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INTRODUCTION TO RIESZ SPACES

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PREFACE

There exist four standard text-books on Riesz spaces, viz., the books written by Fremlin, Luxemburg-Zaanen, Schaefer and Vulikh. (They are the items [6],[10],[12] and [15] in our bibliography). In each one of them, several aspects of the theory are chosen and treated in considerable detail, the results being somewhat discouraging to the beginner. Fremlin and Schaefer deal mainly with specialised subjects (topological Riesz spaces and operator theory, respectively), while the other two books are encyclopedic in character, containing details and generalisations that are of interest to the expert but not to the amateur.

In the present work, which reflects a course given in 1975 at the Catholic University at Nijmegen, we address ourselves to the beginners. We want to give them an idea of what Riesz spaces are and how they occur "in nature".

Trying to steer a course between the Scylla of specialisation and the Charybdis of generality we have come to a compromise that seems to be very reasonable. The book consists of three parts. The first (Chapters I, II and III) is an introduction to the basic concepts of Riesz spaces, ideals, bands and conjugate spaces. Secondly, Chapter IV contains various representation theorems, entailing digressions on Riesz spaces of continuous functions. There are also some applications to Riesz spaces and to other fields of Functional Analysis. (The feed-back to Riesz space theory is sadly missing in [10] and [15]). In the third part (Chapter V) we consider normed Riesz spaces consisting of functions on a measure space, such as Orlicz spaces. (For other topological Riesz spaces, see [6],[15]).

In accordance with our intention of not writing an encyclopedia but only preparing the reader for further study we have kept our bibliography as short as possible. Our book essentially does not contain new results: practically everything in it can be found in one of the four books we have mentioned. There seemed to be no point in giving all the references. The interested reader will find everything he wants in the extensive bibliographies of [10] and [12].

INTRODUCTION

The reader is assumed to be familiar with the notions of a partially ordered set and a lattice. (We view lattices as special cases of ordered sets). Furthermore, we presuppose some knowledge of topology and functional analysis. To give an idea of the extent of this knowledge we present a short list of terms and theorems which we shall use without definition or proof.

From topology : Hausdorff space, compactness, the Urysohn Lemma.

From Banach space theory : the Hahn-Banach Theorem, Alaoglu's Theorem, inner product space, Schwarz' Inequality.

From integration theory : σ-finite measure, L_p-space (at least for $p=1,2,\infty$), Hölder's inequality.

We assume the Axiom of Choice.

Among the theorems we prove with the aid of Riesz space theory are the Riesz Representation Theorem, The Spectral Theorem for Hermitian operators in a real Hilbert space, the reflexivity of L_p-spaces ($1<p<\infty$).

In several sections blanket assumptions are made, such as "X is a compact Hausdorff space" or "(X,Γ,μ) is a σ-finite measure space". We have tried to indicate these assumptions very distinctly. With two exceptions (pages 7 and 89) each assumption is dropped at the end of the section in which it is made.

Within each section the definitions, theorems and corollaries are numbered consecutively. There are also a considerable number of examples and exercises (called A,B,C,...). Some of these are meant to illustrate the preceding theory, others are essential parts of the theory itself and will be referred to in the sequel.

A few standard notations : if A and B are subsets of a set X, we put

$A^c = \{x\in X : x\notin A\}$,

$A\setminus B = A\cap B^c$,

$A\Delta B = (A\setminus B)\cup(B\setminus A)$,

χ_A is the characteristic function of A.

Instead of χ_X (which is the constant function on X whose value is 1) we often write $\underline{1}$.

CHAPTER I. RIESZ SPACES

In this chapter we present a survey of several definitions. Furthermore, a variety of elementary properties of Riesz spaces will be derived.

1. PRELIMINARIES

Partial orderings will always be denoted as $\leq$ ($\geq$). Furthermore, $x<y$ will mean $x\leq y$ and $x\neq y$ ($>$ is defined similarly). If X is a lattice with respect to $\leq$, then the greatest lower bound (infimum) of the subset $\{x_1,x_2\}$ of X will be denoted by $\inf(x_1,x_2)$ or by $x_1\wedge x_2$, and the least upper bound of the subset $\{x_1,x_2\}$ of X by $\sup(x_1,x_2)$ or by $x_1\vee x_2$.

1.1. DEFINITION. A lattice X is said to be a *Boolean algebra* if

(i) X is a distributive lattice, i.e., $x\wedge(y_1\vee y_2)=(x\wedge y_1)\vee(x\wedge y_2)$ for all $x,y_1,y_2\in X$,

(ii) X has a infimum 0 and a supremum 1,

(iii) any $x\in X$ has a complement x' in X, i.e., for any $x\in X$ there exists an element $x'\in X$ such that $x\wedge x'=0$, $x\vee x'=1$.

1.A. *Exercise.* (i) Show that in a lattice X we have distributivity if and only if $x\vee(y_1\wedge y_2)=(x\vee y_1)\wedge(x\vee y_2)$ for all $x,y_1,y_2\in X$.

(ii) Show that in a Boolean algebra X every $x\in X$ has a unique complement x'.

(iii) Show that in a Boolean algebra X we have $(x\wedge y)'=x'\vee y'$ for all $x,y\in X$.

1.B. *Example.* Let S be a point set and let X be a nonempty collection of subsets of S such that $A,B\in X$ implies $A\cup B$, $A\cap B$, $A^c\in X$. If X is partially ordered by inclusion, then X is a Boolean algebra. (Thus $\mathcal{P}(S)$, the collection of all subsets of S, is a Boolean algebra).

Remark. It can be shown that any Boolean algebra can be represented in the above manner.

1.2. DEFINITION. (i) A real linear space L is called an *ordered vector space,* if L is partially ordered in such a manner that the partial ordering is compatible with the algebraic structure, i.e., if $f,g\in L$, then $f\leq g$ implies $f+h\leq g+h$ for all $h\in L$, and $f\geq 0$ implies $af\geq 0$ for all $a\in \mathbb{R}$, $a\geq 0$.

(ii) A real linear space L is called a *Riesz space* if L is partially ordered in such a manner that

(a) L is a lattice,

(b) L is an ordered vector space.

We present several examples of Riesz spaces. It is left to the reader to prove that all presented examples are indeed Riesz spaces. Also, some terminology and some notations will be standarised.

1.C. The Cartesian space $\mathbb{R}^n$, the partial ordering being coordinatewise, is a Riesz space.

1.D. The Cartesian space $\mathbb{R}^2$, partially ordered by $(x_1,x_2)\leq(y_1,y_2)$ if $x_1<y_1$ or if $x_1=y_1$ and $x_2\leq y_2$ is a Riesz space. This space is called the *lexicographically ordered plane.*

1.E. Let (X,Γ) be a measurable space, i.e., let X be a point set and Γ a σ-algebra of subsets of X. Define M to be the collection of all real-valued Γ-measurable functions on X and define $f\leq g$ for $f,g\in M$ if $f(x)\leq g(x)$ for all $x\in X$. Then M is a Riesz space.

1.F. Let (X,Γ,μ) be a measure space and let M be as above. The functions $f,g\in M$ are said to be equivalent $(f\sim g)$ whenever $f=g$ μ-almost everywhere on

X. Next, let M be the real linear space consisting of all equivalence classes of functions in $\mathcal{M}$ (with respect to $\sim$). Elements in M will be denoted by $[f], [g], \ldots$. Define $[f]\leq[g]$ whenever $f\leq g$ μ-almost everywhere on X. Then M is a Riesz space.

1.G. Let (X,τ) be a topological space and let $C(X)$ be the collection of all continuous real-valued functions on X. Define $f\leq g$ (in $C(X)$) whenever $f(x)\leq g(x)$ for every $x\in X$. Then $C(X)$ is a Riesz space.

1.H. Let (X,Γ) be a measurable space and let $\mathcal{B}$ be the collection of all finitely additive bounded measures on Γ, i.e., the collection of all real-valued functions μ on Γ such that

(i) $\mu(A_1\cup A_2)=\mu(A_1)+\mu(A_2)$ if $A_1\cap A_2=\emptyset$,

(ii) there exists a constant K such that $|\mu(A)|\leq K$ for all $A\in\Gamma$.

Define $\mu_1\leq\mu_2$ (in $\mathcal{B}$) whenever $\mu_1(A)\leq\mu_2(A)$ for all $A\in\Gamma$. Then $\mathcal{B}$ is a Riesz space. (Hint: to show that $\mathcal{B}$ is a lattice prove that

$$(\mu_1\vee\mu_2)(A)=\sup\{\mu_1(B)+\mu_2(A\setminus B):\ B\subset A,\ B\in\Gamma\}$$

for all $A\in\Gamma$).

Remark. We note that in examples 1.E, 1.F and 1.G the formula $f>0$ means $f\geq 0$ for all $x\in X$ and $f\neq 0$. However, there may exist many points $x\in X$ such that $f(x)=0$. Thus f does not have to be strictly positive.

The reader will now be convinced that Riesz spaces occur in a natural way in many places in analysis.

FROM NOW ON L *WILL ALWAYS DENOTE A RIESZ SPACE.*

When studying Riesz spaces, it will turn out that most of the important results can be presented in terms of elements greater than 0. Therefore, we introduce

1.3.DEFINITION. The collection of all $u \in L$ satisfying $u \geq 0$ is called the *positive cone* of L. Elements in the positive cone are called *positive elements*. Notation

$$L^{+}=\{u \in L: u \geq 0\}.$$

Furthermore, for arbitrary $f \in L$, we define

$$f^{+}=f \vee 0 \ , \quad f^{-}=(-f) \vee 0 \ , \quad |f|=(-f) \vee f \ .$$

In the sequel we shall often use without reference one of the following rules.

1.4. THEOREM. *Let* $f,g,h \in L$ *and* $u,v,w \in L^{+}$.

(*i*) $f=0$ *if and only if* $f,-f \in L^{+}$.

(*ii*) $(f+h) \vee (g+h)=(f \vee g)+h$; $(f+h) \wedge (g+h)=(f \wedge g)+h$.

(*iii*) $(af) \vee (ag)=a(f \vee g)$ *for all* $a \in \mathbb{R}^{+}$. *Similar rules hold for the infimum of* f *and* g *and for negative real* a.

(*iv*) $((f \vee g) \vee h)=(f \vee (g \vee h))$. (*Consequently, any finite subset of* L *has a supremum, and similarly also an infimum*).

(*v*) $f^{+},f^{-} \in L^{+}$; $f^{+} \wedge f^{-}=0$; $|f| \in L^{+}$; $f^{+} \vee f^{-}=|f|$.

(*vi*) $-f^{-} \leq f \leq f^{+}$; $|f|=0$ if and only if $f=0$.

(*vii*) $f \vee g+f \wedge g=f+g$; $f \vee g-f \wedge g=|f-g|$.

(*viii*) $2(|f| \vee |g|)=|f+g|+|f-g|$; $2(|f| \wedge |g|)=||f+g|-|f-g||$. (*Therefore,* $|f| \wedge |g|=0$ *if and only if* $|f+g|=|f-g|$).

(*ix*) $||f|-|g|| \leq |f+g| \leq |f|+|g|$.

(*x*) $(f+g)^{+} \leq f^{+}+g^{+}$; $(f+g)^{-} \leq f^{-}+g^{-}$.

(*xi*) $(u \wedge w)+(v \wedge w)-w \leq (u+v) \wedge w \leq (u \wedge w)+(v \wedge w) \leq (u \wedge v)+w$;

$u \vee v \leq (u \vee w)+(v \vee w)-w \leq (u+v) \vee w \leq (u \vee w)+(v \vee w)$, *if* $v\wedge w=0$, *then* $(u+v)\wedge w = u\wedge w$,

if $u\wedge v=0$, *then* $(u+v)\wedge w = (u\wedge w)+(v\wedge w)$.

We give some of the proofs, leaving the rest as exercise to the reader.

(ii) Note that $f+h\le f\vee g+h$ and that $g+h\le f\vee g+h$, so

$$(f+h)\vee(g+h)\le f\vee g+h.$$

To obtain the converse inequality, let $k\in L$ be such that $f+h\le k$ and $g+h\le k$. Then $f\le k-h$ and $g\le k-h$, so $f\vee g\le k-h$ or $k\ge f\vee g+h$ which is the desired result.

(vii) Since $f\vee g=(f-g)\vee 0+g=(f-g)^{+}+g$ and since

$$f\wedge g=f+0\wedge(g-f)=f-(f-g)^{+}=-(f-g)^{-}+g,$$

we have $f\vee g+f\wedge g=f+g$ and $f\vee g-f\wedge g=|f-g|$.

(viii) By (vii) we have $2(f\vee g)=f+g+|f-g|$, so

$$2(|f|\vee|g|)=2(f\vee(-f)\vee g\vee(-g))=(2(f\vee(-g)))\vee(2(g\vee(-f)))=$$
$$(f-g+|f+g|)\vee(g-f+|f+g|)=(f-g)\vee(g-f)+|f+g|=|f-g|+|f+g|.$$

Next, write $p=\frac{1}{2}(f+g)$ and $q=\frac{1}{2}(f-g)$, so $f=p+q$ and $g=p-q$. Using (vii) and the above formula we obtain

$$2(|f|\wedge|g|)=2|f|+2|g|-2(|f|\vee|g|)=2|f|+2|g|-|f+g|-|f-g|=$$
$$2|p+q|+2|p-q|-2|p|-2|q|=4(|p|\vee|q|)-2|p|-2|q|=$$
$$2(|p|+|q|+||p|-|q||)-2|p|-2|q|=2||p|-|q||=$$
$$||2p|-|2q||=||f+g|-|f-g||.$$

1.5. THEOREM. *Let* D *be a nonempty subset of* L *and assume that* $f_0=\sup D$ *exists. If* $g\in L$, *then* $f_0\wedge g=\sup\{f\wedge g\colon f\in D\}$. *Similarly, if* $f_1=\inf D$ *exists, then* $f_1\vee g=\inf\{f\vee g\colon f\in D\}$ *for all* $g\in L$.

Proof. Note that $f_0\ge f$ for all $f\in D$, so $f_0\wedge g\ge f\wedge g$ for all $f\in D$. Hence, $f_0\wedge g$ is an upper bound of $\{f\wedge g\colon f\in D\}$. Next, let $m\in L$ be any upper bound of $\{f\wedge g\colon f\in D\}$. Then $m \geq f\wedge g = f+g-(f\vee g) \geq f+g-(f_0\vee g)$ for all $f\in D$, so

$$m-g+(f_0\vee g) \geq f$$

for all $f\in D$, whence

$$m-g+(f_0\vee g) \geq f_0.$$

Equivalently, $m \geq f_0+g-(f_0\vee g) = f_0\wedge g$. Thus $f_0\wedge g$ is the least upper bound of $\{f\wedge g\colon f\in D\}$.

The second statement is proved similarly.

The following statement is now an immediate consequence.

1.6. COROLLARY. *Any Riesz space is a distributive lattice.*

1.7. THEOREM. (Riesz decomposition property). *Let* $u,z_1,z_2\in L^+$ *be such that* $u \leq z_1+z_2$. *Then there exist* $u_1,u_2\in L^+$ *satisfying* $u = u_1+u_2$, $0 \leq u_1 \leq z_1$ *and* $0 \leq u_2 \leq z_2$.

Proof. Define $u_1 = u\wedge z_1$ and $u_2 = u-u_1$. Obviously we have $0 \leq u_1 \leq z_1$ and $u = u_1+u_2$. Also, since $u_1 \leq u$ it is clear that $u_2\in L^+$. Finally,

$$u_2 = u-u_1 = u-(u\wedge z_1) = (u-z_1)\vee 0 \leq z_2\vee 0 = z_2.$$

It will turn out that the preceding theorem plays an important role in the investigation of the order dual of a Riesz space (chapter 2). To show that not only Riesz spaces have the Riesz decomposition property we present the following exercise.

1.I. *Exercise*. Let $C_1([0,1])$ denote the real vector space consisting of the real-valued continuously differentiable functions on the closed interval $[0,1]$. Set $f \leq g$ for $f,g\in C_1([0,1])$ if $f(x) \leq g(x)$ for all x.

(i) Show that $C_1([0,1])$ is an ordered vector space but not a Riesz space.

(ii) Show that $C_1([0,1])$ has the Riesz decomposition property. Hint: if $f, g_1, g_2 \in C_1([0,1])$ are such that $g_1, g_2 \geq 0$ and $0 \leq f \leq g_1+g_2$, then show that the functions f_i $(i=1,2)$ defined by

$$f_i(x) = \frac{g_i(x)f(x)}{g_1(x)+g_2(x)} \quad \text{if} \quad g_1(x)+g_2(x) \neq 0; \quad f_i(x) = 0 \quad \text{otherwise}$$

satisfy the stated conditions.

2. RIESZ SUBSPACES AND RIESZ HOMOMORPHISMS

In linear analysis an important role is played by linear subspaces of a linear space. It will be clear that in our investigations those linear subspaces of L are important which are Riesz spaces in their own right. The following definition distinguishes between three types of such subspaces.

2.1. DEFINITION. (i) A linear subspace V of L is called a *Riesz subspace* if $f,g \in V$ implies $f \vee g \in V$ (and hence also $f \wedge g \in V$ by 1.4(vii)).

(ii) A linear subspace A of L is called an *order ideal* (or *ideal*) if $f \in A$, $g \in L$ and $|g| \leq |f|$ implies $g \in A$.

(iii) An ideal B of L is called a *band* if it follows from $D \subset B$, $D \neq \emptyset$ and $f_0 = \sup D$ existing in L that $f_0 \in B$.

It is obvious that a band is an ideal and that an ideal is a Riesz subspace. However the converses of these statements do not hold as we shall see in example 2.B. First we observe that Riesz subspaces (and hence also ideals and bands) are Riesz spaces in their own right, so that it makes sense to talk about the positive cone of a Riesz subspace (ideal, band).

2.A. *Exercise*. (i) Let A be a Riesz subspace of L. Show that A is an ideal if and only if it follows from $f \in A^+$, $g \in L^+$ and $g \leq f$ that $g \in A^+$.

(ii) Let B be an ideal of L. Show that B is a band if and only if it follows from $D \subset B^+$, $D \neq \emptyset$ and $f_0 = \sup D$ existing in L^+ that $f_0 \in B^+$.

2.B. *Examples*. Let L be the Riesz space $C([0,1])$ (see 1.G).

(i) Let V be the linear subspace of L consisting of all constant functions on $[0,1]$. Then V is a Riesz subspace but not an ideal of L.

(ii) Let A be the linear subspace of L consisting of all functions f satisfying $f(\frac{1}{2}) = 0$. Then A is an ideal but not a band of L. To show that A is not a band observe that the functions f_n, defined by

$$f_n(x) = 1 \quad \text{for} \quad x \in [0, \tfrac{1}{2} - \tfrac{1}{n}] \cup [\tfrac{1}{2} + \tfrac{1}{n}, 1];$$

$$f_n(x) = -nx + \tfrac{1}{2}n \quad \text{for} \quad x \in [\tfrac{1}{2} - \tfrac{1}{n}, \tfrac{1}{2}];$$

$$f_n(x) = nx - \tfrac{1}{2}n \quad \text{for} \quad x \in [\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{n}] \quad (n+2,3,\ldots.)$$

are all members of A. Furthermore, if $f_0(x) = 1$ for all x, then

$$f_0 = \sup \{f_n : n=2,3,\ldots.\}$$

in the partial ordering of L and $f_0 \notin A$. Thus A is not a band.

(iii) Let B be the linear subspace of L consisting of all f satisfying $f \equiv 0$ on $[0,\frac{1}{2}]$. Then B is a band in L.

(iv) Let V be the linear subspace of L consisting of all polynomials on $[0,1]$. Then V is not even a Riesz subspace of L. However, V is an ordered linear space and V has the Riesz decomposition property (see 1.7).

(v) Let V be the linear subspace of L consisting of all polynomials on $[0,1]$ of degree less or equal than one. Then V is not a Riesz subspace of L. However, V considered in its own right is an ordered vector space, and, with this partial ordering V is even a Riesz space.

We leave the simple verifications of the above statements to the reader.

2.2. THEOREM. *(i)* $\{0\}$ *and* L *are bands in* L.

(ii) Intersections of Riesz subspaces (ideals, bands) are Riesz subspaces (ideals, bands) in L. *(Thus, it makes sense to talk about the Riesz subspace (ideal, band) generated by a subset of* L*).*

(iii) Let A_1 *and* A_2 *be ideals in* L. *Then the algebraic sum* A_1+A_2 *is an ideal in* L. *Moreover, if* $f\in(A_1+A_2)^+$, *then there exist* $f_1\in A_1^+$ *and* $f_2\in A_2^+$ *such that* $f = f_1+f_2$. *In particular, if* $A_1\cap A_2 = \{0\}$ *and if* $f\in(A_1+A_2)^+$, *then* $f = f_1+f_2$, $f_1\in A_1^+$, $f_2\in A_2^+$.

Proof. (i) and (ii) are obvious.

(iii) Note that A_1+A_2 is a linear subspace of L. Next, let $f\in A_1+A_2$, $g\in L$ be such that $|g| \leq |f|$. Then $f = f_1+f_2$, $f_1\in A_1$, $f_2\in A_2$. Hence, since

$$g^+ \leq |g| \leq |f| \leq |f_1|+|f_2|$$

it follows from theorem 1.7 that there exist g', g'' in L such that $g^+ = g'+g''$, $0 \leq g' \leq |f_1|$, $0 \leq g'' \leq |f_2|$. Now, as A_1 and A_2 are ideals it follows that $g'\in A_1$, $g''\in A_2$, so $g^+\in A_1+A_2$. Similarly it follows that $g^-\in A_1+A_2$. Thus A_1+A_2 is an ideal.

Finally, let $f\in(A_1+A_2)^+$ be given. Take $g = f$ in the above proof and it follows that $f = f_1+f_2$, $f_1\in A_1^+$, $f_2\in A_2^+$. The rest of the theorem is now obvious.

In the preceding theorem the reader will miss the statements that the algebraic sum of two Riesz subspaces is a Riesz subspace, and that the algebraic sum of two bands is a band. However, this is not an omission since these statements are false in general. It is left to the reader to find a counterexample for the first statement, and we present a counterexample for the second one. Consider $C([0,1])$ as in example 2.B. Define the bands B_1 and B_2 in $C([0,1]$ by

$$B_1 = \{f \in C([0,1]):\ f \equiv 0 \text{ on } [0,\tfrac{1}{2}]\}$$
$$B_2 = \{f \in C([0,1]):\ f \equiv 0 \text{ on } [\tfrac{1}{2},1]\}.$$

Then

$$B_1+B_2 = \{f \in C([0,1]):\ f(\tfrac{1}{2}) = 0\},$$

which is not a band (although it is an ideal) according to 2.B (ii).

In theorem 2.2 (ii) it was observed that it makes sense to talk about ideals and bands spanned by a subset of L. In the following exercises we give characterizations for two cases.

2.C. *Exercise*. Let $u \in L^+$. The smallest ideal in L containing u is denoted by (u) and is called the *principal ideal* spanned by u. Show that

$$(u) = \{f \in L:\ |f| \leq nu \text{ for some } n \in \mathbb{N}\}.$$

2.D. *Exercise*. Let A be an ideal of L and denote by $[A]$ the smallest band in L containing A. Show that

$$[A]^+ = \{f \in L^+:\ f = \sup D \text{ for some } D \subset A\} =$$
$$\{f \in L^+:\ f = \sup\{g:\ g \in A^+,\ g \leq f\}\}.$$

In the remaining part of this section we shall study linear maps from a Riesz space L into a Riesz space M, and in connection with this, quotients of a Riesz space L relative to an ideal A.

THEREFORE, LET FOR THE REST OF THIS SECTION M BE A RIESZ SPACE.

It will be clear that only those linear maps from L into M are interesting which preserve not only the addition and the scalar multiplication but also the order structure of L. To this end, we define the following

2.3. DEFINITION. (i) A linear map $\phi: L \to M$ is called *positive* (notation

$\phi \geq 0$), if $\phi(L^+) \subset M^+$.

(ii) A linear map ϕ: $L \to M$ is called a *Riesz homomorphism* if $f,g \in L^+$ and $f \wedge g = 0$ implies $\phi(f) \wedge \phi(g) = 0$.

2.4. THEOREM. *Let* ϕ: $L \to M$ *be a linear map. Then the following statements are equivalent.*

(a) ϕ *is a Riesz homomorphism.*

(b) $\phi(f \vee g) = \phi(f) \vee \phi(g)$ *for all* $f,g \in L$.

(c) $\phi(f \wedge g) = \phi(f) \wedge \phi(g)$ *for all* $f,g \in L$.

Proof. It is clear that (c) implies (a) and that (b) and (c) are equivalent (since $f+g = f \vee g + f \wedge g$). To show that (a) implies (c), let ϕ be a Riesz homomorphism and let $f,g \in L$ be given. Setting $h = f \wedge g$ we have

$$(f-h) \wedge (g-h) = 0,$$

so $(\phi(f)-\phi(h)) \wedge (\phi(g)-\phi(h)) = 0$. Equivalently, $\phi(f) \wedge \phi(g) = \phi(h) = \phi(f \wedge g)$.

2.5. THEOREM. *Let* ϕ: $L \to M$ *be a Riesz homomorphism and let* $f,g \in L$ *be given.*

(i) $f \geq g$ *implies* $\phi(f) \geq \phi(g)$, *so* $\phi \geq 0$.

(ii) $\phi(f^+) = (\phi(f))^+$; $\phi(f^-) = (\phi(f))^-$; $\phi(|f|) = |\phi(f)|$.

(iii) If $A_\phi = \{f \in L\colon\ \phi(f) = 0\}$, *then* A_ϕ *is an ideal in* L.

Proof. (i) Assume that $f \geq g$. Then $f = f \vee g$, so $\phi(f) = \phi(f) \vee \phi(g) \geq \phi(g)$.

(ii) $\phi(f^+) = \phi(f \vee 0) = \phi(f) \vee \phi(0) = \phi(f) \vee 0 = (\phi(f))^+$. The rest follows similarly.

(iii) It is clear that A_ϕ is a linear subspace of L. Furthermore, $\phi(|f|) = |\phi(f)|$ for all $f \in L$, so $f \in A_\phi$ if and only if $|f| \in A_\phi$. Hence, if $f \in A_\phi$, $g \in L$ and if $|g| \leq |f|$, then $0 \leq \phi(|g|) \leq \phi(|f|) = 0$, so $|g| \in A_\phi$.

Thus $g \in A_\phi$ so A_ϕ is an ideal.

From the preceding theorems we infer that Riesz homomorphisms preserve all relevant structures of L. Also, Riesz homomorphisms turn out to be positive. The reader will not find it hard to prove that a positive linear map from L into M is not necessarily a Riesz homomorphism (see also 2.E).

2.6. DEFINITION. (i) A bijective Riesz homomorphism is called a *Riesz isomorphism.*

(ii) The Riesz spaces L and M are called *Riesz isomorphic* (notation $L \cong M$) if there exists a Riesz isomorphism $\phi: L \to M$.

It is obvious that if $\phi: L \to M$ is a Riesz isomorphism, then $\phi^{-1}: M \to L$ is a Riesz isomorphism as well.

2.7. THEOREM. *Let* $\phi: L \to M$ *be a positive linear bijective map. Then* ϕ *is a Riesz isomorphism if and only if* ϕ^{-1} *is positive.*

Proof. (i) If ϕ is a Riesz isomorphism then ϕ^{-1} is a Riesz isomorphism, so $\phi^{-1} \geq 0$.

(ii) Assume that $\phi^{-1} \geq 0$. Let $f,g \in L$ be given. Since $f \vee g \geq f$ and $f \vee g \geq g$ it follows that $\phi(f \vee g) \geq \phi(f)$ and $\phi(f \vee g) \geq \phi(g)$. Hence

$$\phi(f \vee g) \geq \phi(f) \vee \phi(g).$$

Using $\phi^{-1} \geq 0$ and using $\phi(f) \vee \phi(g) \geq \phi(f)$, $\phi(f) \vee \phi(g) \geq \phi(g)$, we obtain

$$\phi^{-1}(\phi(f) \vee \phi(g)) \geq f \vee g,$$

so $\phi(f) \vee \phi(g) \geq \phi(f \vee g)$ which shows that ϕ is a Riesz homomorphism and hence a Riesz isomorphism.

2.E. *Exercise.* Let L be the coordinatewise ordered plane and let M be the lexicographically ordered plane (1.C and 1.D). Show that there exists a positive bijective linear map from L onto M but that L and M are not Riesz isomorphic.

Next, we consider factor spaces. To this end,
FOR THE REMAINING PART OF THIS SECTION LET A *BE AN IDEAL OF* L.
Consider the factor space L/A. As usual elements of L/A will be denoted by [f], where $f \in L$. Also, we shall frequently use the notation $g \in [f]$ meaning $g \in L$ and $[g] = [f]$ in L/A. We now give L/A the structure of a Riesz space.

2.8. DEFINITION. Let $[f],[g] \in L/A$. Then we write $[f] \le [g]$ whenever there exist elements $f_1 \in [f]$ and $g_1 \in [g]$ such that $f_1 \le g_1$.

2.F. *Exercise.* Show that for $[f],[g] \in L/A$ the following statements are equivalent.

(a) $[f] \le [g]$.

(b) For all $f_1 \in [f]$ there exists a $g_1 \in [g]$ satisfying $f_1 \le g_1$.

(c) For all $f_1 \in [f]$ and for all $g_1 \in [g]$ there exists a $q \in A$ satisfying $g_1 - f_1 \ge q$.

We shall show now that $\le$ is a partial ordering on L/A. It is clear that $[f] \le [f]$ for all $[f] \in L/A$. To derive transitivity, let $[f],[g],[h] \in L/A$ be such that $[f] \le [g]$ and $[g] \le [h]$. By 2.F there exist $q_1, q_2 \in A$ such that $g-f \ge q_1$, $h-g \ge q_2$. Hence $h-f \ge q_1+q_2$, so $[f] \le [h]$. Finally, let $[f],[g] \in L/A$ be such that $[f] \le [g]$ and $[g] \le [f]$. Again by 2.F

$$g-f \ge q_1 \ , \quad f-g \ge q_2$$

for certain $q_1, q_2 \in A$. This is equivalent to $f-g \leq -q_1$, $g-f \leq -q_2$. Letting $q = (-q_1)\vee(-q_2)$, we have $q \in A$ and $|f-g| \leq q$, so $f-g \in A$. Thus $[f] = [g]$. Thus $\leq$ is symmetric on L/A and therefore L/A is partially ordered by $\leq$.

2.9. THEOREM. L/A *endowed with the partial ordering* $\leq$ *is a Riesz space. In particular,*

$$[f]\vee[g] = [f\vee g] \quad \textit{and} \quad [f]\wedge[g] = [f\wedge g]$$

for all $f,g \in L$.

Proof. It is easy to show that L/A is an ordered vector space with respect to $\leq$, so that part of the proof is left to the reader. It remains to show that L/A is a lattice. To this end, let $[f],[g] \in L/A$ be given. Then obviously

$$[f\vee g] \geq [f] \quad \text{and} \quad [f\vee g] \geq [g]$$

so $[f\vee g]$ is an upper bound of $\{[f],[g]\}$. Let $[h] \in L/A$ be any upper bound of $\{[f],[g]\}$. Then there exist $q_1, q_2 \in A$ such that $h-f \geq q_1$ and $h-g \geq q_2$. Setting $q = q_1 \wedge q_2$ it follows that $q \in A$ and that

$$h-f \geq q, \quad h-g \geq q.$$

Hence $h \geq (f+q)\vee(g+q) = (f\vee g)+q$. Equivalently $h-(f\vee g) \geq q$. Hence, by 2.F, $[h] \geq [f\vee g]$ which implies $[f\vee g] = [f]\vee[g]$.
Using now $f+g = (f\vee g)+(f\wedge g)$ it is clear that $[f\wedge g] = [f]\wedge[g]$ holds for all $f,g \in L$, which completes the proof.

From the previous results it is now clear that the operator $\phi: L \to L/A$ defined by $\phi(f) = [f]$ for all $f \in L$ is a Riesz homomorphism with kernel A. Thus analogously to the theory of rings, we have shown that any ideal of L is the kernel of a Riesz homomorphism on L, and conversely (theorem 2.5

(iii)). It is clear as well, that if $\phi: L \to M$ is a Riesz homomorphism with kernel $A_\phi \subset L$, then $L/A_\phi \cong \phi(L)$ (note that $\phi(L)$ is a Riesz subspace of M).

2.G. *Example*. Let (X,Γ,μ) be a measure space. Consider the Riesz space $\mathcal{M}$ defined in example 1.E. Next, define

$$\mathcal{N} = \{f \in \mathcal{M}: \ f=0 \ \ \mu\text{-almost everywhere on } X\}.$$

It is clear that $\mathcal{N}$ is an ideal in $\mathcal{M}$. Thus $\mathcal{M}/\mathcal{N}$ is a Riesz space in view of theorem 2.9. If M is the Riesz space defined in example 1.F, it is easy to see that $M \cong \mathcal{M}/\mathcal{N}$.

Finally we present a useful extension lemma for linear operators.

2.10. LEMMA. *Let* $\phi: L^+ \to M$ *be such that* $\phi(u+v) = \phi(u)+\phi(v)$ *for all* $u,v \in L^+$ *and* $\phi(au) = a\phi(u)$ *for all* $a \in \mathbb{R}^+$ *and all* $u \in L^+$. *Then* ϕ *has a unique linear extension* Φ *to the whole of* L. *Moreover, if* $\phi(u) \geq 0$ *for all* $u \in L^+$, *then* $\Phi \geq 0$.

Proof. For any $f \in L$, define $\Phi(f) = \phi(f^+)-\phi(f^-)$. Then $\Phi = \phi$ on L^+ and Φ is linear. It remains to show that Φ is unique. To this end, assume that $\Phi_1: L \to M$ is a linear operator such that $\Phi_1 = \phi$ on L^+, and let $f \in L$ be given. Then

$$\Phi(f) = \phi(f^+)-\phi(f^-) = \Phi_1(f^+)-\Phi_1(f^-) = \Phi_1(f^+-f^-) = \Phi_1(f).$$

Thus Φ is unique. The rest is obvious.

3. DISJOINTNESS

Beside the concept of taking suprema and infima in L, there is another important concept. In fact,

3.1. DEFINITION. Let $f,g\in L$ be given. Then f and g are said to be *disjoint* (notation $f\perp g$) whenever $|f|\wedge|g| = 0$. Furthermore, if D is any subset of L, then the *disjoint complement* D^d of D is defined by

$$D^d = \{f\in L:\ f\perp d \text{ for all } d\in D\}.$$

Two subsets D_1 and D_2 of L are said to be *disjoint* ($D_1\perp D_2$) whenever $d_1\perp d_2$ for all $d_1\in D_1$ and for all $d_2\in D_2$.

3.2. LEMMA. (*i*) *If* $f\perp g$ *and* $|h| \le |f|$, *then* $h\perp g$.

(*ii*) $f\perp g$, $a\in\mathbb{R}$ *implies* $af\perp g$.

(*iii*) $f_1\perp g$, $f_2\perp g$ *implies* $(f_1+f_2)\perp g$.

(*iv*) $f\perp g$ *if and only if* $f^+\perp g$ *and* $f^-\perp g$.

(*v*) $f^+\perp f^-$.

(*vi*) *If* $D_1,D_2\subset L$ *satisfy* $D_1\perp D_2$, *then* $D_1\cap D_2\subset\{0\}$. *In particular* $f\perp f$ *implies* $f = 0$.

(*vii*) *Let* D *be a subset of* L *such that* $f_0 = \sup D$ *exists in* L. *Then* $f\perp g$ *for all* $f\in D$ *implies* $f_0\perp g$.

Proof. (i) $0 \le |h|\wedge|g| \le |f|\wedge|g| = 0$.

(ii) Let $u,v\in L^+$ and $b > 0$ in $\mathbb{R}$ be given. Assume that $u\perp v$. Then $bu\perp bv$. Furthermore, note that $f\perp g$ if and only if $|f|\perp|g|$. Hence

$$(|a|+1)|f|\perp(|a|+1)|g|.$$

From $|af| = |a||f| \le (|a|+1)|f|$ it follows that $af\perp(|a|+1)|g|$, and from $|g| \le (|a|+1)|g|$ it follows now that $af\perp g$.

(iii) $0 \le |f_1+f_2|\wedge|g| \le (|f_1|+|f_2|)\wedge|g| \le |f_1|\wedge|g|+|f_2|\wedge|g| = 0$.

(iv) $|f^+| = f^+ \le |f|$ and $|f^-| = f^- \le |f|$, so $f\perp g$ implies $f^+\perp g$ and $f^-\perp g$. Conversely, if $f^+\perp g$ and $f^-\perp g$, then $(f^++f^-)\perp g$, so $|f|\perp g$, or $f\perp g$.

(v) Obvious.

(vi) Assume that $D_1 \cap D_2 \neq \emptyset$. Let $d \in D_1 \cap D_2$. Then $d \perp d$, so

$$|d| = |d| \wedge |d| = 0.$$

(vii) We have $f^+ \perp g$ and $f^- \perp g$ for all $f \in D$. By theorem 1.5

$$f_0^+ = \sup \{f^+ : \ f \in D\}; \quad f_0^- = \inf\{f^- : \ f \in D\},$$

so $f_0^+ \wedge |g| = \sup \{f^+ \wedge |g| : \ f \in D\} = 0$ and $0 \leq f_0^- \wedge |g| \leq f^- \wedge |g| = 0$ for all $f \in D$. Thus $f_0^+ \perp g$ and $f_0^- \perp g$ and hence $f_0 \perp g$.

3.3. THEOREM. *Let* D *be a subset of* L. *Then*

(*i*) D^d *is a band,*

(*ii*) $D \subset D^{dd}$ $(=(D^d)^d)$,

(*iii*) $D^d = D^{ddd}$,

(*iv*) $D^d \cap D^{dd} = \{0\}$.

Proof. (i) Immediate from lemma 3.2.

(ii) Obvious.

(iii) From (ii) it follows that $D^d \subset D^{ddd}$. Furthermore, if $D_1 \subset D_2$ (in L) then $D_1^d \supset D_2^d$. Hence, again by (ii), we have $D^d \supset D^{ddd}$.

(iv) It is clear that $D^d \perp D^{dd}$ and that $0 \in D^d \cap D^{dd}$ (by (i)). Thus

$$D^d \cap D^{dd} = \{0\}$$

by lemma 3.2 (vi).

3.4. LEMMA. *Let* A *and* B *be ideals in* L. *Then* $A \perp B$ *if and only if* $A \cap B = \{0\}$.

Proof. (i) If $A \perp B$, then $A \cap B = \{0\}$ by lemma 3.2 (vi) and the fact that $0 \in A \cap B$.

(ii) Assume that $A \cap B = \{0\}$. Let $f_1 \in A$ and $f_2 \in B$ be given. Then

$$|f_1|\wedge|f_2| \in A\cap B = \{0\}.$$

Hence $f_1 \perp f_2$, so $A \perp B$.

We note that if A and B are Riesz subspaces of L such that $A\cap B = \{0\}$, then we do not necessarily have $A\perp B$. The reader will not find it hard to give counterexamples.

3.A. *Example.* Let (X,Γ,μ) be a measure space and let M be the Riesz space defined in example 1.F, with elements $[f],[g],\ldots,$ where $f,g,\ldots$ are elements of $\mathcal{M}$ (see 1.E and 2.G). Let $f,g\in\mathcal{M}$ be such that $[f]\perp[g]$ in M. Then

$$[|f|\wedge|g|] = |[f]|\wedge|[g]| = 0,$$

so $|f|\wedge|g|\in\mathcal{N}$ (see 2.G). This shows that $f(x) = 0$ for μ-almost every x in X for which $g(x) \neq 0$ and conversely. Thus, if $A\in\Gamma$ is any measurable set and if we define

$$B_A = \{[f]\in M:\ f = 0\ \ \mu\text{-almost everywhere on } A^c\},$$

then B_A^d is a band in M and moreover,

$$B_A^d = \{[f]\in M:\ f = 0\ \mu\text{-almost everywhere on } A\} = B_{A^c}.$$

Hence, since $B_A = B_{A^c}^d$ it follows that B_A is also a band in M. In section 4 we shall see that, for σ-finite measure μ, we can obtain every band in M in the above described way (example 4.I).

4. ARCHIMEDEAN AND DEDEKIND COMPLETE RIESZ SPACES

Although it follows from the previous sections that a Riesz space L has an extremely rich structure, far more can be proved if we require more about the partial ordering in L. Therefore, we introduce

4.1. DEFINITION. A Riesz space L is called *Archimedean* if for all $u \in L^+$

$$\inf\{n^{-1}u:\ n=1,2,\ldots\} = 0.$$

4.2. DEFINITION. A Riesz space L is called *Dedekind complete* if every non-empty subset of L that is bounded from above has a supremum.

Before presenting examples, we derive some properties of Archimedean Riesz spaces and of Dedekind complete Riesz spaces.

4.A. *Exercise*. Show that the following statements are equivalent.

(a) L is Dedekind complete.

(b) Every nonempty subset of L that is bounded from below has an infimum.

(c) Every nonempty subset of L^+ that is bounded from above has a supremum.

(d) Every nonempty subset D of L^+ that is bounded from above and for which $f_1, f_2 \in D$ implies $f_1 \vee f_2 \in D$, has a supremum.

Next, as before, let L be a Riesz space. Let A be an ideal in L. Then A^{dd} is a band in L and obviously $A \subset A^{dd}$. Thus it is clear that $A^{dd} = \{0\}$ if and only if $A = \{0\}$. Now, assume that $A \neq \{0\}$, so $A^{dd} \neq \{0\}$ and let $u \in A^{dd}$ be given such that $u > 0$. Then there exists an element v in A such that $0 < v < u$ (so $0 \leq v \leq u$ and $v \neq 0$, $v \neq u$). Indeed, note that $u \notin A^d$, so there exists an element $w \in A$ such that $u \wedge |w| > 0$. Take $v = \frac{1}{2}(u \wedge |w|)$. Then $v \in A$ since A is an ideal and $0 < v = \frac{1}{2}(u \wedge |w|) \leq \frac{1}{2}u < u$. We are now ready to prove the following important theorem.

4.3. THEOREM. *The following statements are equivalent.*

(a) L *is Archimedean.*

(b) $u, v \in L^+$ *and* $0 \le nv \le u$ *for* $n=1,2,\ldots$ *implies* $v = 0$.

(c) If $v \in L^+$, $v \ne 0$, *then* $\{nv: n=1,2,\ldots\}$ *is not bounded from above.*

(d) $A = A^{dd}$ *for every band* A *in* L.

Proof. (i) It is routine to show that (a), (b) and (c) are equivalent.

(ii) To derive (b) $\Rightarrow$ (d), assume that L is Archimedean and let A be a band in L. We argue by contradiction, so assume that $A \ne A^{dd}$. Let $u \in A^{dd}$ be such that $u > 0$ and $u \notin A$, and consider the set M_u, defined by

$$M_u = \{v \in A: 0 < v < u\}.$$

As shown above, M_u is not empty. Moreover, M_u is bounded from above by u. Now note that u cannot be the supremum of M_u since $M_u \subset A$ and since A is a band (since $u \notin A$). Hence, there exists a $w' \in L$ such that w' is an upper bound of M_u and such that $u \le w'$ does not hold. Letting $w = u \wedge w'$ it follows that w is an upper bound of M_u and that $0 \leqslant w < u$. So $w \in A^{dd}$ and $u-w > 0$. By the same argument as before, there exists a $z \in A$ such that $0 < z < u-w$. Next, for all $v \in M_u$ we have $(z+v) \in A$ and

$$0 < z+v \le z+w < u,$$

so $(z+v) \in M_u$. In particular, since $z \in M_u$ it follows that $0 < nz < u$ holds for $n=1,2,\ldots$. This contradicts (b), so $A = A^{dd}$.

(iii) To show that (d) implies (a), assume that $A = A^{dd}$ for every band A in L and suppose that L is not Archimedean. Then there exist $u, v \in L^+$ such that $v \ne 0$, $0 \le nv \le u$ for $n=1,2,\ldots$. Let (v) be the principal ideal generated by v (see 2.C) and consider the ideal A of L defined by

$$A = (v)+(v)^d.$$

(A is an ideal according to 2.2(iii) and 3.3(i)). Then

$$A^d = ((v)+(v)^d)^d \subset (v)^d \cap (v)^{dd} = \{0\},$$

so $A^d = \{0\}$ and equivalently $A^{dd} = L$. Next, let $[A]$ be the smallest band of L containing A (see 2.D). Then also $[A]^{dd} = L$, so $[A] = L$ by assumption. Once more, define

$$M_u = \{w \in A: \ 0 < w < u\}.$$

If $w \in M_u$, then $w = w_1 + w_2$, $w_1 \in (v)$, $w_2 \in (v)^d$ and $0 \le w_1 \le w$, $0 \le w_2 \le w$ (theorem 2.2(iii)). Since (v) consists of all $f \in L$ satisfying $|f| \le nv$ for some $n \in \mathbb{N}$ (2.C), we have $w_1 \le nv$ for some integer n. Hence

$$w_1 + v \le (n+1)v \le u \text{ and } (w_1+v) \in (v).$$

Also, we have $w_1 \le w \le u$ and $w_2 \perp (w_1+v)$, $w_2 \in (v)^d$. Therefore

$$(w_1+v)+w_2 = (w_1+v) \vee w_2 \le u,$$

so $w = w_1 + w_2 \le u-v < u$. This shows that $w \le u-v$ for all $w \in M_u$. It follows that $u-v$ is an upperbound of M_u. Recalling that $u \in [A] = L$, we have

$$u = \sup M_u$$

in view of 2.D, so $u \le u-v$. This contradicts $v > 0$, so L is Archimedean.

4.4. THEOREM. *Any Dedekind complete Riesz space is Archimedean.*

Proof. Let L be a Dedekind complete Riesz space, and let $u, v \in L^+$ be such that $0 \le nu \le v$ for $n=1,2,\ldots$. Then $u_0 = \sup \{nu: \ n=1,2,\ldots\}$ exists in L^+. Now note that

$$2u_0 = \sup \{2nu: n=1,2,\ldots\} = u_0,$$

so $u_0 = 0$ and hence $u = 0$.

The following exercise shows that Dedekind completeness and the Archimedean property are inherited by suitable subspaces.

4.B. *Exercise.* (i) Let L be Archimedean. Then any Riesz subspace of L is Archimedean.

(ii) Let L be Dedekind complete. Then any ideal of L is Dedekind complete.

In section 1 we have presented several important Riesz spaces. We shall show now which of these examples are Archimedean and which are even Dedekind complete (the examples carry the same letter as in section 1).

Examples.

4.C. The Cartesian space $\mathbb{R}^n$ ordered coordinatewise is clearly Dedekind complete (since $\mathbb{R}$ is Dedekind complete) and hence also Archimedean.

4.D. The lexicographically ordered plane $\mathbb{R}^2$ is not Archimedean and therefore not Dedekind complete. Indeed, if $u = (1,0)$ and $v = (0,1)$, then $0 \le nv \le u$ for all n.

4.E. Let (X,Γ) be a measurable space and consider the Riesz space $\mathcal{M}$ of example 1.E. It is clear that $\mathcal{M}$ is Archimedean. However, in general $\mathcal{M}$ is not Dedekind complete. Indeed, if there exists a subset $A \subset X$ such that $A \notin \Gamma$, and if Γ contains all single points of X, then $\mathcal{M}$ is not Dedekind complete because the set D of $\mathcal{M}$ defined by

$$D = \{\chi_{\{x\}} : x \in A\}$$

is bounded from above in $\mathcal{M}$ by χ_X, but D does not have a supremum in $\mathcal{M}$. (χ_B denotes the characteristic function of the subset B of X).

4.F. Let (X,Γ,μ) be a measure space, and consider the Riesz space $M = \mathcal{M}/\mathcal{N}$ (see 1.F and 2.G). Again it is clear that M is Archimedean, but in the present case we can prove more.

4.F.1. THEOREM. *If* μ *is* σ*-finite, then* M *is Dedekind complete.*

Proof. We divide the proof into three steps.

(a) Assume that $\mu(X) < \infty$. Let $D \subset M^+$ be a nonempty subset such that D is bounded from above by $v \in M^+$ and assume that v is bounded (i.e., if we choose a function f in the class of v, then f is always *essentially bounded* and this means that

$$\inf\{a \in \mathbf{R}^+ : \mu\{x \in X : |f(x)| > a\} = 0\} < \infty).$$

To show that $\sup D$ exists in M^+, we may assume that D is closed under taking suprema of finite subsets (see 4.A). Now note that

$$0 \le \int u \, d\mu \le \int v \, d\mu < \infty$$

holds for all $u \in D$ (integration of an equivalence class means integration of one of its members). Letting

$$P = \sup\{\int u \, d\mu : \ u \in D\}$$

it follows that $P < \infty$. Furthermore, there exists a sequence $u_1, u_2, \ldots.$ in D such that $u_1 \le u_2 \le \ldots.$ and $\int u_n \, d\mu \uparrow P$. Next, pick a function f_n in each class u_n and define

$$f_0(x) = \sup\{f_n(x) : \ n=1,2,\ldots.\}$$

for all $x \in X$. After that, define u_0 to be the member of M generated by $f_0 \in \mathcal{M}$. (note that u_0 does not depend on the special choice of the functions f_n). It is clear that

$$\int u_0 \, d\mu = P \quad \text{and} \quad u \le u_0 \quad \text{for all } u \in D.$$

Finally, it is routine to show that $u_0 = \sup D$.

(b) Assume that $\mu(X) < \infty$. Let $D \subset M^+$ be a nonempty subset such that D is bounded from above by $v \in M^+$. Define for all $u \in D$ and for all $n \in \mathbf{N}$

$$u_n = u \wedge [n\chi_X]; \quad D_n = \{u_n : \ u \in D\}$$

where $[n\chi_X]$ is the class of $n\chi_X$. From (a) it follows that each D_n has a supremum v_n $(n=1,2,\ldots)$. Letting $u_0 = \sup\{v_n : \ n=1,2,\ldots\}$ it is clear that $u_0 = \sup D$ (using the fact that $v_n \le v$ for all n).

(c) The general case. Let again $D \subset M^+$ be a nonempty subset that is bounded from above by $v \in M^+$. Let $X = \bigcup_{n=1}^{\infty} X_n$, $\mu(X_n) < \infty$ for all n, and define

$$D_n = \{u\chi_{X_n} : u \in D\}.$$

Then $u_n = \sup D_n$ exists in M^+ for all n by (b). Letting $u_0 = \sup \{u_n : n=1,2,\ldots\}$ it is clear that $u_0 = \sup D$ which proves that M is Dedekind complete.

Remark. Note that we have proved something more than the Dedekind completeness of M. Indeed, we have shown that if $D \subset M^+$ is a nonempty subset that is bounded from above, then D contains a countable subset D_1 such that $\sup D_1 = \sup D$ exists in M^+. Riesz spaces having the property stated above are sometimes called *super-Dedekind complete* Riesz spaces.

4.G. Let (X,τ) be a topological space and consider the Riesz space $C(X)$. It is clear that $C(X)$ is Archimedean. However $C(X)$ is not necessarily Dedekind complete. For instance the Riesz space $C([0,1])$ is not Dedekind complete. Indeed, for $n=2,3,\ldots$, define $f_n \in C([0,1])$ by

$f_n(x) = 0$ if $0 \le x \le \frac{1}{2}$; $f_n(x) = nx - \frac{1}{2}n$ if $\frac{1}{2} \le x \le \frac{1}{2} + n^{-1}$;

$f_n(x) = 1$ if $\frac{1}{2} + n^{-1} \le x \le 1$.

Then $D \subset C([0,1])$ defined by $D = \{f_n : n=2,3,\ldots\}$ is bounded from above but D does not have a supremum in $C([0,1])$.

In section 12 we shall return to the question for which topological spaces (X,τ) the space $C(X)$ is Dedekind complete.

4.H. Let (X,Γ) be a measurable space and let $\mathcal{B}$ the Riesz space consisting of all finitely additive bounded measures on Γ. Then $\mathcal{B}$ is Dedekind complete. Since the proof of this statement is similar to the proof of theorem 5.6, we shall omit the proof here.

As noted in section 2 after theorem 2.2 the algebraic sum of two disjoint bands of L is not necessarily a band. This situation improves if we assume L to be Dedekind complete. In fact, it follows from the next theorem that in a Dedekind complete Riesz space any band is a so-called *projection band*, i.e., if B is a band in a Dedekind complete Riesz space L, then $L = B+B^d$, so there exists a positive projection $P: L \to B$.

4.5. THEOREM. *Let* L *be Dedekind complete.*

(*i*) *If* A_1, A_2 *are bands in* L *such that* $A_1 \perp A_2$, *then* A_1+A_2 *is a band.*

(*ii*) $L = B+B^d$ *for any band* B *in* L.

Proof. (i) According to theorem 2.2(iii) A_1+A_2 is an ideal. Now, let D be a nonempty subset of $(A_1+A_2)^+$ such that $f_0 = \sup D$ exists in L^+. For all $f \in D$ we have $f = f_1+f_2$ for certain uniquely determined $f_1 \in A_1^+$, $f_2 \in A_2^+$. Furthermore, $0 \le f_i \le f \le f_0$ $(i=1,2)$ for all $f \in D$, so

$$f' = \sup \{f_1 : f \in D\} \text{ and } f'' = \sup \{f_2 : f \in D\}$$

exist in L^+ since L is Dedekind complete. Since A_1 and A_2 are bands, we have $f' \in A_1^+$ and $f'' \in A_2^+$. Now note that

$$0 \le f_0 = \sup \{f_1+f_2 : f \in D\} \le \sup \{f_1 : f \in D\} + \sup \{f_2 : f \in D\} = f'+f''$$

and $f'+f'' \in (A_1+A_2)$, so $f_0 \in (A_1+A_2)$ since A_1+A_2 is an ideal. This shows that A_1+A_2 is a band.

(ii) Let $B \subset L$ be a band. Then by the above $B+B^d$ is a band. Since

$$(B+B^d)^d \subset B^d \cap B^{dd} = \{0\}$$

it follows that $(B+B^d)^d = \{0\}$. Hence by the theorems 4.3 and 4.4,

$$B+B^d = (B+B^d)^{dd} = \{0\}^d = L.$$

As an application of the preceding theorem we shall compute all bands in the Riesz space M, defined in example 1.F.

