ASYMPTOTIC EXPANSIONS AND THE DEFICIENCY CONCEPT IN STATISTICS

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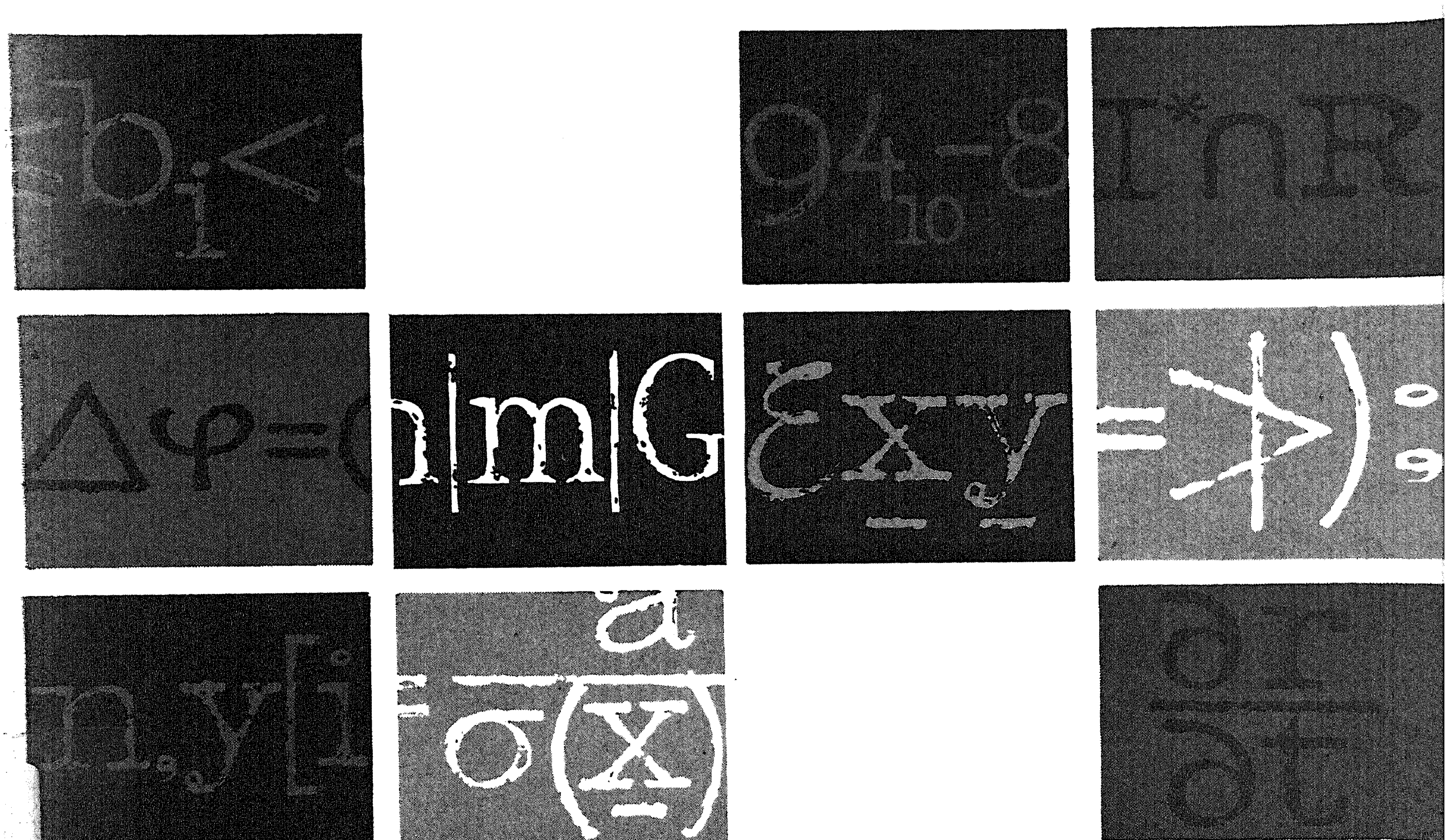

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CONTENTS

CHAPTER 1

INTRODUCTION

As "deficiency" is the unifying topic of this study, we shall start by introducing this concept. Let there be given two statistical procedures A and B. If A is based on N observations, we need k_N observations for B to attain the same level of performance with both procedures.

Usually, A and B are compared by means of the ratio N/k_N. If it exists, $\lim_{N\to\infty} N/k_N$ is called the *asymptotic relative efficiency* (ARE) of B with respect to A and denoted as e. Such efficiency computations are by now almost classical: as early as 1925 Fisher (1925) found $e = 2/\pi$ in comparing the median and the mean for the estimation of normal location. It should always be kept in mind that the information contained in the single number e is of an asymptotic nature. As we are interested in finite, preferably even small sample sizes N, this information becomes more valuable according as the convergence of N/k_N towards e becomes faster. Hence, when e has been found, the natural next step is to investigate this rate of convergence, for example by looking at the behaviour of $ek_N - N$ as $N \to \infty$. This may be done for all cases where $0 < e < \infty$, but we shall always restrict ourselves to the by far most interesting case where $e = 1$. For then we have a second, perhaps even stronger reason for further investigation: from the fact that $e = 1$ we cannot even deduce which of the two procedures A and B is better. Hence a study of the difference $k_N - N$ is not merely useful to get information about the rate of convergence of N/k_N towards 1, but here it may also reveal which of the two procedures is preferable. For $e \neq 1$, $ek_N - N$ only supplies some additional information, but for $e = 1$ this number becomes of importance in its own right.

Although this difference $k_N - N$ seems to be a very natural quantity to examine, historically the ratio N/k_N was preferred by almost all authors in view of its simpler behaviour. The first general investigation of $k_N - N$ was carried out by Hodges and Lehmann (1970). They name $k_N - N$ the *deficiency* of B with respect to A and denote it as d_N. If $\lim_{N\to\infty} d_N$ exists, it is called the *asymptotic deficiency* of B with respect to A and denoted as d. At points where no confusion is likely, we shall simply call d the deficiency of B with respect to A.

Under the assumption $e = 1$ we evaluate d_N and d in the following way. Denote the performance criteria for A and B as $P_{A,N}$ and $P_{B,N}$ respectively. If A and B are tests, $P_{A,N}$ and $P_{B,N}$ may be the powers of these tests, if A and B are point estimators, $P_{A,N}$ and $P_{B,N}$ may be expected squared errors, etc. By definition, $d_N = k_N - N$ may, for each N, be found from

$$(1.1) \qquad P_{A,N} = P_{B,k_N}.$$

In order to solve (1.1), k_N has to be treated as a continuous variable. This can be done in a satisfactory manner by defining P_{B,k_N} for non-integral k_N as

$$P_{B,k_N} = (1-k_N+[k_N])P_{B,[k_N]} + (k_N-[k_N])P_{B,[k_N]+1}$$

(cf. Hodges and Lehmann (1970)).

Generally $P_{A,N}$ and $P_{B,N}$ are not known exactly and we have to use approximations. Here these are obtained by observing that $P_{A,N}$ and $P_{B,N}$ will typically satisfy asymptotic expansions of the form

$$(1.2) \qquad \begin{aligned} P_{A,N} &= \frac{c}{N^r} + \frac{a}{N^{r+s}} + o\left(\frac{1}{N^{r+s}}\right), \\ P_{B,N} &= \frac{c}{N^r} + \frac{b}{N^{r+s}} + o\left(\frac{1}{N^{r+s}}\right), \end{aligned}$$

for certain c, a and b not depending on N and certain constants $s > 0$ and $r \neq 0$. The leading term in both expansions is the same in view of the fact that $e = 1$. From (1.1) and (1.2) it now easily follows that

$$(1.3) \qquad d_N = \frac{(b-a)}{rc} N^{(1-s)} + o(N^{(1-s)}).$$

Hence

$$(1.4) \qquad d = \begin{cases} \pm\infty & , \ 0 < s < 1, \\ \frac{(b-a)}{rc} & , \ s = 1, \\ 0 & , \ s > 1. \end{cases}$$

A useful property of deficiencies is the following: if a third procedure C is given, for which the performance criterion $P_{C,N}$ also has an expansion of the form (1.2), the deficiency d of C with respect to A satisfies $d = d_1+d_2$, where d_1 is the deficiency of C with respect to B and d_2 is the deficiency of B with respect to A.

The situation where s = 1 seems to be the most interesting one. Hodges and Lehmann (1970) demonstrate the use of deficiency in a number of simple examples for which this is the case. One of these problems is the following: consider a sample $X_1,\ldots,X_N$ from a distribution F with mean ξ and variance σ^2. Now σ^2 can be estimated by $M_N = N^{-1}\sum_{i=1}^N (X_i-\xi)^2$, but also by $M'_N = (N-1)^{-1}\sum_{i=1}^N (X_i-\bar{X})^2$, with $\bar{X} = N^{-1}\sum_{i=1}^N X_i$, if we do not know ξ or do not dare to rely on its given value. Both estimators are unbiased and therefore we compare $\sigma^2(M_N)$ and $\sigma^2(M'_N)$

$$(1.5) \qquad \sigma^2(M_N) = \sigma^4 \frac{\gamma}{N}\ , \ \sigma^2(M'_N) = \sigma^4 \frac{\gamma(N-1)+2}{N(N-1)}\ ,$$

where $\gamma+1 = \mu_4/\sigma^4$, the standardized fourth central moment of F. Application of (1.2) and (1.3) to (1.4) shows that $d = 2/\gamma$. If F is normal, $\gamma = 2$ and hence d = 1: the price of not knowing the mean is asymptotically one additional observation. Note that in the normal case not only d = 1, but also $d_N \equiv 1$; in fact, M_N and M'_{N+1} are identically distributed.

The present thesis consists of a number of applications of the deficiency concept. Below we give for each of the problems considered an indication of the problem, of the results, and of the way in which these are obtained.

In chapter 2 the following problem is considered: $X_1,\ldots,X_m$ are independent random variables (r.v.'s), all having distribution P_θ,$Y_1,\ldots,Y_n$ are independent r.v.'s, all having distribution $P_{\tilde{\theta}}$, where θ, $\tilde{\theta} \in \Theta \subset R^1$. From Lehmann (1959) it follows that the test for $\theta = \tilde{\theta}$ against $\theta > \tilde{\theta}$ that rejects the hypothesis for large values of $\sum_{i=1}^m X_i$, conditionally given $\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j$, is uniformly most powerful unbiased (UMPU) for this situation under suitable conditions. These conditions are satisfied for example in the case of the 2×2 table, where $\sum_{i=1}^m X_i$ and $\sum_{j=1}^n Y_j$ are binomial r.v.'s.

Usually the test is performed with equal sample sizes m = n. If the criterion of optimality is the unconditional power of the test, this choice is known to be asymptotically optimal in the sense that the optimal value of $\gamma = m/(m+n)$ satisfies $\gamma_{opt} = \frac{1}{2} + o(1)$ as (m+n), the total number of experiments, tends to infinity. Here we obtain the optimal value of γ to

$o((m+n)^{-1/2})$. Attention is restricted to the case where $|\theta-\tilde{\theta}| \to 0$ as $(m+n) \to \infty$, at such a rate that for a fixed size $\alpha > 0$, β, the error of the second kind, remains bounded away from 0 and $1-\alpha$.

In order to compute deficiencies we need expansions like (1.2) for the powerfunctions. We first expand the conditional distribution function (d.f.) of the test statistic, both under hypothesis and alternative. From this we obtain an expansion for the conditional power. By taking expectations we arrive at an expansion for the unconditional power. Finally, from this expansion γ_{opt} can be determined. Comparison of the expansion of the power for $m = \gamma_{opt}(m+n)$ and $m = \frac{1}{2}(m+n)$, gives the deficiency d_N of the traditional choice $m = n$ with respect to the optimal choice. The asymptotic deficiency d proves to be finite. For the special case where $\alpha = \beta$ we even have $d_N = O(N^{-1/2})$, and hence $d = 0$. For other choices of α and β, d is positive, but usually rather small. For example, let $\sum_{i=1}^{m} X_i$ and $\sum_{j=1}^{n} Y_j$ be binomial r.v.'s with parameters (m,p_1) and (n,p_2) respectively, where $p_1 = p_2$ under the hypothesis and $p_1 > p_2 = p$, $p_1-p_2 = O((m+n)^{-1/2})$ under the alternative. Then d satisfies

$$d = \frac{(2p-1)^2}{36p(1-p)} (u_\alpha - u_\beta)^2.$$

Here $u_\alpha = \Phi^{-1}(1-\alpha)$, $u_\beta = \Phi^{-1}(1-\beta)$, where Φ^{-1} is the inverse of the standard normal d.f. and $\alpha(\beta)$ is the error of the first (second) kind. For $0.03 \le p \le 0.97$ this gives $d \le (u_\alpha - u_\beta)^2$, for $0.01 \le p \le 0.99$, $d \le 3(u_\alpha - u_\beta)^2$. In most applications $u_\alpha - u_\beta$ will satisfy $|u_\alpha - u_\beta| \le 1$. Then we have that for p in the given intervals the price of not using the optimal choice but simply $m = n$ is asymptotically at most 1 or 3 additional observations.

In chapters 3 and 4 we compare various tests for the one sample problem. In its most general form this problem can be formulated in the following way: given a sample $X_1,\dots,X_N$ of independent identically distributed (i.i.d.) r.v.'s with common d.f. G, we have to test the hypothesis H_0 that the distribution determined by G is symmetric about zero, i.e. $G(x) + G(-x) = 1$ for all x, against the alternative H_1 that it is not. H_0 is called the hypothesis of symmetry. In this formulation, H_1 is too large to construct tests, having optimal properties against all points of H_1. Therefore one usually restricts H_1 to some class of interesting alternatives, after which optimal tests against this family are derived. The most common choice is

$\bar{H}_1$: $G(x) = F(x-\theta)$, $\theta > 0$, the family of one-sided location alternatives for a fixed d.f. F that is symmetric about zero.

A well-known class of tests for the one sample problem is the class of linear one sample rank tests, for example Wilcoxon's signed rank test or the absolute normal scores test. Such tests are not only distribution-free, but also relatively easy to compute, a combination which in general is not achieved by other tests for the one sample problem, such as the test based on $\bar{X} = N^{-1}\sum_{i=1}^{N} X_i$, the one sample t-test or the permutation test based on $\bar{X}$. One would expect that the price for these desirable properties would be a loss of efficiency, but, at least asymptotically to first order, this is not the case if one considers contiguous location alternatives H_1^* : $G(x) = F(x-\theta)$, $\theta = \mathcal{O}(N^{-1/2})$. Hájek and Šidák (1967) prove that the asymptotically most powerful rank test against H_1^* has ARE e = 1 with respect to the asymptotically most powerful test. For the normal case this means that the absolute normal scores test has e = 1 with respect to the t-test and the $\bar{X}$-test. The restriction to $\theta = \mathcal{O}(N^{-1/2})$ is rather natural, as for such sequences of alternatives the power remains bounded away from 1.

In view of the above, it seems interesting to know deficiencies of asymptotically most powerful rank tests with respect to the other types of test, as was suggested by Hodges and Lehmann (1970). To this end we need asymptotic expansions as in (1.2) for the power functions of the tests involved. For linear rank tests for the one sample problem these expansions have been obtained by Albers, Bickel and van Zwet (1974); the two sample problem is dealt with by Bickel and van Zwet (1974). A review of asymptotic expansions in nonparametric statistics is given by Bickel (1974).

Chapter 3 is devoted to asymptotic expansions for one sample rank tests. It contains the results of Albers, Bickel and van Zwet (1974) and some extensions; an outline of the proofs is given but we omit a number of technical details for which the interested reader is referred to Albers, Bickel and van Zwet (1974). The idea is that the rank test statistic is a sum of independent random variables, conditionally under the vector Z of order satistics of $|X_1|,\dots,|X_N|$. Hence we can give Edgeworth expansions in this conditional situation.

An unconditional expansion for the distribution of the test statistic, and hence for the power of the test, follows by taking the expectation with respect to Z of the conditional expansion. The evaluation of this expectation is a highly technical matter.

In order to be able to justify the above mentioned Edgeworth expansions we have to exclude cases where the lattice character of the statistic is too pronounced. This occurs for example with the sign test. This test is therefore dealt with seperately.

In chapter 4 similar expansions are derived for several other tests: parametric tests, permutation tests and the randomized rank score tests due to Bell and Doksum (1965). After this, deficiencies can be evaluated of the rank tests with respect to the other types of test. For example, the deficiency of the absolute normal scores test with respect to the t-test and the $\bar{X}$-test satisfies $d_N = \mathcal{O}(\log\log N)$; the asymptotic deficiency of the permutation test based on $\bar{X}$ with respect to the t-test equals zero.

Chapter 5 is devoted to the application of the results of chapters 3 and 4 to estimation of location. Consider again the situation where $X_1,\dots,X_N$ are i.i.d. r.v.'s from $F(x-\theta)$, in which $F(-x) = 1-F(x)$ for all x. For some of the test statistics considered in chapters 3 and 4 there exists a well-known estimator of θ which is closely related to this test statistic in the sense that, for all a,

$$P_{-a}(T < 0) \le P_0(S < a) \le P_{-a}(T \le 0), \tag{1.5}$$

where T is the test statistic, S is the estimator and P_θ denotes probability under θ. From (1.5) it is clear that the expansion for the d.f. of T, obtained in chapter 3 or 4, immediately leads to an expansion for the d.f. of S.

The above correspondence exists for example between the maximum likelihood estimators and the locally most powerful parametric tests of section 4.2 and between the estimators due to Hodges and Lehmann (1956) and the corresponding rank tests of chapter 3. The expansions thus obtained can be used for deficiency comparisons between these estimators. It appears that the deficiency between two estimators equals the deficiency between the corresponding tests for size $\alpha = \frac{1}{2}$.

By using certain generalizations of the Cramér-Rao bound - the so called Bhattacharyya bounds - we obtain a lower bound to order N^{-1} for the variance of an unbiased estimator. We conclude the chapter by evaluating the deficiency of the estimators considered with respect to this lower bound.

Finally, in chapter 6 we give the results of a number of numerical investigations. These give an indication of the extent to which the asymptotic results obtained in chapters 3 and 4, are of value for small to moderate sample sizes.

In the first place we investigate the behaviour of the expansions for the power of the rank tests in chapter 3 as approximations of the finite sample power. For this purpose we have at our disposal a number of exact power results from literature. These are available for rather small sample sizes -5 to 20- for e.g. Wilcoxon's signed rank test and the absolute normal scores test against normal and logistic location alternatives. Comparison of these values with those resulting from our expansions shows that here the expansions perform very well for these sample sizes already. It also shows that they are much better than the usual normal approximations.

One should keep in mind that the optimistic conclusions above depend on the kind of test and alternative under consideration. If we have long-tailed distributions under the alternative, the situation becomes entirely different. For example in the case of Wilcoxon's signed rank test against Cauchy alternatives not only the normal approximation, but also our expansion leads to very bad results for the same range of sample sizes as above.

In section 4.6 we found approximations to $o(1)$ for the deficiencies between various tests. Here we compare some of these asymptotic expressions to deficiency values that are obtained numerically. We consider the absolute normal scores test, the t-test and the $\bar{X}$-test against normal location alternatives for sample sizes 5-10,20 and 50. The results thus obtained show a satisfactory agreement between the numerical and asymptotic values.

CHAPTER 2

THE OPTIMAL RATIO OF SAMPLE SIZES FOR COMPARING TWO DISTRIBUTIONS

2.1. INTRODUCTION

Let $X_1,\dots,X_m$, $Y_1,\dots,Y_n$ be independent r.v.'s, the X_i all having distribution P, the Y_i all having distribution $\tilde{P}$. We assume that P and $\tilde{P}$ belong to a family of distributions $\mathcal{P}$, characterized by a real parameter θ : P, $\tilde{P} \in \mathcal{P} = \{P_\theta\}$, $\theta \in \Theta \subset R^1$.

DEFINITION 2.1.1. Ψ is the class of continuous functions ψ on R^1 with $\psi(0) = 1$, $0 < \psi(t) < 1$ for $t \neq 0$ and $\sup\{\psi(t) : |t| > \pi\} < 1$.

DEFINITION 2.1.2. For $\psi \in \Psi$, $\mathcal{P}_{1\psi}$ is the class of all distributions P on $\{0,1,2,\dots\}$ with characteristic functions ρ satisfying $|\rho(t)| \leq \psi(t)$ for $|t| \leq \pi$. For $\psi \in \Psi$, $\mathcal{P}_{2\psi}$ is the class of all distributions P that are absolutely continuous with respect to Lebesgue measure and satisfy $|\rho(t)| \leq \psi(t)$ for all t.

Throughout this chapter we suppose that the family $\mathcal{P}$ belongs to either $\mathcal{P}_{1\psi}$ or $\mathcal{P}_{2\psi}$ for a suitable choice of ψ.

Before continuing, we shall give an explanation of the definitions above. First we recall some results about lattice distributions (cf. Feller (1966)). A r.v. X has a lattice distribution P if there exist real numbers a and h with $h > 0$ such that all values of X can be represented as $x = a + \nu h$ for some integer ν. If h_0 is the largest number such that all values of X can be represented as $a + \nu h_0$, it is called the span of P. The characteristic function (c.f.) ρ of X is periodic with period $2\pi/h_0$ and $|\rho(t)| < 1$ for $0 < t < 2\pi/h_0$.

Now it is easy to see how we arrive at $\mathcal{P}_{1\psi}$. We are interested in families of distributions that are concentrated on a fixed lattice and that all have the same span. Then it is no loss of generality to assume this lattice to be $\{0,1,2,\dots\}$ and this span to be 1. Note that it is equivalent to assume that the lattice is $\{0,1,2,\dots\}$ and that $|\rho(t)| < 1$ for $0 < |t| \leq \pi$. We arrive at $\mathcal{P}_{1\psi}$ by using the above stronger version of the second assumption. In this way distributions with span larger than 1 and degenerate distributions are not only excluded, but the elements of $\mathcal{P}_{1\psi}$ remain bounded away from such distributions and we can formulate results that are uniformly true for all $P \in \mathcal{P}_{1\psi}$.

A similar explanation can be given for $\mathcal{P}_{2\psi}$. Here the condition on ρ serves to ensure that the elements of $\mathcal{P}_{2\psi}$ remain bounded away from lattice distributions.

We continue our exposition by introducing

$$(2.1.1) \qquad X = \sum_{i=1}^{m} X_i,\ Y = \sum_{j=1}^{n} Y_j,\ T = X+Y,\ N = m+n.$$

Let θ and $\tilde{\theta}$ be the parameter values corresponding to P and $\tilde{P}$, respectively. A possible way to compare P and $\tilde{P}$ on the basis of $X_1,\ldots,X_m$ and $Y_1,\ldots,Y_n$ is to test $H_0 : \theta = \tilde{\theta}$ against $H_1 : \theta > \tilde{\theta}$ by rejecting H_0 for large values of X, conditionally on T = t. This test is in many cases the uniformly most powerful unbiased (UMPU) test for H_0 against H_1, as will be shown in section 2.2.

Usually the test is performed with equal sample sizes $m = n = \frac{1}{2}N$. In this chapter it is investigated which choice of m/N is optimal, given N, as $N \to \infty$. Here the criterion of optimality is the unconditional power of the test. We restrict attention to the Pitman-case: a sequence of alternatives is chosen which converges to the hypothesis at such a rate that for fixed level of significance α the power remains bounded away from α and 1. Then it is shown that m = n is optimal to first order only, but that the deficiency of this choice with respect to the optimal choice is finite.

In section 2.2 conditions are given under which the test is UMPU. Also in this section an expansion is derived for the conditional distribution of X given T = t. In section 2.3 the unconditional power is obtained, which enables us to find an expression for the optimal ratio m/N and for the deficiency of the traditional choice $m = \frac{1}{2}N$. This is done in section 2.4. Finally, section 2.5 contains another application of the results in section 2.3.

2.2. PRELIMINARIES

First we shall show that the test considered in the previous section is UMPU for the case where $\mathcal{P}$ is an exponential family with monotone likelihood ratio.

LEMMA 2.2.1. *Let* $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ *be a family of distributions on* R^1 *having densities* $f_\theta(x) = c(\theta)\, h(x) \exp(Q(\theta)x)$ *with respect to a fixed σ-finite measure* μ. *Suppose that* Θ *is an interval in* R^1 *and that* Q *is continuous and increasing on* Θ. *Let* $X_1,\dots,X_m$, $Y_1,\dots,Y_n$ *be independent r.v.'s, the* X_i *all having distribution* P_θ, *the* Y_i *all having distribution* $P_{\tilde\theta}$. *Finally, let* X, Y, T *and* N *be defined as in* (2.1.1). *Then the test for* $\theta = \tilde\theta$ *against* $\theta > \tilde\theta$ *that rejects the hypothesis for large values of* X, *conditionally on* $T = t$, *is* UMPU.

PROOF. The joint density of X and Y with respect to an appropriately chosen measure ν on R^2 is

$$f_\theta(x,y) = c^m(\theta)\, c^n(\tilde\theta)\, \exp[(Q(\theta)-Q(\tilde\theta))x + Q(\tilde\theta)t],$$

where $t = x+y$. As $\Theta \times \Theta$ is a rectangle and Q is continuous and increasing, $\{(Q(\theta)-Q(\tilde\theta), Q(\tilde\theta)) : \theta, \tilde\theta \in \Theta\}$ is a quadrangle. In view of theorem 4.4.3 of Lehmann (1959) this shows that the test under consideration is UMPU for $H_0 : Q(\theta) = Q(\tilde\theta)$ against $H_1 : Q(\theta) > Q(\tilde\theta)$. Hence, by the monotonicity of Q, the desired result follows. □

REMARK. In the case where Q is decreasing, the test is of course UMPU for $\theta = \tilde\theta$ against $\theta < \tilde\theta$.

The conditions of this lemma are often satisfied for families $\mathcal{P} \subset \mathcal{P}_{1\psi}$ or $\mathcal{P} \subset \mathcal{P}_{2\psi}$. In the first case μ can be taken as counting measure, in the second case as Lebesgue measure. We consider the following examples, where η is a positive constant,

EXAMPLE 2.2.1. $\mathcal{P} = \{P_p\}$ with $P_p\{1\} = 1-P_p\{0\} = p$, $\eta \le p \le 1-\eta$. Hence X and Y are binomial r.v.'s. The c.f. ρ_p of P_p satisfies $|\rho_p(t)| = |1-p+pe^{it}| = \{1-2p(1-p)(1-\cos t)\}^{1/2} \le \{1-2\eta(1-\eta)(1-\cos t)\}^{1/2}$. This shows that $\mathcal{P} \subset \mathcal{P}_{1\psi}$, for a suitable choice of ψ. Furthermore, $Q(p) = \log\{p/(1-p)\}$, which increases.

EXAMPLE 2.2.2. The family of Poisson distributions $\mathcal{P} = \{P_\lambda\}$ with $P_\lambda\{k\} = e^{-\lambda}\lambda^k/(k!)$, $k = 0,1,2,\dots$ and $\lambda \ge \eta$. Again, for a suitable ψ, $\mathcal{P} \subset \mathcal{P}_{1\psi}$ and $Q(\lambda) = \log \lambda$ is increasing.

EXAMPLE 2.2.3. The family of geometric distributions $\mathcal{P} = \{P_p\}$ with $P_p\{k\} = p(1-p)^{(k-1)}$, $k = 1,2,\ldots$ and $0 < p \le 1-\eta$. Again, for a suitable ψ, $\mathcal{P} \subset \mathcal{P}_{1\psi}$, but now $Q(p) = \log(1-p)$ decreases in p (cf. the remark following lemma 2.2.1).

EXAMPLE 2.2.4. The family of exponential distributions $\mathcal{P} = \{P_\lambda\}$ where P_λ is determined by the density $f_\lambda(x) = \lambda e^{-\lambda x}$, $x > 0$ and $0 < \lambda \le \eta$. Now $\mathcal{P} \subset \mathcal{P}_{2\psi}$, for a suitable choice of ψ, and $Q(\lambda) = -\lambda$ decreases.

The examples above can all be placed in the framework of a 2×2-table. In the first example we compare two Bernoulli experiments in the following way: the first experiment, which has probability of success p_1, is performed m times and the second experiment, which has probability of success p_2, is performed n times. The hypothesis $p_1 = p_2$ is tested against the alternative $p_1 > p_2$ on the basis of X, the number of successes obtained with the first experiment, conditionally on X+Y, the total number of successes. It is well-known that the conditional distribution of X under the hypothesis is hypergeometric. The second example can be looked at as a limiting case of the first one for small p-values, as the binomial distribution with parameters $(N,\frac{\lambda}{N})$ tends to the Poisson distribution with parameter λ. The conditional distribution of X under the hypothesis is binomial for this case.

In the third example we also compare two Bernoulli experiments, but now the first (second) experiment is performed until m(n) successes - or, equivalently, failures - have occurred. The above hypothesis is tested here on the basis of the number of trials with the first experiment, conditionally on the total number of trials performed. Finally, the fourth example is the continuous analogue of the third example.

We now return to the general case, where $\mathcal{P} \subset \mathcal{P}_{1\psi}$ or $\mathcal{P}_{2\psi}$. In order to find the optimal ratio m/N, we must compute the unconditional power of the test, which is the expectation with respect to T of the conditional power. For the evaluation of this conditional power, we have to know the conditional distribution of X, given T = t, under H_0, as well as under H_1. First this distribution is found for $\mathcal{P} \subset \mathcal{P}_{1\psi}$; the case $\mathcal{P} \subset \mathcal{P}_{2\psi}$ follows by analogy. As we shall restrict attention to local alternatives, the following lemma is more general than needed here since it holds for general θ and $\tilde{\theta}$. It is given in the present form as it may be of some interest of its own.

LEMMA 2.2.2. *Suppose that* $X_1,\dots,X_m$, $Y_1,\dots,Y_n$ *are independent r.v.'s, the* X_i *with distribution* P_θ, *the* Y_i *with distribution* $P_{\tilde\theta}$, *where* P_θ, $P_{\tilde\theta} \in P \subset P_{1\psi}$, *for some* $\psi \in \Psi$. *Let* X, Y, T *and* N *be defined as in* (2.1.1); *let* μ, σ^2, μ_k $(\tilde\mu, \tilde\sigma^2, \tilde\mu_k)$ *be the expectation, variance and k-th central moment of* X_1 (Y_1), $k = 3,4,\dots$. *Moreover, assume that positive constants* b, B *and* V *exist such that* $\sigma^2 \geq b$, $\tilde\sigma^2 \geq b$, $Ee^{vX_1} \leq B$ *and* $Ee^{vY_1} \leq B$ *for* $|v| \leq V$. *Then for all non-negative integers* m, n *and* t *such that* $|t-m\mu-n\tilde\mu| \leq cN^{2/3}/\log N$ *and* $\varepsilon \leq m/N \leq 1-\varepsilon$ *for some positive constants* c *and* ε, *we have for each non-negative integer* l *the following expansion*

$$P(X \leq l \mid T=t) = \Phi(y_{l+\frac12}) + \phi(y_{l+\frac12})(a_0+a_1y_{l+\frac12}+a_2(y^2_{l+\frac12}+2)) + R, \tag{2.2.1}$$

where

$$|R| \leq A[N^{-1} + N^{-4}(t-m\mu-n\tilde\mu)^6].$$

Here

$$y_l = \left(\frac{m\sigma^2+n\tilde\sigma^2}{mn\sigma^2\tilde\sigma^2}\right)^{1/2}\left(l - \frac{n\tilde\sigma^2 m\mu+m\sigma^2(t-n\tilde\mu)}{m\sigma^2+n\tilde\sigma^2}\right),$$

$$\begin{aligned} a_0 &= \tfrac12(m\sigma^2+n\tilde\sigma^2)^{-1/2}\left[\frac{\mu_3}{\sigma^2}\left(\frac{n\tilde\sigma^2}{m\sigma^2}\right)^{1/2} - \frac{\tilde\mu_3}{\tilde\sigma^2}\left(\frac{m\sigma^2}{n\tilde\sigma^2}\right)^{1/2}\right] + \\ &\quad - \tfrac12(t-m\mu-n\tilde\mu)^2(m\sigma^2+n\tilde\sigma^2)^{-5/2}\left[\mu_3\left(\frac{mn\tilde\sigma^2}{\sigma^2}\right)^{1/2}-\tilde\mu_3\left(\frac{mn\sigma^2}{\tilde\sigma^2}\right)^{1/2}\right], \\ a_1 &= -\tfrac12(t-m\mu-n\tilde\mu)(m\sigma^2+n\tilde\sigma^2)^{-2}\left[\frac{\mu_3 n\tilde\sigma^2}{\sigma^2} + \frac{\tilde\mu_3 m\sigma^2}{\tilde\sigma^2}\right], \\ a_2 &= -\tfrac16(m\sigma^2+n\tilde\sigma^2)^{-3/2}\left[\mu_3\,\frac{n^{3/2}\tilde\sigma^3}{m^{1/2}\sigma^3} - \tilde\mu_3\,\frac{m^{3/2}\sigma^3}{n^{1/2}\tilde\sigma^3}\right], \end{aligned} \tag{2.2.2}$$

and A *depends on* P_θ, $P_{\tilde\theta}$ *in* $P_{1\psi}$, m, n, t *and* l *only through* b, B, V, c, ε *and* ψ.

PROOF. We have

$$P(X \leq l \mid T=t) = \left[\sum_{k=0}^{l} P(X=k)P(Y=t-k)\right]/\left[\sum_{k=0}^{t} P(X=k)P(Y=t-k)\right]. \tag{2.2.3}$$

The procedure is as follows: we give an expansion for $P(X=k)$ for the central k-values. From this expansion we immediately derive a similar expansion for $P(Y=t-k)$, which leads to an expansion for $P(X=k)P(Y=t-k)$, for central k-values. After showing that the $P(X=k)P(Y=t-k)$ for k in the tail of the distribution can be neglected, the sums in (2.2.3) can be evaluated.

First we give an expansion for $P(X=k)$. Let ρ_Z be the characteristic function of a r.v. Z. As $\rho_X(u) = Ee^{iuX} = \sum_{k=0}^{\infty} P(X=k)e^{iuk}$, we have

$$P(X=k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho_X(u)e^{-iuk}du, \qquad k = 0,1,2,\dots . \tag{2.2.4}$$

From the definition of $P_{1\psi}$ it follows that for each $c_1 \in (0,\pi)$ there exists $\varepsilon_1 \in (0,1)$ such that $|\rho_{X_1}(u)| \le 1-\varepsilon_1$ for $c_1 \le |u| \le \pi$, uniformly for all $P \in P_{1\psi}$. This implies that

$$|\rho_X(u)| \le (1-\varepsilon_1)^m, \text{ for } c_1 \le |u| \le \pi. \tag{2.2.5}$$

Using (2.2.5) and the fact that $\varepsilon \le m/N \le 1-\varepsilon$, (2.2.4) becomes

$$P(X=k) = \frac{1}{2\pi m^{1/2}\sigma}\int_{-c_1 m^{1/2}\sigma}^{c_1 m^{1/2}\sigma} \rho_{X-m\mu}\left(\frac{u}{m^{1/2}\sigma}\right)e^{\frac{-iu(k-m\mu)}{m^{1/2}\sigma}} du + O(e^{-\tilde{C}N}), \tag{2.2.6}$$

for some $\tilde{C} > 0$.

Let $w = u+iv$ be a complex number. From the fact that $Ee^{vX_1} \le B$ for $|v| \le V$ it follows that $\rho_{X_1}(w)$ is analytic for $|v| < V$. Also, in analogy to Feller (1966) one may derive a bound for the error in approximating $\rho_{X_1}(w)$ in the region $|v| < V$ by finitely many terms of its power series. For $M = 1,2,\dots$ define

$$p_M(w) = e^{iw} - 1 - \frac{iw}{1!} - \dots\dots\dots\dots - \frac{(iw)^{M-1}}{(M-1)!} .$$

From

$$|p_1(w)| = \left|\int_0^w e^{iz}dz\right| \le \int_0^w e^{-\mathrm{Im}\, z}|dz| \le |w| \max(1,e^{-V}),$$

and

$$p_M(w) = i\int_0^w p_{M-1}(z)dz, \qquad M = 2,3,\dots ,$$

it follows that

$$(2.2.7)\qquad |p_M(w)| \le \frac{|w|^M}{M!}\max(e^{-v},1), \qquad M = 1,2,\ldots .$$

Substitution of wX_1 for w in (2.2.7) and taking expectations shows

$$(2.2.8)\qquad |\rho_{X_1}(w) - 1 - \frac{iw}{1!}EX_1 - \ldots\ldots - \frac{(iw)^{M-1}}{(M-1)!}EX_1^{M-1}| \le$$

$$\le \frac{|w|^M}{M!}E[|X_1|^M \max(1,e^{-vX_1})] = O(\frac{|w|^M}{M!}), \quad M = 1,2,\ldots,$$

for $|v| < V$, as $Ee^{vX_1} \le B$ for $|v| \le V$. As an application of (2.2.8) we have, since $\sigma^2 \ge b > 0$,

$$\rho_{X_1-\mu}(\frac{w}{m^{1/2}\sigma}) = 1 - \frac{w^2}{2m} - \frac{i\mu_3 w^3}{6m^{3/2}\sigma^3} + O(\frac{|w|^4}{m^2}), \text{ for } |v| < m^{1/2}\sigma V.$$

For some positive constant c_2, which we may choose $< V$, $\rho_{X_1-\mu}$ satisfies $\mathrm{Re}\{\rho_{X_1-\mu}(w/(m^{1/2}\sigma))\} > \frac{1}{2}$ for $|w| \le c_2 m^{1/2}\sigma$. Hence, for these w, we may expand $\log \rho_{X_1-\mu}(w/(m^{1/2}\sigma))$ to obtain

$$\log \rho_{X_1-\mu}(\frac{w}{m^{1/2}\sigma}) = -\frac{w^2}{2m} - \frac{i\mu_3 w^3}{6m^{3/2}\sigma^3} + O(\frac{|w|^4}{m^2}).$$

Consequently

$$\log \rho_{X-m\mu}(\frac{w}{m^{1/2}\sigma}) = -\frac{w^2}{2} - \frac{i\mu_3 w^3}{6m^{1/2}\sigma^3} + O(\frac{|w^4|}{m}),$$

for $|w| \le c_2 m^{1/2}\sigma$. Furthermore, there exists a positive constant c_3, which we may choose $\le c_2$, such that for $|w| \le c_3 m^{1/2}\sigma$

$$(2.2.9)\qquad |\log \rho_{X-m\mu}(\frac{w}{m^{1/2}\sigma}) + \frac{w^2}{2}| \le \frac{|w|^2}{4}.$$

Hence, for these values of w

$$(2.2.10)\qquad \rho_{X-m\mu}(\frac{w}{m^{1/2}\sigma}) = e^{-w^2/2}(1 - \frac{i\mu_3 w^3}{6m^{1/2}\sigma^3} + O([N^{-1}+N^{-1}|w|^6]e^{|w|^2/4})).$$

Using (2.2.9) and (2.2.10) we can now deal with the integral in (2.2.6). Suppose $|k-m\mu| \le CN^{2/3}/\log N$ for some positive constant C and choose $c_1 < c_3$. Then the rectangle with vertices $\pm c_1 m^{1/2}\sigma$ and $\pm c_1 m^{1/2}\sigma - i\frac{(k-m\mu)}{m^{1/2}\sigma}$

is contained in $|w| \le c_3 m^{1/2}\sigma$ for N sufficiently large. By complex integration of $\rho_{X-m\mu}(w/m^{1/2}\sigma)\exp\{-iw(k-m\mu)/(m^{1/2}\sigma)\}$ over this rectangle, we arrive at

$$(2.2.11) \qquad P(X=k) = \frac{1}{2\pi m^{1/2}\sigma}\int_{(-c_1 m^{1/2}\sigma - i(\frac{k-m\mu}{m^{1/2}\sigma}))}^{(c_1 m^{1/2}\sigma - i(\frac{k-m\mu}{m^{1/2}\sigma}))} \rho_{X-m\mu}\left(\frac{w}{m^{1/2}\sigma}\right) e^{-iw(\frac{k-m\mu}{m^{1/2}\sigma})} dw +$$

$$+ O\left(\left|\frac{k-m\mu}{m^{1/2}\sigma}\right| \sup_{0\le\nu\le 1} \left|\rho_{X-m\mu}\left(\pm c_1 - i\nu\frac{(k-m\mu)}{m\sigma^2}\right)\right| + e^{-\tilde{C}N}\right).$$

Application of (2.2.10) to the integral in (2.2.11) and of (2.2.9) to the remainder in (2.2.11) leads to

$$(2.2.12) \qquad P(X=k) = \frac{e^{-\frac{1}{2}\frac{(k-m\mu)^2}{m\sigma^2}}}{2\pi m^{1/2}\sigma}\int_{-c_1 m^{1/2}\sigma}^{c_1 m^{1/2}\sigma} e^{-\frac{1}{2}u^2}\left[1 - \frac{i\mu_3(u - i\frac{(k-m\mu)}{m^{1/2}\sigma})^3}{6m^{1/2}\sigma^3} +\right.$$

$$\left. + O\left([N^{-1} + N^{-1}|u - i\frac{(k-m\mu)}{m^{1/2}\sigma}|^6] e^{\frac{1}{4}(u^2 + \frac{(k-m\mu)^2}{m\sigma^2})}\right)\right] du +$$

$$+ O\left(\left|\frac{k-m\mu}{m^{1/2}\sigma}\right| e^{-\frac{1}{4}c_1^2 m\sigma^2 + \frac{3}{4}\frac{(k-m\mu)^2}{m\sigma^2}} + e^{-\tilde{C}N}\right).$$

Upon evaluation of the integral in (2.2.12), where we use the fact that $|k-m\mu| \le CN^{2/3}/\log N$, we finally arrive at

$$(2.2.13) \qquad P(X=k) = \frac{e^{-\frac{1}{2}\frac{(k-m\mu)^2}{m\sigma^2}}}{(2\pi m\sigma^2)^{1/2}}\left[1 - \frac{\mu_3}{6m^{1/2}\sigma^3}\left\{3\left(\frac{k-m\mu}{m^{1/2}\sigma}\right) - \left(\frac{k-m\mu}{m^{1/2}\sigma}\right)^3\right\} +\right.$$

$$\left. + O(N^{-1} + N^{-4}|k-m\mu|^6)\right],$$

where the remainder only depends on P_θ through b, B, V and ψ.

Since $|t-m\mu-n\tilde{\mu}| \le cN^{2/3}/\log N$, $|k-m\mu| \le CN^{2/3}/\log N$ implies $|t-k-n\tilde{\mu}| \le (C+c)N^{2/3}/\log N$. Hence, for $|k-m\mu| \le CN^{2/3}/\log N$ we also have

$$(2.2.14)\qquad P(Y=t-k) = \frac{e^{-\frac{1}{2}\frac{(t-k-n\tilde{\mu})^2}{n\tilde{\sigma}^2}}}{(2\pi n\tilde{\sigma}^2)^{1/2}}\,[1 - \frac{\tilde{\mu}_3}{6n^{1/2}\tilde{\sigma}^3}\{3(\frac{t-k-n\tilde{\mu}}{n^{1/2}\tilde{\sigma}}) - (\frac{t-k-n\tilde{\mu}}{n^{1/2}\tilde{\sigma}})^3\} +$$

$$+ O(N^{-1} + N^{-4}\,|t-k-n\tilde{\mu}|^6)].$$

As we shall have to sum over k, the expression, obtained for $P(X=k)$. $P(Y=t-k)$ from (2.2.13) and (2.2.14), has to be replaced by an integral over the interval $(k-\frac{1}{2}, k+\frac{1}{2})$. First note that

$$(2.2.15)\qquad k-m\tilde{\mu} = [\frac{mn\sigma^2\tilde{\sigma}^2}{m\sigma^2+n\tilde{\sigma}^2}]^{1/2} y_k + \frac{m\sigma^2}{m\sigma^2+n\tilde{\sigma}^2}\,(t-m\mu-n\tilde{\mu}),$$

$$(2.2.16)\qquad t-k-n\tilde{\mu} = -[\frac{mn\sigma^2\tilde{\sigma}^2}{m\sigma^2+n\tilde{\sigma}^2}]^{1/2} y_k + \frac{n\tilde{\sigma}^2}{m\sigma^2+n\tilde{\sigma}^2}(t-m\mu-n\tilde{\mu}),$$

where

$$(2.2.17)\qquad y_k = [\frac{m\sigma^2+n\tilde{\sigma}^2}{nm\sigma^2\tilde{\sigma}^2}]^{1/2}[k - \frac{n\tilde{\sigma}^2}{m\sigma^2+n\tilde{\sigma}^2}\,m\mu - \frac{m\sigma^2}{m\sigma^2+n\tilde{\sigma}^2}(t-n\tilde{\mu})].$$

As a consequence we have

$$(2.2.18)\qquad \frac{(k-m\mu)^2}{m\sigma^2} + \frac{(t-k-n\tilde{\mu})^2}{n\tilde{\sigma}^2} = y_k^2 + \frac{(t-m\mu-n\tilde{\mu})^2}{m\sigma^2+n\tilde{\sigma}^2}\ .$$

From (2.2.15) it follows that $|k-m\mu| \le CN^{2/3}/\log N$ and $|t-m\mu-n\tilde{\mu}| \le$ $\le cN^{2/3}/\log N$ imply $|y_k| \le \bar{c}N^{1/6}/\log N$, for some $\bar{c} > 0$. Hence, if y_x is defined by (2.2.17) with k replaced by x

$$(2.2.19)\qquad \int_{k-1/2}^{k+1/2} e^{-\frac{1}{2}y_x^2}dx = e^{-\frac{1}{2}y_k^2}\int_{k-1/2}^{k+1/2}[1 - (\frac{m\sigma^2+n\tilde{\sigma}^2}{mn\sigma^2\tilde{\sigma}^2})^{1/2}(x-k)y_k +$$

$$+ O(\frac{m\sigma^2+n\tilde{\sigma}^2}{mn\sigma^2\tilde{\sigma}^2}\,(x-k)^2y_k^2)][1 + O(\frac{m\sigma^2+n\tilde{\sigma}^2}{nm\sigma^2\tilde{\sigma}^2}\,(x-k)^2]dx =$$

$$= e^{-\frac{1}{2}y_k^2}\{1 + O(N^{-1} + N^{-1}y_k^2)\}.$$

In the same way it can be shown that the lower order terms can be replaced by appropriate integrals. For example

$$(2.2.20)\qquad \int_{k-1/2}^{k+1/2} \left(\frac{x-m\mu}{m^{1/2}\sigma}\right)^3 e^{-\frac{1}{2}y_x^2}\,dx =$$

$$= \left[\left(\frac{k-m\mu}{m^{1/2}\sigma}\right)^3 + O\left(\frac{(k-m\mu)^2}{m^{3/2}\sigma^3}\right)\right]\int_{k-1/2}^{k+1/2} e^{-\frac{1}{2}y_x^2}\,dx =$$

$$= e^{-\frac{1}{2}y_k^2}\left[\left(\frac{k-m\mu}{m^{1/2}\sigma}\right)^3 + O(N^{-1/2}+N^{-1/2}y_k^4+N^{-5/2}|t-m\mu-n\tilde{\mu}|^4)\right].$$

From (2.2.18), (2.2.19) and expressions similar to (2.2.20) it follows that every term in the product of the expressions in (2.2.13) and (2.2.14) can be replaced by the corresponding integral. Upon doing this and using (2.2.15) we arrive at

$$(2.2.21)\qquad P(X=k)P(Y=t-k) = \exp\left[\frac{-(t-m\mu-n\tilde{\mu})^2}{2(m\sigma^2+n\tilde{\sigma}^2)}\right]/\left[2\pi(m\sigma^2+n\tilde{\sigma}^2)^{1/2}\right]\Big[g(y_k) +$$

$$+ O\Big([N^{-1}+N^{-1}y_k^6+N^{-4}(t-m\mu-n\tilde{\mu})^6]\int_{y_{k-1/2}}^{y_{k+1/2}} e^{-\frac{1}{2}y^2}\,dy\Big)\Big],$$

for $|k-m\mu| \le CN^{2/3}/\log N$, with

$$(2.2.22)\qquad g(y_k) =$$

$$= \int_{y_{k-1/2}}^{y_{k+1/2}} e^{-\frac{1}{2}y^2}\Big(1 - \frac{\mu_3}{6m^{1/2}\sigma^3}\{3[(\frac{n\tilde{\sigma}^2}{m\sigma^2+n\tilde{\sigma}^2})^{1/2}y + \frac{m^{1/2}\sigma}{m\sigma^2+n\tilde{\sigma}^2}(t-m\mu-n\tilde{\mu})] +$$

$$- [(\frac{n\tilde{\sigma}^2}{m\sigma^2+n\tilde{\sigma}^2})^{1/2}y + \frac{m^{1/2}\sigma}{m\sigma^2+n\tilde{\sigma}^2}(t-m\mu-n\tilde{\mu})]^3\} +$$

$$+ \frac{\tilde{\mu}_3}{6n^{1/2}\tilde{\sigma}^3}\{3[(\frac{m\sigma^2}{m\sigma^2+n\tilde{\sigma}^2})^{1/2}y - \frac{n^{1/2}\tilde{\sigma}}{m\sigma^2+n\tilde{\sigma}^2}(t-m\mu-n\tilde{\mu})] +$$

$$- [(\frac{m\sigma^2}{m\sigma^2+n\tilde{\sigma}^2})^{1/2}y - \frac{n^{1/2}\tilde{\sigma}}{m\sigma^2+n\tilde{\sigma}^2}(t-m\mu-n\tilde{\mu})]^3\}\Big)\,dy.$$

With (2.2.21) and (2.2.22) we have found an approximation for the central k-values; it remains to show that the sums over $|k-m\mu| > CN^{2/3}/\log N$ of both $P(X=k)P(Y=t-k)$ and $g(y_k)$ are sufficiently small.

