CARDINAL FUNCTIONS IN TOPOLOGY

I. JUHÁSZ

IN COLLABORATION WITH

A. VERBEEK

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MATHEMATICAL CENTRE TRACTS

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PREFACE

General topology can be considered as a natural outgrowth of set theory; the simple set theoretic nature of its fundamental notions makes it an appropriate area for the application of set theoretic methods. On the other hand, many set theoretic problems have their roots in topology and this makes the interaction between the two disciplines even more profound. The closeness of their relationship is perhaps most apparent in the work done by the Moscow school of topology in the early twenties.

The last decade has witnessed a very rapid development of set theoretic methods and ideas, the main sources of which were, in our opinion, the following: 1) the independence results of P. Cohen and his followers; 2) the results on "large" cardinals of A. Tarski's school, and 3) the achievements of P. Erdös, R. Rado, A. Hajnal, and others in combinatorial set theory (e.g., partition calculus). Not surprisingly, this has stirred up a renewed interest in the set theoretic aspects of general topology. A number of old problems were settled and many new ones were raised.

The aim of this tract is to present a variety of questions of this kind by centering them around the unifying concept of cardinal functions.

Since a considerable part of the means employed in our investigations are relatively recent and not easily accessible in the literature, we have found it both convenient and timely to include an appendix entirely devoted to the detailed explanation of these methods and ideas of combinatorial set theory.

This tract was written during the second half of 1969, while the author was a guest of the Department of Pure Mathematics of the Mathematical Centre in Amsterdam. The appendix is based on a series of talks given by the author during the same period at the Mathematical Centre under the title "Combinatorial Set Theory".

At this point I wish to express my gratitude toward the Mathematical Centre for their kind hospitality which gave me the opportunity to write this tract, as well as for publishing it. I am particularly grateful to Professors J. de Groot and P.C. Baayen for initiating my invitation and supporting this project.

Special thanks are also due to Albert Verbeek, who took on the difficult task of actually writing the text of the appendix, and did most of the work necessary to turn the crude manuscript into print. I would also like to thank Nelly Kroonenberg, who added A6 to the appendix.

Finally, I am greatly indebted to my friend and collegue A. Hajnal, whose help was essential in acquiring the methods used in this tract.

Budapest, December, 1970. István Juhász.

CONTENTS

(* denotes a minor change or correction in the second edition)

CARDINAL FUNCTIONS IN TOPOLOGY by I. JUHÁSZ

0. Notation and preliminaries

0.1. For the set theoretical notations used here we refer the reader to the appendix (p. 72ff).

0.2. For a topological space X we denote by $\sigma(X)$ the set of all open subsets of X. We use the notation $^-$ to indicate closure and Int for interior.

0.3. $A \subset X$ is called a $G_{\delta,\xi}$ set iff there is an $\mathcal{O}\!\mathcal{l} \subset \sigma(X)$ with $|\mathcal{O}\!\mathcal{l}| \leq \omega_\xi$ and $A = \cap \mathcal{O}\!\mathcal{l}$. The complements of $G_{\delta,\xi}$ sets are called $F_{\sigma,\xi}$.
We put

$$\sigma_\xi(X) = \{A \subset X\colon A \text{ is a } G_{\delta,\xi} \text{ set}\}.$$

Thus e.g. $\sigma_0(X)$ is the set of all G_δ's in X.

0.4. A space S is called <u>right</u> (or <u>left</u>) <u>separated</u> iff there is a well-ordering < of S such that every initial (or final) segment of S under < is open. It is easy to see that X has a right (or left) separated subspace of cardinality α iff it contains a by inclusion increasingly (or decreasingly) well-ordered sequence $\{G_\xi\colon \xi < \alpha\}$ of open sets in X.

0.5. (cf. [11]) The following assertions can be verified easily:

(i) If S is right separated by $<$ which well-orders S in type α, α regular, then S has an open covering $\mathcal{U}$ such that every subcover of $\mathcal{U}$ is of cardinality α.

(ii) If S is left separated by $<$ which well-orders S in type α, α regular, then every dense subset of S is of cardinality α.

0.6. A subset $D \subset X$ is called <u>discrete</u> iff every $p \in D$ has a neighbourhood U_p in X such that $D \cap U_p = \{p\}$. We denote by $D(\alpha)$ the discrete space on α = the set of ordinals smaller than α (see appendix).
A sequence $\{p_\xi : \xi < \lambda\}$ of points of X is called <u>free</u> (cf. [3] iff $\{p_\xi : \xi < \eta\}$ and $\{p_\xi : \eta \leq \xi < \lambda\}$ have disjoint closures for every $\eta < \lambda$. Obviously, every free sequence is discrete.
A, by inclusion, decreasing sequence $\{G_\xi : \xi < \lambda\} \subset \sigma(X)$ is called a <u>strongly decreasing chain</u> iff $\xi < \eta < \lambda$ implies

$$\bar{G}_\eta \subset G_\xi \ .$$

If $\{G_\xi : \xi < \lambda\}$ is as above and

$$p_\xi \in G_\xi \setminus \bar{G}_{\xi+1} \qquad (\text{for } \xi < \lambda),$$

then, obviously, $\{p_\xi : \xi < \lambda\}$ is a free sequence.

0.7. If $F \subset X$, $\mathcal{L} \subset \sigma(X)$ is called a <u>neighbourhood basis</u> for F iff $F \subset G \in \sigma(X)$ imply the existence of a $B \in \mathcal{L}$ with $F \subset B \subset G$.
We put

$$\chi(F,X) = \min\{|\mathcal{L}| : \mathcal{L} \text{ is a neighbourhood basis for } F\}.$$

If $p \in X$, we write $\chi(p,X)$ instead of $\chi(\{p\},X)$.

0.8. If X is a T_1 space, $F \subset X$, we introduce the following definition

$$\psi(F,X) = \min\{\omega_\xi : F \in \sigma_\xi(X)\}.$$

Here too we write $\psi(p,X)$ instead of $\psi(\{p\},X)$.
It is well-known and easy to prove (cf. [1]) that if X is a compact T_2 space and $F \subset X$ is closed, then

$$\psi(F,X) = \chi(F,X).$$

0.9. If $p \in X$, we define

$$\partial(p,X) = \min\{\alpha : p \in \bar{A} \rightarrow \exists B \subset A \text{ with } p \in \bar{B} \text{ and } |B| \leq \alpha\}.$$

0.10. X is called <u>α-Lindelöf</u> iff every open covering of X has a subcover of cardinality $\leq \alpha$.
It can easily be shown that a compact T_2 space X is hereditary ω_ξ-Lindelöf (i.e. every subspace of X is ω_ξ-Lindelöf) iff every closed subset of X is a $G_{\delta,\xi}$ set, or equivalently, every open set is an $F_{\sigma,\xi}$ set.

0.11. X is called <u>α-separable</u> iff it has a dense subset of cordinality $\leq \alpha$.

0.12. X is said to have the <u>α-Baire</u> property iff it is not the union of α nowhere dense sets.

0.13. We say that α is a <u>caliber</u> for X iff for every $\mathcal{U} \subset \sigma(X)$ with $|\mathcal{U}| = \alpha$ there is a $\mathcal{U}' \subset \mathcal{U}$ with $|\mathcal{U}'| = \alpha$ and $\cap\mathcal{U}' \neq \emptyset$.

0.14. The topological product of the spaces R_i, $i \in I$ will be denoted by $R = X\{R_i : i \in I\}$. If I is finite (say $I = \{1,\ldots,k\}$) we also write $R = R_1 \times \ldots \times R_k$.
The projection onto the i^{th} factor is denoted by π_i. If $J \subset I$, π_J denotes the projection onto the partial product $X\{R_i : i \in J\}$.
Open subsets of the product which have the form

$$\pi_{i_1}^{-1}(U_1) \cap \ldots \cap \pi_{i_n}^{-1}(U_n) \qquad (U_s \in \sigma(R_{i_s}))$$

are called <u>elementary open sets</u>.
Similarly, a set is an <u>elementary $G_{\delta,\xi}$ set</u> iff it is the intersection

of $\leq \omega_\xi$ elementary open sets.

0.15. $X \underset{top}{\subset} Y$ (or $X \underset{cl}{\subset} Y$) means that there is a (closed) subspace of Y which is homeomorphic to X.

0.16. We use $\mathcal{T}$ to denote the class of all topological spaces. Similarly, $\mathcal{T}_i$ denotes the class of all T_i spaces. We have $\mathcal{T}_i \supsetneqq \mathcal{T}_j$ if $0 \leq i < j \leq 5$. We denote by $\mathcal{T}_\rho$ the class of all completely regular spaces which are not necessarily T_0. Then $\mathcal{T}_{3\frac{1}{2}} = \mathcal{T}_0 \cap \mathcal{T}_\rho$.
$\mathcal{C}$ denotes the class of all compact T_2 spaces.

