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A THEORETICAL AND COMPUTATIONAL STUDY OF GENERALIZED ALIQUOT SEQUENCES

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CONTENTS

PREFACE

Aliquot sequences are defined according to the following rule: a leading term is given and every subsequent term is the sum of the "aliquot parts" of the preceding term. The aliquot parts of a number > 1 are *all* divisors (including 1) *less* than that number. When a term equals one of the preceding terms, we have a so called cycle. Examples of cycles are perfect numbers (cycle-length=1) and amicable number pairs (cycle-length=2). These sequences were studied already by the Pythagoreans and later on by Euler, Catalan, Dickson, and many others.

The advent of (high-speed) computers has stimulated the renewed interest in aliquot sequences, because the computers made possible the extended computation of "difficult" sequences (i.e. sequences the terms of which become too large for factorization by hand), especially in order to get more statistical information about the asymptotic behaviour of aliquot sequences. This information is interesting, in particular in view of the famous Catalan-Dickson conjecture which states that *all* aliquot sequences are bounded. In fact, very recently and on the basis of much statistical and heuristical material, R.K. Guy has put forward the conjecture that *almost all aliquot sequences with even leading term are unbounded!*

In this monograph a theoretical and computational study of *generalized* aliquot sequences is presented. Generalized aliquot sequences are sequences every term of which (except the leader) is the sum of *certain*, but *not necessarily all* aliquot parts of the preceding term.

In chapter 1 generalized aliquot sequences are defined by use of a set F of arithmetical functions f which determine the aliquot parts to be summed in the computation of a term from the preceding one. For this reason, generalized aliquot sequences will be denoted by f-sequences.

Chapters 2 to 5 mainly present theoretical results. In chapter 2, for any $f \in F$ the existence of f-sequences with arbitrarily many monotonically increasing terms is proved. Moreover, the structure of cycles is investigated, and two construction methods for cycles are discussed. In chapter 3 five classes of functions $f \in F$ are indicated, which in subsequent chapters serve as test-cases for the computational experiments. In chapter 4 the distribution of the values of the functions $f \in F$ is investigated. Chapter 5 presents two methods for the computation of the mean value of the quotient of two subsequent terms of an f-sequence.

Chapters 6 to 9 mainly present computational results and analyses. In chapter 6 we present a selection of the results of systematic computations of f-sequences, for the testcases of chapter 3. The main subjects of chapter 7 are the proof of the existence of unbounded f-sequences, for certain $f \in F$, and the construction of such unbounded sequences. Chapter 8 deals with the computation of cycles for the test-cases of chapter 3. Finally, in chapter 9 we study untouchable numbers, i.e. numbers which can only be leaders of f-sequences.

The author's interest in aliquot sequences was awakened by Dr J.D. Alanen; he is very grateful to him for his interest and encouragement.

PRELIMINARIES AND NOTATION

As usual, $\mathbb{N}$ will denote the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers. Throughout, p will denote an arbitrary prime number, unless explicitly stated otherwise, and for any $r \in \mathbb{N}$, p_r is the r-th prime ($p_1 = 2$).

By $(a_1,a_2,\ldots,a_n)$ $(n \geq 2)$ we mean the greatest common divisor of the positive integers $a_1,a_2,\ldots,a_n$. If $(a_1,a_2,\ldots,a_n) = 1$, we say that $a_1,a_2,\ldots,a_n$ are relatively prime.

By $(a_1,a_2,\ldots,a_n)_k$ $(k \in \mathbb{N})$ we mean the greatest common k-th power divisor of $a_1,a_2,\ldots,a_n$. If $(a_1,a_2,\ldots,a_n)_k = 1$, we say that $a_1,a_2,\ldots,a_n$ are relatively k-prime. For any k the integer 1 is considered to be a k-th power divisor of any positive integer.

A *unitary* divisor d of n is a divisor of n with $(d,n/d) = 1$, i.e., every prime p dividing d does not divide n/d. If d is a unitary divisor of n, we write $d \parallel n$.

A k-*ary* divisor d of n $(k \in \mathbb{N})$ is a divisor of n with $(d,n/d)_k = 1$, i.e., every prime power p^k dividing d does not divide n/d.

A positive integer is k-*free* $(k \in \mathbb{N},\ k \geq 2)$ if it is not divisible by the k-th power of any prime. A 2-free integer is also called squarefree.

A positive integer is k-*full* $(k \in \mathbb{N},\ k \geq 2)$ if any of its prime divisors has multiplicity $\geq k$.

If $f : \mathbb{N} \to \mathbb{N}$ is an arithmetical function, then $n \in \mathbb{N}$ is called f-*abundant*, whenever $f(n) > 2n$.

Let $S = \{n_1,n_2,\ldots\}$ be an infinite set of positive integers and let $S(n)$ $(n \in \mathbb{N})$ be the number of elements of S not exceeding n. Then the lower (asymptotic) density and the upper (asymptotic) density of S are the values of

$$\liminf_{n \to \infty} S(n)/n \quad \text{and} \quad \limsup_{n \to \infty} S(n)/n, \quad \text{respectively.}$$

If the lower and upper density are equal, we say that the (asymptotic) density of S exists, with this common value.

Let f(x) and g(x) be two functions of the real variable x. Then by $f \sim g$ $(x \to \infty)$ we mean that $\lim_{x\to\infty} f/g = 1$.
By $f \asymp g$ we mean that there are constants C_1 and C_2 such that $C_1 g < f < C_2 g$.

The mean value M{f} of an arithmetical function $f : \mathbb{N} \to \mathbb{N}$ is the value of $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n)$, provided that this limit exists.

In the tables factorized numbers will sometimes be given with exponents in parentheses; for example, 2(2)3.5.11(2) means $2^2 3.5.11^2$.

CHAPTER 1

GENERALIZED ALIQUOT SEQUENCES AND THE CLASSICAL CASE

{*Tears of joy over man's tortuous journey to the beyond ...*
Elvin J. Lee}

Let $f : \mathbb{N} \to \mathbb{N}$ be an arithmetical function with the following two properties:

P1. f is multiplicative, i.e., if $(a,b) = 1$, then $f(ab) = f(a)f(b)$.

P2. For any $e \in \mathbb{N}$ a polynomial $W_e^f(x)$ of degree e in x is given, such that for any prime p $f(p^e) := W_e^f(p)$. The coefficients of $W_e^f(x)$ are restricted to the values 0 or 1 and $W_e^f(1) \geq 2$.

The set of all functions f with properties P1 and P2 will be denoted by F. It follows that if $f \in F$, then

$$f(1) = 1, \quad f(p) = p+1,$$

either $f(p^2) = p^2+1$, or $f(p^2) = p^2+p$, or $f(p^2) = p^2+p+1$,

either $f(p^3) = p^3+1$, or $f(p^3) = p^3+p$, or $f(p^3) = p^3+p^2$, or $f(p^3) = p^3+p+1$,

or $f(p^3) = p^3+p^2+1$, or $f(p^3) = p^3+p^2+p$, or $f(p^3) = p^3+p^2+p+1$,

and so on.

EXAMPLE 1.1 If for any $e \in \mathbb{N}$, $W_e^f(x) := x^e + x^{e-1} + \ldots + x + 1$, i.e., *all* coefficients of $W_e^f(x)$ are equal to 1, then f is the sum of the divisors function. It will be denoted, as usual, by σ.

EXAMPLE 1.2 If for any $e \in \mathbb{N}$, $W_e^f(x) := x^e + 1$, then f is the sum of the *unitary* divisors function. It will be denoted, as usual, by σ^*.

It also follows from P1 and P2 that $f(n)$ is the sum of n and *certain* other divisors of n; *which* other divisors depends on the choice of the polynomials $W_e^f(x)$. It is customary to call the divisors of n which are less than n the *aliquot* divisors of n.

DEFINITION 1.1 An *aliquot f-sequence with leader* $n \in \mathbb{N}$ (briefly called an *f-sequence on* n, or n-sequence if this gives no confusion) is a sequence

$n_0, n_1, n_2, \ldots$ of positive integers, such that

$$(1.1) \quad \begin{cases} n_0 = n \text{ and} \\ n_{i+1} = f(n_i) - n_i \end{cases} \qquad (i=0,1,2,\ldots).$$

Since $f(p^e) \geq p^e + 1$, we have $f(n) - n > 0$ for all $n \geq 2$, for any $f \in F$. The term n_i is sometimes denoted by $n : i$ (for typographical convenience). An n-sequence is *terminating* if there exists a value of ℓ for which $n_\ell = 1$, and this ℓ is also denoted by $\ell_f = \ell_f(n)$. An n-sequence is *periodic* if there is an $\ell' > 0$ and a $c > 0$ such that $n : (\ell'+c) = n : \ell'$. The least ℓ' with this property is also denoted by $\ell'_f = \ell'_f(n)$ and the least positive c, corresponding to this ℓ', is the *period* (or cycle length), and will be denoted by $c = c_f = c_f(n)$. The c different numbers $\{n : \ell', n : (\ell'+1), \ldots, n : (\ell'+c-1)\}$ are called an (f-)*cycle of length* c.
If $n < m$ and the two f-sequences on n and m, respectively, have a term in common, which is larger than all previous terms in either sequence, then the f-sequence on m is said to be *tributary* to the f-sequence on n.
A sequence which is not tributary to any other one is called a *main* sequence. Thus a bounded n-sequence is main if n is the least number which leads to its maximum. For the example $f = \sigma$, we have $318 : 4 = 498 : 3 = 798$, and 318 is the least number leading to the maximum $722961 = 318 : 32$, so the σ-sequence with leader 318 is main and the 498-sequence is tributary to it. Both sequences are terminating. The 562-sequence is characterized by the first four terms 562, 220, 284, 220; thus it is periodic, $\ell'_\sigma(562) = 1$ and $c_\sigma(562) = 2$. For the 220-sequence we have $\ell'_\sigma(220) = 0$ and $c_\sigma(220) = 2$.

The classical example of an f-sequence is the case in which $f(n)$ is the sum of *all* divisors of n ($f(n) = \sigma(n)$), so that $f(n) - n = \sigma(n) - n$ is the sum of all aliquot divisors of n.

CATALAN [7] was probably the first one to study this case. He conjectured that every (aliquot) σ-sequence contains either unity or a perfect number. PERROTT [27] gave the counterexample 220, 284, 220,... and DICKSON [10] revised Catalan's conjecture to: *Every (aliquot) σ-sequence contains either unity or a cycle* (which can be a perfect number, or an amicable pair as in Perrott's counterexample, or a cycle of length greater than two). The verification of this conjecture is very cumbersome, in particular when the terms become large, because in order to compute a term n_{k+1}, the complete factorization of n_k is needed.

The σ-sequence with least starting value and *unknown* behaviour is currently the 276-sequence. D.H. LEHMER [18] has recently computed the 433-rd term of this sequence, which is a 36-digit number. At present, there are 98 sequences with leader less than 10^4 whose behaviour is unknown. Most computational results on σ-sequences have been collected by GUY and SELFRIDGE in [18].

Nowadays, many researchers believe that the Catalan-Dickson conjecture is false. A partial result in this direction is LENSTRA's theorem (private communication dated April 10th, 1972): For any given $t \in \mathbb{N}$, σ-sequences can be constructed with at least t monotonically increasing terms. TE RIELE [30] proved the same theorem, but *on the condition* that there are infinitely many even perfect numbers.

CHAPTER 2

GENERAL PROPERTIES OF ALIQUOT f-SEQUENCES

In this chapter some general properties of f-sequences and f-cycles are proved.

PROPOSITION 2.1 *Let* $f \in F$ *and let*

$$am_i, am_{i+1}, \ldots, am_{i+k} \qquad (i \geq 0,\ k \geq 1)$$

be $k+1$ *consecutive terms of an* f*-sequence with* $(a, m_{i+j}) = 1$ *for* $j = 0, 1, \ldots, k-1$. *If* $b \in \mathbb{N}$ *is such that* $f(b)/b = f(a)/a$, $b \neq a$, *and* $(b, m_{i+j}) = 1$ *for* $j = 0, 1, \ldots, k-1$, *then*

$$bm_i, bm_{i+1}, \ldots, bm_{i+k}$$

are also $k+1$ *consecutive terms of an* f*-sequence.*

PROOF. Under the hypotheses, we have

$$\begin{aligned} f(bm_{i+j}) - bm_{i+j} &= f(b)f(m_{i+j}) - bm_{i+j} = \\ &= \frac{b}{a}\,[f(a)f(m_{i+j}) - am_{i+j}] = \\ &= \frac{b}{a}\,[f(am_{i+j}) - am_{i+j}] = \\ &= \frac{b}{a}\cdot am_{i+j+1} = \\ &= bm_{i+j+1} \qquad (j=0,1,\ldots,k-1).\ \square \end{aligned}$$

COROLLARY 2.1 *If in proposition 2.1,* $\{am_i, am_{i+1}, \ldots, am_{i+k-1}\}$ *is an* f*-cycle of length* k, *then* $\{bm_i, bm_{i+1}, \ldots, bm_{i+k-1}\}$ *is also an* f*-cycle of the same length.*

Given an f-cycle, one may try to apply this corollary by looking for numbers a and b, satisfying the conditions of proposition 2.1. Application of this corollary to σ^*-cycles (for the definition of σ^*, see example 1.2 in section 1) yielded several hundred new σ^*-cycles (see TE RIELE [32]).

PROPOSITION 2.2 *Let* $f, g \in F$, $f \neq g$, *and let*

$$am_i, am_{i+1}, \ldots, am_{i+k}$$

be $k+1$ *consecutive terms of an* f*-sequence with* $(a, m_{i+j}) = 1$ *for* $j = 0,1,\ldots,k-1$*; let, moreover,* m_{i+j} *be squarefree for the same values of* j*. If* $b \in \mathbb{N}$ *is such that* $(b, m_{i+j}) = 1$ *for* $j = 0,1,\ldots,k-1$, $b \neq a$, *and* $g(b)/b = f(a)/a$, *then*

$$bm_i, bm_{i+1}, \ldots, bm_{i+k}$$

are also $k+1$ *consecutive terms of a* g*-sequence.*

PROOF. Under the hypotheses, we have

$$\begin{aligned} g(bm_{i+j}) - bm_{i+j} &= g(b)g(m_{i+j}) - bm_{i+j} = \\ &= \frac{b}{a}\,[f(a)g(m_{i+j}) - am_{i+j}] = \\ &= \frac{b}{a}\,[f(a)f(m_{i+j}) - am_{i+j}] = \\ &= \frac{b}{a}\,[f(am_{i+j}) - am_{i+j}] = \\ &= \frac{b}{a} \cdot am_{i+j+1} = \\ &= bm_{i+j+1} \qquad (j=0,1,\ldots,k-1). \quad \Box \end{aligned}$$

COROLLARY 2.2 *If in proposition* 2.2, $\{am_i, am_{i+1}, \ldots, am_{i+k-1}\}$ *is an* f*-cycle of length* k*, then* $\{bm_i, bm_{i+1}, \ldots, bm_{i+k-1}\}$ *is a* g*-cycle of the same length.*

Application of this corollary to known σ-cycles of length 2 (LEE & MADACHY [26]) yielded several hundred new σ^*-cycles (see TE RIELE [32]).

THEOREM 2.1 *Let* $N \in \mathbb{N}$ $(N \geq 3)$ *and* $f \in F$ *be given. Then there exist infinitely many* f*-sequences with at least* N *consecutive increasing terms.*

PROOF. Let $q_1, q_2, \ldots, q_N$ be a sequence of N primes defined by

$$(2.1)\qquad \begin{cases} q_1 = 2, \quad q_2 = 3, \\ q_i^2 \mid q_{i+1} + 1 \qquad (i=2,3,\ldots,N-1). \end{cases}$$

The existence of such a sequence follows from Dirichlet's theorem on the occurrence of an infinitude of primes (hence *certainly one*) in the arithmetic progression $tq_i^2 - 1$ $(t=1,2,\ldots)$. Now choose n_0 such that

$$(2.2)\qquad n_0 = m_0 q_1 q_2 \ldots q_N$$

with $(q_i, m_0) = 1$ for $i=1,2,\ldots,N$.
Let $n_0, n_1, n_2, \ldots$ be the f-sequence with leader n_0. Then

$$n_1 = f(n_0) - n_0 =$$
$$= f(m_0)(q_1+1)(q_2+1)\ldots(q_N+1) - m_0 q_1 q_2 \ldots q_N,$$

which by (2.1) may be written in the form

$$n_1 = m_1 q_1 q_2 \ldots q_{N-1}$$

with $(q_i, m_1) = 1$ for $i=1,2,\ldots,N-1$.
Proceeding in the same way with $n_1, n_2, \ldots, n_{N-2}$, we find that for $k=1,2,\ldots,N-1$

$$n_k = m_k q_1 q_2 \ldots q_{N-k} \;,$$

with $(q_1, m_k) = (q_2, m_k) = \ldots = (q_{N-k}, m_k) = 1$.
Hence $6 \parallel n_k$ $(k=0,1,\ldots,N-2)$ so that

$$n_{k+1} = f(n_k) - n_k =$$
$$= f(2)f(3)f(n_k/6) - n_k =$$
$$= 12f(n_k/6) - n_k$$
$$> 12n_k/6 - n_k = n_k.$$

Hence the N terms $n_0, n_1, \ldots, n_{N-1}$ of the f-sequence with leader n_0 are increasing. The existence of infinitely many such sequences follows from the existence of infinitely many numbers m_0 satisfying (2.2). □

Theorem 2.1 was first proved, in this form, for $f = \sigma$ by LENSTRA (private communication dated April 10th, 1972) and for $f = \sigma^*$ by TE RIELE [33].

Very recently, for $f = \sigma$ some stronger results have been obtained by ERDŐS *) and GUY **). Erdős proved that for *all* leaders $n \in \mathbb{N}$, except a sequence of density 0, and for every $t \in \mathbb{N}$ and $\delta > 0$,

$$(1-\delta)\ (n_1/n)^i < n_i/n < (1+\delta)\ (n_1/n)^i,$$

for $1 \le i \le t$. Guy proved: given any prime p, any $t \in \mathbb{N}$, and any $\rho > 1$, there are aliquot sequences containing t consecutive terms, each greater than ρ times the previous one, but whose only prime divisors exceed p.

THEOREM 2.2 *Let* $f \in F$ *and let* $\{n_1, n_2, \ldots, n_k\}$ *be an* f-*cycle of length* k ($k \ge 1$), *where* k *is odd. If the* k *numbers* n_i ($i=1,2,\ldots,k$) *contain the prime* 2 *to the same power, then*

$$(f(n_1), f(n_2), \ldots, f(n_k)) = 2(n_1, n_2, \ldots, n_k);$$

otherwise

$$(f(n_1), f(n_2), \ldots, f(n_k)) = (n_1, n_2, \ldots, n_k).$$

PROOF. Since $\{n_1, n_2, \ldots, n_k\}$ is an f-cycle, we have

$$f(n_1) = n_1+n_2, f(n_2) = n_2+n_3, \ldots, f(n_{k-1}) = n_{k-1}+n_k, f(n_k) = n_k+n_1. \tag{2.3}$$

Note that, for $i=1,2,\ldots,k$, we have $f(n_{i+k}) = f(n_i)$ and also

$$f(n_i) - f(n_{i+1}) + f(n_{i+2}) - \ldots + (-1)^{k-1} f(n_{i+k-1}) =$$
$$= (n_i+n_{i+1})-(n_{i+1}+n_{i+2})+(n_{i+2}+n_{i+3})-\ldots+(-1)^{k-1}(n_{i+k-1}+n_{i+k}) =$$

*) P. ERDŐS, *On asymptotic properties of aliquot sequences*, Math. Comp., 30(1976) 641-645.

**) R.K. GUY, *Aliquot sequences*, manuscript, 1976.

$$= n_i + (-1)^{k-1} n_{i+k} = n_i(1+(-1)^{k-1}),$$

so that

$$(2.4) \qquad \sum_{j=i}^{i+k-1} (-1)^{j-i} f(n_j) = 2n_i,$$

since k is odd.

