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MATROIDS AND LINKING SYSTEMS

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CONTENTS

INTRODUCTION

This monograph deals with *linking systems* and their close relations to *matroids*. Before proceeding to a more strict discussion (chapters 0-7) we give an informal introduction.

a. MATROIDS

Suppose we have a finite set X and a collection $\mathcal{I}$ of subsets of X. Let furthermore a "capacity" function c from X into the set of nonnegative real numbers be given. We want to find a set A in $\mathcal{I}$ with maximum capacity, that is with

$$\sum_{x \in A} c(x)$$

as large as possible. An "obvious" manner to find such a set could be the following (the so-called *greedy algorithm*), in which we may suppose, without loss of generality, that if $B \subset A \in \mathcal{I}$, then $B \in \mathcal{I}$ ($\mathcal{I}$ is *monotonic*):

(1) choose $x_1 \in X$ such that $\{x_1\} \in \mathcal{I}$ and $c(x_1)$ is as large as possible;
(2) choose $x_2 \in X\setminus\{x_1\}$ such that $\{x_1,x_2\} \in \mathcal{I}$ and $c(x_2)$ is as large as possible;
(3) choose $x_3 \in X\setminus\{x_1,x_2\}$ such that $\{x_1,x_2,x_3\} \in \mathcal{I}$ and $c(x_3)$ is as large as possible;
(4) ....

etcetera, until $\{x_1,x_2,x_3,\ldots,x_k\}$ is a maximal set in $\mathcal{I}$.

However, this algorithm needs not always deliver an optimal set in $\mathcal{I}$. If, for a certain nonempty monotonic collection $\mathcal{I}$, the greedy algorithm leads indeed always (i.e. for every capacity function c) to a set in $\mathcal{I}$ with maximal capacity, then (and only then) the pair $(X,\mathcal{I})$ is what is called a *matroid*.

An alternative ("capacity free") definition of a matroid $(X,\mathcal{I})$ is: $\mathcal{I}$ is a nonempty monotonic collection of subsets of X such that if $A,B \in \mathcal{I}$ and $|A| < |B|$ then $A \cup \{b\} \in \mathcal{I}$ for some $b \in B\backslash A$. This second definition can be interpreted more geometrically; e.g. if $\mathcal{I}$ is the collection of affinely independent subsets of a finite affine space X, then $(X,\mathcal{I})$ is a matroid. In an analogous way projective and linear spaces yield matroids (then consider projectively and linearly independent subsets, respectively). Therefore, given a matroid $(X,\mathcal{I})$, the subsets of X in $\mathcal{I}$ are called the *independent* sets of the matroid.

So first motivations for the concept of matroid can arise from geometry and from optimization theory.

Matroid theory has its origin in the 1930s when VAN DER WAERDEN in his book "Moderne Algebra" first approached linear and algebraic independence axiomatically, and WHITNEY introduced the term matroid and investigated structural properties of matroids, inspired by his graph-theoretical research.

VAN DER WAERDEN proved that (i) the collection of linearly independent subsets of a linear space and (ii) the collection of over F algebraically independent subsets of a field extension of some field F satisfy the axioms of a matroid. This was used by VAN DER WAERDEN to unify his argumentations. WHITNEY showed that those sets of edges of an undirected graph that do not contain any circuit form (the collection of independent sets of) a matroid (the *cycle matroid*); also that those sets of edges that do not contain any cutset form a matroid (the *cocycle matroid*). WHITNEY proved that the cycle matroids of exactly the planar graphs are (isomorphic to) the cocycle matroid of some (other) graph. Also WHITNEY examined, with the help of matroids, the problem: when is a graph completely determined by its collection of circuits? So WHITNEY introduced matroids to obtain insight in graph-theoretical problems.

An important result of RADO (1942) is an extension of the marriage theorem of P. HALL. When does a family of subsets $(A_1,\ldots,A_n)$ of a matroid $(X,\mathcal{I})$ have an independent "system of distinct representatives" (SDR), that is, when does there exist distinct elements $x_1 \in A_1,\ldots,x_n \in A_n$ such that $\{x_1,\ldots,x_n\}$ is in $\mathcal{I}$? RADO showed that this is the case if and only if the rank of each union of sets A_i is at least the number of sets forming that union (the *rank* of a subset A of X is the maximum cardinality of an independent set contained in A).

Except for some isolated work of BIRKHOFF, MACLANE and DILWORTH on the lattice-theoretical and geometric aspects of matroid theory the subject was left practically untouched until TUTTE proved in 1958 and 1959 two difficult theorems that gave characterizations of (i) matroids which are the cycle matroid of a graph, and (ii) matroids which are representable over each field (a matroid is *representable over* a field F if the matroid is embeddable in a vector space over F, with the linearly independent subsets as independent sets). These results are fundamental for the theory. TUTTE also tried to solve the Four Colour problem, inter alia with the help of matroids; this produced some interesting conjectures (for instance, does there exist for each directed graph without bridges a "flow" function from the arrows into $GF(5)\setminus\{0\}$, such that in each point of the graph the sum of the incoming flows is equal to the sum of the outgoing flows? From the Five Colour theorem it follows that the answer is "yes" for planar directed graphs).

A big impulse to the more recent development of matroid theory was given by the work of EDMONDS. In 1965 EDMONDS and FULKERSON discovered a new class of matroids, the so-called *transversal matroids*. These are obtained as follows. Let $(A_1,\ldots,A_n)$ be a family of subsets of a set X and define $\mathcal{I}$ as the collection of partial SDR's, that is $\mathcal{I}$ consists of all sets of the form $\{x_1,\ldots,x_k\}$ such that $x_1 \in A_{i_1},\ldots,x_k \in A_{i_k}$ for some distinct $i_1,\ldots,i_k$. Then $(X,\mathcal{I})$ is a matroid. This insight has been of much importance for the transversal theory.

In 1965 as well EDMONDS proved the "covering theorem" and the "packing theorem". The covering theorem says that a matroid $(X,\mathcal{I})$ can be covered by k independent subsets if and only if for each subset A of X we have

$$k \cdot \mathrm{rank}(A) \geq |A|.$$

The packing theorem asserts that $(X,\mathcal{I})$ has k disjoint bases (a *base* of a matroid is a maximal independent subset) if and only if for each subset A of X

$$k \cdot \mathrm{rank}(A) + |X\setminus A| \geq k \cdot \mathrm{rank}(X).$$

These theorems have as simple consequences (previously difficultly proved) theorems on graphs of TUTTE (1961) and NASH-WILLIAMS (1961,1964) (e.g. determining the maximal number of disjoint spanning trees in an undirected graph), a theorem determining the maximal number of disjoint bases of a vector space due to HORN (1955) and several old and new results in transversal theory.

In 1968 PERFECT discovered the following new class of matroids. Let G be a directed graph with "source" vertex v, and let X be a subset of the vertices. Then the sets of endpoints of paths starting in v, ending in X, and being pairwise disjoint (outside v itself) form a matroid on X; these matroids are called *gammoids*. So matroid theorems are applicable in network analysis as well.

Suppose now we have two matroids (X,I) and (X,J) on the same set X. How many elements can a common independent set have? The "intersection theorem" of EDMONDS (1970) tells that this number is equal to the minimal value of $\text{rank}_I(A) + \text{rank}_J(X\setminus A)$, where $A \subset X$. Of course, this theorem can be applied again in transversal theory (does a common SDR exist?), network analysis, graph theory and linear algebra. The advantage of this theorem as a matroid theorem is that now also "mixed" applications are possible. For instance, if G is a directed graph with points in a vector space the intersection theorem then gives an answer to the question: what is the maximal number of linearly independent vectors attainable from some source via disjoint paths?

The more recent research in matroid theory may be divided into a number of directions (of course with interactions between them) among which are:

(i) the direction of discrete optimization (finding algorithms and "min-max" relations for problems involving matroids; a central problem is how to handle the intersection of *three* matroids);

(ii) a direction investigating the geometric and lattice-theoretical aspects of matroids (a major problem is the so-called *critical problem*: given a matroid, what is the minimal number of hyperplanes such that their intersection is empty (a *hyperplane* is a maximal subset of the matroid with rank one less than the rank of the total matroid); solving this problem yields answers to many other open questions in combinatorics);

(iii) a more or less theoretical direction (e.g. characterizing matroids representable over a certain field).

The investigations described in the present monograph may be considered as partially in the first and partially in the third direction.

The recent book "Matroid theory" of WELSH gives a good introduction and a comprehensive survey of the results obtained in matroid theory.

b. LINKING SYSTEMS

In 1969 PERFECT proved the following theorem. Let $(X,\mathcal{I})$ be a matroid and let (X,Y,E) be a bipartite graph. Define $\mathcal{J}$ as the collection of all subsets Y' of Y such that Y' is matched in the bipartite graph with a set X' in $\mathcal{I}$ (X' and Y' are called *matched* with each other if there are disjoint edges $(x_1,y_1),\ldots,(x_n,y_n)$ such that $X' = \{x_1,\ldots,x_n\}$ and $Y' = \{y_1,\ldots,y_n\}$). Then $(Y,\mathcal{J})$ again is a matroid. A corollary of this theorem is that transversal matroids indeed are matroids. PERFECT's theorem has been a starting point for research in so-called *induced* matroids. BRUALDI (1971) proved that PERFECT's theorem also holds when we have a directed graph instead of a bipartite graph as a "medium" between X and Y. In this case X' and Y' are called *matched* if $|X'| = |Y'|$ and there are $|X'|$ pairwise disjoint paths starting in X' and ending in Y'. Together with MASON he also gave a formula for the rank of the induced matroid $(Y,\mathcal{J})$. BRUALDI also obtained a formula which, given a directed graph (Z,E) and matroids $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ on subsets X and Y of Z, determines the maximal cardinality of a set $X' \in \mathcal{I}$ which is matched in the directed graph with a set Y' in $\mathcal{J}$. This last result is a combination of both MENGER's theorem (by taking trivial matroids) and EDMONDS' intersection theorem (by taking a directed graph without edges).

In 1958 MENDELSOHN and DULMAGE showed the following property of matchings in bipartite graphs. Let (X,Y,E) be a bipartite graph and suppose that X_1 and Y_1 are matched with each other, and similarly, that X_2 and Y_2 are matched with each other ($X_1,X_2 \subset X$; $Y_1,Y_2 \subset Y$). Then one can also find a matched pair X', Y' such that $X_1 \subset X' \subset X_1 \cup X_2$ and $Y_2 \subset Y' \subset Y_1 \cup Y_2$. This property, the Mendelsohn-Dulmage property, holds also for infinite graphs (as ORE showed in 1962), and the well-known Schröder-Bernstein theorem is an early precursor of it.

Again, the Mendelsohn-Dulmage property has been extended (in the obvious way) to matchings in directed graph mediums, namely in 1968 by PERFECT. Indeed the Mendelsohn-Dulmage property is the base of our research in linking systems. We show that many of the theorems on matroids induced by bipartite or directed graphs can be extended to the case where the medium is some system which links sets in a Mendelsohn-Dulmage way. To be more precise, we define a *linking system* as a triple (X,Y,Λ), such that X and Y are finite sets and Λ is a nonempty collection of pairs (X',Y'), consisting of a subset X' of X and a subset Y' of Y, with the properties:

(i) if $(X',Y') \in \Lambda$ and $x \in X'$, then $(X'\backslash\{x\},Y'\backslash\{y\}) \in \Lambda$ for some $y \in Y'$;

(ii) if $(X',Y') \in \Lambda$ and $y \in Y'$, then $(X'\backslash\{x\},Y'\backslash\{y\}) \in \Lambda$ for some $x \in X'$;

(iii) if $(X_1,Y_1) \in \Lambda$ and $(X_2,Y_2) \in \Lambda$ then there exists $(X',Y') \in \Lambda$ such that $X_1 \subset X' \subset X_1 \cup X_2$ and $Y_2 \subset Y' \subset Y_1 \cup Y_2$.

Clearly, condition (iii) corresponds with the Mendelsohn-Dulmage property. So the class Λ of matched pairs of subsets in bipartite and directed graphs satisfy the conditions (i), (ii) and (iii).

Also the following class of pairs satisfies (i), (ii) and (iii). Let X and Y be the row collection and column collection, respectively, of a matrix M. Define Λ as the set of pairs (X',Y') such that $X' \subset X$ and $Y' \subset Y$ and the submatrix $M \mid X' \times Y'$ is nonsingular. Simple linear algebraic methods show that (i), (ii) and (iii) are fulfilled for this class Λ.

So obtaining results for the induction of matroids by linking systems also yields properties for matroids induced by matrices.

Among the theorems obtained on this induction are the following ones.

(1) Let $(X,\mathcal{I})$ be a matroid and let (X,Y,Λ) be a linking system; then $(Y,\mathcal{I}*\Lambda)$ is a matroid, where $\mathcal{I}*\Lambda = \{Y' \mid (X',Y') \in \Lambda \text{ for some } X' \in \mathcal{I}\}$.

(2) Let $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ be matroids and let (X,Y,Λ) be a linking system; then the maximal cardinality of a set $X' \in \mathcal{I}$ such that $(X',Y') \in \Lambda$ for some $Y' \in \mathcal{J}$ equals the minimal value of

$$\text{rank}_{\mathcal{I}}(X\backslash X') + \lambda(X',Y') + \text{rank}_{\mathcal{J}}(Y\backslash Y'),$$

for $X' \subset X$ and $Y' \subset Y$ (where $\lambda(X',Y')$ equals the maximal cardinality of the sets in a pair $(X'',Y'') \in \Lambda$ such that $X'' \subset X'$ and $Y'' \subset Y'$; λ is called the *linking function*).

(3) Again let $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ be matroids and let (X,Y,Λ) be a linking system; suppose $(X_1,Y_1) \in \Lambda$ and $(X_2,Y_2) \in \Lambda$ such that $X_1,X_2 \in \mathcal{I}$ and $Y_1,Y_2 \in \mathcal{J}$; then $(X',Y') \in \Lambda$ for some $X' \in \mathcal{I}$ and $Y' \in \mathcal{J}$ such that $X_1 \subset c\ell_{\mathcal{I}}X'$ and $Y_2 \subset c\ell_{\mathcal{J}}Y'$ (the *closure* $c\ell_{\mathcal{I}}X'$ of a set $X' \in \mathcal{I}$ equals $X' \cup \{x \in X \mid X' \cup \{x\} \notin \mathcal{I}\}$; $c\ell_{\mathcal{J}}$ is defined similarly).

The first theorem is, obviously, an extension of the mentioned results of PERFECT and BRUALDI on matroids and bipartite or directed graphs. The second result generalizes a theorem of BRUALDI as well. As a corollary of (2) for matrices we have, e.g., the following. Let M be a matrix with row collection X and column collection Y; suppose $\mathcal{A} = (A_i \mid i \in I)$ and $\mathcal{B} = (B_i \mid i \in I)$ are (finite) families of subsets of X and Y, respectively. Then $\mathcal{A}$ has an SDR X' and $\mathcal{B}$ has an SDR Y' such that the submatrix $M \mid X' \times Y'$ is nonsingular, if and only if the rank of the submatrix $M \mid (\bigcup_{j \in J} A_j \times \bigcup_{k \in K} B_k)$ is at least $|J| + |K| - |I|$, for each $J,K \subset I$. This extends a result of FORD and FULKERSON on the existence of common SDR's for two families of sets.

Theorem (3) implies results of KUNDU and LAWLER (1973) and MCDIARMID (1976). Clearly, it also extends axiom (iii) for a linking system to a matroid level.

Looking closer to the structure of linking systems the following turns out to be the case. Let (X,Y,Λ) be a linking system. For the moment we suppose that X and Y are disjoint. Define

$$\mathcal{B} = \{(X \backslash X') \cup Y' \mid (X',Y') \in \Lambda\}.$$

Then the set $\mathcal{B}$ always is the collection of bases of a matroid on $X \cup Y$. Since $(\emptyset,\emptyset) \in \Lambda$ we have that also X itself is a base of this matroid. Conversely, if $(X \cup Y, \mathcal{I})$ is a matroid, where X is a base of this matroid and Y is disjoint from X, then the collection

$$\Lambda = \{(X',Y') \mid X' \subset X, Y' \subset Y, (X \backslash X') \cup Y' \text{ is a base of the matroid}\}$$

satisfies the conditions (i), (ii) and (iii) for a linking system. Hence linking systems may be characterized as "matroids with a preferred base". So linking systems do not form any "new" structure: each theorem on linking systems can be transformed into a matroid theorem. The reason why we have persisted in formulating our theorems in terms of linking systems (and matroids) is that most of the results lose a lot of their expressiveness and relevance if formulated only in terms of matroids. A linking system reflects better the "bipartite" structure of the theorems than a matroid does. E.g. transforming the three above mentioned theorems on matroids and linking systems in pure matroid terms would yield rather odd results (but, as must be said, sometimes also trivial results). Nevertheless, our proofs often heavily leans on the matroid-like structure of linking systems; of course, if necessary we use known matroid theorems to obtain results on linking systems.

To get some insight in the intrinsical structure of linking systems define for each linking system (X,Y,Λ) its *underlying* bipartite graph (X,Y,E_Λ) by

$$(x,y) \in E_\Lambda \text{ if and only if } (\{x\},\{y\}) \in \Lambda.$$

The underlying bipartite graph tells something about the linking system itself, e.g.

(4) if $(X',Y') \in \Lambda$ then X' and Y' are matched in the bipartite graph (X,Y,E_Λ);

(5) if there is exactly one matching between X' and Y' in the bipartite graph, then $(X',Y') \in \Lambda$.

Clearly, these results have implications for matroids (with a preferred base) as well: the first result was, in terms of matroids, obtained earlier by BRUALDI (1969), and the second one, independently, by KROGDAHL (1975). These results turn out to be useful in the analysis of algorithms on matroids.

Also results have been obtained on forming "new" linking systems from "old" ones, such as:

(6) let (X,Y,Λ_1) and (Y,Z,Λ_2) be linking systems and let $\Lambda_1 * \Lambda_2$ be the collection of all pairs (X',Z') such that $(X',Y') \in \Lambda_1$ and $(Y',Z') \in \Lambda_2$ for some $Y' \subset Y$; then $(X,Z,\Lambda_1 * \Lambda_2)$ again is a linking system;

(7) let (X,Y,Λ_1) and (X,Y,Λ_2) be linking systems and let $\Lambda_1 \vee \Lambda_2$ consist of all pairs $(X' \cup X'', Y' \cup Y'')$ such that $(X',Y') \in \Lambda_1$, $(X'',Y'') \in \Lambda_2$ and $X' \cap X'' = \emptyset = Y' \cap Y''$; then $(X,Y,\Lambda_1 \vee \Lambda_2)$ again is a linking system.

In all cases where we form new matroids or linking systems from older ones we give an explicit formula for the rank or linking function of the new matroid or linking system, in terms of the older ones. This often yields new min-max relations in terms of old min-max relations. As an example consider theorem (2) above. Let $(X,\mathcal{I})$ be a transversal matroid, say let $\mathcal{I}$ be the collection of partial SDR's of the collection $\mathcal{A} = (A_i \mid i \in I)$ of subsets of X. By the well-known König-Hall theorem the rank function of this matroid is given by:

$$\mathrm{rank}_{\mathcal{I}}(X') = \min_{J\subset I} |I\backslash J| + |X' \cap \bigcup_{j\in J} A_j|.$$

Let $(Y,\mathcal{J})$ be a matroid obtained from a vector space by linear independence; that is let Y be embedded in a vector space and let $\mathcal{J}$ be the collection of linearly independent subsets of Y. Then the $\mathrm{rank}_{\mathcal{J}}(Y')$ of a subset Y' of Y equals the minimal cardinality of a subset Y" of Y' such that Y' is contained in the linear span of Y". Finally let (X,Y,Λ) be a linking system obtained from a directed graph (Z,E), with X and Y subsets of Z. MENGER's theorem implies that the linking function λ is determined by: $\lambda(X',Y')$ is equal to the minimal cardinality of a set of points intersecting each path from X' to Y' (for $X' \subset X$ and $Y' \subset Y$). Now applying theorem (2) on this whole gives us that the maximal cardinality of a partial SDR of $\mathcal{A}$ matched in the directed graph (Z,E) with a linearly independent set of vectors in Y equals the minimal value of

$$|I\backslash J| + |Z'| + |Y'|,$$

where $J \subset I$, $Y' \subset Y$ and Z' intersects each path from $\bigcup_{j\notin J} A_j$ to $Y\backslash[Y']$ (in this [Y'] stands for the linear span of Y'). This result itself, only meant as an illustration, also follows from the above mentioned theorem of BRUALDI.

We have also studied other discrete optimization aspects of linking systems. Linear programming problems are often solved by using techniques on certain convex polyhedra. To handle matroid problems in this way EDMONDS introduced polymatroids as polyhedra with certain "matroid-like" properties. Many matroid theorems have their analogue (in fact: extension) for polymatroids; we shall present a polyhedral analogue of a linking system and we shall show that also many properties of matroids and linking systems hold for their polyhedral extensions.

Finally, we give a rather general, but nevertheless *good* algorithm for solving certain problems involving matroids and linking systems. Let $(X_0,\mathcal{I})$ and $(X_k,\mathcal{J})$ be matroids and let $(X_0,X_1,\Lambda_1),\ldots,(X_{k-1},X_k,\Lambda_k)$ be linking systems. Suppose furthermore we have functions $w_0\colon X_0 \longrightarrow \mathbb{R},\ldots,$ $w_k\colon X_k \longrightarrow \mathbb{R}$. We give an algorithm for finding pairs $(X_0',X_1') \in \Lambda_1,\ldots,$

$(X'_{k-1}, X'_k) \in \Lambda_k$ such that $X'_0 \in \mathcal{I}$ and $X'_k \in \mathcal{J}$ and $w_0(X'_0) + \ldots + w_k(X'_k)$ is as large as possible (where $w_i(X'_i) = \sum_{x \in X'_i} w_i(x)$). This algorithm applied to more restricted cases yields algorithms handling the intersection of two and the union of arbitrarily many matroids, and the induction of matroids by linking systems.

c. FURTHER REMARKS

This tract is divided into eight chapters, numbered from 0 to 7. Chapter 0 gives some general background on graphs and matroids. The chapters 1 up to and including 4 develop the basic theory on linking systems and their relations to matroids. Chapter 1 gives definitions and examples, whereas chapter 2 exhibits the induction of matroids by linking systems. In chapter 3 we study the intrinsical structure of linking systems by means of their underlying bipartite graphs, and in chapter 4 we discuss some operations defined on linking systems forming other ones.

In chapter 5 we have a closer look at some special examples of linking systems. The chapters 6 and 7 deal with some optimization aspects of linking systems. Chapter 6 extends the previous theory to polymatroids and so-called poly-linking systems, and in chapter 7 we display an algorithm for solving problems involving matroids and linking systems.

To obtain a better survey the proofs are printed in a somewhat (15%) smaller type; also remarks which are meant as "asides" and which are not essential for the main discourse are printed in the smaller type. We have tried never to refer in the larger type text to remarks in the smaller type (except, sometimes, between brackets). (However, we do not intend to imply with this that proofs are to be considered as not essential.) We have adopted the handy way of referencing of the book "Matroid theory" of WELSH. E.g. EDMONDS [67b] refers to a paper of EDMONDS from 1967, where the letter is added to avoid confusion. Actually WELSH' book proved to be of great use for the present work. Especially the many recent references he gave have been of great help.

The results partially have appeared or will appear also as report or in journals or proceedings of conferences. We close this introduction with a listing of these papers.

Linking systems, *Report* ZW 29/74, Mathematical Centre, Amsterdam, 1974.

Linking systems II, *Report* ZW 51/75, Mathematical Centre, Amsterdam, 1975.

Linking systems, matroids and bipartite graphs, *Proceedings Fifth British Combinatorial Conference 1975* (C.St.J.A. Nash-Williams & J. Sheehan, eds), p. 541-544, Congressus Numerantium XV, Utilitas, Winnipeg, 1976.

The linking of matroids by linking systems, *Combinatorics* (Proceedings V. Hungarian Colloquium on Combinatorics, Keszthely, 1976), Coll. Math. Soc. János Bolyai 18, North-Holland, Amsterdam, 1977 (to appear).

On the structure of deltoids (Sur la structure des deltoïdes), *Proceedings Colloque International C.N.R.S. Paris-Orsay 1976* (to appear).

Matroids and linking systems, *Journal of Combinatorial Theory (B)* (to appear).

CHAPTER ZERO

PRELIMINARIES

In this treatise we shall assume familiarity with the usual mathematical symbols, terms and techniques. To mention just one of these: GF(q) denotes the field with q elements. Also elementary definitions and results from graph and matroid theory are supposed to be known; nevertheless, we give a review of those parts of graph and matroid theory which are essential for the next chapters, also because there does not exist complete standardization of their terminology.

0a. GRAPHS

We shall frequently make use of bipartite and directed graphs (among the books on graph theory are BONDY & MURTY [76], HARARY [69], WILSON [72]).

A *bipartite graph* is defined as a triple (X,Y,E), where X and Y are disjoint finite sets and $E \subset X \times Y$, that is, E consists of ordered pairs (x,y), with $x \in X$ and $y \in Y$. The elements in X and Y are called the *points* or *vertices*, and the elements of E are named *edges* or *arrows* of the bipartite graph. Sometimes a bipartite graph is represented by points in the plane (representing the vertices of the graph) and lines or arrows between points which together form a pair in E (an arrow (x,y) has as its *tail* x and as its *head* y). An example of a bipartite graph is $(X,Y,X\times Y)$, the *complete* bipartite graph.

A *matching* in a bipartite graph is a collection of arrows $\{(x_1,y_1),\ldots,(x_n,y_n)\}$ such that $x_1,\ldots,x_n,y_1,\ldots,y_n$ are all distinct; the matching then is called *between* (or *from*) $\{x_1,\ldots,x_n\}$ *and* (*to*) $\{y_1,\ldots,y_n\}$. If such a matching exists, $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$ are called *matched in* E (*with* each other).

A theorem of KÖNIG [31] says that, given a bipartite graph (X,Y,E) and subsets X' of X and Y' of Y, the maximal cardinality of a matching between a subset of X' and a subset of Y' equals the minimal value of $|X''| + |Y''|$, where $X'' \subset X'$ and $Y'' \subset Y'$, and each edge in $X' \times Y'$ has its tail in X'' or its head in Y''. It is easy to see that this is equal to

$$\min_{X''\subset X'} |X'\backslash X''| + |E(X'') \cap Y'|,$$

and also to

$$\min_{Y''\subset Y'} |Y'\backslash Y''| + |E^{-1}(Y'') \cap X'|,$$

where $E(X'') = \{y \in Y \mid (x,y) \in E \text{ for some } x \in X''\}$ and $E^{-1}(Y'') = \{x \in X \mid (x,y) \in E \text{ for some } y \in Y''\}$.

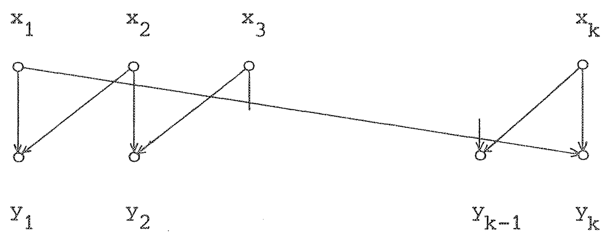
This theorem clearly also yields necessary and sufficient conditions for a pair of subsets to be matched in E.

A theorem of M. HALL [48] implies that if X' and Y' are matched in E and there is exactly one matching between X' and Y' then for some $x \in X'$ there is exactly one $y \in Y'$ such that $(x,y) \in E$ (except when $X' = Y' = \emptyset$).

A *circuit* in a bipartite graph is a set of edges of the form

$$\{(x_1,y_1),(x_2,y_1),(x_2,y_2),(x_3,y_2),\ldots,(x_k,y_{k-1}),(x_k,y_k),(x_1,y_k)\}$$

schematically represented by



A bipartite graph is *connected* if it is not the "disjoint sum" of other bipartite graphs; the (*connected*) *components* of a bipartite graph are the maximal connected sub-bipartite graphs of the bipartite graph.

Closely related to matchings in bipartite graphs are transversals of families of sets (cf. MIRSKY [71]). A *transversal* or *system of distinct representatives* (*SDR*) of a family $\mathcal{A} = (A_i \mid i = 1,\ldots,n)$ of subsets of a set Y, is a set of the form $\{y_1,\ldots,y_n\}$ where $y_1,\ldots,y_n$ are distinct and $y_1 \in A_1,\ldots,y_n \in A_n$. Now in the bipartite graph $(\mathcal{A},Y,E)$, where

$$E = \{(A_i,y) \mid y \in A_i\},$$

we find that $\mathcal{A}$ and a subset Y' of Y are matched iff Y' is a transversal of $\mathcal{A}$.

A *directed graph* or *digraph* is a pair (Z,E) in which Z is a finite set, the set of *points* or *vertices*, and E is a subset of Z × Z, the elements of which are called *edges* or *arrows*. Again, for an edge (x,y), x is called its *tail* and y its *head*. A *(directed) path (from* x_0 *to* x_k*)* in a digraph (Z,E) is an ordered set of vertices $(x_0,\ldots,x_k)$ such that $(x_0,x_1),\ldots,(x_{k-1},x_k)$ are arrows of the digraph (k≥0) and $x_0,\ldots,x_k$ are all distinct, except, possibly, $x_0 = x_k$. In case $x_0 = x_k$ the path is called a *(directed) cycle* (provided k>0). x_0 is the *starting point* and x_k the *end point* of the path.

Two paths are called *(vertex-)disjoint* if they have no point in common. A collection of pairwise vertex-disjoint paths is called a *matching*, being *between* (or: *from*) the set of starting points *and* (*to*) the set of end points of paths in the collection. Again, X' and Y' are called *matched* (*with* each other) (in E) if there is a matching from X' to Y'.

Digraphs form an extension of bipartite graphs; an extension of KŐNIG's theorem is the theorem of MENGER [27]. Let (Z,E) be a digraph and let X,Y ⊂ Z. Then the maximal cardinality of a matching between a subset of X and a subset of Y equals the minimal cardinality of a set of vertices intersecting each path from X to Y. This last is equal to

$$\min_{Z'\subset Z\setminus Y} |(X\cup E(Z'))\setminus Z'|,$$

and to

$$\min_{Z'\subset Z\setminus X} |(Y\cup E^{-1}(Z'))\setminus Z'|,$$

where $E(Z') = \{z \in Z \mid (z',z) \in E \text{ for some } z' \in Z'\}$ and $E^{-1}(Z') = \{z \in Z \mid (z,z') \in E \text{ for some } z' \in Z'\}$.

Finally we give a nice relation between matchings in directed graphs and those in bipartite graphs, found by INGLETON & PIFF [73]. Let (Z,E) be a digraph. Define the bipartite graph $(\bar{Z},\underline{Z},D)$ where $\bar{Z}$ and $\underline{Z}$ are two disjoint copies of Z, by

$$(\bar{x},\underline{y}) \in D \text{ if and only if } x = y \text{ or } (x,y) \in E.$$

Then subsets X and Y of Z are matched in E if and only if $\bar{Z}\backslash\bar{Y}$ and $\underline{Z}\backslash\underline{X}$ are matched in D.

Ob. MATROIDS

A good introduction to and account of the present state of affairs in matroid theory forms the book of WELSH [76].

A *matroid* is a pair $(X,\mathcal{I})$, where X is a finite set and $\mathcal{I}$ is a nonempty collection of subsets of X satisfying the conditions

(i) if $A \subset B \in \mathcal{I}$ then $A \in \mathcal{I}$;
(ii) if $A,B \in \mathcal{I}$ and $|A| < |B|$ then $A \cup \{b\} \in \mathcal{I}$ for some $b \in B\backslash A$.

The elements of $\mathcal{I}$ are the *independent* sets of the matroid.

A matroid determines a *rank* function $\rho\colon \mathcal{P}(X) \to \mathbb{Z}$, where $\rho(X')$ equals the maximal cardinality of an independent subset of X'. One can prove that any function $\rho\colon \mathcal{P}(X) \to \mathbb{Z}$ is the rank function of some matroid if and only if

(i) $0 \le \rho(X') \le |X'|$;
(ii) if $X'' \subset X' \subset X$, then $\rho(X'') \le \rho(X')$;
(iii) if $X', X'' \subset X$, then $\rho(X'\cap X'') + \rho(X'\cup X'') \le \rho(X') + \rho(X'')$ (i.e. ρ is *submodular*).

