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CONTENTS

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INTRODUCTION

In the last three decades the asymptotic theory of rank tests has received considerable attention. The early work in this area concerns the asymptotic normality of linear rank statistics under the hypothesis, i.e. in the case where the sample elements are independent and identically distributed (i.i.d.). Next, asymptotic normality under alternatives, where the sample consists of at most a finite number of independent groups of i.i.d. elements, was proved for fixed alternatives as well as for contiguous alternatives which tend to the hypothesis at a required rate. For the contiguous case these results were extended to e.g. regression alternatives, where the sample elements are independent but each has a different distribution belonging to a paramatric family of distributions. For the results quoted so far we refer to HÁJEK and ŠIDÁK (1967). These results place a severe restriction on the alternatives considered.

The study of the asymptotic behaviour of rank statistics for the general case where the sample elements are independent but may each have a different distribution and where their joint distribution is not necessarily contiguous to the hypothesis, was initiated by HÁJEK (1968) and DUPAČ and HÁJEK (1969). In continuation of this study, but following a different approach, we shall present in this thesis some theorems establishing asymptotic normality of rank statistics in a model which is by far more general than the models one encounters in the literature. We consider rank statistics of a very general type based on sample elements which are allowed to have different multivariate distribution functions.

Our way of dealing with the asymptotic distribution of statistics based on ranks - as they occur in nonparametric statistics - relies on the possibility to express these statistics in terms of empirical distribution functions. In this approach the empirical distribution functions and their properties serve as a probabilistic tool to arrive at results for the rank statistics. However, these properties are known in the i.i.d. case only and our situation requires knowledge of the empirical d.f. in the non-standard situation suggested above.

These fundamental properties of the empirical distribution functions in the non-i.i.d. case will be derived in Chapter I. It is rather striking that these properties carry over from the i.i.d. case to the non-i.i.d. case without any additional condition.

In Chapter II the asymptotic normality is established for standardized versions of rank statistics in the multivariate non-i.i.d. case of the type

$$S_N = N^{-1} \sum_{n=1}^{N} c_{nN} \, a_N(R_{1nN}, R_{2nN}, \ldots, R_{knN}),$$

where, for $n_i = 1,2,\ldots,N$, $i = 1,2,\ldots,k$, the $a_N(n_1,n_2,\ldots,n_k)$ are given real numbers, called scores; for $n = 1,2,\ldots,N$, the c_{nN} are given real constants, called regression constants, and where R_{inN} denotes the rank of the i-th coordinate of the n-th sample element among all i-th coordinates. Here N denotes the sample size and k is the dimension of the sample elements. The statistics S_N are called multivariate linear rank statistics.

For methodological reasons the regression constants c_{nN} will be introduced with the aid of an additional set of random variables, the 0-th coordinates of the sample elements, which are chosen such that with probability one the ranks of these 0-th coordinates are fixed. Clearly these 0-th coordinates then have different distribution functions. However the introduction of this additional randomness does not essentially complicate the problem of establishing the asymptotic normality of standardized versions of the statistics S_N, since we are studying the non-i.i.d. case anyhow. On the other hand the introduction of the dummy random variables has the advantage that it enables us to express the statistics S_N entirely in terms of the multivariate empirical distribution function and its univariate marginal empirical distribution functions.

CHAPTER I

SOME FUNDAMENTAL PROPERTIES OF THE EMPIRICAL DF IN THE NON-I.I.D. CASE

1.0. INTRODUCTION

Let k be a fixed positive integer and for each N = 1,2,..., let $X_{nN} = (X_{1nN}, X_{2nN}, \ldots, X_{knN})$, n = 1,2,...,N, be N mutually independent k-dimensional random vectors with joint distribution functions (d.f.'s)

$$(1.0.1) \qquad F_{nN}(x_1,x_2,\ldots,x_k) = P(X_{1nN} \le x_1, X_{2nN} \le x_2, \ldots, X_{knN} \le x_k), \quad \text{for all } -\infty < x_i < \infty,\ i = 1,2,\ldots,k,$$

and marginal d.f.'s $F_{1nN}, F_{2nN}, \ldots, F_{knN}$, i.e.

$$(1.0.2) \qquad F_{inN}(x) = P(X_{inN} \le x), \qquad \text{for all } -\infty < x < \infty,\ i = 1,2,\ldots,k.$$

All random vectors are supposed to be defined on a single probability space $(\Omega, \mathcal{A}, P)$. For each N, moreover, let us define the joint empirical d.f. $\mathbb{F}_N$ of $X_{1N}, X_{2N}, \ldots, X_{NN}$ by taking $N\mathbb{F}_N(x_1,x_2,\ldots,x_k)$ to be the number of elements in the set $\{X_{nN}: X_{1nN} \le x_1, X_{2nN} \le x_2, \ldots, X_{knN} \le x_k,\ n = 1,2,\ldots,N\}$, for all $-\infty < x_i < \infty$, i = 1,2,...,k, and the averaged d.f.'s $\overline{F}_N$ and $\overline{F}_{iN}$, i = 1,2,...,k, as

$$(1.0.3) \qquad \overline{F}_N(x_1,x_2,\ldots,x_k) = N^{-1} \sum_{n=1}^{N} F_{nN}(x_1,x_2,\ldots,x_k), \quad \text{for } -\infty < x_i < \infty,\ i = 1,2,\ldots,k,$$

$$(1.0.4) \qquad \overline{F}_{iN}(x) = N^{-1} \sum_{n=1}^{N} F_{inN}(x), \qquad \text{for } -\infty < x < \infty.$$

We remark that $\overline{F}_N$ has all the properties of a k-variate d.f. and that its marginal d.f.'s are the $\overline{F}_{iN}$, i = 1,2,...,k.

The classical theory on empirical d.f.'s deals with the case where the N random vectors $X_{1N}, X_{2N}, \ldots, X_{NN}$ are independent and identically distributed (i.i.d.). Our main purpose in this chapter is to derive some fundamental properties of the empirical d.f. in the non-i.i.d. case, where the N sample elements are assumed to be independent, but not necessarily identically distributed. In particular, we shall generalize results obtained by SHORACK (1970), GOVINDARAJULU, LECAM and RAGHAVACHARI (1967), RUYMGAART, SHORACK and VAN ZWET (1972) and VAN ZWET. VAN ZWET's theorem is published in RUYMGAART (1974). It is rather striking that most of the theorems considered in the i.i.d. case remain valid in the non-i.i.d. case without any additional condition. Apart from the fact that the authors mentioned above derived these theorems in the i.i.d. case, they also assumed - with the exception of VAN ZWET - the underlying distribution functions to be continuous. It is our second aim in this chapter to give a rigorous demonstration of the fact that, even in the non-i.i.d. case, most of the theorems considered also hold without this assumption.

Sections 1.1 and 1.2 deal with univariate and multivariate empirical d.f.'s in the case of continuous underlying d.f.'s. In section 1.3 it will be shown that the continuity assumption is superfluous in almost all theorems derived.

The theorems are useful for proving asymptotic normality of rank statistics in a situation where the multivariate sample elements are allowed to have different d.f.'s and where the scores generating functions are allowed to tend to infinity near the boundary of the unit interval and to have a finite number of discontinuities of the first kind. The theorems may also be of interest in their own right.

The basic tools for our study are two related results of HOEFFDING (1956), who showed that in a certain sense the non-i.i.d. case is not less favourable than the i.i.d. case, and a theorem of BILLINGSLEY (1968), page 94, on fluctuations of partial sums of random variables. We shall present these theorems without proofs.

Suppose that Z_n, $1 \leq n \leq N$, are independent random variables (r.v.'s) with

$$(1.0.5) \qquad P(Z_n=1) = 1 - P(Z_n=0) = p_n,$$

and suppose that

$$(1.0.6)\qquad 0 < \bar{p} = N^{-1} \sum_{n=1}^{N} p_n < 1.$$

THEOREM 1.0.1 (HOEFFDING). *If*

$$(1.0.7)\qquad f(k+2) - 2f(k+1) + f(k) > 0, \qquad k = 0,1,\dots,N-2.$$

then

$$(1.0.8)\qquad E\left(f\left(\sum_{n=1}^{N} Z_n\right)\right) \le \sum_{k=0}^{N} f(k)\binom{N}{k} \bar{p}^k (1-\bar{p})^{N-k},$$

where equality holds if and only if $p_1 = p_2 = \dots = p_N = \bar{p}$.

THEOREM 1.0.2 (HOEFFDING). *Let* b *and* c *be two integers such that*

$$0 \le b \le N\bar{p} \le c \le N.$$

Then

$$\sum_{n=b}^{c} \binom{N}{n} \bar{p}^n (1-\bar{p})^{N-n} \le P\left(b \le \sum_{n=1}^{N} Z_n \le c\right) \le 1.$$

Both bounds are attained. The lower bound is attained only if $p_1 = p_2 = \dots$ $\dots = p_N = \bar{p}$ *unless* $b = 0$ *and* $c = N$.

Let $\xi_1,\dots,\xi_m$ be random variables which need not be independent or identically distributed. Let $S_k = \xi_1 + \dots + \xi_k$ $(S_0=0)$, and put

$$(1.0.9)\qquad M_m = \max_{0\le k\le m} |S_k|.$$

THEOREM 1.0.3 (BILLINGSLEY). *Suppose that there exists* $\gamma \ge 0$, $\alpha > 1$, *and nonnegative numbers* $u_1,u_2,\dots,u_m$ *such that*

$$(1.0.10)\qquad E\left(|S_j-S_i|^{\gamma}\right) \le \left(\sum_{\ell=i+1}^{j} u_\ell\right)^{\alpha}, \qquad \text{for } 0 \le i \le j \le m.$$

Then, there exists a positive number $K = K(\gamma,\alpha)$, *only depending on* γ *and* α, *such that for all* $\lambda > 0$,

$$(1.0.11)\qquad P(M_m \ge \lambda) \le \frac{K}{\lambda^{\gamma}} \left(\sum_{\ell=1}^{m} u_\ell\right)^{\alpha}.$$

1.1. PROPERTIES OF THE UNIVARIATE EMPIRICAL DF IN THE CASE OF CONTINUOUS UNDERLYING DF'S

In this section we shall deal with the case that $k = 1$, so that for $N = 1,2,\ldots$, the univariate empirical d.f. $\mathbb{F}_N$ is based on the N random variables $X_{1N}, X_{2N}, \ldots, X_{NN}$, with d.f.'s $F_{1N}, F_{2N}, \ldots, F_{NN}$ respectively. Moreover, for the time being we shall assume the underlying d.f.'s to be continuous. Before stating the theorems we first have to introduce some further notation.

We recall that the averaged univariate d.f. $N^{-1} \Sigma_{n=1}^{N} F_{nN}$ is denoted by $\bar{F}_N$. For the set $X_{1N}, X_{2N}, \ldots, X_{NN}$, let us denote the order statistics by

$$X_{1:N} \leq X_{2:N} \leq \ldots \leq X_{N:N}. \tag{1.1.1}$$

Let F be a d.f. on $(-\infty,\infty)$, which is always taken to be right continuous. Define an inverse of this function by

$$F^{-1}(u) = \inf\{y: F(y) \geq u\}, \qquad \text{for } 0 < u \leq 1, \tag{1.1.2}$$

whereas $F^{-1}(0)$ is defined to be minus infinity. Here by way of exception a function is introduced which may assume an infinite value. According to (1.1.2), $F^{-1}(u)$ is non-decreasing, left continuous and satisfies $F(F^{-1}(u)) \geq u$, for all $0 \leq u \leq 1$, with equality if and only if F is continuous. Furthermore it has the property that $F^{-1}(F(y)) \leq y$, for all $y \in (-\infty,\infty)$, with equality if and only if F is strictly increasing.

We are now in a position to formulate our first theorem. It is a generalization to the non-i.i.d. case of a well-known result of SHORACK (1970), (1972). His result has been applied succesfully to the theory of rank tests in RUYMGAART, SHORACK and VAN ZWET (1972), RUYMGAART (1974) and to linear combinations of order statistics in SHORACK (1972). In Chapter II our generalization of SHORACK's result will be applied similarly in the non-i.i.d. situation for rank tests, whereas the application to linear combinations of order statistics will be discussed in Remark 1.1.3.
In the asymptotic theory the theorem makes it possible to bound certain random functions by other non random functions, see Lemma 2.3.2.

<u>THEOREM 1.1.1</u> *For every* $\beta \in (0,1)$, *every array of continuous underlying d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$ *and for every* $N = 1,2,\ldots$, *we have*

$$(1.1.3)\qquad P\left(\mathbb{F}_N(x) \le \beta^{-1}\,\bar{F}_N(x),\ \text{for } x\in(-\infty,\infty)\right) \ge 1-\frac{2}{3}\pi^2\,\beta(1-\beta)^{-4}$$

and

$$(1.1.4)\qquad P\left(\mathbb{F}_N(x) \ge \beta\,\bar{F}_N(x),\ \text{for } x\in[X_{1:N},\infty)\right) \ge 1-\frac{2}{3}\pi^2\,\beta^2(1-\beta)^{-4}.$$

For $n = 1,2,\ldots,N$ we define the r.v.'s X'_{nN} by $X'_{nN} := -X_{nN}$. Denote by F'_{nN} the d.f. of X'_{nN}, and by $\mathbb{F}'_N$ the empirical d.f. based on $X'_{1N}, X'_{2N},\ldots, X'_{NN}$. We have

$$(1.1.5)\qquad \bar{F}'_N(x) := N^{-1}\sum_{n=1}^{N} F'_{nN}(x) = N^{-1}\sum_{n=1}^{N}(1-F_{nN}(-x)) = 1-\bar{F}_N(-x),$$

$$\text{for } -\infty < x < \infty,$$

and for $n = 1,2,\ldots,N$,

$$(1.1.6)\qquad X'_{n:N} = -X_{N-n+1:N}.$$

Moreover, it is clear from the definitions and from (1.1.6) that the random functions $1-\mathbb{F}'_N(-x)$ and $\mathbb{F}_N(x)$ are simple step functions having jumps of height N^{-1} in the order statistics $X_{1:N}, X_{2:N},\ldots,X_{N:N}$, and that

$$(1.1.7)\qquad 1-\mathbb{F}'_N(-x) = \mathbb{F}_N(x),\qquad \text{for } x\in(-\infty,\infty)-\{X_{1:N},X_{2:N},\ldots,X_{N:N}\}.$$

Since $1-\mathbb{F}'_N(-x)$ is left continuous whereas $\mathbb{F}_N(x)$ is right continuous, it follows from (1.1.7) that

$$(1.1.8)\qquad 1-\mathbb{F}'_N(-x) \le \mathbb{F}_N(x),\qquad \text{for } x\in(-\infty,\infty).$$

In view of (1.1.5) - (1.1.8) and the fact that $\mathbb{F}_N$ is right continuous whereas $\bar{F}_N$ is continuous, we obtain for $0 < \beta < 1$, $N = 1,2,\ldots,$

$$(1.1.9)\qquad \left[\mathbb{F}'_N(x) \le \beta^{-1}\,\bar{F}'_N(x),\ \text{for } x\in(-\infty,\infty)\right] \Leftrightarrow$$

$$\Leftrightarrow \left[1-\mathbb{F}'_N(-x) \ge 1-\beta^{-1}(1-\bar{F}_N(x)),\ \text{for } x\in(-\infty,\infty)\right],$$

$$\Leftrightarrow \left[\mathbb{F}_N(x) \ge 1-\beta^{-1}(1-\bar{F}_N(x)),\ \text{for } x\in(-\infty,\infty)\right],$$

and

$$(1.1.10)\quad \left[\mathbb{F}'_N(x) \ \geq \beta\, \overline{F}'_N(x),\ \text{for } x \in [X'_{1:N},\infty)\right] \Leftrightarrow$$

$$\Leftrightarrow \left[1 - \mathbb{F}'_N(-x) \ \leq 1 - \beta(1-\overline{F}_N(x)),\ \text{for } x \in (-\infty, X_{N:N}]\right] \Leftrightarrow$$

$$\Leftrightarrow \left[\mathbb{F}_N(x) \ \leq 1 - \beta(1-\overline{F}_N(x)),\ \text{for } x \in (-\infty, X_{N:N}] - \{X_{1:N}, X_{2:N}, \ldots, X_{N:N}\}\right] \Leftrightarrow$$

$$\Leftrightarrow \left[\mathbb{F}_N(x) \ \leq 1 - \beta(1-\overline{F}_N(x)),\ \text{for } x \in (-\infty, X_{N:N})\right].$$

From (1.1.9) and (1.1.10) the following corollary of Theorem 1.1.1 is immediate:

COROLLARY 1.1.1 *For every* $\beta \in (0,1)$, *every array of continuous underlying d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$ *and every* $N = 1,2,\ldots$, *we have*

$$(1.1.11)\quad P\left(\mathbb{F}_N(x) \geq 1 - \beta^{-1}(1-\overline{F}_N(x)),\ \textit{for } x \in (-\infty,\infty)\right) \geq 1 - \tfrac{2}{3}\pi^2\, \beta(1-\beta)^{-4},$$

and

$$(1.1.12)\quad P\left(\mathbb{F}_N(x) \ \leq 1 - \beta(1-\overline{F}_N(x)),\ \textit{for } x \in (-\infty, X_{N:N})\right) \geq 1 - \tfrac{2}{3}\pi^2\, \beta^2(1-\beta)^{-4}.$$

REMARK 1.1.1. For $n = 1,2,\ldots,N$ we introduce the r.v.'s $\tilde{X}_{nN} := \overline{F}_N(X_{nN})$. We denote by $\tilde{F}_{nN}$ the d.f. of $\tilde{X}_{nN}$, and by $\tilde{\mathbb{F}}_N$ the empirical d.f. based on $\tilde{X}_{1N}, \tilde{X}_{2N}, \ldots, \tilde{X}_{NN}$. Following SHORACK (1973) we call $\tilde{\mathbb{F}}_N$ the reduced empirical d.f. of $X_{1N}, X_{2N}, \ldots, X_{NN}$. Since the $F_{1N}, F_{2N}, \ldots, F_{NN}$ are assumed to be continuous and are clearly constant on any interval where $\overline{F}_N$ is constant, we have that the $\tilde{F}_{1N}, \tilde{F}_{2N}, \ldots, \tilde{F}_{NN}$ are continuous on $[0,1]$ and in view of the remark below (1.1.2) that

$$(1.1.13)\quad \tilde{F}_{nN}(t) = F_{nN}(\overline{F}_N^{-1}(t)) \qquad \text{for } t \in [0,1],\ n = 1,2,\ldots,N,$$

and

$$(1.1.14)\quad \tilde{\overline{F}}_N(t) := N^{-1} \sum_{n=1}^{N} \tilde{F}_{nN}(t) = N^{-1} \sum_{n=1}^{N} F_{nN}(\overline{F}_N^{-1}(t)) = t,\ \text{for } 0 \leq t \leq 1.$$

DISCUSSION OF THEOREM 1.1.1. Theorem 1.1.1 and Corollary 1.1.1, applied to the reduced empirical d.f. defined in Remark 1.1.1, shows that for every $\varepsilon > 0$, there exist an angle $\alpha = \alpha(\varepsilon)$, $0 < \alpha < \pi/4$, such that for every array of continuous d.f.'s $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$ and for every $N = 1,2,\ldots$, we have, with $\beta = \mathrm{tg}\,\alpha$ (see Fig.1.1.1), that

$$(1.1.15)\quad P\Big(\Big[1-\beta^{-1}(1-t) \le \tilde{\mathbb{F}}_N(t) \le \beta^{-1}t,\ \text{for } t \in [0,1]\Big] \cap \Big[\tilde{\mathbb{F}}_N(t) \le 1-\beta(1-t),\ \text{for } t \in [0,\tilde{X}_{N:N})\Big] \cap \Big[\tilde{\mathbb{F}}_N(t) \ge \beta t,\ \text{for } t \in [\tilde{X}_{1:N},1]\Big]\Big) \ge 1-\varepsilon .$$

Of course, $\tilde{X}_{1:N}$ ($\tilde{X}_{N:N}$) denotes the smallest (largest) order statistic of the set $\tilde{X}_{1N}, \tilde{X}_{2N}, \ldots, \tilde{X}_{NN}$.

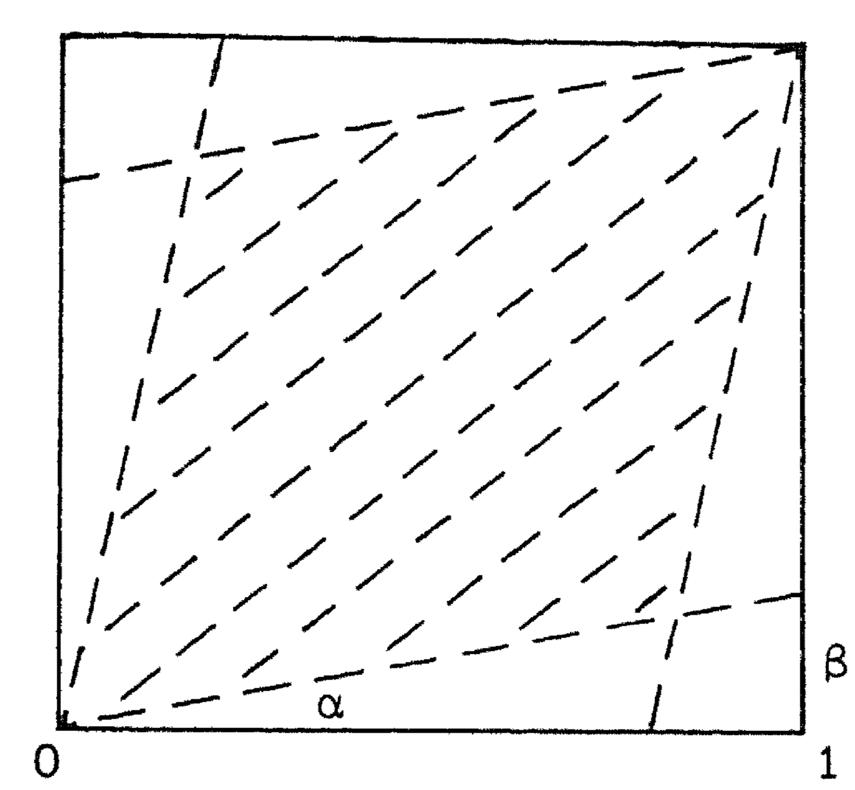

Fig. 1.1.1

Hence, apart from possible exceptions for the intervals $[0, \tilde{X}_{1:N})$ and $[\tilde{X}_{N:N}, 1]$, the angle α can be chosen, no matter the continuous underlying d.f.'s or N, such that with arbitrary high probability the reduced empirical d.f. $\tilde{\mathbb{F}}_N$ moves in the shaded area of Fig. 1.1.1. It is clear that there can be no hope for enlarging in the theorem the interval $[X_{1:N}, \infty)$ to $(-\infty, \infty)$.

PROOF OF THEOREM 1.1.1. Let Z_n, $1 \le n \le N$, be N independent BERNOULLI (p_n) r.v.'s as defined in (1.0.5). Theorem 1.0.1 together with MARKOV's inequality implies that for $j > N\bar{p}$,

$$P\left(\sum_{n=1}^{N} Z_n \geq j\right) \leq$$

$$\leq P\left(\left|\sum_{n=1}^{N} Z_n - N\bar{p}\right| \geq j - N\bar{p}\right) \leq$$

$$\leq (j-N\bar{p})^{-4}\, E\left(\sum_{n=1}^{N} Z_n - N\bar{p}\right)^4 \leq$$

(1.1.16)

$$\leq (j-N\bar{p})^{-4} \sum_{k=0}^{N} (k-N\bar{p})^4 \binom{N}{k} \bar{p}^k (1-\bar{p})^{N-k} =$$

$$= (j-N\bar{p})^{-4}\left\{(\bar{p}(1-\bar{p}))^2(3N^2-6N) + N\bar{p}(1-\bar{p})\right\} \leq$$

$$\leq (j-N\bar{p})^{-4} \min\left((3N^2\bar{p}^2+N\bar{p}),\ (3N^2(1-\bar{p})^2 + N(1-\bar{p}))\right).$$

Choose $N \in \{1,2,...\}$, continuous d.f.'s $F_{1N}, F_{2N}, ..., F_{NN}$ and $\beta \in (0,1)$. First let us prove (1.1.3). From BONFERRONI's inequality and the fact that $\mathbb{F}_N(X_{j:N}) = jN^{-1}$ with probability 1 for $j = 1,2,...,N$, we have that

$$P\left(\mathbb{F}_N(x) \leq \beta^{-1}\, \bar{F}_N(x),\ \text{for } x \in (-\infty,\infty)\right) = \tag{1.1.17}$$

$$= P\left(\mathbb{F}_N(X_{j:N}) \leq \beta^{-1}\, \bar{F}_N(X_{j:N}),\ \text{for } j = 1,2,...,N\right) \geq$$

$$\geq 1 - \sum_{j=1}^{N} P\left(\bar{F}_N(X_{j:N}) < \beta j N^{-1}\right) =$$

$$= 1 - \sum_{j=1}^{N} P\left(\sum_{n=1}^{N} Z_n \geq j\right),$$

where Z_n, $1 \leq n \leq N$, are independent BERNOULLI (p_n) r.v.'s with

$$p_n = F_{nN}\left(\bar{F}_N^{-1}(\beta j N^{-1})\right).$$

From (1.1.16) with $\bar{p} = N^{-1} \Sigma_{n=1}^{N} F_{nN}(\bar{F}_N^{-1}(\beta j N^{-1})) = \bar{F}_N(\bar{F}_N^{-1}(\beta j N^{-1})) = \beta j N^{-1}$, it is now immediate that, for $j = 1,2,...,N$,

$$P\left(\sum_{n=1}^{N} Z_n \geq j\right) \leq \beta(1-\beta)^{-4}\left(3\beta j^{-2} + j^{-3}\right) \leq 4\beta(1-\beta)^{-4} j^{-2}. \tag{1.1.18}$$

Relation (1.1.3) follows from (1.1.17), (1.1.18) and the fact that $\Sigma_{j=1}^{\infty} j^{-2} = \pi^2/6$.

For the proof of (1.1.4) essentially the same method can be used.

We have

$$(1.1.19)\quad P\Big(\mathbb{F}_N(x) \ge \beta\, \overline{F}_N(x),\ \text{for } x \in [X_{1:N},\infty)\Big) =$$

$$= P\Big(\bigcap_{j=2}^{N} \Big[\mathbb{F}_N(X_{j:N}-) \ge \beta\, \overline{F}_N(X_{j:N})\Big]\Big) \ge$$

$$\ge 1 - \sum_{j=2}^{N} P\Big(\overline{F}_N(X_{j:N}) > \beta^{-1}(j-1)N^{-1}\Big) =$$

$$= 1 - \sum_{j=2}^{[\beta N]+1} P\Big(\overline{F}_N(X_{j:N}) > \beta^{-1}(j-1)N^{-1}\Big),$$

where $[\beta N]$ is the greatest integer in βN. The terms with $j > [\beta N] + 1$ may be omitted because $\beta^{-1}(j-1)N^{-1} > 1$ for these terms. Since $j \ge 2$ we are guaranteed that $0 < \beta^{-1}(j-1)N^{-1} \le 1$ for every term.

Now for $j = 2,3,\ldots, [\beta N] + 1$,

$$(1.1.20)\quad P\Big(\overline{F}_N(X_{j:N}) > \beta^{-1}(j-1)N^{-1}\Big) = P\Big(\sum_{n=1}^{N} Z_n \ge N - j + 1\Big),$$

where $Z_1, Z_2, \ldots, Z_N$ are independent BERNOULLI (p_n) r.v.'s with

$$p_n = 1 - F_{nN}\Big(\overline{F}_N^{-1}(\beta^{-1}(j-1)N^{-1})\Big), \qquad \text{for } n = 1,2,\ldots,N.$$

From (1.1.20) and (1.1.16) with $\overline{p} = 1 - \beta^{-1}(j-1)N^{-1}$ it is immediate again that for $j = 2,3,\ldots, [\beta N] + 1$,

$$(1.1.21)\quad P\Big(\overline{F}_N(X_{j:N}) > \beta^{-1}(j-1)N^{-1}\Big) \le 4\beta^{-2}(\beta^{-1}-1)^{-4}(j-1)^{-2} =$$

$$= 4\beta^{2}(1-\beta)^{-4}(j-1)^{-2}.$$

Relation (1.1.4) follows from (1.1.19) and (1.1.21). □

REMARK 1.1.2. SHORACK's proof of (1.1.15) in the i.i.d. case, which is a consequence of Theorem 1.1.1 in the i.i.d. case, is based on the comparison of the empirical process with a POISSON process and on the HÁJEK-RENYI inequality. The present proof of the Theorem 1.1.1 in the non-i.i.d. case is entirely different and although a more general situation is considered, it is more elementary.

REMARK 1.1.3. In Theorem 3 of SHORACK (1973) sufficient conditions are given for asymptotic normality of linear combinations of functions of order statistics in the non-i.i.d. case. The limiting scores generating function J occurring in this theorem has to be bounded. As remarked in SHORACK (1973), the restriction to bounded J could be removed if Theorem 1.1.1 and its consequences for the quantile process were available. Hence our theorem fills the gap in extending the proof of Theorem 3 of SHORACK (1973), so that his Theorem 3 can now be claimed to hold without the assumption $b_1 = b_2 = 0$ (that is, for unbounded J).

Next, let us prove four lemmas, which may be of independent interest and are used to derive generalizations to the non-i.i.d. case of results obtained in GOVINDARAJULU, LECAM and RAGHAVACHARI (1967), and RUYMGAART, SHORACK and VAN ZWET (1972).

The first lemma supplies upper bounds for the central moments of $\Sigma_{n=1}^N Z_n$, where Z_n, $1 \le n \le N$, are independent BERNOULLI (p_n) r.v.'s, defined in (1.0.5). We recall that $\bar{p} = N^{-1}\,\Sigma_{n=1}^N p_n$.

LEMMA 1.1.1. *For every* $\alpha > \frac{1}{2}$, *there exists* $M_\alpha \in (0,\infty)$, *such that for* $N = 1, 2, \ldots$,

$$(1.1.22)\qquad E\left|\sum_{n=1}^N Z_n - N\bar{p}\right|^{2\alpha} \le \begin{cases} M_\alpha N\bar{p}, & \text{for } 0 \le \bar{p} \le N^{-1}, \\ M_\alpha\{N\bar{p}(1-\bar{p})\}^\alpha, & \text{for } N^{-1} \le \bar{p} \le 1 - N^{-1}, \\ M_\alpha N(1-\bar{p}), & \text{for } 1 - N^{-1} \le \bar{p} \le 1. \end{cases}$$

PROOF. Since the assertion is trivial for $\bar{p} = 0$ or $\bar{p} = 1$ and $\alpha > \frac{1}{2}$, Theorem 1.0.1 ensures that it is sufficient to prove Lemma 1.1.1 in the case where $p_1 = p_2 = \ldots = p_N = \bar{p} \in (0,1)$. First let us prove (1.1.22) for $N^{-1} \le \bar{p} \le 1 - N^{-1}$. Let $F_{N\bar{p}}(y)$ be the distribution function of $|\Sigma_{n=1}^N Z_n - N\bar{p}|\,(N\bar{p}(1-\bar{p}))^{-\frac{1}{2}}$. Then using an inequality due to S.N. BERNSTEIN (see e.g. BAHADUR (1966), page 578), we have for $y > 0$ that

$$1 - F_{N\bar{p}}(y) = P\left(\left|\sum_{n=1}^N Z_n - N\bar{p}\right| > y\sqrt{N\bar{p}(1-\bar{p})}\right) \le 2\exp\left(\frac{-y^2}{2 + \frac{2y}{3\sqrt{N\bar{p}(1-\bar{p})}}}\right).$$

Moreover, for $y \ge 1$ and $N^{-1} \le \bar{p} \le 1 - N^{-1}$, $N = 2,3,\ldots$, we have

$$(N\bar{p}(1-\bar{p}))^{-\frac{1}{2}} \le (N/(N-1))^{\frac{1}{2}} \le 2^{\frac{1}{2}},$$

so that then

$$1 - F_{N\bar{p}}(y) \le 2 \exp\left(\frac{-y^2}{4y}\right) = 2 \exp\left(-\frac{y}{4}\right).$$

Hence, for $N^{-1} \le \bar{p} \le 1 - N^{-1}$, $N = 2,3,\ldots,$

$$E\left|\frac{\sum Z_n - N\bar{p}}{\sqrt{N\bar{p}(1-\bar{p})}}\right|^{2\alpha} = \int_0^\infty y^{2\alpha} dF_{N\bar{p}}(y) = 2\alpha \int_0^\infty y^{2\alpha-1}(1-F_{N\bar{p}}(y))dy \le$$

$$\le 2\alpha \int_0^1 dy + 4\alpha \int_1^\infty y^{2\alpha-1} \exp\left(-\frac{y}{4}\right) dy,$$

so that (1.1.22) is proved for $N^{-1} \le \bar{p} \le 1 - N^{-1}$.

