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SIMPLE-PERIODIC AND NON-PERIODIC LAMÉ FUNCTIONS

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PREFACE

This monograph consists of a slightly revised version of my doctoral dissertation and an additional chapter 7. The original work was written under supervision of Prof.dr. G.W. Veltkamp and Prof.dr. C.J. Bouwkamp at the Eindhoven University of Technology.
I want to acknowledge many comments from Prof.dr. J. Boersma.
I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts and all those at the Mathematical Centre who have contributed to its technical realization.

CONTENTS

CHAPTER 0

INTRODUCTION AND SUMMARY

0.0. *Introductory remarks*

The present work is a result of antenna research carried out in the preceding years by the Numerical Mathematics and Service group of the Department of Mathematics and the Theoretical Electrical Engineering group of the Department of Electrical Engineering at the Technological University Eindhoven. The first research object concerned the investigation of a corrugated conical-horn antenna with circular cross-section and large flare angle, the so-called "scalar feed" [1], [2], [3].

This was followed by a contract-research program of the European Space Research and Technology Centre (ESTEC) for the investigation of the propagation and radiation properties of an elliptical waveguide with anisotropic boundary conditions [4], [5]. In the ESTEC report [4] we suggested some topics for further research work, and one of these is the investigation of a corrugated conical-horn feed with elliptical cross-section and large flare angle. This feed illuminates a parabolic satellite reflector which has an elliptical aperture and which is used for telecommunication purposes, e.g., for Western Europe (see figure 0.1) or for time zones in the U.S.A. This problem, however, has so far appeared to be too difficult.

We investigate an easier problem, namely, the electromagnetic field inside a conical horn with an elliptical cross-section and an arbitrary flare angle, bounded by a perfectly conducting rather than an anisotropic surface. The mathematical results of this work and the expertise gained by it will be used as tools for further investigations of horns with anisotropic boundary conditions.

0.1. *Summary*

The first problem in the investigation of the electromagnetic field inside a conical-horn feed with elliptical cross-section is to select a suitable coordinate system, with the following properties:

(1) the boundary of the cone must be a coordinate surface;

(2) the scalar Helmholtz equation must be separable;

(3) the parametric representation of the coordinate system must be chosen so that the solutions of the separated equations are easy to find.

A coordinate system that satisfies these conditions is the sixth coordinate system of Eisenhart, viz., the sphero-conal system parametrically represented by trigonometric functions as described for the first time by Kraus [6] in 1955. Separating the Helmholtz equation, we obtain three equations:

(1) for the r dependence: the differential equation of the spherical Bessel functions;

(2) for the φ dependence: the Lamé differential equation with periodic boundary conditions;

(3) for the θ dependence: the Lamé differential equation with non-periodic boundary conditions.

Up to now there is virtually nothing known about the analytical solutions of the Lamé differential equation with non-periodic boundary conditions. In this work, however, we show that they are connected with the periodic solutions of the Lamé equation.
We observe the same phenomenon in the case of the solutions of the Mathieu equation by separating the Helmholtz equation in the elliptic-cylinder coordinates. Between the solutions of the separated equations of the scalar Helmholtz equation we have now found a relationship in four systems, viz., the cylindrical polar, the spherical polar, the elliptic-cylinder and the sphero-conal coordinate systems. Figure 0.2 displays an overview of these solutions, and it is easy to see that these solutions transform into one another by the corresponding transition of the coordinate systems. As in the spherical polar coordinate system, the electromagnetic field inside a horn can be expressed in terms of two independent scalar Debye potentials. And in the same way as described in the spherical polar coordinate system we give a mode classification of the electromagnetic field.

0.2. *Computational remarks*

We have developed a set of procedures in ALGOL 60 for calculating the periodic and non-periodic solutions of the Lamé equations. These procedures, and directions for use, are obtainable from the author on request.
We have calculated the first forty modes of the electromagnetic field inside a horn with an eccentricity of 0.9 and a flare angle of 60°.
The results of the calculated periodic solutions were compared with the numerical results of the finite-difference method with h^2 extrapolation applied to the Lamé differential equation with periodic boundary conditions. The

calculations agreed to 10 decimals. The results of the computed non-periodic solutions were compared with those of a fifth-order Runge-Kutta method. These calculations agreed to 11 decimals. Both of the calculations mentioned above were performed in double-length arithmetic to guarantee high accuracy. All calculations were performed on the digital Burroughs computer B6700 of the Computer Centre of the Technological University Eindhoven.

0.3. *Appendix*

In this section a new representation of the elliptic coordinates is introduced, and this contains the polar coordinates as a special case by taking the focal distance (2h) zero. At the same time, the equations obtained from the Helmholtz equation on separation tend to the corresponding equations of the polar coordinate system.

The coordinates of the elliptic system denoted by r,φ are related to the Cartesian coordinates x,y by means of

$$x = \sqrt{h^2 + r^2}\cos(\varphi), \quad y = r\sin(\varphi)$$

with $h > 0$ and $0 \le \varphi < 2\pi$, $r \ge 0$.

First of all we observe that for $h = 0$ the polar coordinate system is obtained. The coordinate curves are determined by the following two equations:

$$\frac{x^2}{h^2 + r^2} + \frac{y^2}{r^2} = 1, \quad \frac{x^2}{h^2\cos^2(\varphi)} - \frac{y^2}{h^2\sin^2(\varphi)} = 1 .$$

These equations represent an ellipse and a hyperbola, respectively, with foci $(h,0)$ and $(-h,0)$.

The eccentricity of the ellipse is given by

$$e := \frac{h}{\sqrt{h^2 + r^2}} .$$

If $h = 0$, and consequently $e = 0$, the ellipse becomes a circle and the equation of the hyperbola degenerates into

$$\left(\frac{x}{\cos(\varphi)} - \frac{y}{\sin(\varphi)}\right)\left(\frac{x}{\cos(\varphi)} + \frac{y}{\sin(\varphi)}\right) = 0 ,$$

and this is the equation of a pair of straight lines.

Now we verify whether the coordinate curves are mutually perpendicular at each point in the plane. For that purpose we determine the tangent vectors to the parameter curves $\frac{\partial \underline{x}}{\partial r}$ and $\frac{\partial \underline{x}}{\partial \varphi}$:

$$\frac{\partial \underline{x}}{\partial r} = \begin{pmatrix} (r/\sqrt{h^2 + r^2})\cos(\varphi) \\ \sin(\varphi) \end{pmatrix}, \quad \frac{\partial \underline{x}}{\partial \varphi} = \begin{pmatrix} -(\sqrt{h^2 + r^2})\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix} .$$

It follows indeed that $(\frac{\partial \underline{x}}{\partial r}, \frac{\partial \underline{x}}{\partial \varphi}) = 0$.
The scale factors of this coordinate system are

$$h_r := |\frac{\partial \underline{x}}{\partial r}| = \sqrt{\frac{r^2 + h^2\sin^2(\varphi)}{h^2 + r^2}}\ , \quad h_\varphi := |\frac{\partial \underline{x}}{\partial \varphi}| = \sqrt{r^2 + h^2\sin^2(\varphi)}\ .$$

Again, if $h = 0$ these scale factors are identical with those of the polar coordinate system.

Now we shall investigate the separation of the Helmholtz equation.
We shall suppose that the function $u = u(r,\varphi)$ satisfying the Helmholtz equation

$$\Delta u + k^{*2}u = 0$$

can be factored as

$$u(r,\varphi) = R(r)\,\Phi(\varphi)\ .$$

Then we obtain the following two second-order differential equations:

$$\sqrt{h^2 + r^2}\,\frac{d}{dr}(\sqrt{h^2 + r^2}\,\frac{dR}{dr}) + (k^{*2}r^2 - \nu^2)R = 0\ ,$$

$$\frac{d^2\Phi}{d\varphi^2} + (k^{*2}h^2\sin^2(\varphi) + \nu^2)\Phi = 0,\ \Phi(\varphi) = \Phi(\varphi + 2\pi)\ ,$$

in which ν^2 is the separation constant. Again, if $h = 0$ we obtain the well-known differential equations of the polar coordinate system.
We can divide the φ solutions into four classes and we can expand these functions into trigonometric Fourier series [7;21], [8;187]:

$$ce_{2n}(\varphi;k^*h) = \sum_{\ell=0}^{\infty} A_{2\ell}^{(2n)}\cos(2\ell\varphi)\ ,$$

$$ce_{2n+1}(\varphi;k^*h) = \sum_{\ell=0}^{\infty} A_{2\ell+1}^{(2n+1)}\cos((2\ell + 1)\varphi)\ ,$$

$$se_{2n+1}(\varphi;k^*h) = \sum_{\ell=0}^{\infty} B_{2\ell+1}^{(2n+1)} \sin((2\ell+1)\varphi) ,$$

$$se_{2n+2}(\varphi;k^*h) = \sum_{\ell=0}^{\infty} B_{2\ell+2}^{(2n+2)} \sin((2\ell+2)\varphi) .$$

The corresponding r solutions are [7;158]:

$$Ce_{2n}(r;k^*h) = \frac{ce_{2n}(0;k^*h)}{A_0^{(2n)}} \sum_{\ell=0}^{\infty} A_{2\ell}^{(2n)} J_{2\ell}(k^*r) ,$$

$$Ce_{2n+1}(r;k^*h) = \frac{ce_{2n+1}(0;k^*h)\sqrt{h^2+r^2}}{\frac{1}{2}k^*hrA_1^{(2n+1)}} \sum_{\ell=0}^{\infty} (2\ell+1)A_{2\ell+1}^{(2n+1)} J_{2\ell+1}(k^*r) ,$$

$$Se_{2n+1}(r;k^*h) = \frac{se'_{2n+1}(0;k^*h)}{\frac{1}{2}k^*hB_1^{(2n+1)}} \sum_{\ell=0}^{\infty} B_{2\ell+1}^{(2n+1)} J_{2\ell+1}(k^*r) ,$$

$$Se_{2n+2}(r;k^*h) = \frac{se'_{2n+2}(0;k^*h)\sqrt{h^2+r^2}}{\frac{1}{4}k^{*2}h^2rB_2^{(2n+2)}} \sum_{\ell=0}^{\infty} (2\ell+2)B_{2\ell+2}^{(2n+2)} J_{2\ell+2}(k^*r) .$$

0.4. *References*

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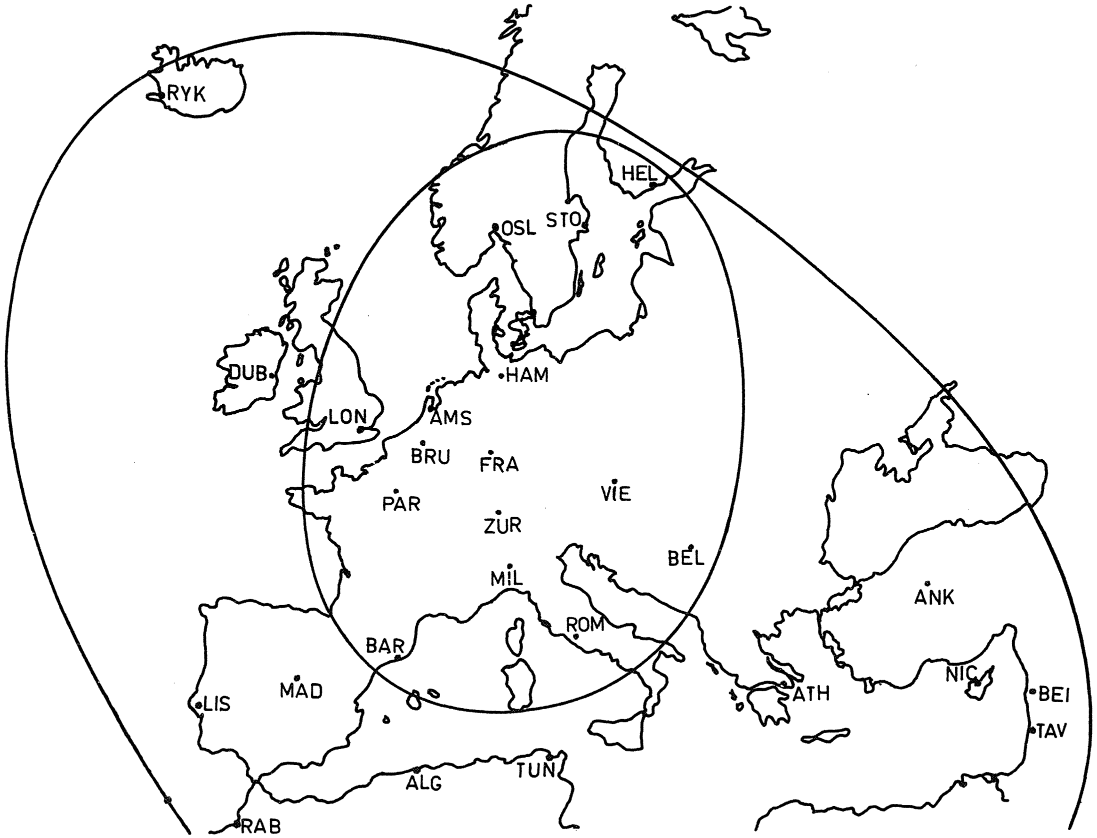
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Figure 0.1.

Solutions of the separated Helmholtz equation $\Delta u + k^{*2}u = 0$.

Elliptic-cylinder coordinate system (r,φ,z)

$$e^{-\gamma_n z}\begin{cases} [\sum_{\ell=0}^{\infty} A_{2\ell}^{(2m)} J_{2\ell}(k_{c_n} r)][\sum_{\ell=0}^{\infty} A_{2\ell}^{(2m)} \cos(2\ell\varphi)], \quad m = 0,1,2,\ldots \\ [g(r)\sum_{\ell=0}^{\infty} (2\ell+1) A_{2\ell+1}^{(2m+1)} J_{2\ell+1}(k_{c_n} r)][\sum_{\ell=0}^{\infty} A_{2\ell+1}^{(2m+1)} \cos((2\ell+1)\varphi)], \quad m = 0,1,2,\ldots \\ [g(r)\sum_{\ell=1}^{\infty} 2\ell B_{2\ell}^{(2m)} J_{2\ell}(k_{c_n} r)][\sum_{\ell=1}^{\infty} B_{2\ell}^{(2m)} \sin(2\ell\varphi)], \quad m = 1,2,\ldots \\ [\sum_{\ell=0}^{\infty} B_{2\ell+1}^{(2m+1)} J_{2\ell+1}(k_{c_n} r)][\sum_{\ell=0}^{\infty} B_{2\ell+1}^{(2m+1)} \sin((2\ell+1)\varphi)], \quad m = 0,1,2,\ldots \end{cases}$$

where $g(r) = \sqrt{h^2 + r^2}/r$.

$$\Big\downarrow \; h \downarrow 0$$

Cylindrical polar coordinate system (r,φ,z)

$$e^{-\gamma_n z}\begin{cases} J_{2m}(k_{c_n} r)\cos(2m\varphi), \quad m = 0,1,2,\ldots \\ J_{2m+1}(k_{c_n} r)\cos((2m+1)\varphi), \quad m = 0,1,2,\ldots \\ J_{2m}(k_{c_n} r)\sin(2m\varphi), \quad m = 1,2,\ldots \\ J_{2m+1}(k_{c_n} r)\sin((2m+1)\varphi), \quad m = 0,1,2,\ldots \end{cases}$$

$$k_{c_n}^2 - k^{*2} = \gamma_n^2$$

Figure 0.2. (continued on page 9)

Solutions of the separated Helmholtz equation $\Delta u + k^{*2}u = 0$.

Sphero-conal coordinate system (r,θ,φ)

$$h^{(1,2)}_{\nu_n}(k^*r)\begin{cases}[\sum_{\ell=0}^{\infty} T(2\ell)A^{(2m)}_{2\ell}P^{2\ell}_{\nu_n}(\cos(\theta))][\sum_{\ell=0}^{\infty} A^{(2m)}_{2\ell}\cos(2\ell\varphi)], & m=0,1,2,\ldots\\ [\sum_{\ell=0}^{\infty} T(2\ell+1)A^{(2m+1)}_{2\ell+1}P^{2\ell+1}_{\nu_n}(\cos(\theta))][\sum_{\ell=0}^{\infty} A^{(2m+1)}_{2\ell+1}\cos((2\ell+1)\varphi)], & m=0,1,2,\ldots\\ [f(\theta;k)\sum_{\ell=1}^{\infty} 2\ell T(2\ell)B^{(2m)}_{2\ell}P^{2\ell}_{\nu_n}(\cos(\theta))][\sum_{\ell=1}^{\infty} B^{(2m)}_{2\ell}\sin(2\ell\varphi)], & m=1,2,\ldots\\ [f(\theta;k)\sum_{\ell=0}^{\infty}(2\ell+1)T(2\ell+1)B^{(2m+1)}_{2\ell+1}P^{2\ell+1}_{\nu_n}(\cos(\theta))]\cdot\\ \quad\cdot[\sum_{\ell=0}^{\infty} B^{(2m+1)}_{2\ell+1}\sin((2\ell+1)\varphi)], & m=0,1,\ldots\end{cases}$$

where $f(\theta;k) = \dfrac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)}$

$$\Big\downarrow\ k' \downarrow 0$$

Spherical polar coordinate system (r,θ,φ)

$$h^{(1,2)}_{\nu_n}(k^*r)\begin{cases}P^{2m}_{\nu_n}(\cos(\theta))\cos(2m\varphi), & m=0,1,2,\ldots\\ P^{2m+1}_{\nu_n}(\cos(\theta))\cos((2m+1)\varphi), & m=0,1,2,\ldots\\ P^{2m}_{\nu_n}(\cos(\theta))\sin(2m\varphi), & m=1,2,\ldots\\ P^{2m+1}_{\nu_n}(\cos(\theta))\sin((2m+1)\varphi), & m=0,1,2,\ldots\end{cases}$$

CHAPTER 1

CONICAL COORDINATE SYSTEMS

1.0. *Introduction*

Of the eleven coordinate systems of Eisenhart [7;656], [8;94] in which the scalar wave equation is separable, we shall need the *sphero-conal system.* However, we shall first study conical coordinate systems in general.

1.1. *The general conical coordinate system*

This system is based on a family of concentric spheres and an orthogonal net of curves on the unit sphere. The conical coordinates, denoted by r,θ,φ, are related to the familiar Cartesian coordinates x,y,z by $\underline{x} = r\underline{f}(\theta,\varphi)$, or

$$x = rf_1(\theta,\varphi),\ y = rf_2(\theta,\varphi),\ z = rf_3(\theta,\varphi)\ ,$$

where f_i $(i = 1,2,3)$ are the Cartesian components of the unit vector $\underline{f}$ defined in a certain domain D of the θ,φ plane to be specified later on. We have

$$(\underline{f},\underline{f}) = 1,\ \text{or}\ f_1^2 + f_2^2 + f_3^2 = 1\ ,$$

and

$$(\frac{\partial\underline{f}}{\partial\theta}\ ,\ \frac{\partial\underline{f}}{\partial\varphi}) = 0,\ \text{or}\ \frac{\partial f_1}{\partial\theta}\frac{\partial f_1}{\partial\varphi} + \frac{\partial f_2}{\partial\theta}\frac{\partial f_2}{\partial\varphi} + \frac{\partial f_3}{\partial\theta}\frac{\partial f_3}{\partial\varphi} = 0\ .$$

The tangent vectors to the parameter curves at the point (r,θ,φ) are given by

$$\frac{\partial\underline{x}}{\partial r} = \underline{f},\ \frac{\partial\underline{x}}{\partial\theta} = r\,\frac{\partial\underline{f}}{\partial\theta}\ ,\ \frac{\partial\underline{x}}{\partial\varphi} = r\,\frac{\partial\underline{f}}{\partial\varphi}\ .$$

The length of these tangent vectors have the nature of scale factors and we define them as:

$$h_r := |\frac{\partial\underline{x}}{\partial r}| = 1,\ h_\theta := |\frac{\partial\underline{x}}{\partial\theta}| = r|\frac{\partial\underline{f}}{\partial\theta}|,\ h_\varphi := |\frac{\partial\underline{x}}{\partial\varphi}| = r|\frac{\partial\underline{f}}{\partial\varphi}|\ .$$

Points for which $h_\theta h_\varphi = 0$ are singular points of the parametric representation. In the vicinity of these points there is no one-to-one mapping on the Cartesian coordinates.

The set of orthogonal unit vectors in the r, θ and φ directions, which vary from point to point, are defined as

$$\underline{e}_r = \frac{\partial \underline{x}}{\partial r} = \underline{f}, \ \underline{e}_\theta = \frac{1}{h_\theta}\frac{\partial \underline{x}}{\partial \theta} = \frac{1}{h^*_\theta}\frac{\partial \underline{f}}{\partial \theta}, \ \underline{e}_\varphi = \frac{1}{h_\varphi}\frac{\partial \underline{x}}{\partial \varphi} = \frac{1}{h^*_\varphi}\frac{\partial \underline{f}}{\partial \varphi},$$

where

$$h^*_\theta := \frac{h_\theta}{r} = \left|\frac{\partial \underline{f}}{\partial \theta}\right|, \ h^*_\varphi := \frac{h_\varphi}{r} = \left|\frac{\partial \underline{f}}{\partial \varphi}\right|.$$

Thus, we can write each vector at the point (r,θ,φ) in a unique way as:

$$\underline{v} = v_r\underline{e}_r + v_\theta\underline{e}_\theta + v_\varphi\underline{e}_\varphi \ .$$

1.2. *Vector operators in the general conical coordinate system*

In this section we shall deduce some vector identities in the general orthogonal curvilinear coordinates [3;298].

Let

$$\underline{F} = \underline{F}(r,\theta,\varphi) = F_r\underline{e}_r + F_\theta\underline{e}_\theta + F_\varphi\underline{e}_\varphi \ .$$

Then

$$\begin{aligned} \operatorname{div} \underline{F} &= \operatorname{div}(F_r\underline{e}_r + (F_\theta\underline{e}_\theta + F_\varphi\underline{e}_\varphi)) \\ &= \frac{1}{h_\theta h_\varphi}\frac{\partial}{\partial r}(h_\theta h_\varphi F_r) + \frac{1}{h_\theta h_\varphi}\left\{\frac{\partial}{\partial \theta}(h_\varphi F_\theta) + \frac{\partial}{\partial \varphi}(h_\theta F_\varphi)\right\} . \end{aligned}$$

For convenience we shall define

$$\operatorname{div} \underline{F} := \operatorname{div}_r \underline{F} + \frac{1}{r}\operatorname{div}_t \underline{F}$$

and

$$\operatorname{div}_r \underline{F} := \operatorname{div} F_r\underline{e}_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) = \frac{\partial F_r}{\partial r} + \frac{2F_r}{r},$$

$$\begin{aligned} \operatorname{div}_t \underline{F} &:= r \operatorname{div}(F_\theta\underline{e}_\theta + F_\varphi\underline{e}_\varphi) \\ &= \frac{1}{h^*_\theta h^*_\varphi}\left\{\frac{\partial}{\partial \theta}(h^*_\varphi F_\theta) + \frac{\partial}{\partial \varphi}(h^*_\theta F_\varphi)\right\} . \end{aligned}$$

The operators with index r are the radial operators and those with index t are the transversal ones, i.e. transversal in relation to r.

Similarly we have for the curl of a vector

$$\text{curl } \underline{F} := \text{curl}_r \underline{F} + \frac{1}{r} \text{curl}_t \underline{F}$$

with

$$\text{curl}_r \underline{F} := (\text{curl } \underline{F})_\theta \underline{e}_\theta + (\text{curl } \underline{F})_\varphi \underline{e}_\varphi$$

$$= \frac{1}{h_\varphi}\left\{\frac{\partial F_r}{\partial \varphi} - \frac{\partial}{\partial r}(h_\varphi F_\varphi)\right\}\underline{e}_\theta + \frac{1}{h_\theta}\left\{\frac{\partial}{\partial r}(h_\theta F_\theta) - \frac{\partial F_r}{\partial \theta}\right\}\underline{e}_\varphi$$

and

$$\text{curl}_t \underline{F} := \frac{r}{h_\theta h_\varphi}\left\{\frac{\partial}{\partial \theta}(h_\varphi F_\varphi) - \frac{\partial}{\partial \varphi}(h_\theta F_\theta)\right\}\underline{e}_r$$

$$= \frac{1}{h_\theta^* h_\varphi^*}\left\{\frac{\partial}{\partial \theta}(h_\varphi^* F_\varphi) - \frac{\partial}{\partial \varphi}(h_\theta^* F_\theta)\right\}\underline{e}_r \; .$$

Let now

$$g = g(r,\theta,\varphi)$$

then

$$\text{grad } g = \frac{\partial g}{\partial r}\, \underline{e}_r + \frac{1}{r}\left\{\frac{1}{h_\theta^*}\frac{\partial g}{\partial \theta}\, \underline{e}_\theta + \frac{1}{h_\varphi^*}\frac{\partial g}{\partial \varphi}\, \underline{e}_\varphi\right\} =$$

$$= \text{grad}_r g + \frac{1}{r}\, \text{grad}_t g \; .$$

In the same way as before we define

$$\text{grad}_r g := \frac{\partial g}{\partial r}\, \underline{e}_r$$

and

$$\text{grad}_t g := \frac{1}{h_\theta^*}\frac{\partial g}{\partial \theta}\, \underline{e}_\theta + \frac{1}{h_\varphi^*}\frac{\partial g}{\partial \varphi}\, \underline{e}_\varphi \; .$$

With Δ the scalar Laplace operator we have

$$\Delta g := \text{div grad } g = (\text{div}_r + \frac{1}{r}\,\text{div}_t)(\text{grad}_r + \frac{1}{r}\,\text{grad}_t) g$$

$$= \text{div}_r \text{grad}_r g + \frac{1}{r^2}\, \text{div}_t \text{grad}_t g \; .$$

This can be written as

$$\Delta g = \Delta_r g + \frac{1}{r^2}\Delta_t g$$

with

$$\Delta_r g := \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial g}{\partial r})$$

and

$$\Delta_t g := \frac{1}{h_\theta^* h_\varphi^*}\{\frac{\partial}{\partial\theta}(\frac{h_\varphi^*}{h_\theta^*}\frac{\partial g}{\partial\theta}) + \frac{\partial}{\partial\varphi}(\frac{h_\theta^*}{h_\varphi^*}\frac{\partial g}{\partial\varphi})\} .$$

In differential geometry the scalar transverse Laplace operator Δ_t is known as the Beltrami operator or the second differentiator of Beltrami [1;225].

1.3. *Trigonometric form of the sphero-conal system*

The sphero-conal system is usually described mathematically in the algebraic form and/or in the elliptic-functional form [7;659], [8;105].
In 1955 however, Kraus described the system with the help of trigonometric functions. This parametric representation is very important to the present work and therefore we shall investigate the trigonometric form [4], [5], [6].

The coordinates of the sphero-conal system, denoted by r,θ,φ, are related to the Cartesian coordinates by

$$x = r\cos(\varphi)\sin(\theta) ,$$

$$y = r\sin(\varphi)\sqrt{1 - k^2\cos^2(\theta)} ,$$

$$z = r\sqrt{1 - k'^2\sin^2(\varphi)}\cos(\theta) ,$$

where

$$0 < k < 1,\ 0 < k' < 1,\ k^2 + k'^2 = 1 ,$$

and

$$r \geq 0,\ D := \{(\theta,\varphi) \mid 0 \leq \theta \leq \pi,\ 0 \leq \varphi < 2\pi\} .$$

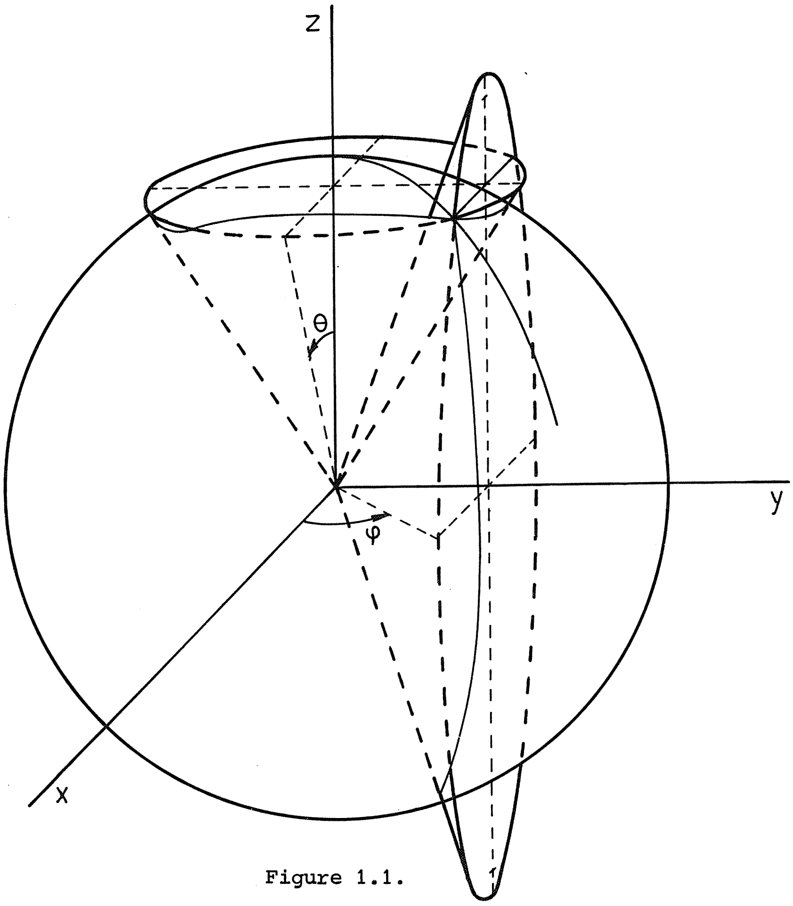

Figure 1.1.

