MATHEMATICAL CENTRE TRACTS 64

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SYMMETRIC CLOSED CATEGORIES

MATHEMATISCH CENTRUM AMSTERDAM 1975

AMS(MOS) subject classification scheme (1970): 18D10, 18D15, 18D99

ISBN 90 6196 112 2

CONTENTS

INTRODUCTION

For a long time it was recognized that in many cases the set $A(XY)$ of morphisms from X into Y in a category A is endowed with an additional structure, such as the structure of an abelian group or a topological space. This observation led to the theory of additive categories and, about 1965, to the theory of enriched catagories. In 1966 S. EILENBERG and G.M. KELLY published their fundamental paper *Closed Categories* ([6]) in which they gave a detailed treatment of the basic theory of closed categories, monoidal closed categories and symmetric monoidal closed categories.

A *closed category* V is a category V_0 equipped with a functor V from V_0 into the category of sets S (in many examples a forgetful functor, but in the general theory of closed categories V need not be faithful) and an internal Hom-functor $[-,-]$: $V_0^* \times V_0 \to V_0$ such that $V[X,Y]$ is the set $V_0(XY)$ of morphisms from X into Y.

A *monoidal category* V is a category V_0 equipped with a bifunctor $-\otimes-$: $V_0 \times V_0 \to V_0$, called the tensor product functor, which is associative up to a natural isomorphism a, and has a two-sided identity I, also up to natural isomorphisms l and r. Although we use the notation and terminology of tensor products, categorical products and even categorical sums give also rise to monoidal structures.

For a closed category V as well as for a monoidal category V one can define the concept of a *V-category* A, that is a "category" whose Hom-functor takes its values $A(XY)$ in the category V_0. With a V-category A is associated an ordinary category A_0, called the underlying category of A, with the same class of objects and such that $A_0(XY) = VA(XY)$. (In order to be able to define the underlying category A_0 in the monoidal case one must assume that V is normalized, i.e. equipped with a functor V: $V_0 \to S$ and a natural isomorphism $\iota_X: VX \to V_0(I,X)$). One can also define V-functors and V-natural transformations, and a great part of the ordinary set-based category theory can be lifted to the V-level.

A *monoidal closed category* is a monoidal category which is also closed, the two structures being related by a natural isomorphism

$$p_{XYZ}: [X\otimes Y,Z] \to [X[YZ]].$$

This isomorphism induces an adjunction of the bifunctors $-\otimes-$ and $[-,-]$:

$$\pi_{XYZ}: V_0(X\otimes Y,Z) \to V_0(X[YZ])$$

and this adjunction induces an interesting interaction between the monoidal structure and the closed structure.

A *symmetric monoidal closed category* is a monoidal closed category with a symmetric monoidal structure, i.e. with a natural isomorphism

$$c_{XY}: X\otimes Y \to Y\otimes X$$

such that $c_{YX} \circ c_{XY} = 1_{X\otimes Y}$.

The exact definitions of all these concepts are to be found in [6]; they will be recalled in the sequel.

This tract could be considered as a supplement to the paper *Closed Categories* of EILENBERG and KELLY ([6]). It contains an exposition of the basic theory of two structures which are closely related to the closed and the monoidal closed categories, and the treatment is along the lines indicated in [6]. Although for the reader's convenience the definitions of the the relevant concepts and some of the theorems will be repeated here, for a good idea of the theory developed in the sequel, some familiarity with the theory of closed categories, as exposed in [6], is desirable.

After a first chapter of introductory character, in chapter II symmetric closed categories are introduced and investigated. A *symmetric closed category* is a closed category with an additional natural isomorphism

$$s_{XYZ}: [X[YZ]] \to [Y[XZ]]$$

such that $s_{YXZ} \circ s_{XYZ} = 1_{[X[YZ]]}$. The existence of a monoidal structure is not assumed. The natural isomorphism s induces a self-adjunction of the internal Hom-functor:

$$\sigma_{XYZ}\colon V_0(X[YZ]) \to V_0(Y[XZ]).$$

Many of the closed categories which appear in mathematics are symmetric closed categories; in fact, any symmetric monoidal closed category is a symmetric closed category. For example, the closed categories $TOP(pot)$ of topological spaces (the function spaces supplied with the point open topology), ΛMOD of modules over a commutative ring Λ, BAN_1 of Banach spaces and linear contractions, and $GRAPH$ of graphs are all symmetric closed categories, with the natural isomorphism s given by

$$((s_{XYZ}f)y)x = (fx)y$$

(where f: $X \to [YZ]$, $x \in X$ and $y \in Y$). All these examples are in fact symmetric monoidal closed categories.

In [15] A. KOCK conjectures that, "as soon as a reasonable definition of (non-monoidal) symmetric closed category has been found" the closed category of T-algebras generated by a commutative monad T will turn out to be symmetric closed. His conjecture is correct, as is shown in a separate paper (W.J. DE SCHIPPER [21]). In KOCK's example a monoidal structure seems to be absent.

The structure of symmetric closed categories is considerably richer than that of closed categories. The symmetry allows us to define the dual A^* of a V-category A and to introduce contravariant V-functors. In this context, the introduction of V-natural transformations, in [6] a rather complicated procedure using a generalized Yoneda lemma, becomes rather simple, because the bifunctor $A(-,-)\colon A_0^* \times A_0 \to V_0$ (where A_0 is the underlying category of the V-category A) can be defined straightforwardly. All the "canonical" natural transformations turn out to be V-natural. It is possible to define a kind of V-natural transformations in the context of symmetric closed categories without reference to the underlying category of a V-category; this is shown in W.J. DE SCHIPPER [20].

In chapter III, semi monoidal closed categories are introduced and studied. Let us first give an example. The cartesian product induces a symmetric monoidal structure in the category TOP of topological spaces. The interaction of this monoidal structure with the closed structure induced by the point open topology is not exactly that of a monoidal closed category. There exists a natural transformation

$$p_{XYZ}: [X \times Y, Z] \longrightarrow [X[YZ]]$$

and all the axioms of a monoidal closed category are fulfilled, but the transformation p fails to be a natural *iso*morphism, as is required in the definition of a monoidal closed category. In the third chapter we give an axiomatic treatment of such structures. A *semi monoidal closed category* is a monoidal category which is also closed, the two structures being related by a natural transformation

$$T_{XYZW}: [XZ] \otimes [YW] \longrightarrow [X \otimes Y, Z \otimes W].$$

One can then *define* a natural transformation

$$p_{XYZ}: [X \otimes Y, Z] \longrightarrow [X[YZ]]$$

and prove all the axioms of a monoidal closed category although p need not be a natural *iso*morphism. One can also define the tensor product $\mathcal{A} \otimes \mathcal{B}$ of the $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{V}$-bifunctors. We also investigate what happens if both the monoidal and the closed structure are symmetric. In that case all the "canonical" natural transformations turn out to be $\mathcal{V}$-natural.

We do not treat the extra structure which arises when the bifunctor ⊗ is the categorical product in $\mathcal{V}_0$, i.e. a right adjoint to the diagonal functor $\Delta: \mathcal{V}_0 \longrightarrow \mathcal{V}_0 \times \mathcal{V}_0$. So in the example $\mathcal{TOP}(pot)$, the theory of semi monoidal closed categories does not describe the whole interaction between the internal Hom-functor and the Cartesian product. For example, the natural isomorphism

$$b_{XYZ}: [X, Y \times Z] \longrightarrow [X,Y] \times [X,Z]$$

is not considered.

In chapter IV we investigate monoidal symmetric closed categories. A *monoidal symmetric closed category* is a monoidal closed category in which the closed part is symmetric. (Recall that a *symmetric monoidal closed category*, defined by EILENBERG and KELLY, is a monoidal closed category in which the monoidal part is symmetric.) In a monoidal symmetric closed category we have two adjunctions

$$\pi_{XYZ}\colon \mathcal{V}_0(X\otimes Y,Z) \longrightarrow \mathcal{V}_0(X[YZ])$$

and

$$\sigma_{XYZ}\colon \mathcal{V}_0(X[YZ]) \longrightarrow \mathcal{V}_0(Y[XZ]).$$

These induce a strong interaction between the monoidal structure and the closed structure; in fact, the monoidal structure is completely determined by the closed structure and the existence of the natural isomorphism p which is subject to one axiom. We first show that monoidal symmetric closed categories and symmetric monoidal closed categories are essentially the same. Next we show that each monoidal symmetric closed category is a symmetric semi-monoidal closed category, so that the results of chapter III are valid for monoidal symmetric closed categories.

The final chapter V contains examples which illustrate the investigated structures and show that they are omnipresent in mathematics. We give an example of a symmetric closed category which is not a monoidal symmetric closed category, and another example (the category of Hausdorff spaces, the function spaces supplied with the compact open topology) shows that a monoidal closed category need not be a semi-monoidal closed category.

The theory of symmetric closed categories and semi monoidal closed categories is developed to roughly the same extent as EILENBERG and KELLY develop the theory of closed and monoidal closed categories in [6]. So topics as $\mathcal{V}$-adjunctions, $\mathcal{V}$-monads, cotensored $\mathcal{V}$-categories etc. are not considered. An application of the theory of symmetric closed categories to the theory of $\mathcal{V}$-monads is given in a separate paper [21]. A coherence theory for the investigated structures is also missing. The coherence results of G.M. KELLY and S. MACLANE [11] for symmetric monoidal closed categories are not used, although application of these results might possibly lead to shorter proofs in the final sections of chapter IV, after we have proved that monoidal symmetric closed categories and symmetric monoidal closed categories are essentially the same. I have preferred to give constructive proofs in these sections, since such proofs provide more insight in the dependencies between the various natural transformations and properties.

CHAPTER I

PRELIMINARIES

1. NOTATIONS AND CONVENTIONS

Since this tract can be considered as an extension of the paper [6] *Closed Categories* by S. EILENBERG and G.M. KELLY, our notation is chosen to agree with theirs, with only a few differences of minor importance which are mentioned below.

If A is a category, $A(XY)$ denotes the Hom-set of all morphisms in A from X into Y. We use A^* for the dual of a category A, and $T^*: A^* \to B^*$ for the dual of a functor $T: A \to B$. The symbol S is reserved for the category of sets. The several natural transformations which occur in S are denoted by symbols in italics (see the propositions II.2.5, III.2.4 and IV.3.4).

Following EILENBERG and KELLY, we often omit brackets and commas. For a Hom-set we usually write $A(XY)$, not $A(X,Y)$ and for a bifunctor T we mostly write T(XY), not T(X,Y). We also often omit indices if no confusion is likely. For example, L^X_{YZ}: [YZ] $\to$ [[XY][XZ]] is often abbreviated to L^X. If $\alpha = (\alpha_X)_{X \in obA}$ is a family of morphisms with α_X: TX $\to$ SX we abbreviate to $\alpha = \alpha_X$: TX $\to$ SX.

The most important difference between our notation and that of EILENBERG and KELLY is that we use the symbol [-,-] for the internal Hom-functor in a closed category, whereas they use the symbol (-,-). Ordered pairs are denoted by angular brackets $\langle -,- \rangle$. Another difference is that we have chosen the symbols d and e for two particular natural transformations, occurring in Chapter IV, section 2, in agreement with more recent publications, whereas they use the symbols u and v, and that we have reserved the symbol f for the middle four interchange isomorphism (Chapter II, section 1) whereas they use the symbol m. We have used the symbols u, v and m for other particular natural transformations.

In the sequel we do not go into the questions concerning foundations.

We have chosen for a theory of categories, based on a set theory with universes. For our purposes it suffices to work with two universes $\mathcal{U}_0$ and $\mathcal{U}_1$ with $\mathcal{U}_0 \in \mathcal{U}_1$. A category $\mathcal{A}$ is called a $\mathcal{U}_0$-category if the class of objects ob $\mathcal{A}$ is a subset of $\mathcal{U}_0$ and if for each pair of objects X and Y the set $\mathcal{A}(XY)$ is an element of $\mathcal{U}_0$. The categories which are considered are supposed to be $\mathcal{U}_0$-categories. For example if we speak about the category of closed categories $\mathcal{CC}$ we mean the category of closed $\mathcal{U}_0$-categories. $\mathcal{CC}$ itself is not a $\mathcal{U}_0$-category, but a $\mathcal{U}_1$-category.

For a discussion of the foundations, and also for the general theory of categories we refer to MACLANE's book [17]. Sometimes we use the language of hypercategories (= 2-categories). For a definition we refer to [6].

In the remaining two sections of this chapter we recall the definition of generalized natural transformations and we formulate two theorems about adjointness of bifunctors which are used in the chapters II and IV.

2. GENERALIZED NATURAL TRANSFORMATIONS

In this section we recall the definition of generalized natural transformations. This concept includes the ordinary natural transformations but also extraordinary kinds of natural transformations. The contents of this section are taken from S. EILENBERG and G.M. KELLY [7]. In that paper the definition of generalized natural transformations is given and the question of their composition is treated. The related notion of dinatural transformations, introduced by E. DUBUC and ROSS STREET, is mentioned in [17].

2.1. DEFINITION. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{E}$ be categories. Consider functors

$$T\colon \mathcal{A} \times \mathcal{B}^* \times \mathcal{B} \longrightarrow \mathcal{E}$$

and

$$S\colon \mathcal{A} \times \mathcal{C}^* \times \mathcal{C} \longrightarrow \mathcal{E}.$$

Let

$$a = a_{XYZ}\colon T(XYY) \longrightarrow S(XZZ)$$

be a family of morphisms in $\mathcal{E}$, indexed by $\langle X,Y,Z\rangle \in \text{ob } \mathcal{A} \times \text{ob } \mathcal{B} \times \text{ob } \mathcal{C}$. This family is called a *generalized natural transformation* if the following three diagrams commute:

(2.1)
$$\begin{array}{ccc}
\mathcal{A}(X,X') & \xrightarrow{T(-YY)} & \mathcal{E}(T(XYY),T(X'YY)) \\
{\scriptstyle S(-ZZ)}\downarrow & & \downarrow{\scriptstyle \mathcal{E}(1,a)} \\
\mathcal{E}(S(XZZ),S(X'ZZ)) & \xrightarrow{\mathcal{E}(a,1)} & \mathcal{E}(T(XYY),S(X'ZZ))
\end{array}$$

(2.2)
$$\begin{array}{ccc}
\mathcal{B}(Y,Y') & \xrightarrow{T(XY'-)} & \mathcal{E}(T(XY'Y),T(XY'Y')) \\
{\scriptstyle T(X-Y)}\downarrow & & \downarrow{\scriptstyle \mathcal{E}(1,a)} \\
\mathcal{E}(T(XY'Y),T(XYY)) & \xrightarrow{\mathcal{E}(1,a)} & \mathcal{E}(T(XY'Y),S(XZZ))
\end{array}$$

(2.3)
$$\begin{array}{ccc}
\mathcal{C}(Z,Z') & \xrightarrow{S(XZ-)} & \mathcal{E}(S(XZZ),S(XZZ')) \\
{\scriptstyle S(X-Z')}\downarrow & & \downarrow{\scriptstyle \mathcal{E}(a,1)} \\
\mathcal{E}(S(XZ'Z'),S(XZZ')) & \xrightarrow{\mathcal{E}(a,1)} & \mathcal{E}(T(XYY),S(XZZ'))
\end{array}$$

If we evaluate these diagrams at $f\colon X \longrightarrow X'$, $g\colon Y \longrightarrow Y'$ and $h\colon Z \longrightarrow Z'$ we obtain the following commutative diagrams:

(2.4)
$$\begin{array}{ccc}
T(XYY) & \xrightarrow{T(f11)} & T(X'YY) \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\
S(XZZ) & \xrightarrow{S(f11)} & S(X'ZZ)
\end{array}$$

(2.5)
$$\begin{array}{ccc}
T(XY'Y) & \xrightarrow{T(11g)} & T(XY'Y') \\
{\scriptstyle T(1g1)}\downarrow & & \downarrow{\scriptstyle a} \\
T(XYY) & \xrightarrow{a} & S(XZZ)
\end{array}$$

(2.6)
$$\begin{array}{ccc}
T(XYY) & \xrightarrow{a} & S(XZZ) \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle S(11h)} \\
S(XZ'Z') & \xrightarrow{S(1h1)} & S(XZZ')
\end{array}$$

These three diagrams can be taken together in a single commuting diagram:

(2.7)
$$\begin{array}{ccccc}
T(XYY) & \xleftarrow{T(1g1)} & T(XY'Y) & \xrightarrow{T(f1g)} & T(X'Y'Y') \\
{\scriptstyle a}\downarrow & & & & \downarrow{\scriptstyle a} \\
S(XZZ) & \xrightarrow{S(f1h)} & S(X'ZZ') & \xleftarrow{S(1h1)} & S(X'Z'Z')
\end{array}$$

In view of the generalization of the notion of natural transformation to V-natural transformation in chapter II we prefer to use the form (2.1) - (2.3) of the naturality conditions, in which the morphisms f,g and h do not occur.

2.2. REMARK. Let I be a category with one object $*$, and one morphism. If $B = I$ we can identify $A \times B^* \times B$ with A, writing TX for T(X**); if also $A = I$, T is given by the object T(***) of E, which we shall also denote by T. By taking two at a time of A, B, C to be I we obtain the following three special cases of generalized natural transformations:

1) a family of morphisms

$$a = a_X\colon TX \longrightarrow SX$$

where T,S: $A \longrightarrow E$, is called natural if the following diagram commutes:

$$\text{(2.8)}\qquad \begin{array}{ccc} A(X,X') & \xrightarrow{\quad T \quad} & E(TX,TX') \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle E(1,a)} \\ E(SX,SX') & \xrightarrow{\quad E(a,1) \quad} & E(TX,SX') \end{array}$$

Evaluation at f: X $\longrightarrow$ X' gives a commutative diagram

$$\text{(2.9)}\qquad \begin{array}{ccc} TX & \xrightarrow{\quad Tf \quad} & TX' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ SX & \xrightarrow{\quad Sf \quad} & SX' \end{array}$$

Hence a: T $\longrightarrow$ S is an ordinary natural transformation.

2) a family of morphisms

$$a = a_Y\colon T(YY) \longrightarrow S$$

where T: $B^* \times B \longrightarrow E$ and $S \in$ ob E, is called natural if the following diagram commutes:

$$
(2.10)\qquad
\begin{array}{ccc}
\mathcal{B}(Y,Y') & \xrightarrow{T(Y'-)} & \mathcal{E}(T(Y'Y),T(Y'Y')) \\
{\scriptstyle T(-Y)}\downarrow & & \downarrow{\scriptstyle \mathcal{E}(1,a)} \\
\mathcal{E}(T(Y'Y),T(YY)) & \xrightarrow{\mathcal{E}(1,a)} & \mathcal{E}(T(Y'Y),S)
\end{array}
$$

Evaluation at g: $Y \longrightarrow Y'$ gives a commutative diagram

$$
(2.11)\qquad
\begin{array}{ccc}
T(Y'Y) & \xrightarrow{T(1g)} & T(Y'Y') \\
{\scriptstyle T(g1)}\downarrow & & \downarrow{\scriptstyle a} \\
T(YY) & \xrightarrow{a} & S
\end{array}
$$

3) a family of morphisms

$$a = a_Z\colon T \to S(ZZ)$$

where $S\colon \mathcal{C}^* \times \mathcal{C} \longrightarrow \mathcal{E}$ and $T \in \text{ob}\ \mathcal{E}$, is called natural if the following diagram commutes:

$$
(2.12)\qquad
\begin{array}{ccc}
\mathcal{C}(Z,Z') & \xrightarrow{S(-Z)} & \mathcal{E}(S(ZZ),S(ZZ')) \\
{\scriptstyle S(-Z')}\downarrow & & \downarrow{\scriptstyle \mathcal{E}(a,1)} \\
\mathcal{E}(S(Z'Z'),S(ZZ')) & \xrightarrow{\mathcal{E}(a,1)} & \mathcal{E}(T,S(ZZ'))
\end{array}
$$

Evaluation at h: $Z \longrightarrow Z'$ gives a commutative diagram

$$
(2.13)\qquad
\begin{array}{ccc}
T & \xrightarrow{a} & S(ZZ) \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle S(1,h)} \\
S(Z'Z') & \xrightarrow{S(h,1)} & S(ZZ')
\end{array}
$$

The latter two types of natural transformations are the so-called extra-ordinary natural transformations. We shall meet them frequently in the sequel, for the first time in the next section. We shall omit the prefix generalized, so that "natural transformation" is sometimes used in the sense of "generalized natural transformation".

3. Adjoint Bifunctors

In this section we formulate for later reference some theorems about adjoint bifunctors. They are the bifunctor analogues of well-known theorems about adjoint functors. For the theory of adjoint functors we refer to [17] chapter IV, where the reader may also find a theorem about adjoint bifunctors (section 7, Theorem 3). We also formulate some statements about bijections between classes of natural transformations in an adjoint situation. These bijections are the prototypes of bijections which play an important role in the chapters II and IV.

3.1. Theorem. (Bifunctor analogue of a theorem about covariant adjoint functors.) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories. Let $T\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ and $H\colon \mathcal{B}^* \times \mathcal{C} \to \mathcal{A}$ be functors.
Then the following conditions are equivalent:

a) There exists a natural isomorphism

$$\pi = \pi_{XYZ}\colon \mathcal{C}(T(XY),Z) \longrightarrow \mathcal{A}(X,H(YZ)).$$

b) There exist natural transformations

$$d = d_{XY}\colon X \longrightarrow H(Y,T(XY))$$

and

$$e = e_{YZ}\colon T(H(YZ),Y) \longrightarrow Z$$

such that the following diagrams are commutative

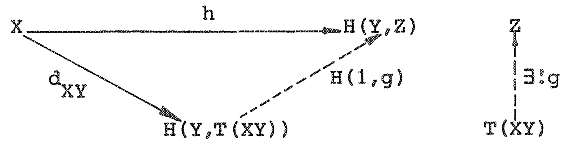
(3.1)
$$\begin{array}{ccc} H(Y,Z) & \xrightarrow{\quad 1 \quad} & H(Y,Z) \\ & {\scriptstyle d_{H(YZ),Y}} \searrow \quad \nearrow {\scriptstyle H(1,e_{YZ})} & \\ & H(Y,T(H(YZ),Y)) & \end{array}$$

(3.2)
$$\begin{array}{ccc} T(X,Y) & \xrightarrow{\quad 1 \quad} & T(X,Y) \\ & {\scriptstyle T(d_{XY},1)} \searrow \quad \nearrow {\scriptstyle e_{Y,T(XY)}} & \\ & T(H(Y,T(XY)),Y) & \end{array}$$

c) There exists a natural transformation

$$d = d_{XY}: X \longrightarrow H(Y,T(XY))$$

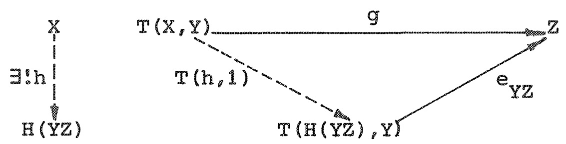
with the following property:
for each morphism $h: X \longrightarrow H(YZ)$ there exists exactly one morphism $g: T(XY) \longrightarrow Z$ such that $H(1,g)d_{XY} = h$.



d) There exists a natural transformation

$$e = e_{YZ}: T(H(YZ),Y) \longrightarrow Z$$

with the following property:
for each morphism $g: T(XY) \longrightarrow Z$ there exists exactly one morphism $h: X \longrightarrow H(YZ)$ such $e_{YZ}T(h,1) = g$.



If the four equivalent conditions are fulfilled, the relations between π, d and e are given by the following formulas:

(3.3) $$d_{XY} = \pi_{XY,T(XY)}(1_{T(XY)})$$

(3.4) $$e_{YZ} = \pi^{-1}_{H(YZ),YZ}(1_{H(YZ)})$$

(3.5) $$\pi_{XYZ}(g) = H(1,g)d_{XY} \quad \text{for } g: T(XY) \longrightarrow Z$$

(3.6) $$\pi^{-1}_{XYZ}(h) = e_{YZ}T(h,1) \quad \text{for } h: X \longrightarrow H(YZ).$$

PROOF. If we keep Y fixed, the theorem is well-known (see for example [17] chapter IV, section 7, theorems 2 and 3). The naturality of π_{XYZ} in the variable Y is equivalent to the (generalized) naturality of d_{XY} and e_{YZ} in the variable Y. The latter means that for each morphism $g\colon Y \to Y'$ the following diagrams commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\quad d \quad} & H(Y,T(XY)) \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle H(1,T(1g))} \\
H(Y',T(XY')) & \xrightarrow{\quad H(g1) \quad} & H(Y,T(XY'))
\end{array}
$$

$$
\begin{array}{ccc}
T(H(Y'Z),Y) & \xrightarrow{\quad T(H(g1),1) \quad} & T(H(YZ),Y) \\
{\scriptstyle T(1,g)}\downarrow & & \downarrow{\scriptstyle e} \\
T(H(Y'Z),Y') & \xrightarrow{\quad e \quad} & Z
\end{array}
$$

□

3.2. DEFINITION. If the equivalent conditions of theorem 3.1 are fulfilled, the ordered triple $\langle T,H,\pi\rangle$ is called an *adjunction of bifunctors*. The bifunctor T is called a *left adjoint* of the bifunctor H and H is called a *right adjoint* of T. π is called the *adjunction-isomorphism*, d is called the *unit* and e is called the *counit* of the adjunction.

3.3. PROPOSITION. ([6], Chapter II, lemma 3.1). Let $\langle T,H,\pi\rangle$ be an adjunction of bifunctors as in theorem 3.1. Let $\mathcal{D}$ be a category, let $P\colon \mathcal{A} \to \mathcal{D}$ and $Q\colon \mathcal{C} \to \mathcal{D}$ be functors and let $Y \in \text{ob}\ \mathcal{B}$.
Commutativity of the diagram

$$
(3.7)\qquad
\begin{array}{ccccc}
\mathcal{C}(T(XY),Z) & & \xrightarrow{\qquad \pi \qquad} & & \mathcal{A}(X,H(YZ)) \\
{\scriptstyle Q}\downarrow & & & & \downarrow{\scriptstyle P} \\
\mathcal{D}(QT(XY),QZ) & \xrightarrow{\mathcal{D}(\alpha,1)} & \mathcal{D}(PX,QZ) & \xleftarrow{\mathcal{D}(1,\beta)} & \mathcal{D}(PX,PH(YZ))
\end{array}
$$

determines a bijection between natural transformations

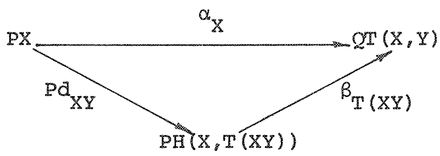
$$\alpha = \alpha_X\colon PX \to QT(XY)$$

and natural transformations

$$\beta = \beta_Z\colon PH(YZ) \longrightarrow QZ.$$

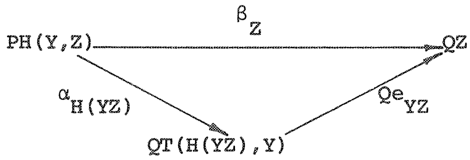
If we take $Z = T(XY)$ and evaluate diagram (3.7) at $1_{T(XY)}$ we see how α depends on β:

(3.8)



If we take $X = H(YZ)$ and evaluate diagram (3.7) at e_{YZ} we see how β depends on α:

(3.9)

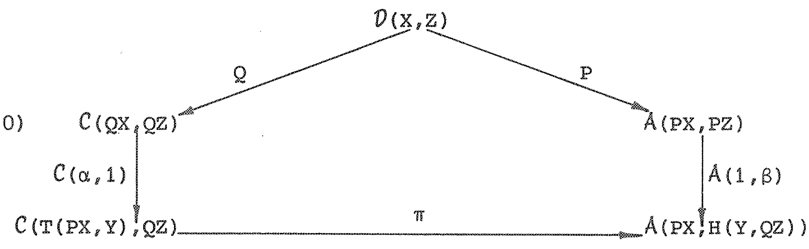


Commutativity of each of the diagrams (3.8) and (3.9) also completely determines the bijection given by commutativity of diagram (3.7). □

3.4. PROPOSITION. Let $\langle T,H,\pi\rangle$ be an adjunction of bifunctors as in theorem 3.1. Let $\mathcal{D}$ be a category, let $P\colon \mathcal{D} \longrightarrow \mathcal{A}$ and $Q\colon \mathcal{D} \longrightarrow \mathcal{C}$ be functors and let $Y \in \text{ob}\,\mathcal{B}$.

Commutativity of the diagram

(3.10)



determines a bijection between natural transformations

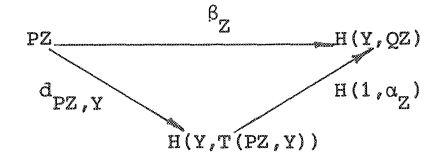
$$\alpha = \alpha_X\colon T(PX,Y) \longrightarrow QX$$

and natural transformations

$$\beta = \beta_Z \colon PZ \longrightarrow H(Y,QZ).$$

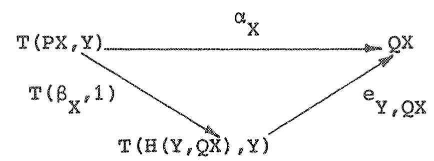
If we take $X = Z$ and evaluate diagram (3.10) at 1_Z we see how β depends on α:

(3.11)



If we take $Z = X$ and evaluate diagram (3.10) at 1_X (after we have replaced π by π^{-1}) we see how α depends on β:

(3.12)



Commutativity of each of the diagrams (3.11) and (3.12) also completely determines the bijection given by commutativity of diagram (3.10). □

3.5. THEOREM. (Bifunctor analogue of a theorem about right adjoint contravariant functors.) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories. Let $H\colon \mathcal{A}^* \times \mathcal{C} \longrightarrow \mathcal{B}$ and $K\colon \mathcal{B}^* \times \mathcal{C} \longrightarrow \mathcal{A}$ be functors.
Then the following conditions are equivalent:

a) There exists a natural isomorphism

$$\sigma = \sigma_{XYZ} \colon \mathcal{A}(X,K(YZ)) \longrightarrow \mathcal{B}(Y,H(XZ)).$$

b) There exist natural transformations

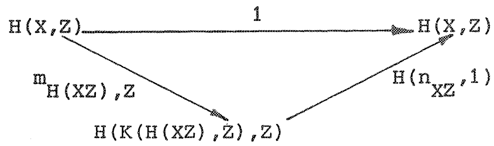
$$m = m_{YZ} \colon Y \longrightarrow H(K(YZ),Z)$$

and

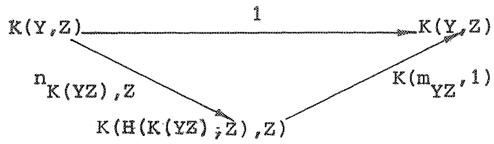
$$n = n_{XZ} \colon X \longrightarrow K(H(XZ),Z)$$

such that the following diagram commutes:

(3.13)



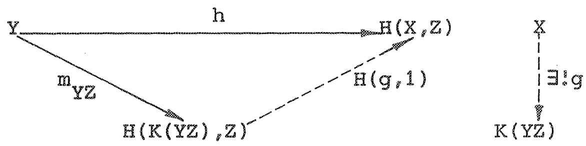
(3.14)



c) There exists a natural transformation

$$m = m_{YZ}: Y \longrightarrow H(K(YZ),Z)$$

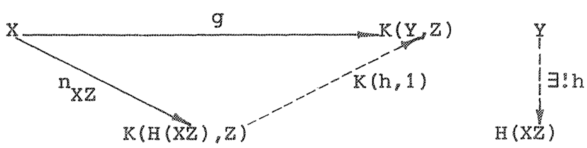
with the following property:
for each morphism $h: Y \longrightarrow H(XZ)$ there exists exactly one morphism $g: X \longrightarrow K(YZ)$ such that $H(g,1)m_{YZ} = h$.



d) There exists a natural transformation

$$n = n_{XZ}: X \longrightarrow K(H(XZ),Z)$$

with the following property:
for each morphism $g: X \longrightarrow K(YZ)$ there exists exactly one morphism $h: Y \longrightarrow H(XZ)$ such that $K(h,1)n_{XZ} = g$.



If the four equivalent conditions are fulfilled, the relations between σ, m and n are given by the following formulas:

(3.15) $$m_{YZ} = \sigma_{K(YZ),YZ}(1_{K(YZ)})$$

(3.16) $$n_{XZ} = \sigma^{-1}_{X,H(XZ),Z}(1_{H(XZ)})$$

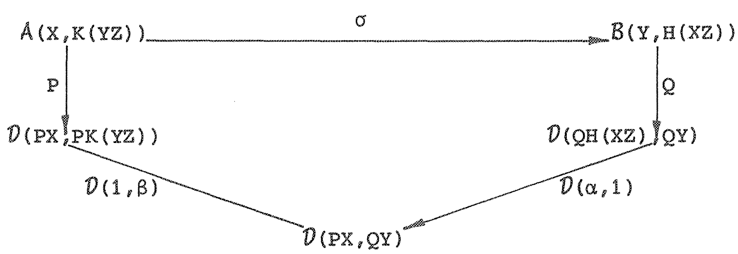
(3.17) $$\sigma_{XYZ}(g) = H(g,1)m_{YZ} \quad \text{for } g\colon X \longrightarrow K(YZ)$$

(3.18) $$\sigma^{-1}_{XYZ}(h) = K(h,1)n_{XZ} \quad \text{for } h\colon Y \longrightarrow H(XZ). \qquad \square$$

3.6. DEFINITION. If the equivalent conditions of theorem 3.5 are fulfilled, the triple $\langle K,H,\sigma\rangle$ is called a *right adjunction of bifunctors*. The bifunctors K and H are called *right adjoint*. σ is called the *adjunction isomorphism*.

3.7. PROPOSITION. Let $\langle K,H,\sigma\rangle$ be a right adjunction of bifunctors as in theorem 3.5. Let $\mathcal{D}$ be a category, let $P\colon \mathcal{A} \longrightarrow \mathcal{D}$ and $Q\colon \mathcal{B}^* \longrightarrow \mathcal{D}$ be functors and let $Z \in \text{ob } \mathcal{C}$.
Commutativity of the diagram

(3.19)



determines a bijection between natural transformations

$$\alpha = \alpha_X\colon PX \longrightarrow QH(XZ)$$

and natural transformations

$$\beta = \beta_Y\colon PK(YZ) \longrightarrow QY.$$

If we take $X = K(YZ)$ and evaluate diagram (3.19) at $1_{K(YZ)}$ we see how β depends on α:

(3.20)
$$\begin{array}{ccc} PK(Y,Z) & \xrightarrow{\beta_Y} & QY \\ {\scriptstyle \alpha_{K(YZ)}} \searrow & & \nearrow {\scriptstyle Qm_{YZ}} \\ & QH(K(YZ),Z) & \end{array}$$

If we take $Y = H(XZ)$ and evaluate diagram (3.19) at n_{XZ} we see how α depends on β:

(3.21)
$$\begin{array}{ccc} PX & \xrightarrow{\alpha_X} & QH(X,Z) \\ {\scriptstyle Pn_{XZ}} \searrow & & \nearrow {\scriptstyle \beta_{H(XZ)}} \\ & PK(H(XZ),Z) & \end{array}$$

Commutativity of each of the diagrams (3.20) and (3.21) also completely determines the bijection, given by commutativity of diagram (3.19). □

3.8. PROPOSITION. Let $\langle K,H,\sigma\rangle$ be a right adjunction of bifunctors as in theorem 3.5. Let $\mathcal{D}$ be a category, let $P: \mathcal{D} \to \mathcal{A}$ and $Q: \mathcal{D}^* \to \mathcal{B}$ be functors and let $Z \in$ ob $\mathcal{C}$.
Commutativity of the diagram

(3.22)
$$\begin{array}{ccc} & \mathcal{D}(X,Y) & \\ {\scriptstyle P} \swarrow & & \searrow {\scriptstyle Q} \\ \mathcal{A}(PX,PY) & & \mathcal{B}(QY,QX) \\ \downarrow {\scriptstyle \mathcal{A}(1,\alpha)} & & \downarrow {\scriptstyle \mathcal{B}(1,\beta)} \\ \mathcal{A}(PX,K(QY,Z)) & \xrightarrow{\sigma} & \mathcal{B}(QY,H(PX,Z)) \end{array}$$

determines a bijection between natural transformations

$$\alpha = \alpha_Y: PY \to K(QY,Z)$$

and natural transformations

$$\beta = \beta_X: QX \to H(PX,Z).$$

If we take $Y = X$ and evaluate diagram (3.22) at 1_X we see how β depends on α:

(3.23)
$$\begin{array}{ccc} QX & \xrightarrow{\qquad\beta_X\qquad} & H(PX,Z) \\ & {\scriptstyle m_{QX,Z}}\searrow \qquad \nearrow{\scriptstyle H(\alpha_X,1)} & \\ & H(K(QX,Z),Z) & \end{array}$$

If we take X = Y and evaluate diagram (3.22) at 1_Y (after we have replaced σ by σ^{-1}) we see how α depends on β:

(3.24)
$$\begin{array}{ccc} PY & \xrightarrow{\qquad\alpha_Y\qquad} & K(QY,Z) \\ & {\scriptstyle n_{PY,Z}}\searrow \qquad \nearrow{\scriptstyle K(\beta_Y,1)} & \\ & K(H(PY,Z),Z) & \end{array}$$

Commutativity of each of the diagrams (3.23) and (3.24) also completely determines the bijection given by commutativity of diagram (3.22). □

CHAPTER II

SYMMETRIC CLOSED CATEGORIES

1. Closed Categories

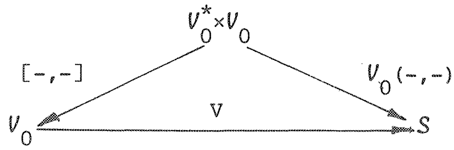
In this section we recall the definition and some properties of a closed category. For more information about closed categories, the reader is referred to [6], chapter I.

1.1. DEFINITION. A *closed category* is an ordered 7-tuple $V = \langle V_0, V, [-,-], I, i, j, L \rangle$ consisting of:

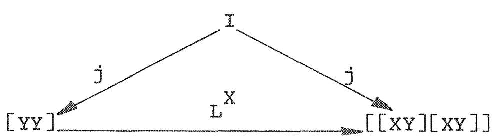
(i) a category V_0 (called the *underlying category of* V);
(ii) a functor $V\colon V_0 \to S$ (called the *basic functor of* V);
(iii) a functor $[-,-]\colon V_0^* \times V_0 \to V_0$ (called the *internal Hom functor*);
(iv) an object I of V_0;
(v) a natural isomorphism $i = i_X\colon X \to [IX]$ in V_0;
(vi) a natural transformation $j = j_X\colon I \to [XX]$ in V_0;
(vii) a natural transformation $L = L^X_{YZ}\colon [YZ] \to [[XY][XZ]]$ in V_0.

These data are to satisfy the following six axioms:

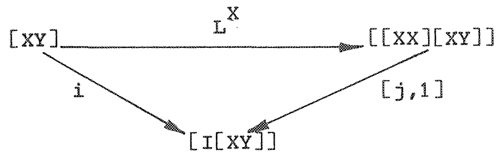
CC0. The following diagram of functors commutes:



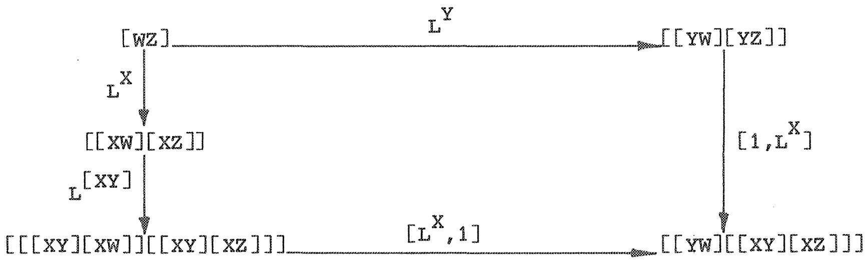
CC1. The following diagram commutes:



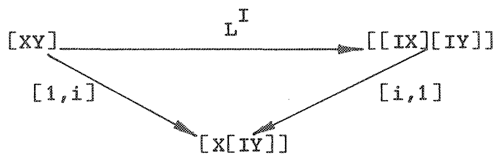
CC2. The following diagram commutes:



CC3. The following diagram commutes:



CC4. The following diagram commutes:



CC5. $Vi_{[XX]}(1_X) = j_X\colon I \longrightarrow [XX]$.

1.2. DEFINITION. Let $V = \langle V_0, V, [-,-], I, i, j, L \rangle$ and $V' = \langle V_0', V', [-,-]', I', i', j', L' \rangle$ be closed categories; we write [XY] for [XY]'. A *closed functor* $\Phi\colon V \longrightarrow V'$ is an ordered triple $\Phi = \langle \phi, \hat{\phi}, \phi^0 \rangle$ consisting of:

(i) a functor $\phi\colon V_0 \longrightarrow V_0'$;
(ii) a natural transformation $\hat{\phi} = \hat{\phi}_{XY}\colon \phi[XY] \longrightarrow [\phi X, \phi Y]$ in V_0';
(iii) a morphism $\phi^0\colon I' \longrightarrow \phi I$ in V_0'.

These data are to satisfy the following three axioms:

CF1. The following diagram commutes:

$$\begin{array}{ccc} I' & \xrightarrow{\quad j' \quad} & [\phi X, \phi X] \\ {\scriptstyle \phi^0}\big\downarrow & & \big\uparrow{\scriptstyle \hat{\phi}} \\ \phi I & \xrightarrow{\quad \phi j \quad} & \phi[X\ X] \end{array}$$

CF2. The following diagram commutes:

$$\begin{array}{ccc} \phi X & \xrightarrow{\quad i' \quad} & [I', \phi X] \\ {\scriptstyle \phi i}\big\downarrow & & \big\uparrow{\scriptstyle [\phi^0, 1]} \\ \phi[I\ X] & \xrightarrow{\quad \hat{\phi} \quad} & [\phi I, \phi X] \end{array}$$

CF3. The following diagram commutes:

$$\begin{array}{ccc} \phi[Y,Z] & \xrightarrow{\quad \phi L^X \quad} & \phi[[XY],[XZ]] \\ {\scriptstyle \hat{\phi}}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\phi}} \\ [\phi Y, \phi Z] & & [\phi[XY], \phi[XZ]] \\ {\scriptstyle L'^{\phi X}}\big\downarrow & & \big\downarrow{\scriptstyle [1, \hat{\phi}]} \\ [[\phi X, \phi Y],[\phi X, \phi Z]] & \xrightarrow{\quad [\hat{\phi}, 1] \quad} & [\phi[XY],[\phi X, \phi Z]] \end{array}$$

1.3. DEFINITION. Let $\Phi = \langle \phi, \hat{\phi}, \phi^0 \rangle$ and $\Psi = \langle \psi, \hat{\psi}, \psi^0 \rangle$ be closed functors $V \longrightarrow V'$. A *closed natural transformation* $\eta\colon \Phi \longrightarrow \Psi\colon V \longrightarrow V'$ is a natural transformation $\eta\colon \phi \longrightarrow \psi\colon V_0 \longrightarrow V'_0$ satisfying the following two axioms:
CN1. The following diagram commutes:

$$\begin{array}{ccccc} & & I' & & \\ & {\scriptstyle \phi^0}\swarrow & & \searrow{\scriptstyle \psi^0} & \\ \phi I & & \xrightarrow{\quad \eta \quad} & & \psi I \end{array}$$

CN2. The following diagram commutes:

$$\begin{array}{ccc} \phi[X\ Y] & \xrightarrow{\quad\eta\quad} & \psi[X\ Y] \\ & & \downarrow{\scriptstyle\hat{\psi}} \\ \downarrow{\scriptstyle\hat{\phi}} & & [\psi X,\psi Y] \\ & & \downarrow{\scriptstyle[\eta,1]} \\ [\phi X,\phi Y] & \xrightarrow{\quad[1,\eta]\quad} & [\phi X,\psi Y] \end{array}$$

1.4. THEOREM. ([6] Theorem I.3.1 and theorem I.4.2). Closed categories, closed functors and closed natural transformations form a hypercategory CC, if we define the composite of $\Phi = \langle \phi,\hat{\phi},\phi^0 \rangle : V \to V'$ and $\Psi = \langle \psi,\hat{\psi},\psi^0 \rangle : V' \to V''$ to be $X = \langle \chi,\hat{\chi},\chi^0 \rangle : V \to V''$, where

(i) χ is the composite $V_0 \xrightarrow{\phi} V'_0 \xrightarrow{\psi} V''_0$;

(ii) $\hat{\chi}_{XY}$ is the composite $\psi\phi[XY] \xrightarrow{\psi\hat{\phi}} \psi[\phi X,\phi Y] \xrightarrow{\hat{\psi}} [\psi\phi X,\psi\phi Y]$;

(iii) χ^0 is the composite $I'' \xrightarrow{\psi^0} \psi I' \xrightarrow{\psi\phi^0} \psi\phi I$;

if we define the composite of $\eta\colon \Phi \to \Phi'$ and $\zeta\colon \Phi' \to \Phi''$ to be $\zeta\eta$ and if for $\Psi\colon V' \to V$, $X\colon W \to W'$ and $\eta\colon \Phi \to \Phi'\colon V \to W$, we define $\eta\Psi$ to be $\eta\psi$ and $X\eta$ to be $\chi\eta$. □

1.5. PROPOSITION. ([6], lemma I.2.2). Let V_0 be a category. Let $V\colon V_0 \to S$ and $[-,-]\colon V_0^* \times V_0 \to V_0$ be functors satisfying CC0. Let $I \in$ ob V_0, let $i = i_X\colon X \to [IX]$ and $j = j_X\colon I \to [XX]$ be natural transformations satisfying CC5.
Then for any $g\colon [XX] \to Y$ in V_0, the composite

$$I \xrightarrow{j} [XX] \xrightarrow{g} Y$$

is the image of $1_X \in V_0(XX) = V[XX]$ under the composite map

$$V[XX] \xrightarrow{Vg} VY \xrightarrow{Vi} V[IY].$$ □

1.6. PROPOSITION. ([6], proposition I.2.3). Let V_0 be category. Let $V\colon V_0 \to S$ and $[-,-]\colon V_0^* \times V_0 \to V_0$ be functors satisfying CC0. Let $I \in$ ob V_0, let $i = i_X\colon X \to [IX]$ be a natural isomorphism and let $j = j_X\colon I \to [XX]$ be a natural transformation satisfying CC5. Let $L = L^X_{YZ}\colon [YZ] \to [[XY][XZ]]$ be a natural transformation. Then the axiom CC1 is equivalent to any of the following:

CC1'. $(VL^X_{YY})(1_Y) = 1_{[XY]}$;

CC1". $(VL^X_{YZ})g = [1,g] \in V_0([XY][XZ])$ for $g \in V_0(YZ)$;

CC1"'. $VL^X_{YZ} = [X-]: V_0(YZ) \longrightarrow V_0([XY][XZ])$.

In particular, in a closed category the properties CC1', CC1" and CC1"' hold. □

1.7. PROPOSITION. ([6], propositions I.3.4 and 3.5). Let V and V' be closed categories and let $\phi: V_0 \longrightarrow V'_0$ be a functor.
Commutativity of the diagram

(1.1)
$$\begin{array}{ccccc} & & V[IX] & & \\ & \swarrow^{(Vi)^{-1}} & & \searrow^{\phi} & \\ VX & & & & V'[\phi I,\phi X] \\ \downarrow{\scriptstyle \phi_0} & & & & \downarrow{\scriptstyle V'[\phi^0,1]} \\ V'\phi X & & \xrightarrow{\quad V'i' \quad} & & V'[I',\phi X] \end{array}$$

sets up a bijection between morphisms $\phi^0: I' \longrightarrow \phi I$ and natural transformations $\phi_0: V \longrightarrow V'\phi: V_0 \longrightarrow S$.

If ϕ^0 and ϕ_0 are related by (1.1) and if $\hat{\phi} = \hat{\phi}_{XY}: \phi[XY] \longrightarrow [\phi X,\phi Y]$ is a natural transformation, then property CF1 is equivalent to the commutativity of the following diagram:

(1.2)
$$\begin{array}{ccccc} & & V[XY] & & \\ & \swarrow^{\phi_0} & & \searrow^{\phi} & \\ V'\phi[XY] & & \xrightarrow{\quad V'\hat{\phi} \quad} & & V'[\phi X,\phi Y] \end{array}$$
□

1.8. DEFINITION. A closed functor $\Phi = \langle \phi,\hat{\phi},\phi^0 \rangle : V \longrightarrow V$ is called *normal* if $V = V'\phi$ and $\phi_0 = 1: V \longrightarrow V'\phi$.

2. Symmetric Closed Categories

2.1. DEFINITION. A *symmetric closed category* is an ordered 8-tuple $\mathcal{V} = \langle \mathcal{V}_0, V, [-,-], I, i, j, L, s \rangle$ consisting of:

(i) a category $\mathcal{V}_0$ (called the *underlying category* of $\mathcal{V}$);

(ii) a functor $V: \mathcal{V}_0 \longrightarrow \mathcal{S}$ (called the *basic functor* of $\mathcal{V}$);

(iii) a functor $[-,-]: \mathcal{V}_0^* \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ (called the *internal Hom functor*);

(iv) an object I of $\mathcal{V}_0$;

(v) a natural isomorphism $i = i_X: X \longrightarrow [IX]$ in $\mathcal{V}_0$;

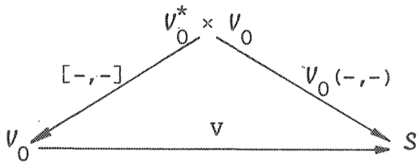
(vi) a natural transformation $j = j_X: I \longrightarrow [XX]$ in $\mathcal{V}_0$;

(vii) a natural transformation $L = L^X_{YZ}: [YZ] \longrightarrow [[XY][XZ]]$ in $\mathcal{V}_0$;

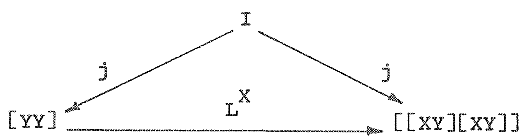
(viii) a natural isomorphism $s = s_{XYZ}: [X[YZ]] \longrightarrow [Y[XZ]]$ in $\mathcal{V}_0$.

These data are to satisfy the following seven axioms:

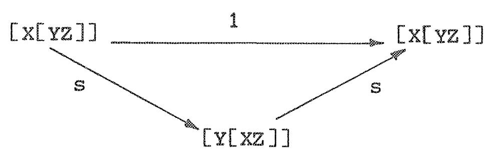
CC0. The following diagram of functors commutes:



CC1. The following diagram commutes:



SCC1. The following diagram commutes:



SCC2. The following diagram commutes:

$$\begin{array}{ccc} [X\,[Y[ZW]]] & \xrightarrow{[1,s]} & [X\,[Z[YW]]] \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ [Y\,[X[ZW]]] & & [Z\,[X[YW]]] \\ \downarrow{\scriptstyle [1,s]} & & \downarrow{\scriptstyle [1,s]} \\ [Y\,[Z[XW]]] & \xrightarrow{s} & [Z\,[Y[XW]]] \end{array}$$

SCC3. The following diagram commutes:

$$\begin{array}{ccc} [Y\,[WZ]] & \xrightarrow{s} & [W\,[YZ]] \\ \downarrow{\scriptstyle L^X} & & \\ [[XY]\,[X[WZ]]] & & \downarrow{\scriptstyle [1,L^X]} \\ \downarrow{\scriptstyle [1,s]} & & \\ [[XY]\,[W[XZ]]] & \xrightarrow{s} & [W\,[[XY][XZ]]] \end{array}$$

SCC4. The following diagram commutes:

$$\begin{array}{ccc} [X\ Y] & \xrightarrow{L^X} & [[XX]\,[XY]] \\ {\scriptstyle [1,i]}\downarrow & & \downarrow{\scriptstyle [j,1]} \\ [X\,[IY]] & \xrightarrow{s} & [I\,[XY]] \end{array}$$

CC5. $Vi_{[XX]}(1_X) = j_X\colon I \longrightarrow [XX]$.

2.2. DEFINITION. Let $V = \langle V_0, V, [-,-], I, i, j, L, s\rangle$ and $V' = \langle V'_0, V', [-,-]', I', i', j', L', s'\rangle$ be two symmetric closed categories; we write $[XY]$ for $[XY]'$.

A *symmetric closed functor* $\Phi\colon V \longrightarrow V'$ is an ordered triple $\Phi = \langle \phi, \hat{\phi}, \phi^0\rangle$ consisting of:

(i) a functor $\phi\colon V_0 \longrightarrow V'_0$;

(ii) a natural transformation $\hat{\phi} = \hat{\phi}_{XY}\colon \phi[XY] \longrightarrow [\phi X, \phi Y]$ in V'_0;

(iii) a morphism $\phi^0\colon I' \longrightarrow \phi I$ in V'_0.

These data are to satisfy the following two axioms:

CF1. The following diagram commutes:

$$\begin{array}{ccc} I' & \xrightarrow{\quad j' \quad} & [\phi X, \phi X] \\ \Big\downarrow \phi^0 & & \Big\uparrow \hat{\phi} \\ \phi I & \xrightarrow{\quad \phi j \quad} & \phi[X\ X] \end{array}$$

SCF3. The following diagram commutes:

$$\begin{array}{ccc} \phi[X\ [YZ]] & \xrightarrow{\qquad \phi s \qquad} & \phi[Y\ [XZ]] \\ \Big\downarrow \hat{\phi} & & \Big\downarrow \hat{\phi} \\ [\phi X, \phi[YZ]] & & [\phi Y, \phi[XZ]] \\ \Big\downarrow [1,\hat{\phi}] & & \Big\downarrow [1,\hat{\phi}] \\ [\phi X\ [\phi Y, \phi Z]] & \xrightarrow{\qquad s' \qquad} & [\phi Y\ [\phi X, \phi Z]] \end{array}$$

2.3. DEFINITION. Let $\Phi = \langle \phi, \hat{\phi}, \phi^0 \rangle$ and $\Psi = \langle \psi, \hat{\psi}, \psi^0 \rangle$ be symmetric closed functors $V \longrightarrow V'$. A *symmetric closed natural transformation*

$$\eta\colon \Phi \longrightarrow \Psi\colon V \longrightarrow V'$$

is a natural transformation $\eta\colon \phi \longrightarrow \psi\colon V_0 \longrightarrow V_0'$ satisfying the axioms CN1 and CN2 of definition 1.3.

2.4. THEOREM. Symmetric closed categories, symmetric closed functors and symmetric closed natural transformations form a hypercategory SCC, if we define the several sorts of composites as in theorem 1.4.

PROOF. A straightforward verification of the axioms. □

2.5. PROPOSITION. *The symmetric closed category of sets.*
We obtain a symmetric closed category, denoted by S, if we define: $V_0 = S$ (the category of sets); $V = 1_S$; $[-,-] = S(-,-)\colon S^* \times S \to S$; for I we take a one point set $\{*\}$, chosen once for all;

$$(L^X_{YZ}h)g = hg \qquad (h \in S(YZ) \text{ and } g \in S(XY));$$

$$((s_{XYZ}g)y)x = (gx)y \qquad (g \in S(X,S(YZ));\ x \in X;\ y \in Y);$$

$$(i_X x)* = x \qquad (x \in X);$$

$$j_X * = 1_X.$$

CONVENTION. Throughout this tract the canonical transformations in the symmetric closed category S are denoted by symbols in italics. (See also the propositions III.2.4 and IV.3.4). □

3. RELATIONS BETWEEN NATURAL TRANSFORMATIONS IN THE FIRST BASIC SITUATION

In this section we investigate the relations between the data for a symmetric closed category. For that purpose we define the *first basic situation,* which consists of:

(i) a category V_0;

(ii) a functor $[-,-]\colon V_0^* \times V_0 \longrightarrow V_0$;

(iii) an object I of V_0;

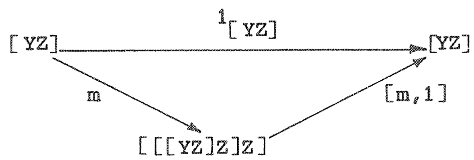
(iv) a natural isomorphism $\sigma = \sigma_{XYZ}\colon V_0(X[YZ]) \longrightarrow V_0(Y[XZ])$ satisfying

(3.1) $$\sigma_{YXZ} \circ \sigma_{XYZ} = 1_{V_0(X[YZ])}.$$

The existence of a natural isomorphism $\sigma_{XYZ}\colon V_0(X[YZ]) \longrightarrow V_0(Y[XZ])$ with (3.1) is equivalent (cf. theorem I.3.5) to the existence of a natural transformation

$$m = m_{YZ}\colon Y \longrightarrow [[YZ]Z]$$

with the property that the following diagram commutes:

(3.2)


The relation between σ and m is given by the equations

(3.3) $$m_{YZ} = \sigma_{[YZ]YZ}(1_{[YZ]})$$

(3.4) $$\sigma_{XYZ}(g) = [g,1]m_{YZ} \qquad \text{for } g\colon X \longrightarrow [YZ].$$

The natural transformation m has the following property: for each morphism $h\colon Y \longrightarrow [XZ]$ there exists exactly one morphism

$g\colon X \longrightarrow [YZ]$ such that $[g,1]m_{YZ} = h$.

(3.5)
$$\begin{array}{ccc} Y & \xrightarrow{\quad h \quad} & [XZ] \\ & {\scriptstyle m}\searrow \qquad \nearrow{\scriptstyle [g,1]} & \\ & [[YZ]Z] & \end{array}$$

We have already seen in section I.3 that a right adjunction of bifunctors induces bijections between several classes of natural transformations. We now describe the bijections which are of importance for the theory of symmetric closed categories.

In the first basic situation, commutativity of the diagram

(3.6)
$$\begin{array}{ccccc} & & \mathcal{V}_0(Y[WZ]) & & \\ & {\scriptstyle [X,-]}\swarrow & & \searrow{\scriptstyle \sigma} & \\ \mathcal{V}_0([XY]\,[X[WZ]]) & & & & \mathcal{V}_0(W\,[YZ]) \\ \downarrow{\scriptstyle \mathcal{V}_0(1,s_{XWZ})} & & & & \downarrow{\scriptstyle \mathcal{V}_0(1,L^X_{YZ})} \\ \mathcal{V}_0([XY]\,[W[XZ]]) & & \xrightarrow{\qquad \sigma \qquad} & & \mathcal{V}_0(W\,[[XY][XZ]]) \end{array}$$

sets up a bijection between natural transformations

$$s = s_{XWZ}\colon [X[WZ]] \longrightarrow [W[XZ]]$$

and natural transformations

$$L = L^X_{YZ}\colon [YZ] \longrightarrow [[XY][XZ]].$$

If we take $W = [YZ]$ and evaluate at m_{YZ} we see how L depends on s:

(3.7)
$$\begin{array}{ccc} [Y\,Z] & \xrightarrow{\qquad L^X \qquad} & [[XY]\,[XZ]] \\ \downarrow{\scriptstyle m} & & \uparrow{\scriptstyle [[1,m],1]} \\ [[[YZ][XZ]]\,[XZ]] & \xrightarrow{\quad [s,1] \quad} & [[X[[YZ]Z]]\,[XZ]] \end{array}$$

If we take $Y = [WZ]$ and evaluate at $1_{[WZ]}$ we see how s depends on L:

$$(3.8)\qquad \begin{array}{ccc} [X[WZ]] & \xrightarrow{\quad s \quad} & [W[XZ]] \\ \Big\downarrow m & & \Big\uparrow [m,1] \\ [[[X[WZ]][XZ]][XZ]] & \xrightarrow{\quad [L^X,1] \quad} & [[[WZ]Z][XZ]] \end{array}$$

Commutativity of each of the diagrams (3.7) and (3.8) also completely determines the bijection given by commutativity of (3.6).

Also in the first basic situation, commutativity of the diagram

$$(3.9)\qquad \begin{array}{ccccc} & & V_0(YW) & & \\ & \swarrow [X,-] & & [-,Z] \searrow & \\ V_0([XY][XW]) & & & & V_0([WZ][YZ]) \\ \Big\downarrow V_0(1,R^Z_{XW}) & & & & \Big\downarrow V_0(1,L^X_{YZ}) \\ V_0([XY][[WZ][XZ]]) & & \xrightarrow{\quad \sigma \quad} & & V_0([WZ][[XY][XZ]]) \end{array}$$

sets up a bijection between natural transformations

$$L = L^X_{YZ}\colon [YZ] \longrightarrow [[XY][XZ]]$$

and natural transformations

$$R = R^Z_{XW}\colon [XW] \longrightarrow [[WZ][XZ]].$$

If we take $W = Y$ and evaluate at 1_Y we obtain the formula

$$(3.10)\qquad L^X_{YZ} = \sigma(R^Z_{XY})$$

and hence also

$$(3.11)\qquad R^Z_{XY} = \sigma(L^X_{YZ}).$$

Each of the two formulas (3.10) and (3.11) also completely determines the bijection given by commutativity of (3.9).