The *rank of the matroid* $(X,\mathcal{I})$ is, by definition, $\rho(X)$.

A *base* of a matroid is an independent subset not contained in another one. A nonempty collection $\mathcal{B}$ of subsets of a set X is the collection of bases of a matroid if and only if

(i) no set in $\mathcal{B}$ is contained in another set in $\mathcal{B}$;
(ii) if $B_1,B_2 \in \mathcal{B}$ and $x \in B_1$ then $(B_1\backslash\{x\}) \cup \{y\} \in \mathcal{B}$ for some $y \in B_2$.

It follows that all bases have the same cardinality. Clearly, each matroid is determined by its rank function, and also by its collection of bases.

The *dual* matroid M^* of a matroid $M = (X,\mathcal{I})$ is the matroid on X which one obtains by taking as bases for M^* all sets $X\backslash B$, where B is a base of M. The rank function ρ^* of M^* satisfies

$$\rho^*(X') = |X'| + \rho(X\backslash X') - \rho(X)$$

for $X' \subset X$ (ρ being the rank function of M).

Given a matroid $(X,\mathcal{I})$ (with rank function ρ) and a subset X' of X, the *restriction* $M \mid X'$ of M to X' is the matroid $(X', \mathcal{I} \cap \mathcal{P}(X'))$. The rank function of $M \mid X'$ clearly equals $\rho \mid \mathcal{P}(X')$. The *contraction* $M \cdot X'$ equals the matroid $(M^* \mid X')^*$; it follows from the above that the rank function $\rho \cdot X'$ is given by

$$(\rho \cdot X')(X'') = \rho((X \backslash X') \cup X'') - \rho(X \backslash X')$$

for $X'' \subset X'$. Also it is easy to see that, given a maximal independent subset X''' of $X \backslash X'$, a subset X'' of X' is independent in $M \cdot X'$ if and only if $X'' \cup X''' \in \mathcal{I}$. A *minor* of a matroid is a contraction of a restriction of the matroid (contraction and restriction commute).

The *circuits* and *cocircuits* of a matroid $M = (X,\mathcal{I})$ are the minimal non-independent sets of M and M^*, respectively. It can be proved that, given a base B of a matroid $(X,\mathcal{I})$ and $x \in X \backslash B$, there is a unique circuit contained in $B \cup \{x\}$. A *(co)loop* is an element of X which is, as a singleton, a (co)circuit.

The *closure* operator $c\ell_{\mathcal{I}}$ of a matroid $M = (X,\mathcal{I})$ (with rank function ρ) is a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$, given by

$$c\ell_{\mathcal{I}}(X') = \{x \in X \mid \rho(X' \cup \{x\}) = \rho(X')\}$$

for subsets X' of X. Then $\rho(c\ell_{\mathcal{I}}(X')) = \rho(X')$ and $c\ell_{\mathcal{I}}(c\ell_{\mathcal{I}}(X')) = c\ell_{\mathcal{I}}(X')$. Furthermore, if $X' \in \mathcal{I}$ and $x \notin X'$, then $x \in c\ell_{\mathcal{I}}(X') \Longleftrightarrow X' \cup \{x\} \notin \mathcal{I}$.

A matroid $(X,\mathcal{I})$ is *connected* if for each two different elements in X there exists a circuit containing both of them. If M is connected, M^* also is connected. The *(connected) components* are the maximal connected minors of a matroid.

The s-*truncation* of a matroid $(X,\mathcal{I})$ is the matroid $(X,\mathcal{I}')$ where $\mathcal{I}'$ consists of all sets in $\mathcal{I}$ with cardinality at most s.

So far the theory has been developed mainly by WHITNEY [35].

Examples and *classes* of matroids $M = (X,\mathcal{I})$ now follow.

(1) *Uniform matroids* $U_{k,n}$ (X is a set with n elements and each subset with at most k elements is in $\mathcal{I}$).

(2) *Graphic matroids* (X being the set of edges of an undirected graph and

I consists of all collections of edges which do not contain a circuit; M also is called the *cycle matroid* of the graph (WHITNEY [35])).

(3) *Cographic matroids* (X being the set of edges of an undirected graph and I consists of all sets of edges which do not contain a cutset; in this case one calls M the *cocycle matroid* of the graph (WHITNEY [35])).

(4) *Matroids (linearly) representable over some field* F (X is the collection of columns of a matrix over F and I consists of all linearly independent sets of columns; the matrix is called a *(matrix-)representation of the* matroid (WHITNEY [35], VAN DER WAERDEN [37])).

(5) *Binary matroids* (matroids representable over GF(2)).

(6) *Regular matroids* (matroids representable over each field).

(7) *Gammoids* (X being a subset of the set of vertices of a directed graph (Z,E), with special set $Z_0 \subset Z$, whereas I consists of all subsets X' of X such that there is a matching from some subset of Z_0 to X' (PERFECT [68])).

(8) *Transversal matroids* (in this case there is a bipartite graph (X,Y,E) and I consists of all subsets X' of X which are matched in E with an $Y' \subset Y$ (EDMONDS & FULKERSON [65])).

(9) *Strict grammoids* (as gammoids, but with X = Z).

(10) *(Strict) deltoids* ($X = X_1 \cup X_2$ for some bipartite graph (X_1,X_2,E) and I consists of all subsets $(X_1\backslash X_1') \cup X_2'$ such that some subset of X_1' is matched with X_2'; the base X_1 is called a *principal base*).

(11) *Algebraic matroids* (X being a subset of a field extension of some field F and I being the collection of all over F algebraically independent subsets of X (VAN DER WAERDEN [37])).

(12) *The Fano matroid* (X is the set of points of the projective plane of order 2 and the collection of bases consists of all sets with 3 points, except the seven lines of the plane).

(13) *The Vamos matroid* ($X = \{a_1,b_1,a_2,b_2,a_3,b_3,a_4,b_4\}$ and bases are all sets with 4 elements except $\{a_1,b_1,a_2,b_2\}$, $\{a_1,b_1,a_3,b_3\}$, $\{a_1,b_1,a_4,b_4\}$, $\{a_2,b_2,a_3,b_3\}$, $\{a_2,b_2,a_4,b_4\}$).

There are many interrelations between the several classes and examples. The cocycle matroids are exactly the duals of the cycle matroids. The classes of uniform matroids, over F linearly representable matroids, binary matroids, regular matroids, gammoids, respectively, are closed under taking duals and minors. The class of strict deltoids is closed under taking duals. The classes of graphic matroids and cographic matroids are closed under taking minors. Each graphic or cographic matroid is regular and hence binary.

A matroid M is binary iff M does not contain $U_{2,4}$ as a minor (TUTTE [65]). A matroid M is regular iff M is binary and does not contain as a minor the Fano matroid or its dual (TUTTE [58,59]). The Fano matroid is representable only over fields of characteristic 2.

Each matroid linearly representable over F is algebraic over F (PIFF [69]). The Vamos matroid is non-algebraic (INGLETON & MAIN [75]). Strict deltoids, transversal matroids, strict gammoids and gammoids are representable over almost every field (PIFF & WELSH [70], MASON [72]). The class of transversal matroids is closed under taking restrictions, while the class of strict gammoids is exactly the class of duals of transversal matroids (INGLETON & PIFF [73]). Hence the class of strict gammoids is closed under taking contractions. Since the classes of gammoids and transversal matroids, respectively are exactly the classes of restrictions of strict gammoids and strict deltoids, respectively, it follows that the class of gammoids is the class of minors of strict deltoids. Also it follows, that the classes of gammoids and strict gammoids, respectively, are equal to the classes of contractions of transversal matroids, and of strict deltoids, respectively.

The following two matroid theorems we shall frequently use.

The *(matroid) union theorem* (EDMONDS & FULKERSON [65], EDMONDS [65a], NASH-WILLIAMS [66]) says the following. Let be given matroids $M_1 = (X_1,\mathcal{I}_1),\ldots,M_k = (X_k,\mathcal{I}_k)$ on not necessarily disjoint sets $X_1,\ldots,X_k$, and set

$$X = X_1 \cup\ldots\cup X_k,$$
$$\mathcal{I} = \mathcal{I}_1 \vee\ldots\vee \mathcal{I}_k = \{X_1' \cup\ldots\cup X_k' \mid X_1' \in \mathcal{I}_1,\ldots,X_k' \in \mathcal{I}_k\}.$$

Then $M_1 \vee\ldots\vee M_k = (X,\mathcal{I})$ again is a matroid, the *union* of the matroids $M_1,\ldots,M_k$. The rank of this union is given by

$$(\rho_1\vee\ldots\vee\rho_k)(X') = \min_{X''\subset X'} \left(\sum_{i=1}^{k} \rho_i(X''\cap X_i) + |X'\setminus X''|\right),$$

$(X'\subset X)$, where ρ_i is the rank function of M_i $(i = 1,\ldots,k)$.

(Edmonds') intersection theorem (EDMONDS [70]) asserts that given two matroids $(X,\mathcal{I}_1)$ and $(X,\mathcal{I}_2)$, with rank functions ρ_1 and ρ_2, respectively, the maximum cardinality of a common independent set (i.e. a set in $\mathcal{I}_1\cap\mathcal{I}_2$) equals

$$\min_{X'\subset X} (\rho_1(X')+\rho_2(X\setminus X')).$$

It is possible to deduce simply these two theorems from each other.

CHAPTER ONE

LINKING SYSTEMS

A linking system is a system which "links" sets in a way to be made more precise in section 1a. This kind of linking appears in bipartite and directed graphs and in matrices (a matrix links the set of rows with the set of columns); this is shown in section 1b. The third section of this chapter gives the connection of linking systems with a somewhat weaker structure, the tabloid (introduced by HOCQUENGHEM).

1a. DEFINITIONS

In this section we define the concept of a linking system by means of axioms for the "collection of linked pairs". As will be seen in theorem 1.2 an alternative definition in terms of a "linking function" is possible (comparable with the alternative definition of a matroid in terms of its rank function).

DEFINITION 1.1. A *linking system* is a triple (X,Y,Λ), where X and Y are finite sets and $\emptyset \neq \Lambda \subset \mathcal{P}(X) \times \mathcal{P}(Y)$, such that

(i) if $(X',Y') \in \Lambda$ and $x \in X'$ then $(X'\setminus\{x\},Y'\setminus\{y\}) \in \Lambda$ for some $y \in Y'$;

(ii) if $(X',Y') \in \Lambda$ and $y \in Y'$ then $(X'\setminus\{x\},Y'\setminus\{y\}) \in \Lambda$ for some $x \in X'$;

(iii) if $(X_1,Y_1) \in \Lambda$ and $(X_2,Y_2) \in \Lambda$ then there exists an $(X',Y') \in \Lambda$ such that $X_1 \subset X' \subset X_1 \cup X_2$ and $Y_2 \subset Y' \subset Y_1 \cup Y_2$.

The elements of Λ are called *linked pairs*. A *maximal linked pair* in (X',Y'), for $X' \subset X$ and $Y' \subset Y$, is a linked pair (X_1,Y_1) such that $X_1 \subset X'$, $Y_1 \subset Y'$ and, in addition, if $X_1 \subset X_1' \subset X'$, $Y_1 \subset Y_1' \subset Y'$ and $(X_1',Y_1') \in \Lambda$ then $X_1 = X_1'$ and $Y_1 = Y_1'$.

From these axioms it follows easily that

(iv) if $(X',Y') \in \Lambda$ then $|X'| = |Y'|$;

(v) if $(X',Y') \in \Lambda$ and $X'' \subset X'$ then $(X'',Y'') \in \Lambda$ for some $Y'' \subset Y'$;

(vi) if $(X',Y') \in \Lambda$ and $Y'' \subset Y'$ then $(X'',Y'') \in \Lambda$ for some $X'' \subset X'$;

(vii) if $X' \subset X$, $Y' \subset Y$ and (X_1,Y_1) and (X_2,Y_2) are maximal linked pairs in (X',Y') then $(X_1,Y_2) \in \Lambda$ (in particular $|X_1| = |Y_2|$);

(viii) $(\emptyset,\emptyset) \in \Lambda$.

In fact the properties (iv, (v), (vi) and (vii) together are sufficient to characterize a linking system.

Each linking system (X,Y,Λ) determines a *linking function* $\lambda: \mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathbb{Z}$, defined by

$$\lambda(X',Y') = \max\{|X''| \mid (X'',Y'') \in \Lambda \text{ for some } X'' \subset X' \text{ and } Y'' \subset Y'\}$$

for $X' \subset X$ and $Y' \subset Y$. Conversely a linking system is determined by its linking function, since, clearly,

$$(X',Y') \in \Lambda \quad \text{if and only if} \quad \lambda(X',Y') = |X'| = |Y'|.$$

A characterization of linking functions is given by the following theorem, which provides actually an alternative definition of linking systems.

<u>THEOREM 1.2</u>. *Let* X *and* Y *be finite sets and let* $\lambda: \mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathbb{Z}$. *Then* λ *is the linking function of a linking system* (X,Y,Λ) *if and only if*

(ix) $0 \le \lambda(X',Y') \le \min\{|X'|,|Y'|\}$ $(X'\subset X, Y'\subset Y)$;

(x) *if* $X'' \subset X'$ *and* $Y'' \subset Y'$ *then* $\lambda(X'',Y'') \le \lambda(X',Y')$ $(X'\subset X, Y'\subset Y)$;

(xi) $\lambda(X'\cap X'',Y'\cup Y'') + \lambda(X'\cup X'',Y'\cap Y'') \le \lambda(X',Y') + \lambda(X'',Y'')$ $(X',X''\subset X;\ Y',Y''\subset Y)$.

<u>PROOF</u>. (1) Let (X,Y,Λ) be a linking system, with linking function λ. Then λ satisfies clearly the properties (ix) and (x).
To prove property (xi) let (X_1,Y_1) be a maximal linked pair in $(X'\cap X'', Y'\cup Y'')$ and let (X_2,Y_2) be a maximal linked pair in $(X'\cup X'', Y'\cap Y'')$, such that

$$|X_1| = |Y_1| = \lambda(X'\cap X'', Y'\cup Y'')$$

and

$$|X_2| = |Y_2| = \lambda(X'\cup X'', Y'\cap Y'').$$

Now, by axiom (iii) of definition 1.1, there exists an $(X_3,Y_3)\in\Lambda$ with

$$X_1 \subset X_3 \subset X_1 \cup X_2 \subset X' \cup X''$$

and

$$Y_2 \subset Y_3 \subset Y_1 \cup Y_2 \subset Y' \cup Y''.$$

Using property (v) there is a $Y_4 \subset Y_3$ with the property

$$(X_3\cap X', Y_4) \in \Lambda.$$

Property (vi) ensures the existence of an $X_4 \subset X_3 \cap X'$ satisfying

$$(X_4, Y_4\cap Y') \in \Lambda.$$

Since $X_4 \subset X'$ and $Y_4 \cap Y' \subset Y'$ it is true that

$$|X_4| = |Y_4\cap Y'| \le \lambda(X',Y').$$

Now we have

$$\lambda(X',Y') \ge |Y_4\cap Y'| = |Y_4| - |Y_4\backslash Y'| = |X_3\cap X'| - |Y_4\backslash Y'| \ge |X_3\cap X'| - |Y_3\backslash Y'|.$$

(Note that $|Y_4| = |X_3\cap X'|$ since $(X_3\cap X',Y_4) \in \Lambda$, and that $Y_4\subset Y_3$.)

The same method applied to X'' and Y'' instead of X' and Y' results in

$$\lambda(X'',Y'') \ge |X_3\cap X''| - |Y_3\backslash Y''|.$$

Hence

$$\lambda(X',Y') + \lambda(X'',Y'') \ge |X_3\cap X'| - |Y_3\backslash Y'| + |X_3\cap X''| - |Y_3\backslash Y''| =$$

$$|X_3| + |X_3\cap X'\cap X''| - |Y_3| + |Y_3\cap Y'\cap Y''| = |X_3\cap X'\cap X''| + |Y_3\cap Y'\cap Y''| \ge$$

$$|X_1| + |Y_2| = \lambda(X'\cap X'', Y'\cup Y'') + \lambda(X'\cup X'', Y'\cap Y'').$$

(Note that $X_3 \subset X' \cup X''$; $Y_3 \subset Y' \cup Y''$; $|X_3| = |Y_3|$; $X_1 \subset X_3 \cap X' \cap X''$; $Y_2 \subset Y_3 \cap Y' \cap Y''$.)

(2). Let $\lambda: P(X) \times P(Y) \to \mathbb{Z}$ satisfy the properties (ix), (x) and (xi); let furthermore $\Lambda = \{(X',Y') \mid \lambda(X',Y') = |X'| = |Y'|\}$.
We first prove that if $X' \subset X$ and $Y' \subset Y$, then there are $X'' \subset X'$ and $Y'' \subset Y'$ such that $(X'', Y'') \in \Lambda$ and $\lambda(X',Y') = |X''| = |Y''|$.
It is immediately clear, that for fixed $X' \subset X$ the function $\sigma: P(Y) \to \mathbb{Z}$ with $\sigma(Y') = \lambda(X',Y')$ is the rank function of a matroid. Therefore, for $X' \subset X$ and $Y' \subset Y$, there is a $Y'' \subset Y$ such that

$$\lambda(X',Y') = |Y''| = \lambda(X',Y'').$$

By a similar argument one proves the existence of an $X'' \subset X'$ such that

$$\lambda(X',Y'') = |X''| = \lambda(X'', Y'').$$

In this manner one finds the required $(X'', Y'') \in \Lambda$.

Since the axioms (i) and (ii) are symmetric we need only prove axioms (i) and (iii).
To prove axiom (i) let $(X',Y') \in \Lambda$ and $x \in X'$. By properties (ix) and (xi) it follows that

$$\lambda(X'\backslash\{x\},Y') \geq \lambda(X',Y') + \lambda(\emptyset,Y') - \lambda(\{x\},Y') \geq \lambda(X',Y') + 0 - 1 = |X'\backslash\{x\}|.$$

Since also $\lambda(X'\backslash\{x\},Y') \leq |X'\backslash\{x\}|$, we have

$$\lambda(X'\backslash\{x\},Y') = |X'\backslash\{x\}|.$$

Therefore there exists a $Y'' \subset Y'$ such that $(X'\backslash\{x\},Y'') \in \Lambda$ and $\lambda(X'\backslash\{x\},Y') = |X'\backslash\{x\}| = |Y''|$. This implies that $Y'' = Y'\backslash\{y\}$ for some $y \in Y'$.

Finally we prove axiom (iii). Let $(X_1,Y_1) \in \Lambda$ and $(X_2,Y_2) \in \Lambda$; without loss of generality we may suppose that both (X_1,Y_1) and (X_2,Y_2) are maximal linked pairs in $(X_1 \cup X_2, Y_1 \cup Y_2)$.
We first prove that

$$|X_1| = |X_2| = |Y_1| = |Y_2| = \lambda(X_1 \cup X_2, Y_1 \cup Y_2).$$

From the maximality of (X_1,Y_1) we conclude that for each $x \in X_2$ and $y \in Y_2$

$$\lambda(X_1 \cup \{x\}, Y_1 \cup \{y\}) = |X_1| = |Y_1|.$$

By induction on $|X'|$ and using property (xi) we find, for each $X' \subset X_2$ and $y \in Y_2$

$$\lambda(X_1 \cup X', Y_1 \cup \{y\}) = |X_1| = |Y_1|,$$

and hence for each $y \in Y_2$

$$\lambda(X_1 \cup X_2, Y_1 \cup \{y\}) = |X_1| = |Y_1|.$$

Proceeding in the same manner one arrives at

$$\lambda(X_1 \cup X_2, Y_1 \cup Y_2) = |X_1| = |Y_1|.$$

Using similar arguments for X_2 and Y_2 instead of X_1 and Y_1, it follows also that

$$\lambda(X_1 \cup X_2, Y_1 \cup Y_2) = |X_2| = |Y_2|.$$

Hence indeed

$$\lambda(X_1 \cup X_2, Y_1 \cup Y_2) = |X_1| = |X_2| = |Y_1| = |Y_2|.$$

Secondly we have that $\lambda(X_1, Y_2) = |X_1| = |Y_2|$, since

$$\lambda(X_1, Y_2) \geq \lambda(X_1, Y_1 \cup Y_2) + \lambda(X_1 \cup X_2, Y_2) - \lambda(X_1 \cup X_2, Y_1 \cup Y_2)$$
$$= |X_1| + |Y_2| - |X_1|. \qquad \Box$$

Let (X,Y,Λ) be a linking system with linking function λ. Its *dual linking system* is the system (Y,X,Λ^*), with

$$\Lambda^* = \{(Y',X') \mid (X',Y') \in \Lambda\},$$

which clearly is again a linking system. The dual linking function λ^* then is given by

$$\lambda^*(Y',X') = \lambda(X',Y'),$$

for $Y' \subset Y$ and $X' \subset X$.

If $X_0 \subset X$ and $Y_0 \subset Y$ let

$$\Lambda_0 = \{(X',Y') \in \Lambda \mid X' \subset X_0 \text{ and } Y' \subset Y_0\}.$$

Then (X_0,Y_0,Λ_0) forms a *sub linking system* of (X,Y,Λ) (of course, this is again a linking system); its linking function is, clearly, $\lambda \mid \mathcal{P}(X_0) \times \mathcal{P}(Y_0)$.

1b. EXAMPLES

We now give three examples of linking systems.

(1) *Deltoid linking systems*

Let (X,Y,E) be a bipartite graph (i.e. X and Y are finite sets and $E \subset X \times Y$). X' and Y' are called *matched* in E if there is a *matching* in E between X' and Y', that is a bijection σ: $X' \longrightarrow Y'$ such that $(x,\sigma(x)) \in E$ for all $x \in X'$. We define Δ_E by

$$\Delta_E = \{(X',Y') \mid X' \text{ and } Y' \text{ are matched in } E\}.$$

Then (X,Y,Δ_E) is a linking system. Clearly, Δ_E satisfies the axioms (i) and (ii) of definition 1.1; a theorem of MENDELSOHN & DULMAGE [58] implies axiom (iii) (the *Mendelsohn-Dulmage property*).

The history of this property goes back to the well-known Schröder-Bernstein theorem proved in 1898 by F. BERNSTEIN (cf. FRAENKEL [23] p. 58). The "Mapping theorem" of BANACH [24] is slightly stronger than the Schröder-Bernstein theorem: if θ: $X \longrightarrow Y$ and ψ: $Y \longrightarrow X$ are injective functions then $\theta[X_1] = Y\backslash Y_1$ and $\psi[Y_1] = X\backslash X_1$ for some $X_1 \subset X$ and $Y_1 \subset Y$. The Mendelsohn-Dulmage property was extended by ORE [62] to the infinite case and PERFECT & PYM [66] generalized both BANACH's and ORE's result to the theorem: if $X' \subset X$, $Y' \subset Y$ and θ: $X' \longrightarrow Y$ is an injective mapping and ψ: $Y' \longrightarrow X$ is an arbitrary mapping then $\theta[X_1] = Y_0\backslash Y_1$ and $\psi[Y_1] = X_0\backslash X_1$, for certain sets X_1,X_0,Y_1,Y_0 such that $X_1 \subset X' \subset X_0 \subset X$ and $Y_1 \subset Y' \subset Y_0 \subset Y$. MIRSKY & PERFECT [67] observed that the theorem also holds when θ is not injective. (See also MIRSKY [71] p. 9-13.) A short proof of the Mendelsohn-Dulmage property (which also works for the other theorems just mentioned), in essence due to BIRKHOFF & MACLANE [48] p. 340, is as follows: suppose $(X_1,Y_1) \in \Delta_E$ and $(X_2,Y_2) \in \Delta_E$ and colour red a matching between X_1 and Y_1 and blue a matching between X_2 and Y_2; now consider the components of the subgraph of all red and blue edges; take the red edges from the components with an endpoint in X_1 and the blue edges from the other components; in this manner we get a matching between sets $X' \subset X_1 \cup X_2$ and $Y' \subset Y_1 \cup Y_2$ such that $X_1 \subset X'$ and $Y_2 \subset Y'$, as required. (For a linear algebraic proof see PERFECT [66].)

Denote the linking function of (X,Y,Δ_E) by δ_E. The well-known theorem of KÖNIG [31] (cf. BONDY & MURTY [76]) states that $\delta_E(X',Y')$ equals the minimal cardinality of a subset of $X \cup Y$ meeting each edge between X' and Y', or, in formula,

$$\delta_E(X',Y') = \min_{X''\subset X'} (|X'\backslash X''| + |E(X'')\cap Y'|)$$

$$= \min_{Y''\subset Y'} (|Y'\backslash Y''| + |E^{-1}(Y'')\cap X'|).$$

(In this formula, $E(X'') = \{y \in Y \mid (x,y) \in E \text{ for some } x \in X''\}$ and $E^{-1}(Y'') = \{x \in X \mid (x,y) \in E \text{ for some } y \in Y''\}$.)

In fact it is easy to prove directly that δ_E satisfies the properties (ix), (x) and (xi) of theorem 1.2. Properties (ix) and (x) are straightforward; in order to prove property (xi), let $X_1,X_2 \subset X$ and $Y_1,Y_2 \subset Y$. Then

$$\delta_E(X_1,Y_1) + \delta_E(X_2,Y_2) = \min_{X'\subset X_1} \{|X_1\backslash X'| + |E(X')\cap Y_1|\} +$$

$$\min_{X''\subset X_2} \{|X_2\backslash X''| + |E(X'')\cap Y_2|\} =$$

$$\min_{\substack{X'\subset X_1\\ X''\subset X_2}} \{|X_1\backslash X'| + |X_2\backslash X''| + |E(X')\cap Y_1| + |E(X'')\cap Y_2|\} \geq$$

$$\min_{\substack{X'\subset X_1\\ X''\subset X_2}} \{|(X_1\cap X_2)\backslash(X'\cap X'')| + |(X_1\cup X_2)\backslash(X'\cup X'')| + |E(X'\cap X'')\cap (Y_1\cup Y_2)| + |E(X'\cup X'')\cap (Y_1\cap Y_2)|\} \geq$$

$$\min_{\substack{A\subset X_1\cap X_2\\ B\subset X_1\cup X_2}} \{|(X_1\cap X_2)\backslash A| + |E(A)\cap(Y_1\cup Y_2)| + |(X_1\cup X_2)\backslash B| + |E(B)\cap(Y_1\cap Y_2)|\} =$$

$$\delta_E(X_1\cap X_2,Y_1\cup Y_2) + \delta_E(X_1\cup X_2,Y_1\cap Y_2).$$

Hence this, together with KÖNIG's theorem and theorem 1.2, provides us with a second proof of the fact that (X,Y,Δ_E) is a linking system.

Linking systems obtained in this way we shall call *deltoid linking systems* (following MIRSKY & PERFECT [67] in the term "deltoid").

(2) *Gammoid linking systems*

Let (Z,E) be a finite directed graph (so $E \subset Z \times Z$). In this case two subsets X' and Y' of Z are called *matched* in E if $|X'| = |Y'|$ and there are $|X'|$ pairwise vertex-disjoint paths from X' to Y'. A path may consist of only one vertex, so X' and Y' are not necessarily disjoint.
Let X and Y be two fixed subsets of Z. Let Γ_E be the collection

$$\Gamma_E = \{(X',Y') \mid X' \subset X,\ Y' \subset Y,\ X' \text{ and } Y' \text{ are matched in } E\}.$$

Then (X,Y,Γ_E) is a linking system. Again the axioms (i) and (ii) are easily verified; axiom (iii) is the "linkage theorem" of PERFECT [68].

Clearly, this property is a generalization of the Mendelsohn-Dulmage property on bipartite graphs (example (1)). It generalizes also a result of MIRSKY [68]: if $\mathcal{U}$ and $\mathcal{V}$ are finite collections of sets such that $\mathcal{U}_1 \subset \mathcal{U}$ and $\mathcal{V}_1 \subset \mathcal{V}$ have a common transversal, and, similarly, $\mathcal{U}_2 \subset \mathcal{U}$ and $\mathcal{V}_2 \subset \mathcal{V}$ have a common transversal, then some $\mathcal{U}_0 \subset \mathcal{U}$ and $\mathcal{V}_0 \subset \mathcal{V}$ have a common transversal, where $\mathcal{U}_1 \subset \mathcal{U}_0$ and $\mathcal{V}_2 \subset \mathcal{V}_0$. PERFECT's result itself is extended to the infinite case by PYM [69b] (cf. PYM [69a], BRUALDI & PYM [71] and MCDIARMID [75] for other proofs).

Let γ_E be the linking function of (X,Y,Γ_E); from MENGER's theorem [27] (cf. BONDY & MURTY [76]) we know, for $X' \subset X$ and $Y' \subset Y$, that $\gamma_E(X',Y')$ equals the minimal cardinality of a subset of Z meeting each path from a point in X' to a point in Y'; in formula

$$\gamma_E(X',Y') = \min_{Z' \subset Z \setminus Y'} |(X' \cup E(Z')) \setminus Z'|,$$

where $E(Z') = \{ z \in Z \mid (z',z) \in E \text{ for some } z' \in Z'\}$ for $Z' \subset Z$ (cf. MCDIARMID [72]).

Again it is easy to show directly that $\gamma_E(X',Y')$ has the properties (ix), (x), (xi) of theorem 1.2. (ix) and (x) are easily seen; to establish (xi), take $X_1,X_2 \subset X$ and $Y_1,Y_2 \subset Y$, and suppose $\gamma_E(X_1,Y_1) = |(X_1 \cup E(Z_1)) \setminus Z_1|$ and $\gamma_E(X_2,Y_2) = |(X_2 \cup E(Z_2)) \setminus Z_2|$, where $Z_1 \subset Z \setminus Y_1$ and $Z_2 \subset Z \setminus Y_2$.
Now consider the four sets

$$(X_1 \cup E(Z_1)) \setminus Z_1,$$
$$(X_2 \cup E(Z_2)) \setminus Z_2,$$
$$((X_1 \cap X_2) \cup E(Z_1 \cap Z_2)) \setminus (Z_1 \cap Z_2),$$
$$((X_1 \cup X_2) \cup E(Z_1 \cup Z_2)) \setminus (Z_1 \cup Z_2).$$

It is easy to see that the union of the last two sets is contained in the union of the first two sets, and, similarly, that the intersection of the last two sets is contained in the intersection of the first two sets. Hence, the sum of the cardinalities of the last two sets is not greater than the sum of the cardinalities of the first two sets; this last sum equals $\gamma_E(X_1,Y_1) + \gamma_E(X_2,Y_2)$. The first sum is at least $\gamma_E(X_1 \cap X_2, Y_1 \cup Y_2) + \gamma_E(X_1 \cup X_2, Y_1 \cap Y_2)$. Hence (xi) holds. So theorem 1.2 together with MENGER's theorem implies PERFECT's linkage theorem.

Linking systems constructed in this way we call *gammoid linking systems* (following PYM [69b] in the term "gammoid"); clearly, each deltoid linking system is a gammoid linking system.

(3) *Representable linking systems*

Let $M = (X,Y,\phi)$ be a matrix over some field F (i.e. ϕ is an F-valued function defined on $X \times Y$; X and Y are the collections of rows and columns, respectively). Let Λ_ϕ be the collection

$$\Lambda_\phi = \{(X',Y') \mid M|X' \times Y' \text{ is nonsingular}\},$$

where $M|X' \times Y'$ is the submatrix of M with rows X' and columns Y', that is, $M|X' \times Y' = (X',Y',\phi|X'{\times}Y')$.

Then (X,Y,Λ_ϕ) is a linking system; using simple arguments from linear algebra one proves that the axioms of definition 1.1 are valid (cf. the lemmata of PERFECT [66]).

Sketch of proof of axiom (iii):
Let $(X_1,Y_1) \in \Lambda_\phi$ and $(X_2,Y_2) \in \Lambda_\phi$. We may suppose that the set X_1 represents a maximal linearly independent collection of rows in $M|(X_1 \cup X_2) \times (Y_1 \cup Y_2)$ and, similarly, that Y_2 represents a maximal linearly independent collection of columns in $M|(X_1 \cup X_2) \times (Y_1 \cup Y_2)$. Hence each row in $M|(X_1 \cup X_2) \times Y_2$ is a linear combination of rows in $M|X_1 \times Y_2$. Since also $|X_1| = |Y_2|$ we find that $M|X_1 \times Y_2$ is nonsingular, i.e. $(X_1,Y_2) \in \Lambda_\phi$.