4.I. *Example*. Let (X,Γ,μ) be a σ-finite measure space and let M be the Riesz space defined in 1.F (see also 2.G, 3.A and 4.F). In 3.A we have shown that if $A\in\Gamma$, then B_A, defined by

$$B_A = \{[f]\in M:\ f\in \mathcal{M},\ f=0\ \ \mu\text{-almost everywhere on } A^c\}$$

is a band in M. We shall show now that each band in M can be obtained in this way. To this end, let B be a band in M. Since M is Dedekind complete (4.F.1) we have $M = B+B^d$ (theorem 4.5(ii)). In particular, if $e = [\chi_X]\in M^+$, then e has a unique decomposition $e = e_1+e_2$, where $e_1\in B$, $e_2\in B^d$ (and $e_1\perp e_2$). Hence, χ_X has a decomposition $\chi_X = f_1+f_2$ for certain $f_1,f_2\in \mathcal{M}^+$ satisfying $[f_1] = e_1$, $[f_2] = e_2$. Since $e_1\perp e_2$ it follows that

$$\mu\{x\in X:\ \min(f_1(x),f_2(x)) \neq 0\} = 0,$$

so there exists a set $A\in\Gamma$ such that $f_1 = \chi_A$, $f_2 = \chi_{A^c}$ μ-almost everywhere on X, so

$$e_1 = [\chi_A],\ \ e_2 = [\chi_{A^c}]$$

for this set A. We shall show now that $B = B_A$. First, let $f\in B^+$ be given. Since $f\wedge e_2 = 0$ it follows that $f\in B_A^+$, so $B\subset B_A$. Similarly, if $f\in B^d$, then $f\in B_{A^c}$, so $B^d\subset B_{A^c}$. This implies that $B = B^{dd} \supset B_{A^c}^d = B_A$, so $B = B_A$.

Finally we observe that if $A_1,A_2\in\Gamma$, then

$$B_{A_1} = B_{A_2} \ \text{ if and only if } \ \mu(A_1\Delta A_2) = 0.$$

Thus we have a one-one correspondence between the collection $\mathcal{B}$ of all bands in M and the collection $[\Gamma]$ consisting of all equivalence classes of sets $A\in\Gamma$ where $A_1 \sim A_2$ if and only if $\mu(A_1\Delta A_2) = 0$.

By th. 4.5 any Dedekind complete Riesz space has the projection property. Also, it can be shown that any Riesz space that has the projection property is Archimedean. The converses of this two statements are false in general. To this end, we present the following exercise.

4.J. *Exercise.* (i) Show that any Riesz space that has the projection property is Archimedean.

(ii) Consider the Riesz space $C([0,1])$ (see 1.G and 4.G). Show that $C([0,1])$ is Archimedean but that it does not have the projection property.

(iii) Let L be the space consisting of all sequences $(x_1,x_2,\ldots.)$ $(x_i \in \mathbb{R})$ such that for any sequence $(x_1,x_2,\ldots.)$ in L there exists a number N_0 such that $x_n = x_{n+1}$ for all $n \geq N_0$ (N_0 depends on the choice of the sequence). The space L is ordered coordinatewise. Show that L is a Riesz space that has the projection property, but that L is not Dedekind complete.

Next, by way of an exercise, we show that if L is Archimedean and if A is an ideal in L, then L/A is not necessarily Archimedean. However, if A is not only an ideal, but even a band, this situation improves.

4.K. *Exercise.* (i) Let L be the Riesz space $C([0,1])$ (see 1.G, 4.G and 4.J(ii)). Let A be the subset of L consisting of all $f \in C([0,1])$ for which there exists an $\varepsilon_f > 0$ (in $\mathbb{R}$) such that $f \equiv 0$ on $[0,\varepsilon_f]$. Show that A is an ideal in L and that L/A is not Archimedean.

(ii) Let L be an Archimedean Riesz space and let B be a band in L. Show that L/B is Archimedean.

Hint: Let ϕ be the canonical Riesz homomorphism from L into L/B and let $u,v \in L^+$ be such that $n\phi(v) \leq \phi(u)$ for $n=1,2,\ldots$. Then

$$0 = (n\phi(v)-\phi(u))^+ = \phi((nv-u)^+)$$

for all n, so $(nv-u)^+ \in B$ for all n. Observe that

$$0 \leq v-(v-n^{-1}u)^+ = |v^+-(v-n^{-1}u)^+| \leq$$
$$|v-(v-n^{-1}u)| = n^{-1}u$$

For all n. Letting $w_n = (v-n^{-1}u)^+$ show that $0 \leq w_1 \leq w_2 \leq \ldots . \leq v$ and that $\sup\{w_n: n=1,2,\ldots\} = v$. Also show that $w_n \in B$ for all n. Conclude that $v \in B$ and that L/B is Archimedean.

Finally, we construct a Boolean algebra $\mathcal{B}$ using an Archimedean Riesz space L. First, let L be any Riesz space and define

$$\mathcal{B}[L] = \{B \subset L: \ B \text{ is a band}\} \ (=\mathcal{B})$$

Then $\mathcal{B}$ is partially ordered by inclusion, i.e., $B_1 \le B_2$ whenever B_1 is contained in B_2. Moreover $\mathcal{B}$ has an infimum (viz. $\{0\}$) and a supremum (viz. L). Also, for any pair $B_1, B_2 \in \mathcal{B}$ the infimum $B_1 \wedge B_2$ exists in $\mathcal{B}$, since $B_1 \cap B_2 = B_1 \wedge B_2$ is a band in L.

4.6. THEOREM. *If* L *is Archimedean, then* $\mathcal{B}$ *is a Boolean algebra.*

Proof. (a) First we show that $\mathcal{B}$ is a lattice. As noted above, $B_1 \wedge B_2$ exists in $\mathcal{B}$ for any pair $B_1, B_2 \in \mathcal{B}$. Defining for $B_1, B_2 \in \mathcal{B}$

$$B_1 \vee B_2 = [B_1 + B_2]$$

(see 2.D) it is immediate that $B_1 \vee B_2$ is the supremum of B_1 and B_2 with respect to the partial ordering in $\mathcal{B}$. We note that $(A+B)^d = A^d \cap B^d$ for $A, B \in \mathcal{B}$ and therefore

$$B_1 \vee B_2 = [B_1 + B_2] = [B_1 + B_2]^{dd} = (B_1 + B_2)^{dd} = (B_1^d \cap B_2^d)^d$$

for $B_1, B_2 \in \mathcal{B}$ since L is Archimedean.

(b) Next we show that $\mathcal{B}$ is distributive. To this end, let $A, B_1, B_2 \in \mathcal{B}$ be given. Since $\mathcal{B}$ is a lattice it is clear that

$$(A \wedge B_1) \vee (A \wedge B_2) \le A \wedge (B_1 \vee B_2).$$

For the converse direction, let $f \in (A \cap (B_1 + B_2))^+$ be given. Then

$$f = f_1 + f_2, \quad f_1 \in B_1^+, \quad f_2 \in B_2^+, \quad f_1 \in A^+, \quad f_2 \in A^+.$$

Hence $f_1 \in A \cap B_1$ and $f_2 \in A \cap B_2$, so $f \in (A \cap B_1 + A \cap B_2)$. This implies that

$$A \wedge (B_1 \vee B_2) = A \cap [B_1 + B_2] = [A \cap (B_1 + B_2)] \subset [(A \cap B_1) + (A \cap B_2)] =$$

$$(A\wedge B_1)\vee(A\wedge B_2).$$

This shows that $\mathcal{B}$ is distributive.

(c) Finally, we show that each element of $\mathcal{B}$ has a complement. Let $B\in\mathcal{B}$ be given. We assert that B^d is a complement of B. Indeed,

$$B\wedge B^d = B\cap B^d = \{0\}$$

and

$$B\vee B^d = (B^d\cap B^{dd})^d = \{0\}^d = L.$$

Remark. Note that the only place where we actually need that L is Archimedean is in part (c) of the proof.

4.7. DEFINITION. A Boolean algebra X is called *Dedekind complete* if every nonempty subset has a supremum.

4.L. *Exercise.* Show that a Boolean algebra X is Dedekind complete if and only if every nonempty subset of X has an infimum.

4.M. *Exercise.* Show that the Boolean Algebra $\mathcal{B}$ of theorem 4.6 is Dedekind complete.

Hint: Use theorem 2.2(ii).

CHAPTER II. THE ORDER DUAL

As before, let L be any Riesz space. By a *linear functional* on L we shall always mean a linear map from L into $\mathbb{R}$. Also, a linear functional ϕ on L is said to be *positive* ($\phi \geq 0$) if $\phi(L^+) \subset \mathbb{R}^+$ (compare definition 2.3(i)). The collection of all linear functionals on L is called the *algebraic dual* of L and is denoted by $L^\#$. We note that $L^\#$ contains at least one positive linear functional (viz. the null functional). In this chapter we shall study the linear subspace $L^\sim$ of $L^\#$ spanned by the positive linear functionals. It will turn out that $L^\sim$ is in a natural way an (even Dedekind complete) Riesz space.

5. ORDER BOUNDED LINEAR FUNCTIONALS

We begin with a definition.

5.1. DEFINITION. An element $\phi \in L^\#$ is called *order bounded* if for all $u \in L^+$

$$\sup\{|\phi(f)|: \ f \in L, \ |f| \leq u\}$$

is finite. The collection of all order bounded linear functionals on L will be denoted by $L^\sim$.

It is clear that for a $\phi \in L^\#$ we have $\phi \geq 0$ if and only if $f \leq g$ in L implies $\phi(f) \leq \phi(g)$ in $\mathbb{R}$. Moreover, if $\phi \geq 0$, then $\phi \in L^\sim$ since for $u \in L^+$ and for any $f \in L$ satisfying $|f| \leq u$, we have $f^+ \leq u$ and $f^- \leq u$, so

$$|\phi(f)| = |\phi(f^+)-\phi(f^-)| \leq \phi(f^+)+\phi(f^-) \leq 2\phi(u).$$

This implies that positive linear functionals on L are order bounded. Finally, it is clear that $L^\sim$ is a linear subspace of $L^\#$.

5.2. DEFINITION. The space $L^\sim$ is called the *order dual* of L.

In the remaining part of this section we shall show that $L^\sim$ is a Dedekind complete Riesz space. It will turn out that the positive cone of $L^\sim$ is precisely the collection of all positive linear functionals on L. A first step in this direction is the following theorem, which for obvious reasons is called the *Jordan decomposition theorem*.

5.3. THEOREM. *Let* $\phi \in L^\#$ *be given. Then* $\phi \in L^\sim$ *if and only if* $\phi = \phi_+ - \phi_-$, $\phi_+ \geq 0$, $\phi_- \geq 0$, $\phi_+, \phi_- \in L^\#$.

Proof. (i) If $\phi = \phi_+ - \phi_-$ where both ϕ_+ and ϕ_- are positive, then ϕ_+ and ϕ_- are in $L^\sim$, so $\phi \in L^\sim$.

(ii) For the converse direction, assume that $\phi \in L^\sim$. Define

$$\phi_+'(u) = \sup \{\phi(v) : 0 \leq v \leq u\}$$

for all $u \in L^+$. Clearly $\phi_+'(u) \geq 0$ for all $u \in L^+$. It is also clear that $\phi_+'(au) = a\phi_+'(u)$ for all $u \in L^+$ and for all $a \in \mathbf{R}^+$. Next we show that ϕ_+' is additive on L^+. To this end, let $u, u' \in L^+$ be given. Then

$$\begin{aligned} \phi_+'(u+u') &= \sup \{\phi(w) : 0 \leq w \leq u+u'\} = \\ &\sup \{\phi(v)+\phi(v') : 0 \leq v+v' \leq u+u', \\ &\qquad w = v+v', \ 0 \leq v \leq u, \ 0 \leq v' \leq u'\} \leq \\ &\sup \{\phi(v)+\phi(v') : 0 \leq v \leq u, \ 0 \leq v' \leq u'\} = \\ &\phi_+'(u) + \phi_+'(u') \end{aligned}$$

by theorem 1.7. Conversely, we have

$$\begin{aligned} &\phi_+'(u) + \phi_+'(u') = \\ &\sup \{\phi(v) : 0 \leq v \leq u\} + \sup \{\phi(v') : 0 \leq v' \leq u'\} = \\ &\sup \{\phi(v+v') : 0 \leq v \leq u, \ 0 \leq v' \leq u'\} \leq \\ &\sup \{\phi(w) : 0 \leq w \leq u+u'\} = \phi_+'(u+u'), \end{aligned}$$

so $\phi_+'(u+u') = \phi_+'(u) + \phi_+'(u')$ for all $u, u' \in L^+$. Thus, by lemma 2.10, ϕ_+' has a unique extension ϕ_+ to the whole of L. It is obvious that $\phi_+ \geq 0$. Moreover, $\phi_+(u) = \phi'_+(u) \geq \phi(u)$ for all $u \in L^+$ and hence, defining $\phi_- = \phi_+ - \phi$, it follows that $\phi_- \in L^\#$, $\phi_- \geq 0$, which is the desired result

The positive linear functionals ϕ_+ and ϕ_-, occuring in the preceding proof are called the *positive variation* and the *negative variation* of ϕ respectively. Moreover, $\phi_+ + \phi_-$ is called the *total variation* of ϕ. It is clear that for any $\phi \in L^\sim$, we have $\phi_+, \phi_- \geq 0$, so $\phi_+, \phi_- \in L^\sim$. Also, if $\phi \geq 0$, then $\phi_+ = \phi$ and $\phi_- = 0$. Finally, we note for use in the sequel, that if $\phi, \phi' \in L^\sim$ are such that $\phi - \phi' \geq 0$, then $\phi_+ - \phi'_+ \geq 0$.

Next, we define a partial ordering on $L^\sim$ by setting $\phi_1 \leq \phi_2$ whenever $\phi_2 - \phi_1 \geq 0$. It is routine to show that $L^\sim$ thus becomes a partially ordered linear space. Moreover, in the next theorem, we prove that $L^\sim$ is even a Riesz space and that ϕ_+ and ϕ_- coincide with the positive and the negative part of ϕ respecticely.

5.4. THEOREM. (*i*) $L^\sim$ *is a Riesz space with respect to* $\leq$.

(*ii*) $\phi_+ = \phi \vee 0$, $\phi_- = (-\phi) \vee 0$ *for all* $\phi \in L^\sim$. (*So* $\phi_+ = \phi^+$ *and* $\phi_- = \phi^-$ *in the Riesz space structure of* $L^\sim$).

Proof. (i) We have to show that any pair $\phi, \phi' \in L^\sim$ has a supremum in $L^\sim$. To this end, let $\phi, \phi' \in L^\sim$ and define

$$\psi'(u) = \sup \{\phi(f) + \phi'(u-f) : \ 0 \leq f \leq u\}$$

for all $u \in L^+$. Then clearly $\phi(u) \leq \psi'(u)$ and $\phi'(u) \leq \psi'(u)$ for all $u \in L^+$ (take $f = u$ and $f = 0$ respectively). Furthermore, we have $\psi'(au) = a\psi'(u)$ for all $u \in L^+$ and for all $a \in \mathbf{R}^+$. Now note that

$$\psi'(u) = \sup \{\phi(f) + \phi'(u-f) : \ 0 \leq f \leq u\} =$$
$$\sup \{ (\phi - \phi')(f) : \ 0 \leq f \leq u\} + \phi'(u) =$$
$$(\phi - \phi')_+(u) + \phi'(u),$$

so ψ' is additive on L^+. Again by lemma 2.10 it follows that ψ' has a unique extension $\psi \in L^\#$. It is easy to see that $\psi \in L^\sim$ and that $\phi \le \psi$ and $\phi' \le \psi$. Finally, let $\psi_1 \in L^\sim$ be such that $\phi \le \psi_1$, $\phi' \le \psi_1$. If $u, f \in L^+$ are such that $0 \le f \le u$, then

$$\phi(f)+\phi'(u-f) \le \psi_1(f)+\psi_1(u-f) = \psi_1(u).$$

Hence $\psi(u) \le \psi_1(u)$ for all $u \in L^+$, so $\psi \le \psi_1$ in $L^\sim$. This shows that ψ is the supremum of ϕ and ϕ'. The existence of the infimum of ϕ and ϕ' in $L^\sim$ is shown similarly.

(ii) Using part (i), we obtain

$$(\phi \vee 0)(u) = \sup\{\phi(v): \ 0 \le v \le u\} = \phi_+(u)$$

for all $u \in L^+$, so $\phi_+ = \phi \vee 0 = \phi^+$. It is clear now that

$$\phi_- = \phi_+ - \phi = \phi^+ - \phi = \phi^- = (-\phi)\vee 0.$$

From now on, the positive and the negative variation of an element $\phi \in L^\sim$ will be denoted by ϕ^+ and ϕ^- respectively. Since $L^\sim$ is a Riesz space, it follows that for any $\phi \in L^\sim$ the absolute value $|\phi| = \phi \vee (-\phi)$ exists in $L^\sim$. Moreover, from the preceding results, we find that $|\phi|$ is exactly the total variation $\phi_+ + \phi_-$ of ϕ.

5.5. LEMMA. *Let* $\phi \in L^\sim$ *be given. Then*

(*i*) $\phi^+(u) = \sup\{\phi(v): \ 0 \le v \le u\}$ *for all* $u \in L^+$.

(*ii*) $|\phi|(u) = \sup\{\phi(f): \ |f| \le u\} = \sup\{|\phi(f)|: \ |f| \le u\}$ *for all* $u \in L^+$.

(*iii*) $|\phi(f)| \le |\phi|(|f|)$ *for all* $f \in L$.

Proof. (i) Has already been proved.

(ii) Let $u \in L^+$ be given. Then

$$\phi^-(u) = \phi^+(u) - \phi(u) = \sup\{\phi(v-u): \ 0 \le v \le u\} =$$

$$\sup \{\phi(w): \ -u \le w \le 0\}.$$

Hence,

$$|\phi|(u) = \phi^+(u)+\phi^-(u) = \sup \{\phi(v_1+v_2): \ 0 \le v_1 \le u, \ -u \le v_2 \le 0\}.$$

Now, let $f = v_1+v_2$ where $0 \le v_1 \le u$, $-u \le v_2 \le 0$. Then $f \le v_1 \le u$ and $-f = -v_1-v_2 \le -v_2 \le u$, so $|f| \le u$. Thus we obtain

$$|\phi|(u) \le \sup \{\phi(f): \ |f| \le u\}.$$

On the other hand, if $f \in L$ is such that $|f| \le u$, then $f = f^+-f^-$ such that $0 \le f^+ \le u$, $-u \le -f^- \le 0$. Hence

$$\sup \{\phi(f): \ |f| \le u\} \le$$

$$\sup \{\phi(v_1+v_2): \ 0 \le v_1 \le u, \ -u \le v_2 \le 0\} = |\phi|(u).$$

The rest of the proof is now easy and left to the reader.

(iii) Obvious from part (ii).

5.6. THEOREM. $L^{\sim}$ *is Dedekind complete.*

Proof. Let Φ be a nonempty subset of $L^{\sim+}$ such that $\phi_1,\phi_2 \in \Phi$ implies $\phi_1 \vee \phi_2 \in \Phi$ and such that there exists a $\phi_0 \in L^{\sim+}$ with $\phi \le \phi_0$ for all $\phi \in \Phi$. By 4.A it suffices to show that $\sup \Phi$ exists in $L^{\sim+}$. To this end, define for all $u \in L^+$

$$\psi(u) = \sup \{\phi(u): \ \phi \in \Phi\}.$$

Then $0 \le \psi(u) \le \phi_0(u) < \infty$ for all $u \in L^+$ and it is obvious that $\psi(au) = a\psi(u)$ for all $u \in L^+$ and for all $a \in \mathbb{R}^+$. Also,

$$\psi(u+v) = \sup \{\phi(u+v): \phi \in \Phi\} \le \psi(u)+\psi(v)$$

for all $u,v \in L^+$. On the other hand, if $u,v \in L^+$ and $\varepsilon > 0$ (in $\mathbb{R}$) are given, then there exist $\phi_1,\phi_2 \in \Phi$ such that

$$\phi_1(u) \le \psi(u) \le \phi_1(u)+\tfrac{1}{2}\varepsilon; \quad \phi_2(v) \le \psi(v) \le \psi_2(v)+\tfrac{1}{2}\varepsilon.$$

Hence, defining $\phi_\varepsilon \in \Phi$ by $\phi_\varepsilon = \phi_1 \vee \phi_2$, then

$$\phi_1(u) \le \phi_\varepsilon(u) \le \psi(u) \le \psi_1(u)+\tfrac{1}{2}\varepsilon \le \phi_\varepsilon(u)+\tfrac{1}{2}\varepsilon,$$

$$\phi_2(v) \le \phi_\varepsilon(v) \le \psi(v) \le \phi_2(v)+\tfrac{1}{2}\varepsilon \le \phi_\varepsilon(v)+\tfrac{1}{2}\varepsilon.$$

Hence

$$\psi(u)+\psi(v) \le \phi_\varepsilon(u)+\phi_\varepsilon(v)+\varepsilon = \phi_\varepsilon(u+v)+\varepsilon \le \psi(u+v)+\varepsilon.$$

This holds for all $\varepsilon > 0$, so $\psi(u+v) = \psi(u)+\psi(v)$ for all $u,v\in L^+$. Extending, as before by lemma 2.10, ψ to the whole of L, it is clear that $0 \le \psi \le \phi_0$, so $\psi\in L^\sim$. Finally, it is obvious that $\psi = \sup \Phi$ holds, so $L^\sim$ is Dedekind complete.

Finally, by way of an example, we show that even if L is Dedekind complete it can happen that $L^\sim = \{0\}$ (so the only positive linear functional on L is the null functional).

5.A. *Example and exercise.* (i) Consider the Riesz space M (defined in 1.F) consisting of the equivalence classes of measurable real-valued functions, defined on the measure space $[0,1]$ (provided with Lebesgue measure). As shown in 4.F.1, M is a Dedekind complete Riesz space. *In this case we have* $M^\sim = \{0\}$. We shall not give a detailed proof of this statement, but we do give some hints.

Argue by contradiction, so suppose that $M^\sim$ contains a non-trivial element. In view of theorem 5.3 there exist $\phi\in M^\sim$, $\phi \ge 0$ and $f\in M^+$ such that $\phi(f) > 0$. Show that there exist measurable sets $A_1 \supset A_2 \supset\ldots\ldots$ such that for every n $\mu(A_n) \le 2^{-n}$, $\phi(f\chi_{A_n}) > 0$. For $n\in\mathbb{N}$ let $a_n\in\mathbb{R}^+$ be such that $a_n\phi(f\chi_{A_n}) = 1$. Now $\Sigma a_n f\chi_{A_n}$ converges almost everywhere, so there exists a $g\in M$, $g = \Sigma a_n f\chi_{A_n}$ almost everywhere. For every N we have

$$g \ge \sum_{n\le N} a_n f\chi_{A_n},$$

so

$$\phi(g) \ge \sum_{n\le N} a_n\phi(f\chi_{A_n}) = N.$$

This is the desired contradiction.

(ii) The situation changes if we consider the Riesz space $C([0,1])$ (see 1.E). Indeed, let $a \in [0,1]$ and define

$$\phi_a(f) = f(a)$$

for all $f \in C([0,1])$. Then $\phi_a \in (C([0,1]))^{\sim}$ and $\phi_a \geq 0$, $\phi_a \neq 0$. Thus, for this Riesz space (which is not Dedekind complete), there exist many non-trivial order bounded linear functionals.

6. EXTENSION THEOREMS

In this section we show that under some additional conditions order bounded linear functionals that are defined on a Riesz subspace of L can be extended to the whole of L. First, we state without proof the classical Hahn-Banach theorems. For a proof we refer the reader to 2 , 5 , 14 or 16.

6.1. THEOREM. (*Hahn-Banach*). *Let* V *be any linear space and let* V_1 *be a linear subspace of* V. *Furthermore, let* p *be a sub-linear functional on* V *(i.e.,* $p(v) \in \mathbf{R}$ *for all* $v \in V$, $p(v_1+v_2) \leq p(v_1)+p(v_2)$ *for all* $v_1, v_2 \in V$ *and* $p(av) = ap(v)$ *for all* $v \in V$ *and for all* $a \in \mathbf{R}^+$*). If* ϕ *is a linear functional on* V_1 *such that* $\phi(v) \leq p(v)$ *for all* $v \in V_1$, *then there exists a linear functional* Φ *on* V *such that* $\Phi(v) = \phi(v)$ *for all* $v \in V_1$ *and such that* $\Phi(v) \leq p(v)$ *for all* $v \in V$.

6.2. THEOREM. *Let* V *be any normed linear space and let* V_1 *be a linear subspace of* V. *By* V^* *we denote the conjugate space (norm dual, Banach dual) of* V. *If* $\phi \in V_1^*$, *then there exists a* $\Phi \in V^*$ *such that* $\Phi(v) = \phi(v)$ *for all* $v \in V_1$ *and such that* $\|\Phi\| = \|\phi\|$

Especially the second Hahn-Banach theorem has important consequences in functional analysis. For instance, if V is a normed linear space, it follows that V^* contains many non-trivial elements. In fact, the elements of V^* separate the elements of V. Another important consequence is stated in the next corollary. For a proof we refer the reader to 16.

6.3. COROLLARY. *Let* V *be a normed linear space and let* V^{**} *be its bidual (the conjugate space of* V^**). Then* V *can be imbedded isometrically in* V^{**}*, i.e., there exists a linear norm preserving map* J *from* V *into* V^{**}.

For use in the sequel, we present the map $J: V \to V^{**}$. To this end, let $v \in V$ be given and define for all $\phi \in V^*$

$$\tilde{v}(\phi) = \phi(v).$$

It turns out that $\tilde{v} \in V^{**}$ and that J is defined by $J(v) = \tilde{v}$ for all $v \in V$. The map J is called the *canonical map* from V into V^{**}. We note that normed linear spaces V for which $J(V) = V^{**}$ (so J is surjective) are called *reflexive*. Since the norm dual of a linear space is always a Banach space, it follows that reflexive spaces are Banach spaces. The converse does not hold.

We return to the Riesz space theory. As before, let L be a Riesz space and let $L^{\sim}$ be its order dual.

6.4. DEFINITION. Let ρ be a function from L into $\mathbb{R}^+$ satisfying

(a) $\rho(0) = 0$, $\rho(f+g) \leq \rho(f)+\rho(g)$, $\rho(af) = |a|\rho(f)$ for all $f,g \in L$ and for all $a \in \mathbb{R}$.

(b) $f,g \in L$ and $|f| \leq |g|$ implies $\rho(f) \leq \rho(g)$.

Then ρ is called a *Riesz semi-norm* on L.

Now, assume that ρ is a function from L^+ into $\mathbf{R}^+$ satisfying $\rho(0) = 0$, $\rho(au) = a\rho(u)$ for all $a\in\mathbf{R}^+$ and for all $u\in L^+$, $\rho(u+v) \le \rho(u)+\rho(v)$ for all $u,v\in L^+$, and $\rho(u) \le \rho(v)$ if $0 \le u \le v$ in L^+. If we define $\rho(f) = \rho(|f|)$ for all $f\in L$ it is clear that ρ is a Riesz semi-norm on L. Hence, since any Riesz semi-norm ρ on L satisfies $\rho(f) = \rho(|f|)$ for all $f\in L$, it follows that Riesz semi-norms on L are completely determined by their values on the positive cone L^+ of L.

First we show that the existence of a Riesz semi-norm on L is equivalent to the existence of a non-trivial positive linear functional on L.

6.5. THEOREM. *The following statements are equivalent.*

(*a*) $L^\sim \ne \{0\}$.

(*b*) *There exists a non-trivial positive linear functional on* L.

(*c*) *There exists a non-trivial Riesz semi-norm* ρ *on* L *(i.e., there exists an* $u_0\in L^+$ *such that* $0 < \rho(u_0) < \infty$*).*

Proof. (i) (a) $\Leftrightarrow$ (b) is obvious from theorem 5.3.

(ii) (b) $\Rightarrow$ (c). Let ϕ be a non-trivial positive linear functional on L. Hence, there exists an $u_0\in L^+$ such that $\phi(u_0) > 0$. Setting

$$\rho(f) = \phi(|f|)$$

for all $f\in L$, it is clear that ρ is a non-trivial Riesz semi-norm on L.

(iii) (c) $\Rightarrow$ (a). Let ρ be a non-trivial Riesz semi-norm on L and let $u_0\in L^+$ be such that $\rho(u_0) > 0$. Let $[u_0]$ be the linear subspace of L spanned by u_0 and define ψ on $[u_0]$ by

$$\psi(au_0) = a\rho(u_0)$$

for all $a\in\mathbf{R}$. It is clear that ψ is linear on $[u_0]$ and that $\psi(u) \le \rho(u)$ for all $u\in[u_0]$. Since a semi-norm is a sub-linear functional it follows by theorem 6.1 that there exists a linear functional ϕ on L such that

$\phi = \psi$ on $[u_0]$ and such that $|\phi(f)| \le \rho(f)$ for all $f \in L$. Note that ϕ is non-trivial since $\phi(u_0) = \psi(u_0) = \rho(u_0) > 0$. Also $\phi \in L^{\sim}$ since

$$\sup \{|\phi(f)|: \ |f| \le u\} \le \sup \{\rho(f): \ |f| \le u\} = \rho(u) < \infty$$

for all $u \in L^+$.

Next, we present the announced extension theorems.

6.6. THEOREM. *Let* ρ *be a Riesz semi-norm on* L *and let* K *be a Riesz subspace of* L *(so* K *is a Riesz space in its own right). Furthermore, let* $\phi \in K^{\sim}$, $\phi \ge 0$ *such that* $|\phi(f)| \le \rho(f)$ *for all* $f \in K$. *Then there exists a* $\psi \in L^{\sim}$, $\psi \ge 0$ *such that* $\psi = \phi$ *on* K *and such that* $|\psi(f)| \le \rho(f)$ *for all* $f \in L$.

Proof. By theorem 6.1 it is clear that ϕ can be extended to the whole of L. However, the positivity of this extension is not assured in that case. Therefore, we introduce the sub-linear functional p on L as follows. For all $f \in L$, define $p(f) = \rho(f^+)$. Then obviously p is sub-linear. Furthermore,

$$\phi(f) \le \phi(f^+) \le \rho(f^+) = p(f)$$

holds for all $f \in K$ since $\phi \ge 0$. Using theorem 6.1 it follows that there exists a linear functional ψ on L such that $\psi = \phi$ on K and such that $\psi(f) \le p(f)$ for all $f \in L$. Let $u \in L^+$ be given. Then

$$\psi(-u) \le p(-u) = \rho((-u)^+) = \rho(0) = 0,$$

so $\psi(u) \ge 0$, hence $\psi \ge 0$. Using lemma 5.5 and the positivity of ψ, we obtain

$$|\psi(f)| \le |\psi|(|f|) = \psi(|f|) \le p(|f|) = \rho(|f|^+) = \rho(|f|) = \rho(f)$$

for all $f \in L$, which completes the proof.

We note that in view of theorem 6.5, it is natural to require the existence of a Riesz semi-norm on L in theorem 6.6.

6.7. COROLLARY. *Let* K *be a Riesz subspace of* L*. Furthermore, let* (K) *denote the smallest ideal in* L *containing* K*. If* ϕ *is a positive linear functional on* K*, then there exists a positive linear functional* ψ *on* (K) *such that* $\psi = \phi$ *on* K.

Proof. It is easy to see that for all $f \in (K)^+$ there exists an $f' \in K^+$ such that $f \le f'$ (see also 2.C). Thus we are allowed to define

$$\rho(f) = \inf \{\phi(f'): \; f' \in K, \; f' \ge f\}$$

for all $f \in (K)^+$. Defining $\rho(f) = \rho(|f|)$ for arbitrary $f \in (K)$ it is immediate that ρ is a Riesz semi-norm on (K) and that $|\phi(f)| \le \rho(f)$ for all $f \in K$. Hence, the existence of ψ follows from theorem 6.6.

Finally, we consider normed Riesz spaces. To this end, we restrict ourselves to appropriate norms.

6.8. DEFINITION. A norm ρ on L is called a *Riesz norm* if ρ is a norm as well as a Riesz semi-norm. A *normed Riesz space* is always a Riesz space provided with a Riesz norm.

If L is a normed Riesz space with Riesz norm ρ, then L^* will denote its conjugate space. The norm in L^* will be denoted by ρ^*. Although it is not clear on this moment that there exist non-trivial positive normbounded linear functionals on a normed Riesz space, we do state and prove the following theorem. In chapter 3 we shall show that there are indeed many non- trivial positive normbounded linear functionals on any normed Riesz space.

6.9. THEOREM. *Let* L *be a normed Riesz space with Riesz norm* ρ *and let* K *be a Riesz subspace of* L. *Let* ϕ *be a positive normbounded linear functional on* K. *Then there exists a* $\psi \in L^*$ *such that* $\psi = \phi$ *on* K, $\psi \geq 0$ *and* $\rho^*(\psi) = \rho^*(\phi)$.

Proof. Define $\rho_1(f) = \rho^*(\phi)\rho(f)$ for all $f \in L$. Then ρ_1 is a Riesz seminorm on L and for all $f \in K$ we have $\phi(f) \leq |\phi(f)| \leq \rho_1(f)$. According to theorem 6.6, there exists a $\psi \in L^\sim$ such that $\psi = \phi$ on K and such that $\psi \geq 0$. Moreover $|\psi(f)| \leq \rho_1(f)$ holds for all $f \in L$. Hence $\rho^*(\psi) \leq \rho^*(\phi)$, and therefore $\rho^*(\psi) = \rho^*(\phi)$ since ψ extends ϕ.

6.A. *Example.* Let L be the Riesz space M of example 1.F, where the measure space is taken the interval $[0,1]$ with Lebesgue measure. In 5.A we have shown that there do not exist non-trivial positive linear functionals on L, i.e., $L^\sim = \{0\}$. Next, let K be the Riesz space $C([0,1])$. After obvious identifications we can think of K as a Riesz subspace of L. Also in 5.A we showed that $K^\sim \neq \{0\}$. Thus, if $\phi \in K^{\sim +}$, $\phi \neq 0$, it is impossible to extend ϕ to the whole of L.

Next, we study some important subspaces of $L^\sim$.

7. INTEGRALS AND SINGULAR FUNCTIONALS

As before, L will be an arbitrary Riesz space and $L^\sim$ will be its order dual. Before we continue with the investigation of $L^\sim$ we introduce a notation. Let $f_1, f_2, \ldots$ be a sequence in L and let $f \in L$. If we have

$f_1 \le f_2 \le \ldots . \le f$ and $f = \sup\{f_n: n=1,2,\ldots\}$ we shall write from now on $f_n \uparrow f$. Similarly we define $f_n \downarrow f$.

In general there is a large class of functionals in $L^{\sim}$ that satisfy a useful continuity property. In this connection, we define

7.1. DEFINITION. A functional $\phi \in L^{\sim}$ is called an *integral* if $u_n \downarrow 0$ implies $\lim \phi(u_n) = 0$ (as $n \to \infty$). The collection of all integrals on L will be denoted by $L_c^{\sim}$.

7.A. *Example*. Let (X, Γ, μ) be a measure space and let $L_1(\mu)$ denote the collection of all real-valued μ-integrable functions on X. It is clear that $L_1(\mu)$ is an ideal of the Riesz space M defined in 1.E. Next, for all $f \in L_1(\mu)$ define

$$\phi(f) = \int_X f \, d\mu.$$

Obviously ϕ is a positive linear functional on $L_1(\mu)$, so $\phi \in (L_1(\mu))^{\sim +}$. Also, by Lebesgue's theorem on dominated convergence of integrals it follows that ϕ is an integral on $L_1(\mu)$.

We shall show that $L_c^{\sim}$ is a band in $L^{\sim}$ but first we present an auxiliary result.

7.2. LEMMA. *Let* $\phi \in L^{\sim}$ *be given. Then the following statements are equivalent.*

(*a*) $\phi \in L_c^{\sim}$.

(*b*) $\phi^+, \phi^- \in L_c^{\sim}$.

(*c*) $|\phi| \in L_c^{\sim}$.

Proof. (i) (a) $\Rightarrow$ (b). Let $u_n \downarrow 0$ (in L^+) and choose $v \in L$ such that

$0 \le v \le u_1$. Then

$$v-(v\wedge u_n) = v\wedge u_1 - v\wedge u_n \le v\wedge(u_1-u_n) \le u_1-u_n.$$

Since $\phi^+ \ge \phi$ we obtain

$$\phi(v-(v\wedge u_n)) \le \phi^+(v-(v\wedge u_n)) \le \phi^+(u_1-u_n),$$

so

$$0 \le \phi^+(u_n) \le \phi^+(u_1)+\phi(v\wedge u_n)-\phi(v).$$

Since $u_n \downarrow 0$, also $v\wedge u_n \downarrow 0$. Hence, $\phi\in L_c^\sim$ implies

$$0 \le \lim_{n\to\infty}\phi^+(u_n) \le \phi^+(u_1)-\phi(v).$$

This holds for all $v\in L^+$ such that $v \le u_1$, so

$$0 \le \lim_{n\to\infty}\phi^+(u_n) \le \phi^+(u_1)-\sup\{\phi(v): 0 \le v \le u_1\} =$$
$$\phi^+(u_1)-\phi^+(u_1) = 0.$$

Therefore $\phi^+\in L_c^\sim$. Since $L_c^\sim$ is obviously a linear subspace of $L^\sim$ we also have

$$\phi^- = \phi^+-\phi \in L_c^\sim.$$

(ii) (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are obvious.

(iii) (c) $\Rightarrow$ (b). Let $\psi\in L^{\sim+}$ be such that $0 \le \psi \le |\phi|$. Then $|\phi|\in L_c^\sim$ implies $\psi\in L_c^\sim$ by definition, so $L_c^\sim$ is an ideal in $L^\sim$. Observing that $0 \le \phi^+ \le |\phi|$ and $0 \le \phi^- \le |\phi|$ it is clear that $|\phi|\in L_c^\sim$ implies $\phi^+,\phi^-\in L_c^\sim$.

We have already observed that $L_c^\sim$ is an ideal in $L^\sim$ (part (iii) of the pre-preceding proof), but we can prove more.

7.3. THEOREM. $L_c^\sim$ *is a band in* $L^\sim$.

Proof. It suffices to show that any nonempty subset of $L_c^{\sim+}$ that has a supremum in $L^\sim$ assumes this supremum in $L_c^\sim$. Therefore, let $\Phi\subset L_c^{\sim+}$ be such that $\phi_0 = \sup\Phi$ exists in $L^\sim$. As before (theorem 5.6) we may assume that Φ is closed under the operation of taking suprema of finite sets, i.e.

we may assume that $\phi_1, \phi_2 \in \Phi$ implies $\phi_1 \vee \phi_2 \in \Phi$. As shown in the proof of 5.6 ϕ_0 satisfies

$$\phi_0(u) = \sup \{\phi(u) : \phi \in \Phi\}$$

for all $u \in L^+$. Now, let $\varepsilon > 0$ (in $\mathbb{R}$) be given and let $u_n \downarrow 0$ (in L^+). Then there exists a $\phi \in \Phi$ such that

$$0 \le (\phi_0 - \phi)(u_1) < \varepsilon$$

and hence $0 \le (\phi_0 - \phi)(u_n) < \varepsilon$ for all n, since $\phi_0 \ge \phi$ and since $u_n \le u_1$. Since $\phi \in \Phi \subset L_c^{\sim +}$ it follows that $\lim \phi(u_n) = 0$. Thus

$$0 \le \lim_{n \to \infty} \phi_0(u_n) < \varepsilon.$$

This holds for all $\varepsilon > 0$ and for all sequences $u_n \downarrow 0$ in L^+. Hence $\phi_0 \in L_c^{\sim}$, so $L_c^{\sim}$ is a band.

Next, we define another type of linear functionals on L.

7.4. DEFINITION. A functional $\phi \in L^{\sim}$ is called *singular* if $\phi \in (L_c^{\sim})^d$ (so $|\phi| \wedge |\psi| = 0$ for all $\psi \in L_c^{\sim}$). The collection of all singular functionals on L will be denoted by $L_s^{\sim}$.

Combining the results of 3.3(i), 4.5(ii), 5.6 and 2.2(iii), we obtain immediately the following results.

7.5. THEOREM. (*i*) $L_s^{\sim}$ *is a band in* $L^{\sim}$.

(*ii*) $L^{\sim} = L_c^{\sim} + L_s^{\sim}$, *i.e., if* $\phi \in L^{\sim}$, *then there exists a unique decomposition* $\phi = \phi_c + \phi_s$, $\phi_c \in L_c^{\sim}$, $\phi_s \in L_s^{\sim}$. *Moreover, if* $\phi \ge 0$, *then* $\phi_c \ge 0$ *and* $\phi_s \ge 0$.

The functionals ϕ_c and ϕ_s occuring in the preceding theorem are called the *integral part* and the *singular part* of ϕ. Our next step will be to compute the integral part of a given element $\phi \in L^{\sim}$. Before doing so, we first observe that integrals are not only additive but even σ-additive. To this end we introduce the following notation. Let $u_1, u_2, \ldots.$ be a sequence in L^+ and let $u \in L^+$. If we have

$$\sum_{n=1}^{k} u_n \uparrow u \text{ (as } k \to \infty),$$

then we shall write $u = \Sigma\, u_n$ $(= \sum_{n=1}^{\infty} u_n)$.

7.6. LEMMA. *Let* $\phi \in L^{\sim +}$. *Then* $\phi \in L_c^{\sim}$ *if and only if* $\phi(\Sigma\, u_n) = \Sigma\, \phi(u_n)$ *for any sequence* $u_1, u_2, \ldots.$ *in* L^+ *for which* $\Sigma\, u_n$ *exists in* L^+.

Proof. (i) Assume that $\phi \in L_c^{\sim}$, and let $u_1, u_2, \ldots.$ in L^+ be such that $u = \Sigma\, u_n$ exists in L^+. Defining

$$s_k = \sum_{n=1}^{k} u_n$$

it follows that $s_k \uparrow u$, so $u - s_k \downarrow 0$. Hence $\phi(u-s_k) \downarrow 0$ (in $\mathbb{R}^+$) and therefore

$$\phi(u) = \lim \phi(s_k) = \lim \sum_{n=1}^{k} (u_n) = \Sigma\, \phi(u_n) \quad (k \to \infty),$$

which shows that ϕ is σ-additive on L^+.

(ii) Assume that ϕ is σ-additive on L^+. In order to show that ϕ is an integral, let $f_1, f_2, \ldots.$ in L^+ be such that $f_n \downarrow 0$. Defining

$$u_n = f_n - f_{n+1}$$

for $n=1,2,\ldots.$, we have $u_n \in L^+$ for all n. Moreover, if we set

$$s_k = \sum_{n=1}^{k} u_n = f_1 - f_{k+1}$$

for $k=1,2,\ldots.$, it follows that $s_k \uparrow f_1$, so $\Sigma\, u_n = f_1$. Hence,

$$\phi(f_1) = \phi(\Sigma\, u_n) = \lim \sum_{n=1}^{k-1} \phi(u_n) = \phi(f_1) - \lim \phi(f_k),$$

so $\lim \phi(f_k) = 0$. Thus $\phi \in L_c^{\sim}$.

To compute the integral part ϕ_c of $\phi \in L^{\sim}$ we introduce a functional ϕ_L as follows. Let $\phi \in L^{\sim +}$ be given and define for all $u \in L^+$

$$\phi_L(u) = \inf \{\lim \phi(u_n) : \ 0 \le u_n \uparrow u\}.$$

By the proof of lemma 7.6 (part (ii)) we have

$$\phi_L(u) = \inf \{ \Sigma\phi(v_n): \ v_n \in L^+, \ \Sigma v_n = u \}$$

for all $u \in L^+$. Next, we extend ϕ_L to the whole of L by setting

$$\phi_L(f) = \phi_L(f^+) - \phi_L(f^-)$$

for all $f \in L$.

7.7. LEMMA. *Let* $\phi \in L^{\sim+}$. *Then* $\phi_L \in L^{\sim+}$ *and* $0 \leq \phi_L \leq \phi$.

Proof. It suffices to prove that $\phi_L(u+v) = \phi_L(u)+\phi_L(v)$ for all $u,v \in L^+$. To this end, let $u,v \in L^+$ be given. Now, let $\varepsilon > 0$ be given (in $\mathbb{R}$) and let $u_1,u_2,\ldots\ldots$ and $v_1,v_2,\ldots\ldots$ be sequences in L^+ satisfying $u_n \uparrow u$, $v_n \uparrow v$ and

$$\phi_L(u) > \lim \phi(u_n) - \tfrac{1}{2}\varepsilon; \quad \phi_L(v) > \lim \phi(v_n) - \tfrac{1}{2}\varepsilon.$$

Obviously we have $0 \leq (u_n+v_n) \uparrow (u+v)$, so since

$$\lim \phi(u_n+v_n) \leq \phi_L(u)+\phi_L(v)+\varepsilon$$

we have $\phi_L(u+v) \leq \phi_L(u)+\phi_L(v)$. Conversely, since

$$\phi_L(u+v) = \inf \{\lim \phi(w_n): \ 0 \leq w_n \uparrow u+v\}$$

there exists a sequence $w_1,w_2,\ldots\ldots$ in L^+ such that $0 \leq w_n \uparrow u+v$ and

$$\lim \phi(w_n) < \phi_L(u+v)+\varepsilon.$$

Defining $u_n = w_n \wedge u$ and $v_n = w_n - u_n$, we have $0 \leq u_n \uparrow u$ and $0 \leq v_n \uparrow v$ (since $u_{n+1}-u_n = w_{n+1} \wedge u - w_n \wedge u \leq (w_{n+1}-w_n) \wedge u \leq w_{n+1}-w_n$, so $v_{n+1} \geq v_n$ for all n). Hence

$$\phi_L(u)+\phi_L(v) \leq \lim \phi(u_n) + \lim \phi(v_n) = \lim \phi(w_n) < \phi_L(u+v)+\varepsilon.$$

This holds for all $\varepsilon > 0$, so we obtain $\phi_L(u+v) = \phi_L(u)+\phi_L(v)$ which is the desired result.

We are now ready to prove the main result of this section.

7.8. THEOREM. *Let* $\phi \in L^{\sim +}$. *Then* $\phi_c = \phi_L$.

Proof. Let $u \in L^+$ and $\varepsilon > 0$ (in $\mathbb{R}$) be given. Furthermore, let $v_n \in L^+$ ($n = 1,2,\ldots.$) be such that $u = \Sigma\, v_n$. For any natural number n there exists a sequence $v_{n1}, v_{n2}, \ldots.$ in L^+ such that $\sum_k v_{nk} = v_n$ and such that

$$\sum_k \phi(v_{nk}) < \phi_L(v_n) + \varepsilon 2^{-n}.$$

Defining $w_n = \sum_{j=1}^{n} \sum_{k=1}^{n} v_{jk}$ for $n=1,2,\ldots.$, it follows that $0 \le w_n \uparrow u$ and that

$$\phi(w_n) = \sum_{j=1}^{n} \sum_{k=1}^{n} \phi(v_{jk}) < \Sigma\, \phi_L(v_j) + \varepsilon.$$

This holds for all n, so $\phi_L(u) \le \Sigma\, \phi_L(v_j) + \varepsilon$, and since this holds for all $\varepsilon > 0$, we obtain $\phi_L(u) \le \Sigma\, \phi_L(v_j)$. On the other hand, from the positivity of ϕ_L it is clear that $\phi_L(u) \ge \Sigma\, \phi_L(v_j)$. Thus ϕ_L is σ-additive, so ϕ_L is an integral.

Next, since $\phi_L \in L_c^{\sim +}$ and since $\phi_L \le \phi$, it follows that $\phi_L \le \phi_c$. Now observe that any $\psi \in L_c^{\sim +}$ satisfies $\psi_L = \psi$. Using this fact and using $\phi_c \le \phi$, we obtain

$$\phi_c = (\phi_c)_L \le \phi_L,$$

so $\phi_c = \phi_L$.

As an immediate consequence we obtain the following characterization of singular functionals.

7.8. COROLLARY. *Let* $\phi \in L^{\sim +}$ *be given. Then* $\phi \in L_s^{\sim}$ *if and only if there exists for all* $\varepsilon > 0$ *(in* $\mathbb{R}$*) and for all* $u \in L^+$ *a sequence* $u_1, u_2, \ldots.$ *in* L^+ *satisfying* $u_n \uparrow u$ *and* $0 \le \phi(u_n) < \varepsilon$ *for all* n.

Proof. This is an immediate consequence of

$$\phi_c(u) = \inf\,\{\lim \phi(u_n) : \; 0 \le u_n \uparrow u\}$$

for all $u \in L^+$ and for all $\phi \in L^{\sim +}$ and the fact that $\phi_c = 0$ for any $\phi \in L_s$. we leave the straightforward proof to the reader.

7.B. *Example*. Consider the Riesz space $C([0,1])$ (1.G). Define the positive linear functional ϕ on $C([0,1])$ by setting

$$\phi(f) = f(0)$$

for all $f \in C([0,1])$ (see 5.A(ii)). Then ϕ is singular on $C([0,1])$. Indeed if $u \in C([0,1])^+$ is given, define for $n=1,2,\ldots$.

$$u_n(x) = \min(nx, u(x))$$

for all $x \in [0,1]$. Then clearly $u_n \uparrow u$ and $\phi(u_n) = 0$ for all n. Hence ϕ is singular.

We note that it can be proved that any order bounded linear functional on $C([0,1])$ is singular. However, we omit the rather technical proof. In particular it follows that the Riemann integral (or the Lebesgue integral) on $C([0,1])$ is singular.

8. ANNIHILATORS AND ABSOLUTELY CONTINUOUS ELEMENTS

Let L and $L^\sim$ be as before.

8.1. DEFINITION. Let A be a subset of L. The *annihilator* A° of A is the subset of $L^\sim$ defined by

$$A^\circ = \{\phi \in L^\sim : \ \phi(f) = 0 \ \text{ for all } f \in A\}.$$

Conversely, if B is a subset of L, then the *inverse annihilator* ${}^\circ B$ of B is the subset of L defined by

$${}^\circ B = \{f \in L: \ \phi(f) = 0 \ \text{ for all } \phi \in B\}.$$

The sets A° and ${}^\circ B$ are linear subspaces of $L^\sim$ and L respectively.

The proofs of these facts are so simple that they can be left confidently to the reader.

8.2. THEOREM. *(i) If* A *is an ideal in* L*, then* A° *is a band in* $L^\sim$.

(ii) If B *is an ideal in* $L^\sim$*, then* ${}^\circ B$ *is an ideal in* L.

Proof. (i) Let $\phi \in A^\circ$ and $u \in A^+$ be given. If $f \in L$ is such that $|f| \leq u$, then $f \in A$ so $\phi(f) = 0$. Hence

$$|\phi|(u) = \sup \{|\phi(f)| : \ |f| \leq u\} = 0$$

by lemma 5.5(ii), so $|\phi| \in A^\circ$. Next, let $\psi \in L^\sim$, $\phi \in A^\circ$ be such that $|\psi| \leq |\phi|$. Since $|\phi| \in A^\circ$ it is clear that $|\psi| \in A^\circ$. Therefore,

$$0 \leq |\psi(u)| \leq |\psi|(u) = 0$$

holds for all $u \in A^+$, so $\psi \in A^\circ$. This shows that A° is an ideal in $L^\sim$. The proof that A° is even a band is similar to the proof of theorem 7.3 and therefore it is left to the reader.

(ii) Let $f \in {}^\circ B$ and $\phi \in B^+$ be given. If $\psi \in L^\sim$ is such that $|\psi| \leq |\phi|$, then $\psi \in B$, so $\psi(f) = 0$. Now observe that analogously to lemma 5.5(ii), we have

$$\phi(|f|) = \sup \{|\psi(f)| : \ |\psi| \leq \phi\},$$

so $f \in {}^\circ B$ implies $|f| \in {}^\circ B$. Finally, if $g \in L$, $f \in {}^\circ B$ are such that $|g| \leq |f|$, then $|f| \in {}^\circ B$, so clearly $|g| \in {}^\circ B$. Since $|\phi(g)| \leq \phi(|g|)$ holds for all ϕ in $L^{\sim +}$, it follows that $\phi(g) = 0$ for all $\phi \in B^+$, so $g \in {}^\circ B$. Thus ${}^\circ B$ is an ideal.

We note that although the annihilator of an ideal in L is always a band in $L^\sim$ it is not true in general that the inverse annihilator of an ideal in $L^\sim$ is a band in L. Even the inverse annihilator of a band in $L^\sim$ (which is an ideal in L by 8.2(ii)) is not necessarily a band in L. This will become clear from the remaining part of this section.

In the preceding section we have introduced the bands $L_c^{\sim}$ and $L_s^{\sim}$ of $L^{\sim}$. It would be interesting to know what the inverse annihilators of $L_c^{\sim}$ and $L_s^{\sim}$ look like in L. It turns out that not much can be said about $^{\circ}(L_c^{\sim})$. However $^{\circ}(L_s^{\sim})$ can be nicely represented in L. To this end, we define the following.

8.3. DEFINITION. An element $f \in L$ is called *absolutely continuous* if for any sequence $u_1, u_2, \ldots$ in L^+ satisfying $|f| \geq u_n$ for all n and $u_n \downarrow 0$ (from now on denoted by $|f| \geq u_n \downarrow 0$) we have $\lim \phi(u_n) = 0$ for all ϕ in $L^{\sim}$. The collection of all absolutely continuous elements in L is denoted by L^{α}.

To give the reader an idea what L^{α} looks like, we present the following exercise.

8.A. *Exercise*. (i) Let $L = C([0,1])$. Show that $L^{\alpha} = \{0\}$.

(ii) Let L be ℓ_{∞} (the collection of all sequences $(x_1, x_2, \ldots)$, $x_n \in \mathbb{R}$ for all n, such that $\sup |x_n| < \infty$, the partial ordering being coordinatewise). Show that $L^{\alpha} = c_0$ (the set of all null-sequences).

(iii) Let L be ℓ_1 (the collection of all sequences $(x_1, x_2, \ldots)$, $x_n \in \mathbb{R}$ for all n, such that $\Sigma |x_n| < \infty$, the partial ordering being coordinate wise). Show that $L^{\alpha} = L$.

