial x^2}\} + \frac{\partial u}{\partial t} + u|u|^\nu = 0, \qquad (x,t) \in D \\[2mm] u(x,0;\varepsilon) = f(x), \qquad -\infty < x < \infty \\[2mm] u_t(x,0;\varepsilon) = g(x), \qquad -\infty < x < \infty. \end{cases}$$

As we pointed out in subsection 4.1 this is no serious limitation as long as we are concerned with approximations of u, since the reduced problem

$$\begin{cases} a(x,t)\frac{\partial w}{\partial t} + b(x,t)\frac{\partial w}{\partial x} + d_1(x,t)w + d_2(x,t)w|w|^\nu = 0, \qquad (x,t) \in D \\[2mm] w(x,0) = f(x), \qquad -\infty < x < \infty \end{cases}$$

can always be transformed into an equivalent problem of the type

(60)
$$\begin{cases} \frac{\partial w}{\partial t} + w|w|^\nu = 0, \qquad (x,t) \in D \\[2mm] w(x,0) = f(x), \qquad -\infty < x < \infty \end{cases}$$

by means of a LIOUVILLE transformation followed by a transition to characteristic coordinates.

We shall now establish an approximation for the solution u of (59).

Firstly, let us remark that for even values of ν the theory of section 2

may be applied in a straightforward manner, since we have in that case $d(x,t,u) \in C^\infty(\bar{D} \times \mathbb{R})$ and the solution w of (60), given by

$$(61) \qquad w(x,t) = \frac{f(x)}{\{1 + \nu t|f(x)|^\nu\}^{1/\nu}},$$

belongs to $C^\infty(\bar{D})$. So we obtain from corollary 2 of theorem 2

$$u(x,t;\varepsilon) = \frac{f(x)}{\{1 + \nu t|f(x)|^\nu\}^{1/\nu}} + O(\varepsilon), \qquad \varepsilon \downarrow 0$$

for even ν, uniformly valid in each compact subset $\Omega \subset \bar{D}$.

In order to deal with other values of ν we shall assume for the moment that the function f has no zeroes. Then the solution (61) of the reduced problem belongs again to $C^\infty(\bar{D})$. Starting with this function as a first approximation $\underset{\sim}{w}_0$ it is possible to construct for any $n \in \mathbb{N} \cup \{0\}$ a formal approximation $\underset{\sim}{u}$ for u in the form

$$(62) \qquad \underset{\sim}{u}(x,t;\varepsilon) = \sum_{k=0}^{n} \varepsilon^k w_k(x,t) + \sum_{k=0}^{n} \varepsilon^{k+1} v_k(x,\tau)$$

in the same way as we did in section 2. Note that the higher derivatives of the function $u|u|^\nu$ remain bounded for u in a neighborhood of w_0 due to the fact that w_0 has no zeroes.

The proof of the correctness of the expansion is similar to the proof given in section 2. The only difference is that in the application of theorem 1 to problem (34) the Lipschitz exponent α mentioned in (10, iii) is now given by

$$\alpha = \min(\nu,1)$$

instead of $\alpha = 1$, since the function $d^*(x,t,\underset{\sim}{z};\varepsilon)$ defined by (35) satisfies in this case

$$\frac{\partial d^*}{\partial \underset{\sim}{z}}(x,t,\underset{\sim}{z};\varepsilon) = (1 + \nu)|\underset{\sim}{z} + \underset{\sim}{u} - \varepsilon^{n+1} v_n(x,0)|^\nu.$$

As in section 2 the application of theorem 1 to problem (34) leads to the existence of a solution $\underset{\sim}{z}$ satisfying (37), provided that (cf. (14))

(63) $n + 1 > \lambda + \dfrac{1}{4}$.

Here the constant λ, according to (13), is given by

$$\lambda = \frac{1}{4 \min(\nu,1)} \; .$$

The condition (63) can also be written as

(64) $\min(\nu,1) > \dfrac{1}{4n + 3}$.

Hence we may conclude that an increase of the number of terms in (62) yields not only better approximations for u but it also enables us to consider smaller values of ν. In particular, letting n tend to infinity we obtain for all $\nu > 0$

$$u(x,t;\varepsilon) = \frac{f(x)}{\{1 + \nu t |f(x)|^\nu\}^{1/\nu}} + O(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, provided that f has no zeroes.

Let us now investigate what happens if the function f has zeroes. Starting with (61) as a first approximation w_0 it is not possible to construct a formal approximation of arbitrarily high order, because for any fixed value of ν

(i)   the higher derivatives of $u|u|^\nu$ may become unbounded in a neighborhood of w_0, since w_0 may be equal to zero,

(ii) the higher derivatives of w_0 with respect to x become unbounded in a neighborhood of a zero of f.

As for the derivatives of w_0, a calculation shows that

(65)

$$\frac{\partial w_0}{\partial x} = \frac{f'(x)}{\{1 + \nu t |f(x)|^\nu\}^{1+1/\nu}}$$

$$\frac{\partial^2 w_0}{\partial x^2} = \frac{f''(x)}{\{1 + \nu t |f(x)|^\nu\}^{1+1/\nu}} - \frac{\nu(\nu + 1)t|f(x)|^\nu\{f'(x)\}^2}{f(x)\{1 + \nu t |f(x)|^\nu\}^{2+1/\nu}}$$

so if f has in $x = x_0$ a zero with multiplicity k ($k \in \mathbb{N}$) then it follows that in a neighborhood of $x = x_0$

$$\frac{\partial^2 w_0}{\partial x^2} \text{ is } \begin{cases} \text{unbounded if } 0 < \nu < 1, \ k = 1 \\ \\ \text{bounded otherwise.} \end{cases}$$

N.B1. $\dfrac{\partial^2 w_0}{\partial x^2}$ has a jump discontinuity in $x = x_0$ if $\nu = k = 1$.

N.B2. If $f(x) \equiv 0$ in a neighborhood of $x = x_0$ then we can think of x_0 as a zero with infinite multiplicity.

On the other hand, w_0 is for all positive values of ν infinitely continuously differentiable with respect to t.

From the fact that $\dfrac{\partial^2 w_0}{\partial x^2}$ is unbounded for $0 < \nu < 1$, $k = 1$ it apparently follows that for small values of ν we cannot even start the formal approximation with a first term w_0 given by (61). Although it will turn out that we can do something about this by some regularization technique it will be clear that we may not expect to obtain expansions of high order. Of course, the highest possible order of the approximation increases as ν increases, but since we want our results to be valid for as many values of ν as possible, including small ones, we shall from now on restrict ourselves to first order approximations. Putting

$$\tilde{u}(x,t;\varepsilon) = w_0(x,t) + \varepsilon v_0(x,\tfrac{t}{\varepsilon})$$

$$= \frac{f(x)}{\{1 + \nu t |f(x)|^\nu\}^{1/\nu}} - \varepsilon\{g(x) + f(x)|f(x)|^\nu\}\exp(-\tfrac{t}{\varepsilon})$$

and excluding for a moment the combination $0 < \nu < 1$, $k = 1$ we obtain as in section 2 (formula (31) with $n = 0$)

$$\begin{cases} L^{(\varepsilon)}[\tilde{u}] = R(x,t;\varepsilon), \qquad (x,t) \in D \\ \\ \tilde{u}(x,0;\varepsilon) = f(x) - \varepsilon\{g(x) + f(x)|f(x)|^\nu\}, \qquad -\infty < x < \infty \\ \\ \tilde{u}_t(x,0;\varepsilon) = g(x), \qquad -\infty < x < \infty \end{cases}$$

where $R(x,t;\varepsilon)$ is a function satisfying

$$R(x,t;\varepsilon) = O(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly in each compact subset $\Omega \subset \bar{D}$.
Applying the estimate (37) with $n = 0$ we get

$$u(x,t;\varepsilon) = \frac{f(x)}{\{1 + \nu t|f(x)|^{\nu}\}^{1/\nu}} + O(\varepsilon^{\frac{3}{4}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, provided that (64) holds with $n = 0$, that is

$$\nu > \frac{1}{3} .$$

Note that for $0 < \nu \leq \frac{1}{3}$, $k > 1$ approximations of higher order are required because of (64); in that case one must also investigate the regularity of the functions $\frac{\partial^n w_0}{\partial x^n}$ ($n \geq 3$). Since this involves a number of tedious calculations which do not lead to an essential improvement of the results we shall not go further into this matter.

We shall now examine the case $0 < \nu < 1$, $k = 1$.
Let us suppose that f has a certain number of zeroes with multiplicity $k = 1$ which are nowhere dense in $\mathbb{R}$. Since we only consider approximations in bounded subsets of $\bar{D}$ we may assume without loss of generality that this number is finite, say m, and we denote the zeroes by x_i ($i = 1,2,\ldots,m$) where $x_i < x_j$ for $i < j$.
    Since $\frac{\partial^2 w_0}{\partial x^2}$ is unbounded in a neighborhood of each of the zeroes we shall consider instead of (59) the modified problem

$$(66) \quad \begin{cases} \varepsilon\{\dfrac{\partial^2 U}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 U}{\partial x^2}\} + \dfrac{\partial U}{\partial t} + U|U|^{\nu} = 0, & (x,t) \in D \\[2ex] U(x,0;\varepsilon) = \tilde{f}(x;\varepsilon), & -\infty < x < \infty \\[2ex] U_t(x,0;\varepsilon) = g(x), & -\infty < x < \infty \end{cases}$$

with $0 < \nu < 1$ and

$$(67) \qquad \tilde{f}(x;\varepsilon) = f(x) \prod_{i=1}^{m} [1 - \exp\{-\frac{(x - x_i)^2}{\varepsilon^{2p}}\}] .$$

Here p is some positive constant to be determined later on.

Note that the zeroes of $\tilde{f}$ have multiplicity $k \geq 2$. Moreover, we obtain from (67)

$$f(x) - \tilde{f}(x;\varepsilon) = f(x)[1 - \prod_{i=1}^{m} [1 - \exp\{-\frac{(x - x_i)^2}{\varepsilon^{2p}}\}]]$$

(68)

$$= f(x) \sum_{i=1}^{m} \exp\{-\frac{(x - x_i)^2}{\varepsilon^{2p}}\} + O(\varepsilon^N), \qquad \varepsilon \downarrow 0$$

uniformly valid in every compact subset of $\mathbb{R}$, where N is an arbitrarily large positive number. Since x_i $(i = 1,2,\ldots,m)$ is a zero of f with multiplicity $k = 1$ it follows that

(69) $f(x) - \tilde{f}(x;\varepsilon) = O(\varepsilon^p), \qquad \varepsilon \downarrow 0$

uniformly valid in every compact subset of $\mathbb{R}$.

For small values of ε the initial functions f and $\tilde{f}$ differ only slightly, so we may expect the solution U of (66) to differ slightly from the solution u of (59). We shall now try to approximate u by constructing an approximation for U.

A first approximation for U is given by

(70) $\tilde{U}(x,t;\varepsilon) = W_0(x,t;\varepsilon) + \varepsilon V_0(x,\frac{t}{\varepsilon};\varepsilon)$

where V_0 is the boundary layer function

$$V_0(x,\frac{t}{\varepsilon};\varepsilon) = -\{g(x) + \tilde{f}(x;\varepsilon)|\tilde{f}(x;\varepsilon)|^\nu\}\exp(-\frac{t}{\varepsilon})$$

and W_0 is the solution of the reduced problem, viz.

(71) $W_0(x,t;\varepsilon) = \dfrac{\tilde{f}(x;\varepsilon)}{\{1 + \nu t|\tilde{f}(x;\varepsilon)|^\nu\}^{1/\nu}}$.

This function is now twice continuously differentiable with respect to x. Putting

(72) $\qquad Z(x,t;\varepsilon) = U(x,t;\varepsilon) - \tilde{U}(x,t;\varepsilon) + \varepsilon V_0(x,0;\varepsilon)$

we obtain as in section 2 (cf. formula (34))

(73)
$$\begin{cases} \varepsilon\{\dfrac{\partial^2 Z}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 Z}{\partial x^2}\} + \dfrac{\partial Z}{\partial t} + d^*(x,t,Z;\varepsilon) = -R^*(x,t;\varepsilon) \qquad (x,t) \in D \\[2em] Z(x,0;\varepsilon) = Z_t(x,0;\varepsilon) = 0, \qquad -\infty < x < \infty \end{cases}$$

where

$$\begin{cases} d^*(x,t,Z;\varepsilon) = \{Z + \tilde{U} - \varepsilon V_0(x,0;\varepsilon)\}|Z + \tilde{U} - \varepsilon V_0(x,0;\varepsilon)|^\nu \\[1.5em] \qquad\qquad - \{\tilde{U} - \varepsilon V_0(x,0;\varepsilon)\}|\tilde{U} - \varepsilon V_0(x,0;\varepsilon)|^\nu \\[1.5em] R^*(x,t;\varepsilon) = \varepsilon\{\dfrac{\partial^2 W_0}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 W_0}{\partial x^2}\} \\[2em] \qquad\qquad + \varepsilon^2 c^2(x,t)\{\dfrac{d^2}{dx^2} V_0(x,0;\varepsilon) - \dfrac{\partial^2 V_0}{\partial x^2}(x,t;\varepsilon)\} \\[2em] \qquad\qquad + \{\tilde{U} - \varepsilon V_0(x,0;\varepsilon)\}|\tilde{U} - \varepsilon V_0(x,0;\varepsilon)|^\nu - W_0|W_0|^\nu \ . \end{cases}$$

From (67) it follows that

$$\begin{cases} \tilde{f}'(x;\varepsilon) = 0(1), \qquad \varepsilon \downarrow 0 \\[1em] \tilde{f}''(x;\varepsilon) = 0(\varepsilon^{-p}), \qquad \varepsilon \downarrow 0 \end{cases}$$

uniformly valid in each compact subset of $\mathbb{R}$, so using (65) we obtain

$$R^*(x,t;\varepsilon) = 0(\varepsilon^{1-p}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$.
Hence, applying theorem 1 to (73), we obtain

$$Z(x,t;\varepsilon) = 0(\varepsilon^{-p+\frac{3}{4}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, provided that

$$(74) \qquad 1 - p > \frac{1}{4\nu} + \frac{1}{4} \; .$$

Using (70) and (72) we get

$$U(x,t;\varepsilon) = W_0(x,t;\varepsilon) + O(\varepsilon^{-p+\frac{3}{4}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$.
From (61), (69) and (71) we easily derive that

$$W_0(x,t;\varepsilon) = w_0(x,t) + O(\varepsilon^p), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, whence

$$(75) \qquad U(x,t;\varepsilon) = w_0(x,t) + O(\varepsilon^\kappa), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, with

$$\kappa = \min(p, -p + \frac{3}{4}) \; .$$

In order to estimate the difference between the solution u of the original problem (59) and the solution U of the modified problem (66) we put

$$(76) \qquad z(x,t;\varepsilon) = u(x,t;\varepsilon) - U(x,t;\varepsilon) - f(x) + \tilde{f}(x;\varepsilon).$$

Then it follows from (59) and (66) that

$$(77) \quad \begin{cases} \varepsilon\{\dfrac{\partial^2 z}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 z}{\partial x^2}\} + \dfrac{\partial z}{\partial t} + \tilde{d}(x,t,z;\varepsilon) = -\tilde{R}(x,t;\varepsilon), \qquad (x,t) \in D \\[3mm] z(x,0;\varepsilon) = z_t(x,0;\varepsilon) = 0, \qquad -\infty < x < \infty \end{cases}$$

where

$$\left\{\begin{array}{l} \tilde{d}(x,t,z;\varepsilon) = (z + U + f - \tilde{f})|z + U + f - \tilde{f}|^{\nu} \\[2ex] \qquad\qquad - (U + f - \tilde{f})|U + f - \tilde{f}|^{\nu} \\[3ex] \tilde{R}(x,t;\varepsilon) = (U + f - \tilde{f})|U + f - \tilde{f}|^{\nu} - U|U|^{\nu} \\[2ex] \qquad\qquad + \varepsilon c^2(x,t)\{\tilde{f}''(x;\varepsilon) - f''(x)\}. \end{array}\right.$$

We shall examine the right-hand side function $\tilde{R}$.

Since the function w_0 defined by (61) has the same zeroes as f it follows from (68) that

$$|w_0(x,t)|^{\nu}\{f(x) - \tilde{f}(x;\varepsilon)\} = O(\varepsilon^{p+\nu p}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$.
Using this result together with the formulae (69) and (75) and the inequality

$$(x + y)^{\alpha} \le x^{\alpha} + y^{\alpha} \quad \text{for } x \ge 0,\ y \ge 0 \text{ and } 0 < \alpha \le 1$$

we obtain for ε sufficiently small

$$\begin{aligned} \left| (U+f-\tilde{f})|U+f-\tilde{f}|^{\nu} - U|U|^{\nu} \right| &= (\nu+1)|f-\tilde{f}| \cdot |U + \Theta(f-\tilde{f})|^{\nu} \\[2ex] &\le (\nu+1)|f-\tilde{f}|\{|U|^{\nu} + |f-\tilde{f}|^{\nu}\} \\[2ex] &\le (\nu+1)|f-\tilde{f}|\{|w_0|^{\nu} + |f-\tilde{f}|^{\nu}\} + C\varepsilon^{p+\kappa\nu} \\[2ex] &\le C\varepsilon^{p+\kappa\nu} \end{aligned}$$

with $0 < \Theta < 1$ and C a generic constant independent of ε.
Hence it follows that

$$\tilde{R}(x,t;\varepsilon) = O(\varepsilon^{\lambda}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, where

$$(78) \qquad \lambda = \min\{1 - p,\ p + \nu p,\ p + \nu(\tfrac{3}{4} - p)\}\ .$$

Applying theorem 1 to (77) we obtain

$$z(x,t;\varepsilon) = O(\varepsilon^{\lambda-\frac{1}{4}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, provided that

$$(79) \qquad \lambda > \frac{1}{4\nu} + \frac{1}{4} .$$

Hence, using (69), (75) and (76), we obtain

$$(80) \qquad u(x,t;\varepsilon) = w_0(x,t) + O(\varepsilon^{\mu-\frac{1}{4}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, where

$$\mu = \min\{1 - p, \; p + \frac{1}{4}, \; p + \nu p, \; p + \nu(\frac{3}{4} - p)\}.$$

Next we shall determine the positive constant p. This constant has to be chosen such that the inequalities (74) and (79) are satisfied, where λ is given by (78). A straightforward calculation shows that these inequalities can only be fulfilled for some positive value of p if we restrict ourselves to values of ν satisfying

$$\nu > \frac{2}{3} .$$

In that case an optimal value $\mu = \frac{5}{8}$ is obtained if we choose $p = \frac{3}{8}$. So finally we obtain from (80)

$$u(x,t;\varepsilon) = w_0(x,t) + O(\varepsilon^{3/8}), \qquad \varepsilon \downarrow 0$$

uniformly valid in each compact subset $\Omega \subset \bar{D}$, provided that $\nu > \frac{2}{3}$.

We summarize the results in

THEOREM 4. *Under each of the following restrictions on ν and f there exists for sufficiently small values of ε a solution u of problem* (59) *under the conditions* (58), *which satisfies*

$$u(x,t;\varepsilon) = \frac{f(x)}{\{1 + \nu t|f(x)|^\nu\}^{1/\nu}} + O(\varepsilon^\alpha), \qquad \varepsilon \downarrow 0$$

uniformly in each compact subset $\Omega \subset \bar{D}$:

(i) $\nu = 2k$ ($k \in \mathbb{N}$); *in this case* $\alpha = 1$

(ii) $\nu > 0$ *and* f *has no zeroes; in this case* $\alpha = 1$

(iii) $\nu \geq 1$; *in this case* $\alpha = \dfrac{3}{4}$

(iv) $\dfrac{1}{3} < \nu < 1$ *and zeroes of* f *have multiplicity* $k \geq 2$; *in this case* $\alpha = \dfrac{3}{4}$

(v) $\dfrac{2}{3} < \nu < 1$; *in this case* $\alpha = \dfrac{3}{8}$.

