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**TREELIKE SPACES AND
RELATED CONNECTED
TOPOLOGICAL SPACES**

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I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts and all those at the Mathematical Centre who have contributed to its technical realization.

This booklet describes the structure of various types of connected topological spaces. Our main interest is in spaces that are "thin" or "one-dimensional" or "bush-like" or "easily disconnected" in some intuitive sense. Although the spaces discussed here differ widely in appearance and properties, the method to study them is always the same: look what happens when you remove one or more connected subsets. Our basic tools are lemmas 1 - 4 of chapter 0.

In chapter I we analyse connected spaces in which each connected subset has at most one end point (i.e. non-cut point). Knowledge of the structure of such spaces is very useful if one wants to characterize a connected linearly orderable space as a space where each connected subset has at most two end points that satisfies some additional conditions.

In chapter II we study a class of spaces introduced by WHYBURN containing the treelike spaces, biconnected spaces and spaces with connected intersection property. The main point here is finding minimal conditions assuring that such spaces are treelike.

In chapter III we give the basic theory of treelike spaces and prove the results on locally peripherally compact treelike spaces announced by GURIN. Chapter IV contains a complete structure analysis for connected spaces such that the complement of a connected subset has at most two components.

This work and its companion volume (MC Tract 49: H.KOK, *Connected orderable spaces*, 1974) essentially represent the outcome of a collaboration with H. KOK in the years 1970/1971.

CONTENTS

iii

<i>Preface</i>		<i>i</i>
<i>Contents</i>		<i>iii</i>
Chapter 0	PRELIMINARIES	1
Chapter I	ON V -SPACES	5
	0. Introduction	5
	1. On (V_p)	6
	2. The relation between (V) and (V_1)	6
	3. The partial order of V -spaces	8
	4. Subspaces	13
	5. Topological properties of V -spaces	19
	6. Construction of V -spaces	22
	7. On the structure of H -spaces	28
	8. Maps between V -spaces	34
Chapter II	ON W -SPACES	37
	1. Definition and separation properties	37
	2. Relation with (INT_2) and (S)	39
	3. Some other properties of W -spaces	40
	4. The locally connected non-Hausdorff case	47
CHAPTER III	ON TREELIKE SPACES	55
	1. Definitions and elementary properties	55
	2. The locally connected case	61
	3. The locally peripherally compact case	65
	4. The locally compact case	68
	5. A characterization of connected (weakly) orderable spaces	73
	6. On (\mathcal{B})	76
	7. Weaker properties	82
	7.1. On (NS)	82
	7.2. On (INT_2) and some variants of it	83
	7.3. Example of a treelike space satisfying (INT_2) and the first axiom of countability but without the intersection property	85
	8. Supercompactness of compact treelike spaces	88

CHAPTER IV CLASSIFICATION OF THE CONNECTED TOPOLOGICAL SPACES THAT
CANNOT BE BROKEN UP INTO MORE THAN TWO COMPONENTS BY
REMOVAL OF A CONNECTED SUBSET 91

REFERENCES 107

CHAPTER 0

PRELIMINARIES

Although this booklet is independent and self-contained, it is more or less a sequel to H. KOK's "Connected orderable spaces" [18]. We use the same notations and terminology, some of which we explain below:

If the topological space X is the topological sum of its subspaces A and B (that is, $X = A \cup B$ and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$) and $p \in A$ and $q \in B$ then we may write $X = \underset{p}{A} + \underset{q}{B}$. Besides point denotations also names of sets may occur written below a summand; of course this denotes set inclusion.

Throughout the book X will denote the topological space under discussion; most of the time it will be a connected T_1 -space with at least two points.

If C is connected then a point $p \in C$ is called an *end point*, resp. a *cut point* if $C \setminus p$ is connected resp. is not connected. If $C \setminus p$ has exactly two components then p is called a *strong cut point*. If $C \setminus p$ has at least three components then p is called a *ramification point* of C .

[In contexts like $C \setminus p$ we shall almost always omit the braces; sometimes we even write $C \setminus p, q$ instead of $C \setminus \{p, q\}$.]

A subset S of X is called a *segment (of p)* if S is a component of $X \setminus p$ for some point $p \in X$.

Two points p and q are called *conjugated* when there does not exist a point $r \in X$ such that r separates p and q (i.e., such that $X \setminus r = \underset{p}{A} + \underset{q}{B}$).

We now list for some properties of a topological space X the abbreviations which are used in this book.

- (O) X is (weakly) orderable.
- (H) Every connected subset of X has at most two end points.
- (V) Every connected subset of X has at most one end point.
- (INT) The intersection of an arbitrary collection of connected

subsets of X is connected.

- (INTC) The intersection of an arbitrary collection of closed connected subsets of X is connected.
- (INT2) The intersection of two connected sets is connected.
- (INTC2) The intersection of two closed connected sets is connected.
- (S) X is treelike, that is, no two points of X are conjugated.
- (W) The boundary of any component of the complement of a connected subset of X contains at most one point.
- (B) X does not contain three mutually disjoint segments.
- (B') Each cut point is a strong cut point.
- (NS) Any two points of X can be separated by an open connected set.
- (S') Among any three points of X exactly one separates the other two.

Many other properties occur only locally; for $(\forall x)$ with $x \in \{0,1,m,a\}$ see page 11.

Apart from these we have the usual notation for separation axioms (e.g., (T_1)) and countability axioms (e.g., (C_1)).

Basic tools in what follows are the lemma's below; of course they are well-known, although it is difficult to find explicit statements of them in the literature.

Let X be a connected topological space, and C a connected subset of X .

LEMMA 1. *If A is clopen (= closed-and-open) in $X \setminus C$ then $A \cup C$ is connected.*

PROOF. Let $X \setminus C = A + B$ and suppose $A \cup C = S + T$ where $C \subset S$. Then $X = (B \cup S) + T$, hence $T = \emptyset$. \square

LEMMA 2. *If S is a component of $X \setminus C$ then $X \setminus S$ is connected.*

PROOF. Suppose $X \setminus S = A + B$ and $C \subset A$. Then $B \cup S$ is connected and contained in $X \setminus C$ hence $B \cup S = S$ and $B = \emptyset$. \square

LEMMA 3. *If Q is a quasicomponent of $X \setminus C$ then $X \setminus Q$ is connected.*

PROOF. $Q = \cap \{B_\alpha \mid B_\alpha \text{ is clopen in } X \setminus C \text{ and } Q \subset B_\alpha\}$, so

$$X \setminus Q = \cup \{X \setminus B_\alpha \mid B_\alpha \text{ clopen in } X \setminus C \text{ and } Q \subset B_\alpha\}.$$

Each $X \setminus B_\alpha$ is connected and contains C , so $X \setminus Q$ is connected. \square

LEMMA 4. Let X be a connected T_1 -space, $p \in X$. Let B be a subset of $X \setminus p$ such that B is clopen in $X \setminus p$ or B is a component in $X \setminus p$ or B is a quasi-component in $X \setminus p$. Then, if $Y = X \setminus B$, the following holds:

- (i) Y is a connected T_1 -space.
- (ii) The components of $Y \setminus p$ are exactly the components of $X \setminus p$ contained in Y .
- (iii) If $q \in Y \setminus p$ then the components of $Y \setminus q$ are exactly the intersections of the components of $X \setminus q$ with Y .

PROOF. Straightforward. \square

The following lemma is of some use in the construction of counterexamples.

LEMMA 5. Let (X, T) be a connected topological space and let D be a dense subset of X . Give X a new topology by making D open (i.e., take $T \cup \{D\}$ as a subbase for a topology T' on X). Then with the new topology X is still connected.

PROOF. $T' = \{U \cup (V \cap D) \mid U, V \in T\}$. Now if $X = (U_0 \cup (V_0 \cap D)) \cup (U_1 \cup (V_1 \cap D))$ and $(U_0 \cup (V_0 \cap D)) \cap (U_1 \cup (V_1 \cap D)) = \emptyset$ then $(U_0 \cup V_0) \cap (U_1 \cup V_1) \cap D = \emptyset$, hence, since D is dense, $(U_0 \cup V_0) \cap (U_1 \cup V_1) = \emptyset$. But this means that $U_i \cup V_i = U_i \cup (V_i \cap D)$ ($i = 0, 1$) and $X = (U_0 \cup V_0) + (U_1 \cup V_1)$, a contradiction. \square

CHAPTER I

ON V -SPACES

0. INTRODUCTION

For connected T_1 -spaces we consider the following properties:

- (H) : Every connected subset of the space has at most two end points.
- (V) : Every connected subset of the space has at most one end point.
- (V1) : The space contains a point x_0 (called the lowest point or the base point of the space) such that every connected set containing x_0 is closed.

[Note: a $V1$ -space cannot contain two base points, for if x_0 and x_1 are two different base points of a $V1$ -space X then the component S of $X \setminus \{x_0\}$ containing x_1 is closed, but its complement $X \setminus S$ is connected and contains x_0 hence is closed too, which contradicts the connectedness of X .]

In [14] H. HERRLICH proved that a connected, locally connected H -space is strictly orderable, and in [18] it is shown that a connected T_1 -space is orderable if it satisfies (H) and the condition that a segment is never closed (i.e., if C is a component of $X \setminus \{p\}$ then $p \in \overline{C}$).

The question whether (H) alone implies the orderability was solved in the negative by J.L. HURSCH and A. VERBEEK-KROONENBERG who gave an example of a countable connected Hausdorff $V1$ -space (which necessarily is non-orderable) in [15]. (We shall see that (V1) implies (V), while obviously (V) implies (H).)

In this chapter we determine the structure of V -spaces, thus preparing the characterization of non-orderable H -spaces.

REMARK. For reasons that will become clear later on we have interchanged the meaning of (V) and (V1) compared to [6] and [18].

Unless the contrary is stated explicitly X will designate a connected T_1 -space satisfying (V) .

1. ON (Vp)

Let (Hp) be the property: "each connected *proper* subset of X has at most two end points". In [18] spaces not satisfying (H) but with property (Hp) are characterized as spaces which are cyclically orderable but not orderable. Analogously, we define (Vp) by: "each connected *proper* subset of X has at most one end point". In contrast to the result concerning (H) and (Hp) referred to we have:

THEOREM 1. $(V) \leftrightarrow (Vp)$ (for connected T_1 -spaces).

PROOF. Obviously $(V) \rightarrow (Vp)$. Conversely, let X be a connected T_1 -space satisfying (Vp) and suppose that x and y are two distinct end points of X .

- (i) If $X \setminus \{x, y\} = A + B$, $A \neq \emptyset$ then both $A \cup \{x\}$ and $A \cup \{y\}$ are connected and hence $A \cup \{x, y\}$ is connected and has two end points. This contradicts (Vp) unless B is empty. It follows that $X \setminus \{x, y\}$ is connected.
- (ii) Choose $z \in X \setminus \{x, y\}$. If $X \setminus z$ is connected then by (i) $X \setminus \{x, z\}$ and $X \setminus \{y, z\}$ are connected and hence $X \setminus z$ has two end points, which contradicts (Vp) . Hence $X \setminus z$ is not connected.

If $X \setminus z$ can be separated with x and y on the same side of the separation then $X \setminus z = \underset{x, y}{A} + B$ and $\bar{A} = A \cup \{z\} = X \setminus B$ is a connected subset of X with the two end points x and y (see ch.0, lemma 4 (iii)).

If $X \setminus z$ cannot be separated in such a way then the separation is clearly unique, that is, $X \setminus z$ has exactly two components: $X \setminus z = \underset{x}{A} + \underset{y}{B}$. But now $\bar{A} = A \cup \{z\}$ has the two end points x and z .

In all cases we arrive at a contradiction with (Vp) . \square

2. THE RELATION BETWEEN (V) and $(V1)$

PROPOSITION 1. Let X be a V -space and let $C \subset X$ be connected. Then either each component of $X \setminus C$ is open or all components but one are open; in the latter case if C_1 is the one that is not open then $C_1 \setminus C_1^\circ$ is a singleton.

PROOF. If C_1 and C_2 were two non-open components of $X \setminus C$ and $x_i \in C_i \setminus C_i^\circ$ ($i = 1, 2$) then the connected set $\overline{X \setminus (C_1 \cup C_2)}$ would have the two end points x_1 and x_2 (for: $X \setminus (C_1 \cup C_2)$ is connected), a contradiction of (V).
If $\{x_1, x_2\} \subset C_1 \setminus C_1^\circ$ then $\overline{X \setminus C_1}$ has two end points which again contradicts (V). \square

THEOREM 2. Let X be a V -space containing an end point x_0 . Then X is a V_1 -space with x_0 as base point.

[Note: the base point of a V_1 -space is not necessarily an end point. Theorem 2 is a special case of the result to be proved in the next section (after the introduction of a partial order on V -spaces) that a V -space is a V_1 -space with base point x_0 if and only if it contains a smallest point x_0 .]

PROOF. (i) We first prove:

If $p \neq x_0$ and C is the component of $X \setminus p$ containing x_0 then C is closed.

For that purpose, suppose that C is not closed, that is, $\overline{C} = C \cup \{p\}$.

x_0 cannot be an end point of \overline{C} since in that case \overline{C} would have the end points x_0 and p . Hence x_0 is cut point of \overline{C} and $\overline{C} \setminus x_0 = \underset{p}{A} + B$ ($B \neq \emptyset$). If C is open then B is open too, and, since B clearly is also closed in $X \setminus x_0$, we have $X \setminus x_0 = ((X \setminus C) \cup A) + B$, which contradicts the assumption that x_0 is an end point of X . Therefore, C is not open and (by proposition 1) $C \setminus C^\circ = \{x_1\}$ for some point x_1 .

First suppose $x_1 \neq x_0$. If $\overline{C} \setminus x_0$ can be separated with p and x_1 on the same side of the separation, i.e., if $\overline{C} \setminus x_0 = \underset{p, x_1}{A} + B$ ($B \neq \emptyset$), then B is open and it follows as above that x_0 is a cut point of X . Hence, only one separation is possible: $\overline{C} \setminus x_0 = \underset{p}{A} + \underset{x_1}{B}$ where A and B are connected. But now $A \cup \{x_0\}$ is connected and has the two end points p and x_0 (since $C \setminus x_0 = (A \setminus p) + B$ we have that $(A \setminus p) \cup \{x_0\} = (A \cup \{x_0\}) \setminus p$ is connected), a contradiction.

Therefore, we must have $x_0 = x_1$; but in this case B is open and again it follows from $X \setminus x_0 = ((X \setminus C) \cup A) + B$ that x_0 is a cut point, which is impossible.

(ii) Now we can prove theorem 2:

Let C be connected with $x_0 \in C$, and suppose that C is not closed. Then precisely one of the components of $X \setminus C$ is not open; let this one be C_1 and let $\{x_1\} = C_1 \setminus C_1^\circ$. Notice that $x_1 \neq x_0$. Let S be the component of $X \setminus x_1$ containing x_0 . $x_0 \in X \setminus C_1$ and $X \setminus C_1$ is connected, so $X \setminus C_1 \subset S$. But S is closed according to (i) and $x_1 \in \overline{X \setminus C_1} \subset \overline{S} = S$ gives a contradiction.

Hence, C is closed. \square

THEOREM 3. $(V1) \rightarrow (V)$.

PROOF. Let X satisfy $(V1)$ with x_0 as base point. Suppose C is connected and has two distinct end points p and q . Then $x_0 \notin C$; for, if $x_0 \in C$ and, e.g., $x_0 \neq p$ then $C \setminus p$ contains x_0 and is connected but not closed. The component of $X \setminus p$ containing x_0 is closed, and each other component of $X \setminus p$ is open (for if C_1 is a segment of p and $x_0 \notin C_1$ then $X \setminus C_1$ is connected, contains x_0 and hence is closed). In particular p is a cut point. The connected set $C \setminus p$ is contained in some component T of $X \setminus p$. Since $p \in \overline{T}$, T is not closed and, therefore, $x_0 \notin T$. Hence, T is an open component of $X \setminus p$ and we may write $X \setminus p = \underset{x_0}{A} + T$. But now $A \cup \{p\} \cup (C \setminus q)$ is connected, contains x_0 but is not closed, which is impossible. \square

COROLLARY. A $V1$ -space contains at most one end point (which then is the base point).

3. THE PARTIAL ORDER OF V -SPACES

In [15] a partial order for $V1$ -spaces with base point x_0 is defined by

$$(*) \quad \begin{cases} x_0 <_1 y & \text{for all } y \in X \setminus x_0 \\ x <_1 y & \text{iff } x \text{ separates } x_0 \text{ and } y. \end{cases}$$

For V -spaces this can be generalized as follows:

$$x < y \quad \text{iff } y \text{ belongs to some open segment of } x.$$

THEOREM 4. $<$ is a partial order.

PROOF. (i) $<$ is antisymmetric:

Suppose that both $x < y$ and $y < x$. Then $X \setminus x = A + \underset{y}{S}$ and $X \setminus y = B + \underset{x}{T}$ where S and T are open and connected. Observe that $B \cup \{y\}$ is connected and contained in $X \setminus x$ so $B \cup \{y\} \subset S$ and similarly $A \cup \{x\} \subset T$. Now $S \setminus y = B + (S \cap T)$ and $T \setminus x = A + (S \cap T)$ and hence $(S \cap T) \cup \{x\}$ and $(S \cap T) \cup \{y\}$ are connected; since $S \cap T \neq \emptyset$ [otherwise we would have $x = (A \cup \{x\}) + (B \cup \{y\})$ contrary to the connectivity of X] it follows that $(S \cap T) \cup \{x, y\}$ is a connected set with two end points, which contradicts (V) .

(ii) $<$ is transitive:

Let $x < y$ and $y < z$. Certainly $x \neq z$ since $<$ is antisymmetric. Now $X \setminus x = A + \frac{S}{y}$ and $X \setminus y = B + \frac{T}{z}$, where S and T are open and connected.

If $z \in A$ then since $A \cup \{x\}$ is connected in $X \setminus y$ we have $A \cup \{x\} \subset T$ and, consequently, $x \in T$. But this means $y < x$ contradicting the antisymmetry of $<$. It follows that $z \in S$ and hence $x < z$. \square

PROPOSITION 2. For each $x \in X$ the subset $\{y \mid y < x\}$ of X is linearly ordered.

PROOF. Suppose $u < x$ and $v < x$; we must prove that u and v are comparable. Suppose not, then $X \setminus u = \frac{A}{v} + \frac{S}{x}$ and $X \setminus v = \frac{B}{u} + \frac{T}{x}$, where S and T are open and connected. But now $S \cup \{u\} = \overline{S}$ is connected and contained in $X \setminus v$ which is impossible since it contains both u and x . \square

PROPOSITION 3. In V_1 -spaces the partial order $<$ coincides with the partial order $<_1$ defined by (*).

PROOF. Let $x < y$ in a V_1 -space X with base point x_0 . If $x = x_0$ then $x <_1 y$ by definition. So assume that $x \neq x_0$. Then $X \setminus x = A + \frac{S}{y}$ where S is open and connected. Since x_0 does not belong to any proper open connected subset of X it follows that $x_0 \in A$ and hence that x separates y from x_0 . Consequently, $x <_1 y$.

Conversely: let $x <_1 y$ and $x \neq x_0$; then x separates y from x_0 . Now the segment S of x which contains x_0 is closed and hence not open, and obviously $y \notin S$. All other segments of x are open by proposition 1; y belongs to such a segment, i.e., $x < y$.

Finally $x_0 < y$ for each $y \neq x_0$ because the complement of the segment of x_0 containing y is connected and, therefore, closed. \square

REMARK. Since we did not use the fact that $<_1$ is a partial order, proposition 3 (together with theorems 3 and 4) constitutes an independent proof hereof.

It is convenient to introduce the notations:

$$P_x := \{y \mid x \leq y\}$$

and

$$Q_x := \{y \mid x \not\leq y\} = X \setminus P_x.$$

If the space has a smallest point x then $P_x = X$ and $Q_x = \emptyset$. Otherwise $P_x \neq X$ and Q_x is the only component of $X \setminus x$ which is not open, P_x is its complement and P_x is not closed. Clearly in all cases both P_x and Q_x are connected. Observe that if X is a $V1$ -space then Q_x (is either empty or contains the base point and hence) is closed and consequently P_x is open.

PROPOSITION 4.

- (i) Let X be a V -space and let $x \in X$. Then P_x is a $V1$ -space with base point x .
- (ii) A V -space with a smallest point x is a $V1$ -space with x as a base point.

PROOF. Clearly property (V) is hereditary for connected subspaces. Also each component of $P_x \setminus x$ is open (in X hence also in P_x), i.e., x is the smallest point of P_x (in its intrinsic order). Therefore, it suffices to prove (ii).

Suppose C is a connected set such that $x \in C$ and $q \in \overline{C} \setminus C$. Since $x < q$ we have $X \setminus x = A + \bigcup_q S$ where S is open and connected (and possibly $A = \emptyset$). Now $C \setminus x = (C \cap A) + (C \cap S)$ and $q \notin \overline{C \cap A}$ hence $q \in \overline{C \cap S}$. But by theorem 2 $S \cup \{x\}$ is a $V1$ -space and $(C \cap S) \cup \{x\}$ is a connected subspace containing the base point hence is closed (in $S \cup \{x\} = \overline{S}$, hence also in X). Contradiction. \square

COROLLARY. A V -space is a $V1$ -space iff it has exactly one minimal point.

PROOF. The 'if' part is given by (ii) of the above proposition. The 'only if' part follows from the definition of the partial order $<_1$ on $V1$ -spaces and the fact that $<$ and $<_1$ coincide (proposition 3). \square

REMARK. Let X be a V -space and Y a connected subspace. Then we have two partial orderings defined on Y : first the intrinsic order $<_Y$ on Y -as-a- V -space, and second the restriction $(<_X) \cap (Y \times Y)$ of the partial order on X to Y . Unfortunately, these two do not coincide in general. We shall see, however, that in the important special case that X is a $V1$ -space $<_Y = (<_X) \cap (Y \times Y)$ for each connected subspace Y of X (proposition 14). In the sequel we shall sometimes use the notation $<_X$ for the partial order in the V -space X . If confusion seems unlikely we shall stick to our custom to drop the index X .

PROPOSITION 5. Let y be a non-minimal point of X . Then P_y is open and Q_y is closed. Conversely, if P_y is open and $P_y \neq X$ then y is not minimal; indeed, if U is an open connected subset of X and $U \neq X$ then U does not contain any point that is minimal in X .

PROOF. Let $x < y$. Let $X \setminus x = A + \underset{y}{S}$ where S is open and connected (and possibly $A = \emptyset$). We then have

$$(S \cup \{x\}) \setminus y = (P_y \setminus y) \cup (Q_y \cap (S \cup \{x\})),$$

since $P_y \subset S$ (because P_y intersects S and does not contain x). Note that $A \cup \{x\} \subset Q_y$. Furthermore,

$$Q_y \setminus x = (Q_y \cap A) + (Q_y \cap S)$$

and consequently $Q_y \cap (S \cup \{x\}) = (Q_y \cap S) \cup \{x\}$ is connected.

By theorem 2 $S \cup \{x\}$ is a $V1$ -space with base point x and so $Q_y \cap (S \cup \{x\})$ is closed (in $S \cup \{x\} = \bar{S}$ hence also in X). It follows that $Q_y = (A \cup \{x\}) \cup (Q_y \cap (S \cup \{x\}))$ is closed and thus P_y is open.

Conversely, if U is a proper open connected subset of X then $\bar{U} = U \cup \{q\}$ for some q and $X \setminus q = U + X \setminus \bar{U}$, so for each $u \in U$ we have $q < u$. \square

PROPOSITION 6. *Each nonempty subset $A \subset X$ which is directed downwards and which is bounded from below has a (unique) infimum.*

PROOF. Suppose A does not have a minimum. Then for each $a \in A$ P_a is open. Let $A^* = \bigcup_{a \in A} P_a$. A^* is open and different from X since it does not contain a lower bound for A . Also it is connected because A is directed and each P_a is connected. Therefore, its boundary is a singleton: $\partial A^* = \{x\}$. We will show that $x = \inf A$.

- (i) If P_x is open then it certainly intersects A^* . Choose $a^* \in P_x \cap A^*$ and choose $a \in A$ such that $a \leq a^*$. If a_1 is an arbitrary point of A then choose $a_2 \in A$ such that both $a_2 \leq a_1$ and $a_2 \leq a$. By proposition 2 $\{z \mid z \leq a^*\}$ is linearly ordered; since this set contains both x and a_2 it follows that either $x < a_2$ or $a_2 \leq x$. In the latter case, however, we would have $x \in P_{a_2} \subset A^*$, which is impossible. Therefore, $x < a_2 \leq a_1$, i.e., x is a lower bound for A .
- (ii) If P_x is not open then (by proposition 5) x is a minimal element of X . Now by assumption A has a lower bound y , that is, $A \subset P_y$ and the connected set A^* is contained in precisely one (open) component S of $X \setminus y$. Since $x \in \overline{A^*} \subset \bar{S} = S \cup \{y\}$ it follows that $x \geq y$ and hence $x = y$.

Thus in both cases x is a lower bound for A . Now suppose $y < a$ for each $a \in A$. As under (ii) it is seen that $x \geq y$, i.e., $x = \inf A$. \square

PROPOSITION 7. *If y is not a minimal point of X then y has an immediate predecessor y' ; moreover, $\{y'\} = \overline{P_y} \setminus P_y = \partial P_y$.*

PROOF. If y is not minimal then P_y is open and connected and $P_y \neq X$, so ∂P_y is a singleton. Let $\partial P_y = \{y'\}$. Now if $z < y$ then y belongs to some open component S of $X \setminus z$; it follows that $P_y \subset S$ and hence $y' \in \overline{P_y} \subset \overline{S} = S \cup \{z\}$, thus $z \leq y'$. Consequently, $y' = \max\{z \mid z < y\}$. \square

PROPOSITION 8. *If X does not have a minimal point then X is directed downwards.*

PROOF. Let A be a maximal linearly ordered subset of X and let $P = \bigcup_{a \in A} P_a$. Then P is open because each P_a is open.

Let $z \in \overline{P}$ and choose $y \in P_z \cap P$. Let $a \in A$ be such that $a \leq y$. Since $\{x \mid x \leq y\}$ is linearly ordered, a is comparable with z . Hence either $z \in P_a \subset P$ or $z \in \{x \mid x < a\} \subset A \subset P$ by the maximality of A . Thus $z \in P$, i.e., P is closed.

Since X is connected it follows that $P = X$. \square

PROPOSITION 9. *Let X contain at least one minimal point. Then each nonempty subset $A \subset X$ which is directed downwards is bounded from below. In particular below each $x \in X$ there is a minimal point.*

PROOF. Without loss of generality we may assume that A has no minimal element. Let $P = \bigcup_{a \in A} P_a$. P is open and connected and does not contain a minimal point (such a point would also be a minimal element of A). Hence, in any case $P \neq X$. Let $\partial P = \{z\}$ then P is an open component of $X \setminus z$. Hence, $A \subset P \subset P_z$ and consequently z is a lower bound of A . \square

We shall have to distinguish more than once between V -spaces with no, one or more than one minimal point. To this end we define:

A V -space X is said to satisfy

- (V0) iff it does not contain minimal points,
- (V1) iff it contains exactly one minimal point,
- (Vm) iff it contains more than one minimal point,
- (Va) iff it contains at most one minimal point.

[Note that because of the corollary to proposition 4 this definition of (V1) coincides with that given earlier.]

Hausdorff examples of each of these types do exist, and will be given in section 6 of this chapter.

PROPOSITION 10. Let A be a linearly ordered subset of X with order type α . If A is bounded above then $\alpha = \beta^*$ for some ordinal β . If A is not bounded above in X then $\alpha = \sum_{n < \omega_0} \beta_n^*$ for some suitable countable set of ordinals $\{\beta_n\}_{n < \omega_0}$.

(Here if S is an ordered set of order type α then the order type of the inversely ordered set is denoted by α^* .)

PROOF. If $A \neq \emptyset$ is bounded above (by z , say) then A has a largest element, namely $\max A = \inf\{y \mid \forall a \in A : a \leq y \leq z\}$.

(For: the infimum exists by proposition 6 (and proposition 2), and if it would not belong to A then its immediate predecessor - which exists by proposition 7 - would still be an upper bound for A .)

Hence A is well ordered by $>$. If A is not bounded above then there exists an infinite strictly increasing sequence B in A . B is cofinal with A (otherwise B would be bounded and then would have a largest element, contradicting the fact that B is strictly increasing). Thence it follows that $\alpha = \sum_{n < \omega_0} \beta_n^*$ for some suitable countable set of ordinals $\{\beta_n\}$. \square

4. SUBSPACES

In this section we investigate for connected subspaces Y of X the relation between $<_Y$ and $<_X$ restricted to Y . In all cases where $<$ or \min or \max does not have an index, the index X is meant.

PROPOSITION 11. If $C \subset X$ is connected and has an end point x then $x = \min C$ or x is a minimal point of X (or both).

PROOF. $C \setminus x$ is entirely contained in one component of $X \setminus x$. Suppose $x \neq \min C$. Then $C \setminus x \subset Q_x$. (Otherwise $C \setminus x \subset P_x$, but $x = \min P_x$.) Now if x is not minimal then Q_x is closed and $C = (C \setminus x) + \{x\}$ would be disconnected. \square

REMARK. The second alternative may indeed occur: if X is a (V_m) -space and z is a minimal point of X then P_z is not closed, i.e., $\overline{P_z} = P_z \cup \{x\}$ for some x . Here x is not comparable with z and the proposition says that x is another minimal point of X .

PROPOSITION 12. Let X be a V -space and let Y be a connected subspace. Let $x \in X$ and let S be an open component of $X \setminus x$. Then $Y \cap \overline{S}$ is connected. Hence, also $Y \cap P_x$ is connected.

PROOF. We may suppose $x \in Y$ (otherwise $Y \cap \bar{S}$ is empty or equals Y).

$X \setminus x = S + (X \setminus \bar{S})$ hence $Y \setminus x = (Y \cap S) + (Y \setminus \bar{S})$ and thus $(Y \cap S) \cup \{x\}$ is connected. \square

PROPOSITION 13. Let X be a V -space and let Y be a connected subspace. Then $\langle_{X} \rangle \cap (Y \times Y) \subset \langle_{Y} \rangle$.