Let $a = (n_1,n_2,\ldots,n_k)$ and $b = (f(n_1),f(n_2),\ldots,f(n_k))$. From (2.3) it follows that $a \mid f(n_i)$ $(i=1,2,\ldots,k)$, so that $a \mid b$. On the other hand, (2.4) implies that $b \mid 2n_i$ $(i=1,2,\ldots,k)$, so that

$$(2.5) \qquad \text{either} \quad b = a \quad \text{or} \quad b = 2a.$$

If every n_i contains 2 to the same power, then n_i/a is odd and $n_i/a + n_{i+1}/a = f(n_i)/a$ is *even*; thus in (2.5) we can only have $b = 2a$. If *not* every n_i contains 2 to the same power, then there is an index j such that n_j contains the least power of 2 and n_{j+1} contains a higher one. For that index j we have $n_j/a + n_{j+1}/a = f(n_j)/a$ is *odd*, so that in (2.5) we can only have $b = a$. □

This theorem generalizes a theorem of BORHO [4].

COROLLARY 2.3 *Let* $\{n_1,n_2,\ldots,n_k\}$ *be an f-cycle of length* $k > 1$ *with* k *odd and let* $(n_1,n_2,\ldots,n_k) = a > 1$. *Then from theorem* 2.2 *it follows that*

$$(a, n_i/a) = 1 \qquad (i=1,2,\ldots,k)$$

is impossible.

Suppose contrariwise that $(a, n_i/a) = 1$ for $i=1,2,\ldots,k$.
If a is odd and at least one of the n_i/a is even, then we have by theorem 2.2:

$$(f(n_1),\ldots,f(n_k)) = (n_1,\ldots,n_k),$$

so that

$$f(a)(f(n_1/a),\ldots,f(n_k/a)) = a.$$

This is impossible, since $f(a) > a$.

If a is even, or if n_i is odd for all i=1,2,...,k, then we have by theorem 2.2:

$$(f(n_1),\dots,f(n_k)) = 2(n_1,\dots,n_k) ,$$

so that

$$f(a)(f(n_1/a),\dots,f(n_k/a)) = 2a.$$

Hence f(a) = 2a; this implies that $n_{i+1} \geq n_i$, for all i=1,2,...,k, so that k = 1, a contradiction.

REMARK 2.1 DICKSON [10] proved this corollary for $f = \sigma$.

REMARK 2.2 In [24], LAL, TILLER & SUMMERS remark that (we quote) "for unitary sociable groups, it appears that no regular groups of order >2 exist". In our terminology: a regular unitary group of order k is a σ^*-cycle $\{n_1,n_2,\dots,n_k\}$, for which $(n_1,n_2,\dots,n_k) = a > 1$ and $(a, n_i/a) = 1$ for i=1,2,...,k. *Corollary* 2.3 *implies that no regular unitary sociable groups of* odd *order* > 2 *exist.*

Next we prove a theorem about the finiteness of the number of f-cycles of certain form, but we first give two lemmas.

LEMMA 2.1 *If* $f \in F$, $a \in \mathbb{N}$, *and* p *is a prime number, then there exist positive integers* $x_1,x_2,\dots,x_g$, *such that*

$$\frac{f(p^a)}{p^a} = \left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) \dots \left(1 + \frac{1}{x_g}\right),$$

where g = g(a) *is the number of coefficients equal to* 1 *in the polynomial* $W_a^f(y) - y^a$, *i.e.*, $g = W_a^f(1) - 1$. *In particular, when*

$$f(p^a) = p^a + \sum_{i=1}^{g} p^{a_i}$$

with $a > a_1 > a_2 > \dots > a_{g-1} > a_g \geq 0$, *we may take*

$$(2.6)\qquad x_j = \frac{p^a + \sum_{i=1}^{g-j} p^{a_i}}{p^{a_{g-j+1}}} \qquad \textit{for } j=1,2,\dots,g.$$

Before proving this lemma we give an example.

If

$$f(p^5) = p^5 + p^3 + p^2 + 1,$$

then we have

$$\frac{f(p^5)}{p^5} = \frac{p^5+p^3+p^2+1}{p^5} = \frac{p^5+p^3+p^2+1}{p^5+p^3+p^2} \cdot \frac{p^5+p^3+p^2}{p^5} =$$

$$= \left(1 + \frac{1}{p^5+p^3+p^2}\right) \frac{p^3+p+1}{p^3+p} \cdot \frac{p^3+p}{p^3} =$$

$$= \left(1 + \frac{1}{p^5+p^3+p^2}\right) \left(1 + \frac{1}{p^3+p}\right) \left(1 + \frac{1}{p^2}\right),$$

so that $x_1 = p^5+p^3+p^2$, $x_2 = p^3+p$ and $x_3 = p^2$.

PROOF of lemma 2.1. By (2.6) we have

$$\prod_{j=1}^{g}\left(1 + \frac{1}{x_j}\right) = \frac{x_1+1}{x_g} \prod_{j=1}^{g-1} \frac{x_{j+1}+1}{x_j} =$$

$$= \frac{p^{a-a_g}+p^{a_1-a_g}+\ldots+p^{a_{g-1}-a_g}+1}{p^{a-a_1}} \prod_{j=1}^{g-1} \frac{p^{a-a_{g-j}}+\sum_{i=1}^{g-j-1} p^{a_i-a_{g-j}}+1}{p^{a-a_{g-j+1}}+\sum_{i=1}^{g-j} p^{a_i-a_{g-j+1}}} =$$

$$= \frac{f(p^a)}{p^{a-a_1+a_g}} \prod_{j=1}^{g-1} \frac{p^{-a_{g-j}}}{p^{-a_{g-j+1}}} =$$

$$= \frac{f(p^a)}{p^{a-a_1+a_g}}\, p^{a_g-a_{g-1}+a_{g-1}-a_{g-2}+\ldots+a_2-a_1} =$$

$$= \frac{f(p^a)}{p^a}. \qquad \square$$

LEMMA 2.2 (BORHO [3]). *The equation*

$$\prod_{i=1}^{k}\left[\prod_{j=1}^{t_i}\left(1 + \frac{1}{x_{ij}}\right) - 1\right] = 1,$$

where $k, t_1, t_2, \ldots, t_k$ *are given, has only finitely many solutions in positive integers* $x_{11}, x_{12}, \ldots, x_{kt_k}$.

PROOF. See [3]. □

THEOREM 2.3 *Let* $f \in F$ *and let there be given positive integers* $k, s_1, s_2, \ldots, s_k, e_{11}, e_{12}, \ldots, e_{1s_1}, e_{21}, e_{22}, \ldots, e_{2s_2}, \ldots, e_{k1}, e_{k2}, \ldots, e_{ks_k}$. *Then there exists only a finite number of f-cycles* $\{n_1, n_2, \ldots, n_k\}$ *where* n_i *has the canonical factorization*

$$n_i = p_{i1}^{e_{i1}} p_{i2}^{e_{i2}} \cdots p_{is_i}^{e_{is_i}} \qquad (i=1,2,\ldots,k).$$

PROOF. The numbers $n_1, n_2, \ldots, n_k$ form an f-cycle of length k. It follows that

$$1 = \frac{n_2}{n_1} \cdot \frac{n_3}{n_2} \cdot \ldots \cdot \frac{n_k}{n_{k-1}} \cdot \frac{n_1}{n_k} =$$

$$= \left(\frac{f(n_1)}{n_1} - 1\right)\left(\frac{f(n_2)}{n_2} - 1\right) \cdots \left(\frac{f(n_k)}{n_k} - 1\right) =$$

$$= \prod_{i=1}^{k} \left[\left(\prod_{j=1}^{s_i} \frac{f\left(p_{ij}^{e_{ij}}\right)}{p_{ij}^{e_{ij}}}\right) - 1\right].$$

By lemma 2.1 $f\left(p_{ij}^{e_{ij}}\right) \Big/ p_{ij}^{e_{ij}}$ may be written in the form

$$\left(1 + y_1^{-1}\right)\left(1 + y_2^{-1}\right) \cdots \left(1 + y_g^{-1}\right),$$

for some positive integers $y_1, \ldots, y_g$, where $g = g(e_{ij})$.
Hence, on the assumption that

$$\prod_{j=1}^{s_i} \frac{f\left(p_{ij}^{e_{ij}}\right)}{p_{ij}^{e_{ij}}} = \prod_{j=1}^{t_i} \left(1 + \frac{1}{x_{ij}}\right),$$

with $t_i = \sum_{j=1}^{s_i} g(e_{ij})$, we have

$$1 = \prod_{i=1}^{k} \left[\left(1 + x_{i1}^{-1}\right)\left(1 + x_{i2}^{-1}\right) \cdots \left(1 + x_{it_i}^{-1}\right) - 1\right],$$

for some positive integers $x_{11}, x_{12}, \ldots, x_{1t_1}, \ldots, x_{k1}, \ldots, x_{kt_k}$.

By lemma 2.2 this equation can have only finitely many solutions in positive integers. □

COROLLARY 2.4 By choosing $f = \sigma$ and $f = \sigma^*$, respectively, the following two theorems of BORHO [3] follow easily from theorem 2.3:

There are only finitely many aliquot σ*-cycles of length* k, *with less than* L ($L \in \mathbb{N}$) *prime factors (in the product of the* k *terms of the cycle).*

There are only finitely many aliquot σ^**-cycles of length* k, *with less than* L ($L \in \mathbb{N}$) distinct *prime factors (in the product of the* k *terms of the cycle).*

CHAPTER 3

TEST-CASES FOR THE COMPUTATIONAL EXPERIMENTS

In chapter 1 we saw that for every $f \in F$, $f(n)$ is the sum of *certain* divisors of n. Here we consider some particular f by specifying *which* divisors are to be summed. It is easily verified that these functions f have property P1 (multiplicativity) and property P2 (existence of the polynomials $W_e^f(x)$ for all $e \in \mathbb{N}$) so that $f \in F$. The proofs are omitted, but the polynomials W are included.

EXAMPLE 3.1 If $f = \sigma$ (the sum of *all* divisors of n), then

$$W_e^{\sigma}(x) = x^e + x^{e-1} + \ldots + x + 1 \qquad (e=1,2,\ldots).$$

The number of divisors to be summed is $\prod_{p^e \| n} (e+1)$.

EXAMPLE 3.2 For $k \in \mathbb{N}_0$ we define $M_k(n)$ as the sum of the (k+1)-ary divisors of n, so that

$$W_e^{M_k}(x) = \begin{cases} x^e + x^{e-1} + \ldots + x + 1 & (e \le 2k), \\ x^e + \ldots + x^{e-k} + x^k + \ldots + x + 1 & (e > 2k). \end{cases}$$

In this case, the number of divisors to be summed is $\prod_{p^e \| n} \min(e+1, 2k+2)$.

EXAMPLE 3.3 For $k \in \mathbb{N}$ we define $\Psi_k(n)$ as the sum of those divisors d of n for which n/d is (k+1)-free, so that

$$W_e^{\Psi_k}(x) = \begin{cases} x^e + x^{e-1} + \ldots + x + 1 & (e \le k), \\ x^e + x^{e-1} + \ldots + x^{e-k} & (e > k). \end{cases}$$

In this case, the number of divisors to be summed is $\prod_{p^e \| n} \min(e+1,k+1)$.

EXAMPLE 3.4 For $k \in \mathbb{N}_0$ we define $L_k(n)$ as the sum of those divisors d of n, such that any prime p which divides d has an exponent which is at most k less than that of p in n. For convenience, we define the integer 1 to be such a divisor of any $n \in \mathbb{N}$. It easily follows that

$$W_e^{L_k}(x) = \begin{cases} x^e + x^{e-1} + \ldots + x + 1 & (e \le k), \\ x^e + x^{e-1} + \ldots + x^{e-k} + 1 & (e > k). \end{cases}$$

The number of divisors to be summed here is $\prod_{p^e \| n} \min(e+1,k+2)$.

EXAMPLE 3.5 For $k \in \mathbb{N}_0$ we define $R_k(n)$ as the sum of those divisors d of n, such that any prime p which divides n/d has an exponent, which is at most k less than that of p in n. In this case we have

$$W_e^{R_k}(x) = \begin{cases} x^e + x^{e-1} + \ldots + x + 1 & (e \le k), \\ x^e + x^k + x^{k-1} + \ldots + x + 1 & (e > k), \end{cases}$$

and the number of divisors to be summed here is the same as in example 3.4, $f = L_k$.

REMARK 3.1 We have

$$M_0 = L_0 = R_0 = \sigma^* ,$$

where σ^* denotes the usual "sum of the unitary divisors" function.

These five examples of (classes of) functions will serve as test-cases for our computational experiments. Some of them are well-known, like σ and σ^*. The function Ψ_1 (also known as the Dedekind function) plays an important role in WALL's study [41]. The other functions given here, have never been used, as far as we know, to generate aliquot sequences.

CHAPTER 4

THE DISTRIBUTION OF THE VALUES OF f

In this chapter we investigate the (natural) density of the values of the function $f \in F$, *counting multiplicity*.

Since $f(n) \geq n$, the number of all $n \in \mathbb{N}$ such that $f(n) \leq N$ is *finite* for any $N \in \mathbb{N}$. The number of n satisfying $f(n) \leq N$ is denoted by $\#(f,N)$.

<u>THEOREM 4.1</u> *If* $f \in F$, *then* $\Delta f = \lim_{N\to\infty} \frac{\#(f,N)}{N}$ *exists and*

$$\Delta f = \prod_p \left\{ (1 - \frac{1}{p}) \sum_{e=0}^{\infty} \frac{1}{f(p^e)} \right\} . \tag{4.1}$$

<u>PROOF</u>. According to the definition of F, for any $f \in F$, $e \in \mathbb{N}$ and prime p, $f(p^e)$ can be written as

$$f(p^e) = \sum_{i=0}^{e} c_{e,i} p^{e-i},$$

where $c_{e,0} = 1$ and $c_{e,i} = 0$ or 1 $(i=1,2,\dots,e)$. By the multiplicativity of f, we have for any $n \in \mathbb{N}$

$$f(n) = n \prod_{p^e \| n} \sum_{i=0}^{e} c_{e,i} p^{-i}.$$

Now for $r,k \in \mathbb{N}$ we introduce the function $f_{r,k} : \mathbb{N} \to \mathbb{N}$, defined by

$$f_{r,k}(n) = n \prod_{\substack{p^e \| n \\ p \leq p_r}} \sum_{i=0}^{\min(e,k)} c_{e,i} p^{-i}.$$

We first give two lemmas.

<u>LEMMA 4.1</u> *For any* $r,k,N \in \mathbb{N}$ *we have*

$$\#(f_{r,k},N) \leq N \prod_{j=1}^{r} \left\{ (1-p_j^{-1}) \sum_{e=0}^{k} \frac{1}{f(p_j^e)} + p_j^{-k-1} \right\} + (k+1)^r \prod_{j=1}^{r} p_j, \tag{4.2}$$

and

(4.3)
$$\#(f_{r,k},N) \ge N \prod_{j=1}^{r} \left\{(1-p_j^{-1}) \sum_{e=0}^{k} \frac{1}{f(p_j^e)} + (p_j^{k+1}+p_j^k+\dots+p_j)^{-1}\right\} - (k+1)^r \prod_{j=1}^{r} p_j.$$

PROOF of lemma 4.1. For every r-tuple $(t_1,t_2,\dots,t_r)$ with $0 \le t_j \le k+1$ $(j=1,2,\dots,r)$, define $A(t_1,t_2,\dots,t_r)$ to be the set of positive integers n with $p_j^{t_j} \| n$ for $t_j < k+1$ and $p_j^{t_j} | n$ for $t_j = k+1$. For example, if $r = 4$ and $k = 2$, then $A(1,0,3,2)$ is the set of all numbers $n \in \mathbb{N}$ of the form $n = 2.5^3 7^2 m$, where $(2.3.7,m) = 1$.

If $n \in A(t_1,t_2,\dots,t_r)$, then by the definition of $f_{r,k}$ we have

$$f_{r,k}(n) = n \prod_{t_j \le k} f_{r,k}(p_j^{t_j})/p_j^{t_j} \prod_{t_j=k+1} f_{r,k}(p_j^{e(t_j)})/p_j^{e(t_j)},$$

where $e(t_j)$ is the exponent such that $p_j^{e(t_j)} \| n$. Hence,

$$n\,\Pi_1 \le f_{r,k}(n) \le n\,\Pi_1\,\Pi_2,$$

where

$$\Pi_1 = \prod_{t_j \le k} f(p_j^{t_j})/p_j^{t_j},$$

and

$$\Pi_2 = \prod_{t_j=k+1} \sum_{i=0}^{k} p_j^{-i} \quad (\ge 1).$$

It follows that for $N \in \mathbb{N}$ we have

$$n \le N\,\Pi_1^{-1}\,\Pi_2^{-1} \Rightarrow f_{r,k}(n) \le N$$

and

$$n > N\,\Pi_1^{-1} \Rightarrow f_{r,k}(n) > N.$$

From the definition of $A(t_1,t_2,\dots,t_r)$ it follows that among any $\prod_{j+1}^{r} p_j^{t_j} \prod_{t_j \le k} p_j$ consecutive numbers, precisely $\prod_{t_j \le k} (p_j - 1)$ belong to

$A(t_1,t_2,\dots,t_r)$. Hence, the number of positive integers $n \in A(t_1,t_2\dots,t_r)$ satisfying $f_{r,k}(n) \le N$ is not less than

$$(4.4) \qquad N\,\Pi_1^{-1}\,\Pi_2^{-1} \prod_{j=1}^{r} p_j^{-t_j} \prod_{t_j \le k} (1-p_j^{-1}) - \prod_{t_j \le k} (p_j-1),$$

but not greater than

$$(4.5) \qquad N\,\Pi_1^{-1} \prod_{j=1}^{r} p_j^{-t_j} \prod_{t_j \le k} (1-p_j^{-1}) + \prod_{t_j \le k} (p_j-1).$$

For different r-tuples $(t_1,t_2,\dots,t_r)$ the sets $A(t_1,t_2,\dots,t_r)$ are disjoint and their union (over all t_j with $0 \le t_j \le k+1$, $j=1,2,\dots,r$) is $\mathbb{N}$. Hence, in order to find an upperbound and a lowerbound for the *total* number of $n \in \mathbb{N}$ satisfying $f_{r,k}(n) \le N$ (i.e. $\#(f_{r,k},N)$), we must sum the upperbound (4.4) and the lowerbound (4.5) over all r-tuples $(t_1,t_2,\dots,t_r)$. The inequalities (4.2) and (4.3) then follow after some (simple) calculations.