Let us next concentrate on the proof of (1.1.22) for $0 < \bar{p} \le N^{-1}$. For $k = 0,1,\ldots,N$, we have $P(\Sigma_{n=1}^N Z_n = k) \le \frac{(N\bar{p})^k}{k!}$ and $k \le e^k$, so that for $\alpha > \frac{1}{2}$, $0 < \bar{p} \le N^{-1}$, $N = 1,2,\ldots,$

$$E\left|\sum_{n=1}^N Z_n - N\bar{p}\right|^{2\alpha} =$$

$$= \sum_{k=0}^N |k - N\bar{p}|^{2\alpha} P\left(\sum_{n=1}^N Z_n = k\right) \le$$

$$\le \sum_{k=0}^\infty |k - N\bar{p}|^{2\alpha} \frac{(N\bar{p})^k}{k!} \le N\bar{p}\left[1 + \sum_{k=1}^\infty \frac{k^{2\alpha}}{k!}\right] \le$$

$$\le N\bar{p}\left[1 + \sum_{k=1}^\infty \frac{e^{2\alpha k}}{k!}\right] = N\bar{p} \sum_{k=0}^\infty \frac{e^{2\alpha k}}{k!} =$$

$$= N\bar{p} \exp(\exp(2\alpha)).$$

Relation (1.1.22) for $1 - N^{-1} \le \bar{p} \le 1$ follows from (1.1.22) for $0 \le \bar{p} \le N^{-1}$ by symmetry. □

With $\tilde{\mathbb{F}}_N$ as given in Remark 1.1.1, we define the reduced empirical process X_N by

$$(1.1.23) \qquad X_N(t) = N^{\frac{1}{2}}(\tilde{\mathbb{F}}_N(t)-t), \qquad \text{for } 0 \le t \le 1.$$

From this definition of X_N and Lemma 1.1.1 we obtain:

LEMMA 1.1.2. *For every* $\alpha > \frac{1}{2}$ *there exists* $M_\alpha \in (0,\infty)$ *such that for every array of continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every* $N = 1,2,\ldots$ *and every pair* $s,t \in [0,1]$,

$$(1.1.24)\qquad E|X_N(t) - X_N(s)|^{2\alpha} \le \begin{cases} M_\alpha N^{1-\alpha}|t-s|, & \text{if } 0 \le |t-s| \le N^{-1}, \\ M_\alpha |t-s|^\alpha (1-|t-s|)^\alpha, & \text{if } N^{-1} \le |t-s| \le 1-N^{-1}, \\ M_\alpha N^{1-\alpha}(1-|t-s|), & \text{if } 1-N^{-1} \le |t-s| \le 1. \end{cases}$$

PROOF. Let $\chi(S)$ denote the indicator function of a set S and let $\chi(S;s)$ denote the value of this function at the point s. Without loss of generality take $s < t$. Then,

$$(1.1.25)\qquad E|X_N(t) - X_N(s)|^{2\alpha} = N^{-\alpha}E|N\tilde{\mathbb{F}}_N(t) - N\tilde{\mathbb{F}}_N(s) - Nt + Ns|^{2\alpha} =$$

$$= N^{-\alpha}E|\sum_{n=1}^{N} \chi((s,t];\, \tilde{X}_{nN}) - N(t-s)|^{2\alpha} = N^{-\alpha}E|\sum_{n=1}^{N} Z_n - N(t-s)|^{2\alpha},$$

where Z_n, $1 \le n \le N$, are independent BERNOULLI (p_n) r.v.'s, with (see (1.1.3))

$$p_n = \tilde{F}_{nN}(t) - \tilde{F}_{nN}(s),$$

and hence $\bar{p} = t - s$. Relation (1.1.24) follows from (1.1.25) and Lemma 1.1.1. □

For $0 < \delta \le \frac{1}{2}$ we define the function q_δ as

$$(1.1.26)\qquad q_\delta(t) = \{t(1-t)\}^{\frac{1}{2}-\delta}, \qquad \text{for } 0 \le t \le 1.$$

Lemma 1.1.3, which will be derived from Lemma 1.1.2, tells us what happens with the upper bound in (1.1.24) if one replaces the process X_N by the process X_N/q_δ. Throughout this chapter $\frac{0}{0}$ is defined to be zero.

LEMMA 1.1.3. *For every* $\alpha > \frac{1}{2}$ *there exists* $\tilde{M}_\alpha \in (0,\infty)$ *such that for every array of continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every* $N = 1,2,\ldots$, *every pair* $s,t \in [N^{-1}, 1-N^{-1}] \cup \{0\} \cup \{1\}$ *with* $|t-s| \ge N^{-1}$, *and every* $\delta \in (0,\frac{1}{2}]$,

$$(1.1.27)\qquad E\left|\frac{X_N(t)}{q_\delta(t)}-\frac{X_N(s)}{q_\delta(s)}\right|^{2\alpha}\le \tilde{M}_\alpha|t-s|^{2\alpha\delta}.$$

PROOF. Without loss of generality take $0 \le s < t \le \frac{1}{2}$. The c_r-inequality and Lemma 1.1.2 yield for $N^{-1} \le s < t \le \frac{1}{2}$, $t-s \ge N^{-1}$,

$$E\left|\frac{X_N(t)}{q_\delta(t)}-\frac{X_N(s)}{q_\delta(s)}\right|^{2\alpha}\le 2^{2\alpha-1}\left\{E\left|\frac{X_N(t)-X_N(s)}{q_\delta(t)}\right|^{2\alpha}+\right.$$

$$\left.+\,E\left|X_N(s)\left(\frac{1}{q_\delta(s)}-\frac{1}{q_\delta(t)}\right)\right|^{2\alpha}\right\}\le 2^{2\alpha-1}M_\alpha\left\{\frac{(t-s)^\alpha}{(t/2)^{2\alpha(\frac{1}{2}-\delta)}}+\right.$$

$$\left.+\,s^\alpha\left(\frac{1}{q_\delta(s)}-\frac{1}{q_\delta(t)}\right)^{2\alpha}\right\}\le 2^{2\alpha-1}M_\alpha\{2^\alpha(t-s)^{2\alpha\delta}+2^\alpha(t-s)^{2\alpha\delta}\}=$$

$$=2^{3\alpha}M_\alpha(t-s)^{2\alpha\delta},$$

because

$$s^{\frac{1}{2}}\left(\frac{1}{q_\delta(s)}-\frac{1}{q_\delta(t)}\right)\le 2^{\frac{1}{2}}(t-s)^\delta,\qquad \text{for } 0\le s<t\le \tfrac{1}{2}.$$

For $s = 0$, $t - s \ge N^{-1}$ implies $t \ge N^{-1}$ and although now $X_N(s) = 0$ the proof is still formally correct. □

LEMMA 1.1.4. *For every* $\alpha > \frac{1}{2}$ *there exist* $M^* \in (0,\infty)$ *and* $M^*_\alpha \in (0,\infty)$ *such that for every array of continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 2,3,\ldots$, *every* $N = 2,3,\ldots$, *every* $\delta \in (0,\frac{1}{2}]$ *and every* $c > 0$,

$$(1.1.28)\qquad P\left(\sup_{|t-\frac{k}{N}|\le N^{-1}}\left|\frac{X_N(t)}{q_\delta(t)}-\frac{X_N(k/N)}{q_\delta(\frac{k}{N})}\right|\ge c\right)\le\begin{cases}M^*_\alpha(cN^\delta)^{-2\alpha}, & \text{for } k=2,3,\ldots,N-2,\\ M^*(cN^\delta)^{-1}, & \text{for } k=1,\ N-1,\end{cases}$$

and

$$(1.1.29)\qquad P\left(\sup_{|t-\frac{k}{N}|\le N^{-1}}\left|X_N(t)-X_N\left(\frac{k}{N}\right)\right|\ge c\right)\le M^*_\alpha(cN^{\frac{1}{2}})^{-2\alpha},\quad \text{for } k=1,2,\ldots,N-1.$$

PROOF. We assume $k + 1 \le \frac{1}{2}N$; the proof for other values of k requires only minor modifications.

Suppose first that $2 \le k \le \frac{1}{2}N-1$. Then

$$(1.1.30)\qquad \sup_{|t-kN^{-1}|\le N^{-1}} \left|\frac{X_N(t)}{q_\delta(t)} - \frac{X_N(kN^{-1})}{q_\delta(kN^{-1})}\right| \le$$

$$\le \sup_{|t-kN^{-1}|\le N^{-1}} \left|\frac{X_N(t)-X_N(kN^{-1})}{q_\delta(t)}\right| + |X_N(kN^{-1})|\left(\frac{1}{q_\delta\left(\frac{k-1}{N}\right)} - \frac{1}{q_\delta\left(\frac{k}{N}\right)}\right) \le$$

$$\le \frac{\left|X_N\left(\frac{k+1}{N}\right) - X_N\left(\frac{k-1}{N}\right)\right| + 4N^{-\frac{1}{2}}}{q_\delta\left(\frac{k-1}{N}\right)} + |X_N(kN^{-1})|\left(\frac{1}{q_\delta\left(\frac{k-1}{N}\right)} - \frac{1}{q_\delta\left(\frac{k}{N}\right)}\right).$$

Since $4\left\{N^{\frac{1}{2}}q_\delta\left(\frac{k-1}{N}\right)\right\}^{-1} \le 2^{5/2}\, N^{-\delta}$, the reasoning in the proof of Lemma 1.1.3 shows that for $\alpha > \frac{1}{2}$,

$$(1.1.31)\qquad E\left(\sup_{|t-kN^{-1}|\le N^{-1}} \left|\frac{X_N(t)}{q_\delta(t)} - \frac{X_N(kN^{-1})}{q_\delta(kN^{-1})}\right|\right)^{2\alpha} \le M'_\alpha\, N^{-2\alpha\delta}.$$

Application of MARKOV's inequality proves (1.1.28) for $2 \le k \le \frac{1}{2}N-1$; taking $\delta = \frac{1}{2}$ we also obtain (1.1.29) for $2 \le k \le \frac{1}{2}N-1$.

For $k = 1$ we note from Theorem 1.1.1 that for $0 < \beta \le \frac{1}{2}$,

$$P\left(\sup_{t\le N^{-1}} \frac{|X_N(t)|}{t} \ge (\beta^{-1}-1)N^{\frac{1}{2}}\right) \le 2^7\beta,$$

so that

$$P\left(\sup_{t\le N^{-1}} \frac{|X_N(t)|}{q_\delta(t)} \ge 2^{\frac{1}{2}}(\beta^{-1}-1)N^{-\delta}\right) \le 2^7\beta$$

and this proves (1.1.28) for $k = 1$ and $c \ge 2^{\frac{1}{2}}N^{-\delta}$ and hence for all $c > 0$.

Finally we note that for $\alpha > \frac{1}{2}$,

$$(1.1.32)\qquad E\left(\sup_{t\le N^{-1}} |X_N(t)|\right)^{2\alpha} \le E\left(|X_N(N^{-1})| + 2N^{-\frac{1}{2}}\right)^{2\alpha} \le M''_\alpha\, N^{-\alpha},$$

and the MARKOV inequality proves (1.1.29) for $k = 1$. □

Combination of Lemma 1.1.4 with Theorem 1.0.3 leads to the following two fundamental theorems:

<u>THEOREM 1.1.2</u>. *For every* $\alpha > \frac{1}{2}$ *there exists* $\overline{M}_\alpha > 0$ *such that for every array of continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every* $N = 1,2,\ldots$,

every $0 \le a < b \le 1$ *and every* $c > 0$,

$$(1.1.33)\qquad P\Big(\sup_{s,t\in[a,b]} |X_N(t) - X_N(s)| \ge c\Big) \le \begin{cases} \bar{M}_\alpha\, c^{-2\alpha}\, N^{1-\alpha}(b-a), & \text{if } b-a \le N^{-1}, \\ \bar{M}_\alpha\, c^{-2\alpha}(b-a)^\alpha, & \text{if } b-a > N^{-1}. \end{cases}$$

PROOF. If $b-a \le N^{-1}$, Lemma 1.1.2 and the c_r-inequality imply that for $\alpha > \frac{1}{2}$,

$$E\Big(\sup_{s,t\in[a,b]} |X_N(s) - X_N(t)|\Big)^{2\alpha} \le E\Big(|X_N(b) - X_N(a)| + 2N^{\frac{1}{2}}(b-a)\Big)^{2\alpha} \le$$

$$\le 2^{2\alpha-1}\Big(M_\alpha N^{1-\alpha}(b-a) + 2^{2\alpha}N^\alpha(b-a)^{2\alpha}\Big) \le 2^{2\alpha-1}(M_\alpha+2^{2\alpha})N^{1-\alpha}(b-a),$$

and application of MARKOV's inequality proves the first part of the theorem. If $b-a > N^{-1}$, let k and $k+m$ be the smallest and largest integers in $[aN,bN]$, so that $m \le (b-a)N$. Define $S_i = X_N\big(\frac{k+i}{N}\big) - X_N\big(\frac{k}{N}\big)$, $i = 0,1,\ldots,m$. Then $S_0 = 0$ and from Lemma 1.1.2 we have

$$E|S_j - S_i|^{2\alpha} \le M_\alpha\Big(\frac{j-i}{N}\Big)^\alpha, \qquad \text{for } 0 \le i \le j \le m.$$

It follows from Theorem 1.0.3 that for $\alpha > 1$,

$$P\Big(\max_{0\le i\le m}\Big|X_N\Big(\frac{k+i}{N}\Big) - X_N\Big(\frac{k}{N}\Big)\Big| > c\Big) \le M'_\alpha\, c^{-2\alpha}\Big(\frac{m}{N}\Big)^\alpha \le M'_\alpha\, c^{-2\alpha}(b-a)^\alpha.$$

Combining this with the second part of Lemma 1.1.4 we find for $\alpha > 1$,

$$P\Big(\sup_{s,t\in[a,b]} |X_N(t) - X_N(s)| \ge c\Big) \le 2P\Big(\sup_{t\in[a,b]}\Big|X_N(t) - X_N\Big(\frac{k}{N}\Big)\Big| \ge c/2\Big) \le$$

$$\le 2M^*_\alpha\Big(\frac{cN^{\frac{1}{2}}}{4}\Big)^{-2\alpha}(m+1) + 2M'_\alpha\Big(\frac{c}{4}\Big)^{-2\alpha}(b-a)^\alpha \le$$

$$\le M^*_\alpha\, 2^{4\alpha+2}c^{-2\alpha}N^{-\alpha}(b-a)N + M'_\alpha\, 2^{4\alpha+1}c^{-2\alpha}(b-a)^\alpha \le$$

$$\le \Big(M^*_\alpha\, 2^{4\alpha+2} + M'_\alpha 2^{4\alpha+1}\Big)\, c^{-2\alpha}(b-a)^\alpha.$$

Since a probability is bounded by 1, the result remains true for $\alpha > \frac{1}{2}$ if we take $\overline{M}_\alpha \geq 1$. □

<u>THEOREM 1.1.3</u>. *For every* $\alpha > 0$ *and every* $\delta \in (0,\frac{1}{2}]$ *there exist* $\overline{M} > 0$ *and* $\overline{M}_{\alpha,\delta} > 0$ *such that for every array of continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every* $N = 1,2,\ldots$ *and every* $c > 0$,

$$(1.1.34)\qquad P\left(\sup_{t\in[0,1]} \left|\frac{X_N(t)}{q_\delta(t)}\right| \geq c\right) \leq \overline{M}_{\alpha,\delta} c^{-2\alpha} + \overline{M}\, c^{-1} N^{-\delta}.$$

<u>PROOF</u>. Define $S_i = X_N(iN^{-1})/q_\delta(iN^{-1})$, for $i = 1,2,\ldots,N$, $S_0 = 0$. Lemma 1.1.3 ensures that for $\alpha > \frac{1}{2}$,

$$E|S_j - S_i|^{2\alpha} \leq \tilde{M}_\alpha\left(\frac{j-i}{N}\right)^{2\alpha\delta}, \qquad \text{for } 0 \leq i \leq j \leq N.$$

Theorem 1.0.3 implies, for $\alpha > (2\delta)^{-1}$,

$$P\left(\max_{0\leq k\leq N} \left|\frac{X_N(kN^{-1})}{q_\delta(kN^{-1})}\right| \geq c\right) \leq M_{\alpha,\delta} c^{-2\alpha}.$$

Application of Lemma 1.1.4 yields

$$P\left(\sup_{t\in[0,1]} \left|\frac{X_N(t)}{q_\delta(t)}\right| \geq c\right) \leq 2^{2\alpha} M_{\alpha,\delta} c^{-2\alpha} + 2^{2\alpha} N M_\alpha^* (cN^\delta)^{-2\alpha} + 4M^* (cN^\delta)^{-1},$$

which proves the theorem for $\alpha > (2\delta)^{-1}$ and hence for every $\alpha > 0$. □

The following corollary is immediate from Theorem 1.1.3:

<u>COROLLARY 1.1.2</u>. *For every* $\varepsilon > 0$ *and every* $\delta \in (0,\frac{1}{2}]$, *there exists* $M = M(\varepsilon,\delta)$, *such that for every array of continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *and every* $N = 1,2,\ldots$,

$$(1.1.35)\qquad P\left(\sup_{-\infty<x<\infty} \frac{N^{\frac{1}{2}}|\mathbb{F}_N(x) - \overline{F}_N(x)|}{q_\delta(\overline{F}_N(x))} \geq M\right) \leq \varepsilon.$$

<u>PROOF</u>. Since $\overline{F}_N$ is assumed to be continuous, we have

$$(1.1.36)\qquad \sup_{0\leq t\leq 1} \frac{N^{\frac{1}{2}}|\tilde{\mathbb{F}}_N(t) - t|}{q_\delta(t)} = \sup_{-\infty<x<\infty} \frac{N^{\frac{1}{2}}|\tilde{\mathbb{F}}_N(\overline{F}_N(x)) - \overline{F}_N(x)|}{q_\delta(\overline{F}_N(x))}.$$

Moreover, $\tilde{\mathbb{F}}_N \circ \bar{F}_N = \mathbb{F}_N$ with probability 1, so that (1.1.35) follows from (1.1.34) and (1.1.36). □

Corollary 1.1.2 is basic in the asymptotic theory of rank statistics in the case where the sample elements are allowed to have different d.f.'s. In particular this corollary can be used to counterbalance the growth of the scores generating functions near the boundary of the unit interval. In the i.i.d. case Corollary 1.1.2 is proved for the first time in GOVINDARAJULU, LECAM and RAGHAVACHARI (1967). PYKE and SHORACK (1968) gave a simpler proof with the aid of the POISSON process and the BIRNBAUM-MARSHALL inequality. The result in the non-i.i.d. case for continuous underlying d.f.'s is already given in SEN (1970). However, it is clear from SHORACK (1973) that the proof given by SEN is incorrect. The proof given here is different from the methods used by the authors mentioned above.

In order to formulate a corollary of Theorem 1.1.2 let us introduce for every positive integer m the function I_m on [0,1] defined by $I_m(1) = 1$ and

$$(1.1.37) \qquad I_m(u) = \frac{i-1}{m} \qquad \text{for } \frac{i-1}{m} \le u < \frac{i}{m}, \qquad i = 1,2,\ldots,m.$$

<u>COROLLARY 1.1.3</u>. *For every* $\varepsilon > 0$ *and every* $c > 0$, *there exist* $N_0 = N_0(\varepsilon,c)$ *and* $m_0 = m_0(\varepsilon,c)$, *such that for every array of continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every* $N \ge N_0$ *and every* $m \ge m_0$,

$$(1.1.38) \qquad P\left(\sup_{0\le t\le 1} |X_N(I_m(t)) - X_N(t)| \ge c\right) \le \varepsilon.$$

<u>PROOF</u>. Note that Theorem 1.1.2 implies that for every $\alpha > \frac{1}{2}$,

$$P\left(\sup_{0\le t<1} |X_N(I_m(t))-X_N(t)|\ge c\right) = P\left(\max_{k=1,2,\ldots,m} \sup_{\frac{k-1}{m}\le t<\frac{k}{m}} |X_N(I_m(t))-X_N(t)|\ge c\right) \le$$

$$\le \sum_{k=1}^{m} P\left(\sup_{\frac{k-1}{m}\le t<\frac{k}{m}} \left|X_N\left(\frac{k-1}{m}\right)-X_N(t)\right|\ge c\right) \le \bar{M}_\alpha\, c^{-2\alpha}\{\min(m,N)\}^{1-\alpha}. \qquad \square$$

Corollary 1.1.3 is a generalization to the non-i.i.d. case of a theorem due to RUYMGAART, SHORACK and VAN ZWET (1972). This result is especially useful in the asymptotic theory of rank statistics when one wants to replace certain integrals with respect to the measure induced by the empirical d.f. by the corresponding integrals with respect to the measure induced by the averaged d.f..

A second consequence of Theorem 1.1.2 is Corollary 1.1.4. It is a stronger statament than Theorem 1.2.1 for k = 1.

COROLLARY 1.1.4. *For every* $\varepsilon > 0$ *there exists* $M = M(\varepsilon)$, *such that for every array of continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every* $N = 1,2,\ldots$, *every* $0 \le a < b \le 1$,

$$(1.1.39)\qquad P\left(\sup_{s,t\in[a,b]} |X_N(t) - X_N(s)| \ge M(b-a)^{\frac{1}{2}}\right) \le \varepsilon.$$

PROOF. Apply Theorem 1.1.2 with $\alpha = 1$ and $c = M(b-a)^{\frac{1}{2}}$. □

The last theorem in this section is also of much help in the asymptotic theory of rank statistics in the non-i.i.d. case. For instance, it is useful when one wants to replace Theorem 1.1.1 and Corollary 1.1.2, which supply bounds for the empirical d.f. $\mathbb{F}_N$, by similar statements where bounds are given for the modified empirical d.f. $\mathbb{F}_N^*$, defined as $\mathbb{F}_N^* = \frac{N}{N+1}\mathbb{F}_N$ (see Lemma 2.3.1).

THEOREM 1.1.4. *For* $N \in \{1,2,\ldots\}$, *continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$ *and* $\alpha \in (0,N)$, *we have*

$$(1.1.40)\qquad P\left(\bar{F}_N(X_{N:N}) \le 1 - \alpha/N\right) \le (1-\alpha/N)^N \le e^{-\alpha},$$

$$(1.1.41)\qquad P\left(\bar{F}_N(X_{1:N}) \ge \alpha/N\right) \le (1-\alpha/N)^N \le e^{-\alpha}.$$

For α *restricted to the interval* $(0,1)$, *we have, even if the sample elements are not independent,*

$$(1.1.42)\qquad P\left(\bar{F}_N(X_{N:N}) \le 1 - \alpha/N\right) \ge 1 - \alpha,$$

$$(1.1.43)\qquad P\left(\bar{F}_N(X_{1:N}) \ge \alpha/N\right) \ge 1 - \alpha.$$

PROOF. Note that

$$(1.1.44)\qquad P\left(\bar{F}_N(X_{N:N}) \le 1 - \alpha/N\right) = P\left(X_{N:N} \le \bar{F}_N^{-1}(1-\alpha/N)\right) = \prod_{n=1}^{N} F_{nN}\left(\bar{F}_N^{-1}(1-\alpha/N)\right).$$

Hence, from the concavity of log y and JENSEN's inequality we obtain,

(1.1.45) $$\frac{1}{N}\sum_{n=1}^{N} \log F_{nN}\left(\bar{F}_N^{-1}(1-\alpha/N)\right) = \frac{1}{N}\log \prod_{n=1}^{N} F_{nN}\left(\bar{F}_N^{-1}(1-\alpha/N)\right) =$$

$$= \frac{1}{N}\log P\left(F_N(X_{N:N}) \le 1-\alpha/N\right) \le \log(1-\alpha/N),$$

which proves (1.1.40). For the proof of (1.1.41) observe that application of (1.1.40) to the random variables X'_{nN}, $n = 1,2,\ldots,N$, defined above (1.1.5), together with (1.1.6) and (1.1.5), shows that

$$P\left(\bar{F}_N(X_{1:N}) \ge \alpha/N\right) = P\left(1 - \bar{F}_N(X_{1:N}) \le 1-\alpha/N\right) =$$

$$= P\left(1 - \bar{F}_N(-X'_{N:N}) \le 1-\alpha/N\right) = P\left(\bar{F}'_N(X'_{N:N}) \le 1-\alpha/N\right) \le (1-\alpha/N)^N.$$

In order to prove (1.1.42) we remark that BONFERRONI's inequality implies that

$$P\left(\bar{F}_N(X_{N:N}) \le 1-\alpha/N\right) = P\left(\bigcap_{n=1}^{N}\left[X_{nN} \le \bar{F}_N^{-1}(1-\alpha/N)\right]\right) =$$

$$= 1 - P\left(\bigcup_{n=1}^{N}\left[X_{nN} > \bar{F}_N^{-1}(1-\alpha/N)\right]\right) \ge$$

$$\ge 1 - \sum_{n=1}^{N} P\left(X_{nN} > \bar{F}_N^{-1}(1-\alpha/N)\right) =$$

$$= 1 - \sum_{n=1}^{N}\left(1 - P\left(X_{nN} \le \bar{F}_N^{-1}(1-\alpha/N)\right)\right) =$$

$$= 1 - \sum_{n=1}^{N}\left(1 - F_{nN}\left(\bar{F}_N^{-1}(1-\alpha/N)\right)\right) = 1 - N + N\bar{F}_N\left(\bar{F}_N^{-1}(1-\alpha/N)\right) =$$

$$= 1 - N + N - \alpha = 1-\alpha.$$

Finally, (1.1.43) can be proved again from (1.1.42) with the aid of the r.v.'s X'_{nN} defined above (1.1.5). □

REMARK 1.1.4. The bounds derived in Theorem 1.1.4 are sharp in the sense that one can construct examples where these bounds are attained. If $F_{1N} = F_{2N} = \dots = F_{NN} = \bar{F}_N$ then $P\left(\bar{F}_N(X_{N:N}) \le 1 - \alpha/N\right) = (1-\alpha/N)^N$. Moreover, if F_{1N} is chosen such that $1 - F_{1N}\left(\bar{F}_N^{-1}(1-\alpha/N)\right) = \alpha$ and $F_{2N}, F_{3N}, \dots, F_{NN}$ such that $1 - F_{nN}\left(\bar{F}_N^{-1}(1-\alpha/N)\right) = 0$ for $n = 2,3,\dots,N$, then

$$P\left(\bar{F}_N(X_{N:N}) \le 1 - \alpha/N\right) = 1-\alpha.$$

REMARK 1.1.5. The continuity of the underlying d.f.'s is essential for the relations (1.1.41) and (1.1.42), as the following counterexample shows. Take $N = 2$, $\alpha = \frac{1}{2}$ and for $a < b$, $F_1(x) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{elsewhere} \end{cases}$, $F_2(x) = \begin{cases} 0 & \text{for } x < b \\ 1 & \text{elsewhere} \end{cases}$.

We conclude this section by remarking that MEHRA and RAO (1975) also used BILLINGSLEY's Theorem 1.0.3 fruitfully in their study of the one-dimensional empirical process, divided by certain q-functions, in the situation where the sample elements do have a common d.f., but where they are not necessarily independent.

1.2. A PROPERTY OF THE MULTIVARIATE EMPIRICAL DF IN THE CASE OF CONTINUOUS UNDERLYING DF'S

In this section k is an arbitrary positive integer, so that for N = 1, 2,..., the multivariate d.f. $\mathbb{F}_N$ is based on the N random vectors $X_{nN} = (X_{1nN}, X_{2nN}, \ldots, X_{knN})$, $n = 1,2,\ldots,N$, with d.f.'s $F_{1N}, F_{2N}, \ldots, F_{NN}$ respectively. Assuming for the moment again continuity of these underlying d.f.'s, we shall present a generalization of a slightly weaker version of a theorem due to VAN ZWET (Lemma 4.4 in RUYMGAART (1974)). See also BAHADUR (1966). In fact VAN ZWET proved that, in the i.i.d. case, Theorem 1.2.1 below holds, without the factor $(\log(N+1))^{\frac{1}{2}}$ in (1.2.1). We conjecture that one can dispense with this factor in the non-i.i.d. case too. This conjecture is clearly true for k = 1, where the theorem follows from Corollary 1.1.4. However, the present Theorem 1.2.1 is strong enough for our purposes in Chapter II, where it makes it possible to handle problems, connected with discontinuities in the scores generating functions of rank statistics.

By an abuse of notation we write $\mathbb{F}_N$ and $\bar{F}_N$ for the measure induced by the d.f.'s, thus $\mathbb{F}_N\{B\} = \int_B d\mathbb{F}_N$, $\bar{F}_N\{B\} = \int_B d\bar{F}_N$ for a Borel set B in $\mathbb{R}^k$. An interval in $\mathbb{R}^k$ is defined as the product set of k intervals, closed, open or half open, on the line.

<u>THEOREM 1.2.1</u>. *Let* I *be an interval in* $\mathbb{R}^k$ *and let* $\mathcal{I} = \{I^*: I^*$ *is an interval contained in* $I\}$. *For every* $\varepsilon > 0$ *and every positive integer* k, *there exist* $M = M(\varepsilon,k)$, *such that for every array of* k*-variate continuous d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every interval* I *and every* $N = 1,2,\ldots$,

$$P\left(\sup_{I^* \in \mathcal{I}} |\mathbb{F}_N\{I^*\} - \bar{F}_N\{I^*\}| \le M\left(\frac{\log(N+1)\bar{F}_N\{I\}}{N}\right)^{\frac{1}{2}} \right) \ge 1 - \varepsilon. \tag{1.2.1}$$

Before presenting the proof of this theorem, we shall prove a lemma which supplies an upper bound for $\sup_{I^* \in \mathcal{I}} |\mathbb{F}_N\{I^*\} - \bar{F}_N\{I^*\}|$ in terms of a maximum over a finite number of sets.

By [a] we denote the largest integer in the number a.