First $g(y_k)$ is considered. Choose $C > 2c$, then $|k-m\mu| > CN^{2/3}/\log N$ and $|t-m\mu-n\tilde{\mu}| \le cN^{2/3}/\log N$ imply in view of (2.2.15) that $|y_k| > \tilde{c}N^{1/6}/\log N$, for some constant $\tilde{c} > 0$. Hence

$$(2.2.23) \qquad \sum_{|k-m\mu|>CN^{2/3}/\log N} |g(y_k)| \le \sum_{|y_k|>\tilde{c}N^{1/6}/\log N} |g(y_k)| =$$

$$= O\left(\int_{|y|>\tilde{c}N^{1/6}/\log N} (1+|y|^3)e^{-\frac{1}{2}y^2} dy\right) = O(N^{-1}).$$

For $P(X=k)P(Y=t-k)$ we proceed in the following way. According to (2.2.21) to (2.2.23) we have

$$(2.2.24) \qquad \sum_{m\mu-CN^{2/3}/\log N}^{m\mu+CN^{2/3}/\log N} P(X=k)P(Y=t-k) = \exp\left[-\frac{(t-m\mu-n\tilde{\mu})^2}{2(m\sigma^2+n\tilde{\sigma}^2)}\right]/[2\pi(m\sigma^2+n\tilde{\sigma}^2)^{1/2}]$$

$$[1 + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6)].$$

As X+Y is a sum of independent r.v.'s, an expansion similar to (2.2.14) holds for $P(X+Y=t)$ for $|t-m\mu-n\tilde{\mu}| \le cN^{2/3}/\log N$. In particular, $P(X+Y=t)$ equals the right side of (2.2.24). This implies that

$$(2.2.25) \qquad \sum_{|k-m\mu|>CN^{2/3}/\log N} P(X=k)P(Y=t-k) =$$

$$= O\left([N^{-3/2}+N^{-9/2}(t-m\mu-n\tilde{\mu})^6]\exp\left[-\frac{1}{2}\frac{(t-m\mu-n\tilde{\mu})^2}{(m\sigma^2+n\tilde{\sigma}^2)}\right]\right).$$

The sums in (2.2.3) can now be approximated. For any l we have

$$(2.2.26) \qquad \sum_{k=0}^{l} P(X=k)P(Y=t-k) = \exp\left[-\frac{(t-m\mu-n\tilde{\mu})^2}{2(m\sigma^2+n\tilde{\sigma}^2)}\right]/[2\pi(m\sigma^2+n\tilde{\sigma}^2)^{1/2}]\left\{\sum_{k=0}^{l} g(y_k) +\right.$$

$$\left. + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6)\right\},$$

where we use (2.2.21) for $|k-m\mu| \le CN^{2/3}/\log N$ and (2.2.24) and (2.2.25) for $|k-m\mu| > CN^{2/3}/\log N$. From (2.2.26) it immediately follows that

(2.2.27) $$P(X\leq l\,|\,T=t) = [\sum_{k=0}^{l} g(y_k) + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6)]/[\sum_{k=0}^{t} g(y_k) + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6)].$$

It remains to insert (2.2.22) in (2.2.27). We can write $g(y_k)$ as

$$g(y_k) = \int_{y_{k-1/2}}^{y_{k+1/2}} e^{-\frac{1}{2}y^2}(1+b-a_0y-a_1y^2-a_2y^3)dy,$$

where b, a_0, a_1 and a_2 are all $O(N^{-1/2}+N^{-3/2}(t-m\mu-n\tilde{\mu})^2)$ and a_0, a_1 and a_2 are given in (2.2.2). This implies

$$\sum_{k=0}^{l} g(y_k) = \Phi(y_{l+1/2})(1+b-a_1) + \phi(y_{l+1/2})[a_0+a_1y_{l+1/2}+a_2(y_{l+1/2}^2+2)].$$

In particular, $\sum_{k=0}^{t} g(y_k) = 1+b-a_1+O(N^{-1})$. Substitution of these results in (2.2.27) leads to the desired expressions (2.2.1) and (2.2.2). Finally the uniformity of the O-symbol in m, l and t is evident from the method of proof. □

COROLLARY 2.2.1. *Under the conditions of lemma* 2.2.2 *we have for each* $\delta \in (0,1]$ *an expansion to* $O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6)$ *for* $P(X<l\,|\,T=t) + \delta P(X=l\,|\,T=t)$ *if we replace* $y_{l+1/2}$ *by* $y_{l+\delta-1/2}$ *in* (2.2.1).

PROOF. The result follows immediately from lemma 2.2.2 by noting that $P(X<l\,|\,T=t) + \delta P(X=l\,|\,T=t) = (1-\delta)P(X\leq l-1\,|\,T=t) + \delta P(X\leq l\,|\,T=t)$ and that $(1-\delta)\Phi(y_{l-1/2}) + \delta\Phi(y_{l+1/2}) = \Phi(y_{l+\delta-1/2}) + O(N^{-1})$. □

In the second case, where $P \subset P_{2\psi}$, a similar expansion holds.

LEMMA 2.2.3. *Let* $\rho_{X_1}(\rho_{Y_1})$ *be the characteristic function of* $X_1(Y_1)$ *and suppose that* $\int|\rho_{X_1}(u)|^{\nu}du \leq B$, $\int|\rho_{Y_1}(u)|^{\nu}du \leq B$ *for some* $\nu \geq 1$. *Let* P_θ, $P_{\tilde{\theta}} \in P \subset P_{2\psi}$ *for some* $\psi \in \Psi$, *let* t *and* l *be real numbers and suppose that the remaining conditions of lemma* 2.2.2 *continue to hold.*

Then the conclusion in (2.2.1) *continues to hold, if* $y_{l+1/2}$ *is replaced by* y_l.

PROOF. As X_1 and Y_1 have densities $f_{X_1}(x)$ and $f_{Y_1}(y)$, the conditional density of $X = \sum_{i=1}^{m} X_i$ given $T = t$ is given by

$$f_{X|T=t}(x) = \begin{cases} \dfrac{f_X(x).f_Y(t-x)}{f_T(t)} & \text{if } f_T(t) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $P(X \leq l | T=t) = \int_{-\infty}^{l} f_X(x) f_Y(t-x)dx / \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)dx$. For $m \geq \nu$, $|\rho_X| = |\rho_{X_1}|^m \leq |\rho_{X_1}|^\nu$ and therefore $\int_{-\infty}^{\infty} |\rho_X(u)| du < B$. Under this condition, the Fourier inversion theorem yields

$$(2.2.28) \qquad f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_X(u) e^{-iux} du =$$

$$= \frac{1}{2\pi m^{1/2}\sigma} \int_{-c_1 m^{1/2}\sigma}^{c_1 m^{1/2}\sigma} \rho_{(X-m\mu)}\left(\frac{u}{m^{1/2}\sigma}\right) e^{-iu\left(\frac{x-m\mu}{m^{1/2}\sigma}\right)} du +$$

$$+ \frac{1}{2\pi} \int_{|u|>c_1} \rho_X(u) e^{-iux} dx.$$

From the definition of $P_{2\psi}$ it follows that for each $c_1 > 0$ there exists $\varepsilon_1 \in (0,1)$ such that $|\rho_{X_1}(u)| \leq 1-\varepsilon_1$ for $|u| \geq c_1$, uniformly for all $P \in P_{2\psi}$. This implies that

$$|\rho_X(u)| = |\rho_{X_1}(u)|^m \leq (1-\varepsilon_1)^{m-\nu} |\rho_{X_1}(u)|^\nu,$$

for $|u| > c_1$. As $|\rho_{X_1}(u)|^\nu$ is summable, it follows that the second integral in (2.2.28) is exponentially small.

Since (2.2.10) still holds in the present case, the first integral in (2.2.28) can be handled in the same way as in the proof of lemma 2.2.2. Hence, for $|x-m\mu| \leq CN^{2/3}/\log N$, $f_X(x)$ can be approximated by the right side of (2.2.14), upon changing k into x. The remaining part of the proof is analogous to and simpler than the proof of lemma 2.2.2. □

2.3. AN EXPANSION FOR THE UNCONDITIONAL POWER

The conditional power of the test of $\theta = \tilde{\theta}$ against $\theta > \tilde{\theta}$ (or $\theta < \tilde{\theta}$) that rejects the hypothesis for large values of X, given T = t, is denoted as $\pi_{t\alpha}$. Corollary 2.2.1 and lemma 2.2.3 enable us to find an expansion for $\pi_{t\alpha}$ for those t that satisfy $|t-m\mu-n\tilde{\mu}| \le cN^{2/3}/\log N$. This will suffice to find an expansion for the unconditional power, which is denoted as π_α, under the assumption of local alternatives. In the following lemma we give such an expansion.

LEMMA 2.3.1. *Let* P_θ, $P_{\tilde{\theta}} \in \mathcal{P}$, *where* $\mathcal{P} \subset \mathcal{P}_{1\psi}$ *or* $\mathcal{P} \subset \mathcal{P}_{2\psi}$ *for some* $\psi \in \Psi$. *Suppose there exist positive constants* ε, b, C, V *and* $0 < \alpha < 1$ *such that* $\varepsilon \le m/N \le 1-\varepsilon$, $Ee^{vX_1} \le C$, $Ee^{vY_1} \le C$ *for* $|v| \le V$, $\sigma^2, \tilde{\sigma}^2 \ge b$, $|\mu-\tilde{\mu}| \le CN^{-1/2}$, $|\sigma^2-\tilde{\sigma}^2| \le CN^{-1/2}$ *and* $|\mu_3-\tilde{\mu}_3| \le CN^{-1/2}$. *If* $\mathcal{P} \subset \mathcal{P}_{2\psi}$ *then also assume that* $\int|\rho_{X_1}|^\nu \le C$, $\int|\rho_{Y_1}|^\nu \le C$ *for some* $\nu \ge 1$.

Then the unconditional power π_α *satisfies*

$$|\pi_\alpha - g(u_\alpha)| \le AN^{-1}, \tag{2.3.1}$$

where

$$\begin{aligned} g(u_\alpha) = {} & 1-\Phi[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})]-\phi[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})] \\ & [u_\alpha\{\frac{\tilde{\mu}_3}{6\tilde{\sigma}^3}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})(\frac{n+N}{N}) - \frac{n}{2N}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2})\} + \\ & + (\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})\{\frac{n}{2N}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2}) - \frac{\tilde{\mu}_3}{6\tilde{\sigma}^3}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})(\frac{n-m}{N})\}], \end{aligned} \tag{2.3.2}$$

and where A *depends on* P_θ, $P_{\tilde{\theta}}$ *in* $\mathcal{P}_{1\psi}$ *or* $\mathcal{P}_{2\psi}$, m *and* n *only through* ε, b, C, V *and* ψ.

PROOF. We give the proof for the case where $\mathcal{P} \subset \mathcal{P}_{1\psi}$. The proof for the other case is similar but requires some minor notational changes which are left to the reader. As long as nothing is said to the contrary, we restrict attention to those t that satisfy $|t-m\mu-n\tilde{\mu}| \le cN^{2/3}/\log N$, for some $c > 0$. According to corollary 2.2.1 we have under H_1

(2.3.3) $$P_1(X<l|T=t) + \delta P_1(X=l|T=t) = \Phi(y_{l+\delta-1/2}) + \phi(y_{l+\delta-1/2})[a_0 + a_1 y_{l+\delta-1/2} + a_2(y^2_{l+\delta-1/2}+2)] + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6),$$

with y_l, a_0, a_1 and a_2 as defined in (2.2.2). Under H_0 we have

(2.3.4) $$P_0(X<l|T=t) + \delta P_0(X=l|T=t) = \Phi(\tilde{y}_{l+\delta-1/2}) + \phi(\tilde{y}_{l+\delta-1/2})[\tilde{a}_0 + \tilde{a}_1\tilde{y}_{l+\delta-1/2} + \tilde{a}_2(\tilde{y}^2_{l+\delta-1/2}+2)] + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6),$$

where $\tilde{y}_l$, $\tilde{a}_0$, $\tilde{a}_1$ and $\tilde{a}_2$ are derived from y_l, a_0, a_1 and a_2 by replacing μ, σ^2 and μ_3 by $\tilde{\mu}$, $\tilde{\sigma}^2$ and $\tilde{\mu}_3$ everywhere.

Let $l_{t\alpha}$ and $\delta_{t\alpha}$ be such that $P_0(X<l_{t\alpha}|T=t) + \delta_{t\alpha}P_0(X=l_{t\alpha}|T=t) = 1-\alpha = \Phi(u_\alpha)$. From (2.3.4) and the fact that $u_\alpha = O(1)$ it follows that

(2.3.5) $$\tilde{y}_{l_{t\alpha}+\delta_{t\alpha}-1/2} = u_\alpha - [\tilde{a}_0+\tilde{a}_1 u_\alpha+\tilde{a}_2(u_\alpha^2+2)] + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6),$$

as $\tilde{a}_0$, $\tilde{a}_1$ and $\tilde{a}_2$ are $O(N^{-1/2}+N^{-3/2}(t-m\mu-n\tilde{\mu})^2)$. In order to find $\pi_{t\alpha}$, we must know $y_{l_{t\alpha}+\delta_{t\alpha}-1/2}$. To this end, note that

$$\frac{n\tilde{\sigma}^2 m\mu+m\sigma^2(t-n\tilde{\mu})}{m\sigma^2+n\tilde{\sigma}^2} = (t-n\tilde{\mu}) - \frac{n\tilde{\sigma}^2}{m\sigma^2+n\tilde{\sigma}^2}(t-m\mu-n\tilde{\mu}),$$

and hence

$$\frac{n\tilde{\sigma}^2 m\mu+m\sigma^2(t-n\tilde{\mu})}{m\sigma^2+n\tilde{\sigma}^2} - \frac{n\tilde{\sigma}^2\, m\tilde{\mu}+m\tilde{\sigma}^2(t-n\tilde{\mu})}{m\tilde{\sigma}^2+n\tilde{\sigma}^2} = \frac{mn}{N}(\mu-\tilde{\mu}) + [\frac{mn}{N^2}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2}) + O(N^{-1})](t-m\mu-n\tilde{\mu}).$$

Combining this result and (2.3.5) we get

(2.3.6) $$y_{l_{t\alpha}+\delta_{t\alpha}-1/2} = [\tilde{y}_{l_{t\alpha}+\delta_{t\alpha}-1/2} -(\frac{N}{mn\tilde{\sigma}^2})^{1/2}(\frac{mn}{N}(\mu-\tilde{\mu}) + \{\frac{mn}{N^2}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2}) + O(N^{-1})\}(t-m\mu-n\tilde{\mu}))](\frac{N\sigma^2}{m\sigma^2+n\tilde{\sigma}^2})^{-1/2} =$$
$$= [u_\alpha-(\frac{mn}{N})^{1/2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})][1 - \frac{n}{2N}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2})] - (\frac{mn}{N})^{1/2}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^3})\frac{(t-m\mu-n\tilde{\mu})}{N} + - [\tilde{a}_0+\tilde{a}_1 u_\alpha+\tilde{a}_2(u_\alpha^2+2)] + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6),$$

as $|\mu-\tilde{\mu}| = O(N^{-1/2})$ and $|\sigma^2-\tilde{\sigma}^2| = O(N^{-1/2})$. From (2.3.3) it now follows that

$$(2.3.7)\qquad 1-\pi_{t\alpha} = \Phi\{[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})][1-\tfrac{n}{2N}(\tfrac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2})]\} +$$
$$+\ \phi[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})]\Big[-(\tfrac{mn}{N})^{1/2}(\tfrac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^3})\tfrac{(t-m\mu-n\tilde{\mu})}{N} +$$
$$+\ a_0+a_1[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})] + a_2\{[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})]^2+2\} +$$
$$-\ \tilde{a}_0-\tilde{a}_1u_\alpha-\tilde{a}_2(u_\alpha^2+2)\Big] + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6).$$

From (2.2.2) it follows that $|a_0-\tilde{a}_0| = O(N^{-1}+N^{-2}(t-m\mu-n\tilde{\mu})^2)$, $|a_2-\tilde{a}_2| = O(N^{-1})$,

$$a_1-\tilde{a}_1 = \frac{m}{2N}\frac{\tilde{\mu}_3}{\tilde{\sigma}^3}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}}) + O(N^{-1}+N^{-3/2}(t-m\mu-n\tilde{\mu})).$$

Inserting these results and the expressions for a_1 and a_2 from (2.2.2), we get

$$(2.3.8)\qquad \pi_{t\alpha} = g(u_\alpha)+\phi[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})](\tfrac{mn}{N})^{1/2}\ \frac{(t-m\mu-n\tilde{\mu})}{N\tilde{\sigma}}[(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2}) +$$
$$-\ \frac{1}{2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})\ \frac{\tilde{\mu}_3}{\tilde{\sigma}^3}] + O(N^{-1}+N^{-4}(t-m\mu-n\tilde{\mu})^6),$$

where $g(u_\alpha)$ is defined by (2.3.2).

The expansion in (2.3.8) only holds for $|t-m\mu-n\tilde{\mu}| < cN^{2/3}/\log N$, but this suffices to find $\pi_\alpha = E\pi_{T\alpha}$. Let S be a set satisfying

$$S \subset \{t : |t-m\mu-n\tilde{\mu}| < cN^{2/3}/\log N\}$$

and let $I_S(t)$ be the indicator function of S. Then

$$(2.3.9)\qquad \pi_\alpha-g(u_\alpha) = E(\pi_{T\alpha}-g(u_\alpha)) = E[(T-m\mu-n\tilde{\mu})I_S(T)]O(N^{-1}) +$$
$$+\ O(N^{-1}+N^{-4}E[(T-m\mu-n\tilde{\mu})^6 I_S(T)]) + O(E[|\pi_{T\alpha}-g(u_\alpha)|I_{(S^c)}(T)]).$$

As $ET = m\mu+n\tilde{\mu}$, $E[(T-m\mu-n\tilde{\mu})I_S(T)] = -E[(T-m\mu-n\tilde{\mu})I_{(S^c)}(T)]$. Because $Ee^{vX_1} \le B$, $Ee^{vY_1} \le B$ for $|v| \le V$ and $T = \sum_{i=1}^m X_i + \sum_{j=1}^n Y_j$, we have $E|T-m\mu-n\tilde{\mu}|^r = O(N^{r/2})$ for all positive real r, by the Marcinkievitz-Zygmund-

Chung-inequality (cf. Chung (1951)). In particular, $E(T-m\mu-n\tilde{\mu})^6 = O(N^3)$. Furthermore, $\pi_{T\alpha}$ and $g(u_\alpha)$ are both bounded. In view of these facts, (2.3.9) simplifies to

$$(2.3.10) \qquad |\pi_\alpha - g(u_\alpha)| = O(N^{-1} + E\{[|T-m\mu-n\tilde{\mu}|+1]I_{(S^c)}(T)\}).$$

Application of Hölder's inequality shows that for all $r,s > 1$ with $1/r+1/s = 1$

$$(2.3.11) \qquad E\{[|T-m\mu-n\tilde{\mu}|+1]I_{(S^c)}(T)\} \le \{E[|T-m\mu-n\tilde{\mu}|+1]^r\}^{1/r}\{E[I_{(S^c)}(T)]^s\}^{1/s} = O(N^{1/2}[P(T \in S^c)]^{1/s}).$$

From Chebyshev's inequality we have, if we choose $S = \{t: |t-m\mu-n\tilde{\mu}| \le N^{1/2+\tau}\}$, for some $0 < \tau < 1/6$

$$(2.3.12) \qquad P(T \in S^c) = O(E|T-m\mu-n\tilde{\mu}|^p N^{-p(1/2+\tau)}) = O(N^{-p\tau}),$$

for all $p > 0$. Now (2.3.1) follows from (2.3.10), (2.3.11) and (2.3.12). The uniformity in P_θ, $P_{\tilde{\theta}}$ and m follows again from the method of proof. □

For exponential families with monotone likelihood ratio the expression for $g(u_\alpha)$ in (2.3.2) can be simplified by using the result in the next lemma.

LEMMA 2.3.2. *Let* $\mathcal{P} = \{P_\theta : \theta \in \Theta \subset R^1\}$ *be a family of distributions on* R^1 *having densities* $f_\theta(x) = \exp(Q(\theta)x)/c(\theta)$ *with respect to a σ-finite measure* ν. *Let* μ, σ^2 and μ_3 ($\tilde{\mu}$, $\tilde{\sigma}^2$ *and* $\tilde{\mu}_3$) *be the expectation, variance and third central moment of a r.v. with distribution* $P_\theta(P_{\tilde{\theta}})$. *Let* $\theta, \tilde{\theta} \in \Theta$ *be such that* $|Q(\theta)-Q(\tilde{\theta})| \le CN^{-1/2}$ *and* $0 < \tilde{c} \le \int \exp\{(Q(\theta)+\varepsilon)x\}d\nu \le \tilde{C} < \infty$, $0 < \tilde{c} \le \int \exp\{(Q(\theta)-\varepsilon)x\}d\nu \le \tilde{C} < \infty$, *for constants* $\tilde{c}$, C, $\tilde{C}$ *and* $\varepsilon > 0$. *Then we have, uniformly for fixed* $\tilde{c}$, C, $\tilde{C}$ *and* ε

$$(2.3.13) \qquad (\mu-\tilde{\mu})\mu_3 - (\sigma^2-\tilde{\sigma}^2)\sigma^2 = O(N^{-1}).$$

PROOF. Let $\tau = Q(\theta)$ and define $d(\tau) = \int e^{\tau x} d\nu(x)$. Then $\int x^k f_\theta(x) d\nu(x) = d^{(k)}(\tau)/d(\tau)$, $k = 1,2,\ldots$, and therefore the expression on the left side of (2.3.13) may be expressed in terms of the values of the functions $d^{(k)}$ and

$\ddot{d}$ at the points $Q(\theta)$ and $Q(\tilde{\theta})$, for $k = 1,2,3$. Under the conditions above on θ, $d^{(k)}$ and d are uniformly bounded away from zero and infinity at the point $Q(\theta)$. As $|Q(\theta)-Q(\tilde{\theta})| \le CN^{-1/2}$, this will also hold at the point $Q(\tilde{\theta})$ for N sufficiently large. Furthermore,

$$d^{(k)}(Q(\tilde{\theta})) = d^{(k)}(Q(\theta)) + \{Q(\tilde{\theta})-Q(\theta)\}d^{(k+1)}(Q(\theta)) + O(N^{-1}).$$

Inserting this in $(\mu-\tilde{\mu})\mu_3 - (\sigma^2-\tilde{\sigma}^2)\sigma^2$ leads to the desired result. □

If (2.3.13) holds, $g(u_\alpha)$ simplifies to

$$(2.3.14)\qquad g(u_\alpha) = 1-\Phi[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})] - \phi[u_\alpha-(\tfrac{mn}{N})^{1/2}(\tfrac{\mu-\tilde{\mu}}{\tilde{\sigma}})] [\frac{\tilde{\mu}_3}{6\tilde{\sigma}^3}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})(\frac{N-2n}{N})u_\alpha + (\frac{mn}{N})^{1/2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})^2\ \frac{\tilde{\mu}_3}{6\tilde{\sigma}^3}(\frac{n+N}{N})].$$

We now apply the results of this section to the four examples given in section 2.2. Here lemma's 2.2.1 and 2.3.2 clearly apply. Hence π_α is the power of the UMPU test for these cases and under the conditions of lemma 2.3.1 it may be approximated by (2.3.14). Let η_1 and η_2 be positive constants.

EXAMPLE 2.3.1. $P(X_1=1) = 1-P(X_1=0) = p_1$, $P(Y_1=1) = 1-P(Y_1=0) = p_2 = p$, where $\eta_2 \le p_2 < p_1 \le 1-\eta_2$ and $p_1 = p_2+bN^{-1/2}$, $b = O(1)$. Then $\mu = p_1$, $\sigma^2 = p_1(1-p_1)$, $\mu_3 = (1-2p_1)p_1(1-p_1)$ and therefore $\mu-\tilde{\mu} = bN^{-1/2}$, $\sigma^2-\tilde{\sigma}^2 = (1-2p)bN^{-1/2} + O(N^{-1})$, $\mu_3-\tilde{\mu}_3 = O(N^{-1/2})$. Furthermore, $Ee^{vX_1} = (1-p_1)+p_1e^v$ for all v. It follows that we may apply lemma 2.3.1. Hence

$$(2.3.15)\qquad \pi_\alpha = 1-\Phi(u_\alpha-[\frac{mn}{p(1-p)}]^{1/2}\frac{b}{N}) - \phi(u_\alpha-[\frac{mn}{p(1-p)}]^{1/2}\frac{b}{N}) [\frac{(1-2p)(N-2n)}{6p(1-p)N^{3/2}}bu_\alpha + [\frac{mn}{p(1-p)}]^{1/2}\frac{(1-2p)(n+N)}{6p(1-p)N^{5/2}}b^2] + O(N^{-1}).$$

EXAMPLE 2.3.2. $X_1(Y_1)$ has a Poisson distribution with parameter $\lambda_1(\lambda_2)$, where $\eta_2 \le \lambda_2 < \lambda_1 \le \eta_1$, $\lambda_2 = \lambda$, $\lambda_1 = \lambda+bN^{-1/2}$, $b = O(1)$. Then $\mu = \sigma^2 = \mu_3 = \lambda_1$, and therefore $\mu-\tilde{\mu} = \sigma^2-\tilde{\sigma}^2 = \mu_3-\tilde{\mu}_3 = bN^{-1/2}$. Furthermore, $Ee^{vX_1} = e^{-\lambda_1+\lambda_1e^v}$ for all v. Hence

$$(2.3.16)\qquad \pi_\alpha = 1-\Phi(u_\alpha-[\frac{mn}{\lambda}]^{1/2}\frac{b}{N})-\phi(u_\alpha-[\frac{mn}{\lambda}]^{1/2}\frac{b}{N})[\frac{(N-2n)}{6\lambda N^{3/2}}bu_\alpha +$$

$$+[\frac{mn}{\lambda}]^{1/2}\frac{(n+N)}{6\lambda N^{5/2}}b^2] + O(N^{-1}).$$

EXAMPLE 2.3.3. $X_1(Y_1)$ has a geometric distribution with parameter $p_1(p_2)$, with $\eta_2 \le p_1 < p_2 \le 1-\eta_2$, $p_2 = p$, $p_1 = p+bN^{-1/2}$, $b = O(1)$. Then $\mu = \frac{1}{p_1}$, $\sigma^2 = \frac{(1-p_1)}{p_1^2}$, $\mu_3 = \frac{(1-p_1)(2-p_1)}{p_1^3}$ and therefore $\mu-\tilde{\mu} = -\frac{bN^{-1/2}}{p^2} + O(N^{-1})$, $\sigma^2-\tilde{\sigma}^2 = -\frac{(2-p)}{p^3}bN^{-1/2} + O(N^{-1})$, $\mu_3-\tilde{\mu}_3 = O(N^{-1/2})$. Furthermore, $Ee^{vX_1} = p_1e^v/(1-[1-p_1]e^v)$, for $v < \log(\frac{1}{1-p_1})$. Hence

$$(2.3.17)\qquad \pi_\alpha = 1-\Phi(u_\alpha+[\frac{mn}{1-p}]^{1/2}\frac{b}{pN})-\phi(u_\alpha+[\frac{mn}{1-p}]^{1/2}\frac{b}{pN})[\frac{(p-2)(N-2n)}{6p(1-p)N^{3/2}}bu_\alpha +$$

$$+[\frac{mn}{1-p}]^{1/2}\frac{(2-p)(n+N)}{6p^2(1-p)N^{5/2}}b^2] + O(N^{-1}).$$

EXAMPLE 2.3.4. $X_1(Y_1)$ has an exponential distribution with parameter $\lambda_1(\lambda_2)$, with $\eta_2 \le \lambda_1 < \lambda_2 \le \eta_1$, $\lambda_2 = \lambda$, $\lambda_1 = \lambda+bN^{-1/2}$, $b = O(1)$. Then $\mu = \frac{1}{\lambda_1}$, $\sigma^2 = \frac{1}{\lambda_1^2}$, $\mu_3 = \frac{2}{\lambda_1^3}$, and therefore $\mu-\tilde{\mu} = -\frac{b}{\lambda^2}N^{-1/2} + O(N^{-1})$, $\sigma^2-\tilde{\sigma}^2 = -\frac{2b}{\lambda}N^{-1/2} + O(N^{-1})$, $\mu_3-\tilde{\mu}_3 = O(N^{-1/2})$. Furthermore, $Ee^{vX_1} = \lambda/(\lambda-v)$, for $v < \lambda$, and $|\rho_{X_1}(u)| = (1+\frac{u^2}{\lambda^2})^{-1/2}$. Hence

$$(2.3.18)\qquad \pi_\alpha = 1-\Phi(u_\alpha+\frac{(mn)^{1/2}b}{\lambda N})-\phi(u_\alpha+\frac{(mn)^{1/2}b}{\lambda N})[-\frac{(N-2N)}{3\lambda N^{3/2}}bu_\alpha +$$

$$+\frac{(mn)^{1/2}(N+n)}{3\lambda^2N^{5/2}}b^2] + O(N^{-1}).$$

2.4. OPTIMAL RATIO AND DEFICIENCIES

In this section we solve the main problem of this chapter: which choice of $\gamma = m/N$ is optimal as $N \to \infty$? From (2.3.2) it is evident that $\gamma = \frac{1}{2} + O(N^{-1/2})$. In order to find the second order term we set $\gamma = \frac{1}{2} + fN^{-1/2}$, with $f = O(1)$. For a given f, denote the power at level α by $\pi_{\alpha,f}$ and the expression in (2.3.2) by $g(u_\alpha,f)$. Under the conditions of lemma 2.3.1 we have

(2.4.1) $$\pi_{\alpha,f}-\pi_{\alpha,0} = g(u_\alpha,f)-g(u_\alpha,0) + O(N^{-3/2}),$$

as changes in f cause changes of $O(N^{-3/2})$ in $\pi_{\alpha,f}-g(u_\alpha,f)$. From (2.3.2) and (2.4.1) it follows that

(2.4.2) $$\pi_{\alpha,f}-\pi_{\alpha,0} = N^{-1}\phi[u_\alpha - \frac{N^{1/2}(\mu-\tilde{\mu})}{2\tilde{\sigma}}][-f^2 \frac{N^{1/2}(\mu-\tilde{\mu})}{\tilde{\sigma}} + $$
$$+ fu_\alpha\{\frac{\tilde{\mu}_3}{6\tilde{\sigma}^3}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})N^{1/2} - \frac{1}{2}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2})N^{1/2}\} +$$
$$+ f\frac{N^{1/2}(\mu-\tilde{\mu})}{\tilde{\sigma}}\{\frac{N^{1/2}}{4}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2}) - \frac{\tilde{\mu}_3 N^{1/2}}{6\tilde{\sigma}^3}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})\}] + O(N^{-3/2}).$$

Obviously, (2.4.2) reaches its maximum for

(2.4.3) $$f_0 = [u_\alpha\{\frac{\tilde{\mu}_3}{6\tilde{\sigma}^3}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}}) - \frac{1}{2}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2})\} + N^{1/2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})\{\frac{1}{4}(\frac{\sigma^2-\tilde{\sigma}^2}{\tilde{\sigma}^2}) - \frac{\tilde{\mu}_3}{6\tilde{\sigma}^3}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})\}]/$$
$$[2(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})] + O(N^{-1/2}).$$

Consequently,

(2.4.4) $$\pi_{\alpha,f_0}-\pi_{\alpha,0} = N^{-1/2}\phi[u_\alpha - \frac{N^{1/2}(\mu-\tilde{\mu})}{2\tilde{\sigma}}](\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})]f_0^2 + O(N^{-3/2}).$$

Define $b_N^* = \frac{N^{1/2}}{2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})$. Note that b_N^* is bounded. As α is a constant, with $0 < \alpha < 1$, it follows that

(2.4.5)
$$\pi_{\alpha,f_0} = 1-\Phi[u_\alpha-b_N^*+A(u_\alpha,b_N^*)N^{-1/2} - 2b_N^* f_0^2 N^{-1} + O(N^{-3/2})],$$
$$\pi_{\alpha,0} = 1-\Phi[u_\alpha-b_N^*+A(u_\alpha,b_N^*)N^{-1/2} + O(N^{-3/2})],$$

where $|A(u_\alpha,b_N^*)| \le C$ for some $C > 0$. Application of (1.2) to (2.4.5) shows that the deficiency d_N of the choice $m = n$ with respect to the optimal choice $m/N = \frac{1}{2} + f_0 N^{-1/2}$, satisfies

(2.4.6) $$d_N = (2f_0)^2 + O(N^{-1/2}).$$

Hence the asymptotic deficiency d equals $(2f_0)^2$. This is finite, which may seem surprising in view of the fact that the difference between the optimal choice γ_0 and $\frac{1}{2}$ is $O(N^{-1/2})$ rather than $O(N^{-1})$. This suggests the asymptotic deficiency to be infinite. That this is not the case is explained as follows: the leading term in $g(u_\alpha, f)$ is symmetric around $\gamma = \frac{1}{2}$, in fact $\pi_\alpha = 1-\Phi(u_\alpha-[N\gamma(1-\gamma)]^{1/2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})) + O(N^{-1/2})$, and $\gamma = \frac{1}{2} + O(N^{-1/2})$ implies $\gamma(1-\gamma) = \frac{1}{4} + O(N^{-1})$.

We finally apply these results to our examples. In the first place, (2.3.13) holds in all examples considered, and hence (2.4.3) simplifies to

$$(2.4.7) \qquad f_0 = -\frac{\tilde{\mu}_3}{24\tilde{\sigma}^3}[4u_\alpha - N^{1/2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})] + O(N^{-1/2}).$$

The expression in (2.4.7) becomes more transparant if we eliminate $N^{1/2}(\mu-\tilde{\mu})/\tilde{\sigma}$ by using its relation to the power of the test. Denote $1-\pi_\alpha$, the error of the second kind, as β. From $\beta = \Phi(u_\alpha - \frac{N^{1/2}}{2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}})) + O(N^{-1/2})$ it then follows that

$$\frac{N^{1/2}}{2}(\frac{\mu-\tilde{\mu}}{\tilde{\sigma}}) = (u_\alpha+u_\beta) + O(N^{-1/2}).$$

Hence

$$(2.4.8) \qquad f_0 = -\frac{\tilde{\mu}_3}{12\tilde{\sigma}^3}(u_\alpha-u_\beta) + O(N^{-1/2}),$$

$$(2.4.9) \qquad d_N = \frac{\tilde{\mu}_3^2}{36\tilde{\sigma}^6}(u_\alpha-u_\beta)^2 + O(N^{-1/2}).$$

In most applications $|u_\alpha-u_\beta| < 1$, which makes $(\frac{\tilde{\mu}_3}{6\tilde{\sigma}^3})^2$ a reasonable upper bound for d. In the special case where the errors of the first and second kind are equal, $f_0 = O(N^{-1/2})$ and the choice of m = n is therefore optimal to $O(N^{-1})$. We conclude this section by making some remarks on each of the examples of section 2.3 separately.

EXAMPLE 2.4.1. For example 2.3.1, (2.4.8) specializes to

$$f_0 = \frac{(2p-1)}{12[p(1-p)]^{1/2}}(u_\alpha-u_\beta).$$

Remembering that $m/N = \gamma_0 = \frac{1}{2} + f_0 N^{-1/2}$, we may conclude that, if the error of the first kind has to be smaller (larger) than the error of the second kind, one should perform more (less) often the experiment whose probability of success differs most from $\frac{1}{2}$. Furthermore, $(2p-1)^2/(36p(1-p)) =$

$= (2p-1)^2/[9(1-(2p-1)^2)]$. Hence, if $|u_\alpha - u_\beta| < 1$, then $d \le D$ for $|p-1/2| \le \frac{1}{2}[9D/(9D+1)]^{1/2} \approx \frac{1}{2} - \frac{1}{36D}$. This gives for example $d \le 1$ for $0.03 \le p \le 0.97$ and $d \le 3$ for $0.01 \le p \le 0.99$.

EXAMPLE 2.4.2. For example 2.3.2 we find $f_0 = -(u_\alpha - u_\beta)/(12\sqrt{\lambda})$. If one wants to have $\alpha < \beta$, then one should perform the experiment with the smallest parameter more often. Furthermore, $d \le D$ if $\lambda \le \frac{D}{36}$ and $|u_\alpha - u_\beta| < 1$.

EXAMPLE 2.4.3. For example 2.3.3 $f_0 = -(2-p)(u_\alpha - u_\beta)/[12(1-p)^{1/2}]$.

EXAMPLE 2.4.4. For example 2.3.4 $f_0 = -(u_\alpha - u_\beta)/6$.

2.5. COMPARISON OF SAMPLING RULES FOR BERNOULLI EXPERIMENTS

In the previous section the results of section 2.3 were used to solve the main problem of this chapter. Here we briefly discuss another application of these results.

Consider two Bernoulli experiments, with probability of success p_1 and p_2, respectively. We shall give an asymptotic comparison of the performance of two sampling rules in testing the hypothesis $p_1 = p_2$ against the alternative $p_2 = p$, $p_1 = p + \Delta$, $\Delta > 0$. The first of these sampling rules is the "Vector-at-a-Time" (VT) rule, which simply states that m and n, the numbers of trials with both experiments, are equal. (cf. Sobel and Weiss (1970)). From example 2.3.1 it follows that for this rule the power π_α of the UMPU test satisfies

$$(2.5.1) \qquad \pi_\alpha = 1-\Phi\{u_\alpha - \Delta\left(\frac{n}{2p(1-p)}\right)^{1/2}\left(1 - \frac{(1-2p)\Delta}{4p(1-p)}\right)\} + O(n^{-1}),$$

for $\Delta = O(n^{-1/2})$ and $\varepsilon \le p_2 < p_1 \le 1-\varepsilon$ for some constant $\varepsilon > 0$.

The second sampling rule we consider is the "Play-the-Winner" (PW) rule, which prescribes that one continues with the same experiment after each success and that one switches to the opposite experiment after each failure. As soon as r failures have occurred with both experiments, sampling is terminated. In this situation there also exists an UMPU test for $H_0 : p_1 = p_2$ against $H_1 : p_2 = p$, $p_1 = p + \Delta$, $\Delta > 0$. An approximation to the power π^*_α of this test is supplied by example 2.3.3, if we interchange p and 1-p in (2.3.17). We find for $\Delta = O(r^{-1/2})$ and $\varepsilon \le p_2 < p_1 \le 1-\varepsilon$

$$(2.5.2) \qquad \pi^*_\alpha = 1-\Phi\{u_\alpha - \frac{\Delta}{(1-p)}(\frac{r}{2p})^{1/2}(1 - \frac{(1+p)\Delta}{4p(1-p)}\} + O(r^{-1}).$$

In order to make the PW rule and the VT rule comparable, it seems reasonable to choose r, for each n, in such a way that the powers of the two tests under consideration are equal. From (2.5.1) and (2.5.2) it follows that $|\pi_\alpha - \pi^*_\alpha| = O(n^{-1})$ for

$$(2.5.3) \qquad r = n(1-p)(1 + \frac{3}{2(1-p)}\Delta) + O(1).$$

Now there are various criteria according to which we can compare the two sampling rules. For example, we may prefer the rule that has the lowest expected number of trials on the poorer experiment, i.e. the experiment with the smallest probability of success. Another criterion is the expected number of failures that could have been avoided by using the better experiment throughout. Finally, a third criterion is the expected total number of trials.

We consider the first criterion. For the VT rule the expected number of trials on the poorer experiment obviously equals n. From (2.5.3) we obtain that for the corresponding PW rule this expectation is

$$n(1 + \frac{3}{2(1-p)}\Delta) + O(1).$$

Hence the PW rule requires in expectation $3n\Delta/\{2(1-p)\}$ additional observations on the poorer experiment.

As concerns the other criteria mentioned above, we note that the second criterion is equivalent to the first, whereas the third criterion will certainly not prefer the PW rule if the first criterion prefers the VT rule. Hence, according to all criteria, the PW rule is asymptotically worse than the VT rule for the problem of this section, uniformly in p. Apparently the fact that the PW rule has a tendency to use the better experiment more often, is outweighed by the fact that the negative binomial distribution has a larger skewness than the binomial distribution.

CHAPTER 3

ASYMPTOTIC EXPANSIONS FOR NONPARAMETRIC TESTS FOR THE ONE SAMPLE PROBLEM

3.1. INTRODUCTION

In this chapter we shall give asymptotic expansions for the distribution functions of one sample linear rank statistics and also for the power functions of the corresponding tests. These have been derived by Albers, Bickel and van Zwet (1974); the present chapter contains the results of this paper and two extensions. We only sketch the proofs.

Our starting point in establishing the above expansions will be the so called Edgeworth expansions (cf. Cramér (1946), Feller (1966)). For the distribution function (d.f.) $R(x)$ of any r.v. X with mean 0 and variance 1 we can give a formal Edgeworth expansion $\tilde{R}(x)$ in powers of $N^{-\frac{1}{2}}$. For example, to $O(N^{-1})$ this looks like

$$(3.1.1) \qquad \tilde{R}(x) = \Phi(x) - \phi(x)\left[\frac{N^{-\frac{1}{2}}\kappa_3}{6}(x^2-1) + \frac{N^{-1}\kappa_4}{24}(x^3-3x) + \right. \left. + \frac{N^{-1}\kappa_3^2}{72}(x^5-10x^3+15x)\right],$$

where κ_3 and κ_4 are the third and fourth cumulant of X, multiplied by $N^{\frac{1}{2}}$ and N, respectively, and Φ and ϕ denote the d.f. and the density of the standard normal distribution.

Such expansions have been used for rank tests before, for example by Hodges and Fix (1955), Fellingham and Stoker (1964), Sundrum (1954), Witting (1960) and Rogers (1971). These authors, however, restrict attention to the special case of the one or two sample Wilcoxon test. Furthermore, with the exception of Rogers, they do not bother to show that (3.1.1) is a valid expansion, but merely recommend it as an approximation on purely numerical grounds. Rogers gives an Edgeworth expansion $\tilde{R}(x)$ to $O(N^{-1})$ for the two sample Wilcoxon distribution $R(x)$ under the hypothesis, and proves that $\sup_x |R(x) - \tilde{R}(x)| = o(N^{-1})$.

Here we shall justify expansions to $O(N^{-1})$, not only under the hypothesis, but also under contiguous alternatives, for quite general test scores. In section 3.2. we give a basic expansion for the d.f. of a linear rank statistic, without any assumptions at all about the alternative. In the next section we restrict attention to contiguous alternatives and obtain a more explicit form for this expansion. This expression still involves sums of functions of moments of order statistics. These are replaced by appropriate integrals in section 3.4 under the assumption of smooth scores and contiguous location alternatives. Finally, in section 3.5, we consider the sign test as a separate case, since it cannot be handled by the general methods of this chapter, because of its pronounced lattice character.

3.2. THE BASIC EXPANSION

Let $X_1,\dots,X_N$ be i.i.d. r.v.'s with common d.f. G and density g, and let $0 < Z_1 < Z_2 < \dots < Z_N$ denote the order statistics of the absolute values of $X_1,\dots,X_N$. If $|X_{R_j}| = Z_j$, define

$$(3.2.1) \qquad V_j = \begin{cases} 1 & \text{if } X_{R_j} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We introduce a vector of scores $a = (a_1,\dots,a_N)$ and define the statistic

$$(3.2.2) \qquad T = \sum a_j V_j .$$

Throughout this section $\sum$ always means $\sum_{j=1}^{N}$, unless stated otherwise. We shall be concerned with obtaining an asymptotic expansion for the distribution of T as $N \to \infty$.

Our notation strongly suggests that we are considering a fixed underlying d.f. G and perhaps also a fixed infinite sequence of scores as $N \to \infty$. However, this is merely a matter of notational convenience and our main concern will in fact be the case where the d.f. depends on N and the scores form a triangular array $a_{j:N}$, $j = 1,\dots,N$, $N = 1,2,\dots$.

The r.v. T is of course the general linear rank statistic for testing the hypothesis that g is symmetric about zero. Under this hypothesis, $V_1,\dots,V_N$ are i.i.d. with $P(V_j=1) = 1/2$. For general G, $V_1,\dots,V_N$ are not independent. However, one easily verifies that conditionally on $Z = (Z_1,\dots,Z_N)$, the r.v.'s $V_1,\dots,V_N$ are independent with

$$(3.2.3) \qquad P_j = P(V_j=1|Z) = \frac{g(Z_j)}{g(Z_j)+g(-Z_j)} .$$

As independence allows us to obtain expansions of Edgeworth type, we shall carry out the following program to arrive at an expansion for the distribution of T. First we obtain an Edgeworth expansion for the distribution of $\sum a_j W_j$, where $W_1,\dots,W_N$ are independent with $p_j = P(W_j=1) = 1 - P(W_j=0)$. Having done this, we substitute the random vector $P = (P_1,\dots,P_N)$ defined in (3.2.3) for $p = (p_1,\dots,p_N)$ in this expansion. The expected value of the resulting expression will then give us an expansion for the distribution of T.