LEMMA 4.2 *For any* $r,k,N \in \mathbb{N}$ *satisfying* $k \le r-1$ *and* $N < (k+2)^r \prod_{j=1}^{r} p_j$, *we have*

$$(4.6) \qquad \#(f,N) \ge \#(f_{r-1,k}, NS_{r-1,k}),$$

where

$$S_{r-1,k} = \prod_{j=1}^{r-1} \left(1+\frac{1}{p_j^k(p_j-1)}\right)^{-1} \prod_{j=r}^{3r-1} (1-p_j^{-1}).$$

PROOF of lemma 4.2. Let $T_{n,r,k} := f_{r,k}(n)/f(n)$. If y is an arbitrary positive real number, then we clearly have

$$f_{r,k}(n) \le y \Rightarrow f(n) \le y/T_{n,r,k}.$$

Replacing r by $r-1$ and y by $NT_{n,r-1,k}$, we get

$$f_{r-1,k}(n) \le NT_{n,r-1,k} \Rightarrow f(n) \le N,$$

so that

$$(4.7) \qquad \#(f,N) \ge \#(f_{r,k}, NT_{n,r-1,k}).$$

If some $n \in \mathbb{N}$ satisfies $f_{r-1,k}(n) \le NT_{n,r-1,k}$, it follows that

$$f(n) \le N < (k+2)^r \prod_{j=1}^{r} p_j < \prod_{j=1}^{2r} p_j,$$

since $k+2 \le r+1$. Hence the number of different prime factors of n is certainly less than 2r. Now we have for $T_{n,r-1,k}$:

$$1 \ge T_{n,r-1,k} = \frac{f_{r-1,k}(n)}{f(n)} =$$

$$= \prod_{\substack{p^e \| n \\ p \le p_{r-1} \\ e > k}} \left(\sum_{i=0}^{k} c_{e,i}p^{-i}\right)\left(\sum_{i=0}^{e} c_{e,i}p^{-i}\right)^{-1} \prod_{\substack{p^e \| n \\ p > p_{r-1}}} \left(\sum_{i=0}^{e} c_{e,i}p^{-i}\right)^{-1}$$

$$= \prod_{\substack{p^e \| n \\ p \le p_{r-1} \\ e > k}} \left(1 + \sum_{i=k+1}^{\infty} p^{-i}\right)^{-1} \prod_{\substack{p^e \| n \\ p > p_{r-1}}} \left(\sum_{i=0}^{\infty} p^{-i}\right)^{-1} =$$

$$= \prod_{\substack{p^e \| n \\ p \le p_{r-1} \\ e > k}} \left(1 + \frac{1}{p^k(p-1)}\right)^{-1} \prod_{\substack{p^e \| n \\ p > p_{r-1}}} \left(1 - p^{-1}\right) .$$

Since the number of different prime factors of n is less than 2r, the value of this last form is certainly greater than

$$\prod_{j=1}^{r-1} \left(1 + \frac{1}{p_j^k(p_j-1)}\right)^{-1} \prod_{j=r}^{3r-1} \left(1 - p_j^{-1}\right) = S_{r-1,k}.$$

So we have $1 \ge T_{n,r-1,k} > S_{r-1,k}$. Combining this with (4.7) yields (4.6). □

The proof of theorem 4.1 proceeds as follows. Clearly, for any $r,k,n \in \mathbb{N}$ we have $f(n) \ge f_{r,k}(n)$, so that for any $N \in \mathbb{N}$ $\#(f,n) \le \#(f_{r,k},N)$. Hence,

$$\limsup_{N\to\infty} \#(f,N)/N \le \limsup_{N\to\infty} \#(f_{r,k},N)/N.$$

Since $(k+1)^r \prod_{j=1}^{r} p_j$ is bounded for fixed r and k, it follows from lemma 4.1, (4.2), that

$$\limsup_{N\to\infty} \#(f_{r,k},N)/N \le \prod_{j=1}^{r} \left\{(1-p_j^{-1}) \sum_{e=0}^{k} \frac{1}{f(p_j^e)} + p_j^{-k-1}\right\},$$

for any fixed $r,k \in \mathbb{N}$. From the inequalities $p^e < f(p^e) < (p+1)^e$ it easily follows that

$$\lim_{r,k\to\infty} \prod_{j=1}^{r} \left\{ (1-p_j^{-1}) \sum_{e=0}^{k} \frac{1}{f(p_j^e)} + p_j^{-k-1} \right\} = \prod_p \left\{ (1-p^{-1}) \sum_{e=0}^{\infty} \frac{1}{f(p^e)} \right\} .$$

Hence,

$$(4.8) \qquad \limsup_{N\to\infty} \#(f;N)/N \le \prod_p \left\{ (1-p^{-1}) \sum_{e=0}^{\infty} \frac{1}{f(p^e)} \right\} .$$

If we can prove, on the other hand, that

$$(4.9) \qquad \liminf_{N\to\infty} \#(f,N)/N \ge \prod_p \left\{ (1-p^{-1}) \sum_{e=0}^{\infty} \frac{1}{f(p^e)} \right\} ,$$

then theorem 4.1 clearly follows.

From now on, we assume that $r,k,N \in \mathbb{N}$ are such that $k \le r-1$, k large, and

$$(4.10) \qquad (k+1)^r \prod_{j=1}^{r} p_j \le N < (k+2)^r \prod_{j=1}^{r} p_j .$$

By lemma 4.2 we have

$$(4.11) \qquad \#(f,N) \ge \#(f_{r-1,k}, NS_{r-1,k}),$$

where

$$S_{r-1,k} = \prod_{j=1}^{r-1} \left(1 + \frac{1}{p_j^k(p_j-1)} \right)^{-1} \prod_{j=r}^{3r-1} (1-p_j^{-1}).$$

From the theorem of Mertens

$$\prod_{p\le x} (1 - p^{-1}) \sim \frac{e^{-\gamma}}{\log x} \qquad (x \to \infty),$$

where γ is Euler's constant, and from the theorem of Tchebychef:

$$\pi(x) \asymp x/\log x,$$

it follows that $\lim_{r\to\infty} \prod_{j=r}^{3r-1} (1 - p_j^{-1}) = 1$.

Furthermore, we have

$$1 > \prod_{j=1}^{r-1} \left(1 + \frac{1}{p_j^k(p_j-1)} \right)^{-1} > \prod_{j=1}^{r-1} \left(1 - p_j^{-k} \right) > \zeta^{-1}(k) \qquad (k>1),$$

which tends to 1 for $k\to\infty$. Hence, $S_{r-1,k}$ tends to 1 from below when k and r tend to infinity. Now by lemma 4.1, (4.3), with r replaced by r-1 and N by $NS_{r-1,k}$ we have

$$\#(f_{r-1,k}, NS_{r-1,k}) \geq NS_{r-1,k} \prod_{j=1}^{r-1} \left\{ (1-p_j^{-1}) \sum_{e=0}^{k} \frac{1}{f(p_j^e)} + \left(p_j^{k+1} + p_j^k + \ldots + p_j \right)^{-1} \right\} - (k+1)^{r-1} \prod_{j=1}^{r-1} p_j .$$

From (4.10) it follows that $(k+1)^{r-1} \prod_{j=1}^{r-1} p_j \leq \frac{N}{(k+1)p_r}$. Using this and (4.11) gives

$$\#(f,N) \geq NS_{r-1,k} \prod_{j=1}^{r-1} \left\{ (1-p_j^{-1}) \sum_{e=0}^{k} \frac{1}{f(p_j^e)} + \left(p_j^{k+1} + p_j^k + \ldots + p_j \right)^{-1} \right\} - \frac{N}{(k+1)p_r} .$$

Dividing by N and letting N, k and r tend to infinity gives (4.9). □

<u>REMARK 4.1</u> Three proofs of this theorem have been given for the special case $f = \sigma$. In the first one ERDÖS [13] used analytic results of SCHOENBERG, but did not give the value of $\Delta\sigma$. DRESSLER [11] was the second one to prove this theorem for $f = \sigma$. His elementary proof also gives the value of $\Delta\sigma$. Our proof of the more general theorem 4.1 is based on DRESSLER's method. BATEMAN [2] proved theorem 4.1 for $f = \sigma$ using the WIENER-IKEHARA theorem.

In table 4.1 we give the (approximate) value of Δf for some $f \in F$, where the absolute error in this value is always less than 2.10^{-5}. The accuracy of this table is justified by theorem 4.2.

TABLE 4.1

Some values of Δf

f	Δf
σ	.67274
M_0 (= σ^*)	.76872
M_1	.67887
Ψ_1	.70444
Ψ_2	.67848
L_1	.68618
L_2	.67541
R_1	.71070
R_2	.68950

THEOREM 4.2 *Let $\varepsilon > 0$ be a (small) number and let Q be a (large) prime. Let* $(1 - \frac{1}{p})\sum_{e=0}^{\infty} \frac{1}{f(p^e)} =: 1 - a_p$, $f \in F$. *If the series*

$$S = \sum_p \log(1 - a_p)$$

is approximated by

$$\tilde{S}_Q = \sum_{p \le Q} \log(1 - \tilde{a}_p) ,$$

where

$$(4.12) \qquad |a_p - \tilde{a}_p| < \varepsilon \qquad \textit{for } p=2,3,5,\ldots,Q ,$$

then

$$|S - \tilde{S}_Q| < \frac{4}{3Q} + 2\varepsilon\pi(Q) ,$$

where $\pi(Q)$ is the number of primes $\le Q$.

PROOF We show that, if

$$S_Q = \sum_{p \le Q} \log(1 - a_p) ,$$

then

$$\text{(i) } |S - S_Q| < \frac{4}{3Q} \qquad \text{and} \qquad \text{(ii) } |S_Q - \tilde{S}_Q| < 2\varepsilon\pi(Q),$$

from which the theorem follows.

$$\text{(i)} \qquad |S - S_Q| = \Big|\sum_{p>Q} \log(1 - a_p)\Big| < \sum_{p>Q} |\log(1 - a_p)| .$$

From the definition of f it follows that

$$1 - a_p = \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{f(p)} + \frac{1}{f(p^2)} + \ldots\right)$$
$$< \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right) = 1 .$$

On the other hand,

$$1 - a_p \geq \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p+1} + \frac{1}{p^2+p+1} + \ldots\right) =$$

$$= \left(1 - \frac{1}{p}\right)\left(1 + \frac{p-1}{p^2-1} + \frac{p-1}{p^3-1} + \ldots\right)$$

$$> \left(1 - \frac{1}{p}\right)\left(1 + \frac{p-1}{p^2} + \frac{p-1}{p^3} + \ldots\right), \quad \text{or}$$

$$(4.13) \qquad 1 - a_p > \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right) = 1 - \frac{1}{p^2},$$

so that

$$0 < \left|\log\left(1 - a_p\right)\right| < \left|\log\left(1 - \frac{1}{p^2}\right)\right| .$$

By using the inequality $|\log(1-x)| < \frac{x}{1-x}$, for $0 < x < 1$, we get

$$0 < |\log(1 - a_p)| < \frac{1}{p^2-1} \leq \frac{4}{3p^2} .$$

Hence,

$$|S - S_Q| < \sum_{p>Q} |\log(1-a_p)| < \frac{4}{3} \sum_{p>Q} \frac{1}{p^2} < \frac{4}{3} \int_{Q+1}^{\infty} \frac{dx}{(x-1)^2} = \frac{4}{3Q} .$$

(ii)
$$|S_Q - \tilde{S}_Q| = \left|\sum_{p\leq Q}\left\{\log(1-\tilde{a}_p) - \log(1-a_p)\right\}\right|$$

$$\leq \sum_{p\leq Q}\left|\log\left(1 + \frac{a_p - \tilde{a}_p}{1 - a_p}\right)\right| .$$

By (4.12) and (4.13) we have

$$\left|\frac{a_p - \tilde{a}_p}{1 - a_p}\right| < \frac{\varepsilon}{1 - \frac{1}{p^2}} < \frac{4}{3}\varepsilon ,$$

since $p \geq 2$. Hence

$$\left|\log\left(1 + \frac{a_p - \tilde{a}_p}{1 - a_p}\right)\right| < \frac{\frac{4}{3}\varepsilon}{1 - \frac{4}{3}\varepsilon} < 2\varepsilon ,$$

for $\varepsilon < \frac{1}{4}$. From this we deduce that

$$\left|S_Q - \tilde{S}_Q\right| \leq \sum_{p \leq Q} 2\varepsilon = 2\varepsilon\pi(Q). \qquad \square$$

<u>REMARK 4.2</u> It is easy to approximate

$$a_p = \left(\frac{1}{p} - \frac{1}{f(p)}\right) + \left(\frac{1}{pf(p)} - \frac{1}{f(p^2)}\right) + \ldots$$

by

$$\tilde{a}_p = \left(\frac{1}{p} - \frac{1}{f(p)}\right) + \ldots + \left(\frac{1}{pf(p^{i-1})} - \frac{1}{f(p^i)}\right)$$

with an accuracy prescribed by (4.12), by choosing i large enough. In fact, we have

$$\left|\frac{1}{pf(p^{j-1})} - \frac{1}{f(p^j)}\right| = \left|\frac{f(p^j) - pf(p^{j-1})}{pf(p^{j-1})f(p^j)}\right|$$

$$< \frac{p^j + p^{j-1} + \ldots + p + 1 - p^j}{p.p^{j-1}.p^j} =$$

$$< \frac{1}{p^j(p-1)} \qquad \text{for } j=1,2,\ldots ,$$

so that

$$|a_p - \tilde{a}_p| < \left|\frac{1}{pf(p^i)} - \frac{1}{f(p^{i+1})}\right| + \left|\frac{1}{pf(p^{i+1})} - \frac{1}{f(p^{i+2})}\right| + \ldots$$

$$\leq \frac{1}{p^{i+1}(p-1)} + \frac{1}{p^{i+2}(p-1)} + \ldots = \frac{1}{p^i(p-1)^2}.$$

In order to obtain the values of Δf given in table 4.1, we chose $Q = 10^5$ and for every $p \leq Q$ we determined $i = i_p$ such that $\frac{1}{p^i(p-1)^2} < \varepsilon = 10^{-10}$.

CHAPTER 5

THE MEAN VALUE OF f(n)/n

For any $f \in F$ let

$$\bar{f}(n) := f(n) - n , \qquad (n \in \mathbb{N}),$$

so that

$$\frac{\bar{f}(n_i)}{n_i} = \frac{f(n_i) - n_i}{n_i} = \frac{n_{i+1}}{n_i} ,$$

where n_i and n_{i+1} are two consecutive terms of an f-sequence.

The purpose of this section is to determine the mean value $M\left\{\frac{\bar{f}(n)}{n}\right\}$ of $\frac{\bar{f}(n)}{n}$. Note that

$$M\left\{\frac{\bar{f}(n)}{n}\right\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\bar{f}(n)}{n} =$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\frac{f(n)}{n} - 1\right) =$$

$$= \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{f(n)}{n}\right) - 1 = M\left\{\frac{f(n)}{n}\right\} - 1 .$$

The mean value of an arithmetical function g may be determined by the following two theorems.

<u>THEOREM 5.1</u> *If* g *is an arithmetical function and* $h = g * \mu$, *i.e.*,

(5.1) $$h(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right) \qquad (n \in \mathbb{N}),$$

where μ *denotes the Möbius function, then*

(5.2) $$M\{g\} = \sum_{n=1}^{\infty} \frac{h(n)}{n} ,$$

provided that this series is absolutely convergent.

PROOF By the Möbius inversion formula,

$$g(n) = \sum_{d|n} h(d),$$

so that

$$\frac{1}{N}\sum_{n=1}^{N} g(n) = \frac{1}{N}\sum_{n=1}^{N}\sum_{d|n} h(d) = \frac{1}{N}\sum_{d=1}^{N} h(d)\left[\frac{N}{d}\right] =$$

$$= \sum_{d=1}^{\infty}\frac{h(d)}{d} - \sum_{d=N+1}^{\infty}\frac{h(d)}{d} - \frac{1}{N}\sum_{d=1}^{N} h(d)\left(\frac{N}{d} - \left[\frac{N}{d}\right]\right).$$

Clearly

$$\lim_{N\to\infty}\sum_{d=N+1}^{\infty}\frac{h(d)}{d} = 0.$$

Next observe that

$$\left|\frac{1}{N}\sum_{d=1}^{N} h(d)\left(\frac{N}{d} - \left[\frac{N}{d}\right]\right)\right| \le \frac{1}{N}\sum_{d=1}^{N}|h(d)| = \frac{1}{N}\sum_{d=1}^{N} d\left|\frac{h(d)}{d}\right| .$$

From the absolute convergence of $\sum_{d=1}^{\infty}\frac{h(d)}{d}$, and a well-known theorem of Kronecker (see KNOPP [23], p.129), it follows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{d=1}^{N} d\left|\frac{h(d)}{d}\right| = 0 .$$

□

We apply this theorem to the function $g(n) = \frac{M_k(n)}{n}$ $(k=0,1,2,\ldots)$, and we first show that

$$h(n) = \sum_{d|n}\frac{M_k(d)}{d}\mu\left(\frac{n}{d}\right) = O(n^{-\frac{1}{2}}) \qquad (n\to\infty).$$

We have $h(1) = 1$ and for any prime p and $e \in \mathbb{N}$

$$h(p^e) = \frac{M_k(p^e)}{p^e} - \frac{M_k(p^{e-1})}{p^{e-1}} .$$

By the definition of M_k

$$h(p^e) = \begin{cases} p^{-e}, & 1 \le e \le 2k+1, \\ p^{-e}(1 - p^{k+1}), & e > 2k+1, \end{cases}$$

from which it is easily seen that

$$|h(p^e)| \le p^{-e/2} .$$

Because of the multiplicativity of h, it follows that

$$h(n) = O(n^{-1/2}) \qquad (n \to \infty),$$

and from this it is clear that we may apply theorem 5.1.

Because of the absolute convergence of $\sum_{n=1}^{\infty} \frac{h(n)}{n}$ and the multiplicativity of h, theorem 286 of [22] gives

$$\sum_{n=1}^{\infty} \frac{h(n)}{n} = \prod_p \left\{1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \ldots\right\} ,$$

so that

$$\begin{aligned} M\left\{\frac{M_k(n)}{n}\right\} &= \prod_p \left\{1 + \frac{1}{p}\left(\frac{M_k(p)}{p} - 1\right) + \frac{1}{p^2}\left(\frac{M_k(p^2)}{p^2} - \frac{M_k(p)}{p}\right) + \ldots\right\} = \\ &= \prod_p \left\{(1 - \frac{1}{p}) \sum_{j=0}^{\infty} \frac{M_k(p^j)}{p^{2j}}\right\} = \\ &= \prod_p \left[(1 - \frac{1}{p})\left\{\sum_{j=0}^{2k} \frac{p^j+p^{j-1}+\ldots+p+1}{p^{2j}} + \right.\right. \\ &\qquad \left.\left. + \sum_{j=2k+1}^{\infty} \frac{p^j+\ldots+p^{j-k}+p^k+\ldots+1}{p^{2j}}\right\}\right] = \\ &= \prod_p \left\{(1 - \frac{1}{p})\left(\frac{p^3 - p^{-3k}}{(p-1)^2(p+1)}\right)\right\} = \\ &= \prod_p \left\{(1 - p^{-2})^{-1}(1 - p^{-3k-3})\right\} = \\ &= \frac{\zeta(2)}{\zeta(3k+3)} \qquad (k=0,1,2,\ldots), \end{aligned}$$

yielding

COROLLARY 5.1

$$M\left\{\frac{M_k(n)}{n}\right\} = \frac{\zeta(2)}{\zeta(3k+3)} \qquad (k=0,1,2,\ldots).$$

We may determine the mean value of the functions $\Psi_k(n)/n$, $L_k(n)/n$, and $R_k(n)/n$ in the same way as the mean value of $M_k(n)/n$ was determined. However, we shall perform this in another way, namely by combining the next theorem ([25]) with theorem 5.1.

THEOREM 5.2 *If*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(n)$$

exists, then the generating Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

converges for $s > 1$, *and moreover*

$$\lim_{s\downarrow 1} (s-1)G(s) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(n) \ . \tag{5.3}$$

Under the hypothesis of theorem 5.1, $M\{g\}$ exists, so that theorem 5.2 applies. Therefore, we should like to know the generating Dirichlet series of $g(n)$.
The functions g which we shall consider ($g(n) = \Psi_k(n)/n$, $L_k(n)/n$, $R_k(n)/n$ and for the sake of completeness $M_k(n)/n$), partly coïncide with $\sigma(n)/n$. Hence, we first compute the multiplicative function $g_2(n)$, implicitly defined by the convolution product

$$g = g_1 * g_2 \ , \tag{5.4}$$

where $g_1(n) = \sigma(n)/n \quad (n \in \mathbb{N})$. It is well-known that $G(s)$ is then determined by

$$G(s) = G_1(s)G_2(s), \tag{5.5}$$

where $G_1(s)$ and $G_2(s)$ are the generating Dirichlet series of $g_1(n)$ and

$g_2(n)$, respectively. Now it is readily seen that

$$(5.6) \qquad G_1(s) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} . n^{-s} = \zeta(s)\zeta(s+1) \qquad (s > 1).$$

From (5.5) and (5.6) we infer that

$$\lim_{s\downarrow 1}(s-1)G(s) = \lim_{s\downarrow 1}(s-1)G_1(s)G_2(s) =$$

$$= \zeta(2) \lim_{s\downarrow 1} G_2(s) \; .$$

Hence, by theorem 5.2, we finally have

$$(5.7) \qquad M\{g\} = \zeta(2) \lim_{s\downarrow 1} G_2(s) \; .$$

For each of the considered functions g, table 5.1 presents the order of magnitude of $h(n)$ (so that theorem 5.1 applies), the multiplicative function $g_2(n)$, its generating Dirichlet series $G_2(s)$, and, finally, the mean value $M\{g\}$ according to (5.7).