<u>LEMMA 1.2.1</u>. *Let for* $N = 1,2,\ldots$ *the* k*-dimensional d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$ *be continuous and let* I *be an interval in* $\mathbb{R}^k$ *with* $\bar{F}_N\{I\} > 0$, *for*

$N = 1,2,\ldots$. *Define* $\bar{F}_{iN}^{-1}(1+a) = \infty$, *for* $a > 0$, *where* $\bar{F}_{iN} = N^{-1}\ \Sigma_{n=1}^{N}\ F_{inN}$, *and let*

$$I = \{I^*:\ I^* \text{ is an interval contained in } I\},$$

$$(1.2.2)\qquad \tilde{I}_N = \left\{\tilde{I}_N:\ \tilde{I}_N = I \cap \prod_{i=1}^{k} \left(\bar{F}_{iN}^{-1}\left(\frac{n_{i1}}{N}\,\bar{F}_N\{I\}\right),\ \bar{F}_{iN}^{-1}\left(\frac{n_{i2}}{N}\,\bar{F}_N\{I\}\right)\right],\right.$$

for k *pairs of integers* (n_{i1}, n_{i2}), *with* $n_{i1} < n_{i2}$ *and*

$$\left. n_{ij} \in \left\{0,1,2,\ldots,\left[\frac{N}{\bar{F}_N\{I\}}\right] + 1\right\},\qquad \text{for } i = 1,2,\ldots,k,\ j = 1,2\right\}.$$

Then, for every $\omega \in \Omega$, $N = 1,2,\ldots$, $k = 1,2,\ldots$ *we have*

$$(1.2.3)\qquad \sup_{I^* \in I} |\mathbb{F}_N\{I^*\} - \bar{F}_N\{I^*\}| \le \max_{\tilde{I}_N \in \tilde{I}_N} |\mathbb{F}_N\{\tilde{I}_N\} - \bar{F}_N\{\tilde{I}_N\}| + 2kN^{-1}\bar{F}_N\{I\}.$$

PROOF. Let I^* be an arbitrary interval in I. Define

$$\bar{I}_N^* = \bigcap_{\substack{\tilde{I}_N \in \tilde{I}_N \\ \tilde{I}_N \supset I^*}} \tilde{I}_N, \qquad \underline{I}_N^* = \bigcup_{\substack{\tilde{I}_N \in \tilde{I}_N \\ \tilde{I}_N \subset I^*}} \tilde{I}_N.$$

Note that $\bar{I}_N^*$ and $\underline{I}_N^*$ are elements of $\tilde{I}_N \cup \emptyset$ and that

$$(1.2.4)\qquad \bar{F}_N\{\bar{I}_N^*\} - \bar{F}_N\{\underline{I}_N^*\} \le 2kN^{-1}\bar{F}_N\{I\}.$$

If I^* is such that $\mathbb{F}_N\{I^*\} - \bar{F}_N\{I^*\} \ge 0$, we have using (1.2.4),

$$|\mathbb{F}_N\{I^*\} - \bar{F}_N\{I^*\}| = \mathbb{F}_N\{I^*\} - \bar{F}_N\{I^*\} \le \mathbb{F}_N\{\bar{I}_N^*\} - \bar{F}_N\{\underline{I}_N^*\} \le$$

$$\le \mathbb{F}_N\{\bar{I}_N^*\} - \bar{F}_N\{\bar{I}_N^*\} + 2kN^{-1}\bar{F}_N\{I\} \le$$

$$\le |\mathbb{F}_N\{\bar{I}_N^*\} - \bar{F}_N\{\bar{I}_N^*\}| + 2kN^{-1}\bar{F}_N\{I\}$$

and if $\mathbb{F}_N\{I^*\} - \bar{F}_N\{I^*\} < 0$, we have

$$|\mathbb{F}_N\{I^*\} - \overline{F}_N\{I^*\}| = \overline{F}_N\{I^*\} - \mathbb{F}_N\{I^*\} \le \overline{F}_N\{\overline{I}_N^*\} - \mathbb{F}_N\{\underline{I}_N^*\} \le$$

$$\le \overline{F}_N\{\underline{I}_N^*\} - \mathbb{F}_N\{\underline{I}_N^*\} + 2kN^{-1}\overline{F}_N\{I\} \le$$

$$\le |\mathbb{F}_N\{\underline{I}_N^*\} - F_N\{\underline{I}_N^*\}| + 2kN^{-1}\overline{F}_N\{I\}. \quad \square$$

<u>PROOF OF THEOREM 1.2.1</u>. If $\overline{F}_N\{I\} = 0$, the theorem follows immediately. It proves to be convenient to consider the cases $0 < \overline{F}_N\{I\} \le \frac{8\log(N+1)}{\varepsilon N}$ and $\overline{F}_N\{I\} > \frac{8\log(N+1)}{\varepsilon N}$, for fixed $0 < \varepsilon < 1$, separately. Compare with RUYMGAART (1973), page 19.

First suppose that $0 < \overline{F}_N\{I\} \le 8(\varepsilon N)^{-1}\log(N+1)$, and choose $M = M_1(\varepsilon) = (2/\varepsilon)^{3/2}$. Then

$$M\left(\frac{\log(N+1)\overline{F}_N\{I\}}{N}\right)^{\frac{1}{2}} \ge M\left(\frac{\varepsilon(\overline{F}_N\{I\})^2}{8}\right)^{\frac{1}{2}} \ge \frac{\overline{F}_N\{I\}}{\varepsilon}. \tag{1.2.5}$$

Moreover, since

$$\sup_{I^*\in\mathcal{I}} |\mathbb{F}_N\{I^*\} - \overline{F}_N\{I^*\}| \le \max(\mathbb{F}_N\{I\}, \overline{F}_N\{I\}), \tag{1.2.6}$$

we have from (1.2.5), (1.2.6) and MARKOV's inequality that the left-hand side of (1.2.1) is bounded below by

$$P(\max(\mathbb{F}_N\{I\}, \overline{F}_N\{I\}) \le \overline{F}_N\{I\}/\varepsilon) = P(\mathbb{F}_N\{I\} \le \overline{F}_N\{I\}/\varepsilon) \ge 1 - \varepsilon.$$

Next we suppose that $\overline{F}_N\{I\} > 8(\varepsilon N)^{-1}\log(N+1)$. Application of Lemma 1.2.1 shows that for $M > M_2(k) = 4k(\log 2)^{-\frac{1}{2}}$ and $N = 1,2,\ldots$, the left hand side of (1.2.1) is bounded below by

$$P\left(\max_{\tilde{I}_N\in\tilde{\mathcal{I}}_N} |\mathbb{F}_N\{\tilde{I}_N\} - \overline{F}_N\{\tilde{I}_N\}| \le M\left(\frac{\log(N+1)\overline{F}_N\{I\}}{N}\right)^{\frac{1}{2}} - 2k\,\frac{\overline{F}_N\{I\}}{N}\right) \ge \tag{1.2.7}$$

$$\ge P\left(\max_{\tilde{I}_N\in\tilde{\mathcal{I}}_N} |\mathbb{F}_N\{\tilde{I}_N\} - \overline{F}_N\{\tilde{I}_N\}| \le \tfrac{1}{2}M\left(\frac{\log(N+1)\overline{F}_N\{I\}}{N}\right)^{\frac{1}{2}}\right) \ge$$

$$\ge 1 - \sum_{\tilde{I}_N\in\tilde{\mathcal{I}}_N} P\left(|\mathbb{F}_N\{\tilde{I}_N\} - \overline{F}_N\{\tilde{I}_N\}| > \tfrac{1}{2}M\left(\frac{\log(N+1)\overline{F}_N\{I\}}{N}\right)^{\frac{1}{2}}\right).$$

Since $\frac{1}{2}M(N \log(N+1)\overline{F}_N\{I\})^{\frac{1}{2}} \ge 1$ for $M \ge M_3 = \frac{1}{2}\sqrt{2}(\log 2)^{-\frac{1}{2}}$, Theorem 1.0.2 is applicable, so that we may assume that $N\,\mathbb{F}_N\{\tilde{I}_N\}$ in (1.2.7) is a binomial r.v. with parameters N and $\overline{F}_N\{\tilde{I}_N\}$.

With the aid of BERNSTEIN's inequality (see e.g. BAHADUR (1966), page 578) we find, using $\max(\overline{F}_N\{\tilde{I}_N\}, 1-\overline{F}_N\{\tilde{I}_N\}) \le 1$, that for $N = 1,2,\ldots,$ and $M > 0$,

$$(1.2.8) \qquad P\left(\left|\mathbb{F}_N\{\tilde{I}_N\} - \overline{F}_N\{\tilde{I}_N\}\right| \ge \tfrac{1}{2}M\left(\frac{\log(N+1)\overline{F}_N\{I\}}{N}\right)^{\frac{1}{2}}\right) \le$$

$$\le 2\exp\left(-\frac{\frac{1}{4}M^2 N \log(N+1)\overline{F}_N\{I\}}{2N\,\overline{F}_N\{\tilde{I}_N\} + \frac{1}{3}M(N\log(N+1)\overline{F}_N\{I\})^{\frac{1}{2}}}\right).$$

Moreover, since $\overline{F}_N\{I\} > 8(\varepsilon N)^{-1}\log(N+1) > 8N^{-1}\log(N+1)$ and $\overline{F}_N\{\tilde{I}_N\} \le \overline{F}_N\{I\}$, we obtain the following upper bound for (1.2.8),

$$(1.2.9) \qquad 2\exp\left(-\frac{\frac{3}{2}M^2\sqrt{2}\log(N+1)}{12\sqrt{2}+M}\right) \le 2\exp(-\tfrac{3}{4}M\log(N+1)),$$

$$\text{for } M \ge M_4 = 12\sqrt{2}.$$

Noting that the number of elements in $\tilde{\mathcal{I}}_N$ is bounded above by

$$\left(\frac{N}{\overline{F}_N\{I\}} + 2\right)^{2k} \le \left(\frac{N^2\varepsilon}{8\log(N+1)} + 2\right)^{2k} \le (5N^2)^{2k},$$

we obtain from (1.2.6)-(1.2.9) that for $M \ge \max(M_1, M_2, M_3, M_4, 5\frac{1}{3}k)$, $N = 1,2, \ldots,$

$$(1.2.10) \qquad P\left(\sup_{I^* \in \mathcal{I}} \left|\mathbb{F}_N\{I^*\} - \overline{F}_N\{I^*\}\right| \le M\left(\frac{\log(N+1)\overline{F}_N\{I\}}{N}\right)^{\frac{1}{2}}\right) \ge$$

$$\ge 1 - (5N^2)^{2k}\,2(N+1)^{-\frac{3}{4}M} \ge 1 - 2.5^{2k}(N+1)^{4k-\frac{3}{4}M} \ge$$

$$\ge 1 - 2.5^{2k}.2^{4k-\frac{3}{4}M},$$

which completes the proof of the theorem. □

REMARK 1.2.1. If in Theorem 1.2.1 we take $I = \mathbb{R}^k$, we obtain the following result which is a kind of GLIVENKO-CANTELLI theorem:

For every $\varepsilon > 0$ and every positive integer k, there exists $M = M(\varepsilon,k)$ such that for every array of k-variate continuous d.f.'s $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, and every $N = 1,2,\ldots$,

$$(1.2.11) \quad P\left(\sup_{-\infty<x_1,x_2,\ldots,x_k<\infty} |\mathbb{F}_N(x_1,x_2,\ldots,x_k) - \bar{F}_N(x_1,x_2,\ldots,x_k)| \leq M\, N^{-\frac{1}{2}} (\log(N+1))^{\frac{1}{2}} \right) \geq 1 - \varepsilon.$$

1.3. DISCONTINUOUS UNDERLYING DF'S

In this section we shall establish a theorem which makes it clear that, without any additional condition, the most important results from the foregoing sections remain valid without the restriction of continuous underlying d.f.'s. For related results see e.g. BEHNEN (1976) and CONOVER (1973).

An interval $I \subset \mathbb{R}^k$ is defined in the introduction of section 1.2; the corresponding definition of the class of intervals I is given in Theorem 1.2.1. Given a set S, S^c will denote its complement, $\chi(S)$ its indicator function and $\chi(S;s)$ the value of this function at the point s, i.e.

$$\chi(S;s) = \begin{cases} 1 \text{ for } s \in S, \\ 0 \text{ for } s \in S^c. \end{cases} \tag{1.3.1}$$

THEOREM 1.3.1. *Let* k *be a positive integer and let* $\mathbb{F}_N$ *be the empirical d.f. based on* N k*-variate sample elements* $X_{nN} = (X_{1nN}, X_{2nN}, \ldots, X_{knN})$, $n = 1,2,\ldots,N$, *where the* X_{nN} *are distributed independently according to given, possibly discontinuous d.f.'s* F_{nN}. *Let us denote for* $i = 1,2,\ldots,k$ *by* F_{inN} *the* i^{th} *marginal d.f. of* F_{nN}, *let* $\bar{F}_{iN} = N^{-1} \Sigma_{n=1}^{N} F_{inN}$, *let* $\{\xi_\nu^{(i)}, \nu = 1,2,\ldots\}$ *be the countable set of discontinuity points of* $\bar{F}_{iN}$ *and let* $p_\nu^{(i)}$ *be the height of the jump at* $\xi_\nu^{(i)}$ *of* $\bar{F}_{iN}$. *Finally let* I *be an interval in* $\mathbb{R}^k$. *There exist* N k*-variate random vectors* $Y_{nN} = (Y_{1nN}, Y_{2nN}, \ldots, Y_{knN})$, $n = 1,2,\ldots,N$, *where the* Y_{nN} *are distributed independently according to continuous d.f.'s* G_{nN}, *and an interval* $\tilde{I} \subset \mathbb{R}^k$, *such that*

$$\bar{F}_N(x_1,x_2,\ldots,x_k) = \tag{1.3.2}$$

$$= \bar{G}_N\Big(x_1 + \sum_\nu p_\nu^{(1)} \chi([\xi_\nu^{(1)},\infty);x_1),\ldots,x_k + \sum_\nu p_\nu^{(k)} \chi([\xi_\nu^{(k)},\infty);x_k)\Big)$$

and with probability one

$$\mathbb{F}_N(x_1,x_2,\ldots,x_k) = \tag{1.3.3}$$

$$= \mathbb{G}_N\Big(x_1 + \sum_\nu p_\nu^{(1)} \chi([\xi_\nu^{(1)},\infty);x_1),\ldots,x_k + \sum_\nu p_\nu^{(k)} \chi([\xi_\nu^{(k)},\infty);x_k)\Big)$$

and

$$(1.3.4)\qquad \sup_{I^*\in\mathcal{I}} \frac{|\mathbb{F}_N\{I^*\} - \bar{F}_N\{I^*\}|}{(\bar{F}_N\{I\})^{\frac{1}{2}}} \le \sup_{\tilde{I}^*\in\tilde{\mathcal{I}}} \frac{|\mathbb{G}_N\{\tilde{I}^*\} - \bar{G}_N\{\tilde{I}^*\}|}{(\bar{G}_N\{\tilde{I}\})^{\frac{1}{2}}},$$

where $\mathbb{G}_N$ *denotes the empirical d.f. based on the* Y_{nN}, $n = 1,2,\dots,N$ *and* $\bar{F}_N = N^{-1}\,\Sigma_{n=1}^N\, F_{nN}$, $\bar{G}_N = N^{-1}\,\Sigma_{n=1}^N\, G_{nN}$.

PROOF. Let $\{U_\nu^{(in)},\ i = 1,2,\dots,k,\ n = 1,2,\dots,N,\ \nu = 1,2,\dots\}$ be a set of uniform (0,1) distributed r.v.'s, mutually independent and also independent of the random vectors X_{nN}, $n = 1,2,\dots,N$. Note that $\{\xi_\nu^{(i)},\ \nu = 1,2,\dots\}$ contains the discontinuity points of each F_{inN}, $n = 1,2,\dots,N$.

Since $\Sigma_\nu p_\nu^{(i)} \le 1$ for $i = 1,2,\dots,k$, we can define for $n = 1,2,\dots,N$ the random vector $Y_{nN} = (Y_{1nN},Y_{2nN},\dots,Y_{knN})$ as follows:

$$(1.3.5)\qquad Y_{inN} = X_{inN} + \sum_\nu p_\nu^{(i)}\ \chi((\xi_\nu^{(i)},\infty);X_{inN}) + \sum_\nu p_\nu^{(i)}\ U_\nu^{(in)}\ \chi(\{\xi_\nu^{(i)}\};X_{inN}),$$

for $i = 1,2,\dots,k$, so that X_{nN} is transformed stochastically to Y_{nN}. Let G_{nN} be the d.f. of Y_{nN} and let $\mathbb{G}_N$ be the empirical d.f. based on Y_{1N}, $Y_{2N},\dots,Y_{NN}$. It is clear that all the marginal d.f.'s of G_{nN} are continuous and hence G_{nN} is continuous. From definition (1.3.5) it is immediate that for $n = 1,2,\dots,N$ and $i = 1,2,\dots,k$,

$$(1.3.6)\qquad X_{inN} + \sum_\nu p_\nu^{(i)}\ \chi((\xi_\nu^{(i)},\infty);X_{inN}) \le Y_{inN} \le X_{inN} + \sum_\nu p_\nu^{(i)}\ \chi([\xi_\nu^{(i)},\infty);X_{inN}),$$

and hence

$$(1.3.7)\qquad [X_{inN} \le x_i] \Leftrightarrow [Y_{inN} \le x_i + \sum_\nu p_\nu^{(i)}\ \chi([\xi_\nu^{(i)},\infty);x_i)],$$

$$(1.3.8)\qquad [X_{inN} < x_i] \Leftrightarrow [Y_{inN} < x_i + \sum_\nu p_\nu^{(i)}\ \chi((\xi_\nu^{(i)},\infty);x_i)].$$

From (1.3.7) it is obvious that (with $\bar{G}_N = N^{-1}\Sigma_{n=1}^N G_{nN}$), the equalities (1.3.2) and (1.3.3) hold.

Next, let us construct from the given interval $I \subset \mathbb{R}^k$ an interval $\tilde{I} \subset \mathbb{R}^k$, such that (1.3.4) is satisfied. Therefore, we define for $i = 1,2,\dots,k$ the functions f_i and g_i as follows:

$$(1.3.9)\qquad f_i(x) = x + \sum_\nu p_\nu^{(i)}\ \chi([\xi_\nu^{(i)},\infty);x), \qquad \text{for } x \in (-\infty,\infty),$$

$$g_i(x) = x + \sum_{\nu} p_{\nu}^{(i)} \chi((\xi_{\nu}^{(i)},\infty);x), \qquad \text{for } x \in (-\infty,\infty). \tag{1.3.10}$$

Let $I = \Pi_{i=1}^{k} I_i$ and let for $i = 1,2,\dots,k$, a_i and b_i be the end points of the interval $I_i \subset \mathbb{R}$, with $a_i \le b_i$. Let

$$\tilde{a}_i = \begin{cases} g_i(a_i), & \text{for } a_i \in I_i, \\ f_i(a_i), & \text{elsewhere,} \end{cases} \quad \text{and} \quad \tilde{b}_i = \begin{cases} g_i(b_i), & \text{for } b_i \in I_i^c, \\ f_i(b_i), & \text{elsewhere.} \end{cases} \tag{1.3.11}$$

We define $\tilde{I} = \Pi_{i=1}^{k} \tilde{I}_i$, where for $i = 1,2,\dots,k$, $\tilde{I}_i$ is the interval in $\mathbb{R}$ with end points $\tilde{a}_i$ and $\tilde{b}_i$ and $\tilde{a}_i \in \tilde{I}_i$ iff $a_i \in I_i$, $\tilde{b}_i \in \tilde{I}_i$ iff $b_i \in I_i$. With the aid of (1.3.7) and (1.3.8) it can be verified that

$$\overline{F}_N\{I\} = \overline{G}_N\{\tilde{I}\} \quad \text{and} \quad \mathbb{F}_N\{I\} = \mathbb{G}_N\{\tilde{I}\}. \tag{1.3.12}$$

Since, analogously we can construct for every interval $I^* \subset I$ an interval $\tilde{I}^* \subset \tilde{I}$ satisfying (1.3.12) with $I = I^*$ and $\tilde{I} = \tilde{I}^*$, the proof is completed.

□

COROLLARY 1.3.1. *Theorem* 1.1.1, *Corollary* 1.1.1, *Corollary* 1.1.2, (1.1.40), (1.1.43) *and Theorem* 1.2.1 *also hold without the restriction to continuous underlying d.f.'s.*

PROOF. The assertion for Theorem 1.2.1 is immediate from (1.3.4). For $k = 1$, we denote by $Y_{1:N}$, $Y_{N:N}$ the first and last order statistic of the random variables $Y_{1N}, Y_{2N}, \dots, Y_{NN}$, which are constructed in the proof of Theorem 1.3.1 (cf. (1.3.5)). In view of (1.3.7) we obtain

$$x \ge X_{1:N} \Leftrightarrow x + \sum_{\nu} p_{\nu}^{(1)} \chi([\xi_{\nu}^{(1)},\infty);x) \ge Y_{1:N},$$

$$x < X_{N:N} \Leftrightarrow x + \sum_{\nu} p_{\nu}^{(1)} \chi([\xi_{\nu}^{(1)},\infty);x) < Y_{N:N},$$

so that (1.3.2) and (1.3.3) imply that with probability one

$$\sup_{x \ge X_{1:N}} \frac{\overline{F}_N(x)}{\mathbb{F}_N(x)} \le \sup_{x \ge Y_{1:N}} \frac{\overline{G}_N(x)}{\mathbb{G}_N(x)}, \tag{1.3.13}$$

$$\sup_{x < X_{N:N}} \frac{1-\overline{F}_N(x)}{1-\mathbb{F}_N(x)} \le \sup_{x < Y_{N:N}} \frac{1-\overline{G}_N(x)}{1-\mathbb{G}_N(x)}, \tag{1.3.14}$$

$$(1.3.15)\qquad \sup_{-\infty<x<\infty} \frac{\mathbb{F}_N(x)}{\overline{F}_N(x)} \le \sup_{-\infty<x<\infty} \frac{\mathbb{G}_N(x)}{\overline{G}_N(x)},$$

$$(1.3.16)\qquad \sup_{-\infty<x<\infty} \frac{1-\mathbb{F}_N(x)}{1-\overline{F}_N(x)} \le \sup_{-\infty<x<\infty} \frac{1-\mathbb{G}_N(x)}{1-\overline{G}_N(x)},$$

$$(1.3.17)\qquad \sup_{-\infty<x<\infty} \frac{|\mathbb{F}_N(x)-\overline{F}_N(x)|}{q_\delta(\overline{F}_N(x))} \le \sup_{-\infty<x<\infty} \frac{|\mathbb{G}_N(x)-\overline{G}_N(x)|}{q_\delta(\overline{G}_N(x))}.$$

Moreover, with the aid of (1.3.2) one can show that

$$(1.3.18)\qquad \overline{F}_N(X_{N:N}) = \overline{G}_N(X_{N:N}+\sum_\nu p_\nu^{(1)}\chi([\xi_\nu^{(1)},\infty);X_{N:N})) \ge \overline{G}_N(Y_{N:N}),$$

$$(1.3.19)\qquad \overline{F}_N(X_{1:N}) = \overline{G}_N(X_{1:N}+\sum_\nu p_\nu^{(1)}\chi([\xi_\nu^{(1)},\infty);X_{1:N})) \ge \overline{G}_N(Y_{1:N}).$$

The proof can be completed from (1.3.13)-(1.3.19). □

<u>REMARK 1.3.1</u>. From Corollary 1.1.4, the proof of Corollary 1.1.2 and (1.3.4) it is immediate that for k = 1 Theorem 1.2.1 even holds without the factor $(\log(N+1))^{\frac{1}{2}}$ in (1.2.1) and without the restriction to continuous underlying d.f.'s.

<u>REMARK 1.3.2</u>. Of course, as in the proof of Corollary 1.4.1, one can show that also the transformed versions (cf. (1.1.36)) of Theorem 1.1.2 and Theorem 1.1.3 remain valid without the restriction to continuous d.f.'s.

CHAPTER II

ASYMPTOTIC THEORY OF RANK STATISTICS

2.0. INTRODUCTION

There exists a variety of theorems on asymptotic normality of both univariate and multivariate rank statistics. Although these results are obviously related, separate proofs are given and in general different techniques are used. It is our purpose to give a unifying approach to these various results. We shall present three theorems establishing asymptotic normality for a general class of multivariate rank statistics and, apart from regularity conditions, almost arbitrary underlying continuous distribution functions (d.f.'s) which may correspond to the null hypothesis or to local or fixed alternatives. As such these theorems are more general than existing results. As special cases they contain or extend many of the results found in the literature and include e.g. asymptotic normality for simple linear rank statistics as well as rank statistics for independence, under the null hypothesis and under alternatives. The technique used in the proofs appears to be generally applicable in problems of this kind and is based on the properties of empirical distribution functions which are derived in Chapter I. Specializing our theorems to particular cases it turns out that the present conditions are rather close to the best conditions that appear in the literature, although they are occasionally slightly stronger. The study in this chapter is a continuation of previous work by F.H. RUYMGAART and the author.

Let k be a fixed positive integer and for each N = 1,2,... let $X_{nN} = (X_{1nN},X_{2nN},\dots,X_{knN})$, n = 1,2,...,N, be N independent k-dimensional random vectors with joint continuous distribution function F_{nN} and marginal d.f.'s $F_{1nN},F_{2nN},\dots,F_{knN}$. For each N, moreover, let $\mathbb{F}_N$ be the joint empirical d.f. based on the N random vectors $X_{1N},X_{2N},\dots,X_{NN}$ and, for i = 1,2,...,k, denote the marginal empirical d.f. of the independent random variables $X_{i1N}, X_{i2N}, \dots, X_{iNN}$ by $\mathbb{F}_{iN}$ and the ranks of these r.v.'s by $R_{i1N}, R_{i2N}, \dots, R_{iNN}$. We have the relations

$$R_{inN} = N\mathbb{F}_{iN}(X_{inN}) \qquad \text{for } i = 1,2,\dots,k. \tag{2.0.1}$$

All random vectors are supposed to be defined on a single probability space $(\Omega, \mathcal{A}, P)$. We recall that $\bar{F}_N = N^{-1} \Sigma_{n=1}^{N} F_{nN}$ and $\bar{F}_{iN} = N^{-1} \Sigma_{n=1}^{N} F_{inN}$ for $i = 1,2,\dots,k$ (cf. (1.0.3) and (1.0.4)).

The rank statistics that we are interested in are called multivariate linear rank statistics; these are of the type

$$S_N = N^{-1} \sum_{n=1}^{N} c_{nN}\, a_N(R_{1nN}, R_{2nN}, \dots, R_{knN}). \tag{2.0.2}$$

Here, for $n_i = 1,2,\dots,N$, $i = 1,2,\dots,k$, the $a_N(n_1, n_2, \dots, n_k)$ are given real numbers, called scores, and the c_{nN}, for $n = 1,2,\dots,N$, are given real constants, called regression constants. For this terminology see HÁJEK and ŠIDÁK (1967). An important sub-class of the statistics of the form (2.0.2) are those for which the scores have product structure, viz.

$$T_N = N^{-1} \sum_{n=1}^{N} c_{nN} \prod_{i=1}^{k} a_{iN}(R_{inN}), \tag{2.0.3}$$

where, for $n = 1,2,\dots,N$ and $i = 1,2,\dots,k$, the $a_{iN}(n)$ are the scores. Statistics of the more general form

$$\sum_{j=1}^{m} \lambda_j\, T_{jN}, \tag{2.0.4}$$

with $\lambda_1, \lambda_2, \dots, \lambda_m$ real constants and each T_{jN} of the type (2.0.3) occupies an intermediate position between (2.0.2) and (2.0.3).

To motivate the study of the statistics mentioned in (2.0.3) or (2.0.4), let us observe that most of the rank statistics considered in the literature are of this form. In PURI and SEN (1969), (1971), functions of statistics of the type (2.0.3) are proposed as permutationally (conditionally) distribution-free tests for some specified problems; in SHIRAHATA (1973) it is shown that in many natural multivariate models locally most powerful rank tests are based on such rank statistics. To get an insight into the situations that are covered in the present set-up, we shall consider some examples.

EXAMPLE 2.0.1 (simple linear rank statistics):

Choosing k = 1, (2.0.3) reduces to

$$(2.0.5) \qquad T_{1N} = N^{-1} \sum_{n=1}^{N} c_{nN}\, a_{1N}(R_{1nN}).$$

Statistics of this type are called simple linear rank statistics and are of general importance. In particular they are locally most powerful for testing the null hypotheses of randomness against regression in location. For this terminology see HÁJEK and ŠIDÁK (1967), page 216. Under the null hypothesis the distribution of T_{1N} is independent of the underlying univariate continuous d.f. F and its limiting (normal) distribution can be found e.g. in HÁJEK and ŠIDÁK (1967). In the general case where the F_{nN} are almost arbitrary, asymptotic normality has been studied in HÁJEK (1968) and DUPAČ and HÁJEK (1969). The special case where each F_{nN} equals one of the arbitrary continuous univariate d.f.'s F_1 or F_2, both independent of N, has been investigated e.g. in CHERNOFF and SAVAGE (1958), GOVINDARAJULU, LECAM and RAGHAVACHARI (1967) and PYKE and SHORACK (1968) (two-sample problem).

EXAMPLE 2.0.2 (rank statistics for independence):

Choosing k = 2 and $c_{nN} = 1$, for $n = 1,2,\ldots,N$, (2.0.3) reduces to

$$(2.0.6) \qquad T_{2N} = N^{-1} \sum_{n=1}^{N} a_{1N}(R_{1nN})\, a_{2N}(R_{2nN}).$$

Statistics of this type are particularly well suited for testing the null hypothesis of independence against alternatives with an underlying bivariate d.f. exhibiting a positive (negative) stochastic dependence (see e.g. RUYMGAART (1974)). Under the null hypothesis the distribution of T_{2N} is independent of the underlying bivariate continuous d.f. Moreover, it is well-known that, under the null hypothesis, the distribution of T_{2N} is equal to that of T_{1N}, provided we take c_{nN} in (2.0.5) equal to $a_{2N}(n)$ in (2.0.6). An example of a fixed alternative arises when each F_{nN} equals an arbitrary bivariate continuous d.f. F, independent of N, which is not of product type. In this case the limiting (normal) d.f. of T_{2N} has been derived in BHUCHONGKUL (1964), RUYMGAART, SHORACK and VAN ZWET (1972) and RUYMGAART (1973), (1974).

EXAMPLE 2.0.3 (generalization of HÁJEK's model):

Let $k \geq 2$. We consider a generalization to the k-dimensional "regression" case of HÁJEK's model, proposed in HÁJEK and ŠIDÁK (1967), page 75. For k = 2 see also SHIRAHATA (1973). Let $X_{nN} = (X_{1nN}, X_{2nN}, \dots, X_{knN})$, $n = 1,2,\dots,N$, be random vectors defined by

$$X_{inN} = X^*_{inN} + c_{nN}\,\Delta Z_{nN}, \qquad i = 1,2,\dots,k,$$

where $\{X^*_{inN}\}_{n=1}^N$, for $i = 1,2,\dots,k$ and $\{Z_{nN}\}_{n=1}^N$ are mutually independent and each sequence is an i.i.d. sequence of random variables, the c_{nN} are known constants and Δ is an unknown parameter. For $i = 1,2,\dots,k$, let f_{iN} denote the density function of X^*_{inN}, $f^{(1)}_{iN}$ the derivative of f_{iN} and let M_N be the d.f. of Z_{nN}. Then the density function of X_{nN} is given by

$$h_{nN\Delta}(x_1,x_2,\dots,x_k) = \int_{-\infty}^{\infty} \prod_{i=1}^{k} f_{iN}(x_i - c_{nN}\Delta z)\,dM_N(z).$$

Using results of SHIRAHATA (1973), we find that under regularity conditions the locally most powerful rank test for testing $\Delta = 0$ (independence) against $\Delta > 0$ is based on the rank statistic

$$(2.0.7) \qquad T_{3N} = EZ_{nN} \sum_{n=1}^{N} c_{nN} \sum_{i=1}^{k} E_0\left[\frac{f^{(1)}_{iN}(X_{inN})}{f_{iN}(X_{inN})} \,\Big|\, R_{inN}\right],$$

which is of type (2.0.4), where the product in each T_{jN} is a trivial product, consisting of only one factor.