The two bijections mentioned above induce a bijection between natural transformations

$$s = s_{XWZ}\colon [X[WZ]] \longrightarrow [W[XZ]]$$

and natural transformations

$$R = R^{Z}_{XY} \colon [XY] \longrightarrow [[YZ][XZ]].$$

The same bijection is given by commutativity of the following diagram:

(3.12)
$$\begin{array}{ccc}
V_0(Y\ [WZ]) & \xrightarrow{\ \sigma\ } & V_0(W\ [YZ]) \\
\downarrow{\scriptstyle [X,-]} & & \downarrow{\scriptstyle [-,[XZ]]} \\
V_0([XY]\ [X[WZ]]) & & V_0([[YZ][XZ]]\ [W[XZ]]) \\
{\scriptstyle V_0(1,s_{XWZ})} \searrow & & \swarrow {\scriptstyle V_0(R^{Z}_{XY},1)} \\
& V_0([XY][W[XZ]]) &
\end{array}$$

If we take $Y = [WZ]$ and evaluate at $1_{[WZ]}$ we see how s depends on R:

(3.13)
$$\begin{array}{ccc}
[X[WZ]] & \xrightarrow{\ s\ } & [W[XZ]] \\
{\scriptstyle R^{Z}} \searrow & & \nearrow {\scriptstyle [m,1]} \\
& [[[WZ]Z][XZ]] &
\end{array}$$

and if we take $W = [YZ]$ and evaluate at m_{YZ} we see how R depends on s:

(3.14)
$$\begin{array}{ccc}
[XY] & \xrightarrow{\ R^{Z}\ } & [[YZ][XZ]] \\
{\scriptstyle [1,m]} \searrow & & \nearrow {\scriptstyle s} \\
& [X[[YZ]Z]] &
\end{array}$$

Commutativity of each of the diagrams (3.13) and (3.14) also completely determines the bijection given by commutativity of diagram (3.12).

Finally, commutativity of the diagram

(3.15)
$$\begin{array}{ccc}
V_0(X\ Y) & \xrightarrow{\ [X,-]\ } & V_0([XX]\ [XY]) \\
\downarrow{\scriptstyle V_0(1,i_Y)} & & \downarrow{\scriptstyle V_0(j_X,1)} \\
V_0(X\ [IY]) & \xrightarrow{\ \sigma\ } & V_0(I\ [XY])
\end{array}$$

sets up a bijection between natural transformations

$$i = i_Y \colon Y \longrightarrow [IY]$$

and natural transformations

$$j = j_X \colon I \longrightarrow [XX].$$

If we take $Y = X$ and evaluate at 1_X we obtain the formula

(3.16) $\quad j_X = \sigma(i_X).$

3.1. PROPOSITION. Let V be a symmetric closed category. Define a natural isomorphism $\sigma = \sigma_{XYZ} \colon V_0(X[YZ]) \longrightarrow V_0(Y[XZ])$ by $\sigma_{XYZ} = Vs_{XYZ}$, using CC0. Then we obtain the first basic situation, L and s are related by (3.6) and i and j are related by (3.15).

PROOF. From CC0 and SCC1 it follows that we obtain the first basic situation. From CC0, CC1 and CC5 it follows that $VL^X_{YZ} = [X,-] \colon V_0(YZ) \longrightarrow V_0([XY][XZ])$ (proposition 1.6). If we apply V to SCC3 and use CC0 we obtain (3.6); if we apply V to SCC4 and use CC0 we obtain (3.15). □

4. RELATIONS BETWEEN SOME PROPERTIES IN THE FIRST BASIC SITUATION

4.1. PROPOSITION. Suppose that in addition to the first basic situation we have natural isomorphisms

$$s = s_{XYZ} \colon [X[YZ]] \longrightarrow [Y[XZ]]$$

with

$$s_{YXZ} \circ s_{XYZ} = 1_{[X[YZ]]}$$

and

$$i = i_X \colon X \longrightarrow [IX],$$

and natural transformations

$$L = L^X_{YZ} \colon [YZ] \longrightarrow [[XY][XZ]]$$

$$R = R^Z_{XY} \colon [XY] \longrightarrow [[YZ][XZ]]$$

and

$$j = j_X \colon I \longrightarrow [XX],$$

which are connected by (3.6), (3.9) and (3.15).

Then:

(a) CC1 is equivalent to CC4.

(b) CC2 is equivalent to commutativity of the following diagram:

(4.1)
$$\begin{array}{ccc} & I & \\ {\scriptstyle j}\swarrow & & \searrow{\scriptstyle j} \\ [XX] & \xrightarrow{\ R^{Y}\ } & [[XY][XY]] \end{array}$$

(c) CC3 is equivalent to SCC3 and to commutativity of any of the following diagrams:

(4.2)
$$\begin{array}{ccc} [Y\ W] & \xrightarrow{\qquad R^{Z}\qquad} & [[WZ]\ [YZ]] \\ \downarrow{\scriptstyle L^{X}} & & \\ [[XY]\ [XW]] & & \downarrow{\scriptstyle [1,L^{X}]} \\ \downarrow{\scriptstyle [1,R^{Z}]} & & \\ [[XY]\ [[WZ][XZ]]] & \xrightarrow{\qquad s\qquad} & [[WZ]\ [[XY][XZ]]] \end{array}$$

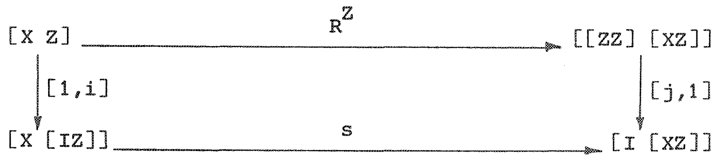
(4.3)
$$\begin{array}{ccc} [Y\ [WZ]] & \xrightarrow{\qquad s\qquad} & [W\ [YZ]] \\ & & \downarrow{\scriptstyle R^{[XZ]}} \\ \downarrow{\scriptstyle L^{X}} & & [[[YZ][XZ]]\ [W[XZ]]] \\ & & \downarrow{\scriptstyle [R^{Z},1]} \\ [[XY]\ [X[WZ]]] & \xrightarrow{\qquad [1,s]\qquad} & [[XY]\ [W[XZ]]] \end{array}$$

PROOF. First we note that (3.9) (together with (3.7) and (3.15)) implies the commutativity of the following diagrams:

(4.4)
$$\begin{array}{ccc} [W\ [XY]] & \xrightarrow{\qquad R^{[XZ]}\qquad} & [[[XY][XZ]]\ [W[XZ]]] \\ \downarrow{\scriptstyle [1,R^{Z}]} & & \downarrow{\scriptstyle [L^{X},1]} \\ [W\ [[YZ][XZ]]] & \xrightarrow{\qquad s\qquad} & [[YZ]\ [W[XZ]]] \end{array}$$

(4.5)
$$\begin{array}{ccc} [W\ [YZ]] & \xrightarrow{\qquad R^{[XZ]}\qquad} & [[[YZ][XZ]]\ [W[XZ]]] \\ \downarrow{\scriptstyle [1,L^{X}]} & & \downarrow{\scriptstyle [R^{Z},1]} \\ [W\ [[XY][XZ]]] & \xrightarrow{\qquad s\qquad} & [[XY]\ [W[XZ]]] \end{array}$$

(4.6)



(a) Apply σ to each leg of CC1:

$$\sigma(L^X \circ j) = [j,1]\sigma(L^X) \qquad \text{by the naturality of } \sigma$$

$$= [j,1]R^Y \qquad \text{by (3.11)}$$

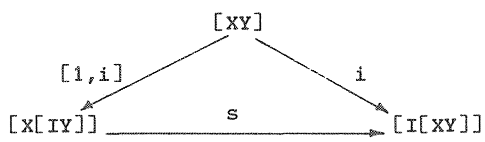
$$= s[1,i] \qquad \text{by (4.6)}$$

and

$$\sigma(j) = i \qquad \text{by (3.15).}$$

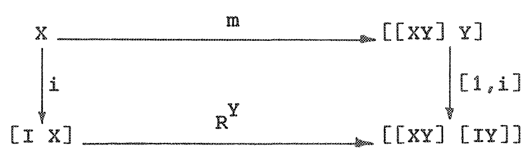
Hence CC1 is equivalent to commutativity of the following diagram:

(4.7)

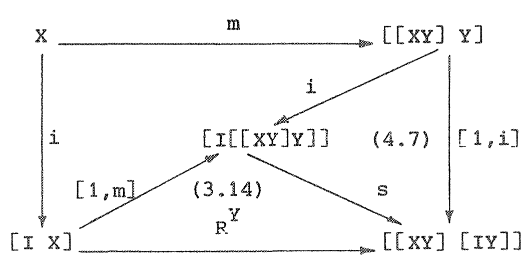


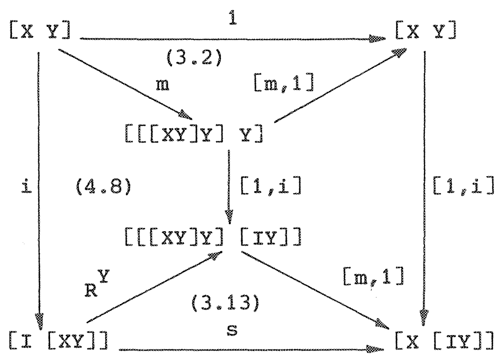
Commutativity of diagram (4.7) is equivalent to commutativity of the following diagram:

(4.8)



as can be seen from the following two diagrams:





Finally an application of σ to each leg of (4.8) shows that the commutativity of this diagram is equivalent to CC4:

$\sigma([1,i]m) = [1,i]\sigma(m)$ by the naturality of σ

$= [1,i]$ by (3.3)

and

$\sigma(R^{Y}i) = [i,1]\sigma(R^{Y})$ by the naturality of σ

$= [i,1]L^{I}$ by (3.10).

(b) Apply σ to each leg of CC2:

$\sigma([j,1]L^{X}) = \sigma(L^{X})\circ j$ by the naturality of σ

$= R^{Y}\circ j$ by (3.11)

and

$\sigma(i) = j$ by (3.15)

Hence CC2 is equivalent to commutativity of diagram (4.1).

(c) Apply σ to each leg of CC3:

$\sigma([1,L^{X}]L^{Y}) = [1,L^{X}]\sigma(L^{Y})$ by the naturality of σ

$= [1,L^{X}]R^{Z}$ by (3.11).

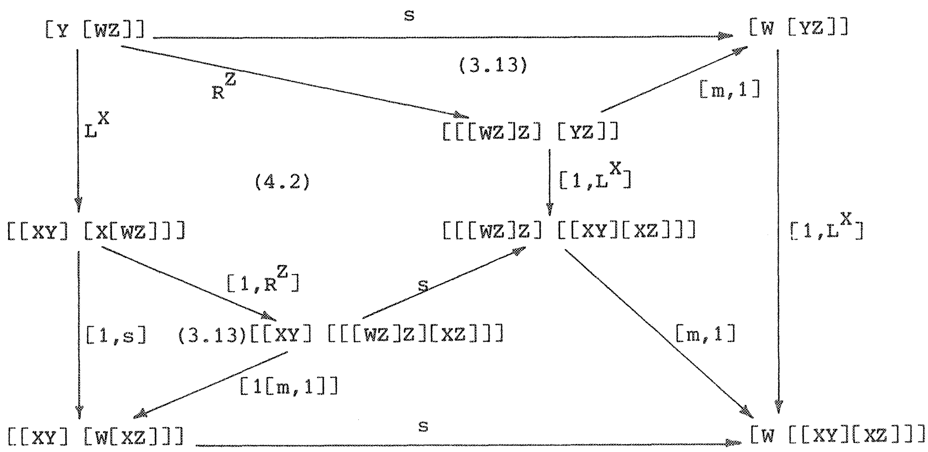
and

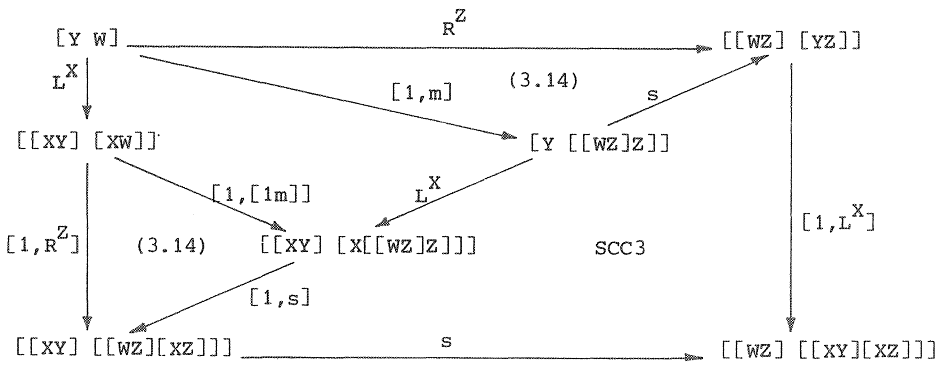
$$\sigma([L^X,1]L^{[XY]}L^X) = [L^X,1]\sigma(L^{[XY]})L^X \quad \text{by the naturality of } \sigma$$

$$= [L^X,1]R^{[XZ]}L^X \quad \text{by (3.11)}$$

$$= s[1,R^Z]L^X \quad \text{by (4.4).}$$

Hence CC3 is equivalent to commutativity of diagram (4.2). The following two diagrams show that this property is equivalent to SCC3:





Finally, commutativity of diagram (4.5) together with the assumption that $s \circ s = 1$ implies the equivalence of SCC3 to commutativity of diagram (4.3). □

4.2. **THEOREM**. Let $\mathcal{V} = \langle \mathcal{V}_0, V, [-,-], I, i, j, L, s \rangle$ be a symmetric closed category. Then ${}^{c}\mathcal{V} := \langle \mathcal{V}_0, V, [-,-], I, i, j, L \rangle$ is a closed category.

PROOF. If we define $\sigma = Vs$ then we are in the first basic situation. L and s are related by (3.6) and i and j are related by (3.15) (proposition 3.1). By proposition 4.1 we have CC1 $\Rightarrow$ CC4 and SCC3 $\Rightarrow$ CC3. Finally, SCC4 together with the commutativity of (4.7) implies CC2. □

4.3. **PROPOSITION**. Suppose that besides the first basic situation $\mathcal{V}_0$ etc. with natural isomorphisms s and i and natural transformations L,R and j connected by (3.6), (3.9) and (3.15), we have a second instance $\mathcal{V}'_0$ etc. with natural isomorphisms s',i' and natural transformations L',R' and j', again connected by (3.6), (3.9) and (3.15). Let $\phi\colon \mathcal{V}_0 \to \mathcal{V}'_0$ be a functor, let $\phi^0\colon I' \to \phi I$ be a morphism in $\mathcal{V}'_0$ and let $\hat{\phi} = \hat{\phi}_{XY}\colon \phi[XY] \to [\phi X, \phi Y]$ be a natural transformation in $\mathcal{V}'_0$ such that the following diagram commutes:

$$
\text{(4.9)}\qquad
\begin{array}{ccc}
\mathcal{V}_0(X\,[YZ]) & \xrightarrow{\quad\sigma\quad} & \mathcal{V}_0(Y\,[XZ]) \\
\big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi} \\
\mathcal{V}'_0(\phi X, \phi[YZ]) & & \mathcal{V}'_0(\phi Y, \phi[XZ]) \\
\big\downarrow{\scriptstyle \mathcal{V}'_0(1,\hat{\phi})} & & \big\downarrow{\scriptstyle \mathcal{V}'_0(1,\hat{\phi})} \\
\mathcal{V}'_0(\phi X\,[\phi Y, \phi Z]) & \xrightarrow{\quad\sigma'\quad} & \mathcal{V}'_0(\phi Y\,[\phi X, \phi Z])
\end{array}
$$

Then we have the following implications:

(a) CF1 $\Longleftrightarrow$ CF2;

(b) CF3 $\Longleftrightarrow$ SCF3 and these properties are also equivalent to commutativity of the following diagram:

$$
\text{(4.10)}\qquad
\begin{array}{ccc}
\phi[X\,Y] & \xrightarrow{\quad\phi R^{Z}\quad} & \phi[[YZ]\,[XZ]] \\
\big\downarrow{\scriptstyle \hat{\phi}} & & \big\downarrow{\scriptstyle \hat{\phi}} \\
[\phi X, \phi Y] & & [\phi[YZ], \phi[XZ]] \\
\big\downarrow{\scriptstyle R'^{\phi Z}} & & \big\downarrow{\scriptstyle [1,\hat{\phi}]} \\
[[\phi Y, \phi Z]\,[\phi X, \phi Z]] & \xrightarrow{\quad[\hat{\phi},1]\quad} & [\phi[YZ], [\phi X, \phi Z]]
\end{array}
$$

PROOF. If we evaluate diagram (4.9) at $g \in V_0(X[YZ])$ we obtain a commutative diagram

$$(4.11) \qquad \begin{array}{ccc} \phi Y & \xrightarrow{\ \phi\sigma(g)\ } & \phi[X\ Z] \\ {\scriptstyle \sigma'\hat{\phi}}\downarrow & \searrow{\scriptstyle \sigma'(\hat{\phi}\cdot\phi g)} & \downarrow{\scriptstyle \hat{\phi}} \\ [\phi[YZ],\phi Z] & \xrightarrow{\ [\phi g,1]\ } & [\phi X,\phi Z] \end{array}$$

In particular, if we take $X = [YZ]$ and $g = 1_{[YZ]}$ we obtain:

$$(4.12) \qquad \begin{array}{ccc} \phi Y & \xrightarrow{\ \phi m\ } & \phi[[YZ]\ Z] \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle \hat{\phi}} \\ [[\phi Y,\phi Z],\phi Z] & \xrightarrow{\ [\hat{\phi},1]\ } & [\phi[YZ],\phi Z] \end{array}$$

(a) Apply σ' to each leg of diagram CF1:

$$\sigma'(\hat{\phi}_{XX}\circ\phi j\circ\phi^0) = [\phi^0,1]\sigma'(\hat{\phi}_{XX}\circ\phi j) \qquad \text{by the naturality of } \sigma'$$

$$= [\phi^0,1]\hat{\phi}\circ\phi i \qquad \text{by (4.11) and (3.16)}$$

and

$$\sigma'(j') = i' \qquad \text{by (3.16).}$$

This proves CF1 $\Longleftrightarrow$ CF2.

(b) Apply σ' to each leg of diagram CF3:

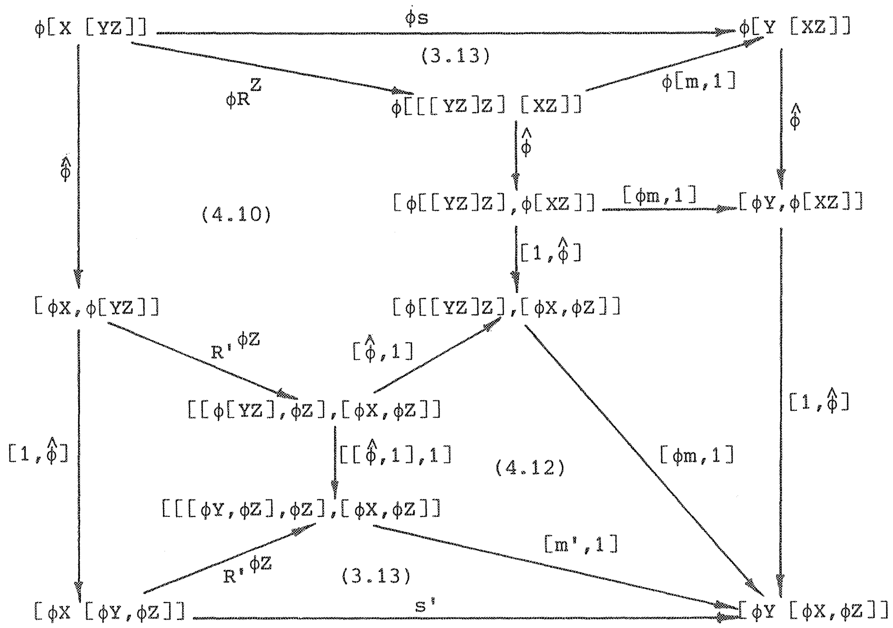
$$\sigma'([1,\hat{\phi}]\circ\hat{\phi}\circ\phi L^X) = [1,\hat{\phi}]\sigma(\hat{\phi}\circ\phi L^X) \qquad \text{by the naturality of } \sigma'$$

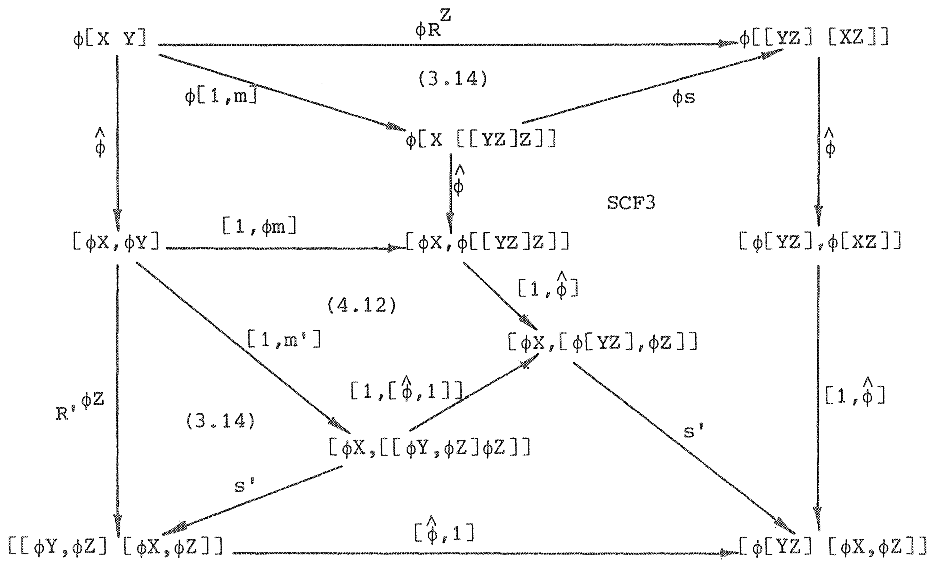
$$= [1,\hat{\phi}]\circ\hat{\phi}\circ\phi R^Z \qquad \text{by (4.11) and (3.11)}$$

and

$$\sigma'([\hat{\phi},1]\circ L'^{\phi X}\circ\hat{\phi}) = [\hat{\phi},1]R'^{\phi Z}\circ\hat{\phi} \qquad \text{by the naturality of } \sigma' \text{ and (3.11).}$$

Hence CF3 is equivalent to commutativity of diagram (4.10). The following two diagrams show that this property is equivalent to SCF3:

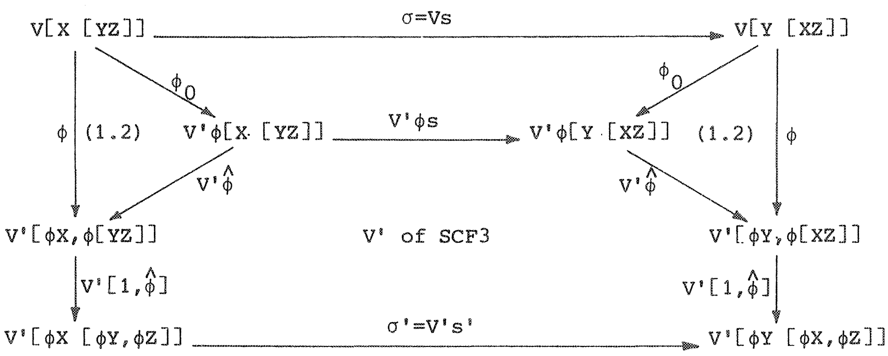

φ[X [YZ]]
φs
φ[Y [XZ]]
(3.13)
φR^Z
φ[[[YZ]Z] [XZ]]
φ[m,1]
φ̂
[φ[[YZ]Z],φ[XZ]]
[φm,1]
[φY,φ[XZ]]
(4.10)
[1,φ̂]
[φX,φ[YZ]]
[φ[[YZ]Z],[φX,φZ]]
R'^φZ
[φ̂,1]
[[φ[YZ],φZ],[φX,φZ]]
[1,φ̂]
[[φ̂,1],1]
(4.12)
[φm,1]
[[[φY,φZ],φZ],[φX,φZ]]
[m',1]
R'^φZ
(3.13)
[φX [φY,φZ]]
s'
[φY [φX,φZ]]


φ[X Y]
φR^Z
φ[[YZ] [XZ]]
(3.14)
φ[1,m]
φs
φ̂
φ[X [[YZ]Z]]
SCF3
[φX,φY]
[1,φm]
[φX,φ[[YZ]Z]]
[φ[YZ],φ[XZ]]
[1,φ̂]
(4.12)
[1,m']
[φX,[φ[YZ],φZ]]
[1,[φ̂,1]]
R'^φZ
(3.14)
[φX,[[φY,φZ]φZ]]
s'
[1,φ̂]
s'
[[φY,φZ] [φX,φZ]]
[φ̂,1]
[φ[YZ] [φX,φZ]]

□

4.4. THEOREM. If $\Phi = \langle \phi, \hat{\phi}, \phi^0 \rangle \colon \mathcal{V} \longrightarrow \mathcal{V}'$ is a symmetric closed functor. Then ${}^c\Phi := \langle \phi, \hat{\phi}, \phi^0 \rangle \colon {}^c\mathcal{V} \longrightarrow {}^c\mathcal{V}'$ is a closed functor.

PROOF. Let $\Phi \colon \mathcal{V} \longrightarrow \mathcal{V}'$ be a symmetric closed functor. The following diagram shows that diagram (4.9) commutes:



Proposition 4.3 then implies that CF2 and CF3 are consequences of CF1 and SCF3. □

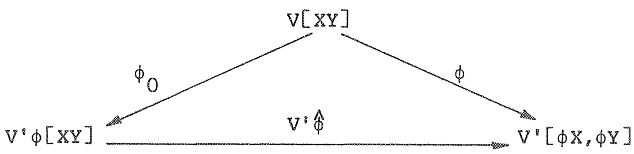
4.5. PROPOSITION. (cf. [6], proposition I.3.8). Let $\mathcal{V}$ and $\mathcal{V}'$ be two symmetric closed categories. Let $\phi \colon \mathcal{V}_0 \longrightarrow \mathcal{V}'_0$ be a functor and let $\hat{\phi} = \hat{\phi}_{XY} \colon \phi[XY] \longrightarrow [\phi X, \phi Y]$ and $\phi_0 = \phi_{0X} \colon VX \longrightarrow V'\phi X$ be natural transformations. Suppose that the following four properties hold:

(i) V' is faithful;

(ii) Diagram (4.9) commutes;

(ii) The following diagram commutes:

(1.2)



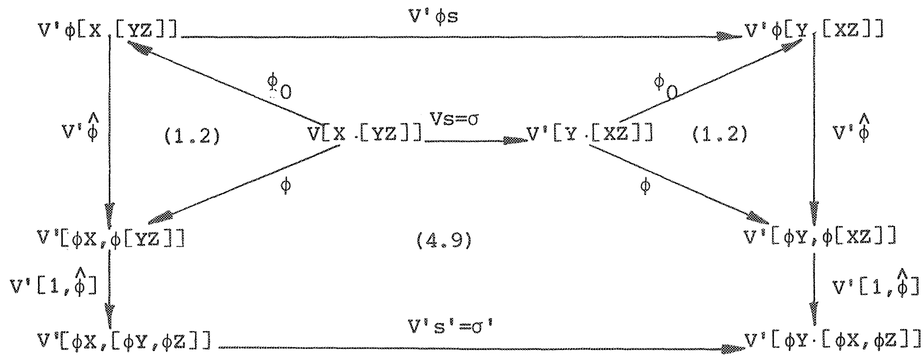
(iv) $\phi_{0X} \colon VX \longrightarrow V'\phi X$ is an epimorphism, for each $X \in$ ob $\mathcal{V}_0$.

Define a morphism $\phi^0 \colon I' \to \phi I$ by

$$(4.13) \qquad \phi^0 = (V'i'_{\phi I} \cdot \phi_{0I} \cdot Vi_I^{-1})\ 1_I.$$

Then $\Phi = \langle \phi, \hat{\phi}, \phi^0 \rangle \colon \mathcal{V} \longrightarrow \mathcal{V}'$ is a symmetric closed functor.

PROOF. If ϕ^0 is defined by (4.13) then the commutativity of (1.2) is equivalent to CF1 (proposition 1.7). Consider the image of diagram SCF3 under V':



Since $\phi_{0[X[YZ]]}$ is an epimorphism, the exterior diagram is commutative; since V' is faithful this implies SCF3. □

5. THE NATURAL TRANSFORMATION R^Z_{XY}: $[XY] \longrightarrow [[YZ][XZ]]$

Let V be a symmetric closed category. According to (3.9) and (3.11) we define in V_0 a natural transformation

$$R = R^Z_{XY}\colon [XY] \longrightarrow [[YZ][XZ]]$$

by

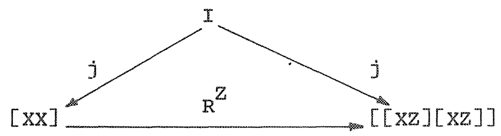
$$(3.11) \qquad R^Z_{XY} = \sigma(L^X_{YZ}).$$

Then the relation between R and s is given by (3.13) and (3.14).

5.1. PROPOSITION. Let V_0 be a category; let $V\colon V_0 \longrightarrow S$ and $[-,-]\colon V_0^* \times V_0 \longrightarrow V_0$ be functors satisfying CC0; let $I \in$ ob V_0, let $i = i_X\colon X \longrightarrow [IX]$ be a natural isomorphism and let $j = j_X\colon I \longrightarrow [XX]$ be a natural transformation such that CC5 is fulfilled. Then for a natural transformation $R = R^Z_{XY}\colon [XY] \longrightarrow [[YZ][XZ]]$ in V_0 the following conditions are equivalent:

(a) The following diagram commutes:

(5.1)



(b) $(VR^Z_{XX})1_X = 1_{[XZ]}$;

(c) $(VR^Z_{XY})g = [g,1] \in V_0([YZ][XZ])$ for $g \in V_0(XY)$;

(d) $VR^Z_{XY} = [-Z]: V_0(XY) \longrightarrow V_0([YZ][XZ])$.

Since in a symmetric closed category V diagram (5.1) commutes, the properties (b), (c) and (d) also hold in V.

PROOF. The proof is similar to that of proposition 1.6 and depends on CC5 and proposition 1.5. □

5.2. PROPOSITION. Suppose that in addition to the first basic situation we have a functor $V: V_0 \longrightarrow S$ satisfying CC0, a natural isomorphism $s = s_{XYZ}: [X[YZ]] \longrightarrow [Y[XZ]]$ and a natural transformation $R = R^Z_{XY}: [XY] \to [[YZ][XZ]]$ which are connected by (3.12). Then we have:

$$(VR^Z_{XX})1_X = 1_{[XZ]} \quad \text{iff} \quad Vs_{XYZ} = \sigma_{XYZ}.$$

PROOF.

(i) Suppose $(VR^Z_{XX})1_X = 1_{[XZ]}$.
Apply V to diagram (3.13) and use CC0; take $X = [WZ]$ and evaluate at $1_{[WZ]} \in V_0([WZ][WZ])$:

$$\begin{aligned}(Vs_{XWZ})1_{[WZ]} &= (V[m,1]\circ VR^Z)1_{[WZ]} \\ &= V[m,1]1_{[[WZ]Z]} \qquad \text{by assumption} \\ &= m_{WZ}.\end{aligned}$$

Hence $Vs_{XWZ} = \sigma_{XWZ}$ by (3.3).

(ii) Suppose $Vs_{XYZ} = \sigma_{XYZ}$.
Apply V to diagram (3.14) and use CC0; take $Y = X$ and evaluate at 1_X:

$$
\begin{aligned}
(VR^{Z}_{XX})1_X &= (Vs\circ V[1,m])1_X \\
&= (Vs)m \\
&= \sigma(m) && \text{by assumption} \\
&= 1_{[XZ]}. && \text{by (3.3) and (3.4).}
\end{aligned}
$$
□

5.3. PROPOSITION. Suppose that in addition to the first basic situation we have a functor V: $V_0 \longrightarrow S$ satisfying CC0. Then the following conditions are equivalent:

(a) The following diagram commutes:

$$
(5.2)\qquad
\begin{array}{ccc}
V_0(X\ [YZ]) & \xrightarrow{\quad\sigma\quad} & V_0(Y\ [XZ]) \\
\downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} \\
S(VX,V[YZ]) & & S(VY,V[XZ]) \\
\downarrow{\scriptstyle S(1,V_{YZ})} & & \downarrow{\scriptstyle S(1,V_{XZ})} \\
S(VX,S(VY,VZ)) & \xrightarrow{\quad s\quad} & S(VY,S(VX,VZ))
\end{array}
$$

(b) The following diagram commutes:

$$
(5.3)\qquad
\begin{array}{ccc}
VY & \xrightarrow{\quad Vm\quad} & V[[YZ]\ Z] \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle V} \\
S(S(VY,VZ),VZ) & \xrightarrow{\quad S(V_{YZ},1)\quad} & S(V[YZ],VZ)
\end{array}
$$

In a symmetric closed category these conditions hold (if we define σ by $\sigma_{XYZ} = Vs_{XYZ}$).

Note that if we evaluate (5.2) at $g \in V_0(X[YZ])$ we obtain the formula

$$(5.4)\qquad V((V(\sigma_{XYZ}g))y)x = V((Vg)x)y \qquad \text{for } x \in VX,\ y \in VY$$

and if we evaluate (5.3) at $y \in VY$ we obtain the formula

$$(5.5)\qquad (V((Vm_{YZ})y))g = (Vg)y \qquad \text{for } g \in V[YZ] = V_0(YZ).$$

PROOF.

(a) $\Rightarrow$ (b). Let $Y,Z \in$ ob $\mathcal{V}_0$; $g \in \mathcal{V}_0(YZ)$ and $y \in VY$. Then:

$$
\begin{aligned}
(V((Vm)y))g &= (V(((V\sigma)1)y))g && \text{by (3.3)}\\
&= V((V1)g)y && \text{by (5.4)}\\
&= (Vg)y && \text{by CC0.}
\end{aligned}
$$

(b) $\Rightarrow$ (a). Let $X,Y,Z \in$ ob $\mathcal{V}_0$, $g \in \mathcal{V}_0(X[YZ])$, $x \in VX$ and $y \in VY$. Then:

$$
\begin{aligned}
V((V(\sigma g))y)x &= (V((V([g,1]m))y))x && \text{by (3.4)}\\
&= (V((V[g,1]\circ Vm)y))x && \\
&= (V((Vm)y\circ g))x && \text{by CC0}\\
&= (V((Vm)y)\circ Vg)x && \\
&= (V((Vm)y))(Vg)x && \\
&= (V((Vg)x))y && \text{by (5.5).}
\end{aligned}
$$

If $\mathcal{V}$ is a symmetric closed category, then the proof of proposition 4.1 implies the commutativity of diagram (4.8). If we apply V to that diagram and use CC0 and proposition 5.1 we obtain the following commutative diagram:

$$
\begin{array}{ccc}
VY & \xrightarrow{\quad Vm \quad} & \mathcal{V}_0([YZ]Z)\\
{\scriptstyle Vi}\downarrow & & \downarrow{\scriptstyle \mathcal{V}_0(1,i)}\\
\mathcal{V}_0(IY) & \xrightarrow{\quad [-Z] \quad} & \mathcal{V}_0([YZ][IZ])
\end{array}
$$

Evaluation at $y \in VY$ gives

$$i\circ(Vm)y = [(Vi)y,1]\colon [YZ] \longrightarrow [IZ].$$

If we apply V and evaluate at $g \in \mathcal{V}_0(YZ)$ we obtain

$$(V(i\circ(Vm)y))g = Vi\circ(V((Vm)y))g$$

and

$$\mathcal{V}_0((Vi)y,1)g = g\circ(Vi)y$$

$$= (\mathcal{V}_0(1,g)\circ Vi)y$$

$$= (Vi\circ Vg)y \qquad \text{by the naturality of Vi}$$

$$= Vi((Vg)y).$$

Since Vi is a natural isomorphism, we have proved (5.5). □

5.4. THEOREM. (cf. [6], proposition I.3.11).
The symmetric closed functor V: $\mathcal{V} \to \mathcal{S}$.
If $\mathcal{V}$ is a symmetric closed category, the functor V: $\mathcal{V}_0 \to \mathcal{S}$ admits a unique extension to a normal symmetric closed functor $\langle V,\hat{V},V^0\rangle$: $\mathcal{V} \to \mathcal{S}$ which we still shall denote by V. We have:

$$\hat{V}_{XY} = V_{XY}\colon V[XY] \to \mathcal{S}(VX,VY)$$

and

$$V^0 * = (Vi_I^{-1})1_I.$$

PROOF: Consequence of the propositions 5.3 and 4.5. □

6. CATEGORIES OVER A SYMMETRIC CLOSED CATEGORY

In this section we first recall the definition of a $\mathcal{V}$-category, given in [6], for the case that $\mathcal{V}$ is a closed category. Then we investigate the extra structure in the case that $\mathcal{V}$ is a symmetric closed category.

6.1. DEFINITION. Let $\mathcal{V}$ be a closed category. A *$\mathcal{V}$-category* $\mathcal{A}$ consists of the following four data:

(i) a class ob $\mathcal{A}$ of "objects";
(ii) for each X,Y $\in$ ob $\mathcal{A}$ an object $\mathcal{A}(XY)$ of $\mathcal{V}_0$;
(iii) for each X $\in$ ob $\mathcal{A}$ a morphism j_X: I $\to \mathcal{A}(XX)$ in $\mathcal{V}_0$;

(iv) for each $X,Y,Z \in \text{ob}\,\mathcal{A}$ a morphism

$$L^X_{YZ}\colon \mathcal{A}(YZ) \longrightarrow [\mathcal{A}(XY),\mathcal{A}(XZ)] \text{ in } \mathcal{V}_0.$$

These data are to satisfy the following three axioms:

VC1. The following diagram commutes:

$$\begin{array}{ccc} & I & \\ {\scriptstyle j}\swarrow & & \searrow{\scriptstyle j} \\ \mathcal{A}(YY) & \xrightarrow{\quad L^X \quad} & [\mathcal{A}(XY),\mathcal{A}(XY)] \end{array}$$

VC2. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}(XY) & \xrightarrow{\quad L^X \quad} & [\mathcal{A}(XX),\mathcal{A}(XY)] \\ {\scriptstyle i}\searrow & & \swarrow{\scriptstyle [j,1]} \\ & [I,\mathcal{A}(XY)] & \end{array}$$

VC3. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}(WZ) & \xrightarrow{\qquad L^Y \qquad} & [\mathcal{A}(YW),\mathcal{A}(YZ)] \\ \downarrow{\scriptstyle L^X} & & \\ [\mathcal{A}(XW),\mathcal{A}(XZ)] & & \downarrow{\scriptstyle [1,L^X]} \\ \downarrow{\scriptstyle L^{\mathcal{A}(XY)}} & & \\ [[\mathcal{A}(XY),\mathcal{A}(XW)]\,[\mathcal{A}(XY),\mathcal{A}(XZ)]] & \xrightarrow{\quad [L^X,1] \quad} & [\mathcal{A}(YW),[\mathcal{A}(XY),\mathcal{A}(XZ)]] \end{array}$$

6.2. DEFINITION. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. A *$\mathcal{V}$-functor* $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ consists of the following two data:

(i) a function $T\colon \text{ob}\,\mathcal{A} \longrightarrow \text{ob}\,\mathcal{B}$;

(ii) for each $X,Y \in \text{ob}\,\mathcal{A}$ a morphism $T_{XY}\colon \mathcal{A}(XY) \longrightarrow \mathcal{B}(TX,TY)$ in $\mathcal{V}_0$.

These data are to satisfy the following two axioms:

VF1. The following diagram commutes

$$\begin{array}{ccc} & I & \\ {}^{j}\swarrow & & \searrow^{j} \\ \mathcal{A}(YY) & \xrightarrow{\quad T_{YY} \quad} & \mathcal{B}(TY,TY) \end{array}$$

VF2. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}(WZ) & \xrightarrow{\qquad L^{Y} \qquad} & [\mathcal{A}(YW),\mathcal{A}(YZ)] \\ \downarrow T_{WZ} & & \\ \mathcal{B}(TW,TZ) & & \downarrow [1,T_{YZ}] \\ \downarrow L^{TY} & & \\ [\mathcal{B}(TY,TW),\mathcal{B}(TY,TZ)] & \xrightarrow{\quad [T_{YW},1] \quad} & [\mathcal{A}(YW),\mathcal{B}(TY,TZ)] \end{array}$$

6.3. PROPOSITION. ([6], theorem I.5.1) $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category $\mathcal{V}_*$ if we define the composite of T: $\mathcal{A} \to \mathcal{B}$ and S: $\mathcal{B} \to \mathcal{C}$ to be P: $\mathcal{A} \to \mathcal{C}$ where PX = STX (X $\in$ ob $\mathcal{A}$) and P_{XY} is the composite

$$\mathcal{A}(XY) \xrightarrow{T_{XY}} \mathcal{B}(TX,TY) \xrightarrow{S_{TX,TY}} \mathcal{C}(STX,STY). \qquad \square$$

6.4. PROPOSITION. ([6], theorem I.5.2). If $\mathcal{V}$ is a closed category we get a $\mathcal{V}$-category, also denoted by $\mathcal{V}$ if we take ob $\mathcal{V}$ = ob $\mathcal{V}_0$, $\mathcal{V}(XY) = [X,Y]$ and take for j and L those of the closed category $\mathcal{V}$. Moreover, if $\mathcal{A}$ is any $\mathcal{V}$-category and X $\in$ ob $\mathcal{A}$ we get a $\mathcal{V}$-functor L^X: $\mathcal{A} \to \mathcal{V}$ if we take $L^XY = \mathcal{A}(XY)$ and $(L^X)_{YZ} = L^X_{YZ}$. $\square$

6.5. PROPOSITION. ([6], proposition I.5.4). An $\mathcal{S}$-category $\mathcal{A}$ may be identified with an ordinary category $\mathcal{A}$ if we identify the image of j: $* \to \mathcal{A}(XX)$ with 1_X and identify $(L^X_{YZ}g)h$ with the composite gh, where g $\in$ $\mathcal{A}$(YZ) and h $\in$ $\mathcal{A}$(XY). An $\mathcal{S}$-functor is then an ordinary functor, and in particular the functor L^A: $\mathcal{A} \to \mathcal{S}$ is the left represented functor $\mathcal{A}$(X-). $\square$

6.6. PROPOSITION. Let $\mathcal{V}$ be a symmetric closed category. Suppose that the following data are given:

(i) a class ob $\mathcal{A}$ of "objects";

(ii) for each X,Y $\in$ ob $\mathcal{A}$ an object $\mathcal{A}$(XY) of $\mathcal{V}_0$.

Then there is a bijection between morphisms

$$L^X_{YZ}\colon \mathcal{A}(YZ) \longrightarrow [\mathcal{A}(XY),\mathcal{A}(XZ)]$$

and morphisms

$$R^Z_{XY}\colon \mathcal{A}(XY) \longrightarrow [\mathcal{A}(YZ),\mathcal{A}(XZ)]$$

given by

$$R^Z_{XY} = \sigma(L^X_{YZ}). \tag{6.1}$$

Suppose that for each $X,Y,Z \in \operatorname{ob} \mathcal{A}$ we have such morphisms L^X_{YZ} and R^Z_{XY}, related by (6.1), and that for each $X \in \operatorname{ob} \mathcal{A}$ we have a morphism $j_X\colon I \longrightarrow \mathcal{A}(XX)$. Then:

(a) VC1 is equivalent to commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{A}(XY) & \xrightarrow{\quad R^Y \quad} & [\mathcal{A}(YY),\mathcal{A}(XY)] \\ & {\scriptstyle i} \searrow \quad \swarrow {\scriptstyle [j,1]} & \\ & [I,\mathcal{A}(XY)] & \end{array} \tag{6.2}$$

(b) VC2 is equivalent to commutativity of the following diagram:

$$\begin{array}{ccc} & I & \\ & {\scriptstyle j} \swarrow \quad \searrow {\scriptstyle j} & \\ \mathcal{A}(XX) & \xrightarrow{\quad R^Y \quad} & [\mathcal{A}(XY),\mathcal{A}(XY)] \end{array} \tag{6.3}$$

(c) VC3 is equivalent to commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{A}(XW) & \xrightarrow{\qquad R^Y \qquad} & [\mathcal{A}(WY),\mathcal{A}(XY)] \\ \downarrow {\scriptstyle R^Z} & & \\ [\mathcal{A}(WZ),\mathcal{A}(XZ)] & & \downarrow {\scriptstyle [1,R^Z]} \\ \downarrow {\scriptstyle L^{\mathcal{A}(YZ)}} & & \\ [[\mathcal{A}(YZ),\mathcal{A}(WZ)]\ [\mathcal{A}(YZ),\mathcal{A}(XZ)]] & \xrightarrow{\quad [R^Z,1] \quad} & [\mathcal{A}(WY)\ [\mathcal{A}(YZ),\mathcal{A}(XZ)]] \end{array} \tag{6.4}$$

Consequently, if $\mathcal{A}$ is a $\mathcal{V}$-category and R^Z_{XY} is defined by (6.1) then these properties hold.

PROOF.

(a) : Apply σ to each leg of VC1.

(b) : Apply σ to each leg of VC2.

(c) : Just as (3.9) implies commutativity of (4.4) and (4.5), (6.1) implies commutativity of the following diagrams:

$$
(6.5)\qquad
\begin{array}{ccc}
[\acute{A}(PQ),\acute{A}(XY)] & \xrightarrow{\;R^{\acute{A}(XZ)}\;} & [[\acute{A}(XY),\acute{A}(XZ)][\acute{A}(PQ),\acute{A}(XZ)]] \\
\downarrow{\scriptstyle [1,R^{Z}_{XY}]} & & \downarrow{\scriptstyle [L^{X}_{YZ},1]} \\
[\acute{A}(PQ)\ [\acute{A}(YZ),\acute{A}(XZ)]] & \xrightarrow{\;s\;} & [\acute{A}(YZ)[\acute{A}(PQ),\acute{A}(XZ)]]
\end{array}
$$

$$
(6.6)\qquad
\begin{array}{ccc}
[\acute{A}(PQ),\acute{A}(YZ)] & \xrightarrow{\;R^{\acute{A}(XZ)}\;} & [[\acute{A}(YZ),\acute{A}(XZ)][\acute{A}(PQ),\acute{A}(XZ)]] \\
\downarrow{\scriptstyle [1,L^{X}_{YZ}]} & & \downarrow{\scriptstyle [R^{Z}_{XY},1]} \\
[\acute{A}(PQ)\ [\acute{A}(XY),\acute{A}(XZ)]] & \xrightarrow{\;s\;} & [\acute{A}(XY)[\acute{A}(PQ),\acute{A}(XZ)]]
\end{array}
$$

A computation similar to the one in the proof of proposition 4.1 shows that VC3 is equivalent to commutativity of the following diagram:

$$
(6.7)\qquad
\begin{array}{ccc}
\acute{A}(YW) & \xrightarrow{\;R^{Z}\;} & [\acute{A}(WZ),\acute{A}(YZ)] \\
\downarrow{\scriptstyle L^{X}} & & \\
[\acute{A}(XY),\acute{A}(XW)] & & \downarrow{\scriptstyle [1,L^{X}]} \\
\downarrow{\scriptstyle [1,R^{Z}]} & & \\
[\acute{A}(XY)\ [\acute{A}(WZ),\acute{A}(XZ)]] & \xrightarrow{\;s\;} & [\acute{A}(WZ)\ [\acute{A}(XY),\acute{A}(XZ)]]
\end{array}
$$

By axiom SCC1 we may reverse the direction of the bottom arrow. If we apply σ to each leg of the resulting diagram we obtain (6.4):

$$
\begin{aligned}
\sigma(s[1,L^{X}]R^{Z}) &= \sigma([R^{Z},1]R^{\acute{A}(XZ)}R^{Z}) && \text{by (6.6)} \\
&= [R^{Z},1]\sigma(R^{\acute{A}(XZ)})R^{Z} && \text{by the naturality of } \sigma \\
&= [R^{Z},1]L^{\acute{A}(XY)}R^{Z} && \text{by (6.1),}
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma([1,R^{Z}]L^{X}) &= [1,R^{Z}]\sigma(L^{X}) && \text{by the naturality of } \sigma \\
&= [1,R^{Z}]R^{X} && \text{by (6.1).}
\end{aligned}
$$

□

6.7. PROPOSITION. Let $\mathcal{V}$ be a symmetric closed category; let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. Suppose that the following data are given:

(i) a function T: ob $\mathcal{A} \longrightarrow$ ob $\mathcal{B}$;

(ii) for each X,Y $\in$ ob $\mathcal{A}$ a morphism $T_{XY}\colon \mathcal{A}(XY) \longrightarrow \mathcal{B}(TX,TY)$ in $\mathcal{V}_0$.

Then VF2 is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathcal{A}(YW) & \xrightarrow{\quad R^{Z} \quad} & [\mathcal{A}(WZ),\mathcal{A}(YZ)] \\
\downarrow T_{YW} & & \\
\mathcal{B}(TY,TW) & & \downarrow [1,T_{YZ}] \\
\downarrow R^{TZ} & & \\
[\mathcal{B}(TW,TZ),\mathcal{B}(TY,TZ)] & \xrightarrow{\quad [T_{WZ},1] \quad} & [\mathcal{A}(WZ),\mathcal{B}(TY,TZ)]
\end{array} \tag{6.8}$$

Consequently, if T is a $\mathcal{V}$-functor then this property hold.

PROOF. Diagram (6.8) is the image of VF2 under σ. □

6.8. PROPOSITION AND DEFINITION. (cf. [6], propositions III.2.1 and III.2.2). If $\mathcal{V}$ is a symmetric closed category and $\mathcal{A}$ is a $\mathcal{V}$-category, the following data define a $\mathcal{V}$-category $\mathcal{A}^*$, called *the dual of* $\mathcal{A}$:

(i) ob $\mathcal{A}^*$ = ob $\mathcal{A}$;

(ii) $\mathcal{A}^*(XY) = \mathcal{A}(YX)$;

(ii) $j^*_X\colon I \longrightarrow \mathcal{A}^*(XX)$ is $j_X\colon I \longrightarrow \mathcal{A}(XX)$;

(iv) $L^{*X}_{YZ}\colon \mathcal{A}^*(YZ) \longrightarrow [\mathcal{A}^*(XY),\mathcal{A}^*(XZ)]$ is $R^X_{ZY}\colon \mathcal{A}(ZY) \rightarrow [\mathcal{A}(YX),\mathcal{A}(ZX)]$.

If $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a $\mathcal{V}$-functor then the following data define a $\mathcal{V}$-functor $T^*\colon \mathcal{A}^* \longrightarrow \mathcal{B}^*$:

(i) $T^*X = TX$;

(ii) $T^*_{YZ}\colon \mathcal{A}^*(YZ) \longrightarrow \mathcal{B}^*(T^*Y,T^*Z)$ is $T_{ZY}\colon \mathcal{A}(ZY) \longrightarrow \mathcal{B}(TZ,TY)$.

PROOF. Immediate consequence of the propositions 6.6 and 6.7: VC1 for $\mathcal{A}^*$ is (6.3); VC2 for $\mathcal{A}^*$ is (6.2); VC3 for $\mathcal{A}^*$ is (6.4). VF1 for T^* is VF1 for T; VF2 for T^* is (6.8). □

6.9. PROPOSITION. (cf. [6], proposition III.2.3). If V is a symmetric closed category the assignments $A \mapsto A^*$ and $T \mapsto T^*$ constitute an involutory functor $D: V_* \to V_*$. □

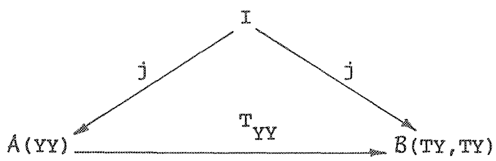
6.10. REMARK. Let A be a V-category and let A^* be its dual. Define $R^{*Z}_{XY}: A^*(XY) \to [A^*(YZ), A^*(XZ)]$ by $R^{*Z}_{XY} = \sigma(L^{*X}_{YZ})$. Then $R^{*Z}_{XY} = L^{Z}_{YX}: A(YX) \to [A(ZY), A(ZX)]$. □

6.11. DEFINITION. Let A and B be V-categories. A *V-cofunctor* $T: A \to B$ is a V-functor $T: A^* \to B$.

So a V-cofunctor $T: A \to B$ consists of a function T: ob $A \to$ ob B and a family of morphisms $T_{XY}: A(YX) \to B(TX,TY)$ $(X,Y \in$ ob $A)$ satisfying the following two axioms:

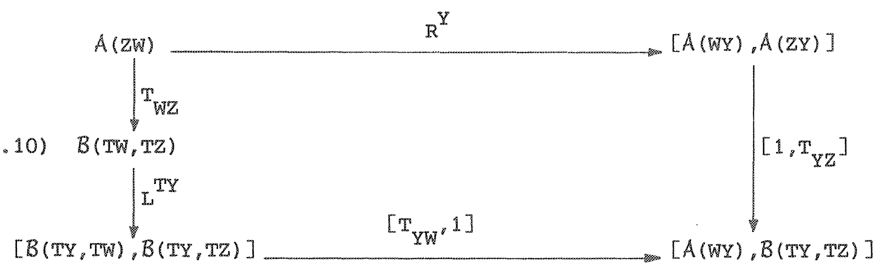
$VF^{0}1$ = VF1. The following diagram commutes:

(6.9)



$VF^{0}2$. The following diagram commutes:

(6.10)



6.12. PROPOSITION. If A is any V-category and $Z \in$ ob A we obtain a V-cofunctor $R^Z: A \to V$ if we take $R^Z X = A(XZ)$ and $(R^Z)_{YX} = R^Z_{XY}: A(XY) \to [A(YZ), A(XZ)] = [R^Z Y, R^Z X]$.

PROOF. $VF^{0}1$ for R^Z is (6.3) and $VF^{0}2$ for R^Z is (6.4). □

6.13. DEFINITION. Let V be a symmetric closed category and let A,B and C be V-categories. A *quasi-V-bifunctor* $P: \langle A,B \rangle \to C$ is an ordered pair $P = \langle S,T \rangle$ consisting of

(i) a family $S = S^Y\colon \mathcal{A} \longrightarrow \mathcal{C}$ of $\mathcal{V}$-functors, indexed by $Y \in \mathrm{ob}\,\mathcal{B}$;

(ii) a family $T = T^X\colon \mathcal{B} \longrightarrow \mathcal{C}$ of $\mathcal{V}$-functors, indexed by $X \in \mathrm{ob}\,\mathcal{A}$.

These data are to satisfy the following two axioms:

QVF1. $S^Y X = T^X Y$ for each $X \in \mathrm{ob}\,\mathcal{A}$ and $Y \in \mathrm{ob}\,\mathcal{B}$;

we denote this object by $P(XY)$.

QVF2. The following diagram commutes:

(6.11)
$$\begin{array}{ccc}
\mathcal{A}(XZ) & \xrightarrow{\quad S^W \quad} & \mathcal{C}(P(XW),P(ZW)) \\
\downarrow{\scriptstyle S^Y} & & \downarrow{\scriptstyle L^{P(XY)}} \\
\mathcal{C}(P(XY),P(ZY)) & & [\mathcal{C}(P(XY),P(XW)),\mathcal{C}(P(XY),P(ZW))] \\
\downarrow{\scriptstyle R^{P(ZW)}} & & \downarrow{\scriptstyle [T^X,1]} \\
[\mathcal{C}(P(ZY),P(ZW)),\mathcal{C}(P(XY),P(ZW))] & \xrightarrow{\quad [T^Z,1] \quad} & [\mathcal{B}(YW),\mathcal{C}(P(XY),P(ZW))]
\end{array}$$

Note that QVF2 is equivalent to commutativity of the following diagram:

(6.12)
$$\begin{array}{ccc}
\mathcal{B}(YW) & \xrightarrow{\quad T^Z \quad} & \mathcal{C}(P(ZY),P(ZW)) \\
\downarrow{\scriptstyle T^X} & & \downarrow{\scriptstyle L^{P(XY)}} \\
\mathcal{C}(P(XY),P(XW)) & & [\mathcal{C}(P(XY),P(ZY)),\mathcal{C}(P(XY),P(ZW))] \\
\downarrow{\scriptstyle R^{P(ZW)}} & & \downarrow{\scriptstyle [S^Y,1]} \\
[\mathcal{C}(P(XW),P(ZW)),\mathcal{C}(P(XY),P(ZW))] & \xrightarrow{\quad [S^W,1] \quad} & [\mathcal{A}(XZ),\mathcal{C}(P(XY),P(ZW))]
\end{array}$$

The $\mathcal{V}$-functors $S^Y\colon \mathcal{A} \longrightarrow \mathcal{C}$ and $T^X\colon \mathcal{B} \longrightarrow \mathcal{C}$ are called the *partial $\mathcal{V}$-functors* of the quasi-$\mathcal{V}$-bifunctor P.

6.14. PROPOSITION. Let $\mathcal{V}$ be a symmetric closed category, and let $\mathcal{A}$ be a $\mathcal{V}$-category. The $\mathcal{V}$-functors $R^Z\colon \mathcal{A}^* \longrightarrow \mathcal{V}$ and $L^X\colon \mathcal{A} \longrightarrow \mathcal{V}$ are the partial $\mathcal{V}$-functor of a quasi-$\mathcal{V}$-bifunctor

$$\mathrm{Hom}\,\mathcal{A}\colon \langle \mathcal{A}^*,\mathcal{A}\rangle \twoheadrightarrow \mathcal{V}.$$

On the objects we have $\mathrm{Hom}\,\mathcal{A}(XZ) = L^X Z = R^Z X = \mathcal{A}(XZ)$.

PROOF. QVF1 holds by the preceding line. It remains to prove that QVF2 holds. Axiom SCC2 for $\mathcal{V}$ implies the commutativity of the following diagram (use (4.4) and (3.14)):

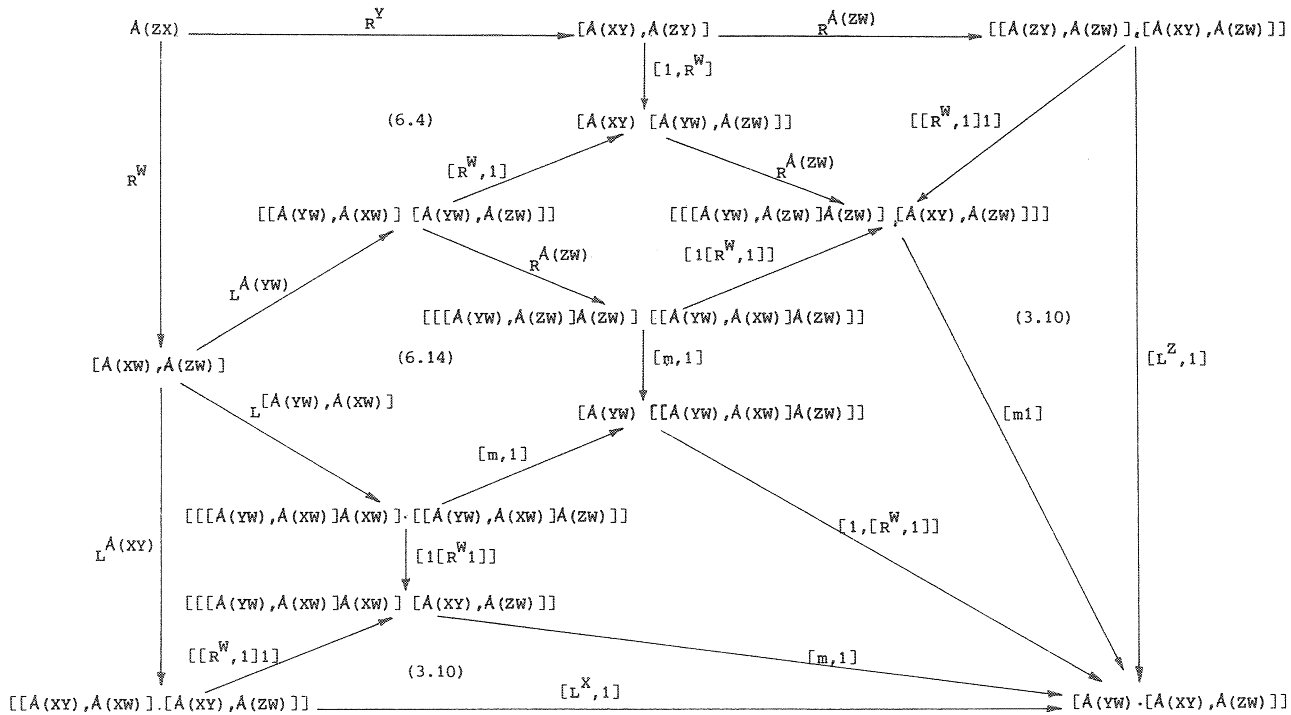
$$
(6.13)\qquad
\begin{array}{ccc}
[X[YZ]] & \xrightarrow{\quad s \quad} & [Y[XZ]] \\
\downarrow{\scriptstyle R^{[YW]}} & & \downarrow{\scriptstyle R^{[XW]}} \\
[[[YZ][YW]][X[YW]]] & & [[[XZ][XW]][Y[XW]]] \\
\downarrow{\scriptstyle [L^{Y},1]} & & \downarrow{\scriptstyle [L^{X},1]} \\
[[ZW][X[YW]]] & \xrightarrow{\quad [1,s] \quad} & [[ZW][Y[XW]]]
\end{array}
$$

If we apply V to this diagram, taking X = [YZ] and evaluate at $1_{[YZ]}$ we obtain the following commutative diagram (we have used CC0, proposition 5.1, and (3.13)):

$$
(6.14)\qquad
\begin{array}{ccc}
[ZW] & \xrightarrow{\quad L^{[YZ]} \quad} & [[[YZ]Z][[YZ]W]] \\
\downarrow{\scriptstyle L^{Y}} & & \\
[[YZ][YW]] & \searrow{\scriptstyle s} & \downarrow{\scriptstyle [m,1]} \\
\downarrow{\scriptstyle R^{W}} & (3.13) & \\
[[[YW]W][[YZ]W]] & \xrightarrow{\quad [m,1] \quad} & [Y[[YZ]W]]
\end{array}
$$

Now we are prepared to prove QVF2:

(6.15)


Á(ZX)
R^Y
[Á(XY),Á(ZY)]
R^Á(ZW)
[[Á(ZY),Á(ZW)],[Á(XY),Á(ZW)]]
[1,R^W]
(6.4)
[Á(XY) [Á(YW),Á(ZW)]]
[[R^W,1]1]
R^W
[R^W,1]
R^Á(ZW)
[[Á(YW),Á(XW)] [Á(YW),Á(ZW)]]
[[[Á(YW),Á(ZW)]Á(ZW)] [Á(XY),Á(ZW)]]]
R^Á(ZW)
[1[R^W,1]]
L^Á(YW)
[[[Á(YW),Á(ZW)]Á(ZW)] [[Á(YW),Á(XW)]Á(ZW)]]
(3.10)
[Á(XW),Á(ZW)]
(6.14)
[m,1]
[L^Z,1]
L^[Á(YW),Á(XW)]
[Á(YW) [[Á(YW),Á(XW)]Á(ZW)]]
[m1]
[m,1]
[[[Á(YW),Á(XW)]Á(XW)] [[Á(YW),Á(XW)]Á(ZW)]]
[1,[R^W,1]]
L^Á(XY)
[1[R^W1]]
[[[Á(YW),Á(XW)]Á(XW)] [Á(XY),Á(ZW)]]
[[R^W,1]1]
[m,1]
(3.10)
[L^X,1]
[[Á(XY),Á(XW)].[Á(XY),Á(ZW)]]
[Á(YW).[Á(XY),Á(ZW)]]

7. THE UNDERLYING CATEGORY OF A V-CATEGORY

In [6], section I.6 it is shown that a closed functor Φ: $V \to V'$ (V and V' closed categories) induces a functor Φ_*: $V_* \to V'_*$. In [6], section I.7 this is applied to the particular closed functor V: $V \to S$. We restrict ourselves to this special case. V is a closed category.