Finally we remark that nonlinear initial-boundary value problems in more than one space variable, which are governed by a differential equation of the type discussed in this section, have been studied by GENET & MADAUNE [17] and MADAUNE [30]. It was shown amongst other things that the solution u of these kind of problems converges in some integral norm to the solution w_0 of the reduced problem as ε tends to zero.

A discussion of nonlinearities of the type $u|u|^{\nu}$ is also given by LIONS [29].

CHAPTER V

INITIAL – BOUNDARY VALUE PROBLEMS FOR

LINEAR HYPERBOLIC DIFFERENTIAL EQUATIONS

In this chapter we shall discuss the initial-boundary value problem (singular perturbation problem)

$$(1) \quad \begin{cases} \varepsilon L_2[u] + L_1^{(\varepsilon)}[u] = f(x,t;\varepsilon), & (x,t) \in D \\[2mm] u(x,0;\varepsilon) = g(x;\varepsilon), & 0 \le x \le \ell \\[2mm] u_t(x,0;\varepsilon) = h(x;\varepsilon), & 0 \le x \le \ell \\[2mm] u(0,t;\varepsilon) = k_1(t;\varepsilon), & 0 \le t < \infty \\[2mm] u(\ell,t;\varepsilon) = k_2(t;\varepsilon), & 0 \le t < \infty \end{cases}$$

where ε is a small positive parameter, i.e. $0 < \varepsilon \ll 1$, the region D is the strip

$$D = \{(x,t) \mid 0 < x < \ell,\ 0 < t < \infty\}$$

and the differential operators L_2 and $L_1^{(\varepsilon)}$ are defined by

$$\begin{cases} L_2[u] = \dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2} \\[4mm] L_1^{(\varepsilon)}[u] = a(x,t;\varepsilon)\dfrac{\partial u}{\partial t} + b(x,t;\varepsilon)\dfrac{\partial u}{\partial x} + d(x,t;\varepsilon)u. \end{cases}$$

The functions a, b, d, f, g, h, k_1 and k_2 depend on the variables x and/or t and on the parameter ε; the function c satisfies

$$c(x,t) > 0 \qquad \text{in } \bar{D},$$

which implies that L_2 is a hyperbolic operator.

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It will be assumed that the boundary functions g, k_1 and k_2 are continuous, no exception being made for the corners of the strip; so we require in particular

$$\begin{cases} g(0;\varepsilon) = k_1(0;\varepsilon) \\ \\ g(\ell;\varepsilon) = k_2(0;\varepsilon) \end{cases}$$

valid for all ε with $0 < \varepsilon \ll 1$.

Furthermore we shall impose the growth conditions discussed in section 1 of chapter III (cf. formula (7) of that chapter), that is, we assume the coefficient $a(x,t;\varepsilon)$ to be strictly positive and we require that the functions a, b and c are such that for sufficiently small values of ε the sub-characteristics of $L_1^{(\varepsilon)}$ are timelike with respect to the characteristic directions connected with L_2.

There exists few literature on singularly perturbed initial-boundary value problems for hyperbolic differential equations. For problems in $\mathbb{R}^2$ with constant coefficients an analysis based on Laplace transforms is given in WHITHAM [56, chapter 10]. Initial-boundary value problems with more than one space variable have been investigated by DZAVADOV [8] and GENET & PUPION [18] (linear problems) and by GENET & MADAUNE [17] and MADAUNE [30] (nonlinear problems).

Using a modified version of the energy integral method developed in chapter III we shall derive in section 1 a priori estimates for the solution u of problem (1). In section 2 we shall construct an approximation for the solution u of a problem of the type (1), in which the parameter ε occurs only as a factor in front of the operator L_2.

REMARK. It is possible to consider problem (1) with a more general hyperbolic operator L_2 defined by

$$L_2[u] = \frac{\partial^2 u}{\partial t^2} + \{c_1(x,t) + c_2(x,t)\}\frac{\partial^2 u}{\partial x \partial t} + c_1(x,t)c_2(x,t)\frac{\partial^2 u}{\partial x^2}$$

with $c_1(x,t) < 0 < c_2(x,t)$ in $\bar{D}$. However, since all essential features of the general theory become manifest in the special case where

$$-c_1(x,t) \equiv c_2(x,t) \equiv c(x,t) > 0 \qquad \text{in } \bar{D}$$

we prefer to discuss the latter for reasons of simplicity.

1. A priori estimates

We consider the initial-boundary value problem (1) under the following conditions

$$(2) \qquad \begin{cases} c(x,t) \in C^1(\bar{D}) \\\\ c(x,t) > 0 \qquad \text{in } \bar{D} \end{cases}$$

$$(3) \begin{cases}
\text{for each compact subset } \Omega \subset \bar{D} \text{ there exists a positive constant } \varepsilon_1 \text{ (de-} \\
\text{pending on } \Omega) \text{ such that} \\\\
\text{(i)} \quad a(x,t;\varepsilon) \in C^1(\Omega), \quad b(x,t;\varepsilon) \in C^1(\Omega) \\
\qquad d(x,t;\varepsilon) \in C^0(\Omega), \quad f(x,t;\varepsilon) \in C^0(\Omega) \\\\
\qquad \text{valid for every value of } \varepsilon \text{ satisfying } 0 < \varepsilon \leq \varepsilon_1 \\\\
\text{(ii)} \quad \text{the functions } a(x,t;\varepsilon), \ b(x,t;\varepsilon) \text{ and their first derivatives, as} \\
\qquad \text{well as the function } d(x,t;\varepsilon), \text{ are uniformly bounded in } \Omega \text{ for} \\
\qquad 0 < \varepsilon \leq \varepsilon_1 \\\\
\text{(iii)} \quad a(x,t;\varepsilon) \geq a_0 > 0 \\\\
\qquad \text{valid for all } (x,t) \in \Omega \text{ and all } \varepsilon \in (0,\varepsilon_1], \text{ where } a_0 \text{ is some con-} \\
\qquad \text{stant depending on } \Omega \text{ and } \varepsilon_1 \text{ but independent of } \varepsilon \\\\
\text{(iv)} \quad \begin{cases} a(x,t;\varepsilon)c(x,t) + b(x,t;\varepsilon) \geq p > 0 \\ a(x,t;\varepsilon)c(x,t) - b(x,t;\varepsilon) \geq p > 0 \end{cases} \\\\
\qquad \text{valid for all } (x,t) \in \Omega \text{ and all } \varepsilon \in (0,\varepsilon_1], \text{ where } p \text{ is some con-} \\
\qquad \text{stant depending on } \Omega \text{ and } \varepsilon_1 \text{ but independent of } \varepsilon
\end{cases}$$

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$\Big\{$ for each positive constant T there exists a positive constant ε_2 (depending on T) such that

$$g(x;\varepsilon) \in C^1[0,\ell], \quad h(x;\varepsilon) \in C^0[0,\ell]$$

$$k_1(t;\varepsilon) \in C^1[0,T], \quad k_2(t;\varepsilon) \in C^1[0,T]$$

(4) $\Big\{$ and

$$\begin{cases} g(0;\varepsilon) = k_1(0;\varepsilon) \\ g(\ell;\varepsilon) = k_2(0;\varepsilon) \end{cases}$$

valid for every value of ε satisfying $0 < \varepsilon \le \varepsilon_2$.

In order to derive a priori estimates for the solution $u(x,t;\varepsilon)$ of (1) in some compact subset $S \subset \bar{D}$ we enclose S by a rectangle

$$\Omega = \{(x,t) \mid 0 \le x \le \ell, \ 0 \le t \le T\}$$

where $T \ge \sup_{(x,t)\in S} t$ (see figure 1).
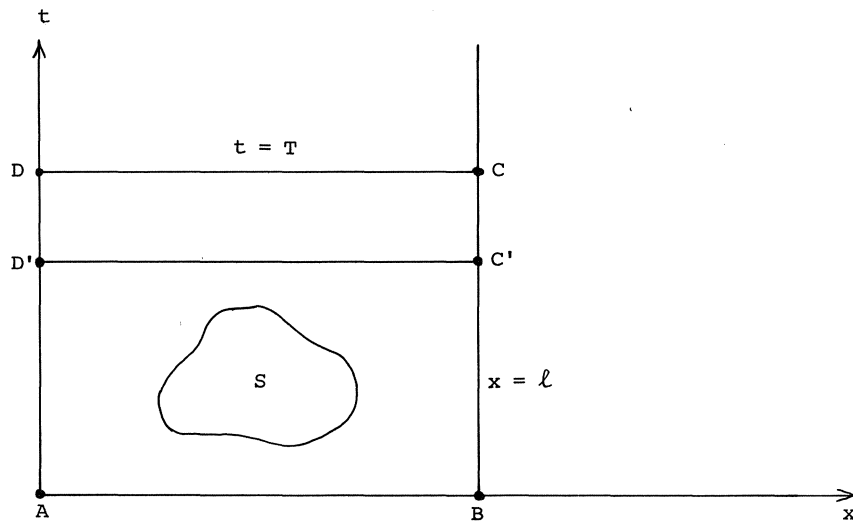
Figure 1

Let α be some positive constant *(independent of ε)* to be determined later on and let $\beta = \beta(x,t;\varepsilon)$ and $\gamma = \gamma(x,t;\varepsilon)$ be continuously differentiable functions (depending on the variables x and t and on the parameter ε) likewise to be specified later on.

We multiply the differential equation (1) by $2\alpha au$ to obtain

$$\frac{\partial}{\partial t}\{\alpha a^2 u^2 + 2\varepsilon\alpha a uu_t\} + \frac{\partial}{\partial x}\{\alpha abu^2 - 2\varepsilon\alpha ac^2 uu_x\} =$$

$$(5) \qquad u^2\{-2\alpha ad + \frac{\partial(\alpha a^2)}{\partial t} + \frac{\partial(\alpha ab)}{\partial x}\} + 2\varepsilon uu_t\frac{\partial(\alpha a)}{\partial t} - 2\varepsilon uu_x\frac{\partial(\alpha ac^2)}{\partial x}$$

$$+ 2\varepsilon\alpha au_t^2 - 2\varepsilon\alpha ac^2 u_x^2 + 2\alpha afu$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0 = \min(\varepsilon_1,\varepsilon_2)$, the values of ε_1 and ε_2 being determined by the region Ω (see the conditions (3,i) and (4)).

Multiplication of (1) by $2\beta u_t$ yields

$$\frac{\partial}{\partial t}\{\varepsilon\beta u_t^2 + \varepsilon\beta c^2 u_x^2\} - \frac{\partial}{\partial x}\{2\varepsilon\beta c^2 u_t u_x\} =$$

$$(6) \qquad \varepsilon u_t^2\frac{\partial\beta}{\partial t} - 2\varepsilon u_t u_x\frac{\partial(\beta c^2)}{\partial x} + \varepsilon u_x^2\frac{\partial(\beta c^2)}{\partial t}$$

$$- 2\beta duu_t - 2\beta au_t^2 - 2\beta bu_t u_x + 2\beta fu_t$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$.

Finally, multiplication of (1) by $2\gamma u_x$ leads to

$$\frac{\partial}{\partial t}\{2\varepsilon\gamma u_t u_x\} - \frac{\partial}{\partial x}\{\varepsilon\gamma u_t^2 + \varepsilon\gamma c^2 u_x^2\} =$$

$$(7) \qquad - \varepsilon u_t^2\frac{\partial\gamma}{\partial x} + 2\varepsilon u_t u_x\frac{\partial\gamma}{\partial t} - \varepsilon u_x^2\frac{\partial(\gamma c^2)}{\partial x}$$

$$- 2\gamma duu_x - 2\gamma au_t u_x - 2\gamma bu_x^2 + 2\gamma fu_x$$

likewise valid for $(x,t) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$.

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Adding (5), (6) and (7) we obtain

(8) $\qquad \frac{\partial}{\partial t} Q_1 + \frac{\partial}{\partial x} Q_2 = Q_3 + Q_4$

where the expressions Q_i (i = 1,2,3,4) are given by

(9)
$$Q_1 = \alpha a^2 u^2 + 2\epsilon\alpha a u u_t + \epsilon\beta u_t^2 + 2\epsilon\gamma u_t u_x + \epsilon\beta c^2 u_x^2$$

$$Q_2 = -\epsilon\gamma c^2 u_x^2 - 2\epsilon\alpha a c^2 u u_x - 2\epsilon\beta c^2 u_t u_x + \alpha a b u^2 - \epsilon\gamma u_t^2$$

$$Q_3 = u^2 \{-2\alpha a d + \frac{\partial(\alpha a^2)}{\partial t} + \frac{\partial(\alpha a b)}{\partial x}\} + 2\epsilon u u_t \frac{\partial(\alpha a)}{\partial t} - 2\epsilon u u_x \frac{\partial(\alpha a c^2)}{\partial x}$$

$$+ \epsilon u_t^2 \{\frac{\partial\beta}{\partial t} - \frac{\partial\gamma}{\partial x}\} + 2\epsilon u_t u_x \{\frac{\partial\gamma}{\partial t} - \frac{\partial(\beta c^2)}{\partial x}\} + \epsilon u_x^2 \{\frac{\partial(\beta c^2)}{\partial t} - \frac{\partial(\gamma c^2)}{\partial x}\}$$

$$Q_4 = -2\beta d u u_t - 2\gamma d u u_x + 2\alpha a f u + 2\beta f u_t + 2\gamma f u_x$$

$$- 2a(\beta - \epsilon\alpha)u_t^2 - 2(\gamma a + \beta b)u_t u_x - 2(\gamma b + \epsilon\alpha a c^2)u_x^2 .$$

Integrating (8) over the rectangle Ω and using GREEN's theorem in the plane
we obtain (see figure 1)

(10) $\qquad \int_D^C Q_1 dx = \int_A^B Q_1 dx + \int_A^D Q_2 dt - \int_B^C Q_2 dt + \iint_\Omega (Q_3 + Q_4) dx dt .$

We shall now specify the functions β and γ.

Let δ be an arbitrarily small positive number (independent of ϵ) sat-
isfying $0 < \delta < \frac{1}{2}\ell$. Let $\Psi(x)$ be some C^∞-function defined for $0 \le x \le \ell$
which satisfies

(11)
$$\Psi(x) = \begin{cases} 1 & \text{for } \delta \le x \le \frac{1}{2}\ell \\ 0 & \text{for } 0 \le x \le \frac{1}{2}\delta \end{cases}$$

$\Psi(x)$ monotonically increasing for $\frac{1}{2}\delta \le x \le \delta$

$\Psi(x) = \Psi(\ell - x) \qquad \text{for } \frac{1}{2}\ell \le x \le \ell$

(see figure 2a) and let the C^∞-function $\Phi(x;\varepsilon)$ by defined by

(12) $\Phi(x;\varepsilon) = \varepsilon + (1 - \varepsilon)\Psi(x)$ for $0 \leq x \leq \ell$
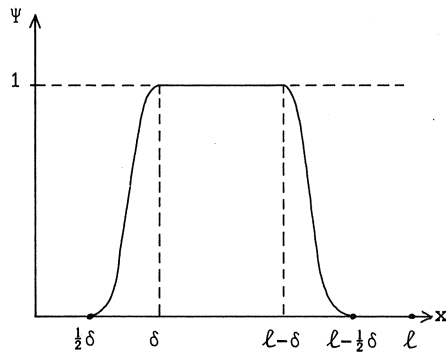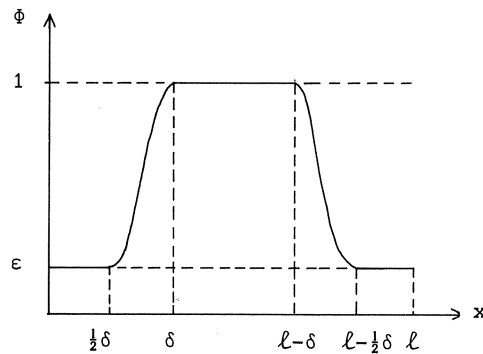
(see figure 2b).

Figure 2a   Figure 2b

We now choose the functions β and γ as follows

(13)
$$\begin{cases} \beta = \beta(x,t;\varepsilon) = a(x,t;\varepsilon)\Phi(x;\varepsilon) + 2\varepsilon\alpha = a\Phi + 2\varepsilon\alpha \\ \\ \gamma = \gamma(x,t;\varepsilon) = b(x,t;\varepsilon)\Phi(x;\varepsilon) + \varepsilon\phi(x) = b\Phi + \varepsilon\phi \end{cases}$$

where $\phi(x)$ is some C^∞-function defined for $0 \leq x \leq \ell$, which will be specified later on. This leaves the constant α and the function ϕ to be determined.

Now that we have chosen β and γ we shall have a closer look at the expressions Q_i ($i = 1,2,3,4$). Let us start with Q_2. The integrals along AD and BC in the right-hand side of (10) contain in their integrands the function u_x that we do not know a priori along AD and BC. This difficulty can be overcome by choosing the function ϕ such that $\gamma \geq \varepsilon$ along AD and $\gamma \leq -\varepsilon$ along BC. Since we have from (11), (12) and (13)

$$\begin{cases} \gamma(0,t;\epsilon) = \epsilon b(0,t;\epsilon) + \epsilon\phi(0) & \text{along AD} \\ \\ \gamma(\ell,t;\epsilon) = \epsilon b(\ell,t;\epsilon) + \epsilon\phi(\ell) & \text{along BC} \end{cases}$$

this can be achieved by choosing ϕ such that

$$\begin{cases} \phi(0) \geq B + 1 \\ \\ \phi(\ell) \leq -B - 1 \end{cases}$$

where the ϵ-independent nonnegative constant B is defined by

$$B = \sup_{\substack{(x,t)\in\Omega \\ 0<\epsilon\leq\epsilon_0}} |b(x,t;\epsilon)|$$

(cf. the condition (3,ii)). A possible choice for ϕ is given by

$$\phi(x) = (B + 1)(1 - \frac{2x}{\ell}) \qquad \text{for } 0 \leq x \leq \ell.$$

With this choice we obtain along AD, using (9), (13) and the inequality $\gamma(0,t;\epsilon) \geq \epsilon$

$$
\begin{aligned}
Q_2 &\leq -\epsilon^2 c^2 u_x^2 - 2\epsilon\alpha ac^2 uu_x - 2\epsilon^2(a + 2\alpha)c^2 u_t u_x + \alpha abu^2 - \epsilon^2 u_t^2 \\
&= -2c^2(\tfrac{1}{2}\epsilon u_x + \alpha au)^2 - 2\epsilon^2 c^2\{\tfrac{1}{2}u_x + (a + 2\alpha)u_t\}^2 \\
&\quad + (\alpha ab + 2\alpha^2 a^2 c^2)u^2 + \epsilon^2\{2(a + 2\alpha)^2 c^2 - 1\}u_t^2 \\
&\leq M_1(u^2 + \epsilon^2 u_t^2)
\end{aligned}
$$
(14)

and similarly along BC, using (9), (13) and the inequality $\gamma(\ell,t;\epsilon) \leq -\epsilon$

$$
\begin{aligned}
Q_2 &\geq \epsilon^2 c^2 u_x^2 - 2\epsilon\alpha ac^2 uu_x - 2\epsilon^2(a + 2\alpha)c^2 u_t u_x + \alpha abu^2 + \epsilon^2 u_t^2 \\
&= 2c^2(\tfrac{1}{2}\epsilon u_x - \alpha au)^2 + 2\epsilon^2 c^2\{\tfrac{1}{2}u_x - (a + 2\alpha)u_t\}^2 \\
&\quad + (\alpha ab - 2\alpha^2 a^2 c^2)u^2 + \epsilon^2\{1 - 2(a + 2\alpha)^2 c^2\}u_t^2 \geq
\end{aligned}
$$
(15)

$$\geq -M_2(u^2 + \varepsilon^2 u_t^2)$$

for certain positive constants M_1 and M_2 depending on Ω but independent of ε.