First of all we observe that if k = 1, and consequently k' = 0, this coordinate system reduces to the spherical polar coordinate system [8;99]. Now we verify whether the coordinate curves are mutually perpendicular at each point in space. For that purpose we determine the tangent vectors to the parameter curves $\frac{\partial \underline{x}}{\partial r}$, $\frac{\partial \underline{x}}{\partial \theta}$ and $\frac{\partial \underline{x}}{\partial \varphi}$ [3;298],

$$\frac{\partial \underline{x}}{\partial r} = \begin{pmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sqrt{1 - k^2\cos^2(\theta)} \\ \sqrt{1 - k'^2\sin^2(\varphi)}\cos(\theta) \end{pmatrix},$$

$$\frac{\partial \underline{x}}{\partial \theta} = \begin{pmatrix} r\cos(\varphi)\cos(\theta) \\ rk^2\sin(\varphi)\cos(\theta)\sin(\theta)/\sqrt{1-k^2\cos^2(\theta)} \\ -r\sin(\theta)\sqrt{1-k'^2\sin^2(\varphi)} \end{pmatrix},$$

$$\frac{\partial \underline{x}}{\partial \varphi} = \begin{pmatrix} -r\sin(\varphi)\sin(\theta) \\ r\cos(\varphi)\sqrt{1-k^2\cos^2(\theta)} \\ -rk'^2\cos(\theta)\sin(\varphi)\cos(\varphi)/\sqrt{1-k'^2\sin^2(\varphi)} \end{pmatrix}.$$

It follows that, indeed,

$$\left(\frac{\partial \underline{x}}{\partial \theta}, \frac{\partial \underline{x}}{\partial \varphi}\right) = 0 .$$

We also find that the vector product

$$\frac{\partial \underline{x}}{\partial r} \times \frac{\partial \underline{x}}{\partial \theta}$$

is a positive multiple of the vector $\frac{\partial \underline{x}}{\partial \varphi}$ and hence r,θ,φ form, in this order, a right-handed system of coordinates. The scale factors of this coordinate system are

$$h_r := \left|\frac{\partial \underline{x}}{\partial r}\right| = 1 ,$$

$$h_\theta := \left|\frac{\partial \underline{x}}{\partial \theta}\right| = r\sqrt{\frac{k'^2\cos^2(\varphi) + k^2\sin^2(\theta)}{1-k^2\cos^2(\theta)}} ,$$

$$h_\varphi := \left|\frac{\partial \underline{x}}{\partial \varphi}\right| = r\sqrt{\frac{k'^2\cos^2(\varphi) + k^2\sin^2(\theta)}{1-k'^2\sin^2(\varphi)}} .$$

We observe again that if $k = 1$ these scale factors are identical with those of the spherical polar coordinate system.

The coordinate surfaces are determined by the following equations

(1.1) $$x^2 + y^2 + z^2 = r^2 ,$$

(1.2) $$\frac{x^2}{\sin^2(\theta)} + \frac{k^2y^2}{1-k^2\cos^2(\theta)} = \frac{z^2}{\cos^2(\theta)} ,$$

(1.3) $$\frac{k'^2z^2}{1-k'^2\sin^2(\varphi)} + \frac{x^2}{\cos^2(\varphi)} = \frac{y^2}{\sin^2(\varphi)} .$$

Equation (1.1) represents a sphere with centre at the origin. If $\theta \neq \pi/2$ equation (1.2) represents a cone with the vertex at the origin. The cross-section of this cone with a plane $z = z_0 \neq 0$ is an ellipse satisfying the equation

$$\frac{x^2}{z_0^2\tan^2(\theta)} + \frac{y^2}{z_0^2(\sec^2(\theta)/k^2 - 1)} = 1 .$$

The major axis, lying in the y,z plane, is denoted by 2a, in which

$$a = |z_0|\sqrt{\sec^2(\theta)/k^2 - 1}$$

and the minor axis, lying in the x,z plane,is denoted by 2b, in which

$$b = |z_0\tan(\theta)| .$$

The eccentricity is

$$e := \frac{\sqrt{a^2 - b^2}}{a} = \frac{k'}{\sqrt{1 - k^2\cos^2(\theta)}} .$$

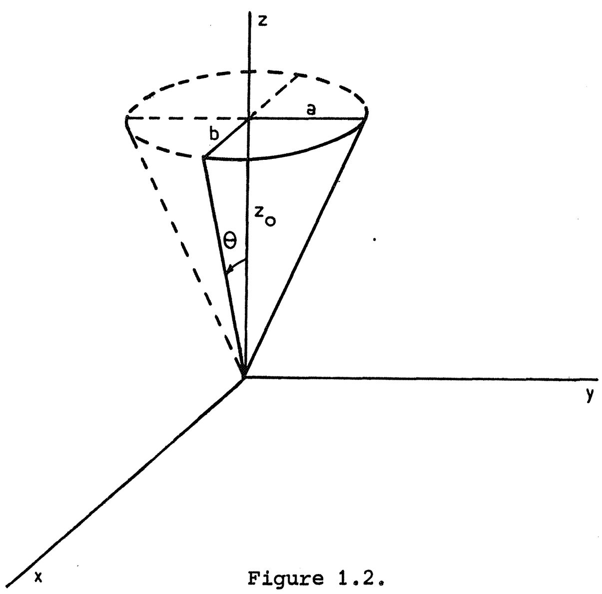

Figure 1.2.

If k = 1, and consequently e = 0, the elliptic cone becomes a circular cone.

Equation (1.3) represents an elliptic cone with the vertex at the origin. The cross-section of this cone with a plane $y = y_0 \neq 0$ is an ellipse which satisfies the equation

$$\frac{z^2}{y_0^2(\csc^2(\varphi)/k'^2 - 1)} + \frac{x^2}{y_0^2\cot^2(\varphi)} = 1, \ \varphi \neq 0,\pi \ .$$

The major axis, lying in the y,z plane, is denoted by 2a, in which

$$a = |y_0|\sqrt{\csc^2(\varphi)/k'^2 - 1} \ ,$$

and the minor axis, lying in the x,y plane, is denoted by 2b, in which

$$b = |y_0\cot(\varphi)| \ .$$

The eccentricity is

$$e = \frac{k}{\sqrt{1 - k'^2\sin^2(\varphi)}} \ .$$

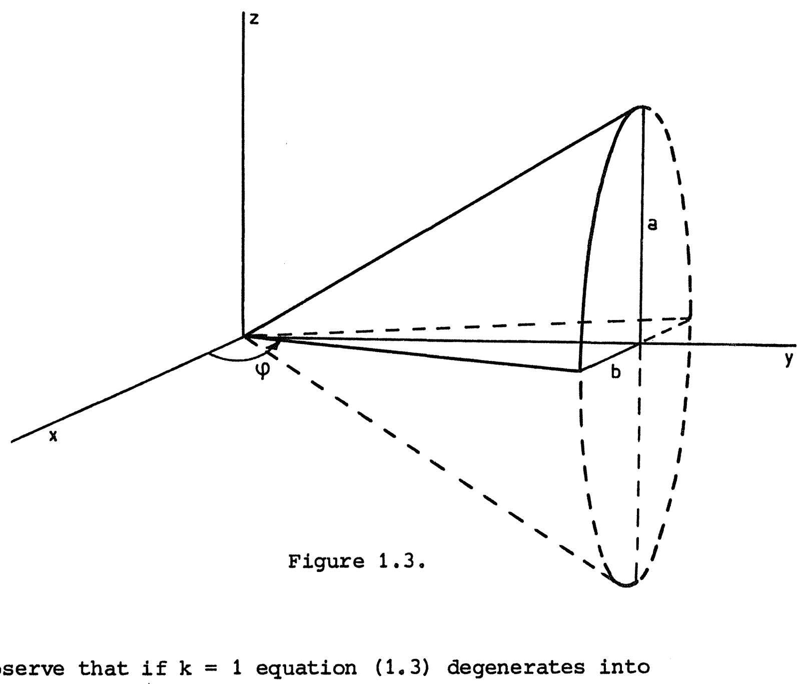

Figure 1.3.

We observe that if $k = 1$ equation (1.3) degenerates into

$$\left(\frac{x}{\cos(\varphi)} - \frac{y}{\sin(\varphi)}\right)\left(\frac{x}{\cos(\varphi)} + \frac{y}{\sin(\varphi)}\right) = 0 \ ,$$

and this is the equation of a pair of planes.

Now we investigate the one-one correspondence between the Cartesian coordinate system and the sphero-conal system. For that purpose we consider the functional determinant

$$\det\left(\frac{\partial \underline{x}}{\partial r}, \frac{\partial \underline{x}}{\partial \theta}, \frac{\partial \underline{x}}{\partial \varphi}\right) = h_r h_\theta h_\varphi = r^2 \frac{k'^2\cos^2(\varphi) + k^2\sin^2(\theta)}{\sqrt{(1 - k^2\cos^2(\theta))(1 - k'^2\sin^2(\varphi))}} .$$

If $r = 0$ *or* $(\cos(\varphi) = 0$ *and* $\sin(\theta) = 0)$, the functional determinant is zero and we have locally no one-to-one mapping on the Cartesian coordinate system.

Each point $(0,\theta,\varphi)$ is mapped onto the origin of the Cartesian coordinate system.

$(\cos(\varphi) = 0$ *and* $\sin(\theta) = 0)$ holds if:

$\theta = 0$, $\varphi = \pi/2$; this corresponds to the half-line $k'z - ky = 0$, $y \geq 0$, $x = 0$;

$\theta = 0$, $\varphi = 3\pi/2$; corresponding to $k'z + ky = 0$, $y \leq 0$, $x = 0$;

$\theta = \pi$, $\varphi = \pi/2$; corresponding to $k'z + ky = 0$, $y \geq 0$, $x = 0$;

$\theta = \pi$, $\varphi = 3\pi/2$; corresponding to $k'z - ky = 0$, $y \leq 0$, $x = 0$.

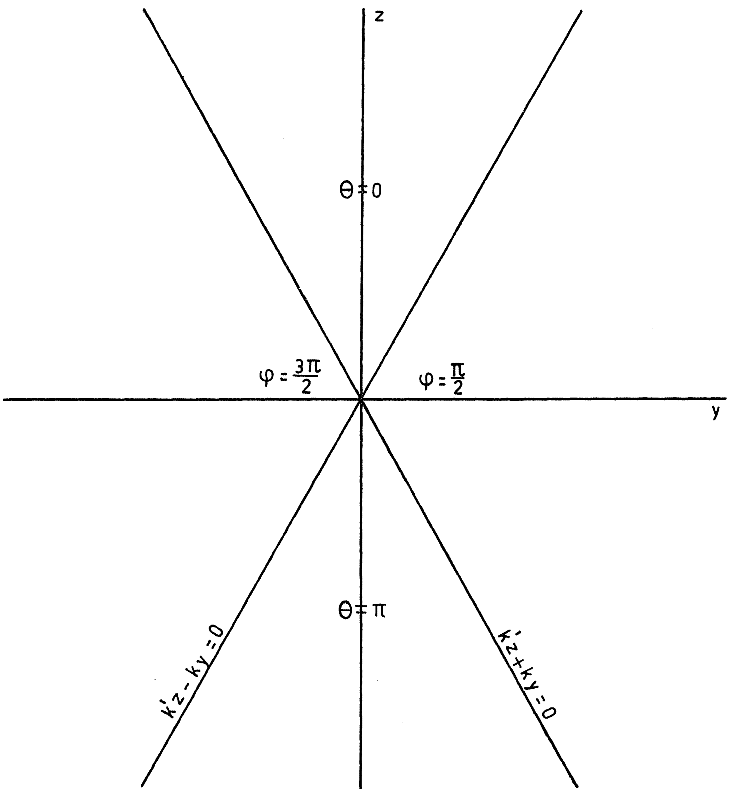

Figure 1.4.

We observe that if $\theta = 0$ the elliptic cone (1.2) degenerates into a sector of the y,z plane determined by the conditions $x = 0$, $|y| \le (k'/k)z$.
To each point inside this sector there exist two coordinate triples, viz., $(r,0,\varphi)$ and $(r,0,\pi-\varphi)$.
If $\varphi = \pi/2$ the elliptic cone (1.3) degenerates into a sector of the y,z plane determined by the conditions $x = 0$, $|z| \le (k/k')y$.
At each point inside this sector, however, the mapping from (r,θ,φ) to (x,y,z) is one-to-one.
We observe that if $k = 1$ the sectors corresponding to $\theta = 0$ and $\theta = \pi$ degenerate into the z axis (this is also true in the spherical polar coordinate system).
If $\theta = \pi/2$ the elliptic cone (1.2) degenerates into the whole x,y plane, and if $\varphi = 0$ the elliptic cone (1.3) degenerates into the whole x,z plane.

1.4. *Single-valued functions in the sphero-conal system*

It is convenient to enlarge the domain D of definition to $-\infty < \theta,\varphi < \infty$.
First of all, we observe that in this extended domain the following relations hold:

(i) $\underline{x}(r,\theta,\varphi) = \underline{x}(r,\theta,\varphi+2\pi)$, periodicity relation.

(ii) $\underline{x}(r,\theta,\varphi) = \underline{x}(r,-\theta,\pi-\varphi)$, reflection relation with respect to the point $(r,0,\pi/2)$.

(iii) $\underline{x}(r,\theta,\varphi) = \underline{x}(r,2\pi-\theta,\pi-\varphi)$, reflection relation with respect to the point $(r,\pi,\pi/2)$.

It is evident that if $F(x,y,z)$ is a single-valued function in the whole $\mathbb{R}^3$ space, then

$$f(r,\theta,\varphi) := F(x(r,\theta,\varphi),y(r,\theta,\varphi),z(r,\theta,\varphi))$$

obeys the following relations:

(i) $f(r,\theta,\varphi) = f(r,\theta,\varphi+2\pi)$,

(ii) $f(r,\theta,\varphi) = f(r,-\theta,\pi-\varphi)$,

(iii) $f(r,\theta,\varphi) = f(r,2\pi-\theta,\pi-\varphi)$.

In addition, if $(x,y,z) = (0,0,0)$ then $f(0,\theta,\varphi)$ is independent of θ and φ.

Now let $f(r,\theta,\varphi)$ be continuously differentiable in $r \geq 0$, $-\infty < \theta,\varphi < \infty$. Then, after some analysis, it turns out that the relations (i), (ii) and (iii) are sufficient conditions to guarantee that f corresponds to a continuously differentiable function $F(x,y,z)$ in the whole $\mathbb{R}^3$ space.

We observe that the function $f(r,\theta,\varphi)$ is doubly periodic with respect to θ and φ, that is, periodic in both θ and φ with period 2π. Further, the points $(r,0,\pi/2)$ and $(r,\pi,\pi/2)$ are centres of symmetry of $f(r,\theta,\varphi)$ in the extended domain $-\infty < \theta,\varphi < \infty$.

THEOREM 1.1. Let $f(r,\theta,\varphi)$ be a continuously differentiable function in the domain $r \geq 0$, $-\infty < \theta,\varphi < \infty$. Then f is a single-valued continuously differentiable function of the point (x,y,z) if and only if f satisfies the following conditions:

(i) $f(0,\theta,\varphi)$ is independent of θ and φ.

(ii) $f(r,\theta,\varphi) = f(r,\theta,\varphi+2\pi)$, periodicity condition.

(iii) $\left.\begin{aligned} f(r,\theta,\varphi) &= f(r,-\theta,\pi-\varphi) \\ &= f(r,2\pi-\theta,\pi-\varphi) \end{aligned}\right\}$ reflection conditions. □

1.5. *References*

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CHAPTER 2

THE SCALAR HELMHOLTZ EQUATION IN THE CONICAL COORDINATE SYSTEM

2.0. *Introduction*

There are a number of problems in physics and engineering defined in a conical domain and formulated in terms of potentials satisfying the scalar Helmholtz equation. For a simple mathematical description of these problems it is recommendable that the boundary of the domain is a coordinate surface, in order that separation of variables may be successful.

From now on we take the origin of coordinates at the apex of the cone. It should be understood that our cone is actually a half-cone in the sense of mathematics. Thus the cone C is defined by a set of straight half-lines from the origin through the points of a simple closed piecewise-smooth curve on the unit sphere. It is natural to define the interior G^* of the cone corresponding to the interior of the curve on the unit sphere. We define $\dot{G}$ as part of the cone C between two concentric spheres with radii r_0 and r_1 $(0 < r_0 < r_1)$ centred about the origin. The domain G is part of G^* between and on the two concentric spheres, and $\bar{G} := G \cup \dot{G}$.
In this chapter we shall investigate the scalar Helmholtz equation

$$(2.1) \qquad \Delta u + k^{*2} u = 0, \ \underline{x} \in G, \ u \in C^0(\bar{G}), \ u \in C^2(G), \ u \not\equiv 0$$

with boundary conditions, either

$$(2.2) \qquad u = 0, \ \underline{x} \in \dot{G} \ \text{(Dirichlet condition)} ,$$

or

$$(2.3) \qquad \frac{\partial u}{\partial n} = 0, \ \underline{x} \in \dot{G} \ \text{(Neumann condition)} .$$

Here n is the outward normal, k^* is the wave number defined by $k^* = \omega/c$ in which $\omega/2\pi$ is the frequency and c is the phase velocity.

2.1. *Separation of the variable* r

Let r,θ,φ be general conical coordinates in the sense of section 1.1. Because the boundary conditions (2.2) and (2.3) are independent of r we shall first separate the r dependence. For that purpose we suppose

$$u(r,\theta,\varphi) = R(r)v(\theta,\varphi) .$$

The Helmholtz equation (2.1) is then transformed into

$$v(\theta,\varphi)\Delta_r R(r) + \frac{1}{r^2} R(r)\Delta_t v(\theta,\varphi) + k^{*2} R(r)v(\theta,\varphi) = 0 ,$$

where

$$\Delta_r := \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) \quad \text{and} \quad \Delta_t := \frac{1}{h_\theta^* h_\varphi^*}(\frac{\partial}{\partial\theta}\frac{h_\varphi^*}{h_\theta^*}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{h_\theta^*}{h_\varphi^*}\frac{\partial}{\partial\varphi}) ,$$

as in chapter 1.
It follows that

$$(2.4) \qquad r^2 \frac{(\Delta_r R(r) + k^{*2}R(r))}{R(r)} = - \frac{\Delta_t v(\theta,\varphi)}{v(\theta,\varphi)} = \nu(\nu+1) = \mu^*$$

in which $\nu(\nu+1) = \mu^*$ is the separation constant.
For the r dependence we now obtain the following equation:

$$\frac{d}{dr}(r^2 \frac{dR}{dr}) + (k^{*2}r^2 - \nu(\nu+1))R = 0 .$$

We observe that this is the differential equation of the "spherical" Bessel functions with the linearly independent solutions

$$h_\nu^{(1)}(k^*r) = \sqrt{\frac{\pi}{2k^*r}}\, H_{\nu+\frac{1}{2}}^{(1)}(k^*r)$$

and

$$h_\nu^{(2)}(k^*r) = \sqrt{\frac{\pi}{2k^*r}}\, H_{\nu+\frac{1}{2}}^{(2)}(k^*r) .$$

Here, $H_{\nu+\frac{1}{2}}^{(1)}$ and $H_{\nu+\frac{1}{2}}^{(2)}$ are the Hankel functions of the first and second kinds; $h_\nu^{(1)}$ and $h_\nu^{(2)}$ are called the "spherical" Hankel functions [1;437].

2.2. *The transverse dependence*

From (2.4) we obtain for the θ,φ dependence

$$\Delta_t v + \mu^* v = 0, \quad (\theta,\varphi) \in \Omega,\ v \in C^0(\bar{\Omega}),\ v \in C^2(\Omega),\ v \not\equiv 0$$

with boundary condition, either

$$v = 0,\ (\theta,\varphi) \in \dot{\Omega} \text{ (Dirichlet condition)} ,$$

or

$$\frac{\partial v}{\partial n} = 0,\ (\theta,\varphi) \in \dot{\Omega} \text{ (Neumann condition)} .$$

Here

$$\dot{\Omega} := \{(\theta,\varphi) \mid (\theta,\varphi) \text{ is on a simple closed piecewise-smooth curve on the unit sphere}\} ,$$

$$\Omega := \{(\theta,\varphi) \mid (\theta,\varphi) \text{ is the interior of } \dot{\Omega} \text{ on the unit sphere}\}$$

and

$$\bar{\Omega} := \Omega \cup \dot{\Omega} .$$

Consequently, we have to investigate the eigenvalues and the eigenfunctions of the Dirichlet and Neumann problems for the Beltrami operator Δ_t in a domain Ω on the unit sphere. It is easy to see that the Beltrami operator with either the Dirichlet or the Neumann boundary condition is a Hermitian operator with respect to the inner product

$$(2.5) \qquad (u,v) = \iint_{\Omega} u(\theta,\varphi)v(\theta,\varphi)h_\theta^*(\theta,\varphi)h_\varphi^*(\theta,\varphi)d\theta d\varphi \ .$$

Moreover, if h_θ^* and h_φ^* are both positive and bounded in $\bar{\Omega}$, Δ_t is uniformly elliptic here. From the spectral theory of elliptic Hermitian operators [4] it is not difficult to see that the following theorem holds.

THEOREM 2.1. Consider the two eigenvalue problems

$$\Delta_t v + \mu^* v = 0, \quad (\theta,\varphi) \in \Omega,\ v \in C^0(\bar{\Omega}),\ v \in C^2(\Omega),\ v \not\equiv 0$$

with boundary condition, either

$$v = 0,\ (\theta,\varphi) \in \dot{\Omega} \text{ (Dirichlet condition)} ,$$

or

$$\frac{\partial v}{\partial n} = 0,\ (\theta,\varphi) \in \dot{\Omega} \text{ (Neumann condition)} .$$

If h_θ^* and h_φ^* are continuous and nonzero in $\bar{\Omega}$, either eigenvalue problem admits a denumerable set of eigenvalues having the following properties. The eigenvalues μ_n^* are real and form an infinite sequence (with ∞ as the only accumulation point) such that $0 < \mu_1^* \le \mu_2^* \le \ldots$ in the Dirichlet case, and $0 = \mu_0^* < \mu_1^* \le \ldots$ in the Neumann case.

The corresponding eigenfunctions $v_k(\theta,\varphi)$ can be chosen such that they form a complete set of orthogonal eigenfunctions with respect to the inner product (2.5). □

To indicate a proof of this theorem we shall first transform the Beltrami operator with the aid of stereographic projection and conformal mapping into the two-dimensional Laplace operator multiplied by a function positive on the unit disk and show that this operator has a compact inverse. The theorem then follows from the well-known theory of compact Hermitian operators. We do this in the following steps.

(i) The unit sphere is given by the equation

$$x^2 + y^2 + z^2 = 1$$

and we identify the north pole with the point (0,0,1). We choose the parameter representation of the unit sphere so that the north pole lies outside $\bar{\Omega}$.

We then consider the stereographic projection [2;20] from the north pole on the complex z^* plane that coincides with the x,y plane in the Cartesian coordinate system. This transformation is given by

$$z^* = x^* + iy^* = \frac{x + iy}{1 - z} \ , \ \underline{x} \in \bar{\Omega} \ ,$$

where x^* and y^* are functions of θ and φ.

$$\frac{\partial x^*}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{1}{1 - z} + \frac{x}{(1 - z)^2} \frac{\partial z}{\partial \theta} ,$$

$$\frac{\partial y^*}{\partial \theta} = \frac{\partial y}{\partial \theta} \frac{1}{1 - z} + \frac{y}{(1 - z)^2} \frac{\partial z}{\partial \theta} ,$$

$$\frac{\partial x^*}{\partial \varphi} = \frac{\partial x}{\partial \varphi} \frac{1}{1 - z} + \frac{x}{(1 - z)^2} \frac{\partial z}{\partial \varphi} ,$$

$$\frac{\partial y^*}{\partial \varphi} = \frac{\partial y}{\partial \varphi} \frac{1}{1 - z} + \frac{y}{(1 - z)^2} \frac{\partial z}{\partial \varphi} .$$

Let

$$h_\theta^{**} := \left|\frac{\partial z^*}{\partial \theta}\right| \text{ and } h_\varphi^{**} := \left|\frac{\partial z^*}{\partial \varphi}\right|$$

then, after some calculation,

$$h_\theta^{**} = \frac{h_\theta^*}{1 - z} \text{ and } h_\varphi^{**} = \frac{h_\varphi^*}{1 - z} .$$

The operator Δ_t transforms into the operator

$$\frac{1}{(1-z)^2}\left(\frac{\partial^2}{\partial x^{*2}}+\frac{\partial^2}{\partial y^{*2}}\right)$$

defined in the domain Ω^* bounded by the simple closed curve $\dot{\Omega}^*$ as stereographic projection of Ω and $\dot{\Omega}$, respectively.
This operator can also be written as

$$\left(\frac{\tilde{R}^2}{2}\right)^2\left(\frac{\partial^2}{\partial x^{*2}}+\frac{\partial^2}{\partial y^{*2}}\right) ,$$

in which $\tilde{R}$ is the distance from the north pole to the point z^*.

(ii) According to the Riemann mapping theorem [2;172] we can map Ω^* with the aid of a conformal mapping

$$\zeta = g(z^*) = \xi + i\eta$$

on the unit disk

$$B := \{(\xi,\eta) \mid \xi^2 + \eta^2 < 1\} .$$

$\dot{\Omega}^*$ is mapped on

$$\dot{B} := \{(\xi,\eta) \mid \xi^2 + \eta^2 = 1\} .$$

We have now shown the equivalence of the eigenvalue problems

$$\Delta_t v + \mu^* v = 0, \quad (\theta,\varphi) \in \Omega$$

with boundary condition, either

$$v(\theta,\varphi) = 0, \ (\theta,\varphi) \in \dot{\Omega}$$

or

$$\frac{\partial}{\partial n} v(\theta,\varphi) = 0, \ (\theta,\varphi) \in \dot{\Omega} ,$$

and the eigenvalue problems [2;175]

$$f^{-2}(\xi,\eta)\Delta_\zeta v + \mu^* v = 0, \ (\xi,\eta) \in B ,$$

with boundary condition, either

$$v(\xi,\eta) = 0, \ (\xi,\eta) \in \dot{B} \text{ (Dirichlet condition)} ,$$

or

$$\frac{\partial}{\partial n} v(\xi,\eta) = 0, \ (\xi,\eta) \in \dot{B} \text{ (Neumann condition)} .$$

Here $f^{-2}(\xi,\eta)$ is defined as

$$f^{-2}(\xi,\eta) := (\tfrac{1}{2}\tilde{R}^2|g'(z^*)|)^2, \ z^* \in \bar{\Omega}^* .$$

We observe that the inner product (2.5) becomes

$$(u,v) = \iint_B f^2(\xi,\eta)u(\xi,\eta)v(\xi,\eta)d\xi d\eta .$$

(iii) Invoking theorems (2.28) and (2.35) of the appendix the proof of our theorem is complete.