7.1. DEFINITION. ([6], section I.7). Each V category A determines an ordinary category $V_*A = A_0$, called the *underlying category* of the V-category A. This category (considered as an S-category (cf. proposition 6.5)) is defined as follows:

(i) ob A_0 = ob A;

(ii) $A_0(XY) = VA(XY)$ $(X,Y \in \text{ob } A_0)$;

(iii) The j' of A_0 is the composite

$$(7.1) \qquad * \xrightarrow{V^0} VI \xrightarrow{Vj} VA(XX)$$

so that $1_X \in VA(XX)$ is the image of * under (7.1);

(iv) The L' of A_0 is the composite

$$(7.2) \qquad VA(XZ) \xrightarrow{VL^X} V[A(XY),A(XZ)] \xrightarrow{V} S(VA(XY),VA(XZ))$$

so that the composite in A_0 of $g \in VA(XY)$ and $h \in VA(YZ)$ is

$$(7.3) \qquad hg = (V((VL^X)h))g.$$

Each V-functor T: $A \to B$ determines an ordinary functor $V_*T = T_0$: $A_0 \to B_0$ called *the underlying functor of the V-functor* T. This functor (considered as an S-functor) is defined by:

(i) $T_0X = TX$ $(X \in \text{ob } A_0)$;

(ii) (7.4) $T_0f = (VT_{XY})f$ for $f \in A_0(XY)$.

7.2. PROPOSITION. ([6], proposition I.7.2) $V_*V = V_0$. □

7.3. DEFINITION AND REMARK. ([6], section I.7). For the underlying functor of the $\mathcal{V}$-functor L^X: $\mathcal{A} \longrightarrow \mathcal{V}$ EILENBERG and KELLY adopt the special notation $\mathcal{A}(X-)$: $\mathcal{A}_0 \longrightarrow \mathcal{V}_0$, so that

$$(7.5) \qquad \mathcal{A}(X-) = V_* L^X .$$

The value of $\mathcal{A}(X-)$ on the object Y is $\mathcal{A}(XY)$, and its value $\mathcal{A}(Xf)$ on the morphism $f \in V\mathcal{A}(YZ)$ is given by

$$(7.6) \qquad \mathcal{A}(Xf) = (VL^X)f.$$

They also show the commutativity of the following diagram of functors:

$$(7.7) \qquad \begin{array}{ccccc} & & \mathcal{A}_0 & & \\ & \swarrow^{\mathcal{A}(X-)} & & \searrow^{\mathcal{A}_0(X-)} & \\ \mathcal{V}_0 & & \xrightarrow{\quad V \quad} & & \mathcal{S} \end{array}$$

In the particular case $\mathcal{A} = \mathcal{V}$ one has $\mathcal{V}(X-) = [X-]$: $\mathcal{V}_0 \longrightarrow \mathcal{V}_0$.

7.4. REMARK. THE SYMMETRIC CASE.

Now we assume that $\mathcal{V}$ is a symmetric closed category. If $\mathcal{A}$ is a $\mathcal{V}$-category with underlying category $\mathcal{A}_0$ then one can prove that the R' of $\mathcal{A}_0$ is the composite

$$(7.8) \qquad V\mathcal{A}(XY) \xrightarrow{\; VR^Z \;} V[\mathcal{A}(YZ), \mathcal{A}(XZ)] \xrightarrow{\; V \;} \mathcal{S}(V\mathcal{A}(YZ), V\mathcal{A}(XZ))$$

so that the composite in $\mathcal{A}_0$ of $g \in V\mathcal{A}(XY)$ and $h \in V\mathcal{A}(YZ)$ is

$$(7.9) \qquad hg = (V((VR^Z)g))h.$$

From (7.8) it follows that $(\mathcal{A}^*)_0 = (\mathcal{A}_0)^*$. Consequently, if T: $\mathcal{A}^* \longrightarrow \mathcal{B}$ is a $\mathcal{V}$-functor, then its underlying functor is T_0: $(\mathcal{A}^*)_0 = (\mathcal{A}_0)^* \longrightarrow \mathcal{B}_0$; so we can consider the underlying functor of a $\mathcal{V}$-cofunctor T: $\mathcal{A} \longrightarrow \mathcal{B}$ as a cofunctor T_0: $\mathcal{A}_0 \longrightarrow \mathcal{B}_0$.

For the underlying functor of the $\mathcal{V}$-cofunctor R^Z: $\mathcal{A} \longrightarrow \mathcal{V}$ we adopt the notation $\mathcal{A}(-Z)$: $\mathcal{A}_0^* \longrightarrow \mathcal{V}_0$, so that

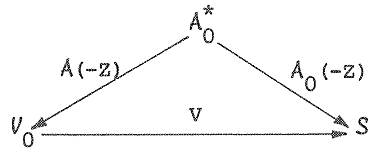
(7.10) $\quad A(-Z) = V_*R^Z$.

The value of $A(-Z)$ on the object Y is $A(YZ)$ and its value $A(fZ)$ on the morphism $f \in A_0(XY)$ is given by

(7.11) $\quad A(fZ) = (VR^Z)f$.

Analogous to (7.7) we now have a commutative diagram of functors

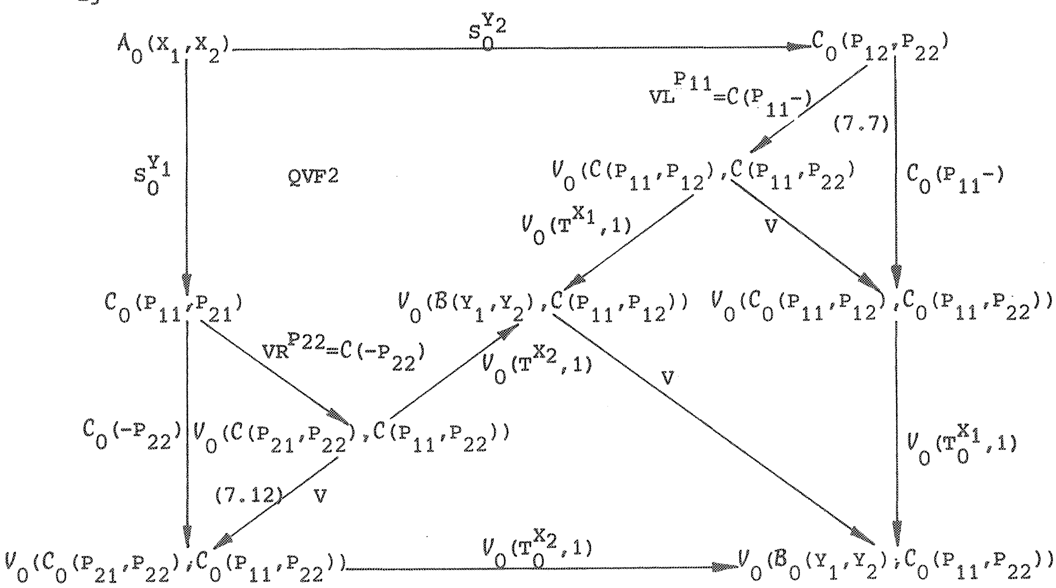
(7.12)



In the particular case $A = V$ we have $V(-Z) = [-Z]\colon V_0^* \longrightarrow V_0$.

7.5. PROPOSITION. Let A, B and C be V-categories and let $P = \langle S,T\rangle\colon \langle A,B\rangle \longrightarrow C$ be a quasi-V-bifunctor. The underlying functors $S_0^Y\colon A_0 \longrightarrow C_0$ and $T_0^X\colon B_0 \longrightarrow C_0$ are the partial functors of a bifunctor $P_0\colon A_0 \times B_0 \longrightarrow C_0$ which is called the *underlying bifunctor* of the quasi-V-bifunctor P.

PROOF. QVF1 implies that $S_0^YX = T_0^XY$ $(=P_0(XY))$. QVF2, (7.7) and (7.12) imply the commutativity of the following diagram: (we abbreviate $P(X_i,Y_j)$ by P_{ij} $(i,j\in\{1,2\})$):



If we evaluate this diagram at $g \in A_0(X_1X_2)$ and $h \in B_0(Y_1Y_2)$ we obtain $S_0^{Y_2}g \cdot T_0^{X_1}h = T_0^{X_2}h \cdot S_0^{Y_1}g\colon P(X_1Y_1) \to P(X_2Y_2)$. This means that the functors S_0^Y and T_0^X are the partial functors of a bifunctor $P_0\colon A_0 \times B_0 \to C_0$. □

7.6. PROPOSITION. Let A be a V-category. The underlying bifunctor of the quasi-V-bifunctor Hom A: $\langle A^*, A\rangle \to V$ is the bifunctor

$$\text{hom } A\colon A_0^* \times A_0 \to V_0$$

which is defined by

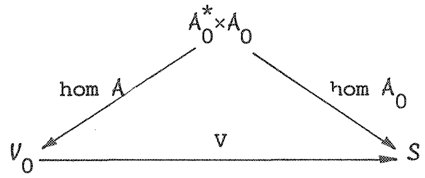
(i) hom $A(XY) = A(XY)$;

(ii) hom $A(gh) = A(X_2h)A(gY_1) = A(gY_2)A(X_1h)$
for $g \in A_0(X_2X_1)$ and $h \in A_0(Y_1Y_2)$.

We shall write $A(gh)$ for hom $A(gh)$.

The following diagram of functors is commutative:

(7.13)



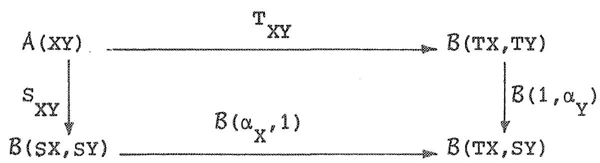
In the particular case $A = V$ we have hom $V = [-,-]\colon V_0^* \times V_0 \to V_0$.

PROOF. Consequence of proposition 6.13, remark 7.4 and proposition 7.5. □

7.7. DEFINITION. ([6], section I.10). Let A and B be V categories and let T and S: $A \to B$ be V-functors. A *V-natural transformation* $\alpha\colon T \to S\colon A \to B$ is a family of morphisms $\alpha_X\colon TX \to SX$ is B_0, indexed by the objects of A, satisfying the following axiom:

VN. The following diagram commutes:

(7.14)



7.8. REMARK. In [6] this definition is given for a closed category $\mathcal{V}$. In that case the definition of the functor $\mathcal{B}(-,SY)\colon \mathcal{B}_0^* \to \mathcal{V}_0$ (appearing in the bottom arrow of VN) is rather complicated. If $\mathcal{V}$ is a symmetric closed category then the definition of $\mathcal{B}(-,SY)$ is much easier (see remark 7.4).

7.9. DEFINITION. (cf. [6], section III.5). Let $\mathcal{V}$ be a symmetric closed category; let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories, let $P = \langle S,T\rangle\colon \langle \mathcal{A}^*,\mathcal{A}\rangle \to \mathcal{B}$ be a quasi-$\mathcal{V}$-bifunctor and let Y be a fixed object of $\mathcal{B}$. A family of morphisms in $\mathcal{B}_0$

$$\gamma = \gamma_X\colon Y \to P(XX) \quad (X \in \text{ob}\ \mathcal{A})$$

is said to be *$\mathcal{V}$-natural* if the following axiom is satisfied:
VN'. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}(XX') & \xrightarrow{\ T^X\ } & \mathcal{B}(P(XX),P(XX')) \\ \downarrow{\scriptstyle S^{X'}} & & \downarrow{\scriptstyle \mathcal{B}(\gamma_X,1)} \\ \mathcal{B}(P(X'X'),P(XX')) & \xrightarrow{\ \mathcal{B}(\gamma_{X'},1)\ } & \mathcal{B}(Y,P(XX')) \end{array}$$

Similarly, a family of morphisms in $\mathcal{B}_0$

$$\delta = \delta_X\colon P(XX) \to Y \quad (X \in \text{ob}\ \mathcal{A})$$

is said to be *$\mathcal{V}$-natural* if the following axiom is satisfied:
VN". The following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}(XX') & \xrightarrow{\ S^X\ } & \mathcal{B}(P(X'X),P(XX)) \\ \downarrow{\scriptstyle T^{X'}} & & \downarrow{\scriptstyle \mathcal{B}(1,\delta_X)} \\ \mathcal{B}(P(X'X),P(X'X')) & \xrightarrow{\ \mathcal{B}(1,\delta_{X'})\ } & \mathcal{B}(P(X'X),Y) \end{array}$$

With this definition of two extraordinary kinds of $\mathcal{V}$-natural transformations we are able to formulate the following theorem:

7.10. THEOREM. If $\mathcal{V}$ is a symmetric closed category the families of morphisms i, j, L, R, s and m are $\mathcal{V}$-natural in every variable. Moreover, if $\mathcal{A}$ is a $\mathcal{V}$-category, the families of morphisms j, L and R are $\mathcal{V}$-natural in every variable.

PROOF. Let $\mathcal{A}$ be a $\mathcal{V}$-category.

(i) The $\mathcal{V}$-naturality of j is expressed by the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathcal{A}(XY) & \xrightarrow{\quad L^X \quad} & [\mathcal{A}(XX),\mathcal{A}(XY)] \\
\downarrow{\scriptstyle R^Y} & & \downarrow{\scriptstyle [j,1]} \\
[\mathcal{A}(YY),\mathcal{A}(XY)] & \xrightarrow{\quad [j,1] \quad} & [I,\mathcal{A}(XY)]
\end{array}$$

The commutativity of this diagram follows from VC2 and (6.2).

(ii) The $\mathcal{V}$-naturality of L^X_{YZ} in the variable Z is expressed by VC3 and in the variable X by (6.15). If we apply σ to VC3 we obtain the following commutative diagram which expresses the $\mathcal{V}$-naturality in Y:

$$(7.14)\qquad \begin{array}{ccc}
\mathcal{A}(YY') & \xrightarrow{\quad R^Z \quad} & [\mathcal{A}(Y'Z),\mathcal{A}(YZ)] \\
\downarrow{\scriptstyle L^X} & & \\
[\mathcal{A}(XY),\mathcal{A}(XY')] & & \downarrow{\scriptstyle [1,L^X]} \\
\downarrow{\scriptstyle R^{\mathcal{A}(XZ)}} & & \\
[[\mathcal{A}(XY'),\mathcal{A}(XZ)]\ [\mathcal{A}(XY),\mathcal{A}(XZ)]] & \xrightarrow{\quad [L^X,1] \quad} & [\mathcal{A}(Y'Z)\ [\mathcal{A}(XY),\mathcal{A}(XZ)]]
\end{array}$$

(iii) The $\mathcal{V}$-naturality of R^Z_{XY} in the variable X is expressed by (6.4). If we apply σ to this diagram we obtain the following commutative diagram, which expresses the $\mathcal{V}$-naturality of R^Z_{XY} in Y:

$$(7.15)\qquad \begin{array}{ccc}
\mathcal{A}(Y'Y) & \xrightarrow{\quad L^X \quad} & [\mathcal{A}(XY'),\mathcal{A}(XY)] \\
\downarrow{\scriptstyle R^Z} & & \\
[\mathcal{A}(YZ),\mathcal{A}(Y'Z)] & & \downarrow{\scriptstyle [1,R^Z]} \\
\downarrow{\scriptstyle R^{\mathcal{A}(XZ)}} & & \\
[[\mathcal{A}(Y'Z),\mathcal{A}(XZ)]\ [\mathcal{A}(YZ),\mathcal{A}(XZ)]] & \xrightarrow{\quad [R^Z,1] \quad} & [\mathcal{A}(XY')\ [\mathcal{A}(YZ),\mathcal{A}(XZ)]]
\end{array}$$

If we apply σ to diagram (6.15) we obtain a commutative diagram which expresses the $\mathcal{V}$-naturality of R^Z_{XY} in Z:

$$
(7.16)\qquad
\begin{array}{ccc}
\mathcal{A}(ZZ') & \xrightarrow{\quad L^X \quad} & [\mathcal{A}(XZ),\mathcal{A}(XZ')] \\
\big\downarrow{\scriptstyle L^Y} & & \big\downarrow{\scriptstyle L^{\mathcal{A}(YZ)}} \\
[\mathcal{A}(YZ),\mathcal{A}(YZ')] & & [[\mathcal{A}(YZ),\mathcal{A}(XZ)]\,[\mathcal{A}(YZ),\mathcal{A}(XZ')]] \\
\big\downarrow{\scriptstyle R^{\mathcal{A}(XZ')}} & & \big\downarrow{\scriptstyle [R^Z,1]} \\
[[\mathcal{A}(YZ'),\mathcal{A}(XZ')]\,[\mathcal{A}(YZ),\mathcal{A}(XZ')]] & \xrightarrow{\quad [R^{Z'},1] \quad} & [\mathcal{A}(XY)\,[\mathcal{A}(YZ),\mathcal{A}(XZ')]]
\end{array}
$$

(iv) The $\mathcal{V}$-naturality of i is expressed by CC4.

(v) If we apply σ to each leg of diagram (4.3) we obtain the following commutative diagram, which expresses the $\mathcal{V}$-naturality of s_{XYZ} in X:

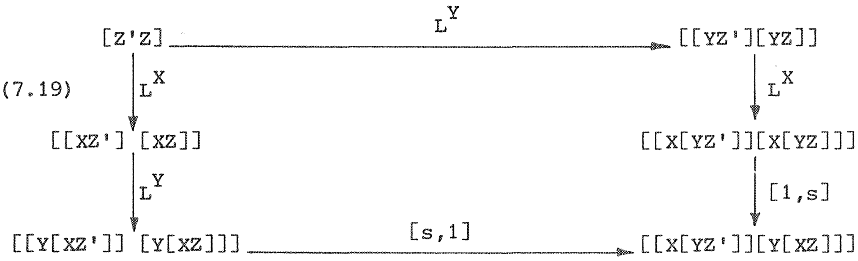
$$
(7.17)\qquad
\begin{array}{ccc}
[XX'] & \xrightarrow{\quad R^{[YZ]} \quad} & [[X'[YZ]][X[YZ]]] \\
\big\downarrow{\scriptstyle R^Z} & & \\
[[X'Z][XZ]] & & \big\downarrow{\scriptstyle [1,s]} \\
\big\downarrow{\scriptstyle L^Y} & & \\
[[Y[XZ]][Y[X'Z]]] & \xrightarrow{\quad [s,1] \quad} & [[X'[YZ]][Y[XZ]]]
\end{array}
$$

If we reverse the direction of the bottom arrow in diagram (4.3) (which is allowed by axiom SCC1), and if we again apply σ to the resulting diagram, we obtain a commutative diagram which expresses the $\mathcal{V}$-naturality of s in Y:

$$
(7.18)\qquad
\begin{array}{ccc}
[YY'] & \xrightarrow{\quad R^Z \quad} & [[Y'Z][YZ]] \\
 & & \big\downarrow{\scriptstyle L^X} \\
\big\downarrow{\scriptstyle R^{[XZ]}} & & [[X[Y'Z]][X[YZ]]] \\
 & & \big\downarrow{\scriptstyle [1,s]} \\
[[Y'[XZ]][Y[XZ]]] & \xrightarrow{\quad [s,1] \quad} & [[X[Y'Z]][Y[XZ]]]
\end{array}
$$

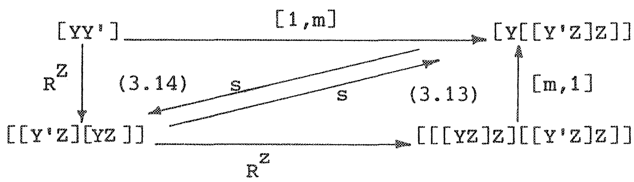
As we have seen, axiom SCC2 implies the commutativity of diagram (6.13). If we apply σ to each leg of (6.13) we obtain a commutative diagram which expresses the $\mathcal{V}$-naturality of s in Z:

(7.19)



(vi) The V-naturality of m_{YZ} in the variable Y is expressed by the commutativity of the following diagram, which is a consequence of SCC1, (3.13) and (3.14):

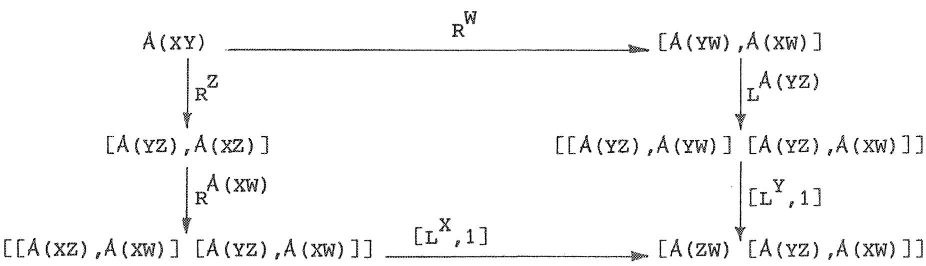
(7.20)



The V-naturality of m_{YZ} in the variable Z is expressed by the commutativity of diagram (6.14). □

7.11. PROPOSITION. (cf. [6], proposition I.8.4). Let A be a V-category, let $f \in A_0(XY)$ and $h \in A_0(YZ)$. The morphisms $A(fZ)\colon A(YZ) \longrightarrow A(XZ)$ are the components of a V-natural transformation $L^f\colon L^Y \longrightarrow L^X\colon A \longrightarrow V$ and the morphisms $A(Xh)\colon A(XY) \longrightarrow A(XZ)$ are the components of a V-natural transformation $R^h\colon R^Y \longrightarrow R^Z\colon A^* \longrightarrow V$.

PROOF. We rewrite the diagrams (6.15) and (7.16), changing the letters:



$$
\begin{array}{ccc}
\mathcal{A}(YZ) & \xrightarrow{\quad L^{X} \quad} & [\mathcal{A}(XY),\mathcal{A}(XZ)] \\
\downarrow{\scriptstyle L^{W}} & & \downarrow{\scriptstyle L^{\mathcal{A}(WY)}} \\
[\mathcal{A}(WY),\mathcal{A}(WZ)] & & [[\mathcal{A}(WY),\mathcal{A}(XY)]\,[\mathcal{A}(WY),\mathcal{A}(XZ)]] \\
\downarrow{\scriptstyle R^{\mathcal{A}(XZ)}} & & \downarrow{\scriptstyle [R^{Y},1]} \\
[[\mathcal{A}(WZ),\mathcal{A}(XZ)]\,[\mathcal{A}(WY),\mathcal{A}(XZ)]] & \xrightarrow{\quad [R^{Z},1] \quad} & [\mathcal{A}(XW)\,[\mathcal{A}(WY),\mathcal{A}(XZ)]]
\end{array}
$$

If we apply V to these diagrams, and evaluate at $f \in \mathcal{A}_0(XY)$ and $h \in \mathcal{A}_0(YZ)$ we obtain the following commutative diagrams:

$$
\begin{array}{ccc}
\mathcal{A}(ZW) & \xrightarrow{\quad L^{Y} \quad} & [\mathcal{A}(YZ),\mathcal{A}(YW)] \\
\downarrow{\scriptstyle L^{X}} & & \downarrow{\scriptstyle [1,\mathcal{A}(fW)]} \\
[\mathcal{A}(XZ),\mathcal{A}(XW)] & \xrightarrow{\quad [\mathcal{A}(fZ),1] \quad} & [\mathcal{A}(YZ),\mathcal{A}(XW)]
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{A}(XW) & \xrightarrow{\quad R^{Y} \quad} & [\mathcal{A}(WY),\mathcal{A}(XY)] \\
\downarrow{\scriptstyle R^{Z}} & & \downarrow{\scriptstyle [1,\mathcal{A}(Xh)]} \\
[\mathcal{A}(WZ),\mathcal{A}(XZ)] & \xrightarrow{\quad [\mathcal{A}(Wh),1] \quad} & [\mathcal{A}(WY),\mathcal{A}(XZ)]
\end{array}
$$

□

CHAPTER III

SEMI MONOIDAL CLOSED CATEGORIES

1. MONOIDAL CATEGORIES

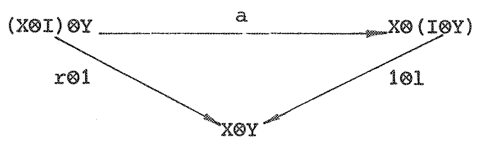
In this section we recall the definition and some properties of a monoidal category. We conform to the numeration of the axioms in [6], section II.1. For the definition of a monoidal category and for some examples we also refer to [17].

1.1. DEFINITION. *A monoidal category* is an ordered 6-tuple $V = \langle V_0, \otimes, I, r, l, a \rangle$ consisting of:

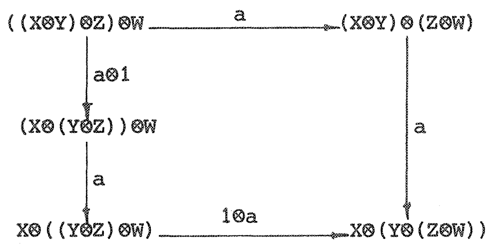
(i) a category V_0 (called the *underlying category* of V);
(ii) a functor $-\otimes-$: $V_0 \times V_0 \longrightarrow V_0$ (called the *tensor product functor*);
(iii) an object I of V_0;
(iv) a natural isomorphism $r = r_X$: $X\otimes I \longrightarrow X$;
(v) a natural isomorphism $l = l_X$: $I\otimes X \longrightarrow X$;
(vi) a natural isomorphism $a = a_{XYZ}$: $(X\otimes Y)\otimes Z \longrightarrow X\otimes(Y\otimes Z)$.

These data are to satisfy the following two axioms:

MC2. The following diagram commutes:



MC3. The following diagram commutes:



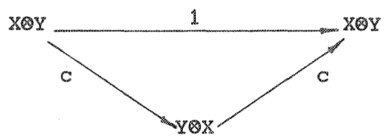
1.2. DEFINITION. A *symmetric monoidal category* is an ordered pair $V = \langle {}^{m}V, c\rangle$ consisting of

(i) a monoidal category ${}^{m}V = \langle V_0, \otimes, I, r, l, a\rangle$;

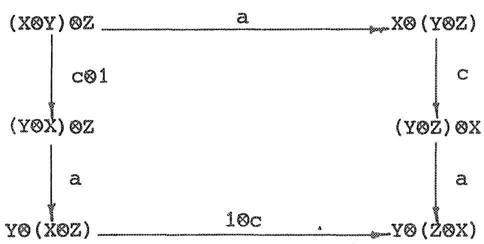
(ii) a natural isomorphism $c = c_{XY}: X\otimes Y \to Y\otimes X$.

These data are to satisfy the following two axioms (above MC2 and MC3):

MC6. The following diagram commutes:

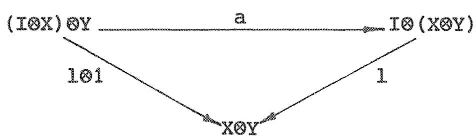


MC7. The following diagram commutes:

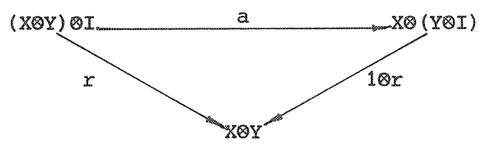


1.3. PROPOSITION ([10]). In a monoidal category V the following diagrams commute:

MC1.

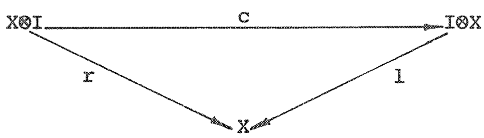


MC4.



Moreover, $l_I = r_I$: $I \otimes I \to I$ (property MC5).
In a symmetric monoidal category the following diagram commutes:

MC8.



□

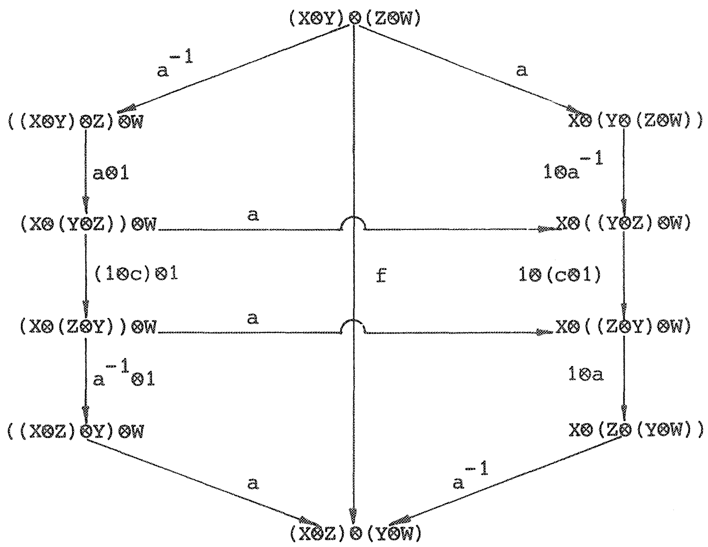
1.4. REMARK. For monoidal and symmetric monoidal categories S. MACLANE [13] has proved coherence theorems. A coherence theorem states that every diagram of a certain class commutes. In this case the class of diagrams consists of those diagrams which are built up from instances of units, a, r and l and, in the symmetric case, c, by multiplications ⊗. For an exact description of the meaning of coherence we refer to [16] or to [17], section VII.2. In the sequel we will use the coherence of a,r,l and c several times.

1.5. DEFINITION. In a symmetric monoidal category V one can construct, suitably combining a, a^{-1} and c (the details being irrelevant by coherence) a unique natural isomorphism

$$f = f_{XYZW}\colon (X \otimes Y) \otimes (Z \otimes W) \to (X \otimes Z) \otimes (Y \otimes W)$$

called the *middle four interchange isomorphism*.
The following diagram shows some possible constructions of f:

(1.1)

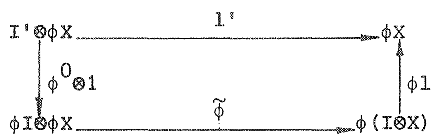


1.6. DEFINITION. Let $V = \langle V_0, \otimes, I, r, l, a \rangle$ and $V' = \langle V'_0, \otimes', I', r', l', a' \rangle$ be monoidal categories; we write $\otimes$ for $\otimes'$. *A monoidal functor* $\Phi: V \longrightarrow V'$ is an ordered triple $\Phi = \langle \phi, \tilde{\phi}, \phi^0 \rangle$ consisting of

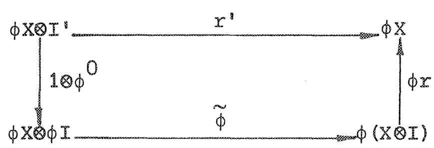
(i) a functor $\phi: V_0 \longrightarrow V'_0$;
(ii) a natural transformation $\tilde{\phi} = \tilde{\phi}_{XY}: \phi X \otimes \phi Y \rightarrow \phi(X \otimes Y)$;
(iii) a morphism $\phi^0: I' \rightarrow \phi I$.

These data are to satisfy the following three axioms:

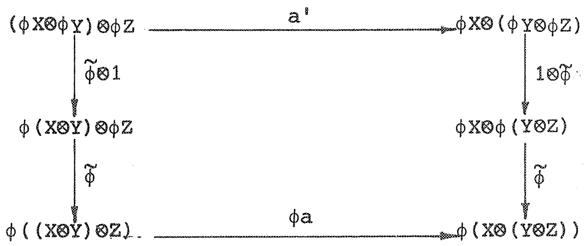
MF1. The following diagram commutes:



MF2. The following diagram commutes:

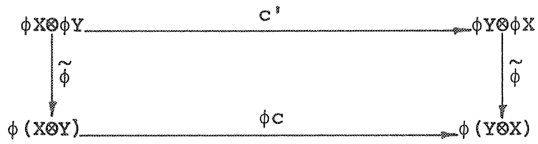


MF3. The following diagram commutes:



Let $V = \langle {}^{m}V, c\rangle$ and $V' = \langle {}^{m}V', c'\rangle$ be symmetric monoidal categories. A *symmetric monoidal functor* $\Phi\colon V \to V'$ is a monoidal functor $\Phi = \langle \phi, \tilde{\phi}, \phi^{0}\rangle\colon {}^{m}V \to {}^{m}V'$ which satisfies the additional axiom:

MF4. The following diagram commutes:

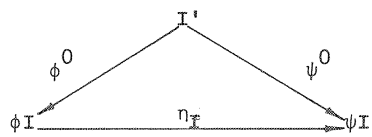


1.7. DEFINITION. Let $\Phi = \langle \phi, \tilde{\phi}, \phi^{0}\rangle\colon V \to V'$ and $\Psi = \langle \psi, \tilde{\psi}, \psi^{0}\rangle\colon V \to V'$ be monoidal functors. A *monoidal natural transformation*

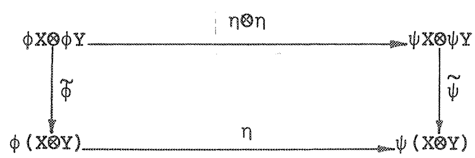
$$\eta\colon \Phi \to \Psi\colon V \to V'$$

is a natural transformation $\eta\colon \phi \to \psi\colon V_0 \to V'_0$ satisfying the following two axioms:

MN1. The following diagram commutes:



MN2. The following diagram commutes:



If Φ and Ψ: $V \longrightarrow V'$ are symmetric monoidal functors, a *symmetric monoidal natural transformation* η: $\Phi \longrightarrow \Psi$: $V \longrightarrow V'$ is simply a monoidal natural transformation η: $\Phi \longrightarrow \Psi$: ${}^{m}V \longrightarrow {}^{m}V'$.

1.8. THEOREM. ([6] theorem II.1.3). Monoidal categories, monoidal functors and monoidal natural transformations form a hypercategory *Mon*, and symmetric monoidal categories, symmetric monoidal functors and symmetric monoidal natural transformations form a hypercategory *SMon*.
For the rules of composition we refer to [6], theorem II.1.3 and to theorem II.1.4 of this tract. □

2. SEMI MONOIDAL CLOSED CATEGORIES

2.1. DEFINITION. A *semi monoidal closed category* is an ordered quadruple $V = \langle {}^{m}V, {}^{c}V, \tilde{V}, T \rangle$ consisting of

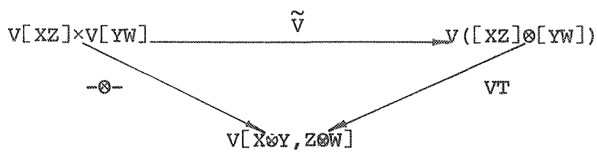
(i) a monoidal category ${}^{m}V = \langle V_0, \otimes, I, r, l, a \rangle$;

(ii) a closed category ${}^{c}V = \langle V_0, V, [-,-], I, i, j, L \rangle$ with the same V_0 and I as ${}^{m}V$;

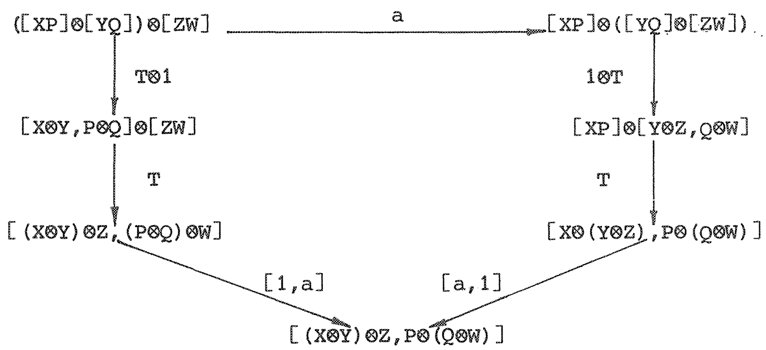
(iii) a natural transformation $\tilde{V} = \tilde{V}_{XY}$: $VX \times VY \longrightarrow V(X \otimes Y)$ in S;

(iv) a natural transformation $T = T_{XYZW}$: $[XZ] \otimes [YW] \longrightarrow [X \otimes Y, Z \otimes W]$.

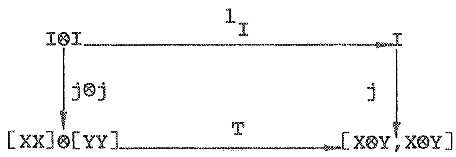
These data are to satisfy the following seven axioms:
PMCC0. The following diagram commutes:



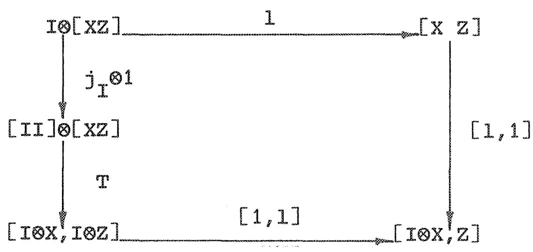
PMCC1. The following diagram commutes:



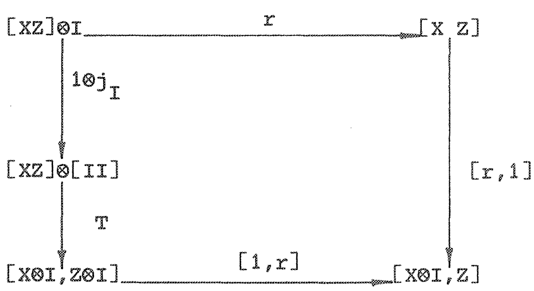
PMCC2. The following diagram commutes:



PMCC3. The following diagram commutes:



PMCC4. The following diagram commutes:



PMCC5. The following diagram commutes:

$$\begin{array}{ccc} X\otimes Y & \xrightarrow{\quad i \quad} & [I,X\otimes Y] \\ {\scriptstyle i\otimes i}\big\downarrow & & \big\downarrow{\scriptstyle [1,1]} \\ [IX]\otimes[IY] & \xrightarrow{\quad T \quad} & [I\otimes I,X\otimes Y] \end{array}$$

PMCC6. The following diagram commutes:

$$\begin{array}{ccc} [PZ]\otimes[QW] & \xrightarrow{\quad L^X\otimes L^Y \quad} & [[XP][XZ]]\otimes[[YQ][YW]] \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ [P\otimes Q,Z\otimes W] & & [[XP]\otimes[YQ],[XZ]\otimes[YW]] \\ {\scriptstyle L^{X\otimes Y}}\big\downarrow & & \big\downarrow{\scriptstyle [1,T]} \\ [[X\otimes Y,P\otimes Q],[X\otimes Y,Z\otimes W]] & \xrightarrow{\quad [T,1] \quad} & [[XP]\otimes[YQ],[X\otimes Y,Z\otimes W]] \end{array}$$

2.2. DEFINITION. A *symmetric semi monoidal closed category* is an ordered quadruple $V = \langle {}^{sm}V, {}^{sc}V, \tilde{V}, T \rangle$ consisting of

(i) a symmetric monoidal category ${}^{sm}V = \langle V_0, \otimes, I, r, l, a, c \rangle$;

(ii) a symmetric closed category ${}^{sc}V = \langle V_0, V, [-,-], I, i, j, L, s \rangle$ with the same V_0 and I as ${}^{sm}V$;

(iii) a natural transformation $\tilde{V} = \tilde{V}_{XY}: VX \times VY \to V(X\otimes Y)$;

(iv) a natural transformation $T = T_{XYZW}: [XZ]\otimes[YW] \to [X\otimes Y, Z\otimes W]$.

These data are to satisfy the axioms PMCC0, PMCC1, PMCC2, PMCC3, and the following two axioms:

PMCC7. The following diagram commutes:

$$\begin{array}{ccc} [XZ]\otimes[YW] & \xrightarrow{\quad c \quad} & [YW]\otimes[XZ] \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ [X\otimes Y,Z\otimes W] & \xrightarrow{\quad [c,c] \quad} & [Y\otimes X,W\otimes Z] \end{array}$$

PMCC8. The following diagram commutes:

$$\begin{array}{ccc}
[Z[XP]]\otimes[W[YQ]] & \xrightarrow{\hat{s}\otimes s} & [X[ZP]]\otimes[Y[WQ]] \\
\downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\
[Z\otimes W,[XP]\otimes[YQ]] & & [X\otimes Y,[ZP]\otimes[WQ]] \\
\downarrow{\scriptstyle [1,T]} & & \downarrow{\scriptstyle [1,T]} \\
[Z\otimes W,[X\otimes Y,P\otimes Q]] & \xrightarrow{s} & [X\otimes Y,[Z\otimes W,P\otimes Q]]
\end{array}$$

2.3. THEOREM. In a symmetric semi monoidal closed category the properties PMCC4, PMCC5 and PMCC6 hold, as well as the following properties:

(a) The following diagram commutes:

$$\begin{array}{ccc}
X\otimes Y & \xrightarrow{m} & [[X\otimes Y,Z\otimes W]\ Z\otimes W] \\
\downarrow{\scriptstyle m\otimes m} & & \downarrow{\scriptstyle [T,1]} \\
[[XZ]Z]\otimes[[YW]W] & \xrightarrow{T} & [[XZ]\otimes[YW],Z\otimes W]
\end{array} \qquad (2.1)$$

(b) The following diagram commutes:

$$\begin{array}{ccc}
[XP]\otimes[YQ] & \xrightarrow{R^{Z}\otimes R^{W}} & [[PZ][XZ]]\otimes[[QW][YW]] \\
\downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\
[X\otimes Y,P\otimes Q] & & [[PZ]\otimes[QW],[XZ]\otimes[YW]] \\
\downarrow{\scriptstyle R^{Z\otimes W}} & & \downarrow{\scriptstyle [1,T]} \\
[[P\otimes Q,Z\otimes W]\ [X\otimes Y,Z\otimes W]] & \xrightarrow{[T,1]} & [[PZ]\otimes[QW]\ [X\otimes Y,Z\otimes W]]
\end{array} \qquad (2.2)$$

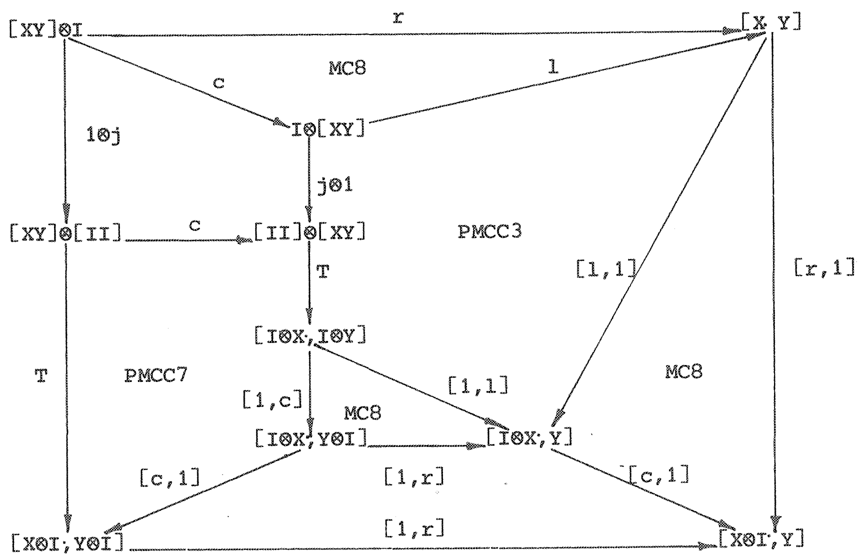
(c) The following diagram commutes:

$$\begin{array}{ccc}
([XP]\otimes[ZR])\otimes([YQ]\otimes[WS]) & \xrightarrow{f} & ([XP]\otimes[YQ])\otimes([ZR]\otimes[WS]) \\
\downarrow{\scriptstyle T\otimes T} & & \downarrow{\scriptstyle T\otimes T} \\
[X\otimes Z,P\otimes R]\otimes[Y\otimes W,Q\otimes S] & & [X\otimes Y,P\otimes Q]\otimes[Z\otimes W,R\otimes S] \\
\downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\
[(X\otimes Z)\otimes(Y\otimes W),(P\otimes R)\otimes(Q\otimes S)] & & [(X\otimes Y)\otimes(Z\otimes W),(P\otimes Q)\otimes(R\otimes S)] \\
& \searrow{\scriptstyle [f,1]} \quad {\scriptstyle [1,f]}\swarrow & \\
& [(X\otimes Y)\otimes(Z\otimes W),(P\otimes R)\otimes(Q\otimes S)] &
\end{array} \qquad (2.3)$$

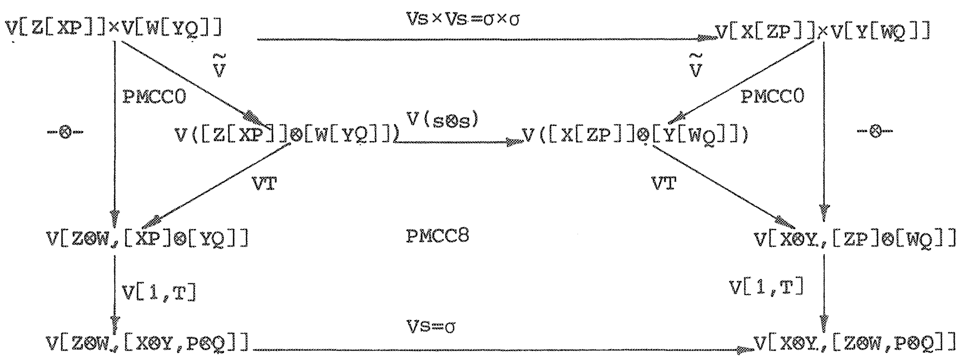
Consequently, if $V = \langle {}^{sm}V, {}^{sc}V, \tilde{V}, T\rangle$ is a symmetric semi monoidal closed category, if ${}^{m}V$ is the 'underlying' monoidal category of ${}^{sm}V$ and if ${}^{c}V$ is the 'underlying' closed category of ${}^{sc}V$ then $\langle {}^{m}V, {}^{c}V, \tilde{V}, T\rangle$ is a semi monoidal closed category.

PROOF.

(i) PMCC4 is a consequence of PMCC3 and PMCC7 (together with MC6 and MC8):

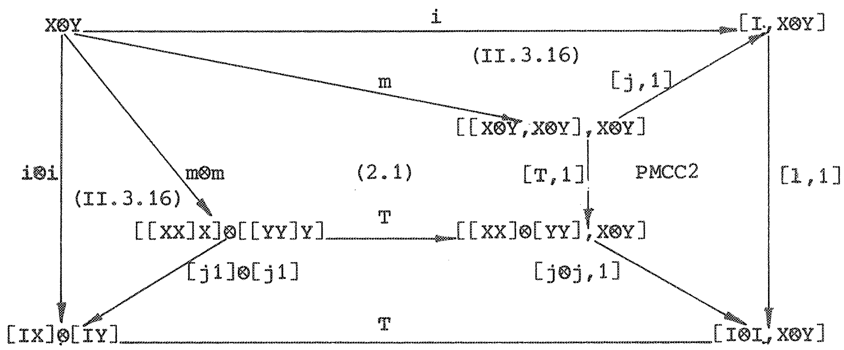


(ii) The commutativity of diagram (2.1) follows from PMCC0 and PMCC8. If we apply V to diagram PMCC8 and use PMCC0 we obtain the following diagram:

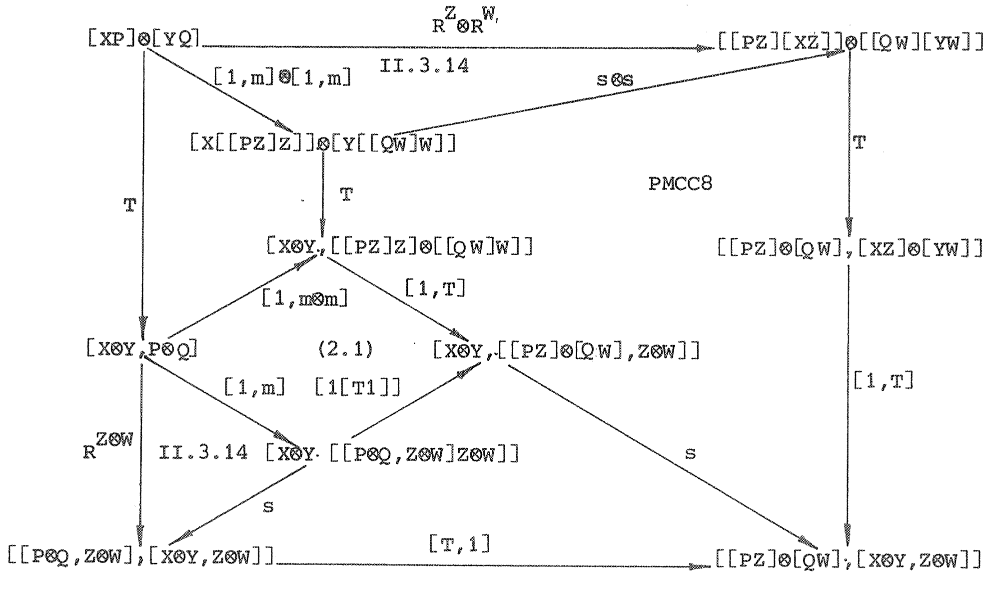


If we take Z = [XP] and W = [YQ] and evaluate at ⟨1,1⟩ we obtain diagram (2.1)

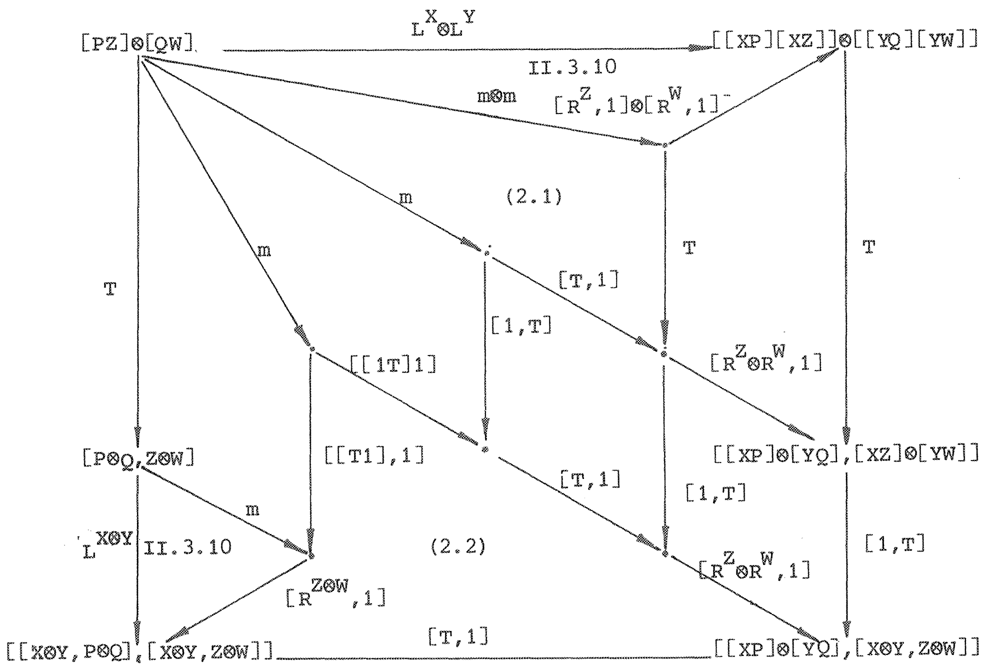
(iii) PMCC5 is a consequence of PMCC2 and (2.1):



(iv) The commutativity of (2.2) follows from PMCC8 and (2.1):



(v) PMCC6 is a consequence of (2.1) and (2.2):



(vi) The commutativity of (2.3) is a consequence of PMCC1 and PMCC7. The proof, for which one needs a rather big diagram, is left to the reader. □

2.4. PROPOSITION. *The symmetric semi monoidal closed category of sets.* We obtain a symmetric semi monoidal closed category, denoted by S, if we supply the category of sets with the symmetric closed structure defined in proposition II.2.5, with the symmetric monoidal structure induced by the cartesian product, and if we define $T_{XYZW}\colon [XZ] \times [YW] \to [X{\times}Y, Z{\times}W]$ by $T\langle g,h\rangle\langle x,y\rangle = \langle gx,hy\rangle$ $(g \in [XZ];\ h \in [YW];\ x \in X;\ y \in Y)$. □

2.5. DEFINITION. Let V and V' be semi monoidal closed categories. A *semi monoidal closed functor* $\Phi\colon V \to V'$ is an ordered quadruple $\Phi = \langle \phi,\tilde{\phi},\hat{\phi},\phi^0\rangle$ consisting of:

(i) a functor $\phi\colon V_0 \to V'_0$;
(ii) a natural transformation $\tilde{\phi} = \tilde{\phi}_{XY}\colon \phi X\otimes\phi Y \to \phi(X\otimes Y)$;
(iii) a natural transformation $\hat{\phi} = \hat{\phi}_{XY}\colon \phi[XY] \to [\phi X,\phi Y]$;
(iv) a morphism $\phi^0\colon I' \to \phi I$,

such that:

(1) ${}^m\Phi := \langle \phi,\tilde{\phi},\phi^0\rangle\colon {}^mV \to {}^mV'$ is a monoidal functor;
(2) ${}^c\Phi := \langle \phi,\hat{\phi},\phi^0\rangle\colon {}^cV \to {}^cV'$ is a closed functor,

and in addition:

PMCF. The following diagram commutes:

$$
\begin{array}{ccc}
\phi[XZ]\otimes\phi[YW] & \xrightarrow{\hat{\phi}\otimes\hat{\phi}} & [\phi X,\phi Z]\otimes[\phi Y,\phi W] \\
\downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle T'} \\
\phi([XZ]\otimes[YW]) & & [\phi X\otimes\phi Y,\phi Z\otimes\phi W] \\
\downarrow{\scriptstyle \phi T} & & \\
\phi[X\otimes Y,Z\otimes W] & & \downarrow{\scriptstyle [1,\tilde{\phi}]} \\
\downarrow{\scriptstyle \hat{\phi}} & & \\
[\phi(X\otimes Y),\phi(Z\otimes W)] & \xrightarrow{[\tilde{\phi},1]} & [\phi X\otimes\phi Y,\phi(Z\otimes W)]
\end{array}
$$

Let V and V' be symmetric semi monoidal closed categories;
a *symmetric semi monoidal closed functor* Φ: $V \to V'$ is an ordered quadruple $\Phi = \langle \phi,\tilde{\phi},\hat{\phi},\phi^0 \rangle$ as above, such that

(1) ${}^{sm}\Phi := \langle \phi,\tilde{\phi},\phi^0 \rangle$: ${}^{sm}V \to {}^{sm}V'$ is a symmetric monoidal functor;

(2) ${}^{sc}\Phi := \langle \phi,\hat{\phi},\phi^0 \rangle$: ${}^{sc}V \to {}^{sc}V'$ is a symmetric closed functor;

(3) property PMCF holds.

2.6. PROPOSITION. *The semi monoidal closed functor* V: $V \to S$.
Let V be a (symmetric) semi monoidal closed category. We obtain a (symmetric) semi monoidal closed functor V: $V \to S$ if we define $V = \langle V,\tilde{V},\hat{V},V^0 \rangle$ where

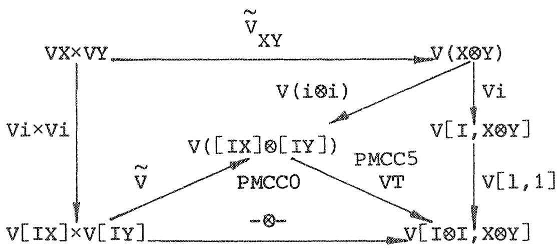
(i) V: $V_0 \to S$ is the basic functor of ${}^{c}V$;

(ii) $\tilde{V} = \tilde{\tilde{V}}_{XY}$: VX × VY → V(X⊗Y) is the natural transformation given in the definition of V;

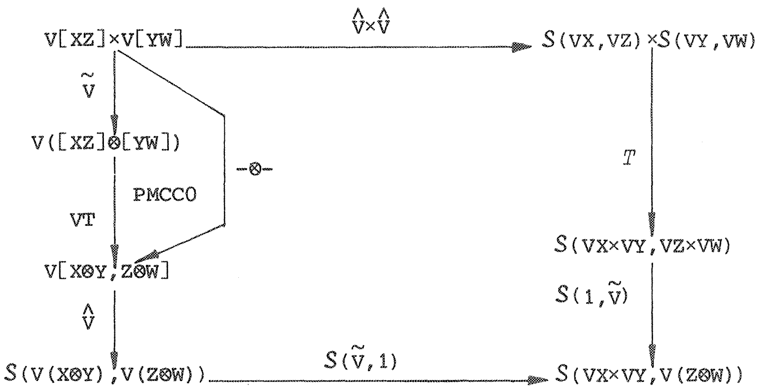
(iii) $\hat{V} = \hat{V}_{XY}$: V[XY] → S(VX,VY) is V_{XY};

(iv) V^0: {*} → VI is $V^0 * = (Vi_I)^{-1}1$.

PROOF. By [6] proposition I.3.11 the functor V admits an extension to a closed functor ${}^{c}V := \langle V,\hat{V},V^0 \rangle$: ${}^{c}V \to {}^{c}S$. In order to prove that ${}^{m}V := \langle V,\tilde{V},V^0 \rangle$: ${}^{m}V \to {}^{m}S$ is a monoidal functor we note that the following diagram is commutative:



This diagram agrees with [6],II.8.4. Hence from [6] proposition II.8.1 it follows that ${}^{m}V$: ${}^{m}\mathcal{V} \longrightarrow {}^{m}S$ is a monoidal functor. Next we prove PMCF for V:



Evaluate this diagram at $\langle g,h\rangle \in V[XZ]\times V[YW]$:

$$(S(1,\tilde{V})\circ T\circ \hat{V}\times\hat{V})\ \langle g,h\rangle = \tilde{V}(Vg\times Vh)$$

and

$$(S(\tilde{V},1)\circ\hat{V}\circ VT\circ\tilde{V})\ \langle g,h\rangle = S(\tilde{V},1)\hat{V}(g\otimes h) \qquad \text{by PMCC0}$$

$$= V(g\otimes h)\tilde{V},$$

hence the diagram commutes by the naturality of $\tilde{V}$. If $\mathcal{V}$ is a symmetric semi monoidal closed category then ${}^{sc}V = \langle V,\hat{V},V^0\rangle$: ${}^{sc}\mathcal{V} \longrightarrow {}^{sc}S$ is a symmetric closed functor (theorem II.5.4) and ${}^{sm}V = \langle V,\tilde{V},V^0\rangle$: ${}^{sm}\mathcal{V} \longrightarrow {}^{sm}S$ is a symmetric monoidal functor ([6], proposition III,1,3). □

2.7. DEFINITION. Let $\Phi = \langle \phi,\tilde{\phi},\hat{\phi},\phi^0\rangle$ and $\Psi = \langle \psi,\tilde{\psi},\hat{\psi},\psi^0\rangle$: $\mathcal{V} \longrightarrow \mathcal{V}'$ be (symmetric) semi monoidal closed functors. A *(symmetric) semi monoidal closed natural transformation* η: $\Phi \longrightarrow \Psi$: $\mathcal{V} \longrightarrow \mathcal{V}'$ is a natural transformation η: $\phi \longrightarrow \psi$: $\mathcal{V}_0 \longrightarrow \mathcal{V}'_0$ which is a (symmetric) closed natural transformation

$\eta\colon {}^c\Phi \to {}^c\Psi\colon {}^cV \to {}^cV'$ and a (symmetric) monoidal natural transformation $\eta\colon {}^m\Phi \to {}^m\Psi\colon {}^mV \to {}^mV'$. This means that η satisfies the three axioms MN1 = CN1, MN2 and CN2.