In carrying out the first part of this program, we shall indicate any dependence on $p = (p_1,\dots,p_N)$ in our notation. Consider the r.v.

$$(3.2.4)\qquad \frac{\sum a_j(W_j-p_j)}{\tau(p)},$$

where

$$(3.2.5)\qquad \tau^2(p) = \sum p_j(1-p_j)a_j^2$$

denotes the variance of $\sum a_j W_j$. Obviously (3.2.4) has expectation 0 and variance 1. Let $R(x,p)$ and $\rho(t,p)$ denote the d.f. and the characteristic function (c.f.) of (3.2.4) respectively. Denote the Edgeworth expansion to $O(N^{-1})$ for $R(x,p)$ as $\tilde{R}(x,p)$ (cf. (3.1.1)). Let $\tilde{r}(x,p)$ be $\frac{\partial}{\partial x}\tilde{R}(x,p)$ and $\tilde{\rho}(t,p) = \int_{-\infty}^{\infty} \exp(itx)\, \tilde{r}(x,p)dx$, the Fourier transform of $\tilde{r}$.

To justify a formal Edgeworth expansion $\tilde{R}$, i.e. to show that $|R - \tilde{R}|$ is indeed $o(N^{-1})$, one usually invokes the following result (Feller (1966)).

LEMMA 3.2.1. *Let* R *be a d.f. with vanishing expectation and c.f.* ρ. *Suppose that* $R - \tilde{R}$ *vanishes at* $\pm\infty$ *and that* $\tilde{R}$ *has a derivative* $\tilde{r}$ *such that* $|\tilde{r}| \le m$. *Finally, suppose that* $\tilde{r}$ *has a continuously differentiable Fourier transform* $\tilde{\rho}$ *such that* $\tilde{\rho}(0) = 1$ *and* $\tilde{\rho}'(0) = 0$. *Then, for all* x *and* $T > 0$,

$$(3.2.6)\qquad |R(x) - \tilde{R}(x)| \le \frac{1}{\pi}\int_{-T}^{T} \left|\frac{\rho(t)-\tilde{\rho}(t)}{t}\right| dt + \frac{24m}{\pi T}.$$

PROOF. See Feller (1966).

To prove that $|R - \tilde{R}| = o(N^{-1})$, it suffices to show that e.g. for $T = b \,.\, N^{3/2}$, the integral in (3.2.6) is $o(N^{-1})$. For the case we are considering, this may be done largely in the standard manner (Feller (1966)), by splitting the integral into several parts. Require that for some positive constants C and c, $\sum a_j^4 \le CN$ and $\tau^2(p) = \sum p_j(1-p_j)a_j^2 \ge cN$. As this implies that $(\max_j a_j)/\tau(p) \le (Cc^{-2})^{1/4}N^{-1/4}$, we can expand

$$\rho(t,p) = \exp\left[\sum \log \rho_{W_j-p_j}\ (a_j t/\tau(p))\right]$$

around $t = 0$ for $|t| \le c_1 N^{1/4}$, for some positive c_1, depending on C and c. Comparison of this expansion with $\tilde{\rho}(t,p)$ shows that $|\rho(t,p) - \tilde{\rho}(t,p)|$ is sufficiently small on this interval to ensure that

$$\int_{|t|\le c_1 N^{1/4}} \left|\frac{\rho(t,p)-\tilde{\rho}(t,p)}{t}\right| dt = O(N^{-5/4}).$$

On $c_1 N^{1/4} \le |t| \le b.N^{3/2}$ we cannot expand ρ anymore, but as ρ and $\tilde{\rho}$ are both small here, we simply use

$$\int\left|\frac{\rho(t,p)-\tilde{\rho}(t,p)}{t}\right| dt \le \int\left|\frac{\rho(t,p)}{t}\right| dt + \int\left|\frac{\tilde{\rho}(t,p)}{t}\right| dt.$$

The last integral on the right is shown to be sufficiently small for $|t| \ge \log(N+1)$ without any difficulties. As it can be shown that $|\rho(t,p)| \le \exp\,[-\frac{1}{2}t^2+Ct^4/(96c^2N)]$, we also have that the following integral is sufficiently small

$$\int_{c_1 N^{1/4}\le|t|\le b_1 N^{1/2}} \left|\frac{\rho(t,p)}{t}\right| dt,$$

for some positive constant b_1. Hence it remains to estimate

$$\int_{b_1 N^{1/2}\le|t|\le bN^{3/2}} \left|\frac{\rho(t,p)}{t}\right| dt. \tag{3.2.7}$$

Here one usually makes what Feller calls the extravagantly luxurious assumption that the c.f.'s of all summands are uniformly bounded away from 1 in absolute value outside every neighbourhood of 0. Obviously, this condition is not satisfied in our case where the summands $a_j W_j$ are lattice r.v.'s. Weaker sufficient conditions of this type are known, but all seem to imply at the very least that the sum itself is non-lattice. In our case this would exclude for instance both the sign test and the Wilcoxon test. On the other hand, it is clear that one has to exclude cases where the sum (3.2.4) can only assume relatively few different values. As $\tilde{R}$ is continuous, one cannot allow R to have jumps of $O(N^{-1})$ or larger. Thus the sign test, where jumps of order $N^{-1/2}$ occur, will certainly have to be excluded.

However, it is exactly the simple lattice character of this statistic that makes it easily amenable to other methods of expansion (see section 3.5). For the Wilcoxon statistic on the other hand, all jumps are $O(N^{-3/2})$ and the assumptions we shall make will not rule out this case.

For $\rho(t,p)$ we have

$$(3.2.8) \qquad |\rho(t,p)| = \prod_{j=1}^{N} \left\{1-2p_j(1-p_j)(1-\cos \frac{a_j t}{\tau(p)})\right\}^{1/2}.$$

This is exponentially small if for a positive fraction of indices j the following two conditions are simultaneously satisfied : $\varepsilon \le p_j \le 1-\varepsilon$ and $a_j t/\tau(p)$ differs at least η from the nearest multiple of 2π, for some $0 < \varepsilon < 1$, $\eta > 0$. For our purpose this must hold for all t with $b_1 N^{1/2} \le |t| \le b.N^{3/2}$. If this is the case, then obviously (3.2.7) is sufficiently small, and the Edgeworth expansion is justified.

We summarize this result in the following theorem, where the two conditions above are replaced by one weaker, but less intuitive condition.

THEOREM 3.2.1. *Suppose that positive numbers* c, C, δ *and* ε *exist such that* $\sum a_j^4 \le CN$, $\sum a_j^2 p_j(1-p_j) \ge cN$ *and* $\gamma(\varepsilon,\zeta,p) = \lambda\{x \mid \exists_j \ |x-a_j| < \zeta,$ $\varepsilon \le p_j \le 1-\varepsilon\} \ge \delta N\zeta$ *for some* $\zeta \ge N^{-3/2}\log N$, *where* λ *is Lebesgue measure. Then*

$$(3.2.9) \qquad \sup_x |R(x,p) - \tilde{R}(x,p)| \le A.N^{-5/4},$$

where A *depends on* N, a *and* p *only through* c, C, δ *and* ε.

PROOF. For a formal and detailed proof see Albers, Bickel and van Zwet (1974). □

REMARK. If we require $\sum |a_j|^5 \le CN$ instead of $\sum a_j^4 \le CN$, we get $A.N^{-3/2}$ instead of $A.N^{-5/4}$ in (3.2.9). This is the "natural" order of the remainder.

Before we replace p by the random vector $P = (P_1,\dots,P_N)$ defined in (3.2.3) and compute the unconditional distribution of T by taking the expected value, two modifications must be performed.

In the first place, we have to change the standardization of $\sum a_j W_j$ into one that does not involve p. As before, let $W_1,\dots,W_N$ be i.i.d. with $P(W_j=1) = 1 - P(W_j=0) = p_j$, let $\tilde{p} = (\tilde{p}_1,\dots,\tilde{p}_N)$ be a vector with $0 \le \tilde{p}_j \le 1$ for all j, and consider

$$(3.2.10)\qquad R^*(x,p,\tilde{p}) = P\left(\frac{\sum a_j(W_j-\tilde{p}_j)}{\tau(\tilde{p})} \le x\right).$$

Here $\tau^2(\tilde{p}) = \sum \tilde{p}_j(1-\tilde{p}_j)a_j^2$ in accordance with (3.2.5). From the fact that

$$R^*(x,p,\tilde{p}) = R\left(\left[x - \frac{\sum(p_j-\tilde{p}_j)a_j}{\tau(\tilde{p})}\right]\frac{\tau(\tilde{p})}{\tau(p)},\ p\right),$$

we can immediately derive an expansion for R^* by means of (3.2.9).

The second modification is that everywhere in this expansion we expand $\tau(\tilde{p})/\tau(p)$ in powers of $[\tau^2(p) - \tau^2(\tilde{p})]/\tau^2(\tilde{p})$; the reasons for this will become clear in the sequel. The result of these steps is the following lemma.

LEMMA 3.2.2. *If the conditions of theorem* 3.2.1 *are satisfied and in addition* $\sum \tilde{p}_j(1-\tilde{p}_j)a_j^2 \ge cN$, *we have*

$$(3.2.11)\qquad \sup_x |R^*(x,p,\tilde{p}) - \tilde{R}^*(x,p,\tilde{p})| \le$$

$$\le A\{N^{-5/4} + N^{-3/2}\sum(p_j-\tilde{p}_j)^2|a_j|^3 + N^{-3}|\tau^2(p) - \tau^2(\tilde{p})|^3\},$$

where $A > 0$ *depends on* N, a, p, $\tilde{p}$ *only through* c, C, δ *and* ε *and where*

$$(3.2.12)\qquad \tilde{R}^*(x,p,\tilde{p}) = \tilde{R}(x-u,\tilde{p}) - \phi(x-u)\ \Big\{\frac{1}{2}\,\frac{\tau^2(p)-\tau^2(\tilde{p})}{\tau^2(\tilde{p})}\,(x-u) +$$

$$+ \frac{1}{6}\,\frac{\sum(p_j-\tilde{p}_j)(1-6\tilde{p}_j+6\tilde{p}_j^2)a_j^3}{\tau^3(\tilde{p})}\,[(x-u)^2-1] +$$

$$+ \frac{1}{8}\left(\frac{\tau^2(p)-\tau^2(\tilde{p})}{\tau^2(\tilde{p})}\right)^2[(x-u)^3-3(x-u)] +$$

$$+ \frac{\kappa_3(\tilde{p})}{12N^{\frac{1}{2}}}\,\frac{\tau^2(p)-\tau^2(\tilde{p})}{\tau^2(\tilde{p})}\,[(x-u)^4-6(x-u)^2+3]\Big\},$$

with $u = [\sum(p_j-\tilde{p}_j)a_j]/\tau(\tilde{p})$.

PROOF. See Albers, Bickel and van Zwet (1974).

We shall now replace p by $P = (P_1,\dots,P_N)$ in $\tilde{R}^*(x,p,\tilde{p})$ and take expectations. Define the vector $\pi = (\pi_1,\dots,\pi_N)$ by

$$(3.2.13) \qquad \pi_j = EP_j, \qquad\qquad j = 1,\dots,N.$$

It will play the role of $\tilde{p}$. Then the following theorem can be formulated.

THEOREM 3.2.2. *Let* $X_1,\dots,X_N$ *be i.i.d. with common d.f.* G *and density* g, *and let* T, P *and* π *be defined by* (3.2.2), (3.2.3) *and* (3.2.13). *Suppose that positive numbers* c, C, δ, δ' *and* ε *exist with* $\delta' < \min(\delta/2, c^2C^{-1})$ *and such that*

$$(3.2.14) \qquad \sum a_j^2 \geq cN,\ \sum a_j^4 \leq CN,$$

$$(3.2.15) \qquad \gamma(\zeta) = \lambda\{x \mid \exists_j |x-a_j| < \zeta\} \geq \delta N\zeta \text{ for some } \zeta \geq N^{-3/2}\log N,$$

$$(3.2.16) \qquad P\{\varepsilon \leq \frac{g(X_1)}{g(X_1)+g(-X_1)} \leq 1-\varepsilon\} \geq 1-\delta'.$$

Then there exists $A > 0$ *depending on* N, a *and* G *only through* c, C, δ, δ' *and* ε *and such that*

$$(3.2.17) \qquad \sup_x |P(\frac{T-\sum a_j\pi_j}{\tau(\pi)}) - E\tilde{R}^*(x,P,\pi)| \leq$$

$$\leq A\Big\{N^{-5/4}+N^{-3/4}[\sum\{E(P_j-\pi_j)^2\}^{5/2}]^{2/5}+N^{-3/2}[\sum\{E|P_j-\pi_j|^3\}^{2/3}]^{3/2}\Big\}.$$

PROOF. Here we only sketch the proof. For a complete proof see Albers, Bickel and van Zwet (1974). Using (3.2.16) one can show that for $\delta'' \in (\delta', \min(\delta/2, c^2C^{-1}))$

$$P(E) \leq e^{-2N(\delta''-\delta')^2},$$

where $E = \{P \mid \varepsilon \leq P_j \leq 1-\varepsilon$ for less than $(1-\delta'')N$ indices $j\}$. On E^c, $\sum a_j^2 \geq cN$ implies both $\sum a_j^2 P_j(1-P_j) \geq c^*N$ and $\sum a_j^2\pi_j(1-\pi_j) \geq c^*N$, for some c^* depending only on c, C, δ'', δ' and ε and satisfying $0 < c^* < c$. Also $\gamma(\zeta) \geq \delta N\zeta$ implies $\gamma(\varepsilon,\zeta,P) \geq (\delta-2\delta'')N\zeta$ on E^c, where $\delta-2\delta''$ is positive by assumption. Hence, under the conditions of this theorem, a, P and π satisfy on E^c the conditions for a, p and $\tilde{p}$ in lemma 3.2.2, if c, C, δ and ε are replaced there by positive numbers c^*, C, $\delta-2\delta''$ and ε, depending only on c, C, δ, δ' and ε. In dealing with the set E it will suffice to note that $\tilde{R}^*(x,P,\pi)$ is bounded. Of course, $R^*(x,P,\pi)$, being a probability, is also bounded.

As $P([T-\sum a_j\pi_j]/\tau(\pi) \le x) = ER^*(x,P,\pi)$, the left side of (3.2.17) is bounded above by

$$E \sup_x |R^*(x,P,\pi) - \tilde{R}^*(x,P,\pi)|. \tag{3.2.18}$$

Applying lemma 3.2.2 on E^c and using the boundedness of $|R^*(x,P,\pi) - \tilde{R}^*(x,P,\pi)|$ together with $P(E) = O(N^{-5/4})$ we find that (3.2.18) is

$$O(N^{-5/4} + N^{-3/2}\sum E(P_j-\pi_j)^2|a_j|^3 + N^{-3}E|\tau^2(P) - \tau^2(\pi)|^3),$$

where the order symbol is uniform for fixed c, C, δ, δ' and ε. This expression can be shown to be of the order of the right side in (3.2.17). □

We note that the boundedness of $\tilde{R}^*(x,P,\pi)$ on E plays an important role in the above proof. Because $\tau(P)$ may be arbitrarily small on E, this explains why we had to remove $\tau(p)$ from the denominator of the expansion in lemma 3.2.2 by expanding $\tau(\tilde{p})/\tau(p)$ in powers of $[\tau^2(p)-\tau^2(\tilde{p})]/\tau^2(\tilde{p})$.

Although theorem 3.2.2 is formally stated as a result for a fixed, but arbitrary value of N, it is of course meaningless for fixed N because we do not investigate the way in which A depends on c, C, δ, δ' and ε. In fact the theorem is a purely asymptotic result. Let us for a moment indicate dependence on N by a superscript. Thus for N = 1,2,..., consider the distribution of the statistic $T^{(N)}$ based on a vector of scores $a^{(N)} = (a_1^{(N)},\dots,a_N^{(N)})$ when the underlying d.f. is $G^{(N)}$. Fix positive values of c, C, δ, δ' and ε with $\delta' < \min(\delta/2, c^2C^{-1})$. The theorem asserts that if for each N, $a^{(N)}$ and $G^{(N)}$ satisfy (3.2.14) - (3.2.16) for these fixed c, C, δ, δ' and ε, then the error of the approximation $E\tilde{R}^*(x,P^{(N)},\pi^{(N)})$ is

$$O(N^{-5/4} + N^{-3/4}[\sum\{E(P_j^{(N)}-\pi_j^{(N)})^2\}^{5/2}]^{2/5} + N^{-3/2}[\sum\{E|P_j^{(N)}-\pi_j^{(N)}|^3\}^{2/3}]^{3/2}),$$

as $N \to \infty$. Moreover, the order of the remainder is uniform for all such sequences $a^{(N)}$, $G^{(N)}$, $N = 1,2,\ldots$.

Assumption (3.2.15) may need some clarification. It is clear from the sketch of the proof of theorem 3.2.1 that the role of the $\gamma(\varepsilon,\zeta,p)$ and $\gamma(\zeta)$ conditions in theorem 3.2.1 and theorem 3.2.2 respectively, is to ensure that the a_j do not cluster too much around too few points. Assumption (3.2.15) is certainly satisfied if for some $k \geq \delta N/2$, indices $j_1, j_2, \ldots, j_k$ exist such that $a_{j_{i+1}} - a_{j_i} \geq 2N^{-3/2} \log N$ for $i = 1,\ldots,k$. Under condition (3.2.14) this will typically be the case. Consider for instance the important case where $a_j = EJ(U_{j:N})$, where $U_{1:N} < U_{2:N} < \ldots < U_{N:N}$ are order statistics from the uniform distribution on (0,1) and J is a continuously differentiable, nonconstant function on (0,1) with $\int J^4 < \infty$. Here both (3.2.14) and (3.2.15) are satisfied for all N with fixed c, C and δ. The same is true if $a_j = J(j/(N+1))$, provided that J is monotone near 0 and 1.

For a large class of underlying d.f.'s G, the right side of (3.2.17) is uniformly $o(N^{-1})$. Still theorem 3.2.2 does not yet provide an explicit expansion to order N^{-1} for the distribution of T since we are still left with the task of computing the expected value of $\tilde{R}^*(x,P,\pi)$. This is of course a trivial matter under the hypothesis that g is symmetric about zero and, more generally, in the case where, for some $\eta > 0$, $g(x)/g(-x) = \eta$ for all $x > 0$. In this case $P_j = \eta(1+\eta)^{-1}$ with probability 1 for all j and an expansion for the distribution of T is already contained in theorem 3.2.1. For fixed alternatives in general, however, the computation of $E\tilde{R}^*(x,P,\pi)$ presents a formidable problem that we shall not attempt to solve here. It would seem that what is needed, is an expansion for the distribution of a linear combination of functions of order statistics.

In the remaining part of this chapter we shall restrict attention to sequences of alternatives that are contiguous to the hypothesis. Heuristically the situation is now as follows. Since $g(x)/(g(x)+g(-x)) = \frac{1}{2} + O(N^{-1/2})$, $P_j - \frac{1}{2}$ and $\pi_j - \frac{1}{2}$ will be $O(N^{-1/2})$, whereas $P_j - \pi_j$ will be $O(N^{-1})$ instead of $O(N^{-1/2})$ as before. In the first place this allows us to simplify $E\tilde{R}^*(x,P,\pi)$ considerably as a number of terms may now be relegated to the remainder and functions of π_j may be expanded about the point $\pi_j = \frac{1}{2}$. Much more important, however, is the fact that $U^* = \tau^{-1}(\pi) \sum (P_j - \pi_j) a_j$ will now be $O(N^{-1/2})$ and that we may therefore expand $\tilde{R}^*(x,P,\pi)$ in powers of U^*. This means that we shall be dealing with low moments of linear combinations of functions of order statistics rather than with their distributions. We need hardly point out that a heuristic argument like this can be entirely misleading and that the actual order of the remainder in our expansion will of course have to be investigated.

Define

$$(3.2.19)\qquad \tilde{K}(x) = \Phi(x) + \phi(x)\ \{\frac{\sum a_j^2\ E(2P_j-1)^2 - 4\sigma^2(\sum a_j P_j)}{2\sum a_j^2}\ x +$$

$$+ \frac{\sum a_j^3(2\pi_j-1)}{3(\sum a_j^2)^{3/2}}\ (x^2-1) + \frac{\sum a_j^4}{12(\sum a_j^2)^2}\ (x^2-3x)\},$$

where $\sigma^2(Z)$ denotes the variance of a r.v. Z. Carrying out the type of computation outlined above we arrive at the following simplified version of theorem 3.2.2.

THEOREM 3.2.3. *Theorem* 3.2.2 *continues to hold if* (3.2.17) *is replaced by*

$$(3.2.20)\qquad \sup_x \left| P(\frac{2T-\sum a_j}{(\sum a_j^2)^{1/2}} \le x) - \tilde{K}(x - \frac{\sum a_j(2\pi_j-1)}{(\sum a_j^2)^{1/2}}) \right| \le$$

$$\le A\left\{N^{-5/4} + \sum\{E(2P_j-1)^4\}^{5/4} + N^{-3/4}[\sum\{E|P_j-\pi_j|^3\}^{4/9}]^{9/4}\right\}.$$

PROOF. See Albers, Bickel and van Zwet (1974).

Theorem 3.2.3 provides the basic expansion for the distribution of T under contiguous alternatives. In section 3.3 we shall be concerned with a further simplification of this expansion and a precise evaluation of the order of the remainder term.

3.3. CONTIGUOUS ALTERNATIVES

We first consider the case of contiguous location alternatives. Let F be a d.f. with a density f that is positive on R^1, symmetric about zero and four times differentiable with derivatives $f^{(i)}$, i = 1,...,4. Define functions

$$(3.3.1)\qquad \psi_i = \frac{f^{(i)}}{f}\ ,\ i = 1,\dots,4,$$

and suppose that positive numbers ε and C exist such that for

$$(3.3.2)\qquad \begin{aligned} &m_1 = 6,\ m_2 = 3,\ m_3 = \tfrac{4}{3},\ m_4 = 1, \\ &\sup\left\{\int_{-\infty}^{\infty} |\psi_i(x+y)|^{m_i} f(x)dx \; : \; |y| \le \varepsilon\right\} \le C,\ i = 1,\dots,4. \end{aligned}$$

Let $X_1, X_2, \dots, X_N$ be i.i.d. with common d.f. $G(x) = F(x-\theta)$ where

$$(3.3.3)\qquad 0 \le \theta \le CN^{-1/2}$$

for some positive C. Note that (3.3.2) and (3.3.3) together imply contiguity. Let $0 < Z_1 < \dots < Z_N$ again denote the order statistics of $|X_1|, \dots, |X_N|$, and let T be defined by (3.2.2). Probabilities, expected values and variances under G will be denoted by P_θ, E_θ and σ_θ^2; under F they will be indicated as P_0, E_0 and σ_0^2. Define

$$(3.3.4)\qquad \begin{aligned} K_\theta(x) = \Phi(x) + \phi(x)\ \Big\{&\frac{\sum a_j^4}{12(\sum a_j^2)^2}(x^3-3x) - \theta\frac{\sum a_j^3 E_0\psi_1(Z_j)}{3(\sum a_j^2)^{3/2}}(x^2-1) + \\ &+ \frac{\theta^2}{2\sum a_j^2}[\sum a_j^2 E_0\psi_1^2(Z_j) - \sigma_0^2(\sum a_j\psi_1(Z_j))]x + \\ &+ \frac{\theta^3}{6(\sum a_j^2)^{1/2}}\sum a_j E_0[3\psi_1^3(Z_j) - 6\psi_1(Z_j)\psi_2(Z_j) + \psi_3(Z_j)]\Big\}, \end{aligned}$$

$$(3.3.5)\qquad \eta = -\theta\frac{\sum a_j E_0\psi_1(Z_j)}{(\sum a_j^2)^{1/2}}.$$

We shall show that $K_\theta(x-\eta)$ is an expansion to order N^{-1} for the d.f. of $(2T-\sum a_j)/(\sum a_j^2)^{1/2}$. The expansion will be established in theorem 3.3.1 and an evaluation of the order of the remainder will be given in theorem 3.3.2.

Let $\pi(\theta)$ denote the power of the one-sided level α test based on T for the hypothesis of symmetry against the alternative $G(x) = F(x-\theta)$. Suppose that for some $\varepsilon > 0$

$$(3.3.6)\qquad \varepsilon \le \alpha \le 1-\varepsilon.$$

We prove that an expansion for $\pi(\theta)$ is given by

$$(3.3.7)\qquad \tilde{\pi}(\theta) = 1 - K_\theta(u_\alpha - \eta) + \phi(u_\alpha - \eta)\,\frac{\sum a_j^4}{12(\sum a_j^2)^2}\,(u_\alpha^3 - 3u_\alpha),$$

where $u_\alpha = \Phi^{-1}(1-\alpha)$ denotes the upper α-point of the standard normal distribution.

THEOREM 3.3.1. *Suppose that positive numbers* c, C, δ *and* ε *exist such that* (3.2.14), (3.2.15), (3.3.2) *and* (3.3.3) *are satisfied. Then there exists* $A > 0$ *depending on* N, a, F *and* θ *only through* c, C, δ *and* ε *and such that*

$$(3.3.8)\qquad \begin{aligned} &\sup_x \left| P_\theta\left(\frac{2T - \sum a_j}{(\sum a_j^2)^{1/2}} \le x\right) - K_\theta(x-\eta)\right| \le \\ &\le A\{N^{-5/4} + N^{-3/4}\theta^3[\sum\{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^3\}^{4/9}]^{9/4}\}, \end{aligned}$$

$$(3.3.9)\qquad |\eta| \le A,$$

$$(3.3.10)\qquad \begin{aligned} &\theta\,\frac{|\sum a_j^3 E_0\psi_1(Z_j)|}{(\sum a_j^2)^{3/2}} \le AN^{-1},\ \theta^2\,\frac{\sum a_j^2 E_0\psi_1^2(Z_j)}{\sum a_j^2} \le AN^{-1}, \\ &\frac{\theta^3}{(\sum a_j^2)^{1/2}}\,|\sum a_j E_0[3\psi_1^3(Z_j) - 6\psi_1(Z_j)\psi_2(Z_j) + \psi_3(Z_j)]| \le AN^{-1}. \end{aligned}$$

If, in addition, (3.3.6) is satisfied there exists $A' > 0$ depending on N, a, F, θ and α only through c, C, δ and ε and such that

$$(3.3.11)\qquad |\pi(\theta) - \tilde{\pi}(\theta)| \le A'\{N^{-5/4} + N^{-3/4}\theta^3[\sum\{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^3\}^{4/9}]^{9/4}\}.$$

PROOF. The first step is to show that the conditions of the present theorem imply the conditions of theorem 3.2.3. This comes down to the verification of (3.2.16), which is easily done by applying (3.3.2) and (3.3.3). Hence theorem 3.2.3 holds and we must show that (3.3.8) is implied by (3.2.20). This is achieved by Taylor expansion with respect to θ, which is a highly technical and laborious procedure; the interested reader is referred to appendix 1 of Albers, Bickel and van Zwet (1974). The main problem is that not only $P_j = f(Z_j-\theta)/[f(Z_j-\theta)+f(Z_j+\theta)]$, but also its distribution depends on θ because Z_j is the j-th absolute order statistic of a sample from $F(x-\theta)$.

Once the expansion for $\tilde{K}(x - [\sum a_j(2\pi_j - 1)]/[(\sum a_j^2)^{1/2}])$ has been established, it remains to show that $|\tilde{K}\ (x - [\sum a_j(2\pi_j-1)]/[(\sum a_j^2)^{1/2}]) - K_\theta(x-\eta)|$ and the remainder in (3.2.20) are of the order of the remainder in (3.3.8). Then (3.3.8) is proved.

As concerns (3.3.9) and (3.3.10) we note that both are immediate consequences of the results of appendix 1 in Albers, Bickel and van Zwet (1974). The one-sided level α test based on T rejects the hypothesis if $(2T-\sum a_j)/(\sum a_j^2)^{1/2} \geq \xi_\alpha$, with possible randomization if equality occurs. Taking $\theta = 0$ in (3.3.8) we find that

$$1 - \Phi(\xi_\alpha) - \phi(\xi_\alpha)\,\frac{\sum a_j^4}{12(\sum a_j^2)^2}\,(\xi_\alpha^3-3\xi_\alpha) = \alpha + O(N^{-5/4}),$$

and hence because of (3.2.14) and (3.3.6)

$$\xi_\alpha = u_\alpha - \frac{\sum a_j^4}{12(\sum a_j^2)^2}\,(u_\alpha^3-3u_\alpha) + O(N^{-5/4}). \tag{3.3.12}$$

The power of this test against the alternative $F(x-\theta)$ is

$$\begin{aligned}\pi(\theta) &= 1 - K_\theta(\xi_\alpha-\eta) + \\ &\quad + O(N^{-5/4}+N^{-3/4}\theta^3[\sum\{E_0|\psi_1(Z_j)-E_0\psi_1(Z_j)|^3\}^{4/9}]^{9/4}).\end{aligned} \tag{3.3.13}$$

In (3.3.13) we expand $K_\theta(\xi_\alpha-\eta)$ around $u_\alpha-\eta$. Noting that $|\xi_\alpha-u_\alpha| = O(N^{-1})$ and using (3.2.14) and (3.3.10) we find (3.3.11). □

REMARK. As we shall see in section 3.4, $\sigma_0^2(\sum a_j\psi_1(Z_j))$ is typically $O(N)$ and therefore our expansion is of the form $K_\theta(x) = \Phi(x) + N^{-1}A(x) + o(N^{-1})$, for a certain bounded $A(x)$. There is no term of the order $N^{-1/2}$ because of a certain symmetry in the situation.

For i = 1,2,3, define functions Ψ_i on (0,1) by

$$\Psi_i(t) = \psi_i(F^{-1}(\tfrac{1+t}{2})) = \frac{f^{(i)}(F^{-1}(\frac{1+t}{2}))}{f(F^{-1}(\frac{1+t}{2}))}\ . \tag{3.3.14}$$

THEOREM 3.3.2. *Suppose that positive numbers* C *and* δ *exist such that* (3.3.3) *is satisfied and that* $|\Psi_1'(t)| \leq C(t(1-t))^{-4/3+\delta}$ *for all* $0 < t < 1$. *Then there exists* $A'' > 0$ *depending on* N, F *and* θ *only through* C *and* δ *and such that*

$$N^{-3/4}\theta^3[\sum\{E_0|\psi_1(Z_j)-E_0\psi_1(Z_j)|^3\}^{4/9}]^{9/4} \le A''N^{-5/4}.$$

For the highly technical proof of this result the reader is referred to appendix 2 of Albers, Bickel and van Zwet (1974).

The more general case where $X_1,X_2,\ldots,X_N$ are i.i.d. r.v.'s with common d.f. $G(x) = F(x,\theta)$ can be dealt with analogously; the computations become even more laborious but no new techniques are needed. For this reason we only give the results and omit the proofs.

Suppose that (3.3.3) holds and that

$$f_{ij}(x,\theta) = \frac{\partial^{i+j+1}}{\partial x^{i+1}\partial\theta^j} F(x,\theta)$$

exists for $i+j \le 4$, $i \ge -1$, $j \ge 0$. We shall simply write $f(x,\theta)$ for the density $f_{00}(x,\theta)$ and assume that $f(x,0) = f(-x,0)$ for all x. Next we define

$$\text{(3.3.15)} \qquad \begin{aligned} \tilde{\psi}_{ij} &= \frac{f_{ij}}{f}, \\ \psi_{ij}(x) &= \tilde{\psi}_{ij}(x,0), \end{aligned} \qquad \text{for } i+j \le 4,\ i \ge -1,\ j \ge 0.$$

Assume that positive numbers ε and C exist such that for

$$m_1 = 6,\ m_2 = 3,\ m_3 = \frac{4}{3},\ m_4 = 1,\ i \ge -1,\ j \ge 0,\ 1 \le i+j \le 4,$$

$$\text{(3.3.16)} \qquad \sup\ \{\int_{-\infty}^{\infty} |\tilde{\psi}_{ij}(x,y)|^{m_{i+j}}[1+\tilde{\psi}_{-11}^{6}(x,y)]f(x,0)dx\ :\ |y| \le \varepsilon\} \le C.$$

Under the additional assumption

$$\int_{-\infty}^{\infty} f_{0i}(x,\theta)dx = 0,\ i = 1,2,\ 0 \le \theta \le CN^{-1/2},$$

it may be shown that (3.3.3) and (3.3.16) together imply contiguity.

Define

$$(3.3.17)\qquad \bar{K}_\theta(x) = \Phi(x)+\phi(x)\{\frac{\sum a_j^4}{12(\sum a_j^2)^2}(x^3-3x)+\theta\frac{\sum a_j^3E_0[\psi_{01}(Z_j)-\psi_{01}(-Z_j)]}{6(\sum a_j^2)^{3/2}}(x^2-1)+$$

$$+\theta^2\frac{\sum a_j^2E_0[\psi_{01}(Z_j)-\psi_{01}(-Z_j)]^2-\sigma_0^2(\sum a_j[\psi_{01}(Z_j)-\psi_{01}(-Z_j)])}{8\sum a_j^2}x+$$

$$-\frac{\theta^4}{2\sum a_j^2}(\sum a_jE_0\tilde{p}_{02}(Z_j))^2x-\frac{\theta^2}{(\sum a_j^2)^{1/2}}\sum a_jE_0\tilde{p}_{02}(Z_j)+$$

$$-\frac{\theta^3}{3(\sum a_j^2)^{1/2}}\sum a_jE_0\tilde{p}_{03}(Z_j)\},$$

where

$$(3.3.18)\qquad \tilde{p}_{02}(x) = \tfrac{1}{4}[\psi_{02}(x)-\psi_{02}(-x)-\psi_{01}^2(x)+\psi_{01}^2(-x)+$$

$$-(\psi_{-11}(x)-\psi_{-11}(-x))(\psi_{11}(x)+\psi_{11}(-x)-\psi_{10}(x)[\psi_{01}(x)-\psi_{01}(-x)])],$$

$$(3.3.19)\qquad \tilde{p}_{03}(x) = \tfrac{1}{4}[\{\psi_{03}(x)-\psi_{03}(-x)-3\psi_{01}(x)\psi_{02}(x)+3\psi_{01}(-x)\psi_{02}(-x)+$$

$$+\tfrac{3}{2}(\psi_{01}(x)+\psi_{01}(-x))(\psi_{01}^2(x)-\psi_{01}^2(-x))+$$

$$-\tfrac{3}{2}(\psi_{11}(x)+\psi_{11}(-x)-\psi_{10}(x)[\psi_{01}(x)-\psi_{01}(-x)])(\psi_{-12}(x)-\psi_{-12}(-x))\}+$$

$$+3\{\psi_{12}(x)+\psi_{12}(-x)-3\psi_{01}(x)\psi_{11}(x)-3\psi_{01}(-x)\psi_{11}(-x)-\psi_{11}(x)\psi_{01}(-x)+$$

$$-\psi_{01}(x)\psi_{11}(-x)-\psi_{10}(x)(\psi_{02}(x)-\psi_{02}(-x))+$$

$$+3\psi_{10}(x)(\psi_{01}^2(x)-\psi_{01}^2(-x))\}\xi_{01}(x)+3\{\psi_{21}(x)-\psi_{21}(-x)+$$

$$-3\psi_{10}(x)(\psi_{11}(x)+\psi_{11}(-x))-\psi_{20}(x)(\psi_{01}(x)-\psi_{01}(-x))+$$

$$+3\psi_{10}^2(x)(\psi_{01}(x)-\psi_{01}(-x))\}\xi_{01}^2(x)].$$

Here $\xi_{01}(x) = -\frac{1}{2}[\psi_{-11}(x)-\psi_{-11}(-x)]$. Finally define

$$(3.3.20)\qquad \bar{\eta} = \frac{\theta}{2}\frac{\sum a_jE_0[\psi_{01}(Z_j)-\psi_{01}(-Z_j)]}{(\sum a_j^2)^{1/2}},$$

$$(3.3.21)\qquad \bar{\pi}(\theta) = 1 - \bar{K}_\theta(u_\alpha-\bar{\eta}) + \phi(u_\alpha-\bar{\eta})\,\frac{\sum a_j^4}{12(\sum a_j^2)^2}\,(u_\alpha^3-3u_\alpha).$$

Now we have in analogy to theorem 3.3.1

THEOREM 3.3.3. *Suppose that positive numbers* c, C, δ *and* ε *exist such that* (3.2.14), (3.2.15), (3.3.3) *and* (3.3.16) *are satisfied. Then there exists* $A > 0$ *depending on* N, a, F *and* θ *only through* c, C, δ *and* ε *and such that*

$$(3.3.22)\qquad \sup_x |P_\theta(\frac{2T-\sum a_j}{(\sum a_j^2)^{1/2}} \le x) - \bar{K}_\theta(x-\bar{\eta})| \le$$

$$\le A\{N^{-5/4}+N^{-3/4}\theta^3[\sum\{E_0|\psi_{01}(Z_j)-E_0\psi_{01}(Z_j)|^3\}^{4/9}]^{9/4} +$$

$$+ N^{-3/4}\theta^3[\sum\{E_0|\psi_{01}(-Z_j)-E_0\psi_{01}(-Z_j)|^3\}^{4/9}]^{9/4} +$$

$$+ N^{-1/4}\theta^3\sigma_0(\sum a_j[\psi_{01}(Z_j)-\psi_{01}(-Z_j)])\}.$$

If, in addition, (3.3.6) *is satisfied, there exists* $A' > 0$ *depending on* N, a, F, θ *and* α *only through* c, C, δ *and* ε *and such that*

$$(3.3.23)\qquad |\pi(\theta)-\bar{\pi}(\theta)| \le A'\{N^{-5/4}+N^{-3/4}\theta^3[\sum\{E_0|\psi_{01}(Z_j)-E_0\psi_{01}(Z_j)|^3\}^{4/9}]^{9/4} +$$

$$+ N^{-3/4}\theta^3[\sum\{E_0|\psi_{01}(-Z_j)-E_0\psi_{01}(-Z_j)|^3\}^{4/9}]^{9/4} +$$

$$+ N^{-1/4}\theta^3\sigma_0(\sum a_j[\psi_{01}(Z_j)-\psi_{01}(-Z_j)])\}.$$

We conclude this section with some remarks on the relation between the general and the location case. In the first place, condition (3.3.16) is a straightforward generalization of condition (3.3.2) except for the $\tilde{\psi}^6_{-11}$-term. In the location case, $|\tilde{\psi}_{-11}(x,\theta)| = |\frac{\partial}{\partial\theta} F(x-\theta)/\frac{\partial}{\partial x} F(x-\theta)| = 1$, which explains why it does not occur in (3.3.2).

In the location case $f(x,\theta)$ is not only symmetric in x about $x = \theta$, but also in θ about $\theta = x$. Then $\psi_{01}(x) = -\psi_{01}(-x) = \psi_1(x)$, $\tilde{p}_{02}(x) \equiv 0$, $\xi_{01}(x) \equiv 0$ and $\tilde{p}_{03}(x) = \frac{1}{2}[3\psi_1^3(x)-6\psi_1(x)\psi_2(x)+\psi_3(x)]$. Inserting these results in (3.3.17) and (3.3.20) we again obtain (3.3.4) and (3.3.5). In view of these facts, the main difference between $K_\theta(x)$ and $\bar{K}_\theta(x)$ is the presence of the terms

$$- \frac{\theta^2}{(\sum a_j^2)^{1/2}} \sum a_j E_0 \tilde{p}_{02}(Z_j) - \frac{\theta^4}{2\sum a_j^2} (\sum a_j E_0 \tilde{p}_{02}(Z_j))^2 x$$

in (3.3.17). The first of these terms is in general not $\mathcal{O}(N^{-1})$ but only $\mathcal{O}(N^{-1/2})$.

The first three terms on the right side of (3.3.22) and (3.3.23) are again generalizations of the remainder in (3.3.8) and (3.3.11). The last term in (3.3.22) and (3.3.23), however, is new. It is due to the fact that $\tilde{p}_{02}(x) \equiv 0$ does not hold in general. As $\sigma_0^2(\sum a_j[\psi_{01}(Z_j)-\psi_{01}(-Z_j)])$ is typically $\mathcal{O}(N)$, this remainder term is of the right order, i.e. $\mathcal{O}(N^{-5/4})$. To be more precise: with the aid of theorem 3.3.2 it can be shown to be $\mathcal{O}(N^{-7/6})$. If in this theorem $|\Psi_1'(t)| = \mathcal{O}((t(1-t))^{-5/4+\delta})$ instead of $\mathcal{O}((t(1-t))^{-4/3+\delta})$, we get $\mathcal{O}(N^{-5/4})$.

This concludes our treatment of the general case; in the next section we again restrict attention to contiguous location alternatives.

3.4. EXACT AND APPROXIMATE SCORES AND CONTIGUOUS LOCATION ALTERNATIVES

The expansions given in section 3.3 for contiguous location alternatives can be simplified further if we make certain assumptions about the scores a_j. Consider a continuous function J on (0,1) and let $U_{1:N} < U_{2:N} < \ldots < U_{N:N}$ denote order statistics of a sample of size N from the uniform distribution on (0,1). For N = 1,2,... we define the exact scores generated by J by

$$(3.4.1) \qquad a_j = a_{jN} = EJ(U_{j:N}), \; j = 1,2,\ldots,N,$$

and the approximate scores generated by J by

$$(3.4.2) \qquad a_j = a_{jN} = J(\tfrac{j}{N+1}), \quad j = 1,\ldots,N.$$

For almost all well known linear rank tests the scores are of one of these two types. The locally most powerful rank test against location alternatives of type F is based on exact scores generated by the function $-\Psi_1$, where Ψ_1 is defined in (3.3.14).

So far, we have systematically kept the order of the remainder in our expansions down to $\mathcal{O}(N^{-5/4})$. From this point on,however, we shall be content with a remainder that is $o(N^{-1})$, because otherwise we would have to impose rather restrictive conditions. In the previous sections, we have also consistently stressed the fact that the remainder depends on a and F only through certain constants occurring in our conditions, thus in effect indicating classes of scores and distributions for which the expansion holds uniformly. As the number of these constants is becoming rather large, we prefer to formulate our results from here on for a fixed score function J and a fixed d.f. F. The reader can easily construct uniformity classes for himself by using the results of section 3.3 and tracing the development of appendix 2 of Albers, Bickel and van Zwet (1974).