If either $EZ_{nN} = 0$ or $c_{nN} = 1$ for $n = 1,2,\dots,N$, then (2.0.7) reduces to a constant and hence is useless for testing purposes. In that case the locally most powerful rank test for testing $\Delta = 0$ against $\Delta \neq 0$ is based on the rank statistic

$$\tilde{T}_{3N} = 2\,\mathcal{V}ar(Z_{nN}) \sum_{n=1}^{N} c_{nN} \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{k} E_0\left[\frac{f^{(1)}_{iN}(X_{inN})}{f_{iN}(X_{inN})} \,\Big|\, R_{inN}\right] E_0\left[\frac{f^{(1)}_{jN}(X_{jnN})}{f_{jN}(X_{jnN})} \,\Big|\, R_{jnN}\right]\right),$$

which is of type (2.0.4), where the product in each T_{jN} consists of two factors.

<u>EXAMPLE 2.0.4</u> (generalization of FARLIE's model):

Let $k \geq 2$. We consider a generalization to the k-dimensional "regression" case of FARLIE's model, proposed in FARLIE (1960). For k = 2 see also SHIRAHATA (1973). Let the sample elements $X_{nN} = (X_{1nN}, X_{2nN}, \ldots, X_{knN})$ have d.f. $F_{nN\Delta}$, $n = 1,2,\ldots,N$, where

$$F_{nN\Delta}(x_1, x_2, \ldots, x_k) = \prod_{i=1}^{k} F_{iN}(x_i) \, \{1 + c_{nN} \, \Delta \prod_{i=1}^{k} g_{iN}(F_{iN}(x_i))\},$$

$$\Delta \geq 0,$$

where for $i = 1,2,\ldots,k$, F_{iN} is a distribution function, g_{iN} is a function on [0,1], and the c_{nN} are known constants. Let $g_{iN}^{(1)}$ denote the derivative of g_{iN}. Again using SHIRAHATA (1973), we find that under certain regularity conditions the locally most powerful rank test for testing $\Delta = 0$ against $\Delta > 0$ is based on the rank statistic

$$\text{(2.0.8)} \qquad T_{4N} = \sum_{n=1}^{N} c_{nN} \prod_{i=1}^{k} E_0\Big[g_{iN}(F_{iN}(X_{inN})) + F_{iN}(X_{inN})\, g_{iN}^{(1)}(F_{iN}(X_{inN})) \;\Big|\; R_{inN}\Big],$$

which is exactly of type (2.0.3).

If $g_{iN}(s) = 1 - s$, for $s \in [0,1]$, and $c_{nN} = 1$, for $n = 1,2,\ldots,N$, then (2.0.8) reduces to

$$\tilde{T}_{4N} = \sum_{n=1}^{N} \prod_{i=1}^{k} \left(1 - \frac{2R_{inN}}{N+1}\right).$$

In this way we obtain a generalization to the multivariate case of Spearman's statistic.

<u>EXAMPLE 2.0.5</u> (generalization of a model of WITTING and NÖLLE):

Let $k \geq 2$. We consider a generalisation to the k-dimensional "regression" case of a model proposed in WITTING and NÖLLE (1970), page 130. Let the sample elements $X_{nN} = (X_{1nN}, X_{2nN}, \ldots, X_{knN})$ have d.f. $F_{nN\Delta}$, $n = 1,2,\ldots,N$, where

$$F_{nN\Delta}(x_1, x_2, \ldots, x_k) = (1 - c_{nN}\Delta) \prod_{i=1}^{k} F_{iN}(x_i) + c_{nN}\,\Delta \prod_{i=1}^{k} F_{iN}^2(x_i),$$

$$0 \leq \Delta < 1,$$

where for $i = 1,2,\ldots,k$, F_{iN} is a distribution function and the c_{nN} are known constants. Under certain regularity conditions, we find (see SHIRAHATA (1973)) that the locally most powerful rank test for testing $\Delta = 0$ against $\Delta > 0$ is based on the rank statistic

$$T_{5N} = \sum_{n=1}^{N} c_{nN} \left(2^k \prod_{i=1}^{k} \frac{R_{inN}}{N+1} - 1\right),$$

which is of the type (2.0.3).

Let us now return to the statistic T_N. It is well-known that locally optimal scores can be determined if one has in mind particular parametric alternatives. In many such cases (see also the examples given) these optimal scores are so-called exact scores derived from suitable functions J_i on (0,1) according to

$$\text{(2.0.9)} \qquad a^*_{iN}(n) = EJ_i(\xi_{n:N}), \qquad \text{for } i = 1,2,\ldots,k,\ n = 1,2,\ldots,N,$$

where $\xi_{n:N}$ is the n-th order statistic of a sample of size N from the uniform distribution on (0,1). These exact scores, however, are not only hard to compute, but also hard to manipulate in the asymptotic theory. For this reason one frequently uses the scores

$$\text{(2.0.10)} \qquad a_{iN}(n) = J_i(E(\xi_{n:N})) = J_i(\tfrac{n}{N+1}), \qquad i = 1,2,\ldots,k,\ n = 1,2,\ldots,N,$$

called the approximate scores derived from J_i. Under a suitable condition ((2.0.17) below) approximate scores are as good as exact scores in the sense of PITMAN-efficiency. The regression constants c_{nN} can always be generated by some function J_{0N} according to

$$\text{(2.0.11)} \qquad c_{nN} = J_{0N}(\tfrac{n}{N+1}), \qquad n = 1,2,\ldots,N.$$

Note that in constrast to the scores, the regression constants are generated by a function which is allowed to depend on N. This has the advantage that we also contain in our theory rank statistics used for the regression problem and the k-sample problem. In fact this dependence is already needed to cover the two-sample situation.

For methodological reasons it will be convenient to introduce the regression constants with the aid of the additional set of mutually independent r.v.'s $X_{01N}, X_{02N}, \ldots, X_{0NN}$, independent of all random vectors considered so far and also defined on the same probability space. Let $\mathcal{U}_{a,b}$ denote the uniform d.f. on the interval (a,b) and let us assume that the d.f. F_{0nN} of X_{0nN} satisfies

$$(2.0.12) \qquad F_{0nN} = \mathcal{U}_{(n-1)/N,n/N}, \qquad \text{for } n = 1,2,\ldots,N.$$

For the ranks of these r.v.'s this entails that

$$(2.0.13) \qquad R_{0nN} = n, \qquad \text{for } n = 1,2,\ldots,N,$$

with probability 1. For n = 1,2,...,N the joint d.f. of the (k+1)-dimensional random vector $(X_{0nN}, X_{1nN}, \ldots, X_{knN})$ will be written as G_{nN}, the corresponding (k+1)-dimensional empirical d.f. by $\mathbb{G}_N$ and its first marginal empirical d.f. (based on $X_{01N}, X_{02N}, \ldots, X_{0NN}$) by $\mathbb{F}_{0N}$. It should be observed that

$$(2.0.14) \qquad G_{nN} = F_{0nN} \times F_{nN} = \mathcal{U}_{(n-1)/N,n/N} \times F_{nN}, \qquad \text{for } n = 1,2,\ldots,N,$$

and that $N^{-1} \Sigma_{n=1}^{N} \mathcal{U}_{(n-1)/N,n/N} = \mathcal{U}_{0,1}$, the uniform d.f. on (0,1). Analogous to previous notation we shall write $\bar{G}_N = N^{-1} \Sigma_{n=1}^{N} G_{nN}$.

In order to give an alternative expression for T_N in the case of approximate scores we have to introduce the modified marginal empirical d.f.'s

$$(2.0.15) \qquad \mathbb{F}^*_{iN} = [N/(N+1)]\mathbb{F}_{iN}, \qquad \text{for } i = 0,1,\ldots,k.$$

Combining (2.0.3) with (2.0.1), (2.0.10), (2.0.11), (2.0.13) and (2.0.15), it follows that T_N equals

$$(2.0.16) \qquad T_N = \int J_{0N}(\mathbb{F}^*_{0N}) \prod_{j=1}^{k} J_j(\mathbb{F}^*_{jN})\, d\mathbb{G}_N,$$

with probability 1. Here the integration is extended over the (k+1)-dimensional number space. The extension of each of the original k-dimensional random vectors with a 1-dimensional dummy random coordinate, each having one of the uniform d.f.'s in (2.0.12), has the effect that the statistic T_N can be entirely expressed in terms of empirical d.f.'s.

Our main result - Theorem 2.1.1 in section 2.1 - is the asymptotic normality of a suitably standardized version of T_N for approximate scores, where the next three points should be kept in mind. In the first place we remark that the generating functions are allowed to tend to infinity near 0 and 1, and to have a finite number of discontinuities of the first kind. The price for allowing these discontinuities is a local differentiability condition on the underlying d.f.'s. In the second place there appears to be a natural balance between the respective orders of magnitude of the generating functions near 0 and 1. In the particular case (2.0.5) e.g. this leads to quite a spectrum of possible orders of magnitude of J_{0N} and J_1 near 0 and 1, whereas in HÁJEK (1968) and DUPAČ and HÁJEK (1969) only two possibilities are considered. In the third place the asymptotic normality is established for almost arbitrary triangular arrays of underlying d.f.'s. Hence asymptotic normality for a triangular array corresponding to a set of local alternatives is included as a special case. From the latter result we can immediately derive the asymptotic power of the corresponding tests, which is used for the computation of asymptotic relative efficiencies. It is worthwhile noting that in contrast to e.g. the theorems in CHERNOFF and SAVAGE (1958) and RUYMGAART (1973) we do not need uniformity of the convergence on a subclass of arrays of underlying d.f.'s to achieve the computation of the limiting distribution under local alternatives.

The proof of the asymptotic normality of the statistic considered will be given by way of a decomposition in a sum of leading terms, which is asymptotically normally distributed, and a remainder term, which is asymptotically negligible. In section 2.2 this decomposition for the standardized version of T_N for approximate scores is presented and the asymptotic normality of the leading terms is established. The proof of the asymptotic negligibility of the corresponding remainder term - given in section 2.4 - will rely almost completely on properties of the empirical d.f.'s as is suggested by the representation of T_N in (2.0.16). Apart from a component due to the introduction of the dummy random variables X_{01N}, $X_{02N},..,X_{0NN}$, and apart from the dimension, the components of this remainder term are very similar to the higher order terms in RUYMGAART (1973), (1974), the main difference being that in the present case we have N possibly different underlying d.f.'s, whereas in RUYMGAART (1973), (1974) there is one single fixed underlying d.f.. The proof of the asymptotic negligibility, however, can be given in essentially the same way, because it turns out that all the lemmas used in RUYMGAART (1973), (1974) remain valid, properly modified if necessary, under the present circumstances with not necessarily identical underlying d.f.'s and with the averaged d.f. in the role of the single fixed underlying d.f.. These lemmas are summarized in section 2.3 and based on the properties of the empirical d.f. in the non-i.i.d. case, which are obtained in Chapter I.

Under the assumption that

$$(2.0.17) \qquad N^{-\frac{1}{2}} \sum_{n=1}^{N} c_{nN}\left[\prod_{i=1}^{k} a^{*}_{iN}(R_{inN}) - \prod_{i=1}^{k} a_{iN}(R_{inN})\right] = o_P(1), \qquad \text{as } N \to \infty,$$

one immediately derives an asymptotic result for the statistic T_N in the case of exact scores from the corresponding Theorem 2.1.1 on approximate scores. Condition (2.0.17) is well known in the literature (see e.g. BHUCHONGKUL (1964), CHERNOFF and SAVAGE (1958) and RUYMGAART (1973)). A verification of the condition is a problem in itself (see e.g. RUYMGAART (1973)). In general an additional condition on the generating functions is needed. More attention will be paid to this matter in section 2.5, where the asymptotic normality of the standardized statistic T_N for exact scores will be established.

Our third result, presented and proved in section 2.6, is the asymptotic normality of a suitably standardized version of S_N (see 2.0.2), in the case where the scores $a_N(n_1,n_2,\ldots,n_k)$ are generated by some continuous function J on $(0,1)^k$ according to

$$(2.0.18) \qquad a_N(n_1,n_2,\ldots,n_k) = J\left(\frac{n_1}{N+1}, \frac{n_2}{N+1}, \ldots, \frac{n_k}{N+1}\right), \qquad n_i = 1,2,\ldots,N, \quad i = 1,2,\ldots,k.$$

Finally, in section 2.7 some further possible extensions will be discussed.

REMARK 2.0.1. Following the PYKE-SHORACK approach, RÜSCHENDORF derived in 1976 the asymptotic distribution of certain multivariate rank statistics under an assumption concerning the weak convergence of the reduced multivariate sequential empirical process (cf. RÜSCHENDORF (1976), Theorem 5.1).

2.1. STATEMENT OF THE MAIN THEOREM

Before presenting the theorem let us introduce some more notation and conventions, to be used throughout the present and the subsequent sections. Let the inverse of a univariate d.f. F be defined as in (1.1.2) and let us denote the standard normal d.f. by

$$(2.1.1) \qquad \mathcal{N}(y) = (2\pi)^{-\frac{1}{2}} \int_{(-\infty,y]} \exp(-z^2/2)dz, \qquad \text{for } y \in (-\infty,\infty).$$

For convenience we shall only use q-functions and reproducing u-shaped functions (for a definition see the appendix in SHORACK (1972)) of a special but common type, based on the function

$$(2.1.2) \qquad r(t) = \{t(1-t)\}^{-1}, \qquad \text{for } t \in (0,1).$$

For an arbitrary positive integer m the m-fold Cartesian product of a set S with itself will be denoted by S^m. For each m, moreover, let us define

$$(2.1.3) \qquad \mathcal{F}_m = \{F\colon F \text{ is an m-variate d.f. which is continuous on } \mathbb{R}^m\}.$$

In the theorem the d.f's F_{nN} will be restricted to $\mathcal{F}_k$.

With respect to the generating functions we shall assume that the J_{0N} $(N=1,2,\ldots)$ and J_i $(i=1,2,\ldots,k)$ have a finite number of discontinuities of the first kind only. Without loss of generality it can and will be assumed that these generating functions are right-continuous.

For any finite set S let $\#S$ denote the number of elements in S and for any function f the i-th derivative is written as $f^{(i)}$ $(f^{(0)}=f)$.

<u>ASSUMPTION 2.1.1</u> (generating functions):
(a) *For* $N = 1,2,\ldots$ *the function* J_{0N} *has discontinuities of the first kind only and a continuous derivative* $J_{0N}^{(1)}$ *on the set* $(0,1) - \mathcal{D}_{0N}$.
(b) *For* $i = 1,2,\ldots,k$ *the function* J_i *has discontinuities of the first kind only and a continuous derivative* $J_i^{(1)}$ *on the set* $(0,1) - \mathcal{D}_i$.
(c) *There exist positive numbers* $\ell_0,\ell_1,\ldots,\ell_k$ *and* τ *such that for* $N = 1,2,\ldots$ *and* $i = 1,2,\ldots,k$,

$$\mathcal{D}_{0N} \subset (\tau,1-\tau), \; \#\mathcal{D}_{0N} \le \ell_0 \quad \textit{and} \quad \mathcal{D}_i \subset (\tau,1-\tau), \; \#\mathcal{D}_i \le \ell_i,$$

(d) *There exist positive numbers* $a_0, a_1, \dots, a_k$ *and* K_1, *satisfying* $a := \Sigma_{j=0}^{k} a_j < \frac{1}{2}$, *such that, with* r *defined in* (2.1.2), *we have for* $\nu = 0,1$, $N = 1,2,\dots$ *and* $i = 1,2,\dots,k$,

$$(2.1.4) \qquad |J_{0N}^{(\nu)}| \le K_1 \, r^{a_0+\nu} \quad \text{and} \quad |J_i^{(\nu)}| \le K_1 \, r^{a_i+\nu},$$

wherever these functions are defined on (0,1).

The price for discontinuities in the scores generating functions is a kind of local differentiability condition on the transformations

$$(2.1.5) \qquad \Phi_{nN} = F_{nN}(\bar{F}_{1N}^{-1}, \bar{F}_{2N}^{-1}, \dots, \bar{F}_{kN}^{-1})$$

of the F_{nN} to the k-dimensional unit cube $[0,1]^k$ for $n = 1,2,\dots,N$. We shall say that Φ_{nN} possesses a density ϕ_{nN} (with respect to LEBESGUE measure on $[0,1]^k$) on the BOREL set $B_0 \subset [0,1]^k$ if, for each BOREL set $B \subset B_0$, we have

$$(2.1.6) \qquad \int_B d\Phi_{nN} = \int_B \phi_{nN}(t_1, t_2, \dots, t_k)\, dt_1 dt_2 \dots dt_k.$$

To formulate the assumption on the underlying d.f.'s, let us define for $\eta > 0$,

$$(2.1.7) \qquad \mathcal{Q}_{\eta,i} = \bigcup_{s \in \tilde{\mathcal{D}}_i} (s-\eta, s+\eta), \qquad \text{for } i = 1,2,\dots,k,$$

where $\tilde{\mathcal{D}}_i$ is the set of discontinuity points of J_i. Note that $\tilde{\mathcal{D}}_i \subset \mathcal{D}_i$.

<u>ASSUMPTION 2.1.2</u> (underlying d.f.'s):

There exist positive numbers $\eta, b_1, b_2, \dots, b_k$ *and* K_2 *such that for* $N = 1,2,\dots$, $n = 1,2,\dots,N$ *and* $i = 1,2,\dots,k$, Φ_{nN} *(see* (2.1.5)) *has a continuous density* ϕ_{nN} *on* $(0,1)^{i-1} \times \mathcal{Q}_{\eta,i} \times (0,1)^{k-i}$, *satisfying*

$$(2.1.8) \qquad |\phi_{nN}(t_1, t_2, \dots, t_k)| \le K_2 \prod_{\substack{j=1 \\ j \ne i}}^{k} \{r(t_j)\}^{b_j},$$

for $(t_1, t_2, \dots, t_k)$ *in this set. Moreover, for every* $(t_1, t_2, \dots, t_{i-1}, t_{i+1}, \dots, t_k) \in (0,1)^{k-1}$, *every* $t_i \in \tilde{\mathcal{D}}_i$ *(see* 2.1.7) *and every* $i = 1,2,\dots,k$, (2.1.9) $$\sup_{n,N} |\phi_{nN}(t_1,\ldots,t_{i-1},t,t_{i+1},\ldots,t_k) - \phi_{nN}(t_1,\ldots,t_k)| \to 0,$$

as $t \to t_i$.

REMARK 2.1.1. If J_j is continuous, then $\tilde{\mathcal{D}}_j = \emptyset$ and $\mathcal{Q}_{n,j} = \emptyset$ so that Assumption 2.1.2 is vacuous for $i = j$.

To standardize the location of the statistics T_N we shall use the quantities

(2.1.10) $$\mu_N = \mu_N(F_{1N},F_{2N},\ldots,F_{NN}) = \int J_{0N}(\bar{F}_{0N}) \prod_{j=1}^{k} J_j(\bar{F}_{jN})\, d\bar{G}_N.$$

The quantity μ_N arises in the fundamental decomposition of T_N in (2.2.10). The quantities used to standardize the scale of the T_N will be given in the implicit form

(2.1.11) $$\sigma_N^2 = \sigma_N^2(F_{1N},F_{2N},\ldots,F_{NN}) = \mathcal{V}ar(A_N + \sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd}),$$

where A_N and the A_{iNc} and A_{iNd} also arise in (2.2.10). Under the conditions of the theorem below these quantities are well defined.

THEOREM 2.1.1. *Let an arbitrary triangular array of underlying d.f.'s* $F_{nN} \in \mathcal{F}_k$, $n = 1,2,\ldots,N$, $N = 1,2,\ldots$ *be given, such that for the resulting triangular array of transformed d.f.'s* Φ_{nN} *Assumption* 2.1.2 *is fulfilled. Let the generating functions satisfy Assumption* 2.1.1 *and let the constants* a_j *(appearing in Assumption* 2.1.1*) and the constants* b_j *(appearing in Assumption* 2.1.2*) satisfy* $a_j + b_j < 1$ *for* $j = 1,2,\ldots,k$. *Then the quantities* μ_N *and* σ_N^2, *defined in* (2.1.10) *and* (2.1.11) *are finite. If, moreover,* $\liminf_{N\to\infty} \sigma_N^2 > 0$, *we have*

(2.1.12) $$\sup_{-\infty<z<\infty} |P(N^{\frac{1}{2}}(T_N-\mu_N)/\sigma_N \le z) - N(z)| \to 0, \quad \textit{as } N \to \infty,$$

for T_N *as in* (2.0.16), *i.e. the case of approximate scores.*

2.2. ASYMPTOTIC NORMALITY OF THE LEADING TERMS

Before writing down the leading terms of the standardized version of the statistic T_N for approximate scores let us make some introductory remarks.

We introduce for N = 1,2,... a (k+1)-dimensional random vector

$$(2.2.1) \qquad (Y_{0N}, Y_{1N}, \ldots, Y_{kN}), \qquad \text{with joint d.f. } \overline{G}_N,$$

where $\overline{G}_N$ is defined below (2.0.14). Besides the transformed d.f.'s in (2.1.5) it will be convenient to have at our disposal the transformation

$$(2.2.2) \qquad \overline{\Psi}_N := \overline{G}_N(\overline{F}_{0N}^{-1}, \overline{F}_{1N}^{-1}, \ldots, \overline{F}_{kN}^{-1}) = N^{-1} \sum_{n=1}^{N} \mathcal{U}_{(n-1)/N, n/N} \times \Phi_{nN}.$$

The transformed random vector $(\overline{F}_{0N}(Y_{0N}), \overline{F}_{1N}(Y_{1N}), \ldots, \overline{F}_{kN}(Y_{kN}))$ has joint d.f. $\overline{\Psi}_N$ because of the continuity of the underlying d.f.'s and by definition all the univariate marginal d.f.'s of $\overline{\Psi}_N$ are $\mathcal{U}_{0,1}$. If Assumption 2.1.2 holds one can show that $\overline{\Psi}_N$ has, for i = 1,2,...,k, a density $\overline{\psi}_N$ (with respect to LEBESGUE measure on $(0,1)^{k+1}$) on the set $(0,1)^i \times \mathcal{Q}_{n,i} \times (0,1)^{k-i}$, where $\mathcal{Q}_{n,i}$ is defined in (2.1.7). We have for n = 1,2,...,N, i = 1,2,...,k,

$$(2.2.3) \qquad \overline{\psi}_N(t_0, t_1, \ldots, t_k) = \phi_{nN}(t_1, t_2, \ldots, t_k),$$

for

$$(t_0, t_1, \ldots, t_k) \in ((n-1)/N, n/N) \times (0,1)^{i-1} \times \mathcal{Q}_{n,i} \times (0,1)^{k-i}.$$

Anticipating the finiteness of all expectations and integrals involved let us consider for N = 1,2,..., $i \in \{1,2,\ldots,k\}$ and $t_i \in (0,1)$ the conditional expectation

$$(2.2.4) \qquad E\left(J_{0N}(\overline{F}_{0N}(Y_{0N})) \prod_{\substack{j=1 \\ j \neq i}}^{k} J_j(\overline{F}_{jN}(Y_{jN})) \;\middle|\; \overline{F}_{iN}(Y_{iN}) = t_i\right).$$

Under Assumption 2.1.2, again, one of the possible determinations of

(2.2.4) equals $h_{iN}(t_i)$, where

$$h_{iN}(t_i) = \sum_{n=1}^{N} \left(\int_{(\frac{n-1}{N},\frac{n}{N})} J_{0N}(t_0)dt_0 \right) \times \tag{2.2.5}$$

$$\times \int_{(0,1)^{k-1}} \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(t_j)\phi_{nN}(t_1,t_2,\ldots,t_k)dt_1,\ldots,dt_{i-1},dt_{i+1},\ldots,dt_k,$$

provided t_i is restricted to $\mathcal{Q}_{\eta,i}$.

Throughout the sequel the symbol M will be employed as a generic constant, independent of N.

LEMMA 2.2.1. *Let the function* h_{iN} *be defined as in* (2.2.5). *Under the conditions of Theorem* 2.1.1 *we have for* $N = 1,2,\ldots$ *and* $i = 1,2,\ldots,k$ *that* $|h_{iN}(t_i)| \le M_i$, *for* $t_i \in \mathcal{Q}_{\eta,i}$, *where* M_i *is a number independent of* N. *Moreover, for* $N = 1,2,\ldots$ *and* $i = 1,2,\ldots,k$, h_{iN} *is a continuous function of* t_i *for* $t_i \in \mathcal{Q}_{\eta,i}$, *and for each* i *the set of functions* $\{h_{iN},\ N = 1,2,\ldots\}$ *is equicontinuous on* $\tilde{\mathcal{D}}_i$ (cf. 2.1.7).

PROOF. From the assumptions in Theorem 2.1.1 it is immediate that

$$|h_{iN}(t_i)| \le$$

$$\le M \sum_{n=1}^{N} \left(\int_{(\frac{n-1}{N},\frac{n}{N})} r^{a_0}(t_0)dt_0 \right) \int_{(0,1)^{k-1}} \prod_{\substack{j=1 \\ j\neq i}}^{k} r^{a_j+b_j}(t_j)dt_1,\ldots,dt_{i-1},dt_{i+1},\ldots,dt_k =$$

$$= M \int_0^1 r^{a_0}(t_0)dt_0 \prod_{\substack{j=1 \\ j\neq i}}^{k} \int_0^1 r^{a_j+b_j}(t_j)dt_j = M_i.$$

For the second statement it suffices to show that for $n = 1,2,\ldots,N$,

$$\int_{(0,1)^{k-1}} \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(t_j)\phi_{nN}(t_1,\ldots,t_k)dt_1,\ldots,dt_{i-1},dt_{i+1},\ldots,dt_k \tag{2.2.6}$$

is a continuous function of t_i, for $t_i \in \mathcal{Q}_{\eta,i}$. Let $t_i, t_i + \xi \in \mathcal{Q}_{\eta,i}$. Because of Assumption 2.1.2 in Theorem 2.1.1 we have that

$$\phi_{nN}(t_1,\ldots,t_{i-1},t_i+\xi,t_{i+1},\ldots,t_k) - \phi_{nN}(t_1,\ldots,t_k) \to 0 \qquad \text{as } \xi \to 0, \tag{2.2.7}$$

for each $(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_k) \in (0,1)^{k-1}$. The continuity of h_{iN} follows from (2.1.4), (2.1.8), (2.2.7) and the dominated convergence theorem since $a_j + b_j < 1$ for $j = 1,2,\ldots,k$. Analogously, the equicontinuity can be established with the aid of (2.1.9). □

In view of Assumption 2.1.1 and the way in which we shall conduct the proof of Theorem 2.1.1 it is no loss of generality to assume that for $i = 1,2,\ldots,k$ the generating functions J_i have only one discontinuity (say in s_i), so that

$$J_i(t) = J_{ic}(t) + \Lambda_i\, c(t-s_i), \tag{2.2.8}$$

where J_{ic} is the continuous part of J_i and where

$$c(z) = \begin{cases} 1 & \text{for } z \in [0,\infty), \\ 0 & \text{elsewhere.} \end{cases} \tag{2.2.9}$$

We are now in a position to give the basic decomposition, which holds with probability 1,

$$N^{\frac{1}{2}}(T_N-\mu_N) = A_N + \sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd} + E_N, \tag{2.2.10}$$

where

$$A_N = N^{\frac{1}{2}} \int J_{0N}(\bar{F}_{0N}) \prod_{j=1}^{k} J_j(\bar{F}_{jN})\, d(\mathbb{G}_N-\bar{G}_N), \tag{2.2.11}$$

$$A_{iNc} = N^{\frac{1}{2}} \int (\mathbb{F}_{iN}-\bar{F}_{iN})\, J_{iN}^{(1)}(\bar{F}_{iN})\, J_{0N}(\bar{F}_{0N}) \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(\bar{F}_{jN})\, d\bar{G}_N, \tag{2.2.12}$$

$$A_{iNd} = N^{\frac{1}{2}}\, \Lambda_i\, h_{iN}(s_i) \left(\mathbb{F}_{iN}(\bar{F}_{iN}^{-1}(s_i))-s_i\right), \tag{2.2.13}$$

and E_N is a remainder term which is of second order as will be proved in section 2.4. Remark that for $\Lambda_i > 0$ the conditional expectation $h_{iN}(s_i)$ is well defined; if $\Lambda_i = 0$ then A_{iNd} is defined to be zero. This section is devoted to establishing the asymptotic normality of the A-terms, i.e. under the conditions of Theorem 2.1.1 we shall show, with σ_N defined in (2.1.11), that

$$\text{(2.2.14)} \qquad \sup_{-\infty<z<\infty} \left| P\left(\left(A_N + \sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd}\right) \Big/ \sigma_N \le z\right) - N(z) \right| \to 0, \quad \text{as } N \to \infty.$$

We begin by noting that with probability 1,

$$\text{(2.2.15)} \qquad A_N + \sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd} = N^{-\frac{1}{2}} \sum_{n=1}^{N} Z_{nN},$$

where

$$\text{(2.2.16)} \qquad Z_{nN} = A_{nN} + \sum_{i=1}^{k} A_{inNc} + \sum_{i=1}^{k} A_{inNd},$$

and

$$\text{(2.2.17)} \qquad A_{nN} = J_{ON}(\overline{F}_{ON}(X_{OnN})) \prod_{j=1}^{k} J_j(\overline{F}_{jN}(X_{jnN})) - \mu_N,$$

$$\text{(2.2.18)} \qquad A_{inNc} = \int \left[c(\overline{F}_{iN} - \overline{F}_{iN}(X_{inN})) - \overline{F}_{iN}\right] J_{iN}^{(1)}(\overline{F}_{iN}) J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(\overline{F}_{jN}) d\overline{G}_N,$$

$$\text{(2.2.19)} \qquad A_{inNd} = \Lambda_i \, h_{iN}(s_i) \left[c(s_i - \overline{F}_{iN}(X_{inN})) - s_i\right].$$

It should be observed that the r.v. Z_{nN} depends on the random vector X_{nN} only. Consequently these r.v.'s $Z_{1N}, Z_{2N}, \ldots, Z_{NN}$ are mutually independent.

Next we show that there exists a $\delta > 0$ such that

$$\text{(2.2.20)} \qquad \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} E|Z_{nN}|^{2+\delta} < \infty.$$

This will be achieved by proving the stronger assertion that

$$\text{(2.2.21)} \qquad \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} E|A_{nN}|^{2+\delta} < \infty,$$

and that for $i = 1,2,\ldots,k$,

$$\text{(2.2.22)} \qquad \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} E|A_{inNc}|^{2+\delta} < \infty,$$

$$\text{(2.2.23)} \qquad \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} E|A_{inNd}|^{2+\delta} < \infty.$$

We note in passing that this result will ensure the finiteness of the expectations and integrals considered so far. The proof relies on HÖLDER's inequality in the form

$$(2.2.24)\qquad \int \left|\prod_{i=0}^{k} f_i(\overline{F}_{iN})\right| d\overline{G}_N \le \prod_{i=0}^{k}\left[\int_0^1 |f_i(s_i)|^{\xi_i} ds_i\right]^{1/\xi_i},$$

where $f_0, f_1, \ldots, f_k$ are measurable functions on $(0,1)$ such that the above integrals exist and where $\xi_0, \xi_1, \ldots, \xi_k > 1$ satisfy $\Sigma_{i=0}^{k} \xi_i^{-1} = 1$.