2.8. THEOREM. (Symmetric) semi monoidal closed categories, (symmetric) semi monoidal closed functors, and (symmetric) semi monoidal closed natural transformations form, with the obvious rules of composition, a hypercategory *PMCC* (*SPMCC* respectively). □

3. TENSOR PRODUCTS OF V-CATEGORIES

In this section V is a semi monoidal closed category. If we suppose V to be symmetric, this will be mentioned explicitely. V-categories, V-functors and V-natural transformations are defined to be ${}^cV$-categories, ${}^cV$-functors and ${}^cV$-natural transformations (cf. [6] sections I.5 and I.10, and this tract, sections II.6 and II.7), the monoidal structure of V playing no part in these definitions. However, the monoidal structure of V enables us to define a tensor product of V-categories, and as a next step, V-bifunctors. We develop the theory parallel to the theory of tensor products of categories over a symmetric monoidal category in [6], chapter III, sections 3, 4 and 5, as far as we need it. In chapter IV, section 11 we shall show that these two concepts in fact coincide in the case that V is a symmetric monoidal closed category.

3.1. PROPOSITION. If A and B are V-categories, the following data define a V-category $A\otimes B$:

(i) $\text{ob}\, A\otimes B = \text{ob}\, A \times \text{ob}\, B$;

(ii) $(A\otimes B)(\langle XY\rangle\langle X'Y'\rangle) = A(XX')\otimes B(YY')$;

(iii) $j_{\langle XY\rangle}\colon I \to (A\otimes B)(\langle XY\rangle\langle XY\rangle)$ is the composite

$$(3.1)\qquad I \xrightarrow{\;l_I^{-1}\;} I\otimes I \xrightarrow{\;j\otimes j\;} A(XX)\otimes B(YY)$$

(iv) $L^{\langle XY\rangle}_{\langle X'Y'\rangle\langle X''Y''\rangle}\colon (A\otimes B)(\langle X'Y'\rangle\langle X''Y''\rangle) \to$

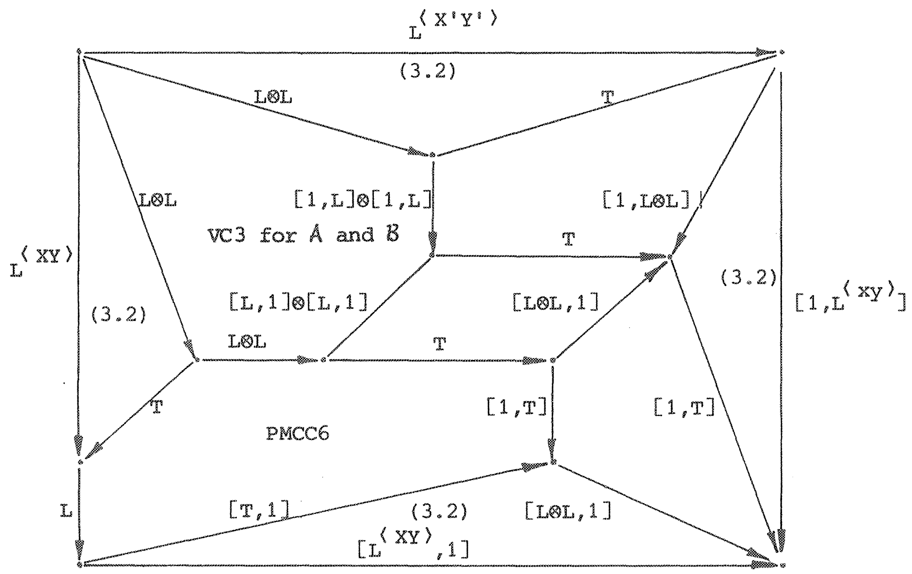
$\qquad\to [(A\otimes B)(\langle XY\rangle\langle X'Y'\rangle), (A\otimes B)(\langle XY\rangle\langle X''Y''\rangle)]$

is the composite

$$A(X'X'')\otimes B(Y'Y'') \xrightarrow{L^X_{X'X''}\otimes L^Y_{Y'Y''}} [A(XX'),A(XX'')]\otimes[B(YY'),B(YY'')] \tag{3.2}$$

$$\xrightarrow{\quad T \quad} [A(XX')\otimes B(YY'),A(XX'')\otimes B(YY'')]$$

PROOF. VC3 for $A\otimes B$ follows from VC3 for A and B and PMCC6 for V:



In a similar way, VC1 follows from VC1 for A and B and PMCC2 for V; and VC2 follows from VC2 for A and B and PMCC5 for V. □

3.2. PROPOSITION. If $T\colon A \to C$ and $S\colon B \to D$ are V-functors, the following data define a V-functor $T\otimes S\colon A\otimes B \to C\otimes D$:

(i) $(T\otimes S)\langle XY\rangle = \langle TX,SY\rangle$;

(ii) $(T\otimes S)_{\langle XY\rangle\langle X'Y'\rangle}\colon (A\otimes B)(\langle XY\rangle\langle X'Y'\rangle) \to (C\otimes D)(\langle TX,SY\rangle,\langle TX',SY'\rangle)$ is $T_{XX'}\otimes S_{YY'}\colon A(XX')\otimes B(YY') \to C(TX,TX')\otimes D(SY,SY')$.

The proof is straightforward and is left to the reader. □

3.3. PROPOSITION. The assignments $\langle A,B\rangle \mapsto A\otimes B$ and $\langle T,S\rangle \mapsto T\otimes S$ constitute a functor $\otimes\colon V_* \times V_* \to V_*$. □

3.4. PROPOSITION. Let $\mathcal{V}$ be a symmetric semi monoidal closed category, and let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. Define $R^{\langle X''Y''\rangle}_{\langle XY\rangle\langle X'Y'\rangle}$:
$(\mathcal{A}\otimes\mathcal{B})(\langle XY\rangle\langle X'Y'\rangle) \longrightarrow [(\mathcal{A}\otimes\mathcal{B})(\langle X'Y'\rangle\langle X''Y''\rangle),(\mathcal{A}\otimes\mathcal{B})(\langle XY\rangle\langle X''Y''\rangle)]$ by

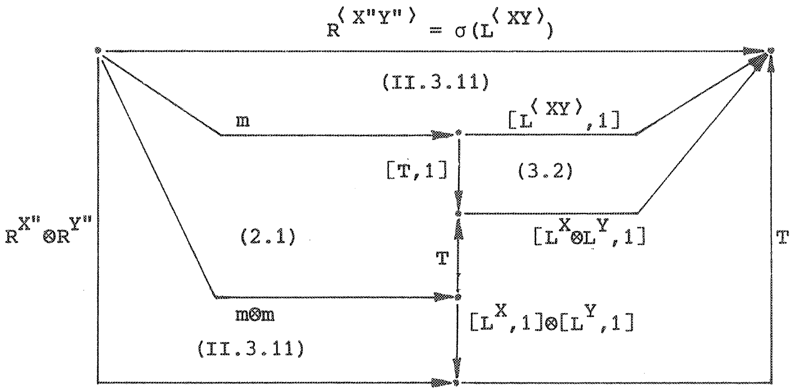
$$R^{\langle X''Y''\rangle}_{\langle XY\rangle\langle X'Y'\rangle} = \sigma(L^{\langle XY\rangle}_{\langle X'Y'\rangle\langle X''Y''\rangle}).$$

Then $R^{\langle X''Y''\rangle}_{\langle XY\rangle\langle X'Y'\rangle}$ is the composite

$$\mathcal{A}(XX')\otimes\mathcal{B}(YY') \xrightarrow{R^{X''}_{XX'}\otimes R^{Y''}_{YY'}} [\mathcal{A}(X'X''),\mathcal{A}(XX'')]\otimes[\mathcal{B}(Y'Y''),\mathcal{B}(YY'')]$$

$$\xrightarrow{T} [\mathcal{A}(X'X'')\otimes\mathcal{B}(Y'Y''),\mathcal{A}(XX'')\otimes\mathcal{B}(YY'')]$$

PROOF.



□

3.5. COROLLARY. If $\mathcal{V}$ is a symmetric semi monoidal closed category then

$$(\mathcal{A}\otimes\mathcal{B})^* = \mathcal{A}^*\otimes\mathcal{B}^*$$

and

$$(T\otimes S)^* = T^*\otimes S^*.$$

□

We recall [6], proposition I.5.3: If $\mathcal{V}$ is a closed category, we obtain a $\mathcal{V}$-category $\mathcal{I}$ with a single object $*$ if we take $\mathcal{I}(**) = I$, take $j\colon I \to \mathcal{I}(**)$ to be $1_I\colon I \to I$, and take $L\colon \mathcal{I}(**) \longrightarrow [\mathcal{I}(**),\mathcal{I}(**)]$ to be $i_I\colon I \to [II]$. Moreover, if $\mathcal{A}$ is a $\mathcal{V}$-category and $X \in \text{ob}\,\mathcal{A}$ we get a $\mathcal{V}$-functor $J^X\colon \mathcal{I} \longrightarrow \mathcal{A}$ if we set $J^X* = X$ and take $J^X\colon \mathcal{I}(**) \longrightarrow \mathcal{A}(XX)$ to be $j_X\colon I \longrightarrow \mathcal{A}(XX)$.

3.6. PROPOSITION. A V-functor a_{ABC}: $(A\otimes B)\otimes C \longrightarrow A\otimes(B\otimes C)$ is defined by

$$a\langle\langle XY\rangle Z\rangle = \langle X\langle YZ\rangle\rangle$$

and

$$a:\ ((A\otimes B)\otimes C)(\langle\langle XY\rangle Z\rangle,\langle\langle X'Y'\rangle Z'\rangle) \longrightarrow (A\otimes(B\otimes C))(\langle X\langle YZ\rangle\rangle,\langle X'\langle Y'Z'\rangle\rangle)$$

is

$$a:\ (A(XX')\otimes B(YY'))\otimes C(ZZ') \longrightarrow A(XX')\otimes(B(YY')\otimes C(ZZ')).$$

A V-functor r_A: $A\otimes I \longrightarrow A$ is defined by

$$r\langle X,*\rangle = X$$

and

$$r:\ (A\otimes I)(\langle X,*\rangle,\langle X'*\rangle) \longrightarrow A(XX')$$

is

$$r:\ A(XX')\otimes I \longrightarrow A(XX').$$

A V-functor l_A: $I\otimes A \longrightarrow A$ is defined similarly by

$$l\langle *,X\rangle = X$$

and

$$l:\ (I\otimes A)(\langle *,X\rangle,\langle *,X'\rangle) \longrightarrow A(XX')$$

is

$$l:\ I\otimes A(XX') \longrightarrow A(XX').$$

r,l and a are natural isomorphisms in the category V_* and $\langle V_*,\otimes,I,r,l,a\rangle$ is a monoidal category.
If V is a symmetric semi monoidal closed category then a V-functor c_{AB}: $A\otimes B \longrightarrow B\otimes A$ is defined by

$$c\langle XY\rangle = \langle YX\rangle$$

and

$$c:\ (A\otimes B)(\langle XY\rangle\langle X'Y'\rangle) \longrightarrow (B\otimes A)(\langle YX\rangle\langle Y'X'\rangle)$$

is

$$c:\ A(XX')\otimes B(YY') \longrightarrow B(YY')\otimes A(XX').$$

c is also a natural isomorphism in the category V_* and $\langle V_*,\otimes,I,r,l,a,c\rangle$ is a symmetric monoidal category.

PROOF. The proofs of VF1 and VF2 for a, l, r and c are straightforward verifications. Clearly, a, r, l and c are natural isomorphisms. The axioms MC2 and MC3 and, in the symmetric case, MC6 and MC7, are immediate consequences of the corresponding axioms for ${}^{m}\mathcal{V}$. □

3.7. DEFINITION. Let $\mathcal{A}$ $\mathcal{B}$ and $\mathcal{C}$ be $\mathcal{V}$-categories. A $\mathcal{V}$-functor $S\colon \mathcal{A}\otimes\mathcal{B} \to \mathcal{C}$ is also called a *$\mathcal{V}$-bifunctor*. Given a $\mathcal{V}$-bifunctor $S\colon \mathcal{A}\otimes\mathcal{B} \to \mathcal{C}$ we define for each $X \in \mathrm{ob}\,\mathcal{A}$ a $\mathcal{V}$-functor $S(X-)\colon \mathcal{B} \to \mathcal{C}$ as the composite

$$\mathcal{B} \xrightarrow{\;l^{-1}\;} \mathcal{I}\otimes\mathcal{B} \xrightarrow{\;J^{X}\otimes 1\;} \mathcal{A}\otimes\mathcal{B} \xrightarrow{\;S\;} \mathcal{C} .$$

Similarly, for each $Y \in \mathrm{ob}\,\mathcal{B}$ we define a $\mathcal{V}$-functor $S(-Y)\colon \mathcal{A} \to \mathcal{C}$ as the composite

$$\mathcal{A} \xrightarrow{\;r^{-1}\;} \mathcal{A}\otimes\mathcal{I} \xrightarrow{\;1\otimes J^{Y}\;} \mathcal{A}\otimes\mathcal{B} \xrightarrow{\;S\;} \mathcal{C} .$$

The $\mathcal{V}$-functors $S(X-)$ and $S(-Y)$ are called the *partial $\mathcal{V}$-functors* of the $\mathcal{V}$-bifunctor S.

From this definition it follows that:

(i) $S(X-)Y = S(-Y)X = S(XY)$ for all $X \in \mathrm{ob}\,\mathcal{A}$ and $Y \in \mathrm{ob}\,\mathcal{B}$;

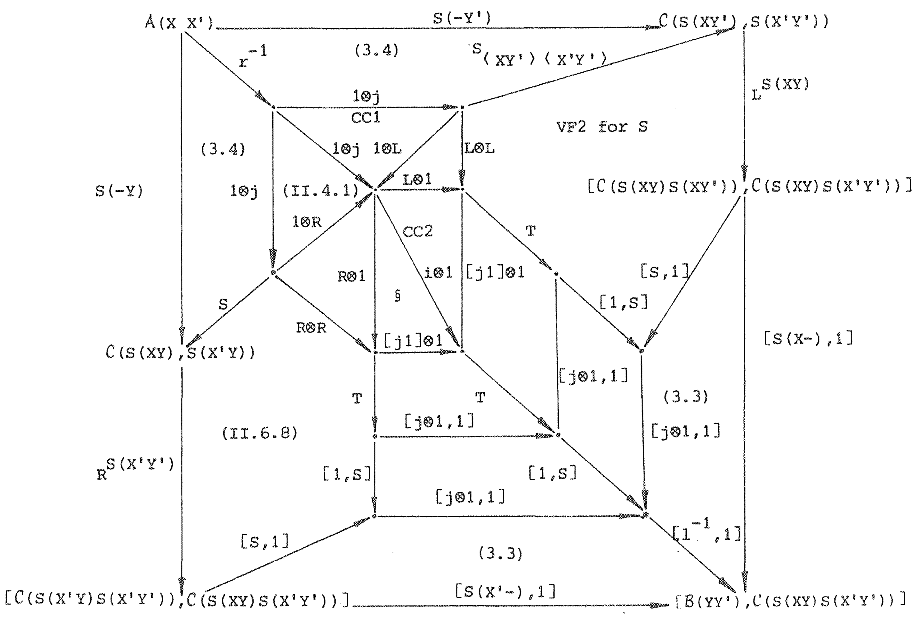
(ii) the following two diagrams in $\mathcal{V}_0$ are commutative:

(3.3)
$$\begin{array}{ccc}
\mathcal{B}(Y\,Y') & \xrightarrow{\;S(X-)_{YY'}\;} & \mathcal{C}(S(XY),S(XY')) \\
\Big\downarrow{\scriptstyle l^{-1}} & & \Big\uparrow{\scriptstyle S_{\langle XY\rangle\langle XY'\rangle}} \\
I\otimes\mathcal{B}(YY') & \xrightarrow{\;j\otimes 1\;} & \mathcal{A}(XX)\otimes\mathcal{B}(YY')
\end{array}$$

(3.4)
$$\begin{array}{ccc}
\mathcal{A}(X\,X') & \xrightarrow{\;S(-Y)_{XX'}\;} & \mathcal{C}(S(XY),S(X'Y)) \\
\Big\downarrow{\scriptstyle r^{-1}} & & \Big\uparrow{\scriptstyle S_{\langle XY\rangle\langle X'Y\rangle}} \\
\mathcal{A}(XX')\otimes I & \xrightarrow{\;1\otimes j\;} & \mathcal{A}(XX')\otimes\mathcal{B}(YY)
\end{array}$$

3.8. PROPOSITION. Let $\mathcal{V}$ be a symmetric semi monoidal closed category, and let $S\colon \mathcal{A}\otimes\mathcal{B} \to \mathcal{C}$ be a $\mathcal{V}$-functor. Let $S_1 = (S(-Y))_{Y \in \mathrm{ob}\,\mathcal{B}}$ and $S_2 = (S(X-))_{X \in \mathrm{ob}\,\mathcal{A}}$ be the families of partial $\mathcal{V}$-functors. Then $\langle S_1,S_2\rangle\colon \langle \mathcal{A},\mathcal{B}\rangle \to \mathcal{C}$ is a quasi-$\mathcal{V}$-bifunctor.

PROOF. QVF2 for $\langle S_1, S_2 \rangle$:



(§ commutes since it is the image of CC1 under σ). $\square$

4. THE V-BIFUNCTOR TEN: $V\otimes V \longrightarrow V$ AND ITS PARTIAL V-FUNCTORS.

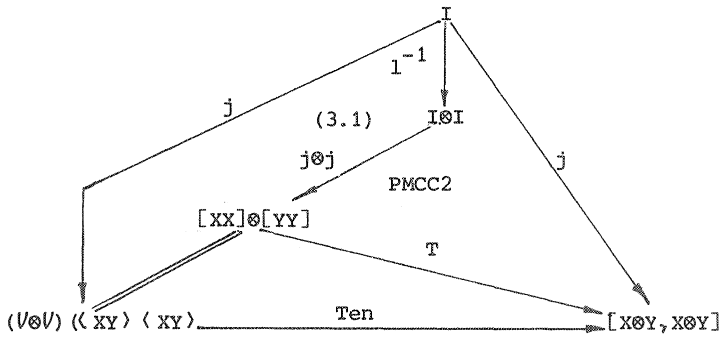
4.1. PROPOSITION. A V-bifunctor Ten: $V\otimes V \longrightarrow V$ is defined by

(i) $\mathrm{Ten}\langle XY\rangle = X\otimes Y$;

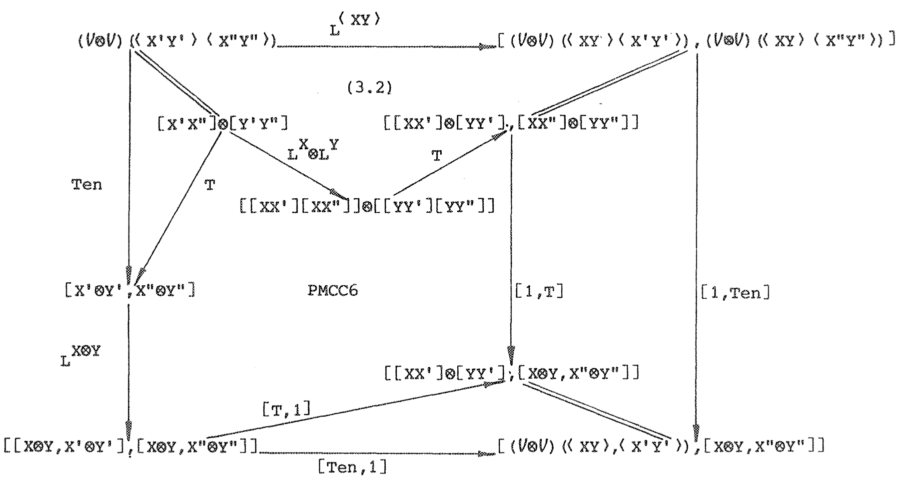
(ii) $\mathrm{Ten}_{\langle XY\rangle\langle X'Y'\rangle}: (V\otimes V)(\langle XY\rangle\langle X'Y'\rangle) \longrightarrow V(\mathrm{Ten}\langle X,Y\rangle, \mathrm{Ten}\langle X'Y'\rangle)$

is $\quad T_{XYX'Y'}: [XX']\otimes[YY'] \longrightarrow [X\otimes Y, X'\otimes Y']$.

PROOF. VF1 for Ten:



VF2 for Ten:



□

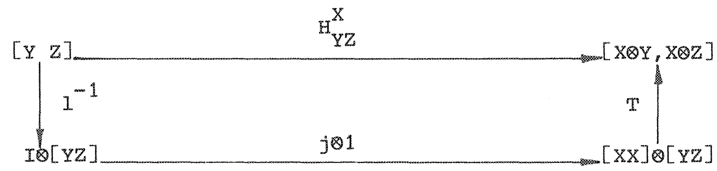
4.2. DEFINITION. For the partial V-functors of the V-bifunctor Ten we adopt the special notation H^X = Ten ⟨X-⟩ and K^Y = Ten ⟨-Y⟩. So the V-functors H^X: $V \longrightarrow V$ and K^Y: $V \longrightarrow V$ (X,Y $\in$ ob V_0) are defined by:

(i) $H^X Y = K^Y X = X \otimes Y$

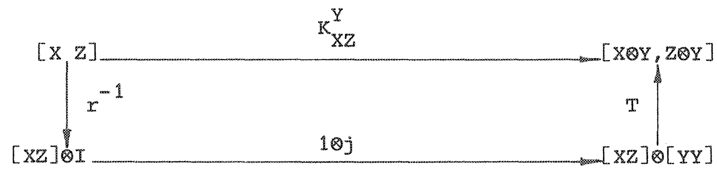
and

(ii) H^X_{YZ}: $[YZ] \to [X\otimes Y, X\otimes Z]$ and K^Y_{XZ}: $[XZ] \to [X\otimes Y, Z\otimes Y]$
are determined by the commutativity of the following diagrams:

(4.1)

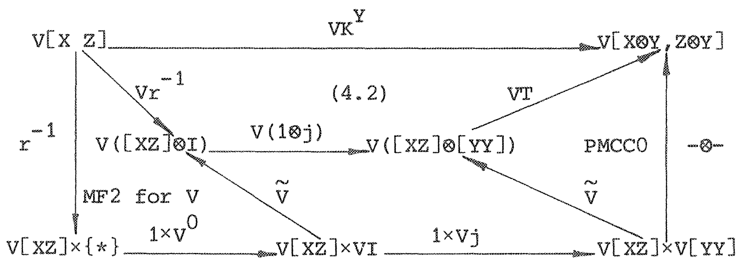


(4.2)



4.3. PROPOSITION. The underlying functor $V_* H^X$: $V_0 \to V_0$ of the V-functor H^X: $V \to V$ is $X\otimes -$; the underlying functor $V_* K^Y$: $V_0 \to V_0$ of the V-functor K^Y: $V \to V$ is $-\otimes Y$.

PROOF. Consider the following diagram:



Evaluate this diagram at $g \in V[XZ]$:

$(VK^Y)g = (VK^Y \circ r)\langle g,*\rangle$

$= g\otimes(Vj \circ V^0)*$ by the commutativity of the diagram

$= g\otimes 1$ by [6] proposition I.7.2.

Thus we have proved $V_* K^Y = -\otimes Y$.

Similarly, $V_* H^X = X\otimes -$. □

4.4. PROPOSITION (The V-naturality of T).

The morphisms T_{XYZW}: $[XZ]\otimes[YW] \to [X\otimes Y, Z\otimes W]$ are V-natural in the variables Z and W; if V is a symmetric semi monoidal closed category then the morphisms T_{XYZW} are V-natural in all variables. The V-naturality of T_{XYZW} is expressed by commutativity of the following four diagrams:

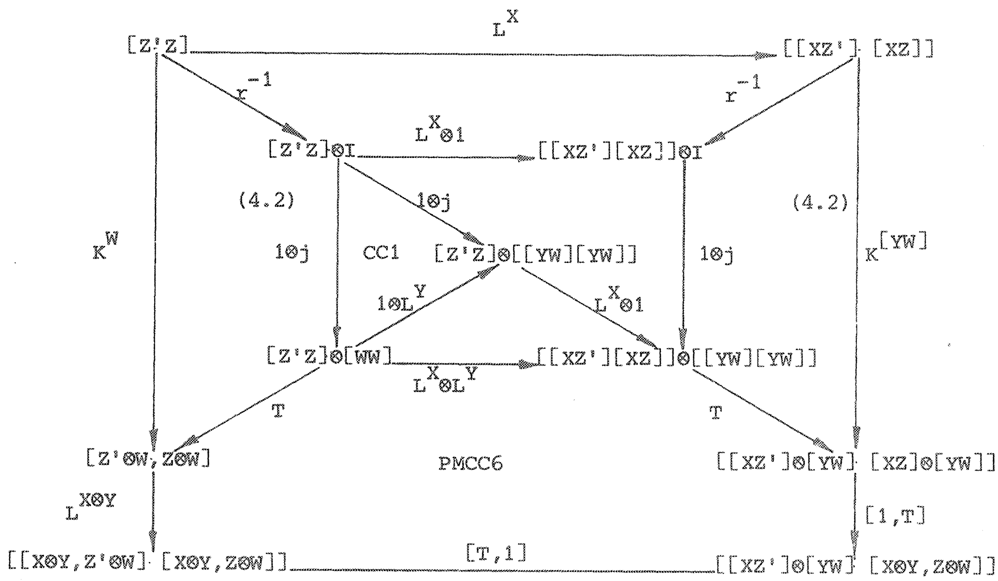
$$
(4.3)\qquad
\begin{array}{ccc}
[W'W] & \xrightarrow{\;L^{Y}\;} & [[YW']\,[YW]] \\
\downarrow{\scriptstyle H^{Z}} & & \downarrow{\scriptstyle H^{[XZ]}} \\
[Z\otimes W', Z\otimes W] & & [[XZ]\otimes[YW']\,[XZ]\otimes[YW]] \\
\downarrow{\scriptstyle L^{X\otimes Y}} & & \downarrow{\scriptstyle [1,T]} \\
[[X\otimes Y, Z\otimes W']\,[X\otimes Y, Z\otimes W]] & \xrightarrow{\;[T,1]\;} & [[XZ]\otimes[YW']\,[X\otimes Y, Z\otimes W]]
\end{array}
$$

$$
(4.4)\qquad
\begin{array}{ccc}
[Z'Z] & \xrightarrow{\;L^{X}\;} & [[XZ']\,[XZ]] \\
\downarrow{\scriptstyle K^{W}} & & \downarrow{\scriptstyle K^{[YW]}} \\
[Z'\otimes W, Z\otimes W] & & [[XZ']\otimes[YW]\,[XZ]\otimes[YW]] \\
\downarrow{\scriptstyle L^{X\otimes Y}} & & \downarrow{\scriptstyle [1,T]} \\
[[X\otimes Y, Z'\otimes W]\,[X\otimes Y, Z\otimes W]] & \xrightarrow{\;[T,1]\;} & [[XZ']\otimes[YW]\,[X\otimes Y, Z\otimes W]]
\end{array}
$$

$$
(4.5)\qquad
\begin{array}{ccc}
[Y\,Y'] & \xrightarrow{\;R^{W}\;} & [[Y'W]\,[YW]] \\
\downarrow{\scriptstyle H^{X}} & & \downarrow{\scriptstyle H^{[XZ]}} \\
[X\otimes Y, X\otimes Y'] & & [[XZ]\otimes[Y'W]\,[XZ]\otimes[YW]] \\
\downarrow{\scriptstyle R^{Z\otimes W}} & & \downarrow{\scriptstyle [1,T]} \\
[[X\otimes Y', Z\otimes W]\,[X\otimes Y, Z\otimes W]] & \xrightarrow{\;[T,1]\;} & [[XZ]\otimes[Y'W]\,[X\otimes Y, Z\otimes W]]
\end{array}
$$

$$
(4.6)\qquad
\begin{array}{ccc}
[X\,X'] & \xrightarrow{\;R^{Z}\;} & [[X'Z]\,[XZ]] \\
\downarrow{\scriptstyle K^{Y}} & & \downarrow{\scriptstyle K^{[YW]}} \\
[X\otimes Y, X'\otimes Y] & & [[X'Z]\otimes[YW]\,[XZ]\otimes[YW]] \\
\downarrow{\scriptstyle R^{Z\otimes W}} & & \downarrow{\scriptstyle [1,T]} \\
[[X'\otimes Y, Z\otimes W]\,[X\otimes Y, Z\otimes W]] & \xrightarrow{\;[T,1]\;} & [[X'Z]\otimes[YW]\,[X\otimes Y, Z\otimes W]]
\end{array}
$$

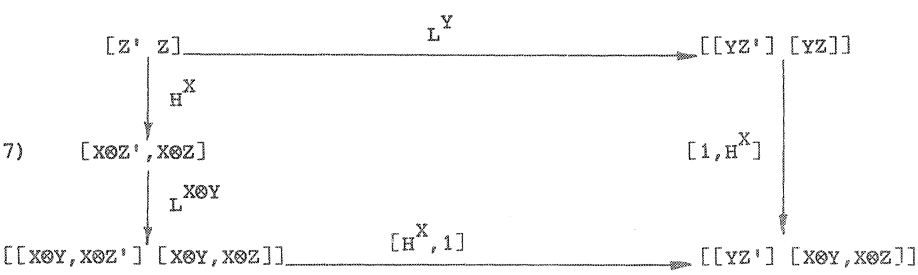
PROOF. Proof of the commutativity of diagram (4.4):



The commutativity of diagram (4.3) follows in a similar way from PMCC6, the commutativity of (4.5) and (4.6) is a consequence of the commutativity of (2.2). □

4.5. PROPOSITION (The V-naturality of K and H).
The morphisms H^X_{YZ}: $[YZ] \longrightarrow [X\otimes Y, X\otimes Z]$ and K^Y_{XZ}: $[XZ] \longrightarrow [X\otimes Y, Z\otimes Y]$ are V-natural in the variable Z; if V is a symmetric semi monoidal closed category then H^X_{YZ} and K^Y_{XZ} are V-natural in every variable. The V-naturality of H^X_{YZ} and K^Y_{XZ} is expressed by commutativity of the following six diagrams:

(4.7)



$$\begin{array}{ccc}
[Y\ Y'] & \xrightarrow{R^Z} & [[Y'Z]\ [YZ]] \\
\downarrow{\scriptstyle H^X} & & \\
[X\otimes Y, X\otimes Y'] & & \downarrow{\scriptstyle [1,H^X]} \\
\downarrow{\scriptstyle R^{X\otimes Z}} & & \\
[[X\otimes Y', X\otimes Z]\ [X\otimes Y, X\otimes Z]] & \xrightarrow{[H^X,1]} & [[Y'Z]\ [X\otimes Y, X\otimes Z]]
\end{array} \tag{4.8}$$

$$\begin{array}{ccc}
[X\ X'] & \xrightarrow{K^Z} & [X\otimes Z, X'\otimes Z] \\
\downarrow{\scriptstyle K^Y} & & \downarrow{\scriptstyle L^{X\otimes Y}} \\
[X\otimes Y, X'\otimes Y] & & [[X\otimes Y, X\otimes Z]\ [X\otimes Y, X'\otimes Z]] \\
\downarrow{\scriptstyle R^{X'\otimes Z}} & & \downarrow{\scriptstyle [H^X,1]} \\
[[X'\otimes Y, X'\otimes Z]\ [X\otimes Y, X'\otimes Z]] & \xrightarrow{[H^{X'},1]} & [[YZ]\ [X\otimes Y, X'\otimes Z]]
\end{array} \tag{4.9}$$

$$\begin{array}{ccc}
[Z'Z] & \xrightarrow{L^X} & [[XZ']\ [XZ]] \\
\downarrow{\scriptstyle K^Y} & & \\
[Z'\otimes Y, Z\otimes Y] & & \downarrow{\scriptstyle [1,K^Y]} \\
\downarrow{\scriptstyle L^{X\otimes Y}} & & \\
[[X\otimes Y, Z'\otimes Y]\ [X\otimes Y, Z\otimes Y]] & \xrightarrow{[K^Y,1]} & [[XZ']\ [X\otimes Y, Z\otimes Y]]
\end{array} \tag{4.10}$$

$$\begin{array}{ccc}
[X\ X'] & \xrightarrow{R^Z} & [[X'Z]\ [XZ]] \\
\downarrow{\scriptstyle K^Y} & & \\
[X\otimes Y, X'\otimes Y] & & \downarrow{\scriptstyle [1,K^Y]} \\
\downarrow{\scriptstyle R^{Z\otimes Y}} & & \\
[[X'\otimes Y, Z\otimes Y]\ [X\otimes Y, Z\otimes Y]] & \xrightarrow{[K^Y,1]} & [[X'Z]\ [X\otimes Y, Z\otimes Y]]
\end{array} \tag{4.11}$$

$$\begin{array}{ccc}
[Y\ Y'] & \xrightarrow{H^Z} & [Z\otimes Y, Z\otimes Y'] \\
\downarrow{\scriptstyle H^X} & & \downarrow{\scriptstyle L^{X\otimes Y}} \\
[X\otimes Y, X\otimes Y'] & & [[X\otimes Y, Z\otimes Y]\ [X\otimes Y, Z\otimes Y']] \\
\downarrow{\scriptstyle R^{Z\otimes Y'}} & & \downarrow{\scriptstyle [K^Y,1]} \\
[[X\otimes Y', Z\otimes Y']\ [X\otimes Y, Z\otimes Y']] & \xrightarrow{[K^{Y'},1]} & [[XZ]\ [X\otimes Y, Z\otimes Y']]
\end{array} \tag{4.12}$$

PROOF. (4.7) is VF2 for H^X, (4.8) is the image of (4.7) under σ; (4.10) is VF2 for K^Y, (4.11) is the image of (4.10) under σ (cf. diagram II.6.8) (4.12) is QVF2 for H and K, (4.9) is the image of (4.12) under σ (cf. diagram II.6.12). □

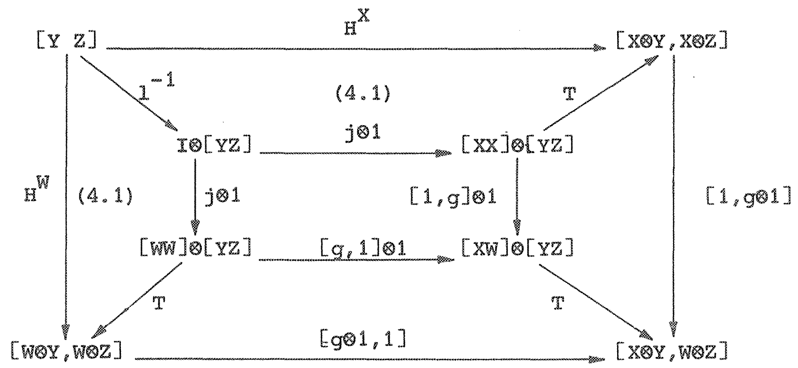
4.6. PROPOSITION. If $g\colon X \longrightarrow W$ then the morphisms $g\otimes 1\colon X\otimes Y \longrightarrow W\otimes Y$ are the Y-components of a V-natural transformation

$$H^g\colon H^X \longrightarrow H^W\colon V \longrightarrow V.$$

If $h\colon Y \longrightarrow W$ then the morphisms $1\otimes h\colon X\otimes Y \longrightarrow X\otimes W$ are the X-components of a V-natural transformation

$$K^h\colon K^Y \longrightarrow K^W\colon V \longrightarrow V.$$

PROOF. VN for H^g:



The proof of VN for K^h is similar. □

4.7. PROPOSITION. (The V-naturality of a,l,r and c)
The isomorphisms $a_{XYZ}\colon (X\otimes Y)\otimes Z \longrightarrow X\otimes(Y\otimes Z)$, $l_X\colon I\otimes X \longrightarrow X$ and $r_X\colon X\otimes I \longrightarrow X$ are V-natural in every variable. If V is a symmetric semi monoidal closed category then the isomorphism $c_{XY}\colon X\otimes Y \longrightarrow Y\otimes X$ is V-natural in every variable. These V-naturalities are expressed by the commutativity of the following diagrams:

(4.13)
$$
\begin{array}{ccc}
[X'X] & \xrightarrow{K^{Y}} & [X'\otimes Y, X\otimes Y] \\
\Big\downarrow K^{Y\otimes Z} & & \Big\downarrow K^{Z} \\
 & & [(X'\otimes Y)\otimes Z, (X\otimes Y)\otimes Z] \\
 & & \Big\downarrow [1,a] \\
[X'\otimes(Y\otimes Z), X\otimes(Y\otimes Z)] & \xrightarrow{[a,1]} & [(X'\otimes Y)\otimes Z, X\otimes(Y\otimes Z)]
\end{array}
$$

(4.14)
$$
\begin{array}{ccc}
[Y',Y] & \xrightarrow{H^{X}} & [X\otimes Y', X\otimes Y] \\
\Big\downarrow K^{Z} & & \Big\downarrow K^{Z} \\
[Y'\otimes Z, Y\otimes Z] & & [(X\otimes Y')\otimes Z, (X\otimes Y)\otimes Z] \\
\Big\downarrow H^{X} & & \Big\downarrow [1,a] \\
[X\otimes(Y'\otimes Z), X\otimes(Y\otimes Z)] & \xrightarrow{[a,1]} & [(X\otimes Y')\otimes Z, X\otimes(Y\otimes Z)]
\end{array}
$$

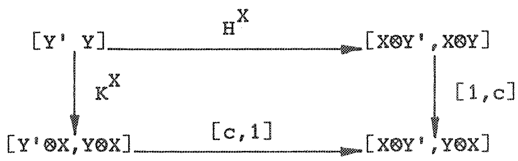
(4.15)
$$
\begin{array}{ccc}
[Z',Z] & \xrightarrow{H^{X\otimes Y}} & [(X\otimes Y)\otimes Z', (X\otimes Y)\otimes Z] \\
\Big\downarrow H^{Y} & & \Big\downarrow \\
[Y\otimes Z', Y\otimes Z] & & [1,a] \\
\Big\downarrow H^{X} & & \Big\downarrow \\
[X\otimes(Y\otimes Z'), X\otimes(Y\otimes Z)] & \xrightarrow{[a,1]} & [(X\otimes Y)\otimes Z', X\otimes(Y\otimes Z)]
\end{array}
$$

(4.16)
$$
\begin{array}{ccc}
[X'X] & \xrightarrow{H^{I}} & [I\otimes X', I\otimes X] \\
 & {\scriptstyle [1,1]}\searrow \quad \swarrow{\scriptstyle [1,1]} & \\
 & [I\otimes X', X] &
\end{array}
$$

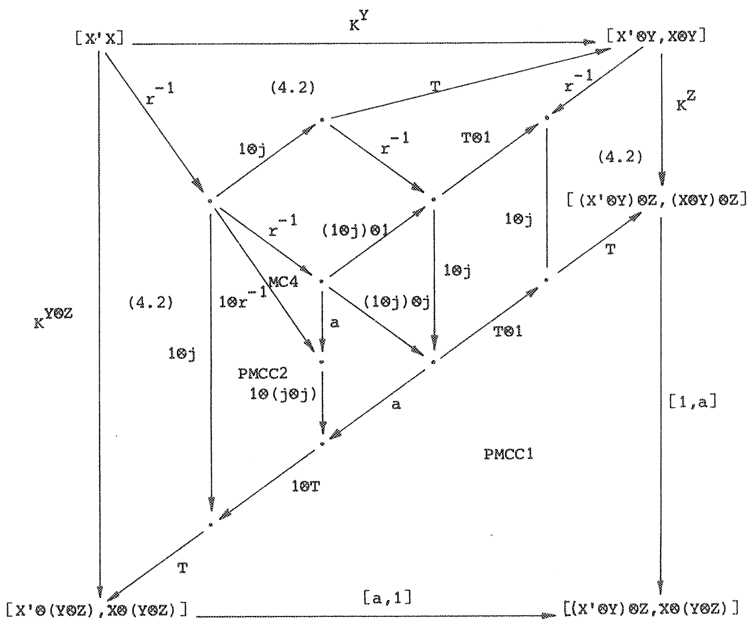
(4.17)
$$
\begin{array}{ccc}
[X'X] & \xrightarrow{K^{I}} & [X'\otimes I, X\otimes I] \\
 & {\scriptstyle [r,1]}\searrow \quad \swarrow{\scriptstyle [1,r]} & \\
 & [X'\otimes I, X] &
\end{array}
$$

(4.18)
$$
\begin{array}{ccc}
[X'X] & \xrightarrow{K^{Y}} & [X'\otimes Y, X\otimes Y] \\
\Big\downarrow H^{Y} & & \Big\downarrow [1,c] \\
[Y\otimes X', Y\otimes X] & \xrightarrow{[c,1]} & [X'\otimes Y, Y\otimes X]
\end{array}
$$

(4.19)

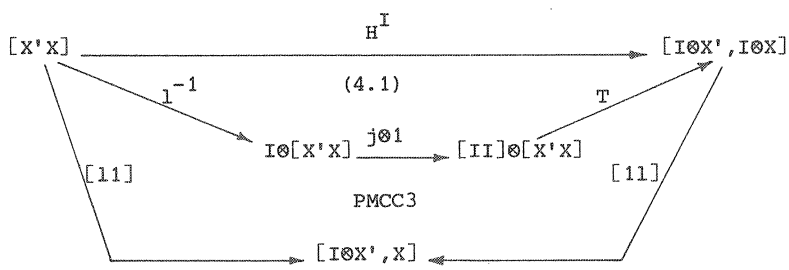


PROOF. Proof of the commutativity of diagram (4.13)



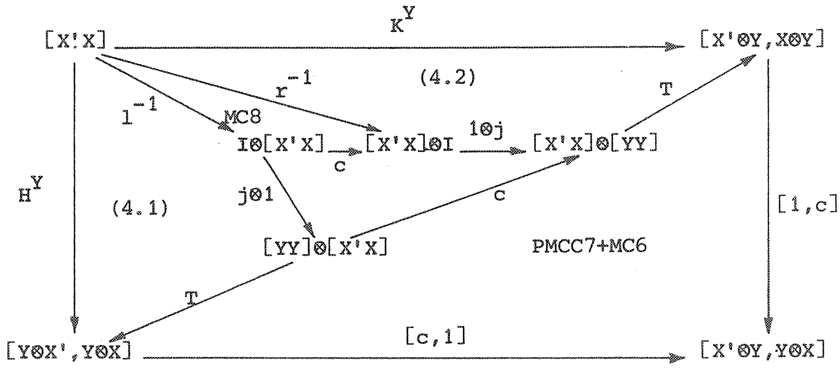
The commutativity of the diagrams (4.14) and (4.15) follows in a similar way from PMCC1.

Proof of the commutativity of diagram (4.16):



The commutativity of diagram (4.17) follows in a similar way from PMCC4

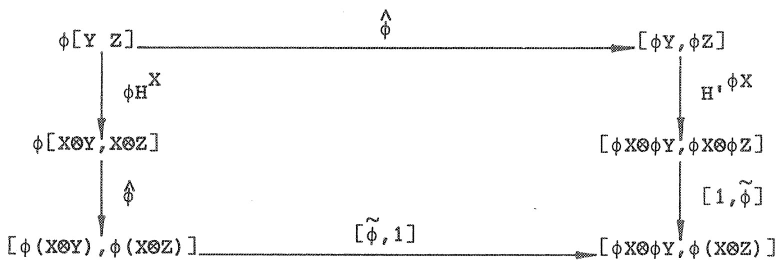
Proof of the commutativity of diagram (4.18):



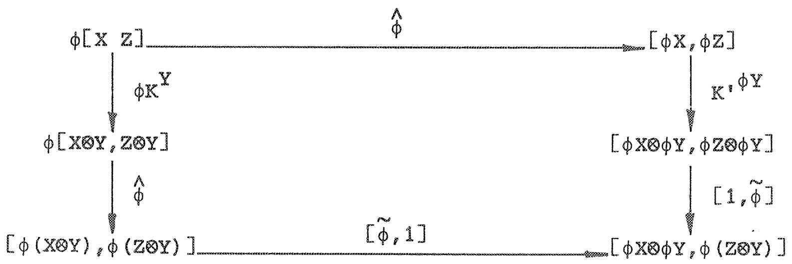
The commutativity of diagram (4.19) follows from (4.18) and MC6.

4.8. PROPOSITION. Let $\Phi = \langle \phi, \tilde{\phi}, \hat{\phi}, \phi^0 \rangle : V \longrightarrow V'$ be a semi monoidal closed functor. Then the following diagrams commute:

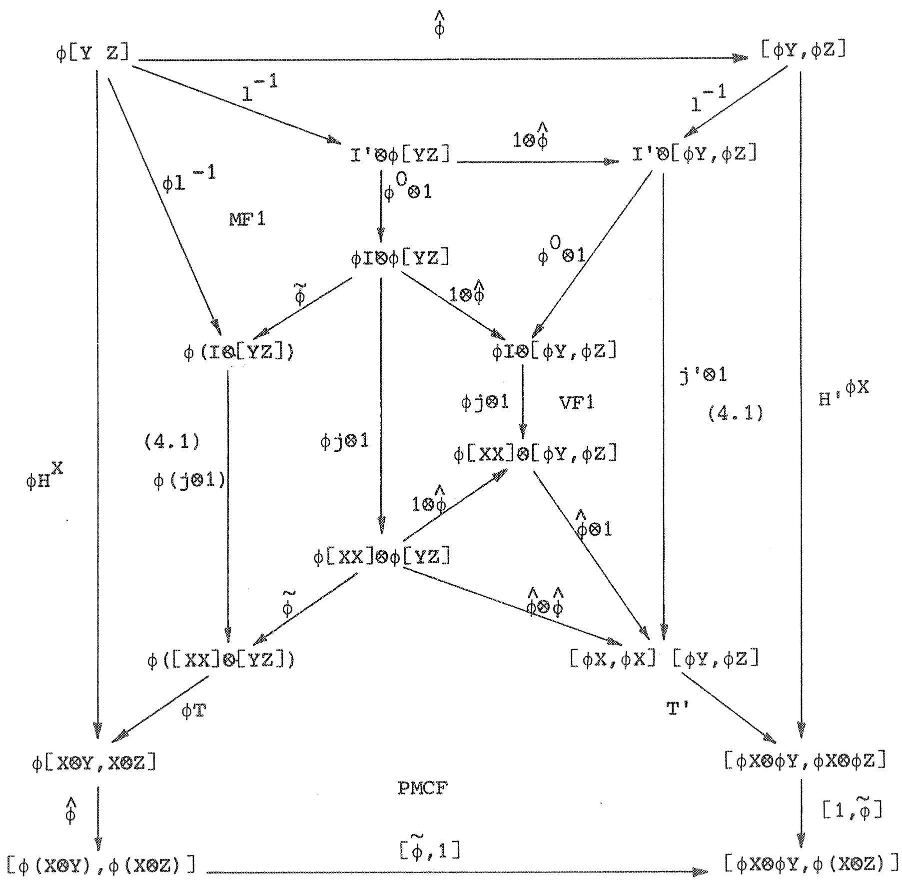
(4.20)



(4.21)



PROOF. Proof of the commutativity of diagram (4.20):



The proof of the commutativity of (4.21) is similar. □

5. The Natural Transformations $d_{XY}: X \longrightarrow [Y,X\otimes Y]$ and $u_{YX}: Y \longrightarrow [X,X\otimes Y]$

5.1. DEFINITION. The natural transformations

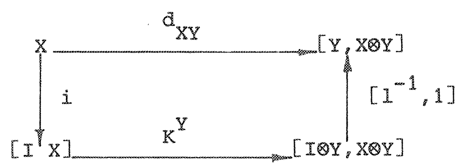
$$d = d_{XY}: X \longrightarrow [Y,X\otimes Y]$$

and

$$u = u_{YX}: Y \longrightarrow [X,X\otimes Y]$$

are defined by the following diagrams:

(5.1)



(5.2)

$$\begin{array}{ccc} Y & \xrightarrow{u_{YX}} & [X, X\otimes Y] \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle [r^{-1},1]} \\ [I,Y] & \xrightarrow{H^X} & [X\otimes I, X\otimes Y] \end{array}$$

5.2. PROPOSITION. (The $\mathcal{V}$-naturality of d and u)
d_{XY} is $\mathcal{V}$-natural in the variable X and u_{YX} is $\mathcal{V}$-natural in the variable Y. If $\mathcal{V}$ is a symmetric semi monoidal closed category then d and u are $\mathcal{V}$-natural in both variables. The $\mathcal{V}$-naturality of d and u is expressed by the commutativity of the following diagrams:

(5.3)

$$\begin{array}{ccc} [X',X] & \xrightarrow{[1,d]} & [X',[Y,X\otimes Y]] \\ {\scriptstyle K^Y}\downarrow & & \uparrow{\scriptstyle [d,1]} \\ [X'\otimes Y, X\otimes Y] & \xrightarrow{L^Y} & [[Y,X'\otimes Y],[Y,X\otimes Y]] \end{array}$$

(5.4)

$$\begin{array}{ccc} [Y,Y'] & \xrightarrow{H^X} & [X\otimes Y, X\otimes Y'] \\ & & \downarrow{\scriptstyle L^Y} \\ {\scriptstyle R^{X\otimes Y'}}\downarrow & & [[Y,X\otimes Y],[Y,X\otimes Y']] \\ & & \downarrow{\scriptstyle [d,1]} \\ [[Y',X\otimes Y'],[Y,X\otimes Y']] & \xrightarrow{[d,1]} & [X,[Y,X\otimes Y']] \end{array}$$

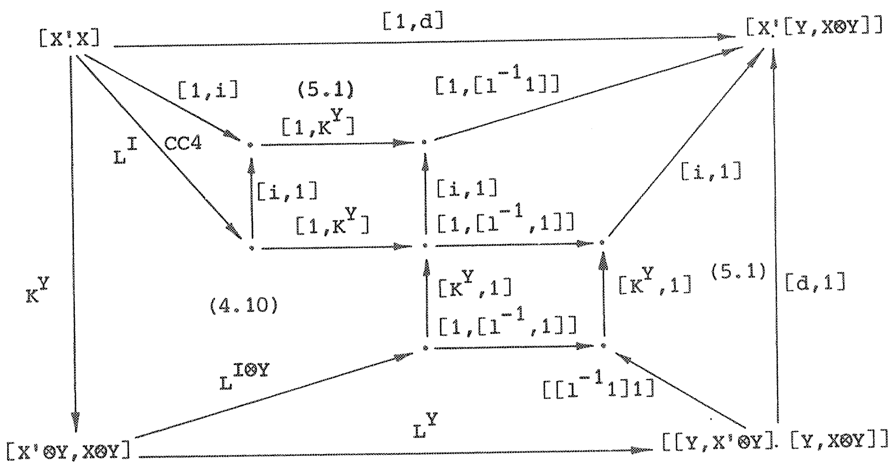
(5.5)

$$\begin{array}{ccc} [Y',Y] & \xrightarrow{[1,u]} & [Y',[X,X\otimes Y]] \\ {\scriptstyle H^X}\downarrow & & \uparrow{\scriptstyle [u,1]} \\ [X\otimes Y', X\otimes Y] & \xrightarrow{L^X} & [[X,X\otimes Y'],[X,X\otimes Y]] \end{array}$$

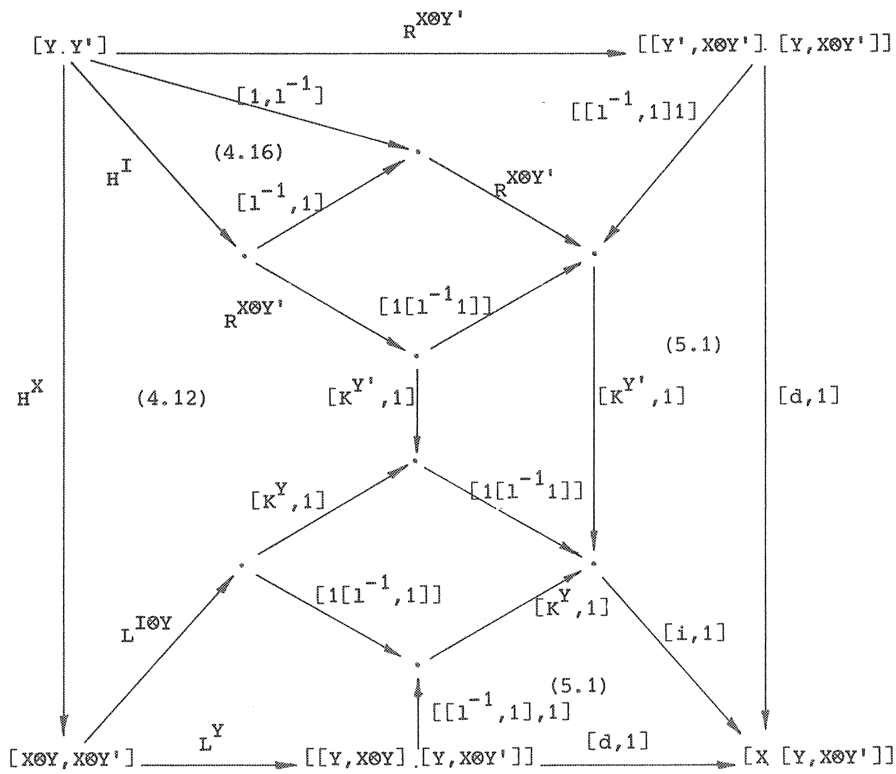
(5.6)

$$\begin{array}{ccc} [X,X'] & \xrightarrow{K^Y} & [X\otimes Y, X'\otimes Y] \\ & & \downarrow{\scriptstyle L^X} \\ {\scriptstyle R^{X'\otimes Y}}\downarrow & & [[X,X\otimes Y],[X,X'\otimes Y]] \\ & & \downarrow{\scriptstyle [u,1]} \\ [[X',X'\otimes Y],[X,X'\otimes Y]] & \xrightarrow{[u,1]} & [Y,[X,X'\otimes Y]] \end{array}$$

PROOF. Proof of the commutativity of diagram (5.3):



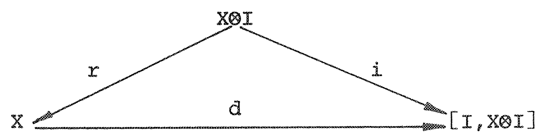
Proof of the commutativity of diagram (5.4):



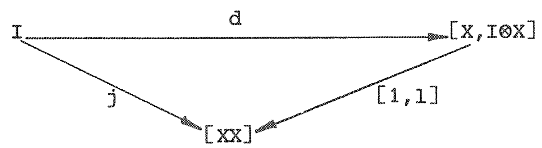
The commutativity of the diagram (5.5) and (5.6) is proved similarly. □

5.3. PROPOSITION. The following diagrams are commutative:

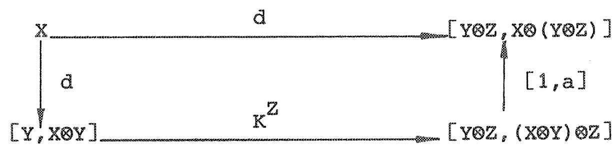
(5.7)



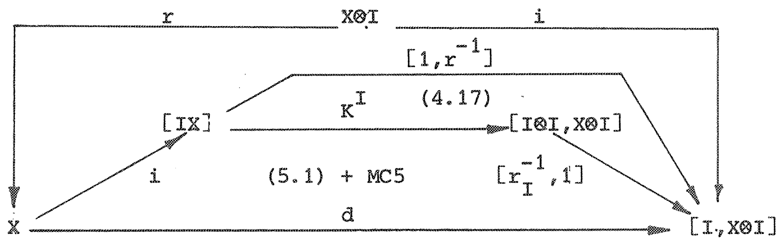
(5.8)



(5.9)

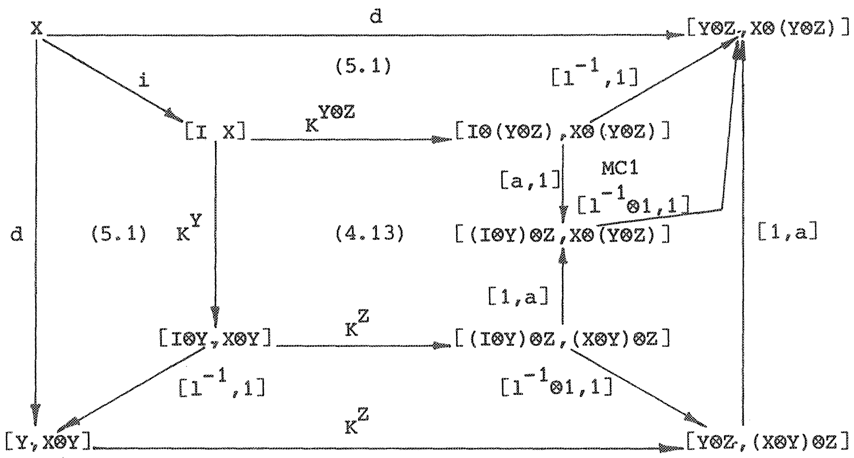


PROOF. Proof of the commutativity of diagram (5.7):



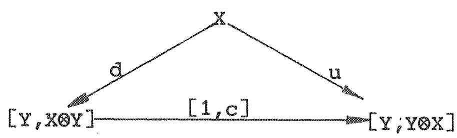
The commutativity of diagram (5.8) follows in a similar way from VF1 for K and (5.1).

