COMPACT ORDERED SPACES

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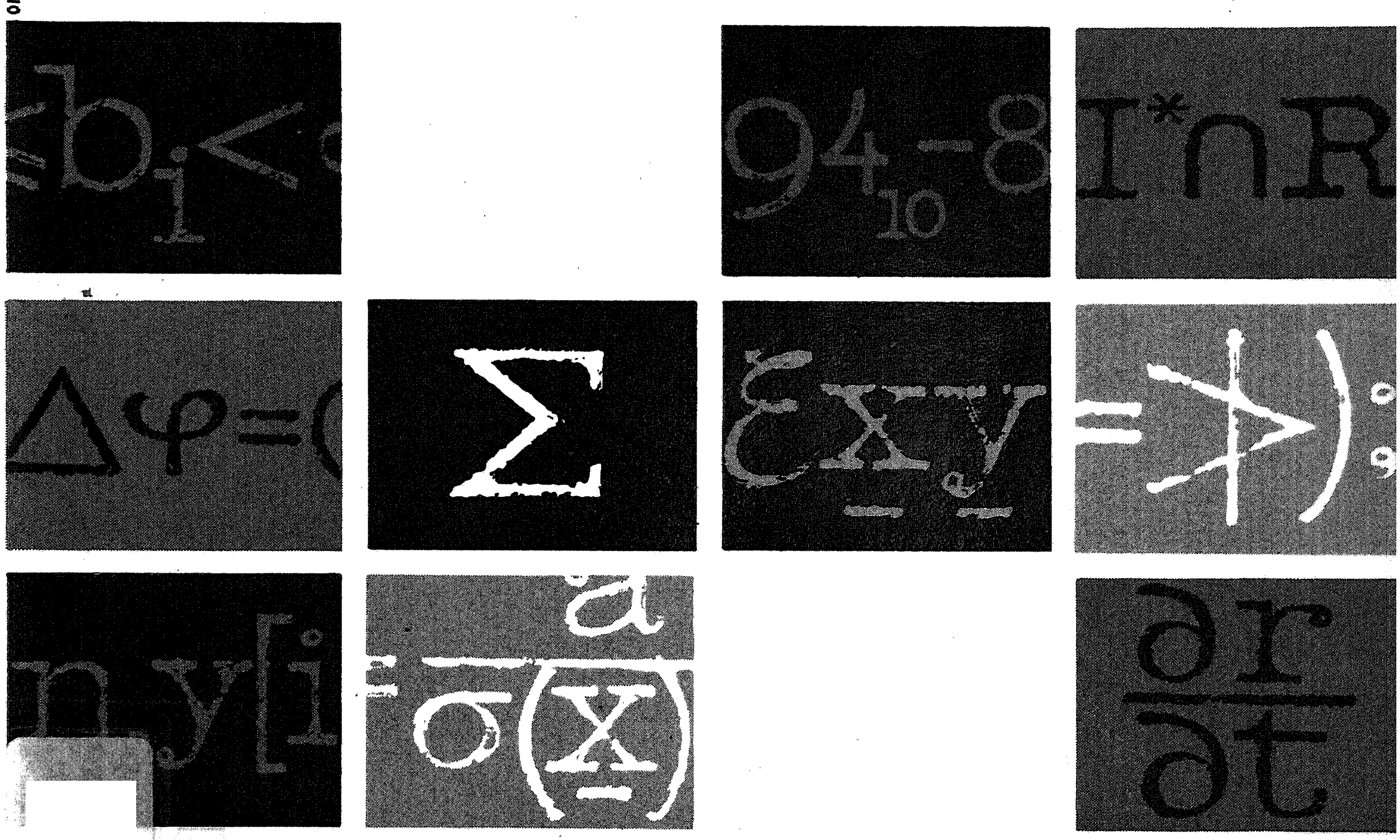

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6

COMPACT ORDERED SPACES

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CONTENTS

INTRODUCTION

This thesis deals with totally ordered compact topological spaces (supplied with interval topology). A compact ordered space will be called a "cor".

In chapter I some fundamental concepts are developed. For each cor X the notion of a θ-sequence is introduced; this is, roughly speaking, a transfinitely continued subdivision into closed left and right intervals, where a subdivision into disjoint intervals is preferred to a subdivision into intervals with a common end point. If V is a θ-sequence for a cor X, then $\theta(V)$ is the least ordinal μ with the property that all intervals of the subdivision of order μ are one-point intervals. For each cor X the "splitting degree" $\Theta(X)$ will be the least ordinal in the class of all $\theta(V)$. It is shown that $\Theta(X)$ is a topological invariant. For instance, if $Z_\alpha = \{0,1\}^\alpha$ denotes the lexicographically ordered product of α factors $\{0,1\}$, where α is an ordinal number, then $\Theta(Z_\alpha) = \alpha$; this means that all Z_α are different topological spaces.

Finally the relation between $\Theta(X)$ and the occurrence of sequences of certain type in X is investigated. In some of these results the generalized continuum hypothesis is used. Theorems, which rest on this hypothesis are marked by an asterisk (*).

In chapter II it is shown that all Z_{ω^α} are homogeneous, where α is a countable ordinal, whereas all other Z_β $(\beta>\omega)$ are not homogeneous. Also $Z_{\omega+1}$ minus isolated points is homogeneous. It is not known if there are any other homogeneous cor's with infinitely many points.

In chapter III the relation between the splitting degree, the weight and the density of a cor X is investigated. It is shown that the weight (the density) of a zero-dimensional or a connected cor equals the cardinal number $\aleph$ if and only if $\Theta(X) = \omega_\aleph$ ($\Theta(X) = \omega_\aleph$ or $= \omega_\aleph + 1$), where $\omega_\aleph$ denotes the least ordinal number of which the cardinal number is $\aleph$.

In chapter IV a survey of the literature is given.

I am grateful to the Mathematical Centre, Amsterdam, which gave me the opportunity to carry on the investigations which are dealt with in this treatise. Here I wish to thank also Miss L.J. Noordstar and her staff and Mr. D. Zwarst for typing and printing the manuscript.

LIST OF SYMBOLS AND NOTATIONS

1. Greek letters and sometimes also small latin letters denote ordinal numbers;
   gothic letters like $\mathfrak{m}$, $\mathfrak{n}$ etc. and $\aleph$ denote cardinal numbers.

2. If X is a set, then $|X|$ denotes the cardinal number of X;
   if μ is an ordinal number, then $|\mu|$ denotes the cardinal number of μ.

3. (i) In the class of ordinal numbers, ω_i denotes the initial number with ordinal index i; also $\omega_0 = \omega$, $\omega_1 = \Omega$
   (ii) If $\aleph$ is a cardinal number, then $\omega_\aleph$ will denote the least ordinal number μ, such that $|\mu| = \aleph$;
   we write : $\aleph_i = |\omega_i|$;
   also: $\aleph_o = \mathfrak{a}$
   $\aleph_1 = \mathfrak{c}$ (continuum hypothesis).

4. If α is an ordinal number, then

$$W_\alpha = W(\alpha) = \{\mu \mid \mu < \alpha\}$$
$$\widetilde{W}_\alpha = \widetilde{W}(\alpha) = \{\mu \mid \mu \leq \alpha\}$$

5. If α is an ordinal number, then α^* denotes the inverse order type.

6. If $p = (p_i)_{i<\alpha}$ is a sequence of type α , then

$$p \mid \beta = (p_i)_{i<\beta}$$

   if $\beta \leq \alpha$.

7. If $p = (p_i)_{i<\alpha}$ and $q = (q_i)_{i<\beta}$, then

$$pq = (s_i)_{i<\alpha+\beta},$$

   where
$$s_i = p_i \quad \text{if } i < \alpha$$
$$s_i = q_i \quad \text{if } \alpha \leq i < \alpha + \beta .$$

8. If X and Y are linearly ordered sets, then

$$X \backsim Y$$

means that X and Y are similar (i.e. there is a one to one map f of X onto Y which is monotone: $x_1 < x_2$ implies $f(x_1) < f(x_2)$).

9. If X is a linearly ordered set, and $a, b \in X$ $(a \leq b)$, then

(i) $I = [a,b] = \{x \mid a \leq x \leq b\}$

is a <u>closed interval</u>; $l(I) = a$, $r(I) = b$.

(ii) $J = (a,b) = \{x \mid a < x < b\}$

is an <u>open interval</u>.

If K is both an open and a closed interval, then K is called a <u>clopen interval</u>.

10. An ordered pair of elements a and b (first coordinate a, second coordinate b) is also denoted by (a,b); if confusion with an open interval is possible, we write $\overline{a,b}$ for the ordered pair.

11. Theorems which are proved with the aid of the (generalized) continuum hypothesis are marked with an asterisk (*).

12. If A and B are sets, then $A \subsetneq B$ means that $A \subset B$ and $A \neq B$.

CHAPTER I

Fundamental examples and fundamental properties of compact ordered spaces

§1.

1.1. A "linearly ordered set" is a pair $(X,<)$ where X is a set, and $<$ is a subset of $X \times X$, with the properties

(i) $\forall x \in X$: $(x,x) \notin <$

(ii) $\forall x,y,z \in X$: $[(x,y) \in <$ and $(y,z) \in <] \rightarrow (x,z) \in <$

(iii) $\forall x,y \in X$: $x=y$ or $(x,y) \in <$ or $(y,x) \in <$.

$<$ is called the "ordering" of $(X,<)$.

In the following the linearly ordered set $(X,<)$ will mostly be denoted by X.

Instead of $(x,y) \in <$ we shall always write $x < y$.

For definitions and properties of the notions "order type", "well-ordered set", "ordinal number" etc. see for instance Hausdorff [1] or Sierpinski [3].

1.2. If X is a linearly ordered set, and $A \subset X$, then by $<$ an ordering $<_A$ is induced in A.

For definitions and properties of the notions "supremum (infimum) of A", "A is bounded", "X is complete" etc. see for instance Kelley [1], Chapter O.

1.3. Suppose for each ordinal number α which is less than a given ordinal number μ, we are given a linearly ordered set $X_\alpha = (X_\alpha, <_\alpha)$.
Then the "lexicographically ordered product" $\prod_{\alpha<\mu} X_\alpha$ is defined as the set of all sequences $x = (x_\alpha)_{\alpha<\mu}$ $(x_\alpha \in X_\alpha$ for all $\alpha<\mu)$ with an ordering $<$ which is given by

$x < y \leftrightarrow$ (if β is the least ordinal $<\mu$ such that $x_\beta \neq y_\beta$, then $x_\beta <_\beta y_\beta$).

In particular, if X is a linearly ordered set, then X^{μ} is the lexicographically ordered product $\prod_{\alpha<\mu} X_{\alpha}$, where $X_{\alpha} = X$ for all $\alpha<\mu$; and if both X and Y are linearly ordered sets, then $X \cdot Y$ is the lexicographically ordered product $\prod_{\alpha<2} X_{\alpha}$, where $X_0=X$ and $X_1=Y$.
It is clear, that

$$(X^{\mu})^{\nu} \backsim X^{\mu\nu}$$

$$X^{\mu} \cdot X^{\nu} \backsim X^{\mu+\nu}$$

1.4. In the following the sets $\{0,1\}^{\alpha}$ will be denoted by Z_{α}. It is easy to see that Z_{ω} is similar to the Cantorset.

§2.

2.1. A "linearly ordered topological space" is a pair $(X,\mathcal{T}_<)$, where $X = (X,<)$ is a linearly ordered set, in which a topology $\mathcal{T}_<$ is defined by the subbase consisting of all sets $\{x|x<a\}$ and $\{x|x>b\}$ $(a,b \in X)$.

In the following the space $(X,\mathcal{T}_<)$ will mostly be denoted by X.
It is known that a linearly ordered space is completely normal; cf. Bourbaki [1] .

A topological space $(T,\mathcal{T})$ is said to be "orderable" if there exists an ordering $<$ of T, such that $(T,\mathcal{T}_<)$ and $(T,\mathcal{T})$ are homeomorphic.

2.2. If X is a linearly ordered space, and $A \subset X$, then the relative topology which is induced in A by $\mathcal{T}_<$ will be denoted by $\mathcal{T}_<^{(A)}$.
In general it is not true that $(A,\mathcal{T}_<^{(A)})$ is homeomorphic to $(A,\mathcal{T}_{<_A})$; even not if A is closed in X.

Example:

$$X = \{x|x \text{ irrational}; -\sqrt{2} \leq x \leq \sqrt{2}\}$$

$$A = \{x|x \text{ irrational}; -\sqrt{2} \leq x < 0\} \cup \{\tfrac{1}{2}\sqrt{2}\} \;;$$

A is closed in X, but $(A,\mathcal{T}_<^{(A)})$ is not homeomorphic to $(A,\mathcal{T}_{<_A})$ (the first space has an isolated point; the second has not).

2.3. If A is a compact subset of $(X,\mathcal{T}_<)$ then A is closed in $(X,\mathcal{T}_<)$ and bounded in $(X,<)$;
and if $(X,\mathcal{T}_<)$ itself is compact, then $(X,<)$ has both a least and a greatest element.

If A is a compact subset of $(X, \mathcal{T}_<)$, then $(A, <_A)$ is complete. On the other hand it is possible that A is closed and bounded in X, and that $(A, <_A)$ is complete, whereas $(A, \mathcal{T}_<^{(A)})$ is not compact;

Example:

$$X = \{x \mid -1 \leq x \leq +1\} \setminus \{0\}$$

$$A = \{x \mid -1 \leq x < 0\} .$$

Theorem 1: The assertions "$(X,<)$ is complete" and "Any bounded closed subset of $(X, \mathcal{T}_<)$ is compact" are equivalent.

Proof: see Kelley [1], Chapter V, problem C.

Corollary: The assertions "$(X,<)$ is complete and has both a least and a greatest element" and "$(X, \mathcal{T}_<)$ is compact" are equivalent.

If $(X, \mathcal{T}_<)$ is connected, then clearly $(X,<)$ is complete. Consequently each connected linearly ordered space is locally compact.

Theorem 2: If A is a compact subset of $X = (X, \mathcal{T}_<)$, then $\mathcal{T}_{<_A} = \mathcal{T}_<^{(A)}$.

Proof:

(i) It is clear, that $\mathcal{T}_{<_A} \subset \mathcal{T}_<^{(A)}$

(ii) Now take $O \in \mathcal{T}_<^{(A)}$; then for each $p \in O$ there exists an interval $I = (r,s)$, $I \in \mathcal{T}_<$, such that

$$p \in A \cap I \subset O.$$

Since A is compact, $b = \inf\{x \mid p < x,\ x \in A\}$ exists and $b \in A$;
if $b=p$, then choose $a_2 \in A$ in such a way that $p < a_2 < s$;
if $b > p$, then let $a_2 = b$.
Choose a_1 in an analogous way.
If now one puts $I' = (a_1, a_2)$, it follows that

$$p \in A \cap I' \subset O,\ I' \in \mathcal{T}_{<_A} .$$

This means that $O \in \mathcal{T}_{<_A}$.

2.4. Theorem 3: $Z_\alpha = \{0,1\}^\alpha$ is compact and zero-dimensional for all α.

Proof:

(i) Let $A \subset Z_\alpha$; define $b = (b_i)_{i<\alpha}$ by transfinite induction in the following way:

$$\begin{cases} b_0 = 0 \text{ if } a_0 = 0 \text{ for all } a = (a_i)_{i<\alpha} \in A \\ b_0 = 1 \text{ else;} \end{cases}$$

if b_i is defined for all $i < \nu$, then let

$$\begin{cases} b_\nu = 0, \text{ if } a_\nu = 0 \text{ for all } a = (a_i)_{i<\alpha} \in A \text{ with the property that} \\ \qquad a_i = b_i \text{ for } i < \nu \\ b_\nu = 1 \text{ else.} \end{cases}$$

It is clear that $b = \sup A$.

This means that $(Z_\alpha, <)$ is complete, and so $(Z_\alpha, \mathcal{J}_<)$ is compact.

(ii) Let $A \subset Z_\alpha$, $a = (a_i)_{i<\alpha} \in A$, $b = (b_i)_{i<\alpha} \in A$.
If $a < b$ and if i_0 is the least index i with the property $a_i \neq b_i$ (so that $a_{i_0} = 0$, $b_{i_0} = 1$),

then define $p = (p_i)_{i<\alpha}$ by $\begin{cases} p_i = a_i = b_i & \text{if } i < i_0 \\ p_{i_0} = 0 & \\ p_i = 1 & \text{if } i > i_0 \end{cases}$

and $q = (q_i)_{i<\alpha}$ by $\begin{cases} q_i = a_i = b_i & \text{if } i < i_0 \\ q_{i_0} = 1 & \\ q_i = 0 & \text{if } i > i_0. \end{cases}$

Then $a \leq p < q \leq b$ and $\{x \mid p < x < q\} = \emptyset$.

This means that Z_α is totally disconnected and consequently is zero-dimensional.

<u>Remark</u>: In the following the phrase "compact linearly ordered topological space" will always be abbreviated to "cor".

§3.

3.1. Let X be a cor.

Two elements $a, b \in X$ will be called "neighbours" (and a is a "left neighbour (of b)", b is a "right neighbour (of a)") if $a < b$ and $\{x \mid a < x < b\} = \emptyset$. Both a and b are also referred to as "jump points".

If in X, for any increasing or decreasing sequence $\{x_i\}_{i \leq \alpha}$ with the property that x_i and x_{i+1} are neighbours for all $i < \alpha$, all elements of the same sequence are identified, then the resulting space is denoted by X^*.

It is obvious that X^* is a connected cor.

Theorem 4: (i) A clopen subset of a cor X is the union of a finite number of disjoint clopen intervals.

(ii) A cor X is not connected if and only if there are two neighbours in X.

Proof: obvious.

