Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

MATHEMATICAL CENTRE TRACTS 45

A.A. BALKEMA

MONOTONE TRANSFORMATIONS AND LIMIT LAWS

MATHEMATISCH CENTRUM AMSTERDAM 1973

AMS (MOS) subject classification scheme (1970): 60B10, 60F05, 62G30

Acknowledgements

This monograph is a slightly revised version of my thesis which was written under the supervision of Professor Dr. J.Th. Runnenburg. I thank him for his careful reading of the various versions of the manuscript and his many helpful suggestions for improving the readability.

The work owes its existence to a number of stimulating discussions with L. de Haan.

Part of the material was presented in 1972 in the Colloquium on Probability Theory organized jointly by the Institute for Applications of Mathematics of the University of Amsterdam and by the Mathematical Centre.

Contents

Introduction

Probability theory studies convergence of distribution types rather than distribution functions. Recall that two distribution functions F and G are of the same type if there exist $a > 0$ and $b \in \mathbb{R}$ such that

$$G(x) = F(ax + b) \quad \text{for all } x \in \mathbb{R}.$$

The distributions of partial sums, averages and maxima of a sequence of random variables tend to diverge to defective or degenerate distributions. It is only by the use of norming constants that we obtain interesting limit relations.

The basic result on the convergence of distribution types is due to B.V. Gnedenko [1943] and A. Ya Khinchine [1938]. See chapter 14, p.152. It states that the limit in type of a sequence of random variables or distribution functions is unique. That is, if $F_n^{(1)}$ is of the same type as $F_n^{(2)}$ for $n = 1,2,\dots$ and if $F_n^{(i)}$ converges weakly to a non-degenerate distribution $F^{(i)}$ for $i = 1,2$, then $F^{(1)}$ is of the same type as $F^{(2)}$. (See theorem 14.1.)

In the following chapters we shall see that under quite general circumstances the following assertion holds.

If a sequence of random variables $\underline{x}_n$ converges in type to a limit random variable $\underline{u}$, i.e. if there exists a sequence of positive constants a_n and a sequence of real constants b_n such that $\alpha_n\underline{x}_n := a_n\underline{x}_n + b_n$ converges to $\underline{u}$ in distribution, and if the sequence $\underline{y}_n = f(\underline{x}_n)$ with f non-decreasing, converges in type to a limit random variable $\underline{v}$, then up to an affine transformation of the limit variables either $\underline{v}$ is distributed like exp $\underline{u}$, like log $\underline{u}$ or like some power of $\underline{u}$.

In its full generality the assertion is obviously false. One need only take $\underline{x}_n = \underline{x}$ for all n and $f(x) = \text{arctg } x$ for instance.

In the case that $\underline{x}_n$ is the sum of n i.i.d. random variables and $\alpha_n\underline{x}_n := (\underline{x}_n - n)/\sqrt{n}$ converges in distribution to a random variable $\underline{u}$ with a normal distribution, Resnick [1973] has shown that the only possible non-degenerate limits in type for $f(\underline{x}_n)$ with f non-decreasing are $\underline{v} = \underline{u}$ and $\underline{v} = \exp \underline{u}$ i.e. $\underline{v}$ is either normal or lognormal (up to an affine transformation of the variables $\underline{u}$ and $\underline{v}$). This result seemed to be sufficiently surprising to warrant closer attention.

In chapter 2 we shall see that the problem of determining the class of possible pairs of limit types of the sequences $(\underline{x}_n)$ and $(f(\underline{x}_n))$ reduces to the following analytical question.

Suppose that f is a non-decreasing function on $\mathbb{R}$ and

$$a_n'f(a_n x + b_n) + b_n'$$

converges on a set $X \subset \mathbb{R}$ (where $a_n, a_n' > 0$ and $b_n, b_n' \in \mathbb{R}$). What further information is needed to conclude that the limit (on X) is one of the functions e^x, $\log x$ or x^λ?

The main part of the tract, chapters 3 - 13, treats this analytical problem. To emphasize the probabilistic results, these have been listed as theorems whereas the analytical theory is developed in propositions.

The variable x of our function f will often be subject to a probability distribution. We find it convenient to adhere to the Dutch custom and underline the variable in that case.

The opening chapter contains the definitions and notations which are needed in the ensuing chapters. To prevent the reader from falling asleep they have been interspersed with a number of exercises. These take the place of the usual "it is easy to see"-formulations. They contain additional background information and may be bypassed by the reader. A more detailed summary of the contents of the book is given in the last pages of chapter 1.

We now give an example of the probabilistic situation sketched above.

Observe that a random variable $\underline{y}$ with distribution function F is distributed like $f(\underline{x})$ where $\underline{x}$ is homogeneous on (0, 1) and f is the inverse function of F. Let $\underline{x}_{nk}$ denote the kth order statistic from a sample of size n from the homogeneous distribution on the interval (0, 1), and similarly let $\underline{y}_{nk}$ denote the kth order statistic from a sample of size n from the distribution F. Then obviously $\underline{y}_{nk}$ is distributed as $f(\underline{x}_{nk})$. It is known that for $n \to \infty$ and $k/n \to p \in (0, 1)$ the variables $\underline{x}_{nk}$ are asymptotically normal, i.e. $\underline{x}_{nk}$ converges in type to the normal distribution. What can one say about the asymptotic behaviour of the random variables $\underline{y}_{nk}$? (See Smirnov [1949].)

We shall return to this example in the final chapter, which contains some applications of the theory.

1 Notation and main theorems

Throughout this monograph we shall be concerned with the following basic situation

$$(1.1)\qquad \begin{aligned} &\alpha_n \underline{x}_n \to \underline{u} \text{ in distribution}\\ &\beta_n \underline{y}_n \to \underline{v} \text{ in distribution}\\ &\underline{y}_n \overset{M}{=} f(\underline{x}_n) \quad n = 1,2,\ldots\\ &\alpha_n \to \infty. \end{aligned}$$

Here $\underline{u}$, $\underline{v}$, $\underline{x}_n$ and $\underline{y}_n$ are real-valued random variables and α_n and β_n are positive affine transformations on the real line $\mathbb{R}$, i.e. of the form $\alpha_n x = a_n x + b_n$ with $a_n > 0$ and $b_n \in \mathbb{R}$. Further $\alpha_n \to \infty$ means that $|\log a_n| + |b_n| \to \infty$, f is a fixed non-decreasing function defined on an open interval in $\mathbb{R}$ and $\overset{M}{=}$ denotes equality of the corresponding distributions. (Since we shall not distinguish between the right and the left continuous version of a monotone function, the symbol $\overset{M}{=}$ in (1.1) only makes sense if $\underline{x}_n$ is a continuity point of f with probability 1. In the sequel, see definitions 1.6 and 1.3, we shall extend the definition of $\overset{M}{=}$ to cover arbitrary non-decreasing functions. M for monotone!)

It is our aim to give conditions under which the basic situation (1.1) implies

$$(1.2)\qquad \underline{v} \overset{M}{=} \phi(\underline{u}) \quad \text{for some } \phi \in \Phi$$

where Φ is a small set of functions. See definition 1.7.

EXERCISE 1.1 On the divergence of α_n. Suppose for the moment that the basic situation (1.1) holds except that the sequence (α_n) does not diverge. Assume moreover that f is a continuous, strictly increasing function on $\mathbb{R}$ and that neither $\underline{u}$ nor $\underline{v}$ is constant. Then $\underline{v} \overset{M}{=} \beta f(\alpha^{-1}\underline{u})$ for a pair of positive affine transformations α and β. (Hint. Let α be a limit point of (α_n). For convenience assume $\alpha_k \to \alpha$. Then $\underline{x}_k \to \underline{x} = \alpha^{-1}\underline{u}$ and $f(\underline{x}_k) \to f(\underline{x})$ in distribution. By Khinchine's theorem, mentioned in the introduction, $\underline{v}$ and $f(\underline{x})$ are of the same type.) We see that the condition $\alpha_n \to \infty$ cannot be entirely dispensed with if we want to prove (1.2).

EXERCISE 1.2 Let $\rho_n \to \rho$ and $\sigma_n \to \sigma$ be convergent sequences of positive affine transformations. Set $\alpha'_n = \rho_n\alpha_n$ and $\beta'_n = \sigma_n\beta_n$. Then

$$\alpha'_n\underline{x}_n \to \underline{u}' = \rho\underline{u} \quad \text{in distribution}$$

$$\beta'_n\underline{y}_n \to \underline{v}' = \sigma\underline{v} \quad \text{in distribution}$$

and if $\underline{v} \overset{M}{=} \phi(\underline{u})$, then $\underline{v}' \overset{M}{=} \sigma\phi(\rho^{-1}\underline{u}')$. Hence we may expect that together with $\phi \in \Phi$ also $\sigma\phi\rho^{-1} \in \Phi$. Give similar arguments for expecting $\phi \in \Phi$ to imply $\phi^* \in \Phi$ where $\phi^*(x) = -\phi(-x)$.

We now introduce some notation needed in presenting the main results.

DEFINITION 1.1 G is defined to be the group of positive affine transformations γ on $\mathbb{R}$, i.e. $\gamma x = ax + b$ with $a > 0$ and $b \in R$. The group G is not commutative. For example $2(x + 1) \neq (2x) + 1$. The identity element of G is denoted by ε. It is the identity map, $\varepsilon x = x$ for all $x \in \mathbb{R}$.

It is sometimes convenient to think of the elements of G as points in the plane, where we associate with the positive affine transformation $\gamma x = ax + b$ the point $(\log a, b) \in \mathbb{R}^2$. This gives a one-to-one correspondence between the elements of G and the points of the plane.

Suppose $\gamma_n x = a_n x + b_n$ for $n = 0,1,2,\ldots$ Convergence $\gamma_n \to \gamma_0$ may be described by any of the following equivalent statements

$$\log a_n \to \log a_0 \text{ and } b_n \to b_0$$

$$a_n \to a_0 \text{ and } b_n \to b_0$$

$$\gamma_n x_i \to \gamma_0 x_i \quad \text{for two distinct reals } x_1 \text{ and } x_2$$

$$\gamma_n x \to \gamma_0 x \quad \text{for all } x \in \mathbb{R}.$$

Multiplication and inversion are continuous operations, as is obvious by writing out the formulas

$$\gamma_2\gamma_1 x = a_2(a_1 x + b_1) + b_2$$

$$\gamma x = ax + b = y \quad \text{implies} \quad x = a^{-1}(y - b) = \gamma^{-1}y.$$

Since, as we have seen above, convergence of a sequence in G is equivalent to convergence of the corresponding sequence of points in the plane, we shall use the geometry of the plane to describe subsets of G. Thus a set $B \subset G$ is bounded if the corresponding set

$$\{(\log a, b) \mid \gamma x = ax + b, \gamma \in B\}$$

is a bounded subset in $\mathbb{R}^2$, and we write $\gamma_n \to \infty$ where $\gamma_n x = a_n x + b_n$ if the corresponding sequence in $\mathbb{R}^2$ diverges to infinity, i.e. if $|\log a_n| + |b_n| \to \infty$.

EXERCISE 1.3 If $\alpha_n \to \infty$ in G, then $\alpha_n^{-1} \to \infty$.

EXERCISE 1.4 For each $\alpha \in G$ there exists a unique element $\alpha^{\frac{1}{2}} \in G$ such that $\alpha^{\frac{1}{2}}.\alpha^{\frac{1}{2}} = \alpha$. More generally for each $\alpha \in G$ there exists a unique continuous function $t \mapsto \alpha^t$ from $\mathbb{R}$ into G such that $\alpha^t\alpha^s = \alpha^{t+s}$ for all s and t in $\mathbb{R}$ and $\alpha^1 = \alpha$. (Hint. If α is a translation, $\alpha x = x + b$, then the assertion is easily verified, $\alpha^t x = x + bt$ is a solution and is the unique solution for rational t. If α is not a translation, then we write $\alpha x = e^c(x - x_0) + x_0$ with $c \neq 0$, and α is a multiplication with centre x_0. Hence $\alpha^t x = e^{ct}(x - x_0) + x_0$. By choosing appropriate affine coordinates on $\mathbb{R}$ we may even obtain that the centre of multiplication $x_0 = 0$, in which case $\alpha^t x = e^{ct}x$.) The set $\{\alpha^t \mid t \in \mathbb{R}\}$ is the one-parameter subgroup of G generated by α.

DEFINITION 1.2 $\Delta = \Delta(\alpha)$ is a subset of G which depends on the sequence (α_n) in (1.1) and is defined as follows:

$\sigma \in \Delta$ if there exist sequences of positive integers $k_n \to \infty$ and $l_n \to \infty$ such that $\alpha_{k_n}\alpha_{l_n}^{-1} \to \sigma$.

EXERCISE 1.5

1. Construct Δ for the sequences

$$\alpha_n x = x + n \qquad \alpha_n x = x + n^2$$

$$\alpha_n x = nx \qquad \alpha_{2n+i} x = x + n^2 + i \quad i = 0, 1.$$

$$\alpha_n x = \frac{x - n}{\sqrt{n}}$$

Hint. In one of the five cases Δ is the set of all translations.

2. Let (σ_n) be a bounded sequence in G. Define

$$\alpha_{2n}x = x + n^2$$
$$\alpha_{2n+1}x = \sigma_n(x + n^2)$$

and determine Δ.

3. If α_n converges then Δ consists of the identity element.

4. Δ is a closed, symmetric (i.e. $\sigma \in \Delta$ implies $\sigma^{-1} \in \Delta$) subset of G and $\varepsilon \in \Delta$; conversely any closed, symmetric subset of G which contains ε is the Δ of an appropriate divergent sequence α_n. (Compare exercise 1.5.2 above.)

DEFINITION 1.3 M will denote the set of all curves $\{(x(t), y(t)) \mid t \in \mathbb{R}\}$ in the plane, where

1. $x(t)$ and $y(t)$ are continuous non-decreasing functions of t,
2. $x(t) + y(t) = t$ for all $t \in \mathbb{R}$.

By abuse of language we shall sometimes call elements of M non-decreasing functions.

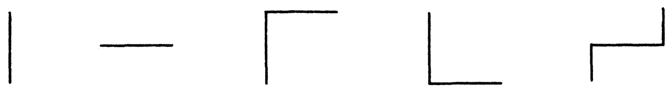
Let $g(x)$ be a non-decreasing function. (We shall always assume that a monotone function has as its domain a non-empty connected subset of $\mathbb{R}$.) There exists a unique element $g_1 \in M$ which consists of the graph of g augmented with a countable number of vertical line segments in the discontinuity points of g and if need be in the endpoints of the domain of definition of g. For instance, the function $g(x) = 6$ on the subset $\{0\} \subset \mathbb{R}$ gives rise to the curve $g_1 = \{(0, t) \mid t \in \mathbb{R}\} \in M$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a function with the graph $H = \{(x, h(x)) \mid x \in \mathbb{R}\}$. Let $\alpha, \beta \in G$. Then the function $\beta h \alpha^{-1}$ has obviously the graph $\{(\alpha x, \beta y) \mid (x, y) \in H\}$. In view of this we give a similar definition for $\beta g \alpha^{-1}$ for $g \in M$.

DEFINITION 1.4 For $g \in M$ we define

$$\beta g \alpha^{-1} = \{(\alpha x, \beta y) \mid (x, y) \in g\}.$$

EXERCISE 1.6 Call two curves h and g in M equivalent if $h = \beta g \alpha^{-1}$ for some $\alpha, \beta \in G$. If $g(x)$ is a constant non-decreasing function then the corresponding element in M is equivalent to one of five inequivalent curves



(Hint. The domain of definition of g is a one point set, the real line, a right half line, a left half line or a bounded non-degenerate interval.)

DEFINITION 1.5 M_0 denotes the subset of M of all curves which correspond to one of the non-decreasing functions g(x) with

$$g(x) = c \quad \text{for all } x \text{ in the domain of definition}$$

where $c \in \mathbb{R}$ and the domain of definition is a non-empty connected subset of $\mathbb{R}$. (See exercise 1.6.)

EXERCISE 1.7 Suppose $g \in M$ satisfies $g = \beta g$ for $\beta \in G$, $\beta \neq \varepsilon$. Then $g \in M_0$. (Hint. Else there exist continuity points x_1 and x_2 of g such that $g(x_1) \neq g(x_2)$. If $g = \beta g$ then β leaves the two points $g(x_i)$ invariant and we conclude that $\beta = \varepsilon$.)

Let λ be a probability measure which lives on the graph H of a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$, i.e. λ is a probability measure on the Borel sets of the x,y-plane and $\lambda(H) = 1$. The coordinate functions x and y are measurable real-valued functions on the probability space $(\mathbb{R}^2, \lambda)$ and hence may be regarded as random variables which we denote by $\underline{x}$ and $\underline{y}$. Since λ is concentrated on the graph of h we have $\underline{y} = h(\underline{x})$ almost surely and hence in particular $\underline{y} = h(\underline{x})$ in distribution.

DEFINITION 1.6 For $g \in M$ and random variables $\underline{x}$ and $\underline{y}$ we define the relation

$$\underline{y} \overset{M}{=} g(\underline{x})$$

to mean that there exists a probability measure λ on g having the probability distributions of $\underline{x}$ and $\underline{y}$ as marginals.

This definition coincides with the usual definition of equality in distribution if the vertical line segments of g carry no positive probability mass. In particular this is the case if g is the graph of a continuous function, or if the probability distribution of $\underline{x}$ is continuous. The definition above has the advantage that the role of $\underline{x}$ and $\underline{y}$ is symmetric, $\underline{y} \overset{M}{=} g(\underline{x})$ if and only if $\underline{x} \overset{M}{=} g^{-1}(\underline{y})$ where g^{-1} is the inverse of g obtained by reflecting g in the diagonal. It has the disadvantage, that $\underline{y}_1 \overset{M}{=} g(\underline{x})$ and $\underline{y}_2 \overset{M}{=} g(\underline{x})$ need not imply $\underline{y}_1 \overset{M}{=} \underline{y}_2$. Indeed let g be the vertical axis and $\underline{x} = 0$ almost surely, then $\underline{y} \overset{M}{=} g(\underline{x})$ holds for any random variable $\underline{y}$.

DEFINITION 1.7 Φ denotes the smallest subset of M which has the properties,

1. $M_0 \subset \Phi$

2. Φ contains the elements in M correponding to the non-decreasing functions

$$
\begin{aligned}
y(x) &= x && \text{on } \mathbb{R} \\
y(x) &= e^x && \text{on } \mathbb{R} \\
y(x) &= \log x && \text{on } (0, \infty) \\
y(x) &= -x^{-\lambda} && \text{on } (0, \infty) \\
y(x) &= x^{\lambda} && \text{on } (0, \infty) \\
y(x) &= x^{\lambda} && \text{on } [0, \infty) \\
&= c(-x)^{\lambda} && \text{on } (-\infty, 0) \\
y(x) &= \text{sign } x && \text{on } \mathbb{R}
\end{aligned}
$$

for each $\lambda > 0$ and $c \leq 0$.

3. $\phi \in \Phi$, $\alpha \in G$ implies $\phi\alpha^{-1} \in \Phi$, $\alpha\phi \in \Phi$ and $\phi^* \in \Phi$ where ϕ^* is defined by the condition $(x, y) \in \phi^*$ if and only if $(-x, -y) \in \phi$.

Observe that Φ is closed with respect to inversion (reflection in the diagonal).

Condition 3. in the definition of Φ states that the set Φ does not depend on a particular choice of coordinates on the axes. (The seven curves listed under 2. obviously do depend on the coordinates.) If we introduce new coordinates x' and y' where either $x' = \sigma x$, $y' = \tau y$ with $\sigma, \tau \in G$, or $x' = -x$, $y' = -y$, then Φ also contains the curves obtained by substituting x' and y' for x and y in the seven expressions under 2.

The reason for listing together this apparently disparate collection of functions will become clear in proposition 1.1.

EXERCISE 1.8 Let Φ_0 be the smallest subset of M which contains the curves corresponding to the functions $\phi(x) = x$, $\phi(x) = e^x$ and $\phi(x) = -e^{-x}$, and which satisfies the condition that $\beta\phi\alpha^{-1} \in \Phi_0$ whenever $\phi \in \Phi_0$ and $\alpha, \beta \in G$.

1. $\Phi_0 \subset \Phi$.
2. Every element $\phi \in \Phi_0$ corresponds to a function

$$\phi(x) = \beta e_\lambda(x)$$

with $\beta \in G$ and $\lambda \in \mathbb{R}$, where $e_\lambda(x)$ is defined by

$$\begin{aligned} e_\lambda(x) &= \lambda^{-1}(e^{\lambda x} - 1) \qquad & \lambda \neq 0 \\ &= x & \lambda = 0. \end{aligned}$$

3. Φ_0 is homeomorphic to $\mathbb{R}^3$. (The representation $\phi(x) = \beta e_\lambda(x)$ is unique and $\beta_n \to \beta$, $\lambda_n \to \lambda$ imply $\phi_n \to \phi$ weakly and vice versa.)
4. Let α be the translation $\alpha x = x + 1$. Let $\beta \in G$ and $g \in M$. If g satisfies

$$\beta^t g \alpha^{-t} = g \quad \text{for all } t \in \mathbb{R}$$

then g is a horizontal line or $g \in \Phi_0$. (Hint. Suppose $(x_0, y_0) \in g$. Then also $(\alpha^t x_0, \beta^t y_0) = (x_0+t, \beta^t y_0) \in g$ for all $t \in \mathbb{R}$, i.e.

$$g(x_0 + t) = \beta^t y_0.$$

Now substitute y_0 for x in the expression for $\beta^t x$ in exercise 1.4.)

PROPOSITION 1.1 Suppose $\alpha, \beta \in G$ and $(\alpha, \beta) \neq (\varepsilon, \varepsilon)$. Let $g \in M$ satisfy

$$\beta^t g \alpha^{-t} = g \quad \text{for all } t \in \mathbb{R}, \tag{1.3}$$

then $g \in \Phi$. Conversely for every $\phi \in \Phi$ there exist α and β in G not both equal to ε, such that (1.3) holds.

PROOF The proof is elementary but rather cumbersome because of the many particular instances which we have to consider.

It is useful to introduce the sets

$$\Phi(\alpha, \beta) = \{g \in M \mid (1.3)\}. \tag{1.4}$$

Proposition 1.1 states that Φ is the union of all $\Phi(\alpha, \beta)$ with $(\alpha, \beta) \neq$ $\neq (\varepsilon, \varepsilon)$.

To prove that $\Phi \subset \cup\, \Phi(\alpha, \beta)$ with $(\alpha, \beta) \neq (\varepsilon, \varepsilon)$ it suffices to check

1. $M_0 \subset \cup\, \Phi(\varepsilon, \beta)$ with $\beta \neq \varepsilon$
2. each of the seven functions listed in the definition of Φ lies in a set $\Phi(\alpha, \beta)$, (in fact $\alpha^t x = a^t x$ or $\alpha^t x = x + bt$ and similarly for β^t)
3. $g \in \Phi(\alpha, \beta)$ implies $\gamma g \in \Phi(\alpha, \gamma\beta\gamma^{-1})$
   $g \in \Phi(\alpha, \beta)$ implies $g\gamma^{-1} \in \Phi(\gamma\alpha\gamma^{-1}, \beta)$
   $g \in \Phi(\alpha, \beta)$ implies $g^* \in \Phi(\alpha^*, \beta^*)$, where $g^*(x) := -g(-x)$,

and similarly for α^* and β^*. (Note that the equality $(\alpha^t)^*(\alpha^s)^* = (\alpha^{t+s})^*$ implies that $(\alpha^t)^*$ is the unique continuous one parameter subgroup generated by α^* and hence $(\alpha^t)^* = (\alpha^*)^t$. See exercise 1.4.)

To prove the first part of the theorem, i.e. $\Phi(\alpha, \beta) \subset \Phi$ if $(\alpha, \beta) \neq$ $\neq (\varepsilon, \varepsilon)$ we first observe that

$\Phi(\varepsilon, \beta) \subset M_0$ for $\beta \neq \varepsilon$ (see exercise 1.7)

$\Phi(\alpha, \beta) \subset \Phi_0 \subset \Phi$ if α is a translation (see parts 1 and 4 of exercise 1.8) or Φ is constant

$g \in \Phi(\alpha, \beta)$ implies $g^{-1} \in \Phi(\beta, \alpha)$.

Thus it suffices to check that $\Phi(\alpha, \beta) \subset \Phi$ if α and β are non-trivial multiplications. For convenience we assume that $\alpha^t x = a^t x$ and $\beta^t x = b^t x$, where a and b are positive constants not equal to 1. Observe that $(x_0, y_0) \in g$ for $g \in \Phi(\alpha, \beta)$ implies that g contains the whole curve $(a^t x_0, b^t y_0)$. For $(x_0, y_0) \neq (0, 0)$ this curve is either one of the half axes (if $x_0 = 0$ or $y_0 = 0$), or it is the graph of one of the functions

$$y(x) = cx^\lambda \quad \text{on } (0, \infty) \text{ with } c\lambda > 0$$

$$y(x) = c(-x)^\lambda \quad \text{on } (-\infty, 0) \text{ with } c\lambda < 0$$

where λ is uniquely determined by

$$b^t y_0 = y(a^t x_0) = ca^{\lambda t} x_0^\lambda .$$

Hence $b = a^{\lambda}$. Either g is this curve (if $\lambda < 0$) or the curve has (0, 0) as endpoint and g is the union of two such curves and the origin.

REMARK A very simple description of Φ can be given as follows (see Ince [1926], chapter 4). Let α^t, $t \in \mathbb{R}$, be a one-parameter subgroup of G. The infinitesimal generator $\dot{\alpha}$ is defined by

$$\dot{\alpha}x := \lim_{t \to 0} \frac{\alpha^t x - x}{t} .$$

The maps $(x, y) \mapsto (\alpha^t x, \beta^t y)$ form a continuous transformation group on $\mathbb{R}^2$. The corresponding infinitesimal transformation is

$$(x, y) \mapsto (x + dx, y + dy) = (x + \dot{\alpha}x\, dt, y + \dot{\beta}y\, dt).$$

The elements of $\Phi(\alpha, \beta)$ are the non-decreasing invariant curves and satisfy the differential equation

$$\dot{\beta}y\, dx = \dot{\alpha}x\, dy .$$

EXAMPLE 1.1 Every $\phi \in \Phi$ does occur in the limit relation (1.2) for an appropriate choice of α_n, β_n and f in (1.1).

First suppose $\alpha, \beta \in G$ and $\alpha \neq \varepsilon$. Let $f = \phi \in \Phi(\alpha, \beta)$ and let λ be an arbitrary probability measure on f with marginals $\underline{u}$ and $\underline{v} \overset{M}{=} \phi(\underline{u})$. Define $\underline{x}_n$ and $\underline{y}_n$ by

$$\alpha^n \underline{x}_n = \underline{u}$$
$$\beta^n \underline{y}_n = \underline{v}.$$

Then

$$\underline{y}_n \overset{M}{=} f(\underline{x}_n) \quad \text{since } \beta^n f \alpha^{-n} = f$$
$$\alpha^n \to \infty \qquad \text{since } \alpha \neq \varepsilon.$$

Now if $\phi \in \Phi(\varepsilon, \beta)$ then ϕ is one of the constant functions (see exercise 1.7). These functions also lie in $\Phi(\alpha, \varepsilon)$ except for the function $\phi = c$ on I where I is a bounded interval. That this functions also occurs as a limit follows from example 1.4 below on interchanging the x and y-axis.

EXAMPLE 1.2 Counter example showing that the monotonicity of f is essential.

Let $\underline{u}$ be a random variable with a continuous probability density and set $\underline{x}_n := n\underline{u}$. Choose $\alpha_n x = n^{-1}x$, $\beta_n y = y$ and $f(x) = \cos x$. Then $\alpha_n \to \infty$, $\alpha_n \underline{x}_n \to \underline{u}$ and $\underline{y}_n = \cos \underline{x}_n$ converges in distribution to $\cos \underline{w}$, where $\underline{w}$ is homogeneous on $[0, \pi]$.

Obviously we could have used any periodic function instead of the cosine and obtained a similar result.

EXAMPLE 1.3 On the divergence of (β_n).

The limit variables $\underline{u}$ and $\underline{v}$ will have distribution functions $F(u) = pF_1(u) + qF_2(u)$ and $G(v) = pG_1(v) + qG_2(v)$ where $p, q > 0$, $p + q = 1$, and

F_1 is standard normal

F_2 is degenerate in zero

G_1 has point mass $\frac{1}{2}$ in $-\frac{\pi}{2}$ and in $\frac{\pi}{2}$

G_2 is the uniform distribution on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

One readily checks that $\underline{u}$ and $\underline{v}$ are marginals of a probability measure which lives on $\phi \in \Phi$ where

$$\phi(x) = \frac{\pi}{2} \operatorname{sign} x.$$

(Mix the uniform distribution on the vertical part with the two halves of the normal distribution on each of the horizontal halflines in ϕ.)

Choose $f(x) = \operatorname{arctg} x$ on $\mathbb{R}$. Let λ_n be the (unique) probability measure on $f \in M$ such that the marginal $\underline{x}_n$ has a normal $N(0, n^2)$ distribution. The distribution of $n^{-1}\underline{x}_n$ converges to the standard normal F_1 and the distribution of the marginal $\underline{y}_n$ converges to G_1. Similarly, if κ denotes the probability measure on f whose marginal $\underline{y}$ is uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ then $n^{-1}\underline{x} \to 0$. Setting $\mu_n = p\lambda_n + q\kappa$, $\alpha_n x = n^{-1}x$ and $\beta_n y = y$ we obtain the announced result.

Example 1.3 above shows that $\alpha_n \to \infty$ need not imply $\beta_n \to \infty$. In proposition 7.5 we shall see that if β_n does not diverge to ∞ then the basic situation (1.1) implies that $\underline{u} \overset{M}{=} \phi(\underline{v})$ where $\phi \in M_0$. For the sake of symmetry we could add the condition $\beta_n \to \infty$ to the basic situation (1.1). We shall not do so since in applications the given situation will in general be asymmetric. (For instance in the application mentioned in the introduction, the sequence (α_n) is known.)

EXAMPLE 1.4 On the definition of $\overset{M}{=}$ and the necessity of allowing vertical line segments in the graphs of discontinuous functions.

In the example above we had $\underline{v} \overset{M}{=} \frac{\pi}{2}$ sign $\underline{u}$ where $\underline{u}$ had positive mass q in 0. The fact that $\beta_n = \varepsilon$ is not essential since we can obtain the same limit function $\phi_0(x) = \frac{\pi}{2}$ sign x starting with $f(x) = \text{arctg } x.\log(1 + |x|)$ and using the norming transformations $\alpha_n x = n^{-1}x$ and $\beta_n y = (\log n)^{-1}y$. Then $\beta_n f\alpha_n^{-1}x \to \frac{\pi}{2}$ sign $x = \phi_0(x)$ and it is possible to obtain any desired probability measure on $\phi_0 \in \Phi$ as the limit of an appropriate sequence of distributions on $g_n := \beta_n f\alpha_n^{-1}$. This construction will be developed in full generality in the next chapter.

We shall now give a summary of the contents of chapters 2 - 14.

Chapter 2 contains a description of the topology of M. In this topology M is a locally compact, metrizable space, and the sequence $\beta_n f\alpha_n^{-1}$, with α_n, β_n and f as in (1.1), is relatively compact. We shall prove two theorems on monotone functions of random variables.

1. Given two random variables, $\underline{u}$ and $\underline{v}$, there exists a non-decreasing function $g \in M$ such that $\underline{v}$ is distributed like $g(\underline{u})$, i.e. $\underline{v} \overset{M}{=} g(\underline{u})$.

2. Let the sequence $\underline{u}_n$ and the sequence $\underline{v}_n \overset{M}{=} g_n(\underline{u}_n)$ converge in distribution to $\underline{u}$ and $\underline{v} \overset{M}{=} g(\underline{u})$. Let $\Lambda \subset g$ be the support of the measure λ on g with marginals $\underline{u}$ and $\underline{v}$ (see definition 1.6). Then the sequence g_n converges onto the set Λ (in the sense of definition 2.1).

We shall apply these two theorems to the sequence $\beta_n f\alpha_n^{-1}$ in M.

The two theorems will enable us to reformulate the basic situation (1.1) in purely analytical terms (2.1), as "$\beta_n f\alpha_n^{-1}$ converges onto Λ".

In particular, if the distribution function of the limit random variable $\underline{u}$ in the basic situation (1.1) is strictly increasing on $\mathbb{R}$, then (2.1) implies that the sequence of non-decreasing functions $\beta_n f\alpha_n^{-1}$ converges weakly on $\mathbb{R}$ to a non-decreasing function h, and that the limit variables $\underline{u}$ and $\underline{v}$ in the basic situation (1.1) satisfy $\underline{v} \overset{M}{=} h(\underline{u})$. In chapter 3 it will be shown that this function h satisfies a functional equation of the form

$\tau h = h\sigma$ with $\tau \in G$ for every $\sigma \in \Delta$. (This functional equation is a variant of the wellknown functional equation $h(x) = h(x + p)$ for periodic functions.) Table 3.2 gives a complete classification of the possible limit functions in terms of the set Δ. If the set Δ is sufficiently large, then any solution h of the system of functional equations $\tau h = h\sigma$, with $\sigma \in \Delta$, is an element of Φ, and even of $\Phi(\sigma, \tau)$. (See (1.4) for the definition of $\phi(\sigma, \tau)$). This implies that if Δ contains two elements which do not commute, then h is constant or affine (either $\underline{v}$ is constant or $\underline{v}$ is of the same type as $\underline{u}$). If all elements of Δ are integral powers of a common element $\sigma \in G$, $\sigma \neq \varepsilon$, then h is the composition of an element of Φ and a function $k(x) = \lambda x + c + \pi(x)$ where π is periodic modulo σ (see table 3.1). Finally if $\Delta = \{\varepsilon\}$, i.e. if the sequence (α_n) diverges fast, then every non-decreasing function h is possible in the relation $\underline{v} \overset{M}{=} h(\underline{u})$ (for a suitably chosen $f \in M$ and sequence (β_n) in G). The proof of this statement occupies the greater part of chapter 4. (We give an explicit construction of f for a given sequence (α_n) with $\alpha_n \to \infty$ and $\Delta = \{\varepsilon\}$, and given $h \in M$.)

We describe this case (the distribution function of $\underline{u}$ strictly increasing on the whole real line) in some detail since it occurs in most applications. The theory in the first part of the book, chapters 3 - 6, is developed under the more general assumption that

(1.6) the distribution function of $\underline{u}$ is strictly increasing on an open interval I and $P\{\underline{u} \in I\} = 1$.

Because of this condition there will be little need to distinguish between non-decreasing functions and elements of M in these 4 chapters and we shall use the classical theory of non-decreasing functions and weak convergence. The basic situation (1.1) together with condition (1.6) implies weak convergence of the sequence $\beta_n f \alpha_n^{-1}$ on the interval I.

For a bounded interval I the system of functional equations $\tau h = h\sigma$ is less easy to handle than in the case $I = \mathbb{R}$ which we considered above, and we shall only prove theorem 3.1.

If ε is a condensation point of Δ, then $\underline{v} \overset{M}{=} \phi(\underline{u})$ for some $\phi \in \Phi$.

Note that for a bounded interval also the functions log x and $-x^{-\lambda}$ can occur in relation (1.2).

In chapter 5 we define the domain of attraction of a function $h \in M$ for a given sequence (α_n) in G as the set of all $f \in M$ for which there exists a sequence (β_n) such that $\beta_n f \alpha_n^{-1} \to h$. We give some examples for the case that h is the identity function. It will be shown that a continuously differentiable function f, which satisfies the condition that $\lim_{|x|\to\infty} f'(x)$ exists and is positive, lies in the domain of attraction of the identity function for every sequence (α_n) which diverges to ∞.

In chapter 6 we introduce the extra restrictions

> the sequence $(\alpha_{n+1}\alpha_n^{-1})$ is bounded,
> Δ is a subset of a one-parameter subgroup $G(\gamma) = \{\gamma^t \mid t \in \mathbb{R}\}$ of G.

If $\beta_n f \alpha_n^{-1}$ converges weakly to a non-constant element $\phi \in \Phi$ on an open interval I, and if $I \cap \sigma I$ is non-empty for every limit point σ of the sequence $(\alpha_{n+1}\alpha_n^{-1})$, then $\beta_n f \alpha_n^{-1}$ converges weakly to ϕ on the half line (c, ∞) or $(-\infty, c)$ containing I (if γ is a multiplication with centre c) or on the whole real line (if γ is a translation). See proposition 6.1.

In this case we can embed the sequence (α_n) in a continuous function $\alpha : [0, \infty) \to G$ such that $\alpha_n = \alpha(t_n)$ where $t_n \to \infty$, $t_{n+1} - t_n$ is bounded, and for all $s \in \mathbb{R}$

$$\alpha(t + s)\alpha(t)^{-1} \to \gamma^s \qquad \text{for } t \to \infty.$$

A similar statement holds for the sequence (β_n).

The results of chapter 6 should be seen as an introduction to the second part of the book, chapters 7 - 13, where we replace the restriction (1.6) on the random variable $\underline{u}$ by the restriction $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$ on the sequence (α_n).

The condition $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$ allows us to replace the sequences (α_n) and (β_n) by continuous functions α and β from $[0, \infty)$ into G. In this second half of the book we shall employ to the full the geometrical interpretation of a non-decreasing function as a curve in the x,y-plane which we introduced in definition 1.3.

In chapter 7 we introduce a compactification of G and we give a complete analysis of the basic sutuation (1.1) under the condition that the sequence (β_n) does not diverge to infinity. (Compare example 1.3 and the remarks following this example.)

Now let us consider a limit point g of the sequence $\beta_n f \alpha_n^{-1}$ in M. Then $\underline{v} \overset{M}{=} g(\underline{u})$. This implies by the definition of $\overset{M}{=}$ that $\underline{u}$ and $\underline{v}$ are marginals of a probability measure λ supported by a closed subset $\Lambda \subset g$. In chapter 8 we shall see that there exists an unbounded, connected, closed subset $C \subset G\times G$, the "guide set" of g for Λ, such that $\tau^{-1}\Lambda\sigma \subset g$ for all $(\sigma, \tau) \in C$ (see proposition 8.1). This is the geometrical analogue of the functional equation $\tau h = h\sigma$ which we derived in chapter 3. We shall discuss some simple consequences of this inclusion in chapter 8.

It was the aim of these investigations to derive the implication "(1.1) and $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$ implies (1.2)". We have not been able to prove this implication, nor have we been able to construct a counter-example. A proof of the implication under the extra condition that $\underline{u}$ should have an absolutely continuous distribution function will be published elsewhere in a joint paper with L. de Haan. In the present work we place an additional restriction on the sequence (α_n).

One rather striking consequence of the condition, $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$, is that the set Δ contains a one-parameter subgroup $\{\gamma^t \mid t \in \mathbb{R}\}$ of G, with $\gamma \neq \varepsilon$ (see proposition 7.2). We shall be particularly interested in the case that Δ is equal to this one-parameter subgroup. It is then possible to introduce a continuous function $\alpha : [0, \infty) \to G$, such that $\alpha_n = \alpha(t_n)$ where $t_n \to \infty$ and $t_{n+1} - t_n \to 0$, and which satisfies the relation (see proposition 9.7)

$$\lim_{t\to\infty} \alpha(t + s)\, \alpha(t)^{-1} = \gamma^s \quad \text{for all } s \in \mathbb{R}.$$

This equation leads us to a theory of regular variation on separable, metrizable groups. (Recall that a function U from $[0, \infty)$ to $(0, \infty)$ is said to vary regularly in the additive formulation if there exists a constant $\rho \in \mathbb{R}$ such that $\lim_{t\to\infty} U(t + s)\, U(t)^{-1} = e^{\rho s}$ for all $s \in \mathbb{R}$.) Many theorems in the classical theory of regular variation remain valid in the more general setting of separable metrizable topological groups as will be shown in chapter 9.

If Δ is the one-parameter subgroup of the translations, (this is the case if for instance $\alpha_n x = (x - n)/\sqrt{n}$, see exercise 1.5.1), and if Λ contains three points no two of which lie on the same horizontal or vertical line, then the inclusion $\sigma^{-1}\Lambda\tau \subset g$ for all (σ, τ) in the guide set C, which was established in chapter 8, implies that g, after a simple normalization, satisfies a functional equation

$$g(x + \theta) - g(x) = C(g(x + 1) - g(x))$$

for positive constants θ and C less than 1. It will be shown in chapter 10 that the solutions of this equation satisfy a pair of equations $\tau_i g = g\sigma_i$ for i = 1, 2. (See the corollary to proposition 10.3.) In particular if θ is irrational, then $g \in \Phi$. In this case g is uniquely determined by the two functional equations $\tau_i g = g\sigma_i$, i = 1, 2, and the condition that $\Lambda \subset g$. This implies that g is the only limit point of the sequence $\beta_n f\alpha_n^{-1}$, and hence, that $\beta_n f\alpha_n^{-1} \to g$ in M. One can say even more in this case. Also the norming function $\beta(t)$ varies regularly. (Note the similarity with the results of chapter 6.)

As an interesting corollary we obtain the corollary to proposition 10.4

Let f be a non-decreasing function on $\mathbb{R}$ and let x_0, x_1 and x_2 be real numbers such that $x_0 < x_1 < x_2$ and $(x_2 - x_0)/(x_1 - x_0)$ is irrational. If

$$\frac{f(x_2 + t) - f(x_0 + t)}{f(x_1 + t) - f(x_0 + t)} \to c \quad \text{for } t \to \infty$$

with c > 1, then

$$\frac{f(x + t) - f(x_0 + t)}{f(x_1 + t) - f(x_0 + t)} \to \phi(x) \quad \text{weakly for } t \to \infty$$

where $\phi \in \Phi_0$ (see exercise 1.8).

We are now able to derive the main result of this part of the book. This is theorem 11.1 which states.

If in addition to the basic situation (1.1) it is given that

$\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$, and

(1.7) Δ is contained in a one-parameter subgroup $\{\sigma^t \mid t \in \mathbb{R}\}$ in G,

then there exists an element $\phi \in \Phi$, such that

$$\underline{v} \overset{M}{=} \phi(\underline{u}).$$

Moreover $\phi \in M_0$ or $\phi \in \Phi(\sigma, \tau)$ for some $\tau \in G$, where $\Phi(\sigma, \tau)$ is defined in (1.4).

In chapter 12 we consider the domain of attraction of the identity function, now for a given sequence (α_n) which satisfies $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$ and (1.7) with $\sigma x = x + 1$.

In chapter 13, theorem 13.1, we show that (1.1), together with the condition $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$, implies that either $\underline{v} \overset{M}{=} \phi(\underline{u})$ for some $\phi \in \Phi$ or $\underline{v} \overset{M}{=} h(\underline{u})$ where $h \in M$ is the graph of a homeomorphism of an open interval.

The final chapter gives some applications. As an example we mention the wellknown fact that if $\underline{u}$ and $\underline{v}$ are each limit in type of a sequence of maxima of i.i.d. random variables, i.e. $\underline{u}$ and $\underline{v}$ are each distributed according to one of Gnedenko's limit laws in extreme value theory, then $\underline{v} \overset{M}{=} \phi(\underline{u})$ for some $\phi \in \Phi$.

2 Monotone functions of random variables

In this chapter we take a closer look at the space M of non-decreasing functions introduced in chapter 1 (definition 1.3) and we prove the following two well known assertions.

Let $\underline{u}$ and $\underline{v}$ be random variables. Then there exists an element $g \in M$ such that $\underline{v} \overset{M}{=} g(\underline{u})$, i.e. there exists a probability measure $\lambda = \lambda(\underline{u}, \underline{v})$ which lives on the curve g and which has marginals $\underline{u}$ and $\underline{v}$. This probability measure λ is unique (though g in general is not). Moreover if $\underline{u}_n \to \underline{u}$ and $\underline{v}_n \to \underline{v}$ in distribution then the corresponding curves g_n converge onto the support $\Lambda \subset g$ of the measure $\lambda = \lambda(\underline{u}, \underline{v})$, where convergence onto is defined as follows.

DEFINITION 2.1 Let (g_n) be a sequence in M and let Λ be a subset of an element of M. Then g_n converges onto Λ if for each point $P \in \Lambda$ there exists a sequence $P_n \in g_n$ such that $P_n \to P$.