8.4. THEOREM. $L^{\alpha} = {}^{\circ}(L_s^{\sim})$.

Proof. First, let $f \in L^{\alpha}$ and $\phi \in L^{\sim +}$ be given. If $0 \leq u_n \uparrow |f|$ in L^+, then $|f|-u_n \downarrow 0$. Hence $\phi(u_n) \uparrow \phi(|f|)$ since $f\in L^{\alpha}$. In view of theorem 7.8 we have

$$\phi_c(|f|) = \inf\{\lim \phi(u_n): \ 0 \le u_n \uparrow |f|\} = \phi(|f|),$$

so $\phi_s(|f|) = 0$. This holds for all $f\in L^{\alpha}$ and for all $\phi\in L^{\sim +}$. Especially, if $\phi\in L_s^{\sim +}$ we have $\phi = \phi_s$, so $\phi(|f|) = 0$ for all $f\in L^{\alpha}$. This shows that $|f|\in {}^{\circ}(L_s^{\sim})$ and therefore $f\in {}^{\circ}(L_s^{\sim})$ since ${}^{\circ}(L_s^{\sim})$ is an ideal. Thus L^{α} is a subset of ${}^{\circ}(L_s^{\sim})$.

Conversely, let $f\in {}^{\circ}(L_s^{\sim})$ be given. Then $|f|\in {}^{\circ}(L_s^{\sim})$, so $\phi(|f|) = 0$ for all $\phi\in L_s^{\sim}$. In particular $\phi(|f|) = 0$ for all $\phi\in L_s^{\sim +}$. Let now $u_1,u_2,\ldots$ be a sequence in L^+ such that $|f| \ge u_n \downarrow 0$. Then $\phi(u_n) = 0$ for all n and for all $\phi\in L_s^{\sim +}$. Thus, if $\phi\in L^{\sim +}$, $\phi = \phi_c+\phi_s$, we obtain

$$\lim \phi(u_n) = \lim \phi_s(u_n)+\lim \phi_c(u_n) = 0+\lim \phi_c(u_n) = 0.$$

It follows that $f\in L^{\alpha}$, so ${}^{\circ}(L_s^{\sim}) \subset L^{\alpha}$. This completes the proof.

The following corollaries are now obvious.

8.5. COROLLARY. L^{α} *is an ideal in* L.

8.6. COROLLARY. *The following assertions are equivalent.*

(*a*) $L^{\alpha} = L$.

(*b*) $L^{\sim} = L_c^{\sim}$.

(*c*) $L_s^{\sim} = \{0\}$.

Finally we note that it is now clear that the inverse annihilator of a band in $L^{\sim}$ is not necessarily a band in L. Indeed, let L be ℓ_{∞} (see 8.A). Then $L_s^{\sim}$ is a band in $L^{\sim}$, but ${}^{\circ}(L_s^{\sim}) = L^{\alpha} = c_0$. It is clear that c_0 is an ideal in ℓ_{∞}, but c_0 is not a band in ℓ_{∞}.

CHAPTER III. NORMED RIESZ SPACES

In section 6 we have already presented a theorem in which a normed Riesz space played a role. In this chapter we shall study normed Riesz spaces and their duals. We recall that a norm ρ on L is called a Riesz norm whenever $|f| \leq |g|$ in L implies that $\rho(f) \leq \rho(g)$, and that a normed Riesz space is always a Riesz space provided with a Riesz norm (see definition 6.8). If a normed Riesz space is normcomplete we shall call it a *Banach lattice*.

9. NORMED RIESZ SPACES

First we present some examples of normed Riesz spaces.

Examples.

9.A. Let $1 \leq p < \infty$. By ℓ_p we denote the collection of all sequences $(x_1, x_2, \ldots.)$ with $x_n \in \mathbb{R}$ for all n and such that

$$\|(x_1, x_2, \ldots..)\|_p = (\Sigma\ |x_n|^p)^{1/p} < \infty.$$

The partial ordering on ℓ_p is defined coordinatewise. Then ℓ_p is a Banach lattice.

9.B. Consider ℓ_∞ (see 8.A(ii)). For all $(x_1, x_2, \ldots.) \in \ell_\infty$ we define

$$\|(x_1, x_2, \ldots..)\|_\infty = \sup_n |x_n|.$$

Then ℓ_∞ is a Banach lattice. It is easy to see that c_0 is a normcomplete ideal of ℓ_∞, so c_0 is a Banach lattice in its own right.

9.C. Denote by c_{00} the collection of all sequences $(x_1, x_2, \ldots.)$, $x_n \in \mathbb{R}$ for all n, such that for each sequence $(x_1, x_2, \ldots.)$ of c_{00} there exists a natural number N such that $x_n = 0$ for all $n \geq N$. If c_{00} is ordered coordinatewise it becomes a Riesz space. Next, define

$$\|(x_1,x_2,\ldots\ldots)\|_\infty = \max\{|x_i|: i=1,2,\ldots.\}.$$

Then $\|.\|_\infty$ is a Riesz norm on c_{00}, so c_{00} is a normed Riesz space. It is obvious that c_{00} is a normed ideal of c_0 (and of ℓ_∞) and that c_{00} is not a Banach lattice.

9.D. Let (X,τ) be a topological Hausdorff space. Denote by $BC(X)$ the collection of all bounded continuous real-valued functions on X. If the elements of $BC(X)$ are partially ordered by setting $f \le g$ if $f(x) \le g(x)$ for all $x \in X$, then $BC(X)$ becomes a Riesz space. Moreover, if for all $f \in BC(X)$, we define

$$\rho(f) = \sup\{|f(x)|: \ x \in X\},$$

then $BC(X)$ is a Banach lattice.

We note that if X is compact, then $BC(X) = C(X)$, so $C(X)$ provided with the above *sup-norm* is a Banach lattice whenever X is compact.

THROUGHOUT THIS CHAPTER L *WILL BE A NORMED RIESZ SPACE. THE NORM ON* L *WILL BE DENOTED BY* ρ.

We collect some simple properties of Riesz norms in an exercise.

9.E. *Exercise*. (i) Any normed Riesz space is Archimedean.

(ii) $f \perp g$ in L implies $\rho(f+g) = \rho(f-g)$ (hint: use 1.4(viii)).

(iii) For all $f,g,f',g' \in L$ we have

$$\rho(f \vee g - f' \vee g') \le \rho(f-f') + \rho(g-g').$$

(iv) $\vee$ and $\wedge$ are continuous maps from $L \times L$ into L. The maps

$$f \mapsto f^+; \quad f \mapsto f^-; \quad f \mapsto |f|$$

are continuous from L into L^+.

(v) Each band in L is closed. In particular, if $D \subset L$ then D^d is closed. (Note that ideals are not necessarily closed. For instance, c_{00} is an ideal in ℓ_∞ which is not closed).

Finally, we observe that we have to distinguish between "normcompleteness" and "Dedekind completeness". Indeed,

9.F. *Example*. (i) The Riesz space $C([0,1])$ provided with the sup-norm is a Banach lattice but it is not Dedekind complete.

(ii) The normed Riesz space c_{00} of 9.C is Dedekind complete but not normcomplete.

We leave the straightforward verifications to the reader.

10. DUAL SPACES

First we observe that L now has two dual spaces, viz. $L^\sim$, its order dual and L^*, its norm dual. The norm in L^* will be denoted by ρ^*. It follows from the next theorem that $L^\sim$ and L^* are closely related.

10.1. THEOREM. *(i)* L^* *is an ideal of* $L^\sim$. *(In particular, it follows that* $L^{\sim+}$ *separates the points of* L*).*

(ii) L^* *is a Dedekind complete Banach lattice.*

(iii) If L *is a Banach lattice, then* $L^* = L^\sim$.

Proof. (i) Let $\phi \in L^*$ be given. For all $u \in L^+$ we have

$$\sup\{|\phi(f)|:\ |f| \le u\} \le \sup\{\rho^*(\phi)\rho(f):\ |f| \le u\} = \rho^*(\phi)\rho(u),$$

so ϕ is order bounded. Hence L^* is a linear subspace of $L^\sim$. Furthermore if $\phi \in L^*$, $\psi \in L^\sim$ are such that $|\psi| \le |\phi|$, then for all $f \in L$ we have

$$|\psi(f)| \le |\psi|(|f|) \le |\phi|(|f|) = \sup\{|\phi(g)|:\ |g| \le |f|\} \le$$

$$\sup\{\rho^*(\phi)\rho(g): \ |g| \le |f|\} = \rho^*(\phi)\rho(f),$$

so $\psi \in L^*$. It also follows that $\rho^*(\psi) \le \rho^*(\phi)$, so ρ^* is a Riesz norm on L^* and L^* is an ideal of $L^\sim$.

(ii) By theorem 5.6 $L^\sim$ is Dedekind complete, so by 4.B(ii) any ideal of $L^\sim$ is Dedekind complete. In particular this implies that L^* is Dedekind complete (by part (i)). The rest is obvious.

(iii) Assume that L is a Banach lattice. To show that $L^* = L^\sim$ we argue by contradiction. Therefore assume that there exists a $\phi \in L^{\sim +}$ such that $\phi \notin L^*$. Then there exists a sequence $f_1, f_2, \ldots$ in L such that $\rho(f_n) \le 1$, but

$$|\phi(f_n)| \ge 4^n$$

for all n. Since

$$\Sigma\ \rho(2^{-n}|f_n|) < \infty,$$

and since L is normcomplete, it follows that $f = \Sigma 2^{-n}|f_n|$ exists in L. For all n we now have $f \ge 2^{-n}|f_n|$, so

$$\phi(f) \ge 2^{-n}\phi(|f_n|) \ge 2^{-n}|\phi(f_n)| \ge 2^n,$$

which is the desired contradiction.

We have shown that L^* is a Banach lattice. Hence, the norm bidual L^{**} of L is a Banach lattice as well. From corollary 6.3 it follows that L can be considered as a linear subspace of L^{**} (and hence also as a linear subspace of $L^{*\sim} = L^{**}$) under the canonical map $J: L \to L^{**}$. The following theorem gives more information.

10.2. THEOREM. *The map* J *is an isometrical Riesz isomorphism from* L *onto a Riesz subspace of* L^{**}. *(Thus any normed Riesz space can be considered as a Riesz subspace of a Dedekind complete Banach lattice).*

Proof. It is clear that J is injective. Hence, it suffices to show that

$$J(f \vee g) = (Jf) \vee (Jg)$$

holds for all $f,g \in L$ (since in that case $J(L)$ is a Riesz subspace of L^{**} and J becomes a Riesz isomorphism from L onto $J(L)$). Since

$$f \vee g = f+(g-f)^+$$

for all $f,g \in L$ it is already sufficient to show that $J(f^+) = (Jf)^+$ holds for all $f \in L$. Therefore, let $f \in L$ be given. For all $\phi \in L^{*+}$ we have

$$(J(f^+))(\phi) = \phi(f^+) \geq 0,$$

so $J(f^+) \geq 0$. Also, for all $\phi \in L^{*+}$ we have

$$(J(f^+-f))(\phi) = \phi(f^+)-\phi(f) \geq 0,$$

so $J(f^+)-J(f) \geq 0$. This shows that

$$J(f^+) \geq (Jf) \vee 0 = (Jf)^+.$$

For the converse inequality, let $f \in L$ and $\omega \in L^{*+}$ be given. Define

$$K = \{af^+ + bf^- : \ a,b \in \mathbf{R}\}.$$

Then K is a Riesz subspace of L. Furthermore, set

$$\sigma(g) = \omega(|g|)$$

for all $g \in L$. Then σ is a Riesz semi-norm on L. Applying theorem 6.6 it follows that there exists a $\psi \in L^{\sim +}$ such that

$$\psi(af^+ + bf^-) = \omega(af^+)$$

for all $a,b \in \mathbf{R}$ and such that $|\psi(g)| \leq \sigma(g) = \omega(|g|)$ for all $g \in L$. Hence

$$(J(f^+))(\omega) = \omega(f^+) = \psi(f^+-f^-) = \psi(f) = (Jf)(\psi) \leq$$
$$(Jf)^+(\psi) \leq (Jf)^+(\omega).$$

This holds for all $\omega \in L^{*+}$, so $J(f^+) \leq (Jf)^+$. Thus the proof is complete.

The following exercise shows that L is in general not a "very nice" Riesz subspace of L^{**}.

10.A. *Exercise*. Let L be $C([0,1])$ provided with the sup-norm. Show that

L has a subset $\{f_n: n=1,2,....\}$ such that $f = \sup\{f_n: n=1,2,.....\}$ exists in L and such that $\theta = \sup\{Jf_n: n=1,2,....\}$ exists in L^{**} but $Jf \neq \theta$ (in fact $Jf > \theta$).

Hint: Let $f_n(x) = x^{1/n}$ for all $x \in [0,1]$ $(n=1,2,....)$. Then $f = \chi_{[0,1]}$. Next, define for all $\phi \in L^*$

$$\theta(\phi) = \lim \phi(f_n).$$

Show that $\theta \in L^{**}$, that $\theta = \sup\{Jf_n: n=1,2,....\}$ (in L^{**}) and that $\theta <$ Jf.

Remark. In particular it follows that a Banach lattice L is in general not an ideal in L^{**} (after identification).

Next we state in an exercise an important property for the norm of a positive bounded linear functional on L.

10.B. *Exercise*. Let $\phi \in L^{*+}$ be given. Show that

$$\rho^*(\phi) = \sup\{\phi(f): f \in L^+, \rho(f) \leq 1\}.$$

(Hence, the norm of ϕ is completely determined by the behavior of ϕ on the positive cone L^+).

Finally we consider the case that a Banach lattice carries several Riesz norms.

10.3. THEOREM. (*i*) *Let* X *be a subset of* L^+. *Then* X *is normbounded if and only if every element of* L^* *is bounded on* X *(i.e., for every* $\phi \in L^*$ *there exists a number* M *such that* $|\phi(f)| \leq M$ *for all* $f \in X$*).*

(*ii*) *Let* ρ_1 *and* ρ_2 *be Riesz norms on* L *and assume that* L *is* ρ_2-*complete. Then there exists a number* C *such that* $\rho_1(f) \leq C\rho_2(f)$ *for all* $f \in L$. *If* L *is both* ρ_1-*complete and* ρ_2-*complete, then* ρ_1 *and* ρ_2 *are* *equivalent norms.*

Proof. (i) If X is normbounded there exists a number K such that $\rho(f) \leq K$ for all $f \in X$. For every $\phi \in L^*$ we have $|\phi(f)| \leq \rho^*(\phi)K$ $(f \in X)$. Thus every element of L^* is bounded on X.

Conversely, suppose X is not normbounded: We construct a $\phi \in L^*$ that is unbounded on X. There exist $f_1, f_2, \ldots \in X$ for which $\rho(f_n) \geq 4^n$ $(n=1,2, \ldots.)$. By theorem 6.9, for each n there exists a $\phi_n \in L^*$ such that

$$\phi_n(f_n) = 1 \;;\quad \rho^*(\phi_n) = (\rho(f_n))^{-1} \leq 4^{-n}.$$

Then the series $\Sigma\, 2^n |\phi_n|$ is normconvergent in L^*. Let ϕ be its sum. For each $n \in \mathbb{N}$,

$$\phi(f_n) \geq 2^n |\phi_n|(f_n) \geq 2^n \phi_n(f_n) = 2^n.$$

Thus ϕ is not bounded on X.

(ii) For $i = 1,2$ we denote by L_i the Riesz space L under the norm ρ_i. Both dual spaces L_1^* and L_2^* are subsets of $L^\sim$ (theorem 10.1(ii)), and even $L_2^* = L^\sim$ (theorem 10.1(iii)). Thus $L_1^* \subset L_2^*$. By part (i) it follows that the ρ_2-bounded set $\{f \in L^+ : \rho_2(f) \leq 1\}$ is ρ_1-bounded. This means that there exists a number C such that $\rho_1(f) \leq C$ for every $f \in L^+$ for which $\rho_2(f) \leq 1$. As $\rho_i(|g|) = \rho_i(g)$ $(g \in L,\ i=1,2)$ it follows easily that $\rho_1(f) \leq C\rho_2(f)$ for all $f \in L$.

The second part of (ii) is a consequence of the first part.

11. BOUNDED INTEGRALS AND BOUNDED SINGULAR FUNCTIONALS

As shown in section 7, the order dual $L^\sim$ of L can be decomposed into the band of integrals $L_c^\sim$ and the band of singular functionals $L_s^\sim$. Setting

$$L_c^* = L_c^\sim \cap L^* ;\quad L_s^* = L_s^\sim \cap L^*$$

it follows from the fact that L^* is an ideal of $L^\sim$ that L_s^* and L_c^* are now ideals in $L^\sim$ and bands in L^*. Furthermore, it is trivial to see, but important to observe that L_c^* and L_s^* are Dedekind complete Banach lattices in their own right and that $L^* = L_c^* + L_s^*$.

We recall that L^α is the ideal of L consisting of all absolutely continuous elements and that $L^\alpha = {}^\circ(L_s^\sim)$ (theorem 8.4). We shall compute now the inverse annihilator ${}^\circ(L_s^*)$ of L_s^*. Note already that since $L_s^* \subset L_s^\sim$ it follows that ${}^\circ(L_s^*) \supset {}^\circ(L_s^\sim) = L^\alpha$.

11.1. DEFINITION. An element $f \in L$ is said to have an *absolutely continuous norm* if $|f| \geq u_n \downarrow 0$ implies $\lim \rho(u_n) = 0$. The collection of all elements of L having an absolutely continuous norm is denoted by L^a. If $L^a = L$, then ρ is said to be an *absolutely continuous norm.*

The following theorem shows that there exists a characterization of elements in L^a similar to the definition of elements in L^α.

11.2. THEOREM. *Let* $f \in L$ *be given. Then* $f \in L^a$ *if and only if* $|f| \geq u_n \downarrow 0$ *implies* $\lim \phi(u_n) = 0$ *for all* $\phi \in L^*$.

Proof. (a) Assume that $f \in L^a$. Let $|f| \geq u_n \downarrow 0$ and let $\phi \in L^*$ be given. Then

$$|\phi(u_n)| \leq \rho^*(\phi)\rho(u_n),$$

so $\lim \phi(u_n) = 0$.

(b) Consider the Banach lattice ℓ_∞ (provided with the sup-norm $\|.\|_\infty$). Let c be the Riesz subspace of ℓ_∞ consisting of all convergent sequences. Next, define

$$\omega'\{(x_1,x_2,\ldots\ldots)\} = \lim x_n$$

for all $(x_1,x_2,\ldots\ldots)\in c$. It is clear that $\omega'\in c^{*+}$ and that $\|\omega'\| = 1$. It follows now by theorem 6.9 that there exists an element $\omega\in\ell_\infty^{*+}$ such that $\|\omega\| = 1$ and such that $\omega = \omega'$ on c.

Next, let $f\in L$ be such that $|f| \geq u_n \downarrow 0$ implies $\phi(u_n) \to 0$ for all $\phi\in L^*$. Suppose that $f\notin L^a$. Then there exists a sequence $u_1,u_2,\ldots\ldots$ in L^+ such that

$$|f| \geq u_n \downarrow 0 \; ; \quad \rho(u_n) \geq s > 0 \quad (n=1,2,\ldots.).$$

Now, there exist $\phi_n\in L^{*+}$ such that $\rho^*(\phi_n) = 1$ and such that $\phi_n(u_n) = \rho(u_n)$ for all n. Note that if $g\in L$, then $|\phi_n(g)| \leq \rho(g)$ for all n, hence, we are allowed to define

$$\phi(g) = \omega(\phi_1(g),\phi_2(g),\ldots\ldots)$$

for all $g\in L$, since $(\phi_1(g),\phi_2(g),\ldots\ldots)\in\ell_\infty$. It is clear that $\phi\in L^{\sim+}$. Moreover,

$$|\phi(g)| = |\omega(\phi_1(g),\phi_2(g),\ldots\ldots)| \leq$$
$$\omega(|\phi_1(g)|,|\phi_2(g)|,\ldots\ldots) \leq \omega(\rho(g),\rho(g),\ldots\ldots) = \rho(g),$$

so $\phi\in L^{*+}$ and $\rho^*(\phi) \leq 1$. Next, fix $k\in\mathbb{N}$. If $n \geq k$, then

$$\phi_n(u_k) \geq \phi_n(u_n)$$

since $u_n \downarrow 0$ and since $\phi_n \geq 0$. Thus we obtain

$$\phi(u_k) = \omega(\phi_1(u_k),\phi_2(u_k),\ldots\ldots) \geq$$
$$\omega(\phi_1(u_k),\ldots.,\phi_k(u_k),\phi_{k+1}(u_{k+1}),\ldots\ldots) =$$
$$\omega(\phi_1(u_k),\ldots.,\rho(u_k),\rho(u_{k+1}),\ldots\ldots) \geq$$
$$\omega(\phi_1(u_k),\ldots.,s,s,\ldots\ldots) = s.$$

Hence $\lim \phi(u_k) \geq s > 0$, which is a contradiction. Hence $f\in L^a$.

Remark. Using a well-known theorem of S. Mazur it is possible to give a shorter proof of the preceding theorem. However, it seems better to us to present a proof based on the Riesz space theory we have already developed.

As an immediate corollary of the preceding theorem we obtain

11.3. COROLLARY. *(i)* $L^{\alpha} \subset L^{a}$.

(ii) If L *is a Banach lattice, then* $L^{a} = L^{\alpha}$.

Next, we state a theorem that is the norm analogon of theorem 8.4. Using theorem 11.2 the proof of this theorem is similar to the proof of theorem 8.4 as well, so the proof is omitted.

11.4. THEOREM. $L^{a} = {}^{\circ}(L_{s}^{*})$.

Observing that L_{s}^{*} is a band in L^{*} and that L^{*} is an ideal in $L^{\sim}$ it follows that L_{s}^{*} is an ideal in $L^{\sim}$. Also, since obviously ${}^{\circ}A$ is a norm closed linear subspace of L for every subset A of L^{*}, we obtain immediately

11.5. COROLLARY. L^{a} *is a norm closed ideal of* L.

11.6. COROLLARY. *The following assertions are equivalent.*

(*a*) $L^{a} = L$.

(*b*) $L^{*} = L_{c}^{*}$.

(*c*) $L_{s}^{*} = \{0\}$.

Proof. Obvious.

We note that there exist normed Riesz spaces L for which $L^{\alpha} \neq L^{a}$ (so L^{α} is a proper subset of L^{a}). An example will be presented in chapter 5 (example 25.F). However, although L is a Banach lattice implies $L^{\alpha}=L^{a}$, it is not true that $L^\alpha = L^a$ implies that L is a Banach lattice. Indeed,

11.A. *Example*. Consider the normed Riesz space c_{00} as defined in example 9.C. Then c_{00} is <u>not</u> a Banach lattice but we have $c_{00}^\alpha = c_{00}^a$. This can be seen as follows. Since $c_0^\alpha = c_0$ (see 8.A(ii)) we also have $c_{00}^\alpha = c_{00}$. Thus $c_{00}^\alpha \subset c_{00}^a$ implies $c_{00}^\alpha = c_{00}^a = c_{00}$.

Next, we consider again the embedding of L under J into L^{**} $(=L^{*\sim})$. As shown in theorem 10.2 L can be considered as a Riesz subspace of L^{**}. There can be proved more.

11.7. LEMMA. $J(L) \subset (L^{*})^{*}_{c}$ *(so* L *can be considered as a Riesz subspace of* $(L^{*})^{*}_{c}$*).*

Proof. Let $u \in L^+$ be given and define $u^{**} = J(u) \in L^{**}$. Furthermore, let $\phi_1, \phi_2, \ldots$ in L^* be such that $\phi_n \downarrow 0$. We have to show that $\lim u^{**}(\phi_n) = 0$, or equivalently, by the definition of J, that $\lim \phi_n(u) = 0$. This is however clear from the fact that $\phi_n \downarrow 0$.

In exercise 10.A we have already observed that suprema (or infima) of countable subsets of L are not necessarily preserved under the embedding J. We present a result dealing with this problem.

11.8. THEOREM. *The following assertions are equivalent.*

(*a*) *Suprema and infima of countable systems in* L *are preserved under* J.

(*b*) $L = L^a$.

(*c*) $L^* = L^*_c$.

Proof. (i) (b) $\Leftrightarrow$ (c) is shown in corollary 11.6.

(ii) (a) $\Rightarrow$ (c). Let $u_n \downarrow 0$ in L. Then $J(u_n) \downarrow 0$ in $(L^*)^*_c$, so

$$\lim \phi(u_n) = 0$$

for all $\phi \in L^*$. This shows that $L^* \subset L^*_c$, so $L^* = L^*_c$.

(iii) (c) $\Rightarrow$ (b). Let first $u_n \uparrow u$ in L. Then $J(u_1) \le J(u_2) \le \ldots \le J(u)$ in L^{**}, so $J(u_n) \uparrow u''$ for some $u'' \in L^{**}$ by the Dedekind completeness of L^{**}. It follows that $\phi(u_n) \uparrow u''(\phi)$ for all $\phi \in L^{*+}$. On the other hand, by assumption, we have $\phi(u_n) \uparrow \phi(u)$ for all $\phi \in L^{*+}$, so $u''(\phi) = \phi(u)$ for all $\phi \in L^*$. This shows that $u'' = J(u)$, i.e. $J(u_n) \uparrow J(u)$ in L^{**}. The same holds for decreasing sequences. Now, if $u_1, u_2, \ldots$ is a not necessarily monotone sequence in L with $u = \sup u_n$ in L and if we set $v_n = \sup \{u_1, \ldots, u_n\}$ then $v_n \uparrow u$, so $J(v_n) \uparrow J(u)$. Hence

$$J(u) = \sup_n J(v_n) = \sup_n \{J(u_1) \vee \ldots \vee J(u_n)\} = \sup_n J(u_n).$$

Finally we derive some properties of reflexive Riesz spaces.

11.9. LEMMA. *Assume that* L *is reflexive, i.e.,* $J(L) = L^{**}$. *Then*

(*i*) $L^a = L$ *(equivalently* $L^*_c = L^*$*),*

(*ii*) $(L^*)^a = L^*$ *(equivalently* $(L^*)^*_c = L^{**}$ *and even* $L^{**} = (L^*_c)^*_c$ *by (i)),*

(*iii*) *if* D *is a non-empty subset of* L^+ *such that* $u_1, u_2 \in D$ *implies* $u_1 \vee u_2 \in D$ *and such that* $\sup \{\rho(u) : u \in D\} < \infty$, *then* $\sup D$ *exists in* L^+. *(In particular, if* $0 \le u_1 \le u_2 \le \ldots$ *in* L^+ *is such that* $\sup \rho(u_n) < \infty$, *then* $u_n \uparrow u_0$ *for some* $u_0 \in L^+$*).*

Proof. (i) Since $J(L) = L^{**}$ J preserves suprema and infima of arbitrary subsets of L, so $L^a = L$ by theorem 11.8.

(ii) This follows from (i) since L^* is reflexive as well.

(iii) Set $D'' = \{J(u): u \in D\}$. Then $u_1'', u_2'' \in D''$ implies $u_1'' \vee u_2'' \in D''$ and $\sup \{\rho(u): u \in D\} = \sup \{\rho^{**}(u''): u'' \in D''\} < \infty$. For brevity, define

$$\alpha = \sup \{\rho(u): u \in D\}.$$

Clearly we have $\sup \{\phi(u): u \in D\} \leq \alpha\rho^*(\phi)$ for all $\phi \in L^{*+}$, thus, defining

$$u_0''(\phi) = \sup \{\phi(u): u \in D\}$$

for all $\phi \in L^{*+}$ and $u_0''(\phi) = u_0''(\phi^+) - u_0''(\phi^-)$ for arbitrary $\phi \in L^*$, it follows by similar methods to that used in the proof of theorem 5.6 that $u_0'' \in L^{**+}$ and that $u_0'' = \sup D''$. Hence, if $u_0 \in L^+$ is such that $u_0'' = J(u_0)$, then

$$u_0 = \sup D.$$

It can be shown that the conditions (i), (ii) and (iii) of lemma 11.9 are not only necessary but also sufficient for L to be reflexive. However, the rather involved proof of this theorem is far beyond the scope of this book.

CHAPTER IV. REPRESENTATION THEOREMS

In the theory of Riesz spaces there are several theorems stating that Riesz spaces or Banach lattices of certain types are isomorphic to spaces of functions. A fundamental result in this direction is Yosida's Representation Theorem 13.11. We shall prove some of these theorems and give a few applications, both to the general Riesz space theory and, in Section 17, to the theory of Hermitian operators in a Hilbert space.

12. THE RIESZ SPACE $C(X)$

THROUGHOUT THIS SECTION, X *IS A COMPACT HAUSDORFF SPACE.*

$C(X)$ denotes the vector space of all continuous real-valued functions on X. Under pointwise ordering,

$$f \le g \quad \text{if} \quad f(x) \le g(x) \quad \text{for all} \quad x \in X,$$

$C(X)$ is an Archimedean Riesz space. (See Ex.1.G). We have

$$\begin{aligned} (f\vee g)(x) &= f(x)\vee g(x) \\ (f\wedge g)(x) &= f(x)\wedge g(x) \qquad (f,g\in C(X),\ x\in X) \\ |f|(x) &= |f(x)| \end{aligned}$$

$C(X)$ also carries a natural norm, the supremum-norm $\|\cdot\|_\infty$, defined by

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \qquad (f\in C(X))$$

Under this norm, $C(X)$ is a Banach lattice. (See 9.D).

We shall frequently use the following topological theorem.

12.1. URYSOHN'S LEMMA. *If* A *and* B *are closed disjoint subsets of* X*, then there exists a continuous* $f:X \to [0,1]$ *such that* $f\equiv 0$ *on* A *while* $f\equiv 1$ *on* B.

For a proof we refer the reader to [9].

12.2. THEOREM. *For every* $a\in X$ *define* $\phi_a : C(X) \to \mathbb{R}$ *by*

$$\phi_a(f) = f(a) \qquad (f\in C(X))$$

Then ϕ_a *is a Riesz homomorphism and* $\phi_a(\underline{1})=1$. *Conversely, for every Riesz homomorphism* $\phi : C(X) \to \mathbb{R}$ *with* $\phi(\underline{1})=1$ *there exists a unique* $a\in X$ *such that* $\phi=\phi_a$.

Proof. The first part of the theorem is obvious. Now let ϕ be a Riesz homomorphism $C(X) \to \mathbb{R}$ and $\phi(\underline{1})=1$. Suppose that for every $a\in X$ there exists an $f_a\in C(X)$ for which $\phi(f_a)\neq\phi_a(f_a)$: we derive a contradiction. For each a, set $g_a = |f_a-\phi(f_a)\underline{1}|$. Then $g_a(a) = |\phi_a(f_a)-\phi(f_a)| > 0$ while $\phi(g_a) = |\phi(f_a)-\phi(f_a)\phi(\underline{1})| = 0$. Because X is compact and each g_a is continuous, there exist $a_1,\dots,a_m\in X$ such that

$$X = \bigcup_i \{x\in X : g_{a_i}(x) > 0\}.$$

Now let $g = g_{a_1}\vee\dots\vee g_{a_m}$. Then $g(x)>0$ for all $x\in X$. As g is continuous and X is compact, there must exist a $\delta>0$ such that $g(x)\geq\delta$ for all $x\in X$. Then $\phi(g) \geq \phi(\delta\underline{1}) = \delta > 0$. On the other hand,

$$\phi(g) = \phi(g_{a_1})\vee\phi(g_{a_2})\vee\dots\vee\phi(g_{a_m}) = 0$$

and we have a contradiction.

Thus, there must indeed exist an $a\in X$ such that $\phi=\phi_a$. The uniqueness of a follows from Urysohn's Lemma: if $x,y\in X$ and $x\neq y$, then there exists an $f\in C(X)$ such that $f(x)=0$ and $f(y)=1$, i.e. $\phi_x(f)=0$, $\phi_y(f)=1$.

If ϕ is a continuous map of a compact Hausdorff space Y into a compact Hausdorff space X, then $\Phi : f \mapsto f\circ\phi$ is a Riesz homomorphism of $C(X)$ into $C(Y)$ with $\Phi(\underline{1})=\underline{1}$. Conversely, we have the following.

12.3. COROLLARY. *Let* X *and* Y *be compact Hausdorff spaces and* Φ *a Riesz homomorphism of* $C(X)$ *into* $C(Y)$ *such that* $\Phi(\underline{1})=\underline{1}$. *Then there exists a unique continuous map* $\phi:Y \to X$ *such that*

$$\Phi f = f\circ\phi \qquad (f\in C(X)).$$

Proof. For every $y\in Y$, by applying the preceding theorem to the map $f \mapsto (\Phi f)(y) \quad (f\in C(X))$ we see that there exists a unique element $\phi(y)$ of X such that $(\Phi f)(y) = f(\phi(y))$ for every $f\in C(X)$. Thus we obtain a $\phi:Y \to X$ with the property

$$\Phi f = f\circ\phi \qquad (f\in C(X))$$

It remains to prove that ϕ is continuous. Let $U \subset X$ be open and let $b \in \phi^{-1}(U) \subset Y$. By Urysohn's Lemma (12.1) there exists an $f\in C(X)$ such that $f(\phi(b))=1$ while f vanishes on $X\backslash U$. Setting $g=\Phi f$ we have $g\in C(Y)$, $g(b)=f(\phi(b))=1$ and g vanishes on $Y\backslash\phi^{-1}(U)$. Now $\{y\in Y : g(y)>0\}$ is open in Y and $b \in \{y\in Y : g(y)>0\} \subset \phi^{-1}(U)$. Hence, $\phi^{-1}(U)$ is open in X and therefore ϕ is continuous.

12.4. COROLLARY. (Banach-Stone). *Let* X *and* Y *be compact Hausdorff spaces. If* $C(X)$ *and* $C(Y)$ *are Riesz isomorphic, then* X *and* Y *are homeomorphic.*

Proof. Let Φ be a Riesz isomorphism of $C(X)$ onto $C(Y)$, let $u = \Phi\underline{1}$ and take $v\in C(X)$ such that $\Phi v = \underline{1}$. There exists a positive number c for which $v \le c\underline{1}$. Then $\underline{1} = \Phi v \le c\Phi\underline{1} = cu$, so $u(y)>0$ for every $y\in Y$. The formula

$$(\Psi f)(y) = \frac{(\Phi f)(y)}{u(y)} \qquad (f\in C(X),\ y\in Y)$$

can be used to define a Riesz isomorphism Ψ of $C(X)$ onto $C(Y)$ such that $\Psi\underline{1} = \underline{1}$. By the above corollary there exist continuous $\phi_1:Y \to X$ and

$\phi_2 : X \to Y$ with the properties

$$\Psi f = f \circ \phi_1 \qquad (f \in C(X))$$

and

$$\Psi^{-1} g = g \circ \phi_2 \qquad (g \in C(Y).$$

If $x \in X$, then for every $f \in C(X)$, $f(x) = (\Psi^{-1}\Psi f)(x) = (f \circ (\phi_1 \circ \phi_2))(x)$, so (by Urysohn's Lemma) $x = (\phi_1 \circ \phi_2)(x)$. Similarly, $y = (\phi_2 \circ \phi_1)(y)$ for each $y \in Y$. Thus, ϕ_1 and ϕ_2 are each other's inverses. Then X and Y are homeomorphic.

This corollary is the key to several other theorems stating that two compact Hausdorff spaces X and Y are homeomorphic as soon as $C(X)$ and $C(Y)$ are in some sense isomorphic.

12.5. DEFINITION. An *algebra* (over $\mathbb{R}$) is a vector space V provided with a multiplication $V \times V \to V$ (which we denote by juxtaposition) such that for every $f \in V$ the maps $g \mapsto fg$ and $g \mapsto gf$ are linear.

For every topological space Z, $C(Z)$ is an algebra under pointwise operations.

Two algebras, V and W, are said to be *isomorphic* if there exists a linear bijection $\Phi : V \to W$ such that $\Phi(fg) = (\Phi f)(\Phi g)$ for all $f, g \in V$.

12.6. COROLLARY. *Let* X *and* Y *be compact Hausdorff spaces.*

(i) *If* $C(X)$ *and* $C(Y)$ *are isomorphic algebras, then* X *and* Y *are homeomorphic.*

(ii) *If* $C(X)$ *and* $C(Y)$ *are isomorphic as Banach spaces, then* X *and* Y *are homeomorphic.*

Proof. (i) Let Φ be a linear bijection $C(X) \to C(Y)$ that preserves the multiplication. An element of $C(X)$ or of $C(Y)$ is ≥ 0 if and only if it is a square. Hence, for $f \in C(X)$ we have $f \geq 0$ if and only if $\Phi f \geq 0$. Now apply Cor.12.4.

(ii) Let Φ be a linear bijection $C(X) \to C(Y)$ that preserves the norm. We first make a linear isometry Ψ of $C(X)$ onto $C(Y)$ for which $\Psi\underline{1}=\underline{1}$. Let $u = \Phi\underline{1}$. Then $\|u\|_\infty=1$, so $-\underline{1}\leq u\leq \underline{1}$. Set $h = \frac{1}{2}(u^2-\underline{1})$. Then we have $u+h = \frac{1}{2}(u+\underline{1})^2-\underline{1}$, whence $\|u+h\|_\infty \leq 1$ and $\|\underline{1}+\Phi^{-1}h\|_\infty = \|\Phi^{-1}(u+h)\|_\infty \leq 1$. Similarly, $u-h = \underline{1}-\frac{1}{2}(u-\underline{1})^2$, $\|u-h\|_\infty \leq 1$ and $\|\underline{1}-\Phi^{-1}h\|_\infty \leq 1$. However, if $\|\underline{1}+\Phi^{-1}h\|_\infty \leq 1$ and $\|\underline{1}-\Phi^{-1}h\|_\infty \leq 1$, then $\Phi^{-1}h=0$, so $h=0$, $u^2=\underline{1}$ and the only values taken by u are 1 and -1. Then we can define a linear isometry Ψ of $C(X)$ onto $C(Y)$ by

$$(\Psi f)(y) = \frac{(\Phi f)(y)}{u(y)} \qquad (f \in C(X),\ y \in Y)$$

and we obtain $\Psi\underline{1} = \underline{1}$.

For $f \in C(X)$ or $f \in C(Y)$ one has $f \geq 0$ if and only if

$$\| f - \|f\|_\infty \underline{1} \|_\infty \leq \|f\|_\infty$$

Hence, for $f \in C(X)$ one has $f \geq 0$ if and only if $\Psi f \geq 0$. Apparently, Ψ is a Riesz isomorphism and again we can apply 12.4.

12.7. Take $f,g \in C(X)$. We have $f \perp g$ if and only if for every $x \in X$ either $f(x)=0$ or $g(x)=0$, i.e. if and only if $g\equiv 0$ on $\{x \in X : f(x)\neq 0\}$.

Let $D \subset C(X)$ and set $U = \{x \in X :$ there is an $f \in D$ with $f(x)\neq 0\}$. Then clearly $D^d = \{g \in C(X) : g\equiv 0$ on $U\}$. Since $C(X)$ is Archimedean, every band of $C(X)$ is of the form D^d (Th.4.3). Thus, for every band A of $C(X)$ there exists an open $U \subset X$ with $A = \{g \in C(X) : g\equiv 0$ on $U\}$.

Conversely, every open subset of X determines a band:

12.A. *Exercise*. For every open subset U of X, $\{g\in C(X) : g\equiv 0 \text{ on } U\}$ is a band in $C(X)$.

However, in general the above does not give us a one-to-one correspondence between the open subsets of X and the bands of $C(X)$. In fact, it is clear that any two open subsets of X that have the same closure determine the same band. (E.g., the open subsets $[0,\frac{1}{2})\cup(\frac{1}{2},1]$ and $[0,1]$ of $[0,1]$).

In the following exercise we shall see that for every open $U \subset X$ there exists a largest open subset of X that has the same closure as U.

12.B. *Exercise*. For every open subset U of X let $U^{\square}$ denote the interior of $\overline{U}$. For every open $U \subset X$ the following is true.

(i) $U^{\square}$ is an open set whose closure is $\overline{U}$.

(ii) $U^{\square}$ contains every open subset of X that is contained in $\overline{U}$.

(iii) $U^{\square\square} = U^{\square}$.

12.8. DEFINITION. An open subset U of X is called *regular* if $U^{\square} = U$.

By Ex.12.B(iii), for every open $U \subset X$ the open set $U^{\square}$ is regular. In fact, $U^{\square}$ is the smallest regular open set containing U. (If W is a regular open set and if $W \supset U$, then $W = W^{\square} \supset U^{\square}$).

An example of a non-regular open subset of $[0,1]$ is $[0,\frac{1}{2})\cup(\frac{1}{2},1]$. Since $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$ are regular, we see that a union of two regular open sets may fail to be regular.

12.C. *Exercise*. The regular open subsets of X, ordered by inclusion, form a complete Boolean algebra $\mathcal{A}$. For $U_1, U_2 \in \mathcal{A}$ we have

$$U_1 \wedge U_2 = U_1 \cap U_2, \quad U_1 \vee U_2 = (U_1 \cup U_2)^{\square}, \quad U_1' = U_1^{c}$$

It can now be proved that the formula $U \mapsto \{g \in C(X) : g \equiv 0 \text{ on } U\}$ yields a one-to-one correspondence between the regular open sets in X and the bands of $C(X)$. Now both the regular open sets and the bands form Boolean algebras (12.C and 4.6). Unfortunately, the correspondence we have just mentioned reverses the ordering: large open sets will yield small bands. In order to obtain an isomorphism of Boolean algebras we introduce a complementation in the following way.

12.9. THEOREM. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be the Boolean algebra of all regular open subsets of X and let $\mathcal{B}[C(X)]$ be the Boolean algebra of all bands of $C(X)$. (See Th.4.6). The formula*

$$U \mapsto \{g \in C(X) : g \equiv 0 \text{ on } X \backslash \overline{U}\}$$

defines a bijection $\mathcal{A} \to \mathcal{B}[C(X)]$ which is a lattice isomorphism.

Proof. For $U \in \mathcal{A}$ let $\phi(U) = \{g \in C(X) : g \equiv 0 \text{ on } X \backslash \overline{U}\}$: then $\phi(U) \in \mathcal{B}[C(X)]$. (See Ex.12.A). For every $A \in \mathcal{B}[C(X)]$, as we already know, there exists a regular open $W \subset X$ such that $A = \{g \in C(X) : g \equiv 0 \text{ on } W\} = \phi(W^{c})$. Thus, ϕ is surjective $\mathcal{A} \to \mathcal{B}[C(X)]$.

Trivially, if $U_1, U_2 \in \mathcal{A}$ and $U_1 \subset U_2$, then $\phi(U_1) \subset \phi(U_2)$. On the other hand, if $U_1, U_2 \in \mathcal{A}$ and $U_1 \not\subset U_2$, then $U_1 \not\subset \overline{U_2}$ (12.B(ii)), so $X \backslash \overline{U_2} \not\subset X \backslash U_1$. By Urysohn's Lemma there exists an $f \in C(X)$ that vanishes on $X \backslash U_1$ but not on $X \backslash \overline{U_2}$. Then $f \in \phi(U_1)$, $f \notin \phi(U_2)$, so $\phi(U_1) \neq \phi(U_2)$.

Thus, for $U_1, U_2 \in \mathcal{A}$ we have $U_1 \subset U_2$ if and only if $\phi(U_1) \subset \phi(U_2)$. Consequently, ϕ is injective and a lattice isomorphism.

Now assume that $C(X)$ is Dedekind complete.

Let U be a regular open subset of X and let A be the corresponding band of $C(X)$, i.e. $A = \{g \in C(X) : g \equiv 0 \text{ on } U^c\}$. By the above theorem, $A^d = \{g \in C(X) : g \equiv 0 \text{ on } U^{cc}\} = \{g \in C(X) : g \equiv 0 \text{ on } U\}$. The Dedekind completeness of $C(X)$ implies (Th.4.5(ii)) that $\underline{1} \in A+A^d$. It follows that $X = U^c \cup U = (X \backslash \overline{U}) \cup U$. But then $\overline{U} = U$, so U is not only open but also closed.

12.10. DEFINITION. A subset of X is said to be *clopen* if it is both open and closed.

Clearly every clopen set is a regular open set.

12.11. DEFINITION. X is called *extremally disconnected* if every regular open subset of X is clopen.

We have just proved that *if* $C(X)$ *is Dedekind complete, then* X *is extremally disconnected*. In Theorem 12.16 we shall see that the converse is also true. For that we need more knowledge about extremally disconnected spaces.

Let X be extremally disconnected. If a,b are points of X and $a \neq b$, there exist open sets $U, V \subset X$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Setting $W = U^{\square}$ (see 12.B) we see: if X is extremally disconnected, then for any two distinct points, a and b, of X there exists a clopen set $W \subset X$ such that $a \in W$, $b \notin W$.

In particular, X does not contain any connected subset that consists of more than one element. This observation may serve as a partial explanation of the term "extremally disconnected".

We can, in fact, prove more. Let X be any compact Hausdorff space. Let $a \in X$ and let S be a neighbourhood of a. There exists an open set U such that $a \in U \subset \overline{U} \subset S$. Consequently, $a \in U^{\square} \subset S$. Thus, the regular open sets form a base for the topology of X. In the terminology of 12.12, *if* X *is extremally disconnected, then it is zerodimensional.*

12.12. DEFINITION. X is called *zerodimensional* if the clopen subsets of X form a base for the topology of X.

12.D. *Exercise.* Let X be the subset $\{0\} \cup \{n^{-1} : n \in \mathbb{N}\}$ of $\mathbb{R}$. Under the usual topology, X is zerodimensional but not extremally disconnected.

12.E. *Exercise.* (Unpleasant behaviour of extremally disconnected spaces). If X is extremally disconnected and metrizable, then X is a finite set. (Hint. Take $a \in X$: it suffices to prove that $\{a\}$ is open. If $\{a\}$ is not open, there exist clopen sets $W_1 \supset W_2 \supset \ldots$ whose intersection is $\{a\}$ while $W_1 \neq W_2 \neq \ldots$ Let $U = \bigcup\{W_n : n$ is even$\}$. Then $\overline{U} = U \cup \{a\}$. Derive a contradiction).

12.F. *Exercise.* The following conditions are equivalent.

(a) X is extremally disconnected.

(b) If U,V are mutually disjoint open subsets of X, then $\overline{U} \cap \overline{V} = \emptyset$.

(c) Every open subset of X has clopen closure. (In the literature this property is usually chosen to define extremal disconnectedness).

12.13. DEFINITION. A real-valued function f on X is said to be *lower semicontinuous* if for every $s\in\mathbb{R}$ the set $\{x\in X : f(x)>s\}$ is open: f is called *upper semicontinuous* if for every s the set $\{x\in X : f(x)<s\}$ is open. Instead of "semicontinuous" we often use the abbreviation "s.c.".

Clearly, a function is continuous if and only if it is both lower and upper semicontinuous.

12.G. *Exercise.* (i) The characteristic function of a subset V of X is lower s.c. if and only if V is open.

(ii) The sum of two lower s.c. functions is lower s.c.

(iii) Every lower s.c. function on X is bounded from below.

(iv) Let F be a set of lower s.c. functions on X such that for every $x\in X$, $\sup\{f(x) : f\in F\}$ is finite. Then the function $x\mapsto \sup\{f(x) : f\in F\}$ is lower s.c.

(v) Every lower s.c. function f_0 on X is pointwise supremum of a set F of continuous functions. (For F one may take $\{f\in C(X) : f\le f_0\}$).

12.14. DEFINITION. For any function $f:X\to\mathbb{R}$ that is bounded from below and for any $a\in X$ we define

$$f^{\downarrow}(a) = \sup\{\inf_{x\in U} f(x) : U \text{ is a neighbourhood of } a\}$$

Note that $(\inf f)\le f^{\downarrow}(a)\le f(a)$, so that $f^{\downarrow}(a)\in\mathbb{R}$. Thus, we have made a function $f^{\downarrow}:X\to\mathbb{R}$. Similarly, we define $f^{\uparrow}$ if f is bounded from above.

12.H. *Exercise.* Let f and g be functions $X\to\mathbb{R}$ that are bounded from below. Then

(i) $f^{\downarrow}\le f$.

(ii) $f^{\downarrow}$ is bounded from below; if fact, $\inf_{x\in X} f^{\downarrow}(x) = \inf_{x\in X} f(x)$.

(iii) if $f \le g$, then $f^{\downarrow} \le g^{\downarrow}$.

(iv) $f^{\downarrow}$ is lower s.c. It is the largest lower s.c. function that is $\le f$.

(v) f is lower s.c. if and only if $f^{\downarrow} = f$.

(vi) $f^{\downarrow\downarrow} = f^{\downarrow}$.

12.15. LEMMA. *Let* X *be extremally disconnected and let* $f:X \to \mathbb{R}$ *be bounded and lower s.c. Then* $f^{\uparrow}$ *is continuous.*

Proof. By Ex.12.H(iv), $f^{\uparrow}$ is upper s.c. In order to prove that it is also lower s.c., take $s \in \mathbb{R}$ and $W = \{x : f^{\uparrow}(x) > s\}$: we show W to be open. For every $\varepsilon>0$ let V_ε be the set $\{x : f(x) > s+\varepsilon\}$. This V_ε is open: then so is $\overline{V_\varepsilon}$, since X is extremally disconnected. (See Ex.12.F). We prove that $W = \bigcup_{\varepsilon>0} \overline{V_\varepsilon}$: then W must be open.

If $a \in W$, there is an $\varepsilon>0$ for which $f^{\uparrow}(a) > s+\varepsilon$. By the definition of $f^{\uparrow}(a)$, every neighborhood of a contains a point x with $f(x) > s+\varepsilon$, i.e. $a \in \overline{V_\varepsilon}$. Thus, $W \subset \bigcup_{\varepsilon>0} \overline{V_\varepsilon}$.

Conversely, let $\varepsilon>0$, $a \in \overline{V_\varepsilon}$. Then, by the definition of $f^{\uparrow}(a)$ we have $f^{\uparrow}(a) \ge s+\varepsilon$ and $a \in W$. Hence, $\bigcup_{\varepsilon>0} \overline{V_\varepsilon} \subset W$.

Now we can prove a result announced above.

12.16. THEOREM. (H.Nakano). $C(X)$ *is Dedekind complete if and only if* X *is extremally disconnected.*

Proof. We already have the "only if": see the lines following Def.12.11. Now assume that X is extremally disconnected. Let F be a non-empty subset of $C(X)^+$ having an upper bound in $C(X)$. Then we can define a bounded lower s.c. function f_0 on X by setting $f_0(x) = \sup\{f(x) : f \in F\}$. (See

Ex.12.G(iv)). Then $f_0^{\dagger} \in C(X)$, according to Lemma 12.15. It is now easy to see that, in the sense of the Riesz space $C(X)$, $f_0^{\dagger} = \sup F$. By Ex.4.A(c), $C(X)$ is Dedekind complete.

12.I. *Exercise*. (An infinite extremally disconnected space). We make a compact Hausdorff space X such that $C(X)$ is Riesz isomorphic to ℓ_∞. Then by Nakano's Theorem 12.16, X is extremally disconnected.

Let $A = [0,1]^{\mathbb{N}}$, $B = [0,1]^A$. Under the product topology, B is a compact Hausdorff space. Define $b: \mathbb{N} \to B$ by

$$b(n)(a) = a(n) \qquad (a \in A;\ n \in \mathbb{N})$$

The closure of $b(\mathbb{N})$ is a compact Hausdorff space X.

Every $f \in C(X)$ determines an element Φf of ℓ_∞ by

$$\Phi f = f \circ b.$$

In this way one obtains a Riesz isomorphism Φ of $C(X)$ onto ℓ_∞. (To prove surjectivity, take $a \in A$ and define $f \in C(X)$ by $f(x)=x(a)$ $(x \in X)$: then $\Phi f = a$. Thus, the range space of Φ contains A).

(For a more direct proof of the extremal disconnectedness of X, take a regular open subset U of X. Let u be the characteristic function of $\{n \in \mathbb{N} : b(n) \in U\}$. Then $u \in A$. Put $Y = \{x \in X : u(x)=1\}$. As

$$b(\mathbb{N}) \subset \{x \in B : x(u)=1 \text{ or } x(u)=0\},$$

it follows that for every $x \in X$ we either have $x(u)=1$ or $x(u)=0$, so that Y is clopen in X. Now both Y and U are regular open subsets of X, $b(\mathbb{N})$ is dense in X and $Y \cap b(\mathbb{N}) = U \cap b(\mathbb{N})$. Consequently, $U = Y$).

13. THE YOSIDA REPRESENTATION THEOREM

IN THE REST OF THIS BOOK L *IS AN ARCHIMEDEAN RIESZ SPACE.*

13.1. LEMMA. *If* $L \neq \{0\}$ *while* $\{0\}$ *and* L *are the only ideals of* L, *then* L *is Riesz isomorphic to* $\mathbb{R}$.

Proof. All we really have to prove is that L is one-dimensional. We are done if for any $f_1, f_2 \in L^+$ we can show f_1 and $f_1 \vee f_2$ to be linearly dependent.

Let $f, g \in L^+$, $0 < f \leq g$: it suffices to prove that $f = \alpha g$ for some $\alpha \in \mathbb{R}$. Let $\Lambda = \{\lambda \in \mathbb{R} : \lambda g \leq f\}$. Observe that $0 \in \Lambda$ and that $\lambda \in \Lambda$ as soon as $\lambda \leq \lambda'$ for some $\lambda' \in \Lambda$. As L is Archimedean there is a $\lambda \in \mathbb{R}$ such that $\lambda \notin \Lambda$. Then $\Lambda \subset (-\infty, \lambda)$. Now we can define $\alpha \in \mathbb{R}^+$ by $\alpha = \sup \Lambda$. For every $m \in \mathbb{N}$, $\alpha - \frac{1}{m} \in \Lambda$, so that $\alpha g - f \leq \frac{1}{m} g$: hence, $\alpha g - f \leq 0$ and $f - \alpha g \in L^+$. The principal ideal generated by $f - \alpha g$ is either $\{0\}$ or L. In the first case, $f - \alpha g = 0$, i.e. $f = \alpha g$. In the second case, by 2.C there must exist an $n \in \mathbb{N}$ such that $g \leq n(f - \alpha g)$, whence $\alpha + \frac{1}{n} \in \Lambda$ which contradicts the definition of Λ. Therefore, $f = \alpha g$.

13.2. DEFINITION. An ideal I in L is said to be *proper* if $I \neq L$. A *maximal ideal* is a proper ideal that is maximal among the proper ideals.