As for the expression Q_3, it is easily established that there exist positive constants M_3 (depending on Ω and α but independent of ε) and M_4 (depending on Ω but *independent of α and ε*) such that

$$(16) \qquad Q_3 \leq M_3(u^2 + \varepsilon^2 u_t^2 + \varepsilon^2 u_x^2) + \varepsilon M_4(u_t^2 + u_x^2)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$. The independence of M_4 with respect to α follows from the observation that the terms $a\Phi$ and $b\Phi$ in (13) are independent of α.

Next we consider Q_4. Substituting (13) into (9) we obtain after some calculations

$$Q_4 = -2\Phi\{(au_t + bu_x)^2 + (du - f)(au_t + bu_x)\}$$

$$- 4\varepsilon\alpha duu_t - 2\varepsilon\phi duu_x + 2\alpha afu + 4\varepsilon\alpha fu_t + 2\varepsilon\phi fu_x$$

$$- 2\varepsilon\alpha\{au_t^2 + (2b + \frac{a\phi}{\alpha})u_t u_x + (ac^2 + \frac{b\phi}{\alpha})u_x^2\}.$$

Using (3,iv) it is not difficult to verify that there exists a positive constant p_0, *independent of α and ε* and satisfying

$$0 < p_0 < 1 - \sup_{\substack{(x,t)\in\Omega \\ 0 < \varepsilon \leq \varepsilon_0}} \frac{|b|}{ac} \quad ,$$

such that for sufficiently large α, say $\alpha > \alpha_1$

$$au_t^2 + (2b + \frac{a\phi}{\alpha})u_t u_x + (ac^2 + \frac{b\phi}{\alpha})u_x^2 \geq p_0(au_t^2 + ac^2 u_x^2)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

Hence it follows that there exist for $\alpha > \alpha_1$ positive constants M_5 (depending on Ω and α but independent of ε) and M_6 (depending on Ω but *independent of α and ε*) such that

(17) $\qquad Q_4 \leq M_5(u^2 + \varepsilon^2 u_t^2 + \varepsilon^2 u_x^2 + f^2) - \varepsilon \alpha M_6(u_t^2 + u_x^2)$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

Taking α such that

$$\alpha > \alpha_2 = \max(\alpha_1, \frac{M_4}{M_6})$$

we obtain from (16) and (17), using (12)

(18)
$$Q_3 + Q_4 \leq (M_3 + M_5)(u^2 + \varepsilon^2 u_t^2 + \varepsilon^2 u_x^2 + f^2)$$

$$\leq (M_3 + M_5)(u^2 + \varepsilon \Phi u_t^2 + \varepsilon \Phi u_x^2 + f^2)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

Finally we investigate the expression Q_1. Substituting (13) into (9) we obtain

$$Q_1 = \alpha a^2 u^2 + 2\varepsilon \alpha a u u_t + 2\varepsilon^2 \alpha u_t^2 + 2\varepsilon^2 \phi u_t u_x + 2\varepsilon^2 \alpha c^2 u_x^2$$

$$+ \varepsilon a \Phi u_t^2 + 2\varepsilon b \Phi u_t u_x + \varepsilon a c^2 \Phi u_x^2$$

$$= \frac{1}{4}\alpha a^2 u^2 + \frac{3}{4}\alpha (au + \frac{4}{3}\varepsilon u_t)^2 + 2\varepsilon^2(\frac{1}{3}\alpha u_t^2 + \phi u_t u_x + \alpha c^2 u_x^2)$$

$$+ \varepsilon \Phi (a u_t^2 + 2b u_t u_x + a c^2 u_x^2).$$

It follows from the conditions (3,iii) and (3,iv) that there exists a positive constant m^*, depending on Ω but independent of ε, such that

$$a u_t^2 + 2b u_t u_x + a c^2 u_x^2 \geq m^*(u_t^2 + u_x^2)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$. Moreover, the expression

$$\frac{1}{3}\alpha u_t^2 + \phi u_t u_x + \alpha c^2 u_x^2$$

is definite positive if we choose α sufficiently large, say $\alpha > \alpha_3$. So taking for the constant α

$$\alpha = \alpha_0 = \max(\alpha_2, \alpha_3)$$

and using the condition (3,iii) we obtain the inequality

$$(19) \qquad Q_1 \geq m(u^2 + \varepsilon\Phi u_t^2 + \varepsilon\Phi u_x^2)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$, where m is some positive constant depending on Ω but independent of ε.
It is also easily established that

$$(20) \qquad Q_1 \leq M_7(u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)$$

valid for $(x,t) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$ with M_7 some positive constant depending on Ω but independent of ε.

Putting

$$M = \max(M_1, M_2, M_3 + M_5, M_7)$$

we obtain from (10), (14), (15), (18), (19) and (20)

$$(21) \qquad m \int_D^C (u^2 + \varepsilon\Phi u_t^2 + \varepsilon\Phi u_x^2)dx \leq M \iint_{ABCD} (u^2 + \varepsilon\Phi u_t^2 + \varepsilon\Phi u_x^2)dxdt + MK^2(\Omega,\varepsilon)$$

valid for $0 < \varepsilon \leq \varepsilon_0$ where the positive constant $K(\Omega,\varepsilon)$, depending on Ω and ε, is defined by

$$K(\Omega,\varepsilon) = \|f\|_\Omega + \|g\|_{[0,\ell]} + \|k_1\|_{[0,T]} + \|k_2\|_{[0,T]}$$
$$+ \varepsilon^{\frac{1}{2}}\|g_x\|_{[0,\ell]} + \varepsilon^{\frac{1}{2}}\|h\|_{[0,\ell]} + \varepsilon\|k_{1_t}\|_{[0,T]} + \varepsilon\|k_{2_t}\|_{[0,T]}$$

with $\|\cdot\|_\Omega$ denoting the L_2-norm over Ω, etc.
Since (21) with the same values of m and M is also valid for every rectangle $\Omega' = ABC'D'$ inside $\Omega = ABCD$ (see figure 1) we obtain as in section 2 of chapter III, using GRONWALL's lemma, the estimate

$$\int_0^\ell (u^2 + \varepsilon\Phi u_t^2 + \varepsilon\Phi u_x^2)dx \leq C(\Omega)K^2(\Omega,\varepsilon)$$

(cf. formula (26) of chapter III), whence particularly

(22) $$\int_0^{\ell} (u^2 + \varepsilon^2 u_t^2 + \varepsilon^2 u_x^2)\,dx \leq C(\Omega)K^2(\Omega,\varepsilon)$$

and

(23) $$\int_{\delta}^{\ell-\delta} (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)\,dx \leq C(\Omega)K^2(\Omega,\varepsilon)$$

valid for $0 \leq t \leq T$ and $0 < \varepsilon \leq \varepsilon_0$; the constant $C(\Omega)$, depending on Ω but independent of ε, is given by

$$C(\Omega) = \frac{M}{m} \exp\left(\frac{MT}{m}\right).$$

Pointwise estimates for $u(x,t;\varepsilon)$ can be obtained from (22) and (23) by applying SOBOLEV's inequality (cf. AGMON [1, p.32]; see also formula (33a) of chapter III). We formulate the results in

THEOREM 1. *Let* $u(x,t;\varepsilon)$ *be the solution of the initial-boundary value problem* (1) *under the conditions* (2), (3) *and* (4), *and let* Ω *be the closed rectangle* $0 \leq x \leq \ell$, $0 \leq t \leq T$. *Then we have for* ε *sufficiently small, say* $0 < \varepsilon \leq \varepsilon_0$ *with* ε_0 *depending on* Ω, *the following estimates in* L_2-*norm*

$$\int_0^{\ell} (u^2 + \varepsilon^2 u_t^2 + \varepsilon^2 u_x^2)\,dx \leq C(\Omega)K^2(\Omega,\varepsilon) \qquad \text{valid for } 0 \leq t \leq T$$

$$\int_{\delta}^{\ell-\delta} (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)\,dx \leq C(\Omega)K^2(\Omega,\varepsilon) \qquad \text{valid for } 0 \leq t \leq T.$$

We have also the following pointwise estimates

(24a) $$\sup_{\Omega} |u(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-\frac{1}{2}}K(\Omega,\varepsilon)$$

(24b) $$\sup_{\substack{\delta \leq x \leq \ell-\delta \\ 0 \leq t \leq T}} |u(x,t;\varepsilon)| \leq C(\Omega)\varepsilon^{-\frac{1}{4}}K(\Omega,\varepsilon).$$

Here $C(\Omega)$ *is some positive constant depending on* Ω *but independent of* ε, δ *is an arbitrarily small positive constant independent of* ε *and satisfying*

$0 < \delta < \frac{1}{2}\ell.$

The positive constant $K(\Omega,\epsilon)$, *depending on both* Ω *and* ϵ, *is given by*

(25)
$$K(\Omega,\epsilon) = \|f\|_{\Omega} + \|g\|_{[0,\ell]} + \|k_1\|_{[0,T]} + \|k_2\|_{[0,T]}$$
$$+ \epsilon^{\frac{1}{2}}\|g_x\|_{[0,\ell]} + \epsilon^{\frac{1}{2}}\|h\|_{[0,\ell]} + \epsilon\|k_{1_t}\|_{[0,T]} + \epsilon\|k_{2_t}\|_{[0,T]}$$

with $\|\cdot\|_{\Omega}$ *denoting the* L_2-*norm over* Ω, *etc.*

REMARKS.

1. It is possible to obtain pointwise estimates for the derivatives u_x and u_t by taking into account integrals along characteristics (cf. chapter III, section 2). It follows after tedious calculations that

$$\begin{cases} \sup_{\Omega} \; |\frac{\partial u}{\partial x}(x,t;\epsilon)| \; \leq \; C(\Omega)\epsilon^{-2}K_1(\Omega,\epsilon) \\[2ex] \sup_{\Omega} \; |\frac{\partial u}{\partial t}(x,t;\epsilon)| \; \leq \; C(\Omega)\epsilon^{-2}K_1(\Omega,\epsilon), \end{cases}$$

the positive constant $K_1(\Omega,\epsilon)$ being given by

$$K_1(\Omega,\epsilon) = \sup_{\Omega} \; |f| + \|g\|_{[0,\ell]} + \|k_1\|_{[0,T]} + \|k_2\|_{[0,T]}$$
$$+ \epsilon \sup_{[0,\ell]} |g_x| + \epsilon \sup_{[0,\ell]} |h| + \epsilon \sup_{[0,T]} |k_{1_t}| + \epsilon \sup_{[0,T]} |k_{2_t}|$$

(cf. the formulae (33b), (33c) of chapter III).

2. Other estimates are possible if it is given that

$$b(0,t;\epsilon) \geq b_0 > 0 \quad \text{for } 0 \leq t \leq T \text{ and } 0 < \epsilon \leq \epsilon_0$$

where b_0 is some constant depending on T and ϵ_0 but independent of ϵ; this condition implies the absence of a boundary layer at $x = 0$, $t \geq 0$. Replacing (11) by

$$\begin{cases} \Psi(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \ell-\delta \\[1ex] 0 & \text{for } \ell - \frac{1}{2}\delta \; \leq x \leq \ell \end{cases} \\[3ex] \Psi(x) \text{ monotonically decreasing for } \ell-\delta \leq x \leq \ell - \frac{1}{2}\delta \end{cases}$$

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and adjusting the proof of theorem 1 we obtain

(26a)
$$\int_0^{\ell-\delta} (u^2 + \varepsilon u_t^2 + \varepsilon u_x^2)\,dx \le C(\Omega)K_2^2(\Omega,\varepsilon) \qquad \text{valid for } 0 \le t \le T$$

(26b)
$$\sup_{\substack{0 \le x \le \ell-\delta \\ 0 \le t \le T}} |u(x,t;\varepsilon)| \le C(\Omega)\varepsilon^{-\frac{1}{4}}K_2(\Omega,\varepsilon)$$

with

(27)
$$K_2(\Omega,\varepsilon) = \|f\|_\Omega + \|g\|_{[0,\ell]} + \|k_1\|_{[0,T]} + \|k_2\|_{[0,T]}$$
$$+ \varepsilon^{\frac{1}{2}}\|g_x\|_{[0,\ell]} + \varepsilon^{\frac{1}{2}}\|h\|_{[0,\ell]} + \varepsilon^{\frac{1}{2}}\|k_{1_t}\|_{[0,T]} + \varepsilon\|k_{2_t}\|_{[0,T]}$$

and δ an arbitrarily small positive constant independent of ε and satisfying $0 < \delta < \ell$.

Similar refinements are possible if

$$b(\ell,t;\varepsilon) \le -b_0 < 0 \qquad \text{for } 0 \le t \le T \text{ and } 0 < \varepsilon \le \varepsilon_0.$$

2. Approximations

As an application of the a priori estimates that we derived in the preceding section we shall now construct an approximation for the solution $u(x,t;\varepsilon)$ of the initial-boundary value problem

(28)
$$\begin{cases} \varepsilon L_2[u] + L_1[u] = f(x,t), & (x,t) \in D \\[2mm] u(x,0;\varepsilon) = g(x), & 0 \le x \le \ell \\[2mm] u_t(x,0;\varepsilon) = h(x), & 0 \le x \le \ell \\[2mm] u(0,t;\varepsilon) = k_1(t), & 0 \le t < \infty \\[2mm] u(\ell,t;\varepsilon) = k_2(t), & 0 \le t < \infty \end{cases}$$

where $0 < \varepsilon \ll 1$ and

$$\begin{cases} L_2[u] = \dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2} \\ \\ L_1[u] = a(x,t)\dfrac{\partial u}{\partial t} + b(x,t)\dfrac{\partial u}{\partial x} + d(x,t)u. \end{cases}$$

In order not to complicate the calculations it will be assumed that all data are C^∞-functions which in addition satisfy

(29a) $\quad\qquad a(x,t) > 0, \quad c(x,t) > 0 \qquad$ in $\bar{D}$

(29b) $\quad\qquad 0 < b(x,t) < a(x,t)c(x,t) \qquad$ in $\bar{D}$

(29c) $\quad\qquad g(0) = k_1(0), \quad g(\ell) = k_2(0).$

REMARK. The assumption $b(x,t) > 0$ has been made for reasons of simplicity only. We emphasize that the results to be obtained in this section remain valid if less stringent conditions are imposed on the coefficient $b(x,t)$. These conditions will be specified at the end of this section. For the moment we confine ourselves to the remark that any problem with

$$-a(x,t)c(x,t) < b(x,t) < 0 \qquad \text{in } \bar{D}$$

instead of (29b) can be reduced to a problem of the above mentioned type by means of a transformation of coordinates, namely $x' = \ell-x$, $t' = t$.

As a first approximation for the solution u of (28) we could take the solution w of the reduced problem

(30) $\quad \begin{cases} a(x,t)\dfrac{\partial w}{\partial t} + b(x,t)\dfrac{\partial w}{\partial x} + d(x,t)w = f(x,t), \qquad (x,t) \in D \\ \\ w(x,0) = g(x), \qquad 0 \le x \le \ell \\ \\ w(0,t) = k_1(t), \qquad 0 \le t < \infty. \end{cases}$

However, the solution of this problem is generally not twice differentiable in D, so the quantity $L_2[w]$ cannot be evaluated. It will be clear that this difficulty has to be overcome before we can start with the construction of a formal approximation.

2.1. A lemma

It is known that the solution of a problem of the type (30) belongs to the class $C^n(\bar{D})$ if the data satisfy certain compatibility conditions (see for instance DZAVADOV [8, p.1402]). In this subsection we state a condition which is necessary and sufficient for the solution w of problem (30) to be continuously differentiable in $\bar{D}$. A consequence is that under this condition the second derivatives of w (and hence also the expression $L_2[w]$) remain bounded in compact subsets of $\bar{D}$. In that case w can be considered as a first approximation for the solution u of (28), since the quantity $\varepsilon L_2[w]$ is of order ε, uniformly in compact subsets of $\bar{D}$. The necessary and sufficient condition to be stated is closely related to the compatibility conditions mentioned before, but by the form in which it is written it has the advantage of being easier applicable. Since a proof of the correctness of compatibility conditions is not to be found in most text-books we shall for reasons of completeness state the results in the form of a lemma and give a detailed proof.

LEMMA 1. *Let* u(x,y) *be the solution of the boundary value problem*

$$
(31) \quad
\begin{cases}
a(x,y)\dfrac{\partial u}{\partial x} + b(x,y)\dfrac{\partial u}{\partial y} + c(x,y)u = f(x,y), \qquad (x,y) \in D \\[2ex]
u(x,0) = g(x), \qquad 0 \le x < \infty \\[1ex]
u(0,y) = h(y), \qquad 0 \le y < \infty
\end{cases}
$$

where the region D *is given by*

$$
D = \{(x,y) \mid 0 < x, \; y < \infty\}
$$

and all data are supposed to be C^∞*-functions which in addition satisfy the conditions*

$$
\begin{cases}
a(x,y) > 0, \quad b(x,y) > 0 \qquad \text{in } \bar{D} \\[2ex]
g(0) = h(0).
\end{cases}
$$

Then the function u(x,y) *belongs to* $C^1(\bar{D})$ *iff*

(32) $a(0,0)g'(0) + b(0,0)h'(0) + c(0,0)g(0) = f(0,0).$

In that case the second derivatives of $u(x,y)$ *exist in* $\bar{D}$ *and are continuous, with possible exception of the characteristic through the origin where the second derivative in the direction perpendicular to this characteristic may exhibit a finite jump.*

PROOF. From the theory of first order partial differential equations it is known that the solution u of (31) under the above mentioned conditions is infinitely differentiable in $\bar{D}$, with the exception of the characteristic through the origin. Therefore we only have to investigate the behaviour of u in a neighborhood of this characteristic.

The necessity of the condition (32) follows immediately if we let the point (x,y) tend to (0,0) in the differential equation (31). In order to prove the sufficiency we shall first assume that $c(x,y) \equiv 0$ in $\bar{D}$. In that case problem (31) reduces to

(33)
$$\begin{cases} a(x,y)\dfrac{\partial u}{\partial x} + b(x,y)\dfrac{\partial u}{\partial y} = f(x,y) & \text{in D} \\[2mm] u(x,0) = g(x), & 0 \le x < \infty \\[2mm] u(0,y) = h(y), & 0 \le y < \infty. \end{cases}$$

Let $\tilde{a}(x,y)$ and $\tilde{b}(x,y)$ be positive functions defined in

$$\bar{D} \cup \{(x,y) \mid -\infty < x < 0,\ 0 \le y \le 2\}$$

which satisfy

$$\tilde{a}(x,y) \equiv a(x,y), \quad \tilde{b}(x,y) \equiv b(x,y) \qquad \text{in } \bar{D}.$$

It is possible to choose these functions such that they are n times continuously differentiable (n an arbitrarily large positive integer). A possible choice for $\tilde{a}(x,y)$ is for instance

$$\tilde{a}(x,y) = \begin{cases} \Psi(-3\delta-x) + \Psi(x)\displaystyle\sum_{k=0}^{n} \frac{x^k}{k!}\frac{\partial^k a}{\partial x^k}(0,y) & \text{for } -\infty<x\le0,\ 0\le y\le2 \\[3mm] a(x,y) & \text{for } 0 \le x,\ y < \infty \end{cases}$$

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where δ is a sufficiently small positive constant and $\Psi(x)$ is some C^∞-function defined in $\mathbb{R}$ which satisfies

$$\begin{cases} \Psi(x) = \begin{cases} 1 & \text{for } x \geq -\delta \\ 0 & \text{for } x \leq -2\delta \end{cases} \\ \\ \Psi(x) \text{ monotonically increasing for } -2\delta \leq x \leq -\delta. \end{cases}$$

A similar choice is possible for $\tilde{b}(x,y)$.