The two theorems above allow us to dispense with the probabilistic flavour of the basic situation (1.1) and to formulate it more simply in purely analytical terms. Indeed on introducing the new variables $\underline{u}_n := \alpha_n \underline{x}_n$ and $\underline{v}_n := \beta_n \underline{y}_n$ the basic situation (1.1) may be formulated as

$$\underline{u}_n \to \underline{u} \quad \text{in distribution}$$
$$\underline{v}_n \to \underline{v} \quad \text{in distribution}$$
$$\underline{v}_n \overset{M}{=} \beta_n f(\alpha_n^{-1} \underline{u}_n)$$
$$\alpha_n \to \infty.$$

Let λ be the probability measure with marginals $\underline{u}$ and $\underline{v}$ and with support $\Lambda \subset g \in M$, i.e. $\underline{v} \overset{M}{=} g(\underline{u})$. Now apply the second theorem formulated above to the curves $g_n = \beta_n f \alpha_n^{-1}$ and the sequences $\underline{u}_n \to \underline{u}$ and $\underline{v}_n \to \underline{v}$. We find that the basic situation (1.1) implies

$$\text{(2.1)} \qquad \beta_n f \alpha_n^{-1} \quad \text{converges onto } \Lambda.$$

In order to prove that the limit random variables $\underline{u}$ and $\underline{v}$ of the basic situation (1.1) satisfy (1.2) it suffices to show that the analytic basic situation, (2.1) together with $\alpha_n \to \infty$, implies

$$(2.2) \qquad \Lambda \subset \phi$$

The investigation of conditions on Λ and on the sequence (α_n) which ensure

$$(2.1) \Rightarrow (2.2)$$

is the subject matter of the thesis.

Since (2.1) is a statement about a sequence in M it seems proper to give some attention to this space. We shall give a number of alternative description of the set M and introduce a topology on this set. This exposition is not strictly essential for the remainder of the book and may be regarded as background material. In particular chapters 3 - 6, where Λ is the graph of a non-decreasing function defined on an open interval, may be read in the context of the classical definition of non-decreasing functions and weak convergence. However, for the proof of the two assertions of the opening paragraph of this chapter the space M is the natural setting.

DEFINITION 2.2 For $a = (a_1, a_2) \in \mathbb{R}^2$ define

$$a_{\ulcorner} = \{(x, y) \in \mathbb{R}^2 \mid x > a_1, \; y < a_2\}.$$

The set $a_{\ulcorner}$ is the open lower right quadrant with vertex a. For $A \subset \mathbb{R}^2$ we define

$$A_{\ulcorner} = \bigcup_{a \in A} a_{\ulcorner}.$$

Similarly ${}^{\lrcorner}a := (-\infty, a_1) \times (a_2, \infty)$ and ${}^{\lrcorner}A := \bigcup_{a \in A} {}^{\lrcorner}a$. Observe that $A_{\ulcorner}$ is open and $A_{\ulcorner} = A_{\ulcorner\ulcorner}$.

THEOREM 2.1 Let $\underline{x}$ and $\underline{y}$ be real-valued random variables. There exists a probability measure λ such that

λ lives on a curve $g \in M$

λ has marginals $\underline{x}$ and $\underline{y}$.

The measure λ is uniquely determined by these two conditions.

PROOF Let $F(x) = P\{\underline{x} \le x\}$ be the distribution function of $\underline{x}$ and let $G(y)$ be the distribution function of $\underline{y}$.

1. Existence of λ

Set

$$(2.3) \qquad H(x, y) := \min F(x), G(y).$$

H is a probability distribution on $\mathbb{R}^2$ and has marginals $F(x)$ and $G(y)$.
Let λ be the associated probability measure on $\mathbb{R}^2$.
Set $A = \{(x, y) \mid F(x) \ge G(y)\}$. Then, denoting closure by $\overline{}$,

$$A_{\ulcorner} \subset A \subset \overline{A_{\ulcorner}}$$

$$\lambda A_{\ulcorner} = 0.$$

Similarly, setting $B = \{(x, y) \mid F(x) \le G(y)\}$, we find

$${}^{\lrcorner}B \subset B \subset \overline{{}^{\lrcorner}B}$$

$$\lambda\, {}^{\lrcorner}B = 0.$$

Since $A \cup B = \mathbb{R}^2$, the measure λ lives on $\partial A \cap \partial B \subset \partial A$, the boundary of A.

2. Uniqueness of λ

Let λ satisfy the two conditions of the theorem. Then $\lambda(g_{\ulcorner}) = \lambda({}^{\lrcorner}g) = 0$.
Let $H(x, y)$ be the distribution function of λ. For $(x, y) \in g$ we have $\lambda((x, y)_{\ulcorner}) = 0$ and, hence, for $(x, y) \in g_{\ulcorner}$ we have

$$H(x, y) = H(\infty, y) = G(y)$$

$$H(x, y) \le H(x, \infty) = F(x).$$

This proves that (2.3) holds on $g_{\ulcorner}$. A similar argument shows that (2.3) holds on ${}^{\lrcorner}g$. Since the union $g_{\urcorner} \cup {}^{\lrcorner}g$ is dense in $\mathbb{R}^2$, the relation (2.3) defines H.

COROLLARY 1 The measure λ in theorem 2.1 has the distribution function given by (2.3).

COROLLARY 2 Suppose that the distribution function F of $\underline{x}$ is strictly increasing on an open interval I, and that $P\{\underline{x} \in I\} = 1$. Suppose $\underline{y} \overset{M}{=} g(\underline{x})$ with $g \in M$, and let $h \in M$. Then $\underline{y} \overset{M}{=} h(\underline{x})$ if and only if $g_{|I} = h_{|I}$ where $g_{|I}$ denotes the restriction of the curve g to the vertical strip IxR in the x,y-plane.

PROOF Let Λ be the support of the measure λ defined in theorem 2.1. Then $\underline{y} \overset{M}{=} h(\underline{x})$ if and only if $h \supset \Lambda$. The condition on the distribution of $\underline{x}$ ensures that I is the interior of the projection of Λ on the x-axis. If h agrees with g on I, then $\Lambda \subset h$; if $\Lambda \subset h$, then h agrees with g on I.

REMARK Fréchet [1951] has studied the distribution function H(x, y) defined in (2.3) and shown that for F and G continuous and strictly increasing the curve g is unique and is defined by F(x) = G(y).

EXAMPLE 2.1 Let $\underline{u}$ be homogeneous on (0, 1). Then $\underline{x} \overset{M}{=} F^{-1}(\underline{u})$ where F is the distribution function of $\underline{x}$.

THEOREM 2.2 Let $\underline{x}$, $\underline{y}$, $\underline{x}_n$ and $\underline{y}_n$ be real-valued random variables such that

$$\underline{x}_n \to \underline{x} \quad \text{in distribution}$$

$$\underline{y}_n \to \underline{y} \quad \text{in distribution}$$

$$\underline{y}_n \overset{M}{=} g_n(\underline{x}_n) \quad \text{with } g_n \in M \text{ for } n = 1,2,\ldots$$

$$\underline{y} \overset{M}{=} g(\underline{x}) \quad \text{with } g \in M.$$

Let Λ be the support of the probability measure λ on g with marginals $\underline{x}$ and $\underline{y}$. Then g_n converges onto Λ.

PROOF For n = 1,2,... let λ_n be the probability measure on g_n with marginals $\underline{x}_n$ and $\underline{y}_n$. Let $F_n(x)$ be the distribution function of $\underline{x}_n$, $G_n(y)$ of $\underline{y}_n$, and $H_n(x, y)$ of λ_n. Then by theorem 2.1, corollary 1,

$$H_n(x, y) = \min (F_n(x), G_n(y)).$$

Since the right hand side converges, so does the left hand side (pointwise on a dense subset). We obtain

$$\lim H_n(x, y) = \min (F(x), G(y)).$$

The associated probability measure on $\mathbb{R}^2$ satisfies the two conditions of theorem 2.1. Hence it is λ by uniqueness.

In order to prove convergence of g_n onto Λ, take an arbitrary point $(x, y) \in \Lambda$. Let U be an open neighbourhood of (x, y). Then $\lambda U > 0$. Now $\lambda_n \to \lambda$ implies $\liminf \lambda_n U \geq \lambda U$ for open sets U. Hence $\lambda_n U > 0$ for $n \geq n_0$, i.e. g_n intersects U for $n \geq n_0$. Q.E.D.

If the limit distribution function F of $\underline{x}$ is strictly increasing on an open interval I and if $P\{\underline{x} \in I\} = 1$, then the conditions of theorem 2.2 imply that the sequence of non-decreasing functions g_n converges weakly to g on I. In order to prove this we shall define a topology on M. With this topology M becomes a locally compact, metrizable space. We shall also introduce a compact space M^*, which may be viewed as a two-point compactification of M in the same way in which the closed interval [0, 1], which is homeomorphic to the extended real line $[-\infty, \infty]$, may be considered to be the two-point compactification of $\mathbb{R}$.

Table 2.1 lists five representations of the same space M^* and four representations of M. The reader will observe that there are obvious bijections between the different representations and that M may be regarded as a subset of M^*, the complement $M^*\setminus M$ consisting of two elements which are the 0 and 1 of the Boolean algebra M^*.

Before discussing the topology on these spaces we prove a well known result on weak convergence of non-decreasing functions.

PROPOSITION 2.1 Let $g_0, g_1, \ldots$ be non-decreasing functions on an open interval I. Let C be the set of continuity points of g_0 on I and define the function G on $I \times \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$ by

$$G(x, \tfrac{1}{n}) := g_n(x) \quad n = 1,2,\ldots, \quad x \in I$$

$$G(x, 0) := g_0(x) \quad\quad x \in I.$$

Then weak convergence of g_n to g_0 is equivalent to each of the following

1. for each x in some dense set $D \subset I$ there exists a sequence $x_n \to x$ such that $g_n(x_n) \to g_0(x)$

2. g_n converges pointwise to g_0 on some dense subset of I

3. $g_n(x)$ converges to $g_0(x)$ for all $x \in C$

4. g_n converges uniformly to g_0 on compact subsets of C

5. G is continuous in each point of $C \times \{0\}$.

PROOF It suffices to prove that 1. implies 5. since 5. ⟹ 4. ⟹ 3. ⟹ 2. ⟹ ⟹ 1. is obvious. Hence suppose $c \in C$ and $\varepsilon > 0$. Choose x' and x'' in the dense set D such that $x' < c < x''$ and $g_0(x'') - g_0(x') < \varepsilon$. Choose $\delta > 0$ such that $x' + \delta < c < x'' - \delta$. Let x'_n and x''_n be the sequences mentioned in 1. converging to x' and x''. Choose n_0 such that $x'_n < c - \delta$, $c + \delta < x''_n$, $|g_n(x'_n) - g_0(x')| < \varepsilon$ and $|g_n(x''_n) - g_0(x'')| < \varepsilon$ for $n \geq n_0$. Then $|g_n(x'_n) - g_0(c)| < 2\varepsilon$ and $|g_n(x''_n) - g_0(c)| < 2\varepsilon$. Hence for $n \geq n_0$ and $x \in (c - \delta, c + \delta) \subset [x'_n, x''_n]$ we have $|g_n(x) - g_0(c)| < 2\varepsilon$. Q.E.D.

The usual topologies (Lévy metric or Hausdorff metric on M^*3, weak star topology on M^*1, weak convergence on M^*2, M^*3) make these sets into compact metrizable spaces. The obvious bijections are homeomorphisms.

The set M will be regarded as a subset of this compact space. The complement of M in M^* consists of the minimal and maximal element of the Boolean algebra M^*. (Compare M^*5 with M3.) As a subspace of M^* the space M is a locally compact metrizable space.

DEFINITION 2.2 The symbol M will henceforth denote a topological space. The underlying set is the set M1 of table 2.1, the topology is described above.

EXERCISE 2.1 The set of increasing homeomorphisms of $\mathbb{R}$ is dense in M.

PROPOSITION 2.2 Let $g_0, g_1, g_2, \ldots$ be elements of M with $g_n = \{(x_n(t), y_n(t)) \mid t \in \mathbb{R}\}$ in the representation of definition 1.3 for $n = 0,1,2,\ldots$ Then the following statements are equivalent

1. $g_n \to g_0$ in M

2. $x_n \to x_0$ weakly

3. for each $P_0 \in g_0$ there exists a sequence $P_n \in g_n$ which converges to P_0.

TABLE 2.1

M^*1. All probability measures on $[0, 1]$

2. All $f : (0, 1) \to [0, 1]$ non-decreasing and right-continuous

3. All curves $(x, y) : [0, 1] \to [0, 1]^2$ such that
   - x and y are continuous and non-decreasing
   - $x(t) + y(t) = 2t$ for $t \in [0, 1]$

4. All $f : \mathbb{R} \to [-\infty, \infty]$ non-decreasing and right-continuous

5. All sets A_Γ with $A \subset \mathbb{R}^2$

M1. All curves $\{(x(t), y(t)) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$ with
   - x and y continuous and non-decreasing
   - $x(t) + y(t) = t$ for all $t \in \mathbb{R}$

2. All non-decreasing real-valued functions f defined on some non-empty connected set $S \subset \mathbb{R}$ where we identify two functions (f_1, S_1) and (f_2, S_2) if
   - the closures of S_1 and of S_2 are equal
   - $\{f_1 = f_2\}$ is dense in the common interior

3. All sets A_Γ, $A \subset \mathbb{R}^2$, with A non-empty and $A_\Gamma \neq \mathbb{R}^2$

4. All sets ∂A_Γ, $A \subset \mathbb{R}^2$, with ∂A_Γ, the boundary of A_Γ, non-empty

PROOF We first prove that 3. implies 2. Set $P_n = (x_n(t_n), y_n(t_n))$. Then $x_n(t_n) \to x_0(t_0)$ and $y_n(t_n) \to y_0(t_0)$ imply $2t_n = x_n(t_n) + y_n(t_n) \to 2t_0$. This implies weak convergence of x_n (by the first criterium of proposition 2.1). Similarly, using criterium 3, one proves that 2. implies 3.

Now given a fixed increasing homeomorphism of $\mathbb{R}$ onto (0, 1), say the standard normal distribution function, there exists for each $g_n \in M$ a unique curve $g_n^* \in M^*3$ in $[0, 1] \times [0, 1]$.

The corresponding conditions 1*., 2*. and 3*. for the sequence g_n^* in M^*3 are equivalent. (If we extend x_n^* to $\mathbb{R}$ by setting $x_n^*(t) = 0$ for $t < 0$ and $x_n^*(t) = 1$ for $t > 1$, then proposition 2.1 applies to the extended functions and on comparing conditions 3. and 4. of that proposition we see that pointwise convergence of the sequence x_n^* on [0, 1] is equivalent to uniform convergence on [0, 1], i.e. to convergence in the Lévy metric. The equivalence of 2*. and 3*. is proved as above.)

Now 1*. is equivalent to 1. by definition of the topology of M and 3*. equivalent to 3. is obvious.

REMARK If $g_n \in M$ converges onto a non-empty set Λ, then the sequence (g_n) is relatively compact and the set of limit points is a closed subset of the compact set $M(\Lambda) = \{g \in M \mid \Lambda \subset g\}$.

PROOF The sequence (g_n) has a limit point g^* in M^*. Obviously this limit point is not the 0 or 1 of the Boolean algebra M^*, since g_n converges onto Λ. Hence $g^* \in M$. Now observe that $M(\Lambda)$ is a closed subset of M^* which does not contain the 0 or 1 of M^*.

PROPOSITION 2.3 Let h_n be a function defined and non-decreasing on an open interval I_n for $n = 0,1,2,\ldots$ Let $g_n \in M$ contain the graph of h_n. Let Λ_0 be the closure of the graph of h_0 and let Λ_1 be the closure of the restriction of g_0 to $I_0 \times \mathbb{R}$. Then the following are equivalent,

$h_n \to h_0$ weakly on I_0

g_n converges onto Λ_0

g_n' converges onto Λ_1.

PROOF Suppose $h_n \to h_0$ weakly on I_0. Let $x \in I_0$ be a continuity point of h_0, then $h_n(x) \to h_0(x)$. Hence g_n converges onto the set

$$\Lambda = \{(x, h_0(x)) \mid x \in I_0 \text{ continuity point of } h_0\}.$$

Let g be limit point of the relatively compact sequence (g_n) in M. (See the remark preceeding this proposition.) Then $g \supset \Lambda$, hence $g \supset \Lambda_1$. This proves that g_n converges onto Λ_1.

Suppose g_n converges onto Λ_0. Let I be an open interval such that $\overline{I} \subset I_0$. Then h_n is defined on I for $n \geq n_0$. By criterium 1 of proposition 2.1 the sequence h_n converges weakly to h_0 on I.

LEMMA 2.1 Suppose $g_n \to g$ in M and $\alpha_n \to \alpha$, $\beta_n \to \beta$ in G. Then $\beta_n g_n \alpha_n^{-1} \to \beta g \alpha^{-1}$ in M.

PROOF Observe that $x_n \to x$ implies

$$\alpha_n x_n = a_n x_n + b_n \;\to\; ax + b = \alpha x.$$

Suppose $P \in h$. There exist $P_n \in g_n$ such that $P_n \to P$. Then $\beta_n P_n \alpha_n^{-1} \to \beta P \alpha^{-1}$. Hence $\beta_n g_n \alpha_n^{-1} \to \beta g \alpha^{-1}$ by proposition 2.2 part 3.

PROPOSITION 2.4 Φ is a closed subset of M.

PROOF Suppose $\phi_n \to g$ in M with $\phi_n \in \Phi(\alpha_n, \beta_n)$, where $\Phi(\alpha_n, \beta_n)$ is defined as in (1.4), and $(\alpha_n, \beta_n) \neq (\varepsilon, \varepsilon)$. Define the equivalence relation $\sim$ on $(G \times G)\backslash\{(\varepsilon, \varepsilon)\}$ by $(\alpha, \beta) \sim (\alpha^t, \beta^t)$ for $t \neq 0$. The quotient space is the three dimensional real projective space, hence compact, and since $\Phi(\alpha, \beta) = \Phi(\alpha', \beta')$ if $(\alpha, \beta) \sim (\alpha', \beta')$, the sequence (α_n, β_n) may be chosen to be relatively compact in $(G \times G)\backslash\{(\varepsilon, \varepsilon)\}$. Let the subsequence (α_k, β_k) converge to $(\alpha, \beta) \neq (\varepsilon, \varepsilon)$ for $k \to \infty$. Then for $t \in R$ for $k \to \infty$, by lemma 2.1,

$$\phi_k \to g$$
$$\beta_k^t \phi_k \alpha_k^{-t} \to \beta^t g \alpha^{-t}.$$

Since the left hand sides agree, the right hand sides agree. Hence $g \in \Phi(\alpha, \beta) \subset \Phi$.

3 The equation $\tau h = h\sigma$

In this chapter we study the basic situation (1.1) under the extra condition that the support of the random variable $\underline{u}$ is the closure of an open interval I in $\mathbb{R}$ and that $P\{\underline{u} \in I\} = 1$, i.e. every non-empty open sub-interval of I contains positive mass and the endpoints of I carry no mass.

By theorem 2.1 there exists $g \in M$ such that $\underline{v} \overset{M}{=} g(\underline{u})$, i.e. $\underline{u}$ and $\underline{v}$ are marginals of a probability measure λ which lives on g. Let h be the right-continuous, non-decreasing function on I whose graph is contained in g. By the condition above on $\underline{u}$ the support Λ of the probability measure λ contains the graph of h. By theorem 2.2 the sequence $\beta_n f \alpha_n^{-1}$ in M converges onto Λ, and by proposition 2.3 this implies weak convergence to h on I of the corresponding sequence of non-decreasing functions.

Hence we may reformulate the basic situation (1.1) as

$$\beta_n f(\alpha_n^{-1}x) \to h(x) \text{ weakly on I} \qquad (3.1)$$
$$\alpha_n \to \infty.$$

In order to prove (1.2) it suffices to show that h is the restriction to I of a function $\phi \in \Phi$.

Now suppose $\sigma \in \Delta$, i.e. there exist sequences $k_n \to \infty$ and $l_n \to \infty$ such that

$$\alpha_{k_n}\alpha_{l_n}^{-1} = : \sigma_n \to \sigma.$$

Setting

$$\beta_{k_n}\beta_{l_n}^{-1} = : \tau_n, \text{ and}$$

$$g_n : = \beta_n f \alpha_n^{-1}$$

we may write

$$g_{l_n} = \beta_{l_n} f \alpha_{l_n}^{-1} = \tau_n^{-1} g_{k_n} \sigma_n$$

or equivalently

(3.2) $\qquad a_n g_{l_n}(x) + b_n = g_{k_n}(\sigma_n x)$

where $a_n y + b_n := \tau_n y$.

We assume h non-constant on $I \cap \sigma^{-1}I$. It is then possible to find $x_1, x_2 \in C \cap \sigma^{-1}C$ such that $h(x_1) \neq h(x_2)$, where C is the set of continuity points of h in I.

Substituting x_1 and x_2 in equation 3.2 and subtracting we find

$$a_n(g_{l_n}(x_2) - g_{l_n}(x_1)) = g_{k_n}(\sigma_n x_2) - g_{k_n}(\sigma_n x_1).$$

For n tending to infinity, this becomes

$$a(h(x_2) - h(x_1)) = h(\sigma x_2) - h(\sigma x_1).$$

In particular $a_n \to a \geq 0$ since $h(x_2) \neq h(x_1)$. Similarly substituting x_1 in (3.2) we find that b_n converges to a real number b, and for all $x \in C \cap \sigma^{-1}C$ we have

$$ah(x) + b = h(\sigma x).$$

If we assume h to be right continuous, then this equality holds throughout the interval $I \cap \sigma^{-1}I$. If $a = 0$, then h is constant $\equiv b$ on this interval. Hence if we also assume that h is not constant on $I \cap \sigma I$ then $a > 0$, and setting $\tau y = ay + b$ we obtain

(3.3) $\qquad \tau h = h\sigma,$

i.e. for each x for which both the right and left hand side are defined, equality holds.

Thus we have proved the following proposition.

PROPOSITION 3.1 Let f be a non-decreasing function, let α_n and β_n be positive affine transformations such that $\alpha_n \to \infty$ and let h be a right-continuous, non-decreasing function defined on the open interval I, such that

$$\beta_n f(\alpha_n^{-1}x) \to h(x) \quad \text{weakly on I.}$$

Let $\sigma \in \Delta = \Delta(\alpha)$ be such that $I \cap \sigma I$ is non-empty and that h is non-constant on $I \cap \sigma I$ and on $I \cap \sigma^{-1}I$. Then there exists $\tau \in G$ such that

$$\tau h(x) = h(\sigma x) \quad \text{if } x \text{ and } \sigma x \in I$$

and such that

$$\alpha_{k_n}\alpha_{l_n}^{-1} \to \sigma \quad \text{implies} \quad \beta_{k_n}\beta_{l_n}^{-1} \to \tau.$$

PROOF See above.

Observe that the condition about h being non-constant on $I \cap \sigma I$ and on $I \cap \sigma^{-1}I$ is fulfilled in each of the following cases

1. $I = \mathbb{R}$ and $\underline{v}$ is non-constant
2. $I = (c, \infty)$ and the probability distribution function $G(v)$ of $\underline{v}$ is continuous in its upper endpoint $\sup \{v \mid G(v) < 1\}$
3. the probability distribution of $\underline{v}$ is continuous.

This follows from the inequality

$$P\{\underline{v} = c\} \geq P\{\underline{u} \in J_c\}$$

where $\underline{v} \overset{M}{=} g(\underline{u})$, $g \in M$, and $g = c$ on the open interval J_c.

In the particular case that the support of $\underline{u}$ is the closure of an open interval I and $P\{\underline{u} \in I\} = 1$, the problem of finding a functional relation between the limit random variables $\underline{u}$ and $\underline{v}$ leads us thus to the problem of solving a set of functional equations of the form (3.3).

Although one could also derive the functional equation if $\underline{u}$ has a continuous distribution function F, it would only hold on the set X obtained by deleting the closed intervals of constancy of F. In this generality the equations are quite untractable.

The problem of giving necessary and sufficient conditions on Δ and I such that the system of equations

$$\tau h = h\sigma$$

(where $\tau = \tau(\sigma) \in G$ is not known and σ varies over Δ), implies that h is the restriction to I of some $\phi \in \Phi$, is difficult. We shall limt ourselves to two particular cases

1. I is unbounded
2. ε is a condensation point of Δ.

First let us settle the question of determining all solutions h of (3.3) for a fixed pair (σ, τ) of positive affine transformations.

Consider the simplest case in which both σ and τ are translations. The functional equation (3.3) then has the form

$$h(x + p) = q + h(x). \tag{3.4}$$

We are interested in finding all functions h defined on the given interval I which satisfy (3.4). To avoid trivialities we shall assume $p \neq 0$ and $|I| > |p|$. Note that

1. if h_1 and h_2 are solutions of (3.4), then $h_2 - h_1$ is periodic modulo p,
2. the function $h(x) = qp^{-1}x$ on I is a solution.

Thus every right-continuous non-decreasing solution of (3.4) has a representation

$$h(x) = \lambda x + c + \pi(x) \tag{3.5}$$

where $\lambda = qp^{-1}$, π is periodic modulo p, bounded (since h is bounded over a period) and upper semi-continuous (since h is), and c is chosen so that $\max \pi(x) = 0$.

DEFINITION 3.1 The upper envelope $\overline{h}$ of h with respect to equation (3.4) is the function $\overline{h}(x) := \lambda x + c$ on I where λ and c are defined by (3.5).

Note that the upper envelope in general depends on p and q in equation (3.4). Note too that the set $\{\overline{h} = h\}$ is periodic modulo p and non-empty (it is the set $\{\pi = 0\}$).

If h satisfies two equations

(3.6) $\qquad h(x + p_i) = q_i + h(x) \quad i = 1, 2$

then we have two representations

$$h(x) = \bar{h}_i(x) + \pi_i(x) \quad i = 1, 2$$

where $\bar{h}_i(x) = q_i p_i^{-1} x + c_i$ $(i = 1, 2)$.

If I is sufficiently large, in fact if $|I| > |p_1| + |p_2|$ then one can prove that $\bar{h}_1 = \bar{h}_2$, hence $\pi_1 = \pi_2$ is periodic modulo p_1 and modulo p_2. In particular if p_1/p_2 is irrational, then π is constant and $h = \bar{h}_1 = \bar{h}_2$ is an affine function.

To prove that $\bar{h}_1 = \bar{h}_2$ we need three points, $x_0, x_1, x_2 \in I$ such that $x_0 < x_1 < x_2$ and

$$\bar{h}_1(x_0) = h(x_0)$$
$$\bar{h}_2(x_1) = h(x_1)$$
$$\bar{h}_1(x_2) = h(x_2)$$

(or the same equations with the $\bar{h}_1$ and $\bar{h}_2$ interchanged). Existence follows from the periodicity of the sets $\{\bar{h}_1 = h\}$ and $\{\bar{h}_2 = h\}$. Since we know that h is majorized by its upper envelope, the three equations above imply that the affine function $d(x) = \bar{h}_1(x) - \bar{h}_2(x)$ has at least two zero's on I, hence d vanishes indentically and $\bar{h}_1 = \bar{h}_2$.

Now consider equation (3.3) for general σ and τ. We choose the origin of our x-axis and y-axis to be the centre of multiplication of the transformation σ and of τ (unless these are translations). Thus we may assume that $\sigma x = x + p$ or $\sigma x = e^p x$ and $\tau y = y \pm q$ or $\tau y = e^{\pm q} y$ with $q > 0$ or $\tau y = y$. Since (3.3) is equivalent to

$$h\sigma^{-1} = \tau^{-1} h$$

we may assume p to be positive. (If $p = 0$ then h is constant or the equation is trivial.) By a suitable transformation of the form

$$h_1(x) = \log h(x)$$
$$h_1(\xi) = h(e^{\xi})$$

or $\qquad h_1(\xi) = \log h(e^{\xi})$

we may reduce all these cases to equation (3.4). This yields the non-constant solutions of table 3.1.

DEFINITION 3.2 The upper envelope of $h = \psi(k)$ in table 3.1 with respect to equation (3.3) is $\overline{h} = \psi(\overline{k})$ where $\overline{k}$ is the upper envelope of k.

The upper envelope is the restriction to I of an element of $\Phi(\sigma, \tau)$ as can be checked with some patience (see (1.4) for the definition of $\Phi(\sigma, \tau)$).

LEMMA 3.1 Suppose $\phi \in \Phi$ is defined on the open interval I_1 and has a strictly positive (finite) derivative in every point of I_1. Then there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\frac{d}{dx} \log \frac{d}{dx} \phi(x) = \frac{\mu - \lambda}{1 + \lambda(x - \lambda)} .$$

TABLE 3.1

σx	τy	$h(x)$		
$x + p$	$y + q$	$k(x) := c + \lambda x + \pi(x)$		
	$e^q y$	$e^{k(x)}$		
	$e^{-q} y$	$-e^{-k(x)}$		
$e^p x$	$y + q$	$k(\log x)$	$x > 0$	
	$y - q$	$-k(\log -x)$	$x < 0$	
	$e^q y$	$\begin{cases} c_1 e^{k_1(\log x)} \\ -c_2 e^{k_2(\log -x)} \end{cases}$	$x > 0$ $x < 0$	$c_1, c_2 \geq 0$ $c_1 + c_2 > 0$
	$e^{-q} y$	$-e^{k(\log x)}$	$x > 0$	
	$e^{-q} y$	$e^{k(\log -x)}$	$x < 0$	
	y	$c_1 + c_2 \operatorname{sign} x$		$c_2 > 0$

PROOF Set $L = \frac{d}{dx} \log \frac{d}{dx}$. Then $L\alpha\phi = L\phi$, $L(\phi\alpha) = a.(L\phi)(\alpha)$ if $\alpha x = ax + b$ and $L\psi(x) = -(L\phi)(-x)$ if $\psi(x) = -\phi(-x)$. It suffices to check the formula for the functions x, e^x, $\log x$ and x^α. And indeed $Lx = 0$, $Le^x = 1$, $L \log x = -\frac{1}{x}$, $Lx^\alpha = \frac{\alpha - 1}{x}$.

COROLLARY Suppose $\phi_1, \phi_2 \in \Phi$ are defined and differentiable on the open interval I_1 and their derivatives are strictly positive on I_1. If ϕ_1 and ϕ_2 agree in four points of I_1 (counted with proper multiplicities), then they coincide on I_1.

PROOF The derivatives ϕ_1' and ϕ_2' agree in three points, so too $\log \phi_1'$ and $\log \phi_2'$. The functions $L\phi_1$ and $L\phi_2$ agree in two points ($L = \frac{d}{dx} \log \frac{d}{dx}$ as above). Hence they coincide. (Either both vanish on I_1 or the algebraic inverse is affine.)

PROPOSITION 3.2 Suppose h is non-decreasing non-constant on the open interval I. Suppose $(\sigma_i, \tau_i) \neq (\varepsilon, \varepsilon)$ and

$$(3.7) \qquad \tau_i h = h\sigma_i \quad \text{for } i = 1, 2.$$

Let $\bar{h}_1$ and $\bar{h}_2$ be the upper envelopes of h with respect to equation (3.7). Then $\bar{h}_1 = \bar{h}_2$ if one of the following conditions holds

a) I is unbounded

b) I is bounded and σ_1 and σ_2 lie in the neighbourhood of ε defined by

$$|\sigma x - x| < \frac{1}{15} |I| \text{ for all } x \in I.$$

PROOF We only prove b). The proof of a) is similar.
The upper envelope $\bar{h}_i$ is the restriction to I of some $\phi_i \in \Phi(\sigma_i, \tau_i)$. Note that ϕ_1 and ϕ_2 are not constant on the whole interval I since this would imply that h is constant on I and this case is explicitly excluded.

Suppose one of the functions is constant on part of the interval, say $\phi_1 = c$ on $I_0 = I \cap (-\infty, x_0)$ with $x_0 \in I$ and x_0 maximal. Then $h = c$ on I_0 (the periodic part vanishes on the left side of x_0). If $|I_0| < \frac{2}{15} |I|$, then there exist x_1 and $x_2 = \sigma_2 x_1$ in I_0 such that $h(x_i) = \phi_2(x_i)$ for $i = 1, 2$ and $\phi_2 = h = c$ on (x_1, x_2). Then $\phi_2 = c$ on a maximal halfline (with endpoint $x_3 \in I$) containing (x_1, x_2). In particular σ_2 is a multiplication with centre x_3. If $x_3 \in I_0$ then $\phi_2' = c$ would hold on a left and a right neighbourhood of x_3 and hence $\phi_2 = c$ would hold throughout I. This case, as we have seen above is excluded. Hence $x_3 \geq x_0$. By symmetry of argument $x_0 \geq x_3$. Therefore $x_0 = x_3$ and we obtain either $h = c_1 > c$ on $(x_0, \infty) \cap I$ (and $\phi_1 = \phi_2 = h$) or there exist functions k_1 and k_2 such that

$$(3.8) \qquad h(x) = c + e^{k_i(\log(x - x_0))} \qquad \text{for } x > x_0$$

(see table 3.1). Set $\xi := \log(x - x_0)$. Then $k_1(\xi) = k_2(\xi)$ on a neighbourhood of $-\infty$. The upper envelopes $\bar{k}_1$ and $\bar{k}_2$ coincide by the argument used in treating the system (3.5). Hence $\phi_1 = \phi_2$ on $(x_0, \infty) \cap I$ by definition of upper envelope of h.

If neither of the functions ϕ_i is constant on part of I or if $|I_0| < \frac{2}{15}|I|$, then there exists a subinterval I_1 of length $\geq \frac{1}{3}|I|$ on which both ϕ_1 and ϕ_2 have strictly positive continuous derivatives. Any subinterval of length $\geq \frac{1}{15}|I|$ contains a zero of $\phi_i - h$ for i = 1, 2 and we may choose $x_0 < y_1 < x_1 < y_2 < x_2$ in I_1 such that $\phi_1 - h$ vanishes in x_0, x_1, x_2 and such that $\phi_2 - h$ vanishes in y_1 and y_2. Since $\phi_i - h \geq 0$ for i = 1, 2, the function $\phi_1 - \phi_2$ is non-positive in x_0, x_1 and x_2 and non-negative in y_1 and y_2. Hence it has at least four zeros (counted with their proper multiplicities) on the closed interval $[x_0, x_2]$. By the corollary to lemma 3.1 this implies that $\phi_1 = \phi_2$ on I_1. Now if $\phi_1 \neq \phi_2$ on the whole interval I, then $\phi_1 = \phi_2$ on $I_1 \cap L$ where L is a maximal halfline containing I_1 with endpoint $x_4 \in I$. Then ϕ_1' vanishes in x_4 or becomes infinite and hence so too ϕ_2' (or vice versa). On $I \setminus L$ we again have a representation of h of the form (3,8) and the argument used there may be repeated to prove that also on the set $I \setminus L$ the functions ϕ_1 and ϕ_2 coincide.

COROLLARY 1 If h is non-constant non-decreasing on the open interval I and if

$$\tau_n h = h\sigma_n$$

holds for a sequence $\sigma_n \to \varepsilon$ with $\sigma_n \neq \varepsilon$, then h is the restriction to I of a function $\phi \in \Phi$.

PROOF Either I is unbounded or I is bounded and σ_n satisfies condition b) of proposition 3.2 for $n \geq n_0$. In either case the upper envelope ψ_n of h with respect to σ_n and τ_n does not depend on n for $n \geq n_0$ (by proposition 3.2). Let us denote this common upper envelope by ψ (which is the restriction to I of an element ϕ of Φ).

The set $\{\psi = h\}$ is periodic modulo σ_n for $n \geq n_0$, and $\sigma_n \to \varepsilon$ implies that the set is dense in I. Since ψ and h are both non-decreasing on I, they coincide (in their continuity points) on I.

COROLLARY 2 Under the conditions of proposition 3.2, condition a) and b) each imply that either h is the restriction to I of an element $\phi \in \Phi$ or that there exist integers n_1 and n_2 and elements $\sigma, \tau \in G$ such that $\sigma_i = \sigma^{n_i}$, $\tau_i = \tau^{n_i}$ and $\tau h = h\sigma$.

PROOF If $h = \overline{h}_1$ ($= \overline{h}_2$) we are done since $\overline{h}_1$ is the restriction to I of an element $\phi \in \Phi$. Hence assume $h \neq \overline{h}_1$. Then $\phi \in \Phi(\sigma_i, \tau_i)$ for i = 1, 2 implies either ϕ is affine or $\Phi(\sigma_1, \tau_1) = \Phi(\sigma_2, \tau_2)$ as one easily checks.

In the latter case

$$(3.9) \qquad \sigma_2 = \sigma_1^t \quad , \quad \tau_2 = \tau_1^t \quad \text{for some } t \neq 0$$

and $h = \psi(k_i)$ (see table 3.1) where $k_i(x) = \lambda x + c + \pi_i$ with π_i periodic modulo p_i for i = 1, 2. Since h, ψ, λ and c do not depend on i, we obtain $\pi_1 = \pi_2$ is periodic modulo p_1 and $p_2 = tp_1$ (by (3.9)). If t is irrational then this periodic part vanishes and $h = \overline{h}_1$. Else $t = n_2/n_1$ for integral n_1 and n_2, π is periodic modulo $p = p_1/n_1 = p_2/n_2$ and $\tau h = h\sigma$ if we set $\sigma^{n_1} = \sigma_1$ and $\tau^{n_1} = \tau_1$.

In the former case $\overline{h}_1$ is affine, say $\overline{h}_1 = \gamma_{|I}$ with $\gamma \in G$. Then

$$(3.10) \qquad \tau_i\gamma = \gamma\sigma_i \qquad i = 1, 2.$$

Introduce the lower envelopes $\underline{h}_1$ and $\underline{h}_2$ in the obvious way. If σ_1 is a translation, then $\underline{h}_1$ and $\overline{h}_1$ are parallel lines; if σ_1 is a multiplication with centre c, then $\underline{h}_1$ and $\overline{h}_1$ are each of the form $c_0(x - c) + c_1|x - c|$ and intersect in x = c. Since $\underline{h}_1 = \underline{h}_2$ holds for the lower envelopes as well, either σ_1 and σ_2 are both translations or both multiplications with the same centre c. Together with (3.10) this implies (3.9) and the argument proceeds as above.

THEOREM 3.1 Suppose that in addition to the basic situation (1.1) it is known that

1. the support of $\underline{u}$ is the closure of an open interval I and $P\{\underline{u} \in I\} = 1$

2. ε is a condensation point of Δ,

then

$$\underline{v} = \phi(\underline{u}) \quad \text{for some } \phi \in \Phi.$$

PROOF We may assume that the function h on I defined in (3.1) is not constant. There exists an open neighbourhood U of ε in G such that h is non-constant on $I \cap \gamma^{-1}I$ and on $I \cap \gamma I$ for each $\gamma \in U$. For each $\sigma \in \Delta \cap U$ there exists by proposition 3.1 a unique $\tau \in G$ such that

$$\tau h = h\sigma.$$

From corollary 1 to proposition 3.2 above it follows that h is the restriction of some element $\phi \in \Phi(\sigma, \tau)$.

COROLLARY If in addition to condition 1 and 2 it is known that

3. $\underline{v}$ is non-constant

4. every neighbourhood of ε in Δ contains elements σ_1 and σ_2 which do not commute i.e. $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$, then

$$\underline{v} = \gamma\underline{u} \quad \text{for some } \gamma \in G, \text{ and}$$

$$\alpha_{k_n}\alpha_{l_n}^{-1} \to \sigma$$

and $I \cap \sigma I$ is non-empty implies

$$\beta_{k_n}\beta_{l_n}^{-1} \to \gamma\sigma\gamma^{-1}.$$

PROOF Since $\phi \in \Phi(\sigma_1, \tau_1)$ and $\phi \in \Phi(\sigma_2, \tau_2)$ and σ_1 and σ_2 do not commute, ϕ is affine, i.e. $\phi = \gamma$ for some $\gamma \in G$. Hence h is non-constant on $I \cap \sigma I$ and on $I \cap \sigma^{-1}I$ whenever $I \cap \sigma I$ is non-empty. By proposition 3.1 the sequence $\beta_{k_n}\beta_{l_n}^{-1}$ converges to an element $\tau \in G$ and $\tau h = h\sigma$. Setting $h = \gamma$ gives the desired result.

THEOREM 3.2 Suppose that in addition to the basic situation (1.1) it is known that the support of $\underline{u}$ is $\mathbb{R}$. Then there exists a unique element $g \in M$ such that $\underline{v} \overset{M}{=} g(\underline{u})$ and, unless $\underline{v}$ is degenerate, for each $\sigma \in \Delta$ there exists a unique $\tau \in G$ such that

$$(3.11) \qquad \tau g = g\sigma$$

$$(3.12) \qquad \alpha_{k_n}\alpha_{l_n}^{-1} \to \sigma \text{ implies } \beta_{k_n}\beta_{l_n}^{-1} \to \tau.$$

PROOF Combine theorem 2.1, theorem 2.2 and proposition 3.1.

In the case that $\underline{v}$ is non-degenerate, i.e. g is non-constant, we can give a complete classification of the possible situations which can occur in the case that the support of $\underline{u}$ is $\mathbb{R}$. To this end we introduce the closed subgroup H of G generated by $\Delta = \Delta(\alpha)$. Observe that case 4 occurs if Δ contains two elements which do not commute, else H is contained in a one parameter subgroup $\{\sigma^t \mid t \in \mathbb{R}\}$ for some $\sigma \neq \varepsilon$. The classification is given in table 3.2 on page 39.

PROOF of the classification in table 3.2.

Suppose $\sigma_i \in \Delta = \Delta(\alpha)$. By proposition 3.1 there exists $\tau_i \in \Delta(\beta)$ such that (3.11) and (3.12). By proposition 3.2 the upper and lower envelopes $\overline{g}$ and $\underline{g}$ are independent of the choice of σ_i. By corollary 2 to this proposition either $g = \overline{g} \in \Phi(\sigma_i, \tau_i)$ for all pairs (σ_i, τ_i) or there exist σ, τ and integers $n(i)$ such that $\sigma_i = \sigma^{n(i)}$, $\tau_i = \tau^{n(i)}$ and (3.11) holds.

Hence if Δ contains two elements σ_1 and σ_2 which do not commute then $g \in \Phi(\sigma_1, \tau_1) \cap \Phi(\sigma_2, \tau_2)$ and by checking the different possibilities it is clear that g is an affine function γ and $\tau\gamma = \gamma\sigma$. This proves 4.

Else $\Delta \subset \{\sigma^t \mid t \in R\}$ for some $\sigma \neq \varepsilon$. Since $\Phi(\sigma^t, \tau^t) = \Phi(\sigma, \tau)$ for $t \neq 0$, we may write $\sigma = \sigma_i^{t(i)}$ (if $\sigma_i \neq \varepsilon$) and

$$\overline{g} \in \Phi(\sigma, \tau_i^{t(i)})$$

for all i and hence $\tau_i^{t(i)} = \tau$ is independent of i ($g = \tau g$ with $\tau \neq \varepsilon$ implies $g \in M_0$, see exercise 1.7). This proves 3, 2 and 1.

In chapter 4 we shall see that indeed any non-constant non-decreasing function g is possible if $\Delta = \{\varepsilon\}$ whatever the sequence (α_n).

TABLE 3.2

The possible relations $\underline{v} \overset{M}{=} g(\underline{u})$, with $g \in M$, between the limit variables $\underline{u}$ and $\underline{v}$ in the basic situation (1.1) under the condition that the support of $\underline{u}$ is $\mathbb{R}$ and that $\underline{v}$ is non-constant.

H is the closed subgroup of G generated by Δ, and $k_n \to \infty$.

1. $H = \{\varepsilon\}$ (degenerate case)

   a. $\alpha_{k_n}\alpha_{l_n}^{-1} \to \varepsilon$ implies $\beta_{k_n}\beta_{l_n}^{-1} \to \varepsilon$

   b. g may be an arbitrary non-constant non-decreasing function on $\mathbb{R}$.

2. $H = \{\sigma^k \mid k \text{ integral}\}$ for some $\sigma \neq \varepsilon$ (discrete case)

   There exists $\tau \in G$ such that

   a. $\alpha_{k_n}\alpha_{l_n}^{-1} \to \sigma^k$ implies $\beta_{k_n}\beta_{l_n}^{-1} \to \tau^k$

   b. g satisfies the functional equation

   $$\tau g = g\sigma.$$

3. $H = \{\sigma^t \mid t \in \mathbb{R}\}$ for some $\sigma \neq \varepsilon$

   There exists $\tau \in G$ such that

   a. $\alpha_{k_n}\alpha_{l_n}^{-1} \to \sigma^t$ implies $\beta_{k_n}\beta_{l_n}^{-1} \to \tau^t$

   b. $g \in \Phi(\sigma, \tau)$.