If M is a maximal ideal in L, then L/M is a Riesz space L_0. Let Φ denote the natural surjection $L \to L_0$. For any ideal I of L_0, $\Phi^{-1}(I)$ is an ideal in L that contains M. Then $\Phi^{-1}(I)$ is either M or L; so that I is either $\{0\}$ or L_0.

Before we can apply the above lemma to L_0 we have to show that L_0 is Archimedean. This is not hard to do. Let $f,g\in L_0^+$ be such that $mf \le g$ for all $m\in\mathbb{N}$. Set $J = \{h\in L_0 : |mh| \le g$ for all $m\in\mathbb{N}\}$. Then J is an ideal in L_0, so either $J=\{0\}$ or $J=L_0$. In the first case, trivially $f=0$; in the second case $g\in J$, hence, $g=0$ and $f=0$.

We have proved:

13.3. LEMMA. *If* M *is a maximal ideal in* L, *then* L/M *is Riesz isomorphic to* $\mathbb{R}$.

13.A. *Exercise*. (i) The kernel of any non-zero Riesz homomorphism $L \to \mathbb{R}$ is a maximal ideal.

(ii) Every maximal ideal of L is the kernel of a non-zero Riesz homomorphism $L \to \mathbb{R}$.

(iii) If ϕ and ψ are non-zero Riesz homomorphisms $L \to \mathbb{R}$ with the same kernel, then they are scalar multiples of each other.

13.B. *Example*. Let X be a compact Hausdorff space. For $a\in X$ define $M_a = \{f\in C(X) : f(a)=0\}$. Every M_a is a maximal ideal in $C(X)$. By the above and by Th.12.2, every maximal ideal is an M_a.

13.4. Let $\mathcal{M}$ denote the set of all maximal ideals of L. For any subset A of L^+ put

$$A^\Delta = \{M\in\mathcal{M} : A \subset M\}$$

Then

(i) $\emptyset = (L^+)^\Delta$, $\mathcal{M} = \emptyset^\Delta$.

(ii) If $(A_\sigma)_{\sigma\in\Sigma}$ is any family of subsets of L^+, then $\bigcap_{\sigma\in\Sigma} A_\sigma^\Delta = (\bigcup_{\sigma\in\Sigma} A_\sigma)^\Delta$.

(iii) If A and B are subsets of L^+ and if $C = \{f\wedge g : f\in A,\ g\in B\}$, then $A^\Delta\cup B^\Delta = C^\Delta$. In fact, the inclusion $A^\Delta\cup B^\Delta \subset C^\Delta$ is perfectly clear. Now take an $M\in\mathcal{M}$ such that $M\notin A^\Delta$ and $M\notin B^\Delta$: we prove that $M\notin C^\Delta$. There must exist $f\in A$ and $g\in B$ with $f\notin M$, $g\notin M$. By 13.A, M is the kernel of a Riesz homomorphism $\phi:L\to\mathbb{R}$. Then $\phi(f)\neq 0$ and $\phi(g)\neq 0$, and therefore $\phi(f\wedge g) = \phi(f)\wedge\phi(g) \neq 0$ and $f\wedge g \notin M$. But $f\wedge g \in C$. Consequently, $C \not\subset M$, i.e. $M\notin C^\Delta$.

By (i),(ii) and (iii) there exists a topology on $\mathcal{M}$ such that the sets A^Δ $(A \subset L^+)$ are just the closed subsets of $\mathcal{M}$.

13.5. DEFINITION. This topology is known as the *hull-kernel topology*. The topological space we have just made is called the *maximal ideal space* of L. We denote it by $\mathcal{M}(L)$ or by $\mathcal{M}$.

13.6. LEMMA. *$\mathcal{M}(L)$ is a Hausdorff space.*

Proof. Let $M,N\in\mathcal{M}$, $M\neq N$. There exist $f \in M^+\setminus N$ and $g \in N^+\setminus M$. Put

$$f' = f - f\wedge g,\quad g' = g - f\wedge g.$$

As $f\wedge g \in M\cap N$, we have $f' \in M^+\setminus N$ and $g' \in N^+\setminus M$. In particular we obtain $N \notin \{f'\}^\Delta$ and $M \notin \{g'\}^\Delta$. Furthermore, (see (iii), above).

$$\{f'\}^\Delta\cup\{g'\}^\Delta = \{f'\wedge g'\}^\Delta = \{0\}^\Delta = \mathcal{M}$$

Thus, $\mathcal{M}\setminus\{f'\}^\Delta$ and $\mathcal{M}\setminus\{g'\}^\Delta$ are disjoint open sets containing N and M, respectively.

13.C. *Exercise.* Let X be a compact Hausdorff space. For $a\in X$, define $M_a = \{f\in C(X) : f(a)=0\}$. Then $a\mapsto M_a$ is a homeomorphism $X \to \mathcal{M}(C(X))$.

13.D. *Example*. In Ex. 5.A(i) we have found a non-trivial Archimedean Riesz space M for which $M^{\sim} = \{0\}$. It is clear that the maximal ideal space of such an M is empty.

13.E *Example*. ($\mathcal{M}(L)$ is not always locally compact).

The theory of maximal ideals in Riesz spaces closely resembles the one of maximal ideals in commutative Banach algebras over $\mathbb{C}$. There too one has the connection between maximal ideals and scalar-valued homomorphisms, and there exists a topologized maximal ideal space. For Banach algebras these maximal ideal spaces are always locally compact. This example is intended to show that in this respect our theory is different.

Let X be any subset of $\mathbb{R}$ and let $C(X)$ be the Riesz space of all continuous functions $X \to \mathbb{R}$. We show $\mathcal{M}(C(X))$ to be homeomorphic to X itself. Every element a of X determines an $M_a \in \mathcal{M}(C(X))$ by

$$M_a = \{f \in C(X) : f(a)=0\}.$$

Then $a \mapsto M_a$ is an injective map $X \to \mathcal{M}(C(X))$ which is easily seen to be a homeomorphism if only it is surjective. Therefore, take an $M \in \mathcal{M}(C(X))$: we shall find an $a \in X$ for which $M = M_a$.

M is the kernel of a Riesz homomorphism $\phi: C(X) \to \mathbb{R}$. There exists an $h \in C(X)^+$ with $\phi(h) > 0$. For all $x \in X$, set

$$h^*(x) = \frac{h(x)+1}{\phi(h+\underline{1})}$$

Then $h^* \in C(X)$, $\phi(h^*) = 1$ and $h^*(x)>0$ for all $x \in X$. Define $\psi: C(X) \to \mathbb{R}$ by $\psi(f) = \phi(fh^*)$ $(f \in C(X))$. Then ψ is a Riesz homomorphism, its kernel is M, and $\psi(\underline{1}) = 1$. The identity map $X \to \mathbb{R}$ is an element j of $C(X)$. Let $a = \psi(j)$. We prove that $a \in X$ and that $\psi(f)=f(a)$ for all $f \in C(X)$: then the kernel of ψ will be M_a, i.e. $M = M_a$.

If $g \in C(X)^+$ and $g(x)>0$ for all $x \in X$, then $\frac{1}{g} \in C(X)$. For all $n \in \mathbb{N}$ we have

$$1 = \psi(\underline{1}) \leq \psi(\ ng \vee \frac{1}{ng}\ ) = n\psi(g) \vee \frac{1}{n}\psi(\frac{1}{g}),$$

so $\psi(g)>0$. In other words, if $g\in C(X)^+$ and $\psi(g)=0$, then g must take the value 0 somewhere.

Now let $f\in C(X)$. Set $g = |j-a\underline{1}|\vee|f-\psi(f)\underline{1}|$. Then $g\in C(X)^+$ and

$$\psi(g) = |\psi(j)-a\psi(\underline{1})|\vee|\psi(f)-\psi(f)\psi(\underline{1})| = 0,$$

so, by the above, there must exist an $x\in X$ with $g(x)=0$. For this x we have $j(x)=a$ and $f(x)=\psi(f)$, so $a\in X$ and $f(a)=\psi(f)$.

As in the case of the Banach algebras, we need a unit to guarantee the existence of sufficiently many maximal ideals.

13.7. DEFINITION. A (*strong*) *unit* in L is an element e of L^+ with the property that the principal ideal generated by e is L itself. This is the case if and only if for every $f\in L^+$ there exists an $n\in\mathbb{N}$ with $f\leq ne$.

L is said to be *unitary* if it has a unit.

13.F. *Examples*. Under coordinatewise ordering, $\mathbb{R}^n$ $(n\in\mathbb{N})$ is unitary. For a compact Hausdorff space X, $C(X)$ is unitary. For a non-compact space X, in general $C(X)$ will not be unitary. (If e is a unit in $C(X)$, then $e^2\leq ne$ for some $n\in\mathbb{N}$, so that e is bounded. Then every element of the principal ideal generated by e is bounded).

ℓ_1 is not unitary. (Take any $e\in\ell_1^+$. There exist positive integers $n_1<n_2<\ldots$ such that $e_{n_k}\leq 4^{-k}$ for each $k\in\mathbb{N}$. There exists an $a\in\ell_1^+$ such that $a_{n_k} = 2^{-k}$ for each k. Then there is no $n\in\mathbb{N}$ with $a\leq ne$).

13.8. THEOREM. *Let* L *be unitary.*

(i) *An ideal* I *of* L *is proper if and only if it contains no unit of* L.

(ii) *Every proper ideal of* L *is contained in a maximal ideal. Hence, if* $L\neq\{0\}$, *then there exist maximal ideals.*

(iii) *If* $f\in L$, $f>0$, *then there exists a Riesz homomorphism* $\phi:L\to\mathbb{R}$ *such that* $\phi(f)>0$.

(iv) *If* $g,h\in L$ *and* $g\neq h$, *there is a Riesz homomorphism* $\phi:L\to\mathbb{R}$ *for which* $\phi(g)\neq\phi(h)$.

(v) $\mathcal{M}(L)$ *is compact.*

Proof. (i) If I is an ideal containing a unit e, then $L=(e)\subset I$, so a proper ideal cannot contain units. Conversely, an ideal that contains no unit trivially differs from L.

(ii) Let E be the set of all units of L. If I is a proper ideal of L, then $I\subset L\backslash E$. By Zorn's Lemma, among the ideals of L that contain I and are subsets of $L\backslash E$ there is a maximal one, M, say. Then $I\subset M\in\mathcal{M}(L)$.

(iii) Let $f\in L$, $f>0$. Let e be a unit of L. As L is Archimedean, there exists a positive number s such that $f\not\leq se$, i.e. $(f-se)^+\neq 0$. If I is the principal ideal generated by $(f-se)^-$, then $(f-se)^+\perp I$, so I is proper. By (ii), I is contained in a maximal ideal M, which is the kernel of a Riesz homomorphism $\phi:L\to\mathbb{R}$. Now $(f-se)^-\in I\subset M$, so that

$$0=\phi((f-se)^-)=(\phi(f)-s\phi(e))^-,$$

whence $\phi(f)\geq s\phi(e)$. But by part (i), $e\notin M$, so $\phi(e)>0$. It follows that $\phi(f)>0$.

(iv) Set $f=|g-h|$ and apply (iii).

(v) Let $(A_\sigma)_{\sigma\in\Sigma}$ be a family of subsets of L^+ such that (in the terminology of 13.4) the collection $\{A_\sigma^\Delta:\sigma\in\Sigma\}$ of closed subsets of $\mathcal{M}(L)$ has the finite intersection property: we prove that its intersection is not empty. To do this we have to make an $M\in\mathcal{M}(L)$ such that $A_\sigma\subset M$ for every $\sigma\in\Sigma$, i.e. we have to prove that the union of all the sets A_σ is contained in a maximal ideal M of L. Let A be the union of the sets A_σ. Without restriction we suppose that $tf \in A$ for all $f \in A$ and $t \in \mathbb{R}^+$. Let I be the set of all elements f of L for which there exist $n \in \mathbb{N}$ and $f_1,\ldots,f_n \in A$ such that $|f| \le f_1 \vee \ldots \vee f_n$. Then I is an ideal and $A \subset I$: by (i) and (ii) we are done if I contains no units of L. Suppose L has a unit e with $e \in A$. Then $e \le f_1 \vee \ldots \vee f_n$ for certain $f_1,\ldots,f_n \in A$. For each i there is a $\sigma(i) \in \Sigma$ such that $f_i \in A_{\sigma(i)}$. Now $A_{\sigma(1)}^{\Delta} \cap \ldots \cap A_{\sigma(n)}^{\Delta}$ is not empty, so L has a maximal ideal N for which $N \supset A_{\sigma(1)} \cup \ldots \cup A_{\sigma(n)}$. But then $f_i \in N$ for each i, so that $e \in N$. By (i) this is impossible.

13.9. Part (iv) of the above theorem yields a simple technique (due to A.I.Yudin) to prove elementary identities in Archimedean Riesz spaces. As an example we show that for all $f,g,h \in L$,

(*) $\quad (f \wedge g) \vee (g \wedge h) \vee (h \wedge f) = (f \vee g) \wedge (g \vee h) \wedge (h \vee f)$

Let $f,g,h \in L$. First, we assume that L is unitary. By (iv) it suffices to prove that every Riesz homomorphism $L \to \mathbb{R}$ assigns the same value to both sides of (*). Let $\phi : L \to \mathbb{R}$ be Riesz homomorphism and let $\alpha = \phi(f)$, $\beta = \phi(g)$, $\gamma = \phi(h)$. We are done if

(**) $\quad (\alpha \wedge \beta) \vee (\beta \wedge \gamma) \vee (\gamma \wedge \alpha) = (\alpha \vee \beta) \wedge (\beta \vee \gamma) \wedge (\gamma \vee \alpha)$

But the truth of (**) (for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$) is easily seen.

In case L is not unitary, in the above one replaces L by the principal ideal L_0 generated by $|f|+|g|+|h|$. This L_0 is unitary, since $|f|+|g|+|h|$ is a unit in L_0.

13.10. Let e be a unit in L. For every maximal ideal M of L, by 13.A and 13.8(i), M is the kernel of a unique Riesz homomorphism $\phi_M : L \to \mathbb{R}$ that maps e onto 1. For every $f \in L$, $M \mapsto \phi_M(f)$ is a function $\hat{f}$ on $\mathcal{M}(L)$. We have

$$\hat{f}(M) = \phi_M(f) \qquad (f\in L,\ M\in \mathcal{M})$$

Then $\phi_M(f - \hat{f}(M)e) = \phi_M(f) - \hat{f}(M)\phi_M(e) = 0$, so $f - \hat{f}(M)e \in M$. Conversely, if $s\in\mathbb{R}$ and $f-se \in M$, then $0 = \phi_M(f-se) = \phi_M(f)-s\phi_M(e) = \hat{f}(M)-s$. Thus, $\hat{f}(M)$ is the (only) real number s for which $f-se \in M$.

The functions $\hat{f}$ are continuous. In fact, for $f\in L$ and $s\in\mathbb{R}$ we have

$$\{M\in\mathcal{M} : \hat{f}(M) \geq s\} = \{M : \phi_M(f-se) \geq 0\} =$$
$$= \{M : \phi_M((f-se)^-) = 0\} = \{(f-se)^-\}^\Delta$$

(We use $^\Delta$ as in 10.4). Thus, $\{M : \hat{f}(M) \geq s\}$ is closed in $\mathcal{M}$. So is, of course, $\{M : \hat{f}(M) \leq s\}$ $(f\in L,\ s\in\mathbb{R})$.

Now $f \mapsto \hat{f}$ is a map $L \to C(\mathcal{M})$. It is easily seen to be a Riesz homomorphism. The image of e is $\underline{1}$. If $f,g\in L$ and $f\neq g$, then by Th.13.8(iv) there is a (non-zero) Riesz homomorphism $\phi: L \to \mathbb{R}$ with $\phi(f)\neq\phi(g)$. If M is the kernel of ϕ, then ϕ_M is a non-zero scalar multiple of ϕ, so that $\phi_M(f)\neq\phi_M(g)$. This means that $\hat{f}(M)\neq\hat{g}(M)$ and therefore $\hat{f}\neq\hat{g}$.

We have now proved the following.

13.11. REPRESENTATION THEOREM. (K.Yosida, S.Kakutani, M.and S.Krein, H.Nakano). *Let* e *be a unit in* L. *For all* $f\in L$ *and* $M\in\mathcal{M}(L)$ *there exists a unique real number* $\hat{f}(M)$ *such that* $f - \hat{f}(M)e \in M$. *For every* $f\in L$, $\hat{f}$ *is a continuous function on the compact Hausdorff space* $\mathcal{M} = \mathcal{M}(L)$. *The map* $f \to \hat{f}$ *is a Riesz isomorphism of* L *onto a Riesz subspace* $\hat{L}$ *of* $C(\mathcal{M})$ *and* $\hat{e} = \underline{1}$.

This theorem is to be continued in 13.13, 13.17, 13.22, 13.23, 13.25, 13.28 and 13.32.

13.12. THEOREM. (Stone-Weierstrass). *Let* X *be a compact Hausdorff space containing at least two points. Let* L *be a Riesz subspace of* $C(X)$ *such that for any two points,* x *and* y, *of* X *there exists an* $f\in L$ *for which*

$f(x)=0$, $f(y)\neq 0$. *Then* L *is a norm dense subset of* $C(X)$.

Proof. Let $g\in C(X)$ and $\varepsilon>0$: we construct an $f\in L$ that satisfies the inequality $g-\varepsilon\underline{1} \le f \le g+\varepsilon\underline{1}$.

Let $a\in X$. From the given property of L it follows that for every $b\in X$ there is an $f_b\in L$ such that $f_b(a)=g(a)$, $f_b(b)=g(b)$. By the compactness of X there exist $b_1,\dots,b_n\in X$ for which

$$X = \bigcup_i \{x : f_{b_i}(x) - g(x) < \varepsilon\}$$

Set $f^a = f_{b_1}\wedge\dots\wedge f_{b_n}$. Then $f^a\in L$, $f^a(a)=g(a)$, $f^a \le g+\varepsilon\underline{1}$

For every $a\in X$ we can make such an f^a. There exist $a_1,\dots,a_m\in X$ for which

$$X = \bigcup_j \{x : f^{a_j}(x) - g(x) > -\varepsilon\}$$

Now set $f = f^{a_1}\vee\dots\vee f^{a_m}$. Then $f\in L$ and $g-\varepsilon\underline{1} \le f \le g+\varepsilon\underline{1}$.

In the following we let $\hat{L}$ be $\{\hat{f} : f\in L\}$ and we set $\mathcal{M} = \mathcal{M}(L)$.

If $M,N\in\mathcal{M}$ and $M\neq N$, then there exists an $f\in L$ with $f\in M$, $f\notin N$. Then $\hat{f}(M)=0$, $\hat{f}(N)\neq 0$. The Stone-Weierstrass Theorem now yields Cor.13.13.

13.13. COROLLARY. (Sequel to Th.13.11). $\hat{L}$ *is norm-dense in* $C(\mathcal{M})$.

13.14. COROLLARY. (A.I.Yudin). *Let* $n\in\mathbb{N}$ *and let* L *be* n*-dimensional as a vector space over* $\mathbb{R}$. *Then* L *is Riesz-isomorphic to* $\mathbb{R}^n$ *(where* R^n *is ordered coordinatewise)*.

Proof. Choose a base $f_1,\dots,f_n$ in L: set $e = |f_1|+\dots+|f_n|$. Then e is a unit in L. (For all $s_1,\dots,s_n\in\mathbb{R}$, $|\Sigma s_i f_i| \le (\Sigma|s_i|)e$). Thus, L is isomorphic to a norm dense subspace $\hat{L}$ of $C(\mathcal{M})$. Every finite dimensional normed vector space being complete, $\hat{L}$ is closed in $C(M)$ and $\hat{L} = C(M)$. Then $C(M)$ is n-dimensional, so that M contains exactly n elements. It follows that $C(M)$ is Riesz isomorphic to $\mathbb{R}^n$.

(Remark. For $n=1,2,3,4,5$ the number of isomorphism classes of (not necessarily Archimedean) n-dimensional Riesz spaces is 1,2,3,8,18, resp.)

There is another sense in which $\hat{L}$ is a dense subset of $C(M)$:

13.15. DEFINITION. Let L_1 be an Archimedean Riesz space, L_0 a Riesz subspace of L_1. We say that L_0 is *order dense* in L_1 if for every $f \in L_1$ with $f>0$ there exists a $g \in L_0$ for which $0<g\leq f$.

13.16. LEMMA. *Let* X *be a compact Hausdorff space. Every norm dense Riesz subspace of* $C(X)$ *is order dense.*

Proof. Let L_0 be a norm dense Riesz subspace of $C(X)$. Let $f \in C(X)$, $f>0$. Take $\varepsilon = \frac{1}{3}\|f\|_\infty$. There exists a $g \in L_0$ for which $\|(f-\varepsilon\underline{1})-g\|_\infty < \varepsilon$. Then $f-2\varepsilon\underline{1} \leq g \leq f$. Because $2\varepsilon < \|f\|_\infty$, we have $0 < (f-2\varepsilon\underline{1})^+ \leq g^+ \leq f$. Now note that $g^+ \in L_0$.

(Observe that $\{f \in C([0,1]) : f(0)=0\}$ is an order dense Riesz subspace of $C([0,1])$ that is not norm dense).

13.17. COROLLARY. (Sequel to 13.11). $\hat{L}$ *is order dense in* $C(M)$.

Before we can go into the consequences of this corollary we have to study order denseness.

13.18. LEMMA. *For an ideal* L_0 *of* L *the following conditions are equivalent.*

(a) L_0 *is order dense in* L.

(b) *Every element of* L^+ *is the supremum of a subset of* L_0.

(c) *The band of* L, *generated by* L_0, *is* L *itself.*

Proof. (a) is equivalent to $(L_0^{\,d})^+ = \{0\}$, hence to $L_0^{\,d} = \{0\}$, which, by the implication (a) $\Rightarrow$(d) of Th.4.3, is equivalent to (c). Besides, it is evident that (b) implies (a), while we know (Ex.2.D) that (b) is implied by (c).

13.19. Let L be the Riesz space of all bounded functions on $[0,1]$. The continuous functions on $[0,1]$ form a Riesz subspace L_0 of L. Let F be the subset of L_0 consisting of the functions $x \mapsto 1 \wedge nx$ $(n \in \mathbb{N})$. In the Riesz space L, this set F has a supremum, viz. the characteristic function of $(0,1]$. In the smaller Riesz space L_0, F also has a supremum, but this is $\underline{1}$. We see that in this context the expression "sup F" is ambiguous and may lead to confusion. Therefore we shall occasionally use notations like "L-sup F" and "L_0-sup F". The meaning of such a notation will always be clear.

For a Riesz subspace L_0 of L and for any $F \subset L_0$ the following two facts are obvious.

(i) *If* L-sup F *and* L_0-sup F *both exist, then* $L\text{-sup } F \le L_0\text{-sup } F$.

(ii) *If* L-sup F *exists and belongs to* L_0, *then* $L\text{-sup } F = L_0\text{-sup } F$.

13.20. DEFINITION. A Riesz subspace L_0 of L is said to be *normal* in L if $L\text{-sup } F = L_0\text{-sup } F$ for every subset F of L_0 for which L_0-sup F exists. By (i) and (ii) above, every ideal of L is normal in L.

13.21. LEMMA. *Let* L_0 *be an order-dense Riesz subspace of* L.

(i) L_0 *is normal in* L.

(ii) *If* L_0 *is a Dedekind complete Riesz space, then it is an ideal in* L.

(If $L = C([0,1])$ and if L_0 is the Riesz subspace of L consisting of the constant functions, then L_0 is normal but not order-dense in L: furthermore, it is normal in L and Dedekind complete but it is not an ideal in L).

Proof. (i) Let F be a non-empty subset of L_0 and let $f_0 = L_0\text{-sup } F$. Of course f_0 is an upper bound of F in L. Let $f_1 \in L$ be any upper bound of F: we have to prove that $f_1 \geq f_0$. Suppose $f_1 \not\geq f_0$. Then we have $f_0 \neq f_1 \wedge f_0$, so $f_0 - f_1 \wedge f_0 > 0$ and there exists a $g \in L_0$ such that $0 < g \leq f_0 - (f_1 \wedge f_0)$. But then $f_0 - g \geq f_1 \wedge f_0$, so $f_0 - g$ (which is an element of L_0) is an upper bound of F. Then $f_0 - g \geq L_0\text{-sup } F = f_0$, which is a contradiction.

(ii) Let $f_0 \in L_0$ and let $f_1 \in L$ be such that $0 \leq f_1 \leq f_0$: it suffices to prove that then necessarily $f_1 \in L_0$. Let $F = \{f \in L_0^+ : f \leq f_1\}$. This F is a non-empty subset of L_0, having the upper bound f_0 in L_0. By the Dedekind completeness of L_0, $L_0\text{-sup } F$ exists. But according to Lemma 13.18, $f_1 = L\text{-sup } F$. Now L_0 is normal in L. Consequently, $f_1 = L_0\text{-sup } F \in L_0$.

13.22. COROLLARY. (Sequel to 13.11). (i) $\hat{L}$ *is normal in* L.

(ii) *If* L *is Dedekind complete, then* $\hat{L} = C(M)$. *In this case, the maximal ideal space* M *of* L *is extremally disconnected.* (See Th.12.16).

13.G. *Exercise* on normality.

(i) If L_0 is a normal Riesz subspace of L and if L_1 is a Riesz sub-

space of L containing L_0, then L_0 is normal in L_1.
(ii) Every ideal in L is normal in L.

If A and B are disjoint closed subsets of M, then by Urysohn's Lemma there exists a $g \in C(M)$ such that $g \equiv -1$ on A while $g \equiv 2$ on B. As $\hat{L}$ is norm-dense in $C(M)$ we have an $h \in \hat{L}$ with $\|g-h\|_\infty \leq 1$. Then $h \leq 0$ on A but $h \geq 1$ on B. Setting $f = 1 \wedge h^+$ we have the first part of the following lemma. The second part is an immediate consequence of it.

13.23. LEMMA. (Sequel to 13.11). (i) *If* A *and* B *are disjoint closed subsets of* M, *there exists an* $f \in \hat{L}$ *such that* $f \equiv 0$ *on* A, $f \equiv 1$ *on* B.
(ii) $\hat{L}$ *contains the characteristic functions of all clopen subsets of* M.

13.24. DEFINITION. A function f on a topological space X is said to be *locally constant* if every element of X has a neighbourhood on which f is constant.

13.H. *Exercise*. Let X be a compact Hausdorff space.
(i) A function $f: X \to \mathbb{R}$ is locally constant if and only if it is a linear combination of characteristic functions of clopen sets.
(ii) The locally constant functions on X form a Riesz subspace of $C(X)$.
(iii) If X is zerodimensional, this subspace is order-dense in $C(X)$.

13.25. FREUDENTHAL SPECTRAL THEOREM. (Sequel to 13.11). *Suppose that* $B+B^d = L$ *for every principal band* B *of* L. (See 13.I). *Then* M *is zerodimensional and* $\hat{L}$ *contains all locally constant functions on* M.

Proof. Let $a \in M$ and let W be an open neighbourhood of a. We construct a clopen set U such that $a \in U \subset W$: then X is zerodimensional and the rest of the theorem follows from Ex.13.H(i) and Lemma 13.23(ii).

There is an open $V \subset M$ such that $a \in V \subset \overline{V} \subset W$. By Lemma 13.23(i) there exist $f,g \in L^+$ such that $\hat{f}(a)=1$, $\hat{f}\equiv 0$ on $M\setminus V$, $\hat{g}\equiv 1$ on $M\setminus W$ and $\hat{g}\equiv 0$ on $\overline{V}$. Let B be the principal band generated by f. As $B+B^d = L$ there exists a $u \in B$ with $e-u \in B^d$. Now $u \perp (e-u)$, so $\hat{u} \perp (\underline{1}-\hat{u})$. It follows that $\hat{u}$ is the characteristic function of a subset U of M. Since $\hat{u}$ is continuous, U must be clopen.

We have $(e-u) \perp f$, so $(\underline{1}-\hat{u}) \perp \hat{f}$. Furthermore, $\hat{f}(a)=1$. Consequently, $(\underline{1}-\hat{u})(a) = 0$, which means that $\hat{u}(a)=1$ and $a \in U$. On the other hand, $g \perp f$, so $g \perp B$, $g \perp u$ and $\hat{g} \perp \hat{u}$. As $\hat{g}\equiv 1$ on $M\setminus W$ it follows that $\hat{u}\equiv 0$ on $M\setminus W$, so $U \subset W$.

The condition " $B+B^d = L$ for all principal bands B " is satisfied if L is Dedekind complete. (Th.4.5(ii)). More generally:

13.I. *Exercise*. Assume that every countable non-empty bounded subset of L has a supremum. (Such an L is called *Dedekind σ-complete*). Then $B+B^d = L$ for every principal band B of L. (See also 13.M).

Our use of the term "Freudenthal Spectral Theorem" to denote Theorem 13.25 is poetic license. The "classical" formulation of the Freudenthal Theorem involves abstract integration theory rather than topology and looks quite different. The justification for our unorthodox approach is that our version fits better into the given context and has the classical form as an immediate corollary. In order to appease the Muse of History we now give the better known formulation, leaving its proof to the reader.

13.26. Let L be as in Th.13.25, let $f\in L$. For every $s\in\mathbb{R}$ let B_s be the band generated by $(se-f)^+$. For each s there is a unique $u_s\in B_s$ such that $e-u_s \in B_s^d$. If $s,t\in\mathbb{R}$ and $s\le t$, then $u_s\le u_t$. Choose $a,b\in\mathbb{R}$ for which $ae \le f \le (b-\alpha)e$ for some $\alpha>0$. Then $u_s=0$ if $s\le a$ while $u_s=e$ for $s\ge b$. Now

$$f = \int_a^b s \, du_s$$

in the following sense.

(i) For every $\varepsilon>0$ there exists a $\delta>0$ with the following property. If $n\in\mathbb{N}$, $a=s_0\le t_1\le s_1\le t_2\le \ldots \le t_n\le s_n=b$ and if $s_i-s_{i-1} \le \delta$ for each i, then

$$| f - \sum_i t_i(u_{s_i} - u_{s_{i-1}}) | \le \varepsilon e$$

(ii) For every $\phi\in L^\sim$ the function $s \to \phi(u_s)$ is of bounded variation and

$$\phi(f) = \int_a^b s \, d\phi(u_s)$$

13.27. DEFINITION. If L is unitary, a unit u in L defines a function σ_u from L into $[0,\infty)$ by

$$\sigma_u(f) = \inf\{s\in[0,\infty) : |f| \le su\} \qquad (f\in L)$$

It is easy to see that σ_u is a Riesz norm on L.

For every $\varepsilon>0$, $|f| \le (\sigma_u(f)+\varepsilon)u$, i.e. $|f|-\sigma_u(f)u \le \varepsilon u$. From the fact that L is Archimedean it follows that $|f|-\sigma_u(f)u \le 0$. Thus,

$$|f| \le \sigma_u(f)u \qquad (f\in L;\ u \text{ a unit in } L)$$

If u and v are units of L, then from this formula we may infer that $\sigma_v(f) \le \sigma_u(f)\sigma_v(u)$ for all $f\in L$, i.e. $\sigma_v \le \sigma_v(u)\sigma_u$. Similarly, $\sigma_u \le \sigma_u(v)\sigma_v$. Apparently, the norms σ_u and σ_v are equivalent. We see that all the norms σ_u (where u runs through the set of all units of L) determine the same topology on L.

L is said to be *uniformly complete* if for some unit u every

σ_u-Cauchy sequence is σ_u-convergent. Then, of course, for every unit v all σ_v-Cauchy sequences are σ_v-convergent.

13.28. THEOREM. (Sequel to 13.11). *For every* $f \in L$,

$$\sigma_e(f) = \|\hat{f}\|_\infty$$

If L *is uniformly complete, then* $\hat{L} = C(M)$.

13.29. COROLLARY. *The following two properties of* L *are equivalent*

(a) *There exists a compact Hausdorff space* X *such that* L *is Riesz isomorphic to* C(X).

(b) L *is unitary and uniformly complete.*

From 13.22 and 13.29 we derive the following result (which is not hard to prove directly, either).

13.30. COROLLARY. *Every unitary Dedekind complete Riesz space is uniformly complete.*

We have seen (Ex.4.H(i)) that quotients of Archimedean Riesz spaces may fail to be Archimedean. The following exercise suggests that unitary Riesz spaces all of whose quotients are Archimedean, are rare phenomena.

13.J. *Exercise.* Let L be unitary and assume that all quotient Riesz spaces of L are Archimedean.

(i) (In the terminology of our representation theorem). M is zerodimensional and $\hat{L}$ is just the space of all locally constant functions on M.

(ii) If L is uniformly complete, then L is a finite dimensional vector space, hence is Riesz isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

(Hint for (i). By 13.H(i) and 13.23(ii), all locally constant functions belong to L. For the converse, let $f \in \hat{L}$, $a \in \mathcal{M}$ and $f(a)=0$. Consider now the images of $\underline{1}$ and f in the quotient $\hat{L}/J$ where J is the set of all elements of L that vanish identically on some neighbourhood of a).

13.31. DEFINITION. A *Riesz algebra* is a Riesz space L_0 which is also an algebra (under the same vector space structure) such that

$$\text{if } f \in L_0^+ \text{ and } g \in L_0^+, \text{ then } fg \in L_0^+.$$

A Riesz algebra L_0 is said to be *unitary* if it has a two-sided identity element (relative to the multiplication) which is also a strong unit in the Riesz space. The prime example is, of course, $C(X)$ where X is a compact space.

Two Riesz algebras, L_1 and L_2, are called *isomorphic* if there exists a Riesz isomorphism of L_1 onto L_2 that is an algebra homomorphism.

13.K. *Exercise*. Let L be a Riesz algebra.

(i) If $f_1, f_2 \in L$, $f_1 \le f_2$ and $g \in L^+$, then $f_1 g \le f_2 g$ and $gf_1 \le gf_2$.

(ii) For all $f, g \in L$ we have $|fg| \le |f||g|$.

13.32. THEOREM. (Sequel to 13.11). *Let* L *be a unitary Riesz algebra whose identity element is* e. *Then the multiplication of* L *is commutative. All Riesz ideals in* L *are algebra ideals. The map* $f \mapsto \hat{f}$ *is an algebra homomorphism.*

Proof. Let J be a Riesz ideal in L. Let $f \in L$, $g \in J$. There is an $n \in \mathbb{N}$ for which $|f| \le ne$. Then $|fg| \le |f||g| \le ne|g| = n|g|$, so $fg \in J$. Similarly, $gf \in J$. Thus, J is a two-sided ring ideal in L.

Let $M \in \mathcal{M}$, $f \in L$, $g \in L$. Then $f - \hat{f}(M)e \in M$. By the above,

$$fg - \hat{f}(M)g = (f-\hat{f}(M)e)g \in M.$$

This means that

$$0 = (fg - \hat{f}(M)g)^{\wedge}(M) = \widehat{fg}(M) - \hat{f}(M)\hat{g}(M).$$

Apparently, $\widehat{fg} = \hat{f}\hat{g}$ for all $f,g \in L$. In other words, the map $f \to \hat{f}$ is an algebra homomorphism. Then for all f and g we have $\widehat{fg} = \widehat{gf}$, so the multiplication of L is commutative.

13.L. *Example*. Let ℓ_∞ be as in 8.A(ii) and 9.B. Under coordinatewise operations ℓ_∞ is a Dedekind complete Riesz algebra. The constant sequence $(1,1,\ldots)$, which we call $\underline{1}$, is the identity element for the multiplication and also a strong unit in the sense of the Riesz theory. Thus, ℓ_∞ is a unitary Riesz algebra. The norm $\sigma_{\underline{1}}$ induced by $\underline{1}$ is the sup-norm.

Taking $e = \underline{1}$. from the preceding theory we obtain a compact Hausdorff space M and an isometric multiplicative Riesz isomorphism of ℓ_∞ onto $C(M)$. According to Cor.12.3, M is homeomorphic to the space X of 12.I. In the language of 13.33, $M = \mathbb{N}^\beta$.

13.M. *Example*. Let (X,Γ) be a measurable space. Under pointwise operations the bounded Γ-measurable functions on X form a unitary Riesz algebra that we call $L_\infty(X,\Gamma)$. Its identity element is $\underline{1}$. The norm $\sigma_{\underline{1}}$ is the sup-norm. $L_\infty(X,\Gamma)$ is uniformly complete. Every non-empty countable bounded subset of $L_\infty(X,\Gamma)$ has a supremum (see Ex.13.I), but in general $L_\infty(X,\Gamma)$ is not Dedekind complete. (E.g., let X be $\mathbb{R}$ and let Γ be the Borel σ-algebra).

Now let μ be a measure on Γ. By identifying functions that are μ-a.e. equal, from $L_\infty(X,\Gamma)$ we obtain the space $L_\infty(\mu)$. This $L_\infty(\mu)$ in a natural way becomes a (uniformly complete) unitary Riesz algebra. If μ is slightly decent (e.g., if μ is σ-finite), $L_\infty(\mu)$ is Dedekind complete.

13.N. *Example.* Let X be any topological space. The bounded continuous functions on X form an Archimedean Riesz algebra $BC(X)$. The constant function $\underline{1}$ is the identity element of the algebra and a strong unit of the Riesz space. $\sigma_{\underline{1}}$ is the supremum-norm $\| \|_\infty$:

$$\|f\|_\infty = \sup\{|f(x)| : x\in X\} \qquad (f\in BC(X))$$

$BC(X)$ is a Banach space relative to this norm, so $BC(X)$ is uniformly complete. (Ex.13.L is the special case $X=\mathbb{N}$).

This example deserves a little more attention:

13.33. DEFINITION. The maximal ideal space of $BC(X)$ is called the *Stone-Čech compactification* of X. We denote it by βX or X^β.

In this context, for $f\in BC(X)$ the function $\hat{f}$ on X^β is indicated by βf or f^β.

Every $x\in X$ determines a maximal ideal $\{f\in BC(X) : f(x)=0\}$ in $BC(X)$. We denote this maximal ideal by $\beta(x)$: thus we obtain a map $\beta : X \to X^\beta$. Using the notations of 13.4, for all subsets A of $BC(X)^+$ we have

$$\beta^{-1}(A^\Delta) = \{x\in X : f(x)=0 \text{ for every } f\in A\}.$$

Therefore, β is continuous.

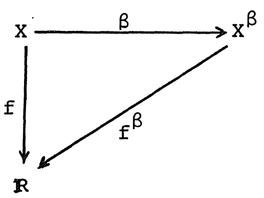
For $f\in BC(X)$ and $x\in X$, by the definition of f^β, the maximal ideal $\beta(x)$ of $BC(X)$ contains $f - f^\beta(\beta(x))\underline{1}$, i.e. $f(x) - f^\beta(\beta(x)) = 0$. Thus,

$$f^\beta \circ \beta = f \qquad (f\in BC(X))$$

But now it is easy to prove that $\beta(X)$ is a dense subset of X^β. In fact, if $a\in X^\beta$ and $a \notin \overline{\beta(X)}$, there exists a $g\in C(X^\beta)$ such that $g(a)=1$ while $g\equiv 0$ on $\beta(X)$. This g is f^β for some $f\in BC(X)$. But then $f = g\circ\beta$, so $f=0$ although $f^\beta(a) = g(a) \neq 0$: contradiction.

Summing up, we have the following theorem.

13.34. THEOREM. *Let* X *be a topological space, let* X^β *be the maximal ideal space of* $BC(X)$. *For every* $x \in X$ *let* $\beta(x)$ *be the maximal ideal* $\{f : f(x)=0\}$ *of* $BC(X)$. *Then* β *is a continuous map of* X *onto a dense subset of* X^β. *For every* $f \in BC(X)$ *there exists a unique* $f^\beta \in C(X^\beta)$ *with*



$f = f^\beta \circ \beta$. *The correspondence* $f \to f^\beta$ *is an isometric and multiplicative Riesz isomorphism of* $BC(X)$ *onto* $C(X^\beta)$.

13.P. *Exercise.* Let X be a topological space.

(i) The following properties of X are equivalent.

(a) β is a homeomorphism of X onto $\beta(X)$.

(b) X is homeomorphic to a subset of some compact Hausdorff space.

(c) If A is a closed subset of X and if $a \in X$, $a \notin A$, then there exists a continuous function f on X for which $f(a)=0$ while $f \equiv 1$ on A.

(For such an X one usually identifies X with $\beta(X)$, viewing f^β as an extension of f).

(ii) X has the properties (a),(b),(c) if it is metrizable.

(iii) If X is discrete, then for every $x \in X$ the set $\{\beta(x)\}$ is open in X^β.

13.Q. *Exercise*. Let σ be the absolute value topology on $[0,1]$.

$$\tau = \{U_1 \cup (U_2 \cap \mathbb{Q}) : U_1, U_2 \in \sigma\}.$$

Then τ is a Hausdorff topology that is strictly stronger than σ. The topologies σ and τ give us two topological spaces with $[0,1]$ as underlying point sets; we call them S and T, respectively. S is compact, T is not.

(i) If f is a τ-continuous function on $[0,1]$, then f is σ-continuous. (Take $a \in [0,1]$, $\varepsilon > 0$: we want to make a $W \in \sigma$ with $a \in W$ and $|f(x)-f(a)| < \varepsilon$ for all $x \in W$. We may assume $a \in \mathbb{Q}$. There exists a $W \in \sigma$ such that $a \in W \cap \mathbb{Q}$ and $|f(x)-f(a)| < \frac{1}{2}\varepsilon$ for all $x \in W \cap \mathbb{Q}$. Now take $x \in W \setminus \mathbb{Q}$. There is a $V \in \sigma$ with $x \in V$ and with $|f(y)-f(x)| < \frac{1}{2}\varepsilon$ for all $y \in V \cap \mathbb{Q}$. Observe that there exists a $y \in V \cap W \cap \mathbb{Q}$. Then $|f(x)-f(a)| \leq |f(y)-f(x)| + |f(y)-f(a)| < \varepsilon$).

(ii) $BC(T) = C(S)$.

(iii) The map $\beta: T \to T^{\beta}$ is bijective but not a homeomorphism.

13.R. *Exercise*. (The Stone-Čech compactification of $(0,1]$).

Let X be the space $(0,1]$ under the absolute value topology.

(i) β is a homeomorphism of X onto $\beta(X)$.

(ii) X^{β} is not metrizable. (Suppose X^{β} is metrizable. By (i), there is an $a \in X^{\beta}$, $a \notin \beta(X)$. There exist $x_1, x_2, \ldots \in X$ such that $a = \lim x_n$ and $x_n \neq x_m$ as soon as $n \neq m$. There is a bounded continuous function f on X such that $f(x_n) = (-1)^n$ for every n).

(iii) X^{β} is not homeomorphic to $[0,1]$.

14. THE RIESZ SPACE B(X)

IN THIS SECTION, X *IS AGAIN A COMPACT HAUSDORFF SPACE.*

14.1. DEFINITION. A subset A of X is said to be *meagre* if there exists a (countable) sequence $A_1, A_2, \ldots$ of subsets of X such that

(i) $A \subset \bigcup A_n$,

(ii) Every A_n is closed and has empty interior.

Every subset of a meagre set is meagre. A union of countably many meagre sets is meagre.

$\mathbb{Q} \cap [0,1]$ is meagre as a subset of $[0,1]$. So is $\{0\}$, although of course $\{0\}$ is not meagre as a subset of $\{0\}$.

14.2. BAIRE'S CATEGORY THEOREM. *Let* $X \neq \emptyset$.

(i) Every meagre subset of X *has empty interionr.*

(ii) A closed subset of X *is meagre if and only if its interior is empty.*

(iii) X *is not meagre.*

Proof. (ii) and (iii) are immediate consequences of (i).

To prove (i), assume that X has a non-empty meagre open subset U_0. There exist closed subsets $A_1, A_2, \ldots$ of X for which $U_0 \subset \bigcup A_n$ while each A_n has empty interior. In particular, $U_0 \not\subset A_1$, so there exists an $a_1 \in U_0 \setminus A_1$. Then there is an open $U_1 \subset X$ such that $a_1 \in U_1 \subset \overline{U_1} \subset U_0 \setminus A_1$. Now $U_1 \not\subset A_2$, so there exist an $a_2 \in U_1 \setminus A_2$ and an open set U_2 for which $a_2 \in U_2 \subset \overline{U_2} \subset U_1 \setminus A_2$.

Proceeding in this fashion one obtains a sequence $U_1, U_2, \ldots$ of non-empty open sets such that $\overline{U_n} \subset U_{n-1} \setminus A_n$ for every $n \in \mathbb{N}$. Then $\overline{U_1} \supset \overline{U_2} \supset \ldots$ so, by the compactness of X, $\bigcap \overline{U_n}$ contains an element a. Now, on the one hand, $a \in \overline{U_1} \subset U_0 \subset \bigcup A_n$, while on the other hand $a \in \bigcap \overline{U_n} \subset X \setminus \bigcup A_n$: contradiction.

14.3. DEFINITION. If for every $x \in X$ there is given a proposition $P(x)$, we say that $P(x)$ holds for *almost every* $x \in X$ (or " P holds *almost everywhere*", " P is true *a.e.*") in case the set $\{x \in X : P(x) \text{ is false}\}$ is meagre.

Thus, because $\mathbb{Q} \cap [0,1]$ is meagre in $[0,1]$ we may say that "almost every element of $[0,1]$ is irrational".

This topological "almost everywhere" is very similar to the "almost everywhere" used in Measure Theory, the meagre sets taking over the role of the sets of measure 0.

From 14.2(i) we deduce directly:

14.4. COROLLARY. *If* $f,g \in C(X)$ *and* $f=g$ *a.e., then* $f=g$.

For a while we have to return to the semicontinuous functions. (See 12.13-12.15).

14.5. LEMMA. *If* $f : X \to \mathbb{R}$ *is bounded and lower s.c., then* $f=f^{\uparrow}$ *a.e.*

Proof. $\{x : f(x) \neq f^{\uparrow}(x)\} = \{x : f(x) < f^{\uparrow}(x)\} = \bigcup \{S_{\alpha,\beta} : \alpha,\beta \in \mathbb{Q},\ \alpha<\beta\}$ if we define $S_{\alpha,\beta} = \{x : f(x) \leq \alpha < \beta \leq f^{\uparrow}(x)\} = \{x : f(x) \leq \alpha\} \cap \{x : f^{\uparrow}(x) \geq \beta\}$. Every $S_{\alpha,\beta}$ is closed. Hence, we are done if we prove that every $S_{\alpha,\beta}$ has empty interior. Assume that a is an interior point of some $S_{\alpha,\beta}$. Then $S_{\alpha,\beta}$ is a neighbourhood of a and $f \leq \alpha$ on $S_{\alpha,\beta}$. By the definition of $f^{\uparrow}(a)$ it follows that $f^{\uparrow}(a) \leq \alpha$. But $f^{\uparrow}(a) \geq \beta$ because $a \in S_{\alpha,\beta}$: contradiction.

14.6. THEOREM. *For a bounded real function* f *on* X *the following conditions are equivalent.*

(a) There exists a meagre set $A \subset X$ *such that the restriction of* f *to* $X\backslash A$ *is continuous.*

(b) There exists an upper s.c. function $g:X \to \mathbb{R}$ *such that* $f=g$ *a.e.*

(c) There exists a lower s.c. function $g:X \to \mathbb{R}$ *such that* $f=g$ *a.e.*

(d) There exists a Borel measurable $g:X \to \mathbb{R}$ *such that* $f=g$ *a.e.*

Proof. We prove the implications (b)⇒(c), (c)⇒(b), (b)∧(c)⇒(a), (a)⇒(d) and (d)⇒(c).

(b)⇒(c). If g is upper s.c. and $f=g$ a.e., then $g^{\downarrow}$ is lower s.c. (see 12.H(iv)) and $g^{\downarrow}=g=f$ a.e. (14.5).

(c)⇒(b) is proved similarly.

(b)∧(c)⇒(a). There exist meagre sets A_1 and A_2, an upper s.c. function g_1 and a lower s.c. function g_2 such that $f\equiv g_1$ off A_1, $f\equiv g_2$ off A_2. The restriction of f to $X\backslash(A_1 \cup A_2)$ is both upper and lower s.c., hence is continuous.

(a)⇒(d). Let A be a meagre set for which the restriction of f to $X\backslash A$ is continuous. We may assume that A is a union of countably many closed sets. Then $f\chi_{X\backslash A}$ is Borel measurable and $f\chi_{X\backslash A} = f$ a.e.

(d)⇒(c). Let Γ be the collection of all subsets A of X for which there exists an open set $U \subset X$ with $\chi_A=\chi_U$ a.e. If $A_1,A_2,\ldots \in \Gamma$, then $\bigcup A_n \in \Gamma$. If V is an open subset of X, then $\overline{V}\backslash V$ is meagre, hence $\chi_{X\backslash V} = \chi_{X\backslash \overline{V}}$ a.e. and $X\backslash V \in \Gamma$. Therefore, if $A \in \Gamma$ then $X\backslash A \in \Gamma$. Thus, Γ is a σ-algebra that contains all open sets. But then it must contain all Borel sets.

Now assume (d). Without restriction we can assume that $0\le f\le \underline{1}$, $0\le g\le \underline{1}$. For $s \in \mathbb{Q}\cap[0,1]$ let $A_s = \{x: g(x)>s\}$. By the above for each s there exists an open set U_s such that $\chi_{A_s} = \chi_{U_s}$ a.e. Define $h:X \to [0,1]$ by

$$h(x) = \sup \{s\chi_{U_s}(x) : s \in \mathbb{Q}\cap[0,1]\} \qquad (x\in X)$$

Then h is lower s.c. (12.G(iv)). For every $x\in X$,

$$g(x) = \sup \{s\chi_{A_s}(x) : s \in \mathbb{Q}\cap[0,1]\}$$

Hence, $f = g = h$ a.e. and f has property (c).

14.7. DEFINITION. The bounded Borel functions on X form, under pointwise operations, an Archimedean Riesz algebra L. In this L, the functions that vanish almost everywhere on X form a Riesz ideal N.

By $B(X)$ we denote the quotient Riesz space L/N. If f is an element of L, we indicate by $\overline{f}$ the image of f under the quotient map $L \to B(X)$. Often we identify f with $\overline{f}$.

The pointwise multiplication in L induces a multiplication in $B(X)$. In this way, $B(X)$ becomes a unitary Riesz algebra.

The natural embedding $C(X) \to L$ yields an injective (Cor.14.4!) and multiplicative Riesz homomorphism of $C(X)$ into $B(X)$. Identifying an element f of $C(X)$ with the corresponding element $\overline{f}$ of $B(X)$, we shall view $C(X)$ as a subspace of $B(X)$.

Now let X be extremally disconnected. Then this map $C(X) \to B(X)$ is bijective. In fact, let $f\in B(X)$. By Th.14.6 there exists a lower s.c. function g on X such that $f=g$ a.e. According to Lemma 14.5, $f=g^{\uparrow}$ a.e. while $g^{\uparrow}$ is continuous. (Lemma 12.15). We have proved the following.

14.8. THEOREM. *If* X *is extremally disconnected, then for every bounded Borel function* f *on* X *there is a continuous function* f' *on* X *such that* $f=f'$ *a.e.* (*Thus, the natural map* $C(X) \to B(X)$ *is bijective*).

14.A. *Exercise*. X is extremally disconnected if and only if for every Borel subset A of X there exists a clopen subset U of X such that $A\setminus U$ and $U\setminus A$ are meagre.

If X is not extremally disconnected, then the natural map of C(X) into B(X) is not surjective. Its range space is still order dense, though, as we shall now prove.

14.9. THEOREM. B(X) *is Archimedean. In fact,* B(X) *is a Dedekind complete unitary Riesz algebra, containing* C(X) *as an order dense Riesz subspace.*

Proof. For $f,g \in L$ we have $\overline{f} \le \overline{g}$ if and only if $f \le g$ a.e. It follows easily that B(X) is Archimedean. It is practically obvious that B(X) is a unitary Riesz algebra whose identity element is $\overline{\underline{1}}$.

Next, we prove C(X) to be order dense in B(X). Let $f \in L$, $\overline{f} > 0$: we construct an $h \in C(X)$ such that $0 < \overline{h} \le \overline{f}$. By Th.14.6, we may assume that f is lower s.c. Certainly there is an $a \in X$ with $f(a) > 0$. Define

$$A = \{x : f(x) \le \tfrac{1}{2}f(a)\}.$$

This set A is closed and does not contain a. By Urysohn's Lemma there is an $h \in C(X)$ for which $0 \le h \le \frac{1}{2}f(a)\underline{1}$, $h(a) = \frac{1}{2}f(a)$ and $h \equiv 0$ on A. Then $0 < \overline{h} \le \overline{f}$.

It remains to prove the Dedekind completeness. Let F be a non-empty bounded subset of $B(X)^+$. Without restriction we may assume that $f \le \overline{\underline{1}}$ for all $f \in F$. There exists a subset G of L such that $F = \{\overline{g} : g \in G\}$ and we may assume all elements of G to be lower s.c. For every $g \in G$, $g \wedge \underline{1}$ is lower s.c. and $\overline{g \wedge \underline{1}} = \overline{g} \wedge \overline{\underline{1}} = \overline{g}$. Thus, we may assume $g \le \underline{1}$ for all $g \in G$. Then the formula $g_0(x) = \sup\{g(x) : g \in G\}$ defines a bounded lower s.c. function g_0 on X. Clearly, $\overline{g_0} = \sup F$.

It follows that $C(X)$ is a normal subspace of $B(X)$. In general, however, $C(X)$ is not normal in the space of all bounded Borel functions on X. (A counterexample is easily obtained from 13.19).

By combining 12.16, 13.22(ii), 14.8 and 14.9 we obtain

14.10. COROLLARY. *The following conditions on* L *are equivalent.*

(*a*) L *is unitary and Dedekind complete.*

(*b*) *There exists a compact Hausdorff space* Y *such that* L *is Riesz isomorphic to* $B(Y)$.

(*c*) *There exists an extremally disconnected compact Hausdorff space* Z *such that* L *is Riesz isomorphic to* $C(Z)$.

14.11. DEFINITION. A *Dedekind completion* of an Archimedean Riesz space L is a pair (M,Φ) consisting of a Riesz space M and a Riesz isomorphism Φ of L onto a Riesz subspace of M, such that

(i) M is Dedekind complete,

(ii) $\Phi(L)$ is a normal subspace of M,

(iii) M has no Dedekind complete proper Riesz subspace that contains $\Phi(L)$.

(Later on (Cor.15.7, 15.13) we shall prove general existence and uniqueness theorems).