Let $x = \Phi_1(y)$, $y \geq 0$ denote the monotonically increasing solution of

$$\frac{dx}{dy} = \frac{\tilde{a}(x,y)}{\tilde{b}(x,y)}, \quad x(1) = 0.$$

The graph of this solution intersects the x-axis at $x = -x_0$ (see figure 3). Let $x = \Phi_2(y)$, $y \geq 0$ denote the monotonically increasing solution of

$$\frac{dx}{dy} = \frac{a(x,y)}{b(x,y)}, \quad x(0) = x_0.$$

Finally, let $x = \Phi(y)$, $y \geq 0$ denote the solution of

$$\frac{dx}{dy} = \frac{a(x,y)}{b(x,y)}, \quad x(0) = 0.$$

This function is likewise monotonically increasing for $y \geq 0$ (see figure 3).

We are going to apply a coordinate transformation to the region Ω defined by

$$\begin{cases} \Omega = \Omega_- \cup \Omega_+ \\ \\ \Omega_- = \{(x,y) \mid \max(0,\Phi_1(y)) < x \leq \Phi(y), \ y > 0\} \\ \\ \Omega_+ = \{(x,y) \mid \Phi(y) \leq x < \Phi_2(y), \ y > 0\} \end{cases}$$

(see figure 3). To this end we introduce functions ϕ and ψ which are the solutions of the boundary value problems

$$(34) \quad \begin{cases} \tilde{a}(x,y)\dfrac{\partial\phi}{\partial x} + \tilde{b}(x,y)\dfrac{\partial\phi}{\partial y} = 0, \quad \Phi_1(y) < x < \Phi_2(y), \ y > 0 \\[3mm] \phi(x,0) = x, \quad -x_0 \le x \le x_0 \end{cases}$$

and

$$(35) \quad \begin{cases} b(x,y)\dfrac{\partial\psi}{\partial x} - a(x,y)\dfrac{\partial\psi}{\partial y} = 0 \quad \text{in } \Omega \\[3mm] \psi(\Phi(y),y) = y, \quad y \ge 0. \end{cases}$$

Note that the functions Φ, Φ_2 and ψ are C^∞-functions. The functions Φ_1 and ϕ can be made arbitrarily smooth by choosing the coefficients $\tilde{a}$ and $\tilde{b}$ sufficiently smooth.

We define new coordinates ξ and η by

$$\begin{cases} \xi = \phi(x,y) \quad \text{for } (x,y) \in \bar{\Omega} \\[2mm] \eta = \psi(x,y) \quad \text{for } (x,y) \in \bar{\Omega}. \end{cases}$$

This transformation transforms the characteristics of the operator

$$a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

and their orthogonal trajectories into lines ξ = constant and η = constant. The characteristic through the origin is transformed into the line $\xi = 0$ (see figure 3).
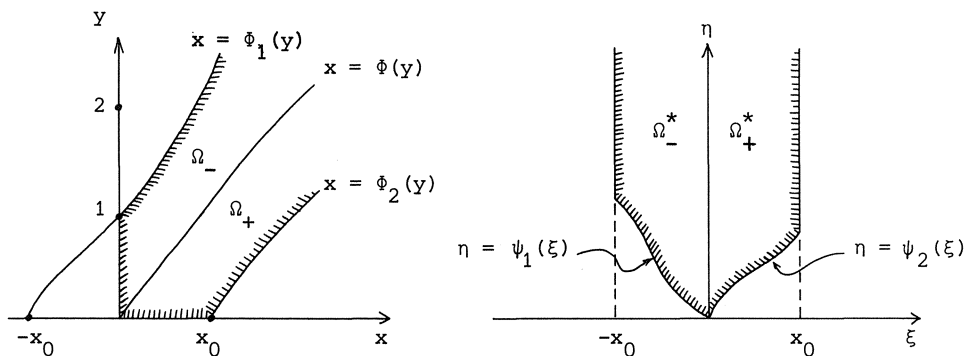

Figure 3

Problem (33) (restricted to the region $\bar{\Omega}$) becomes after the above mentioned transformation

(36)
$$\begin{cases} (a\psi_x + b\psi_y)\dfrac{\partial u}{\partial \eta} = f & \text{for } (\xi,\eta) \in \Omega^* \\[2mm] u(\xi,\psi_1(\xi)) = h(\phi_1^{-1}(\xi)), & -x_0 \le \xi \le 0 \\[2mm] u(\xi,\psi_2(\xi)) = g(\phi_2^{-1}(\xi)), & 0 \le \xi \le x_0 \end{cases}$$

where the region Ω^* is given by

$$\begin{cases} \Omega^* = \Omega_-^* \cup \Omega_+^* \\[2mm] \Omega_-^* = \{(\xi,\eta) \mid -x_0 < \xi \le 0, \ \eta > \psi_1(\xi)\} \\[2mm] \Omega_+^* = \{(\xi,\eta) \mid 0 \le \xi < x_0, \ \eta > \psi_2(\xi)\}. \end{cases}$$

The function ϕ_1^{-1} is the inverse of the monotonically decreasing function $\phi_1(y) = \phi(0,y)$, $0 \le y \le 1$; the function ϕ_2^{-1} is the inverse of the monotonically increasing function $\phi_2(x) = \phi(x,0)$, $0 \le x \le x_0$. The functions ψ_1 and ψ_2 (monotonically decreasing for $-x_0 \le \xi \le 0$ resp. monotonically increasing for $0 \le \xi \le x_0$) are defined by

$$\begin{cases} \psi_1(\xi) = \psi(0,\phi_1^{-1}(\xi)), & -x_0 \le \xi \le 0 \\[2mm] \psi_2(\xi) = \psi(\phi_2^{-1}(\xi),0), & 0 \le \xi \le x_0. \end{cases}$$

Since $a\psi_x + b\psi_y \neq 0$ in $\bar{D}$ we obtain from (36) the solution

(37)
$$u(\xi,\eta) = \begin{cases} \displaystyle\int_{\psi_1(\xi)}^{\eta} \left(\frac{f}{a\psi_x+b\psi_y}\right)(\xi,\tau)\,d\tau + h(\phi_1^{-1}(\xi)) & \text{in } \overline{\Omega_-^*} \\[6mm] \displaystyle\int_{\psi_2(\xi)}^{\eta} \left(\frac{f}{a\psi_x+b\psi_y}\right)(\xi,\tau)\,d\tau + g(\phi_2^{-1}(\xi)) & \text{in } \overline{\Omega_+^*}. \end{cases}$$

From this formula it follows immediately that u is continuous in $\overline{\Omega^*}$ and infinitely differentiable with respect to η. It also follows that u is n times differentiable with respect to ξ, provided that $\xi \neq 0$ (n an arbitrarily

large positive integer). We obtain in $\overline{\Omega_-^*}$

$$\frac{\partial u}{\partial \xi} = \int_{\psi_1(\xi)}^{\eta} \frac{\partial}{\partial \xi}\left(\frac{f}{a\psi_x + b\psi_y}\right)(\xi,\tau)d\tau + \frac{h'(\phi_1^{-1}(\xi))}{\phi_y(0,\phi_1^{-1}(\xi))}$$

(38)

$$- \frac{\psi_y(0,\phi_1^{-1}(\xi))}{\phi_y(0,\phi_1^{-1}(\xi))}\left(\frac{f}{a\psi_x + b\psi_y}\right)(\xi,\psi_1(\xi))$$

and in $\overline{\Omega_+^*}$

$$\frac{\partial u}{\partial \xi} = \int_{\psi_2(\xi)}^{\eta} \frac{\partial}{\partial \xi}\left(\frac{f}{a\psi_x + b\psi_y}\right)(\xi,\tau)d\tau + \frac{g'(\phi_2^{-1}(\xi))}{\phi_x(\phi_2^{-1}(\xi),0)}$$

(39)

$$- \frac{\psi_x(\phi_2^{-1}(\xi),0)}{\phi_x(\phi_2^{-1}(\xi),0)}\left(\frac{f}{a\psi_x + b\psi_y}\right)(\xi,\psi_2(\xi)).$$

Using the smoothness of the solutions ϕ and ψ we obtain from (34) and (35) after a straightforward calculation

$$\begin{cases} \phi_x(0,0) = 1 \\ \phi_y(0,0) = -\frac{a(0,0)}{b(0,0)} \\ \psi_x(0,0) = \frac{a(0,0)b(0,0)}{a^2(0,0) + b^2(0,0)} \\ \psi_y(0,0) = \frac{b^2(0,0)}{a^2(0,0) + b^2(0,0)} \end{cases}$$

Hence it follows from (38) and (39) that

$$\frac{\partial u}{\partial \xi}(+0,\eta) - \frac{\partial u}{\partial \xi}(-0,\eta) = \left[\frac{\psi_y(0,0)}{\phi_y(0,0)} - \frac{\psi_x(0,0)}{\phi_x(0,0)}\right]\left(\frac{f}{a\psi_x + b\psi_y}\right)(0,0)$$

$$+ \frac{g'(0)}{\phi_x(0,0)} - \frac{h'(0)}{\phi_y(0,0)}$$

$$= \frac{1}{a(0,0)}\{a(0,0)g'(0) + b(0,0)h'(0) - f(0,0)\}$$

$$= 0$$

because of (32) (with $c(0,0) = 0$); so $\frac{\partial u}{\partial \xi}$ is continuous in $\overline{\Omega}^*$.

It has now been shown that u_ξ and u_η are continuous in $\overline{\Omega}^*$, whence it follows that u_x and u_y are continuous in $\overline{\Omega}$. Thus we have established the sufficiency of (32) in case $c(x,y) \equiv 0$ in $\overline{D}$. The other statements of the lemma follow directly from (38) and (39) by differentiation; in particular, it follows that the second derivative $u_{\xi\xi}$ in the direction perpendicular to the characteristic through the origin may exhibit a finite jump across this characteristic. The second derivatives $u_{\xi\eta}$ and $u_{\eta\eta}$ are continuous in $\overline{\Omega}^*$.

If $c(x,y) \not\equiv 0$ we put

$$u = w \, e^v$$

where w and v are the solutions of

$$\begin{cases} a(x,y)\dfrac{\partial v}{\partial x} + b(x,y)\dfrac{\partial v}{\partial y} = -c(x,y), \qquad (x,y) \in D \\[2ex] v(x,0) = -\dfrac{c(0,0)}{a(0,0)}\,x, \qquad 0 \le x < \infty \\[2ex] v(0,y) = 0, \qquad 0 \le y < \infty \end{cases}$$

and

$$(40) \quad \begin{cases} a(x,y)\dfrac{\partial w}{\partial x} + b(x,y)\dfrac{\partial w}{\partial y} = f(x,y)\exp[-v(x,y)], \qquad (x,y) \in D \\[2ex] w(x,0) = g(x)\exp[\dfrac{c(0,0)}{a(0,0)}\,x], \qquad 0 \le x < \infty \\[2ex] w(0,y) = h(y), \qquad 0 \le y < \infty. \end{cases}$$

These boundary value problems are both of the type (33). It follows from the preceding analysis that $v(x,y) \in C^1(\overline{D})$ and also, because of the condition (32), that $w(x,y) \in C^1(\overline{D})$ (note that the function $f(x,y)\exp[-v(x,y)]$ in (40) does not belong to $C^\infty(\overline{D})$ but has second derivatives which may exhibit a finite jump across the characteristic through the origin; one easily verifies that this does not affect the proof that we gave for the case $c(x,y) \equiv 0$). Hence we obtain $u(x,y) \in C^1(\overline{D})$ which completes the proof of the lemma.

REMARK. The condition of positiveness of the coefficients $a(x,y)$ and $b(x,y)$ can be disposed of. It can be shown in a similar way as before that the statements of the lemma remain valid if instead of this condition the following requirements are fulfilled:

(i) $a^2(x,y) + b^2(x,y) \neq 0$ in $\bar{D}$

(ii)   the characteristics are nowhere tangent to the boundary, so $a(0,y) \neq 0$ for $y \geq 0$ and $b(x,0) \neq 0$ for $x \geq 0$

(iii) the characteristic through the origin has a positive slope in $(0,0)$, so $a(0,0)/b(0,0) > 0$

(iv) through each point in $\bar{D}$ there passes exactly one characteristic and this characteristic has exactly one point in common with the boundary of D.

Also the condition of infinite smoothness of the data can be weakened.

Now that we have established a condition which is necessary and sufficient in order that the solution w of the reduced problem (30) has bounded second derivatives we can start with the construction of a formal approximation. We might of course impose a compatibility condition of the type (32) on the singular perturbation problem (28) and then take the solution w of (30) as the first term of the formal approximation. However, it is also possible to construct an approximation for the solution u of (28) without imposing such a condition. This construction is carried out in the next subsection.

2.2. Construction of an approximation

We consider instead of (30) the so-called modified reduced problem

$$(41) \quad \begin{cases} a(x,t)\dfrac{\partial \tilde{w}}{\partial t} + b(x,t)\dfrac{\partial \tilde{w}}{\partial x} + d(x,t)\tilde{w} = f(x,t), & (x,t) \in D \\[2mm] \tilde{w}(x,0;\varepsilon) = g(x), & 0 \leq x \leq \ell \\[2mm] \tilde{w}(0,t;\varepsilon) = K(t;\varepsilon), & 0 \leq t < \infty \end{cases}$$

where the function $K(t;\varepsilon)$ is defined by

$$K(t;\varepsilon) = k_1(t) - C\,t\exp(-\frac{t}{\varepsilon^p}), \qquad 0 \leq t < \infty,$$

the constant C is given by

$$C = \frac{a(0,0)k_1'(0)+b(0,0)g'(0)+d(0,0)g(0)-f(0,0)}{a(0,0)}$$

and p is some positive constant to be determined later on.

Note that the expression $L_2[\tilde{w}]$ is defined because it follows from lemma 1 that the solution $\tilde{w}(x,t;\varepsilon)$ of (41) is twice differentiable. Moreover, (41) differs only slightly from (30) because of

$$\sup_{0\leq t<\infty} t\exp(-\frac{t}{\varepsilon^p}) = 0(\varepsilon^p), \qquad \varepsilon \downarrow 0$$

so we may expect the difference between $\tilde{w}$ and w to be small, too. Therefore it is reasonable to take the solution $\tilde{w}$ of the modified reduced problem (41) as a first approximation for the solution u of (28).

In order to cope with the remaining boundary conditions in (28) we shall look for an approximation $\tilde{u}(x,t;\varepsilon)$ of $u(x,t;\varepsilon)$ in the form

(42) $\qquad \tilde{u}(x,t;\varepsilon) = \tilde{w}(x,t;\varepsilon) + v_1(\xi,t;\varepsilon) + \varepsilon\{v_2(\xi,t;\varepsilon) + v_3(x,\tau;\varepsilon)\}$

where

$$\xi = \frac{\ell-x}{\varepsilon}, \qquad \tau = \frac{t}{\varepsilon}.$$

Expanding the operator $L^{(\varepsilon)} = \varepsilon L_2 + L_1$ into powers of ε we obtain

$$L^{(\varepsilon)}[\tilde{u}] = f(x,t) - \varepsilon^{-1}\{c^2(\ell,t)\frac{\partial^2 v_1}{\partial\xi^2} + b(\ell,t)\frac{\partial v_1}{\partial\xi}\}$$

$$- \{c^2(\ell,t)\frac{\partial^2 v_2}{\partial\xi^2} + b(\ell,t)\frac{\partial v_2}{\partial\xi} - \phi(\xi,t;\varepsilon)\}$$

(43)

$$+ \{\frac{\partial^2 v_3}{\partial\tau^2} + a(x,0)\frac{\partial v_3}{\partial\tau}\} + \varepsilon\psi(x,t;\varepsilon)$$

$$+ \varepsilon^2\{\frac{\partial^2 v_2}{\partial t^2} - c^2(x,t)\frac{\partial^2 v_3}{\partial x^2}\}$$

where

(44) $$\phi(\xi,t;\epsilon) = 2\xi(cc_x)(\ell,t)\frac{\partial^2 v_1}{\partial\xi^2} + a(\ell,t)\frac{\partial v_1}{\partial t} + \xi b_x(\ell,t)\frac{\partial v_1}{\partial\xi} + d(\ell,t)v_1$$

and

$$\begin{aligned}
\psi(x,t;\epsilon) &= \frac{\partial^2\tilde{w}}{\partial t^2} - c^2(x,t)\frac{\partial^2\tilde{w}}{\partial x^2} + \frac{\partial^2 v_1}{\partial t^2} - \xi^2(cc_x)_x(\ell+\Theta_1(x-\ell),t)\frac{\partial^2 v_1}{\partial\xi^2} \\
&\quad - \xi a_x(\ell+\Theta_2(x-\ell),t)\frac{\partial v_1}{\partial t} - \frac{1}{2}\xi^2 b_{xx}(\ell+\Theta_3(x-\ell),t)\frac{\partial v_1}{\partial\xi}
\end{aligned}$$

(45)
$$\begin{aligned}
&\quad - \xi d_x(\ell+\Theta_4(x-\ell),t)v_1 + 2\xi(cc_x)(\ell+\Theta_5(x-\ell),t)\frac{\partial^2 v_2}{\partial\xi^2} \\
&\quad + a(x,t)\frac{\partial v_2}{\partial t} + \xi b_x(\ell+\Theta_6(x-\ell),t)\frac{\partial v_2}{\partial\xi} + d(x,t)v_2 \\
&\quad + \tau a_t(x,\Theta_7 t)\frac{\partial v_3}{\partial\tau} + b(x,t)\frac{\partial v_3}{\partial x} + d(x,t)v_3
\end{aligned}$$

with $0 < \Theta_i < 1$, $i = 1,2,\ldots,7$.