4. Δ contains two elements which do not commute

   There exists $\gamma \in G$ such that

   a. $\alpha_{k_n}\alpha_{l_n}^{-1} \to \sigma$ implies $\beta_{k_n}\beta_{l_n}^{-1} \to \gamma\sigma\gamma^{-1}$

   b. $g = \gamma$.

4 Existence theorems

This chapter is devoted almost entirely to the construction (for given sequence (α_n) in G and given $g \in M$) of a sequence (β_n) in G and an increasing homeomorphism f on $\mathbb{R}$ such that

(4.1) $$\beta_n f \alpha_n^{-1} \to g \quad \text{in } M.$$

We assume here that g lives on an open interval I (i.e. I is the interior of the projection of g on the x-axis), which may be unbounded, and that σI and I are disjoint for all $\sigma \in \Delta$, $\sigma \neq \varepsilon$.

The proofs of proposition 4.1 and 4.3 are rather involved. In order to ease the reading, the proofs have been cut up into several parts, A, B, ..., most of which consist of a statement, followed by a proof of this statement. Reading the statements A, B, ... should give the reader a bird's eye view of the proof.

Before entering on the proof of theorem 4.1, or rather its analytic counterpart, proposition 4.3, let us consider a simple particular case.

Let ψ be a continuous, strictly increasing, bounded function on the open interval I = (0, 1) and let $\alpha_1, \alpha_2, \ldots$ be a sequence of translations, $\alpha_n x = x - t_n$.

If $\alpha_n \to \infty$, then $|t_n| \to \infty$ and the sequence of t_n's may be indexed anew to be non-decreasing

$$\ldots \leq t_{-1} \leq t_0 \leq t_1 \leq \ldots$$

where the index now runs through an infinite set of consecutive integers. We shall assume that the index set is the set of all integers and that $t_{k+1} - t_k > \frac{1}{2}$ for all k.

Note that Δ consists of translations $\sigma x = x + s$ where s is limit point of the double sequence $(t_k - t_m)$. If $I \cap \sigma I$ is empty for all $\sigma \in \Delta \setminus \{\varepsilon\}$, then $\sigma \in \Delta$, $\sigma x = x + s$ with $s \neq 0$, implies $|s| \geq 1$. Hence $\liminf(t_{k+1} - t_k) \geq 1$.

Suppose first $t_{k+1} - t_k \geq 1$ for all k. Then the intervals $J_k := \alpha_k^{-1} I = (t_k, 1 + t_k)$ are disjoint. We define f_0 on $\cup J_k$ by

(4.2) $$f_0(x) := c_k + \psi(x - t_k) \quad \text{for } x \in J_k$$

where we choose the constants c_k such that f_0 may be extended to a continuous strictly increasing function f on $\mathbb{R}$, which is a homeomorphism since $\lim_{x\to\infty} f(x) = \infty = -\lim_{x\to-\infty} f(x)$. On setting $\beta_k y = y - c_k$ we obtain

$$\beta_k f(\alpha_k^{-1}x) = f_0(t_k + x) - c_k = \psi(x) \quad \text{for } x \in I.$$

Thus we have constructed for the given function ψ and the given sequence (α_k) a sequence (β_k) and a function f such that

$$\beta_k f\alpha_k^{-1} \to \psi \quad \text{on } I = (0, 1).$$

In general we only know that $\liminf(t_{k+1} - t_k) \geq 1$. The intervals $\alpha_k^{-1}I = (t_k, 1 + t_k)$ need not be disjoint. However, there exist subintervals $I_k \subset I$ such that $I_k \to I$ and such that the intervals $J_k := \alpha_k^{-1}I_k$ are disjoint. The construction of f then proceeds as above.

If instead of a continuous strictly increasing bounded function ψ on I, we want f to satisfy (4.1) where $g \in M$ lives on I, then we first construct $I_k \uparrow I$ such that the intervals $J_k := \alpha_k^{-1}I_k$ are disjoint, and continuous strictly increasing bounded functions $\psi_k : I_k \to \mathbb{R}$ such that $\psi_k \to g$, and

$$\text{(4.3)} \qquad \sup_{x\in I_k} \psi_k(x) \to \infty \quad , \quad \inf_{x\in I_k} \psi_k(x) \to -\infty.$$

As in (4.2) we define f_0 on $\cup J_n$ by

$$f_0(x) := c_k + \psi_k(x - t_k) \quad \text{for } x \in J_k,$$

extend f_0 to a homeomorphism f on $\mathbb{R}$ and observe that for $\beta_k y = y - c_k$ we obtain for $x \in I_m$, $|k| \geq m$,

$$\beta_k f(\alpha_k^{-1}x) = f_0(t_k + x) - c_k = \psi_k(x) \to g(x).$$

This together with (4.3) implies $\beta_k f\alpha_k^{-1} \to g$ in M.

This construction can also readily be adapted to the case that I is an unbounded interval.

However, if we do not restrict the α_n to lie in a one parameter subgroup of G, the construction becomes more involved. In the case sketched above it was obvious that we could replace the sequence $\alpha_k^{-1}I$ by a sequence

$J_k = \alpha_k^{-1} I_k$ of disjoint intervals such that $I_k \to I$. In the general case we need the following proposition.

PROPOSITION 4.1 Let Λ be a set of bounded open intervals I, such that

$$(4.4) \qquad \frac{|I_1 \cap I_2|}{|I_1| + |I_2|} \to 0 \quad \text{for } I_1 \neq I_2 \text{ and } I_1, I_2 \in \Lambda.$$

(I.e. for any $\delta > 0$ there are only finitely many pairs (I_1, I_2) with $I_1 \neq I_2$ for which the quotient above exceeds δ.) Then for each interval I there exists an open interval I^* such that

$$(4.5) \qquad I^* \subset I$$

$$(4.6) \qquad |I^*|/|I| \to 1$$

(4.7) If $I_1^* \cap I_2^*$ is non empty then either $I_1^* \subset I_2^*$ or $I_2^* \subset I_1^*$.

PROOF The proof consists of eight parts.

A. Λ is countable

Relation (4.4) implies that the set of pairs (I, J') with $I \neq J'$ for which $|I \cap J'| > 0$,is countable. On Λ we define the equivalence relation R by

IRJ' if there exist $I_0 = I, I_1, \ldots, I_n = J'$ in Λ such that $I_{i-1} \cap I_i$ is non-empty for $i = 1,\ldots,n$.

For each $I \in \Lambda$ the equivalence class R(I) is a countable subset of Λ. If I and J' are not equivalent the open intervals $\cup R(I)$ and $\cup R(J')$ are disjoint. Hence there are only countably many equivalence classes.

B. We may assume that no two intervals in Λ have the same length.

We reduce each interval I to a subinterval I' such that $|I'|/|I| \to 1$ and such that these new intervals all have different lengths. Relation (4.4) holds for the set of intervals I' and if (4.5) and (4.6) hold for I' instead of I, they also hold for I.

C. Set

(4.8) $\rho(I) = \sup \{|I \cap J|/|I| \mid |J| < |I|\}$.

Then $\rho(I) \to 0$ by (4.4).

D. We may assume that $\rho(I) < \frac{1}{4}$ for all $I \in \Lambda$.

Set $I^* = \emptyset$ whenever $\rho(I) \geq \frac{1}{4}$.

E. For each interval $(l, r) = I \in \Lambda$ define

$$A = [l, l + \rho\lambda] \qquad E = [r - \rho\lambda, r]$$
$$B = [l + \rho\lambda, l + 2\rho\lambda] \qquad D = [r - 2\rho\lambda, r - \rho\lambda]$$
$$C = (l + 2\rho\lambda, r - 2\rho\lambda)$$

where $\lambda = |I| = r - l$ and $\rho = \rho(I)$. C is called the core of the interval I. Then the following assertion holds (for the proof see below under G and H).

For each interval I there exists $x = x(I)$ in $A = A(I)$ and $y = y(I)$ in E such that neither lies in the core $C(J)$ of any interval J shorter than I.

F. Define $X(I) = \{x(J), y(J) \mid J \in \Lambda \text{ and } |J| \geq |I|\}$. Because of the assertion above the core C(I) is disjoint from X(I). Let I^* be the largest open interval which contains C and is disjoint from X. Then

$$C \subset I^* \subset (x(I), y(I)) \subset I$$

which proves (4.5) and (4.6).

If $|I_1| < |I_2|$ and $u \in I_1^* \cap I_2^*$, then I_1^* is the largest interval which contains u and is disjoint from $X(I_1)$ and I_2^* is the largest interval which contains u and is disjoint from $X(I_2)$. Since $X(I_2) \subset X(I_1)$ we have $I_1^* \subset I_2^*$ which proves (4.7). Note that we have even proved that

(4.9) $$\frac{|I \setminus I^*|}{|I|} \leq 4\rho(I).$$

G. For any $x \in \mathbb{R}$ the set $\{|I| \mid x \in I \in \Lambda\}$ is a discrete subset of $(0, \infty)$.

Indeed, suppose I, I', I" $\in \Lambda$, $x \in I \cap I' \cap I''$ and $\frac{1}{2}\lambda < \lambda'' < \lambda' < \lambda$. Then either $x + \frac{\lambda}{4}$ or $x - \frac{\lambda}{4}$ lies in two of there intervals, say J_1 and J_2. We assume $|J_1| < |J_2|$. Then by D.

$$|J_1 \cap J_2| > \frac{1}{4}\lambda > \rho(J_2).|J_2|$$

which contradicts the definition of $\rho(J_2)$.

H. Proof of assertion E

Suppose $I_0 \in \Lambda$. Let x_0 be the left endpoint of I_0. We shall construct a point $x \in A_0$ such that $x \notin C(I)$ whenever $|I| \le |I_0|$. If x_0 already has this property we define $x := x_0$. Else we choose $I_1 \in \Lambda$ of maximal length (see G) such that $|I_1| < |I_0|$ and $x_0 \in B_1 \cup C_1 \cup D_1$. Let x_1 be the right endpoint of I_1. Then

1) $E_1 \subset A_0$
2) if $I \in \Lambda$ and $|I_1| < |I| \le |I_0|$ then I_1 and $C = C(I)$ are disjoint.

To prove 1) note that $(x_0, x_1) \subset I_0 \cap I_1$, hence $x_1 - x_0 \le \lambda_0\rho_0$ by definition of ρ_0. Similarly if $I \in \Lambda$ is such that $|I_1| < |I| \le |I_0|$ and I_1 meets C then by definition of $\rho(I)$ we have $I_1 \subset B \cup C \cup D$ and hence $x_0 \in B \cup C \cup D$ contradicting the maximality of $|I_1|$.

Now we recursively choose x_n and I_n such that

(4.10) $\qquad A_0 \supset E_1 \supset A_2 \supset E_3 \supset \dots$

(4.11) if $I \in \Lambda$ and $|I_{n+1}| < |I| \le |I_n|$, then I_{n+1} and $C = C(I)$ are disjoint.

This construction either can be repeated indefinitely, or there exists an integer n such that $x_n \notin C$ for any interval I for which $|I| < |I_n|$.

In the former case G and (4.10) imply that $|I_n| \to 0$. The sequence (4.10) determines a unique point $x \in A_0$. Now suppose $x \in I \in \Lambda$, $|I| < |I_0|$. There exists $n \ge 0$ such that

$$|I_{n+1}| < |I| \le |I_n|.$$

Then I_{n+1} is disjoint from C by (4.11) and since $x \in I_{n+1}$ this implies that $x \notin C$.

The proof in the latter case is similar. Q.E.D.

Note that the set $\Lambda^* = \{I^*\}$ also satisfies (4.4) and that we may replace (4.6) by the stronger result

$$(4.9) \qquad |I \setminus I^*|/|I| \leq 4\rho(I)$$

where $\rho(I)$ is defined in (4.8).

DEFINITION 4.1 Suppose that Λ has the properties (4.4) and (4.7). An element $I \in \Lambda$ is maximal if $I \subset J \in \Lambda$ implies $J = I$. If I is not maximal, then it has a successor $I' \in \Lambda$, that is

$$I \subset I'$$

$$I \neq I'$$

$$I \subset J \subset I' \text{ with } J \in \Lambda \text{ implies } J = I \text{ or } J = I'.$$

This follows readily from (4.4). The successor is unique because of (4.7).

It is possible that each element $I \in \Lambda$ is contained in a maximal element. Else there exists a sequence $I_1, I_2, \ldots$ such that I_{n+1} is the successor of I_n for $n = 1,2,\ldots$ If $J_1, J_2, \ldots$ is another such sequence then either $\cup I_n$ and $\cup J_n$ are disjoint or the symmetric difference of the sets $\{I_1, I_2, \ldots\}$ and $\{J_1, J_2, \ldots\}$ is finite. (Indeed I_k intersects J_m implies that the one lies in the other, say $I_k \subset J_m$. Then $I_l = J_m$ for some $l \geq k$ and hence $I_{l+n} = J_{m+n}$ for $n = 1,2,\ldots$) Also $|I_n|/|I_{n+1}| \leq \rho_{n+1} = \rho(I_{n+1}) \to 0$. Hence $\cup I_n$ is unbounded. It follows hat Λ does not contain three mutually disjoint successor sequences.

PROPOSITION 4.2 Let Λ^* be a collection of bounded open intervals J^* such that

$$(4.7) \quad \text{if } J_1^* \cap J_2^* \text{ is non-empty then } J_1^* \subset J_2^* \text{ or } J_2^* \subset J_2^*$$

$$(4.12) \qquad \rho(J^*) \to 0, \text{ where}$$

(4.13) $\rho(J^*) = \max\ \{|I^*|/|J^*|\ |\ I^* \subset J^* \text{ and } I^* \neq J^*\}$.

Then each J^* contains an open interval J such that Λ, the set of intervals J, satisfies (4.7), (4.12),

(4.14) $|J^* \setminus J|/|J^*| \leq 2\rho(J^*)$

and has the following property

(4.15) either each element $J \in \Lambda$ is contained in a maximal element, or there exists a successor sequence $J_n = (a_n, b_n)$ such that

(4.15a) $\cup J_n = \mathbb{R}$

(4.15b) $a_{n+1} < a_n - |J_n|$ and $b_{n+1} > b_n + |J_n|$.

PROOF If Λ^* contains no successor sequence we are done (choose $\Lambda := \Lambda^*$ and $I := I^*$). Else Λ^* contains at most two disjoint successor sequences, say $I_n^* = (c_n^*, d_n^*)$ and J_n^*. We define $I_n = (c_n, d_n)$ as follows for $n = 1,2,\ldots$

$$\begin{aligned} I_1 &= I_1^* \\ I_{n+1} &= I_{n+1}^* \quad \text{if } c_{n+1}^* < c_n - |I_n^*| \text{ and } d_{n+1}^* > d_n + |I_n^*| \\ &= (d_n, d_{n+1}^*) \quad \text{if } c_{n+1}^* \geq c_n - |I_n^*| \\ &= (c_{n+1}^*, c_n) \quad \text{else} \end{aligned}$$

and J_n similarly. Then (4.7) remains valid and

$$\frac{|I_{n+1}^* \setminus I_{n+1}|}{|I_{n+1}^*|} \leq \frac{2|I_n^*|}{|I_{n+1}^*|} \leq 2\rho(I_{n+1}^*).$$

Hence (4.14) holds. (We set $J = J^*$ for all other elements of Λ^*.)

Now suppose $I_n \neq I_n^*$ for $n \geq n_0$. (This is the case if $\cup I_n^*$ is a half line, say (a, ∞), for then the left endpoints of I_n^* converge to a and $|I_n^*| \to \infty$.) Then the intervals $I_{n_0}, I_{n_0+1}, \ldots$ are disjoint and I_n is maximal for $n \geq n_0$. (If $I_n \subset J$, then $I_n^* \subset J^* = J$ since $J^* \subset I_n^*$ implies $|I_n| \leq |J^*| < \frac{1}{4}|I_n^*|$ for large n. This contradicts $\rho(I_n) \to 0$.) Let $K_1, K_2, \ldots$ be a successor sequence in Λ. Then K_n is not maximal and hence $K_n \notin \{I_1, I_2, \ldots, J_1, J_2, \ldots\}$ for $n \geq n_2$. Then $K_n^* = K_n$ for $n \geq n_2$ and K_n^*, $n \geq n_2$, is a new disjoint successor sequence in Λ^*. This contradiction shows that Λ does not contain a successor sequence.

If $I_n = I_n^*$ infintely often, then $UI_n = UI_n^* = \mathbb{R}$ and the subsequence $I_{k_n} = (a_n, b_n)$ for which $I_{k_n} = I_{k_n}^*$ is a successor sequence which satisfies (4.15b).

THEOREM 4.1 Suppose $\alpha_n \underline{x}_n \to \underline{u}$ in distribution where α_n is a sequence in G which diverges to ∞. If $\Delta = \{\varepsilon\}$ then for each random variable $\underline{v}$ there exists an increasing homeomorphism f on $\mathbb{R}$ and a sequence of positive affine transformations β_n such that $\beta_n f(\underline{x}_n) \to \underline{v}$ in distribution.

PROOF The theorem follows from proposition 4.3 below if we choose g such that $\underline{v} \overset{M}{=} g(\underline{u})$.

PROPOSITION 4.3 Suppose $g \in M$ lives on I. (That is, I is the largest open interval on which g is finite.) Let α_n be a sequence in G which diverges to ∞ such that $I \cap \sigma I$ is empty for each $\sigma \in \Delta$, $\sigma \neq \varepsilon$. Then there exists an increasing homeomorphism f on $\mathbb{R}$ and a sequence β_n in G such that

$$(4.16) \qquad \beta_n f \alpha_n^{-1} \to g \quad \text{in M.}$$

The set of such homeomorphisms f is dense in M.

PROOF The proof consists of seven parts. The actual construction of the homeomorphism occurs in part F. We shall first construct a sequence of subintervals I_n of I which converge to I (every point $x \in I$ lies in I_n for $n \geq n(x)$) such that the associated sequence of intervals $J_n = \alpha_n^{-1} I_n$ has property (4.4) (they are "asymptotically disjoint"). This is done in B for bounded I and in C for unbounded I.

A. Let us call a sequence γ_n uniformly discrete if there exists a neighbourhood U of ε such that $\gamma_n \gamma_m^{-1} \in U$ implies $m = n$. We show here that we may assume α_n to be uniformly discrete.

Let U be a symmetric compact neighbourhood of ε (i.e. $\gamma \in U$ implies $\gamma^{-1} \in U$) such that γI intersects I for all $\gamma \in U$. Suppose there exist subsequences k_n and l_n such that $\alpha_{k_n} \alpha_{l_n}^{-1} \in U$ for $n = 1,2,\ldots$ Then $\alpha_{k_n} \alpha_{l_n}^{-1} \to \varepsilon$. (Indeed since U is compact it suffices to prove that ε is the only limit point. Let σ be a limit point. Then $\sigma \in \Delta$ and $\sigma \in U$, hence $\sigma I \cap I$ non-empty, implies $\sigma = \varepsilon$.)

Define the sequence γ_n by

$$\begin{aligned} \gamma_1 &= \alpha_1 \\ \gamma_n &= \gamma_k \quad \text{if} \quad \alpha_n\gamma_k^{-1} \in U \quad \text{with } k < n \text{ minimal} \\ &= \alpha_n \quad \text{if} \quad \alpha_n\gamma_k^{-1} \notin U \quad \text{for } k = 1,\dots,n-1. \end{aligned}$$

The argument above proves that $\alpha_n\gamma_n^{-1} \to \varepsilon$. Hence (4.16) is equivalent to

$$\beta_n f\gamma_n^{-1} \to g \quad \text{in } M$$

and this remains true if we replace (γ_n) by the subsequence of all distinct terms. By construction this subsequence is uniformly discrete (with respect to the compact neighbourhood U).

B. If I is bounded, then setting $J_n = \alpha_n^{-1}I$

$$\lim_{n\neq m} \frac{|J_n \cap J_m|}{|J_n| + |J_m|} = 0 \tag{4.17}$$

as we prove below.

Suppose $\delta \in (0, \frac{1}{2})$ and

$$|J_{k_n} \cap J_{l_n}| \geq \delta(|J_{k_n}| + |J_{l_n}|).$$

With $\sigma_n = \alpha_{k_n}\alpha_{l_n}^{-1}$ we obtain

$$|I \cap \sigma_n I| \geq \delta(|I| + |\sigma_n I|). \tag{4.18}$$

The set V of $\sigma \in G$ which satisfy (4.18) is a compact neighbourhood of ε. Hence σ_n has a limit point σ which satisfies (4.18) and lies in Δ. This means that $\sigma = \varepsilon$. Since we assume α_n to be uniformly discrete, we must have $k_n = l_n$ for $n \geq n_0$. This proves (4.17).

In the construction of f we shall need the relation

$$\frac{|J_n \cap J_m|}{|J_n| + |J_m|} \, |I_n| \to 0 \quad \text{for } n \neq m.$$

where $I_n \to I$. This follows from (4.17) for bounded intervals I. For unbounded intervals we have to refine our construction of the sequence J_n. This we shall do in part C.

C. If I is unbounded there exists a sequence of bounded open subintervals $I_1 \subset I_2 \subset \ldots$ such that $I = \cup I_n$ and such that

$$\lim_{n \neq m} \eta(n, m) = 0$$

where

$$\eta(n, m) := |J_n \cap J_m| \, \frac{|I_n| + |I_m|}{|J_n| + |J_m|}, \quad J_n = \alpha_n^{-1} I_n.$$

Indeed let I(1),I(2),... be an increasing sequence of open intervals such that $I = \cup I(n)$ and $|I(n)| = n$. Define

$$\eta_k(n, m) := \frac{|\alpha_n^{-1} I(k) \cap \alpha_m^{-1} I(k)|}{|\alpha_n^{-1} I(k)| + |\alpha_m^{-1} I(k)|} \cdot 2|I(k)|.$$

Then B implies that for fixed k

$$\lim_{n \neq m} \eta_k(n, m) = 0.$$

Choose n_k such that

$$\eta_k(n, m) \leq \frac{1}{k^2} \quad \text{for } \max(n, m) \geq n_k, \; n \neq m.$$

We assume that the sequence $n_1, n_2, \ldots$ is strictly increasing and define

$$I_n = I(k) \quad \text{for } n_k \leq n < n_{k+1}.$$

Suppose $m < n$ with $n_k \leq n < n_{k+1}$. Then

$$|I_m| \geq 1 = \frac{1}{k} |I(k)|$$

and hence

$$|\alpha_n^{-1}I_n| + |\alpha_m^{-1}I_m| \geq \frac{1}{k}\,(|\alpha_n^{-1}I(k)| + |\alpha_m^{-1}I(k)|)$$

which implies that

$$\eta(n, m) \leq k.\eta_k(n, m) \leq \frac{1}{k}.$$

D. There exists a collection Λ of open intervals $J_n = \alpha_n^{-1}I_n$ such that

(4.19) $I_n \subset I$ and $I_n \to I$

(4.20) $J_n \cap J_m$ non-empty implies $J_n \subset J_m$ or $J_m \subset J_n$

(4.21) $\rho_n . |I_n| \to 0$

where ρ_n is defined by

(4.22) $\rho_n = \max \{|J_m|/|J_n| \mid J_m \subset J_n \text{ and } J_m \neq J_n\}$, and

(4.23) either each $J_n \in \Lambda$ is contained in a maximal element of Λ or we have a successor sequence $J^{(n)} = (a_n, b_n)$ in Λ such that

(4.23a) $\cup J^{(n)} = \mathbb{R}$

(4.23b) $a_{n+1} < a_n - |J^{(n)}|$, $b_{n+1} > b_n + |J^{(n)}|$ $\quad n = 1,2,\ldots$

Note that (4.21) is a stronger version of (4.12).

In parts B and C we have constructed a collection Λ^0 of intervals $J_n^0 = \alpha_n^{-1}I_n^0$ such that (4.19) holds and

(4.24) $$\lim_{n \neq m} |J_n^0 \cap J_m^0| \frac{|I_n^0| + |I_m^0|}{|J_n^0| + |J_m^0|} = 0.$$

As in the proof of proposition 4.1 (part B) we may assume that the intervals J_n^0 have different lengths. Define $\rho_n^0 = \rho(J_n^0)$ as in (4.8). Then (4.24) implies

$$\rho_n^0 . |I_n^0| \to 0$$

and hence certainly

$$\rho_n^0 \to 0.$$

For the subintervals $J_n^* \subset J_n^0$ constructed in proposition 4.1 we have by (4.9) that

$$(4.25) \qquad \frac{|J_n^0 \setminus J_n^*|}{|J_n^0|} \cdot |I_n^0| \leq 4\rho_n^0 . |I_n^0| \to 0$$

and hence $|I_n^0 \setminus I_n^*| \to 0$. The collection J_n^* satisfies (4.19) up to (4.22).

Now apply proposition 4.2 to obtain the desired collection Λ of intervals J_n. (Convergence in (4.19) follows from (4.14) and the analogous form of inequality (4.25).)

E. There exists a sequence of strictly increasing continuous functions ψ_n defined on $\overline{I}_n$ such that

$$\psi_n \to g \quad \text{weakly on } I$$

(4.26) $$\sup_{x \in I_n} \psi_n(x) \to \infty \quad , \quad \inf_{x \in I_n} \psi_n(x) \to -\infty.$$

(4.27) the increase of $\psi_n \alpha_n$ over any subinterval $J_m \subset J_n$, $m \neq n$, is less then one half of the total increase of $\psi_n \alpha_n$ over J_n.

F. The construction of f

Recall that we constructed f in the introduction to the chapter by setting

$$f = \beta_n^{-1} \psi_n \alpha_n \quad \text{on } J_n = \alpha_n^{-1} I_n$$

where the β_n were chosen so as to ensure that f should be a homeomorphism. Since the intervals J_n are no longer disjoint in the general situation, we have to be more careful. We shall define f as the limit of a sequence f_n.

There are two distinct cases to consider according to whether there exists a successor sequence in Λ or not.

a. First assume that each interval in Λ is contained in a maximal interval in Λ. We enumerate the intervals in Λ such that

$$J_n \subset J_m \text{ implies } n \geq m,$$

i.e. either J_n is maximal or it has a successor J_k with $k < n$.

Let f_0 be an arbitrary increasing homeomorphism of R. If the homeomorphisms $f_1,\dots,f_{n-1}$ on $\mathbb{R}$ have been constructed, we define

$$\begin{aligned} f_n(x) &:= f_{n-1}(x) && x \notin J_n \\ &:= \beta_n^{-1}\psi_n(\alpha_n x) && x \in \overline{J}_n \end{aligned}$$

where β_n is the unique element of G such that f_n is well defined in the endpoints of J_n. This is possible since f_{n-1} is strictly increasing. The function f_n is a homeomorphism on $\mathbb{R}$.

b. Now assume not every interval $J \in \Lambda$ is contained in a maximal interval. Then there exists a successor sequence $J_n = (a_n, b_n) \in \Lambda$ which satisfies (4.23a/b).

Define

$$h_1(x) := \psi_1(\alpha_1 x) \quad \text{on } \overline{J}_1.$$

If $h_1,\dots,h_{m-1}$ have been defined, we define h_m on $\overline{J}_m$ by

$$\begin{aligned} h_m(x) &:= h_{m-1}(x) && x \in \overline{J}_{m-1} \\ &:= \beta_m^{-1}\psi_m(\alpha_m x) && x \in \overline{J}_m \setminus J_{m-1} \end{aligned}$$

where β_m is the unique element in G such that h_m is well defined in the endpoints of J_{m-1}.

Clearly $h_n(x) = h_m(x)$ on J_m for $n > m$ and it follows that h_n converges to a strictly increasing continuous function h on $\mathbb{R}$.

The function h need not be a homeomorphism since it may be bounded. However, we have some freedom in defining the sequence ψ_n, which we shall now use to ensure that h is a homeomorphism.

We alter ψ_m into a continuous function ψ_m^* such that

$$\begin{aligned} &\psi^*_{m+1}\alpha_{m+1} \quad \text{is affine on the interval } J'_m = (a_m - |J_m|, b_m + |J_m|) \\ &\psi^*_{m+1}\alpha_{m+1} \quad \text{coincides with } \psi_{m+1}\alpha_{m+1} \text{ outside } J'_m. \end{aligned}$$

Note that $J'_m \subset J_{m+1}$ by (4.23b) and $|J'_m| = 3|J_m|$ and hence (4.26) holds and $|J'_m|.|I_{m+1}|.|J_{m+1}|^{-1} \to 0$ by (4.21). This implies

$$\psi_m^* \to g \text{ weakly on I.}$$

Finally since h_{m+1}^* is affine on J_m' we find that $h_{m+1}^*(J_{m+1}) \supset h_{m+1}^*(J_m') =$ $= (c - d, c + 2d)$ if $h_m^*(J_m) = (c, c + d)$. Hence $h^* = \lim h_m^*$ is a homeomorphism.

In $\Lambda \setminus \{J_1,J_2,\ldots\}$ every interval is contained in a maximal interval and hence we can use the construction of part a) starting with $f_0 = h$ (which may break off after a finite number of steps, if $\Lambda \setminus \{J_1,J_2,\ldots\}$ is finite).

We shall now prove that the countable collection of functions $h_1,h_2,\ldots,f_0,f_1,\ldots$ converges. It suffices to prove that f_n converges.

Define the set

$$E = \bigcap_n \bigcup \{J \in \Lambda \mid |J| < \tfrac{1}{n}\}.$$

The complement of E is dense in $\mathbb{R}$. It contains the endpoints of all intervals $J \in \Lambda$.

If $x \notin E$ then the sequence $f_n(x)$ is constant for $n \geq n(x)$. This proves that f_n converges on a dense set, that the limit f is strictly increasing and that $\sup f(x) = -\inf f(x) = \infty$ since $f_n(x) = f_0(x)$ in the endpoints of maximal intervals of $\Lambda \setminus$ {successor sequence}.

Condition (4.27) ensures that f is continuous. (If $J_1 \supset J_2 \supset J_3 \supset \ldots$ then the increase of f over J_{n+1} is less than one half of the increase of f over J_n.)

Let f^* be a given homeomorphism. We may ensure that f is close to f^* on a given bounded interval by altering a finite number of the functions ψ_n. This shows that the set of such homeomorphisms f as constructed above is dense in M.

G. Convergence of $\beta_n f\alpha_n^{-1}$

Let $x \in I$ be a continuity point of g. Consider $\xi_n := \beta_n f\alpha_n^{-1}$ on $\underline{I_n}$. By construction of f we have $\psi_n = \beta_n f_n \alpha_n^{-1}$ on I_n. Also $f = f_n$ for all $y \in \overline{J}_n$ which do not lie in an interval $J \in \Lambda$ having J_n as successor. Hence $\xi_n(x) =$ $= \psi_n(x)$ unless $x \in \alpha_n J = (x_n', x_n'')$ say, where $J \in \Lambda$ has successor J_n. In that case $x_n'' - x_n' = |J|.|I_n|.|J_n|^{-1} \to 0$. Hence $x_m' \to x$ and $\xi_n(x_n') = \psi_n(x_n') \to$ $\to g(x)$ by definition of the sequence ψ_n. (Set $x_n' = x$ if $x \notin \alpha_n J$ for some J with successor J_n.) Thus $\psi_n \to g$ weakly on I implies $\xi_n \to g$ weakly on I.

Since $\xi_n = \psi_n$ in the endpoints of I_n condition (4.26) implies that $|\beta_n f\alpha_n^{-1}| \to \infty$ outside I and hence $\beta_n f\alpha_n^{-1} \to g$ in M.

Here ends the proof of proposition 4.3.

One might conclude from theorem 4.1 and the theory of the previous chapter that the set Δ contains complete information on the class of $g \in M$ which can occur in the relation $\underline{v} \overset{M}{=} g(\underline{u})$ for appropriately chosen $f \in M$ and sequence (β_n) in G.

This is not the case. If $\alpha_n x = x + n$, then $\Delta = \{\sigma \in G \mid \sigma x = x + k \text{ with } k \text{ integral}\}$. The possible limit functions, by table 3.1, are

$$k(x) = \lambda x + c + \pi(x)$$

with π periodic modulo 1, and

$$h_1(x) = b + e^{k(x)}$$
$$h_2(x) = b - e^{-k(x)}.$$

For the sequence α_n above they are realized as limit of the sequence $\beta_n f\alpha_n^{-1}$ with $f = k$ or $f = h_i$ and β_n chosen appropriately. The example below shows that there exist sequences (α_n) on the other hand, having the same set Δ, such that the functions h_1 and h_2 are not possible as the limit of a sequence $\beta_n f\alpha_n^{-1}$ for any $f \in M$ and any sequence (β_n) in G.

EXAMPLE of a sequence (α_n) such that

(4.28) $$\Delta = \{\sigma \in G \mid \sigma x = x + k, \ k \text{ integral}\},$$

and for which there exist no $f \in M$ and no sequence (β_n) in G such that $\beta_n f\alpha_n^{-1} \to \phi$ with $\phi(x) = e^x$.

Consider the set

$$\{\alpha_{nk} \mid k \text{ integral}, \ |k| \le (n!)^2, \ n = 1,2,\ldots\}$$

where $\alpha_{nk}x := n!x - k$. Observe that $\alpha_{nk} \to \infty$ and that $\alpha_{nk} \to \infty$ implies $n \to \infty$. Now consider the quotient

$$\alpha_{m1}\alpha_{nk}^{-1}x = \frac{m!}{n!}\,x + \frac{m!}{n!}\,k - 1.$$

If a sequence of such quotients converges to $\sigma \in G$, and at the same time the numerator and denominator diverge to ∞, then $\sigma x = x + j$ for some integer j. Hence (4.28).

Suppose $\beta_{nk}f\alpha_{nk}^{-1} \to \phi$ weakly on $\mathbb{R}$. Then

$$\frac{f(\alpha_{nk}^{-1}2) - f(\alpha_{nk}^{-1}1)}{f(\alpha_{nk}^{-1}1) - f(\alpha_{nk}^{-1}0)} \to \frac{\phi(2) - \phi(1)}{\phi(1) - \phi(0)} = e.$$

Hence for $n \geq n_0$ and all k we have

$$\frac{f(\frac{2 + k}{n!}) - f(\frac{1 + k}{n!})}{f(\frac{1 + k}{n!}) - f(\frac{k}{n!})} \geq 2.$$

Let $x_0 < x_1$ be continuity points of f. Fix $n \geq n_0$ and add the nominators and the denumerators for $k = k_0, k_0+1, \ldots, k_1$, where k_i is the integral part of $n!x_i$ for $i = 0,\ 1$. Then

$$\frac{f\left(\frac{2 + k_1}{n!}\right) - f\left(\frac{1 + k_0}{n!}\right)}{f\left(\frac{1 + k_1}{n!}\right) - f\left(\frac{1 + k_0}{n!}\right)} \geq 2$$

For $n \to \infty$ this fraction converges to

$$\frac{f(x_1) - f(x_0)}{f(x_1) - f(x_0)} = 1.$$

Hence $1 \geq 2$. Contradiction.

5 Domains of attraction I

Up to now we have been primarily concerned with determining the possible limit functions ϕ if it is given that the sequence $\beta_n f\alpha_n^{-1}$ converges weakly on an interval I. One can also ask the following question.
Given a function ϕ and an interval I, determine the class of non-decreasing functions f such that $\beta_n f\alpha_n^{-1} \to \phi$ weakly on I.

We do not propose to give a complete answer to this question. We shall only make some general remarks on the subject and give a number of examples.

Let us start with some examples. Let $\phi = \varepsilon$ be the identity function on $\mathbb{R}$. Suppose f is a strictly increasing function on $\mathbb{R}$ which is affine on the intervals $(n^2 - 1, n^2 + 1)$, for $n = 1,2,\ldots$ If we choose $\alpha_n^{-1}x = x + n^2$, then $f\alpha_n^{-1}$ is affine on $(-1, 1)$ for $n = 1,2,\ldots$ and $\beta_n f\alpha_n^{-1} \to \varepsilon$ on $(-1, 1)$ for a suitably chosen sequence (β_n). If we choose $\alpha_n^{-1}x = n^{-1}x + n^2$, then $f\alpha_n^{-1}$ is affine on $(-n, n)$ for $n = 1,2,\ldots$ and $\beta_n f\alpha_n^{-1} \to \varepsilon$ on $\mathbb{R}$ for a suitably chosen sequence (β_n). Similarly $\beta_n f\alpha_n^{-1} \to \varepsilon$ for properly chosen sequences (α_n) and (β_n) if f is affine on any sequence of non-empty open intervals $(x_n - \delta_n, x_n + \delta_n)$, if f has a positive derivative in a sequence of points x_n, or even if f has a positive derivative in only one point x_0. On the other hand also the step function $f(x) = [x]$, the integral part of x, tends to ε with suitably chosen norming sequences (α_n) and (β_n), say $\alpha_n x = \beta_n x = n^{-1}x$.

In order to obtain interesting results, we reformulate the problem. For a given non-decreasing function h on I and a given sequence (α_n) in G determine all non-decreasing functions f and all sequences (β_n) in G such that

$$(5.1) \qquad \beta_n f\alpha_n^{-1} \to h \quad \text{weakly on I.}$$

If f, h, I and (α_n) are known in relation (5.1), then finding the sequence (β_n) for this given f is no problem. Indeed, suppose (f_n) is a sequence of non-decreasing functions and $\beta_n f_n$ converges weakly to a non-constant limit h on I. Let $x_0, x_1 \in$ I be fixed continuity points of h such that $h(x_0) < h(x_1)$. Then

$$(5.2) \qquad \frac{f_n(x) - f_n(x_0)}{f_n(x_1) - f_n(x_0)} = \frac{\beta_n f_n(x) - \beta_n f_n(x_0)}{\beta_n f_n(x_1) - \beta_n f_n(x_0)} \to \frac{h(x) - h(x_0)}{h(x_1) - h(x_0)}$$

weakly on I. Hence instead of (β_n) we may use the norming sequence $(\gamma^{-1}\beta_n^*)$

where

$$\gamma y := \frac{y - h(x_0)}{h(x_1) - h(x_0)}, \quad \beta_n^* y := \frac{y - f_n(x_0)}{f_n(x_1) - f_n(x_0)} \quad \text{for } n = 1,2,\ldots$$

By Khinchine's theorem on the convergence of types, see theorem 14.1, it follows that $\gamma^{-1}\beta_n^*$ is asymptotic to β_n.

DEFINITION 5.1 Suppose $g \in M$ and (α_n) is a sequence in G. Then $f \in M$ lies in the domain of attraction of g for the sequence (α_n) and we write $f \in D =$ $= D(g, \alpha)$ if there exists a sequence (β_n) in G such that

$$\beta_n f \alpha_n^{-1} \to g.$$

With this notation we may formulate the main result of the previous chapter, proposition 4.4, as follows. If $g \in M$ and $\alpha_n \to \infty$ such that $\Delta = \{\varepsilon\}$, then $D(g, \alpha)$ is dense in the set of all increasing homeomorphisms of $\mathbb{R}$ on $\mathbb{R}$.

We shall now give sufficient conditions for f to lie in the domain of attraction of ε, the identity on $\mathbb{R}$, for various classes of sequences (α_n). In the examples below we shall use the following notation,

f is a non-decreasing function defined on $\mathbb{R}$

(α_n) is a sequence in G and $\alpha_n \to \infty$ (hence $\alpha_n^{-1} \to \infty$)

$\alpha_n^{-1}x = a_n x + b_n$, hence $\alpha_n x = a_n^{-1}(x - b_n)$

$D = D(\varepsilon, \alpha)$.

1. If $a_n \to \infty$ and $f(x) - x$ is bounded, then $f \in D$.

PROOF

$$\frac{f(a_n x + b_n) - b_n}{a_n} = \frac{a_n x + c_n(x)}{a_n} \to x$$

since $c_n(x)$ is bounded (in x and n).

2. If $a_n \geq q > 0$ for all n and $f(x) = x + o(1)$ for $|x| \to \infty$, then $f \in D$.

PROOF Set $d(x) := f(x) - x$. The function d is bounded, say $|d(x)| \le M$ for all x, and for each $\varepsilon > 0$ there exists $L > 0$ such that $|d(x)| < \varepsilon$ for $|x| \ge L$. For x fixed we have

$$\frac{f(a_n x + b_n) - b_n}{a_n} - x = \frac{d(a_n x + b_n)}{a_n}.$$

The right hand side tends to zero since $a_n + |b_n| \to \infty$. (For sufficiently large n either $a_n \ge \varepsilon^{-1}M$, or else $|b_n| \ge L + \varepsilon^{-1}M|x|$ and then $|a_n x + b_n| \ge$ $\ge L$.)

3. If $a_n \to 0$ and f is differentiable, f' is positive and log f' is uniformly continuous, then $f \in D$.

PROOF

$$\frac{f(a_n x + b_n) - f(b_n)}{a_n f'(b_n)} = \frac{a_n x f'(\xi_n)}{a_n f'(b_n)}$$

where ξ_n lies between b_n and $b_n + a_n x$. Since log f' is uniformly continuous, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\xi_n - b_n| < \delta \quad \text{implies} \quad \left|\frac{f'(\xi_n)}{f'(b_n)} - 1\right| < \varepsilon.$$

For fixed x the condition $|\xi_n - b_n| < \delta$ is satisfied for $n \ge n_0$ since $a_n \to 0$.

4. If log a_n is bounded, $b_n \to \infty$, and if f satisfies

$$\frac{f(x + t) - f(t)}{f(1 + t) - f(t)} \to x \quad \text{for } t \to \infty, \tag{5.3}$$

then $f \in D$.

REMARK The condition (5.3) has been extensively studied in de Haan [1970, p. 31 and def. 1.4.1] in the multiplicative version. See also chapter 12 below.

PROOF The relation (5.3) is uniform on bounded x-intervals, the limit function being continuous. Hence it implies

$$\frac{f(a_n x + b_n) - f(b_n)}{f(a_n + b_n) - f(b_n)} \to x \quad \text{as } n \to \infty.$$

5. If f is differentiable and the derivative f' is positive and continuous and converges to a positive constant ρ as $|x| \to \infty$, then $f \in D(\varepsilon, \alpha)$ for every sequence (α_n) in G which diverges to ∞.

PROOF It suffices to prove that each subsequence of (α_n) contains a subsubsequence, say (α_k), such that $\beta_k f \alpha_k^{-1} \to \varepsilon$. (See remark after proposition 2.2.) If there is a subsubsequence with $a_k \to 0$, then convergence follows from example 3. above. Else there is a subsubsequence with $a_k \geq q > 0$ and then we refine the argument used in the proof in example 2. above, as follows. Since $f'(x)$ tends to $\rho > 0$ for $|x| \to \infty$, the set $\{|f' - \rho| > \varepsilon\}$ is bounded for each $\varepsilon > 0$, and hence for $s < t$ we have

$$\int_s^t f'(x)dx \sim \rho(t - s) \quad \text{for } \max(|t|, |s|) \to \infty.$$

This implies that

$$\frac{f(a_k x + b_k) - f(b_k)}{\rho a_k} \to x.$$

6. The conditions in 5. above are sufficient but not necessary. Consider $f(x) = -x \log|x|$ on $(-\frac{1}{3}, \frac{1}{3})$ and extend this function to the whole real line so as to satisfy the conditions of 5. for $|x| > \frac{1}{4}$. Then $f \in D(\varepsilon, \alpha)$ for every sequence (α_n) in G for which $\alpha_n \to \infty$.

PROOF In view of 5. we need only consider the case that $a_n \to 0$, $b_n \to 0$. By going over on subsubsequences we may assume that either $b_n \sim c a_n$ (in which case we may even assume that $b_n = c a_n$ since α_n is asymptotic to $\tilde{\alpha}_n$ where $\tilde{\alpha}_n x = a_n^{-1} x - c$) or that $|b_n a_n^{-1}| \to \infty$.

Consider the quotient

$$Q_n(x) := \frac{f(a_n x + b_n) - f(b_n)}{f(a_n + b_n) - f(b_n)}.$$

If $c = 0$, then $b_n = 0$ and $Q_n(x) \to x$. Else consider

$$\frac{f(a_n x + b_n)}{f(b_n)} - 1 = \frac{a_n x + b_n}{b_n} \cdot \frac{\log|a_n x + b_n|}{\log|b_n|} - 1 =$$

$$= (1 + \frac{x}{c})(1 + o(1)) - 1 = \frac{x}{c} + o(1) \quad \text{if } b_n = ca_n,\ c \neq 0$$

$$= (1 + \frac{a_n x}{b_n})(1 + o(\frac{a_n x}{b_n})) - 1 = \frac{a_n x}{b_n}(1 + o(1)) \quad \text{if } |b_n a_n^{-1}| \to \infty.$$

Hence $Q_n(x)$, which is the quotient of two such terms converges to x.