3.2. Let X be a cor.

By a θ-sequence for X we mean a (transfinite) sequence $V = \{V_\gamma\}_\gamma$ of θ-decompositions $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ of X, which by transfinite induction is defined as follows:

(i) $V_0 = \{X^{(0)}\}$, $X^{(0)} = X$

(ii) If V_γ has been defined for $\gamma<\delta$, then V_δ is defined in the following way:

a. if $\delta = \varepsilon + 1$ and $|X_p^{(\varepsilon)}| = 1$

then $X_{p0}^{(\delta)} = X_{p1}^{(\delta)} = X_p^{(\varepsilon)}$

b. if $\delta = \varepsilon + 1$ and $X_p^{(\varepsilon)}$ is not connected

then $X_{p0}^{(\delta)} = \{x \mid x \leq a\} \cap X_p^{(\varepsilon)}$

$X_{p1}^{(\delta)} = \{x \mid x \geq b\} \cap X_p^{(\varepsilon)}$,

for two neighbours a and b $(a < b)$

c. if $\delta = \varepsilon + 1$ and $X_p^{(\varepsilon)}$ is connected

then $X_{p0}^{(\delta)} = \{x \mid x \leq a\} \cap X_p^{(\varepsilon)}$

$X_{p1}^{(\delta)} = \{x \mid x \geq a\} \cap X_p^{(\varepsilon)}$,

for an a such that $\inf X_p^{(\varepsilon)} < a < \sup X_p^{(\varepsilon)}$

d. if δ is a limit number

then
$$X_p^{(\delta)} = \bigcap_{\gamma<\delta} X_{p|\gamma}^{(\gamma)}$$

(cf. Novak [2] , where for the case of a connected cor a "dyadic partition" P is defined; such a "dyadic partition" can be considered as the system of non-degenerate intervals which are the elements of the members of a certain θ-sequence V_P).

It is clear that for every θ-sequence:

(i) $\forall\alpha\ \forall p \in Z_\alpha : X_p^{(\alpha)}$ is a closed interval $\neq \emptyset$

(ii) $\forall\alpha : \bigcup_{p \in Z_\alpha} X_p^{(\alpha)} = X$

(iii) $\forall\alpha\ \forall x,y,p,q : [(p < q, x \in X_p^{(\alpha)}, y \in X_q^{(\alpha)}) \rightarrow x \leqq y]$.

If X is a cor, then for every θ-sequence V and for every $x \in X$ there exists an ordinal number

$$\mu_x = \mu_x(V) = \inf \{\mu \mid \exists p \in Z_\mu ; X_p^{(\mu)} = \{x\}\} .$$

We put

$$\theta = \theta(V) = \sup_x \mu_x .$$

In the case of a connected cor the definition of the order of a dyadic partition P as given by Novak coincides with $\theta(V_P)$. For a connected cor the following theorem is also contained in Novak [3] .

Theorem 5: If V is a θ-sequence for the cor X, and $\theta = \theta(V)$, then

$$|\theta| \leqq |X| \leqq 2^{|\theta|} .$$

Proof:

(i) Take $x \in X$.

Now consider a sequence $\{X_{p(\alpha)}^{(\alpha)}\}_{\alpha<\mu}$ $(p(\alpha) \in Z_\alpha)$ such that

$$x \in X_{p(\alpha)}^{(\alpha)} \subset X_{p(\beta)}^{(\beta)} \text{ for } \alpha > \beta,$$

and suppose that $|X_{p(\alpha)}^{(\alpha)}| \geqq 2$ for all $\alpha < \nu$ so that

$$X_{p(\alpha+1)}^{(\alpha+1)} \subsetneq X_{p(\alpha)}^{(\alpha)} \quad \text{for } \alpha+1 \leqq \nu .$$

Consequently

$$\bigcup_{\alpha<\nu} \left(X_{p(\alpha)}^{(\alpha)} \setminus X_{p(\alpha+1)}^{(\alpha+1)} \right)$$

is a subset of X, which is the union of $|\nu|$ disjoint, non-void sets; this means that $|\nu| \leqq |X|$.

So for every $x \in X$ there exists an ordinal ν_x, with the properties:

$$|\nu_x| \leq |X|$$
$$\{x\} = X_p^{(\nu_x)} \quad \text{for some } p \in Z_{\nu_x} .$$

It is clear that

$$\theta = \sup_x \mu_x \leq \sup_x \nu_x ,$$

and so

$$|\theta| \leq |X| \cdot |X| = |X| .$$

(ii) From the definition of θ it follows that

$$\forall x \in X \ \exists p = p(x) \in Z_\theta : \{x\} = X_p^{(\theta)} .$$

Then

$$f : x \to p(x)$$

is a 1-1-map of X into Z_θ ;
this means that
$$|X| \leq |Z_\theta| = 2^{|\theta|} .$$

Theorem 6: If X is a cor and V is a θ-sequence for X, then

$$\theta = \theta(V) = \inf\{\gamma \mid \forall p \in Z_\gamma : |X_p^{(\gamma)}| = 1\} .$$

Proof:

(i) It is clear that $\theta \leq \inf\{\gamma \mid \forall p \in Z_\gamma : |X_p^{(\gamma)}| = 1\}$

(ii) One can easily prove (by transfinite induction), that a disconnected $X_p^{(\gamma)}$ is disjoint with all $X_q^{(\gamma)}$ $(q \neq p)$. Now, if $|X_p^{(\gamma)}| \geq 2$ and $X_p^{(\gamma)}$ is disconnected, then it follows from the above that $\mu_x > \gamma$ for all $x \in X_p^{(\gamma)}$;
if, on the other hand, $|X_p^{(\gamma)}| \geq 2$ and $X_p^{(\gamma)}$ is connected then for all x such that $\inf X_p^{(\gamma)} < x < \sup X_p^{(\gamma)}$: $\mu_x > \gamma$;
consequently in both cases $\theta > \gamma$.
This means that $\theta \geq \inf\{\gamma \mid \forall p \in Z_\gamma : |X_p^{(\gamma)}| = 1\}$.

Definition: If X is a cor, then

$$\Theta = \Theta(X) = \inf \{\theta(V) \mid V \text{ is } \theta\text{-sequence for } X\}$$

is called the splitting degree of X.

It is clear that $\Theta(X)$ is invariant under similarity maps of $X=(X,<)$. We shall show, however, introducing a topological invariant (ordinal number) $\tau(X)$ - which is proved to be equal to $\Theta(X)$ - that $\Theta(X)$ is also a topological invariant; that is, if two cor's $(X,\mathfrak{T}_{<'})$ and $(Y,\mathfrak{T}_{<''})$ are homeomorphic, then $\Theta(X) = \Theta(Y)$; we can formulate this also in the following way: if a compact Hausdorff space is orderable in more than one way, then the splitting degree is the same in all cases.

3.3. Let T be a compact Hausdorff space.
By a τ-sequence for T we mean a (transfinite) sequence $U = \{U_\gamma\}_\gamma$ of τ-decompositions $U_\gamma = \{T_p^{(\gamma)}\}_{p \in Z_\gamma}$ of T, which by transfinite induction is defined as follows:

(i) $U_0 = \{T^{(0)}\}$, $T^{(0)} = T$

(ii) If U_γ has been defined for $\gamma < \delta$, then V_δ is defined in the following way:

a. if $\delta = \varepsilon+1$ and $|T_p^{(\varepsilon)}| = 1$ then $T_{p0}^{(\delta)} = T_{p1}^{(\delta)} = T_p^{(\varepsilon)}$

b. if $\delta = \varepsilon+1$ and $T_p^{(\varepsilon)}$ is not connected, then let $T_{p0}^{(\delta)}$ and $T_{p1}^{(\delta)}$ be two disjoint, non-void subsets of $T_p^{(\varepsilon)}$, which are clopen in $T_p^{(\varepsilon)}$ and the union of which is $T_p^{(\varepsilon)}$

c. if $\delta = \varepsilon+1$ and $T_p^{(\varepsilon)}$ is connected, then let $T_{p0}^{(\delta)}$ and $T_{p1}^{(\delta)}$ be two non-void proper subsets of $T_p^{(\varepsilon)}$, which are closed in $T_p^{(\varepsilon)}$, and which moreover have the properties that $T_{p0}^{(\delta)} \cup T_{p1}^{(\delta)} = T_p^{(\varepsilon)}$ and that $|T_{p0}^{(\delta)} \cap T_{p1}^{(\delta)}|$ is minimal.

d. if δ is a limit number then

$$T_p^{(\delta)} = \bigcap_{\gamma<\delta} T_{p|\gamma}^{(\gamma)} \quad .$$

It is clear that for every τ-sequence:

(i) $\forall\alpha \quad \forall p \in Z_\alpha \; : T_p^{(\alpha)}$ is closed and $\neq \emptyset$

(ii) $\forall\alpha \; : \bigcup_{p \in Z_\alpha} T_p^{(\alpha)} = T.$

If T is a compact Hausdorff space, then for every τ-sequence U and for every $t \in T$ there exists an ordinal number

$$\mu_t = \mu_t(U) = \inf\{\mu \mid \exists p \in Z_\mu : T_p^{(\mu)} = \{t\}\}.$$

We put

$$\tau = \tau(U) = \sup_t \mu_t .$$

<u>Theorem 7</u>: If U is a τ-sequence for the compact Hausdorff space T and $\tau = \tau(X)$, then

$$|\tau| \leq |T| \leq 2^{|\tau|} .$$

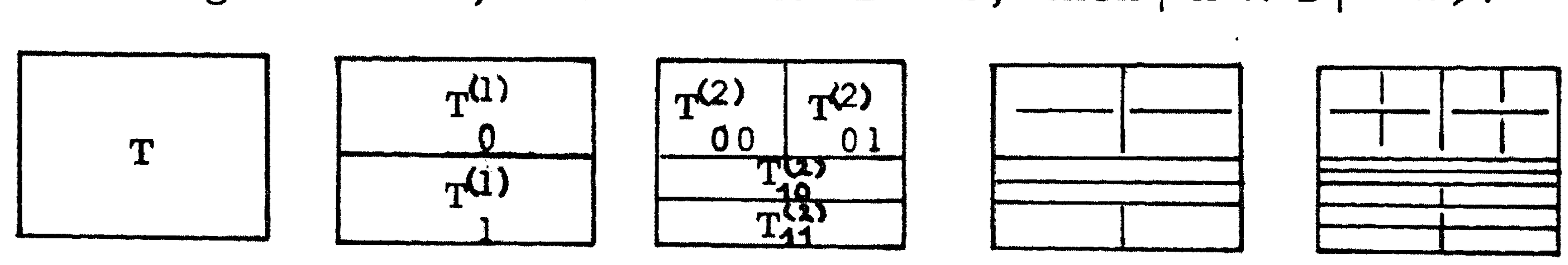
<u>Proof</u>: analogous to the proof of theorem 5.

<u>Theorem 8</u>: If T is a cor and U is a τ-sequence for T then

$$\tau = \tau(U) = \inf\{\gamma \mid \forall p \in Z_\gamma : |T_p^{(\gamma)}| = 1\}.$$

<u>Proof</u>: analogous to the proof of theorem 6.

Theorem 8 does not hold for arbitrary compact Hausdorff spaces. A counterexample is obtained if one defines a τ-sequence U for the unit square T in R^2, which is most clearly suggested by the following sequence of pictures (observe that the sequence of subdivisions is indeed a τ-sequence: if A en B are two non-void closed proper subsets of a rectangle S in R^2, such that $A \cup B = S$, then $|A \cap B| = \mathfrak{c}$):



It is clear that $\tau(U) = \omega$, whereas $\inf\{\gamma \mid \forall p \in Z_\gamma : |T_p^{(\gamma)}| = 1\} \geq \omega + \omega$.

<u>Definition</u>: If T is a compact Hausdorff space, then we define:

$$\tau = \tau(T) = \inf \{\tau(U) \mid U \text{ is a } \tau\text{-sequence for } T\} .$$

It is clear that $\tau(T)$ is a topological invariant.

3.4. <u>Lemma</u>: Let X be a cor.
Let $U = \{U_\gamma\}_\gamma$ — $U_\gamma = \{T_p^{(\gamma)}\}_{p \in Z_\gamma}$ — be a τ-sequence for X, such that $\tau(U) = \tau(X) = \tau$.
Suppose $\tau \geq \omega$ and let $\tau = \mu_0 + \nu_0$, where μ_0 is a limit ordinal and ν_0 is an integer ≥ 0.
Then there exists a θ-sequence $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — for X, with the property that for every limit number $\mu \leq \mu_0$ and for every $p \in Z_\mu$ there is a $q = q(p) \in Z_\mu$ such that

(i) $q(p \mid \nu) = q(p) \mid \nu$ if ν is a limit number $< \mu$

(ii) $X_p^{(\mu)} \subset T_{q(p)}^{(\mu)}$.

<u>Proof</u>:

1. Let $\mu = \omega$.

<u>a</u>. If X is connected, then

$$T_{i_0}^{(1)} = \{x \mid x \leq a\} \text{ and } T_{i_1}^{(1)} = \{x \mid x \geq a\}$$

— where $(i_0, i_1) = (0,1)$ or $= (1,0)$ — for some $a \in X$.

Then take $X_0^{(1)} = T_{i_0}^{(1)}$, $X_1^{(1)} = T_{i_1}^{(1)}$.

<u>b</u>. If X is not connected, then both $T_0^{(1)}$ and $T_1^{(1)}$ are the union of a finite number of disjoint clopen intervals:

$$T_0^{(1)} = I_1 \cup I_2 \cup \ldots \cup I_k$$
$$T_1^{(1)} = J_1 \cup J_2 \cup \ldots \cup J_l ;$$

without loss of generality we may suppose

$$I_1 < J_1 < I_2 < J_2 < \ldots$$

(all elements of I_1 are less than all elements of J_1 etc.)

Now define

$$X_0^{(1)} = I_1,\ X_1^{(1)} = J_1 \cup I_2 \cup J_2 \cup \dots$$

$$\begin{cases} (X_{00}^{(2)},\ X_{01}^{(2)}) \text{ is an arbitrary } \theta\text{-decomposition of } X_0^{(1)} \\ X_{10}^{(2)} = J_1,\ X_{11}^{(2)} = I_2 \cup J_2 \cup \dots \end{cases}$$

$$\begin{cases} (X_{000}^{(3)},\ X_{001}^{(3)}),\ (X_{010}^{(3)}, X_{011}^{(3)}),\ (X_{100}^{(3)}, X_{101}^{(3)}) \text{ are arbitrary } \theta\text{-} \\ \qquad \text{decompositions of } X_{00}^{(2)}, X_{01}^{(2)}, X_{10}^{(2)} \text{ respectively} \\ X_{110}^{(3)} = I_2,\ X_{111}^{(3)} = J_2 \cup I_3 \cup \dots \end{cases}$$

etc.

c. In both cases a and b one finds an integer $\gamma_1 \geqq 1$ ($\gamma_1 = 1$ in case a and $\gamma_1 = k+l-1$ in case b) such that V_γ is defined for $\gamma \leqq \gamma_1$ and moreover

$$\forall p \in Z_{\gamma_1} : \left[X_p^{(\gamma_1)} \subset T_0^{(1)} \text{ or } X_p^{(\gamma_1)} \subset T_1^{(1)} \right]$$

d. Now suppose that a non-decreasing sequence of integers $\gamma_m \geqq m$ ($m=1,2,\dots,n$) has been found and that V_γ has been defined for all $\gamma \leqq \gamma_n$ ($n \leqq \gamma_n < \omega$) in such a way that for all $m \leqq n$

$$(1)\quad \begin{cases} \forall p \in Z_{\gamma_m} : \exists q = q(p) \in Z_m : X_p^{(\gamma_m)} \subset T_{q(p)}^{(m)} \\ q(p \mid \gamma_k) = q(p) \mid k \text{ if } p \in Z_{\gamma_m} \text{ and } k \leqq m. \end{cases}$$

Now, if $p \in Z_{\gamma_n}$, let

$$Y_0^{(1)}(p) = X_p^{(\gamma_n)} \cap T_{q0}^{(n+1)} \text{ and } Y_1^{(1)}(p) = X_p^{(\gamma_n)} \cap T_{q1}^{(n+1)}$$

(i) if $X_p^{(\gamma_n)} = Y_i^{(1)}(p)$ for i=0 or 1, then take $\delta'(p) = 0$

(ii) if $Y_i^{(1)} \subsetneq X_p^{(\gamma_n)}$ for i=0 and 1, then, according to c, there exists an integer $\delta'(p) \geqq 1$ such that a θ-sequence $\{V'_\varepsilon(p)\}_{\varepsilon \leqq \delta'(p)}$ — $V'_\varepsilon = \{(X_p^{(\gamma_n)})_u^{(\varepsilon)}\}_{u \in Z_\varepsilon}$ — for $X_p^{(\gamma_n)}$ can be defined with the property that

$$\forall t \in Z_{\delta'} : \left[((X_p^{(\gamma_n)})_t^{(\delta')} \subset Y_0^{(1)} \subset T_{q0}^{(n+1)} \text{ or } (X_p^{(\gamma_n)})_t^{(\delta')} \subset Y_1^{(1)} \subset T_{q1}^{(n+1)} \right];$$

if $\gamma'_{n+1}(p) = \gamma_n + \delta'(p)$ this means that

$$\forall r \in Z_{\gamma'_{n+1}} : \left[r \mid \gamma_n = p \rightarrow \exists s \in Z_{n+1} : [s \mid n = q \text{ and } X_r^{(\gamma'_{n+1})} \subset T_s^{(n+1)}] \right]$$

(iii) Put $\delta = \max_{p \in Z_{\gamma_n}} (1, \delta'(p))$, $\gamma_{n+1} = \max_{p \in Z_{\gamma_n}} (n+1, \gamma'_{n+1}(p))$;

then also $\gamma_{n+1} = \gamma_n + \delta$

(iv) Now we have defined the intervals

$$X_{pt}^{(\gamma'_{n+1})} \qquad (p \in Z_{\gamma_n}, \ t \in Z_{\delta'}, \ pt \in Z_{\gamma'_{n+1}}).$$

If for some $p \in Z_{\gamma_n}$: $\delta'(p) = \delta - 1$, then define

$$X_{pt0}^{(\gamma_{n+1})} \text{ and } X_{pt1}^{(\gamma_{n+1})}$$

by an arbitrary θ-decomposition of $X_{pt}^{(\gamma'_{n+1})}$.