TABLE 5.1

Mean value $M\{g\}$ and intermediate results for various g

$g(n)$	$h(n)$ $(n \to \infty)$	$g_2(n)$ (g_2 is multiplicative)	$G_2(s)$ $(s > 1)$	$M\{g\}$ $= \lim_{s \downarrow 1} G_2(s)\zeta(2)$
$M_k(n)/n$ $(k=0,1,2,...)$	$O(n^{-1/2})$	$g_2(p^{2k+2}) = -p^{-(k+1)}$ $g_2(p^i) = 0,\ i \in \mathbb{N},\ i \neq 2k+2$	$\frac{1}{\zeta((k+1)(2s+1))}$	$\frac{\zeta(2)}{\zeta(3(k+1))}$
$\Psi_k(n)/n$ $(k=1,2,...)$	$O(n^{-1})$	$g_2(p^{k+1}) = -p^{-(k+1)}$ $g_2(p^i) = 0,\ i \in \mathbb{N},\ i \neq k+1$	$\frac{1}{\zeta((k+1)(s+1))}$	$\frac{\zeta(2)}{\zeta(2(k+1))}$
$L_k(n)/n$ $(k=0,1,2,...)$	$O(n^{-1/2})$	$g_2(p^{k+2}) = -p^{-(k+1)}$ $g_2(p^i) = 0,\ i \in \mathbb{N},\ i \neq k+2$	$\frac{1}{\zeta((k+2)s+k+1)}$	$\frac{\zeta(2)}{\zeta(2k+3)}$
$R_k(n)/n$ $(k=0,1,2,...)$	$O(n^{-1/(k+1)})$	$g_2(p^{k+2}) = -p^{-1}$ $g_2(p^i) = 0,\ i \in \mathbb{N},\ i \neq k+2$	$\frac{1}{\zeta((k+2)s+1)}$	$\frac{\zeta(2)}{\zeta(k+3)}$

CHAPTER 6

COMPUTATIONAL RESULTS ON ALIQUOT f-SEQUENCES WITH LEADER $n \leq 1000$

In order to get some insight into the behaviour of aliquot f-sequences, we have carried out some computer calculations on the functions f, described in chapter 3. From the definitions it is clear that, with increasing k, the M_k-, Ψ_k-, L_k- and R_k-sequences coïncide more and more with the σ-sequences. Therefore, we have computed these sequences only for some *small* values of k.

For $f = M_k$ (k=1,2), $f = \Psi_k$ (k=1,2,3,4), $f = L_k$ (k=1,2,3,4) and $f = R_k$ (k=1,2,3,4) we have computed all n-sequences for $1 \leq n \leq 1000$, stopping after reaching a term greater than 10^8. Table 6.1 gives frequency counts of the number of sequences *incomplete* at the bound 10^8, and (in parentheses) the corresponding number of incomplete main sequences; next the number of *periodic* sequences and the number of *terminating* sequences. In chapter 7 some of the incomplete Ψ_1-, Ψ_2- and Ψ_3-sequences will be proved to be *unbounded*. The last column of table 6.1 gives the number of these sequences with the corresponding number of (unbounded) main sequences (in parentheses). For purposes of comparison the corresponding results for $f = \sigma$ and $f = \sigma^*$ are included in table 6.1.

Table 6.2 gives the first term greater than 10^8 in all incomplete main sequences with first term ≤ 1000, of which the behaviour is unknown to us.

TABLE 6.1

Frequency counts of the (aliquot) f-sequences on $n \leq 1000$, for various choices of f

f	number of (main) sequences, incomplete at bound 10^8		number of periodic sequences	number of terminating sequences	number of incomplete (main) sequences proved to be unbounded (in chapter 7)	
σ	30	(11)	22	948		
σ^*	0		86	914		
M_1	38	(9)	17	945		
M_2	28	(11)	23	949		
Ψ_1	15	(3)	151	834	15	(3)
Ψ_2	8	(4)	457	535	7	(3)
Ψ_3	94	(23)	143	763	45	(11)
Ψ_4	34	(11)	31	935		
L_1	8	(3)	56	936		
L_2	47	(12)	18	935		
L_3	17	(7)	21	962		
L_4	42	(8)	23	935		
R_1	0		34	966		
R_2	34	(5)	24	942		
R_3	16	(4)	21	963		
R_4	35	(9)	22	943		

TABLE 6.2

10^8 bounds of incomplete main sequences

f = σ

138	:	69	=	147793668
276	:	32	=	121129260
552	:	36	=	114895284
564	:	22	=	196505388
660	:	50	=	144750606
702	:	21	=	139130668
720	:	69	=	132775020
840	:	15	=	139098120
858	:	30	=	159862836
936	:	26	=	111494688
966	:	35	=	181027656

f = M_1

120	:	30	=	100491408
216	:	43	=	155349264
402	:	32	=	124353480
462	:	45	=	161499768
570	:	43	=	108977466
642	:	23	=	115388280
660	:	23	=	103608720
840	:	15	=	139098120
966	:	43	=	121249806

f = M_2

180	:	30	=	121823520
276	:	32	=	121129260
552	:	36	=	114895284
564	:	84	=	166139664
570	:	107	=	109946862
600	:	73	=	123828888
720	:	48	=	137975796
840	:	15	=	139098120
864	:	28	=	197379960
936	:	21	=	102579864
966	:	35	=	119896080

f = Ψ_2

756	:	20	=	208430376

f = Ψ_3

180	:	26	=	131598960
282	:	62	=	102277120
318	:	34	=	152730624
360	:	43	=	127848510
462	:	36	=	154178412
564	:	23	=	102691584
702	:	17	=	199796580
714	:	36	=	181993620
720	:	92	=	113704960
840	:	15	=	139098120
852	:	42	=	100106240
936	:	36	=	105164730

f = Ψ_4

120	:	23	=	124250364
276	:	32	=	121129260
564	:	62	=	124774110
570	:	56	=	143028208
600	:	65	=	148695936
642	:	41	=	107321286
702	:	29	=	116227422
840	:	15	=	139098120
858	:	29	=	113150496
936	:	21	=	130295840
966	:	39	=	125235882

f = L_1

282	:	94	=	108787260
750	:	51	=	124400724
858	:	77	=	215879274

f = L_2

180	:	71	=	160477212
282	:	31	=	107259180
360	:	42	=	117609900
474	:	32	=	114583824
480	:	71	=	229226172
660	:	84	=	120023082
702	:	39	=	162230796
720	:	31	=	154052736
840	:	15	=	139098120
936	:	33	=	126864192
960	:	105	=	101902724
966	:	32	=	171433320

f = L_3

120	:	32	=	121129260
552	:	86	=	126294174
570	:	80	=	141073044
840	:	15	=	139098120
896	:	45	=	188579412
966	:	49	=	102182706
1000	:	50	=	134757462

f = L_4

138	:	21	=	139098120
180	:	108	=	173393484
276	:	32	=	121129260
448	:	37	=	114895284
564	:	24	=	125050980
858	:	33	=	133562928
864	:	30	=	104767338
966	:	34	=	102297492

f = R_2

282	:	53	=	136831950
318	:	38	=	106216404
504	:	18	=	139098120
570	:	35	=	109215852
720	:	19	=	119423880

f = R_3

138	:	46	=	121129260
600	:	67	=	116465076
720	:	46	=	144750606
840	:	15	=	139098120

f = R_4

138	:	22	=	122945760
180	:	89	=	105128120
276	:	32	=	121129260
480	:	30	=	135688812
552	:	36	=	114895284
570	:	53	=	114809502
840	:	15	=	139098120
864	:	37	=	164699262
966	:	38	=	158510148

CHAPTER 7

UNBOUNDED ALIQUOT Ψ_k-SEQUENCES

In the preceding chapter we mentioned the discovery of unbounded f-sequences. As table 6.1. shows, unbounded sequences were found only in the cases $f = \Psi_1$, $f = \Psi_2$ and $f = \Psi_3$. *How* these sequences were found may be best illustrated by the data given in table 7.1. Our attention was immediately attracted to the regular pattern in the prime factors of the terms from 318 : 12 onwards. More explicitly,

$$\begin{array}{ll}
318 : 31 = 3^3 \,.\, (318 : 12) \;, & 318 : 50 = 3^3 \,.\, (318 : 31) \;, \\
318 : 32 = 3^3 \,.\, (318 : 13) \;, & 318 : 51 = 3^3 \,.\, (318 : 32) \;, \\
\vdots & \vdots \\
318 : 47 = 3^3 \,.\, (318 : 28) \;, & 318 : 66 = 3^3 \,.\, (318 : 47) \;. \\
318 : 48 = 3^3 \,.\, (318 : 29) \;, & \\
318 : 49 = 3^3 \,.\, (318 : 30) \;, &
\end{array}$$

Therefore, the 67 terms in table 7.1, together with their prime factorizations, strongly suggest the unboundedness of the sequence. A precise proof follows easily from the following discussion.

Let $n_0, n_1, \ldots, n_\ell$ be $\ell+1$ ($\ell > 0$) consecutive terms of a Ψ_k-sequence, and suppose that for $i=0,1,\ldots,\ell-1$ we have

$$(7.1) \qquad n_i = q_1^{e_{i1}} \ldots q_s^{e_{is}} m_i \;,$$

where $q_1, q_2, \ldots, q_s$ are s (> 0) different primes, $(q_1 q_2 \ldots q_s, m_i) = 1$ and $e_{ij} \geq k$ for $j=1,2,\ldots,s$. Let us write n_ℓ as

$$(7.2) \qquad n_\ell = q_1^{e_{\ell 1}} \ldots q_s^{e_{\ell s}} m_\ell \;,$$

TABLE 7.1

The aliquot Ψ_1-sequence with leader 318

rank	term	factorization	rank	term	factorization
0	318	2.3. 53	34	674406	2.3(4) 23.181
1	330	2.3. 5. 11	35	740826	2.3(4) 17.269
2	534	2.3. 89	36	833814	2.3(4) 5147
3	546	2.3. 7. 13	37	834138	2.3(4) 19.271
4	798	2.3. 7. 19	38	928422	2.3(4) 11.521
5	1122	2.3. 11. 17	39	1101114	2.3(4) 7.971
6	1470	2.3. 5. 7(2)	40	1418310	2.3(4) 5. 17.103
7	2562	2.3. 7. 61	41	2220858	2.3(4)13709
8	3390	2.3. 5.113	42	2221182	2.3(4)13711
9	4818	2.3. 11. 73	43	2221506	2.3(5) 7.653
10	5838	2.3. 7.139	44	2863998	2.3(5) 71. 83
11	7602	2.3. 7.181	45	3014658	2.3(5) 6203
12	9870	2.3. 5. 7. 47	46	3015630	2.3(5) 5. 17. 73
13	17778	2.3. 2963	47	4752594	2.3(5) 7. 11.127
14	17790	2.3. 5.593	48	7191342	2.3(5)14797
15	24978	2.3. 23.181	49	7192314	2.3(6) 4933
16	27438	2.3. 17.269	50	7195230	2.3(7) 5. 7. 47
17	30882	2.3. 5147	51	12960162	2.3(7) 2963
18	30894	2.3. 19.271	52	12968910	2.3(7) 5.593
19	24386	2.3. 11.521	53	18208962	2.3(7) 23.181
20	40782	2.3. 7.971	54	20002302	2.3(7) 17.269
21	52530	2.3. 5. 17.103	55	22512978	2.3(7) 5147
22	82254	2.3. 13709	56	22521726	2.3(7) 19.271
23	82266	2.3. 13711	57	25067394	2.3(7) 11.521
24	82278	2.3(2) 7.653	58	29730078	2.3(7) 7.971
25	106074	2.3(2) 71. 83	59	38294370	2.3(7) 5. 17.103
26	111654	2.3(2) 6203	60	59963166	2.3(7)13709
27	111690	2.3(2) 5. 17. 73	61	59971914	2.3(7)13711
28	176022	2.3(2) 7. 11.127	62	59980662	2.3(8) 7.653
29	266346	2.3(2)14797	63	77327946	2.3(8) 71. 83
30	266382	2.3(3) 4933	64	81395766	2.3(8) 6203
31	266490	2.3(4) 5. 7. 47	65	81422010	2.3(8) 5. 17. 73
32	480006	2.3(4) 2963	66	128320038	2.3(8) 7. 11.127
33	480330	2.3(4) 5.593			

where $(q_1 q_2 \ldots q_s, m_\ell) = 1$ and $e_{\ell j} \geq 0$ for $j=1,2,\ldots,s$.

Moreover, suppose that

(7.3) $\quad m_0 = m_\ell$, and, if $\ell > 1$, then $m_0 \neq m_j$, for $j=1,2,\ldots,\ell-1$.

Now four possible cases may be distinguished.

Case 1. $e_{\ell j} \geq e_{0j}$, for $j=1,2,\ldots,s$, with strict inequality for at least one j. Then by (7.2), (7,3) and (7.1),

$$n_\ell = \left(\prod_{j=1}^{s} q_j^{e_{\ell j}}\right) m_\ell = \left(\prod_{j=1}^{s} q_j^{e_{\ell j}}\right) m_0 = \left(\prod_{j=1}^{s} q_j^{e_{\ell j}-e_{0j}}\right)\left(\prod_{j=1}^{s} q_j^{e_{0j}}\right) m_0 =$$

$$= an_0 ,$$

where $a = \prod_{j=1}^{s} q_j^{e_{\ell j}-e_{0j}}$.
Now observe that

$$n_{\ell+1} = \Psi_k(n_\ell) - n_\ell = \Psi_k(an_0) - an_0 =$$

$$= a\{\Psi_k(n_0) - n_0\} = an_1 ,$$

so that $n_{\ell+1} = an_1$. Similarly,

$$\begin{aligned} n_{\ell+2} &= an_2 , \\ &\vdots \\ n_{2\ell-1} &= an_{\ell-1} , \text{ and} \\ n_{2\ell} &= an_\ell = a^2 n_0 . \end{aligned}$$

By induction, we infer that for $r=1,2,\ldots$

$$n_{r\ell+j} = a^r n_j , \qquad (j=0,1,\ldots,\ell-1),$$

so that the Ψ_k-sequence with leader n_0 is increasing (since $a > 1$), and hence unbounded. We propose to call a the *multiplier* of this unbounded sequence. Furthermore, observe that it is periodic in the sense that for $r=1,2,\ldots$ we have

$$(q_1\ldots q_s)^k m_0 \quad \text{divides } n_{r\ell} ,$$

$$(q_1\ldots q_s)^k m_1 \quad \text{divides } n_{r\ell+1} ,$$

$$\vdots$$

$$(q_1\ldots q_s)^k m_{\ell-1} \text{ divides } n_{r\ell+\ell-1} .$$

Therefore, we propose to call ℓ the *semi-period* of the unbounded sequence. The example in table 7.1 has $\ell = 19$ and $a = 27$.

In table 7.2 we have drawn the directed graphs of the unbounded f-sequences mentioned in table 6.1, for $f = \Psi_1, \Psi_2$ and Ψ_3. Every number ≤ 1000 for which the f-sequence is found to be unbounded appears in one of the digraphs. Every first term of the "semi-periodic" part of the sequence is marked with an asterisk. The semi-period ℓ and the multiplier a are given at the foot of the sequence. Details of the semi-periodic parts of the unbounded sequences can be found in table 7.3.

Case 2. $e_{\ell j} \leq e_{0j}$, for $j=1,2,\ldots,s$, with strict inequality for at least one j. Then by (7.1), (7,3) and (7.2),

$$n_0 = \left(\prod_{j=1}^{s} q_j^{e_{0j}}\right)m_0 = \left(\prod_{j=1}^{s} q_j^{e_{0j}}\right)m_\ell = \prod_{j=1}^{s}\left(q_j^{e_{0j}-e_{\ell j}}\right)\left(\prod_{j=1}^{s} q_j^{e_{\ell j}}\right)m_\ell =$$

$$= an_\ell ,$$

where $a = \Pi_{j=1}^{s}\ q_j^{e_{0j}-e_{\ell j}}$.
Now observe that

$$\Psi_k(an_{\ell-1}) - an_{\ell-1} = a\{\Psi_k(n_{\ell-1}) - n_{\ell-1}\} =$$

$$= an_\ell = n_0 ,$$

so that $an_{\ell-1}$ is a predecessor of n_0. Therefore we choose $n_{-1} = an_{\ell-1}$.

TABLE 7.2

Directed graphs of unbounded f-sequences

f = Ψ_1

318(2.3.53)			942(2.3.157)
330(2.3.5.11)	498(2.3.83)	978(2.3.163)	954(2.3(2)53)
534(2.3.89)	510(2.3.5.17)		990(2.3(2)5.11)
546(2.3.7.13)	786(2.3.131)		1602(2.3(2)89)
798(2.3.7.19)			1638(2.3(2)7.13)
1122(2.3.11.17)	636(2(2)3.53)		2394(2.3(2)7.19)
1470(2.3.5.7(2))	660(2(2)3.5.11)	996(2(2)3.83)	3366(2.3(2)11.17)
2562(2.3.7.61)	1068(2(2)3.89)	1020(2(2)3.5.13)	4410(2.3(2)5.7(2))
3390(2.3.5.113)	1092(2(2)3.7.13)	1572(2(2)3.131)	7686(2.3(2)7.61)
4818(2.3.11.73)	1596(2(2)3.7.19)		10170(2.3(2)5.113)
5838(2.3.7.139)	2244(2(2)3.11.17)		14454(2.3(2)11.73)
7602(2.3.7.181)	2940(2(2)3.5.7(2))		17514(2.3(2)7.139)
* 9870(2.3.5.7.47)	5124(2(2)3.7.61)		22806(2.3(2)7.181)
l=19,a=27	6780(2(2)3.5.113)		* 29610(2.3(2)5.7.47)
	9636(2(2)3.11.73)		l=19,a=27
	11676(2(2)3.7.139)		
	15204(2(2)3.7.181)		
	* 19740(2(2)3.5.7.47)		
	l=19,a=27		

f = Ψ_2

* 252(2(2)3(2)7)	* 504(2(3)3(2)7)	852(2(2)3.71)
476(2(2)7.17)	952(2(3)7.17)	1164(2(2)3.97)
532(2(2)7.19)	1064(2(3)7.19)	1580(2(2)5.79)
588(2(2)3.7(2))	1176(2(3)3.7(2))	* 1780(2(2)5.89)
* 1008(2(4)3(2)7)	* 2016(2(5)3(2)7)	l=6,a=8
l=4,a=4	l=4,a=4	

TABLE 7.2 (continued)

```
                                  f = Ψ3

 *120(2(3)3.5)    216(2(3)3(3))                   252(2(2)3(2)7)
    ↓                ↓                               ↓
 *240(2(4)3.5)  *384(2(7)3)                       476(2(2)7.17)
    ↓                ↓                               ↓
 *480(2(5)3.5)    576(2(6)3(2))                   532(2(2)7.19)    408(2(3)3.17)
    ↓                ↓                               ↓                ↓
 *960(2(6)3.5)    984(2(3)3.41) 864(2(5)3(3))     588(2(2)3.7(2))  672(2(5)3.7)
    ↓                ↓ ←────────┘                    ↓                ↓
*1920(2(7)3.5)  *1536(2(9)3)                     1008(2(4)3(2)7)  1248(2(5)3.13)
    .                .                               ↓ ←──────────────┘
    .                .                          *2112(2(6)3.11)
 l=1,a=2          l=3,a=4                            .
                                                     .
                                                 l=13,a=1024

 276(2(2)3.23)   306(2.3(2)17)                      *552(2(3)3.23)   642(2.3.107)
  ↓ ←────────────┘                                    ↓                ↓
 396(2(2)3(2)11)                   *336(2(4)3.7)     888(2(3)3.37)    654(2.3.109)
  ↓                                   ↓               ↓                ↓
*696(2(3)3.29)   504(2(3)3(2)7)    624(2(4)3.13)   1392(2(4)3.29)    666(2.3(2)37)
  ↓               └──────────────→  ↓                 .                ↓
1104(2(4)3.23)                    1056(2(5)3.11)      .               816(2(4)3.17)
  .                                   .             l=3,a=4            ↓
  .                                   .                             *1344(2(6)3.7)
l=3,a=4                           l=13,a=1024                          .
                                                                       .
                                                                   l=13,a=1024

                                 996(2(2)3.83)   660(2(2)3.5.11)  828(2(2)3(2)23)
                                  └────────────→  ↓ ←──────────────┘
                                                 1356(2(2)3.113)
                                                  ↓
                 402(2.3.67)     762(2.3.127)    1836(2(2)3(3)17)
                  ↓               ↓               ↓
                 414(2.3(2)23)   774(2.3(2)43)   3204(2(2)3(2)89)
                  ↓               ↓               ↓
432(2(4)3(3))    522(2.3(2)29)   942(2.3.157)    4986(2.3(2)277)
 ↓                ↓               ↓               ↓
768(2(8)3)       648(2(3)3(4))   954(2.3(2)53)   5856(2(5)3.61)
 └──────────────→ ↓ ←─────────────┘               ↓
               *1152(2(7)3(2))                   9024(2(6)3.47)
                  .                               ↓
                  .                             14016(2(6)3.73)
               l=3,a=4                            ↓
                                               *21504(2(10)3.7)
                                                  .
                                                  .
                                                l=13,a=1024
```

TABLE 7.2 (concluded)

$f = \Psi_3$

726(2.3.11(2))	570(2.3.5.19)	858(2.3.11.13)
→	↓	↓
	870(2.3.5.29)	1158(2.3.193)
	↓	↓
	1290(2.3.5.43)	1170(2.3(2)5.13)
	↓	↓
	1878(2.3.313)	2106(2.3(4)13)
	↓	↓
	1890(2.3(3)5.7)	2934(2.3(2)163)
	↓	↓
	3870(2.3(2)5.43)	3462(2.3.577)
	↓	↓
	6426(2.3(3)7.17)	3474(2.3(2)193)
	↓	↓
	10854(2.3(4)67)	4092(2(2)3.11.31)
	↓	↓
	13626(2.3(2)757)	6660(2(2)3(2)5.37)
	↓	↓
	15936(2(6)3.83)	14088(2(3)5.587)
	↓	↓
	24384(2(6)3.127)	21192(2(3)3.883)
	↓	↓
	37056(2(6)3.193)	31848(2(3)3.1327)
	↓	↓
	56064(2(8)3.73)	47832(2(3)3.1993)
	↓	↓
	* 86016(2(12)3.7)	* 71808(2(7)3.11.17)
	.	.
	.	.
	l=13,a=1024	l=13,a=1024

Similarly,

$$n_{-2} = an_{\ell-2} ,$$
$$\vdots$$
$$n_{-\ell+1} = an_1 , \text{ and}$$
$$n_{-\ell} = an_0 = a^2 n_\ell .$$

By induction we infer that for $r=1,2,\ldots$

$$n_{-r\ell+j} = a^r n_j , \qquad (j=0,1,\ldots,\ell-1),$$

so that we now have a decreasing (since $a > 1$) sequence of infinitely many predecessors of n_0. Again, we call ℓ the semi-period and a the multiplier of this sequence.