7. Let f be a non-decreasing function defined on an open neighbourhood of [0, 1]. Suppose that f has a strictly positive continuous derivative on $I = (0, 1)$ and that $d_0 = \lim_{x\to 0+} f'(x)$ and $d_1 = \lim_{x \to 1-0} f'(x)$ exist and are positive. Define

$$A = \{\alpha \in G \mid \alpha I \supset I\}.$$

For $\alpha_n \in A$, $\alpha_n \to \infty$ with $\alpha_n x = a_n^{-1}(x - b_n)$ we define $\beta_n y = (a_n f'(b_n))^{-1}(y - f(b_n))$. Then for $x \in (0, 1)$ we have

$$\beta_n f(\alpha_n^{-1} x) = \frac{f(a_n x + b_n) - f(b_n)}{a_n f'(b_n)} = x\,\frac{f'(a_n \xi_n + b_n)}{f'(b_n)} \to x$$

where $\xi_n \in I$ and $f'(b_n)$ is interpreted as the left or right hand derivative if $b_n = 1$ or $b_n = 0$.

If f is continuous in 0 and also the left hand derivative say d_0^* of f exists in 0, then for the sequence α_n with $\alpha_n x = nx$, we find

$$\beta_n f(\alpha_n^{-1} x) = \frac{f(n^{-1}x) - f(0)}{n^{-1} d_0} \to \begin{cases} x & \text{for } x > 0 \\ d_0^* d_0^{-1} x & \text{for } x < 0. \end{cases}$$

Hence we have convergence for all $x \in \mathbb{R}$ for every sequence $\alpha_n \in A$ with $\alpha_n \to \infty$ if f is also differentiable in the endpoints of I.

If f is not differentiable in 0 nor in 1, then we can choose (α_n) with $\alpha_n \in A$ such that $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$, $\liminf b_n = 0$ and $\limsup b_n = 1$. In this case $\beta_n f \alpha_n^{-1} x \to x$ only for $x \in [0, 1]$.

6 Continuation theorems

In this chapter we introduce the new condition that $(\alpha_{n+1}\alpha_n^{-1})$ is bounded. In a sense this is a much more stringent condition on the sequence (α_n) then any condition on Δ can be. (Even if $\Delta = G$, the sequence (α_n) may have very large gaps. Take for instance $\alpha_{2n} = \gamma_n$ and $\alpha_{2n+1} = \sigma_n\gamma_n$ with (σ_n) dense in G. Then $\Delta = G$ whatever the sequence (γ_n).)

Under certain circumstances this new condition on the sequence (α_n) has the consequence that

$$\beta_n f\alpha_n^{-1} \to \phi \quad \text{on I implies} \quad \beta_n f\alpha_n^{-1} \to \phi \quad \text{on } I^*$$

where I^* is an unbounded interval.

The most simple case is where α_n is a translation for each n, say $\alpha_n x = x + t_n$ with $(t_{n+1} - t_n)$ bounded. Then Δ is a set of translations. Set

$$s_0 := \inf \{s > 0 \mid \sigma \in \Delta \text{ and } \sigma x = x + s\}$$

and let I be an open interval of length $|I| > s_0$. We are then able to prove the following. If

$$\beta_n f\alpha_n^{-1} \to \phi \quad \text{weakly on I}$$

with $\phi \in \Phi$, ϕ non-constant on I, then

$$\beta_n f\alpha_n^{-1} \to \phi \quad \text{weakly on } \mathbb{R}.$$

A condition like "$(\alpha_{n+1}\alpha_n^{-1})$ is bounded" obviously is necessary in order to prove such a continuation theorem. That this condition is not sufficient is shown in the last lines of example 7. of the previous chapter (where $\alpha_{n+1}\alpha_n^{-1}$ even converges to ε).

Therefore we assume in this chapter that the sequence (α_n) in G satisfies the following three conditions.

(6.1) $\quad \alpha_n \to \infty$

(6.2) $\quad (\alpha_{n+1}\alpha_n^{-1})$ is bounded

(6.3) Δ is contained in a one-parameter subgroup

$$G(\gamma) = \{\gamma^t \mid t \in \mathbb{R}\}$$

with $\gamma \in G$, and obviously $\gamma \neq \varepsilon$.

This chapter may serve as an introduction to the second part of the book, chapters 7 - 13. There we shall replace (6.2) by the stronger condition

$$(6.4) \qquad \alpha_{n+1}\alpha_n^{-1} \to \varepsilon,$$

and obtain similar results, even though we drop the condition that $\beta_n f \alpha_n^{-1}$ converges weakly on an open interval.

DEFINITION 6.1 Let I be an open interval and $\gamma \in G$. The γ-invariant extension of I is the smallest open interval J with the properties

$I \subset J$

$\gamma J = J.$

REMARK This terminology is only used in this chapter. Clearly J exists and

$J = I$ if $\gamma = \varepsilon$

$J = \mathbb{R}$ if γ is a non-trivial translation

$J = (-\infty, c)$ or $J = (c, \infty)$ or $J = \mathbb{R}$ if γ is a multiplication with centre c.

PROPOSITION 6.1 Suppose that

$$\beta_n f \alpha_n^{-1} \to h \text{ weakly on } I$$

where I is an open interval, h is defined and non-decreasing on I and as usual $f \in M$ and $\alpha_n, \beta_n \in G$. Assume moreover that in addition to (6.1), (6.2) and (6.3) the following two conditions are satisfied,

1. h is non-constant on $I \cap \sigma I$ and on $I \cap \sigma^{-1} I$ for each limit point σ of the sequence $(\alpha_{n+1}\alpha_n^{-1})$,
2. the function h extends to a function h_1 on I_1, the γ-invariant extension of I, which for some $\tau \in G$ satisfies the functional equation

$$h_1\gamma^t = \tau^t h_1$$

for all $\gamma^t \in \Delta$. Then

$$\beta_n f\alpha_n^{-1} \to h_1 \quad \text{weakly on } I_1.$$

PROOF We may assume I to be the maximal open interval on which $\beta_n f\alpha_n^{-1}$ converges to h_1.

By taking a subsequence and re-indexing if need be, we may ensure that all limit points of the sequence $\sigma_n := \alpha_{n+1}\alpha_n^{-1}$ have the form γ^t with $0 < c \le t \le c_1$ (or that they all have the form γ^{-t} with $0 < c \le t \le c_1$), where c_1 is so small that h is non-constant on $I \cap \gamma^{c_1}I$ and on $I \cap \gamma^{-c_1}I$.

(For an exact proof of this assertion we need proposition 9.7 which states that the conditions (6.1), (6.2) and (6.3) are sufficient to construct a continuous function $\alpha : [0, \infty) \to G$ and a sequence $t_n \to \infty$ such that

$$\alpha_n = \alpha(t_n) \quad \text{for all } n$$

$$\alpha(t + s)\alpha(t)^{-1} \to \psi(s) \quad \text{for } t \to \infty \text{ for all } s \in \mathbb{R},$$

where either $\psi(s) = \gamma^s$ for all s or $\psi(s) = \gamma^{-s}$ for all s. We may as well assume that the former is the case. Now set $a := \limsup (t_{n+1} - t_n)$. Then γ^a is a limit point of the sequence $(\alpha_{n+1}\alpha_n^{-1})$. Hence h is non-constant on $I \cap \gamma^a I$ and on $I \cap \gamma^{-a}I$. This implies that h is non-constant on $I \cap \gamma^b I$ and on $I \cap \gamma^{-b}I$ for some $b > a$. Now construct the subsequence $t'_n = t_{k_n}$ as follows. Set $k_1 = 1$. For given $k_1, \ldots, k_n$ choose $k_{n+1} > k_n$, minimal, and so that

$$t'_{n+1} \ge t'_n + b - a.$$

Obviously $\liminf (t'_{n+1} - t'_n) \ge b - a$ and $\limsup (t'_{n+1} - t'_n) \le b$. Set $c_1 = b$ and $c = b - a$ to obtain the desired result.)

Set $\tau_n := \beta_{n+1}\beta_n^{-1}$, $g_n := \beta_n f\alpha_n^{-1}$ and $\sigma_n = \alpha_{n+1}\alpha_n^{-1}$ as above. The sequence (σ_n) need not converge. However, each subsequence contains a subsubsequence (k_n) such that $\sigma_{k_n} \to \gamma^t$ for some $t \in [c, c_1]$. For this particular subsubsequence we have

$$\sigma_{k_n} \to \gamma^t$$

$$g_{k_n} \to h_1 \quad \text{on } I$$

(6.5) $$g_{k_n+1} \to h_1 \quad \text{on } I$$

(6.6) $$g_{k_n+1} = \tau_{k_n} g_{k_n} \sigma_{k_n}^{-1}$$

and since h_1 is non-constant on $I \cap \gamma^{-t}I$ and on $I \cap \gamma^t I$ we may use the argument of the opening section of chapter 3 to conclude that $\tau_{k_n} \to \tau^t$ for some $\tau \in G$. The relations (6.5) and (6.6) imply

$$g_{k_n} = \tau_{k_n}^{-1} g_{k_n+1} \sigma_{k_n} \to \tau^{-t} h_1 \gamma^t = h_1 \quad \text{on } \gamma^{-t}I.$$

Hence $g_{k_n} \to h_1$ on $I \cup \gamma^{-t}I$ (which is an interval since $I \cap \gamma^{-t}I$ is non-empty). In particular $g_{k_n} \to h_1$ on $I \cup \gamma^{-c}I$. This holds for all suitable sub-subsequences and hence for the whole sequence (g_n). Since I is maximal we have $I \supset \gamma^{-c}I$. Similarly $I \supset \gamma^c I$. This proves the proposition.

COROLLARY We use the notation of proposition 6.1. Suppose

$$\beta_n f \alpha_n^{-1} \to h \quad \text{weakly on } I.$$

If (6.1), (6,3) and (6.4) (in stead of (6.2)) are satisfied, and h is non-constant on I, then there exists $\tau \in G$ and $\phi \in \Phi(\gamma, \tau)$ (see (1.4) for definition) such that

$$\beta_n f \alpha_n^{-1} \to \phi \quad \text{weakly on } I_1.$$

PROOF By corollary 1 to proposition 3.2 the function h is the restriction to I of a function $\phi \in \Phi(\gamma, \tau)$. Hence condition 1 and 2 of proposition 6.1 are fulfilled with $h_1 = \phi$.

EXAMPLE Let the function $\psi : \mathbb{R} \to \mathbb{R}$ satisfy

$$\psi(x) = 0 \quad |x| > 1$$

$$x + \psi(x) \quad \text{is non-decreasing.}$$

Set

$$f(x) = x + \sum_{n=1}^{\infty} \psi(x - x_n)$$

where (x_n) is a sequence of positive reals such that $5 \leq x_{n+1} - x_n \leq 7$. Set $\alpha_n x = \beta_n x = x - x_n$. Then for $x \in I = (-4, 4)$ we have

$$\beta_n f(\alpha_n^{-1} x) = x + \psi(x).$$

The sequence $\beta_n f \alpha_n^{-1}$ obviously converges on I. The sequence will only converge on a substantially larger interval, say (-7, 7) if $x_{n+1} - x_n$ converges. The example shows that condition 2. in proposition 6.1 cannot be omitted altogether.

DEFINITION 6.2 Let (α_n) and (β_n) be sequences in G. Then $\alpha_n \sim \beta_n$ and we say that α_n is asymptotic to β_n if $\beta_n \alpha_n^{-1} \to \varepsilon$.

PROPOSITION 6.2 Suppose

$$\beta_n f \alpha_n^{-1} \to h \quad \text{weakly on I.}$$

Here $f \in M$, $\beta_n \in G$, the sequence (α_n) in G satisfies (6.1), (6.2) and

$$\Delta \subset \{\gamma^k \mid k \text{ integral}\}$$

for some $\gamma \in G$, I is an open interval such that $\gamma I = I$ and h is non-constant on I, and satisfies the equation

$$(6.7) \qquad \tau h = h\gamma$$

for some $\tau \in G$. Then there exist sequences $(\tilde{\alpha}_n)$ and $(\tilde{\beta}_n)$ such that

$$\tilde{\alpha}_{n+1}\tilde{\alpha}_n^{-1} \to \sigma \qquad \tilde{\beta}_{n+1}\tilde{\beta}_n^{-1} \to \tau$$
$$\tilde{\beta}_n f \tilde{\alpha}_n^{-1} \to h \quad \text{weakly on I,}$$

and there exists a function n(k) from the positive integers to the positive integers such that

$$\alpha_k \sim \tilde{\alpha}_{n(k)} \qquad \beta_k \sim \tilde{\beta}_{n(k)}.$$

PROOF There exists a bounded sequence of integers k_n such that

$$\alpha_{n+1} \sim \gamma^{k_n}\alpha_n.$$

Then also for some $\tau \in G$ we have

$$\beta_{n+1} \sim \tau^{k_n}\beta_n.$$

For $I = \mathbb{R}$ this follows from part 2 in table 3.2. If $I \neq \mathbb{R}$, then (6.7) implies that h is non-constant on each unbounded subinterval of I and we obtain this asymptotic relation from proposition 3.1 and corollary 2 to proposition 3.2.

By rearranging the sequence (α_n) we may assume that $k_n \geq 0$ for all n (or $k_n \geq 0$ for all n). (Use proposition 9.7 for an exact proof.) For convenience we assume that k_n is strictly positive. We form the sequence $(\tilde{\alpha}_n)$ by setting $\tilde{\alpha}_1 = \alpha_1$ and inserting the elements $\gamma^j\alpha_n$, $j = 1,\dots,k_n-1$, between α_n and α_{n+1}. We thus obtain the sequence

$$\alpha_1, \gamma\alpha_1, \gamma^2\alpha_1, \dots, \gamma^{k_1-1}\alpha_1, \alpha_2, \gamma\alpha_2, \dots \ .$$

Similarly for the sequence $(\tilde{\beta}_n)$.

In order to prove convergence of $\tilde{\beta}_n f\tilde{\alpha}_n^{-1}$ we have to use the boundedness of the sequence (k_n). First note that for each n there exist j(n) and l(n) such that

$$\tilde{\beta}_n f\tilde{\alpha}_n^{-1} = \tau^{j(n)}\beta_{l(n)} f\alpha_{l(n)}^{-1}\gamma^{-j(n)}$$

and $(j(n))$ is bounded. Hence it suffices to prove convergence for subsequences with constant exponents j(n), and this is trivial.

PROPOSITION 6.3 Suppose

$$\beta_n f\alpha_n^{-1} \to \phi \quad \text{weakly on I.}$$

where $f \in M$, $\beta_n \in G$, the sequence (α_n) satisfies (6.1), (6.2) and (6.3), I is a non-empty open interval, $\gamma I = I$ and $\phi \in \Phi$ is non-constant on I. Then there exist $\tau \in G$, continuous functions α and β from $[0, \infty)$ into G, and a sequence $t_n \to \infty$ such that

$$\alpha_n = \alpha(t_n) \qquad \beta_n = \beta(t_n)$$

(6.8) $$\alpha(t+s)\alpha(t)^{-1} \to \gamma^s \qquad \beta(t+s)\beta(t)^{-1} \to \tau^s \quad \text{for } t \to \infty$$

(6.9) $$\beta(t)f\alpha(t)^{-1} \to h \quad \text{on } I \text{ for } t \to \infty.$$

PROOF Existence of these functions α and β follows from proposition 9.7 as in proposition 6.2.

The sequence $(t_{n+1} - t_n)$ is bounded by the remark after proposition 9.7. This implies convergence in (6.9) if we use that convergence in (6.8) is uniform on bounded intervals by proposition 9.3.

7 Some consequences of the condition $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$

A basic feature of the central limit law for sums of random variables is that the contribution of each single random variable to the sum is asymptotically negligable. Although the distribution functions of the partial sums diverge, the distributions of the nth and of the n+1st partial sums lie close to one another as n tends to infinity. To be more explicit we consider the special case where $\underline{x}_n$ is the sum of n elements of a sequence of independent identically distributed random variables with expectation μ and variance $\sigma^2 > 0$. Let α_n be the usual norming transformation for the nth partial sum,

$$\alpha_n x = \frac{x - \mu n}{\sigma\sqrt{n}} ,$$

then

(7.1) $$\alpha_{n+1}\alpha_n^{-1} \to \varepsilon.$$

Indeed

$$\alpha_{n+1}\alpha_n^{-1}x = \frac{\sigma.\sqrt{n}x + \mu.n - \mu.(n + 1)}{\sigma.\sqrt{n + 1}} = \sqrt{\frac{n}{n + 1}}\, x - \frac{\mu}{\sigma.\sqrt{n + 1}} \to x.$$

DEFINITION 7.1 The sequence (α_n) is asymptotic to (γ_n), with $\alpha_n \in G$ and $\gamma_n \in G$, and we write

$$\alpha_n \sim \gamma_n$$

if $\alpha_n\gamma_n^{-1} \to \varepsilon$.

Note that $\sim$ is an equivalence relation and that (7.1) may be formulated as $\alpha_{n+1} \sim \alpha_n$.

Condition (7.1) seems to be a quite natural one to make. One does not in general use norming constants to tame a sequence of wildly diverging distribution functions, but rather to keep control of a sequence of distribution functions which, though apparently well-behaved, exhibits a tendency to drift away to a defective or degenerate distribution.

Note too that condition (7.1) depends on the order of the index set. In chapters 2 - 5 the index set could have been an arbitrary countable set and the positive integers were used only to conform with standard usage.

In exercise 1.2 we have seen that the sequence (α_n) may be replaced by any sequence (α_n') which is asymptotic to the given sequence. This does not alter convergence or the limit distributions in the basic situation (1.1).

Nor does it alter the set Δ. Hence if (7.1) holds and $\alpha_n \underline{x}_n \to \underline{u}$ in distribution, then also

(7.2) $\qquad \alpha(t)\underline{x}_t \to \underline{u}$ in distribution

where we define for $t = n + \theta$, $0 \le \theta < 1$, $n = 0,1,2,\ldots$

(7.3) $\qquad \alpha(n + \theta) := (\alpha_{n+1}\alpha_n^{-1})^\theta \alpha_n$

and $\underline{x}_t := \underline{x}_n$.

In (7.2) the norming constants depend continuously on a parameter t which varies over the non-negative reals. This allows us to employ the theory of functions of a real variable to obtain interesting results. In the chapters 9, 10 and 12 we shall see that in particular Karamata's theory of regular variation is a very useful tool in certain situations (if Δ is a one-parameter subgroup of G).

In this chapter we prove a number of loosely connected results, the most important being proposition 7.1 which states that under the condition (7.1) we may replace the sequences (α_n) and (β_n) in (2.1) by continuous functions α and β from $[0, \infty)$ into G where for $t = n + \theta$, $0 \le \theta < 1$, we define

(7.3a) $\qquad \alpha(t) := (\alpha_{n+1}\alpha_n^{-1})^\theta \alpha_n$

(7.3b) $\qquad \beta(t) := (\beta_{n+1}\beta_n^{-1})^\theta \beta_n$.

Furthermore we shall prove in proposition 7.2 that if $\alpha_n \to \infty$ and $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$, then the set Δ contains a one-parameter subgroup of G. (Compare this with exercise 1.4.4.) In the ensuing chapters we shall be particularly interested in the case that Δ is equal to a one-parameter subgroup of G.

Finally we introduce a compactification G^* of G, which is homeomorphic to the closed disk in the plane. With the aid of this compactification we shall be able to give a simple analysis of the basic situation (1.1) or (2.1) in the case that the sequence (β_n) does not diverge to ∞.

PROPOSITION 7.1 Suppose

$$\beta_n f \alpha_n^{-1} \text{ converges onto } \Lambda$$

and $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$. (We assume as usual $\alpha_n \in G$, $\beta_n \in G$, $f \in M$ and that Λ is a closed subset of some element of M. We do not assume that $\alpha_n \to \infty$.) Then

$$\beta(t)f\alpha(t)^{-1} \text{ converges onto } \Lambda$$

where α and β are the continuous functions from $[0, \infty)$ into G defined in (7.3a) and (7.3b).

PROOF Set $g_t := \beta(t)f\alpha(t)^{-1}$. Then for $\theta \in [0, 1]$

$$g_{n+\theta} = \beta(n + \theta)\beta_n^{-1}g_n\alpha_n\alpha(n + \theta)^{-1} = \tau_n^{\theta}g_n\sigma_n^{-\theta}$$

where $\tau_n = \beta_{n+1}\beta_n^{-1}$ and $\sigma_n = \alpha_{n+1}\alpha_n^{-1}$. Since g_{n+1} converges onto Λ and $\sigma_n \to \varepsilon$ we find that both g_n and $\tau_n g_n$ converge onto Λ. It suffices to prove that for any sequence $\theta_n \in [0, 1]$ also $\tau_n^{\theta_n}g_n$ converges onto Λ.

Suppose $P \in \Lambda$. There exist $P_n \in g_n$ such that $P_n \to P$ and $Q_n \in g_n$ such that $\tau_n Q_n \to P$. Since $\tau^{\theta}y$ lies between y and τy if θ lies between 0 and 1 we can find R_n which lies between P_n and Q_n on g_n such that $\tau_n^{\theta_n}R_n \to P$.

This proves the proposition.

We shall now show, see proposition 7.2, that the conditions $\alpha_n \to \infty$ and $\alpha_{n+1} \sim \alpha_n$ imply that the set Δ contains a one-parameter subgroup of G.

DEFINITION 7.2 For any unbounded set $A \subset G$ we define $\Delta(A)$ to be the set of all $\sigma \in G$ for which there exist divergent sequences (α_n) and (β_n) in A such that $\beta_n\alpha_n^{-1} \to \sigma$.

Let (α_n) be a sequence in G such that $\alpha_{n+1} \sim \alpha_n \to \infty$. One easily verifies that $\Delta = \Delta(A)$ where

(7.4) $\qquad A = \{\alpha(t) \mid t \geq 0\}$

and $\alpha(t)$ is defined by (7.3a).

In this case the set Δ is unbounded. Indeed $A\alpha_n^{-1}$ intersects the circle $\{\gamma \in G \mid \gamma x = e^c x + b \text{ and } c^2 + b^2 = r^2\}$ in a point $\sigma_n = \alpha(t_n)\alpha_n^{-1}$, since A is connected and unbounded. The circle is compact. Hence the sequence (σ_n) has a limit point σ on the circle which belongs to Δ.

LEMMA 7.1 Let B be a closed connected subset of the plane and τ a translation such that B and τB are disjoint. Then so are B and $\tau^k B$ for all integers $k \neq 0$.

PROOF We may assume that B is a subset of the complex plane and that $\tau z = z + 2\pi i$. Let $R \supset B$ be a region (i.e. an open connected subset of $\mathbb{C}$) such that R and τR are disjoint. It suffices to prove that the exponential function $w(z) := e^z$ is injective on R.

Suppose $z_1, z_2 \in R$, $z_1 \neq z_2$ and $w(z_1) = w(z_2)$. Let Γ be a smooth curve in R connecting z_1 and z_2. We may assume w to be injective on $\Gamma \setminus \{z_2\}$. Then $w(\Gamma)$ is a simple closed curve in the image plane. Hence

$$z_2 - z_1 = \int_\Gamma dz = \int_{w(\Gamma)} \frac{dw}{w} = 2k\pi i$$

with $k \in \{-1,0,1\}$. Since $z_2 \neq z_1$ we have $z_2 - z_1 = \pm 2\pi i$ and hence R and τR intersect. This contradiction proves the lemma.

REMARK The proof makes implicit use of Jordan's theorem that a simple closed curve divides the plane into two disjoint regions. For a more topological proof see Hopf [1936].

COROLLARY Let A be a closed connected subset of G and let β be an element of G such that A and βA are disjoint. Then so are A and $\beta^k A$ for all integers $k \neq 0$.

PROOF Choose $\alpha \in G$ such that $\alpha\beta \neq \beta\alpha$ and either α or β is a translation. The map $\beta^s\alpha^t \mapsto s + it$ is a homeomorphism of G onto the complex plane. If B is the image of A, then $\tau^k B$ is the image of $\beta^k A$ where τ is the translation $z + 1$ in the complex plane.

PROPOSITION 7.2 Suppose (α_n) is a sequence in G, $\alpha_n \to \infty$ and $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$. Then Δ contains a one-parameter subgroup $G(\tau) = \{\tau^t \mid t \in \mathbb{R}\}$ for some $\tau \in G$, $\tau \neq \varepsilon$. Moreover if $\Delta \cap U = G(\tau) \cap U$ for some neighbourhood U of ε, then $\Delta = G(\tau)$.

PROOF Define $A = \{\alpha_t \mid t \geq 0\}$ as in (7.4). We first prove

(7.5) $\gamma \notin \Delta$ implies $\gamma^k \notin \Delta$ for all integers k.

Suppose $\gamma \notin \Delta$. Then there exists a neighbourhood V of γ such that $A \cap VA$ is bounded. Choose $T \in \mathbb{R}$ such that $A_1 = \{\alpha_t \mid t \geq T\}$ and VA_1 are disjoint. Clearly $\Delta(A_1) = \Delta$. By the corollary to lemma 7.1 the sets A_1 and $\beta^k A_1$ are disjoint for all $\beta \in V$ and all integers $k \neq 0$. For $k \neq 0$ the set $V_k = \{\beta^k \mid \beta \in V\}$ is a neighbourhood of γ^k. Moreover A_1 and $V_k A_1$ are disjoint. Hence $\gamma^k \notin \Delta$.

Suppose for a $\tau \neq \varepsilon$ we have $\Delta \supset G(\tau)$ and $\Delta \cap U = G(\tau) \cap U$ for some neighbourhood U of ε. If $\gamma \notin G(\tau)$, then $\gamma_1 := \gamma^{1/n} \in U \setminus G(\tau)$ for some sufficiently large n. Since $\Delta \cap U = G(\tau) \cap U$ we have $\gamma_1 \notin \Delta$, and hence $\gamma = \gamma_1^n \notin \Delta$ by (7.5). This proves the last part of the proposition.

It only remains to prove that Δ contains a one-parameter subgroup $G(\tau)$ for some $\tau \neq \varepsilon$.

Set $S = \{\gamma \in G \mid \gamma x = e^c x + b,\ c^2 + b^2 = 1\}$. For each $\gamma \neq \varepsilon$ there exists $t > 0$ such that $\gamma^t = \tilde{\gamma} \in S$. Let (γ_n) be a divergent sequence in Δ. Let τ be a limit point of $\tilde{\gamma}_n$ in S. We may assume that $\tilde{\gamma}_n \to \tau$. We shall prove that $G(\tau) \subset \Delta$.

Suppose $\tau_1 = \tau^s \notin \Delta$ for some $s > 0$. Then V_1 is disjoint from Δ for some neighbourhood V_1 of τ_1. We also know that $\gamma_n^{r_n} \to \tau_1$ for some sequence $r_n \to 0$. This implies that $\gamma_n^{1/m} \in V_1$ for n sufficiently large where m is the integral part of r_n^{-1}. By (7.5) we obtain $\gamma_n^{1/m} \in \Delta$. This contradiction proves the proposition.

We shall now consider the following situation,

$$g_n \to g \quad \text{in } M$$

$$\sigma_n \to \sigma \quad \text{in } G$$

$$\tau_n g_n \sigma_n^{-1} \quad \text{converges onto } \Lambda.$$

The reader may recall that a similar situation in the opening pages of chapter 3, where Λ was the closure of the graph of a function h, defined and non-decreasing on the open interval I, led us to the very useful functional equation $h\sigma = \tau h$.

It will be convenient to introduce a compactification G^* of G which is homeomorphic to the closed disk in the plane.

The group G is isomorphic to a subgroup of the projective transformations of the projective real line

$$x \mapsto ax + b \quad \text{corresponds to} \quad \binom{x}{1} \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\binom{x}{1} = \binom{ax + b}{1}.$$

The matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} ap & bp \\ 0 & p \end{pmatrix}$ with $p > 0$ define the same projective transformation. Hence we may choose the matrix $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ to satisfy

$$a^2 + b^2 + c^2 = 1 \qquad a > 0 \text{ and } c > 0.$$

The set of these matrices is homeomorphic to G. The closure of this set in $\mathbb{R}^3$ is a closed quarter sphere. It is the set of all matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for which

$$a^2 + b^2 + c^2 = 1 \qquad a \geq 0 \text{ and } c \geq 0.$$

This compact set determines a compactification G^* of G.

Let us consider sequences T_n which converge to an element on the boundary (we assume $a > 0$ and $c > 0$)

$$T_n \to \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix} \quad \text{implies} \quad T_n\binom{x}{1} \to \binom{\pm\infty}{1}$$

$$T_n \to \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \quad \text{implies} \quad T_n\binom{x}{1} \to \binom{b/c}{1}$$

$$T_n \to \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \quad \text{implies} \quad T_n\binom{x}{1} \to \binom{\infty}{1} \quad \text{for } x > \frac{-b}{a}$$

$$\to \binom{-\infty}{1} \quad \text{for } x < \frac{-b}{a}.$$

The corresponding limits in G^* will be denoted by $\pm\infty$, $\cdot 0$ and $\cdot\infty$. Often we shall also mention the centre of multiplication in the second and third case. Observe that $\alpha_n \to \infty$ and $\alpha_n \to +\infty$ have very different meaning.

Note that each one-parameter subgroup

$$G(\gamma) = \{\gamma^t \mid t \in \mathbb{R}\}$$

with $\gamma \neq \varepsilon$ can be extended with two boundary elements γ^{∞} and $\gamma^{-\infty}$ in G^*, that G^* is homeomorphic to the closed disk and that G^* may also be realized as the closure of G in the two point compactification M^* of M introduced in chapter 2. Then $\cdot 0$ is a horizontal line, $\cdot\infty$ is a vertical line and $+\infty$ and $-\infty$ are respectively the 1 and 0 of the Boolean algebra M^*.

DEFINITION 7.3 By G^* we denote the compactification of G introduced above. Moreover

$$\alpha_n \to +\infty \quad \text{means} \quad \alpha_n x \to +\infty \quad \text{for all } x \in \mathbb{R}$$

$$\alpha_n \to -\infty \quad \text{means} \quad \alpha_n x \to -\infty \quad \text{for all } x \in \mathbb{R}$$

$$\alpha_n \to \cdot 0 \text{ (with centre } c) \quad \text{means} \quad \alpha_n x \to c \quad \text{for all } x \in \mathbb{R}$$

$$\alpha_n \to \cdot\infty \text{ (with centre } c) \quad \text{means} \quad \begin{cases} \alpha_n x \to +\infty & \text{for } x > c \\ \alpha_n x \to -\infty & \text{for } x < c. \end{cases}$$

We now introduce some terminology which should speak for itself.

DEFINITION 7.4 We say that h_1 lies to the left of h_2 where $h_1, h_2 \in M$ if $(x_1, y_1) \in h_1$, $(x_2, y_2) \in h_2$ implies $x_1 \le x_2$. Similarly for two connected subsets I_1 and I_2 of $\mathbb{R}$ we say that I_1 lies to the left of I_2 if $x_1 \le x_2$ whenever $x_1 \in I_1$ and $x_2 \in I_2$. Thus, if I_i denotes the projection of h_i on the x-axis for i = 1, 2, then h_1 lies to the left of h_2 if and only if I_1 lies to the left of I_2.

Furthermore we shall say that $h \in M$ lives on the connected subset I if the closure of I contains the projection of h on the x-axis.

Finally $\{h = c\}$ is shorthand for $\{x \in \mathbb{R} \mid (x, c) \in h\}$.

PROPOSITION 7.3 Suppose g, g_n and h lie in M, and $\tau_n \in G$. If

$$g_n \to g$$

$$\tau_n g_n \to h$$

then, with the notation of definition 7.3 and 7.4,

1. $\tau_n \to \tau \in G$ implies $\tau g = h$
2. $\tau_n \to +\infty$ implies h lies to the left of g
3. $\tau_n \to -\infty$ implies g lies to the right of h
4. $\tau_n \to \cdot 0$ (with centre c) implies g lives on $\{h = c\}$
5. $\tau_n \to \cdot\infty$ (with centre c) implies h lives on $\{g = c\}$.

PROOF The third implication follows from the second and the fifth from the fourth by writing $h_n = \tau_n^{-1} g_n$.

1. See lemma 2.1.
2. Suppose $(x_n, y_n) \in g_n$ converges to $(x, y) \in g$. Then $\tau_n y_n \to \infty$. Suppose $(x', y') \in h$ and $x < x'$. Then $x_n < x'$ for $n \geq n_0$ and hence $\tau_n y_n < y' + 1$ for $n \geq n_1$. This contradicts $\tau_n y_n \to \infty$.
4. For $(x, y) \in g$ there exists $(x_n, y_n) \in g_n$ such that $(x_n, y_n) \to (x, y)$. Then $\tau_n y_n \to c$. Hence $(x, c) \in h$.

DEFINITION 7.5 J denotes the interior of the smallest connected subset of the x-axis which contains the projection of Λ and I is the interior of the projection of g on the x-axis.

Usually Λ will be a closed subset of g, and g will be a limit point of the sequence $\beta_n f \alpha_n^{-1}$ in M. Recall from chapter 2 that M is a locally compact metrizable space and that the sequence $(\beta_n f \alpha_n^{-1})$ is relatively compact if $\beta_n f \alpha_n^{-1}$ converges onto a non-empty set Λ of the x,y-plane.

Note that Λ is constant on J if and only if $\Lambda \subset h$ for some $h \in M_0$.

PROPOSITION 7.4 Suppose $g, g_n \in M$, $\Lambda \subset g$, and $\sigma, \sigma_n, \tau_n \in G$. If

$$(7.6) \qquad \begin{aligned} & g_n \to g \\ & \sigma_n \to \sigma \\ & \tau_n g_n \sigma_n^{-1} \text{ converges onto } \Lambda \end{aligned}$$

then, in the notation of definitions 7.3, 7.4 and 7.5,

1. $\tau_n \to \tau \in G$ implies $\Lambda \subset \tau g \sigma^{-1}$ implies $J \subset \sigma I$
2. $\tau_n \to +\infty$ implies J lies to the left of σI
3. $\tau_n \to -\infty$ implies J lies to the right of σI
4. $\tau_n \to \cdot 0$ (with centre c) implies $\Lambda = c$ on σI
5. $\tau_n \to \cdot\infty$ (with centre c) imples $g = c$ on $\sigma^{-1} J$.

PROOF It suffices to prove the proposition for subsequences $\tau_k g_k \sigma_k^{-1}$ which converge to some element h in M. Now apply proposition 7.3 to $g_k \sigma_k^{-1} \to g\sigma^{-1}$ and $\tau_k g_k \sigma_k^{-1} \to h$.

COROLLARY If in addition to (7.6) it is given that

$J \cap \sigma I$ is non-empty
Λ is non-constant on σI
g is non-constant on $\sigma^{-1}J$

then the sequence (τ_n) is bounded.

Proposition 7.3 has an interesting application in the particular case that (g_n) is a constant sequence. We formulate this as a separate proposition.

PROPOSITION 7.5 Suppose $f \in M$, $h \in M$, $\tau_n \in G$ and $\tau_n f \to h$. Let L be the projection of f on the x-axis. L is a connected subset of $\mathbb{R}$ with endpoints $l_1 \leq l_2$ which may be infinite.

1. $\tau_n \to \tau \in G$ implies $h = \tau f$.

2. $\tau_n \to +\infty$ implies that h is the vertical line through l_1. In particular l_1 is finite.

3. $\tau_n \to -\infty$ implies that h is the vertical through l_2.

4. $\tau_n \to \cdot 0$ (with centre c) implies that $h \in M_0$ is the constant function on L,

$$h(x) = c \qquad x \in L.$$

5. $\tau_n \to \cdot\infty$ (with centre c) implies that $h \in M_0$. Moreover h is the constant function on $\{f = c\}$ or h is a vertical line through one of the endpoints of $\{f = c\}$. In this case the corresponding endpoint has to be finite.

PROOF As for proposition 7.3.

Note that this proposition gives a fairly complete analysis of the basic situation

$$(7.7) \qquad \beta_n f \alpha_n^{-1} \text{ converges onto } \Lambda$$

in the case that the sequences (α_n) and (β_n) do not both diverge to ∞. Indeed for convenience assume $\beta_n \to \infty$ and $\alpha_k \to \alpha$. (The case where $\alpha_n \to \infty$ is obtained from this by reflecting f in the diagonal.) Then $\alpha_k \sim \alpha$ and (7.7) implies

$$\beta_k f \alpha^{-1} \text{ converges onto } \Lambda.$$

Thus if Λ contains three points (0, 0), (1, 1) and (p, q) with $0 < p, q < 1$, then every limit point h of the sequence $\beta_k f \alpha^{-1}$ has to have the form $h(x) = q$ on (0, 1). If follows from the proposition above that for $f \neq h$ every limit point α of the sequence (α_n) satisfies

either $\alpha[0, 1] = \{f = c\}$ for some $c \in \mathbb{R}$ (and then $\alpha_k \to \alpha$ implies $\beta_k \to \cdot\infty$ with centre c),

or $\alpha(0, 1)$ is the interior of the projection of f on the x-axis (and then $\alpha_k \to \alpha$ implies $\beta_k \to \cdot 0$ with centre q).

For the sake of completeness we formulate

THEOREM 7.1 If in addition to the basic situation (1.1) it is given that β_n does not diverge to ∞ then $\underline{u} \overset{M}{=} \phi(\underline{v})$ where $\phi \in M_0$.

8 The functional equation $\tau^{-1}\Lambda\sigma \subset g$

In this chapter we shall prove the proposition below and discuss some of its consequences.

PROPOSITION 8.1 Suppose $f \in M$ and α and β are continuous functions from $[0, \infty)$ into G such that $\alpha(t) \to \infty$ for $t \to \infty$ and

$$(8.1) \qquad g_t := \beta(t)f\alpha(t)^{-1} \text{ converges onto } \Lambda \text{ for } t \to \infty$$

where Λ is a non-empty closed subset of some element of M. (See definition 2.1.) Let $g \in M$ be a limit point of g_t for $t \to \infty$. Then there exists an unbounded closed connected subset $C \subset G^2$, containing $(\varepsilon, \varepsilon)$, such that

$$(8.2) \qquad \tau^{-1}\Lambda\sigma \subset g$$

for all $(\sigma, \tau) \in C$.

PROOF Consider for $s \geq 0$ the set

$$D(s) = \{(\alpha(t)\alpha(s)^{-1}, \beta(t)\beta(s)^{-1}) \in G^2 \mid t \geq 0\}.$$

This is a closed, connected, unbounded subset of G^2 which contains the element $(\varepsilon, \varepsilon)$.

Any sequence $r_n \to \infty$ contains a subsequence $s_n \to \infty$ such that the sequence $D_n := D(s_n)$ converges to a set $D \subset G^2$. Indeed this holds for any sequence of subsets of a separable metrizable space. See Whyburn [1942, theorem 1.7.1]. By convergence $D_n \to D$ we mean that every point $x \in D$ is limit of a sequence $x_n \in D_n$ and that D contains the limit points of any sequence $x_n \in D_n$.

We now show that every component of D is unbounded.

Suppose $x \in D$, $x = \lim x_n$ with $x_n \in D_n$. Let B be an open ball in G^2 containing x and let K_n be the component of x_n in $\overline{B} \cap D_n$ where $\overline{B}$ denotes the closure of B in G^2. Then K_n contains a point of the boundary ∂B of B. Also K, the topological limsup of K_n, i.e. the set of all limit points of all sequences $y_n \in K_n$, is a compact connected set (Whyburn [1942, 1.9.12]), contains x, contains a point of ∂B and is contained in D. Hence the component of x in D contains points on the boundary of any ball B containing x.

(A slightly different proof can be given by observing that $D'_s := D_s \cup \{\infty\}$ is a connected closed subset of the one-point Hausdorff compactification $G^2 \cup \{\infty\}$ of G^2, which is homeomorphic to the sphere S^4. The set of connected closed subsets of a compact metric space is itself a compact metric space (in the Hausdorff metric), see Montgomery and Zippin [1955, chapter 1.10]. Hence a subsequence $D'_n \to D'$. The set D' is connected and contains ∞, hence every component of $D' \setminus \{\infty\}$ is unbounded. Now use the fact that for compact spaces convergence in the Hausdorff metric is equivalent to the convergence defined above. See Whyburn [1942, corollary to 1.7.2].)

Note that $g_{s_n} \to g$ if we start with a sequence $r_n \to \infty$ such that $g_{r_n} \to g$.

Let C be the component of $(\varepsilon, \varepsilon)$ in D. Then for $(\sigma, \tau) \in C$ there exist $t_n \to \infty$ such that

$$\alpha(t_n)\alpha(s_n)^{-1} =: \sigma_n \to \sigma$$
$$\beta(t_n)\beta(s_n)^{-1} =: \tau_n \to \tau$$

and since

$$g_{t_n} = \tau_n g_{s_n} \sigma_n^{-1}$$

the right hand side converging to $\tau g \sigma^{-1}$ by lemma 2.1 and the left hand side converging onto Λ by (8.1) we obtain

$$\Lambda \subset \tau g \sigma^{-1}$$

which proves (8.2).

DEFINITION 8.1 The set C of proposition 8.1 will be called a guide set of g for Λ.

To give some indication of the far-reaching consequences of equation (8.2), assume for a moment that σ and τ are translations (instead of arbitrary positive affine transformations).

Relations (8.2) states that the set Λ can be moved continuously along the curve g using only transformations σ^{-1} in the horizontal and transformations τ^{-1} in the vertical direction. If we only allow translations σ and τ, then the set Λ is moved along the curve g as a rigid body. If Λ contains two points, say (0, 0) and (p, q) with p and q positive, it will only be possible to move Λ continuously along g if g is either affine with slope $\lambda = qp^{-1}$ or if g is the sum of such an affine function and a periodic function $\pi(x)$ with period p. We obtain for g the same representation

$$g(x) = \lambda x + c + \pi(x)$$

which we obtained for the solutions of the functional equation $g(x + p) = g(x) + q$ in chapter 3. Since we know that the guide set C is unbounded the representation holds on one of the half lines $(-\infty, 0)$ or $(0, \infty)$. Clearly g will be affine on this half line if Λ is sufficiently large. For instance this will be the case if Λ contains three points (0, 0), (p_1, q_1) and (p_2, q_2) with $0 < p_1 < p_2$, $0 < q_1 < q_2$ and p_1/p_2 irrational.

If we do not restrict σ and τ to be translations, the inclusion $\tau^{-1}\Lambda\sigma \subset g$ for each pair (σ, τ) in the guide set C, may be formulated analytically in various ways. In order to avoid trivialities we assume that Λ contains two points (x_i, y_i) for i = 0, 1 such that $x_0 < x_1$ and $y_0 < y_1$.

Let (x_2, y_2) be a third point in Λ and let $(\sigma, \tau) \in C$ be such that $\sigma^{-1}x_i$ is a continuity point of g for i = 0, 1, 2. Then $g(\sigma^{-1}x_i) = \tau^{-1}y_i$ for i = 0, 1, 2, and hence

$$(8.3)\qquad \frac{g(\sigma^{-1}x_2) - g(\sigma^{-1}x_0)}{g(\sigma^{-1}x_1) - g(\sigma^{-1}x_0)} = \frac{\tau^{-1}y_2 - \tau^{-1}y_0}{\tau^{-1}y_1 - \tau^{-1}y_0} = \frac{y_2 - y_0}{y_1 - y_0} .$$

Thus for fixed $(x^*, y^*) \in \Lambda$ we are able to express $g(\sigma^{-1}x^*)$ in terms of $g(\sigma^{-1}x_0)$ and $g(\sigma^{-1}x_1)$ as

$$(8.4)\qquad g(\sigma^{-1}x^*) = g(\sigma^{-1}x_0) + c^*(g(\sigma^{-1}x_1) - g(\sigma^{-1}x_0))$$

where $c^* = (y^* - y_0)/(y_1 - y_0)$.

In particular we may choose coordinates such that $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (1, 1)$. Define

$$S = \{(s_0, s_1) = (\sigma^{-1}0, \sigma^{-1}1) \mid (\sigma, \tau) \in C\}.$$

For $(p, q) \in \Lambda$ we have, with $\sigma^{-1}p = (1 - p)s_0 + ps_1$, (8.5) $$g((1 - p)s_0 + ps_1) = (1 - q)g(s_0) + qg(s_1)$$

for all $(s_0, s_1) \in S$ for which s_0, s_1 and $(1 - p)s_0 + ps_1$ are continuity points of g. Observe that this is a generalized version of Cauchy's functional equation

$$g(s_0 + s_1) = g(s_0) + g(s_1) \qquad s_0, s_1 \in \mathbb{R}.$$

We may also write (8.2) as

(8.6) $$\Lambda \subset \tau g \sigma^{-1}$$

for $(\sigma, \tau) \in C$. This brings us back to the original basic situation (2.1), with g instead of f. The inclusion (8.6) evidently implies

(8.7) $\tau g \sigma^{-1}$ converges onto Λ for $(\sigma, \tau) \to \infty$ in C.