Proof of the commutativity of diagram (5.9):

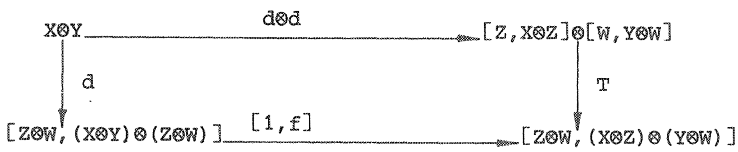


5.4. PROPOSITION. Let $\mathcal{V}$ be a symmetric semi monoidal closed category. Then the following diagrams commute:

(5.10)



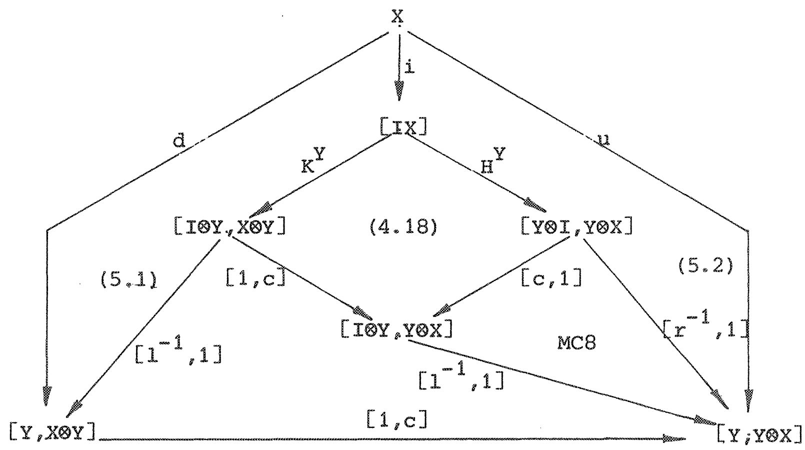
(5.11)



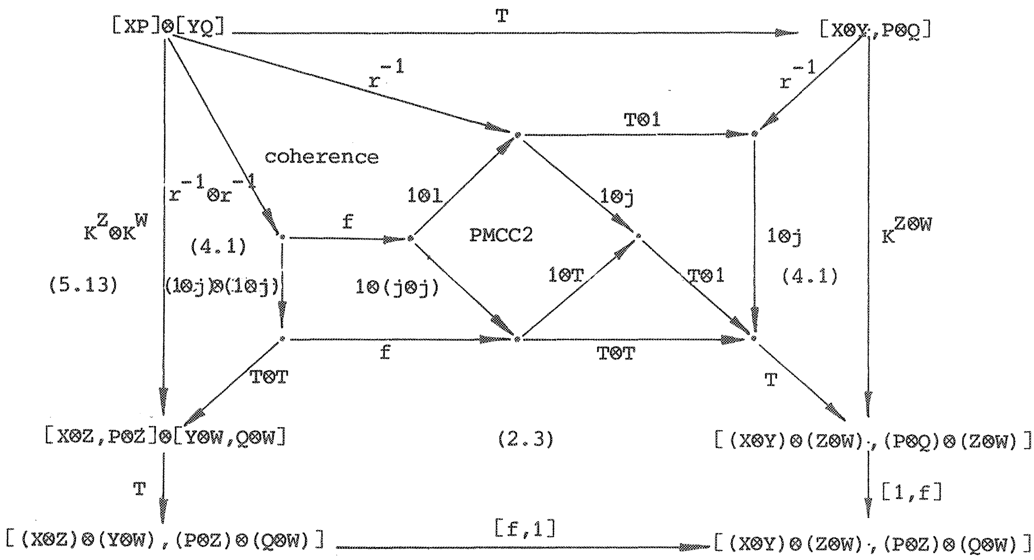
Moreover, we have

(5.12) $\quad d_{XY} = \sigma(u_{YX})$

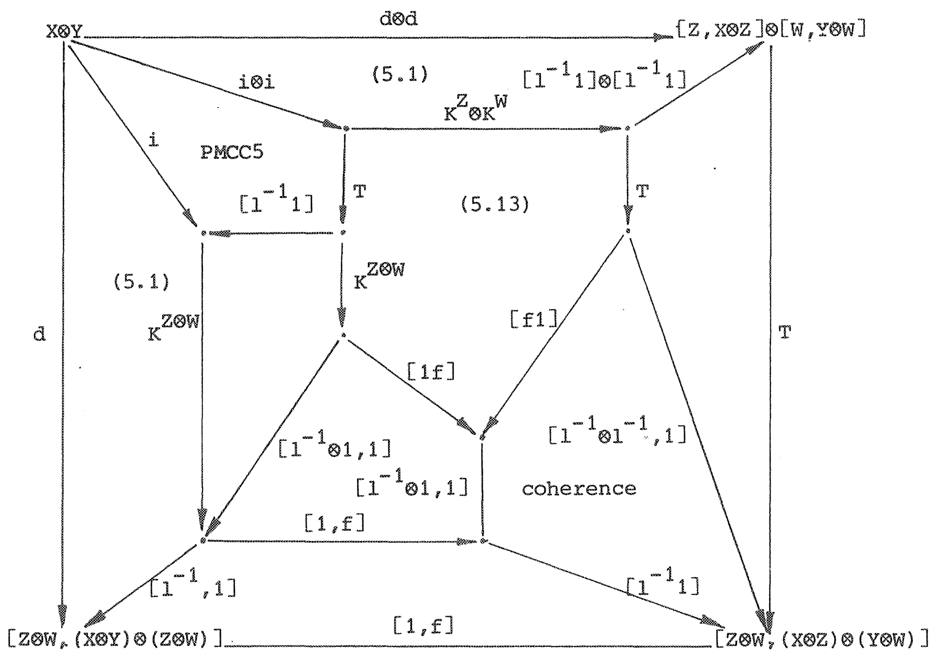
PROOF. Proof of the commutativity of diagram (5.10):



Next we prove the commutativity of the following diagram:

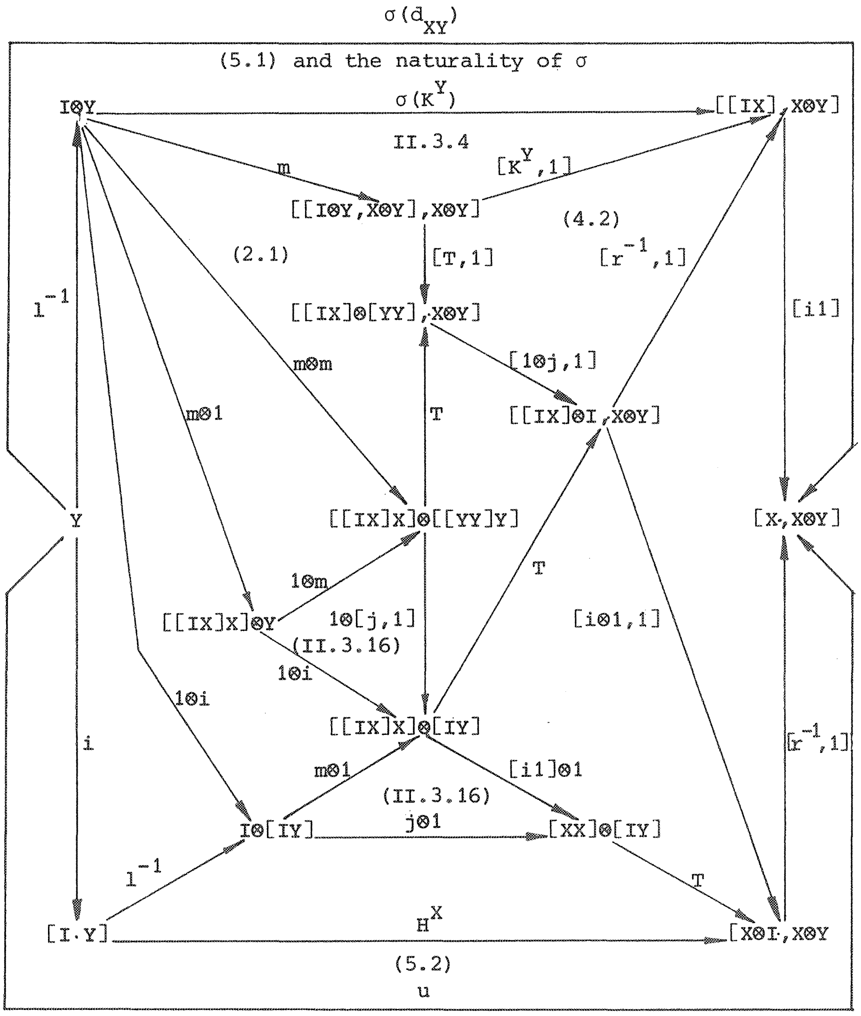


The commutativity of (5.13) enables us to prove the commutativity of (5.11):



□

The following diagram shows that $u_{YX} = \sigma(d_{XY})$:



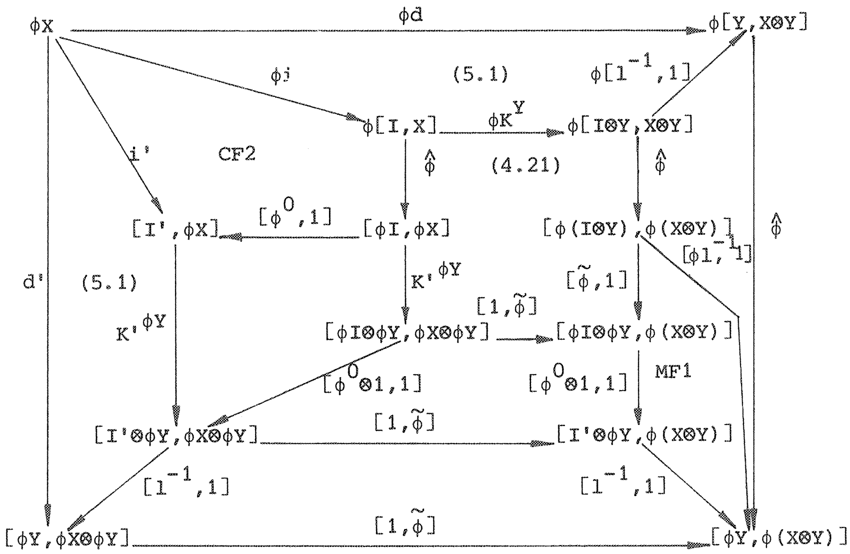
□

5.5. PROPOSITION. Let $\Phi = \langle \phi, \tilde{\phi}, \hat{\phi}, \phi^0 \rangle : V \longrightarrow V'$ be a semi monoidal closed functor. Then the following diagrams commute:

$$\begin{array}{ccc} \phi X & \xrightarrow{\phi d} & \phi[Y, X\otimes Y] \\ {\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle \hat{\phi}} \\ [\phi Y, \phi X \otimes \phi Y] & \xrightarrow{[1,\tilde{\phi}]} & [\phi Y, \phi(X\otimes Y)] \end{array} \tag{5.14}$$

$$\begin{array}{ccc} \phi Y & \xrightarrow{\phi u} & \phi[X, X\otimes Y] \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle \hat{\phi}} \\ [\phi X, \phi X \otimes \phi Y] & \xrightarrow{[1,\tilde{\phi}]} & [\phi X, \phi(X\otimes Y)] \end{array} \tag{5.15}$$

PROOF. Proof of the commutativity of diagram (5.14):



The commutativity of (5.15) is proved in a similar way using (4.20). □

6. THE NATURAL TRANSFORMATIONS p_{XYZ}: $[X\otimes Y,Z] \longrightarrow [X[YZ]]$ AND t_{XYZ}: $[X\otimes Y,Z] \longrightarrow [Y[XZ]]$.

6.1. DEFINITION. The natural transformations

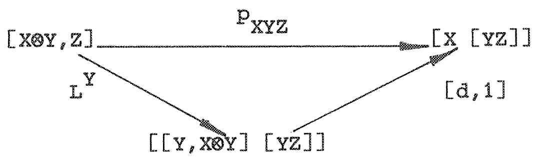
$$p = p_{XYZ}\colon [X\otimes Y,Z] \longrightarrow [X[YZ]]$$

and

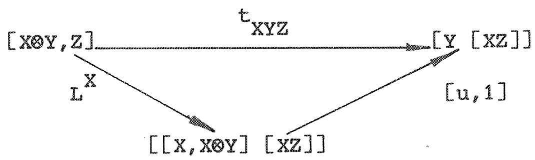
$$t = t_{XYZ}\colon [X\otimes Y,Z] \longrightarrow [Y[XZ]]$$

are defined by the following diagrams:

(6.1)



(6.2)



6.2 REMARK. Define natural transformations

$$\pi = \pi_{XYZ}\colon V_0(X\otimes Y,Z) \longrightarrow V_0(X[YZ])$$

and

$$\tau = \tau_{XYZ}\colon V_0(X\otimes Y,Z) \longrightarrow V_0(Y[XZ])$$

by

$$\pi_{XYZ} = Vp_{XYZ} \quad \text{and} \quad \tau_{XYZ} = Vt_{XYZ} \quad \text{(using CC0)}.$$

If we apply V to the diagrams (6.1) and (6.2) and use CC0 and CC1 we get

$$d_{XY} = \pi_{XY,X\otimes Y}(1_{X\otimes Y})$$

and

$$u_{YX} = \tau_{XY,X\otimes Y}(1_{X\otimes Y}).$$

Conversely, if $g \in V_0(X\otimes Y,Z)$ then

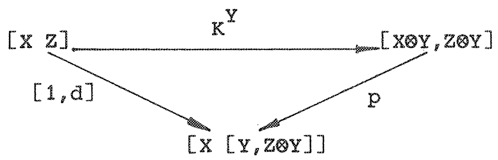
(6.3) $\pi_{XYZ}(g) = [1,g]d_{XY}$

and

(6.4) $\tau_{XYZ}(g) = [1,g]u_{YX}$.

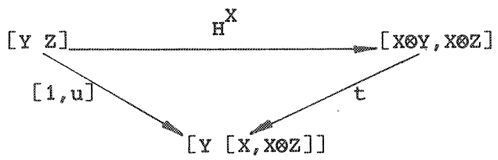
Combination of diagram (6.1) with diagram (5.3) gives a commutative diagram:

(6.5)



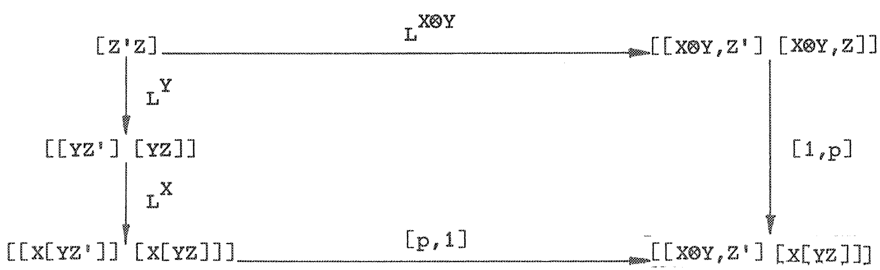
and combination of diagram (6.2) with diagram (5.5) gives a commutative diagram:

(6.6)



6.3. PROPOSITION. (The V-naturality of p and t)
The morphisms p_{XYZ}: [X⊗Y,Z] ⟶ [X[YZ]] and t_{XYZ}: [X⊗Y,Z] ⟶ [Y[XZ]] are V-natural in the variable Z. If V is a symmetric semi monoidal closed category then p and t are V-natural in every variable. The V-naturality of p and t is expressed by the commutativity of the following six diagrams:

(6.7)



$$
(6.8)\qquad
\begin{array}{ccc}
[Y\ Y'] & \xrightarrow{\quad H^{X}\quad} & [X\otimes Y, X\otimes Y'] \\
\downarrow{\scriptstyle R^{Z}} & & \downarrow{\scriptstyle R^{Z}} \\
[[Y'Z]\ [YZ]] & & [[X\otimes Y',Z]\ [X\otimes Y,Z]] \\
\downarrow{\scriptstyle L^{X}} & & \downarrow{\scriptstyle [1,p]} \\
[[X[Y'Z]]\ [X[YZ]]] & \xrightarrow{\quad [p,1]\quad} & [[X\otimes Y',Z]\ [X[YZ]]]
\end{array}
$$

$$
(6.9)\qquad
\begin{array}{ccc}
[X\ X'] & \xrightarrow{\quad K^{Y}\quad} & [X\otimes Y, X'\otimes Y] \\
 & & \downarrow{\scriptstyle R^{Z}} \\
\downarrow{\scriptstyle R^{[YZ]}} & & [[X'\otimes Y,Z]\ [X\otimes Y,Z]] \\
 & & \downarrow{\scriptstyle [1,p]} \\
[[X'[YZ]]\ [X[YZ]]] & \xrightarrow{\quad [p,1]\quad} & [[X'\otimes Y,Z]\ [X[YZ]]]
\end{array}
$$

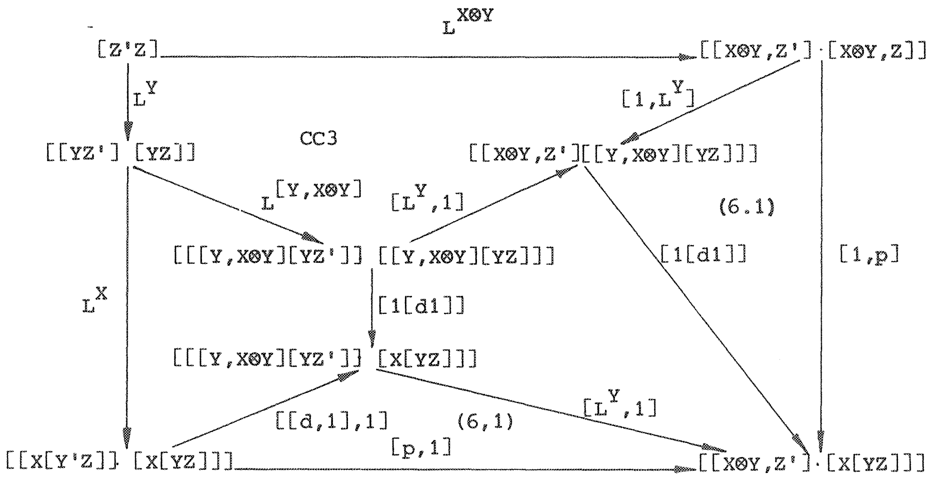
$$
(6.10)\qquad
\begin{array}{ccc}
[Z'Z] & \xrightarrow{\quad L^{X\otimes Y}\quad} & [[X\otimes Y,Z']\ [X\otimes Y,Z]] \\
\downarrow{\scriptstyle L^{X}} & & \\
[[XZ']\ [XZ]] & & \downarrow{\scriptstyle [1,t]} \\
\downarrow{\scriptstyle L^{Y}} & & \\
[[Y[XZ']]\ [Y[XZ]]] & \xrightarrow{\quad [t,1]\quad} & [[X\otimes Y,Z']\ [Y[XZ]]]
\end{array}
$$

$$
(6.11)\qquad
\begin{array}{ccc}
[Y\ Y'] & \xrightarrow{\quad H^{X}\quad} & [X\otimes Y, X\otimes Y'] \\
 & & \downarrow{\scriptstyle R^{Z}} \\
\downarrow{\scriptstyle R^{[XZ]}} & & [[X\otimes Y',Z]\ [X\otimes Y,Z]] \\
 & & \downarrow{\scriptstyle [1,t]} \\
[[Y'[XZ]]\ [Y[XZ]]] & \xrightarrow{\quad [t,1]\quad} & [[X\otimes Y',Z]\ [Y[XZ]]]
\end{array}
$$

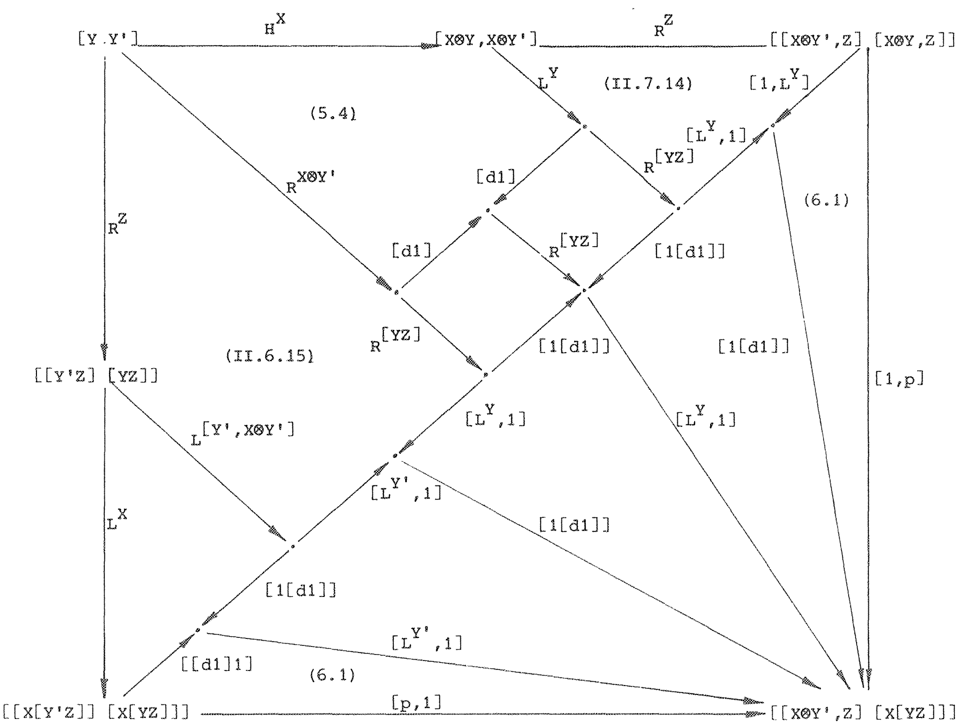
$$
(6.12)\qquad
\begin{array}{ccc}
[X\ X'] & \xrightarrow{\quad K^{Y}\quad} & [X\otimes Y, X'\otimes Y] \\
\downarrow{\scriptstyle R^{Z}} & & \downarrow{\scriptstyle R^{Z}} \\
[[X'Z]\ [XZ]] & & [[X'\otimes Y,Z]\ [X\otimes Y,Z]] \\
\downarrow{\scriptstyle L^{Y}} & & \downarrow{\scriptstyle [1,t]} \\
[[Y[X'Z]]\ [Y[XZ]]] & \xrightarrow{\quad [t,1]\quad} & [[X'\otimes Y,Z]\ [Y[XZ]]]
\end{array}
$$

PROOF. We prove the V-naturality of p; the V-naturality of t is proved similarly.

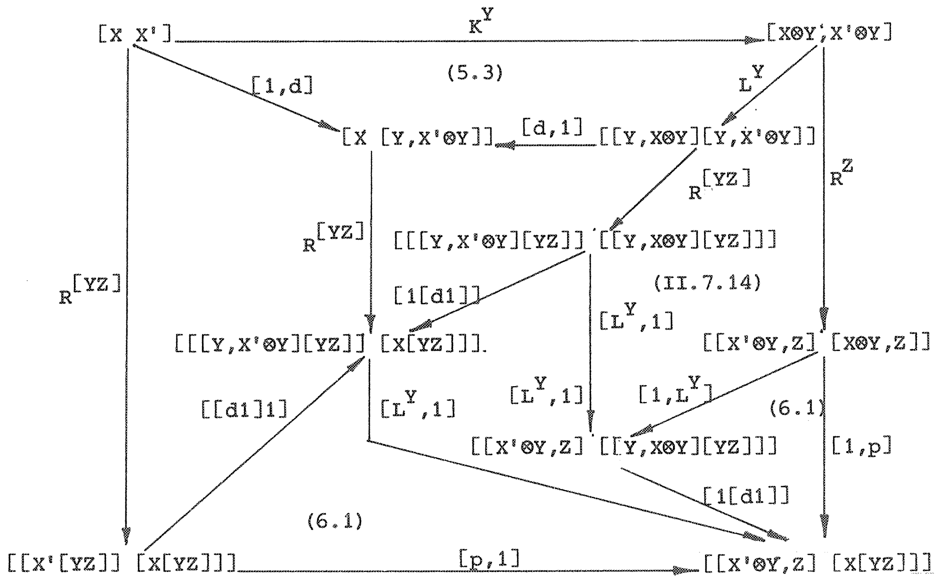
Proof of the commutativity of diagram (6.7):



Proof of the commutativity of (6.8):



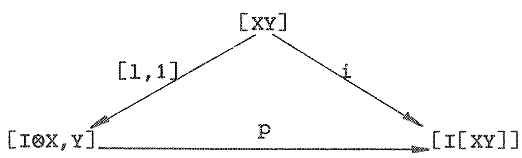
Proof of the commutativity of (6.9):



6.4. THEOREM. In a semi monoidal closed category the following four diagrams commute:

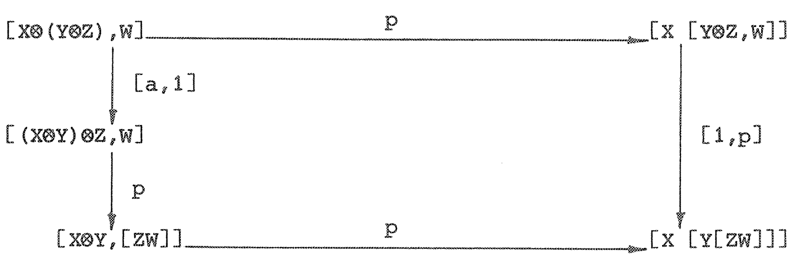
MCC2:

(6.13)



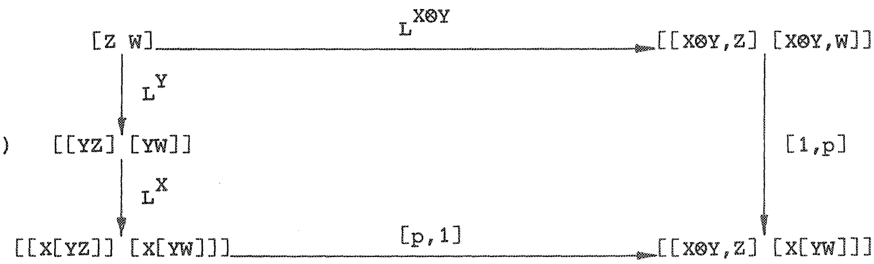
MCC3:

(6.14)



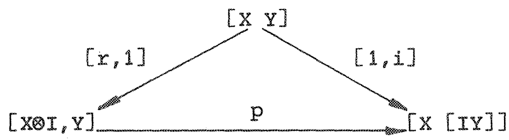
MCC3'

(6.15)



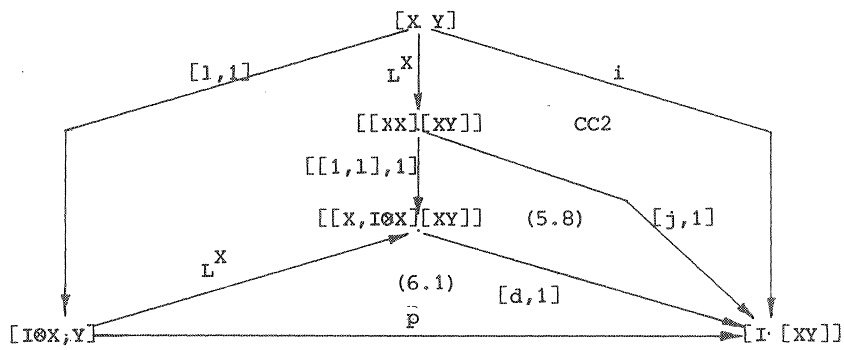
MCC4':

(6.16)



Consequently all the axioms of a monoidal closed category (see definition IV.1.2) hold; however, the natural transformation p need not be a natural *iso*morphism.

PROOF. Proof of the commutativity of diagram (6.13):



Proof of the commutativity of diagram (6.14):

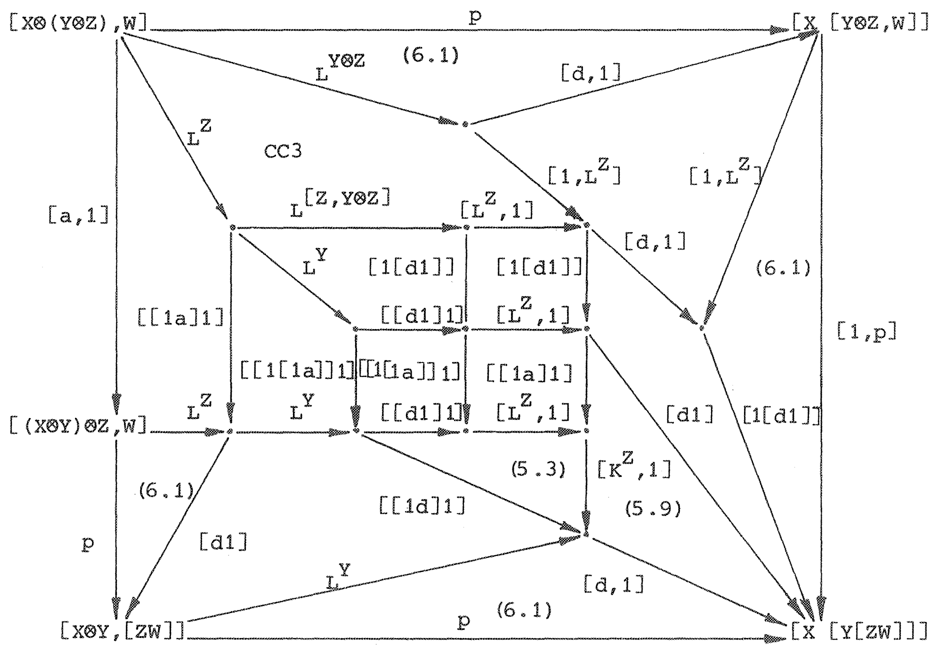
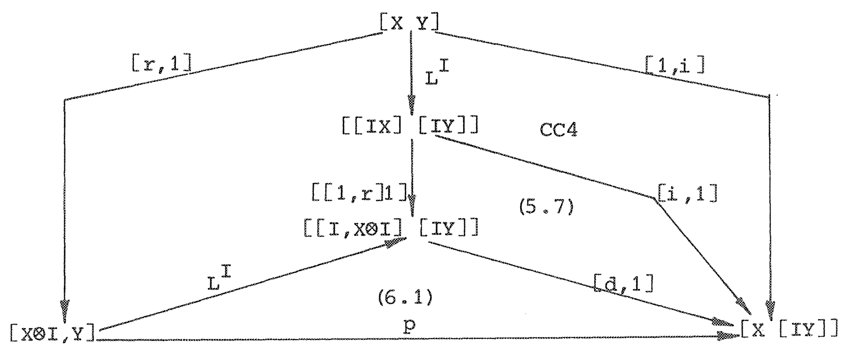


Diagram (6.15) is the V-naturality of p_{XYZ} in the variable Z (proposition 6.3; diagram (6.7)).

Proof of the commutativity of diagram (6.16):

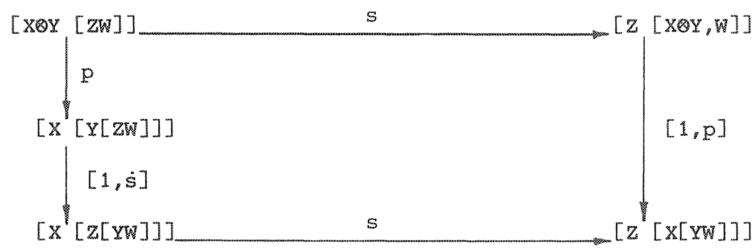


The fact that p need not be a natural *iso*morphism is shown by several examples, for which we refer to chapter V. □

6.5. THEOREM. If V is a symmetric semi monoidal closed category then the following two diagrams commute:

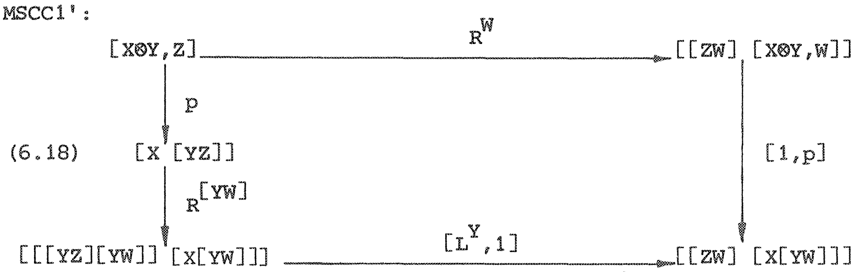
MSCC1:

(6.17)



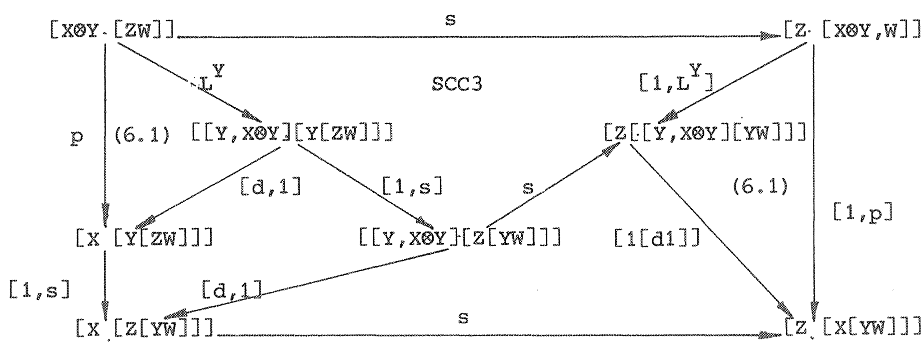
MSCC1':

(6.18)

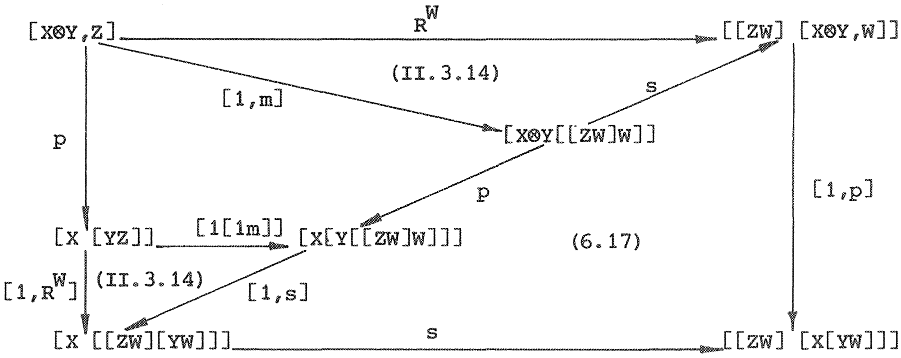


Consequently all the axioms of a monoidal symmetric closed category (see definition IV.3.1) hold; however, the natural transformation p need not be a natural *iso*morphism.

PROOF. Proof of the commutativity of diagram (6.17):



Proof of the commutativity of diagram (6.18):



□

The fact that p need not be a natural *iso*morphism is shown by several examples in chapter V. □

6.6. PROPOSITION. If V is a symmetric semi monoidal closed category then

(6.19) $$t_{XYZ} = s_{XYZ} \circ p_{XYZ}$$

PROOF. First we show that

(6.20) $$\tau_{XYZ} = \sigma_{XYZ}\ \pi_{XYZ}$$

Let $g \in V_0(X \otimes Y, Z)$; then:

$$(\sigma_{XYZ} \circ \pi_{XYZ})g = \sigma_{XYZ}([1,g]d_{XY}) \quad \text{by (6.3)}$$

$$= [1,g]\sigma_{XYZ}(d_{XY}) \quad \text{by the naturality of } \sigma$$

$$= [1,g]u_{YX} \quad \text{by (5.12)}$$

$$= \tau_{XYZ}g \ . \quad \text{by (6.4).}$$

Next we note that the commutativity of the following diagram is a consequence of SCC2 and MSCC1':

$$
\begin{array}{ccccc}
[X\otimes Y,Z] & \xrightarrow{\quad p \quad} & [X[YZ]] & \xrightarrow{\quad s \quad} & [Y[XZ]] \\
 & (6.18) & \downarrow R^{[YW]} & (II.6.13) & \downarrow R^{[XW]} \\
\downarrow R^{W} & & [[[YZ][YW]][X[YW]]] & & [[[XZ][XW]][Y[XW]]] \\
 & & \downarrow [L^{Y},1] & & \downarrow [L^{X},1] \\
[[ZW][X\otimes Y,W]] & \xrightarrow{\quad [1,p] \quad} & [[ZW][X[YW]]] & \xrightarrow{\quad [1,s] \quad} & [[ZW][Y[XW]]]
\end{array}
$$

If we apply V to this diagram, and use CC0, propositions II.1.6 and II.5.1. and the fact (just proved) that $\tau = \sigma\pi$ we obtain the commutativity of the following diagram:

$$
\begin{array}{ccc}
V_0(X\otimes Y,Z) & \xrightarrow{\quad \tau \quad} & V_0(Y[XZ]) \\
\downarrow [-W] & & \downarrow [-[XW]] \\
V_0([ZW][X\otimes Y,W]) & & V_0([[XZ][XW]][Y[XW]]) \\
\downarrow V_0(1,p) & & \downarrow V_0(L^{X},1) \\
V_0([ZW][X[YW]]) & \xrightarrow{\quad V_0(1,s) \quad} & V_0([ZW][Y[XW]])
\end{array}
$$

If we take $Z = X\otimes Y$ and evaluate at $1_{X\otimes Y}$ we obtain by the definition of t:

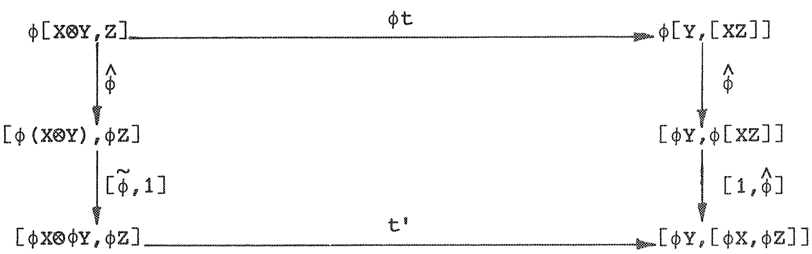
$$s_{XYZ} \circ p_{XYZ} = [u,1]\, L^{X} = t_{XYZ}. \qquad \square$$

6.7. THEOREM. Let $\Phi = \langle \phi, \tilde{\phi}, \hat{\phi}, \phi^{0} \rangle : V \longrightarrow V'$ be a semi monoidal closed functor. Then the following diagrams commute:

MCF3:

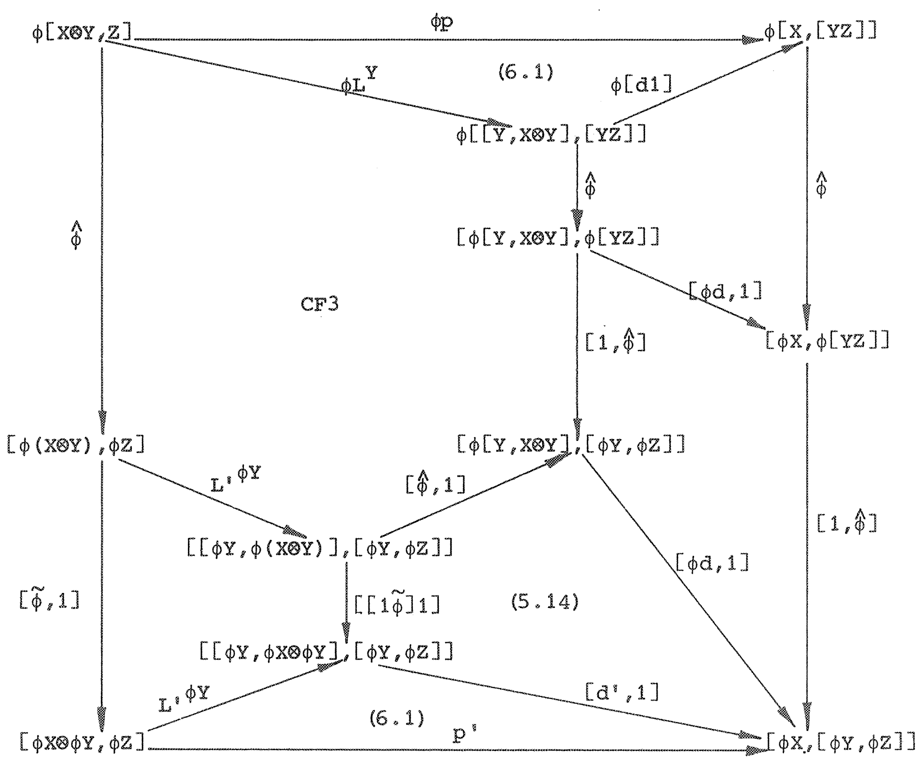
$$
\begin{array}{ccc}
\phi[X\otimes Y,Z] & \xrightarrow{\quad \phi p \quad} & \phi[X,[YZ]] \\
\downarrow \hat{\phi} & & \downarrow \hat{\phi} \\
(6.21)\ [\phi(X\otimes Y),\phi Z] & & [\phi X,\phi[YZ]] \\
\downarrow [\tilde{\phi},1] & & \downarrow [1,\hat{\phi}] \\
[\phi X\otimes\phi Y,\phi Z] & \xrightarrow{\quad p' \quad} & [\phi X,[\phi Y,\phi Z]]
\end{array}
$$

(6.22)



Hence a semi monoidal closed functor fulfils all the axioms of a monoidal closed functor (see definition IV.1.3).

PROOF. Proof of the commutativity of diagram (6.21):



(6.22) follows in a similar way from CF3 and (5.15). □

CHAPTER IV

MONOIDAL SYMMETRIC CLOSED CATEGORIES

1. Monoidal Closed Categories

In this section we recall the definition and some properties of a monoidal closed category and a symmetric monoidal closed category. These definitions are taken from [6], sections II.2 and III.1.

1.1. Definition. A *monoidal closed category* is an ordered triple $\mathcal{V} = \langle {}^m\mathcal{V}, {}^c\mathcal{V}, p\rangle$ consisting of:

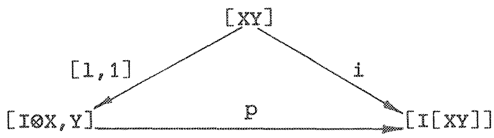
(i) a monoidal category ${}^m\mathcal{V} = \langle \mathcal{V}_0, \otimes, I, r, l, a\rangle$;

(ii) a closed category ${}^c\mathcal{V} = \langle \mathcal{V}_0, V, [-,-], I, i, j, L\rangle$ with the same $\mathcal{V}_0$ and I as ${}^m\mathcal{V}$;

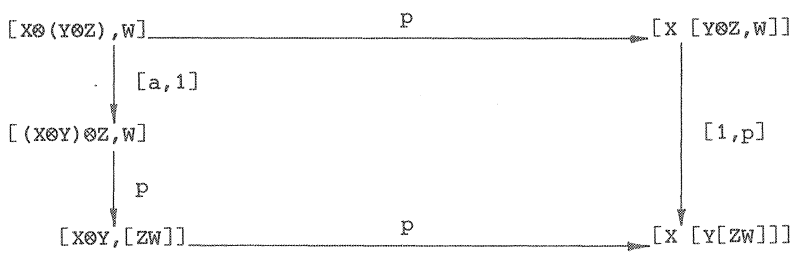
(iii) a natural isomorphism $p = p_{XYZ}\colon [X\otimes Y, Z] \longrightarrow [X[YZ]]$.

These data are to satisfy the following four axioms:

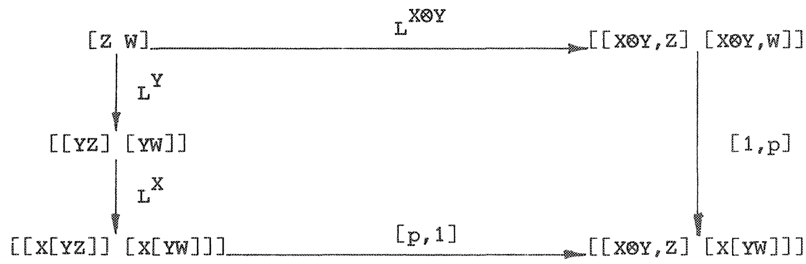
MCC2. The following diagram commutes:



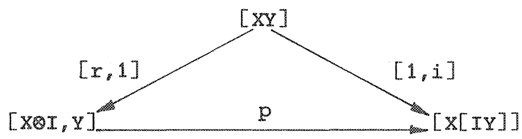
MCC3. The following diagram commutes:



MCC3'. The following diagram commutes:



MCC4. The following diagram commutes:



1.2. DEFINITION. A *symmetric monoidal closed category* is an ordered triple $V = \langle {}^{sm}V, {}^{c}V, p\rangle$ consisting of

(i) a symmetric monoidal category ${}^{sm}V = \langle {}^{m}V, c\rangle$,

(ii) a closed category ${}^{c}V$ and

(iii) a natural isomorphism $p = p_{XYZ}\colon [X\otimes Y, Z] \longrightarrow [X[YZ]]$

such that $\langle {}^{m}V, {}^{c}V, p\rangle$ is a monoidal closed category.

In other words, a symmetric monoidal closed category is a monoidal closed category, together with a symmetry $c_{XY}\colon X\otimes Y \longrightarrow Y\otimes X$ for the monoidal structure.

1.3. DEFINITION. Let $V = \langle {}^{m}V, {}^{c}V, p\rangle$ and $V' = \langle {}^{m}V', {}^{c}V', p'\rangle$ be monoidal closed categories. A *monoidal closed functor* $\Phi\colon V \longrightarrow V'$ is an ordered quadruple $\Phi = \langle \phi, \tilde{\phi}, \hat{\phi}, \phi^0\rangle$ such that

(i) ${}^{m}\Phi := \langle \phi, \tilde{\phi}, \phi^0\rangle\colon {}^{m}V \longrightarrow {}^{m}V'$ is a monoidal functor;

(ii) ${}^{c}\Phi := \langle \phi\ \hat{\phi}\ \phi^0\rangle\colon {}^{c}V \longrightarrow {}^{c}V'$ is a closed functor;

(iii) the following axiom MCF3 is satisfied:

MCF3. The following diagram commutes:

$$\begin{array}{ccc}
\phi[X\otimes Y,Z] & \xrightarrow{\phi p} & \phi[X,[YZ]] \\
\downarrow{\scriptstyle \hat{\phi}} & & \downarrow{\scriptstyle \hat{\phi}} \\
[\phi(X\otimes Y),\phi Z] & & [\phi X,\phi[YZ]] \\
\downarrow{\scriptstyle [\tilde{\phi},1]} & & \downarrow{\scriptstyle [1,\hat{\phi}]} \\
[\phi X\otimes\phi Y,\phi Z] & \xrightarrow{p'} & [\phi X,[\phi Y,\phi Z]]
\end{array}$$

If $V = \langle {}^{sm}V, {}^{c}V, p\rangle$ and $V' = \langle {}^{sm}V', {}^{c}V', p'\rangle$ are symmetric monoidal closed categories, a *symmetric monoidal closed functor* $\Phi\colon V \to V'$ is an ordered quadruple $\Phi = \langle \phi,\tilde{\phi},\hat{\phi},\phi^0\rangle$ such that

(i) ${}^{sm}\Phi := \langle \phi,\tilde{\phi},\phi^0\rangle\colon {}^{sm}V \to {}^{sm}V'$ is a symmetric monoidal functor;

(ii) ${}^{c}\Phi := \langle \phi,\hat{\phi},\phi^0\rangle\colon {}^{c}V \to {}^{c}V'$ is a closed functor;

(iii) the axiom MCF3 is satisfied.

1.4. DEFINITION. Let Φ and $\Psi\colon V \to V'$ be monoidal closed functors. A *monoidal closed natural transformation* $\eta\colon \Phi \to \Psi\colon V \to V'$ is a natural transformation $\eta\colon \phi \to \psi\colon V_0 \to V'_0$ which is both a monoidal natural transformation $\eta\colon {}^{m}\Phi \to {}^{m}\Psi\colon {}^{m}V \to {}^{m}V'$ and a closed natural transformation $\eta\colon {}^{c}\Phi \to {}^{c}\Psi\colon {}^{c}V \to {}^{c}V'$. If Φ and Ψ are symmetric monoidal closed functors, a monoidal closed natural transformation $\eta\colon \Phi \to \Psi\colon V \to V'$ will be called a *symmetric monoidal closed natural transformation*.

1.5. THEOREM ([6], Theorem II.2.2). Monoidal closed categories, monoidal closed functors and monoidal closed natural transformations form a hypercategory MCC. Symmetric monoidal closed categories, symmetric monoidal closed functors and symmetric monoidal closed natural transformations form a hypercategory $SMCC$. For the rules of composition we refer to [6]. □

2. THE SECOND BASIC SITUATION.

Both the defining data and the axioms for a monoidal closed category are somewhat redundant. The dependence relations between the data and between the axioms are analyzed in great detail in [6], chapter II, sections 3 and 4. In this section we summarize the results of these sections.

The *second basic situation* consists of:

(i) a category V_0;

(ii) functors $[-,-]\colon V_0^* \times V_0 \longrightarrow V_0$ and $-\otimes-\colon V_0 \times V_0 \longrightarrow V_0$;

(iii) an object I of V_0;

(iv) a natural isomorphism $\pi = \pi_{XYZ}\colon V_0(X\otimes Y,Z) \longrightarrow V_0(X[YZ])$.

The existence of a natural isomorphism

$$\pi = \pi_{XYZ}\colon V_0(X\otimes Y,Z) \longrightarrow V_0(X[YZ])$$

is equivalent (cf. theorem I.3.1) to the existence of natural transformations

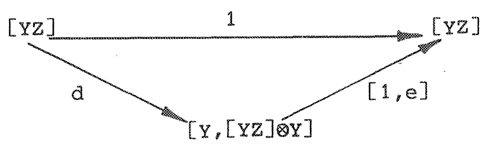
$$d = d_{XY}\colon X \longrightarrow [Y,X\otimes Y]$$

and

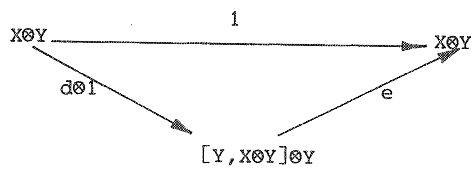
$$e = e_{YZ}\colon [YZ]\otimes Y \longrightarrow Z$$

with the property that the following diagrams commute:

(2.1)



(2.2)



The relations between π,d and e are given by the following formulas:

(2.3) $$d_{XY} = \pi_{XY,X\otimes Y}(1_{X\otimes Y}),$$

(2.4) $$e_{YZ} = \pi^{-1}_{[YZ]\,YZ}(1_{[YZ]}),$$

(2.5) $$\pi_{XYZ}(g) = [1,g]d_{XY} \quad \text{for } g\colon X\otimes Y \longrightarrow Z$$

(2.6) $$\pi^{-1}_{XYZ}(h) = e_{YZ}(h\otimes 1) \quad \text{for } h\colon X \longrightarrow [YZ].$$

The adjunction π induces (cf. section I.3) bijections between several classes of natural transformations.

Commutativity of the diagram

$$(2.7)\qquad \begin{array}{ccccc} V_0(X\otimes Y,Z) & & \xrightarrow{\quad\pi\quad} & & V_0(X,[YZ]) \\ \downarrow{\scriptstyle [-,W]} & & & & \downarrow{\scriptstyle [-,[YW]]} \\ V_0([ZW],[X\otimes Y,W]) & & & & V_0([[YZ][YW]],[X[YW]]) \\ & {\scriptstyle V_0(1,p_{XYW})}\searrow & & \swarrow{\scriptstyle V_0(L^Y_{ZW},1)} & \\ & & V_0([ZW][X[YW]]) & & \end{array}$$

sets up a bijection between natural transformations

$$p = p_{XYW}\colon [X\otimes Y,W] \to [X[YW]]$$

and natural transformations

$$L = L^Y_{ZW}\colon [ZW] \to [[YZ][YW]].$$

If we evaluate diagram (2.7) at $x \in V_0(X\otimes Y,Z)$ we obtain a diagram

$$(2.8)\qquad \begin{array}{ccc} [Z\,W] & \xrightarrow{\quad[x,1]\quad} & [X\otimes Y,W] \\ \downarrow{\scriptstyle L^Y} & & \downarrow{\scriptstyle p} \\ [[YZ]\,[YW]] & \xrightarrow{\quad[\pi x,1]\quad} & [X\,[YW]] \end{array}$$

If we take $Z = X\otimes Y$ and $x = 1_{X\otimes Y}$ we see how p depends on L:

$$(2.9)\qquad \begin{array}{ccccc} [X\otimes Y,W] & & \xrightarrow{\quad p\quad} & & [X\,[YW]] \\ & {\scriptstyle L^Y}\searrow & & \nearrow{\scriptstyle [d,1]} & \\ & & [[Y,X\otimes Y][YW]] & & \end{array}$$

If we take $X = [YZ]$ and $x = e_{YZ}$ we see how L depends on p:

(2.10)
$$\begin{array}{ccc} [ZW] & \xrightarrow{\quad L^Y \quad} & [[YZ][YW]] \\ & {\scriptstyle [e,1]}\searrow \quad \nearrow{\scriptstyle p} & \\ & [[YZ]\otimes Y,W] & \end{array}$$

Commutativity of each of the diagram (2.9) and (2.10) also completely determines the bijection given by commutativity of (2.8).
Commutativity of the diagram

(2.11)
$$\begin{array}{ccccc} V_0(X\otimes(Y\otimes Z),W) & & \xrightarrow{\quad\pi\quad} & & V_0(X\,[Y\otimes Z,W]) \\ \Big\downarrow{\scriptstyle V_0(a,1)} & & & & \Big\downarrow{\scriptstyle V_0(1,p)} \\ V_0((X\otimes Y)\otimes Z,W) & \xrightarrow{\quad\pi\quad} & V_0(X\otimes Y[ZW]) & \xrightarrow{\quad\pi\quad} & V_0(X\,[Y[ZW]]) \end{array}$$

sets up a bijection between natural isomorphisms

$$a = a_{XYZ}\colon\ (X\otimes Y)\otimes Z \longrightarrow X\otimes(Y\otimes Z)$$

and natural isomorphisms

$$p = p_{YZW}\colon\ [Y\otimes Z,W] \longrightarrow [Y[ZW]].$$

If we evaluate (2.11) at $x \in V_0(X\otimes(Y\otimes Z),W)$ we obtain a diagram

(2.12)
$$\begin{array}{ccc} X & \xrightarrow{\quad \pi\pi(xa) \quad} & [Y[ZW]] \\ & {\scriptstyle \pi x}\searrow \quad \nearrow{\scriptstyle p} & \\ & [Y\otimes Z,W] & \end{array}$$

If we take $X = [Y\otimes Z,W]$ and $x = e_{Y\otimes Z,W}$ we see how p depends on a:

(2.13) $$p_{YZW} = \pi\pi(e_{Y\otimes Z,W}\circ a_{[Y\otimes Z,W]YZ}).$$

If we replace in diagram (2.11) π by π^{-1} (reversing the arrows), take $W = X\otimes(Y\otimes Z)$ and evaluate at $d_{X,Y\otimes Z}$ we see how a depends on p:

(2.14) $$a_{XYZ} = \pi^{-1}\pi^{-1}\,(p_{YZW}\circ d_{X,Y\otimes Z}).$$

Commutativity of the diagram

(2.15)

$$\begin{array}{ccccc} & & V_0(XY) & & \\ & \swarrow^{V_0(r,1)} & & \searrow^{V_0(1,i)} & \\ V_0(X\otimes I,Y) & & \xrightarrow{\quad\pi\quad} & & V_0(X[IY]) \end{array}$$

sets up a bijection between natural transformations

$$r = r_X \colon X\otimes I \longrightarrow X$$

and natural transformations

$$i = i_X \colon X \longrightarrow [IX]$$

If we take $Y = X$ and evaluate at 1_X we obtain

(2.16) $\quad i_X = \pi r_X$

Moreover, if r and i are related by (2.15) then r is a natural isomorphism if and only if i is a natural isomorphism.
Commutativity of the diagram

(2.17)

$$\begin{array}{ccc} V_0(XY) & \xrightarrow{\quad [X-] \quad} & V_0([XX]\,[XY]) \\ \Big\downarrow{\scriptstyle V_0(l,1)} & & \Big\downarrow{\scriptstyle V_0(j,1)} \\ V_0(I\otimes X,Y) & \xrightarrow{\quad \pi \quad} & V_0(I\,[XY]) \end{array}$$

sets up a bijection between natural transformations

$$l = l_X \colon I\otimes X \longrightarrow X$$

and natural transformations

$$j = j_X \colon I \longrightarrow [XX].$$

If we take $Y = X$ and evaluate at 1_X we obtain

(2.18) $\quad j_X = \pi 1_X.$

Now suppose that besides the second basic situation V_0 etc. we have another such situation V_0' etc. and that we have a functor $\phi\colon V_0 \longrightarrow V_0'$. Commutativity of the diagram

(2.19)
$$\begin{array}{ccc}
V_0(X\otimes Y,Z) & \xrightarrow{\quad\pi\quad} & V_0(X,[YZ]) \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\
V_0'(\phi(X\otimes Y),\phi Z) & & V_0'(\phi X,\phi[YZ]) \\
\downarrow{\scriptstyle V_0'(\tilde{\phi},1)} & & \downarrow{\scriptstyle V_0'(1,\hat{\phi})} \\
V_0'(\phi X\otimes\phi Y,\phi Z) & \xrightarrow{\quad\pi'\quad} & V_0'(\phi X,[\phi Y,\phi Z])
\end{array}$$

sets up a bijection between natural transformations

$$\tilde{\phi} = \tilde{\phi}_{XY}\colon \phi X\otimes\phi Y \longrightarrow \phi(X\otimes Y)$$

and natural transformations

$$\hat{\phi} = \hat{\phi}_{YZ}\colon \phi[YZ] \longrightarrow [\phi Y,\phi Z].$$

If we take $X = [YZ]$ and evaluate at e_{YZ} we see how $\hat{\phi}$ depends on $\tilde{\phi}$:

(2.20) $\quad \hat{\phi}_{YZ} = \pi'(\phi e_{YZ}\cdot\tilde{\phi}_{[YZ]Y}).$

If in diagram (2.19) we replace π by π^{-1} and π' by π'^{-1}, take $Z = X\otimes Y$ and evaluate at d_{XY} we see how $\tilde{\phi}$ depends on $\hat{\phi}$:

(2.21) $\quad \tilde{\phi}_{XY} = \pi'^{-1}(\hat{\phi}_{Y,X\otimes Y}\cdot\phi d_{XY}).$

If we take $Z = X\otimes Y$ and evaluate at $1_{X\otimes Y}$ we obtain a diagram

(2.22)
$$\begin{array}{ccc}
\phi X & \xrightarrow{\quad\phi d\quad} & \phi[Y,X\otimes Y] \\
{\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle\hat{\phi}} \\
[\phi Y,\phi X\otimes\phi Y] & \xrightarrow{\quad[1,\tilde{\phi}]\quad} & [\phi Y,\phi(X\otimes Y)]
\end{array}$$

If we replace in diagram (2.19) π by π^{-1} and π' by π'^{-1}, take $X = [YZ]$ and evaluate at $1_{[YZ]}$ we obtain a commutative diagram

$$
(2.23) \qquad
\begin{array}{ccc}
\phi[YZ]\otimes\phi Y & \xrightarrow{\quad\tilde{\phi}\quad} & \phi([YZ]\otimes Y) \\
{\scriptstyle \hat{\phi}\otimes 1}\downarrow & & \downarrow{\scriptstyle \phi e} \\
[\phi Y,\phi Z]\otimes\phi Y & \xrightarrow{\quad e'\quad} & \phi Z
\end{array}
$$

2.1. PROPOSITION ([6], proposition II.3.2). Let V be a monoidal closed category. Define a natural isomorphism π by

$$\pi_{XYZ} = Vp_{XYZ}\colon V_0(X\otimes Y,Z) \longrightarrow V_0(X[YZ])$$

(using CC0). Then the second basic situation obtains, and the natural transformations L, p, a; i, r; j and l are related by (2.7), (2.11), (2.15) and (2.17). If $\Phi = \langle \phi,\tilde{\phi},\hat{\phi},\phi^0\rangle\colon V \longrightarrow V'$ is a monoidal closed functor, then $\tilde{\phi}$ and $\hat{\phi}$ are connected by (2.19). □

2.2. PROPOSITION ([6], proposition II.4.1). Suppose that in addition to the second basic situation we have natural isomorphisms a, r, l, p, i and natural transformations j and L connected by (2.7), (2.11), (2.15) and (2.17). Then the following implications hold between the axioms MC, MCC and CC:

(i) MC1 $\Longleftrightarrow$ CC1;

(ii) MC2 $\Longleftrightarrow$ MCC2 $\Longleftrightarrow$ CC2;

(iii) MC3 $\Longleftrightarrow$ MCC3 $\Longleftrightarrow$ MCC3' $\Longleftrightarrow$ CC3;

(iv) MC4 $\Longleftrightarrow$ MCC4 $\Longleftrightarrow$ CC4. □

2.3. PROPOSITION ([6], proposition II.4.3). Let V and V' be monoidal closed categories, $\phi\colon V_0 \longrightarrow V'_0$ a functor, $\phi^0\colon I' \longrightarrow \phi I$ a morphism, and let $\tilde{\phi} = \tilde{\phi}_{XY}\colon \phi X\otimes\phi Y \longrightarrow \phi(X\otimes Y)$ and $\hat{\phi} = \hat{\phi}_{YZ}\colon \phi[YZ] \longrightarrow [\phi Y,\phi Z]$ be natural transformations, connected by (2.19). Then the following implications hold between the axioms CF, MCF and MF:

(i) MF1 $\Longleftrightarrow$ CF1

(ii) MF2 $\Longleftrightarrow$ CF2

(iii) MF3 $\Longleftrightarrow$ MCF3 $\Longleftrightarrow$ CF3. □

3. Monoidal Symmetric Closed Categories

In this section we introduce the concept of monoidal symmetric closed category. In the sections 6 and 7 we shall show that this concept coincides with the concept of a symmetric monoidal closed category (definition 1.2), so that in fact we give an alternative definition for the same structure.

3.1. Definition. A *monoidal symmetric closed category* is an ordered triple $V = \langle {}^{sc}V, \otimes, p \rangle$ consisting of:

(i) a symmetric closed category ${}^{sc}V = \langle V_0, V, [-,-], I, i, j, L, s \rangle$;

(ii) a functor $\otimes$: $V_0 \times V_0 \to V_0$;

(iii) a natural isomorphism $p = p_{XYZ}$: $[X\otimes Y, Z] \to [X[YZ]]$.

These data are to satisfy the following axiom (in addition to the axioms for a symmetric closed category):

MSCC1. The following diagram commutes:

$$\begin{array}{ccc}
[X\otimes Y\,[ZW]] & \xrightarrow{\quad s \quad} & [Z\,[X\otimes Y, W]] \\
\downarrow{\scriptstyle p} & & \\
[X\,[Y[ZW]]] & & \downarrow{\scriptstyle [1,p]} \\
\downarrow{\scriptstyle [1,s]} & & \\
[X\,[Z[YW]]] & \xrightarrow{\quad s \quad} & [Z\,[X[YW]]]
\end{array}$$

3.2. Definition. Let V and V' be monoidal symmetric closed categories. A *monoidal symmetric closed functor* Φ: $V \to V'$ is an ordered quadruple $\Phi = \langle \phi, \tilde{\phi}, \hat{\phi}, \phi^0 \rangle$ consisting of:

(i) a functor ϕ: $V_0 \to V'_0$;

(ii) a natural transformation $\tilde{\phi} = \tilde{\phi}_{XY}$: $\phi X \otimes \phi Y \to \phi(X\otimes Y)$;

(iii) a natural transformation $\hat{\phi} = \hat{\phi}_{XY}$: $\phi[XY] \to [\phi X, \phi Y]$;

(iv) a morphism ϕ^0: $I' \to \phi I$.

These data are to satisfy the axioms CF1, SCF3 and MCF3.

3.3. Definition. Let Φ and Ψ: $V \to V'$ be monoidal symmetric closed functors. A *monoidal symmetric closed natural transformation* η: $\Phi \to \Psi$: $V \to V'$ is a natural transformation η: $\phi \to \psi$: $V_0 \to V'_0$ satisfying the axioms CN1 and CN2.