Case 3. $e_{\ell j} = e_{0j}$ for $j=1,2,\ldots,s$. In this case, obviously, $n_\ell = n_0$, so that the numbers $n_0, n_1, \ldots, n_{\ell-1}$ form a Ψ_k-cycle of length ℓ.

Case 4. There are indices $j_1, j_2 \in \{1,2,\ldots,s\}$ so that $e_{\ell j_1} < e_{0j_1}$ and $e_{\ell j_1} > e_{0j_2}$. Now it is no longer possible to construct unbounded sequences of the kind described in cases 1 and 2, but yet it is still possible to construct arbitrarily long increasing or decreasing sequences, according as $n_\ell/n_0 > 1$ or $n_\ell/n_0 < 1$. Again, ℓ is called the semi-period of the sequence.

According to table 7.2, the Ψ_3-sequence of $120 = 2^3 3.5$ is unbounded with semi-period 1 and multiplier 2. Also, 120 is a multiply perfect number because $\sigma(120) = 3.120$. The following theorem gives a method to construct unbounded Ψ_k-sequences of semi-period 1 from multiply perfect numbers.

<u>THEOREM 7.1</u> *If* N *is a multiply perfect number, i.e.,* $\sigma(N) = sN$ *for some positive integer* $s > 2$, *if* $s-1 = p^a$ *for some prime* p *and some positive integer* a, *and if* $N = p^k N_1$, *where* $(p,N_1) = 1$, N_1 *is* $(k+1)$*-free and* k *is some positive integer* > 1, *then the aliquot* Ψ_k*-sequence with leader* N *is unbounded with semi-period* 1 *and multiplier* p^a.

PROOF. Since $N = p^k N_1$ is (k+1)-free, we have $\Psi_k(N) = \sigma(N)$, so that

$$\Psi_k(N) - N = \sigma(N) - N = sN - N = p^a N \ .$$

Furthermore, from the definition of Ψ_k (chapter 3) it follows that

$$\begin{aligned}\Psi_k(p^a N) - p^a N &= \Psi_k(p^{a+k})\Psi_k(N_1) - p^a N = \\ &= p^a \Psi_k(p^k)\Psi_k(N_1) - p^a N = \\ &= p^a[\sigma(N) - N] = \\ &= Np^{2a} \ .\end{aligned}$$

By induction we infer that

$$\Psi_k(p^{ja}N) - p^{ja}N = Np^{(j+1)a} \qquad (j=0,1,\ldots).\square$$

In all, except two, of the multiply perfect numbers in the lists [5], [6], [16], [17] and [29], the highest exponent occurs as exponent of 2. Hence, for these numbers the condition $N = p^k N_1$, with $(p,N_1) = 1$ and N_1 is (k+1)-*free*, can only be satisfied if we choose p = 2, but then s-1 must be a power of 2. Application of theorem 7.1 yields

COROLLARY 7.1 *Every multiply perfect number* N *in the lists cited above, satisfying* $\sigma(N) = 3N$, *resp.* $\sigma(N) = 5N$, *is the starting value of an unbounded* $\Psi_{k(N)}$*-sequence with period* 1 *and multiplier* 2, *resp.* 4, *where* k(N) *is the exponent of* 2 *in the canonical factorization of* N. *(There are* 6 *cases with* $\sigma(N) = 3N$ *and* 66 *cases with* $\sigma(N) = 5N$.*)*

The two exceptional multiply perfect numbers mentioned above are

$$N = 2^2 3^2 5.7^2 13.19 \quad \text{and}$$
$$N = 2^7 3^{10} 5.17.23.137.547.1093 \ .$$

Both satisfy $\sigma(N) = 4N$. Application of theorem 7.1 to these numbers yields

<u>COROLLARY 7.2</u> *For all positive integers* $m,n \geq 2$ *the* Ψ_2*-sequence with leader* $2^m 3^2 5.7^n 13.19$ *is unbounded with semi-period* 1 *and multiplier* 3.

<u>COROLLARY 7.3</u> *The* Ψ_{10}*-sequence with leader* $2^7 3^{10} 5.17.23.137.547.1093$ *is unbounded with semi-period* 1 *and multiplier* 3.

A computer search for Ψ_k-sequences, described in the cases 1 - 4 above, was undertaken. Let $Q = \{q_1, q_2, \ldots, q_s\}$ $(s > 0)$ be a set of different prime numbers, let $m_0 > 1$ be some integer such that $(m_0, q_1 \ldots q_s) = 1$, and let $c = (q_1 \ldots q_s)^k$. The sequence $m_0, m_1, \ldots$ is defined as follows:

m_{i+1} is obtained from the number
$\Psi_k(cm_i) - cm_i = \Psi_k(c)\Psi_k(m_i) - cm_i$
by dropping all prime factors
$q_1, q_2, \ldots, q_s$ from it,
so that $(m_{i+1}, q_1 \ldots q_s) = 1$.

$\Big\}\quad i=0,1,2,\ldots$

If this sequence is periodic, i.e., if there are indices i_1, i_2 with $0 \leq i_1 < i_2$ so that

$$m_{i_2} = m_{i_1} ,$$

then from the definition of Ψ_k it follows that the Ψ_k-sequence of

$$q_1^{e_1} q_2^{e_2} \ldots q_s^{e_s} m_{i_1} = n_0$$

contains a term

$$q_1^{e_1'} q_2^{e_2'} \ldots q_s^{e_s'} m_{i_2} = n_{i_2 - i_1} \qquad (e_j' \geq 0,\ j=1,2,\ldots,s),$$

provided that the exponents $e_1, \ldots, e_s$ are chosen sufficiently large. In this way, we arrive at precisely one of the four cases discussed above, according as

$e'_j \geq e_j$ for $j=1,2,\ldots,s$ with strict inequality for at least one j (case 1),

$e'_j \leq e_j$ for $j=1,2,\ldots,s$ with strict inequality for at least one j (case 2),

$e'_j = e_j$ for $j=1,2,\ldots,s$ (case 3), or

$\exists j_1, j_2 \in \{1,2,\ldots,s\}$ with $e'_{j_1} < e_{j_1}$ and $e'_{j_2} > e_{j_2}$ (case 4).

For k=1,2,3 and for the sets $Q = \{2\},\{3\},\{5\},\{2,3\},\{2,5\},\{3,5\}$ and $\{2,3,5\}$ we have computed the sequences $m_0, m_1, \ldots$ for all $m_0 \leq 1000$, until we found a term m_{i_0} with

(i) $m_{i_0} = m_j$ for some $j < i_0$, or

(ii) $m_{i_0} = 1$, or

(iii) m_{i_0} has a prime factor $> 10^8$, or two prime factors $> 10^4$.

After finding a periodic sequence, the corresponding Ψ_k-sequence was computed. In table 7.3 we have listed all special Ψ_k-sequences found in this way. The sequences belonging to case 3 (Ψ_k-cycles) are listed in chapter 8, table 8.3, where general f-cycles are treated.

<u>EXAMPLE</u> $k = 2$, $Q = \{2\}$,

$$\begin{aligned} m_0 &= \ \ 63 = 3^2 7 , \\ m_1 &= 119 = 7.17 , \\ m_2 &= 133 = 7.19 , \\ m_3 &= 147 = 3.7^2 , \\ m_4 &= \ \ 63 = m_0 . \end{aligned}$$

The corresponding Ψ_2-sequence with leader $2^e m_0$ $(e \geq 2)$ is

$$\begin{aligned} n_0 &= 2^e 3^2 7 , \\ n_1 &= 2^e 7.17 , \\ n_2 &= 2^e 7.19 , \\ n_3 &= 2^e 3.7^2 , \\ n_4 &= 2^{e+2} 3^2 7 = 2^2 n_0 . \end{aligned}$$

It is clear that we can choose e = 2 and e = 3, so that we have found two unbounded Ψ_2-sequences, both with semi-period $\ell = 4$ and multiplier a = 4. The general terms are

$$\begin{aligned} n_{4j} &= 2^{e+2j}3^2 7 \\ n_{4j+1} &= 2^{e+2j}7.17 \\ n_{4j+2} &= 2^{e+2j}7.19 \\ n_{4j+3} &= 2^{e+2j}3.7^2 \;. \end{aligned} \qquad (j=0,1,\ldots;\ e=2 \text{ or } e=3)$$

These sequences are listed in table 7.3 as follows:

terms	characteristics
$2^m 3^2 7$	$m \geq 2$
$2^m 7.17$	monotonically increasing
$2^m 7.19$	case 1
$2^m 3.7^2$	$\ell = 4$
======	a = 4
$2^{m+2}3^2 7$	

In the first column, the terms of the periodic part are given, together with the first term of the next period, so that the behaviour of the sequence is completely determined.

Some characteristics of the sequence are given in the next column, namely

- the admitted values of the parameter(s),
- whether the sequence is (monotonically) increasing or decreasing,
- the case to which the sequence belongs,
- the semi-period ℓ,
- the multiplier a.

TABLE 7.3

Special aliquot Ψ_k-sequences (k=1,2,3) belonging to the cases 1, 2 and 4

k = 1

terms	characteristics
5(m)31	m ≥ 8
5(m-1)37	mon. decr.
5(m-2)43	case 2
5(m-3)7(2)	l = 8
5(m-4)7.13	a = 5(6)
5(m-5)7.31	
5(m-6)11.41	
5(m-7)769	
===========	
5(m-6)31	

terms	characteristics
2(m)3(n)5(i)281	m,n ≥ 1, i ≥ 3
2(m)3(n)5(i-1)1979	mon. incr.
2(m)3(n)5(i-1)47.59	case 1
2(m)3(n)5(i-1)4139	l = 8
2(m)3(n)5(i-1)11.17.31	a = 5(2)
2(m)3(n)5(i-2)53959	
2(m)3(n)5(i-1)29.521	
2(m)3(n)5(i+1)29.31	
======================	
2(m)3(n)5(i+2)281	

terms	characteristics
5(m)11.13	m ≥ 9
5(m-1)293	mon. decr.
5(m-2)13.23	case 2
5(m-3)521	l = 10
5(m-4)17.31	a = 5(8)
5(m-5)821	
5(m-6)827	
5(m-7)7(2)17	
5(m-8)7.269	
5(m-8)709	
============	
5(m-8)11.13	

terms	characteristics
3(m)5(n)7	m ≥ 1, n ≥ 8
3(m)5(n-1)29	mon. decr.
3(m)5(n-1)19	case 4
3(m)5(n-1)13	l = 15
3(m)5(n-2)47	
3(m)5(n-3)149	
3(m)5(n-3)7.13	
3(m+2)5(n-4)7(2)	
3(m+2)5(n-5)7.29	
3(m+2)5(n-5)181	
3(m+2)5(n-6)19.29	
3(m+2)5(n-6)409	
3(m+2)5(n-6)13.19	
3(m+3)5(n-6)67	
3(m+3)5(n-7)11.19	
=================	
3(m+3)5(n-5)7	

terms	characteristics
2(m)3(n)5.7.47	m,n ≥ 1
2(m)3(n)2963	mon. incr.
2(m)3(n)5.593	case 1
2(m)3(n)23.181	l = 19
2(m)3(n)17.269	a = 3(3)
2(m)3(n)5147	
2(m)3(n)19.271	
2(m)3(n)11.521	
2(m)3(n)7.971	
2(m)3(n)5.17.103	
2(m)3(n)13709	
2(m)3(n)13711	
2(m)3(n+1)7.653	
2(m)3(n+1)71.83	
2(m)3(n+1)6203	
2(m)3(n+1)5.17.73	
2(m)3(n+1)7.11.127	
2(m)3(n+1)14797	
2(m)3(n+2)4933	
=================	
2(m)3(n+3)5.7.47	

TABLE 7.3 (continued)

k = 2

terms	characteristics	terms	characteristics
2(m)3(n)5.7(i)13.19 ==================== 2(m)3(n+1)5.7(i)13.19	m,n,i ≥ 2 mon. incr. case 1 l = 1 a = 3	5(m)103 5(m-2)11.59 5(m-3)23.53 5(m-5)89.109 ============ 5(m-3)103	m ≥ 7 mon. decr. case 2 l = 4 a = 5(3)
2(m)3(n)11.13 2(m)3(n-1)5.13(2) ================= 2(m-1)3(n+2)11.13	m ≥ 2, n ≥ 3 mon. incr. case 4 l = 2	3(m)5.7 3(m-1)103 3(m-3)5(2)17 3(m-2)127 3(m-4)521 3(m-4)233 ============ 3(m-3)5.7	m ≥ 6 mon. decr. case 2 l = 6 a = 3(3)
2(m)3(2)7 2(m)7.17 2(m)7.19 2(m)3.7(2) ========== 2(m+2)3(2)7	m ≥ 2 mon. incr. case 1 l = 4 a = 2(2)	2(m)5.89 2(m+2)5(3) 2(m)3(2)5.13 2(m+1)3.13.17 2(m+1)3.367 2(m+1)5(2)59 ============= 2(m+3)5.89	m ≥ 2 mon. incr. case 1 l = 6 a = 2(3)
2(m)3(n)7(2)43 2(m+1)3(n-1)7.907 2(m+1)3(n-3)5.7.3089 2(m+1)3(n-1)5.7(2)11(2) ======================= 2(m)3(n+3)7(2)43	m ≥ 2, n ≥ 5 mon. incr. case 1 l = 4 a = 3(3)		
3(m)13.743 3(m-1)11.13.113 3(m)5.13.59 3(m)5.13.53 3(m)13.239 3(m-1)13(2)31 =============== 3(m-2)13.743	m ≥ 3 mon. decr. case 2 l = 6 a = 3(2)	3(m)7.101 3(m-1)5.283 3(m-2)43.73 3(m-4)7.2011 3(m-6)5.11.19.79 3(m-6)5.41.409 3(m-6)5.11.29.41 3(m-6)5.19.691 3(m-7)5.31.1051 3(m-8)386549 ================ 3(m-3)7.101	m ≥ 10 mon. decr. case 2 l = 10 a = 3(3)
3(m)13.2459 3(m-1)13.11.373 3(m-2)13.5.11.157 3(m-2)13(2)17.41 3(m-2)13.6311 3(m-3)13.17.619 3(m-2)13.43.53 3(m-2)13(2)109 ================= 3(m-3)13.2459	m ≥ 5 mon. decr. case 2 l = 8 a = 3(3)		

TABLE 7.3 (concluded)

k = 3

terms	characteristics	terms	characteristics
2(m)3.5	m ≥ 3	2(m)3.7	m ≥ 4
=======	mon. incr.	2(m)3.13	mon. incr.
2(m+1)3.5	case 1	2(m+1)3.11	case 1
	l = 1	2(m+1)3.19	l = 13
	a = 2	2(m+1)3.31	a = 2(10)
		2(m+1)3.7(2)	
2(m)3	m ≥ 7	2(m)3.11.17	
2(m-1)3(2)	mon. incr.	2(m)3.353	
2(m-4)3.41	case 1	2(m+2)3.7.19	
==========	l = 3	2(m+2)3(2)89	
2(m+2)3	a = 2(2)	2(m)3(2)619	
		2(m-1)3.6361	
2(m)3.23	m ≥ 3	2(m+2)3.1193	
2(m)3.37	mon. incr.	============	
2(m+1)3.29	case 1	2(m+10)3.7	
==========	l = 3		
2(m+2)3.23	a = 2(2)		

terms	charact.	terms	terms
2(m)2137	m ≥ 22	2(m-19)7(2)11.159311	2(m-12)79.1693
2(m-2)7487	decreasing	2(m-19)23.53.97169	2(m-12)61.1973
2(m-2)6553	(not	2(m-19)13.281.32213	2(m-13)218249
2(m-4)22943	monotonic.)	2(m-18)13(2)29.12323	2(m-15)763879
2(m-4)17.1181	case 2	2(m-11)13.59.677	2(m-13)167099
2(m-5)39631	l = 96	2(m-11)548591	2(m-14)292427
2(m-5)34679	a = 2(8)	2(m-11)480019	2(m-15)17.30103
2(m-4)15173		2(m-12)11.76367	2(m-15)7(2)41.251
2(m-6)53113		2(m-12)79.11117	2(m-14)7(2)6397
2(m-8)185903		2(m-12)379.2083	2(m-16)7.211619
2(m-8)47.3461		2(m-12)67.97.107	2(m-16)1692967
2(m-8)148913		2(m-12)654067	2(m-14)11.131.257
2(m-10)17.23.31.43		2(m-13)1144621	2(m-14)11.35993
2(m-10)619277		2(m-15)43.151.617	2(m-13)139.1489
2(m-12)13.137.1217		2(m-15)3743539	2(m-12)92077
2(m-9)280591		2(m-16)439.14923	2(m-14)29.11113
2(m-9)245519		2(m-16)151.38153	2(m-15)31.19541
2(m-9)214831		2(m-16)29.176303	2(m-15)439.1291
2(m-9)11.23.743		2(m-16)97.49529	2(m-15)499151
2(m-9)71.3011		2(m-17)2683.3203	2(m-15)31.73.193
2(m-9)79.2441		2(m-17)7.47(2)487	2(m-15)424601
2(m-9)89.1949		2(m-17)1367.6577	2(m-17)11.135101
2(m-10)311203		2(m-17)7.1125973	2(m-15)11.35311
2(m-11)13.41893		2(m-17)2129.4231	2(m-15)73.5563
2(m-12)43.25819		2(m-17)2539.3109	2(m-14)182953
2(m-12)59(2)293		2(m-14)7(3)2521	2(m-16)11.23.2531
2(m-14)3728173		2(m-14)43.23879	2(m-16)17.61.701
2(m-16)23.31.18301		2(m-14)943303	2(m-15)37(2)271
2(m-16)13.67.15277		2(m-10)79.653	2(m-15)113.3067
2(m-16)13964963		2(m-10)193.241	2(m-14)349.443
2(m-17)11.2221699		2(m-11)79.1051	================
2(m-17)25549561		2(m-11)74771	2(m-8)2137
2(m-19)23.569.6833		2(m-12)19.71.97	

CHAPTER 8

ALIQUOT f-CYCLES

The subject of this chapter is the study of (aliquot) f-cycles, for special choices of f. This chapter is divided into three sections: section 8.1 deals with f-cycles of length 1 (also called f-*perfects*), in section 8.2 we treat f-cycles of length 2 (also called f-*amicable pairs*) and in section 8.3 we study f-cycles of length $\ell > 2$. We notice that it follows from the definitions in chapter 3 that any (2k+2)-free σ-cycle is an M_k-cycle (k=0,1,2,...), that any (k+1)-free σ-cycle is a Ψ_k-cycle (k=1,2,...), and that any (k+2)-free σ-cycle is an L_k-cycle (k=0,1,2,...) and also an R_k-cycle (k=0,1,2,...).