0.17. Let $(L,\leq)$ be a linearly ordered set. We denote by (a,b), $[a,b)$, $(a,b]$ and $[a,b]$ respectively the open, half open and closed intervals of L. The <u>order topology</u> for L is the one for which the open intervals form a basis.
We denote by $\tilde{L}$ the <u>Dedekind completion</u> of L (including the degenerate cuts $\emptyset$ and L as first and last element):

$$\tilde{L} = \{A \subset L:\ A = \cup\{\{b \in L:\ b < a\}:\ a \in A\} \text{ and } A \in \sigma(L)\},$$

$\tilde{L}$ being (linearly) ordered by inclusion. L is embedded in $\tilde{L}$ by mapping $a \in L$ onto $\{b \in L:\ b < a\} \in \tilde{L}$. Then $\tilde{L} \in \mathcal{C}$ and as can easily be seen, the subspace topology of L in $\tilde{L}$ coincides with the original order topology. (This is in general false for subspaces of ordered spaces!)
$\mathcal{L}$ denotes the class of all linearly ordered spaces.

0.18. A space X is called <u>dispersed</u> iff every subspace $S \subset X$ has isolated points. We denote by $\mathfrak{D}$ the class of all dispersed spaces.

0.19. A T_3 space X is called <u>cocompact</u> (cf. [9], [33]) iff there is an open basis $\mathcal{B}$ for X such that if $\mathcal{F} \subset \mathcal{B}$, and $\mathcal{F}$ has the finite intersection property, then $\cap\{\bar{F} : F \in \mathcal{F}\} \neq \emptyset$. (Note that the def. in [9] is not equivalent!)

0.20. A T_2 space X is called <u>strongly Hausdorff</u> iff from every infinite subset $A \subset X$ we can choose a sequence $\{p_n : n < \omega\}$ such that the p_n have pairwise disjoint neighbourhoods in X. We denote by $\mathcal{H}$ the class of all strongly Hausdorff spaces. It can be shown (cf. [12]) that

$$\mathcal{T}_2 \subsetneqq \mathcal{H} \subsetneqq \mathcal{T}_3.$$

1. Cardinal functions

As we have mentioned above, the aim of this work is to present a systematic study of certain cardinality problems arising in the theory of topological spaces. To achieve this, we shall introduce the notion of a cardinal function by means of which most of the questions we are concerned with can be given a more or less unified treatment.
A function ϕ defined on a class $\mathcal{C}$ of topological spaces is called a <u>cardinal function</u> if it assigns to each member $X \in \mathcal{C}$ a (usually infinite) cardinal number $\phi(X)$.
Now we shall list the cardinal functions to be examined in what follows:

1.1. <u>Weight</u>

$w(X) = \min^{*}\{|\mathcal{B}| : \mathcal{B}$ is an open basis (or: open subbasis) for $X\}$.

1.2. <u>π-weight</u>

$\pi(X) = \min^{**}\{|\mathcal{P}| : \mathcal{P}$ is a π-basis of $X\}$,

where $\mathcal{P}$ is a <u>π-basis</u> for X iff

$\mathcal{P} \subset \sigma(X)\setminus\{\emptyset\}$ and $(\forall U \in \sigma(X)\setminus\{\emptyset\})(\exists V \in \mathcal{P},\ V \subset U)$.

1.3. Uniform weight

$u(X) = \min^{*}\{|\mathfrak{U}| : \mathfrak{U}$ is a (sub)base for a uniform structure compatible with $\sigma(X)\}$.

Here, of course, $X \in \mathcal{T}_\rho$ is assumed.

1.4. Density

$d(X) = \min^{*}\{|S| : S \subset X,\ \bar{S} = X\}$.

1.5. Cellularity

$c(X) = \sup^{*}\{|\mathfrak{V}| : \mathfrak{V} \subset \sigma(X), \mathfrak{V}$ disjoint$\}$,

and analogously

$c_\xi(X) = \sup^{*}\{|\mathfrak{V}| : \mathfrak{V} \subset \sigma_\xi(X), \mathfrak{V}$ disjoint$\}$.

1.6. Spread

$s(X) = \sup^{*}\{|D| : D \subset X$, D discrete as a subspace$\}$.

1.7. Height

$h(X) = \sup^{*}\{|M| : M \underset{top}{\subset} X$, M is right-separated$\}$.

1.8. Width

$z(X) = \sup^{*}\{|Z| : Z \underset{top}{\subset} X$, Z is left-separated$\}$.

1.9. Depth

$k(X) = \sup^{*}\{|\mathfrak{V}| : \mathfrak{V}$ is a strongly decreasing chain in $X\}$.

1.10. Lindelöf degree

$L(X) = \min^{*}\{\alpha : X$ is α-Lindelöf$\}$.

1.11. Character

$\chi(X) = \sup\{\chi(p,X) : p \in X\}$.

1.12. Pseudo-character

$\psi(X) = \sup\{\psi(p,X) : p \in X\}$.

1.13. Tightness

$\partial(X) = \sup\{\partial(p,X) : p \in X\}$

here, $X \in \mathcal{T}_1$ is assumed.

Remark

In the above definitions

$$\min^{*}\{.\} = \omega.\min\{.\}$$

and

$$\sup^{*}\{.\} = \omega.\sup\{.\}.$$

If ϕ is one of the functions χ, ψ or ∂, then $\phi(X) = 1 \leftrightarrow X$ is discrete. In every other possible case, however, each occuring function is infinite.

2. Interrelations between cardinal functions

2.1. Trivial inequalities.

a) $k(X) \leq c(X) \leq d(X) \leq \pi(X) \leq w(X)$

b) $w(X) \leq \exp|X|$; $d(X) \leq |X|$

c) $c(X) \leq s(X) \leq \min\{h(X), z(X)\} \leq h(X).z(X) \leq \min\{|X|, w(X)\}$
moreover
$c(X) \leq c_\xi(X) \leq c_\eta(X)$, if $\xi \leq \eta$

d) $\psi(X) \leq \min\{|X|, \chi(X)\}$
$\partial(X) \leq \min\{\chi(X), \sup\{d(Y): Y \subset X\}\} \leq |X|$
$\chi(X) \leq w(X) \leq \chi(X).|X|$ and $\chi(X) \leq u(X)$
$\pi(X) \leq d(X).\chi(X)$

2.2. If $X \in \mathcal{T}_0$, then $|X| \leq \exp w(X)$. Indeed, assume that $\mathcal{B}$ is a basis for X, $|\mathcal{B}| \leq w(X)$. Then $x, y \in X$, $x \neq y$ imply

$$\{B \in \mathcal{B} : x \in B\} \neq \{B \in \mathcal{B} : y \in B\},$$

since X is T_0, hence there is a 1-1 map of X into $\mathcal{P}(\mathcal{B})$.

Remark

A.V. Arhangel'skiĭ [2] proved that for a rather large class of spaces X, which includes metric and Čech-complete spaces, $w(X) \leq |X|$ holds.

2.3. If $X \in \mathcal{T}_3$, $S \subset X$ is dense in X and $p_0 \in S$, then

(i) $w(X) \leq \pi(X)^{c(X)} \leq \exp d(X)$

(ii) $\pi(S) = \pi(X)$

(iii) $\chi(p_0,S) = \chi(p_0,X)$.

Proof

For $A \subset X$ and $\mathfrak{U} \subset P(X)$ write $\tilde{A} = \mathrm{Int}(\bar{A})$ and $\tilde{\mathfrak{U}} = \{\tilde{A} : A \in \mathfrak{U}\}$. The set A is called regular open if $A = \tilde{A}$. Three observations are crucial: $\tilde{\tilde{A}} = \tilde{A}$; the family $\mathcal{R}(X)$ of regular open sets forms a basis for X; and if A is regular open then $A = \widetilde{A \cap S}$. Hence, if $|S| = d(X)$,

$$w(X) \leq |\mathcal{R}(X)| \leq |P(S)| = \exp d(X).$$

Next it is easy to check that if $\mathfrak{U}$ is a π-basis or a neighbourhood of p_0 basis in the subspace S, then $\tilde{\mathfrak{U}}$ is a π-basis or a neighbourhood basis of p_0 in X. Hence (ii) and (iii) hold.

Now assume that $\mathfrak{U}$ is a π-basis of X and $|\mathfrak{U}| = \pi(X)$. Define a function $\phi : \mathcal{R}(X) \to P(\mathfrak{U})$ such that

$\phi(A)$ is a maximal, disjoint subfamily of $\mathfrak{U} \cap P(A)$.