Often we identify an element of L with its image under Φ, viewing L as a subspace of M. Then we call M itself a *Dedekind completion* of L.

14.12. THEOREM. $B(X)$ *is a Dedekind completion of* $C(X)$.

Proof. After Th.14.8, this is a direct consequence of Lemma 14.13.

14.13. LEMMA. *Let* M *be a Dedekind complete Riesz space containing* L *as an order dense Riesz subspace such that* L *is not contained in any proper ideal of* M. *Then* M *is a Dedekind completion of* L.

Proof. By Lemma 13.21(i), L is a normal subspace of M. Let M' be a Riesz subspace of M that is Dedekind complete and contains L: we want to show that M'=M. Now L is order dense in M. Then, a fortiori, M' is order dense. According to 13.21(ii), M' is an ideal in M. But L is not contained in any proper ideal, so M'=M.

It is easy to generalize Th.14.12 a little and prove that actually B(X) is a Dedekind completion of any order dense Riesz subspace of C(X) that contains $\underline{1}$. Then our representation theorem 13.11 yields 14.14.

14.14. THEOREM. *If* L *is unitary, then* B($\mathcal{M}$) *is a Dedekind completion of* L *(where* $\mathcal{M} = \mathcal{M}$(L)).

By applying Th.13.11, Cor.13.22(ii) and Th.13.32 to B(X) one finds:

14.15. THEOREM. *Let* Z *be the maximal ideal space of* B(X). *Then* Z *is an extremally disconnected compact Hausdorff space and* B(X) *is (as a Riesz algebra) isomorphic to* C(Z). *In particular,* C(Z) *is a Dedekind completion of* C(X). (See also Th.14.19).

In order to obtain another description of Z we digress for a moment and consider the relation between the Boolean algebras and the zerodimensional compact Hausdorff spaces, expressed in the beautiful Stone Representation Theorem.

14.16. DEFINITION. If S is a zerodimensional compact Hausdorff space, the clopen subsets of S, under inclusion, form a Boolean algebra: we denote this Boolean algebra by $b(S)$. For $U,V \in b(S)$ we have

$$U \vee V = U \cup V, \quad U \wedge V = U \cap V, \quad U' = U^c.$$

If $\mathcal{A}$ is a Boolean algebra we call S a *Stone space* of $\mathcal{A}$ if $b(S)$ and $\mathcal{A}$ are isomorphic lattices. As an example, from Th.12.9 and from the definition of extremal disconnectedness, we see that, *if* X *is extremally disconnected, then* X *is a Stone space of* $\mathcal{B}[C(X)]$.

14.B. *Exercise*. $b(X)$ is complete (as a Boolean algebra) if and only if X is extremally disconnected.

14.C. *Exercise*. Let S be a discrete topological space. Then the Stone-Čech compactification of S is a Stone space of the Boolean algebra $\mathcal{P}(S)$ that consists of all subsets of S.

14.17. STONE REPRESENTATION THEOREM. *Let* $\mathcal{A}$ *be a Boolean algebra. Then* $\mathcal{A}$ *has a Stone space. All Stone spaces of* $\mathcal{A}$ *are homeomorphic.*

Proof. Put $0 = \inf \mathcal{A}$, $1 = \sup \mathcal{A}$. We assume $0 \neq 1$. For $x \in \mathcal{A}$, by x' we denote the complement of x.

A subset J of $\mathcal{A}$ is called a *filter* if

(i) $1 \in J$,

(ii) $0 \notin J$,

(iii) for $x,y \in \mathcal{A}$ we have $x \wedge y \in J$ if and only if $x \in J$ and $y \in J$.

Let S be the set of all maximal filters.

Take $J \in S$, $x \in \mathcal{A}$, $x \notin J$. Then $\{z \in \mathcal{A} : z \vee x' \in J\}$ has properties (i) and (iii) of a filter, but it contains $J \cup \{x\}$. By the maximality of J we must have $0 \in \{z\in A : z\vee x' \in J\}$, i.e. $x'\in J$. Thus,

(1) if $J\in S$, $x\in A$ and $x\notin J$, then $x'\in J$.

Conversely, for $J\in S$ and $x\in A$ we have $x\wedge x' = 0 \notin J$, so

(2) if $J\in S$ and $x\in J$, then $x'\notin J$.

For all $x\in A$ let $S_x = \{J\in S : x\in J\}$. By (iii) we have

(3) $S_{x\wedge y} = S_x \cap S_y \qquad (x,y\in A)$

From (1) and (2) we infer

(4) $(S_x)^c = S_{x'} \qquad (x\in A)$

so that

(5) $S_{x\vee y} = S_x \cup S_y \qquad (x,y\in A)$

Obviously,

(6) $S_0 = \emptyset$, $S_1 = S$.

It follows from (3) and (6) that $\{S_x : x\in A\}$ is a base for a topology on S. By (4) every S_x is clopen and the topology is zerodimensional. If J_1 and J_2 are distinct elements of S, there exists an $x \in J_1\backslash J_2$: then $J_1\in S_x$, $J_2\in S_{x'}$ while $S_x\cap S_{x'} = \emptyset$. (See (2)). Hence, S is a Hausdorff space.

Next, we show S to be compact. It suffices to prove that every subset A_1 of A for which $S = \bigcup\{S_x : x\in A_1\}$ contains a finite set A_2 such that $S = \bigcup\{S_x : x\in A_2\}$. Thus, let $A_1 \subset A$ be such that $S \neq \bigcup\{S_x : x\in A_2\}$ for all finite subsets A_2 of A_1 : we prove $S \neq \bigcup\{S_x : x\in A_1\}$. Define $I = \{z\in A$: there exists a finite $A_2 \subset A_1$ for which $z \geq (\sup A_2)'\}$. Then this I has properties (i) and (iii) of a filter. Furthermore, for every finite $A_2 \subset A_1$ by (5) we have

$$S_{\sup A_2} = \bigcup\{S_x : x\in A_2\} \neq S$$

so $\sup A_2 \neq 1$ and $(\sup A_2)' \neq 0$. Hence, I also has property (ii) of a filter. By Zorn's Lemma, I is contained in a maximal filter J. Then $J\in S$ and for every $x\in A_1$ we see that $x' \in I \subset J$, i.e. $J\notin S_x$.

We have now proved that S is a zerodimensional compact Hausdorff space. By (3) and (5), $x \mapsto S_x$ is a lattice homomorphism $\mathcal{A} \to b(S)$. To show that S is a Stone space of $\mathcal{A}$, we prove this homomorphism to be bijective. First, let $U \in b(S)$. As U is open, there exists an $\mathcal{A}_1 \subset \mathcal{A}$ such that $U = \bigcup\{S_x : x \in \mathcal{A}_1\}$: as U is compact, we may assume that $\mathcal{A}_1$ is finite: but then $U = S_{\sup \mathcal{A}_1}$. For the injectivity, let $x,y \in \mathcal{A}$, $x \neq y$. We may suppose $x \not\leq y$. By Zorn's Lemma, the filter $\{z \in \mathcal{A} : z \vee y \geq x\}$ is contained in a maximal filter J. Then $y' \in J$ and $x \in J$, so $J \in S_x \cap S_{y'} = S_x \setminus S_y$, whence $S_x \neq S_y$.

Thus, S is a Stone space of $\mathcal{A}$.

Now let T be another Stone space of $\mathcal{A}$. Let $x \mapsto T_x$ be a lattice isomorphism of $\mathcal{A}$ onto $b(T)$. Then $S_x \mapsto T_x$ is a lattice isomorphism of $b(S)$ onto $b(T)$. It is now easy to see that there exists a map $\phi : S \to T$ with the property that

$$\phi(s) \in T_x \text{ if and only if } s \in S_x \qquad (s \in S;\ x \in \mathcal{A})$$

and that this ϕ is actually a homeomorphism of S onto T.

Theorem 14.17 enables us to speak of "the" Stone space of a Boolean algebra.

14.18. THEOREM. *The space* Z *mentioned in* Th.14.15 *is the Stone space of the Boolean algebra* $\mathcal{B}[C(X)]$ *(and also of* $\mathcal{B}[B(X)]$ *).*

Proof. By Th.12.9 and the fact that Z is extremally disconnected, Z is the Stone space of $\mathcal{B}[C(Z)]$, i.e. of $\mathcal{B}[B(X)]$. The theorem follows from Lemma 14.19 and Th.14.15.

14.19. LEMMA. *Let* L_0 *be an order dense Riesz subspace of* L.

(*i*) *If* B *is a band in* L, *then* $B\cap L_0$ *is a band in* L_0.

(*ii*) *The map* $B \to B\cap L_0$ $(B \in \mathcal{B}[L])$ *is a lattice isomorphism of* $\mathcal{B}[L]$ *onto* $\mathcal{B}[L_0]$.

Proof. (i) Let B be a band in L, let F be a non-empty subset of $B\cap L_0$ and let $f = L_0\text{-sup } F$. We have to prove that $f \in B\cap L_0$. Now trivially $f\in L_0$. On the other hand, as L_0 is normal in L (13.21(i)), $f = L\text{-sup } F \in B$.
(ii) For $B \in \mathcal{B}[L]$ let $\Omega(B) = B\cap L_0$. Thus, Ω is a map of $\mathcal{B}[L]$ into $\mathcal{B}[L_0]$. We prove that Ω is bijective and an isomorphism of ordered sets.

Obviously, if $B_1 \le B_2$ in $\mathcal{B}[L]$, then $\Omega(B_1) \le \Omega(B_2)$ in $\mathcal{B}[L_0]$. Conversely, let $B_1, B_2 \in \mathcal{B}[L]$ and $B_1 \not\le B_2$. As (in the language of Boolean algebras) we always have

$$B_1 = B_1\wedge(B_2\vee B_2') = (B_1\wedge B_2)\vee(B_1\wedge B_2') \le B_2\vee(B_1\wedge B_2')$$

and $B_2' = B_2^d$, it follows that $B_1\cap B_2^d \ne \{0\}$. Now L_0 is order dense, so its intersection with any non-zero band (or ideal) can never be $\{0\}$. Hence, $B_1\cap B_2^d\cap L_0 \ne \{0\}$. Then certainly $B_1\cap L_0 \not\le B_2\cap L_0$, i.e. $\Omega(B_1) \not\le \Omega(B_2)$.

We see that for $B_1, B_2 \in \mathcal{B}[L]$ we have $B_1 \le B_2$ if and only if $\Omega(B_1) \le \Omega(B_2)$. Therefore, Ω is injective. We are done if it is surjective.

Let $C \in \mathcal{B}[L_0]$. Let B be the band of L, generated by C. Of course, $B\cap L_0 \supset C$. Conversely, take $f \in (B\cap L_0)^+$. Then by the definition of B (see also Ex.2.D) $f = L\text{-sup}\{g\in C^+ : g\le f\}$. But $f\in L_0$ and, consequently, we also have $f = L_0\text{-sup}\{g\in C^+ : g\le f\}$. We see that $(B\cap L_0)^+ \subset C$. Then $B\cap L_0 \subset C$, and $C = B\cap L_0 = \Omega(B)$. Therefore, Ω is surjective.

15. THE RIESZ SPACE $C^{\infty}(X)$

IN THIS SECTION, X *IS AN EXTREMALLY DISCONNECTED COMPACT HAUSDORFF SPACE.*

In the above we have proved that every unitary Riesz space can be embedded in a space of continuous functions on a compact Hausdorff space. We are now going to prove a generalization of this result to arbitrary Riesz spaces.

15.1. DEFINITION. We extend $\mathbb{R}$ to a set $\overline{\mathbb{R}}$ by adjoining to it the symbols ∞ and $-\infty$. In $\overline{\mathbb{R}}$ we introduce in the usual way an ordering and a topology. The tangent function is an order isomorphism and also a homeomorphism of $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ onto $\overline{\mathbb{R}}$.

15.2. DEFINITION. By $C^{\infty}(X)$ we denote the set of all continuous functions $f: X \to \overline{\mathbb{R}}$ that are a.e. finite, i.e. for which the closed sets $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ have empty interior. (See Th.14.2(ii)).

15.3. LEMMA. *Let* A *be a meagre closed subset of* X *and let* f *be a continuous function* $X \backslash A \to \mathbb{R}$*. Then* f *has exactly one extension* $X \to \overline{\mathbb{R}}$ *that is an element of* $C^{\infty}(X)$.

Proof. Define $g: X \to \mathbb{R}$ by

$$g(x) = \operatorname{arctg} f(x) \qquad (x \in X \backslash A),$$

$$g(x) = -\tfrac{1}{2}\pi \qquad (x \in A)$$

Then g is lower semicontinuous. By Lemma 12.15, $g^{\uparrow}$ is continuous, while $g^{\uparrow} = g$ a.e. according to Lemma 14.5. Let $f^{*}(x) = \operatorname{tg} g^{\uparrow}(x)$ $(x \in X)$. Then f^{*}

is continuous $X \to \overline{\mathbb{R}}$ and $f=f^*$ a.e. In particular, f^* is a.e. finite, so $f^* \in C^\infty(X)$. The set $\{x \in X\backslash A : f(x) \neq f^*(x)\}$ is open in $X\backslash A$, hence open in X. As $f=f^*$ a.e., this set must be empty. Thus, f^* is an extension of f.

As $X\backslash A$ is a dense subset of X, f has at most one continuous extension $X \to \overline{\mathbb{R}}$.

15.4. Let $f,g \in C^\infty(X)$. The set A of all points of X where either f or g takes an infinite value is closed and meagre: on its complement, the function $x \to f(x)+g(x)$ is (well-defined and) continuous. Therefore, there exists exactly one element $f+g$ of $C^\infty(X)$ such that

$$(f+g)(x) = f(x)+g(x)$$

for almost every $x \in X$. Similarly, for all $f,g \in C^\infty(X)$ there is a unique $fg \in C^\infty(X)$ for which

$$(fg)(x) = f(x)g(x)$$

for almost every $x \in X$.

15.A. *Exercise.*With these definitions and with the natural ordering and scalar multiplication, $C^\infty(X)$ is an Archimedean Riesz space and a commutative Riesz algebra. $C(X)$ is an order dense Riesz subspace of $C^\infty(X)$.

15.B. *Exercise.* $C^\infty(X)$ is Dedekind complete.
(Hint. $f \to \operatorname{arctg} f$ is an order-preserving injection $C^\infty(X) \to C(X)$).

15.C. *Exercise.* Let S be any set, $\mathbb{R}^S$ the Riesz algebra of all functions $S \to \mathbb{R}$. Then there exists an extremally discinnected compact Hausdorff space Y such that $\mathbb{R}^S$ and $C^\infty(Y)$ are isomorphic Riesz algebras. (Hint. Give

S the discrete topology and let $Y=S^{\beta}$. (See Th.13.34). By 13.P(iii), $\beta(S)$ is an open subset of S^{β} whose complement is meagre. The restriction topology on $\beta(S)$ is discrete and β is a bijection $S \to \beta(S)$. Now apply 15.3).

Spaces of the type $C^{\infty}(X)$ are of interest because of the following.

15.5. THEOREM. (Maeda-Ogasawara). *Let* X *be the Stone space of* $\mathcal{B}[L]$. *Then there exists a Riesz isomorphism* Φ *of* L *onto an order dense Riesz subspace of* $C^{\infty}(X)$.

For the proof we need

15.6. LEMMA. *Let* L_0 *be a Riesz subspace of* L *that is order dense in the ideal* (L_0). *(E.g.,* L_0 *is order dense in* L *or* L_0 *is an ideal of* L). *Let* Θ *be a Riesz isomorphism of* L_0 *onto an order dense Riesz subspace of* $C^{\infty}(X)$. *Then* Θ *can be extended to a Riesz homomorphism* $\Phi: L \to C^{\infty}(X)$ *whose kernel is* $(L_0)^d$.

Proof. Take $f \in L^+$. By Th.12.16 and its proof, the set

$$\{\text{arctg } (\Theta w) \; : \; w \in L_0^+, \; w \leq f\}$$

has a supremum g in $C(X)$ and for almost every $x \in X$ we have

$$g(x) = \sup\{\text{arctg}(\Theta w)(x) \; : \; w \in L_0^+, \; w \leq f\}.$$

Therefore, there exists a continuous $\overline{\Theta}f: X \to \overline{\mathbb{R}}$ such that

$$(\overline{\Theta}f)(x) = \sup\{(\Theta w)(x) \; : \; w \in L_0^+, \; w \leq f\}$$

for almost every $x \in X$.

Now take $u \in L_0^+$ such that $\Theta u \leq \overline{\Theta}f$: we prove $u \leq f$. Suppose $u \not\leq f$. Then $u-(u \wedge f) > 0$ so that, by the given property of L_0, there exists a non-zero $u' \in L_0^+$ for which $u-(u \wedge f) \geq u'$. Now for almost every $x \in X$,

$$(\Theta u)(x) = \sup\{(\Theta u)(x)\wedge(\Theta w)(x) : w\in L_0^+,\ w\le f\} =$$

$$= \sup\{[\Theta(u\wedge w)](x) : w\in L_0^+,\ w\le f\} \le [\Theta(u-u')](x)$$

Thus, $\Theta u \le \Theta(u-u')$, whence $u \le u-u'$. But this is impossible since $u'>0$.

We can use this observation to prove that $\overline{\Theta}f \in C^\infty(X)$. In fact, suppose that $\overline{\Theta}f \notin C^\infty(X)$. Then the closed set $\{x : (\overline{\Theta}f)(x) = \infty\}$ contains a non-empty clopen subset W of X. As $\Theta(L_0)$ is order dense in $C^\infty(X)$, there exists a non-zero $u\in L_0^+$ for which $\Theta u \le \chi_W$. For all $n\in\mathbb{N}$ we now have $\Theta(nu) = n\Theta(u) \le n\chi_W \le \overline{\Theta}f$, hence, $nu \le f$. But this is contradictory because L is Archimedean and $u>0$.

Thus, for every $f\in L^+$ we obtain $\overline{\Theta}f \in C^\infty(X)$.

Let $f_1,f_2\in L^+$. If $w_1,w_2\in L_0^+$ and $w_1\le f_1$, $w_2\le f_2$, then $w_1+w_2 \le f_1+f_2$ and

$$(\Theta w_1) + (\Theta w_2) = \Theta(w_1+w_2) \le \overline{\Theta}(f_1+f_2)$$

Therefore, $(\overline{\Theta}f_1)+(\overline{\Theta}f_2) \le \overline{\Theta}(f_1+f_2)$. To prove the converse inequality, take $w\in L_0^+$, $w \le f_1+f_2$: it suffices to prove that $\Theta w \le (\overline{\Theta}f_1)+(\overline{\Theta}f_2)$. Set $w_1 = w\wedge f_1$, $w_2 = w-w_1$: then $w_1,w_2\in L^+$, $w = w_1+w_2$, $w_1\le f_1$, $w_2\le f_2$. By the assumption on L_0 we have $w_i =\{\sup u_i\in L_0^+ : u_i\le w_i\}$ $(i=1,2)$ and therefore

$$w = \sup\{u_1+u_2 : u_1,u_2\in L_0^+ : u_1\le w_1,\ u_2\le w_2\}$$

Consequently,

$$\Theta w = \sup\{ \Theta u_1+\Theta u_2 : u_1,u_2\in L_0^+,\ u_1\le w_1,\ u_2\le w_2\} \le$$

$$\le \sup\{\Theta u_1 : u_1\in L_0^+,\ u_1\le f_1\} + \sup\{\Theta u_2 : u_2\in L_0^+,\ u_2\le f_2\} \le$$

$$\le \overline{\Theta}f_1 + \overline{\Theta}f_2$$

We see that $\overline{\Theta}(f_1+f_2) = (\overline{\Theta}f_1)+(\overline{\Theta}f_2)$ for all $f_1,f_2\in L_0^+$. As, clearly, $\overline{\Theta}(\lambda f) = \lambda\overline{\Theta}(f)$ for all $\lambda\ge 0$ and $f\in L^+$, $\overline{\Theta}$ can be extended to a linear map $\Phi:L \to C^\infty(X)$. It is easy to see that $\overline{\Theta}(f_1\wedge f_2) = (\overline{\Theta}f_1)\wedge(\overline{\Theta}f_2)$ for all $f_1,f_2\in L^+$. Therefore, Φ is a Riesz homomorphism. Besides, the kernel of Φ is L_0^d because $\{f\in L^+ : \overline{\Theta}f = 0\} = L^+\cap L_0^d$.

Now we have enough machinery to prove Th.15.5.

Let E be a maximal subset of $L\setminus\{0\}$ whose elements are pairwise disjoint. Let L_0 be the smallest ideal of L that contains E. (L_0 is the algebraic sum of the principal ideals (e) generated by the elements e of E). By the maximality of E we have $L_0^{\,d} = \{0\}$.

Under coordinatewise operations the Cartesian product $\prod_{e\in E}(e)$ is an Archimedean Riesz space M. Let $M_0 = \{g\in M : g_e\neq 0$ for only finitely many elements e of $E\}$. The formula

$$\Omega f = \sum_{e\in E} f_e \qquad (f\in M_0)$$

defines a Riesz isomorphism Ω of M_0 onto L_0.

Now M has a unit, viz. the element e_M defined by

$$(e_M)_e = e \qquad (e\in E)$$

By Th.13.11 and Th.14.15 there exist an extremally disconnected compact Hausdorff space Y and a Riesz isomorphism Ψ of M onto an order dense Riesz subspace of $C(Y)$ such that $\Psi(e_M) = \underline{1}$. Then $\Psi\Omega^{-1}$ is a Riesz isomorphism Θ of L_0 into $C(Y)$. Now M_0 is order dense in M, $\Psi(M)$ is order dense in $C(Y)$ and $C(Y)$ is order dense in $C^{\infty}(Y)$: thus, $(\Psi\Omega^{-1})(L_0)$ is order dense in $C^{\infty}(Y)$. By our lemma (and by the equality $L_0^{\,d} = \{0\}$) there exists a Riesz isomorphism Φ of L into $C^{\infty}(Y)$ that is an extension of $\Psi\Omega^{-1}$.

In particular, $\Phi(L) \supset (\Psi\Omega^{-1})(L_0)$, so $\Phi(L)$ is order dense in $C^{\infty}(Y)$. The Boolean algebras $b(Y)$, $\mathcal{B}[C(Y)]$, $\mathcal{B}[C^{\infty}(Y)]$, $\mathcal{B}[\Phi(L)]$ and $\mathcal{B}[L]$ are lattice isomorphic (Lemma 14.20), so that Y is a Stone space of $\mathcal{B}(L)$.

15.7. DEFINITION. An element u of L^+ is called a *weak unit* or *Freudenthal unit* of L if the principal band generated by u is L itself. (Cf. Def.13.7). Trivially, every strong unit is a weak unit.

15.D. *Exercise*. For every $u \in L^+$ the following properties are equivalent.

(a) u is a weak unit of L.

(b) $\{u\}^d = \{0\}$, i.e. for every non-zero $v \in L^+$ we have $u \wedge v \neq 0$.

(c) For every $f \in L^+$, $f = \sup\{f \wedge nu : n \in \mathbb{N}\}$.

15.E. *Examples*. In the Riesz space $L_1(\mathbb{R})$ the function $x \mapsto e^{-|x|}$ is a weak unit. More generally, if (Z,Γ,μ) is a σ-finite measure space, then $L_1(\mu)$ has a weak unit. (Take any element $e \in L_1(\mu)$ for which $e(z) > 0$ for μ-almost every $z \in Z$). If (Z,Γ,μ) is any measure space and if L is the Riesz space of all Γ-measurable functions modulo μ-negligible sets (Example 1.F), then $\underline{1}$ is a weak unit of L.

If S is any topological space, $\underline{1}$ is a weak unit in the Riesz space $C(S)$ of all continuous functions on S. In $C^\infty(X)$, $\underline{1}$ is a weak unit. c_{00} has no weak unit.

For a Riesz space with a weak unit we can make our representation theorem a little more precise:

15.8. COROLLARY. (Sequel to Th.15.5). *If* u *is a weak unit of* L, *then* Φ *can be chosen such that* $\Phi(u) = \underline{1}$.

Proof. In the proof of Th.15.5, choose $E = \{u\}$.

15.9. THEOREM. *Let* L *be a Riesz algebra with a weak unit* u *that is a two-sided identity element for the multiplication. If* X *and* Φ *are as in Th.15.5 and if* $\Phi(u)=\underline{1}$ (see Cor.15.8), *then* Φ *is multiplicative.*

Proof. Take $w\in L$, $w\geq u$.

By Ex.13.K, the principal ideal (u) generated by u is a unitary Riesz algebra. Let $f,f'\in(u)$, $f\perp f'$. It follows from Th.13.32 that $ff'=0$. Then by Ex.13.K(ii), $((wf)\wedge f')^2 \leq (wf)f' = w(ff') = 0$, so that (apply 13.32 once more) $wf\wedge f' = 0$ and $wf\perp f'$. Thus, for any $f\in(u)$, wf is an element of the band generated by f. From this it follows that, if $f,f'\in(u)$ and $f\perp f'$, then $wf \perp wf'$. In particular, for every $f\in(u)$ we have $wf^+ \perp wf^-$. As $wf^+ + wf^- = wf$ and $wf^+ \geq 0$, $wf^- \geq 0$, we see that $wf^+ = (wf)^+$ and $wf^- = (wf)^-$. Hence, the map $f \mapsto wf$ is a Riesz homomorphism $(u) \to L$. For every $f\in(u)$ with $f\geq 0$ we have $wf \geq uf = f$, so that $f \mapsto wf$ really is a Riesz isomorphism. Now $\Phi(w) \geq \Phi(u) = \underline{1}$. Thus, we can define a Riesz isomorphism $\Psi:(u) \to C^{\infty}(X)$ by

$$\Psi(f) = \frac{\Phi(wf)}{\Phi(w)} \qquad (f\in(u))$$

Note that $\Psi(u) = \underline{1}$.

We proceed to prove that $\Phi\equiv\Psi$ on (u). Assume $\Phi\not\equiv\Psi$. Then there exist $f\in(u)$, $x\in X$ and $s\in\mathbb{R}$ such that $(\Phi f)(x) < s < (\Psi f)(x)$. We may assume $s=0$. (Otherwise replace f by $f-su$). Let U be a clopen subset of X such that $\Phi f < 0$ on U, $\Psi f > 0$ on U, and set $B = \{g\in L : \Phi g \equiv 0 \text{ on } U\}$. Then B is a band in L and $f^+\in B$. As wf^+ is an element of the band generated by f^+, we have $wf^+ \in B$, $\Phi(wf^+) \equiv 0$ on U and $\Psi(f^+) \equiv 0$ on U. But we had chosen U such that $\Psi f > 0$ on U : contradiction.

Thus,

$$(*) \qquad \Phi(f) = \frac{\Phi(wf)}{\Phi(w)} \qquad (f\in(u);\ w\in L,\ w\geq u)$$

Now take $v,w\in L$, $v\geq u$ and $w\geq u$. As the image of (u) under Φ is order-dense in $C^{\infty}(X)$ there exists a subset F of (u) such that

$$\frac{1}{\Phi(w)} = \sup_{f\in F} \Phi(f)$$

Then we obtain the following identities. (The second one is not trivial but it is easily established).

$$\underline{1} = \Phi(w)\cdot\sup_{f\in F}\Phi(f) = \sup_{f\in F}\Phi(w)\Phi(f) = \sup_{f\in F}\Phi(wf)$$

For every $f\in F$ we have $\Phi(wf) \le \underline{1}$, hence $wf \in (u)$ and (according to (*)) $\Phi(vwf) = \Phi(v)\Phi(wf)$. Therefore,

$$\Phi(v) = \Phi(v)\cdot\sup_{f\in F}\Phi(wf) = \sup_{f\in F}\Phi(v)\Phi(wf) =$$

$$= \sup_{f\in F}\Phi(vwf) = \sup_{f\in F}\Phi(vw)\Phi(f) = \Phi(vw)\cdot\frac{1}{\Phi(w)}$$

and we obtain

$$\Phi(v)\Phi(w) = \Phi(vw) \qquad (v,w\in L;\ v\ge u,\ w\ge u)$$

As every element of L is a difference of two elements that are $\ge u$ it follows that Φ is multiplicative.

15.10. COROLLARY. *Let* L *and* u *be as above.*

(i) *The multiplication in* L *is commutative.*

(ii) *For* $f,g\in L$ *one has* $f\perp g$ *if and only if* $fg=0$.

(iii) *If* $f,g\in L$, *then* $|fg| = |f||g|$.

(iv) *Let* $g\in L^+$. *Then the map* $f\mapsto gf$ *is a Riesz homomorphism. Moreover, if* g *is not a divisor of zero, then* $f\mapsto gf$ *is a Riesz isomorphism of* L *onto an order-dense Riesz subspace of* L.

If, in addition, L *is Dedekind complete, then*

(v) *Every element of* L^+ *has a unique square root in* L^+.

(vi) *If* $f\in L^+$ *and* $f\ge u$, *then* f *has an inverse.*

15.F. *Exercise*. (Characterization of $C^\infty(X)$ among the Riesz algebras). Let L,u be as in Th.15.9. The following conditions are equivalent.

(a) There exists an extremally disconnected compact Hausdorff space Y such that L and $C^\infty(Y)$ are isomorphic as Riesz algebras.

(b) L is Dedekind complete and every weak unit of L is invertible.

15.G. *Exercise*. (Application of 15.F). In each of the following cases the Riesz algebra L is isomorphic to $C^{\infty}(Y)$ where Y is the (extremally disconnected) Stone space of $\mathcal{B}[L]$.

(i) Let S be any set, L the Riesz algebra of all real-valued functions on S. (See Ex.15.C).

(ii) Let (Z,Γ,μ) be a σ-finite measure space, L the Riesz algebra of all Γ-measurable functions on Z modulo μ-negligible sets. (Ex.1.F).

(iii) Let Z be a compact Hausdorff space, let L_0 be the Riesz algebra of all Borel measurable functions on Z and let L be the quotient space $L_0/\{f\in L_0 : f\equiv 0 \text{ a.e.}\}$. (The relation between this L and B(Z) is quite similar to the one between $C^{\infty}(X)$ and C(X) (X extremally disconnected) or between the L of 15.G(ii) and $L_{\infty}(\mu)$).

Let X and Φ be as in Th.15.5. Let M be the ideal of $C^{\infty}(X)$ generated by $\Phi(L)$. As $C^{\infty}(X)$ is Dedekind complete, so is M. In view of Lemma 14.13, M is a Dedekind completion of L. We have proved:

15.11. COROLLARY. L *has a Dedekind completion.*

Once we know that a Dedekind completion exists, it is not too hard to prove its uniqueness. (Cor.15.17).

15.12. DEFINITION. A *Dedekind cut* of L is a pair (A:B) of non-empty subsets A,B of L such that $A = \{f\in L : f$ is a lower bound of $B\}$ and $B = \{g\in L : g$ is an upper bound of $A\}$.

We denote by $L^{=}$ the set of all Dedekind cuts of L. Every element f of L determines a Dedekind cut $f^{=}$ by $f^{=} = (f+L^{-}:f+L^{+})$.

15.13. We shall give $L^{=}$ the structure of a Riesz space in such a way that $L^{=}$ becomes a Dedekind completion of L. The basic idea is the following. We take a Dedekind completion M of L and construct a natural bijection $L^{=} \to M$. Then we use this bijection to lift the Riesz space operations from M to $L^{=}$, rendering $L^{=}$ a Dedekind completion of L. Next, we show that these Riesz space operations on $L^{=}$ can be formulated in terms of $L^{=}$ itself without mention of M. The result will be not only that all Dedekind completions of L are isomorphic but also an intrinsic description of the Dedekind completion. (In [10] and [15] this description is used for the existence proof).

First we have to introduce some terminology. For subsets A, A_1, A_2 of L, for $h \in L$ and for $s \in \mathbb{R}$ we define

$$-A = \{-f : f \in A\}$$
$$A_1 + A_2 = \{f_1 + f_2 : f_1 \in A_1,\ f_2 \in A_2\}$$
$$h + A = \{h + f : f \in A\}$$
$$sA = \{sf : f \in A\}$$

Now let M be any Dedekind complete Riesz space containing L as a normal Riesz subspace. (M is not necessarily a Dedekind completion of L).

Let $(A:B) \in L^{=}$. A is a non-empty subset of M, having an upper bound, so $\sup A$ exists in M. Similarly, $\inf B$ exists. Furthermore, it is clear that $\sup A \le \inf B$. We proceed to prove that actually $\sup A = \inf B$. First, observe that $\inf B - \sup A = \inf B + \inf(-A) = \inf(B+(-A))$. By the normality of L we are done if $0 = L\text{-}\inf(B+(-A))$. Now certainly 0 is a lower bound for $B+(-A)$ in L. If h is any lower bound for $B+(-A)$ in L, then $B-h$ consists of upper bounds of A, so $B-h \subset B$. But then $B-nh \subset B$ for all $n \in \mathbb{N}$. Taking $f \in A$ and $g \in B$ we have $nh \le g-f$ for all $n \in \mathbb{N}$, whence $h \le 0$ because L is Archimedean. Thus, 0 is the greatest lower bound of $B+(-A))$ in L, and $\inf B - \sup A = 0$.

We can now define a map $\Omega : L^{=} \to M$ by

(1) $\Omega(A:B) = \sup A = \inf B \qquad ((A:B) \in L^{=})$

Observe that

(2) $\Omega f^{=} = f \qquad (f \in L)$

For $(A:B) \in L^{=}$ and $(A':B') \in L^{=}$ we have $\Omega(A:B) \leq \Omega(A':B')$ if and only if $\sup A \leq \inf B'$, i.e. if and only if all elements of A are lower bounds for B'. Thus,

(3) $\Omega(A:B) \leq \Omega(A':B')$ if and only if $A \subset A'$.

In particular, if $\Omega(A:B) = \Omega(A':B')$, then $A=A'$. Therefore,

(4) Ω is injective.

If $(A:B) \in L^{=}$, then $(-B:-A) \in L^{=}$ and $(sA:sB) \in L^{=}$ for all $s \in (0,\infty)$. It is easy to see that

(5) $$\left.\begin{aligned} \Omega(-B:-A) &= -\Omega(A:B) \\ \Omega(sA:sB) &= s\Omega(A:B) \end{aligned}\right] \qquad ((A:B) \in L^{=};\ s>0)$$

Now let $(A':B') \in L^{=}$ and $(A'':B'') \in L^{=}$: we construct an element $(A:B)$ of $L^{=}$ whose image under Ω is $\Omega(A':B')+\Omega(A'':B'')$. For A we take the set of all elements of L that are lower bounds of $B'+B''$: then $A \neq \emptyset$ as $A \supset A'+A''$. Let B be the set of all upper bounds of A in L : then $B \neq \emptyset$ as $B \supset B'+B''$. By the latter relation we have $A \supset \{f \in L : f$ is a lower bound of $B\}$. But the converse inclusion follows immediately from the definition of B. Hence, $(A:B) \in L^{=}$. The relations $A'+A'' \subset A$ and $B'+B'' \subset B$ now imply that $\Omega(A':B')+\Omega(A'':B'') = \sup A' + \sup A'' = \sup(A'+A'') \leq \sup A = \inf B \leq \inf(B'+B'') = \inf B' + \inf B'' = \Omega(A':B')+\Omega(A'':B'')$. Hence, we have $\Omega(A:B) = \Omega(A':B')+\Omega(A'':B'')$. (Moreover, we see that $\sup(A'+A'') = \sup A$, so $\{g \in L : g$ is an upper bound of $A'+A''\} = \{g \in L : g$ is an upper bound of $A\} = B$).

It follows that the range space of Ω is a linear subspace of M.

15.14. If L is an order dense subspace of M, we can do better: in that case, $\Omega(L^{=})$ is just the ideal (L) of M generated by L. In fact, if $(A:B)\in L^{=}$, we can choose $f\in A\subset L$ and $g\in B\subset L$: then $f\leq\Omega(A:B)\leq g$, so $\Omega(A:B)\in(L)$. Conversely, if $h\in(L)$, choose $A=\{f\in L : f\leq h\}$ and $B=\{g\in L : g\geq h\}$. By the order-denseness of L, A and B are non-empty, and $\sup A = h = \inf B$. It follows that $(A:B)\in L^{=}$ and $h = \Omega(A:B)$.

For a first application we only observe that there actually exists a Dedekind complete Riesz space M containing (a subspace isomorphic to) L as a normal Riesz subspace. Construct Ω as above. Then Ω is a bijection of $L^{=}$ onto a Riesz subspace of M. The Riesz space operations on $\Omega(L^{=})$ can be transplanted to $L^{=}$ by way of Ω. We obtain:

15.15. THEOREM. *Introduce an addition, a scalar multiplication and a binary relation* $\leq$ *on* $L^{=}$ *by*

(*i*) $(A':B')+(A'':B'') = (A:B)$ *where* $A=\{f\in L :$ f *is a lower bound of* $B'+B''\}$ *and* $B=\{g\in L :$ g *is an upper bound of* $A'+A''\}$.

(*ii*) $s(A:B) = (sA:sB)$ *if* $s>0$; $s(A:B) = (sB:sA)$ *if* $s<0$; *while* $s(A:B) = (L^{-}:L^{+})$ *if* $s=0$.

(*iii*) $(A':B')\leq(A'':B'')$ *if* $A'\subset A''$.

Under these definitions, $L^{=}$ *becomes a Riesz space. The map* $f\mapsto f^{=}$ *is a Riesz isomorphism of* L *into* $L^{=}$.

From here on, by $L^{=}$ we shall denote the Riesz space that is introduced in 15.15.

Then for any M, the Ω constructed in 15.13 is a Riesz isomorphism. By the lines leading up to Cor.15.11, L has a Dedekind completion M in which L is an order dense ideal. But then $\Omega(L^{=})$ is the ideal generated by L, i.e. $\Omega(L^{=}) = M$. Thus:

15.16. COROLLARY. $L^{=}$ *is a Dedekind completion of* L.

In particular, $L^{=}$ is Dedekind complete. Returning to the above construction (15.13) we let M be any Dedekind completion of L. Then we see that $L \subset \Omega(L^{=}) \subset M$, while $\Omega(L^{=})$, being isomorphic to $L^{=}$, is Dedekind complete. By the definition of "Dedekind completion" (Def.14.11) it follows that $\Omega(L^{=}) = M$. We have now proved:

15.17. COROLLARY. *Every Dedekind completion of* L *is isomorphic to* $L^{=}$. *More precisely, if* (M,Φ) *is a Dedekind completion of* L, *there exists a Riesz isomorphism* Ω *of* $L^{=}$ *onto* M *such that* $\Omega(f^{=}) = \Phi(f)$ *for every* $f\in L$. (The latter condition completely determines Ω).

Every Dedekind complete Riesz space containing L as a normal Riesz subspace contains a Dedekind completion of L:

15.18. THEOREM. *Let* L *be a normal Riesz subspace of a Dedekind complete Riesz space* M. *There exists a unique Riesz homomorphism* Ω *of* $L^{=}$ *into* M *such that* $\Omega(f^{=}) = f$ *for all* $f\in L$. *This* Ω *is given by the formula*

$$\Omega(A:B) = \sup A = \inf B \qquad ((A:B)\in L^{=}).$$

Ω *is actually a Riesz isomorphism of* $L^{=}$ *onto a normal Riesz subspace* $\Omega(L^{=})$ *of* M. $\Omega(L^{=})$ *is a Dedekind completion of* L *and is the smallest Dedekind complete Riesz subspace of* M *that contains* L. *We have*

$$(*) \qquad \Omega(L^{=}) = \{M\text{-}\sup F : F \text{ is a non-empty subset of } L \text{ with an upper bound in } L\}.$$

If L *is order dense in* M, *then* $\Omega(L^{=})$ *is the ideal generated by* L.

Proof. We already have an injective Riesz homomorphism $\Omega: L^{=} \to M$ with

$$\Omega(f^{=}) = f \qquad (f \in L)$$

$$\Omega(A:B) = \sup A = \sup B \qquad ((A:B) \in L^{=})$$

Let us first prove (*). Take a non-empty subset A' of L such that the set B, consisting of all upper bounds of A' in L, is non-empty: then A' has a supremum in M and it suffices to prove that $M\text{-}\sup A' \in \Omega(L^{=})$. It follows from the last ten lines of page 130 that

$$M\text{-}\sup A' = M\text{-}\inf B.$$

If A is the set of all lower bounds of B in L, then $A \neq \emptyset$ because $A \supset A'$. We then have $(A:B) \in L^{=}$ and $M\text{-}\inf B = \Omega(A:B) \in \Omega(L^{=})$. Therefore, $M\text{-}\sup A' \in \Omega(L^{=})$. This proves (*).

Next, we prove the normality of $\Omega(L^{=})$ in M. To this end, take a non-empty subset F of $L^{=}$ that has an $L^{=}$-supremum $(A_0:B_0)$. We are done if we can show that $M\text{-}\sup \Omega(F) \in \Omega(L^{=})$. For $f \in F$ define the subsets A_f and B_f of L by $f = (A_f:B_f)$. Let $A' = \bigcup\{A_f : f \in F\}$. Then

$$M\text{-}\sup \Omega(F) = M\text{-}\sup_{f \in F}(M\text{-}\sup A_f) = M\text{-}\sup A'$$

Now A' has upper bounds in L, since $B_0 \neq \emptyset$. By (*), $M\text{-}\sup \Omega(F) \in \Omega(L^{=})$.

To prove the uniqueness of Ω, let Ω' be any Riesz homomorphism of $L^{=}$ into M such that $\Omega'(f^{=}) = f$ for all $f \in L$. Take $(A:B) \in L^{=}$. For all $f \in A$ we see that $(A:B) \geq f^{=}$, so $\Omega'(A:B) \geq \Omega'(f^{=}) = f$. It follows that $\Omega'(A:B) \geq M\text{-}\sup A = \Omega(A:B)$. But, similarly, $\Omega'(A:B) \leq \inf B = \Omega(A:B)$. Thus, $\Omega = \Omega'$.

If M' is any Dedekind complete Riesz subspace of M that contains L, by 15.13 there exists a Riesz homomorphism $\Omega': L^{=} \to M' \subset M$ for which $\Omega'(f^{=}) = f$ $(f \in L)$. But, as we have just shown, such an Ω' must be the same as Ω. Hence, $M' \supset \Omega'(L^{=}) = \Omega(L^{=})$. Thus, $\Omega(L^{=})$ is the smallest Dedekind complete Riesz subspace of M that contains L.

It is known (see, e.g., [4]) that any partially ordered set S has a "conditional completion" $S^{=}$. Here the elements of $S^{=}$ are "Dedekind cuts" $(A:B)$, defined in complete analogy to our Def.15.12 and they are ordered by

$$(A':B') \le (A'':B'') \quad \text{if} \quad A' \subset A''.$$

Thus, the construction we gave in 15.11 for a Riesz space L amounts to introducing a Riesz space structure for the conditional completion of the underlying ordered set L. The existence proofs for the Dedekind completion of a Riesz space given in [10] and [15] start from the conditional completion of an ordered set.

We close this section by giving two characterizations of Riesz spaces that are Riesz isomorphic to spaces of the type $C^{\infty}(X)$.

15.20. DEFINITION. L is said to be *universally complete* if every non-empty subset of L^{+} that consists of pairwise disjoint elements has a supremum. (An easy example is $\mathbb{R}^{S}$ for any set S).

15.21. DEFINITION. L is called *inextensible* if it has the following property. If M_0 is an ideal in a Riesz space M such that M_0 is Riesz isomorphic to L, then M_0 is a projection band in M.

15.22. THEOREM. *The following conditions on* L *are equivalent.*

(a) *There exists an extremally disconnected compact Hausdorff space* X *such that* L *is Riesz isomorphic to* $C^{\infty}(X)$.

(b) L *is Dedekind complete and universally complete.*

(c) L *is Dedekind complete and inextensible.*

Proof. (a)⇒(b). Let F be a non-empty collection of pairwise disjoint elements of $C^{\infty}(X)^{+}$. For every $f \in F$, set

$$W_f = \{x \in X : 0 < f(x) < \infty\}.$$

On the clopen set $X \setminus \overline{W_f}$ we have $f \equiv 0$. Let W be the closure of the union of all sets W_f $(f \in F)$: then W is clopen. There exists a unique continuous function g on the set $(X \setminus W) \cup \bigcup \{W_f : f \in F\}$ such that

$g \equiv 0$ on $X \setminus W$,

for every $f \in F$, $g \equiv f$ on W_f,

By Lemma 15.3, g extends uniquely to an element $\overline{g}$ of $C^{\infty}(X)$. For every f we have $f \leq g$ a.e.. Thus, $\overline{g}$ is an upper bound of F. It clearly is the least upper bound of F.

(b)⇒(a). By the Maeda-Ogasawara Theorem we may assume that L is an order dense Riesz subspace of $C^{\infty}(X)$ for some extremally disconnected compact Hausdorff space X. By Lemma 13.21(ii), L is an ideal in $C^{\infty}(X)$. Take $g \in C^{\infty}(X)$: we show that $g \in L$.

Let $\mathcal{U}$ be a collection of clopen subsets of X, maximal relative to the following two properties (i) and (ii).

(i) $U \cap V = \emptyset$ if $U, V \in \mathcal{U}$ and $U \neq V$,

(ii) $g\chi_U \in L$ for every $U \in \mathcal{U}$.

Let W be the closure of $\bigcup \{U : U \in \mathcal{U}\}$: then W is clopen.

By the universal completeness, $\{g\chi_U : U \in \mathcal{U}\}$ has a supremum f in L. As L is order dense in $C^{\infty}(X)$, $f = C^{\infty}(X)\text{-sup}\,\{g\chi_U : U \in \mathcal{U}\}$, i.e. $f = g\chi_W$. We are done if we can prove that $g = g\chi_W$, so assume $g \neq g\chi_W$. Again using the order-denseness of L, we find a non-zero element h of L for which $h \leq g - g\chi_W = g\chi_{X \setminus W}$. As $g < \infty$ a.e., it follows easily that there exist a non-empty clopen set U and real numbers s and t such that

$h \geq s > 0$ on U and $g \leq t$ on U.

Now $g\chi_U \leq ts^{-1}h$. As $h \in L$ and L is an ideal in $C^{\infty}(X)$, it follows that $g\chi_U \in L$. But $U \subset X \setminus W$: we have a contradiction with the maximality of $\mathcal{U}$.

(a)⇒(c). Suppose we have an ideal M_0 in a Riesz space M and a Riesz isomorphism Θ of M_0 onto $C^\infty(X)$. By Lemma 15.6, Θ extends to a Riesz homomorphism Φ of M onto $C^\infty(X)$ whose kernel is M_0^d. Now for every $f \in M$ we have $\Theta^{-1}\Phi f \in M_0$ and $f - \Theta^{-1}\Phi f \in M_0^d$ (since $f = \Phi\Theta^{-1}\Phi f$) : hence, $f \in M_0 + M_0^d$.

(c)⇒(a). By the Maeda-Ogasawara Theorem 15.5 and by 13.21(ii), we may assume that L is an order dense ideal in $C^\infty(X)$ for some extremally disconnected compact Hausdorff space X. According to Lemma 13.18 the band generated by L is $C^\infty(X)$. But Condition (c) certainly implies that L is a band. Therefore, $L = C^\infty(X)$.

15.H. *Exercise*. The conditions (a),(b),(c) are equivalent to the following condition (d).

(d) If M_0 is an order dense Riesz subspace of a Riesz space M such that M_0 is Riesz isomorphic to L, then $M_0 = M$.

15.I. *Exercise*. The locally constant functions on any infinite zerodimensional compact Hausdorff space form a universally complete Riesz space that is not Dedekind complete.

16. L-SPACES AND M-SPACES

We are now going to consider Banach lattices. There are two simple special cases, viz. $C(X)$ (where X is a compact Hausdorff space) and $L_1(\mu)$ (where μ is a measure). We shall obtain characterizations of these spaces among the Banach lattices and apply our knowledge to the theory of measures on topological spaces.

UP TO PAGE 149, L *IS A BANACH LATTICE WITH NORM* ρ.

16.1. THEOREM. *Let* L *have a strong unit* e. *Then the norm* ρ *is equivalent to the norm* σ_e *induced by* e. (See Def.13 27). *In the terminology of the Representation Theorem* 13.11, $\hat{L} = C(M)$ *and* ρ *is equivalent to the norm* $f \to \|\hat{f}\|_\infty$.

Proof. We only have to prove the equivalence of ρ and σ_e : the rest is a direct consequence of Th.13.28. Now for every $f \in L$ we have $|f| \leq \sigma_e(f)e$ (definition of $\sigma_e(f)$), so $\rho(f) \leq \sigma_e(f)\rho(e)$. Thus, $\rho \leq \rho(e)\sigma_e$. On the other hand, by Th.10.3(ii) there exists a number C such that $\sigma_e \leq C\rho$.

By strengthening the conditions in Th.16.1 we can obtain equality of ρ and σ_e:

16.2. THEOREM. *Let the set* $\{f \in L^+ : \rho(f) \leq 1\}$ *have a supremum* e *for which* $\rho(e) \leq 1$. *Then* e *is a strong unit,* $\rho(e)=1$ *and* $\rho = \sigma_e$. *In the language of* Th.13.11, *the map* $f \mapsto \hat{f}$ *is an isometric Riesz isomorphism of* L *onto* $C(M)$.

Proof. Set $W = \{f \in L^+ : \rho(f) \le 1\}$. Of course, $e \in L^+$ and (if $L \ne \{0\}$) $e \ne 0$. For every $f \in L^+ \backslash \{0\}$, $\rho(f)^{-1} f \in W$, so $\rho(f)^{-1} f \le e$ and $f \le \rho(f)e$. It follows that e is a strong unit and that $\sigma_e \le \rho$. In particular, $1 = \sigma_e(e) \le \rho(e)$, so $\rho(e)=1$. But for every $f \in L^+$ we have $f \le \sigma_e(f)e$, and therefore $\rho(f) \le \sigma_e(f)\rho(e) = \sigma_e(f)$. Thus, $\rho \le \sigma_e$. This essentially proves the theorem.

16.A. *Examples* are $BC(X)$ (X is a topological space); every norm-closed Riesz subspace of such a $BC(X)$; $B(X)$ (X is a compact Hausdorff space); and $L_\infty(\mu)$ (where μ is a measure).

For every topological space X we have

$$\|f \vee g\|_\infty = \|f\|_\infty \vee \|g\|_\infty \qquad (f,g \in BC(X)^+)$$

For every measure space (X,Γ,μ),

$$\|f+g\|_1 = \|f\|_1 + \|g\|_1 \qquad (f,g \in L_1(\mu)^+)$$

where $\|\cdot\|_1$ denotes the natural norm on $L_1(\mu)$.

These formulas lead us to a definition.

16.3. DEFINITION. L is said to be an (*abstract*) *M-space* if

$$\rho(f \vee g) = \rho(f) \vee \rho(g) \qquad (f,g \in L^+)$$

L is called an (*abstract*) *L-space* if

$$\rho(f+g) = \rho(f) + \rho(g) \qquad (f,g \in L^+)$$

(In either case one may replace the condition "$f,g \in L^+$" by "$f,g \in L^+$ and $f \perp g$" without changing the concept of an M-space or an L-space. See [12]).

16.B. *Exercise*. Let (X,Γ) be a measurable space. The bounded additive functions $\Gamma \to \mathbb{R}$ form a Riesz space L. The definition

$$\rho(\mu) = |\mu|(X) \qquad (\mu \in L)$$

renders L an L-space. The σ-additive elements of L form a norm-closed band in L which is also an L-space.

16.4. THEOREM. *(i)* L *is an M-space if and only if* L^* *is an L-space.* *(ii)* L *is an L-space if and only if* L^* *is an M-space.*

Proof. Part A. Let L be an M-space. Let $\phi,\psi\in L^{*+}$. Of course we have $\rho^*(\phi+\psi) \le \rho^*(\phi)+\rho^*(\psi)$. Let $\varepsilon>0$. We prove that $\rho^*(\phi+\psi) \ge \rho^*(\phi)+\rho^*(\psi)-2\varepsilon$: it follows then that L is an L-space. There exist $f,g\in L$ such that $\rho(f)\le 1$, $|\phi(f)| \ge \rho^*(\phi)-\varepsilon$, $\rho(g)\le 1$, $|\psi(g)| \ge \rho^*(\psi)-\varepsilon$. Let $h = |f|\vee|g|$. As L is an M-space, $\rho(h) \le \rho(f)\vee\rho(g) \le 1$. Therefore,

$$\rho^*(\phi+\psi) \ge (\phi+\psi)(h) = \phi(h)+\psi(h) \ge \phi(|f|)+\psi(|g|) \ge$$
$$\ge |\phi(f)|+|\psi(g)| \ge \rho^*(\phi)+\rho^*(\psi)-2\varepsilon.$$

Part B. Let L be an L-space. There exists a unique linear function η on L such that $\eta(f) = \rho(f)$ for all $f\in L^+$. For every $g\in L$ we have

$$|\eta(g)| \le |\eta(g^+)|+|\eta(g^-)| = \rho(g^+)+\rho(g^-) =$$
$$= \rho(g^++g^-) = \rho(|g|) = \rho(g).$$

Hence, $\eta\in L^*$ and $\rho^*(\eta)\le 1$. If $\phi\in L^{*+}$ and $\rho^*(\phi)\le 1$, then for all $g\in L^+$ we obtain $\phi(g) \le \rho^*(\phi)\rho(g) \le \rho(g) = \eta(g)$, so $\phi \le \eta$. Apparently,

$$\eta = \sup\{\phi\in L^{*+} : \rho^*(\phi)\le 1\}.$$

Applying Th.16.2 we see that there exists a compact Hausdorff space M such that L^* is isometrically Riesz isomorphic to $C(M)$. Thus, L^* is an M-space.

Part C. If L^* is an L-space, then by Part B of this proof, L^{**} is an M-space. It follows from Th.10.2 that L is an M-space.

Part D. If L^* is an M-space, by part A, L^{**} is an L-space. Then so is L. (Apply Th.10.2 again).

16.5. COROLLARY. *Let* L *be an* L*-space. There exists an extremally disconnected compact Hausdorff space* X *such that* L^* *is isometrically Riesz isomorphic to* C(X).

Proof. Combine Part B of the above proof with Th.10.1(ii) and Th.12.16.

A direct consequence of 16.5 is one of Kakutani's representation theorems. (The other one is Th.16.8):

16.6. THEOREM. (S.Kakutani). *For every* M*-space* L *there exists a compact Hausdorff space* X *such that* L *is isometrically Riesz isomorphic to a* (*norm closed*) *Riesz subspace of* C(X). (Of course, every norm closed Riesz subspace of a C(X) is an M-space).