8.1 f-PERFECTS

8.1.1 $f = \sigma$

24 even σ-perfects are known, the smallest being $N = 6$ and the largest being $N = 2^{p-1}(2^p-1)$ with p = 19937 [38]. Whether there exists any odd perfect number is not known at present. If one exists, it must exceed 10^{50} [19] *) and contain at least eight different prime factors [20].

8.1.2 $f = \sigma^*$

5 even σ^*-perfects are known, the smallest being N = 6 and the largest being $N = 2^{18}3.5^4 7.11.13.19.37.79.109.157.313$ [36], [39]. It is easy to prove that odd σ^*-perfects do no exist.

8.1.3 $f = \Psi_1$

There are infinitely many Ψ_1-perfects, namely $N = 2^m 3^n$ (m,n=1,2,...), and there are no other ones [41].

*) Recently, this bound has been replaced by 10^{100}. See M. BUXTON & B. STUFFLEFIELD, On odd perfect numbers, Notices Amer.Math.Soc., 22 (1975) A-543.

8.1.4 $f = \Psi_2$

THEOREM 8.1 *The only Ψ_2-perfects are 6 and $2^m 7$ (m=2,3,...).*

PROOF. From the definition of Ψ_2 it follows that

$$N = p_1 p_2 \ldots p_r q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$$

($p_1, \ldots, p_r, q_1, \ldots, q_s$ are different primes, all $\alpha_i \geq 2$) is a Ψ_2-perfect, if and only if

$$\overline{N} := p_1 p_2 \ldots p_r q_1^2 q_2^2 \ldots q_s^2 \qquad (8.1)$$

is Ψ_2-perfect. But $\overline{N}$ is 3-free, so that $\Psi_2(\overline{N}) = \sigma(\overline{N})$.
Therefore, we look for numbers $\overline{N}$ of the form (8.1) which satisfy $\sigma(\overline{N}) = 2\overline{N}$. The only even numbers with this property are 6 and 28. If $\overline{N}$ is odd, then it is well-known that $r = 1$ and $p_1 \equiv 1 \pmod 4$. Since STEUERWALD [35] proved that these numbers $\overline{N} = p_1 q_1^2 \ldots q_s^2$ cannot be σ-perfect, our proof is complete. □

8.1.5 $f = \Psi_3$

THEOREM 8.2 *The only Ψ_3-perfects are 6 and 28.*

PROOF. By the same argument as in the proof of theorem 8.1 we look for the 4-free σ-perfects. It is easy to see that there are only two numbers of this kind, namely 6 and 28. □

8.1.6 $f = \Psi_k$

By the same argument as in the case $f = \Psi_2$ we can prove the general

THEOREM 8.3 *The even Ψ_k-perfects ($k \geq 1$) are*

(i) *the even (k+1)-free σ-perfects, and*

(ii) *the numbers $2^{k+i}(2^{k+1}-1)$, for i=1,2,..., provided that $2^k(2^{k+1}-1)$ is σ-perfect.*

We cannot answer the question whether there exist any odd Ψ_k-perfects for $k \geq 4$.

8.1.7 $f = M_k$

We present a general theorem about even M_k-perfects, but we first prove

LEMMA 8.1 *If* $m|n$ $(1 < m \le n)$, *then*

$$\frac{M_k(n)}{n} \ge 1 + \frac{1}{m} \qquad (k=1,2,\ldots).$$

PROOF. Suppose the canonical prime factorization of n is given by $n = p_1^{e_1}\ldots p_s^{e_s}$ $(e_i > 0,\ i=1,2,\ldots,s)$. Then the divisor m of n must be of the form $m = p_1^{\alpha_1}\ldots p_s^{\alpha_s}$ $(0 \le \alpha_i \le e_i,\ i=1,2,\ldots,s$, where at least one α_i is positive). Hence

$$\frac{M_k(n)}{n} = \frac{M_k(p_1^{e_1})}{p_1^{e_1}} \cdots \frac{M_k(p^{e_s})}{p^{e_s}} =$$

$$\ge (1 + \frac{1}{p_1})\ldots(1 + \frac{1}{p_s}) >$$

$$> 1 + \frac{1}{p_1^{\alpha_1} p_2^{\alpha_2}\ldots p_s^{\alpha_s}} = 1 + \frac{1}{m}. \qquad \square$$

THEOREM 8.4 *There are no even* M_k*-perfects* N *such that the exponent of* 2 *in the canonical factorization of* N *is* $\ge 2k + 1$.

PROOF. Suppose contrariwise that $N = 2^a N_1$ (N_1 odd and $a \ge 2k+1$) is M_k-perfect. Then we have

$$(8.2a) \qquad (2^{k+1} - 1)(2^{a-k} + 1)\, M_k(N_1) = 2^{a+1}\, N_1, \qquad \text{so that}$$

$$(8.2b) \qquad \frac{M_k(N_1)}{N_1} = \frac{2^{a+1}}{(2^{k+1} - 1)(2^{a-k} + 1)}.$$

From (8.2a) it follows that $2^{k+1}-1|N_1$ and from lemma 8.1 we infer that

$$\frac{M_k(N_1)}{N_1} \ge 1 + \frac{1}{2^{k+1} - 1} >$$

$$> \frac{2^{k+1}}{2^{k+1} - 1}\,\frac{2^{a-k}}{2^{a-k} + 1} = \frac{2^{a+1}}{(2^{k+1} - 1)(2^{a-k} + 1)},$$

which contradicts (8.2b). $\square$

THEOREM 8.5 *There are no odd* M_1*-perfects.*

PROOF. Suppose $N = p_1^{e_1} \ldots p_s^{e_s}$ is an odd M_1-perfect, so that

$$M_1(p_1^{e_1}) \ldots M_1(p_s^{e_s}) = 2p_1^{e_1} \ldots p_s^{e_s} . \qquad (8.3)$$

None of the exponents e_i can be greater than 2 because, if so, then $M_1(p_i^{e_i}) = (p_i+1)(p_i^{e_i-1}+1)$ would have at least two prime factors 2, whereas the right hand side of (8.3) contains exactly one prime 2. Hence, N is 3-free, which implies that $M_1(N) = \sigma(N)$. But in the proof of theorem 8.1 we showed that there are no 3-free odd σ-perfects. □

We do not know whether there is an odd M_k-perfect for $k \geq 2$.

8.1.8 $f = L_k$ and $f = R_k$

We have not found general theorems for $f = L_k$ and $f = R_k$ *) as we did for $f = \Psi_k$ and $f = M_k$. Table 8.1 gives a list of L_k-perfects for k=1,2,3,4 and table 8.2 gives a list of R_k-perfects for k=1,2,3,4. These perfects were computed by trial and error.

TABLE 8.1

Some L_k-perfects for k=1,2,3,4, found by trial and error

k	L_k-perfects
1	$2.3,\ 2^2 7,\ 2^3 7.13,\ 2^4 5^2 31,\ 2^4 5^3 19.31.151$
2	$2.3,\ 2^2 7$.
3	$2.3,\ 2^2 7,\ 2^4 31,\ 2^5 31.61$
4	$2.3,\ 2^2 7,\ 2^4 31$

*) with the following exception: if $p = 3.2^{k+1}-1$ $(k \in \mathbb{N}_0)$ is a prime, then $2^{k+2}3.p$ is an R_k-perfect. A table of all k's ≤ 1000 for which p is prime may be found in [34].

TABLE 8.2

Some R_k-perfects for k=1,2,3,4, found by trial and error

k	R_k-perfects	k	R_k-perfects
1	2.3 $2^{2}7$ $2^{3}3.11$ $2^{4}3.5.19$ $2^{5}3.5.7$ $2^{6}3^{2}7.13.17.67$ $2^{7}3^{2}7.11.13.131$ $2^{8}3.5.7.19.37$ $2^{9}3.5.7.13.103$ $2^{10}3.5.7.13.79$ $2^{12}3.5^{2}7.31.41.4099$ $2^{13}3^{2}5^{4}7.11.13.79.149.631$ $2^{16}3^{2}5^{4}7.13.19.29.79.113.631.65539$	2	2.3 $2^{2}7$ $2^{4}3.23$ $2^{5}3.7.13$ $2^{6}3^{2}7.13.71$ $2^{7}3^{3}5^{2}31$ $2^{8}3^{2}7.11.13.263$ $2^{9}3^{3}5^{2}29.31.173$ $2^{10}3^{2}7.11.13.43.1031$ $2^{11}3^{3}5^{2}23.31.137$ $2^{12}3^{4}5.7.11^{2}17.19.47.373$ $2^{13}3^{3}5^{2}19.31.911$ $2^{15}3^{4}5^{3}7.13.19.23.47$ $2^{16}3^{3}5^{2}19.31.683.2731.65543$
3	2.3 $2^{2}7$ $2^{4}31$ $2^{5}3.47$ $2^{6}3.5.79$ $2^{7}3.7.11.13$ $2^{8}3^{2}7.13.17.271$ $2^{9}3^{2}7.11.13.527$ $2^{10}3.5.7.13.1039$ $2^{11}3^{2}7.11.13.43.2063$ $2^{12}3^{2}7.11.13.43.257.4111$ $2^{13}3^{3}5^{2}29.31.71.283$ $2^{14}3.5.7.23^{2}31.79$ $2^{15}3^{3}5^{2}19.31.683.32783$ $2^{16}3.5.7.11^{3}17^{2}79.241.307.65551$	4	2.3 $2^{2}7$ $2^{4}31$ $2^{6}5^{2}19.31$ $2^{7}3^{6}5^{2}17.31.53$ $2^{10}3^{3}5^{2}31.53.211$ $2^{11}3^{6}5^{2}7.11.17.31$ $2^{12}3^{2}7.11.13.43.4127$ $2^{13}3^{3}5^{2}23.31.229.457.2741$

8.2 f-AMICABLE PAIRS

8.2.1 $f = \sigma$.

More than 1100 σ-amicable pairs are known [26], the smallest pair being {220,284}. The four largest known pairs (with 32-, 40-, 81- and 152-digit numbers) were recently computed by TE RIELE [31]. In the lists of f-amicable pairs (for $f \neq \sigma$) given in the sequel, those f-amicable pairs, that are also σ-amicable pairs, are omitted.

8.2.2 $f = \sigma^* \ (= M_0 = L_0 = R_0)$.

In 1970, WALL [41] found more than 600 σ^*-amicable pairs. HAGIS in 1971 and TE RIELE in 1973 also investigated σ^*-amicable pairs, both unaware of WALL's thesis. HAGIS [21] computed all σ^*-amicable pairs $\{m,n\}$ with $m < n$ and $m \le 10^6$ [21]. TE RIELE [32] published a list of more than 1100 σ^*-amicable pairs, including nearly all those pairs published by Wall. For some other new σ^*-amicable pairs, see [24].

8.2.3 $f = \Psi_k$ $(k=1,2,\ldots)$.

Many Ψ_k-amicable pairs may be constructed from the known σ-amicable pairs [26] as follows. Suppose the pair $\{m,n\}$ is σ-amicable and $m = p^k m_1$ and $n = p^k n_1$ where $k > 0$, $(p,m_1) = 1$, $(p,n_1) = 1$, and m_1 and n_1 are $(k+1)$-free. Then it follows from the definition of Ψ_k that the pairs $\{p^a m, p^a n\}$, $(a=0,1,2,\ldots)$ are Ψ_k-amicable. In our list of Ψ_k-amicable pairs (table 8.3, pp. 56-58) we have not included these pairs, in order to save space. The pairs given in table 8.3 were found partly by the method described in chapter 7, partly by a systematic computer search for all pairs, the smallest element of which does not exceed 10^4, partly by use of one of the three following lemma's and partly by trial and error.

LEMMA 8.2 *If the two positive integers* $p = 2^{k+i}+2^k-1$ *and* $q = 2^{k-i}+2^k-1$ *are primes, then the pairs*

$$\{2^a p, 2^{a+i} q\} \qquad (a=k,k+1,\ldots)$$

are Ψ_k*-amicable* $(k=2,3,\ldots;\ i=1,2,\ldots,k-1)$.

LEMMA 8.3 *Suppose*

$$AB = 2^k(2^k-1) + 2^{k-i}, \qquad A \neq B,$$

is a factorization of the positive integer $2^k(2^k-1)+2^{k-i}$. *If the three positive integers* $p = 2^k-1+A$, $q = 2^k-1+B$ *and* $r = 2^i(p+1)(q+1)-1$ *are primes, then the pairs*

$$\{2^{a+i}pq, 2^a r\} \qquad \textit{where } a=k,k+1,\ldots,$$

are Ψ_k*-amicable* $(k=2,3,\ldots;\ i=1,2,\ldots,k-1)$.

<u>LEMMA 8.4</u> *Suppose*

$$AB = 2^k(2^k-1) + 2^{k+i}, \qquad A \neq B,$$

is a factorization of the positive integer $2^k(2^k-1)+2^{k+i}$. *If the three positive integers* $p = 2^k-1+A$, $q = 2^k-1+B$ *and* $r = \frac{(p+1)(q+1)}{2^i} - 1$ *are primes, then the pairs*

$$\{2^a pq, 2^{a+i} r\} \qquad \textit{where } a=k,k+1,\ldots,$$

are Ψ_k*-amicable* $(k=2,3,\ldots;\ i=1,2,\ldots,k-1)$.

The proof of these lemma's follows easily by solving the equations

$$\begin{cases} \Psi_k(m) = \Psi_k(n) \\ \Psi_k(m) = m + n \end{cases}$$

for the pairs $\{m,n\}$ given in the lemma's.

8.2.4 $f = M_k$, $f = L_k$ and $f = R_k$.

Table 8.4 gives M_k- $(k=1,2)$, L_k- $(k=1,2,3,4)$ and R_k - $(k=1,2,3,4)$ amicable pairs, which are not at the same time σ-amicable pairs. They were found partly by a computer search for all pairs $\{m,n\}$ with $m < n$ and $m \leq 10^4$, and partly by trial and error.

TABLE 8.3

Some Ψ_k-amicables for k=1,2,3,4, found by various methods (see text)

k	Ψ_k-amicable pairs	
1	$2^m5^n7.19 = (2^{m-1}5^{n-1})1330$ $2^m5^{n+1}31 = (2^{m-1}5^{n-1})1550$	(m,n≥1)
	$2^m5^n7.11 = (2^{m-1}5^{n-2})3850$ $2^m5^{n-1}479 = (2^{m-1}5^{n-2})4790$	(m≥1, n≥2)
	$2^m7^n5.23 = (2^{m-1}7^{n-2})11270$ $2^m7^{n-1}13.71 = (2^{m-1}7^{n-2})12922$	(m≥1, n≥2)
	$2^m5^n13.23 = (2^{m-1}5^{n-2})14950$ $2^m5^{n-1}11.139 = (2^{m-1}5^{n-2})15290$	(m≥1, n≥2)
	$2^m5^n7.53 = (2^{m-1}5^{n-2})18550$ $2^m5^{n-1}19.107 = (2^{m-1}5^{n-2})20330$	(m≥1, n≥2)
	$2^m5^n11.23.29 = (2^{m-1}5^{n-1})73370$ $2^m5^{n+1}31.53 = (2^{m-1}5^{n-1})82150$	(m,n≥1)
	$3^m5^n7^i13.23 = (3^{m-1}5^{n-2}7^{i-1})156975$ $3^m5^{n-1}7^i19.83 = (3^{m-1}5^{n-2}7^{i-1})165585$	(m,i≥1, n≥2)
	$2^m7^n11^i13.109 = (2^{m-1}7^{n-1}11^{i-1})218218$ $2^m7^{n+1}11^{i+1}19 = (2^{m-1}7^{n-1}11^{i-1})225302$	(m,n,i≥1)
	$2^m5^n19^i11.113 = (2^{m-1}5^{n-1}19^{i-1})236170$ $2^m5^n19^{i+1}71 = (2^{m-1}5^{n-1}19^{i-1})256310$	(m,n,i≥1)
	$2^m5^n11^i43.89 = (2^{m-1}5^{n-1}11^{i-1})420970$ $2^m5^n11^{i+1}359 = (2^{m-1}5^{n-1}11^{i-1})434390$	(m,n,i≥1)
	$3^m5^n7^i11.17 = (3^{m-1}5^{n-2}7^{i-2})687225$ $3^m5^{n-1}7^{i-1}29.251 = (3^{m-1}5^{n-2}7^{i-2})764295$	(m≥1, n,i≥2)
	$2^m5^n31^i13.29 = (2^{m-1}5^{n-1}31^{i-2})3622970$ $2^m5^{n+1}31^{i-1}41.61 = (2^{m-1}5^{n-1}31^{i-2})3876550$	(m,n≥1, i≥2)
2	$2^m3 = (2^{m-2})12$ $2^{m+2} = (2^{m-2})16$	(m≥2)
	$2^m5 = (2^{m-3})40$ $2^{m-1}11 = (2^{m-3})44$	(m≥3)

TABLE 8.3 (continued)

k	Ψ_k-amicable pairs	
2 (cont.)	$2^m 5.13 = (2^{m-2})260$	$(m \geq 2)$
	$2^{m+1}41 = (2^{m-2})328$	
	$3^m 5.7.13 = (3^{m-3})12285$	$(m \geq 3)$
	$3^{m-1}7.13.17 = (3^{m-3})13923$	
	$3^m 5.7.13.23 = 3^{m-2}(94185)$	$(m \geq 2)$
	$3^{m+1}7.13.47 = 3^{m-2}(115479)$	
	$2.5^m 7.59 = 5^{m-3}(103250)$	$(m \geq 3)$
	$2.5^{m-1}2399 = 5^{m-3}(119950)$	
3	$2^m 7 = (2^{m-3})56$	$(m \geq 3)$
	$2^{m+3} = (2^{m-3})64$	
	$2^m 11 = (2^{m-4})176$	$(m \geq 4)$
	$2^{m-1}23 = (2^{m-4})184$	
	$2^m 13.19 = (2^{m-3})1976$	$(m \geq 3)$
	$2^{m+1}139 = (2^{m-3})2224$	
	$2^m 11.29 = (2^{m-3})2552$	$(m \geq 3)$
	$2^{m+2}89 = (2^{m-3})2848$	
	$2^m 13.17 = (2^{m-4})3536$	$(m \geq 4)$
	$2^{m-1}503 = (2^{m-4})4024$	
	$3^m 5.7.19 = (3^{m-5})161595$	$(m \geq 5)$
	$3^{m-2}5.29.47 = (3^{m-5})184005$	
	$3^m 5^n 7.109 = (3^{m-5}5^{n-4})115880625$	$(m \geq 5,\ n \geq 4)$
	$3^{m-2}5^{n-1}59.659 = (3^{m-5}5^{n-4})131223375$	
	$3^m 5^n 7.199.967 = (3^{m-5}5^{n-3})40916066625$	$(m \geq 5,\ n \geq 3)$
	$3^{m-2}5^n 47.290399 = (3^{m-5}5^{n-3})46064541375$	
4	$2^{m+1}23 = (2^{m-4})736$	$(m \geq 4)$
	$2^m 47 = (2^{m-4})752$	
	$2^{m+2}19 = (2^{m-4})1216$	$(m \geq 4)$
	$2^m 79 = (2^{m-4})1264$	

TABLE 8.3 (concluded)

k	Ψ_k-amicable pairs	
4 (cont.)	$\begin{cases} 2^m 19.83 = (2^{m-4})25232 \\ 2^{m+1} 839 = (2^{m-4})26848 \end{cases}$	(m≥4)
	$\begin{cases} 2^m 19.107 = (2^{m-4})32528 \\ 2^{m+3} 269 = (2^{m-4})34432 \end{cases}$	(m≥4)
	$\begin{cases} 2^m 17.151 = (2^{m-4})41072 \\ 2^{m+1} 1367 = (2^{m-4})43744 \end{cases}$	(m≥4)
	$\begin{cases} 2^m 17.199 = (2^{m-4})54128 \\ 2^{m+3} 449 = (2^{m-4})57472 \end{cases}$	(m≥4)
	$\begin{cases} 2^{m+1} 17.139 = (2^{m-4})75616 \\ 2^m 5039 = (2^{m-4})80624 \end{cases}$	(m≥4)

TABLE 8.4

The M_k-, L_k- and R_k-amicable pairs $\{m,n\}$ such that $m<n$ and $m \le 10^4$, and some pairs, found by trial and error

$f = M_k$

k	f-amicable pairs	
1	$3608(2^3 11.41)$ $3952(2^4 13.19)$	$9520(2^4 5.7.17)$ $13808(2^4 863)$
2	none with $m \le 10^4$	

$f = L_k$

k	f-amicable pairs	
1	$168(2^3 3.7)$ $248(2^3 31)$	$1548(2^2 3^2 43)$ $2456(2^3 307)$
	$920(2^3 5.23)$ $952(2^3 7.17)$	$5720(2^3 5.11.13)$ $7384(2^3 13.71)$
	$1008(2^4 3^2 7)$ $1592(2^3 199)$	$16268(2^2 7^2 83)$ $17248(2^5 7^2 11)$
2	$8272(2^4 11.47)$ $8432(2^4 17.31)$	
3, 4	none with $m \le 10^4$	

$f = R_k$

k	f-amicable pairs	
1	$366(2.3.61)$ $378(2.3^3 7)$	$16104(2^3 3.11.61)$ $16632(2^3 3^3 7.11)$
	$3864(2^3 3.7.23)$ $4584(2^3 3.191)$	
2	$26448(2^4 3.19.29)$ $28752(2^4 3.599)$	
3	none with $m \le 10^4$	
4	$10194(2.3.1699)$ $10206(2.3^6 7)$	

8.3 f-CYCLES OF LENGTH $\ell > 2$

8.3.1 $f = \sigma$.