If for some $p \in Z_{\gamma_n}$: $\delta'(p) = \delta - 2$, then define

$$X_{pt00}^{(\gamma_{n+1})}, \ X_{pt01}^{(\gamma_{n+1})}, \ X_{pt10}^{(\gamma_{n+1})}, \ X_{pt11}^{(\gamma_{n+1})}$$

by 2 arbitrary θ-decompositions of $X_{pt}^{(\gamma'_{n+1})}$.

Etcetera.

Then it follows that V_γ is defined for $\gamma \leqq \gamma_{n+1}$ $(n+1 \leqq \gamma_{n+1} < \omega)$ and moreover

$$\begin{cases} \forall r \in Z_{\gamma_{n+1}} : \exists s = s(r) \in Z_{n+1} : X_r^{(\gamma_{n+1})} \subset T_{s(r)}^{(n+1)} \\ s(r \mid \gamma_n) = s(r) \mid n. \end{cases}$$

Then clearly ① is also satisfied if $m = n+1$.

(v) We can take together the foregoing in the following way:

There is a (beginning of) a θ-sequence $\{V_\gamma\}_\gamma$ for X — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — and there is a non-decreasing sequence of integers $\gamma_n \geqq n$, with the property that for all $n < \omega$ and for all $p \in Z_{\gamma_n}$ there exists a $q=q(p) \in Z_n$ such that

$$\begin{cases} q(p \mid \gamma_m) = q(p)|m & \text{if } m < n \\ X_p^{(\gamma_n)} \subset T_{q(p)}^{(n)}. \end{cases}$$

e. Now take $p \in Z_\omega$ and define $q = q(p) \in Z_\omega$ by

$$q|n = q(p \mid \gamma_n) \text{ for } n < \omega \ ;$$

then

$$X_p^{(\omega)} = \bigcap_{n<\omega} X_{p|\gamma_n}^{(\gamma_n)} \subset \bigcap_{n<\omega} T_{q|n}^{(n)} = T_q^{(\omega)}.$$

2. Let μ be a limit ordinal, and let V_γ be defined for all γ with the property that there exists a limit number $\nu < \mu$ such that $\gamma \leqq \nu < \mu$; and let for all limit numbers $\nu < \mu$

$$② \begin{cases} \forall p \in Z_\nu \ : \ \exists q = q(p) \in Z_\nu \ : \ X_p^{(\nu)} \subset T_{q(p)}^{(\nu)} \\ q(p|\lambda) = q(p)|\lambda \quad \text{if } \lambda \text{ is a limit number } < \nu. \end{cases}$$

a. Let $\mu = \nu + \omega$.

Take $p' \in Z_\nu$.

From 1. it follows that there exists a θ-sequence $\{V_\gamma(p')\}_{\gamma \leqq \omega}$ — $V_\gamma(p') = \{(X_{p'}^{(\nu)})_n^{(\gamma)}\}_{n \in Z_\gamma}$ — for $X_{p'}^{(\nu)}$ such that

$$\forall r \in Z_\omega \ \exists s = s(r) \in Z_\omega \ : \ (X_{p'}^{(\nu)})_r^{(\omega)} \subset T_{q(p')s}^{(\nu+\omega)} \cap X_{p'}^{(\nu)}$$

and so (if $p'r = p$, $q(p')s = q(p)$)

$$\forall p \in Z_\mu : \Big[p|\nu = p' \rightarrow \exists q(p) \in Z_\mu : \big[q(p)|\nu = q(p') \text{ and } X_p^{(\mu)} \subset T_{q(p)}^{(\mu)} \big] \Big];$$

this holds for every $p' \in Z_\nu$; consequently V_γ is defined for all $\gamma \leq \mu$ and clearly (2) is also satisfied if $\nu = \mu$.

b. If μ is not of the form $\nu+\omega$, then μ is the limit of a transfinite sequence of limit ordinals $\{\nu+\omega\}_{\nu<\mu}$.
In this case V_γ is defined already for all $\gamma < \mu$.
Now take $p \in Z_\mu$ and define $q = q(p) \in Z_\mu$ by

$$q\,|\,(\nu+\omega) = q(p\,|\,(\nu+\omega)) \text{ for } \nu < \mu;$$

then V_μ can be defined by

$$X_p^{(\mu)} = \bigcap_{\nu<\mu} X_{p|(\nu+\omega)}^{(\nu+\omega)} \subset \bigcap_{\nu<\mu} T_{q|(\nu+\omega)}^{(\nu+\omega)} = T_q^{(\mu)}$$

and it is clear that (2) is also satisfied for $\nu = \mu$.

3. Now the lemma is proved by transfinite induction.

Theorem 9: If X is a cor then $\Theta(X) = \tau(X)$.

Proof:

Without loss of generality we may suppose that both $\Theta(X)$ and $\tau(X) \geqq \omega$.

(i) Each θ-sequence is a τ-sequence; hence $\Theta(X) \geqq \tau(X)$;

(ii) Now take a τ-sequence $U = \{U_\gamma\}_\gamma$ — $U_\gamma = \{T_q^{(\gamma)}\}_{q \in Z_\gamma}$ — such that $\tau(U) = \tau(X)$.

Let $\tau = \mu_o + \nu_o$, where μ_o is a limit ordinal and ν_o is an integer $\geqq 0$.
Then there exists a θ-sequence $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — with the property

$$\forall p \in Z_{\mu_o} : \exists q \in Z_{\mu_o} : X_p^{(\mu_o)} \subset T_q^{(\mu_o)}.$$

For all $q \in Z_{\mu_o}$ at most $|\nu_o|$ τ-decompositions are needed for splitting up $T_q^{(\mu_o)}$ into points. This means that $T_q^{(\mu_o)} \leqq 2^{|\nu_o|}$ and consequently (for all $p \in Z_{\mu_o}$) $|X_p^{(\mu_o)}| \leqq 2^{|\nu_o|}$. So also at most $|\nu_o|$ θ-decompositions are needed for splitting up $T_q^{(\mu_o)}$ into points.
That means $\theta(V) \leqq \mu_o + \nu_o = \tau$, and so $\Theta(X) \leqq \tau(X)$.

Corollary: $\Theta(X)$ is a topological invariant.

<u>Theorem 10</u>: If both X and Y are cor's, and $X \subset Y$, then

$$\Theta(X) \leq \Theta(Y)$$

if Y is zero-dimensional or if X is connected.

<u>Proof</u>: clear.

<u>Remark</u>: If X and Y are cor's and $X \subset Y$ then it may happen that

$$\Theta(X) > \Theta(Y);$$

example: $Y = [0,2]$

$$X = \bigcup_{n=2}^{\infty} \{1-\frac{1}{n}\} \cup [1,2]$$

$$\Theta(X) = \omega+\omega > \omega = \Theta(Y)$$

3.5. For Z_α we define the "regular θ-sequence" $W = \{W_\gamma\}_\gamma$ — $W_\gamma = \{Z_p^{(\gamma)}\}_{p \in Z_\gamma}$ — in the following way:

(i) $W_o = \{Z_\alpha\}$

(ii) if $\gamma \geq 1$ and $p \in Z_\gamma$ then

$$Z_p^{(\gamma)} = \{x \mid \overbrace{p_0 p_1 p_2 \ldots}^{p} \overrightarrow{0000\ldots} \leq x \leq \overbrace{p_0 p_1 p_2 \ldots}^{p} \overrightarrow{1111\ldots}\}$$

It is clear that W indeed is a θ-sequence for Z_α.

If, when $\gamma < \alpha$, ξ is determined in such a way that $\gamma + \xi = \alpha$, then for all $p \in Z_\gamma$, $Z_p^{(\gamma)}$ is similar to Z_ξ.

This means that $|Z_p^{(\gamma)}| > 1$ for all $p \in Z_\gamma$ if $\gamma < \alpha$, whereas $|Z_p^{(\alpha)}| = 1$ for all $p \in Z_\alpha$.

For Z_α^* we define the "regular θ-sequence" $W^* = \{W_\gamma^*\}_\gamma$ — $W_\gamma^* = \{Z_p^{*(\gamma)}\}_{p \in Z_\gamma}$ — in an analogous way:

(i) $W_o^* = \{Z_\alpha^*\}$

(ii) if $\gamma \geq 1$ and $p \in Z_\gamma$ then

$$Z_p^{*(\gamma)} = \{x \mid \overbrace{p_0 p_1 p_2 \ldots}^{p} \overrightarrow{0000\ldots} \leq x \leq \overbrace{p_0 p_1 p_2 \ldots}^{p} \overrightarrow{1111\ldots}\}$$

It is clear that W is indeed a θ-sequence for Z_α^*.

If, when $\gamma < \alpha$, ξ is determined in such a way that $\gamma + \xi = \alpha$, then for all $p \in Z_\gamma$, $Z_p^{*(\gamma)}$ is similar to Z_ξ^*.

This means - if one writes $\alpha = \nu + n$ where ν is a limit ordinal (or 0) and n is an integer $\geqq 0$ - that $|Z_p^{*(\gamma)}| > 1$ for all $p \in Z_\gamma$ if $\gamma < \nu$ whereas $|Z_p^{*(\gamma)}| = 1$ for all $p \in Z_\gamma$ if $\gamma \geqq \nu$.

<u>Lemma</u>: 1. If $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — is an arbitrary θ-sequence for Z_α, then

$$\forall \gamma \leqq \alpha \quad \exists p \in Z_\gamma : Z_p^{(\gamma)} \subset X_p^{(\gamma)}$$

2. If $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — is an arbitrary θ-sequence for Z^*_α, then

$$\forall \gamma \leqq \alpha \quad \exists p \in Z_\gamma : Z_p^{*(\gamma)} \subset X_p^{(\gamma)}.$$

<u>Proof</u>:

1. If $\gamma = 0$ the assertion is obvious.

Let the assertion be proved for $\gamma < \delta$ $(\delta \leqq \alpha)$

(i) If $\delta = \delta_1 + 1$ there exists a $p' \in Z_{\delta_1}$, such that

$$Z_{p'}^{(\delta_1)} \subset X_{p'}^{(\delta_1)} ;$$

$Z_{p'0}^{(\delta)}$ and $Z_{p'1}^{(\delta)}$ are obtained from $Z_{p'}^{(\delta_1)}$ by splitting up this interval into a left interval and a right interval; in the same way $X_{p'0}^{(\delta)}$ and $X_{p'1}^{(\delta)}$ are obtained from $X_{p'}^{(\delta_1)}$.

Then $Z_{p'i}^{(\delta)} \subset X_{p'i}^{(\delta)}$ for at least one of the two possibilities i=1,2; for instance for i=1.

If one puts $p'1 = p$, then $Z_p^{(\delta)} \subset X_p^{(\delta)}$.

(ii) If δ is a limit number, there is a sequence $\{p(\varepsilon)\}_{\varepsilon < \delta}$ $(p(\varepsilon) \in Z_\varepsilon)$ such that

$$p(\varepsilon)|\eta = p(\eta) \quad \text{if } \eta < \varepsilon < \delta$$

$$Z_{p(\varepsilon)}^{(\varepsilon)} \subset X_{p(\varepsilon)}^{(\varepsilon)} ;$$

now, if one defines $p \in Z_\delta$ such that

$$p|\varepsilon = p(\varepsilon) \quad \text{for all } \varepsilon < \delta,$$

then

$$Z_p^{(\delta)} = \bigcap_{\varepsilon<\delta} Z_{p|\varepsilon}^{(\varepsilon)} \subset \bigcap_{\varepsilon<\delta} X_{p|\varepsilon}^{(\varepsilon)} = X_p^{(\delta)}.$$

2. The proof is completely analogous to 1.

Corollaries: In case 1 : $\theta(W) \leqq \theta(V)$

In case 2 : $\theta(W^*) \leqq \theta(V)$.

Theorem 11: 1. $\Theta(Z_\alpha) = \alpha$

2. $\Theta(Z_\alpha^*) = \nu$, if $\alpha=\nu+n$, where ν is a limit number (or 0) and n is an integer $\geqq 0$.

Proof:

1. $\theta(W) = \alpha$, so $\Theta(Z_\alpha) \leqq \alpha$.

On the other hand if V is an arbitrary θ-sequence for Z_α, then $\alpha = \theta(W) \leqq \theta(V)$.

Consequently $\Theta(Z_\alpha) = \alpha$.

2. Proof is analogous to 1.

Remark: In general it is not true that $\Theta(X) = \alpha$ — where $\alpha=\nu+n$, ν is a limit number and n is an integer $\geqq 0$ — implies $\Theta(X^*) = \nu$.

Example: $X = \overset{\backsim}{W}(\Omega) \rightarrow \Theta(X) = \Omega$, $\Theta(X^*) = 0$.

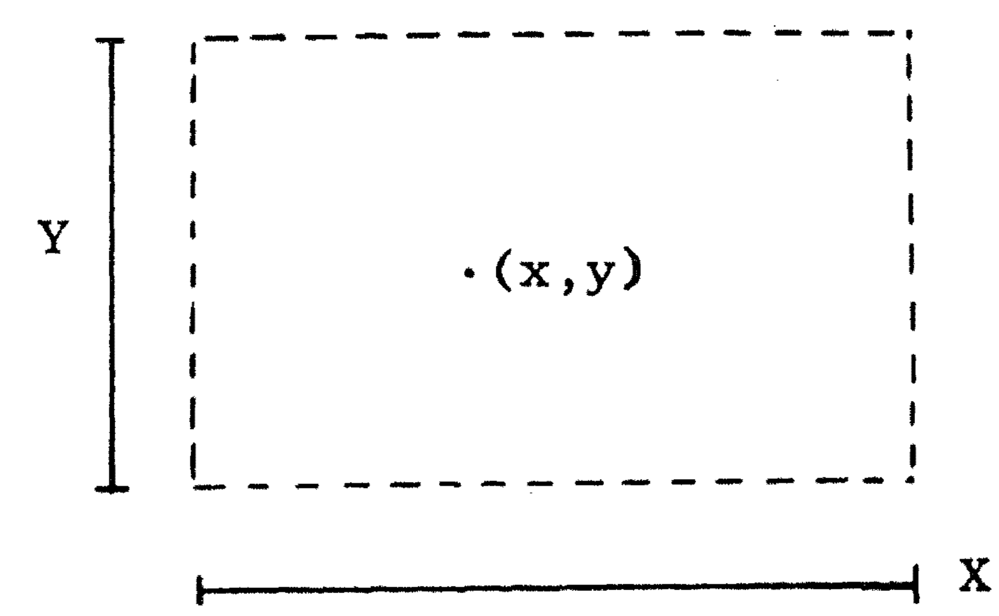
Thus we have the following theorem:

Theorem 12: 1. If $\alpha \neq \beta$, then Z_α and Z_β are different topological spaces.

2. If $\alpha=\nu+n$, $\beta=\mu+m$, where ν,μ are limit numbers (or 0) and n,m are integers $\geqq 0$, then Z_α^* and Z_β^* are different topological spaces if $\nu \neq \mu$.

§4.