2.3. *Separation of the variables* θ *and* φ

In the previous section we considered the spectral properties of the transverse Laplace operator on a domain Ω of the unit sphere with either Dirichlet or Neumann boundary conditions. The choice of the coordinates θ and φ was relatively unimportant there, so long as the operator Δ_t remains uniformly elliptic in Ω. Now we shall be more specific. If Ω corresponds to a cone with elliptic cross-section, we want to choose θ and φ such that the boundary $\dot{\Omega}$ becomes a curve $\theta = \theta_0$ = constant, so that we are able to separate the coordinates θ and φ. Hence we choose for θ and φ those of the sphero-conal coordinates introduced in section 1.3, and consider the domain Ω corresponding to the parameter values $0 < \theta < \theta_0$, $-\infty < \varphi < \infty$. Since we want to consider only functions that are regular in Ω, we now have to adjoin the regularity conditions of section 1.4.

We now reconsider the eigenvalue problem (v,μ^*) (which we will call the Beltrami problem):

$$(2.6) \qquad \frac{\sqrt{1-k^2\cos^2(\theta)}\sqrt{1-k'^2\sin^2(\varphi)}}{k'^2\cos^2(\varphi)+k^2\sin^2(\theta)}\left[\frac{\partial}{\partial\theta}\left(\sqrt{\frac{1-k^2\cos^2(\theta)}{1-k'^2\sin^2(\varphi)}}\,\frac{\partial}{\partial\theta}\right)+\right.$$

$$\left.\frac{\partial}{\partial\varphi}\left(\sqrt{\frac{1-k'^2\sin^2(\varphi)}{1-k^2\cos^2(\theta)}}\,\frac{\partial}{\partial\varphi}\right)\right]v+\mu^*v=0, \quad 0<\theta<\theta_0, \ -\infty<\varphi<\infty, \ v\not\equiv 0$$

$$(2.7) \qquad v(\theta,2\pi) = v(\theta,0), \ \frac{\partial v}{\partial\varphi}(\theta,2\pi) = \frac{\partial v}{\partial\varphi}(\theta,0) ,$$

$$(2.8) \qquad v(0,\pi-\varphi) = v(0,\varphi), \ \frac{\partial v}{\partial\theta}(0,\pi-\varphi) = -\frac{\partial v}{\partial\theta}(0,\varphi) ,$$

either

(2.9) $v(\theta_0,\varphi) = 0$ (Dirichlet condition)

or

(2.10) $\frac{\partial v}{\partial\theta}(\theta_0,\varphi) = 0$ (Neumann condition) .

The conditions at $\varphi = 0,2\pi$ may be replaced by $v(\theta,2\pi+\varphi) = v(\theta,\varphi)$ if we extend v to a 2π-periodic function and $v(\theta,\pi-\varphi) = v(-\theta,\varphi)$ if we extend v below the line $\theta = 0$.

It should be stressed that the existence of eigenvalues and eigenfunctions for Ω has already been shown; these functions when considered as functions of θ and φ of the sphero-conal system certainly satisfy the regularity conditions. We shall now, by separation of variables, construct a set of so-called separable solutions of the above boundary-value problem and show that from these solutions in this way all eigenfunctions of the transverse Laplace operator in Ω (with either Dirichlet or Neumann conditions) can be obtained by finite linear combinations.
The selection of the sphero-conal coordinate system and the boundary and regularity conditions leads to separating the eigenfunctions $v(\theta,\varphi)$ as

$$v(\theta,\varphi) = \Theta(\theta)\Phi(\varphi) .$$

Then the equation (2.6) separates into the following two Lamé equations:

(2.11) $$\sqrt{1-k^2\cos^2(\theta)}\frac{d}{d\theta}(\sqrt{1-k^2\cos^2(\theta)}\ \frac{d\Theta}{d\theta}) + (\mu^*\ k^2\sin^2(\theta) - \lambda^*)\Theta = 0$$

and

(2.12) $$\sqrt{1-k'^2\sin^2(\varphi)}\frac{d}{d\varphi}(\sqrt{1-k'^2\sin^2(\varphi)}\ \frac{d\Phi}{d\varphi}) + (\mu^*\ k'^2\cos^2(\varphi) + \lambda^*)\Phi = 0$$

where λ^* is the separation constant.
For simplicity we put $\lambda = \lambda^* + k'^2\mu^*$. Then equations (2.11) and (2.12) are transformed into

(2.13) $$\sqrt{1-k^2\cos^2(\theta)}\frac{d}{d\theta}(\sqrt{1-k^2\cos^2(\theta)}\ \frac{d\Theta}{d\theta}) + (\mu^*(1 - k^2\cos^2(\theta)) - \lambda)\Theta = 0$$

and

(2.14) $$\sqrt{1-k'^2\sin^2(\varphi)}\frac{d}{d\varphi}(\sqrt{1-k'^2\sin^2(\varphi)}\ \frac{d\Phi}{d\varphi}) +$$
$$+ (\mu^*(1 - k'^2\sin^2(\varphi)) - (\mu^* - \lambda))\Phi = 0$$

respectively.

DEFINITION 2.2. A function $\Phi(\varphi)$ is called *periodic* if $\Phi(\varphi) = \Phi(2\pi+\varphi)$. A function $\Phi(\varphi)$ is called *even symmetric* if $\Phi(\pi-\varphi) = \Phi(\varphi)$. A function $\Phi(\varphi)$ is called *odd symmetric* if $\Phi(\pi-\varphi) = -\Phi(\varphi)$. □

DEFINITION 2.3. A function $v(\theta,\varphi)$ is called *separable* if $v(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$. A function $v(\theta,\varphi)$ is called *strongly separable* if $v(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$ with $\Phi(\varphi)$ symmetric (that is, even or odd symmetric). □

LEMMA 2.4. If an eigenfunction $v(\theta,\varphi)$ of the Beltrami problem is separable then $v(\theta,\varphi)$ is either strongly separable or the sum of two independent strongly separable eigenfunctions.

PROOF. If $v(\theta,\varphi)$ is an eigenfunction of the Beltrami problem then, since the coefficients of the Beltrami operator are even functions of φ that have period π, $v(\theta,\pi-\varphi)$ is also an eigenfunction. It follows that

$$w_1(\theta,\varphi) := \tfrac{1}{2}(v(\theta,\varphi) + v(\theta,\pi-\varphi))$$

and

$$w_2(\theta,\varphi) := \tfrac{1}{2}(v(\theta,\varphi) - v(\theta,\pi-\varphi))$$

are also solutions of the Beltrami problem. These functions are independent, unless one of them is identically zero. We observe that $w_1(\theta,\varphi) = w_1(\theta,\pi-\varphi)$ and $w_2(\theta,\varphi) = -w_2(\theta,\pi-\varphi)$:
If $v(\theta,\varphi)$ is separable, i.e., $v(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$ then

$$w_1(\theta,\varphi) = \tfrac{1}{2}\Theta(\theta)(\Phi(\varphi) + \Phi(\pi-\varphi))$$

and

$$w_2(\theta,\varphi) = \tfrac{1}{2}\Theta(\theta)(\Phi(\varphi) - \Phi(\pi-\varphi))$$

and these functions are, obviously, strongly separable. □

We now have to find appropriate auxiliary conditions for Θ and Φ.
From the periodicity condition (2.7) it follows that $v(\theta,\varphi) = v(\theta,\varphi+2\pi)$, hence the solutions of the Lamé equation (2.14) must satisfy the periodicity condition $\Phi(\varphi) = \Phi(\varphi+2\pi)$ or, equivalently, $\Phi(0) = \Phi(2\pi)$ and $\Phi'(0) = \Phi'(2\pi)$. The right-hand boundary condition belonging to equation (2.13) follows directly from (2.9) and (2.10):

$$\Theta(\theta_0) = 0 \text{ (Dirichlet problem)}$$

or

$$\frac{d\theta}{d\theta}(\theta_0) = 0 \text{ (Neumann problem)} .$$

In order to find a boundary condition for the θ equation at the left-hand end point $\theta = 0$, we observe that from the regularity conditions (2.8) it follows for a separable eigenfunction $v(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$ that

$$\Theta(0)\Phi(\varphi) = \Theta(0)\Phi(\pi-\varphi)$$

and

$$\frac{d\Theta}{d\theta}(0)\Phi(\varphi) = -\frac{d\Theta}{d\theta}(0)\Phi(\pi-\varphi) .$$

Hence for strongly separable eigenfunctions we have

$$\text{(2.15)} \qquad \frac{d\Theta(0)}{d\theta} = 0 \text{ if } \Phi(\varphi) \text{ is even symmetric ,}$$

$$\text{(2.16)} \qquad \Theta(0) = 0 \text{ if } \Phi(\varphi) \text{ is odd symmetric .}$$

Conversely, we can find eigenvalues and strongly separable eigenfunctions of the two-dimensional Beltrami problem by looking for non-trivial solutions of the φ and θ equations (2.14) and (2.13) with the same values of μ^* and λ and satisfying the conditions

(i) $\Phi(\varphi)$ is periodic and symmetric .

(ii) $\begin{cases} \frac{d\Theta}{d\theta}(0) = 0 \text{ if } \Phi \text{ is even symmetric ,} \\ \Theta(0) = 0 \text{ if } \Phi \text{ is odd symmetric .} \end{cases}$

(iii) $\begin{cases} \Theta(\theta_0) = 0 \text{ in the Dirichlet case ,} \\ \frac{d\Theta(\theta_0)}{d\theta} = 0 \text{ in the Neumann case .} \end{cases}$

The above considerations may be summarized in the following theorem:

THEOREM 2.5. $v(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$ is a strongly separable eigenfunction corresponding to the eigenvalue μ^* of the Beltrami problem if and only if there exists a λ such that

(i) $\Phi(\varphi)$ satisfies (2.14) and is periodic and symmetric,

(ii) $\Theta(\theta)$ satisfies (2.13) and the boundary conditions

$$\frac{d\Theta(0)}{d\theta} = 0 \text{ if } \Phi \text{ is even symmetric ,}$$

$\Theta(0) = 0$ if Φ is odd symmetric ,

$\Theta(\theta_0) = 0$ in the Dirichlet case ,

$\frac{d\Theta}{d\theta}(\theta_0) = 0$ in the Neumann case . □

From lemma 2.4 it is obvious that the strongly separable eigenfunctions span the space of all separable eigenfunctions; we will show presently that they even span the space of all eigenfunctions.

LEMMA 2.6. If $\Theta(\theta)\Phi(\varphi)$ is a strongly separable eigenfunction with eigenvalue μ^* then the separation constant λ satisfies

$$0 < \lambda < \mu^* = \nu(\nu + 1) .$$

PROOF. If Θ ($\neq 0$) satisfies equation (2.13), then

$$\int_0^{\theta_0} \frac{d}{d\theta}\left(\sqrt{1 - k^2\cos^2(\theta)}\frac{d\Theta}{d\theta}\right)\bar{\Theta}(\theta)\,d\theta + (\mu^* - \lambda)\int_0^{\theta_0} \frac{\Theta(\theta)\bar{\Theta}(\theta)}{\sqrt{1 - k^2\cos^2\theta}}\,d\theta +$$

$$- \mu^* k^2 \int_0^{\theta_0} \frac{\cos^2(\theta)\Theta(\theta)\bar{\Theta}(\theta)}{\sqrt{1 - k^2\cos^2(\theta)}}\,d\theta = 0 .$$

By integrating by parts and using the boundary conditions (2.15) and (2.16) respectively, as well as $(\Theta \times \frac{d\Theta}{d\theta})_{\theta=\theta_0} = 0$, it follows that $\lambda < \mu^*$.
If Φ satisfies the equation (2.14), then

$$\int_0^{2\pi} \frac{d}{d\varphi}\left(\sqrt{1 - k'^2\sin^2(\varphi)}\frac{d\Phi}{d\varphi}\right)\bar{\Phi}(\varphi)\,d\varphi - k'^2\mu^* \int_0^{2\pi} \frac{\sin^2(\varphi)\Phi(\varphi)\bar{\Phi}(\varphi)}{\sqrt{1 - k'^2\sin^2(\varphi)}}\,d\varphi +$$

$$+ \lambda \int_0^{2\pi} \frac{\Phi(\varphi)\bar{\Phi}(\varphi)}{\sqrt{1 - k'^2\sin^2(\varphi)}}\,d\varphi = 0 .$$

By integrating by parts and using the periodicity conditions, it follows that $\lambda > 0$.

REMARK. The complex conjugate of $\Theta(\theta)$ and $\Phi(\varphi)$ is denoted by $\bar{\Theta}(\theta)$ and $\bar{\Phi}(\varphi)$.□

We now investigate the spectrum of the φ problem for a given $\mu^* = \nu(\nu+1)$. Let $\Phi(\varphi;\lambda)$ be a solution of equation (2.14) that satisfies the periodicity conditions

$$(2.17) \qquad \Phi(0;\lambda) = \Phi(2\pi;\lambda)$$

and

$$(2.18) \qquad \frac{d\Phi}{d\varphi}(0;\lambda) = \frac{d\Phi}{d\varphi}(2\pi;\lambda) \quad .$$

Then (since the coefficients of the differential equation have period π) $\Phi(\varphi+\pi;\lambda)$ is also a solution of equation (2.14) that satisfies the periodicity conditions (2.17) and (2.18). It follows that

$$\psi_1(\varphi;\lambda) := \tfrac{1}{2}(\Phi(\varphi;\lambda) + \Phi(\varphi+\pi;\lambda))$$

and

$$\psi_2(\varphi;\lambda) := \tfrac{1}{2}(\Phi(\varphi;\lambda) - \Phi(\varphi+\pi;\lambda))$$

also satisfy (2.14), (2.17) and (2.18); at least one of them is non-trivial. We observe that $\psi_1(\varphi;\lambda) = \psi_1(\varphi+\pi;\lambda)$ and hence $\psi_1(\varphi;\lambda)$ is a solution with period π. Also, $\psi_2(\varphi;\lambda)$ is a solution with period 2π, for which $\psi_2(\varphi;\lambda) = -\psi_2(\varphi+\pi;\lambda)$. From this we may conclude that (2.14) with the periodicity conditions (2.17) and (2.18) is equivalent to (2.14) on the interval $(0,\pi)$ with the boundary conditions

$$\Phi(0;\lambda) = \Phi(\pi;\lambda) \quad \text{and} \quad \frac{d\Phi}{d\varphi}(0;\lambda) = \frac{d\Phi}{d\varphi}(\pi;\lambda)$$

or

$$\Phi(0;\lambda) = -\Phi(\pi;\lambda) \quad \text{and} \quad \frac{d\Phi}{d\varphi}(0;\lambda) = -\frac{d\Phi}{d\varphi}(\pi;\lambda) \quad .$$

Hence we have to investigate the following Sturm-Liouville eigenvalue problems.

PROBLEM A.

$$\sqrt{1-k'^2\sin^2(\varphi)}\frac{d}{d\varphi}\left(\sqrt{1-k'^2\sin^2(\varphi)}\frac{d\Phi}{d\varphi}\right) + (\lambda - \mu^* k'^2\sin^2(\varphi))\Phi = 0$$

with the periodicity conditions

$$\Phi(0) = \Phi(\pi), \; \frac{d\Phi}{d\varphi}(0) = \frac{d\Phi}{d\varphi}(\pi) \quad .$$

PROBLEM B.

$$\sqrt{1-k'^2\sin^2(\varphi)}\frac{d}{d\varphi}\left(\sqrt{1-k'^2\sin^2(\varphi)}\frac{d\tilde{\Phi}}{d\varphi}\right) + (\lambda \; - \mu^* \; k'^2\sin^2(\varphi))\tilde{\Phi} = 0$$

with the periodicity condition

$$\tilde{\Phi}(0) = -\tilde{\Phi}(\pi), \; \frac{d\tilde{\Phi}}{d\varphi}(0) = -\frac{d\tilde{\Phi}}{d\varphi}(\pi) \; .$$

With the aid of lemma 2.6 and theorem 3.1 from [3;214] we can formulate the following theorem:

THEOREM 2.7. For any $\mu^* > 0$ the eigenvalues λ_i, $i \geq 0$, of problem A and the eigenvalues $\tilde{\lambda}_i$, $i \geq 1$, of problem B, form infinite sequences (with ∞ as the sole accumulation point) such that

$$0 < \lambda_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 < \lambda_1 \leq \lambda_2 < \tilde{\lambda}_3 \leq \tilde{\lambda}_4 < \lambda_3 \leq \lambda_4 \; \ldots \; .$$

For $\lambda = \lambda_0$ there exists a unique eigenfunction without any zero in $[0,\pi]$. For $\lambda = \lambda_{2i+1}$ and $\lambda = \lambda_{2i+2}$, $i \geq 0$ there exist eigenfunctions $\Phi_{2i+1}(\varphi)$ and $\Phi_{2i+2}(\varphi)$ respectively with precisely $2i+2$ zeros in $[0,\pi)$. For $\lambda = \tilde{\lambda}_{2i+1}$ and $\lambda = \tilde{\lambda}_{2i+2}$ there exist eigenfunctions $\tilde{\Phi}_{2i+1}(\varphi)$ and $\tilde{\Phi}_{2i+2}(\varphi)$ respectively with precisely $2i+1$ zeros in $[0,\pi)$. The eigenfunctions $\Phi_i(\varphi)$, $i \geq 0$ and $\tilde{\Phi}_i(\varphi)$, $i \geq 1$ together can be chosen such that they form a complete set of orthonormal eigenfunctions with the inner product

$$(u,v) = \int_0^\pi \frac{u(\varphi)\,v(\varphi)}{\sqrt{1-k'^2\sin^2(\varphi)}}\,d\varphi \; . \qquad \square$$

LEMMA 2.8. If $\Phi(\varphi)$ is an eigenfunction corresponding to the eigenvalue λ of the φ problem then $\Phi(\varphi)$ satisfies either $\Phi(\pi+\varphi) = \Phi(\varphi)$ or $\Phi(\pi+\varphi) = -\Phi(\varphi)$.

PROOF. The functions $\psi_1(\varphi,\lambda)$ and $\psi_2(\varphi,\lambda)$ are independent, unless one of them is trivial. Consequently, if the eigenvalue λ is simple then one of these functions must be trivial. If, however, λ has multiplicity 2 then from theorem 2.7 it follows that the corresponding eigenfunctions both belong either to problem A or to problem B; in the first case $\psi_2(\varphi;\lambda)$ is zero, in the other case $\psi_1(\varphi;\lambda)$ is zero. $\square$

LEMMA 2.9. Each eigenspace of the φ problem has an orthonormal basis consisting of symmetric eigenfunctions.

PROOF. If $\Phi(\varphi)$ is an eigenfunction of the φ problem with eigenvalue λ then, since the coefficients of the differential equation are even and have period π, $\Phi(\pi-\varphi)$ is also an eigenfunction of the φ problem.
It follows that

$$\chi_1(\varphi) := \tfrac{1}{2}(\Phi(\varphi) + \Phi(\pi-\varphi))$$

and

$$\chi_2(\varphi) := \tfrac{1}{2}(\Phi(\varphi) - \Phi(\pi-\varphi))$$

are also solutions of the φ problem. It follows from substitution of $\varphi \to (\pi-\varphi)$ at appropriate places in the integrand that $(\chi_1,\chi_2) = 0$. Hence χ_1 and χ_2 are orthogonal and independent unless one of them is trivial. We observe that $\chi_1(\varphi) = \chi_1(\pi-\varphi)$ and $\chi_2(\varphi) = -\chi_2(\pi-\varphi)$.

From theorem 2.7 we know that an eigenvalue λ has at most multiplicity 2. If λ is simple then one of the functions χ_1 and χ_2 must be trivial. Hence $\Phi(\varphi)$ is symmetric.
Let now λ have multiplicity 2 and let $\Phi(\varphi)$, $\hat{\Phi}(\varphi)$ be an orthonormal basis for the corresponding eigenspace. If for one of the functions Φ and $\hat{\Phi}$ both χ_1 and χ_2 are non-trivial then because $(\chi_1,\chi_2) = 0$ we can choose these functions χ_1 and χ_2 as an orthogonal basis for the eigenspace corresponding to λ. In the other case the functions Φ and $\hat{\Phi}$ are both symmetric, and since they are orthogonal they can be chosen as a basis. □

The results arrived at above may be summarized in the following theorem:

THEOREM 2.10. For any $\mu^* > 0$ the eigenvalues of the φ problem form an infinite sequence $\lambda_0,\lambda_1,\ldots$ (with ∞ as the only accumulation point) such that

$$0 < \lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \ldots .$$

The eigenfunctions Φ_i, $i \ge 0$, can be chosen such that

(i) they satisfy either $\Phi_i(\pi+\varphi) = \Phi_i(\varphi)$ or $\Phi_i(\pi+\varphi) = -\Phi_i(\varphi)$,
(ii) they are symmetric,
(iii) if two eigenvalues ($\lambda_i = \lambda_{i+1}$) are equal then either

$$\Phi_i(\pi+\varphi) = \Phi_i(\varphi) \wedge \Phi_{i+1}(\pi+\varphi) = \Phi_{i+1}(\varphi) ,$$

or

$$\Phi_i(\pi+\varphi) = -\Phi_i(\varphi) \wedge \Phi_{i+1}(\pi+\varphi) = -\Phi_{i+1}(\varphi) .$$

Moreover Φ_i is even symmetric and Φ_{i+1} is odd symmetric, or vice versa,

(iv) they form a complete set of orthonormal eigenfunctions with respect to the inner product

$$(u,v) = \int_0^{2\pi} \frac{u(\varphi)v(\varphi)}{\sqrt{1 - k'^2\sin^2(\varphi)}} d\varphi . \qquad \square$$

REMARK. In comparison with theorem 2.7 the eigenvalues are numbered so that the following statement about zeros of the eigenfunctions holds: Φ_0 has no zero in $[0,2\pi)$. Φ_{4i-1} and Φ_{4i} have precisely 4i zeros in $[0,2\pi)$, $i = 1,2,\ldots$. Φ_{4i+1} and Φ_{4i+2} have precisely $4i+2$ zeros in $[0,2\pi)$, $i = 0,1,2,\ldots$.

THEOREM 2.11. Independent strongly separable eigenfunctions of the Beltrami problem are orthogonal with respect to the inner product (2.5):

$$(u,v) = \int_0^{2\pi}\int_0^{\theta_0} \frac{u(\theta,\varphi)v(\theta,\varphi)(k'^2\cos^2(\varphi) + k^2\sin^2(\theta))}{\sqrt{1 - k'^2\sin^2(\varphi)}\sqrt{1 - k^2\cos^2(\theta)}} d\varphi d\theta .$$

PROOF. Let μ^* be an eigenvalue of the Beltrami problem to which one or more strongly separable eigenfunctions belong. If the eigenvalue μ^* is simple then the corresponding strongly separable eigenfunction is orthogonal to all other eigenfunctions, a consequence of theorem 2.1. Let μ^* now be multiple and let $u = \Theta_i\Phi_i$ and $v = \Theta_j\Phi_j$ be independent strongly separable eigenfunctions then

$$(u,v) = \int_0^{2\pi} \frac{\Phi_i(\varphi)\Phi_j(\varphi)k'^2\cos^2(\varphi)}{\sqrt{1 - k'^2\sin^2\varphi}} d\varphi \int_0^{\theta_0} \frac{\Theta_i(\theta)\Theta_j(\theta)}{\sqrt{1 - k^2\cos^2(\theta)}} d\theta +$$

$$+ \int_0^{2\pi} \frac{\Phi_i(\varphi)\Phi_j(\varphi)}{\sqrt{1 - k'^2\sin^2(\varphi)}} d\varphi \int_0^{\theta_0} \frac{\Theta_i(\theta)\Theta_j(\theta)}{\sqrt{1 - k^2\cos^2(\theta)}} k^2\sin^2(\theta)d\theta .$$

From theorem 2.7 it follows that two independent eigenfunctions Φ_i and Φ_j of the φ problem satisfy

$$(\Phi_i,\Phi_j) = \int_0^{2\pi} \frac{\Phi_i(\varphi)\Phi_j(\varphi)}{\sqrt{1 - k'^2\sin^2(\varphi)}}\, d\varphi = 0 \ .$$

If $\lambda_i \neq \lambda_j$ then it follows from (2.13) and the boundary conditions that

$$(\theta_i,\theta_j) = \int_0^{\theta_0} \frac{\theta_i(\theta)\theta_j(\theta)}{\sqrt{1 - k^2\cos^2(\theta)}}\, d\theta = 0 \ .$$

Hence $\theta_i\Phi_i$ and $\theta_j\Phi_j$ are orthogonal.

If $\lambda_i = \lambda_j$ then according to theorem 2.10 Φ_i is even symmetric and Φ_j is odd symmetric, or vice versa.
Consequently

$$\int_0^{2\pi} \frac{\cos^2(\varphi)\Phi_i(\varphi)\Phi_j(\varphi)}{\sqrt{1 - k'^2\sin^2(\varphi)}}\, d\varphi = 0 \ .$$

Hence also in this case $\theta_i\Phi_i$ and $\theta_j\Phi_j$ are orthogonal. □

THEOREM 2.12. The eigenspace corresponding to an eigenvalue μ^* of the Beltrami problem can be spanned by a finite number of mutually orthogonal strongly separable eigenfunctions.

PROOF. Let $v(\theta,\varphi)$ be an eigenfunction corresponding to μ^*. Since the functions $\Phi_n(\varphi)$ constitute a complete orthonormal set (see theorem 2.10), we can expand $v(\theta,\varphi)$ in a Fourier series

$$v(\theta,\varphi) = \sum_{n=0}^{\infty} \theta_n^*(\theta)\Phi_n(\varphi)$$

with

$$\theta_n^*(\theta) := \int_0^{2\pi} \frac{v(\theta,\varphi)\Phi_n(\varphi)}{\sqrt{1 - k'^2\sin^2(\varphi)}}\, d\varphi \ . \tag{2.19}$$

From the equation

$$\int_0^{2\pi} \frac{[(\Delta_t + \mu^*)\ v(\theta,\varphi)]\Phi_n(\varphi)}{\sqrt{1 - k'^2\sin^2(\varphi)}} [k'^2\cos^2(\varphi) + k^2\sin^2(\theta)]d\varphi = 0$$

it follows after some calculations that $\theta_n^*(\theta)$, $n = 0,1,\ldots$ is a solution of the equation

$$\sqrt{1-k^2\cos^2(\theta)}\frac{d}{d\theta}\left(\sqrt{1-k^2\cos^2(\theta)}\frac{d\theta_n^*}{d\theta}\right) + (\mu^*(1-k^2\cos^2(\theta)) - \lambda_n)\theta_n^* = 0 .$$

From the boundary conditions (2.9) or (2.10) and the definition of θ_n^* it follows that θ_n^* satisfies the right-hand boundary condition

$$\theta_n(\theta_0) = 0 \text{ (Dirichlet condition)}$$

or

$$\frac{d\theta_n}{d\theta}(\theta_0) = 0 \text{ (Neumann condition) .}$$

The appropriate boundary condition for θ_n^* at $\theta = 0$ can be found as follows: $v(\theta,\varphi)$ satisfies the regularity conditions

$$v(0,\varphi) = v(0,\pi-\varphi)$$

and

$$\frac{\partial v}{\partial\theta}(0,\varphi) = -\frac{\partial v}{\partial\theta}(0,\pi-\varphi) .$$

We know, however, that $\Phi_n(\varphi)$ is symmetric. Combining these facts it follows from (2.19) that θ_n^* satisfies the left-hand boundary condition

$$\theta_n(0) = 0 \text{ if } \Phi_n \text{ is odd symmetric}$$

or

$$\frac{d\theta_n}{d\theta}(0) = 0 \text{ if } \Phi_n \text{ is even symmetric .}$$

Comparison with theorem 2.5 shows that each eigenfunction $v(\theta,\varphi)$ of the Beltrami problem is a (possibly infinite) linear combination of strongly separable eigenfunctions $\theta_n^*\Phi_n(\varphi)$. We shall show in two independent ways that these sums are necessarily finite.

(i) For each n, for which $\theta_n^* \not\equiv 0$, the expression $\theta_n^*\Phi_n(\varphi)$ is a non-trivial eigenfunction of the Beltrami problem corresponding to the eigenvalue μ^*. Theorem 2.1 states that each eigenvalue μ^* has finite multiplicity, however.