Application of (2.2.24) with $\xi_i = a/a_i$ (here $a = \Sigma_{i=0}^{k} a_i$) yields

$$(2.2.25)\qquad N^{-1}\sum_{n=1}^{N} E(|A_{nN}|^{2+\delta}) \le$$

$$\le M \int r^{a_0(2+\delta)}(\overline{F}_{0N}) \prod_{j=1}^{k} r^{a_j(2+\delta)}(\overline{F}_{jN})\, d\overline{G}_N \le$$

$$\le M \prod_{i=0}^{k}\left[\int_0^1 r^{(2+\delta)a}(s)\,ds\right]^{a_i/a} < \infty,$$

provided $\delta > 0$ is chosen sufficiently small to ensure that $(2+\delta)a < 1$. Since $a < \frac{1}{2}$ by Assumption 2.1.1, this can always be achieved. Apparently the bound in (2.2.25) is independent of N so that (2.2.21) is proved.

To prove (2.2.22) for arbitrary $i \in \{1,2,\ldots,k\}$ we note that for $\delta \in (0,\frac{1}{2}]$ and $u,v \in (0,1)$, (see RUYMGAART (1973), page 27)

$$(2.2.26)\qquad |c(u-v) - u| \le M[r(v)]^{\frac{1}{2}-\delta}[r(u)]^{-\frac{1}{2}+\delta}.$$

From (2.2.26) and Assumption 2.1.1 we find,

$$N^{-1}\sum_{n=1}^{N} E\left(|A_{inNc}|^{2+\delta}\right) \le$$

$$\le N^{-1}\sum_{n=1}^{N} E\left[M\left(r(\overline{F}_{iN}(X_{inN}))\right)^{\frac{1}{2}-\delta}\int (r(\overline{F}_{iN}))^{-\frac{1}{2}+\delta}(r(\overline{F}_{iN}))^{a_i+1}\prod_{\substack{j=0\\ j\neq i}}^{k}(r(\overline{F}_{jN}))^{a_j}d\overline{G}_N\right]^{2+\delta} \le$$

$$\le M\int_0^1 (r(s))^{(\frac{1}{2}-\delta)(2+\delta)}ds\left[\int \prod_{\substack{j=0\\ j\neq i}}^{k}(r(\overline{F}_{jN}))^{a_j}(r(\overline{F}_{iN}))^{a_i+\frac{1}{2}+\delta}d\overline{G}_N\right]^{2+\delta}.$$

Since for every $\delta > 0$, $(\frac{1}{2}-\delta)(2+\delta) < 1$ it suffices to consider the last factor in the last bound, which is bounded above by

$$(2.2.27)\quad \prod_{\substack{j=0\\ j\neq i}}^{k} \left\{ \int_0^1 [r(s_i)]^{a_j/[a_j+(\frac{1}{2}-a-2\delta)/k]} ds_j \right\}^{a_j+(\frac{1}{2}-a-2\delta)/k} \times$$
$$\times \left\{ \int_0^1 [r(s_i)]^{(a_i+\frac{1}{2}+\delta)/(a_i+\frac{1}{2}+2\delta)} ds_i \right\}^{a_i+\frac{1}{2}+2\delta} < \infty .$$

This follows from an application of (2.2.24) with $\xi_j^{-1} = a_j + (\frac{1}{2}-a-2\delta)/k$ for $j \in \{0,1,\ldots,k\}$ but $j \neq i$, and $\xi_i^{-1} = a_i + \frac{1}{2} + 2\delta$. Because $a < \frac{1}{2}$ we have for $0 < 2\delta < \frac{1}{2} - a$ that $\xi_i > 1$ for $j = 0,1,\ldots,k$. The bound in (2.2.27) is independent of N, so that (2.2.22) is proved.

Finally let us note that because of Lemma 2.2.1 for $\Lambda_i > 0$,

$$(2.2.28)\quad N^{-1} \sum_{n=1}^{N} E\left(|A_{inNd}|^{2+\delta} \right) \leq M\Lambda_i \; |h_{iN}(s_i)|^{2+\delta} \leq M\Lambda_i \; M_i^{2+\delta},$$

so that the contribution due to the purely discrete part of the generating functions is bounded by a finite constant independent of N. It is obvious that the minimum over the finite number of δ's considered so far is a δ for which (2.2.21), (2.2.22) and (2.2.23) are simultaneously satisfied and hence we have proved (2.2.20). Moreover, from the proof of (2.2.20) and FUBINI's theorem it follows that

$$(2.2.29)\quad E \sum_{n=1}^{N} z_{nN} = 0.$$

Asymptotic normality of the A-terms (2.2.14) follows by a version of the central limit theorem due to ESSEEN (1945), using (2.2.20), (2.2.29) and the fact that the σ_N^2 are given to be bounded away from zero for N sufficiently large.

2.3. SOME LEMMAS ON EMPIRICAL DF'S

In this section we produce lemmas on empirical d.f.'s needed in section 2.4 for the proof of the asymptotic negligibility of the remainder term E_N in (2.2.10). These lemmas are based on the fundamental properties of the empirical d.f.'s, which are derived in Chapter I. Theorem 1.2.1 is a result on k-variate empirical d.f.'s and will be applied directly in section 2.4, so that we shall not repeat this theorem here. The six lemmas that are given in this section concern properties of univariate empirical d.f.'s based on real valued independent r.v.'s possessing not necessarily identical, but continuous d.f.'s. We shall adhere to the notation introduced in section 2.0, so that for the d.f.'s, averaged and empirical d.f.'s in question we shall use the notation $F_{i1N},\dots,F_{iNN}$, $\bar{F}_{iN}$ and $\mathbb{F}_{iN}$, $i = 1,2,\dots,k$. We denote the set of order statistics of the independent r.v.'s $X_{i1N},X_{i2N},\dots,X_{iNN}$ by $X^{(i)}_{1:N} \le X^{(i)}_{2:N} \le \dots \le X^{(i)}_{N:N}$. The function r is defined in (2.1.2), the random function $\mathbb{F}^{*}_{iN}$ is defined in (2.0.15).

LEMMA 2.3.1. *For every* $\varepsilon > 0$ *there exists a* $\beta = \beta(\varepsilon) \in (0,1)$, *such that for every positive integer* k, *every array of continuous* k*-variate d.f.'s* $F_{1N},F_{2N},\dots,F_{NN}$, $N = 1,2,\dots$, *every* $N = 1,2,\dots$ *and every* $i \in \{1,2,\dots,k\}$,

$$(2.3.1)\qquad P\Big(\beta\bar{F}_{iN}(x)\le\mathbb{F}_{iN}(x)\le 1-\beta(1-\bar{F}_{iN}(x)),\quad \textit{for } x \in [X^{(i)}_{1:N},X^{(i)}_{N:N})\Big)\ge 1-\varepsilon,$$

$$(2.3.2)\qquad P\Big(\beta\bar{F}_{iN}(x)\le\mathbb{F}^{*}_{iN}(x)\le 1-\beta(1-\bar{F}_{iN}(x)),\quad \textit{for } x \in [X^{(i)}_{1:N},X^{(i)}_{N:N}]\Big)\ge 1-\varepsilon.$$

PROOF. The first part of the lemma is immediate from Theorem 1.1.1 and Corollary 1.1.1. Moreover, it is clear from the definition of $\mathbb{F}^{*}_{iN}$ and from (2.3.1) that (2.3.2) holds with $[X^{(i)}_{1:N},X^{(i)}_{N:N}]$ replaced by $[X^{(i)}_{1:N},X^{(i)}_{N:N})$, so that it remains to be shown that

$$(2.3.3)\qquad P\Big(\beta\bar{F}_{iN}(X^{(i)}_{N:N}) \le \frac{N}{N+1} \le 1 - \beta(1-\bar{F}_{iN}(X^{(i)}_{N:N}))\Big) \ge 1 - \varepsilon.$$

Now

$$(2.3.4)\qquad P\Big(\beta\bar{F}_{iN}(X^{(i)}_{N:N})\le \frac{N}{N+1}\Big) \ge P\Big(\bar{F}_{iN}(X^{(i)}_{N:N})\le \frac{1}{2\beta}\Big) = 1 \qquad \text{for } \beta < \tfrac{1}{2},$$

and

$$(2.3.5) \qquad P\left(\frac{N}{N+1} \le 1 - \beta(1-\bar{F}_{iN}(X_{N:N}^{(i)}))\right) = 1 - P\left(\bar{F}_{iN}(X_{N:N}^{(i)}) < 1 - \alpha/N\right),$$

with $\alpha = \frac{N}{N+1}\frac{1}{\beta}$. If $\alpha \ge N$ then (2.3.5) equals 1, so that we may assume $\alpha < N$. In view of (1.1.40) we obtain for (2.3.5) the following lower bound

$$1 - e^{-\frac{N}{\beta(N+1)}} \ge 1 - e^{-\frac{1}{2\beta}} \to 1 \qquad \text{as } \beta \downarrow 0. \qquad \square$$

<u>LEMMA 2.3.2</u>. *For every* $\varepsilon > 0$ *and* $\delta > 0$ *there exists* $M = M(\varepsilon,\delta)$, *such that for every positive integer* k, *every array of continuous* k*-variate d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every* $N = 1,2,\ldots$ *and every* $i \in \{1,2,\ldots,k\}$,

$$(2.3.6) \qquad P\left(\sup_{x \in [X_{1:N}^{(i)}, X_{N:N}^{(i)})} \left(r(\mathbb{F}_{iN}(x))/r(\bar{F}_{iN}(x))\right)^{\delta} \le M(\varepsilon,\delta)\right) \ge 1 - \varepsilon,$$

$$(2.3.7) \qquad P\left(\sup_{x \in [X_{1:N}^{(i)}, X_{N:N}^{(i)}]} \left(r(\mathbb{F}_{iN}^{*}(x))/r(\bar{F}_{iN}(x))\right)^{\delta} \le M(\varepsilon,\delta)\right) \ge 1 - \varepsilon.$$

<u>PROOF</u>. The proof follows along the lines of the proof of Lemma 2.3.2 in RUYMGAART (1973) and relies on the present Lemma 2.3.1. $\square$

<u>LEMMA 2.3.3</u>. *For every* $\varepsilon > 0$ *and* $\delta \in (0,\frac{1}{2}]$ *there exists* $M = M(\varepsilon,\delta)$, *such that for every positive integer* k, *every array of* k*-variate d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, $N = 1,2,\ldots$, *every* $N = 1,2,\ldots$ *and every* $i \in \{1,2,\ldots,k\}$,

$$(2.3.8) \qquad P\left(\sup_{x \in (-\infty,\infty)} N^{\frac{1}{2}} |\mathbb{F}_{iN}(x) - \bar{F}_{iN}(x)| \left(r(\bar{F}_{iN}(x))\right)^{\frac{1}{2}-\delta} \le M(\varepsilon,\delta)\right) \ge 1 - \varepsilon,$$

$$(2.3.9) \qquad P\left(\sup_{x \in [X_{1:N}^{(i)}, X_{N:N}^{(i)}]} N^{\frac{1}{2}} |\mathbb{F}_{iN}^{*}(x) - \bar{F}_{iN}(x)| \left(r(\bar{F}_{iN}(x))\right)^{\frac{1}{2}-\delta} \le M(\varepsilon,\delta)\right) \ge 1 - \varepsilon.$$

<u>PROOF</u>. The first assertion is immediate from Corollary 1.1.2. The probability in (2.3.9) is bounded below by

$$(2.3.10) \qquad P\left(\sup_{x \in [X_{1:N}^{(i)}, X_{N:N}^{(i)}]} \left\{N^{\frac{1}{2}} |\mathbb{F}_{iN}(x) - \bar{F}_{iN}(x)| \left(r(\bar{F}_{iN}(x))\right)^{\frac{1}{2}-\delta} + N^{-\frac{1}{2}}\left(r(\bar{F}_{iN}(x))\right)^{\frac{1}{2}-\delta}\right\} \le M(\varepsilon,\delta)\right).$$

Theorem 1.1.4 implies for every $\varepsilon > 0$ the existence of a $\beta = \beta(\varepsilon) \in (0,1)$, such that for every positive integer k, every array of continuous k-variate d.f.'s $F_{1N},F_{2N},\dots,F_{NN}$, N = 1,2,..., every N = 1,2,... and every $i \in \{1,2,\dots,k\}$,

$$(2.3.11)\qquad P\left(\beta/N \le \bar{F}_{iN}(X_{1:N}^{(i)}) \le \bar{F}_{iN}(X_{N:N}^{(i)}) \le 1-\beta/N\right) \ge 1-\varepsilon.$$

Assertion (2.3.9) follows from (2.3.8), (2.3.10), (2.3.11) and the fact that for fixed β, $N^{-\frac{1}{2}}(r(\beta/N))^{\frac{1}{2}-\delta}$ is bounded for all N. □

<u>LEMMA 2.3.4</u>. *For every* $\varepsilon > 0$ *there exists* $M = M(\varepsilon)$, *such that for every positive integer* k, *every array of continuous* k-*variate d.f.'s* $F_{1N},F_{2N},\dots,F_{NN}$, N = 1,2,..., *every* N = 1,2,... *and every* $i \in \{1,2,\dots,k\}$,

$$(2.3.12)\qquad P\left(\sup_{m=1,2,\dots,N} N^{\frac{1}{2}}|\bar{F}_{iN}(X_{m:N}^{(i)})-mN^{-1}| \le M(\varepsilon)\right) \ge 1-\varepsilon.$$

<u>PROOF</u>. The assertion is immediate from (2.3.8) with $\delta = \frac{1}{2}$. □

For any positive integer N and real number $u \in (0,1)$ the positive integer N_u is uniquely determined by

$$(2.3.13)\qquad (N+1)u \le N_u < (N+1)u + 1.$$

<u>LEMMA 2.3.5</u>. *For every* $\varepsilon > 0$ *there exists* $M = M(\varepsilon)$, *such that for every positive integer* k, *every array of continuous* k-*variate d.f.'s* $F_{1N},F_{2N},\dots,F_{NN}$, N = 1,2,..., *every* N = 1,2,..., *every* $i \in \{1,2,\dots,k\}$ *and every* $u \in (0,1)$,

$$(2.3.14)\qquad P\left(N^{\frac{1}{2}}|\mathbb{F}_{iN}^{*}(\bar{F}_{iN}^{-1}(u))-u| \le M(\varepsilon)\right) \ge 1-\varepsilon,$$

$$(2.3.15)\qquad P\left(N^{\frac{1}{2}}|\bar{F}_{iN}(X_{N_u:N}^{(i)})-u| \le M(\varepsilon)\right) \ge 1-\varepsilon.$$

<u>PROOF</u>. For fixed integer $k \ge 1$, we have for $M > 0$, N = 1,2,..., $u \in (0,1)$ and $i \in \{1,2,\dots,k\}$,

$$P\left(N^{\frac{1}{2}}|\mathbb{F}_{iN}^{*}(\bar{F}_{iN}^{-1}(u))-u| \le M\right) \ge P\left(\sup_{u\in(0,1)} N^{\frac{1}{2}}|\mathbb{F}_{iN}^{*}(\bar{F}_{iN}^{-1}(u))-u| \le M\right) =$$

$$= P\left(\sup_{x\in\mathbb{R}} N^{\frac{1}{2}}|\mathbb{F}_{iN}^{*}(x)-\bar{F}_{iN}(x)| \le M\right) \ge$$

$$\geq P\Big(\sup_{x\in\mathbb{R}} N^{\frac12}\Big(\big|\tfrac{N}{N+1}\mathbb{F}_{iN}(x) - \mathbb{F}_{iN}(x)\big| + \big|\mathbb{F}_{iN}(x) - \overline{F}_{iN}(x)\big|\Big) \leq M\Big) \geq$$

$$\geq P\Big(\sup_{x\in\mathbb{R}} N^{\frac12}\,|\mathbb{F}_{iN}(x) - \overline{F}_{iN}(x)| \leq M - \frac{N^{\frac12}}{N+1}\Big) \geq$$

$$\geq P\Big(\sup_{x\in\mathbb{R}} N^{\frac12}\,|\mathbb{F}_{iN}(x) - \overline{F}_{iN}(x)| \leq M - \tfrac12\Big),$$

so that (2.3.14) follows from (2.3.8) with $\delta = \frac12$. Moreover, for $M > 0$, $N = 1,2,\ldots$, $u \in (0,1)$, $i \in \{1,2,\ldots,k\}$,

$$P\Big(N^{\frac12}|\overline{F}_{iN}(X^{(i)}_{N_u:N}) - u| \leq M\Big) \geq$$

$$\geq P\Big(N^{\frac12}\Big(|\overline{F}_{iN}(X^{(i)}_{N_u:N}) - N_uN^{-1}| + |N_uN^{-1} - u|\Big) \leq M\Big) \geq$$

$$\geq P\Big(N^{\frac12}|\overline{F}_{iN}(X^{(i)}_{N_u:N}) - N_uN^{-1}| \leq M - \frac{2N^{\frac12}}{N}\Big) \geq$$

$$\geq P\Big(N^{\frac12}|\overline{F}_{iN}(X^{(i)}_{N_u:N}) - N_uN^{-1}| \leq M - 2\Big),$$

so that (2.3.15) follows from Lemma 2.3.4. □

For $N = 1,2,\ldots$, $i = 1,2,\ldots,k$, we define the reduced empirical process U_{iN} as

$$(2.3.16)\qquad U_{iN}(s) = N^{\frac12}\Big(\mathbb{F}_{iN}(\overline{F}^{-1}_{iN}(s)) - s\Big), \qquad \text{for } 0 \leq s \leq 1.$$

<u>LEMMA 2.3.6</u>. *Let the reduced empirical processes* U_{iN} *be defined as in* (2.3.16) *and let the function* I_m *be defined as in* (1.1.37). *For every* $\varepsilon > 0$ *and* $c > 0$, *there exist* $N_0 = N_0(\varepsilon,c)$ *and* $m_0 = m_0(\varepsilon,c)$, *such that for every positive integer* k, *every array of continuous d.f.'s* $F_{1N},F_{2N},\ldots,F_{NN}$, $N = 1,2,\ldots$, *every* $i \in \{1,2,\ldots,k\}$ *and* $N \geq N_0$, $m \geq m_0$,

$$(2.3.17)\qquad P\Big(\sup_{s\in(0,1)} |U_{iN}(I_m(s)) - U_{iN}(s)| \leq c\Big) \geq 1 - \varepsilon.$$

<u>PROOF</u>. The random function $\mathbb{F}_{iN}(\overline{F}^{-1}_{iN})$ is with probability 1 the empirical d.f. $\tilde{\mathbb{F}}_{iN}$ (say) of the set of independent r.v.'s $\overline{F}_{iN}(X_{i1N}),\ldots,\overline{F}_{iN}(X_{iNN})$ (cf. Remark 1.1.1). Hence with probability one,

$$U_{iN}(s) = N^{\frac{1}{2}}(\tilde{\mathbb{F}}_{iN}(s) - s), \qquad \text{for } 0 \le s \le 1, \tag{2.3.18}$$

so that Lemma 2.3.6 is immediate from Corollary 1.1.3. □

2.4. ASYMPTOTIC NEGLIGIBILITY OF THE REMAINDER TERM

Before going into the details of the proof of the asymptotic negligibility of the remainder term E_N in (2.2.10) we have to introduce some notation. We recall (see (2.2.8)) that for $i = 1,2,\dots,k$, s_i is the only discontinuity point of the scores generating function J_i. In view of Assumption 2.1.1 and the way in which we shall conduct the proof it is no loss of generality to assume that, for $i = 1,2,\dots,k$, there exists only one continuity point $\tilde{s}_i$ where J_i is either not differentiable or its derivative is not continuous and that $\tilde{s}_i < s_i$. The same assumption is made for J_{0N}, $N = 1,2,\dots$, where these points are denoted by s_{0N} and $\tilde{s}_{0N}$ respectively.

For $N = 1,2,\dots$, $i = 1,2,\dots,k$, we define the reduced empirical process U_{iN}, the modified reduced empirical process U^*_{iN}, the closed random set Δ_{iN}, the set 0_{iN} and for small positive γ, the set $S_{iN\gamma}$ as follows:

(2.4.1)
$$U_{iN}(s) = N^{\frac{1}{2}}\left(\mathbb{F}_{iN}(\bar{F}_{iN}^{-1}(s))-s\right), \qquad \text{for } 0 \le s \le 1,$$

$$U^*_{iN}(s) = N^{\frac{1}{2}}\left(\mathbb{F}^*_{iN}(\bar{F}_{iN}^{-1}(s))-s\right), \qquad \text{for } 0 \le s \le 1,$$

$$\Delta_{iN} = \left[X^{(i)}_{1:N},\ X^{(i)}_{N:N}\right],$$

$$\Delta_{0N} = \left[X^{(0)}_{1:N},\ X^{(0)}_{N:N}\right],$$

$$0_{iN} = \left\{x\colon \bar{F}_{iN}(x) \in \left[s_i - MN^{-\frac{1}{2}},\ s_i + MN^{-\frac{1}{2}}\right]\right\},$$

$$S_{iN\gamma} = \left\{x\colon \bar{F}_{iN}(x) \in \left[\gamma,\ \tilde{s}_i - \gamma\right] \cup \left[\tilde{s}_i + \gamma,\ s_i - \gamma\right] \cup \left[s_i + \gamma,\ 1-\gamma\right]\right\},$$

$$S_{0N\gamma} = \left\{x\colon \bar{F}_{0N}(x) \in \left[\gamma,\ \tilde{s}_{0N} - \gamma\right] \cup \left[\tilde{s}_{0N} + \gamma,\ s_{0N} - \gamma\right] \cup \left[s_{0N} + \gamma,\ 1-\gamma\right]\right\}.$$

Moreover, for $N = 1,2,\dots$, $i = 1,2,\dots,k$, let

(2.4.2)
$$S_{N\gamma} = \prod_{j=0}^{k} S_{jN\gamma},$$

$$T_{iN\gamma} = \prod_{\substack{j=0\\ j\neq i}}^{k} S_{jN\gamma},$$

$$\tilde{S}_{iN\gamma} = \prod_{j=0}^{i-1} S_{jN\gamma} \times \mathbb{R} \times \prod_{j=i+1}^{k} S_{jN\gamma},$$

$$\Delta_N = \prod_{j=0}^{k} \Delta_{jN}.$$

Since $\overline{F}_{iN}$ is constant on an interval if and only if $F_{i1N},\ldots,F_{iNN}$ are constant on this interval (cf. Remark 1.1.1), we have for every $x \in \mathbb{R}$ that

$$(2.4.3) \qquad F_{inN}\Big(\overline{F}_{iN}^{-1}(\overline{F}_{iN}(x))\Big) = F_{inN}(x), \qquad n = 1,2,\ldots,N,\ i = 0,1,\ldots,k,$$

no matter what the form of the d.f.'s $F_{i1N},\ldots,F_{iNN}$ is (continuous or not). Denoting by

$$(2.4.4) \qquad \Omega_0 = \Big\{\omega : \mathbb{F}_{iN}\Big(\overline{F}_{iN}^{-1}(\overline{F}_{iN}(x))\Big) = \mathbb{F}_{iN}(x),\ \text{for all } x \in \mathbb{R},\ i = 0,1,\ldots,k \text{ and } N = 1,2,\ldots\Big\},$$

we have from (2.4.3) that $P(\Omega_0) = 1$.

For small $\gamma > 0$ we adopt the notation

$$(2.4.5) \qquad \Omega^*_{\gamma N} = \Big(\bigcap_{j=1}^{k} \Big\{\omega : \sup |\mathbb{F}^*_{jN} - \overline{F}_{jN}| < \gamma/2\Big\}\Big) \cap \Omega_0,$$

and remark that for $\omega \in \Omega^*_{\gamma N}$, we have for $i = 1,2,\ldots,k$, $U_{iN}(\overline{F}_{iN}) = N^{\frac{1}{2}}(\mathbb{F}_{iN} - \overline{F}_{iN})$, $U^*_{iN}(\overline{F}_{iN}) = N^{\frac{1}{2}}(\mathbb{F}^*_{iN} - \overline{F}_{iN})$ and

$$(2.4.6) \qquad N^{\frac{1}{2}} J_{ic}(\mathbb{F}^*_{iN}) = N^{\frac{1}{2}} J_{ic}(\overline{F}_{iN}) + U^*_{iN}(\overline{F}_{iN})\, J^{(1)}_{ic}(\tilde{\Phi}_{iN}),$$

for all $x \in \Delta_{iN} \cap S_{iN\gamma}$, where the random number $\tilde{\Phi}_{iN}$ lies in the open interval with end points $\overline{F}_{iN}$ and $\mathbb{F}^*_{iN}$.

Next, let us introduce for $N = 1,2,\ldots$, $i = 1,2,\ldots,k$ and given numbers s_i, the positive integers $N_i = N_{s_i}$ uniquely defined as

$$(2.4.7) \qquad (N+1)\, s_i \le N_i < (N+1)\, s_i + 1,$$

and the random sets Γ_{iN} as

$$(2.4.8) \qquad \Gamma_{iN} = \Big\{x:\ \min\Big(X^{(i)}_{N_i:N},\ \overline{F}^{-1}_{iN}(s_i)\Big) \le x < \max\Big(X^{(i)}_{N_i:N},\ \overline{F}^{-1}_{iN}(s_i)\Big)\Big\}.$$

LEMMA 2.4.1. *For every* $\varepsilon > 0$ *and every positive integer* k*, there exists* $M = M(\varepsilon,k)$ *such that for every array of continuous* k*-variate d.f.'s* $F_{1N},F_{2N},\ldots,F_{NN}$, $N = 1,2,\ldots$*, every* $N = 1,2,\ldots$ *and every* $i \in \{1,2,\ldots,k\}$,

$$(2.4.9)\qquad P\Big(\sup_{I^*_{i1},I^*_{i2}} \Big|\mathbb{G}_N\Big\{I^*_{i1}\times\Gamma_{iN}\times I^*_{i2}\Big\} - \overline{G}_N\Big\{I^*_{i1}\times\Gamma_{iN}\times I^*_{i2}\Big\}\Big| \le \frac{M(\log(N+1))^{\frac{1}{2}}}{N^{\frac{3}{4}}}\Big) \ge 1-\varepsilon.$$

Here the supremum is taken over all intervals $I^*_{i1} \subset \mathbb{R}^i$, $I^*_{i2} \subset \mathbb{R}^{k-i}$, *with the obvious convention that* I^*_{i2} *does not occur for* $i = k$.

<u>PROOF</u>. Choose $\varepsilon > 0$, the integer $k \ge 1$, $N \in \{1,2,\ldots\}$, $i \in \{1,2,\ldots,k\}$ and continuous k-variate d.f.'s $F_{1N},F_{2N},\ldots,F_{NN}$. Lemma 2.3.5 implies the existence of a finite positive number $M_1 = M_1(\varepsilon)$ such that

$$(2.4.10)\qquad P\Big(\overline{F}_{iN}(X^{(i)}_{N_i:N}) \in \Big[s_i - M_1N^{-\frac{1}{2}},\ s_i + M_1N^{-\frac{1}{2}}\Big]\Big) \ge 1 - \tfrac{1}{2}\varepsilon.$$

Let

$$\widetilde{O}_{iN} = \Big\{x\colon \overline{F}_{iN}(x) \in \Big[s_i - M_1N^{-\frac{1}{2}},\ s_i + M_1N^{-\frac{1}{2}}\Big]\Big\}\ ,$$

and apply Theorem 1.2.1 in (k+1) dimensions, with $I = I_N = \mathbb{R}^i \times \widetilde{O}_{iN} \times \mathbb{R}^{k-i}$ and hence with $\overline{G}_N\{I\} = \overline{F}_{iN}\{\widetilde{O}_{iN}\} = 2M_1N^{-\frac{1}{2}}$. We find that there exists a number $M_2 = M_2(\varepsilon,k)$ such that

$$(2.4.11)\qquad P\Big(\sup_{I^*_{i1},I^*_{i},I^*_{i2}} \Big|\mathbb{G}_N\{I^*_{i1}\times I^*_{i}\times I^*_{i2}\} - \overline{G}_N\{I^*_{i1}\times I^*_{i}\times I^*_{i2}\}\Big| \le \frac{(2M_1)^{\frac{1}{2}}M_2(\log(N+1))^{\frac{1}{2}}}{N^{\frac{3}{4}}}\Big) \ge$$
$$\ge 1 - \tfrac{1}{2}\varepsilon.$$

Here the supremum is taken over all intervals $I^*_{i1} \subset \mathbb{R}^i$, $I^*_{i2} \subset \mathbb{R}^{k-i}$, $I^*_i \subset \widetilde{O}_{iN}$. From (2.4.10) and (2.4.11) it is now immediate that (2.4.9) holds, since

$$X^{(i)}_{N_i:N} \in \widetilde{O}_{iN} \Rightarrow \Gamma_{iN} \subset \widetilde{O}_{iN}.\quad \square$$

Let $\chi(S)$ denote the indicator function of a set S as defined in (1.3.1).

<u>LEMMA 2.4.2</u>. *Let* N_i, Γ_{iN}, O_{iN}, Δ_{iN}, U_{iN}, U^*_{iN}, Ω_0 *and the function* r *be defined as in* (2.4.7), (2.4.8), (2.4.1), (2.4.4) *and* (2.1.2). *Let* $I_{i1} = \{I^*_{i1}\colon I^*_{i1}$ is an interval contained in $\mathbb{R}^i\}$ *and let* $I_{i2} = \{I^*_{i2}\colon I^*_{i2}$ is an interval contained in $\mathbb{R}^{k-i}\}$. *Denote* $\Omega_{N\delta} = \Omega_0 \cap (\cap^3_{j=1}\Omega_{jN\delta}) \cap (\cap^6_{j=4}\Omega_{jN})$,

with, for positive M, δ, a_i, i = 1,2,...,k, ξ_N, N = 1,2,..., *and positive integer* k,

$$\Omega_{1N\delta} = \bigcap_{i=1}^{k} \left[\left| U_{iN}(\overline{F}_{iN}) \right| \le M\, r^{-\frac{1}{2}+\delta}(\overline{F}_{iN}) \text{ on } (-\infty,\infty) \right], \tag{2.4.12}$$

$$\Omega_{2N\delta} = \bigcap_{i=1}^{k} \left[\left| U^*_{iN}(\overline{F}_{iN}) \right| \le M\, r^{-\frac{1}{2}+\delta}(\overline{F}_{iN}) \text{ on } \Delta_{iN} \right],$$

$$\Omega_{3N\delta} = \bigcap_{i=1}^{k} \left[\left| U^*_{iN}(\overline{F}_{iN}) - U_{iN}(\overline{F}_{iN}) \right| \le \xi_N\, r^{-\frac{1}{2}+\delta}(\overline{F}_{iN}) \text{ on } \Delta_{iN} \right],$$

$$\Omega_{4N} = \bigcap_{i=1}^{k} \left[r^{a_i}(\mathbb{F}^*_{iN}) \le M\, r^{a_i}(\overline{F}_{iN}),\ r^{a_i+1}(\mathbb{F}^*_{iN}) \le M\, r^{a_i+1}(\overline{F}_{iN}) \text{ on } \Delta_{iN} \right],$$

$$\Omega_{5N} = \bigcap_{i=1}^{k} \left[s_i - MN^{-\frac{1}{2}} \le \mathbb{F}^*_{iN}(\overline{F}^{-1}_{iN}(s_i)) \le s_i + MN^{-\frac{1}{2}},\ X^{(i)}_{N_i:N} \in O_{iN} \right],$$

$$\Omega_{6N} = \bigcap_{i=1}^{k} \left[\sup_{\substack{I^*_{i1} \in \mathcal{I}_{i1} \\ I^*_{i2} \in \mathcal{I}_{i2}}} \left| \mathbb{G}_N\{I^*_{i1} \times \Gamma_{iN} \times I^*_{i2}\} - \overline{G}_N\{I^*_{i1} \times \Gamma_{iN} \times I^*_{i2}\} \right| \le \frac{M(\log(N+1))^{\frac{1}{2}}}{N^{\frac{3}{4}}} \right].$$

For every ε > 0, *every positive integer* k, *every* δ ∈ (0,½] *and every positive* a_i, i = 1,2,...,k, *there exists a number* $M = M(\varepsilon,k,\delta,a_1,\ldots,a_k) \ge 1$ *and a sequence* $\xi_N = \xi_N(\varepsilon,k,\delta)$, *decreasing to zero as* N *tends to infinity, such that the set* $\Omega_{N\delta}$ *has probability* $P(\Omega_{N\delta}) \ge 1 - \varepsilon$, *for* N = 1,2,... *and every array of* k*-variate continuous underlying d.f.'s* $F_{1N}, F_{2N}, \ldots, F_{NN}$, N = 1,2,... .