Since $\phi(A)$ is a disjoint open family $|\phi(A)| \leq c(X)$, and thus there are at most $\pi(X)^{c(X)}$ such subfamilies in $\mathfrak{U}$. Next, ϕ is 1-1: if $A,B \in \mathcal{R}(X)$ and $A \neq B$, then $A\setminus\bar{B}$ or $B\setminus\bar{A}$ is nonempty and hence $\phi(A) \neq \phi(B)$. This shows that

$$w(X) \leq |\mathcal{R}(X)| = |\mathrm{Image}\ \phi| \leq \pi(X)^{c(X)}.$$

Finally $\pi(X) \leq w(X)$ (trivially), $w(X) \leq \exp d(X)$ (see above) and $c(X) \leq d(X)$ (trivially). This implies

$$\pi(X)^{c(X)} \leq (2^{d(X)})^{c(X)} = \exp d(X).$$

2.4. For each $X \in \mathcal{T}_2$

$$|X| \leq \exp \exp d(X).$$

<u>Proof</u>
Let $S \subset X$ be dense, $|S| \leq d(X)$. For $x_0 \in X$ we put

$$\mathfrak{S}_{x_0} = \{G \cap S : x_0 \in G \in \sigma(X)\} \subset \sigma(S).$$

Now $x \neq y \to \mathfrak{S}_x \neq \mathfrak{S}_y$, since X is Hausdorff, hence $\mathfrak{S}$ is a 1-1 map of X into $\mathcal{P}(\sigma(S))$, which proves our assertion.

<u>Corollary</u>
If $X \in \mathcal{T}_2$, then

$$w(X) \leq \exp \exp \exp d(X).$$

This is immediate from 2.2 and 2.4. As is shown in [64] this inequality is best possible, namely for any infinite α there is $X \in \mathcal{T}_2$ such that $d(X) = \alpha$ and $w(X) = \text{expexpexp}\ \alpha$.

2.5. (cf. [10]) If $X \in \mathcal{T}_2$, we have

$$|X| \leq \exp h(X).$$

<u>Proof</u> (Cf. the Remark on p. 25).
Assume $|X| > \exp \alpha$. By transfinite induction we define sets $X_{(\varepsilon_0,\ldots,\varepsilon_\eta,\ldots)_{\eta<\xi}}$, as follows. Let us put

$$X = X_{(0)} \cup X_{(1)},$$

where $X_{(\varepsilon)}$ are proper closed subsets of X.
If the sets $X_{(\varepsilon_0,\dots,\varepsilon_\eta,\dots)_{\eta<\xi}}$ have been defined for all $\xi < \rho$, we put

$$X_{(\varepsilon_\eta)_{\eta<\rho}} = \bigcap_{\xi<\rho} X_{(\varepsilon_\eta)_{\eta<\xi}},$$

if ρ is a limit ordinal, and if $\rho = \sigma + 1$, we put

$$X_{(\varepsilon_\eta)_{\eta<\sigma}} = X_{(\vec{\varepsilon}_\eta,0)} \cup X_{(\vec{\varepsilon}_\eta,1)},$$

where the sets on the right-hand side are proper closed subsets of the left-hand side, if the latter has at least two points.
Now there must be a sequence $(\varepsilon_\eta)_{\eta<\alpha^+}$ such that $|X_{(\varepsilon_\eta)_{\eta<\xi}}| \geq 2$ for every $\xi < \alpha^+$, since otherwise

$$|X| \leq |\{(\varepsilon_\eta)_{\eta<\xi} : \xi < \alpha^+ \wedge \varepsilon_\eta = 0 \vee 1\}| \leq \sum_{\xi<\alpha^+} 2^{|\xi|} = 2^\alpha$$

would hold. Hence we have a decreasing sequence of closed sets of length α^+, which, by 0.4, implies $h(X) \geq \alpha^+$. This completes the proof.

2.6. For every $X \in \mathcal{T}$

(i) $h(X) = \sup\{L(S) : S \subset X\} = \min\{\alpha : X$ is hereditary α-Lindelöf$\}$

(ii) $z(X) = \sup\{d(S) : S \subset X\} = \min\{\alpha : X$ is hereditary α-separable$\}$.

Ad (i). We saw in 0.5 (i) that if $|S| = \alpha^+$, which is a regular cardinal, and S is right separated in type α^+ then S is not β-Lindelöf for any $\beta \leq \alpha$. This obviously implies $h(X) \leq \alpha$, if X is hereditary α-Lindelöf. Conversely, if X is not hereditary α-Lindelöf, then we can find a $\mathcal{U} \subset \sigma(X)$, $|\mathcal{U}| > \alpha$ such that for $\mathcal{U}_0 \subset \mathcal{U}$, $|\mathcal{U}_0| \leq \alpha$ we have

(+) $\quad \cup\mathcal{U}_0 \neq \cup\,\mathcal{U}$.

Let $\mathcal{G} = \{G_\xi : \xi < \beta\}$, where $\beta = |\mathcal{G}|$. If the points x_ρ and their neighbourhoods $G_{\xi(\rho)} \in \mathcal{G}$ have been defined for $\rho < \nu < \beta$, then $H_\nu = \cup\{G_{\xi(\rho)} : \rho < \nu\} \neq \cup\mathcal{G}$ by (+), hence we can choose a point $x_\nu \in \cup\mathcal{G} \setminus H_\nu$ and its neighbourhood $G_{\xi(\nu)} \in \mathcal{G}$. Obviously, $\{x_\rho : \rho < \beta\}$ is right-separated, hence $h(X) \geq \beta > \alpha$.

Ad (ii). Since every left-separated space S whose order-type is a (regular) cardinal α^+, has density α^+ (see 0.5 (ii)), we have $z(X) \leq \sup\{d(S) : S \subset X\}$. On the other hand, if $d(S) = \alpha$, we can easily define a monotone increasing sequence of closed sets in S of length α, using an obvious transfinite induction. This completes the proof.

<u>Remark</u>.
2.5 and 2.6 (i) obviously imply that e.g., every hereditary Lindelöf T_2-space has at most 2^ω points. And 2.1.d+2.6.(ii) imply $\partial(X) \leq z(X)$.

<u>Problem</u>.
It is known to be consistent with the usual axioms of set theory that there exist hereditarily separable normal spaces (even topological groups) of cardinality expexp ω (cf. 62). It is not known, however, whether $X \in \mathcal{B}$ and $z(X) = \omega$ imply $|X| \leq 2^\omega$ or not.

2.7. (cf. [13]) If $X \in \mathcal{T}_2$, $d(X) \leq \exp s(X)$.

<u>Proof</u>
Suppose we have $d(X) > \exp \alpha$. Then, by 2.6 (ii) there is a left-separated subspace $S \subset X$ such that $|S| = (\exp \alpha)^+$. Using 2.5 we obtain a right-separated subspace $T \subset S$, $|T| > \alpha$. Now T is both right and left-separated, and we claim this implies the existence of a $D \subset T$ with $|D| = |T| > \alpha$ such that D is discrete.
Indeed let $\prec_1$ and $\prec_2$ be two wellorderings of T which separate T right and left respectively. Let us define a partition of $[T]^2$ (def of $[T]^2$: p.100) into two classes I and II as follows:

$\{x,y\} \in$ I iff $\prec_1$ and $\prec_2$ coincide on $\{x,y\}$;

$\{x,y\} \in$ II iff $\prec_1$ and $\prec_2$ are opposite on $\{x,y\}$.

Now, if $H \subset T$ is infinite, $[H]^2 \subset II$ cannot hold, otherwise we would have an infinite decreasing sequence in the sense of $\prec_1$ or $\prec_2$, which is absurd, since both are well-orderings. Thus, by Erdös ' theorem A4.7 we obtain a $D \subset T$, $|D| = |T|$, such that $[D]^2 \subset I$. This, however, means that $\prec_1$ and $\prec_2$ coincide on the set D, and this joint well-ordering both right and left separates D, hence D is obviously discrete.

2.8. (cf. [12], [25] or [32]) If $X \in \mathcal{L}$, then we have

(i) $d(X) \leq c(X)^+$.

(ii) If X contains a discrete subspace of power α, it also contains α pairwise disjoint intervals.

(iii) $h(X) = c(X)$.