PROPOSITION 8.2 Suppose (8.1) holds. Let C be a guide set of g for Λ. See definition 8.1. If C contains a sequence (σ_n, τ_n) such that (σ_n) is bounded and $\tau_n \to \infty$, then g_t has a limit point in M_0 for $t \to \infty$. If C contains a sequence (σ_n, τ_n) such that $\sigma_n \to \infty$ and (τ_n) is bounded, then g_t has a limit point ϕ, with $\phi^{-1} \in M_0$, for $t \to \infty$.

PROOF The first part follows from proposition 7.5 applies to $f = g\sigma^{-1}$ with σ a limit point of the sequence (σ_n), and to appropriate subsequences (τ_k) and $\sigma_k \to \sigma$. Compare the text following proposition 7.5. The second part follows from this by symmetry.

Now let us return to (8.7) or (8.6). The set $C \subset G^2$ being unbounded, the projection of C on one of the two factor spaces G will be unbounded. In view of proposition 8.2 we shall assume that for any sequence $(\sigma_n, \tau_n) \in C$ the relation $\sigma_n \to \infty$ implies $\tau_n \to \infty$ and vice versa. Even so the set C need not contain a continuous curve $(\sigma(t), \tau(t))$, $t \geq 0$, such that $\sigma(t) \to \infty$ for $t \to \infty$. However it will contain a sequence (σ_n, τ_n) such that $\sigma_{n+1} \sim \sigma_n$, $\tau_{n+1} \sim \tau_n$ and $\sigma_n \to \infty$ (or equivalently $\tau_n \to \infty$).

Indeed, G^2 is homeomorphic to $\mathbb{R}^4$. Cover G^2 by an increasing sequence of open balls B_n. Let C'_n be the component of ∞ in $(C \cup \{\infty\}) \setminus B_n$. Then $C'_n \downarrow \{\infty\}$. Choose $x_n \in C_n := C'_n \setminus \{\infty\}$, and for each n let

$$x_{n0} := x_n, x_{n1}, \ldots, x_{nk} := x_{n+1}$$

be a finite sequence of points $x_{nj} = (\sigma_{nj}, \tau_{nj}) \in C_n$ such that $\sigma_{nj}\sigma_{nj-1}^{-1} \in U_n$, $\tau_{nj}\tau_{nj-1}^{-1} \in U_n$ for $j = 1,\ldots,k$, where (U_n) is a fixed neighbourhood base of ε. The concatenation of these finite sequences yields the desired sequence.

DEFINITION 8.2 For $\sigma \in G$, $\sigma x = ax + b$, define

$$\chi(\sigma) := \log a.$$

Observe that χ is a continuous homomorphism of G onto the additive group of $\mathbb{R}$.

PROPOSITION 8.3 Let $\alpha : [0, \infty) \to G$ be continuous and $\alpha(t) \to \infty$ for $t \to \infty$. Set $A = \{\alpha(t) \mid t \geq 0\}$. Let H be either one of the halfplanes $\{\sigma \in G \mid \chi(\sigma) \leq 0\}$ or $\{\chi \geq 0\}$. There exists a sequence $t_n \to \infty$ such that the sequence of sets $A\alpha(t_n)^{-1}$ converges to a set $A_0 \subset G$, and such that the component of ε in $H \cap A_0$ is unbounded.

PROOF The existence of A_0 is proved as in proposition 8.1.

We shall prove the second statement for the halfplane $H = \{\chi \leq 0\}$.

Suppose there exists a sequence $r_n \to \infty$ such that $\chi(\alpha(r_n)) \to \infty$. Then there exists a sequence $s_n \to \infty$ such that $\chi(\alpha(s)) \leq \chi(\alpha(s_n))$ for $0 \leq s \leq s_n$. Set

$$A_n = \{\alpha(s)\alpha(s_n)^{-1} \mid 0 \leq s \leq s_n\}.$$

Then $\varepsilon \in A_n \subset H$, A_n is connected and $\alpha(0)\alpha(s_n)^{-1} \in A_n$ diverges to ∞. By the arguments of proposition 8.1 a subsequence of the sequence A_n converges to a set in H every component of which is unbounded.

If $\chi(\alpha(t))$ is bounded from above there exists a sequence $s_n \to \infty$ such that $\chi(\alpha(s)) \leq \chi(\alpha(s_n)) + \frac{1}{n}$ for $s \geq s_n$. Setting

$$A_n = \{\alpha(s)\alpha(s_n)^{-1} \mid s \geq s_n\}$$

we see that $A_n \subset \{\chi \leq \frac{1}{n}\}$. Let C be the component of ε in a limit point of the sequence A_n. Then C is unbounded (again by the same arguments) and $C \subset \{\chi \leq \frac{1}{n}\}$ for all n. Hence $C \subset H$.

DEFINITION 8.3 Λ is a closed subset of some element of M and J as in definition 7.5 is the interior of the smallest connected subset of the x-axis which contains the projection of Λ. We define

$$U := \{\rho \in G \mid J \cap \rho J \text{ is non-empty and } \Lambda \text{ is non-constant on } \rho J \text{ and on } \rho^{-1}J\}.$$

PROPOSITION 8.4 Suppose $\sigma_n, \tau_n \in G$, $g_n \in M$ and let Λ be a closed subset of some element of M. If

$$g_n \text{ converges onto } \Lambda$$
$$\tau_n g_n \sigma_n^{-1} \text{ converges onto } \Lambda$$
$$\sigma_n \to \sigma$$

and if $\sigma \in U$, then (τ_n) is bounded.

PROOF Suppose not. Choose a subsequence $\tau_k \to \infty$, such that $g_k \to g \in M$. The three conditions in the corollary to proposition 7.4 are satisfied, since $J \subset I$, where I is the interior of the projection of g on the x-axis. Hence (τ_k) is bounded.

DEFINITION 8.4 For $\Lambda \subset g \in M$ we define the set $\Omega \subset G^2$ by

$$\Omega = \{(\sigma, \tau) \mid \tau^{-1}\Lambda\sigma \subset g\}.$$

Note that Ω contains each guide set C of g for Λ.

PROPOSITION 8.5 Suppose (8.1) holds. Let $s_n \to \infty$ such that

$$g_{s_n} \to g \text{ in } M$$
$$\{\alpha(t)\alpha(s_n)^{-1} \mid t \geq 0\} \to A.$$

Then Ω_1, the projection on the first coordinate of the set Ω, see definition 8.4, and U, see definition 8.3, satisfy

$$(8.8) \qquad A \cap U\Omega_1 \subset \Omega_1.$$

PROOF Suppose $t_n \to \infty$ and $\sigma_n := \alpha(t_n)\alpha(s_n)^{-1} \to \sigma \in A$. Suppose $\sigma = \rho\gamma$ with $\rho \in U$ and $(\gamma, \delta) \in \Omega$. Then with $\sigma_n =: \rho_n\gamma$, $\beta(t_n)\beta(s_n)^{-1} =: \tau_n =: \pi_n\delta$ and $g_n := \delta g_{s_n}\gamma^{-1}$ we obtain

$$g_{t_n} = \tau_n g_{s_n}\sigma_n^{-1} = \pi_n g_n \rho_n^{-1}$$

and we apply proposition 8.4 with ρ_n and π_n instead of σ_n and τ_n to obtain that the sequence (π_n) and hence also the sequence (τ_n) is bounded. Hence $\Lambda \subset \tau g\sigma^{-1}$ for every limit point τ. This implies that $\sigma \in \Omega_1$.

COROLLARY If Λ is non-constant on J, then U is an open neighbourhood of ε and (8.8) implies that Ω_1 contains the component of ε in A.

DEFINITION 8.5 Let α and β be continuous functions from $[0, \infty)$ into G and let $\alpha(t) \to \infty$ for $t \to \infty$. $\Gamma \subset G^2$ is the set of all limit points in G^2 of sequences

$$(8.9) \qquad (\alpha(t_n)\alpha(s_n)^{-1}, \beta(t_n)\beta(s_n)^{-1}) = (\sigma_n, \tau_n)$$

with $s_n \to \infty$ and $t_n \to \infty$.

Observe that Γ is the two-dimensional analogue of Δ. The set Γ too is symmetric, i.e. $(\sigma, \tau) \in \Gamma$ implies $(\sigma^{-1}, \tau^{-1}) \in \Gamma$, closed and unbounded.

Note also that if $g_t = \beta(t)f\alpha(t)^{-1}$ converges onto Λ for $t \to \infty$, then Γ is the union of all guide sets C of g for Λ for all limit points g of g_t for $t \to \infty$. Moreover if C is a guide set and $(\sigma_i, \tau_i) \in C$ for $i = 1, 2$ then $(\sigma_2\sigma_1^{-1}, \tau_2\tau_1^{-1}) \in \Gamma$.

PROPOSITION 8.6 Let J_c be an open interval. Suppose Λ contains the set $J_c \times \{c\}$ and let $(\sigma, \tau) \in \Gamma$ be an element such that $J_c \cap \sigma J_c$ is non-empty. Then τ is a multiplication with centre c.

PROOF Assume $(\sigma, \tau) = \lim (\sigma_n, \tau_n)$ where (σ_n, τ_n) is defined in (8.9). Assume moreover that $g_{s_n} \to g$ and $g_{t_n} = \tau_n g_{s_n}\sigma_n^{-1} \to h$ (with t_n and s_n as in (8.9)). Then $\Lambda \subset h = \tau g\sigma^{-1}$ and hence

$$\Lambda\sigma \cap \Lambda \subset \tau g \cap g$$

which implies that $\tau g = g$ on $J_c \cap \sigma^{-1}J_c$ and hence τ is a multiplication with centre c.

COROLLARY Suppose for convenience that $c = 0$ and that $J_0 \subset (0, \infty)$ and $\Lambda \ni Q = (0, -1)$. If $J_0 \cap \sigma J_0$ is non-empty and $\sigma 0 > 0$, then $(\sigma, \tau) \in \Gamma$ implies that $\tau y = c.y$ with $c \leq 1$.

PROOF $\tau^{-1}Q\sigma = (\sigma^{-1}0, \tau^{-1}(-1)) \in g$. Since $\sigma^{-1}0 < 0$ and g is non-decreasing we have $\tau^{-1}(-1) \leq -1$.

In the remainder of this chapter we shall consider a number of specific examples of sets Λ. Table 8.1 on page 89 lists the most simple non-trivial cases. In chapter 13 we shall see that if Λ contains a horizontal or vertical line segment, then (8.1) implies that $\Lambda \subset \phi$ for some $\phi \in \Phi$.

DEFINITION 8.6 The set Λ is normal if Λ is closed and if, whenever Λ contains two points on the same horizontal or vertical line, it contains the connecting line segment.

Clearly if g_n converges onto Λ, and Λ contains two points on the same horizontal or vertical line, then g_n converges onto $\Lambda \cup L$ where L is the horizontal or vertical line segment joining the two given points. (Any limit point g of the sequence (g_n) contains L.) Thus we may assume that Λ is normal without loss of generality.

We shall now treat some of the 6 cases in table 8.1 in greater detail. We assume that (8.1) holds, that g is a limit point in M of g_t for $t \to \infty$, and that C is a guide set of g for Λ.

Case 6a. The set Λ is the union of two horizontal intervals $J_1 x\{c_1\}$ and $J_2 x\{c_2\}$ (with $c_1 \neq c_2$ and J_1 and J_2 disjoint open intervals).

Let $C \subset G^2$ be a guide set for Λ (see definition 8.1). Consider

$$C_0 := \{(\sigma, \tau) \in C \mid \tau = \varepsilon\}.$$

We shall prove that C_0 is open-and-closed in C and hence $C_0 = C$. This implies that the projection of C on the second coordinate is bounded and hence by proposition 8.2 the set g_t has a limit point in Φ for $t \to \infty$ if g_t

converges onto Λ for $t \to \infty$.

Suppose $(\sigma_0, \varepsilon) \in C_0$ and $(\sigma, \tau) \in C$, with $\sigma = \rho\sigma_0$. Then $(\rho, \tau) \in \Gamma$. Hence if ρ is sufficiently close to ε then $J_i \cap \rho J_i$ is non-empty for $i = 1, 2$ and by proposition 8.6 we find that τ is a multiplication with centre c_1 and with centre $c_2 \neq c_1$. Hence $\tau = \varepsilon$.

Case 3a. Let J_0 be the interior of the projection of the horizontal line segment in Λ on the x-axis and let J as usual denote the interior of the smallest set containing the projection of Λ on the x-axis.

Let C be a guide set of g for Λ, and recall that C is connected.

Suppose $(\sigma, \tau) \in C$. Then $\tau^{-1}\Lambda\sigma \subset g$. Hence g is constant on $\sigma^{-1}J_0$ and since $\Lambda \subset g$ and Λ is non-constant, this implies that $\sigma^{-1}J_0 \subset J$. Similarly $\Lambda \subset g$ implies that g is constant on J_0 and hence $\sigma^{-1}J \supset J_0$ (if $\tau^{-1}\Lambda\sigma \subset g$).

The set $\{\rho \in G \mid J_0 \subset \rho J \;\&\; \rho J_0 \subset J\}$ is compact for bounded open intervals J_0 and J.

As in case 6a the projection of C on the first coordinate is bounded and g_t has a limit point in Φ for $t \to \infty$ by proposition 8.2.

Case 4b. We assume that Λ is a subset of the coordinate axes, to be even more explicit we assume that $\Lambda = (\{0\}x(y_1, y_2)) \cup ((x_1, x_2)x\{0\})$ where $y_1 < y_2 < 0$ and $0 < x_1 < x_2$. Let C be the guide set of g for Λ where g is a limit point of $\beta(t)f\alpha(t)^{-1}$ for $t \to \infty$. We assume that both coordinates of C are unbounded.

Let C_0 be the set of all $(\sigma, \tau) \in C$ such that both σ and τ are multiplications with centre 0. Then $(\varepsilon, \varepsilon) \in C_0$. Suppose $(\sigma_0, \tau_0) \in C_0$ and $(\sigma, \tau) \in C$. If (σ, τ) is sufficiently close to (σ_0, τ_0), then the intervals J_0 and $\sigma\sigma_0^{-1}J_0$ intersect (with $J_0 = (x_1, x_2)$), and by proposition 8.6 the element $\tau\tau_0^{-1}$ and hence also τ is a multiplication with centre 0. A similar argument for the interval (y_1, y_2) proves that σ is a multiplication with centre 0. Hence C_0 is both open and closed in C. Hence $C_0 = C$, we may write $(\sigma x, \tau y) = (e^s x, e^t y)$ for $(\sigma, \tau) \in C$ and

$$\tau^{-1}\Lambda\sigma = (\{0\}x(e^{-t}y_1, e^{-t}y_2)) \cup ((e^{-s}x_1, e^{-s}x_2)x\{0\})$$

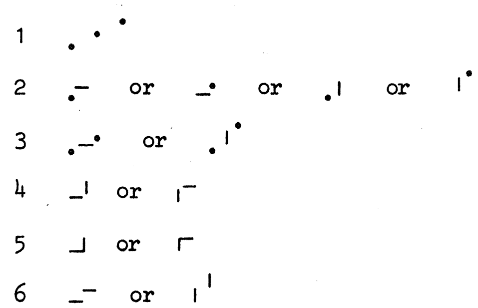
is contained in g for all $(\sigma, \tau) \in C$.

Now suppose $(\sigma, \tau) \to \infty$ in C such that $s \to \infty$ and $t \to \infty$. Then

$$g \text{ contains } (\{0\}x(y_1, 0]) \cup ([0, x_2)x\{0\})$$

TABLE 8.1

Examples of small normal sets Λ



Sets which can be obtained from a given Λ by reflection in the diagonal or by a change of sign on both axes are listed together.

Only two sets Λ in the list have the property that they are non-constant on J, examples 3b and 6a.

(since g is closed),

$$\tau g\sigma^{-1} \text{ contains } (\{0\}x(e^t y_1, 0]) \cup ([0, e^s x_2)x\{0\})$$

and hence $\tau g\sigma^{-1}$ converges to the constant function, $\phi(x) = 0$ on $(0, \infty)$ for $(\sigma, \tau) \to \infty$, $(\sigma, \tau) \in C$.

A similar argument holds in the other three cases $s \to \infty$ and $t \to -\infty$ or $s \to -\infty$ and $t \to \pm\infty$. In view of proposition 8.2 we thus obtain that g_t has a limit point $\phi \in \Phi$ for $t \to \infty$.

PROPOSITION 8.7 Suppose $f \in M$, α and β are continuous functions from $[0, \infty)$ into G, $\alpha(t) \to \infty$ for $t \to \infty$ and

$$g_t = \beta(t)f\alpha(t)^{-1} \text{ converges onto } \Lambda \text{ for } t \to \infty$$

where Λ is a closed subset of an element of M. Suppose moreover Λ is normal (see definition 8.6) and contains two non-degenerate line segments, which do not lie on the same line, and are parallel to the axes.

Then $\Lambda \subset \phi$ for some $\phi \in \Phi$.

PROOF Λ contains one of the sets 4a, 4b, 6a or 6b in table 8.1. Denote this set by Λ_0. (If Λ contains a set 5a (or 5b), it contains a set 4a (or 4b).) Then g_t converges onto Λ_0 for $t \to \infty$. Hence g_t has a limit point $\phi \in \Phi$ for $t \to \infty$. (See the commentary on the cases 6a and 4b above.)

REMARK Obviously either $\phi \in M_0$ or $\phi^{-1} \in M_0$. (These are the only elements of Φ which can possibly contain Λ.)

9 Regular variation in topological groups

DEFINITION 9.1 Throughout this chapter H will denote a topological group with a countable base.

The theory of regular variation which originated in two papers by Karamata, [1930] and [1933], has recently played an increasingly important role in probability theory. See Bingham, Seneta & Teugels [1974].

As an example consider the following situation

$$\frac{f(x+t)-b(t)}{a(t)} \to h(x) \quad \text{weakly for } t \to \infty \tag{9.1}$$

where f and h are non-decreasing functions on $\mathbb{R}$, $a(t) > 0$ and $b(t) \in \mathbb{R}$ are norming functions. Note that this is a particular case of the basic situation (2.1) with which we are concerned. In fact de Haan's work on this equation [1970], and his enthusiastic presentation of the theory of regular variation in a seminar in Amsterdam, initiated my own interest in this subject.

Equation (9.1) may be simplified by assuming either $a(t) \equiv 1$ and hence

$$f(x+t) - b(t) \to h(x) \tag{9.2}$$

or $b(t) \equiv 0$ and hence

$$\frac{f(x+t)}{a(t)} \to h(x). \tag{9.3}$$

Note that the limit relation (9.3) may be translated into (9.2) by taking logarithms. Note too that we may choose $b(t) := f(c+t)$ for the norming function in (9.2) if $x = c$ is a continuity point of h. (Compare (5.2).) Indeed (9.2) implies

$$f(x+t) - f(c+t) \to h(x) - h(c). \tag{9.4}$$

A simple transformation $U = \exp f \log$ leads us to the most commonly used definition for regular variation

$$\frac{U(ys)}{U(s)} \to h(y) \quad \text{for } s \to \infty, \tag{9.5}$$

where in general U and h are assumed to be a measurable positive function on $(0, \infty)$ and convergence is pointwise. See de Haan [1970, theorem 1.1.1].

In de Haan [1970, section 1.4] it is shown that the possible limit functions of (9.1) are the affine functions, i.e. the limit functions of (9.2), and the exponential functions, i.e. the limit functions in (9.3), to which a constant is added. In the latter case any function f which satisfies (9.1) is of the form $f(x) = c + f_0(x)$ where f_0 satisfies (9.3). Thus to a great extent the study of equation (9.1) reduces to the classical theory of regular variation.

Since, as we have seen above, we may replace the norming constant b(t) in (9.2) by f(t + c) to obtain (9.4), we may as well study the norming constants instead of the function f. This point of view, applied to (9.1), leads us to consider a theory of regular variation in the group G of positive affine transformations on $\mathbb{R}$. This theory is very similar to the theory of regular variation in the additive group of the reals, based on relation (9.2), and to the theory of regular variation in the multiplicative group of the positive reals, based on relation (9.3).

It will be convenient to develop this theory of regular variation in the slightly more general setting of a topological group H with a countable base.

The countability condition ensures that the group H is a separable metrizable space, see Montgomery and Zippin [1955, section 1.22]. Although we shall not make explicit use of this metric, it allows us to work with sequences instead of filters. The theory of measure for separable metric spaces is by now well-established, see for instance Parthasarathy [1967]. In particular we shall use the well known fact that every measurable function $f : \mathbb{R} \to H$ is λ - a.e. equal to the pointwise limit of a sequence of continuous functions (where λ is Lebesgue-measure on $\mathbb{R}$, and f is measurable with respect to the Baire σ-algebras on $\mathbb{R}$ and H). Indeed, this is true if λ is the standard normal probability distribution on $\mathbb{R}$, since the simple functions, and hence the continuous functions from $\mathbb{R}$ to H are dense in the metric of convergence in probability, and every sequence which converges in probability contains an a.s. convergent subsequence.

PROPOSITION 9.1 For $\alpha : [0, \infty) \to H$ let S be the set of all $s \in \mathbb{R}$ for which

(9.6) $$\lim_{t\to\infty} \alpha(t + s)\alpha(t)^{-1} = : \psi(s)$$

exists. Then S is an additive subgroup of $\mathbb{R}$ and $\psi : S \to H$ is a homomorphism.

PROOF Clearly $0 \in S$. Suppose $s \in S$. Set $\tau := t + s$ and invert both sides of (9.6). Then $\tau \to \infty$ and

$$\alpha(\tau - s)\alpha(\tau)^{-1} \to \psi(s)^{-1}.$$

Hence $-s \in S$ and $\psi(-s) = \psi(s)^{-1}$. Similarly if $s_1, s_2 \in S$, then

$$\alpha(t+s_1+s_2)\alpha(t)^{-1} = (\alpha(t+s_1+s_2)\alpha(t+s_1)^{-1})(\alpha(t+s_1)\alpha(t)^{-1})$$

and for $t \to \infty$ (and hence $t + s_1 \to \infty$) the right hand side converges to $\psi(s_2)\psi(s_1)$. Hence $s_2 + s_1 \in S$ and $\psi(s_2 + s_1) = \psi(s_2)\psi(s_1)$.

COROLLARY If S contains a set of positive Lebesgue measure, then $S = \mathbb{R}$.

PROOF This is the theorem of Steinhaus, see Hewitt and Stromberg [1965, p. 143]. (It is a simple consequence of the fact that the set $S - S$ contains an open neighbourhood of 0.)

DEFINITION 9.2 Let $\psi : \mathbb{R} \to H$ be a homomorphism. A function $\alpha : [0, \infty) \to H$ varies like ψ if

1. α is measurable (with respect to the Baire σ-algebras)
2. for all $s \in \mathbb{R}$ one has

$$\alpha(t + s)\alpha(t)^{-1} \to \psi(s) \quad \text{for } t \to \infty.$$

A function $\alpha : [0, \infty) \to G$ is said to vary like γ, with $\gamma \in G$, $\gamma \neq \varepsilon$, if α is measurable and for all $s \in \mathbb{R}$ one has

$$\alpha(t + s)\alpha(t)^{-1} \to \gamma^s \quad \text{for } t \to \infty.$$

We now give an example which will be used later in proposition 9.7.

EXAMPLE 9.1 Let $\psi : \mathbb{R} \to H$ be a continuous homomorphism, let (s_n) be a sequence of positive reals bounded away from zero and let (γ_n) be a sequence in H which is asymptotic to $\psi(s_n)$, i.e. $\psi(s_n)\gamma_n^{-1} \to \varepsilon$, the identity in H.

Define

$$t_0 := 0 \quad , \quad t_n := s_n + s_{n-1} + \dots + s_1$$
$$\alpha(t) := \psi(s)\gamma_n\gamma_{n-1}\dots\gamma_1 \quad \text{for } t = t_n + s,\ 0 \le s < s_{n+1}.$$

Then α varies like ψ.

PROOF Suppose $x > 0$. Set $\gamma_n =: \varepsilon_n\psi(s_n)$. Then

$$\alpha(t + x)\alpha(t)^{-1} = \psi(u)\gamma_m\gamma_{m-1}\dots\gamma_1(\psi(v)\gamma_n\dots\gamma_1)^{-1} =$$
$$= \psi(u)\gamma_m\dots\gamma_{n+2}\varepsilon_{n+1}\psi(s_{n+1} - v)$$

where $t + x = t_m + u < t_{m+1}$, $t = t_n + v < t_{n+1}$ and u and v are non-negative. Hence $x = u + s_m + \dots + s_{n+1} - v$. Since the s_k are bounded away from zero, the number of factors in the last product above, $m - n + 2$, is bounded for x fixed. From lemma 9.1 below it follows that for $t \to \infty$, $\alpha(t + x)\alpha(t)^{-1}$ is asymptotic to $\psi(u)\psi(s_m)\dots\psi(s_{n+2})\psi(s_{n+1} - v) = \psi(x)$.

LEMMA 9.1 Let K be a compact subset of H and let n be a positive integer. For any neighbourhood U of ε in H, there exists a neighbourhood V of ε such that

$$\alpha_k \in K \qquad k = 1,\dots,n$$
$$\beta_k \in V\alpha_k \qquad k = 1,\dots,n$$

implies

$$\beta_n \dots \beta_1 \in U\alpha_n \dots \alpha_1. \tag{9.7}$$

PROOF Standard. For fixed $\alpha_1,\dots,\alpha_n$ existence of V such that (9.7) holds, follows from the continuity of product: $H^n \to H$. Now use the fact that K^n is a compact subset of H^n and uniformize.

PROPOSITION 9.2 Let $\psi : \mathbb{R} \to H$ be a measurable homomorphism. Then ψ is uniformly continuous on $\mathbb{R}$.

PROOF We use a simple adaptation of Banach's [1920] proof that the measurable solutions of Cauchy's functional equation $f(x + y) = f(x) + f(y)$ are continuous.

First observe that Lusin's theorem holds, there exists a compact set K with positive Lebesgue measure, λK, such that the restriction of ψ to K is continuous. Indeed, ψ being measurable, is limit λ-a.e. of a sequence of continuous functions ψ_n. See above. This sequence converges uniformly on a compact set with positive Lebesgue measure by Egorov's theorem, proof as for real-valued functions.

Let U be a neighbourhood of ε in H. There exists $\delta > 0$ such that $\psi(y)\psi(x)^{-1} \in U$ whenever $x, y \in K$ and $|x - y| < \delta$ and also such that $K - K$ contains the interval $(-\delta, \delta)$. (See proof corollary to proposition 9.1.) Hence for each $s \in (-\delta, \delta)$ there exists $x_0 \in K$ such that also $x_0 + s \in K$. Then

$$\psi(x + s)\psi(x)^{-1} = \psi(s) = \psi(x_0 + s)\psi(x_0)^{-1} \in U$$

for all $x \in \mathbb{R}$. This proves the theorem.

Of fundamental importance in the theory of regular variation is the following theorem which states that regular variation implies uniform convergence on bounded intervals. The proof given here is a variant of that given by van Aardenne-Ehrenfest, de Bruijn and Korevaar [1949] for the case that H is the additive group of the reals.

PROPOSITION 9.3 If α varies like ψ, then ψ is continuous and $\alpha(t + x)\alpha(t)^{-1} \to \psi(x)$ for $t \to \infty$ uniformly on bounded intervals.

PROOF Set $\psi_n(x) := \alpha(n + x)\alpha(n)^{-1}$. Then $\psi_n \to \psi$. The functions ψ_n are measurable by the definition of regular variation. Hence ψ is measurable and ψ is continuous by proposition 9.2.

We shall prove uniform convergence on $[-1, 1]$.

Let V be a neighbourhood of ε in H. Choose a symmetric neighbourhood U of ε such that

$$U\psi(x)U\psi(-x) \subset V \quad \text{for all } x \in [-1, 1].$$

This is possible by lemma 9.1 since ψ is continuous and hence $\{\psi(x) \mid |x| \le 1\}$ is compact.

For $t > 2$ define

$$R(t) = \{r \in [-2, 2] \mid \alpha(t)\alpha(t + r)^{-1}\psi(r) \in U\}.$$

$R(t)$ is a measurable set for each t and $\lambda R(t) \ge 3$ for $t \ge t_0$. (Indeed else $\lambda R(t_n) < 3$ for some sequence $t_n \to \infty$. However $\alpha(t)\alpha(t + r)^{-1}\psi(r) \to \varepsilon$ for all $r \in R$ implies that $[-2, 2] \subset \liminf R(t_n)$. This leads to the contradiction

$$4 = \lambda[-2, 2] \le \lambda \liminf R(t_n) \le \liminf \lambda R(t_n) \le 3.)$$

If $t \ge t_0 + 1$ and $|x| \le 1$ then there exists $r \in R(t)$ such that $r - x \in R(t + x)$. (Indeed else $R(t)$ and $x + R(t + x)$ are disjoint. This implies

$$6 \le \lambda\big(R(t) \cup (x + R(t + x)) \le \lambda([-2, 2] \cup [x-2, x+2]) \le 5.)$$

Thus we obtain

$$\begin{aligned}\alpha(t+x)\alpha(t)^{-1} &= \alpha(t+x)\alpha(t+r)^{-1}.\alpha(t+r)\alpha(t)^{-1} \\ &\in U\psi(x-r).\psi(r)U \subset V\psi(x).\end{aligned}$$

PROPOSITION 9.4 If α varies like ψ and $\beta(t) \sim \alpha(t)$ for $t \to \infty$, then β varies like ψ.

PROOF $\beta(t+s).\beta(t)^{-1} = \varepsilon(t+s)\alpha(t+s).\alpha(t)^{-1}\varepsilon(t)^{-1} \to \psi(s)$.

PROPOSITION 9.5 If α and β vary like ψ and $\alpha(n) \sim \beta(n)$, then $\alpha(t) \sim \beta(t)$ for $t \to \infty$.

PROOF Set $t = n_t + \theta_t$ where n_t is an integer and $0 \le \theta_t < 1$. Then, because of uniform convergence on $[0, 1)$

$$\beta(t) \sim \psi(\theta_t)\beta(n_t) \sim \psi(\theta_t)\alpha(n_t) \sim \alpha(t) \quad \text{as } t \to \infty.$$

PROPOSITION 9.6 (Representation theorem). If α varies like ψ there exists a sequence $\gamma_n \to \psi(1)$ such that

(9.8) $\qquad \alpha(t) \sim \psi(\theta)\gamma_n\gamma_{n-1}\cdots\gamma_1$ with $t = n + \theta$, $0 \le \theta < 1$.

PROOF Define $\gamma_n = \alpha(n)\alpha(n-1)^{-1}$ and $\gamma_1 = \alpha(1)$. The right hand side of (9.8) varies like ψ (as was proved in the example earlier in this chapter). Since (9.8) is an equality for integral values of $t > 0$ the result follows from proposition 9.5.

In proposition 7.2 it was shown that the set Δ contains a one-parameter subgroup $G(\gamma) = \{\gamma^t \mid t \in \mathbb{R}\}$ with $\gamma \neq \varepsilon$, if $\alpha_n \to \infty$ and $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$. In proposition 9.7 below we shall see that if Δ is equal to this one-parameter subgroup $G(\gamma)$ of G, then the sequence (α_n) can be embedded in a function α from $[0, \infty)$ to G which varies like γ or like γ^{-1}.

PROPOSITION 9.7 Suppose that

H is a locally compact topological group with a countable base,

(α_n) is a divergent sequence in H (i.e. any compact subset of H contains only finitely many elements of the sequence), such that the sequence $(\alpha_{n+1}\alpha_n^{-1})$ is relatively compact,

L is a subgroup of H which is isomorphic to the additive topological group $\mathbb{R}$.

If every limit point of the double sequence $(\alpha_n\alpha_m^{-1})$ lies in L, then there exist

an isomorphism $\psi : \mathbb{R} \to L$,
a function $\alpha : [0, \infty) \to H$ which varies like ψ,
a sequence $x_n \to \infty$ such that $\alpha_n = \alpha(x_n)$ for $n = 1,2,\ldots$

PROOF Let ψ be an isomorphism $\mathbb{R} \to L$. (Then any other isomorphism ψ_0 necessarily has the form $\psi_0(t) = \psi(ct)$ for some $c \neq 0$.) For $t > 0$ we define

$$L(t) := \{\psi(s) \mid |s| \le t\}.$$

We may and do assume that ψ is chosen so that all limit points of $\alpha_{n+1}\alpha_n^{-1}$

lie in $L(1)$.

If the points α_n of our sequence were to lie on L, then we could write $\alpha_n = \psi(t_n)$ with $t_n \in \mathbb{R}$. Since we have in fact assumed that $\limsup |t_{n+1} - t_n| \leq 1$ and that α_n and hence t_n diverges, either $t_n \to \infty$ (or $t_n \to -\infty$). Then $\alpha(t) = \psi(t)$ (or $\alpha(t) = \psi(-t)$) would be the desired function. In the general case the construction of α is more complicated and it is convenient to select first a subsequence n_k such that the corresponding subsequence of (t_n) is strictly increasing and such that the sequence of successive differences is bounded away from zero.

The construction makes use of the fact that for any neighbourhood U of ε and any compact set $K \subset H$ there exists an integer k such that

$$(9.9) \qquad \alpha_n\alpha_m^{-1} \in K \text{ implies } \alpha_n\alpha_m^{-1} \in UL \text{ if } n, m \geq k.$$

(Indeed else (for U open) the double sequence $\alpha_n\alpha_m^{-1}$ restricted to K would have a limit point in $K \setminus UL$.)

We shall now specify U, K and k.

Let U_1 be a compact symmetric neighbourhood of ε such that $U_1^2 \cap L \subset L(\frac{1}{2})$. We choose a compact symmetric neighbourhood U of ε in U_1 such that

$$(9.10) \qquad U\gamma_1 U\gamma \subset U_1\gamma_1\gamma$$

for all $\gamma_1 \in L(3)$, see lemma 9.1 above, and we define $K := U_1L(3)$ and k such that (9.9) holds and

$$(9.11) \qquad \alpha_{n+1}\alpha_n^{-1} \in UL(1) \qquad \text{for } n \geq k.$$

Consider subsets $B = \{\beta_j \mid j \in J\} \subset \{\alpha_k, \alpha_{k+1}, \ldots\}$ indexed by a set J of consecutive integers (not necessarily non-negative), such that

$$\beta_0 = \alpha_k$$
$$\beta_j \in U\psi(c_j)\beta_{j-1} \qquad \text{for some } c_j \in [2, 3\tfrac{1}{2}] \text{ where } j-1, j \in J.$$

The class of all such subsets B is ordered in a natural way, $B \subset B'$ if $J \subset J'$ and $\beta_j' = \beta_j$ for all $j \in J$. Let B be maximal, i.e. $B \subset B'$ implies $B' = B$. We prove

(9.12) if $n \geq k$ and $\alpha_n \in UL(1)UL(2)B$, then $\alpha_n \in UL(2)B$.

Indeed suppose

$$\alpha_n = \varepsilon_1\psi(p)\varepsilon_2\psi(q)\beta_j$$

with ε_1, $\varepsilon_2 \in U$, $|p| \leq 1$, $|q| \leq 2$. There exist $\varepsilon_0 \in U_1$ and $\varepsilon_3 \in U$ such that

(9.13a) $\alpha_n\beta_j^{-1} = \varepsilon_0\psi(p + q)$ by (9.10)

(9.13b) $\alpha_n\beta_j^{-1} = \varepsilon_3\psi(r)$ by (9.9).

If $|r| \leq 2$, then $\alpha_n = \varepsilon_3\psi(r)\beta_j \in UL(2)B$ and (9.12) is proved. Hence suppose $r > 2$. Now (9.13a,b) gives $\psi(p + q - r) = \varepsilon_0^{-1}\varepsilon_3 \in U_1^2$. Hence $|p + q - r| \leq \frac{1}{2}$ by definition of U_1. This implies $2 < r \leq 3\frac{1}{2}$. In particular j is not the maximal element of J since then $B \cup \{\alpha_n\}$ would be an extension of B and B is supposed to be maximal. Hence β_{j+1} exists and we may consider

$$\begin{aligned} \alpha_n\beta_{j+1}^{-1} &= \alpha_n\beta_j^{-1}\psi^{-1}(c_{j+1})\varepsilon_4 && \text{with } \varepsilon_4 \in U \\ &= \varepsilon_3\psi(r - c_{j+1})\varepsilon_4 && \text{by (9.13b)} \\ &= \varepsilon_5\psi(r - c_{j+1}) && \text{with } \varepsilon_5 \in U_1 \text{ by (9.10)} \\ &= \varepsilon_6\psi(s) && \text{with } \varepsilon_6 \in U \text{ by (9.9).} \end{aligned}$$

As above we find $|r - c_{j+1} - s| \leq \frac{1}{2}$ hence $|s| \leq 2$ and $\alpha_n \in UL(2)B$. For $r < -2$ the proof is similar, and we obtain $\alpha_n\beta_{j-1}^{-1} \in UL(2)$.

In fact we have proved more than (9.12). If $\alpha_n \in UL(1)UL(2)\beta_j$ and $n \geq k$, then α_n lies in at least one of the three sets $UL(2)\beta_j$, $UL(2)\beta_{j+1}$ or $UL(2)\beta_{j-1}$. For each $n \geq k$ we now choose an integer $j(n) \in J$ such that $\alpha_n \in UL(2)\beta_{j(n)}$ and $|j(n + 1) - j(n)| \leq 1$. This is possible since $n \geq k$ and $\alpha_n \in UL(2)\beta_j$ imply that $\alpha_{n+1} = (\alpha_{n+1}\alpha_n^{-1})\alpha_n \in UL(1)UL(2)\beta_j$.

Since $UL(2)\beta_j$ is compact for each j, it contains only finitely many terms of the sequence (α_n) and hence $j(n) \to \infty$ or $j(n) \to -\infty$. In the latter case we replace ψ by ψ_* where $\psi_*(t) = \psi(-t)$ in the foregoing. Then $j(n) \to \infty$. Hence we may and do assume that J has the form $J = \{j_0, j_0+1, \ldots\}$ and we may assume that $j_0 = 0$ by an appropriate choice of k.

Relation (9.9) implies that a relatively compact sequence of quotients $\alpha_p\alpha_q^{-1}$ is asymptotically equal to a sequence $\psi(s_n)$ where (s_n) is bounded.

We apply this to the sequence $\beta_n\beta_{n-1}^{-1}$ and obtain $\beta_n\beta_{n-1}^{-1} \sim \psi(s_n)$ with $1\frac{1}{2} \le s_n \le 4$. (Since $|s_n - c_n| \le \frac{1}{2}$.) Hence the function

$$\beta(t) := \psi(s)\beta_n \qquad \text{for } t = t_n + s,\ 0 \le s < s_{n+1}$$

and with $t_n = s_1 + \dots + s_n$, varies like ψ. (See example 9.1.)

We now define $\alpha(t)$.

Since the sequence $\beta_{j(n)}\alpha_n^{-1}$ is relatively compact we have as above

$$\alpha_n \sim \psi(p_n)\beta_{j(n)} \qquad \text{with } |p_n| \le 2\tfrac{1}{2}. \tag{9.14}$$

We may choose p_n such that the numbers $x_n := p_n + t_{j(n)}$ are positive, distinct for distinct α_n and equal for equal α_n. Then $x_n \to \infty$ and by (9.14) we obtain, since β varies like ψ,

$$\alpha(x_n) := \alpha_n \sim \psi(p_n)\beta_{j(n)} = \psi(p_n)\beta(t_{j(n)}) \sim \beta(x_n). \tag{9.15}$$

Now let $y_1, y_2, \dots$ be a non-decreasing rearrangement of the sequence (x_n) and define $\alpha : (0, \infty) \to H$ by

$$\alpha(y) := \psi(p)\alpha(y_n) \qquad \text{for } y = y_n + p < y_{n+1},\ \ p \ge 0.$$

Then

$$\alpha(y) = \psi(p)\alpha(y_n) \sim \psi(p)\beta(y_n) \qquad \text{by (9.15)}$$
$$\sim \beta(p + y_n) = \beta(y)$$

and since β varies like ψ so does α (by proposition 9.4).

This proves the proposition.

REMARK 1 The sequence $(x_{n+1} - x_n)$ is bounded. (Indeed $1\frac{1}{2} \le s_n \le 4$ and $|p_n| \le 2\frac{1}{2}$, see (9.14).)

REMARK 2 If $H = G$ is the group of positive affine transformations on $\mathbb{R}$, we may choose the function α to be continuous.

PROOF Since α varies like ψ and $(y_{n+1} - y_n)$ is bounded, we have

$$\alpha(y_{n+1})\alpha(y_n)^{-1} \sim \psi(y_{n+1} - y_n).$$

We define

$$\tilde{\alpha}(y) := (\alpha(y_{n+1})\alpha(y_n)^{-1})^{\theta}\alpha(y_n) \qquad \text{for } y = y_n + \theta(y_{n+1} - y_n)$$

with $0 \le \theta \le 1$. Then

$$\tilde{\alpha}(y)\alpha(y_n)^{-1} \sim (\psi(y_{n+1} - y_n))^{\theta} = \psi(y - y_n)$$

and

$$\tilde{\alpha}(y) \sim \psi(y - y_n)\alpha(y_n) \sim \alpha(y).$$

REMARK 3 In the statement of proposition 9.7 we need only assume that L is the image of a continuous injective homomorphism $\psi_0 : \mathbb{R} \to H$.

PROOF If $\psi_0(\mathbb{R})$ is closed, then ψ_0 is an isomorphism, see Pontrjagin [1957, Satz 12]. Else the closure of $\psi_0(\mathbb{R})$ is compact, see Pontrjagin [1957, Par. 39, Hilfsatz 1]. Let W be a relatively compact open neighbourhood of the closure of $\psi_0(\mathbb{R})$, and A a compact set containing ε, such that $\alpha_{n+1}\alpha_n^{-1} \in A$ for $n \ge n_0$. Set $K = A\overline{W}$. For each $m \ge n_0$ there exists m' such that $\alpha_{m'}\alpha_m^{-1} \in K \setminus W$. (Indeed $\alpha_m\alpha_m^{-1} = \varepsilon \in K$ and $\alpha_n\alpha_m^{-1} \to \infty$ for $n \to \infty$ implies that there exists a least integer $m' \ge m$ such that $\alpha_{m'+1}\alpha_m^{-1} \notin K$. If $\alpha_{m'}\alpha_m^{-1} \in W$, then $\alpha_{m'+1}\alpha_m^{-1} \in AW$. Hence $\alpha_{m'}\alpha_m^{-1} \in K \setminus W$.) Since $K \setminus W$ is compact, we find $\Delta \cap (K \setminus W)$ is non-empty. This contradicts $\Delta \subset \psi(\mathbb{R}) \subset W$.

EXAMPLE 9.2 Let H be Hilbert space with the orthonormal base $e_1, e_2, \ldots$ and define

$$\alpha_n = \sum_{k=1}^{n} \frac{e_k}{\sqrt{k}}.$$

Then α_n diverges and Δ, the set of limit points of $\alpha_n - \alpha_m$ is $\{0\}$, since $\alpha_n - \alpha_m \perp e_1, \ldots, e_m$ for $n \ge m$.

EXAMPLE 9.3 Let H be the multiplicative group of complex numbers $\ne 0$, and $\alpha(t) = te^{2\pi it}$, $t \ge 0$. Then α varies like ψ, where $\psi(t) = e^{2\pi it}$ and $\Delta = H$.

EXAMPLE 9.4 Suppose $\alpha_n x = (x - n)/\sqrt{n}$. Then Δ is the one-parameter subgroup of all translation. Set

$$\alpha_t x = (x - t)/\sqrt{t} \qquad \text{for } t \ge 1,$$

then,

$$\alpha_{t+s}\alpha_t^{-1}x = \frac{\sqrt{t}}{\sqrt{t + s}}\, x - \frac{s}{\sqrt{t + s}}$$

and evidently α_t does not vary like a translation. However, a change of variable yields a function which does vary like a translation. We set

$$\alpha(t)x = (x - t^2)/t \qquad \text{for } t \geq 1,$$

then

$$\alpha(t + s)\alpha(t)^{-1}x = \frac{t}{t + s}\, x - \frac{(t + s)^2 - t^2}{t + s} \to x - 2s.$$

For a more detailed analysis see de Haan [1970, section 2.5].