Writing λ_ϕ for the linking function of (X,Y,Λ_ϕ) we have that $\lambda_\phi(X',Y')$ equals the rank of $M|X' \times Y'$.

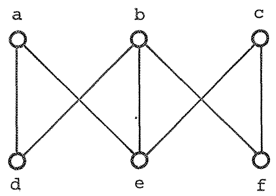
Linking systems obtained in this way will be called *(linearly) representable over* F. A *binary* linking system is a linking system linearly representable over GF(2).

All three examples are self-dual: the dual linking system of a deltoid linking system (or gammoid linking system, or linking system representable over a field F) is again a deltoid linking system (or gammoid linking system, or linking system representable over F, respectively).

The following two constructions in general do not yield linking systems. First, let (X,Y,E) be a bipartite graph and let

$\Lambda = \{(X',Y') \mid X' \subset X, Y' \subset Y$, there is exactly one matching in E between X' and $Y'\}$.

Then in general (X,Y,Λ) is not a linking system. Consider e.g. the bipartite graph (X,Y,E) schematically represented by the figure:



where $X = \{a,b,c\}$ and $Y = \{d,e,f\}$. Now we have $(\{a,b\},\{e,f\}) \in \Lambda$ and $(\{b,c\},\{d,e\}) \in \Lambda$, with Λ defined as above. Axiom (iii) applied to these two pairs would imply that either $(\{a,b\},\{d,e\}) \in \Lambda$ or $(X,Y) \in \Lambda$, but neither is the case.

A second non-example is obtained by taking a matrix $M = (X,Y,\phi)$ over some field F and putting

$$\Lambda = \{(X',Y') \mid X' \subset X, Y' \subset Y, |X'| = |Y'|, \text{per } (M|X' \times Y') \neq 0\}.$$

Once again, in general (X,Y,Λ) is not a linking system. Clearly, in case char $F = 2$ the system (X,Y,Λ) is a linking system, since in this case the notions of determinant and permanent coincide (cf. example (3)). Also, if $F = \mathbb{Q}$ and all entries in the matrix are nonnegative, (X,Y,Λ) is a linking system, namely a deltoid linking system (cf. example (1)). But in general (X,Y,Λ) fails to be a linking system. E.g. let F be the field with 3 elements and let M be the matrix

$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} ,$$

with $X = \{x_1,x_2,x_3,x_4\}$ and $Y = \{y_1,y_2,y_3,y_4\}$. Construct Λ as indicated above. Now $(\{x_1,x_2,x_3\}, \{y_1,y_2,y_3\}) \in \Lambda$ and $(\{x_2,x_3,x_4\}, \{y_2,y_3,y_4\}) \in \Lambda$, but neither $(\{x_1,x_2,x_3\}, \{y_2,y_3,y_4\}) \in \Lambda$ nor $(X,Y) \in \Lambda$, contradicting axiom (iii) of definition 1.1.

1c. TABLOIDS

HOCQUENGHEM [74] defines the notion of a "tabloid", which is closely related to the concept of a linking system; in fact each linking system produces a tabloid. HOCQUENGHEM (private communication) gave a necessary and sufficient condition for a tabloid to be obtained from a linking system (see theorem 1.4). HOCQUENGHEM's definition is as follows.

DEFINITION. A *tabloid* is a triplet (X,Y,λ) such that

(i) X and Y are finite sets and $\lambda\colon \mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathbb{Z}$;

(ii) for each $X' \subset X$ the function $\lambda_{X'}\colon \mathcal{P}(Y) \longrightarrow \mathbb{Z}$, with $\lambda_{X'}(Y') = \lambda(X',Y')$, is the rank-function of a matroid on Y;

(iii) for each $Y' \subset Y$ the function $\lambda_{Y'}\colon \mathcal{P}(X) \longrightarrow \mathbb{Z}$, with $\lambda_{Y'}(X') = \lambda(X',Y')$, is the rank-function of a matroid on X.

The notions of linking system and tabloid are interrelated but not the same, as the following proposition shows.

PROPOSITION 1.3. *If* (X,Y,Λ) *is a linking system with linking function* λ, *then* (X,Y,λ) *is a tabloid. There is a tabloid* (X,Y,λ) *which is not obtained from a linking system in this way.*

PROOF. By theorem 1.2, λ satisfies the axioms in the definition of a tabloid. For an example that the converse does not hold, let $X = Y = \{a,b\}$, and define $\lambda\colon \mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathbb{Z}$ by

$$\lambda(X',Y') = \min\{1,|X' \cap Y'|\}.$$

Then (X,Y,λ) is a tabloid but λ is not the linking function of a linking system. $\square$

HOCQUENGHEM (private communication) showed furthermore:

THEOREM 1.4. (HOCQUENGHEM). *Let* (X,Y,λ) *be a tabloid. Then the following conditions are equivalent:*

(1) *for each* $X' \subset X$, $Y' \subset Y$, $x \in X\setminus X'$, $y \in Y\setminus Y'$ *it is true that* $\lambda(X,Y\cup\{y\}) + \lambda(X\cup\{x\},Y) \le \lambda(X,Y) + \lambda(X\cup\{x\},Y\cup\{y\})$;

(2) λ *is the linking function of a linking system.*

PROOF. (2) $\Longrightarrow$ (1): Straightforward from the definition of a linking system;
(1) $\Longrightarrow$ (2): We have only to prove: if $X', X'' \subset X$ and $Y', Y'' \subset Y$ then:

$$\lambda(X'\cup X'',Y'\cap Y'') + \lambda(X'\cap X'',Y'\cup Y'') \le \lambda(X',Y') + \lambda(X'',Y'').$$

Let $A \subset X$, $B \subset Y$, $\{a_1,\ldots,a_m\} \subset X\setminus A$, $\{b_1,\ldots,b_n\} \subset Y\setminus B$. Then

$$\begin{array}{llll}
\lambda(A,B\cup\{b_1\}) & -\lambda(A,B) & \le \lambda(A\cup\{a_1\},B\cup\{b_1\}) & -\lambda(A\cup\{a_1\},B), \\
\vdots & \vdots & \vdots & \vdots \\
\lambda(A,B\cup\{b_1,\dots,b_n\}) & -\lambda(A,B\cup\{b_1,\dots,b_{n-1}\}) & \le \lambda(A\cup\{a_1\},B\cup\{b_1,\dots,b_n\}) & -\lambda(A\cup\{a_1\}, B\cup\{b_1,\dots,b_{n-1}\}).
\end{array}$$

Hence

$$\lambda(A,B\cup\{b_1,\dots,b_n\}) - \lambda(A,B) \le \lambda(A\cup\{a_1\},B\cup\{b_1,\dots,b_n\}) - \lambda(A\cup\{a_1\},B).$$

So

$$\lambda(A\cup\{a_1\},B) - \lambda(A,B) \le \lambda(A\cup\{a_1\},B\cup\{b_1,\dots,b_n\}) - \lambda(A,B\cup\{b_1,\dots,b_n\}).$$

By the same method

$$\lambda(A\cup\{a_1,\dots,a_m\},B) - \lambda(A,B) \le \lambda(A\cup\{a_1,\dots,a_m\},B\cup\{b_1,\dots,b_n\}) - \lambda(A,B\cup\{b_1,\dots,b_n\})$$

or

$$\lambda(A\cup\{a_1,\dots,a_m\},B) + \lambda(A,B\cup\{b_1,\dots,b_n\}) \le \lambda(A,B) + \lambda(A\cup\{a_1,\dots,a_m\},B\cup\{b_1,\dots,b_n\}). \qquad \dots(*).$$

Now we have

$\lambda(X'\cap X'',Y'\cup Y'') + \lambda(X'\cup X'',Y'\cap Y'') \le$ (since (X,Y,λ) is a tabloid)
$\lambda(X'\cap X'',Y') + \lambda(X'\cap X'',Y'') - \lambda(X'\cap X'',Y'\cap Y'') +$
$\lambda(X',Y'\cap Y'') + \lambda(X'',Y'\cap Y'') - \lambda(X'\cap X'',Y'\cap Y'') \le$ (using (*))
$\lambda(X',Y') + \lambda(X'',Y'')$. □

CHAPTER TWO

MATROIDS AND LINKING SYSTEMS

Now linking systems has been introduced, what can we do with them? This chapter shows that linking systems do not link only sets but also matroids (defined on these sets) in a natural way. We first establish a one-to-one correspondence between linking systems and "matroids with a fixed base" (section 2a), which will turn out to be very helpful in the continuation.

In section 2b we consider the case where a linking system links a matroid and a set, and in section 2c we look at linking systems linking two matroids.

2a. LINKING SYSTEMS AND MATROIDS

There are close relations between the concepts of matroid and linking system. A simple relation is given in the following proposition.

PROPOSITION 2.1. *Let* (X,Y,Λ) *be a linking system. Let* $\mathcal{J}$ *be the collection of all* $Y' \subset Y$ *with* $(X',Y') \in \Lambda$ *for some* $X' \subset X$. *Then* $\mathcal{J}$ *is the collection of all independent sets of a matroid* $(Y,\mathcal{J})$; *the rank function* σ *of this matroid is given by* $\sigma(Y') = \lambda(X,Y')$ *for* $Y' \subset Y$.

PROOF. The function σ is indeed the rank function of a matroid; also if $Y' \subset Y$ one has

$$\sigma(Y') = |Y'| \text{ if and only if } (X',Y') \in \Lambda \text{ for some } X' \subset X,$$

whence $\mathcal{J}$ is the corresponding collection of independent sets. □

A corollary of this proposition is a theorem of EDMONDS & FULKERSON [65] (extended to the infinite case by MIRSKY & PERFECT [67], BRUALDI & SCRIMGER [68] and BRUALDI [70], cf. BRUALDI & MASON [72]): *if* (X,Y,E) *is a bipartite graph and*

$$\mathcal{J} = \{Y' \subset Y \mid \textit{there is a matching between some subset of } X \textit{ and } Y'\},$$

then $(Y,\mathcal{J})$ *is a matroid*. These matroids are the transversal matroids; using proposition 2.1, we obtain them from deltoid linking systems. A second corollary is the following generalization of EDMONDS & FULKERSON's theorem, due to PERFECT [68] (she did not restrict herself to the finite case):
if (Z,E) *is a directed graph,* X *and* Y *are subsets of* Z *and*

$$\mathcal{J} = \{Y' \subset Y \mid \textit{there are } |Y'| \textit{ pairwise vertex-disjoint paths starting in } X \textit{ and ending in } Y'\},$$

then $(Y,\mathcal{J})$ *is a matroid*. These matroids are the gammoids (cf. MASON [72]); they can be obtained from gammoid linking systems.

Furthermore, matroids obtained as in proposition 2.1 from linking systems representable over some field are exactly the matroids which are representable over that field. In particular the linking system (V,E,Λ_ϕ) obtained from the incidence matrix (V,E,ϕ) (over GF(2)) of a graph (V,E) produces on E the cycle matroid of the graph.

Proposition 2.1 tells us that each linking system produces a matroid. Actually, the correspondence between linking systems and matroids is much closer: each linking system may be understood as a matroid with a fixed base and conversely, as shown in the following theorem.

<u>THEOREM 2.2</u>. *Let* X *and* Y *be disjoint finite sets. Then there is a one-to-one relation between*
(1) *linking systems* (X,Y,Λ),
and
(2) *matroids* $(X\cup Y,\mathcal{I})$ *with the property:* $X \in \mathcal{B}$ *($\mathcal{B}$ is the collection of bases of the matroid),*
given by

$$(X',Y') \in \Lambda \textit{ if and only if } (X\setminus X') \cup Y' \in \mathcal{B}, \textit{ for } X' \subset X \textit{ and } Y' \subset Y.$$

The corresponding linking function λ *and rank-function* ρ *are related by*

$$\rho(X'\cup Y') = \lambda(X\setminus X',Y') + |X'|, \qquad \textit{for } X' \subset X \textit{ and } Y' \subset Y.$$

PROOF.

(1) Let (X,Y,Λ) be a linking system, with linking function λ, and define

$$\mathcal{B} = \{(X\setminus X') \cup Y' \mid (X',Y') \in \Lambda\},$$

and

$$\rho(X'\cup Y') = \lambda(X\setminus X',Y') + |X'|, \qquad \text{for } X' \subset X \text{ and } Y' \subset Y.$$

The fact that ρ is a rank-function of a matroid follows easily from theorem 1.2. The rank of this matroid is

$$\rho(X\cup Y) = \lambda(\emptyset,Y) + |X| = |X|.$$

In order to prove that $\mathcal{B}$ is the collection of bases of this matroid it is sufficient to prove that

$$X'' \cup Y' \in \mathcal{B} \text{ if and only if } \rho(X''\cup Y') = |X''\cup Y'| = |X|,$$

for $X'' \subset X$ and $Y' \subset Y$, or, putting $X' = X\setminus X''$,

$$(X',Y') \in \Lambda \text{ if and only if } \lambda(X',Y') + |X''| = |X''| + |Y'| = |X|.$$

Now this last equality holds if and only if $\lambda(X',Y') = |Y'| = |X'|$, and this is true (by the definition of a linking function) if and only if $(X',Y') \in \Lambda$.

(2) Let $(X\cup Y,\mathcal{I})$ be a matroid with collection of bases $\mathcal{B}$, rank-function ρ and $X \in \mathcal{B}$. Define

$$\Lambda = \{(X',Y') \mid (X\setminus X') \cup Y' \in \mathcal{B},\ X' \subset X \text{ and } Y' \subset Y\},$$

and

$$\lambda(X',Y') = \rho((X\setminus X')\cup Y') - |X\setminus X'|, \qquad \text{for } X' \subset X \text{ and } Y' \subset Y.$$

Now the fact that λ is the linking function of a linking system follows from theorem 1.2 and the axioms for the rank-function of a matroid. Again, it is easy to prove that

$$(X',Y') \in \Lambda \text{ if and only if } \lambda(X',Y') = |X'| = |Y'|. \qquad \square$$

The relation between linking systems and matroids with a fixed base is stable under taking duals: dual linking systems are related with dual matroids, with complementary fixed bases.

Clearly, using theorem 2.2 one can obtain all types of linking systems. For instance, let F be a field, let $X = \{x_1,\dots,x_n\}$ be a set of independent transcendentals over F, and let $y_1,\dots,y_m$ be algebraic over $F(x_1,\dots,x_n)$. Set $Y = \{y_1,\dots,y_m\}$ and define

$$\Lambda = \{(X',Y') \mid (X\backslash X') \cup Y' \text{ is a set of independent transcendentals over } F \text{ and the elements of } (Y\backslash Y') \cup X' \text{ are algebraic over } F((X\backslash X') \cup Y')\}.$$

Then by theorem 2.2 (X,Y,Λ) is a linking system (cf. VAN DER WAERDEN [37]). We shall call linking systems obtained in this way *algebraic* or *algebraically representable* (*over* F).

Under the relation between linking systems and "based" matroids, gammoid linking systems correspond with gammoids (here a theorem of MASON [72] (cf. section 5b) is used, we do not get the gammoid in the form of section 0b), and linking systems linearly representable over some field correspond with matroids representable over that field. The deltoid linking systems (X,Y,Λ) are related with "fundamental transversal matroids" (or "principal transversal matroids" or "strict deltoids") with principal basis X (cf. BONDY & WELSH [71] and INGLETON & PIFF [73]; see also section 5b).

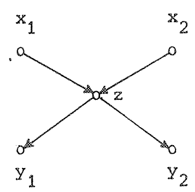
Also, known matroid theorems can now be translated to linking systems. A theorem of MASON [72] (cf. PIFF & WELSH [70] and INGLETON & PIFF [73]) implies that each gammoid linking system is representable over all but a finite number of finite fields. For more details on this representability see section 5c. From PIFF [69] it follows that each linking system representable over a field F is algebraic over F (note that a vector space V over F always can be embedded in a transcendental extension of F, in such a way that algebraic independence within V coincides with linear independence). Hence the classes of

deltoid linking systems,
gammoid linking systems,
linearly representable linking systems,
algebraically representable linking systems,
all linking systems,

are chain-wise contained in each other (such that a class is contained in another class if this last class is lower on this list).

These inclusions are proper:

the gammoid linking system $(\{x_1,x_2\},\{y_1,y_2\},\Gamma_E)$ induced by the directed graph



is not a deltoid linking system;

the linking system, representable over GF(2), obtained from the GF(2)-matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

(i.e. a linking system obtained from the Fano matroid) is only representable over fields of characteristic 2, hence it is not a gammoid linking system (these are linearly representable over almost every field);

the linking system (X,Y,Λ) algebraically represented over GF(2) by $X = \{x,y,z\}$, $Y = \{xy,xz,yz,xyz,x+y,x+z,y+z,x+y+z\}$ (hence $Y \subset GF(2)(X)$) is not linearly representable over any field (if $Y' = \{xy,xz,yz,xyz\}$, then the sub-linking systems $(X,Y\setminus Y',\Lambda')$ and (X,Y',Λ'') are linearly representable only over fields of characteristic 2, resp. only over fields not of characteristic 2) (this example is due to INGLETON [71]);

the linking system with

$$\begin{aligned} X &= \{a_3,b_3,a_4,b_4\}, \\ Y &= \{a_1,b_1,a_2,b_2\}, \\ \Lambda &= \{(X',Y') \mid X' \subset X, Y' \subset Y, |X'| = |Y'| \le 3 \text{ and } (X',Y') \ne (\{a_i,b_i\},\{a_j,b_j\}) \\ &\quad \text{for } i = 3,4,\ j = 1,2\}, \end{aligned}$$

(i.e. a linking system obtained from the Vamos matroid) is not algebraically representable, as was pointed out by INGLETON & MAIN [75].

It is not known whether the dual of an algebraic matroid is again algebraic, or, equivalently, whether the dual of an algebraic linking system is again algebraic.

2b. THE LINKING OF MATROIDS BY LINKING SYSTEMS

Using theorem 2.2 we shall show how a matroid can be linked with a linking system, forming a new matroid. In this way a generalization is obtained of theorems of RADO [42], PERFECT [69], BRUALDI [71c] and MASON [72].

THEOREM 2.3. *Let* $(X,\mathcal{I})$ *be a matroid (with rank function* ρ*) and let* (X,Y,Λ) *be a linking system (with* λ *as linking function). Put*

$$\mathcal{I}*\Lambda = \{Y' \subset Y \mid (X',Y') \in \Lambda \text{ for some } X' \in \mathcal{I}\}.$$

Then $(Y,\mathcal{I}*\Lambda)$ *is again a matroid, with rank-function* $\rho*\lambda$ *given by*

$$(\rho*\lambda)(Y') = \min_{X'\subset X} (\rho(X\backslash X') + \lambda(X',Y')), \qquad \text{for } Y' \subset Y.$$

PROOF. We may suppose that X and Y are disjoint sets (otherwise take disjoint copies of X and Y). Let $(X\cup Y,\mathcal{J})$ be the matroid related with (X,Y,Λ) as in theorem 2.2, let $\mathcal{B}$ its bases collection and let ρ' be its rank-function. Let M be the matroid union of the matroids $(X,\mathcal{I})$ and $X\cup Y,\mathcal{J})$; this matroid is defined on $X \cup Y$ by taking as independent sets all unions of an independent set of $(X,\mathcal{I})$ and an independent set of $(X\cup Y,\mathcal{J})$. We now prove the contraction $M\cdot Y$ of M to Y (i.e. we contract X) has as collection of independent sets the collection $\mathcal{I}*\Lambda$, as defined above.
Let Y' be an independent set of $M\cdot Y$. Then, since X is independent in M, we have that $X \cup Y'$ is independent in M. Then there exists an $X' \subset X$ such that $X' \in \mathcal{I}$ and $(X\backslash X') \cup Y' \in \mathcal{B}$, or $(X',Y') \in \Lambda$; hence $Y' \in \mathcal{I}*\Lambda$. Following the same steps in the reverse order one proves: if $Y' \in \mathcal{I}*\Lambda$, then Y' is independent in $M\cdot Y$.
Let $\rho*\lambda$ be the rank-function of the matroid $(Y,\mathcal{I}*\Lambda)$, i.e. of the matroid $M\cdot Y$, and let ρ'' be the rank-function of the matroid M. Then

$$\begin{aligned}
&(\rho*\lambda)(Y') = \rho''(X\cup Y') - \rho''(X) = \\
&\min \{\rho(X') + \rho'(X'\cup Y'') + |X\backslash X'| + |Y'\backslash Y''| - |X| \mid X' \subset X \text{ and } Y'' \subset Y'\} = \\
&\min \{\rho(X') + \lambda(X\backslash X',Y'') + |X'| + |X\backslash X'| + |Y'\backslash Y''| - |X| \mid X' \subset X \text{ and } Y'' \subset Y'\} = \\
&\min \{\rho(X') + \lambda(X\backslash X',Y'') + |Y'\backslash Y''| \mid X' \subset X \text{ and } Y'' \subset Y'\} = \\
&\min_{X'\subset X} (\rho(X') + \lambda(X\backslash X',Y')) = \\
&\min_{X'\subset X} (\rho(X\backslash X') + \lambda(X',Y')).
\end{aligned}$$

In this derivation we have used well-known theorems on the rank of the contraction of a matroid and on the rank of the union of two matroids (cf. section 0b). □

Proposition 2.1 is an immediate consequence of theorem 2.3; in addition we have the following corollaries.

COROLLARY 2.3a. (PERFECT [69], RADO [42]). *Let* (X,Y,E) *be a bipartite graph* *and let* $(X,\mathcal{I})$ *be a matroid. Then the set*

$$\mathcal{J} = \{Y' \subset Y \mid Y' \text{ is matched with some independent subset of } X\},$$

forms the collection of independent subsets of a matroid. This matroid has rank-function

$$\sigma(Y') = \min_{Y''\subset Y'} (\rho(E^{-1}(Y'')) + |Y'\setminus Y''|), \qquad \text{for } Y' \subset Y.$$

(Here, as usual, $E^{-1}(Y'') = \{x \in X \mid (x,y) \in E \text{ for some } y \in Y''\}$.)

PROOF. Clearly, $\mathcal{J} = \mathcal{I}*\Delta_E$, hence $(Y,\mathcal{J})$ is a matroid (cf. example (1) section 1b). We prove that the rank-function σ is as above. By theorem 2.3, we have

$$\sigma(Y') = (\rho*\delta_E)(Y') = \min_{X'\subset X}(\rho(X\setminus X') + \delta_E(X',Y')),$$

for $Y' \subset Y$. Now

$$\delta_E(X',Y') = \min_{Y''\subset Y'}(|Y'\setminus Y''| + |E^{-1}(Y'')\cap X'|) .$$

Hence

$$\sigma(Y') = \min_{X'\subset X}(\rho(X\setminus X') + \min_{Y''\subset Y'}(|Y'\setminus Y''| + |E^{-1}(Y'')\cap X'|)).$$

It is easy to check that

$$\min_{X'\subset X}(\rho(X\setminus X') + |E^{-1}(Y'')\cap X'|) = \rho(E^{-1}(Y'')),$$

hence

$$\sigma(Y') = \min_{Y''\subset Y'}(|Y'\setminus Y''| + \rho(E^{-1}(Y''))). \qquad \square$$

COROLLARY 2.3b. (BRUALDI [71c], MASON [72], for other proofs see INGLETON & PIFF [73], WOODALL [75]). *Let* (Z,E) *be a directed graph and* $X,Y \subset Z$. *Furthermore, let* $(X,\mathcal{I})$ *be a matroid, with rank function* ρ.
Then the set

$$\mathcal{J} = \{Y' \subset Y \mid \text{there are } |Y'| \text{ pairwise vertex-disjoint paths starting in some independent subset of } X \text{ and ending in } Y'\}$$

forms the collection of independent subsets of a matroid on Y. *This matroid has rank function* σ, *given, for* $Y' \subset Y$, *by*

$$\sigma(Y') = \min\{\rho(X') + |Z'| \mid X' \subset X,\ Z' \subset Z, \text{ every path from } X\backslash X' \text{ to } Y' \text{ intersects } Z'\} = \min_{Z'\subset Z\backslash Y'} \rho(X\backslash(E(Z')\cup Z')) + |E(Z')\backslash Z'|.$$

(Here, as usual, $E(Z') = \{y \in Z \mid (x,y) \in E \text{ for some } x \in Z'\}$.)

PROOF. The corollary follows from theorem 2.3 by taking as linking system the system (X,Y,Γ_E) as in example (2) of section 1b. Then $J = I*\Gamma_E$ and $\sigma = \rho * \gamma_E$. Here

$$(\rho * \gamma_E)(Y') = \min_{X'\subset X}(\rho(X\backslash X') + \gamma_E(X',Y'))$$

for $Y' \subset Y$. Now

$\gamma_E(X',Y')$ = the minimal number of elements in a set Z' of vertices in Z with the property that Z' intersects every path from X' to Y' = $\min_{Z'\subset Z\backslash Y'}((X'\cup E(Z'))\backslash Z')$,

for $X' \subset X$, $Y' \subset Y$. Hence $\sigma = \rho * \gamma_E$ is as formulated in the corollary. Observe that

$$\min_{X'\subset X}(\rho(X\backslash X') + |(X'\cup E(Z'))\backslash Z'|) = \rho(X\backslash(E(Z')\cup Z')) + |E(Z')\backslash Z'|$$

for $Z' \subset Z$. □

COROLLARY 2.3c. *Let* $M = (X,Y,\phi)$ *be a matrix with row collection* X *and column collection* Y *and let* (X,I) *be a matroid on* X, *with rank function* ρ. *Define*

$J = \{Y' \subset Y \mid$ *for some* $X' \in I$ *the submatrix of* M *generated by the rows* X' *and columns* Y' *is nonsingular*$\}$.

Then (Y,J) *is again a matroid, with rank function* σ *given by, for* $Y' \subset Y$,

$$\sigma(Y') = \min_{X'\subset X}(|X'| + \rho(X\backslash[X'])),$$

where $[X'] = \{x \in X \mid$ *the row* x *in* $M|X\times Y'$ *is a linear combination of rows in* $M|X'\times Y'\}$, *for* $X' \subset X$.

PROOF. Straightforward from theorem 2.3 applied to example (3) of section 1b. □

We shall call $(Y,\mathcal{I}*\Lambda)$ the *product* of the matroid $(X,\mathcal{I})$ and the linking system (X,Y,Λ). Clearly, the product of a gammoid and a gammoid linking system is again a gammoid.

PIFF & WELSH [70] have proved that for each pair of matroids there exists a natural number N such that if F is a field with $|F| \geq N$ and both matroids are representable over F then the union of the matroids is again representable over F. In the light of the proof of theorem 2.3 this implies the following. *Let* $(X,\mathcal{I})$ *be a matroid and let* (X,Y,Λ) *be a linking system; then there is a natural number* N *such that: if* F *is a field with at least* N *elements and both* $(X,\mathcal{I})$ *and* (X,Y,Λ) *are linearly representable over* F*, then also the matroid* $(Y,\mathcal{I}*\Lambda)$ *is representable over* F.

The next theorem is a generalization of theorems of MASON [70] and BRUALDI [71c] on the product of a matroid and a bipartite or directed graph (cf. the corollaries).

THEOREM 2.4. *Let* (X,Y,Λ) *be a linking system and let* $(X,\mathcal{I})$ *be a matroid. Suppose* $(X_1,Y_1) \in \Lambda$, $(X_2,Y_2) \in \Lambda$, $X_1 \in \mathcal{I}$ *and* $X_2 \in \mathcal{I}$. *Then there is an* $(X',Y') \in \Lambda$ *such that* $X' \in \mathcal{I}$, $X_1 \subset c\ell_{\mathcal{I}}X'$, $X' \subset X_1 \cup X_2$ *and* $Y_2 \subset Y' \subset Y_1 \cup Y_2$.

PROOF. We may assume that $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Let Y' be a base of the matroid $(Y,\mathcal{I}*\Lambda)$ such that $Y_2 \subset Y'$. Take $X' \in \mathcal{I}$ such that $(X',Y') \in \Lambda$ and $|X_1 \cap X'|$ is maximal. If we prove that $\rho(X_1 \cup X') = |X'|$ we are ready (ρ is the rank function of $(X,\mathcal{I})$). Therefore suppose that $\rho(X_1 \cup X') > |X'|$ and let $x \in X_1 \backslash X'$ such that $X' \cup \{x\} \in \mathcal{I}$. Then we have

$$\lambda((X_1 \cap X') \cup \{x\},Y') \geq \lambda(X' \cup \{x\},Y') + \lambda((X_1 \cap X') \cup \{x\},Y_1 \cup Y') -$$

$$\lambda(X' \cup \{x\},Y_1 \cup Y') = |Y'| + |(X_1 \cap X') \cup \{x\}| - |Y'| = |(X_1 \cap X') \cup \{x\}|.$$

Hence, for some $Y'' \subset Y'$ we have $((X_1 \cap X') \cup \{x\},Y'') \in \Lambda$. Since also $(X',Y') \in \Lambda$ there exists (by axiom (iii) of definition 1.1) an $X''' \subset X' \cup \{x\}$ such that $(X_1 \cap X') \cup \{x\} \subset X'''$ and $(X''',Y') \in \Lambda$. But then

$$X''' \in \mathcal{I} \quad \text{and} \quad |X''' \cap X_1| > |X' \cap X_1|,$$

contradicting the conditions on X'. □

In particular, if B is a base of $(Y, I*\Lambda)$ and $A \in I$ such that $(A,B) \in \Lambda$, then $Y' \in I * \Lambda$ if and only if $(X',Y') \in \Lambda$ for some $X' \in I$ with $X' \subset c\ell_I A$ (this is true since if $A_1, A_2 \in I$ and $|A_1| = |A_2|$ then $A_1 \subset c\ell_I A_2$ iff $A_2 \subset c\ell_I A_1$). Hence we have the following corollaries.

COROLLARY 2.4a. (MASON [70,70a]). *Let* (X,Y,E) *be a bipartite graph and let* (X,I) *be a matroid. Let*

$$J = \{Y' \subset Y \mid \text{there is a matching between some } X' \in I \text{ and } Y'\}.$$

If B *is a base of the matroid* (Y,J) *and* B *is matched with the independent set* A, *then*

$$J = \{Y' \subset Y \mid Y' \text{ is matched with some } X' \subset c\ell_I A \text{ with } X' \in I\}.$$

PROOF. Straightforward from theorem 2.4. □

COROLLARY 2.4b. (BRUALDI [71c]). *Let* (Z,E) *be a directed graph and let* X *and* Y *be subsets of* Z. *Let* (X,I) *be a matroid on* X *and put*

$$J = \{Y' \subset Y \mid \text{there are } |Y'| \text{ pairwise vertex-disjoint paths starting in some independent subset of } X \text{ and ending in } Y'\}.$$

Let B *be a base of the matroid* (Y,J) *and* $A \in I$ *such that there are* $|B|$ *pairwise vertex-disjoint paths from* A *to* B. *Then:*

$$J = \{Y' \subset Y \mid \text{there are } |Y'| \text{ pairwise vertex-disjoint paths starting in some independent subset of } c\ell_I A \text{ and ending in } Y'\}.$$

PROOF. Straightforward from theorem 2.4. □

2c. DOUBLY INDEPENDENT LINKED PAIRS

In this last section of chapter 2 we give a generalization of theorems of BRUALDI [70a,71a] on matroids and graphs. In this case on both sides of the linking system matroids are defined and we determine the maximal cardinality of the sets in a "doubly independent linked pair", i.e. in a linked pair in which both elements are independent. Furthermore we give a symmetrized version of theorem 2.4.