3.4. THEOREM. Monoidal symmetric closed categories, monoidal symmetric closed functors and monoidal symmetric closed natural transformations form a hypercategory $MSCC$. For the rules of composition, see [6], theorem I.3.1 and theorem II.1.3. □

3.4. PROPOSITION. *The monoidal symmetric closed category of sets.* The symmetric closed category of sets S (see proposition II.2.5) becomes a monoidal symmetric closed category, also denoted by S, if we take the cartesian product $X \times Y$ for $X \otimes Y$ and if we define p by

$$((p_{XYZ}g)x)y = g\langle x,y\rangle \qquad (g \in S(X\times Y,Z);\ x \in X;\ y \in Y) \tag{3.1}$$ □

4. THE THIRD BASIC SITUATION.

In this section we analyze the dependence relations between the properties CC, MC, MCC and MSCC in the third basic situation.
The *third basic situation* consists of:

(i) a category V_0;
(ii) functors $[-,-]: V_0^* \times V_0 \longrightarrow V_0$ and $-\otimes-: V_0 \times V_0 \longrightarrow V_0$;
(iii) an object I of V_0;
(iv) a natural isomorphism.

$$\sigma = \sigma_{XYZ}: V_0(X[YZ]) \longrightarrow V_0(Y[XZ])$$

such that

$$\sigma_{YXZ} \circ \sigma_{XYZ} = 1_{V_0(X[YZ])}$$

and a natural isomorphism

$$\pi = \pi_{XYZ}: V_0(X\otimes Y,Z) \longrightarrow V_0(X[YZ]).$$

Hence the third basic situation is the conjunction of the first and the second basic situation (sections II.2 and IV.2).
Define a natural isomorphism

$$\tau = \tau_{XYZ}: V_0(X\otimes Y,Z) \longrightarrow V_0(Y[XZ]) \tag{4.1}$$

by

$$\tau_{XYZ} = \sigma_{XYZ} \circ \pi_{XYZ}.$$

The existence of this natural isomorphism is equivalent to (cf. theorem I.3.1) the existence of natural transformations

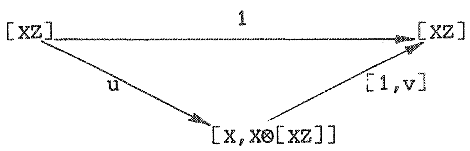
$$u = u_{YX}\colon Y \longrightarrow [X, X\otimes Y]$$

and

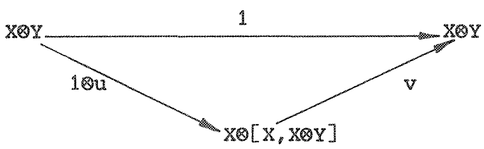
$$v = v_{XZ}\colon X\otimes[XZ] \longrightarrow Z$$

with the property that the following diagrams commute:

(4.2)



(4.3)



The relations between τ, u and v are given by the following formulas:

(4.4) $\quad u_{YX} = \tau_{XY,X\otimes Y}(1_{X\otimes Y}),$

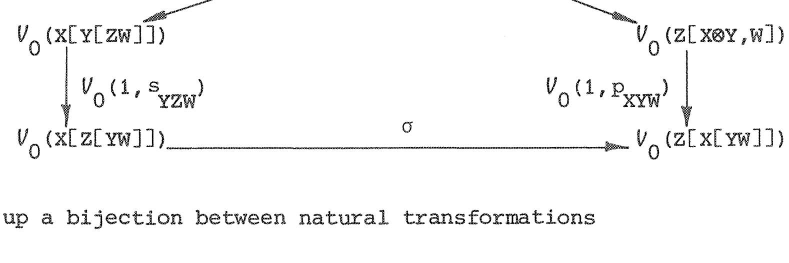
(4.5) $\quad v_{XZ} = \tau^{-1}_{X[XZ]Z}(1_{[XZ]}),$

(4.6) $\quad \tau_{XYZ}(g) = [1,g]u_{YX} \quad$ for $g\colon X\otimes Y \longrightarrow Z,$

(4.7) $\quad \tau^{-1}_{XYZ}(h) = v_{XZ}\circ 1\otimes h \quad$ for $h\colon Y \longrightarrow [XZ].$

Commutativity of the diagram

(4.8)



sets up a bijection between natural transformations

$$s = s_{YZW}\colon [Y[ZW]] \to [Z[YW]]$$

and natural transformations

$$p = p_{XYW}\colon [X\otimes Y, W] \to [X[YW]].$$

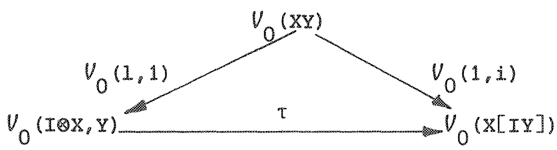
If p and s are related by (4.8) then p is a natural isomorphism if and only if s is a natural isomorphism. Moreover, if we have natural transformations s and p, and a natural transformation

$$L = L^X_{YZ}\colon [YZ] \to [[XY][XZ]]$$

then two of the bijections II.3.6, IV.2.7 and IV.4.8 determine the third.

Commutativity of the diagram

(4.9)



sets up a bijection between natural transformations

$$l = l_X\colon I\otimes X \to X$$

and natural transformations

$$i = i_Y\colon Y \to [IY].$$

If we take Y = X and evaluate at 1_X we obtain

(4.10) $\quad i_X = \tau l_X.$

If i and l are related by (4.9) then l is a natural isomorphism if and only if i is a natural isomorphism. Moreover, if we have natural transformations l and i, and a natural transformation

$$j = j_X\colon I \to [XX]$$

then two of the bijections II.3.15, IV.2.18 and IV.4.9 determine the third.

Commutativity of the diagram

$$(4.11)\qquad \begin{array}{ccc} V_0(X\ Y) & \xrightarrow{[X-]} & V_0([XX]\ [XY]) \\ {\scriptstyle V_0(r,1)}\downarrow & & \downarrow{\scriptstyle V_0(j,1)} \\ V_0(X\otimes I, Y) & \xrightarrow{\ \tau\ } & V_0(I\ [XY]) \end{array}$$

sets up a bijection between natural transformations

$$r = r_X\colon X\otimes I \longrightarrow X$$

and natural transformations

$$j = j_X\colon I \longrightarrow [XX].$$

If we take $Y = X$ and evaluate at 1_X we obtain

$$(4.12)\qquad j_X = \tau r_X.$$

If we have natural transformations r and j, and a natural transformation

$$i = i_X\colon X \longrightarrow [IX]$$

then two of the bijections II.3.15, IV.2.15 and IV.4.11 determine the third.

4.1. PROPOSITION. Let V be a monoidal symmetric closed category. Define σ and π by $\sigma_{XYZ} = Vs_{XYZ}$ and $\pi_{XYZ} = Vp_{XYZ}$. (using CC0). Then the third basic situation obtains and s and p are related by (4.8). Consequently L and p are related by (2.7).

PROOF: (4.8) is the image of MSCC1 under V. □

4.2. PROPOSITION. Suppose that in addition to the third basic situation we have natural isomorphisms a, l, r, p, s and i and natural transformations L, p and j connected by II.3.6; II.3.9; II.3.15; 2.11; 2.15; 2.17 and 4.8. Then the following implications hold:

(i) MC1 $\Leftrightarrow$ CC1 $\Leftrightarrow$ CC4 $\Leftrightarrow$ MCC4 $\Leftrightarrow$ MC4;

(ii) MC2 $\Leftrightarrow$ MCC2 $\Leftrightarrow$ CC2;

(iii) MC3 $\Leftrightarrow$ MCC3 $\Leftrightarrow$ MCC3' $\Leftrightarrow$ CC3 $\Leftrightarrow$ SCC3 $\Leftrightarrow$ MSCC1 $\Leftrightarrow$ MSCC1'.

PROOF. (i) and (ii) follow from proposition II.4.1 and proposition 2.2. From these propositions it also follows that MC3 $\Leftrightarrow$ MCC3 $\Leftrightarrow$ MCC3' $\Leftrightarrow$ CC3 $\Leftrightarrow$ SCC3. From the proof of theorem III.6.5 it follows that SCC3 $\Rightarrow$ MSCC1 $\Rightarrow$ MSCC1'. An application of σ to each leg of MSCC1' shows that MSCC1' $\Leftrightarrow$ MCC3'. This completes the proof. □

4.3. THEOREM. Let $V = \langle {}^{sc}V, \otimes, p\rangle$ be a monoidal symmetric closed category. Define σ and π by $\sigma_{XYZ} = Vs_{XYZ}$ and $\pi_{XYZ} = Vp_{XYZ}$. If we define natural transformations a, r and l by (2.14), (2.16) and (2.18) then ${}^{m}V :=$ $= \langle V_0, \otimes, I, r, l, a\rangle$ is a monoidal category and ${}^{mc}V := \langle {}^{m}V, {}^{c}V, p\rangle$ is a monoidal closed category.

PROOF. By proposition 4.1 we are in the third basic situation. From the definition it follows that a and r are natural isomorphisms. Since $l_X = \pi^{-1} j_X$ and $j_X = \sigma i_X$ it follows that $l_X = \tau^{-1} i_X$ and hence l is also a natural isomorphism. The propositions II.2.1, 4.1 and 4.2 imply the second part of assertion. □

4.4. REMARK. Proposition 4.2 implies the possibility of an alternative definition of a monoidal symmetric closed category which looks more like the definition of a symmetric monoidal closed category, in which the monoidal structure is a part of the definition, and the axioms guarantee that the natural transformations appearing in the definition are suitably related. We give such an alternative definition:
A monoidal symmetric closed category is an ordered triple $V = \langle {}^{m}V, {}^{sc}V, p\rangle$ consisting of:

(i) a monoidal category ${}^{m}V = \langle V_0, \otimes, I, r, l, a\rangle$;

(ii) a symmetric closed category ${}^{sc}V = \langle V_0, V, [-,-], I, i, j, L, s\rangle$ with the same V_0 and I as ${}^{m}V$;

(iii) a natural isomorphism $p = p_{XYZ}\colon [X\otimes Y, Z] \longrightarrow [X[YZ]]$.

These data are to satisfy the axioms MCC2, MCC3, MCC4 and MSCC1.

4.5. PROPOSITION. Suppose that besides the third basic situation V_0 etc. with natural isomorphisms a, l, r, p, s and i and natural transformations L and j connected by II.3.6; II.3.15; 2.11; 2.15; 2.17 and 4.8 we have a a second one $\mathcal{V}_0'$ etc., with natural isomorphisms a',l',r',p',s',i' and natural transformations L' and j' connected in the same way. Let $\phi\colon \mathcal{V}_0 \to \mathcal{V}_0'$ be a functor, let $\phi^0\colon I' \to \phi I$ be a morphism in $\mathcal{V}_0'$ and let $\hat{\phi} = \hat{\phi}_{YZ}\colon \phi[YZ] \to [\phi Y,\phi Z]$ and $\tilde{\phi} = \tilde{\phi}_{XY}\colon \phi X\otimes\phi Y \to \phi(X\otimes Y)$ be natural transformations with the property that the diagrams II.4.9 and 2.19 commute. Then the following implications hold between the axioms CF, MF, MCF and SCF:

(i) MF1 $\Longleftrightarrow$ CF1 $\Longleftrightarrow$ CF2 $\Longleftrightarrow$ MF2;

(ii) MF3 $\Longleftrightarrow$ MCF3 $\Longleftrightarrow$ CF3 $\Longleftrightarrow$ SCF3.

PROOF. Consequence of the propositions II.4.3 and IV.2.3. □

4.6. REMARK. The preceding propositions shows that CF1 and SCF3 determine the other properties. In fact, it is possible to define a monoidal symmetric closed functor $\Phi\colon \mathcal{V} \to \mathcal{V}'$ to be a symmetric closed functor $\Phi = \langle \phi,\hat{\phi},\phi^0\rangle\colon {}^{sc}\mathcal{V} \to {}^{sc}\mathcal{V}'$ and then to *define* the natural transformation $\tilde{\phi} = \tilde{\phi}_{XY}\colon \phi X\otimes\phi Y \to \phi(X\otimes Y)$ by (2.21). Then MCF3 holds, so $\langle \phi,\tilde{\phi},\hat{\phi},\phi^0\rangle$ is a monoidal symmetric closed functor in the sense of definition 3.2. As a corollary to this remark we have:

4.7. PROPOSITION. *The monoidal symmetric closed functor* V: $\mathcal{V} \to \mathcal{S}$.
The functor V: $\mathcal{V}_0 \to \mathcal{S}$ admits a unique extension to a monoidal symmetric closed functor V: $\mathcal{V} \to \mathcal{S}$. According to theorem II.5.4 and the preceding remark, the natural transformation

$$\tilde{V} = \tilde{V}_{XY}\colon VX \times VY \to V(X\otimes Y)$$

is defined by

$$\tilde{V}_{XY} = \pi'^{-1}(V_{Y,X\otimes Y}\cdot Vd_{XY}) \tag{4.13}$$

or, equivalently, by the following diagram:

$$\begin{array}{ccc} VX\times VY & \xrightarrow{\quad \tilde{V}_{XY} \quad} & V(X\otimes Y) \\ \downarrow{\scriptstyle Vd\times 1} & & \uparrow{\scriptstyle e'} \\ V[Y,X\otimes Y]\times VY & \xrightarrow{\quad V\times 1 \quad} & \mathcal{S}(VY,V(X\otimes Y))\times VY \end{array}$$

If $x \in VX$ and $y \in VY$ then

$$\tilde{V}_{XY}\langle x,y\rangle = V((Vd)x)y.$$

□

4.8. REMARK. Let $\Phi: V \to V'$ be a monoidal symmetric closed functor. By proposition 4.5 the diagrams II.4.9 and 2.19 commute. Conjunction of these diagrams gives the following commutative diagram:

$$
(4.14)\qquad
\begin{array}{ccc}
V_0(X\otimes Y,Z) & \xrightarrow{\quad\tau\quad} & V_0(Y,[XZ]) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
V'_0(\phi(X\otimes Y),\phi Z) & & V'_0(\phi Y,\phi[XZ]) \\
\downarrow{\scriptstyle V'_0(\tilde{\phi}_{XY},1)} & & \downarrow{\scriptstyle V'_0(1,\hat{\phi}_{XZ})} \\
V'_0(\phi X\otimes\phi Y,\phi Z) & \xrightarrow{\quad\tau'\quad} & V'_0(\phi Y,[\phi X,\phi Z])
\end{array}
$$

If we take $Z = X\otimes Y$ and evaluate at $1_{X\otimes Y}$ we obtain the following diagram:

$$
(4.15)\qquad
\begin{array}{ccc}
\phi Y & \xrightarrow{\quad\phi u\quad} & \phi[X,X\otimes Y] \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle \hat{\phi}} \\
[\phi X,\phi X\otimes\phi Y] & \xrightarrow{\quad[1,\tilde{\phi}]\quad} & [\phi X,\phi(X\otimes Y)]
\end{array}
$$

If in diagram (4.14) we replace τ by τ^{-1} and τ' by τ'^{-1}, take $Y = [XZ]$ and evaluate at $1_{[XZ]}$ we obtain the following diagram:

$$
(4.16)\qquad
\begin{array}{ccc}
\phi X\otimes\phi[XZ] & \xrightarrow{\quad\tilde{\phi}\quad} & \phi(X\otimes[XZ]) \\
\downarrow{\scriptstyle 1\otimes\hat{\phi}} & & \downarrow{\scriptstyle \phi v} \\
\phi X\otimes[\phi X,\phi Z] & \xrightarrow{\quad v'\quad} & \phi Z
\end{array}
$$

5. THE NATURAL ISOMORPHISM $t_{XYZ}: [X\otimes Y,Z] \to [Y[XZ]]$

5.1. DEFINITION. Let V be a monoidal symmetric closed category. A natural transformation

$$t = t_{XYZ}: [X\otimes Y,Z] \to [Y[XZ]]$$

is defined by the following diagram:

(5.1)
$$\begin{array}{ccc} [X\otimes Y,Z] & \xrightarrow{\;t_{XYZ}\;} & [Y[XZ]] \\ {\scriptstyle L^X}\searrow & & \nearrow{\scriptstyle [u,1]} \\ & [[X,X\otimes Y][XZ]] & \end{array}$$

5.2. REMARK. In the third basic situation, commutativity of the diagram

(5.2)
$$\begin{array}{ccccc} \mathcal{V}_0(X\otimes Y,Z) & & \xrightarrow{\;\tau\;} & & \mathcal{V}_0(Y\,[XZ]) \\ \downarrow{\scriptstyle [-,W]} & & & & \downarrow{\scriptstyle [-,[XW]]} \\ \mathcal{V}_0([ZW]\,[X\otimes Y,W]) & & & & \mathcal{V}_0([[XZ][XW]]\,[Y[XW]]) \\ & {\scriptstyle \mathcal{V}_0(1,t_{XYW})}\searrow & & \swarrow{\scriptstyle \mathcal{V}_0(L^X_{ZW},1)} & \\ & & \mathcal{V}_0([ZW][Y[XW]]) & & \end{array}$$

sets up a bijection between natural transformations

$$t = t_{XYW}\colon [X\otimes Y,W] \longrightarrow [Y[XW]]$$

and natural transformations

$$L = L^X_{ZW}\colon [ZW] \longrightarrow [[XZ][XW]].$$

If we evaluate diagram (5.2) at $x \in \mathcal{V}_0(X\otimes Y,Z)$ we obtain a diagram

(5.3)
$$\begin{array}{ccc} [Z\;W] & \xrightarrow{\;\bar{L}^X\;} & [[XZ]\;[XW]] \\ \downarrow{\scriptstyle [x,1]} & & \downarrow{\scriptstyle [\tau x,1]} \\ [X\otimes Y,W] & \xrightarrow{\;t\;} & [Y\;[XW]] \end{array}$$

If we take $Z = X\otimes Y$ and $x = 1$ we obtain diagram (5.1); if we take $Y = [XZ]$ and $x = v_{XZ}$ we see how L depends on t:

(5.4)
$$\begin{array}{ccc} [ZW] & \xrightarrow{\;L^X\;} & [[XZ][XW]] \\ {\scriptstyle [v,1]}\searrow & & \nearrow{\scriptstyle t} \\ & [X\otimes[XZ],W] & \end{array}$$

Each of the diagrams (5.1) and (5.4) also completely determines the bijection given by commutativity of diagram (5.2). Diagram (5.3) together with (4.10 and (4.12) implies the commutativity of the following two diagrams:

(5.5)
$$\begin{CD} [X,Y] @>{L^X}>> [[XX],[XY]] \\ @V{[r,1]}VV @VV{[j,1]}V \\ [X\otimes I,Y] @>{t}>> [I,[XY]] \end{CD}$$

(5.6)
$$\begin{CD} [X,Y] @>{L^I}>> [[IX],[IY]] \\ @V{[l,1]}VV @VV{[i,1]}V \\ [I\otimes X,Y] @>{t}>> [X,[IY]] \end{CD}$$

5.3. PROPOSITION. Let V be a monoidal symmetric closed category. Then

(5.7) $\quad t = s \circ p$

PROOF. In a monoidal symmetric closed category we have by definition $\tau = \sigma \circ \pi$, and by propositions 4.1 and 4.2, MSCC1' holds. In the proof of proposition III.6.6 it is shown how these facts, together with SCC2, imply that $t = sp$. □

5.4. COROLLARY. Let V be a monoidal symmetric closed category. Then $t = t_{XYZ}: [X\otimes Y,Z] \to [Y[XZ]]$ is a natural isomorphism and $Vt_{XYZ} = \tau_{XYZ}$. □

5.5. PROPOSITION. Let V be a monoidal symmetric closed category. Then the following diagrams commute:

(5.8)
$$\begin{CD} [W,Z] @>{L^{X\otimes Y}}>> [[X\otimes Y,W],[X\otimes Y,Z]] \\ @V{L^X}VV @. \\ [[XW],[XZ]] @. @VV{[1,t]}V \\ @V{L^Y}VV @. \\ [[Y[XW]],[Y[XZ]]] @>{[t,1]}>> [[X\otimes Y,W],[Y[XZ]]] \end{CD}$$

$$
(5.9)\qquad
\begin{array}{ccc}
[X\otimes Y,Z] & \xrightarrow{\quad t \quad} & [Y[XZ]] \\
 & & \big\downarrow R^{[XW]} \\
\big\downarrow R^{W} & & [[[XZ][XW]][Y[XW]]] \\
 & & \big\downarrow [L^{X},1] \\
[[ZW][X\otimes Y,W]] & \xrightarrow{\quad [1,t] \quad} & [[ZW][Y[XW]]]
\end{array}
$$

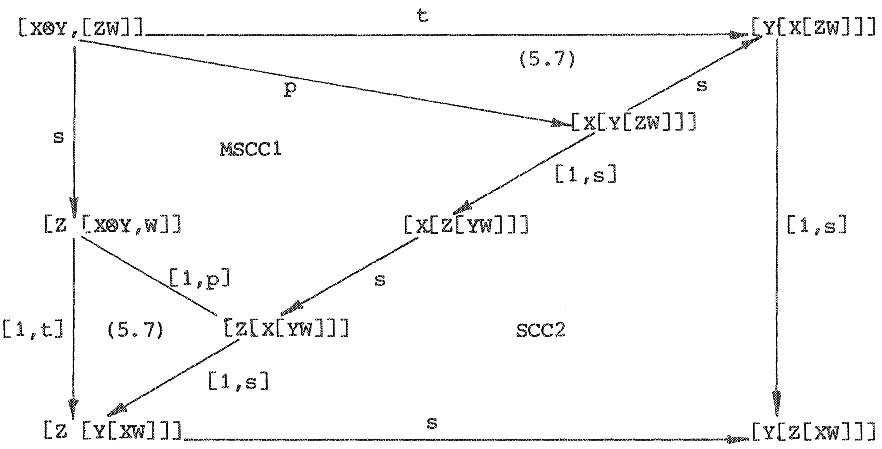
$$
(5.10\qquad
\begin{array}{ccc}
[X\otimes Y[ZW]] & \xrightarrow{\quad t \quad} & [Y[X[ZW]]] \\
\big\downarrow s & & \\
[Z[X\otimes Y,W]] & & \big\downarrow [1,s] \\
\big\downarrow [1,t] & & \\
[Z[Y[XW]]] & \xrightarrow{\quad s \quad} & [Y[Z[XW]]]
\end{array}
$$

$$
(5.11)\qquad
\begin{array}{ccc}
[X\otimes(Y\otimes Z),W] & \xrightarrow{\quad [a,1] \quad} & [(X\otimes Y)\otimes Z,W] \\
\big\downarrow t & & \big\downarrow p \\
[Y\otimes Z[XW]] & & [X\otimes Y[ZW]] \\
\big\downarrow p & & \big\downarrow t \\
[Y[Z[XW]]] & \xrightarrow{\quad [1,s] \quad} & [Y[X[ZW]]]
\end{array}
$$

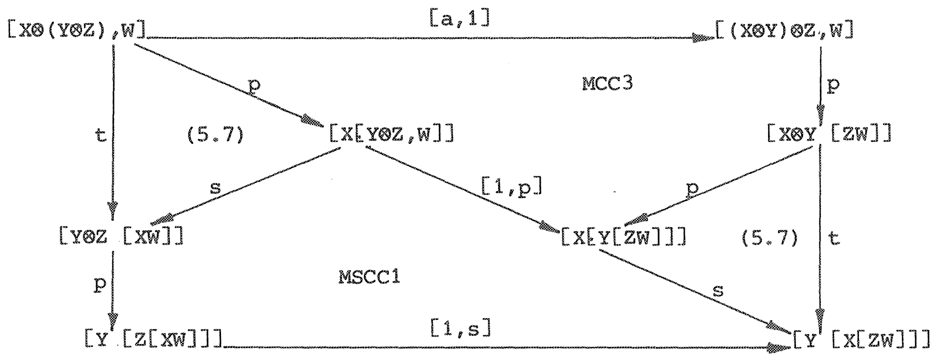
$$
(5.12)\qquad
\begin{array}{ccc}
[X\otimes(Y\otimes Z),W] & \xrightarrow{\quad [a,1] \quad} & [(X\otimes Y)\otimes Z,W] \\
\big\downarrow t & & \\
[Y\otimes Z[XW]] & & \big\downarrow t \\
\big\downarrow t & & \\
[Z[Y[XW]]] & \xrightarrow{\quad [1,t^{-1}] \quad} & [Z[X\otimes Y,W]]
\end{array}
$$

PROOF. The commutativity of diagram (5.9) is a consequence of the commutativity of the diagram II.6.13 and III.6.18 (MSCC1'), and the fact that $t = s\circ p$ (see the proof of proposition III.6.6). Diagram (5.8) is the image of diagram (5.9) under σ.

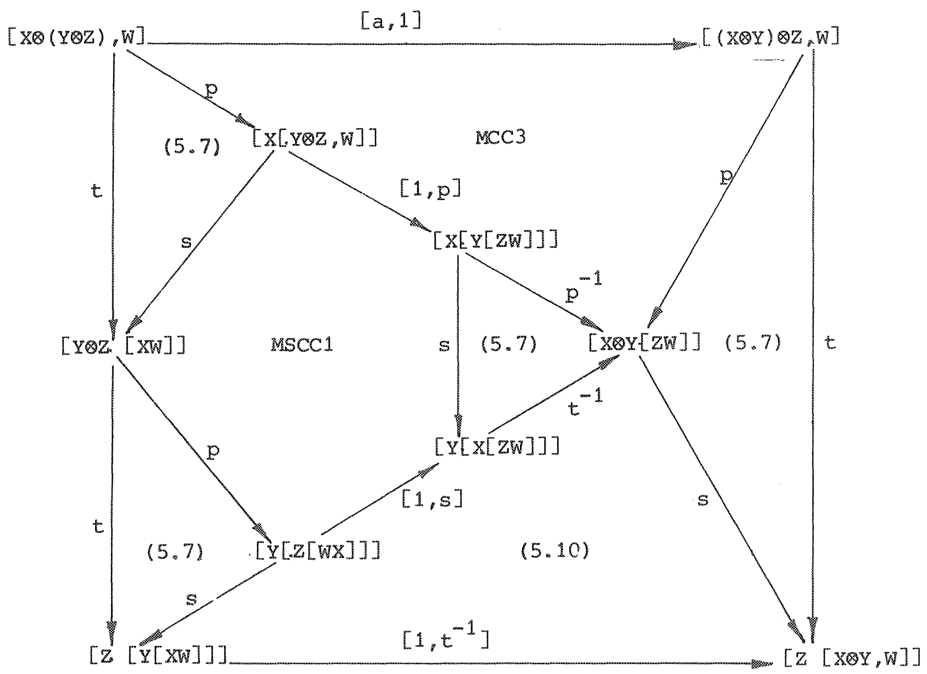
Proof of the commutativity of diagram (5.10):



Proof of the commutativity of diagram (5.11):



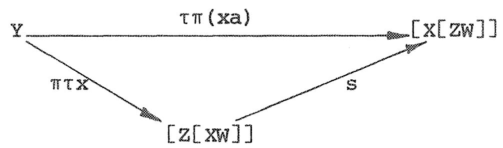
Proof of the commutativity of diagram (5.12):



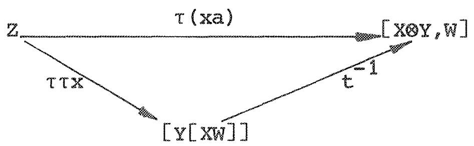
□

5.6. REMARK. If we apply V to the diagrams (5.11) and (5.12) and evaluate at $x \in V_0(X\otimes(Y\otimes Z),W)$ we obtain the following diagrams (see also diagram (2.12)):

(5.13)

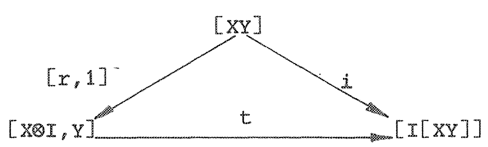


(5.14)



5.7. PROPOSITION. Let V be a monoidal symmetric closed category. Then the following diagrams commute:

(5.15)



(5.16)
$$\begin{array}{ccc} & [XY] & \\ {}^{[l,1]}\swarrow & & \searrow^{[1,i]} \\ [I\otimes X,Y] & \xrightarrow{\quad t \quad} & [X[IY]] \end{array}$$

PROOF. The commutativity of (5.15) follows from CC2 and (5.5); the commutativity of (5.16) follows from CC4 and (5.6). □

5.8. PROPOSITION. Let $\Phi = \langle \phi, \tilde{\phi}, \hat{\phi}, \phi^0 \rangle : V \longrightarrow V'$ be a monoidal symmetric closed functor. Then the following diagram commutes:

(5.17)
$$\begin{array}{ccc} \phi[X\otimes Y,Z] & \xrightarrow{\qquad \phi t \qquad} & \phi[Y,[XZ]] \\ \downarrow{\scriptstyle \hat{\phi}} & & \downarrow{\scriptstyle \hat{\phi}} \\ [\phi(X\otimes Y),\phi Z] & & [\phi Y,\phi[XZ]] \\ \downarrow{\scriptstyle [\tilde{\phi},1]} & & \downarrow{\scriptstyle [1,\hat{\phi}]} \\ [\phi X\otimes\phi Y,\phi Z] & \xrightarrow{\qquad t' \qquad} & [\phi Y,[\phi X,\phi Z]] \end{array}$$

PROOF. The commutativity of diagram (5.17) follows from proposition 5.3 and the axioms SCF3 and MCF3. It follows also from the definition of t, CF3 and (4.15). □

6. THE NATURAL ISOMORPHISM c_{XY}: $X\otimes Y \longrightarrow Y\otimes X$

6.1. DEFINITION. Let V be a monoidal symmetric closed category. A natural transformation

$$c = c_{XY}: X\otimes Y \longrightarrow Y\otimes X$$

is defined by

(6.1) $$c_{XY} = \pi^{-1}(u_{YX}) = \tau^{-1}(d_{XY})$$

or, equivalently, by the following diagram:

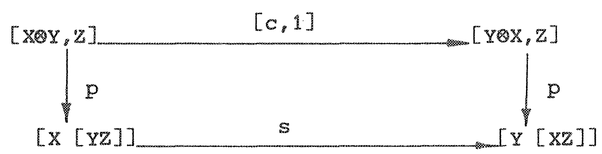
(6.2)
$$\begin{array}{ccc} X\otimes Y & \xrightarrow{\qquad 1\otimes d \qquad} & X\otimes[X,Y\otimes X] \\ \downarrow{\scriptstyle u\otimes 1} & \searrow^{c_{XY}} & \downarrow{\scriptstyle v} \\ [Y,Y\otimes X]\otimes Y & \xrightarrow{\qquad e \qquad} & Y\otimes X \end{array}$$

(Note that, since $\tau = \sigma \circ \pi$, we have

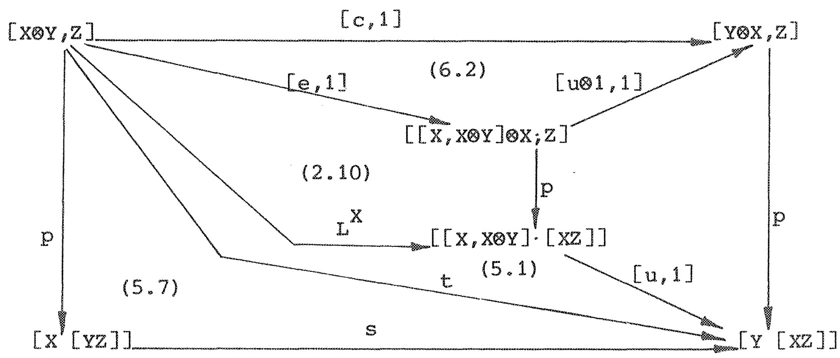
$$\pi^{-1}(u_{YX}) = \pi^{-1}\tau(1_{X\otimes Y}) = \pi^{-1}\sigma\pi(1_{X\otimes Y}) = \tau^{-1}\pi(1_{X\otimes Y}) = \tau^{-1}(d_{XY})).$$

6.2. THEOREM. Let V be a monoidal symmetric closed category. Then c is a natural isomorphism satisfying MC6 and MC7. Consequently ${}^{sm}V = \langle {}^{m}V, c\rangle$ is a symmetric monoidal category and $\langle {}^{sm}V, {}^{c}V, p\rangle$ is a symmetric monoidal closed category. Moreover, the following diagram commutes:

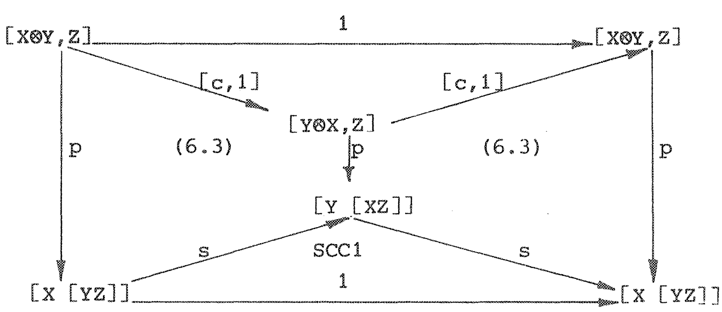
(6.3)



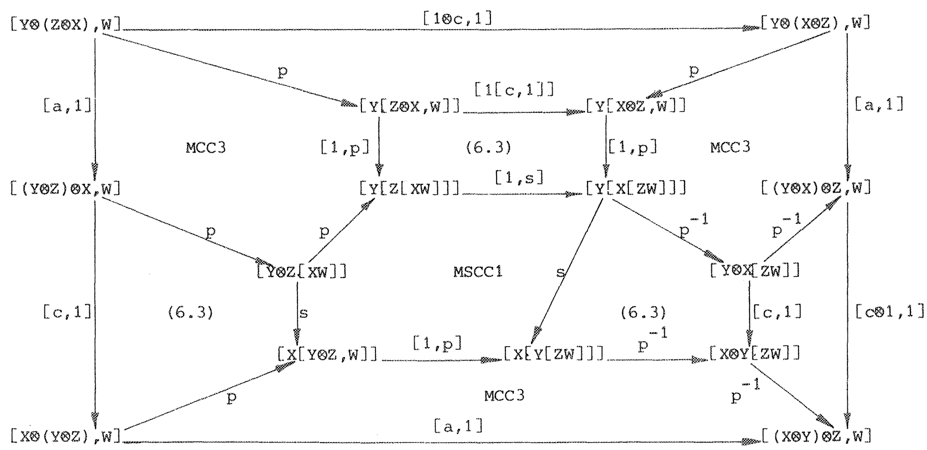
PROOF. First we prove the commutativity of diagram (6.3):



Next, consider the following diagram



The outer rectangle commutes, the lower triangle commutes and the two inner squares commute, hence the upper triangle commutes since p is an isomorphism. If we apply V to this diagram, take $Z = X\otimes Y$ and evaluate at $1_{X\otimes Y}$ we obtain MC6. The validity of MC6 implies that c is a natural isomorphism. For the proof MC7, consider the following diagram:



If we apply V to the outer diagram, take $W = Y\otimes(Z\otimes X)$ and evaluate at $1_{Y\otimes(Z\otimes X)}$ we obtain diagram MC7. □

6.3. THEOREM. Let $\Phi: V \to V'$ be a monoidal symmetric closed functor. Then MF4 holds. Hence Φ is a symmetric monoidal closed functor.

PROOF. If we apply π^{-1} to both legs of the commutative diagram (4.15) we get:

$$\pi^{-1}([1,\tilde{\phi}]u') = \tilde{\phi}\circ\pi^{-1}u' \qquad \text{by the naturality of } \pi^{-1}$$

$$= \tilde{\phi}\circ c \qquad \text{by (6.1)}$$

and

$$\pi^{-1}(\hat{\phi}\cdot\phi u) = \phi(\pi^{-1}u)\cdot\tilde{\phi} \quad \text{by (2.19)}$$

$$= \phi c\cdot\tilde{\phi} \quad \text{by (6.1).} \qquad \square$$

7. SYMMETRIC MONOIDAL CLOSED CATEGORIES.

In the preceding section we have shown that a monoidal symmetric closed category is a symmetric monoidal closed category (see definitions 1.2 and 3.1). In this section we show that the converse is also true, so that these two concepts may be identified.

7.1. THEOREM. Let $V = \langle {}^{sm}V, {}^{c}V, p\rangle$ be a symmetric monoidal closed category. Define a natural transformation

$$s = s_{XYZ}\colon [X[YZ]] \to [Y[XZ]]$$

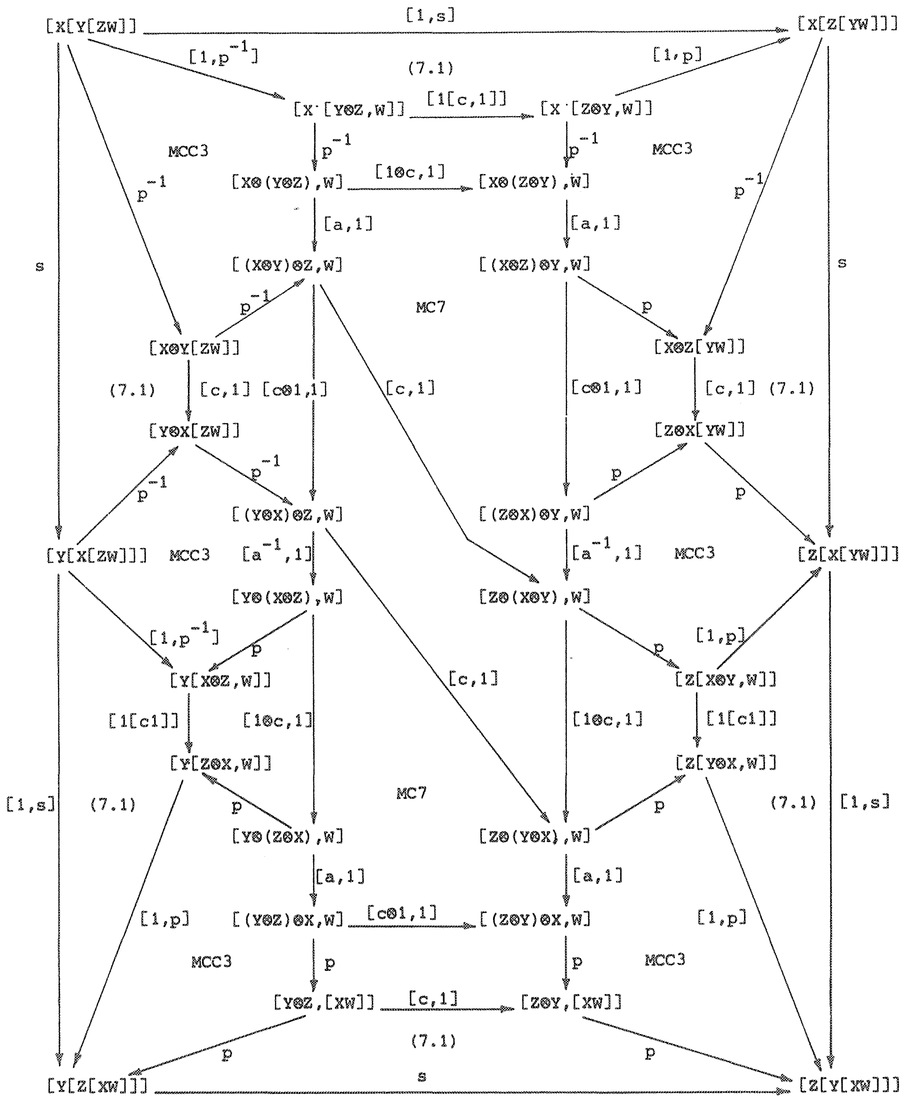
by the following diagram:

(7.1)
$$\begin{array}{ccc} [X\,[YZ]] & \xrightarrow{\quad s \quad} & [Y\,[XZ]] \\ \Big\downarrow{\scriptstyle p^{-1}} & & \Big\uparrow{\scriptstyle p} \\ [X\otimes Y, Z] & \xrightarrow{\quad [c,1] \quad} & [Y\otimes X, Z] \end{array}$$

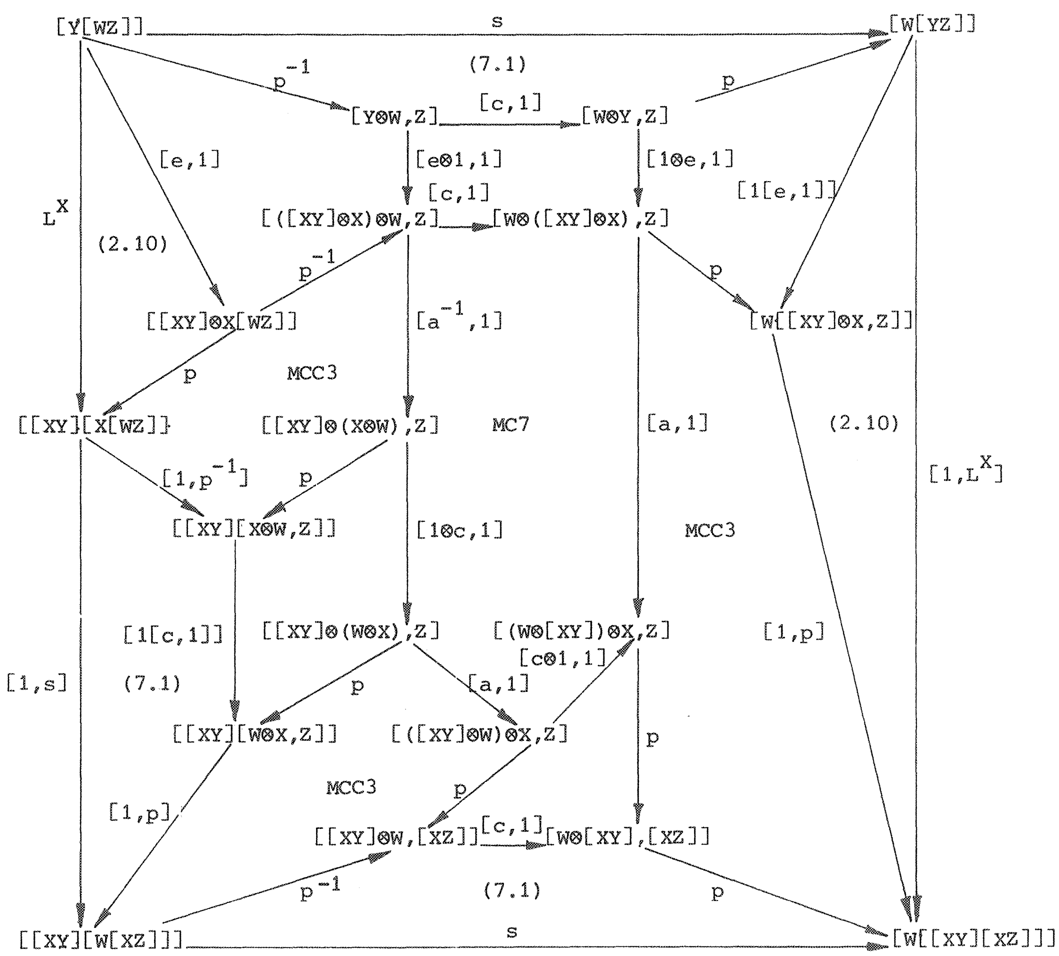
Then ${}^{sc}V = \langle {}^{c}V, s\rangle$ is a symmetric closed category, and $\langle {}^{sc}V, \otimes, p\rangle$ is a monoidal symmetric closed category.

PROOF. We have to prove the axioms SCC1, SCC2, SCC3, SCC4 and MSCC1. SCC1 is a simple consequence of the definition of s and axiom MC6.

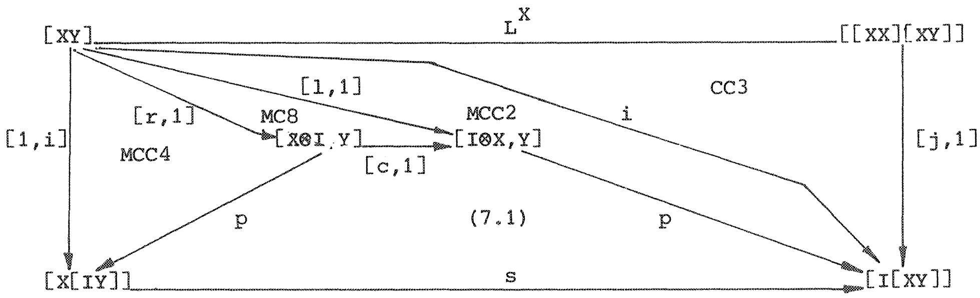
Proof of SCC2:



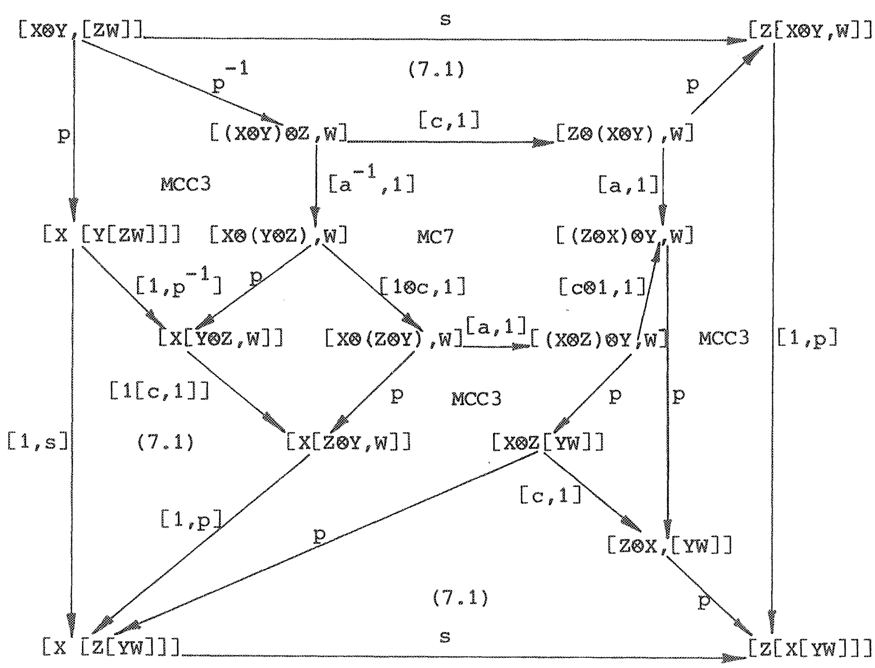
Proof of SCC3:



Proof of SCC4:

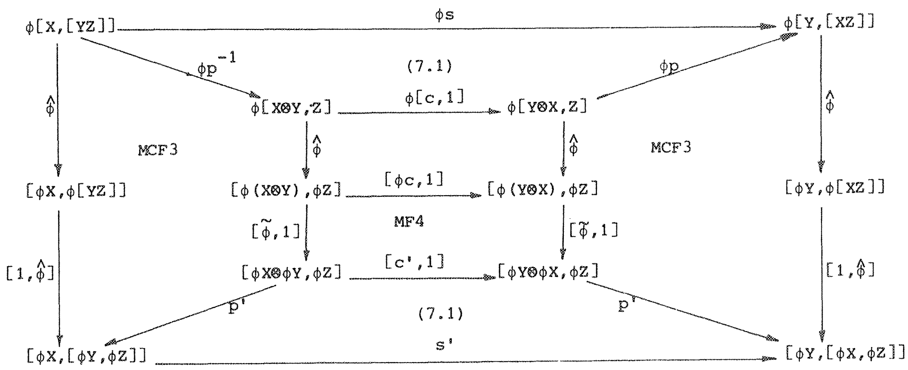


Proof of MSCC1:



7.2. THEOREM. Let $\Phi\colon V \to V'$ be a symmetric monoidal closed functor. Then SCF3 holds; hence Φ is a monoidal symmetric closed functor.

PROOF:



8. CATEGORIES OVER A MONOIDAL SYMMETRIC CLOSED CATEGORY

8.1. DEFINITION. ([6] section II.6). Let V be a monoidal category. A *V-category* A consists of the following four data:

(i) a class ob A of "objects";

(ii) for each $X,Y \in$ ob A an object $A(XY)$ of V_0;

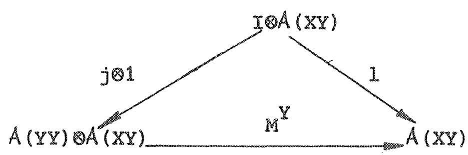
(iii) for each $X \in$ ob A a morphism $j_X\colon I \to A(XX)$ in V_0;

(iv) for each $X,Y,Z \in$ ob A a morphism

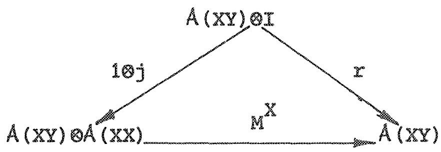
$$M^Y_{XZ}\colon A(YZ)\otimes A(XY) \to A(XZ) \text{ in } V_0.$$

These data are to satisfy the following three axioms:

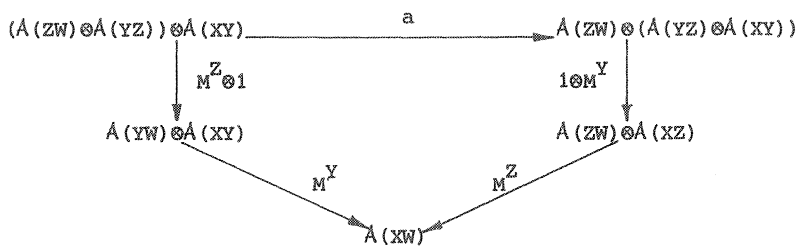
VC1'. The following diagram commutes:



VC2'. The following diagram commutes:



VC3'. The following diagram commutes:



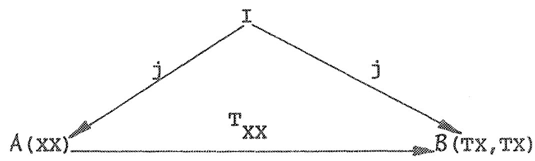
8.2. DEFINITION. ([6] section II.6). Let V be a monoidal category and let A and B be V-categories. A *V-functor* T: $A \to B$ consists of the following two data:

(i) a function T: ob $A \to$ ob B;

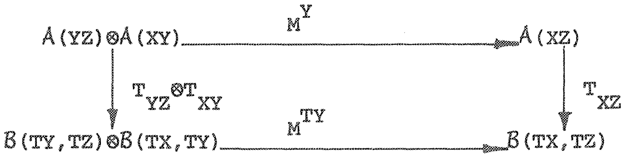
(ii) for each X,Y ∈ ob A a morphism T_{XY}: $A(XY) \to B(TX,TY)$ in V_0.

These data are to satisfy the following two axioms:

VF1'. The following diagram commutes:



VF2'. The following diagram commutes:



8.3. PROPOSITION. ([6], proposition II.6.1). If $\mathcal{V}$ is a monoidal category, $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category $\mathcal{V}_*$. □

8.4. THEOREM. ([6], theorem II.6.4). Let $\mathcal{V}$ be a monoidal symmetric closed category. Then the categories ${}^m\mathcal{V}_*$ and ${}^c\mathcal{V}_*$ coincide if we identify the ${}^m\mathcal{V}$-category $\mathcal{A} = \langle \text{ob } \mathcal{A}, \mathcal{A}(XY), j, M\rangle$ with the ${}^c\mathcal{V}$-category $\mathcal{A} = \langle \text{ob } \mathcal{A}, \mathcal{A}(XY), j, L\rangle$ where

$$(8.1) \qquad L^X_{YZ} = \pi M^Y_{XZ}$$

(here π is $\pi = \mathcal{V}p\colon \mathcal{V}_0(\mathcal{A}(YZ)\otimes\mathcal{A}(XY),\mathcal{A}(XZ)) \to \mathcal{V}_0(\mathcal{A}(YZ),[\mathcal{A}(XY),\mathcal{A}(XZ)]))$ (In fact, EILENBERG and KELLY show that VC1 $\Longleftrightarrow$ VC1'; VC2 $\Longleftrightarrow$ VC2'; VC3 $\Longleftrightarrow$ VC3' and VF2 $\Longleftrightarrow$ VF2'). □

8.5. DEFINITION. Let $\mathcal{V}$ be a monoidal symmetric closed category, and let $\mathcal{A}$ be a $\mathcal{V}$-category. For each $X,Y,Z \in \text{ob } \mathcal{A}$ define a morphism

$$N^Y_{XZ}\colon \mathcal{A}(XY)\otimes\mathcal{A}(YZ) \to \mathcal{A}(XZ) \quad \text{in } \mathcal{V}_0$$

by

$$(8.2) \qquad N^Y_{XZ} = \pi^{-1}(R^Z_{XY})$$

(note that by (II.6.1) $R^Z_{XY} = \sigma(L^X_{YZ})\colon \mathcal{A}(XY) \to [\mathcal{A}(YZ),\mathcal{A}(XZ)]$).