PROOF. Let $y_1, y_2 \in Y$, $y_1 \langle_X y_2$. Let $X \setminus y_1 = \frac{S}{Y_2} + A$ where S is connected and open (and A may be empty). $Y \cap (S \cup \{y_1\})$ is a connected subspace of the $V1$ -space $S \cup \{y_1\}$ containing its base point hence itself a $V1$ -space with base point y_1 . (From the original definition of $(V1)$ it is immediately seen that $(V1)$ is hereditary for connected subspaces containing the base point of the space.) Therefore, all components of $Y \cap S$ are open (in $Y \cap S$ and hence in Y). It follows that $y_1 \langle_Y y_2$. \square

PROPOSITION 14. Let X be a $V\alpha$ -space and let Y be a connected subspace. Then

(i) $\langle_X \rangle \cap (Y \times Y) = \langle_Y \rangle$, that is, the partial order \langle_Y on Y is the restriction of the partial order \langle_X on X to Y .

(ii) Y has at most one minimal point, i.e., $(V\alpha)$ is hereditary for connected subspaces.

PROOF. (i) In view of proposition 13 it suffices to show $\langle_Y \subset \langle_X \rangle \cap (Y \times Y)$.

For that purpose suppose $y_1 \langle_Y y_2$, say $Y \setminus y_1 = \frac{T}{Y_2} + B$, where T is open and connected in Y (and B may be empty). Now all components of $X \setminus y_1$ are either open or closed. If y_2 belongs to some open component of $X \setminus y_1$ then $y_1 \langle_X y_2$. On the other hand if $y_2 \in Q_{Y_1}$ then $T \subset Q_{Y_1}$ which is impossible since $y_1 \in \bar{T}$ but $y_1 \notin \overline{Q_{Y_1}} = Q_{Y_1}$.

(ii) If both y_1 and y_2 are minimal under \langle_Y then $P'_{Y_1} = P_{Y_1} \cap Y$ is a proper clopen subset of Y : it is open in Y since P_{Y_1} is open in X ; it is closed in Y since a point $y \in \partial_Y P'_{Y_1}$ would be strictly smaller than y_1 ; it is proper since it does not contain y_2 . Contradiction. \square

PROPOSITION 15. Let X be a V -space and $x \in X$. Then $\langle_{P_x} \rangle = \langle_X \rangle \cap (P_x \times P_x)$, that is, the intrinsic partial order on P_x is the restriction of the partial order on X to P_x .

PROOF. By proposition 13 $\langle_{P_x} \rangle \supset \langle_X \rangle \cap (P_x \times P_x)$. Conversely, suppose $x_1 \langle_{P_x} x_2$. Then $P_x \setminus x_1 = \frac{A}{X} + \frac{S}{x_2}$ where S is connected and open in P_x and does not contain x , hence is open in X . Therefore, $X \setminus x_1 = (Q_x \cup A) + S$, i.e., $x_1 \langle_X x_2$. \square

PROPOSITION 16. Let X be a Vm -space. Let Y be the collection of the minimal points of X (so that $|Y| \geq 2$). Let y_0 be some point in Y . By transfinite induction we define y_α for $\alpha > 0$ by:

$$\{y_\alpha\} = \overline{\bigcup_{\beta < \alpha} P_{Y_\beta}} \setminus \bigcup_{\beta < \alpha} P_{Y_\beta}, \quad \text{if } \bigcup_{\beta < \alpha} P_{Y_\beta} \text{ is not closed.}$$

Then:

- (1) $y_\alpha \in Y$ and $\bigcup_{\beta < \alpha} P_{Y_\beta}$ is connected.
 (2) If α_0 is the first ordinal such that $\bigcup_{\beta < \alpha_0} P_{Y_\beta}$ is closed, then

$$\bigcup_{\beta < \alpha_0} P_{Y_\beta} = X \quad \text{and} \quad Y = \{y_\alpha \mid \alpha < \alpha_0\}.$$

- (3) For each $\gamma < \alpha_0$ we have

$$\{y_0\} = \overline{\bigcup_{\gamma \leq \beta < \alpha_0} P_{Y_\beta}} \setminus \bigcup_{\gamma \leq \beta < \alpha_0} P_{Y_\beta},$$

that is, Y is cyclically well ordered.

PROOF. (1) (By transfinite induction): Let the assertion be proved for all α with $1 \leq \alpha < \gamma$, where γ is some ordinal less than α_0 . If γ is a limit number then certainly

$$\bigcup_{\beta < \gamma} P_{Y_\beta} = \bigcup_{\alpha < \gamma} \bigcup_{\beta < \alpha} P_{Y_\beta}$$

is connected. If $\gamma = \delta + 1$, then

$$\bigcup_{\beta < \gamma} P_{Y_\beta} = \bigcup_{\beta < \delta} P_{Y_\beta} \cup P_{Y_\delta},$$

which is connected, since

$$y_\delta \in \overline{\bigcup_{\beta < \delta} P_{Y_\beta}}.$$

Next,

$$\overline{\bigcup_{\beta < \gamma} P_{Y_\beta}} \setminus \bigcup_{\beta < \gamma} P_{Y_\beta}$$

consists of precisely one point, y_γ say. Since certainly y_γ is not the mini-

mum of $\bigcup_{\beta < \gamma} P_{Y_\beta}$ (each y_β is minimal!) it follows from proposition 11 that y_γ is a minimal point of X , i.e., $y_\gamma \in Y$.

(2) We now have to prove that if $P = \bigcup_{\beta < \alpha_0} P_{Y_\beta}$ is closed then $P = X$. If $X \setminus P \neq \emptyset$ then by proposition 9 there is a minimal point y of X with $y \in (X \setminus P) \cap Y$. By proposition 5 Q_y is not closed; hence $Z := \overline{Q_y} = Q_y \cup \{y\}$ is a V_1 -space with base point y (by theorem 2). Of course P_y is not closed since Q_y is not open by definition; so $\overline{P_y} = P_y \cup \{z\}$ for some $z \in Z$.

For $u \in Z$, let P'_u and Q'_u denote the sets $\{v \in Z \mid v \geq_Z u\}$ and $Z \setminus P'_u$ respectively.

Claim (i) z is a minimal point in X .

(ii) Let $u \in Z \setminus y$. If $u <_Z z$ and S'_u is the component of $Z \setminus u$ containing z then $P_u = P'_u \setminus S'_u$. If $u \not<_Z z$ then $P_u = P'_u$. (In this case we write $S'_u = \emptyset$.)

(iii) $z \geq_Z y_0$ (and, for that matter, $z \geq_Z y_\alpha$ ($\alpha < \alpha_0$)).

For: (i) If z were not minimal then by proposition 5 P_z would be open and intersect P_y in some point u . But then y and z are both in the linearly ordered set $\{v \mid v \leq u\}$ hence are comparable, and by the minimality of y : $y \leq z$, which is impossible since $z \notin P_y$ by definition.

(ii) First of all, by proposition 13, we have $P_u \cap Z \subset P'_u$. P_u is connected and does not contain y so $P_u \subset Z$, i.e., $P_u \subset P'_u$. $Z \setminus \{y, z\} = X \setminus \overline{P_y}$ is open in X , so each component of $Z \setminus u$ not containing z and open in Z (and hence not containing y) is open in X (and again a component of $X \setminus u$). On the other hand, a component of $Z \setminus u$ containing z cannot be contained in an open component of $X \setminus u$ since z is minimal. Hence $P_u =$ union of open components of $X \setminus u =$ union of open components of $Z \setminus u$ not containing $z = P'_u \setminus S'_u$.

(iii) P_{y_0} is not open in X but P'_{y_0} is open in Z so either $P_{y_0} \neq P'_{y_0}$ or $z \in P'_{y_0}$ (since certainly $y \notin P'_{y_0}$; remember that y is the base point of Z). But also in the former case we have $z \in P'_{y_0}$ (by (ii)); so in all cases $z \geq_Z y_0$.

This proves the claim.

Now if we put $Q := P \cup S'_{y_0}$ then, since P is closed, $\overline{Q} = P \cup \{y_0\} \cup S'_{y_0} = Q$. But on the other hand Q must be open in Z : Define for $\gamma \leq \alpha_0$

$$Q_\gamma := S'_{y_0} \cup \bigcup_{\beta < \gamma} P_{Y_\beta}.$$

By transfinite induction it is seen that

- (iv) Q_γ is connected and open in Z , and
 (v) $Q_\gamma = S'_{y_\gamma}$.

For: If $\gamma = 0$ then $Q_\gamma = S'_{Y_\gamma}$ by definition, and this is an open connected subset of Z (and nonempty because of (iii)). If γ is a successor ordinal then $Q_\gamma = Q_{\gamma-1} \cup P_{Y_{\gamma-1}} = S'_{Y_{\gamma-1}} \cup P_{Y_{\gamma-1}} = P'_{Y_{\gamma-1}}$ by (ii), so Q_γ is connected and open in Z .

$\overline{Q_\gamma} = P_{Y_{\gamma-1}} \cup \{y_\gamma\} \cup S'_{Y_{\gamma-1}} = Q_\gamma \cup \{y_\gamma\}$, hence Q_γ is the component of $Z \setminus Y_\gamma$, containing z , i.e., $Q_\gamma = S'_{Y_\gamma}$. If γ is a limit ordinal then $Q_\gamma = \bigcup_{\beta < \gamma} Q_\beta$ is connected and open in Z . Moreover, $\overline{Q_\gamma} = S'_{Y_0} \cup \bigcup_{\beta < \gamma} P_{Y_\beta} \cup \{y_\gamma\} = Q_\gamma \cup \{y_\gamma\}$, so again $Q_\gamma = S'_{Y_\gamma}$.

In particular $Q = Q_{\alpha_0}$ is open in Z . But since Z is connected we must have either $Q = \emptyset$ (which is impossible since $y_0 \in Q$) or $Q = Z$ (which is impossible since $y \in Z \setminus Q$). This contradiction proves that $P = X$ and hence $Y = \{y_\beta \mid \beta < \alpha_0\}$.

(3) Finally, if $Y = \{y_\beta \mid 0 \leq \beta < \alpha_0\}$ then if we start with y instead of y_0 , we must find the same sequence of points and, finally, $y_0 = y_{\alpha_0}$ (because if $\gamma < \delta$ then

$$\overline{\bigcup_{\beta < \delta} P_{Y_\beta}} \setminus \bigcup_{\beta < \delta} P_{Y_\beta} = \overline{\bigcup_{\gamma \leq \beta < \delta} P_{Y_\beta}} \setminus \bigcup_{\gamma \leq \beta < \delta} P_{Y_\beta} = \{y_\delta\},$$

i.e., each point is already determined by a cofinal set of its predecessors. \square

PROPOSITION 17. Let Y be a V_m -space and assume that Y is a subspace of a V -space X . Then

- (i) each minimal point of Y is minimal in X ,
- (ii) each minimal point of X is element of Y (and minimal in Y because of (iv)),
- (iii) Y is closed in X and all components of $X \setminus Y$ are open,
- (iv) $<_Y$ is the restriction of $<_X$ to Y , and the cyclic well orderings on the minimal points coincide.

PROOF. (i) Suppose $x < y$, where y is minimal in Y . Then by proposition 13 if $x \in Y$ then $x <_Y y$. But if $x \notin Y$ then $Y \subset P_x$ contradicting proposition 14.

(ii) Let x be a minimal point of X not in Y . Then Q_x is not closed (by proposition 5; note that because of (i) X is a V_m -space) hence $Q_x \cup \{x\}$ is a V_1 -space containing Y contradicting proposition 14. [Note: at this point we do not know yet whether x must be a minimal point of Y .]

(iii) If $x \in \overline{Y} \setminus Y$ then by (i) Y is not contained in an open component of $X \setminus x$,

i.e., $Y \subset Q_x$. But then Q_x is not closed, i.e., x is a minimal point in x , and by (ii) $x \in Y$.

If C is a non-open component of $X \setminus Y$ then $C \setminus C^\circ = \{z\}$ (by proposition 1). $X \setminus C$ is connected and $\overline{X \setminus C} = X \setminus C \cup \{z\}$, so by proposition 11 either $z = \min \overline{X \setminus C}$ or z is a minimal point of X (or both). But $z \notin Y$ so by (ii) z is not minimal in X and, consequently, $z = \min \overline{X \setminus C}$. However, $Y \subset X \setminus C$ implies that z is smaller than each point of Y , contradicting (i).

(iv) Let $\{x_\gamma \mid \gamma < \alpha_0\}$ be a canonical well ordering of the set of minimal points of X . Suppose that for some $\alpha < \alpha_0$, $Y \cap \bigcup_{\beta < \alpha} P_{x_\beta}$ is closed. Then (by a suitable renumbering of the x_α) we might as well suppose that $Y \cap \bigcup_{\gamma \leq \beta < \alpha_0} P_{x_\beta}$ is closed for some $\gamma > 0$ (and hence for all $\gamma < \alpha_0$). [Note: the value of α_0 might have been changed in the process of renumbering.] This means that all sets $Y \cap \bigcup_{\delta < \gamma} P_{x_\delta}$ are open in Y . Also these sets are connected: by proposition 12 each $Y \cap P_{x_\delta}$ is connected, and by (ii) it is nonempty. Now for $0 < \delta < \gamma$, $Y \cap \bigcup_{\beta < \delta} P_{x_\beta}$ is not closed (since Y is connected), so for each such δ $x_\delta \in \overline{Y \cap \bigcup_{\beta < \delta} P_{x_\beta}}$ and it follows that $Y \cap \bigcup_{\delta < \gamma} P_{x_\delta}$ is connected for each $\gamma < \alpha_0$. But an open connected set different from the entire space (in casu Y) does not contain a minimal point (proposition 5), i.e., x_δ is not minimal in Y for $\delta < \gamma < \alpha_0$ and this means that Y contains at most one minimal point. Contradiction. Therefore, for no $\alpha < \alpha_0$ $Y \cap \bigcup_{\beta < \alpha} P_{x_\beta}$ is closed, so $x_\alpha \in \overline{Y \cap \bigcup_{\beta < \alpha} P_{x_\beta}}$ which means that $Q_{x_\gamma} \cap Y = \bigcup_{\beta \neq \gamma} P_{x_\beta} \cap Y$ is connected and not closed for each $\gamma < \alpha_0$. This proves that if x is minimal in X then x is minimal in Y and $x < y$ is equivalent to $x <_Y y$ for $y \in Y$. If $y_1 <_Y y_2$ then y_2 is element of an open component of $Y \setminus y_1$, i.e., $y_2 \in P_{y_1} \cap Y$ and $y_1 < y_2$. This (together with proposition 13) proves everything. \square

While V_m -spaces are maximal, V_0 -spaces can be extended with a lowest point:

PROPOSITION 18. *If X is a V_0 -space then it can be made into a V_1 -space by adjoining a (closed) lowest point x_0 with basic (open) neighbourhoods $\{x_0\} \cup \left(X \setminus \bigcup_{i=1}^n P_{x_i} \right)$ for $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$. If X satisfies a T_i separation axiom ($i=1, 2, 2\frac{1}{2}, 3$) then $Y = X \cup \{x_0\}$ is T_i too.*

PROOF. Observe that we indeed defined a topology on Y and that X is inbedded in Y as an open subset; in particular each open subset of X is open in Y . Next, since X has no minimal points and since $\overline{P_x} \subset P_x$, for $x \in X$ (see proposition 7), it is immediately seen that $Y = \overline{X}$ is connected and T_i ($i=1, 2, 2\frac{1}{2}, 3$).

$[T_{2\frac{1}{2}}: \{x_0\} \cup (X \setminus P_x)$ and $\overline{P_x} = P_x \cup \{x\}$ are closed disjoint nbds of x_0 and x .

T_3 : Let $F \subset Y$ be closed, and $y \notin F$. If $y = x_0$ then $y \notin \overline{F} = F$ yields

$F \cap \bigcup_{i=1}^n \overline{P_{x_i}}$ for certain points x_i ($1 \leq i \leq n$). Now

$$y \in \{x_0\} \cup \left(X \setminus \bigcup_{i=1}^n \overline{P_{x_i}} \right) \subset \overline{\{x_0\} \cup \left(X \setminus \bigcup_{i=1}^n \overline{P_{x_i}} \right)} \subset \\ \subset \{x_0\} \cup \left(X \setminus \bigcup_{i=1}^n \overline{P_{x_i}} \right) \subset Y \setminus F.$$

If $y \neq x_0$ and U and V are disjoint open nbds of y and $F \cap X$ in X then $U \cap P_y$ and $V \cup (\{x_0\} \cup X \setminus \overline{P_y})$ are disjoint open nbds of y and F in Y .]

Furthermore, Y is a V_1 -space:

Suppose $x_0 \in C$, C connected but not closed, say $p \in \overline{C} \setminus C$. Then P_p (which is an open set in X and hence in Y) intersects C ; let $q \in P_p \cap C$. Let S be the (open) component of $X \setminus p$ containing q , then $\overline{S}^Y = \overline{S}^X = S \cup \{p\}$ and $C = (C \cap S) + (C \setminus \overline{S}_{x_0})$, a contradiction. \square

Note: we do not know whether there exists a V -space satisfying $T_{2\frac{1}{2}}$.

5. TOPOLOGICAL PROPERTIES OF V -SPACES

THEOREM 5. A V -space containing more than one point cannot be:

- (a) countably compact,
- (b) locally countably compact,
- (c) locally peripherally compact and T_2 .

PROOF. First observe that a V -space X containing more than one point does not contain maximal elements: if x is maximal and not minimal then $P_x = \{x\}$ is clopen which is impossible; therefore, a maximal element must be minimal too, and since $Q_x = X \setminus x$ is open it follows from the definition of Q_x that $Q_x = \emptyset$ and $x = \{x\}$.

(a) Let $\{u_n\}_{n \in \mathbb{N}}$ be a strictly increasing sequence in X . For $i \in \mathbb{N}$ let $U_i := P_{u_i} \setminus \overline{P_{u_{i+2}}}$. Let $Q = X \setminus \overline{P_{u_1}}$. Then $\{Q\} \cup \{U_i \mid i \in \mathbb{N}\}$ is a countable open covering of X (for: $\{u_n\}$ is cofinal in X by proposition 10). $u_i \notin U_j$ for $j = i, i-1$; hence this covering has no finite subcovering. Therefore, no subset of X containing a strictly increasing sequence is countably compact; in particular X itself is not countably compact.

(b) When U is open in X and when U contains a maximal element x then as before either $P_x \cap U = \{x\}$ is clopen, which is impossible, or x is minimal

in X such that $Q_x \cup \{x\}$ is a neighbourhood of x . But this means $P_x = \{x\}$, $Q_x = \emptyset$, $X = \{x\}$. Therefore, no open subset of X contains a maximal element and by (a) X is not locally countably compact.

(c) Let U and V be disjoint open sets such that $x \in U$, $x' \in V$, $U \subset P_x$ and ∂U is compact (for some non-minimal point $x \in X$). $\overline{P_x} = P_x \cup \{x'\}$ and $x', x \notin \partial U$; hence $\partial U \subset P_x \setminus \{x\}$. Let $X \setminus x = Q_x \cup \bigcup_{\alpha} C_{\alpha}$ be the decomposition of $X \setminus x$ in components. Now $\{C_{\alpha}\}_{\alpha}$ is an open covering of ∂U consisting of disjoint sets. Let $\{C_{\alpha_1}, \dots, C_{\alpha_n}\}$ be a finite subcovering. Then $C_{\alpha} \cap \partial U = \emptyset$ for $\alpha \neq \alpha_1, \dots, \alpha_n$. Since $x \in U$ and $\overline{C_{\alpha}} = C_{\alpha} \cup \{x\}$ is connected it follows that $\bigcup_{\alpha \neq \alpha_1, \dots, \alpha_n} C_{\alpha} \subset U$. But $x' \in \overline{P_x} = \overline{\bigcup_{\alpha} C_{\alpha}}$ hence each nbd of x' meets $\bigcup_{\alpha} C_{\alpha}$ and - since $\{x\} \cup C_{\alpha_1} \cup \dots \cup C_{\alpha_n}$ is closed and does not contain x' - also $\bigcup_{\alpha \neq \alpha_1, \dots, \alpha_n} C_{\alpha}$. (In particular there exist indices $\alpha \neq \alpha_1, \dots, \alpha_n$.)

It follows that each nbd of x' meets U . Contradiction. \square

REMARK. In the next section examples of locally peripherally compact T_1 V -spaces will be given.

THEOREM 6. Let X be a V -space. Let Y be a dense subset of X . Then $|Y| = |X|$.

PROOF. Let D be a maximal set of pairwise incomparable elements of X such that no $d \in D$ is minimal in X . Since all P_d ($d \in D$) are disjoint it follows that $|D| \leq |Y|$. Let E be a chain in X and choose for $e \in E$ an open component of $X \setminus e$ disjoint from E . Since all these open sets are disjoint it follows that $|E| \leq |Y|$. In particular, if $B_D = \{y \in X \mid \exists d \in D: y \leq d\}$ then $|B_D| \leq |Y|^2 = |Y|$. Now define by induction sets D_i with $D_0 = D$ and D_{i+1} a maximal set of pairwise incomparable elements of $X \setminus \bigcup_{j \leq i} B_{D_j}$. Then $X = \bigcup_{i=0}^{\infty} B_{D_i}$ since a strictly ascending sequence in X does not have an upper bound. Therefore, $|X| \leq \aleph_0 \cdot |Y| = |Y|$. Since obviously $|Y| \leq |X|$ this proves $|Y| = |X|$. \square

REMARK. Note that in fact we proved $|X| = c(X)$, the degree of cellularity of X .

DEFINITION. Let X be a V -space. The *depth* of X is the supremum of all ordinals α such that X contains a strictly decreasing sequence $\{u_{\gamma}\}_{\gamma < \alpha}$.

THEOREM 7. Let X be a V -space with depth ω_0 . Then each continuous $f: X \rightarrow I$ is constant.

PROOF. If f is constant on each $P_x \subset X$ then it is easily seen that f is also constant on X . Therefore, we may assume that X is a V_1 -space with base point x_0 . Since X has depth ω_0 we may write $X = \bigcup_{k=0}^{\infty} Y_k$ where

$$(*) \quad Y_0 = \{x_0\} \quad \text{and} \quad Y_{k+1} = \{y \mid y' \in Y_k\}.$$

Arguing by contradiction we assume that f is not constant.

Without loss of generality we may assume that $f(x_0) = r < s = f(x_1)$ for some $x_1 \in Y_1$. We will exhibit a separation $X = A + B$ by defining inductively sets A_k and B_k and functions $\phi_k, \psi_k: X \rightarrow I$ as follows:

Set $A_0 = \{x_0\}$ and $B_0 = \emptyset$. Choose t with $r < t < s$ and set $\phi_0(x) = t$,

$\psi_0(x) = 1$ for each $x \in X$. Assume A_i, B_i, ϕ_i and ψ_i defined for $i < k$ in such a way that

- (a) $A_i \cap B_i = \emptyset$ and $A_i \cup B_i = Y_i$,
- (b) (if $i > 0$ then) $\forall x \in X: \phi_{i-1}(x) \leq \phi_i(x) < \psi_i(x) \leq \psi_{i-1}(x)$,
- (c) ϕ_i and ψ_i are constant on P_y for each $y \in Y_i$,
- (d) $Y_i \cap \bigcup_{y \in X} (P_y \cap f^{-1}([0, \phi_i(y)))) \subset A_i$ and $Y_i \cap \bigcup_{y \in X} (P_y \cap f^{-1}((\psi_i(y), 1])) \subset B_i$.

Set $A_k = \{y \in Y_k \mid f(y) \leq \phi_{k-1}(y)\}$ and $B_k = \{y \in Y_k \mid f(y) > \phi_{k-1}(y)\}$.

Define ϕ_k by:

$$\phi_k|_{Y_i} = \phi_{k-1}|_{Y_i} \quad (i < k),$$

and for $z \in P_y$ with $y \in Y_k$:

$$\phi_k(z) = \begin{cases} \phi_{k-1}(y) & \text{if } f(y) \neq \phi_{k-1}(y), \\ \frac{1}{2}(\phi_{k-1}(y) + \psi_{k-1}(y)) & \text{if } f(y) = \phi_{k-1}(y); \end{cases}$$

and likewise ψ_k :

$$\psi_k|_{Y_i} = \psi_{k-1}|_{Y_i} \quad (i < k),$$

and for $z \in P_y$ with $y \in Y_k$:

$$\psi_k(z) = \begin{cases} \psi_{k-1}(y) & \text{if } f(y) \leq \phi_{k-1}(y) \text{ or } f(y) > \psi_{k-1}(y), \\ \frac{1}{2}(\phi_{k-1}(y) + f(y)) & \text{if } \phi_{k-1}(y) < f(y) \leq \psi_{k-1}(y). \end{cases}$$

Now clearly (a), (b) and (c) are satisfied for $i = k$. Also $z \geq y, z \in Y_k, y \in Y_i$ ($i \leq k$) and $f(z) < \phi_k(y)$ imply $z \in A_k$ (observe that $f(z) < \phi_k(y) =$

$\phi_i(y) = \phi_i(z) \leq \phi_{k-1}(z)$ if $i < k$; and $\phi_k(y) = \phi_{k-1}(y)$ or $y \in A_k$ if $i = k$.
Likewise, if $f(z) > \psi_k(y)$ then $f(z) > \psi_k(y) = \psi_i(y) = \psi_i(z) > \phi_{k-1}(z)$,
hence $z \in B_k$.

This proves (d) and therewith the induction hypothesis for the next step.

Let $A = \bigcup_{i=0}^{\infty} A_i$ and $B = \bigcup_{i=0}^{\infty} B_i$.

Claim. $X = \overset{A}{x_0} + \overset{B}{x_1}$.

For: by (*) and (a) it follows that $X = A \cup B$ and $A \cap B = \emptyset$. (b) and (d)

together yield: for $y \in A_i$: $y \in (P_Y \cap f^{-1}([0, \phi_i(y)])) \subset A$

[for: since $y \in A_i$ it follows that $f(y) \leq \phi_{i-1}(y) \leq \phi_i(y)$ and equality cannot hold both times]. But this means that A is open. Likewise B is open,

for if $y \in B_i$ then $y \in (P_Y \cap f^{-1}((\psi_i(y), 1])) \subset B$

[for: since $y \in B_i$ it follows that $f(y) > \phi_{i-1}(y)$ and hence $f(y) > \psi_i(y)$].

Finally $x_0 \in A$ and $x_1 \in B$ by definition of A_0 and B_0 . This proves the claim, and since X is connected we have derived a contradiction. Therefore, f is constant. \square

6. CONSTRUCTION OF V -SPACES

Up to this point the reader may have wondered whether there exist any V -spaces. The theorem below assures us of the existence of a V -space with prescribed partial ordering (provided this partial order is admissible in view of the structure theory developed in the preceding sections).

THEOREM 8. *Let a set X be partially ordered by a relation $<$ such that*

- (i) $\forall x \in X$ the set $\{z \mid z \leq x\}$ is well ordered by $>$.
- (ii) $\forall x \in X$ the set $\{z \mid z > x\}$ is the union of infinitely many mutually disjoint sets E_x^α ($\alpha \in A_x$) such that points from E_x^α for different α are incomparable, while each set E_x^α is directed downwards.
- (iii) If $A \subset X$ is closed both upwards and downwards (i.e., each element comparable to some point in A belongs to A), and $A \neq X$ then A contains a minimal point of X .

Let Y be the collection of minimal points in X (this may well be an empty set) and fix a well ordering of it:

$$Y = \{y_\alpha \mid \alpha < \alpha_0\}.$$

Let

$$P_x = \{z \mid z \geq x\} \text{ and } O_x^\alpha = \{z \mid z \neq x \text{ \& } z \notin E_x^\alpha\} \quad (x \in X, \alpha \in A_x).$$

When we take the collection

$$\begin{aligned} & \{P_x \mid x \in X \setminus Y\} \cup \{O_x^\alpha \mid x \in X, \alpha \in A_x\} \cup \\ & \cup \{X \setminus (\bigcup_{\gamma \leq \beta < \alpha} P_{Y_\beta} \cup \{y_\alpha\}) \mid \gamma < \alpha < \alpha_0\} \cup \\ & \cup \{X \setminus (\bigcup_{\gamma \leq \beta < \alpha_0} P_{Y_\beta} \cup \{y_0\}) \mid \gamma < \alpha_0\} \end{aligned}$$

as a subbase for a topology on X , we get a T_1 - V -space the natural partial order of which coincides with $<$. Conversely each V -space has a partial order $<$ satisfying conditions (i)-(iii), and the elements of the above collection are open, so the topology constructed here is in fact the minimal T_1 V -topology with the given order $<$.

PROOF. 1. X is T_1 since $\{x\} = (X \setminus O_x^{\alpha_1}) \cap (X \setminus O_x^{\alpha_2})$ is closed for $\alpha_1 \neq \alpha_2$ in A_x .

2. X is connected: Let $x \in X \setminus Y$ and let x' denote the largest element of $\{z \mid z < x\}$. (Such an element exists because of (i).) We first prove:

2A. x and x' do not have disjoint neighbourhoods. For: $E_x^\alpha = U\{P_y \mid y \in E_x^\alpha\}$ is open for each $x \in X, \alpha \in A_x$ (note that $y \in E_x^\alpha$ implies that y is not minimal in X). A basic nbd of x is $P_x \cap \bigcap_{i=1}^n O_{x_i}^{\alpha_i}$ with $x_i > x$ and $\alpha_i \in A_{x_i}$. (For, $\forall y \in X \forall \beta \in A_y$: [either $x \notin P_y$ or $P_x \subset P_y$] and if $y \not\leq x$ [either $x \notin O_y^\beta$ or $O_y^\beta \supset P_x$] and if $x \in X \setminus (\bigcup_{\delta} P_{Y_\delta} \cup \{y_\gamma\})$ then $P_x \subset X \setminus (\bigcup_{\delta} P_{Y_\delta} \cup \{y_\gamma\})$ since x is not minimal.)