<u>THEOREM 2.5</u>. *Let* (X,Y,Λ) *be a linking system and let* $(X,\mathcal{I})$ *and* $(Y,\mathcal{J})$ *be matroids, with rank functions* ρ *and* σ *respectively. Then:*

$$\max\{|Y'| \mid X' \in \mathcal{I},\ Y' \in \mathcal{J} \text{ and } (X',Y') \in \Lambda\} = \min_{\substack{X'\subset X\\ Y'\subset Y}} (\rho(X') + \sigma(Y') + \lambda(X\backslash X', Y\backslash Y')).$$

<u>PROOF</u>. As is done by WELSH [70] in proving BRUALDI's theorem, we use EDMONDS' intersection theorem (cf. section 0b), applied here to the matroids $(Y,\mathcal{I}*\Lambda)$ and $(Y,\mathcal{J})$. This gives us that

$$\max\{|Y'| \mid X' \in \mathcal{I},\ Y' \in \mathcal{J} \text{ and } (X',Y') \in \Lambda\} =$$
$$\max\{|Y'| \mid Y' \in \mathcal{J} \text{ and } Y' \in \mathcal{I}*\Lambda\} =$$
$$\min_{Y'\subset Y} (\sigma(Y') + (\rho*\lambda)(Y\backslash Y')) =$$
$$\min_{Y'\subset Y} (\sigma(Y') + \min_{X'\subset X} (\rho(X') + \lambda(X\backslash X', Y\backslash Y')).$$

The last equality follows from theorem 2.3. □

As corollaries we have the following two results of BRUALDI. The first one is the "symmetrized version of RADO's theorem".

<u>COROLLARY 2.5a</u>. (BRUALDI [70a]). *Let* (X,Y,E) *be a bipartite graph and let* $(X,\mathcal{I})$ *and* $(Y,\mathcal{J})$ *be matroids, with rank functions* ρ *and* σ*, respectively. Then:*

$$\max\{|Y'| \mid Y' \in \mathcal{J} \text{ and there is a matching in } E \text{ between some } X' \in \mathcal{I} \text{ and } Y'\} = \min_{X'\subset X} (\sigma(E(X')) + \rho(X\backslash X')).$$

<u>PROOF</u>. The bipartite graph (X,Y,E) generates a linking system (X,Y,Λ_E) (cf. example (1) in section 1b). In this linking system the linking function δ_E is given by

$$\delta_E(X',Y') = \min_{X''\subset X'} (|X'\backslash X''| + |E(X'')\cap Y'|).$$

Hence, by theorem 2.5,

$$\max\{|Y'| \mid Y' \in \mathcal{J} \text{ is matched in E with an } X' \in \mathcal{I}\} = \min_{\substack{X' \subset X \\ Y' \subset Y}} (\rho(X\backslash X') + \sigma(Y\backslash Y') + \delta_E(X',Y')) =$$

$$\min\{\rho(X\backslash X') + \sigma(Y\backslash Y') + |X'\backslash X''| + |E(X'')\cap Y'| \mid X'' \subset X' \subset X \text{ and } Y' \subset Y\}.$$

Now, for fixed $X'' \subset X$, the minimum is reached if $X' = X''$ and $Y' = Y\backslash E(X'')$. Hence the expression equals

$$\min_{X'' \subset X} (\rho(X\backslash X'') + \sigma(E(X''))). \qquad \square$$

COROLLARY 2.5b. (BRUALDI [71a], cf. MCDIARMID [75]). *Let* (Z,E) *be a directed graph and let* X *and* Y *be subsets of* Z. *Let* $(X,\mathcal{I})$ *and* $(Y,\mathcal{J})$ *be matroids, with rank functions* ρ *and* σ, *respectively. Then the maximal cardinality of a set* Y' *in* $\mathcal{J}$ *such that there are* |Y'| *pairwise vertex-disjoint paths starting in an* X' *in* $\mathcal{I}$ *and ending in* Y' *equals*

$$\min\{\rho(Z_1) + |Z_0| + \sigma(Z_2) \mid Z_0 \subset Z,\ Z_1 \subset X,\ Z_2 \subset Y;\ Z_0 \textit{ meets every path from } X\backslash Z_1 \textit{ to } Y\backslash Z_2\}.$$

PROOF. The subsets X and Y of Z generate a linking system (X,Y,Γ_E) (cf. example (2) in section 1b). In this linking system the linking function γ_E is given by

$$\gamma_E(X',Y') = \min\{|Z_0| \mid Z_0 \subset Z;\ Z_0 \text{ meets every path from } X' \text{ to } Y'\}.$$

This and theorem 2.5 imply the assertion straightforwardly. □

There are several other forms for this last corollary (cf. WELSH [76] or MCDIARMID [75]), e.g.:

(i) Let (Z,E) be a directed graph, X, Y ⊂ Z and let $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ be matroids with rank functions ρ and σ, respectively. Then the maximal cardinality of a set in $\mathcal{I}$ which is matched onto a set in $\mathcal{J}$ equals

$$\min_{Z' \subset Z} (\rho(X\backslash(E(Z')\cup Z')) + |E(Z')\backslash Z'| + \sigma(Z'\cap Y)).$$

(ii) Let (Z,E) be a directed graph and let $(Z,\mathcal{I})$ and $(Z,\mathcal{J})$ be matroids with rank functions ρ and σ, respectively, such that ρ(Z) = σ(Z). Then there is a base of the matroid $(Z,\mathcal{I})$ matched onto a base of the matroid $(Z,\mathcal{J})$ if and only if for all Z' ⊂ Z

$$\rho^*(E(Z')\cup Z') + \sigma(Z') \geq |Z'|.$$

Both of these results follow easily from corollary 2.5b.

Another consequence of theorem 2.5 is the following: let M be a matrix with row collection X and column collection Y; let furthermore $\mathcal{A} = (A_i | i \in I)$ and $\mathcal{B} = (B_i | i \in I)$ be collections of subsets of X and Y, respectively. Then $\mathcal{A}$ and $\mathcal{B}$ have transversals $X' \subset X$ and $Y' \subset Y$ such that $M|X' \times Y'$ is nonsingular, if and only if for all $J \subset I$ and $K \subset I$ we have

$$\mathrm{rank}(M| \bigcup_{j\in J} A_j \times \bigcup_{k\in K} B_k) \geq |J| + |K| - |I|.$$

In case X = Y and M is the unity-matrix this result becomes a theorem of FORD & FULKERSON [58]: let $\mathcal{A} = (A_i | i \in I)$ and $\mathcal{B} = (B_i | i \in I)$ be collections of subsets of a set X; then $\mathcal{A}$ and $\mathcal{B}$ have a common transversal if and only if for all $J \subset I$ and $K \subset I$ we have

$$| \bigcup_{j\in J} A_j \cap \bigcup_{k\in K} B_k | \geq |J| + |K| - |I|.$$

Just as theorem 2.5 is a symmetrized version of theorem 2.3, the following theorem is a symmetrized version of theorem 2.4, having as corollaries results of KUNDU & LAWLER [73] and MCDIARMID [76].

THEOREM 2.6. *Let (X,Y,Λ) be a linking system and let $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ be matroids. Furthermore, let (X_1,Y_1), $(X_2,Y_2) \in \Lambda$ such that $X_1, X_2 \in \mathcal{I}$ and $Y_1, Y_2 \in \mathcal{J}$. Then there are $X' \in \mathcal{I}$ and $Y' \in \mathcal{J}$ such that $(X',Y') \in \Lambda$ and $X_1 \subset c\ell_{\mathcal{I}}X'$, $X' \subset X_1 \cup X_2$, $Y_2 \subset c\ell_{\mathcal{J}}Y'$, $Y' \subset Y_1 \cup Y_2$.*

PROOF. We may assume that $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Let $(X',Y') \in \Lambda$ such that $X' \in \mathcal{I}$, $Y' \in \mathcal{J}$, $X_1 \subset c\ell_{\mathcal{I}} X'$, and $|Y' \cap Y_2|$ is as large as possible. We shall prove that $Y_2 \subset c\ell_{\mathcal{J}} Y'$. Suppose $y \in Y_2 \setminus c\ell_{\mathcal{J}} Y'$; this implies that $Y' \cup \{y\} \in \mathcal{J}$. Now $(Y' \cap Y_2) \cup \{y\} \in \mathcal{I} * \Lambda$. Since also $(X',Y') \in \Lambda$ and $X' \in \mathcal{I}$ by theorem 2.4 there exists a pair $(X'',Y'') \in \Lambda$ such that $X'' \in \mathcal{I}$, $X' \subset c\ell_{\mathcal{I}} X''$ and $(Y' \cap Y_2) \cup \{y\} \subset Y'' \subset Y' \cup \{y\}$. As $Y' \cup \{y\} \in \mathcal{J}$, also $Y'' \in \mathcal{J}$. Furthermore $X_1 \subset c\ell_{\mathcal{I}} X' \subset c\ell_{\mathcal{I}} X''$, but

$$|Y'' \cap Y_2| \geq |(Y' \cap Y_2) \cup \{y\}| > |Y' \cap Y_2|,$$

contradicting the maximality of $|Y' \cap Y_2|$. □

COROLLARY 2.6a. (KUNDU & LAWLER [73]). *Let $(Z,\mathcal{I})$ and $(Z,\mathcal{J})$ be matroids and suppose Z_1, Z_2 are elements of both $\mathcal{I}$ and $\mathcal{J}$. Then there exists a Z' in $\mathcal{I} \cap \mathcal{J}$ such that $Z_1 \subset c\ell_{\mathcal{I}} Z'$, $Z_2 \subset c\ell_{\mathcal{J}} Z'$ and $Z' \subset Z_1 \cup Z_2$.*

PROOF. Apply theorem 2.6 to the linking system (Z,Z,Λ) with $\Lambda = \{(Z_0,Z_0) | Z_0 \subset Z\}$. □

COROLLARY 2.6b. (MCDIARMID [76]). *Let (Z,E) be a directed graph and let $X,Y \subset Z$; let (X,I) and (Y,J) be matroids and suppose $X_1 \in I$ is matched in E onto $Y_1 \in J$, and, similarly, $X_2 \in I$ is matched in E onto $Y_2 \in J$. Then there exist $X' \in I$ and $Y' \in J$ such that X' is matched in E onto Y' and $X_1 \subset c\ell_I X'$ and $Y_2 \subset c\ell_J Y'$.*

PROOF. Straightforward from theorem 2.6. □

Clearly, theorem 2.4 also follows from theorem 2.6.

Actually MCDIARMID [76] proved a theorem more general than corollary 2.6b; this more general assertion also can be extended to linking systems. MCDIARMID calls a nonempty collection I of subsets of a set X *hereditary* if $X'' \subset X' \in I$ implies $X'' \in I$. With such a collection I an operator $c\ell_I$ on X is associated, given by

$$c\ell_I X' = X' \cup \{y \in X\backslash X' \mid X'' \cup \{y\} \notin I \text{ for some } X'' \subset X', X'' \in I\}$$

for subsets X' of X. In case (X,I) is a matroid $c\ell_I$ coincides with the usual closure operator associated with matroids. MCDIARMID observed that, given a hereditary collection I on a set X, the pair (X,I) is a matroid if and only if $c\ell_I c\ell_I X' = c\ell_I X'$ for all $X' \subset X$. As this last identity does not hold in general it makes sense to define a closure operator $\overline{c\ell}_I$ on X by

$$\overline{c\ell}_I X' = \bigcup_{n=1}^{\infty} c\ell_I^n X',$$

for $X' \subset X$, where $c\ell_I^n X'$ arises from X' by applying n times in succession the closure operator $c\ell_I$.
First we have an extension of theorem 2.4.

THEOREM 2.7. *Let (X,Y,Λ) be a linking system and let I be a hereditary collection of subsets of X. Let furthermore (X_1,Y_1), $(X_2,Y_2) \in \Lambda$ such that $X_1,X_2 \in I$. Then there is a pair $(X',Y') \in \Lambda$ such that $X' \in I$, $X_1 \subset \overline{c\ell}_I X'$, $X' \subset X_1 \cup X_2$ and $Y_2 \subset Y' \subset Y_1 \cup Y_2$.*

PROOF. We may suppose that $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.
Choose $(X',Y') \in \Lambda$ such that $X' \in I$, $Y_2 \subset Y'$ and $|X' \cap X_1|$ is as large as possible (such a pair (X',Y') exists since $(X_2,Y_2) \in \Lambda$, $X_2 \in I, Y_2 \subset Y_2$). We prove that $X_1 \subset \overline{c\ell}_I X'$; to this end suppose that $x \in X_1 \backslash \overline{c\ell}_I X'$. This implies that $X' \cup \{x\} \in I$.

Since $((X' \cap X_1) \cup \{x\}, Y_1') \in \Lambda$, for some $Y_1' \subset Y_1$, and $(X',Y') \in \Lambda$ there exists a pair $(X'',Y'') \in \Lambda$ such that

$$(X' \cap X_1) \cup \{x\} \subset X'' \subset X' \cup \{x\} \quad \text{and} \quad Y' \subset Y''.$$

Now $\quad X'' \in I$ (for $X'' \subset X' \cup \{x\} \in I$) and $Y_2 \subset Y' \subset Y''$. But

$$|X'' \cap X_1| \geq |(X' \cap X_1) \cup \{x\}| > |X' \cap X_1|$$

contradicting our choice of (X',Y'). □

Now we can prove

THEOREM 2.8. *Let (X,Y,Λ) be a linking system and let I and J be hereditary collections of subsets of X and Y, respectively. Let furthermore $(X_1,Y_1),(X_2,Y_2) \in \Lambda$ such that $X_1,X_2 \in I$ and $Y_1,Y_2 \in J$. Then there is a pair $(X',Y') \in \Lambda$ such that $X' \in I$, $Y' \in J$, $X_1 \subset c\ell_I X'$, $X' \subset X_1 \cup X_2$, $Y_2 \subset \overline{c\ell}_J Y'$, $Y' \subset Y_1 \cup Y_2$.*

PROOF. Again we may suppose that $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Choose $(X',Y') \in \Lambda$ such that $X' \in I$, $Y' \in J$, $Y_2 \subset \overline{c\ell}_J Y'$ and $|X' \cap X_1|$ is as large as possible. We prove that $X_1 \subset c\ell_I X'$; again suppose that $x \in X_1 \setminus c\ell_I X'$. Hence $X' \cup \{x\} \in I$. Since now $((X' \cap X_1) \cup \{x\}, Y_1') \in \Lambda$ for some $Y_1' \subset Y_1$, $(X',Y') \in \Lambda$, $Y_1' \in J$, $Y' \in J$ there exists, by theorem 2.7, a pair $(X'',Y'') \in \Lambda$ such that

$$Y'' \in J,\ (X' \cap X_1) \cup \{x\} \subset X'' \subset X' \cup \{x\},\ Y' \subset c\ell_J Y''.$$

Now $X'' \in I$ (for $X'' \subset X' \cup \{x\} \in I$), $Y'' \in J$, $Y_2 \subset \overline{c\ell}_J Y' \subset \overline{c\ell}_J\, c\ell_J Y'' = \overline{c\ell}_J Y''$. But

$$|X'' \cap X_1| \geq |(X' \cap X_1) \cup \{x\}| > |X' \cap X_1|$$

contradicting the choice of (X',Y'). □

MCDIARMID obtained this result (inter alia) for gammoid linking systems.

PERFECT posed the following question: given matroids (X,I) and (Y,J), with the same rank, when does there exist a bipartite graph (X,Y,E) such that (Y,J) is the matroid induced by (X,I) and (X,Y,E) on Y (i.e. $J = I * \Delta_E$) and (X,I) is the matroid induced by (Y,J) and (Y,X,E^{-1}) on X (i.e. $I = J * \Delta_E^*$)? This problem is trivially solved when we ask for an arbitrary linking system as a medium instead of a deltoid linking system. In that case, let

$$\Lambda = \{(X',Y') \mid X' \in I, Y' \in J \text{ and } |X'| = |Y'|\}.$$

Then (X,Y,Λ) is a linking system and $\mathcal{J} = \mathcal{I}*\Lambda$ and $\mathcal{I} = \mathcal{J}*\Lambda^*$. PERFECT [73] proves that, given two transversal matroids $(X,\mathcal{I})$ and $(Y,\mathcal{J})$, both of rank r, there exists a bipartite graph (X,Y,E) such that $\mathcal{I} = \{X'\subset X \mid X'$ matched in E with a $Y'\subset Y\}$ and $\mathcal{J} = \{Y'\subset Y \mid Y'$ is matched in E with an $X'\subset X\}$ if and only if the total number of coloops in $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ is at least r. Clearly, this produces a deltoid linking system (X,Y,Δ_E) such that $\mathcal{J} = \mathcal{I}*\Delta_E$ and $\mathcal{I} = \mathcal{J}*\Delta_E^*$.

CHAPTER THREE

LINKING SYSTEMS AND BIPARTITE GRAPHS

We know already that each bipartite graph (X,Y,E) produces a deltoid linking system (X,Y,Δ_E). Now for each linking system (X,Y,Λ) we define its *underlying* bipartite graph (X,Y,E_Λ) by

$$(x,y) \in E_\Lambda \text{ if and only if } (\{x\},\{y\}) \in \Lambda.$$

(Without loss of generality we may suppose that X and Y are disjoint sets.) We will give some relations between a linking system and its underlying bipartite graph. Since each linking system can be understood as a matroid with a fixed base (theorem 2.2) this also leads to results for matroids. In this direction related work has been done by KROGDAHL [75]; the underlying bipartite graph is the same as KROGDAHL's dependence graph of the fixed base. He uses this notion to analyse some algorithms on matroids (cf. KROGDAHL [76,76a] and section 7).

The two basic results of this section are the following ones:

(i) if $X' \subset X$ and $Y' \subset Y$ are such that there is exactly one matching in (X,Y,E_Λ) between X' and Y', then $(X',Y') \in \Lambda$;

(ii) if $(X',Y') \in \Lambda$ then there is at least one matching in (X,Y,E_Λ) between X' and Y'.

Clearly, a linking system (X,Y,Λ) is a deltoid linking system if and only if $\Lambda = \Delta_{E_\Lambda}$.

Let $(X \cup Y,\mathcal{B})$ be the matroid with fixed base X, related to the linking system (X,Y,Λ) (cf. theorem 2.2). Then for each $y \in Y$ the set

$$\{y\} \cup \{x \in X \mid (x,y) \in E_\Lambda\}$$

is the unique circuit of the matroid contained in $X \cup \{y\}$. Similarly, for each $x \in X$ the set

$$\{x\} \cup \{y \in Y \mid (x,y) \in E_\Lambda\}$$

is the unique cocircuit contained in $Y \cup \{x\}$.

First we prove a theorem, which was inspired by the following result of GREENE [73] and BRYLAWSKI [73a]: given two bases B_1 and B_2 of a matroid and a partition $B_1 = X_1 \cup Y_1$, there is a partition $B_2 = X_2 \cup Y_2$ such that both $X_1 \cup X_2$ and $Y_1 \cup Y_2$ are bases of the matroid (in case X_1 is a singleton the result was first proved by BRUALDI [69]; for other proofs and extensions of the above result see WOODALL [74] and MCDIARMID [75a]; see WELSH [76] p. 124 for a generalization to arbitrary partitions).

THEOREM 3.1. *Let (X,Y,Λ) be a linking system and let $(X',Y') \in \Lambda$. Furthermore let $X'' \subset X'$. Then $(X'',Y'') \in \Lambda$ and $(X'\backslash X'',Y'\backslash Y'') \in \Lambda$ for some $Y'' \subset Y'$.*

We imitate the method of proof of WOODALL [74] and MCDIARMID [75a].

PROOF. Let $M_1 = (Y',\mathcal{J}_1)$ be the matroid on Y' with

$$\mathcal{J}_1 = \{Y_0' \subset Y' \mid (X_0',Y_0') \in \Lambda \text{ for some } X_0' \subset X''\}.$$

Similarly, let $M_2 = (Y',\mathcal{J}_2)$ be the matroid on Y' with

$$\mathcal{J}_2 = \{Y_0' \subset Y' \mid (X_0',Y_0') \in \Lambda \text{ for some } X_0' \subset X'\backslash X''\}.$$

If we have that Y' is a base of the union $M_1 \vee M_2$ of M_1 and M_2, then there exists a $Y'' \subset Y'$ such that

$$(X'',Y'') \in \Lambda \text{ and } (X'\backslash X'',Y'\backslash Y'') \in \Lambda.$$

The matroid union theorem implies that Y' is a base of $M_1 \vee M_2$ if and only if

$$\rho_1(Y_0') + \rho_2(Y_0') \geq |Y_0'|, \text{ for each } Y_0' \subset Y',$$

where ρ_1 and ρ_2 are the rank-functions of M_1 and M_2, respectively.
We shall prove that this last inequality holds. Let $Y_0' \subset Y'$. Then, by axiom (ii) of definition 1.1, $(X_0',Y_0') \in \Lambda$ for some $X_0' \subset X'$. Now it is easy to see that

$$\rho_1(Y_0') \geq |X_0' \cap X''| \text{ and } \rho_2(Y_0') \geq |X_0'\backslash X''|.$$

Hence

$$\rho_1(Y_0') + \rho_2(Y_0') \geq |X_0' \cap X''| + |X_0' \setminus X''| = |X_0'| = |Y_0'|.$$

As this is true for each $Y_0' \subset Y'$, we have shown that Y' is a base of $M_1 \vee M_2$. □

We use theorem 3.1 to prove the following theorem, which was first obtained in terms of matroids by BRUALDI [69]: if B_1 and B_2 are bases of a matroid then there is an injection $\sigma: B_1 \to B_2$ such that $(B_2 \setminus \{\sigma(e)\}) \cup \{e\}$ is a base for each e in B_1 (see also KROGDAHL [75]).

THEOREM 3.2. *Let* (X,Y,Λ) *be a linking system and* (X,Y,E_Λ) *its underlying bipartite graph. Then for each pair* $(X',Y') \in \Lambda$ *there exists a matching in* (X,Y,E_Λ) *between* X' *and* Y'.

PROOF. We proceed by induction on $|X'|$. If $X' = \emptyset$ the result is trivial. Let $X' \neq \emptyset$ and suppose the theorem holds for all pairs $(X'',Y'') \in \Lambda$ with $|X''| < |X'|$. Take $x \in X'$. Then, by theorem 3.1, we can find a $y \in Y'$ such that $(\{x\},\{y\}) \in \Lambda$ and $(X'\setminus\{x\},Y'\setminus\{y\}) \in \Lambda$. Now, by induction, there is a matching in (X,Y,E_Λ) between $X'\setminus\{x\}$ and $Y'\setminus\{y\}$; since $(x,y) \in E_\Lambda$, also a matching exists between X' and Y'. □

Here we proved that for a linking system (X,Y,Λ) we always have $\Lambda \subset \Delta_{E_\Lambda}$ (or $\lambda \leq \delta_{E_\Lambda}$, where λ is the linking function of the linking system). It means that Δ_E is the maximum (under inclusion) of all linking systems with underlying bipartite graph (X,Y,E).

It is in general not true that, given a linking system (X,Y,Λ) and a linked pair $(X',Y') \in \Lambda$, there is an injection $\sigma: X' \to Y'$ such that for all $x \in X'$ we have that

$$(\{x\},\{\sigma(x)\}) \in \Lambda \quad \text{and} \quad (X'\setminus\{x\},Y'\setminus\{\sigma(x)\}) \in \Lambda.$$

Clearly this holds for deltoid and gammoid linking systems, but in general not for linking systems representable over a field, as can be seen easily by observing the linking system represented over GF(2) by the following matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Base orderable matroids are matroids with the property that for all bases B_1 and B_2 there is an injection $\sigma: B_1 \to B_2$ such that $B_1\setminus\{e\}\cup\{\sigma(e)\}$ and $B_2\setminus\{\sigma(e)\}\cup\{e\}$

are bases for all e in B_1. Now if the matroid obtained from a linking system (X,Y,Λ) in the sense of theorem 2.2 is base orderable then for each $(X',Y') \in \Lambda$ there is a bijection $\sigma\colon X' \to Y'$ such that

$$(\{x\},\{\sigma(x)\}) \in \Lambda \quad \text{and} \quad (X'\backslash\{x\},Y'\backslash\{\sigma(x)\}) \in \Lambda$$

for all $x \in X'$. The reverse does not hold: the linking system represented over GF(2) by the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

has a bijection as required for each linked pair, but the corresponding matroid is not base orderable.

Let (X,Y,Λ) be a linking system and let $(X',Y') \in \Lambda$. By an application of KÖNIG's theorem on matchings in bipartite graphs, there exists a bijection $\sigma\colon X' \to Y'$ such that

$$(\{x\},\{\sigma(x)\}) \in \Lambda \quad \text{and} \quad (X'\backslash\{x\},Y'\backslash\{\sigma(x)\}) \in \Lambda$$

for all $x \in X'$ if and only if for all $X'' \subset X'$ and $Y'' \subset Y'$ such that $|X''| + |Y''| > |X'|$ there are $x \in X''$ and $y \in Y''$ with the property

$$(\{x\},\{y\}) \in \Lambda \quad \text{and} \quad (X'\backslash\{x\},Y'\backslash\{y\}) \in \Lambda.$$

The above example shows that this condition is not always met, but a slightly weaker property is true: if $X'' \subset X'$ and $Y'' \subset Y'$ such that $|X''| + |Y''| > |X'|$, there are *non-empty* subsets X''' of X'' and Y''' of Y'' such that

$$(X''',Y''') \in \Lambda \quad \text{and} \quad (X'\backslash X''',Y'\backslash Y''') \in \Lambda.$$

This is equivalent to another exchange property for bases of GREENE [74b]: if B_1 and B_2 are bases of a matroid and $C \subset B_1$, $D \subset B_2$ such that $|C| + |D| > |B_1|$, then there are non-empty subsets C_0 of C and D_0 of D such that $(B_1\backslash C_0) \cup D_0$ and $(B_2\backslash D_0) \cup C_0$ are bases (this was pointed out to me by A. FRANK).

Next we prove that if there is exactly one matching in the underlying bipartite graph between two sets X' and Y' then (X',Y') is a linked pair. In matroid language this result was also found by KROGDAHL [75]. We first need a lemma.

LEMMA. *Let* (X,Y,Λ) *be a linking system with underlying bipartite graph* (X,Y,E_Λ). *Suppose* X' *and* X'' *are disjoint subsets of* X, *and* Y' *and* Y'' *are disjoint subsets of* Y. *Suppose furthermore:* $|X'| = |Y'|$ *and* $|X''| = |Y''|$, *and there is no edge between* X' *and* Y'' *(i.e.* $E_\Lambda \cap (X'\times Y'') = \emptyset$*). Then*

$$((X'\cup X''),(Y'\cup Y'')) \in \Lambda \text{ if and only if } (X',Y') \in \Lambda \text{ and } (X'',Y'') \in \Lambda.$$

PROOF.

(1) Suppose $(X'\cup X'',Y'\cup Y'') \in \Lambda$. By theorem 3.1 there exists a subset Y_0 of $Y' \cup Y''$ with the properties:

$$(X',Y_0) \in \Lambda \text{ and } (X'',(Y'\cup Y'')\backslash Y_0) \in \Lambda.$$

By theorem 3.2 there is a matching in E_Λ between X' and Y_0; since there is no edge between X' and Y'' it follows that $Y_0 \subset Y'$. But $|Y'| = |X'| = |Y_0|$; hence $Y_0 = Y'$ and

$$(X',Y') \in \Lambda \text{ and } (X'',Y'') \in \Lambda.$$

(2) Suppose $(X',Y') \in \Lambda$ and $(X'',Y'') \in \Lambda$. By axiom (iii) of definition 1.1 there is a $(X_0,Y_0) \in \Lambda$ such that:

$$X' \subset X_0 \subset X' \cup X'' \text{ and } Y'' \subset Y_0 \subset Y' \cup Y''.$$

According to theorem 3.2 there is a matching in E_Λ between X_0 and Y_0. Since there is no edge between X' and Y'', one has: $|X_0\backslash X'| \geq |Y''|$, or $X_0 = X' \cup X''$ and $Y_0 = Y' \cup Y''$. Therefore: $(X'\cup X'',Y'\cup Y'') \in \Lambda$. □

As a consequence we have

THEOREM 3.3. *Let* (X,Y,Λ) *be a linking system with underlying bipartite graph* (X,Y,E_Λ). *Let* $X' \subset X$ *and* $Y' \subset Y$ *be such that there is exactly one matching in* (X,Y,E_Λ) *between* X' *and* Y'. *Then* $(X',Y') \in \Lambda$.

PROOF. Again, we prove the theorem by induction on $|X'|$. If $X' = \emptyset$ the theorem is trivial. Let $X' \neq \emptyset$, and suppose the theorem holds for all pairs (X'',Y'') with $|X''| < |X'|$. Since there is exactly one matching between X' and Y', there exists, by a theorem of HALL [48] on the number of matchings in a bipartite graph, an $x \in X'$ such that there is only one $y \in Y'$ with $(x,y) \in E_\Lambda$. Consequently, there is exactly one matching between $X'\backslash\{x\}$ and $Y'\backslash\{y\}$. By induction we know

$$(X'\backslash\{x\},Y'\backslash\{y\}) \in \Lambda.$$

Also $(\{x\},\{y\}) \in \Lambda$ and there is no edge in E_Λ between $\{x\}$ and $Y'\backslash\{y\}$. Hence, by the foregoing lemma, $(X',Y') \in \Lambda$. □

In general it is not true that there is a minimum of all linking systems with underlying bipartite graph (X,Y,E); in particular the set of all pairs (X',Y') with the properties:

> $X' \subset X$, $Y' \subset Y$ and there is exactly one matching in (X,Y,E) between X' and Y',

in general does not form the set of linked pairs of a linking system (see the first non-example in section 1b).

We mention another consequence of the lemma, which says that a linking system is completely determined by the sub linking systems on the connected components of the underlying bipartite graph. This notion of component coincides with the usual concept of a component in the matroid, related to the linking system (in the sense of theorem 2.2). (This phenomenon was also noticed by KROGDAHL [75].)

THEOREM 3.4. *Let* (X,Y,Λ) *be a linking system with underlying bipartite graph* (X,Y,E_Λ). *Let* $X_1 \cup Y_1,\dots,X_n \cup Y_n$ *be the connected components of this bipartite graph (where* $X_1,\dots,X_n \subset X$ *and* $Y_1,\dots,Y_n \subset Y$*). For each* $i = 1,\dots,n$, *let* $X_i' \subset X_i$ *and* $Y_i' \subset Y_i$. *Then*

$$\left(\bigcup_{i=1}^{n} X_i', \bigcup_{i=1}^{n} Y_i'\right) \in \Lambda \text{ if and only if } (X_i',Y_i') \in \Lambda \text{ for each } i = 1,\dots,n.$$

PROOF. The theorem is an easy consequence of the lemma above. □

Let M be a matrix with column collection Z. Suppose X is a maximal set of linearly independent columns of M and put $Y = Z\backslash X$. Express each column in Y in terms of the columns in X, say

$$y = \sum_{x\in X} \phi(x,y)\cdot x.$$

The class of all matrices (X,Y,ϕ) obtained from some fixed matrix M in this way is called a *combivalence class* or a *pivotal system*; the matrices in a class can be obtained from each other by the pivot exchange procedure of linear programming (cf. TUCKER [63]).
The class of linking systems obtained from the matrices in a combivalence class forms the class of all linking systems with one and the same underlying matroid (in the sense of theorem 2.2).

IRI [69] proved the following: *the maximum rank for matrices in a combivalence class equals the minimum term rank for these matrices; moreover, for some matrix in the class, rank and term rank coincide*. Here the *term rank* of a matrix is the maximal number of non-zero entries which are pairwise in different rows and in different columns. This clearly equals the matching number of the underlying bipartite graph of the linking system obtained from the matrix.
FULKERSON observed and MAURER [75a] proved that IRI's result also holds for more general structures, namely, in some sense, for matroids. We state this in terms of linking systems.

THEOREM 3.5. *Let* M *be a matroid and let* $\mathbb{L}$ *be the class of all linking systems with underlying matroid* M. *Then the maximal size of the linked pairs of linking systems in* $\mathbb{L}$ *equals the minimal matching number of the underlying bipartite graphs of the linking systems in* $\mathbb{L}$. *Furthermore, every linking system in* $\mathbb{L}$ *with a linked pair of the maximum size has an underlying bipartite graph with minimal matching number.*

For a proof we refer to MAURER [75a].

CHAPTER FOUR

OPERATIONS ON LINKING SYSTEMS

The preceding chapter dealt with the intrinsical structure of linking systems; we now consider how linking systems are related to each other.