DEFINITION 3.4.1. $\mathcal{J}$ is the class of functions J on (0,1) that are twice continuously differentiable and nonconstant on (0,1) and satisfy

$$\int_0^1 J^4(t)dt < \infty. \tag{3.4.3}$$

$$\limsup_{t\to 0,1} t(1-t)\left|\frac{J''(t)}{J'(t)}\right| < \frac{3}{2}. \tag{3.4.4}$$

$\mathcal{F}$ is the class of d.f.'s F on R^1 with positive densities f that are symmetric about zero, four times differentiable and such that for $\psi_i = f^{(i)}/f$, $\Psi_i(t) = \psi_i(F^{-1}(\frac{1+t}{2}))$, $m_1 = 6$, $m_2 = 3$, $m_3 = \frac{4}{3}$, $m_4 = 1$,

$$\limsup_{y\to 0} \int_{-\infty}^{\infty} |\psi_i(x+y)|^{m_i} f(x)dx < \infty, \qquad i = 1,\dots,4, \tag{3.4.5}$$

$$\limsup_{t\to 0,1} t(1-t)\left|\frac{\Psi_1''(t)}{\Psi_1'(t)}\right| < \frac{3}{2}. \tag{3.4.6}$$

For $J \in \mathcal{J}$ and $F \in \mathcal{F}$, let

$$\begin{aligned}\tilde{K}_\theta(x) = \Phi(x)+\phi(x)\{&\frac{N^{-1}\int_0^1 J^4(t)dt}{12(\int_0^1 J^2(t)dt)^2}(x^3-3x) + \\ &- N^{-1/2}\theta\frac{\int_0^1 J^3(t)\Psi_1(t)dt}{3(\int_0^1 J^2(t)dt)^{3/2}}(x^2-1) + \frac{\theta^2}{2\int_0^1 J^2(t)dt}\cdot \\ &\cdot\Big[\int_0^1 J^2(t)\Psi_1^2(t)dt - \int_0^1\int_0^1 J(s)\Psi_1'(s)J(t)\Psi_1'(t)(s\wedge t-st)dsdt\Big]x + \\ &+ \frac{N^{1/2}\theta^3}{6(\int_0^1 J^2(t)dt)^{1/2}}\int_0^1 J(t)[3\Psi_1^3(t) - 6\Psi_1(t)\Psi_2(t)+\Psi_3(t)]dt\},\end{aligned} \tag{3.4.7}$$

(3.4.8)
$$K_{\theta,1}(x) = \tilde{K}_\theta(x)+\phi(x)\,\frac{N^{-1/2}\theta}{2(\int_0^1 J^2(t)dt)^{1/2}}\Big\{\frac{\int_0^1 J(t)\Psi_1(t)dt}{\int_0^1 J^2(t)dt}\sum_{j=1}^{N}\sigma^2(J(U_{j:N})) + - 2\sum_{j=1}^{N}\operatorname{cov}(J(U_{j:N}),\Psi_1(U_{j:N}))\Big\},$$

(3.4.9)
$$K_{\theta,2}(x) = \tilde{K}_\theta(x)+\phi(x)\,\frac{N^{-1/2}\theta}{2(\int_0^1 J^2(t)dt)^{1/2}}\Big\{\frac{\int_0^1 J(t)\Psi_1(t)dt}{\int_0^1 J^2(t)dt}\int_{1/N}^{1-1/N}(J'(t))^2 t(1-t)dt + - 2\int_{1/N}^{1-1/N} J'(t)\Psi_1'(t)t(1-t)dt\Big\},$$

(3.4.10)
$$\tilde{\eta} = - N^{1/2}\theta\,\frac{\int_0^1 J(t)\Psi_1(t)dt}{(\int_0^1 J^2(t)dt)^{1/2}},$$

(3.4.11)
$$\pi_i(\theta) = 1 - K_{\theta,i}(u_\alpha-\tilde{\eta}) + \phi(u_\alpha-\tilde{\eta})N^{-1}\,\frac{\int_0^1 J^4(t)dt}{12(\int_0^1 J^2(t)dt)^2}\,(u_\alpha^3-3u_\alpha),$$

for $i = 1,2$. Then, in the notation of section 3.3, we have for contiguous location alternatives and exact scores

THEOREM 3.4.1. *Let* $F \in \mathcal{F}$, $J \in \mathcal{J}$, $a_j = EJ(U_{j:N})$ *for* $j = 1,\ldots,N$, *and let* $0 \le \theta \le CN^{-1/2}$, $\varepsilon \le \alpha \le 1-\varepsilon$ *for positive* C *and* ε. *Then, for every fixed* J, F, C *and* ε, *there exist positive numbers* $A, \delta_1, \delta_2, \ldots$ *such that* $\lim_{N\to\infty}\delta_N = 0$ *and for every* N

(3.4.12)
$$\sup_x \left| P_\theta\Big(\frac{2T-\sum a_j}{(\sum a_j^2)^{1/2}} \le x\Big) - K_{\theta,1}(x-\tilde{\eta})\right| \le \delta_N\, N^{-1},$$

(3.4.13)
$$\sup_x \left| P_\theta\Big(\frac{2T-\sum a_j}{(\sum a_j^2)^{1/2}} \le x\Big) - K_{\theta,2}(x-\tilde{\eta})\right| \le \delta_N\, N^{-1} + + A.N^{-3/2}\int_{1/N}^{1-1/N} |J'(t)|(|J'(t)| + |\Psi_1'(t)|)(t(1-t))^{1/2}dt,$$

(3.4.14)
$$|\pi(\theta)-\pi_1(\theta)| \le \delta_N\, N^{-1},$$

$$(3.4.15)\qquad |\pi(\theta)-\pi_2(\theta)| \le \delta_N N^{-1} + + AN^{-3/2}\int_{1/N}^{1-1/N} |J'(t)|(|J'(t)|+|\Psi_1'(t)|)(t(1-t))^{1/2}dt.$$

PROOF. For fixed $J \in \mathcal{J}$, positive constants c, C, and δ exist for which (3.2.14) and (3.2.15) hold for all N (cf. one of the remarks following the proof of theorem 3.2.2). Similarly, for fixed $F \in \mathcal{F}$, (3.3.2) is satisfied and it follows that the conclusions of theorem 3.3.1 hold with A and A' depending only on F, J, C and ε. From appendix 2 of Albers, Bickel and van Zwet (1974) it is clear that (3.4.5) and (3.4.6) imply that the conclusion of theorem 3.3.2 holds with A" depending only on F and C. In this appendix it is also shown that

$$(3.4.16)\qquad \frac{1}{N}\sum_{j=1}^{N} a_j^2 = \int_0^1 J^2(t)dt + o(1),$$

$$(3.4.17)\qquad \frac{1}{N}\sum_{j=1}^{N} a_j^k\, E\Psi_1^{4-k}(U_{j:N}) = \int_0^1 J^k(t)\Psi_1^{4-k}(t)dt + o(1),\ k = 1,\dots,4,$$

$$(3.4.18)\qquad \frac{1}{N}\sum_{j=1}^{N} a_j\, E\Psi_1(U_{j:N})\Psi_2(U_{j:N}) = \int_0^1 J(t)\Psi_1(t)\Psi_2(t)dt + o(1),$$

$$(3.4.19)\qquad \frac{1}{N}\sum_{j=1}^{N} a_j\, E\Psi_3(U_{j:N}) = \int_0^1 J(t)\Psi_3(t)dt + o(1),$$

$$(3.4.20)\qquad \frac{1}{N}\sigma^2\Big(\sum_{j=1}^{N} a_j\Psi_1(U_{j:N})\Big) = \int_0^1\int_0^1 J(s)J(t)\Psi_1'(s)\Psi_1'(t)[s\wedge t-st]dsdt+o(1),$$

and furthermore

$$(3.4.21)\qquad \frac{N^{-1/2}\sum_{j=1}^{N} a_j\, E\Psi_1(U_{j:N})}{\Big(\sum_{j=1}^{N} a_j^2\Big)^{1/2}} =$$

$$= \frac{\int_0^1 J(t)\Psi_1(t)dt}{(\int_0^1 J^2(t)dt)^{1/2}} - \frac{1}{N}\,\frac{\sum_{j=1}^{N}\operatorname{covar}(J(U_{j:N}),\Psi_1(U_{j:N}))}{(\int_0^1 J^2(t)dt)^{1/2}} +$$

$$+ \frac{1}{2N}\,\frac{\int_0^1 J(t)\Psi_1(t)dt}{(\int_0^1 J^2(t)dt)^{3/2}}\sum_{j=1}^{N}\sigma^2(J(U_{j:N})) +$$

$$+ o(N^{-1}) = \frac{\int_0^1 J(t)\Psi_1(t)dt}{(\int_0^1 J^2(t)dt)^{1/2}} - \frac{1}{N}\frac{\int_{1/N}^{1-1/N} J'(t)\Psi_1'(t)t(1-t)dt}{(\int_0^1 J^2(t)dt)^{1/2}} +$$

$$+ \frac{1}{2N}\frac{\int_0^1 J(t)\Psi_1(t)dt}{(\int_0^1 J^2(t)dt)^{3/2}}\int_{1/N}^{1-1/N}(J'(t))^2 t(1-t)dt +$$

$$+ o(N^{-1}) + O(N^{-3/2}\int_{1/N}^{1-1/N}|J'(t)|(|J'(t)| + |\Psi_1'(t)|)(t(1-t))^{1/2}dt).$$

From (3.4.16) to (3.4.20) it is clear that $K_\theta(x)$ in (3.3.4) satisfies $K_\theta(x) = \tilde{K}_\theta(x) + o(N^{-1})$. Applying this to the expansions $K_\theta(x-\eta)$ and $\tilde{\pi}(\theta)$ in theorem 3.3.1 and expanding these functions of η around the point $\eta = \tilde{\eta}$, we get the desired results in view of (3.4.21). □

In general the expansions given in theorem 3.4.1 will not hold if the exact scores are replaced by approximate scores $a_j = J(\frac{j}{N+1})$, because $\eta - \tilde{\eta}$ will then give rise to a different term of order N^{-1}. If $J = -\Psi_1$, however, it is clear from appendix 2 of Albers, Bickel and van Zwet (1974), that expansions (3.4.13) and (3.4.15) are valid for approximate as well as exact scores. Also for $J = -\Psi_1$, these expansions may be simplified because $F \in \mathcal{F}$ implies that by partial integration (cf. lemma 4.2.1)

$$(3.4.22)\qquad \int_0^1\int_0^1 \Psi_1(s)\Psi_1'(s)\Psi_1(t)\Psi_1'(t)(s\wedge t-st)dsdt = \frac{1}{4}\int_0^1\Psi_1^4(t)dt - \frac{1}{4}(\int_0^1\Psi_1^2(t)dt)^2,$$

$$(3.4.23)\qquad \int_0^1 \Psi_1(t)[6\Psi_1(t)\Psi_2(t)-\Psi_3(t)]dt = \frac{10}{3}\int_0^1\Psi_1^4(t)dt + \int_0^1\Psi_2^2(t)dt.$$

It follows that in this case $\tilde{\eta}$, $K_{\theta,2}(x-\tilde{\eta})$ and $\pi_2(\theta)$ reduce to

$$(3.4.24)\qquad \eta_1 = N^{1/2}\theta[E_0\psi_1^2(X_1)]^{1/2},$$

$$(3.4.25)\qquad L_\theta(x) = \Phi(x-\eta_1) + \frac{\phi(x-\eta_1)}{72N}\{\eta_2[6(x^3-3x) + 6\eta_1(x^2-1) - 3\eta_1^2 x - 5\eta_1^3] + 12\eta_3\eta_1^3 + 9\eta_1^2(x-\eta_1) + 36\frac{\int_{1/N}^{1-1/N}(\Psi_1'(t))^2 t(1-t)dt}{E_0\psi_1^2(X_1)}\eta_1\},$$

where

$$(3.4.26)\qquad \begin{aligned}\eta_2 &= E_0\psi_1^4(X_1)/[E_0\psi_1^2(X_1)]^2,\\ \eta_3 &= E_0\psi_2^2(X_1)/[E_0\psi_1^2(X_1)]^2,\end{aligned}$$

and

$$(3.4.27) \qquad \pi^*(\theta) = 1-\Phi(u_\alpha-\eta_1) + \frac{\eta_1\phi(u_\alpha-\eta_1)}{72N}\{-6\eta_2(u_\alpha^2-1)+3(\eta_2-3)\eta_1 u_\alpha + (5\eta_2-12\eta_3+9)\eta_1^2 - 36\frac{\int_{1/N}^{1-1/N}(\Psi_1'(t))^2 t(1-t)dt}{E_0\Psi_1^2(X_1)}\} .$$

Finally we note that for $F \in \mathcal{F}$, $-\Psi_1$ cannot be constant on $(0,1)$ because the density $f(x) = \frac{1}{2}\lambda\exp(-\lambda|x|)$ of the double exponential distribution is not differentiable at zero. It follows that $-\Psi_1 \in \mathcal{J}$ for every $F \in \mathcal{F}$. We have shown

THEOREM 3.4.2. *Let* $F \in \mathcal{F}$ *and let either* $a_j = -E\Psi_1(U_{j:N})$ *for* $j = 1,\dots,N$ *or* $a_j = -\Psi_1(\frac{j}{N+1})$ *for* $j = 1,2,\dots,N$. *Suppose that* $0 \le \theta \le CN^{-1/2}$ *and* $\varepsilon \le \alpha \le 1-\varepsilon$ *for positive* C *and* ε. *Then, for every fixed* F, C *and* ε, *there exist positive numbers* A, δ_1, δ_2, ... *such that* $\lim_{N\to\infty}\delta_N = 0$ *and for every* N

$$(3.4.28) \qquad \sup_x \left|P_\theta\left(\frac{2T-\sum a_j}{(\sum a_j^2)^{1/2}} \le x\right) - L_\theta(x)\right| \le \delta_N N^{-1} + AN^{-3/2}\int_{1/N}^{1-1/N}(\Psi_1'(t))^2(t(1-t))^{1/2}dt,$$

$$(3.4.29) \qquad |\pi(\theta) - \pi^*(\theta)| \le \delta_N N^{-1} + AN^{-3/2}\int_{1/N}^{1-1/N}(\Psi_1'(t))^2(t(1-t))^{1/2}dt.$$

At this point it may be useful to make some remarks concerning the assumptions in theorems 3.4.1 and 3.4.2. Conditions (3.4.4) and (3.4.6) ensure that J' and Ψ_1' do not oscillate to wildly near 0 and 1. They also limit the growth of these functions near 0 and 1, but in this respect conditions (3.4.3) and (3.4.5) for $i = 1$ are typically much stronger. Together with (3.4.4) and (3.4.6) they imply that $J'(t) = o((t(1-t))^{-5/4})$ and $\Psi_1'(t) = o((t(1-t))^{-7/6})$ near 0 and 1.

For expansions (3.4.13), (3.4.15), (3.4.28) and (3.4.29) to be meaningful rather than just formally correct, even stronger growth conditions have to be imposed. Consider, for example, expansion (3.4.29) and suppose, as is typically the case, that Ψ_1' remains bounded near 0. If $\Psi_1'(t) = o((1-t)^{-1})$ near 1, then the right side in (3.4.29) is $o(N^{-1})$ and the expansion makes sense. However, if $\Psi_1'(t)$ is of exact order $(1-t)^{-1}$, the expansion reduces to

$$\pi(\theta) = 1-\Phi(u_\alpha-\eta_1) - \frac{\eta_1\phi(u_\alpha-\eta_1)}{2N}\frac{\int_0^{1-1/N}(\Psi_1'(t))^2t(1-t)dt}{\int_0^1\Psi_1^2(t)dt} + O(N^{-1}),$$

and if $\Psi_1'(t) \sim (1-t)^{-1-\delta}$ near 1 for some $0 < \delta < \frac{1}{6}$, all we have left in (3.4.29) is $\pi(\theta) = 1-\Phi(u_\alpha-\eta_1) + O(N^{-1+2\delta})$.

We conclude this section with a few applications of theorems 3.4.1 and 3.4.2. The tedious computations will be omitted almost completely. First we consider Wilcoxon's signed rank test (W), which is based on the scores $a_j = \frac{j}{N+1}$. This is the locally most powerful rank test against logistic (L) location alternatives $G(x) = F(x-\theta)$, with $F(x) = 1/(1+e^{-x})$ and $\theta = O(N^{-1/2})$. As in this case $-\Psi_1(t) = t$ the conditions of theorem 3.4.2 are easily verified and we get

$$(3.4.30)\qquad \pi_{W,L}(\theta) = 1-\Phi(u_\alpha-\eta_1) - \frac{\phi(u_\alpha-\eta_1)\eta_1}{20N}\{3u_\alpha^2+u_\alpha\eta_1+2+\eta_1^2\} + o(N^{-1}),$$

where $\eta_1 = (\frac{N}{3})^{1/2}\theta$. The power of W against normal (N) alternatives $G(x) = \Phi(x-\theta)$, $\theta = O(N^{-1/2})$ may be found from theorem 3.4.1. We now have $\Psi_1(t) = -\Phi^{-1}(\frac{1+t}{2})$ and therefore

$$\lim_{t\to 1}(1-t)\left|\frac{\Psi_1''(t)}{\Psi_1'(t)}\right| = \lim_{t\to 1}(1-t)\frac{\Phi^{-1}(\frac{1+t}{2})}{2\phi(\Phi^{-1}(\frac{1+t}{2}))} =$$

$$= \lim_{x\to\infty}\frac{x(1-\Phi(x))}{\phi(x)} = 1 < \frac{3}{2},$$

as for positive x

$$\phi(x)(\frac{1}{x}-\frac{1}{x^3}) < 1-\Phi(x) < \frac{\phi(x)}{x}.$$

The remaining conditions of theorem 3.4.1 are verified easily and we may apply (3.4.15). The expression thus obtained can be simplified further by evaluating for $k = 0,1,2,3,4$

$$\int_0^1 y^k[\Phi^{-1}(\frac{1+y}{2})]^{4-k}dy.$$

This can be done by partial integration. In doing so, only two integrals occur that are not entirely trivial: $\int_0^\infty \phi^2(x)\Phi(x)dx$ and $\int_0^\infty \phi^2(x)\Phi^2(x)dx$. For the first one we note

$$\int_0^\infty \phi^2(x)\Phi(x)dx = (2\pi)^{-3/2}\int_0^\infty\int_{-\infty}^x e^{-x^2-\frac{1}{2}y^2}dydx =$$

$$= (2\pi)^{-3/2}\int_{-\pi/2}^{\pi/4}\int_0^\infty e^{-\frac{1}{2}r^2(2\cos^2\phi+\sin^2\phi)}rdrd\phi =$$

$$= (2\pi)^{-3/2}\int_{-\pi/2}^{\pi/4}\frac{d\phi}{1+\cos^2\phi} = (2\pi)^{-3/2}\int_{-\infty}^1\frac{dt}{2+t^2} =$$

$$= \frac{1}{4\sqrt{\pi}}\left(\frac{\arctan\frac{1}{2}\sqrt{2}}{\pi}+\frac{1}{2}\right).$$

Furthermore, for $x > 0$, we have

$$\Phi^2(x) = \frac{1}{2\pi}\int_{-\infty}^x\int_{-\infty}^x e^{-\frac{1}{2}(y^2+z^2)}dydz = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/4}(1-e^{-\frac{x^2}{2\cos^2\phi}})d\phi +$$

$$+\frac{1}{2\pi}\int_{\pi/4}^{\pi}(1-e^{-\frac{x^2}{2\sin^2\phi}})d\phi + \frac{1}{2\pi}\int_{\pi}^{3\pi/2}d\phi = 1-\frac{1}{\pi}\int_{\pi/4}^{\pi}e^{-\frac{x^2}{2\sin^2\phi}}d\phi.$$

Hence,

$$\int_0^\infty \phi^2(x)\Phi^2(x)dx = \int_0^\infty \phi^2(x)dx - \frac{1}{2\pi^2}\int_{\pi/4}^{\pi}\int_0^\infty e^{-\frac{x^2}{2}(2+\frac{1}{\sin^2\phi})}dxd\phi =$$

$$= \frac{1}{4\sqrt{\pi}} - \frac{1}{2\pi\sqrt{\pi}}\int_{\pi/4}^{\pi}(2+\frac{1}{\sin^2\phi})^{-1/2}d\phi = \frac{1}{8\sqrt{\pi}}.$$

Application of these results leads to

$$(3.4.31)\quad \pi_{W,N}(\theta) = 1-\Phi(u_\alpha-\tilde{\eta}) - \frac{\tilde{\eta}\phi(u_\alpha-\tilde{\eta})}{N}\{u_\alpha^2(\frac{6\arctan\frac{1}{4}\sqrt{2}}{\pi}-\frac{9}{20}) +$$

$$+ u_\alpha\tilde{\eta}(\frac{49}{20}-\frac{2}{3}\sqrt{3}-\frac{12\arctan\frac{1}{4}\sqrt{2}}{\pi}) + (\frac{44}{20}-\sqrt{2}-\frac{6\arctan\frac{1}{4}\sqrt{2}}{\pi}) +$$

$$+ \tilde{\eta}^2(\frac{6\arctan\frac{1}{4}\sqrt{2}}{\pi}+\frac{2}{3}\sqrt{3}-\frac{43}{20}+\frac{\pi}{9})\} + o(N^{-1}),$$

where $\tilde{\eta} = (\frac{3N}{\pi})^{1/2}\theta$.

The second test we consider is the one sample normal scores test (NS) which is based on the scores $a_j = E\Phi^{-1}(\frac{1+U_{j:N}}{2})$. Its power against the normal and logistic alternatives described above satisfies

$$(3.4.32)\qquad \pi_{NS,N}(\theta) = 1-\Phi(u_\alpha-\eta_1) - \frac{\eta_1\phi(u_\alpha-\eta_1)}{4N}\{u_\alpha^2-1 + 2\int_0^{\Phi^{-1}(1-\frac{1}{2N})} \frac{(2\Phi(x)-1)(1-\Phi(x))}{\phi(x)}\,dx\} + o(N^{-1}),$$

where $\eta_1 = N^{1/2}\theta$ and

$$(3.4.33)\qquad \pi_{NS,L}(\theta) = 1-\Phi(u_\alpha-\tilde{\eta}) - \frac{\tilde{\eta}\phi(u_\alpha-\tilde{\eta})}{N}\{\frac{u_\alpha^2}{12} + u_\alpha\tilde{\eta}(-\frac{5}{12}+\frac{\pi}{6}) + (\frac{23}{12}-\surd 2) + \tilde{\eta}^2(\arctan\frac{23}{20}\surd 2 + \frac{1}{12} - \frac{\pi}{3}) - \frac{1}{2}\int_0^{\Phi^{-1}(1-\frac{1}{2N})} \frac{(2\Phi(x)-1)(1-\Phi(x))}{\phi(x)}\,dx\} + o(N^{-1}),$$

where now $\tilde{\eta} = (\frac{N}{\pi})^{1/2}\theta$. We note that theorem 3.4.2 ensures that (3.4.32) will also hold for van der Waerden's one sample test which is based on the approximate scores $a_j = \Phi^{-1}[(N+1+j)/(2(N+1))]$. To evaluate the integral in (3.4.32) and (3.4.33) we write

$$(3.4.34)\qquad \int_0^{\Phi^{-1}(1-\frac{1}{2N})} \frac{(2\Phi(x)-1)(1-\Phi(x))}{\phi(x)}\,dx =$$

$$= \frac{1}{2}\log\log N + \frac{1}{2}\log 2 - 2\int_0^\infty \log x\,\phi(x)dx +$$

$$+ \int_0^\infty \frac{(2\Phi(x)-1)\{(1-\Phi(x))x-\phi(x)\}}{\phi(x)}\,dx + o(1) = \frac{1}{2}\log\log N + \frac{1}{2}\log 2 +$$

$$+ 0.05832\ldots\ldots + o(1),$$

where the final result is obtained by numerical integration.

3.5. THE SIGN TEST

The method developped in this chapter cannot be used for the sign test, as was pointed out in section 3.2. The lattice character of the sign test is too pronounced, which is caused by the fact that all scores are equal for this test. However, it is exactly this equality of scores that yields another, very simple method of finding and justifying a powerexpansion for the sign test: it makes it possible to write T as

$$T = \sum_{X_j > 0} 1,$$

the number of positive elements in the sample. This has a binomial distribution, both under the hypothesis and under alternatives. With the aid of a well-known expansion for the binomial distribution function, this leads to the desired expansion for the power of the sign test.

Let $X_1, X_2, \ldots, X_N$ be i.i.d. r.v.'s with continuous d.f. $F(x-\theta)$, where $F(-x) = 1 - F(x)$ for all x. Define

$$\tau = N^{1/2}(2F(\theta)-1). \tag{3.5.1}$$

Let $0 < \alpha < 1$ and let $\pi(\theta)$ be the power of the sign test with size α for $H_0 : \theta = 0$ against $H_1 : \theta > 0$. Define

$$\tilde{\pi}(\theta) = 1 - \Phi(u_\alpha - \tau) - \frac{\tau\phi(u_\alpha - \tau)}{12N}\{u_\alpha^2 + u_\alpha\tau - 3\tau^2 + 24\gamma_\alpha(1-\gamma_\alpha) - 3\}. \tag{3.5.2}$$

Here

$$\gamma_\alpha = \frac{(N+1)}{2} + \frac{N^{1/2}u_\alpha}{2} - [\frac{(N+1)}{2} + \frac{N^{1/2}u_\alpha}{2}] + O(N^{-1/2}), \tag{3.5.3}$$

where [y] is the integer part of y.

LEMMA 3.5.1. *Suppose that* $|\tau| \le C$ *for some constant* $C > 0$. *Then there exists* $A > 0$ *depending on* N, θ *and* F *only through* C *such that*

$$|\pi(\theta) - \tilde{\pi}(\theta)| \le AN^{-3/2}.$$

PROOF. Let Y have a binomial distribution with parameters N and p, where $0 < p < 1$. Define $q = 1-p$, $\sigma^2 = Npq$, $u_k = (k-Np)/\sigma$ for $k = 0,1,\ldots,N$. If $u_k = O(1)$, the following expansion holds

$$P(Y \le k) = \tag{3.5.4}$$

$$= \Phi\{u_{k+1/2} + \frac{p-q}{6\sigma}(u_{k+1/2}^2 - 1) + \frac{(5-14pq)u_{k+1/2}^3 + (-2+2pq)u_{k+1/2}}{72\sigma^2}\} +$$

$$+ O(N^{-3/2}).$$

Expansions of this type are given by Molenaar (1970).

Clearly $T = \sum_{X_j>0} 1$ has a binomial distribution with parameters N and p, with $p = P_\theta(X_1 > 0) = P_0(X_1 > -\theta) = 1 - F(-\theta) = F(\theta) = \frac{1}{2}(1+\tau N^{-1/2})$. For this choice of p,

$$\frac{p-q}{6\sigma} = \frac{\tau}{3N} + O(N^{-2}),$$

and hence (3.5.4) simplifies to

$$(3.5.5) \qquad P_\theta(T \le k) = \Phi\{u_{k+1/2} + \frac{\tau}{3N}(u^2_{k+1/2} - 1) + \frac{1}{12N}(u^3_{k+1/2} - u_{k+1/2})\} + O(N^{-3/2}).$$

For power computations, we need an expansion for $P_\theta(T < k) + \gamma P_\theta(T = k)$, for $0 < \gamma \le 1$. Note that

$$P_\theta(T < k) + \gamma P_\theta(T = k) = \gamma P_\theta(T \le k) + (1-\gamma)P_\theta(T \le k-1)$$

and

$$\gamma\Phi(u_{k+1/2}) + (1-\gamma)\Phi(u_{k-1/2}) =$$

$$= \Phi(u_{k+\gamma-1/2}) + \frac{\gamma(1-\gamma)}{2\sigma^2}\,\Phi'(u_{k+\gamma-1/2}) + O(N^{-3/2}) =$$

$$= \Phi(u_{k(\gamma)}) + O(N^{-3/2}),$$

with

$$(3.5.6) \qquad u_{k(\gamma)} = \frac{k+\gamma - Np - \frac{1}{2}}{\sigma}\left(1 - \frac{\gamma(1-\gamma)}{2\sigma^2}\right).$$

Hence

$$(3.5.7) \qquad P_\theta(T < k) + \gamma P_\theta(T = k) =$$

$$= \Phi\{u_{k(\gamma)} + \frac{\tau}{3N}(u^2_{k(\gamma)} - 1) + \frac{(u^3_{k(\gamma)} - u_{k(\gamma)})}{12N}\} + O(N^{-3/2}).$$

Under H_0 $\tau = 0$. Let k_α and γ_α be such that $P_0(T < k_\alpha) + \gamma_\alpha P_0(T = k_\alpha) = 1-\alpha = \Phi(u_\alpha)$, with k_α an integer and $0 < \gamma_\alpha \le 1$. As $0 < \alpha < 1$, α constant, we have $u_\alpha = O(1)$ and we may use (3.5.7). This leads to

$$(3.5.8)\qquad \frac{k_\alpha+\gamma_\alpha-1/2N-1/2}{1/2N^{1/2}}\left(1-\frac{2\gamma_\alpha(1-\gamma_\alpha)}{N}\right)=u_\alpha-\frac{(u_\alpha^3-u_\alpha)}{12N}+O(N^{-3/2}),$$

which shows that γ_α satisfies (3.5.3). Under H_1 we have $\tau > 0$ and $1-\pi(\theta) = P_\theta(T < k_\alpha) + \gamma_\alpha P_\theta(T = k_\alpha)$. Before applying (3.5.7) again we note that

$$(3.5.9)\qquad \frac{k_\alpha+\gamma_\alpha-1/2N(1+\tau N^{-1/2})-1/2}{1/2N^{1/2}}\left(1-\frac{2\gamma_\alpha(1-\gamma_\alpha)}{N}\right)\left(1-\frac{\tau^2}{N}\right)^{-1/2}=$$

$$=\left(\frac{k_\alpha+\gamma_\alpha-1/2N-1/2}{1/2N^{1/2}}-\tau\right)\left(1-\frac{2\gamma_\alpha(1-\gamma_\alpha)}{N}\right)\left(1+\frac{\tau^2}{2N}+O(N^{-2})\right)=$$

$$=(u_\alpha-\tau)\left(1+\frac{\tau^2}{2N}\right)+\frac{2\tau\gamma_\alpha(1-\gamma_\alpha)}{N}-\frac{(u_\alpha^3-u_\alpha)}{12N}+O(N^{-3/2}).$$

Inserting this result in (3.5.7), we get $\pi(\theta) = \tilde{\pi}(\theta) + O(N^{-3/2})$. The uniformity of the O-symbol is evident from the proof. □

COROLLARY 3.5.1. *If* $|\tau| \leq C$ *and* F *has a density* f *that is three times differentiable with* $\sup_{|x|\leq\varepsilon} |f^{(3)}(x)| \leq c$, *for some constants* ε, C *and* $c > 0$, *we may replace* $\tilde{\pi}(\theta)$ *in lemma* 3.5.1 *by*

$$\tilde{\pi}(\theta) = 1-\Phi(u_\alpha-2N^{1/2}\theta f(0)) - \frac{\theta f(0)}{6\sqrt{N}}\,\phi(u_\alpha-2N^{1/2}\theta f(0))\{u_\alpha^2 +$$

$$+\,2N^{1/2}\theta f(0)u_\alpha-2N\theta^2\left(\frac{f''(0)}{f(0)}+6(f(0))^2\right)+24\gamma_\alpha(1-\gamma_\alpha)-3\}.$$

The constant A *depends on* N, θ *and* F *only through* ε, C *and* c *for this choice of* $\tilde{\pi}(\theta)$.

PROOF. Immediate. □

The conditions of the lemma and its corollary are satisfied for e.g. the normal or the logistic d.f.; the double-exponential d.f. satisfies the conditions of the lemma, but not of its corollary.

CHAPTER 4

DEFICIENCIES BETWEEN VARIOUS TESTS FOR THE ONE SAMPLE PROBLEM

4.1. INTRODUCTION

One of the results of the previous chapter is an expansion to $o(N^{-1})$ for the power of the locally most powerful (LMP) rank test for the one sample problem. In this chapter we obtain similar expansions for various other types of test that are in some sense optimal for this one sample problem. Using the expansions thus obtained, we can evaluate deficiencies between any pair of the tests involved. Such evaluations make sense since the LMP rank test has asymptotic relative efficiency 1 with respect to all tests considered in this chapter.

In section 4.2 we investigate two parametric tests; in section 4.3 permutation tests are dealt with. Scale invariant tests are considered in section 4.4, the randomized rank score tests due to Bell and Doksum (1965) in section 4.5. Finally, the deficiency evaluations mentioned above, take place in section 4.6.

4.2. PARAMETRIC TESTS

Let $X_1, X_2, \ldots, X_N$ be i.i.d. r.v.'s with d.f. $F(x-\theta)$, where F is known and has a density f that is positive on R^1 and symmetric about zero. For the testing problem $H_0 : \theta = 0$ against $H_1 : \theta > 0$ the Neyman-Pearson lemma asserts that the test that rejects H_0 for large values of $\pi_{j=1}^{N}[f(X_j-\theta_1)/f(X_j)]$ is most powerful against the alternative $\theta = \theta_1$. It follows that for every $\theta > 0$, the envelope power at θ equals the power at θ of the test that rejects H_0 for large values of $S = \sum_{j=1}^{N} S_j$, where $S_j = \frac{1}{\theta} \log [f(X_j-\theta)/f(X_j)]$.

In general, a uniformly most powerful (UMP) test against H_1 does not exist and no single test attains the envelope power for all $\theta > 0$. If this is the case, one may consider the use of the LMP test in the sense of Lehmann (1959), p. 342. A test ϕ_0 is LMP if, given any other test ϕ with the same level, there exists $\Delta > 0$ such that the powers satisfy $\pi_{\phi_0}(\theta) \geq \pi_{\phi}(\theta)$ for all $0 < \theta < \Delta$. A LMP test of $\theta = 0$ against $\theta > 0$ exists and is defined by the fact that it maximizes $\pi'(0)$ among all tests with the same level. It follows that in the present case the LMP test is based on the teststatistic $S^* = \sum_{j=1}^N S_j^*$, with $S_j^* = -f'(X_j)/f(X_j)$.

In the sequel we shall derive power expansions to $o(N^{-1})$ for the envelope power and for the test based on S^*. First we shall deal with the envelope power; the result for S^* is proved in a similar fashion.

Since S is a sum of i.i.d. r.v.'s, the obvious thing to do is to establish Edgeworth expansions to $o(N^{-1})$ for the d.f. of S, both under F(x) and F(x-θ). Before we can evaluate the standardized cumulants required for such expansions, the following preliminaries are needed. Assume that f is five times continuously differentiable and define

$$\text{(4.2.1)}\quad \begin{aligned} \psi_i(x) &= \frac{d^i f(x)}{dx^i} \Big/ f(x), \qquad & i = 1,2,\ldots,5,\\ \zeta_i(x) &= \frac{d^i}{dx^i}(\log f(x)), \qquad & i = 0,1,\ldots,5, \end{aligned}$$

where $\zeta_0 = \log f$. The connection between the ψ_i and ζ_i is as follows

$$\text{(4.2.2)}\quad \begin{aligned} &\zeta_1 = \psi_1,\ \zeta_2 = \psi_2-\psi_1^2,\ \zeta_3 = \psi_3-3\psi_1\psi_2+2\psi_1^3,\\ &\zeta_4 = \psi_4-3\psi_2^2-4\psi_1\psi_3+12\psi_1^2\psi_2-6\psi_1^4,\\ &\zeta_5 = \psi_5-5\psi_1\psi_4-10\psi_2\psi_3+30\psi_1\psi_2^2+20\psi_1^2\psi_3-60\psi_1^3\psi_2+24\psi_1^5. \end{aligned}$$

Next we give two lemmas. For lemma 4.2.1, compare lemma a of Hájek and Šidák (1967), p.19; the second lemma is proved in Albers, Bickel and van Zwet (1974) and is an application of Taylor's formula with Cauchy's form of the remainder.

LEMMA 4.2.1. *Let* h *be a real differentiable function with derivative* h'. *If* |h| *and* |h'| *are both summable, we have* $\lim_{x\to\pm\infty} h(x) = 0$.

PROOF. $\int_{-\infty}^{\infty}|h'(x)|dx < \infty$ and hence by the dominated convergence theorem $\lim_{\nu\to\infty}\int_{c'_\nu}^{c_\nu} h'(x)dx = 0$ for any pair of sequences $\{c_\nu\}$ and $\{c'_\nu\}$ with $c_\nu \to \infty$, $c'_\nu \to \infty$ for $\nu \to \infty$ and $c'_\nu \le c_\nu$ for all ν. This implies that $h(c_\nu)-h(c'_\nu) \to 0$ for $\nu \to \infty$, i.e. $\lim_{x\to\infty} h(x)$ exists. Since h itself is also summable it follows that $\lim_{x\to\infty} h(x) = 0$. In the same way of course $\lim_{x\to-\infty} h(x) = 0$. □

LEMMA 4.2.2. *Let* $q(x,t)$ *be a function of two variables possessing derivatives of order* $\leq k+1$ *in* t *in a neighbourhood of* 0. *Then if* S *is any r.v. and* $m \geq 1$

$$(4.2.3)\qquad E|q(S,t) - \sum_{j=0}^{k} q_{0,j}(S,0)\frac{t^j}{j!}|^m \leq$$

$$\leq \left(\frac{|t|^{k+1}}{(k+1)!}\right)^m \sup\{E|q_{0,k+1}(S,\nu t)|^m : 0 \leq \nu \leq 1\}.$$

Suppose moreover that for $j = 0,1,\ldots,k$, $Eq_{0,j}(S,0)$ *exists and is finite. Then*

$$(4.2.4)\qquad E|\{q(S,t)-Eq(S,t)\} - \sum_{j=0}^{k}\{q_{0,j}(S,0)-Eq_{0,j}(S,0)\}\frac{t^j}{j!}|^m \leq$$

$$\leq 2^m\left(\frac{|t|^{k+1}}{(k+1)!}\right)^m \sup\{E|q_{0,k+1}(S,\nu t)|^m : 0 \leq \nu \leq 1\}.$$

PROOF. We have (c.f. J. Dieudonné (1960), p.186, Titchmarsh (1939), p.368)

$$q(S,t) = \sum_{j=0}^{k} q_{0,j}(S,0)\frac{t^j}{j!} + \frac{t^{k+1}}{(k+1)!}\int_0^1 (k+1)(1-\nu)^k q_{0,k+1}(S,\nu t)d\nu,$$

provided that the integral converges. Hence the left side of (4.2.3) is bounded by

$$\left(\frac{|t|^{k+1}}{(k+1)!}\right)^m E\left|\int_0^1 (k+1)(1-\nu)^k q_{0,k+1}(S,\nu t)d\nu\right|^m.$$

This obviously remains true even if the integral diverges for some values of S. An application of Lyapunov's inequality and Fubini's theorem completes the proof of (4.2.3) and a similar argument disposes of (4.2.4). □

We denote expectation and variance under $F(x-\theta)$ by E_θ and σ_θ^2 and under $F(x)$ by E_0 and σ_0^2. In the same way, the third and fourth cumulants of $(S-E_\theta S)/\sigma_\theta S$, multiplied by $N^{1/2}$ and N respectively, are denoted by κ_3^θ and κ_4^θ or by κ_3^0 and κ_4^0.

Now we can evaluate the necessary moments and cumulants.

LEMMA 4.2.3. *Let* $\{\delta_N\}$ *be a sequence of positive real numbers with* $\lim_{N\to\infty} \delta_N = 0$. *Let for some* N

(4.2.5) $$0 < \theta < \delta_N.$$

Suppose that positive constants ε *and* C *exist such that*

(4.2.6) $$\sup\{\int_{-\infty}^{\infty} |\psi_i(x+y)|^{5/i} f(x)dx : |y| \le \varepsilon\} \le C, \quad i = 1,\dots,5.$$

Then there exists $A > 0$ *depending on* N, θ *and* F *only through* $\{\delta_N\}$, ε *and* C *and such that*

(4.2.7)
$$|E_0 S - \{-\frac{N\theta}{2} E_0\psi_1^2(X_1) + \frac{N\theta^3}{24}[E_0\psi_2^2(X_1) - \frac{2}{3}E_0\psi_1^4(X_1)]\}| \le AN\theta^4,$$

$$|\sigma_0^2(S) - \{NE_0\psi_1^2(X_1) + \frac{N\theta^2}{36}[5E_0\psi_1^4(X_1) - 3E_0\psi_2^2(X_1) - 9(E_0\psi_1^2(X_1))^2]\}| \le AN\theta^3,$$

$$|\kappa_3^0 - \frac{\theta}{2(E_0\psi_1^2(X_1))^{3/2}}[3(E_0\psi_1^2(X_1))^2 - E_0\psi_1^4(X_1)]| \le A\theta^2,$$

$$|\kappa_4^0 - \{\frac{E_0\psi_1^4(X_1)}{(E_0\psi_1^2(X_1))^2} - 3\}| \le A\theta.$$

PROOF. As $E_0S = NE_0S_1$ we have to find E_0S_1. Expansion around $\theta = 0$ shows that

(4.2.8) $$S_1 = \frac{1}{\theta}[\zeta_0(X_1-\theta) - \zeta_0(X_1)] = -\zeta_1(X_1) + \frac{\theta}{2}\zeta_2(X_1) +$$

$$-\frac{\theta^2}{6}\zeta_3(X_1) + \frac{\theta^3}{24}\zeta_4(X_1) + \frac{1}{\theta}[\zeta_0(X_1-\theta) - \sum_{j=0}^{4}\zeta_j(X_1)\frac{(-\theta)^j}{j!}].$$

Application of lemma 4.2.2 with $q(x,t) = \zeta_0(x-t)$ gives

$$E_0|\frac{1}{\theta}[\zeta_0(X_1-\theta) - \sum_{j=0}^{4}\zeta_j(X_1)\frac{(-\theta)^j}{j!}]| \le$$

$$\le \frac{\theta^4}{5!}\sup\{E_0|\zeta_5(X_1-\nu\theta)|;\ 0 \le \nu \le 1\}.$$

Conditions (4.2.5) and (4.2.6) ensure that $E_0|\psi_i^{5/i}(X_1-\theta)|$ is bounded for

$i = 1,\dots,5$ and for N sufficiently large. Hence by Hölder's inequality and the triangle inequality, we get from (4.2.2) that $|E_0\zeta_i^{5/i}(X_1-\theta)|$ is also bounded. This shows that the expectation of the last term in (4.2.8) is $O(\theta^4)$.

The remaining terms in (4.2.8) are dealt with in the following way. First we note that $E_0\zeta_{(2i-1)}(X_1) = 0$, $i = 1,2,3$, by virtue of the symmetry of f. From (4.2.6) we have $\int_{-\infty}^{\infty} |f^{(i)}(x)|dx < \infty$, $i = 0,1,\dots,5$. Hence by lemma 4.2.1, $\lim_{x\to\pm\infty} f^{(i-1)}(x) = 0$, $i = 1,\dots,5$, where $f^{(0)} = f$, and thus $E_0\psi_i(X_1) = \int_{-\infty}^{\infty} f^{(i)}(x)dx = 0$, $i = 1,\dots,5$. These two steps show that

$$E_0S_1 = -\frac{\theta}{2}E_0\psi_1^2(X_1) + \frac{\theta^3}{24}E_0[-3\psi_2^2(X_1)-4\psi_1(X_1)\psi_3(X_1)+12\psi_1^2(X_1)\psi_2(X_1) + \\ - 6\psi_1^4(X_1)] + O(\theta^4).$$

By partial integration and another application of lemma 4.2.1 we observe that $E_0\psi_1^2(X_1)\psi_2(X_1) = 2/3\ E_0\psi_1^4(X_1)$ and $E_0\psi_1(X_1)\psi_3(X_1) = 2/3\ E_0\psi_1^4(X_1)-E_0\psi_2^2(X_1)$. With this the statement for E_0S is proved.

The formula for $\sigma_0^2(S)$ is proved analogously; the only point that needs some explanation is the expansion of S_1^2.

$$S_1^2 = -\zeta_1(X_1)S_1 + \frac{\theta}{2}\zeta_2(X_1)S_1 - \frac{\theta^2}{6}\zeta_3(X_1)S_1 + \\ + \frac{1}{\theta}[\zeta_0(X_1-\theta) - \sum_{j=0}^{3}\zeta_j(X_1)\frac{(-\theta)^j}{j!}]S_1 = \\ = \zeta_1^2(X_1) - \theta\zeta_1(X_1)\zeta_2(X_1) + \frac{\theta^2}{3}\zeta_1(X_1)\zeta_3(X_1) + \frac{\theta^2}{4}\zeta_2^2(X_1) + \\ + \frac{1}{\theta}[\zeta_0(X_1-\theta)-\sum_{j=0}^{3}\zeta_j(X_1)\frac{(-\theta)^j}{j!}][-\zeta_1(X_1) + \frac{1}{\theta}\{\zeta_0(X_1-\theta)-\zeta_0(X_1)\}] + \\ + \frac{1}{\theta}[\zeta_0(X_1-\theta)-\sum_{j=0}^{2}\zeta_j(X_1)\frac{(-\theta)^j}{j!}]\frac{\theta}{2}\zeta_2(X_1) + \\ - \frac{1}{\theta}[\zeta_0(X_1-\theta)-\sum_{j=0}^{1}\zeta_j(X_1)\frac{(-\theta)^j}{j!}]\frac{\theta^2}{6}\zeta_3(X_1).$$

By expanding S_1^2 in this way it is possible to deal with the remainder terms without using more than (4.2.6).

As concerns κ_3^0 and κ_4^0, we have by definition

$$\kappa_3^0 = N^{1/2}[\sum_{j=1}^{N} E_0(S_j-E_0S_j)^3]/[\sum_{j=1}^{N} E_0(S_j-E_0S_j)^2]^{3/2} =$$

$$= E_0(S_1-E_0S_1)^3/\sigma_0^3(S_1),$$

$$\kappa_4^0 = N[\sum_{j=1}^{N}\{E_0(S_j-E_0S_j)^4-3\sigma_0^4(S_j)\}]/[\sum_{j=1}^{N}\sigma_0^2(S_j)]^2 =$$

$$= E_0(S_1-E_0S_1)^4/\sigma_0^4(S_1)-3.$$

The third and fourth central moment of S_1 are found in the same way as E_0S_1 and $\sigma_0^2(S_1)$. We get $E_0(S_1-E_0S_1)^3 = (\theta/2)[3\{E_0\psi_1^2(X_1)\}^2-E_0\psi_1^4(X_1)] + O(\theta^2)$ and $E_0(S_1-E_0S_1)^4 = E_0\psi_1^4(X_1) + O(\theta)$. From this the required expressions for κ_3^0 and κ_4^0 follow immediately. □

Let R_0 and ρ_0 be the d.f. and c.f., respectively, of $(S-E_0S)/\sigma_0(S)$ under $F(x)$. Let $\tilde{R}_0$ be the Edgeworth expansion to $o(N^{-1})$ for R_0 (cf. (3.1.1)) and let $\tilde{\rho}_0$ be the Fourier transform of $\tilde{R}_0'$. Define

$$(4.2.9)\qquad \tilde{R}_0^*(x) = \Phi(x)+\phi(x)\Big[-\frac{\theta}{12N^{1/2}(E_0\psi_1^2(X_1))^{3/2}}\{3(E_0\psi_1^2(X_1))^2 + - E_0\psi_1^4(X_1)\}(x^2-1) - \frac{1}{24N}\Big(\frac{E_0\psi_1^4(X_1)}{(E_0\psi_1^2(X_1))^2} - 3\Big)(x^3-3x)\Big].$$

The following lemma gives an upper bound for $|R_0-\tilde{R}_0^*|$.

LEMMA 4.2.4. *Let* $\{\delta_N\}$ *be a sequence of positive real numbers with* $\lim_{N\to\infty}\delta_N = 0$ *and suppose that* (4.2.5) *and* (4.2.6) *hold. Suppose in addition that there exist positive constants* c *and* η *such that*

$$(4.2.10)\qquad |\frac{d}{dy}\psi_1(F^{-1}(y))| \geq c, \qquad y \in \tau \subset (0,1),$$

where the interval τ *has length at least* η. *Then there exists* $A > 0$ *depending on* N, θ *and* F *only through* $\{\delta_N\}$, ε, C, c *and* η *and such that*

(4.2.11) $$\sup_x |R_0(x)-\tilde{R}_0^*(x)| \le A\{N^{-3/2}+N^{-1/2}\theta^2\}.$$

PROOF. Upon inserting the expressions for κ_3^0 and κ_4^0 from (4.2.7) in the Edgeworth expansion $\tilde{R}_0$, we obtain in view of (4.2.9) that

$$\sup_x |\tilde{R}_0(x)-\tilde{R}_0^*(x)| \le A\{N^{-3/2}+N^{-1/2}\theta^2\}.$$

Hence it remains to show that $\sup_x |R_0(x)-\tilde{R}_0(x)| \le A\{N^{-3/2}+N^{-1/2}\theta^2\}$. As the proof of this result is almost standard (cf. Feller (1966)), we do not give many details here. The emphasis will be on showing that, under condition (4.2.10), the distribution of S_1 satisfies a strong non-lattice condition, i.e. that the modulus of its c.f. remains bounded away from 1 outside a neighbourhood of 0.

From lemma 3.2.1 it is clear that it is sufficient to prove that

$$\int_{-T}^{T} \left|\frac{\rho_0(t)-\tilde{\rho}_0(t)}{t}\right| dt \le A\{N^{-3/2}+N^{-1/2}\theta^2\},$$

where $T = bN^{1/2}\min(N,\theta^{-2})$, for some $b > 0$. In analogy to the rank test case we bound the integral by

(4.2.12) $$\int_{|t|\le c_1N^{1/2}} \left|\frac{\rho_0(t)-\tilde{\rho}_0(t)}{t}\right| dt + \int_{c_1N^{1/2}\le|t|\le bN^{1/2}\min(N,\theta^{-2})} \left|\frac{\rho_0(t)}{t}\right| dt +$$

$$+ \int_{c_1N^{1/2}\le|t|\le bN^{1/2}\min(N,\theta^{-2})} \left|\frac{\tilde{\rho}_0(t)}{t}\right| dt,$$

and show that these three integrals are sufficiently small.

First consider the interval $|t| \le c_1N^{1/2}$. Let $w_0(t)$ be the c.f. of $S_1-E_0S_1$ under H_0, then $\rho_0(t) = \exp[N \log w_0\{t/(N^{1/2}\sigma_0(S_1))\}]$. From (4.2.10) it follows that $E_0\psi_1^2(X_1) \ge \bar{c}$, for some $\bar{c} > 0$, depending on c and η. According to (4.2.7), $\sigma_0^2(S_1) = E_0\psi_1^2(X_1) + O(\theta^2)$ and hence $\sigma_0^2(S_1) \ge \bar{c}/2$ for N sufficiently large. This proves that for some $c_1 > 0$, $\mathrm{Re}[w_0\{t/(N^{1/2}\sigma_0(S_1))\}] \ge \frac{1}{2}$ for $|t| \le c_1N^{1/2}$. Therefore $\log w_0\{t/(N^{1/2}\sigma_0(S_1))\}$, and hence $\log \rho_0(t)$, can be expanded around $t = 0$, for $|t| \le c_1N^{1/2}$. Doing so, we find that $\rho_0(t) = \tilde{\rho}_0(t)+M(t)$, where $|M(t)| \le N^{-3/2}|t|^5Q(|t|)\exp(-t^2/4)$ for

$|t| \le c_1 N^{1/2}$, and Q is a polynomial with coefficients that depend only on $\{\delta_N\}$, ε, C, c and η. From this it is clear that the first integral in (4.2.12) is $O(N^{-3/2})$.

Just as in the sketch of the proof of theorem 3.2.1, one can show that

$$\int_{|t|\ge \log(N+1)} \left|\frac{\tilde{\rho}_0(t)}{t}\right| dt$$

is sufficiently small. Hence it remains to show that the second integral in (4.2.12) is $O(N^{-3/2}+N^{-1/2}\theta^2)$. Using the fact that ζ_0 is an even and ζ_1 is an odd function, we find

(4.2.13) $$|w_0(t)| = \Bigg|\int_0^\infty [e^{it\zeta_1(x)} + e^{-it\zeta_1(x)}]e^{\frac{it}{\theta}[\zeta_0(x-\theta)-\zeta_0(x)+\theta\zeta_1(x)]} +$$

$$+ e^{it\zeta_1(x)}[e^{\frac{it}{\theta}[\zeta_0(x+\theta)-\zeta_0(x)-\theta\zeta_1(x)]} +$$

$$- e^{\frac{it}{\theta}[\zeta_0(x-\theta)-\zeta_0(x)+\theta\zeta_1(x)]}]f(x)dx\Bigg| \le$$

$$\le 2\int_0^\infty |\cos t\psi_1(x)| f(x)dx + \int_0^\infty \left| e^{it[\frac{\zeta_0(x+\theta)-\zeta_0(x-\theta)}{\theta} - 2\zeta_1(x)]} - 1\right| f(x)dx.$$

The last integral in (4.2.13) is less than or equal to

(4.2.14) $$|t|\int_0^\infty \left|\frac{\zeta_0(x+\theta)-\zeta_0(x-\theta)}{\theta} - 2\zeta_1(x)\right| f(x)dx \le$$

$$\le \frac{|t|\theta^2}{12} E_0|2\zeta_3(X_1+\nu\theta)| \le \theta^2|t|\tilde{C},$$

for $|\nu| \le 1$ and some $\tilde{C} > 0$. The first inequality is a consequence of lemma 4.2.2, the second of (4.2.2) and (4.2.6).