(ii) From lemma 2.6 it follows that if $\theta_n^* \not\equiv 0$ then the corresponding λ_n satisfies $0 < \lambda_n < \mu^*$. Since ∞ is the only accumulation point of the sequence $\lambda_0, \lambda_1, \ldots$ only a finite number of θ_n^* are not identically zero.

Let now μ^* have multiplicity M with independent eigenfunctions

$$v_1(\theta,\varphi), v_2(\theta,\varphi), \ldots, v_M(\theta,\varphi) \ .$$

Let $\Phi_j\theta_j$, $j = 1,\ldots,N$ be strongly separable Beltrami solutions, occurring with nonzero coefficient in at least one of the Fourier expansions of $v_1,\ldots,v_M$. Then these functions, which by theorem 2.11 are mutually orthogonal, span an N-dimensional eigenspace corresponding to μ^* which contains the space spanned by $v_1,\ldots,v_M$. Hence N = M. □

COROLLARY 2.13. The strongly separable eigenfunctions of the Beltrami problem span the same space as the collection of all eigenfunctions of the Beltrami problem. □

Consequently, when in the future we consider eigenfunctions of the Beltrami problem we shall restrict ourselves, without loss of generality, to the strongly separable eigenfunctions of the Beltrami problem.

2.4. *Appendix*

2.4.0. *Introductory remarks*

Let throughout this section

$$x := (x_1,x_2), \ |x| = \sqrt{x_1^2 + x_2^2} \ ,$$

$$B := \{x \mid |x| < 1\} \ ,$$

$$\dot{B} := \{x \mid |x| = 1\} \ ,$$

$$\bar{B} = B \cup \dot{B} \ ,$$

$$f(x) \in C^0(\bar{B}) \wedge f(x) \neq 0 \ .$$

DEFINITION 2.14. We note the set of all square Lebesgue integrable complex valued functions on B as the space $L_2(B)$, which will be considered as a Hilbert space with the inner product defined by

$$(u,v) := \int_B f^2(x)\,u(x)\,\bar{v}(x)\,dx$$

and the norm by

$$\|u\|^2 := (u,u) = \int_B f^2(x)\,u(x)\,\bar{u}(x)\,dx \; . \qquad \square$$

DEFINITION 2.15. $L_2^*(B) := \{u(x) \mid u(x) \in L_2(B) \wedge (u,1) = 0\}$. □

LEMMA 2.16. With the inner product and the norm such as defined in definition 2.14, $L_2^*(B)$ is a separable complex Hilbert space [5;27]. □

Now we consider the two-dimensional eigenvalue problems

$$f^{-2}\Delta u + \lambda u = 0,\; x \in B,\; u \in C^0(\bar{B}),\; u \in C^2(B),\; u \not\equiv 0$$

with the boundary condition, either

$$u = 0,\; x \in \dot{B} \text{ (Dirichlet condition)}$$

or

$$\frac{\partial u}{\partial n} = 0,\; x \in \dot{B} \text{ (Neumann condition)} \; .$$

Δ is the two-dimensional Laplace operator $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. Here n is the outward normal.

LEMMA 2.17. Let $u \in C^1(\bar{B})$, $u \in C^2(B)$ then

$$\int_B \Delta u \; dx = \int_{\dot{B}} \frac{\partial u}{\partial n}\, ds \; ,$$

where ds is the element of the arc of $\dot{B}$. □

LEMMA 2.18. Let $u \in C^0(\bar{B})$, $u \in C^1(B)$ and $v \in C^1(\bar{B})$, $v \in C^2(B)$ then

$$\int_B (u\Delta v + (\mathrm{grad}(u),\mathrm{grad}(v)))dx = \int_{\dot{B}} u \frac{\partial v}{\partial n} ds \ .$$

This is Green's first identity. □

LEMMA 2.19. Let $u \in C^1(\bar{B})$, $u \in C^2(B)$, $v \in C^1(\bar{B})$, $v \in C^2(B)$ then

$$\int_B (u\Delta v - v\Delta u)\, dx = \int_{\dot{B}} (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}) ds \ .$$

This is Green's second identity. □

DEFINITION 2.20. Let $S(x;y)$ be a fundamental solution of the Laplace equation with unit source at y, then

$$-\Delta_x S(x;y) = \delta(x - y) \ .$$ □

LEMMA 2.21. Let $u \in C^1(\bar{B})$, $u \in C^2(B)$, $y \in B$ and $S(x;y)$ be a fundamental solution of the Laplace equation with unit source at y, then

$$u(y) = - \int_B S(x;y)\Delta u(x)\, dx + \int_{\dot{B}} (S(x;y)\frac{\partial u(x)}{\partial n} - u(x)\frac{\partial S(x;y)}{\partial n}) ds_x \ .$$

This is Green's third identity. □

LEMMA 2.22. Let $u \in C^1(\bar{B})$, $u \in C^2(B)$, $u \not\equiv 0$ be a solution of Dirichlet's eigenvalue problem

$$f^{-2}\Delta u + \lambda u = 0$$

with the boundary condition

$$u = 0, \ x \in \dot{B} \ ,$$

then

$$\lambda = \frac{\int_B (\mathrm{grad}(u),\mathrm{grad}(u))\, dx}{\| u \|^2} > 0$$ □

LEMMA 2.23. Let $u \in C^1(\bar{B})$, $u \in C^2(B)$, $u \not\equiv 0$ be a solution of Neumann's eigenvalue problem

$$f^{-2}\Delta u + \lambda u = 0$$

with the boundary condition

$$\frac{\partial u}{\partial n} = 0, \ x \in \dot{B} ,$$

then

$$\lambda = \frac{\int_B (\mathrm{grad}(u), \mathrm{grad}(u))\, dx}{\| u \|^2} .$$

Hence, if u = constant, then $\lambda = 0$, else $\lambda > 0$. □

The last two lemmas are a consequence of Green's first identity.

2.4.1. *Dirichlet's eigenvalue problem*

DEFINITION 2.24. Green's function $G(x;y)$, $x \in \bar{B}$, $y \in B$ is defined as follows:

(1) $G(x;y)$ is a fundamental solution of the Laplace equation with unit source at y.
(2) $G(x;y) = 0$, $x \in \dot{B}$. □

LEMMA 2.25. For the domain B, Green's function is given by

$$G(x;y) = -\frac{1}{2\pi}\log(|x - y|) + \frac{1}{2\pi}\log(|y|\cdot|x - y^*|)$$

where $x \in \bar{B}$, $y \in B$ and $y^* := \frac{1}{|y|^2}\, y$. □

We now consider Dirichlet's eigenvalue problem

$$f^{-2}\Delta u + \lambda u = 0, \ u \in C^0(\bar{B}), \ u \in C^2(B)$$

with the boundary condition $u = 0$, $x \in \dot{B}$. With the aid of Green's third identity and the property of symmetry of $G(x;y)$ we obtain

$$u(x) = \lambda \int_B G(x;y) f^2(y) u(y)\, dy, \ x \in B .$$

This representation also applies if $u \in C^0(\bar{B})$, $u \in C^2(B)$ [4;225]. Let $\lambda = \frac{1}{\mu}$, then

$$\int_B G(x;y)f^2(y)u(y)\,dy = \mu u(x),\ x \in B\ .$$

Let now $u \in L_2(B)$, then, with the aid of Weyl's lemma [4;225-226,199], the following equivalence theorem holds.

THEOREM 2.26. Dirichlet's eigenvalue problem

$$f^{-2}\Delta u + \lambda u = 0,\ x \in B,\ u \in C^0(\bar{B}),\ u \in C^2(B),\ f \in C^0(\bar{B}),\ u \not\equiv 0 \wedge f \not\equiv 0$$

with the boundary condition $u = 0$, $x \in \dot{B}$, is equivalent to the eigenvalue problem

$$\int_B G(x;y)f^2(y)u(y)\,dy = \mu u(x),\ x \in B,\ \mu = \frac{1}{\lambda},\ u \in L_2(B),\ u \not\equiv 0\ .\ \square$$

For the sake of convenience we shall write this eigenvalue problem in the operator notation

$$Tu = \mu u,\ u \in L_2(B)$$

in which the integral operator T is defined by

$$(Tu)(x) := \int_B G(x;y)f^2(y)u(y)\,dy\ .$$

LEMMA 2.27. The integral operator T is a linear, Hermitian, compact operator; Hermitian with relation to the inner product from definition 2.14.

PROOF. Linearity is trivial. Because $G(x;y) = G(y;x)$,

$$(Tu,v) = (u,Tv) = \int_B \int_B f^2(x)f^2(y)G(x;y)u(y)v(x)\,dxdy\ .$$

Moreover $f(x)G(x;y)f(y)$ is square integrable over $B \times B$, which means

$$\int_B \int_B f^2(x)G^2(x;y)f^2(y)\,dxdy < \infty\ .$$

So T is a Hilbert-Schmidt operator [5;86,182], hence compact. $\square$

With the aid of the spectral theorem of compact Hermitian operators [5;202], and from lemma 2.27 and theorem 2.26 we obtain

THEOREM 2.28. Dirichlet's eigenvalue problem

$$f^{-2}\Delta u + \lambda u = 0,\ x \in B,\ u \in C^0(\bar{B}),\ u \in C^2(B),\ u \not\equiv 0$$

with the boundary conditions $u = 0$, $x \in \dot{B}$ has denumerably many positive eigenvalues λ_j, $j = 1,2,\ldots$ with corresponding orthonormal eigenfunctions $u_j \in C^0(\bar{B})$, $u_j \in C^2(B)$. □

2.4.2. *Neumann's eigenvalue problem*

To solve Neumann's eigenvalue problem we shall introduce two Neumann functions, namely $N(x;y)$ and $N^*(x;y)$.

DEFINITION 2.29. Neumann's function $N(x;y)$, $x \in \bar{B}$, $y \in B$ is defined as follows:

(1) $N(x;y)$ is, as a function of x, a fundamental solution of the Laplace equation with the unit source at y.

(2) $$\frac{\partial N}{\partial n_x}(x;y) = -\frac{1}{2\pi},\ x \in \dot{B}.$$

(3) $$\int_{\dot{B}} N(x;y)\,ds_x = 0.$$ □

LEMMA 2.30. Neumann's function $N(x;y)$ for the domain B is given by

$$N(x;y) = -\frac{1}{2\pi}\log(|x-y|) - \frac{1}{2\pi}\log(|y|\cdot|x-y^*|)$$

where $x \in \bar{B}$, $y \in B$ and $y^* := \frac{1}{|y|^2}\,y$. □

DEFINITION 2.31. Neumann's function $N^*(x;y)$, $x \in \bar{B}$, $y \in B$ is defined as follows:

(1) $N^*(x;y)$ is, as a function of x, solution of

$$\Delta_x N^*(x;y) = -\delta(x-y) + \frac{f^2(x)}{\|1\|^2}.$$

(2) $$\frac{\partial N^*}{\partial n_x}(x;y) = 0,\ x \in \dot{B} .$$

(3) $$\int_{\dot{B}} N^*(x;y)ds_x = 0 .$$ □

Since the Neumann function as defined above is perhaps not conventional, the following explanation may be given.
With the aid of lemma 2.17 we obtain

$$\int_{\dot{B}} \frac{\partial N^*}{\partial n_x}(x;y)ds_x = \int_B \Delta_x N^*(x;y)dx = \int_B \frac{f^2(x)}{\|1\|^2}\,dx - 1 = 0$$

and therefore it is possible to postulate $\frac{\partial N^*}{\partial n}(x;y) = 0,\ x \in \dot{B}$. We observe that $N^*(x;y)$ is determined uniquely but for a solution of a Neumann problem. For convenience we suppose that $N^{**}(x;y)$ satisfies the first two conditions of definition 2.31. Then

$$N^*(x;y) = N^{**}(x;y) + g(x;y)$$

where $g(x;y)$ is a solution of Neumann's problem

$$\Delta g(x;y) = 0,\ x \in B$$

with the boundary condition

$$\frac{\partial g}{\partial n}(x;y) = 0,\ x \in \dot{B} .$$

For fixed y, this problem has as solution $g(x;y) = c(y)$. This constant is defined by the third condition of definition 2.31. Hence $N^*(x;y)$ is uniquely determined.

LEMMA 2.32. Neumann's function $N^*(x;y)$ for the domain B is given by

$$N^*(x;y) = N(x;y) - \int_B N(x;\tilde{y})\frac{f^2(\tilde{y})}{\|1\|^2}\,d\tilde{y} .$$ □

Now we consider Neumann's eigenvalue problem

$$f^{-2}\Delta u + \lambda u = 0,\ u \in C^0(\bar{B}),\ u \in C^2(B),\ u \not\equiv 0$$

with the boundary condition

$$\frac{\partial u}{\partial n} = 0,\ x \in \dot{B} .$$

With the aid of Green's second identity we obtain

$$u(x) = \lambda \int_B N^*(x;y) f^2(y) u(y) dy + \int_B \frac{f^2(y)}{\|1\|^2} u(y) dy .$$

From lemma 2.17

$$\lambda \int_B f^2(y) u(y) dy = 0 .$$

We shall now exclude $\lambda = 0$ with the corresponding eigenfunction $u \equiv 1$. Hence

$$\int_B f^2(y) u(y) dy = 0$$

and it follows that

$$u(x) = \lambda \int_B N^*(x;y) f^2(y) u(y) dy .$$

THEOREM 2.33. The "restricted" Neumann eigenvalue problem

$$f^{-2}\Delta u + \lambda u = 0 , \quad x \in B, \ u \in C^1(\bar{B}), \ u \in C^2(B), \ u \not\equiv \text{constant}$$

with the boundary condition $\frac{\partial u}{\partial n} = 0$, $x \in \dot{B}$ is equivalent to the eigenvalue problem

$$\int_B N^*(x;y) f^2(y) u(y) dy = \frac{1}{\mu} u, \ x \in B$$

$$\mu = \frac{1}{\lambda} , \ u \in L_2^*(B) , \ u \not\equiv 0 .$$

For simplicity we write this eigenvalue problem in operator notation

$$Su = \mu u, \ u \in L_2^*(B)$$

in which the integral operator S is defined by

$$(Su)(x) := \int_B N^*(x;y) f^2(y) u(y) dy .$$

LEMMA 2.34. The integral operator S is a linear, Hermitian, compact operator.

PROOF. Analogous to that of lemma 2.27. □

Using the spectral theorem of compact Hermitian operators we infer:

THEOREM 2.35. The "restricted" Neumann eigenvalue problem

$$f^{-2}\Delta u + \lambda u = 0,\ x \in B,\ u \in C^1(\bar{B}),\ u \in C^2(B),\ u \not\equiv \text{constant}$$

with the boundary condition $\frac{\partial u}{\partial n} = 0$, $x \in \dot{B}$, has denumerably many positive eigenvalues λ_j, $j = 1,2,\ldots$ with corresponding orthonormal eigenfunctions $u_j \in C^1(\bar{B})$, $u_j \in C^2(B)$. □

2.5. *References*

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CHAPTER 3

LAMÉ EQUATIONS

3.0. *Introduction*

This chapter is most important to our whole work. In fact, we here show that the solutions of the φ Lamé equation are related to those of the θ Lamé equation.

Results about solutions of the φ Lamé equation are known since a long time (Ince, [4], [5]). For the θ Lamé equation very little has been published, however.

Sometimes Levine is quoted as to have obtained various results on the solutions of the θ Lamé equation, but his report [8] although announced (in [7]) has not appeared.

Kong [6] in his doctoral thesis says that for lack of a better method of computing the θ Lamé solution he used a numerical approach, viz., a 4th-order Runge-Kutta method.

In 1948 Erdélyi investigated the φ Lamé solutions by representing them by a series, infinite in general, of associated Legendre functions of the first kind [3].

In 1956 Sleeman expressed the Lamé solutions associated with the corresponding Lamé polynomials by means of series of associated Legendre functions of the second kind [11].

These results for the φ case have led us to the idea of representing the solutions of the θ Lamé equation in terms of series of Legendre functions. The result obtained in this way, viz., the connection between the coefficients of the solution of the φ equation and those of the solution of the corresponding θ equation seems to be new (but similar to existing results for the periodic and non-periodic solutions of the Mathieu equation [9]).

3.1. *Solutions of the φ Lamé equation*

In the previous chapter we were led to Lamé's equation

$$\text{(3.1)}\qquad \sqrt{1-k'^2\sin^2(\varphi)}\frac{d}{d\varphi}\left(\sqrt{1-k'^2\sin^2(\varphi)}\frac{d\Phi}{d\varphi}\right) + (\lambda - \nu(\nu+1)k'^2\sin^2(\varphi))\Phi = 0$$

$$0 < k' < 1,\ 0 \le \varphi < 2\pi ,$$

with periodicity condition $\Phi(\varphi) = \Phi(\varphi+2\pi)$ for the φ solutions. Here, ν is a fixed parameter and λ is the eigenvalue. Let $(\lambda,\Phi(\varphi))$ be a solution of this eigenvalue problem. In this section we shall show that to each such eigenfunction $\Phi(\varphi)$ there corresponds uniquely an eigenvector u of a certain infinite tridiagonal matrix, corresponding to the same eigenvalue λ.

In the previous chapter we proved that we can restrict ourselves, without loss of generality, to 2π-periodic eigenfunctions $\Phi(\varphi)$ satisfying

(i) either $\Phi(\pi+\varphi) = \Phi(\varphi)$ or $\Phi(\pi+\varphi) = -\Phi(\varphi)$

(ii) either $\Phi(\pi-\varphi) = \Phi(\varphi)$ or $\Phi(\pi-\varphi) = -\Phi(\varphi)$.

Starting from these properties we can divide the eigenfunctions into four classes and we may expand these eigenfunctions into trigonometric Fourier series, namely

I: $$L_{c\nu}^{(2n)}(\varphi) := \sum_{r=0}^{\infty} A_{2r}^{(2n)}\cos(2r\varphi),\ n = 0,1,2,\ldots$$

$$\{\Phi(\varphi) = \Phi(\pi+\varphi) = \Phi(\pi-\varphi)\} .$$

II: $$L_{c\nu}^{(2n+1)}(\varphi) := \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}\cos((2r+1)\varphi),\ n = 0,1,2,\ldots$$

$$\{\Phi(\varphi) = -\Phi(\pi+\varphi) = -\Phi(\pi-\varphi)\} .$$

III: $$L_{s\nu}^{(2n)}(\varphi) := \sum_{r=1}^{\infty} B_{2r}^{(2n)}\sin(2r\varphi),\ n = 1,2,3,\ldots$$

$$\{\Phi(\varphi) = \Phi(\pi+\varphi) = -\Phi(\pi-\varphi)\} .$$

IV: $$L_{s\nu}^{(2n+1)}(\varphi) := \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}\sin((2r+1)\varphi),\ n = 0,1,2,\ldots$$

$$\{\Phi(\varphi) = -\Phi(\pi+\varphi) = \Phi(\pi-\varphi)\} .$$

We remark that the upper indices of the four classes of periodic Lamé solutions are related to the ordinal numbers of the corresponding eigenvalues. Substitution of these formal series into the differential equation (3.1) shows that the coefficients must satisfy the following recurrence relations in which $\Lambda := \lambda - \frac{1}{2}\nu(\nu+1)k'^2$ (for the sake of convenience we have omitted the upper index)

I:
$$-\Lambda A_0 - \frac{k'^2}{4}(\nu-1)(\nu+2)A_2 = 0\ ,$$

$$-\frac{k'^2}{2}\nu(\nu+1)A_0 + [4(1-\tfrac{1}{2}k'^2) - \Lambda]A_2 - \frac{k'^2}{4}(\nu-3)(\nu+4)A_4 = 0\ ,$$

$$-\frac{k'^2}{4}(\nu-2r+2)(\nu+2r-1)A_{2r-2} + [(2r)^2(1-\tfrac{1}{2}k'^2) - \Lambda]A_{2r} -$$

$$-\frac{k'^2}{4}(\nu-2r-1)(\nu+2r+2)A_{2r+2} = 0,\ r = 2,3,\ldots\ .$$

II:
$$[(1-\tfrac{1}{2}k'^2) - \frac{\nu(\nu+1)k'^2}{4} - \Lambda]A_1 - \frac{k'^2}{4}(\nu-2)(\nu+3)A_3 = 0\ ,$$

$$-\frac{k'^2}{4}(\nu-2r+1)(\nu+2r)A_{2r-1} + [(2r+1)^2(1-\tfrac{1}{2}k'^2) - \Lambda]A_{2r+1} -$$

$$-\frac{k'^2}{4}(\nu-2r-2)(\nu+2r+3)A_{2r+3} = 0,\ r = 1,2,\ldots\ .$$

III:
$$[4(1-\tfrac{1}{2}k'^2) - \Lambda]B_2 - \frac{k'^2}{4}(\nu-3)(\nu+4)B_4 = 0\ ,$$

$$-\frac{k'^2}{4}(\nu-2r+2)(\nu+2r-1)B_{2r-2} + [(2r)^2(1-\tfrac{1}{2}k'^2) - \Lambda]B_{2r} -$$

$$-\frac{k'^2}{4}(\nu-2r-1)(\nu+2r+2)B_{2r+2} = 0,\ r = 2,3,\ldots\ .$$

IV:
$$[(1-\tfrac{1}{2}k'^2) + \nu(\nu+1)\frac{k'^2}{4} - \Lambda]B_1 - \frac{k'^2}{4}(\nu-2)(\nu+3)B_3 = 0\ ,$$

$$-\frac{k'^2}{4}(\nu-2r+1)(\nu+2r)B_{2r-1} + [(2r+1)^2(1-\tfrac{1}{2}k'^2) - \Lambda]B_{2r+1} -$$

$$-\frac{k'^2}{4}(\nu-2r-2)(\nu+2r+3)B_{2r+3} = 0,\ r = 1,2,3,..$$

We remark that all coefficients satisfy the homogeneous three-term recurrence relation of the type

$$b_r y_{2r-2} + a_r y_{2r} + c_r y_{2r+2} = 0$$

with

$$\lim_{r\to\infty} \frac{b_r}{c_r} = 1 \quad \text{and} \quad \lim_{r\to\infty} \frac{a_r}{c_r} = 2\,\frac{1+k^2}{1-k^2} .$$

By virtue of Perron's theorem (see theorem 3.7) the three-term recurrence relation has two linearly independent solutions $\{y^{(1)}\}_{2r}$ and $\{y^{(2)}\}_{2r}$, say, which satisfy

$$\lim_{r\to\infty} \frac{y^{(1)}_{2r+2}}{y^{(1)}_{2r}} = \frac{k-1}{k+1} \quad \text{and} \quad \lim_{r\to\infty} \frac{y^{(2)}_{2r+2}}{y^{(2)}_{2r}} = \frac{k+1}{k-1} .$$

Hence, it is possible to choose the coefficients in such a way that

$$\lim_{r\to\infty} \frac{A^{(2n)}_{2r+2}}{A^{(2n)}_{2r}} = \lim_{r\to\infty} \frac{A^{(2n+1)}_{2r+1}}{A^{(2n+1)}_{2r-1}} = \lim_{r\to\infty} \frac{B^{(2n)}_{2r+2}}{B^{(2n)}_{2r}} = \lim_{r\to\infty} \frac{B^{(2n+1)}_{2r+1}}{B^{(2n+1)}_{2r-1}} = \frac{k-1}{k+1} .$$

We observe that only for special values of Λ, the so-called eigenvalues, do the above conditions hold, because the coefficients must also satisfy the initial two-term (recurrence) relation.
By this choice of the coefficients the trigonometric Fourier series with their derivatives converge uniformly on $[0,2\pi]$ and satisfy the differential equation (3.1). Summarizing we have the following theorem:

THEOREM 3.1. If the coefficients of the series for $L^{(2n)}_{c\nu}(\varphi)$, $L^{(2n+1)}_{c\nu}(\varphi)$, $L^{(2n)}_{s\nu}(\varphi)$ and $L^{(2n+1)}_{s\nu}(\varphi)$ satisfy

(i) the corresponding recurrence relations,

(ii) $$\lim_{r\to\infty} \frac{A^{(2n)}_{2r+2}}{A^{(2n)}_{2r}} = \lim_{r\to\infty} \frac{A^{(2n+1)}_{2r+1}}{A^{(2n+1)}_{2r-1}} = \lim_{r\to\infty} \frac{B^{(2n)}_{2r+2}}{B^{(2n)}_{2r}} = \lim_{r\to\infty} \frac{B^{(2n+1)}_{2r+1}}{B^{(2n+1)}_{2r-1}} = \frac{k-1}{k+1} ,$$

then these series and their derivatives converge uniformly in $[0,2\pi]$ and satisfy the differential equation (3.1). □

The relations I, II, III and IV can also be written in matrix notation, and then we obtain the following eigenvalue equations for infinite tridiagonal matrices.

I: $$\begin{bmatrix} a_0 & c_0 & & & \\ b_0 & a_2 & c_2 & & \\ & & & & \\ & b_{2r-2} & a_{2r} & c_{2r} & \\ & & & & \end{bmatrix} \begin{bmatrix} A_0 \\ A_2 \\ \vdots \\ A_{2r} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} A_0 \\ A_2 \\ \vdots \\ A_{2r} \\ \vdots \end{bmatrix},$$

II: $$\begin{bmatrix} a_1 & c_1 & & & \\ b_1 & a_3 & c_3 & & \\ & & & & \\ & b_{2r-1} & a_{2r+1} & c_{2r+1} & \\ & & & & \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ \vdots \\ A_{2r+1} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} A_1 \\ A_3 \\ \vdots \\ A_{2r+1} \\ \vdots \end{bmatrix},$$

III: $$\begin{bmatrix} \tilde{a}_2 & \tilde{c}_2 & & & \\ \tilde{b}_2 & \tilde{a}_4 & \tilde{c}_4 & & \\ & & & & \\ & \tilde{b}_{2r-2} & \tilde{a}_{2r} & \tilde{c}_{2r} & \\ & & & & \end{bmatrix} \begin{bmatrix} B_2 \\ B_4 \\ \vdots \\ B_{2r} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} B_2 \\ B_4 \\ \vdots \\ B_{2r} \\ \vdots \end{bmatrix},$$

IV: $$\begin{bmatrix} \tilde{a}_1 & \tilde{c}_1 & & & \\ \tilde{b}_1 & \tilde{a}_3 & \tilde{c}_3 & & \\ & & & & \\ & \tilde{b}_{2r-1} & \tilde{a}_{2r+1} & \tilde{c}_{2r+1} & \\ & & & & \end{bmatrix} \begin{bmatrix} B_1 \\ B_3 \\ \vdots \\ B_{2r+1} \\ \vdots \end{bmatrix} = \Lambda \begin{bmatrix} B_1 \\ B_3 \\ \vdots \\ B_{2r+1} \\ \vdots \end{bmatrix},$$

with elements

$a_0 := 0,\ a_1 := (1-\tfrac{1}{2}k'^2) - \nu(\nu+1)k'^2/4,\ \tilde{a}_1 := (1-\tfrac{1}{2}k'^2) + \nu(\nu+1)k'^2/4;$

$a_r = \tilde{a}_r := r^2(1-\tfrac{1}{2}k'^2),\ r = 2,3,\ldots;$

$b_0 := -\nu(\nu+1)k'^2/2;$

$b_r = \tilde{b}_r := -(\nu-r)(\nu+r+1)k'^2/4,\ r = 1,2,\ldots;$

$c_0 := -(\nu-1)(\nu+2)k'^2/4;$

$c_r = \tilde{c}_r := -(\nu-r-1)(\nu+r+2)k'^2/4,\ r = 1,2,\ldots\ .$

Analogously to the Mathieu function [9;188] we can normalize the eigenvectors so that

$$\frac{1}{\pi}\int_0^{2\pi} (L_{c\nu}^{(m)}(\varphi))^2 d\varphi = \frac{1}{\pi}\int_0^{2\pi} (L_{s\nu}^{(m)}(\varphi))^2 d\varphi = 1$$

and

$$\lim_{k'\downarrow 0} L_{c\nu}^{(m)}(\varphi) = \cos(m\varphi), \lim_{k'\downarrow 0} L_{s\nu}^{(m)}(\varphi) = \sin(m\varphi) .$$

Hence,

$$2(A_0^{(2n)})^2 + \sum_{r=1}^{\infty} (A_{2r}^{(2n)})^2 = 1, \sum_{r=0}^{\infty} A_{2r}^{(2n)} > 0 ,$$

$$\sum_{r=0}^{\infty} (A_{2r+1}^{(2n+1)})^2 = 1, \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} > 0 ,$$

$$\sum_{r=1}^{\infty} (B_{2r}^{(2n)})^2 = 1, \sum_{r=1}^{\infty} 2r\, B_{2r}^{(2n)} > 0 ,$$

$$\sum_{r=0}^{\infty} (B_{2r+1}^{(2n+1)})^2 = 1, \sum_{r=0}^{\infty} (2r+1)B_{2r+1}^{(2n+1)} > 0 .$$

In the next chapter we shall give algorithms for calculating the eigenvalues and the corresponding eigenvectors of these infinite tridiagonal matrices.