(iv) $d(X) = z(X)$.[1)]

Proof

Ad (i). Assume $X \in \mathcal{L}$ and $d(X) \geq \alpha^+$. We want to show that X contains α pairwise disjoint intervals. This will evidently imply (i).
Now let $\prec$ be an arbitrary well-ordering of X. A point $p \in X$ is called normal, if p is the $\prec$-smallest element of some neighbourhood U_p of p. We put

$$N = \{p \in X: p \text{ is normal}\}.$$

First we show that N is dense in X. Indeed, if $G \in \sigma(X)$ and p_0 is the $\prec$-smallest element of G, then p_0 is obviously normal. Thus we have $|N| \geq d(X) \geq \alpha^+$.
For each $p \in N$ let I_p denote the union of all open intervals containing p as their first element by $\prec$. Now, if $p, p' \in N$, $p \prec p'$ then either $I_p \cap I_{p'} = \emptyset$, or $I_{p'} \subset I_p$, which follows immediately from the maximality of the I_p.
Now, if there are α pairwise disjoint I_p, we are done. If not, let us put

$$N_0 = \{p \in N: I_p \text{ is not contained in any other } I_{p'}\};$$

since p, $p' \in N_0 \to I_p \cap I_{p'} = \emptyset$, we have $|N_0| < \alpha$.
Similarly, by transfinite induction, we define

[1)] added in proof. Moreover it is easy to prove $\psi(X) = \chi(X) \leq c(X)$, as was observed by Nelly Kroonenberg.

$$N_\xi = \{p \in H_\xi \colon I_p \text{ is not contained in other } I_{p'} \text{ for } p' \in H_\xi\},$$

where $H_\xi = N \setminus \bigcup\{N_\eta : \eta < \xi\}$. Then, again, $|N_\xi| < \alpha$, hence $|\cup\{N_\xi : \xi < \alpha\}| \leq \alpha$. This implies $H_\alpha \neq \emptyset$.
Let $p' \in H_\alpha$. This means that for each $\xi < \alpha$ there is a $p_\xi \in N_\xi$ such that $I_{p'} \subset I_{p_\xi}$, hence $\{I_{p_\xi} : \xi < \alpha\}$ is a decreasing chain. For each $\xi < \alpha$ we can choose an $x_\xi \in I_{p_\xi} \setminus I_{p_{\xi+1}}$.
We put $K = \{x_\xi : \xi < \alpha\}$ and

$$K^l = \{x_\xi : x_\xi \prec p_{\xi+1}\},\ K^r = \{x_\xi : x_\xi \succ p_{\xi+1}\},$$

where $\prec$ denotes the original ordering of X. The convexity of the I_p implies that $x_\xi \prec x_\eta$ holds, if $x_\xi \in K^l$, $\eta > \xi$ and $x_\xi \succ x_\eta$, if $x_\xi \in K^r$, $\eta > \xi$.
Now we have $|K^l| = \alpha$ or $|K^r| = \alpha$. In the first case we have an increasingly well-ordered, in the second a decreasingly well-ordered subset of type α of X, which immediately gives us α disjoint intervals.
This proves (i).

<u>Remark</u>

A Suslin continuum, whose existence is consistent with the usual axioms of set theory, (cf. [18] or [34]) yields us a compact ordered space X, for which

$$c(X) = \omega \text{ and } d(X) = \omega_1.$$

Ad (ii) Let $X \in \mathcal{L}$, $D \subset X$ discrete, $\alpha = |D| \geq \omega$. For each $p \in D$ we can choose an interval $I_p = (a_p, b_p)$ such that $I_p \cap D = \{p\}$. If D contains α isolated points of X, we are done. If not, we can assume that no point p of D is isolated in X, hence either $(a_p, p) \neq \emptyset$ or

$(p,b_p) \neq \emptyset$. Thus we have α points in D for which either the intervals (a_p,p) or the intervals (p,b_p) are pairwise disjoint and non-empty.

Ad (iii) Since $c(X) \leq h(X)$ is trivial, we have only to show that $h(X) \leq c(X) = \alpha$, i.e., by 2.6 (i) that X is hereditary α-Lindelöf. Since $c(X) = c(\tilde{X})$ (cf. 0.17) and the order topology of X coincides with its subspace topology in $\tilde{X}$, and finally $\tilde{X} \in \mathcal{B}$, it suffices to prove the hereditary α-Lindelöfness of a compact $X \in \mathcal{L}$ with $c(X) = \alpha$. So in what follows we assume that X is compact.

By 0.10, X is hereditary α-Lindelöf (with $\alpha = \omega_\xi$) iff every open subset $G \in \sigma(X)$ is an $F_{\sigma,\xi}$ set. It is well known that every $G \in \sigma(X)$ is the disjoint union of open intervals in X, whose number, by $c(X) = \alpha$, is at most α. Thus it suffices to show that every $(a,b) \subset X$ is an $F_{\sigma,\xi}$ set.

Now if a has no immediate successor and b has no immediate predecessor then we can choose decreasingly and increasingly well-ordered sequences $\{a_\eta : \eta < \gamma_a \leq \alpha\} \subset (a,b)$ and $\{b_\nu : \nu < \gamma_b \leq \alpha\} \subset (a,b)$, respectively, such that they converge to a and b. ($\gamma_a, \gamma_b \leq \alpha$ follows from $c(X) = \alpha$.) Then

$$(a,b) = \bigcup\{[a_\eta, b_\nu] : (\eta,\nu) \in \gamma_a \times \gamma_b\},$$

hence (a,b) is the union of $\leq \alpha$ closed intervals, and thus is an $F_{\sigma,\xi}$ set.

It is obvious how to modify the above construction in the cases where a has an immediate successor or b has an immediate predecessor.

Ad (iv) Suppose $d(X) = \alpha$. We want to show (cf. 2.6 (ii)) that for every $S \subset X$, $d(S) \leq \alpha$.

Let A be a dense subset of X with $|A| = \alpha$. We put $A_S = \{(x,y) : x, y \in A \text{ and } (x,y) \cap S \neq \emptyset\}$, furthermore if $(x,y) \in A_S$ we choose a point $p_{(x,y)} \in (x,y) \cap S$, and put

$$A_S^* = \{p_{(x,y)} : (x,y) \in A_S\}.$$

Obviously, $|A_S^*| \leq \alpha$. Since $c(X) \leq d(X) = \alpha$, by (iii), X is hereditary

α-Lindelöf. Therefore if I_S is the set of all isolated points of S, then $|I_S| \leq \alpha$. We claim that $D_S = A_S^* \cup I_S$ is dense in S. Since $|D_S| \leq \alpha$, this will complete the proof.

It is enough to show that if a, b $\in$ X, $(a,b) \cap S \neq \emptyset$ then $(a,b) \cap S \cap D_S \neq \emptyset$. If $(a,b) \cap S$ contains an isolated point of S, then we are done.

If not, then $|(a,b) \cap S| \geq \omega$, hence we can choose five points $x_1, \ldots, x_5 \in (a,b) \cap S$ such that $x_i \prec x_j$ if $i < j$. Then $x_2 \in (x_1,x_3) \neq \emptyset$ and $x_4 \in (x_3,x_5) \neq \emptyset$. Hence there are y_1, $y_2 \in A$ such that $y_1 \in (x_1,x_3)$ and $y_2 \in (x_3,x_5)$. Consequently $x_3 \in (y_1,y_2) \cap \cap S \neq \emptyset$, and therefore $(y_1,y_2) \in A_S$.

Now, obviously, $p_{(y_1,y_2)} \in (y_1,y_2) \cap S \subset (a,b) \cap S$, hence $D_S \cap (a,b) \cap S \neq \emptyset$, which was to be shown. This completes the proof.

<u>Remark</u>

We do not know whether $d(X) \leq (s(X))^+$ holds for a larger class of spaces than $\mathcal{L}$, say for $\mathcal{T}_3(!)$, independently of GCH, of course. (cf. 2.7.)

2.9. (cf. [13]) For each $X \in \mathcal{T}_2$ we have

$$|X| \leq \exp \exp s(X).$$

<u>Proof</u> (see p.100 for the definition of $[X]^r$).

Assume $|X| > \exp \exp \alpha$ and let $<$ be a well-ordering of X. Since $X \in \mathcal{T}_2$, for each pair $\{x,y\} \in [X]^2$ with $x < y$ we can choose neighbourhoods $U(x,y)$ and $V(x,y)$ of x and y respectively, such that $U(x,y) \cap V(x,y) = \emptyset$.

Now we define a partition of $[X]^3$ as follows:

If $\{x,y,z\} \in [X]^3$, $x < y < z$ then we put

$$\{x,y,z\} \in I_{(\varepsilon_1,\varepsilon_2)} \ (\varepsilon_i = 0,1)$$

according to the following rules:

$\varepsilon_1 = 0$, if $x \in U(y,z)$;

$\varepsilon_1 = 1$, if $x \notin U(y,z)$;

$\varepsilon_2 = 0$, if $z \in V(x,y)$;

$\varepsilon_2 = 1$, if $z \notin V(x,y)$.