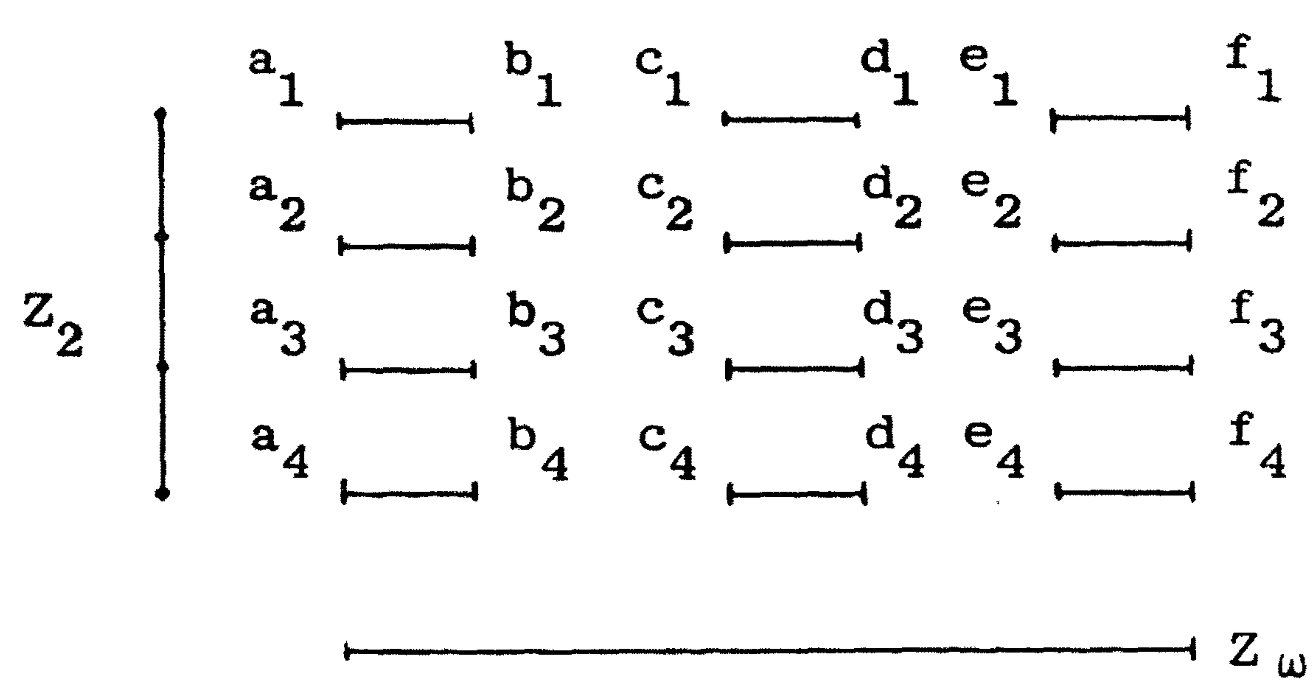
4.1. The lexigrographically ordered product X·Y will in the following sometimes be denoted by a figure

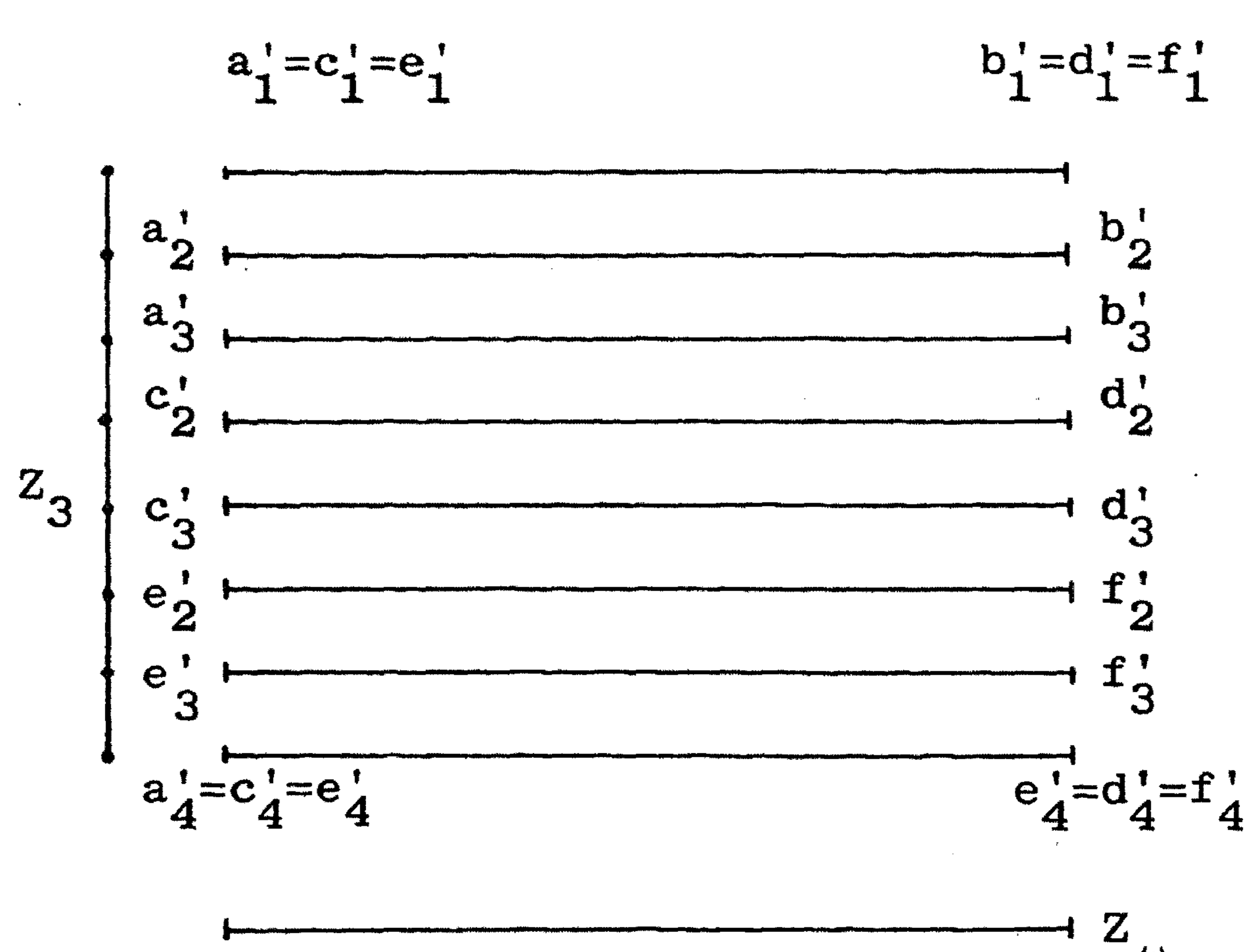


where the pairs (x,y) are thought to be ordered as described in

4.2. If X and Y are cor's and Y is the image of X under a conti
map, then it may happen that $\Theta(X) < \Theta(Y)$.

Example:





The map which is, for shortness sake, denoted by the following

$$\begin{cases} a_i b_i \to a_i' b_i' & (i=1,2,3,4) \\ c_i d_i \to c_i' d_i' & (i=1,2,3,4) \\ e_i f_i \to e_i' f_i' & (i=1,2,3,4) \end{cases}$$

is obviously a continuous map of $Z_{\omega+2}$ onto $Z_{\omega+3}$; but $\Theta(Z_{\omega+2}) = \omega+2 < \omega+3 = \Theta(Z_{\omega+3})$.

Theorem 13: If X is a connected cor and f is a continuous map of X onto the cor Y, then $\Theta(Y) \leqq \Theta(X)$.

Proof:

(i) The image of a closed interval is clearly a closed interval.

(ii) If μ is a limit number, let $\{X_i\}_{i<\mu}$ be a sequence of closed intervals in X, such that $X_i \subset X_j$ if $i > j$; and let $X^+ = \bigcap_{i<\mu} X_i$.

Then
$$f[X^+] = \bigcap_{i<\mu} f[X_i].$$

For suppose that
$$f[X^+] \subsetneq \bigcap_{i<\mu} f[X_i];$$

then take
$$u \in \bigcap_{i<\mu} f[X_i] \setminus f[X^+],$$

so
$$\forall i < \mu \quad \exists x_i \in X_i \setminus X^+ : u = f(x_i);$$

in each neighbourhood of X^+ there is an x_i; this means that at least one of the two points $v = \inf X^+$, $w = \sup X^+$ is an accumulation point of the set $\{x_i\}_{i<\mu}$, for instance v has this property; since however $f(x_i) = u$ for all i and since f is a continuous map, it follows that $f(v) = u$; consequently $u \in f[X^+]$.

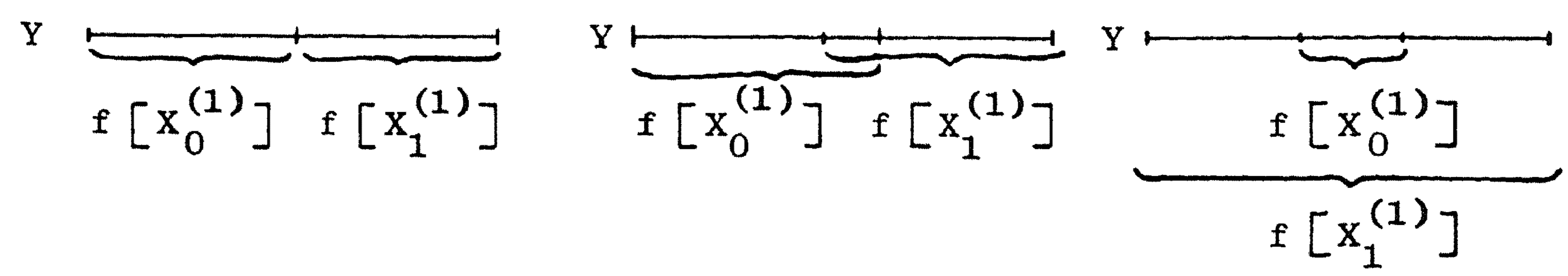
(iii) Now let $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p\in Z_\gamma}$ — be a θ-sequence for X, such that $\theta(V) = \Theta(X)$.

We show that, by transfinite induction, a θ-sequence $W = \{W_\gamma\}_\gamma$ — $W_\gamma = \{Y_p^{(\gamma)}\}_{p\in Z_\gamma}$ — for Y can be defined such that for all γ

$$\begin{cases} \forall p \in Z_\gamma \quad \exists q = q(p) \in Z_\gamma : Y_p^{(\gamma)} \subset f[X_q^{(\gamma)}] \\ q(p|\varepsilon) = q(p)|\varepsilon \quad \text{for all } \varepsilon < \gamma. \end{cases}$$

1. $Y^{(0)} = Y = f[X] = f[X^{(0)}]$.

Since $f[X_0^{(1)}]$ and $f[X_1^{(1)}]$ are closed intervals in Y, with union =Y, one of the following situations occurs (if necessary by changing the letters)



In all cases $Y_0^{(1)}$ and $Y_1^{(1)}$ can be defined in such a way that

$$\forall i (i=1,2)\ \exists j (j=1,2) : Y_i^{(1)} \subset f[X_j^{(1)}]$$

2. Now suppose that W_γ is defined for $\gamma<\delta$ such that for all those γ

$$(1) \quad \begin{cases} \forall p \in Z_\gamma \quad \exists q = q(p) \in Z_\gamma : Y_p^{(\gamma)} \subset f[X_q^{(\gamma)}] \\ q(p|\varepsilon) = q(p)|\varepsilon . \end{cases}$$

2.1. Let $\delta = \delta_1 + 1$.

$\forall p \in Z_{\delta_1} : \exists q \in Z_{\delta_1} : Y_p^{(\delta_1)} \subset f[X_q^{(\delta_1)}]$.

Since $f[X_{q0}^{(\delta)}]$ and $f[X_{q1}^{(\delta)}]$ are closed intervals in $f[X_q^{(\delta_1)}]$ with union $f[X_q^{(\delta_1)}]$ it is clear that in all possible situations $Y_{p0}^{(\delta)}$ and $Y_{p1}^{(\delta)}$ can be defined in such a way that

$$\forall i\ (i=1,2)\ \exists j\ (j=1,2) : Y_{pi}^{(\delta)} \subset f[X_{qj}^{(\delta)}] .$$

And this can be done for all $p \in Z_{\delta_1}$.

Consequently W_δ can be defined in such a way that (1) is satisfied for $\gamma = \delta$ too.

2.2. Let δ be a limit number.

Take $p \in Z_\delta$ and define $q = q(p)$ by

$$q|\varepsilon = q(p|\varepsilon) \quad \text{for all } \varepsilon < \delta .$$

Then it follows that

$$Y_p^{(\delta)} = \bigcap_{\varepsilon<\delta} Y_{p|\varepsilon}^{(\varepsilon)} \subset \bigcap_{\varepsilon<\delta} f\left[X_{q(p|\varepsilon)}^{(\varepsilon)}\right] = f\left[X_q^{(\delta)}\right] .$$

Consequently W_δ can be defined in such a way that ① is satisfied for $\gamma = \delta$.

(iii) If $\mu = \Theta(X)$, then for the θ-sequence W which was defined above

$$\forall p \in Z_\mu \ \exists q \in Z_\mu : Y_p^{(\mu)} \subset f\left[X_q^{(\mu)}\right] .$$

As $|X_q^{(\mu)}| = 1$ and so $|f[X_q^{(\mu)}]| = 1$ for all $q \in Z_\mu$, it follows that $|Y_p^{(\mu)}| = 1$ for all $p \in Z_\mu$.

This means that $\theta(W) \leqq \mu = \Theta(X)$.

Consequently $\Theta(Y) \leqq \Theta(X)$.

<u>Theorem 14</u>: If $\alpha \leqq \beta$ then Z_α is a continuous image of Z_β .

<u>Proof</u>:

The natural map $f : p \to p|\alpha$ of Z_β onto Z_α is obviously continuous.

<u>Theorem 15</u>: If X is a cor there exists a least α, say α_o, such that X is a continuous image of Z_α. Moreover $\alpha_o \leqq \Theta(X)$.

<u>Proof</u>:

If $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — is a θ-sequence for X, and $\mu = \theta(V)$, then $|X_p^{(\mu)}| = 1$ for all $p \in Z_\mu$. Say $X_p^{(\mu)} = \{x_p^{(\mu)}\}$ for all $p \in Z_\mu$.

Then $\phi : p \to x_p^{(\mu)}$ is a continuous map of Z_μ onto X.

<u>Remark</u>: It may happen that $\alpha_o < \Theta(X)$.

Example: $\alpha_o(Z_{\omega+3}) \leqq \omega+2 < \omega+3 = \Theta(Z_{\omega+3})$.

4.3. Let X be a cor.

Let $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — be a θ-sequence for X.

Let

$$D_\lambda = D_\lambda(V) = \{ l(X_p^{(\lambda)}),\ r(X_p^{(\lambda)}) \mid p \in Z_\lambda \} .$$

It is clear that

(i) $p \in Z_\mu, \sigma<\mu \implies l(X_{p|\sigma}^{(\sigma)}) \leq l(X_p^{(\mu)}) \leqq r(X_p^{(\mu)}) \leqq r(X_{p|\sigma}^{(\sigma)})$

(ii) $\tau > \nu \Longrightarrow D_\tau \supset D_\nu$

(iii) $\tau \geq \nu \Longrightarrow D_\nu \cap (l(X_p^{(\tau)}),\ r(X_p^{(\tau)})) = \emptyset$

(iv) $|D_\tau| \leq 2^{|\tau|}$.

<u>Theorem 16</u>: a. D_τ is closed in X

b. If τ is a limit number, then $D_\tau = \overline{\bigcup_{\nu<\tau} D_\nu}$.

<u>Proof</u>:

a. Without loss of generality we may suppose that $D_\tau \subseteq X$.
If $y \in X \setminus D_\tau$ there exists a $p \in Z_\tau$, such that $y \in X_p^{(\tau)}$; since $y \neq l(X_p^{(\tau)}),\ r(X_p^{(\tau)})$, it follows that

$$y \in (l(X_p^{(\tau)}),\ r(X_p^{(\tau)})) \subset X \setminus D_\tau .$$

Consequently $X \setminus D_\tau$ is open and D_τ is closed.

b. Since $\bigcup_{\nu<\tau} D_\nu \subset D_\tau$, it follows that also

$$\overline{\bigcup_{\nu<\tau} D_\nu} \subset \overline{D}_\tau = D_\tau \ ;$$

now take, if possible,

$$x \in D_\tau \setminus \bigcup_{\nu<\tau} D_\nu \ ;$$

then for some $p \in Z_\tau$

$$x = l(X_p^{(\tau)}) \text{ or } x = r(X_p^{(\tau)}).$$

Since

$$X_p^{(\tau)} = \bigcap_{\nu<\tau} X_{p|\nu}^{(\nu)} ,$$

one has

$$l(X_p^{(\tau)}) = \sup_{\nu<\tau} l(X_{p|\nu}^{(\nu)}),\ r(X_p^{(\tau)}) = \inf_{\nu<\tau} r(X_{p|\nu}^{(\nu)});$$

hence

$$x \in \overline{\bigcup_{\nu<\tau} D_\nu} .$$

If $\theta = \theta(V)$ there does not necessarily exist an $x \in X$ with the property that $\mu_x = \theta$.

Example:

Let f be a 1-1-map of a subset A of Z_ω onto $W(\Omega)$.

Let H be the set of all pairs

$$\begin{cases} (a,x_a) & \text{if } a \in A,\ x_a \in Z_{f(a)} \\ (a,0) & \text{if } a \in Z_\omega \setminus A \end{cases}$$

ordered by

$$\begin{cases} (a,u) < (b,v) & \text{if } a < b \text{ in } Z_\omega \\ (a,u) < (a,v) & \text{if } u=0,\ v\neq 0 \text{ or if } u < v \text{ in } Z_{f(a)}. \end{cases}$$

It is clear that H is a cor

(i) Since $Z_\mu \subset H$ for all $\mu < \Omega$, it follows that $\Theta(H) \geq \Theta(Z_\mu) = \mu$ for all $\mu < \Omega$, and so $\Theta(H) \geq \Omega$.

On the other hand there is a θ-sequence V for H with the property $\theta(V) = \Omega$ (namely "the regular θ-sequence for Z_ω, for each $a \in A$ continued by the regular θ-sequence for $Z_{f(a)}$").

Consequently $\Theta(H) = \Omega$.

(ii) If V is the θ-sequence for H which is mentioned in (i), there does not exist an $x \in H$ such that $\mu_x(V) = \Omega$.

(iii) H satisfies the first axiom of countability.

<u>Theorem 17</u>: Let V be a θ-sequence for the cor X; let $\mu_x = \mu_x(V)$, $\theta = \theta(V)$.

If for some $x \in X$ it is true that

$$\mu_x \geq \omega_{\varkappa}$$

(and this is certainly the case, if $\theta > \omega_{\varkappa}$), then there exists a (decreasing or increasing) sequence of type $\omega_{\varkappa}$ in X.