In addition we have

(46)
$$\begin{cases}
\tilde{u}(x,0;\epsilon) = g(x) + v_1(\xi,0;\epsilon) + \epsilon\{v_2(\xi,0;\epsilon) + v_3(x,0;\epsilon)\}, & 0 \le x \le \ell \\[2mm]
\tilde{u}_t(x,0;\epsilon) = \tilde{w}_t(x,0;\epsilon) + \frac{\partial v_1}{\partial t}(\xi,0;\epsilon) + \epsilon\frac{\partial v_2}{\partial t}(\xi,0;\epsilon) + \frac{\partial v_3}{\partial\tau}(x,0;\epsilon), & 0 \le x \le \ell \\[2mm]
\tilde{u}(0,t;\epsilon) = K(t;\epsilon) + v_1(\frac{\ell}{\epsilon},t;\epsilon) + \epsilon\{v_2(\frac{\ell}{\epsilon},t;\epsilon) + v_3(0,\tau;\epsilon)\}, & 0 \le t < \infty \\[2mm]
\tilde{u}(\ell,t;\epsilon) = \tilde{w}(\ell,t;\epsilon) + v_1(0,t;\epsilon) + \epsilon\{v_2(0,t;\epsilon) + v_3(\ell,\tau;\epsilon)\}, & 0 \le t < \infty.
\end{cases}$$

We shall now specify the functions v_i ($i = 1,2,3$). For the function v_1 we take the solution of

(47)
$$\begin{cases}
c^2(\ell,t)\frac{\partial^2 v_1}{\partial\xi^2} + b(\ell,t)\frac{\partial v_1}{\partial\xi} = 0, & 0 < \xi < \infty, \ 0 \le t < \infty \\[2mm]
v_1(0,t;\epsilon) = k_2(t) - \tilde{w}(\ell,t;\epsilon) + \Theta\Phi(t-t_0;\epsilon), & 0 \le t < \infty \\[2mm]
\lim_{\xi\to\infty} v_1(\xi,t;\epsilon) = 0, & 0 \le t < \infty
\end{cases}$$

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where t_0 is the t-coordinate of the point of intersection between the line $x = \ell$ and the subcharacteristic through the origin (see figure 4), the ε-dependent constant Θ is given by

(48) $\qquad \Theta = \frac{1}{2}\{\frac{\partial^2 \tilde{w}}{\partial t^2}(\ell,t_0+0;\varepsilon) - \frac{\partial^2 \tilde{w}}{\partial t^2}(\ell,t_0-0;\varepsilon)\}$

and the function $\Phi(t;\varepsilon)$ is defined by

(49) $\qquad \Phi(t;\varepsilon) = \begin{cases} \frac{1}{2}t^2 \exp(-\frac{t}{\varepsilon^q}), & t \geq 0 \\[2ex] -\frac{1}{2}t^2 \exp(\frac{t}{\varepsilon^q}), & t \leq 0 \end{cases}$
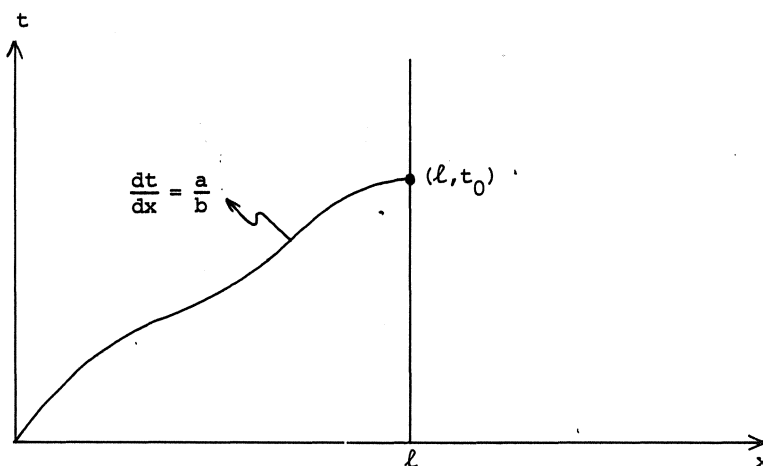
with q some positive constant to be determined later on.

Figure 4

It will turn out in the course of the analysis that the prescribed boundary function $v_1(0,t;\varepsilon)$ should be at least three times differentiable. Therefore the term $\Theta\Phi(t-t_0;\varepsilon)$ in (47) cannot be omitted, since the remaining function $k_2(t) - \tilde{w}(\ell,t;\varepsilon)$ is only twice differentiable. One easily verifies that the boundary function

$\qquad k_2(t) - \tilde{w}(\ell,t;\varepsilon) + \Theta\Phi(t-t_0;\varepsilon)$

is indeed three times differentiable.

It follows from (47) that

$$(50) \qquad v_1(\xi,t;\varepsilon) = \{k_2(t) - \tilde{w}(\ell,t;\varepsilon) + \Theta\Phi(t - t_0;\varepsilon)\}\exp[- \frac{b(\ell,t)}{c^2(\ell,t)} \xi]$$

valid for $0 \leq \xi, t < \infty$.

REMARK. If the subcharacteristic through the origin does not intersect the line $x = \ell$ the analysis can be simplified. We leave the details to the interested reader.

For the function v_2 we take the solution of

$$(51) \qquad \begin{cases} c^2(\ell,t)\dfrac{\partial^2 v_2}{\partial\xi^2} + b(\ell,t)\dfrac{\partial v_2}{\partial\xi} = \phi(\xi,t;\varepsilon), & 0 < \xi < \infty,\ 0 \leq t < \infty \\[2em] v_2(0,t;\varepsilon) = \lim_{\xi\to\infty} v_2(\xi,t;\varepsilon) = 0, & 0 \leq t < \infty. \end{cases}$$

Using (44) and (50) we can write this problem as

$$\begin{cases} \dfrac{\partial^2 v_2}{\partial\xi^2} + \dfrac{b(\ell,t)}{c^2(\ell,t)} \dfrac{\partial v_2}{\partial\xi} = \{\dfrac{\alpha(t;\varepsilon)}{c^2(\ell,t)} \xi + \dfrac{\beta(t;\varepsilon)}{c^2(\ell,t)}\}\exp[- \dfrac{b(\ell,t)}{c^2(\ell,t)} \xi],\ 0<\xi<\infty, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\leq t<\infty \\[2em] v_2(0,t;\varepsilon) = \lim_{\xi\to\infty} v_2(\xi,t;\varepsilon) = 0, & 0 \leq t < \infty \end{cases}$$

with

$$\begin{cases} \alpha(t;\varepsilon) = \{\tilde{w}(\ell,t;\varepsilon) - \Theta\Phi(t - t_0;\varepsilon) - k_2(t)\} \times \\[1em] \qquad\qquad \times \{a(\ell,t)\dfrac{d}{dt} \dfrac{b(\ell,t)}{c^2(\ell,t)} + \dfrac{b(\ell,t)b_x(\ell,t)}{c^2(\ell,t)} - \dfrac{2(cc_x)(\ell,t)b^2(\ell,t)}{c^4(\ell,t)}\} \\[2em] \beta(t;\varepsilon) = d(\ell,t)\{k_2(t) - \tilde{w}(\ell,t;\varepsilon) + \Theta\Phi(t - t_0;\varepsilon)\} \\[1em] \qquad\qquad + a(\ell,t)\{k_2'(t) - \tilde{w}_t(\ell,t;\varepsilon) + \Theta\Phi'(t - t_0;\varepsilon)\}. \end{cases}$$

Hence we obtain

$$(52) \quad v_2(\xi,t;\varepsilon) = \{-\frac{\alpha(t;\varepsilon)}{2b(\ell,t)}\xi^2 - \frac{\alpha(t;\varepsilon)c^2(\ell,t)+\beta(t;\varepsilon)b(\ell,t)}{b^2(\ell,t)}\xi\} \times$$

$$\times \exp[-\frac{b(\ell,t)}{c^2(\ell,t)}\xi]$$

valid for $0 \leq \xi, t < \infty$.

Note that the derivative $\dfrac{\partial^2 v_2}{\partial t^2}$ occurring in the right-hand side of (43) exists thanks to the fact that the boundary function $v_1(0,t;\varepsilon)$ in (47) is three times differentiable.

Finally, we take for the function v_3 the solution of

$$(53) \quad \begin{cases} \dfrac{\partial^2 v_3}{\partial \tau^2} + a(x,0)\dfrac{\partial v_3}{\partial \tau} = 0, & 0 \leq x \leq \ell, \ 0 < \tau < \infty \\[2em] \dfrac{\partial v_3}{\partial \tau}(x,0;\varepsilon) = h(x) - \tilde{w}_t(x,0;\varepsilon), & 0 \leq x \leq \ell \\[2em] \lim_{\tau \to \infty} v_3(x,\tau;\varepsilon) = 0, & 0 \leq x \leq \ell \end{cases}$$

which is given by

$$(54) \quad v_3(x,\tau;\varepsilon) = \frac{\tilde{w}_t(x,0;\varepsilon)-h(x)}{a(x,0)} \exp[-a(x,0)\tau], \quad 0 \leq x \leq \ell, \ 0 \leq \tau < \infty.$$

Note that v_1, v_2 and v_3 are boundary layer functions.

It follows from (43), (46), (47), (50), (51) and (53) that

$$(55) \quad \begin{cases} L^{(\varepsilon)}[\tilde{u}] = f(x,t) + \varepsilon\psi(x,t;\varepsilon) + \varepsilon^2\{\dfrac{\partial^2 v_2}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 v_3}{\partial x^2}\}, \quad (x,t) \in D \\[2em] \tilde{u}(x,0;\varepsilon) = g(x) + \varepsilon\{v_2(\xi,0;\varepsilon) + v_3(x,0;\varepsilon)\} \\[1em] \qquad\qquad + \Theta\Phi(-t_0;\varepsilon)\exp[-\dfrac{b(\ell,0)}{c^2(\ell,0)}\xi], \quad 0 \leq x \leq \ell \\[2em] \tilde{u}_t(x,0;\varepsilon) = h(x) + \dfrac{\partial v_1}{\partial t}(\xi,0;\varepsilon) + \varepsilon\dfrac{\partial v_2}{\partial t}(\xi,0;\varepsilon), \quad 0 \leq x \leq \ell \\[2em] \tilde{u}(0,t;\varepsilon) = K(t;\varepsilon) + v_1(\dfrac{\ell}{\varepsilon},t;\varepsilon) + \varepsilon\{v_2(\dfrac{\ell}{\varepsilon},t;\varepsilon) + v_3(0,\tau;\varepsilon)\}, \ 0 \leq t < \infty \\[2em] \tilde{u}(\ell,t;\varepsilon) = k_2(t) + \varepsilon v_3(\ell,\tau;\varepsilon) + \Theta\Phi(t-t_0;\varepsilon), \quad 0 \leq t < \infty. \end{cases}$$

Defining the remainder term $z(x,t;\varepsilon)$ by

$$(56) \qquad z(x,t;\varepsilon) = u(x,t;\varepsilon) - \tilde{u}(x,t;\varepsilon)$$

we obtain from (28) and (55)

$$(57) \quad \begin{cases} L^{(\varepsilon)}[z] = -\varepsilon\psi(x,t;\varepsilon) + \varepsilon^2\{c^2(x,t)\dfrac{\partial^2 v_3}{\partial x^2} - \dfrac{\partial^2 v_2}{\partial t^2}\}, \qquad (x,t) \in D \\[2em] z(x,0;\varepsilon) = -\varepsilon\{v_2(\xi,0;\varepsilon) + v_3(x,0;\varepsilon)\} \\[1em] \qquad\qquad\qquad\qquad + \Theta\Phi(t_0;\varepsilon)\exp[-\dfrac{b(\ell,0)}{c^2(\ell,0)}\xi], \qquad 0 \le x \le \ell \\[2em] z_t(x,0;\varepsilon) = -\dfrac{\partial v_1}{\partial t}(\xi,0;\varepsilon) - \varepsilon\dfrac{\partial v_2}{\partial t}(\xi,0;\varepsilon), \qquad 0 \le x \le \ell \\[2em] z(0,t;\varepsilon) = Ct\,\exp(-\dfrac{t}{\varepsilon^p}) - v_1(\dfrac{\ell}{\varepsilon},t;\varepsilon) \\[2em] \qquad\qquad\qquad\qquad - \varepsilon\{v_2(\dfrac{\ell}{\varepsilon},t;\varepsilon) + v_3(0,\tau;\varepsilon)\}, \qquad 0 \le t < \infty \\[2em] z(\ell,t;\varepsilon) = -\varepsilon v_3(\ell,\tau;\varepsilon) - \Theta\Phi(t-t_0;\varepsilon), \qquad 0 \le t < \infty. \end{cases}$$

By writing the solution $\tilde{w}$ of (41) in explicit form (cf. the proof of lemma 1) it is easily established that the ε-dependence of $\tilde{w}$ and its derivatives is completely determined by the ε-dependence of the boundary function $K(t;\varepsilon)$ and its derivatives (cf. the formulae (37), (38) and (39)). Bearing this in mind we obtain after tedious calculations

$$(58) \quad \begin{cases} \|\dfrac{\partial^2 \tilde{w}}{\partial t^2}\|_\Omega = O(\varepsilon^{-\frac{1}{2}p}), \qquad \varepsilon \downarrow 0 \\[2em] \|\dfrac{\partial^2 \tilde{w}}{\partial x^2}\|_\Omega = O(\varepsilon^{-\frac{1}{2}p}), \qquad \varepsilon \downarrow 0 \\[2em] \sup_\Omega |\dfrac{\partial^2 \tilde{w}}{\partial t^2}| = O(\varepsilon^{-p}), \qquad \varepsilon \downarrow 0 \end{cases}$$

valid for any compact subset $\Omega \subset \bar{D}$.

Similarly we obtain from (50) and (52) the results

$$(59) \quad \begin{cases} \| \dfrac{\partial^2 v_1}{\partial t^2} \|_\Omega = 0(\varepsilon^{\lambda_1}), & \varepsilon \downarrow 0 \\[3mm] \| \dfrac{\partial^2 v_2}{\partial t^2} \|_\Omega = 0(\varepsilon^{\lambda_2}), & \varepsilon \downarrow 0 \end{cases}$$

valid for any compact subset $\Omega \subset \bar{D}$, where

$$\begin{cases} \lambda_1 = \min(-\dfrac{1}{2}p, -p + \dfrac{1}{2}q) \\[3mm] \lambda_2 = \min(-\dfrac{3}{2}p, -p - \dfrac{1}{2}q). \end{cases}$$

Hence we obtain from (45)

$$(60) \qquad \| \psi \|_\Omega = 0(\varepsilon^{\lambda_1}), \qquad \varepsilon \downarrow 0$$

valid for any compact subset $\Omega \subset \bar{D}$.

Applying theorem 1 (and the second remark following that theorem) to (57) it is possible to obtain estimates for the function z and its derivatives. Pointwise estimates for z follow from the formulae (24a), (25) and (26b), (27). Using the formulae (58), (59) and (60) it can be shown that the constants $K(\Omega,\varepsilon)$ and $K_2(\Omega,\varepsilon)$ defined by (25) and (27) satisfy

$$\begin{cases} K(\Omega,\varepsilon) = 0(\varepsilon^\mu), & \varepsilon \downarrow 0 \\[3mm] K_2(\Omega,\varepsilon) = 0(\varepsilon^\mu), & \varepsilon \downarrow 0 \end{cases}$$

with

$$\mu = \min(1 - \dfrac{1}{2}p, 1 - p + \dfrac{1}{2}q, \ 2 - \dfrac{3}{2}p, \ 2 - p - \dfrac{1}{2}q, \dfrac{3}{2}p, -p + \dfrac{5}{2}q).$$

It is not difficult to show that the value of μ becomes optimal (that is, maximal) if we choose $p = q = \dfrac{1}{2}$. In that case we obtain $\mu = \dfrac{3}{4}$, so that according to the formulae (24a) and (26b)

$$(61) \quad \begin{cases} \sup_{\substack{0 \le x \le \ell \\ 0 \le t \le T}} |z(x,t;\varepsilon)| = O(\varepsilon^{\frac{1}{4}}), & \varepsilon \downarrow 0, \\[2ex] \sup_{\substack{0 \le x \le \ell - \delta \\ 0 \le t \le T}} |z(x,t;\varepsilon)| = O(\varepsilon^{\frac{1}{2}}), & \varepsilon \downarrow 0. \end{cases}$$

An estimate in L_2-norm for the derivatives z_x and z_t is furnished by formula (26a). We obtain

$$(62) \quad \int_0^{\ell-\delta} (z_x^2 + z_t^2)\,dx = O(\varepsilon^{\frac{1}{2}}), \quad \varepsilon \downarrow 0$$

uniformly valid for $0 \le t \le T$.

REMARK. Note that the right-hand side function of the differential equation (57) does not satisfy the continuity condition (3,i) of theorem 1 but has instead finite jump discontinuities. One easily verifies that this does not affect the validity of the theorem.

It follows from the formulae (48), (49), (52) and (54) that

$$\begin{cases} \sup_{\substack{0 \le \xi < \infty \\ 0 \le t \le T}} |v_2(\xi,t;\varepsilon)| = O(1), & \varepsilon \downarrow 0 \\[2ex] \sup_{\substack{0 \le x \le \ell \\ 0 \le \tau < \infty}} |v_3(x,\tau;\varepsilon)| = O(1), & \varepsilon \downarrow 0 \\[2ex] \sup_{0 \le t \le T} |\Theta\Phi(t - t_0;\varepsilon)| = O(\varepsilon^{\frac{1}{2}}), & \varepsilon \downarrow 0. \end{cases}$$

Hence we obtain from (42), (50), (56) and (61)

$$(63) \quad u(x,t;\varepsilon) = \tilde{w}(x,t;\varepsilon) + \{k_2(t) - \tilde{w}(\ell,t;\varepsilon)\}\exp[\frac{b(\ell,t)}{c^2(\ell,t)} \cdot \frac{x-\ell}{\varepsilon}] + O(\varepsilon^{\frac{1}{4}}), \quad \varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$, and also

$$(64) \quad u(x,t;\varepsilon) = \tilde{w}(x,t;\varepsilon) + O(\varepsilon^{\frac{1}{2}}), \quad \varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \{(x,t) \mid 0 \le x \le \ell-\delta, \; 0 \le t < \infty\}$ with δ an arbitrarily small positive constant independent of ε and satisfying $0 < \delta < \ell$.

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Putting

(65) $Z(x,t;\varepsilon) = w(x,t) - \tilde{w}(x,t;\varepsilon)$

we obtain from (30) and (41)

(66)
$$
\begin{cases}
a(x,t)\dfrac{\partial Z}{\partial t} + b(x,t)\dfrac{\partial Z}{\partial x} + d(x,t)Z = 0, & (x,t) \in D \\[2ex]
Z(x,0;\varepsilon) = 0, & 0 \le x \le \ell \\[2ex]
Z(0,t;\varepsilon) = C\, t \exp(-\dfrac{t}{\sqrt{\varepsilon}}), & 0 \le t < \infty.
\end{cases}
$$

Multiplying this differential equation by $2Z$, integrating along characteristics and applying GRONWALL's lemma we get

(67) $Z(x,t;\varepsilon) = O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$

uniformly valid in any compact subset $\Omega \subset \bar{D}$.

Hence we conclude from (65) and (67) that the formulae (63) and (64) remain valid if we replace $\tilde{w}$ by w.   •

From (62) it follows that

(68)
$$
\begin{cases}
\displaystyle\int_0^{\ell-\delta} (u_x - \tilde{u}_x)^2\, dx = O(\varepsilon^{\frac{1}{2}}), & \varepsilon \downarrow 0 \\[3ex]
\displaystyle\int_0^{\ell-\delta} (u_t - \tilde{u}_t)^2\, dx = O(\varepsilon^{\frac{1}{2}}), & \varepsilon \downarrow 0
\end{cases}
$$

uniformly valid for $0 \le t \le T$. Evaluating $\tilde{u}_x$ and $\tilde{u}_t$ from (42) and using the formulae

$$
\begin{cases}
\displaystyle\int_0^{\ell} (w_x - \tilde{w}_x)^2\, dx = \int_0^{\ell} z_x^2\, dx = O(\varepsilon^{\frac{1}{2}}), & \varepsilon \downarrow 0 \\[3ex]
\displaystyle\int_0^{\ell} (w_t - \tilde{w}_t)^2\, dx = \int_0^{\ell} z_t^2\, dx = O(\varepsilon^{\frac{1}{2}}), & \varepsilon \downarrow 0
\end{cases}
$$

uniformly valid for $0 \leq t \leq T$ (these formulae can be established by writing the solution Z of (66) in explicit form, cf. (37), (38) and (39)) we obtain from (68) after a straightforward calculation the estimates

$$\int_0^{\ell-\delta} \left[\frac{\partial u}{\partial x}(x,t;\varepsilon) - \frac{\partial w}{\partial x}(x,t)\right]^2 dx = O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$$

and

$$\int_0^{\ell-\delta} \left[\frac{\partial u}{\partial t}(x,t;\varepsilon) - \frac{\partial w}{\partial t}(x,t) + \{\frac{\partial w}{\partial t}(x,0) - h(x)\}\exp[-a(x,0)\frac{t}{\varepsilon}]\right]^2 dx = O(\varepsilon^{\frac{1}{2}}), \quad \varepsilon \downarrow 0$$

uniformly valid for $0 \leq t \leq T$.