Now for each i ($1 \leq i \leq n$) there is a $\beta_i \in A_{x_i}$ with $x_i \in E_{x_i}^{\beta_i}$ so $O_{x_i}^{\alpha_i} \supset X \setminus E_{x_i}^{\beta_i}$ and consequently the set E_x^α is contained in any given basic nbd of x for almost every $\alpha \in A_x$. A basic nbd of x' is $P_{x'} \cap \bigcap_{j=1}^m O_{z_j}^{\gamma_j}$ with $z_j > x'$ and $\gamma_j \in A_{z_j}$ if x' is not minimal. $P_{x'}$ contains all $E_{x'}^\alpha$, and $O_{z_j}^{\gamma_j}$ contains all $E_{z_j}^\alpha$ unless $z_j \geq x$ in which case it contains all $E_{z_j}^\alpha$ except one. Therefore, if x' is not minimal, the intersection of a neighbourhood of x and a neighbourhood of x' contains almost all E_x^α and in particular it is not empty. If x' is minimal, say $x' = y_\alpha$, then a basic nbd of it is

$$\bigcap_{j=1}^m O_{z_j}^{\gamma_j} \cap (\bigcup_{\beta \leq \gamma \leq \alpha} P_{Y_\gamma} \setminus \{y_\beta\}) \quad \text{with } \beta < \alpha \text{ and } z_j > x'$$

(provided $\alpha > 0$; if $\alpha = 0$ then the second term in the intersection must be replaced by $P_{Y_0} \cup (\bigcup_{\gamma \geq \beta} P_{Y_\gamma} \setminus \{y_\beta\})$, $\beta > 0$ if $|Y| > 1$, and must be deleted altogether if $|Y| = 1$). Again this nbd contains almost all E_x^α . This proves that x and x' do not have disjoint neighbourhoods.

Now, suppose X is not connected. Then there exists a separation $X = A + B$ with nonempty A and B . Let $b \in B$ and let a be the largest element of $A \cap \{z \mid z \leq b\}$ if this last set is nonempty. Let $a \in P_a \cap \bigcap_{j=1}^m O_{x_j}^{\alpha_j} \subset A$ with $x_j > a$ and $a_j \in A_{x_j}$. [This is a basic nbd of a if a is not minimal, and contained in a basic nbd in any case.] Then for some x_j, α_j we have $b \notin O_{x_j}^{\alpha_j}$, i.e., $b \in \{x_j\} \cup E_{x_j}^{\alpha_j}$; but then $a < x_j \leq b$ and it follows that $x_j \notin A$ hence $x_j \in B$ by definition of a . But then, by 2A, also $x_j' \in B$. Now if x_{j_0} is a minimal element of (the finite set) $\{x_j \mid b \notin O_{x_j}^{\alpha_j}\}$ then $x_{j_0}' \in P_a \cap \bigcap_{j=1}^m O_{x_j}^{\alpha_j}$; hence because of the previous argument, $x_{j_0}' \in P_a \cap \bigcap_{j=1}^m O_{x_j}^{\alpha_j} \cap B$. Contradiction. Therefore, $A \cap \{z \mid z \leq b\}$ is empty and it follows that both A and B are closed downwards (and, therefore, also upwards). By requirement (iii) A and B contain minimal points y_α and y_β , respectively. We may suppose that $\alpha < \beta$ and that β is the first element of $\{\gamma > \alpha \mid y_\gamma \in B\}$. Then $U\{P_{Y_\gamma} \mid \alpha \leq \gamma < \beta\} \subset A$, but each nbd of y_β intersects this set, contradiction. Therefore, X is connected.

3. Let C be a connected subset of X which is not closed. Let $x \in \bar{C} \setminus C$. Then $x = \min \bar{C}$ or $x \in Y$. For: Since E_x^α is open and $E_x^\alpha \cup \{x\}$ is closed (the former was noted in the first line of the proof of 2A, while the latter follows immediately from the definition of the topology) the set E_x^α is clopen in $X \setminus x$ for each $\alpha \in A_x$. Therefore, if C intersects some E_x^α then $C \subset E_x^\alpha$ and $x = \min \bar{C}$. On the other hand if $C \cap \bigcup_{\alpha \in A_x} E_x^\alpha = C \cap P_x = \emptyset$ then $\bar{C} \cap P_x = \{x\}$ and if $x \notin Y$ it follows that $\{x\}$ is clopen in \bar{C} , which is impossible.

4. X is a V -space. For: Suppose C is a connected subset of X with two endpoints x_1 and x_2 . By 3. both x_1 and x_2 are minimal points of X , say $x_1 = y_\alpha$, $x_2 = y_\beta$ with $\alpha < \beta$. Since

$$X \setminus y_\alpha = \left(\bigcup_{\gamma \in A_{y_\alpha}} E_{y_\alpha}^\gamma \right) \cup (X \setminus P_{y_\alpha}),$$

where each $E_{y_\alpha}^\gamma$ is clopen in $X \setminus y_\alpha$ while $y_\beta \in X \setminus P_{y_\alpha}$, it follows that

$$C \cap (P_{y_\alpha} \setminus \{y_\alpha\}) = C \cap \left(\bigcup_{\gamma} E_{y_\alpha}^\gamma \right) = \emptyset.$$

Likewise

$$C \cap (P_{Y_\beta} \setminus \{y_\beta\}) = \emptyset.$$

Now $\bigcup_{\alpha \leq \gamma \leq \beta} P_{Y_\gamma} \setminus \{y_\alpha\}$ is open and $\bigcup_{\alpha+1 \leq \gamma < \beta} P_{Y_\gamma} \cup \{y_\beta\}$ is closed; hence

$$C \cap \left(\bigcup_{\alpha \leq \gamma \leq \beta} P_{Y_\gamma} \setminus \{y_\alpha\} \right) = C \cap \left(\bigcup_{\alpha+1 \leq \gamma < \beta} P_{Y_\gamma} \cup \{y_\beta\} \right)$$

is clopen in C and contains y_β hence equals C . But this set does not contain y_α , while C does. Contradiction.

5. X has partial order $<$. For: by the same argument used to prove the connectedness of X it follows that each E_x^α ($x \in X$, $\alpha \in A_x$) is connected. Therefore, by (ii) the set $\{z \mid z > x\}$ is a union of open components of $X \setminus x$. And hence, it follows that the natural partial order contains $<$.

Next, if x is not minimal, then $<$ restricted to the set $X \setminus P_x$ satisfies the conditions of the theorem and gives the subspace topology to $X \setminus P_x$. Therefore, $X \setminus P_x$ is connected in this case (but not open since its complement is open).

Also if x is minimal. then by definition of the neighbourhoods of minimal points it follows that $X \setminus P_x$ is connected (in fact it is a V -space with at most one minimal point), while $X \setminus P_x$ is not open.

Therefore, $y > x \iff y$ belongs to an open component of $X \setminus x$, that is, $>$ is the partial order of the V -space X .

6. The converse part of the theorem readily follows from propositions 2, 10, 1, 8, 9 and 16 and the definition of the partial order of a V -space. \square

PROPOSITION 19. *The space X defined in the previous theorem is locally peripherally compact. Its subset Y is compact. If Y is nonempty then X is Lindelöf.*

PROOF. (i) As was seen in the proof of the previous theorem, the sets $P_x \cap \bigcap_{i=1}^n O_{x_i}^{\alpha_i}$ with $x_i > x$, $\alpha_i \in A_{x_i}$ form a collection of basic nbds of x when $x \in X \setminus Y$, while if $x \in Y$, say $x = y_\alpha$, such a collection is given by the sets

$$\bigcap_{j=1}^m O_{z_j}^{\gamma_j} \cap \left(\bigcup_{\beta \leq \gamma \leq \alpha} P_{Y_\gamma} \setminus \{y_\beta\} \right),$$

with $z_j > x$ and $\gamma_j \in A_{z_j}$ and $\beta < \alpha$, provided $\alpha > 0$.

(For $\alpha = 0$ in the second term of the intersection $\beta \leq \gamma \leq \alpha$ must be replaced by $\alpha = 0$ or $\beta \leq \gamma$ where this time $\beta > \alpha$, unless $Y = \{y_0\}$ in which case the

second term in the intersection is to be deleted altogether.) Now local peripheral compactness follows immediately, since this basis shows that X is even locally peripherally finite:

$$\partial\{P_x \cap \bigcap_{i=1}^n O_{x_i}^{\alpha_i}\} \subset \{x', x_1, \dots, x_n\}$$

and

$$\partial\left\{\bigcap_{j=1}^m O_{z_j}^{\gamma_j} \cap \left(\bigcup_{\beta \leq \gamma \leq \alpha} P_{y_\gamma} \setminus \{y_\beta\}\right)\right\} \subset \{z_1, \dots, z_m, y_\beta, y_{\alpha+1}\}.$$

(ii) In the relative topology Y is homeomorphic to the ordinal space $\{\alpha \mid 1 \leq \alpha \leq \alpha_0\}$ which is compact.

(iii) From (ii) and the definition of the topology of X it follows that in order to prove that X is Lindelöf it suffices to show that each P_y ($y \in Y$) is Lindelöf. But by induction one easily constructs a countable subcover of a given cover of P_y : first take a set covering $\{y\}$. After some stage k in the induction all of P_y is covered except for a finite union

$$\bigcup_{i=1}^{n_k} (E_{x_{i,k}}^{\alpha_{i,k}} \cup \{x_{i,k}\}).$$

Then in stage $k+1$ choose sets from the cover containing the points $x_{i,k}$ ($i \leq n_k$) and add them to the subcover being constructed. Since a strictly increasing sequence in X cannot have an upper bound we indeed obtain a subcover in this way. \square

REMARK. In fact if $Y = \emptyset$ then X is Lindelöf iff there is a countable sequence cointial with X .

PROOF. If $\{u_i \mid i \in \mathbb{N}\}$ is cointial in X then $X = \bigcup P_{u_i}$ is Lindelöf by the previous proposition. Conversely, let X be Lindelöf and let $\{u_\alpha \mid \alpha < \alpha_1\}$ be a strictly decreasing sequence cointial with X . Since $\{P_{u_\alpha} \mid \alpha < \alpha_1\}$ is an open cover of X it has a countable subcover $\{P_{u_{\alpha_j}} \mid j \in \mathbb{N}\}$. But this means that $\{u_{\alpha_j} \mid j \in \mathbb{N}\}$ is a countable sequence cointial with X . \square

PROPOSITION 20. *If the set X in theorem 8 is order homogeneous then X becomes a homogeneous V_0 -space, that is, given two points $x, y \in X$ there exists a homeomorphism ψ of X onto itself with $\psi(x) = y$. \square*

THEOREM 9. *The topology defined in theorem 8 can be strengthened in such*

a way that a Hausdorff V -space results. (That is, each partial order which is admissible for T_1 V -spaces does occur also among the T_2 V -spaces.)

PROOF. This proof is a generalisation of the construction of J.L. HURSCH and A. VERBEEK-KROONENBERG who were the first to give an example of a countable Hausdorff V_1 -space [15]. The idea is the following: add for each x a new nbd U_x to the topology (in such a way that $y \in U_x \Rightarrow U_y \subset U_x$). Two points of X do already have disjoint nbds unless one is the immediate predecessor of the other; therefore, if $U_x \cap U_{x'} = \emptyset$ the new topology will be Hausdorff. In order to get a V -space we have to ensure the connectedness of X in this new topology; to this end we construct the new topology in such a way that if V_x and $V_{x'}$ are nbds of x and x' then $\bar{V}_x \cap \bar{V}_{x'} \neq \emptyset$. If, moreover, for each minimal y $U_y \cup P_y = X$, that is, $X \setminus P_y \subset U_y$, and if also for each x, y if $x < y$ and x has no immediate successor in $\{u \mid x \leq u \leq y\}$ then $x \in \overline{\{u \mid x < u \leq y\}}$, then X is indeed connected. For, if $X = A + B$ and $a \in A$, then it follows that $\{u \mid u < a\} \subset A$, that is, A and B are directed downwards and, therefore, also upwards. By condition (iii) of theorem 7 A and B both contain points of Y . Since each nbd of a minimal point y_α still intersects all P_{y_β} for $\gamma \leq \beta < \alpha$ (for some $\gamma < \alpha$) and each P_{y_β} is entirely contained in either A or B it follows that all minimal points and then also all points of X are contained in say A . Hence X is connected. So we have to find a collection of subsets U_x such that:

- (i) $x \in U_x$,
- (ii) $y \in U_x \rightarrow U_y \subset U_x$,
- (iii) $U_x \cap U_{x'} = \emptyset$,
- (iv) $\overline{U_x} \cap \overline{U_{x'}} \cap P_x$ contains infinitely many pairwise incomparable points,
- (v) y minimal $\Rightarrow X \setminus P_y \subset U_y$,
- (vi) if $x < y$ and if x has no immediate successor in $\{u \mid x \leq u \leq y\}$ then $U_x \cap \{u \mid x < u \leq y\} \neq \emptyset$.

If we have found $U_x \cap P_x$ then we let $U_x = U_x \cap P_x$ if x is not minimal and $U_x = (X \setminus P_x) \cup (U_x \cap P_x)$ if x is minimal in order to satisfy (v). Therefore, we may forget about restriction (v) and assume that X has at most one minimal point.

Restriction (iv) may be strengthened to

- (iv'): $\overline{U_x} \cap \overline{U_{x'}}$ intersects each E_x^α ($\alpha \in A_x$).

Now let us introduce an equivalence relation on X namely the symmetric and transitive closure of $x \sim x'$ (i.e., $x \sim y$ iff $\exists k, l \in \mathbb{N}: x^{(k)} = y^{(l)}$)

where $x^{(k)}$ denotes the k -th predecessor of x .

Restriction (vi) is now satisfied if:

(vi') U_x intersects each equivalence class lying entirely above x .

If E is an equivalence class, and $e \in E$ is a fixed point in E then we can define a function $\phi_e: E \rightarrow \mathbb{Z}$ by $\phi_e(e) = 0$ and $\forall x \in E: \phi_e(x') = \phi_e(x) - 1$. By choosing a point from each equivalence class we get a function $\phi: X \rightarrow \mathbb{Z}$. Define $f: X \rightarrow \{n \in \mathbb{N} \mid n \geq 2\} = M$ by $f(x) = 2 + |\phi(x)|$, then

- (1) $\forall x \in X \ |f(x) - f(x')| = 1$, and
- (2) if E is an equivalence class then $f[E] = M$, and
- (3) if $x \in X$ and $n > f(x)$ then $\exists y \in P_x: n = f(y)$.

Now let $U_x = \{u \in P_x \mid p \mid f(u) \Rightarrow p \mid f(x) \text{ for each prime } p\}$. (i), (ii) and (iii) are immediately verified, (vi') follows from property (2) of f , and (iv') follows from property (3) of f by taking $n = m \cdot f(x) \cdot f(x')$, for if $y > x$, $f(y) = m \cdot f(x) \cdot f(x')$ then U_y intersects both U_x and $U_{x'}$.

This completes the (sketch of a) proof of theorem 9. \square

7. ON THE STRUCTURE OF H -SPACES

The property (H) (a connected set has at most two end points) was introduced by HERRLICH [14] in order to characterize connected orderable spaces. Unfortunately, in the absence of local connectedness, it is not strong enough to imply (weak) orderability since for instance each V -space satisfies (H).

In general, an H -space looks as follows: it has a linearly ordered backbone (not necessarily connected) some points of which are the base point of an embedded V -space.

V -spaces, which have an other position with respect to the backbone, can occur also; for instance filling up a gap or a jump of the backbone.

There are several inequivalent ways to define this backbone, two of which will be examined in some more detail below. First a lemma showing why a non-orderable H -space is full of embedded V_1 -spaces:

LEMMA. *Let X be an H -space and let $x \in X$. If $X \setminus x$ has more than two components: $X \setminus x = A + B + C$ then at most two of \bar{A} , \bar{B} , \bar{C} can contain a non-closed connected subset containing x . (So at least one of them is a V_1 -space with base point x .)*

(Proof: if each of \bar{A} , \bar{B} and \bar{C} would contain a nonclosed connected subset containing x then the union S of three such subsets would be a connected subset of X , such that $|\bar{S} \setminus S| \geq 3$, contradicting (H).)

For a first approximation to a backbone we take the set of all points t such that each component of $X \setminus t$ is open:

THEOREM 10. *Let X be an H -space. Let*

$$T = \{t \mid \text{all components of } X \setminus t \text{ are open}\}.$$

Then T is linearly ordered by an ordering which coincides with the separation order (or, formally, its inverse) "between" each two points of T .

PROOF. The only thing we have to show is that among each three points of T one of them separates the other two. Suppose the contrary, and let $t_i \in T$ ($i = 1, 2, 3$) be the distinct points of T such that none of them separates the other two. Let S_i be the (open) component of $X \setminus t_i$ containing $\{t_1, t_2, t_3\} \setminus \{t_i\}$. Then $\overline{S_1} \cap \overline{S_2} \cap \overline{S_3}$ is connected and has three end points (by applying lemma 4, it first follows that $\overline{S_1} \cap \overline{S_2}$ is connected and then that $\overline{S_1} \cap \overline{S_2} \cap \overline{S_3}$ is connected; similarly after interchanging the indices), a contradiction. Therefore, the separation order is a linear order. \square

This theorem immediately implies the well-known (and also some new) orderability results involving (H):

COROLLARY.

- (i) *Let X be an H -space such that either*
- (a) *each point of X is a strong cut point, or*
 - (b) *for each $x \in X$ all components of $X \setminus x$ are open, or*
 - (c) *for each $x \in X$ no component of $X \setminus x$ is closed, or*
 - (d) *X is separable metric, or*
 - (e) *X is locally countably compact,*
- then X is (weakly) orderable.*
- (ii) *Let X be an H -space such that either*
- (a) *X is locally connected, or*
 - (b) *X is locally compact, or*
 - (c) *X is T_2 and locally peripherally compact,*
- then X is strongly orderable.*

PROOF. Each of the conditions mentioned prohibit embedded V_1 -spaces as found in the lemma. Therefore, each cut point of X is a strong cut point, i.e., $X = T$ and is orderable by the previous theorem. \square

A second way to define a backbone is given by

$$T' := \{t \mid \exists s \in X: s \text{ belongs to an open component of } X \setminus t \\ \text{and } t \text{ belongs to an open component of } X \setminus s\}.$$

THEOREM 11.

- (i) T' is linearly ordered (by the separation order).
- (ii) $T \subset T'$ unless $T = \{t_0\}$ and $T' = \emptyset$ for some $t_0 \in X$.
- (iii) T does not contain gaps but may well contain jumps.

PROOF. (i) Let $t_i \in T'$ ($i = 1, 2, 3$) and suppose that none of the t_i separates the other two. Let s_i be a point associated with t_i according to the definition of T' : s_i belongs to an open component of $X \setminus t_i$ and vice versa. A contradiction follows as in the proof of theorem 10 unless one of the points, say t_3 , does not belong to an open component of the complement of one of the other two points, say t_1 . Then

$$X \setminus t_1 = \overset{A}{s_1} + \overset{B}{t_3},$$

where A is open and connected and $t_2 \in B$ since t_1 does not separate t_2 and t_3 . But now s_1, t_2, t_3 are three points in T' , none separating the other two (clearly s_1 does not separate t_2 and t_3 , since $\bar{B} = B \cup \{t_1\}$ is connected and contained in $X \setminus s_1$; next if

$$X \setminus t_2 = \overset{E}{s_1} + \overset{F}{t_3}$$

then $\bar{A} = A \cup \{t_1\} \subset E$ and t_2 would separate t_1 and t_3), and t_2 and t_3 belong to the same open component of $X \setminus s_1$ (namely that containing t_1). After repeating this argument at most three times, a contradiction follows.

(ii) If $|T| \geq 2$ then obvious $T \subset T'$. So suppose $T = \{t_0\}$. If $T' \neq \emptyset$ then T' contains two points t_1, t_2 where each belongs to an open segment of the other; but these open components cover X , hence t_0 belongs to an open component of $X \setminus t_1$, say. By definition of T it follows that t_1 belongs to an open component of $X \setminus t_0$ and, therefore, $\{t_0, t_1\} \subset T'$. Consequently, if $T \not\subset T'$ then $T' = \emptyset$.

(iii) Fix one of the two possible separation orders on T' and denote it by $<$. Suppose $T' = T_1 + T_2$ is a gap in T' , i.e., $t_1 < t_2$ for each $t_1 \in T_1$ and $t_2 \in T_2$, and T_1 has no last element and T_2 has no first element. Let for $t \in T'$ $X \setminus t = A_t + B_t$ where $\{s \in T' \mid s < t\} \subset A_t$ and $\{s \in T' \mid s > t\} \subset B_t$, and B_t is connected if possible.

[We shall see that in all relevant cases B_t can be taken connected.]

Let $A = U\{A_t \mid t \in T_1\}$ then since each A_t is open and each $\overline{A_t}$ is connected and contained in A , and T_1 has no last element, it follows that A is open and connected. If ∂A contains two points and if $t \in T_1$ is such that B_t is open and connected, then $A \cap B_t$ is connected and $\partial(A \cap B_t)$ contains three points, a contradiction.

But such a t exists: let $r \in T_1$ and s be a point such that r and s belong to open segments of each other. If $r < s$ then B_r is connected and we can let $t = r$; if $s < r$ then $s \in T_1$ and B_s is connected so that we may take $t = s$. Therefore, ∂A contains at most one point. If $\partial A = \{a\}$ then again let $t \in T_1$ be a point such that B_t is connected. By definition of T' we have $\{a, t\} \subset T'$. But then a is the first element of T_2 , impossible. Therefore, $\partial A = \emptyset$ and since X is connected it follows that $A = \emptyset$ and $T_1 = \emptyset$. \square

A third characteristic subset of the space is defined by

$$T'' := U\{\overline{C} \setminus C \mid C \subset X \text{ connected, } |\overline{C} \setminus C| = 2\}.$$

THEOREM 12. $T' \subset T''$.

This theorem immediately follows from the following lemma:

LEMMA. Let X be an H -space with end points s and t . Then for some connected $C \subset X$ we have $\overline{C} \setminus C = \{s, t\}$.



PROOF. Suppose not, and let $X \setminus \{s, t\} = \bigcup_{\alpha \in A} C_\alpha$ be the decomposition of $X \setminus \{s, t\}$ into components. First observe that if Y is an H -space and C is a connected subset of Y then $Y \setminus C$ has at most two non-open components. In particular if $A_0 = \{\alpha \mid C_\alpha \text{ not open}\}$ then $|A_0| \leq 2$ and $X' := X \setminus \bigcup_{\alpha \in A_0} C_\alpha$ is an H -space satisfying all assumptions made on X . But now

$$X' = U\{\overline{C}_\alpha \mid s \in \overline{C}_\alpha\} + U\{\overline{C}_\alpha \mid t \in \overline{C}_\alpha\}$$


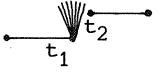
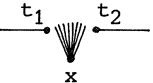
yields a contradiction. \square

Unfortunately, T'' is not weakly ordered (by the separation order) in general (see example 5 below).

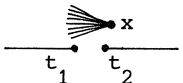
REMARKS. Below some counterexamples are given in the form of a picture of the space; I could have given a formal description of each space but all intuition is lost that way. The building blocks are as follows:

- : denotes a point.
- : denotes a copy of the unit interval.
-  : denotes a V_1 -space with base point v .
-  : denotes a V_1 -space with end point x .

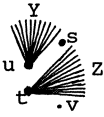
Now some examples:

1.  : a strong cut point need not belong to T'
(here $T = \{x\}$, $T' = \emptyset$).
2.  : (Here t_2 is a limit point of the components of $X \setminus t_1$):
If t_1 and t_2 are points in T' then it is not necessarily true that there is a connected set with end points t_1 and t_2 .
3.  : A point separating two points of T' need not belong to T' .

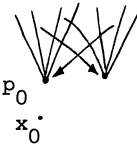
The last two examples also show that T' may have jumps. Likewise it is seen that T may have both gaps and jumps.

4.  : A V_1 -space need not be canonically embedded (i.e., within a larger V_1 -space or with base point in T').
In $X \setminus x$ there is a quasicomponent consisting of two components.

5. : $X = Y \cup Z \cup \{s,v\}$, where Y has base point u and $Y \cup Z$ is a V -space with base point $t = u'$, v is a limit point of the components of $Z \setminus t$ and s is a limit point both of the components of $Y \setminus u$ and of the components of $Z \setminus t$. Here $T = \{s,v\} = T'$, $T'' = \{s,t,v\}$; $\bar{Y} \setminus Y = \{s,t\}$ and $\overline{Y \cup Z} \setminus (Y \cup Z) = \{s,v\}$ but for no connected C is $\bar{C} \setminus C = \{t,v\}$. The topology can be defined in such a way that t does not separate s and v so that T'' is not ordered by the separation order.



6. : If $T' = \emptyset$ then X is not necessarily a V -space: attach a lowest point x_0 under a V_m -space (as follows: let p_0 be a minimal point of the V_m -space, and take the sets $P_{p_0} \setminus \{\text{finite set including } p_0\} \cup \{x_0\}$ as nbds for x_0). This gives an H -space with $T' = \emptyset$, $T = \{x_0\}$.



In view of the above examples it is rather difficult to describe the structure of an H -space relative to its backbone. Trivially we have:

$$X = T \cup \{v \in X \mid v \text{ is base point of a (non-degenerated) embedded } V_1\text{-space}\},$$

that is, X indeed consists of a backbone together with a lot of V_1 -spaces, but nothing is said about the position of those V_1 -spaces.

One may define a partial order for H -spaces (coinciding with the previously defined one if X happens to be a V -space) as follows:

$$x < y \iff y \text{ belongs to an open component } S \text{ of } X \setminus x \text{ such that } S \cap T' = \emptyset.$$

Then we have that if $P_x := \{z \in X \mid z \geq x\}$ then $|\overline{P_x} \setminus P_x| \leq 2$ and if $Z = \{z \mid |\overline{P_z} \setminus P_z| = 2\}$ then Z is an antichain. "Above" this antichain X looks decent, but between the antichain and the backbone all kinds of complications are possible, which we will not try to describe.

For some further definitions and results on (H)-like properties I refer to KOK [18], chapter III.

8. MAPS BETWEEN V -SPACES

LEMMA. Let X be a V -space, $x, x_1, x_2 \in X$, $x < x_1$ and $x < x_2$. Then x separates x_1 and x_2 iff both

- (i) x_1 and x_2 are not comparable, and
- (ii) $x = \max\{y \mid y < x_1 \text{ and } y < x_2\}$.

PROOF. Suppose x separates x_1 and x_2 . Then (i) if $x_1 < x_2$ then P_{x_1} is a connected set containing x_1 and x_2 but not x , contradiction; and (ii) if $x < y$, $y < x_1$, $y < x_2$ then P_y is such a connected set which is again impossible.

Conversely, assume (i) and (ii). Now $U_i = U\{P_z \mid x < z \leq x_i\}$ ($i = 1, 2$) are disjoint clopen sets in $X \setminus x$ containing x_1 and x_2 respectively.

[For: if $z > x$ then P_z is open, so U_i is open; also $\overline{U_i} \setminus \{x\}$ is connected hence contained in $P_x \setminus \{x\}$. But then if $u \in \partial U_i \setminus \{x\}$ it follows that P_u is open and disjoint from U_i , a contradiction.] \square

PROPOSITION 21. Let X, Y be two V -spaces and $f: X \rightarrow Y$ a continuous injection.

Then

- (i) f^{-1} : $f[X] \rightarrow X$ is isotonic (order-preserving).
- (ii) If Y is a V_0 - or V_1 -space then f is isotonic, hence an order isomorphism from X onto $f[X]$ (which is again a V_0 - or V_1 -space).

PROOF. Since $f[X]$ is connected it is again a V -space, and we may suppose $Y = f[X]$.

(i) Let $x \in X$, $y \in Y$, $y = f(x)$. We will show that $f^{-1}Q_y = Q_x$. If U is clopen in $Y \setminus y$ then $f^{-1}U$ is clopen in $X \setminus x$. Therefore, if $y_0 \in Q_y$ then

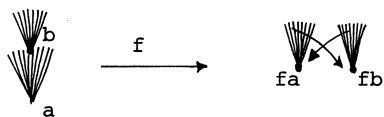
$$f^{-1}Q_y = \Omega\{f^{-1}U \mid U \text{ clopen in } Y \setminus y, y_0 \in U\}$$

is a union of quasicomponents in $X \setminus x$. But $x \notin \overline{f^{-1}Q_y}$ while x is in the closure of all but one of the components of $X \setminus x$. Therefore, $f^{-1}Q_y = Q_x$, the only remaining component.

(ii) Since f^{-1} is isotonic we have to show that if y_1 and y_2 are not comparable in Y then $f^{-1}y_1$ and $f^{-1}y_2$ are not comparable in X . And indeed, by propositions 8 and 10 and the above lemma we can find a point y separating y_1 and y_2 such that $y \leq y_1$ and $y \leq y_2$. It follows that $f^{-1}y$ separates

$f^{-1}y_1$ and $f^{-1}y_2$ and by the above lemma $f^{-1}y_1$ and $f^{-1}y_2$ are not comparable. \square

REMARK. In general f need not be isotonic:



The image of a $V1$ -space may be a Vm -space; this example also shows that f^{-1} need not preserve connectedness. Even in $V1$ -spaces

f^{-1} need not preserve connectedness: if $\{C_\alpha\}_{\alpha \in A}$ is the collection of components of $X \setminus x$ then for infinite $B \subset A$ $\{x', x\} \cup \bigcup_{\alpha \in B} C_\alpha$ may or may not be connected. Also if f is only supposed to be a surjection then neither it nor its inverse need preserve order.

CHAPTER II

ON \mathcal{W} -SPACES

0. INTRODUCTION

For connected topological spaces we consider the following properties:

- (INT) : The intersection of an arbitrary collection of connected subsets is again connected.
- (INT2): The intersection of two connected sets is connected.
- (S) : No two points are conjugated.
- (\mathcal{W}) : The boundary of each component of the complement of a nonempty connected proper subset of the space is a singleton.

(\mathcal{W}) is a rather weak property, inspired by the concept of A -set. [A closed set A in a connected T_1 space X is called an A -set provided that $X \setminus A$ is the union of a collection of open sets each bounded by a single point of A (WHYBURN [28]).] When the space is T_2 and locally connected or locally compact however, (\mathcal{W}) is sufficient to imply (S) and (INT).