3.2. *Solutions of the* θ *Lamé equation*

As proved in the previous chapter, we can restrict ourselves to the strongly separable eigenfunctions $\Theta(\theta)\Phi(\varphi)$ of the Beltrami problem (see corollary 2.13).

We were led to the θ Lamé equation

$$(3.2) \qquad \sqrt{1-k^2\cos^2(\theta)}\frac{d}{d\theta}\left(\sqrt{1-k^2\cos^2(\theta)}\frac{d\Theta}{d\theta}\right) + (\mu^*(1 - k^2\cos^2(\theta)) - \lambda)\Theta = 0$$

with the left-hand boundary conditions

$$\frac{d\Theta}{d\theta}(0) = 0, \text{ if } \Phi(\pi-\varphi) = \Phi(\varphi) \text{ (i.e., } \Phi \text{ is even symmetric)}$$

and

$$\Theta(0) = 0, \text{ if } \Phi(\pi-\varphi) = -\Phi(\varphi) \text{ (i.e., } \Phi \text{ is odd symmetric) .}$$

The θ equation (3.2) can be obtained from the φ equation (3.1) by substitution of

$$\varphi = \arccos(\frac{ik \ \sin(\theta)}{k'}) = \frac{\pi}{2} + i\psi$$

with

$$\psi = \log\left[\frac{k'}{\sqrt{1 - k^2\cos^2(\theta)} + k \sin(\theta)}\right] .$$

For then

$$k'\cos(\varphi) = ik \sin(\theta)$$

from which follows

$$\frac{id\varphi}{\sqrt{1 - k'^2\sin^2(\varphi)}} = \frac{d\theta}{\sqrt{1 - k^2\cos^2(\theta)}} .$$

Hence from the φ solutions of the Lamé equation (3.1) we can obtain formal θ solutions of the Lamé equation (3.2) corresponding to the four classes of the periodic Lamé solutions:

I: $$L_{C\nu}^{(2n)}(\theta) := \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \cosh(2r\psi) , \quad n = 0,1,2,\dots .$$

II: $$L_{S\nu}^{(2n+1)}(\theta) := \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} \sinh((2r+1)\psi) , \quad n = 0,1,2,\dots .$$

III: $$L_{S\nu}^{(2n)}(\theta) := \sum_{r=0}^{\infty} (-1)^r B_{2r}^{(2n)} \sinh(2r\psi) , \quad n = 1,2,\dots .$$

IV: $$L_{C\nu}^{(2n+1)}(\theta) := \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \cosh((2r+1)\psi) , \quad n = 0,1,2,\dots .$$

and

$$\psi := \log\left[\frac{k'}{\sqrt{1 - k^2\cos^2(\theta)} + k \sin(\theta)}\right] .$$

These solutions satisfy the respective left-hand boundary conditions.

THEOREM 3.2. The series for $L_{C\nu}^{(2n)}(\theta)$, $L_{S\nu}^{(2n+1)}(\theta)$, $L_{S\nu}^{(2n)}(\theta)$ and $L_{C\nu}^{(2n+1)}(\theta)$ converge uniformly on any closed subinterval $[0,\theta_0]$ of $[0,\pi/2]$ and satisfy the differential equation (3.2).

PROOF. With the aid of theorem (3.1) it follows that

$$\lim_{m\to\infty} \frac{(-1)^{m+1}A_{2m+2}^{(2n)}\cosh((2m+2)\psi)}{(-1)^m A_{2m}^{(2n)}\cosh(2m\psi)} = \frac{1-k}{1+k}\,e^{2|\psi|} =$$

$$= \frac{1-k}{1+k}\left(\frac{\sqrt{1-k^2\cos^2(\theta)}+k\sin(\theta)}{k'}\right)^2 .$$

In the other three cases we obtain the same limit; this limit is a monotonically increasing function of θ, $\theta \in [0,\pi/2)$ with range $[\frac{1-k}{1+k}, 1)$. □

The series converge very slowly for θ near $\pi/2$. For that reason we now try to find series that converge faster.
If $k = 1$, the θ Lamé equation (3.2) reduces to the differential equation of Legendre. Consequently, the solutions of class I transform into $P_\nu^{2n}(\cos(\theta))$ and those of class II into $P_\nu^{2n+1}(\cos(\theta))$, $n = 0,1,2,\ldots$.
Our conjecture is now that, if $k < 1$, the non-periodic solutions of class I can be written as:

$$\Theta(\theta) = \sum_{m=0}^{\infty} c_{2m} P_\nu^{2m}(\cos(\theta))$$

and, similarly for class II

$$\Theta(\theta) = \sum_{m=0}^{\infty} c_{2m+1} P_\nu^{2m+1}(\cos(\theta)) .$$

From the properties of the Legendre functions we find that the left-hand boundary conditions are satisfied.
Just as with the periodic solutions, we substitute

$$\Theta(\theta) = \sum_{m=0}^{\infty} c_m P_\nu^{m}(\cos(\theta))$$

in the Lamé equation (3.2).
Using lemma 3.15 and Legendre's differential equation we obtain the following general three-term recurrence relation:

$$(3.3)\qquad \frac{k'^2}{4}c_{m-2} + (m^2(1-\tfrac{1}{2}k'^2) - \Lambda)c_m +$$

$$+ \frac{k'^2}{4}(\nu-m)(\nu+m+1)(\nu-m-1)(\nu+m+2)c_{m+2} = 0,\ m \ge 3$$

with $\Lambda = \lambda - \frac{1}{2}\nu(\nu+1)k'^2$.

If $m = 0$, we obtain the two-term relation

$$-\Lambda c_0 + \frac{k'^2}{4}\nu(\nu+1)(\nu-1)(\nu+2)c_2 = 0 \ .$$

Using the relation

$$P_\nu^1(\cos(\theta)) = -\nu(\nu+1)P_\nu^{-1}(\cos(\theta)) \qquad [2;144]$$

we obtain for $m = 1$

$$((1-\tfrac{1}{2}k'^2) - \frac{k'^2}{4}\nu(\nu+1) - \Lambda)c_1 + \frac{k'^2}{4}(\nu-1)(\nu+2)(\nu-2)(\nu+3)c_3 = 0.$$

If $m = 2$, we make use of the relation

$$P_\nu^2(\cos(\theta)) = \nu(\nu-1)(\nu+1)(\nu+2)P_\nu^{-2}(\cos(\theta)) \qquad [2;144]$$

and then we obtain

$$\frac{k'^2}{2}c_0 + (4(1-\tfrac{1}{2}k'^2) - \Lambda)c_2 + \frac{k'^2}{4}(\nu-3)(\nu-2)(\nu+3)(\nu+4)c_4 = 0 \ .$$

We now observe that these recurrence relations can be transformed into the recurrence relations of the periodic Lamé solutions, by substitution

$$c_m = T(m)A_m$$

where $T(m)$ has to satisfy the recurrence relation

$$T(m) = -(\nu-m)(\nu+m+1)T(m+2), \ m = 0,1,2,\dots \ .$$

$T(m)$ is defined unequivocally except for an arbitrary multiplicative constant. For instance we can take

$$T(m) := \frac{2^{-m}\Gamma(\frac{\nu+1}{2})\Gamma(-\frac{\nu}{2})}{\Gamma(\frac{\nu+m+1}{2})\Gamma(\frac{m-\nu}{2})} \ .$$

Conversely, if $\{A_{2m}\}_{m=0}^{\infty}$ and $\{A_{2m+1}\}_{m=0}^{\infty}$ are the solutions of the recurrence relations corresponding to the classes I and II of the periodic Lamé solutions then

$$\sum_{m=0}^{\infty} T(2m)A_{2m}P_\nu^{2m}(\cos(\theta))$$

and

$$\sum_{m=0}^{\infty} T(2m + 1)A_{2m+1}P_{\nu}^{2m+1}(\cos(\theta))$$

formally satisfy the θ Lamé equation (3.2) and the corresponding left-hand boundary conditions.

Before investigating the convergence properties of these series we shall first construct the θ Lamé solutions corresponding to the classes III and IV of the φ Lamé equations. Again, if $k = 1$, the solutions of class III reduce to $P_{\nu}^{2m}(\cos(\theta))$, $m = 1,2,\ldots$ and the same applies to class IV, namely $P_{\nu}^{2m}(\cos(\theta))$, $m = 0,1,2,\ldots$.

However, simple series of Legendre functions alone will not do in this case. It is perhaps more natural to expect that, if $k < 1$, the non-periodic solutions of the classes III and IV can be written as series of Legendre functions multiplied by an odd function $f(\theta;k)$:

$$\text{III:} \qquad \Theta(\theta) := f(\theta;k) \sum_{m=1}^{\infty} d_{2m}P_{\nu}^{2m}(\cos(\theta))$$

and

$$\text{IV:} \qquad \Theta(\theta) := f(\theta;k) \sum_{m=0}^{\infty} d_{2m+1}P_{\nu}^{2m+1}(\cos(\theta)) \ .$$

For $\theta \to 0$ we have $P_{\nu}^{2m}(\cos(\theta)) \sim (\sin(\theta))^{2m}$ (viz., lemma 3.12). Since we know that for class III $\Theta(0) = 0$ and $\frac{d\Theta(0)}{d\theta} \neq 0$ it follows that we must require

$$f(\theta;k) \sim \theta^{-1} \text{ as } \theta \to 0 \ .$$

From the left-hand boundary condition of class IV we obtain the same asymptotic condition for $f(\theta;k)$. In the degenerated case we must require

$$f(\theta;1) = 1 \ .$$

An appropriate choice of $f(\theta;k)$ which satisfies these requirements is

$$f(\theta;k) := \frac{\sqrt{1 - k^2\cos^2(\theta)}}{\sin(\theta)} \ .$$

Now we substitute

$$\Theta(\theta) = \frac{\sqrt{1 - k^2\cos^2(\theta)}}{\sin(\theta)}\,\tilde{\Theta}(\theta)$$

in the equation (3.2) and it follows that $\tilde{\Theta}$ satisfies the equation

$$\sqrt{1-k^2\cos^2(\theta)}\frac{d}{d\theta}\left(\sqrt{1-k^2\cos^2(\theta)}\frac{d\tilde{\Theta}}{d\theta}\right) + (\mu^*(1 - k^2\cos^2(\theta)) - \lambda)\tilde{\Theta} -$$

$$- 2k'^2\cot(\theta)\frac{d\tilde{\Theta}}{d\theta} + \frac{k'^2(1 + \cos^2(\theta))}{\sin^2(\theta)}\,\tilde{\Theta} = 0$$

and we observe that, if $k = 1$, this equation reduces to the equation of Legendre. Just as in the previous cases we substitute

$$\tilde{\Theta}(\theta) = \sum_{m=1}^{\infty} d_m P_\nu^m(\cos(\theta))$$

in the above-mentioned equation.
With the aid of lemma 3.16 and the differential equation of Legendre we obtain the general three-term recurrence relation

$$\frac{k'^2}{4}\,\frac{m}{m-2}\,d_{m-2} + (m^2(1-\tfrac{1}{2}k'^2) - \Lambda)d_m +$$

$$+\frac{k'^2}{4}\,\frac{m}{m+2}(\nu-m)(\nu+m+1)(\nu-m-1)(\nu+m+2)d_{m+2} = 0,\; m \geq 3.$$

If $m = 2$, we obtain the two-term relation

$$(4(1-\tfrac{1}{2}k'^2) - \Lambda)d_2 + \frac{k'^2}{4}\,\frac{(\nu-3)(\nu+4)(\nu-2)(\nu+3)}{2}\,d_4 = 0\ .$$

Using the relation

$$P_\nu^1(\cos(\theta)) = -\nu(\nu+1)P_\nu^{-1}(\cos(\theta)) \qquad [2;144]$$

we obtain for $m = 1$

$$((1-\tfrac{1}{2}k'^2) + \frac{k'^2}{4}\,\nu(\nu+1) - \Lambda)d_1 + \frac{k'^2}{4}\,\frac{(\nu-2)(\nu+3)(\nu-1)(\nu+2)}{3}\,d_3 = 0.$$

Again, we observe that these recurrence relations can be transformed into the recurrence relations of the periodic Lamé solutions in this way by substitution

$$d_m = mT(m)B_m\ .$$

Conversely, if $\{B_{2m}\}_{m=1}^{\infty}$ and $\{B_{2m+1}\}_{m=0}^{\infty}$ are the solutions of the recurrence relations corresponding to the classes III and IV of the periodic Lamé solutions then

$$\frac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)} \sum_{m=1}^{\infty} (2m)T(2m)B_{2m}P_{\nu}^{2m}(\cos(\theta))$$

and

$$\frac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)} \sum_{m=0}^{\infty} (2m+1)T(2m+1)B_{2m+1}P_{\nu}^{2m+1}(\cos(\theta))$$

formally satisfy the θ Lamé equation (3.2) and the corresponding left-hand boundary conditions.

Consequently, corresponding to the four classes of the periodic Lamé solutions, we have now four classes of formal θ solutions of the Lamé equation (3.2):

$$\text{I:} \qquad L_{cp\nu}^{(2n)}(\theta) := \sum_{m=0}^{\infty} T(2m)A_{2m}^{(2n)}P_{\nu}^{2m}(\cos(\theta)),\ n = 0,1,2,\ldots$$

$$\text{II:} \qquad L_{cp\nu}^{(2n+1)}(\theta) := \sum_{m=0}^{\infty} T(2m+1)A_{2m+1}^{(2n+1)}P_{\nu}^{2m+1}(\cos(\theta)),\ n = 0,1,2,\ldots$$

$$\text{III:} \qquad L_{sp\nu}^{(2n)}(\theta) := \frac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)} \sum_{m=1}^{\infty} (2m)T(2m)B_{2m}^{(2n)}P_{\nu}^{2m}(\cos(\theta)),\ n = 1,2,\ldots$$

$$\text{IV:} \qquad L_{sp\nu}^{(2n+1)}(\theta) := \frac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)} \sum_{m=0}^{\infty} (2m+1)T(2m+1)B_{2m+1}^{(2n+1)} \cdot P_{\nu}^{2m+1}(\cos(\theta)),\ n = 0,1,2,\ldots .$$

Since a solution of the θ Lamé equation is, up to a multiplicative constant, uniquely defined by the eigenvalue and the initial condition at $\theta = 0$, these solutions are the same as those derived before, but for a normalization factor.

We remark that all the recurrence relations of $P_{\nu}^{m}(\cos(\theta))$, $m = 0,1,2,\ldots$, $\nu > 0$ we have used, also hold for the associated Legendre functions of the second kind $Q_{\nu}^{m}(\cos(\theta))$, $m = 0,1,2,\ldots$, taking into account the essential restriction $\nu > 0$. Hence, we have another four classes of non-periodic solutions of the Lamé's equation, corresponding to the four classes of periodic solutions, namely

I: $$L^{(2n)}_{cq\nu}(\theta) := \sum_{m=0}^{\infty} T(2m)A^{(2n)}_{2m}Q^{2m}_{\nu}(\cos(\theta)),\ n = 0,1,2,\ldots$$

II: $$L^{(2n+1)}_{cq\nu}(\theta) := \sum_{m=0}^{\infty} T(2m+1)A^{(2n+1)}_{2m+1}Q^{2m+1}_{\nu}(\cos(\theta)),\ n = 0,1,2,\ldots$$

III: $$L^{(2n)}_{sq\nu}(\theta) := \frac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)} \sum_{m=1}^{\infty} (2m)T(2m)B^{(2n)}_{2m}Q^{2m}_{\nu}(\cos(\theta)),\quad n = 1,2,\ldots$$

IV: $$L^{(2n+1)}_{sq\nu}(\theta) := \frac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)} \sum_{m=0}^{\infty} (2m+1)T(2m+1)B^{(2n+1)}_{2m+1} \cdot Q^{2m+1}_{\nu}(\cos(\theta)),\ n = 0,1,2,\ldots .$$

These solutions are not bounded at $\theta = 0$. We observe that these solutions and the previous ones are pairwise independent solutions of the θ Lamé equation (3.2).

THEOREM 3.3. The series for $L^{(2n)}_{cp\nu}(\theta)$, $L^{(2n+1)}_{cp\nu}(\theta)$, $L^{(2n)}_{sp\nu}(\theta)$ and $L^{(2n+1)}_{sp\nu}(\theta)$ converge uniformly on any closed subinterval $[0,\theta_0]$ of $[0,2\arctan(\sqrt{\frac{1+k}{1-k}}))$ and satisfy the differential equation (3.2).

PROOF. With the aid of theorem 3.1 and lemma 3.18 it follows that

$$\lim_{m\to\infty} \frac{A^{(2n)}_{2m+2}T(2m+2)P^{2m+2}_{\nu}(\cos(\theta))}{A^{(2n)}_{2m}T(2m)P^{2m}_{\nu}(\cos(\theta))} = \frac{k-1}{k+1}\tan^2(\tfrac{1}{2}\theta) .$$

In the other three cases we obtain the same limit. □

THEOREM 3.4. The series for $L^{(2n)}_{cq\nu}(\theta)$, $L^{(2n+1)}_{cq\nu}(\theta)$, $L^{(2n)}_{sq\nu}(\theta)$ and $L^{(2n+1)}_{sq\nu}(\theta)$ converge uniformly on any closed subinterval $[\theta_1,\theta_0]$ of $(2\arctan(\sqrt{\frac{1-k}{1+k}}),\ 2\arctan(\sqrt{\frac{1+k}{1-k}}))$ and satisfy the differential equation (3.2).

PROOF. With the aid of theorem 3.1 and lemma 3.20 it follows that

$$\lim_{m\to\infty} \frac{A^{(2n)}_{2m+2}T(2m+2)Q^{2m+2}_{\nu}(\cos(\theta))}{A^{(2n)}_{2m}T(2m)Q^{2m}_{\nu}(\cos(\theta))} = \frac{k-1}{k+1}\begin{cases}\cot^2(\tfrac{1}{2}\theta), & \theta \le \pi/2\\ \tan^2(\tfrac{1}{2}\theta), & \theta > \pi/2\end{cases}$$

In the other three cases we obtain the same limit. □

3.3. *The infinite tridiagonal matrices*

In this section we shall derive some properties of the infinite tridiagonal matrices corresponding to the classes I to IV of the φ Lamé solutions.
Let L be defined by

$$L := -\sqrt{1-k'^2\sin^2(\varphi)}\frac{d}{d\varphi}(\sqrt{1-k'^2\sin^2(\varphi)}\frac{d}{d\varphi}) + \nu(\nu+1)k'^2\sin^2(\varphi) \ ,$$

then the φ Lamé eigenvalue problem can be written as

$$L\Phi = \lambda\Phi \ , \ \Phi(\varphi) = \Phi(\varphi+2\pi) \ .$$

It is easy to verify that L is a Hermitian operator with respect to the inner product

$$(u,v) := \int_0^{2\pi} \frac{u(\varphi)\overline{v(\varphi)}}{\sqrt{1-k'^2\sin^2(\varphi)}}\, d\varphi \ .$$

It follows from well-known results that the inverse operator L^{-1} is a compact (integral) operator in $L_2(0,2\pi)$.
Now consider the set of functions

$$\{\varepsilon_m\cos(2m\varphi)\}_{m=0}^{\infty} \cup \{\cos((2m+1)\varphi)\}_{m=0}^{\infty} \cup \{\sin(2m\varphi)\}_{m=1}^{\infty} \cup$$

$$\cup \{\sin((2m+1)\varphi)\}_{m=0}^{\infty} \quad (\varepsilon_0 = \tfrac{1}{2}\sqrt{2},\ \varepsilon_m = 1,\ m = 1,2,\ldots) \ ,$$

which is orthonormal with respect to the inner product

$$\langle u,v\rangle := \frac{1}{\pi}\int_0^{2\pi} u(\varphi)\overline{v(\varphi)}\,d\varphi \ .$$

We remark that this inner product is equivalent to the inner product (,).
In section 3.1 we expanded the eigenfunctions of the φ Lamé eigenvalue problem into a Fourier series in terms of this basis. Before proceeding we shall first prove a lemma.

LEMMA 3.5. Let H be a separable Hilbert space with two equivalent inner products (,) and < , >.
Let L be a linear operator defined in a dense subspace $D_L \subset H$, with a compact inverse L^{-1}.
Let L be self-adjoint with respect to the inner product (,) with eigenvalues λ_n, n = 1,2,... and corresponding eigenfunctions f_n, n = 1,2,...,

$f_n \in D_L$.
Let $\{\varphi_i\}_{i=1}^{\infty}$ be a complete orthonormal system, with respect to $<\ ,\ >$.
Let A be the infinite matrix defined by

$$A_{ij} := <L\varphi_j, \varphi_i>, \quad i,j = 1,2,\ldots .$$

Then A possesses the following properties:

(i) A has a compact inverse.

(ii) The eigenvalues of A are λ_n, $n = 1,2,\ldots$.

(iii) The components of the corresponding eigenvectors $\underline{x}_n \in \ell^2$ satisfy

$$(\underline{x}_n)_j = <f_n, \varphi_j>, \quad j = 1,2,\ldots .$$

These eigenvectors are orthonormal with respect to the inner product

$$[\underline{x}_n, \underline{x}_m] := \underline{x}_n^H C \underline{x}_m, \; C_{ij} := (\varphi_j, \varphi_i) .$$

The matrix A maps a dense subspace of ℓ_2 into ℓ_2.

PROOF. Consider the equations

$$<(L - \lambda_n I) f_n, \varphi_i> = 0, \text{ n fixed, } i = 1,2,\ldots$$

or

$$<(L - \lambda_n I) \sum_{j=1}^{\infty} <f_n, \varphi_j> \varphi_j, \varphi_i> = 0 ,$$

and this results into

$$\sum_{j=1}^{\infty} <L\varphi_j, \varphi_i><f_n, \varphi_j> = \lambda_n <f_n, \varphi_i>$$

or abbreviated

$$A\underline{x}_n = \lambda_n \underline{x}_n$$

with

$$A_{ij} := <L\varphi_j, \varphi_i>, \; (\underline{x}_n)_j := <f_n, \varphi_j> .$$

Since $f_1, f_2, \ldots$ span H, $\underline{x}_1, \underline{x}_2, \ldots$ span ℓ^2. Hence the matrix A maps a dense subspace D_A of ℓ^2 into ℓ^2 and $(\lambda_1, \underline{x}_1), (\lambda_2, \underline{x}_2), \ldots$ are a set of eigenpairs of this mapping. Conversely, if $(\lambda, \underline{x})$ is an eigenpair of A, then (λ, f) where $f = \sum_{i=1}^{\infty} (\underline{x})_i \varphi_i$ is an eigenpair of L.

The matrix A has a compact inverse A^{-1} defined by

$$(A^{-1})_{ij} := \langle L^{-1}\varphi_j,\varphi_i\rangle, \quad i,j = 1,2,\ldots .$$

Hence A has a point spectrum that coincides with that of L. (We observe that in general A is not Hermitian.)
Because L is self-adjoint relative to the inner product (,), we can normalize the eigenfunctions f_n in such a way that

$$(f_m,f_n) = \delta_{nm}, \quad n,m = 1,2,\ldots .$$

Consequently,

$$\delta_{nm} = \Big(\sum_{i=1}^{\infty} \langle f_m,\varphi_i\rangle\varphi_i, \sum_{j=1}^{\infty} \langle f_n,\varphi_j\rangle\varphi_j\Big) =$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle f_m,\varphi_i\rangle(\varphi_i,\varphi_j)\overline{\langle f_n,\varphi_j\rangle}$$

or abbreviated

$$\underline{x}_n^H C \underline{x}_m = \delta_{nm}$$

in which C is defined as

$$C_{ij} = (\varphi_j,\varphi_i), \quad i,j = 1,2,3,\ldots .$$

With the aid of this lemma and the results of theorem 2.10 of chapter 2 the following theorem holds.

THEOREM 3.6. For $\nu > 0$ and $0 < k' < 1$ the infinite tridiagonal matrices corresponding to the four classes of the periodic Lamé solutions have the following properties:

(i) They can be considered as mappings of dense subspaces of ℓ^2 into ℓ^2, having a compact inverse.
(ii) They have a real point spectrum, consisting of simple eigenvalues only.

□

3.4. *Appendix*

THEOREM 3.7. (Perron's theorem [10]). Let

$$y_{n+1} + a_n y_n + b_n y_{n-1} = 0, \quad n = 1,2,3,\ldots$$

be a three-term recurrence relation with $b_n \neq 0$, $n = 1,2,\ldots$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b \neq 0$.

If the characteristic polynomial $t^2 + at + b$ has two different zeros t_1 and t_2 with $|t_1| > |t_2|$, then the recurrence relation has two independent solutions $y_n^{(1)}$ and $y_n^{(2)}$ satisfying

$$\lim_{n\to\infty} \frac{y_{n+1}^{(1)}}{y_n^{(1)}} = t_1$$

and

$$\lim_{n\to\infty} \frac{y_{n+1}^{(2)}}{y_n^{(2)}} = t_2$$

respectively. □

DEFINITION 3.8. Pochhammer's symbol $(a)_n$ is defined by

$$(a)_0 := 1$$

$$(a)_n := a(a+1)(a+2)\ldots(a+n-1), \quad n = 1,2,3,\ldots \text{ [1;256].}$$ □

LEMMA 3.9.