In section 4a we show how two linking systems can be connected to form a product and a union. The notions of nonsingular linking systems and their inverses are the extensions of the analogous matrix-concepts; they are introduced in section 4b.

4a. THE PRODUCT AND UNION OF LINKING SYSTEMS

In this section we define the notions of product and sum of two linking systems. The product of two linking systems is analogous to the product of a matroid and a linking system.

THEOREM 4.1. *Let* (X,Y,Λ_1) *and* (Y,Z,Λ_2) *be two linking systems, with linking functions* λ_1 *and* λ_2*, respectively. Define*

$$\Lambda_1*\Lambda_2 = \{(X',Z') \mid (X',Y') \in \Lambda_1 \text{ and } (Y',Z') \in \Lambda_2 \text{ for some } Y' \subset Y\}.$$

Then $(X,Z,\Lambda_1*\Lambda_2)$ *is again a linking system, with linking function given by*

$$(\lambda_1*\lambda_2)(X',Z') = \min_{Y'\subset Y}(\lambda_1(X',Y') + \lambda_2(Y\backslash Y',Z')).$$

PROOF. It is easy to show that $\lambda_1*\lambda_2$ as defined above satisfies the properties (ix), (x) and (xi) of theorem 1.2. Hence $\lambda_1*\lambda_2$ is indeed the linking function of a linking system, say of (X,Z,Λ). Now

$$(X',Z') \in \Lambda \quad \text{iff} \quad (\lambda_1*\lambda_2)(X',Z') = |X'| = |Z'|,$$

or

$$(X',Z') \in \Lambda \quad \text{iff} \quad \lambda_1(X',Y') + \lambda_2(Y\setminus Y',Z') \geq |X'| = |Z'|, \quad \text{for all } Y' \subset Y.$$

By an application of EDMONDS' intersection theorem to the matroids $(Y,\mathcal{P}(X')*\Lambda_1)$ and $(Y,\mathcal{P}(Z')*\Lambda_2^*)$ (cf. theorem 2.3) it follows that this last is the case iff there is an $Y' \subset Y$ such that

$$\lambda_1(X',Y') = \lambda_2(Y',Z') = |Y'| = |X'| = |Z'|,$$

i.e. iff $(X',Y') \in \Lambda_1$ and $(Y',Z') \in \Lambda_2$ for some $Y' \subset Y$. □

We shall call the linking system $(X,Z,\Lambda_1*\Lambda_2)$ the *product* of (X,Y,Λ_1) and (Y,Z,Λ_2).

In case the systems (X,Y,Λ_1) and (Y,Z,Λ_2) are deltoid linking systems, theorem 4.1 reduces to the following two results.

(i) (FORD & FULKERSON [58]) Two finite families $\mathcal{A} = (A_i \mid i\in I)$ and $\mathcal{B} = (B_i \mid i\in I)$ of sets have a common transversal if and only if for all $J \subset I$, $K \subset I$:

$$\left|\bigcup_{j\in J} A_j \cap \bigcup_{k\in K} B_k\right| \geq |J| + |K| - |I|;$$

(ii) (MIRSKY [68]) Let $\mathcal{A} = (A_i \mid i\in I)$ and $\mathcal{B} = (B_j \mid j\in J)$ be two finite families of sets and let $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subset \mathcal{B}$. Suppose $\mathcal{A}'$ and a subfamily of $\mathcal{B}$ have a common transversal, and similarly, $\mathcal{B}'$ and a subfamily of $\mathcal{A}$ have a common transversal. Then there are $\mathcal{A}'' \supset \mathcal{A}'$ and $\mathcal{B}'' \supset \mathcal{B}'$ with a common transversal.

(These theorems follow also from MENGER's graph theorem and from PERFECT's linkage theorem, cf. example (2) of section 1b.)

It is evident that if $(X,\mathcal{I})$ is a matroid and (X,Y,Λ_1), (Y,Z,Λ_2) and (Z,U,Λ_3) are linking systems, then

$$(\mathcal{I}*\Lambda_1)*\Lambda_2 = \mathcal{I}*(\Lambda_1*\Lambda_2)$$

and

$$(\Lambda_1*\Lambda_2)*\Lambda_3 = \Lambda_1*(\Lambda_2*\Lambda_3).$$

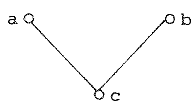
It is not difficult to show that the matroid corresponding to the linking system $(X,Z,\Lambda_1*\Lambda_2)$ (in the sense of theorem 2.2) equals the contraction to $X\cup Z$ of the union of the matroids corresponding to (X,Y,Λ_1) and (Y,Z,Λ_2)

(here we may assume that X,Y and Z are pairwise disjoint). PIFF & WELSH [70] have proved that for each pair of matroids there exists a natural number N such that if F is a field with more than N elements and both matroids are representable over F, then the union of the two matroids again is representable over F. In the light of the assertion above this result implies: *if* (X,Y,Λ_1) *and* (Y,Z,Λ_2) *are linking systems, then there is a natural number* N *such that: if* F *is a field with more than* N *elements and both* (X,Y,Λ_1) *and* (Y,Z,Λ_2) *are representable over* F*, then also the linking system* $(X,Z,\Lambda_1*\Lambda_2)$ *is representable over* F.

In general it is not true that the product of two linking systems generated by two matrices equals the linking system generated by the product of the two matrices. It is not even true that the product of two linking systems representable over a field F is again representable over F. To show this, take the linking system (X,Y,Λ) represented over GF(2) by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

Now $(X,X,\Lambda*\Lambda^*)$ is not representable over GF(2).
Also it is not true that the product of the deltoid linking systems (X,Y,Δ_{E_1}) and (Y,Z,Δ_{E_2}), say generated by the bipartite graphs (X,Y,E_1) and (Y,Z,E_2), equals $(X,Z,\Delta_{E_1E_2})$ (where (X,Z,E_1E_2) is the bipartite graph such that $(x,z) \in E_1E_2$ iff $(x,y) \in E_1$ and $(y,z) \in E_2$ for some $y \in Y$). That is, the product of two deltoid linking systems is not always again a deltoid linking system. E.g. let (X,Y,Λ) be the deltoid linking system obtained from the bipartite graph schematically represented by:



where $X = \{a,b\}$ and $Y = \{c\}$. Now $(X,X,\Lambda*\Lambda^*)$ is not a deltoid linking system. Of course, the product of two gammoid linking systems is again a gammoid linking system.
Furthermore we always have $E_{\Lambda_1*\Lambda_2} = E_{\Lambda_1}E_{\Lambda_2}$.

Now let (X_1,Y_1,Λ_1) and (X_2,Y_2,Λ_2) be linking systems (we have no requirement for the disjointness of X_1 and X_2, nor for the disjointness of Y_1 and Y_2). Put $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ and define

$$\Lambda_1 \vee \Lambda_2 = \{(X_1' \cup X_2', Y_1' \cup Y_2') \mid X_1' \cap X_2' = \emptyset = Y_1' \cap Y_2' \text{ and } (X_1', Y_1') \in \Lambda_1 \text{ and } (X_2', Y_2') \in \Lambda_2\}.$$

We call $(X,Y,\Lambda_1 \vee \Lambda_2)$ the *union* of (X_1,Y_1,Λ_1) and (X_2,Y_2,Λ_2). That this union is again a linking system is proved in the following theorem, in which we assume, without loss of generality, that $X_1 = X_2$ and $Y_1 = Y_2$.

THEOREM 4.2. *Let (X,Y,Λ_1) and (X,Y,Λ_2) be linking systems, with linking functions λ_1 and λ_2, respectively.*

Then $(X,Y,\Lambda_1 \vee \Lambda_2)$ is again a linking system, with linking function $\lambda_1 \vee \lambda_2$ given by

$$(\lambda_1 \vee \lambda_2)(X',Y') = \min_{\substack{X'' \subset X' \\ Y'' \subset Y'}} (|X' \setminus X''| + |Y' \setminus Y''| + \lambda_1(X'',Y'') + \lambda_2(X'',Y'')),$$

for $X' \subset X$ and $Y' \subset Y$.

PROOF. Let $\bar{X}$ and $\underline{X}$ be two disjoint copies of X, and, similarly, let $\bar{Y}$ and $\underline{Y}$ be disjoint copies of Y. Let $(\bar{X},\bar{Y},\bar{\Lambda}_1)$ and $(\underline{X},\underline{Y},\underline{\Lambda}_2)$ be the corresponding linking systems. It is easy to see that $(\bar{X}\cup\underline{X},\bar{Y}\cup\underline{Y},\bar{\Lambda}_1 \vee \underline{\Lambda}_2)$ is a linking system, with linking function λ given by

$$\lambda(\bar{X}_1 \cup \underline{X}_2, \bar{Y}_1 \cup \underline{Y}_2) = \lambda_1(X_1,Y_1) + \lambda_2(X_2,Y_2),$$

for $X_1,X_2 \subset X$ and $Y_1,Y_2 \subset Y$.
Define in addition the linking systems $(X,\bar{X}\cup\underline{X},\Lambda_3)$ and $(\bar{Y}\cup\underline{Y},Y,\Lambda_4)$ by

$$\Lambda_3 = \{(X_1 \cup X_2, \bar{X}_1 \cup \underline{X}_2) \mid X_1,X_2 \subset X,\ X_1 \cap X_2 = \emptyset\}$$

and

$$\Lambda_4 = \{(\bar{Y}_1 \cup \underline{Y}_2, Y_1 \cup Y_2) \mid Y_1,Y_2 \subset Y,\ Y_1 \cap Y_2 = \emptyset\}.$$

The corresponding linking functions λ_3 and λ_4, are given by

$$\lambda_3(X', \bar{X}_1 \cup \underline{X}_2) = |X' \cap (X_1 \cup X_2)|,$$

for $X',X_1,X_2 \subset X$, and

$$\lambda_4(\bar{Y}_1 \cup \underline{Y}_2, Y') = |Y' \cap (Y_1 \cup Y_2)|,$$

for $Y', Y_1, Y_2 \subset Y$. Now $(X,Y,\Lambda_1 \vee \Lambda_2) = (X,Y,\Lambda_3 * (\bar{\Lambda}_1 \vee \underline{\Lambda}_2) * \Lambda_4)$, hence the union of two linking systems is again a linking system. Furthermore, its linking function follows from straightforward calculations. □

Since the union operator is associative, the union of several linking systems is again a linking system. We can apply this to the case where all linking systems are the same. Let (X,Y,Λ) be a linking system, with linking function λ and $|X| = |Y|$; we want to find the minimal number of disjoint linked pairs, necessary to cover X and Y. That is, the minimal k such that there are $(X_1,Y_1),\ldots,(X_k,Y_k) \in \Lambda$ with the property that $(X_1,\ldots,X_k)$ is a partition of X and $(Y_1,\ldots,Y_k)$ is a partition of Y. Now it is possible to split up X and Y in this way if and only if $(X,Y) \in \underbrace{\Lambda \vee \ldots \vee \Lambda}_{k \text{ times}}$, or iff

$$|X| = |Y| = \min_{\substack{X' \subset X \\ Y' \subset Y}} (|X \setminus X'| + |Y \setminus Y'| + k \cdot \lambda(X',Y'));$$

this last identity holds if and only if $|X'| + |Y'| - |X| \leq k \cdot \lambda(X',Y')$ for all $X' \subset X$ and $Y' \subset Y$. Hence the minimal k with the required property equals

$$\max \left\{ \left\lceil \frac{|X'|+|Y'|-|X|}{\lambda(X',Y')} \right\rceil \;\middle|\; X' \subset X,\ Y' \subset Y,\ |X'| + |Y'| > |X| \right\}.$$

(Here $\lceil x \rceil$ denotes the least integer not less than x; notice that k is finite iff $(X,Y) \in \Delta_{E_\Lambda}$.)

4b. THE INVERSE OF A NONSINGULAR LINKING SYSTEM

In this section we lift the notion of nonsingularity of matrices to linking systems. This can be accomplished by defining a linking system (X,Y,Λ) to be *nonsingular* if $(X,Y) \in \Lambda$. It follows that in a nonsingular linking system (X,Y,Λ) one always has $|X| = |Y|$. The concept of the inverse of a non-singular matrix is reflected in the following definition. The *inverse* system of the nonsingular linking system (X,Y,Λ) is the system (Y,X,Λ^{-1}), where

$$\Lambda^{-1} = \{(Y',X') \mid (X \setminus X', Y \setminus Y') \in \Lambda\}$$

We first prove that such an inverse system is again a linking system.

THEOREM 4.3. *Let* (X,Y,Λ) *be a nonsingular linking system, with linking function* λ. *Then* (Y,X,Λ^{-1}) *is again a linking system, with linking function* λ^{-1} *given by*

$$\lambda^{-1}(Y',X') = \lambda(X\backslash X',Y\backslash Y') + |X'| + |Y'| - |X|,$$

for $Y' \subset Y$ *and* $X' \subset X$.

PROOF. λ^{-1} satisfies properties (ix), (x) and (xi) of theorem 1.2. Hence λ^{-1} is the linking function of a linking system. Furthermore, $\lambda^{-1}(Y',X') = |Y'| = |X'|$ if and only if $\lambda(X\backslash X',Y\backslash Y') = |X\backslash X'| = |Y\backslash Y'|$. Hence the corresponding linking system is (Y,X,Λ^{-1}). □

An alternative method of proof consists of an application of theorem 2.2, observing that Y is also a base of the matroid corresponding to the nonsingular linking system (X,Y,Λ).

Clearly, the inverse of a nonsingular linking system is always nonsingular again, and $(\Lambda^{-1})^{-1} = \Lambda$.

The notion of inverse of a nonsingular linking system indeed generalizes the notion of inverse of a nonsingular matrix. Let $M = (X,Y,\phi)$ be a nonsingular matrix over F and let $M^{-1} = (Y,X,\phi^{-1})$ its inverse. That is, for all $x \in X$ we have

$$\sum_{y\in Y} \phi(x,y)\cdot\phi^{-1}(y,x) = 1$$

and for all $x_1 \neq x_2$ in X

$$\sum_{y\in Y} \phi(x_1,y)\cdot\phi^{-1}(y,x_2) = 0.$$

We prove that $\Lambda_{\phi^{-1}} = (\Lambda_\phi)^{-1}$, or, what is the same, $M|X'\times Y'$ is nonsingular if and only if $M^{-1}|(Y\backslash Y') \times (X\backslash X')$ is nonsingular. Suppose $M|X'\times Y'$ is nonsingular. Write

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where $M_1 = M|X'\times Y'$. Similarly, write

$$M^{-1} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} ,$$

where $N_4 = M^{-1}|(Y\backslash Y') \times (X\backslash X')$. Now $M \circ M^{-1}$ is an identity matrix, hence $M_1N_2 + M_2N_4$ is a zero-matrix and $M_3N_2 + M_4N_4$ is an identity matrix. Let M_1^{-1} be the inverse matrix of the nonsingular M_1; then $N_2 = - M_1^{-1}M_2N_4$ and $-M_3M_1^{-1}M_2N_4 + M_4N_4$ is an identity matrix. Therefore N_4 is nonsingular. The proof of the converse is similar.

To investigate nonsingular deltoid and gammoid linking systems we introduce the notion of a *strict gammoid linking system*. Let (Z,E) be a digraph. Then the linking system (Z,Z,Γ_E) is called a strict gammoid linking system. These systems are always nonsingular, and each gammoid linking system is a sub linking system of a strict one.

Let (Z,E) be a digraph. Define the bipartite graph $(\bar{Z},\underline{Z},D)$, where $\bar{Z}$ and $\underline{Z}$ are two disjoint copies of Z, by

$$(\bar{x},\underline{y}) \in D \quad \text{iff} \quad x = y \text{ or } (x,y) \in E.$$

The fundamental lemma (the linkage lemma) of INGLETON & PIFF [73] now says the following for X, Y $\subset$ Z:

X and Y are matched in E iff $\bar{Z}\backslash\bar{Y}$ and $\underline{Z}\backslash\underline{X}$ are matched in D,

or

$$(X,Y) \in \Gamma_E \text{ iff } (\bar{Z}\backslash\bar{Y},\underline{Z}\backslash\underline{X}) \in \Delta_D.$$

This means that the deltoid linking system $(\bar{Z},\underline{Z},\Delta_D)$ is isomorphic to the inverse of the strict gammoid linking system (Z,Z,Γ_E). Conversely, each nonsingular deltoid linking system may be understood as induced by a bipartite graph derived from a directed graph in the above manner. Thus we have that a linking system is a strict gammoid linking system iff it is the inverse of a nonsingular deltoid linking system. Hence to prove that each gammoid linking system is representable over some field it is sufficient to prove that each deltoid linking system is representable over some field (see section 5c).

An easy generalization of the proof of theorem 3.7 of INGLETON & PIFF [73] (cf. WELSH [76] p. 225) shows the following. Let $M = (X,\mathcal{I})$ be a matroid, with dual $M^* = (X,\mathcal{I}^*)$, and let (X,Y,Λ) be a nonsingular linking system. Then the dual of the matroid $(Y,\mathcal{I}*\Lambda)$ equals the matroid $(Y,\mathcal{I}^**(\Lambda^{-1})^*)$ (this is clear by the trivial observation that $(X',Y') \in \Lambda$ with X' a base of M iff $(Y\backslash Y',X\backslash X') \in \Lambda^{-1}$ with $X\backslash X'$ a base of M^*).

BONDY & WELSH [71] introduced the notion of *identically self-dual* matroids. These are matroids M such that $M^* = M$, i.e. if B is a base of M then also the complement of B is a base of M. Therefore the linking system associated with an identically self-dual matroid is nonsingular and the inverse linking system equals the dual linking system. The search for identically self-dual matroids becomes then the search for nonsingular linking systems (X,Y,Λ) with the property that $\Lambda^{-1} = \Lambda^*$, or

$$(X',Y') \in \Lambda \quad \text{iff} \quad (X\backslash X',Y\backslash Y') \in \Lambda.$$

GRAVER [75] determined all k for which there exists a connected binary identically self-dual matroid on 2k elements (i.e. of rank k). He found that these exists for all k except k = 2,3 or 5. In the light of the above and of theorem 3.4 this problem is the same as the one of determining all k for which there is a $k \times k$-matrix M over GF(2) such that $M \cdot M^T = I$ and the underlying bipartite graph of the linking system obtained from M is connected. For k even, $k \geq 4$ we can take $M = J_k - I_k$ (where J_k is the all-one matrix and I_k the identity matrix of size k). For k odd, $k \geq 7$ we can take

$$M = \begin{pmatrix} & & & 0 & 0 & 0 & 1 \\ & & & \cdot & \cdot & \cdot & \cdot \\ & J_t - I_t & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & 0 & 0 & 0 & 1 \\ 1 & \cdots & 1 & 0 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 0 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

where $t = k - 4$.

A $k \times k$-matrix M over GF(2) satisfies the required properties if and only if the rows of the matrix (I_k,M) generate a linear code C which is not the direct sum of other codes and which is such that the dual code $C^{\perp}$ equals C.

CHAPTER FIVE

DELTOIDS, GAMMOIDS AND REPRESENTABILITY

Up to now we have considered mainly general linking systems. In this chapter we investigate properties of particular examples of linking systems, namely of deltoid, gammoid and representable linking systems. Section 5a deals with representable linking systems, section 5b with deltoid and gammoid linking systems and section 5c with the representability of deltoid and gammoid linking systems.

5a. REPRESENTABILITY

Recall that a linking system (X,Y,Λ) is called representable over a field F if $\Lambda = \Lambda_\phi$ for some F-matrix (X,Y,ϕ), and that (X,Y,Λ) is binary if it is representable over GF(2). As is the case with matroids, we call a linking system *regular* if it is representable over every field.

TUTTE [58,59] proved that a matroid M is regular iff M is binary and has no minor isomorphic either to the Fano matroid or to its dual. TUTTE [65] also proved that a matroid M is regular iff M has a totally unimodular matrix representation (over $\mathbb{Q}$).

(A matrix is a *representation* for M if M is isomorphic to the matroid induced on the set of columns by linear independence. A matrix is *totally unimodular* if every square submatrix has determinant 0, +1 or -1.) Furthermore, TUTTE [65] showed that a matroid M is binary iff M has no minor isomorphic to $U_{2,4}$ (that is, the uniform matroid of rank 2 on a 4-set). From TUTTE's first characterization of regularity it follows that a matroid M is regular iff M is representable over GF(2) and some other field not of characteristic 2 (since the Fano-matroid is representable only over fields of characteristic 2).

Now we shall look at TUTTE's results in the light of linking systems. TUTTE's forbidden minor characterizations are not easy to translate to linking systems, since the concept of a minor does not have an easy translation. If we take the sub linking system (X',Y',Λ') of a linking system (X,Y,Λ), then in the sense of the underlying matroid, we contract $X\backslash X'$ and remove $Y\backslash Y'$; but restrictions of X are not simple to represent in terms of the linking system, and the same holds for contractions of Y. Hence we confine our attention to forbidden sub linking system characterizations instead of forbidden minor characterizations.

For a forbidden sub linking system characterization for binary linking systems we need the following two collections. $\mathcal{M}$ is the collection of all singular square matrices over GF(2) such that the minor of each element is nonsingular. That is, $\mathcal{M}$ consists of all matrices M of the form

$$M = \left(\begin{array}{c|c} & a_1 \\ & \vdots \\ M' & \vdots \\ & a_{n-1} \\ \hline b_1 \cdots\cdots b_{n-1} & c \end{array}\right)$$

where M' is a nonsingular $(n-1) \times (n-1)$-matrix over GF(2), a_i equals the sum of the elements in row i of M', b_i equals the sum of the elements of column i $(1 \le i \le n-1)$, and c equals the sum of all elements of M' (modulo 2). The set $\mathcal{L}$ consists of all linking systems (X,Y,Λ) such that there is a linking system (X,Y,Λ_1) induced by a matrix in $\mathcal{M}$ with the property that $\Lambda = \Lambda_1 \cup \{(X,Y)\}$. It is easy to see that each matrix in $\mathcal{M}$ generates in this way a linking system. We use this to characterize binary linking systems.

THEOREM 5.1. *A linking system* (X,Y,Λ) *is binary iff it has no sub linking system isomorphic to an element in* $\mathcal{L}$.

PROOF. Let (X,Y,Λ) be a binary linking system. Since each sub linking system is again binary and no element of $\mathcal{L}$ is binary (a binary linking system is determined by its underlying bipartite graph) (X,Y,Λ) has no sub linking system isomorphic to an element in $\mathcal{L}$.

Suppose now that (X,Y,Λ) is not binary, but each proper sub linking system (X,Y,Λ) is binary. Let $M = (X,Y,\phi)$ be the matrix over GF(2) with $\phi(x,y) = 1$ iff $(\{x\},\{y\}) \in \Lambda$, for $x \in X$ and $y \in Y$, and let (X,Y,Λ_ϕ) be the linking system generated by the matrix M. Now for $(X',Y') \neq (X,Y)$ we have $(X',Y') \in \Lambda$ if and only if $(X',Y') \in \Lambda_\phi$, since the sub linking system (X',Y',Λ') of (X,Y,Λ) is binary and hence is generated by the matrix $M|X' \times Y'$.
But $\Lambda \neq \Lambda_\phi$, for (X,Y,Λ) itself is not binary.
First suppose $(X,Y) \in \Lambda_\phi$ and $(X,Y) \notin \Lambda$. As in this case (X,Y,ϕ) is nonsingular, (X,Y,ϕ) has an inverse (Y,X,ψ). Choose distinct y_1 and y_2 in Y and distinct x_1 and x_2 in X such that $\psi(y_1,x_1) = \psi(y_2,x_2) = 1$. This means: $(X\backslash\{x_1\},Y\backslash\{y_1\}) \in \Lambda_\phi$ and $(X\backslash\{x_2\},Y\backslash\{y_2\}) \in \Lambda_\phi$, and hence both pairs are also in Λ. But since $(X,Y) \notin \Lambda$, we have $(X\backslash\{x_1\},Y\backslash\{y_2\}) \in \Lambda$ and $(X\backslash\{x_2\},Y\backslash\{y_1\}) \in \Lambda$ (axiom (iii) for a linking system). These two pairs are therefore also in Λ_ϕ and hence $\psi(y_1,x_2) = \psi(y_2,x_1) = 1$. Since x_1,x_2,y_1,y_2 were chosen arbitrarily we find that (Y,X,ϕ) has the form

$$\begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & & 1 & 0 & & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

which is (almost) impossible for a nonsingular matrix.
Therefore we may assume that $(X,Y) \in \Lambda$ and $(X,Y) \notin \Lambda_\phi$. We are ready once we have proved that $M \in \mathcal{M}$. Since for each proper subset X' of X there is a $Y' \subset Y$ such that $(X',Y') \in \Lambda_\phi$, and for each proper subset Y' of Y there is an $X' \subset X$ such that $(X',Y') \in \Lambda_\phi$, we have that the sum of the rows equals the zero vector and the sum of the columns equals the zero vector. Also for some $x \in X$ and $y \in Y$ $M|(X\backslash\{x\}) \times (Y\backslash\{y\})$ is nonsingular. Therefore $M \in \mathcal{M}$. □

The next theorem follows straightforwardly from TUTTE's results mentioned above. However, we give a proof using the following theorem on totally unimodular matrices proved by GOMORY (see CAMION [65]): a matrix with entries 0, +1 or −1 is totally unimodular (over $\mathbb{Q}$) iff no square submatrix has determinant equal to +2 or −2.

<u>THEOREM 5.2</u>. *Let* (X,Y,Λ) *be a linking system. Then the following assertions are equivalent:*

(i) (X,Y,Λ) *is regular,*

(ii) (X,Y,Λ) *is representable over* GF(2) *and* GF(3);

(iii) (X,Y,Λ) *is representable over* $\mathbb{Q}$ *by a totally unimodular matrix.*

PROOF. (i) $\rightarrow$ (ii) is obvious.
(ii) $\rightarrow$ (iii). Let $M = (X,Y,\phi)$ be a matrix representation over GF(2) and let $M' = (X,Y,\psi)$ be a matrix representation over GF(3) for (X,Y,Λ). Conceiving M' as a matrix over $\mathbb{Q}$, we prove that M' is totally unimodular and that (X,Y,Λ) is represented over $\mathbb{Q}$ by M'.
Suppose $M'|X' \times Y'$ has determinant equal to +2 or -2 (in $\mathbb{Q}$). Since $\phi(x,y) = 0$ if and only if $\psi(x,y) = 0$, the determinant of $M|X' \times Y'$ equals 0 (in GF(2)), but the determinant of $M'|X' \times Y'$ in GF(3) is not 0. This is in contradiction with the fact that both M and M' are representations for (X,Y,Λ). Hence, by GOMORY's result, M' is totally unimodular.
Let $(X',Y') \in \Lambda$. Then $\det(M'|X'\times Y') \neq 0$ (in GF(3)) and therefore $\det(M'|X'\times Y') \neq 0$ (in $\mathbb{Q}$), so $M'|X' \times Y'$ is nonsingular (in $\mathbb{Q}$).
If $M'|X' \times Y'$ is nonsingular (in $\mathbb{Q}$), then $\det(M'|X'\times Y') = \pm 1$ (in $\mathbb{Q}$) and hence $\det(M'|X'\times Y') \neq 0$ (in GF(3)).
So M' is a totally unimodular matrix representation for (X,Y,Λ).
(iii) $\rightarrow$ (i). For each prime number p the totally unimodular matrix representation for (X,Y,Λ) also is a matrix representation over GF(p) since for totally unimodular matrices M we have

$$\det(M|X'\times Y') = 0 \quad \text{iff} \quad \det(M|X'\times Y') \equiv 0 \pmod{p}. \qquad \square$$

In order to give a forbidden sub linking system characterization for regularity we can restrict ourselves to finding minimal binary linking systems which are not regular, since if the forbidden sub linking system is not binary it is an element of L and conversely (cf. theorem 5.1). Hence we have to find minimal (0,1)-matrices such that we can not change some ones in -1 to get a totally unimodular matrix. E.g. the following matrices are of this type:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

We do not know a characterization of these matrices.

5b. DELTOID AND GAMMOID LINKING SYSTEMS

We first recall some definitions and results. Let (X,Y,E) be a bipartite graph. Following INGLETON & PIFF [73] we call the matroid on $X \cup Y$ with collection of bases

$$\mathcal{B} = \{(X\backslash X') \cup Y' \mid X' \text{ and } Y' \text{ are matched in } E\}$$

a *(strict) deltoid*. The base X is called a *principal base* of the deltoid. Deltoids are called by BONDY & WELSH [71] *fundamental transversal matroids*, by BRUALDI & DINOLT [75] *principal pregeometries* and by BRYLAWSKI [75a] *principal transversal matroids*.

Deltoids can be characterized as transversal matroids, the independent sets of which are the partial transversals of the collection of fundamental cocircuits with respect to some cobase.

Clearly, the dual matroid of a deltoid again is a deltoid (cf. LAS VERGNAS [70]) and a matroid is a transversal matroid iff it is a restriction of a deltoid. Moreover, a matroid is a deltoid iff it is the underlying matroid of a deltoid linking system (in the sense of theorem 2.2). BONDY & WELSH [71] showed that the collection of bases of a matroid always is the intersection of bases-collections of deltoids. For some other results on deltoids we refer to DOWLING & KELLY [74], BRUALDI [74], BRUALDI & DINOLT [75] and BRYLAWSKI [75a].

Now let (Z,E) be a directed graph and let X and Y be subsets of Z. Let $\mathcal{J}$ be the collection

$$\mathcal{J} = \{Y' \subset Y \mid Y' \text{ is matched in E with some subset of } X\}.$$

Then, as PERFECT [68] proved, $(Y,\mathcal{J})$ is a matroid and following MASON [72] the matroids obtained in this way are called *gammoids*. In case $Y = Z$ the matroid is called a *strict gammoid*. Hence a matroid is a gammoid iff it is a restriction of a strict gammoid. INGLETON & PIFF [73] proved:

(i) a matroid is a strict gammoid iff its dual is a transversal matroid;
(ii) a matroid is a strict gammoid iff it is a contraction of a deltoid;
(iii) a matroid is a gammoid iff it is a minor of a deltoid.

(Partial results in this direction were obtained earlier by MASON [72].)

The class of gammoids is the class of matroids induced by a gammoid linking system (X,Y,Γ_E) on the set Y in the sense of proposition 2.1. But this class is the same as the class of matroids obtained from gammoid linking systems in the sense of theorem 2.2. That is, without restriction on the generality we may suppose that $X \subset Y$ in the above definition of a gammoid. Furthermore, each base of the gammoid can play the rôle of X, i.e. let $(W,\mathcal{J})$ be a gammoid and let $X \subset W$ be a base of $(W,\mathcal{J})$; then there exists a directed graph (Z,E) such that $W \subset Z$ and $(W,\mathcal{J})$ is the matroid obtained from the gammoid linking system $(X,W\backslash X,\Gamma_E)$ (in the sense of theorem 2.2).

This follows from a result of MASON [72] which says that for each base X of a strict gammoid $(Z,\mathcal{I})$ there is a directed graph (Z,E) such that

$$\mathcal{I} = \{Z' \subset Z \mid \exists X' \subset X\colon X' \text{ and } Z' \text{ are matched in } E\}.$$

Using the methods developed by INGLETON & PIFF [73] this is not difficult to prove. Now let $(W,\mathcal{J})$ be a gammoid and X one of its bases. Suppose $(W,\mathcal{J})$ is the restriction of the strict gammoid $(Z,\mathcal{I})$. X is contained in a base of $(Z,\mathcal{I})$, say in X_0; it follows that $X_0 \cap W = X$. MASON's result implies that there exists a directed graph (Z,E) such that

$$\mathcal{I} = \{Z' \subset Z \mid \exists X' \subset X_0\colon X' \text{ and } Z' \text{ are matched in } E\};$$

therefore

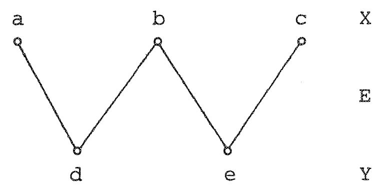
$$\mathcal{J} = \{W' \subset W \mid \exists X' \subset X_0\colon X' \text{ and } W' \text{ are matched in } E\}.$$

We prove that

$$\mathcal{J} = \{W' \subset W \mid \exists X' \subset X\colon X' \text{ and } W' \text{ are matched in } E\},$$

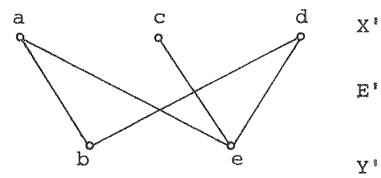
whence $(W,\mathcal{J})$ is the matroid obtained from the gammoid linking system $(X,W\backslash X,\Gamma_E)$ (in the sense of theorem 2.2). Assume X' and W' are matched in E for some $X' \subset X_0$. So there are $|W'|$ pairwise vertex-disjoint paths starting in X_0 and ending in W'. We may assume that every path intersects X_0 exactly once (namely at the start). Suppose one of the paths starts in $x \in X_0\backslash X$; its endpoint y is an element of $W\backslash X$. But then $X \cup\{y\} \in \mathcal{I}$ since $X \cup\{x\}$ and $X \cup\{y\}$ are matched in E. As $(W,\mathcal{J})$ is the restriction of $(Z,\mathcal{I})$ and $X \cup\{y\} \subset W$ it follows that $X \cup\{y\} \in \mathcal{J}$ contradicting the fact that X is a base of $(W,\mathcal{J})$.