Since (\mathcal{W}) is weaker than (S) and (INT) we study it first in order to apply the results in the next chapter, which will be devoted to (S) and (INT).

1. DEFINITION AND SEPARATION PROPERTIES

H. KOK [18] has given an equivalent definition:

PROPOSITION 1. X is a \mathcal{W} -space iff for any two disjoint connected sets $A, B \subset X$ we have $|\bar{A} \cap \bar{B}| \leq 1$.

PROOF. Let X be a \mathcal{W} -space and let $\{x, y\} \subset \bar{A} \cap \bar{B}$. If C is the component of $X \setminus A$ containing B then $\{x, y\} \subset \partial C$; hence $x = y$. Conversely, let S be a com-

ponent of $X \setminus C$, $\emptyset \neq C \neq X$, C connected. Since $X \setminus S$ is connected, the hypothesis implies $|\overline{S} \cap \overline{X \setminus S}| = |\partial S| \leq 1$. But if $\partial S = \emptyset$ then X would be disconnected. \square

From this characterization one sees immediately that (W) is hereditary:

PROPOSITION 2. *A connected subspace of a W -space is a W -space.* \square

Dropping for a moment the restriction that all spaces considered are (T_1) , we investigate the separation properties of a W -space. To this end recall the following definitions:

A space X is said to satisfy:

(T_0) iff for all $x, y \in X$: if $\{x, y\} \subset \{x\}^- \cap \{y\}^-$ then $x = y$.

(T_Y) iff for all $x, y \in X$: if $x \neq y$ then $|\{x\}^- \cap \{y\}^-| \leq 1$.

(T_D) iff for each $x \in X$: $\{x\}' = \{x\}^- \setminus \{x\}$ is closed.

(T_F) iff given a point $x \in X$ and a finite subset $F \subset X$ with $x \notin F$ we have

$$x \notin \overline{F} \text{ or } F \cap \{x\}^- = \emptyset.$$

(T_1) iff for each $x \in X$: $\{x\}' = \emptyset$, i.e., $\{x\}^- = \{x\}$.

One has the following implications:

$$(T_1) \Rightarrow (T_Y) \Rightarrow (T_F) \Rightarrow (T_0)$$

and

$$(T_1) \Rightarrow (T_D) \Rightarrow (T_0).$$

(T_F) is equivalent to: each point in the derived set of another point is closed.

(T_D) is equivalent to: each point is the intersection of an open and a closed set.

(For proofs and more properties like these between (T_0) and (T_1) see C.E. AULL & W.J. THRON [4].)

It is easily seen that a W -space is T_0 :

THEOREM 1a. $(W) \Rightarrow (T_0)$.

PROOF. Let p, q be distinct points of X and suppose $q \in \{p\}^-$. Let C be the component of $X \setminus q$ containing p . Then $q \in \{p\}^- \subset \overline{C}$ and $q \notin C$, hence $q \in \partial C$ so $\{q\} = \partial C$. Since ∂C is closed, $\{q\}$ is closed, and in particular $p \notin \{q\}^-$.

A \mathcal{W} -space need not satisfy (T_1) : Let $X = \mathbb{N}$ with topology $\mathcal{O} = \{U \mid U = \emptyset \text{ or } 1 \in U\}$. Then X is a connected locally \mathcal{W} -space satisfying (INT) . X is locally compact in the sense that X has a basis of compact open sets. But X does not satisfy (T_1) since $\{1\}$ is not closed. However, we do have:

THEOREM 1b. $(\mathcal{W}) \Rightarrow (T_D)$.

PROOF. We shall show that $\{x\}$ is open in $\{x\}^-$ for each $x \in X$. Since (\mathcal{W}) is hereditary (see proposition 2), $Y := \{x\}^-$ is also a \mathcal{W} -space. Now each nonempty open subset of Y contains x and is connected. Let C be an arbitrary component of $Y \setminus x$. Then, by (\mathcal{W}) , $\partial_Y C = \{c\}$ for some c . If for some such C $c = x$ then $Y = \{x\}^- = \overline{\partial_Y C} = \partial_Y C = \{x\}$ so that $\{x\}$ is open in Y . If on the other hand $c \neq x$ for all C then $C \setminus c = C \setminus \partial_Y C$ is open in Y and does not contain x ; hence, for all C , $C \setminus c = \emptyset$; that is, $Y \setminus x$ is totally disconnected. Therefore, if $Y \setminus x$ is also connected, it is a singleton $\{c\}$, and since $\{c\} = \partial_Y \{c\}$ is closed it follows that $\{x\}$ is open in Y . If $Y \setminus x$ is not connected let

$$Y \setminus x = \underset{a}{A} + \underset{b}{B} .$$

Then $A \cup \{x\}$ is open in Y (for, if B is not closed in Y , then $\bar{B} = B \cup \{x\}$, and hence A is open in Y ; but this is impossible since $x \notin A$) and, likewise, $B \cup \{x\}$ is open in Y so $(A \cup \{x\}) \cap (B \cup \{x\}) = \{x\}$ is open in Y . Hence for each $x \in X$, $\{x\}$ is open in $\{x\}^-$, i.e., X is a T_D -space. \square

Finally, from proposition 1 we immediately get:

THEOREM 1c. $(\mathcal{W}) \Rightarrow (T_Y)$. \square

2. RELATION WITH (INT_2) AND (S)

In this section we show that (INT_2) and (S) imply (\mathcal{W}) .

THEOREM 2. *In connected T_1 -spaces (INT_2) implies (\mathcal{W}) .*

PROOF. Let X be a T_1 -space and suppose $p, q \in \partial T$ where T is a component of $X \setminus C$, C a connected subset of X . Then $T \cup \{p, q\}$ and $(X \setminus T) \cup \{p, q\}$ are connected (since $X \setminus T$ is connected) and by (INT_2) $\{p, q\}$ is connected, and hence $p = q$. \square

REMARK. In T_F -spaces (*INT2*) implies that the boundary of a component of the complement of a connected subset of X consists of at most two points.

PROOF. In the above proof we found that $p, q \in T \rightarrow \{p, q\}$ connected. But if $\{p, q\}$ is connected $p \in \{q\}^-$ or $q \in \{p\}^-$ and by (T_F) p or q is closed (respectively). Now suppose ∂T contains three points p, q, r . Then $\{p, q\}$, $\{p, r\}$ and $\{q, r\}$ are connected and also at least two of the three points are closed; but this is a contradiction. \square

In T_0 - or T_D -spaces (*INT2*) need not imply (W) as is shown by the following example:

Let $X = \{0, 1, 2\}$ and $\mathcal{O} = \{\emptyset, \{0\}, \{0, 1\}, X\}$, then X with topology \mathcal{O} is a T_D -space satisfying (*INT*) (since all subsets are connected) but not (W) (for $\{1, 2\}$ is the component of $X \setminus \{0\}$, but $\partial\{1, 2\} = \{1, 2\}$ contains more than one point).

Also, if $X = [0, 1] \cup \{p\}$ with Euclidean topology on $I = [0, 1]$ and basic nbds $(1-\epsilon, 1] \cup \{p\}$ for p ($0 < \epsilon \leq 1$), then each point in X is closed except 1, and $\{1\}^- = \{1, p\}$. Therefore, X satisfies (T_D) and (T_Y) and (*INT2*) but X does not satisfy (W) .

THEOREM 3. $(S) \Rightarrow (T_2)$ and (W) .

PROOF. For the (trivial) proof of $(S) \Rightarrow (T_2)$ we refer to chapter III, proposition 1.

Suppose $C \subset X$ is connected, T is a component of $X \setminus C$ and $\{p, q\} \subset \partial T$. By (S) there is a point r separating p and q . Since $T \cup \{p, q\}$ is connected it follows that $r \in T$. On the other hand $(X \setminus T) \cup \{p, q\}$ is connected and hence $r \notin T$. Contradiction. \square

3. SOME OTHER PROPERTIES OF W -SPACES

PROPOSITION 3. If X is a connected W -space and p_1, \dots, p_n are distinct end points of X , then $X \setminus \{p_1, \dots, p_n\}$ is connected.

PROOF. By proposition 2 it suffices to prove that p_2, \dots, p_n are end points of $X \setminus p_1$. Suppose that, to the contrary, for instance p_2 is a cut point of $X \setminus p_1$, i.e., $X \setminus \{p_1, p_2\} = A + B$ where both A and B are nonempty. Then $A \cup \{p_1\}$ and $B \cup \{p_2\}$ are connected, hence $B \cup \{p_2\}$ is the (only) component of

$X \setminus (A \cup \{p_1\})$, and by (W) $\partial(BU\{p_2\})$ must be a singleton. But $\partial(BU\{p_2\}) \supset \{p_1, p_2\}$. Contradiction. \square

THEOREM 4. *Let X be a connected W -space. Then among every three distinct points of X there is at least one which belongs to a connected set separating the other two.*

PROOF. Suppose none of the three distinct points p, q, r belongs to a connected set separating the other two. If q and r belong to different components C_1 and C_2 of $X \setminus p$ then $X \setminus (C_1 \cup C_2)$ is connected and separates q and r , contrary to the hypothesis; hence q and r belong to the same component C_p of $X \setminus p$. Let $\partial C_p = \{p_1\}$.

If $p_1 = q$ then $C_p \setminus q$ is clopen in $X \setminus q$ and hence q separates p and r , which is contrary to the hypothesis. Therefore, $p_1 \neq q$ and, likewise, $p_1 \neq r$.

Now let C_q be the component of p_1 and r in $\overline{C_p} \setminus q$. (Again p_1 and r belong to the same component of $\overline{C_p} \setminus q$ since otherwise the - connected - complement (in C_p) of the component K of r in $\overline{C_p} \setminus q$ would separate p and r . In fact if $p \neq p_1$ then $\overline{C_p} = C_p$ and $p_1 \in C_p \setminus K$ so that $X \setminus (C_p \setminus K) = (X \setminus C_p) + K$; on the other hand if $p = p_1$ then K is the component of r in $C_p \setminus q$ so that $C_p \setminus K$ is connected and again it follows that $X \setminus (C_p \setminus K) = (X \setminus C_p) + K$ is a separation of p and r by a connected set containing q .)

If $p \neq p_1$ then $X \setminus C_q = (X \setminus C_p) + (C_p \setminus C_q)$ so that r belongs to a connected set (C_q) separating p and q .

Therefore, assume $p = p_1$. Since $(X \setminus C_p) \cup C_q$ is a component of $X \setminus q$ we may set $\partial((X \setminus C_p) \cup C_q) = \{q_1\}$, where $q_1 \in \overline{C_q}$.

If $q_1 = p$ then p separates q and r :

$$X \setminus p = (X \setminus \overline{C_p}) + (\overline{C_p} \setminus C_q) + (C_q \setminus p).$$

If $q_1 = r$ then r separates p and q :

$$X \setminus r = ((X \setminus \overline{C_p}) \cup (C_q \setminus r)) + \overline{C_p} \setminus C_q.$$

Hence we may suppose $q_1 \neq p, r$.

If $q_1 \neq q$ then $q_1 \in C_q$ and it follows that $C_q \setminus p$ is connected (for otherwise $C_q \setminus p = \underset{q_1 \neq \emptyset}{E} + F$ and F would be clopen in $X \setminus p$ which is impossible since F is strictly contained in C_p). But this means that $C_q \setminus p$ is a connected set containing r and separating p and q , contrary to the hypothesis.

Therefore, we have to assume that $q_1 = q$, that $C_q \setminus p$ is not connected and that $q_1 = q$ belongs to the closure of each clopen subset of $C_q \setminus p$. But this means that $\overline{C_q \setminus p} = (C_q \cup \{q\}) \setminus p$ is connected so that $C = C_q \cup \{q\}$ is a connected set with two end points p and q while $C \setminus \{p, q\}$ is disconnected. But this contradicts proposition 3 (since (W) is hereditary). \square

THEOREM 5. In connected T_1 -spaces (Va) implies (W) .

PROOF. Let C be a (nonempty) connected subset of a V_0 -space X . For $x \in X \setminus C$ let C_x be the component of $X \setminus C$ which contains x .

(i) Let C_x be a component of $X \setminus C$ containing a point x such that $y < x$ for some $y \in C$. Then $P_x = \{z \mid z \geq x\} \subset C_x$ (for: P_x is connected and $P_x \cap C = \emptyset$ since x separates each point of $P_x \setminus x$ from the points below x). Next we show that C_x is open. Suppose, to the contrary, that $z \in C_x \setminus C_x^\circ$; then (since P_z is open) $P_z \not\subset C_x$, hence $P_z \cap C \neq \emptyset$ (since P_z is connected and otherwise would be contained in C_x). Let $p \in P_z \cap C$. Since C_x is connected and $y' < y < x$ and $x \in C_x$, $y' \notin C_x$ it follows that $y' \notin C_x$. But $X \setminus y' = P_y + X \setminus \overline{P_y}$ and C_x meets P_y hence $C_x \subset P_y$. (If X has a lowest point x_0 and $y = x_0$ then y' is not defined but obviously $C_x \subset P_y = X$ holds also in this case.) Now $p > z > y$, p and y in C and $z \in C_x \subset X \setminus C$, a contradiction. This proves that C_x is open. Since each connected set in X has at most one end point it follows that $\overline{C_x} = C_x \cup \{z\}$ for some $z \in X$ (and then C_x is a - clopen - component of $X \setminus z$); hence $\partial C_x = \{z\}$, as was to be proved.

(ii) Now assume C_x is a component of $X \setminus C$ such that for no point $z \in C_x$ and $y \in C$ we have $y < z$. Let z be a boundary point of C_x (such a point of course exists since X is connected). Again $P_z \not\subset C_x$ and hence $P_z \cap C \neq \emptyset$; also $z \notin C$ since, otherwise, z would be a point in C smaller than some point in C_x . Therefore, $C \subset P_z$.

If y is another boundary point of C_x , then also $C \subset P_y$ and in particular $P_y \cap P_z \neq \emptyset$, so that y and z are comparable. Therefore, we may suppose $y < z$.

Now P_z is a component of $X \setminus z'$ and hence $X \setminus P_z$ is connected. (If $z = x_0$ then z' is undefined but certainly $X \setminus P_z = \emptyset$ is connected.) Moreover, since $C \subset P_z \setminus z$ it follows that $C \subset C^*$ where C^* is a component of $X \setminus z$; also $z \in C_x$ (since $z \notin C$ and $z \in \overline{C_x}$) so that $X \setminus C^*$ is a connected set containing z and disjoint from C ; hence $X \setminus C^* \subset C_x$. But $\overline{C^*} = C^* \cup \{z\}$ hence $y \in X \setminus \overline{C^*} \subset (C_x)^\circ$, a contradiction. Hence also in this case $\partial C_x = \{z\}$. \square

REMARK. It is easy to verify that no V_m -space satisfies (W) , so $(Va) \Rightarrow (V)$ and (W) .

From the foregoing one might have got the impression that (W) is an extremely weak property. In conjunction with local connectedness, however, it implies (S) (in Hausdorff spaces):

THEOREM 6. *A locally connected Hausdorff space satisfying (W) is treelike (i.e., satisfies (S)).*

PROOF. Let p, q be two points of X and let U and V be disjoint connected neighbourhoods of p resp. q ; let A be the component of q in $X \setminus \bar{U}$. Then A is open, hence $q \notin \partial A$. Therefore, if $\partial A = \{r\}$ then r separates p and q . \square

Concerning the relation between (W) , (S) and (INT) : in the next chapter we will see $(INT) \Rightarrow (S)$, and we saw already that both (S) and $(INT2) + (T_1)$ imply (W) . In [28] WHYBURN proves for connected and locally connected T_1 -spaces: $(S) \Leftrightarrow (T_2) + (INT2) \Leftrightarrow (INT)$. By theorem 6 it then follows that the conditions (W) , $(INT2)$, (S) and (INT) are all equivalent in connected locally connected Hausdorff spaces. In the next section we shall prove the equivalence of (W) and $(INT2)$ in connected locally connected T_1 -spaces. WHYBURN gives examples to show that the local connectedness condition is essential, but gives no example of a connected and locally connected T_1 -space satisfying $(INT2)$ but not (S) . Such spaces do, however, exist as will be shown by the next example.

Let X be the set of natural numbers \mathbb{N} with the following topology: if $\{B_\alpha\}_{\alpha \in A}$ is a free ultrafilter on \mathbb{N} we take for open sets the empty set and the elements B_α of the ultrafilter. X is (T_1) , connected and locally connected and satisfies (W) and $(INT2)$ but not (S) . For each $A \subset X$ containing at least two points the following three conditions are equivalent:

- (i) $A^\circ = \emptyset$,
- (ii) $A^- = X$,
- (iii) A is connected.

In particular each closed connected set is a singleton or coincides with X . [Note: this space satisfies the (H_1) separation axiom between (T_1) and (T_2) as introduced and studied by WHYBURN [28].]

The above example shows the existence of a T_1 -space without cut points satisfying (W). Under assumption of the continuum hypothesis (CH) it is possible to show the existence of a nontrivial T_2 -space with this property, for we have:

THEOREM 7.

- (i) *Each biconnected set is a W-space. Conversely*
- (ii) *a W-space without cut points containing at least three points is a biconnected set without dispersion point.*

[REMARK. Assuming (CH) E.W. MILLER [23] proved the existence of a non-degenerated biconnected set without dispersion point in the plane.]

PROOF. Remember that a biconnected set C is a connected set which is not the union of two nondegenerate proper connected subsets, while a dispersion point p of a set C is a point such that $C \setminus p$ is totally disconnected. Each connected space with a dispersion point is biconnected but as noted above there exist biconnected sets without dispersion points.

(i) Let $C \subset X$ be connected and let S be a component of $X \setminus C$. Since $X \setminus S$ is connected it follows from $X = S \cup (X \setminus S)$ and the biconnectedness of X that either $|S| = 1$ or $|X \setminus S| = 1$. In the first case we have $|\partial S| = |S| = 1$ while in the second case $|\partial S| = |C| = 1$, hence X satisfies (W).

(ii) Suppose $X = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$, C_1 and C_2 connected. Then C_1 is the (only) component of $X \setminus C_2$ hence $\partial C_1 = \{p\}$. Since X has no cut points while $X \setminus p = (C_1 \setminus p) + (C_2 \setminus p)$ we have either $C_1 = \{p\}$ or $C_2 = \{p\}$. Also, since $|X| \geq 3$ a dispersion point is a cut point, so X does not possess a dispersion point. \square

COROLLARY. *Let X be connected and $|X| \geq 3$. X is a W-space without cut points iff X is a biconnected set without dispersion point.*

PROOF. It is easily seen that a cut point of a biconnected set must be a dispersion point. \square

THEOREM 8. *A locally compact, connected Hausdorff space X satisfying (W) is treelike.*

PROOF. Let p and q be two points of X . Then we have to find a point separating p and q . Let V and W be two disjoint compact neighbourhoods of p and q , respectively.

(i) Suppose that the component P of V containing p does not meet ∂V . Since P is the intersection of all clopen (in P) sets containing p , there is for each $x \in \partial V$ a separation $V = A_x + B_x$ of V between p and x . ∂V is compact

and A_x is open in V hence finitely many A_x cover ∂V . Let U be the intersection of the corresponding B_x . Then U is clopen in V and does not meet ∂V hence U is clopen in X . Contradiction.

In a similar way it is seen that the component Q of W containing q meets ∂W .

(ii) We now can exhibit a point r separating p and q as follows:

Let C be the component of $X \setminus Q$ containing p (clearly $C \supset P$). By (W) we have $\partial C = \{r\}$ and r separates p and q unless $r=p$ or $r=q$.

If $r = p$ then $(X \setminus C) \cup \{p\}$ is a closed, connected and locally compact W -subspace of X , and in this subspace $(V \setminus C) \cup \{p\} = V \setminus (C \setminus \{p\})$ is a compact neighbourhood of p . Moreover, the component containing p of this neighbourhood is $\{p\}$ and hence does not meet the boundary of $(V \setminus C) \cup \{p\}$ in $(X \setminus C) \cup \{p\}$. But then - according to (i) - $(X \setminus C) \cup \{p\}$ cannot be connected. Contradiction.

If $r = q$ then $C \cup \{q\}$ is a closed, connected and locally compact W -subspace of X , and in this subspace $(W \setminus C) \cup \{q\}$ is a compact nbd of q . Since the component of this nbd containing q is $\{q\}$ we again arrive at a contradiction. \square

We have seen that both local connectivity and local compactness are sufficient to imply that a W -space is treelike. Rimcompactness, however, is not as is shown by the following example:

Let

$$X = \{(x,y) \in \mathbb{R}^2 \mid (x=0 \text{ and } -1 < y < +1 \text{ and } y \in \mathbb{Q}) \\ \text{or } (x > 0 \text{ and } y = \sin \frac{1}{x})\}$$

with subspace topology of the plane. X is $(T_5), (CII), (INT2)$ and hence (W), locally peripherally compact but not locally compact or treelike.

Also the condition (T_2) is necessary in the previous theorem:

Let

$$X = \{(x,y) \in \mathbb{R}^2 \mid (y=0 \text{ and } 0 < x \leq 1) \\ \text{or } (y=1 \text{ and } 0 \leq x < 1) \\ \text{or } (y=2 \text{ and } 0 < x < 1)\},$$

with basic neighbourhood system:

$$U_i(a,0) = \{(a,0)\} \cup [a - \frac{1}{i}, a) \times \{0,1\}$$

$$U_i(a,1) = \{(a,1)\} \cup (a, a + \frac{1}{i}] \times \{0,1\}$$

$$U_i(a,2) = \{(a,2)\} \cup ([a - \frac{1}{i}, a) \cup (a, a + \frac{1}{i}]) \times \{0,1\}$$

($i \in \mathbb{N}$, sufficiently large so that $U_i(p) \subset X$).

These sets are Hausdorff and compact, that is, X satisfies the following version of local compactness: X has a base of compact (T_2) sets. [Of course the $U_i(p)$ are not open.]

X is connected and if $Y = \pi_2^{-1}[\{0,1\}]$ and $Z = \pi_2^{-1}[\{2\}]$ then: Y is homeomorphic with the double-arrow space (cf. ALEXANDROFF [2]), and Z is closed and discrete. Y is the collection of all end points of X , and Z is the collection of all cut points in X . X satisfies (W) and is locally orderable, locally (T_2) and (T_1) . X is not locally connected or locally peripherally compact and does not satisfy $(INT2)$ or (S) .

A variant of this example showing that (T_2) is essential also in theorem 6 is the following:

Let X be the same set as in the previous example but with basic neighbourhood system:

$$V_i(a,0) = \{(a - \frac{1}{i}, 1), (a,0)\} \cup (a - \frac{1}{i}, a) \times \{0,1,2\}$$

$$V_i(a,1) = \{(a,1), (a + \frac{1}{i}, 0)\} \cup (a, a + \frac{1}{i}) \times \{0,1,2\}$$

$$V_i(a,2) = \{(a - \frac{1}{i}, 1), (a + \frac{1}{i}, 0)\} \cup \\ \cup (a - \frac{1}{i}, a + \frac{1}{i}) \times \{0,1,2\} \setminus \{(a,0), (a,1)\}.$$

(The intelligent reader will have no difficulty in deciding in which cases the symbol $(,)$ denotes an ordered pair and when an open interval.)

X is compact and hence locally peripherally compact, connected, locally connected, (T_1) and (CI) and satisfies (W) and $(INT2)$, but is not treelike.

4. THE LOCALLY CONNECTED NON-HAUSDORFF CASE

In this section we answer a question of H. KOK by proving

THEOREM 9. *Let X be a connected, locally connected (T_1) (W) -space. Then X satisfies $(INT2)$.*

Before giving the proof we first present two typical examples.

1. Let $X = (\mathbb{N}, U \cup \{\emptyset\})$ where U is a free ultrafilter on \mathbb{N} . Then X is connected, locally connected and (T_1) (for: each two non-empty open sets intersect and no point of X is open).

$C \subset X$ is connected iff $\bar{C} = X$ or $|C| = 1$ (for: $\bar{C} = X$ is equivalent with $C = C^\circ \neq \emptyset$, while $\bar{C} \neq X$ implies that C is totally disconnected).

It follows that X satisfies (W) , $(INT2)$ and $(INTC)$.

Moreover, X has no cut points.

2. Let $X = \{(x,y) \in \mathbb{R}^2 \mid (x \leq 0 \ \& \ y^2 = 1) \vee (x > 0 \ \& \ y = 0)\}$, with Euclidean topology on these three rays and basic nbds in $(0, \pm 1)$:

$$U_n((0,1)) = \{(x,y) \in X \mid (-\frac{1}{n} < x \leq 0 \ \& \ y = 1) \vee (0 < x < \frac{1}{n})\},$$

(for $n \in \mathbb{N}$), and

$$U_n((0,-1)) \text{ likewise.}$$

Then X is connected, locally connected, (T_1) and satisfies (W) , $(INT2)$ but not $(INTC)$.

Each point of X is a strong cut point.

It will be seen that the overall structure of X is something as a tree-like space, while the connected cyclic elements are ultrafilter based parts like in the first example, and the disconnected cyclic elements are the boundary of an open connected set with many cut points like in the second example.

All this will be made precise in the course of the proof.

PROPOSITION 4. *Let X be a connected and locally connected (W) -space. Let C be a closed connected subset and let S be a connected subset of X . Then $C \cap S$ is connected.*

PROOF. Suppose $C \cap S = A_1 + A_2$, A_1 and A_2 nonempty. Let $X \setminus C = \sum C_\alpha$, where

each C_α is open and connected. We then show that

$$S = A_1 \cup (S \cap \bigcup \{C_\alpha \mid \partial C_\alpha \subset A_1\}) + A_2 \cup (S \cap \bigcup \{C_\alpha \mid \partial C_\alpha \subset A_2\}).$$

If we have verified this equation then a contradiction is immediate since S is connected.

- (i) Obviously S contains the right-hand side.
- (ii) Conversely if $s \in S \setminus C$ then $s \in C_\alpha$ for some α . Now certainly the singleton ∂C_α is contained in C . Also, since S is connected and intersects both C_α and $X \setminus C_\alpha$, it follows that S contains ∂C_α . Consequently, $\partial C_\alpha \subset S \cap C = A_1 + A_2$. It follows that s is an element of the right-hand side. Therefore, S equals the right-hand side.
- (iii) Obviously both summands of the right-hand side are disjoint and nonempty.
- (iv) Finally we have to justify the $+$.

A_1 and A_2 are separated and so are the open sets $\bigcup \{C_\alpha \mid \partial C_\alpha \subset A_1\}$ and $\bigcup \{C_\alpha \mid \partial C_\alpha \subset A_2\}$. Suppose $a_2 \in A_2 \cap \bigcup \{C_\alpha \mid \partial C_\alpha \subset A_1\}$.

Let U be a connected nbd of a_2 disjoint with A_1 . Now, for some C_α with $\partial C_\alpha \subset A_1$, U intersects C_α as well as its complement but not its boundary. Contradiction.

By symmetry this means that the above equation is indeed a separation of S . \square

Note that this proposition is almost what we want; only the restriction that C must be closed has to be removed. Still in spaces like the first example this proposition is nearly vacuous: there the only closed connected sets are \emptyset , singletons and X .

COROLLARY. *Let X be a connected and locally connected (W) -space. Let Y be a closed connected subspace. Then Y is locally connected and (W) .*

PROOF. If \mathcal{U} is a basis of open connected sets for X then $\{U \cap Y \mid U \in \mathcal{U}\}$ is a basis of open connected sets for Y . Also Y satisfies (W) since (W) is hereditary for connected subspaces. \square

We now look at the situation of the first example:

PROPOSITION 5. *Let X be a connected and locally connected (W) -space without cut points. Then the topology on X (minus the empty set) is an ultrafilterbase, and X satisfies $(INT2)$.*

PROOF. (i) If U and V are two disjoint nonempty open and connected subsets of X then by (W) the component S containing V in $X \setminus U$ is bounded by a singleton $\{p\}$. But then p separates U and V , i.e., p is a cut point. Therefore, $U \cap V \neq \emptyset$ and it follows that the topology-minus-the-empty-set \mathcal{O} is a filter-base.

(ii) To show that \mathcal{O} is an ultrafilterbase, suppose that P is a set such that each nonempty open set intersects both P and $X \setminus P$. This implies that both P and $X \setminus P$ are connected and dense (for if $P = P_1 + P_2$ then $P_1 = U_1 \cap P$ and $P_2 = U_2 \cap P$ for some open sets U_1, U_2 in X ; but now $(U_1 \cap U_2) \cap P = P_1 \cap P_2 = \emptyset$, a contradiction). But this contradicts (W): the boundary of the only component of $X \setminus P$ would be the whole of X instead of a singleton. Therefore, for each nonempty P either P or $X \setminus P$ is in the filter generated by \mathcal{O} . Thus \mathcal{O} is an ultrafilterbase. Let \mathcal{U} be the ultrafilter generated by it. We can characterize the connected subsets of X as follows:

C is connected iff $C \in \mathcal{U}$ or $|C| \leq 1$.

For: if $C \in \mathcal{U}$ then C is connected by the same argument used above for P . Conversely, let C be connected and $C \notin \mathcal{U}$. There is some open connected set U disjoint with C . By (W) and the fact that there are no cut points it follows that the connected set C is totally disconnected, i.e., is a singleton or empty.

From this characterization it is obvious that X satisfies (INT2).

Now the PROOF OF THE THEOREM:

Let C_1 and C_2 be connected and suppose that there exists a separation

$$C_1 \cap C_2 = \begin{matrix} A & + & B \\ a & & b \end{matrix} .$$

(i) First we show that we may suppose a and b to be conjugate points of X . First observe, that $E(a,b) \subset A + B$. Now, if $E(a,b) \neq \emptyset$, then let $E = E(a,b) \cap A$ and $F = E(a,b) \cap B$. For each $u \in E(a,b)$ we take a separation $X \setminus u = A_u + B_u$ where B_u is connected (this is possible: X is (T_1) and locally connected). If $E \neq \emptyset$ let $U = \bigcup_{u \in E} A_u$. U is open and connected and contains a . If S is the component of b in $X \setminus U$ then $\partial S = \{s\}$ by (W). $s \notin U$ (since U is open) but $s \in \bar{U}$ (since X is locally connected and S is maximal connected in $X \setminus U$). If $s \in B$ then a connected nbd of s disjoint with A intersects A_u and $X \setminus A_u$ for some $u \in E$ without containing u . This is impossible, thus $s \in A$, i.e., s is the last element (in the separation ordering) of

$S(a,b) \cap A$. On the other hand if $E = \emptyset$ then a is the last element of $S(a,b) \cap A$. Therefore, in each case $S(a,b) \cap A$ has a last element, and by symmetry $S(a,b) \cap B$ has a first element; clearly these two points are conjugate. (In fact we showed the following: if $S(a,b)$ is not connected then it contains a jump.)