Moreover, on $\Omega_{N\delta}$ *we have for* i = 1,2,...,k,

$$|c(\mathbb{F}^*_{iN}(x)-s_i) - c(\overline{F}_{iN}(x)-s_i)| \le \chi(\Gamma_{iN};x) \le \chi(O_{iN};x). \tag{2.4.13}$$

<u>PROOF</u>. The first assertion is immediate from the Lemmas 2.3.2 and 2.3.3, the fact that for every $\omega \in \Omega_0$ we have $|U^*_{iN}(\overline{F}_{iN}) - U_{iN}(\overline{F}_{iN})| \le N^{-\frac{1}{2}}$, (2.3.11) and the Lemmas 2.3.5 and 2.4.1. The second assertion follows from Lemma 3.3.4 in RUYMGAART (1973). □

Let us notice the following property of the set Ω_{4N}. For i = 1,2, ...,k let, for each ω, $\tilde{\Phi}_{iN} = \tilde{\Phi}(\omega)$ be a function defined on Δ_{iN}, satisfying

$$\min(\overline{F}_{iN},\mathbb{F}^*_{iN}) \le \tilde{\Phi}_{iN} \le \max(\overline{F}_{iN},\mathbb{F}^*_{iN}) \quad \text{on } \Delta_{iN}. \tag{2.4.14}$$

Then, independently of the continuous d.f.'s $F_{1N}, F_{2N}, \ldots, F_{NN}$, we have for $i = 1,2,\ldots,k$ on Δ_{iN},

$$(2.4.15)\qquad r^{a_i}(\tilde{\Phi}_{iN}) \le M\, r^{a_i}(\bar{F}_{iN}), \qquad r^{a_i+1}(\tilde{\Phi}_{iN}) \le M\, r^{a_i+1}(\bar{F}_{iN}),$$

for each $\omega \in \Omega_{4N}$.

The last auxiliary result we need is the following statement:

LEMMA 2.4.3. *Suppose that the numbers* α_{ij}, $1 \le i \le k$, $1 \le j \le N$, *satisfy*

$$0 \le \alpha_{i1} \le \alpha_{i2} \le \ldots \le \alpha_{iN}, \qquad \textit{for } 1 \le i \le k.$$

Then

$$(2.4.16)\qquad \sum_{j=1}^{N} \alpha_{1\pi_1(j)}\alpha_{2\pi_2(j)}\cdot\ldots\cdot\alpha_{k\pi_k(j)} \le \sum_{j=1}^{N} \alpha_{1j}\alpha_{2j}\cdot\ldots\cdot\alpha_{kj},$$

for every set of k *permutations* $\{(\pi_1(1),\ldots,\pi_1(N)),\ (\pi_2(1),\ldots,\pi_2(N)),\ldots,$ $(\pi_k(1),\ldots,\pi_k(N))\}$ *of the numbers* $1,2,\ldots,N$.

PROOF. The lemma can be proved by induction on k. For $k = 1$ the assertion (2.4.16) is trivially true. Suppose that (2.4.16) holds for a fixed $k \ge 1$. We shall show that

$$(2.4.17)\qquad \sum_{j=1}^{N} \alpha_{1\pi_1(j)}\alpha_{2\pi_2(j)}\cdot\ldots\cdot\alpha_{k+1\,\pi_{k+1}(j)} \le$$
$$\le \sum_{j=1}^{N} \alpha_{1j}\alpha_{2j}\alpha_{3\pi_3'(j)}\cdot\ldots\cdot\alpha_{k+1\,\pi_{k+1}'(j)},$$

for a set of (k-1) permutations $\{(\pi_3'(1),\ldots,\pi_3'(N)),\ (\pi_4'(1),\ldots,\pi_4'(N)),\ldots,$ $(\pi_{k+1}'(1),\ldots,\pi_{k+1}'(N))\}$ of the numbers $1,2,\ldots,N$.

First we therefore prove that

$$(2.4.18)\qquad \sum_{j=1}^{N} \alpha_{1\pi_1(j)}\alpha_{2\pi_2(j)}\cdot\ldots\cdot\alpha_{k+1\,\pi_{k+1}(j)} \le$$
$$\le \sum_{j=1}^{N-1} \alpha_{1\pi_1'(j)}\alpha_{2\pi_2'(j)}\cdot\ldots\cdot\alpha_{k+1\,\pi_{k+1}'(j)} + \alpha_{1N}\alpha_{2N}\alpha_{3\pi_3'(N)}\cdot\ldots\cdot\alpha_{k+1\,\pi_{k+1}'(N)},$$

for a set of (k-1) permutations $\{(\pi_3'(1),\ldots,\pi_3'(N)),\ (\pi_4'(1),\ldots,\pi_4'(N)),\ldots,$ $(\pi_{k+1}'(1),\ldots,\pi_{k+1}'(N))\}$ of the numbers $1,2,\ldots,N$ and two permutations

$(\pi_1'(1),\ldots,\pi_1'(N-1))$, $(\pi_2'(1),\ldots,\pi_2'(N-1))$ of the numbers $1,2,\ldots,N-1$. Namely, if $\pi_1(j) = \pi_2(j) = N$ for some j, then (2.4.18) is trivially true. So suppose

$$\pi_1(\ell) = \pi_2(j) = N, \qquad \text{for some } j \text{ and } \ell \text{ with } j \neq \ell,$$

and denote

$$\beta = \alpha_{3\pi_3(j)}\alpha_{4\pi_4(j)}\cdot\ldots\cdot\alpha_{k+1\ \pi_{k+1}(j)},$$

$$\gamma = \alpha_{3\pi_3(\ell)}\alpha_{4\pi_4(\ell)}\cdot\ldots\cdot\alpha_{k+1\ \pi_{k+1}(\ell)}.$$

For $\beta \geq \gamma$ we have

$$\alpha_{1N}\ \alpha_{2N}\beta + \alpha_{1\pi_1(j)}\alpha_{2\pi_2(\ell)}\gamma \geq \alpha_{1\pi_1(j)}\alpha_{2N}\beta + \alpha_{1N}\alpha_{2\pi_2(\ell)}\gamma =$$

$$= \alpha_{1\pi_1(j)}\alpha_{2\pi_2(j)}\cdot\ldots\cdot\alpha_{k+1\ \pi_{k+1}(j)} + \alpha_{1\pi_1(\ell)}\alpha_{2\pi_2(\ell)}\cdot\ldots\cdot\alpha_{k+1\ \pi_{k+1}(\ell)},$$

because $(\alpha_{1N}-\alpha_{1\pi_1(j)})(\alpha_{2N}\beta-\alpha_{2\pi_2(\ell)}\gamma) \geq 0$. Analougsly we find for $\beta < \gamma$ that

$$(\alpha_{2N}-\alpha_{2\pi_2(\ell)})(\alpha_{1N}\gamma-\alpha_{1\pi_1(j)}\beta) \geq 0$$

and hence

$$\alpha_{1N}\ \alpha_{2N}\gamma + \alpha_{1\pi_1(j)}\alpha_{2\pi_2(\ell)}\beta \geq \alpha_{1\pi_1(j)}\alpha_{2N}\beta + \alpha_{1N}\ \alpha_{2\pi_2(\ell)}\gamma,$$

so that (2.4.18) is proved.
Applying the same reasoning to $\Sigma_{j=1}^{N-1}\ \alpha_{1\pi_1'(j)}\alpha_{2\pi_2'(j)}\cdot\ldots\cdot\alpha_{k+1\pi_{k+1}'(j)}$, N-1 times, gives us (2.4.17).

The lemma now follows directly from (2.4.17) using the induction hypothesis, because

$$\sum_{j=1}^{N} \alpha_{1\pi_1(j)}\alpha_{2\pi_2(j)}\cdot\ldots\cdot\alpha_{k+1\ \pi_{k+1}(j)} \leq$$

$$\leq \sum_{j=1}^{N} (\alpha_{1j}\alpha_{2j})\ \alpha_{3\pi_3'(j)}\cdot\ldots\cdot\alpha_{k+1\ \pi_{k+1}'(j)} \leq$$

$$\leq \sum_{j=1}^{N} (\alpha_{1j}\alpha_{2j})\ \alpha_{3j}\cdot\ldots\cdot\alpha_{k+1\ j}. \qquad \square$$

Since we now have collected the basic tools for the proof of the asymptotic negligibility of the remainder term, let us return to the statistic T_N, defined in (2.0.16). Writing

$$B_N = \sum_{i=1}^{k} \left[J_i(\mathbb{F}^*_{iN}) - J_i(\overline{F}_{iN}) \right] J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(\overline{F}_{jN}), \tag{2.4.19}$$

$$C_N = J_{ON}(\mathbb{F}^*_{ON}) \prod_{j=1}^{k} J_j(\mathbb{F}^*_{jN}) - J_{ON}(\overline{F}_{ON}) \prod_{j=1}^{k} J_j(\overline{F}_{jN}) - B_N, \tag{2.4.20}$$

it is immediate from (2.0.16), (2.1.10), (2.2.11), (2.4.19) and (2.4.20) that with probability one,

$$N^{\frac{1}{2}}(T_N - \mu_N) = A_N + N^{\frac{1}{2}} \int_{\Delta_N} B_N \, d\mathbb{G}_N + N^{\frac{1}{2}} \int_{\Delta_N} C_N \, d\mathbb{G}_N. \tag{2.4.21}$$

LEMMA 2.4.4. *Under the conditions of Theorem* 2.1.1 *there exists for every* $\varepsilon > 0$ *and every positive integer* k *a positive integer* N_0, *depending on* ε, k *and the constants in Assumptions* 2.1.1 *and* 2.1.2, *such that for every* $N \geq N_0$ *we have*

$$P\left(N^{\frac{1}{2}} \int_{\Delta_N} C_N d\mathbb{G}_N \leq \varepsilon \right) \geq 1 - \varepsilon. \tag{2.4.22}$$

PROOF. We remark that

$$N^{\frac{1}{2}} \int_{\Delta_N} C_N \, d\mathbb{G}_N = N^{\frac{1}{2}} \int_{\Delta_N} C_{ON} \, d\mathbb{G}_N + N^{\frac{1}{2}} \int_{\Delta_N} \sum_{i=1}^{k-1} C_{iN} \, d\mathbb{G}_N,$$

where

$$C_{ON} = [J_{ON}(\mathbb{F}^*_{ON}) - J_{ON}(\overline{F}_{ON})] \prod_{j=1}^{k} J_j(\mathbb{F}^*_{jN}), \tag{2.4.23}$$

$$C_{1N} = [J_1(\mathbb{F}^*_{1N}) - J_1(\overline{F}_{1N})] J_{ON}(\overline{F}_{ON}) \left[\prod_{j=2}^{k} J_j(\mathbb{F}^*_{jN}) - \prod_{j=2}^{k} J_j(\overline{F}_{jN}) \right], \tag{2.4.24}$$

and for $i = 2,3,\ldots,k-1$,

$$C_{iN} = [J_i(\mathbb{F}^*_{iN}) - J_i(\overline{F}_{iN})] J_{ON}(\overline{F}_{ON}) \prod_{j=1}^{i-1} J_j(\overline{F}_{jN}) \left[\prod_{j=i+1}^{k} J_j(\mathbb{F}^*_{jN}) - \prod_{j=i+1}^{k} J_j(\overline{F}_{jN}) \right]. \tag{2.4.25}$$

First let us deal with the asymptotic negligibility of the term $N^{\frac{1}{2}}\int_{\Delta_N} \Sigma_{i=1}^{k-1} C_{iN}\, d\mathbb{G}_N$. From (2.2.8), Assumption 2.1.1, Lemma 2.4.2, possibly step-wise application of the mean value theorem, together with (2.4.15), it follows that for every $\omega \in \Omega_{N\delta}$ and $i = 1,2,\ldots,k$ we have on Δ_{iN},

$$|J_i(\mathbb{F}^*_{iN}) - J_i(\overline{F}_{iN})| \le M\left(N^{-\frac{1}{2}} r^{a_i+\frac{1}{2}+\delta}(\overline{F}_{iN}) \wedge r^{a_i}(\overline{F}_{iN})\right) + M\chi(O_{iN}). \tag{2.4.26}$$

Moreover, from (2.4.24), (2.4.25), (2.4.26) and Assumption 2.1.1 we have

$$E\left(\chi(\Omega_{N\delta})\ \left|N^{\frac{1}{2}}\int_{\Delta_N}\sum_{i=1}^{k-1} C_{iN}\, d\mathbb{G}_N\right|\right) \le \tag{2.4.27}$$

$$\le ME\left(\chi(\Omega_{N\delta})\sum_{i=1}^{k-1}\ \sum_{j=i+1}^{k} N^{\frac{1}{2}}\int_{\Delta_N} |J_i(\mathbb{F}^*_{iN})-J_i(\overline{F}_{iN})|\,|J_j(\mathbb{F}^*_{jN})-J_j(\overline{F}_{jN})| \prod_{\substack{h=0\\ h\neq i,j}}^{k} r^{a_h}(\overline{F}_{hN})\, d\mathbb{G}_N\right) \le$$

$$\le M\sum_{i=1}^{k-1}\ \sum_{j=i+1}^{k}\int r^{a_i+\frac{1}{2}+\delta}(\overline{F}_{iN})\left(N^{-\frac{1}{2}} r^{a_j+\frac{1}{2}+\delta}(\overline{F}_{jN}) \wedge r^{a_j}(\overline{F}_{jN})\right)\prod_{\substack{h=0\\ h\neq i,j}}^{k} r^{a_h}(\overline{F}_{hN})\, d\overline{G}_N +$$

$$+ M\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int r^{a_j+\frac{1}{2}+\delta}(\overline{F}_{jN})\ \chi(O_{iN})\prod_{\substack{h=0\\ h\neq i,j}}^{k} r^{a_h}(\overline{F}_{hN})\, d\overline{G}_N +$$

$$+ M\sum_{i=1}^{k-1}\ \sum_{j=i+1}^{k} N^{\frac{1}{2}}\int \chi(O_{iN})\ \chi(O_{jN})\prod_{\substack{h=0\\ h\neq i,j}}^{k} r^{a_h}(\overline{F}_{hN})\, d\overline{G}_N.$$

From HÖLDER's inequality (see 2.2.24), using the same ξ's as in (2.2.27), it is straightforward that the first two terms in the upper bound above converge to zero as N tends to infinity. As far as the last term in the upper bound is concerned we remark that from Assumption 2.1.2 and from (2.2.3) we obtain for $N \ge \tilde{N}_0 = \tilde{N}_0(\eta)$, with η as in Assumption 2.1.2 and $\overline{\Psi}_N$ as in (2.2.2), that

$$N^{\frac{1}{2}}\int \chi(O_{iN})\ \chi(O_{jN})\prod_{\substack{h=0\\ h\neq i,j}}^{k} r^{a_h}(\overline{F}_{hN})\, d\overline{G}_N = \tag{2.4.28}$$

$$= N^{\frac{1}{2}}\int \chi([s_i-MN^{-\frac{1}{2}}, s_i+MN^{-\frac{1}{2}}];t_i)\ \chi([s_j-MN^{-\frac{1}{2}}, s_j+MN^{-\frac{1}{2}}];t_j)\ \times$$

$$\times \prod_{\substack{h=0\\ h\neq i,j}}^{k} r^{a_h}(t_h)\ d\overline{\Psi}_N(t_0,\ldots,t_k) \le$$

$$\leq K_2\, N^{\frac{1}{2}}(2MN^{-\frac{1}{2}}) \int_{(0,1)^k} \chi([s_j-MN^{-\frac{1}{2}}, s_j+MN^{-\frac{1}{2}}]; t_j) \prod_{\substack{h=0\\ h\neq i,j}}^{k} r^{a_h}(t_h) \times$$

$$\times \prod_{\substack{h=1\\ h\neq i}}^{k} r^{b_h}(t_h)\; dt_0,\ldots,dt_{i-1},dt_{i+1},\ldots,dt_k,$$

which converges to zero as $N \to \infty$, as can be seen from the dominated convergence theorem ($a_j+b_j<1$). Compare with SHIRAHATA (1975), Remark 2.

Secondly let us prove the asymptotic negligibility of the term $N^{\frac{1}{2}}\int_{\Delta_N} C_{ON}\, d\mathbb{G}_N$ (see (2.4.23), due to the introduction of the dummy random variables $X_{01N}, X_{02N}, \ldots, X_{0NN}$. Denoting by J_{0Nc} the continuous part of J_{0N}, and by Λ_{0N} the height of the jump in s_{0N}, $N = 1,2,\ldots$, we have

$$(2.4.29) \qquad \left|N^{\frac{1}{2}} \int_{\Delta_N} C_{0N}\, d\mathbb{G}_N\right| \leq$$

$$\leq N^{-\frac{1}{2}} \sum_{n=1}^{N} |J_{0N}(\mathbb{F}^*_{0N}(X_{0nN})) - J_{0N}(X_{0nN})| \prod_{j=1}^{k} |J_j(\mathbb{F}^*_{jN}(X_{jnN}))| =$$

$$= N^{-\frac{1}{2}} \sum_{n=1}^{N} |J_{0N}(\tfrac{n}{N+1}) - J_{0N}(X_{0nN})| \prod_{j=1}^{k} \left|J_j\left(\frac{R_{jnN}}{N+1}\right)\right| =$$

$$= N^{-\frac{1}{2}}\Lambda_{0N} \sum_{n=1}^{N} |c(\tfrac{n}{N+1}-s_{0N}) - c(X_{0nN}-s_{0N})| \prod_{j=1}^{k} \left|J_j\left(\frac{R_{jnN}}{N+1}\right)\right| +$$

$$+ N^{-\frac{1}{2}} \sum_{n=1}^{N} \left|J_{0Nc}\left(\frac{n}{N+1}\right) - J_{0Nc}(X_{0nN})\right| \prod_{j=1}^{k} \left|J_j\left(\frac{R_{jnN}}{N+1}\right)\right| .$$

Let n_0 be the index such that $(n_0-1)N^{-1} < s_{0N} \leq n_0 N^{-1}$. With probability one we have that the first term in the upper bound in (2.4.29) is bounded above by

$$N^{-\frac{1}{2}}M \left| c\left(\frac{n_0}{N+1}-s_{0N}\right) - c(X_{0n_0N}-s_{0N}) \right| \prod_{j=1}^{k} \left|J_j\left(\frac{R_{jn_0N}}{N+1}\right)\right| \leq$$

$$\leq 2M\, N^{-\frac{1}{2}} \prod_{j=1}^{k} \left| r^{a_j}\left(\frac{N}{N+1}\right)\right| ,$$

which converges to zero as N tends to infinity since $\Sigma_{j=1}^{k}\, a_j < \frac{1}{2}$.

Next we consider the sum $N^{-\frac{1}{2}}\Sigma_{n=1}^{N}\left|J_{0Nc}\left(\frac{n}{N+1}\right)-J_{0Nc}(X_{0nN})\right|\prod_{j=1}^{k}\left|J_j\left(\frac{R_{jnN}}{N+1}\right)\right|$.

We recall that $\tilde{s}_{0N}$ is the only continuity point of J_{0Nc}, where $J_{0Nc}^{(1)}$ either does not exist or is not continuous. Let $\tilde{n}_0$ be the index such that $(\tilde{n}_0-1)N^{-1} < \tilde{s}_{0N} \le \tilde{n}_0 N^{-1}$. For sufficiently large N we have $\tilde{n}_0 \neq 1,N$. Since it is not hard to show that every single term in the sum above is asymptotically negligible, we restrict attention to the sum

$$N^{-\frac{1}{2}}\sum_{\substack{n=2\\ n\neq\tilde{n}_0}}^{N-1}\left|J_{0Nc}\left(\frac{n}{N+1}\right)-J_{0Nc}\left(X_{0nN}\right)\right|\prod_{j=1}^{k}\left|J_j\left(\frac{R_{jnN}}{N+1}\right)\right|,$$

which, in view of Assumption 2.1.1 and Lemma 2.4.3, is bounded above by

$$M\,N^{-\frac{3}{2}}\sum_{n=1}^{N} r^{a_0+1}\left(\frac{n}{N+1}\right)\prod_{j=1}^{k} r^{a_j}\left(\frac{R_{jnN}}{N+1}\right) \le$$

$$\le M\,N^{-\frac{3}{2}}\sum_{n=1}^{N} r^{a_0+1}\left(\frac{n}{N+1}\right)\prod_{j=1}^{k} r^{a_j}\left(\frac{n}{N+1}\right) =$$

$$= M\,N^{-\frac{3}{2}}\sum_{n=1}^{N} r^{a+1}\left(\frac{n}{N+1}\right) \le M\,N^{\frac{1}{2}a-\frac{1}{4}} \to 0, \qquad \text{as } N\to\infty.$$

This completes the proof of the asymptotic negligibility of the C_{0N}-term. The reader should note that this proof is a matter of straightforward calculus only. This could be expected as the appearance of the r.v. C_{0N} is merely due to the introduction of the dummy uniformly distributed r.v.'s X_{0nN} and hence ought not give rise to any serious trouble. □

Our next aim is to show that the term $N^{\frac{1}{2}}\int_{\Delta_N} B_N\, d\mathbb{G}_N$ in (2.4.21) can be approximated by the term $\Sigma_{i=1}^{k}(A_{iNc}+A_{iNd})$ in (2.2.10).
In view of (2.2.8) we have

$$B_N = \sum_{i=1}^{k}(B_{iNc}+B_{iNd}),$$

where, for i = 1,2,...,k,

$$\text{(2.4.30)}\qquad B_{iNc} = \left(J_{ic}(\mathbb{F}_{iN}^{*}) - J_{ic}(\overline{F}_{iN})\right) J_{0N}(\overline{F}_{0N}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN}),$$

$$(2.4.31)\qquad B_{iNd} = \Lambda_i\left[c(\mathbb{F}^*_{iN}-s_i) - c(\overline{F}_{iN}-s_i)\right] J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN}).$$

From now on in this section let $i \in \{1,2,\dots,k\}$ be fixed. We have the following decompositions

$$(2.4.32)\qquad N^{\frac{1}{2}} \int_{\Delta_N} B_{iNc}\, d\mathbb{G}_N - A_{iNc} = \sum_{j=1}^{6} D_{jNi},$$

$$(2.4.33)\qquad N^{\frac{1}{2}} \int_{\Delta_N} B_{iNd}\, d\mathbb{G}_N - A_{iNd} = \sum_{j=7}^{10} D_{jNi},$$

where, with $\widetilde{\Phi}_{iN}$, $\Omega^*_{\gamma N}$, U_{iN}, U^*_{iN}, $S_{N\gamma}$, $\widetilde{S}_{iN\gamma}$ as defined in (2.4.6), (2.4.5), (2.4.1) and (2.4.2),

$$D_{1Ni} = \chi(\Omega^*_{\gamma N}) \int_{S_{N\gamma}} U^*_{iN}(\overline{F}_{iN})\left[J^{(1)}_{ic}(\widetilde{\Phi}_{iN}) - J^{(1)}_{ic}(\overline{F}_{iN})\right] J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN})\, d\mathbb{G}_N,$$

$$D_{2Ni} = \chi(\Omega^*_{\gamma N}) \int_{S_{N\gamma}} \left[U^*_{iN}(\overline{F}_{iN}) - U_{iN}(\overline{F}_{iN})\right] J^{(1)}_{ic}(\overline{F}_{iN})\, J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN})\, d\mathbb{G}_N,$$

$$D_{3Ni} = \chi(\Omega^*_{\gamma N}) \int_{S_{N\gamma}} U_{iN}(\overline{F}_{iN})\, J^{(1)}_{ic}(\overline{F}_{iN})\, J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN})\, d(\mathbb{G}_N-\overline{G}_N),$$

$$D_{4Ni} = -\chi(\Omega^*_{\gamma N}) \int_{S^c_{N\gamma}} U_{iN}(\overline{F}_{iN})\, J^{(1)}_{ic}(\overline{F}_{iN})\, J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN})\, d\overline{G}_N,$$

$$D_{5Ni} = \chi(\Omega^*_{\gamma N}) N^{\frac{1}{2}} \int_{S^c_{N\gamma}} \left[J_{ic}(\mathbb{F}^*_{iN}) - J_{ic}(\overline{F}_{iN})\right] J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN})\, d\mathbb{G}_N,$$

$$D_{6Ni} = \chi(\Omega^{*c}_{\gamma N})\left(N^{\frac{1}{2}} \int_{\Delta_N} B_{iNc}\, d\mathbb{G}_N - A_{iNc}\right),$$

$$D_{7Ni} = N^{\frac{1}{2}}\Lambda_i \int \left[c(\mathbb{F}^*_{iN}-s_i) - c(\overline{F}_{iN}-s_i)\right] J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN})\, d\overline{G}_N -$$

$$- \Lambda_i h_{iN}(s_i)\, U_{iN}(s_i),$$

$$D_{8Ni} = N^{\frac{1}{2}}\Lambda_i \int_{\tilde{S}_{iN\gamma}} \left[c(\mathbb{F}^*_{iN}-s_i) - c(\bar{F}_{iN}-s_i)\right] J_{0N}(\bar{F}_{0N}) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(\bar{F}_{jN})\, d(\mathbb{G}_N-\bar{G}_N),$$

$$D_{9Ni} = -N^{\frac{1}{2}}\Lambda_i \int_{\tilde{S}^c_{iN\gamma}} \left[c(\mathbb{F}^*_{iN}-s_i) - c(\bar{F}_{iN}-s_i)\right] J_{0N}(\bar{F}_{0N}) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(\bar{F}_{jN})\, d\bar{G}_N,$$

$$D_{10Ni} = N^{\frac{1}{2}}\Lambda_i \int_{\tilde{S}^c_{iN\gamma}} \left[c(\mathbb{F}^*_{iN}-s_i) - c(\bar{F}_{iN}-s_i)\right] J_{0N}(\bar{F}_{0N}) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(\bar{F}_{jN})\, d\mathbb{G}_N.$$

<u>LEMMA 2.4.5</u>. *Under the conditions of Theorem* 2.1.1 *there exists for every* $\varepsilon > 0$ *and every positive integer* k *a positive* γ_0*, depending on* ε*,* k *and the constants in Assumptions* 2.1.1 *and* 2.1.2*, such that for every* $0 < \gamma < \gamma_0$*, and every* $N = 1,2,\ldots$ *we have*

$$(2.4.34) \qquad P\left(|D_{hNi}| \le \varepsilon\right) \ge 1 - \varepsilon, \qquad \textit{for } h = 4,5,9,10.$$

<u>PROOF</u>. From Assumption 2.1.1 and Lemma 2.4.2 it is immediate that

$$\chi(\Omega_{N\delta})\ |D_{4Ni}| \le M \int_{S^c_{N\gamma}} r^{a_i+\frac{1}{2}+\delta}(\bar{F}_{iN}) \prod_{\substack{j=0\\j\neq i}}^{k} r^{a_j}(\bar{F}_{jN})\, d\bar{G}_N.$$

As far as D_{5Ni} is concerned, step-wise application of the mean value theorem (see also (2.4.26)) implies that $\chi(\Omega_{N\delta})\ |D_{5Ni}| \le \tilde{D}_{N\gamma}$, where

$$\tilde{D}_{N\gamma} = M \int_{S^c_{N\gamma}} r^{a_i+\frac{1}{2}+\delta}(\bar{F}_{iN}) \prod_{\substack{j=0\\j\neq i}}^{k} r^{a_j}(\bar{F}_{jN})\, d\mathbb{G}_N,$$

and hence

$$E\tilde{D}_{N\gamma} = M \int_{S^c_{N\gamma}} r^{a_i+\frac{1}{2}+\delta}(\bar{F}_{iN}) \prod_{\substack{j=0\\j\neq i}}^{k} r^{a_j}(\bar{F}_{jN})\, d\bar{G}_N.$$

Now, for sufficiently small $\delta > 0$, we obtain with the aid of HÖLDER's inequality and of (2.2.27),

$$E\tilde{D}_{N\gamma} \le$$

$$\le \left[\int \left\{r^{a_i+\frac{1}{2}+\delta}(\bar{F}_{iN}) \prod_{\substack{j=0\\j\neq i}}^{k} r^{a_j}(\bar{F}_{jN})\right\}^{1+\delta} d\bar{G}_N\right]^{(1+\delta)^{-1}} \left[\int_{S^c_{N\gamma}} d\bar{G}_N\right]^{\delta(1+\delta)^{-1}} \le$$

$$\leq M^{(1+\delta)^{-1}} \left[\sum_{i=0}^{k} \int_{S^c_{iN\gamma}} d\bar{F}_{iN} \right]^{\delta(1+\delta)^{-1}} \leq$$

$$\leq M^{(1+\delta)^{-1}} \{6(k+1)\gamma\}^{\delta(1+\delta)^{-1}},$$

which converges to zero as $\gamma \downarrow 0$, uniformly in N.

With respect to the term D_{9Ni} we have, in view of Lemma 2.4.2 and Assumption 2.1.1, that

$$(2.4.35) \qquad \chi(\Omega_{N\delta}) \; |D_{9Ni}| \leq MN^{\frac{1}{2}} \int \chi(0_{iN}) \; \chi(T^c_{iN\gamma}) \prod_{\substack{j=0 \\ j\neq i}}^{k} r^{a_j}(\bar{F}_{jN}) \; d\bar{G}_N.$$

From Assumption 2.1.2 and HÖLDER's inequality again it follows that the upper bound in (2.4.35) converges to zero as $\gamma \downarrow 0$, uniformly in N. Since $E(\chi(\Omega_{N\delta})|D_{10Ni}|)$ has the same upper bound the proof of the lemma is completed. □

LEMMA 2.4.6. *Under the conditions of Theorem* 2.1.1 *there exists for every* $\varepsilon > 0$, *every* $0 < \gamma < \tau/2$ *and every positive integer* k *a positive integer* N_0, *depending on* ε, γ, k *and the constants in Assumptions* 2.1.1 *and* 2.1.2, *such that for every* $N \geq N_0$ *we have*

$$(2.4.36) \qquad P(|D_{hNi}| \leq \varepsilon) \geq 1 - \varepsilon, \qquad \textit{for } h = 1,2,3,6,7,8.$$

PROOF. Choose $\varepsilon > 0$, $0 < \gamma < \tau/2$, and the integer $k \geq 1$. The lemma is immediate from the remarks we shall make for the different cases corresponding to different values of h.

Case h = 6

We note that

$$P\Big(\{\omega : \sup_x |\mathbb{F}^*_{iN} - \bar{F}_{iN}| \geq \gamma/2\}\Big) \leq$$

$$\leq P\Big(\{\omega : \sup_x [\,|\mathbb{F}^*_{iN}(x) - \mathbb{F}_{iN}(x)| + |\mathbb{F}_{iN}(x) - \bar{F}_{iN}(x)|\,] \geq \gamma/2\}\Big) \leq$$

$$\leq P\Big(\{\omega : \sup_x |\mathbb{F}^*_{iN}(x) - \bar{F}_{iN}(x)| \geq \gamma/2 - (N+1)^{-1}\}\Big),$$

which converges to zero as N tends to infinity because of (2.3.8) with

$\delta = \frac{1}{2}$. Hence $P(\Omega^*_{\gamma N}) \to 1$ as $N \to \infty$ since $P(\Omega_0) = 1$.