In the deltoid case we meet with a different state of affairs. Not every base of a deltoid is a principal base. For example, let (X,Y,E) be the bipartite graph schematically represented by the figure



Let $\mathcal{B}$ be the set of bases of the deltoid generated by this bipartite graph. Take X' = {a,c,d} and Y' = {b,e}. Then $X' \in \mathcal{B}$ but X' is not a principal base. If X' were a principal base there would exist a bipartite graph (X',Y',E') generating the deltoid.

Since {b,c,d}, {c,d,e}, {a,d,e}, {a,b,c}, {a,c,e} are in $\mathcal{B}$, it follows that (a,b), (a,e), (c,e), (d,b), (d,e) are in E'.



But now {a,d} and {b,e} are matched in E', hence {b,c,e} should be a base of the deltoid, which is not the case since {a} and {e} are not matched in E.

Below we characterize the principal bases of a deltoid, given one principal base and the corresponding bipartite graph. To this end let (X,Y,E) be a bipartite graph. Colour *red* each edge $(x,y) \in E$ with the following property:

$$\text{if } (x',y) \in E \quad \text{and} \quad (x,y') \in E \quad \text{then} \quad (x',y') \in E.$$

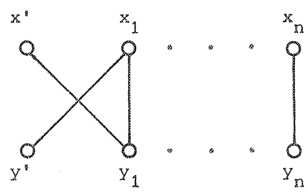
Let E_R be the set of all red edges of (X,Y,E). So (X,Y,E_R) is a sub bipartite graph of (X,Y,E). We call a matching in (X,Y,E_R), that is a matching consisting of red edges, a *red* matching.

The bipartite graph (X,Y,E_R) is such that each component is a complete bipartite graph. (This can be proved straightforwardly; some other properties of the red sub bipartite graph will be dealt with following theorem 5.3.)

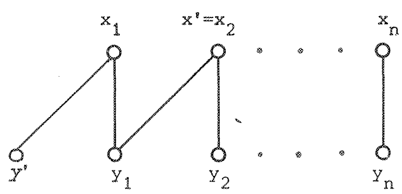
The next theorem characterizes principal bases of a deltoid.

<u>THEOREM 5.3</u>. *Let* (X,Y,E) *be a bipartite graph, and let* $X' \subset X$ *and* $Y' \subset Y$. *Then* $(X\backslash X') \cup Y'$ *is a principal base of the deltoid generated by* (X,Y,E) *if and only if there is a red matching between* X' *and* Y'.

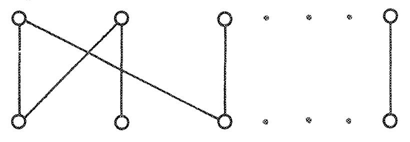
<u>PROOF</u>. Let $\mathcal{B}$ be the collection of bases of the deltoid.
(i) First suppose $(X\backslash X') \cup Y'$ is a principal base of the deltoid and let $E_0 \subset E$ be a matching between X' and Y'. We shall prove $E_0 \subset E_R$. Define $\bar{X} = (X\backslash X') \cup Y'$ and $\bar{Y} = (Y\backslash Y') \cup X'$. Since $\bar{X}$ is a principal base there exists a bipartite graph $(\bar{X},\bar{Y},\bar{E})$ generating the deltoid. Write $E_0 = \{(x_1,y_1),\ldots,(x_n,y_n)\}$. It follows that $\{(y_1,x_1),\ldots,(y_n,x_n)\} \subset \bar{E}$ (since $(\bar{X}\backslash\{y_i\}) \cup \{x_i\} = (X\backslash(X'\backslash\{x_i\})) \cup (Y'\backslash\{y_i\}) \in \mathcal{B}$ for $i = 1,\ldots,n$, as $X'\backslash\{x_i\}$ and $Y'\backslash\{y_i\}$ are matched in E). Now take $(x,y) \in E_0$, say $(x,y) = (x_1,y_1)$, and suppose $(x',y_1) \in E$ and $(x_1,y') \in E$. In order to prove $(x',y') \in E$ we distinguish four cases.
(a) $x' \notin X'$ and $y' \notin Y'$.



In this case $X' \cup \{x'\}$ and $Y' \cup \{y'\}$ are matched in E; hence $(X\backslash(X'\cup\{x'\})) \cup Y' \cup\{y'\} = (\bar{X}\backslash\{x'\}) \cup\{y'\} \in \mathcal{B}$. This implies that $\{x'\}$ and $\{y'\}$ are matched in $\bar{E}$, i.e. $(x',y') \in \bar{E}$. Therefore $Y' \cup\{x'\}$ and $X' \cup\{y'\}$ are matched in $\bar{E}$, or $(\bar{X}\backslash(Y'\cup\{x'\})) \cup X' \cup\{y'\} = (X\backslash\{x'\}) \cup\{y'\} \in \mathcal{B}$. Hence $(x',y') \in E$.
(b) $x' \in X'$ and $y' \notin Y'$, say $x' = x_2$.



Now X' and $(Y'\backslash\{y_2\})\cup\{y'\}$ are matched in E, hence $(X\backslash X') \cup (Y'\backslash\{y_2\}) \cup\{y'\} = (\bar{X}\backslash\{y_2\}) \cup\{y'\} \in \mathcal{B}$. Then $(y_2,y') \in \bar{E}$; therefore Y' and $(X'\backslash\{x_2\}) \cup\{y'\}$ are matched in $\bar{E}$. This implies $(\bar{X}\backslash Y') \cup (X'\backslash\{x_2\}) \cup\{y'\} = (X'\backslash\{x_2\}) \cup\{y'\} \in \mathcal{B}$, from which it follows that $(x_2,y') = (x',y') \in E$.
(c) The case $x' \notin X'$ and $y' \in Y'$ is treated similarly to case (b).
(d) $x' \in X'$ and $y' \in Y'$, say $x' = x_2$ and $y' = y_3$ (if $(x',y') \in E_0$ we would be ready).



In this case $X'\setminus\{x_3\}$ and $Y'\setminus\{y_2\}$ are matched in E, hence $(X\setminus(X'\setminus\{x_3\})) \cup (Y'\setminus\{y_2\}) = (\bar{X}\setminus\{y_2\}) \cup\{x_3\} \in \mathcal{B}$. So $(y_2,x_3) \in \bar{E}$; therefore $Y'\setminus\{y_3\}$ and $X'\setminus\{x_2\}$ are matched in $\bar{E}$. This implies that $(\bar{X}\setminus(Y'\setminus\{y_3\})) \cup (X'\setminus\{x_2\}) = (X\setminus\{x_2\}) \cup\{y_3\} \in \mathcal{B}$, from which it follows that $(x_2,y_3) = (x',y') \in E$.

(ii) Secondly, suppose that there exists a red matching

$$E_0 = \{(x_1,y_1),\dots,(x_n,y_n)\} \subset E_R$$

between X' and Y'. Again, let $\bar{X} = (X\setminus X') \cup Y'$ and $\bar{Y} = (Y\setminus Y') \cup X'$. Let $\bar{E}$ be the set of pairs

(x,y),	if	$x \notin X'$, $y \notin Y'$, and $(x,y) \in E$,
(x,x_i),	if	$x \notin X'$, and $(x,y_i) \in E$,
(y_i,y),	if	$y \notin Y'$, and $(x_i,y) \in E$,
(y_i,x_j),	if	$(x_i,y_j) \in E$.

Notice that $(\bar{X},\bar{Y},\bar{E})$ is isomorphic to (X,Y,E); they pass into each other by interchanging the labels x_i and y_i $(i = 1,\dots,n)$.

We prove that the deltoid generated by $(\bar{X},\bar{Y},\bar{E})$ is the same as the deltoid generated by (X,Y,E). So we prove that $\bar{X}'$ and $\bar{Y}'$ are matched in $\bar{E}$ iff $(\bar{X}\setminus\bar{X}') \cup \bar{Y}' \in \mathcal{B}$, for $\bar{X}' \subset \bar{X}$ and $\bar{Y}' \subset \bar{Y}$.

Therefore, take $X'' \subset X\setminus X'$, $Y_1 \subset Y'$, $Y'' \subset Y\setminus Y'$ and $X_2 \subset X'$, and suppose $X'' \cup Y_1$ and $Y'' \cup X_2$ are matched in $\bar{E}$. Putting X_1 as the set of all x_i such that $y_i \in Y_1$, and Y_2 as the set of all y_i such that $x_i \in X_2$ $(i = 1,\dots,n)$, we have, by definition of $\bar{E}$, that $X'' \cup X_1$ and $Y'' \cup Y_2$ are matched in E.

We have to prove that this is the case if and only if $(\bar{X}\setminus(X''\cup Y_1)) \cup (Y''\cup X_2) = (X\setminus(X''\cup(X'\setminus X_2))) \cup (Y'' \cup (Y'\setminus Y_1)) \in \mathcal{B}$, that is, if and only if $X''\cup(X'\setminus X_2)$ and $Y''\cup(Y'\setminus Y_1)$ are matched in E.

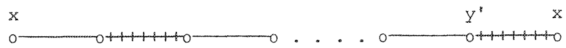
So we have to prove:

$X'' \cup X_1$ and $Y'' \cup Y_2$ are matched in E iff $X''\cup(X'\setminus X_2)$ and $Y''\cup(Y'\setminus Y_1)$ are matched in E.

Let $E_1 \subset E$ be a matching between $X'' \cup X_1$ and $Y'' \cup Y_2$, and consider the components of the bipartite graph $(X,Y,E_0 \cup E_1)$.

There are five possible types of components.

(a)

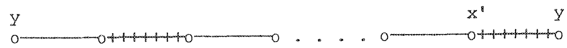


with $x \in X''$, $y' \in Y_2 \setminus Y_1$, $x' \in X_2 \setminus X_1$.

(In this schematical representation a drawn line is an edge of E_1, while a crossed line is a (red) edge of E_0.)

Now colour green the edge $(x,y') \in E$ (the red edges of the component enforce that (x,y') indeed is in E).

(b)



with $y \in Y''$, $x' \in X_1 \setminus X_2$, $y' \in Y_1 \setminus Y_2$. In this case colour green the edge $(x',y) \in E$.

(c)

with $x \in X''$ and $y \in Y''$ (possibly $(x,y) \in E_1$). Colour green $(x,y) \in E$.

(d)

with $x \in X_2 \setminus X_1$, $y \in Y_2 \setminus Y_1$, $x' \in X_1 \setminus X_2$, $y' \in Y_1 \setminus Y_2$.

Colour green the edge $(x',y) \in E$.

In case the component consists of one edge of E_0 (re)colour green this edge.

(e) The component is a circuit. Do not colour any edge.

The green edges together form a matching in E between $X'' \cup (X' \setminus X_2)$ and $Y'' \cup (Y' \setminus Y_1)$.

In the same way one can prove, conversely, that if $X'' \cup (X' \setminus X_2)$ and $Y'' \cup (Y' \setminus Y_1)$ are matched in E then $X'' \cup X_1$ and $Y'' \cup Y_2$ are matched in E. □

Note that we also have proved that a base $(X \setminus X') \cup Y'$ is principal iff *every* matching between X' and Y' is red.

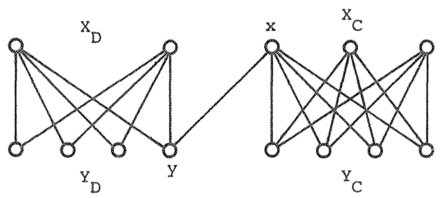
Theorem 5.3 implies that the principal bases of a deltoid form again (the collection of bases of) a matroid, namely the deltoid generated by the red sub bipartite graph of the original graph. In particular, if each base of a deltoid is a principal base then every component of any generating bipartite graph is a complete bipartite graph. This gives us the following result of BRUALDI [74]: *if each base of a connected deltoid M is principal then M is a uniform matroid* (BRUALDI states this in terms of fundamental transversal matroids). For more results concerning principal bases cf. BRUALDI & DINOLT [75].

We next investigate the structural properties of the red subgraph of bipartite graphs.

Let (X,Y,E) be a bipartite graph with red subgraph (X,Y,E_R). Let $\mathcal{C}$ be the set of components of (X,Y,E_R) and for each $C \in \mathcal{C}$ define X_C as the set of points in X of component C and Y_C as the set of points in Y of C. Clearly, $\{X_C \mid C \in \mathcal{C}\}$ and $\{Y_C \mid C \in \mathcal{C}\}$ are partitions of X and Y, respectively. Since each component is a complete bipartite graph we also have

$$E_R = \bigcup_{C \in \mathcal{C}} (X_C \times Y_C).$$

Let $C, D \in \mathcal{C}$ and suppose there is an edge (x,y) in E between X_C and Y_D.



Then $(x,y') \in E$ for all $y' \in Y_D$ and $(x',y) \in E$ for all $x' \in X_C$, since each edge between X_D and Y_D is red and also each edge between X_C and Y_C is red. It follows that $(x',y') \in E$ for all $x' \in X_C$ and $y' \in Y_D$.

So we proved:

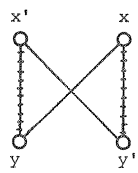
$$(X_C \times Y_D) \cap E = \emptyset \quad \text{or} \quad X_C \times Y_D \subset E$$

for all components $C, D \in \mathcal{C}$.

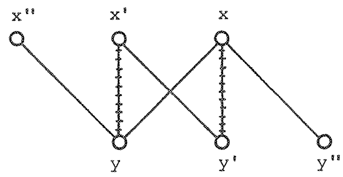
Now define a relation $\le$ on $\mathcal{C}$ by

$$C \le D \quad \text{iff} \quad C = D \quad \text{or} \quad (X_C \times Y_D) \cap E \ne \emptyset.$$

(Since X_C or Y_D can be empty the clause $C = D$ is not superfluous.) We prove that $\le$ is a partial order on $\mathcal{C}$. Clearly $C \le C$ for all $C \in \mathcal{C}$. Suppose $C \le D \le C$ and $C \ne D$. Then we know: $(x,y) \in E$ for all $x \in X_C \cup X_D$ and $y \in Y_C \cup Y_D$. We prove that $(x,y) \in E_R$ for all $x \in X_C \cup Y_D$ and $y \in Y_C \cup Y_D$. Take $x \in X_C$ and $y \in Y_D$. We prove that (x,y) is red. To this end take $x' \in X_D$ and $y' \in Y_C$ (this is possible since $X_D \times Y_C \ne \emptyset$). Since $(x',y') \in E$ we have:



in which a red edge is crossed. To prove that (x,y) is red let $(x,y'') \in E$ and $(x'',y) \in E$. We have to prove that $(x'',y'') \in E$.
In picture:



This is straightforward: $(x'',y') \in E$ since (x',y) is red and $(x'',y'') \in E$ since (x,y') is red.
Thus $X_C \times Y_D \subset E_R$; in the same way one proves that $X_D \times Y_C \subset E_R$. As $X_C \times Y_D \neq \emptyset$, C and D cannot be different components of (X,Y,E_R). Hence $\leq$ is anti-symmetric. The transitivity of $\leq$ follows easily.

Clearly, the bipartite graph (X,Y,E) is determined by the set $\mathcal{C}$ of components of (X,Y,E_R) and the partial order $\leq$ on $\mathcal{C}$. Note that in case $X_C = \emptyset$ then Y_C is a singleton and C is a maximal element of $(\mathcal{C},\leq)$. Likewise, if $Y_C = \emptyset$ then X_C is a singleton and C is a minimal element of $(\mathcal{C},\leq)$.
The deltoid generated by the bipartite graph is determined by the collection of vertex sets $X_C \cup Y_C$ of components C in $\mathcal{C}$, by the partial order $\leq$ and by the *cardinality* of X_C for each component C (this follows from theorem 5.3; we do not need the graphical structure of the components).

5c. REPRESENTABILITY OF DELTOIDS AND GAMMOIDS

We now consider the representability of deltoid and gammoid linking systems.

To this end, let (X,Y,E) be a bipartite graph and let (X,Y,Δ_E) be the deltoid linking system generated by (X,Y,E). Then (X,Y,Δ_E) is representable over a field F iff there exists a matrix $M = (X,Y,\phi)$ such that:

$$X' \text{ and } Y' \text{ are matched in } E \iff M \mid X' \times Y' \text{ is nonsingular,}$$

for all $X' \subset X$ and $Y' \subset Y$. Clearly, M is such that $\phi(x,y) \neq 0$ iff $(x,y) \in E$. Hence $M = (X,Y,\phi)$ is a matrix representation of (X,Y,Δ_E) iff $\phi(x,y) = 0$ for $(x,y) \notin E$ and $M \mid X' \times Y'$ is nonsingular whenever $(X',Y') \in \Delta_E$.

Now, let F be a field and let $(z_e \mid e \in E)$ be a collection of algebraically independent indeterminates over F. Define $\phi\colon X \times Y \longrightarrow F(z_e \mid e \in E)$ by:

$$\phi(x,y) = z_{(x,y)} \quad \text{if } (x,y) \in E,$$
$$\phi(x,y) = 0 \quad \text{otherwise.}$$

Then $M = (X,Y,\phi)$ is a matrix representation over $F(z_e | e \in E)$ of (X,Y,Δ_E). This idea, due to PERFECT [66], EDMONDS [67] and MIRSKY & PERFECT [67], shows that each deltoid and hence each transversal matroid is representable over some extension of F (for each field F).

PIFF & WELSH [70] proved that each deltoid is representable over all but a finite number of finite fields and ATKIN [72] showed that a transversal matroid of rank r on a set of n elements is representable over any field F with

$$|F| > n + \binom{n}{r-1}.$$

To establish his result, ATKIN's proof produces no explicit construction of a representation. A construction due to H.W. LENSTRA JR. is as follows.

Let (X,Y,E) be a bipartite graph and let $n = |X|$ and $m = |Y|$. Furthermore let F be a field with at least $2^{n^{2m}}$ elements. Then there exists a $\beta \in F$ such that no non-empty sum of pairwise distinct and non-opposite elements from

$$\pm \beta^0, \pm \beta^1, \ldots, \pm \beta^{n^m - 1}$$

is zero.

If $\text{char}(F) = 0$ or $\text{char}(F) \geq 2^{n^m}$ we can take $\beta = 2$. If $|F| = p^\alpha$ such that p is prime and $\alpha \geq n^m$, let β be a primitive element of F (in this case the minimal polynomial of β over the prime field of F has degree at least n^m). If F is an infinite field of non-zero characteristic then either F has an element, algebraically independent over the prime field of F (we can take this element as β) or F has a subfield F' with p^α elements (p prime, $\alpha \geq n^m$), for which the above applies. At least one of these cases applies to F.

Let $X = \{x_0,\ldots,x_{n-1}\}$ and let $Y = \{y_1,\ldots,y_{m-1}\}$. Define $\phi\colon X \times Y \to F$ as follows:

$$\phi(x_i,y_j) = \beta^{i.n^j} \quad \text{if } (x_i,y_j) \in E,$$
$$\phi(x_i,y_j) = 0 \quad \text{otherwise.}$$

Now $M = (X,Y,\phi)$ is a matrix representation for (X,Y,Δ_E). For suppose $(X',Y') \in \Delta_E$, that is, X' and Y' are matched in E. The determinant of $M \mid X' \times Y'$ is a non-empty sum of terms of the form

$$\pm \beta^{i_1 \cdot n^{j_1}} \cdots\cdots\cdots \beta^{i_k \cdot n^{j_k}};$$

it follows from the choice of β that this sum is nonzero.

Unfortunately, the bound $2^{n^{2m}}$ is in general much larger than ATKIN's bound, which is $n + m + \binom{n+m}{n-1}$ in this case.

Let (X,Y,E) be a bipartite graph. The following assertions are easily verified:

(i) (X,Y,Δ_E) *is representable over* GF(2) *iff* (X,Y,E) *has no circuits;*

(ii) (X,Y,Δ_E) *is representable over* GF(3) *iff no two vertices of* (X,Y,E) *are connected by three pairwise vertex-disjoint paths.*

EDMONDS (see BONDY [72a]) proved that a transversal matroid is binary iff it has a presentation such that the associated bipartite graph is a forest. Together with (i) this implies that each binary transversal matroid is the restriction of a binary deltoid.

Next we turn our attention to gammoid linking systems. Let (Z,E) be a directed graph and let $X,Y \subset Z$. The gammoid linking system (X,Y,Γ_E) is representable over the field F iff there exists a matrix $M = (X,Y,\phi)$ such that

$$X' \text{ and } Y' \text{ are matched in } E \iff M \mid X' \times Y' \text{ is nonsingular,}$$

for all $X' \subset X$ and $Y' \subset Y$.

MASON [72] proved that each field F has an extension over which (X,Y,Γ_E) is representable. In order to prove this, let $(z_e \mid e \in E)$ be again a collection of algebraically independent indeterminates over F. Furthermore, we define $\phi\colon X \times Y \to F(z_e \mid e \in E)$ in the following manner:

$$\phi(x,y) = \sum\{z_{e_1} \cdot z_{e_2} \cdot \ldots z_{e_k} \mid k \geq 0 \text{ and } e_1,\ldots,e_k \text{ are the edges of a path in } (Z,E) \text{ from } x \text{ to } y, \text{ using no point twice}\}.$$

Then (X,Y,ϕ) is a matrix representation of (X,Y,Γ_E). LINDSTRÖM [73] also proved this result and he showed in addition that each gammoid linking system (X,Y,Γ_E) is representable over any field F such that $|F| \geq 2^{|X|+|Y|}$. To obtain an explicit construction of a matrix-representation we use the fact that each strict gammoid linking system is the inverse of a nonsingular deltoid linking system (cf. section 4b). So LENSTRA's construction, together with a calculation of the inverse of the obtained matrix, yields a construction of a matrix representation for gammoid linking systems.

LINDSTRÖM [73] also observed the following. Take any directed graph (Z,E) and subsets $X,Y \subset Z$ and let (X,I) be a matroid, representable over some field F. Let $\psi: X \to V$ be a representation of (X,I) in a vector space V over F. Take $\phi: X \times Y \to F(z_e | e \in E)$ as described above. Define $\chi: Y \to V'$ by

$$\chi(y) = \sum_{x \in X} \phi(x,y) \cdot \psi(x)$$

where V' is the extension of V to a vector space over $F(z_e | e \in E)$ (χ can be conceived as a matrix product).

Then:

(i) χ is a representation of $(Y, I*\Gamma_E)$ over $F(z_e | e \in E)$;

(ii) if $|F| \geq 2^{|Y|}$ then $(Y, I*\Gamma_E)$ is representable over F.

CHAPTER SIX

POLYMATROIDS AND POLY-LINKING SYSTEMS

The concept of polymatroid was introduced by EDMONDS [70] as a generalization of the notion of matroid. This chapter will deal with polymatroids and with poly-linking systems, a corresponding generalization of linking systems.

In section 6a we present the definition of a polymatroid and of a submodular function and we review some results on polymatroids and submodular functions, mainly due to EDMONDS [70].

In section 6b we introduce the notions of poly-linking systems and bisubmodular functions and derive some properties, using a relation between poly-linking systems and polymatroids, analogous to the relation between linking systems and matroids in the sense of theorem 2.2. Moreover we give examples.

Sections 6c and 6d treat the linking of polymatroids by poly-linking systems; this linking is an extension of the linking of matroids by linking systems as treated in sections 2b and 2c. Finally, in section 6e we extend some operations on linking systems (product and union; cf. section 4a) to operations on poly-linking systems.

6a. PRELIMINARIES ON POLYMATROIDS

As announced we present in this section a short survey of the theory of polymatroids and submodular functions; most of the results here are due to EDMONDS [70] (for proofs see WELSH [76]).

The pair (X,P) is a *polymatroid* if X is a finite set and P is a nonempty compact collection of vectors in $\mathbb{R}_+^X$ such that

(i) if $\underline{0} \le u \le v \in P$ then $u \in P$;

(ii) if $u, v \in P$ and $|u| < |v|$ then $u < w \le u \vee v$ for some $w \in P$.

Here, as usual, $u \le v$ means that no coordinate of u is greater than the corresponding coordinate of v; $\underline{0}$ is the all-zero vector, $|u|$ is the sum of the coordinates of u, and $u \vee v$ ($u \wedge v$, respectively) is the supremum (infimum, respectively) of u and v (with respect to $\le$). Moreover, $u < v$ iff $u \le v$ and $u \ne v$. We identify $\mathbb{R}_+^X$ (the set of functions from X into $\mathbb{R}_+$) with the corresponding set of vectors.

Let (X,P) be a polymatroid. For $u \in \mathbb{R}_+^X$ define the *rank* of u as

$$r(u) = \max\{|v| \mid v \in P,\ v \le u\},$$

which value exists since P is compact. It is easy to see that if $w \le u \in \mathbb{R}_+^X$ and $w \in P$ then there is a $v \in P$ such that $w \le v \le u$ and $r(u) = |v|$.

The maximal vectors in P (with respect to $\le$) are called *basis vectors*. Choosing $u \in \mathbb{R}_+^X$ such that $v \le u$ for all $v \in P$ one finds that $|w| = r(u)$ for each basis vector w of P. Hence all basis vectors have the same modulus. Clearly, a polymatroid is determined by its set of basis vectors. A *basis vector of* $u \in \mathbb{R}_+^X$ is a vector $w \in P$ such that $w \le u$ and $r(u) = |w|$.

A *submodular function* on a (finite) set X is a function $\rho\colon \mathcal{P}(X) \longrightarrow \mathbb{R}_+$ such that

(i) $\rho(\emptyset) = 0$;
(ii) if $X'' \subset X' \subset X$ then $\rho(X'') \le \rho(X')$;
(iii) if $X', X'' \subset X$ then $\rho(X' \cap X'') + \rho(X' \cup X'') \le \rho(X') + \rho(X'')$.

(This definition is not standard; sometimes only axiom (iii) is required.) EDMONDS proved that the following one-to-one relation exists between polymatroids (X,P) and submodular functions ρ on X:

(i) $\rho(X') = \max\{u(X') \mid u \in P\}$ is submodular whenever (X,P) is a polymatroid;
(ii) $P = \{u \in \mathbb{R}_+^X \mid u(X') \le \rho(X')$ for all $X' \subset X\}$ defines a polymatroid (X,P) whenever ρ is a submodular function on X.

As usual, $u(X') = \sum_{x \in X'} u_x$, for $u \in \mathbb{R}_+^X$ and $X' \subset X$.

That is, each polymatroid determines uniquely a submodular function, and conversely. If a polymatroid and a submodular function are related as above, we shall say that they *correspond* to each other.

Let (X,P) be a polymatroid with corresponding submodular function ρ. From the relation (ii) above it follows that P is a convex polyhedron in $\mathbb{R}_+^X$. It is possible to express the rank r(u) of elements $u \in \mathbb{R}_+^X$ in terms of ρ:

$$r(u) = \min_{X' \subset X} (u(X') + \rho(X \setminus X')).$$

Let $u \in \mathbb{R}_+^X$. The *restriction* of (X,P) to u is the polymatroid $(X,P|u)$ where

$$P|u = \{v \in P \mid v \leq u\}.$$

(It is easy to see that this is again a polymatroid.) The submodular function $\rho|u$ corresponding to $P|u$ then is

$$(\rho|u)(X') = \min_{X'' \subset X'} (u(X'') + \rho(X' \setminus X''))$$

for $X' \subset X$.

Since P is a convex polyhedron, P is completely determined by its set of vertices. (X,P) (or P) is called *integral* if each vertex of P is integer-valued, i.e. an element of $\mathbb{Z}_+^X$. EDMONDS showed (by means of the greedy algorithm) that P is integral iff ρ is integer-valued, and also iff for each $u \in \mathbb{Z}_+^X$ there is a $v \in P$ such that: $v \leq u$, $r(u) = v$ and $v \in \mathbb{Z}_+^X$.

Obviously, if P is integral and $u \in \mathbb{Z}_+^X$ then also $P|u$ is integral. Moreover, in case P is integral and $u \in P$ is integer-valued then $u \leq v$ for some integer-valued basis-vector of P.

Let (X,P_1) and (X,P_2) be polymatroids, with corresponding submodular functions ρ_1 and ρ_2, respectively. Let

$$P_1 + P_2 = \{u+v \mid u \in P_1, v \in P_2\}.$$

Then (X,P_1+P_2) is again a polymatroid, with corresponding submodular function $\rho_1+\rho_2$, defined by

$$(\rho_1+\rho_2)(X') = \rho_1(X') + \rho_2(X'),$$

for $X' \subset X$. Therefore, if P_1 and P_2 both are integral then also P_1+P_2 is integral. Moreover, in this case if $w \in P_1+P_2$ and $w \in \mathbb{Z}_+^X$ then $w = u + v$ for integer-valued $u \in P_1$ and $v \in P_2$. In a straightforward way, this notion of sum of two polymatroids extends to the sum of more polymatroids.

One of the most important results in this field is EDMONDS' intersection-theorem for polymatroids, yielding properties of the intersection $P_1 \cap P_2$. It states, inter alia:

(i) $\max\{|u| \mid u \in P_1, u \in P_2\} = \min_{X' \subset X}(\rho_1(X') + \rho_2(X \setminus X'))$;

(ii) if P_1 and P_2 are integral, then $P_1 \cap P_2$ is again a convex polyhedron with integer-valued vertices (but, in general, not a polymatroid);

(iii) (in particular) if P_1 and P_2 are integral, then the maximum in (i) is attained by an integer-valued u.

This theorem has (as yet) no natural generalization to more than two polymatroids.

In what sense is the theory above an extension of matroid theory? Let $(X,\mathcal{I})$ be a matroid. The rank function ρ of this matroid is a submodular function on X. The corresponding polymatroid is the set of all convex combinations of characteristic vectors of independent sets of $(X,\mathcal{I})$.

A submodular function ρ on X is the rank function of a matroid iff ρ is integer-valued and $\rho(X') \leq |X'|$ for all $X' \subset X$. A polymatroid (X,P) can be obtained from a matroid in the above manner iff P is integral and $u \leq \underline{1}$ for all $u \in P$. ($\underline{1}$ is the all-one vector in $\mathbb{R}_+^X$.)

Conversely, each integral polymatroid (X,P), with corresponding submodular function ρ, determines (but not one-to-one) a matroid $(X,\mathcal{I})$ with rank function ρ', in the following way:

$X' \in \mathcal{I}$ iff the characteristic vector of X' is in P, that is iff $|X''| \leq \rho(X'')$ for all $X'' \subset X'$;

$$\rho'(X') = \min_{X'' \subset X'}(|X' \setminus X''| + \rho(X'')).$$

This matroid corresponds (in the above sense) with the restriction of (X,P) to $\underline{1}$.

Now the union and intersection theorems for matroids turn out to be special cases of the theorems above. For more results on submodular functions and matroids see PYM & PERFECT [70], MCDIARMID [73,75c], DUNSTAN [76].

6b. POLY-LINKING SYSTEMS AND POLYMATROIDS

Now we generalize the concept of a linking system to that of a poly-linking system.