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

where $\Gamma(x)$ is the gamma function [1;256]. □

DEFINITION 3.10. The hypergeometric function is defined by

$$F(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1, \; c \neq 0,-1,-2,\ldots \text{ [1;556].}$$ □

LEMMA 3.11. The associated Legendre function of the first kind $P_\nu^m(x)$, $-1 \le x \le 1$, is defined by

$$P_\nu^m(x) = \frac{(-1)^m}{2^m m!}\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}(1-x^2)^{\frac{1}{2}m}F(1+m+\nu, m-\nu; 1+m; \tfrac{1}{2}(1-x)),$$

$$m = 0,1,2,\ldots,\ \nu > 0\ [2;148]. \quad \square$$

LEMMA 3.12.

$$P_\nu^m(x) = \frac{(-1)^m}{2^m m!}\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}(1-x^2)^{\frac{1}{2}m}[1+\frac{(m+\nu+1)(m-\nu)}{2(m+1)}(1-x) +$$

$$+ \mathcal{O}(1-x)^2],\ m = 0,1,2,\ldots,\ \nu > 0\ (x \uparrow 1)\ . \quad \square$$

With the aid of the recurrence relation

$$(1-x^2)\frac{dP_\nu^m(x)}{dx} = (\nu+1)xP_\nu^m(x) - (\nu-m+1)P_{\nu+1}^m(x)\ [2;161]$$

and lemma 3.12 follows

LEMMA 3.13.

$$\sqrt{1-x^2}\,\frac{dP_\nu^m(x)}{dx} = \frac{(-1)^m}{2^m m!}\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}(1-x^2)^{\frac{1}{2}(m-1)}\ \cdot$$

$$\cdot\ \{-m + \frac{1}{2(m+1)}[(\nu+1)(m+\nu+1)(m-\nu) -$$

$$- 2(\nu+1)(m+1) - (\nu+m+1)(\nu+m+2)(m-\nu-1)]\ \cdot$$

$$\cdot\ (1-x) + \mathcal{O}(1-x)^2\} \quad (x \uparrow 1)\ . \quad \square$$

LEMMA 3.14.

$$P_\nu^0(1) = 1$$

$$P_\nu^m(1) = 0, \quad m = 1,2,3,\ldots$$

$$\left.\frac{dP_\nu^0(\cos(\theta))}{d\theta}\right|_{\theta=0} = 0$$

$$\left.\frac{dP_\nu^1(\cos(\theta))}{d\theta}\right|_{\theta=0} = -\frac{\nu(\nu+1)}{2}$$

$$\left.\frac{dP_\nu^m(\cos(\theta))}{d\theta}\right|_{\theta=0} = 0, \quad m = 2,3,\ldots \; . \qquad \square$$

LEMMA 3.15.

$$x\,\frac{dP_\nu^m(x)}{dx} = \tfrac{1}{4}(\nu-m+2)(\nu+m-1)(\nu-m+1)(\nu+m)P_\nu^{m-2}(x) + $$

$$+ \tfrac{1}{2}(\nu(\nu+1) + m^2)P_\nu^m(x) + \tfrac{1}{4}P_\nu^{m+2}(x) - \frac{m^2}{1-x^2}P_\nu^m(x) \; .$$

PROOF. [3]. Using the recurrence relation

$$(1-x^2)\frac{dP_\nu^m(x)}{dx} = -\sqrt{1-x^2}\,P_\nu^{m+1}(x) - mxP_\nu^m(x)$$

we obtain

$$x\,\frac{dP_\nu^m(x)}{dx} = \frac{-x}{\sqrt{1-x^2}}P_\nu^{m+1}(x) - \frac{mx^2}{1-x^2}P_\nu^m(x)$$

$$= \frac{x}{\sqrt{1-x^2}}\left[-P_\nu^{m+1}(x) + \frac{m(m-1)x}{\sqrt{1-x^2}}P_\nu^m(x)\right] - \frac{m^2x^2}{1-x^2}P_\nu^m(x) \; .$$

Using the recurrence relation

$$(3.4) \qquad \frac{2mx}{\sqrt{1-x^2}}P_\nu^m(x) = -P_\nu^{m+1}(x) - (\nu-m+1)(\nu+m)P_\nu^{m-1}(x) \qquad [2;161]$$

we have

$$(3.5) \qquad x\,\frac{dP_\nu^m(x)}{dx} = \frac{-x}{\sqrt{1-x^2}}\Big[P_\nu^{m+1}(x) + \frac{m-1}{2}(P_\nu^{m+1}(x) + $$

$$+ (\nu-m+1)(\nu+m)P_\nu^{m-1}(x))\Big] - \frac{m^2x^2}{1-x^2}P_\nu^m(x) \; .$$

Substituting (3.4) in (3.5) once more we obtain

$$x\,\frac{dP_\nu^m(x)}{dx} = \tfrac{1}{4}(P_\nu^{m+2}(x) + (\nu-m)(\nu+m+1)P_\nu^m(x)) + $$

$$+ \tfrac{1}{4}(\nu-m+1)(\nu+m)(P_\nu^m(x) + (\nu-m+2)(\nu+m-1)P_\nu^{m-2}(x)) - $$

$$- \frac{m^2x^2}{1-x^2} P_\nu^m(x)$$

$$= \tfrac{1}{4}(\nu - m + 1)(\nu + m - 1)(\nu - m + 2)(\nu + m) P_\nu^{m-2}(x) +$$

$$+ \tfrac{1}{2}(\nu(\nu+1) + m^2) P_\nu^m(x) + \tfrac{1}{4} P_\nu^{m+2}(x) - \frac{m^2}{1-x^2} P_\nu^m(x) \quad . \qquad \square$$

LEMMA 3.16.

$$3xm \frac{dP_\nu^m(x)}{dx} = \tfrac{1}{4}(m-2)(\nu - m + 1)(\nu + m)(\nu - m + 2) \cdot$$

$$\cdot (\nu + m - 1) P_\nu^{m-2}(x) + \tfrac{1}{2}(\nu(\nu+1) + m^2 + 2) m P_\nu^m(x) +$$

$$+ \tfrac{1}{4}(m+2) P_\nu^{m+2}(x) - \frac{(m^2+2)}{1-x^2} m P_\nu^m(x) \quad .$$

PROOF. This is analogous to the proof of lemma 3.15, [3]. $\square$

LEMMA 3.17. The associated Legendre function of the first kind $P_\nu^m(\cos(\theta))$, $0 \le \theta < \pi$ can also be defined as

$$P_\nu^m(\cos(\theta)) = \frac{(-1)^m}{m!} \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} (\tan(\tfrac{1}{2}\theta))^m \cdot$$

$$\cdot F(-\nu,\ \nu+1;\ 1+m;\ (\sin(\tfrac{1}{2}\theta))^2) \qquad [2;147],\ [2;144]. \quad \square$$

With the aid of this lemma we can investigate the asymptotic behaviour of $P_\nu^m(\cos(\theta))$ $(m \to \infty)$.

LEMMA 3.18. The associated Legendre function of the first kind $P_\nu^m(\cos(\theta))$, $0 \le \theta < \pi$, $\nu > 0$, $m = 0,1,2,\ldots$ has the following asymptotic expansion:

$$P_\nu^m(\cos(\theta)) = \frac{(-1)^m}{m!} \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} \ (\tan(\tfrac{1}{2}\theta))^m (1 + \mathcal{O}(\tfrac{1}{m})) \quad (m \to \infty) \quad . \qquad \square$$

LEMMA 3.19. The associated Legendre function of the second kind $Q_\nu^m(\cos(\theta))$, $0 < \theta < \pi$ is defined in terms of hypergeometric series by

$$Q_\nu^m(\cos(\theta)) := \frac{(-1)^m}{2m!}\Gamma(\nu+m+1)\Gamma(m-\nu)(\cot(\tfrac{1}{2}\theta))^m \cdot$$
$$\cdot [F(-\nu,\ \nu+1;\ 1+m;\ (\cos(\tfrac{1}{2}\theta))^2) -$$
$$- (-1)^m\cos(\pi\nu)(\tan(\tfrac{1}{2}\theta))^{2m}F(-\nu,\ \nu+1; 1+m;$$
$$(\sin(\tfrac{1}{2}\theta))^2)] \quad [2;141],\ [2;143]. \qquad \square$$

With the aid of this lemma we can investigate the asymptotic behaviour of $Q_\nu^m(\cos(\theta))$ $(m \to \infty)$.

LEMMA 3.20. The associated Legendre function of the second kind $Q_\nu^m(\cos(\theta))$, $0 < \theta < \pi$, $\nu > 0$, $m = 0,1,2,\ldots$ has the following asymptotic expansion:

$$Q_\nu^m(\cos(\theta)) = \frac{(-1)^m}{2m!}\Gamma(\nu+m+1)\Gamma(m-\nu)((\cot(\tfrac{1}{2}\theta))^m -$$
$$- (-1)^m\cos(\pi\nu)(\tan(\tfrac{1}{2}\theta))^m + \mathcal{O}(\tfrac{1}{m})),\ (m \to \infty)\ . \qquad \square$$

3.5. *References*

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CHAPTER 4

EIGENVALUES AND EIGENVECTORS OF THE PERIODIC LAMÉ SOLUTIONS

4.0. *Introduction*

In the previous chapter we obtained four infinite tridiagonal matrices corresponding to the four classes of the φ solutions. We also derived a few spectral properties of the matrices.

Starting from these properties we shall develop algorithms for the computation of the eigenvalues and the corresponding eigenvectors of the tridiagonal matrices. These algorithms are closely connected to the continued-fraction algorithms devised by Blanch [1] and Bouwkamp [2], [3].

4.1. *Calculation of eigenvalues*

For the sake of convenience we shall only investigate the eigenvalues λ (rather than Λ, see section 3.1) of the (redefined) infinite tridiagonal matrix A corresponding to class I:

$$A := \begin{bmatrix} a_0 & c_0 & & & \\ b_0 & a_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-1} & a_n & c_n \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

with elements

$$a_n := (2n)^2(1-\tfrac{1}{2}k'^2) + \tfrac{1}{2}\nu(\nu+1)k'^2, \; n = 0,1,2,\ldots ,$$

$$b_0 := -\tfrac{1}{2}k'^2\nu(\nu+1) ,$$

$$b_n := -\tfrac{1}{4}k'^2(\nu-2n)(\nu+2n+1), \; n = 1,2,\ldots ,$$

$$c_n := -\tfrac{1}{4}k'^2(\nu-2n-1)(\nu+2n+2), \; n = 0,1,2,\ldots .$$

From the previous chapter we know that the matrix A (when considered as the matrix of a mapping from a dense subspace $D_A \subset \ell^2$ in ℓ^2 relative to the natural basis in ℓ^2) has a compact inverse and a positive point spectrum. It follows that the matrix A^T has the same spectrum as A and has also a compact inverse. Since the matrix A^T is diagonally dominant (see appendix) we shall speak in terms of the eigenvalues of A^T rather than those of A.

In chapter 2 we proved that of all eigenvalues λ of A, we only need those eigenvalues satisfying

$$0 < \lambda < \nu(\nu+1) \ .$$

Let, for a given $\nu > 0$, the integer N be so that

$$2N-1 < \nu \le 2N+1 \ .$$

Then, for $j \neq N$, $b_j c_j > 0$, but $b_N c_N$ may be negative (see appendix). Therefore, we partition the matrix A^T as follows:

$$A^T := \left[\begin{array}{c|c} A_1^T & \begin{matrix} & \bigcirc \\ b_N & \end{matrix} \\ \hline \begin{matrix} & c_N \\ \bigcirc & \end{matrix} & A_2^T \end{array}\right]$$

in which

$$A_1^T := \begin{bmatrix} a_0 & b_0 & & \bigcirc \\ c_0 & a_1 & b_1 & \\ & & & b_{N-1} \\ \bigcirc & & c_{N-1} & a_N \end{bmatrix} \ , \quad A_2^T := \begin{bmatrix} a_{N+1} & b_{N+1} & & \bigcirc \\ c_{N+1} & a_{N+2} & b_{N+2} & \\ & \ddots & \ddots & \ddots \\ \bigcirc & & \ddots & \ddots \end{bmatrix} .$$

We first show that for $Re(\lambda) \le \nu(\nu+1)$, $A_2^T - \lambda I_2$ has a compact inverse. Since A^T has a compact inverse, the matrix A_2^T and also, for each λ, $A_2^T - \lambda I_2$ satisfies the Fredholm alternative,
either

$$\exists_{x_2 \neq 0} \ (A_2^T - \lambda I_2) x_2 = 0$$

or

$$A_2^T - \lambda I_2 \text{ has a compact inverse .}$$

It is easy, to verify that the matrix $A_2^T - \lambda I_2$ is strictly diagonally dominant for all λ satisfying $Re(\lambda) \le \nu(\nu+1)$ (see appendix). Consequently, from $(A_2^T - \lambda I_2)x_2 = 0$ it follows that $x_2 = 0$ (see appendix), hence $A_2^T - \lambda I_2$ has a compact inverse for $Re(\lambda) \le \nu(\nu+1)$.

Now let λ, $0 < \lambda < \nu(\nu+1)$, be an eigenvalue of the matrix A^T with eigenvector $x = (x_1^T \mid x_2^T)^T$. Then

$$(4.1)\qquad \left[\begin{array}{c|c} A_1^T & b_N \\ \hline c_N & A_2^T \end{array}\right]\begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix}$$

or, written differently,

$$A_1^T x_1 + b_N(\tilde{e}_{N+1}^T x_2)e_N = \lambda x_1 \ ,$$

$$c_N(e_N^T x_1)\tilde{e}_{N+1} + A_2^T x_2 = \lambda x_2 \ ,$$

where $e_N \in \mathbb{R}^{N+1}$, $(e_N)_i = \delta_{N,i}$, $i = 0,1,\dots,N$, and $\tilde{e}_{N+1} \in \mathbb{R}^\infty$, $(\tilde{e}_{N+1})_i = \delta_{N+1,i}$, $i = N+1, N+2, \dots$. Taking into account that $(A_2^T - \lambda I_2)^{-1}$ is compact it follows that $x_1 \neq 0$ and

$$(4.2)\qquad (A_1^T - b_N c_N \alpha(\lambda) e_N e_N^T) x_1 = \lambda x_1 \ ,$$

where

$$\alpha(\lambda) := \tilde{e}_{N+1}^T (A_2^T - \lambda I_2)^{-1} \tilde{e}_{N+1} \ .$$

Since (4.2) implies (4.1) the following theorem holds:

THEOREM 4.1. λ, $0 < \lambda < \nu(\nu+1)$, is an eigenvalue of A if and only if

$$\det(A_1^T - \lambda I_1 - b_N c_N \alpha(\lambda) e_N e_N^T) = 0 \ . \qquad \square$$

Now we investigate the function $\alpha(\lambda)$. Consider (for some $\lambda \le \nu(\nu+1)$) the equation

$$(A_2^T - \lambda I_2)\tilde{y} = \tilde{e}_{N+1}$$

in which $\tilde{y} \in D_{A_2} \subset \ell^2$ with components $\tilde{y}_i$, $i = N+1, \dots$. Then $\alpha(\lambda) = \tilde{y}_{N+1}$. Let η_ℓ be defined as

$$\eta_\ell := -\frac{\tilde{y}_{\ell+1}}{\tilde{y}_\ell} \ , \ \ell = N+1, N+2, \dots \ .$$

It is easy to verify that the η_ℓ satisfy the recurrence relation

$$\eta_{\ell-1} = \frac{c_{\ell-1}}{a_\ell - \lambda - b_\ell \eta_\ell}\ , \quad \ell = N+2,\ldots$$

and that

$$\alpha(\lambda) = \frac{1}{a_{N+1} - \lambda - b_{N+1}\eta_{N+1}}\ .$$

As proved in the previous chapter, we have (with the aid of Perron's theorem 3.7) that

$$\lim_{\ell\to\infty} \frac{\tilde{y}_{\ell+1}}{\tilde{y}_\ell} = \frac{k-1}{k+1}\ .$$

Hence there is an integer M such that $0 < \eta_m < 1$ for $m > M$. Starting from this result and the property that the matrix $A_2^T - \lambda I_2$ is strictly diagonally dominant for all λ satisfying $\lambda \le \nu(\nu+1)$ (see appendix), it is easy to prove by induction that $0 < \eta_\ell < 1$, for $\ell = N+1, N+2, \ldots$.
Consequently

$$0 < \alpha(\lambda) < \frac{1}{a_{N+1} - \lambda - b_{N+1}} < \frac{1}{c_N}\ .$$

We observe that if $b_N c_N = 0$ we do not need $\alpha(\lambda)$. Now let

$$p_0(\lambda) := 1\ ,$$

$$p_1(\lambda) := a_0 - \lambda\ ,$$

$$p_{i+1}(\lambda) := (a_i - \lambda)p_i(\lambda) - b_{i-1}c_{i-1}p_{i-1}(\lambda)\,,\ i = 1,2,\ldots,N-1\ ,$$

$$p_{N+1}(\lambda) := (a_N - \lambda - b_N c_N \alpha(\lambda))p_N(\lambda) - b_{N-1}c_{N-1}p_{N-1}(\lambda)\ ,$$

be the principal minors of the matrix

$$A_1^T - \lambda I_1 - b_N c_N \alpha(\lambda) e_N e_N^T\ .$$

We shall prove that these functions form a Sturm sequence on $[0,\nu(\nu+1)]$, i.e. that they satisfy:

(i) $p_0(\lambda)$ does not vanish on $[0,\nu(\nu+1)]$;

(ii) if $p_i(\lambda) = 0$ for some $\lambda \in [0,\nu(\nu+1)]$, $1 \le i \le N$, then $p_{i-1}(\lambda)p_{i+1}(\lambda) < 0$;

(iii) $\mathrm{sign}(p_i(0)) = 1$ and $\mathrm{sign}\ p_i(\nu(\nu+1)) = (-1)^i$, $i = 0,1,2,\ldots,N+1$;

(iv) $p_{N+1}(\lambda)$ has at most simple zeros in $[0,\nu(\nu+1)]$.

Since $b_i c_i > 0$, $i = 0,1,\dots,N-1$ (see appendix), it is well known that

$$p_0(\lambda),\dots,p_N(\lambda),p^*_{N+1}(\lambda) := (a_N - \lambda)p_N(\lambda) - b_{N-1}c_{N-1}p_{N-1}(\lambda)$$

constitute a Sturm sequence on $(-\infty,\infty)$. It follows that the requirements (i) and (ii) are fulfilled. (Also for $i = N$ since the difference between $p_{N+1}(\lambda)$ and $p^*_{N+1}(\lambda)$ is proportional to $p_N(\lambda)$.)
In order to prove (iii) we observe (see appendix) that

$$A_1^T - b_N c_N \alpha(0) e_N e_N^T - \lambda I_1$$

and its principal minors are diagonally dominant for $\mathrm{Re}(\lambda) \le 0$.
The same holds for

$$A_1^T - b_N c_N \alpha(\nu(\nu+1)) e_N e_N^T - \lambda I_1$$

in case $\mathrm{Re}(\lambda) \ge \nu(\nu+1)$.
It follows that $\mathrm{sign}(p_i(0)) = 1$ and $\mathrm{sign}(p_i(\nu(\nu+1))) = (-1)^i$, $i = 0,1,..N+1$.

Finally, we have to prove (iv). Since $p_0,\dots,p_N,p^*_{N+1}$ is a Sturm sequence and $p^*_{N+1}(\lambda)$ is a polynomial of degree $N+1$ it follows that $p_{N+1}(\lambda)$ has $N+1$ simple zeros on $[0,\nu(\nu+1)]$ which are strictly separated by the zeros of $p_N(\lambda)$. Consider now $q_i := p_i/p_{i-1}$, $i = 1,2,\dots,N+1$ and $q^*_{N+1} := p^*_{N+1}/p_N$.
Then $q_{N+1}(\lambda) = q^*_{N+1}(\lambda) - b_N c_N \alpha(\lambda)$.
Graphs of $q^*_{N+1}(\lambda)$ and $b_N c_N \alpha(\lambda)$ against λ are given below.

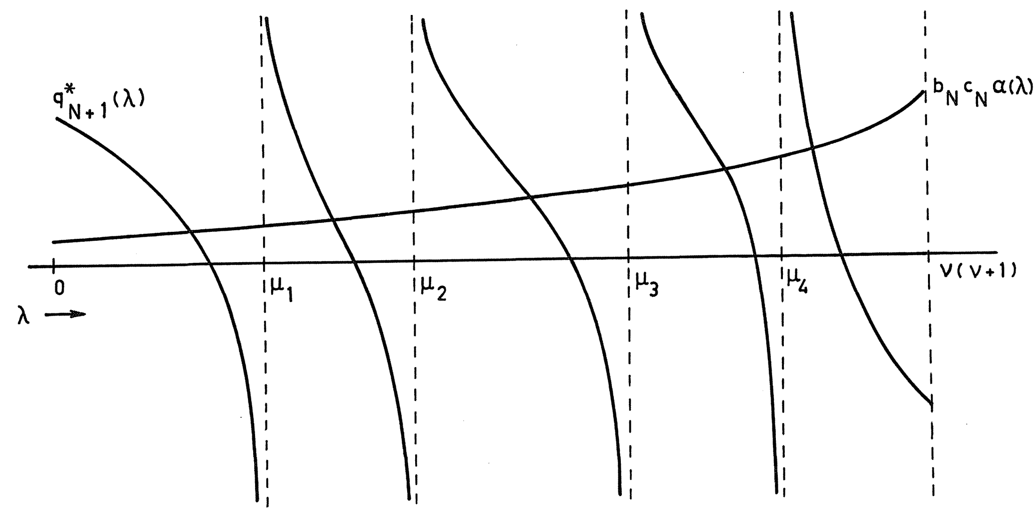

Figure 4.1.

The function $q^*_{N+1}(\lambda)$ is a monotonically decreasing function with simple poles at the zeros μ_i, $i = 1,2,\dots,N$ of $p_N(\lambda)$ and one simple zero in each of the intervals $[0,\mu_1)$, (μ_i,μ_{i+1}), $i = 1,2,\dots,N-1$ and $(\mu_N,\nu(\nu+1))$ (being the zeros of $p^*_{N+1}(\lambda)$).
From (iii) it follows that $q_{N+1}(0) > 0$ and $q_{N+1}(\nu(\nu+1)) < 0$ and it follows that $q^*_{N+1}(\nu(\nu+1)) < b_N c_N \alpha(\nu(\nu+1))$.
Now we prove that for $0 \le \mathrm{Re}(\lambda) < \nu(\nu+1)$ the function $q_{N+1}(\lambda)$ has precisely $N+1$ zeros. We observe that the number of poles μ_i, $i = 1,2,\dots,N$ of the function $q_{N+1}(\lambda)$ that lie in the region $0 \le \mathrm{Re}(\lambda) \le \nu(\nu+1)$ is independent of $b_N c_N$. If $b_N c_N = 0$ then $q_{N+1}(\lambda) = q^*_{N+1}(\lambda)$ has $N+1$ simple zeros in $[0,\nu(\nu+1))$. Consequently, for $b_N c_N \neq 0$ there are $N+1$ zeros in $0 \le \mathrm{Re}(\lambda) \le \nu(\nu+1)$. We know that $q_{N+1}(\lambda)$ has at least one zero in each interval $[0,\mu_1)$, (μ_i,μ_{i+1}), $i = 1,2,\dots,N-1$ and $(\mu_N,\nu(\nu+1))$. Hence, the zeros of $q_{N+1}(\lambda)$ are simple.

Summarizing, we have proved that the functions $p_i(\lambda)$, $i = 0,1,2,\dots,N+1$ form a Sturm sequence on $[0,\nu(\nu+1)]$.
Hence we calculate the eigenvalues λ of A^T satisfying $0 < \lambda < \nu(\nu+1)$ by the method of bisection, as described in [6;302].

4.1.0. *Calculation of* $\alpha(\lambda)$

In this section we shall construct an algorithm to calculate $\alpha(\lambda)$. Let r_ℓ be defined as $r_\ell := \eta_\ell / c_\ell$. Then we proved that

$$\alpha(\lambda) = \frac{1}{a_{N+1} - \lambda - b_{N+1} c_{N+1} r_{N+1}}$$

in which r_{N+1} satisfies the recurrence relation

$$r_{\ell-1} = \frac{1}{a_\ell - \lambda - b_\ell c_\ell r_\ell} \quad , \ \ell = N+2, N+3, \dots$$

with (since $c_\ell \to \infty$)

$$\lim_{\ell\to\infty} r_\ell = 0 \ .$$

These recurrence relations may be interpreted as the fundamental recurrence formulas for the following continued fraction [1]:

$$\alpha(\lambda) = \frac{1|}{|a_{N+1} - \lambda} - \frac{b_{N+1}c_{N+1}|}{|a_{N+2} - \lambda} - \frac{b_{N+2}c_{N+2}|}{|a_{N+3} - \lambda} - \dots .$$

Let $r_\ell^{(m)}$ and α_m, $m \geq N+1$ be defined as follows:

$$r_m^{(m)} := 0,\ r_{\ell-1}^{(m)} := \frac{1}{a_\ell - \lambda - b_\ell c_\ell r_\ell^{(m)}},\ \ell = m, m-1, \dots, N+2,$$

$$\alpha_m := \frac{1}{a_{N+1} - \lambda - b_{N+1}c_{N+1}r_{N+1}^{(m)}},$$

or in terms of a continued fraction

$$\alpha_m = \frac{1|}{|a_{N+1} - \lambda} - \frac{b_{N+1}c_{N+1}|}{|a_{N+2} - \lambda} - \dots - \frac{b_{m-1}c_{m-1}|}{|a_m - \lambda}.$$

Let us denote the approximants α_m of $\alpha(\lambda^2)$ by

$$\alpha_m := A_m/B_m .$$

It is easy to verify by induction [1] that the A_m satisfy the recurrence relation

$$A_{m+1} = (a_{m+1} - \lambda)A_m - b_m c_m A_{m-1},\ m = N+1, \dots ,$$

with

$$A_N = 0 \quad \text{and} \quad A_{N+1} = 1 .$$

The denominators B_m satisfy the recurrence relation

$$B_{m+1} = (a_{m+1} - \lambda)B_m - b_m c_m B_{m-1},\ m = N+1, \dots,$$

with

$$B_N = 1 \quad \text{and} \quad B_{N+1} = a_{N+1} - \lambda .$$

Consider

$$\varepsilon_{m+1} := \alpha_{m+1} - \alpha_m = \frac{A_{m+1}}{B_{m+1}} - \frac{A_m}{B_m} = \frac{D_{m+1}}{B_{m+1}B_m}$$

where

$$D_{m+1} := A_{m+1}B_m - B_{m+1}A_m .$$

It is easy to verify that

$$D_{m+1} = b_m c_m D_m$$

and consequently

$$\varepsilon_{m+1} = b_m c_m \frac{B_{m-1}}{B_{m+1}} \varepsilon_m = (\delta_{m+1} - 1)\varepsilon_m$$

where

$$\delta_{m+1} := (a_{m+1} - \lambda) B_m / B_{m+1} .$$

Consequently, the δ_m satisfy the recurrence relation

$$\delta_{m+1} = \cfrac{1}{1 - \cfrac{b_m c_m}{(a_{m+1} - \lambda)(a_m - \lambda)} \delta_m} , \quad m = N+1, \ldots ,$$

with starting value $\delta_{N+1} = 1$.

Summarizing, we have the following continued-fraction algorithm:

$$f_m := \frac{b_m c_m}{(a_m - \lambda)(a_{m+1} - \lambda)} , \quad m = N+1, N+2, \ldots ,$$

$$\delta_{m+1} := \frac{1}{1 - f_m \delta_m} , \quad \delta_{N+1} := 1 ,$$

$$\varepsilon_{m+1} := \frac{f_m \delta_m}{1 - f_m \delta_m} \varepsilon_m = (\delta_{m+1} - 1)\varepsilon_m , \quad \varepsilon_{N+1} = \frac{1}{a_{N+1} - \lambda} ,$$

$$\alpha_{m+1} := \alpha_m + \varepsilon_{m+1} , \quad \alpha_{N+1} := \frac{1}{a_{N+1} - \lambda}$$

$$\alpha(\lambda) := \lim_{m\to\infty} \alpha_m .$$

The convergence requirement is [1], that from an index M onward, $0 < f_m < \frac{1}{4}$, $m \geq M$. In our case,

$$\lim_{m\to\infty} f_m = \frac{k'^4}{4(2 - k'^2)^2} < \frac{1}{4} .$$

It is easy to verify that

$$f_m < \frac{1}{3}\frac{k'^4}{(2-k'^2)^2}\ ,\ m = N+1,N+2,\ldots\ .$$

Hence, for all practical values of k'^2, viz. $k'^2 \le \dfrac{2\sqrt{3}}{2+\sqrt{3}} \doteq 0.93$, $f_m < \frac{1}{4}$, $m = N+1,N+2,\ldots$.
It appears, however, that for $k'^2 > 0.93$, after some terms, f_m also satisfies $f_m < \frac{1}{4}$. Consequently, it follows that $0 < f_m\delta_m < \frac{1}{2}$ and numerical stability is ensured.

4.2. *Calculation of eigenvectors*

Let λ be an eigenvalue of the matrix A. The corresponding eigenvector $x = (x_1^T|x_2^T)^T$ satisfies

$$\left[\begin{array}{c|c} A_1 & c_N \\ \hline b_N & A_2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or, written differently,

$$(A_1 - \lambda I_1)x_1 + c_N(\tilde{e}_{N+1}^T x_2)e_N = 0\ ,$$

$$(4.3)\qquad (A_2 - \lambda I_2)x_2 = -b_N(e_N^T x_1)\tilde{e}_{N+1}\ .$$

Using the definition of $\alpha(\lambda)$ it follows that x_1 satisfies the equation

$$(4.4)\qquad (A_1 - \lambda I_1 - b_N c_N \alpha(\lambda) e_N e_N^T)x_1 = 0\ .$$

In the previous section we derived an algorithm for the computation of the eigenvalues $\lambda \le \nu(\nu+1)$, using bisection and Sturm sequences.
Let now $\tilde{\lambda}$ be such a computed eigenvalue, and $\tilde{x}$ the corresponding eigenvector.
First, we shall calculate the first $N+1$ components of the vector $\tilde{x}$ with the aid of equation (4.4). To determine the ratios of the components of the eigenvector we only need N equations. Hence we can omit one equation.
We take as omitted equation that equation for which $|a_i - \tilde{\lambda}|$, $i=0,1,\ldots,N$ takes its minimal value. Let s be the index of the omitted equation. Since the a_i are monotonically increasing we have $a_i - \tilde{\lambda} < 0$, $(i < s)$ and

$a_i - \tilde{\lambda} > 0$ $(i > s)$.