Proof. By 16.4(ii) and 16.5, L^{**} is isometrically Riesz isomorphic to some C(X). Now apply Th.10.2.

16.7. Let $L = \{f \in C([0,1]) : f(0) = 2f(1)\}$. This L is a uniformly complete unitary Riesz space, hence (Th.13.28) Riesz isomorphic to C(X) for some compact Hausdorff space X. Indeed, if $T = \{z \in \mathbb{C} : |z| = 1\}$, then the formula

$$(\Phi f)(e^{2\pi ix}) = (1+x)f(x) \qquad (f \in L;\ x \in [0,1])$$

establishes a Riesz isomorphism of L onto C(T).

From this and from Kakutani's Theorem one might conjecture that L is isometrically Riesz isomorphic to some C(X). However, this is false, because the set $\{f \in L : \|f\|_\infty \le 1\}$ has no supremum in L.

This example raises the question, how to describe all unitary M-spaces. Every unitary M-space is a uniformly complete unitary Riesz space and is therefore, by 13.28, Riesz isomorphic to some $C(X)$. Thus, we can formulate the question this way: given a compact Hausdorff space X, what norms on $C(X)$ turn $C(X)$ into an M-space ? We answer this question in Ex.16.C.

16.C. *Exercise*. Let X be a compact Hausdorff space.

(i) Let s be a bounded function $X \to [0,\infty)$ such that $\{x \in X : s(x) > 0\}$ is dense in X. Then the formula

$$\tau_s(f) = \sup_{x \in X} |f(x)|s(x) \qquad (f \in C(X))$$

defines a Riesz norm τ_s on $C(X)$, rendering $C(X)$ an M-space.

(ii) Conversely, let τ be a Riesz norm on $C(X)$ under which $C(X)$ is an M-space. By the Kakutani Theorem 16.6 there exist a compact Hausdorff space Y and a Riesz isomorphism Ω of $C(X)$ onto $C(Y)$ such that $\tau(f) = \|f\|_\infty$ for all $f \in C(X)$. Let $h = \Omega\underline{1}$, $Y_0 = \{y \in Y : h(y) > 0\}$. Then by 12.3 for every $y \in Y_0$ there exists a unique $\omega(y) \in X$ for which

$$(\Omega f)(y) = h(y)f(\omega(y)) \qquad (f \in C(X))$$

Define $s: X \to [0,\infty)$ by

$$s(x) = 0 \quad \text{if} \quad x \notin \omega(Y_0),$$

$$s(x) = \sup\{h(y) : y \in Y_0,\ \omega(y) = x\} \quad \text{if} \quad x \in \omega(Y_0).$$

Then $\tau = \tau_s$ where τ_s is as in (i). The set $\{x \in X : s(x) > 0\}$ is dense in X. (s is upper semicontinuous).

Now we turn to the L-spaces.

16.8. THEOREM. (S.Kakutani). *Let* L *be an* L-*space. There exists a measure space* (X,Γ,μ) *such that there is an isometric Riesz isomorphism* Φ *of* L *onto* $L_1(\mu)$.

(X,Γ,μ) *can be chosen such that* X *is an extremally disconnected* *compact Hausdorff space,* Γ *is the Borel* σ*-algebra of* X*, and* μ *is a topological measure in the sense of Def.*16.9.

If L *has a weak unit* u *, then, in addition to the above, for* μ *we can take a finite measure and* Φ *can be chosen such that* $\Phi u = \underline{1}$.

16.9. DEFINITION. Let X be a compact Hausdorff space, Γ the Borel σ-algebra of X, μ a measure on Γ. We say that μ is *topological* if a Borel set A is μ-negligible if and only if it is meagre.

Proof of Th.16.8. We first prove L to be Dedekind complete. Let F be a non-empty bounded subset of L^+ such that $f \vee g \in F$ for all $f,g \in F$. (See Ex.4.A(d)). Let $s = \sup\{\rho(f) : f \in F\}$: then s is finite. There exist $h_1, h_2, \ldots$ in F with $\lim \rho(h_n) = s$. Setting $f_n = h_1 \vee h_2 \vee \ldots \vee h_n$ $(n \in \mathbb{N})$ we have $f_n \in F$, $f_1 \le f_2 \le \ldots$ and $\lim \rho(f_n) = s$. For $m \le n$, by the L-space property of L we see that $\rho(f_n - f_m) = \rho(f_n) - \rho(f_m) \le s - \rho(f_m)$. Therefore, $f_1, f_2, \ldots$ is a ρ-Cauchy sequence in L, converging to some element f of L. Then $\rho(f) = \lim \rho(f_n) = s$. If g is any element of F, then we have $f_n \vee g \in F$ and therefore $\rho(f_n \vee g) \le s$ for each n. Thus,

$$\rho(g - f \wedge g) = \lim \rho(g - f_n \wedge g) = \lim \rho(f_n \vee g - f_n) =$$
$$= \lim[\rho(f_n \vee g) - \rho(f_n)] \le s - s = 0,$$

so $g = f \wedge g \le f$. Apparently, f is an upper bound of F. On the other hand, if h is any upper bound of F, then $h - f = \lim(h - f_n) \in L^+$ as $h - f_n \ge 0$ for every $n \in \mathbb{N}$. Thus, $f = \sup F$.

Therefore, L is Dedekind complete. By the Maeda-Ogasawara Theorem (15.5) and by Lemma 13.21(ii) there exist an extremally disconnected compact Hausdorff space X and a Riesz isomorphism Φ of L onto an order dense Riesz ideal of $C^\infty(X)$. If u is a weak unit in L, then we can choose Φ such that $\Phi u = \underline{1}$. (See 15.8).

Let B be the set of all Borel measurable functions $X \to [0,\infty)$. It follows from 14.8 that for every $f \in B$ there exists a unique $f' \in C^\infty(X)^+$ such that $f = f'$ a.e. Conversely, of course, for every $f \in C^\infty(X)^+$ there is a $g \in B$ such that $f = g$ a.e.

Define $J: B \to [0,\infty]$ by

$$J(f) = \rho(g) \quad \text{if} \quad g \in L \quad \text{and} \quad \Phi g = f \text{ a.e.},$$

$$J(f) = \infty \quad \text{if there is no} \quad g \in \Phi(L) \quad \text{with} \quad g = f \text{ a.e.}$$

Clearly,

(i) $J(f_1) \le J(f_2)$ if $f_1 \le f_2$ a.e. $\quad (f_1, f_2 \in B)$

(ii) $J(f) = 0$ if and only if $f = 0$ a.e. $\quad (f \in B)$

(iii) $J(sf) = sJ(f) \quad (f \in B;\ s \in (0,\infty))$

From the facts that L is an L-space and that $\Phi(L)$ is an ideal in $C^\infty(X)$ it follows easily that

(iv) $J(f_1+f_2) = J(f_1)+J(f_2) \quad (f_1, f_2 \in B)$

Harder to prove is

(v) if $h, f_1, f_2, \ldots \in B$ and $h = \sum_n f_n$ a.e., then $J(h) = \sum_n J(f_n)$.

To show the validity of (v), let $h, f_1, f_2, \ldots \in B$ and $h = \Sigma f_n$ a.e. For all $N \in \mathbb{N}$, $J(h) \ge J(f_1+\ldots+f_N) = J(f_1)+\ldots+J(f_N)$. Thus, $J(h) \ge \Sigma J(f_n)$. For the converse inequality we may assume that $\Sigma J(f_n)$ is finite. Then for each n we have a $g_n \in L^+$ with $\Phi g_n = f_n$ a.e. As $\rho(g_n) = J(f_n)$ and L is norm complete, the series Σg_n is norm convergent in L : let g be its sum. Then $g \ge g_1+\ldots+g_N$ for every $N \in \mathbb{N}$, whence $h \le \Phi g$ a.e.. Therefore,

$$J(h) \le \rho(g) \le \sup_{N \in \mathbb{N}} \rho(g_1+\ldots+g_N) = \sum_{n \in \mathbb{N}} \rho(g_n) = \sum_{n \in \mathbb{N}} J(f_n)$$

which proves (v).

For a Borel set $A \subset X$, define $\mu(A) = J(\chi_A)$. By (ii) and (v), μ is a topological σ-additive measure on the Borel σ-algebra of X. From (iii), (iv), (v) it is easy to prove that $J(f) = \int f \, d\mu$ for every $f \in B$. It follows that a Borel function f on X is μ-integrable if and only if there is a $g\in L$ with $f = \Phi g$ a.e., and for such f and g we have $\int|f|d\mu = \rho(g)$. Besides, a Borel measurable function is μ-negligible if and only if it vanishes almost everywhere.

Thus, Φ induces an isometric Riesz isomorphism of L onto $L_1(\mu)$. If $u\in L$ is a weak unit and $\Phi u = \underline{1}$, then of course μ is finite.

The theorems 16.10 and 16.13 and Ex.16.E are applications of Th.16.8.

16.10. RIESZ REPRESENTATION THEOREM. (See also Th.16.13). *Let* Z *be a compact Hausdorff space and let* $\phi \in C(Z)^{\sim}$, $\phi \geq 0$. *Then there exists a finite measure* ν *on the Borel* σ*-algebra of* Z *such that*

$$\phi(f) = \int f\,d\nu \qquad (f\in C(Z))$$

(Conversely, if ν is a finite measure on the Borel sets of Z, then every element of $C(Z)$ is ν-integrable and $f \mapsto \int f\,d\nu$ is an element of $C(Z)^{\sim}$).

Proof. For all $f\in C(Z)$, put $\tau(f) = \phi(|f|)$. Thus we have defined a Riesz semi-norm τ. The elements f of $C(Z)$ for which $\tau(f)=0$ form a Riesz ideal N in $C(Z)$. The quotient space $C(Z)/N$ in a natural way (Th.2.9) becomes a Riesz space, which we call L_0. Let P be the quotient map of $C(Z)$ onto L_0. Define a Riesz norm ρ_0 on L_0 by

$$\rho_0(Pf) = \tau(f) \qquad (f\in C(Z)).$$

If $f,g\in L_0^+$, then from the linearity of P one obtains the identity $\rho_0(f+g) = \rho_0(f)+\rho_0(g)$.

Let J denote the natural map of L_0 into L_0^{**}. (Th.10,2). This J is an isometric Riesz isomorphism. The closure L of $J(L_0)$ in L_0^{**} is a Banach lattive, which, by the above, is an L-space. The band generated by $J(L_0)$ in L_0^{**} is norm closed, hence contains L. It follows that if we set $u = JP\underline{1}$, then u is a weak unit in L.

For the L-space L with the weak unit u, let X,Γ,μ,Φ be as in Kakutani's theorem. (We want $\Phi u = \underline{1}$). Then we have the following chain of Riesz homomorphisms.

$$C(Z) \xrightarrow{\quad P \quad} C(Z)/N = L_0 \xrightarrow{\quad J \quad} L \xrightarrow{\quad \Phi \quad} C^{\infty}(X)$$

ΦJP is a Riesz homomorphism $C(Z) \to C^{\infty}(X)$ mapping $\underline{1}$ into $\underline{1}$: hence, it maps $C(Z)$ into $C(X)$. By Cor.12.3, there exists a continuous $\omega: X \to Z$ such that $(\Phi JP)(f) = f\circ\omega$ for all $f\in C(Z)$. We can now define a measure ν on the Borel σ-algebra of Z by the formula

$$\nu(A) = \mu(\omega^{-1}(A)) = \int(\chi_A\circ\omega)\,d\mu$$

For $f\in C(Z)^+$ we obtain

$$\int f\,d\nu = \int(f\circ\omega)\,d\mu = \int(\Phi JP)(f)\,d\mu = \|(\Phi JP)(f)\|_1 =$$
$$= \rho_0(Pf) = \tau(f) = \phi(|f|) = \phi(f)$$

Therefore, $\int f\,d\nu = \phi(f)$ for all $f\in C(Z)$.

The measure ν of the preceding theorem is, in general, not unique. We can, however, artificially create uniqueness by restricting μ to the class of the so-called "regular" measures.

16.11. DEFINITION. Let Γ be the Borel σ-algebra of a compact Hausdorff space X. A measure μ on Γ is said to be *regular* if

(i) μ is finite,

(ii) for every Borel set A and every $\varepsilon>0$ there exists an open subset U of X with $U \supset A$ and such that $\mu(U) \le \mu(A)+\varepsilon$.

By complementation one sees that (for finite μ) (ii) is equivalent to

(ii') for every Borel set A and every $\varepsilon>0$ there exists a compact subset C of X with $C \subset A$ and $\mu(C) \ge \mu(A)-\varepsilon$.

16.D. *Exercise*. Let X and Γ be as above.

(i) For every $a \in X$ the point measure $\mu_a : A \to \chi_A(a)$ $(A \in \Gamma)$ is regular.

(ii) If μ, ν are measures on Γ, if $\nu \le \mu$ and if μ is regular, then so is ν.

(iii) If $\mu_1, \mu_2, \ldots$ are regular measures on Γ and if $\Sigma\mu_n(X) < \infty$, then $\Sigma\mu_n$ is regular.

16.12. LEMMA. *Let* Γ *be the Borel* σ*-algebra of an extremally disconnected compact Hausdorff space* X. *Then every finite topological measure on* Γ *is regular.*

Proof. Let μ be a finite topological measure on Γ. Let $A \in \Gamma$, $\varepsilon > 0$. We make an open subset U of X such that $U \supset A$, $\mu(U) \le \mu(A) + \varepsilon$. By Ex.14.A (or Th.14.8) there exists a clopen set $W \subset X$ such that $W \setminus A$ and $A \setminus W$ are meagre. In particular, $A \setminus W$ is contained in a union of countably many closed meagre sets, $A_1, A_2, \ldots$, say. Now $\mu(A) = \mu(W)$ and $A \subset W \cup A_1 \cup A_2 \cup \ldots$ Thus, we are done if for each $n \in \mathbb{N}$ we can find an open set U_n containing A_n and such that $\mu(U_n) \le \varepsilon 2^{-n}$. Considering the fact that $\mu(A_n) = 0$ for every n this means that we may assume A to be closed and meagre.

For such an A, let $t = \inf\{\mu(U) : U \text{ clopen}, U \supset A\}$. There exists a sequence $U_1 \supset U_2 \supset \ldots$ of clopen sets such that for each n, $U_n \supset A$ and $\mu(U_n) \le t + \frac{1}{n}$. If the closed set $\bigcap U_n$ is not meagre, then it contains a non-empty clopen set U'. There is an $m \in \mathbb{N}$ with $\mu(U') \le \frac{1}{m}$. But then we have $\mu(U_m \setminus U') < t$, in contradiction to the definition of t. Therefore, $\bigcap U_n$ must be meagre, so $0 = \mu(\bigcap U_n) = \lim \mu(U_n)$ and $\mu(U_n) \le \varepsilon$ for some m.

16.13. THEOREM. *Let* Z *and* ϕ *be as in* Th.16.10. *There exists exactly one regular measure* ν *on the Borel* σ*-algebra of* Z *such that* $\phi(f) = \int f d\nu$ *for every* $f \in C(Z)$.

Proof. We use the terminology of the proof of Th.16.10.

Let A be a Borel set of Z, let $\varepsilon>0$: we make a compact set $C \subset A$ for which $\nu(C) \geq \nu(A)-\varepsilon$. As μ is regular (Lemma 16.12), there is a compact $C_0 \subset \omega^{-1}(A)$ such that $\mu(C_0) \geq \mu(\omega^{-1}(A))-\varepsilon$. Set $C = \omega(C_0)$. Then C is compact, $C \subset A$ and $\nu(C) \geq \mu(C_0) \geq \nu(A)-\varepsilon$.

To prove the uniqueness, let π be a regular measure on the Borel σ-algebra of Z such that $\phi(f) = \int f d\pi$ $(f \in C(Z))$: we show that $\nu = \pi$. Let U be an open subset of Z and $\varepsilon>0$. There exist a compact set $C \subset U$ with $\pi(C) \geq \pi(U)-\varepsilon$ and an $f \in C(Z)$ with $\chi_C \leq f \leq \chi_U$. (Urysohn Lemma). Then $\nu(U) \geq \int f d\nu = \phi(f) = \int f d\pi \geq \pi(C) \geq \pi(U)-\varepsilon$. Thus, $\nu(U) \geq \pi(U)$ for every open set U. By the regularity of ν, for every Borel set A we have

$$\nu(A) = \inf \{\nu(U) : U \text{ open}, U \supset A \} \geq$$

$$\geq \inf \{\pi(U) : U \text{ open}, U \supset A \} \geq \pi(A).$$

Similarly, $\nu(X\backslash A) \geq \pi(X\backslash A)$. However,

$$\nu(A)+\nu(X\backslash A) = \nu(X) = \phi(\underline{1}) = \pi(X) = \pi(A)+\pi(X\backslash A).$$

Therefore, $\nu(A) = \pi(A)$ for every Borel set A.

16.E. *Exercise*. Let π be a finite measure on the Borel σ-algebra of a compact Hausdorff space Z.

(i) If for every open $U \subset Z$ and every $\varepsilon>0$ there exists a compact set $C \subset U$ such that $\pi(C) \geq \pi(U)-\varepsilon$, then π is regular. (Hint. Re-read the above proof).

(ii) If Z is metrizable, then π is regular. (Every open subset of Z is a union of countably many compact sets).

Finally we show that the norm dual of a normed Riesz space often contains a large L-space. For this purpose we drop part of the assumption on L we made at the beginning of this section : *ON THE REMAINING PAGES OF THIS SECTION,* L *IS A RIESZ SPACE WITH A RIESZ NORM* ρ. Thus, we no longer require L to be norm complete.

16.14. DEFINITION. The space L is said to be a *semi*-M-*space* if it has the following property. If $u_1,u_2 \in L^+$, if $\rho(u_1)=\rho(u_2)=1$ and if $v_1,v_2,\ldots$ is a sequence in L^+ satisfying

$$u_1 \vee u_2 \geq v_n \downarrow 0,$$

then $\lim \rho(v_n) \leq 1$.

It is clear that every M-space is a semi-M-space. Also, if L has absolutely continuous norm (see Def.11.1), then L is a semi-M-space. However, there are less obvious examples of semi-M-spaces. Some of these will be presented in the next chapter. (See 24.1). The following theorem shows the importance of semi-M-spaces.

16.15. THEOREM. L *is a semi*-M-*space if and only if* L_s^* *is an* L-*space*.

Proof. (i) Assume that L is a semi-M-space. To show that L_s^* is an L-space, let $\phi_1,\phi_2 \in L_s^*$ be given such that $\phi_1,\phi_2 \geq 0$. Furthermore, let ε be a positive real number. Then there exist elements u_1,u_2 of L^+ such that $\rho(u_1)=\rho(u_2)=1$ and

$$\phi_i(u_i) > \rho^*(\phi_i) - \tfrac{1}{2}\varepsilon \quad (i=1,2)$$

Setting $u = u_1 \vee u_2$ and $\phi = \phi_1+\phi_2$ it is clear that $\phi \in (L_s^*)^+$ and $u \in L^+$. Hence, by Cor.7.9, there exists a sequence $w_1,w_2,\ldots$ in L^+ such that $w_n \uparrow u$ and $\phi(w_n)<\varepsilon$ for all n. Defining $v_n = u-w_n$ for all n, the sequence $v_1,v_2,\ldots$ satisfies $u \geq v_n \downarrow 0$. Hence, $\lim \rho(v_n) \leq 1$ as $n \to \infty$.

Thus there exists a number n_0 such that $\rho(v_n) < 1+\varepsilon$ for all $n \geq n_0$. For these n,

$$\rho^*(\phi_1+\phi_2) = \rho^*(\phi) \geq \phi((1+\varepsilon)^{-1}v_n) = \phi((1+\varepsilon)^{-1}(u-w_n)) \geq$$
$$\geq \phi((1+\varepsilon)^{-1}u) - (1+\varepsilon)^{-1}\varepsilon \geq (1+\varepsilon)^{-1}\{\phi_1(u_1)+\phi_2(u_2)-\varepsilon\} \geq$$
$$\geq (1+\varepsilon)^{-1}\{\rho^*(\phi_1)+\rho^*(\phi_2)-2\varepsilon\}$$

The resulting inequality $\rho^*(\phi_1+\phi_2) \geq (1+\varepsilon)^{-1}\{\rho^*(\phi_1)+\rho^*(\phi_2)-2\varepsilon\}$ holds for all $\varepsilon>0$, so

$$\rho^*(\phi_1+\phi_2) \geq \rho^*(\phi_1)+\rho^*(\phi_2).$$

The inverse inequality is obvious, so $\rho^*(\phi_1+\phi_2) = \rho^*(\phi_1)+\rho^*(\phi_2)$ holds for all $\phi_1,\phi_2 \in (L_s^*)^+$. Since L_s^* is norm complete (it is a band in L^*), it follows that L_s^* is an L-space.

(ii) For the converse direction, assume that L is not a semi-M-space. Then there exist $u_1,u_2 \in L^+$, $\rho(u_1)=\rho(u_2)=1$ and there exists a sequence $v_1,v_2,\ldots$ in L^+ such that $u_1 \vee u_2 \geq v_n \downarrow 0$ and $\lim \rho(v_n) = \alpha > 0$. Now, for all $n \in \mathbb{N}$ there exists a $\phi_n \in L^*$ satisfying

$$\rho^*(\phi_n) = 1, \quad \phi_n \geq 0, \quad \phi_n(v_n) = \rho(v_n)$$

by one of the Hahn-Banach Theorems (Th.6.9). Since the unit ball of L^* is weak* compact, the sequence $\phi_1,\phi_2,\ldots$ has a weak* cluster point ϕ_0. It is clear that $\phi_0 \geq 0$ and that $\rho^*(\phi_0) \leq 1$. Furthermore, it is obvious that

$$0 \leq \lim \phi_0(v_n) \leq \lim \rho(v_n) = \alpha \quad \text{as } n\to\infty.$$

Next, fix n and let $\varepsilon>0$ be given, Considering the weak* open neighbourhood

$$U = \{\phi : |\phi(v_n)-\phi_0(v_n)| < \varepsilon\}$$

we see that $\phi_m \in U$ for infinitely many values of m. Since $m \geq n$ implies

$$\phi_m(v_n) \geq \phi_m(v_m) = \rho(v_m) \geq \alpha,$$

it follows that $\rho(v_m) \leq \phi_0(v_n)+\varepsilon$ for those $m \geq n$ for which $\phi_m \in U$. Thus,

$$\lim \rho(v_n) \le \lim \phi_0(v_n) + \varepsilon \qquad (n\to\infty).$$

Therefore, $\lim \phi_0(v_n) = \alpha$. Now observe that ϕ_0 has a decomposition

$$\phi_0 = \phi_c + \phi_s\ , \quad 0 \le \phi_c \in L_c^*\ , \quad 0 \le \phi_s \in L_s^*\ .$$

Moreover, $\lim \phi_c(v_n) = 0$, so $\lim \phi_s(v_n) = \alpha > 1$. Especially it follows that $\phi_s(u_1 \vee u_2) \ge \alpha > 1$. Let now ϕ_1 be defined by

$$\phi_1(u) = \sup\{\phi_s(u \wedge n(u_1-u_2)^+) : n \in \mathbb{N}\}$$

for all $u \in L^+$, and $\phi_1(f) = \phi_1(f^+) - \phi_1(f^-)$ for arbitrary $f \in L$. Then $\phi_1 \in L^*$, $0 \le \phi_1 \le \phi_s$, so $\phi_1 \in L_s^*$. Moreover, $\phi_1((u_1-u_2)^+) = \phi_s((u_1-u_2)^+)$ and $\phi_1((u_1-u_2)^-) = 0$. Set $\phi_2 = \phi_s - \phi_1$. Then

$$\rho^*(\phi_1)+\rho^*(\phi_2) \ge \phi_1(u_1)+\phi_2(u_2) = \phi_1(u_1)+\phi_s(u_2)-\phi_1(u_2) =$$

$$= \phi_s(u_2) + \phi_1(u_1-u_2) =$$

$$= \phi_s(u_2) + \phi((u_1-u_2)^+) =$$

$$= \phi_s(u_1 \vee u_2) \ge \alpha > 1 \ge \rho^*(\phi_s) = \rho^*(\phi_1+\phi_2).$$

Thus, in this case L_s^* is not an L-space.

17. HERMITIAN OPERATORS

Our purpose in this section is, to use the theory we have developed and prove a form of the Spectral Theorem for Hermitian operators in a Hilbert space over $\mathbb{R}$.

We assume that the reader is familiar with real inner products, real Hilbert spaces, Schwarz' inequality, orthogonality, orthogonal complements of closed linear subspaces of a Hilbert space, projections, orthogonal bases in finite dimensional Hilbert spaces, and with the fact that $L_2(\mu)$ is a Hilbert space for every measure μ.

THROUGHOUT THIS SECTION, H *IS A REAL HILBERT SPACE WITH AN INNER PRODUCT* $(\ ,\ )$ *AND A NORM* $\|\ \|$. *BY* $\mathcal{L}(H)$ *WE DENOTE THE BANACH SPACE OF ALL CONTINUOUS LINEAR MAPS* $H \to H$.

17.1. DEFINITION. An element T of $\mathcal{L}(H)$ for which

$$(Tx,y) = (x,Ty) \qquad (x,y\in H)$$

is said to be *Hermitian*. The Hermitian elements of $\mathcal{L}(H)$ form a vector space $\mathcal{H}$ that is a closed subspace of $\mathcal{L}(H)$. Thus, $\mathcal{H}$ is a Banach space.

17.A. *Exercise*. (i) An element T of $\mathcal{L}(\mathbb{R}^n)$ $(n\in\mathbb{N})$ is Hermitian if and only if its matrix relative to the standard base of $\mathbb{R}^n$ is symmetric.
(ii) If (X,Γ,μ) is a measure space, then every element h of $L_\infty(\mu)$ induces a Hermitian element T_h of $\mathcal{L}(L_2(\mu))$ by

$$(T_h f)(x) = h(x)f(x) \qquad (x\in X;\ f\in L_2(\mu))$$

17.B. *Exercise*. An element P of $\mathcal{L}(H)$ is a projection if and only if $P\in\mathcal{H}$ and $P = P^2$.

17.C. *Exercise.* (The Hellinger-Toeplitz Theorem). If T is a linear map $H \to H$ such that $(Tx,y) = (x,Ty)$ for all $x,y \in H$, then T is continuous and, consequently, Hermitian.

(Hint. Assume that T is not continuous. For every finite dimensional linear subspace A of H and every $s \in \mathbb{R}$ there exists an $x \in H$ for which $x \perp A$ and $\|Tx\| \geq s\|x\|$. Therefore, there exist $e_1, e_2, \ldots$ in H such that, for every $n \in \mathbb{N}$, $\|e_n\| \leq 1$, $\|Te_n\| = 2^n$ and (if $n \geq 2$) e_n is perpendicular to the linear hull of $T^2e_1, \ldots, T^2e_{n-1}$. Then $Te_n \perp Te_m$ for $n \neq m$. The sum $\Sigma\, 3^{-n}Te_n$ converges to some $a \in H$. For every n, $(Ta,e_n) = (a,Te_n) = (\frac{4}{3})^n$, but $(Ta,e_n) \leq \|Ta\|$).

17.2. DEFINITION. For $S,T \in \mathcal{L}(H)$ we write $S \square T$ if $ST = TS$.

Of course, if $S \square T$, then $T \square S$. If $S \square T_1$ and $S \square T_2$, then $S \square T_1T_2$.

17.D. *Exercise.* Let $S,T \in \mathcal{H}$. Then $S \square T$ is and only if $ST \in \mathcal{H}$.

17.3. DEFINITION. For $S,T \in \mathcal{H}$ we define $S \leq T$ if

$$(Sx,x) \leq (Tx,x) \qquad (x \in H)$$

17.4. THEOREM. $\leq$ *is an ordering, rendering* $\mathcal{H}$ *an ordered vector space.*

Proof. The only thing that is not perfectly obvious is the fact that the inequality $S \leq T \leq S$ implies $S = T$. In other words, all we have to prove is that, if $W \in \mathcal{H}$ and if $(Wx,x)=0$ for all $x \in H$, then $W=0$. Now for such a W and for all $x,y \in H$ we have

$$2(Wx,y) = (W(x+y),x+y) - (Wx,x) - (Wy,y) = 0$$

so that $(Wx,y)=0$ for all x and y. Taking $y=Wx$ we obtain (for every $x \in H$) $\|Wx\|^2 = (Wx,Wx) = 0$, so $Wx=0$. Then $W=0$.

17.5. DEFINITION. Set $H^+ = \{T \in H : T \geq 0\}$.

If $S, T \in H^+$, then $S+T \in H^+$. If $S \in H^+$ and $c \geq 0$, then $cS \in H^+$.

17.6. LEMMA. *(i) If* $T \in H$, *then* $T^2 \in H^+$.

(ii) If $S \in H^+$ *and* $T \in H$, *then* $TST \in H^+$.

(iii) If $S \in H^+$, *then* $S^n \in H^+$ *for every* $n \in \mathbb{N}$.

Proof. (i) For all $x \in H$ we have $(T^2x, x) = (Tx, Tx) \geq 0$.

(ii) If $x \in H$, then $(TSTx, x) = (STx, Tx) \geq 0$.

(iii) follows from (i) and (ii).

17.7. LEMMA. *If* $T \in H^+$, *then*

$$\|Tx\|^2 \leq \|T\|(Tx, x) \qquad (x \in H)$$

Proof. The formula

$$(x,y)_T = (Tx,y) \qquad (x,y \in H)$$

defines a positive, symmetric bilinear form $(\; , \;)_T$ on H. By Schwarz' inequality, for all $x \in H$ one has

$$(x, Tx)_T{}^2 \leq (x,x)_T (Tx,Tx)_T$$

i.e.

$$\|Tx\|^4 \leq (Tx,x)(T^2x,Tx) \leq (Tx,x)\|T^2x\|\|Tx\| \leq (Tx,x)\|T\|\|Tx\|\|Tx\|$$

The lemma follows.

17.8. COROLLARY. *(i) If* $T \in H^+$, *then*

$$\|T\| = \sup_{\|x\| \leq 1} (Tx, x)$$

(ii) For $T \in H^+$ *we have* $T \leq sI$ *if and only if* $\|T\| \leq s$.

Proof. (i) Put $B = \{x\in H : \|x\|\le 1\}$. Let $T\in \mathcal{H}^+$. For $x\in B$ we have

$$(Tx,x) \le \|Tx\|\|x\| \le \|T\|\|x\|^2 \le \|T\|.$$

Hence, $\sup_{x\in B} (Tx,x) \le \|T\|$. Conversely, by 17.7,

$$\|T\|^2 = \sup_{x\in B} \|Tx\|^2 \le \|T\| \sup_{x\in B} (Tx,x)$$

and therefore $\|T\| \le \sup_{x\in B} (Tx,x)$.

(ii) follows.

17.9. COROLLARY. *If* $S,T \in \mathcal{H}^+$ *and* $S \le T$, *then* $\|S\| \le \|T\|$.

17.10. COROLLARY. *For every* $T\in\mathcal{H}$ *we have* $\|T^2\| = \|T\|^2$.

Proof. Again, put $B = \{x\in H : \|x\|\le 1\}$. Let $T\in\mathcal{H}$. As we know, $T^2\in\mathcal{H}^+$ (17.6(i)). Hence,

$$\|T^2\| = \sup_{x\in B} (T^2x,x) = \sup_{x\in B} (Tx,Tx) = (\sup_{x\in B} \|Tx\|)^2 = \|T\|^2$$

17.11. THEOREM. *Every* $T\in\mathcal{H}^+$ *has exactly one square root in* $\mathcal{H}^+$.

Proof. If p is any real polynomial $\alpha_0+\alpha_1X+\ldots+\alpha_nX^n$ and if $S\in\mathcal{H}$, then by $p(S)$ we denote the element $\alpha_0I+\alpha_1S+\ldots+\alpha_nS^n$ of $\mathcal{H}$. Further, for such a polynomial p we set $p\gg 0$ if $\alpha_k>0$ for every k.

If $p\gg 0$ and $S\in\mathcal{H}^+$, then $p(S) \in \mathcal{H}^+$. (17.6(iii)). Moreover, if $p\gg 0$ and $S\in\mathcal{H}$, then $\|p(S)\| \le p(\|S\|)$. In particular, if $p\gg 0$, and $S\in\mathcal{H}$, $\|S\|\le 1$, then $\|p(S)\| \le p(1)$.

If p,q are polynomials and if $(p-q)\gg 0$, then we also write $p\gg q$ or $q\ll p$. In that case, $p(S) \ge q(S)$ for every $S\in\mathcal{H}^+$.

Now consider the sequence of polynomials $p_0,p_1,p_2,\ldots$ defined by

$$(*)\quad \begin{cases} p_0 = 0, \\ 2p_{n+1}(X) = X + p_n(X)^2 \qquad (n=0,1,2,\dots) \end{cases}$$

Observe that $p_n >> 0$ for every n. Furthermore,

$$2(p_{n+1}-p_n) = (p_n-p_{n-1})(p_n+p_{n-1}) \qquad (n\in\mathbb{N})$$

Consequently, $p_{n+1}-p_n >> 0$ for every n, i.e.

$$0<<p_1<<p_2<<\dots$$

In particular, $0 \le p_1(x) \le p_2(x) \le \dots$ for every $x\in\mathbb{R}^+$. Inductively one sees that $p_n(x) \le 1$ for every $x \in [0,1]$. Thus, for every $x \in [0,1]$, $\lim p_n(x)$ exists. We denote this limit by $g(x)$. From (*) we infer that $2g(x) = x + g(x)^2$ for each $x \in [0,1]$, whence

$$1-g(x) = \sqrt{1-x} \qquad (0\le x\le 1)$$

Now we return to the T of Th.17.11. Without restriction we assume that $\|T\|\le 1$. Put $S = I-T$. As $0 \le T \le I$, it follows that $0 \le S \le I$, so that $S\in H^+$ and $\|S\|\le 1$. (Cor.17.8(ii)). As $p_n >> 0$, for every $n\in\mathbb{N}$ we have $p_n(S) \in H^+$. Moreover, for $m\ge n$ we know that $p_m-p_n >> 0$ and therefore

$$\|p_m(S)-p_n(S)\| = \|(p_m-p_n)(S)\| \le (p_m-p_n)(\|S\|) \le$$
$$\le (p_m-p_n)(1) = p_m(1)-p_n(1).$$

Now $\lim p_n(1)$ exists. Hence, $p_1(S), p_2(S), \dots$ is a Cauchy sequence in H^+. Let $W = I - \lim p_n(S)$.

It follows from (*) that $2p_{n+1}(S) = S + p_n(S)^2$ for each n. Thus, $2(I-W) = (I-T) + (I-W)^2$, i.e.

$$W^2 = T$$

For every $n\in\mathbb{N}$ we have $p_n(S) \in H^+$, so $I-W \in H^+$. Furthermore,

$$\|I-W\| = \lim \|p_n(S)\| \le \lim p_n(1) = g(1) = 1.$$

By Cor.17.8(ii), $I-W \le I$ and

$$W \in H^+$$

This completes the existence proof. For the moment we put off the proof of the uniqueness. Meanwhile it will be useful to give a meaning to the symbol $\sqrt{T}$ for $T \in \mathcal{H}^+$:

17.12. DEFINITION. Let $p_0, p_1, \ldots$ be as above. Let $T \in \mathcal{H}^+$. We define

$$\sqrt{T} = I - \lim p_n(I-T) \quad \text{if} \quad \|T\|=1,$$
$$\sqrt{T} = \sqrt{t} \cdot \sqrt{t^{-1}T} \quad \text{if} \quad T \neq 0 \quad \text{and} \quad t = \|T\|,$$
$$\sqrt{T} = 0 \quad \text{if} \quad T=0$$

In any case, $\sqrt{T} \in \mathcal{H}^+$ and $\sqrt{T}^2 = T$.

17.13. COROLLARY. *Let* $T \in \mathcal{H}^+$.

(*i*) *There exists a sequence* $q_0, q_1, \ldots$ *of real polynomials without constant terms and such that* $\sqrt{T} = \lim q_n(T)$.

(*ii*) *If* $S \in \mathcal{H}$ *and* $S \square T$, *then* $S \square \sqrt{T}$.

(*iii*) *If* $V \in \mathcal{H}$, $V \square T$ *and* $V^2 \leq T$, *then* $V \leq \sqrt{T}$.

Before proving this corollary we mention:

17.14. COROLLARY. (*i*) *If* $S,T \in \mathcal{H}^+$ *and* $S \square T$, *then* $ST \in \mathcal{H}^+$.

(*ii*) *If* $S,T \in \mathcal{H}^+$, $S \leq T$ *and* $S \square T$, *then* $S^2 \leq T^2$ *and* $\|Sx\| \leq \|Tx\|$ *for all* $x \in H$.

(*iii*) *If* $S,T \in \mathcal{H}^+$, $S \leq T$ *and* $S \square T$, *then* $\sqrt{S} \leq \sqrt{T}$.

(*iv*) *If* $S,T \in \mathcal{H}$, $V \in \mathcal{H}^+$, $V \square S$, $V \square T$ *and* $S \leq T$, *then* $VS \leq \epsilon VT$.

Now we prove Cor.17.13 and Cor.17.14 simultaneously.

17.13(i). If $\|T\|=1$, take $q_n(X) = p_n(1) - p_n(1-X)$. More generally, if $T \neq 0$ and $t = \|T\|$, take $q_n(X) = t \cdot [p_n(1) - p_n(1-t^{-1}X)]$.

17.13(ii) follows directly from 17.13(i).

17.14(i). $ST = S\sqrt{T}\sqrt{T} = \sqrt{T}S\sqrt{T}$ (by 17.13(ii)). Now apply 17.6(ii).

17.14(ii). By 17.14(i), $T^2 - S^2 = (T-S)(T+S) \in \mathcal{H}^+$, so $S^2 \leq T^2$. For every $x \in H$ we see that $\|Sx\|^2 = (S^2x,x) \leq (T^2x,x) = \|Tx\|^2$.

17.13(iii). We may assume $\|T\|=1$. Inductively one proves that $p_n(I-T) \leq I-V$ for every $n \in \mathbb{N}$. (Indeed, if $p_n(I-T) \leq I-V$, then $p_n(I-T)^2 \leq (I-V)^2$ by 17.14(ii), so

$$2p_{n+1}(I-T) = I-T+p_n(I-T)^2 \leq I-V^2+(I-V)^2 = 2(I-V)$$

whence $p_{n+1}(I-T) \leq I-V$). It follows that $I-\sqrt{T} \leq I-V$, i.e. $V \leq \sqrt{T}$.

17.14(iii). Apply 17.13(iii) to $V=\sqrt{S}$.

17.14(iv). By 17.14(i), $VT - VS = V(T-S) \in H^+$.

The commutativity conditions mentioned in Cor.17.13 and Cor.17.14 are not redundant. To see this, compare 17.14(ii) to the following.

17.E. *Exercise*. Let $H = \mathbb{R}^2$, $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $S,T \in H^+$ and $S \leq T$, but not $S^2 \leq T^2$.

Now we can prove the uniqueness part of Th.17.11. Let $T \in H^+$, $V \in H^+$ and $V^2=T$: we prove that $V=\sqrt{T}$. By 17.13(ii), $V \square \sqrt{T}$. Hence, $\sqrt{T}-V \geq 0$, so that (by 17.14(iv))

$$0 \leq (\sqrt{T}-V)(\sqrt{T}-V) \leq (\sqrt{T}-V)(\sqrt{T}+V) =$$

$$= T-V\sqrt{T}+\sqrt{T}V-V^2 = T-V^2 = 0$$

Then for every $x \in H$, $\|(\sqrt{T}-V)x\|^2 = ((\sqrt{T}-V)^2 x,x) = 0$, so $\sqrt{T}=V$.

17.15. LEMMA. *Let* $T \in H$.

(i) $\sqrt{T^2} \geq T$ *and* $\sqrt{T^2} \geq -T$.

(ii) If $S \in H$, $S \square T$, $S \geq T$ *and* $S \geq -T$, *then* $S \geq \sqrt{T^2}$.

Proof. (i) follows from Cor.17.13(iii).

(ii) $2S \geq T+(-T)$, so $S \in H^+$ and $S = \sqrt{S^2}$. As $S \square T$, by 17.14(i) we have $S^2-T^2 = (S-T)(S+T) \geq 0$, so $S^2 \geq T^2$ and $S = \sqrt{S^2} \geq \sqrt{T^2} = T$. (17.14(iii)).

17.16. DEFINITION. A *subalgebra* of H is a linear subspace A of H such that $ST \in A$ for all $S,T \in A$. (The terminology is not very good, as H itself is not an algebra. However, every subalgebra of H is an algebra and is even commutative (17.D)).

17.F. *Exercise*. The closure of a subalgebra of H is a subalgebra of H.

17.G. *Exercise*. Let $T \in H$. Let A be the closure of $\{p(T) : p$ is a real polynomial$\}$. Then A is a subalgebra of H that contains I.

17.H. *Exercise*. Let $X \subset H$ be such that $S \square T$ for all $S,T \in X$. Define $X^{\square} = \{S \in H : S \square T$ for all $S,T \in X\}$ and $X^{\square\square} = \{T \in H : S \square T$ for all $S \in X^{\square}\}$. Then $X^{\square\square}$ is a closed subalgebra of H, $I \in X^{\square\square}$ and $X \subset X^{\square\square}$.

17.17. THEOREM. *Let A be a closed subalgebra of H. Then A is a Banach lattice. For all $T \in A$ we have $|T| = \sqrt{T^2}$.*

Proof. Let $S \in A$. Then $p(S) \in A$ for every polynomial p without constant term. Hence, (see 17.13(i)) if $S \in A \cap H^+$, then $\sqrt{S} \in A$. In particular, $\sqrt{T^2} \in A$ for every $T \in A$. From Lemma 17.15 and from the commutativity of the ring A, it follows that $\sqrt{T^2} = T \vee (-T)$ in the ordered vector space A. For arbitrary $V,W \in A$ set $S = \frac{1}{2}(V+W)$, $T = \frac{1}{2}(V-W)$. Then (in A) we have $\sqrt{T^2} + S = [T \vee (-T)] + S = (T+S) \vee (-T+S) = V \vee W$. Thus, A is a Riesz space.

By Cor.17.9 and Cor.17.10, the norm of A is a Riesz norm.

Now let A be a closed subalgebra of H such that $I \in A$. By 17.8(ii). I is a strong unit in A and $I = \sup\{T \in H : \|T\| \leq 1\}$. An application of Th.16.2 and of Th.13.32 proves the following.

17.18. SPECTRAL THEOREM. *Let* $\mathcal{A}$ *be a closed subalgebra of* $\mathcal{H}$ *with* $I\in\mathcal{A}$. *Then there exist a compact Hausdorff space* X *and a linear bijection* $T\mapsto\hat{T}$ *of* $\mathcal{A}$ *onto* C(X) *such that for all* $S\in\mathcal{A}$,

(*i*) $S\leq T$ *if and only if* $\hat{S}\leq\hat{T}$.

(*ii*) $\widehat{ST}=\hat{S}\hat{T}$, $\hat{I}=\underline{1}$.

(*iii*) $\|S\|=\|\hat{S}\|_\infty$.

17.19. COROLLARY. *If* H *is finite dimensional and if* $T\in\mathcal{H}$, *then* H *has a base consisting of pairwise orthogonal eigenvectors of* T.

Proof. Let $\mathcal{A}=\{p(T)\ :\ p$ is a polynomial$\}$. Then $\mathcal{A}$ is a finite dimensional (hence closed) subalgebra of $\mathcal{H}$ and $I\in\mathcal{A}$. Let X be as in Th.17.18. Then $\dim C(X)=\dim\mathcal{A}<\infty$ so that X consists of finitely many points $a_1,\ldots,a_n$ ($a_i\neq a_j$ if $i\neq j$). Let $\alpha_i=\hat{T}(a_i)$. For every $i\in\{1,\ldots,n\}$ there exists a $P_i\in\mathcal{A}$ for which $\hat{P}_i$ is the characteristic function of $\{a_i\}$. As $\hat{P}_i=(\hat{P}_i)^2$ we have $P_i=P_i^{\,2}$, so P_i is the projection of H onto some linear subspace H_i of H. (Ex.17.B). As $P_iP_j=\hat{P}_i\hat{P}_j=o$ if $i\neq j$ we see that the spaces $H_1,\ldots,H_n$ are pairwise orthogonal: furthermore, $\Sigma H_i=H$ because $(\Sigma P_i)^\wedge=\Sigma\hat{P}_i=\underline{1}=\hat{I}$. For every i we obtain

$$(TP_i)^\wedge=\hat{T}\hat{P}_i=\alpha_i\hat{P}_i=(\alpha_iP_i)^\wedge$$

so $Tx=\alpha_ix$ for all $x\in H_i$. It is now easy to prove the corollary by choosing an orthogonal base in each H_i.

For infinite dimensional H we can generalize the above. The followin exercise shows that we shall have to be cautious.

17.I. *Exercise*. Let μ denote the Lebesgue measure on $[0,1]$, let $H=L_2(\mu)$. The formula

$$(Tf)(x) = xf(x) \qquad (f\ H;\ 0\le x\le 1)$$

defines a $T\in\mathcal{H}$ that has no eigenvectors at all.

17.20. THEOREM. *Let* $\mathcal{X}$ *be a subset of* $\mathcal{H}$ *such that* $S\square T$ *for all* $S,T\in\mathcal{X}$. *Set* $\mathcal{A} = \mathcal{X}^{\square\square}$.(See Ex.17.H). *Then, as a Riesz space,* $\mathcal{A}$ *is Dedekind complete.*

Proof. Let X and ^ be as in the Spectral Theorem 17.18. We prove X to be extremally disconnected. (See 12.16). To this end, take an open subset U of X: we show that its closure is open. (Ex.12.F).

Set $\mathcal{U} = \{T\in\mathcal{A}^+ : \hat{T}\le\chi_U\}$, let D be the closed linear hull of the set $\{T(H) : T\in\mathcal{U}\}$ and let P be the projection of H onto D.

Take $S\in\mathcal{X}$. For all $T\in\mathcal{U}$ we have $S(T(H)) = T(S(H)) \subset T(H) \subset D$. It follows that $S(D)\subset D$. But then $SP = PSP \in \mathcal{H}$, so $S\square P$ (17.D). We see that $P\in\mathcal{A}$. Now $P^2=P$, so $\hat{P}$ is the characteristic function of some subset V of X, which, by the continuity of P, must be clopen.

For every element a of U there is a $T\in\mathcal{U}$ with $\hat{T}(a)=1$. Then we have $T(H)\subset D = P(H)$, $PT=T$, $\hat{P}\hat{T}=\hat{T}$, $\hat{P}(a)=1$ and $a\in V$. Thus, $U\subset V$ and therefore $\overline{U}\subset V$. On the other hand, for every element b of $\overline{U}^c$ there exists an $R\in\mathcal{A}$ with $\hat{R}(b)=1$ and $\hat{R}\equiv 0$ on $\overline{U}$. Then for all $T\in\mathcal{U}$ we see that $\hat{R}\hat{T}=0$, $RT=0$ and $T(H)$ is contained in the kernel of R. It follows that D is contained in the kernel of R, so $RP=0$, $\hat{R}\hat{P}=0$, $\hat{R}\equiv 0$ on V, and $b\notin V$. Hence, $\overline{U}=V$ and U has clopen closure.

As an application, we prove the following extension of 17.19, which does not seem to have anything to do with Riesz spaces.

17.21. THEOREM. *Let* $T\in\mathcal{H}$, $\varepsilon>0$. *There exist a positive integer* n, *closed linear subspaces* $H_1,\dots,H_n$ *of* H *and real numbers* $s_1,\dots,s_n$ *such that* *(i)* $H_1,\dots,H_n$ *are pairwise orthogonal and* $H_1+\dots+H_n = H$,

(ii) $T(H_i) \subset H_i$ $(i=1,\dots,n)$,

(iii) if $i\in\{1,\dots,n\}$, *then* $\|Tx-s_ix\| \le \varepsilon$ *for all* $x\in H_i$,

(iv) if for each i, P_i *is the projection onto* H_i, *then* $\|T - \Sigma s_iP_i\| \le \varepsilon$.

Proof. Set $A = \{T\}^{\square\square}$: then $T\in A$. Let X be as in the Spectral Theorem. By the above we know that X is extremally disconnected: then it is zero-dimensional. There exist pairwise disjoint clopen subsets $X_1,\dots,X_n$ of X whose union is X and such that $|\hat{T}(x)-\hat{T}(y)| \le \varepsilon$ as soon as x and y lie in the same X_i. Choose $s_1,\dots,s_n\in\mathbb{R}$ such that $|\hat{T}(x)-s_i| \le \varepsilon$ for all $x\in X_i$ $(i=1,\dots,n)$: then $\|\hat{T} - \Sigma s_i\chi_{X_i}\|_\infty \le \varepsilon$. There exist $P_1,\dots,P_n\in A$ with $\hat{P}_i = \chi_{X_i}$ for each i: then $\|T - \Sigma s_iP_i\| \le \varepsilon$. Just as in the proof of 17.19 one shows P_i to be the projection on some closed linear subspace H_i of H. It is easy to finish the proof of the theorem.

In this proof we have not fully exploited the extremal disconnectedness of X: it would have been enough to know that X is zerodimensional and that the characteristic functions of the clopen subsets of X all lie in $\{\hat{S} : S\in A\}$. With the aid of Freudenthal's Spectral Theorem 13.25 and Ex.13.I one can give a somewhat simpler proof of 17.21 by using the following exercise. (This proof requires less knowledge about Hilbert spaces).

17.J. *Exercise*. Let A be a closed subalgebra of H. Suppose that A has the following property.

If $A\in H$, $A_1,A_2,\dots\in A$ and if $Ax = \lim A_nx$ for all $x\in H$, then $A\in A$.

Then every countable bounded subset of A has a supremum in A.

Hint. Let $B,A_1,A_2,\dots\in A$; let $0\le A_1\le A_2\le\dots$ and $A_n\le B$ for all n. Define

$$s(x) = \sup_{n\in N} (A_n x, x) \qquad (x\in H)$$

Use Lemma 17.7 to prove that

$$\|A_n x - A_m x\|^2 \le \|B\|[s(x)-(A_m x,x)] \qquad (x\in H;\ n,m\in \mathbb{N};\ n\ge m)$$

Show that $\lim A_n x$ exists for every $x\in H$.

17.K. *Exercise*. The *spectrum* of an element T of $L(H)$ is defined to be the set $\sigma(T) = \{s\in R : T-sI$ has no inverse in the ring $L(H)\}$.

Let $T\in H$. Let A be as in Ex.17.G, X as in 17.18.

(i) If H is finite dimensional, $\sigma(T)$ is just the set of all eigenvalues of T.

(ii) For $s\in\mathbb{R}$ the following conditions are equivalent.

(a) $s \in \sigma(T)$.

(b) $T-sI$ has no inverse in A.

(c) There is an $x\in X$ for which $s=\hat{T}(x)$.

Hint for the implication (c)$\Rightarrow$(a). Suppose $s \in T(X)$ while $T-sI$ has an inverse S in $L(H)$. Let $0<\varepsilon<\|S\|^{-1}$. There exists a $g\in C(X)$ such that $\|g\|_\infty = 1$, $\|(\hat{T}-s)g\|_\infty \le \varepsilon$. There exists a $V\in A$ such that $\hat{V} = g$.

(iii) The map

$$x \mapsto \hat{T}(x) \qquad (x\in X)$$

is a homeomorphism of X onto $\sigma(T)$. Hence, $\sigma(T)$ is compact and non-empty.

17.L. *Exercise*. Let exp denote the exponential function $\mathbb{R} \to \mathbb{R}$.

(i) For every $T\in H$ we can define $\exp T \in H$ by

$$\exp T = I + \frac{1}{1!}T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3 + \ldots.$$

(ii) Let A and X be as in the Spectral Theorem. If $T\in A$, then $\exp T$ is an element of A and $(\exp T)\hat{} = \exp \circ \hat{T}$.

CHAPTER V. NORMED KÖTHE SPACES

In this chapter we shall investigate an important class of normed Riesz spaces. More precisely, we study spaces consisting of (equivalence classes) of measurable functions on a measure space. Well-known members of this class turn out to be the L_p-spaces $(1 \le p \le \infty)$.

First we fix some terminology. From now on (X,Γ,μ) will be a fixed σ-finite measure space. We shall assume that the Carathéodory extension procedure has already been applied to μ (so in particular it follows that all μ-null sets are in Γ). By $\overline{M}$ we denote the collection of all μ-measurable functions on X which take their values in the set of extended real numbers $\overline{\mathbf{R}}$ $(=\mathbf{R}\cup(\infty)\cup(-\infty))$, and by M we denote as before (1.E) the collection of all real-valued μ-measurable functions on X. In the sequel members of $\overline{M}$ which are finite μ-almost everywhere on X will also be regarded as members of M. Furthermore, N will denote the collection of all μ-null functions on X. If M is partially ordered by setting $f \le g$ whenever $f(x) \le g(x)$ for all $x\in X$, then M is a Riesz space and it is clear that N is an ideal of M. Moreover, setting $\mathrm{M} = M/N$ it follows that M is a (super) Dedekind complete Riesz space (see 4.F). The Riesz spaces considered in this chapter will all be order ideals of the above defined Riesz space M.

18. FUNCTION SEMI-NORMS AND KÖTHE SPACES

Let $\overline{M}$ be as above. If K is any subset of $\overline{M}$, then K^+ will denote the collection of all functions in K assuming only non-negative values.

18.1. DEFINITION. A function $\rho\colon \overline{M}^+ \to \overline{\mathbf{R}}^+$ is called a *function semi-norm* on $\overline{M}^+$ if

(i) $\rho(f) = 0$ for all $f\in N^+$,

(ii) $\rho(af) = a\rho(f)$ for all $a\in\mathbb{R}^+$ and for all $f\in\overline{M}^+$ $(0.\infty = \infty.0 = 0)$,

(iii) $\rho(f+g) \le \rho(f)+\rho(g)$ for all $f,g\in\overline{M}^+$,

(iv) $f,g\in\overline{M}^+$ and $f \le g$ μ-almost everywhere on X implies $\rho(f) \le \rho(g)$.

If, in addition, we have

(i)' $\rho(f) = 0$ if and only if $f\in N^+$,

then ρ is called a *function norm* on $\overline{M}^+$.