(ii) Suppose now that a and b are conjugate. Let $Y = \{y \mid y \text{ conjugate to both } a \text{ and } b\}$. (Then $\{a,b\} \subset Y$.) For $u \in X$ let

$$X \setminus u = \begin{matrix} A_u & + & B_u \\ \text{a or b} & & \end{matrix},$$

where A_u is connected and contains a and b (unless $u = a$ or $u = b$), and B_u is possibly empty. Then

$$Y = \bigcap_u (X \setminus B_u) = \bigcap_u \overline{A_u}$$

is closed and contains a and b .

A. Suppose Y is connected.

In this case Y is a connected, locally connected \mathcal{W} -space by the corollary above. Moreover, Y has no cut points: if $y \in Y$ then $Y \setminus y \subset A_y$ and $Y \setminus y = Y \cap A_y$ is connected by the first proposition above. But then $(Y \cap C_1) \cap (Y \cap C_2) = (Y \cap A) + (Y \cap B)$ gives a contradiction since we already established (INT2) for the case where the space has no cut points.

B. Y is not connected: $Y = Y_1 + Y_2$.

Let $X \setminus Y = \bigcup_\alpha T_\alpha$ be the decomposition of $X \setminus Y$ into open components. Since X is connected the statement

$$X = (Y_1 \cup \bigcup \{T_\alpha \mid \partial T_\alpha \subset Y_1\}) + (Y_2 \cup \bigcup \{T_\alpha \mid \partial T_\alpha \subset Y_2\})$$

must be false. The right-hand side is indeed a disconnected set, hence it is different from X ; this means that there is a component T of $X \setminus Y$ such that ∂T intersects both Y_1 and Y_2 .

Now consider two different (open) components T' and T'' of $X \setminus Y$. First of all from (W) it follows that T' and T'' are separated by a point t :

$X \setminus t = A_t + B_t$ where for instance $T' \subset B_t$. We find

$$\begin{matrix} X \setminus t & = & A_t & + & B_t \\ & & Y \setminus t & & T' \end{matrix}$$

Since $\emptyset \neq \partial T' \subset Y$ we conclude that $\partial T' = \{t\}$.

This proves that of any two components T_α of $X \setminus Y$ at most one has a boundary

that is not a singleton. Therefore, T must be the only component of $X \setminus Y$ such that $|\partial T| > 1$.

Now let Y_0 be any component of Y . Then it follows from (W), that there is a point t separating T and $Y_0 \setminus t$. But if $X \setminus t = A_t + B_t$ then $Y \setminus t \subset A_t$, hence - since $\partial T \cap (Y \setminus t) \neq \emptyset - T \subset A_t$, that is, T and $Y_0 \setminus t$ are contained in the same component of $X \setminus t$ which means that $Y_0 \setminus t = \emptyset$.

This proves that Y is totally disconnected.

Also $Y = \partial T$ (and so $\overline{T} = T \cup Y$). For certainly $\partial T \subset Y$. On the other hand, if $y \in Y \setminus \partial T$ then y has an open connected nbd U disjoint from T ; let u be a point separating U from T : $X \setminus u = E + F$, $U \subset E$, $T \subset F$, then $Y \setminus u \subset E$ and $\partial T \subset Y \cap \overline{F} \subset \{u\}$ which contradicts the definition of T .

Next if C is connected and $C \cap T = \emptyset$ then $C \cap Y$ is connected for otherwise $C \cap Y = E_1 + E_2$ and

$$C = (E_1 \cup U\{T_\alpha \cap C \mid \partial T_\alpha \subset E_1\}) + (E_2 \cup U\{T_\alpha \cap C \mid \partial T_\alpha \subset E_2\})$$

would be a separation of C .

Now look at the assumption $C_1 \cap C_2 = \frac{A}{a} + \frac{B}{b}$. Since C_1 and C_2 each intersect Y in at least two points and since Y is totally disconnected it follows from the previous observation that both C_1 and C_2 must intersect T ; choose $t_i \in C_i \cap T$ ($i = 1, 2$).

While C_1 and C_2 intersect T , $C_1 \cap C_2$ does not:

CLAIM. $(A \cup B) \cap T = \emptyset$.

For suppose $x_0 \in (A \cup B) \cap T = C_1 \cap C_2 \cap T$. x_0 is not conjugate to a and b so there is a point x_1 separating x_0 from a and b (observe that if x_1 separates a and x_0 then $x_1 \neq b$). Necessarily $x_1 \in C_1 \cap C_2 = A \cup B$. Also $x_1 \in T$ since $T \cup \{a, b\}$ is connected and contains x_0 , a and b .

We continue by transfinite induction, so suppose that x_α has been defined for $\alpha < \alpha_0$ such that x_α separates x_β from a and b for $\beta < \alpha$, $x_\alpha \in C_1 \cap C_2 \cap T$. If α_0 is a successor then as above we can take a point x_{α_0} separating x_{α_0-1} from a and b . If α_0 is a limit ordinal then consider

$$\bigcup_{\alpha < \alpha_0} B_{x_\alpha} =: E.$$

E is open and connected (since $\bigcup_{\alpha < \alpha_0} B_{x_\alpha} = \bigcup_{\alpha < \alpha_0} \overline{B_{x_\alpha}}$). If $a \notin \partial E$ then by (W)

there is a point x_{α_0} separating E from a and then also from b , i.e., $b \notin \partial E$. If $a \in \partial E$ (and thus also $b \in \partial E$) then the construction is finished. We then continue as follows: let U be an open connected nbd of a not intersecting B and let V be an open connected nbd of b disjoint with A . Both U and V contain all points x_α from some index onward. Since - by construction - $\{x_\alpha\}_\alpha \subset S(a, x_0) \cap S(b, x_0) \subset C_1 \cap C_2$, we have $\{x_\alpha \mid \alpha < \alpha_0\} \cap U \cap V \subset (A \cup B) \cap U \cap V = \emptyset$, a contradiction.

This proves the claim.

Now let $S_i := U\{B_t \mid t_i \in B_t\}$ ($i = 1, 2$), then S_1 and S_2 are nonempty, open, connected and contained in T . (Nonempty since $t_i \in T$ so $t_i \notin Y$ and $t_i \in S_i$; connected since $S_i = U\{\overline{B_t} \mid t_i \in B_t\}$: for each $t \in T$ there is a t' separating t from a and b , and $\overline{B_{t'}} \subset B_{t'}$; contained in T since it intersects T but not Y .)

If $S_1 \cap S_2 = \emptyset$ then some point s separates S_1 and S_2 , hence either S_1 or S_2 is contained in B_s . But from $S_2 \subset B_s$ it follows by definition of S_2 that $S_2 = B_s \subset \overline{B_s}$, a contradiction.

Therefore $S_1 \cap S_2 \neq \emptyset$ and we may choose $s \in S_1 \cap S_2$. Let $s_i \in S_i$ such that $s \in B_{s_i}$ and $t_i \in B_{s_i}$ ($i = 1, 2$). Then $s_i \in C_i$ ($i = 1, 2$) (since C_i is connected and contains a and t_i , whereas it follows from $a \in A_{s_i}$ and $t_i \in B_{s_i}$ that s_i separates a and t_i).

Since $S(a, s)$ is linearly ordered and contains s_1 and s_2 either $s_1 \leq s_2$ or $s_1 < s_2$ in this order, and we may suppose $s_1 \leq s_2$. But then, since C_2 is connected and $\{a, s_2\} \subset C_2$, we must have $s_1 \in C_2$, i.e., $s_1 \in C_1 \cap C_2 \cap T = (A \cup B) \cap T = \emptyset$.

This contradiction proves the theorem. \square

REMARK. Since generally in T_1 -spaces $(INT2) \rightarrow (W)$ (see theorem 2) the theorem can be expressed by saying that in connected locally connected T_1 -spaces the properties (W) and $(INT2)$ are equivalent.

In [18] p.69 KOK asks the following three questions:

- (i) Is it true that for connected, locally connected T_1 -spaces (W) implies $(INT2)$?
- (ii) Is it true that for connected, locally connected T_1 -spaces (W) and $(INTC)$ together imply $(INT2)$?
- (iii) Is it false that for connected T_1 -spaces (W) and $(INTC)$ together imply $(INT2)$?

Question (i) and, a fortiori, question (ii) are answered affirmatively by the above theorem; as KOK already suspected MILLER's widely connected biconnected subset of the plane [23] provides an affirmative answer to the third question. (Hence the four questionmarks in the table of [18] p.85 should all be minus signs.)

[Let us indicate briefly why Miller's example fails to satisfy (INT2): using his notations the space under discussion is $M = \Delta \cup \bigcup \{M_\mu \mid \mu < \Omega_c\}$ where Ω_c denotes the smallest ordinal with cardinality c . Here Δ is a countable set, and each M_μ is either empty or a singleton, where a singleton is chosen (on a certain continuum B_μ) if otherwise that continuum would separate M . Now it is possible to partition Ω_c into two disjoint sets of ordinals A_1 and A_2 in such a way that $S_i = \Delta \cup \bigcup \{M_\mu \mid \mu \in A_i\}$ is connected ($i = 1, 2$). But $S_1 \cap S_2 = \Delta$ is totally disconnected. That suitable sets A_i can indeed be found follows by again using the continuum hypothesis:

By transfinite induction assign M_α to either S_1 or S_2 (i.e., put $\alpha \in A_1$ or $\alpha \in A_2$). Now if we choose to let $\alpha \in A_1$, and B_α does not meet S_2 yet, then choose some index $\mu > \alpha$ such that M_μ is still free and such that $M_\mu \subset B_\alpha$ and put $\mu \in A_2$. By the continuum hypothesis, at each stage during the transfinite induction all sets M_μ are free, except for at most countably many; on the other hand there are uncountably many indices μ for which $M_\mu \subset B_\alpha$ since each continuum occurs essentially uncountably often in the collection $\{B_\alpha \mid \alpha < \Omega_c\}$.]

CHAPTER III

ON TREELIKE SPACES

1. DEFINITIONS AND ELEMENTARY PROPERTIES

We recall the following definitions:

A connected topological space is said to be *treelike* or to satisfy (S) if no two of its points are conjugated, that is, if for any two distinct points in the space there is a third point which separates them.

A connected topological space is said to have the *intersection property* or to satisfy (INT) if the intersection of any collection of connected subsets is again connected.

Also we have the following notations:

$$E(a,b) := \{x \in X \mid x \text{ separates } a \text{ and } b\},$$

$$S(a,b) := \{a\} \cup E(a,b) \cup \{b\},$$

$$C(a,b) := \bigcap \{T \subset X \mid T \text{ connected and } \{a,b\} \subset T\}.$$

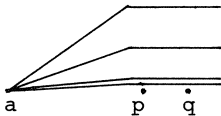
In an arbitrary connected topological space X $E(a,b)$ and $S(a,b)$ are linearly ordered in a natural way ("separation ordering"):

For $p \in E(a,b)$ set $a < p$, $p < b$ and if q is another point in $E(a,b)$ then $p < q$ iff there is a separation of $X \setminus p$ with a and q on different sides (or, which amounts to the same, iff there is a separation of $X \setminus q$ with p and b on different sides).

[Antisymmetry and transitivity are readily checked; see e.g. WHYBURN [29].]

$E(a,b)$ is nonempty if and only if a and b are not conjugated.

$S(a,b) \subset C(a,b)$ but $C(a,b)$ may be strictly larger:



$$S(p,q) = \{p,q\}$$

$$C(p,q) = \{a,p,q\}.$$

In fact we have:

$$C(a,b) = \{x \mid a \text{ and } b \text{ belong to different components of } X \setminus x\} \cup \{a,b\}$$

$$S(a,b) = \{x \mid a \text{ and } b \text{ belong to distinct quasicomponents of } X \setminus x\} \cup \{a,b\},$$

hence

$$C(a,b) \setminus S(a,b) = \{x \mid \text{in } X \setminus x \text{ } a \text{ and } b \text{ belong to different components but to the same quasicomponent}\}.$$

For sake of completeness we now formulate and prove some properties of treelike spaces, of which in any case propositions 1 - 4 are well-known.

PROPOSITION 1. *A treelike space is Hausdorff.*

PROOF. First of all X is (T_1) since two points cannot be separated if one is contained in the closure of the other. Next, if p and q are two points in X then they are separated by a third point r : $X \setminus r = A \cup B$, and since $\{r\}$ is closed, A and B are disjoint open nbds of p and q . \square

PROPOSITION 2. *Each segment in a treelike space is open.*

PROOF. Let S be a component of $X \setminus r$. If t is a non-interior point of S then let p separate r and t . $\{t\} \cup (X \setminus S)$ is connected and contains r and t , hence $p \in X \setminus S$. But now $X \setminus p = \underset{t,S}{A} \cup \underset{r}{B}$ and $A \cup \{p\}$ is a connected subset of $X \setminus r$ strictly containing the component S . Contradiction. \square

PROPOSITION 3. *A treelike space satisfies (W).*

PROOF. See chapter II, theorem 3. \square

PROPOSITION 4. *In a treelike space $S(a,b)$ is continuously ordered, i.e., has no jumps and no gaps.*

PROOF. If $a \leq p < q \leq b$ in $S(a,b)$ and if r is a point separating p and q then r separates a and b and $p < r < q$, hence (p,q) is not a jump. If $E(a,b) = E \cup F$, E an initial interval of $E(a,b)$ without last element and F a final interval of $E(a,b)$ without first element, then this gap in $S(a,b)$ induces a separation of X : if for each $t \in E(a,b)$ A_t resp. B_t is the component of $X \setminus t$ containing a resp. b , then it is checked easily that $X = \bigcup_{t \in E} A_t \cup \bigcup_{t \in F} B_t$. Therefore, either E or F is empty. \square

COROLLARY. If $a \neq b$ then $|S(a,b)| \geq 2^{\aleph_0}$.

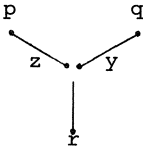
THEOREM 1. Let X be treelike and $p,q,r \in X$. Then $S(p,q) \cap S(p,r) \cap S(q,r)$ is a singleton.

PROOF. (i) Suppose y and z are two distinct points in $S(p,q) \cap S(p,r) \cap S(q,r)$. If $r \in S(p,q)$ then $S(p,q) = S(p,r) \cup S(r,q)$ and $S(p,r) \cap S(r,q) = \{r\}$ and hence $S(p,q) \cap S(p,r) \cap S(q,r) = \{r\}$. Therefore, we may suppose $y \notin \{p,q,r,z\}$.

Now e.g. $X \setminus y = \underset{p,z}{A} + \underset{q}{B} + \underset{r}{C}$, but then $B \cup \{y\} \cup C$ is connected in $X \setminus z$ so that z does not separate q and r .

(ii) Suppose $S(p,q) \cap S(p,r) \cap S(q,r) = \emptyset$. Now $S(p,q) = ((S(p,q) \cap S(p,r)) \cup (S(p,q) \cap S(r,q)))$, where both intersections are convex subsets of $S(p,q)$. But $S(p,q)$ is continuously ordered, hence this cut determines a point z , say the last element of $S(p,q) \cap S(p,r)$.

Likewise $S(r,q) = (S(r,q) \cap S(r,p)) \cup (S(r,q) \cap S(p,q))$ determines a point y , say the first element of $S(r,q) \cap S(p,q)$.



Now let u be a point in X separating z and y . Then $u \in S(p,q)$ and $z < u < y$. Hence from the definition of z and y it follows that $u \notin S(p,r)$ and $u \notin S(r,q)$ which contradicts the fact that $S(p,q) \subset S(p,r) \cup S(r,q)$ for all triples $p,q,r \in X$. \square

COROLLARY. [KOK, 19]. A treelike space without ramification points is (weakly) orderable.

PROOF. Let X be a treelike space without ramification points. Then among every three points p,q,r in X there is (at least) one which separates the other two: indeed, since $S(p,q) \cap S(p,r) \cap S(q,r)$ is not a ramification point it is one of the sets $\{p\}$, $\{q\}$ or $\{r\}$. But this property characterizes (weakly) orderable spaces (see for instance R. DUDA [11] who ascribes this remark to mrs. D. ZAREMBA-SZCZEPKOWICZ; for a detailed proof see H. KOK [18], theorem 3 p.16). \square

Theorem 1 enables us to introduce the "projection onto an interval": let X be treelike, $a,b \in X$, then we define the projection

$$\pi_{ab}: X \rightarrow S(a,b)$$

by

$$\{\pi_{ab}(x)\} = S(a,b) \cap S(a,x) \cap S(x,b).$$

[Note: π_{ab} is not continuous in general, not even if $S(a,b)$ is connected:



A sufficient condition for the continuity is that $S(a,b)$ is connected and locally connected (see proposition 8).]

PROPOSITION 5. *Let X be treelike. Then $S(a,b)$ is the intersection of all closed connected sets containing a and b . In particular: $S(a,b) = C(a,b)$ and $S(a,b)$ is closed.*

PROOF. Let $K(a,b) = \bigcap \{T \subset X \mid T \text{ closed, connected, } \{a,b\} \subset T\}$. Obviously $S(a,b) \subset C(a,b) \subset K(a,b)$. On the other hand, let $p \notin S(a,b)$. Let $q = \pi_{ab}(p)$. If $q \notin \{a,b\}$ then $X \setminus q = \frac{A}{a} + \frac{B}{b} + \frac{C}{p}$ and $A \cup \{q\} \cup B$ is a closed connected set containing a and b but not p . If $q \in \{a,b\}$ then we can write the same decomposition but with $A = \emptyset$ or $B = \emptyset$. Therefore, $K(a,b) \subset S(a,b)$. \square

Defining (INTC) by the requirement that the intersection of an arbitrary collection of closed connected sets be connected we have:

COROLLARY. [KOK, pp.61-64]. $(S) + (INTC) \Rightarrow (INT)$.

PROOF. (INTC) is equivalent with $\forall a,b K(a,b)$ is connected.

(INT) is equivalent with $\forall a,b C(a,b)$ is connected.

But by the previous proposition $C(a,b)$ and $K(a,b)$ are the same in spaces satisfying (S). \square

PROPOSITION 6. *If X is treelike and if S and C are connected subsets of X and $p \in S \subset C$ while p is an end point of C then p is an end point of S .*

PROOF. Clearly treelikeness is hereditary for connected subsets, hence it is no restriction to assume $C = X$.

Now the proposition is equivalent to: If Y is a connected subspace of a treelike space X then for each $a,b \in Y$ we have $S_Y(a,b) = S_X(a,b)$. Since any point separating a and b in X obviously must belong to Y we of course always have: $Y \subset X \Rightarrow S_Y(a,b) \supset S_X(a,b)$, and it is easily seen that the

orders in both sets are compatible.

[It suffices to verify that for all $u, v \in S_X(a, b)$ we have $u <_X v \Rightarrow u <_Y v$, but this follows immediately from the definition of the separation order.]

Now suppose $p \in S_Y(a, b) \setminus S_X(a, b)$ and define (in $S_Y(a, b)$):

$$q = \text{lub } S_X(a, b) \cap S_Y(a, p),$$

$$r = \text{glb } S_X(a, b) \cap S_Y(p, b).$$

Since $S_X(a, b)$ is continuously ordered either $q \in S_X(a, b)$ or $r \in S_X(a, b)$ but not both. In particular $q \neq r$. But $E_X(q, r) = \emptyset$, hence q and r are conjugated in X . Contradiction. \square

REMARK. Trivially this proposition remains true if "treelike" is replaced by "(INT2)".

THEOREM 2. In T_1 -spaces (INT) implies (S).

PROOF. Let X satisfy (INT).

(i) If $p \in X$ then in $X \setminus p$ components coincide with quasicomponents. For: let Q be a quasicomponent in $X \setminus p$; then for some $q \in Q$ we have:

$$Q = \cap \{C_\alpha \mid C_\alpha \text{ clopen in } X \setminus p, q \in C_\alpha\}.$$

By (INTC)

$$Q \cup \{p\} = \cap \{C_\alpha \cup \{p\} \mid C_\alpha \text{ clopen in } X \setminus p, q \in C_\alpha\}$$

is connected. Also

$$X \setminus Q = \cup \{X \setminus C_\alpha \mid C_\alpha \text{ clopen in } X \setminus p, q \in C_\alpha\}$$

is connected. If $X \setminus Q$ were not closed then

$$((X \setminus Q) \cup \{t\}) \cap (Q \cup \{p\}) = \{p, t\}$$

would contradict (INT2) for some $t \in \overline{X \setminus Q} \cap Q$. Hence Q is open and, therefore, connected.

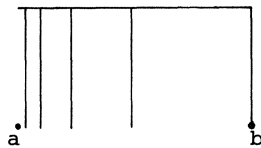
(ii) Now if $a, b \in X$ then by (i) and the observation before proposition 1 we have $S(a, b) = C(a, b)$ is connected. Therefore, $S(a, b) \neq \{a, b\}$, i.e., X is treelike. \square

REMARK. This theorem has also been proved by H. KOK [18] pp. 57-60, who even shows:

If in a T_1 -space X the closure of the intersection of an arbitrary collec-

tion of connected subsets of X is connected ((INT^*)), then X is treelike. This follows by the same argument since under (i) we need only $(INT2^*)$ instead of $(INT2)$ while under (ii) the knowledge $S(a,b) = C(a,b)$ and $\overline{C(a,b)}$ is connected suffices to infer $S(a,b) \neq \{a,b\}$. Having proved this KOK proceeds to show $(INT) \Leftrightarrow (INT^*)$ (ibid. p.64) by the sequence of implications $(INT) \Rightarrow (INT^*) \Rightarrow (S) + (INTC) \Rightarrow (INT)$.

REMARK. The converse of theorem 2 is not true, that is, in a treelike space it may happen that $S(a,b)$ is disconnected:



DEFINITION. A connected topological space X is called weakly treelike (WS) if $\forall a,b \in X: |C(a,b)| \neq 2$, in other words, if for any two distinct points a and b there is a point p such that a and b belong to different components of $X \setminus p$.

[Note that trivially both (INT) and (S) imply (WS) .]

Although, in general, separation is a stronger concept than cutting, in this case they coincide:

THEOREM 3. $(S) \Leftrightarrow (WS)$.

PROOF. It clearly suffices to prove that if X satisfies (WS) then X has open segments. But suppose $p \in X$, C a non-open component of $X \setminus p$, say with $q \in C \setminus C^\circ$. If $X \setminus p = \cup_{\alpha} C_{\alpha}$ is the decomposition of $X \setminus p$ in components then for each index β

$$S_{\beta} := \{p,q\} \cup \cup_{\alpha \neq \beta} C_{\alpha}$$

is a connected set and

$$\cap S_{\beta} = \{p,q\},$$

so $C(p,q) = \{p,q\}$, contrary to the hypothesis. \square

REMARK. Since trivially $(INT) \Rightarrow (WS)$ (and $(INT^*) \Rightarrow (WS)$) this provides a new proof of theorem 2.

2. THE LOCALLY CONNECTED CASE

THEOREM 4. *Let X be locally connected. Equivalent are:*

- (i) X is treelike,
- (ii) X satisfies (T_1) and (INT) ,
- (iii) X satisfies (T_2) and (W) ,
- (iv) X is Hausdorff and such that if $C \subset X$ is connected and E is a collection of end points of C then $C \setminus E$ is connected.

PROOF. (i) \iff (ii) was proved by WHYBURN [28] p.387, theorem 9.3.

(i) \iff (iii) has already been proved in chapter II (theorems 3 and 6).

(ii) \Rightarrow (iv): Since (ii) \Rightarrow (i), X is certainly Hausdorff; furthermore

$$C \setminus E = \bigcap_{p \in E} (C \setminus p)$$

is connected by (INT) .

(iv) \Rightarrow (i): Let p and q be two points of X and suppose p and q are not separated by a third point. Let U and V be disjoint open connected neighbourhoods of p and q respectively. For each $r \in \partial U$ let $X \setminus r = A_r + B_r$ where A_r is the (open) component of $X \setminus r$ containing U and V (and possibly $B_r = \emptyset$). Define

$$Y := X \setminus \bigcup_{r \in \partial U} B_r.$$

[Observe that $\bar{U} \cup \bar{V} \subset Y$.] Then:

(1) Y is connected.

For: Suppose $Y = A + B$, where $\bar{U} \subset A$ and B is nonempty. Let P be a component of B (and hence of Y). Y is closed so P is closed and, therefore, not open. Take $z \in P \setminus P^\circ$ and let W be an open and connected neighbourhood of z contained in $X \setminus \bar{U}$. If W intersects B_s for some s , then $W \subset B_s$ (since $s \notin W$) which is impossible since $z \notin B_s$. Therefore, $W \subset Y$. W is connected and intersects P so $W \subset P$. But then $z \in W \subset P^\circ$, contradiction.

(2) For each $s \in \partial U$ $Y \setminus s$ is connected.

For: Suppose $Y \setminus s = A + B$ where $U \subset A$ and $B \neq \emptyset$. B is closed in $Y \setminus s$, hence also in $X \setminus s$ and, therefore, B cannot be open in X (otherwise B would be open in $X \setminus s$, hence clopen in $X \setminus s$, and since $B \cap B_s = \emptyset$ it would follow

that $B = A_s$ which is impossible). Take $z \in B \setminus B^\circ$ and let W be an open connected neighbourhood of z disjoint with $\bar{A} = A \cup \{s\}$. If W intersects B_r for some r then $W \subset B_r$ which is impossible (as above). Therefore, $W \subset Y$. W is connected, intersects B and does not contain s ; so $W \subset B$, hence $z \in W \subset B^\circ$, a contradiction.

At this moment we use the (second) condition in the hypothesis and conclude that $Y \setminus \partial U$ is connected. On the other hand, $Y \setminus \partial U = \bigcup_p U + \bigcup_q Y \setminus \bar{U}$, a contradiction. \square

COROLLARY. *In a locally connected treelike space each set $S(p,q)$ is connected.* \square

REMARK. (1) Without local connectivity neither (i) \Rightarrow (iv) nor (iv) \Rightarrow (i) holds:

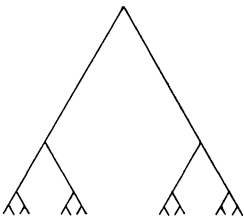
(A) Let X be the sine curve together with two limit points:



$$X = \{(x,y) \in \mathbb{R}^2 \mid (x > 0 \ \& \ y = \sin \frac{1}{x}) \text{ or} \\ (x,y) = (0,-1) \text{ or} \\ (x,y) = (0,+1)\}.$$

Then X satisfies (iii) and (iv) but not (i) or (ii).

(B)



Let C be the Cantor set and C_0 the countable subset of C consisting of all the end points of the deleted intervals.

Let X be the minimal compact subset of the plane containing the set

$$\{(x,y) \in \mathbb{R}^2 \mid (x+y=1 \ \& \ \frac{1}{2} \leq x \leq 1) \text{ or } (x-y=0 \ \& \ 0 \leq x \leq \frac{1}{2})\}$$

and such that $\phi_1[X] \subset X$ and $\phi_2[X] \subset X$ where ϕ_1 and ϕ_2 are the mappings $(x,y) \mapsto (\frac{1}{3}x, \frac{1}{3}y)$ and $(x,y) \mapsto (\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y)$, respectively. Note that $C \subset X$. [X looks like a binary tree over the Cantor set.]

Then X is a compact locally connected treelike space.

Now strengthen the topology on X somewhat by adding $C \setminus C_0$ to a subbase of X , and call the resulting space Y . Y is a treelike space in which the collection of end points (i.e. C) has a nonempty interior (namely $C \setminus C_0$). This space satisfies (i) and (iii) but not (ii) or (iv). Note that this space is not locally peripherally compact.

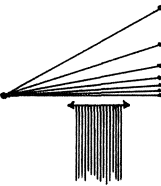
(2) In locally compact spaces (iv) \Rightarrow (i) does not hold:



Let $X = \{(x,y) \in \mathbb{I}^2 \mid (y=0 \text{ or } (\exists n \in \mathbb{N}: x=ny))\}$,

then X is compact and satisfies (iv) but not (i), (ii) or (iii).

The implication (iv) \Rightarrow (i) does not even hold in locally compact spaces in which each point is a cut point:



Let $X = \{(x,y) \in \mathbb{R}^2 \mid (0 \leq x \leq 1 \text{ \& } \exists n \in \mathbb{N}: x=ny) \text{ or } (\frac{1}{2} < x < 1 \text{ \& } y \leq 0)\}$,

with the topology defined as follows:

$X \cap \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ has Euclidean topology.

$X \cap \{(x,y) \in \mathbb{R}^2 \mid y < 0\}$ is the topological sum

of the lines $\{(a,y) \in \mathbb{R}^2 \mid y < 0\}$ with $\frac{1}{2} < a < 1$, where each line has Euclidean topology.

Finally, a point $(a,0) \in X$ has basic neighbourhoods

$$\{(x,y) \in X \mid y \leq 0 \text{ and } |x+y-a| < \frac{1}{2} \text{ and } x \notin F\}$$

in

$$X \cap \{(x,y) \in \mathbb{R}^2 \mid y \leq 0\},$$

where $i \in \mathbb{N}$ and F is a finite set. This construction defines a connected locally compact topological space X in which each point is a cut point.

X satisfies (iv) but is not treelike.

(3) Also the requirement that X be Hausdorff is indispensable as is easily seen by splitting an end point of the unit interval into two points. This gives a locally connected compact T -space which satisfies (iii) and (iv) but not (i) or (ii).