DEFINITION 6.1. A *poly-linking system* is a triple (X,Y,L) where X and Y are finite sets and L is a non-empty compact subset of $\mathbb{R}_+^X \times \mathbb{R}_+^Y$ such that

(i) if $(u,v) \in L$ then $|u| = |v|$;

(ii) if $(u,v) \in L$ and $0 \le u' \le u$ then $(u',v') \in L$ for some $v' \le v$;

(iii) if $(u,v) \in L$ and $0 \le v' \le v$ then $(u',v') \in L$ for some $u' \le u$;

(iv) if (u',v'), $(u'',v'') \in L$ then there exists a $(u,v) \in L$ such that $u' \le u \le u' \vee u''$ and $v'' \le v \le v' \vee v''$.

This definition will turn out to be an extension of definition 1.1 of a linking system. The concept of a linking function is generalized by the following definition.

DEFINITION 6.2. Let X and Y be finite sets. A function $\lambda\colon P(X) \times P(Y) \to \mathbb{R}_+$ is called *bi-submodular* (on X and Y) if it has the properties

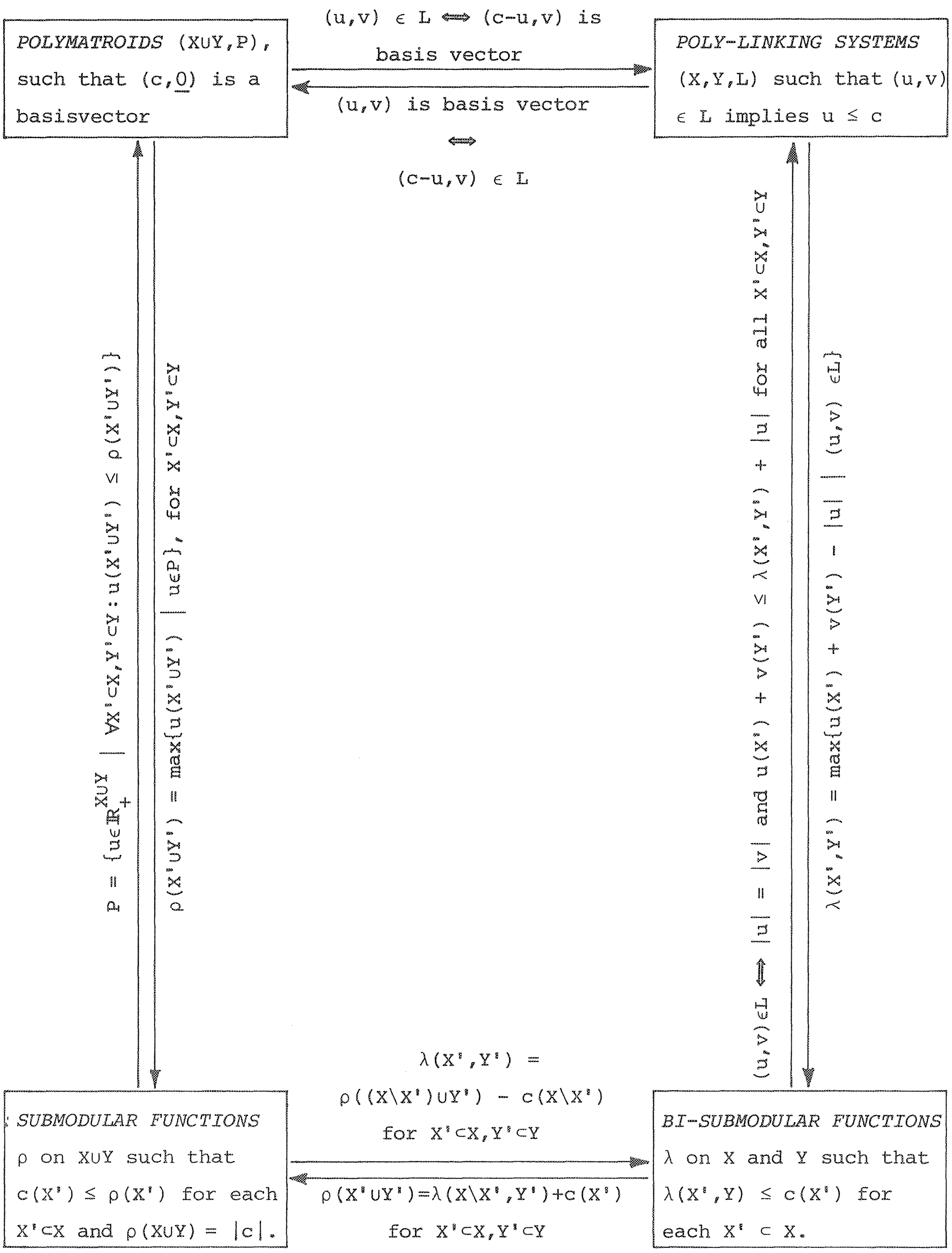
(i) $\lambda(\emptyset,Y) = \lambda(X,\emptyset) = 0$;

(ii) if $X'' \subset X' \subset X$ and $Y'' \subset Y' \subset Y$ then $\lambda(X'',Y'') \le \lambda(X',Y')$;

(iii) if $X', X'' \subset X$ and $Y', Y'' \subset Y$ then $\lambda(X'\cap X'',Y'\cup Y'') + \lambda(X'\cup X'',Y'\cap Y'') \le \lambda(X',Y') + \lambda(X'',Y'')$.

Obviously, the linking function of a linking system is bi-submodular. Relations between polymatroids, poly-linking systems, submodular and bi-submodular functions (cf. theorems 1.2 and 2.2) are traced in the following theorem. (For disjoint sets X and Y we identify $\mathbb{R}_+^{X\cup Y}$ and $\mathbb{R}_+^X \times \mathbb{R}_+^Y$.)

THEOREM 6.3. *Let X and Y be finite disjoint sets and $c \in \mathbb{R}_+^X$. Then there exist the following relations between the indicated structures.*


POLYMATROIDS (X∪Y,P), such that (c,0) is a basisvector
(u,v) ∈ L ⟺ (c−u,v) is basis vector
(u,v) is basis vector ⟺ (c−u,v) ∈ L
POLY-LINKING SYSTEMS (X,Y,L) such that (u,v) ∈ L implies u ≤ c
P = {u∈ℝ+^X∪Y | ∀X'⊂X,Y'⊂Y: u(X'∪Y') ≤ ρ(X'∪Y')}
ρ(X'∪Y') = max{u(X'∪Y') | u∈P}, for X'⊂X,Y'⊂Y
(u,v)∈L ⟺ |u| = |v| and u(X') + v(Y') ≤ λ(X',Y') + |u| for all X'⊂X,Y'⊂Y
λ(X',Y') = max{u(X') + v(Y') − |u| | (u,v) ∈L}
SUBMODULAR FUNCTIONS ρ on X∪Y such that c(X') ≤ ρ(X') for each X'⊂X and ρ(X∪Y) = |c|.
λ(X',Y') = ρ((X\X')∪Y') − c(X\X') for X'⊂X,Y'⊂Y
ρ(X'∪Y')=λ(X\X',Y')+c(X') for X'⊂X,Y'⊂Y
BI-SUBMODULAR FUNCTIONS λ on X and Y such that λ(X',Y) ≤ c(X') for each X' ⊂ X.

PROOF. The vertical arrows on the left (between polymatroids and submodular functions) follow from polymatroid theory (cf. section 6a). The lower horizontal arrows (between submodular and bi-submodular functions) are easy to check. We prove only the relation between polymatroids and poly-linking systems (the upper horizontal arrows); the right vertical arrows then follow.

Therefore, let $(X \cup Y, P)$ be a polymatroid, with basis vector $(c, \underline{0})$ and define

$$L = \{(u,v) \in \mathbb{R}_+^X \times \mathbb{R}_+^Y \mid (c-u,v) \text{ is a basis vector of } (X \cup Y, P)\}.$$

We prove that (X,Y,L) is a poly-linking system, such that if $(u,v) \in L$ then $u \leq c$. The latter statement is clear; axiom (i) of definition 6.1 follows from the fact that all basis vectors of a polymatroid have the same modulus.
In order to prove axiom (ii), let $(u,v) \in L$ and $\underline{0} \leq u' \leq u$. Hence $(c-u,v) \in P$. Since also $(c-u',\underline{0}) \in P$ there exists a basis vector $(c-u'',v')$ such that $(c-u',\underline{0}) \leq (c-u'',v') \leq (c-u',v)$ (since the rank of $(c-u',v)$ is $|c|$). So $u' = u''$, $(u',v') \in L$ and $v' \leq v$. For proving axiom (iii) take $(u,v) \in L$ and $\underline{0} \leq v' \leq v$. This means, that $(c-u,v)$ is a basisvector; hence $(c-u,v') \in P$. Since also $(c,\underline{0})$ is a basis vector there exists a basis vector $(c-u',v'') \in P$ such that $(c-u,v') \leq (c-u',v'') \leq (c,v')$. Therefore $v'' = v'$, $(u',v') \in L$ and $u' \leq u$.
Finally we prove axiom (iv). Let $(u',v') \in L$ and $(u'',v'') \in L$. That is, $(c-u',v')$ and $(c-u'',v'')$ are basisvectors of P. Then $(c-(u' \vee u''),v'') \leq (c-u'',v'')$, hence there is a basisvector $(c-u,v)$ such that $(c-(u' \vee u''),v'') \leq (c-u,v) \leq (c-u',v' \vee v'')$. But then $(u,v) \in L$, $u' \leq u \leq u' \vee u''$ and $v'' \leq v \leq v' \vee v''$.

Conversely, let (X,Y,L) be a poly-linking system such that $(u,v) \in L$ implies $u \leq c$. Define

$$P = \{(s,t) \in \mathbb{R}_+^{X \cup Y} \mid (s,t) \leq (c-u,v) \quad \text{for some} \quad (u,v) \in L\}.$$

We prove that $(X \cup Y, P)$ is a polymatroid such that $(c,\underline{0})$ is a basisvector. This last is clear, just as the fact that P is a non-empty compact set satisfying axiom (i) of the definition of a polymatroid (section 6a). To prove axiom (ii) let (s,t), $(s',t') \in P$ such that $|s| + |t| < |s'| + |t'|$. We prove that there is a $(s'',t'') \in P$ such that $(s,t) < (s'',t'') \leq (s \vee s', t \vee t')$.
Since $(s,t) \in P$ and $(s',t') \in P$ we may suppose (by axiom (iii) of definition 6.1) that $(s,t) \leq (c-u,t)$ and $(s',t') \leq (c-u',t')$ for certain $(u,t) \in L$ and $(u',t') \in L$.

By axiom (ii) of definition 6.1 there exists a $q \leq t'$ such that $((c-s) \wedge u', q) \in L$. Applying axiom (iv) on the pairs $((c-s) \wedge u', q)$ and (u,t) in L yields a pair (u'',t'') in L such that

$$(c-s) \wedge u' \leq u'' \leq ((c-s) \wedge u') \vee u = (c-s) \wedge (u \vee u')$$

and

$$t \le t'' \le t \vee q.$$

Now let $s'' = (c-u'') \wedge (s \vee s')$. Now clearly $(s'',t'') \in P$ since $(s'',t'') \le (c-u'',t'')$, and also

$$(s,t) \le (s'',t'') \le (s \vee s', t \vee t')$$

(observe that $s \le c-u''$, since $u'' \le (c-s) \wedge (u \vee u') \le c-s$, and that $t'' \le t \vee q \le t \vee t'$). We prove that $(s,t) \neq (s'',t'')$ by proving that $|s'| + |t'| \le |s''| + |t''|$.
Since $(c-s) \wedge u' \le u''$ we have $c-u'' \le s \vee (c-u')$ and hence

$$|(c-u'') \vee s \vee s'| \le |(c-u') \vee s| \tag{1}$$

since also $s' \le c-u'$. The latter inequality also implies that

$$|s'| \le |(c-u') \wedge (s \vee s')| \tag{2}$$

Clearly,

$$|(c-u') \vee s| \le |(c-u') \vee s \vee s'| \tag{3}$$

Using the property that $|x \wedge y| + |x \vee y| = |x| + |y|$ we obtain

$$|(c-u') \wedge (s \vee s')| + |(c-u') \vee (s \vee s')| \le |c-u'| + |s \vee s'| \tag{4}$$

and, since $s'' = (c-u'') \wedge (s \vee s')$

$$|c-u''| + |s \vee s'| \le |(c-u'') \vee (s \vee s')| + |s''|. \tag{5}$$

Since $|u'| = |t'|$ and $|u''| = |t''|$ we have

$$|c-u'| + |t'| \le |c-u''| + |t''|. \tag{6}$$

Summing up the inequalities (1), (2), (3), (4), (5) and (6) we get

$$|s'| + |t'| \le |s''| + |t''|. \quad \square$$

We shall say that a poly-linking system and a bi-submodular function *correspond* with each other if the relation between both is such as indicated in theorem 6.3.

Let (X,Y,L) be a poly-linking system with corresponding bi-submodular function λ. Furthermore, let $d \in \mathbb{R}_+^X$ and $e \in \mathbb{R}_+^Y$. Define

$$L|(d,e) = \{(u,v) \in L \mid u \leq d,\ v \leq e\}.$$

From definition 6.1 it is clear that $(X,Y,L|(d,e))$ is again a poly-linking system. The corresponding submodular function $\lambda|(d,e)$ is given by

$$(\lambda|(d,e))(X',Y') = \min_{\substack{X''\subset X'\\ Y''\subset Y'}} (d(X'') + \lambda(X'\backslash X'',Y'\backslash Y'') + e(Y'')).$$

This can be proved straightforwardly in the following way: (i) $\lambda|(d,e)$ (as given above) is bi-submodular; (ii) the poly-linking system corresponding to $\lambda|(d,e)$ equals $(X,Y,L|(d,e))$.
Define the *linking function* ℓ of (X,Y,L) by

$$\ell(d,e) = \max\{|u| \mid (u,v) \in L,\ u \leq d,\ v \leq e\},$$

for $d \in \mathbb{R}_+^X$ and $e \in \mathbb{R}_+^Y$. So $(u,v) \in L$ iff $\ell(u,v) = |u| = |v|$. It follows from theorem 6.3 that

$$\lambda(X,Y) = \max\{|u| \mid (u,v) \in L\}.$$

Hence

$$\ell(d,e) = (\lambda|(d,e))(X,Y) = \min_{\substack{X'\subset X\\ Y'\subset Y}} (d(X') + \lambda(X\backslash X',Y\backslash Y') + e(Y'))$$

for $d \in \mathbb{R}_+^X$ and $e \in \mathbb{R}_+^Y$.

(X,Y,L) is called *integral* if for all $(u,v) \in L$ there are $(u^i,v^i) \in L$ and $\lambda_i \in \mathbb{R}_+$ $(i = 1,\ldots,k$, for some $k)$ such that $u^i \in \mathbb{Z}_+^X$, $v^i \in \mathbb{Z}_+^Y$ $(i = 1,\ldots,k)$, $\sum_i \lambda_i = 1$ and $\sum_i (\lambda_i u^i, \lambda_i v^i) = (u,v)$; that is, each pair in L is a convex combination of integer-valued pairs in L.

Choose $c \in \mathbb{Z}_+^X$ such that $u \leq c$ for all $(u,v) \in L$, and let P and ρ be related to L (and λ) and c in the sense of theorem 6.3 (we may suppose that X and Y are disjoint). Then

$$(X,Y,L) \text{ integral} \iff (X\cup Y,P) \text{ integral} \iff \rho \text{ integer-valued} \iff \lambda \text{ integer-valued}.$$

It follows that if (X,Y,L) is integral and $d \in \mathbb{Z}_+^X$ and $e \in \mathbb{Z}_+^Y$ then $(X,Y,L|(d,e))$ is again integral (since $\lambda|(d,e)$ is integer-valued). Therefore, if (X,Y,L) is integral,

$$\ell(d,e) = \max\{|u| \mid (u,v) \in L,\ u \in \mathbb{Z}_+^X, v \in \mathbb{Z}_+^Y,\ u \le d,\ v \le e\}$$

for all $d \in \mathbb{Z}_+^X$ and $e \in \mathbb{Z}_+^Y$ (since the polyhedron $L|(d,e)$ has its vertices in $\mathbb{Z}_+^X \times \mathbb{Z}_+^Y$).

Conversely, if $\ell(d,e)$ is as above for all $d \in \mathbb{Z}_+^X$ and $e \in \mathbb{Z}_+^Y$, then λ is integer-valued since $\lambda(X',Y') = \ell(d,e)$ for some $d \in \mathbb{Z}_+^X$ and $e \in \mathbb{Z}_+^Y$ (choose d (e, respectively) large enough on X' (Y', respectively) and zero on $X\backslash X'$ ($Y\backslash Y'$, respectively)). So in that case (X,Y,L) is integral. Moreover, if (X,Y,L) is integral and (u_1,v_1), $(u_2,v_2) \in L$ are integer-valued then $u_1 \le u' \le u_1 \vee u_2$ and $v_2 \le v' \le v_1 \vee v_2$ for some integer-valued $(u',v') \in L$ (cf. the proof of theorem 6.3).

The linking function λ of a linking system always is bi-submodular. Obviously, λ is the linking function of a linking system (X,Y,Λ) iff λ is bi-submodular on X and Y, λ is integer-valued and $\lambda(X',Y') \le \min\{|X'|,|Y'|\}$ for all $X' \subset X$ and $Y' \subset Y$. The poly-linking system (X,Y,L) corresponding to the linking function of a linking system (X,Y,Λ) is given by: $(u,v) \in L$ iff (u,v) is a convex combination of (characteristic functions of) pairs in Λ.

Conversely, each integral poly-linking system (X,Y,L) determines (but not one-to-one) a linking system (X,Y,Λ) by

$$(X',Y') \in \Lambda \quad \text{iff} \quad (u_{X'},v_{Y'}) \in L,$$

where $u_{X'}$ ($v_{Y'}$ respectively) is the characteristic function of X' (Y' respectively), for $X' \subset X$ and $Y' \subset Y$. If λ is the bi-submodular function corresponding to (X,Y,L) then the linking function λ_0 of (X,Y,Λ) equals $\lambda \mid (1_X,1_Y)$ (where 1_X (1_Y respectively) is the one-vector on X (Y respectively)); that is

$$\lambda_0(X',Y') = \min_{\substack{X''\subset X'\\ Y''\subset Y'}} (|X''| + \lambda(X'\backslash X'',Y'\backslash Y'') + |Y''|),$$

for $X' \subset X$ and $Y' \subset Y$.

EXAMPLES OF POLY-LINKING SYSTEMS

We now present some examples of poly-linking systems. We just have seen that each linking system (X,Y,Λ) (with linking function λ) yields a poly-linking system (X,Y,L) (with corresponding bi-submodular function λ).

A second example is an extension of example (2) of section 1b (gammoid linking systems). Let (Z,E) be a directed graph and let $X,Y \subset Z$. Let furthermore be given a "*capacity*" function $c: Z \longrightarrow \mathbb{R}_+$. Define L as the collection of all pairs $(u,v) \in \mathbb{R}_+^X \times \mathbb{R}_+^Y$ such that there is a *flow* in (Z,E), respecting the capacity function c, with *imput vector* u and *output vector* v. By this is meant a "flow" function $f: E \longrightarrow \mathbb{R}_+$ such that for all $z \in Z$

$$u(z) + \sum_{(z',z)\in E} f(z',z) = v(z) + \sum_{(z,z')\in E} f(z,z') \le c(z),$$

in which we put $u(z) = 0$ for $z \in Z\backslash X$, and $v(z) = 0$ for $z \in Z\backslash Y$. For $X' \subset X$ and $Y' \subset Y$ let $\lambda(X',Y')$ be the maximal flow from X' to Y' subject to the capacity restriction c, or, which is the same by the "max-flow min-cut theorem" of FORD & FULKERSON [58], the minimal capacity of an (X',Y')-cutset. That is

$$\lambda(X',Y') = \min_{Z'\subset Z\backslash Y'} c((X'\cup E(Z'))\backslash Z'),$$

where $E(Z') = \{z\in Z \mid (z',z) \in E \text{ for some } z' \in Z'\}$.

Now (X,Y,L) is a poly-linking system with corresponding bi-submodular function λ.

In order to prove this, first observe that λ as given above is bi-submodular (cf. the proof in example (2) of section 1b). For $d \in \mathbb{R}_+^X$ and $e \in \mathbb{R}_+^Y$ define

$$\ell(d,e) = \min_{\substack{X'\subset X\\ Y'\subset Y}} (d(X') + \lambda(X\backslash X',Y\backslash Y') + e(Y')).$$

Let (X,Y,L') be the poly-linking system corresponding with λ. We show that $L' = L$. That is, we show that

$$\ell(d,e) = |d| = |e| \quad \text{iff} \quad (d,e) \in L,$$

for all $d \in \mathbb{R}_+^X$ and $e \in \mathbb{R}_+^Y$. Now, as can be seen easily (using the max-flow min-cut theorem), $\ell(d,e)$ equals the maximal flow from X to Y respecting the capacity function c on Z, and also respecting the functions d on X and e on Y. Hence $\ell(d,e) = |d| = |e|$ iff $(d,e) \in L$.

Clearly, if c is integer-valued then also λ is integer-valued and hence (X,Y,L) is integral.

As special cases of this example we have, inter alia:

(i) Let (Z,E) be a digraph and let X and Y be disjoint subsets of Z. Then the function λ given by

$$\lambda(X',Y') = \text{maximal number of edge-disjoint paths from } X' \text{ to } Y'$$

$(X'\subset X, Y'\subset Y)$ is bi-submodular on X and Y.

This can be seen by inserting new points on the edges of (Z,E), giving these new points a capacity 1, and giving the other points a large capacity (e.g. larger than the degree). Now $\lambda(X',Y')$ as defined above equals the maximal value of a flow from X' to Y' respecting the capacities, and therefore λ is bi-submodular.

By MENGER's theorem $\lambda(X',Y')$ equals the minimal cardinality of an (X',Y')-edge cutset.
The collection L of the corresponding poly-linking system (X,Y,L) consists of all pairs $(u,v) \in \mathbb{R}_+^X \times \mathbb{R}_+^Y$ such that there is a flow in (Z,E), with input vector u and output vector v, which is not greater than 1 in any edge.

(ii) Let (X,Y,E) be a bipartite graph. Vectors $u \in \mathbb{R}_+^X$ and $v \in \mathbb{R}_+^Y$ are *matched* in E if there exists a "flow" function $f: E \to \mathbb{R}_+$ such that

$$u(x) = \sum_{(x,y)\in E} f(x,y) \quad \text{and} \quad v(y) = \sum_{(x,y)\in E} f(x,y)$$

for $x \in X$ and $y \in Y$. It follows easily from HALL's marriage theorem (cf. MIRSKY [71] or McDIARMID [75c]) that u and v are matched in E iff $|u| = |v|$ and $u(X') \le v(E(X'))$ for all $X' \subset X$. Now let $d \in \mathbb{R}_+^X$ and $e \in \mathbb{R}_+^Y$ and put

$$L = \{(u,v) \in \mathbb{R}_+^X \times \mathbb{R}_+^Y \mid u \text{ and } v \text{ are matched in E and } u \le d,\ v \le e\}.$$

Then (X,Y,L) is a poly-linking system, with corresponding bi-submodular function λ given by

$$\lambda(X',Y') = \min_{X''\subset X'} (d(X'\setminus X'') + e(E(X'') \cap Y')).$$

If $d \in \mathbb{Z}_+^X$ and $e \in \mathbb{Z}_+^Y$ then (X,Y,L) is integral.

For other types of linking in graphs yielding poly-linking systems, cf. WOODALL [75].

6c. THE LINKING OF POLYMATROIDS AND POLYLINKING SYSTEMS

Theorem 2.3 and 2.5 exhibit the linking of matroids by linking systems; the present section is devoted to "poly"-extensions of these theorems.

First we extend theorem 2.3.

THEOREM 6.4. *Let* (X,Y,L) *be a poly-linking system (with corresponding bi-submodular function* λ*) and let* (X,P) *be a polymatroid (with corresponding submodular function* ρ*). Define furthermore*

$$P{*}L = \{v \mid (u,v) \in L \text{ for some } u \in P\}.$$

Then (Y,P*L) *is again a polymatroid, with corresponding submodular function* $\rho{*}\lambda$ *given by*

$$(\rho{*}\lambda)(Y') = \min_{X'\subset X} (\rho(X\setminus X') + \lambda(X',Y'))$$

for $Y' \subset Y$.

If P *and* L *are integral then also* P*L *is integral; in that case moreover, if* $v \in P{*}L$ *is integer-valued then* $(u,v) \in L$ *for some integer-valued* $u \in P$.

PROOF. Define for each $v \in \mathbb{R}_+^Y$

$$P_v = \{u \mid (u,v') \in L \text{ for some } v' \le v\}.$$

So $u \in P_v$ if and only if $(\lambda|(u,v))(X,Y) = |u|$, that is, if and only if

$$u(X') \le \min_{Y'\subset Y} (\lambda(X',Y') + v(Y\setminus Y')) \qquad (*)$$

for each $X' \subset X$. Defining

$$\rho_v(X') = \min_{Y'\subset Y} (\lambda(X',Y') + v(Y\setminus Y'))$$

$(v\in \mathbb{R}_+^Y, X'\subset X)$, we find that ρ_v is submodular and by (*) the corresponding polymatroid equals (X,P_v).
Now we arrive at a series of equivalent assertions. First let $v \in P * L$. This is equivalent to

$$|v| \leq \max \{|u| \mid u \in P_v, u \in P\}. \qquad (**)$$

By the intersection theorem for polymatroids this is the same as

$$|v| \leq \min_{X'\subset X} (\rho_v(X') + \rho(X\setminus X')) = \min_{X'\subset X, Y'\subset Y} (\rho(X\setminus X') + \lambda(X',Y') + v(Y\setminus Y')).$$

Let $\rho * \lambda$ be as given above. Then the inequality passes into

$$|v| \leq \min_{Y'\subset Y} ((\rho*\lambda)(Y') + v(Y\setminus Y')).$$

That is

$$v(Y') \leq (\rho*\lambda)(Y')$$

for each $Y' \subset Y$.
So we have proved

$$v \in P * L \Longleftrightarrow v(Y') \leq (\rho*\lambda)(Y') \quad \text{for all} \quad Y' \subset Y.$$

Since clearly $\rho * \lambda$ is submodular, $P * L$ and $\rho * \lambda$ form a corresponding pair of a polymatroid and a submodular function.

If P and L are integral then ρ and λ are integer-valued and hence $\rho * \lambda$ is integer-valued; hence also $P * L$ is integral. Moreover if $v \in P * L$ and v is integer-valued then ρ_v is integer-valued and hence P_v is integral. Then, again by the polymatroid-intersection theorem, the maximum in (**) is attained at an integer-valued u; so $(u,v) \in L$ and $u \in P$ for some integer-valued u. □

As said before, theorem 2.3 is a special case of theorem 6.4. Other corollaries are the following results of MCDIARMID [75c].

COROLLARY 6.4a (MCDIARMID [75c]). *Let* (X,Y,E) *be a bipartite graph and let* (X,P) *be a polymatroid. Let* Q *be the set of vectors* $v \in \mathbb{R}_+^X$ *such that for some* $u \in P$ *the vectors* u *and* v *are matched in* E. *Then* (Y,Q) *is again a polymatroid, with corresponding submodular function* σ *given by*

$$\sigma(Y') = \rho(E^{-1}(Y')),$$

for $Y' \subset Y$, *where* ρ *is the submodular function corresponding with* (X,P). *If* P *is integral then also* Q *is integral; in that case for each integer-valued* $v \in Q$ *there is an integer-valued* $u \in P$ *such that* u *and* v *are matched in* E.

PROOF. Apply theorem 6.4 to the last example of section 6b; we have to give integer-valued, sufficiently large capacities (e.g. larger than $\rho(x)$) to the vertices of the bipartite graph. □

It follows that, given a bipartite graph (X,Y,E) and a polymatroid (X,P) (with corresponding submodular function ρ), a vector $v \in \mathbb{R}_+^Y$ is the output vector of a flow in (X,Y,E) with input vector u in P if and only if

$$v(Y') \leq \rho(E^{-1}(Y'))$$

for all $Y' \subset Y$. MCDIARMID [75c] called this result "Rado's theorem for polymatroids".

Replacing the bipartite graph by a network we obtain

COROLLARY 6.4b. (MCDIARMID [75c]). *Let* (Z,E) *be a digraph and let* $c \in \mathbb{R}_+^Z$. *Let furthermore* $X, Y \subset Z$ *and let* (X,P) *be a polymatroid, with corresponding submodular function* ρ, *such that* $u_x \leq c_x$ *for all* $u \in P$ *and* $x \in X$. *Let* Q *be the set of output-vectors* v *in* $\mathbb{R}_+^Y$ *of flows respecting the capacity* c *and having input-vector in* P. *Then* (Y,Q) *is a polymatroid, with corresponding submodular function* σ *given by*

$$\sigma(Y') = \min_{Z' \subset Z \setminus Y'} (\rho(X \setminus (E(Z') \cup Z')) + c(E(Z') \setminus Z')),$$

for $Y' \subset Y$.

Furthermore, if c *is integer-valued and* P *is integral then* Q *is integral, and each integer-valued* $v \in Q$ *is the output-vector of a flow with integer-valued input-vector* $u \in P$.

PROOF. Apply theorem 6.4 to the network-example in section 6b. □

Let the data of corollary 6.4b be given.
Since, for $v \in \mathbb{R}_+^Y$,

$$v(Y') \leq \min_{Z' \subset Z \setminus Y'} (\rho(X \setminus (E(Z') \cup Z')) + c(E(Z') \setminus Z'))$$

for each $Y' \subset Y$, if and only if

$$v(Y\setminus Z') \le \rho(X\setminus(E(Z')\cup Z')) + c(E(Z')\setminus Z'))$$

for each $Z' \subset Z$, it follows that $v \in Q$ if and only if

$$v(Y\cap Z') \le \rho((X\cap Z')\setminus E(Z\setminus Z')) + c(Z'\cap E(Z\setminus Z'))$$

for all $Z' \subset Z$ (cf. theorem 6 of McDIARMID [75c]).

Using the polymatroid-intersection theorem we obtain easily the following extension of theorem 2.5.

THEOREM 6.5. *Let* (X,Y,L) *be a poly-linking system (with corresponding bi-submodular function* λ*) and let* (X,P) *and* (Y,Q) *be polymatroids (with corresponding submodular functions* ρ *and* σ*, respectively). Then*

$$\max\{|u| \mid u \in P,\ (u,v) \in L,\ v \in Q\} = \min_{X'\subset X, Y'\subset Y} (\rho(X\setminus X') + \lambda(X',Y') + \sigma(Y\setminus Y')).$$

Furthermore, if P, L *and* Q *are integral then the maximum is attained by integer-valued* u *and* v.

PROOF. An easy application of the polymatroid-intersection theorem to the polymatroids (Y,P*L) and (Y,Q) yields

$$\max\{|v| \mid u \in P,\ (u,v) \in L,\ v \in Q\} = \max\{|v| \mid v \in P * L \text{ and } v \in Q\} =$$

$$\min_{Y'\subset Y} ((\rho*\lambda)(Y') + \sigma(Y\setminus Y')) = \min_{X'\subset X, Y'\subset Y} (\rho(X\setminus X') + \lambda(X',Y') + \sigma(Y\setminus Y'))$$

If P, L and Q are integral then P * L is integral, hence, again by the polymatroid-intersection theorem, the maximum is attained by an integer-valued v. By theorem 6.4 for such v there exists an integer-valued $u \in P$ such that $(u,v) \in L$. □

Clearly, theorem 2.5 follows from this theorem. Another consequence is the following result of McDIARMID [75c].

COROLLARY 6.5a. (McDIARMID [75c]) *Let* (Z,E) *be a digraph and let* $c \in \mathbb{R}_+^Z$. *Let furthermore* (Z,P) *and* (Z,Q) *be polymatroids (with corresponding submodular functions* ρ *and* σ*, respectively) such that* $u \le c$ *and* $v \le c$ *for all* $u \in P$ *and*

$v \in Q$. *Then the maximal modulus* $|u|$ *of an input-vector* $u \in P$ *of a flow, respecting the capacity* c *and having output-vector* $v \in Q$, *equals*

$$\min_{Z' \subset Z} (\rho(Z\setminus(E(Z')\cup Z')) + c(E(Z')\setminus Z') + \sigma(Z')).$$

Furthermore, if c *is integer-valued and* P *and* Q *are integral then the maximum is attained by integer-valued* u *and* v.