It is clear that the collection of all function semi-norms on $\overline{M}^+$ can be partially ordered. Indeed, if ρ_1 and ρ_2 are function semi-norms, set $\rho_1 \le \rho_2$ whenever $\rho_1(f) \le \rho_2(f)$ for all $f\in\overline{M}^+$. With respect to this partial ordering there exists a smallest function semi-norm (viz. $\rho(f) = 0$ for all $f\in\overline{M}^+$) on $\overline{M}^+$ as well as a largest function semi-norm (viz. $\rho(f) = 0$ if $f\in N^+$, $\rho(f) = \infty$ if $f\in\overline{M}^+\setminus N^+$) on $\overline{M}^+$. The following exercise shows that each collection of function semi-norms on $\overline{M}^+$ has a supremum.

18.A. *Exercise*. Let $\{\rho_\tau: \tau\in T\}$ be a collection of function semi-norms on $\overline{M}^+$. Set

$$\rho(f) = \sup\{\rho_\tau(f): \tau\in T\}$$

for all $f\in\overline{M}^+$. Show that ρ is a function semi-norm on $\overline{M}^+$. Also, show that if at least one ρ_{τ_0} $(\tau_0\in T)$ is a function norm, then ρ is a function norm. Finally show by a counterexample that the converse of the preceding statement does not have to hold.

The following result plays a key role in our investigations.

18.2. THEOREM. *Let* ρ *be a function norm on* $\overline{M}^+$ *and let* $f\in\overline{M}^+$ *be such that* $\rho(f)< \infty$. *Then* $f\in M^+$.

Proof. Let $E = \{x\in X: f(x)=\infty\}$. We have to show that $\mu(E) = 0$. To this end, observe that $\chi_E \le n^{-1}f$ on X for $n=1,2,\dots$, so

$$\rho(\chi_E) \le n^{-1}\rho(f)$$

for all n. Since $\rho(f) < \infty$, this shows that $\rho(\chi_E) = 0$. Hence, since ρ is a function norm it follows that $\chi_E\in N$, so $\mu(E) = 0$.

In the sequel our main interest lies in those functions $f\in\overline{M}^+$ for which $\rho(f) < \infty$ for some given function norm ρ. The preceding theorem shows that, in that case, the domain of ρ may be restricted to M^+. Indeed, if necessary, we can always define $\rho(f) = \infty$ for all $f\in\overline{M}^+\setminus M^+$, thus having extended ρ uniquely to the whole of $\overline{M}^+$.

Now, let ρ be a function norm on M^+. It is easy to see that ρ can be extended to the whole of M. Indeed, define $\rho(f) = \rho(|f|)$ for all $f\in M$. From now on, function norms will always be assumed to be extended to M in the above manner. Next, consider

$$L_\rho = \{f\in M: \rho(f) < \infty\}.$$

It is clear that L_ρ is an ideal of M and that $N \subset L_\rho$. Also, it will be clear that ρ is a Riesz semi-norm on L_ρ and that the null-space of ρ is precisely the ideal N of L_ρ (since ρ is a function norm). Hence, setting

$$\mathrm{L}_\rho = L_\rho/N$$

it follows that L_ρ is a Riesz space and that $\tilde{\rho}$ defined by

$$\tilde{\rho}([f]) = \rho(f)$$

for all $[f]\in\mathrm{L}_\rho$ ($[f]$ is the equivalence class containing the element $f\in L_\rho$) is a Riesz norm on L_ρ. Furthermore, L_ρ is an (order) ideal of M, so it follows that L_ρ is a Dedekind complete normed Riesz space. In the sequel L_ρ will be called a *(normed) Köthe space (generated by* ρ*)*. As usual, we shall from now on identify an $f\in L_\rho$ with its equivalence class $[f]\in\mathrm{L}_\rho$

and conversely. Also we shall drop the class notation for elements of M. Finally, the Riesz norm $\tilde{\rho}$ on L_ρ will from now on be denoted by ρ again. Thus, from now on function norms will be thought of as defined on M, unless stated otherwise.

19. BANACH FUNCTION SPACES

Let, in this section, ρ be a fixed function norm and let L_ρ be the Köthe space generated by ρ. If L_ρ is a Banach space with respect to the norm ρ, then L_ρ is called a *Banach funtion space*. In this section we shall investigate under which additional conditions on ρ L_ρ is a Banach function space. First observe the following. If $u_1, u_2, \ldots.$ is a sequence in L_ρ^+ (and hence in M^+), then $u = \Sigma\, u_n$ means $\sum_{n=1}^{k} u_n \uparrow u$ $(k \to \infty)$, where $u \in M^+$ (or $u \in L_\rho^+$) (see section 7 after theorem 7.5). In our new situation we can give another interpretation of $\Sigma\, u_n$. Indeed, if we think of u_n as a member of L_ρ^+ (and hence of $\overline{M}^+$) $\Sigma\, u_n$ becomes a member of $\overline{M}^+$ as well. In the case that $\Sigma\, u_n \in M^+$ we can again think of $\Sigma\, u_n$ as a member of M. Now observe that this construction does not depend on the choice of the representants and that both readings of $\Sigma\, u_n$ coincide (provided $\Sigma\, u_n \in M$). In the sequel we shall use both readings. Finally, by the second reading, if $u_1, u_2, \ldots.$ is a sequence in L_ρ^+ and if we think of $\Sigma\, u_n$ as a member of $\overline{M}^+$, then $\rho(\Sigma\, u_n)$ is unambigiously determined. Having this in mind we introduce the following.

19.1. DEFINITION. The function norm ρ is said to have the *Riesz-Fischer property (R-F property)*, if for any sequence $u_1, u_2, \ldots.$ in L_ρ^+ satisfying $\Sigma\, \rho(u_n) < \infty$, we have that $\rho(\Sigma\, u_n) < \infty$ (i.e., $\Sigma\, u_n \in L_\rho^+$).

19.2. THEOREM. *The following assertions are equivalent.*

(*a*) ρ *has the R-F property.*

(*b*) *For any sequence* $u_1, u_2, \ldots$ *in* L_ρ^+ *we have* $\rho(\Sigma\, u_n) \leq \Sigma\, \rho(u_n)$.

Proof. (i) (b) $\Rightarrow$ (a). Obvious.

(ii) (a) $\Rightarrow$ (b). Assume that ρ has the R-F property. We argue by contradiction, so suppose that there exists a sequence $u_1, u_2, \ldots$ in L_ρ^+ such that $\Sigma\, \rho(u_n) < \infty$ and such that $\rho(\Sigma\, u_n) > \Sigma\, \rho(u_n)$. Then there exists a real number $\varepsilon > 0$ such that

$$\rho(\Sigma\, u_n) > \varepsilon + \Sigma\, \rho(u_n).$$

Multiplication by $k\varepsilon^{-1}$ $(k=1,2,\ldots)$ furnishes us sequences $u_{1k}, u_{2k}, \ldots$ in L_ρ^+ such that

$$\rho(\textstyle\sum_n u_{nk}) > k + \sum_n \rho(u_{nk})\ ;\quad \sum_n \rho(u_{nk}) < \infty.$$

On the other hand, for all r we have $\rho(\sum_{n<r} u_{nk}) \leq \sum_{n<r} \rho(u_{nk})$, so

$$\rho(\textstyle\sum_{n\geq r} u_{nk}) > k + \sum_{n\geq r} \rho(u_{nk})$$

for $r=1,2,\ldots$ and $k=1,2,\ldots$. For all k there exists an r_k such that

$$\textstyle\sum_{n\geq r_k} \rho(u_{nk}) < k^{-2}.$$

Thus, dropping a finite number of elements of each sequence $u_{1k}, u_{2k}, \ldots$ we obtain sequences $v_{1k}, v_{2k}, \ldots$ in L_ρ^+ such that

$$\textstyle\sum_n \rho(v_{nk}) < k^{-2}\ ;\quad \rho(\sum_n v_{nk}) > k + \sum_n \rho(v_{nk}) \geq k$$

for $k=1,2,\ldots$. Next, reindex the double sequence $(v_{nk})_{n\in\mathbb{N};k\in\mathbb{N}}$ to obtain the sequence $w_1, w_2, \ldots$ in L_ρ^+. Note that

$$\textstyle\sum_k \rho(w_k) < \sum_k k^{-2} < \infty,$$

so $\Sigma\, w_k \in L_\rho^+$ since ρ has the R-F property. However,

$$\rho(\Sigma\, w_n) \geq \rho(\textstyle\sum_n v_{nk}) \geq k$$

holds for all k, which implies $\rho(\Sigma\, w_n) = \infty$. This is the desired contradiction.

Next, we state and prove the main theorem of this section.

19.3. THEOREM. L_ρ *is a Banach function space if and only if* ρ *has the R-F property.*

Proof. (i) Assume that ρ has the R-F property. Let $f_1, f_2, \ldots$ be a Cauchy sequence in L_ρ. Then there exists a subsequence $g_1, g_2, \ldots$ of $f_1, f_2, \ldots$ such that

$$\sum_n \rho(g_{n+1}-g_n) < \infty.$$

Hence

$$\sum_n \rho(|g_{n+1}-g_n|) = \sum_n \rho(g_{n+1}-g_n) < \infty.$$

Since ρ has the R-F property it follows that $\rho(\sum_n |g_{n+1}-g_n|) < \infty$ and hence, thinking for a moment of g_n $(n=1,2,\ldots)$ as being an element of M, it follows that

$$\sum_n |g_{n+1}(x)-g_n(x)| < \infty$$

for μ-almost every $x \in X$. Thus, setting

$$f(x) = g_1(x)+\sum_n (g_{n+1}(x)-g_n(x))$$

for all $x \in X$, it follows that $f \in M$. Next, consider f as an element of M. Then $f \in L_\rho$ in view of

$$\rho(f) = \rho(|g_1+\Sigma(g_{n+1}-g_n)|) \leq \rho(|g_1|)+\rho(\Sigma\ |g_{n+1}-g_n|) \leq$$
$$\rho(g_1)+\Sigma\ \rho(g_{n+1}-g_n) < \infty$$

(since ρ has the R-F property). Since

$$f-g_p = \sum_{n\geq p}(g_{n+1}-g_n)$$

for $p=1,2,\ldots$, it follows that

$$\rho(f-g_p) \leq \sum_{n\geq p}\rho(g_{n+1}-g_n) \to 0 \quad \text{as} \quad p \to \infty,$$

so

$$\rho(f-f_n) \leq \rho(f-g_p)+\rho(g_p-f_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus L_ρ is a Banach space.

(ii) Conversely, assume that L_ρ is a Banach function space. Let the sequence $u_1,u_2,....$ in L_ρ^+ be given such that $\Sigma\,\rho(u_n) < \infty$. Setting $s_n = u_1+....+u_n$ for all $n\in\mathbb{N}$, it follows that

$$\rho(s_m-s_n) \le \sum_{m+1}^{n}\rho(u_k) \to 0 \quad (m,n\to\infty),$$

so $s_1,s_2,....$ is a Cauchy sequence in L_ρ^+. Since L_ρ is a Banach space and since L_ρ^+ is closed (9.E(v)) it follows that there exists an $f\in L_\rho^+$ such that

$$\rho(f-s_k) \to 0 \quad (k\to\infty).$$

If $n \ge k$, then

$$0 \le s_k - f\wedge s_k = s_n\wedge s_k - f\wedge s_k \le (s_n-f)\wedge s_k \le |s_n-f|,$$

so $\rho(s_k-(f\wedge s_k)) = 0$ for all k. This implies $f \ge s_k$ for all k, so $f \ge \Sigma\, u_n$. Thus

$$\rho(\Sigma\, u_n) \le \rho(f) < \infty,$$

so ρ has the R-F property.

Next, we introduce properties that function semi-norms can have and which are in general easier to check than the R-F property. Before doing so we first introduce a type of convergence in $\overline{M}^+$. Let $u_1,u_2,....$ be a sequence in $\overline{M}^+$ and let $u\in\overline{M}^+$. We shall write $u_n \uparrow u$ if $u_n(x)\uparrow u(x)$ for μ-almost every $x\in X$. Observe that if $u\in M^+$ and if $u_n\uparrow u$ in $\overline{M}^+$, then $u_n\in M^+$ for all n. Moreover, considering u_n $(n=1,2,....)$ and u as elements of M the meaning of $u_n\uparrow u$ coincides with the definition presented in section 7.

19.4. DEFINITION. Let ρ be a function semi-norm on $\overline{M}^+$.

(i) ρ is called a *Fatou semi-norm* if $0 \le u_n \uparrow u$ (in $\overline{M}^+$) implies $\rho(u_n)\uparrow\rho(u)$ (in $\overline{\mathbb{R}}^+$).

(ii) ρ is called a *weak Fatou semi-norm* if $0 \le u_n \uparrow u$ (in $\overline{M}^+$) and

$\lim \rho(u_n) < \infty$ implies $\rho(u) < \infty$.

19.5. THEOREM. *(i) If* ρ *is a Fatou semi-norm, then* ρ *is a weak Fatou semi-norm.*

(ii) If ρ *is a weak Fatou norm, then* ρ *has the R-F property.*

Proof. (i) Obvious.

(ii) Let $u_1, u_2, \ldots$ in L_ρ^+ be such that $\Sigma\, \rho(u_n) < \infty$. Letting $s_n = u_1 + \ldots u_n$ for all n, it is clear that

$$0 \leq s_n \uparrow \sum_k u_k,$$

and

$$\rho(s_n) \leq \sum_1^n \rho(u_k) \leq \sum_k \rho(u_k) < \infty.$$

Hence $\lim \rho(s_n) < \infty$, so $\rho(\Sigma\, u_k) < \infty$. Thus ρ has the R-F property.

The following is now obvious.

19.6. COROLLARY. *If* ρ *is a function norm which is either a Fatou norm or a weak Fatou norm, then* L_ρ *is a Banach function space.*

By means of exercises we now show that the converses of theorem 19.5 do not hold.

19.A. *Exercise.* Let X be the set of natural numbers and let the measure μ be such that $\mu(S)$ equals the number of elements of S for any subset S of X. Any $u \in \overline{M}$ is now of the form $u = (u_1, u_2, \ldots\ldots)$ where $u_i \in \overline{\mathbb{R}}$ for all i.

(i) Set $\rho(u) = \sup \{u_n : n=1,2,\ldots\} + \limsup \{u_n : n=1,2,\ldots.\}$ for all $u \in \overline{M}^+$. Show that ρ is a function norm such that ρ is weak Fatou but not

Fatou.

(ii) Set $\rho(u) = \sup\{u_n: n=1,2,\ldots\}$ if $u_n \to 0$ $(n \to \infty)$ and $\rho(u) = \infty$ otherwise, for all $u \in \overline{M}^+$. Show that ρ is a function norm, that $L_\rho = c_0$ (see 8.A(ii), 9.B), that ρ has the R-F property, but that ρ is not a weak Fatou norm.

Finally, for use in the sequel, we state and prove the following.

19.7. LEMMA. *Let* $\{\rho_\tau: \tau \in T\}$ *be a collection of Fatou semi-norms on* $\overline{M}^+$. *If* $\rho = \sup\{\rho_\tau: \tau \in T\}$, *then* ρ *is a Fatou semi-norm on* $\overline{M}^+$.

Proof. In view of 18.A ρ is a function semi-norm. Next, let $0 \le u_n \uparrow u$ in $\overline{M}^+$ and set $\alpha = \lim \rho(u_n)$. Since $\alpha \le \rho(u)$ is obvious we have to show that $\rho(u) \le \alpha$ holds. Hence, we may assume that $\alpha < \infty$. Next, if $\alpha = 0$, then obviously $\rho(u) = 0$, so we have done. Therefore, assume that $0 < \alpha < \infty$. Now, let $\beta \in \mathbf{R}^+$ be such that $\beta < \rho(u)$. Then there exists a $\tau_0 \in T$ such that $\rho_{\tau_0}(u) > \beta$, so $\rho_{\tau_0}(u_n) > \beta$ if n is large enough because ρ_{τ_0} is a Fatou semi-norm. Then also $\rho(u_n) > \beta$ if n is large enough. This shows that $\alpha > \beta$ and thus $\rho(u) = \alpha$.

20. ORLICZ SPACES

In this section we present an important class of Banach function spaces the so-called Orlicz spaces.

20.1. DEFINITION. A function $\Phi: \mathbf{R}^+ \to \overline{\mathbf{R}}^+$ is called an *Orlicz function* if

(i) $\Phi(0) = 0$, $\Phi(x) \ge 0$ if $x \ge 0$,

(ii) Φ is convex (i.e., $0 \le x < y$ and $0 \le \lambda \le 1$ implies

$$\Phi(\lambda x+(1-\lambda)y) \le \lambda\Phi(x)+(1-\lambda)\Phi(y),$$

(iii) Φ is continuous from the left for all $x > 0$,

(iv) there exist $x_1, x_2 > 0$ such that $\Phi(x_1) < \infty$ and $\Phi(x_2) > 0$.

We make some remarks. First note that by condition (iv) the cases $\Phi \equiv 0$ on $[0,\infty)$ and $\Phi(0) = 0$, $\Phi \equiv \infty$ on $(0,\infty)$ are excluded. Next, note that by condition (ii) an Orlicz function Φ can have at most one point x_0 of discontinuity and that if Φ is discontinuous at x_0, then $x_0 > 0$. Also, in this case, we have $\Phi(x) < \infty$ if $0 \le x < x_0$ and $\Phi(x) = \infty$ if $x > x_0$. By condition (iii), we have $\Phi(x_0) = \lim_{x\uparrow x_0} \Phi(x)$. Finally, we observe that any Orlicz function Φ is non-decreasing on $[0,\infty)$ and that $\lim \Phi(x) = \infty$ as $x \to \infty$.

Next, by means of an Orlicz function we define a function norm.

20.2. DEFINITION. Let Φ be an Orlicz function.

(i) For all $f \in \overline{M}$, define

$$M_\Phi(f) = \int_X \Phi(|f(x)|)\, d\mu(x).$$

(ii) For all $u \in \overline{M}^+$, define

$$\rho_\Phi(u) = \inf\{k > 0: \ M_\Phi(k^{-1}u) \le 1\},$$

where it is to be understood that $\inf \emptyset = +\infty$. Then ρ_Φ is called an *Orlicz norm* on $\overline{M}^+$ (this terminology will be justified by theorem 20.3).

Observe that $M_\Phi(k^{-1}u)$ is a non-increasing function in k for all $u \in \overline{M}^+$. Hence, it follows that for $u \in \overline{M}^+$ we have $M_\Phi(k^{-1}u) \le 1$ for all $k > \rho_\Phi(u)$.

20.A. *Exercise*. Let Φ be an Orlicz function and let $u \in \overline{M}^+$ be such that $0 < \rho_\Phi(u) < \infty$. Show that $M_\Phi(u/\rho_\Phi(u)) \le 1$.

20.3. THEOREM. *Let* Φ *be an Orlicz function. Then* ρ_Φ *is a Fatou norm.*

Proof. We divide the proof into five parts.

(a) Let $u,v \in \overline{M}^+$ be such that $u \le v$ μ-almost everywhere on X. If $k > 0$ is given, then $0 \le M_\Phi(k^{-1}u) \le M_\Phi(k^{-1}v)$, so $\rho_\Phi(u) \le \rho_\Phi(v)$.

(b) Let $u \in N^+$ be given and let $k > 0$. Then $k^{-1}u \in N^+$, so $M_\Phi(k^{-1}u) = 0$. Thus $\rho_\Phi(u) = 0$.

Conversely, let $A \in \Gamma$ be such that $\mu(A) > 0$ and let $k > 0$ be given. Then

$$M_\Phi(k^{-1}\chi_A) = \Phi(k^{-1})\mu(A).$$

If $k \downarrow 0$, then $\Phi(k^{-1})\mu(A) \to \infty$, so there exists a $k_0 > 0$ such that

$$M_\Phi(k_0^{-1}\chi_A) > 1.$$

Thus $\rho_\Phi(\chi_A) \ge k_0 > 0$. Using part (a) this shows that $\rho_\Phi(u) = 0$ implies $u \in N^+$.

(c) It is clear that $\rho_\Phi(au) = a\rho_\Phi(u)$ for all $a \in \mathbb{R}^+$ and for all $u \in \overline{M}^+$.

(d) Let $u,v \in \overline{M}^+$ be given. We have to show that $\rho_\Phi(u+v) \le \rho_\Phi(u)+\rho_\Phi(v)$ holds. If either $\rho_\Phi(u) = \infty$ or $\rho_\Phi(v) = \infty$, this is obvious. Next, if $\rho_\Phi(u) = 0$, then $u+v = v$ μ-almost everywhere by part (b), so there is nothing to prove. Hence, we may assume that $0 < \rho_\Phi(u) < \infty$ and that $0 < \rho_\Phi(v) < \infty$. Next, let $\alpha = \rho_\Phi(u)+\rho_\Phi(v)$ for brevity and let $\varepsilon > 0$ (in $\mathbb{R}$) be given. Then

$$M_\Phi((u+v)/(\alpha+\varepsilon)) \le$$

$$\frac{\rho_\Phi(u)+\frac{1}{2}\varepsilon}{\alpha+\varepsilon} M_\Phi\left(\frac{u}{\rho_\Phi(u)+\frac{1}{2}\varepsilon}\right) + \frac{\rho_\Phi(v)+\frac{1}{2}\varepsilon}{\alpha+\varepsilon} M_\Phi\left(\frac{v}{\rho_\Phi(v)+\frac{1}{2}\varepsilon}\right) \le$$

$$\frac{\rho_\Phi(u)+\frac{1}{2}\varepsilon+\rho_\Phi(v)+\frac{1}{2}\varepsilon}{\alpha+\varepsilon} = 1,$$

so $\rho_\Phi(u+v) \le \alpha+\varepsilon = \rho_\Phi(u)+\rho_\Phi(v)+\varepsilon$. This holds for all $\varepsilon > 0$, so we have done.

(e) Finally, we have to show that ρ_Φ is Fatou. To this end, let $u_n \uparrow u$ in $\overline{M}^+$. It is clear that $\lim \rho_\Phi(u_n) \le \rho_\Phi(u)$. Now, set $\alpha = \lim \rho_\Phi(u_n)$. If $\alpha = \infty$ we have $\rho_\Phi(u) = \infty$ so we are ready. Assume that $\alpha < \infty$ and let

$\varepsilon > 0$ be given. Then $\rho_\Phi(u_n) < \alpha+\varepsilon$ for all n. Moreover, we have

$$\Phi(u_n/(\alpha+\varepsilon)) \uparrow \Phi(u/(\alpha+\varepsilon))$$

μ-almost everywhere on X. Hence

$$M_\Phi(u/(\alpha+\varepsilon)) = \lim M_\Phi(u_n/(\alpha+\varepsilon)) \le 1.$$

This shows that $\rho_\Phi(u) \le \alpha+\varepsilon$. This holds for all $\varepsilon > 0$, so

$$\rho_\Phi(u) = \lim \rho_\Phi(u_n).$$

The normed Köthe space generated by the Fatou norm ρ_Φ (which, in its turn, is generated by an Orlicz function Φ) will be denoted by L_Φ and will be called an *Orlicz space*. By the previous results, any Orlicz space is a Banach function space.

Next, we present some explicit examples of Orlicz spaces.

20.B. *Example*. Let $1 \le p < \infty$ be given and define $\Phi(x) = x^p$ for all $x \ge 0$. It is obvious that Φ is a continuous Orlicz function. Now, let $u \in \overline{M}^+$ be given. Then

$$\rho_\Phi(u) = \inf \{k > 0: \int (k^{-1}u)^p d\mu \le 1\} =$$
$$\inf \{k > 0: \int u^p \, d\mu \le k^p\} = (\int u^p \, d\mu)^{1/p}.$$

Setting, for all $f \in M$

$$\|f\|_p = (\int |f|^p \, d\mu)^{1/p},$$

it follows that $\rho_\Phi(f) = \|f\|_p$ for all $f \in M$. Hence, the Orlicz norm ρ_Φ equals in this case the well-known L_p-norm $\|.\|_p$ and thus the Orlicz space L_Φ equals the well-known space L_p. Thus we have proved that L_p provided with the norm $\|.\|_p$ is an Orlicz space. In particular it follows that L_p is a Dedekind complete Banach lattice.

20.C. Define $\Phi(x) = 0$ for $0 \le x \le 1$, $\Phi(x) = \infty$ for $x > 1$. Again it is obvious that Φ is an Orlicz function. Let $u \in \overline{M}^+$ be given. We recall that

$\|u\|_\infty$ = ess sup $\{u(x):\ x\in X\}$ =

inf $\{\alpha > 0:\ u \le \alpha$ μ-almost everywhere on $X\}$.

Observe that if $0 < \|u\|_\infty < \infty$, then $u/\|u\|_\infty \le 1$ μ-almost everywhere on X, and if $k < \|u\|_\infty$, then there exists a set $A\in\Gamma$ satisfying $\mu(A) > 0$ and $u/k > \chi_A$. It follows that if $u\in\overline{M}^+$ is given, then $\rho_\Phi(u) = 0$ if and only if $\|u\|_\infty = 0$, and $\rho_\Phi(u) = \infty$ if and only if $\|u\|_\infty = \infty$. Also, the above observations imply that $\rho_\Phi(u) = \|u\|_\infty$ for all $u\in\overline{M}^+$, $0 < \|u\|_\infty < \infty$. Thus $\rho_\Phi(u) = \|u\|_\infty$ holds for all $u\in\overline{M}^+$. Therefore, L_Φ equals the well-known space L_∞ and $\rho_\Phi(f) = \|f\|_\infty$ for all $f\in M$. Hence, L_∞ is a Dedekind complete Banach lattice.

Remark. The reader who is not familiar with the theory of L_p-spaces $(1 \le p \le \infty)$ can take the examples 20.B and 20.C as a definition for these spaces.

We collect the preceding results in a theorem.

20.4. *Let* $1 \le p \le \infty$. *Then* L_p *is a Dedekind complete Banach lattice under the* L_p*-norm* $\|.\|_p$. *Moreover,* L_p *is an Orlicz space and hence a Banach function space.*

21. USELESS SETS AND SATURATED FUNCTION SEMI-NORMS

In the sequel a function semi-norm ρ is called *trivial* if either $\rho(u) = 0$ or $\rho(u) = \infty$ for all $u\in\overline{M}^+$. A trivial function norm ρ is, therefore, a function norm ρ such that $\rho(u) = 0$ if and only if $u\in N^+$ and $\rho(u) = \infty$ elsewhere on $\overline{M}^+$. If ρ is a trivial function norm, then $L_\rho = \{0\}$, so, when studying L_ρ, the measure space (X,Γ,μ) is in fact not involved. Also, if ρ is non-trivial (so $0 < \rho(u) < \infty$ for some $u\in M^+$), it can happen that the measure space (X,Γ,μ) is not "optimally used". We illustrate this by an example. Let (X,Γ,μ) be the interval $[0,2]$ provided with Lebesgue measure and set

$$\rho(u) = \|u\chi_{[0,1]}\|_1 + \infty\|u\chi_{(1,2]}\|_\infty$$

for all $u\in\overline{M}^+$. It is easily verified that ρ is a Fatou function norm. However, if $f\in L_\rho$, so $\rho(f) < \infty$, then $f\chi_{(1,2]} = 0$ μ-almost everywhere. Hence, considering the smaller measure space $[0,1]$ provided with Lebesgue measure and considering the function norm ρ_1, defined by

$$\rho_1(u) = \|u\|_1$$

for all measurable u on $[0,1]$ it follows that $L_\rho \overset{\sim}{=} L_{\rho_1}$ $(=L_1)$. Roughly spoken, the interval $(1,2]$ does not have any influence on the space L_ρ, so $(1,2]$ is a "useless" set when studying L_ρ. In this section we shall prove that such useless sets can be removed.

21.1. DEFINITION. Let ρ be a function semi-norm. A set $E\in\Gamma$ is called *useless (with respect to* ρ*)* if

(i) $\mu(E) > 0$,

(ii) $\rho(\chi_F) = \infty$ for all $F \subset E$, $\mu(F) > 0$.

21.2. LEMMA. *Let* ρ *be a function semi-norm and let* $E\in\Gamma$ *be such that* $\mu(E) > 0$. *Then* E *is useless if and only if for all* $u\in\overline{M}^+$ *such that* $\rho(u) < \infty$ *we have* $u \equiv 0$ μ*-almost everywhere on* E *(i.e.,* $u\chi_E\in N^+$*).*

Proof. (i) Assume that E is useless and let $u\in\overline{M}^+$ be such that $\rho(u) < \infty$. Supposing that $u\chi_E\notin N^+$ it follows that there exist an $\varepsilon > 0$ and a set $F \subset E$ satisfying $\mu(F) > 0$ such that $u \geq \varepsilon\chi_F$. Hence

$$\infty = \rho(\varepsilon\chi_F) \leq \rho(u) < \infty$$

which is a contradiction.

(ii) Assume that $u\chi_E \in N^+$ for all $u \in \overline{M}^+$ with $\rho(u) < \infty$. Supposing that E is not useless, the existence of a set $F \subset E$ with $\mu(F) > 0$ and $\rho(\chi_F) < \infty$ follows. Letting $u_0 = \chi_F$ we have $u_0\chi_E \notin N$ and $\rho(u_0) < \infty$. Contradiction.

The following theorem states, up to a μ-null set, that there exists a maximal useless set.

21.3. THEOREM. *Let* ρ *be a function semi-norm. Then there exists a set* X_∞ *in* Γ *such that* $X \backslash X_\infty$ *does not contain any useless sets and such that either* $X_\infty = \emptyset$ *or* X_∞ *is a useless set. The set* X_∞ *is* μ*-uniquely determined. Moreover, if* ρ *is non-trivial, then* $\mu(X \backslash X_\infty) > 0$.

Proof. First assume that $\mu(X) < \infty$ and set

$$\alpha = \sup \{\mu(E): \ E \text{ is useless}\},$$

where it is to be understood that $\sup \emptyset = 0$ in this case. If $\alpha = 0$ there is nothing to prove, so assume that $\alpha > 0$. Then there exists a sequence of sets $E_1, E_2, \ldots.$ in Γ such that $\mu(E_n) \uparrow \alpha$ and such that E_n is useless for all n. Letting

$$E_\infty = \cup E_n,$$

it is clear that E_∞ is useless and that $\mu(E_\infty) = \alpha$. Hence $X \backslash E_\infty$ cannot contain a useless set. Finally, it is also clear that E_∞ is μ-uniquely determined.

Next, assume that $\mu(X) = \infty$. Then $X = \cup X_n$ with $\mu(X_n) < \infty$ $(n=1,2,\ldots)$ and $X_n \cap X_m = \emptyset$ if $m \neq n$ (since μ is σ-finite). If X_k contains a useless set, then X_k contains a maximal useless set E_k by what was proved above. If X_k does not contain a useless set, define $E_k = \emptyset$. Setting

$$X_\infty = \cup E_k$$

it follows that X_∞ is a useless set or that $X_\infty = \emptyset$. If $F \subset X\backslash X_\infty$ is a useless set, then $\mu(F) > 0$ has to hold. Hence $\mu(F\cap X_k) > 0$ for some k. It follows that $F\cap X_k$ is useless and that $(F\cap X_k)\cap E_k = \emptyset$. This is impossible, so X_∞ is a maximal useless set. It is clear again that X_∞ is μ-uniquely determined.

Finally, assume that ρ is non-trivial. Then there exists an element $u\in\overline{M}^+$ such that $0 < \rho(u) < \infty$. Letting $E = \{x\in X: u(x) > 0\}$ it follows from $\rho(u) > 0$ that $\mu(E) > 0$. Furthermore, lemma 21.2 implies that E cannot contain a useless set, so $E \subset X\backslash X_\infty$. This shows that

$$\mu(X\backslash X_\infty) \geq \mu(E) > 0.$$

Now, let ρ be a function norm. Lemma 21.2 shows that if E is a useless set and if $f\in L_\rho$ (so $\rho(f) < \infty$), then $f\chi_E\in N$. Furthermore, theorem 21.3 shows that there exists a maximal useless set X_∞, so for any $f\in L_\rho$ we have $f\chi_{X_\infty}\in N$. Considering the restricted measure space $(X\backslash X_\infty, \Gamma_\infty, \mu_\infty)$ where

$$\Gamma_\infty = \{A\cap(X\backslash X_\infty): A\in\Gamma\},$$

and $\mu_\infty(A) = \mu(A)$ for all $A\in\Gamma_\infty$, it follows that the spaces $L_\rho(X)$ and $L_\rho(X\backslash X_\infty)$ can be identified, so also $L_\rho(X) \cong L_\rho(X\backslash X_\infty)$. Furthermore, the measure space $(X\backslash X_\infty, \Gamma_\infty, \mu_\infty)$ is now "optimally used" when studying L_ρ. Therefore, we define

21.4. DEFINITION. A function semi-norm is called *saturated* if there are no useless sets (i.e., $X_\infty = \emptyset$).

21.5. LEMMA. *Let* Φ *be an Orlicz function. Then* ρ_Φ *is saturated.*

Proof. We have to show that there do not exist useless sets. Let $A\in\Gamma$ be such that $0 < \mu(A) < \infty$. Then $0 < \rho_\Phi(\chi_A) < \infty$. Indeed, if $k > 0$ is given, then

$$M_\Phi(k^{-1}\chi_A) = \Phi(k^{-1})\mu(A).$$

Since $\Phi(k^{-1}) \to 0$ as $k \to \infty$ it follows that $\Phi(k^{-1})\mu(A) \le 1$ for k large enough, so $\rho_\Phi(\chi_A) < \infty$. It is obvious that $\rho_\Phi(\chi_A) > 0$. Thus, since measurable subsets of X contain a set of finite measure it follows that ρ_Φ is saturated.

We make a final remark. Given the non-trivial function norm ρ it follows by the observations made above that if we "restrict" ρ to the space $(X\backslash X_\infty, \Gamma_\infty, \mu_\infty)$, then ρ becomes a saturated function norm and the structure of L_ρ has not been changed.

22. ASSOCIATE FUNCTION SEMI-NORMS

In the study of the dual space of a Köthe space L_ρ it turns out that the so-called associate norms are important. Let ρ be a function semi-norm and define $\rho^{(0)} = \rho$ and

$$\rho^{(n)}(u) = \sup \{\int uv\, d\mu:\ \rho^{(n-1)}(v) \le 1,\ v\in\overline{M}^+\}$$

for $n=1,2,\dots.$, and for all $u\in\overline{M}^+$. In stead of $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(3)}$ we shall also write ρ', ρ'' and ρ''' respectively. The function $\rho^{(n)}$ will be called the n^{th} *associate semi-norm* of ρ. This terminology is justified by the following theorem.

22.1. THEOREM. *If* ρ *is a semi-norm on* $\overline{M}^+$, *then* $\rho^{(n)}$ *is a Fatou semi-norm on* $\overline{M}^+$ $(n=1,2,\dots.)$.

Proof. It is sufficient to show that ρ' is a Fatou semi-norm since the rest is then obvious by induction. Now, let $v\in\overline{M}^+$ be such that $\rho(v) \leq 1$ and set

$$\rho_v(u) = \int uv \, d\mu$$

for all $u\in\overline{M}^+$. It is clear that ρ_v is a Fatou semi-norm, so in view of lemma 19.7 it follows that

$$\rho' = \sup \{\rho_v: \; v\in\overline{M}^+, \; \rho(v) \leq 1\}$$

is a Fatou semi-norm.

Next, we show that there is a close relation between the associate semi-norms that can be derived from a given function semi-norm.

22.2. THEOREM. *(i) (Hölder's inequality). If* $u,v\in\overline{M}^+$ *are such that* $\rho(u)$ *and* $\rho'(v)$ *are finite, then*

$$\int uv \, d\mu \leq \rho(u)\rho'(v).$$

(ii) $\rho'' \leq \rho$ *and* $\rho^{(n+2)} = \rho^{(n)}$ *for all* $n \geq 1$.

Proof. (i) If $0 < \rho(u) < \infty$, then the stated inequality is clear from the definition of $\rho'(v)$. Hence assume that $\rho(u) = 0$. If $u\in N^+$, then also $uv\in N^+$, so $\int uv \, d\mu = 0$ and we are done. Therefore, assume that the set $E\in\Gamma$ defined by $E = \{x\in X: \; u(x) > 0\}$ is such that $\mu(E) > 0$. Define

$$E_n = \{x\in X: \; u(x) \geq n^{-1}\}$$

for $n=1,2,\ldots.$. Then $E_n \uparrow E$ so there exists an n_0 such that $\mu(E_n) > 0$ for all $n \geq n_0$. We shall show that E_n is a useless set with respect to ρ' for all $n \geq n_0$. To this end, let $n \geq n_0$ and let $F \subset E_n$ with $\mu(F) > 0$ be given. Since $u \geq n^{-1}$ on F it follows that $\chi_F \leq nu$, so $\rho(\chi_F) = 0$. Hence, $\rho(k\chi_F) = 0$ for $k=1,2,\ldots.$, so

$$\rho'(\chi_F) = \sup \{\int w\chi_F \, d\mu: \; w\in\overline{M}^+, \; \rho(w) \leq 1\} \geq$$
$$\sup \{k\int \chi_F \, d\mu: \; k=1,2,\ldots.\} = \infty.$$

This implies that E_n is useless with respect to ρ' for all $n \geq n_0$. Hence E is useless with respect to ρ' in view of $E = \cup E_n$. Since $\rho'(v)$ is finite it follows that $v\chi_E \in N$, so $uv \in N$. Hence $\int uv \, d\mu = 0$, so the inequality is clear.

(ii) First we show that $\rho'' \leq \rho$. Therefore, let $u \in \overline{M}^+$ be given. If $\rho(u) = \infty$ we have $\rho''(u) \leq \rho(u)$, so assume that $\rho(u) < \infty$. Then

$$\rho''(u) = \sup \{\int uv \, d\mu: \; v \in \overline{M}^+, \; \rho'(v) \leq 1\} \leq$$

$$\sup \{\rho(u)\rho'(v): \; v \in \overline{M}^+, \; \rho'(v) \leq 1\} = \rho(u)$$

by part (i). Thus $\rho'' \leq \rho$.

Applying this inequality on ρ', we obtain $(\rho')'' = \rho''' \leq \rho'$. On the other hand, if ρ_1, ρ_2 are function semi-norms on $\overline{M}^+$ satisfying $\rho_1 \leq \rho_2$ then clearly $\rho_1' \geq \rho_2'$, so $\rho'' \leq \rho$ implies $\rho''' \geq \rho'$. Thus $\rho''' = \rho'$. By induction it is now clear that $\rho^{(n+2)} = \rho^{(n)}$ for all $n \geq 1$.

The preceding theorem shows that a given function semi-norm ρ has at most three different associate norms, viz. ρ', ρ'' and ρ itself.

Next we study the behaviour of useless sets. Let $E \in \Gamma$ be given and assume that E is useless with respect to ρ'' (so in particular $\mu(E) > 0$). Since $\rho'' \leq \rho$ it follows that E is also useless with respect to ρ. Conversely, assume that E is useless with respect to ρ (again $\mu(E) > 0$ is a consequence). Then $u\chi_E \in N^+$ for all $u \in \overline{M}^+$ satisfying $\rho(u) < \infty$. Hence

$$\rho'(\infty\chi_E) = \sup \{\int \infty u \chi_E \, d\mu: \; u \in \overline{M}^+, \; \rho(u) \leq 1\} = 0.$$

Letting $F \subset E$ be such that $\mu(F) > 0$ it follows that

$$\rho''(\chi_F) \geq \int \infty \chi_E \chi_F \, d\mu = \infty,$$

so E is useless with respect to ρ''. Thus we have proved the following lemma.

22.3. LEMMA. *Let* $E \in \Gamma$, $\mu(E) > 0$ *be given. Then* E *is useless with respect* *to* ρ *if and only if* E *is useless with respect to* ρ''.

The following theorem gives sufficient and necessary conditions so that ρ' is not only a function semi-norm but even a function norm.

22.4. THEOREM. *The following statements are equivalent.*

(*a*) ρ *is a saturated function semi-norm.*

(*b*) ρ' *is a function norm (not necessarily saturated).*

(*c*) ρ'' *is a saturated function semi-norm.*

Proof. (i) (a) $\Leftrightarrow$ (c) is immediate from lemma 22.3.

(ii) (a) $\Rightarrow$ (b). Assume that ρ is saturated but that ρ' is not a function norm. Then there exists a set $E \in \Gamma$, $\mu(E) > 0$ such that $\rho'(\chi_E) = 0$. Since ρ' is a Fatou semi-norm it follows that $\rho'(\infty\,\chi_E) = 0$. Now, similarly as in the proof of lemma 22.3 it follows that E is useless with respect to ρ'' and hence with respect to ρ. This contradicts the assumption that ρ is saturated, so ρ' is a function norm.

(iii) (b) $\Rightarrow$ (a). Assume that ρ' is a function norm but that ρ is not saturated. Then there exists a useless set E (with respect to ρ). Similarly as above we obtain $\rho'(\chi_E) = 0$ and this contradicts $\mu(E) > 0$ since ρ' is a function norm. Thus ρ is saturated.

We note that if ρ is a function norm (not necessarily saturated), then, by similar arguments as above, it follows that ρ' is a saturated semi-norm. Thus, it follows that if ρ is a saturated function norm, then ρ' as well as ρ'' are saturated function norms with the Fatou property. The following theorem is now obvious.

22.5. THEOREM. *(Hölder's inequality). Let* ρ *be a saturated function norm.*

(*i*) $\int uv\, d\mu \le \rho''(u)\rho'(v) \le \rho(u)\rho'(v)$ *for all* $u,v\in\overline{M}^{+}$.

(*ii*) $\int |fg|\, d\mu \le \rho''(f)\rho'(g) \le \rho(f)\rho'(g)$ *for all* $f,g\in M$.

In the remaining part of this section we show the importance of the first associate norm, when studying the norm dual space of a Köthe space. To this end,

Let, for the rest of this chapter, ρ *be a fixed saturated function norm.* As shown in section 21 the requirement that ρ be saturated is no restriction at all. It follows that ρ' is now a Fatou norm on M (saturated). The normed Köthe space (even Banach function space) generated by ρ' will be denoted by L'_ρ. This space is called *the first associate space* of L_ρ. It is clear that

$$\rho'(g) = \sup \{\int |fg|\, d\mu: \ \rho(f) \le 1\}$$

holds for all $g\in M$ (or all $g\in M$ after identifications). First we derive a slightly different formula for elements in L'_ρ. Let $g\in L'_\rho$ be given. Let $f\in L_\rho$ be such that $\rho(f) \le 1$. Then

$$\int |fg|\, d\mu \le \rho(f)\rho'(g) < \infty$$

by theorem 22.5. Hence $fg\in L_1$ (see 20.B and theorem 20.4), so $\int fg\, d\mu$ exists. Next, note that

$$\sup \{|\int fg\, d\mu|: \ \rho(f) \le 1\} \le \sup \{\int |fg|\, d\mu: \ \rho(f) \le 1\} = \rho'(g).$$

Now, let $\varepsilon > 0$ be given and let $f_1\in L_\rho$ be such that $\rho(f_1) \le 1$,

$$\int |f_1 g|\, d\mu > \rho'(g)-\varepsilon.$$

Defining $f = |f_1|.\text{sgn } g$ (where $\text{sgn } g(x) = 1$ if $g(x) \ge 0$, $\text{sgn } g(x) = -1$ if $g(x) < 0$ after we have chosen a representant of g in M), it follows that $|f| = |f_1|$, so $\rho(f) = \rho(f_1) \le 1$ and

$$0 \le \int fg\, d\mu = \int |f_1 g|\, d\mu.$$

This shows that

$$|\int fg\, d\mu| > \rho'(g)-\varepsilon,$$

so

$$\rho'(g) \le \sup \{|\int fg\, d\mu|: \ \rho(f) \le 1\}+\varepsilon.$$

Since this holds for all $\varepsilon > 0$ we have proved the following lemma.

22.6. LEMMA. *For all* $g\in L'_\rho$ *we have*

$$\rho'(g) = \sup \{|\int fg\, d\mu|: \ f\in M, \ \rho(f) \le 1\}.$$

As observed above, for any $f\in L_\rho$ and for any $g\in L'_\rho$ $\int fg\, d\mu$ exists as a finite number ($fg\in L_1$). Hence, fixing $g\in L'_\rho$ and setting

$$\phi_g(f) = \int fg\, d\mu$$

for all $f\in L_\rho$, it is clear that ϕ_g is linear on L_ρ. However, more can be said. Let L^*_ρ denote the Banach dual of L_ρ and let ρ^* be the norm in L^*_ρ.

22.7. THEOREM. *Let* $g\in L'_\rho$ *be given. Then* $\phi_g\in L^*_\rho$ *and* $\rho^*(\phi_g) = \rho'(g)$.

Proof. Using lemma 22.6, we obtain

$$\rho^*(\phi_g) = \sup \{|\phi_g(f)|: \ \rho(f) \le 1\} = \sup \{|\int fg\, d\mu|: \ \rho(f) \le 1\} = \rho'(g).$$

Next, defining the operator $I: L'_\rho \to L^*_\rho$ by $I(g) = \phi_g$ for all $g\in L'_\rho$ it is clear that I is a positive linear norm preserving operator from L'_ρ into L^*_ρ. Hence, we can think of L'_ρ as a linear subspace of L^*_ρ. Moreover, since ρ' is a Fatou norm (which implies that L'_ρ is a Banach lattice), it follows that L'_ρ is even a norm closed linear subspace of L^*_ρ (after applying I). We shall show now that the image of L'_ρ under I is precisely $L^*_{\rho,c}$, the set of integrals on L_ρ (definition 7.1). Thus I

turns out to be a norm preserving Riesz isomorphism from L'_ρ onto $L^*_{\rho,c}$.

First observe the following.

22.8. LEMMA. *Let* $g \in M$ *be given. Then the following assertions are equivalent*

(*a*) $g \in L'_\rho$.

(*b*) $fg \in L_1$ *for all* $f \in L_\rho$ *and* $\phi_g \in L^*_\rho$.

Proof. (i) (a) ⇒ (b). Has already been shown.

(ii) (b) ⇒ (a). Let $f \in L_\rho$ be given and set $f_1 = f.\text{sgn } g$. Then $f_1 \in L_\rho$ since $|f_1| = |f|$, so

$$\int |fg| \, d\mu = \int f_1 g \, d\mu = \phi_g(f_1) \le \rho^*(\phi_g)\rho(f_1) = \rho^*(\phi_g)\rho(f).$$

This shows that

$$\rho'(g) \le \rho^*(\phi_g) < \infty,$$

so $g \in L'_\rho$ (and hence we also obtain $\rho'(g) = \rho^*(\phi_g)$);

Next we state and prove the main theorem of this section.

22.9. THEOREM. *The operator* I *is a norm preserving Riesz isomorphism from* L'_ρ *onto* $L^*_{\rho,c}$, *i.e.* $L'_\rho \cong L^*_{\rho,c}$.

Proof. (i) Let $g \in L'_\rho$ be given and let $\phi_g = I(g)$ as before. Then $\phi_g \in L^*_\rho$ and $\rho^*(\phi_g) = \rho'(g)$ by theorem 22.7. Now, let $f_n \downarrow 0$ in L^+_ρ. Then $|f_1 g| \in L_1$ and

$$|f_1 g| \ge |f_n g| \downarrow 0,$$

so by the theorem on dominated convergence of integrals we obtain

$$\lim \phi_g(f_n) = \lim \int f_n g \, d\mu = 0 \quad (n \to \infty).$$

This shows that $\phi_g \in L^*_{\rho,c}$, so I is a positive linear norm preserving map from L'_ρ into $L^*_{\rho,c}$.

(ii) Conversely, let $\phi \epsilon L^*_{\rho,c}$ be given. First assume that $\phi \geq 0$. We have to find a $g \epsilon (L'_\rho)^+$ for which $\phi_g = \phi$. To this end, define

$$\Gamma_0 = \{A \epsilon \Gamma : \ \chi_A \epsilon L_\rho\}.$$

Then Γ_0 contains many non-trivial elements since ρ is saturated. Moreover Γ_0 is a semi-ring of subsets of X. Next, define the function $\nu_\phi : \Gamma_0 \to \mathbb{R}^+$ by

$$\nu_\phi(A) = \phi(\chi_A)$$

for all $A \epsilon \Gamma_0$. Then $\nu_\phi(\emptyset) = 0$ and if $A_1, A_2, \ldots$ is a sequence in Γ_0 such that $A_n \cap A_m = \emptyset$ $(m \neq n)$ and $\cup A_n \in \Gamma_0$, then

$$\nu_\phi(\cup A_n) = \phi(\chi_{\cup A_n}) = \phi(\Sigma \chi_{A_n}) = \Sigma \phi(\chi_{A_n}) = \Sigma \nu_\phi(A_n)$$

by lemma 7.6. Hence ν_ϕ is a pre-measure on Γ_0. Next, let Γ_1 be the σ-algebra of subsets of X consisting of the ν_ϕ-measurable sets. Then $\Gamma \subset \Gamma_1$. Indeed, if $B \epsilon \Gamma$, $A \epsilon \Gamma_0$ then $B \cap A \in \Gamma_0$ and $B^c \cap A \in \Gamma_0$, so

$$\nu_\phi(A) = \nu_\phi(B \cap A) + \nu_\phi(B^c \cap A).$$

This shows that $B \epsilon \Gamma_1$, so $\Gamma \subset \Gamma_1$. Hence, applying the Carathéodory extension procedure on ν_ϕ, we can consider ν_ϕ as a measure on Γ. It is clear that ν_ϕ is absolutely continuous with respect to μ $(\nu_\phi << \mu)$. Hence, (by the Radon-Nikodym theorem), there exists a μ-uniquely dtermined μ-measurable positive function g such that

$$\nu_\phi(A) = \int g \chi_A \, d\mu$$

for all $A \epsilon \Gamma$. In particular it follows that

$$\nu_\phi(\chi_A) = \int g \chi_A \, d\mu$$

for all $A \epsilon \Gamma_0$. Hence, if t is any step function in L^+_ρ then

$$\phi(t) = \int gt \, d\mu.$$

Next, if $f \epsilon L^+_\rho$ is given, there exists a sequence of stepfunctions $t_1, t_2, \ldots$ in L^+_ρ such that $0 \leq t_n \uparrow f$. Using again the fact that ϕ is an integral, we obtain

$$\phi(f) = \lim \phi(t_n) = \lim \int g t_n \, d\mu = \int fg \, d\mu,$$

since $gt_n \uparrow gf$. It is now obvious that $\phi(f) = \int fg\,d\mu$ holds for all $f \in L_\rho$ (see also lemma 2.10). Using lemma 22.8 it follows that $g \in (L_\rho')^+$ and that $I(g) = \phi_g = \phi$.

Finally, if $\phi \in L_{\rho,c}^*$ is not necessarily positive, set $\phi = \phi^+ - \phi^-$. Then there exists unique $g_1, g_2 \in (L_\rho')^+$ such that

$$\phi^+(f) = \int fg_1\,d\mu \quad \text{and} \quad \phi^-(f) = \int fg_2\,d\mu$$

holds for all $f \in L_\rho$. Hence, setting $g = g_1 - g_2$ we have $I(g) = \phi$. Thus we have shown that I maps L_ρ' onto $L_{\rho,c}^*$ and that I^{-1} is a positive map from $L_{\rho,c}^*$ into L_ρ', so the theorem is proved (see theorem 2.7).

In the next section we shall compute the first associate space of an Orlicz space.

23. THE FIRST ASSOCIATE SPACE OF AN ORLICZ SPACE

Let, in this section, Φ be a fixed Orlicz function. Furthermore, let ρ_Φ and L_Φ be as defined in section 20. Since ρ_Φ is a saturated Fatou norm it follows that ρ_Φ' is a saturated Fatou norm as well (theorem 22.4). In this section we shall show that ρ_Φ' is equivalent to an Orlicz norm ρ_Ψ, where Ψ is an Orlicz function called the *complementary Orlicz function* of Φ. To this end, define

$$\Psi(x) = \sup\,\{\, xy - \Phi(y) : \; y \geq 0\}$$

for all $x \geq 0$. Observe that

$$\Psi(0) = \sup\,\{-\Phi(y) : \; y \geq 0\} = \Phi(0) = 0,$$

and

$$\Psi(x) \geq \{x.0 - \Phi(0)\} = 0.$$

Setting, for $y \geq 0$ fixed, $f_y(x) = xy - \Phi(y)$ for all $x \geq 0$, we see that

$$\Psi(x) = \sup\,\{f_y(x) : \; y \geq 0\}$$

for all $x \geq 0$ and that f_y is convex. Hence, Ψ is a convex function on $[0,\infty)$. More can be proved.

23.1. THEOREM. Ψ *is an Orlicz function on* $[0,\infty)$.

Proof. We have to verify the conditions (iii) and (iv) of definition 20.1.

Let $x_1 > 0$ be such that $\Phi(x_1) < \infty$. Then there exists an $n \in \mathbb{N}$ such that $\Phi(x_1) < nx_1$. Thus

$$\Psi(n) \geq nx_1 - \Phi(x_1) > 0.$$

This shows the existence of a number $y_1 > 0$ for which $\Psi(y_1) > 0$. Next, observe that there exist real numbers a,b such that $a > 0$, $b \leq 0$ and

$$\Phi(y) \geq ay+b$$

for all $y \geq 0$ (since Φ is non-decreasing and convex). Hence, if $0 < x < a$, then

$$\Psi(x) = \sup\{(xy-\Phi(y): y \geq 0\} \leq$$

$$\sup\{(xy-ay-b): y \geq 0\} = -b < \infty.$$

This shows that there exists a number $x_2 > 0$ such that $\Psi(x_2) < \infty$. Thus condition (iv) of definition 20.1 is verified.

To verify condition (iii), let $x_0 > 0$ be given. First assume that $\Psi(x_0) < \infty$. Then $\{f_y: y \geq 0\}$ is a collection of continuous functions on $[0,x_0]$ which are bounded from above by $\Psi(x_0)$. Hence, by 12.H(iv) Ψ is a lower semi-continuous function on $[0,x_0]$. Since Ψ is non-decreasing on $[0,x_0]$ it follows that Ψ is continuous from the left at x_0. Next, assume that $\Psi(x_0) = \infty$. If $\Psi(x) = \infty$ for some $x < x_0$ there is nothing to prove, so assume in addition that $\Psi(x) < \infty$ for all $x < x_0$. Let $x_1,x_2,\ldots$ be a sequence in $(0,x_0)$ such that $x_n \uparrow x_0$. Since

$$\Psi(x_n) = \sup\{f_y(x_n): y \geq 0\}$$

and since

$$\infty = \Psi(x_0) = \sup \{f_y(x_0): \ y \geq 0\}$$

and since all f_y $(y \geq 0)$ are continuous it is clear that $\Psi(x_n) \uparrow \infty$ as $n \to \infty$. Thus the theorem is proved.