Next we investigate the behaviour of $\psi_1(F^{-1}(y))$ on τ. This interval has the form $\tau = (a, a+\tilde{\eta})$, where $0 < a < 1-\tilde{\eta}$ and $\eta \le \tilde{\eta} < 1$. Let δ and $\tilde{c}$ be constants with $0 < \delta < \frac{\pi}{2}$ and $\tilde{c} > 0$ and define, for a fixed $|t| \ge \tilde{c}$, the set $\tilde{\tau}$ by

$$\tilde{\tau} = \{y \mid \exists \text{ integer } k,\ k\pi+\delta \le t\psi_1(F^{-1}(y)) \le (k+1)\pi-\delta\} \cap \tau.$$

Let λ be Lebesgue measure and denote $\lambda(\tilde{\tau})$ as $\tilde{\lambda}$. We shall show that $\tilde{\lambda} \geq \eta/2$ for δ sufficiently small and depending only on c, $\tilde{c}$, C and η. Obviously, we have $\lambda(\tau \cap \tilde{\tau}^c) = \tilde{\eta}-\tilde{\lambda}$ and therefore

$$(4.2.15) \qquad \lambda(\{y|a + \frac{\tilde{\eta}-\tilde{\lambda}}{4} \leq y \leq a+\tilde{\eta} - \frac{\tilde{\eta}-\tilde{\lambda}}{4}\} \cap \tilde{\tau}^c) \geq \frac{\tilde{\eta}-\tilde{\lambda}}{2}.$$

Condition (4.2.10) implies that $|\frac{d}{dy} t\psi_1(F^{-1}(y))| \geq |t|c$ on τ. From the differentiability conditions on f and the assumption that f(x) is positive for all x, it follows that $\frac{d}{dy}\psi_1(F^{-1}(y))$ is continuous on τ. Hence (4.2.10) also implies that $\psi_1(F^{-1}(y))$ is monotone on τ. This gives, together with (4.2.15), that $t\psi_1(F^{-1}(y))$ traverses at least $[\frac{(\tilde{\eta}-\tilde{\lambda})c|t|}{4\delta}]$ intervals of the form $(k\pi-\delta, k\pi+\delta)$ as y increases from $a+(\tilde{\eta}-\tilde{\lambda})/4$ to $a+\tilde{\eta}-(\tilde{\eta}-\tilde{\lambda})/4$. Here [z] stands for the integer part of z. Because $t\psi_1(F^{-1}(y))$ is monotone on τ

$$(4.2.16) \qquad |t||\psi_1(F^{-1}(a+\tilde{\eta} - \frac{\tilde{\eta}-\tilde{\lambda}}{4}))-\psi_1(F^{-1}(a + \frac{\tilde{\eta}-\tilde{\lambda}}{4}))| \geq \{[\frac{(\tilde{\eta}-\tilde{\lambda})c|t|}{4\delta}] -1\}\pi.$$

This inequality leads to the desired lower bound for $\tilde{\lambda}$. Take $\delta = \min\{\eta c\tilde{c}/16, c\pi\eta^2/(128C)\}$. If $(\tilde{\eta}-\tilde{\lambda})c\tilde{c} \leq 8\delta$, then $\tilde{\lambda} \geq \tilde{\eta}-8\delta/(c\tilde{c}) \geq \tilde{\eta}-\eta/2 \geq \eta/2$. If $(\tilde{\eta}-\tilde{\lambda})c\tilde{c} \geq 8\delta$, (4.2.16) implies that

$$|\psi_1(F^{-1}(a+\tilde{\eta} - \frac{\tilde{\eta}-\tilde{\lambda}}{4}))-\psi_1(F^{-1}(a + \frac{\tilde{\eta}-\tilde{\lambda}}{4}))| \geq \frac{(\tilde{\eta}-\tilde{\lambda})c\pi}{8\delta},$$

and hence

$$(4.2.17) \qquad \int_0^1 |\psi_1(F^{-1}(y))|dy \geq \frac{(\tilde{\eta}-\tilde{\lambda})^2 c\pi}{32\delta}.$$

According to (4.2.6) the integral in (4.2.17) is at most C and therefore $\tilde{\lambda} \geq \tilde{\eta}-\{32\delta C/(c\pi)\}^{1/2} \geq \eta/2$.

In view of the above, there exists a positive constant $\tilde{\delta}$, depending only on c, $\tilde{c}$, C and η and such that, for all $|t| \geq \tilde{c}$, we have $\lambda(\tilde{\tau}) = \tilde{\lambda} \geq \eta/2$ and on $\tilde{\tau}$

$$|\cos\{t\psi_1(F^{-1}(y))\}| \leq 1-\tilde{\delta}.$$

Hence, for $|t| \geq \tilde{c}$,

$$(4.2.18) \qquad 2\int_0^\infty |\cos\{t\psi_1(x)\}|f(x)dx = \int_0^1 |\cos\{t\psi_1(F^{-1}(y))\}|dy \leq 1 - \frac{\tilde{\delta}\eta}{2}.$$

Combination of (4.2.14) and (4.2.18) leads to

$$|w_0\{t/(N^{1/2}\sigma_0(S_1))\}| \le 1 - \frac{\tilde{\delta}\eta}{2} + \theta^2|t|\tilde{C}/(N^{1/2}\sigma_0(S_1)),$$

for $|t| \ge \tilde{c}N^{1/2}\sigma_0(S_1)$. Now $\sigma_0^2(S_1)$ is bounded above because of (4.2.6). Therefore, $\tilde{c}\sigma_0(S_1) \le c_1$ if $\tilde{c}$ is chosen sufficiently small. On the other hand, $\sigma_0^2(S_1) \ge \bar{c}/2$. Together this gives

$$|w_0\{t/(N^{1/2}\sigma_0(S_1))\}| \le 1 - \frac{\tilde{\delta}\eta}{2} + N^{-1/2}\theta^2|t|\tilde{C}(\frac{\bar{c}}{2})^{-1/2},$$

for $|t| \ge c_1N^{1/2}$. Hence, there exists a constant $b > 0$ such that for $c_1N^{1/2} \le |t| \le bN^{1/2}\theta^{-2}$ we have $|w_0\{t/(N^{1/2}\sigma_0(S_1))\}| \le 1-\tilde{\delta}\eta/4$. But then $|\rho_0(t)| \le (1-\tilde{\delta}\eta/4)^N$ and

$$\int_{c_1N^{1/2}\le|t|\le bN^{1/2}\min(N,\theta^{-2})} |\frac{\rho_0(t)}{t}|dt \le \frac{b}{c_1}N(1-\frac{\tilde{\delta}\eta}{4})^N = O(N^{-3/2}).$$

□

Let R_θ be the d.f. of $(S-E_\theta S)/\sigma_\theta(S)$ under $F(x-\theta)$. If X_1 has d.f. $F(x-\theta)$, the r.v. $Y_1 = X_1-\theta$ clearly has d.f. $F(x)$. Furthermore,

$$S_1 = (1/\theta)\log\{f(X_1-\theta)/f(X_1)\} = (1/(-\theta))\log\{f(Y_1-(-\theta))/f(Y_1)\}.$$

Hence by changing θ into $(-\theta)$ in the right side of (4.2.9), we get an expansion $\tilde{R}_\theta^*$ for R_θ, such that $\sup_x |R_\theta(x)-\tilde{R}_\theta^*(x)| \le A\{N^{-3/2}+N^{-1/2}\theta^2\}$ under the conditions of lemma 4.2.4.

From $\tilde{R}_0^*$ and $\tilde{R}_\theta^*$ an expansion $\tilde{\pi}_S(\theta)$ for the envelope power $\pi_S(\theta)$ can be found easily. Define

(4.2.19) $$\eta_1 = N^{1/2}\theta[E_0\psi_1^2(X_1)]^{1/2},\ \eta_2 = \frac{E_0\psi_1^4(X_1)}{[E_0\psi_1^2(X_1)]^2},\ \eta_3 = \frac{E_0\psi_2^2(X_1)}{[E_0\psi_1^2(X_1)]^2},$$

(cf. (3.4.24) and (3.4.26)) and

(4.2.20) $$\tilde{\pi}_S(\theta) = 1-\Phi(u_\alpha-\eta_1)+\phi(u_\alpha-\eta_1)[\frac{\eta_1(\eta_2-3)}{24N}(u_\alpha^2-\eta_1u_\alpha-1) + \frac{\eta_1^3}{72N}(2\eta_2-3\eta_3)].$$

LEMMA 4.2.5. *Let* $\{\delta_N\}$ *be a sequence of positive real numbers with* $\lim_{N\to\infty} \delta_N = 0$ *and assume that* (4.2.5), (4.2.6) *and* (4.2.10) *hold and suppose in addition that* $\varepsilon' \le \alpha \le 1-\varepsilon'$ *for some constant* $\varepsilon' > 0$. *Then there exists* $A > 0$, *depending on* N, θ, α *and* F *only through* $\{\delta_N\}$, ε, C, c, η *and* ε' *and such that*

$$(4.2.21)\qquad |\pi_S(\theta)-\tilde{\pi}_S(\theta)| \le A\{N^{-3/2}+N^{-1/2}\theta^2\}.$$

PROOF. From (4.2.9) it follows that the critical value ξ_α satisfies

$$(4.2.22)\qquad \xi_\alpha = u_\alpha - \frac{\eta_1(\eta_2-3)}{12N}(u_\alpha^2-1) + \frac{(\eta_2-3)}{24N}(u_\alpha^3-3u_\alpha) + O(N^{-3/2}+N^{-1/2}\theta^2).$$

Furthermore, we have for $\pi_S(\theta)$

$$(4.2.23)\qquad \pi_S(\theta) = 1-R_\theta([\xi_\alpha + \frac{E_0S-E_\theta S}{\sigma_0(S)}]\frac{\sigma_0(S)}{\sigma_1(S)}).$$

E_0S and $\sigma_0(S)$ are given in (4.2.7), $E_\theta S$ and $\sigma_\theta(S)$ follow by changing θ into $(-\theta)$ in these expressions. Application of this and (4.2.22) to (4.2.23) leads to (4.2.21), with an additional remainder term of order $N^{1/2}\theta^4\phi(N^{1/2}\theta)$. This term can be omitted, as $N^{1/2}\theta^4\phi(N^{1/2}\theta) = O(N^{-1/2}\theta^2)$. □

REMARKS. 1. Conditions (4.2.6) and (4.2.10) determine a class of d.f.'s F for which expansion (4.2.20) holds uniformly. If we restrict attention to a fixed F, condition (4.2.6) can be weakened to

$$\limsup_{y\to 0} \int_{-\infty}^{\infty} |\psi_i^{5/i}(x+y)|f(x)dx < \infty, \qquad i = 1,\ldots,5.$$

Furthermore, condition (4.2.10) can be omitted: if no uniformity is required, we only need that $\psi_1(F^{-1}(y))$ is non-constant on (0,1). This condition is always satisfied, since $\psi_1(F^{-1}(y))$ is constant on (0,1) only for uniform d.f.'s. But these d.f.'s are already ruled out by the fact that their density is not everywhere positive and differentiable.

2. We have treated location alternatives with $\theta = o(1)$, since we shall need the results for such θ in section 5.2. The typical case of interest, however, remains the case of contiguous location alternatives, where $\theta = O(N^{-1/2})$. Then η_1, and hence $1/\{1-\Phi(u_\alpha-\eta_1)\}$, are $O(1)$.

As announced at the beginning of this section, we shall also give a power-expansion for the LMP test based on $S^* = \sum_{j=1}^{N} S_j^*$, where $S_j^* = -\psi_1(X_j)$. Under $F(x)$ we have

$$E_0(S_1^*)^{2k} = E_0\psi_1^{2k}(X_1),\ E_0(S_1^*)^{2k-1} = 0,\ \text{for } k = 1,2.$$

Under $F(x-\theta)$ the necessary moments can be found by using

$$S_1^* = -\psi_1(X_1) = -\zeta_1(X_1) = -\zeta_1(Y_1) - \theta\zeta_2(Y_1) - \frac{\theta^2}{2}\zeta_3(Y_1) +$$

$$-\frac{\theta^3}{6}\zeta_4(Y_1) + [-\zeta_1(X_1) + \sum_{j=0}^{3}\zeta_{j+1}(Y_1)\frac{\theta^j}{j!}],$$

where $Y_1 = X_1 - \theta$ has d.f. $F(x)$. Proceeding in this way, we obtain expansions similar to (4.2.9), which are justified in a manner analogous to lemma 4.2.4. The final result is the following expansion $\tilde{\pi}_{S^*}(\theta)$ for the power $\pi_{S^*}(\theta)$:

$$(4.2.24)\qquad \tilde{\pi}_{S^*}(\theta) = 1-\Phi(u_\alpha-\eta_1)+\phi(u_\alpha-\eta_1)[\frac{\eta_1(\eta_2-3)}{24N}(u_\alpha^2-\eta_1u_\alpha-1) + \frac{\eta_1^3}{72N}(5\eta_2-12\eta_3+9)].$$

Under the assumptions of lemma 4.2.5 we have $|\pi_{S^*}(\theta)-\tilde{\pi}_{S^*}(\theta)| \le A\{N^{-3/2} + N^{-1/2}\theta^2\}$. Comparison of (4.2.20) and (4.2.24) gives

$$\tilde{\pi}_S(\theta)-\tilde{\pi}_{S^*}(\theta) = \frac{\eta_1^3}{24N}(3\eta_3-\eta_2-3)\phi(u_\alpha-\eta_1) =$$

$$= \frac{N^{1/2}\theta^3\sigma_0^2(\psi_1'(X_1))}{8[E_0\psi_1^2(X_1)]^{1/2}}\phi(u_\alpha-\eta_1) \ge 0,$$

where equality only occurs if $\psi_1'(X_1)$ is constant a.s., i.e. if X_1 is normally distributed. The fact that $\tilde{\pi}_S(\theta) = \tilde{\pi}_{S^*}(\theta)$ here is obvious, since S and S^* are equivalent if X_1 comes from $\Phi(x-\theta)$.

4.3. PERMUTATION TESTS

We shall start by showing how permutation tests can be derived in a natural way (cf. Lehmann (1959)). Consider again the one sample problem: $X_1, X_2, \ldots, X_N$ are i.i.d. r.v.'s. Under the hypothesis they come from a d.f. F, which has a density f that is positive on R^1 and symmetric around zero, under the alternative they come from a d.f. G which has a density g that is not symmetric around zero. Let $X = (X_1, \ldots, X_N)$. In the previous section we considered LMP tests for this problem, which, of course, are not distribution-free. Here we shall restrict attention to distribution-free tests and search for the most powerful one in this restricted class.

If ϕ is a distribution-free test, then ϕ must have the same size α for all $F \in \mathcal{P}_X$, the family of all d.f.'s with a continuous symmetric density, i.e. ϕ has to be similar with respect to $\mathcal{P}_X$. A concept related to similarity is Neyman Structure (NS): a test ϕ has NS with respect to a statistic T if T is sufficient for X with respect to $\mathcal{P}_X$ and $E(\phi(X)|T) = \alpha$ a.s. under $\mathcal{P}_X$. If ϕ has NS, it is similar; the converse holds if T is also complete with respect to $\mathcal{P}_X$. Let $Z = (Z_1, \ldots, Z_N)$ denote again the vector of absolute order statistics of the X_j. As Z is sufficient and complete with respect to $\mathcal{P}_X$, the class of all similar tests for the one sample problem coincides with the class of all tests having $E(\phi(X)|Z) = \alpha$ a.s. under H_0. This last condition can also be stated as

$$(4.3.1) \qquad (2^N N!)^{-1} \sum_{y \in S(x)} \phi(y) = \alpha \text{ a.s.},$$

where S(x) is the set of all $y = (y_1, \ldots, y_N)$, giving rise to the same z as x.

Any test satisfying (4.3.1) is called a permutation test. In particular, every rank test satisfies (4.3.1) and therefore rank tests form a subclass of the family of permutation tests. This implies that the most powerful permutation test for a certain alternative is always at least as good as the most powerful rank test for that alternative. As they both possess the desirable property of being distribution-free, permutation tests are superior to rank tests. The only reason to prefer rank tests over permutation tests is of a practical nature: rank tests are much easier to apply.

In view of this relation between rank tests and permutation tests, it seems interesting to make deficiency comparisons between them. To this end we shall derive in this section a power expansion for the most powerful permutation test. First we shall derive the explicit form of its test-statistic if we consider a fixed alternative, under which the X_i come from a d.f.

G. Then we have for the conditional power

$$E(\phi(X)|Z) = \sum_{y\in S(x)} \phi(y) \frac{P(Z=z,R=r,\text{sign } Y=\text{sign } y)}{P(Z=z)} =$$

$$= \sum_{y\in S(x)} \phi(y) \frac{\prod_{j=1}^{N} g(y_j)}{\sum_{y\in S(x)} \prod_{j=1}^{N} g(y_j)},$$

where $R = (R_1,\ldots,R_N)$ is the vector of ranks for $(|Y_1|,\ldots,|Y_N|)$ and sign $Y = (\text{sign } Y_1,\ldots,\text{sign } Y_N)$. Conditionally under (4.3.1), $E(\phi(X)|Z)$ is maximal if

$$\phi(x) = \begin{cases} 1 & \text{for } \prod_{j=1}^{N} g(x_j) \geq c(Z), \\ 0 & \text{otherwise,} \end{cases}$$

where $c(Z)$ depends on Z only. As $\prod_{j=1}^{N} f(x_j)$ is constant over $S(x)$, the most powerful permutation test rejects for large values of $\prod_{j=1}^{N}\{g(x_j)/f(x_j)\}$, conditionally under Z.

We now restrict attention to contiguous location alternatives $g(x) = f(x-\theta)$, with $0 \leq \theta \leq DN^{-1/2}$, for some positive constant D. Then the most powerful permutation test can be based on $(1/\theta)\sum_{j=1}^{N} \log\{f(X_j-\theta)/f(X_j)\}$. As $\sum_{j=1}^{N}\log\{f(-|X_j|-\theta)/f(-|X_j|)\} = \sum_{j=1}^{N}\log\{f(Z_j+\theta)/f(Z_j)\}$ is constant given $Z = z$, we can equivalently use

$$\frac{1}{2\theta} \sum_{j=1}^{N} [\log \frac{f(X_j-\theta)}{f(X_j)} - \log \frac{f(-|X_j|-\theta)}{f(-|X_j|)}] = \frac{1}{2\theta} \sum_{X_j>0} \log \frac{f(X_j-\theta)}{f(X_j+\theta)}.$$

A drawback of the above test is the fact that it is only optimal against one particular alternative θ_1. For a composite hypothesis we may therefore prefer the LMP permutation test, which is based on

$$U = -\sum_{X_j>0} \psi_1(X_j),$$

with ψ_1 as defined in (4.2.1). The relation between U and the statistic of the most powerful permutation test is

$$\frac{1}{2\theta} \sum_{X_j>0} \log \frac{f(X_j-\theta)}{f(X_j+\theta)} = U - \frac{\theta^2}{12} \sum_{X_j>0} [\zeta_3(X_j-\nu_1\theta)+\zeta_3(X_j+\nu_2\theta)],$$

with $0 \le \nu_1, \nu_2 \le 1$ and ζ_3 as defined in (4.2.1). This relation is the same as between S and S^* in the previous section. Note, however, that $S-S^* = O(\theta)$ instead of $O(\theta^2)$ as is the case here. This explains why $\pi_{S^*}(\theta)-\pi_S(\theta) = O(N^{-1})$, while the difference in the power of the two permutation tests is $O(N^{-5/4})$, as is shown in lemma 4.3.5. Hence, as concerns deficiencies, both tests perform equally well.

We shall proceed to give a power expansion for the test based on U. The unconditional distribution of U can be found easily, but since the critical value depends on Z, this does not lead to the power. We have to go through the following procedure: the distribution of U, conditionally on Z, has to be found, whereupon the conditional power can be evaluated as a function of Z. Then the unconditional power is found by taking the expectation with respect to Z.

In order to find the conditional distribution of U, it is useful to represent U as $U = -\sum_{j=1}^N \psi_1(Z_j)V_j$, where $V_j = 1$ if the X_i corresponding to Z_j is positive, and $V_j = 0$ otherwise. With this representation we can apply various results from Chapter 3. Just as in this chapter we have, conditionally on Z, the following situation: the $-\psi_1(Z_j)$ are constant and $V_1,\ldots,V_N$ are independent with

$$(4.3.2) \qquad P_\theta(V_j=1) = 1-P_\theta(V_j=0) = \frac{f(Z_j-\theta)}{f(Z_j+\theta)+f(Z_j-\theta)},\ j = 1,\ldots,N.$$

We introduce the following notation

$$A_j = -\psi_1(Z_j), \quad a_j = -\psi_1(z_j),$$

$$(4.3.3) \qquad P_j = P_\theta(V_j=1|Z),\ p_j = P_\theta(V_j=1|z),$$

$$Q_j = 2P_j-1,\ q_j = 2p_j-1,$$

and A, a, P, p, Q, q are the corresponding N-dimensional vectors. Conditionally under $Z = z$, 2U has variance

$$(4.3.4) \qquad \tau^2 = \sum_{j=1}^N (1-q_j^2)a_j^2,$$

and the statistic $2(U - \sum_{j=1}^{N} a_j p_j)/\tau$ has mean 0 and variance 1. Its third and fourth cumulants, multiplied by $N^{1/2}$ and N respectively, are given by

$$(4.3.5)\qquad \kappa_3 = -2N^{1/2}\Big(\sum_{j=1}^{N} q_j(1-q_j^2)a_j^3\Big)/\tau^3,\ \kappa_4 = -2N\Big(\sum_{j=1}^{N}(1-q_j^2)(1-3q_j^2)a_j^4\Big)/\tau^4.$$

Define

$$(4.3.6)\qquad R_\theta(x|z) = P_\theta\left(\frac{2(U - \sum_{j=1}^{N} a_j p_j)}{\tau} \le x|z\right),$$

$$\gamma(\varepsilon,\zeta,q) = \lambda\{x|\exists \text{ integer } j\ |x-a_j| < \zeta,\ |q_j| \le 1-\varepsilon\},$$

where $0 < \varepsilon < 1/2$, $\zeta > 0$ and λ is Lebesgue measure.

The following lemma supplies an expansion for $R_\theta(x|z)$. From this point on, summation always runs from $j = 1$ to $j = N$, unless stated otherwise.

LEMMA 4.3.1. *Let* z *and* θ *be such that there exist positive numbers* c, C, δ *and* ε *for which the following conditions hold*

$$\frac{1}{N}\sum(1-q_j^2)a_j^2 \ge c,\ \frac{1}{N}\sum a_j^4 \le C,\ \gamma(\varepsilon,\zeta,q) \ge \delta N\zeta,$$

for some $\zeta \ge N^{-3/2}\log N$. *Then there exists* $A > 0$, *depending on* N, z *and* θ *only through* c, C, δ *and* ε *and such that*

$$(4.3.7)\qquad R_\theta(x|z) = \Phi(x)+\phi(x)\Big\{\frac{\sum q_j a_j^3}{3(NB)^{3/2}}(x^2-1) + \frac{\sum a_j^4}{12(NB)^2}(x^3-3x)\Big\} + R,$$

where B *is an arbitrary positive constant and*

$$(4.3.8)\qquad |R| \le A\{N^{-5/4}+N^{-5/2}|\sum q_j a_j^3||\tau^2-NB|+N^{-3/2}|\sum q_j^3 a_j^3| +$$
$$+ N^{-2}|\tau^2-NB|+N^{-2}|\sum q_j^2 a_j^4|+N^{-2}|\sum q_j a_j^3|\}.$$

PROOF. Under the above conditions, theorem 3.2.1 can be applied to $R_\theta(x|z)$. This means that

$$(4.3.9)\qquad R_\theta(x|z) = \tilde{R}_\theta(x|z) + O(N^{-5/4}),$$

where $\tilde{R}_\theta$ is the Edgeworth expansion to $o(N^{-1})$ for R_θ, i.e.

$$(4.3.10) \qquad \tilde{R}_\theta(x|z) = \Phi(x)-\phi(x)\{\frac{N^{-1/2}\kappa_3}{6}(x^2-1) + \frac{N^{-1}\kappa_4}{24}(x^3-3x) + \frac{N^{-1}\kappa_3^2}{72}(x^5-10x^3+15x)\}.$$

From (4.3.5) we obtain

$$\kappa_3 = -2N^{1/2}\left(\frac{\sum q_j a_j^3 - \sum q_j^3 a_j^3}{(NB)^{3/2}}\right)\left(\frac{NB}{\tau^2}\right)^{3/2} =$$

$$= -2\,\frac{\sum q_j a_j^3}{NB^{3/2}} + O(N^{-2}|\sum q_j a_j^3|\;|\tau^2-NB| + N^{-1}|\sum q_j^3 a_j^3|),$$

$$\kappa_4 = -2N\left(\frac{\sum a_j^4 - 4\sum q_j^2 a_j^4 + 3\sum q_j^4 a_j^4}{(NB)^2}\right)\left(\frac{NB}{\tau^2}\right)^2 = -\frac{2\sum a_j^4}{NB^2} +$$

$$+ O(N^{-1}|\tau^2-NB| + N^{-1}\sum q_j^2 a_j^4),$$

$$\kappa_3^2 = \kappa_3\kappa_3 = O(1)O(N^{-1}|\sum q_j a_j^3| + N^{-1}|\sum q_j^3 a_j^3|) = O(N^{-1}|\sum q_j a_j^3|).$$

Substitution of these expressions in (4.3.10) leads, in combination with (4.3.9), to the desired result. □

REMARK. The expansion in (4.3.7) is formally correct for any choice of $B > 0$. In future applications we shall choose B such that the expression in (4.3.8) is small. This is achieved by taking $B = E_0\psi_1^2(X_1)$, which is the obvious thing to do in view of the definitions (4.3.3) and (4.3.4).

Let $\pi(\theta|z)$ be the conditional power of the test based on U and let α be the size of the test. In the next lemma an expansion for $\pi(\theta|z)$ is given.

LEMMA 4.3.2. *Let* z *and* θ *be such that the conditions of lemma 4.3.1 are satisfied and such that* $|\sum q_j a_j| \le c'N^{1/2}$ *for some constant* $c' > 0$. *If furthermore* $0 \le \theta \le DN^{-1/2}$, $\varepsilon' \le \alpha \le 1-\varepsilon'$ *for positive constants* D *and* ε', *we have*

$$\pi(\theta|z) = \tilde{\pi}(\theta|z) + R,$$

where

$$(4.3.11)\qquad \tilde{\pi}(\theta|z) = 1-\Phi(u_\alpha-\theta(NB)^{1/2})-\phi(u_\alpha-\theta(NB)^{1/2})\Big[\frac{\sum q_j a_j^3}{3(NB)^{3/2}}\{(u_\alpha + -\theta(NB)^{1/2})^2-1\}+\frac{\sum a_j^4}{12(NB)^2}(-3u_\alpha^2\theta(NB)^{1/2}+3u_\alpha\theta^2 NB-\theta^3(NB)^{3/2} + + 3\theta(NB)^{1/2}) + \frac{1}{2}\frac{\sum q_j^2 a_j^2}{NB}(u_\alpha-\theta(NB)^{1/2}) + \frac{1}{2}\frac{(\sum a_j^2-NB)}{NB}\theta(NB)^{1/2} + - \frac{1}{8}\frac{(\sum a_j^2-NB)^2}{(NB)^2}\theta(NB)^{1/2}\{3+\theta(NB)^{1/2}(u_\alpha-\theta(NB)^{1/2})\} + - \frac{\sum q_j a_j-\theta NB}{(NB)^{1/2}} - \frac{1}{2}\frac{(\sum q_j a_j-\theta NB)^2}{NB}(u_\alpha-\theta(NB)^{1/2}) + + \frac{1}{2}\frac{(\sum q_j a_j-\theta NB)(\sum a_j^2-NB)}{(NB)^{3/2}}\{1+\theta(NB)^{1/2}(u_\alpha-\theta(NB)^{1/2})\}\Big],$$

with B *an arbitrary positive constant and*

$$(4.3.12)\qquad |R| \le A\{N^{-5/4}+N^{-3}|\sum a_j^2-NB|^3+N^{-3/2}|\sum q_j a_j-\theta NB|^3+N^{-3/2}(\sum q_j^2 a_j^2)^{3/2} + + N^{-9/4}|\sum q_j a_j^3|^{3/2} + N^{-3/2}|\sum q_j^3 a_j^3| + N^{-2}|\sum q_j^2 a_j^4|\},$$

where $A > 0$ *depends on* N, α, θ *and* z *only through* c, C, δ, ε, c', D *and* ε'.

PROOF. The conditions of lemma 4.3.1 are satisfied for some $\theta \ge 0$. Therefore they must be satisfied also for $\theta = 0$, because this is the most favourable case, as then all q_j are 0. From (4.3.7) and (4.3.8) it follows that

$$(4.3.13)\qquad R_0(x|z) = \Phi(x)+\phi(x)\frac{\sum a_j^4}{12(NB)^2}(x^3-3x) + O(N^{-5/4}+N^{-2}|\sum a_j^2-NB|).$$

Since $\varepsilon' \le \alpha \le 1-\varepsilon'$, $u_\alpha = O(1)$ and therefore we derive from (4.3.13) that the conditional critical value ξ_α of the test based on $2(U-\sum p_j a_j)/\tau$, satisfies

$$(4.3.14)\qquad \xi_\alpha = u_\alpha - \frac{\sum a_j^4}{12(NB)^2}(u_\alpha^3-3u_\alpha) + O(N^{-5/4}+N^{-2}|\sum a_j^2-NB|).$$

The connection between $\pi(\theta|z)$ and ξ_α is $\pi(\theta|z) = P_\theta([2U-\sum a_j]/(\sum a_j^2)^{1/2} > \xi_\alpha|z)$. Together with (4.3.6) this gives

$$(4.3.15) \qquad 1-\pi(\theta|z) = R_\theta(\xi_\alpha(\frac{\sum a_j^2}{\tau^2})^{1/2} - \frac{\sum q_j a_j}{\tau}|z).$$

Using (4.3.14) we obtain

$$\xi_\alpha(\frac{\sum a_j^2}{\tau^2})^{1/2} - \frac{\sum q_j a_j}{\tau} = u_\alpha - \theta(NB)^{1/2} + \frac{1}{2} u_\alpha \frac{\sum q_j^2 a_j^2}{NB} +$$

$$- \frac{1}{12} (u_\alpha^3 - 3u_\alpha) \frac{\sum a_j^4}{(NB)^2} + \frac{1}{2}\theta(NB)^{1/2} \frac{\tau^2 - NB}{NB} +$$

$$- \frac{3}{8}\theta(NB)^{1/2} \frac{(\tau^2-NB)^2}{(NB)^2} - \frac{\sum q_j a_j - \theta NB}{(NB)^{1/2}} +$$

$$+ \frac{1}{2} \frac{(\sum q_j a_j - \theta NB)(\tau^2 - NB)}{(NB)^{3/2}} + R_1,$$

where

$$R_1 = \mathcal{O}\{N^{-2}(\sum q_j^2 a_j^2)^2 + N^{-2}|\sum q_j^2 a_j^2||\sum a_j^2 - NB| + N^{-2}\sum q_j^2 a_j^2 + N^{-5/4} +$$

$$+ N^{-2}|\sum a_j^2 - NB| + N^{-3}|\tau^2 - NB|^3 + N^{-5/2}|\sum q_j a_j - \theta NB|(\tau^2 - NB)^2\}.$$

By the following relations, R_1 can be simplified considerably

$$N^{-5/2}|\sum q_j a_j - \theta NB|(\tau^2-NB)^2 \le N^{-3/2}|\sum q_j a_j - \theta NB|^3 + N^{-3}|\tau^2 - NB|^3,$$

$$N^{-2}|\sum a_j^2 - NB| \le N^{-3/2} + N^{-3}|\sum a_j^2 - NB|^3,$$

$$N^{-2}\sum q_j^2 a_j^2 |\sum a_j^2 - NB| \le N^{-3/2}(\sum q_j^2 a_j^2)^{3/2} + N^{-3}|\sum a_j^2 - NB|^3,$$

$$N^{-3}(\sum q_j^2 a_j^2)^3 + N^{-2}(\sum q_j^2 a_j^2)^2 + N^{-2}\sum q_j^2 a_j^2 = \mathcal{O}(N^{-3/2}(\sum q_j^2 a_j^2)^{3/2}).$$

We also have

$$N^{-2}(\tau^2 - NB)^2 = N^{-2}(\sum a_j^2 - NB)^2 + \mathcal{O}(N^{-2}\sum q_j^2 a_j^2 |\sum a_j^2 - NB| + N^{-2}(\sum q_j^2 a_j^2)^2),$$

$$N^{-3/2}(\sum q_j a_j - \theta NB)(\tau^2 - NB) = N^{-3/2}(\sum q_j a_j - \theta NB)(\sum a_j^2 - NB) +$$

$$+ O(N^{-3/2}|\sum q_j a_j - \theta NB|\sum q_j^2 a_j^2) = N^{-3/2}(\sum q_j a_j - \theta NB)(\sum a_j^2 - NB) +$$

$$+ O(N^{-3/2}(\sum q_j^2 a_j^2)^{3/2} + N^{-3/2}|\sum q_j a_j - \theta NB|^3).$$

Application of these results leads to

$$\xi_\alpha\left(\frac{\sum a_j^2}{\tau^2}\right)^{1/2} - \frac{\sum q_j a_j}{\tau} = u_\alpha - \theta(NB)^{1/2} + \tfrac{1}{2}(u_\alpha - \theta(NB)^{1/2})\,\frac{\sum q_j^2 a_j^2}{NB} + \tag{4.3.16}$$

$$- \frac{1}{12}(u_\alpha^3 - 3u_\alpha)\frac{\sum a_j^4}{(NB)^2} + \frac{1}{2}\theta(NB)^{1/2}\,\frac{\sum a_j^2 - NB}{NB} - \frac{3}{8}\theta(NB)^{1/2}\,\frac{(\sum a_j^2 - NB)^2}{(NB)^2} +$$

$$- \frac{\sum q_j a_j - \theta NB}{(NB)^{1/2}} + \frac{1}{2}\,\frac{(\sum q_j a_j - \theta NB)(\sum a_j^2 - NB)}{(NB)^{3/2}} + R_2,$$

$$R_2 = O\{N^{-5/4} + N^{-3}|\sum a_j^2 - NB|^3 + N^{-3/2}|\sum q_j a_j - \theta NB|^3 + N^{-3/2}(\sum q_j^2 a_j^2)^{3/2}\}. \tag{4.3.17}$$

According to (4.3.15), $\pi(\theta|z)$ can be found by substituting expansion (4.3.16) in expansion (4.3.7) for $R_\theta(x|z)$. We shall first consider the $\Phi(x)$-term, and after this the $x^k\phi(x)$ terms, for $k = 0,1,2,3$. The following expansion is used

$$\Phi\left(\xi_\alpha\left(\frac{\sum a_j^2}{\tau^2}\right)^{1/2} - \frac{\sum q_j a_j}{\tau}\right) = \Phi(u_\alpha - \theta(NB)^{1/2}) + \tag{4.3.18}$$

$$+ \phi(u_\alpha - \theta(NB)^{1/2})\left[\xi_\alpha\left(\frac{\sum a_j^2}{\tau^2}\right)^{1/2} - \frac{\sum q_j a_j}{\tau} - u_\alpha + \theta(NB)^{1/2} +\right.$$

$$\left.- \frac{1}{2}(u_\alpha - \theta(NB)^{1/2})\left\{\xi_\alpha\left(\frac{\sum a_j^2}{\tau^2}\right)^{1/2} - \frac{\sum q_j a_j}{\tau} - u_\alpha + \theta(NB)^{1/2}\right\}^2\right] +$$

$$+ O\left(\left|\xi_\alpha\left(\frac{\sum a_j^2}{\tau^2}\right)^{1/2} - \frac{\sum q_j a_j}{\tau} - u_\alpha + \theta(NB)^{1/2}\right|^3\right).$$

After some calculations this leads to

$$(4.3.19)\qquad \Phi(\xi_\alpha(\frac{\sum a_j^2}{\tau^2})^{1/2}-\frac{\sum q_j a_j}{\tau}) = \Phi(u_\alpha-\theta(NB)^{1/2}) +$$

$$+ \phi(u_\alpha-\theta(NB)^{1/2})\Big[\tfrac{1}{2}(u_\alpha-\theta(NB)^{1/2})\ \frac{\sum q_j^2a_j^2}{NB} - \tfrac{1}{12}(u_\alpha^3-3u_\alpha)\ \frac{\sum a_j^4}{(NB)^2} +$$

$$+ \tfrac{1}{2}\theta(NB)^{1/2}\ \frac{\sum a_j^2-NB}{NB} - \tfrac{1}{8}\theta(NB)^{1/2}\{3+\theta(NB)^{1/2}(u_\alpha-\theta(NB)^{1/2})\}\ \cdot$$

$$\frac{(\sum a_j^2-NB)^2}{(NB)^2} - \frac{\sum q_j a_j-\theta NB}{(NB)^{1/2}} - \tfrac{1}{2}(u_\alpha-\theta(NB)^{1/2})\frac{(\sum q_j a_j-\theta NB)^2}{NB} +$$

$$+ \tfrac{1}{2}\{1+\theta(NB)^{1/2}(u_\alpha-\theta(NB)^{1/2})\}\frac{(\sum q_j a_j-\theta NB)(\sum a_j^2-NB)}{(NB)^{3/2}}\Big] + O(R_2).$$

The remainder is still $O(R_2)$ by virtue of the conditions $|\sum q_j a_j| \le c'N^{1/2}$, $0 \le \theta \le DN^{-1/2}$, B constant and $\sum a_j^4 \le C$. They ensure that every term occurring in (4.3.16) and (4.3.17), is $O(1)$.

A simpler version of (4.3.16) suffices for the $x^k\phi(x)$ terms

$$\xi_\alpha(\frac{\sum a_j^2}{\tau^2})^{1/2} - \frac{\sum q_j a_j}{\tau} = u_\alpha-\theta(NB)^{1/2} + R_3,$$

$$R_3 = O(N^{-1}+N^{-1}\sum q_j^2a_j^2+N^{-1}|\sum a_j^2-NB|+N^{-1/2}|\sum q_j a_j-\theta NB|).$$

It follows that

$$(4.3.20)\qquad (\xi_\alpha(\frac{\sum a_j^2}{\tau^2})^{1/2} - \frac{\sum q_j a_j}{\tau})^k\ \phi(\zeta_\alpha\ (\frac{\sum a_j^2}{\tau^2})^{1/2} - \frac{\sum q_j a_j}{\tau}) =$$

$$= (u_\alpha-\theta(NB)^{1/2})^k\phi(u_\alpha-\theta(NB)^{1/2}) + O(R_3).$$

Inserting (4.3.19) and (4.3.20) in (4.3.7) leads to the required expression (4.3.11) for $\pi(\theta|z)$, with remainder

$$O\{R + (N^{-1}+N^{-3/2}|\sum q_j a_j^3|)R_3+N^{-5/2}|\sum q_j a_j^3|\,|\tau^2-NB|+N^{-3/2}|\sum q_j^3a_j^3| +$$

$$+ N^{-2}|\sum q_j^2a_j^4|+N^{-2}|\sum q_j a_j^3|\}.$$

Finally observe that

$$N^{-1}R_3 = O(R),$$

$$N^{-5/2}|\sum q_j a_j^3||\tau^2-NB| \le N^{-9/4}|\sum q_j a_j^3|^{3/2}+N^{-3}|\tau^2-NB|^3,$$

$$N^{-2}|\sum q_j a_j^3||\sum q_j a_j-\theta NB| \le N^{-9/4}|\sum q_j a_j^3|^{3/2}+N^{-3/2}|\sum q_j a_j-\theta NB|^3,$$

$$N^{-5/2}|\sum q_j a_j^3||\sum q_j^2 a_j^2| \le N^{-3}(\sum q_j a_j^3)^2+N^{-2}(\sum q_j^2 a_j^2)^2,$$

$$N^{-3}(\sum q_j a_j^3)^2+N^{-2}|\sum q_j a_j^3| = O(N^{-3/2}+N^{-9/4}|\sum q_j a_j^3|^{3/2}).$$

This shows that the remainder (4.3.12) has the required order. □

The next step is to show that under θ, the set of Z-values for which the conditions of the previous lemma are not satisfied, has a sufficiently small probability. The following definition is analogous to (4.3.3)

$$\tilde{A}_j = -\psi_1(X_j),\ \tilde{P}_j = \frac{f(X_j-\theta)}{f(X_j-\theta)+f(X_j+\theta)},\ \tilde{Q}_j = 2\tilde{P}_j-1,\ j = 1,\dots,N.$$

LEMMA 4.3.3. *Let* f *be symmetric around zero, positive on* R^1 *and twice continuously differentiable. Assume that positive constants* ε', D, η, $\tilde{c}$, $\tilde{C}$ *and* η' *exist, such that*

(4.3.21) $\quad 0 \le \theta \le DN^{-1/2},$

(4.3.22) $\quad \varepsilon' \le \alpha \le 1-\varepsilon',$

(4.3.23) $\quad \sup\{\int_{-\infty}^{\infty} \psi_1^{10}(x+y)f(x)dx : |y| \le \eta\} \le \tilde{C},$

(4.3.24) $\quad |\frac{d}{dy}\psi_1(F^{-1}(y))| \ge \tilde{c},$

for y *in a subinterval* τ *of* $(0,1)$ *with length at least* η'. *Then there exists* $A > 0$, *depending on* N, θ, F *and* α *only through* ε', D, $\tilde{c}$, $\tilde{C}$ *and* η' *and such that* (4.3.11) *and* (4.3.12) *hold for all* z*-values, except for those* *in a set* B *with* $P_\theta(B) \le AN^{-5/4}$.

PROOF. We have to verify that the conditions of lemma 4.3.2 are satisfied, except on a set of probability of order $N^{-5/4}$, i.e. we have to show that there exist constants c, C, δ, ε and c' such that

$$\frac{1}{N}\sum A_j^4 \le C, \tag{4.3.25}$$

$$\frac{1}{N}\sum (1-Q_j^2)A_j^2 \ge c, \tag{4.3.26}$$

$$\gamma(\varepsilon,\zeta,Q) \ge \delta N\zeta, \text{ for some } \zeta \ge N^{-3/2}\log N, \tag{4.3.27}$$

$$|\sum Q_j A_j| \le c'N^{1/2}, \tag{4.3.28}$$

except on a set of probability $O(N^{-5/4})$.

First consider (4.3.25). As ψ_1 is odd, we have $\sum A_j^4 = \sum \tilde{A}_j^4$. The $\tilde{A}_j^4$ are i.i.d. r.v.'s and in view of (4.3.23) we have

$$\sup_{0\le\theta\le DN^{-\frac{1}{2}}} E_\theta|\tilde{A}_j^{10}| \le \tilde{C}.$$

From Chebyshev's inequality we obtain

$$P_\theta\left(\left|\frac{1}{N}\sum(\tilde{A}_j^4 - E_\theta\tilde{A}_j^4)\right| \ge d\right) \le d^{-5/2}E_\theta\left|\frac{1}{N}\sum(\tilde{A}_j^4-E_\theta\tilde{A}_j^4)\right|^{5/2}. \tag{4.3.29}$$

At this point we use an inequality which is given by Chung (1951) and due to Marcinkievitz, Zygmund and Chung: if $Y_1,\ldots,Y_N$ are independent r.v.'s, all having mean zero, we have, for all $p \ge 1$,

$$E|\sum Y_j|^{2p} \le \hat{C}N^{p-1}\sum E|Y_j|^{2p},$$

where the constant $\hat{C}$ only depends on p. By taking $Y_j = \tilde{A}_j^4 - E_\theta\tilde{A}_j^4$, $j = 1,\ldots,N$ and $p = 5/4$, it follows that

$$E_\theta\left|\frac{1}{N}\sum(\tilde{A}_j^4-E_\theta\tilde{A}_j^4)\right|^{5/2} \le \hat{C}N^{-9/4}\sum E_\theta|\tilde{A}_j^4-E_\theta\tilde{A}_j^4|^{5/2} = \hat{C}N^{-5/4}E_\theta|\tilde{A}_1^4-E_\theta\tilde{A}_1^4|^{5/2}.$$

Together with (4.3.23) and (4.3.29) this implies that $\sum\tilde{A}_j^4 \le N(\tilde{C}+d) = NC$, except on a set B_1 with $P_\theta(B_1) = O([d^2N]^{-5/4})$.

The next condition to verify is (4.3.26). As was mentioned in theorem 3.3.1, (4.3.21) and (4.3.23) imply (3.2.16). For the present case this means that $P_\theta(\varepsilon \le \tilde{P}_1 \le 1-\varepsilon) > 1-\delta'$ for some δ' with $0 < \delta' < \min(\delta/2, c^2C^{-1})$. In theorem 3.2.2 it was shown that under this condition (4.3.26) holds, except on a set of probability $O(N^{-5/4})$, provided that $\sum A_j^2 \ge c^* N$, for some $c^* > c$. By applying Chebyshev's inequality one shows that $\sum A_j^2 = \sum \tilde{A}_j^2 \ge N(E_\theta \tilde{A}_1^2 - d)$, except on a set B_2, with $P_\theta(B_2) = O([d^2N]^{-5/4})$. Hence it only remains to prove that $E_\theta \tilde{A}_1^2$ is positive. Since $\theta \to 0$ as $N \to \infty$, (4.3.24) implies that

$$(4.3.30) \qquad \left|\frac{d}{dy}\, \psi_1(\theta + F^{-1}(y))\right| \ge \tilde{c},$$

on a subinterval of (0,1) with length at least $\eta'/2$. Therefore

$$E_\theta \tilde{A}_1^2 = \int_{-\infty}^{\infty} \psi_1^2(x) f(x-\theta)dx = \int_0^1 \psi_1^2(\theta + F^{-1}(y))dy > 0.$$

The third condition we have to deal with is (4.3.27). Inspection of the sketch of the proof of theorem 3.2.1 shows that this condition only serves to prove that

$$(4.3.31) \qquad \int_{b'N^{1/2} \le |t| \le bN^{3/2}} \left|\frac{\rho(t)}{t}\right| dt = O(N^{-3/2}),$$

where $|\rho|$ is given by (cf. (3.2.8))

$$(4.3.32) \qquad |\rho(t)| \le \exp\{-\sum P_j(1-P_j)(1 - \frac{\cos A_j t}{\tau})\} =$$

$$= \exp\{-2\sum \tilde{P}_j(1-\tilde{P}_j) \sin^2 \frac{\tilde{A}_j t}{2\tau}\}.$$

Instead of verifying (4.3.27) we shall prove (4.3.31) directly. From (4.3.30) and a similar argument as in lemma 4.2.4 it follows that

$$E_\theta \sin^2 \tilde{A}_1 t = \int_0^1 \sin^2 \{t\psi_1(\theta + F^{-1}(y))\}dy > 0,$$

for $|t| \ge \bar{c}$, where $\bar{c}$ is some positive constant. As $\sin^2 \tilde{A}_1 t$ is bounded, all moments exist and an application of Chebyshev's inequality shows that for some positive constant c_1, with probability $1 - O(N^{-5/4})$,

$$\frac{1}{N} \sum \sin^2 \tilde{A}_j t \geq c_1,$$

for all $|t| \geq \bar{c}$. Furthermore we note that, with probability $1 - O(N^{-5/4})$, $\tau^2/N = \{\sum(1-Q_j^2)A_j^2\}/N$ is bounded from below as well as from above. Hence, there exists a constant $c_2 > 0$, such that for all $|t| \geq c_2 N^{1/2}$ with probability $1 - O(N^{-5/4})$

$$\frac{1}{N} \sum \sin^2 \frac{\tilde{A}_j t}{2\tau} \geq c_1.$$

Now there exists $\tilde{\varepsilon} > 0$, depending on c_1, such that at most a fraction $c_1/2$ of the $\tilde{P}_j$ does not lie in the interval $(\tilde{\varepsilon}, 1-\tilde{\varepsilon})$, again with probability $1 - O(N^{-5/4})$. Hence, for all $|t| \geq c_2 N^{1/2}$,

$$\sum \tilde{P}_j(1-\tilde{P}_j)\sin^2 \frac{\tilde{A}_j t}{2\tau} \geq \tilde{\varepsilon}(1-\tilde{\varepsilon}) \left[\sum \sin^2 \frac{\tilde{A}_j t}{2\tau} - \frac{c_1 N}{2}\right] \geq \frac{\tilde{\varepsilon}(1-\tilde{\varepsilon})c_1 N}{2},$$

with probability $1 - O(N^{-5/4})$. Together with (4.3.32), this proves the validity of (4.3.31), except on a set B_3 with $P_\theta(B_3) = O(N^{-5/4})$.