<u>Proof</u>:

In all cases there exists a $p \in Z_{\omega_{\varkappa}}$ such that

$$\text{①} \qquad X_p^{(\omega_{\varkappa})} = \bigcap_{\nu < \omega_{\varkappa}} X_{p|\nu}^{(\nu)}$$

$$\text{②} \qquad X_{p|\nu}^{(\nu)} \subset X_{p|\tau}^{(\tau)} \quad \text{if } \tau < \nu < \omega_{\varkappa}$$

If one puts

$$\begin{cases} a^{(\tau)} = l(X_{p|\tau}^{(\tau)}),\ b^{(\tau)} = r(X_{p|\tau}^{(\tau)}) & \text{if } \tau < \omega_{\varkappa} \\ a^* = l(X_p^{(\omega_{\varkappa})}),\ b^* = r(X_p^{(\omega_{\varkappa})}), \end{cases}$$

then it follows from (2) that all elements of the sequence of ordered pairs

$$\{ \overline{a^{(\tau)}, b^{(\tau)}} \}_{\tau < \omega_{\aleph}}$$

are different; thus

$$\left| \{ \overline{a^{(\tau)}, b^{(\tau)}} \}_{\tau < \omega_{\aleph}} \right| = \aleph .$$

Now define the sequence $\{ a^{(\tau_\mu)} \}_\mu$ by transfinite induction in the following way

$a^{(\tau_0)} = a^{(0)}$,

if $a^{(\tau_\nu)}$ has been defined for all $\nu < \mu$, and if λ is the least index such that $a^{(\lambda)} \neq$ all $a^{(\tau_\nu)}$ $(\nu < \mu)$, then let $a^{(\tau_\mu)} = a^{(\lambda)}$;

define the sequence $\{ b^{(\tau_\nu)} \}_\nu$ in an analogous way.
If both the type of $\{ a^{(\tau_\mu)} \}_\mu$ and the type of $\{ b^{(\tau_\nu)} \}_\nu$ are less than $\omega_{\aleph}$, then it follows that

$$\left| \{ a^{(\tau_\mu)} \}_\mu \right| = \aleph_1 < \aleph , \quad \left| \{ b^{(\tau_\nu)} \}_\nu \right| = \aleph_2 < \aleph$$

so

$$\left| \{ a^{(\mu)} \}_{\mu < \omega_{\aleph}} \right| = \aleph_1 \quad , \quad \left| \{ b^{(\nu)} \}_{\nu < \omega_{\aleph}} \right| = \aleph_2$$

and so

$$\left| \{ \overline{a^{(\mu)}, b^{(\nu)}} \}_{\mu < \omega_{\aleph} ,\ \nu < \omega_{\aleph}} \right| = \aleph_1 \cdot \aleph_2 < \aleph$$

and a forteriori

$$\left| \{ \overline{a^{(\tau)}, b^{(\tau)}} \}_{\tau < \omega_{\aleph}} \right| < \aleph ;$$

this is a contradiction.
Consequently at least one of the sequences $\{ a^{(\tau_\mu)} \}_\mu$, $\{ b^{(\tau_\nu)} \}_\nu$ has the type $\omega_{\aleph}$; for instance this holds for $\{ a^{(\tau_\mu)} \}_\mu$. Then a^* is the limit of a sequence of type $\omega_{\aleph}$.

Theorem 18: If $|X| > 2^{\aleph_i^{\aleph}}$ and V is a θ-sequence for X, then $\theta(V) \geqq \omega_{i+1}$.

If $\theta(V) = \omega_{i+1}$ then moreover there exists an $x \in X$ such that $\mu_x(V) = \omega_{i+1}$.

Proof:

If $\tau < \omega_{i+1}$ then

$$|D_\tau| \leq 2^{|\tau|} \leq 2^{\aleph_i},$$

$$\left|\bigcup_{\tau<\omega_{i+1}} D_\tau\right| \leq \aleph_{i+1} \; 2^{\aleph_i} \leq 2^{\aleph_i},$$

(1) $$\bigcup_{\tau<\omega_{i+1}} D_\tau \subset X,$$

$$\theta(V) \geq \omega_{i+1}.$$

If $\theta(V) = \omega_{i+1}$ then it follows from (1) that $\mu_x = \omega_{i+1}$ for every $x \in X \setminus \bigcup_{\tau<\omega_{i+1}} D_\tau$.

Theorem 19: If $|X| > 2^{\aleph_i}$ there is a point $x \in X$ that is the limit of a sequence of type ω_{i+1}, or there is a point $y \in X$ that is the limit of a sequence of type ω^*_{i+1}.

Proof: Follows from theorem 17 and theorem 18.

Corollary: If $|X| > \mathfrak{c} = 2^{\aleph_0}$ then X does not satisfy the first axiom of countability.

Assertions in which the (generalized) continuum hypothesis is used will in the following be denoted by an asterisk. This will be the case among others if one of the following assumptions is used:

(i) $2^{\mathfrak{m}}$ is the least cardinal number $> \mathfrak{m}$;
also: $\mathfrak{m} < \mathfrak{n} \to 2^{\mathfrak{m}} \leq \mathfrak{n}$

(ii) if $\mathfrak{n}$ is a limit cardinal and $\mathfrak{m} < \mathfrak{n}$ then also $2^{\mathfrak{m}} < \mathfrak{n}$

(iii) $2^{\mathfrak{m}}$ is not a limit cardinal and $\omega_{2^{\mathfrak{m}}}$ is regular.

*Theorem 20: If $|X| > \aleph$ and V is a θ-sequence for X, then $\theta(V) \geq \aleph$.

If $\theta(V) = \aleph$ then moreover there exists an $x \in X$ such that $\mu_x(V) = \omega_\aleph$.

Proof:

If $\tau < \omega_\aleph$ then

$$|D_\tau| \leq 2^{|\tau|} \overset{*}{\leq} \aleph,$$

$$\left|\bigcup_{\tau<\omega_\aleph} D_\tau\right| \leq \aleph \cdot \aleph = \aleph,$$

etc. (cf. the proof of theorem 18).

*Theorem 21: If $|X| > \aleph$ there is a point $x \in X$ that is the limit of a sequence of type $\omega_\aleph$, or there is a point $y \in X$ that is the limit of a sequence of type $\omega_\aleph^*$.

Proof: Theorem 17 and theorem 20.

Theorem 22: If in X there exists a sequence of type $\omega_\aleph$ or a sequence of type $\omega_\aleph^*$, then

$$\theta(V) \geq \omega_\aleph$$

for every θ-sequence V for X.[1]

Proof:

Let $\{x_i\}_{i < \omega_\aleph}$ be an increasing sequence of type $\omega_\aleph$ in X and let $y = \sup_{i < \omega_\aleph} x_i$.

1. Let $\omega_\aleph$ be regular.

Let $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — be a θ-sequence for X.

(i) Now consider the set D of all $\gamma < \theta(V)$ with the property that there exists a $q(\gamma) \in Z_\gamma$ such that

$$l(X_{q(\gamma)}^{(\gamma)}) < y \leq r(X_{q(\gamma)}^{(\gamma)});$$

it is clear that $D = \{\gamma | \gamma < \delta\}$ for some $\delta > 0$.

If $X_p^{(\delta)}$ is the intersection of all $X_{q(\gamma)}^{(\gamma)}$ (so $p|\gamma = q(\gamma)$ for all $\gamma < \delta$), then one has $l(X_p^{(\delta)}) = y$.

Since $y = \lim_{i<\omega_\aleph} x_i$ is not a left neighbour if follows that δ is a limit number; moreover, since $|X_{p|\gamma}^{(\gamma)}| \geq 2$ if $\gamma<\delta$, it follows that $\theta(V) \geq \delta$.

(ii) If now x_{i_γ} is the least x_i such that $x_i > l(X_p^{(\gamma)})$, then $\{x_{i_\gamma}\}_{\gamma<\delta}$ is a non-decreasing sequence and $\lim_{\gamma<\delta} x_{i_\gamma} = y$. Because of the regularity of $\omega_\aleph$ one concludes that $\delta \geq \omega_\aleph$.

Consequently $\theta(V) \geq \omega_\aleph$.

2. Let $\omega_\aleph$ be singular.

Then $\omega_\aleph$ is the limit of a sequence of regular ordinals $\omega_{\alpha+1} < \omega_\aleph$:

$$\lim_{\alpha < \lambda} \omega_{\alpha+1} = \omega_\aleph.$$

Since for all $\alpha < \lambda$ there exists an increasing sequence $\{x_i\}_{i < \omega_{\alpha+1}}$ in

1) In the case of a connected cor this theorem is also contained in Novak [3].

X it follows from 1. that $\theta(V) \geqq \omega_{\alpha+1}$ for all $\alpha<\lambda$. This means that $\theta(V) \geqq \omega$.

Finally it should be observed, that there exist cor's with the property that two θ-sequences V and V' can be constructed such that $|\theta(V)| \neq |\theta(V')|$.

Essentially following Novak [3] , p.383 an example can be obtained as follows.

For every countable α there exists a θ-sequence $V(\alpha)$ for Z^{*}_{ω} such that $\theta(V(\alpha)) \geqq \alpha$. Now let f be a 1-1-map of $W(\Omega)$ into Z^{*}_{ω}. Then define a θ-sequence V for $X = X_1 \cdot X_2$ ($X_1 = X_2 = Z^{*}_{\omega}$) such that V is the regular θ-sequence for X_1, "for each $a \in f[W(\Omega)]$ continued by the θ-sequence $V(f^{-1}(a))$ for X_2". Clearly $\theta(V) = \Omega$.

On the other hand it is easy to construct a θ-sequence V' for X, such that $\theta(V') = \omega + \omega$.

§ 5.

Let A_n be the set of positive integers $\leqq n$ in natural ordering.

Let I be the unit interval $[0,1]$.

Define X_n by $X_n = I . A_n$.

<u>Theorem 24</u>: X_n and X_m are different topological spaces if $n \neq m$.

<u>Proof</u>:

Suppose $n > m$.

If $n \geqq 2$, $m=1$ then X_n is totally disconnected and X_m is connected.

If $n \geqq 3$, $m=2$ then X_n has continuously many isolated points and X_m has two isolated points.

Now suppose $n > m > 2$.

A set $\{(a,2),(a,3),\dots,(a,n-1)\}$ $(a \in I)$ of n-1 successive isolated points in X_n will be denoted by $B^{(n)}_a$; and $B^{(m)}_a$ $(\subset X_m)$ is defined in an analogous way.

(i) If S and T are two disjoint sets of isolated points in X_n with the property that for all $a \in I$

$$S \cap B^{(n)}_a \neq \emptyset \Longleftrightarrow T \cap B^{(n)}_a \neq \emptyset$$

then it is clear that

$$\overline{S} \setminus S = \overline{T} \setminus T$$

(ii) Now suppose there is a topological map f of X_n onto X_m.

a. If p and q ($p < q$) are points in X_m such that the set $\{x \mid p < x < q\}$ is infinite, then there exists a $B_c^{(n)}$ such that $p < r < q$ for all $r \in f[B_c^{(n)}]$.

We can show this in the following way: Take an infinite sequence $\{y_i\}_{i<\omega}$ of points in X_m between p and q; now, if the assertion is not true, for every y_i there exists a z_i such that $z_i \leqq p$ or $z_i \geqq q$ whereas $f^{-1}(y_i)$ and $f^{-1}(z_i)$ belong to the same $B_a^{(n)}$; then the sets $\{y_i\}_{i<\omega}$ and $\{z_i\}_{i<\omega}$ have different accumulation points, whereas the sets $\{f^{-1}(y_i)\}_{i<\omega}$ and $\{f^{-1}(z_i)\}_{i<\omega}$ have the same accumulation points.

b. Take a $B_{a_1}^{(n)}$; since $n > m$ there are two points p_1 and q_1 ($p_1 < q_1$) in $f[B_{a_1}^{(n)}]$ such that $\{x \mid p_1 < x < q_1\}$ is an infinite set. Now choose $B_{a_2}^{(n)}$ in such a way that $p_1 < r < q_1$ for all $r \in f[B_{a_2}^{(n)}]$. There exist two points p_2 and q_2 ($p_2 < q_2$) in $f[B_{a_2}^{(n)}]$ such that $\{x \mid p_2 < x < q_2\}$ is an infinite set.

Etcetera.

We thus obtain two sequences $\{p_i\}_{i<\omega}$ and $\{q_i\}_{i<\omega}$ in X_m which have different accumulation points, whereas the sets $\{f^{-1}(p_i)\}_{i<\omega}$ and $\{f^{-1}(q_i)\}_{i<\omega}$ have the same accumulation points.

CHAPTER II

<u>On the homogeneity of a compact ordered space</u>

§1.

A topological space T is called <u>homogeneous</u>, if for every $p,q \in T$ there exists an autohomeomorphism f of T with the property $f(p)=q$.

<u>Theorem 1</u>: A homogeneous cor X satisfies the first axiom of countability.

<u>Proof</u>:

Since X is compact, every countable infinite set $\{x_i\}_{i<\omega}$ has an accumulation point, say y. Then y is the limit of a countable sequence, and so, since X is homogeneous, also $a=\inf X$ is the limit of a countable sequence. Consequently in a there is a countable local base. Because of the homogeneity of X this means that X satisfies the first axiom of countability.

<u>Theorem 2</u>: If X is a cor, and $|X| > \mathfrak{c}$, then X is not homogeneous.

<u>Proof</u>: Chapter I, theorem 19, corollary and theorem 1.

<u>Theorem 3</u>: A homogeneous cor X is zero-dimensional.

<u>Proof</u>:

Let Y be a component of X. If $|Y| > 1$, let $a=\inf Y$, $b=\sup Y$ and take c such that $a < c < b$. If now C_x denotes the component of X to which x belongs, then obviously $C_a \setminus \{a\} = Y \setminus \{a\}$ is a connected subspace of X, whereas $C_c \setminus \{c\} = Y \setminus \{c\}$ is a disconnected subspace of X. This means that X is not homogeneous. Consequently $|Y| = 1$ and X is zero-dimensional.

§2.

The following lemma presumably will be known.

<u>Lemma</u>: If α and $\beta=\omega^\delta$ are countable limit ordinals, and $\alpha<\beta$, then there exists an increasing sequence $(\mu_i)_{i<\alpha}$ of type α, such that $\lim_{i<\alpha} \mu_i = \beta$.

Proof:

1. We first observe that for every countable limit ordinal τ there exists a sequence $(\sigma_i)_{i<\omega}$ of type ω, such that $\lim_{i<\omega} \mu_i = \tau$: if the set $W(\tau)$ is well ordered like a sequence of type ω,

$$W(\tau) = (\nu_i)_{i<\omega},$$

then for the sequence $(\sigma_i)_{i<\omega}$ one can take an increasing subsequence of $(\nu_i)_{i<\omega}$.

2. Now we show that it is sufficient to prove the lemma for ordinal numbers α of the form $\alpha = \omega^\gamma$ $(1 \leq \gamma < \delta)$.

Let
$$\alpha = \omega^{\gamma_1}.n_1 + \omega^{\gamma_2}.n_2 + \dots + \omega^{\gamma_\kappa}.n_\kappa$$

$(n_i > 0$ if $i=1,2,\dots,\kappa; \delta > \gamma_1 > \gamma_2 > \dots > \gamma_\kappa > 0)$, and thus

$$\alpha = \alpha' + \omega^{\gamma_\kappa}.$$

Now, if $(\nu_i)_{i<\omega^{\gamma_\kappa}}$ is an increasing sequence with limit ω^δ, we define

$$\begin{cases} \mu_i = i & \text{if } i < \alpha' \\ \mu_i = \alpha' + \nu_i & \text{if } \alpha' \leq i < \alpha \ ; \end{cases}$$

then also $(\mu_i)_{i<\alpha}$ is an increasing sequence with limit ω^δ.

3. We now prove the lemma by transfinite induction with respect to δ.

(i) if $\delta=1$ the assertion is obvious

(ii) suppose the lemma is proved for $\delta < \varepsilon$

(ii,1) Let $\varepsilon = \delta_1 + 1$.

Then $\beta = \omega^\varepsilon = \omega^{\delta_1+1} = \omega^{\delta_1}.\omega = \beta_1.\omega$.

Since $\alpha = \omega^\gamma < \beta$ we have $\alpha \leq \beta_1$.

Now β_1 is the limit both of an increasing sequence $(\nu_i)_{i<\alpha}$ and of an increasing sequence $(\lambda_j)_{j<\omega}$; if we define

$$\begin{cases} \mu_i = \nu_i & \text{for all i such that } \nu_i < \lambda_0 \\ \mu_i = \beta_1 j + \nu_i & \text{for all i such that } \lambda_{j-1} \leq \nu_i < \lambda_j, \end{cases}$$

then $(\mu_i)_{i<\alpha}$ is an increasing sequence with limit β.

(ii,2) Let ε be a limit number.

Then ε is the limit of an increasing sequence $(\varepsilon_n)_{n<\omega}$, and

$$\beta = \omega^{\varepsilon} = \lim_{n<\omega} \omega^{\varepsilon_n}.$$

Since $\beta > \alpha$ there exists an integer $N < \omega$ such that $\omega^{\varepsilon_n} > \alpha$ if $N \leqq n < \omega$; without loss of generality we may suppose that N=0.

Now, if $V_0 = \{\nu \mid \nu < \omega^{\varepsilon_0}\}$ and $V_n = \{\nu \mid \omega^{\varepsilon_{n-1}} \leqq \nu < \omega^{\varepsilon_n}\}$ (n=1,2,3,...), then each ω^{ε_n} is the limit of a sequence $(\nu_i^{(n)})_{i<\alpha}$ of elements $\nu_i^{(n)} \in V_n$. And α itself is the limit of a sequence $(\lambda_j)_{j<\omega}$.

Now, putting

$$\begin{cases} \mu_i = \nu_i^{(0)} & \text{if } i < \lambda_0 \\ \mu_i = \nu_i^{(n)} & \text{if } \lambda_{n-1} \leqq i < \lambda_n \quad (n=1,2,3,\ldots), \end{cases}$$

we find a sequence $(\mu_i)_{i<\alpha}$ with the limit β.

In the following, we denote by $Z'_{\omega+1}$ the cor which is obtained from $Z_{\omega+1}$ by removing the isolated points. Clearly $Z'_{\omega+1}$ is similar to $I.\{0,1\} \setminus \{(0,0),(1,1)\}$, where I is the unit interval $[0,1]$.

<u>Theorem 4</u>: Let $X = Z_{\omega^\alpha}$, $|\alpha| \leqq \aleph_0$; or let $X = Z'_{\omega+1}$.

1. If p is not a jump point, or if p is a left neighbour, then $\{x \mid x \leqq p\} \backsim Z_{\omega^\alpha}$.
2. If p is not a jump point, or if p is a right neighbour, then $\{x \mid p \leqq x\} \backsim Z_{\omega^\alpha}$.

<u>Proof</u>: (for the case that $X = Z_{\omega^\alpha}$, $|\alpha| \leqq \aleph_0$)

a. Let $L = \{p \mid \exists i_0 < \omega^\alpha: \; p_i = 1 \text{ if } i \geqq i_0\}$
$R = \{p \mid \exists i_0 < \omega^\alpha: \; p_i = 0 \text{ if } i \geqq i_0\}$.

In both cases we suppose that i_0 is the least index with the required property.