The Orlicz space L_Ψ (with norm ρ_Ψ) is called the *complementary space* of L_Φ.

23.A. *Exercise.* Show that the complementary Orlicz function of Ψ is Φ again (thus the complementary space of L_Ψ is L_Φ).

The preceding exercise allows us to talk about the pair of complementary Orlicz functions Φ and Ψ without mentioning whether Ψ is derived from Φ or conversely.

LET, FOR THE REMAINING PART OF THIS SECTION Φ AND Ψ BE A PAIR OF COMPLEMENTARY ORLICZ FUNCTIONS.

Let ρ_Φ and ρ_Ψ be the Orlicz function norms derived from Φ and Ψ respectively. Furthermore, let ρ'_Φ be the first associate norm of ρ_Φ. We shall show now that ρ'_Φ and ρ_Ψ are equivalent norms on L_Ψ, which implies that the first associate space L'_Φ of L_Φ is Riesz isomorphic (in fact equal) to L_Ψ (but not necessarily isometric). First note that for any $f \in L_\Phi$ we have

$$\rho_\Phi(f) \leq 1 \ \text{ if and only if } \ M_\Phi(f) \leq 1.$$

Next, it follows from the construction of Ψ that for all $x, y \geq 0$ we have

$$xy \leq \Phi(y) + \Psi(x).$$

Thus, if $f \in M$, then

$$\rho'_\Phi(f) = \sup \{\int |fg| \, d\mu: \ \rho_\Phi(g) \leq 1\} = \sup \{\int |fg| \, d\mu: \ M_\Phi(g) \leq 1\} \leq$$

$$\sup \{M_\Phi(g)+M_\Psi(f):\ M_\Phi(g) \le 1\} \le$$

$$1+M_\Psi(f) = 1+\int\Psi(|f|)\,d\mu.$$

Hence, if $f\in L_\Psi$, $\rho_\Psi(f) > 0$, then

$$\rho'_\Phi(f)/\rho_\Psi(f) \le 1+M_\Psi(f)/\rho_\Psi(f) \le 2,$$

so $\rho'_\Phi(f) \le 2\rho_\Psi(f)$. It is clear that this inequality also holds for those $f\in M$ for which $\rho_\Psi(f) = 0$ or $\rho_\Psi(f) = \infty$. Let us collect the results so far obtained in a lemma.

23.2. LEMMA. *(i) (Young's inequality). For all* $x,y \ge 0$ *we have*

$$xy \le \Phi(x)+\Psi(y).$$

(ii) (Amemiya's inequality). For all $f\in M$ *we have*

$$\rho'_\Phi(f) \le 1+\int\Psi(|f|)\,d\mu = 1+M_\Psi(f).$$

(iii) For all $f\in M$ *we have*

$$\rho'_\Phi(f) \le 2\rho_\Psi(f).$$

(The lemma also holds with Φ *and* Ψ *interchanged).*

Next, we shall show that $\rho_\Psi(f) \le \rho'_\Phi(f)$ for all $f\in M$. Before doing so we have to make some observations.

Since Φ is convex, it follows that the left derivative Φ' of Φ exists for all points $x > 0$ for which $\Phi(x+\varepsilon) < \infty$ for some $\varepsilon > 0$. Therefore, let the function $\phi: [0,\infty) \to [0,\infty]$ be defined by

$\phi(0) = 0,$

$\phi(x) = \Phi'(x)$ for all $x > 0$, $\Phi(x+\varepsilon) < \infty$ for some $\varepsilon > 0$,

$\phi(x_0) = \lim_{x\uparrow x_0} \Phi'(x)$ if x_0 is such that $\Phi(x) < \infty$ for all $x \le x_0$, $\Phi(x) = \infty$ for all $x > x_0$,

$\phi(x) = \infty$ for all $x > 0$ where $\Phi(x) = \infty$.

We leave it to the reader to show that ϕ is a non-decreasing left continuous function on $(0,\infty)$. The function ψ related to Ψ is now defined similarly.

23.B. *Exercise*. Let $x_1, x_2 \geq 0$ be given such that $x_1 < x_2$ and such that $\Phi(x_2) < \infty$. Show that

$$\phi(x_1) \leq (\Phi(x_2)-\Phi(x_1))/(x_2-x_1) \leq \phi(x_2).$$

A similar result holds for Ψ and ψ.

23.3. LEMMA. *For all* $x \geq 0$ *we have* $x\phi(x) = \Phi(x)+\Psi(\phi(x))$ *and similarly with* Φ *and* Ψ *interchanged.*

Proof. It is clear that $0\phi(0) = \Phi(0)+\Psi(\phi(0))$, so to prove the equality, let $x > 0$ be given. If x is such that $\Phi(x) = \infty$, then also $\phi(x) = \infty$ so we are done. Next, assume that x is such that $\Phi(x)<\infty$ but $\phi(x) = \infty$. Again the equality is clear since $\Psi(\infty) = \infty$. Thus we can assume that both $\Phi(x)$ and $\phi(x)$ are finite. By lemma 23.2(i) it is clear that

$$x\phi(x) \leq \Phi(x)+\Psi(\phi(x)).$$

To prove equality, let $y \geq 0$, $y \neq x$ be given. If $\Phi(y) = \infty$, then

$$y\phi(x)-\Phi(y) = -\infty < x\phi(x)-\Phi(x).$$

Furthermore, if $\Phi(y) < \infty$, then exercise 23.B shows that

$$y\phi(x)-\Phi(y) \leq x\phi(x)-\Phi(x)$$

(consider the cases $y < x$ and $y > x$). Hence, by the definition of Ψ,

$$\Psi(\phi(x)) = \sup \{(y\phi(x)-\Phi(y)): y \geq 0\} \leq x\phi(x)-\Phi(x),$$

so

$$x\phi(x) \geq \Phi(x)+\Psi(\phi(x)),$$

which proves the lemma.

Before proving our main theorem we derive some auxiliary results.

23.4. LEMMA. (*i*) *If* Φ *is discontinuous at* $x_0 > 0$, *then* $\psi(x) \le x_0$ *for all* $x \ge 0$. *The same holds for* Ψ *and* ϕ.

(*ii*) *If* Ψ *is discontinuous at* $x_0 > 0$, *and if* $f \in L_\infty$, *then*

$$\|f\|_\infty \le x_0 \rho'_\Phi(f).$$

Proof. (i) By the definition of Ψ it follows that $\Psi(x) \le x_0 x$ for all $x \ge 0$. Hence, since Ψ is convex, we obtain $\psi(x) \le x_0$ for all $x \ge 0$.

(ii) Let $f \in L_\infty$ be given. If $\|f\|_\infty = 0$ the statement is clear, so assume that $0 < \|f\|_\infty < \infty$. Now, let ε be such that $0 < \varepsilon < \|f\|_\infty$ and define

$$A = \{x \in X: \ |f(x)| > \|f\|_\infty - \varepsilon\}.$$

Then $\mu(A) > 0$. Setting $f_\varepsilon = f\chi_A$ and using part (i), we obtain

$$\rho'_\Phi(f) \ge \rho'_\Phi(f_\varepsilon) = \sup \{\int |f_\varepsilon g| \, d\mu: \ \rho_\Phi(g) \le 1\} \ge$$
$$\sup \{\int |f_\varepsilon g| \, d\mu: \ x_0 \int |g| \, d\mu \le 1\} \ge$$
$$\sup \{(\|f\|_\infty - \varepsilon) \int |g| \, d\mu: \ x_0 \int |g| \, d\mu \le 1\} = x_0^{-1}(\|f\|_\infty - \varepsilon).$$

This holds for all ε, $0 < \varepsilon < \|f\|_\infty$, so $x_0 \rho'_\Phi(f) \ge \|f\|_\infty$.

23.5. THEOREM. *We have* $\rho_\Psi(f) \le \rho'_\Phi(f)$ *for all* $f \in M$. *The same holds if* Φ *and* Ψ *are interchanged.*

Proof. Let $f \in M$ be given. If $\rho'_\Phi(f) = 0$ or if $\rho'_\Phi(f) = \infty$ there is nothing to prove, so assume that $0 < \rho'_\Phi(f) < \infty$. We divide the proof into two parts.

(a) Assume that $|f|$ is bounded (i.e., $\|f\|_\infty < \infty$) and that $f \equiv 0$ outside a set $A \in \Gamma$, $\mu(A) < \infty$. Moreover, let $\varepsilon > 0$ be given. Next, introduce f_0 by setting

$$f_0(x) = \psi(|f(x)|/(\rho'_\Phi(f)+\varepsilon)).$$

It follows from lemma 23.4 that $\|f_0\|_\infty < \infty$ (oonsider the cases Ψ continuous and Ψ discontinuous). Also, since $f_0 \equiv 0$ outside A, it follows that $M_\Phi(f_0) < \infty$ (if Φ is continuous this is clear, if Φ is discontinuous the statement follows from lemma 23.4). Next, let $g \in M$ be such that $M_\Phi(g) < \infty$. If $M_\Phi(g) \le 1$, then $\rho_\Phi(g) \le 1$, so by Hölder's inequality,

$$\int |fg|\, d\mu \le \rho'_\Phi(f)\rho_\Phi(g) \le \rho'_\Phi(f).$$

If $1 < M_\Phi(g) < \infty$, then

$$M_\Phi(g/M_\Phi(g)) \le M_\Phi(g)/M_\Phi(g) = 1$$

by the convexity of Φ and the fact that $\Phi(0) = 0$. Hence

$$\int |fg|\, d\mu \le \rho'_\Phi(f)M_\Phi(g)$$

in this case. Setting $M'_\Phi(g) = \max\{M_\Phi(g), 1\}$ it follows that

$$\int |fg|\, d\mu \le \rho'_\Phi(f)M'_\Phi(g).$$

This inequality holds for all $g \in M$ satisfying $M_\Phi(g) < \infty$. Taking $g = f_0$, and using lemma 23.3, we obtain

$$M_\Psi(f/(\rho'_\Phi(f)+\varepsilon))+M_\Phi(f_0) = \int |ff_0|/(\rho'_\Phi(f)+\varepsilon)\, d\mu \le$$

$$\rho'_\Phi(f)M'_\Phi(f_0)/(\rho'_\Phi(f)+\varepsilon) < M'(f_0).$$

Thus

$$M_\Psi(f/(\rho'_\Phi(f)+\varepsilon)) < M'_\Phi(f_0)-M_\Phi(f_0) \le 1.$$

Hence $\rho_\Psi(f) \le \rho'_\Phi(f)+\varepsilon$. This holds for all $\varepsilon > 0$, so $\rho_\Psi(f) \le \rho'_\Phi(f)$.

(b) Let $f \in L'_\Phi$ be arbitrary. Now, since μ is σ-finite, there exists a sequence $X_1, X_2, \ldots\ldots$ in Γ such that $X_n \uparrow X$ and such that $\mu(X_n) < \infty$ for all n. Defining

$$f_n = \sup\{n\chi_{X_n}, |f|\chi_{X_n}\}$$

for all n it follows that $0 \le f_n \uparrow f$ as $n \to \infty$, so, since ρ'_Φ and ρ_Ψ are Fatou norms, we obtain

$$\rho_\Psi(f) = \lim \rho_\Psi(f_n) \le \lim \rho'_\Phi(f_n) = \rho'_\Phi(f)$$

(by part (a)). Thus the proof is complete.

The following corollary is now immediate.

23.6. COROLLARY. *Let* $f\in M$ *be given.*

(*i*) $\rho_\Psi(f) \le \rho'_\Phi(f) \le 2\cdot\rho_\Psi(f)$. Thus, *the norms* ρ'_Φ *and* ρ_Ψ *are equivalent and* $L'_\Phi = L_\Psi$ *when regarded as point sets.*

(*ii*) *If* $0 < \rho'_\Phi(f) < \infty$, *then* $M_\Phi(f/\rho'_\Phi(f)) \le 1$.

The same holds with Φ *and* Ψ *interchanged.*

We have shown that the first associate space of L_Φ (with norm ρ_Φ) is its complementary space L_Ψ (with norm ρ'_Φ equivalent to ρ_Ψ). As customary, from now on we denote the Fatou norm ρ'_Φ by $\|.\|_\Psi$ (and ρ'_Ψ by $\|.\|_\Phi$). It is also clear that the first associate space of L_Ψ (with norm ρ_Ψ) is now its complementary space L_Φ (with norm $\|.\|_\Phi$).

Once more consider the Orlicz space L_Φ but now endowed with the norm $\|.\|_\Phi$. It is natural to ask what the first associate norm $\|.\|'_\Phi$ of $\|.\|_\Phi$ looks like. Note that since $\|.\|_\Phi$ is equivalent to ρ_Φ it follows that $\|.\|'_\Phi$ is equivalent to $\rho'_\Phi = \|.\|_\Psi$ and hence also to ρ_Ψ. Moreover, since $\|.\|_\Phi = \rho'_\Psi$ it follows that $\|.\|'_\Phi = \rho''_\Psi$, so by theorem 22.2(ii) it follows that $\|.\|'_\Phi \le \rho_\Psi$. However, it can be shown that $\|.\|'_\Phi$ equals ρ_Ψ (and hence also $\|.\|'_\Psi = \rho_\Phi$). This will be done in section 26.

24. THE DUAL OF AN ORLICZ SPACE

In this section we study the dual L^*_Φ of a given Orlicz space L_Φ. Furthermore, applications for L_p-spaces will be presented.

Throughout this section Φ will be a fixed Orlicz function, ρ_Φ and L_Φ will be as before. Moreover, L^*_Φ will denote the dual space of L_Φ and the norm in L_Φ will be denoted by ρ^*_Φ. As observed in section 11 we have

$$L^*_\Phi = L^*_{\Phi,c} + L^*_{\Phi,s}.$$

Moreover, by theorem 22.9 we have $L^*_{\Phi,c} \overset{\sim}{=} L'_\Phi$, so if Ψ is the complementary function of Φ, we obtain $L_{\Phi,c}^{*} \overset{\sim}{=} L_{\Psi}$ where the norm in L_{Ψ} is now $\|.\|_{\Psi}$. Thus we have completely characterized the integral part of L_{Φ}^{*}. It is not so easy to characterize the singular part $L_{\Phi,s}^{*}$ of L_{Φ}^{*}. However, there is something we can say.

24.1. THEOREM. L_{Φ} *is a semi-M-space (so, by theorem 16.15,* $L_{\Phi,s}^{*}$ *is an abstract* L-*space).*

Proof. Let $f_1,f_2 \in L_{\Phi}^{+}$ be given such that $\rho_{\Phi}(f_1) = \rho_{\Phi}(f_2) = 1$. Furthermore, set $g = f_1 \vee f_2$, and let $g_1,g_2,\ldots.$ in L_{Φ}^{+} be such that $g \geq g_n \downarrow 0$. We have to show that $\lim \rho_{\Phi}(g_n) \leq 1$. To this end, note that $M_{\Phi}(f_1) \leq 1$ and $M_{\Phi}(f_2) \leq 1$. Next, let

$$A = \{x \in X\colon f_1(x) \geq f_2(x)\}.$$

Then

$$M_{\Phi}(g) = \int \Phi(f_1\chi_A)\, d\mu + \int \Phi(f_2\chi_{A^c})\, d\mu \leq$$

$$M_{\Phi}(f_1) + M_{\Phi}(f_2) \leq 2.$$

Therefore, by Lebesgue's theorem on dominated convergence, we have

$$\lim M_{\Phi}(g_n) = 0.$$

Hence, combining lemma 23.2(ii) and theorem 23.5 with this result, we obtain

$$\lim \rho_{\Phi}(g_n) \leq 1 + \lim M_{\Phi}(g_n) = 1,$$

which is the desired result.

Remark. If L_{Φ} is endowed with the norm $\|.\|_{\Phi}$, then it follows similarly as above that L_{Φ} is a semi-M-space as well. The easy verifications are left to the reader.

Next, we shall study under what conditions we have $L_{\Phi}^{*} \overset{\sim}{=} L_{\Psi}$ (L_{Φ}^{*} provided with the norm ρ_{Φ}^{*} and L_{Ψ} provided with the norm $\|.\|_{\Psi}$). By corollary 11.6 we have $L_{\Phi}^{*} \overset{\sim}{=} L_{\Psi}$ ($\overset{\sim}{=} L_{\Phi,c}^{*}$) if and only if $L_{\Phi}^{a} = L_{\Phi}$. Thus we have to find conditions so that $L_\Phi^a = L_\Phi$. Before solving this problem, we first note the following.

24.2. LEMMA. *Assume that* Φ *is continuous. If* $A\epsilon\Gamma$ *is such that* $\mu(A) < \infty$, *then* $\chi_A \epsilon L_\Phi^a$.

Proof. Let $A\epsilon\Gamma$ be given such that $\mu(A) < \infty$. In the proof of lemma 21.5 we have shown that $\chi_A \epsilon L_\Phi^+$. To show that $\chi_A \epsilon L_\Phi^a$, let $\varepsilon > 0$ be given and let $f_1, f_2, \ldots$ in L_Φ^+ be such that $\chi_A \geq f_n \downarrow 0$. Letting

$$A_n = \{x\epsilon A: \ f_n(x) > \varepsilon\}$$

it is clear that

$$f_n \leq \chi_{A_n} + \varepsilon\chi_{A_n^c}$$

for all n. Hence

$$\rho_\Phi(f_n) \leq \rho_\Phi(\chi_{A_n}) + \varepsilon\rho_\Phi(\chi_{A_n^c}).$$

Now note that $\lim \mu(A_n) = 0$ since $\mu(A) < \infty$ and since $f_n \downarrow 0$. Hence,

$$\lim \rho_\Phi(\chi_{A_n}) = 0$$

(using the formula for $\rho_\Phi(\chi_{A_n})$, derived in lemma 21.5). Here we use the continuity of Φ, which implies that $\Phi^{-1}(x) \to \infty$ as $x \to \infty$. Thus

$$\lim \rho_\Phi(f_n) \leq \varepsilon \lim \rho_\Phi(\chi_{A_n^c}) = \varepsilon\rho_\Phi(\chi_A).$$

This holds for all $\varepsilon > 0$, so $\lim \rho_\Phi(f_n) = 0$. Hence $\chi_A \epsilon L_\Phi^a$.

Since L_Φ^a is an order ideal of L_Φ it follows that if $f\epsilon M$ is a bounded function satisfying $\mu\{x\epsilon X: \ f(x) \neq 0\} < \infty$, then $f\epsilon L_\Phi^a$ provided that Φ is continuous.

24.3. DEFINITION. Let $x_0 \geq 0$. The Orlicz function Φ is said to satisfy the $\Delta_2(x_0)$*-condition*, whenever there exists a constant $M > 0$ such that $\Phi(2x) < M\Phi(x)$ for all $x \geq x_0$.

We note that if Φ satisfies the $\Delta_2(x_0)$-condition for some $x_0 \geq 0$, then Φ has to be continuous. Also, since $\Phi(0) = 0$, Φ is convex and increasing, the constant M occuring in the definition satisfies $M \geq 2$. Let us present an example.

24.A. *Example*. Let $1 \leq p < \infty$ and set $\Phi(x) = x^p$ for all $x \geq 0$. Then Φ is an Orlicz function (see 20.B). Also Φ satisfies the $\Delta_2(0)$-condition. Indeed,

$$\Phi(2x) = 2^p x^p < M\Phi(x)$$

for all $x \geq 0$ if $M = 2^p+1$.

Next, we show the usefulness of the $\Delta_2(x_0)$-condition.

24.4. THEOREM. (*i*) *If* Φ *satisfies the* $\Delta_2(0)$*-condition, then* $L^a_\Phi = L_\Phi$.

(*ii*) *If* $\mu(X) < \infty$ *and if* Φ *satisfies the* $\Delta_2(x_0)$*-condition for some* $x_0 \geq 0$, *then* $L^a_\Phi = L_\Phi$.

Proof. (i) Let $f \in L^+_\Phi$ and $\varepsilon > 0$ be given. We shall prove the existence of a function $f_\varepsilon \in L^a_\Phi$ such that $\rho_\Phi(f-f_\varepsilon) < \varepsilon$. Since this can be done for all $\varepsilon > 0$ and since L^a_Φ is norm closed it follows then that $f \in L^a_\Phi$, so $L^a_\Phi = L_\Phi$.

To construct $f_\varepsilon \in L^a_\Phi$, let $X_1, X_2, \ldots.$ be a sequence in Γ such that $X_n \uparrow X$ and such that $\mu(X_n) < \infty$ for all n, and define

$$f_n = (f\chi_{X_n}) \wedge (n\chi_{X_n})$$

for all n. Since Φ satisfies the $\Delta_2(0)$-condition it follows that Φ is continuous, so by lemma 24.2 we have $f_n \in L^a_\Phi$ for all n. Furthermore, it is clear that $f_n \uparrow f$. Now we may assume that $\rho_\Phi(f) > 0$, otherwise $f \in L^a_\Phi$ so we have done. Note that $f-f_n \downarrow 0$ and that

$$0 \leq (f-f_n)/\rho_\Phi(f) \leq f/\rho_\Phi(f)$$

for all n. This implies that

$$M_\Phi((f-f_n)/\rho_\Phi(f)) \to 0 \quad \text{as} \quad n \to \infty$$

since $M_\Phi(f/\rho_\Phi(f)) \le 1 < \infty$. Next, choose a positive integer k such that

$$\rho_\Phi(f)2^{-k} < \varepsilon.$$

Since Φ satisfies the $\Delta_2(0)$-condition there exists a constant $M > 0$ such that $\Phi(2x) < M\Phi(x)$ for all $x \ge 0$. Hence

$$0 \le \lim M_\Phi(2^k(f-f_n)/\rho_\Phi(f)) \le M^k \lim M_\Phi((f-f_n)/\rho_\Phi(f)) = 0,$$

so there exists an n_0 such that $M_\Phi(2^k(f-f_n)/\rho_\Phi(f)) \le 1$ for all $n \ge n_0$. Taking $f_\varepsilon = f_{n_0}$, we obtain

$$\rho_\Phi(f-f_\varepsilon) \le 2^{-k}\rho_\Phi(f) < \varepsilon,$$

which is the desired result.

(ii) Let $f \in L_\Phi^+$ and $\varepsilon > 0$ be given. As in the preceding part we shall prove the existence of a function $f_\varepsilon \in L_\Phi^a$ such that $\rho_\Phi(f-f_\varepsilon) < \varepsilon$. To this end, note that there exist constants $x_0 \ge 0$ and $M > 0$ such that $\Phi(2x) < M\Phi(x)$ for all $x \ge x_0$. Furthermore, it is obvious that if we set

$$f_n(x) = f(x) \text{ if } f(x) \le n \text{ ; } f_n(x) = 0 \text{ if } f(x) > n$$

for $n=1,2,\ldots.,$ then $f_n \in L_\Phi^a$ since $\mu(X) < \infty$ and since Φ is continuous (lemma 24.2). Next, choose an integer $k \ge 0$ such that

$$2^{-k}\rho_\Phi(f) < \varepsilon,$$

and let n_0 be an integer such that $n_0 \ge x_0\rho_\Phi(f)$. Finally, note that if $n \ge n_0$ and if x is a point at which $f \ne f_n$, then

$$(f-f_n)(x)/\rho_\Phi(f) \ge x_0.$$

Now, as in part (i) of this theorem, we have

$$\lim M_\Phi((f-f_n)/\rho_\Phi(f)) = 0.$$

Hence

$$0 \le \lim M_\Phi(2^k(f-f_n)/\rho_\Phi(f)) \le M^k \lim M_\Phi((f-f_n)/\rho_\Phi(f)) = 0,$$

since $(f(x)-f_n(x))/\rho_\Phi(f) \ge x_0$ if $f \ne f_n$ and $n \ge n_0$. Thus, similarly as in part (i) it follows that $\rho_\Phi(f-f_n) < \varepsilon$ if n is large enough.

The following corollary is now immediate from the preceding results.

24.5. COROLLARY. *(i) If* Φ *satisfies the* $\Delta_2(0)$*-condition, then* $L_\Phi^* \stackrel{\sim}{=} L_\Psi$ *(the norm in* L_Ψ *being* $\|.\|_\Psi$*).*

(ii) If $\mu(X) < \infty$ *and if* Φ *satisfies the* $\Delta_2(x_0)$*-condition for some* $x_0 \geq 0$*, then* $L_\Phi^* \stackrel{\sim}{=} L_\Psi$ *(the norm in* L_Ψ *again being* $\|.\|_\Psi$*).*

In the following exercise we state that the $\Delta_2(x_0)$-condition is not only sufficient but for many measure spaces also a necessary condition for $L_\Phi^a = L_\Phi$ to hold.

24.B. *Exercise*. Let (X,Γ,μ) be an atomless measure space (i.e., for any $A\epsilon\Gamma$, $\mu(A) > 0$ there exists a set $B\epsilon\Gamma$, $B \subset A$, $0 < \mu(B) < \mu(A)$). Assume that Φ is such that $L_\Phi^a = L_\Phi$. Show the following

(i) if $\mu(X) < \infty$, then Φ satisfies the $\Delta_2(x_0)$-condition for some $x_0 \geq 0$.

(ii) if $\mu(X) = \infty$, then Φ satisfies the $\Delta_2(0)$-condition.

Next, we consider the L_p-spaces. To this end let $1 \leq p < \infty$ be given. Then L_p is generated by the function norm $\|.\|_p$, defined by

$$\|f\|_p = \{\int |f|^p \, d\mu\}^{1/p}$$

for all $f\epsilon M$. As shown in example 20.B we have $\|.\|_p = \rho_\Phi$ if we define $\Phi(x) = x^p$ for all $x \geq 0$. In view of example 24.A and corollary 24.5 we now have $L_p^* \stackrel{\sim}{=} L_\Psi$, where Ψ is the complementary function of Φ and L_Ψ is endowed with the norm $\|.\|_\Psi$. Let, from now on in this section $\Phi(x) = x^p$ for all $x \geq 0$. We shall compute the complementary function .

24.6. LEMMA. *(i) If* $p = 1$*, then* $\Psi(x) = 0$ *if* $0 \leq x \leq 1$*,* $\Psi(x) = \infty$ *if*

$x > 1$.

(ii) If $1 < p < \infty$, *then* $\Psi(x) = p^{1-q}q^{-1}x^q$ *for all* $x \geq 0$, *where* q *satisfies* $p^{-1}+q^{-1} = 1$.

Proof. (i) We have $\Phi(x) = x$ for all $x \geq 0$. Hence, if $0 \leq x \leq 1$, then

$$\Psi(x) = \sup\{(xy-y): y \geq 0\} = 0$$

and if $x > 1$, then

$$\Psi(x) = \sup\{(xy-y): y \geq 0\} = \infty.$$

(ii) Let $1 < p < \infty$. Then $\phi(y) = py^{p-1}$ for all $y \geq 0$. Let $x \geq 0$ be given, and set

$$y = (x/p)^{1/(p-1)}$$

Then $\phi(y) = x$, so, by lemma 23.3,

$$\Psi(x) = \Psi(\phi(y)) = y\phi(y)-\Phi(y) = (x/p)^{1/(p-1)}x - (x/p)^{p/(p-1)} = (p/q)(x/p)^q,$$

where $q = p/(p-1)$.

It follows that for $1 < p < \infty$ $\Psi(x) = cx^q$ (where $c > 0$ is some constant). Hence, as point sets, the spaces L_Ψ and L_q are equal. Moreover, since $\|.\|_\Psi$ and $\|.\|_q$ are both Riesz norms on L_Ψ (or L_q) under which L_Ψ is a Banach space it follows that $\|.\|_\Psi$ and $\|.\|_q$ are equivalent norms (theorem 10.3(ii)). We shall show now that these norms are even equal. First we state the classical Hölder inequality.

24.7. THEOREM. *(Hölder's inequality). Let* $1 \leq p \leq \infty$ *be given and let* q *be defined by* $p^{-1}+q^{-1} = 1$ *(*$q = \infty$ *if* $p = 1$, $q = 1$ *if* $p = \infty$*). Then for all* $f,g \in M$ *we have*

$$\int |fg|\, d\mu \leq \|f\|_p \|g\|_q.$$

For a proof we refer the reader to 14.

24.8. THEOREM. *(i) If* $1 < p < \infty$ *and if* $\Phi(x) = x^p$ *for all* $x \geq 0$, *then* $\|f\|_\Psi = \|f\|_q$ *for all* $f \in M$.

(ii) If $\Phi(x) = x$ *for all* $x \geq 0$, *then* $\|f\|_\Psi = \|f\|_\infty$ *for all* $f \in M$.

(iii) If $\Phi(x) = 0$ *for* $0 \leq x \leq 1$, $\Phi(x) = \infty$ *for* $x > 1$, *then* $\Psi(x) = x$ *for all* $x \geq 0$ *and* $\|f\|_\Psi = \|f\|_1$ *for all* $f \in M$.

Proof. Let $1 \leq p \leq \infty$ and let q be as in theorem 24.7. In view of the examples 20.B and 20.C we now have

$$\|f\|_\Psi = \sup \{\int |fg| \, d\mu : \|g\|_p \leq 1\} \leq \|f\|_q,$$

where $\Phi(x) = x^p$ for $1 \leq p < \infty$ and $\Phi(x) = 0$ $0 \leq x \leq 1$, $\Phi(x) = \infty$ $x > 1$ if $p = \infty$. Now we shall prove the separate parts.

(i) Let $f \in M$ be such that $0 < \|f\|_q < \infty$, and set

$$g_0 = (|f|^{q-1} \operatorname{sgn} f)/\|f\|_q^{q-1}.$$

Then

$$\|g_0\|_p^p = \int |g_0|^p \, d\mu = (\int |f|^q \, d\mu)/\|f\|_q^q = 1,$$

so $\|g_0\|_p = 1$. Hence

$$\|f\|_\Psi \geq \int |fg_0| \, d\mu = (\int |f|^q \, d\mu)/\|f\|_q^{q-1} = \|f\|_q.$$

Thus $\|f\|_\Psi = \|f\|_q$ for all $f \in M$ satisfying $0 < \|f\|_q < \infty$. Since $\|.\|_\Psi$ and $\|.\|_q$ are equivalent it is clear that this equality holds for all $f \in M$.

(ii) By lemma 24.6(i) we have

$$\Psi(x) = 0 \text{ if } 0 \leq x \leq 1 \; ; \; \Psi(x) = \infty \text{ if } x > 1.$$

Thus, by example 20.C, ρ_Ψ is now equal to $\|.\|_\infty$. Hence $\|.\|_\Psi$ and $\|.\|_\infty$ are equivalent Fatou norms. Moreover, by what was shown above we have $\|f\|_\Psi \leq \|f\|_\infty$ for all $f \in M$. To prove the converse inequality, let $f \in M$ be given such that $0 < \|f\|_\infty < \infty$. Furthermore, let $\varepsilon \in \mathbb{R}$ be such that $0 < \varepsilon < \|f\|_\infty$. Next, let $A \in \Gamma$ be defined by

$$A = \{x \in X: \quad |f(x)| > \|f\|_\infty - \varepsilon\}.$$

Then $\mu(A) > 0$, so A contains a subset $B \in \Gamma$ satisfying $0 < \mu(B) < \infty$. Next, set $g_0 = (\mu(B))^{-1}\chi_B$. Then obviously $\rho_\Phi(g_0) = \|g_0\|_1 = 1$, so

$$\|f\|_\Psi \geq \int |fg_0| \, d\mu \geq (\mu(B))^{-1}(\|f\|_\infty - \varepsilon)\mu(B) = \|f\|_\infty - \varepsilon,$$

since $|f| > \|f\|_\infty - \varepsilon$ on the whole of B. This holds for all ε, $0 < \varepsilon < \|f\|_\infty$ so it follows that $\|f\|_\infty = \|f\|_\Psi$, if $0 < \|f\|_\infty < \infty$. Again it is clear that we now have $\|f\|_\Psi = \|f\|_\infty$ for all $f \in M$.

(iii) It is easily computed that $\Psi(x) = x$ for all $x \geq 0$ in this case, so that part of the proof is left to the reader. To show that $\|f\|_\Psi = \|f\|_1$ holds for all $f \in M$, note already that $\|f\|_\Psi \leq \|f\|_1$ by what was observed at the beginning of this proof. Also it is clear that $\|.\|_\Psi$ and $\|.\|_1$ are e-quivalent norms. To show that we have equality, let $f \in M$ be given. Note that $\rho_\Phi(\chi_X) = \|\chi_X\|_\infty = 1$, so $\|f\|_\Psi \geq \int |f|\chi_X \, d\mu = \|f\|_1$, which completes the proof.

Combining all preceding results, we obtain

24.9. COROLLARY. *Let* $1 \leq p \leq \infty$ *and let* q *be as before. Then*

(*i*) $L^*_{p,c} \cong L_q$,

(*ii*) *If* $1 \leq p < \infty$, *then* $L^*_p = L^*_{p,c} \cong L_q$.

(the space L_p *is provided with the* $\|.\|_p$*-norm and* L_q *with the* $\|.\|_q$*-norm).*

Thus, for $1 \leq p < \infty$ we have completely characterized the dual space L^*_p of L_p. In the next section we shall describe L^*_∞.

25. THE DUAL SPACE OF L_∞

We have already seen that $L_\infty^* = L_{\infty,c}^* + L_{\infty,s}^*$ and that $L_{\infty,c}^* \stackrel{\sim}{=} L_1$. In this section we shall present a characterization of the space $L_{\infty,s}^*$. First we shall show that $L_{\infty,s}^*$ generally contains non-trivial elements.

25.1. THEOREM. *Assume that* (X,Γ,μ) *is such that there exists a sequence* $A_1, A_2, \ldots$ *in* Γ *such that* $A_n \downarrow \emptyset$, $\mu(A_n) > 0$ *for all* n. *Then* $L_{\infty,s}^* \neq \{0\}$.

Proof. It suffices to show that $L_\infty^a \neq L_\infty$. To this end, let $A_1, A_2, \ldots$ in Γ be as described in the theorem. It is clear that $\chi_{A_n} \in L_\infty$ for all n. Moreover, $\chi_{A_1} \geq \chi_{A_n} \downarrow 0$ as $n \to \infty$ in L_∞, and $\|\chi_{A_n}\|_\infty = 1$ for all n. Thus $\chi_{A_1} \in L_\infty, \chi_{A_1} \notin L_\infty^a$.

By means of an exercise we give necessary and sufficient conditions for $L_{\infty,s}^* = \{0\}$ to hold.

25.A. *Exercise*. We recall that a non-null set $A \in \Gamma$ is called an atom if $B \subset A$, $B \in \Gamma$ implies $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Show that the following statements are equivalent.

(a) $L_{\infty,s}^* = \{0\}$.

(b) $L_\infty^a = L_\infty$.

(c) $\dim L_\infty < \infty$.

(d) Γ consists of a finite number of atoms and possibly a μ-null set.

(e) There does not exist a sequence $A_1, A_2, \ldots$ in Γ such that $A_n \downarrow \emptyset$, $\mu(A_n) > 0$ for all n.

FOR THE REMAINING PART OF THIS SECTION WE SHALL ASSUME THAT $L_\infty^a \neq L_\infty$.

It follows that none of the statements (a), (b), (c), (d) or (e) of 25.A is satisfied.

To compute L_∞^* we introduce the following. By $\mathcal{B}$ we shall denote the collection of all real-valued set functions ν on Γ satisfying

(i) $\nu(A_1 \cup A_2) = \nu(A_1)+\nu(A_2)$ if $A_1 \cap A_2 = \emptyset$, $A_1, A_2 \in \Gamma$ (i.e., ν is additive),

(ii) $\sup\{|\nu(A)|:\ A\in\Gamma\} < \infty$ (i.e., ν is bounded),

(iii) $\mu(A) = 0$ implies $\nu(A) = 0$ (i.e., $\nu \ll \mu$).

It is easily shown that $\mathcal{B}$, endowed with its obvious addition, scalar multiplication and partial ordering is a Riesz space (see also 1.H). Since $\nu\in\mathcal{B}$ implies $|\nu|\in\mathcal{B}$, so $|\nu|(X) < \infty$, we can define a norm on $\mathcal{B}$ by setting

$$\rho(\nu) = |\nu|(X)$$

for all $\nu\in\mathcal{B}$.

25.B. *Exercise*. Show that ρ is a Riesz norm on $\mathcal{B}$ and that $\mathcal{B}$ is an L-space (in particular $\mathcal{B}$ is a Banach lattice).

Next, the subset $CA(\mathcal{B})$ of $\mathcal{B}$ is defined by

$$CA(\mathcal{B}) = \{\nu\in\mathcal{B}:\ \nu(\cup A_n) = \Sigma\, \nu(A_n) \text{ whenever } A_1, A_2, \ldots \text{ is a pairwise disjoint sequence in } \Gamma\},$$

so $CA(\mathcal{B})$ consists of the σ-additive elements of $\mathcal{B}$. We leave to the reader to show that $CA(\mathcal{B})$ is a Riesz subspace of $\mathcal{B}$. More can be said.

25.2. THEOREM. *(Radon-Nikodym).* $L_1 \cong CA(\mathcal{B})$ *(as normed Riesz spaces).*

Proof. Let $f\in L_1$ and define

$$\nu_f(A) = \int f\chi_A\, d\mu$$

for all $A \in \Gamma$. It is clear that $\nu_f \in CA(\mathcal{B})$ and that $\rho(\nu_f) = \|f\|_1$. Moreover, if $f \in L_1^+$ then $\nu_f \in \mathcal{B}^+$.

Conversely, if $\nu \in CA(\mathcal{B})$ is given, then ν is actually a real measure on Γ that is absolutely continuous with respect to μ. By the theorem of Radon-Nikodym there exists an $f_\nu \in L_1$ such that

$$\nu(A) = \int f_\nu \chi_A \, d\mu$$

for all $A \in \Gamma$. This shows that $L_1 \cong CA(\mathcal{B})$ as normed Riesz spaces.

Thus, a part of L_∞^* (viz. $L_{\infty,c}^*$) can be considered as a Riesz subspace of $\mathcal{B}$. It is our purpose to show that $L_\infty^* \cong \mathcal{B}$ when considered as normed Riesz spaces. To this end, let $\phi \in (L_\infty^*)^+$ be given and set

$$\nu_\phi(A) = \phi(\chi_A)$$

for all $A \in \Gamma$. It is clear that $\nu_\phi \in \mathcal{B}^+$. Moreover,

25.C. *Exercise*. Show the following.

(i) If $\phi \in (L_{\infty,c}^*)^+$, then $\nu_\phi \in (CA(\mathcal{B}))^+$. Hint: use lemma 7.6.

(ii) $\nu_{a\phi} = a\nu_\phi$ for all $a \in \mathbf{R}^+$ and for all $\phi \in (L_\infty^*)^+$.

(iii) $\nu_{\phi+\psi} = \nu_\phi + \nu_\psi$ for all $\phi, \psi \in (L_\infty^*)^+$.

(iv) If $0 \le \phi \le \psi$ in L_∞^*, then $0 \le \nu_\phi \le \nu_\psi$ in $\mathcal{B}^+$.

(v) $\rho(\nu_\phi) = \|\phi\|_\infty^*$ for all $\phi \in (L_\infty^*)^+$.

It is immediate from this exercise that if we define the map $I: L_\infty^* \to \mathcal{B}$ by

$$I(\phi) = \nu_{\phi^+} - \nu_{\phi^-}$$

for all $\phi \in L_\infty^*$, then I is a linear positive norm preserving map from L_∞^* into $\mathcal{B}$ such that $I(L_{\infty,c}^*) \subset CA(\mathcal{B})$.

Next, we show that the image of L_∞^* under I is the whole of $\mathcal{B}$. To this end, let T be the collection of all step functions in M.

25.D. *Exercise*. Show that T is an order dense Riesz subspace of L_∞.

Letting $\nu \in \mathcal{B}$, $t \in T$, $t = \sum_1^n a_i \chi_{A_i}$, define

$$\phi_\nu(t) = \sum_1^n a_i \nu(A_i).$$

Then ϕ_ν is a linear functional on T (it is easily seen that ϕ_ν is independent of the choice of the representation of t). Moreover,

$$|\phi_\nu(t)| = |\sum_1^n a_i \nu(A_i)| \le \sum_1^n |a_i||\nu(A_i)| \le \|t\|_\infty \sum_1^n |\nu(A_i)| \le \|t\|_\infty |\nu|(X) = \|t\|_\infty \rho(\nu),$$

so $\phi_\nu \in T^*$ and $\|\phi_\nu\|_\infty^* \le \rho(\nu)$. It is easily seen that if $\nu \in \mathcal{B}^+$, then

$$\|\phi_\nu\|_\infty^* = \rho(\nu),$$

since $\|\phi_\nu\|_\infty^* = \phi_\nu(\chi_X) = \nu(X) = \rho(\nu)$ in that case. In the following exercise we present some additional properties of ϕ_ν.

25.E. *Exercise*. Show that

(i) $\nu \in \mathcal{B}^+$ implies $\phi_\nu \in (T^*)^+$.

(ii) $\phi_{a\nu} = a\phi_\nu$, $\phi_{\nu_1+\nu_2} = \phi_{\nu_1} + \phi_{\nu_2}$ for all $a \in \mathbf{R}$ and for all $\nu, \nu_1, \nu_2 \in \mathcal{B}$.

(iii) $\|\phi_\nu\|_\infty^* = \rho(\nu)$ for all $\nu \in \mathcal{B}$.

Again let $\nu \in \mathcal{B}$ and consider $\phi_\nu \in T^*$. Since T is norm dense in L_∞ it follows that ϕ_ν has a unique extension to the whole of L_∞. Denoting this extension again by ϕ_ν it follows that $\phi_\nu \in L_\infty^*$ and $\|\phi_\nu\|_\infty^* = \rho(\nu)$. Since T is also a Riesz subspace of L_∞ it follows that if $\phi_\nu \ge 0$ holds with respect to T, then ϕ_ν becomes a positive linear functional on L_∞ (after extension), since any positive linear functional has a positive extension (theorem 6.9). Next, define the map $J: \mathcal{B} \to L_\infty^*$ by setting

$$J(\nu) = \phi_\nu$$

for all $\nu \in \mathcal{B}$. From the above results it is clear that J is a positive linear norm preserving map from $\mathcal{B}$ into L_∞^*. It is now easy to prove the main result of this section.

25.3. THEOREM. $L_\infty^* \cong \mathcal{B}$.

Proof. In view of theorem 2.7, it suffices to show that I and J are each other's inverses. To this end, let $t\in T$ and $\phi\in L_\infty^*$ be given. Then

$$\{J(I(\phi))\}(t) = \phi_{(\nu_\phi)}(t) = \sum_1^n a_i\nu_\phi(A_i) =$$
$$\sum_1^n a_i\phi(\chi_{A_i}) = \phi(\sum_1^n a_i\chi_{A_i}) = \phi(t)$$

where $t = \sum_1^n a_i\chi_{A_i}$. Thus $J(I(\phi))$ and ϕ coincide on T and hence on L_∞. Since both I and J are norm preserving the statement is clear.

As observed before, we have $L_\infty^* = L_{\infty,c}^* + L_{\infty,s}^*$ and also

$$L_{\infty,c}^* \cong L_1 \cong CA(\mathcal{B}).$$

Thus, since $L_{\infty,c}^*$ is a band in L_∞^* it follows that $CA(\mathcal{B})$ is a band in $\mathcal{B}$. Moreover, after identifications, we have

$$L_{\infty,s}^* = (L_1)^d = (CA(\mathcal{B}))^d.$$

Elements $\nu\in(CA(\mathcal{B}))^d$ are sometimes called *purely finitely additive measures* and the set $(CA(\mathcal{B}))^d$ (which is a band in $\mathcal{B}$) is denoted by $PFA(\mathcal{B})$. Collecting the results, we have proved the following:

(i) $L_\infty^* \cong \mathcal{B}$.

(ii) $L_1 \cong L_{\infty,c}^* \cong CA(\mathcal{B})$.

(iii) $L_{\infty,s}^* \cong (CA(\mathcal{B}))^d = PFA(\mathcal{B})$.

(iv) $\mathcal{B} = CA(\mathcal{B})+PFA(\mathcal{B})$ (Yosida-Hewitt decomposition of a finitely additive measure into a σ-additive and a purely finitely additive measure).

We make a final remark concerning these representations. Let ρ be a saturated function norm. Then $L_{\rho,c}^* \cong L_\rho'$ by theorem 22.9. Thus we have characterized the integral part of L_ρ^*. Similarly as above, it is in many cases possible to represent $L^*_{\rho,s}$ as a band in PFA($\mathcal{B}$). In fact, the following can be proved.

$L^*_{\rho,s}$ *is isometric and Riesz isomorphic to a band in* PFA($\mathcal{B}$) *if and only if* L_ρ *is a semi-M-space.*

It is beyond the scope of this book to present the proof of this theorem.

Finally, we recall that if L is a normed Riesz space, then L^a denotes the set of all elements having an absolutely continuous norm and L^α denotes the set of all absolutely continuous elements (see sections 8 and 11). In section 11 we have shown that always $L^\alpha \subset L^a$ and that $L^\alpha = L^a$ whenever L is a Banach lattice (but $L^\alpha = L^a$ can also occur if L is <u>not</u> a Banach lattice). We are now able to present an example of a normed Riesz space for which L^α is a proper subset of L^a.

25.F. *Example*. Let (X,Γ,μ) be a measure space such that $L^*_{\infty,s} \neq \{0\}$ (for instance, take $X = [0,1]$ provided with Lebesgue measure). Next, define for all $f \in M$

$$\rho(f) = \|f\|_1 \text{ if } \|f\|_\infty < \infty; \quad \rho(f) = \infty \text{ if } \|f\|_\infty = \infty.$$

(Here it is also assumed that $\mu(X) < \infty$). It is clear that ρ is a function norm on M and that $L_\rho = L_\infty$ when considered as point sets. Now note that L_ρ is norm and order dense in the Banach lattice L_1. Hence

$$L^*_\rho = L^*_1 \cong L_\infty.$$

Since $L^a_1 = L_1$ it follows that $L^a_\rho = L_\rho$ has to hold as well.

Next, consider $L^\sim_\rho$ (the order dual of L_ρ). Since L_∞ is a Banach lattice it follows that $L^*_\infty = L^\sim_\infty$. Now, the norm on L_ρ does not influence the order dual of L_ρ. Thus, since $L_\rho = L_\infty$ as point sets (and hence also as Riesz spaces), we obtain $L^\sim_\rho = L^\sim_\infty = L^*_\infty$. Therefore,

$$L^\alpha_\rho = L^\alpha_\infty = L^a_\infty = {}^\circ(L^*_{\infty,s}) \neq L_\infty = L_\rho = L^a_\rho.$$

26. THE SECOND ASSOCIATE NORM

Throughout this section let ρ be a fixed saturated function norm on M. Furthermore, let ρ' and ρ'' be its associate norms (see theorem 22.2 (ii)). As proved in theorem 22.2(ii) we always have $\rho'' \leq \rho$. In this section we shall show that if ρ is a Fatou norm, then $\rho = \rho''$. Since ρ'' is a Fatou norm it follows that the converse statement holds as well. A first step is the following.

26.1. LEMMA. *Assume that* ρ *is a Fatou norm and assume that* $\rho(u) = \rho''(u)$ *for all* $u \in M^+$ *satisfying* $\|u\|_\infty < \infty$, $\mu\{x \in X: \ u(x) \neq 0\} < \infty$. *Then* $\rho = \rho''$.

Proof. It suffices to show that $\rho(f) = \rho''(f)$ for all $f \in M^+$. Therefore, let $f \in M^+$ be given. Now, let $X_1, X_2, \dots$ be a sequence in Γ such that $X_n \uparrow X$ and such that $\mu(X_n) < \infty$ for all n. Defining

$$f_n = (f\chi_{X_n}) \wedge (n\chi_{X_n})$$

for all n it follows that $0 \leq f_n \uparrow f$. By assumption we have $\rho(f_n) = \rho''(f_n)$ for all n. Hence, since both ρ and ρ'' are Fatou norms,

$$\rho(f) = \lim \rho(f_n) = \lim \rho''(f_n) = \rho''(f).$$

In the following lemma L_2 will be as defined in example 20.B (p=2) and the norm in L_2 will be the $\|.\|_2$-norm.

26.2. LEMMA. *If* ρ *is a Fatou norm, then* U *defined by*

$$U = \{f \in L_2: \ \rho(f) \leq 1\}$$

is a closed convex subset of L_2 *and if* $f, g \in L_2$, $g \in U$, $0 \leq |f| \leq |g|$, *then* $f \in U$.

Proof. Since U is the intersection of L_2 with the unit ball of L_ρ, U is convex. To show that U is norm closed in L_2 it suffices to show that if $f \in L_2^+$ and $f_1, f_2, \ldots$ in U^+ are such that $\lim \|f-f_n\|_2 = 0$, then $f \in U$. Setting $g_n = f \wedge f_n$ for all $n \in \mathbb{N}$ it follows that

$$0 \leq |f-g_n| = f-(f \wedge f_n) \leq |f-f_n|$$

for all n, so $\lim \|f-g_n\|_2 = 0$. Since $0 \leq g_n \leq f$ for all n this implies that $f = \sup \{g_n: n=1,2,\ldots\}$. Using the Dedekind completeness of L_2 it follows that we can define $h_n = \inf \{g_k: k \geq n\}$ for all n and it follows that $h_n \uparrow f$ in L_2^+. Now note that

$$0 \leq h_n \leq g_n \leq f_n$$

for all n, so $\rho(h_n) \leq 1$ and $h_n \in L_2^+$, so $h_n \in U^+$ for all n. Moreover, since ρ is a Fatou norm

$$\rho(f) = \lim \rho(h_n) \leq 1,$$

so $f \in U^+$. The rest is now obvious.

Before proving our main result we note the following. Let $U \subset L_2$ be as in the preceding lemma and let $f_0 \in L_2^+$, $f_0 \notin U$. By the Hahn-Banach theorem there exists a functional $\phi \in L_2^*$ such that $|\phi(u)| \leq 1$ for all $u \in U$ and $\phi(f_0) > 1$. Observing that $L_2^* \cong L_2$ (corollary 24.9(ii)), it follows that there exists an $h \in L_2$ such that $\phi(f) = \int fh \, d\mu$ for all $f \in L_2$, so $|\int uh \, d\mu| \leq 1$ for all $u \in U$ and $|\int f_0 h \, d\mu| > 1$ for this function $h \in L_2$. Setting $v_0 = |h|$ and observing that U has the property that $u \in U$ implies $|u| \operatorname{sgn} h \in U$, we obtain already that $\int |u| v_0 \, d\mu \leq 1$ for all $u \in U$ and $\int f_0 v_0 \, d\mu > 1$.

26.3. THEOREM. *The following statements are equivalent.*

(*a*) ρ *is a Fatou norm.*

(*b*) $\rho = \rho''$.

Proof. (i) (b) ⇒ (a). Obvious.

(ii) (a) ⇒ (b). Let $u_0 \in M^+$ be given such that $\|u_0\|_\infty < \infty$ and such that $\mu\{x \in X: u_0(x) \neq 0\} < \infty$. If $\rho(u_0) \leq 1$ then clearly $\rho''(u_0) \leq 1$. On the other hand, if $\rho(u_0) > 1$, let $U \subset L_2$ be as above. Then $u_0 \notin U$. Now, let $v_0 \in L_2$ be such that $\int |u| v_0 \, d\mu \leq 1$ for all $u \in U$ and $\int u_0 v_0 \, d\mu > 1$. It will be shown that $\int |f| v_0 \, d\mu \leq 1$ for all $f \in L_\rho$, $\rho(f) \leq 1$. To this end, let f be such that $\rho(f) \leq 1$, and let $X_1, X_2, \ldots.$ be a sequence in Γ such that $X_n \uparrow X$, $\mu(X_n) < \infty$ for all n. Setting

$$f_n = |f| \wedge n\chi_{X_n}$$

for all n, it follows that $f_n \in U$ for all n and that $f_n \uparrow |f|$. Hence

$$\int |f| v_0 \, d\mu = \lim \int f_n v_o \, d\mu \leq 1.$$

This implies that

$$0 < \rho'(v_0) = \sup \{\int |f| v_0 \, d\mu: \ \rho(f) \leq 1\} \leq 1.$$

Now, by theorem 22.5,

$$0 < \rho'(v_0) \leq 1 < \int u_0 v_0 \, d\mu \leq \rho''(u_0)\rho'(v_0),$$

so $\rho''(u_0) > 1$. Thus we have shown that $\rho''(u_0) \leq 1$ if and only if $\rho(u_0) \leq 1$. It follows that $\rho''(u) = \rho(u)$ for all $u \in M$ such that $\|u\|_\infty < \infty$, $\mu\{x \in X: u(x) \neq 0\} < \infty$. Hence $\rho'' = \rho$ by lemma 26.1.

As an application we present the following theorem.

26.4. THEOREM. *Let* ρ *be a saturated Fatou norm. Then* $L_\rho \overset{\sim}{=} (L^*_{\rho,c})^*_c$.

Proof. We have $L^*_{\rho,c} \overset{\sim}{=} L'_\rho$. Hence

$$(L^*_{\rho,c})^*_c \overset{\sim}{=} (L'_\rho)^*_c \overset{\sim}{=} (L'_\rho)' = L''_\rho = L_\rho.$$

Theorem 26.3 has also applications in the theory of Orlicz spaces.

26.5. THEOREM. *Let* Φ *and* Ψ *be a pair of complementary Orlicz functions.*

(*i*) $\|.\|_{\Phi}' = \rho_{\Psi}$; $\|.\|_{\Psi}' = \rho_{\Phi}$.

(*ii*) *If* L_{Φ} *is provided with the norm* $\|.\|_{\Phi}$ *and* L_{Ψ} *with the norm* ρ_{Ψ}, *then* $L_{\Phi,c}^{*} \cong L_{\Psi}$.

Proof. It suffices to show that $\|.\|_{\Phi}' = \rho_{\Psi}$. We know already that $\rho_{\Psi}' = \|.\|_{\Phi}$. Hence, since ρ_{Ψ} is Fatou,

$$\rho_{\Psi} = \rho_{\Psi}'' = \|.\|_{\Phi}'.$$

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