Let $\tilde{r}_i$ be defined by $\tilde{r}_i := -\tilde{x}_i/(c_i\tilde{x}_{i+1})$. We determine $\tilde{r}_0,\ldots,\tilde{r}_{s-1}$ from the equations

$$\tilde{r}_0 = 1/(a_0 - \tilde{\lambda}) \quad ,$$

$$\tilde{r}_i = 1/(a_i - \tilde{\lambda} - b_{i-1}c_{i-1}\tilde{r}_{i-1}), \; i = 1,\ldots,s-1 \; .$$

Let r_i^* be defined by $r_i^* := -\tilde{x}_{i+1}/(b_i\tilde{x}_i)$. We determine $r_{N-1}^*,\ldots,r_{s-1}^*$ from the equations

$$r_{N-1}^* = 1/(a_N - \tilde{\lambda} - \alpha(\tilde{\lambda})c_Nb_N) \; ,$$

$$r_{i-1}^* = 1/(a_i - \tilde{\lambda} - b_ic_ir_i^*), \; i = N-1,\ldots,s+1 \; .$$

The required $N+1$ components of the eigenvector are

$$\tilde{x}_0,\tilde{x}_1,\ldots,\tilde{x}_{s-1},1,\tilde{x}_{s+1},\ldots,\tilde{x}_N \; ,$$

where

$$\tilde{x}_s := 1, \; \tilde{x}_i = -\tilde{r}_ic_i\tilde{x}_{i+1}, \; i = s-1,\ldots,0$$

$$\tilde{x}_i = -r_{i-1}^*b_{i-1}\tilde{x}_{i-1}, \; i = s+1,\ldots,N \; .$$

If we do not need more than $N+1$ components of the eigenvector, there is nothing more to do. But in most cases we do need considerably more components, depending on the desired accuracy of the Lamé functions. Suppose we do need M $(M > N)$ components of the eigenvector. With the aid of (4.3) we calculate the ratios of the components of the remaining components of the eigenvector. Let now r_i be defined by $r_i := -\tilde{x}_{i+1}/(b_i\tilde{x}_i)$. Then the r_i satisfy the backward recurrence relation

$$r_i = 1/(a_{i+1} - \tilde{\lambda} - b_{i+1}c_{i+1}r_{i+1}), \; i = N,N+1,\ldots$$

with

$$\lim_{i\to\infty} r_i = 0 \; .$$

We use now Miller's algorithm [4]: Take $N_1 > M$ and set $\tilde{r}_{N1}^{(N1)} := 0$. Calculate

$$\tilde{r}_i^{(N1)} := 1/(a_{i+1} - \tilde{\lambda} - b_{i+1}c_{i+1}\tilde{r}_{i+1}^{(N1)}), \; i = N_1-1,\ldots,N-1 \; .$$

Then we calculate

$$r_i^{(N1)} := \frac{r^*_{N-1}}{\tilde{r}^{(N1)}_{N-1}} \tilde{r}_i^{(N1)}, \; i = N,\dots,M .$$

After this we select another integer $N_2 > N1$. If now

$$|r_i^{(N1)} - r_i^{(N2)}| < \varepsilon r_i^{(N2)}, \; i = N,N+1,\dots,M$$

then we take $r_i^{(N2)}$ as an approximation of r_i, $i = N,\dots,M$.

Let ε be the desired relative tolerance for $\tilde{x}_M$. In the previous chapter we proved, with the aid of Perron's theorem, that

$$\lim_{i\to\infty} \frac{\tilde{x}_{i+1}}{\tilde{x}_i} = \frac{k-1}{k+1} .$$

A good choice of the integer N_1 is such that

$$\left(\frac{1-k}{1+k}\right)^{N_1-M} < \varepsilon .$$

The above-defined algorithm depends on the three indices N,M and s.
The index N is determined by the relation $2N-1 < \nu \le 2N+1$.
The index M depends on the desired accuracy of the Lamé functions.
The index s has to be chosen so that the algorithm has good numerical stability.
Since $\hat{\lambda}$ is computed by using bisection and Sturm sequences we may expect that we obtain an acceptable eigenvalue in the sense that

$$|\tilde{\lambda} - \lambda| \le \eta \| A_1^T \|$$

in which η is of the order of the relative machine precision.
We say that the corresponding computed eigenvector $\tilde{x}$ is an acceptable one, if

$$\frac{\| A\tilde{x} - \tilde{\lambda}\tilde{x} \|}{\| \tilde{x} \|} \le \eta \| A_1^T \|.$$

We observe that the above-given algorithm for the computation of eigenvectors is essentially one inverse iteration corresponding to the initial vector e_s.

Starting from this initial vector we have

$$(A - \tilde{\lambda} I)\tilde{\tilde{x}} = e_s$$

so that we determine $\tilde{\tilde{x}}$ and then normalize it to give

$$\tilde{x} := \tilde{\tilde{x}}/(e_s^T\tilde{\tilde{x}}) \quad ,$$

satisfying

$$(A - \tilde{\lambda} I)\tilde{x} = e_s/(e_s^T\tilde{\tilde{x}}) \quad .$$

The residual corresponding to $\tilde{x}$ is very small if $|e_s^T\tilde{\tilde{x}}|$ is very large. From [8] we know that there exists an index s such that $|e_s^T\tilde{\tilde{x}}|$ is very large. In our case, however, the "best" value of s is that index for which

$$|\tilde{\tilde{x}}_s| = \max_{i\geq 0} |\tilde{\tilde{x}}_i| \quad ,$$

because the components of the eigenvector have a reasonably sharp maximum at index s.

Since it happens that all rows, except one (for which $|a_i - \lambda|$ is minimal), of the matrix $(A^T - \lambda I)$ are diagonally dominant it follows that our choice of s, viz. that value for which $|a_i - \tilde{\lambda}|$ is minimal, is a good one.

Moreover in one iteration the enrichment of the required eigenvector relative to the others is directly dependent on the smallness of $\tilde{\lambda} - \lambda$ relative to the other $\tilde{\lambda} - \lambda_i$.

Since in our case $|\tilde{\lambda} - \lambda| \leq \eta \| A_1^T \|$ and the other $\tilde{\lambda} - \lambda_i$ are of order 1 we conclude that the computed eigenvector $\tilde{x}$ is acceptable in the above-defined sense.

Finally, it follows from [7] that by this choice of the index s all the components of the eigenvector have a relative error at most of the order of η.

4.3. *Appendix*

$$a_n := (2n)^2(1 - \tfrac{1}{2}k'^2) + \tfrac{1}{2}\nu(\nu+1)k'^2, \; n = 0,1,2,\ldots \; ;$$

$$b_0 := -\tfrac{1}{2}k'^2\nu(\nu+1) \; ;$$

$$b_n := -\tfrac{1}{4}k'^2(\nu - 2n)(\nu + 2n + 1), \; n = 1,2,\ldots \; ;$$

$$= \tfrac{1}{2}k'^2 n(2n+1) - \tfrac{1}{4}k'^2 \nu(\nu+1) \;;$$

$$c_n := -\tfrac{1}{4}k'^2(\nu - 2n - 1)(\nu + 2n + 2),\; n = 0,1,2,\ldots\,,$$

$$= \tfrac{1}{2}k'^2 (n+1)(2n+1) - \tfrac{1}{4}k'^2 \nu(\nu+1) \;.$$

Let, for a given $\nu > 0$, the integer N be so that

$$2N - 1 < \nu \le 2N + 1 \;.$$

We observe that:

(i) $a_n > 0$, $n = 0,1,2,\ldots$;

(ii) $b_n < 0$, $n \le N-1$ and $b_n > 0$, $n \ge N+1$;

(iii) $c_n < 0$, $n \le N-1$ and $c_n \ge 0$, $n \ge N$;

(iv) $b_N \ge 0$, $2N-1 < \nu \le 2N$; $b_N < 0$, $2N < \nu \le 2N+1$.

It is easy to verify the following (in)equalities:

(i) $|a_0| - |b_0| = 0$, $|a_0| + |b_0| = k'^2\nu(\nu+1) < \nu(\nu+1)$;

(ii) $|a_n| - |b_n| - |c_{n-1}| = 4n^2 > 0$, $1 \le n \le N-1$

$|a_n| + |b_n| + |c_{n-1}| = (4n^2 - \nu(\nu+1))k^2 + \nu(\nu+1) < \nu(\nu+1)$, $1 \le n \le N-1$;

(iii) $|a_n| - |b_n| - |c_{n-1}| = (4n^2 - \nu(\nu+1))k^2 + \nu(\nu+1) > \nu(\nu+1)$, $n \ge N+1$;

(iv) $|a_N| - |b_N| - |c_{N-1}| = 4N^2(1 - \tfrac{1}{2}k'^2) + \tfrac{1}{2}k'^2(\nu(\nu+1) - 2N) > 0$;

$|a_N| + |b_N| + |c_{N-1}| = (1 - \tfrac{1}{2}k'^2)(4N^2 - \nu(\nu+1)) + k'^2 N + \nu(\nu+1)$,

$2N - 1 < \nu \le 2N$;

(v) $|a_N| - |b_N| - |c_{N-1}| = 4N^2 > 0$;

$|a_N| + |b_N| + |c_{N-1}| = (4N^2 - \nu(\nu+1))k^2 + \nu(\nu+1) < \nu(\nu+1)$,

$2N < \nu \le 2N+1$.

From these results we may conclude:

(i) A^T is diagonally dominant;

(ii) $A_1^T - b_N c_N \alpha(0) e_N e_N^T - \lambda I_1$ and its principal minors are diagonally dominant for $\lambda < 0$;

(iii) $A_1^T - b_N c_N \alpha(\nu(\nu+1)) e_N e_N^T - \lambda I_1$ and its principal minors are diagonally dominant for $\lambda \ge \nu(\nu+1)$;

(iv) $A_2^T - \lambda I_2$ is diagonally dominant for $\lambda \le \nu(\nu+1)$.

THEOREM 4.2. Let B be an infinite complex matrix strictly diagonally dominant. From $x \in \ell_2$ and $Bx = 0$ it follows $x = 0$.

PROOF. By definition

$$x_i = \frac{1}{B_{ii}} \sum_{j \neq i} B_{ij} x_j, \quad i = 1,2,\ldots .$$

In particular, if $x_r \neq 0$ is the largest component in absolute value then it follows that

$$|x_r| < |x_r| .$$

Contradiction. Hence $x = 0$. □

4.4. *References*

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CHAPTER 5

ELECTROMAGNETIC FIELDS IN THE SPHERO-CONAL SYSTEM

5.0. *Introduction*

In this chapter we apply the theory of Lamé functions to electromagnetic fields inside a perfectly conducting cone with elliptical cross-section. In analogy to the case of the circular cone, the electromagnetic fields will now be analysed in terms of two types of partial fields which are due to the transverse electric waves (TE-fields) and transverse magnetic waves (TM-fields) [4;483]. The papers of Debye [2], Bouwkamp and Casimir [1], and Wilcox [5] are essential for this chapter.

5.1. *Electromagnetic field in the interior of a cone with elliptical cross-section*

We consider a cone with the vertex at the origin of the Cartesian coordinate system and main axis along the z axis. In the sphero-conal system we describe the main cone by the parameter k and the angle θ_0 (see chapter 1). Throughout this section we consider a linear homogeneous non-conducting isotropic medium, free of charges and currents with permittivity ε and permeability μ, bounded by two concentric spheres $|\underline{x}| = r_0$ and $|\underline{x}| = r_1$, $0 < r_0 < r_1$, and the surface of the cone, $\theta = \theta_0$.
We shall confine ourselves to the requirements imposed by a perfectly conducting boundary surface of the cone, that is, $E_r = 0$ and $E_\varphi = 0$ for $r_0 < r < r_1$ and $\theta = \theta_0$.
For this region we have [4;5]:

$$\underline{D} = \varepsilon\underline{E},\ \underline{B} = \mu\underline{H},\ \rho = 0 \text{ and } \underline{J} = \underline{0}\ .$$

The Maxwell equations now are

$$\operatorname{curl} \underline{E} + \mu \frac{\partial \underline{H}}{\partial t} = 0,$$

$$\operatorname{curl} \underline{H} - \varepsilon \frac{\partial \underline{E}}{\partial t} = 0,$$

$$\operatorname{div} \underline{E} = 0,\ \operatorname{div} \underline{H} = 0\ .$$

Throughout this section a time dependence $e^{-i\omega t}$ is assumed but always suppressed. We can represent the fields as

$$\underline{E} = \mathrm{Re}(\tilde{\underline{E}}e^{-i\omega t}) \quad \text{and} \quad \underline{H} = \mathrm{Re}(\tilde{\underline{H}}e^{-i\omega t}) \ .$$

We observe that $\tilde{\underline{E}}$ and $\tilde{\underline{H}}$ are complex functions independent of time.

THEOREM 5.1. Let us consider a space domain D bounded by two concentric spheres $|\underline{x}| = r_0$ and $|\underline{x}| = r_1$, $0 < r_0 < r_1$ and by a perfectly conducting surface of an elliptical cone, $r_0 < r < r_1$, $\theta = \theta_0$, $0 \le \varphi < 2\pi$. Let there be in D an analytic electromagnetic field $\tilde{\underline{E}}$, $\tilde{\underline{H}}$ with vanishing radial components $\tilde{E}_r$ and $\tilde{H}_r$. It then follows that $\tilde{\underline{E}}$ and $\tilde{\underline{H}}$ are identically zero within D.

PROOF. Starting from the Maxwell's equations and from $\tilde{E}_r = \tilde{H}_r = 0$ we obtain the following equations:

$$(5.1) \qquad \frac{\partial}{\partial\theta}(h_\varphi^*\tilde{H}_\varphi) = \frac{\partial}{\partial\varphi}(h_\theta^*\tilde{H}_\theta) \ , \quad \frac{\partial}{\partial\theta}(h_\varphi^*\tilde{E}_\varphi) = \frac{\partial}{\partial\varphi}(h_\theta^*\tilde{E}_\theta) \ ,$$

$$i\omega\varepsilon r\tilde{E}_\theta = \frac{\partial}{\partial r}(r\tilde{H}_\varphi) \ , \qquad -i\omega\mu r\tilde{H}_\theta = \frac{\partial}{\partial r}(r\tilde{E}_\varphi) \ ,$$

$$i\omega\varepsilon r\tilde{E}_\varphi = -\frac{\partial}{\partial r}(r\tilde{H}_\theta) \ , \qquad i\omega\mu r\tilde{H}_\varphi = \frac{\partial}{\partial r}(r\tilde{E}_\theta) \ ,$$

in which h_θ^* and h_φ^* are the scale factors of the sphero-conal system (see chapter 1). The integration with respect to r can be carried out immediately. From the last four equations we get

$$\sqrt{\varepsilon}r\tilde{E}_\theta = A_1(\theta,\varphi)e^{ik^*r} + B_1(\theta,\varphi)e^{-ik^*r} \ ,$$

$$\sqrt{\mu}r\tilde{H}_\varphi = A_1(\theta,\varphi)e^{ik^*r} - B_1(\theta,\varphi)e^{-ik^*r} \ ,$$

$$\sqrt{\mu}r\tilde{H}_\theta = A_2(\theta,\varphi)e^{ik^*r} - B_2(\theta,\varphi)e^{-ik^*r} \ ,$$

$$\sqrt{\varepsilon}r\tilde{E}_\varphi = -A_2(\theta,\varphi)e^{ik^*r} - B_2(\theta,\varphi)e^{-ik^*r} \ ,$$

in which $k^{*2} = \omega^2\mu\varepsilon$.

Introducing the new variables

$$\tilde{\theta} := \int_0^\theta \frac{dt}{\sqrt{1 - k^2\cos^2(t)}} \ , \ \tilde{\varphi} := \int_0^\varphi \frac{dt}{\sqrt{1 - k'^2\sin^2(t)}}$$

and the functions

$$(5.2)\quad \begin{aligned} \tilde{A}_i &:= \sqrt{k'^2\cos^2(\varphi) + k^2\sin^2(\theta)}\,A_i, \quad i = 1,2,\\ \tilde{B}_i &:= \sqrt{k'^2\cos^2(\varphi) + k^2\sin^2(\theta)}\,B_i, \quad i = 1,2, \end{aligned}$$

we readily get from (5.1)

$$\Delta\tilde{A}_i = 0,\ \Delta\tilde{B}_i = 0,\ i = 1,2, \quad (\Delta := \frac{\partial^2}{\partial\tilde{\theta}^2} + \frac{\partial^2}{\partial\tilde{\varphi}^2}) .$$

From the boundary condition $\tilde{E}_\varphi(r,\theta_0,\varphi) = 0$ we conclude, using the maximum principle of harmonic functions, that $\tilde{A}_2$ and $\tilde{B}_2$ are identically zero. Hence $\tilde{H}_\theta$ and $\tilde{E}_\varphi$ are identically zero. Substituting these results in equations (5.1) we obtain

$$\frac{\partial\tilde{A}_1}{\partial\tilde{\theta}} = 0,\ \frac{\partial\tilde{A}_1}{\partial\tilde{\varphi}} = 0,\ \frac{\partial\tilde{B}_1}{\partial\tilde{\theta}} = 0,\ \frac{\partial\tilde{B}_1}{\partial\tilde{\varphi}} = 0 .$$

It thus follows that $\tilde{A}_1$ and $\tilde{B}_1$ are constant. Since $\tilde{H}_\varphi$ and $\tilde{E}_\theta$ are uniformly bounded and consequently so are A_1 and B_1, we may conclude by taking $\theta = 0$, $\varphi = \pi/2$ in (5.2), that $\tilde{A}_1$ and $\tilde{B}_1$ are identically zero. Hence $\tilde{H}_\varphi$ and $\tilde{E}_\theta$ are identically zero. This completes the proof. □

REMARK. If we have a coaxial elliptical cone then there exists an electromagnetic field with non-vanishing components

$$\tilde{E}_\theta = (a_1e^{ik^*r} + b_1e^{-ik^*r})/[\sqrt{\varepsilon}r\sqrt{k'^2\cos^2(\varphi) + k^2\sin^2(\theta)}]$$

and

$$\tilde{H}_\varphi = (a_1e^{ik^*r} - b_1e^{-ik^*r})/[\sqrt{\mu}r\sqrt{k'^2\cos^2(\varphi) + k^2\sin^2(\theta)}] ,$$

the so-called TEM-fields, in which a_1 and b_1 are constant.

DEFINITION 5.2. A region in space Ω is simply connected if every continuous closed surface and every continuous closed curve lying in Ω can be contracted to a point without passing outside of Ω. □

In analogy to the case of the spherical waves [4;483], we can express the electromagnetic fields in a simply connected region in space by means of two scalar functions, the so-called Debye potentials.

THEOREM 5.3. Any electromagnetic field in a simply connected space domain can be written as

$$\underline{\tilde{E}} = i\omega \text{ curl}(\Pi_1 r\underline{e}_r) + \text{curl curl}(\Pi_2 r\underline{e}_r) \ ,$$

$$\underline{\tilde{H}} = -i\omega\varepsilon \text{ curl}(\Pi_2 r\underline{e}_r) + \frac{1}{\mu} \text{curl curl}(\Pi_1 r\underline{e}_r) \ ,$$

where Π_1 and Π_2 satisfy the reduced wave equations:

$$\Delta\Pi_1 + \omega^2\mu\varepsilon\Pi_1 = 0, \ \Delta\Pi_2 + \omega^2\mu\varepsilon\Pi_2 = 0 \ .$$

PROOF. Take Jones's proof [4;483] for the analogue of spherical Debye potentials, and replace the scale factors of the spherical polar coordinates by those of the sphero-conal system. It should be remarked that Jones's proof implicitly assumes certain connectivity properties of the domain. If the domain is simply connected, the proof is correct. □

This proof again demonstrates how easy it is to generalize results known in spherical polar coordinates to corresponding results in sphero-conal coordinates. We observe that we can decompose the electromagnetic fields in a simply connected domain in space into two partial fields:

(i) The TE-fields, with vanishing radial component $\tilde{E}_r$:

$$\underline{\tilde{E}} = i\omega \text{ curl}(\Pi_1 r\underline{e}_r), \ \underline{\tilde{H}} = \frac{1}{\mu} \text{curl curl}(\Pi_1 r\underline{e}_r) \ .$$

(ii) The TM-fields, with vanishing radial component $\tilde{H}_r$:

$$\underline{\tilde{E}} = \text{curl curl}(\Pi_2 r\underline{e}_r), \ \underline{\tilde{H}} = -i\omega\varepsilon \text{ curl}(\Pi_2 r\underline{e}_r) \ .$$

We now investigate the electromagnetic fields in a space domain bounded by two concentric spheres $|\underline{x}| = r_0$ and $|\underline{x}| = r_1$, $0 < r_0 < r_1$ and by a perfectly conducting surface of an elliptical cone, $r_0 < r < r_1$, $\theta = \theta_0$. This domain is simply connected. Hence we can decompose the electromagnetic fields into TE-fields and TM-fields, as described before. Now we must require that the electric field components tangential to the cone surface vanish. This means:

(i) $$\tilde{E}_\varphi = \frac{-i\omega}{h_\theta^*} \frac{\partial}{\partial\theta} \Pi_1 = 0 \qquad \text{(TE-fields)}$$

(ii) $$\tilde{E}_r = \frac{\partial^2}{\partial r^2}(\Pi_2 r) + \omega^2\mu\varepsilon\Pi_2 r = 0, \qquad \text{(TM-fields)}$$
$$\tilde{E}_\varphi = \frac{1}{rh_\varphi^*} \frac{\partial^2}{\partial r\partial\varphi}(\Pi_2 r) = 0,$$

in which $r_0 < r < r_1$, $\theta = \theta_0$, $0 \le \varphi < 2\pi$.
Condition (ii) implies that either $\Pi_2(r,\theta_0,\varphi) = 0$ or $\tilde{E}_r$ is identically zero. However, in the latter case the field is identically zero, as follows from the uniqueness theorem 5.1.

THEOREM 5.4. Let us consider a space domain bounded by two concentric spheres $|\underline{x}| = r_0$ and $|\underline{x}| = r_1$, $0 < r_0 < r_1$ and by a perfectly conducting surface of an elliptical cone, $r_0 < r < r_1$, $\theta = \theta_0$, $0 \le \varphi < 2\pi$.
Any electromagnetic field can be decomposed into a TE-field and a TM-field. The TE-field is given by

$$\underline{\tilde{E}} = i\omega \operatorname{curl}(\Pi_1 r\underline{e}_r), \ \underline{\tilde{H}} = \frac{1}{\mu} \operatorname{curl} \operatorname{curl}(\Pi_1 r\underline{e}_r)$$

in which Π_1 is a solution of the Helmholtz equation:

$$\Delta\Pi_1 + \omega^2\mu\varepsilon\Pi_1 = 0$$

with boundary (Neumann) condition

$$\frac{\partial\Pi_1}{\partial\theta}(r,\theta_0,\varphi) = 0 .$$

The TM-field is given by

$$\underline{\tilde{E}} = \operatorname{curl} \operatorname{curl}(\Pi_2 r\underline{e}_r), \ \underline{\tilde{H}} = -i\omega\varepsilon \operatorname{curl}(\Pi_2 r\underline{e}_r)$$

in which Π_2 is a solution of the Helmholtz equation:

$$\Delta\Pi_2 + \omega^2\mu\varepsilon\Pi_2 = 0$$

with boundary (Dirichlet) condition

$$\Pi_2(r,\theta_0,\varphi) = 0 . \qquad \square$$

In chapter 2 we proved that the solutions of the Dirichlet and the Neumann problem, respectively, form a complete set of orthogonal functions. This implies that, given $\tilde{E}_r$ and $\tilde{H}_r$ on part of the sphere $|\underline{x}| = r_0$, $0 \le \theta < \theta_0$, $0 \le \varphi < 2\pi$ and a radiation condition, the electromagnetic field inside the cone is determined in a unique way.

5.2. *Classification of the modes*

We shall classify the TE and TM modes in such a way that if the elliptical cross-section degenerates into a circular one, that means $k^2 \to 1$, TE and TM modes are transformed into the well-known spherical TE and TM modes [3;280].

The TE modes are classified as

$$ {}_{e}TE^{(1,2)}_{mn} : h^{(1,2)}_{\nu_n}(k^* r) L^{(m)}_{cp\nu_n}(\theta) L^{(m)}_{c\nu_n}(\varphi), \; m = 0,1,2,\ldots $$

$n = 1,2,3,\ldots$, where ν_n is the nth positive root of the equation $L^{(m)}_{cp\nu}(\theta_0) = 0$; θ_0 defines the boundary surface of the cone.

$$ {}_{o}TE^{(1,2)}_{mn} : h^{(1,2)}_{\nu_n}(k^* r) L^{(m)}_{sp\nu_n}(\theta) L^{(m)}_{s\nu_n}(\varphi), \; m = 1,2,3,\ldots $$

$n = 1,2,3,\ldots$, where ν_n is the nth positive root of the equation $L^{m}_{sp\nu}(\theta_0) = 0$; θ_0 defines the boundary surface of the cone.

The TM modes are classified as

$$ {}_{e}TM^{(1,2)}_{mn} : h^{(1,2)}_{\nu_n}(k^* r) L^{(m)}_{cp\nu_n}(\theta) L^{(m)}_{c\nu_n}(\varphi), \; m = 0,1,2,\ldots $$

$n = 1,2,3,\ldots$, where ν_n is the nth positive root of the equation $\frac{d}{d\theta} L^{(m)}_{cp\nu}(\theta) \Big|_{\theta=\theta_0} = 0$; θ_0 defines the boundary surface of the cone.

$$ {}_{o}TM^{(1,2)}_{mn} : h^{(1,2)}_{\nu_n}(k^* r) L^{(m)}_{sp\nu_n}(\theta) L^{(m)}_{s\nu_n}(\varphi), \; m = 1,2,\ldots $$

$n = 1,2,3,\ldots$, where ν_n is the nth positive root of the equation $\frac{d}{d\theta} L^{(m)}_{sp\nu}(\theta) \Big|_{\theta=\theta_0} = 0$; θ_0 defines the boundary surface of the cone.

We observe that if $k^2 \to 1$, $L^{(m)}_{cp\nu}(\theta)$ and $L^{(m)}_{sp\nu}(\theta)$ reduce to $P^{m}_{\nu}(\cos(\theta))$. $L^{(m)}_{c\nu}(\varphi)$ and $L^{(m)}_{s\nu}(\varphi)$ reduce to $\cos(m\varphi)$ and $\sin(m\varphi)$.

5.3. *References*

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CHAPTER 6

COMPUTATION OF THE ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

6.0. *Introduction*

For the computation of the θ Lamé solutions to calculate the modes in a perfectly conducting cone as described in the previous chapter, we need the associated Legendre functions of the first kind, $P_\nu^m(\cos(\theta))$, $0 \le \theta < \frac{\pi}{2}$, $\nu > 0$, $m = 0,1,2,\ldots$. Because in the literature with which the author is familiar no algorithms at all are available to compute these Legendre functions, we shall derive a stable algorithm.

6.1. *Computational aspects of the three-term recurrence relations*

To calculate the associated Legendre functions of the first kind, $P_\nu^m(\cos(\theta))$, $0 \le \theta < \frac{\pi}{2}$, $m = 0,1,2,\ldots$, we take the three-term recurrence relation

$$(6.1) \qquad P_\nu^{m+1}(x) + \frac{2mx}{\sqrt{1-x^2}} P_\nu^m(x) + (\nu - m + 1)(\nu + m) P_\nu^{m-1}(x) = 0, \quad m = 1,2,3,\ldots$$

where $x = \cos(\theta)$.

For computational reasons we transform the functions $P_\nu^m(\cos(\theta))$ so that the maximum magnitude is smaller than or equal to 1.