EXAMPLE 9.5 H is the group of complex affine transformations, $\gamma z = az + b$, with a and b complex numbers and $a \neq 0$. Set

$$\beta_n z = w_n z + 1 \qquad \text{with } w_n = \exp\,(2\pi i/n).$$

Then

$$\beta_n^k z = w_n^k z + w_n^{k-1} + \ldots + w_n + 1$$

and since $(w_n^{n-1} + w_n^{n-2} + \ldots + 1)(w_n - 1) = w_n^n - 1 = 0$, we have $\beta_n^n = \varepsilon$. Now define (γ_n) by

$$\gamma_n = \beta_k \qquad \text{for } n = k^2 + j \text{ with } -k < j \leq k,$$

and set

$$\alpha(t) = \gamma_{n+1}^{(\theta)}\gamma_n \ldots \gamma_1 \qquad \text{for } t = n + \theta \text{ with } 0 \leq \theta < 1$$

where $\beta_k^{(\theta)}z = z.\exp(2\pi i\theta/k) + \theta$. Then α is continuous, varies like a translation, see example 9.1, but $\alpha(n) \not\to \infty$ since $\alpha(k^2) = \varepsilon$ for $k = 1,2,\ldots$

LEMMA 9.2 Suppose $\gamma \in G$, $\gamma x = ax + b$ and $M > 1$. If $a \geq 1 - (4M)^{-1}$ and $b \geq \frac{1}{2}$, then

$$0 \leq x \leq M \text{ implies } \gamma x \geq x + \tfrac{1}{4}$$
$$M \leq x \text{ implies } \gamma x \geq M.$$

PROOF Trivial.

PROPOSITION 9.8 Suppose $\alpha : [0, \infty) \to G$ varies like γ, where $\gamma \in G$, $\gamma \neq \varepsilon$. If (t_n) and (s_n) are sequences of positive numbers and $(\alpha(t_n)\alpha(s_n)^{-1})$ is bounded, then so is $(t_n - s_n)$.

PROOF Suppose $t_n - s_n \to \infty$. We prove that then $\alpha(t_n)\alpha(s_n)^{-1} \to \infty$. Because of proposition 9.3 it suffices to prove this for integer sequences (t_n) and (s_n).

Hence set $\alpha_{n+1}\alpha_n^{-1} = \gamma_n$. Then $\gamma_n \to \gamma$.

If $\gamma x = ax + b$, with $a < 1$, then $\gamma_n x = a_n x + b_n$ and $a_n \le q < 1$ for $n \ge n_0$. Hence $\alpha(t_n)\alpha(s_n)^{-1}x = c_n x + d_n$ with $\log c_n \le (t_n - s_n)\log q + O(1) \to \to \neq\infty$ and hence $\alpha(t_n)\alpha(s_n)^{-1} \to \infty$. Similarly if $a > 1$.

Now suppose γx is a translation. For convenience assume $\gamma x = x + 1$. Then $\gamma_n x = a_n x + b_n$ with $b_n \ge \frac{1}{2}$ for $n \ge n_1$ and $a_n \to 1$. Thus $\alpha(t_n)\alpha(s_n)^{-1}0 \ge 0$ for $t_n \ge s_n \ge n_1$ and lemma 9.2 yields that for any $M > 1$ we have

$$\alpha(t_n)\alpha(s_n^*)^{-1}0 \ge M$$

if $s_n^* = \max(s_n, n_1)$, $t_n \ge n_2 + 4M$, $t_n - s_n \ge 4M$ where n_2 is chosen so that $a_n \ge 1 - (4M)^{-1}$ for $n \ge n_2$. Since n_1 is fixed and M arbitrary, this implies that $\alpha(t_n)\alpha(s_n)^{-1} \to \infty$.

COROLLARY If $\alpha : [0, \infty) \to G$ varies like γ, with $\gamma \in G$, $\gamma \neq \varepsilon$, then $\Delta = \{\gamma^t \mid t \in \mathbb{R}\}$, and $\alpha(t) \to \infty$ for $t \to \infty$.

The group G can be represented by the matrix group $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$ and b real. Hence one can talk about differentiable functions of $\mathbb{R}$ into G.

PROPOSITION 9.9 Let H be a locally compact matrix group and A a continuous function from $[0, \infty)$ into the Lie algebra H_0 of H, such that $A(t) \to A_0$ for $t \to \infty$. Let $\alpha : [0, \infty) \to H$ satisfy the differential equation

$$\frac{d}{dt}\alpha(t) = A(t)\alpha(t) \qquad \alpha(0) = \alpha_0,$$

and let $\psi : \mathbb{R} \to H$ satisfy

$$\frac{d}{dt}\psi(t) = A_0\psi(t) \qquad \psi(0) = \varepsilon.$$

Then ψ is a homomorphism and α varies like ψ.

PROOF Let $A_n : \mathbb{R} \to H_0$ be a sequence of continuous functions such that

(9.16) $A_n \to A_0$ uniformly on bounded intervals.

Let $\alpha_n : \mathbb{R} \to H$ satisfy the differential equation

$$\frac{d}{dt} \alpha_n(t) = A_n(t)\alpha_n(t) \qquad \alpha_n(0) = \varepsilon.$$

Then (9.16) implies that $\alpha_n \to \psi$ uniformly on bounded intervals. See Dieudonné [1969, (10.7.2)], Pontrjagin [1958, Satz 58].

Consider $\alpha(p + t)\alpha(p)^{-1} =: \beta_p(t)$. This function satisfies the differential equation

$$\frac{d}{dt} \beta_p(t) = A(p + t)\beta_p(t)$$

and since $A_p(t) := A(p + t) \to A_0$ uniformly on bounded intervals of $\mathbb{R}$ for $p \to \infty$, we have $\beta_p \to \psi$ uniformly on bounded intervals for $p \to \infty$.

PROPOSITION 9.10 Suppose $\alpha : [0, \infty) \to G$ varies like ψ. Then there exists $\beta : [0, \infty) \to G$ such that

$$\beta \text{ is } C^\infty$$

$$\beta(t) \sim \alpha(t) \qquad \text{for } t \to \infty$$

$$B(t) = (\frac{d}{dt} \beta(t))\beta(t)^{-1} \to \frac{d}{dt} \psi(0) = B_0 \qquad \text{for } t \to \infty$$

$$\frac{d}{dt} B(t) \to 0 \qquad \text{for } t \to \infty.$$

PROOF For each continuous homomorphism ψ there exists a probability measure with density $m(s)$ such that $m(s)$ is C^∞, vanishes for $|s| \geq 1$ and satisfies

(9.17) $\int \psi(-s)m(s)ds = \varepsilon.$

Indeed this is obvious for $\psi(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} e^{\lambda s} & 0 \\ 0 & 1 \end{pmatrix}$ and hence it is true for all ψ. Note that (9.17) implies that

(9.18) $\psi * m = \psi$

where $(\psi * m)(t) := \int \psi(t - s)m(s)ds$.

Since we are only interested in the behaviour of $\alpha(t)$ for $t \to \infty$ and since $\alpha(t+s)\alpha(t)^{-1} \to \psi(s)$ uniformly on bounded s-intervals for $t \to \infty$, we may as well assume that α is locally integrable and that $\alpha(t) = \varepsilon$ for $t \leq 0$. Then $\beta := \alpha * m$ is C^{∞} and is an element of G for all t. Moreover

$$\beta(t)\alpha(t)^{-1} = \int \alpha(t-s)\alpha(t)^{-1}m(s)ds \to \varepsilon \qquad \text{for } t \to \infty$$

since

$$\alpha(t-s)\alpha(t)^{-1} \to \psi(-s) \quad \text{uniformly for } |s| \leq 1$$

and hence $\beta(t) \sim \alpha(t)$ for $t \to \infty$.

Also

$$\dot{\beta}(t) = \frac{d}{dt}\int \alpha(s)m(t-s)ds = \int \alpha(s)\dot{m}(t-s)ds = \\ = \int \alpha(t-s)\dot{m}(s)ds$$

and

$$\dot{\beta}(t)\alpha(t)^{-1} = \int \alpha(t-s)\alpha(t)^{-1}\,\dot{m}(s)ds \\ \to \int \psi(-s)\dot{m}(s)ds = \dot{\psi}(0) \qquad \text{for } t \to \infty$$

by differentiation of (9.18).

Hence

$$B(t) = \dot{\beta}(t)\beta(t)^{-1} \to \dot{\psi}(0) = B_0$$

Similarly $\ddot{\beta} = \alpha * \ddot{m} = \dot{B}\beta + B^2\beta$, and

$$\ddot{\beta}(t)\beta(t)^{-1} \to \ddot{\psi}(0) = B_0^2 \qquad \text{for } t \to \infty$$

which implies $\dot{B}(t) \to 0$ for $t \to \infty$.

REMARK If above we define the density m so that $\int \psi(s)m(s)ds = \varepsilon$, and $\gamma^{-1} := \alpha^{-1} * m$, then γ is C^{∞},

$$\alpha(t)\gamma^{-1}(t) = \int \alpha(t)\alpha(t-s)^{-1}m(s)ds \to \varepsilon \qquad \text{for } t \to \infty$$

$$\alpha(t)\frac{d}{dt}\,\gamma^{-1}(t) \to \int\psi(s)m(s)ds = \frac{d}{dt}\,\psi^{-1}(0) \qquad \text{for } t \to \infty.$$

10 The functional equation $h(x + p) - h(x) = C.(h(x + 1) - h(x))$

This chapter treats the basic situation (1.1) under the conditions that $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$ and that Δ is the one-parameter subgroup of all translations. In the previous chapter, proposition 9.7, we have seen that we may then embed the sequence (α_n) in a continuous function $\alpha : [0, \infty) \to G$ which varies like a translation. Equation (8.5) now takes the particularly simple form of a difference equation. Every limit point h in M of $g_t = \beta(t)f\alpha(t)^{-1}$ for $t \to \infty$ has to satisfy the functional equation in the heading of this chapter (unless $\Lambda \subset \phi$ for some $\phi \in M_0$).

The chapter falls apart in three sections. This is best illustrated with the particular case that Λ contains three points (0, 0), (θ, θ) and (1, 1) with $0 < \theta < 1$ and θ irrational. We first prove, if $\Lambda \subset h$, then the identity is the only solution of the associated difference equation (10.1), in the second section we show that this implies that the identity is the only limit point in M of g_t for $t \to \infty$, and in the third section it is shown that this implies that also $\beta(t)$ varies like a translation.

PROPOSITION 10.1 Let θ be an irrational number and let h be a continuous non-negative function on $\mathbb{R}$ which satisfies the functional equation

$$(10.1) \qquad \theta(h(x + 1) - h(x)) = h(x + \theta) - h(x).$$

Then h is constant.

PROOF The functional equation (10.1) states that the three points $(x, h(x))$, $(x + \theta, h(x + \theta))$ and $(x + 1, h(x + 1))$ are collinear. From the geometric picture it follows that we may assume that $\theta \in (0, 1)$. Let L_x be the line segment in $\mathbb{R}^2$ with endpoints $(x, h(x))$ and $(x + 1, h(x + 1))$, let S be the band

$$S := \bigcup_{x \in \mathbb{R}} L_x$$

swept out in the upper halfplane by moving the line segment L_x along the graph, and let $\psi(x) := \inf\{t \mid (x, t) \in S\}$ be the lower edge of this band.

It is not difficult to see that S is closed and that ψ is continuous. (For fixed y define $h(y, x)$ to be continuous, linear on $[y, y+1]$ and equal to h(x) outside this interval [y, y+1]. Then $\psi(x) = \min_y h(y, x)$. Since $h(y, x)$ is continuous on $\mathbb{R}^2$ and as a function in y, for x_0 fixed, $h(y, x_0)$ is constant $= h(x_0)$ for $y \notin (x_0-1, x_0)$, the function ψ is continuous.)

Now suppose $y \in \mathbb{R}$. The point $(y, \psi(y))$ lies on some line segment L_x. It even lies in the interior of some line segment L_x. (If it is an endpoint, then $\psi(y) = h(y)$ and $(y, \psi(y))$ is interior point of $L_{y-\theta}$.) Since L_x lies above the graph of ψ (by definition of ψ), the function ψ is concave. (Indeed, suppose A is affine and $\psi(x_i) = A(x_i)$ for i = 1, 2 and $\psi(x) < A(x)$ for some $x \in (x_1, x_2)$. Then $\psi(x) - A(x)$ attains a negative minimum in $x_0 \in (x_1, x_2)$. Choose x_0 minimal. Then obviously $(x_0, \psi(x_0))$ cannot be interior point of a line segment which lies above or on the graph of ψ. Hence we see that the equation $\psi(x_i) = A(x_i)$ for i = 1, 2 implies $\psi(x) \geq A(x)$ on (x_1, x_2), i.e. ψ is concave.) However, ψ is also non-negative. If follows that ψ is constant.

Define E by

$$E = \{x \in \mathbb{R} \mid \psi(x) = h(x)\}.$$

Suppose again $y \in \mathbb{R}$ and $(y, \psi(y))$ is an interior point of the line segment L_x. Since ψ is constant, the whole line segment L_x lies in the graph of ψ. In particular x, $x + \theta$ and $x + 1$ lie in E. In general if $y = u + \theta \in E$, then also $y - \theta$ and $y + 1 - \theta \in E$.

We now use the fact that θ is irrational to conclude that E is dense in R, and hence $E = \mathbb{R}$ (h and ψ being continuous) and $h = \psi$ is constant.

REMARK If θ is rational, say $\theta = pq^{-1}$ with (p, q) = 1 then h is periodic modulo q^{-1}.

PROOF The proof is similar to the proof above except that we consider the restriction of h to some coset $\{a + kq^{-1} \mid k \text{ integral}\}$. Note that now we do not need any continuity properties of h.

Equation (10.1) is a simple case of the homogeneous linear difference equation with constant coefficients

$$(10.2) \qquad \sum_{k=0}^{n} c_k h(x + \theta_k) = 0$$

which has been investigated in the more general setting of a linear difference-differential equation by Hilb [1918]. See also Bellman and Cooke [1963, p. 215] and Doetsch [1956, Kapitel 22] for further references.

We mention a few simple properties of the solution of (10.2). For convenience we assume that $0 = \theta_0 < \theta_1 < \dots < \theta_n = 1$ and $c_0 c_1 \dots c_n \neq 0$.

If h_0 is an arbitrary function on $[0, 1)$ then there exists a unique extension to a solution h of (10.2) on $\mathbb{R}$.

If a solution of (10.2) is continuous on $[0, 1]$ it is continuous everywhere.

The set of solutions of (10.2) is a linear space. It is closed for pointwise limits and for translations. This implies that for any locally integrable solution h, also $h * \psi$ is a solution, where ψ is a continuous function with compact support. In particular any locally integrable solution may be approximated by C^∞ solutions. (If ψ is C^∞, then so is $h * \psi$.)

If h is bounded on $[0, 1)$ then it is of finite exponential growth, i.e. there exist constants M and C such that

$$(10.3) \qquad |h(x)| \le Me^{C|x|} \qquad \text{for all } x \in \mathbb{R}.$$

Hence in this case (if h is bounded on $[0, 1)$ and measurable) we may define the Laplace transform

$$\tilde{h}(s) = \int_0^\infty h(x)e^{-sx}dx$$

and the integral converges absolutely for all s with $\operatorname{Re} s > C$ by (10.3). On taking Laplace transforms of both sides of (10.2) we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{n} c_k \int_0^\infty h(x+\theta_k)e^{-sx}dx \\ &= \sum_{k=0}^{n} c_k e^{s\theta_k} \int_{\theta_k}^\infty h(y)e^{-sy}dy \\ &= \sum_{k=0}^{n} c_k e^{s\theta_k}\,\tilde{h}(s) - \sum_{k=0}^{n} c_k e^{s\theta_k} \int_0^{\theta_k} h(y)e^{-sy}dy \\ &= \lambda(s).\tilde{h}(s) - \gamma(s) \quad \text{say.} \end{aligned}$$

The Laplace transform has the simple form

$$(10.4) \qquad \tilde{h}(s) = \frac{\gamma(s)}{\lambda(s)}$$

where $\lambda(s) = \sum_{k=0}^{n} c_k e^{s\theta_k}$ is an exponential polynomial all of whose zero's lie in some vertical strip $x_1 \le \mathrm{Re}\, s \le x_2$ and which converges to c_0 as $\mathrm{Re}\, s \to -\infty$.

By using contour integration and the well known inversion formula, with $c_1 > C$, where C is the constant in (10.3), see Widder [1946, p. 69],

$$H(x) := \int_0^x h(t)dt = \lim_{t\to\infty} \frac{1}{2\pi i} \int_{c_1-it}^{c_1+it} \frac{\tilde{h}(s)}{s} e^{sx} ds$$

we obtain a series development of H of the form

$$H(x) \sim \sum_{z_k} a_k e^{z_k x}$$

where z_k runs through the zeros of $z\lambda(z)$ and $a_k e^{z_k x}$ is the residue of the integrand in z_k.

The characteristic function λ of the difference equation in the heading of this chapter is

$$\lambda(s) = Ce^{s} - e^{sp} + 1 - C.$$

The function λ has two real zeros, $s_0 = 0$ and s_1. This yields two non-decreasing solutions $h(x)$, viz.

$$h(x) = 1$$
$$h'(x) = e^{s_1 x}.$$

We shall see, proposition 10.3, that for irrational p any non-decreasing solution of the difference equation is a linear combination of these two solutions.

Let us now first consider the following variant of (10.1),

$$\frac{h(t) - h(at)}{1 - a} = \frac{h(bt) - h(t)}{b - 1}, \tag{10.5}$$

with $0 < a < 1 < b$ and h continuous and non-decreasing on $(0, \infty)$. The points $(at, h(at))$, $(t, h(t))$ and $(bt, h(bt))$ on the graph of h are collinear. As in the proof of proposition 10.1 we introduce line segments. Here the line segment $L(t)$ has endpoints $(at, h(at))$ and $(bt, h(bt))$. Hence $(t, h(t))$ is an interior point of $L(t)$. We define the band

$$S = \bigcup_{t>0} L(t).$$

As in the proof of proposition 10.1 it follows that the functions

(10.6a) $\psi(x) := \min \{y \mid (x, y) \in S\}$

(10.6b) $\phi(x) := \max \{y \mid (x, y) \in S\}$

are well defined and continuous on $(0, \infty)$. Moreover ψ is concave, ϕ is convex and

$$\psi(t) \leq h(t) \leq \phi(t) \quad \text{on } (0, \infty).$$

The proof that h is affine (if log a/log b is irrational), is somewhat more involved than in the case of equation (10.1). The function ψ is now only defined on the half line $(0, \infty)$. The main idea of the proof is as follows.

The function h cannot fluctuate too much since it is monotone. For large values of t the function h will approach ψ (and ϕ) closely at rather regular intervals. If $h(t)$ is close to $\psi(t)$, then also both endpoints of $L(t)$ will lie near the graph of ψ since the line segment $L(t)$ does not intersect the graph of ψ. Hence also $h(at)$ and $h(bt)$ lie near $\psi(at)$ and $\psi(bt)$. Similarly for $h(a^2t)$, $h(abt)$, $h(b^2t)$ and more generally for $h(a^m t)$, $h(a^{m-1}bt), \ldots, h(b^m t)$. For large values of m the fixed interval (a^2t, b^2t) contains a large number of points $a^k b^l t$ with k and l non-negative integers and $k + l \leq m$. Moreover the distance between successive points in this interval will be small (independently of t if we work with a logarithmic scale). This implies that h is close to ψ throughout the interval (a^2t, b^2t), and hence so is ϕ. In particular we show that $t^{-1}(\phi(t) - \psi(t)) \to 0$ for $t \to \infty$. Since also $\phi(t) - \psi(t) \to C$ for $t \to 0+$, it follows that $\phi = \psi$ is affine.

PROPOSITION 10.2 Let h be a continuous non-decreasing function on $(0, \infty)$ which satisfies the functional equation

$$(10.5) \qquad \frac{h(x) - h(ax)}{1 - a} = \frac{h(bx) - h(x)}{b - 1}$$

with $0 < a < 1 < b$ and log a/log b irrational. Then h is affine.

PROOF We define $L(x)$ as the line segment with endpoints $(ax, h(ax))$ and $(bx, h(bx))$, and $S = \cup\{L(x) \mid x > 0\}$. Then ψ, the lower edge of S, is concave, and ϕ, the upper edge of S, is convex. See above, equation (10.6a).

We define the constant $c := ba^{-1}$. Then $c > 1$.

The proof consists of eight parts. Our main tool will be the implication A.

A. If $h(x) \geq A(x)$ on $(c^{-1}u, cu)$ for some affine function A, then $\psi(u) \geq A(u)$. (Similarly $h \leq A$ on $(c^{-1}u, cu)$ implies $\phi(u) \leq A(u)$.)

Proof of A. Suppose $(u, v) \in S$. Then (u, v) lies on a line segment $L(w)$ both of whose endpoints lie on or above the graph of A. Hence $v \geq A(u)$. Then also $\psi(u) = \inf\{v \mid (u, v) \in S\} \geq A(u)$.

B. The functions ψ and ϕ are non-decreasing.

Indeed $\psi(x) = \inf\{h(y, x) \mid y > 0\}$ where for fixed $y > 0$ the function $h(y, x)$ is the continuous non-decreasing function which is affine on (ay, by) and equal to $h(x)$ outside this interval. (Compare the proof of the continuity of ψ in proposition 10.1.) Similarly ϕ is non-decreasing as a limit of non-decreasing functions.

C. $\psi(0) = h(0) = \phi(0)$ is finite.

The functions ψ and ϕ are non-decreasing by B hence the limits for $x \to 0+$ exist. The limits may equal $-\infty$. By definition we have

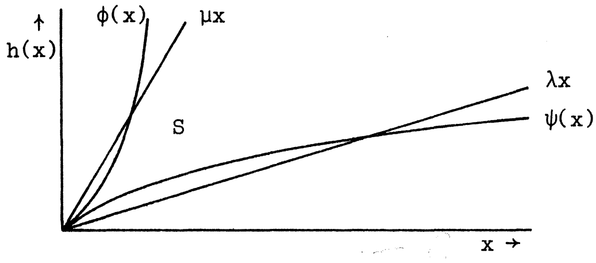
$$\psi(0) \leq h(0) \leq \phi(0).$$

Using part A with a constant affine function $A(x) = c_0 > h(0)$ and u sufficiently small we obtain $h(x) \leq A(x)$ on $(0, cu)$ and hence $\phi(0) \leq \phi(u) \leq A(u) = c_0$. Similarly $\psi(0) = h(0)$. Finally ϕ convex implies $\phi(0) > -\infty$.

D. We may and shall assume that $\psi(0) = h(0) = \phi(0) = 0$.

The solutions of (10.5) form a linear space which contains the constant functions.

We shall now consider halflines λx, with λ positive, which intersect the band S. (See illustration.)



E. Suppose $\psi(x) < \lambda x < \mu x < \phi(x)$ for some $x > 0$. Then $\mu < c^2\lambda$.

Proof of E. Since ψ is concave and ϕ is convex, we have

$$\psi(t) < \lambda t < \mu t < \phi(t) \quad \text{for } t \geq x.$$

By part A with $A(x) = \lambda x$ or $A(x) = \mu x$ we find that each interval (y_1, y_2) with $y_2 \geq c^2 y_1$ and $y_1 \geq x$ contains points x_1 and x_2 with $h(x_1) < \lambda x_1$ and $h(x_2) > \mu x_2$. Now choose y_1 and y_2 such that

$$x \leq y_1 < y_2$$
$$\lambda y_2 = \mu y_1$$
$$h(y_2) < \lambda y_2.$$

For each $y \in (y_1, y_2)$ we have

$$h(y) \leq h(y_2) < \lambda y_2 = \mu y_1 < \mu y.$$

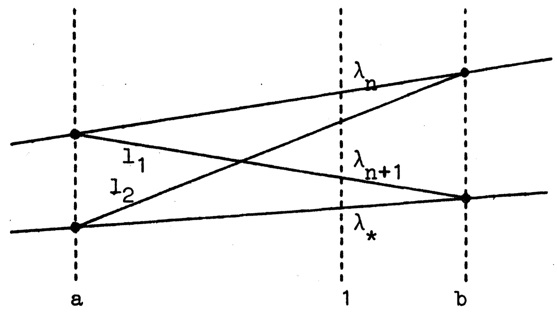
By the argument above $y_2 < c^2 y_1$. Hence also $\mu\lambda^{-1} = y_2 y_1^{-1} < c^2$.

F. Construction of the sequence (λ_n).

Set $\lambda_* := \inf\{\lambda \geq 0 \mid \lambda x \text{ intersects } S\}$. Suppose there exists $\lambda_0 > \lambda_*$ such that $\lambda_0 x$ intersects S. We define the sequence $\lambda_n \downarrow \lambda_*$ inductively. Suppose $\lambda_0, \ldots, \lambda_n$ have been defined.

Let $l_1(x)$ be the affine function on $[a, b]$ with the values $l_1(a) = \lambda_n a$ and $l_1(b) = \lambda_* b$ in the endpoints, and $l_2(x)$ the affine function on $[a, b]$

with the values $l_2(a) = \lambda_* a$ and $l_2(b) = \lambda_n b$ in the endpoints. Define $\lambda_{n+1} = \min\{l_1(1), l_2(1)\}$.



Note that $\lambda_{n+1} \in (\lambda_*, \lambda_n)$ and that $h(u) < \lambda_{n+1}u$ implies that the line segment $L(u)$ lies below the halfline $\lambda_n x$ (since it has to lie on or above the halfline $\lambda_* x$). In particular $h(au)$ and $h(bu)$ will lie below the halfline $\lambda_n x$; $h(a^2u)$, $h(abu)$, $h(b^2u)$ below $\lambda_{n-1}x$; etc.

G. Define $r = \lambda_0\lambda_1^{-1} > 1$. There exists a positive integer q such that each interval (x, rx), which intersects the interval $(1, c^2)$, contains an element of the set $A_q := \{a^l b^k \mid l, k \geq 0, l + k \leq q\}$.

Indeed since $\log a/\log b$ is irrational and negative the set $\{l \log a + k \log b \mid l, k \text{ non-negative integers}\}$ is dense in $\mathbb{R}$ and $\{a^l b^k \mid l, k \geq 0\}$ is dense in $(0, \infty)$.

H. Choose $u > 0$ such that $h(u) < \lambda_{q+1}u$ and $\phi(u) > \lambda_0 u$. Then $h(x_1) > \lambda_0 x_1$ for some $x_1 \in (u, c^2u)$ by A. There exist two successive elements $c_1, c_2 \in A_q$ such that

$$c_1u \leq x_1 \leq c_2u \leq rc_1u.$$

Then

$$h(x_1) \leq h(c_2u) < \lambda_1c_2u \leq \lambda_1rc_1u = \lambda_0c_1u \leq \lambda_0x_1.$$

Contradiction. The assumption that there exists $\lambda_0 > \lambda_*$ which intersects S is untenable. Hence $\phi(x) = \psi(x) = h(x) = \lambda_* x$ for all $x > 0$. This proves the proposition.

REMARK If log a/log b is rational, say

$$a = b^{-p/q}$$

with p and q positive integers, and (p, q) = 1, then we may write $a = r^{-p}$ and $b = r^q$ for some $r > 1$. If h is a non-decreasing function on $(0, \infty)$ which satisfies (10.5) then h is affine on each coset $\{r^k s \mid k \text{ integral}\}$ with $s > 0$. The proof is similar to the one given above. See remark above.

PROPOSITION 10.3 Suppose the function h is non-decreasing and non-constant on $\mathbb{R}$ and satisfies the functional equation

$$(10.7) \qquad h(x + p) - h(x) = C(h(x + 1) - h(x))$$

with p irrational and $C \in (0, 1)$. Then $h \in \Phi_0$, see exercise 1.8. That is, h is differentiable and

$$h'(x) = ae^{\lambda x}$$

with a positive and λ real.

If $C = p$, then $\lambda = 0$. Else λ is the non-zero solution of the characteristic equation

$$(10.8) \qquad e^{\lambda p} - 1 = C(e^{\lambda} - 1).$$

PROOF Let ψ be C^∞ with compact support. Then $h_1 := h * \psi$ is C^∞ and satisfies (10.7). It suffices to prove that h_1 has the desired properties. Hence we shall assume h to be C^∞.

If $p = C$, then the derivative h' satisfies (10.7) and is non-negative. Apply proposition 10.1 with $C = p = \theta$ to h' to obtain h' is constant.

Suppose $p \neq C$. We may write (10.7) and (10.8) as

$$h(x + q) - h(x) = C_1(h(x) - h(x - p))$$
$$s(\lambda) := e^{q\lambda} - 1 - C_1(1 - e^{-p\lambda}) = 0$$

where $q = 1 - p$ and $C_1 = C^{-1} - 1$. Then $s(0) = 0$ and $s'(\lambda) = qe^{q\lambda} - pC_1e^{-p\lambda}$. Hence s is convex and since $s'(0) \neq 0$ for $p \neq C$, and $s(\lambda) > 0$ for $|\lambda|$ large, it follows that (10.8) determines λ uniquely.

Now define g by $g(e^{\lambda x}) = \lambda h(x)$. Then

$$g(e^{\lambda q}e^{\lambda x}) - g(e^{\lambda x}) = C_1(g(e^{\lambda x}) - g(e^{-\lambda p}e^{\lambda x}))$$

with g a non-decreasing, non-constant C^{∞} function on $(0, \infty)$. Setting $t = e^{\lambda x}$, $a = e^{-\lambda p}$, $b = e^{\lambda q}$ we obtain $C_1 = (b - 1)/(1 - a)$ and

$$\frac{g(bt) - g(t)}{b - 1} = \frac{g(t) - g(at)}{1 - a} .$$

Since $\log b/\log a = -q/p$ is irrational if p is, the function g is affine by proposition 10.2, and

$$h(x) = \lambda^{-1}ae^{\lambda x} + b.$$

COROLLARY Suppose $x_0 < x_1 < x_2$ and h is a non-decreasing, non-constant function on $\mathbb{R}$ which satisfies the functional equation

$$h(x + x_1) - h(x + x_0) = C(h(x + x_2) - h(x + x_0)).$$

Then there exist positive affine transformations τ_1 and τ_2 such that

$$\tau_i h = h\sigma_i$$

where $\sigma_i x = x + x_i - x_0$ for $i = 1, 2$ and $\tau_1 = \tau_2^p$ with $p = (x_1 - x_0)/(x_2 - x_0)$.

PROOF We first prove that $0 < C < 1$. Obviously the monotonicity of h implies $0 \le C \le 1$. If $C = 0$, then $h(x + x_1) = h(x + x_0)$ and h is constant. If $C = 1$ then $h(x + x_2) = h(x + x_1)$ and h is constant.

If p is irrational, then $h \in \Phi_0$ by proposition 10.3 and hence h satisfies

$$\tau^t h = h\sigma^t$$

for some $\tau \in G$, $\tau \neq \varepsilon$. See exercise 1.8.

If p is rational, set

$$X = \{k_1(x_1 - x_0) + k_2(x_2 - x_0) \mid k_1, k_2 \text{ integral}\}.$$

Then on each coset s + X the function h has the form

$$h(x) = \lambda^{-1}ae^{\lambda x} + b \qquad C \neq p$$
$$= a_0x + b_0 \qquad C = p.$$

(See remark following proposition 10.2.) Since λ is uniquely determined by C, and h is non-decreasing we see that the constant b (respectively a_0) is independent of the particular coset. Hence h is an element of table 3.1, and solution of the equation

$$\tau h = h\sigma$$

and $\tau_i = \tau^{n(i)}$, $\sigma_i = \sigma^{n(i)}$ with n(1) and n(2) integral, (n(1), n(2)) = 1, and p = n(1)/n(2).

PROPOSITION 10.4 Suppose that

$$\beta_{(t)}f\alpha(t)^{-1} \text{ converges onto } \Lambda, \text{ for } t \to \infty$$

where $f \in M$ and α and β are continuous functions from $[0, \infty)$ into G, and

$$\alpha(t + s)\alpha(t)^{-1}x \to x + s \qquad \text{for all x and s as } t \to \infty.$$

Then one of the following holds.

1) Λ is contained in one of the following constant functions in M_0

$\phi(x) = c$ for all x

$\phi(x) = c$ for x > a

$\phi(x) = c$ for x < a

$\phi(x) = c$ for x = a (i.e. vertical line through (a, c))

for some a and $c \in \mathbb{R}$. (Note that we exclude the constant function on a bounded open interval.)

2) There exists a differentiable function $\phi \in \Phi$ such that

$$\Lambda \subset \phi$$

$$\phi'(x) = ae^{bx} \quad \text{for some } a > 0,\ b \in \mathbb{R}. \tag{10.9}$$

In the second case there are two possibilities. If the projection of Λ on the x-axis is contained in a periodic set $Z = \{c + kd \mid k \text{ integral}\}$ with d maximal then

$$\beta_{(t)} f\alpha(t)^{-1} \text{ converges onto } \{(z, \phi(z)) \mid z \in Z\} \text{ for } t \to \infty$$

else

$$\beta_{(t)} f\alpha(t)^{-1} \text{ converges weakly onto } \phi \text{ for } t \to \infty.$$

PROOF The proof consists of seven parts.

A. There is nothing to prove if 1) holds. Hence we shall assume that Λ is not contained in any of the elements ϕ mentioned under 1). In particular we assume that Λ contains at least three points and that Λ contains two points with distinct x- and distinct y-coordinates. Without loss of generality we may assume these points to be (0, 0) and (1, 1).

B. The proof depends on the results derived in chapter 8. We therefore commence by recalling some of the pertinent notations.

$g_t = \beta(t) f\alpha(t)^{-1}$.

g is a fixed limit point of the set g_t for $t \to \infty$. Hence $\Lambda \subset g$.

J is the interior of the smallest connected subset of $\mathbb{R}$ which contains the projection of Λ on the x-axis. $J \supset (0, 1)$ if we assume $(0, 0)$, $(1, 1) \in \Lambda$.

$\Delta = \{\sigma^t \mid t \in \mathbb{R}\}$, with $\sigma x = x - 1$.

$\Omega = \{(\gamma_1, \gamma_2) \in G^2 \mid \gamma_2^{-1} \Lambda \gamma_1 \subset g\}$ and

Ω_1 is the projection of Ω on the first coordinate.

$U = \{\rho \in G \mid J \cap \rho J$ non-empty and Λ non-constant on ρJ and on $\rho^{-1} J\}$.

C. Since α varies like a translation we have $\alpha(t) \to \infty$ for $t \to \infty$. See proposition 9.8.

D. Our main tool is the following result. If Λ is non-constant on $\sigma^s J$ for $0 < |s| < c$, for some $c > 0$, then for each $s \in \mathbb{R}$ there exists $\tau_s \in G$ such that

$$\tau_s^{-1}\Lambda\sigma^s \subset g.$$

Indeed, the condition on Λ implies that $\sigma^s \in U$ for $0 < |s| < c$. By proposition 8.5 we have

$$\Delta \cap U\Omega_1 \subset \Omega_1$$

since $A = \Delta$ if α varies like a translation. Hence if $\sigma^t \in \Omega_1$, then $\sigma^{t+s} \in \Omega_1$ for $|s| < c$. Since $\varepsilon = \sigma^0 \in \Omega_1$, we have $\Delta \subset \Omega_1$.

E. Distinct points in Λ have distinct x- and distinct y-coordinates.

1. If Λ contains two points P_1 and P_2 on the same vertical line $\{x = x_0\}$ then Λ also contains two points Q_1 and Q_2 with distinct y-coordinates which do not lie on this vertical. It follows that Λ is non-constant on $\sigma^s J$ for $0 < |s| < c$ for some $c > 0$. (Either $x_0 \in \sigma^s J$ or both Q_1 and Q_2 lie above $\sigma^s J$.) Hence by D for each $s \in \mathbb{R}$ there exists $\tau_s \in G$ such that $\tau_s^{-1}\Lambda\sigma^s \subset g$. This implies that g contains the two curves $\tau_s^{-1}P_1\sigma^s$ and $\tau_s^{-1}P_2\sigma^s$ which lie strictly above each other! Contradiction.

2. If Λ contains two points P_1 and P_2 on the same horizontal line, then it contains two points Q_1 and Q_2 with distinct x-coordinates which do not lie on this horizontal. Again Λ is non-constant on $\sigma^s J$ for $0 < |s| < c$ for some $c > 0$ (since one of the points P_0 or P_1 and one of the points Q_1 or Q_2 will lie above $\sigma^s J$). By D, for each $s \in \mathbb{R}$ there exists $\tau_s \in G$ such that $\tau_s^{-1}\Lambda\sigma^s \subset g$. In particular $\tau_s^{-1}P_1\sigma^s$ and $\tau_s^{-1}P_2\sigma^s$ lie on g for all s. For each s these two points lie on the same horizontal. Besides they maintain a constant distance, say $d > 0$, apart as s ranges over $\mathbb{R}$. Hence g is constant over each interval of length d. Hence g is constant, and Λ is contained in a horizontal line. Contradiction.

F. Let g be a limit point of g_t for $t \to \infty$. Suppose $x_0 < x_1 < x_2$, $y_0 < y_1 < y_2$ and $(x_i, y_i) \in \Lambda$ for $i = 0, 1, 2$. By (8,3), for the right-continuous version of g,

$$g(s + x_1) - g(s + x_0) = C.((g(s + x_2) - g(s + x_0))$$

with $C = (y_1 - y_0)/(y_2 - y_0)$. (Since Λ is non-constant on $\sigma^s J$ for

$0 < |s| < c = \min(p, 1-p)$ with $p = (x_1 - x_0)/(x_2 - x_0)$, there exists by part D for each $s \in \mathbb{R}$ an element $\tau_s \in G$ such that $\tau_s^{-1}\Lambda\sigma^s \subset g$.)

By the corollary to proposition 10.3 there exist $\tau_i \in G$ such that

$$\tau_i g(x) = g(x + x_i - x_0) = g(\sigma_i x) \qquad i = 1, 2.$$

If (x_2, y_2) varies over Λ we obtain a system of such equations. By table 3.2 this implies either $g \in \Phi_0$, see exercise 1.8, if the closed subgroup generated by the σ_i is the group of all translations, or $\Lambda_1 \subset Z$ and $g_{|Z} = \phi_{|Z}$ for some $\phi \in \Phi_0$, where Λ_1 is the projection of Λ on the x-axis, and Z is a periodic set $\{c + kd \mid k \text{ integral}\}$ with d maximal.

G. The set Z and the function $\phi \in \Phi_0$ which agrees with g on Z are completely determined by Λ and do not depend on the particular choice of the limit point g in F. This proves the last two statements in the proposition.

REMARK Except for the last two statements, the proposition remains valid if we only assume that

$$\beta(t)f\alpha(t)^{-1} \text{ converges onto } \Lambda \qquad \text{for } t \to \infty$$

$$\alpha(t) \to \infty \qquad \text{for } t \to \infty$$

with $f \in M$ and α and β continuous functions from $[0, \infty)$ into G, and if we assume that for some sequence $s_n \to \infty$ the sequence of sets

$$A_n = \{\alpha(t)\alpha(s_n)^{-1} \mid t \geq 0\}$$

converges to $A = \{\sigma^t \mid t \in \mathbb{R}\}$ where σ is a translation. (In the proposition we assume that A is the only limit point of the collection $A(s) = \{\alpha(t)\alpha(s)^{-1} \mid t \geq 0\}$ for $s \to \infty$.)

PROOF Part D of the proof remains valid, and hence so does the remainder.

COROLLARY Let f be a non-decreasing function on $[0, \infty)$ and let x_0, x_1 and x_2 be real numbers such that $x_0 < x_1 < x_2$ and $(x_2 - x_0)/(x_1 - x_0)$ is irrational. If

$$\frac{f(x_2 + t) - f(x_0 + t)}{f(x_1 + t) - f(x_0 + t)} \to c \qquad \text{for } t \to \infty$$

with $c > 1$, then

$$\frac{f(x + t) - f(x_0 + t)}{f(x_1 + t) - f(x_0 + t)} \to \phi(x) \quad \text{weakly} \qquad \text{for } t \to \infty$$

where $\phi \in \Phi_0$ (see exercise 1.8).

PROOF Choose $\beta(t)y = (y - f(x_0 + t))(f(x_1 + t) - f(x_0 + t))^{-1}$ and $\alpha(t)x = x - t$. Then

$$\beta(t)f\alpha(t)^{-1} \text{ converges onto } \Lambda$$

where $\Lambda = \{(x_0, 0), (x_1, 1), (x_2, c)\}$. Now apply proposition 10.4.

Leaving aside for the moment the case that Λ is contained in a horizontal or vertical line, there remain four distinct cases in the proposition above. We assume Λ to be normal, see definition 8.6.

Either Λ contains a line segment, and then

1. Λ contains a vertical line segment, or
2. Λ contains a horizontal line segment,

or Λ contains no line segment. Let Λ_1 be the projection of Λ on the x-axis. There are two cases

3. Λ_1 is contained in a periodic set $Z = \{c + kd \mid k \text{ integral}\}$,
4. Λ_1 is not contained in a periodic set.

In case 4 we know from proposition 10.4 that $\beta(t)f\alpha(t)^{-1} \to \phi$ and hence by table 3.2, case 3, there exists $\gamma \in G$ such that $\beta(t + s)\beta(t)^{-1} \to \gamma^s$ for $t \to \infty$ for all $s \in \mathbb{R}$. In the remaining propositions of this chapter we shall consider the possible limit points of $g_t = \beta(t)f\alpha(t)^{-1}$ for $t \to \infty$ and the behaviour of $\beta(t)$ for $t \to \infty$ in the cases 1, 2 and 3 in greater detail.

In the next propositions we make the following assumptions.

$f \in M$

α and β are continuous functions from $[0, \infty)$ into G

$\alpha(t) \to \infty$ for $t \to \infty$

$g_t = \beta(t)f\alpha(t)^{-1}$ converges onto Λ for $t \to \infty$

$s_n \to \infty$ such that

$\beta(s_n)f\alpha(s_n)^{-1} \to g \in M$

$A_n = \{\alpha(t)\alpha(s_n)^{-1} \mid t \geq 0\} \to A$

$A \supset T = \{\gamma^t \mid t \in \mathbb{R}\}$ where $\gamma x = x - 1$

$\alpha(t_n)\alpha(s_n)^{-1} = \sigma_n \to \sigma = \gamma^s$

$\beta(t_n)\beta(s_n)^{-1} = \tau_n$

I is the interior of the projection of g on the x-axis

J is the interior of the smallest connected subset containing the projection of Λ on the x-axis.

We recall proposition 7.4, which we shall apply repeatedly with $g_n = g_{s_n}$.

Suppose that

$g_n \to g$

$\tau_n g_n \sigma_n^{-1}$ converges onto Λ

$\sigma_n \to \sigma$

then, recall that $\sigma x = \gamma^s x = x - s$,

$\tau_n \to \tau \in G$ implies $\Lambda \subset \tau g \sigma^{-1}$ and hence $J + s \subset I$

$\tau_n \to \pm\infty$ implies $J + s$ and I are disjoint

$\tau_n \to \cdot 0$ (with centre c) implies $\Lambda = c$ on $I - s$

$\tau_n \to \cdot\infty$ (with centre c) implies $g = c$ on $J + s$.

PROPOSITION 10.5 If

$$\Lambda = \{(0, -c_2), (0, -c_1), (1, 0)\} \text{ with } c_2 > c_1 \geq 0,$$

then g is the constant function, $g(x) = 0$ on $(0, \infty)$.

PROOF Let S be the set of discontinuity points of g, i.e. $s \in S$ if $g \cap \{x = s\}$ contains at least two points. The set S is countable.

Suppose (τ_n) has a limit point $\tau \in G$. Then $\tau^{-1}\Lambda\sigma \subset g$, hence g has a discontinuity in s and $s \in S$.

Let $x_1 \le x_2$ be the endpoints of $\{g = 0\}$. Then clearly $0 \le x_1 \le 1 \le x_2 \le \le \infty$. Suppose $s \in (0, x_2) \setminus S$. Then $\tau_n \to \infty$. By proposition 7.4 we can only have

$$\tau_n \to \cdot\infty \text{ (with centre 0), and}$$

$$g = 0 \quad \text{on } (s, 1 + s).$$

This implies that $g = 0$ on $(0, x_2 + 1)$, hence $x_2 = \infty$ and $\{g = 0\} = [0, \infty)$.

Suppose $I = (x_0, \infty)$ with $x_0 < 0$. Choose $s \in (x_0, 0) \setminus S$ and $s > -1$. By proposition 7.4 this is impossible. Hence $I = (0, \infty)$.

COROLLARY If $\sigma 0 > 0$, then $\tau_n \to \cdot 0$ (with centre 0),
if $\sigma 0 < 0$, then $\tau_n \to \cdot\infty$ (with centre 0).