By A4.5 there is a subset $H \subset X$, $|H| = \alpha^+$ such that for a fixed pair (η_1,η_2) $(\eta_i=0,1)$ we have $[H]^3 \subset I_{(\eta_1,\eta_2)}$. Suppose now $y \in H$ and y has both an immediate predecessor and an immediate successor in H by $<$, say x and z respectively, i.e. $x < y < z$. We shall show that

$$H \cap U(y,z) \cap V(x,y) = \{y\},$$

hence y is isolated in H. Since H obviously contains α^+ such points y, this yields a discrete subspace D of H and hence X, of cardinality α^+ and proves our proposition.
Assume now that $p \in H \cap V(x,y) \cap U(y,z)$ and $p \neq y$. Since $p \neq x$ and $p \neq z$ are obvious, we have either $p < x$ or $z < p$. In the first case $p \in U(y,z)$, hence the triple $\{p,y,z\}$ gives us $\eta_1 = 0$. This, in turn implies $p \in U(x,y)$, looking at the triple $\{p,x,y\} \in [H]^3$, and thus $p \notin V(x,y)$, which is a contradiction. A similar contradiction arises if $p > z$ is assumed. This completes the proof.

Problem

Can one exp be omitted in 2.9 if $X \in \mathcal{B}$? (cf. [62]).

2.10. For $X \in \mathcal{L}$ we have

$$|X| \leq \exp c(X).$$

Proof

This follows immediately from 2.5 and 2.8 (iii). A direct proof goes as follows:

Let $\prec$ be an arbitrary well-ordering of X, while $<$ is the ordering which defines the topology of X. We put for any $\{x,y\} \in [X]^2$

$$\{x,y\} \in I_s \text{ resp. } \{x,y\} \in I_{op},$$

according to whether $\prec$ orders $\{x,y\}$ in the same, or in the opposite way as $<$ does.
Now, if $|X| > 2^{\alpha}$, by A4.4 we have a $H \subset X$, $|H| = \alpha^+$ such that $[H]^2 \subset I_s$ or $[H]^2 \subset I_{op}$. Thus in the first case H is increasingly well-ordered and in the second case decreasingly well-ordered by its original ordering $<$. In either case, X contains α^+ pairwise disjoint intervals, hence $c(X) \geq \alpha^+$. This completes the proof.

2.11. If $X \in \mathcal{B}$, for each ξ we have

$$c_{\xi}(X) \leq \exp(\omega_{\xi}.c(X)).$$

Proof

First we show that, in any regular space Y, each $H \in \sigma_{\xi}(Y)$ contains a closed $H' \in \sigma_{\xi}(Y)$, where $H' \neq \emptyset$ if $H \neq \emptyset$. Indeed let $p \in H \in \sigma_{\xi}(Y)$. Then $H = \cap \{H_{\rho} : \rho < \omega_{\xi}\}$, where $H_{\rho} \in \sigma(Y)$ for each $\rho < \omega_{\xi}$. Now because Y is regular, for any fixed $\rho < \omega_{\xi}$ we can define by induction open sets $H_{\rho}^{(n)}$ such that $H_{\rho}^{(0)} = H_{\rho}$ and for $0 < n < \omega$ we have

$$p \in H_{\rho}^{(n)} \subset \overline{H_{\rho}^{(n)}} \subset H_{\rho}^{(n-1)} \subset H_{\rho} \qquad (H_{\rho}^{(n)} \in \sigma(Y)).$$

Let us put

$$H' = \cap \{H_{\rho}^{(n)} : \rho < \omega_{\xi} \wedge n < \omega\} = \cap \{\overline{H_{\rho}^{(n)}} : \rho < \omega_{\xi} \wedge n < \omega\}.$$

This shows that $p \in H' \in \sigma_{\xi}(Y)$ and H' is closed, which was to be shown.
Thus, to prove our proposition, it is enough to show that X does not contain more than $\exp(\omega_{\xi}.c(X))$ pairwise disjoint closed $G_{\delta,\xi}$ sets. Assume, on the contrary, that $\mathcal{A}$ is such a disjoint sub-family of

$\sigma_\xi(X)$ and $|\mathcal{A}| > \exp(\omega_\xi . c(X))$. Since X is compact, for each $A \in \mathcal{A}$ we have $\psi(A,X) = \chi(A,X) \leq \omega_\xi$ (cf. 0.8), so we can choose a basis of neighbourhoods

$$\mathscr{L}_A = \{G_A^{(\rho)} : \rho < \omega_\xi\}$$

of each $A \in \mathcal{A}$. Now, the normality of X implies that for $\{A_1, A_2\} \in [\mathcal{A}]^2$ we can choose $\rho_1, \rho_2 < \omega_\xi$ such that

$$G_{A_1}^{(\rho_1)} \cap G_{A_2}^{(\rho_2)} = \emptyset.$$

This induces a partition of $[\mathcal{A}]^2$ into $|\omega_\xi \times \omega_\xi| = \omega_\xi$ classes as follows

$$\{A_1, A_2\} \in I_{(\rho_1,\rho_2)} \longleftrightarrow G_{A_1}^{(\rho_1)} \cap G_{A_2}^{(\rho_2)} = \emptyset.$$

Since $|\mathcal{A}| > \exp(\omega_\xi . c(X))$, by A4.4 we have a subsystem $\mathcal{C} \subset \mathcal{A}$ and a fixed pair $(\bar\rho_1, \bar\rho_2) \in \omega_\xi \times \omega_\xi$ such that $|\mathcal{C}| > \omega_\xi + c(X) \geq c(X)$, and for all $\{C_1, C_2\} \in [\mathcal{C}]^2$

$$G_{C_1}^{(\bar\rho_1)} \cap G_{C_2}^{(\bar\rho_2)} = \emptyset.$$

Now, if we put $G_C = G_C^{(\bar\rho_1)} \cap G_C^{(\bar\rho_2)}$ for each $C \in \mathcal{C}$, the family of open sets $\{G_C : C \in \mathcal{C}\}$ is obviously disjoint. This, however, is a contradiction, because $|\mathcal{C}| > c(X)$.

<u>Remark</u>

A completely regular space X is called a $G_{\delta\Sigma}$ space, if X is an arbitrary union of G_δ sets in some compactification cX (w.r.t. cX). Thus, e.g., Arhangel'skiĭ p-spaces mentioned in 2.2 are $G_{\delta\Sigma}$ spaces. It is an easy corollary of 2.11 that $c_\xi(X) \leq \exp(\omega_\xi . c(X))$ holds true for arbitrary $G_{\delta\Sigma}$ spaces as well.

2.12. (cf. [20]) For $X \in \mathcal{T}_\rho$ we have

$$w(X) \leq u(X) . c(X).$$

Proof

Let us first note that if X is a pseudometrisable space (i.e. $u(X) = \omega$), then we have

$$w(X) = c(X).$$

Indeed, this follows immediately from R.H. Bing's pseudometrisation theorem, namely the existence of a σ-disjoint base.

Now, if $X \in \mathcal{T}_\rho$, there is a family $\mathcal{P}$ of pseudometrics with $|\mathcal{P}| = u(X)$ which generates the topology of X. For each $\sigma \in \mathcal{P}$ let X_σ denote the pseudometric space on X determined by σ.

If $u(X) = |\mathcal{P}| \geq w(X)$, we are done. If not, i.e. $|\mathcal{P}| < w(X)$, then

$$w(X) = \sum_{\sigma \in \mathcal{P}} w(X_\sigma) = \sup \{w(X_\sigma) : \sigma \in \mathcal{P}\}$$

and therefore for each $\alpha < w(X)$ we have a $\sigma_0 \in \mathcal{P}$ such that

$$w(X_{\sigma_0}) = c(X_{\sigma_0}) > \alpha.$$

This, however shows

$$w(X) = \sup_{\sigma \in \mathcal{P}} c(X_\sigma) = c(X).$$

2.13. $X \in \mathcal{T}_{3\frac{1}{2}}$ implies

$$u(X) \leq w(X).$$

Proof

Evidently, $Y \subset X$ implies $u(Y) \leq u(X)$ and this shows that it suffices to prove $u(X) \leq w(X)$ for compact spaces, because every $Y \in \mathcal{T}_{3\frac{1}{2}}$ has a compactification of the same weight as Y.

Now, if $X \in \mathcal{B}$ and $\mathcal{L}$ is a base for the topology of X with $|\mathcal{L}| \leq w(X)$, then, as can easily be checked, all finite coverings of X with members of $\mathcal{L}$ yield a basis for the unique uniformity of X. The number of these finite coverings, however, is equal to $|\mathcal{L}| \leq w(X)$, hence

$u(X) \leq w(X)$ does hold.