Using lemma 6.3, we put

$$\psi_m := \frac{\Gamma(\nu + 1)}{\Gamma(\nu + m + 1)} P_\nu^m(\cos(\theta))$$

and the three-term recurrence relation (6.1) then becomes

$$(6.2) \qquad \psi_{m+1} + \frac{2m}{\nu + m + 1} \cot(\theta) \psi_m + \frac{\nu - m + 1}{\nu + m + 1} \psi_{m-1} = 0, \quad m = 1,2,3,\ldots .$$

Let

$$p(t) = t^2 + at + b$$

be the characteristic polynomial of the recurrence relation (6.2), with

$$a = \lim_{m \to \infty} \frac{2m \cot(\theta)}{\nu + m + 1} = 2 \cot(\theta)$$

and

$$b = \lim_{m \to \infty} \frac{\nu - m + 1}{\nu + m + 1} = -1 .$$

The zeros of p(t) are

$$t_1 = \tan(\tfrac{1}{2}\theta)$$

and

$$t_2 = -\cot(\tfrac{1}{2}\theta) \ .$$

By virtue of Perron's theorem (see theorem 3.7) the three-term recurrence relation has two linearly independent solutions $\{f_m\}$ and $\{g_m\}$, say, which satisfy

$$\lim_{m\to\infty} \frac{f_{m+1}}{f_m} = \tan(\tfrac{1}{2}\theta)$$

and

$$\lim_{m\to\infty} \frac{g_{m+1}}{g_m} = -\cot(\tfrac{1}{2}\theta) \ .$$

We observe that lemma 3.18 implies

$$\lim_{m\to\infty} \frac{\psi_{m+1}}{\psi_m} = \tan(\tfrac{1}{2}\theta)$$

and hence $\{\psi_m\}$ is a minimal solution to (6.2).
To calculate the minimal solution $\{\psi_m\}$ with backward recursion, we need a normalization relation of the form

$$(6.3) \qquad \sum_{m=0}^{\infty} \alpha_m \psi_m = s, \ s \neq 0 \ ,$$

where α_m are given constants and s is a given non-vanishing function. The normalization relation (6.3) must satisfy the following conditions:

(i) $|s|$ must not be small compared to the first non-vanishing term $|\alpha_m \psi_m|$, because dangerous cancellation of figures will occur.

(ii) The normalization relation must converge as fast as or faster than the backward recursion, because otherwise we should have to calculate too many terms.

From lemma 6.8 we obtain normalization relations which satisfy (i) namely

$$(6.4) \qquad \psi_0 + 2\sum_{m=1}^{\infty} (-1)^m \psi_{2m} = \cos(\nu\theta), \ |\cos(\nu\theta)| \geq \tfrac{1}{2}\sqrt{2}$$

and

(6.5) $$-2\sum_{m=0}^{\infty}(-1)^m\psi_{2m+1} = \sin(\nu\theta)\,,\quad |\sin(\nu\theta)| \geq \tfrac{1}{2}\sqrt{2}\ .$$

We observe that these series converge very slowly for θ near $\frac{\pi}{2}$. Let now

$$r_n := \frac{\psi_{n+1}}{\psi_n}$$

and

$$s_n := \frac{1}{\psi_n}\sum_{m=n+1}^{\infty}\alpha_m\psi_m\ ,$$

then Miller's algorithm [2;37] is as follows.
Select an integer N:
Calculate:

$$r_N^{(N)} := 0,\ r_{n-1}^{(N)} := \frac{-b_n}{a_n + r_n^{(N)}}\ ,\ n = N,N-1,\dots,1\ ,$$

$$s_N^{(N)} := 0,\ s_{n-1}^{(N)} = r_{n-1}^{(N)}(\alpha_n + s_n^{(N)})\,,\ n = N,N-1,\dots,1\ ,$$

$$\psi_0^{(N)} := \frac{s}{\alpha_0 + s_0^{(N)}}\ ,\ \psi_n^{(N)} := r_{n-1}^{(N)}\psi_{n-1}^{(N)},\ n = 1,2,\dots,N\ ,$$

with

$$a_n := \frac{2n}{\nu + n + 1}\cot(\theta)$$

and

$$b_n := \frac{\nu - n + 1}{\nu + n + 1}\ . \qquad \square$$

Let M be the number of Legendre functions we need for the calculation of the θ Lamé functions for a desired degree of accuracy.
Initially, we select an integer N_1 such that

$$(\tan(\tfrac{1}{2}\theta)^{N_1-M}) < \varepsilon$$

where ε is the desired relative accuracy. After this we select another integer $N_2 > N_1$. If

$$|\psi_i^{(N_1)} - \psi_i^{(N_2)}| \leq \varepsilon|\psi_i^{(N_2)}|$$

with $i = 0,1,2,\dots,M$, then we take $\psi_i^{(N_2)}$ as an approximation of ψ_i,

$i = 0,1,\ldots,M$, else we redefine $\psi_i^{(N_1)} := \psi_i^{(N_2)}$ and further we increase N_2 and repeat the algorithm as often as necessary.
We observe that this algorithm converges very slowly for θ near $\frac{\pi}{2}$.
For this, three reasons can be given, namely:

(i) $\lim_{m\to\infty} \frac{\psi_{m+1}}{\psi_m} = \tan(\frac{1}{2}\theta)$ and hence the minimal solution decreases very slowly;

(ii) the difference between the minimal solution and the dominant one is very small;

(iii) the normalization relations (6.4) and (6.5) converge very slowly.

The algorithm was tested with the aid of lemma 6.11, for $0 \le \theta \le 85^\circ$, because there are no accurate tables to refer to. The range of θ is sufficient for the application to the theory of electromagnetism as treated in the previous chapter.

6.2. *Appendix*

LEMMA 6.1.

$$P_\nu^{m+1}(x) + \frac{2mx}{\sqrt{1-x^2}} P_\nu^m(x) + (\nu - m + 1)(\nu + m) P_\nu^{m-1}(x) = 0 ,$$

$$-1 < x < 1,\ m = 1,2,3,\ldots,\ \nu \text{ real} \quad [1;161]. \qquad \square$$

LEMMA 6.2.

$$P_\nu^m(\cos(\theta)) = \frac{i^m\Gamma(\nu+m+1)}{2\pi\Gamma(\nu+1)} \int_0^{2\pi} (\cos(\theta) + i\sin(\theta)\cos(\varphi))^\nu \cos(m\varphi)\,d\varphi$$

$$0 \le \theta \le \frac{\pi}{2},\ \nu > -1,\ m = 0,1,2,\ldots$$

[1;159]. $\square$

LEMMA 6.3.

$$|P_\nu^m(\cos(\theta))| \le \frac{\Gamma(\nu+m+1)}{\Gamma(\nu+1)} ,\ 0 \le \theta \le \frac{\pi}{2} ,\ m = 0,1,2,\ldots,\ \nu > 0 . \qquad \square$$

DEFINITION 6.4. Let $y_{n+1} + a_n y_n + b_n y_{n-1} = 0$, $n = 1,2,\ldots$ be a three-term recurrence relation with two linearly independent solutions f_n and g_n. If

$$\lim_{n\to\infty} \frac{f_n}{g_n} = 0$$

then f_n is said to be the minimal solution and g_n a dominant one of the three-term recurrence relation. □

LEMMA 6.5.

$$P_\nu(z) + 2 \sum_{m=1}^{\infty} \frac{\Gamma(\nu+1)}{\Gamma(\nu+m+1)} P_\nu^m(z)\cos(m\psi) = (z + (z^2-1)^{\frac{1}{2}}\cos(\psi))^\nu ,$$

in which ψ and ν are real and $\mathrm{Re}(z) > 0$ [1;166]. □

LEMMA 6.6.

$$P_\nu^m(x+i0) = i^{-m}P_\nu^m(x), \; -1 < x < 1, \; m = 0,1,2,\ldots, \; \nu \text{ real } [1;143].$$

□

Using lemma 6.5 and lemma 6.6 we obtain:

LEMMA 6.7.

$$P_\nu(x) + 2 \sum_{m=1}^{\infty} \frac{(-1)^m\Gamma(\nu+1)}{\Gamma(\nu+2m+1)} P_\nu^{2m}(x)\cos(2m\psi) -$$

$$- 2i \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\nu+1)}{\Gamma(\nu+2m+2)} P_\nu^{2m+1}(x)\cos((2m+1)\psi) =$$

$$= (x + i\sqrt{1-x^2}\cos(\psi))^\nu, \; 0 < x < 1, \; \psi \text{ and } \nu \text{ real} .$$ □

If $\psi = 0$, then from lemma 6.7 we obtain:

LEMMA 6.8.

$$P_\nu(\cos(\theta)) + 2 \sum_{m=1}^{\infty} \frac{(-1)^m\Gamma(\nu+1)}{\Gamma(\nu+2m+1)} P_\nu^{2m}(\cos(\theta)) = \cos(\nu\theta)$$

and

$$-2 \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\nu+1)}{\Gamma(\nu+2m+2)} P_\nu^{2m+1}(\cos(\theta)) = \sin(\nu\theta), \; 0 \le \theta \le \frac{\pi}{2}, \; \nu \text{ real}.$$

□

If $\psi = \frac{\pi}{2}$, however, we obtain from lemma 6.7:

LEMMA 6.9.

$$P_\nu(x) + 2 \sum_{m=1}^{\infty} \frac{\Gamma(\nu+1)}{\Gamma(\nu+2m+1)} P_\nu^{2m}(x) = x^\nu, \ 0 < x < 1, \ \nu \text{ real} . \qquad \Box$$

LEMMA 6.10.

$$P_\nu^m(0) = \frac{2^m}{\sqrt{\pi}} \cos(\tfrac{1}{2}\pi(\nu+m)) \frac{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}m)}{\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}m)} \quad [1;145] . \qquad \Box$$

LEMMA 6.11.

$$P_\nu(\cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\varphi)) = P_\nu(\cos(\theta))P_\nu(\cos(\theta')) +$$

$$+ 2 \sum_{m=1}^{\infty} \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(\cos(\theta))P_\nu^m(\cos(\theta'))\cos(m\varphi)$$

$$0 \le \theta < \pi, \ 0 \le \theta' < \pi, \ \theta + \theta' < \pi, \ \varphi \text{ real} \quad [1;169]. \qquad \Box$$

LEMMA 6.12.

$$P_\nu^m(-x) = (-1)^m[P_\nu^m(x)\cos(\pi\nu) - \frac{2}{\pi} Q_\nu^m(x)\sin(\pi\nu)] ,$$

and

$$Q_\nu^m(-x) = (-1)^{m+1}[Q_\nu^m(x)\cos(\pi\nu) + \frac{\pi}{2} P_\nu^m(x)\sin(\pi\nu)]$$

$$0 < x < 1, \ m = 0,1,2,\dots \quad [1;144] . \qquad \Box$$

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CHAPTER 7

FURTHER INVESTIGATIONS INTO THE LAMÉ FUNCTIONS

7.0. *Introduction*

This chapter supplements the original doctoral dissertation.
In section 3.2 we derived solutions of the θ Lamé equation in terms of Legendre functions in a heuristic way. We proved that these series converge uniformly in any closed subinterval of $[0, 2\arctan(\sqrt{(1+k)/(1-k)})$, and this is sufficient for our practical applications in conical waveguides. However, Prof. Boersma [2] remarks, that for example in diffraction problems we need solutions defined in any closed subinterval of $[0,\pi)$. For that reason he had found new series for $L^{(2n)}_{cp\nu}(\theta)$ and $L^{(2n+1)}_{sp\nu}(\theta)$, namely:

$$L^{(2n)}_{cp\nu}(\theta) = \sum_{m=0}^{\infty} (-1)^m T(2m) A^{(2n)}_{2m} P^{2m}_{\nu}(k\cos(\theta)) ,$$

$$L^{(2n+1)}_{sp\nu}(\theta) = \sum_{m=0}^{\infty} (-1)^m T(2m+1) B^{(2n+1)}_{2m+1} P^{2m+1}_{\nu}(k\cos(\theta)) .$$

On the same way as described in section 3.2 we obtained new series for $L^{(2n+1)}_{cp\nu}(\theta)$ and $L^{(2n)}_{sp\nu}(\theta)$:

$$L^{(2n+1)}_{cp\nu}(\theta) = \frac{\sin(\theta)}{\sqrt{1-k^2\cos^2(\theta)}} \sum_{m=0}^{\infty} (-1)^m (2m+1) T(2m+1) A^{(2n+1)}_{2m+1} P^{2m+1}_{\nu}(k\cos(\theta)) ,$$

$$L^{(2n)}_{sp\nu}(\theta) = \frac{\sin(\theta)}{\sqrt{1-k^2\cos^2(\theta)}} \sum_{m=1}^{\infty} (-1)^m (2m) T(2m) B^{(2n)}_{2m} P^{2m}_{\nu}(k\cos(\theta)) .$$

Up to now all the θ series are found in a heustic way.
Inspired by the existing results for the periodic and non-periodic solutions of the Mathieu equation, we shall derive the former θ solutions (in terms of Legendre functions) with the aid of integral equations of the periodic Lamé solutions.

7.1. *Integral representations of θ solutions*

Let

$$\Phi(\varphi) = \lambda \int_0^{2\pi} \frac{N(\varphi,\phi)}{\sqrt{1-k'^2\sin^2(\phi)}} \Phi(\phi)\,d\phi$$

be any of the many known integral equations for the Lamé φ solutions (cf. Arscott [1]).
Then by substitution

$$k'\cos(\varphi) = ik\sin(\theta)$$

(see section 3.2) we obtain an integral representation of a θ Lamé function:

$$\Theta(\theta) = \frac{1}{\pi}\int_0^{2\pi} \frac{K(\theta,\phi)}{\sqrt{1 - k'^2\sin^2(\phi)}}\,\Phi(\phi)\,d\phi \; .$$

Starting from appropriate kernels and φ solutions we shall now derive θ solutions.
Arscott [1] deduced 24 kernels which can be classified into eight types.
Four our purpose there are four appropriate kernels, namely:

(i) $P_\nu(\frac{ik'}{k}\cos(\varphi)\cos(\phi))$,

(ii) $k'\sin(\varphi)\sin(\phi)P'_\nu(\frac{ik'}{k}\cos(\varphi)\cos(\phi))$,

(iii) $P_\nu(k'\sin(\varphi)\sin(\phi))$,

(iv) $\frac{ik'}{k}\cos(\varphi)\cos(\phi)P'_\nu(k'\sin(\varphi)\sin(\phi))$.

7.2. *Periodic φ solutions*

In section 3.1 we expanded the periodic eigenfunctions of the φ problem into trigonometric Fourier series. But it should be remarked that these eigenfunctions can also be expanded into trigonometric Fourier series multiplied by the function

$$\sqrt{1 - k'^2\sin^2(\varphi)} \quad [4;65],\ [5]\ .$$

We then obtain the same four classes, namely:

$$L^{(2n)}_{c\nu}(\varphi) = \sqrt{1 - k'^2\sin^2(\varphi)}\sum_{m=0}^{\infty} C^{(2n)}_{2m}\cos(2m\varphi) \; ,$$

$$L^{(2n+1)}_{c\nu}(\varphi) = \sqrt{1 - k'^2\sin^2(\varphi)}\sum_{m=0}^{\infty} C^{(2n+1)}_{2m+1}\cos((2m+1)\varphi) \; ,$$

$$L^{(2n)}_{s\nu}(\varphi) = \sqrt{1 - k'^2\sin^2(\varphi)}\sum_{m=1}^{\infty} D^{(2n)}_{2m}\sin(2m\varphi) \; ,$$

$$L_{s\nu}^{(2n+1)}(\varphi) = \sqrt{1 - k'^2\sin^2(\varphi)} \sum_{m=0}^{\infty} D_{2m+1}^{(2n+1)} \sin((2m+1)\varphi) \ .$$

We now observe that the recurrence relations for the coefficients C_m and D_m can be transformed into the previous recurrence relations, by substitution

$$C_m = T^*(m) A_m \quad \text{and} \quad D_m = T^*(m) B_m$$

where $T^*(m)$ has to satisfy the recurrence relation

$$T^*(m) := \frac{(m+\nu+1)(m-\nu)}{(m-\nu+1)(m+\nu+2)} T^*(m+2)$$

and the appropriate initial values. We can take

$$T^*(m) := \frac{P_\nu^0(0)}{P_\nu^m(0) P_\nu^{-m}(0)} \ .$$

We remark that for $T(m)$ defined in section 3.2, we have

$$T(m) := \frac{P_\nu^0(0)}{P_\nu^m(0)}$$

(see lemma 7.1). This we shall assume to be done.

7.3. *The* θ *solutions*

With the aid of the integral representations given in section 7.1 and the periodic φ solutions given in section 7.2, it is easy to derive the corresponding θ solutions in terms of Legendre functions in the following way.

i) Starting from the kernel

$$P_\nu\left(\frac{ik'}{k}\cos(\varphi)\cos(\phi)\right)$$

the function $L_{cp\nu}^{(2n)}(\theta)$ is expressible as

$$L_{cp\nu}^{(2n)}(\theta) = \frac{1}{\pi} \int_0^{2\pi} \frac{P_\nu(\sin(-\theta)\cos(\varphi))}{\sqrt{1 - k'^2\sin^2(\varphi)}} L_{c\nu}^{(2n)}(\varphi)\, d\varphi$$

$$= \sum_{m=0}^{\infty} T^*(2m) A_{2m}^{(2n)} \frac{1}{\pi} \int_0^{2\pi} P_\nu(\sin(-\theta)\cos(\varphi))\cos(2m\varphi)\, d\varphi \ .$$

Using lemma 7.4 we obtain

$$L_{cp\nu}^{(2n)}(\theta) = \sum_{m=0}^{\infty} T(2m) A_{2m}^{(2n)} P_{\nu}^{2m}(\cos(\theta)) \ .$$

Likewise, $L_{cp\nu}^{(2n+1)}(\theta)$ is expressible as

$$L_{cp\nu}^{(2n+1)}(\theta) = \sum_{m=0}^{\infty} T(2m+1) A_{2m+1}^{(2n+1)} P_{\nu}^{2m+1}(\cos(\theta)) \ .$$

ii) Starting from the kernel

$$k'\sin(\varphi)\sin(\phi) P_{\nu}'(\frac{ik'}{k}\cos(\varphi)\cos(\phi))$$

the function $L_{sp\nu}^{(2n)}(\theta)$ is expressible as

$$L_{sp\nu}^{(2n)}(\theta) = \sqrt{1-k^2\cos^2(\theta)}\ \frac{1}{\pi}\int_0^{2\pi} \frac{\sin(\varphi) P_{\nu}'(\sin(-\theta)\cos(\varphi))}{\sqrt{1-k'^2\sin^2(\varphi)}} L_{s\nu}^{(2n)}(\varphi)\, d\varphi$$

$$= \sqrt{1-k^2\cos^2(\theta)} \sum_{m=1}^{\infty} T^*(2m) B_{2m}^{(2n)} \ \cdot$$

$$\cdot\ \frac{1}{\pi}\int_0^{2\pi} \sin(\varphi) P_{\nu}'(\sin(-\theta)\cos(\varphi))\sin(2m\varphi)\, d\varphi \ .$$

Using lemma 7.8 we obtain

$$L_{sp\nu}^{(2n)}(\theta) = \frac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)} \sum_{m=1}^{\infty} (2m) T(2m) B_{2m}^{(2n)} P_{\nu}^{2m}(\cos(\theta)) \ .$$

Likewise, $L_{sp\nu}^{(2n+1)}$ is expressible as

$$L_{sp\nu}^{(2n+1)}(\theta) = \frac{\sqrt{1-k^2\cos^2(\theta)}}{\sin(\theta)} \sum_{m=0}^{\infty} (2m+1) T(2m+1) B_{2m+1}^{(2n+1)} P_{\nu}^{2m+1}(\cos(\theta)) .$$

iii) Starting from the kernel

$$P_{\nu}(k'\sin(\varphi)\sin(\phi))$$

the function $L_{cp\nu}^{(2n)}(\theta)$ is expressible as

$$L_{cp\nu}^{(2n)}(\theta) = \frac{1}{\pi}\int_0^{2\pi} \frac{P_\nu(\sqrt{1-k^2\cos^2(\theta)}\sin(\varphi))}{\sqrt{1-k'^2\sin^2(\varphi)}} L_{c\nu}^{(2n)}(\varphi)\,d\varphi$$

$$= \sum_{m=0}^{\infty} T^*(2m)A_{2m}^{(2n)} \frac{1}{\pi}\int_0^{2\pi} P_\nu(\sqrt{1-k^2\cos^2(\theta)}\sin(\varphi))\cos(2m\varphi)\,d\varphi .$$

Using lemma 7.5 we obtain

$$L_{cp\nu}^{(2n)}(\theta) = \sum_{m=0}^{\infty} (-1)^m T(2m)A_{2m}^{(2n)} P_\nu^{2m}(k\cos(\theta)) .$$

Likewise, $L_{sp\nu}^{(2n+1)}(\theta)$ is expressible as

$$L_{sp\nu}^{(2n+1)}(\theta) = \sum_{m=0}^{\infty} (-1)^m T(2m+1)B_{2m+1}^{(2n+1)} P_\nu^{2m+1}(k\cos(\theta)) .$$

iv) Starting from the kernel

$$\frac{ik'}{k}\cos(\varphi)\cos(\phi)P_\nu'(k'\sin(\varphi)\sin(\phi))$$

the function $L_{cp\nu}^{(2n+1)}(\theta)$ is expressible as

$$L_{cp\nu}^{(2n+1)}(\theta) = -\sin(\theta)\frac{1}{\pi}\int_0^{2\pi} \frac{\cos(\varphi)P_\nu'(\sqrt{1-k^2\cos^2(\theta)}\sin(\varphi))}{\sqrt{1-k'^2\sin^2(\varphi)}} \cdot L_{c\nu}^{(2n+1)}(\varphi)\,d\varphi$$

$$= -\sin(\theta)\sum_{m=0}^{\infty} T^*(2m+1)A_{2m+1}^{(2n+1)} \cdot \frac{1}{\pi}\int_0^{2\pi} \cos(\varphi)P_\nu'(\sqrt{1-k^2\cos^2(\theta)}\sin(\varphi))\cos((2m+1)\varphi)\,d\varphi .$$

Using lemma 7.9 we obtain

$$L_{cp\nu}^{(2n+1)}(\theta) = \frac{\sin(\theta)}{\sqrt{1-k^2\cos^2(\theta)}} \sum_{m=0}^{\infty} (-1)^m(2m+1)T(2m+1)A_{2m+1}^{(2n+1)} \cdot P_\nu^{2m+1}(k\cos(\theta)) .$$

Likewise, $L_{sp\nu}^{(2n)}(\theta)$ is expressible as

$$L^{(2n)}_{sp\nu}(\theta) = \frac{\sin(\theta)}{\sqrt{1-k^2\cos^2(\theta)}} \sum_{m=1}^{\infty} (-1)^m (2m)\,T(2m)\,B^{(2n)}_{2m} P^{2m}_\nu(k\cos(\theta)) \;.$$

It is easy to verify that also the following kernels are valid:

i) $Q_\nu(\frac{ik'}{k}\cos(\varphi)\cos(\phi))$,

ii) $k'\sin(\varphi)\sin(\phi)Q'_\nu(\frac{ik'}{k}\cos(\varphi)\cos(\phi))$,

iii) $Q_\nu(k'\sin(\varphi)\sin(\phi))$,

iv) $\frac{ik'}{k}\cos(\varphi)\cos(\phi)Q'_\nu(k'\sin(\varphi)\sin(\phi))$.

Consequently, with the aid of the lemmas 7.12 and 7.13 and the above kernels it is easy to obtain series expansions for the functions $L^{(2n)}_{cq\nu}(\theta)$, $L^{(2n+1)}_{cq\nu}(\theta)$, $L^{(2n)}_{sq\nu}(\theta)$ and $L^{(2n+1)}_{sq\nu}(\theta)$.

7.4. *Appendix*

LEMMA 7.1.

$$P^m_\nu(0) = \frac{-2^{m-1}}{\pi^{3/2}}\sin(\pi\nu)\,\Gamma(\frac{m+\nu+1}{2})\,\Gamma(\frac{m-\nu}{2}) \qquad [3;145]. \qquad \square$$

LEMMA 7.2.

$$P_\nu(\cos(\psi)\cos(\theta) + \sin(\psi)\sin(\theta)\cos(\varphi)) =$$

$$P_\nu(\cos(\psi))P_\nu(\cos(\theta)) + 2\sum_{m=1}^{\infty}(-1)^m P^{-m}_\nu(\cos(\psi))P^m_\nu(\cos(\theta))\cos(m\varphi)$$

[3;168] . □

Taking $\psi = \pi/2$ we obtain

LEMMA 7.3.

$$P^m_\nu(\sin(\theta)\cos(\varphi)) = P_\nu(0)P_\nu(\cos(\theta)) +$$

$$+ 2\sum_{m=1}^{\infty}(-1)^m P^{-m}_\nu(0)P^m_\nu(\cos(\theta))\cos(m\varphi) \;. \qquad \square$$

Using this lemma we obtain

LEMMA 7.4.

$$\frac{1}{\pi}\int_0^{2\pi} P_\nu(\sin(-\theta)\cos(\varphi))\cos(m\varphi)\,d\varphi = (-1)^m P_\nu^{-m}(0)\,P_\nu^m(\cos(\theta)),$$
$$m = 0,1,2,\ldots . \quad \square$$

LEMMA 7.5.

$$\frac{1}{\pi}\int_0^{2\pi} P_\nu(\sqrt{1-k^2\cos^2(\theta)}\sin(\varphi))\cos(2m\varphi)\,d\varphi = (-1)^{m+1} P_\nu^{-2m}(0)\,P_\nu^{2m}(k\cos(\theta)),$$
$$m = 0,1,2,\ldots . \quad \square$$

LEMMA 7.6.

$$\frac{1}{\pi}\int_0^{2\pi} P_\nu(\sqrt{1-k^2\cos^2(\theta)}\sin(\varphi))\sin((2m+1)\varphi)\,d\varphi =$$
$$= (-1)^m P_\nu^{-(2m+1)}(0)\,P_\nu^{2m+1}(k\cos(\theta)),\ m = 0,1,2,\ldots . \quad \square$$

By differentiating of the formula of lemma 7.3 with respect to φ we obtain

LEMMA 7.7.

$$\sin(\theta)\sin(\varphi)P_\nu'(\sin(\theta)\cos(\varphi)) =$$
$$2\sum_{m=1}^{\infty} (-1)^m m P_\nu^{-m}(0)\,P_\nu^m(\cos(\theta))\sin(m\varphi) . \quad \square$$

Using this lemma we obtain

LEMMA 7.8.

$$\frac{1}{\pi}\int_0^{2\pi} \sin(\varphi)P_\nu'(\sin(-\theta)\cos(\varphi))\sin(m\varphi)\,d\varphi =$$
$$\frac{(-1)^m}{\sin(\theta)}\, m P_\nu^{-m}(0)\,P_\nu^m(\cos(\theta)),\ m = 0,1,2,\ldots . \quad \square$$

LEMMA 7.9.

$$\frac{1}{\pi}\int_0^{2\pi} \cos(\varphi)P_\nu'(\sqrt{1-k^2\cos^2\theta}\ \sin(\varphi))\cos((2m+1)\varphi)\,d\varphi =$$
$$\frac{(-1)^m(2m+1)}{\sqrt{1-k^2\cos^2(\theta)}}\, P_\nu^{-(2m+1)}(0)\,P_\nu^{2m+1}(k\cos(\theta)),\ m=0,1,2,\ldots . \quad \square$$

LEMMA 7.10.

$$\frac{1}{\pi}\int_0^{2\pi} \cos(\varphi)P_\nu'(\sqrt{1-k^2\cos^2(\theta)}\sin(\varphi))\sin(2m\varphi)\,d\varphi =$$

$$\frac{(-1)^{m+1}(2m)}{\sqrt{1-k^2\cos^2(\theta)}}\,P_\nu^{-2m}(0)P_\nu^{2m}(k\,\cos(\theta))\,,\quad m=1,2,\ldots\ . \qquad \square$$

LEMMA 7.11.

$$Q_\nu(\cos(\psi)\cos(\theta) + \sin(\psi)\sin(\theta)\cos(\varphi)) =$$

$$P_\nu(\cos(\psi))Q_\nu(\cos(\theta)) + 2\sum_{m=1}^{\infty} (-1)^m P_\nu^{-m}(\cos(\psi))Q_\nu^m(\cos(\theta))\cos(m\varphi)$$

[3;169] . □

Taking $\psi = \pi/2$ we obtain

LEMMA 7.12.

$$Q_\nu(\sin(\theta)\cos(\varphi)) = P_\nu(0)Q_\nu(\cos(\theta)) +$$

$$+ 2\sum_{m=1}^{\infty} (-1)^m P_\nu^{-m}(0)Q_\nu^m(\cos(\theta))\cos(m\varphi)\ . \qquad \square$$

By differentiating of this formula with respect to φ we obtain

LEMMA 7.13.

$$\sin(\theta)\sin(\varphi)Q_\nu'(\sin(\theta)\cos(\varphi)) =$$

$$2\sum_{m=1}^{\infty} (-1)^m m P_\nu^{-m}(0)P_\nu^m(\cos(\theta))\sin(m\varphi)\ . \qquad \square$$

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