PROOF. Insert the network-example in theorem 6.5. The expression for the minimum above is obtained by interchanging the order of minima. □

6d. EXCHANGING IN LINKED POLYMATROIDS

Axiom (iii) of definition 1.1, theorem 2.4 and theorem 2.6 together are chain-wise connected by implications: axiom (iii) follows from theorem 2.4, which itself follows from theorem 2.6. At the other hand, axiom (iii) was used in proving theorem 2.4, and this theorem itself was used in proving theorem 2.6. We proceed in the same manner to obtain the following generalization of 2.6.

THEOREM 6.6. *Let* (X,Y,L) *be a poly-linking system and let* (X,P) *and* (Y,Q) *be polymatroids. Furthermore, let* (u_1,v_1), $(u_2,v_2) \in L$ *such that* $u_1, u_2 \in P$ *and* $v_1, v_2 \in Q$. *Then there are* $u' \in P$ *and* $v' \in Q$ *such that* $(u',v') \in L$, $u' \leq u_1 \vee u_2$, $v' \leq v_1 \vee v_2$, u' *is a basis vector of* $u' \vee u_1$ *and* v' *is a basis vector of* $v' \vee v_2$.

If P, L *and* Q *are integral and* u_1, u_2, v_1, v_2 *are integer-valued we can have also* u' *and* v' *integer-valued.*

PROOF. The proof follows the same steps as the proofs of the theorem 2.4 and 2.6.

(i) We first prove the following generalization of theorem 2.4:
Let (X,Y,L) be a poly-linking system and let (X,P) be a polymatroid; furthermore, let (u_1,v_1), $(u_2,v_2) \in L$ such that $u_1, u_2 \in P$. Then there exists a $(u',v') \in L$ such that $u' \in P$, $u' \leq u_1 \vee u_2$, $v_2 \leq v' \leq v_1 \vee v_2$ and u' is a basisvector of $u' \vee u_1$ in (X,P). In case P and L are integral and u_1, u_2, v_1, v_2 are integer-valued we can have also u' and v' integer-valued.
To prove this, we may suppose that $u \leq u_1 \vee u_2$ for all $u \in P$.
Let v' be a basisvector of $v_1 \vee v_2$ in the polymatroid $(Y,P*L)$ (cf. theorem 6.4) such that $v_2 \leq v'$. Choose $u' \in P$ such that $(u',v') \in L$ and $|u' \wedge u_1|$ is as large as possible.

We are ready if we have proved that u' is a basisvector of $u' \vee u_1$ in (X,P); suppose to the contrary that $u' < u'' \le u' \vee u_1$ for some $u'' \in P$.
Since $(u_1,v_1) \in L$ a $v'' \le v_1$ exists such that $(u''\wedge u_1, v'') \in L$. As $(u',v') \in L$ there exists a $(u''',v''') \in L$ such that

$$u'' \wedge u_1 \le u''' \le (u''\wedge u_1) \vee u' = u''$$

and

$$v' \le v''' \le v' \vee v''.$$

Now from $u'' \in P$ it follows that $u''' \in P$ and hence $v''' \in P * L$. As v' is a basisvector of $v_1 \vee v_2$ in $(Y,P*L)$ and

$$v' \le v''' \le v' \vee v'' \le v_1 \vee v_2$$

we have $v' = v'''$. But

$$|u''' \wedge u_1| \ge | u'' \wedge u_1 | > | u' \wedge u_1|,$$

contradicting the maximality of $|u'\wedge u_1|$.
If P and L are integral and u_1,v_1,u_2,v_2 are integer-valued we can choose u',v',u'',v'',u''',v''' above integer-valued (cf. section 6b).

(ii) We now prove theorem 6.6. We may suppose that $u \le u_1 \vee u_2$ for all $u \in P$ and that $v \le v_1 \vee v_2$ for all $v \in Q$.
Let $(u',v') \in L$ be such that $u' \in P$, $v' \in Q$, u' is a basisvector of $u' \vee u_1$ and $|v' \wedge v_2|$ is as large as possible.
We are ready if we have proved that v' is a basisvector of $v' \vee v_2$ in the polymatroid (Y,Q); suppose to the contrary that $v' < v'' \le v' \vee v_2$ for some $v'' \in Q$.
Since $v_2 \in P * L$ also $v'' \wedge v_2 \in P * L$, say $(u'',v''\wedge v_2) \in L$ for $u'' \in P$. From (i), applied to (u',v') and $(u'',v''\wedge v_2)$ it follows that $(u''',v''') \in L$ exists such that $u''' \in P$, u''' is a basisvector of $u' \vee u'''$ in (X,P) and

$$v'' \wedge v_2 \le v''' \le (v''\wedge v_2) \vee v' = v''.$$

Since $v'' \in Q$ also $v''' \in Q$. The facts that u''' is a basisvector of $u' \vee u'''$ and that u' is a basisvector of $u' \vee u_1$ imply that u''' is a basisvector of $u''' \vee u_1$. But

$$|v''' \wedge v_2| \ge |v'' \wedge v_2| > | v' \wedge v_2|,$$

contradicting the maximality of $|v'\wedge v_2|$.
If P, L and Q are integral and u_1,v_1,u_2,v_2 are integer-valued we can choose u', v', u'', v'', u''', v''' integer-valued. □

As a consequence we have the following. Let (X,Y,L) be a poly-linking system and let (X,P) be a polymatroid. Let furthermore v_0 be a basisvector of $(Y,P*L)$, and $(u_0,v_0) \in L$ with $u_0 \in P$. Define

$$P_0 = \{u \in P \mid u_0 \text{ is a basis vector of } u \vee u_0\}.$$

Then $P*L = P_0*L$.

6e. THE PRODUCT AND SUM OF POLY-LINKING SYSTEMS

Finally we extend some operations on linking systems, namely the product and union (section 4a), to operations on poly-linking systems.

THEOREM 6.7. *Let (X,Y,L_1) and (Y,Z,L_2) be poly-linking systems, with corresponding bi-submodular functions λ_1 and λ_2. Define*

$$L_1*L_2 = \{(u,w) \mid (u,v) \in L_1 \text{ and } (v,w) \in L_2 \text{ for some } v\}.$$

Then (X,Z,L_1*L_2) is again a poly-linking system, with corresponding bi-submodular function given by

$$(\lambda_1*\lambda_2)(X',Z') = \min_{Y'\subset Y}(\lambda_1(X',Y') + \lambda_2(Y\backslash Y',Z'))$$

for $X'\subset X$ and $Z'\subset Z$.

If L_1 and L_2 are integral then also L_1*L_2 is integral.

PROOF. $\lambda_1 * \lambda_2$ as defined above clearly is bi-submodular. Let (X,Z,L) be the corresponding poly-linking system. We prove that $L_1 * L_2 = L$. Define for $u \in \mathbb{R}_+^X$ a polymatroid (Y,P_u) by

$$P_u = \{v \in \mathbb{R}_+^Y \mid (u',v) \in L_1 \text{ for some } u' \le u\}.$$

The corresponding submodular function ρ_u is given by

$$\rho_u(Y') = \min_{X'\subset X}(u(X\backslash X') + \lambda_1(X',Y'))$$

for $Y' \subset Y$. Similarly, define for $w \in \mathbb{R}_+^Z$ a polymatroid (Y,Q_w) by

$$Q_w = \{v \in \mathbb{R}_+^Y \mid (v,w') \in L_2 \text{ for some } w' \le w\}.$$

The corresponding submodular function σ_w is given by

$$\sigma_w(Y') = \min_{Z' \subset Z} (w(Z \setminus Z') + \lambda_2(Y',Z'))$$

for $Y' \subset Y$. Now $(u,w) \in L_1 * L_2$ if and only if the polymatroids P_u and Q_w have a common vector v, such that $|u| = |v| = |w|$. By EDMONDS' intersection theorem for polymatroids the latter is the case iff

$$\min_{Y' \subset Y} (\rho_u(Y') + \sigma_w(Y \setminus Y')) \geq |u| = |w|.$$

But this is true if and only if $|u| = |w|$ and for all $X' \subset X$ and $Z' \subset Z$ we have

$$u(X') + w(Z') \leq (\lambda_1 * \lambda_2)(X',Z') + |u|.$$

By definition of L, this implies and is implied by: $(u,w) \in L$.
In case λ_1 and λ_2 are integer-valued $\lambda_1 * \lambda_2$ is again integer-valued. □

If L_1 and L_2 are integral and $(u,w) \in L_1 * L_2$ is integer-valued, then $(u,v) \in L_1$ and $(v,w) \in L_2$ for some integer-valued v (this follows also from EDMONDS' theorem).

We call the poly-linking system $(X,Z,L_1 * L_2)$ the *product* of (X,Y,L_1) and (Y,Z,L_2). The *sum* of the poly-linking systems (X,Y,L_1) and (X,Y,L_2) is the system (X,Y,L_1+L_2), where, as usual,

$$L_1 + L_2 = \{(u_1+u_2,v_1+v_2) \,|\, (u_1,v_1) \in L_1,\ (u_2,v_2) \in L_2\}.$$

The fact that (X,Y,L_1+L_2) is again a poly-linking system is stated in the following theorem.

THEOREM 6.8. *Let* (X,Y,L_1) *and* (X,Y,L_2) *be poly-linking systems (with corresponding bi-submodular functions* λ_1 *and* λ_2*, respectively). Then* (X,Y,L_1+L_2) *again is a poly-linking system, with corresponding bi-submodular function* $\lambda_1 + \lambda_2$*. If* L_1 *and* L_2 *are integral then also* $L_1 + L_2$ *is integral.*

PROOF. Left to the reader (cf. theorem 4.2). □

Again, if L_1 and L_2 are integral and $(u,v) \in L_1 + L_2$ is integer-valued, then $(u,v) = (u_1+u_2,v_1+v_2)$ for integer-valued $(u_1,v_1) \in L_1$ and $(u_2,v_2) \in L_2$.

CHAPTER SEVEN

ALGORITHMS

An important motivation for investigating matroids comes from combinatorial optimization. In fact, matroids are characterized as those structures for which the greedy algorithm always yields optimal solutions. EDMONDS [70] showed that there exists also a good algorithm (i.e. an algorithm such that the required number of elementary steps is bounded by a polynomial in the size of the problem) that finds an optimal common independent set in two matroids. Since each linking system may be understood as a matroid with a fixed base (theorem 2.2) it is not surprising that also for certain optimization problems involving linking systems there are good algorithms.

In this chapter we give one, rather general, good algorithm, handling the following *cascade problem*. Let $(X_1,X_2,\Lambda_1),\ldots,(X_{k-1},X_k,\Lambda_{k-1})$ be linking systems and let $(X_1,\mathcal{I})$ and $(X_k,\mathcal{J})$ be matroids. Furthermore, let "weight" functions $w_1\colon X_1 \to \mathbb{R},\ldots,w_k\colon X_k \to \mathbb{R}$ be given. The problem which the algorithm solves is: given a natural number s, find a k-tuple $(X_1',\ldots,X_k')$ such that:

(i) $s = |X_1'| = \ldots = |X_k'|$;

(ii) $X_1' \in \mathcal{I}$, $X_k' \in \mathcal{J}$;

(iii) $(X_1',X_2') \in \Lambda_1,\ldots,(X_{k-1}',X_k') \in \Lambda_{k-1}$;

(iv) $w_1(X_1') +\ldots+ w_k(X_k')$ is as small as possible.

In particular, the algorithm determines whether a k-tuple with the properties (i), (ii) and (iii) exists. Note that weights may be negative, and that the algorithm also can be used to find *maximum* weighted k-tuples. In a simple way other good algorithms can be deduced from this algorithm, e.g. for the following problems.

(a) Given matroids $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ and a linking system (X,Y,Λ), find sets $X' \in \mathcal{I}$ and $Y' \in \mathcal{J}$ such that $(X',Y') \in \Lambda$ and $|X'|$ is as large as possible (cf. theorem 2.5).

(b) Given a matroid $(X,\mathcal{I})$, a linking system (X,Y,Λ) and a subset Y' of Y, determine whether $Y' \in \mathcal{I}*\Lambda$ (cf. theorem 2.3).

(c) Given linking systems (X,Y,Λ_1) and (Y,Z,Λ_2) and subsets $X' \subset X$ and $Z' \subset Z$, determine whether $(X',Z') \in \Lambda_1*\Lambda_2$ (cf. theorem 4.1).

(d) Given linking systems (X,Y,Λ_1) and (X,Y,Λ_2) and subsets $X' \subset X$ and $Y' \subset Y$, determine whether $(X',Y') \in \Lambda_1 \vee \Lambda_2$ (cf. theorem 4.2).

(e) Given matroids $(X_1,\mathcal{I}_1),\ldots,(X_m,\mathcal{I}_m)$, find $X_1' \in \mathcal{I}_1,\ldots,X_m' \in \mathcal{I}_m$ such that $|X_1' \cup \ldots \cup X_m'|$ is as large as possible (matroid partition; EDMONDS [67b], cf. KNUTH [73], GREENE & MAGNANTI [74]).

(f) Given matroids $(X,\mathcal{I})$ and $(X,\mathcal{J})$, find an $X' \in \mathcal{I} \cap \mathcal{J}$ such that $|X'|$ is as large as possible (matroid intersection; EDMONDS [70]).

(g) Given matroids $(X,\mathcal{I})$ and $(X,\mathcal{J})$ and a function $w:X \to \mathbb{R}$, find an $X' \in \mathcal{I} \cap \mathcal{J}$ such that $w(X')$ is as large as possible (EDMONDS [70], LAWLER [70,75a]).

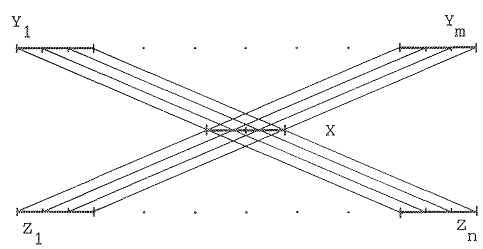
(h) Given matroids $(X,\mathcal{I}_1),\ldots,(X,\mathcal{I}_m)$, $(X,\mathcal{J}_1),\ldots,(X,\mathcal{J}_n)$ and real-valued functions $u_1,\ldots,u_m$, $v_1,\ldots,v_n$ on X, find pairwise disjoint sets $X_1' \in \mathcal{I}_1,\ldots,X_m' \in \mathcal{I}_m$, and, similarly, pairwise disjoint sets $X_1'' \in \mathcal{J}_1,\ldots,X_n'' \in \mathcal{J}_n$ such that

(i) $X_1' \cup \ldots \cup X_m' = X_1'' \cup \ldots \cup X_m''$;

(ii) $u_1(X_1') + \ldots + u_m(X_m') + v_1(X_1'') + \ldots + v_n(X_n'')$ is as large as possible

(KROGDAHL [74,76a]).

In order to obtain from our algorithm an algorithm for this last problem, first make disjoint copies $Y_1,\ldots,Y_m$, $Z_1,\ldots,Z_n$ of X. Let $(Y_1\cup\ldots\cup Y_m,X,\Lambda_1)$ be the linking system obtained from the bipartite graph $(Y_1\cup\ldots\cup Y_m,X,E_1)$ which arises by joining each element of $Y_1\cup\ldots\cup Y_m$ with the corresponding element in X by an edge. Similarly, let $(X,Z_1\cup\ldots\cup Z_n,\Lambda_2)$ be the linking system obtained from the bipartite graph $(X,Z_1\cup\ldots\cup Z_n,E_2)$ which arises by joining each element of X with the corresponding elements in $Z_1\cup\ldots\cup Z_n$ by an edge. A schematical representation of the systems is:



Form, in the obvious way, on $Y_1 \cup \ldots \cup Y_m$ the disjoint sum of the matroids $(X,I_1),\ldots,(X,I_m)$, and similarly, form on $Z_1 \cup \ldots \cup Z_n$ the disjoint sum of the matroids $(X,J_1),\ldots,(X,J_n)$. Furthermore, define the real-valued functions w_1 on $Y_1 \cup \ldots \cup Y_m$ and w_3 on $Z_1 \cup \ldots \cup Z_n$ by taking the disjoint unions of $-u_1,\ldots,-u_m$ and of $-v_1,\ldots,-v_n$, respectively. Finally, let w_2 be the zero function on X. Now our cascade algorithm applied to this structure gives an answer to problem (h).

Clearly, KROGDAHL's algorithm for problem (h) also solves (f) and (g). KROGDAHL's algorithm has been of great help in forming our algorithm for the cascade problem; some of his terminologies and ideas has been taken over. Actually, it can be shown that, conversely, KROGDAHL's algorithm gives an answer to the cascade problem.

Briefly we mention some aspects of combinatorial optimization which are relevant in the present approach; for a good survey on this subject we refer to the recent book of LAWLER [76].

A rather classical algorithm in discrete optimization (sometimes attributed to HALL [56] or to FORD & FULKERSON [56]) is the one for finding a matching with maximum cardinality in a bipartite graph.

This algorithm is based on the following step. Let $E' \subset E$ be a matching in the bipartite graph (X,Y,E), say between X' and Y'. Turn the direction of the arrows in E' (i.e. they now have tail in Y' and head in X'), and leave the direction of the other arrows in E unchanged. Find a path P in the so-obtained digraph, beginning in (some point of) $X\backslash X'$ and ending in $Y\backslash Y'$. The symmetric difference of E' and (the set of edges in) P is a matching with cardinality one greater than $|E'|$. If no such path P exists, E' has maximum cardinality.

This algorithm has been extended in (at least) three directions, namely to good algorithms for finding maximum:

(a) matchings in an arbitrary (undirected) graph (EDMONDS [65b]);
(b) common independent sets in two matroids (EDMONDS [70]);
(c) collections of disjoint paths between two specified subsets of a directed graph (FORD & FULKERSON [56]).

Since the third extension is of special interest for linking systems we give, in brief, the basic step of this algorithm. Let be given a directed graph $G = (Z,E)$, subsets $X' \subset X \subset Z$ and $Y' \subset Y \subset Z$ and a collection Π of disjoint paths from X' to Y', such that $|X'| = |Y'| = |\Pi|$. We may assume that each point of G has either in-degree 1 or out-degree 1. Let G' be the graph arising from G by reversing the direction of the edges occuring in paths of Π (and by leaving the remaining edges unchanged). Find a path P in G' from $X\setminus X'$ to $Y\setminus Y'$. Then the symmetric difference of the sets of edges in Π and in P forms a set of edges of a path collection Π' (between X and Y) such that $|\Pi'| = |\Pi| + 1$. If no such path P exists, Π has maximum cardinality.

Also good algorithms for finding optimal *weighted* solutions for the problems (i), (ii) and (iii) above have been found.

Very often a step in the algorithm consists of finding a path P from a subset X to a subset Y in a directed graph (Z,E) with weight function $w: Z \to \mathbb{R}$, such that w(P) is as small as possible (where w(P) is the sum of the weights of the vertices occurring in P). Evidently, an algorithm using such a step can only be good if there is a good algorithm for finding such a path. In case (Z,E) has no (directed) cycles C with negative weight w(C), there is indeed a good algorithm to this purpose, namely the *Bellman-Ford algorithm* (cf. BELLMAN [58] and FORD [56]).

This algorithm runs as follows. Define for i = 1,2,... the function $f_i: Z \to \mathbb{R} \cup \{\infty\}$ by

$$f_i(z) = \inf \{w(P) \mid P \text{ is a path from } X \text{ to } z \text{ with at most } i \text{ points}\}$$

for $z \in Z$. Let $|Z| = n$. Clearly, if $f_n(z) = \infty$ there is no path from X to z. If $f_n(z) \neq \infty$, the functions f_n and w together determine a path from X to z with minimal weight (in case the digraph has no negative cycles). Now calculate f_i inductively. In case i = 1 we have, obviously, $f_1(z) = w(z)$ if $z \in X$ and $f_1(z) = \infty$ if $z \notin X$. For $i > 1$ the function f_i satisfies, if there are no negative cycles,

$$f_i(z) = \min \{f_{i-1}(z), \min_{(z',z)\in E} (f_{i-1}(z') + w(z))\}.$$

for $z \in Z$. For directed graphs without negative cycles this algorithm finds paths with minimal weight. In particular, the algorithm determines *whether* a path from X to any vertex exists. The algorithm is good as long as there is a good algorithm for determining whether there is an edge from one point to another.

We now turn our attention to the algorithm for a weighted cascade of linking systems. First we observe that it does not matter, from the point of view of good algorithms, whether a linking system is given by its set Λ of linked pairs or by its linking function λ. As can be seen easily, there is a good algorithm for determining a specific value of λ whenever there is a good algorithm for determining whether a certain pair is in Λ, and conversely. Above we saw that there are indeed good algorithms to determine whether a certain pair is a linked pair of a deltoid or gammoid linking system, given the bipartite or directed graph, respectively. Also, given a matrix, a good algorithm (transforming it into echelon form) exists to determine whether a given submatrix is nonsingular.

We recall some results of chapter 3, in a somewhat different terminology. Let $M = (X,\mathcal{I})$ be a matroid and let B be a base of M. The *dependence graph* of M with respect to B is the bipartite graph $G = (B,X\backslash B,E)$ in which

$$(x,y) \in E \quad \text{iff} \quad (B\backslash\{x\})\cup\{y\} \text{ is a base of } M$$

for $x \in B$ and $y \in X\backslash B$. Clearly, G is the "underlying graph of the linking system corresponding with the matroid M with fixed base B". According to theorem 2.2, the theorems 3.3 and 3.2 assert the following:

(i) *if there is exactly one matching in* G *between* $B' \subset B$ *and* $C' \subset X\backslash B$, *then* $(B\backslash B') \cup C'$ *is a base of* M;

(ii) *if* $B' \subset B$, $C' \subset X\backslash B$ *and* $(B\backslash B') \cup C'$ *is a base of* M *then there exists a matching in* G *between* B' *and* C'.

The second result was first obtained by BRUALDI [69]; both results were also found, independently, by KROGDAHL [75].

Now, let (X_i,X_{i+1},Λ_i) be linking systems, for $i = 1,\ldots,k-1$, and let $(X_1,\mathcal{I})$ and $(X_k,\mathcal{J})$ be matroids. Furthermore, let weight functions $w_i : X_i \longrightarrow \mathbb{R}$ $(i = 1,\ldots,k)$ be given. Without loss of generality we may suppose that $X_1,\ldots,X_k$ are pairwise disjoint.

For a k-tuple $\mathcal{X}' = (X_1',\ldots,X_k')$ with $X_1' \subset X_1,\ldots,X_k' \subset X_k$, define $w(\mathcal{X}')$ by

$$w(\mathcal{X}') = w_1(X_1') + \ldots + w_k(X_k').$$

An *s-solution* is a k-tuple $\mathcal{X}' = (X_1',\ldots,X_k')$ such that

(i) $s = |X_1'| = \ldots = |X_k'|$;

(ii) $X_1' \in \mathcal{I}$, $X_k' \in \mathcal{J}$;

(iii) $(X_i',X_{i+1}') \in \Lambda_i$ for $i = 1,\ldots,k-1$.

An s-solution $\mathcal{X}'$ is *minimum* if $w(\mathcal{X}')$ is as small as possible for s-solutions. The problem for which we give an algorithm is to find, for each $s \geq 0$, a minimum s-solution.

Clearly, it is easy to find a minimum 0-solution; the only one is $(\emptyset,\dots,\emptyset)$. Suppose now, we have found a minimum s-solution $X' = (X_1',\dots,X_k')$; we give an algorithm for finding a minimum (s+1)-solution. To this end we define a directed graph on $X = X_1 \cup \dots \cup X_k$.

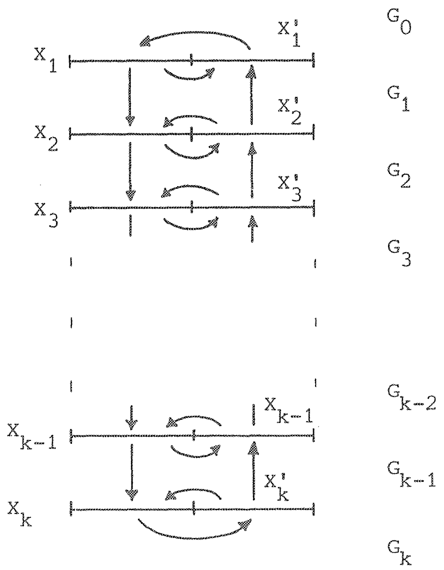
Since $(X_i', X_{i+1}') \in \Lambda_i$, we can make the dependence graph G_i of the matroid underlying $(X_i, X_{i+1}, \Lambda_i)$ with respect to the base $(X_i \setminus X_i') \cup X_{i+1}'$ (for $i = 1,\dots,k-1$). Moreover, let the bipartite graphs $G_0 = (X_1', X_1 \setminus X_1', E_0)$ and $G_k = (X_k \setminus X_k', X_k', E_k)$ be defined by

$$(x,y) \in E_0 \quad \text{iff} \quad (X_1' \setminus \{x\}) \cup \{y\} \in \mathcal{I}$$

for $x \in X_1'$ and $y \in X_1 \setminus X_1'$, and

$$(x,y) \in E_k \quad \text{iff} \quad (X_k' \setminus \{y\}) \cup \{x\} \in \mathcal{J}$$

for $x \in X_k \setminus X_k'$ and $y \in X_k'$. That is, G_0 is the dependence graph of the s-truncation of the matroid $(X_1, \mathcal{I})$ with respect to the base X_1', and G_k is the "converse" of the dependence graph of the s-truncation of the matroid $(X_k, \mathcal{J})$ with respect to the base X_k'. Let G be the union (on the vertex set $X = X_1 \cup \dots \cup K_k$) of the graphs $G_0, G_1, \dots, G_{k-1}, G_k$. The general form of this graph is shown in the figure, where each arrow stands for possible edges in the respective graphs.



Define a second weight functions v on $X = X_1 \cup \ldots \cup X_k$ by

$$v(x) = w_i(x) \quad \text{if} \quad x \in X_i \backslash X_i',$$
$$v(x) = -w_i(x) \quad \text{if} \quad x \in X_i',$$

for $i = 1,\ldots,k$. Let P be a directed path (or cycle) in G (i.e. without repeated vertices). Define the k-tuple $\mathcal{X}' \,\Delta\, P = (X_1'',\ldots,X_k'')$ by

$$X_i'' = (X_i' \backslash \{x \in X_i' \mid x \text{ occurs in } P\}) \cup \{x \in X_i \backslash X_i' \mid x \text{ occurs in } P\}$$

for $i = 1,\ldots,k$. Let v(P) be equal to

$$v(P) = \sum_{x \text{ in } P} v(x).$$

So we have $w(\mathcal{X}'\Delta P) = w(\mathcal{X}') + v(P)$.

In order to apply the Bellman-Ford algorithm to the digraph G with weight function v we have to prove that G has no negative cycles.

PROPOSITION. *G has no cycle C such that* $v(C) < 0$.

PROOF. Suppose C is a cycle in G such that $v(C) < 0$, and that C has as few points as possible. Then $\mathcal{X}' \,\Delta\, C$ is an s-solution, contradicting the fact that $w(\mathcal{X}')$ is as small as possible for s-solutions. That $\mathcal{X}' \,\Delta\, C$ is an s-solution can be proved by observing that, for each $i = 0,\ldots,k$, the arrows in C from G_i form a unique matching between the set of tails and the set of heads of those arrows (if the matching is not unique, there would exist a smaller negative cycle). Now theorem 3.3, in the form given above, implies that $\mathcal{X}' \,\Delta\, C$ is an s-solution. □

Finally we present our

ALGORITHM *to find a minimum (s+1)-solution, given the minimum s-solution* $\mathcal{X}'$.

Step 1. *Find a path* $P = (x_0,\ldots,x_p)$ *in G such that*

(i) $x_0 \in X_1 \backslash c\ell_I X_1'$ *and* $x_p \in X_k \backslash c\ell_J X_k'$;

(ii) $v(P)$ *is as small as possible for paths with property* (i);

(iii) *each ("cut off") path* $P' = (x_0,\ldots,x_{i-1},x_{j+1},\ldots,x_p)$ *from* $X_1 \backslash c\ell_I X_1'$ *to* $X_k \backslash c\ell_J X_k'$, *where* $0 \le i \le j \le p$, *has* $v(P') > v(P)$.

Step 2. $\mathcal{X}' \,\Delta\, P$ *is a minimum (s+1)-solution.*

Using the good Bellman-Ford algorithm step 1 can be carried out within polynomial time (provided there is a good algorithm for determining whether a pair is in Λ_i $(i = 1,\ldots,k-1)$, and whether a set is in I, or in J respectively).

<u>PROOF OF THE CORRECTNESS OF THE ALGORITHM</u>. To prove correctness it is sufficient to establish the following two assertions:

(a) if $P = (x_0,\dots,x_p)$ is a path from $X_1\backslash c\ell_I X_1'$ to $X_k\backslash c\ell_J X_k'$ without ("cut off") paths $P' = (x_0,\dots,x_{i-1},x_{j+1},\dots,x_p)$ such that $0 \le i \le j \le p$ and $v(P') \le v(P)$ then $\mathcal{X}' \,\Delta\, P$ is an $(s+1)$-solution;

(b) if $\mathcal{X}''$ is an $(s+1)$-solution then there is a path P from $X_1\backslash c\ell_I X_1'$ to $X_k\backslash c\ell_J X_k'$ such that $v(P) \le w(\mathcal{X}'') - w(\mathcal{X}')$.

Proof of (a). Let $P = (x_0,\dots,x_p)$ be a path from $X_1\backslash c\ell_I X_1'$ to $X_k\backslash c\ell_J X_k'$ without cut off paths. Now, for $i = 0,\dots,k$, the arrows in P from G_i form a unique matching between the set of tails and the set of heads of those arrows (otherwise, as can be checked easily, there would exist a negative cycle or a cut off path; note that x_0 and x_p are the only points of P in $X_1\backslash c\ell_I X_1'$ and $X_k\backslash c\ell_J X_k'$, respectively). Theorem 3.3 in the form given above implies that $\mathcal{X}' \,\Delta\, P$ is an $(s+1)$-solution.

Proof of (b). Let $\mathcal{X}'' = (X_1'',\dots,X_k'')$ be an $(s+1)$-solution. Now, for $i = 1,\dots,k-1$, according to theorem 3.2, there is a matching in G_i between $(X_{i+1}'\backslash X_{i+1}'') \cup (X_i''\backslash X_i')$ and $(X_{i+1}''\backslash X_{i+1}') \cup (X_i'\backslash X_i'')$. Take $x_0 \in X_1''\backslash c\ell_I X_1'$ and $x_p \in X_k''\backslash c\ell_J X_k'$. Again by theorem 3.2 there are matchings in G_0 and in G_k, between $X_1'\backslash X_1''$ and $X_1''\backslash(X_1'\cup\{x_0\})$, and between $X_k''\backslash(X_k'\cup\{x_p\})$ and $X_k'\backslash X_k''$, respectively. These matchings together form a digraph, consisting of a path P from x_0 to x_p, and a number of disjoint cycles, say $C_1,\dots,C_t$. Obviously

$$w(\mathcal{X}'') = w(\mathcal{X}') + v(P) + v(C_1) + \dots + v(C_t).$$

Since G has no negative cycles it follows that $v(P) \le w(\mathcal{X}'') - w(\mathcal{X}')$. □

We remark that in case $(X_i, X_{i+1}, \Lambda_i)$ is a deltoid or gammoid linking system we may replace G_i by the bipartite or directed graph with the direction of some of the arrows reversed (just as in the algorithms of HALL and FORD & FULKERSON, see above). The algorithm above, applied to a cascade of one bipartite graph with matroids on both sides (with weight functions the all zero functions, for instance) then passes into an algorithm developed by AIGNER & DOWLING [71a] (cf. VAN LINT [74], where one of the matroids is trivial). In the case of a directed graph medium as a linking system it is possible to give the points (or the edges) of the digraph non-negative "cost" values, and then to search for s-solutions such that the sum of the weights and costs in all points passed through is as small as possible. The algorithm (and the proof that the algorithm works) is similar to the one above.

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