This shows that for $X \in \mathfrak{T}_{3\frac{1}{2}}$ $\leq$ can be replaced by $=$ in 2.12

2.14. For each $X \in \mathfrak{D}$ we have

$$h(X) = |X|.$$

Proof

It is well-known that every dispersed space X can be written as a disjoint union of the form

$$X = \cup\{L_\xi : \xi < \rho\}, \quad (L_\xi \neq \emptyset)$$

where for each $\xi_0 < \rho$, L_{ξ_0} is the set of all isolated points of the closed subspace

$$X_{\xi_0} = \cup\{L_\xi : \xi_0 \leq \xi < \rho\}.$$

Thus we have $|L_\xi| \leq s(X) \leq h(X)$ for all $\xi < \rho$. On the other hand, choosing a point $p_\xi \in L_\xi$ from each level L_ξ, the resulting set $H = \{p_\xi : \xi < \rho\}$ is obviously right separated, hence $|H| = |\rho| \leq h(X)$ holds as well. This however, shows that

$$|X| = \sum_{\xi<\rho} |L_\xi| \leq |\rho| . h(X) = h(X),$$

hence

$$|X| = h(X).$$

2.15. (cf. [13]) Suppose $X \in \mathfrak{T}_1$. Then

$$|X| \leq \exp(\psi(X).s(X)).$$

Proof

It is enough to show that $\psi(X) \leq \alpha$ and $|X| > \exp \alpha$ imply the existence of a discrete subspace of X of cardinality α^+. To show this, let $\prec$ be a linear ordering of X and choose for each $p \in X$ a sequence of its neighbourhoods

$$\mathcal{W}_p = \{V_\xi(p) : \xi < \alpha\}$$

such that $\cap\, \mathcal{W}_p = \{p\}$. Now, for $\xi, \eta < \alpha$ let us put
$I_{(\xi,\eta)} = \{\{p,q\} : p \prec q$ and $q \notin V_\xi(p)$ and $p \notin V_\eta(q)\}$.
obviously we have

$$[X]^2 = \cup\{I_{(\xi,\eta)} : (\xi,\eta) \in \alpha \times \alpha\},$$

i.e., a partition of $[X]^2$. By A4.4 there is a subset $D \subset X$, $|D| = \alpha^+$ such that

$$[D]^2 \subset I_{(\bar{\xi},\bar{\eta})}$$

holds for a fixed pair $(\bar{\xi},\bar{\eta})$. Now it is obvious from our construction that

$$D \cap (V_{\bar{\xi}}(p) \cap V_{\bar{\eta}}(p)) = \{p\}$$

holds for each $p \in D$, i.e., p is isolated in D and thus D is discrete. This completes the proof.

2.16. (cf. [13] or [21]) Assume $X \in \mathcal{T}_2$, $A \subset X$, $|A| > 2^\alpha$, furthermore $\chi(p,X) \leq \alpha$ for each $p \in A$. Then

$$c(X) > \alpha.$$

The proof of 2.15 can be applied after having made the following changes:
For $p \in A$ $\mathcal{W}_p$ is a basis of neighbourhoods in X and we form a partition of $[A]^2$ by putting

$$I_{(\xi,\eta)} = \{\{p,q\} : p \prec q \text{ and } V_\xi(p) \cap V_\eta(q) = \emptyset\}$$

<u>Corollary</u>
If $X \in \mathcal{T}_2$ then

$$|X| \leq \exp(\chi(X).c(X)).$$

2.17. For every $X \in \mathcal{T}_2$

$$\psi(X) \leq h(X)$$

holds.

Proof

Since $X \in \mathcal{T}_2$, for each $p \in X$ we can choose a system $\mathcal{W}_p$ of closed neighbourhoods of p such that $\cap\, \mathcal{W}_p = \{p\}$. We can assume that $\mathcal{W}_p$ is of minimal cardinality among such systems, say $|\mathcal{W}_p| = \alpha_p$. Then, of course, $\alpha_p \geq \psi(p,X)$.

Now fix $p \in X$. We define members V_ξ of $\mathcal{W}_p$ and points x_ξ by transfinite induction as follows:

Let $V_0 \in \mathcal{W}_p$ and $x_0 \in X \setminus V_0$ arbitrary. Suppose $\xi < \alpha_p$ and for every $\eta < \xi$ the $V_\eta \in \mathcal{W}_p$ and point x_η have already been defined. Then, because of the minimality of α_p,

$$\bigcap_{\eta<\xi} V_\eta \supsetneqq \{p\},$$

hence there is an $x_\xi \in \bigcap_{\eta<\xi} V_\eta \setminus \{p\}$ and a $V_\xi \in \mathcal{W}_p$ such that $x_\xi \notin V_\xi$, since $\cap\, \mathcal{W}_p = \{p\}$.

Now let us put $F_\xi = \bigcap_{\eta<\xi} V_\eta$ for $\xi < \alpha_p$. Then, obviously, $x_\xi \in F_\eta \setminus F_\xi$ and $F_\eta \supset F_\xi$, if $\eta < \xi$, hence $\{F_\xi : \xi < \alpha_p\}$ is a monotone decreasing sequence of closed sets in X. This implies

$$h(X) \geq \alpha_p \geq \psi(p,X)$$

for all $p \in X$, hence $h(X) \geq \psi(X)$.

Problem

For what spaces does

$$z(X) \geq \psi(X)$$

hold?

Remark

Since $s(X) \leq h(X)$ always holds, from 2.17 and 2.15, we immediately obtain another proof of 2.5.

2.18. (cf. 6.4) If X is connected, then

$$k(X) \leq [\chi(X)]^+.$$

Proof

In fact, we shall prove that if

$$\mathcal{G} = \{G_\xi : \xi < \mu\}$$

is a strongly decreasing chain in X, then

$$\mu \leq \alpha^+, \text{ where } \alpha = \chi(X).$$

Assume, on the contrary, that $\mu > \alpha^+$ and put $\mathcal{G}_0 = \{G_\xi : \xi < \alpha^+\}$. Then $\cap \mathcal{G}_0 \supset G_{\alpha^+} \neq \emptyset$. Since $\mathcal{G}_0$ is strongly decreasing, we have

$$H = \cap \mathcal{G}_0 = \cap \{\bar{G}_\xi : \xi < \alpha^+\},$$

hence H is a non empty closed proper subset. Since X is connected, H cannot be open, therefore we can choose a boundary point $p_0 \in H$. We claim that $\chi(p_0,X) \geq \alpha^+$, which is a contradiction.

Indeed, if $\{U_\eta : \eta < \alpha\}$ were a basis of neighbourhoods of p_0, then for each $\xi < \alpha^+$ we could choose an $\eta_\xi < \alpha$ such that

$$p_0 \in U_{\eta_\xi} \subset G_\xi$$

hold. Now, since α^+ is regular, there is a cofinal subsequence $\{G_{\xi_\nu} : \nu < \alpha^+\}$ of $\mathcal{G}_0$ and an $\bar{\eta} < \alpha$ such that

$$\eta_{\xi_\nu} = \bar{\eta}$$

holds for each $\nu < \alpha^+$. This implies

$$p_0 \in U_{\bar{\eta}} \subset \cap\{G_{\xi_\nu} : \nu < \alpha^+\} = \cap\{G_\xi : \xi < \alpha^+\} = H,$$

which is in contradiction to the assumption that p_0 is a boundary point of H. This completes the proof.

Remark

J. Gerlits [61] has shown that for Kowalski's "hedgehog space" X with ω_1 needles we have $k(X) = \omega_1 = \chi(X)^+$, while X is of course a connected metrizable space.

2.19. For every X we have

$$k(X) \leq L(X).\partial(X).$$

Proof

(cf. [3]) Let us put $L(X).\partial(X) = \alpha$. We shall prove a somewhat stronger result, namely that every free sequence in X is of length $< \alpha^+$ (cf. 0.6). We shall need this stronger result in the proof of 2.21.