PROOF The sequence $\beta(t_n)f\alpha(t_n)^{-1}$ converges onto g, the constant function, $g(x) = 0$ on $(0, \infty)$, since

$$\{\alpha(t)\alpha(t_n)^{-1} \mid t \ge 0\} = A_n\sigma_n^{-1} \to A\sigma^{-1} \supset T.$$

Now apply proposition 7.3 with $g_{s_n}\sigma_n^{-1} \to g\sigma^{-1}$ and $\tau_n g_{s_n}\sigma_n^{-1} \to g$.

PROPOSITION 10.6 If

$$\Lambda = \{(0, -1), (1, 0), (c, 0)\} \quad \text{with } c > 1$$

then there exists $\theta \in [0, 1]$ such that

$$g = 0 \quad \text{on } (\theta, \infty)$$

$$I \subset (\theta - 1, \infty).$$

PROOF Set $J_0 = (1, c)$ and let I_0 be the interior of $\{g = 0\}$. Suppose $s > 0$ and $J_0 + s$ intersects I_0, then $J_0 + s \subset I_0$. (If (τ_n) has a limit point $\tau \in G$, then $\sigma^{-1}J_0 \subset I_0$ as in proposition 8.6. If $\tau_n \to \infty$, then by proposition 7.4

only $\tau_n \to \cdot\infty$ (with centre 0) is possible, and then $g = 0$ on $J_0 + s$.) This proves that $g = 0$ on $(1, \infty)$.

Set $I_0 = (\theta, \infty)$. Obviously $0 \le \theta \le 1$. Suppose $s \in I$ such that $1 < \theta - s < c$. By proposition 7.4 the sequence (τ_n) is bounded. Let τ be a limit point. Then $\tau^{-1}\Lambda\sigma \subset g$, and since τ is a multiplication with centre 0 by proposition 8.6, we have $(1 + s, 0) \in g$. This implies $\theta \le 1 + s$, contradicting the choice of s. Hence $I \subset (\theta - 1, \infty)$.

COROLLARY Suppose $I = (x_0, \infty)$.

If $x_0 < s < \theta$, then (τ_n) is bounded, and every limit point of (τ_n) is a multiplication with centre 0,

if $s < x_0$, then $\tau_n \to \cdot 0$ (with centre 0),

if $s > \theta$, then $\tau_n \to \cdot\infty$ (with centre 0).

PROOF Apply proposition 7.4, and, for the second part of the first statement, proposition 8.6.

We shall now consider the case that Λ_1 is contained in a periodic set. Suppose

$$\Lambda = \{(x_i, y_i) \mid i = 0,1,2\} \text{ with } x_0 < x_1 < x_2,\ y_0 < y_1 < y_2.$$

Suppose $(x_1 - x_0)(x_2 - x_0)^{-1}$ is rational. For convenience we assume that $x_0 = 0$, x_1 and x_2 are integral and $(x_1, x_2) = 1$. From part F of proposition 10.4 above we know that there exists $\tau \in G$, $\tau \ne \varepsilon$, such that

$$(10.10) \qquad \tau g = g\gamma.$$

Let S denote the set of discontinuity points of g. The set S is periodic modulo γ. If $s \notin S$, then

$$\tau_{(i)}^{-1}\Lambda\gamma^s \subset g \qquad \text{for } i = 1, 2$$

implies $\tau_{(1)} = \tau_{(2)}$. By proposition 7.4 the sequence (τ_n) is bounded. Since each limit point $\tau_{(i)}$ satisfies the inclusion above, the sequence converges to an element $\tau_s \in G$, and

$$(10.11) \qquad \tau_s^{-1}\Lambda\gamma^s \subset g.$$

Then $\Lambda \subset \tau_s g\gamma^{-s} = g_1$ and also g_1 satisfies a functional equation $\tau' g_1 = g_1\gamma$. Moreover g_1 is continuous in 0. Hence we may as well assume that g is continuous in 0.

Since g is continuous in $m = x_1$, and

$$\tau^{-m}P_0\gamma^m = (x_1, \tau^{-m}0) \in g$$

$$P_1 = (x_1, y_1) \in g$$

we have

$$P_1 = \tau^{-m}P_0\gamma^m.$$

In the three propositions below we shall see that the functional relation (10.10) for g induces an analogous functional relation for τ_s, where τ_s for $s \notin S$ is defined by (10.11), and conversely.

In particular we shall see that the elements τ_s commute, i.e. $\tau_s\tau_t = \tau_t\tau_s$, and that $\tau_{s+1} = \tau_s\tau$. Thus if α varies like γ, i.e. for all $s \in \mathbb{R}$ $\alpha(t + s)\alpha(t)^{-1} \to \gamma^s$ for $t \to \infty$, then for the function β we obtain that for integral k

$$\beta(t + k)\beta(t)^{-1} \to \tau^k \qquad \text{for } t \to \infty.$$

The guide set C of g for Λ, see definition 8.1, has the structure

$$C = \{(\tau^t, \gamma^s) \mid (s, t) \in h\}$$

where h is an element of M which satisfies

$$h\gamma = \gamma h.$$

This formulation reflects the symmetric role of the two axes. We can choose a new parameter such that β varies like γ and the function α satisfies for integral k

$$\alpha(t + k)\alpha(t)^{-1} \to \gamma^k \qquad \text{for } t \to \infty.$$

PROPOSITION 10.7 Let $\sigma \in G$ be the translation $\sigma x = x + 1$. Suppose $g \in M$ and $\tau \in G$ satisfy

$$\tau g = g\sigma.$$

Let $P_1, P_2 \in g$ where $P_2 = \tau^{-k}P_1\sigma^k$ for some integer k. Then there exists $h \in M$ such that

$$h\sigma = \sigma h$$

$$\tau^{-h(t)}P_i\sigma^t \in g \quad \text{for } i = 1,2 \text{ and all continuity points } t \text{ of } h.$$

This element h is unique.

PROOF Suppose $P_1 = (x_1, y_1)$. The equation

$$\tau^{-h(t)}y_1 = g(x_1 - t)$$

determines h(t) uniquely for t for which $x_1 - t$ is a continuity point of g. The right hand side depends continuously on t, hence so does the left hand side and so does h. Moreover

$$\tau^{-1}\tau^{-h(t)}y_1 = \tau^{-1}g(x_1 - t) = g\sigma^{-1}(x_1 - t) = g(x_1 - t - 1)$$

which implies $h(t + 1) = h(t) + 1$ if $x_1 + t$ is a continuity point of g. Hence $h\sigma = \sigma h$.

By definition of h we have $\tau^{-h(t)}P_1\sigma^t \in g$ for all t. Hence also

$$\tau^{-h(t)}P_2\sigma^t = \tau^{-h(t)}\tau^{-k}P_1\sigma^k\sigma^t = \tau^{-h(t+k)}P_1\sigma^{k+t} \in g.$$

PROPOSITION 10.8 Suppose $h \in M$ satisfies

$$h\sigma = \sigma h$$

where $\sigma \in G$ is the translation $\sigma x = x + 1$. Suppose moreover that $g \in M$, $\tau \in G$, $P \in g$ satisfy

$$\tau^{-h(t)}P\sigma^t \in g \quad \text{for all continuity points } t \text{ of } h.$$

Then

$$\tau g = g\sigma.$$

PROOF Let x be a continuity point of g. Then

$$(x,\ g(x)) = \tau^{-h(t)}P\sigma^{t}$$

for some $t \in \mathbb{R}$. Also t is a continuity point of h, and

$$(\sigma x,\ \tau g(x)) = (\tau\tau^{-h(t)}P\sigma^{t}\sigma^{-1})$$
$$= (\tau^{-h(t-1)}P\sigma^{t-1}) \in g.$$

PROPOSITION 10.9 Suppose $h \in M$ satisfies

(10.12) $$h\sigma = \sigma h$$

where $\sigma \in G$ is the translation $\sigma x = x + 1$. Suppose moreover that $g \in M$, $\tau \in G$, P_1, $P_2 \in g$ and

$$\tau^{-h(t)}P_i\sigma^{t} \in g \quad \text{for } i = 1,2 \text{ and all continuity points } t \text{ of } h.$$

Then

$$P_2 = \tau^{-s}P_1\sigma^{s}$$

for some $s \in \mathbb{R}$ and

$$h\sigma^{s} = \sigma^{s}h.$$

PROOF We have

$$P_2 = \tau^{-c_2}P_1\sigma^{c_1}$$

for some $(c_1,\ c_2) \in h$ (since $\tau^{-s_2}P_1\sigma^{s_1}$ varies over g as $(s_1,\ s_2)$ varies over h). Hence whenever $x_2 - t$ is a continuity point of g

$$\tau^{-h(t)}P_2\sigma^{t} = \tau^{-h(t)}\tau^{-c_2}P_1\sigma^{c_1}\sigma^{t}$$
$$= \tau^{-h(t+c_1)}P_1\sigma^{t+c_1}.$$

This implies that

$$h(t + c_1) = h(t) + c_2.$$

In chapter 3 we have seen that then

$$h(t) = c_2 c_1^{-1} t + \pi_1(t)$$

with π_1 periodic modulo c_1. By (10.12) also

$$h(t) = t + \pi(t)$$

with π periodic modulo 1. Letting $t \to \infty$ we find $c_2 = c_1$. Setting $s = c_1$ completes the proof.

11 Regular variation and limit laws

In this chapter we shall prove the following theorem.

THEOREM 11.1 Suppose that in addition to the basic situation (1.1)

$$\alpha_n \underline{x}_n \to \underline{u} \quad \text{in distribution}$$

$$\beta_n \underline{y}_n \to \underline{v} \quad \text{in distribution}$$

$$\underline{y}_n \overset{M}{=} f(\underline{x}_n) \qquad n = 1,2,\ldots$$

$$\alpha_n \to \infty$$

we are given that

$$\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$$

$$\Delta \subset \{\gamma^t \mid t \in \mathbb{R}\} \qquad \text{for some } \gamma \in G.$$

Then

$$\underline{v} \overset{M}{=} \phi(\underline{u})$$

for some $\phi \in \Phi$. Moreover there exists $\delta \in G$ such that one of the following holds

(11.1) $\quad \delta^t\phi = \phi\gamma^t \quad$ for all $t \in \mathbb{R}$

(11.2) $\quad \phi \in M_0 \quad$ and $\quad \gamma I \subset I \quad$ or $\quad \gamma^{-1}I \subset I$

where I is the interior of the projection of ϕ on the x-axis.

The proof of this theorem occupies the greater part of this chapter. We shall need two lemmas.

LEMMA 11.1 Let $\Lambda \subset g \in M$ contain two points $P_i = (x_i, y_i)$, $i = 1,2$, with $0 < x_1$ and $y_1 < y_2$. For each $t > 0$ let $\tau_t \in G$ satisfy

$$\tau_t^{-1}\Lambda\sigma^t \subset g$$

where $\sigma x = e.x$, where $e = 2.718\ldots$

If g(0+) is finite, then

$$\tau_t^{-1} \to \cdot 0 \text{ (with centre } g(0+)) \text{ for } t \to \infty.$$

PROOF $\tau_t^{-1} P_i \sigma^t = (e^{-t} x_i, \tau_t^{-1} y_i) \in g$ for $i = 1, 2$ and $t \in \mathbb{R}$. Hence

$$\lim_{t\to\infty} \tau_t^{-1} y_i = g(0+) \qquad \text{for } i = 1, 2.$$

LEMMA 11.2 Suppose $0 < x_1 < x_2$ and $y_1 < y_2$. For each $y < y_1$ there exist unique $a > 0$ and $\rho > 0$ such that

$$a x_i^\rho = y_i - y \qquad \text{for } i = 1, 2.$$

The exponent $\rho = \rho(y)$ is a strictly increasing continuous function from $(-\infty, y_1)$ into $(0, \infty)$.

PROOF The set of linear equations

$$\log a + \rho \log x_i = \log (y_i - y) \qquad i = 1, 2$$

has a unique solution $(\log a, \rho)$ and

$$\rho = \frac{\log(y_2 - y) - \log(y_1 - y)}{\log x_2 - \log x_1} > 0.$$

Moreover $\dfrac{y_2 - y}{y_1 - y} = \dfrac{y_2 - y_1}{y_1 - y} + 1$ is strictly increasing and continuous from $(-\infty, y_1)$ into $(0, \infty)$. Hence so is ρ.

PROOF of theorem 11.1 By theorem 2.1 there exists a unique probability measure λ with support Λ contained in some element of M such that λ has marginals $\underline{u}$ and $\underline{v}$. By theorem 2.2 the sequence $\beta_n f \alpha_n^{-1}$ in M converges onto Λ. (See definition 2.1.) Because of the definition of $\overset{M}{=}$ we need only prove that $\Lambda \subset \phi$ for some $\phi \in \Phi$ which satisfies one of the two relations (11.1), (11.2).

By proposition 9.7 the inclusion $\Delta \subset \{\gamma^t \mid t \in \mathbb{R}\}$ implies that there exists a continuous function $\alpha : [0, \infty) \to G$ which varies like γ^t (or like γ^{-t}) and a sequence $t_n \to \infty$ such that $\alpha_n = \alpha(t_n)$. By reordering the sequence α_n we may suppose that the t_n are non-decreasing and that $\alpha(t) = (\alpha_{n+1}\alpha_n^{-1})^\theta \alpha_n$

for $t = t_n + \theta(t_{n+1} - t_n)$ with $\theta \in [0, 1)$. By proposition 7.1 there exists a continuous function $\beta : [0, \infty) \to G$, defined by $\beta(t) = (\beta_{n+1}\beta_n^{-1})^{\theta}\beta_n$ with t and θ as above) such that

$$g_t := \beta(t)f\alpha^{-1}(t) \quad \text{converges onto } \Lambda \text{ for } t \to \infty.$$

If γ is a translation, then theorem 11.1 follows from proposition 10.4. If γ is not a translation we may and do assume that $\gamma x = e.x$ (or $\gamma x = e^{-1}.x$). This may be realized by an appropriate choice of the origin on the x-axis and if need be a transformation $t' = \lambda t$, with $\lambda > 0$, of the argument of the functions α and β.

The case $\gamma^t x = e^t.x$ is in many respects similar to the case $\gamma^t x = x + t$ which has been treated in the previous chapter. The main difference results from the fact that $\mathbb{R}$ contains three γ-invariant subsets

$$(-\infty, 0) \quad , \quad \{0\} \quad , \quad (0, \infty) \quad ,$$

if γ is a multiplication with centre 0, and that there is a certain amount of independence between the three parts of the limit function on these sets. This is already visible in the sixth function in the list in the definition of Φ in chapter 1.

From the analytical version of the inclusion $\tau_t^{-1}\Lambda\gamma^t \subset g$ in chapter 8, see equation (8.3),

$$\frac{g(\gamma^{-t}x_2) - g(\gamma^{-t}x_0)}{g(\gamma^{-t}x_1) - g(\gamma^{-t}x_0)} = \frac{y_2 - y_0}{y_1 - y_0}$$

where (x_i, y_i) for $i = 0,1,2$ lie in Λ and we assume $x_2 < 0 < x_0 < x_1$ and $y_0 < y_1$, it follows by varying t over $\mathbb{R}$, that the value of g for any $x = \gamma^{-t}x_2 < 0$, with the exception of a countable set, may be determined from the values of g on the positive axis.

The proof of the theorem is complicated by the fact that there is a host of particular cases which we have to consider separately. In general there are two courses open to us if we wish to decide whether Λ is contained in some element of Φ.

1. Λ is so small that we can find an element $\phi \in \Phi$ which contains Λ. For instance Λ may consist of two points. In this case also every subset of Λ is contained in this element ϕ. These cases will be called trivial cases.

2. Λ is so large, that convergence of g_t onto Λ for $t \to \infty$ implies that g_t converges to an element $\phi \in \Phi$ for $t \to \infty$, that g_t has a limit point in Φ for $t \to \infty$ or that every limit point g of g_t for $t \to \infty$ satisfies certain functional equations. Obviously if g_t converges onto a set containing Λ for $t \to \infty$, it will also have these same properties.

We shall now specify the different cases which we shall consider in more detail. We shall assume that Λ is normal (see definition 8.6). Φ_1 will denote the subset of Φ consisting of all elements $\phi \in \Phi$ which satisfy one of the relations (11.1) or (11.2).

Outline of the different cases in the proof of theorem 11.1

I. $\Lambda \subset \{x \geq 0\}$ or $\Lambda \subset \{x \leq 0\}$,

II. Λ contains points in both open half planes $\{x > 0\}$ and $\{x < 0\}$. In view of the obvious symmetry, $(x, y) \mapsto (-x, -y)$, we need only distinguish five cases.

A $\Lambda \cap \{x > 0\}$ contains three points, not all on the same horizontal or vertical line.

B Λ contains two line segments, which do not both lie on the same horizontal or vertical line.

C $\Lambda \cap \{x = 0\}$ is empty, and Λ is divided over the two half planes as follows.

1. (2 points, 2 points)
2. (2 points, horizontal line segment)
3. (2 points, vertical line segment).

D $\Lambda \cap \{x = 0\}$ consists of one point, and Λ is divided over the two open half planes as follows.

1. (1 point, 2 points)
2. (1 point, horizontal line segment)
3. (1 point, vertical line segment).

E $\Lambda \cap \{x = 0\}$ contains two points.

We now turn to a detailed consideration of these cases.

I. $\Lambda \subset \{x \geq 0\}$

a $\Lambda \subset \{x > 0\}$

The proof that $\Lambda \subset \phi$ for some $\phi \in \Phi_1$ is similar to that given in proposition 10.4 in the previous chapter for the case that $\alpha(t)$ varies like a translation, and is omitted.

b $\Lambda \subset \{x \geq 0\}$ and $\Lambda \cap \{x = 0\}$ is non-empty

There exists a limit point g of g_t for $t \to \infty$. This limit point g agrees with an element $\phi \in \Phi_1$, at least on a set $\{a^k c \mid k \text{ integral and positive}\}$ with a and c positive and $a < 1$. If Λ contains a point on the y-axis, then $\{\phi(a^k c) \mid k \text{ integral}\}$ is bounded below, say $\lim_{k\to\infty} \phi(a^k c) = y_0$, and we may choose $\phi \in \Phi_1$ to contain the vertical halfline $\{(0, y) \mid y \leq y_0\}$.

II. Λ contains points in both open half planes $\{x > 0\}$ and $\{x < 0\}$

A $\Lambda \cap \{x > 0\}$ contains three points, not all on the the same horizontal or vertical line

Let g be a limit point of g_t for $t \to \infty$.

If Λ does not lie in an element $\phi \in M_0$, then by proposition 8.5, since U is an open neighbourhood of ε, for all $s \in \mathbb{R}$ there exists $\tau_s \in G$ such that

$$\tau_s^{-1}\Lambda\gamma^s \subset g$$

and hence g is finite on the whole real line.

This implies that $\Lambda \subset \phi \in M_0$ if $\Lambda \cap \{x > 0\}$ contains a horizontal or vertical line segment. See propositions 10.5 and 10.6.

If $\Lambda \cap \{x > 0\}$ contains three points, no two of which lie on the same horizontal or vertical line, then either $\beta(t)$ varies like τ for some $\tau \in G$, $\tau \neq \varepsilon$, and $\tau^{-s}\Lambda\gamma^s \subset g$ for all $s \in \mathbb{R}$, which implies $g \in \Phi$, or there exists $h \in M$ such that $h\sigma = \sigma h$ for some translation $\sigma x = x + p$ and

$$\tau^{-h(s)}\Lambda\gamma^s \subset g \quad \text{for all } s.$$

(See proposition 10.7) Then $\tau^p g = g\gamma^p$ (proposition 10.8) and if we choose $p > 0$ and minimal, then any two points of Λ in the same half plane are congruent modulo γ^p (see proposition 10.9). This implies that $g_{|\Lambda_1} = \phi_{|\Lambda_1}$ where Λ_1 is the projection of Λ on the x-axis, and $\phi \in \Phi$ satisfies $\tau^s\phi\gamma^{-s} = \phi$ for all s.

B $\quad\Lambda$ contains two line segments

By proposition 8.6 this implies that $\Lambda \subset \phi$ with $\phi \in M_0$ or $\phi^{-1} \in M_0$. Then $\phi \in \Phi_1$ unless ϕ has the form $\phi(x) = \alpha\ \mathrm{sign}(x - x_0)$ for some $\alpha \in G$ and $x_0 \neq 0$. However, if we cannot choose $x_0 = 0$, then case A applies.

C $\quad\Lambda$ has no points on the y-axis

We may and do assume that Λ does not contain two line segments and that Λ contains no more than two points or one line segment in each of the half planes $\{x > 0\}$ and $\{x < 0\}$. The possible distributions of these over the two half planes are, up to a trivial symmetry,

1. (2 points, 2 points)
2. (2 points, horizontal)
3. (2 points, vertical).

For 1. and 2. we use lemma 11.2 to construct a function which contains Λ and has the form

$$\begin{aligned} y(x) &= y_0 + a_1 x^\rho \qquad & x \geq 0 \\ &= y_0 - a_2 |x|^\rho \qquad & x < 0 \end{aligned}$$

with $a_1 \geq 0$ and a_2 and ρ positive.

For 3. we choose $\phi \in M_0$.

D $\quad\Lambda \cap \{x = 0\} = \{(0, 0)\}$

1. Λ contains four points $\tilde{P}$, P_0, P_1, P_2, with $P_i = (x_i, y_i)$ such that

$$\tilde{x} < x_0 = 0 < x_1 < x_2$$
$$\tilde{y} < y_0 = 0 < y_1 < y_2.$$

Let $\sigma_0 x = x_2 x_1^{-1}.x$, $\tau_0 y = y_2 y_1^{-1}.y$ and let g be a limit point of g_t for $t \to \infty$. We shall prove that g satisfies the functional equation

$$\tau_0 g = g\sigma_0.$$

Indeed, since Λ is not constant on $J = (\tilde{x}, x_2)$, there exists for each $t \in \mathbb{R}$ an element $\tau_t \in G$ such that

$$\tau_t^{-1}\Lambda\gamma^t \subset g.$$

(See proposition 8.4) By lemma 11.1 we have $\tau_t^{-1} \to \cdot 0$ (with centre g(0+)) in G^* for $t \to \infty$. Hence $\tau_t^{-1}Q\gamma^t \to (0, g(0+))$ for $t \to \infty$ for every point $Q \in \Lambda$. In particular for $Q = \tilde{P}$ we obtain g(0-) = g(0+). Hence g is continuous in 0. For $Q = P_0$ we obtain $(\gamma^{-t}0, \tau_t^{-1}g(0)) = (0, 0)$ for all t. Hence $\tau_t 0 = 0$ for all t, and there exists $h \in M$ such that

$$\tau_s = \gamma^{h(s)}$$

in the sense that $(s, t) \in h$ implies

$$\gamma^{-t}\Lambda\gamma^s \subset g.$$

There is a one-one correspondence between the points (α, β) of the guide set C of Λ for g, the points $(s, t) \in h$ and the points of g in the half plane $\{x > 0\}$ as follows

$$(\gamma^s, \gamma^t) \in C$$
$$(s, t) \in h$$
$$(\gamma^{-s}x_1, \gamma^{-t}y_1) \in g$$
$$(\gamma^{-s}x_2, \gamma^{-t}y_2) \in g.$$

Now $(\alpha, \beta) \in C$ implies $(\sigma_0\alpha, \tau_0\beta) \in C$. (Indeed $(\alpha, \beta) \in C$ implies $(\alpha^{-1}x_1, \beta^{-1}y_1) \in g$, equivalently $(\alpha^{-1}\sigma_0^{-1}x_2, \beta^{-1}\tau_0^{-1}y_2) \in g$, and hence $(\sigma_0\alpha, \tau_0\beta) \in C$.) Therefore $(x, y) = (\alpha^{-1}x_1, \beta^{-1}y_1) \in g$ with $(\alpha, \beta) \in C$ implies $(\sigma_0^{-1}x, \tau_0^{-1}y) \in g$. Equivalently $\tau_0 g = g\sigma_0$.

D2. Choose $\phi(x) = \alpha$ sign x.

D3. Choose $\phi \in M_0$.

E Λ contains a vertical line segment on the y-axis

Then $\Lambda \subset \alpha$ sign x for suitable α. See table 8.1, case 3b.

Clearly it is of some interest to know which $f \in M$ can occur in the relation

(12.1) $\beta(t)f\alpha(t)^{-1}$ converges onto Λ for $t \to \infty$

where $\alpha(t)$ varies like γ for some $\gamma \neq \varepsilon$.

If Λ is sufficiently large, then (12.1) implies

$$\beta(t)f\alpha(t)^{-1} \to \phi \qquad \text{for } t \to \infty$$

with $\phi \in \Phi(\gamma, \tau)$ and $\tau \neq \varepsilon$. In this case $\beta(t)$ varies like τ. (See the last statement in proposition 10.4, and table 3.2.)

For convenience we restrict ourselves to the case that $\alpha(t)$ varies like a translation and assume that

(12.2) $\beta(t)f\alpha(t)^{-1} \to \phi \qquad \text{for } t \to \infty$

where $\phi \in \Phi_0$ (see exercise 1.8 for notation).

The class of these f has been investigated by de Haan [1970] in a slightly different setting for the case that

(12.3) $\alpha(t)x = x - t \qquad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$

He has shown, [1970, section 1.4], that if f and β are measurable functions on $[0, \infty)$ and convergence in (12.2) is pointwise, then the following holds.

If $\phi(x) = b + ae^{\lambda x}$ with $b \in \mathbb{R}$ and a and λ positive, then

$$f(t) \to \infty \qquad \text{for } t \to \infty$$

$$\frac{f(t + x)}{f(t)} \to e^{\lambda x} \qquad \text{for } t \to \infty \text{ and all } x.$$

If $\phi(x) = b - ae^{-\lambda x}$ with $b \in \mathbb{R}$ and a and λ positive, then

$$f(t) < \lim_{t\to\infty} f(t) = c < \infty \qquad \text{for } t \geq t_0$$

$$\frac{c - f(t + x)}{c - f(t)} \to e^{-\lambda x} \qquad \text{for } t \to \infty \text{ and all } x.$$

If $\phi(x) = b + ax$ with $b \in \mathbb{R}$ and a positive, then

$$\frac{f(t + x)}{f(t)} \to 1 \qquad \text{for } t \to \infty \text{ and all } x.$$

In the first two cases where ϕ is exponential the converse also holds (trivially), but if ϕ is affine, this is no longer so. For this case we introduce the sets E_0 and Π_0. See de Haan [1970, definition 1.1.1 and definition 1.4.1].

DEFINITION 12.1 E_0 is the set of all measurable functions $g : [0, \infty) \to \mathbb{R}$ which satisfy

$$g(t) > 0 \qquad \text{for } t > t_0$$

$$g(t + x)/g(t) \to 1 \qquad \text{for } t \to \infty.$$

DEFINITION 12.2 Π_0 is the set of all measurable functions $f : [0, \infty) \to \mathbb{R}$ which satisfy

$$f(t + 1) - f(t) > 0 \qquad \text{for } t > t_0$$

$$\frac{f(t + x) - f(t)}{f(t + 1) - f(t)} \to x \qquad \text{for } t \to \infty \text{ for all } x \in \mathbb{R}.$$

Observe that E_0 and Π_0 are convex cones. If $g \in E_0$ is strictly positive, then as a function into the multiplicative group of positive reals it varies like the trivial homomorphism 1.

If ψ is locally integrable on $[0, \infty)$ and $\psi(t) \to 0$ for $t \to \infty$, then $g(x) = \exp \int_0^x \psi(t)dt$ is an element of E_0. If $g \in E_0$ is locally integrable, then $f(x) = \int_0^x g(t)dt$ is an element of Π_0.

We now give a result of de Haan which gives some insight in the relation between Π_0 and E_0. We first need a preliminary result.

PROPOSITION 12A (de Haan [1970, (1.3.8) and (1.3.9)]). Let f and g be locally integrable functions on $[0, \infty)$. Then the following two equalities are equivalent

$$f(x) = g(x) + \int_0^x g(t)dt$$

$$g(x) = f(x) - e^{-x}\int_0^x e^t f(t)dt$$

$$(= \int_0^\infty e^{-s}(f(x) - f(x - s))ds \quad \text{if we set } f(s) = 0 \text{ on } [-\infty, 0))$$

PROPOSITION 12B (de Haan [1970, theorem 1.4.1]). Let f and g be locally integrable functions on $[0, \infty)$ which satisfy one of the equivalent relations in proposition 12A. If $f \in \Pi_0$, then $g \in E_0$. If $g \in E_0$, then $f \in \Pi_0$.

If instead of (12.3) one assumes that α varies like the translation $x - 1$, then (12.2) implies

$$f = f_2 \circ f_1^{-1}$$

with f_1 and $f_2 \in \Pi_0$ (de Haan, oral communication, see also proposition 12.7).

In this chapter we shall give a more geometric treatment of these results.

We shall see that there exists a very simple connection between elements of Π_0 and functions $\alpha(t)$ which vary like the translation $x - 1$. For any $x_0 \in \mathbb{R}$ the function $f(t) = \alpha(t)x_0$ is an element of Π_0.

The relation of asymptotic equality played an important role in the theory of regular variation in chapter 9. Recall that

if $\alpha(t)$ varies like γ and $\tilde{\alpha}(t) \sim \alpha(t)$ for $t \to \infty$, then $\tilde{\alpha}(t)$ varies like γ, see proposition 9.4,

if $\tilde{\alpha}(t)$ varies like γ, there exists $\alpha(t)$ with $\alpha(t) \sim \tilde{\alpha}(t)$ for $t \to \infty$, such that $\alpha(t)$ is C^∞ and $A(t) = \alpha(t)\frac{d}{dt}\alpha(t)^{-1} \to A$ for $t \to \infty$, see proposition 9.10,

if $\beta(t)f\alpha(t)^{-1} \to \phi$ and $\tilde{\alpha}(t) \sim \alpha(t)$ and $\tilde{\beta}(t) \sim \beta(t)$, then $\tilde{\beta}(t)f\tilde{\alpha}(t)^{-1} \to \phi$ for $t \to \infty$.

We shall see that for non-decreasing functions in Π_0 the asymptotic equality

$$\alpha(t) \sim \tilde{\alpha}(t) \qquad \text{for } t \to \infty$$

corresponds to the relation

$$g(t) - \tilde{g}(t) \to 0 \qquad \text{for } t \to t^* - 0$$

where g and $\tilde{g}$ are the inverse functions to the functions $f(t) = \alpha(t)^{-1}x_0$ and $\tilde{f}(t) = \tilde{\alpha}(t)^{-1}x_0$ in Π_0, and $t^* = \lim_{x\to\infty} f(x)$ may be infinite.

We first prove a very general result.

PROPOSITION 12.1 Suppose $f \in M$ and α and β are continuous functions from $[0, \infty)$ into g, such that

$$\beta(t)f\alpha(t)^{-1} \text{ converges onto } \Lambda \text{ for } t \to \infty$$

where $\Lambda = \{(0, 0), (1, 1)\}$. Then there exist continuous functions $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$ asymptotically equal to $\alpha(t)$ and $\beta(t)$ for $t \to \infty$ such that

$$\tilde{\beta}(t)f\tilde{\alpha}(t)^{-1} \text{ contains } \Lambda \text{ for all } t \geq 0.$$

PROOF By definition $f = \{(x(s), y(s)) \mid s \in \mathbb{R}\}$ with x and y continuous and non-decreasing and $x(s) + y(s) = s$ for all $s \in \mathbb{R}$. Choose $t_n \to \infty$ such that $\alpha(t) \sim \alpha(t_n) = \alpha_n$ and $\beta(t) \sim \beta(t_n) = \beta_n$ for $t_n \leq t < t_{n+1}$ and $t \to \infty$. Choose $P_n^i \in f$ such that

$$\beta_n P_n^i \alpha_n^{-1} \to (i, i) \qquad \text{for } i = 0, 1.$$

Choose $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ such that

$$\tilde{\beta}_n P_n^i \tilde{\alpha}_n^{-1} = (i, i) \qquad \text{for } i = 0, 1 \text{ and all } n.$$

Then obviously $\tilde{\alpha}_n \sim \alpha_n$ and $\tilde{\beta}_n \sim \beta_n$.

Now $P_n^i = (x(s_n^i), y(s_n^i))$ and if we define

$$P^i(t) = (x(s^i), y(s^i)) \qquad \text{for } i = 1, 2$$

for $t = t_n + \theta(t_{n+1} - t_n)$ and $s^i = s_n^i + \theta(s_{n+1}^i - s_n^i)$ with $0 \leq \theta < 1$, then $P^i(t)$ is continuous for $i = 1, 2$ and there exist unique continuous functions $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$ such that

$$\tilde{\beta}(t)P^i(t)\tilde{\alpha}(t)^{-1} = (i, i) \quad \text{for } i = 0, 1 \text{ and all } t \geq 0.$$

Moreover $\tilde{\alpha}(t_n) = \tilde{\alpha}_n \sim \tilde{\alpha}_{n+1}$ and hence for $t \in [t_n, t_{n+1})$ and $t \to \infty$ we have $\tilde{\alpha}(t) \sim \tilde{\alpha}(t_n)$ which implies $\tilde{\alpha}(t) \sim \alpha(t)$ for $t \to \infty$. Similarly for $\tilde{\beta}(t)$. This proves the proposition.

Now suppose

$$\beta(t)f\alpha(t)^{-1} \ni P_0 = (x_0, y_0) \quad \text{for all } t \geq 0.$$

We may write this as

$$(12.4) \qquad \{(a(t), b(t)) \mid t \geq 0\} \subset f$$

where $a(t) = \alpha(t)^{-1}x_0$ and $b(t) = \beta(t)^{-1}y_0$. The functions $a(t)$ and $b(t)$ are continuous if $\alpha(t)$ and $\beta(t)$ are. In particular if $a(t)$ is strictly increasing, then the inverse function is well defined, and we may formulate (12.4) as

(12.5) $\quad f$ contains the graph of ba^{-1}.

If (12.4) holds and $\alpha(t)$ and $\beta(t)$ vary like α_0 and β_0, then, setting $P_t = (a(t + s), b(t + s))$ for fixed $s \in \mathbb{R}$,

$$\beta(t)P_t\alpha(t)^{-1} = \beta(t)\beta(t + s)^{-1}P_0\alpha(t + s)\alpha(t)^{-1} \to \beta_0^{-s}P_0\alpha_0^s$$

and hence

$$\beta(t)f\alpha(t)^{-1} \text{ converges onto } \Lambda \text{ for } t \to \infty$$

where Λ is the curve $\{\beta_0^sP_0\alpha_0^s \mid s \in \mathbb{R}\}$, which either is an element of Φ or half an element of Φ (see the proof of proposition 1.1).

PROPOSITION 12.2 Suppose $\alpha : [0, \infty) \to G$ varies like the translation $x - 1$, i.e.

$$(12.6) \qquad \alpha(t + s)\alpha(t)^{-1}x \to x - s \quad \text{for } t \to \infty \text{ for all } s.$$

Set $f(t) := \alpha(t)^{-1}x_0$ with x_0 fixed. Then $f \in \Pi_0$.

PROOF The function f is measurable by definition 9.2. Also for $s \in \mathbb{R}$,

$$\alpha(t)f(t+s) = \alpha(t)\alpha(t+s)^{-1}x_0 \to x_0 + s \qquad \text{for } t \to \infty \tag{12.7}$$

hence $\alpha(t)f(t+1) > \alpha(t)f(t)$ and then $f(t+1) > f(t)$ for $t > t_0$. Finally note that for all $s \in \mathbb{R}$ for $t \to \infty$,

$$\frac{f(t+s) - f(t)}{f(t+1) - f(t)} = \frac{\alpha(t)f(t+s) - \alpha(t)f(t)}{\alpha(t)f(t+1) - \alpha(t)f(t)} \to \frac{x_0 + s - x_0}{x_0 + 1 - x_0} = s.$$

COROLLARY $f \in \Pi_0$ if f is measurable and $\beta(t)f(t+x) \to x$ for $t \to \infty$ for all x, for a function $\beta(t)$.

PROPOSITION 12.3 Suppose $f \in \Pi_0$ and $f(t+1) > f(t)$ for all $t \geq 0$. Define $\alpha(t) \in G$ by

$$\alpha(t)^{-1}0 := f(t)$$

$$\alpha(t)^{-1}1 := f(t+1).$$

Then α is measurable and satisfies (12.6).

PROOF We have $\alpha(t)^{-1}x = (f(t+1) - f(t))x + f(t)$ for $x \in \mathbb{R}$ and hence

$$\alpha(t)y = \frac{y - f(t)}{f(t+1) - f(t)}\ .$$

Substituting $y = \alpha(t+s)^{-1}0$ and $y = \alpha(t+s)^{-1}1$ gives the desired result.

COROLLARY If $f \in \Pi_0$, the convergence

$$\frac{f(t+x) - f(t)}{f(t+1) - f(t)} \to x \qquad \text{for } t \to \infty$$

is uniform on bounded x-intervals.

PROOF This follows from the analogous statement for the function α, proved in proposition 9.3.

PROPOSITION 12.4 Suppose $\alpha(t)$ and $\beta(t)$ vary like the translation $x - 1$. Define $f(t) = \alpha(t)^{-1}0$ and $g(t) = \beta(t)^{-1}0$. Then $\alpha(t) \sim \beta(t)$ for $t \to \infty$ if and only if f and g are related as follows

(12.8) for every $\varepsilon > 0$ there exists t_0 such that for $t \geq t_0$
$g(t - \varepsilon) < f(t) < g(t + \varepsilon)$.

PROOF Suppose (12.8) holds. We may write the inequalities above as

$$\beta(t - \varepsilon)^{-1}0 < \alpha(t)^{-1}0 < \beta(t + \varepsilon)^{-1}0.$$

The inequalities remain valid if we multiply on the left by $\beta(t)$. This gives

$$\beta(t)\beta(t - \varepsilon)^{-1}0 < \beta(t)\alpha(t)^{-1}0 < \beta(t)\beta(t + \varepsilon)^{-1}0. \tag{12.9}$$

The left hand side converges to $-\varepsilon$, the right hand side to $+\varepsilon$ for $t \to \infty$. Since $\varepsilon > 0$ is arbitrary, we find

$$\beta(t)\alpha(t)^{-1}0 \to 0 \quad \text{for } t \to \infty.$$

Similarly, multiplying on the left by $\beta(t - 1)$, we have

$$\beta(t - 1)\alpha(t)^{-1}0 \to 1 \quad \text{for } t \to \infty$$

this implies

$$\beta(t - 1)\alpha(t - 1)^{-1}1 \to 1 \quad \text{for } t \to \infty.$$

Hence $\beta(t) \sim \alpha(t)$ for $t \to \infty$.

Conversely if $\beta(t) \sim \alpha(t)$ for $t \to \infty$, then (12.9) holds for $\varepsilon > 0$ for $t \geq t_0$ and hence (12.8).

REMARK If f and g are non-decreasing we may express (12.8) simply as

$$f^{-1}(t) - g^{-1}(t) \to 0 \quad \text{for } t \to t^*$$

where f^{-1} is the inverse function to f, obtained by reflecting the graph of f in the diagonal, and similarly for g^{-1}, and t^* is the common upper bound of f and g (which may be finite). Compare the relation of tail equivalence introduced in Resnick [1971].

PROPOSITION 12.5 (von Mises [1936]) Let f be twice differentiable on $[0, \infty)$ and f' strictly positive on $[0, \infty)$. If $f''(x)/f'(x) \to 0$ for $x \to \infty$, then $f \in \Pi_0$.

PROOF Use the second remark after definition 12.2 with $\psi = f''/f'$.

PROPOSITION 12.6 Suppose $f \in \Pi_0$. There exists $g \in \Pi_0$ satisfying the conditions of proposition 12.5 such that (12.8) holds. We may even choose g to be C^∞.

PROOF Define $\alpha : [0, \infty) \to G$ as in proposition 12.3. By proposition 9.9 there exists $\beta : [0, \infty) \to G$ such that $\beta(t) \sim \alpha(t)$ for $t \to \infty$ and β is C^∞. This function β is defined in the remark to proposition 9.9 by

$$\beta(t)^{-1} = \int \alpha(t - s)^{-1} m(s)ds$$

where m is a non-negative C^∞-function with compact support, such that $\int m(s)ds = 1$ and $\int sm(s)ds = 0$.

Set $g(x) := \int f(x - s)m(s)ds = \beta(x)^{-1}0$. Then

$$g'(x) = \int f(x - s)m'(s)ds$$

$$g''(x) = \int f(x - s)m''(s)ds$$

and since $\int m'(s)ds = \int m''(s)ds = \int sm''(s)ds = 0$, and

$$\frac{f(x) - f(x - s)}{f(x) - f(x - 1)} = s + O(1) \qquad \text{for } x \to \infty$$

uniformly on bounded s-intervals, we have

$$g'(x).(f(x) - f(x - 1))^{-1} = \int (s + O(1))m'(s)ds \to 1$$

$$g''(x).(f(x) - f(x - 1))^{-1} = \int (s + O(1))m''(s)ds \to 0$$

which proves the proposition.

Compare Balkema and de Haan [1972].

Another more trivial characterization of Π_0 is the following. Let us call a sequence (x_n) normal if

$$x_{n+1} > x_n \qquad \text{for } n \geq n_0$$
$$\frac{x_n - x_{n-1}}{x_{n+1} - x_n} \to 1.$$

Obviously the sequence $f(n)$ is normal if $f \in \Pi_0$. Conversely suppose $(f(n))$ is normal, f is affine on each interval $(n, n+1)$ and f is continuous on $[0, \infty)$. It is not difficult to prove that this implies that $f \in \Pi_0$, and even that for any $g \in \Pi_0$ there exists such a broken linear function f such that (12.8) holds. (Compare propositions 9.5 and 9.6.)

DEFINITION 12.3 F_0 is the set of all $f \in M$ for which there exist continuous functions α and β from $[0, \infty)$ into G, which vary like the translation $x - 1$, such that $\beta(t)f\alpha(t)^{-1} \to \varepsilon$ for $t \to \infty$.

Clearly F_0 is the union of $D(\varepsilon, \alpha)$ over all continuous functions α from $[0, \infty)$ into G, which vary like the translation $x - 1$. Here $D(\varepsilon, \alpha)$ is the domain of attraction of ε with respect to the function α, see definition 5.1.

Since $D(\varepsilon, \alpha) = D(\varepsilon, \tilde{\alpha})$ if $\tilde{\alpha}(t) \sim \alpha(t)$ for $t \to \infty$ (even if $\tilde{\alpha}(t)\alpha(t)^{-1}$ is an arbitrary bounded function!), we may assume α to be the C^∞ function of proposition 12.6.

Suppose $f \in D(\varepsilon, \alpha)$, i.e.

$$\beta(t)f\alpha(t)^{-1} \to \varepsilon$$

and set $a(t) = \alpha(t)^{-1}0$. Set $b(t) = f(a(t))$. Then

$$\beta(t)b(t + x) = \beta(t)f(\alpha(t + x)^{-1}0) \to x$$

since $\alpha(t)\alpha(t + x)^{-1}0 \to x$, and hence $b(t) \in \Pi_0$.

In view of the remarks following proposition 12.1 we obtain the following result.

PROPOSITION 12.7 Suppose $f \in M$. Then $f \in F_0$ if and only if f contains the graph of a function ba^{-1} with $a, b \in \Pi_0$, $a(t)$ strictly increasing (and hence $b(t)$ non-decreasing). We may choose $a(t)$ to be C^∞ and to satisfy $a''(t)/a'(t) \to 0$ for $t \to 0$.

An intuitive geometric characterization of $f \in F_0$ is as follows.

Since $\beta(t)f\alpha(t)^{-1}$ certainly converges onto the set $L = \{(\theta, \theta) \mid 0 \leq \theta \leq 1\}$, one should be able to move this line segment L continuously in the plane, so that

1. the endpoints of L move along f,
2. fluctuations in length and slope of the transformed line segment $\beta(t)^{-1}L\alpha(t)$ should vanish for $t \to \infty$, and
3. asymptotically the transformed line segment should fit the curve.

We make this more explicit in the next proposition.

PROPOSITION 12.8 Suppose $f \in M$ contains a sequence of points $P_n = (x_n, y_n)$ such that

1. the sequences (x_n) and (y_n) are normal, i.e.

$$x_{n+1} > x_n \qquad \text{for } n \geq n_0$$
$$\frac{x_n - x_{n-1}}{x_{n+1} - x_n} \to 1$$

and similarly for (y_n),

2. f is asymptotically affine between P_n and P_{n+1}, i.e.

$$\beta_n f\alpha_n^{-1} \quad \text{converges onto the set } \{(\theta, \theta) \mid 0 \leq \theta \leq 1\}$$

where α_n and β_n are the unique elements of G such that for $n \geq n_0$

$$\beta_n P_n \alpha_n^{-1} = (0, 0)$$
$$\beta_n P_{n+1} \alpha_n^{-1} = (1, 1).$$

Then there exist continuous functions α and β from $[0, \infty)$ into G, which vary like the translation $x - 1$, such that $\alpha(n) = \alpha_n$, $\beta(n) = \beta_n$ and $\beta(t)f\alpha(t)^{-1} \to \varepsilon$ for $t \to \infty$.