8.6. REMARK. Combination of (8.1) and (8.2) with (4.1) and (II.3.11) gives

$$(8.3) \qquad M^Y_{XZ} = \tau^{-1}(R^Z_{XY})$$

and

$$(8.4) \qquad N^Y_{XZ} = \tau^{-1}(L^X_{YZ})$$

Combination of (8.1) - (8.4) with (2.8) and (5.3) gives the commutativity of the following four diagrams:

$$(8.6) \qquad \begin{array}{ccc} [\mathcal{A}(XZ),\mathcal{A}(PQ)] & \xrightarrow{\;L^{\mathcal{A}(XY)}\;} & [[\mathcal{A}(XY),\mathcal{A}(XZ)]\,[\mathcal{A}(XY),\mathcal{A}(PQ)]] \\ \downarrow{\scriptstyle [M^Y,1]} & & \downarrow{\scriptstyle [L^X,1]} \\ [\mathcal{A}(YZ)\otimes\mathcal{A}(XY),\mathcal{A}(PQ)] & \xrightarrow{\;p\;} & [\mathcal{A}(YZ)\,[\mathcal{A}(XY),\mathcal{A}(PQ)]] \end{array}$$

$$(8.7)\qquad \begin{array}{ccc} [\mathcal{A}(XZ),\mathcal{A}(PQ)] & \xrightarrow{L^{\mathcal{A}(YZ)}} & [[\mathcal{A}(YZ),\mathcal{A}(XZ)]\,[\mathcal{A}(YZ),\mathcal{A}(PQ)]] \\ \downarrow{\scriptstyle [N^Y,1]} & & \downarrow{\scriptstyle [R^Z,1]} \\ [\mathcal{A}(XY)\otimes\mathcal{A}(YZ),\mathcal{A}(PQ)] & \xrightarrow{p} & [\mathcal{A}(XY)\,[\mathcal{A}(YZ),\mathcal{A}(PQ)]] \end{array}$$

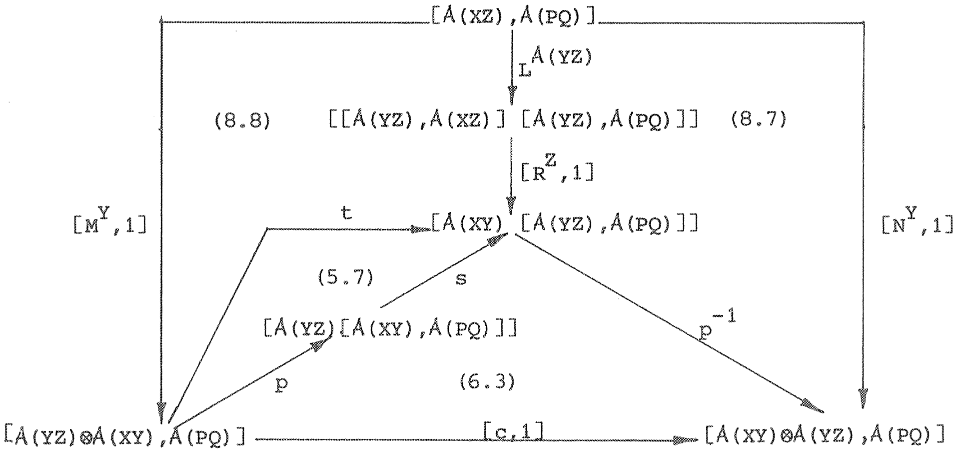
$$(8.8)\qquad \begin{array}{ccc} [\mathcal{A}(XZ),\mathcal{A}(PQ)] & \xrightarrow{L^{\mathcal{A}(YZ)}} & [[\mathcal{A}(YZ),\mathcal{A}(XZ)]\,[\mathcal{A}(YZ),\mathcal{A}(PQ)]] \\ \downarrow{\scriptstyle [M^Y,1]} & & \downarrow{\scriptstyle [R^Z,1]} \\ [\mathcal{A}(YZ)\otimes\mathcal{A}(XY),\mathcal{A}(PQ)] & \xrightarrow{t} & [\mathcal{A}(XY)\,[\mathcal{A}(YZ),\mathcal{A}(PQ)]] \end{array}$$

$$(8.9)\qquad \begin{array}{ccc} [\mathcal{A}(XZ),\mathcal{A}(PQ)] & \xrightarrow{L^{\mathcal{A}(XY)}} & [[\mathcal{A}(XY),\mathcal{A}(XZ)]\,[\mathcal{A}(XY),\mathcal{A}(PQ)]] \\ \downarrow{\scriptstyle [N^Y,1]} & & \downarrow{\scriptstyle [L^X,1]} \\ [\mathcal{A}(XY)\otimes\mathcal{A}(YZ),\mathcal{A}(PQ)] & \xrightarrow{t} & [\mathcal{A}(YZ)\,[\mathcal{A}(XY),\mathcal{A}(PQ)]] \end{array}$$

8.7. PROPOSITION. Let $\mathcal{V}$ be a monoidal symmetric closed category and let $\mathcal{A}$ be a $\mathcal{V}$-category. Then the following diagram commutes:

$$(8.10)\qquad \begin{array}{ccc} \mathcal{A}(XY)\otimes\mathcal{A}(YZ) & \xrightarrow{c} & \mathcal{A}(YZ)\otimes\mathcal{A}(XY) \\ & {\scriptstyle N^Y}\searrow \quad \swarrow{\scriptstyle M^Y} & \\ & \mathcal{A}(XZ) & \end{array}$$

PROOF. The following diagram commutes:



If we apply V to this diagram, use CC0, take $A(PQ) = A(XZ)$ and evaluate at $1_{A(XZ)}$, we obtain diagram (8.10). □

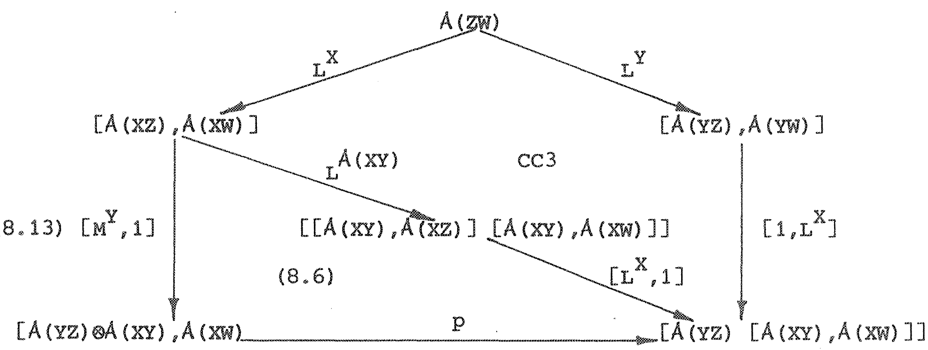
8.8. PROPOSITION. Let V be a monoidal symmetric closed category and let A be a V-category. Let $X,Y,Z \in \text{ob } A$, $h \in A_0(XY)$ and $g \in A_0(YZ)$. Then:

(8.11) $$VM^Y_{XZ} \cdot \tilde{V}_{A(YZ)A(XY)} \langle g,h \rangle = gh$$

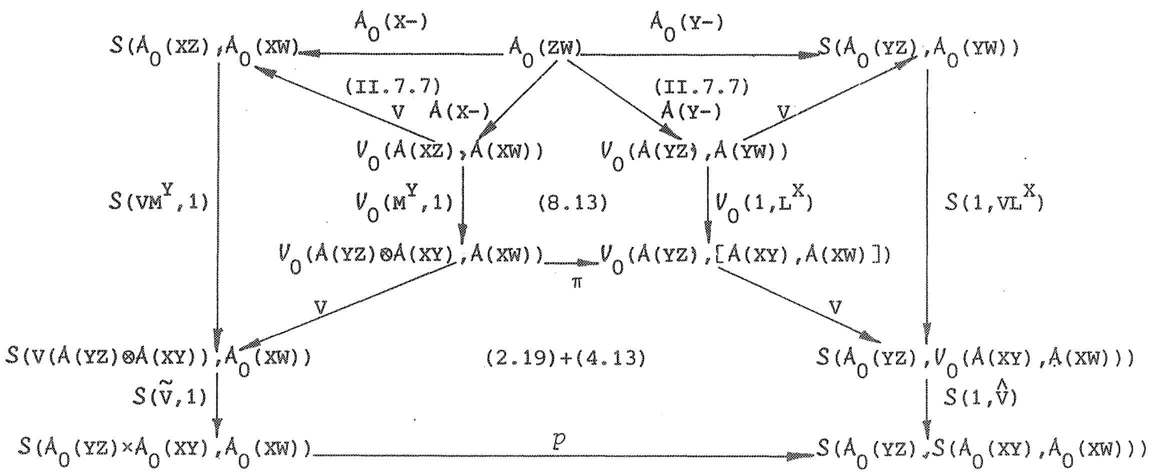
and

(8.12) $$VN^Y_{XZ} \cdot \tilde{V}_{A(XY)A(YZ)} \langle h,g \rangle = gh.$$

PROOF. The following diagram commutes:



The following diagram commutes:



If we take W = Z and evaluate at 1_Z we obtain:

$$(VM^Y \cdot \tilde{V})\langle g,h\rangle = p^{-1}(\hat{V}\cdot VL^X)\langle g,h\rangle$$

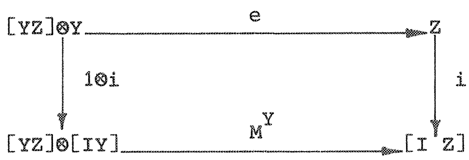
$$= ((\hat{V}\cdot VL^X)g)h \qquad \text{by (3.1)}$$

$$= gh \qquad \text{by (II.7.3).}$$

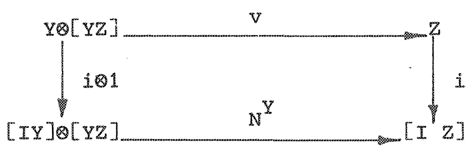
The proof of (8.12) is similar. □

8.9. PROPOSITION. (cf. [6], proposition II.7.3). Let V be a monoidal symmetric closed category. Then the following diagrams commute:

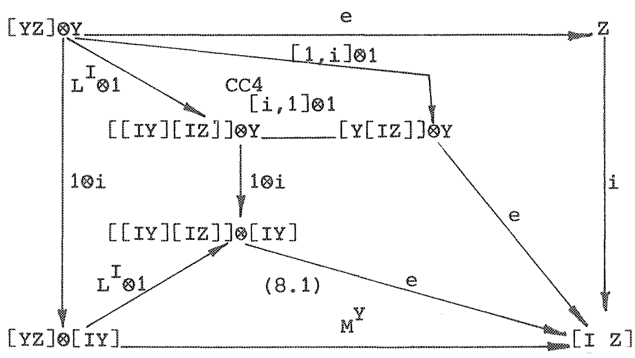
(8.14)



(8.15)



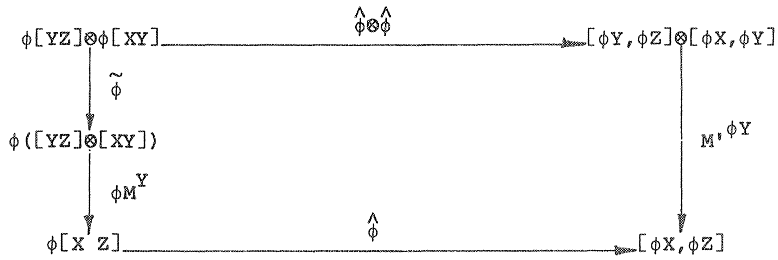
PROOF. Proof of the commutativity of diagram (8.14):



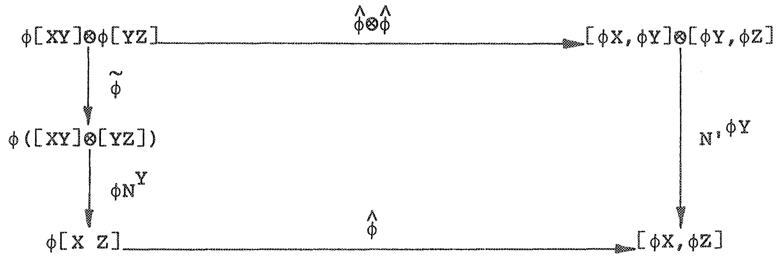
The commutativity of (8.15) follows in a similar way from CC4 and (8.4). □

8.10. PROPOSITION. Let V and V' be two monoidal symmetric closed categories and let Φ: $V \longrightarrow V'$ be a monoidal symmetric closed functor.
Then the following diagrams are commutative:

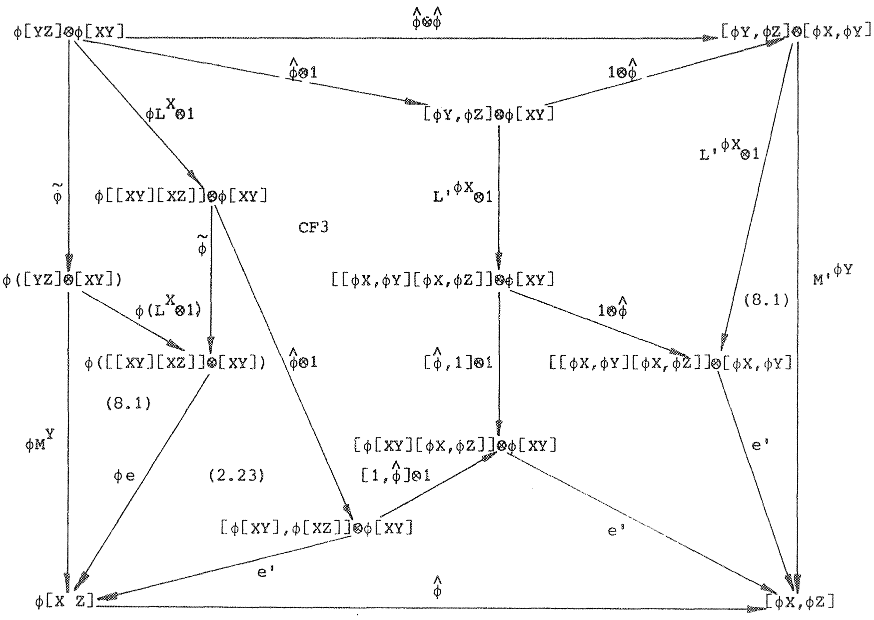
(8.16)



(8.17)



PROOF. Proof of the commutativity of diagram (8.16):



The proof of the commutativity of diagram (8.17) is similar. □

9. THE NATURAL TRANSFORMATIONS $K^Y_{XZ}: [XY] \to [X\otimes Y, Z\otimes Y]$ AND $H^X_{YW}: [YW] \to [X\otimes Y, X\otimes W]$

9.1. DEFINITION. Let V be a monoidal symmetric closed category. We define natural transformations

$$K = K^Y_{XZ}: [XZ] \longrightarrow [X\otimes Y, Z\otimes Y]$$

and

$$H = H^X_{YW}: [YW] \longrightarrow [X\otimes Y, X\otimes W]$$

by means of the following diagrams:

(9.1)
$$\begin{array}{ccc} [XZ] & \xrightarrow{\quad K^Y \quad} & [X\otimes Y, Z\otimes Y] \\ & {\scriptstyle [1,d]} \searrow \qquad \nearrow {\scriptstyle p^{-1}} & \\ & [X[Y, Z\otimes Y]] & \end{array}$$

(9.2)
$$\begin{array}{ccc} [YW] & \xrightarrow{\quad H^X \quad} & [X\otimes Y, X\otimes W] \\ & {\scriptstyle [1,u]} \searrow \qquad \nearrow {\scriptstyle t^{-1}} & \\ & [Y[X, X\otimes W]] & \end{array}$$

9.2. REMARK. In the third basic situation, commutativity of the diagram

(9.3)
$$\begin{array}{ccc} V_0(Z\otimes Y, W) & \xrightarrow{\quad \pi \quad} & V_0(Z, [YW]) \\ \Big\downarrow {\scriptstyle [X\otimes Y,-]} & & \Big\downarrow {\scriptstyle [X,-]} \\ V_0([X\otimes Y, Z\otimes Y], [X\otimes Y, W]) & & V_0([XZ][X[YW]]) \\ & {\scriptstyle V_0(K^Y_{XZ},1)} \searrow \qquad \swarrow {\scriptstyle V_0(1,p'_{XYW})} & \\ & V_0([XZ][X\otimes Y, W]) & \end{array}$$

sets up a bijection between natural transformations

$$p' = p'_{XYW}: [X[YW]] \longrightarrow [X\otimes Y, W]$$

and natural transformations

$$K = K^{Y}_{XZ}\colon [XZ] \longrightarrow [X\otimes Y, Z\otimes Y].$$

If we evaluate diagram (9.3) at $x \in \mathcal{V}_0(Z\otimes Y, W)$ we obtain a diagram

$$\text{(9.4)}\qquad \begin{array}{ccc} [X,Z] & \xrightarrow{\;[1,\pi x]\;} & [X,[YW]] \\ {\scriptstyle K^{Y}}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ [X\otimes Y, Z\otimes Y] & \xrightarrow{\;[1,x]\;} & [X\otimes Y, W] \end{array}$$

If we take $W = Z\otimes Y$ and $x = 1$ we obtain (9.1) (with p' for p^{-1}).
If we take $Z = [YW]$ and $x = e$ we see how p' depends on K:

$$\text{(9.5)}\qquad \begin{array}{ccc} [X[YW]] & \xrightarrow{\;p'\;} & [X\otimes Y, W] \\ & {\scriptstyle K^{Y}}\searrow \quad \nearrow{\scriptstyle [1,e]} & \\ & [X\otimes Y, [YW]\otimes Y] & \end{array}$$

Each of the diagrams (9.1) and (9.5) also completely determines the bijection given by commutativity of diagram (9.3).

Similarly, in the third basic situation, commutativity of the diagram

$$\text{(9.6)}\qquad \begin{array}{ccc} \mathcal{V}_0(X\otimes W, Z) & \xrightarrow{\;\tau\;} & \mathcal{V}_0(W,[XZ]) \\ {\scriptstyle [X\otimes Y,-]}\big\downarrow & & \big\downarrow{\scriptstyle [Y,-]} \\ \mathcal{V}_0([X\otimes Y, X\otimes W],[Y\otimes Y, Z]) & & \mathcal{V}_0([YW][Y[XZ]]) \\ & {\scriptstyle \mathcal{V}_0(H^{X}_{YW},1)}\searrow \quad \swarrow{\scriptstyle \mathcal{V}_0(1,t'_{XYZ})} & \\ & \mathcal{V}_0([YW][X\otimes Y, Z]) & \end{array}$$

sets up a bijection between natural transformations

$$t' = t'_{XYZ}\colon [Y[XZ]] \longrightarrow [X\otimes Y, Z]$$

and natural transformations

$$H = H^{X}_{YW}\colon [YW] \longrightarrow [X\otimes Y, X\otimes W].$$

If we evaluate diagram (9.6) at $y \in \mathcal{V}_0(X\otimes W, Z)$ we obtain a diagram

$$(9.7)\qquad \begin{array}{ccc} [Y\ W] & \xrightarrow{\quad [1,\tau y] \quad} & [Y\ [XZ]] \\ \big\downarrow{\scriptstyle H^X} & & \big\downarrow{\scriptstyle t'} \\ [X\otimes Y, X\otimes W] & \xrightarrow{\quad [1,y] \quad} & [X\otimes Y, Z] \end{array}$$

If we take $Z = X\otimes W$ and $y = 1$ we obtain (9.2) (with t' for t^{-1}).
If we take $W = [XZ]$ and $y = v$ we see how t' depends on H:

$$(9.8)\qquad \begin{array}{ccc} [Y[XZ]] & \xrightarrow{\quad t' \quad} & [X\otimes Y, Z] \\ & {\scriptstyle H^X}\searrow \qquad \nearrow{\scriptstyle [1,v]} & \\ & [X\otimes Y, X\otimes [XZ]] & \end{array}$$

Each of the diagrams (9.2) and (9.8) also completely determines the bijection given by the commutativity of diagram (9.6).

9.3. REMARK. Combination of the formulas (8.1) - (8.4) with the diagram (9.4) and (9.7) gives the following four commutative diagrams:

$$(9.9)\qquad \begin{array}{ccc} [\acute{A}(PQ), \acute{A}(YZ)] & \xrightarrow{\quad [1,L^X] \quad} & [\acute{A}(PQ), [\acute{A}(XY), \acute{A}(XZ)]] \\ \big\downarrow{\scriptstyle K^{\acute{A}(XY)}} & & \big\downarrow{\scriptstyle p^{-1}} \\ [\acute{A}(PQ)\otimes \acute{A}(XY), \acute{A}(YZ)\otimes \acute{A}(XY)] & \xrightarrow{\quad [1,M^Y] \quad} & [\acute{A}(PQ)\otimes \acute{A}(XY), \acute{A}(XZ)] \end{array}$$

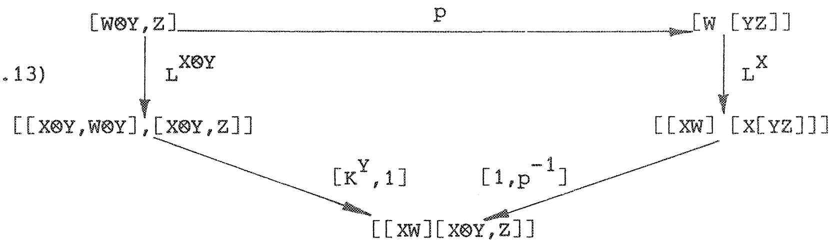
$$(9.10)\qquad \begin{array}{ccc} [\acute{A}(PQ), \acute{A}(XY)] & \xrightarrow{\quad [1,R^Z] \quad} & [\acute{A}(PQ), [\acute{A}(YZ), \acute{A}(XZ)]] \\ \big\downarrow{\scriptstyle K^{\acute{A}(YZ)}} & & \big\downarrow{\scriptstyle p^{-1}} \\ [\acute{A}(PQ)\otimes \acute{A}(YZ), \acute{A}(XY)\otimes \acute{A}(YZ)] & \xrightarrow{\quad [1,N^Y] \quad} & [\acute{A}(PQ)\otimes \acute{A}(YZ), \acute{A}(XZ)] \end{array}$$

$$(9.11)\qquad \begin{array}{ccc} [\acute{A}(PQ), \acute{A}(YZ)] & \xrightarrow{\quad [1,L^X] \quad} & [\acute{A}(PQ), [\acute{A}(XY), \acute{A}(XZ)]] \\ \big\downarrow{\scriptstyle H^{\acute{A}(XY)}} & & \big\downarrow{\scriptstyle t^{-1}} \\ [\acute{A}(XY)\otimes \acute{A}(PQ), \acute{A}(XY)\otimes \acute{A}(YZ)] & \xrightarrow{\quad [1,N^Y] \quad} & [\acute{A}(XY)\otimes \acute{A}(PQ), \acute{A}(XZ)] \end{array}$$

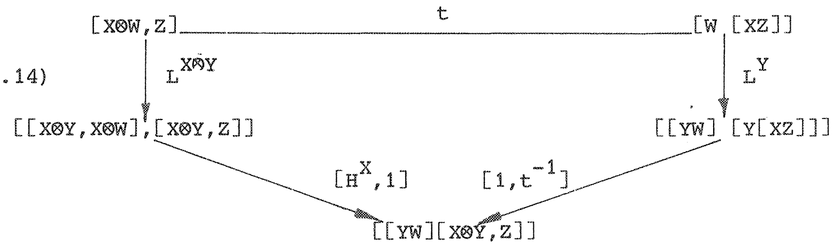
$$(9.12)\qquad \begin{array}{ccc} [\acute{A}(PQ), \acute{A}(XY)] & \xrightarrow{\quad [1,R^Z] \quad} & [\acute{A}(PQ), [\acute{A}(YZ), \acute{A}(XZ)]] \\ \big\downarrow{\scriptstyle H^{\acute{A}(YZ)}} & & \big\downarrow{\scriptstyle t^{-1}} \\ [\acute{A}(YZ)\otimes \acute{A}(PQ), \acute{A}(YZ)\otimes \acute{A}(XY)] & \xrightarrow{\quad [1,M^Y] \quad} & [\acute{A}(YZ)\otimes \acute{A}(PQ), \acute{A}(XZ)] \end{array}$$

9.4. PROPOSITION. Let V be a monoidal symmetric closed category. Then the following diagrams are commutative:

(9.13)

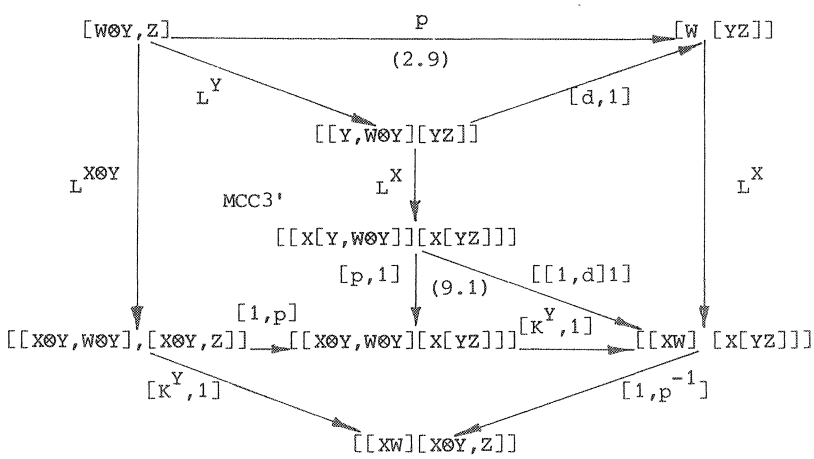


(9.14)

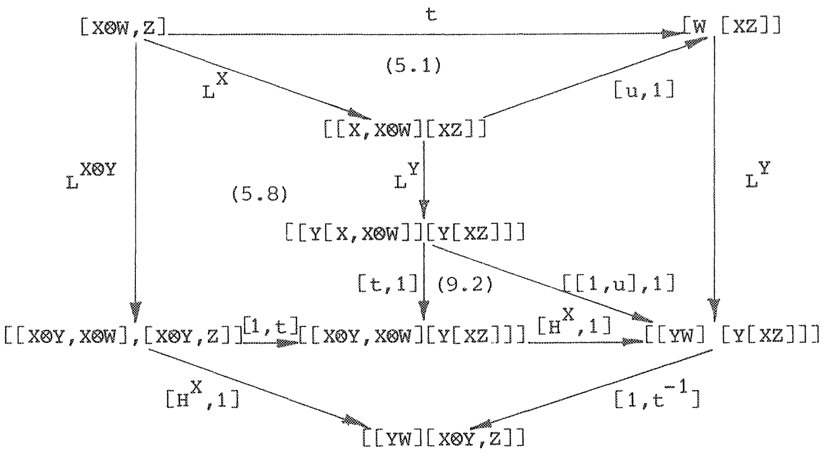


PROOF.

Proof of the commutativity of diagram (9.13):



Proof of the commutativity of diagram (9.14):

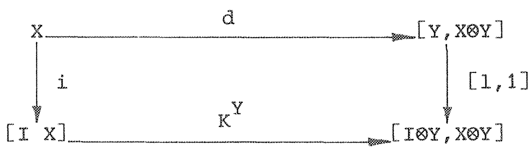


□

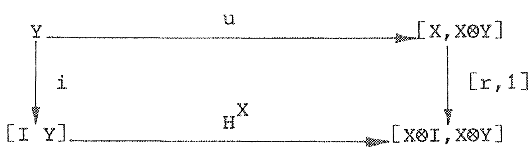
9.5. REMARK. Note that if we apply V to the diagrams (9.13) and (9.14) we obtain the diagrams (9.3) and (9.6).

9.6. PROPOSITION. Let V be a monoidal symmetric closed category. Then the following diagrams are commutative:

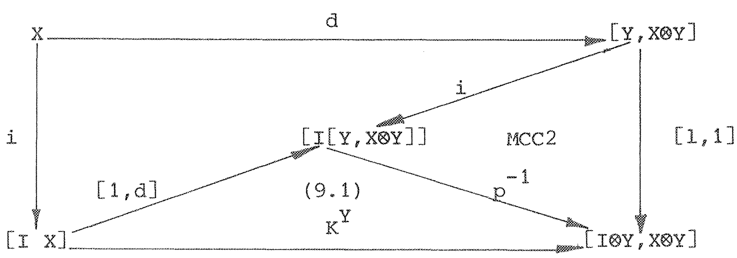
(9.15)

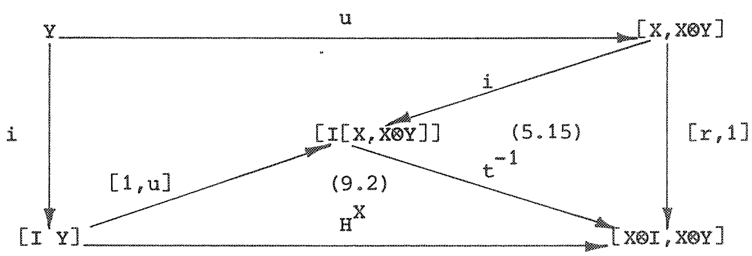


(9.16)



PROOF.



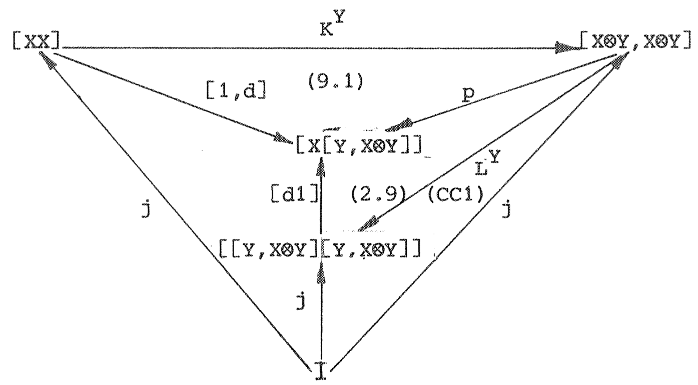


□

9.7. PROPOSITION ([6], theorem II.7.1). Let V be a monoidal symmetric closed category. For each $Y \in$ ob V we obtain a V-functor $K^Y: V \longrightarrow V$ if we define $K^Y X = X \otimes Y$ and $(K^Y)_{XZ} = K^Y_{XZ}$. For each $X \in$ ob V we obtain a V-functor $H^X: V \longrightarrow V$ if we define $H^X Y = X \otimes Y$ and $(H^X)_{YZ} = H^X_{YZ}$. The underlying functor $V_* K^Y$ is $- \otimes Y$, and the underlying functor $V_* H^X$ is $X \otimes -$.

PROOF.

a) VF1 for K^Y:



The outer triangle commutes since p is an isomorphism.

b) VF1 for H^X is proved in a similar way.

c) VF2 for K^Y and for H^X is proved in section 10, theorem 10.1 (diagram (10.27) and diagram (10.30) respectively).

If we apply V to diagram (9.1) and evaluate at $g \in V_0(XZ)$ we obtain:

$$VK^Y_{XZ}(g) = (\pi^{-1} \circ V[1,d])\ (g)$$

$$= \pi^{-1}(dg)$$

$$= \pi^{-1} d \circ (g \otimes 1) \qquad \text{by the naturality of } \pi^{-1}$$

$$= g \otimes 1 \qquad \text{by (2.3).}$$

Hence we have proved

(9.17) $\quad VK^{Y}_{XZ}(g) = g\otimes 1\colon X\otimes Y \longrightarrow Z\otimes Y.$

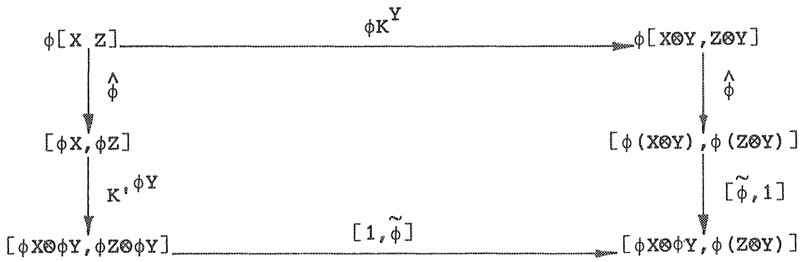
Similarly one can prove

(9.18) $\quad VH^{X}_{YW}(h) = 1\otimes h\colon X\otimes Y \longrightarrow X\otimes W \qquad$ for $h \in V_0(YW)$,

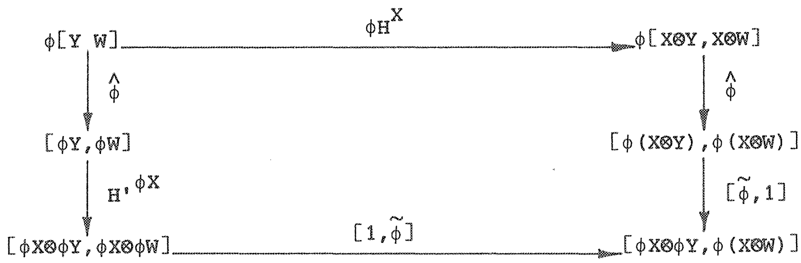
and this proves the second part of the proposition. □

9.8. PROPOSITION. Let $\Phi\colon V \longrightarrow V'$ be a monoidal symmetric closed functor. Then the following diagrams are commutative:

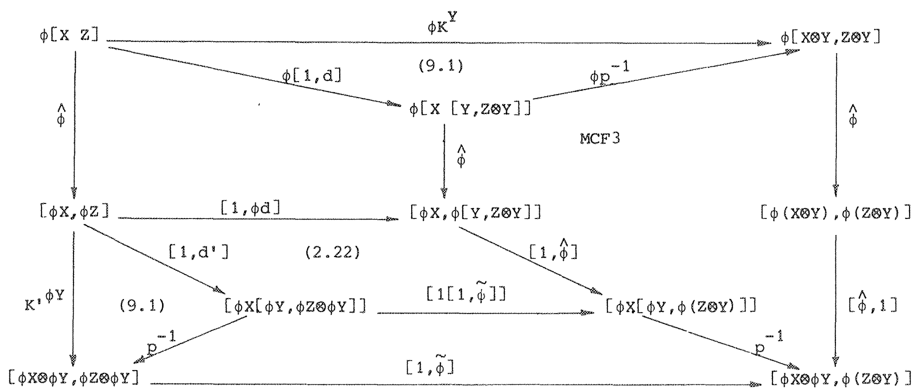
(9.19)



(9.20)



PROOF. Proof of the commutativity of diagram (9.19):



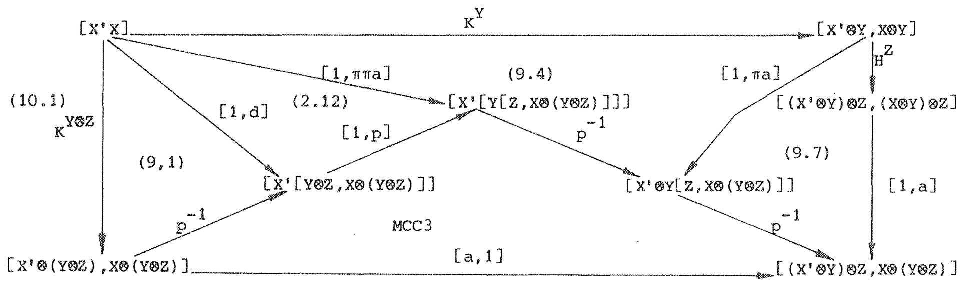
The proof of (9.20) is similar. □

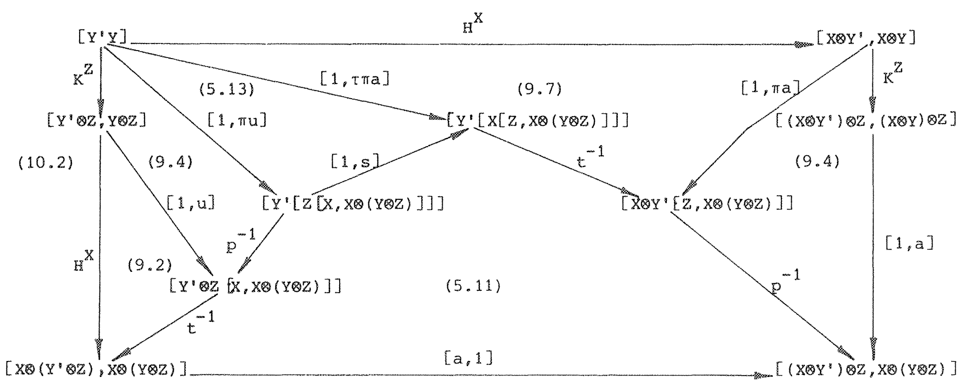
10. THE V-NATURALITY OF THE CANONICAL TRANSFORMATIONS

10.1. THEOREM. (cf. [6], theorem III.7.4). Let V be a monoidal symmetric closed category. Then the morphisms a, r, l, c, s, m, i, p, t, H, K, d, e, u and v are V-natural in every variable. Moreover, if A is a V-category then the morphisms L, R, j, M and N are V-natural is every variable.

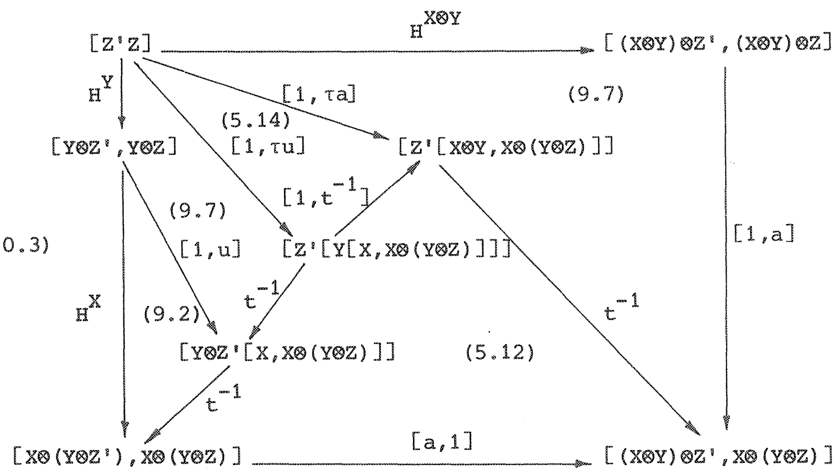
PROOF. Although a proof of the V-naturality of most of the morphisms is to be found in [6], theorem III.7.4, we prefer to give a complete and detailed proof of this theorem, because many of the diagrams, expressing V-naturality, are needed in the next section. The V-naturality of L, R, s, m, i and j has been proved in theorem II.7.10.

a) Proof of the V-naturality of a:



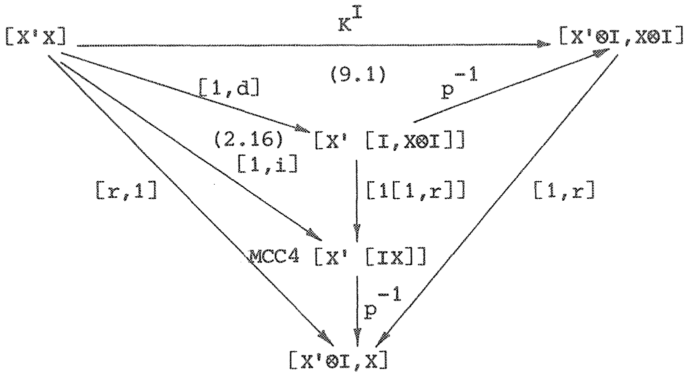


(10.3)



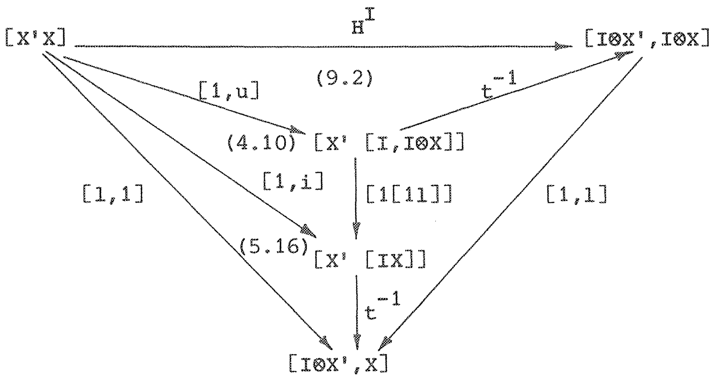
b) Proof of the V-naturality of r:

(10.4)

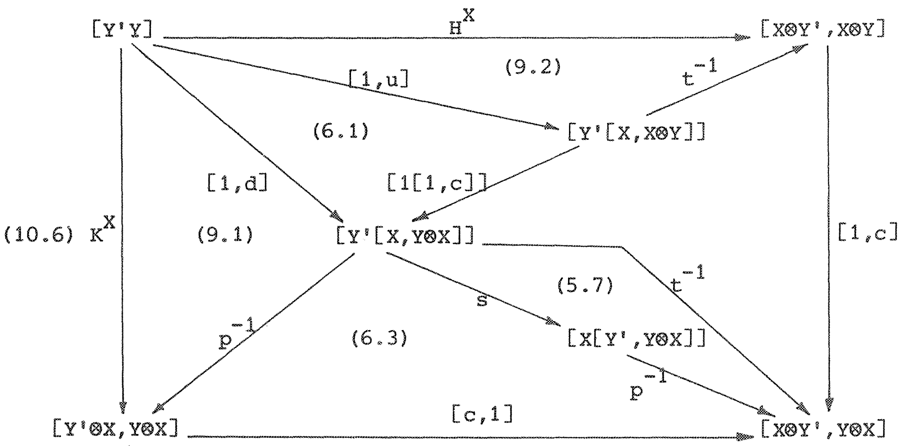


c) Proof of the V-naturality of l:

(10.5)

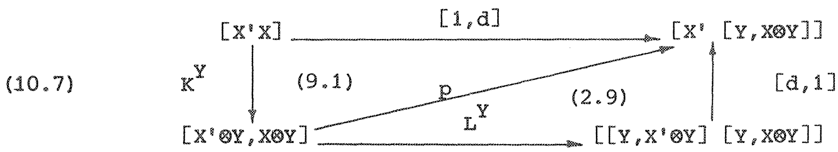


d) Proof of the V-naturality of c:

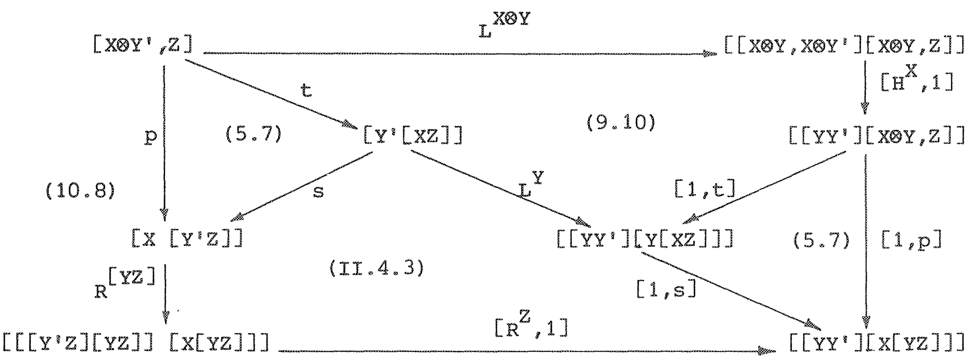


The proof of the V-naturality of c in the other variable is similar.

e) Proof of the V-naturality of d_{XY} is the variable X:



In order to prove the V-naturality of d_{XY} in the variable Y we first note that the following diagram commutes:



If we apply V to this diagram, use CCO, the propositions II.1.6 and II.5.1, take Z = X⊗Y' and evaluate at $1_{X⊗Y'}$, we obtain

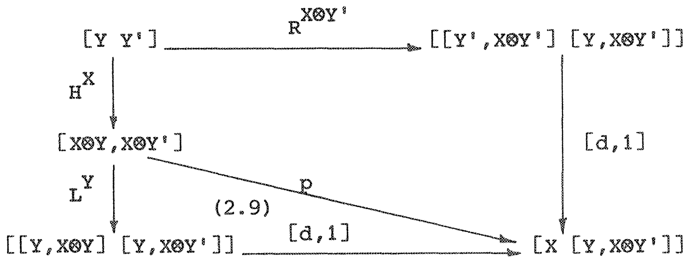
$$(\mathcal{V}_0(H^X,p)\cdot VL^{X\otimes Y})1_{X\otimes Y'} = p\cdot H^X$$

and

$$(\mathcal{V}_0(R^{X\otimes Y'},1)\cdot VR^{[Y,X\otimes Y']}\cdot Vp)1_{X\otimes Y'} = [d,1]R^{X\otimes Y'}.$$

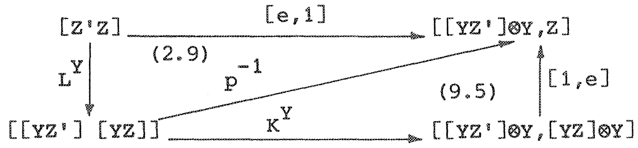
Together with (2.9) this proves the $\mathcal{V}$-naturality of d_{XY} in Y:

(10.9)



f) Proof of the $\mathcal{V}$-naturality of e_{YZ} in the variable Z:

(10.10)



Proof of the $\mathcal{V}$-naturality of e_{YZ} in the variable Y:
Replace in diagram (10.8) p by p^{-1} by reversing two arrows, apply V to the resulting diagram, take X = [Y'Z] and evaluate at $1_{[Y'Z]}$:

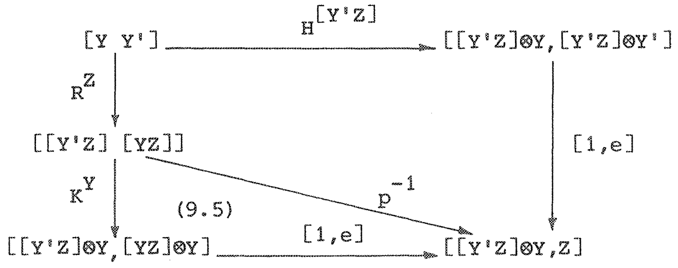
$$(\mathcal{V}_0(H^{[Y'Z]},1)\cdot VL^{[Y'Z]\otimes Y}\cdot\pi^{-1})1_{[Y'Z]} = [1,e]H^{[Y'Z]}$$

and

$$(\mathcal{V}_0(R^Z,p^{-1})\cdot VR^{[YZ]})1_{[Y'Z]} = p^{-1}\ R^Z.$$

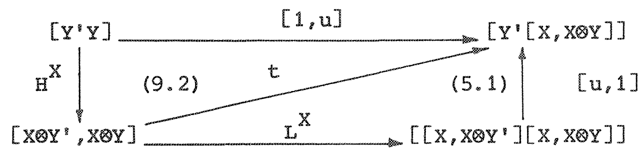
Together with (9.5) this proves the $\mathcal{V}$-naturality of e_{YZ} in the variable Y:

(10.11)



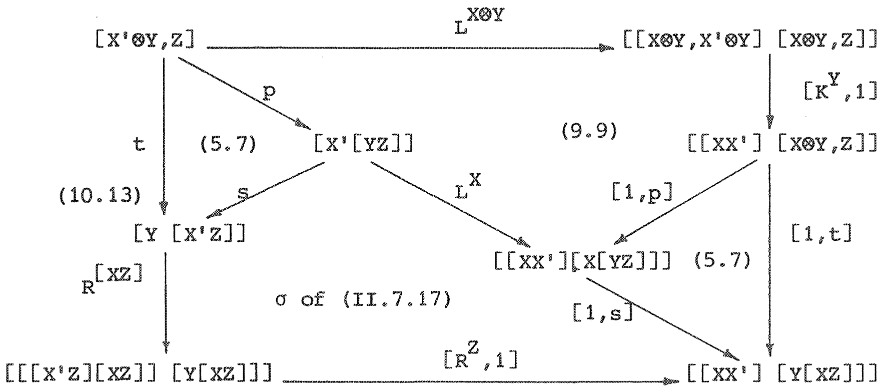
g) Proof of the V-naturality of u_{YX} in the variable Y:

(10.12)



Proof of the V-naturality of u_{YX} in the variable X: first we note that the following diagram commutes:

(10.13)



If we apply V to this diagram, take $Z = X'\otimes Y$ and evaluate at $1_{X'\otimes Y}$ we obtain:

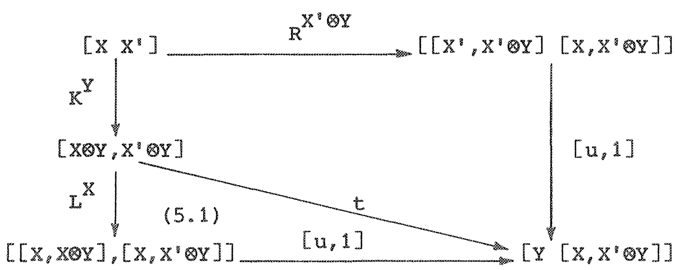
$$(V_0(K^Y,t)\cdot VL^{X\otimes Y})1_{X'\otimes Y} = t\cdot K^Y$$

and

$$(V_0(R^{X'\otimes Y},1)\cdot VR^{[X,X'\otimes Y]}\cdot\tau)1_{X'\otimes Y} = [u,1]R^{X'\otimes Y}.$$

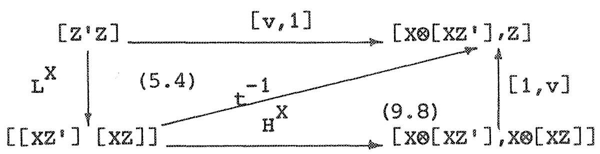
Consequently the following diagram commutes:

(10.14)



h) Proof of the $\mathcal{V}$-naturality of v_{XZ} in the variable Z:

(10.15)



Proof of the $\mathcal{V}$-naturality of v_{XZ} in the variable X: Replace in diagram (10.13) t by t^{-1} by reversing two arrows. Apply V to the resulting diagram, take Y = [X'Z] and evaluate at $1_{[X'Z]}$; then we obtain

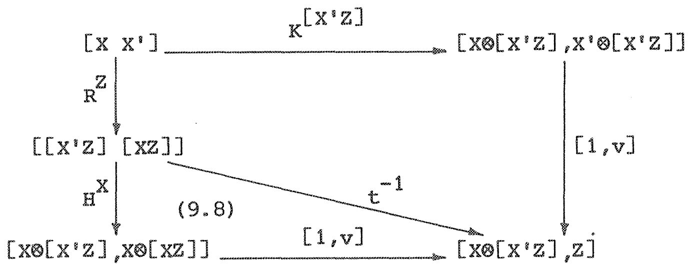
$$(\mathcal{V}_0(K^{[X'Z]},1)\circ VL^{X\otimes[X'Z]}\circ\tau^{-1})1_{[X'Z]} = [1,v]K^{[X'Z]}$$

and

$$(\mathcal{V}_0(R^Z,t^{-1})\circ VR^{[XZ]})1_{[X'Z]} = t^{-1}\,R^Z.$$

Consequently the following diagram commutes:

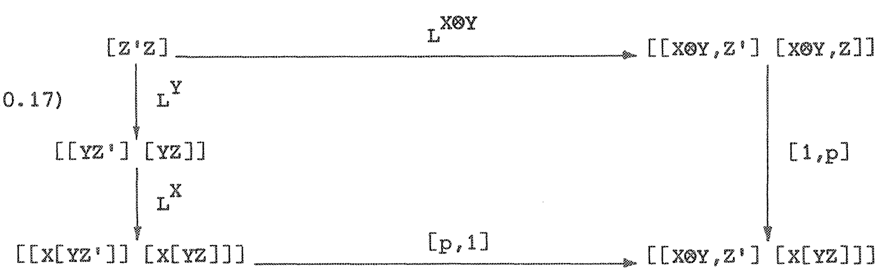
(10.16)



i) Proof of the $\mathcal{V}$-naturality of p:

MCC3' implies the $\mathcal{V}$-naturality of p_{XYZ} in the variable Z:

(10.17)



If we apply σ to diagram (10.8) we obtain the following diagram:

$$
(10.18)\qquad
\begin{array}{ccc}
[Y\ Y'] & \xrightarrow{\quad H^X \quad} & [X\otimes Y, X\otimes Y'] \\
\downarrow{\scriptstyle R^Z} & & \downarrow{\scriptstyle R^Z} \\
[[Y'Z]\ [YZ]] & & [[X\otimes Y', Z]\ [X\otimes Y, Z]] \\
\downarrow{\scriptstyle L^X} & & \downarrow{\scriptstyle [1,p]} \\
[[X[Y'Z]]\ [X[YZ]]] & \xrightarrow{\quad [p,1] \quad} & [[X\otimes Y', Z]\ [X[YZ]]]
\end{array}
$$

If we apply σ to diagram (9.13) (replace p^{-1} by p) we obtain:

$$
(10.19)\qquad
\begin{array}{ccc}
[X\ X'] & \xrightarrow{\quad K^Y \quad} & [X\otimes Y, X'\otimes Y] \\
 & & \downarrow{\scriptstyle R^Z} \\
\downarrow{\scriptstyle R^{[YZ]}} & & [[X'\otimes Y, Z]\ [X\otimes Y, Z]] \\
 & & \downarrow{\scriptstyle [1,p]} \\
[[X'[YZ]]\ [X[YZ]]] & \xrightarrow{\quad [p,1] \quad} & [[X'\otimes Y, Z]\ [X[YZ]]]
\end{array}
$$

j) Proof of the V-naturality of p^{-1}:

If we apply σ to diagram (9.13) (replace p by p^{-1}) we obtain:

$$
(10.20)\qquad
\begin{array}{ccc}
[X\ X'] & \xrightarrow{\quad R^{[YZ]} \quad} & [[X'[YZ]]\ [X[YZ]]] \\
\downarrow{\scriptstyle K^Y} & & \\
[X\otimes Y, X'\otimes Y] & & \downarrow{\scriptstyle [1,p^{-1}]} \\
\downarrow{\scriptstyle R^Z} & & \\
[[X'\otimes Y, Z]\ [X\otimes Y, Z]] & \xrightarrow{\quad [p^{-1},1] \quad} & [[X'[YZ]]\ [X\otimes Y, Z]]
\end{array}
$$

If we apply σ to diagram MSCC1' (replace p by p^{-1} twice) we obtain:

$$
(10.21)\qquad
\begin{array}{ccc}
[Z'Z] & \xrightarrow{\quad L^Y \quad} & [[YZ']\ [YZ]] \\
 & & \downarrow{\scriptstyle L^X} \\
\downarrow{\scriptstyle L^{X\otimes Y}} & & [[X[YZ']]\ [X[YZ]]] \\
 & & \downarrow{\scriptstyle [1,p^{-1}]} \\
[[X\otimes Y, Z']\ [X\otimes Y, Z]] & \xrightarrow{\quad [p^{-1},1] \quad} & [[X[YZ']]\ [X\otimes Y, Z]]
\end{array}
$$

If we apply σ to diagram (10.8) (replace p by p^{-1} twice) we obtain:

$$(10.22)\qquad \begin{array}{ccc}
[Y\ Y'] & \xrightarrow{\quad R^{Z}\quad} & [[Y'Z]\ [YZ]] \\
\downarrow{\scriptstyle H^{X}} & & \downarrow{\scriptstyle L^{X}} \\
[X\otimes Y, X\otimes Y'] & & [[X[Y'Z]]\ [X[YZ]]] \\
\downarrow{\scriptstyle R^{Z}} & & \downarrow{\scriptstyle [1,p^{-1}]} \\
[[X\otimes Y',Z]\ [X\otimes Y,Z]] & \xrightarrow{\quad [p^{-1},1]\quad} & [[X[Y'Z]]\ [X\otimes Y,Z]]
\end{array}$$

k) Proof of the $\mathcal{V}$-naturality of t:

By proposition 5.5 the following diagram commutes:

$$(10.23)\qquad \begin{array}{ccc}
[Z'Z] & \xrightarrow{\quad L^{X\otimes Y}\quad} & [[X\otimes Y,Z']\ [X\otimes Y,Z]] \\
\downarrow{\scriptstyle L^{X}} & & \\
[[XZ']\ [XZ]] & & \downarrow{\scriptstyle [1,t]} \\
\downarrow{\scriptstyle L^{Y}} & & \\
[[Y[XZ']]\ [Y[XZ]]] & \xrightarrow{\quad [t,1]\quad} & [[X\otimes Y,Z']\ [Y[XZ]]]
\end{array}$$

If we apply σ to diagram (10.13) we obtain:

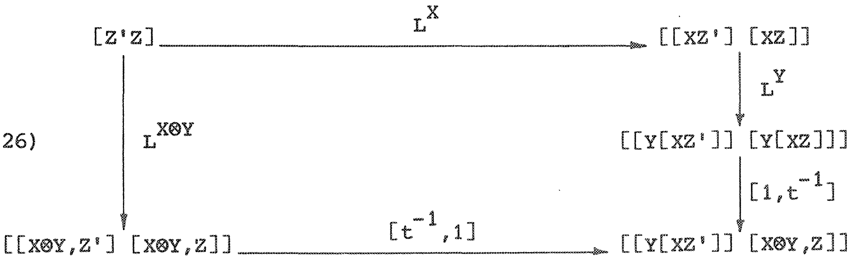
$$(10.24)\qquad \begin{array}{ccc}
[X\ X'] & \xrightarrow{\quad K^{Y}\quad} & [X\otimes Y, X'\otimes Y] \\
\downarrow{\scriptstyle R^{Z}} & & \downarrow{\scriptstyle R^{Z}} \\
[[X'Z]\ [XZ]] & & [[X'\otimes Y,Z]\ [X\otimes Y,Z]] \\
\downarrow{\scriptstyle L^{Y}} & & \downarrow{\scriptstyle [1,t]} \\
[[Y[X'Z]]\ [Y[XZ]]] & \xrightarrow{\quad [t,1]\quad} & [[X'\otimes Y,Z]\ [Y[XZ]]]
\end{array}$$

If we apply σ to diagram (9.14) we obtain:

$$(10.25)\qquad \begin{array}{ccc}
[Y\ Y'] & \xrightarrow{\quad H^{X}\quad} & [X\otimes Y, X\otimes Y'] \\
 & & \downarrow{\scriptstyle R^{Z}} \\
\downarrow{\scriptstyle R^{[XZ]}} & & [[X\otimes Y',Z]\ [X\otimes Y,Z]] \\
 & & \downarrow{\scriptstyle [1,t]} \\
[[Y'[XZ]]\ [Y[XZ]]] & \xrightarrow{\quad [t,1]\quad} & [[X\otimes Y',Z]\ [Y[XZ]]]
\end{array}$$

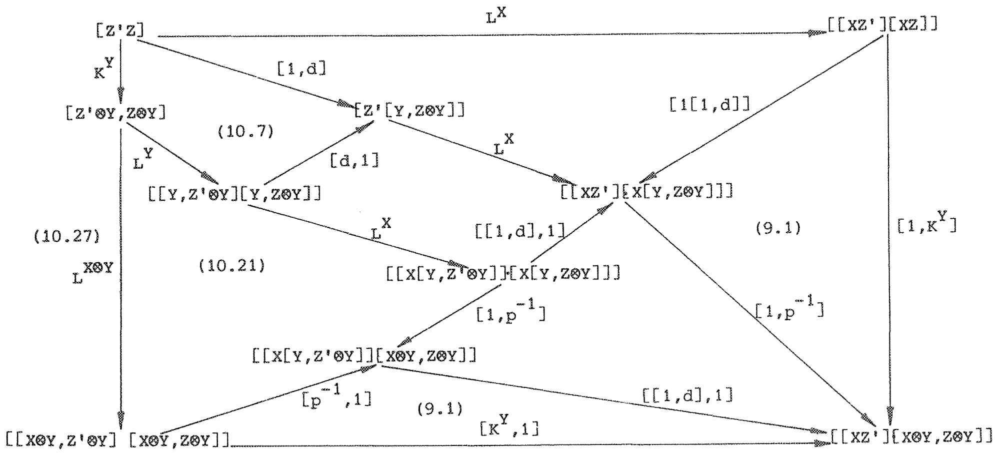
l) The $\mathcal{V}$-naturality of t^{-1} is expressed by the images under σ of the diagram (10.13), (9.14) and (5.9) (replace t by t^{-1}). The $\mathcal{V}$-naturality of t^{-1} in the variable Z is expressed by the following diagram:

(10.26)



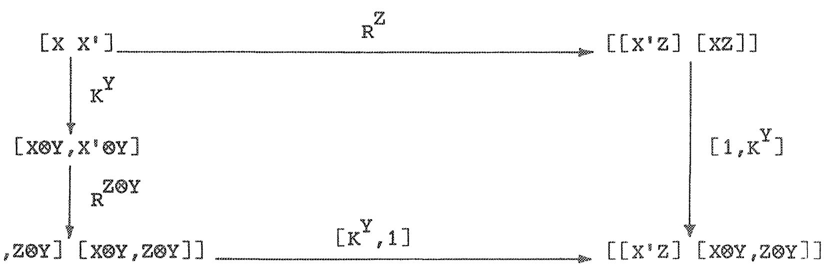
m) Proof of the $\mathcal{V}$-naturality of K.

The following diagram expresses the $\mathcal{V}$-naturality of K^Z_{XY} in the variable Z:

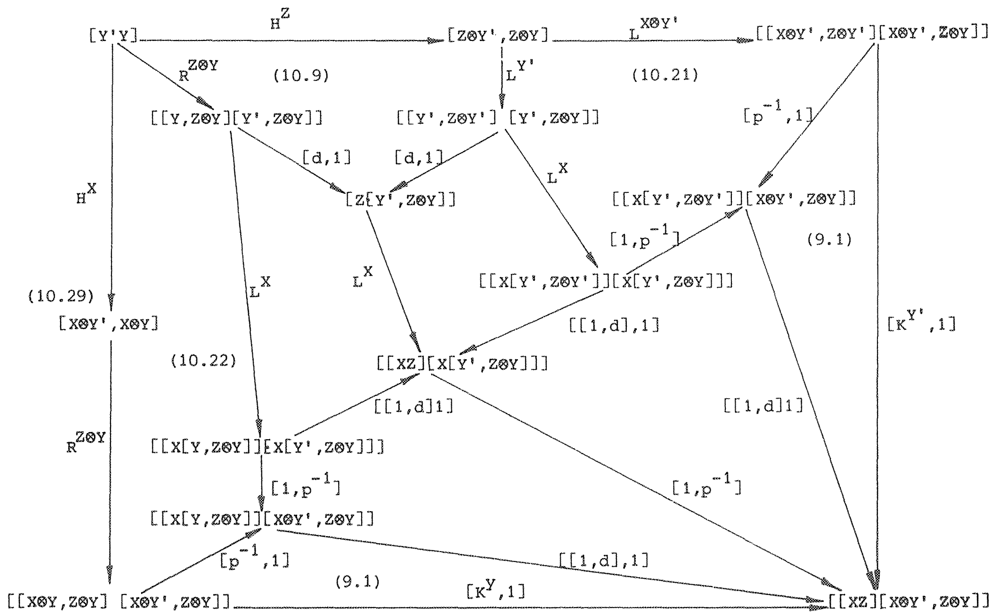


If we apply σ to (10.27) we obtain the following diagram (replace Z' by X'):

(10.28)

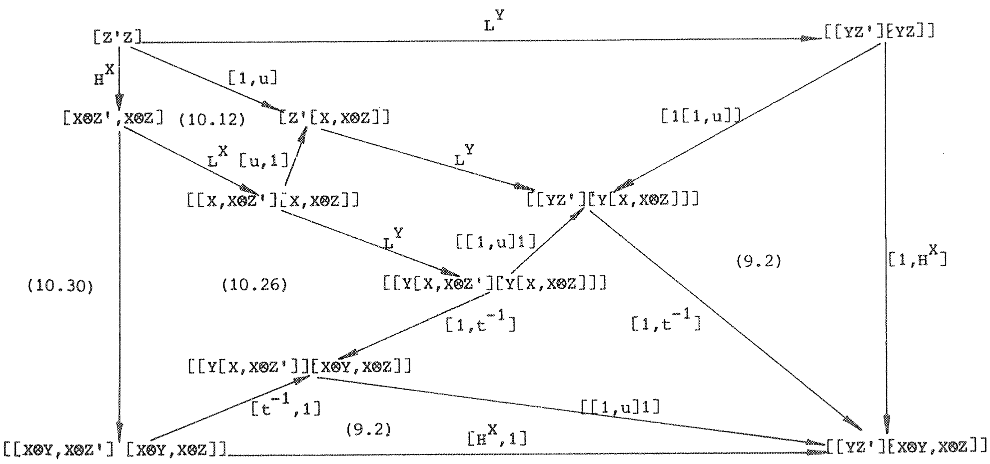


The following diagram expresses the V-naturality of K^{Y}_{XZ} in the variable Y:



n) Proof of the V-naturality of H.

The following diagram expresses the V-naturality of H^{X}_{YZ} in the variable Z:



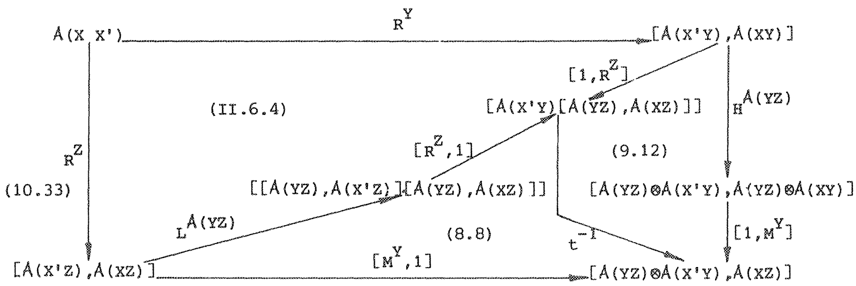
If we apply σ to (10.30) we obtain the following diagram (replace Z' by Y'):

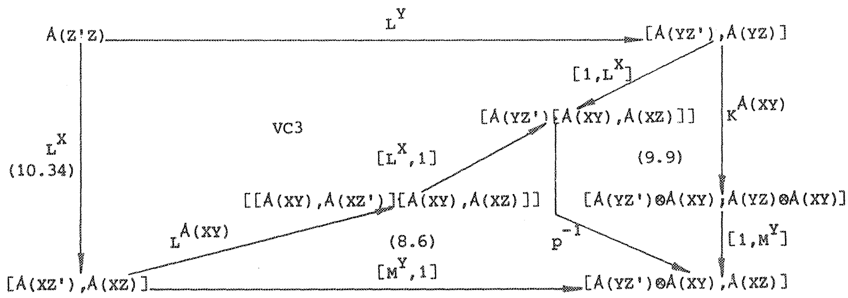
$$
\begin{array}{lcl}
[Y\ Y'] & \xrightarrow{\ R^{Z}\ } & [[Y'Z]\ [YZ]] \\
\downarrow H^{X} & & \\
(10.31)\ [X\otimes Y, X\otimes Y'] & & \downarrow [1,H^{X}] \\
\downarrow R^{X\otimes Z} & & \\
[[X\otimes Y', X\otimes Z]\ [X\otimes Y, X\otimes Z]] & \xrightarrow{\ [H^{X},1]\ } & [[Y'Z]\ [X\otimes Y, X\otimes Z]]
\end{array}
$$

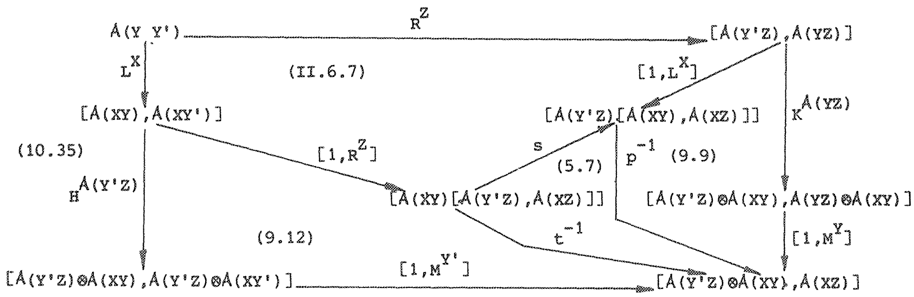
If we apply σ to diagram (10.29) we obtain the following diagram:
(replace Y by Z, Y' by Y and Z by X'):

$$
\begin{array}{lcl}
[X\ X'] & \xrightarrow{\ K^{Z}\ } & [X\otimes Z, X'\otimes Z] \\
\downarrow K^{Y} & & \downarrow L^{X\otimes Y} \\
(10.32)\ [X\otimes Y, X'\otimes Y] & & [[X\otimes Y, X\otimes Z]\ [X\otimes Y, X'\otimes Z]] \\
\downarrow R^{X'\otimes Z} & & \downarrow [H^{X},1] \\
[[X'\otimes Y, X'\otimes Z]\ [X\otimes Y, X'\otimes Z]] & \xrightarrow{\ [H^{X'},1]\ } & [[YZ]\ [X\otimes Y, X'\otimes Z]]
\end{array}
$$

0) Proof of the $\mathcal{V}$-naturality of M:

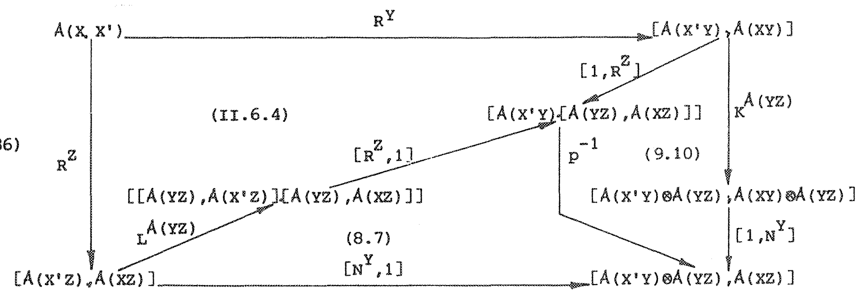




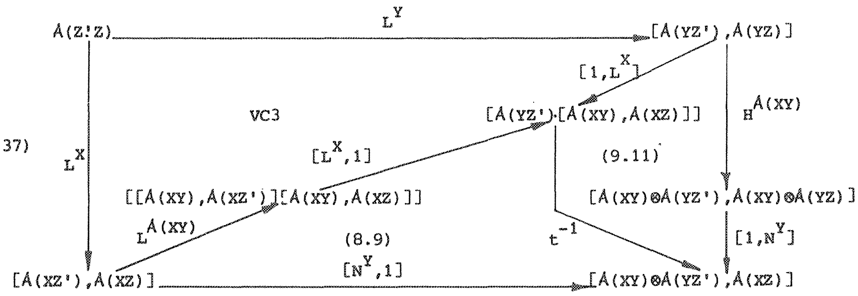


p) Proof of the $\mathcal{V}$-naturality of N:

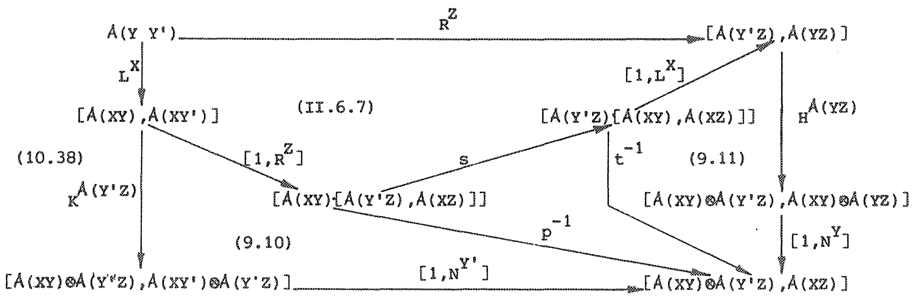
(10.36)



(10.37)



(10.38)



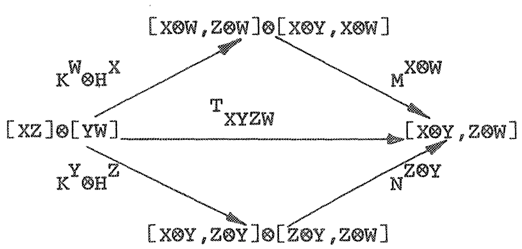
11. THE NATURAL TRANSFORMATION T_{XYZW}: $[XZ]\otimes[YW] \longrightarrow [X\otimes Y, Z\otimes W]$.