It remains to prove (4.3.28). As Q_j and A_j are both odd functions of Z_j, we have $|\sum Q_j A_j| \leq \sum |Q_j A_j| \leq \sum |\tilde{Q}_j \tilde{A}_j|$. Because

$$\tilde{Q}_1 \tilde{A}_1 = -\psi_1(X_1) \frac{f(X_1-\theta)-f(X_1+\theta)}{f(X_1-\theta)+f(X_1+\theta)} =$$

$$= 2\theta\psi_1(X_1) \int_0^1 \frac{f'(X_1+\nu\theta)f(X_1-\nu\theta)+f(X_1+\nu\theta)f'(X_1-\nu\theta)}{(f(X_1+\nu\theta)+f(X_1-\nu\theta))^2} d\nu,$$

it follows that

$$|\tilde{Q}_1 \tilde{A}_1| \leq 2\theta|\psi_1(X_1) \int_0^1 \{\tfrac{1}{2}\psi_1(X_1+\nu\theta) + \tfrac{1}{2}\psi_1(X_1-\nu\theta)\}d\nu| \leq$$

$$\leq \theta\{|\psi_1^2(X_1)| + \int_0^1 \psi_1^2(X_1+\nu\theta)d\nu + \int_0^1 \psi_1^2(X_1-\nu\theta)d\nu\}.$$

As $0 \leq \theta \leq DN^{-1/2}$, it remains to prove that, with probability $1 - O(N^{-5/4})$

$$\sum_{j=1}^{N} \int_0^1 \psi_1^2(X_j+\nu\theta)d\nu \leq \tilde{C}N.$$

This is done in the same way as in which (4.3.25) is proved; therefore we only mention that an application of Fubini's theorem shows that

$$E_\theta|\int_0^1 \psi_1^2(X_1\underline{+}\nu\theta)d\nu|^{5/2} \le \sup_{|\nu|\le 1} E_\theta|\psi_1^5(X_j+\nu\theta)| \le \tilde{C}. \qquad \Box$$

COROLLARY 4.3.1. *Under the conditions of lemma* 4.3.3, *the power* $\pi(\theta)$ *of the permutation test based on* U, *satisfies*

$$(4.3.33) \qquad |\pi(\theta)-E_\theta\tilde{\pi}(\theta|Z)| \le AE_\theta R,$$

where $\tilde{\pi}(\theta|z)$ *and* R *are given in* (4.3.11) *and* (4.3.12) *and* A *is a positive constant, depending on* N, θ, F *and* α *only through* ε', D, η, $\tilde{c}$, $\tilde{C}$ *and* η'.

PROOF. On B^c $\pi(\theta|z)$ satisfies $\pi(\theta|z) = \tilde{\pi}(\theta|z) + R$. Hence, for $\pi(\theta) = E_\theta\pi(\theta|Z)$ we obtain

$$|\pi(\theta)-E_\theta(\tilde{\pi}(\theta|Z)+R)| \le E_\theta|\pi(\theta|Z)I_B(Z)|+E_\theta|(\tilde{\pi}(\theta|Z)+R)I_B(Z)|,$$

where I_B is the indicator function of B. Now $\pi(\theta|z)$, being a probability, is bounded, and therefore $E_\theta|\pi(\theta|Z)I_B(Z)| = O(N^{-5/4})$.

The term $E_\theta|(\tilde{\pi}(\theta|Z)+R)I_B(Z)|$ has to be treated with more care. First, $\Phi(u_\alpha-\theta(NB)^{1/2})$ is bounded, and thus contributes $O(N^{-5/4})$. The next three terms of $\tilde{\pi}(\theta|Z)$ can be split into a bounded part and a part that has the form of one of the remaining terms in $\tilde{\pi}(\theta|Z)$:

$$|N^{-3/2}\textstyle\sum Q_jA_j^3| \le N^{-3/2}\sum A_j^2\sum|Q_jA_j| \le N^{-2}(\sum A_j^2)^2+N^{-1}(\sum|Q_jA_j|)^2 \le$$

$$\le 2(B^2+\theta^2NB^2)+2N^{-2}(\textstyle\sum A_j^2-NB)^2+2N^{-1}(\sum|Q_jA_j|-\theta NB)^2,$$

$$N^{-2}\textstyle\sum A_j^4 \le N^{-2}(\sum A_j^2)^2 \le 2B^2+2N^{-2}(\sum A_j^2-NB)^2,$$

$$N^{-1}\textstyle\sum Q_j^2A_j^2 \le N^{-1}(\sum|Q_jA_j|)^2 \le 2\theta^2NB^2+2N^{-1}(\sum|Q_jA_j|-\theta NB)^2.$$

The remaining terms of $\tilde{\pi}(\theta|Z)$, and the terms of R, can all be treated in the following way. Take for example the first term, $N^{-1}(\sum A_j^2-NB)$. On the part of B where $N^{-1}(\sum A_j^2-NB) \le 1$, the contribution to the expectation is clearly $O(N^{-5/4})$. On the remaining part of B the contribution is

$O(N^{-3}E_\theta|\sum A_j^2-NB|^3)$ because for any r.v. Y with d.f. H, and all $p \geq 1$,

$$\int_{|y|\geq 1} |y|dH(y) \leq \int_{|y|\geq 1} |y|^p dH(y) \leq \int_{-\infty}^{\infty} |y|^p dH(y) = E|Y|^p.$$

Thus the total contribution of this term is $O(N^{-5/4}+N^{-3}\ E_\theta|\sum A_j^2-NB|^3)$. But these terms already occur in $E_\theta R$. Inspection of the other terms shows that their contribution is always $O(E_\theta R)$. □

Our final task is the derivation of simple expressions for $E_\theta\tilde{\pi}(\theta|Z)$ and $E_\theta R$. To this end, $E_\theta\tilde{Q}_1^r\tilde{A}_1^s$ has to be evaluated for various r and s, which is done by expanding $\tilde{Q}_1^r\tilde{A}_1^s$ around $\theta = 0$ to the appropriate order. Note that $E_\theta\tilde{Q}_1^r\tilde{A}_1^s$ has a double dependence on θ: explicitly in $\tilde{Q}_1 = [f(X_1-\theta)-f(X_1+\theta)]/[f(X_1-\theta)+f(X_1+\theta)]$ and implicitly as X_1 comes from $F(x-\theta)$. We introduce the following notation

$$h(x,\theta) = \frac{f(x+2\theta)-f(x)}{f(x+2\theta)+f(x)},\quad q_{r,s}(x,\theta) = h^r(x,\theta)\psi_1^s(x+\theta).$$

Let $q_{r,s}^{(l)}(x,\theta)$, $h^{(l)}(x,\theta)$, $\psi_1^{(l)}(x+\theta)$ denote the l-th derivative of $q_{r,s}(x,\theta)$, $h(x,\theta)$ and $\psi_1(x+\theta)$ respectively, with respect to θ. In the next lemma we evaluate the necessary moments.

LEMMA 4.3.4. *Let* f *be five times differentiable and suppose that positive constants* $\tilde{C}$, D *and* η *exist, such that* $0 \leq \theta \leq DN^{-1/2}$ *and*

$$\sup\left\{\int_{-\infty}^{\infty} |\psi_j(x+y)|^{5/j} f(x)dx : |y| \leq \eta\right\} \leq \tilde{C},\ j = 1,\ldots,5. \tag{4.3.34}$$

Then there exists $A > 0$, *depending on* N, θ *and* F *only through* $\tilde{C}$, D *and* η *and such that*

$$\begin{aligned}
&E_\theta|\tilde{Q}_1^r\tilde{A}_1^{(5-r)}| \leq AN^{-r/2},\quad r = 0,1,2,\\
&|E_\theta\tilde{Q}_1^r\tilde{A}_1^{(4-r)}-\theta^r E_0\tilde{A}_1^4| \leq AN^{-(r+1)/2},\quad r = 0,1,2,\\
&|E_\theta\tilde{A}_1^2-\{E_0\tilde{A}_1^2 + \frac{\theta^2}{3}E_0\psi_1^4(X_1)\}| \leq AN^{-3/2},\\
&|E_\theta\tilde{Q}_1\tilde{A}_1- \{\theta E_0\tilde{A}_1^2 + \frac{\theta^3}{6}E_0(\frac{2}{3}\psi_1^4(X_1)-\psi_2^2(X_1))\}| \leq AN^{-2}.
\end{aligned} \tag{4.3.35}$$

PROOF. For $l = 1,2,3,4$ we have

$$\psi_1^{(1)}(x+\theta) = \psi_2(x+\theta) - \psi_1^2(x+\theta),$$

(4.3.36) $$\psi_1^{(2)}(x+\theta) = \psi_3(x+\theta) - 3\psi_1(x+\theta)\psi_2(x+\theta) + 2\psi_1^3(x+\theta),$$

$$|\psi_1^{(l)}(x+\theta)| \le \bar{C}\sum_{j=1}^{l+1} |\psi_j^{\frac{l+1}{j}}(x+\theta)|,$$

for some $\bar{C} > 0$. Differentiation of h yields that

$$h(x,0) = 0,\ |h(x,\theta)| \le 1,$$

$$h^{(1)}(x,\theta) = 4\,\frac{f'(x+2\theta)}{f(x)} \Big/ \left(\frac{f(x+2\theta)}{f(x)} + 1\right)^2,\quad h^{(1)}(x,0) = \psi_1(x),$$

$$h^{(2)}(x,\theta) = 8\,\frac{f''(x+2\theta)}{f(x)} \Big/ \left(\frac{f(x+2\theta)}{f(x)} + 1\right)^2 +$$

$$- 16\left(\frac{f'(x+2\theta)}{f(x)}\right)^2 \Big/ \left(\frac{f(x+2\theta)}{f(x)} + 1\right)^3,$$

(4.3.37) $$h^{(2)}(x,0) = 2(\psi_2(x)-\psi_1^2(x)),$$

$$h^{(3)}(x,\theta) = 16\,\frac{f^{(3)}(x+2\theta)}{f(x)} \Big/ \left(\frac{f(x+2\theta)}{f(x)} + 1\right)^2 +$$

$$- 96\,\frac{f'(x+2\theta)f''(x+2\theta)}{f^2(x)} \Big/ \left(\frac{f(x+2\theta)}{f(x)} + 1\right)^3 +$$

$$+ 96\left(\frac{f'(x+2\theta)}{f(x)}\right)^3 \Big/ \left(\frac{f(x+2\theta)}{f(x)} + 1\right)^4,$$

$$h^{(3)}(x,0) = 4\psi_3(x) - 12\psi_1(x)\psi_2(x) + 6\psi_1^3(x),$$

$$|h^{(l)}(x,\theta)| \le \bar{C}\sum_{j=1}^{l} |\psi_j^{l/j}(x+2\theta)|,\quad l = 1,2,3,4.$$

Under $F(x-\theta)$, X_1 has the same distribution as $X_1+\theta$ under $F(x)$. Hence

$$E_\theta \tilde{Q}_1^r \tilde{A}_1^s = E_0(-h)^r(X_1,\theta)(-\psi_1)^s(X_1+\theta) = E_0 q_{r,s}(X_1,\theta)(-1)^{r+s}.$$

We now prove the results in the first line of (4.3.35). Note that

$$E_\theta|\tilde{A}_1^5| = E_\theta|\psi_1^5(X_1)| = E_0|\psi_1^5(X_1+\theta)| = O(1),$$

because of (4.3.34) and the fact that $0 \leq \theta \leq DN^{-1/2}$. Application of lemma 4.2.2 gives

$$E_\theta|\tilde{Q}_1\tilde{A}_1^4| = E_0|q_{1,4}(X_1,\theta)| \leq \theta \sup_{0\leq\nu\leq1} E_0|q_{1,4}^{(1)}(X_1,\nu\theta)|,$$

since $h(x,0) = 0$. Furthermore, for some $C^* > 0$,

$$|q_{1,4}^{(1)}| \leq C^*\{|h^{(1)}\psi_1^4| + |h\psi_1^3\psi_1^{(1)}|\}.$$

Using (4.3.36), (4.3.37) and Hölder's inequality, one obtains

$$|q_{1,4}^{(1)}(x,\nu\theta)| \leq C^* \sum_{j=1}^{2} \{|\psi_j^{5/j}(x+\nu\theta)| + |\psi_j^{5/j}(x+2\nu\theta)|\}.$$

Hence, by (4.3.34), $\sup_{0\leq\nu\leq1} E_0|q_{1,4}^{(1)}(X_1,\nu\theta)| \leq C_1$, for some $C_1 > 0$, and therefore $E_\theta|\tilde{Q}_1\tilde{A}_1^4| = \mathcal{O}(N^{-1/2})$. Similarly,

$$E_\theta|\tilde{Q}_1^2\tilde{A}_1^3| \leq \frac{\theta^2}{2} \sup_{0\leq\nu\leq1} E_0|q_{2,3}^{(2)}(X_1,\nu\theta)| \leq$$

$$\leq \frac{\theta^2}{2}C^* \sup_{0\leq\nu\leq1} E_0[\sum_{j=1}^{3}\{|\psi_j^{5/j}(X_1+\nu\theta)| + |\psi_j^{5/j}(X_1+2\nu\theta)|\}] = \mathcal{O}(N^{-1}).$$

The second line of (4.3.35) is proved by continuing in the same way. For $r = 0,1,2$ we have

$$|E_\theta\tilde{Q}_1^r\tilde{A}_1^{(4-r)} - \theta^r E_0\tilde{A}_1^4| = |E_0 q_{r,4-r}(X_1,\theta) - \theta^r E_0\psi_1^4(X_1)| \leq$$

$$\leq \frac{\theta^{r+1}}{(r+1)!} \sup_{0\leq\nu\leq1} |q_{r,4-r}^{(r+1)}(X_1,\nu\theta)| \leq$$

$$\leq \frac{\theta^{(r+1)}}{(r+1)!} C^* \sup_{0\leq\nu\leq1} E_0[\sum_{j=1}^{r+2} \{|\psi_j^{5/j}(X_1+\nu\theta)| +$$

$$+ |\psi_j^{5/j}(X_1+2\nu\theta)|\}] = \mathcal{O}(N^{-\frac{(r+1)}{2}}).$$

$E_\theta\psi_1^2(X_1)$ is evaluated in the following way

$$|E_\theta\psi_1^2(X_1)-\{E_0\psi_1^2(X_1)+2\theta E_0[\psi_1\psi_2(X_1)-\psi_1^3(X_1)] +$$

$$+ \theta^2 E_0[\psi_2^2(X_1)+\psi_1\psi_3(X_1)-5\psi_1^2\psi_2(X_1)+3\psi_1^4(X_1)]\}| \leq$$

$$\leq \frac{\theta^3}{3!} \sup_{0\leq\nu\leq 1} E_0|q_{0,2}^{(3)}(X_1,\nu\theta)| \leq \frac{\theta^3}{3!} C^* \sup_{0\leq\nu\leq 1} E_0[\sum_{j=1}^{4}\{|\psi_j^{5/j}(X_1+\nu\theta)| +$$

$$+ |\psi_j^{5/j}(X_1+\nu\theta)|\}] = O(N^{-3/2}).$$

As $\psi_1^2\psi_2$ and ψ_1^3 are odd, the corresponding moments are zero. Also, in section 4.2 we found by partial integration that $E_0\psi_1^2\psi_2(X_1) = 2/3\ E_0\psi_1^4(X_1)$ and $E_0\psi_1\psi_3(X_1) = 2/3\ E_0\psi_1^4(X_1)-E_0\psi_2^2(X_1)$. Application of these results yields the expression in (4.3.35) for $E_\theta\tilde{A}_1^2 = E_\theta\psi_1^2(X_1)$.

Finally $E_\theta\tilde{Q}_1\tilde{A}_1$ has to be found.

$$|E_\theta\tilde{Q}_1\tilde{A}_1 - \sum_{j=0}^{3} E_0 q_{1,1}^{(j)}(X_1,0)\frac{\theta^j}{j!}| \leq \frac{\theta^4}{4!} \sup_{0\leq\nu\leq 1} E_0|q_{1,1}^{(4)}(X_1,\nu\theta)| \leq$$

$$\leq \frac{\theta^4}{4!} C^* \sup_{0\leq\nu\leq 1} E_0[\sum_{j=1}^{5}\{|\psi_j^{5/j}(X_1+\nu\theta)| + |\psi_j^{5/j}(X_1+2\nu\theta)|\}] = O(N^{-2}).$$

By applying (4.3.36) and (4.3.37), $E_0 q_{1,1}^{(j)}(X_1,0)$ can be obtained, which leads to the desired formula in (4.3.35). □

We are now in a position to give a simple expansion for $\pi(\theta)$.

THEOREM 4.3.1. *Suppose that positive constants* D, η, $\tilde{c}$, $\tilde{C}$ *and* ε' *exist such that* (4.3.21), (4.3.22) *and* (4.3.24) *hold and such that*

$$\sup \{\int_{-\infty}^{\infty} |\psi_j^{m_j}(x+y)|f(x)dx \ :\ |y| \leq \eta\} \leq \tilde{C},\ j = 1,\dots,5,$$

$$m_1 = 10,\ m_j = 5/j,\ j = 2,\dots,5.$$

Then there exists $A > 0$ *depending on* N, θ, F *and* α *only through* D, η, $\tilde{c}$, $\tilde{C}$ *and* ε', *such that*

$$(4.3.38) \qquad |\pi(\theta)-\tilde{\pi}(\theta)| \leq AN^{-5/4},$$

where

$$(4.3.39)\qquad \tilde{\pi}(\theta) = 1-\Phi(u_\alpha-\eta_1) + \frac{\eta_1\phi(u_\alpha-\eta_1)}{N}\Big[-\frac{\eta_2}{12}u_\alpha^2 + \frac{(\eta_2-3)}{24}\eta_1 u_\alpha + - \frac{(\eta_2-3)}{24} - \frac{(12\eta_3-5\eta_2-9)\eta_1^2}{72}\Big],$$

and η_1, η_2 *and* η_3 *as defined in* (4.2.19).

PROOF. The results of lemma 4.3.4 strongly suggest the choice of $B = E_0\psi_1^2(X_1)$ in $E\tilde{\pi}(\theta|Z)$. This is allowed, as $E_0\psi_1^2(X_1)$ is a positive constant as was proved in lemma 4.3.3. The conditions of the present theorem contain the conditions of lemmas 4.3.3 and 4.3.4. Hence we may use (4.3.35) in $E(\tilde{\pi}(\theta|Z))$. After some algebra we obtain $\tilde{\pi}(\theta) = E_\theta\tilde{\pi}(\theta|Z)+O(N^{-3/2})$. Hence, in order to prove (4.3.38), it suffices in view of (4.3.33), to prove that $E_\theta R = O(N^{-5/4})$. Consider the expression for R in (4.3.12). We begin with the last term

$$N^{-2}E_\theta|\sum Q_j^2A_j^4| \le N^{-1}E_\theta|\tilde{Q}_1^2\tilde{A}_1^4| \le N^{-1}E_\theta|\tilde{Q}_1\tilde{A}_1^4| = O(N^{-3/2}),$$

$$N^{-3/2}E_\theta|\sum Q_j^3A_j^3| \le N^{-1/2}E_\theta|\tilde{Q}_1^3\tilde{A}_1^3| \le N^{-1/2}E_\theta|\tilde{Q}_1^2\tilde{A}_1^3| = O(N^{-3/2}),$$

$$N^{-3/2}E_\theta|\sum Q_j^2A_j^2|^{3/2} = N^{-3/2}E_\theta(\sum\tilde{Q}_j^4\tilde{A}_j^4 + \sum_{i\neq j}\sum \tilde{Q}_i^2\tilde{A}_i^2\tilde{Q}_j^2\tilde{A}_j^2)^{3/4} \le$$

$$\le N^{-3/2}E_\theta\sum|\tilde{Q}_j^3\tilde{A}_j^3| + N^{-3/2}(E_\theta\sum_{i\neq j}\sum \tilde{Q}_i^2\tilde{A}_i^2\tilde{Q}_j^2\tilde{A}_j^2)^{3/4} \le$$

$$\le N^{-1/2}E_\theta|\tilde{Q}_1^2\tilde{A}_1^3| + (E_\theta\tilde{Q}_1^2\tilde{A}_1^2)^{3/2} = O(N^{-3/2}),$$

$$N^{-9/4}E_\theta|\sum Q_jA_j^3|^{3/2} \le N^{-3/2} + N^{-5/2}E_\theta(\sum\tilde{Q}_j\tilde{A}_j^3)^2 \le$$

$$\le N^{-3/2} + N^{-3/2}E_\theta\tilde{Q}_1^2\tilde{A}_1^6 + N^{-1/2}(E_\theta\tilde{Q}_1\tilde{A}_1^3)^2 = O(N^{-3/2}).$$

Another application of the inequality due to Marcinkievitz, Zygmund and Chung disposes of the remaining two terms. If $Y_1,\dots,Y_N$ are i.i.d. r.v.'s with mean zero, this inequality implies that

$$(4.3.40)\qquad E|\sum Y_j^3| \le \hat{C}N^{1/2}\sum E|Y_j|^3 = \hat{C}N^{3/2}E|Y_1|^3.$$

Hence

$$N^{-3}E_\theta|\sum A_j^2-NB|^3 = N^{-3}E_\theta|\sum(\tilde{A}_j^2-E_0\tilde{A}_j^2)|^3 = O(N^{-3}E_\theta|\sum(\tilde{A}_j^2-E_\theta\tilde{A}_j^2)|^3 +$$

$$+ |E_\theta\tilde{A}_1^2-E_0\tilde{A}_1^2|^3) = O(N^{-3/2}E_\theta|\tilde{A}_1^2-E_\theta\tilde{A}_1^2|^3 +$$

$$+ N^{-3/2}\{E_\theta(\tilde{A}_1^2-E_\theta\tilde{A}_1^2)^2\}^{3/2} + N^{-3}) = O(N^{-3/2}),$$

and in the same way $N^{-3/2}E_\theta|\sum Q_jA_j-\theta NB|^3 = O(N^{-3/2})$. Finally, the announced uniformity in (4.3.38) is an immediate consequence of the uniformity in the conditions. □

REMARK. The remainder is $O(N^{-5/4})$ instead of $O(N^{-3/2})$ only because of the fact that in lemma 4.3.1 the Edgeworth expansion of theorem 3.2.1 is used, which approximates to $O(N^{-5/4})$. If in theorem 3.2.1 condition $\sum a_j^4 \le CN$ is replaced by $\sum|a_j|^5 \le CN$, the remainder becomes $O(N^{-3/2})$, as was remarked in chapter 3. Hence by changing of $m_1 = 10$ to $m_1 = 25/2$ one obtains a remainder of $O(N^{-3/2})$.

We conclude this section by considering again the permutation test based on $U^* = (1/2\theta)\sum_{X_j>0}\log\{f(X_j-\theta)/f(X_j+\theta)\}$, instead of on U. Let $\pi^*(\theta)$ be the power of the U^*-test and define

$$\tilde{A}_j^* = \frac{1}{2\theta}\log\frac{f(X_j-\theta)}{f(X_j+\theta)},\quad \tilde{a}_j^* = \frac{1}{2\theta}\log\frac{f(x_j-\theta)}{f(x_j+\theta)},$$

$$A_j^* = \frac{1}{2\theta}\log\frac{f(Z_j-\theta)}{f(Z_j+\theta)},\quad a_j^* = \frac{1}{2\theta}\log\frac{f(z_j-\theta)}{f(z_j+\theta)}.$$

In the following lemma we prove that $\pi^*(\theta)$ agrees with $\pi(\theta)$ up to $O(N^{-5/4})$.

LEMMA 4.3.5. *Under the conditions of theorem 4.3.1 we have*

$$|\pi^*(\theta)-\tilde{\pi}(\theta)| \le AN^{-5/4},$$

where $\tilde{\pi}(\theta)$ *is given by* (4.3.39).

PROOF. As $U^* = \sum A_j^* V_j$ is of the same form as $U = \sum A_j V_j$, the proof consists of showing that all lemmas in this section continue to hold in the case of U^*, provided that some minor changes are made. Thus lemma 4.3.1 and 4.3.2 clearly hold for the U^*-test, if a_j is replaced by a_j^* everywhere. Lemma 4.3.3 remains valid without any changes at all, because $\tilde{A}_j^* = -\frac{1}{2}\psi_1(X_j-\nu_1\theta) - \frac{1}{2}\psi_1(X_j+\nu_2\theta)$, $0 \le \nu_1, \nu_2 \le 1$ and therefore (4.3.21) and (4.3.23) ensure that $\sup_{0\le\theta\le DN^{-\frac{1}{2}}} E_\theta|\tilde{A}_1^*|^{10} \le \mathcal{C}$ and $E_\theta \tilde{A}_1^{*2} > 0$. Corollary 4.3.1 also continues to hold in the same form. Hence it finally remains to adapt lemma 4.3.4. Expansion around $\theta = 0$ shows that

$$\tilde{A}_1^* = -\psi_1(X_1) - \frac{\theta^2}{6}\{\psi_3(X_1) - 3\psi_1\psi_2(X_1) + 2\psi_1^3(X_1)\} + \ldots\ldots,$$

(4.3.41)

$$(\tilde{A}_1^*)^2 = \psi_1^2(X_1) + \frac{\theta^2}{3}\{\psi_1\psi_3(X_1) - 3\psi_1^2\psi_2(X_1) + 2\psi_1^4(X_1)\} + \ldots\ldots .$$

In a similar way as in lemma 4.3.4 one shows that the first two statements in (4.3.35) remain valid if $\tilde{A}_1$ is replaced by $\tilde{A}_1^*$. Moreover,

$$E_\theta(\tilde{A}_1^*)^2 = E_\theta\tilde{A}_1^2 + \frac{\theta^2}{3} E_0\psi_1\psi_3(X_1) + O(N^{-3/2}),$$

(4.3.42)

$$E_\theta(\tilde{Q}_1\tilde{A}_1^*) = E_\theta(\tilde{Q}_1\tilde{A}_1) + \frac{\theta^3}{6} E_0\psi_1\psi_3(X_1) + O(N^{-2}).$$

The expressions in (4.3.35) are used to obtain $E_\theta\tilde{\pi}(\theta|Z)$ from (4.3.11). In (4.3.11), $E_\theta(A_1^*)^2$ and $E_\theta\tilde{Q}_1\tilde{A}_1^*$ only occur in

$$E_\theta[\frac{1}{2}\theta(NB)^{1/2}\frac{(\sum A_j^{*2}-NB)}{NB} - \frac{(\sum\tilde{Q}_j\tilde{A}_j^*-\theta NB)}{(NB)^{1/2}}].$$

But in view of (4.3.42), this expectation differs at most $O(N^{-3/2})$ from the expectation of the same expression in $\tilde{A}_j$ instead of $\tilde{A}_j^*$. □

4.4. SCALE INVARIANT TESTS

One of the advantages of the use of rank tests is the fact that such tests are distribution-free. This was the motivation in the previous section to compare their behaviour to that of most powerful distribution-free tests.

Another nice property of rank tests is their scale invariance. In analogy to the previous section it therefore seems worthwhile to compare LMP rank tests with most powerful scale invariant tests. Let again $X_1, X_2, \ldots, X_N$ be i.i.d. r.v.'s from a d.f. $F(x-\theta)$, where $f = F'$ is symmetric around zero and positive on R^1. According to Hájek and Šidák (1967) the most powerful scale invariant test for $\theta = 0$ against a simple alternative $\theta > 0$ rejects the hypothesis for large values of

$$(4.4.1) \qquad \{\int_0^\infty \prod_{j=1}^N f(\lambda X_j - \theta)\lambda^{N-1} d\lambda\}/\{\int_0^\infty \prod_{j=1}^N f(\lambda X_j)\lambda^{N-1} d\lambda\}.$$

In Hájek and Šidák (1967) it is also shown that in the normal case this test is equivalent with Student's one sample t-test. Unfortunately, for general f we are unable to find an expansion for the d.f. of the statistic in (4.4.1).

We conclude this section with a remark on the relation between the most powerful permutation test for the normal case and the t-test, being the most powerful scale invariant test for this case. Note that if f is the normal density, the permutation tests based on U and U^* are both equivalent to the permutation test based on $\sum X_j$. Under the conditions of theorem 4.3.1 its power satisfies, according to (4.3.38) and (4.3.39)

$$(4.4.2) \qquad \pi(\theta) = 1-\Phi(u_\alpha - N^{1/2}\theta) - \frac{\theta u_\alpha^2}{4N^{1/2}} \phi(u_\alpha - N^{1/2}\theta) + O(N^{-5/4}).$$

But this expansion also holds for the power of the t-test, as is shown by Hodges and Lehmann (1970). Hence, in particular, the normal permutation-test has deficiency zero with respect to the t-test if normal location alternatives are considered. This rather striking phenomenon can be made more transparant by looking directly at the two test statistics involved. For this approach the reader is referred to Albers, Bickel and van Zwet (1974).

4.5. RANDOMIZED RANK SCORE TESTS

In the preceding sections of this chapter we considered tests that are only slightly better than the LMP rank test in the sense that the asymptotic relative efficiency (ARE) of the LMP rank test with respect to these tests always equals one. Here we consider the opposite case: randomized rank score (RRS) tests are worse than LMP rank tests, but also have ARE 1 with respect to these tests.

RRS tests have been introduced by Bell and Doksum (1965) for the two sample problem. By proceeding analogously, we define a RRS test for the one sample problem. As before, $X_1,\ldots,X_N$ are i.i.d. r.v.'s from $F(x-\theta)$, where the known d.f. F has a density f that is symmetric around zero and positive on R^1. We also have an auxiliary independent sample $X_1^*,\ldots,X_N^*$ from F. Denote $(X_1,\ldots,X_N)$ as X and $(X_1^*,\ldots,X_N^*)$ as X^*. Let $Z(Z^*)$ be the vector of order statistics for the absolute values $|X_1|,\ldots,|X_N|(|X_1^*|,\ldots,|X_N^*|)$. Now we test the hypothesis of symmetry against the restricted alternative $F(x-\theta)$, $\theta > 0$, by rejecting the hypothesis for large values of

$$L = \sum A_j \bar{V}_j ,$$

where $A_j = -\psi_1(Z_j^*)$, $\bar{V}_j = 1$ if the X_i corresponding to Z_j is positive and $\bar{V}_j = -1$ otherwise. $\sum$ always means $\sum_{j=1}^{N}$.

The statistic of the LMP rank test may be expressed as $T = \sum(EA_j)\bar{V}_j$. Note that a subscript in EA_j is superfluous since the X_j^* always come from F. Hence L can be interpreted as the randomized counterpart of T. An advantage of L over T lies in the fact that its computation requires no tables of EA_j. Moreover, in some cases, L has under the hypothesis a continuous, known and tabulated distribution, e.g. normal or χ^2, whereas special tables are needed for the distribution of T under the hypothesis. One may suspect that the price for these advantages will be a loss of power of the RRS test. However, Bell and Doksum have shown that its ARE with respect to the LMP rank test equals one. Therefore it seems worthwhile to obtain an expansion for the power of the RRS test in order to obtain a comparison to $o(N^{-1})$ instead of $o(1)$.

In deriving such an expansion, we exploit the resemblance between the RRS test, the LMP rank test and the LMP permutation test. For the LMP permutation test the scores are $-\psi_1(Z_j)$. These are also random and of the same form as the A_j used here. The typical difference, however, is that the permutation testscores are based on the sample itself, whereas the RRS testscores are found from a second, independent sample which comes from F under H_0 as well as under H_1. The independence thus obtained between the A_j and the $\bar{V}_j$ enables us to apply the results of chapter 3 to a much larger extent than was possible for permutation tests.

Denote $-\psi_1(z_j^*)$ as a_j, $-\psi_1(X_j^*)$ as $\tilde{A}_j$ and $-\psi_1(x_j^*)$ as $\tilde{a}_j$. Let $V_j = (1+\bar{V}_j)/2$, $j = 1,\ldots,N$. Hence $L = 2\sum A_j V_j - \sum A_j$. Conditionally on $Z^* = z^*$, L is equivalent to $\sum a_j V_j$. Moreover, in view of the independence of A_j and V_j we have $P_\theta(V_j=1|z^*) = P_\theta(V_j=1)$. Hence in the conditional situation we have exactly the same case as in chapter 3. Define

$$R_\theta(x|z^*) = P_\theta\left(\frac{L-\theta NE\tilde{A}_1^2}{(NE\tilde{A}_1^2)^{1/2}} \le x|z^*\right),$$

$$\gamma(\zeta) = \lambda\{x|\exists_j |x-a_j| < \zeta\},$$

where λ denotes Lebesgue measure. The following lemma supplies an expansion for $R_\theta(x|z^*)$.

LEMMA 4.5.1. *Suppose that* z^* *is such that there exist positive constants* c, C *and* δ *for which*

$$(4.5.1) \qquad \sum a_j^4 \le CN,\ \sum a_j^2 \ge cN,\ \gamma(\zeta) \ge \delta N\zeta,$$

for some $\zeta \ge N^{-3/2}\log N$. *Moreover, assume that there are positive constants* D, $\tilde{C}$, ε, $\tilde{c}$ *and* η *such that*

$$(4.5.2) \qquad 0 \le \theta \le DN^{-1/2},$$

$$(4.5.3) \qquad \sup\left\{\int_{-\infty}^{\infty} |\psi_j(x+y)|^{m_j} f(x)dx : |y| \le \varepsilon\right\} \le \tilde{C},\ j = 1,\ldots,4,$$

$$m_1 = 6,\ m_2 = 3,\ m_3 = 4/3,\ m_4 = 1,$$

$$(4.5.4) \qquad \left|\frac{d}{dy}\psi_1(F^{-1}(y))\right| \ge \tilde{c},$$

on a subinterval τ *of* (0,1) *with length at least* η. *Then there exists* A > 0, *depending on* N, z^*, θ *and* F *only through* c, $\bar{C}$, δ, D, $\tilde{C}$, ε, $\tilde{c}$ *and* η, *such that*

$$\sup_x |R_\theta(x|z^*)-K_\theta(y_x)| \le A\{N^{-5/4}+N^{-3/4}\theta^3[\sum\{E|A_j-EA_j|^3\}^{4/9}]^{9/4}\},$$

where $K_\theta(x)$ *is given by* (3.3.4) *and*

$$y_x = [x - \theta \frac{[\sum(a_j EA_j - EA_j^2)]}{(N\widetilde{EA}_1^2)^{1/2}}][\frac{\sum\tilde{a}_j^2}{N\widetilde{EA}_1^2}]^{-1/2} .$$

PROOF. In the first place we note that $\widetilde{EA}_1^2$ remains bounded away from zero in view of (4.5.4). Hence no problems arise from the fact that $\widetilde{EA}_1^2$ occurs in the denominator of $(L-\theta N\widetilde{EA}_1^2)/(N\widetilde{EA}_1^2)^{1/2}$.

Next we observe that $R_\theta(x|z^*) = P_\theta\Big(L/(\sum a_j^2)^{1/2} \le \{x+\theta(N\widetilde{EA}_1^2)^{1/2}\}\{\sum\tilde{a}_j^2/N\widetilde{EA}_1^2\}^{-1/2}\Big)$ and $L = 2\sum a_j V_j - \sum a_j$. In view of the conditions of the present lemma, we can apply theorem 3.3.1, where the argument of K_θ is

$$[x + \theta(N\widetilde{EA}_1)^{1/2}][\frac{\sum\tilde{a}_j^2}{N\widetilde{EA}_1^2}]^{-1/2} - \theta \frac{\sum a_j E_0(-\psi_1(Z_j))}{(\sum a_j^2)^{1/2}} .$$

As $E_0(-\psi_1(Z_j)) = EA_j$, the desired result follows. □

The obvious way to proceed is to show that the conditions of this lemma are satisfied for all z^*, except for those in a set B with sufficiently small probability. Then the unconditional distribution of L follows by taking expectations with respect to Z^*. However, as $K_\theta(y_x)$ is not bounded on B, we first have to replace $K_\theta(y_x)$ by an approximation that can be controlled on B. Define

(4.5.5) $$\tilde{K}_\theta(x) = \Phi(x) + \phi(x)\Big[\frac{\sum a_j^4}{12(N\widetilde{EA}_1^2)^2}(x^3-3x) + \frac{\theta\sum a_j^3 EA_j}{3(N\widetilde{EA}_1^2)^{3/2}}(x^2-1) + \\ + \frac{\theta^2}{2N\widetilde{EA}_1^2}\{\sum a_j^2 EA_j^2 - \sigma^2(\sum a_j A_j)\}x + \frac{\theta^3}{6(N\widetilde{EA}_1^2)^{1/2}}\sum a_j E_0\{3\psi_1^3(Z_j) + \\ - 6\psi_1\psi_2(Z_j) + \psi_3(Z_j)\} - \frac{1}{2}x\frac{\sum(\tilde{a}_j^2 - \widetilde{EA}_1^2)}{N\widetilde{EA}_1^2} - \theta\frac{\sum(a_j EA_j - EA_j^2)}{(N\widetilde{EA}_1^2)^{1/2}} + \\ - \frac{1}{8}\frac{(\sum(\tilde{a}_j^2 - \widetilde{EA}_1^2))^2}{(N\widetilde{EA}_1^2)^2}(x^3-3x) - \frac{\theta}{2}\frac{\sum(\tilde{a}_j^2-\widetilde{EA}_1^2)\sum(a_j EA_j - EA_j^2)}{(N\widetilde{EA}_1^2)^{3/2}}(x^2-1) + \\ - \frac{\theta^2}{2}\frac{(\sum(a_j EA_j - EA_j^2))^2}{N\widetilde{EA}_1^2}x\Big],$$

(4.5.6) $$R = N^{-5/4} + N^{-3}|\sum(\tilde{a}_j^2 - \widetilde{EA}_1^2)|^3 + \theta^3 N^{-3/2}|\sum(a_j EA_j - EA_j^2)|^3.$$

LEMMA 4.5.2. *Under the conditions of lemma* 4.5.1 *we have*

$$\sup_x |K_\theta(y_x)-\tilde{K}_\theta(x)| \le AR.$$

PROOF. As $\sum(\tilde{a}_j^2-E\tilde{A}_1^2)/(NE\tilde{A}_1^2)$ remains bounded away from -1, we get the following expansion for y_x

$$(4.5.7) \qquad y_x = \{x - \theta \frac{\sum(a_jEA_j-EA_j^2)}{(NE\tilde{A}_1^2)^{1/2}}\}\{1 + \frac{\sum(\tilde{a}_j^2-E\tilde{A}_1^2)}{NE\tilde{A}_1^2}\}^{-1/2} =$$

$$= x - \frac{1}{2}x\frac{\sum(\tilde{a}_j^2-E\tilde{A}_1^2)}{NE\tilde{A}_1^2} - \theta\frac{\sum(a_jEA_j-EA_j^2)}{(NE\tilde{A}_1^2)^{1/2}} + \frac{3}{8}x\frac{(\sum(\tilde{a}_j^2-E\tilde{A}_1^2))^2}{(NE\tilde{A}_1^2)^2} +$$

$$+ \frac{\theta}{2}\frac{\sum(a_jEA_j-EA_j^2)\sum(\tilde{a}_j^2-E\tilde{A}_j^2)}{(NE\tilde{A}_1^2)^{3/2}} + O(R[1+|x|]).$$

Under the conditions of lemma 4.5.1, $1/y_x = O(1/x)$ as $|x| \to \infty$, and therefore

$$(4.5.8) \qquad \Phi(y_x) = \Phi(x) + \phi(x)\{(y_x-x) - \frac{1}{2}x(y_x-x)^2\} + O(|y_x-x|^3\phi(\frac{1}{2}x)).$$

As for all $p > 0$, $x^p\phi(x) = O(1)$, (4.5.7) and (4.5.8) together show that

$$(4.5.9) \qquad \Phi(y_x) = \Phi(x) + \phi(x)\Big\{-\frac{1}{2}x\frac{\sum(\tilde{a}_j^2-E\tilde{A}_1^2)}{NE\tilde{A}_1^2} - \theta\frac{\sum(a_jEA_j-EA_j^2)}{(NE\tilde{A}_1^2)^{1/2}} +$$

$$+ \frac{1}{8}(3x-x^3)\frac{(\sum(\tilde{a}_j^2-E\tilde{A}_1^2))^2}{(NE\tilde{A}_1^2)^2} + \frac{\theta}{2}(1-x^2)\frac{\sum(\tilde{a}_j^2-E\tilde{A}_1^2)\sum(a_jEA_j-EA_j^2)}{(NE\tilde{A}_1^2)^{3/2}} +$$

$$- \frac{\theta^2}{2}x\frac{(\sum(a_jEA_j-EA_j^2))^2}{NE\tilde{A}_1^2}\Big\} + O(R).$$

From (4.5.7) it is clear that

$$(4.5.10) \qquad y_x = x + O(N^{-1}x|\sum(\tilde{a}_j^2-E\tilde{A}_1^2)| + \theta N^{-1/2}|\sum(a_jEA_j-EA_j^2)|).$$

By means of Hölder's inequality we show that the coefficients of the second order terms in $K_\theta(x)$ are $O(N^{-1})$. Together with (4.5.10) this leads to

$$(4.5.11)\qquad K_\theta(y_x)-\Phi(y_x) = K_\theta(x)-\Phi(x) + O(N^{-2}|\sum(\tilde{a}_j^2-E\tilde{A}_1^2)| +$$

$$+ \theta N^{-3/2}|\sum(a_jEA_j-EA_j^2)|) = K_\theta(x)-\Phi(x) + O(R).$$

Finally,

$$\theta^{(4-k)}(\sum a_j^2)^{-k/2} = (\theta)^{(4-k)}(NE\tilde{A}_1^2)^{-k/2} + O(N^{-2}|\sum(\tilde{a}_j^2-E\tilde{A}_1^2)|) =$$

(4.5.12)

$$= \theta^{(4-k)}(NE\tilde{A}_1^2)^{-k/2} + O(R), \quad k = 1,2,3,4.$$

Combination of (4.5.9), (4.5.11) and (4.5.12) leads to $\tilde{K}_\theta(x)$. □

Now we can give an expansion for the unconditional distribution of L. Define

$$R_\theta(x) = P_\theta\left(\frac{L-\theta NE\tilde{A}_1^2}{(NE\tilde{A}_1^2)^{1/2}} \le x\right).$$

LEMMA 4.5.3. *Let there be positive constants* D, ε, $\tilde{C}$, $\tilde{c}$ *and* η *such that* (4.5.2), (4.5.3) *and* (4.5.4) *hold. Suppose in addition that* $\int_{-\infty}^{\infty}\psi_1^{10}(x)f(x)dx \le \tilde{C}$. *Then there exists* A > 0 *depending on* N, θ *and* F *only through* D, ε, $\tilde{C}$, $\tilde{c}$ *and* η, *such that*

$$(4.5.13)\qquad \sup_x |R_\theta(x)-E\tilde{K}_\theta(x)| \le A\{N^{-3/4}\theta^3[\sum\{E|A_j-EA_j|^3\}^{4/9}]^{9/4} + ER\},$$

where $\tilde{K}_\theta(x)$ *and* R *are given by* (4.5.5) *and* (4.5.6) *respectively.*

PROOF. The scores A_j we use, have the same distribution as the scores $-\psi_1(Z_j)$ for the LMP permutation test under the hypothesis and therefore we can apply lemma 4.3.3. This shows that under the conditions of the present lemma the results of lemma's 4.5.1 and 4.5.2 hold, except on a set B with $P_\theta(B) = O(N^{-5/4})$.