Assume, on the contrary, that

$$S = \{p_\xi : \xi < \alpha^+\}$$

is a free sequence in X. Since X is α-Lindelöf, there is a point $x_0 \in X$ such that for each neighbourhood U of x_0 we have

$$|U \cap S| = \alpha^+.$$

Indeed, assume that each $x \in X$ has a neighbourhood U_x such that $|U_x \cap S| \leqq \alpha$. We can choose a subcovering

$$\pi(X)^{c(X)} \leq (2^{d(X)})^{c(X)} = \exp\, d(X).$$

$$\mathfrak{V} \subset \{U_x : x \in X\}$$

for which $|\mathfrak{V}| \leq \alpha$. Then, however,

$$S = \cup \{U_x \cap S : U_x \in \mathfrak{V}\},$$

hence $|S| \leq \sum \{|U_x \cap S| : U_x \in \mathfrak{V}\} \leq \alpha.\alpha = \alpha$ would hold, which is a contradiction.
Now, since $x_0 \in \bar{S}$ and $\partial(X) \leq \alpha$, there is a subset $A \subset S$, $|A| \leq \alpha$ such that $x_0 \in \bar{A}$. Since α^+ is regular, there is an ordinal $\xi_0 < \alpha^+$ such that $A \subset S_0 = \{p_\xi : \xi < \xi_0\}$, hence

$$x_0 \in \bar{A} \subset \bar{S}_0.$$

But, S is free, hence $\bar{S}_0 \cap \overline{S \setminus S_0} = \emptyset$. Therefore $U_0 = X \setminus \overline{S \setminus S_0}$ is a neighbourhood of x_0, for which $U_0 \cap S \subset S_0$, hence

$$|U_0 \cap S| \leq \alpha < \alpha^+.$$

This, however, contradicts our choice of x_0, and thus finishes the proof.

2.20. For $X \in \mathfrak{T}_2$ we have

$$|X| \leq d(X)^{\chi(X)}.$$

Proof

Let $S \subset X$ be dense in X, $|S| = d(X)$ and put $\chi(X) = \alpha$. For each $x \in X$ we choose an open neighborhoodbasis $\mathcal{U}_x$ of cardinality α. For each $U \in \mathcal{U}_x$ we take $p(U) \in U \cap S$. Put $N_x = \{p(U) : U \in \mathcal{U}_x\}$. Hence $N_x \in \mathcal{P}_\alpha(S)$, if, for a set A, $\mathcal{P}_\alpha(A)$ is defined as $\mathcal{P}_\alpha(A) = \{B \subset A : (B| \leq \alpha\}$.
Consider the function

$$f: \quad x \longmapsto \{U \cap N_x : U \in \mathcal{U}_x\}$$

which carries X into $\mathcal{P}_\alpha(\mathcal{P}_\alpha(S))$. Because $X \in \mathcal{T}_2$ we find that $\{x\} = \cap\{(U \cap N_x)^- : U \in \mathcal{U}_x\}$. Thus the function f is 1-1, implying that

$$|X| \leq |\mathcal{P}_\alpha(\mathcal{P}_\alpha(S))| = (|S|^\alpha)^\alpha = |S|^\alpha \quad .$$

2.21. For each $X \in \mathcal{T}_2$ we have

$$|X| \leq \exp\,(L(X).\chi(X)).$$

This is a beautiful and quite recent result of A.V. Arhangel'skiĭ, [3], which settled an almost fifty year old conjecture of P.S. Aleksandrov namely that every first countable compactum is of cardinality $\leq 2^\omega$.

First we need two lemmas:

Lemma a)

Assume $X \in \mathcal{T}_2$, $\alpha \geq \omega$, $|X| > \exp\alpha$, furthermore if $A \subset X$, $|A| \leq \alpha$ then

(i) $|\bar{A}| \leq \exp\alpha$

and

(ii) $\psi(\bar{A},X) \leq \exp\alpha$

hold. Then there is a free sequence

$$\{p_\xi : \xi < \alpha^+\} \subset X$$

of length α^+ in X.

Proof

We shall construct a ramification system in the sense of [39], lemma 1, by defining sets $R_{[\rho_0,\ldots,\rho_\xi]}$ and points $p_{[\rho_0,\ldots,\rho_\xi]}$ for certain sequences of ordinals, where $\rho_\eta < 2^\alpha$ and $\xi < \alpha^+$.

First we put $R_0 = X$ and $p_0 \in R_0$ arbitrary; here 0 stands for the empty sequence. Suppose now that $\xi < \alpha^+$ and for all $\eta < \xi$ the sets

$R_{[\rho_0,\ldots,\rho_\eta]}$ and points $p_{[\rho_0,\ldots,\rho_\eta]}$ have been defined for each $[\rho_0,\ldots,\rho_\eta] \in S_{\eta+1}$, where S_ν denotes the set of sequences of type ν of ordinals $< 2^\alpha$.

Let us now choose a sequence $s \in S_\xi$ and put

$$R'_s = \cap \{R_{s|\eta+1} : \eta + 1 \leq \xi\}$$

where $s|\eta+1$ denotes the initial segment of s of type $\eta+1$. Now we distinguish two cases, a) and b):

a) $|R'_s| \leq 2^\alpha$. In this case we put $R_{[s,\rho]} = R'_s$ for all $\rho < 2^\alpha$; here $[s,\rho]$ denotes the sequence $[\rho_0,\ldots,\rho]$ of type $\xi+1$ obtained by augmenting s by ρ. The points $p_{[s,\rho]}$ can be chosen arbitrarily.

b) $|R'_s| > 2^\alpha$. Since $\xi < \alpha^+$, applying (ii) and putting $\overline{\{p_{s|\eta+1} : \eta + 1 \leq \xi\}} = G^{(s)}$ we can write $X \setminus G^{(s)} = \cup \{F^{(s)}_\rho : \rho < 2^\alpha\}$, where the $F^{(s)}_\rho$'s are (not necessarily distinct) closed subsets of X. Next we put

$$R_{[s,\rho]} = R'_s \cap F^{(s)}_\rho$$

for each $\rho < 2^\alpha$ and choose any element of $R_{[s,\rho]}$ as $p_{[s,\rho]}$ if $R_{[s,\rho]} \neq \emptyset$. Otherwise $p_{[s,\rho]}$ can be chosen arbitrarily.

By transfinite induction on ν we can easily show that

$$X = \bigcup\{R'_s : s \in S_\nu\} \cup \bigcup\{G^{(s)} : s \in S_\nu\}$$

holds for each $\nu < \alpha^+$. Next we claim that there exists a sequence $t \in S_{\alpha^+}$ such that

$$|R'_{t|\nu}| > 2^\alpha$$

holds for each $\nu < \alpha^+$. Indeed, let us put

$$\tilde{S}_\nu = \{s \in S_\nu : |R'_s| \leq 2^\alpha\}$$

and

$$S = \cup\{S_\nu : \nu < \alpha^+\},\ \tilde{S} = \cup\{\tilde{S}_\nu : \nu < \alpha^+\}.$$

Then $|\tilde{S}| \leq |S| \leq \sum_{\nu<\alpha^+} 2^{|\nu|} \leq \alpha^+ .\ 2^\alpha = 2^\alpha$, hence we have, by (i) and the choice of $\tilde{S}$

$$|\cup\{G^{(s)} : s \in S\} \cup \cup\{R'_s : s \in \tilde{S}\}| \leq 2^\alpha .\ 2^\alpha + 2^\alpha .\ 2^\alpha = 2^\alpha .$$

Now if x_0 is an arbitrary point in the complement of the above set we can find a sequence $t \in S_{\alpha^+}$ such that

$$x_0 \in R'_{t|\nu}$$

holds for each $\nu < \alpha^+$. Indeed, if t is a maximal sequence such that $x_0 \in R'_{t|\nu}$ holds for each $\nu <$ length of t, then the length of t must be α^+. Because of the choice of x_0, however, we have $t|\nu \in S_\nu \setminus \tilde{S}_\nu$, hence $|R'_{t|\nu}| > 2^\alpha$ for each $\nu < \alpha^+$.
Let us now put $t = [\rho_0, \ldots, \rho_\xi, \ldots]$ and

$$p_\xi = p_{t|\xi+1} = p_{[\rho_0, \ldots, \rho_\xi]}$$

for all $\xi < \alpha^+$. Then for arbitrary $\xi < \alpha^+$ we have

$$\overline{\{p_\eta : \eta < \xi\}} = G^{(t|\xi)}$$

and

$$\{p_\eta : \xi \leq \eta < \alpha^+\} \subset \overline{\{p_\eta : \xi \leq \eta < \alpha^+\}} \subset F^{(t|\xi)}_{\rho_\xi},$$

which shows that $\{p_\xi : \xi < \alpha^+\}$ is a free sequence, because $G^{(t|\xi)} \cap F^{(t|\xi)}_{\rho_\xi} = \emptyset$, by definition. This completes the proof.

<u>Lemma b)</u>

Assume X is an α-Lindelöf T_1 space, $A \subset X$ is closed and $|A| \leq 2^\alpha$, moreover $\psi(p,X) \leq 2^\alpha$ holds for each $p \in A$. Then

$$\psi(A,X) \leq 2^{\alpha}$$

holds too.