We summarize the results in

THEOREM 2. *Let* $u(x,t;\varepsilon)$ *be the solution of the initial-boundary value problem* (28) *where all data are* C^∞*-functions which in addition satisfy* (29). *Then we have*

$$u(x,t;\varepsilon) = w(x,t) + O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \{(x,t) \mid 0 \leq x \leq \ell-\delta, \; 0 \leq t < \infty\}$ *where* $w(x,t)$ *is the solution of the reduced problem* (30) *and* δ *is an arbitrarily small positive constant independent of* ε *and satisfying* $0 < \delta < \ell$. *In addition we have*

$$u(x,t;\varepsilon) = w(x,t) + \{k_2(t) - w(\ell,t)\}\exp\left[\frac{b(\ell,t)}{c^2(\ell,t)} \cdot \frac{x-\ell}{\varepsilon}\right] + O(\varepsilon^{\frac{1}{4}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$, *and also*

$$\int_0^{\ell-\delta} \left[\frac{\partial u}{\partial x}(x,t;\varepsilon) - \frac{\partial w}{\partial x}(x,t)\right]^2 dx = O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$$

and

$$\int_0^{\ell-\delta} \left[\frac{\partial u}{\partial t}(x,t;\varepsilon) - \frac{\partial w}{\partial t}(x,t) + \{\frac{\partial w}{\partial t}(x,0) - h(x)\}\exp[-a(x,0)\frac{t}{\varepsilon}]\right]^2 dx = O(\varepsilon^{\frac{1}{2}}), \quad \varepsilon \downarrow 0$$

uniformly valid for $0 \leq t \leq T$.

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REMARK. The statements of theorem 2 remain valid if the condition (29b) is replaced by

(69) $|b(x,t)| < a(x,t)c(x,t)$ in $\bar{D}$

provided that the following requirements are fulfilled

(i) $b(\ell,t) > 0$ for $t \geq 0$ (this condition implies that there is a boundary layer at $x = \ell$, $t \geq 0$)

(ii) $b(0,t) \neq 0$ for $t \geq 0$

(iii) the subcharacteristic through the origin has a positive slope in $(0,0)$, so $a(0,0)/b(0,0) > 0$

(iv)   through each point in $\bar{D}$ there passes exactly one subcharacteristic and this subcharacteristic has exactly one point in common with the part of the boundary given by $\{(x,t) \mid x = 0, \ t \geq 0\} \cup \{(x,t) \mid t = 0, \ 0 \leq x \leq \ell\}$

(compare these requirements with the conditions (ii) to (iv) mentioned in the remark following the proof of lemma 1). It follows from the timelike condition (69) and the requirements (i) to (iv) that necessarily

$b(0,t) > 0$ for $t \geq 0$.

CHARACTERISTIC BOUNDARY VALUE PROBLEMS FOR

LINEAR HYPERBOLIC DIFFERENTIAL EQUATIONS

In this chapter we shall discuss the characteristic boundary value
problem (singular perturbation problem)

(1)
$$
\begin{cases}
\varepsilon L_2[u] + L_1^{(\varepsilon)}[u] = f(x,y;\varepsilon), & (x,y) \in D \\
\\
u(x,0;\varepsilon) = g(x;\varepsilon), & 0 \le x < \infty \\
u(\phi(y),y;\varepsilon) = h(y;\varepsilon), & 0 \le y < \infty
\end{cases}
$$

where ε is a small positive parameter, i.e. $0 < \varepsilon \ll 1$, and the differential
operators L_2 and $L_1^{(\varepsilon)}$ are defined by

$$
\begin{cases}
L_2[u] = \dfrac{\partial^2 u}{\partial x \partial y} \\
\\
L_1^{(\varepsilon)}[u] = a(x,y;\varepsilon)\dfrac{\partial u}{\partial x} + b(x,y;\varepsilon)\dfrac{\partial u}{\partial y} + c(x,y;\varepsilon)u.
\end{cases}
$$

The functions a, b, c, f, g and h depend on the variables x and/or y and on
the parameter ε.

The region D is defined by

$$
D = \{(x,y) \mid \phi(y) < x < \infty, \; 0 < y < \infty\}
$$

where $\phi(y)$ is some non-decreasing differentiable function defined for
$0 \le y < \infty$ and satisfying $\phi(0) = 0$. Note that the part of the boundary ∂D
given by $y = 0$, $x \ge 0$ is characteristic, whereas the part given by $x = \phi(y)$,
$y \ge 0$ is timelike or characteristic.

It will be assumed that the boundary functions g and h are continuous; we
require in particular

$$g(0;\varepsilon) = h(0;\varepsilon)$$

to be valid for all ε with $0 < \varepsilon << 1$.

In addition we shall assume the functions a and b to be strictly positive.
This condition can be shown to be equivalent to the growth conditions that
were discussed in section 1 of chapter III.

In section 1 we shall derive by means of the energy integral method a
priori estimates for the solution $u(x,y;\varepsilon)$ of problem (1).
In the sections 2 and 3 we shall construct an approximation for the solution
u of a problem of the type (1).

1. A priori estimates

We consider the characteristic boundary value problem (1) under the
following conditions

$$(2) \quad \begin{cases} \phi(y) \in C^1[0,\infty) \\ \phi(0) = 0 \\ \phi'(y) \geq 0 \quad \text{for } 0 \leq y < \infty \end{cases}$$

(3) $\begin{cases}\end{cases}$ for each compact subset $\Omega \subset \bar{D}$ there exists a positive constant ε_1 (de-
pending on Ω) such that

(i) $a(x,y;\varepsilon) \in C^1(\Omega)$, $b(x,y;\varepsilon) \in C^1(\Omega)$

$c(x,y;\varepsilon) \in C^0(\Omega)$, $f(x,y;\varepsilon) \in C^0(\Omega)$

valid for every value of ε satisfying $0 < \varepsilon \leq \varepsilon_1$

(ii) the functions $a(x,y;\varepsilon)$, $b(x,y;\varepsilon)$ and their first derivatives, as
well as the function $c(x,y;\varepsilon)$, are uniformly bounded in Ω for
$0 < \varepsilon \leq \varepsilon_1$

(iii) $a(x,y;\varepsilon) \geq p > 0$, $b(x,y;\varepsilon) \geq p > 0$

valid for all $(x,y) \in \Omega$ and all $\varepsilon \in (0,\varepsilon_1]$, where p is some con-
stant depending on Ω and ε_1 but independent of ε

(4) $\Biggl\{$ for each pair of positive constants X and Y there exists a positive constant ε_2 (depending on X and Y) such that

$\Biggl\{$ $g(x;\varepsilon) \in C^1[0,X]$, $\quad h(y;\varepsilon) \in C^1[0,Y]$

$g(0;\varepsilon) = h(0;\varepsilon)$

valid for every value of ε satisfying $0 < \varepsilon \leq \varepsilon_2$.

In order to derive a priori estimates for the solution $u(x,y;\varepsilon)$ of (1) in some compact subset $S \subset \bar{D}$ we enclose S by a region

(5) $\qquad \Omega = \{(x,y) \mid \phi(y) \leq x \leq S_x + \lambda(S_y - y), \ y \geq 0\}$

where

$$S_x = \sup_{(x,y) \in S} x, \qquad S_y = \sup_{(x,y) \in S} y$$

and λ is some positive constant, independent of ε, to be determined later on (see figure 1).
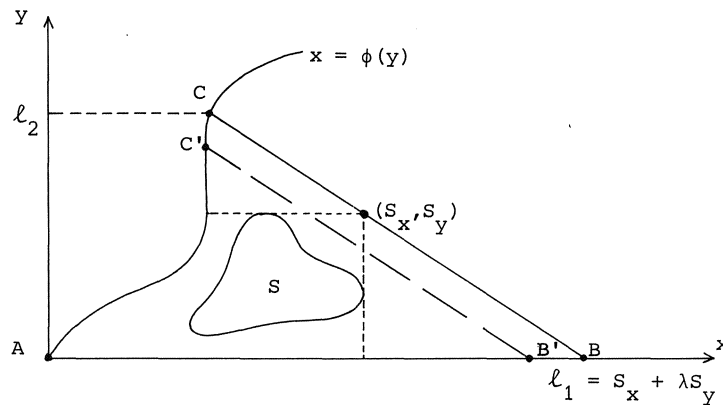

Figure 1: The region Ω

We define the constant ℓ_1 by $\ell_1 = S_x + \lambda S_y$ and we shall denote the y-coordinate of the point C by ℓ_2.

Let $\alpha = \alpha(x,y;\varepsilon)$, $\beta = \beta(x,y;\varepsilon)$ and $\gamma = \gamma(x,y;\varepsilon)$ be continuously differentiable functions (depending on the variables x and y and on the parameter ε) to be specified later on.

We multiply the differential equation (1) by $2\alpha u$ to obtain

$$\frac{\partial}{\partial y}(\alpha b u^2 + \varepsilon\alpha u u_x) + \frac{\partial}{\partial x}(\alpha a u^2 + \varepsilon\alpha u u_y) =$$

(6)
$$\{\frac{\partial(\alpha a)}{\partial x} + \frac{\partial(\alpha b)}{\partial y} - 2\alpha c\}u^2 + \varepsilon\frac{\partial\alpha}{\partial y} uu_x + \varepsilon\frac{\partial\alpha}{\partial x} uu_y$$

$$+ 2\varepsilon\alpha u_x u_y + 2\alpha fu$$

valid for $(x,y) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0 = \min(\varepsilon_1,\varepsilon_2)$, the values of ε_1 and ε_2 being determined by the region Ω (see the conditions (3,i) and (4)).

Multiplication of (1) by $2\beta u_x$ yields

$$\frac{\partial}{\partial y}(\varepsilon\beta u_x^2) = \varepsilon\frac{\partial\beta}{\partial y} u_x^2 - 2\beta c uu_x$$

(7)
$$- 2\beta a u_x^2 - 2\beta b u_x u_y + 2\beta f u_x$$

valid for $(x,y) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$.

Similarly, multiplication by $2\gamma u_y$ leads to

$$\frac{\partial}{\partial x}(\varepsilon\gamma u_y^2) = \varepsilon\frac{\partial\gamma}{\partial x} u_y^2 - 2\gamma c uu_y$$

(8)
$$- 2\gamma b u_y^2 - 2\gamma a u_x u_y + 2\gamma f u_y$$

likewise valid for $(x,y) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$.

Adding (6), (7) and (8) we obtain

(9)
$$\frac{\partial}{\partial y} Q_1 + \frac{\partial}{\partial x} Q_2 = Q_3 + Q_4$$

valid for $(x,y) \in \Omega$ and $0 < \varepsilon \le \varepsilon_0$, where the expressions Q_i (i = 1,2,3,4) are given by

$$
(10)
\begin{cases}
Q_1 = \alpha b u^2 + \varepsilon \alpha u u_x + \varepsilon \beta u_x^2 \\[2ex]
Q_2 = \alpha a u^2 + \varepsilon \alpha u u_y + \varepsilon \gamma u_y^2 \\[2ex]
Q_3 = \{\dfrac{\partial(\alpha a)}{\partial x} + \dfrac{\partial(\alpha b)}{\partial y} - 2\alpha c\} u^2 + \varepsilon \dfrac{\partial \alpha}{\partial y} uu_x + \varepsilon \dfrac{\partial \alpha}{\partial x} uu_y \\[2ex]
\qquad + \varepsilon \dfrac{\partial \beta}{\partial y} u_x^2 + \varepsilon \dfrac{\partial \gamma}{\partial x} u_y^2 \\[2ex]
Q_4 = -2\beta a u_x^2 - 2(\beta b + \gamma a - \varepsilon a) u_x u_y - 2\gamma b u_y^2 \\[2ex]
\qquad + 2\alpha f u + 2(f - cu)(\beta u_x + \gamma u_y).
\end{cases}
$$

Integrating (9) over the region Ω and using GREEN's theorem in the plane we obtain (see figure 1)

$$
(11) \qquad \int_B^C (Q_2 + \lambda Q_1) dy = \int_A^B Q_1 dx + \int_A^C (Q_2 - \phi' Q_1) dy + \iint_\Omega (Q_3 + Q_4) dx dy.
$$

We shall now specify the functions α, β and γ.

Let the variable ρ be defined by

$$
\rho = x - \phi(y)
$$

and let δ be an arbitrarily small positive number which does not depend on ε. Let $\Psi(\rho)$ be some C^∞-function defined for $\rho \geq 0$ which satisfies

$$
(12) \qquad
\begin{cases}
\Psi(\rho) = \begin{cases} 1 & \text{for } \rho \geq \delta \\[1ex] 0 & \text{for } 0 \leq \rho \leq \frac{1}{2}\delta \end{cases} \\[3ex]
\Psi(\rho) \text{ monotonically increasing for } \frac{1}{2}\delta \leq \rho \leq \delta
\end{cases}
$$

(see figure 2a) and let the C^∞-function $\Phi(\rho;\varepsilon)$ be defined by

$$
\Phi(\rho;\varepsilon) = \varepsilon + (1 - \varepsilon)\Psi(\rho), \qquad \rho \geq 0
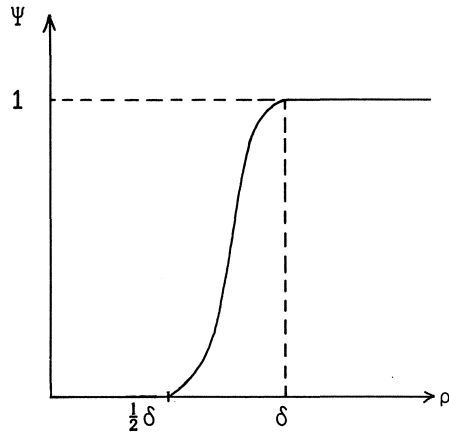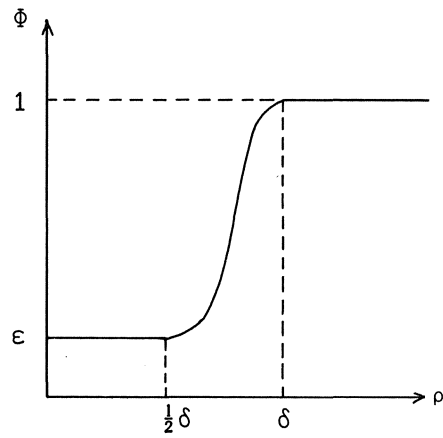$$

(see figure 2b).

Figure 2a

Figure 2b

We now choose the functions α, β and γ as follows

$$(13) \quad \begin{cases} \alpha = \alpha(x,y;\epsilon) = a(x,y;\epsilon) + \lambda b(x,y;\epsilon) = a + \lambda b \\[2mm] \beta = \beta(x,y;\epsilon) = \mu a(x,y;\epsilon)\Phi(\rho;\epsilon) + \epsilon\lambda = \mu a\Phi + \epsilon\lambda \\[2mm] \gamma = \gamma(x,y;\epsilon) = \mu b(x,y;\epsilon)\Phi(\rho;\epsilon) + \epsilon = \mu b\Phi + \epsilon \end{cases}$$

where μ is some positive constant, independent of ϵ, to be determined later on (this leaves λ and μ to be specified).

Now that we have chosen α, β and γ we shall have a closer look at the integrands in (11).

Since we have

$$u_y = \frac{du}{dy} - \phi' u_x \qquad \text{on AC}$$

we obtain along AC, using (10)

$$(14) \quad \begin{aligned} \Omega_2 - \phi'\Omega_1 &= -\epsilon\phi'(\beta - \gamma\phi')u_x^2 - 2\epsilon\alpha\phi'uu_x - 2\epsilon\gamma\phi' \frac{du}{dy} u_x \\ &\quad + \alpha(a - b\phi')u^2 + \epsilon\alpha u \frac{du}{dy} + \epsilon\gamma(\frac{du}{dy})^2. \end{aligned}$$

This expression contains the function u_x that we do not know a priori along AC. However, it is possible to eliminate u_x by an appropriate choice of λ and μ. Let us assume for the moment that λ and μ have been chosen such that

$$(15) \qquad \beta - \gamma\phi' \geq \varepsilon \qquad \text{on AC.}$$

Then it follows from (14) that there exists a positive constant M_1 (depending on Ω, λ and μ but independent of ε) such that

$$(16) \qquad Q_2 - \phi'Q_1 \leq M_1\{u^2 + \varepsilon^2(\frac{du}{dy})^2\} \qquad \text{on AC.}$$

What conditions have to be imposed on λ and μ in order that (15) holds? Using (13) we see that (15) is fulfilled iff

$$(17) \qquad \lambda - \phi' \geq \mu(b\phi' - a) + 1 \qquad \text{on AC.}$$

We now choose λ as follows:

$$\lambda = 2 + \max_{0\leq y\leq \ell_2} \phi'(y).$$

Then it follows that (17) is satisfied if μ is chosen such that

$$0 < \mu < \mu_1 = \frac{1}{1+\Theta}$$

where the positive constant Θ, independent of ε, is given by

$$\Theta = \sup_{\substack{0\leq y\leq \ell_2 \\ 0<\varepsilon\leq\varepsilon_0}} \{a(\phi(y),y;\varepsilon) + b(\phi(y),y;\varepsilon)\phi'(y)\}.$$

As for the expression Q_3, it is easily established that there exist positive constants M_2 (depending on Ω but *independent of μ and* ε) and M_3 (depending on Ω but *independent of μ and* ε) such that

$$(18) \qquad Q_3 \leq M_2(u^2 + \varepsilon^2 u_x^2 + \varepsilon^2 u_y^2) + \varepsilon\mu M_3(u_x^2 + u_y^2)$$

valid for $(x,y) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

Next we consider Q_4. Substituting (13) into (10) we obtain

$$Q_4 = -2\mu\Phi\{(au_x + bu_y)^2 + (cu - f)(au_x + bu_y)\}$$

$$+ 2(a + \lambda b)fu + 2\varepsilon(f - cu)(\lambda u_x + u_y)$$

$$- 2\varepsilon(\lambda au_x^2 + bu_y^2).$$

Using (3,iii) it is not difficult to verify that there exist positive constants M_4 (depending on Ω and μ but independent of ε) and M_5 (depending on Ω but *independent of* μ *and* ε) such that

(19) $\qquad Q_4 \leq M_4(u^2 + \varepsilon^2 u_x^2 + \varepsilon^2 u_y^2 + f^2) - \varepsilon M_5(u_x^2 + u_y^2)$

valid for $(x,y) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$.

For μ sufficiently small, viz.

(20) $\qquad 0 < \mu < \mu_0 = \min(\mu_1, \dfrac{M_5}{M_3})$

it follows from (18) and (19) that

$$Q_3 + Q_4 \leq (M_2 + M_4)(u^2 + \varepsilon^2 u_x^2 + \varepsilon^2 u_y^2 + f^2)$$

(21)

$$\leq (M_2 + M_4)(u^2 + \varepsilon\Phi u_x^2 + \varepsilon\Phi u_y^2 + f^2)$$

valid for $(x,y) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$. From now on it will be assumed that the value of μ is fixed and satisfies (20).

Finally, we consider the expression $Q_2 + \lambda Q_1$. It follows from (10) and (13) that on BC

$$Q_2 + \lambda Q_1 = (a + \lambda b)^2 u^2 + \varepsilon(a + \lambda b)u(\lambda u_x + u_y) + \varepsilon^2\lambda^2 u_x^2 + \varepsilon^2 u_y^2$$

$$+ \varepsilon\lambda\mu a\Phi u_x^2 + \varepsilon\mu b\Phi u_y^2$$

(22)

$$= \{\tfrac{1}{2}(a + \lambda b)u + \varepsilon\lambda u_x\}^2 + \{\tfrac{1}{2}(a + \lambda b)u + \varepsilon u_y\}^2$$

$$+ \tfrac{1}{2}(a + \lambda b)^2 u^2 + \varepsilon\lambda\mu a\Phi u_x^2 + \varepsilon\mu b\Phi u_y^2$$

$$\geq m(u^2 + \varepsilon\Phi u_x^2 + \varepsilon\Phi u_y^2)$$

where m is some positive constant depending on Ω but independent of ε.