Fourteen σ-cycles of length $\ell = 4$ and one each for $\ell = 5$ and $\ell = 28$ are known [18].

8.3.2 $f = \sigma^*$.

One σ^*-cycle of length $\ell = 3$, 8 for $\ell = 4$, one each for $\ell = 25$, $\ell = 39$ and $\ell = 65$ are known [24], [33].

8.3.3 $f = \Psi_3$, $f = L_3$, $f = R_1$.

Table 8.5 gives the only three f-cycles of length $\ell > 2$ (not at the same time being σ-cycles) which are known to us. They were found by trial and error.

TABLE 8.5

Three aliquot f-cycles of length $\ell > 2$, that are not σ-cycles

f	ℓ	aliquot f-cycle		
Ψ_3	4	$\begin{cases} 2^m 3917 \\ 2^{m-2} 11.29.43 \\ 2^{m-2} 11.1453 \\ 2^m 47.89 \end{cases} = 2^{m-5}.\begin{cases} 125344 \\ 109736 \\ 127864 \\ 133856 \end{cases}$		$(m \geq 5)$
L_3	4	$\begin{cases} 4040 (2^3 5.101) \\ 5140 (2^2 5.257) \\ 5696 (2^6 89) \\ 5194 (2.7^2 53) \end{cases}$		
R_1	3	$\begin{cases} 834\ (2.3.139) \\ 846\ (2.3^2 47) \\ 1026 (2.3^3 19) \end{cases}$		

CHAPTER 9

SOLVING THE EQUATION f(x)-x=m

In this chapter we investigate the equation

$$(9.1) \qquad f(x) - x = m ,$$

for $f \in F$ and $m \in \mathbb{N}$. If (9.1) has no solution $x \in \mathbb{N}$ for some m, then m is called f-*untouchable*, otherwise, m is called f-*touchable*.
In [14], ERDÖS proved that the lower density of the σ-untouchables is *positive*. ALANEN [1] found the 570 σ-untouchables ≤ 5000.

THEOREM 9.1 *Let* $f \in F$. *Suppose that* f *satisfies the additional condition*

$$(9.2) \qquad \frac{f(d)}{d} \le \frac{f(n)}{n} ,$$

for all divisors d *of* n. *If* M *is even and* f-*abundant, and if* M' *is an even and* f-*abundant divisor of* M, *then the lower density of the* f-*untouchables* m, *satisfying* $m \equiv M' \pmod M$, *is* $\ge \frac{1}{M}(1 - \frac{M'}{f(M')-M'}) > 0$.

Note that for M' = M, this statement reduces to: if M is even and f-abundant, then the lower density of the f-untouchables m, satisfying $m \equiv 0 \pmod M$, is $\ge \frac{1}{M} - \frac{1}{f(M)-M}$.

Before proving this theorem, we give two lemma's.

LEMMA 9.1 *The number of* 2-*full numbers* $\le x$ *is* $O(\sqrt{x})$, *for* $x \to \infty$.

PROOF. Any 2-full number n can be uniquely represented in the form $n = a^2b^3$, where $a \in \mathbb{N}$ and b is squarefree. If T(x) is the number of 2-full numbers $\le x$, then it follows that

$$T(x) \le \sum_{\substack{b^3 \le x \\ b \text{ is squarefree}}} (x/b^3)^{1/2} < \sqrt{x} \sum_{b=1}^{\infty} \frac{1}{b^{3/2}} = O(\sqrt{x}) , \qquad \text{for } x \to \infty. \ \square$$

The next lemma is a special case of a more general result of Scourfield *).

LEMMA 9.2 *If* $f \in F$, *then for any* $d \in \mathbb{N}$ *the number of positive integers* $n \le x$ *such that* $d \nmid f(n)$, *is* $o(x)$ *for* $x \to \infty$.

PROOF OF THEOREM 9.1. First notice that (9.2) implies that for any prime divisor p of n

(9.3) $\qquad f(n) - n \ge n/p.$

Let A(N) be the number of $n \in \mathbb{N}$ satisfying

(9.4) $\qquad f(n) - n \le N$, and

(9.5) $\qquad f(n) - n \equiv M'(\text{mod } M)$.

This number is *finite* for any $N \in \mathbb{N}$. Indeed, if $n = p$, then $f(n)-n = 1 \not\equiv M'(\text{mod } M)$. If n is not a prime, and if p_1 is the smallest prime divisor of n, then we have $p_1 \le \sqrt{n}$, so that by (9.3) we have $f(n)-n \ge n/p_1 \ge \sqrt{n}$. From (9.4) it follows that $n \le N^2$.

If $A_1(N)$ is the number of *odd* n, satisfying (9.4) and (9.5), if $A_2(N)$ is the number of *even* n, with $n \not\equiv -M'(\text{mod } M)$, satisfying (9.4) and (9.5) and if $A_3(N)$ is the number of *even* n, with $n \equiv M'(\text{mod } M)$, satisfying (9.4) and (9.5), then we obviously have

(9.6) $\qquad A(N) = A_1(N) + A_2(N) + A_3(N)$.

If n is *odd* then by (9.5), f(n) is also odd. Since, for odd p, $f(p) = p+1$ is even, n must be 2-full. Suppose $n = p^2$. Then by (9.3) $f(n)-n \ge p$, so that the

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*) E.J. SCOURFIELD, *Non-divisibility of some multiplicative functions*, Acta Arithmetica, 22(1973) 287-314.

number of odd $n = p^2$, satisfying (9.4) and (9.5), is $\leq \pi(N)$, which is $o(N)$, for $N \to \infty$. If $n \neq p^2$, and if p_1 is the smallest prime divisor of n, then we have $p_1 \leq n^{1/3}$, so that, by (9.3), $f(n)-n \geq n/p_1 \geq n^{2/3}$. From (9.4) it follows that $n \leq N^{3/2}$, and by lemma 9.1, we conclude that the number of odd $n \neq p^2$, satisfying (9.4) and (9.5) is $O(N^{3/4})$, for $N \to \infty$. Hence

$$A_1(N) = o(N) \qquad \text{for } N \to \infty. \tag{9.7}$$

If n is *even*, then (9.3) implies that $f(n)-n \geq n/2$, so that, by (9.4), $n \leq 2N$.

If $n \not\equiv -M'(\text{mod } M)$, then by (9.5) we have $f(n) \not\equiv 0 \ (\text{mod } M)$. It follows from lemma 9.2 that the number of positive integers $n \leq 2N$ such that $f(n) \not\equiv 0(\text{mod } M)$ is $o(N)$, so that

$$A_2(N) = o(N) \qquad \text{for } N \to \infty. \tag{9.8}$$

If $n \equiv -M'(\text{mod } M)$, then, since $M'|M$, we have $M'|n$ and it follows from (9.2) that

$$\frac{f(M')}{M'} \leq \frac{f(n)}{n} ,$$

so that

$$\frac{f(M') - M'}{M'} \leq \frac{f(n) - n}{n} .$$

By use of (9.4) we find that

$$n \leq N \cdot \frac{M'}{f(M') - M'} .$$

Hence

$$A_3(N) \leq \frac{N}{M} \cdot \frac{M'}{f(M') - M'} .$$

Combining this with (9.8), (9.7) and (9.6), we conclude that the upper density of the numbers n satisfying (9.5) is at most $M'/(M(f(M')-M'))$, so that the upper density of the f-touchables m, satisfying $m \equiv M'(\text{mod } M)$, is also at most $M'/(M(f(M')-M'))$. From this we finally conclude that the lower

density of the f-untouchables m, satisfying $m \equiv M'(\mathrm{mod}\ M)$, is at least
$\frac{1}{M} - \frac{M'}{M(f(M')-M')}$. □

Of the examples of f given in chapter 3, only the functions σ and Ψ_k (k=1,2,...) satisfy (9.2), so that theorem 9.1 applies to them.

Since M = 30 is squarefree, we have f(30) = 72 > 60, so that 30 is an f-abundant number *for all* $f \in F$. Therefore, we may apply theorem 9.1 with M = 30, and M' = M, yielding

COROLLARY 9.1 *For all functions* $f \in F$ *which satisfy* (9.2), *the lower density of the* f*-untouchables* m, *which are* $\equiv 0(\mathrm{mod}\ 30)$, *is* $\geq \frac{1}{30}(1 - \frac{30}{42}) = \frac{1}{105}$.

It is not difficult to improve this lower bound when we consider special choices of f. As an example, we shall derive

COROLLARY 9.2 *The lower density of the* σ*-untouchables is* > .0324.

To prove this, we note that every even number belongs to *at most one* of the following congruence classes: 0(mod 24), 12(mod 24), 30(mod 60), 20(mod 60), 40(mod 120), 70(mod 420) and 350(mod 2100). Every class is of the form M'(mod M), where M'|M and both M' and M are even and σ-abundant. Hence theorem 9.1 applies to all these classes, so that the lower density of the even σ-untouchables is at least

$$\frac{1}{72} + \frac{1}{96} + \frac{1}{210} + \frac{1}{660} + \frac{1}{600} + \frac{1}{7770} + \frac{11}{206850} > .0324 \ .$$

Since for all $f \in F$ we have

$$f(pq) - pq = p + q + 1 \ ,$$

for primes p and q ($p \neq q$), and since almost all even numbers can be written as the sum of two prime numbers (proved by VAN DER CORPUT [9], ESTERMANN [15] and TSCHUDAKOFF [37]), it follows that the density of the *odd* f-untouchables is zero, for all $f \in F$.

Corollary 9.1 implies that for all $f \in F$, satisfying (9.2), there are infinitely many f-untouchables. Although Ψ_1 belongs to this class of functions, we shall prove now, in a more constructive way, that there are infinitely many Ψ_1-untouchables. Unfortunately, this proof does not seem to be applicable to other functions $f \in F$.

THEOREM 9.2 *The numbers* $2^n 3.R$ $(n=1,2,\ldots)$, *where* R *is fixed and* $(6,R) = 1$, *are either all* Ψ_1*-touchable or else are all* Ψ_1*-untouchable.*

Before proving this theorem, we derive

LEMMA 9.3 *Any solution* $x = x_0$ *of the equation*

$$\Psi_1(x) - x = 2^n 3.R\,, \qquad (n \in \mathbb{N} \text{ and } (6,R)=1) \tag{9.9}$$

has the form $x_0 = 2^n 3.S$, *where* $(6,S) = 1$.

PROOF. Let x_0 be a solution of (9.9) with canonical factorization $x_0 = \prod_{i=1}^{s} p_i^{e_i}$. Then we have

$$\Psi_1(x_0) - x_0 = \prod_{i=1}^{s}\left(p_i^{e_i} + p_i^{e_i-1}\right) - \prod_{i=1}^{s} p_i^{e_i} = 2^n 3.R\,.$$

Now x_0 must be even, since, if x_0 is odd, then $\Psi_1(x_0)-x_0$ is also odd. This gives, with $p_1 = 2$,

$$2^{e_1-1}3\prod_{i=2}^{s}\left(p_i^{e_i} + p_i^{e_i-1}\right) - 2^{e_1}\prod_{i=2}^{s} p_i^{e_i} = 2^n 3.R\,.$$

Hence $p_2 = 3$ and $s \geq 2$, yielding

$$2^{e_1}3^{e_2}\left[2\prod_{i=3}^{s}\left(p_i^{e_i} + p_i^{e_i-1}\right) - \prod_{i=3}^{s} p_i^{e_i}\right] = 2^n 3.R\,,$$

so that $e_1 = n$ and $e_2 = 1$. □

PROOF OF THEOREM 9.2. Let $a \in \mathbb{N}$ be fixed and let $R \in \mathbb{N}$ so that $(R,6) = 1$. Suppose $2^a 3.R$ is Ψ_1-touchable. According to lemma 9.3, any solution of the equation

$$\Psi_1(x) - x = 2^a 3.R$$

has the form $x_0 = 2^a 3.S$, for some S with $(6,S) = 1$. From the definition of Ψ_1 it follows that

$$\Psi_1(2^e x_0) - 2^e x_0 = 2^e \Psi_1(x_0) - 2^e x_0 = 2^{e+a}3.R ,$$

for all integers $e \geq -a+1$. Hence all numbers $2^n 3.R$ $(n=1,2,\ldots)$ are Ψ_1-touchable.
Now suppose $2^a 3.R$ is Ψ_1-untouchable. Then *all* numbers $2^n 3.R$ $(n=1,2,\ldots)$ must be Ψ_1-untouchable, since if any one of them is Ψ_1-touchable, it follows from the first part of this proof that they are all Ψ_1-touchable. □

According to lemma 9.3, any solution $x = x_0$ of the equation $\Psi_1(x)-x = 6R$, $(6,R) = 1$, must have the form $x_0 = 6S$, $(6,S) = 1$. Now we have

$$\Psi_1(x_0) - x_0 = 12\Psi_1(S) - 6S = 6[2\Psi_1(S) - S] \geq 6S ,$$

with equality if and only if $S = 1$. Hence it follows immediately that $30 = 6.5$ is Ψ_1-untouchable, and that, since $42 = \Psi_1(30)-30$, the number $42 = 6.7$ is Ψ_1-touchable. Application of theorem 9.2 shows that the numbers $2^n 3.5$ $(n=1,2,\ldots)$ are all Ψ_1-untouchable, whereas the numbers $2^n 3.7$ $(n=1,2,\ldots)$ are all Ψ_1-touchable.

In [1] ALANEN has given an algorithm for the computation of every solution x of the equation

(9.10) $$\sigma(x) - x = n \qquad \text{for all } n \leq N,$$

where $N \in \mathbb{N}$ is given (yielding all σ-untouchables $\leq N$). The largest value of N, to which ALANEN applied his algorithm is N = 5000. We have improved the algorithm, with respect to the required amount of memory, as follows. Let $\sigma(x)-x =: s(x)$. The situation occurs that the values of a, s(a), ap_i^e and $s(ap_i^e)$ are *known* $(a,e \in \mathbb{N}$, p_i is the i-th prime and $(a,p_i) = 1)$, whereas the value of $s(ap_i^{e+1})$ *must be computed*. In Alanen's procedure this is done by use of the relation

(9.11) $$s(ap_i^{e+1}) = s(a)s(p_i^{e+2}) + as(p_i^{e+1}) .$$

The values of $s(p_i^{e+2})$ and $s(p_i^{e+1})$ are available in an array TABLE, where

$$\text{TABLE}[i,j] = s(p_i^j) = p_i^{j-1} + p_i^{j-2} + \ldots + p_i + 1 ,$$

for $i=1,2,\ldots, \pi(N)$ and $j=2,3,\ldots, [\log_2 N]+1$. In our procedure, instead of (9.11), we use the relation

$$(9.12) \qquad s(ap_i^{e+1}) = p_i s(ap_i^e) + s(a) + a ,$$

the validity of which may be easily verified. Now we only need to store the primes p_i, for $i=1,2,\ldots, \pi(N)$, so that the required amount of memory for (9.12) is of the order of magnitude of $\pi(N)$, instead of $\pi(N)\log_2 N$ required for (9.11).

With this improvement, we have applied Alanen's algorithm (to $f = \sigma$) with $N = 20000$. With some appropriate modifications, the algorithm could also be adapted for the computation of all solutions of $f(x)-x = n$, for all $n \leq N$, for *other* $f \in F$. In particular, we have applied the modified algorithm with $N = 20000$ to $f = \Psi_1$, Ψ_2, M_1, L_1 and R_0 ($= \sigma^*$).
Results of these computations are collected in tables 9.1, 9.2, 9.3 and 9.4.

Table 9.1 displays (for the functions f above) the number of even and the number of odd f-untouchables ≤ 20000; the number of $n \in \mathbb{N}$ for which $f(n)-n$ is *even* and $f(n)-n \leq 20000$ ($= A_e = A_e(20000)$); the number of $n \in \mathbb{N}$ for which $f(n)-n$ is *odd* and $1 < f(n)-n < 20000$ ($= A_o = A_o(20000)$. Note that, for all $f \in F$, $f(n)-n = 1$, if n is a prime); and, finally, the value of the function

$$10000\left(1 - \frac{1}{10000}\right)^{A_e} .$$

TABLE 9.1

f	number of f-untouchables ≤ 20000		A_e	A_o	$10000(1-\frac{1}{10000})^{A_e}$
	even	odd			
σ	2565	1(5) *)	13434	1454747	2610
Ψ_1	2896	0	13854	1457942	2502
Ψ_2	2360	2(5,7)	13948	1454702	2479
M_1	2485	1(5)	13891	1454829	2493
L_1	2181	1(7)	14468	1454994	2353
R_0	157	3(3,5,7)	47083	1544668	90

*) The *odd* f-untouchables are given in parentheses.

The last column of table 9.1 appears to be a reasonable approximation to the number of even f-untouchables. This may be explained heuristically as follows. When N_1 balls are randomly distributed among N_2 (initially void) boxes, it can be shown, that the *expected* number of void boxes is given by

$$N_2(1 - \frac{1}{N_2})^{N_1} .$$

Hence, on the assumption that the even values of $f(n)-n \le N$ are randomly distributed among the numbers $2,4,6,\dots,N$ (assume N is even), we may expect the function

$$\frac{N}{2}(1 - \frac{2}{N})^{A_e(N)} , \tag{9.13}$$

where $A_e(N)$ is the number of n for which $f(n)-n$ is even and $f(n)-n \le N$, to be a reasonable approximation to the number of even f-untouchables $\le N$. Unfortunately, the value of $A_e(N)$ can not be given *a priori* (the value of $A_e(20000)$ in table 9.1 is a by-product of Alanen's modified algorithm).