Case h = 2

Because of Assumption 2.1.1 there exists a positive number M_γ, not depending on N, such that

$$(2.4.37) \qquad \sup_{S_{N\gamma}} \left| J^{(1)}_{ic}(\overline{F}_{iN})\, J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN}) \right| \le M_\gamma .$$

In view of Lemma 2.4.2 we obtain

$$\chi(\Omega_{N\delta})\, |D_{2Ni}| \le M\, \xi_N\, M_\gamma \to 0, \qquad \text{as } N \to \infty.$$

Case h = 1

Assumption 2.1.1 implies the existence of a positive number $\widetilde{M}_\gamma$, not depending on N, such that

$$(2.4.38) \qquad \sup_{T_{iN\gamma}} \left| J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN}) \right| \le \widetilde{M}_\gamma .$$

Lemma 2.4.2 and (2.4.38) imply that

$$(2.4.39) \qquad \chi(\Omega_{N\delta})\, |D_{1Ni}| \le M\widetilde{M}_\gamma \sup_{S_{iN\gamma}} \left| J^{(1)}_{ic}(\widetilde{\Phi}_{iN}) - J^{(1)}_{ic}(\overline{F}_{iN}) \right| .$$

For fixed $0 < \gamma < \tau/2$ we remark that the function $J^{(1)}_{ic}$ is uniformly continuous on $[\gamma/2, \tilde{s}_i - \gamma/2] \cup [\tilde{s}_i + \gamma/2, s_i - \gamma/2] \cup [s_i + \gamma/2, 1 - \gamma/2]$. Since $|\widetilde{\Phi}_{iN} - \overline{F}_{iN}| \le |\mathbb{F}^*_{iN} - \overline{F}_{iN}|$, assertion (2.3.9) with $\delta = \frac{1}{2}$ yields the convergence to zero in probability of the right-hand side of (2.4.39), as N tends to infinity.

Case h = 3

For positive integers m and N the r.v. $|D_{3Ni}|$ is bounded above by $\Sigma^3_{j=1} D_{3Nimj}$, where

$$D_{3Nim1} = \int_{S_{N\gamma}} \Bigg| U_{iN}(\overline{F}_{iN})\, J^{(1)}_{ic}(\overline{F}_{iN})\, J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\overline{F}_{jN}) -$$

$$- U_{iN}(I_m(\overline{F}_{iN}))\, J^{(1)}_{ic}(I_m(\overline{F}_{iN}))\, J_{ON}(I_m(\overline{F}_{ON})) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(I_m(\overline{F}_{jN})) \Bigg|\, d\mathfrak{C}_N ,$$

$$D_{3Nim2} = \left| \int_{S_{N\gamma}} U_{iN}(I_m(\overline{F}_{iN}))\, J_{ic}^{(1)}(I_m(\overline{F}_{iN}))\, J_{ON}(I_m(\overline{F}_{ON})) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(I_m(\overline{F}_{jN}))\, d(\mathbb{G}_N - \overline{G}_N) \right| ,$$

$$D_{3Nim3} = \int_{S_{N\gamma}} \Big| U_{iN}(I_m(\overline{F}_{iN}))\, J_{ic}^{(1)}(I_m(\overline{F}_{iN}))\, J_{ON}(I_m(\overline{F}_{ON})) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(I_m(\overline{F}_{jN})) -$$

$$- U_{iN}(\overline{F}_{iN})\, J_{ic}^{(1)}(\overline{F}_{iN})\, J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(\overline{F}_{jN}) \Big|\, d\overline{G}_N,$$

and the function I_m on $[0,1]$ is defined in (1.1.37).
It suffices to show that each of the r.v.'s above can be made arbitrarily small with arbitrarily high probability for some common positive integer m, provided N is large enough.

Consider D_{3Nim1} and D_{3Nim3}, which are both bounded by the supremum of the integrand over the closed set $S_{N\gamma}$. From Assumption 2.1.1 it is not hard to check that, for fixed $0 < \gamma < \tau/2$, there exist numbers $\xi_{m\gamma}$, independent of N and with $\xi_{m\gamma} \to 0$ as $m \to \infty$, such that

(2.4.40)

$$\sup_{S_{N\gamma}} \Big| J_{ic}^{(1)}(\overline{F}_{iN}) J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(\overline{F}_{jN}) - J_{ic}^{(1)}(I_m(\overline{F}_{iN})) J_{ON}(I_m(\overline{F}_{ON})) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(I_m(\overline{F}_{jN})) \Big| \le \xi_{m\gamma}.$$

Denoting $M_1 = M \sup_{0\le t\le 1} r^{-\frac{1}{2}+\delta}(t)$, we have from Lemma 2.4.2, (2.4.40) and (2.4.37) that on $\Omega_{N\delta}$,

$$\sup_{S_{N\gamma}} \Big| U_{iN}(\overline{F}_{iN})\, J_{ic}^{(1)}(\overline{F}_{iN})\, J_{ON}(\overline{F}_{ON}) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(\overline{F}_{jN}) -$$

$$- U_{iN}(I_m(\overline{F}_{iN}))\, J_{ic}^{(1)}(I_m(\overline{F}_{iN}))\, J_{ON}(I_m(\overline{F}_{ON})) \prod_{\substack{j=1\\j\neq i}}^{k} J_j(I_m(\overline{F}_{jN})) \Big| \le$$

$$\le \sup_{0\le t\le 1} |U_{iN}(t) - U_{iN}(I_m(t))|\, M_\gamma + M_1 \xi_{m\gamma}.$$

The desired result for the terms D_{3Nim1} and D_{3Nim3} follows after application of Lemma 2.3.6.

Next let us consider D_{3Nim2} for fixed positive integer m. For each ω, the integrand in the expression for this r.v. is a simple step function assuming the value $Z_{imN\ell}(\omega)$ (say) on the rectangle

$$\tilde{S}_{\gamma;\ell_0,\ell_1,\ldots,\ell_k} = \left(\prod_{j=0}^{k}\left\{x_j:\ \overline{F}_{jN}(x_j) \in \left[\frac{\ell_j-1}{m}, \frac{\ell_j}{m}\right)\right\}\right) \cap S_{N\gamma},$$

where $\ell_0,\ell_1,\ldots,\ell_k \in \{1,2,\ldots,m\}$.
Because $|Z_{imN\ell}| \le M(M_\gamma+\xi_{m\gamma})$ on $\Omega_{N\delta}$, with M_γ as in (2.4.37) and $\xi_{m\gamma}$ as in (2.4.40), we have

$$\chi(\Omega_{N\delta})D_{3Nim2} = \chi(\Omega_{N\delta})\left|\sum_{\ell_0,\ell_1,\ldots,\ell_k=1}^{m} Z_{imN\ell}\int_{\tilde{S}_{\gamma;\ell_0,\ell_1,\ldots,\ell_k}} d(\mathbb{G}_N-\overline{G}_N)\right| \le$$

$$\le M(M_\gamma+\xi_{m\gamma})\sum_{\ell_0,\ell_1,\ldots,\ell_k=1}^{m}\left|\mathbb{G}_N\left\{\tilde{S}_{\gamma;\ell_0,\ell_1,\ldots,\ell_k}\right\}-\overline{G}_N\left\{\tilde{S}_{\gamma;\ell_0,\ell_1,\ldots,\ell_k}\right\}\right| \le$$

$$\le 2^{k+1}m^{k+1}M(M_\gamma+\xi_{m\gamma})\ \sup|\mathbb{G}_N-\overline{G}_N| \underset{P}{\to} 0,$$

for fixed γ and m as $N\to\infty$ (Remark 1.2.1).

The asymptotic negligibility of D_{3Ni} follows by straightforward combination of these partial results.

Case h = 8

For every positive integer m and N we can make the decomposition $\chi(\Omega_{N\delta})D_{8Ni} = \Sigma_{j=1}^{3} D_{8Nimj}$, where (see Lemma 3.3.4 in RUYMGAART (1973))

$$D_{8Nim1} = \Lambda_i\chi(\Omega_{N\delta})N^{\frac{1}{2}}\ \mathrm{sgn}(\overline{F}_{iN}^{-1}(s_i)-X^{(i)}_{N_i:N}) \times$$

$$\times \int_{\prod_{h=0}^{i-1} S_{hN\gamma}\times\Gamma_{iN}\times\prod_{h=i+1}^{k} S_{hN\gamma}} J_{0N}(I_m(\overline{F}_{0N}))\prod_{\substack{j=1\\ j\ne i}}^{k} J_j(I_m(\overline{F}_{jN}))\,d(\mathbb{G}_N-\overline{G}_N),$$

$$D_{8Nim2} = \Lambda_i\chi(\Omega_{N\delta})N^{\frac{1}{2}}\int_{\tilde{S}_{iN\gamma}} [c(\mathbb{F}^*_{iN}-s_i) - c(\overline{F}_{iN}-s_i)] \times$$

$$\times\left[J_{0N}(\overline{F}_{0N})\prod_{\substack{j=1\\ j\ne i}}^{k} J_j(\overline{F}_{jN}) - J'_{0N}(I_m(\overline{F}_{0N}))\prod_{\substack{j=1\\ j\ne i}}^{k} J_j(I_m(\overline{F}_{jN}))\right] d\mathbb{G}_N,$$

$$D_{8Nim3} = \Lambda_i\chi(\Omega_{N\delta})N^{\frac{1}{2}}\int_{\tilde{S}_{iN\gamma}} [c(\mathbb{F}^*_{iN}-s_i) - c(\overline{F}_{iN}-s_i)] \times$$

$$\times \left[J_{0N}(I_m(\bar{F}_{0N})) \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(I_m(\bar{F}_{jN})) - J_{0N}(\bar{F}_{0N}) \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(\bar{F}_{jN}) \right] d\bar{G}_N .$$

Because of Assumption 2.1.1 again (compare with (2.4.40)), there exist for fixed $0 < \gamma < \tau/2$, positive numbers $\tilde{\xi}_{m\gamma}$, independent of N and with $\tilde{\xi}_{m\gamma} \to 0$ as $m \to \infty$, such that

$$\text{(2.4.41)} \qquad \sup_{T_{iN\gamma}} \left| J_{0N}(I_m(\bar{F}_{0N})) \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(I_m(\bar{F}_{jN})) - J_{0N}(\bar{F}_{0N}) \prod_{\substack{j=1 \\ j\neq i}}^{k} J_j(\bar{F}_{jN}) \right| \leq \tilde{\xi}_{m\gamma} .$$

Since the marginal d.f.'s of $\bar{G}_N$ are uniform, we obtain with the aid of Lemma 2.4.2 that

$$\text{(2.4.42)} \qquad \chi(\Omega_{N\delta}) \; |D_{8Nim3}| \leq M\chi(\Omega_{N\delta}) N^{\frac{1}{2}} \int \chi(0_{iN}) \tilde{\xi}_{m\gamma} \, d\bar{G}_N \leq M \, \tilde{\xi}_{m\gamma} ,$$

which converges to zero as m tends to infinity.

For the term D_{8Nim2} a similar argument applies.

With respect to the term D_{8Nim1} we remark that the function $J_{0N}(I_m(\bar{F}_{0N})) \prod_{j=1, j\neq i}^{k} J_j(I_m(\bar{F}_{jN}))$ assumes the value $\tilde{Z}_{imN\ell}(\omega)$ (say) on the set $\tilde{S}_{i\gamma\ell m1} \times \tilde{S}_{i\gamma\ell m2}$, where

$$\tilde{S}_{i\gamma\ell m1} = \prod_{j=0}^{i-1} \left\{ x_j : \bar{F}_{jN}(x_j) \in \left[\frac{\ell_j - 1}{m} , \frac{\ell_j}{m} \right) \right\} \cap \prod_{j=0}^{i-1} S_{jN\gamma} ,$$

and

$$\tilde{S}_{i\gamma\ell m2} = \prod_{j=i+1}^{k} \left\{ x_j : \bar{F}_{jN}(x_j) \in \left[\frac{\ell_j - 1}{m} , \frac{\ell_j}{m} \right) \right\} \cap \prod_{j=i+1}^{k} S_{jN\gamma} ,$$

for $\ell_0, \ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_k \in \{1,2,\ldots,m\}$. Note that from (2.4.38) and (2.4.41) it follows that $|\tilde{Z}_{imN\ell}| \leq \tilde{M}_\gamma + \tilde{\xi}_{m\gamma}$. Because $\Omega_{N\delta} \subset \Omega_{6N}$ we have that

$$\text{(2.2.43)} \qquad |D_{8Nim1}| \leq$$

$$\leq M\chi(\Omega_{N\delta}) N^{\frac{1}{2}} \left| \sum_{\ell_0,\ldots,\ell_{i-1},\ell_{i+1},\ldots,\ell_k=1}^{m} \tilde{Z}_{imN\ell} \int \chi\left(\tilde{S}_{i\gamma\ell m1} \times \Gamma_{iN} \times \tilde{S}_{i\gamma\ell m2} \right) d(G_N - \bar{G}_N) \right| \leq$$

$$\leq M\chi(\Omega_{N\delta}) N^{\frac{1}{2}} (\tilde{M}_\gamma + \tilde{\xi}_{m\gamma}) \sum_{\ell_0,\ldots,\ell_{i-1},\ell_{i+1},\ldots,\ell_k=1}^{m} \Big| G_N\left\{ \tilde{S}_{i\gamma\ell m1} \times \Gamma_{iN} \times \tilde{S}_{i\gamma\ell m2} \right\} -$$

$$- \bar{G}_N\left\{ \tilde{S}_{i\gamma\ell m1} \times \Gamma_{iN} \times \tilde{S}_{i\gamma\ell m2} \right\} \Big| \leq$$

$$\leq Mm^k\left(\tilde{M}_\gamma+\tilde{\xi}_{m\gamma}\right)N^{\frac{1}{2}}\ (\log(N+1))^{\frac{1}{2}}\ N^{-\frac{3}{4}} \to 0, \qquad \text{for fixed } m \text{ as } N \to \infty.$$

The asymptotic negligibility of D_{8Ni} follows again by straightforward combination of (2.4.42), the remark below (2.4.42) and (2.4.43).

Case h = 7

Using Lemma 3.3.4 of RUYMGAART (1973) with $u = s_i$ we can write $\chi(\Omega_{N\delta})D_{7Ni} = D_{7Ni1} + D_{7Ni2}$, where

$$D_{7Ni} = \Lambda_i N^{\frac{1}{2}} \int\limits_{\mathbb{R}^i\times\Gamma_{iN}\times\mathbb{R}^{k-i}} \operatorname{sgn}\left(\bar{F}_{iN}^{-1}(s_i)-X_{N_i:N}^{(i)}\right) J_{0N}(\bar{F}_{0N}) \prod_{\substack{j=1\\ j\neq i}}^{k} J_j(\bar{F}_{jN})\ d\bar{G}_N -$$

$$- \Lambda_i h_{iN}(s_i)\ U_{iN}(s_i),$$

$$D_{7Ni1} = \chi(\Omega_{N\delta})\ \left\{\Lambda_i N^{\frac{1}{2}}\left(\int\limits_{2s_i-\mathbb{F}_{iN}^*(\bar{F}_{iN}^{-1}(s_i))}^{s_i} h_{iN}(t_i)dt_i\right) - \Lambda_i h_{iN}(s_i)\ U_{iN}(s_i)\right\},$$

$$D_{7Ni2} = \chi(\Omega_{N\delta})\Lambda_i N^{\frac{1}{2}} \int\limits_{\bar{F}_{iN}(X_{N_i:N}^{(i)})}^{2s_i-\mathbb{F}_{iN}^*(\bar{F}_{iN}^{-1}(s_i))} h_{iN}(t_i)\ dt_i.$$

Let us first consider the r.v. D_{7Ni1}. Since $h_{iN}(t_i)$ is a continuous function of t_i on $\mathcal{Q}_{\eta,i}$ (Lemma 2.2.1) and since $2s_i - \mathbb{F}_{iN}^*(\bar{F}_{iN}^{-1}(s_i)) \in$ $\in [s_i - MN^{-\frac{1}{2}},\ s_i + MN^{-\frac{1}{2}}]$ on $\Omega_{N\delta}$, we can, for N sufficiently large, apply the mean value theorem for integrals. Writing $\Phi_{iN}(s_i)$ for the random point between s_i and $2s_i - \mathbb{F}_{iN}^*(\bar{F}_{iN}^{-1}(s_i))$, we obtain with the aid of Lemma 2.2.1,

$$|D_{7Ni1}| = \chi(\Omega_{N\delta})\Lambda_i\left|N^{\frac{1}{2}}(\mathbb{F}_{iN}^*(\bar{F}_{iN}^{-1}(s_i))-s_i)\ h_{iN}(\Phi_{iN}(s_i)) - U_{iN}(s_i)h_{iN}(s_i)\right| \leq$$

$$\leq \chi(\Omega_{N\delta})\left\{M\left|U_{iN}^*(s_i)\right|\left|h_{iN}(\Phi_{iN}(s_i))-h_{iN}(s_i)\right|+MN^{\frac{1}{2}}\left|\mathbb{F}_{iN}^*(\bar{F}_{iN}^{-1}(s_i))-\mathbb{F}_{iN}(\bar{F}_{iN}^{-1}(s_i))\right|\right\}\leq$$

$$\leq \chi(\Omega_{N\delta})\left\{M|h_{iN}(\Phi_{iN}(s_i))-h_{iN}(s_i)| + MN^{\frac{1}{2}}(N+1)^{-1}\right\}.$$

As $\Omega_{N\delta} \subset \Omega_{5N}$, we have that for each $\omega \in \Omega_{N\delta}$ the random point $\Phi_{iN}(s_i)$ satisfies $|\Phi_{iN}(s_i) - s_i| \leq MN^{-\frac{1}{2}}$, so that Lemma 2.2.1 implies that the upper bound for $|D_{7Ni1}|$ converges to zero as N tends to infinity.

The r.v. D_{7Ni2} is bounded above by

$$\chi(\Omega_{N\delta})MN^{\frac{1}{2}}\left|\mathbb{F}_{iN}(\bar{F}_{iN}^{-1}(s_i)) - \mathbb{F}_{iN}(X_{N_i:N}^{(i)}-)+\bar{F}_{iN}(X_{N_i:N}^{(i)}) - s_i\right| +$$

$$+ \chi(\Omega_{N\delta})MN^{\frac{1}{2}}\left|\mathbb{F}_{iN}^{*}(\bar{F}_{iN}^{-1}(s_i)) - \mathbb{F}_{iN}(\bar{F}_{iN}^{-1}(s_i))+\mathbb{F}_{iN}(X_{N_i:N}^{(i)}-) - s_i\right| \leq$$

$$\leq \chi(\Omega_{N\delta})MN^{\frac{1}{2}}\left|\mathbb{G}_N\left\{\mathbb{R}^i\times\Gamma_{iN}\times\mathbb{R}^{k-i}\right\} - \bar{G}_N\left\{\mathbb{R}^i\times\Gamma_{iN}\times\mathbb{R}^{k-i}\right\}\right| +$$

$$+ \chi(\Omega_{N\delta})MN^{\frac{1}{2}}\left[(N+1)^{-1} + |(N_i-1)N^{-1} - s_i|\right] .$$

On $\Omega_{N\delta}$ this upper bound converges to zero as N tends to infinity as a consequence of the definition of N_i in (2.4.7) and because $\Omega_{N\delta} \subset \Omega_{6N}$. □

Straightforward combination of Lemma 2.4.5 and Lemma 2.4.6 leads to the asymptotic negligibility of the term $N^{\frac{1}{2}}\int_{\Delta_N}(B_{iNc}+B_{iNd})d\mathbb{G}_N - (A_{iNc}+A_{iNd})$ for i = 1,2,...,k (see (2.4.30)-(2.4.33)) and hence of

$$N^{\frac{1}{2}} \int_{\Delta_N} B_N \, d\mathbb{G}_N - \left(\sum_{i=1}^{k} A_{iNc} + \sum_{i=1}^{k} A_{iNd}\right) .$$

This result, together with Lemma 2.4.4, yields the asymptotic negligibility of the term E_N (see (2.2.10) and (2.4.21)), which completes the proof of Theorem 2.1.1.

REMARK 2.4.1. Theorem 2.1.1 can be generalized in the sense that one can allow the scores generating functions J_i to depend also on N. However an equicontinuity condition on $J_{iN}^{(1)}$ is needed to ensure the validity of Lemma 2.4.6 in the cases h = 1, 3 and 8.

2.5. EXACT SCORES

Theorem 2.1.1 is an asymptotic result on rank statistics in the case where approximate scores (cf. (2.0.10)) are used. Clearly, a result like Theorem 2.1.1 also holds in the case where exact scores (cf. (2.0.9)) are used, provided condition (2.0.17) is satisfied. Assumption 2.5.1 is a strengthening of Assumption 2.1.1 which ensures that condition (2.0.17) holds.

ASSUMPTION 2.5.1 (generating functions):

(a) *For* $N = 1,2,\ldots$ *the function* J_{0N} *has discontinuities of the first kind only and a continuous derivative* $J_{0N}^{(1)}$ *on the set* $(0,1) - \mathcal{D}_{0N}$.
(b) *For* $i = 1,2,\ldots,k$ *the function* J_i *is continuous on* $(0,1)$ *and has a second derivative* $J_i^{(2)}$ *on the set* $(0,1) - \mathcal{D}_i^*$.
(c) *There exist positive numbers* $\ell_0,\ell_1,\ldots,\ell_k$ *and* τ *such that for* $N = 1,2,\ldots$ *and* $i = 1,2,\ldots,k$,

$$\mathcal{D}_{0N} \subset (\tau,1-\tau),\ \#\mathcal{D}_{0N} \le \ell_0 \text{ and } \mathcal{D}_i^* \subset (\tau,1-\tau),\ \#\mathcal{D}_i^* \le \ell_i .$$

(d) *There exist positive numbers* $a_0,a_1,\ldots,a_k$ *and* K_1, *satisfying* $a := \Sigma_{j=0}^k a_j < \frac{1}{2}$, *such that, with* r *defined in* (2.1.2), *we have*

$$(2.5.1) \qquad |J_{0N}^{(\nu)}| \le K_1\, r^{a_0+\nu} \qquad \text{for } \nu = 0,1,\ N = 1,2,\ldots,$$

$$|J_i^{(\nu)}| \le K_1\, r^{a_i+\nu} \qquad \text{for } \nu = 0,1,2,\ i = 1,2,\ldots,k,$$

wherever these functions are defined on $(0,1)$.

LEMMA 2.5.1. *Let for* $n = 1,2,\ldots,N$, $N = 1,2,\ldots$, $i = 1,2,\ldots,k$, *the exact scores* $a_{iN}^*(n)$ *and the approximate scores* $a_{iN}(n)$ *be defined as in* (2.0.9) *and* (2.0.10) *respectively. Suppose that Assumption* 2.5.1 *is satisfied. Then, with probability one,*

$$(2.5.2) \qquad N^{-\frac{1}{2}} \sum_{n=1}^{N} c_{nN} \left| \prod_{i=1}^{k} a_{iN}^*(R_{inN}) - \prod_{i=1}^{k} a_{iN}(R_{inN}) \right| \to 0 \quad \text{as } N \to \infty,$$

uniformly in the continuous underlying d.f.'s $F_{1N},F_{2N},\ldots,F_{NN}$, $N = 1,2,\ldots$.

PROOF. First we remark that

$$(2.5.3)\qquad \prod_{i=1}^{k} a^*_{iN}(R_{inN}) - \prod_{i=1}^{k} a_{iN}(R_{inN}) =$$

$$= \sum_{i=1}^{k} \left\{ \prod_{j=1}^{i-1} a_{jN}(R_{jnN}) \, [a^*_{iN}(R_{inN}) - a_{iN}(R_{inN})] \prod_{j=i+1}^{k} a^*_{jN}(R_{jnN}) \right\}.$$

With the aid of Assumption 2.5.1, (2.0.9), (2.0.10), (2.0.11) and the remarks in RUYMGAART (1973), page 87, we find for every $i \in \{1,2,\ldots,k\}$ that with probability one

$$(2.5.4)\qquad N^{-\frac{1}{2}} \sum_{n=1}^{N} c_{nN} \left| \prod_{j=1}^{i-1} a_{jN}(R_{jnN}) [a^*_{iN}(R_{inN}) - a_{iN}(R_{inN})] \prod_{j=i+1}^{k} a^*_{jN}(R_{jnN}) \right| \le$$

$$\le M\, N^{-\frac{1}{2}} \sum_{n=1}^{N} \left| J_{ON}\left(\frac{R_{0nN}}{N+1}\right) \right| \prod_{\substack{j=1\\ j\neq i}}^{k} r^{a_j}\left(\frac{R_{jnN}}{N+1}\right) \left| a^*_{iN}(R_{inN}) - a_{iN}(R_{inN}) \right| \le$$

$$\le M\, N^{-\frac{1}{2}} \sum_{n=1}^{N} \prod_{\substack{j=0\\ j\neq i}}^{k} r^{a_j}\left(\frac{R_{jnN}}{N+1}\right) \left| a^*_{iN}(R_{inN}) - a_{iN}(R_{inN}) \right| =$$

$$= M\, N^{-\frac{1}{2}} \sum_{n=1}^{N} \prod_{\substack{j=0\\ j\neq i}}^{k} r^{a_j}\left(\frac{Q_{jnN}}{N+1}\right) \left| a^*_{iN}(n) - a_{iN}(n) \right|,$$

where, for $j = 0,1,\ldots,k$, $j \neq i$, $(Q_{j1N}, Q_{j2N}, \ldots, Q_{jNN})$ is a random permutation of $(1,2,\ldots,N)$. From the derivation of (7.14) and from (7.25) in CHERNOFF and SAVAGE (1958) it is clear that

$$(2.5.5)\qquad |a^*_{iN}(1) - a_{iN}(1)| \le M\, N^{a_i},$$

and, for $1 < n \le N/2$, that

$$(2.5.6)\qquad |a^*_{iN}(n) - a_{iN}(n)| \le M\, N^{a_i} \left| N\left(\frac{-n^{\frac{1}{2}}}{M}\right) + \frac{1}{N} + \frac{1}{n^{1+a_i}} \right| + \left| J_i\left(\frac{n}{N}\right) - J_i\left(\frac{n}{N+1}\right) \right| ,$$

where the function N is defined in (2.1.1). Hence,

$$M\, N^{-\frac{1}{2}} \sum_{n=1}^{[N/2]} \prod_{\substack{j=0\\ j\neq i}}^{k} r^{a_j}\left(\frac{Q_{jnN}}{N+1}\right) \left| a^*_{iN}(n) - a_{iN}(n) \right| \le$$

$$\le M N^{-\frac12} \prod_{\substack{j=0\\ j\ne i}}^{k} r^{a_j}(\tfrac{N}{N+1}) N^{a_i} +$$

$$+ M N^{-\frac12} \sum_{n=2}^{[N/2]} \prod_{\substack{j=0\\ j\ne i}}^{k} r^{a_j}(\tfrac{N}{N+1}) N^{a_i} \left| N\!\left(\frac{-\sqrt{n}}{M}\right) + N^{-1} + n^{-1-a_i} \right| +$$

$$+ M N^{-\frac12} \sum_{n=2}^{[N/2]} \prod_{\substack{j=0\\ j\ne i}}^{k} r^{a_j}(\tfrac{Q_{jnN}}{N+1}) \left| J_i(\tfrac{n}{N}) - J_i(\tfrac{n}{N+1}) \right| .$$

It is obvious that the first two terms in this expression converge to zero as N tends to infinity. Application of the mean value theorem shows that the last term is bounded above by

$$M N^{-3/2} \sum_{n=2}^{[N/2]} \prod_{\substack{j=0\\ j\ne i}}^{k} r^{a_j}(\tfrac{Q_{jnN}}{N+1})\; r^{a_i+1}(\tfrac{n}{N+1}),$$

which, in view of Lemma 2.4.3, is bounded above by

$$M N^{-3/2} \sum_{n=1}^{N} r^{a+1}(\tfrac{n}{N+1}) \to 0 \qquad \text{as } N \to \infty.$$

By a symmetric argument we can cover the range $N/2 < n \le N$, so that (2.5.4) converges to zero as N tends to infinity. Combination of this with (2.5.3) completes the proof of (2.5.2). □

THEOREM 2.5.1. *Let an arbitrary triangular array of underlying d.f.'s* $F_{nN} \in \mathcal{F}_k$, $n = 1,2,\dots,N$, $N = 1,2,\dots$ *be given and let the generating functions satisfy Assumption* 2.5.1. *Then the quantities* μ_N *and* σ_N^2, *defined in* (2.1.10) *and* (2.1.11) *are finite. If, moreover,* $\liminf_{N\to\infty} \sigma_N^2 > 0$, *we have*

$$(2.5.7) \qquad \sup_{-\infty<z<\infty} \left| P(N^{\frac12}(T_N-\mu_N)/\sigma_N \le z) - N(z) \right| \to 0, \qquad \text{as } N \to \infty,$$

for T_N *as in* (2.0.3) *with* a_{iN} *replaced by* a_{iN}^* *defined in* (2.0.9), *i.e. the case of exact scores.*

PROOF. Immediate from Theorem 2.1.1, Lemma 2.5.1 and the equality

$$N^{\frac12}\sigma_N^{-1}\left(N^{-1}\sum_{n=1}^{k} c_{nN}\prod_{i=1}^{k} a_{iN}^*(R_{inN})-\mu_N\right) = N^{\frac12}\sigma_N^{-1}\left(N^{-1}\sum_{n=1}^{N} c_{nN}\prod_{i=1}^{k} a_{iN}(R_{inN})-\mu_N\right) +$$

$$+ N^{\frac{1}{2}}\sigma_N^{-1} \sum_{n=1}^{N} c_{nN} \left[\prod_{i=1}^{k} a^{*}_{iN}(R_{inN}) - \prod_{i=1}^{k} a_{iN}(R_{inN}) \right] ,$$

for N sufficiently large. □

2.6. SCORES GENERATING FUNCTIONS WHICH ARE CONTINUOUS, BUT NOT NECESSARILY OF PRODUCT TYPE

In this section we shall present a theorem extablishing asymptotic normality of suitably standardized statistics S_N (cf. (2.0.2)) of the type

$$S_N = N^{-1} \sum_{n=1}^{N} c_{nN}\, a_N(R_{1nN}, R_{2nN}, \ldots, R_{knN}). \tag{2.6.1}$$

Here, for $n_i = 1,2,\ldots,N$, $i = 1,2,\ldots,k$, the $a_N(n_1,n_2,\ldots,n_k)$ are given real numbers, called scores, and the c_{nN}, for $n = 1,2,\ldots,N$ are given real constants, called regression constants. Again, we shall suppose these regression constants c_{nN} to be generated by some function J_{0N} according to (2.0.11). However, in contrast to the foregoing sections we shall assume that the scores are generated by a function J on $(0,1)^k$, according to

$$a_N(n_1,n_2,\ldots,n_k) = J\left(\frac{n_1}{N+1}, \frac{n_2}{N+1}, \ldots, \frac{n_k}{N+1}\right), \qquad n_i = 1,2,\ldots,N, \quad i = 1,2,\ldots,k. \tag{2.6.2}$$

With the aid of the dummy r.v.'s $X_{01N}, X_{02N}, \ldots, X_{0NN}$, defined in and above (2.0.12), the statistic S_N can be entirely expressed in terms of empirical d.f.'s. Namely, in the notation of section 2.0, we obtain after combination of (2.6.1) with (2.6.2), (2.0.11), (2.0.1), (2.0.13) and (2.0.15), that

$$S_N = \int J_{0N}(\mathbb{F}^*_{0N})\, J(\mathbb{F}^*_{1N}, \mathbb{F}^*_{2N}, \ldots, \mathbb{F}^*_{kN})\, d\mathbb{G}_N, \tag{2.6.3}$$

where the integration is extended over the (k+1)-dimensional number space (cf. (2.0.16)).