In view of corollary 4.3.1 it remains to prove that $E|K_\theta(x)I_B| = O(ER)$. To this end we note that

$$|\sum a_j^3EA_j| \le \sum a_j^4+\sum(EA_j)^4 \le \sum a_j^4+\sum EA_j^4 = O(\sum a_j^4+N).$$

Likewise $|\sum a_j^2 EA_j^2|$ and $|\sum a_j E_0\{3\psi_1^3(Z_j)-6\psi_1\psi_2(Z_j)+\psi_3(Z_j)\}|$ are of this order and $\sigma^2(\sum a_j A_j) = O((\sum a_j^2)^2+N^2)$. Furthermore, $\sum a_j^4 \leq (\sum a_j^2)^2 \leq 2N^2 E\tilde{A}_1^2 +$ $+ 2\{\sum(\tilde{a}_j^2-(E\tilde{A}_1^2))\}^2$. Hence the contribution of the second, third and fourth term of $\tilde{K}_\theta$ to $E|\tilde{K}_\theta I_B|$ is $O(ER)$. The same result can be proved for the remaining terms of $\tilde{K}_\theta$ in a manner analogous to the proof for the corresponding terms in corollary 4.3.1. □

COROLLARY 4.5.1. *Under the conditions of lemma* 4.5.3 *we have*

$$(4.5.14)\qquad \sup_x |R_\theta(x)-\tilde{R}_\theta(x)| \leq A\Big\{N^{-5/4}+N^{-3/4}\theta^3[\sum\{E|A_j-EA_j|^3\}^{4/9}]^{9/4} +$$

$$+ N^{-1}\theta^2 E[\sum(A_j-EA_j)^2]^2+N^{-1}\theta\{1+\theta\sigma(\sum(EA_j)A_j)\}\{E[\sum(A_j-EA_j)^2]^2\}^{1/2}\Big\},$$

where

$$(4.5.15)\qquad \tilde{R}_\theta(x) = \Phi(x)+\phi(x)\Big\{-\frac{N^{-1}}{24}\Big(\frac{E\tilde{A}_1^4}{(E\tilde{A}_1^2)^2}-3\Big)(x^3-3x) +$$

$$+ \frac{N^{-1/2}\theta}{12}\Big\{\frac{4N^{-1}\sum EA_j^3 EA_j-3E\tilde{A}_1^4}{(E\tilde{A}_1^2)^{3/2}} + 3(E\tilde{A}_1^2)^{1/2}\Big\}(x^2-1) +$$

$$+ \frac{\theta^2}{2E\tilde{A}_1^2}\{\sum(EA_j^2)^2-2\sigma^2(\sum(EA_j)A_j)\}x +$$

$$+ \frac{\theta^3}{6(NE\tilde{A}_1^2)^{1/2}}\sum(EA_j)E_0\{3\psi_1^3(Z_j)-6\psi_1\psi_2(Z_j)+\psi_3(Z_j)\} +$$

$$+ \frac{\theta}{(NE\tilde{A}_1^2)^{1/2}}\sum\sigma^2(A_j)\Big\}.$$

PROOF. In view of lemma 4.5.3 the essence of the proof is the evaluation of $E\tilde{K}_\theta(x)$ and ER from (4.5.5) and (4.5.6). The first term in (4.5.5) that deserves attention is $\sigma^2(\sum a_j A_j)$. Its expectation with respect to Z^* is

(4.5.16) $$E\sigma_0^2(\sum\psi_1(Z_j^*)\psi_1(Z_j)|Z^*) = EE_0[\sum\psi_1(Z_j^*)\{\psi_1(Z_j)-E_0\psi_1(Z_j)\}]^2 =$$

$$= EE_0[\sum E\psi_1(Z_j^*)\{\psi_1(Z_j)-E_0\psi_1(Z_j)\} +$$

$$+ \sum\{\psi_1(Z_j^*)-E\psi_1(Z_j^*)\}\{\psi_1(Z_j)-E_0\psi_1(Z_j)\}]^2 =$$

$$= \sigma^2(\sum(EA_j)A_j)+2EE_0[(\sum E\psi_1(Z_j^*)\{\psi_1(Z_j)-E_0\psi_1(Z_j)\})(\sum\{\psi_1(Z_j^*)-E\psi_1(Z_j^*)\}\cdot$$

$$\cdot \{\psi_1(Z_j)-E_0\psi_1(Z_j)\})]+EE_0[\sum\{\psi_1(Z_j^*)-E\psi_1(Z_j^*)\}\{\psi_1(Z_j)-E_0\psi_1(Z_j)\}]^2.$$

By Schwarz' inequality the last term in (4.5.16) is at most $E[\sum(A_j-EA_j)^2]^2$; by applying the same inequality to the second term we arrive at

$$E\sigma_0^2(\sum\psi_1(Z_j^*)\psi_1(Z_j)|Z^*) = \sigma^2(\sum(EA_j)A_j) +$$

$$+ O(\sigma(\sum(EA_j)A_j)[E\{\sum(A_j-EA_j)^2\}^2]^{1/2} + E\{\sum(A_j-EA_j)^2\}^2).$$

Now we treat the remaining terms in (4.5.5). We begin by noting that

$$E[\sum(\tilde{A}_j^2-E\tilde{A}_j^2)] = 0,\ E[\sum(\tilde{A}_j^2-E\tilde{A}_j^2)]^2 = N(E\tilde{A}_1^4-(E\tilde{A}_1^2)^2),$$

$$E\sum(A_jEA_j-EA_j^2) = -\sum\sigma^2(A_j),\ E|\sum(\tilde{A}_j^2-E\tilde{A}_j^2)|^3 = O(N^{3/2}).$$

Next we observe that $\sum(A_jEA_j-EA_j^2) = \frac{1}{2}\sum(\tilde{A}_j^2-E\tilde{A}_j^2) - \frac{1}{2}\sum(A_j-EA_j)^2 - \frac{1}{2}\sum\sigma^2(A_j)$. This leads to

$$E\{\sum(A_jEA_j-EA_j^2)\sum(\tilde{A}_j^2-E\tilde{A}_j^2)\} = \frac{1}{2}E[\sum(\tilde{A}_j^2-E\tilde{A}_j^2)]^2 +$$

$$-\frac{1}{2}E\{\sum(A_j-EA_j)^2\sum(\tilde{A}_j^2-E\tilde{A}_j^2)\} =$$

$$= \frac{N}{2}\{E\tilde{A}_1^4-(E\tilde{A}_1^2)^2\} + O(N^{1/2}\{E[\sum(A_j-EA_j)^2]^2\}^{1/2}).$$

Furthermore,

$$E[\sum(A_jEA_j-EA_j^2)]^2 = E[\sum(EA_j)(A_j-EA_j) - \sum\sigma^2(A_j)]^2 =$$

$$= \sigma^2(\sum(EA_j)A_j) + (\sum\sigma^2(A_j))^2,$$

$$E|\sum(A_jEA_j-EA_j^2)|^3 = O(E|\sum(EA_j)(A_j-EA_j)|^3 + (\sum\sigma^2(A_j))^3) =$$

$$= O(N\{\sum\sigma^2(A_j)\}^2 + [\sum|EA_j|\{E|A_j-EA_j|^3\}^{1/3}]^3) =$$

$$= O(N^{3/4}[\sum\{E|A_j-EA_j|^3\}^{4/9}]^{9/4} + N\{\sum\sigma^2(A_j)\}^2).$$

Substitution of these results in (4.5.13) gives (4.5.14) and (4.5.15) with an additional term of order $N^{-1}\theta^2(\sum\sigma^2(A_j))^2$. As

$$(\sum\sigma^2(A_j))^2 \le E[\sum(A_j-EA_j)^2]^2,$$

this term may be omitted. □

A further simplification of (4.5.14) and (4.5.15) is achieved by applying theorem 3.3.2 and the results of section 3.4. In doing so, we restrict attention to a fixed d.f. F. One of the consequences of this restriction is that condition (4.5.3) can be given a weaker formulation, whereas condition (4.5.4) can be omitted altogether (cf. the first remark following lemma 4.2.5). Let $\pi(\theta)$ be the power of the test based on L and define

$$(4.5.17) \qquad L_\theta(x) = \Phi(x) + \frac{\phi(x)}{N}\Big[-\frac{(\eta_2-3)}{24}(x^3-3x) + \frac{\eta_1(\eta_2+3)}{12}(x^2-1) + \frac{\eta_1^2(\eta_2+1)}{4}x + \frac{\eta_1^3(3\eta_3+\eta_2)}{18} + \eta_1\frac{\int_{1/N}^{1-1/N}(\psi_1'(t))^2t(1-t)dt}{E_0\psi_1^2(X_1)}\Big],$$

$$(4.5.18)\qquad \tilde{\pi}(\theta) = 1-\Phi(u_\alpha-\eta_1) + \frac{\eta_1\phi(u_\alpha-\eta_1)}{N}\Big[-\frac{(5\eta_2-3)}{24}(u_\alpha^2-1) + \frac{\eta_1(\eta_2-3)}{24}u_\alpha - \frac{(12\eta_3-5\eta_2-9)\eta_1^2}{72} - \frac{\int_{1/N}^{1-1/N}(\Psi_1'(t))^2 t(1-t)dt}{E_0\psi_1^2(X_1)}\Big],$$

where Ψ_1, η_1, η_2, η_3 are defined in (3.3.14), (3.4.24), (3.4.26). Let $\mathcal{F}$ be the class of d.f.'s F, defined by (3.4.5) and (3.4.6). We now arrive at our final result.

THEOREM 4.5.1. *Suppose that* F *is such that* $F \in \mathcal{F}$ *and* $\int_{-\infty}^{\infty}\psi_1^{10}(x)f(x)dx < \infty$. *Let there be positive constants* C *and* ε *such that* $0 \le \theta \le CN^{-1/2}$ *and* $\varepsilon \le \alpha \le 1-\varepsilon$. *Then, for every fixed* F, C *and* ε *there are positive numbers* A, δ_1, δ_2,... *such that* $\lim_{N\to\infty}\delta_N = 0$ *and for every* N

$$(4.5.19)\qquad \sup_x \left|P_\theta\left(\frac{L-\theta NE_0\psi_1^2(X_1)}{[NE_0\psi_1^2(X_1)]^{1/2}} \le x\right) - L_\theta(x)\right| \le \delta_N N^{-1} + AN^{-3/2}\int_{1/N}^{1-1/N}(\Psi_1'(t))^2(t(1-t))^{1/2}dt,$$

$$(4.5.20)\qquad |\pi(\theta)-\tilde{\pi}(\theta)| \le \delta_N N^{-1} + AN^{-3/2}\int_{1/N}^{1-1/N}(\Psi_1'(t))^2(t(1-t))^{1/2}dt.$$

PROOF. We check that under the conditions of this theorem the results of theorem 3.3.2 and lemma 4.5.3 hold. As was mentioned after theorem 3.4.2, the fact that $F \in \mathcal{F}$ implies that $\Psi_1'(t) = o([t(1-t)]^{-7/6})$ near 0 and 1. Hence theorem 3.3.2 holds and

$$N^{-3/4}\theta^3[\sum\{E|A_j-EA_j|^3\}^{4/9}]^{9/4} = O(N^{-5/4}).$$

The other remainder terms in (4.5.14) are dealt with in an analogous fashion. One may show that the condition $|\Psi_1'(t)| = o(t(1-t))^{-5/4}$ near 0 and 1, already suffices to ensure that $\theta\sigma(\sum(EA_j)A_j) = o(1)$, $E[\sum(A_j-EA_j)^2]^2 = o(N)$, and hence that $|R_\theta(x)-\tilde{R}_\theta(x)| = o(N^{-1})$.

The fact that the conditions of the present theorem also imply the results of lemma 4.5.3 and corollary 4.5.1 is verified in the same way as in which it is shown that theorem 3.4.1 implies theorem 3.3.1. Application of the results in (3.4.16)-(3.4.23) to (4.5.15) yields (4.5.19). From this, (4.5.20) follows in the usual way. □

REMARK. Note that under the hypothesis the theorem asserts that

$$P_0\left(\frac{L}{[NE_0\psi_1^2(X_1)]^{1/2}} \le x\right) = \Phi(x)-\phi(x)\,\frac{(n_2-3)}{24N}(x^3-3x) + o(N^{-1}).$$

This agrees with the fact that under H_0 the statistics L and $-\sum_{j=1}^{N}\psi_1(X_j)$ have the same distribution.

4.6. DEFICIENCIES

In this section we obtain the deficiencies among the various tests considered. First we summarize the results of Chapter 3, section 4.2, 4.3 and 4.5. $X_1,\ldots,X_N$ are i.i.d. r.v.'s from a d.f. $F(x-\theta)$, where $f = F'$ is symmetric around zero and positive on R^1. Denote the power

- of the MP parametric test, based on $1/\theta\ \sum\log\{f(X_j-\theta)/f(X_j)\}$ by $\pi_1(\theta)$,
- of the LMP parametric test, based on $-\sum\psi_1(X_j)$ by $\pi_2(\theta)$,
- of the MP permutation test, based on $(1/2\theta)\sum_{X_j>0}\log\{f(X_j-\theta)/f(X_j+\theta)\}$ by $\pi_3(\theta)$,
- of the LMP permutation test, based on $-\sum_{X_j>0}\psi_1(X_j)$ by $\pi_4(\theta)$,
- of the LMP rank test, based on $-\sum E_0\psi_1(Z_j)V_j$ by $\pi_5(\theta)$,
- of the LMP RRS test, based on $-\sum\psi_1(Z_j^*)\bar{V}_j$ by $\pi_6(\theta)$.

Note that $\pi_1(\theta)$ is the envelope power. The following two types of conditions will be imposed:

f is k times differentiable and $\psi_j = \frac{f^{(j)}}{f}$ satisfies

(4.6.1)
$$\limsup_{y\to 0}\int_{-\infty}^{\infty}|\psi_j(x+y)|^{m_j}f(x)dx < \infty,\ j = 1,\ldots,k,$$

where k is a positive integer and $m_j > 0$, $j = 1,\ldots,k$, and

$$\limsup_{t\to 0,1} t(1-t)\left|\frac{\Psi_1''(t)}{\Psi_1'(t)}\right| < \frac{3}{2},\ \text{where } \Psi_1(t) = \psi_1(F^{-1}(\tfrac{1+t}{2})). \qquad (4.6.2)$$

Now we introduce four classes of d.f.'s F, determined by such conditions

$$F_1 = \{F \mid (4.6.1) \text{ holds for } k = 5,\ m_j = \tfrac{5}{j},\ j = 1,\ldots,5\},$$

$$F_2 = F_1 \cap \{F \mid \limsup_{y\to 0}\int_{-\infty}^{\infty}\psi_1^{10}(x+y)f(x)dx < \infty\},$$

$$F_3 = \{F \mid (4.6.1) \text{ holds for } k = 4,\ m_1 = 6,\ m_2 = 3,\ m_3 = \tfrac{4}{3},\ m_4 = 1;\ (4.6.2) \text{ holds}\},$$

$$F_4 = F_3 \cap \{F \mid \int_{-\infty}^{\infty} \psi_1^{10}(x)f(x)dx < \infty\}.$$

Finally, let $K = (k_{ij})$ be the 6×4 matrix

$(\eta_2-3)/24$	$-(\eta_2-3)/24$	$-(\eta_2-3)/24$	$(2\eta_2-3\eta_3)/72$
$(\eta_2-3)/24$	$-(\eta_2-3)/24$	$-(\eta_2-3)/24$	$(5\eta_2-12\eta_3+9)/72$
$-\eta_2/12$	$(\eta_2-3)/24$	$-(\eta_2-3)/24$	$(5\eta_2-12\eta_3+9)/72$
$-\eta_2/12$	$(\eta_2-3)/24$	$-(\eta_2-3)/24$	$(5\eta_2-12\eta_3+9)/72$
$-\eta_2/12$	$(\eta_2-3)/24$	$\eta_2/12-\sum\sigma^2(\Psi_1(U_{j:N}))/\{2E_0\psi_1^2(X_1)\}$	$(5\eta_2-12\eta_3+9)/72$
$-(5\eta_2-3)/24$	$(\eta_2-3)/24$	$(5\eta_2-3)/24-\sum\sigma^2(\Psi_1(U_{j:N}))/\{E_0\psi_1^2(X_1)\}$	$(5\eta_2-12\eta_3+9)/72$

After these preliminaries, the following theorem can be formulated.

THEOREM 4.6.1. *Suppose there are positive constants* c, C *and* ε *such that* $c \le N^{1/2}\theta \le C$ *and* $\varepsilon \le \alpha \le 1-\varepsilon$. *Moreover, assume that* $F \in F_1$ *if* $i = 1,2$, $F \in F_2$ *if* $i = 3,4$, $F \in F_3$ *if* $i = 5$ *and* $F \in F_4$ *if* $i = 6$. *Then, for every fixed* F, c, C *and* ε, *there exists positive numbers* A, δ_1, $\delta_2, \ldots$ *such that* $\lim_{N\to\infty} \delta_N = 0$ *and for every* N *and* $i = 1,2,\ldots,6$

$$(4.6.3) \qquad \pi_i(\theta) = 1-\Phi(u_\alpha-\eta_1) + \frac{\eta_1\phi(u_\alpha-\eta_1)}{N}\{k_{i1}u_\alpha^2+k_{i2}u_\alpha\eta_1+k_{i3}+k_{i4}\eta_1^2\} + R_i,$$

where

$$\eta_1 = \theta(NE_0\psi_1^2(X_1))^{1/2},\quad \eta_2 = \{E_0\psi_1^4(X_1)\}/\{E_0\psi_1^2(X_1)\}^2,$$

$$\eta_3 = \{E_0\psi_2^2(X_1)\}/\{E_0\psi_1^2(X_1)\}^2$$

and

$$|R_1| \le AN^{-3/2},\quad |R_2| \le AN^{-3/2},\quad |R_3| \le AN^{-5/4},$$

$$|R_4| \le AN^{-5/4},\quad |R_5| \le \delta_N N^{-1},\quad |R_6| \le \delta_N N^{-1}.$$

PROOF. The result is immediate in view of theorem 3.4.1, 3.4.2, lemma 4.2.5, theorem 4.3.1, lemma 4.3.5 and theorem 4.5.1. Note that here the result is given for a fixed F, not only in the case of rank tests and RRS tests, but also in the case of parametric and permutation tests (cf. the remark following lemma 4.2.5). □

REMARK. If $F \in \mathcal{F}_3$, the sum of variances occurring in k_{53} and k_{63}, can be written as

$$\sum\sigma^2\{\Psi_1(U_{j:N})\} = \int_{1/N}^{1-1/N} (\Psi_1'(t))^2 t(1-t)dt + R,$$

$$|R| \le \delta_N + AN^{-1/2} \int_{1/N}^{1-1/N} (\Psi_1'(t))^2 (t(1-t))^{1/2} dt.$$

Denote the deficiency of the test with power $\pi_i(\theta)$ with respect to the test with power $\pi_j(\theta)$ as $d_N(i,j)$; if it exists, the corresponding asymptotic deficiency is denoted as $d(i,j)$ $(i,j = 1,\ldots,6)$. In the following theorem we give $d_N(i+1,i)$ and, if possible, $d(i+1,i)$, $i = 1,\ldots,5$. As deficiencies are transitive, this suffices to find $d_N(i,j)$ for all i and j.

THEOREM 4.6.2. *Under the conditions of theorem* 4.6.1 *we have*

$$d(2,1) = \eta_1^2(3\eta_3-\eta_2-3)/12, \quad |d_N(2,1)-d(2,1)| \le AN^{-1/2},$$

$$d(3,2) = (\eta_2-1)u_\alpha^2/4-(\eta_2-3)\eta_1 u_\alpha/6, \quad |d_N(3,2)-d(3,2)| \le AN^{-1/4},$$

$$d(4,3) = 0, \quad |d_N(4,3)| \le AN^{-1/4},$$

$$d_N(5,4) = \{\sum\sigma^2(\Psi_1(U_{j:N}))\}/\{E_0\psi_1^2(X_1)\}-(\eta_2-1)/4 + \delta_N,$$

$$d_N(6,5) = (\eta_2-1)u_\alpha^2/4 + \{\sum\sigma^2(\Psi_1(U_{j:N}))\}/\{E_0\psi_1^2(X_1)\}-(\eta_2-1)/4 + \delta_N.$$

PROOF. As $\Phi\{u_\alpha-\theta(N+d_N)^{1/2}(E_0\psi_1^2(X_1))^{1/2}\} = \Phi(u_\alpha-\eta_1)-\eta_1\phi(u_\alpha-\eta_1)d_N/(2N)+O(N^{-2})$, it follows that $d_N(i+1,i) = \{2N(\pi_i(\theta)-\pi_{i+1}(\theta))\}/\{\eta_1\phi(u_\alpha-\eta_1)\}+O(N^{-1})$. Application of (4.6.3) and the definition of $K = (k_{ij})$ leads to the desired result. □

REMARK. $d(2,1)$ is independent of α, $d_N(5,4)$ only depends on α and θ through

δ_N, $d_N(6,5)$ only depends on θ through δ_N.

APPLICATION 1. We first consider the normal case $F = \Phi$. In section 3.4 it has already been verified that $\Phi \in F_3$. The other F_i also contain Φ, as may be checked in the same way. Hence we may use all results of theorems 4.6.1 and 4.6.2. As $\eta_1 = N^{1/2}\theta$, $\eta_2 = 3$, $\eta_3 = 2$, it follows that

$$d(2,1) = 0,\ d(3,2) = \frac{1}{2}u_\alpha^2\ ,\ d(4,3) = 0,$$

$$d_N(5,4) = \sum\sigma^2(\Phi^{-1}(\frac{1+U_{j:N}}{2})) - \frac{1}{2} + \delta_N,$$

$$d_N(6,5) = \frac{1}{2}(u_\alpha^2 - 1) + \sum\sigma^2(\Phi^{-1}(\frac{1+U_{j:N}}{2})) + \delta_N.$$

According to (3.4.34),

$$\sum\sigma^2(\Phi^{-1}(\frac{U_{j:N}+1}{2})) = \int_0^{\Phi^{-1}(1-1/2N)} \frac{(2\Phi(x)-1)(1-\Phi(x))}{\phi(x)}\,dx + o(1) =$$

$$= 1/2 \log\log N + 1/2 \log 2 + 0.05832 \ldots + o(1).$$

Hence $d(5,4)$ and $d(6,5)$ do not exist, but on the other hand, $d_N(5,4)$ and $d_N(6,5)$ are of order log log N.

In section 4.4 it is mentioned that the asymptotic deficiency $d(3,t)$ of the normal permutation test with respect to the t-test equals zero. This result enables us to compare the t-test with the other tests we are considering. We have for example

$$d(t,1) = \frac{1}{2}u_\alpha^2, \tag{4.6.4}$$

$$d_N(5,t) = \sum\sigma^2(\Phi^{-1}(\frac{1+U_{j:N}}{2})) - \frac{1}{2} + \delta_N. \tag{4.6.5}$$

The first result was already obtained by Hodges and Lehmann (1970).

A final remark on the normal case is that $d_N(6,1) = 2d_N(5,1) + \delta_N$: the deficiency of the RRS test with respect to the envelope power is twice the deficiency of the normal scores test with respect to the envelope power, apart from a term that tends to zero as $N \to \infty$.

APPLICATION 2. As a second example we take the logistic d.f. $F(x) = 1/(1+e^{-x})$.

As $\psi_1 = 1-2F$, one easily verifies that $F \in \cap_{i=1}^{4} F_i$. We have $\eta_1=(N/3)^{1/2}\theta$, $\eta_2 = \eta_3 = 9/5$, $\sum\sigma^2(\Psi_1(U_{j:N})) = 1/6 + o(1)$ and therefore $d(i+1,i)$ exists for $i = 1,\dots,5$:

$$d(2,1) = \frac{N\theta^2}{60}\ ,\ d(3,2) = \frac{u_\alpha^2}{5} + \frac{u_\alpha N^{1/2}\theta\sqrt{3}}{15}\ ,\ d(4,3) = 0,$$

$$d(5,4) = \frac{3}{10},\ d(6,5) = \frac{u_\alpha^2}{5} + \frac{3}{10}\ .$$

$d(2,1)$ and $d(3,2)$ can be made more transparant by using the relation between θ and the power of the test. If we want to achieve a certain power $1-\beta$ at level α, it follows from $\beta = \Phi(u_\alpha-\sqrt{3}N^{1/2}\theta/3) + O(N^{-1})$ that $N^{1/2}\theta = \sqrt{3}(u_\alpha+u_\beta) + O(N^{-1})$ and therefore

$$d(2,1) = \frac{(u_\alpha+u_\beta)^2}{20}\ ,\ d_{3,2} = \frac{u_\alpha(2u_\alpha+u_\beta)}{5}\ .$$

CHAPTER 5

DEFICIENCIES OF SOME RELATED ESTIMATORS

5.1. INTRODUCTION

In this chapter we take advantage of the correspondence between some of the teststatistics we considered in chapters 3 and 4, and some well-known estimators. This correspondence immediately gives expansions to $o(N^{-1})$ for the distribution of these estimators. The expansions can be used to make deficiency comparisons between the estimators.

By applying certain generalizations of the Cramér-Rao bound - the so called Bhattacharyya bounds - we obtain a lower bound to $o(N^{-1})$ for the variance of an unbiased estimator. We also derive the deficiency of the estimators considered with respect to this lower bound.

In section 5.2 maximum likelihood estimators are dealt with; in section 5.3 we consider Hodges-Lehmann estimators.

5.2. MAXIMUM LIKELIHOOD ESTIMATORS

Let $X_1,\ldots,X_N$ be i.i.d. r.v.'s from $F(x-\theta)$, where $f = F'$ is symmetric around zero and positive on R^1. In section 4.2 we considered the test for $\theta = 0$ against $\theta > 0$ based on $S^* = -\sum_{j=1}^N \psi_1(X_j)$ and derived the expansion $\tilde{\pi}_{S^*}(\theta)$ for its power $\pi_{S^*}(\theta)$ in (4.2.24). From this a similar expansion for the distribution of the maximum likelihood estimator (MLE) $\hat{\theta}$ of θ can be derived. Since $\hat{\theta}$ is translation invariant, we may restrict attention to the case $\theta = 0$. Probabilities are then denoted as P_0, otherwise as P_θ. Define

$$H_N(x) = P_0([NE_0\psi_1^2(X_1)]^{1/2}\hat{\theta} \le x), \tag{5.2.1}$$

$$\tilde{H}_N(x) = \Phi(x) + \frac{x\phi(x)}{N}\{-\frac{(\eta_2-3)}{24} + \frac{x^2}{72}(5\eta_2-12\eta_3+9)\}, \tag{5.2.2}$$

where

$$\eta_2 = E_0\psi_1^4(X_1)/\{E_0\psi_1^2(X_1)\}^2 \text{ and } \eta_3 = E_0\psi_2^2(X_1)/\{E_0\psi_1^2(X_1)\}^2.$$

LEMMA 5.2.1. *Suppose that* f *is five times differentiable and*

$$(5.2.3) \qquad \limsup_{y\to 0} \int_{-\infty}^{\infty} |\psi_j(x+y)|^{5/j} f(x)dx < \infty, \; j = 1,\dots,5.$$

Moreover, assume that ψ_1 *is non-increasing. Then for every fixed d.f.* F *there exists* A > 0 *such that for all* x

$$(5.2.4) \qquad |H_N(x)-\tilde{H}_N(x)| \le A\{N^{-3/2}+N^{-3/2}x^2\}.$$

PROOF. By definition, $\hat{\theta}$ is the value of θ for which $\sum_{j=1}^N \psi_1(X_j-\theta) = 0$. As ψ_1 is non-increasing, the events $\{\sum_{j=1}^N \psi_1(X_j-N^{-1/2}x) > 0\}$ and $\{\hat{\theta} < N^{-1/2}x\}$ are essentially the same. Furthermore, under $\theta = 0$, $X_1-N^{-1/2}x$ has the same distribution as X_1 under $\theta = -N^{-1/2}x$. Together this shows that

$$(5.2.5) \qquad P_{(-\frac{x}{\sqrt{N}})}(\sum_{j=1}^N \psi_1(X_j) > 0) \le P_0(\hat{\theta} < \frac{x}{\sqrt{N}}) \le P_{(-\frac{x}{\sqrt{N}})}(\sum_{j=1}^N \psi_1(X_j) \ge 0).$$

Let $\{\delta_N\}$ be a sequence of positive real numbers defined by $\delta_N = b.\ N^{-p}$ for some $b > 0$ and $0 < p < \frac{1}{2}$. We first restrict attention to the case where $|x| < \delta_N N^{1/2}$. Then, for $\theta = -N^{-1/2}x$ and a fixed F, the conditions of lemmas 4.2.4 and 4.2.5 are satisfied (cf. the first remark after lemma 4.2.5). Hence we can use the expansion in (4.2.9) for the probabilities on the left and right side of (5.2.5). The first consequence of this is that these probabilities differ at most $O(N^{-3/2}+N^{-3/2}x^2)$ and therefore

$$(5.2.6) \qquad P_0(\hat{\theta} < \frac{x}{\sqrt{N}}) = P_{(-\frac{x}{\sqrt{N}})}(S^* \le 0) + O(N^{-3/2}+N^{-3/2}x^2).$$

Under $\theta = 0$, the symmetry of S^* gives $P_0(S^* \le 0) = \frac{1}{2} + O(N^{-3/2}+N^{-3/2}x^2)$. Hence, if $\pi_{S^*}(\theta,\alpha)$ denotes the power of the S^* test against the alternative θ at level α, (5.2.6) becomes

$$P_0(\hat{\theta} < \frac{x}{\sqrt{N}}) = 1-\pi_{S^*}(-\frac{x}{\sqrt{N}},\frac{1}{2}) + O(N^{-3/2}+N^{-3/2}x^2).$$

By replacing π_{S^*} by its expansion $\tilde{\pi}_{S^*}$, as given in (4.2.24), for $\alpha = 1/2$, we arrive at

$$P_0(\hat{\theta} < \frac{x}{\sqrt{N}}) = \tilde{H}_N(x(E_0\psi_1^2(X_1))^{1/2}) + O(N^{-3/2}+N^{-3/2}x^2),$$

which proves (5.2.4) for $|x| < \delta_N N^{1/2}$.

From (5.2.2) it follows that $|\tilde{H}_N(-x)| = |1-\tilde{H}_N(x)| \le A\{N^{-3/2}+N^{-3/2}x^2\}$ for $|x| \ge \delta_N N^{1/2}$. As $H_N(x)$ is non-decreasing, this shows that (5.2.4) also holds for $|x| \ge \delta_N N^{1/2}$. □

REMARK. Linnik and Mitrofanova (1965) have given Edgeworth expansions to $O(N^{-k})$ for the distribution of $\hat{\theta}$ under rather restrictive conditions. Recently, Čibišov (1972) has announced results where such expansions are obtained under minimal assumptions.

We now have an expansion to $o(N^{-1})$ for the distribution of $\{NE_0\psi_1^2(X_1)\}^{1/2}\hat{\theta}$. In order to answer the question, whether this also gives an expansion to $o(N^{-1})$ for the variance of $\{NE_0\psi_1^2(X_1)\}^{1/2}\hat{\theta}$, we first consider the following frequently occurring situation (cf. Hodges and Lehmann (1956), Chernoff (1956)). A normal sequence of estimators T_N has an asymptotic distribution with variance τ^2. Call τ^2 the variance of the asymptotic distribution. On the other hand, the variances of T_N tend to a limit σ^2, as N tends to infinity. Call σ^2 the asymptotic variance. Now it is not necessarily true that $\sigma^2 = \tau^2$. We can only assert that $\sigma^2 \geq \tau^2$ and strict inequality may occur. This arises, loosely speaking, when very large errors occur with very small probabilities. If one wants to take this possibility into account, one should use σ^2 as a criterion of performance, otherwise one can use τ^2.

In the present situation we define the following analogue of τ^2 for the normed sequence $\{NE_0\psi_1^2(X_1)\}^{1/2}\hat{\theta}$

(5.2.7) $$\tau_N^2 = \int_{-\infty}^{\infty} x^2 d\tilde{H}_N(x) - \left(\int_{-\infty}^{\infty} x d\tilde{H}_N(x)\right)^2.$$

Application of (5.2.2) shows that

(5.2.8) $$\tau_N^2 = 1 + \frac{(\eta_3 - \frac{1}{3}\eta_2 - 1)}{N}.$$

On the other hand, we of course have the variance of $\{NE_0\psi_1^2(X_1)\}^{1/2}\hat{\theta}$ which we shall denote by σ_N^2. Again one might use either σ_N^2 or τ_N^2 as a criterion of performance. It is therefore desirable to have conditions under which $\sigma_N^2 = \tau_N^2 + o(N^{-1})$. Such conditions are supplied by the following lemma.

LEMMA 5.2.2. *Let there be constants* $k > 10$ *and* $\tilde{c} > 0$ *such that* $E_0|\psi_1(X_1)|^k < \infty$ *and* $\psi_1^2 - \psi_2 \geq \tilde{c}$ *on* R^1. *If condition* (5.2.3) *is also satisfied, we have, for every fixed d.f.* F, *a constant* $A > 0$ *depending on* N *only through* k *and* $\tilde{c}$, *such that*

$$|\sigma_N^2-\tau_N^2| \le AN^{-\{1+\frac{(k-10)}{2(k+2)}\}} .$$

PROOF. As $-\psi_1' = \psi_1^2 - \psi_2 \ge \tilde{c} > 0$, we have for $x > 0$

$$(5.2.9) \qquad 1-H_N(x\{E_0\psi_1^2(X_1)\}^{1/2}) = P_0(\hat{\theta} > \tfrac{x}{\sqrt{N}}) = P_0(\sum_{j=1}^{N} \psi_1(X_j - \tfrac{x}{\sqrt{N}}) < 0) =$$

$$= P_0(\sum_{j=1}^{N} \psi_1(X_j) < \tfrac{x}{\sqrt{N}} \sum_{j=1}^{N} \psi_1'(X_j-\rho_j \tfrac{x}{\sqrt{N}}) \le P_0(\sum_{j=1}^{N} \psi_1(X_j) < -x\sqrt{N}\tilde{c}),$$

where $0 \le \rho_j \le 1$, $j = 1,\ldots,N$. Application of Chebyshev's inequality leads to

$$(5.2.10) \qquad P_0(\sum_{j=1}^{N} \psi_1(X_j) \le -x\sqrt{N}\tilde{c}) \le E_0|N^{-1/2} \sum_{j=1}^{N} \psi_1(X_j)|^k(x\tilde{c})^{-k}.$$

As $\psi_1(X_1),\ldots,\psi_1(X_N)$ are i.i.d. r.v.'s with mean zero and $E_0|\psi_1(X_1)|^k < \infty$, application of the inequality, due to Marcinkievitz, Zygmund and Chung (cf. 4.3.40), leads to

$$(5.2.11) \qquad E_0|N^{-1/2} \sum_{j=1}^{N} \psi_1(X_j)|^k < \infty.$$

Combining (5.2.9) - (5.2.11) we obtain that, for $x > 0$,

$$(5.2.12) \qquad 1-H_N(x) = \mathcal{O}(x^{-k}).$$

In view of the symmetry of f, we have for all x

$$P_0(\hat{\theta} \le \tfrac{x}{\sqrt{N}}) = P_{(-\frac{x}{\sqrt{N}})}(\sum_{j=1}^{N} \psi_1(X_j) \ge 0) =$$

$$= P_{(\frac{x}{\sqrt{N}})}(\sum_{j=1}^{N} \psi_1(X_j) \le 0) = P_0(\hat{\theta} \ge -\tfrac{x}{\sqrt{N}}),$$

and hence, for all x,

$$(5.2.13) \qquad H_N(-x) = 1-H_N(x).$$

From (5.2.12) and (5.2.13) it follows that $\int_{-\infty}^{\infty} x^2 dH_N(x) < \infty$. But then (5.2.13) also implies that $\int_{-\infty}^{\infty} x dH_N(x) = 0$ (i.e., $\hat{\theta}$ is unbiased), and hence $\sigma_N^2 = \int_{-\infty}^{\infty} x^2 dH_N(x)$. By another application of the symmetry of f and by partial integration, this leads to

$$(5.2.14)\qquad \sigma_N^2 = 2\int_0^\infty x^2 dH_N(x) = 4\int_0^\infty x(1-H_N(x))dx.$$

Let $\{\rho_N\}$ be a sequence of positive real numbers with $\rho_N \to \infty$ and $\rho_N = o(N^{1/2})$. In view of lemma 5.2.1 we have $H_N(x)-\tilde{H}_N(x) = O(N^{-3/2}+N^{-3/2}x^2)$ for $|x| < \rho_N$. This leads to

$$\sigma_N^2 = 4\int_0^{\rho_N} x(1-\tilde{H}_N(x))dx + 4\int_{\rho_N}^\infty x(1-H_N(x))dx + O(N^{-3/2}\rho_N^4).$$

From (5.2.2) it follows that, for all x, $\tilde{H}_N(-x) = 1-\tilde{H}_N(x)$ and hence $\tau_N^2 = 4\int_0^\infty x(1-\tilde{H}_N(x))dx$. Furthermore, from (5.2.2) it is also clear that $\int_{\rho_N}^\infty x(1-\tilde{H}_N(x))dx = O(\rho_N^3 \exp\{-\frac{1}{2}\rho_N^2\})$ and therefore

$$(5.2.15)\qquad \sigma_N^2-\tau_N^2 = 4\int_{\rho_N}^\infty x(1-H_N(x))dx + O(\rho_N^3 e^{-\frac{1}{2}\rho_N^2} + N^{-3/2}\rho_N^4).$$

Application of (5.2.12) to the integral in (5.2.15) shows that

$$\sigma_N^2-\tau_N^2 = O(\rho_N^{-k+2}+\rho_N^3 e^{-\frac{1}{2}\rho_N^2}+N^{-3/2}\rho_N^4).$$

For given k, it is most favourable to choose $\rho_N = N^{\frac{3}{2}(k+2)^{-1}}$. Then

$$\sigma_N^2-\tau_N^2 = O(N^{-\frac{3}{2}(\frac{k-2}{k+2})}),$$

which is the desired result. □

REMARK. Linnik and Mitrofanova (1965) prove that $\sigma_N^2 = 1 + o(N^{-1/2})$, under stronger conditions.

In the above lemma we have given conditions under which σ_N^2 and τ_N^2 agree to $o(N^{-1})$. In the sequel we shall no longer consider both kinds of variances but always take τ_N^2 as our criterion of performance.

The expression for τ_N^2 in (5.2.8) can be used for deficiency comparisons between the MLE and other estimators. An application of this kind occurs in the next section, where the MLE is compared to Hodges-Lehmann estimators. In the present section another possibility is treated: here we use (5.2.8) for a comparison of τ_N^2 with certain lower bounds for the variance of an unbiased estimator.

The first lower bound we consider is the well-known Cramér-Rao (CR) bound. The assumptions which are necessary to apply this bound are satisfied if the conditions of lemma 5.2.1 hold. We have for any unbiased estimator $U_N(X_1,\ldots,X_N)$ of θ that

$$(5.2.16) \qquad \sigma^2(\{NE_0\psi_1^2(X_1)\}^{1/2}U_N) \geq 1.$$

The variance of the asymptotic distribution of $\{NE_0\psi_1^2(X_1)\}^{1/2}U_N$ may be smaller than this lower bound, but only on a subset of $\Theta \subset R^1$ of Lebesgue measure zero. (cf. Bahadur (1964) and Rao (1965)). Comparison of (5.2.8) and (5.2.16) shows that the variance of the asymptotic distribution of $\{NE_0\psi_1^2(X_1)\}^{1/2}\hat{\theta}$ is asymptotically equal to the CR bound and hence the MLE is optimal to this extent.

Now Bhattacharyya (1946) has developped a series of refinements of the CR bound, which can be applied if stronger conditions are satisfied than are necessary for the CR bound. The k-th Bhattacharyya (B) bound for $\sigma^2(\{NE_0\psi_1^2(X_1)\}^{1/2}U_N\}$ has the form $1+\sum_{j=1}^{k-1} a_jN^{-j}$: the first B bound is the CR bound and the (k+1)-th B bound is obtained from the k-th B bound by adding a term of order N^{-k}. Hence, for comparison to $o(N^{-1})$, the second B bound is needed. From Davis (1951) we obtain that the assumptions, needed for this bound, are satisfied if the conditions of lemma 5.2.1 hold. Furthermore, it follows that, in view of the symmetry of f, $a_1 \equiv 0$, i.e. the second B bound coincides with the CR bound for the present case.

As $\tau_N^2 = 1+(\eta_3-\frac{1}{3}\eta_2-1)/N$, it follows that in general the MLE $\hat{\theta}$ is not optimal to $o(N^{-1})$. The difference between τ_N^2 and the second B bound equals

$$(5.2.17) \qquad \frac{\eta_3-\frac{1}{3}\eta_2-1}{N} = \frac{\sigma_0^2(\psi_1'(X_1))}{N\{E_0\psi_1^2(X_1)\}^2} \geq 0,$$

where equality occurs only if ψ_1' is constant a.s., and this is the case if X_1 has a normal distribution.

Finally we restate the result in terms of deficiencies. We have

$$\sigma^2(\hat{\theta}) = \frac{\sigma_N^2}{NE_0\psi_1^2(X_1)} \geq \frac{1}{NE_0\psi_1^2(X_1)} + o(N^{-2}),$$

$$\frac{\tau_N^2}{NE_0\psi_1^2(X_1)} = \frac{1}{NE_0\psi_1^2(X_1)} + \frac{\sigma_0^2(\psi_1'(X_1))}{N^2\{E_0\psi_1^2(X_1)\}^3}.$$

This implies that the asymptotic deficiency of the MLE with respect to the second B bound is finite and equals

$$(5.2.18) \qquad d = \frac{\sigma_0^2(\psi_1'(X_1))}{\{E_0\psi_1^2(X_1)\}^2} .$$

5.3. HODGES-LEHMANN ESTIMATORS

Let $X_1,\ldots,X_N$ be i.i.d. r.v.'s from $F(x-\theta)$, where $f = F'$ is symmetric around zero and positive on R^1. Let $Z_1,\ldots,Z_N$ be the absolute order statistics of the $X_1,\ldots,X_N$ and define $V_j = 1$ if the X_i corresponding to Z_j is positive, and $V_j = 0$ otherwise, for $j = 1,\ldots,N$. In chapter 3 we considered the test for the hypothesis of symmetry against $F(x-\theta)$, $\theta > 0$, based on $T = \sum_{j=1}^N a_jV_j$, with $a_j = EZ_j^*$, where $Z_1^*,\ldots,Z_N^*$ are order statistics from a d.f. F^*.

From this rank test, Hodges and Lehmann (1963) have derived an estimator $\tilde{\theta}$ of θ. Define $\mu = \frac{1}{2}\sum_{j=1}^N EZ_j^*$ and $X-\theta = (X_1-\theta,\ldots,X_N-\theta)$. Under the hypothesis, the distribution of T is symmetric around μ. Let $\tilde{\theta}_1 = \sup\{\theta\colon T(X-\theta) > \mu\}$ and $\tilde{\theta}_2 = \inf\{\theta\colon T(X-\theta) < \mu\}$, then $\tilde{\theta} = (\tilde{\theta}_1+\tilde{\theta}_2)/2$ is the Hodges-Lehmann (HL) estimator. Hodges and Lehmann prove that $\tilde{\theta}$ is translation invariant and distributed symmetrically around θ. Hence it suffices to find the distribution of $\tilde{\theta}$ for $\theta = 0$. The close connection between $\tilde{\theta}$ and T makes it easy to find an expansion for the d.f. of $\tilde{\theta}$ from the expansion for the d.f. of T, which we obtained in chapter 3. We restrict attention to the case where T is the locally most powerful rank test, i.e. where $a_j = -E_0\psi_1(Z_j)$. Define

$$K_N(x) = P_0(\{NE_0\psi_1^2(X_1)\}^{1/2}\tilde{\theta} \le x),$$

$$(5.3.1) \qquad \tilde{K}_N(x) = \Phi(x) + \frac{x\phi(x)}{N}\{\frac{\eta_2}{12} - \frac{\sum_{j=1}^N \sigma_0^2(\Psi_1(U_{j:N}))}{2E_0\psi_1^2(X_1)} + \frac{x^2}{72}(5\eta_2-12\eta_3+9)\},$$

where η_2 and η_3 are defined in (5.2.2), $\Psi_1(t) = \psi_1(F^{-1}(\frac{1+t}{2}))$ and $U_{1:N} < \ldots < U_{N:N}$ are order statistics of a sample of size N from the uniform distribution on $(0,1)$. Let $\mathcal{F}$ be the class of d.f.'s introduced in definition 3.4.1. The following lemma gives conditions under which $|K_N-\tilde{K}_N|$ is $o(N^{-1})$.

LEMMA 5.3.1. *Suppose that* $F \in \mathcal{F}$ *and that* ψ_1 *is non-increasing. Then, for every fixed* F *and every positive constant* C*, there exist positive numbers*

$\delta_1, \delta_2, \ldots$ *such that* $\lim_{N\to\infty} \delta_N = 0$ *and for all* $|x| \leq C$

$$|K_N(x) - \tilde{K}_N(x)| \leq \delta_N N^{-1}.$$

PROOF. From the construction of $\tilde{\theta}$ and the monotonicity of ψ_1 it follows that

$$(5.3.2) \qquad P_0(T(X - \tfrac{x}{\sqrt{N}}) < \mu) \leq P_0(\tilde{\theta} \leq \tfrac{x}{\sqrt{N}}) \leq P_0(T(X - \tfrac{x}{\sqrt{N}}) \leq \mu),$$

cf. Hodges and Lehmann (1963). Note that expression (5.3.2) is completely analogous to (5.2.5). The remaining part of the proof is therefore analogous to the proof of lemma 5.2.1. We only mention that under the conditions of the present lemma, theorem 3.4.2 can be applied. □

In analogy to the previous section, we use as a criterion of performance for $\tilde{\theta}$

$$(5.3.3) \qquad \tilde{\tau}_N^2 = \int_{-\infty}^{\infty} x^2 d\tilde{K}_N(x) - \left(\int_{-\infty}^{\infty} x d\tilde{K}_N(x)\right)^2 =$$

$$= 1 + \frac{1}{N}\left\{\eta_3 - \frac{7}{12}\eta_2 - \frac{3}{4} + \frac{\sum_{j=1}^{N} \sigma_0^2(\Psi_1(U_{j:N})}{E_0\psi_1^2(X_1)}\right\}.$$

Comparison of (5.2.8) and (5.3.3) leads to

$$(5.3.4) \qquad \tilde{\tau}_N^2 - \tau_N^2 = \frac{\sum_{j=1}^{N} \sigma_0^2(\Psi_1(U_{j:N}))}{NE_0\psi_1^2(X_1)} - \frac{(\eta_2 - 1)}{4N}.$$

It follows that the deficiency of the HL estimator $\tilde{\theta}$ with respect to the MLE $\hat{\theta}$ equals

$$(5.3.5) \qquad d_N = \frac{\sum_{j=1}^{N} \sigma_0^2(\Psi_1(U_{j:N}))}{E_0\psi_1^2(X_1)} - \frac{(\eta_2 - 1)}{4} + o(1).$$

Note that d_N equals the deficiency of the LMP rank test with respect to the LMP test for the size $\alpha = 1/2$ (cf. Albers, Bickel and van Zwet (1974)). Finally, the deficiency of the HL estimator with respect to the second B bound is the sum of the deficiencies in (5.2.18) and (5.3.5).

CHAPTER 6

FINITE SAMPLE COMPUTATIONS

6.1. INTRODUCTION

In section 3.4 we derived expansions for the power of one sample linear rank tests based on exact or approximate scores against contiguous location alternatives. In section 6.2 to 6.4 we investigate the performance of these expansions as approximations to the finite sample power. In particular we compare these expansions to the usual normal approximations.

In section 4.6 deficiencies of the above rank tests with respect to various other types of tests for the one sample problem have been obtained. Section 6.5 is devoted to the comparison of these expressions with deficiencies for finite sample sizes that are obtained numerically. We focus attention on the normal case and consider the test based on the sample mean, the t-test and the one sample normal scores test.