PROPOSITION 7. [WHYBURN]. *Let X be a locally connected treelike space. Then for $a, b \in X$ $S(a, b)$ is connected, locally connected and compact.*

PROOF. This is corollary 7.2 in WHYBURN [28]. \square

As a partial converse we have:

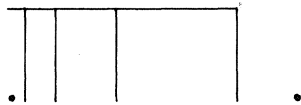
PROPOSITION 7a. *Let X be treelike. If either*
 (i) *For all $a, b \in X$ $S(a, b)$ is connected, or*
 (ii) *For all $a, b \in X$ $S(a, b)$ is compact*
then X satisfies (INT).

PROOF. (i) \Leftrightarrow (INT) is obvious.

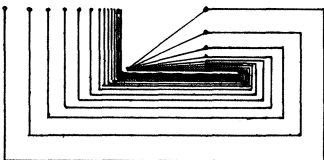
(ii) \Rightarrow (i): $S(a, b)$ is a compact orderable space without jumps and hence is connected. \square

[Observe that from this proof it follows also, that if $S(a, b)$ is compact, it is locally connected.]

Local connectedness of $S(a, b)$ for all a, b does not imply (INT) as is seen in the example given in the second remark after theorem 2:



Compactness of $S(a, b)$ for all a, b does not imply local connectedness of a treelike space X :



Let X be the space obtained by identifying the points $(0, 0)$ and $(1, 0)$ in

$$\{(x, y) \in \mathbb{I}^2 \mid \exists n \in \mathbb{N}: x = ny \text{ or } (x, y) = (1, 0)\}.$$

Then X is treelike, and each set $S(a, b)$ is homeomorphic to I (for $a \neq b$) but X is not locally compact or locally connected.

As we have seen already, the projection function π is not continuous in general. We have, however:

PROPOSITION 8. Let X be a treelike space and $a, b \in X$. Let $\pi = \pi_{ab}: X \rightarrow S(a, b)$ be the projection onto $S(a, b)$. Then

- (i) $\pi \circ \pi = \pi$ (this justifies the name "projection").
- (ii) If $c \in S(a, b)$ and $X \setminus c = \underset{a}{A} + \underset{b}{B} + C$ where A and B are connected (and empty when $a = c$ resp. $b = c$), then $\pi^{-1}(c) = C \cup \{c\}$. That is, $\pi^{-1}(c)$ is a closed connected set (called the stalk at c).
- (iii) If $S(a, b)$ is connected and locally connected then π is continuous. In particular this is the case when X is locally connected or when $S(a, b)$ is compact.

PROOF. (i) follows from $\pi(x) = x$ if $x \in S(a, b)$.

(ii) If $x \in C$ then $S(a, b) \cap S(a, x) \cap S(b, x) \supset \{c\}$ hence $\pi(x) = c$. Conversely if $x \neq \pi(x) = c$ then c separates x from a and b (unless $c = a$ or $c = b$); hence $x \in C$.

(iii) Let $U \subset S(a, b)$ be open in $S(a, b)$. Since $S(a, b)$ is locally connected we may suppose U to be connected; then it is easily seen that $U = E(p, q)$ for some p, q with $a \leq p < q \leq b$.

Now $\partial\pi^{-1}U = \{p, q\}$ - for certainly $\partial\pi^{-1}U \subset S(p, q)$ by (ii); but if $u \in U$ then $u \notin \partial\pi^{-1}U$ since there are points v, w in $S(p, q)$ with $p < v < u < w < q$ and then the component of $X \setminus v, w$ containing u is an open neighbourhood of u contained in $\pi^{-1}U$ - and, therefore, $\pi^{-1}U$ is open.

[Of course connectedness of $S(a, b)$ is used to conclude that $p, q \notin U$.] \square

3. THE LOCALLY PERIPHERALLY COMPACT CASE

PROPOSITION 9. Let X be a locally peripherally compact treelike space. Then for all a, b $S(a, b)$ is compact.

PROOF. Let an open cover \mathcal{U} (in X) of $S(a, b)$ be given and let

$$x = \sup\{u \mid S(a, u) \text{ can be covered with finitely many elements of } \mathcal{U}\}.$$

Suppose that $S(a, x)$ cannot be covered with finitely many elements of \mathcal{U} ,

and let $x \in V \subset U \in \mathcal{U}$ where V is open and ∂V is compact. If ∂V covers an entire interval $S(u,x) \setminus \{x\}$ for some $u \in S(a,x) \setminus \{x\}$ then $S(u,x)$ is compact (since $S(u,x) \cap \partial V$ is closed in ∂V) and hence $S(a,x) = S(a,u) \cup S(u,x)$ has a finite cover with elements of \mathcal{U} . Therefore, we can find an increasing sequence $\{u_i\}_{i < \omega_0}$ of points in $S(a,x)$ such that

$$(1) \forall i: u_i \notin \partial V$$

and

$$(2) \forall i: \pi^{-1}(E(u_i, u_{i+1})) \cap \partial V \neq \emptyset$$

(where $\pi = \pi_{ab}$).

[For: suppose by induction that the points u_0, \dots, u_k are found; if there is no choice for u_{k+1} this means that $\pi^{-1}(E(u_k, x)) \cap \partial V = \emptyset$ and since

$$x \in \overline{\pi^{-1}(E(u_k, x))},$$

it follows that

$$\pi^{-1}(E(u_k, x)) \subset V;$$

but then U covers the entire interval $E(u_k, x)$ contrary to the hypothesis.]

But now the sets

$$\pi^{-1}(\{a\} \cup E(a, u_0)),$$

$$\pi^{-1}(u_i) \setminus \{u_i\},$$

$$\pi^{-1}(x) \setminus \{x\}$$

and

$$\pi^{-1}(E(x, b) \cup \{b\})$$

form an open cover of ∂V with disjoint open sets and because of (2) there is no finite subcover. Contradiction.

If $x \neq b$ then in an entirely analogous way we see that there is a point $v \in S(x, b) \setminus \{x\}$ such that $S(a, v)$ has a finite subcover. Therefore, $x = b$ and $S(a, b)$ is compact. \square

Note: since by proposition 7a $S(a, b)$ is connected and locally connected, this proposition may be formulated as follows:

PROPOSITION 9a. *A locally peripherally compact treelike space is continuum-*

wise connected.

This is a result PROIZVOLOV ascribes to GURIN [13], but in the paper cited GURIN does not prove this statement. R. BENNETT [5] tried to fill this gap and gives an amazingly short proof of proposition 9a. Unfortunately however, he regards as self-evident the fact that X has an open basis consisting of connected sets with finite boundaries, and indeed, if that is true then X is locally connected and according to WHYBURN [28] (see proposition 7) $S(a,b)$ is compact and connected. But local connectedness of X is more difficult to prove than continuumwise connectedness. This is the reason I give this (longer) proof.

THEOREM 5. [GURIN, PROIZVOLOV]. *Let X be a locally peripherally compact treelike space. Then X satisfies (INT), is locally connected and has an open basis consisting of connected sets with finite boundaries. X is $(T_{3\frac{1}{2}})$ and has a unique treelike compactification. This compactification is of the same weight and has zero-dimensional remainder. Conversely, any treelike space which has a treelike compactification has an open basis consisting of connected sets with finite boundaries and, in particular, is locally peripherally compact.*

PROOF. (i) By propositions 9 and 7a X satisfies (INT).

(ii) Let $X \setminus x = \Sigma C_\alpha$ be the decomposition of $X \setminus x$ in open components; let V be an open nbd of x such that ∂V is compact. ∂V intersects only finitely many C_α , hence V contains almost all C_α . We may find an open nbd W of x with $C_\alpha \subset V \Rightarrow C_\alpha \subset W$ and $W \subset V$ and $|\partial W \cap C_\alpha| \leq 1 \quad \forall \alpha$ as follows:

Let C be one of the C_α for which $C_\alpha \not\subset V$, and choose $a \in C$. $S(a,x)$ is compact hence locally connected, so there is an $u \in E(a,x)$ with $E(u,x) \subset V$. If $\pi = \pi_{ax}$ then consider $\pi^{-1}(E(u,x))$: since ∂V is compact it can intersect only finitely many of the sets $\pi^{-1}(v)$ for $v \in E(u,x)$ so we might as well suppose u chosen in such a way that $S := \pi^{-1}(E(u,x)) \subset V$. Now $\partial_C S = \{u\}$, so if we take the finitely many sets S obtained in this way, together with $\{x\}$ and all sets C_α contained in V then we get an open connected set W contained in V and containing x such that ∂W is finite.

(iii) Let \tilde{X} be the set of all maximal centered systems consisting of closed connected subsets of X , and let the collection of all $\tilde{A} = \{\tilde{x} \in \tilde{X} \mid A \in \tilde{x}\}$ be a subbase for the closed sets in \tilde{X} . Then it is not difficult to prove that \tilde{X} is a treelike compactification of X . For the proof of this statement

and the remaining part of the theorem I refer to V.V. PROIZVOLOV [25]. \square

REMARK. K.R. ALLEN [1] showed that this compactification is the Freudenthal compactification of X ; he also shows that it is the GA compactification of X generated by all closed connected sets.

B.J. PEARSON [24] shows that a treelike space has a treelike compactification iff it is continuumwise connected and semi-locally connected [i.e., has a base of open sets V such that $X \setminus V$ has only finitely many components].

This theorem has a surprising corollary:

COROLLARY. *Any treelike space is functionally Hausdorff.*

PROOF. Let X be a treelike space and X_0 the same set, but with a weaker topology: an open subbase of X_0 is given by the components of $X \setminus p$ for $p \in X$. X_0 is connected and treelike and has an open base consisting of sets with finite boundaries. By the previous theorem X_0 has a Hausdorff compactification and hence is completely regular. A fortiori X_0 is functionally Hausdorff; but this last property is preserved when the topology is strengthened, i.e., X is functionally Hausdorff. \square

At the end of his paper [25] PROIZVOLOV asked whether each compact treelike space is the continuous image of an ordered continuum. This question was answered affirmatively and independently by J.L. CORNETTE [10] and the present author [7]; CORNETTE gave the following slightly stronger statement:

THEOREM 6. [CORNETTE]. *A continuum is the continuous image of an ordered continuum iff each of its cyclic elements is.* \square

4. THE LOCALLY COMPACT CASE

In this section we study conditions under which a locally compact connected Hausdorff space is treelike.

THEOREM 7. *A separable connected locally compact Hausdorff space in which each point is a cut point is treelike (and hence locally connected and separable metric).*

A corollary of this theorem is:

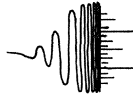
THEOREM 8. *A separable connected locally compact Hausdorff space in which each point is a strong cut point is homeomorphic to the real line.*

This last result was stated by FRANKLIN & KRISHNARAO [12], but their proof is incorrect as it would also apply to prove the same statement with locally compact replaced by rimcompact. This is, however, false as is shown by the separable metric counterexample:

$$X = \{(x,y) \in \mathbb{R}^2 \mid (x < 0 \ \& \ y = \sin \frac{1}{x}) \text{ or}$$

$$\exists n: (0 \leq x < 2^{-n} \ \& \ \exists k \leq 2^{n-1} \ |y| = \frac{2k-1}{2^n})\}$$

with subspace topology.



In fact they ascribe to KOK [19] the fancy-theorem: "In a connected Hausdorff space each point being a strong cut point is equivalent to (S'): given three distinct points, someone separates the other two", against which he in fact gives a counterexample.

The separability is required in theorem 7, and also in theorem 8 the orderability does not follow without separability.

Example.

Let $X = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0\}$ with topology given by the local bases:

$$U_i(x,y,z) = \{x\} \times \{y\} \times (z - \frac{1}{i}, z + \frac{1}{i}) \quad (z \geq \frac{1}{i}),$$

$$U_{i,F}(x,y,0) = \{(u,v,w) \in X \mid ((u+w-x)^2 + (y-v)^2 < \frac{1}{i^2}) \text{ and} \\ \text{(if } v=y \ \& \ w=x-u \text{ then } u \notin F \setminus \{x\})\},$$

where $i \in \mathbb{N}$ and F is a finite set.

Then X is a locally compact connected Hausdorff space in which each point is a strong cut point; but X is not locally connected or treelike.

However, if not only the points but also the compact connected sets separate the space in exactly two pieces then the space is orderable:

THEOREM 9. *A connected locally compact Hausdorff space X is orderable (without end points) if $X \setminus C$ consists of exactly two components for each compact connected subset C of X .*

In order to prove theorems 7-9 we first introduce the concept of a brush. Let X be a connected locally compact Hausdorff space. A compact connected nondegenerated subset C of X is called a *base of a brush* if:

If $p \in C$ then $C \setminus p$ is contained in one component B_p of $X \setminus p$.

If C is the base of a brush, the set $Y := \bigcup_{p \in C} (X \setminus B_p)$ is called the brush determined by C .

The usefulness of this concept stems from the following lemma:

Lemma:

Let X be a connected locally compact Hausdorff space. If X is not treelike then there is a brush in X .

Proof:

Case A: There is a point p such that a component S of $X \setminus p$ is not open.

In this case choose a point $q \in S \cap \overline{X \setminus S}$. $\overline{X \setminus S}$ is a connected locally compact Hausdorff space, hence if V is a compact neighbourhood of q in $\overline{X \setminus S}$ not containing p , then the component C of q in V must reach ∂V . (For: in a connected space the component of a point in a compact neighbourhood V of that point intersects the boundary ∂V of V .) But this component lies entirely in S and hence is the base for a brush. (If $r \in C$ then the component of $X \setminus r$ containing p also contains $X \setminus S$ and, therefore $\overline{(X \setminus S)} \setminus r$ and a fortiori $C \setminus r$.)

Case B: For each point $p \in X$ all components of $X \setminus p$ are open. Since X is not treelike, it contains two points a and b which cannot be separated by a third point. Let for each point $p \in X$ B_p be the component of $X \setminus p$ containing a or b . Let $S_p = X \setminus B_p$, then S_p is closed and connected, and $S_p \setminus (S_p)^\circ = \{p\}$.

Observe that if $S_q \cap S_p \neq \emptyset$ then $S_q \subset S_p$ or $S_p \subset S_q$. (For: otherwise we would have $B_p \not\subset B_q$ and $B_q \not\subset B_p$, i.e., $p \in B_q$ and $q \in B_p$ and hence $S_p \subset B_q$. But then $S_p \cap S_q = \emptyset$.)

Let $W_p := \bigcup \{S_q \mid p \in S_q\}$. If $p \in S_q$ then $W_p = W_q$, hence if $p \in W_q$ then $W_p = W_q$ and, therefore, if $W_p \cap W_q \neq \emptyset$ then $W_p = W_q$. Moreover, each W_p is connected (since the S_q are).

For each set W_p there are two possibilities:

- (i) It is open; this is the case if for each $r \in W_p$ there is a $q \neq r$ such that $r \in S_q$.
- (ii) It contains exactly one non-interior point q ; in this case $W_p = S_q$ and is, therefore, closed.

B(i). Assume first that some W_p is open, then $a, b \notin W_p$. (For: $W_a = S_a$ and $W_b = S_b$.) Since X is connected and $W_p \neq X$ W_p cannot be closed. If $\overline{W_p} \setminus W_p = \{q\}$ then $p \in S_q$, so $q \in W_p$. Contradiction.

Therefore, there are two distinct points $q, r \in \overline{W_p} \setminus W_p$. Now $\overline{W_p}$ is a locally compact connected subspace of X , so we can find two disjoint compact neighbourhoods V_q and V_r of q and r resp. in $\overline{W_p}$.

The components C_q and C_r of q and r in V_q and V_r (resp.) cannot both meet W_p , since if $q_1 \in C_q \cap W_p$ and $r_1 \in C_r \cap W_p$ then there is a point $s \in W_p$ such that $\{q_1, r_1\} \subset S_s$ and, therefore, s separates $(q_1$ and $r_1)$ from $(q$ and $r)$. This, however, is impossible since s cannot lie both in C_q and C_r .

Therefore, we may suppose $C = C_q \subset \overline{W_p} \setminus W_p$. C is non-degenerated by the same argument already used in *Case A* and, therefore, is the base of a brush.

(By the same argument: if $t \in C$ then the component of $X \setminus t$ containing p also contains W_p and hence $\overline{W_p} \setminus t$ and a fortiori $C \setminus t$.)

B(ii). Now suppose that each W_p is closed, i.e., of the form S_q for some q .

Let $Z = \{q \mid W_q = S_q\}$. Z is closed since $X \setminus Z = \bigcup_{q \in Z} S_q^\circ$ is open. Moreover, $\{a, b\} \subset Z$. If Z were connected we could find a non-degenerate compact connected subset C of Z and take $Y = \bigcup_{q \in C} S_q$ to get our brush.

(By the very definition of Z , $Z \setminus t$ is contained in one component of $X \setminus t$, sc. the component containing a or b .)

On the other hand, if $Z = Z_1 + Z_2$ then, since X is connected, either

$$\exists z_2 \in Z_2: z_2 \in \overline{\bigcup_{q \in Z_1} S_q} \quad \text{or} \quad \exists z_1 \in Z_1: z_1 \in \overline{\bigcup_{q \in Z_2} S_q}.$$

Suppose

$$z_2 \in Z_2 \cap \overline{\bigcup_{q \in Z_1} S_q}.$$

Let V be a compact neighbourhood of z_2 in the locally compact space \overline{S} ,

where $S = \bigcup_{q \in Z_1} S_q$, such that $V \cap Z_1 = \emptyset$. Since V cannot contain a clopen neighbourhood of z_2 (each S_q is connected) the component C of z_2 in V must reach ∂V . But this component cannot intersect S , hence $C \subset \bar{S} \subset Z_2$, and again C is the base for the brush $Y = \bigcup_{q \in C} S_q$.

This proves the lemma. \square

PROOF OF THEOREMS 7-9. Let X be a connected locally compact Hausdorff space. Suppose there is a brush in X with base C . Let for each $p \in C$ B_p be the component of $X \setminus p$ containing $C \setminus p$. Let $S_p = X \setminus B_p$. S_p is connected and contains p .

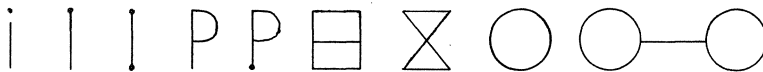
If $p \neq q$, $p, q \in C$ then $q \in C \setminus p \subset B_p$ and $p \in B_q$, so $S_p \subset B_q$, i.e., $S_p \cap S_q = \emptyset$. ["Hairs on the brush are disjoint."] Since X is regular, C contains at least 2^{\aleph_0} points p and, therefore, X contains a collection of 2^{\aleph_0} pairwise disjoint open sets S_p , which are nonempty if each point p is a cut point of X . This implies that X cannot be separable nor satisfy the countable chain condition, which proves theorem 7.

Also $X \setminus C$ decomposes into at least 2^{\aleph_0} components, so the hypothesis of theorem 9 implies that X is treelike and, therefore, since in particular each point is a strong cut point, that X is orderable (see the corollary to theorem 1). This proves theorem 9.

In the same way it follows from the hypothesis of theorem 8 that X is orderable and since it is separable homeomorphic to \mathbb{R} . This proves theorem 8. \square

It is not possible to strengthen theorem 9 to the statement: "A connected locally compact Hausdorff space is orderable iff for each compact connected subset C of X $X \setminus C$ consists of at most two components", for if X is any separable metric indecomposable continuum and C a subcontinuum, then either $C = X$ or C is contained entirely within one component of X and hence in both cases $X \setminus C$ is connected.

Other counterexamples are the locally compact ones among all connected Hausdorff spaces for which the complement of any connected subset has at most two components. These spaces are characterized in the next chapter; we find the following ten types:



5. A CHARACTERIZATION OF CONNECTED (WEAKLY) ORDERABLE SPACES

It is well-known and already mentioned before, that a connected topological space X is (weakly) orderable iff among any three points of X there is exactly one which separates the other two. Since a point (singleton) is connected, this result immediately follows from the more general

THEOREM 10. *A connected T_1 -space is (weakly) orderable iff among any three points of X there is exactly one which lies in a connected set that separates the other two.*

A similar characterization is possible using regions instead of just connected sets:

THEOREM 11. *A connected T_1 -space X is (weakly) orderable iff among any three points of X there is exactly one which has an open connected neighbourhood that separates the other two.*

These two theorems will follow from the next proposition:

PROPOSITION 10. *If the connected T_1 -space X satisfies the following three conditions:*

- (i) *among any three points of X there is at least one which lies in a connected set separating the other two;*
- (ii) *among any three points of X there is at most one which lies in an open connected set that separates the other two points;*
- (iii) *all segments in X are open,*

then X is orderable (and, conversely, an orderable space certainly satisfies (i) - (iii)).

PROOF OF THE PROPOSITION

1. X contains at least one cut point.

For suppose no point of X is a cut point. We consider two cases.

1A. Suppose $X \setminus \{p, q\}$ is disconnected for all $p, q \in X$. Now $X \setminus \{p, q\} = \underset{r}{A} + B$ and $\bar{B} = B \cup \{p, q\}$ is connected. So r cannot lie in a connected set separating p and q . Since p, q and r are arbitrary we arrive at a contradiction with (i).

1B. Let $X \setminus \{p, q\}$ be connected for some fixed pair of distinct points p, q . Now $X \setminus \{p, q\}$ is an open connected neighbourhood of r separating p and q for each point $r \in X \setminus \{p, q\}$. Therefore, by (ii), q cannot have an open connected neighbourhood separating p and some other point r . Hence $X \setminus \{p, r\}$ is not connected for $r \neq q$, i.e., $X \setminus p$ has exactly one end point. Choose r_1, r_2 different from p, q . Let

$$X \setminus \{p, r_1\} = \underset{\neq \emptyset}{A_1} + \underset{r_2}{A_2}$$

and

$$X \setminus \{p, r_2\} = \underset{\neq \emptyset}{B_1} + \underset{r_1}{B_2}.$$

Then $\bar{A}_1 = A_1 \cup \{p, r_1\}$ and $\bar{B}_1 = B_1 \cup \{p, r_2\}$ so r_2 and r_1 cannot lie in a connected set separating r_1 and p respectively r_2 and p . Therefore, by (i), p belongs to a connected set that separates r_1 and r_2 so that $X \setminus \{r_1, r_2\}$ is connected (, otherwise we would have $X \setminus \{r_1, r_2\} = \underset{p}{A} + B$ and $\bar{B} = B \cup \{r_1, r_2\}$ which gives a contradiction).

This proves that in $X \setminus r_1$ all points, except possibly p and q , are end points; on the other hand, by the above argument $X \setminus r_1$ has exactly one end point. Contradiction.

2. X cannot contain exactly one cut point.

For suppose p is the only cut point of X , and

$$X \setminus p = \underset{p_1, p_2}{A} + \underset{p_3}{B}.$$

Then $\bar{A} \setminus p_1$ and $\bar{A} \setminus p_2$ and $\bar{B} \setminus p_3$ are connected and, consequently,

$$(B \setminus p_3) \cup \{p\} \cup (A \setminus p_1)$$

is an open connected neighbourhood of p_j separating p_i and p_3 ($\{i, j\} = \{1, 2\}$). This contradicts (ii).

3. For each $p \in X$ $X \setminus p$ has at most two components.

For suppose

$$X \setminus p = \underset{a}{A} + \underset{b}{B} + \underset{c}{C}; \quad \bar{A} \setminus a = \underset{p}{A_1} + A_2; \quad \bar{B} \setminus b = \underset{p}{B_1} + B_2; \quad \bar{C} \setminus c = \underset{p}{C_1} + C_2$$

(where A_2, B_2, C_2 may be empty).

Since the components of $X \setminus a$ are open (by (iii)), the components of $\bar{A} \setminus a$ are open in \bar{A} , so we may assume that A_1 , B_1 and C_1 are connected.

Now $A \cup B_1 \cup C_1$ is an open connected neighbourhood of a , separating b and c , and $B \cup A_1 \cup C_1$ is an open connected neighbourhood of b , separating a and c , a contradiction.

4. X is orderable.

Choose two cut points $p, q \in X$, and let

$$X \setminus p = \underset{a}{A_p} + \underset{q}{B_p} \quad \text{and} \quad X \setminus q = \underset{p}{A_q} + \underset{b}{B_q}.$$

Let $Y = X \setminus (A_p \cup B_q) = \overline{A_q} \cap \overline{B_p}$. (Then Y is connected.)

If $Y \setminus r = \underset{p,q}{A} + B$ then $X \setminus r = (\underset{p}{A} \cup \underset{q}{B}) + B$, so B and, therefore, A are connected.

Let

$$\overline{A_p} \setminus a = \underset{p}{E_a} + F_a$$

and

$$\overline{B_q} \setminus b = \underset{p}{E_b} + F_b$$

where E_a and E_b are connected and F_a and F_b may be empty. Then

$$Z := E_a \cup A \cup E_b$$

is open and connected, while

$$X \setminus Z = (F_a \cup \{a\}) + (F_b \cup \{b\}) + (B \cup \{r\}).$$

But from this it easily follows that each of the points a , b , r has an open connected nbd separating the other two. Contradiction.

Therefore, each point $r \in Y \setminus \{p, q\}$ separates p and q , i.e., $Y = S(p, q)$.

By extending the order on Y in the obvious way to X we find that X is partially ordered in such a way that its collection of cut points is connected and linearly ordered, and each end point is either maximal or minimal. But if A_p contains two end points u_1 and u_2 then $(\overline{A_q} \setminus u_1) \cup E_b$ is an open connected nbd of u_2 , separating u_1 and b ($\{i, j\} = \{1, 2\}$). This violates (ii). Therefore, A_p and B_q each contain at most one end point of X , and it follows that X is linearly ordered. \square

PROOF OF THE THEOREMS

In both cases it suffices to prove that X satisfies (iii).

Suppose $X \setminus p = \bar{A} + \bar{B} + \bar{C}$. Then \bar{A} , \bar{B} and \bar{C} are connected sets, containing a resp. b resp. c , and separating b and c resp. a and c resp. a and b , yielding a contradiction in the case of the first theorem.

In the other case let U_{ab} be an open connected neighbourhood of p which does not contain a and b (note that such a set exists since a connected nbd of a cannot separate p and b etc.). Now $C \cup U_{ab}$ is an open connected neighbourhood of c , separating a and b . Likewise, a and b have open connected nbds, separating b and c , resp. a and c . Contradiction.

This proves that $X \setminus p$ has at most two components (and a fortiori that all components of $X \setminus p$ are open). \square

REMARK. The third condition in the proposition is needed to ensure the existence of sufficiently many open connected sets without which the second condition would be useless.

For example a V_1 -space satisfies (i) and

- (ii') : the complement of an open connected set is connected (so no open connected set separates any two points), and
- (iii') : for each $p \in X$ at most one segment of p is not open.

6. ON (B)

In this section we consider the following property of a connected topological space X :

(B) : X does not contain three mutually disjoint segments.

[Remember that a segment in X is a component of $X \setminus p$ for some $p \in X$.]

Clearly, a B -space need not be orderable (as is seen by considering an arbitrary connected space with more than one point but without cut points). We can, however, say something about X when $E(X)$, the set of all end points of X , is not too large. First of all we have:

THEOREM 12. [18] *Let X be a B -space without end points. Then X satisfies (S') and hence X is orderable iff X is (T_1) .*

This follows from the more general

THEOREM 13. Let X be a \mathcal{B} -space and let p, q, r be three cut points of X . Then one of them separates the other two. It follows that the separation order induces a linear order on $X \setminus E(X)$. [The latter space is not necessarily connected, however.]

Conversely, if

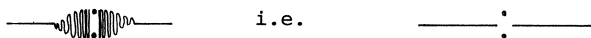
- (i) each cut point is a strong cut point, and
 - (ii) among any three cut points one separates the other two
- then X satisfies (\mathcal{B}) .

PROOF. Immediate from the definitions. \square

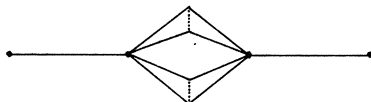
From now on we shall assume that X satisfies (\mathcal{T}_1) .

To get some feeling for the situation we first give some examples.

1. It is not true that if X satisfies (\mathcal{B}) then $X \setminus E(X)$ is connected:



2. In general, there is no connected orderable space Y such that $X \setminus E(X) \subset Y \subset X$, not even when $E(X)$ is totally disconnected:



Take the biconnected set of **KNASTER & KURATOWSKI** [17], attach a line segment to its dispersion point and reflect the set thus obtained in the x -axis.

3. It is possible that $E(X)^\circ \neq \emptyset$ while each set of conjugated end points is finite: Let

$$X = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} \times \{0, 1\}$$

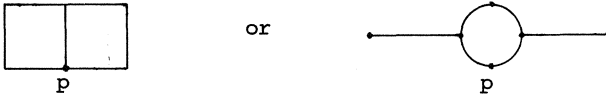
with topology defined by the neighbourhood bases:

$$U_\varepsilon((q, j)) = ((q - \varepsilon, q + \varepsilon) \cap \mathbb{Q}) \times \{0, 1\} \setminus \{(q, 1 - j)\}$$

$$U_\varepsilon(x) = ((x - \varepsilon, x + \varepsilon) \cap \mathbb{Q}) \times \{0, 1\} \cup (x - \varepsilon, x + \varepsilon) \cap (\mathbb{R} \setminus \mathbb{Q}).$$

Then $E(X) = \mathbb{Q} \times \{0, 1\}$ is open and totally disconnected, while each set of conjugated points is a doubleton.

4. If $p \in E(X)$, then $Y = X \setminus p$ does not necessarily satisfy (\mathcal{B}) :

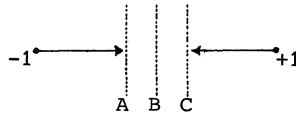


5. Let $X = [-1,0) \cup A \cup B \cup C \cup (0,1]$, where A, B and C are three dense subsets of a copy I' of I such that $A \cup B \cup C$ is totally disconnected (in the Euclidean topology), with the following topology:

- (i) for each $a \in A$ $[-1,0) \cup \{a\}$ is homeomorphic with I ;
- (ii) for each $c \in C$ $\{c\} \cup (0,1]$ is homeomorphic with I ;
- (iii) $[-1,0)$, $(0,1]$ and B are open in X ;
- (iv) $[-1,0)^- = [-1,0) \cup A$,
 $(0,1]^+ = (0,1] \cup C$,
 $B^- = A \cup B \cup C$;
- (v) in $A \cup B \cup C$ basic open sets are $J \cap B$, $J \cap (A \cup B)$ and $J \cap (B \cup C)$ if J is an open interval in I' .