To standardize the location of the statistics S_N we shall use the quantities

$$\mu_N = \mu_N(F_{1N}, F_{2N}, \ldots, F_{NN}) = \int J_{0N}(\bar{F}_{0N})\, J(\bar{F}_{1N}, \bar{F}_{2N}, \ldots, \bar{F}_{kN})\, d\bar{G}_N. \tag{2.6.4}$$

The quantities used to standardize the scale of S_N will be given in the implicit form

$$\sigma_N^2 = \sigma_N^2(F_{1N}, F_{2N}, \ldots, F_{NN}) = \mathcal{Var}\left(A_N + \sum_{i=1}^{k} A_{iN}\right), \tag{2.6.5}$$

where A_N and the A_{iN} arise in the fundamental decomposition of S_N in (2.6.13).

In this section we shall assume the scores generating function J to be continuous. By J^i we denote the partial derivative of J with respect to the i-th coordinate. The function r is defined in (2.1.2).

ASSUMPTION 2.6.1 (generating functions):
(a) *For* $N = 1,2,\dots$ *the function* J_{ON} *has discontinuities of the first kind only and a continuous derivative* $J_{ON}^{(1)}$ *on the set* $(0,1) - \mathcal{D}_{ON}$.
(b) *The function* J *is continuous on* $(0,1)^k$ *and has, for* $i = 1,2,\dots,k$, *a continuous partial derivative* J^i *on the set* $\Pi_{j=1}^k\{(0,1) - \mathcal{D}_i\}$.
(c) *There exist positive numbers* $\ell_0,\ell_1,\dots,\ell_k$ *and* τ *such that for* $N = 1,2,\dots$ *and* $i = 1,2,\dots,k$,

$$\mathcal{D}_{ON} \subset (\tau,1-\tau),\ \#\mathcal{D}_{ON} \le \ell_0 \text{ and } \mathcal{D}_i \subset (\tau,1-\tau),\ \#\mathcal{D}_i \le \ell_i.$$

(d) *There exist positive numbers* $a_0,a_1,\dots,a_k$ *and* K_1, *satisfying* $a := \Sigma_{j=0}^k a_j < \frac{1}{2}$, *such that on* $(0,1)$,

$$(2.6.6)\qquad |J_{ON}^{(\nu)}(t_0)| \le K_1[r(t_0)]^{a_0+\nu}, \qquad \nu = 0,1,\ N = 1,2,\dots,$$

and on $(0,1)^k$, *for* $i = 1,2,\dots,k$,

$$(2.6.7)\qquad |J^i(t_1,\dots,t_k)| \le K_1[r(t_i)]^{a_i+1} \prod_{\substack{j=1\\ j\neq i}}^k [r(t_j)]^{a_j},$$

and

$$(2.6.8)\qquad |J(t_1,\dots,t_k)| \le K_1 \prod_{j=1}^k [r(t_j)]^{a_j},$$

wherever these functions are defined.

THEOREM 2.6.1. *Let an arbitrary triangular array of underlying d.f.'s* $F_{nN} \in \mathcal{F}_k$, $n = 1,2,\dots,N$, $N = 1,2,\dots$ *be given and let the generating functions satisfy Assumption* 2.6.1. *Then the quantities* μ_N *and* σ_N^2, *defined in* (2.6.4) *and* (2.6.5) *are finite. If, moreover,* $\liminf_{N\to\infty} \sigma_N^2 > 0$, *we have with* S_N *defined in* (2.6.1) *that*

$$(2.6.9)\qquad \sup_{-\infty<z<\infty} |P(N^{\frac{1}{2}}(S_N-\mu_N)/\sigma_N \le z) - N(z)| \to 0, \quad \text{as } N \to \infty.$$

PROOF. Without loss of generality we may assume (compare with (2.2.8)) that $\ell_1 = \ell_2 = \ldots = \ell_k = 1$ in Assumption 2.6.1, so that $\mathcal{D}_i = \{s_i\}$, say. For small positive γ we define the sets

$$S_{iN\gamma} = \{x: \overline{F}_{iN}(x) \in [\gamma, s_i - \gamma] \cup [s_i + \gamma, 1 - \gamma]\}. \tag{2.6.10}$$

With $S_{0N\gamma}$ defined in (2.4.1), let

$$\begin{aligned}
\tilde{S}_{N\gamma} &= \prod_{j=1}^{k} S_{jN\gamma}, \\
S_{N\gamma} &= S_{0N\gamma} \times \tilde{S}_{N\gamma}, \\
\tilde{\Delta}_N &= \prod_{j=1}^{k} \left[X_{1:N}^{(i)}, X_{N:N}^{(i)}\right], \\
\Delta_N &= \left[X_{1:N}^{(0)}, X_{N:N}^{(0)}\right] \times \tilde{\Delta}_N,
\end{aligned} \tag{2.6.11}$$

and let $\Omega^*_{\gamma N}, U_{iN}, U^*_{iN}$ be defined as in (2.4.5) and (2.4.1).

For every $\omega \in \Omega^*_{\gamma N}$ the multivariate mean value theorem yields

$$\begin{aligned}
N^{\frac{1}{2}} J(\mathbb{F}^*_{1N}, \mathbb{F}^*_{2N}, \ldots, \mathbb{F}^*_{kN}) &= N^{\frac{1}{2}} J(\overline{F}_{1N}, \overline{F}_{2N}, \ldots, \overline{F}_{kN}) + \\
&+ \sum_{i=1}^{k} U^*_{iN}(\overline{F}_{iN}) \, J^i(\tilde{\Phi}_{1N}, \tilde{\Phi}_{2N}, \ldots, \tilde{\Phi}_{kN}),
\end{aligned} \tag{2.6.12}$$

for all $(x_1, x_2, \ldots, x_k) \in \tilde{\Delta}_N \cap \tilde{S}_{N\gamma}$, where for $i = 1, 2, \ldots, k$, the random number $\tilde{\Phi}_{iN}$ lies in the open interval with end points $\overline{F}_{iN}$ and $\mathbb{F}^*_{iN}$.

From (2.6.3) together with (2.6.4) it is immediate that with probability one

$$N^{\frac{1}{2}}(S_N - \mu_N) = A_N + \sum_{i=1}^{k} A_{iN} + B_N + C_N, \tag{2.6.13}$$

where

$$\begin{aligned}
A_N &= N^{\frac{1}{2}} \int J_{0N}(\overline{F}_{0N}) \, J(\overline{F}_{1N}, \overline{F}_{2N}, \ldots, \overline{F}_{kN}) \, d(\mathbb{G}_N - \overline{G}_N), \\
A_{iN} &= \int U_{iN}(\overline{F}_{iN}) \, J_{0N}(\overline{F}_{0N}) \, J^i(\overline{F}_{1N}, \overline{F}_{2N}, \ldots, \overline{F}_{kN}) \, d\overline{G}_N, \\
B_N &= N^{\frac{1}{2}} \int J_{0N}(\overline{F}_{0N}) \, [J(\mathbb{F}^*_{1N}, \ldots, \mathbb{F}^*_{kN}) - J(\overline{F}_{0N}, \ldots, \overline{F}_{kN})] \, d\mathbb{G}_N - \sum_{i=1}^{k} A_{iN},
\end{aligned}$$

$$C_N = N^{\frac{1}{2}} \int [J_{0N}(\mathbb{F}^*_{0N}) - J_{0N}(\overline{F}_{0N})]\; J(\mathbb{F}^*_{1N},\ldots,\mathbb{F}^*_{kN})\; d\mathbb{G}_N.$$

Moreover, (2.6.12) leads to the following decomposition of B_N,

$$(2.6.14) \qquad B_N = D_{1N} + D_{2N} + \sum_{i=1}^{k} \sum_{j=1}^{5} D_{ijN},$$

where,

$$D_{1N} = \chi(\Omega^{*c}_{\gamma N}) N^{\frac{1}{2}} \int J_{0N}(\overline{F}_{0N}) [J(\mathbb{F}^*_{1N},\ldots,\mathbb{F}^*_{kN}) - J(\overline{F}_{1N},\ldots,\overline{F}_{kN})]\, d\mathbb{G}_N,$$

$$D_{2N} = \chi(\Omega^{*}_{\gamma N}) N^{\frac{1}{2}} \int_{S^c_{N\gamma}} J_{0N}(\overline{F}_{0N}) [J(\mathbb{F}^*_{1N},\ldots,\mathbb{F}^*_{kN}) - J(\overline{F}_{1N},\ldots,\overline{F}_{kN})]\; d\mathbb{G}_N,$$

$$D_{i1N} = \chi(\Omega^{*}_{\gamma N}) \int_{S_{N\gamma}} U^*_{iN}(\overline{F}_{iN}) [J^i(\tilde{\Phi}_{1N},\ldots,\tilde{\Phi}_{kN}) - J^i(\overline{F}_{1N},\ldots,\overline{F}_{kN})]\, J_{0N}(\overline{F}_{0N})\, d\mathbb{G}_N,$$

$$D_{i2N} = \chi(\Omega^{*}_{\gamma N}) \int_{S_{N\gamma}} [U^*_{iN}(\overline{F}_{iN}) - U_{iN}(\overline{F}_{iN})]\, J^i(\overline{F}_{1N},\ldots,\overline{F}_{kN}) J_{0N}(\overline{F}_{0N})\, d\mathbb{G}_N,$$

$$D_{i3N} = \chi(\Omega^{*}_{\gamma N}) \int_{S_{N\gamma}} U_{iN}(\overline{F}_{iN}) J^i(\overline{F}_{1N},\ldots,\overline{F}_{kN}) J_{0N}(\overline{F}_{0N})\, d(\mathbb{G}_N - \overline{G}_N),$$

$$D_{i4N} = -\chi(\Omega^{*}_{\gamma N}) \int_{S^c_{N\gamma}} U_{iN}(\overline{F}_{iN}) J^i(\overline{F}_{1N},\ldots,\overline{F}_{kN}) J_{0N}(\overline{F}_{0N})\, d\overline{G}_N,$$

$$D_{i5N} = -\chi(\Omega^{*c}_{\gamma N}) \int U_{iN}(\overline{F}_{iN}) J^i(\overline{F}_{1N},\ldots,\overline{F}_{kN}) J_{0N}(\overline{F}_{0N})\, d\overline{G}_N.$$

First, let us look at the A-terms in (2.6.13). As in section 2.2 we shall establish the asymptotic normality of these A-terms, i.e. we shall show, with σ_N defined in (2.6.5), that

$$(2.6.15) \qquad \sup_{-\infty<z<\infty} \left| P\left((A_N + \sum_{i=1}^{k} A_{iN})/\sigma_N \le z \right) - N(z) \right| \to 0 \qquad \text{as } N \to \infty.$$

Again we begin by noting that with probability one,

$$(2.6.16) \qquad A_N + \sum_{i=1}^{k} A_{iN} = N^{-\frac{1}{2}} \sum_{n=1}^{N} Z_{nN},$$

where

$$(2.6.17)\qquad Z_{nN} = A_{nN} + \sum_{i=1}^{k} A_{inN},$$

with

$$(2.6.18)\qquad A_{nN} = J_{0N}(\bar{F}_{0N}(X_{0nN}))\ J(\bar{F}_{1N}(X_{1nN}),\dots,\bar{F}_{kN}(X_{knN})) - \mu_N,$$

$$(2.6.19)\qquad A_{inN} = \int [c(\bar{F}_{iN}-\bar{F}_{iN}(X_{inN})) - \bar{F}_{iN}]\ J_{0N}(\bar{F}_{0N})\ J^{i}(\bar{F}_{1N},\dots,\bar{F}_{kN})\ d\bar{G}_N,$$

and the function c defined in (2.2.9). The r.v. Z_{nN} depends on the random vector X_{nN} only, so that the r.v.'s $Z_{1N},Z_{2N},\dots,Z_{NN}$ are mutually independent.

Furthermore, in view of Assumption 2.6.1, one can show by following a similar reasoning as in the proof of (2.2.25) and (2.2.26) that there exists a $\delta > 0$, such that

$$(2.6.20)\qquad \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} E|A_{nN}|^{2+\delta} < \infty,$$

and for $i = 1,2,\dots,k$,

$$(2.6.21)\qquad \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} E|A_{inN}|^{2+\delta} < \infty,$$

Relations (2.6.20), (2.6.21) imply the existence of a $\delta > 0$ such that

$$(2.6.22)\qquad \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} E|Z_{nN}|^{2+\delta} < \infty.$$

Moreover, from the proof of (2.6.22) and FUBINI's theorem it follows that

$$(2.6.23)\qquad E \sum_{n=1}^{N} Z_{nN} = 0.$$

Asymptotic normality of the A-terms (2.6.15) follows by ESSEEN's theorem, using (2.6.22), (2.6.23) and the fact that the σ_N^2 are given to be bounded away from zero for N sufficiently large.

Our next aim is to show the asymptotic negligibility of the terms B_N and C_N in (2.6.13).

The asymptotic negligibility of the C_N-term is immediate from a reasoning similar to the proof of the asymptotic negligibility of the term $N^{\frac{1}{2}}\int_{\Delta_N} C_{0N}\, d\mathbb{G}_N$ on the pages 63 and 64 in Section 2.4, since Assumption 2.6.1 implies that

$$(2.6.24)\qquad J\left(\frac{R_{1nN}}{N+1}, \frac{R_{2nN}}{N+1}, \ldots, \frac{R_{knN}}{N+1}\right) \le K_1 \prod_{j=1}^{k} \left[r\left(\frac{R_{jnN}}{N+1}\right)\right]^{a_j}.$$

As far as the components of the B_N-term in (2.6.14) is concerned we begin by remarking that from Assumption 2.6.1 and Lemma 2.4.2 it follows that

$$(2.6.25)\qquad \chi(\Omega_{N\delta})\ |D_{i4N}| \le M \int_{S^c_{N\gamma}} r^{a_i+\frac{1}{2}+\delta}(\bar{F}_{iN}) \prod_{\substack{j=0\\ j\neq i}}^{k} r^{a_j}(\bar{F}_{jN})\, d\bar{G}_N.$$

Step-wise application of the mean value theorem (see (2.6.12)), together with Assumption 6.1.1 and Lemma 2.4.5 imply that

$$(2.6.26)\qquad \chi(\Omega_{N\delta})\ |D_{2N}| \le \sum_{i=1}^{k} M \int_{S^c_{N\gamma}} r^{a_i+\frac{1}{2}+\delta} \prod_{\substack{j=0\\ j\neq i}}^{k} r^{a_j}(\bar{F}_{jN})\, d\mathbb{G}_N.$$

In Lemma 2.4.5 it is shown that the upper bounds in (2.6.25) and (2.6.26) converge to zero as $\gamma \downarrow 0$, uniformly in N.

Moreover, for fixed γ sufficiently small, we have for every $i \in \{1,2,\ldots,k\}$ that the terms D_{1N}, D_{i1N}, D_{i2N}, D_{i3N} and D_{i5N} converge to zero in probability as N tends to infinity, because of Lemma 2.6.1 that follows. □

LEMMA 2.6.1. *Under the conditions of Theorem* 2.6.1 *there exists for every* $\varepsilon > 0$, *every* $0 < \gamma < \tau/2$ *and every positive integer* k *a positive integer* N_0, *depending on* ε, γ, k *and the constants in Assumption* 2.6.1, *such that for every* $N \ge N_0$ *and every* $i \in \{1,2,\ldots,k\}$, *we have*

$$(2.6.27)\qquad P(|D_{1N}| \le \varepsilon) \ge 1 - \varepsilon,$$

$$(2.6.28)\qquad P(|D_{ihN}| \le \varepsilon) \ge 1 - \varepsilon, \qquad \textit{for } h = 1,2,3,5.$$

PROOF. The proof in Case h = 6 of Lemma 2.4.6 implies (2.6.27) and (2.6.28) for h = 5. The relation (2.6.28) for h = 1,2,3 is immediate from the remarks we shall make for the different cases corresponding to different values of h.

Case h = 2

Practically the same reasoning as in the case h = 2 of Lemma 2.4.6 applies.

Case h = 1

Assumption 2.6.1 and Lemma 2.4.2 imply the existence of a positive number M_γ, not depending on N, such that with $\tilde{S}_{N\gamma}$ and $\tilde{\Delta}_N$ defined in (2.6.11),

$$\chi(\Omega_{N\delta})\ |D_{i1N}| \le M_\gamma \sup_{\tilde{S}_{N\gamma}\cap\tilde{\Delta}_N} |J^i(\tilde{\Phi}_{1N},\ldots,\tilde{\Phi}_{kN}) - J^i(\bar{F}_{1N},\bar{F}_{2N},\ldots,\bar{F}_{kN})|. \tag{2.6.29}$$

For fixed $0 < \gamma < \tau/2$, the function J^i is uniformly continuous on the closed set

$$\prod_{j=1}^{k} \{[\gamma/2,\ s_j-\tau/2] \cup [s_j+\gamma/2,\ 1-\gamma/2]\}.$$

Since, for $i = 1,2,\ldots,k$, $|\tilde{\Phi}_{iN}-\bar{F}_{iN}| \le |\mathbb{F}^*_{iN}-\bar{F}_{iN}|$, assertion (2.3.9) with $\delta = \frac{1}{2}$ yields the convergence to zero in probability of the right-hand side of (2.6.29), as N tends to infinity.

Case h = 3

The proof follows the lines of the proof of Case h = 3 of Lemma 2.4.6, replacing throughout $\Pi_{j=1,\, j\neq i}^{k} J_j(\bar{F}_{jN})\ J_{ic}^{(1)}(\bar{F}_{iN})$ by the function $J^i(\bar{F}_{1N},\ldots,\bar{F}_{kN})$.

□

2.7. SCORES GENERATING FUNCTIONS WHICH ARE NOT NECESSARILY CONTINUOUS OR OF PRODUCT TYPE

In this section we shall sketch how the results of the foregoing sections in this chapter can be combined to obtain a theorem in the case where the scores generating function J on $(0,1)^k$ may exhibit discontinuities on the hyperplanes $t_i = s_j^i$, for $j = 1,2,\dots,\ell_i$ and $i = 1,2,\dots,k$, where $0 < s_j^i < 1$. Throughout this section the points s_j^i, $j = 0,1,\dots,\ell_i + 1$, $i = 1,2,\dots,k$, are fixed elements of the unit interval, satisfying

$$(2.7.1) \qquad 0 \equiv s_0^i < s_1^i < \dots < s_{\ell_i}^i < s_{\ell_i+1}^i \equiv 1, \qquad \text{for } i = 1,2,\dots,k.$$

We write

$$(2.7.2) \qquad \mathcal{D}_i^{**} = \{s_1^i, s_2^i, \dots, s_{\ell_i}^i\}, \qquad i = 1,2,\dots,k.$$

We begin by formulating an assumption on the generating functions.

<u>ASSUMPTION 2.7.1</u> (generating functions):

(a) *For* $N = 1,2,\dots$ *the function* J_{ON} *has discontinuities of the first kind only and a continuous derivative* $J_{ON}^{(1)}$ *on the set* $(0,1) - \mathcal{D}_{ON}$.

(b) *There exist functions* $J_{h_1,h_2,\dots,h_k}$ *on* $(0,1)^k$ *such that the scores generating function* J *on* $(0,1)^k$ *is equal to* $J_{h_1,h_2,\dots,h_k}$ *on the set*

$$\prod_{i=1}^{k} \left\{ \left[s_{h_i-1}^i, s_{h_i}^i \right) \cap (0,1) \right\}, \qquad \textit{for } h_i = 1,2,\dots,\ell_i+1,\ i = 1,2,\dots,k.$$

Here $J_{h_1,h_2,\dots,h_k}$ *is defined and continuous on* $\prod_{i=1}^{k} \left\{ \left[s_{h_i-1}^i, s_{h_i}^i \right] \cap (0,1) \right\}$ *and possesses a continuous partial derivative*

$$J_{h_1,h_2,\dots,h_k}^i(t_1,t_2,\dots,t_k) = \frac{\partial J_{h_1,h_2,\dots,h_k}(t_1,\dots,t_k)}{\partial t_i}.$$

on

$$\prod_{j=1}^{i-1} \left\{ \left[s_{h_j-1}^i, s_{h_j}^i \right] \cap (0,1) \right\} \times \left(s_{h_i-1}^i, s_{h_i}^i \right) \times \prod_{j=i+1}^{k} \left\{ \left[s_{h_j-1}^j, s_{h_j}^j \right] \cap (0,1) \right\},$$

$$\textit{for } h_i = 1,2,\dots,\ell_i+1 \textit{ and } i = 1,2,\dots,k.$$

(c) *There exist positive numbers* ℓ_0 *and* τ *such that* $\mathcal{D}_{0N} \subset (\tau, 1-\tau)$ *and* $\#\mathcal{D}_{0N} \le \ell_0$ *for* $N = 1,2,\ldots$.

(d) *There exist positive numbers* $a_0, a_1, \ldots, a_k$ *and* K_1, *satisfying* $a := \sum_{j=0}^{k} a_j < \frac{1}{2}$, *such that, with* r *defined in* (2.1.2), *we have on* $(0,1)$,

$$(2.7.2) \qquad \left| J_{0N}^{(\nu)}(t_0) \right| \le K_1 [r(t_0)]^{a_0+\nu} \qquad \text{for } \nu = 0,1,\ N = 1,2,\ldots,$$

and on $(0,1)^k$, *for* $i = 1,2,\ldots,k$,

$$(2.7.3) \qquad |J(t_1,t_2,\ldots,t_k)| \le K_1 \prod_{j=1}^{k} [r(t_j)]^{a_j},$$

$$(2.7.4) \qquad |J^i(t_1,t_2,\ldots,t_k)| = \left| \frac{\partial J(t_1,t_2,\ldots,t_k)}{\partial t_i} \right| \le K_1 [r(t_i)]^{a_i+1} \prod_{\substack{j=1 \\ j \ne i}}^{k} [r(t_j)]^{a_j},$$

wherever these functions are defined.

<u>LEMMA 2.7.1</u>. *Suppose that the scores generating function* $J(t_1,\ldots,t_k)$ *satisfies Assumption* 2.7.1. *Then the function* $J(t_1,t_2,\ldots,t_k)$ *can be written as a finite sum of functions of the type*

$$(2.7.5) \qquad K_c\left(t_{i_1}, t_{i_2}, \ldots, t_{i_\alpha}\right) \prod_{j=\alpha+1}^{k} L_j(t_{i_j}),$$

where $(i_1, i_2, \ldots, i_k)$ *is a permutation of the numbers* $1,2,\ldots,k$, $\alpha \in \{0,1,2,\ldots,k\}$, *and* K_c *is not necessarily of product type, but continuous on* $(0,1)^\alpha$. *By convention* $K_c(t_{i_1},\ldots,t_{i_0}) = 1$ *and* $\prod_{j=k+1}^{k} L_j(t_{i_j}) = 1$.

Moreover, for some positive number K_2 *these functions have the following properties.*

(i) *For* $\nu = 1,2,\ldots,\alpha$, *the* ν^{th} *partial derivative* $\dfrac{\partial K_c(t_{i_1}, t_{i_2}, \ldots, t_{i_\alpha})}{\partial t_{i_\nu}} = K_c^{\nu}(t_{i_1}, t_{i_2}, \ldots, t_{i_\alpha})$ *exists and is continuous on* $\prod_{j=1}^{\alpha} \{(0,1) - \mathcal{D}_{i_j}^{**}\}$.

With $a_1,\ldots,a_k$ *as in Assumption* 2.7.1 *these functions satisfy on* $(0,1)^\alpha$,

$$(2.7.6) \qquad |K_c(t_{i_1},\ldots,t_{i_\alpha})| \le K_2 \prod_{j=1}^{\alpha} [r(t_{i_j})]^{a_{i_j}},$$

and for $\nu = 1,2,\ldots,\alpha$,

$$(2.7.7)\qquad \left|K_c^{\nu}(t_{i_1},\ldots,t_{i_\alpha})\right| \le K_2\left(\prod_{\substack{j=1\\ j\neq\nu}}^{\alpha} [r(t_{i_j})]^{a_{i_j}}\right)[r(t_{i_\nu})]^{a_{i_\nu}+1},$$

wherever these functions are defined.

(ii) *The functions* L_j, $j=\alpha+1,\ldots,k$, *are defined on* $(0,1)$ *and can be decomposed into* $L_j = L_{jc} + L_{jd}$. *Here*

$$L_{jd}(t) = \sum_{w=1}^{\ell_{i_j}} \Lambda_w\, c(t-s_w^{i_j}) \qquad \textit{for } t \in (0,1)$$

and numbers $\Lambda_1,\ldots,\Lambda_{i_j}$. *The function* $c(\cdot)$ *is defined in* (2.2.9). *Furthermore,* L_{jc} *is continuous on* $(0,1)$ *and has a continuous derivative* $L_{jc}^{(1)} = L_j^{(1)}$ *on* $(0,1) - \mathcal{D}_{i_j}^{**}$.

With $a_1,\ldots,a_k$ *as in Assumption* 2.7.1 *these functions satisfy*

$$(2.7.8)\qquad |L_j(t)| \le K_2[r(t)]^{a_{i_j}};\ \ |L_j^{(1)}(t)| \le K_2[r(t)]^{a_{i_j}+1},$$

wherever these functions are defined on $(0,1)$.

PROOF. It suffices to prove the representation in (2.7.5) for each of the components of J separately. Hence let us consider

$$\tilde{J} = \prod_{i=1}^{k} \chi((0,s_1^i);t_i)\; J_{1,1,\ldots,1}(t_1,t_2,\ldots,t_k).$$

We write $\tilde{J} = \tilde{J} + J^* - J^*$,
with

$$J^*(t_1,t_2,\ldots,t_k) =$$

$$= \sum_{\substack{A\subset\{1,2,\ldots,k\}\\ A\neq\emptyset}} \prod_{h\in A} \chi([s_1^h,1);t_h) \prod_{h\notin A} \chi((0,s_1^h);t_h) \times$$

$$\times J_{1,1,\ldots,1}(\tilde{t}_1,\tilde{t}_2,\ldots,\tilde{t}_k),$$

where

$$\tilde{t}_h = \begin{cases} t_h, & \text{for } h \notin A,\\ s_1^h, & \text{for } h \in A,\end{cases}$$

so that $J_{1,1,\ldots,1}$ is a function of $(k-j)$ variables.

Now $(\tilde{J}+J^*)$ is continuous on $(0,1)^k$ and J^* is a finite sum of functions of the type

$$\prod_{h\in A} L_h(t_h)J,$$

where J is a function of t_h, $h \notin A$, only. Since J satisfies the assumptions of the lemma we have reduced the dimension of the problem. Since the lemma is true for k = 2 (cf. RUYMGAART (1973), page 39) the decomposition holds for every k by induction. By straightforward verification one finds that the functions obtained enjoy the properties listed under (i) and (ii) respectively. □

According to Lemma 2.7.1 a scores generating function satisfying Assumption 2.7.1 can be expressed as a finite sum of products of scores generating functions of the types which are studied in the sections 2.1 and 2.6 respectively.

Consider the rank statistic $\tilde{S}_N$ corresponding to such a product, where

$$\text{(2.7.9)} \qquad \tilde{S}_N = \int J_{ON}(\mathbb{F}^*_{ON})\, K_c(\mathbb{F}^*_{1N},\ldots,\mathbb{F}^*_{\alpha N}) \prod_{j=\alpha+1}^{k} L_j(\mathbb{F}^*_{jN})\, d\mathbb{G}_N,$$

and let

$$\text{(2.7.10)} \qquad \mu_N = \int J_{ON}(\bar{F}_{ON})\, K_c(\bar{F}_{1N},\ldots,\bar{F}_{\alpha N}) \prod_{j=\alpha+1}^{k} L_j(\bar{F}_{jN})\, d\bar{G}_N.$$

From the basic decompositions in (2.2.10) and (2.6.13) it is not hard to arrive at the analogous decomposition of $N^{\frac{1}{2}}(\tilde{S}_N-\mu_N)$ in leading terms and remainder term. Again assuming only one discontinuity of L_j (say in s_j), we have

$$\text{(2.7.11)} \qquad N^{\frac{1}{2}}(\tilde{S}_N-\mu_N) = A_N + \sum_{i=1}^{\alpha} A_{iN} + \sum_{i=\alpha+1}^{k} A_{iNc} + \sum_{i=\alpha+1}^{k} A_{iNd} + \tilde{E}_N,$$

where

$$A_N = N^{\frac{1}{2}} \int J_{ON}(\bar{F}_{ON})\, K_c(\bar{F}_{1N},\ldots,\bar{F}_{\alpha N}) \prod_{j=\alpha+1}^{k} L_j(\bar{F}_{jN})\, d(\mathbb{G}_N-\bar{G}_N),$$

$$A_{iN} = U_{iN}(\bar{F}_{iN})\, J_{ON}(\bar{F}_{ON})\, K_c^{i}(\bar{F}_{1N},\ldots,\bar{F}_{\alpha N}) \prod_{j=\alpha+1}^{k} L_j(\bar{F}_{jN})\, d\bar{G}_N,$$

$$A_{iNc} = U_{iN}(\bar{F}_{iN})\, J_{ON}(\bar{F}_{ON})\, L_{iN}^{(1)}(\bar{F}_{iN}) \prod_{\substack{j=\alpha+1\\ j\neq i}}^{k} L_j(\bar{F}_{jN})\, K_c(\bar{F}_{1N},\ldots,\bar{F}_{\alpha N})\, d\bar{G}_N,$$

$$A_{iNd} = \Lambda_i \tilde{h}_{iN}(s_i)\ U_{iN}(s_i),$$

with Λ_i the height of the jump of L_i in s_i, U_{iN} defined in (2.4.1) and

$$\tilde{h}_{iN}(s_i) = E\left(J_{0N}(\bar{F}_{0N}(Y_{0N}))K_c(\bar{F}_{1N}(Y_{1N}),\ldots,\bar{F}_{\alpha N}(Y_{\alpha N})) \prod_{\substack{j=\alpha+1 \\ j\neq i}}^{k} L_j(\bar{F}_{jN}(Y_{jN})) \,\Big|\, \bar{F}_{iN}(Y_{iN})=s_i\right),$$

where $(Y_{0N}, Y_{1N}, \ldots, Y_{kN})$ has a joint d.f. $\bar{G}_N$.

Though we have not checked the details, it seems clear that with the aid of the technique of sections 2.2, 2.4 and 2.6, it is possible to show that $N^{\frac{1}{2}}(\tilde{S}_N - \mu_N)$ is asymptotically normal under Assumption 2.1.2 on the underlying d.f.'s. First one shows that

$$(2.7.12)\qquad \sup_{-\infty<z<\infty} \left| P\left(\left(A_N + \sum_{i=1}^{\alpha} A_{iN} + \sum_{i=\alpha+1}^{k} A_{iNc} + \sum_{i=\alpha+1}^{k} A_{iNd}\right)\Big/ \sigma_N \le z\right) - \mathcal{N}(z) \right| \to 0,$$
$$\text{as } N \to \infty$$

where

$$\sigma_N^2 = \mathit{Var}\left(A_N + \sum_{i=1}^{\alpha} A_{iN} + \sum_{i=\alpha+1}^{k} A_{iNc} + \sum_{i=\alpha+1}^{k} A_{iNd}\right),$$

provided $\liminf_{N\to\infty} \sigma_N^2 > 0$. The proof of the asymptotic negligibility of the remainder term $\tilde{E}_N$ can be given, as in the sections 2.4 and 2.6, with the aid of the properties of the empirical d.f. derived in Chapter I.

Finally we remark that if, following the approach from this chapter asymptotic normality can be established of each element of a finite set of standardized statistics, then the asymptotic normality of a suitable standardized version of any linear combination of these statistics will follow.

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