6.2. THE NORMAL SCORES AND THE WILCOXON TEST AGAINST NORMAL AND LOGISTIC ALTERNATIVES

Here we shall consider the one sample normal scores (NS) test and the one sample Wilcoxon (W) test, both against normal (N) location alternatives $G(x) = \Phi(x-\theta)$ and logistic (L) location alternatives $G(x) = 1/[1+\exp(-x+\theta)]$. We assume that θ is non-negative and $\theta = O(N^{-1/2})$. From section 3.4 we have

$$\pi_{NS,N}(\theta) = 1-\Phi(u_\alpha-\eta_1) - \frac{\eta_1\phi(u_\alpha-\eta_1)}{4N}\{u_\alpha^2 + - 1 + 2\sum_{j=1}^{N}\sigma^2(\Phi^{-1}(\frac{1+U_{j:N}}{2}))\} + o(N^{-1}), \tag{6.2.1}$$

$$\pi_{W,L}(\theta) = 1-\Phi(u_\alpha-\eta_2) - \frac{\eta_2\phi(u_\alpha-\eta_2)}{20N}\{3u_\alpha^2+u_\alpha\eta_2+2+\eta_2^2\} + o(N^{-1}), \tag{6.2.2}$$

$$\pi_{NS,L}(\theta) = 1-\Phi(u_\alpha-\eta_3) - \frac{\eta_3\phi(u_\alpha-\eta_3)}{12N}\Big\{u_\alpha^2+u_\alpha\eta_3(-5+2\pi)+(23-12\sqrt{2}) + + \eta_3^2(12\arctan\tfrac{23}{20}\sqrt{2}+1-4\pi) - 6\sum_{j=1}^{N}\sigma^2(\Phi^{-1}(\frac{1+U_{j:N}}{2}))\Big\} + o(N^{-1}), \tag{6.2.3}$$

$$(6.2.4)\qquad \pi_{W,N}(\theta) = 1-\Phi(u_\alpha-\eta_4) - \frac{\eta_4\phi(u_\alpha-\eta_4)}{N}\left\{u_\alpha^2\left(\frac{6\arctan\frac{1}{4}\sqrt{2}}{\pi} - \frac{9}{20}\right) + \right.$$

$$+ u_\alpha\eta_4\left(\frac{49}{20} - \frac{2}{3}\sqrt{3} - \frac{12\arctan\frac{1}{4}\sqrt{2}}{\pi}\right) + \left(\frac{44}{20} - \sqrt{2} - \frac{6\arctan\frac{1}{4}\sqrt{2}}{\pi}\right) +$$

$$\left. + \eta_4^2\left(\frac{6\arctan\frac{1}{4}\sqrt{2}}{\pi} + \frac{2}{3}\sqrt{3} - \frac{43}{20} + \frac{\pi}{9}\right)\right\} + o(N^{-1}),$$

where $U_{1:N} < \ldots < U_{N:N}$ are order statistics from the uniform distribution on $(0,1)$ and

$$(6.2.5)\qquad \eta_1 = N^{1/2}\theta,\ \eta_2 = \left(\frac{N}{3}\right)^{1/2}\theta,\ \eta_3 = \left(\frac{N}{\pi}\right)^{1/2}\theta,\ \eta_4 = \left(\frac{3N}{\pi}\right)^{1/2}\theta,$$

$$(6.2.6)\qquad \sum_{j=1}^{N}\sigma^2\left(\Phi^{-1}\left(\frac{1+U_{j:N}}{2}\right)\right) = \int_0^{\Phi^{-1}(1-\frac{1}{2N})}\frac{(2\Phi(x)-1)(1-\Phi(x))}{\phi(x)}\,dx + o(1) =$$

$$= \frac{1}{2}\log\log N + \frac{1}{2}\log 2 + 0.05832\ldots + o(1).$$

Upon evaluation of the coefficients, (6.2.3) and (6.2.4) become

$$(6.2.7)\qquad \pi_{NS,L}(\theta) = 1-\Phi(u_\alpha-\eta_3) - \frac{\eta_3\phi(u_\alpha-\eta_3)}{N}\{0.08333\,u_\alpha^2 + 0.10693\,u_\alpha\eta_3 +$$

$$+ 0.50245 + 0.05565\,\eta_3^2 - \frac{1}{2}\sum_{j=1}^{N}\sigma^2\left(\Phi^{-1}\left(\frac{1+U_{j:N}}{2}\right)\right)\} + o(N^{-1}).$$

$$(6.2.8)\qquad \pi_{W,N}(\theta) = 1-\Phi(u_\alpha-\eta_4) - \frac{\eta_4\phi(u_\alpha-\eta_4)}{N}\{0.19904\,u_\alpha^2 - 0.00278\,u_\alpha\eta_4 +$$

$$+ 0.13675 + 0.00281\,\eta_4^2\} + o(N^{-1}).$$

We shall now investigate how well the exact power is approximated by (6.2.1) - (6.2.4) for small samples. We shall also compare this approximation to the usual normal approximation, which approximates to $o(1)$ instead of $o(N^{-1})$. The necessary results about exact powers of the tests involved can be found in papers by Klotz (1963) and Thompson, Govindarajulu and Eisenstat (1967).

Klotz gives the small sample power for the normal scores test and the Wilcoxon test against normal alternatives, for sample sizes N = 5(1)10 at significance levels $\alpha = k/2^N$ for various integers k and for shifts

$\theta = 1/4(1/4)3/2(1/2)3$. The non-standard levels $k/2^N$ are necessary to avoid randomization. In short, his method consists of selecting the $k = \alpha.2^N$ orderings $V = (V_1,\dots,V_N)$ that give rise to the largest values of $T = \sum_{j=1}^N EJ(U_{j:N})V_j$ and adding the probabilities associated with these orderings. The evaluation of such probabilities involves the evaluation of an N-dimensional integral. By using a recursive scheme, this problem can be reduced to the computation of N one-dimensional integrals. In this way Klotz can go as far as $N = 10$, obtaining exact results in four decimal places.

For larger sample sizes the amount of computation that is necessary for this method, becomes prohibitive and one has to turn to Monte Carlo methods, as is done in the paper by Thompson et al. (1967).

They give the power for the normal scores and the Wilcoxon test against both normal and logistic alternatives for $N = 10, 20$, $\alpha = 0.01, 0.025$ and 0.05 and $\theta = 1/4(1/4)1,3/2$. In their paper these results are collected in table 4.1, where it is indicated that the test sizes considered are $\alpha = 0.01, 0.025$ and 0.10. However, the last value should be 0.05. This is not only evident from the numerical results obtained, but also from a remark elsewhere in the paper.

The method used by Thompson et al. is the following: first the required critical values are found by using Edgeworth approximations up to an appropriate order under the hypothesis. Then,for each combination of N, θ and α under consideration, 1000 trials are conducted. Each trial consists of drawing a random sample of size N from the standard normal or logistic distribution, shifting it over θ, forming the Wilcoxon and normal scores teststatistics and counting the number of samples for which the tests reject the hypothesis. This procedure results in unbiased power estimates with standard deviation at most 0.016.

In order to compute (6.2.1) to (6.2.4), we can use (6.2.6) for the sum of variances occurring in (6.2.1) and (6.2.3). Another possibility is to use values that are obtained numerically. Klotz (1963) gives a table of $E\Phi^{-1}\{(1+U_{j:N})/2\}$ for $N = 5(1)10$, whereas Thompson et al. (1967) give $\sum_{j=1}^N [E\Phi^{-1}\{(1+U_{j:N})/2\}]^2$ for $N = 10(1)20, 30, 50$ and 100. As

$$\sum_{j=1}^{N} \sigma^2\Phi^{-1}\{(1+U_{j:N})/2\} = N - \sum_{j=1}^{N} [E\Phi^{-1}\{(1+U_{j:N})/2\}]^2,$$

these results enable us to find the sum of variances for various sample sizes. For the sample sizes we shall consider, we list in table 6.2.1 the numerical values and the values supplied by the second approximation in (6.2.6).

Table 6.2.1

Approximations of $\sum_{j=1}^{N} \sigma^2\Phi^{-1}(\frac{1+U_{j:N}}{2})$

N	numerical approximation	$\frac{1}{2}$ log log N + $\frac{1}{2}$ log 2 + 0.05832
5	0.693	0.643
6	0.724	0.696
7	0.752	0.738
8	0.777	0.771
9	0.794	0.798
10	0.810	0.822
20	0.911	0.953
50	1.022	1.087
100	1.080	1.168

Tables 6.2.2 - 6.2.7 (p.128-p.133) give the results of the comparison of the approximations (6.2.1) - (6.2.4) to the normal approximation and to the results of Klotz and Thompson et al. We have used the numerically obtained values for $\sum_{j=1}^{N}\sigma^2\Phi^{-1}\{(1+U_{j:N})/2\}$ in (6.2.1) and (6.2.3). This is only slightly better than using the second approximation in table 6.2.1: for N = 5 or 6 the difference is always less than 0.004, for larger N it is even less than 0.001.

Inspection of these table shows that (6.2.1) - (6.2.4) supply excellent approximations for all N, α and θ under consideration. They always constitute a substantial improvement over the usual normal approximation, which yields values that are systematically too large. This bias is corrected by the $O(N^{-1})$ term, as may also be seen from (6.2.1) to (6.2.4).

6.3. THE WILCOXON TEST AGAINST CAUCHY ALTERNATIVES

In the previous section we have considered cases where the normal approximation already gives reasonable results. Adding terms of order N^{-1} merely constitutes an improvement, however substantial it may be, over an already rather satisfactory approximation. In view of this, it seems more interesting to consider situations where the normal approximation performs very badly. If in such a case the approximation to $o(N^{-1})$ does give reasonable results, we have found an approximation for a situation where none was available yet.

A case in which the normal approximation leads to very bad results occurs for example if Wilcoxon's test is used against location alternatives from a Cauchy distribution. For this case, an expansion to $o(N^{-1})$ can be justified. The standard Cauchy distribution has density $f(x) = 1/\{\pi(1+x^2)\}$. Hence, in the notation of chapter 3, $\psi_1(x) = -2x/(1+x^2)$, $\psi_2(x) = 2(3x^2-1)/(1+x^2)^2$ and $\psi_3(x) = -24x(x^2-1)/(1+x^2)^3$. Furthermore, $F(x) = (\arctan x)/\pi+1/2$ and therefore $F^{-1}\{(1+t)/2\} = \mathrm{tg}(\pi t/2)$, $-\Psi_1(t) = \sin \pi t$. Finally, for Wilcoxon's test, we have $J(t) = t$. From these facts it can easily be verified that the conditions of theorem 3.4.1 are satisfied. After elementary integrations we find that the power $\pi_{W,C}(\theta)$ satisfies $\pi_{W,C}(\theta) = \tilde{\pi}_{W,C}(\theta) + o(N^{-1})$, where

$$(6.3.1) \qquad \tilde{\pi}_{W,C}(\theta) = 1-\Phi(u_\alpha-\eta) - \frac{\eta\phi(u_\alpha-\eta)}{N}\left\{u_\alpha^2\left(\frac{11}{20}-\frac{6}{\pi^2}\right) + u_\alpha\eta\left(\frac{12}{\pi^2}-\frac{21}{20}\right) + \left(\frac{6}{\pi^2}-\frac{4}{5}\right) + \eta^2\left(\frac{\pi^2}{9}+\frac{7}{20}-\frac{6}{\pi^2}\right)\right\},$$

with $\eta = (3N)^{1/2}\theta/\pi$.

The exact power results for this case are obtained from a paper by Arnold (1965). Using the same approach as Klotz (1963), Arnold gives the power for Wilcoxon's test against alternatives from t-distributions with $\frac{1}{2}$, 1, 2 and 4 degrees of freedom. Note that the Cauchy distribution is the t_1-distribution and that the normal distribution is the t_∞-distribution. The sample sizes considered are $N = 5(1)10$, the levels are $\alpha = k/2^N$ for various k and the shifts are $\mu = 1/4$, 1/2, 1, 2 and 3. To obtain a better comparison with the case of normal alternatives, Arnold has scaled all densities f he considers in such a way that $\int_{1.645}^{\infty} f(x)dx = 0.05$. Here $1.645 = \Phi^{-1}(0.95)$. As $\frac{1}{\pi}\int_{6.314}^{\infty} (1+x^2)^{-1}dx = 0.05$, this means for the Cauchy case that the standard density $1/\{\pi(1+x^2)\}$ is replaced by $f(x/\sigma)/\sigma$, with $1/\sigma = 6.314/1.645 = 3.838$. Since the power for a shift μ and a density $f(x/\sigma)/\sigma$ under the hypothesis is the same as the power for a shift μ/σ and a density $f(x)$

under the hypothesis, the results from Arnold's paper and those obtained from (6.3.1) become comparable by inserting $\theta = \mu/\sigma = 3.838\mu$ in (6.3.1), instead of $\theta = \mu$.

In analogy to the previous section we compare the exact results with the normal approximation and with (6.3.1). The normal approximation is very bad. It tends to 1 too fast for increasing μ. However, the expansion in (6.3.1) leads to even worse results: as μ increases, this approximation tends to 0 very fast. A typical result is

Table 6.3.1

$N = 8$, $\alpha = 0.05469 = 14/256$

μ	power		
	exact	normal appr.	appr. (6.3.1)
1/4	.35	.46	.32
1/2	.57	.92	.06
1	.74	1.00	-10^{-9}
2	.86	1.00	-3×10^{-9}
3	.90	1.00	-2×10^{-9}

Apparently we have not succeeded in finding a useful approximation by considering higher order terms. The expansion (6.3.1) obviously has a very local character. It will only give reasonable results for very small values of θ. Here "small" means that these θ-values give rise to values of $\pi(\theta)$ considerably below 0.5, and this region is of little practical interest. The local character of (6.3.1) is borne out by computation of the coefficients in this expansion. We find

$$(6.3.2)\qquad \tilde{\pi}_{W,C}(\theta) = 1-\Phi(u_\alpha-0.55N^{1/2}\theta) - N^{-1/2}\theta\phi(u_\alpha-0.55N^{1/2}\theta)\{-0.03u_\alpha^2 + $$
$$+ 0.05u_\alpha N^{1/2}\theta-0.10+0.14N\theta^2\} \pm 0.00.....$$

This shows that $\tilde{\pi}_{W,C}(\theta)$ reaches its maximum for $\theta \approx (\frac{0.55}{0.42})^{1/2} \approx 1.1$. As any approximation of the power $\pi_{W,C}(\theta)$ should be increasing, it follows that $\tilde{\pi}_{W,C}(\theta)$ is certainly unreliable for $\theta \geq 1.1$. Since $\theta = 3.838\mu$, we have that (6.3.1) is certainly bad for $\mu \geq 0.3$, which agrees with table 6.3.1.

A similar inspection of (6.2.7) and (6.2.8) shows that here the coefficient of $N^{1/2}\theta^3\phi(u_\alpha-\eta)$ is much smaller than is the case in (6.3.2). Hence, these approximations first reach their maximum in θ if θ is large, i.e. if $\pi(\theta)$ is already very close to 1. This explains their excellent performance.

6.4. THE SIGN TEST AGAINST NORMAL, LOGISTIC AND DOUBLE EXPONENTIAL ALTERNATIVES

In section 3.5 an expansion was derived for the power $\pi(\theta)$ of the sign test. Let $\tau = N^{1/2}(2F(\theta)-1)$ and $\gamma_\alpha = (N+1+N^{1/2}u_\alpha)/2-[(N+1+N^{1/2}u_\alpha)/2]$, where [y] denotes the integer part of y. Then, for all θ such that τ is bounded,

$$(6.4.1) \qquad \pi(\theta) = 1-\Phi(u_\alpha-\tau) - \frac{\tau\phi(u_\alpha-\tau)}{12N}\{u_\alpha^2+u_\alpha\tau-3\tau^2+24\gamma_\alpha(1-\gamma_\alpha)-3\} + O(N^{-3/2}).$$

Here we shall investigate the performance of (6.4.1) as an approximation to the exact power, in the case of normal, logistic and double exponential alternatives. The last type of alternatives is considered since the sign test is the locally most powerful rank test against such alternatives. For these three alternatives we have $\tau = (2N/\pi)^{1/2}\int_0^\theta \exp(-x^2/2)dx$, $\tau = N^{1/2}(1-e^{-\theta})/(1+e^{-\theta})$ and $\tau = N^{1/2}(1-e^{-\theta})$, respectively.

The paper by Thompson et al. (1967), discussed in the previous section, also contains Monte Carlo estimates of the sign test power against these three types of alternatives. The values of N, α and θ that are considered, are the same as in the previous section. We compare approximation (6.4.1) with these estimates, and also with the normal approximation. The results are collected in tables 6.4.1-6.4.3 (p.134-p.136). It appears that (6.4.1) is better than the normal approximation, but the improvement is less striking than in section 6.2. For a number of combinations of N, α and θ for which both approximations perform rather well (e.g. where the absolute error is less than 0.03), the normal approximation is even slightly better. However, in cases where larger errors occur (e.g. larger than 0.06), approximation (6.4.1) is always substantially better than the normal approximation. The explanation of the fact that including terms of order N^{-1} is less effective than in section 6.2, probably lies in the pronounced lattice character of the sign test statistic.

6.5. DEFICIENCIES BETWEEN TESTS FOR THE NORMAL CASE

In chapter 4 an approximation to $o(1)$ for the deficiency of the locally most powerful rank test against alternatives $F(x-\theta)$ was found with respect to various other tests that are optimal in some sense for the one sample problem. Here we shall go into the question to what extent such an asymptotic expansion is useful as a prediction of the deficiency for finite sample sizes.

We shall restrict attention to normal alternatives. This case is very interesting, as some of the competitors of the locally most powerful rank test, i.e. the normal scores test, are well-known. In the first place, the parametric tests based on $\sum_{j=1}^{N} \log\{f(X_j-\theta)/f(X_j)\}$ and $\sum_{j=1}^{N} -\psi_1(X_j)$ respectively, both reduce to the test based on the sample mean $\bar{X} = N^{-1}\sum_{j=1}^{N} X_j$. Furthermore, in the normal case the teststatistic of the locally most powerful scale invariant test is explicitly known: it is the t-test statistic. Let $d_N(NS,\bar{X})(d_N(NS,t))$ denote the deficiency of the normal scores test with respect to the $\bar{X}$-test (t-test) based on N observations. Now we have from (4.6.4) and (4.6.5) that

$$(6.5.1) \qquad d_N(NS,\bar{X}) = \tilde{d}_N(NS,\bar{X}) + o(1), \; d_N(NS,t) = \tilde{d}_N(NS,t) + o(1),$$

where

$$(6.5.2) \qquad \tilde{d}_N(NS,\bar{X}) = \tilde{d}_N(NS,t) + \tfrac{1}{2}u_\alpha^2, \; \tilde{d}_N(NS,t) = \sum_{j=1}^{N} \sigma^2\Phi^{-1}\left(\frac{1+U_{j:N}}{2}\right) - \frac{1}{2}.$$

The remaining part of this section is devoted to the comparison of $d_N(NS,\bar{X})$ with $\tilde{d}_N(NS,\bar{X})$ and of $d_N(NS,t)$ with $\tilde{d}_N(NS,t)$. For the approximations $\tilde{d}_N$ we can use the values of $\sum_{j=1}^{N} \sigma^2\Phi^{-1}\{(1+U_{j:N})/2\}$, given in the first column of table 6.2.1. The exact values are obtained as follows: if the power of the normal scores test for a certain sample size N is available, we determine the sample size k_N for which the $\bar{X}$-test (or the t-test) reaches the same power, and this gives $d_{k_N} = N-k_N$. Here the role of N and k_N has been interchanged because of the fact that for the normal scores test only a limited number of exact power values is available, whereas the exact power of the other two tests can be obtained rather easily for any sample size. For convenience we compare d_{k_N} with $\tilde{d}_N$ rather than with $\tilde{d}_{k_N}$. As $d_{k_N}-\tilde{d}_{k_N} = o(1)$ and $\tilde{d}_N-\tilde{d}_{k_N} = O(N^{-1}\log\log N)$, this modification seems rather harmless, which impression is confirmed by the numerical results.

In the above k_N is treated as a continuous variable, which is interpreted as follows: for non-integer k_N, we select sample size $[k_N]$ or $[k_N]+1$ with probability $1-k_N+[k_N]$ and $k_N-[k_N]$ respectively. Here $[y]$ means the integer part of y. This yields an expected sample size k_N and an expected power $(1-k_N+[k_N])\pi_{[k_N]} + (k_N-[k_N])\pi_{[k_N]+1}$.

The first series of comparisons is based on the exact values of the power of the normal scores test, obtained by Klotz (1963) (cf. section 6.2). The power of the $\bar{X}$-test is $1-\Phi(u_\alpha-N^{1/2}\theta)$ and hence $d_{k_N}(NS,\bar{X})$ can be evaluated in a straightforward manner. Klotz also tabulated the exact efficiency $e_{k_N}(NS,t) = k_N/N$ of the normal scores test with respect to the t-test. Now $d_{k_N}(NS,t)$ immediately follows from the relation $d_{k_N}(NS,t) = N\{1-e_{k_N}(NS,t)\}$. The results are collected in table 6.5.1. (p.137). Note that $\tilde{d}_N(NS,\bar{X})$ only depends on N and α and that $\tilde{d}_N(NS,t)$ only depends on N. Here and in the sequel, we use for $\sum_{j=1}^N \sigma^2\Phi^{-1}\{(1+U_{j:N})/2\}$ the numerically obtained values from table 6.2.1. The agreement of the exact and asymptotic results appears to be satisfactory already at these small sample sizes.

The results of Thompson et al. (1967) from section 6.2 for the power of the normal scores test against normal alternatives, can also be used for deficiency computations. Here we deal with Monte Carlo estimates instead of with exact values. This leads to values of $d_{k_N}(NS,\bar{X})$ and $d_{k_N}(NS,t)$ which are also subject to error. As in general $d_{k_N} = N-k_N$ is much smaller than N, the relative error in d_{k_N} will be much larger than the relative error in the power estimates. To give an impression of the reliability of the obtained values of d_N, we also evaluate the values of d_N for the power estimates plus or minus their standarddeviation. We only use the power values for $N = 20$: for $N = 10$ we already have the exact results of Klotz, which are much more informative.

The necessary power values of the $\bar{X}$-test are again immediately given by $1-\Phi(u_\alpha-N^{1/2}\theta)$. For the power values of the t-test we proceed in the following way: the critical values involved are found from Owen (1962). Furthermore, Resnikoff and Lieberman (1957) have tabulated the non-central t-distribution function for various degrees of freedom f and various non-centrality parameters δ. These δ's are of the form $(f+1)^{1/2}u_\alpha$, whereas we need $(f+1)^{1/2}k/4$, for $k = 1,\dots,6$. Hence the necessary power values cannot be found directly from these tables. Using the description of the method of computation that is contained in the introduction to the tables, a program was written to obtain the power values for the values of δ considered here. According to Resnikoff and Lieberman the accuracy is four decimal places, which amply suffices for our purposes. The deficiency results are collected in table 6.5.2. (p.138). They are not very conclusive, as the estimates of the exact deficiency appear to be very crude, but again it seems that the asymptotic results to reasonable predictions.

For sample sizes larger than 20, no results about the power of the normal scores test are available in literature. Yet it seems desirable to have some idea about the agreement between the d_{k_N} and $\tilde{d}_N$ for such sample sizes. Therefore, we use the simulation method described by Thompson et al. (1967) to obtain estimates for the power of the normal scores test for sample sizes larger than 20.

In section 6.2 we already mentioned that this method involves conducting $\tilde{N}$ simulations, each consisting of drawing a random sample of size N from the standard normal distribution, shifting it over θ, computing $T = \sum_{j=1}^{N} E\Phi^{-1}\{(1+U_{j:N})/2\}V_j$ and counting the number of samples for which T exceeds the critical value c_α. Here we supply some more details. In the first place, we restrict attention to the case where $\tilde{N} = 1600$ and $N = 50$. For this value of N, the scores $E\Phi^{-1}\{(1+U_{j:N})/2\}$ can be found in tables by Govindarajulu and Eisenstat (1964). As these values are exact to five decimal places, their contribution to the error in the power estimates can be neglected.

For N as large as 50 it is impracticable to evaluate the exact critical values c_α and we have to use Edgeworth expansions, as advised and tabulated by Thompson et al. Denote these approximate values as c'_α. The portion of the error in the power estimate, due to the use of c'_α, may be estimated as follows: according to Thompson et al. the use of c'_α instead of c_α causes an error of at most 2% in the test size α, i.e.

$$(6.5.3)\qquad |P_0(T > c'_\alpha) - P_0(T > c_\alpha)| \leq 0.02\alpha.$$

The distribution of $(T-E_0T)/\sigma_0(T)$ is asymptotically standard normal and therefore $(c_\alpha-E_0T)/\sigma_0(T) \approx u_\alpha$. Hence

$$(6.5.4)\qquad P_0(T > c'_\alpha) - P_0(T > c_\alpha) \approx \Phi(\frac{c_\alpha-E_0T}{\sigma_0(T)}) - \Phi(\frac{c'_\alpha-E_0T}{\sigma_0(T)}) \approx \frac{c_\alpha-c'_\alpha}{\sigma_0(T)}\,\phi(u_\alpha).$$

Denote the exact power $P_\theta(T > c_\alpha)$ as π, the approximate power $P_\theta(T > c'_\alpha)$ as π' and let $u_\pi = \Phi^{-1}(1-\pi)$. In analogy to (6.5.4) we have

$$(6.5.5) \qquad \pi' - \pi \approx \frac{c_\alpha - c'_\alpha}{\sigma_\theta(T)} \phi(u_\pi).$$

As $\sigma^2_\theta(T) \approx \sigma^2_0(T) = N/4$, (6.5.3)-(6.5.5) lead to the following upper bound for the error that is caused by the use of inexact critical values

$$(6.5.6) \qquad |\pi'-\pi| \le 0.02\alpha \frac{\phi(u_\pi)}{\phi(u_\alpha)} .$$

The main source of error remains of course the fact that we use simulation methods to find π. For $i = 1,\ldots,\tilde{N}$, define the r.v.'s $\hat{\pi}_i$ by

$$\hat{\pi}_i = \begin{cases} 1, & T > c'_\alpha \text{ for the } i^{th} \text{ sample,} \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, define $\hat{\pi} = \tilde{N}^{-1}\sum_{i=1}^{\tilde{N}} \hat{\pi}_i$. $\hat{\pi}$ clearly is an unbiased estimate of π'. If all samples are drawn independently, we have as an unbiased estimate for its variance

$$(6.5.7) \qquad \hat{\sigma}^2(\hat{\pi}) = \frac{\hat{\pi}(1-\hat{\pi})}{\tilde{N}}.$$

The variance of $\hat{\pi}$ can be reduced by using the antithetic method that is also applied by Thompson et al.. Here we independently draw $\tilde{N}/2$ samples $X_1,..,X_N$ from $\Phi(x-\theta)$ and form the other $\tilde{N}/2$ samples by taking $2\theta-X_1,\ldots,2\theta-X_N$ for each of the first $\tilde{N}/2$ samples. Note that if X_1 has d.f. $\Phi(x-\theta)$, this is also the case for $2\theta-X_1$. Now the $\hat{\pi}_i$ form $\tilde{N}/2$ pairs of dependent r.v.'s. Each pair has the same covariance, which we denote as Covar, and the pairs are mutually independent. The variance of $\hat{\pi}$ is $\{\pi'(1-\pi')+\text{Covar}\}/\tilde{N}$. From the construction of the pairs it is clear that Covar < 0 and therefore this method decreases the variance of $\hat{\pi}$.. Computation of Covar is too complicated, but the estimated values of Covar that are obtained from our numerical results, indicate that the reduction is rather small. In view of this, the conjecture of Thompson et al. that $\sigma^2(\hat{\pi})$ becomes about ten times smaller seems far too optimistic. The main advantage of the antithetic method lies here in the reduction of the time, needed to form the $\tilde{N}$ samples.

The expressions in (6.5.2) for $\tilde{d}_N(NS,\bar{X})$ and $\tilde{d}_N(NS,t)$ suggest that the normal scores test is only slightly worse than the $\bar{X}$-test and the t-test. Hence we may expect the critical regions of the three tests to be much the same. As the power of the $\bar{X}$-test and the t-test can be obtained with high accuracy for all N, α and θ, the above resemblance enables us to achieve a further reduction of $\sigma^2(\hat{\pi})$. For each of the $\tilde{N}$ samples that is drawn, we not only note whether $T > c'_\alpha$, but also whether the $\bar{X}$-test and the t-test exceed their respective critical values. To be precise, we define for i,j,k = 0,1 the r.v.'s $\hat{\pi}_{ijk}$ as $\hat{\pi}_{ijk} = \tilde{N}^{-1}\times$the number of samples for which the $\bar{X}$-test does (does not) reject the hypothesis if i = 0(1), for which the t-test does (does not) reject the hypothesis if j = 0(1) and for which the normal scores test does (does not) reject the hypothesis if k = 0(1). Moreover, let $\hat{\pi}_{ij.} = \hat{\pi}_{ij0}+\hat{\pi}_{ij1}$, and define $\hat{\pi}_{i.k}$ and $\hat{\pi}_{.jk}$ analogously. Then we have

$$\hat{\pi} = \hat{\pi}_{000} + \hat{\pi}_{100} + \hat{\pi}_{010} + \hat{\pi}_{110} = \hat{\pi}_t + \hat{\pi}_{.10} - \hat{\pi}_{.01} =$$

(6.5.8)

$$= \hat{\pi}_{\bar{X}} + \hat{\pi}_{1.0} - \hat{\pi}_{0.1},$$

where $\hat{\pi}_t$ and $\hat{\pi}_{\bar{X}}$ are unbiased estimates of the power π_t of the t-test and the power $\pi_{\bar{X}}$ of the $\bar{X}$-test, respectively. Since π_t and $\pi_{\bar{X}}$ can be obtained exactly, we can improve on $\hat{\pi}$ by considering the following two estimates

$$\hat{\pi}_1 = \pi_t + \hat{\pi}_{.10} - \hat{\pi}_{.01} = \hat{\pi}+ (\pi_t-\hat{\pi}_t),$$

(6.5.9)

$$\hat{\pi}_2 = \pi_{\bar{X}} + \hat{\pi}_{1.0} - \hat{\pi}_{0.1} = \hat{\pi}+ (\pi_{\bar{X}} - \hat{\pi}_{\bar{X}}),$$

Note that $\hat{\pi}_1$ and $\hat{\pi}_2$ are also unbiased estimates of π'. From the close resemblance of the critical regions of the three tests it follows that $\hat{\pi}_{.10}$, $\hat{\pi}_{.01}$, $\hat{\pi}_{1.0}$ and $\hat{\pi}_{0.1}$ estimate very small probabilities. Hence, their variances are also small. In view of this and of (6.5.9), we may expect that $\sigma^2(\hat{\pi}_1)$ and $\sigma^2(\hat{\pi}_2)$ are considerably smaller than $\sigma^2(\hat{\pi})$ and hence this approach yields another reduction of $\sigma^2(\hat{\pi})$. Unbiased estimates for the variances of $\hat{\pi}_1$ and $\hat{\pi}_2$ are

$$\hat{\sigma}^2(\hat{\pi}_1) = \tilde{N}^{-1}\{\hat{\pi}_{.10}(1-\hat{\pi}_{.10}) + \hat{\pi}_{.01}(1-\hat{\pi}_{.01}) + 2\hat{\pi}_{.10}\hat{\pi}_{.01}\},$$

(6.5.10)

$$\hat{\sigma}^2(\hat{\pi}_2) = \tilde{N}^{-1}\{\hat{\pi}_{1.0}(1-\hat{\pi}_{1.0}) + \hat{\pi}_{0.1}(1-\hat{\pi}_{0.1}) + 2\hat{\pi}_{1.0}\hat{\pi}_{0.1}\}.$$

As our ultimate estimate we use $\hat{\pi}_3 = (\hat{\pi}_1+\hat{\pi}_2)/2$. An unbiased estimate for $\sigma^2(\hat{\pi}_3)$ is

$$\hat{\sigma}^2(\hat{\pi}_3) = \{\hat{\sigma}^2(\hat{\pi}_1) + \hat{\sigma}^2(\hat{\pi}_2)\}/4 + \{\hat{\pi}_{110} - \hat{\pi}_{.10}\hat{\pi}_{1.0} + \hat{\pi}_{001} +$$

(6.5.11)

$$- \hat{\pi}_{.01}\hat{\pi}_{0.1} + \hat{\pi}_{.10}\hat{\pi}_{0.1} + \hat{\pi}_{1.0}\hat{\pi}_{.01}\}/(2\tilde{N}).$$

Together with (6.5.5) this leads to the following estimated standard deviation of the obtained power estimates.

(6.5.12) $$\{[0.02\alpha \frac{\phi(u_\pi)}{\phi(u_\alpha)}]^2 + \hat{\sigma}^2(\hat{\pi}_3)\}^{1/2}.$$

The numerical results are given in tables 6.5.3 and 6.5.4 (p.139). We also give the values of d_N that are obtained if we use the power estimates plus or minus their standard deviation. Again the agreement between finite and asymptotic results is satisfactory.

The general conclusions of this section are that the asymptotic results seem to provide a reasonable approximation of the exact values and that the normal scores test requires only very few additional observations to attain the same power against normal alternatives as the $\bar{X}$-test or the t-test.

Table 6.2.2

Power of the normal scores test against normal alternatives. The upper, middle and lower numbers give the exact values obtained by Klotz, the approximations (6.2.1), and the normal approximations, respectively.

N	α \ θ	1/4	1/2	3/4	1	5/4	3/2
5	.06250	.145 .146 .165	.278 .284 .339	.450 .465 .557	.629 .654 .759	.781 .810 .896	.888 .913 .966
6	.04688	.125 .126 .144	.263 .268 .326	.450 .465 .564	.646 .670 .780	.807 .834 .917	.911 .933 .977
7	.05469	.155 .156 .174	.332 .336 .390	.556 .566 .649	.760 .775 .852	.897 .910 .956	.966 .973 .991
8	.05469	.167 .168 .186	.369 .373 .426	.614 .623 .698	.818 .829 .890	.936 .944 .973	.983 .987 .996
8	.07422	.212 .213 .230	.438 .441 .488	.684 .692 .750	.866 .875 .917	.958 .964 .982	.991 .993 .997
9	.02734	.104 .104 .121	.271 .274 .337	.515 .524 .629	.750 .765 .860	.904 .917 .966	.973 .979 .995
9	.03711	.132 .133 .150	.326 .329 .388	.581 .590 .679	.803 .816 .888	.932 .942 .975	.983 .988 .997
9	.04883	.164 .165 .182	.380 .384 .438	.642 .650 .724	.847 .857 .910	.953 .960 .982	.990 .992 .998
10	.00098	.006 .006 .011	.025 .027 .064	.077 .092 .234	.178 .229 .526	.327 .439 .804	.501 .669 .950
10	.00977	.048 .048 .061	.158 .160 .225	.362 .373 .514	.612 .636 .796	.820 .845 .947	.937 .954 .992
10	.02441	.102 .102 .119	.283 .285 .348	.544 .553 .656	.785 .798 .883	.928 .938 .976	.983 .987 .997
10	.05273	.186 .186 .204	.431 .434 .485	.706 .712 .774	.894 .901 .939	.974 .978 .990	.996 .997 .999
10	.09668	.288 .289 .305	.572 .575 .610	.820 .825 .858	.950 .953 .969	.991 .992 .996	.999 .999 1.000

Table 6.2.3

Power of the Wilcoxon test against normal alternatives. The upper, middle and lower numbers give the exact values obtained by Klotz, the approximations (6.2.4), and the normal approximations, respectively.

N	α \ θ	1/4	1/2	3/4	1	5/4	3/2
5	.06250	.145 .146 .162	.278 .283 .329	.450 .462 .542	.629 .650 .742	.781 .805 .884	.888 .909 .959
6	.04688	.125 .126 .141	.263 .268 .316	.450 .464 .547	.646 .669 .763	.807 .832 .906	.911 .931 .972
7	.05469	.155 .155 .170	.332 .334 .379	.556 .563 .632	.760 .771 .837	.897 .907 .948	.966 .972 .989
8	.05469	.167 .167 .181	.369 .370 .413	.614 .619 .681	.818 .825 .878	.936 .941 .968	.983 .986 .995
8	.07422	.212 .211 .225	.436 .437 .475	.683 .686 .735	.865 .870 .906	.958 .961 .978	.991 .992 .997
9	.02734	.104 .104 .117	.271 .275 .324	.515 .525 .609	.750 .765 .844	.904 .916 .959	.973 .979 .993
9	.03711	.132 .132 .146	.326 .328 .375	.581 .589 .660	.804 .814 .874	.933 .940 .970	.984 .987 .995
9	.04883	.164 .164 .178	.379 .381 .424	.640 .646 .706	.846 .853 .899	.953 .957 .978	.990 .992 .997
10	.00098	.006 .006 .010	.025 .031 .060	.077 .105 .218	.178 .262 .497	.327 .489 .778	.501 .719 .938
10	.00977	.048 .049 .059	.158 .164 .215	.362 .382 .493	.612 .646 .775	.820 .853 .937	.937 .957 .989
10	.02441	.102 .102 .115	.282 .286 .335	.544 .554 .636	.785 .798 .869	.928 .937 .971	.983 .987 .996
10	.05273	.184 .185 .199	.427 .430 .470	.701 .707 .758	.891 .897 .929	.973 .976 .988	.996 .996 .999
10	.09668	.286 .286 .299	.567 .568 .596	.815 .817 .845	.948 .949 .963	.990 .991 .995	.999 .999 1.000

Table 6.2.4

Power of the normal scores test against normal alternatives. The upper, middle and lower numbers give the Monte Carlo powers obtained by Thompson et al., the approximations (6.2.1), and the normal approximations, respectively.

N	α \ θ	1/4	1/2	3/4	1	3/2
10	.01	.041 .049 .062	.160 .163 .228	.365 .377 .518	.622 .640 .798	.954 .956 .992
10	.025	.086 .104 .121	.285 .289 .352	.554 .559 .660	.794 .802 .885	.990 .988 .997
10	.05	.178 .179 .196	.425 .423 .475	.697 .702 .766	.896 .895 .935	.998 .997 .999
20	.01	.112 .098 .113	.406 .396 .464	.781 .778 .848	.955 .964 .984	- 1.000 1.000
20	.025	.199 .182 .200	.551 .558 .609	.874 .885 .918	.988 .988 .994	- 1.000 1.000
20	.05	.296 .282 .299	.683 .689 .723	.940 .941 .956	.995 .996 .998	- 1.000 1.000

Table 6.2.5

Power of the Wilcoxon test against normal alternatives. The upper, middle and lower numbers give the Monte Carlo powers obtained by Thompson et al., the approximations (6.2.4), and the normal approximations, respectively.

N	θ / α	1/4	1/2	3/4	1	3/2
10	.01	.041 .050 .060	.158 .166 .217	.366 .386 .497	.621 .650 .778	.952 .958 .990
10	.025	.084 .104 .118	.288 .290 .339	.556 .559 .640	.793 .802 .871	.990 .987 .996
10	.05	.171 .178 .192	.412 .419 .460	.690 .697 .749	.893 .891 .926	.998 .996 .999
20	.01	.106 .097 .109	.396 .392 .444	.768 .774 .829	.953 .962 .980	- 1.000 1.000
20	.025	.185 .180 .193	.520 .550 .589	.868 .879 .906	.982 .986 .992	- 1.000 1.000
20	.05	.276 .278 .290	.650 .679 .705	.924 .936 .949	.993 .995 .997	- 1.000 1.000

Table 6.2.6

Power of the normal scores test against logistic alternatives. The upper, middle and lower numbers give the Monte Carlo powers obtained by Thompson et al., the approximations (6.2.3), and the normal approximations, respectively.

N	α \ θ	1/4	1/2	3/4	1	3/2
10	.01	.029 .028 .030	.063 .066 .076	.133 .131 .162	.239 .226 .294	.486 .467 .637
10	.025	.056 .062 .065	.144 .130 .143	.244 .233 .267	.373 .364 .430	.632 .636 .763
10	.05	.122 .112 .115	.220 .212 .226	.353 .346 .380	.507 .498 .555	.755 .762 .849
20	.01	.046 .043 .045	.144 .131 .143	.301 .293 .332	.510 .504 .578	.849 .850 .928
20	.025	.103 .089 .092	.238 .229 .242	.458 .437 .473	.653 .657 .713	.929 .927 .966
20	.05	.166 .152 .155	.358 .337 .351	.560 .568 .598	.763 .771 .810	.963 .965 .984

Table 6.2.7

Power of the Wilcoxon test against logistic alternatives. The upper, middle and lower numbers give the Monte Carlo powers obtained by Thompson et al., the approximations (6.2.2), and the normal approximations, respectively.

N	α \ θ	1/4	1/2	3/4	1	3/2
10	.01	.028 .028 .031	.061 .065 .079	.131 .132 .169	.240 .231 .308	.486 .489 .660
10	.025	.055 .062 .066	.139 .131 .148	.243 .237 .277	.373 .374 .447	.637 .662 .782
10	.05	.121 .112 .117	.230 .215 .232	.356 .355 .391	.516 .512 .572	.770 .785 .863
20	.01	.044 .043 .046	.133 .134 .150	.301 .302 .348	.520 .522 .601	.860 .872 .939
20	.025	.093 .090 .094	.232 .234 .252	.447 .450 .491	.648 .677 .733	.926 .941 .972
20	.05	.152 .154 .159	.338 .345 .362	.557 .583 .615	.762 .789 .826	.964 .973 .987

Table 6.4.1

Power of the sign test against normal alternatives. The upper, middle and lower values give the Monte Carlo powers obtained by Thompson et al., the approximations (6.4.1), and the normal approximations, respectively.

N	α \ θ	1/4	1/2	3/4	1	3/2
10	.01	.042 .041 .044	.128 .123 .132	.284 .270 .275	.487 .455 .433	.832 .741 .660
10	.025	.071 .085 .091	.207 .213 .227	.386 .403 .409	.591 .601 .579	.921 .846 .782
10	.05	.148 .152 .154	.328 .334 .332	.555 .558 .534	.761 .749 .696	.967 .928 .863
20	.01	.070 .071 .074	.263 .264 .270	.558 .567 .547	.823 .815 .766	- .978 .939
20	.025	.137 .138 .141	.403 .408 .402	.719 .721 .686	.915 .908 .863	- .993 .972
20	.05	.211 .220 .223	.524 .535 .527	.826 .818 .788	.961 .951 .920	- .997 .987

Table 6.4.2

Power of the sign test against logistic alternatives. The upper, middle and lower values give the Monte Carlo powers obtained by Thompson et al., the approximations (6.4.1), and the normal approximations, respectively.

N	θ / α	1/4	1/2	3/4	1	3/2
10	.01	.026 .025 .027	.058 .056 .060	.112 .108 .116	.192 .184 .194	.409 .385 .375
10	.025	.049 .056 .059	.110 .110 .118	.177 .191 .204	.288 .296 .309	.533 .530 .519
10	.05	.095 .104 .105	.191 .190 .192	.298 .305 .304	.449 .437 .427	.691 .684 .642
20	.01	.031 .037 .038	.100 .104 .109	.219 .228 .235	.400 .399 .398	.739 .737 .696
20	.025	.079 .079 .080	.189 .191 .194	.351 .364 .360	.552 .561 .542	.875 .856 .811
20	.05	.129 .137 .138	.290 .289 .291	.480 .488 .483	.678 .683 .663	.936 .918 .884

Table 6.4.3

Power of the sign test against double exponential alternatives. The upper, middle and lower numbers give the Monte Carlo powers obtained by Thompson et al., the approximations (6.4.1), and the normal approximations, respectively.

N	α \ θ	1/4	1/2	3/4	1	3/2
10	.01	.050 .048 .052	.135 .130 .140	.261 .249 .255	.404 .381 .372	.662 .604 .552
10	.025	.097 .097 .104	.226 .223 .237	.381 .377 .385	.510 .525 .516	.773 .737 .690
10	.05	.176 .170 .172	.369 .347 .344	.535 .530 .509	.691 .680 .638	.890 .856 .792
20	.01	.072 .086 .091	.256 .280 .285	.501 .528 .513	.712 .732 .692	.951 .926 .874
20	.025	.155 .163 .166	.424 .428 .421	.670 .686 .655	.858 .852 .807	.979 .971 .935
20	.05	.243 .254 .256	.547 .555 .546	.779 .791 .763	.914 .915 .881	.992 .987 .966

Table 6.5.1

Deficiencies under normal alternatives of the normal scores test with respect to the $\bar{X}$-test and the t-test. The upper and lower numbers give $d_{k_N}(NS,\bar{X})$ and $d_{k_N}(NS,t)$, respectively.

N	θ / α	1/4	1/2	3/4	1	5/4	3/2	$\tilde{d}_N(NS,\bar{X})$	$\tilde{d}_N(NS,t)$
5	0.06250	1.364 0.070	1.413 0.080	1.463 0.090	1.501 0.095	1.536 0.100	1.566 0.105	1.370	0.193
6	0.04688	1.582 0.102	1.650 0.120	1.711 0.138	1.771 0.150	1.835 0.162	1.909 0.168	1.629	0.224
7	0.05469	1.480 0.119	1.541 0.133	1.595 0.154	1.642 0.168	1.690 0.175	1.744 0.182	1.534	0.252
8	0.05469	1.520 0.160	1.577 0.184	1.630 0.200	1.677 0.216	1.726 0.224	1.782 0.232	1.559	0.277
8	0.07422	1.310 0.176	1.358 0.200	1.401 0.216	1.440 0.232	1.479 0.240	1.521 0.256	1.321	0.277
9	0.02734	2.036 0.162	2.104 0.180	2.175 0.207	2.240 0.225	2.300 0.243	2.352 0.252	2.141	0.294
9	0.03711	1.818 0.180	1.886 0.198	1.954 0.225	2.031 0.234	2.098 0.261	2.160 0.279	1.888	0.294
9	0.04883	1.623 0.180	1.682 0.198	1.739 0.234	1.796 0.243	1.865 0.261	1.945 0.261	1.666	0.294
10	0.00098	4.629 0.710	4.835 0.800	5.061 0.920	5.293 1.040	5.510 1.180	5.719 1.310	5.104	0.310
10	0.00977	2.804 0.190	2.912 0.210	3.018 0.240	3.110 0.260	3.218 0.280	3.309 0.290	3.037	0.310
10	0.02441	2.125 0.170	2.211 0.200	2.287 0.230	2.357 0.250	2.423 0.280	2.494 0.290	2.252	0.310
10	0.05273	1.574 0.210	1.638 0.230	1.696 0.250	1.753 0.270	1.820 0.290	1.875 0.310	1.621	0.310
10	0.09668	1.171 0.250	1.216 0.280	1.254 0.300	1.294 0.320	1.327 0.340	1.333 -	1.156	0.310

Table 6.5.2

Deficiencies under normal alternatives of the normal scores test with respect to the $\bar{X}$-test and the t-test. The upper numbers give $d_{k_N}(NS,\bar{X})$ if one uses the power estimate plus its standard deviation, the power estimate itself and the power estimate minus its standard deviation, respectively. The lower numbers similarly give $d_{k_N}(NS,t)$.

N	α \ θ	1/4	1/2	3/4	$\tilde{d}_N(NS,\bar{X})$	$\tilde{d}_N(NS,t)$
20	0.01	-2.8,0.2,3.2 -5.3,-2.4,0.6	1.9,2.5,3.2 -0.9,-0.2,0.5	2.2,2.9,3.4 -0.5,0.1,0.7	3.118	0.411
20	0.025	-2.0,0.0,2.1 -3.9,-1.9,0.2	1.8,2.5,3.2 -0.1,0.6,1.2	1.9,2.8,3.8 -0.1,0.8,1.6	2.333	0.411
20	0.05	-1.4,0.3,1.9 -2.7,-1.0,0.6	1.2,2.0,2.7 -0.2,0.6,1.3	0.0,1.7,3.2 -1.5,0.4,1.7	1.764	0.411

Table 6.5.3

Power of the normal scores test against normal alternatives. The first number gives the power estimate, the second gives its standard deviation.

N	α \ θ	0.40		$\frac{3}{10}\sqrt{2}$	
50	0.05	0.871	0.004	0.907	0.004
50	0.025	0.781	0.005	0.833	0.005
50	0.01	0.660	0.006	0.714	0.006

Table 6.5.4

Deficiences under normal alternatives of the normal scores test with respect to the $\bar{X}$-test and the t-test. The organization of this table is the same as for table 6.5.2.

N	α \ θ	0.40	$\frac{3}{10}\sqrt{2}$	$\tilde{d}_N(NS,\bar{X})$	$\tilde{d}_N(NS,t)$
50	0.01	2.6,3.1,3.7 -0.2,0.4,0.9	3.0,3.6,4.1 0.3,0.8,1.2	3.23	0.52
50	0.025	2.5,3.2,3.7 0.7,1.3,1.9	1.7,2.4,3.0 -0.1,0.5,1.1	2.45	0.52
50	0.05	1.1,1.8,2.5 -0.2,0.5,1.2	0.3,1.0,1.8 -0.9,-0.2,0.5	1.87	0.52

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