It is clear that a left neighbour (right neighbour) belongs to L (to R), and moreover, that a point of L (of R) is a left neighbour (right neighbour) if and only if i_0 is a non-limit number.

b. The following notation is used: if

$$ab\ldots c\ldots d\ldots e\ldots$$

denotes a well-ordered sequence $(t_i)_{i<\mu}$ of type μ, then

$$ab\ldots\overset{\beta}{c}\ldots d\ldots\overset{\gamma}{e}\overrightarrow{eee\ldots}$$

means that c is the element with index β in the given sequence (thus $c = t_\beta$) and that $t_i = e$ if $i \geq \gamma$.

(i) Let $p \in L$, and let $(m_\lambda)_\lambda$ be the well-ordered sequence of indices for which $p_{m_\lambda} = 1$; then $(m_\lambda)_\lambda$ is a sequence of type ω^α.
Now define

$$V_0 = \{x \mid x \leq p_0 p_1 \ldots \overset{m_0}{0}\ \overrightarrow{1111\ldots}\}$$

$$V_\lambda = \{x \mid p_0 p_1 p_2 \ldots \overset{m_{\lambda-1}}{1}\ \overrightarrow{000\ldots} \leq x \leq p_0 p_1 p_2 \ldots \overset{m_\lambda}{0}\ \overrightarrow{1111\ldots}\}$$

if λ is a non-limit number

$$= \{x \mid p_0 p_1 p_2 \ldots \overset{n_\lambda}{\overrightarrow{000\ldots}} \leq x \leq p_0 p_1 p_2 \ldots \overset{m_\lambda}{0}\ \overrightarrow{1111\ldots}\}$$

if λ is a limit number and $n_\lambda = \lim_{i<\lambda} m_i$.

Also

$$W_0 = \{x \mid x \leq 0\ \overrightarrow{1111\ldots}\}$$

$$W_\lambda = \{x \mid 111\ldots \overset{\lambda-1}{1}\ \overrightarrow{000\ldots} \leq x \leq 1111\ldots \overset{\lambda}{0}\ \overrightarrow{1111\ldots}\}$$

if λ is a non-limit number

$$= \{x \mid 111\ldots \overset{\lambda}{0}\ \overrightarrow{0000\ldots} \leq x \leq 1111\ldots \overset{\lambda}{0}\ \overrightarrow{1111\ldots}\}$$

if λ is a limit number.

It is clear that all sets V_λ, W_λ $(0 \leq \lambda < \omega^\alpha)$ are similar to Z_{ω^α}.
Then also the ordered unions

$$\bigcup_{\lambda<\omega^\alpha} V_\lambda \quad \text{and} \quad \bigcup_{\lambda<\omega^\alpha} W_\lambda$$

are similar, and consequently

$$\{x \mid x \leq p\} \backsim Z_{\omega^\alpha}.$$

(ii) In the same way it can be proved, that

$$\{x \mid p \leqq x\} \backsim Z_{\omega^\alpha}$$

if $p \in R$

(iii) Now it follows from (i) and (ii) that

$$\{x \mid p \leqq x \leqq q\} \backsim Z_{\omega^\alpha}$$

if $p \in R$, $q \in L$, $p < q$.

c. (i) If β is a countable limit ordinal, and $\beta \leqq \omega^\alpha$, then Z_{ω^α} may be considered as the ordered union of type β of sets A_i $(0 \leqq i \leqq \beta)$, which are such that $A_i \backsim Z_{\omega^\alpha}$ if $0 \leqq i < \omega^\alpha$ and $|A_\beta| = 1$.
For, there exists an increasing sequence $(\mu_i)_{i<\beta}$ with the limit ω^α, and we may take

$$A_0 = \{x \mid x \leqq 111\ldots\overbrace{0}^{\mu_0}\ \overrightarrow{1111\ldots}\ \}$$

$$A_i = \{x \mid 111\ldots\overbrace{1}^{\mu_{i-1}}\ \overrightarrow{000\ldots} \leqq x \leqq 111\ldots\overbrace{0}^{\mu_i}\ \overrightarrow{1111\ldots}\ \}$$

if i is a non-limit number

$$= \{x \mid 111\ldots\overbrace{0}^{\nu_i}\ \overrightarrow{000\ldots} \leqq x \leqq 111\ldots\overbrace{0}^{\mu_i}\ \overrightarrow{1111\ldots}\}$$

if i is a limit number and $\nu_i = \lim\limits_{j<i} \mu_j$

(ii) In the same way we may show: If β is a countable limit ordinal, and $\beta \leqq \omega^\alpha$, then Z_{ω^α} may be considered as the ordered union of type β^* of sets A_i $(0 \leqq i \leqq \beta)$, which are such that $A_i \backsim Z_{\omega^\alpha}$ if $0 \leqq i < \omega^\alpha$ and $|A_\beta| = 1$.

d. (i) Let p be a non-jump point and let $(m_\lambda)_{\lambda<\beta}$ be the well-ordered sequence of indices, for which $p_{m_\lambda} = 1$; then β is a limit number.
Now take

$$B_0 = \{x \mid x \leqq p_0 p_1 \ldots \overbrace{0}^{m_0}\ \overrightarrow{1111\ldots}\ \}$$

$$B_i = \{x \mid p_0 p_1 \ldots \overbrace{1}^{m_{i-1}}\ \overrightarrow{000\ldots} \leqq x \leqq p_0 p_1 \ldots \overbrace{0}^{m_i}\ \overrightarrow{1111\ldots}\ \}$$

if i is a non-limit number

$$= \{x \mid p_0 p_1 \ldots \overbrace{0}^{n_i}\ \overrightarrow{000\ldots} \leqq x \leqq p_0 p_1 \ldots \overbrace{0}^{m_i}\ \overrightarrow{1111\ldots}\ \}$$

if i is a limit number and $n_i = \lim\limits_{j<i} m_j$.

It now easily follows from b that

$$\bigcup_{i<\beta} B_i \backsim \bigcup_{i<\beta} A_i,$$

and so

$$\{x \mid x \leq p\} \backsim Z_{\omega^\alpha}$$

(ii) In the same way it is shown that

$$\{x \mid p \leq x\} \backsim Z_{\omega^\alpha} .$$

Theorem 5: If $|\alpha| \leq \aleph_0$, then Z_{ω^α} is a homogeneous topological space. Also $Z'_{\omega+1}$ is a homogeneous space.

Proof:

(i) If I is a clopen interval which is properly contained in Z_{ω^α}, and which is such that $I \backsim Z_{\omega^\alpha}$, then also $(Z_{\omega^\alpha} \setminus I) \backsim Z_{\omega^\alpha}$. For, if p = inf I, q = sup I, then at most one of the sets $I_p = \{x \mid x < p\}$, $I_q = \{x \mid x > q\}$ is void; if $I_p \neq \emptyset$ (and/or $I_q \neq \emptyset$), then $I_p \backsim Z_{\omega^\alpha}$ (and/or $I_q \backsim Z_{\omega^\alpha}$); and in all three possible cases we have $I_p \cup I_q \backsim Z_{\omega^\alpha}$

(ii) Now take $p,q \in Z_{\omega^\alpha}$; $p < q$.
Then p (respectively q) is the intersection of a decreasing sequence of clopen intervals I_n (respectively J_n). Without loss of generality we may suppose that $I_1 \cap J_1 = \emptyset$.
Let f_0 be an order-preserving map of $Z_{\omega^\alpha} \setminus I_1$ onto $Z_{\omega^\alpha} \setminus J_1$, and let f_n be an order-preserving map of $I_n \setminus I_{n+1}$ onto $J_n \setminus J_{n+1}$ (n=1,2,3,...).
Then the function f, defined by

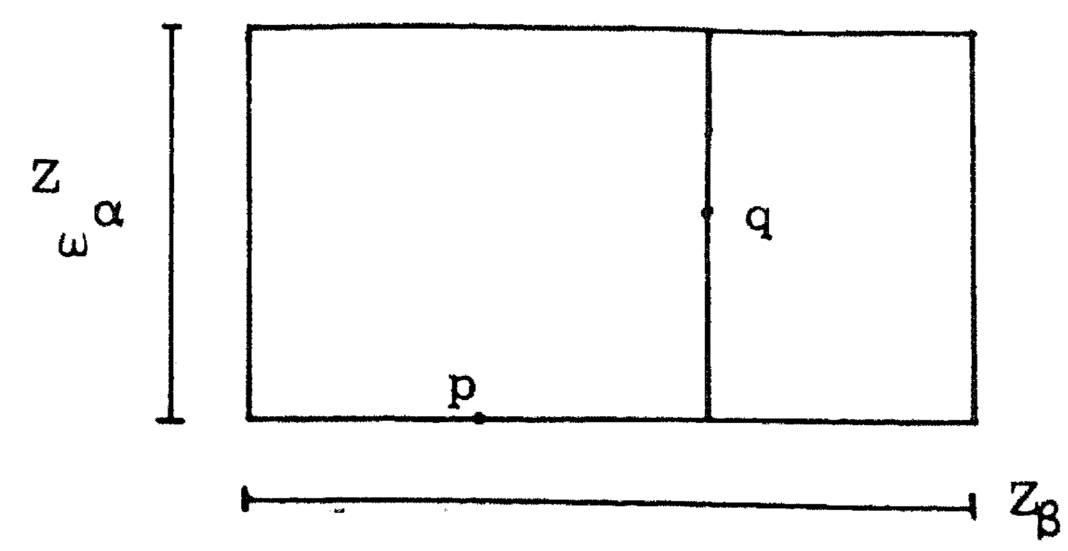
$$\begin{cases} f(x) = f_n(x) & \text{if } x \in I_n \setminus I_{n+1} \\ f(p) = q & \end{cases}$$

is an autohomeomorphism of Z_{ω^α}.
Consequently Z_{ω^α} is homogeneous.

Theorem 6: If $\gamma = \beta + \omega^\alpha$, and $\beta \geq \omega^\alpha$, then Z_γ is not homogeneous.

Proof:

Without loss of generality we may suppose $\beta = \delta + \omega^\varepsilon, \varepsilon \geq \alpha$.



(i) Choose $p = (p_i)_{i<\gamma}$ in such a way that

$$\begin{cases} \forall i<\beta \quad \exists j,k : (i<j,k<\beta \text{ and } p_j=0,\ p_k=1) \\ \forall i \geqq \beta \quad : p_i=0. \end{cases}$$

Then it is clear, that each neighbourhood O_p of p contains a subset which is similar to $Z_{\omega^\varepsilon+\omega^\alpha}$, and so for every neighbourhood O_p of p we have

$$O(O_p) \geqq \omega^\varepsilon + \omega^\alpha$$

(cf. Ch.I, theorem 10)

(ii) Now choose $q = (q_i)_{i<\gamma}$ in such a way that

$$\forall i \quad \exists j,k : (i<j,k \quad \text{and } p_j=0,\ p_k=1).$$

Then there exists neighbourhoods O_q of q, which are similar to Z_{ω^α}, and for which consequently

$$O(O_q) = \omega^\alpha$$

(iii) This means that $Z_{\beta+\omega^\alpha}$ is not homogeneous.

§3.

If X is a connected cor, then X is said to be <u>order-homogeneous</u>, if all closed intervals consisting of more than one point, are similar (and so are similar to X).

<u>Theorem 7</u>: An order-homogeneous connected cor X satisfies the first axiom of countability.

<u>Proof</u>:

Since X is connected, there is an increasing sequence $(x_i)_{i<\omega}$ in X;

and since X is compact $y = \lim_{i<\omega} x_i$ exists. Because of the order-homogeneity it follows that every $z \in X$, $z \neq \inf X$, may be considered as the limit of an increasing sequence of type ω. In the same way it is shown that every $z \in X$, $z \neq \sup X$, may be considered as the limit of a decreasing sequence of type ω .

This means that X satisfies the first axiom of countability.

Theorem 8: If X is a connected cor, and $|X| > \aleph$, then X is not order-homogeneous.

Proof:

Theorem 7 and chapter I, theorem 19.

The following result has been obtained before by Terasaka [1] (cf. also Arens [1]).

Theorem 9: If $|\alpha| \leq \aleph_0$, then $Z^*_{\omega^\alpha}$ is an order-homogeneous topological space.

Proof:

Following the method used in the proof of theorem 4, we can easily show that $\{x|x \leq p\} \backsim Z^*_{\omega^\alpha}$ for all $p > \inf Z^*_{\omega^\alpha}$ and that $\{x|x \geq p\} \backsim Z^*_{\omega^\alpha}$ for all $p < \sup Z^*_{\omega^\alpha}$. From this it immediately follows that $Z^*_{\omega^\alpha}$ is order-homogeneous.

If X is a connected cor, we denote by X^+ the topological space which is obtained from X by identification of inf X and sup X.

Theorem 10: If $\alpha = \nu+n$, $\beta = \mu+m$, where ν and μ are limit numbers (or 0) and n,m are integers ≥ 0, then Z^{*+}_α and Z^{*+}_β are different topological spaces if $\nu \neq \mu$.

Proof: Clearly $\tau(Z^{*+}_\alpha) = \nu$, $\tau(Z^{*+}_\beta) = \mu$.

Theorem 11: If X is an order-homogeneous connected cor, then X^+ is a homogeneous topological space.

Proof:

Let $a = \inf X$, $b = \sup X$; in X^+ we write $c=a=b$ (for sake of simplicity, in the other cases we denote the points of X and those of X^+ by the same letters).

Now take $p,q \in X^+$

(i) If p and q $\neq$ c, then let f be a map of X onto X, such that $f \mid [a,p]$ is a similarity map of $[a,p]$ onto $[a,q]$ and $f \mid [p,b]$ is a similarity map of $[p,b]$ onto $[q,b]$.
This induces an autohomeomorphism $f^{\dagger}$ of $X^{\dagger}$ such that $f^{\dagger}(p) = q$.

(ii) If $p = c$, $q \neq c$, then choose (in X) r such that $a < r < b$.
And let f be a map of X onto X, such that $f \mid [a,r]$ is a similarity map of $[a,r]$ onto $[q,b]$ and such that $f \mid [r,b]$ is a similarity map of $[r,b]$ onto $[a,q]$.
This again induces an autohomeomorphism $f^{\dagger}$ of $X^{\dagger}$ such that $f^{\dagger}(p) = q$.

<u>Corollary</u>: If $|\alpha| \leqq \aleph_0$, then $Z^{*\dagger}_{\omega^{\alpha}}$ is a homogeneous topological space.

CHAPTER III

On the connection between splitting degree, density and weight of a compact ordered space

§1.

By the density of a topological space X we mean

$$d = d(X) = \inf \{\aleph \mid \exists N \subset X : \overline{N} = X, |N| = \aleph\}.$$

By the weight of a topological space X we mean

$$w = w(X) = \inf\{\aleph \mid \exists \text{ base } \mathfrak{B} \text{ for } X : |\mathfrak{B}| = \aleph\}.$$

The following theorem is well-known

Theorem 1: If X is a T_1-space, then

$$|X| \leq 2^w, \quad w \leq 2^{|X|} .$$

Proof:

(i) If $\mathfrak{B}$ is a base of X with the property that $|\mathfrak{B}| = w$ and $I(x)$ is the family of all $O \in \mathfrak{B}$ such that $x \in O$, then

$$\bigcap_{O \in I(x)} O = \{x\} .$$

So $x \to I(x)$ is a one to one map of X into $\mathfrak{P}(\mathfrak{B})$, and consequently

$$|X| \leq 2^{|\mathfrak{B}|} = 2^w$$

(ii) Obvious (every base is a subset of $\mathfrak{P}(X)$).

Theorem 2 (see Arhangelskiĭ [1]): If X is a compact Hausdorff space, then

$$w \leq |X| \leq 2^w.$$

Proof:

If $p,q \in X$, then let the open sets O_{pq} and O_{qp} be such that

$$p \in O_{pq},\ q \in O_{qp},\ O_{qp} \cap O_{pq} = \emptyset.$$

Let $\mathfrak{B}$ be the family of all finite intersections of sets O_{pq}. Then $\mathfrak{B}$ is a base for X. For, if O is an open set in X and if $p \in O$ then $\{O_{qp}\}_{q \in X \setminus O}$ is an open cover of the compact set $X \setminus O$, which has a finite subcover $\{O_{qp}\}_{q=q_1, q_2, \ldots, q_n}$; but then

$$\bigcap_{i=1}^{n} O_{pq_i} \in \mathfrak{B}$$

and

$$p \in \bigcap_{i=1}^{n} O_{pq_i} \subset O$$

Since $|\mathfrak{B}| = |X|$, it follows that $w(X) \leqq |X|$.

<u>Theorem 3</u> (see Pospišil [1]): If X is a Hausdorff space, then

$$d \leqq |X| \leqq 2^{2^d}.$$

<u>Proof</u>:

Let N be a subset of X such that

$$\overline{N} = X,\ |N| = d.$$

Let I(x) be the family of all $A \in \mathfrak{P}(N)$ with the property that $x \in \overline{A}$. Because of the Hausdorff property, we have $I(x) \neq I(y)$ if $x \neq y$. Consequently $x \to I(x)$ is a one to one map of X into $\mathfrak{P}(\mathfrak{P}(N))$. This means that $|X| \leqq 2^{2^d}$.

<u>Theorem 4</u> (see de Groot [1]): If X is a regular T_1-space, then

$$d \leqq w \leqq 2^d.$$

<u>Proof</u>:

(i) A set O in a topological space is said to be regular, if it is equal to the interior of its closure, that is if

$$O = O^{-o}.$$

Now it will be proved that a regular T_1-space has a regular open base (i.e. a base of regular sets).

For, let $\mathfrak{B}$ be the family of all regular sets.

Let U be an open set and let $x \in U$. Then there exists a closed neighbourhood V of x, with the property that $V \subset U$, and such that

$$x \in W \subset V = \overline{V} \subset U$$

for some open W. Then also

$$x \in W \subset W^{-o-o} \subset V \subset U.$$

Putting $B = W^{-o-o}$ we see that $B = B^{-o}$ (cf. Kelley [1] , p.45 above and p.57, exc.E), so that $B \in \mathfrak{B}$.
Hence $\mathfrak{B}$ is a base.