PROOF Normality of the sequence (x_n) implies

$$\frac{0 - \alpha_n x_{n-1}}{1 - 0} = \frac{\alpha_n x_n - \alpha_n x_{n-1}}{\alpha_n x_{n+1} - \alpha_n x_n} = \frac{x_n - x_{n-1}}{x_{n+1} - x_n} \to 1.$$

Hence

$$\alpha_n\alpha_{n-1}^{-1}0 = \alpha_n x_{n-1} \to -1$$
$$\alpha_n\alpha_{n-1}^{-1}1 = \alpha_n x_n \to 0$$

which shows that

$$\alpha_n\alpha_{n-1}^{-1}x \to x - 1$$

and hence in general

$$\alpha_{n+k}\alpha_n^{-1}x = (\alpha_{n+k}\alpha_{n+k-1}^{-1}) \ldots (\alpha_{n+1}\alpha_n^{-1})x \to x - k.$$

The same argument applies to the sequence (β_n) and yields

$$\beta_{n+k}\beta_n^{-1}y \to y - k.$$

Now suppose $t \in \mathbb{R}$. Set $t = k + \theta$ with $k = [t]$ and $0 \le \theta < 1$. There exists a sequence of points $Q_n \in f$ such that

$$\beta_n Q_n \alpha_n^{-1} \to (\theta, \theta)$$

by definition of convergence onto. Hence

$$\beta_{n-k}Q_n\alpha_{n-k}^{-1} = \beta_{n-k}\beta_n^{-1}(\beta_n Q_n \alpha_n^{-1})\alpha_n\alpha_{n-k}^{-1}$$
$$\to (\theta+k, \theta+k) = (t, t).$$

This proves that $\beta_n f \alpha_n^{-1} \to \varepsilon$.

Let α and β be continuous functions such that

$$\alpha(n) = \alpha_n \quad , \quad \beta(n) = \beta_n$$
$$\alpha(t)\alpha_n^{-1} \sim \beta(t)\beta_n^{-1} \sim \gamma^\theta \qquad \text{for } 0 \le \theta < 1,\ t = n+\theta \to \infty$$

where $\gamma x = x - 1$, then $\beta(t)f\alpha(t)^{-1} \to \varepsilon$ for $t \to \infty$ (since $\alpha(t)\alpha_n^{-1}$ and $\beta(t)\beta_n^{-1}$ for $t = n+\theta$, $0 \le \theta < 1$, are bounded for $t \ge t_0$) and $\alpha(t)$ and $\beta(t)$ vary like the translation $x - 1$.

This proves the proposition.

This proposition has a surprising consequence. The set F_0 contains all curves $f \in M$ which have a positive derivative in at least one point.

We only give an outline of the proof.

Suppose $f = \{(x(t), y(t)) \mid t \in \mathbb{R}\}$ with x and y continuous non-decreasing and $x(t) + y(t) = 2t$. Assume $f'(0) = 1$. This implies that for any $c \in (0, 1)$ the curve f is asymptotically affine between the points Q_n and Q_{n+1}, where $Q_n = (x(-c^n), y(-c^n))$. This enables us to construct a sequence (P_n) in f with properties 1 and 2 of the proposition. (We let c tend to 1.)

13 On an equivalence relation for distributions

DEFINITION 13.1 H denotes the set of all curves $g \in M$ which do not contain a horizontal or vertical line segment.

Each element of H is the graph of a homeomorphism of an open interval onto an open interval. Conversely the graph of an increasing homeomorphism of an open interval onto an open interval lies in H if it is a closed subset of $\mathbb{R}^2$.

DEFINITION 13.2 Let $\underline{u}$ and $\underline{v}$ be real-valued random variables. We write

$$\underline{u} \overset{H}{\sim} \underline{v}$$

if there exists an element $h \in H$ such that $\underline{v} \overset{M}{=} h(\underline{u})$. We shall also write $F \overset{H}{\sim} G$ for the corresponding distribution functions.

The relation $\overset{H}{\sim}$ is an equivalence relation on the set of all distribution functions on $\mathbb{R}$. If $F \overset{H}{\sim} G$ then the probability distributions have the same number of discontinuities. These discontinuities are distributed in the same order along $\mathbb{R}$ and corresponding discontinuities have the same height. Also any non-degenerate interval of constancy of F, $\{x \mid F(x) = p\}$, with $0 < p < 1$, corresponds to a non-degenerate interval of constancy of G, $\{x \mid G(x) = p\}$, and vice versa.

THEOREM 13.1 If in addition to the basic situation (1.1) it is given that $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$, then $\underline{v} \overset{H}{\sim} \underline{u}$ or $\underline{v} \overset{M}{=} \phi(\underline{u})$ for some $\phi \in \Phi$.

PROOF The theorem follows from the proposition below.

PROPOSITION 13.1 If in addition to the basic situation (2.1), $\beta_n f \alpha_n^{-1}$ converges onto Λ and $\alpha_n \to \infty$, it is given that $\alpha_{n+1}\alpha_n^{-1} \to \varepsilon$, and if Λ contains two points on the same horizontal or vertical line, then $\Lambda \subset \phi$ for some $\phi \in \Phi$.

PROOF We may replace the sequences (α_n) and (β_n) by continuous functions α and β from $[0, \infty)$ into G, such that $\alpha(t) \to \infty$ for $t \to \infty$. See proposition 7.1.

We may also assume that $\beta(t) \to \infty$ for $t \to \infty$ since else $\Lambda \subset \phi$ with $\phi^{-1} \in M_0 \subset \Phi$ by proposition 7.5.

Because of the symmetry of the conditions we may assume that the two points lie on the same vertical line and that Λ is normal, see definition 8.6. In particular Λ contains the line segment joining the two points. We assume that the line segment is the segment $\{(0, y) \mid y_1 \le y \le y_2\}$ on the y-axis.

If Λ contains another line segment, not on the same line, then $\Lambda \subset \phi$ for some $\phi \in \Phi$ by proposition 8.6. If Λ contains a point in the half plane $\{x > 0\}$ and a point in the half plane $\{x < 0\}$, then $\Lambda \subset \phi$ with $\phi^{-1} \in M_0$, see case 3b of table 8.1 and the remarks on case 3a.

Hence to be specific we assume that $\Lambda \subset \{x \ge 0\}$, that $\Lambda \cap \{x > 0\}$ contains at least three points and that Λ contains no other line segments, then the one described above. (If $\Lambda \cap \{x > 0\}$ consists of one or two points then $\Lambda \subset \phi$ for some $\phi \in M_0$.)

Let g be a limit point of $g_t = \beta_t f \alpha_t^{-1}$ for $t \to \infty$ and let C be a guide set of g for Λ. See definition 8.1. Define

$$C_0 = \{(\sigma, \tau) \in C \mid \sigma 0 = 0\}.$$

Clearly $(\varepsilon, \varepsilon) \in C_0$. Suppose $(\sigma_0, \tau_0) \in C_0$ and $(\sigma, \tau) \in C$ with $\tau\tau_0^{-1}$ close to ε, so that the open intervals (y_1, y_2) and $\tau\tau_0^{-1}(y_1, y_2)$ intersect. Then $\sigma\sigma_0^{-1}0 = 0$ by proposition 8.6 and hence $\sigma 0 = 0$ and $(\sigma, \tau) \in C_0$. The set C_0 is both open and closed in C, it is non-empty, and C is connected. Hence $C_0 = C$.

The projection of C on the first coordinate is contained in the one-parameter subgroup of multiplications with centre 0. Since it is connected and unbounded, it has the form

$$\{\gamma^t \mid t \ge t_0\}$$

where $\gamma \ne \varepsilon$ is a multiplication with centre 0.

In the first part of chapter 8 we have seen that we may choose a sequence $(\sigma_n, \tau_n) \in C$ such that $\sigma_{n+1}\sigma_n^{-1} \to \varepsilon$, $\tau_{n+1}\tau_n^{-1} \to \varepsilon$ and

$$\tau_n g \sigma_n^{-1} \text{ converges onto } \Lambda.$$

(Indeed $\Lambda \subset \tau_n g \sigma_n^{-1}$ for all n.)

Define $\sigma(t) = \gamma^t$. Then $\sigma_n = \sigma(t_n)$ where $t_n \to \infty$ and $t_{n+1} - t_n \to 0$, and

$$\sigma(t) = (\sigma(t_{n+1})\sigma(t_n)^{-1})^{\theta}\sigma(t_n) \qquad \text{for } t = t_n + \theta(t_{n+1} - t_n).$$

If we define $\tau(t)$ similarly, then by proposition 7.1

$$\tau(t)g\sigma(t)^{-1} \text{ converges onto } \Lambda$$

and since σ varies like γ, theorem 11.1 yields $\Lambda \subset \phi$ for some $\phi \in \Phi$.

14 Applications

This chapter consists of five sections.

1 Khinchine's theorem
2 Extreme value theory
3 Limit distributions for giants
4 Order statistics
5 Random variables in a topological interval

The first two sections contain classical results. We present proofs in terms of the theory developed in this book. Acquaintance with the theory of the previous 13 chapters is not needed to understand these proofs. Rather these proofs should be seen as simple examples of the more general theory.

The third and fourth sections give outlines of some further applications. The final section considers the following more philosophical question. If we agree that one cannot say that something is twice as useful or one unit more useful than something else, why does one express utility by real numbers, and what is the sense in using affine norming transformations for such variables as utility, intelligence, sensitivity?

14.1 Khinchine's theorem on the convergence of types

THEOREM 14.1 (Khinchine [1938], Gnedenko [1943]). Suppose that

$$\alpha_n \underline{x}_n \to \underline{u} \quad \text{in distribution}$$
$$\beta_n \underline{x}_n \to \underline{v} \quad \text{in distribution}$$

where $\underline{x}_n$, $\underline{u}$ and $\underline{v}$ are real-valued random variables, and α_n and β_n positive affine transformations on $\mathbb{R}$. If $\underline{u}$ and $\underline{v}$ are non-constant, there exists a positive affine transformation γ on $\mathbb{R}$ such that

$$\beta_n \alpha_n^{-1} \to \gamma$$
$$\underline{v} = \gamma \underline{u} \quad \text{in distribution.}$$

PROOF Let $\underline{w}$ be a random variable with the standard normal distribution (or more generally with a strictly increasing, continuous distribution function). We choose non-decreasing functions f_n, g and h on $\mathbb{R}$ such that

$$\underline{x}_n \overset{M}{=} f_n(\underline{w})$$
$$\underline{u} \overset{M}{=} g(\underline{w})$$
$$\underline{v} \overset{M}{=} h(\underline{w}).$$

This is possible by theorem 2.1. The conditions above imply

$$\alpha_n f_n \to g \quad \text{weakly}$$
$$\beta_n f_n \to h \quad \text{weakly.}$$

See theorem 2.2 and corollary 2 to this theorem. The theorem above now is an immediate consequence of the following proposition.

PROPOSITION 14.1 Suppose

$$\alpha_n f_n \to g \quad \text{weakly}$$
$$\beta_n f_n \to h \quad \text{weakly}$$

with f_n, g and h non-decreasing functions on $\mathbb{R}$, and α_n and β_n positive affine transformations on R. If g and h are non-constant, then there exists a positive affine transformation γ on R such that

$$\beta_n \alpha_n^{-1} \to \gamma$$
$$h = \gamma g.$$

PROOF Let x_1 and x_2 be continuity points of both h and g such that $g(x_1) < g(x_2)$ and $h(x_1) < h(x_2)$. On setting $\beta_n \alpha_n^{-1} x =: a_n x + b_n$, we have

$$(14.1) \qquad a_n \alpha_n f_n(x) + b_n = \beta_n f_n(x).$$

Hence, substituting x_1 and x_2 and subtracting,

$$a_n(\alpha_n f_n(x_2) - \alpha_n f_n(x_1)) = \beta_n f_n(x_2) - \beta_n f_n(x_1).$$

We let n tend to ∞. Then a_n converges to a positive constant a, since $g(x_2) - g(x_1)$ and $h(x_2) - h(x_1)$ are positive, and

$$a(g(x_2) - g(x_1)) = h(x_2) - h(x_1).$$

On substituting $x = x_1$ in (14.1) it follows that $b_n \to b \in R$ and

$$ag(x) + b = h(x).$$

This proves the proposition if we set $\gamma x := ax + b$.

REMARK The reference to Khinchine [1938] is due to Mejzler [1965, page 206]. The second conclusion in the theorem is due to Khinchine, the first one to Gnedenko.

14.2 Extreme value theory

For a random variable $\underline{y}$ let $\underline{y}_{nn}$ denote the maximum of n independent random variables $\underline{y}_1,\dots,\underline{y}_n$, each distributed like $\underline{y}$. Recall that three kinds of limit law are possible for the sequence $\underline{y}_{nn}$ for $n \to \infty$. The distribution functions of these limit laws are usually denoted by Λ, Φ_λ and Ψ_λ, where λ is a positive constant, and, see Gnedenko [1943],

$$\Lambda(x) = e^{-e^{-x}}$$

$$\Phi_\lambda(x) = e^{-x^{-\lambda}} \qquad x > 0$$

$$\Psi_\lambda(x) = e^{-|x|^{\lambda}} \qquad x < 0.$$

We shall here give a new derivation of these limit laws. This derivation formed the source for the theory of the first part of the present work. It may serve as a concrete example of this theory.

We first observe that if $\underline{x}$ is distributed according to the limit distribution $\Lambda(x)$, then $\underline{x}_{nn}$ is distributed like $\underline{x} + \log n$.

$$\begin{aligned} P\{\underline{x}_{nn} \le x\} &= P\{\underline{x}_1 \le x, \ \dots, \ \underline{x}_n \le x\} \\ &= (P\{\underline{x} \le x\})^n \\ &= (\Lambda(x))^n = \Lambda(x - \log n). \end{aligned}$$

Now let $\underline{y}$ be an arbitrary random variable. There exists a non-decreasing function f on $\mathbb{R}$ such that $\underline{y}$ is distributed like $f(\underline{x})$. The function f is uniquely determined in its continuity points by the random variable $\underline{y}$. See theorem 2.1. Moreover the monotonicity of f implies that $\underline{y}_{nn}$ is distributed like $f(\underline{x}_{nn})$, and hence like $f(\underline{x} + \log n)$.

$$\begin{aligned} \underline{y}_{nn} &= \max(\underline{y}_1,\dots,\underline{y}_n) \\ &= \max(f(\underline{x}_1),\dots,f(\underline{x}_n)) \\ &= f(\max(\underline{x}_1,\dots,\underline{x}_n)) \\ &= f(\underline{x}_{nn}) = f(\underline{x} + \log n) \quad \text{in distribution.} \end{aligned}$$

Suppose there exist sequences of norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

(14.2) $\quad a_n \underline{y}_{nn} + b_n \to \underline{v}$ in distribution.

We set $\beta_n y = a_n y + b_n$, and $\alpha_n x = x - \log n$. Then α_n and β_n are positive affine transformations on $\mathbb{R}$, and we have the basic situation (1.1).

$$\alpha_n \underline{x}_{nn} = \underline{x} \text{ in distribution}$$

$$\beta_n \underline{y}_{nn} \to \underline{v} \text{ in distribution}$$

$$\underline{y}_{nn} \overset{M}{=} f(\underline{x}_{nn})$$

$$\alpha_n \to \infty.$$

Since $\Lambda(x)$ is strictly increasing on the whole real line, we may apply table 3.2 with Δ equal to the set of all translations, as one easily verifies. However, it is more instructive to work through the example in greater detail.

We write $\underline{v} \overset{M}{=} h(\underline{x})$ with h non-decreasing on $\mathbb{R}$. Then (14.2) becomes

$$\beta_n f(\underline{x} + \log n) \to h(\underline{x}) \text{ in distribution.}$$

Since $\Lambda(x)$, the distribution function of $\underline{x}$, is strictly increasing on the whole real line, this implies

(14.3) $\quad \beta_n f(x + \log n) \to h(x)$ weakly on $\mathbb{R}$.

See theorem 2.2 and proposition 2.3.

Because $\log(n+1) - \log n \sim \frac{1}{n} \to 0$, we may as well consider the limit relation

$$\beta(t) f(x + t) \to h(x) \text{ weakly on } \mathbb{R} \text{ for } t \to \infty.$$

See exercise 1.2 or proposition 7.1. This step is not essential for the argument, but it simplifies notation. In order to avoid trivialities, we assume h to be non-constant.

On comparing, for $s \in \mathbb{R}$, the two limit relations

$$\beta(t) f(x + t) \to h(x) \qquad \text{for } t \to \infty$$

$$\beta(t + s) f(x - s + t + s) \to h(x - s) \qquad \text{for } t \to \infty$$

we obtain by proposition 14.1 that there exists a positive affine transformation τ_s such that

$$\beta(t+s)\beta(t)^{-1} \to \tau_s \qquad \text{for } t \to \infty$$

$$\tau_s h(x) = h(x-s) \qquad \text{for } x \in \mathbb{R} \text{ such that } x \text{ and } x-s \text{ are}$$

continuity points of h. The first of these two relations forms the basis for the theory of regular variation developed in chapter 9, the second, for fixed $s \in \mathbb{R}$, is a particular case of the functional equation $\tau h = h\sigma$ which is studied in chapter 3. It is not hard to see that in our case the only non-decreasing, non-constant functions h which satisfy such an equation for all $s \in \mathbb{R}$, are

$$h(x) = b + ax$$

$$h(x) = b + ae^{x/\lambda}$$

$$h(x) = b - ae^{-x/\lambda}$$

with $\lambda > 0$, $a > 0$ and $b \in \mathbb{R}$. (For instance note that $\tau_r\tau_s = \tau_{r+s}$. This implies by exercise 1.4 and proposition 9.2 that $\tau_s = \tau^s$ for some $\tau \in G$ and all $s \in \mathbb{R}$. Then $h \in \Phi(\sigma, \tau)$ where σ is a translation, and $\tau \neq \varepsilon$ since h is non-constant. Thus $h \in \Phi_0$, see exercise 1.8.)

Combining the results we obtain that $\underline{v}$ is distributed like $h(\underline{x})$. We are only interested in limit types and may therefore assume $a = 1$ and $b = 0$. This yields for the three cases of h above

$$P\{\underline{v} \le v\} = P\{\underline{x} \le v\} = \Lambda(v),$$

$$P\{\underline{v} \le v\} = P\{e^{\underline{x}/\lambda} \le v\} = P\{\underline{x} \le \lambda\log v\} = \Phi_\lambda(v), \qquad v > 0,$$

$$P\{\underline{v} \le v\} = P\{-e^{-\underline{x}/\lambda} \le v\} = P\{\underline{x} \le -\lambda\log|v|\} = \Psi_\lambda(v), \qquad v < 0,$$

and these are the only non-degenerate limit laws possible for a sequence $(\underline{y}_{nn})$.

We shall now briefly consider limit laws for subsequences $(\underline{y}_{kk})$ where k runs through an unbounded set K of positive integers. The limit types in this case have been obtained by Mejzler [1965].

Instead of (14.3) we now only have convergence of a subsequence. We write this as

(14.4) $\beta_k f(x + \log k) \to h(x)$ weakly on $\mathbb{R}$ for $k \in K$.

It is convenient to introduce the set $\Delta_0 \subset (0, \infty)$, where we define $s \in \Delta_0$ if there exist sequences (k_n) and (k_n') in K such that $k_n \to \infty$ and

$$\log k_n' - \log k_n \to s.$$

(Thus $s \in \Delta_0$ if and only if $\sigma \in \Delta = \Delta(\alpha_k)$, where $\sigma x = x + s$, $s > 0$.)

Now suppose $s \in \Delta_0$. Then there exist sequences (k_n) and (k_n') in K such that $k_n \to \infty$ and $\log k_n' - \log k_n \to s$, and

$$\beta_{k_n} f(x + \log k_n) \to h(x)$$
$$\beta_{k_n'} f(x + \log k_n') \to h(x).$$

If h is non-constant we obtain by a slight modification of the proof of proposition 14.1

$$\beta_{k_n'}\beta_{k_n}^{-1} \to \tau_s$$
$$\tau_s h(x) = h(x - s)$$

with $\tau_s \in G$, $\tau_s \neq \varepsilon$.

We distinguish three cases.

1. Δ_0 is empty. In this case the subsequence $(\underline{y}_{kk})$, $k \in K$, is so thin that any limit is possible in (14.4) as one easily verifies. See also the first pages of chapter 4.

2. Every $s \in \Delta_0$ is an integral multiple of a fixed positive real number q and q is maximal, i.e. g.c.d. $\Delta_0 = q > 0$. In this case h satisfies the functional equation $\tau h(x) = h(x + q)$ for some element $\tau \in G$, with $\tau \neq \varepsilon$. Then h is one of the functions

$$h(x) = b + a(x + \pi(x))$$
$$h(x) = b + ae^{(x + \pi(x))/\lambda}$$
$$h(x) = b - ae^{-(x + \pi(x))/\lambda}$$

with $a > 0$, $\lambda > 0$, $b \in \mathbb{R}$ and π periodic modulo q such that $x + \pi(x)$ is non-decreasing. See table 3.1.

3. The elements of Δ_0 have no common divisor. In this case the periodic part π is a constant function which may be taken to be zero, and we obtain the three Gnedenko limit classes.

EXAMPLE Let $\underline{y}$ have a geometric distribution on the non-negative integers,

$$P\{\underline{y} = n\} = (1 - p)p^n \qquad n = 0,1,2,\ldots$$

with $p \in (0, 1)$ fixed. We shall see that $\underline{y}_{kk}$ converges in type if we take $K = \{k_1,k_2,\ldots\}$ with $k_n \sim p^{-n}$. We first determine f non-decreasing on $\mathbb{R}$ such that

$$\underline{y} \overset{M}{=} f(\underline{x}).$$

Clearly f is a step function which only takes the values $0,1,2,\ldots$. The function f has jumps of height 1 in the points $x_1 < x_2 < \ldots$, and is constant in between. Moreover

$$P\{\underline{x} > x_n\} = P\{f(\underline{x}) \geq n\} = p^n.$$

Hence we can determine the values of x_n from the equation

$$1 - \Lambda(x_n) = p^n$$

and since

$$\Lambda(x) = \exp - e^{-x} = 1 - e^{-x}(1 + O(e^{-x})) \qquad x \to \infty$$

we have

$$p^n = e^{-x_n}(1 + O(e^{-x_n})) \qquad n \to \infty$$

and therefore

$$x_n = -n \log p + O(p^n) \qquad n \to \infty.$$

Roughly speaking f is approximately equal to the integral part of x/λ for large values of x, where $\lambda = -\log p$. More precisely

$$f(x + \lambda n) - n \to [x/\lambda] \quad \text{weakly}$$

where [y] denotes the integral part of y. (Indeed if x is a continuity point of the limit function, then $x = k\lambda + \theta$ with $0 < \theta < \lambda$ and k integral. Then $f(x + \lambda n) = f(\theta + \lambda(n + k)) = n + k$ for $n \geq n_0$ since $m\lambda - x_m \to 0$ for $m \to \infty$.)

Thus if K is the set $\{k_1, k_2, \ldots\}$ with $k_n \sim p^{-n}$, and if $\beta_k y = y - n$ for $k = k_n$, then

$$\beta_k f(x + \log k) \to [x/\lambda] \quad \text{weakly}$$

and hence

$$\beta_k \underline{y}_{kk} \to \underline{v} \quad \text{in distribution}$$

where $\underline{v}$ is distributed like $[\underline{x}/\lambda]$.

14.3 Limit distributions for giants

Given a random variable $\underline{x}$ with distribution function $F(x)$ and tail distribution $R(x) = 1 - F(x)$ we may restrict this random variable $\underline{x}$ by considering only a fraction, e^{-t} say, of the total population. In particular we shall consider the fraction of large values of $\underline{x}$. This new random variable we denote by $\underline{x}_t$. It has tail distribution

$$R_t(x) = \min\ (1,\ e^t R(x)).$$

This situation occurs in practice if we study extreme weather conditions, for instance heat waves or storms. A slightly different problem has been studied by Balkema and de Haan in [1972'].

We shall now determine the possible limit types for $t \to \infty$.

Obviously if $\underline{x}$ has an exponential distribution, say $P\{\underline{x} > x\} = R(x) = e^{-x}$, then $\underline{x}_t = \underline{x} + t$ in distribution for all $t > 0$. Hence the exponential distribution is a limit distribution for $t \to \infty$.

Let $\underline{y}$ be an arbitrary random variable. Set $\underline{y} = f(\underline{x})$ with f non-decreasing on $(0, \infty)$. Then $\underline{y}_t = f(\underline{x}_t) = f(\underline{x} + t)$ in distribution. (Since the function f is non-decreasing, it maps the maximal e^{-t}-fraction of the x-population onto the maximal e^{-t}-fraction of the y-population.)

Suppose $\underline{y}_t$ can be normed to converge to a non-degenerate limit random variable $\underline{v}$ for $t \to \infty$, i.e. there exists a function $\beta : [0, \infty) \to G$ such that

$$\beta(t)\underline{y}_t \to \underline{v} \quad \text{in distribution for } t \to \infty.$$

Then $\underline{v} = h(\underline{x})$ for some h non-decreasing and non-constant on $(0, \infty)$, and

$$\beta(t)f(x + t) \to h(x) \quad \text{weakly on } (0, \infty).$$

For $s \in \mathbb{R}$, compare the two limit relations

$$\beta(t)f(x + t) \to h(x) \qquad x > 0$$

$$\beta(t + s)f(x + t) \to h(x - s) \qquad x - s > 0.$$

By the arguments of proposition 14.1 we have an element $\tau_s \in G$ such that

$$\beta(t+s)\beta(t)^{-1} \to \tau_s \qquad \text{for } t \to \infty$$

$$\tau_s h(x) = h(x-s) \qquad \text{for } x > 0,\ x - s > 0,$$

whenever h is non-constant on $(|s|, \infty)$. As in section 2 we find that $\tau_s = \tau^s$ for some $\tau \in G$, and hence that h is the restriction of ϕ to $(0, \infty)$ for some $\phi \in \Phi_0$.

Thus h is one of the following functions

$$h(x) = b + ax$$

$$h(x) = b + ae^{x/\lambda}$$

$$h(x) = b - ae^{-x/\lambda}$$

with $\lambda > 0$, $a > 0$ and $b \in \mathbb{R}$. For $b = 0$ and $a = 1$ the limit distribution tails have one of the forms

$$P\{\underline{v} > v\} = P\{\underline{x} > v\} = e^{-v} \qquad v > 0$$

$$P\{\underline{v} > v\} = P\{e^{\underline{x}/\lambda} > v\} = P\{\underline{x} > \lambda \log v\} = v^{-\lambda} \qquad v > 1$$

$$P\{\underline{v} > v\} = P\{-e^{-\underline{x}/\lambda} > v\} = P\{\underline{x} > -\lambda \log |v|\} = |v|^{\lambda} \qquad -1 < v < 0.$$

As in section 2 we can also develop the theory for sequences $t_n \to \infty$. We obtain similar results, except that there is one new possible degeneration (due to the fact that the support of the exponential distribution has a finite endpoint). If Δ_0 is the set of positive limit points of the double sequence $(t_n - t_m)$, and if $\inf \Delta_0 = q > 0$, then any non-decreasing function h which is constant on (q, ∞) may occur in the limit.

14.4 Order statistics

In the introduction to this book we considered the random variables $\underline{x}_{nk}$ and $\underline{y}_{nk}$. These were defined as the kth order statistic from a sample of size n from respectively the homogeneous distribution on (0, 1) and the distribution F(y).

It is well known that

$$P\{\underline{x}_{nk} \le x\} = \frac{1}{B(k,\ n+1-k)} \int_0^x t^{k-1}(1 - t)^{n-k}dt.$$

Suppose $n \to \infty$, $k \to \infty$ such that $\delta < \frac{k}{n} < 1-\delta$, where δ is a positive constant. Set

$$\gamma_n x = \frac{x - \mu}{\sigma} \qquad \text{with } \mu = k/n,\ \sigma^2 = \frac{\mu(1 - \mu)}{n},$$

then for the standardized variables $\underline{u}_{nk} = \gamma_n \underline{x}_{nk}$

$$P\{\underline{u}_{nk} \le u\} = c_n \int_{-\mu/\sigma}^{u} (\mu + \sigma s)^{k-1}(1 - \mu - \sigma s)^{n-k}ds$$
$$= c_n' \int_{-\mu/\sigma}^{u} \left(1 + \frac{\sigma s}{\mu}\right)^{k-1} \left(1 - \frac{\sigma s}{1 - \mu}\right)^{n-k} ds$$

and since

$$\psi(s) = \mu\log\left(1 + \frac{\sigma s}{\mu}\right) + (1 - \mu)\log\left(1 - \frac{\sigma s}{1 - \mu}\right)$$
$$= -\frac{s^2}{2n} + O(\sigma^3 s^3) \qquad \text{for } \sigma s \to 0,$$

and ψ is a concave function, the integrand converges boundedly to $e^{-s^2/2}$ and hence the distribution of $\gamma_n \underline{x}_{nk}$ converges to the standard normal distribution.

As noted in the introduction, $\underline{y}_{nk} = f(\underline{x}_{nk})$, where $f \in M$ is the inverse to the distribution function F_0 of $\underline{y}$.

If F has a strictly positive continuous derivative on (x_1, x_2), then f has a strictly positive continuous derivative on $(p_1, p_2) = (F(x_1 + 0), F(x_2 - 0))$. Suppose $n \to \infty$, $k \to \infty$ such that for some $\delta > 0$

$$p_1 + \delta < k/n < p_2 - \delta \qquad \text{for } n \ge n_0.$$

Then $\underline{y}_{nk}$ is asymptotically normal. Compare the last example in chapter 5.

We shall now consider the particular case that $k/n \to p \in (0, 1)$, in more detail. It will ease notation to use the norming transformation

$$\alpha_n x = \sqrt{n}(x - \frac{k}{n})$$

in which case $\alpha_n \underline{x}_{nk}$ converges to the normal random variable with mean zero and variance $p(1 - p)$.

If $F(x_0) = p$ and if F', the derivative, is positive and continuous in x_0, then $\underline{y}_{nk}$ is asymptotically normal $N(F^{-1}(\frac{k}{n}), \frac{p(1 - p)}{\sqrt{n}F'(x_0)})$. Indeed

$$\beta_n f(\alpha_n^{-1} x) = \sqrt{n}(f(\frac{x}{\sqrt{n}} + \frac{k}{n}) - f(\frac{k}{n})) \;/\; f'(p) \to x.$$

Now assume that there exists a sequence of positive affine transformations β_n such that $\beta_n \underline{y}_{nk} \to \underline{v}$ in distribution.

Observe that the convergence $k/n \to p$ does not determine the set Δ.

$$\alpha_m \alpha_n^{-1} x = \sqrt{m}\,(\frac{x}{\sqrt{n}} + \frac{k(n)}{n} - \frac{k(m)}{m})$$

$$(14.5) \qquad = \frac{\sqrt{m}}{\sqrt{n}} x + \frac{\sqrt{m}}{\sqrt{n}} t_n - t_m$$

where we define t_n by

$$k(n) = p.n + t_n \sqrt{n},$$

and hence $t_n = o(\sqrt{n})$ for $n \to \infty$.

Suppose $t_n \to t$, i.e.

$$k(n) = p.n + t\sqrt{n} + o(\sqrt{n}),$$

then every element $\sigma \in \Delta$ has the form

$$\sigma x = c(x + t) - t$$

and hence Δ is the one-parameter subgroup of all multiplications with centre $-t$. In this case $\underline{v} = \beta\phi(\underline{u})$ where $\beta \in G$, $\underline{u}$ is normal $N(0, p(1-p))$ and ϕ is one of the two functions

$$\phi(x) = \text{sign}\ (x + t), \text{ or}$$

$$\phi(x) = c_1(x + t)^\lambda \qquad x + t \geq 0$$

$$= -c_2|x + t|^\lambda \qquad x + t < 0$$

with λ and $c_1 + c_2$ positive, and c_1 and c_2 non-negative.

These limit distributions have been derived by Smirnov [1949]. Comparison with our standard list in the definition of Φ shows that only the function $\phi(x) = e^x$ is missing. (The function ϕ has to be defined on the whole real line, since $\underline{u}$ is normal, and $\phi(x) = x$ is obtained for $c_1 = c_2 = \lambda = 1$, $t = 0$.)

In order to obtain the limit $\phi(x) = e^x$, the set Δ should consist of translations. Hence, in view of (14.5) we must have

(14.6) $$\frac{m}{n} \to c > 1 \quad \text{implies} \quad \left|\frac{\sqrt{m}}{\sqrt{n}}\, t_n - t_m\right| \to \infty.$$

This is the case if $t_n \sim n^q$ with $q \in (0, \frac{1}{2})$. Then $m/n \to c > 1$ implies $t_m/t_n \to c^q$ and

$$\frac{\sqrt{m}}{\sqrt{n}}\, t_n - t_m = \left(\frac{\sqrt{m}}{\sqrt{n}} - \frac{t_m}{t_n}\right)t_n \to \infty$$

since $t_n \to \infty$ and $\frac{\sqrt{m}}{\sqrt{n}} - \frac{t_m}{t_n} \to c^{1/2} - c^q > 0$.

We obtain the following result.

THEOREM 14.2 Let $\underline{y}_{nk}$ be the kth order statistic from a sample of size n drawn from a distribution F(y). Suppose that

1. $n \to \infty$, $k = k(n) \to \infty$ and $\delta > 0$ so that
   a) $\delta < k(n)/n < 1-\delta \qquad$ for all $n \geq n_0$
   b) $k(n+1) - k(n) = o(\sqrt{n}) \qquad$ for $n \to \infty$,
2. there exist $a_n > 0$ and $b_n \in \mathbb{R}$ so that
   $a_n\underline{y}_{nk(n)} + b_n \to \underline{v}$ in distribution, $\underline{v}$ non-constant.

Then

$$\underline{v} = \phi(\underline{u}) \quad \text{in distribution}$$

where $\underline{u}$ has the standard normal distribution and $\phi \in \Phi$ is one of the functions

$$\phi(x) = b + ax$$

$$\phi(x) = b + a \text{ sign } (x - x_0)$$

$$\phi(x) = b + ae^{\lambda x}$$

$$\phi(x) = b - ae^{-\lambda x}$$

$$\begin{aligned} \phi(x) &= b + a_1(x - x_0)^\lambda \qquad x \geq x_0 \\ &= b - a_2(x_0 - x)^\lambda \qquad x < x_0 \end{aligned}$$

with λ, a and $a_1 + a_2$ positive, a_1 and a_2 non-negative and x_0 and b real constants.

PROOF Condition 1 implies $\mu_{n+1} - \mu_n = o(1/\sqrt{n})$, $\sigma^2_{n+1} \sim \sigma^2_n$ and $(\mu_{n+1} - \mu_n)/\sigma_n \to 0$. Hence $\gamma_{n+1}\gamma_n^{-1} \to \varepsilon$. It follows that ε is a condensation point of Δ, see proposition 7.2. Hence, if $\beta_n \underline{y}_{nk} \to \underline{v}$ in distribution, then $\underline{v} \overset{M}{=} \phi(\underline{u})$ with $\phi \in \Phi$, by theorem 3.1.

Smirnov [1949] has shown that the first two and the last function ϕ in our list do indeed occur. Since we can choose k(n) to satisfy

$$k(n)/n \to p \in (0, 1)$$

$$k(n + 1) - k(n) = o(\sqrt{n}) \qquad \text{for } n \to \infty$$

$$k(n) = p.n + t_n\sqrt{n}$$

$$t_n \sim n^{1/3} \qquad \text{for } n \to \infty$$

in which case Δ is the one-parameter subgroup of the translations, also the third and fourth function ϕ in the list above occur.

14.5 Random variables in a topological interval

The reader will have noticed that the applications in the preceeding four sections hardly make use of the theory beyond chapter 3. Moreover the theory is applied in very straight-forward cases. The limit variable has a continuous distribution function, strictly increasing on the whole real line, or the norming transformations α are translations.

The reason for developing the theory in greater generality is partly due to personal curiosity. The question "in how far does the basic situation (1.1) imply the functional relation (1.2)?" seems to be a sensible one to ask. Since the formulation is simple, one might expect a simple answer.

There is a second, more practical reason for doing research in this subject.

The random variable reflects with a high degree of precision the variability or uncertainty of the corresponding quantity in real life. The correspondence itself exhibits a certain amount of arbitrariness. Even if we consider variables like length, time, temperature in the exact sciences, the values of the corresponding random variables depend on the scale which is used. There is obviously a difference in the values of the random variable if we measure temperature in degrees Celsius or in degrees Fahrenheit. A change of units corresponds to an affine or linear transformation of the associated numerical random variable. Hence the significance of the concept of distribution type. The type reflects the behaviour of the physical quantity and is independent on the units employed in measuring the quantity. (A second reason for introducing the concept of type, viz. that it allows us to formulate asymptotic results, has already been mentioned in the introduction to this book.)

As soon as we go beyond such simple random variables as length or time, the situation is more involved. To describe the random variable "size" in a population of potatoes say, we could use the variable $\underline{x}$ = diameter. The variable $\underline{x}^3$ = volume gives as good a description. And also the variables $\underline{x}^{-3}$ = number per cube metre, or log $\underline{x}$ (for reasons of symmetry) are valid as numerical descriptions of the physical quantity "size" in our population. Note that except in trivial cases, the corresponding four distribution functions are all of different type.

The situation becomes even more awkward if we leave the physical sciences and wish to measure quantities like utility, intelligence, eye sight, retention of knowledge, sensitivity to heat, or any other of the many one dimensional variables which one encounters in physiology, psychology, economics or the social sciences. In these cases any strictly increasing continuous function of the random variable is as good a description as any other. (Any two such numerical descriptions are equivalent in the sense of the equivalence relation $\overset{H}{\sim}$ introduced in chapter 13.)

These random variables may be said to take their values in the oriented topological interval T (homeomorphic to a non-empty open interval of $\mathbb{R}$). A numerical description of the variable then is a strictly increasing, continuous, real-valued function on T.

Now suppose we are given a sequence of random variables with values in T. (Since we shall only be interested in the limit behaviour of the sequence of associated probability distributions on T, we may as well assume that we are given a sequence of probability measures on T.) Let $(\underline{x}_n)$ be a numerical description of this sequence of random variables on T, and let $(\underline{y}_n)$ be another numerical description of the same sequence. Then $\underline{y}_n = f(\underline{x}_n)$ for some strictly increasing, continuous function f.

If both these sequences converge in type, to respectively $\underline{u}$ and $\underline{v}$, then, under certain conditions C, we have the relation $\underline{v} \overset{M}{=} \phi(\underline{u})$ with $\phi \in \Phi$. Hence in this case the limit, if it exists, is unique (up to a transformation $\phi \in \Phi$). Moreover there is only a fairly small class of numerical descriptions $x : T \to \mathbb{R}$, such that the sequence $(\underline{x}_n)$ converges in type.

Roughly speaking we can say that to certain (divergent) sequences of probability measures on T we can associate a "limit" probability measure on $\mathbb{R}$, which is unique up to a transformation $\phi \in \Phi$.

We conclude by giving four possible lines of further investigation in this subject.

1 Develop a theory of T-valued random variables. Since we have abandoned the algebraic structure of $\mathbb{R}$ in the definition of T, we cannot add T-valued random variables, nor can we define their expectation. However the maximum as well as the kth order statistic from a sample of size n of independent T-valued random variables is again a well defined T-valued random variable. So too is the restriction of a T-valued random variable to the e^{-t}-subinterval of large values, see section 3 of this chapter.

2 One would like to have the conditions C, under which (1.1) implies (1.2), phrased in terms of the sequence of T-valued random variables. (By Khinchine's theorem we may replace the condition $\alpha_n \to \infty$ by the condition

that the sequence of probability distributions on T should not contain any subsequence which converges to a non-degenerate probability distribution on T. See exercise 1.1.) Also a simple formulation of the condition $\alpha_{n+1} \sim \alpha_n$ and a description of regular variation of the norming constants in terms of probability distributions would be welcome.

3 There seems to be a certain incongruity in using affine norming transformations $\alpha\underline{x} = a\underline{x} + b$ if $\underline{x}$ describes a random variable on T. Since T has no algebraic structure, the transformation $\underline{x} \mapsto \alpha\underline{x}$ can have no physical significance. There are practical reasons for using these transformations to norm the random variables. G is a finitely dimensional group of continuous strictly increasing transformations on $\mathbb{R}$. Are there any other useful norming transformations?

4 Let $(\underline{x}_n)$ be an arbitrary sequence of real-valued random variables. Suppose the sequence contains no subsequence which converges in distribution to a non-constant random variable. Problem: Give conditions which ensure that there exists a non-decreasing function f such that $f(\underline{x}_n)$ converges in type.

References

VAN AARDENNE-EHRENFEST, T., N.G. DE BRUYN & J. KOREVAAR (1949). A note on slowly oscillating functions. Nieuw Arch. Wisk. II 23, 77-86.

BALKEMA, A.A. & L. DE HAAN (1972). On R. von Mises' condition for the domain of attraction of $\exp(-e^{-x})$. Ann. Math. Stat. 43, 1352-1354.

BALKEMA, A.A. & L. DE HAAN (1972'). Residual life time at great age. Technical report 36, Dept. of Statistics, Stanford University.

BANACH, S. (1920). Sur l'équation fonctionelle $f(x+y) = f(x) + f(y)$. Fund. Math. 1, 123-124.

BELLMAN, R. & K.L. COOKE (1963). Differential-difference equations. Academic Press, New York.

BINGHAM, N., E. SENETA & J.L. TEUGELS (1974). Functions of regular variation. Wiley, New York.

DIEUDONNÉ, J. (1969). Foundations of modern analysis. Academic Press, New York.

DOETSCH, G. (1956). Handbuch der Laplace-Transformation III. Birkhäuser, Basel.

FRÉCHET, M. (1951). Sur les tableaux de corrélation dont les marges sont données. Ann. Univ. de Lyon III 14 A, 53-77.

GNEDENKO, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. Ann. of Math. 44, 423-453.

DE HAAN, L. (1971). A form of regular variation and its application to the domain of attraction of the double exponential distribution. Z. Wahrscheinlichkeitstheorie und verw. Gebiete 17, 241-258.

DE HAAN, L. (1970). On regular variation and its application to the weak convergence of sample extremes. Mathematisch Centrum, Amsterdam.

HEWITT, E. & K. STROMBERG (1965). Real and abstract analysis. Springer, Berlin.

HILB, E. (1918). Zur Theorie der linearen funktionalen Differentialgleichungen. Math. Ann. 78, 137-170.

HOPF, H. (1937). Ueber die Sehnen ebener Kontinuen und die Schleifen geschlossener Wege. Comm. Math. Helv. 9, 303-319.

INCE, E.L. (1926). Ordinary differential equations. Longmans, London.

KARAMATA, J. (1930). Sur un mode de croissance régulière de fonctions. Mathematica Cluj 4, 38-53.

KARAMATA, J. (1933). Sur un mode de croissance régulière. Bull. Soc. Math. France 61, 55-62.

KHINCHINE, A.Ya. (1938). Limit distributions for sums of independent random variables. ONTI, in Russian.

MEJZLER, D. (1965). On a certain class of limit distributions and their domain of attraction. Trans. Amer. Math. Soc. 117, 205-236.

VON MISES, R. (1936). La distribution de la plus grande de n valeurs. Amer. Math. Soc. selected papers II, 271-294.

MONTGOMERY, D. & L. ZIPPIN (1955). Topological transformation groups. Interscience, New York.

PARTHASARATHY, K.R. (1967). Probability measures on metric spaces. Academic Press, New York.

PONTRJAGIN, L.S. (1957/8). Topologische Gruppen I & II. Teubner, Leipzig.

RESNICK, S.I. (1973). Limit laws for record values. Stochastic processes and their appl. 1, 67-82.

RESNICK, S.I. (1971). Tail equivalence and applications. J. Appl. Probability 8, 136-156.

SMIRNOV, N.V. (1962). Limit distributions for the terms of a variational series. Amer. Math. Soc. Translations I 11, 82-143. Original publication 1949 in Russian.

WHYBURN, G.T. (1942). Analytic topology. Amer. Math. Soc., Princeton.

WIDDER, D.V. (1946). The Laplace transform. Princeton University Press.