Then X is connected and satisfies (\mathcal{B}) ; $E(X) = A \cup B \cup C = \bar{B}$ so that $E(X)$ is a regular closed set; also $E(X)$ is totally disconnected.

Yet there is no connected subspace Y of X with $E \setminus E(X) \subset Y \subset X$ such that Y is orderable.



Note that this phenomenon cannot occur when $E(X)$ is compact:

if Z is a totally disconnected compact space, and B an open subset of Z , then B contains clopen subsets of Z .

Observing that we can change cut points into end points by doubling them this shows that

Lemma: Let X be a \mathcal{B} -space such that $\overline{E(X)}$ is compact and totally disconnected. Then $E(X)^\circ = \emptyset$.

THEOREM 14. Let X satisfy (\mathcal{B}) , and suppose that either $E(X)$ is totally disconnected, or $E(X)^\circ = \emptyset$.

Let $p \in E(X)$. Then $Y := X \setminus p$ satisfies (\mathcal{B}) and $E(Y) \subset E(X)$.

PROOF. (i) If $q \in E(Y)$ then $X \setminus \{p, q\}$ and, therefore, also $X \setminus q$ is connected,

i.e., $q \in E(X)$.

(ii) Each cut point of Y is a strong cut point:

Let $q \neq p$, $X \setminus \{p, q\} = E + F + G$, $X \setminus q = A_q + B_q$ where B_q may be empty (if $q \in E(X)$). $X \setminus p$ is connected, so $q \in \bar{E} \cap \bar{F} \cap \bar{G}$ and $E \cup \{q\}$, $F \cup \{q\}$ and $G \cup \{q\}$ are connected.

(iia) $q \notin E(X)$.

Now we may suppose $E = A_q$, and then $B_q = F \cup \{p\} \cup G$. It follows that $F \cup \{p\}$ and $G \cup \{p\}$ are connected. Choose

$$f \in F \setminus E(X) = (F \cup \{p\}) \setminus E(X) \quad \text{and} \quad g \in G \setminus E(X).$$

$$\begin{array}{lcl} X \setminus f = A_f + B_f & \text{and} & X \setminus g = A_g + B_g \\ \neq \emptyset & p & \neq \emptyset & p \\ & G \ (p \in \bar{G}) & & F \\ & q \ (q \in \bar{G}) & & q \\ & A_q \ (q \in \bar{A}_q) & & A_q \\ & g \ (g \in G) & & f \\ & A_g \ (f \in B_g) & & A_f \end{array}$$

It is seen that A_f , A_g and A_q are pairwise disjoint segments. Contradiction.

(iib) $q \in E(X)$.

Now $\bar{E} = E \cup \{p, q\}$, $E \cup \{p\}$ and $E \cup \{q\}$ are connected and the same holds for F and G instead of E . If we choose 3 cut points of X : $e \in E$, $f \in F$ and $g \in G$ then in the same way we find three pairwise disjoint segments A_e , A_f and A_g , again a contradiction.

(iii) Y satisfies (\mathcal{B}) :

Let

$$Y \setminus q_i = A_i + B_i \quad (\{i, j, k\} = \{1, 2, 3\}) \\ \qquad \qquad \qquad q_j, q_k$$

and suppose that $p \in \bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$. Then if we choose a point $a_i \in A_i \setminus E(X)$ ($i = 1, 2, 3$), and let

$$X \setminus a_i = E_i + F_i, \\ \qquad \qquad \qquad p \quad \neq \emptyset$$

it follows as above that the segments F_1 , F_2 and F_3 are pairwise disjoint.

Hence, say, $p \notin \overline{B_3}$. Since $A_1 \cup A_2 \subset B_3$ it follows that

$$X \setminus q_1 = A_1 + (B_1 \cup \{p\}) \quad \text{and} \quad X \setminus q_2 = A_2 + (B_2 \cup \{p\}).$$

X satisfies (\overline{B}) and, therefore, A_3 cannot be a segment in X , i.e., $p \in \overline{A_3}$. But also A_1, A_2 and $A_3 \cup \{p\}$ are pairwise disjoint, i.e., $p \in \overline{B_3}$ and $q_3 \in E(X)$. It follows that $A_3 \subset E(X)$, contradiction. \square

COROLLARY. *Let X satisfy (\overline{B}) and suppose $E(X)$ is finite. Then there is a subspace Y such that $X \setminus E(X) \subset Y \subset X$ which is orderable.*

PROOF. Repeatedly apply the theorem until $E(Y)$ is empty. \square

THEOREM 15. *Let X satisfy (\overline{B}) , and suppose that either $\overline{E(X)}$ is totally disconnected, or $E(X)^\circ = \emptyset$.*

Let Y be a segment in X . Then Y satisfies (\overline{B}) and $E(Y) \subset E(X)$.

PROOF. The proof is almost identical to that of the previous theorem. The only point where something has to be changed is in case (iia) where we used that $F \not\subset E(X)$. As is shown by the second example it is well possible that $Y \setminus q$ has many components, almost all of which are contained within $E(X)$, while nevertheless $E(X)$ is totally disconnected. But if $F \subset E(X)$ then the nondegenerated connected set $F \cup \{p\}$ is contained in $\overline{E(X)}$, which contradicts our strengthened hypothesis. \square

THEOREM 16. *Let X be a locally peripherally compact Hausdorff \overline{B} -space such that $E(X)$ is finite. Then X is (strictly) orderable.*

PROOF. By the corollary to theorem 14 X contains a (weakly) orderable connected subspace Y containing all of its cut points. Suppose p and q are two conjugated points in X . Then p and q determine a cut in Y : $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are order-convex subsets of Y , and Y_2 has no first element. We may assume that $\{p, q\} \subset \overline{Y_2}$. Let U resp. V be disjoint open nbds of p resp. q with compact boundaries. Let $\{y_\alpha\}_\alpha$ be a coinitial sequence in $Y_2 \setminus E(X)$ and let for each α $X \setminus y_\alpha = A_\alpha + B_\alpha$ be the (unique) separation.

If $\{B_\alpha\}_\alpha$ is a cover of $\partial U \cap \overline{Y_2}^{\overline{p, q}}$ then for some α_0 the set B_{α_0} already covers $\partial U \cap \overline{Y_2}$, so that U contains all points $y \in Y_2$ with $y < y_{\alpha_0}$. But this means that \overline{U} contains q , a contradiction.

Therefore $\{B_\alpha\}_\alpha$ does not cover $\partial U \cap \overline{Y_2}$, which means that ∂U contains

a point q_1 conjugated to p . Replacing V by $V \cup V_1$ where V_1 is a nbd of q_1 with compact boundary and taking a smaller U we find by the same argument a point q_2 different from q and q_1 and conjugated to p . Continuing in this way we find countably many points q_i , all conjugated to p . But at most one of them can be in Y and hence X has infinitely many end points. \square

REMARK. If we let

$$X = \{(x,y) \in \mathbb{R}^2 \mid (x > 0 \ \& \ y = \sin \frac{1}{x}) \vee \\ (x = 0 \ \& \ y \in \mathbb{Q} \cap [-1,1])\}$$

then X (as subspace of the plane) is a locally peripherally compact Hausdorff \mathcal{B} -space such that $E(X)$ is countable.

THEOREM 17. *Let X be a locally compact Hausdorff \mathcal{B} -space such that $E(X)$ is totally disconnected. Then X is (strictly) orderable.*

PROOF. First observe that Y is dense in X : if $r \in E(X)$ and W is a compact nbd of x then the component R of r in W intersects ∂W and hence is not reduced to a singleton. Hence $R \not\subset E(X)$ so that

$$W \cap Y \supset R \cap Y \neq \emptyset.$$

Now we can mimick the proof of the previous theorem:

Let $Y = X \setminus E(X)$, and write for each $y \in Y$: $X \setminus y = A_y + B_y$ such that $\{y' \in Y \mid y' < y\} \subset A_y$ and $\{y' \in Y \mid y' > y\} \subset B_y$. In order to prove the theorem it suffices to show that X is treelike. Suppose not and let p and q be two conjugate points in X . As in the previous proof p and q determine a cut in Y : $Y = Y_1 \cup Y_2$ where $Y_1 = \{y \mid \{p,q\} \cap B_y \neq \emptyset\}$ and $Y_2 = Y \setminus Y_1$. We may assume that $\{p,q\} \subset \partial U\{B_y \mid y \in Y_2\}$. Let U resp. V be disjoint compact nbds of p resp. q , and let P (resp. Q) be the component of p (resp. q) in U (resp. V). It is impossible that both P and Q intersect $U\{B_u \mid y \in Y_2\}$, for otherwise $P \cap Q$ would contain points of Y_2 . But if P does not intersect $U\{B_y \mid y \in Y_2\}$ then it contains a nondegenerated connected subset entirely contained within the boundary of this set. But this is impossible since $E(X)$ is totally disconnected. \square

7. WEAKER PROPERTIES

7.1. On (NS)

DEFINITION. A connected topological space X is said to satisfy (NS) each pair of its points can be separated by an open connected set.

We shall see that in this definition the word "open" is superfluous. (Assuming this it is obvious that (NS) is indeed weaker than (S).)

PROPOSITION 11. Let X be a connected topological space such that any two of its points can be separated by a connected set. Then all segments in X are open.

PROOF. Suppose C is a component of $X \setminus p$ which is not open. Then there exists a point $r \in C$ such that $r \in \overline{X \setminus C}$. Let Q be a connected set separating p and r . Then we have

$$X \setminus Q = A + B.$$

$\quad p \quad r$

$B \cup Q$ is connected, intersects C and does not contain p , hence $B \cup Q \subset C$ and $X \setminus C \subset A$. Now $r \in \overline{X \setminus C} \subset \overline{A}$, a contradiction. \square

COROLLARY. Let X satisfy (NS). Then for any two points $a, b \in X$ we have $C(a, b) = S(a, b)$.

PROOF. Cf. the remark before proposition 1. \square

THEOREM 18. If in X any two points can be separated by a connected set, then any two points p, q can be separated by an open connected set S such that $\partial S = \{p, q\}$.

PROOF. Let $X \setminus p = Q + R$, where Q is connected. Let $Q \setminus q = \bigcup_{\alpha \in A} C_\alpha$ be the decomposition of $Q \setminus q$ in components. Now let C be a connected set separating p and q . Then $C \subset C_{\alpha_0}$ for some $\alpha_0 \in A$. $Q \setminus C_{\alpha_0}$ is connected and contains q , so $p \notin Q \setminus C_{\alpha_0}$ and hence $p \in C_{\alpha_0}$. $C_{\alpha_0} \cup \{p\} \cup R$ is a segment of q and hence open; also $R \cup \{p\}$ is closed, so C_{α_0} is open. Therefore, taking $S = C_{\alpha_0}$ we satisfy all requirements. \square

COROLLARY. (S) \Rightarrow (NS). \square

THEOREM 19. Let X satisfy (NS). Then among any three distinct points in X there is at least one which lies in an open connected set that separates the other two.

PROOF. Suppose not, and let the points p_1, p_2, p_3 in X provide a counter-example. Let B and C be open connected sets such that

$$X \setminus C = A_1 + B_1 \quad \text{and} \quad X \setminus B = A_2 + B_2$$

$$p_1 \quad p_2, p_3 \qquad \qquad \qquad p_2 \quad p_3, p_1$$

If $A_2 \cup B \cup C$ is connected, it is an open connected set containing p_2 that separates p_1 and p_3 , contrary to the assumption. But $A_2 \cup B$ is connected, so $C \cap (A_2 \cup B) = \emptyset$ and similarly $B \cap (A_1 \cup C) = \emptyset$. Also $(B_1 \cap B_2) \cup B \cup C$ cannot be connected (observe that it is open), hence

$$(B_1 \cap B_2) \cup B \cup C = S + T.$$

B

If $C \subset T$ then $X = (A_2 \cup S) + (A_1 \cup T)$.

If $C \subset S$ then $X = (A_1 \cup A_2 \cup S) + T$.

In both cases we have a contradiction. \square

7.2. ON (INT2) AND SOME VARIANTS OF IT

DEFINITION. A connected topological space X is said to satisfy

(INT2) iff the intersection of any two connected subsets is again connected,

(INT2^{*}) iff the closure of the intersection of two connected sets is connected,

(INTC2) iff the intersection of two closed connected sets is again connected.

Obviously $(INT2) \Rightarrow (INT2^*) \Rightarrow (INTC2)$.

Spaces satisfying (INTC2) are sometimes called "hereditarily unicoherent" (but often this last phrase implies compactness or is used only for metric continua). [A connected space X is called unicoherent if whenever $X = A \cup B$, where A and B are closed connected sets, $A \cap B$ is connected.

(Cf. KURATOWSKI [20].)

In chapter II, theorem 2 we observed $(INT2) \Rightarrow (W)$ and the argument used there also proves $(INT2^*) \Rightarrow (W)$. It is not true that $(INTC2) \Rightarrow (W)$:

The cofinite topology on \mathbb{N} yields a space which satisfies $(INTC)$ but not (W) . Each point of the space is an end point.

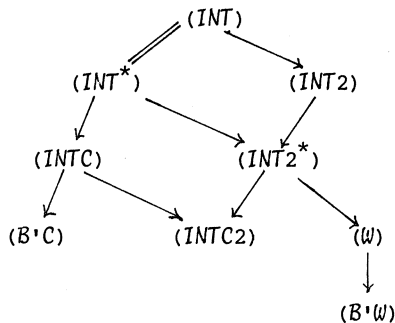
Concerning the properties of segments in X we have the following definitions:

- $(B'O)$: all segments in X are open,
- $(B'C)$: no segment in X is closed,
- $(B'W)$: the boundary of each segment of X is a singleton.

Obviously $(INTC) \Rightarrow (B'C)$ and $(INT2^*) \Rightarrow (W) \Rightarrow (B'W)$ and $(B'O) \Leftrightarrow (B'C)$ and $(B'W)$.

Example 1 of Kok's thesis [18] satisfies $(INTC)$ but not $(B'W)$,
 Example 9 satisfies $(INT2)$ but not $(B'C)$,
 Example 15 satisfies $(B'C)$ and $(INT2)$ but not $(INTC)$,
 Example 18 satisfies (W) and $(B'C)$ but not $(INTC2)$,
 Example 28 satisfies $(B'W)$ and $(INTC)$ but not (W) ,
 Example 30 satisfies $(INT2)$ and $(INTC)$ but not (INT) ,
 Example 31 (Miller's space) satisfies (W) and $(INTC)$ but not $(INT2^*)$
 and a variant of Miller's space provides an example satisfying $(INTC)$
 and $(INT2^*)$ but not $(INT2)$.

That is, the following diagram contains all valid implications:



[Concerning the variant of Miller's example alluded to: in Miller's space X there is a square $ABCD$ such that for any two connected subsets C_1 and C_2 of X , $C_1 \cap C_2$ is dense in $ABCD$; by carrying out his construction without fixing a square beforehand, we obtain a space with the stronger property that for any two connected subsets C_1 and C_2 of X , $C_1 \cap C_2$ is dense in X . In particular this space satisfies $(INTC)$ and $(INT2^*)$. That X does not satisfy $(INT2)$ is seen as before (cf. page 51).]

Among treelike spaces (*INTC*) is equivalent to (*INT*) (by the corollary to proposition 5), and all treelike spaces satisfy (*B'C*) and (*W*), so that we get the smaller diagram

$$(INT) \Rightarrow (INT2) \Rightarrow (INT2^*) \Rightarrow (INTC2).$$

Below we shall construct a treelike space satisfying (*INT2*) and the first axiom of countability, but not (*INT*). I do not know of any treelike space satisfying (*INTC2*) but not (*INT2*).

Sometimes the only way to ensure (*INT2*) is by using some ultrafilter-based construction. In such cases (*INT2*) + (*CI*) is impossible. For example, a *V*-space can satisfy (*INT2*) or can be first countable but not both:

PROPOSITION 12. A *V*-space cannot satisfy (*INT2*) + (*CI*).

PROOF. Suppose *X* is such a *V*-space, and let *x* be a non-minimal point of *X*. Let $\{U_i\}_i$ be a countable local basis at *x*' such that for each *i*:

$$U_{i+1} \subset U_i.$$

Let $P_x \setminus x = \bigcup_{\alpha} C_{\alpha}$ be the decomposition of $P_x \setminus x$ into components. Since $x' \in \overline{P_x}$ but for no α : $x' \in \overline{C_{\alpha}} = C_{\alpha} \cup \{x\}$, we may choose integers i_k and indices α_k such that U_{i_k} meets C_{α_k} , but $U_{i_k} \cap \bigcup_{j < k} C_{\alpha_j} = \emptyset$. Now if

$$S_1 = \{x, x'\} \cup \bigcup_i C_{\alpha_{2i}}$$

and

$$S_2 = \{x, x'\} \cup \bigcup_i C_{\alpha_{2i+1}}$$

then S_1 and S_2 are closed connected subsets of *X* with the disconnected intersection $\{x, x'\}$. \square

For a while I have thought that in the same way (*S*) + (*INT2*) + (*CI*) \Rightarrow (*INT*). This is not true, however. The rather involved counterexample I made was streamlined and simplified by A. SCHRIJVER; in the next subsection I present his version of this example.

7.3. EXAMPLE OF A TREELIKE SPACE SATISFYING (*INT2*) AND THE FIRST AXIOM OF COUNTABILITY BUT WITHOUT THE INTERSECTION PROPERTY

Let $J := [0, \infty)$. Our example *X* is the set $(\mathbb{R} \times \{0\}) \cup (\mathbb{Q} \times J)$ with the following

local basis:

- (i) If $y \neq 0$ then $U_{\frac{1}{i}}((x,y)) = \{(x,z) \in X \mid |y-z| < \frac{1}{i}\}$.
(ii) In order to define neighbourhoods for points $(x,0)$ we need some preparation.

Considering \mathbb{R} and \mathcal{Q} as additive groups we may form the quotient \mathbb{R}/\mathcal{Q} . For $x \in \mathbb{R}$ we write $[x]$ for the coset of \mathcal{Q} containing x . Observe that $|\mathbb{R}/\mathcal{Q}| = c$.

Next let $\mathcal{D} = \{D \subset \mathcal{Q} \mid \bar{D} = \mathbb{R}\}$. Since $|\mathcal{D}| = c$ there exists a bijection $\psi: \mathcal{D} \rightarrow \mathbb{R}/\mathcal{Q}$.

Now we may define neighbourhoods of $(x,0)$ as follows: let $D_x = \psi^{-1}([x])$, and put

$$U_{\frac{1}{i}}((x,0)) = \{(u,v) \in X \mid (u=x) \ \& \ v < \frac{1}{i}\} \text{ or} \\ (0 < |u-x| < \frac{1}{i} \ \& \ u \in D_x \ \& \ v > i)\}.$$

In this way we get a topological space X satisfying (CI) by definition.

CLAIM. The connected subsets of X are the following:

- (i) the sets $\{x\} \times K \subset X$, K a convex subset of J ;
(ii) the sets $C \subset X$ such that

$$\pi_1 C =: L$$

is a nondegenerated convex subset of \mathbb{R} , and for each x :

$$\pi_2(C \cap \{x\} \times J)$$

is a convex set containing 0, and for each k :

$$E_k := \{x \in L \cap \mathcal{Q} \mid (x,k) \notin C\}$$

is nowhere dense in \mathbb{R} .

In particular X itself is connected.

PROOF. Clearly the connected sets C with $|\pi_1 C| \leq 1$ are exactly the sets mentioned under (i), since for each x the subspace $\{x\} \times J$ is homeomorphic to J . Now let $|\pi_1 C| > 1$. We first show that if C is connected it must satisfy the conditions mentioned under (ii). Since

$$X \setminus (x,0) = \{(u,v) \in X \mid u < x\} + \{(u,v) \in X \mid u > x\} + \\ + \{(x,v) \in X \mid v > 0\}$$

and

$$X \setminus (x,y) = \{(x,v) \in X \mid v > y\} + X \setminus \{(x,v) \in X \mid v \geq y\}$$

we have the convexity of L and $\pi_2(C \cap \{x\} \times J)$.

[Note that these separations prove that X is treelike as soon as we know that X is connected.]

Next if U is an open interval in \mathbb{R} contained in the closure of E_k : Let $D = E_k \cup (\emptyset \setminus U)$, then D is dense in \mathbb{R} . Choose $x \in U$ such that $[x] = \psi(D)$, and i such that $(x - \frac{1}{i}, x + \frac{1}{i}) \subset U$; now $U_i((x,0)) \cap C$ contains only points with first coordinate x , so $C \cap \{x\} \times J$ is clopen in C , a contradiction.

Conversely, suppose C satisfies the conditions of (ii) but is, nevertheless, disconnected: $C = A_1 + A_2$. If $(x,y) \in A_i$ then $(x,0) \in A_i$ ($i = 1,2$) since $C \cap \{x\} \times J$ is connected and contains $(x,0)$. Hence we have $L = L_1 \cup L_2$ and $A_i = C \cap \pi_1^{-1}(L_i)$.

Consider a point $(x,0) \in A_1$ and an open nbd $U_i((x,0)) \cap C$ contained in A_1 . Let $(q,0) \in U_i((x,0)) \cap C$ such that $[q] = \psi(Q)$, then for all k

$$U_k((q,0)) \cap U_i((x,0)) \cap C \neq \emptyset,$$

that is, $(q,0) \in A_1$. But now if $(r,0) \in U_i((x,0)) \cap C$ then each nbd of $(r,0)$ in C contains points $(q,0)$ with $[q] = \psi(Q)$ in its closure, i.e. $(r,0) \in A_1$.

This proves that L_1 and L_2 are open in \mathbb{R} , so that $L = L_1 + L_2$. But L is connected. Contradiction.

This proves the claim.

In order to prove (INT2) we merely have to observe that the union of two nowhere dense sets is again nowhere dense, so that the intersection of two connected sets has one of the forms (i) or (ii).

Finally X does not satisfy (INT) since $S((a,0),(b,0)) = [a,b] \times \{0\}$ is totally disconnected (closed and discrete).

Concerning the separation properties of X we may observe that X is functionally Hausdorff but not regular.

8. SUPERCOMPACTNESS OF COMPACT TREELIKE SPACES

THEOREM 20. *A compact treelike space is supercompact.*

[A space is called supercompact if it has an open subbase \mathcal{B} such that each cover of the space with sets from \mathcal{B} has a subcover consisting of (at most) two elements.]

This theorem has been proved independently by J. VAN MILL [21] and A.E. BROUWER & A. SCHRIJVER [9]. Here we shall give a general result of which this is an immediate corollary.

DEFINITION. Let X be a set, \mathcal{A} a collection of subsets.

\mathcal{A} is called *u-binary* if each finite covering of X with elements from \mathcal{A} has a subcovering of at most two elements.

\mathcal{A} is called *i-binary* if $\mathcal{A}^{\odot} := \{X \setminus A \mid A \in \mathcal{A}\}$ is u-binary, i.e. if each finite subset of \mathcal{A} with empty intersection contains two disjoint elements (except for degenerate cases).

\mathcal{A} is called *forestlike* if $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2 = \emptyset$ or $A_1 \subset A_2$ or $A_2 \subset A_1$ or $A_1 \cup A_2 = X$.

THEOREM 21. *Let \mathcal{C} be a collection of connected subsets of a connected topological space X . Let \mathcal{S} be a collection of components of complements of sets from \mathcal{C} . Then*

- (i) *if \mathcal{C} is forestlike then \mathcal{S} is forestlike;*
- (ii) *if \mathcal{C} is i-binary then \mathcal{S} is u-binary.*

COROLLARY. *If the open segments of X form an open subbase and X is compact then X is supercompact. In particular a compact treelike space is supercompact.*

PROOF of theorem 21. (i) Let S_i be a component of $X \setminus C_i$ ($i = 1, 2$).

A. If $C_1 \subset C_2$ then either $S_1 \cap S_2 = \emptyset$ or $S_1 \supset S_2$ and likewise if $C_2 \subset C_1$.

B. If $C_1 \cup C_2 = X$ then $S_1 \cap S_2 = \emptyset$.

C. Let $C_1 \cap C_2 = \emptyset$ and assume $S_1 \cap S_2 \neq \emptyset$, $S_1 \not\subset S_2$, $S_2 \not\subset S_1$. Then $S_1 \cup S_2$ is connected and strictly larger than S_2 hence S_1 intersects C_2 . But then $S_1 \cup C_2$ is connected in $X \setminus C_1$ hence $C_2 \subset S_1$. Now since $X \setminus S_1$ is connected in $X \setminus C_2$ it follows that $X \setminus S_1 \subset S_2$ and $S_1 \cup S_2 = X$.

(ii) Let $\bigcup_{i=1}^k S_i = X$ be a minimal covering of X where $k \geq 3$. Let S_i be a

component of $X \setminus C_i$ ($1 \leq i \leq k$).

If $C_i \cap C_j = \emptyset$ then also $S_i \cap C_j = \emptyset$ since otherwise $C_j \subset S_i$, $X \setminus S_i$ is connected in $X \setminus C_j$ hence either $X \setminus S_i \subset S_j$ i.e. $S_i \cup S_j = X$, or $(X \setminus S_i) \cap S_j = \emptyset$, i.e. $S_j \subset S_i$, but both cases are impossible since the cover was assumed to be minimal.

Define a graph with vertex set $V = \{C_i \mid 1 \leq i \leq k\}$ and edges $\{(C_i, C_j) \mid C_i \cap C_j = \emptyset\}$. Let M be a maximal independent set of vertices. Now

$$\begin{aligned} \bigcap \{C_i \mid C_i \in M\} &\subset \bigcap \{X \setminus S_i \mid C_i \in M\} \cap \bigcap \{C_i \mid C_i \in M\} \subset \\ &\subset \bigcap_{i=1}^k (X \setminus S_i) = \emptyset, \end{aligned}$$

hence (since C is i -binary) two elements of M are disjoint, contradicting the definition of M . \square

PROOF of corollary: (i) $C = \{\{x\} \mid x \in X\}$ is an i -binary collection of connected subsets of X .

(ii) If X is treelike then all segments are open, and the segments separate points, i.e. generate a Hausdorff topology. If X is moreover compact then the segments must form a subbase. \square

REMARK. J. VAN MILL & A. SCHRIJVER [22, thm. 4.3] prove the following (for T_1 -spaces X):

X is compact treelike iff X possesses an i -binary normal connected closed subbase which is forestlike.

This combined with the above corollary yields the following result:

COROLLARY. A connected Hausdorff space X is locally peripherally compact treelike iff the open segments of X form an open subbase. \square

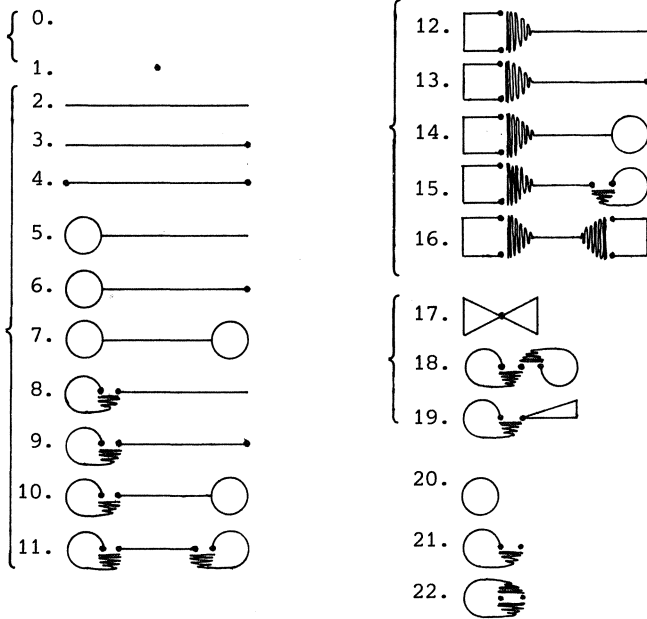
If X is an infinite set with the cofinite topology then the open segments form an open subbase (hence X is supercompact) but X is not treelike. Therefore, the "Hausdorff" requirement is indispensable.

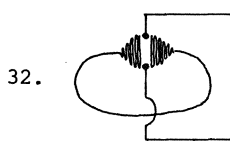
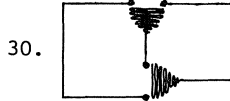
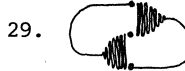
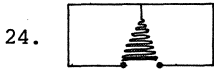
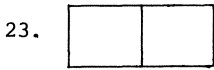
CHAPTER IV

CLASSIFICATION OF THE CONNECTED TOPOLOGICAL SPACES
 THAT CANNOT BE BROKEN UP INTO MORE THAN TWO COMPONENTS
 BY REMOVAL OF A CONNECTED SUBSET

1. THEOREM

Let X be a connected T_1 -space such that $X \setminus C$ decomposes into at most two components whenever C is a connected subset of X . Then X is of one of the following types:





or, what is the same:



These pictures are meant to suggest the *connectivity structure* of the space - rather than the topological structure; i.e., in each picture X is a union of (weakly) orderable connected subsets, joined in the way shown. E.g.,



can be read as:

$[p, q]$ and $[r, s]$ are weakly orderable connected sets,

$[r, s]$ is closed, $[p, q]$ is open and $\partial[p, q] = \{r\}$.

An example of such a space is provided by the subset

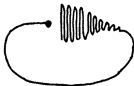
$$\{(x, y) \in \mathbb{R}^2 \mid ((x, y) = (0, 1)) \vee (y = -1 \wedge 0 \leq x \leq 1) \vee (y = \sin \frac{1}{x} \wedge -1 \leq x < 0)\}$$

of the plane, which explains the representation chosen.

This theorem does not say anything about the topology of X but describes the collection of connected subsets of X . [For instance



is an example of type 2, and



is an example of type 20.]

[Note that all concepts occurring in the statement of the theorem can be expressed in terms of connected sets only, without reference to the topology:

- X is T_1 means: for all $x, y \in X$, if $x \neq y$ then $\{x, y\}$ is disconnected;
- $C \subset X$ is orderable and connected means: there is a total order on C such that the connected subsets of C are just the orderintervals;
- $p \in \bar{C}$ for a connected set C means: $C \cup \{p\}$ is connected.]