(ii) Let $N \subset X$ be such that

$$\overline{N} = X, \ |N| = d.$$

Because of the regularity we may conclude that $O_1 \cap N \neq O_2 \cap N$ if $O_1, O_2 \in \mathfrak{B}$, $O_1 \neq O_2$. Consequently $O \to O \cap N$ is a one to one map of $\mathfrak{B}$ into $\mathfrak{P}(N)$. This means that $w \leq |\mathfrak{B}| \leq 2^d$.

§2.

Lemma: If X is a cor and $\{x_i\}_{i < \omega_{\varkappa}}$ is an increasing (decreasing) sequence of type $\omega_{\varkappa}$ in X, then $d(X) \geq \varkappa$.

Proof:

$\{(x_{i+2n}, x_{i+2n+2})\}_{i < \omega_{\varkappa}, n < \omega}$ is a disjoint family of non-void open intervals with cardinal number $\varkappa$.

*Theorem 5: If X is a cor, then

$$d \leq w \leq |X| \leq 2^d \leq 2^w.$$

Proof:

(i) if $w = |X|$, then it follows from $d \leq w \leq 2^d$ that $d \leq |X| \leq 2^d$

(ii) if $w < |X|$, there is a (decreasing or increasing) sequence of type ω_w in X (Chap.I, th.21); this means $d \geq w$ and consequently $d=w$. Then it follows from $w \leq |X| \leq 2^w$ that $d \leq |X| \leq 2^d$.

Theorem 6: Let X be a cor, and let N be dense in X. If V is a θ-sequence for X, then

$$\theta(V) \leq \sup_{x \in N} \mu_x(V) + 1.$$

Proof:

Let $\eta = \eta(V) = \sup_{x \in N} \mu_x(V)$.

Now suppose that $|X_p^{(\eta)}| \geq 3$ for some $p \in Z_\eta$. Then there exists a point c such that $l(X_p^{(\eta)}) < c < r(X_p^{(\eta)})$. This means that $c \notin D_\eta$. Since, however, $N \subset D_\eta = \overline{D}_\eta$ and $\overline{N} = X$, this is a contradiction.
Consequently we have $|X_p^{(\eta)}| \leq 2$ for all $p \in Z_\eta$, and so $\theta(V) \leq \eta+1$.

Remark: It may happen that $\theta(V) = \eta(V) + 1$.
Example: let $Z''_{\omega+1}$ be the cor, which is obtained from $Z_{\omega+1}$ by identification of (a,0) and (a,1) for all rational a; if now V is the regular θ-sequence for $Z''_{\omega+1}$, then $\theta(V) = \omega+1$, $\eta(V) = \omega$.

In the case of a connected cor (an ordered continuum), the following theorem has been obtained before by Novak [3].

Theorem 7: If X is a cor, and V is a θ-sequence for X, then
$|\theta(V)| \leq d$.

Proof:

If $|\mu_x| \geq \aleph$, and thus $\mu_x \geq \omega_\aleph$, for some $x \in X$, then it follows from Chap.I, theorem 17, that there exists a sequence of type $\omega_\aleph$ in X. Then from lemma 1 we may conclude that $d \geq \aleph$. Consequently $|\mu_x| \leq d$ for all $x \in X$. If now N is dense in X, and $|N| = d$, it follows from lemma 2, that

$$|\theta| \leq |\sup_{x \in N} \mu_x + 1| \leq |N| . d = d^2 = d.$$

*Theorem 8: If X is a cor with density d and weight w, and if V is a θ-sequence for X, then

$$|\theta| \leq d \leq w \leq |X| \leq 2^{|\theta|} \leq 2^d \leq 2^w.$$

Moreover, in

$$|\theta| \leq d \leq w \leq |X|$$

at least two of the equality signs hold.

Proof:

The first part of the assertion is an immediate consequence of the foregoing theorems.

Moreover, if in

$$|\theta| \leq d \leq w \leq |X|$$

at least two of the inequality signs hold, we have

$$|X| \geq 2^{2^{|\theta|}},$$

and this is a contradiction, since $|X| \leq 2^{|\theta|}$.

<u>Corollary</u>: For every cor X we have in particular

$$|\Theta| \leq d \leq w \leq |X| \leq 2^{|\Theta|} \leq 2^{d} \leq 2^{w}$$

and in

$$|\Theta| \leq d \leq w \leq |X|$$

at least two of the equality signs hold.

Examples: (i) if $X = Z_{\omega+2}$ then $|\Theta| < d$

(ii) if $X = Z_{\omega+1}$ then $d < w$

(iii) if $X = Z_{\omega}$ then $w < |X|$

(iv) if $X = H$ (see p.34) then $|\Theta| = d = w = |X|$.

<u>Theorem 9</u>: Let X be a cor, with density d, weight w and splitting degree Θ.

1. If $\Theta = \omega_{\aleph}$ or $\Theta = \omega_{\aleph}+1$ then $d = \aleph$
2. If $\Theta = \omega_{\aleph}$ then $w = \aleph$.

<u>Proof</u>:

1. Let V be a θ-sequence for X, such that $\theta(V) = \Theta$.

It is clear that

$$\overline{\bigcup_{\tau<\omega_{\aleph}} D_{\tau}} = D_{\omega_{\aleph}} = D_{\omega_{\aleph}+1} = X ;$$

since

$$\forall \tau < \omega_{\aleph} : \ |D_{\tau}| \leq 2^{|\tau|} \leq \aleph ,$$

it follows that

$$\left| \bigcup_{\tau < \omega_{\aleph}} D_{\tau} \right| \leq \aleph \cdot \aleph = \aleph .$$

So $d \leq \aleph$.

On the other hand it follows from $|\Theta| \leq d$ that $\aleph \leq d$.

Consequently $d = \aleph$.

2. Let V be a θ-sequence for X, such that $\theta(V) = \Theta$, and let

$N = \bigcup_{\tau < \omega_{\aleph}} D_\tau$.

Then the family $\mathfrak{B}$ of all sets

$$\{x \mid a < x < b\} \qquad (a, b \in N)$$

is a base for the topology in X.

For, let O be an open set; without loss of generality we may suppose that

$$O = O_{rs} = \{x \mid r < x < s\} \ .$$

Now take $y \in O$.

(i) if y has both a left neighbour 'y and a right neighbour y', then 'y and y' belong to a certain D_τ $(\tau < \omega_{\aleph})$, and so belong to N; hence

$$y \in O_{'yy'} \subset O$$

$$O_{'yy'} \in \mathfrak{B}$$

(ii) if y has a left neighbour 'y, but no right neighbour, then $'y \in N$ and moreover, since $\overline{N} = X$, there is a $z \in N$ such that $y < z < s$; hence

$$y \in O_{'yz} \subset O$$

$$O_{'yz} \in \mathfrak{B}.$$

Etcetera.

Now we have

$$|\mathfrak{B}| = |N|^2 = \aleph^2 = \aleph$$

and so $w \leq \aleph$.

On the other hand it follows from $|\Theta| \leq d$ that $\aleph \leq d$ and so $\aleph \leq w$.

Consequently $w = \aleph$.

§3.

Let X be a cor.

Let P be the set of jump points in X, and let Q be the set of pairs $\{a,b\}$ in which a and b are neighbours. Clearly $|P| = |Q|$.

*Theorem 10: 1. If $|P| = |X|$, then $w = |X|$

2. If $|P| < |X|$, then $w = d$.

Proof:

1. Let $\mathfrak{B}$ be a base for the topology in X.

Since every set $\{x | x < q,\ q \text{ a right neighbour}\}$ is open, it follows that for every left neighbour p there exists a member of $\mathfrak{B}$ of which p is the greatest element. Hence $|\mathfrak{B}| \geq |P| = |X|$, and so $w = |X|$.

2. Let N be a subset of X such that

$$\bar{N} = X, \ |N| = d.$$

Then clearly the family $\mathfrak{B}$ of all sets

$$\{x | a < x < b\} \qquad (a,b \in P \cup N)$$

is a base for the topology in X.
Since $|P| < |X|$ - and so $|P| \leq d$ - it follows that $|P \cup N| = d$ and consequently $|\mathfrak{B}| \leq d$. Hence $w = d$.

<u>Corollary</u>: If X is connected, then $w = d$ (cf. Mardešić and Papić [1], p.176).

<u>Theorem 11</u>: I. If X is a connected cor, then

$$\theta \leq \omega_d$$

II. If X is a zero-dimensional cor, then

(i) $\theta \leq \omega_d + 1$,

(i)* $\theta \leq \omega_d$ if $|P| < |X|$

(ii)* $\theta \leq \omega_w$.

<u>Remark</u>: Part I of the theorem has been obtained before by Novotny [2].

<u>Proof</u>:

We give the proof of the theorem for case II.

1. We first observe the following: if Y is a zero-dimensional cor, and $p,q,r \in Y$, then there exist two successive θ-decompositions of Y such that no two of the points p,q and r belong to the same $Y_p^{(2)}$.

2. If α is an ordinal number, then we write $\alpha = \nu_\alpha + m_\alpha$, where ν_α is a limit number (or 0) and m_α is an integer ≥ 0. Then let

$$\bar{\alpha} = \nu_\alpha + m_\alpha \cdot 2$$

(so for a limit number we have $\bar{\alpha} = \alpha$).

3. We first prove that $\Theta \leq \omega_d + 1$.

Let N be such, that $N \subset X$, $\overline{N} = X$, $|N| = d$.

Let S be the set of those points s in X, which have both a left neighbour 's and a right neighbour s'; then $S \subset N$.

Let A be the set of all pairs {'s,s} and {s,s'} , and let $R = N \cup A$.

Then $|R| = d$.

Finally suppose that $\{r_i\}_{i < \omega_d}$ is a well-ordering of R.

a. We show that by transfinite induction, a θ-sequence $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — for X can be defined such that for all $\gamma \leq \omega_d$ we have

$$(1) \qquad \forall p \in Z_{\overline{\gamma}} : |X_p^{(\overline{\gamma})} \cap \bigcup_{i<\gamma} t_i| \leq 1,$$

where $t_i = \{r_i\}$ if $r_i \in N$ and $t_i = r_i$ if $r_i \in A$.

(i) Let V_γ be defined for $\gamma \leq \overline{\delta}_1$ and suppose the assertion (1) holds for all those γ.

Put $\delta = \delta_1 + 1$; then $\overline{\delta} = \overline{\delta}_1 + 2$.

Since $|X_p^{(\overline{\delta}_1)} \cap \bigcup_{i<\delta_1} t_i| \leq 1$ and since $|t_{\delta_1}| \leq 2$, two successive θ-decompositions of $X_p^{(\overline{\delta}_1)}$ can be defined in such a way that

$$\forall q \in Z_2 : |X_{pq}^{(\overline{\delta})} \cap \bigcup_{i<\delta} t_i| \leq 1$$

(ii) Let $\delta = \overline{\delta}$ be a limit number and let V_γ be defined for $\gamma < \delta$, such that (1) is satisfied.

Then

$$\forall p \in Z_\delta : |X_p^{(\delta)} \cap \bigcup_{i<\delta} t_i| \leq 1.$$

For, if $|X_p^{(\delta)} \cap \bigcup_{i<\delta} t_i| \geq 2$ for some $p \in Z_\delta$, then there exist points a and b, $a \neq b$, such that

$$a, b \in X_p^{(\delta)} \cap \bigcup_{i<\delta} t_i;$$

this means that $a, b \in \bigcup_{i<\gamma} t_i$ for some $\gamma < \delta$ and so

$$a, b \in X_{p|\overline{\gamma}}^{(\overline{\gamma})} \cap \bigcup_{i<\gamma} t_i,$$

$$\left| X_{p|\bar{\gamma}}^{(\bar{\gamma})} \cap \bigcup_{i<\gamma} t_i \right| \geq 2.$$

This is a contradiction.

b. In particular we have

$$\text{②} \qquad \forall p \in Z_{\omega_d} : \left| X_p^{(\omega_d)} \cap \bigcup_{i<\omega_d} t_i \right| \leq 1$$

Since $\bigcup_{i<\omega_d} t_i$ is dense in X, it follows from ② that $\left|X_p^{(\omega_d)}\right| \leq 3$; however, if $\left|X_p^{(\omega_d)}\right| = 3$, then obviously $X_p^{(\omega_d)} = \{'s,s,s'\}$ for some $s \in S$; this means that $\{s,s'\} = t_i$ for some $i < \omega_d$, so that
$\left|X_p^{(\omega_d)} \cap \bigcup_{i<\omega_d} t_i\right| \geq 2$.
This is a contradiction.
Consequently $\left|X_p^{(\omega_d)}\right| \leq 2$ for all $p \in Z_{\omega_d}$ and so $\theta \leq \omega_d + 1$.

4. We now show that $\theta \leq \omega_d$ if $|P| = |Q| < |X|$.
Let N be such that $N \subset X$, $\bar{N} = X$, $|N| = d$.
Let $R = N \cup Q$; since $|Q| < |X|$ - and so $|Q| \leq d$ - we have $|R| = d$.
Let $\{r_i\}_{i<\omega_d}$ be a well-ordering of R.
In a manner analogous to that used in 3. we can show the existence of a θ-sequence $V = \{V_\gamma\}_\gamma$ — $V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma}$ — for X such that

$$\forall p \in Z_{\bar{\gamma}} : \left| X_p^{(\bar{\gamma})} \cap \bigcup_{i<\gamma} t_i \right| \leq 1,$$

for all $\gamma \leq \omega_d$,
where $t_i = \{r_i\}$ if $r_i \in N$ and $t_i = r_i$ if $r_i \in Q$.
In particular we have

$$\text{③} \qquad \forall p \in Z_{\omega_d} : \left| X_p^{(\omega_d)} \cap \bigcup_{i<\omega_d} t_i \right| \leq 1$$

Since $\bigcup_{i<\omega_d} t_i$ is dense in X, it follows from ③ that $\left|X_p^{(\omega_d)}\right| \leq 3$; if $\left|X_p^{(\omega_d)}\right| = 2$ or 3, then $X_p^{(\omega_d)} = \{a,b\}$ or $X_p^{(\omega_d)} = \{a,b,c\}$ for some pair of neighbours $\{a,b\}$ and -in the second case - $\{b,c\}$; but then

$\{a,b\}= t_i$ for some $i < \omega_d$, so that $\left| X_p^{(\omega_d)} \cap \bigcup_{i<\omega_d} t_i \right| \geq 2$.

From this we conclude that $\left| X_p^{(\omega_d)} \right| = 1$ for all $p \in Z_{\omega_d}$.

So $\theta \leq \omega_d$.

5. Next we prove that $\theta \leq \omega_{|X|}$.

Let $\{x_i\}_{i<\omega_{|X|}}$ be a well-ordering of X.

As in 3. it is shown that there exists a θ-sequence

$$V = \{V_\gamma\}_\gamma \text{ — } V_\gamma = \{X_p^{(\gamma)}\}_{p \in Z_\gamma} \text{ — for X, such that}$$

$$\forall p \in Z_\gamma : \left| X_p^{(\gamma)} \cap \bigcup_{i<\gamma} \{x_i\} \right| \leq 1.$$

So in particular

$$\forall p \in Z_{\omega_{|X|}} : \left| X_p^{(\omega_{|X|})} \cap \bigcup_{i<\omega_{|X|}} \{x_i\} \right| \leq 1,$$

$$\forall p \in Z_{\omega_{|X|}} : \left| X_p^{(\omega_{|X|})} \right| = 1.$$

Consequently $\theta \leq \omega_{|X|}$.

6. Finally we show that $\theta \leq \omega_w$.

If $|P| = |X|$, we have $w = |X|$; so $\theta \leq \omega_{|X|} = \omega_w$.

If $|P| < |X|$, we have $w=d$; so $\theta \leq \omega_d = \omega_w$.

Corollary: If X is a zero-dimensional cor or a connected cor, then

a. $\theta \geq \omega_\aleph + 2$ implies $d > \aleph$

b. $\theta \geq \omega_\aleph + 1$ implies $w > \aleph$.

Example: $d(Z_\alpha) = |\alpha|$ if $\alpha = \omega_{|\alpha|}$ or if $\alpha = \omega_{|\alpha|} + 1$

$> |\alpha|$ if $\alpha \geq \omega_{|\alpha|} + 2$

$w(Z_\alpha) = |\alpha|$ if $\alpha = \omega_{|\alpha|}$

$> |\alpha|$ if $\alpha \geq \omega_{|\alpha|} + 1$

In fact, if L and R are subsets of $Z_{\omega_\aleph}$ defined by

$$L = \{x = (x_i)_{i < \omega_{\aleph}} \mid \exists i_0 < \omega_{\aleph}: x_i = 1 \text{ if } i \geq i_0\}$$

$$R = \{x = (x_i)_{i < \omega_{\aleph}} \mid \exists i_0 < \omega_{\aleph}: x_i = 0 \text{ if } i \geq i_0\} ,$$

then

$$|L| = |R| = \aleph$$

(for instance $L = \bigcup_{i_0 < \omega_{\aleph}} L(i_0)$, where $L(i_0) = \{x \mid x_i = 1 \text{ if } i \geq i_0\}$; and $|L(i_0)| \leq 2^{|i_0|} \leq \aleph^{i_0}$ by the continuum hypothesis).

Moreover

$$\bar{L} = \bar{R} = Z_{\omega_{\aleph}}$$

From this it also follows that

$$\left| Z^{*}_{\omega_{\aleph}} \right| = 2^{\aleph} - \aleph = 2^{\aleph} .$$

<u>Remark</u>: Theorem 11 does not hold for an arbitrary cor.