However, we can give an *asymptotic upper bound* for $A_e(N)$, for any *given* $f \in F$. As an illustration, we will carry this out for $f = \sigma$. We recall that $A_e(N)$ is the number of $n \in \mathbb{N}$, for which $\sigma(n)-n$ is even and $\sigma(n)-n \le N$. As in the proof of theorem 9.1, it is readily seen that the *even* numbers $n \in \mathbb{N}$, which contribute to $A_e(N)$, are $\le 2N$, and that the number of *odd* numbers $n \in \mathbb{N}$ which contribute to $A_e(N)$ is $o(N)$, for $N \to \infty$. Hence, we have

$$A_e(N) \le N + o(N) .$$

Furthermore, it is known (see for instance [40], pp.197-8, exercise 49.7) that the density of the *even* σ-abundant numbers is greater than 0.229, so that asymptotically, for at least $0.229N + o(N)$ of the even numbers n between N and 2N, we have

$$\sigma(n) - n > n > N .$$

Hence,

$$A_e(N) \le N - 0.229N + o(N) = 0.771N + o(N) .$$

From (9.13) we conclude that (under the assumption of the random distribution of the even values of $\sigma(n)-n$ among the numbers $2,4,6,\dots,N$) the number of even σ-untouchables $\le N$ is, asymptotically, greater than

$$\frac{N}{2}\left(1-\frac{2}{N}\right)^{0.771N+o(N)} \approx 0.1069N(1+o(1)) \;.$$

Let $d_f(n)$ be the number of solutions x of the equation $f(x)-x = n$. In table 9.2 we give the values of $n \le 20000$ for which d_f is maximal, and the corresponding maximum. We also list the least number k_0 for which there is no odd number $n \le 20000$, satisfying $d_f(n) = k_0$.

TABLE 9.2

f	n (even)	$d_f(n)$	n (odd)	$d_f(n)$	k_0
σ	11194	10	18481	576	406
	17914	10			
Ψ_1	16384	9	18481	573	393
	17594	9			
	17914	9			
Ψ_2	11194	9	18481	576	374
	17594	9			
	17914	9			
M_1	11194	11	18481	576	387
	17914	11			
L_1	11194	9	18481	576	374
	17594	9			
	17914	9			
R_0	14848	26	18481	588	412

Table 9.3 presents the number of even $n \le 20000$, for which $d_f(n) = k$, for $k=0,1,2,\dots$.

TABLE 9.3

Number of even n ≤ 20000, for which $d_f(n) = k$, k=0,1,2,...

k	$f = \sigma$	$f = \Psi_1$	$f = \Psi_2$	$f = M_1$	$f = L_1$	$f=R_0=\sigma^*$
0	2565	2896	2360	2485	2181	157
1	3655	3299	3662	3598	3627	703
2	2370	2053	2407	2400	2584	1342
3	924	1054	1085	971	1081	1621
4	308	405	329	327	333	1639
5	102	167	90	132	120	1379
6	33	71	35	38	40	1042
7	27	37	18	27	17	673
8	8	15	11	10	14	496
9	6	3	3	7	3	325
10	2			3		200
11				2		145
12						82
13						58
14						43
15						27
16						26
17						20
18						12
19						2
20						2
21						3
22						0
23						1
24						1
25						0
26						1

In table 9.4 all σ^*-untouchables $\leq$ 20000 are given, including their canonical factorizations. These numbers are connected with a conjecture of DE POLIGNAC [28] which states that any odd number > 1 is of the form 2^k+p, where $k \in \mathbb{N}$, and p is either a prime or the number 1. Since, if p is odd, $\sigma^*(2^kp)-2^kp = (2^k+1)(p+1)-2^kp = 2^k+p+1$, the truth of this conjecture would imply that all even numbers > 2 are σ^*-touchable (except perhaps those even numbers which are of the form 2^k+2). However, ERDŐS [12] and VAN DER CORPUT [8] proved that the density of the odd numbers for which DE POLIGNAC's conjecture is false, is *positive*.

TABLE 9.4

The σ^*-untouchables $\leq$ 20000

2(2)	6002(2.3001)	10254(2.3.1709)	15060(2(2)3.5.251)
3(3)	6174(2.3(2)7(3))	10358(2.5179)	15162(2.3.7.19(2))
4(2(2))	6270(2.3.5.11.19)	10620(2(2)3(2)5.59)	15300(2(2)3(2)5(2)17)
5(5)	6404(2(2)1601)	10754(2.19.283)	15350(2.5(2)307)
7(7)	6450(2.3.5(2)43)	10778(2.17.317)	15374(2.7687)
374(2.11.17)	6510(2.3.5.7.31)	10782(2.3(2)599)	15402(2.3.17.151)
702(2.3(3)13)	6758(2.31.109)	11082(2.3.1847)	15958(2.79.101)
758(2.379)	6822(2.3(2)379)	11172(2(2)3.7(2)19)	15998(2.19.421)
998(2.499)	6870(2.3.5.229)	11438(2.7.19.43)	16014(2.3.17.157)
1542(2.3.257)	6884(2(2)1721)	11542(2.29.199)	16118(2.8059)
1598(2.17.47)	7110(2.3(2)5.79)	11772(2(2)3(3)109)	16508(2(2)4127)
1778(2.7.127)	7178(2.37.97)	11790(2.3(2)5.131)	16630(2.5.1663)
1808(2(4)113)	7332(2(2)3.13.47)	11802(2.3.7.281)	16754(2.8377)
1830(2.3.5.61)	7406(2.7.23(2))	11910(2.3.5.397)	16770(2.3.5.13.43)
1974(2.3.7.47)	7518(2.3.7.179)	12234(2.3.2039)	16788(2(2)3.1399)
2378(2.29.41)	7842(2.3.1307)	12252(2(2)3.1021)	17040(2(4)3.5.71)
2430(2.3(5)5)	7902(2.3(2)439)	12372(2(2)3.1031)	17078(2.8539)
2910(2.3.5.97)	8258(2.4129)	12596(2(2)47.67)	17340(2(2)3.5.17(2))
3164(2(2)7.113)	8400(2(4)3.5(2)7)	12806(2.19.337)	17438(2.8719)
3182(2.37.43)	8622(2.3(2)479)	12878(2.47.137)	17468(2(2)11.397)
3188(2(2)797)	8670(2.3.5.17(2))	13092(2(2)3.1091)	17490(2.3.5.11.53)
3216(2(4)3.67)	8790(2.3.5.293)	13298(2.61.109)	17558(2.8779)
3506(2.1753)	8850(2.3.5(2)59)	13352(2(3)1669)	17580(2(2)3.5.293)
3540(2(2)3.5.59)	8862(2.3.7.211)	13410(2.3(2)5.149)	17652(2(2)3.1471)
3666(2.3.13.47)	8916(2(2)3.743)	13800(2(3)3.5(2)23)	17862(2.3.13.229)
3698(2.43(2))	8930(2.5.19.47)	13902(2.3.7.331)	17958(2.3.41.73)
3818(2.23.83)	8982(2.3(2)499)	13962(2.3.13.179)	18210(2.3.5.607)
3846(2.3.641)	9116(2(2)43.53)	14022(2.3(2)19.41)	18566(2.9283)
3986(2.1993)	9518(2.4759)	14048(2(5)439)	18608(2(4)1163)
4196(2(2)1049)	9522(2.3(2)23(2))	14052(2(2)3.1171)	18612(2(2)3(2)11.47)
4230(2.3(2)5.47)	9558(2.3(4)59)	14078(2.7039)	18686(2.9343)
4574(2.2287)	9570(2.3.5.11.29)	14108(2(2)3527)	18846(2.3(3)349)
4718(2.7.337)	9582(2.3.1597)	14142(2.3.2357)	18870(2.3.5.17.37)
4782(2.3.797)	9642(2.3.1607)	14250(2.3.5(3)19)	19058(2.13.733)
5126(2.11.233)	9930(2.3.5.331)	14382(2.3(2)17.47)	19260(2(2)3(2)5.107)
5324(2(2)11(3))	10002(2.3.1667)	14532(2(2)3.7.173)	19358(2.9679)
5610(2.3.5.11.17)	10022(2.5011)	14606(2.67.109)	19362(2.3.7.461)
5738(2.19.151)	10062(2.3(2)13.43)	14612(2(2)13.281)	19632(2(4)3.409)
5918(2.11.269)	10200(2(3)3.5(2)17)	14682(2.3.2447)	19650(2.3.5(2)131)
5952(2(6)3.31)	10238(2.5119)	15038(2.73.103)	19710(2.3(3)5.73)

The even numbers > 2 in table 9.4 cannot be of the form 2^k+p+1 (for some odd prime p and $k \in \mathbb{N}$), and, by inspection, we find that 4 is the only number in this table of the form 2^k+2, so that, if we subtract 1 from all even numbers > 4 in this table, we have a set of numbers, for which DE POLIGNAC's conjecture is false. For the sake of completeness, we give in table 9.5 the remaining exceptions ≤ 20000.

If B(N) is the number of pairs (k,p) for which $2^k+p \le N$ (where $k \in \mathbb{N}$ and p is 1 or an odd prime), then we have

$$B(N) = \sum_{k=1}^{[\log_2 N]} \pi(N - 2^k) .$$

By the same argument used in estimating the number of even f-untouchables, we conclude, under the assumption of the random distribution of the numbers 2^k+p among the odd numbers, that the expected number of exceptions ≤ N to the conjecture of DE POLIGNAC is

$$\frac{N}{2}\left(1 - \frac{2}{N}\right)^{B(N)} .$$

Since B(20000) = 28232, our approximation gives $10000\left(1 - \frac{1}{10000}\right)^{28232} =$ = 594.2, whereas the actual number of exceptions ≤ 20000 is 590.

By using the estimate $B(N) < \pi(N)\log_2 N$, we find for large N that the expected number of exceptions ≤ N is

$$> \frac{N}{2}\left(1 - \frac{2}{N}\right)^{\pi(N)\log_2 N} \approx .0279N(1 + o(1)) .$$

TABLE 9.5

The remaining exceptions ≤ 20000 to the conjecture of DE POLIGNAC

127	2579	4855	7379	9371	11285	13285	15071	16865	18637
149	2669	4889	7387	9391	11317	13393	15101	16867	18719
251	2683	5077	7389	9431	11335	13451	15113	16973	18787
331	2789	5099	7393	9457	11347	13469	15119	17021	18817
337	2843	5143	7417	9473	11411	13589	15121	17047	18881
509	2879	5303	7431	9613	11435	13603	15127	17083	18889
599	2983	5405	7535	9787	11533	13619	15149	17089	18895
809	2993	5467	7547	9809	11549	13679	15187	17113	18897
877	2999	5557	7583	9907	11579	13735	15217	17137	18899
905	3029	5617	7603	9941	11593	13841	15223	17147	18911
907	3119	5729	7747	9959	11627	13859	15247	17229	18959
959	3149	5731	7753	10007	11695	13897	15359	17257	18971
977	3239	5755	7783	10027	11729	13973	15419	17269	19007
1019	3299	5761	7799	10079	11743	14009	15521	17305	19093
1087	3341	5771	7807	10121	11857	14023	15551	17327	19117
1199	3343	5923	7811	10235	11921	14039	15607	17369	19135
1207	3353	6021	7813	10327	11993	14081	15641	17371	19139
1211	3431	6065	7867	10379	12007	14101	15701	17411	19163
1243	3433	6073	7913	10391	12131	14143	15719	17429	19177
1259	3637	6119	7961	10409	12191	14227	15779	17519	19273
1271	3643	6161	8023	10447	12203	14231	15787	17593	19319
1477	3739	6193	8031	10451	12223	14279	15809	17669	19345
1529	3779	6247	8087	10483	12239	14303	15853	17735	19379
1549	3877	6283	8107	10511	12373	14347	15869	17759	19483
1589	3967	6433	8111	10513	12401	14375	15943	17767	19583
1619	4001	6463	8141	10553	12427	14383	16025	17773	19807
1649	4013	6521	8159	10607	12431	14407	16027	17827	19819
1657	4063	6535	8287	10697	12479	14437	16031	17849	19889
1719	4151	6539	8363	10873	12517	14459	16109	17887	19949
1759	4153	6547	8387	10949	12671	14467	16165	17909	19961
1783	4271	6637	8411	10963	12727	14473	16177	17921	
1859	4311	6659	8429	11015	12731	14489	16181	17977	
1867	4327	6673	8467	11023	12733	14533	16213	18033	
1927	4503	6731	8527	11039	12749	14585	16361	18089	
1969	4543	6791	8563	11069	12791	14639	16405	18103	
1985	4567	6853	8587	11083	12881	14765	16409	18155	
2171	4589	6941	8719	11105	12929	14809	16499	18209	
2203	4633	7151	8831	11137	12941	14879	16543	18307	
2213	4649	7169	8873	11141	13001	14917	16559	18359	
2231	4663	7199	8887	11207	13083	14921	16601	18391	
2263	4691	7267	8921	11219	13093	14975	16645	18427	
2279	4811	7289	8923	11227	13099	14981	16727	18487	
2293	4813	7297	9101	11231	13147	15013	16739	18517	
2465	4841	7319	9239	11239	13169	15041	16783	18551	
2503	4843	7343	9307	11279	13217	15043	16849	18613	

REFERENCES

[1] J. ALANEN, *Empirical study of aliquot series*, MR 133/72, Mathematisch Centrum, Amsterdam, July 1972. {61,66} *)

[2] P.T. BATEMAN, *The distribution of values of the Euler function*, Acta Arith., 21 (1972) 329-345. {20}

[3] W. BORHO, *Eine Schranke für befreundete Zahlen mit gegebener Teileranzahl*, Math.Nachr., 63 (1974) 297-301. {10,11,12}

[4] W. BORHO, *Über die Fixpunkte der k-fach iterierten Teilersummenfunktion*, Mitt.Math.Gesells. Hamburg, 9 (1969) 34-48. {8}

[5] A.L. BROWN, *Multiperfect numbers*, Scripta Math., 20 (1954) 103-106.{42}

[6] A.L. BROWN, *Multiperfect numbers, - Cousins of the perfect numbers - No. 1*, Recr.Math.Mag., 14 (1964) 31-39. {42}

[7] E. CATALAN, *Propositions et questions diverses*, Bull.Soc.Math.France, 16 (1887-8) 128-129. {2}

[8] J.G. VAN DER CORPUT, *On the conjecture of de Polignac*, Simon Stevin, 27 (1950) 99-105 (Dutch). {71}

[9] J.G. VAN DER CORPUT, *Sur l'hypothèse de Goldbach pour presque tous les nombres pairs*, Acta Arith., 2 (1937) 266-290. {64}

[10] L.E. DICKSON, *Theorems and tables on the sum of the divisors of a number*, Quart.J.Math., 44 (1913) 264-296. {2}

[11] R.E. DRESSLER, *An elementary proof of a theorem of Erdös on the sum of divisors function*, J.Number Theory, 4 (1972) 532-536. {20}

[12] P. ERDÖS, *On integers of the form 2^k+p and some related problems*, Summa Brasiliensis Math., 2 (1950) 113-123. {71}

[13] P. ERDÖS, *Some remarks on Euler's ϕ function and some related problems*, Bull.Amer.Math.Soc., 51 (1945) 540-544. {20}

[14] P. ERDÖS, *Über die Zahlen der Form $\sigma(n)-n$ und $n-\phi(n)$*, Elem.der Math., 28 (1973) 83-86. {61,62}

[15] TH. ESTERMANN, *On Goldbach's problem: Proof that almost all even positive integers are sums of two primes*, Proc.London Math.Soc., 44 (1938) 307-314. {64}

*)Numbers in curly brackets refer to the page(s) in this thesis where the reference occurs.

[16] B. FRANQUI, M. GARCIA, *Some new multiply perfect numbers*, Amer.Math. Monthly, 60 (1953) 459-462. {42}

[17] B. FRANQUI, M. GARCÍA, *57 new multiply perfect numbers*, Scripta Math., 20 (1954) 169-171. {42}

[18] R.K. GUY, J.L. SELFRIDGE (editors), *Combined report on aliquot sequences*, Proc. 4th annual Manitoba Conf.Numer.Math.,Winnipeg,1974.{3,60}

[19] P. HAGIS, JR., *A lower bound for the set of odd perfect numbers*, Math. Comp., 27 (1973) 951-953. {49}

[20] P. HAGIS, JR., *Every odd perfect number has at least eight prime factors*, Notices Amer.Math.Soc., 22 (1975) A-60. {49}

[21] P. HAGIS, JR., *Unitary amicable numbers*, Math.Comp.,25(1971)915-918.{54}

[22] G.H. HARDY, E.M. WRIGHT, *An introduction to the The Theory of Numbers*, 4th ed., Oxford Univ.Press, New York, 1960. {27}

[23] K. KNOPP, *Theory and application of infinite series* (transl. from the 2nd German ed. by R.C. Young), London, etc., Blackie, 1928. {26}

[24] M. LAL, G. TILLER, T. SUMMERS, *Unitary sociable numbers*, Proc. 2nd annual Manitoba Conf.Numer.Math., Winnipeg, 1972, 211-216. {9,54,60}

[25] E. LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, Reprinted by Chelsea, 1953. {28}

[26] E.J. LEE, J.S. MADACHY, *The history and discovery of amicable numbers*, J.Recr.Math., 5 (1972), Part 1: 77-93, Part 2: 153-173, Part 3: 231-249. {5,54}

[27] J. PERROTT, *Sur une proposition empirique enoncée au Bulletin*, Bull. Soc.Math.France, 17 (1888-9) 155-156. {2}

[28] A. DE POLIGNAC, *Six propositions arithmologiques déduites du crible d'Eratosthène*, Nouv.Ann.Math., 8 (1849) 130-133. {71}

[29] P. POULET, *La chasse aux nombres*, Fascicule I, Bruxelles, 1929. {42}

[30] H.J.J. TE RIELE, *A note on the Catalan-Dickson conjecture*, Math.Comp., 27 (1973) 189-192. {3}

[31] H.J.J. TE RIELE, *Four large amicable pairs*, Math.Comp., 28 (1974) 309-312. {54}

[32] H.J.J. TE RIELE, *Further results on unitary aliquot sequences*, NW2/73, Mathematisch Centrum, Amsterdam, March 1973. {5,54}

[33] H.J.J. TE RIELE, *Unitary aliquot sequences*, MR 139/72, Mathematisch Centrum, Amsterdam, September 1972. {7,60}

[34] H. RIESEL, *Lucasian criteria for the primality of* $N = h.2^n-1$, Math. Comp., 23 (1969) 869-875. {52}

[35] R. STEUERWALD, *Verschärfung einen notwendigen Bedingung für die Existenz einen ungeraden vollkommenen Zahl*, S.-B.Math.-Nat.Abt.Bayer. Akad.Wiss., (1937) 68-72. {50}

[36] M.V. SUBBARAO, L.J. WARREN, *Unitary perfect numbers*, Canad.Math.Bull., 9 (1966) 147-153. {49}

[37] N.G. TSCHUDAKOFF, *Über die Dichte der Menge der geraden Zahlen, welche nicht als Summe zweier ungerader Primzahlen darstellbar sind*, Izv. Akad.Nauk. SSSR, Ser.Mat., (1938) 25-40 (Russian). {64}

[38] B. TUCKERMAN, *The 24th Mersenne Prime*, Proc.Nat.Acad.Sci. USA, 68 (1971) 2319-2320. {49}

[39] C.R. WALL, *A new unitary perfect number*, Notices Amer.Math.Soc., 16 (1969) 825. {49}

[40] C.R. WALL, *Selected topics in elementary number theory*, Univ.of South Carolina Press, Columbia, S.C., 1974. {68}

[41] C.R. WALL, *Topics related to the sum of unitary divisors of an integer*, Ph.D.Thesis, Univ. of Tennessee, March 1970. {14,49,54}

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* MCT 83 L.S. VAN BENTEM JUTTING, *Checking Landau's "Grundlagen" in the automath system*, 1977. ISBN 90 6196 147 5.

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An asterik before the number means "to appear".