11.1. DEFINITION. Let V be a monoidal symmetric closed category. A natural transformation

$$T = T_{XYZW}: [XZ]\otimes[YW] \rightarrow [X\otimes Y, Z\otimes W]$$

is defined by the following commutative diagram:

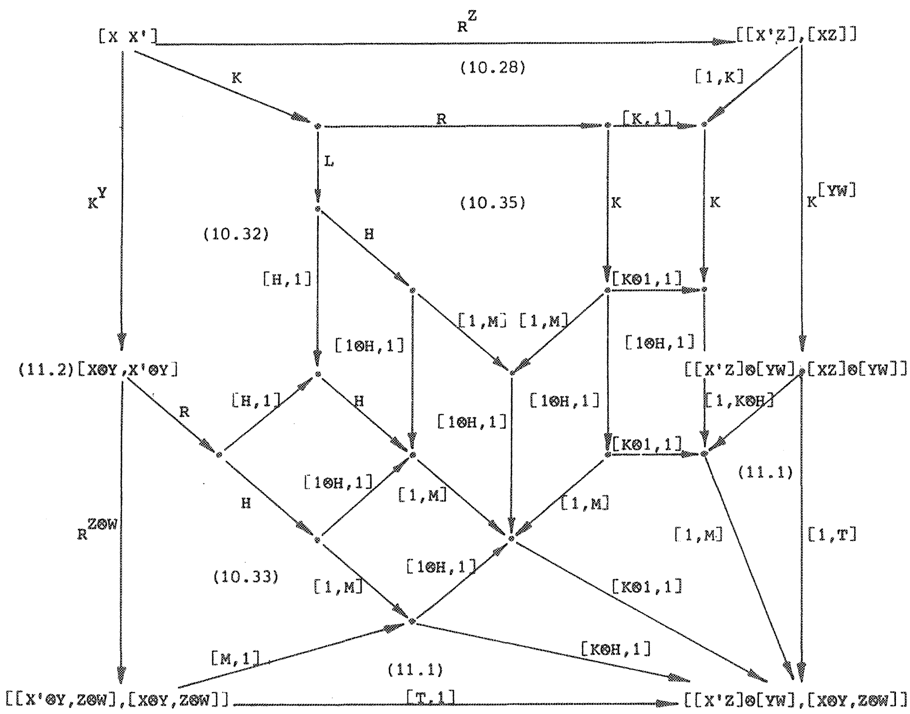
(11.1)



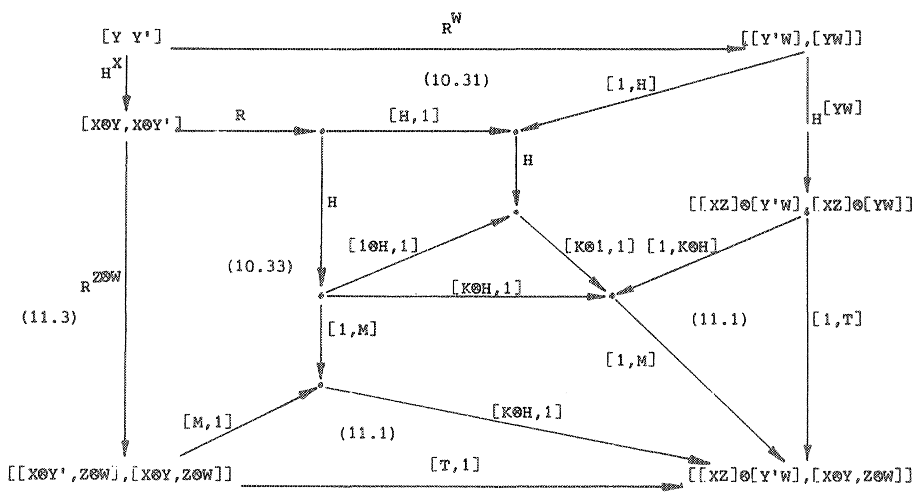
(Note that the outer square commutes since it is the image under π^{-1} of diagram (10.32)).

11.2. PROPOSITION. (The V-naturality of T).
The morphisms T_{XYZW} are V-natural in every variable.

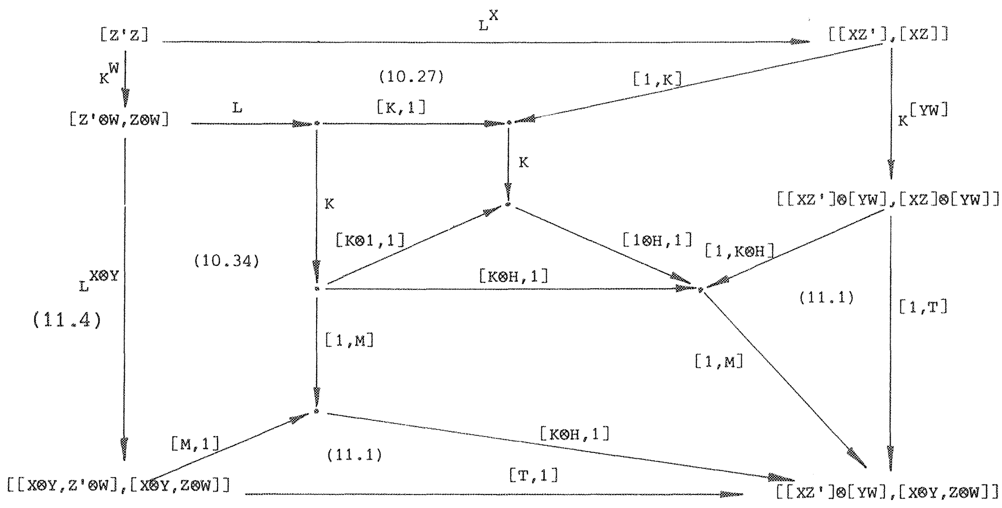
PROOF.

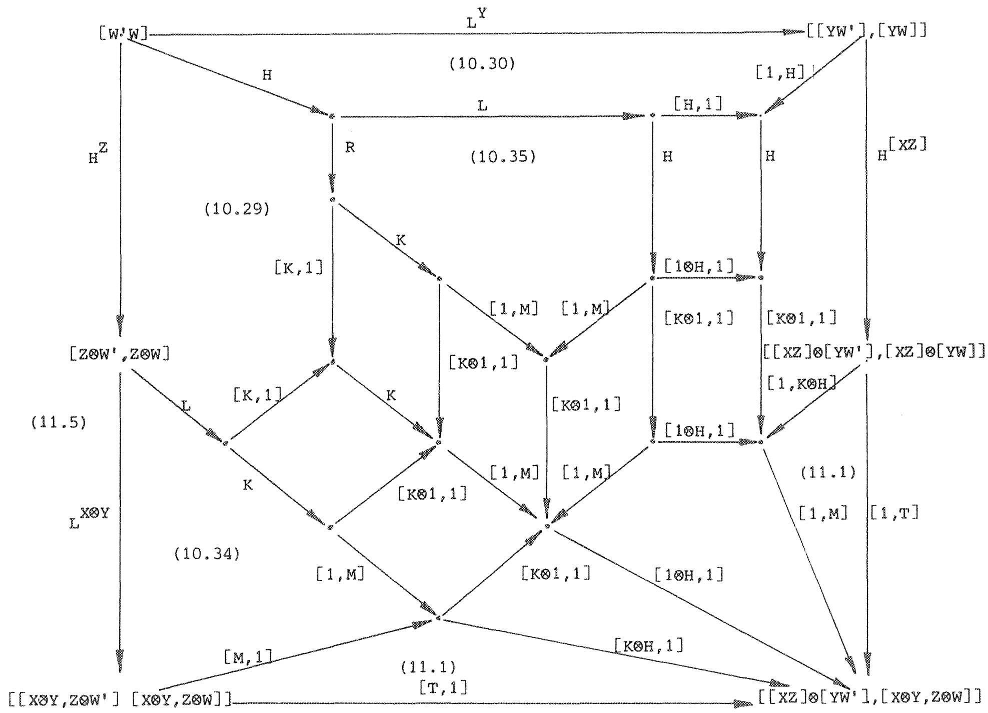


[X X']
R^Z
[[X'Z],[XZ]]
(10.28)
K
[1,K]
R
[K,1]
L
K^Y
(10.35)
K
K
K^[YW]
(10.32)
H
[H,1]
[K⊗1,1]
[1,M] [1,M]
[1⊗H,1]
[1⊗H,1]
(11.2)[X⊗Y,X'⊗Y]
[[X'Z]⊗[YW],[XZ]⊗[YW]]
R
[H,1]
H
[1⊗H,1]
[1⊗H,1]
[1,K⊗H]
[K⊗1,1]
(11.1)
[1⊗H,1]
H
[1,M]
[1,M]
R^Z⊗W
[1,M]
[1,T]
[1⊗H,1]
(10.33)
[1,M]
[K⊗1,1]
[M,1]
[K⊗H,1]
(11.1)
[[X'⊗Y,Z⊗W],[X⊗Y,Z⊗W]]
[T,1]
[[X'Z]⊗[YW],[X⊗Y,Z⊗W]]



[Y Y']
R^W
[[Y'W],[YW]]
H^X
(10.31)
[1,H]
[X⊗Y,X⊗Y']
R
[H,1]
H^[YW]
H
H
[[XZ]⊗[Y'W],[XZ]⊗[YW]]
[1⊗H,1]
[K⊗1,1] [1,K⊗H]
(10.33)
R^Z⊗W
[K⊗H,1]
(11.3)
(11.1)
[1,T]
[1,M]
[1,M]
[M,1]
[K⊗H,1]
(11.1)
[T,1]
[[X⊗Y',Z⊗W],[X⊗Y,Z⊗W]]
[[XZ]⊗[Y'W],[X⊗Y,Z⊗W]]


[Z'Z]
L^X
[[XZ'],[XZ]]
K^W
(10.27)
[1,K]
[Z'⊗W,Z⊗W]
L
[K,1]
K^[YW]
K
K
[[XZ']⊗[YW],[XZ]⊗[YW]]
[K⊗1,1]
[1⊗H,1]
[1,K⊗H]
(10.34)
L^X⊗Y
[K⊗H,1]
(11.1)
[1,T]
(11.4)
[1,M]
[1,M]
[K⊗H,1]
[M,1]
(11.1)
[T,1]
[[X⊗Y,Z'⊗W],[X⊗Y,Z⊗W]]
[[XZ']⊗[YW],[X⊗Y,Z⊗W]]


[W'W]
L^Y
[[YW'],[YW]]
(10.30)
H
[1,H]
L
[H,1]
H^Z
R
(10.35)
H
H
H^[XZ]
(10.29)
K
[K,1]
[1⊗H,1]
[1,M]
[1,M]
[K⊗1,1]
[K⊗1,1]
[K⊗1,1]
[Z⊗W',Z⊗W]
[[XZ]⊗[YW'],[XZ]⊗[YW]]
[1,K⊗H]
[K,1]
K
[K⊗1,1]
L
(11.5)
[1⊗H,1]
[1,M]
[1,M]
(11.1)
K
[K⊗1,1]
L^X⊗Y
[1,M]
[1,T]
(10.34)
[1,M]
[K⊗1,1]
[1⊗H,1]
[K⊗H,1]
[M,1]
(11.1)
[T,1]
[[X⊗Y,Z⊗W'] [X⊗Y,Z⊗W]]
[[XZ]⊗[YW'],[X⊗Y,Z⊗W]]

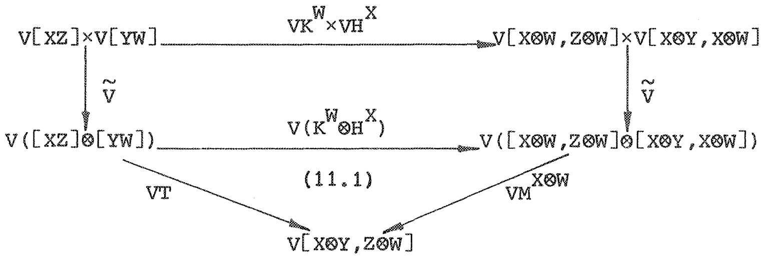
11.3. THEOREM. Let $V = \langle {}^{m}V, {}^{sc}V, p \rangle$ be a monoidal symmetric closed category. Define $\tilde{V} = \tilde{V}_{XY}$: $VX \times VY \longrightarrow V(X \otimes Y)$ by (4.13) and define

$$T = T_{XYZW}: [XZ] \otimes [YW] \longrightarrow [X \otimes Y, Z \otimes W] \quad \text{by (11.1).}$$

Let ${}^{sm}V = \langle {}^{m}V, c \rangle$.
Then ${}^{pmc}V = \langle {}^{sm}V, {}^{sc}V, \tilde{V}, T \rangle$ is a symmetric semi monoidal closed category.

PROOF. We have to prove the properties PMCC0, PMCC1, PMCC2, PMCC3, PMCC7 and PMCC8.

Proof of PMCC0.
Consider the following diagram



This diagram is commutative. Evaluate at $\langle g,h \rangle \in V[XZ] \times V[YW]$:

$$VT \circ \tilde{V} \langle g,h \rangle =$$

$$VM^{X \otimes W} \circ \tilde{V} \circ VK^{W} \times VH^{X} \langle g,h \rangle =$$

$$VM^{X \otimes W} \circ \tilde{V} \langle VK^{W} g, VH^{X} h \rangle =$$

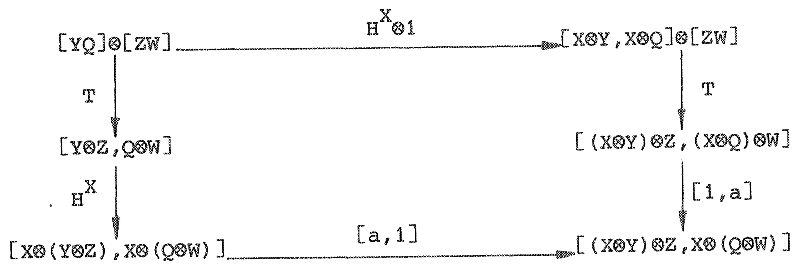
$$VM^{X \otimes W} \circ \tilde{V} \langle g \otimes 1, 1 \otimes h \rangle = \qquad \text{by proposition 9.7}$$

$$(g \otimes 1) \circ (1 \otimes h) = g \otimes h \qquad \text{by proposition 8.8.}$$

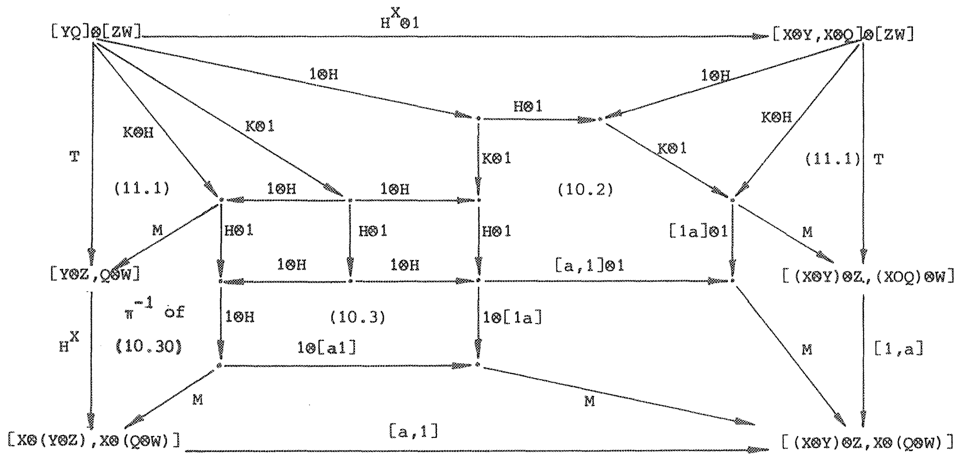
This completes the proof of property PMCC0.

Proof of PMCC1. First we prove the commutativity of the following diagram:

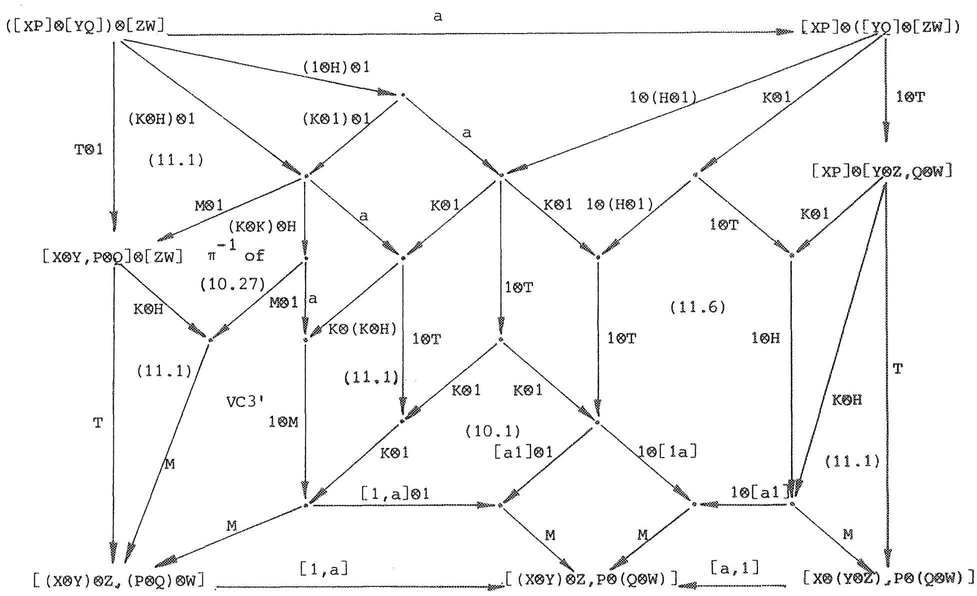
(11.6)



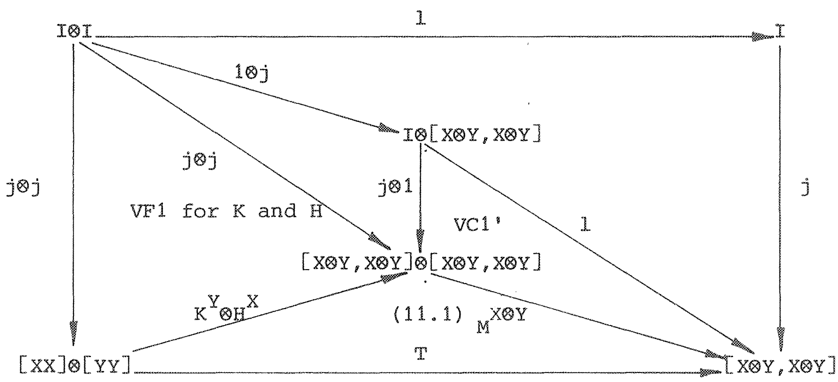
Proof of the commutativity of this diagram:



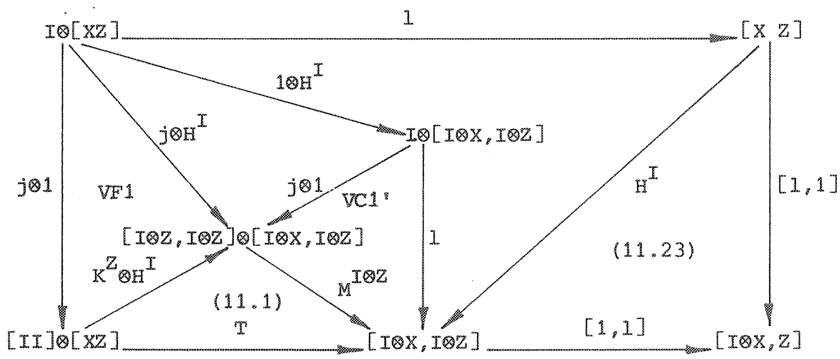
Now we are able to prove PMCC1:



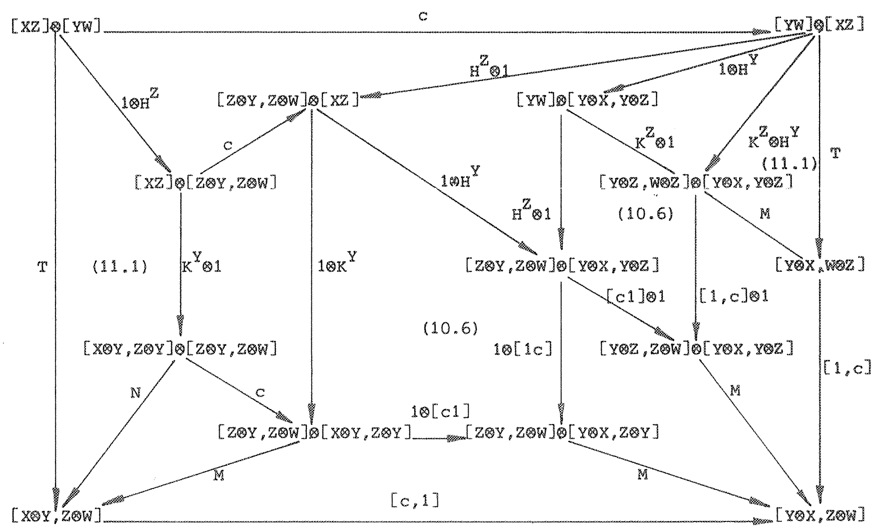
Proof of PMCC2:



Proof of PMCC3:

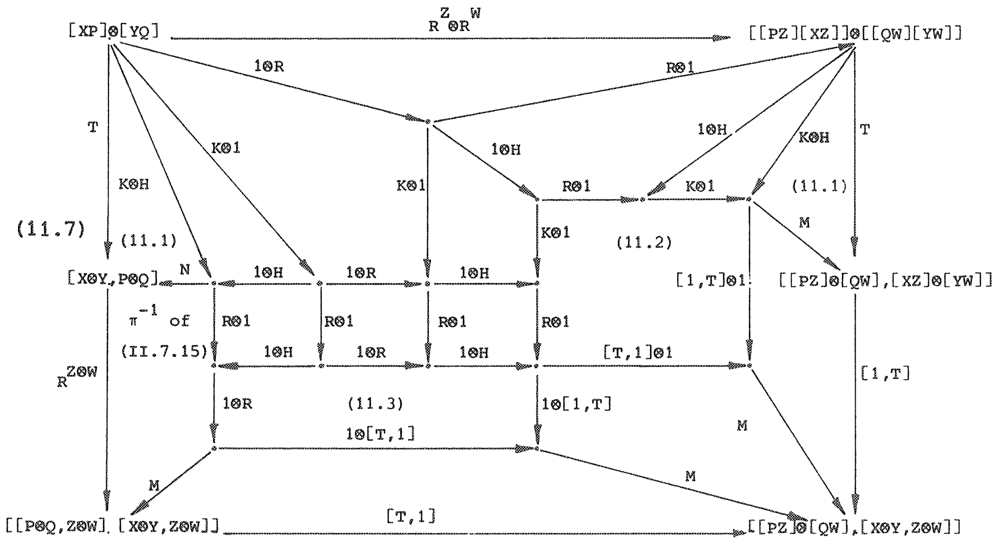


Proof of PMCC7:

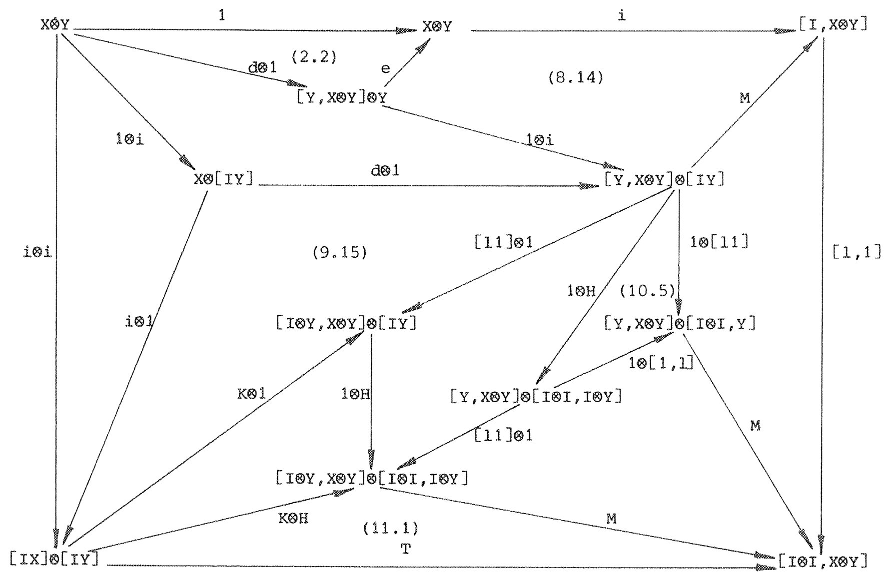


Proof of PMCC8: In order to prove PMCC8 we first prove (in this order) III.2.2, PMCC5 and III.2.1.

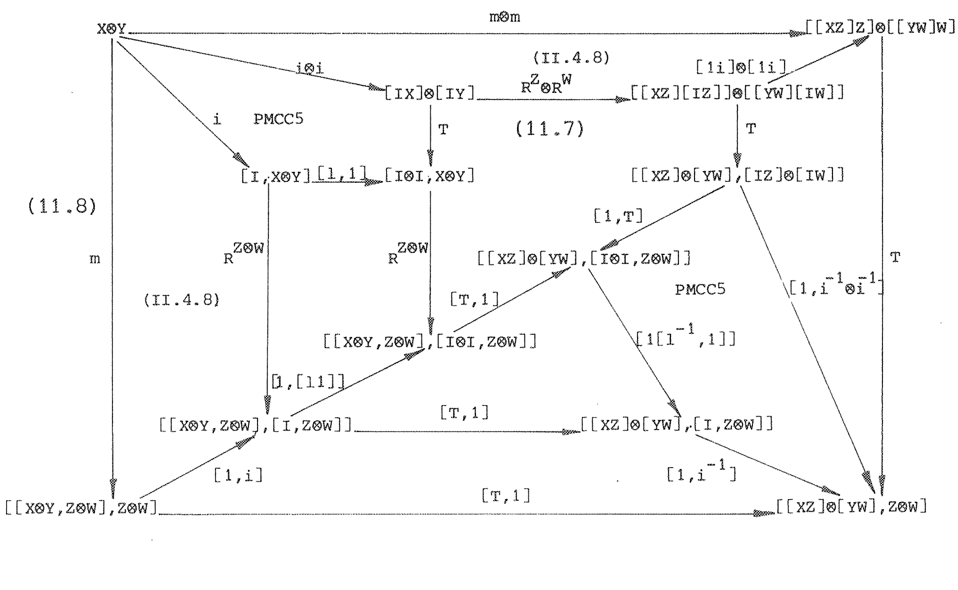
Proof of III.2.2:



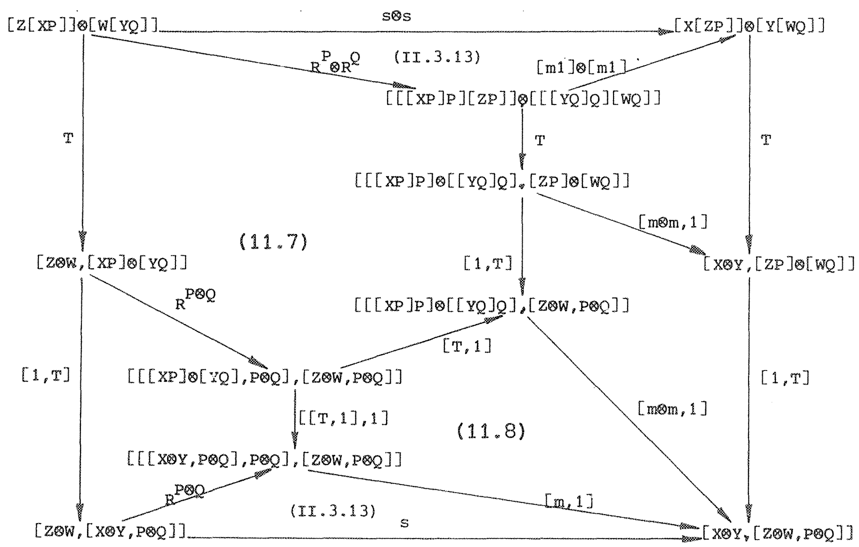
Proof of PMCC5:



Proof of III.2.1:

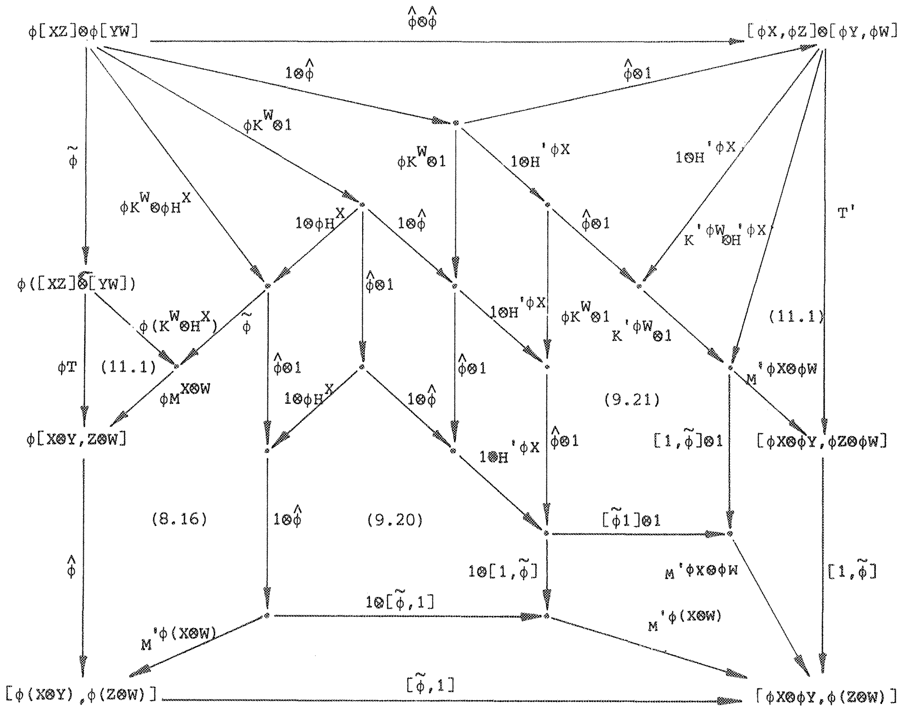


Proof of PMCC8:



11.4. THEOREM. Let $\Phi = \langle \phi, \tilde{\phi}, \hat{\phi}, \phi^0 \rangle : \mathcal{V} \longrightarrow \mathcal{V}'$ be a monoidal symmetric closed functor. Then property (PMCF) holds. Hence Φ is a symmetric semi monoidal closed functor.

PROOF.



□

In chapter III, section 3 we have defined tensor products of V-categories where V is a semi monoidal closed category. EILENBERG and KELLY ([6], section III.2) have defined tensor products of V-categories in the case that V is a symmetric monoidal category. If V is a monoidal symmetric closed category then the ${}^{m}V$-categories and the ${}^{c}V$-categories are essentially the same (theorem 8.4). The following proposition says that the tensor product of ${}^{c}V$-categories is essentially the same as the tensor product of ${}^{m}V$-categories.

11.5. PROPOSITION. Let V be a monoidal symmetric closed category, and let A and B be V-categories. Then the composite

$$A(X'X'')\otimes B(Y'Y'') \xrightarrow{L^X\otimes L^Y} [A(XX'),A(XX'')]\otimes[B(YY'),B(YY'')]$$

$$\xrightarrow{T} [A(XX')\otimes B(YY'),A(XX'')\otimes B(YY'')]$$

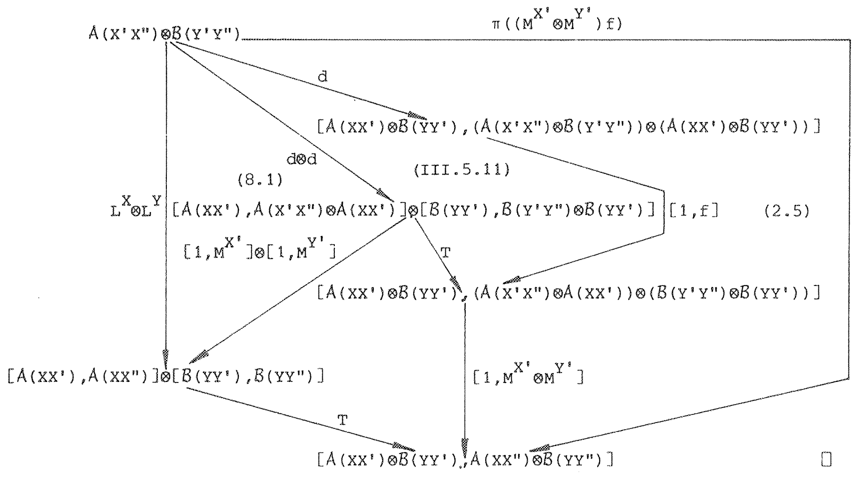
is the image under π of the composite

$$(A(X'X'')\otimes B(Y'Y''))\otimes(A(XX')\otimes B(YY')) \xrightarrow{f}$$

$$(A(X'X'')\otimes A(XX'))\otimes(B(Y'Y'')\otimes B(YY')) \xrightarrow{M^{X'}\otimes M^{Y'}} A(XX'')\otimes B(YY'').$$

Consequently, combination of proposition III.3.1, theorem 8.4 and [6], chapter III, proposition 3.1 implies that if we identify the ${}^{m}V$-categories and the ${}^{c}V$-categories as in theorem 8.4, the two definitions of tensor product of V-categories A and B lead to the same V-category $A\otimes B$.

PROOF.



□

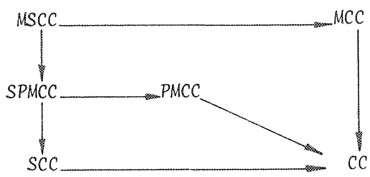
CHAPTER V

EXAMPLES

In this chapter we present a list of examples to illustrate the preceding theory. The aim of this chapter is twofold: on the one hand it will indicate the frequency with which the several structures appear in various parts of mathematics, on the other hand the examples were also chosen to point out the differences between the structures. We use the following abbreviations:

Closed category	:	CC;
Symmetric closed category	:	SCC;
Semi monoidal closed category	:	PMCC;
Symmetric semi monoidal closed category	:	SPMCC;
Monoidal closed category	:	MCC;
Monoidal symmetric closed category	:	MSCC.

As appears from the chapters II, III and IV we have the following scheme of forgetful functors:



Consequently all the examples of MSCC's in the literature are examples of all the structures considered in this paper. Many such examples are to be found in [6], chapter IV. Also, [4] contains a list of examples.

1. Examples of Cartesian Closed Categories

Any category V_0 that admits a categorical product functor × (i.e. a right adjoint to the diagonal functor Δ: $V_0 \longrightarrow V_0 \times V_0$) and a terminal object I (i.e. a right adjoint to the obvious functor $V_0 \longrightarrow I$) admits a structure of a symmetric monoidal category V in which X⊗Y is taken to be X×Y. The existence of the natural isomorphisms a,r,l and c is a consequence of the universal properties of the categorical product and the terminal object. Such a symmetric monoidal category is called cartesian. A cartesian closed category is a symmetric monoidal closed category which is cartesian. Examples of cartesian closed categories are:

1.1. The category of sets.

1.2. The category of small categories.

1.3. The category of compactly generated Hausdorff spaces ([24],[17]).

1.4. The category of k-spaces (in the sense of A. Clark) ([5]).

1.5. The category of quasi-topological spaces ([23]).

1.6. The category of limit-spaces ([1]).

1.7. The category of equivalence relations.

1.8. The category of partial orderings.

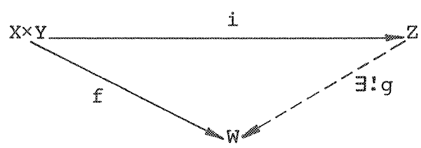
2. TOPOLOGICAL EXAMPLES

2.1. *The category of topological spaces with the point open topology* ([12],[13]). The category of topological spaces *TOP*, becomes an SCC, denoted by *TOP*(*pot*) if we take [XY] to be the space of continuous functions from X into Y with the point open topology. (The other data are defined in an obvious way, see proposition II.2.5; the same holds for the other examples in section 2). The category of T1-spaces, *T1*, the category of Hausdorff spaces, *T2*, and the category of Tychonoff spaces, *TYCH*, are symmetric closed subcategories of *TOP*(*pot*).

If we take X⊗Y to be the topological product X × Y of X and Y we obtain an SPMCC *TOP*(*pot*,×) which is not an MSCC. The mappings T_{XYZW}: [XZ]×[YW] → [X×Y,Z×W] are defined by $T\langle f,g\rangle\langle x,y\rangle = \langle fx,gy\rangle$, and the mappings p_{XYZ}: [X×Y,Z] → [X[YZ]] are given by $((pf)x)y = f\langle x,y\rangle$. It is easy to see that p is not a natural isomorphism, so that *TOP*(*pot*,×) is not an MSCC.

If we take X⊗Y to be the cartesian product of X and Y, with the topology of separate continuity we obtain an MSCC *TOP*(*pot*,⊗). The topology of separate continuity is the finest topology on the set X × Y such that all the mappings $y \mapsto \langle x,y\rangle$ ($x\in X$) and $x \mapsto \langle x,y\rangle$ ($y\in Y$) are continuous. A local base at the point $\langle x,y\rangle$ is given by the collection of all sets of the form $(\{x\}\times V)\cup(U\times\{y\})$ where U runs over all neighborhoods of x in X and V runs over all neighborhoods of y in Y. In the category of topological spaces one can define a *tensor product* of X and Y to be an ordered pair $\langle Z,i\rangle$

consisting of a topological space Z and a separately continuous mapping i: X × Y ⟶ Z with the following universal property: for each topological space W and each separately continuous mapping f: X × Y ⟶ W there exists a unique continuous mapping g: Z ⟶ W such that f = g∘i:



It is not hard to prove that the pair ⟨X⊗Y,i⟩, where i: X × Y ⟶ X⊗Y is the identical mapping, is a tensor product of X and Y in the above sense ([12], theorem 1.2). The universal property makes it easy to prove that p_{XYZ}: [X⊗Y,Z] ⟶ [X[YZ]], defined by ((pf)x)y = f ⟨x,y⟩, is a natural isomorphism, and that *TOP(pot,⊗)* is an MSCC ([12], theorem 1.5; see also [3], remark 1.15). *T1(pot,⊗)* and *T2(pot,⊗)* are monoidal symmetric closed subcategories of *TOP(pot,⊗)*. The tensor product (in *TOP*) of two Tychonoff spaces need not be a Tychonoff space ([12], corollary 3.4) but if $X\tilde{\otimes}Y$ denotes the complete regularization of X⊗Y then $X\tilde{\otimes}Y$ is a tensor product of X and Y in the category of Tychonoff spaces ([13], theorem 5.4) and using this, it is easy to prove that *TYCH(pot,*$\tilde{\otimes}$*)* is an MSCC ([13], theorem 5.6).

2.2. *The category of Hausdorff spaces with the compact open topology* (cf. [2],[3]). The category of Hausdorff spaces *T2*, becomes a CC, denoted by *T2(cot)*, if we take [XY] to be the space of continuous functions from X into Y with the compact open topology. The fact that L^X_{YZ}: [YZ] ⟶ [[XY][XZ]], defined by (Lh)g = hg, is continuous is a consequence of the fact that the evaluation e_{YZ}: [YZ] × Y ⟶ Z, defined by e ⟨g,y⟩ = gy is continuous on sets of the form [YZ] × K, where K is a compact subset of Y ([3], lemma 1.3). Since V: *T2* ⟶ *S* is faithful, it is sufficient to verify the axioms CC0, CC1 and CC5 ([6], proposition I.2.10), which is an easy exercise. Since there exists no natural transformation s_{XYZ}: [X[YZ]] ⟶ [Y[XZ]] satisfying the axioms SCC1 - SCC4, ([3], corollary 1.12), *T2(cot)* is not an SCC.

If we take X⊗Y to be the topological product X × Y of X and Y, we obtain a PMCC *T2(cot,×)* which is not an SPMCC since the closed part is not symmetric. Since the natural transformation p_{XYZ}: [X×Y,Z] ⟶ [X[YZ]] is not an isomorphism ([3], corollary 1.8 and theorem 1.11), *T2(cot×)* is not an MCC.

If we take X⊗Y to be the S-product $X \times_S Y$ of X and Y we obtain an MCC $T2(cot,\times_S)$. The S-product is defined in [2]: $X \times_S Y$ is the cartesian product of X and Y, supplied with the weak topology generated by the subsets $X \times B$ and $\{x\} \times Y$, where B ranges over the compact subsets of Y and x runs through the elements of X, and where $X \times B$ and $\{x\} \times Y$ have the usual product topology. A function $f: X \times_S Y \to Z$ (Z Hausdorff) is continuous if and only if all the restrictions $f \mid X \times B: X \times B \to Z$ and $f \mid \{x\} \times Y: \{x\} \times Y \to Z$ are continuous (for all compact subsets B of Y and all elements x of X). [3], theorem 1.6 implies that $T2(cot,\times_S)$ is an MCC. Since the closed part is not symmetric, it is not an MSCC. $T2(cot,\times_S)$ is not a PMCC. This can be seen in the following way. Since p is a natural isomorphism, the evaluation $e_{YZ} = p^{-1}_{[YZ]YZ}(1): [YZ] \times_S Y \to Z$ is a continuous mapping. The assumption that $T2(cot,\times_S)$ is a PMCC implies the existence of a natural transformation

$$X \times_S Y \xrightarrow{u \times_S 1} [Y, Y \times_S X] \times_S Y \xrightarrow{e} Y \times_S X$$

where u is the natural transformation of section III.5, and it is not hard to show that this composite is the mapping $\langle x,y \rangle \to \langle y,x \rangle$. But this is a contradiction to the fact that the S-product is not commutative ([2], section 5).

If we take X⊗Y to be the weak product $X \times_W Y$ of X and Y we obtain a symmetric monoidal category $T2(\times_W)$, but $T2(cot,\times_W)$ is not an MCC and not a PMCC. The weak product $X \times_W Y$ is the cartesian product of X and Y, supplied with the weak topology, generated by the subsets $A \times B$, where A and B range over the compact subsets of X and Y, respectively, and where $A \times B$ has the usual product topology. The continuous identity mappings $X \times_W Y \to X \times_S Y$ induce a natural transformation $[X \times_S Y, Z] \to [X \times_W Y, Z]$ but since the identity mapping $X \times_W Y \to X \times_S Y$ need not be a homeomorphism ([3], theorem 1.11), the induced natural transformation is not a natural isomorphism, and consequently there exists no natural transformation $p_{XYZ}: [X \times_W Y, Z] \to [X[YZ]]$ satisfying the axioms MCC2 - MCC4. This implies that $T2(cot,\times_W)$ is not an MCC and not a PMCC.

2.3. *The category of uniform spaces with the uniformity of uniform convergence* ([8], chapter III). The category of uniform spaces *UNIF* becomes a CC *UNIF(uc)* if we take [XY] to be the space of uniformly continuous functions from X into Y with the uniformity of uniform convergence.

UNIF(*uc*) is not an SCC, since the function m_{YZ}: Y ⟶ [[YZ]Z] defined by $(m_{YZ}y)g = gy$ is not generally uniformly continuous ([8], p. 44).

If we take X⊗Y to be the uniform product X × Y of X and Y we obtain a PMCC *UNIF*(*uc*,×) which is not an SPMCC, since the closed part is not symmetric, and which is not an MCC since the evaluation e_{YZ}: [YZ] × Y ⟶ Z defined by $e_{YZ}\langle g,y\rangle = gy$ is not generally uniformly continuous ([8], p.44).

If we take X⊗Y to be the semi-uniform product X*Y of X and Y we obtain an MCC *UNIF*(*uc*,*) ([8], theorem 26, p.46). The semi-uniform product X*Y is the cartesian product of the sets X and Y with the weak uniformity induced by all semi-uniform functions from X × Y into uniform spaces whose underlying sets are subsets of X × Y ([8], p.44). A similar argument as in the example *T2*(*cot*,$\times_S$) can be used to show that *UNIF*(*uc*,*) is not a PMCC.

2.4. *The category of uniform spaces with the uniformity of pointwise convergence* ([8], exercise III.7, p.53). The category of uniform spaces *UNIF* becomes an SCC, *UNIF*(*pc*), if we take [XY] to be the space of uniformly continuous functions from X to Y with the uniformity of pointwise convergence.

If we take X⊗Y to be the uniform product X × Y we obtain an SPMCC *UNIF*(*pc*,×), which is not an MSCC.

If we take X⊗Y to be the "tensor product" X # Y of X and Y we obtain an MSCC *UNIF*(*pc*,#). Here X # Y is the cartesian product of the sets X and Y with the uniformity of separate continuity ([8], exercise III.7, p.53).

2.5. *The category of limit spaces* ([1]).

A *limit space* is an ordered pair ⟨X,Λ⟩ consisting of a non-void set X and a mapping Λ from X into the set F(X) of all filters on X, such that the following axioms are satisfied:

Lim 1. if $\Phi \in \Lambda(x)$ and $\Phi \leq \Psi$ then $\Psi \in \Lambda(x)$;
Lim 2. if $\Phi \in \Lambda(x)$ and $\Psi \in \Lambda(x)$ then $\Phi \wedge \Psi \in \Lambda(x)$;
Lim 3. $\langle x\rangle \in \Lambda(x)$ for all $x \in X$.

Here $\Phi \leq \Psi$ means that the filter Ψ is finer then the filter Φ, and ⟨x⟩ is the principal filter generated by x.

A mapping f: ⟨X,Λ⟩ ⟶ ⟨X',Λ'⟩ between limit spaces is said to be *continuous* if $\Phi \in \Lambda(x)$ always implies $f(\Phi) \in \Lambda'(f(x))$. Let *LIM* denote the category of limit spaces and continuous mappings.

If we take [XY] to be the space of continuous mappings from X to Y with the limit structure of continuous convergence and if we take X⊗Y to be the categorical product X × Y of X and Y, the category of limit spaces becomes a cartesian closed category $\mathit{LIM}(cc,\times)$ ([1], section 1).

If we take [XY] to be the space of continuous mappings from X to Y with the limit structure of simple convergence and if we take X⊗Y to be the cartesian product of the sets X and Y with the limit structure of separate continuity, we obtain an MSCC $\mathit{LIM}(sc,\otimes)$ ([1], Satz 26).

2.6. *The category of* L-*spaces* ([18]).

An L-*space* (also called convergence space or Fréchet space) is an ordered pair $\langle X,\Lambda\rangle$ consisting of a set X and a set $\Lambda \subset X^{\mathbb{N}} \times X$, a relation between sequences in X (denoted by (x_n)) and elements of X, such that the following axioms are satisfied:

L0. if $\langle (x_n),x^1\rangle \in \Lambda$ and $\langle (x_n),x^2\rangle \in \Lambda$ then $x^1 = x^2$;

L1. if $x_n = x$ for all $n \in \mathbb{N}$ then $\langle (x_n),x\rangle \in \Lambda$;

L2. if $\langle (x_n),x\rangle \in \Lambda$ and if (x_{k_n}) is a subsequence of (x_n) then $\langle (x_{k_n}),x\rangle \in \Lambda$

L3. if $\langle (x_n),x\rangle \notin \Lambda$ then there is a subsequence (x_{l_n}) of (x_n) such that $\langle (x_{k_{l_n}}),x\rangle \notin \Lambda$ for any subsequence $(x_{k_{l_n}})$ of (x_{l_n}).

A mapping $f\colon \langle X,\Lambda\rangle \to \langle X',\Lambda'\rangle$ between L-spaces is said to be an L-*morphism* if $\langle (x_n),x\rangle \in \Lambda$ always implies that $\langle (f(x_n)),f(x)\rangle \in \Lambda'$. Let LS denote the category of L-spaces.

J. Pavelka [18] shows that LS admits exactly two monoidal symmetric closed structures in which V is the underlying set functor: one is obtained if one takes [XY] to be the set $\mathit{LS}(X,Y)$ with the L-structure of pointwise convergence, and the other if one takes [XY] to be the set $\mathit{LS}(XY)$ with the L-structure of diagonal convergence ([18], theorem 19).

3. Other Examples

3.1. *The category of modules over a commutative ring* Λ.
The category of modules over a commutative ring Λ, $\Lambda\mathit{MOD}$, admits a well-known monoidal symmetric closed structure (cf. [6], section IV.1). In this

case $I = \Lambda$.

If we take $X \otimes Y$ to be the categorical product $X \times Y$ of X and Y, $\Lambda\mathcal{MOD}$ is not a PMCC because the basic object I of the (cartesian) monoidal structure is a one-point module.

3.2. *The category of linear spaces over a field* K, K$\mathcal{VECT}$.

This example is a particular case of the preceding one. The full subcategory of *finite dimensional vector spaces over the field* K, K$\mathcal{FINVECT}$ is an MSCC with several nice properties. For example, the natural transformations $m_{YK}: Y \longrightarrow [[YK]K]$ (where K is the scalar field) and $T_{XYZW}: [XZ]\otimes[YW] \longrightarrow [X\otimes Y, Z\otimes W]$ are natural isomorphisms.

The complexification γ: $\mathbb{R}\mathcal{VECT} \longrightarrow \mathbb{C}\mathcal{VECT}$ admits an extension to a symmetric closed functor, which is an isomorphism. If $X,Y \in$ ob $\mathbb{R}\mathcal{VECT}$ then $\hat{\gamma}_{XY}: \gamma[XY] \to [\gamma X, \gamma Y]$ is given by

$$\hat{\gamma}_{XY} \langle S,T\rangle \langle x,y\rangle = \langle Sx-Ty, Tx+Sy\rangle$$

(where $\langle S,T\rangle \in \gamma[XY]$ and $\langle x,y\rangle \in \gamma X$).

Although this is probably known, I was unable to find a reference.

The real restriction is a functor from $\mathbb{C}\mathcal{VECT}$ to $\mathbb{R}\mathcal{VECT}$ which admits an extension to a symmetric closed functor.

3.3. *The categories of normed linear spaces and of Banach spaces,* K$\mathcal{NVS}_1$ *and* K$\mathcal{BAN}_1$ (*where* K=$\mathbb{R}$ *or* K=$\mathbb{C}$) ([22]). The morphisms of these categories are linear contractions, i.e. bounded linear operators with norm ≤ 1.

If we take V: K$\mathcal{NVS}_1$ (K$\mathcal{BAN}_1$, respectively) $\longrightarrow \mathcal{S}$ to be the unit ball functor and if we take [XY] to be the normed linear space of all bounded linear operators from X to Y, K$\mathcal{NVS}_1$ and K$\mathcal{BAN}_1$ become SCC's.

If we take $X\otimes Y$ to be the projective tensor product $X\otimes_\pi Y$ K$\mathcal{NVS}_1$ becomes an MSCC. If we take $X\otimes Y$ to be the completion of the projective tensor product, $X\hat{\otimes}Y$, K$\mathcal{BAN}_1$ becomes an MSCC.

The Cantor completion can be extended to a symmetric closed functor from K$\mathcal{NVS}_1$ to K$\mathcal{BAN}_1$.

3.4. *The categories of normed linear spaces and of Banach spaces,* $\mathbb{K}NVS_\infty$ *and* $\mathbb{K}BAN_\infty$ *(where* $\mathbb{K}=\mathbb{R}$ *or* $\mathbb{K}=\mathbb{C}$*)* ([22]). The morphisms of these categories are bounded linear operators.

The same assertions as in the preceding example are true, with the modification that for V we have to take the forgetful functor from $\mathbb{K}NVS_\infty$ ($\mathbb{K}BAN_\infty$, respectively) to S.

The complexification admits an extension to a symmetric closed functor from $\mathbb{R}BAN_\infty$ to $\mathbb{C}BAN_\infty$.

3.5. *The category of graphs* GRAPH ([19]).
Objects in the category of graphs are *graphs*, i.e. ordered pairs $\langle X,R_X\rangle$ consisting of a set X and a binary relation $R_X \subset X \times X$, and morphisms are relation-preserving mappings.

If we take [XY] to be the ordered pair $\langle GRAPH(X,Y),R_{[XY]}\rangle$ where $R_{[XY]}$ is the relation defined by

$$R_{[XY]} := \{\langle f,g\rangle \mid \forall x \in X \ \langle fx,gx\rangle \in R_Y\},$$

the category GRAPH becomes an SCC GRAPH ([-,-]).

If we take $X\otimes Y$ to be the ordered pair $\langle X\times Y,R_{X\otimes Y}\rangle$ where $R_{X\otimes Y}$ is the relation defined by

$$R_{X\otimes Y} := \{\langle\langle x_1,y_1\rangle,\langle x_2,y_2\rangle\rangle \mid (\langle x_1,x_2\rangle\in R_X \text{ and } y_1=y_2) \text{ or } (x_1=x_2 \text{ and } \langle y_1,y_2\rangle\in R_Y)\},$$

the category GRAPH becomes an MSCC GRAPH([-,-],$\otimes$). A. Pultr [19] considers the question how many monoidal symmetric closed structures there are on GRAPH.

If we take $X\otimes Y$ to be the categorical product $X \times Y$, i.e. the ordered pair $\langle X\times Y,R_{X\times Y}\rangle$ where $R_{X\times Y}$ is defined by

$$R_{X\times Y} := \{\langle\langle x_1,y_1\rangle,\langle x_2,y_2\rangle\rangle \mid \langle x_1,x_2\rangle \in R_X \text{ and } \langle y_1,y_2\rangle \in R_Y\},$$

we obtain not even a PMCC. This is shown by the following example. Let X,Y and Z be three non-void sets, and consider the graphs $\langle X,\emptyset\rangle$,$\langle Y,Y\times Y\rangle$ and $\langle Z,\emptyset\rangle$. The categorical product of $\langle X,\emptyset\rangle$ and $\langle Y,Y\times Y\rangle$ is $\langle X\times Y,\emptyset\rangle$. Now it is easy to see that $[X\times Y,Z] = \langle S(X\times Y,Z),S(X\times Y,Z)\times S(X\times Y,Z)\rangle$ and

$[X[YZ]] = \langle \emptyset,\emptyset \rangle$. Hence there is no morphism from $[X\times Y,Z]$ to $[X[YZ]]$.

3.6. *The category of strict orderings SO.*

Objects of the category of strict orderings are ordered pairs $\langle X,< \rangle$ consisting of a set X and a strict ordering on X, i.e. a transitive asymmetric relation on X, and morphisms are strictly monotone mappings (so SO is a full subcategory of $GRAPH$).

If we take $[XY]$ to be the ordered pair $\langle SO(X,Y),< \rangle$ where $<$ is the strict ordering defined by

$$f < g :\Longleftrightarrow \forall x \in X: fx < gx,$$

the category SO becomes an SCC $SO([-,-])$, a symmetric closed subcategory of the SCC $GRAPH([-,-])$.

Consider the following three strict orderings on the cartesian product $X \times Y$ of the sets X and Y:

$\langle x_1,y_1 \rangle <_t \langle x_2,y_2 \rangle :\Longleftrightarrow (x_1 \le x_2 \text{ and } y_1 < y_2) \text{ or } (x_1 < x_2 \text{ and } y_1 \le y_2)$;

$\langle x_1,y_1 \rangle < \langle x_2,y_2 \rangle :\Longleftrightarrow (x_1 < x_2 \text{ and } y_1 < y_2)$;

$\langle x_1,y_1 \rangle <_1 \langle x_2,y_2 \rangle :\Longleftrightarrow x_1 < x_2 \text{ or } (x_1=x_2 \text{ and } y_1 < y_2)$.

These orderings induce bifunctors which are on objects given by

$X \otimes Y := \langle X\times Y,<_t \rangle$ (the tensor product of X and Y);

$X \times Y := \langle X\times Y,< \rangle$ (the categorical product of X and Y);

$X \times_1 Y := \langle X\times Y,<_1 \rangle$ (the lexicographic product of X and Y).

$SO([-,-],\otimes)$ is an MSCC; $SO([-,-],\times_1)$ is a PMCC which is not an SPMCC since the monoidal part is not symmetric, and not an MCC, as is shown by the following example: Let $X = Y = \{0,1\}$ with $0 < 1$ and let $Z = \{0,1,2\}$ with $0 < 1 < 2$. The lexicographic ordering $<_1$ on $X \times Y$ is given by $\langle 0,0 \rangle < \langle 0,1 \rangle < \langle 1,0 \rangle < \langle 1,1 \rangle$. Since $X \times_1 Y$ has four elements and is linearly ordered, and Z has three elements there are no strictly monotone mappings from $X \times_1 Y$ to Z: $SO(X\times_1 Y,Z) = \emptyset$. The set $SO(Y,Z)$ consists of three mappings f_1,f_2,f_3, given by $f_1(0) = f_2(0) = 0$; $f_1(1) = f_3(0) = 1$ and $f_2(1) = f_3(1) = 2$ and the ordering on $SO(Y,Z)$ is simply $f_1 < f_3$. Hence $SO(X[YZ])$ contains one element: the mapping $g\colon X \longrightarrow [YZ]$ defined by $g(0) = f_1$ and $g(1) = f_3$. Consequently $[X\times_1 Y,Z]$ and $[X[YZ]]$ are not isomorphic. $SO([-,-],\times)$ is not even a PMCC. The strictly monotone identity mapping $X \times Y \to X \otimes Y$ induces a natural mapping $[X[YZ]] \simeq [X \otimes Y, Z] \longrightarrow [X \times Y, Z]$ which is in general not an isomorphism. Consequently there is no natural transformation $[X \times Y, Z] \longrightarrow [X[YZ]]$ satisfying the properties MCC2 - MCC4 of theorem III.6.4, so that $SO([-,-],\times)$ is not a PMCC.

The assignment $\langle X,<\rangle \longmapsto \langle X,\leq\rangle$ can be extended to a monoidal symmetric closed functor from $SO([-,-],\otimes)$ to $PO([-,-],\times)$, the cartesian closed category of partially ordered sets. (cf.1.8).

3.7. *An example of a symmetric closed category which cannot be extended to a monoidal symmetric closed category.*

Let M be the smallest subset of the set $\mathbb{N}$ of natural numbers with the following three properties: (a) $1 \in M$; (b) $3 \in M$; (c) if $a \in M$ and $b \in M$ then $a^b \in M$ and $b^a \in M$. Let V_0 be the full subcategory of S generated by the objects X with card $X \in M$. If $X,Y \in \text{ob } V_0$ and if $[XY]$ is the set of all mappings from X to Y then card $[XY] = \text{card } Y^{\text{card } X}$, so $[XY] \in \text{ob } V_0$. Consequently, V_0 determines a symmetric closed subcategory of S. It is easy to see that this category cannot be monoidal closed. Suppose that there exists a bifunctor $\otimes: V_0 \times V_0 \to V_0$ such that $[X \otimes Y, Z] \simeq [X[YZ]]$. Let X,Y and Z be objects with three elements. Then card $[X[YZ]] = (3^3)^3 = 3^9$ and card $[X \otimes Y, Z] = 3^{\text{card } X \otimes Y}$ so card $X \otimes Y = 9$. This is a contradiction with the fact that $9 \notin M$.

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