2. LEMMA

Let X be a connected T_1 -space such that for each connected subset C of X , $X \setminus C$ has at most two components. Then for each n -tuple of connected subsets C_i ($1 \leq i \leq n$) of X $X \setminus \bigcup_{i=1}^n C_i$ has at most $n+1$ components.

PROOF. By induction: the case $n = 0$ is trivial and when $n = 1$ the conclusion equals the hypothesis. Let $n \geq 2$ and let C_i be nonempty ($1 \leq i \leq n$). We may assume $C_i \cap C_j = \emptyset$ ($1 \leq i < j \leq n$). Suppose

$$X \setminus \bigcup_{i=1}^{n-1} C_i = \bigcup_{i=1}^k S_i,$$

where each S_i is connected and $C_n \subset S_1$. Let $S_1 \setminus C_n = A_1 + \dots + A_m$ (where the A_j are not necessarily connected since it is not known a priori whether $S_1 \setminus C_n$ has finitely many components or not).

(i) If for some $j \leq m$ and $1 < n$ $C_n \cup A_j \cup C_1$ is connected, then

$$X \setminus \left(\bigcup_{\substack{1 \leq i \leq n-1 \\ i \neq 1}} C_i \cup (C_n \cup A_j \cup C_1) \right) = \bigcup_{i=2}^k S_i + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} A_i,$$

and, therefore, by the induction hypothesis $(k-1) + (m-1) \leq n$. In particular the number of components of $S_1 \setminus C_n$ must be finite and henceforth we may assume that the A_j ($j = 1, \dots, m$) are the components of $S_1 \setminus C_n$. Now

$$X \setminus \bigcup_{i=1}^n C_i = \sum_{i=2}^k S_i + \sum_{i=1}^m A_i$$

has $(k-1) + m \leq n + 1$ components, as required.

(ii) On the other hand, $C_n \cup A_j$ is connected for $1 \leq j \leq m$, so if no $C_n \cup A_j \cup C_1$ is connected this means that $(C_n \cup A_j)^- \cap C_1 = (C_n \cup A_j) \cap C_1^- = \emptyset$ for all $1 < n$ and $j \leq m$. But in that case

$$X \setminus C_n = A_1 + \dots + A_m + \left(\bigcup_{i=2}^k S_i \cup \bigcup_{i=1}^{n-1} C_i \right)$$

and therefore, (since $n \geq 2$ and $C_i \neq \emptyset$ the last term is nonempty) $m \leq 1$. Again it follows that

$$X \setminus \bigcup_{i=1}^n C_i = \sum_{i=2}^k S_i + A_1$$

has at most $k < n + 1$ components. \square

3. PROOF OF THE THEOREM

Let us call a space X a C -space if it satisfies the hypothesis of the theorem. The idea of the proof is that X must be broken up into pieces that have a simple structure (orderable or cyclically orderable), and then the structure of X is determined from the structure of these pieces and the way they are joined together.

I. This process is easy when X has a cut point, and goes as follows:

A. Let Z be the collection of cut points of X . Then Z is a connected orderable subspace of X such that each open order-interval (z_1, z_2) in Z is open in X and each closed order-interval $[z_1, z_2]$ in Z is closed in X , and $\overline{(z_1, z_2)} = [z_1, z_2]$ if $z_1 \neq z_2$.

(PROOF: the statement is trivial if Z contains at most one point; if Z contains two points p, q and $X \setminus p = \underset{q}{A} + \underset{p}{B}$, $X \setminus q = \underset{q}{A} + \underset{p}{B}$ then $Y = \overline{A} \cap \overline{B}$ is

connected, and each point of $Y \setminus \{p, q\}$ separates p and q , for if not then either $Y \setminus z$ is connected and $X \setminus (Y \setminus r) = A_p + \{r\} + B_q$ or $Y \setminus r = \overset{E+F}{p, q \neq \emptyset}$ and $X \setminus (Eu\{r\}) = A_p + F + B_q$; both times a contradiction. Therefore, Y is orderable with end points p and q and $Y \subset Z$, etc.)

B. Let p be a cut point of X , $X \setminus p = A_p + B_p$ (then $\overline{A_p} = A_p \cup \{p\}$ is called the left half of X). Let $Y = (X \setminus Z) \cap A_p$, then since Z is connected, Y is connected. [For: $Z \cap \overline{A_p}$ is connected and $X \setminus (Z \cap \overline{A_p}) = B_p + Y$.]

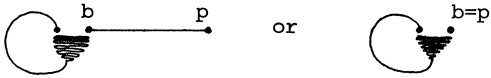
α . If Y is empty the left half of X has the type $\text{---} \bullet p$.

β . If Y consists of a single point, the left half of X has the type $\bullet \text{---} p$.

Let Y contain at least two points. If Z has a smallest point b (possibly $Z = \{b\}$) then Y is a C -space since if $Y \setminus C = Y_1 + Y_2 + Y_3$ then $X \setminus (Cu\{b\}) = Y_1 + Y_2 + Y_3 + B_b$ (where $X \setminus b = Y + B_b$) contradicting the lemma.

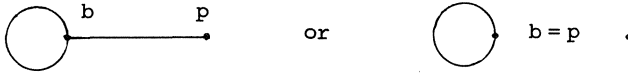
γ . Supposing that Z indeed has a smallest point b , we discern two cases, (γ_1) and (γ_2) depending on whether Y has a cut point or not.

γ_1 . Suppose Y has no cut points. Let $y_1, y_2 \in Y$, then $Y \setminus \{y_1, y_2\}$ is disconnected [otherwise $X \setminus (Y \setminus \{y_1, y_2\}) = \{y_1\} + \{y_2\} + (X \setminus Y)$, hence Y is cyclically orderable (see e.g. KOK [18] p.29). It remains to determine the way b is connected to Y (observe that $b \in \overline{Y}$). Let b' be the intersection of all closed "arcs" in Y which have b in their closure. [If the "arcs" J_1 and J_2 have b in their closure, then also $b \in \overline{J_1 \cap J_2}$, otherwise $(J_1 \setminus J_2) \cup (J_2 \setminus J_1) \cup \{b\}$ is connected and $X \setminus ((J_1 \setminus J_2) \cup (J_2 \setminus J_1) \cup \{b\}) = Y \setminus (J_1 \cup J_2) + (J_1 \cap J_2) + B_b$. Next, $\bigcup_{b \in J} J = \{b'\}$ since this intersection is nonempty (Y is compact in its cyclic order topology) and cannot contain more than one point.] If (a, b') and (b', c) are two disjoint "arcs" in Y , then either $b \in (a, b')^-$ or $b \in (b', c)^-$ but not both, since otherwise $X \setminus ((a, b') \cup \{b\} \cup (b', c)) = \{b'\} + [c, a] + B_b$. Therefore, the left half of X has the type



γ_2 . If Y has a cut point: $Y \setminus c = Y_1 + Y_2$ then c is an end point in $\overline{Y} = Y \cup \{b\}$ [otherwise $c \in Z$], hence $Y_1 \cup \{b\}$ and $Y_2 \cup \{b\}$ are connected. In this case each point of Y is a cut point of Y , for if $Y \setminus d$ is connected with $d \in Y$, then $Y_1 \setminus d$ and hence also $(Y_1 \setminus d) \cup \{b, c\}$ is connected and

$X \setminus ((Y_1 \setminus d) \cup \{b, c\}) = Y_2 + \{d\} + B_b$, a contradiction. Since Y was a C -space, Y is orderable (by part A of this proof) and it is easily verified that $Y \cup \{b\}$ is cyclically orderable. Therefore, the left half of X has the type



δ . Next suppose that Z does not have a smallest point (and $Z \neq \emptyset$).

(i) $|Y \cap \bar{Z}| = 2$.

For: by (A) $Z \cap \bar{A}_p$ is open in \bar{A}_p and Y is closed in X , hence $Y \cap \bar{Z} \neq \emptyset$.

If $Y \cap \bar{Z} = \{b\}$ then b is a cut point of X , which is impossible. Hence $|Y \cap \bar{Z}| \geq 2$. But if $\{x, y, z\} \subset Y \cap \bar{Z}$ ($x \neq y \neq z \neq x$) and $X \setminus \{x, y\} = A + \underset{Z, z}{B}$ then

$$X \setminus (\{x, y\} \cup (\bar{A}_p \cap Z)) = A + \underset{p}{B_p} + \underset{z}{B} \cap Y \setminus \{x, y\},$$

hence $A = \emptyset$, i.e., $X \setminus \{x, y\}$ is connected.

Consequently $\bar{A}_p \setminus \{x, y\}$ is connected and $X \setminus (\bar{A}_p \setminus \{x, y\}) = \{x\} + \{y\} + B_p$, a contradiction. Therefore, $Y \cap \bar{Z} = \{x, y\}$, where $x \neq y$.

(ii) Each point of $Y \setminus \{x, y\}$ separates x and y in Y .

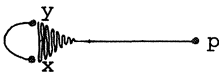
For suppose $z \in Y \setminus \{x, y\}$, z does not separate x and y , then $Y \setminus z$ is connected since otherwise z would be a cut point of X . Also $Y \setminus \{x, y\}$ is connected since otherwise $(\bar{A}_p \cap Z) \cup \{x, y\}$ would separate X into at least three components.

If $Y \setminus \{x, z\}$ is connected then $X \setminus ((Y \setminus \{x, z\}) \cup (\bar{A}_p \cap Z)) = B_p + \{x\} + \{z\}$, a contradiction.

If $Y \setminus \{x, z\} = A + \underset{y}{T}$ then $X \setminus (T \cup \{x, z\}) = A + (X \setminus Y)$ hence A is connected and T is connected; likewise $Y \setminus \{y, z\} = B + S$ where B and S are connected and non-empty. Now $Y \setminus \{x, y, z\} = A + B + (S \cap T)$ where $S \cap T \neq \emptyset$ since otherwise $Y \setminus z = (A \cup \{x\}) + (B \cup \{y\})$. $S = A \cup \{x\} \cup (S \cap T)$ is connected, hence $x \in (S \cap T)^-$ and likewise $y \in (S \cap T)^-$; and since $Y \setminus \{x, y\}$ is connected $z \in (S \cap T)^-$.

Therefore, $(S \cap T) \cup \{x, y, z\}$ is connected and $X \setminus ((S \cap T) \cup \{x, y, z\}) = X \setminus Y + A + B$, contradiction.

This proves assertion (ii) and, therefore, Y is orderable with end points x and y and the left half of X has the type



C. When X has a cut point, both halves of X must have one of the types found above and, therefore, X has one of the types 2 - 19. When X contains at most one point it has type 0 or 1.

II. Now let X contain at least two points, and let each point of X be an end point of X .

D. If for some connected $C \subset X$ $X \setminus C$ is disconnected, then $X \setminus C$ has two components and each is orderable between each pair of its points. [The components need not be orderable themselves since they might be cyclically connected:



PROOF. Let $X \setminus C = S + T$. Observe that

$$\text{if } p \in T \text{ then } C \cup T \setminus p \text{ is connected} \tag{+),}$$

otherwise p would be a cut point of X . If for some $q, r \in T$ $(C \cup T) \setminus \{q, r\}$ is connected then $X \setminus ((C \cup T) \setminus \{q, r\}) = \{q\} + \{r\} + S$. If $(C \cup T) \setminus \{q, r\} = A + B + D$ then $X \setminus \{q, r\} = (A \cup S) + B + D$ hence by the lemma A , B and D are connected; therefore, $A \cup \{q, r\}$ is connected [for: T contains no cut points of $C \cup T$ by (+)], and since $X \setminus (A \cup \{q, r\}) = B + D + S$ it follows that $D = \emptyset$. So we have $(C \cup T) \setminus \{q, r\} = \overset{A}{C} + \overset{B}{\neq \emptyset}$, where A and B are connected, for all pairs q, r ($q \neq r$) in T . Define

$$(q, r) := B \text{ and } [q, r] := \bar{B} = B \cup \{q, r\}.$$

Let $s \in (q, r)$, then $[q, s] \subset [q, r]$ and $[s, r] \subset [q, r]$. By (C) $[q, r] \setminus s$ has at most two components. For, if $[q, r] \setminus s = \overset{E}{q, r} + F$ then $(A \cup \{q, r\}) \cup (E \cup \{s\})$ is connected and has in its complement (which is $S + F$) at most two components, so F is connected. But if $[q, r] \setminus s$ has a component not containing q or r then s is a cut point in X ; therefore, $F = \emptyset$ and $[q, r] \setminus s$ has at most two components.

Now these components have to be $[q, s]$ and $(s, r]$ (where obviously $[a, b]$ means $(a, b) \cup \{a\}$ and $(a, b]$ means $(a, b) \cup \{b\}$).

For: it suffices to show that

- (i) s is a cut point of $[q, r]$, and
- (ii) $[q, r] \setminus s = [q, s] \cup (s, r]$.

Ad (i): Suppose $[q,r]\setminus s$ is connected. By the lemma $X\setminus\{q,r,s\}$ has at most four components, hence we may write

$$(CUT)\setminus\{q,r,s\} = \underset{C}{A} + B_1 + B_2 + B_3 \quad ,$$

where A, B_1, B_2, B_3 are connected (and some of the B_i possibly empty). $A \cup \{q\} \cup B_1 \cup B_2 \cup B_3$ cannot be connected, since its complement is $S + \{r\} + \{s\}$. Therefore, there is some nonempty B_i such that $q \notin \overline{B_i}$. Likewise for some j we have $r \notin \overline{B_j}$. But if $\overline{B_k} \cap \{q,r\} = \emptyset$ then $B_k = \emptyset$ (otherwise s would be a cut point of $C \cup T$, contrary to (+)), so $i \neq j$, say $i = 1$ and $j = 2$. Since $[q,r]\setminus s = B_1 \cup B_2 \cup B_3 \cup \{q,r\}$ is connected by assumption it follows that $\overline{B_3} = B_3 \cup \{q,r,s\}$. But then the connected set $A \cup \{q,r,s\} \cup B_3$ has complement $S + B_1 + B_2$, a contradiction.

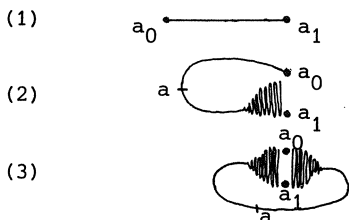
Ad (ii): Using the notations of "ad (i)" we find again $\overline{B_1} = B_1 \cup \{r,s\}$ and $\overline{B_2} = B_2 \cup \{q,s\}$, i.e., $B_1 = (r,s)$ and $B_2 = (q,s)$. We have to prove $B_3 = \emptyset$. If not, then either $\overline{B_3} = B_3 \cup \{q,r,s\}$ which leads to the same contradiction as found above or, say, $\overline{B_3} = B_3 \cup \{q,s\}$ and $B_3 = B_2 = (q,s)$, a contradiction again.

This proves that $[q,r]\setminus s = [q,s] + (s,r]$, i.e., $[q,r]$ is orderable with end points q and r , which justifies the notation.

E. Let for some connected $C \subset X$ $X\setminus C = A + B$ where A and B are nonempty, and A contains at least two points. Then A is either orderable or cyclically orderable (and not orderable).

PROOF. Fixing the order of two points of A , the orders on all subintervals of A induce a total order on A in which all open intervals (a_1, a_2) are open and connected. A is orderable iff also the intervals (\cdot, a) and (a, \cdot) are open. But if A does not have a first element then (\cdot, a) is a union of open sets and hence open. Therefore, assume A has a first element a_0 .

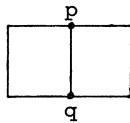
1) If A also has a last element a_1 then A can have the following types:



The second and third possibility can not occur, since if we choose $a \in A \setminus \{a_0, a_1\}$ then $A \setminus \{a, a_1\}$ would be connected. Therefore, in this case A is orderable, and $A \cap \bar{C} = \{a_0, a_1\}$.

2) On the other hand, if A does not have a last element a_1 then either $a_0 \notin (a, \cdot)^-$ and A is orderable or $a_0 \in (a, \cdot)^-$ and A is cyclically orderable. \square

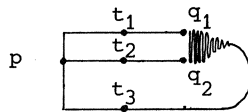
F. Suppose for some $p, q \in X$ $X \setminus \{p, q\} = S_1 + S_2 + S_3$ where $S_i \neq \emptyset$ ($i = 1, 2, 3$). Then X is of type



(type 23).

PROOF. $X \setminus (S_1 \cup \{p, q\}) = S_2 + S_3$ hence S_2 and S_3 and likewise S_1 are orderable between each pair of their points. If $\{p\} \cup S_1 \cup \{q\}$ were not orderable with end points p and q then it would contain a point r not separating p and q . But then $X \setminus ((S_1 \setminus \{r\}) \cup \{p, q\}) = \{r\} + S_2 + S_3$, a contradiction. Henceforth we shall assume that $X \setminus \{p, q\}$ has at most two components for each pair of points p, q .

G. Suppose that for some connected subset C and point p of X $X \setminus C$ is connected while $X \setminus (C \cup \{p\}) = S_1 + S_2 + S_3$ ($S_i \neq \emptyset, i = 1, 2, 3$). Then X has the type



(type 24).

PROOF. (i) By D [and $X \setminus (C \cup S_i \cup \{p\}) = S_j + S_k$ if $\{i, j, k\} = \{1, 2, 3\}$] it follows that S_i is orderable between each pair of its points and, therefore, like above (see F) that $\{p\} \cup S_i$ is orderable with first point p ($i = 1, 2, 3$).

(ii) We may assume that C is closed and $C \cap \bar{S}_i = \{t_i\}$ for some point t_i ($i = 1, 2, 3$) for if this is not already the case, then choose $t_i \in S_i$ and redefine C and S_i by

$$S_i := (p, t_i) \quad (i = 1, 2, 3) ,$$

$$C := X \setminus (S_1 \cup S_2 \cup S_3 \cup \{p\}) .$$

(iii) Let $t \in C \setminus \{T_1, t_2, t_3\}$. If $C \setminus t$ is connected then we get a contradiction with the lemma:

$$X \setminus ((C \setminus t) \cup \{p\}) = \{t\} + S_1 + S_2 + S_3.$$

If

$$C \setminus t = t_1, t_2, t_3 + B$$

then t would be a cut point of X . If

$$C \setminus t = \begin{matrix} A_1 & + & A_2 & + & A_3 \\ t_1 & & t_2 & & t_3 \end{matrix}$$

then

$$X \setminus t, p = (A_1 \cup S_1) + (A_2 \cup S_2) + (A_3 \cup S_3),$$

but this case has been excluded already (see F). Therefore, $C \setminus t$ has exactly two components, one containing two points t_i , the other the third point. Let

$$T_i = \{t_i\} \cup \{t \in C \setminus \{t_1, t_2, t_3\} \mid C \setminus t = t_j, t_k + \frac{B}{t_i}, \\ \{i, j, k\} = \{1, 2, 3\}\}.$$

Then $C = T_1 \cup T_2 \cup T_3$ and $T_i \cap T_j = \emptyset$ ($i \neq j$). If

$$C \setminus t = t_j, t_k + \frac{B}{t_i}$$

then $\{p\} \cup S_1 \cup B \cup \{t\}$ is orderable with end points p and t ; (because of (i) above, taking $C' = X \setminus (S_1 \cup S_2 \cup S_3 \cup \{p, t, t_i\}) \cup B$ instead of C); therefore, each of the T_i is connected and orderable, and each of the $\{p\} \cup S_i \cup T_i$ is connected and orderable. It remains to determine the way the T_i are joined together.

I. If all T_i have a last element q_i and $q_1 \in \overline{T_1 \setminus T_2 \cup T_3}$ then $(T_1 \setminus q_1) \cup T_2 \cup T_3$ is connected [for: p is not a cut point hence $T_1 \cup T_2 \cup T_3$ is connected] and $X \setminus (\{p\} \cup S_1 \cup (T_1 \setminus q_1) \cup T_2 \cup T_3) = S_2 + S_3 + \{q_1\}$, a contradiction. If $q_2, q_3 \in \overline{T_1}$ then again $(T_1 \setminus q_1) \cup T_2 \cup T_3$ is connected. Therefore, we must have say $q_1 \in \overline{T_2 \setminus T_3}$, $q_2 \in \overline{T_3 \setminus T_1}$, $q_3 \in \overline{T_1 \setminus T_2}$,

but from this again follows that $(T_1 \setminus q_1) \cup T_2 \cup T_3$ is connected.

- II. If none of the T_i has a last element then p is a cut point.
- III. If only one of the T_i , say T_1 has a last element q_1 then $X \setminus p, q_1 = (S_1 \cup T_1 \setminus q_1) + (S_2 \cup T_2) + (S_3 \cup T_3)$ contrary to the hypothesis.
- IV. So, e.g., T_1 and T_2 do have last elements q_1 and q_2 while T_3 has no last element. If $q_2 \notin \overline{T_3}$ then $X \setminus p, q_1 = (S_3 \cup T_3) + (S_1 \cup (T_1 \setminus q_1) \cup S_2 \cup T_2)$ and by definition of T_3 $q_1 \in T_3$. Therefore, $\{q_1, q_2\} \subset \overline{T_3}$. If $q_1 \in \overline{T_2}$ then $T_1 \cup (T_2 \setminus q_2) \cup T_3$ would be connected which is impossible as we saw above. So we have:

$$\{q_1, q_2\} \in \overline{T_3}, \quad q_1 \notin \overline{T_2}, \quad q_2 \notin \overline{T_1}$$

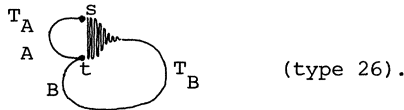
and X has the announced type.

H. Suppose that $X \setminus C = A + B$ where both A and B contain at least two points. Again we may suppose $A = (a_1, a_2)$, $B = (b_1, b_2)$, $|\{a_1, a_2, b_1, b_2\}| = 4$, C is closed, $C \cap \bar{A} = \{a_1, a_2\}$, $C \cap \bar{B} = \{b_1, b_2\}$. If $t \in C \setminus \{a_1, a_2, b_1, b_2\}$ then $C \setminus t$ cannot be connected [by (C)], each component of $C \setminus t$ must intersect $\{a_1, a_2, b_1, b_2\}$, and at least one component of $C \setminus t$ must intersect both $\{a_1, a_2\}$ and $\{b_1, b_2\}$. Hence $C \setminus t$ has at most three components.

α . Suppose for some $t \neq a_1, a_2, b_1, b_2$

$$C \setminus t = \begin{matrix} C_1 & + & C_2 & + & C_3 \\ a_1, b_1 & & a_2 & & b_2 \end{matrix} .$$

Then X has the type



PROOF. (i) Let $S = C_1 \cup \{t\}$, then $X \setminus S = (A \cup C_2) + (B \cup C_3)$ hence [by E] $A \cup C_2$ and $B \cup C_3$ are either orderable or cyclically orderable, where the latter case occurs if and only if $a_1 \in \overline{C_2}$ (resp. $b_1 \in \overline{C_3}$). But $a_1 \in C_1 \subset C \setminus \overline{C_2}$, hence both $A \cup C_2$ and $B \cup C_3$ are orderable: $A \cup C_2 = (a_1, t)$ and $B \cup C_3 = (b_1, t)$.

(ii) If for some r

$$(C_1 \cup \{t\}) \setminus r = \begin{matrix} C_{11} & + & C_{12} & + & C_{13} \\ a_1 & & a_2 & & t \end{matrix}$$

then

$$X \setminus \{t, r\} = (C_{11} \cup A \cup C_2) + (C_{12} \cup B \cup C_3) + (C_{13} \setminus t)$$

contrary to the hypothesis (see F).

If for some r

$$(C_1 \cup \{t\}) \setminus r = \begin{matrix} C_{11} & + & C_{12} \\ a_1, b_1 & & t \end{matrix}$$

then

$$X \setminus C_{11}, t = (C_{12} \cup \{r\} \setminus t) + (A \cup C_2) + (B \cup C_3)$$

contrary to the hypothesis since $X \setminus C_{11}$ is connected (see G).

Now let

$$T_A = \{a_1\} \cup \{r \mid (C_1 \cup \{t\}) \setminus r = \begin{matrix} C_1 & + & C_2 \\ a_1 & & b_1, t \end{matrix}\}$$

and

$$T_B = \{b_1\} \cup \{r \mid (C_1 \cup \{t\}) \setminus r = \begin{matrix} C_1 & + & C_2 \\ b_1 & & a_1, t \end{matrix}\},$$

then

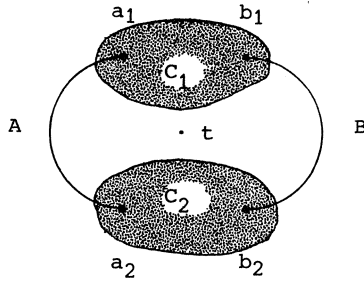
$$C_1 = T_A \cup T_B \cup \{t\} \quad \text{and} \quad T_A \cap T_B = \emptyset.$$

Again both $A \cup T_A$ and $B \cup T_B$ are orderable between each pair of their points. Now it follows that, e.g., T_A has a last point s , and $s, t \in \overline{T_B}$, $t \notin \overline{T_A}$ (since s separates $T_A \setminus s$ from t) and X has the announced type. \square

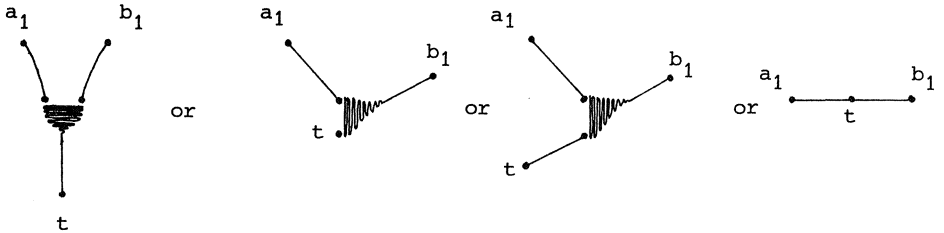
β . Suppose that for no t $C \setminus t$ has 3 components, but

$$C \setminus t = \begin{matrix} C_1 & + & C_2 \\ a_1, b_1 & & a_2, b_2 \end{matrix}$$

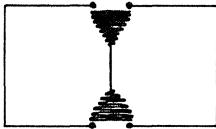
for some t . Now X looks like



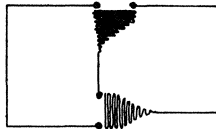
By investigating the behaviour of C_1 when one of its points is deleted one finds that C_1 must have one of the types



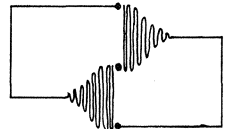
and it follows that X must have one of the types:



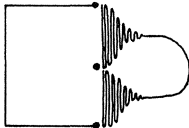
type 27



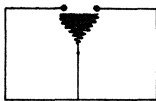
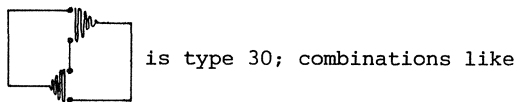
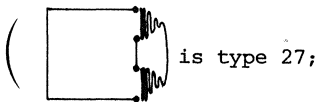
type 30



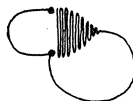
type 29



type 28.



and

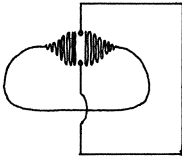


are excluded already.

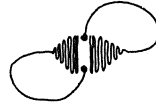
)

γ . If for each $t \neq a_1, a_2, b_1, b_2$ $C \setminus t$ has two components, one of which contains three of the points a_1, a_2, b_1, b_2 while the other contains the fourth point, then like before $C = T_1 \cup T_2 \cup T_3 \cup T_4$, each T_i is orderable and $T_i \cap T_j = \emptyset$. By considering all possible ways of joining the T_i one finds the types:

(i) two of the T_i have a last point.

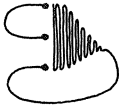


(type 32)

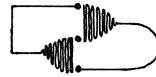


(type 31)

(ii) three of the T_i have a last point.



(type 25)



(type 29)



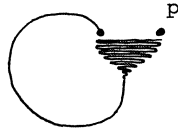
(type 28).

J. Finally, we have to consider the case that for all connected subsets C of X either $X \setminus C$ is connected or $X \setminus C = A + \{p\}$. If always $X \setminus C$ connected then X is cyclically orderable:



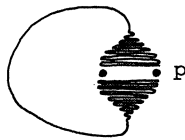
(type 20).

If $X \setminus C = A + \{p\}$ then consider the behaviour of C upon deletion of one point. One finds that X has one of the types



(type 21)

or

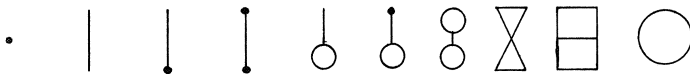


(type 22).

This completes the proof. \square

4. REMARKS

In the presence of conditions like local connectedness and (T_2) connections like \sim : disappear and only the types



remain. If X is moreover separable then one has characterizations like:

1. X is homeomorphic with the circle iff for each connected subset C $C \setminus X$ is connected (see []).
2. X is homeomorphic with the figure eight if it contains exactly one cut point.
3. X is homomomorphic to the square with one diagonal if it has no cut points, and there are two points $p, q \in X$ such that $X \setminus \{p, q\}$ does not have exactly two components.

Of course the only treelike examples of a space satisfying (C) are the orderable ones. More generally we have:

THEOREM. $(W) + (C) \Rightarrow (O)$.

PROOF. Let X satisfy (W) and (C) (in particular X is (T_1) and connected). In [18] it is proved that $(H) + (B') \rightarrow (O)$. Since clearly $(C) \Rightarrow (B')$ it suffices to prove that $(C) + (W) \Rightarrow (H)$.

Suppose S is a connected set with three distinct end points p, q and r . Then by propositions 2.2 and 2.3, $S \setminus \{p, q, r\}$ is connected. By (C) $X \setminus (S \setminus \{p, q, r\})$ has at most two components hence, e.g., p and q belong to the same component. But since $\{p, q\} \subset \overline{S \setminus \{p, q, r\}}$ this contradicts (W) . \square

Clearly for this analysis the assumption that $X \setminus C$ has at most two components was not very essential; more generally if it is given that for each connected subset C of X $X \setminus C$ has at most n components then an analogous classification can be made (where of course the number of types increases sharply with n); in fact it seems quite possible to produce such a classification by computer.

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