Example: $X = W(\Omega) \cup [0,1]$ (ordered union)

$$\begin{aligned} \theta(X) &= \Omega + \omega \\ &> \omega_d + 1 \quad (= \Omega + 1) \\ &> \omega_{|X|} \quad (= \Omega). \end{aligned}$$

CHAPTER IV

Literature and additional remarks

§1

(i) Sierpinski [1] and Cuesta Dutari [5] proved that every ordered set of cardinal number $\leqq \aleph_\nu$ is similar to a subset of Z_{ω_ν}. See also Sierpinski [3], p.460.

This result was already obtained in Hausdorff [1], p.182 (where instead of Z_{ω_ν} a set of the form $\{0,1,2\}^{\omega_\nu}$ is used). In fact, even the following assertion holds: If H_α is the subset of Z_{ω_α} which consists of all sequences $(x_i)_{i<\omega_\alpha}$ with the property that there is some $i_0 < \omega_\alpha$ such that $x_{i_0} = 1$ and $x_i = 0$ for $i > i_0$, then every ordered set of cardinal number $\leqq \aleph_\alpha$ is similar to a subset of H_α (in this case H_α is said to be $\aleph_\alpha$-universal). This was proved by Sierpinski [2] for ordinal numbers α of the first kind and by Gillman [1] for ordinal numbers α of the second kind. (The result of Sierpinski is also a consequence of his theorem, that $H_{\beta+1}$ is an $\eta_{\beta+1}$-set and a theorem proved in Hausdorff [2], p.181, that a $\eta_{\beta+1}$-set is $\aleph_{\beta+1}$-universal.) However, both these facts are proved, in a very short way, by Mendelson [1].

If one uses the generalized continuum hypothesis, it is easy to see that $|H_\alpha| = \aleph_\alpha$.

In this connection it should be observed that in general it is not true that a cor X, such that $|X| \leqq \aleph_\nu$, can be imbedded topologically in Z_{ω_ν} or in $Z^*_{\omega_\nu}$.

For instance, $X = Z_\omega$ cannot be imbedded topologically into Z_{ω_1} or into $Z^*_{\omega_1}$ (in Z_{ω_1} and in $Z^*_{\omega_1}$ there are no points which are the limit both of an increasing sequence of type ω and of a decreasing sequence of type ω).

(ii) It was observed in Ch.I, theorem 14 that each Z_α is a continuous image of Z_β if $\alpha \leqq \beta$. It can also easily be proved that each closed subset of Z_α is a retract, i.e. a continuous image of Z_α.

On the other hand it is by no means true that every compact Hausdorff space is a continuous image of some cor (although every compact metric space is the continuous image of Z_ω; cf. for instance Kelley [1] , p. 166). This can easily be seen by the following argument (de Groot [3]), which might be useful also in other cases;

1. In a cor every sequence has a convergent subsequence.
2. The property "every sequence has a convergent subsequence" is invariant under continuous mappings.
3. In the Stone-Čech compactification βN of the natural numbers N the closure of each infinite subset of N is homeomorphic to βN, and consequently N has no convergent subsequence.

This means that βN is not the continuous image of any cor. The same is true for each space which contains βN as a subset. Thus for instance the topological product X of continuous many spaces X_i ($|X_i| \geq 2$) is not the continuous image of any cor. Taking all $X_i = \{0,1\}$ or all $X_i = [0,1]$ (the unit-interval of real numbers) we obtain a zero-dimensional compact space and a connected, locally connected compact space respectively, which are not the continuous image of any cor.

(iii) The well-known theorem of Hahn-Mazurkiewicz states, that for a space P to be compact, connected, locally connected, and metric, it is necessary and sufficient that P be the image of the unit interval of the real numbers under a continuous mapping into a Hausdorff space (cf. for instance Hocking-Young [1] , p.129). This includes the result that, for locally connected metric compacta, connectedness and pathwise connectedness coincide. According to Mardešić [1] a generalization of these results to non-metric spaces is not possible; i.e.

1. If a space X is said to be "connected by ordered continua" provided, for each pair of points $x_0, x_1 \in X$ there is a connected cor C and a continuous map $\phi : C \to X$ which maps the end-points of C into x_0 and x_1 respectively, then there exists a locally connected compact Hausdorff space which is connected but is not "connected by ordered continua".
2. There exist connected and locally connected compact Hausdorff spaces which are not the continuous image of any connected cor.

An example has been given in (ii). Other examples of such spaces are given in Mardešić [2] .

(iv) Mardešić [2] proves the following theorem: Let X be a continuum (i.e. a connected compact Hausdorff space) and C an ordered continuum (i.e. a connected cor) and let I denote the real line segment; now, if there exists a continuous mapping of C onto $X \times I$, then X has the Suslin property. (A topological space X is said to have the Suslin property if each family of disjoint open sets of X is at most countable.) From this it follows among other things that I is the only non-degenerate ordered continuum C which admits a continuous mapping $C \rightarrow C \times C$ onto its square.
Mardešić and Papić [1] consider the class K of spaces which are continuous images of ordered continua. A characterization is given of those product spaces $\prod_{a \in A} X_a$ (of non-degenerate continua X_a) which belong to K. In fact, in order that such a product space $\prod_{a \in A} X_a$ $(|A| > 1)$ be the continuous image of an ordered continuum it is necessary and sufficient that all X_a be metric Peano continua and that $|A| \leq \aleph_0$; in this case the product space is itself a Peano continuum and thus a continuous image of I.
Treybig [1] generalizes part of this result to the case in which the factors need not be connected; theorem: if each A and B is a compact Hausdorff space which contains infinitely many points and $A \times B$ is the continuous image of a compact ordered space, then both A and B have a countable base (and so are metrizable).

(v) Mardešić [3] proves that the inverse limit of a monotone inverse system of ordered continua is itself an ordered continuum. Moreover each ordered continuum is the inverse limit of a monotone inverse system, consisting only of arcs.

If T is a continuum, then a finite sequence $(U_1, \ldots, U_n)$ of open sets U_i in T is called a "chain", if $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$. T is said to be a "chainable continuum" if every open covering of T admits a chain-refinement $(U_1, \ldots, U_n)$; and if every open covering of T admits a chain-refinement with connected U_i, then T is called "strongly chainable". It is proved that the following three classes of spaces coincide:

(a) ordered continua

(b) strongly chainable continua

(c) locally connected chainable continua.

§2.

(i) Eilenberg [1] says a topological Hausdorff space $(X,\mathcal{T})$ to be an "ordered topological space", if X is an ordered set with ordering $<$, such that $\mathcal{T} \subset \mathcal{T}_<$. He shows that a connected topological space $T=(T,\mathcal{T})$ is "orderable" (i.e. there is an ordering $<$ for T, such that $\mathcal{T} \subset \mathcal{T}_<$) if and only if $T \times T \setminus \{(t,t) \mid t \in T\}$ is not connected.
Moreover two orderings of a connected topological space are equal or inverse to each other.

(ii) Banaschewski [1] considers ordered spaces $(X,\mathcal{T}_<)$ and their compact extensions $\delta(x,\mathcal{T}_<)$ which are obtainable by means of Dedekind cuts. (Remark: Mac Neille [1] ,§ 11 proved that every partially ordered set S can be completed by means of "Dedekind cuts"; cf. also Birkhoff [1] , p.58. Addition of a least and a greatest element, if necessary, then leads to a compactification of S.) $\delta(X,\mathcal{T}_<)$ is connected if and only if $(X,<)$ is dense, i.e. $\forall x,y \in X \ (x < y) \ \exists z \in X\colon x < z < y$. If $(X,<_1)$ and $(X,<_2)$ are dense and $\delta(X,<_1)$ and $\delta(X,<_2)$ are homeomorphic, then $<_1$ and $<_2$ are either identical or inverse to each other. (It should be observed that $(X,<_1)$ and $(X,<_2)$ may be homeomorphic if $(X,<_1)$ is dense and $(X,<_2)$ is not; example: $(X,<_1) = [0,1] \setminus \{\frac{1}{2}\}$, $(X,<_2) = (\frac{1}{2},1] \cup [0,\frac{1}{2})$).

(iii) A "cut point" in a connected space X is a point r such that $X \setminus \{r\} = A \cup B$ where A and B are separated, i.e. $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Following Hocking and Young [1] we denote by $E(p,q)$ the subset of X consisting of the points p en q together with all cut points r of X that separate p and q, i.e. $X \setminus \{r\} = A \cup B$, $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, $p \in A$, $q \in B$. If a relation $<$ is defined in $E(p,q)$ such that, for all $x,y \in E(p,q)$, $x < y$ if and only if $x=p$ or x separates p and y in X, then it is easily proved that $<$ is a simple order in $E(p,q)$. The following theorem is known (cf. for instance Hocking and Young [1], p.53): If X is a compact connected Hausdorff space with just two non-cut points a and b, then X = E(a,b) and the order topology in E(a,b) coincides with the topology in X. In other words: X is orderable.

(iv) Any compact zero-dimensional metric space is homeomorphic to a subset of the Cantor set (cf. for instance Hocking and Young [1], p. 100), and so is orderable.
This in particular holds for countable compact Hausdorff spaces, since these spaces have a countable base (and so are metrizable) and are zero-dimensional; in this connection, it may be remarked that, according to Mazurkiewicz and Sierpinski [1], every countable compact Hausdorff space is homeomorphic to a well-ordered space of a type which has the form $\omega^{\alpha}.n+1$, where α is a countable ordinal number and n is an integer > 0.
Lynn [1] observes that even a zero-dimensional separable metrizable space is orderable.

(v) In Herrlich [1] several conditions are found that a topological space be orderable. A space is called end-finite if no connected subset has more than two non-cut points. From the results obtained by the author the following will be mentioned.

1. A connected T_1-space is orderable if and only if it is end-finite and locally connected.

2. A totally disconnected metric Lindelöf space is orderable if and only if it is zero-dimensional.

3. A countable space is orderable if and only if it is metrizable.

Also conditions are found that a space is locally orderable, which means that every point has an orderable neighbourhood.

§3.

(i) It is known that a linearly ordered topological space is completely normal; cf. Bourbaki [1].
Ball [1] shows that every open covering U of a linearly ordered space X, which is such that each point of X is an element of at most count-

ably many sets of U, has a locally finite refinement. In particular, X is countably paracompact.

(ii) Ball [2] gives three sets of conditions, each of which implies that a connected linearly ordered space is separable.

(iii) A space X is said to have the fixed point property, if every continuous map of X into X leaves a point fixed. It is known that a connected cor has the fixed point property. Cohen [1] shows that the direct product of two connected cor's has the fixed point property.

§4.

(i) Novak [1] constructs six ordered continua of power $2^{\aleph_0}$ containing a dense subset of power $\aleph_1$ but no one of power $\aleph_0$. In these six examples the sets of occurring point characters are $\{c_{00},c_{01}\}$, $\{c_{00},c_{10}\}$, $\{c_{00},c_{01},c_{10},c_{11}\}$, $\{c_{00},c_{01},c_{11}\}$, $\{c_{00},c_{10},c_{11}\}$, $\{c_{00},c_{11}\}$.
(If ω_i and ω_j are regular initial ordinal numbers, then a point is said to have the character c_{ij} if it is the limit of an increasing sequence and of a decreasing sequence of type ω_i and ω_j respectively.)
Misik [1] constructs such a continuum with a set of point characters $\{c_{00},c_{01},c_{10}\}$.
Novotny [1] shows that one of the examples of Novak is similar to the "ultra continuum" constructed by Bernstein [1]. He also gives seven examples of ordered continua of power $2^{\aleph_0}$ and density $2^{\aleph_0}$.

(ii) Novak [2] considers an ordered continuum (that is, a connected cor) C.

a. He calls a system P of closed (non-degenerate) intervals a "dyadic partition" of C, if

1. $\forall X,Y \in P : X \cap Y = Y$ or $X \cap Y = X$ or $|X \cap Y| \leq 1$
2. $C \in P$
3. $\forall X \in P \;\exists X_1, X_2 \in P : X_1 \cup X_2 = X,\ |X_1 \cap X_2| = 1$
4. If $\{X_i\}_{i \in I}$ is a decreasing (transfinite) sequence of intervals $X_i \in P$, then $\bigcap_i X_i \in P$ or $|\bigcap_i X_i| = 1$.

In each dyadic partition the decomposition $\{X_1, X_2\}$ of an interval $X \in P$ according to 3.is clearly unique. If one puts $X_1 \cap X_2 = \{p\}$, then p is called a d-point.

Now, it is easily shown that P is a dyadic partition of C if and only if P is the system of non-degenerate intervals which are elements of the members of a θ-sequence V (cf. theorem 5 of the paper of Novak).

b. If A is an interval in C or a point, which is no d-point, then the subsystem P(A) of P consisting of all $X \in P$ with the property that $A \subset P$, is clearly well-ordered; the ordinal number of this system is called the order of A. If A is a d-point then there are two well-ordered subsystems $P_l(A)$ and $P_r(A)$ of P consisting of intervals which contain A; the greater of the two ordinal numbers of $P_l(A)$ and $P_r(A)$ is called the order of A. The supremum of the orders of all P(X), $X \in P$, is called the order of the dyadic partition P.

Now, it can easily be proved that the order of a dyadic partition P is equal to $\theta(V_P)$, (see Ch.I,p.18), if V_P is the θ-sequence, which corresponds to V.

c. Several other theorems, based on these ideas, are proved. Finally, it is shown that every ordered continuum of power $\aleph_\sigma$ contains at least one point with character c_{00} if and only if $\aleph_\sigma < 2^{\aleph_1}$.

d. Novak does not consider the infimum of all orders of dyadic partitions (which would be the splitting degree Θ, as defined in Ch.I).

Novotny [2] proves for an ordered continuum C the existence of a partition of order at most ω_ν, where $\aleph_\nu$ is the density of C.

(iii) J. Novak [3] defines the following sets of cardinal numbers for an ordered continuum C.

$P = \{ \aleph_\alpha | \exists a \in C\text{: a has point character } c_{\rho\sigma} \text{ and } \aleph_\alpha = \min(\aleph_\rho, \aleph_\sigma)\}$

$Q = \{ \aleph_\alpha | \exists a \in C\text{: a has point character } c_{\rho\sigma} \text{ and } \aleph_\alpha = \max(\aleph_\rho, \aleph_\sigma)\}$

$S = \{ \aleph | \exists \text{ monotone sequence of type } \omega_\aleph \text{ in } C\}$

$I = \{ \aleph | \exists \text{ isolated subset D of C, such that } |D| = \aleph \}$

$I' = \{ \aleph | \exists \text{ disjoint system of non-degenerate intervals in C, with cardinal number } \aleph\}$

$M = \{ \aleph | \exists \text{ subset D of C, such that } \overline{D} = C,\ |D| = \aleph \}$

$R = \{ \aleph | \exists \text{ dyadic partition of C with cardinal number } \aleph\}$.

If now p, q, s, i, i' and r_2 are the respective suprema of the sets P, Q, S, I, I' and R, and if m and r_1 are the respective minima of the sets M and R, then it is proved that

$$p \leq q \leq s \leq i = i' \leq m \leq |C| \leq \min(m^p, 2^{r_1})$$

and

$$s \leq r_1 \leq r_2 \leq m = \max(i, r_1) = \max(i, r_2)$$

and

$$r_2 \leq s^+,$$

where s^+ is the least cardinal number, such that $\aleph < s^+$ for every $\aleph \in S$.

It is shown that $R = \{s\}$ or $R = \{s^+\}$ or $R = \{s, s^+\}$. M. Novotny [3] proves several other relations of this kind; for instance $|C| \leq 2^q$.

(iv) Erdös and Rado [1] prove, using the generalized continuum hypothesis, the following theorem:

A cardinal number $\aleph$ has the property that for every ordered set S of power $\aleph$

1. there is a subset in S of type $\omega_\aleph$

or 2. there is a subset in S of type $\omega_\aleph^*$

or 3. for all $\alpha < \omega_\aleph$ there exist subsets in S, both of type α and of type α^*,

if and only if $\bar{\aleph} = \sup\limits_{\aleph_m < \aleph} \aleph_m$ is regular.

§5.

(i) Arens [1] discusses order-homogeneous connected cor's. For instance, it is proved that the lexicographically ordered product L^ω is an order-homogeneous connected cor, if the same is true for L.

Terasaka [1] proves that all $Z^{*}_{\omega^\alpha}$ are order-homogeneous.

(ii) According to Hausdorff [1] , p.179-181 there exist ordered sets of arbitrary high power with the property that all open intervals are similar.

Vazquez Garcia and Zubieta Russi [3] show that such a set has at most the cardinal number of the continuum if it is complete.

(iii) It is a well-known fact (cf. for instance Kamke [1]) that an ordered set X is similar to the set of the real numbers if it has the following properties.

1. there exists neither a least, nor a greatest element
2. X is complete
3. X has a countable dense subset.

From this it easily follows that an ordered space X is homeomorphic to the space of real numbers if

1. X is homogeneous
2. X is connected
3. X has a countable base.

(iv) It is easily seen that every cor $X(|X| \geq \aleph_0)$ which has a countable base admits continuous many autohomeomorphisms. For, if there are countable many isolated points, the assertion is obvious. In the other case the assertion follows from the fact that there is either a separable connected subspace, which is consequently homeomorphic to an interval of the real numbers, or the space is zero-dimensional and so is homeomorphic to the Cantor set.
Jónsson [1] and Rieger [1] both give an example of an infinite compact ordered zero-dimensional space such that the only homeomorphism of S onto S is the identity mapping.
In this connection it may be observed that de Groot [1] proved the following theorem: There exists a family $\{F_\gamma\}$ of $2^{\aleph}$ zero-dimensional subsets of the real line, such that no F_γ can be mapped locally topologically into or continuously onto itself or any other $F_{\gamma'}$; if F_γ is mapped into itself, we must exclude trivial maps. However, here the occurring sets F_γ are not compact. In de Groot and Maurice [1] the existence will be proved of a cor of continuous power and with continuous weight which is rigid, i.e. which has no autohomeomorphisms except the identity mapping.

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