Since it is easily established that there exists a positive constant M_6 (depending on Ω but independent of ε) such that

$$(23) \qquad Q_1 \leq M_6(u^2 + \varepsilon u_x^2)$$

valid for $(x,y) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$, we obtain from (11), (16), (21), (22) and (23)

$$(24) \qquad m \int_B^C (u^2 + \varepsilon\Phi u_x^2 + \varepsilon\Phi u_y^2)\,dy \leq M \iint_\Omega (u^2 + \varepsilon\Phi u_x^2 + \varepsilon\Phi u_y^2)\,dxdy + MK^2(\Omega,\varepsilon)$$

valid for $0 < \varepsilon \leq \varepsilon_0$, where the positive constant M, depending on Ω but independent of ε, is given by

$$M = \max(M_1, M_2 + M_4, M_6)$$

and the positive constant $K(\Omega,\varepsilon)$, depending on Ω and ε, is defined by

$$(25) \qquad K(\Omega,\varepsilon) = \|f\|_\Omega + \|g\|_{[0,\ell_1]} + \|h\|_{[0,\ell_2]} + \varepsilon^{\frac{1}{2}}\|g_x\|_{[0,\ell_1]} + \varepsilon\|h_y\|_{[0,\ell_2]}.$$

It is easily verified that (24) with the same values of m and M is also valid for every region $\Omega' = AB'C'$ inside $\Omega = ABC$ with $B'C'//BC$ (see figure 1). Hence we obtain

$$(26) \qquad \int_{B'}^{C'} (u^2 + \varepsilon\Phi u_x^2 + \varepsilon\Phi u_y^2)\,dy - \frac{M}{m} \iint_{\Omega'} (u^2 + \varepsilon\Phi u_x^2 + \varepsilon\Phi u_y^2)\,dxdy \leq \frac{M}{m} K^2(\Omega,\varepsilon)$$

valid for $0 < \varepsilon \leq \varepsilon_0$ and every $\Omega' \subset \Omega$ with $B'C'//BC$.

Introducing new variables ξ and η by

$$(27) \qquad \begin{cases} \xi = x + \lambda y \\ \eta = y - \lambda x \end{cases}$$

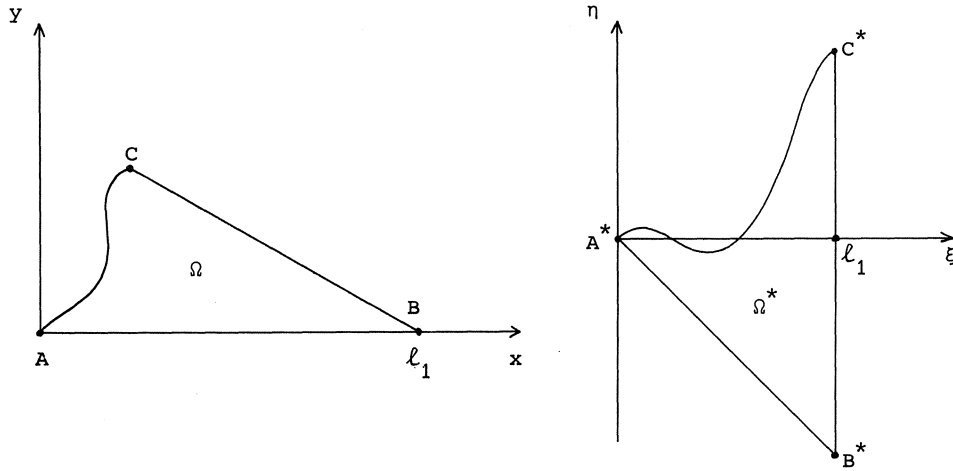we transform the region Ω into a region Ω^* (see figure 3).

Figure 3

The transformed boundaries A^*B^*, B^*C^* and A^*C^* are described by

$$\begin{cases} A^*B^* = \{(\xi,\eta) \mid 0 \le \xi \le \ell_1, \ \eta = -\lambda\xi\} \\ \\ B^*C^* = \{(\xi,\eta) \mid \xi = \ell_1, \ -\lambda\ell_1 \le \eta \le -\lambda\ell_1 + (1 + \lambda^2)\ell_2\} \\ \\ A^*C^* = \{(\xi,\eta) \mid 0 \le \xi \le \ell_1, \ \eta = F(\xi)\} \end{cases}$$

where

$$F(\xi) = -\lambda\xi + (1 + \lambda^2)\psi^{-1}(\xi)$$

with ψ^{-1} the inverse of the monotonically increasing function

$$\psi(y) = \phi(y) + \lambda y, \qquad y \ge 0.$$

We have

$$u_x^2 + u_y^2 = (u_\xi - \lambda u_\eta)^2 + (\lambda u_\xi + u_\eta)^2 = (1 + \lambda^2)(u_\xi^2 + u_\eta^2).$$

Using this result we get from (26)

$$\int_{-\lambda z}^{F(z)} I(z,\eta;\varepsilon)d\eta - \frac{M}{m}\int_0^z \{\int_{-\lambda\xi}^{F(\xi)} I(\xi,\eta;\varepsilon)d\eta\}d\xi \le (1 + \lambda^2)\frac{M}{m}K^2(\Omega,\varepsilon)$$

valid for every $z \in [0,\ell_1]$ and $0 < \varepsilon \le \varepsilon_0$, where

$$I(\xi,\eta;\varepsilon) = u^2 + \varepsilon(1 + \lambda^2)\Phi(u_\xi^2 + u_\eta^2).$$

Applying GRONWALL's lemma we obtain

(28)
$$\int_{-\lambda\xi}^{F(\xi)} (u^2 + \varepsilon\Phi u_\xi^2 + \varepsilon\Phi u_\eta^2)d\eta \le (1 + \lambda^2)\frac{M}{m}K^2(\Omega,\varepsilon)\exp(\frac{M}{m}\xi)$$

$$\le C(\Omega)K^2(\Omega,\varepsilon)$$

valid for $0 \le \xi \le \ell_1$ and $0 < \varepsilon \le \varepsilon_0$, the positive constant $C(\Omega)$ (depending on Ω but independent of ε) being given by

$$C(\Omega) = (1 + \lambda^2)\frac{M}{m}\exp(\frac{M\ell_1}{m}).$$

Since the transformation (27) transforms the part of the curve $x = \phi(y) + \delta$ inside Ω into the curve $\eta = F(\xi - \delta) - \lambda\delta$, $\delta \le \xi \le \ell_1$ inside Ω^* it follows that

(29) $\Phi \equiv 1$ for $-\lambda\xi \le \eta \le G(\xi)$, $\delta \le \xi \le \ell_1$

where $G(\xi) = F(\xi - \delta) - \lambda\delta$. Hence we obtain from (28) and (29)

(30a)
$$\int_{-\lambda\xi}^{F(\xi)} (u^2 + \varepsilon^2 u_\xi^2 + \varepsilon^2 u_\eta^2)d\eta \le C(\Omega)K^2(\Omega,\varepsilon) \qquad \text{for } 0 \le \xi \le \ell_1$$

(30b)
$$\int_{-\lambda\xi}^{G(\xi)} (u^2 + \varepsilon u_\xi^2 + \varepsilon u_\eta^2)d\eta \le C(\Omega)K^2(\Omega,\varepsilon) \qquad \text{for } \delta \le \xi \le \ell_1$$

valid for $0 < \varepsilon \le \varepsilon_0$.

Now that we have established estimates in L_2-norm it is not difficult to derive pointwise estimates. Using (30a) we obtain for $0 \le \xi \le \ell_1$, $-\lambda\xi \le \eta \le F(\xi)$ and $0 < \varepsilon \le \varepsilon_0$

$$|u^2(\xi,\eta;\varepsilon) - u^2(\xi,-\lambda\xi;\varepsilon)| = |2 \int_{-\lambda\xi}^{\eta} u(\xi,\tau;\varepsilon)u_\eta(\xi,\tau;\varepsilon)d\tau|$$

$$\leq 2 \int_{-\lambda\xi}^{F(\xi)} |u(\xi,\tau;\varepsilon)u_\eta(\xi,\tau;\varepsilon)|d\tau$$

$$\leq 2\left[\int_{-\lambda\xi}^{F(\xi)} u^2(\xi,\tau;\varepsilon)d\tau\right]^{\frac{1}{2}}\left[\int_{-\lambda\xi}^{F(\xi)} u_\eta^2(\xi,\tau;\varepsilon)d\tau\right]^{\frac{1}{2}}$$

$$\leq C(\Omega)\varepsilon^{-1}K^2(\Omega,\varepsilon)$$

with $C(\Omega)$ some generic constant depending on Ω but independent of ε. This yields

$$u^2(\xi,\eta;\varepsilon) \leq u^2(\xi,-\lambda\xi;\varepsilon) + C(\Omega)\varepsilon^{-1}K^2(\Omega,\varepsilon)$$

(31)

$$= g^2(\xi;\varepsilon) + C(\Omega)\varepsilon^{-1}K^2(\Omega,\varepsilon).$$

Using SOBOLEV's inequality for estimating g in terms of $\|g\|_{[0,\ell_1]}$ and $\|g_x\|_{[0,\ell_1]}$ and remembering the expression (25) for $K(\Omega,\varepsilon)$ we obtain from (31)

$$u^2(\xi,\eta;\varepsilon) \leq C(\Omega)\varepsilon^{-1}K^2(\Omega,\varepsilon)$$

valid for $(\xi,\eta) \in \Omega^*$ and $0 < \varepsilon \leq \varepsilon_0$, and hence also

$$|u(x,y;\varepsilon)| \leq C(\Omega)\varepsilon^{-\frac{1}{2}}K(\Omega,\varepsilon)$$

valid for $(x,y) \in \Omega$ and $0 < \varepsilon \leq \varepsilon_0$, where $C(\Omega)$ is some positive constant depending on Ω but independent of ε.

In a similar way we obtain from (30b) the estimate

$$u^2(\xi,\eta;\varepsilon) \leq C(\Omega)\varepsilon^{-\frac{1}{2}}K^2(\Omega,\varepsilon)$$

valid for $\delta \leq \xi \leq \ell_1$, $-\lambda\xi \leq \eta \leq G(\xi)$ and $0 < \varepsilon \leq \varepsilon_0$, and hence also

$$|u(x,y;\varepsilon)| \leq C(\Omega)\varepsilon^{-\frac{1}{4}}K(\Omega,\varepsilon)$$

valid for $(x,y) \in \Omega \setminus \{(x,y) \mid \phi(y) \leq x < \phi(y) + \delta, \ 0 \leq y \leq \ell_2\}$ and $0 < \varepsilon \leq \varepsilon_0$, where $C(\Omega)$ is some positive constant depending on Ω but independent of ε.

We summarize the results in

THEOREM 1. *Let $u(x,y;\varepsilon)$ be the solution of the characteristic boundary value problem* (1) *under the conditions* (2), (3) *and* (4) *and let the region $\Omega \subset \bar{D}$ be defined by* (5). *Then we have for ε sufficiently small, say $0 < \varepsilon \leq \varepsilon_0$ with ε_0 depending on Ω, the pointwise estimates*

(32)
$$\begin{cases} \sup_{\Omega} \ |u(x,y;\varepsilon)| \ \leq \ C(\Omega)\varepsilon^{-\frac{1}{2}}K(\Omega,\varepsilon) \\[2ex] \sup_{\tilde{\Omega}} \ |u(x,y;\varepsilon)| \ \leq \ C(\Omega)\varepsilon^{-\frac{1}{4}}K(\Omega,\varepsilon) \end{cases}$$

where the region $\tilde{\Omega}$ is defined by

$$\tilde{\Omega} = \Omega \setminus \{(x,y) \mid \phi(y) \leq x < \phi(y) + \delta, \ 0 \leq y \leq \ell_2\}$$

with δ some arbitrarily small positive constant independent of ε. $C(\Omega)$ is some positive constant depending on Ω but independent of ε, and the positive constant $K(\Omega,\varepsilon)$ depending on Ω and ε is given by

$$K(\Omega,\varepsilon) = \|f\|_{\Omega} + \|g\|_{[0,\ell_1]} + \|h\|_{[0,\ell_2]} + \varepsilon^{\frac{1}{2}}\|g_x\|_{[0,\ell_1]} + \varepsilon\|h_y\|_{[0,\ell_2]}$$

with $\|\cdot\|_{\Omega}$ denoting the L_2-norm over Ω, etc.

REMARKS.

1. A slight improvement (which will turn out to be of importance later on) can be obtained as follows: replacing formula (23) by the inequality

$$Q_1 \leq M_6(u^2 + \varepsilon\Phi u_x^2)$$

of which the validity is easily verified, it follows that the estimates (32) remain valid if the constant $K(\Omega,\varepsilon)$ is replaced by $\tilde{K}(\Omega,\varepsilon)$ with

$$\tilde{K}(\Omega,\varepsilon) = \|f\|_{\Omega} + \|g\|_{[0,\ell_1]} + \|h\|_{[0,\ell_2]}$$

$$+ \ \varepsilon^{\frac{1}{2}}\|\Phi^{\frac{1}{2}}g_x\|_{[0,\ell_1]} + \varepsilon\|h_y\|_{[0,\ell_2]}.$$

We shall use this result in section 3.

2. Another estimate is possible if it is given that there exists for every $Y > 0$ an $\varepsilon_0 > 0$ such that

(33) $\qquad a(\phi(y),y;\varepsilon) - b(\phi(y),y;\varepsilon)\phi'(y) \geq c_0 > 0$

valid for $0 \leq y \leq Y$ and $0 < \varepsilon \leq \varepsilon_0$, where c_0 is some constant depending on Y and ε_0 but independent of ε. The condition (33) implies the absence of a boundary layer along $x = \phi(y)$, $y \geq 0$; this condition is fulfilled if for instance $\phi'(y) \equiv 0$ for $y \geq 0$. Replacing (12) by

$$\Psi(\rho) \equiv 1 \quad \text{for } \rho \geq 0$$

and adjusting the proof of theorem 1 it follows that

$$\sup_{\Omega} |u(x,y;\varepsilon)| \leq C(\Omega)\varepsilon^{-\frac{1}{4}}K_1(\Omega,\varepsilon)$$

where $C(\Omega)$ is some positive constant depending on Ω but independent of ε, and the positive constant $K_1(\Omega,\varepsilon)$ depending on Ω and ε is given by

$$K_1(\Omega,\varepsilon) = \|f\|_{\Omega} + \|g\|_{[0,\ell_1]} + \|h\|_{[0,\ell_2]}$$
$$+ \varepsilon^{\frac{1}{2}}\|g_x\|_{[0,\ell_1]} + \varepsilon^{\frac{1}{2}}\|h_y\|_{[0,\ell_2]}.$$

2. Approximations for the case without boundary layer along $x = \phi(y)$, $y \geq 0$

In this section we shall construct an approximation for the solution $u(x,y;\varepsilon)$ of the characteristic boundary value problem

(34) $\qquad \begin{cases} \varepsilon L_2[u] + L_1^{(\varepsilon)}[u] = f(x,y), & (x,y) \in D \\[2ex] u(x,0;\varepsilon) = g(x), & 0 \leq x < \infty \\[2ex] u(\phi(y),y;\varepsilon) = h(y), & 0 \leq y < \infty \end{cases}$

where $0 < \varepsilon \ll 1$ and

$$\left\{\begin{array}{l} L_2[u] = \dfrac{\partial^2 u}{\partial x \partial y} \\[3mm] L_1^{(\varepsilon)}[u] = \{a_1(x,y) + \varepsilon a_2(x,y)\}\dfrac{\partial u}{\partial x} + \{b_1(x,y) + \varepsilon b_2(x,y)\}\dfrac{\partial u}{\partial y} + c(x,y)u, \end{array}\right.$$

the region D being defined as in the previous section.

In order not to complicate the calculations it will be assumed that all data are C^∞-functions which in addition satisfy

(35) $a_1(x,y) > 0$, $b_1(x,y) > 0$ in $\bar{D}$

and

(36) $\left\{\begin{array}{l} \phi(0) = 0 \\[3mm] \phi'(y) \geq 0 \quad \text{for } 0 \leq y < \infty \\[3mm] g(0) = h(0). \end{array}\right.$

REMARK. Note that the differential equation (34) follows from the equation

(37) $\varepsilon\{\dfrac{\partial^2 u}{\partial t^2} - c^2(x,t)\dfrac{\partial^2 u}{\partial x^2}\} + a(x,t)\dfrac{\partial u}{\partial t} + b(x,t)\dfrac{\partial u}{\partial x} + d(x,t)u = f(x,t)$

(with $c(x,t) > 0$) by a transformation of coordinates. It can be shown that the conditions (35) for the equation (34) correspond to the well-known growth conditions

$a(x,t) > 0$ and $|b(x,t)| < a(x,t)c(x,t)$

for the equation (37).

As a first approximation for the solution $u(x,y;\varepsilon)$ of (34) we shall take the solution $w(x,y)$ of the reduced problem which is governed by the equation

(38) $L_1^{(0)}[w] \equiv a_1(x,y)\dfrac{\partial w}{\partial x} + b_1(x,y)\dfrac{\partial w}{\partial y} + c(x,y)w = f(x,y)$, $(x,y) \in D$.

The number of boundary conditions which can be imposed on solutions of this equation depends on the position of the subcharacteristics with respect to

the part of the boundary ∂D given by $x = \phi(y)$, $y \geq 0$. If it is given that

(39) $a_1(\phi(y),y) - b_1(\phi(y),y)\phi'(y) > 0$ for $0 \leq y < \infty$

then we have the situation of figure 4a and we can impose both boundary conditions on solutions of (38). This case will be discussed in this section. If on the other hand

$a_1(\phi(y),y) - b_1(\phi(y),y)\phi'(y) < 0$ for $0 \leq y < \infty$

then we have a situation as sketched in figure 4b and we can impose only the boundary condition $u(x,0;\varepsilon) = g(x)$ on solutions of (38). We shall discuss this problem in section 3.
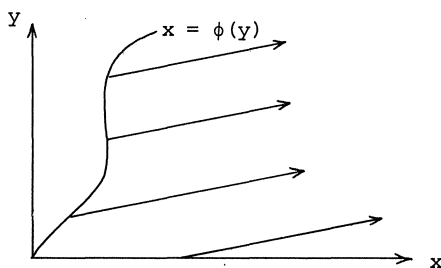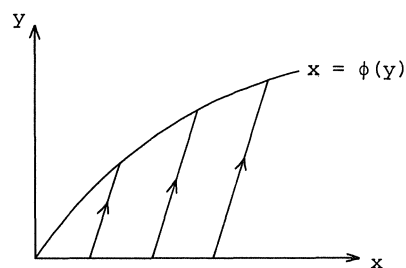

Figure 4a   Figure 4b

Assuming the condition (39) to be fulfilled we take for w the solution of

(40)
$$
\begin{cases}
L_1^{(0)}[w] = f(x,y), & (x,y) \in D \\[2mm]
w(x,0) = g(x), & 0 \leq x < \infty \\[2mm]
w(\phi(y),y) = h(y), & 0 \leq y < \infty.
\end{cases}
$$

As in section 2 of the preceding chapter we cannot evaluate $L_2[w]$ since the solution w of (40) is not necessarily twice differentiable. This is stated in

LEMMA 1. *Let* $u(x,y)$ *be the solution of the boundary value problem*

$$(41) \quad \begin{cases} a(x,y)\dfrac{\partial u}{\partial x} + b(x,y)\dfrac{\partial u}{\partial y} + c(x,y)u = f(x,y), & (x,y) \in D \\[2mm] u(x,0) = g(x), & 0 \le x < \infty \\[2mm] u(\phi(y),y) = h(y), & 0 \le y < \infty \end{cases}$$

where the region D *is given by*

$$D = \{(x,y) \mid \phi(y) < x < \infty, \ 0 < y < \infty\}$$

and all data are supposed to be C^{∞}*-functions which in addition satisfy the conditions*

$$\begin{cases} a(x,y) > 0, \quad b(x,y) > 0 \quad \text{in } \bar{D} \\[2mm] a(\phi(y),y) - b(\phi(y),y)\phi'(y) > 0 \quad \text{for } 0 \le y < \infty \end{cases}$$

and

$$\begin{cases} \phi(0) = 0 \\[2mm] \phi'(y) \ge 0 \quad \text{for } 0 \le y < \infty \\[2mm] g(0) = h(0). \end{cases}$$

Then the function $u(x,y)$ *belongs to* $C^{1}(\bar{D})$ *iff*

$$\{a(0,0) - b(0,0)\phi'(0)\}g'(0) + b(0,0)h'(0) + c(0,0)g(0) = f(0,0).$$

In that case the second derivatives of $u(x,y)$ *exist in* $\bar{D}$ *and are continuous, with possible exception of the characteristic through the origin where the second derivatives may exhibit a finite jump.*

PROOF. Applying the coordinate transformation

$$\begin{cases} \xi = x - \phi(y) \\[2mm] \eta = y \end{cases}$$

to problem (41) we obtain a problem which can be treated in the same way as problem (31) of chapter V (cf. also the remark following the proof of lemma 1 in chapter V).

The difficulty of not being able to evaluate $L_2[w]$ can be avoided in the same way as in subsection 2.2 of chapter V, namely by considering instead of (40) the modified reduced problem

(42)
$$
\begin{cases}
L_1^{(0)}[\tilde{w}] = f(x,y), & (x,y) \in D \\[2mm]
\tilde{w}(x,0;\varepsilon) = g(x), & 0 \le x < \infty \\[2mm]
\tilde{w}(\phi(y),y;\varepsilon) = H(y;\varepsilon), & 0 \le y < \infty
\end{cases}
$$

where the function $H(y;\varepsilon)$ is defined by

$$
H(y;\varepsilon) = h(y) - C\,y\exp(-\frac{y}{\varepsilon^p}), \qquad 0 \le y < \infty,
$$

the constant C is given by

$$
C = \frac{\{a_1(0,0)-b_1(0,0)\phi'(0)\}g'(0)+b_1(0,0)h'(0)+c(0,0)g(0)-f(0,0)}{b_1(0,0)}
$$

and p is some positive constant to be determined later on. According to lemma 1 the solution $\tilde{w}(x,y;\varepsilon)$ of (42) is twice differentiable, so the expression $L_2[\tilde{w}]$ is now defined.

We define the remainder term $z(x,y;\varepsilon)$ by

(43) $z(x,y;\varepsilon) = u(x,y;\varepsilon) - \tilde{w}(x,y;\varepsilon).$

Then we obtain from (34) and (42)

(44)
$$
\begin{cases}
\varepsilon L_2[z] + L_1^{(\varepsilon)}[z] = -\varepsilon\{L_2[\tilde{w}] + a_2(x,y)\frac{\partial\tilde{w}}{\partial x} + b_2(x,y)\frac{\partial\tilde{w}}{\partial y}\}, & (x,y) \in D \\[2mm]
z(x,0;\varepsilon) = 0, & 0 \le x < \infty \\[2mm]
z(\phi(y),y;\varepsilon) = C\,y\exp(-\frac{y}{\varepsilon^p}), & 0 \le y < \infty.
\end{cases}
$$

A calculation shows that

$$\begin{cases} \|L_2[\tilde{w}]\|_\Omega = O(\varepsilon^{-\frac{1}{2}p}), & \varepsilon \downarrow 0 \\[2ex] \|\frac{\partial \tilde{w}}{\partial x}\|_\Omega = O(\varepsilon^{\frac{1}{2}p}), & \varepsilon \downarrow 0 \\[2ex] \|\frac{\partial \tilde{w}}{\partial y}\|_\Omega = O(\varepsilon^{\frac{1}{2}p}), & \varepsilon \downarrow 0 \end{cases}$$

valid for any compact subset $\Omega \subset \bar{D}$.

Hence, applying theorem 1 (and the second remark following that theorem) to (44) we obtain

$$(45) \qquad \sup_\Omega |z(x,y;\varepsilon)| = O(\varepsilon^\mu), \qquad \varepsilon \downarrow 0$$

valid for any compact subset $\Omega \subset \bar{D}$, with

$$\mu = -\frac{1}{4} + \min(1 - \frac{1}{2}p, \frac{3}{2}p, \frac{1}{2} + \frac{1}{2}p).$$

It is easily established that μ becomes optimal if we choose $p = \frac{1}{2}$. In that case we have $\mu = \frac{1}{2}$, so it follows from (43) and (45) that

$$u(x,y;\varepsilon) = \tilde{w}(x,y;\varepsilon) + O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$.

Since the solutions w and $\tilde{w}$ of (40) and (42) satisfy

$$\tilde{w}(x,y;\varepsilon) = w(x,y) + O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$$

uniformly in each compact subset $\Omega \subset \bar{D}$ (cf. chapter V, subsection 2.2) we can formulate

THEOREM 2. *Let* $u(x,y;\varepsilon)$ *be the solution of the characteristic boundary value problem* (34) *where all data are* C^∞-*functions which in addition satisfy* (35), (36) *and* (39). *Then we have*

$$u(x,y;\varepsilon) = w(x,y) + O(\varepsilon^{\frac{1}{2}}), \qquad \varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$, *where* $w(x,y)$ *is the solution*

of the reduced problem (40).

3. <u>Approximations for the case with boundary layer along $x = \phi(y)$, $y \geq 0$</u>

In this section we shall construct an approximation for the solution $u(x,y;\varepsilon)$ of the characteristic boundary value problem (34) where all data are C^{∞}-functions which in addition satisfy

$$(46) \quad \begin{cases} a_1(x,y) > 0, \quad b_1(x,y) > 0 \qquad \text{in } \bar{D} \\ \\ a_1(\phi(y),y) - b_1(\phi(y),y)\phi'(y) < 0 \qquad \text{for } 0 \leq y < \infty \end{cases}$$

and

$$(47) \quad \begin{cases} \phi(0) = 0 \\ \\ \phi'(y) > 0 \qquad \text{for } 0 \leq y < \infty \\ \\ g(0) = h(0). \end{cases}$$

<u>REMARK.</u> It follows from (46) that the part of the boundary ∂D given by $x = \phi(y)$, $y \geq 0$ can no longer be characteristic. Therefore we have to restrict ourselves to problems with a timelike boundary $x = \phi(y)$, $y \geq 0$. Problems of this type are sometimes referred to as Goursat problems.

As a first approximation for $u(x,y;\varepsilon)$ we take the solution $w_1(x,y)$ of the reduced problem

$$(48) \quad \begin{cases} L_1^{(0)}[w_1] = f(x,y), \qquad (x,y) \in D \\ \\ w_1(x,0) = g(x), \qquad 0 \leq x < \infty \end{cases}$$

(see figure 4b). In order to cope with the remaining boundary condition we look for an approximation $\tilde{u}$ of u in the form

$$(49) \qquad \tilde{u}(x,y;\varepsilon) = w_1(x,y) + \varepsilon w_2(x,y) + v_1(\xi,\eta) + \varepsilon v_2(\xi,\eta) + \varepsilon^2 v_3(\xi,\eta)$$

where

(50) $\qquad \xi = \dfrac{x - \phi(y)}{\varepsilon}, \qquad \eta = y$

and the functions w_2, v_1, v_2 and v_3 have to be determined.

Applying the transformation (50) to the operator $L^{(\varepsilon)} = \varepsilon L_2 + L_1^{(\varepsilon)}$ and expanding this operator into powers of ε we obtain after some calculations

$$L^{(\varepsilon)} = \frac{1}{\varepsilon}\{-\phi'(\eta)\frac{\partial^2}{\partial\xi^2} + \lambda_1^{(0)}(\eta)\frac{\partial}{\partial\xi}\} + M_0 + \varepsilon M_1 + \varepsilon^2 M_2^{(\varepsilon)}$$

where the operators M_0, M_1 and $M_2^{(\varepsilon)}$ are defined by

$$
\begin{cases}
M_0 = \dfrac{\partial^2}{\partial\xi\partial\eta} + \{\xi\lambda_1^{(1)}(\eta) + \lambda_2^{(0)}(\eta)\}\dfrac{\partial}{\partial\xi} + b_1(\phi(\eta),\eta)\dfrac{\partial}{\partial\eta} + c(\phi(\eta),\eta) \\[3mm]
M_1 = \{\dfrac{1}{2}\xi^2\lambda_1^{(2)}(\eta) + \xi\lambda_2^{(1)}(\eta)\}\dfrac{\partial}{\partial\xi} + \{\xi\dfrac{\partial b_1}{\partial x}(\phi(\eta),\eta) + b_2(\phi(\eta),\eta)\}\dfrac{\partial}{\partial\eta} \\[3mm]
\qquad + \xi\dfrac{\partial c}{\partial x}(\phi(\eta),\eta) \\[3mm]
M_2^{(\varepsilon)} = \dfrac{1}{6}\xi^3\{\dfrac{\partial^3 a_1}{\partial x^3}(\phi(\eta) + \varepsilon\Theta_1\xi,\eta) - \phi'(\eta)\dfrac{\partial^3 b_1}{\partial x^3}(\phi(\eta) + \varepsilon\Theta_2\xi,\eta)\}\dfrac{\partial}{\partial\xi} \\[3mm]
\qquad + \dfrac{1}{2}\xi^2\{\dfrac{\partial^2 a_2}{\partial x^2}(\phi(\eta) + \varepsilon\Theta_3\xi,\eta) - \phi'(\eta)\dfrac{\partial^2 b_2}{\partial x^2}(\phi(\eta) + \varepsilon\Theta_4\xi,\eta)\}\dfrac{\partial}{\partial\xi} \\[3mm]
\qquad + \{\dfrac{1}{2}\xi^2\dfrac{\partial^2 b_1}{\partial x^2}(\phi(\eta) + \varepsilon\Theta_5\xi,\eta) + \xi\dfrac{\partial b_2}{\partial x}(\phi(\eta) + \varepsilon\Theta_6\xi,\eta)\}\dfrac{\partial}{\partial\eta} \\[3mm]
\qquad + \dfrac{1}{2}\xi^2\dfrac{\partial^2 c}{\partial x^2}(\phi(\eta) + \varepsilon\Theta_7\xi,\eta)
\end{cases}
$$

with $0 < \Theta_i < 1$ ($i = 1,2,\ldots,7$). The functions $\lambda_i^{(j)}(\eta)$ are given by

$$\lambda_i^{(j)}(\eta) = \frac{\partial^j a_i}{\partial x^j}(\phi(\eta),\eta) - \phi'(\eta)\frac{\partial^j b_i}{\partial x^j}(\phi(\eta),\eta) \qquad \text{for } i = 1,2, \\ j = 0,1,2.$$

Hence it follows that

$$L^{(\varepsilon)}[\tilde{u}] = L^{(\varepsilon)}[w_1 + \varepsilon w_2] + L^{(\varepsilon)}[v_1 + \varepsilon v_2 + \varepsilon^2 v_3]$$

$$= L_1^{(0)}[w_1] + \varepsilon\{L_1^{(0)}[w_2] + \frac{\partial^2 w_1}{\partial x \partial y} + a_2(x,y)\frac{\partial w_1}{\partial x} + b_2(x,y)\frac{\partial w_1}{\partial y}\}$$

$$+ \frac{1}{\varepsilon}\{-\phi'(\eta)\frac{\partial^2 v_1}{\partial \xi^2} + \lambda_1^{(0)}(\eta)\frac{\partial v_1}{\partial \xi}\}$$

(51)
$$+ \{-\phi'(\eta)\frac{\partial^2 v_2}{\partial \xi^2} + \lambda_1^{(0)}(\eta)\frac{\partial v_2}{\partial \xi} + M_0[v_1]\}$$

$$+ \varepsilon\{-\phi'(\eta)\frac{\partial^2 v_3}{\partial \xi^2} + \lambda_1^{(0)}(\eta)\frac{\partial v_3}{\partial \xi} + M_1[v_1] + M_0[v_2]\}$$

$$+ \varepsilon^2 R(x,y;\varepsilon)$$

with

$$R(x,y;\varepsilon) = \frac{\partial^2 w_2}{\partial x \partial y} + a_2(x,y)\frac{\partial w_2}{\partial x} + b_2(x,y)\frac{\partial w_2}{\partial y}$$

$$+ M_2^{(\varepsilon)}[v_1] + M_1[v_2] + \varepsilon M_2^{(\varepsilon)}[v_2]$$

$$+ M_0[v_3] + \varepsilon M_1[v_3] + \varepsilon^2 M_2^{(\varepsilon)}[v_3].$$

We now take for the functions w_2, v_1, v_2 and v_3 the solutions of the following problems

(52)
$$\begin{cases} L_1^{(0)}[w_2] = -\frac{\partial^2 w_1}{\partial x \partial y} - a_2(x,y)\frac{\partial w_1}{\partial x} - b_2(x,y)\frac{\partial w_1}{\partial y}, & (x,y) \in D \\ \\ w_2(x,0) = 0, & 0 \le x < \infty \end{cases}$$

(53)
$$\begin{cases} \phi'(\eta)\frac{\partial^2 v_1}{\partial \xi^2} - \lambda_1^{(0)}(\eta)\frac{\partial v_1}{\partial \xi} = 0, & 0 < \xi < \infty, \ 0 \le \eta < \infty \\ \\ v_1(0,\eta) = h(\eta) - w_1(\phi(\eta),\eta), & 0 \le \eta < \infty \\ \\ \lim_{\xi \to \infty} v_1(\xi,\eta) = 0, & 0 \le \eta < \infty \end{cases}$$

$$(54) \quad \begin{cases} \phi'(\eta)\dfrac{\partial^2 v_2}{\partial \xi^2} - \lambda_1^{(0)}(\eta)\dfrac{\partial v_2}{\partial \xi} = M_0[v_1], \qquad 0 < \xi < \infty, \ 0 \le \eta < \infty \\[2ex] v_2(0,\eta) = -w_2(\phi(\eta),\eta), \qquad 0 \le \eta < \infty \\[2ex] \lim_{\xi \to \infty} v_2(\xi,\eta) = 0, \qquad 0 \le \eta < \infty \end{cases}$$

$$(55) \quad \begin{cases} \phi'(\eta)\dfrac{\partial^2 v_3}{\partial \xi^2} - \lambda_1^{(0)}(\eta)\dfrac{\partial v_3}{\partial \xi} = M_1[v_1] + M_0[v_2], \qquad 0 < \xi < \infty, \ 0 \le \eta < \infty \\[2ex] v_3(0,\eta) = 0, \qquad 0 \le \eta < \infty \\[2ex] \lim_{\xi \to \infty} v_3(\xi,\eta) = 0, \qquad 0 \le \eta < \infty. \end{cases}$$

One easily verifies that

$$\begin{cases} v_1(\xi,\eta) = \{h(\eta) - w_1(\phi(\eta),\eta)\}\exp[-\lambda(\eta)\xi], \qquad 0 \le \xi,\eta < \infty \\[2ex] v_2(\xi,\eta) = P_1(\xi;\eta)\exp[-\lambda(\eta)\xi], \qquad 0 \le \xi,\eta < \infty \\[2ex] v_3(\xi,\eta) = P_2(\xi;\eta)\exp[-\lambda(\eta)\xi], \qquad 0 \le \xi,\eta < \infty \end{cases}$$

where the positive function $\lambda(\eta)$ is defined by

$$\lambda(\eta) = -\frac{\lambda_1^{(0)}(\eta)}{\phi'(\eta)} = b_1(\phi(\eta),\eta) - \frac{a_1(\phi(\eta),\eta)}{\phi'(\eta)} \qquad \text{for } 0 \le \eta < \infty$$

and P_1 and P_2 are polynomials in ξ with coefficients depending on η. So v_1, v_2 and v_3 are boundary layer functions.

With the above mentioned choice for the functions w_i $(i = 1,2)$ and v_i $(i = 1,2,3)$ we obtain from (48), (49), (51), (52), (53), (54) and (55)

$$(56) \quad \begin{cases} L^{(\varepsilon)}[\tilde{u}] = f(x,y) + \varepsilon^2 R(x,y;\varepsilon), \qquad (x,y) \in D \\[2ex] \tilde{u}(x,0;\varepsilon) = g(x) + \varepsilon v_2(\tfrac{x}{\varepsilon},0) + \varepsilon^2 v_3(\tfrac{x}{\varepsilon},0), \qquad 0 \le x < \infty \\[2ex] \tilde{u}(\phi(y),y;\varepsilon) = h(y), \qquad 0 \le y < \infty. \end{cases}$$

Defining the remainder term $z(x,y;\varepsilon)$ by

(57) $z(x,y;\varepsilon) = u(x,y;\varepsilon) - \tilde{u}(x,y;\varepsilon)$

we get from (34) and (56)

(58)
$$\begin{cases} L^{(\varepsilon)}[z] = -\varepsilon^2 R(x,y;\varepsilon), & (x,y) \in D \\[2ex] z(x,0;\varepsilon) = -\varepsilon v_2(\tfrac{x}{\varepsilon},0) - \varepsilon^2 v_3(\tfrac{x}{\varepsilon},0), & 0 \le x < \infty \\[2ex] z(\phi(y),y;\varepsilon) = 0, & 0 \le y < \infty. \end{cases}$$

One easily verifies that

$$\|R\|_\Omega = 0(1), \qquad \varepsilon \downarrow 0$$

valid for every compact subset $\Omega \subset \bar{D}$.

Hence, applying theorem 1 (and the first remark following that theorem) to (58) we obtain

(59) $z(x,y;\varepsilon) = 0(\varepsilon), \qquad \varepsilon \downarrow 0$

uniformly valid in any compact subset $\Omega \subset \bar{D}$.

Finally, using the formulae (49), (57) and (59) and also the fact that the functions w_2, v_2 and v_3 are uniformly bounded in compact subsets of $\bar{D}$, we arrive at

THEOREM 3. *Let $u(x,y;\varepsilon)$ be the solution of the Goursat problem (34) where all data are C^∞-functions which in addition satisfy (46) and (47). Then we have*

$$u(x,y;\varepsilon) = w(x,y) + 0(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly valid in any compact subset

$$\Omega \subset \{(x,y) \mid \phi(y) + \delta \le x < \infty, \; 0 \le y < \infty\}.$$

Here $w(x,y)$ *is the solution of the reduced problem*

$$\begin{cases} L_1^{(0)}[w] = f(x,y), & (x,y) \in D \\ \\ w(x,0) = g(x), & 0 \le x < \infty \end{cases}$$

and δ *is an arbitrarily small positive constant independent of* ε. *In addition we have*

$$u(x,y;\varepsilon) = w(x,y) + \{h(y) - w(\phi(y),y)\}\exp[\lambda(y)\frac{\phi(y)-x}{\varepsilon}] + 0(\varepsilon), \qquad \varepsilon \downarrow 0$$

uniformly valid in any compact subset $\Omega \subset \bar{D}$, *where the positive function* $\lambda(y)$ *is defined by*

$$\lambda(y) = b_1(\phi(y),y) - \frac{a_1(\phi(y),y)}{\phi'(y)} \qquad \text{for } 0 \le y < \infty.$$

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