Proof

Let us choose for each $p \in A$ a system of open neighbourhoods of p, say $\mathcal{V}_p$, such that $\cap \mathcal{V}_p = \{p\}$ and $|\mathcal{V}_p| \leq 2^{\alpha}$. Now, if x_0 is an arbitrary point of $X \setminus A$ then for each $p \in A$ there is a $V_p \in \mathcal{V}_p$ such that $x_0 \notin V_p$. Since $\{V_p: p \in A\}$ is a covering of A and X (and A) are α-Lindelöf, there is a subcovering $\mathcal{U}_{x_0} \subset \{V_p: p \in A\}$ such that $|\mathcal{U}_{x_0}| \leq \alpha$. But $x_0 \notin \cup \mathcal{U}_{x_0} \supset A$, which shows that

$$\psi(A,X) \leq |\{\mathcal{U}: \mathcal{U} \subset \bigcup_{p \in A} \mathcal{V}_p \text{ and } |\mathcal{U}| \leq \alpha\}| \leq (2^{\alpha})^{\alpha} = 2^{\alpha},$$

since $|\bigcup_{p \in A} \mathcal{V}_p| \leq 2^{\alpha}.2^{\alpha} = 2^{\alpha}$.

Proof of 2.21

Let us put $\alpha = L(X).\chi(X)$ and suppose that $|X| > \exp \alpha$. Then, by 2.20 and lemma b) respectively, conditions (i) and (ii) of lemma a) are satisfied. Thus, applying the latter we obtain a free sequence of length α^{+} in X. But by the proof of 2.19, the length of any free sequence in X is $\leq L(X).\partial(X) \leq L(X).\chi(X) = \alpha$, which is a contradiction. This completes the proof.

We would like to emphasize the following

Corollary

If X is a first countable, Lindelöf T_2 space, then $|X| \leq \exp \omega$.

Remark

It is interesting to compare this corollary with the following result of S. Mrówka ([29], Theorem 2):

There exists a first countable compact T_1 space of cardinality α iff $D(\alpha) \subseteq_{cl} D(\omega)^{\alpha}$, i.e., α belongs to the class of cardinals M, defined by Mrówka in [28]. It is known e.g., that for each non-measurable β we have $2^{\beta} \in M$ (see [22] or [28] for more details).

2.22. (cf. [4]) Assume $X \in \mathfrak{D}$ and $(\psi(p,X)=)\ \chi(p,X) \geq \alpha$ for each $p \in X$. Then $|X| \geq \exp \alpha$.

<u>Proof</u>

Let J_ν denote the set of all 0-1 sequences of type ν. By transfinite induction on ν we shall define a mapping $V\colon J = \bigcup_{\nu<\alpha} J_\nu \to \sigma(X)$ as follows: We put $V(\emptyset) = X$. Assume that $\nu < \alpha$ and for all $\xi < \nu$, $j \in J_\xi$ $V(j) \in \sigma(X)$ have already been defined in such a way that

(a) For each $\xi < \nu$ the system $\{V(j|\eta)\colon \eta < \xi\}$ has the finite intersection property.

(b) If ξ is of the form $\eta + 1$, $i \in J_\eta$ and $j = [i,\varepsilon]$ $(\varepsilon \in \{0,1\})$, then $\overline{V(j)} \subset V(i)$.

Let $j \in J_\nu$. If ν is limit, we put $i = j$ and $V(j) = X$. If $\nu = \xi + 1$ we have $j = [i,\varepsilon]$ for some $i \in J_\xi$. Notice that in either case

$$H^{(i)} = \cap\{V(j|\eta)\colon \eta \leq \xi\} = \cap\{\overline{V(j|\eta)}\colon \eta \leq \xi\} \neq \emptyset$$

by (a) and (b) and the compactness of X. Also $H^{(i)} \neq \{p\}$ for any $p \in X$, since otherwise we would have $\psi(p,X) \leq |\xi| < \alpha$. Thus we can choose two different points p_ε $(\varepsilon = 0,1)$ such that $p_\varepsilon \in H^{(j)}$ and two open neighbourhoods V_ε of p_ε such that $\overline{V}_\varepsilon \subset V(i)$ and $\overline{V}_0 \cap \overline{V}_1 = \emptyset$ Then we put

$$V([i,\varepsilon]) = V_\varepsilon \quad (\varepsilon = 0,1).$$

Thus $V(j)$ is defined for each $j \in J$.
It follows immediately from the construction that for any $j \in J$

$$\cap\{V(j|\eta)\colon \eta \leq \text{length of } j\} \neq \emptyset$$

and if $j, j' \in J_\alpha$ and $j \neq j'$ then

$$\bigcap_{\eta<\alpha} \{V(j|\eta)\} \cap \bigcap_{\eta<\alpha} \{V(j'|\eta)\} = \emptyset.$$

However, $|J_\alpha| = \exp \alpha$, and this immediately implies $|X| \geq |J_\alpha| \geq \exp \alpha$.

Remark

If in 2.22 instead of $X \in \mathcal{B}$ it is only required that X has a compactification cX with $X \in \sigma_\xi(cX)$ and $\omega_\xi < \alpha$, then according to the proof of 2.11 there is a closed, hence compact, subset $Z \subset X$ which is also a $G_{\delta,\xi}$-set in cX. It is easy to see that $\chi(p,Z) \geq \alpha$ is valid for all $p \in Z$, hence $|X| \geq |Z| \geq \exp \alpha$ by 2.22.

Corollary

If $\chi(p,X) = \alpha$ for each point p of a compact T_2 space X, then $|X| = \exp \alpha$ by 2.21.

If X is a first countable compact T_2 space then either $|X| \leq \omega$ or $|X| = \exp \omega$.

Proof

Assume $|X| > \omega$ and let A be the set of all condensation points of X, i.e.,

$$p \in A \leftrightarrow |U_p| \geq \omega_1$$

for each neighbourhood U_p of p. Obviously, A is closed in X and we assert that A is also dense in itself. In fact, let $p \in A$ and U be an arbitrary neighbourhood of p. We can choose neighbourhoods $V_0 \supset V_1 \supset \ldots \supset V_n \supset \ldots$ $(n < \omega)$ of p such that $U \supset \overline{V}_0$ and $\cap \{V_n : n < \omega\} = \{p\}$. Now, since $V_0 \setminus \{p\} = \cup \{V_n \setminus V_{n+1} : n < \omega\}$ and $|V_0 \setminus \{p\}| \geq \omega_1$, there is an $n_0 < \omega$ such that $|V_{n_0} \setminus V_{n_0+1}| \geq \omega_1$.

Hence if q is a complete accumulation point of $V_{n_0} \setminus V_{n_0+1}$, then

$q \in A$, $q \neq p$ and

$$q \in \overline{V_{n_0}} \subset \overline{V}_0 \subset U.$$

This shows that $(U \setminus \{p\}) \cap A \neq \emptyset$, hence A is dense in itself. Thus $\chi(p,A) = \omega$ holds for each $p \in A$ and by the corollary of 2.22 we have $|A| = \exp \omega$, hence, by 2.21

$$|X| = \exp \omega.$$

2.24. Suppose X is locally compact T_2 and $|X| < \exp \alpha$. Then

$$S = \{p: \chi(p,X) < \alpha\}$$

is dense in X.

Proof

Assume that $G \in \sigma(X)$ and $G \cap S = \emptyset$. Then $\chi(q,X) = \chi(q,G) \geq \alpha$ for each $q \in G$, and hence our Remark made at the end of 2.22 gives us $|G| \geq \exp \alpha > |X|$, which is impossible. This completes the proof.

2.25. Let $X \in \mathfrak{T}$, $\chi(X) = \alpha$ and $d(X) > \alpha$. Then there is a subspace $S \subset X$ such that

$$|S| = d(S) = \alpha^+ \quad \text{and} \quad c(S) \leq c(X).$$

Proof

Let us first choose for each $p \in X$ a basis of neighbourhoods $\{V_\xi(p): \xi < \alpha\}$ and then put

$$f(p,q;\ \xi,\eta) = \begin{cases} \text{a member of } V_p(\xi) \cap V_q(\eta), \text{ if} \\ \qquad V_p(\xi) \cap V_q(\eta) \neq \emptyset; \\ \\ \text{not defined otherwise.} \end{cases}$$

If $H \subset X$ is arbitrary, we define

$$H' = \{f(p,q;\xi,\eta): \{p,q\} \in [H]^2 \wedge (\xi,\eta) \in \alpha \times \alpha\}.$$

Furthermore, we set

$$H^{(0)} = H,\ H^{(n)} = (H^{(n-1)})' \text{ and } Cl(H) = \bigcup\{H^{(n)}: n < \omega\}.$$

Obviously, $|H| \leq \alpha$ implies $|Cl(H)| \leq \alpha$.
Now we define sets $A_\xi \subset X$ for $\xi < \alpha^+$, by transfinite induction as follows:
Let $A_0 = \emptyset$; assume the sets A_ξ have already been defined for each $\xi < \nu$, where $\nu < \alpha^+$ and $|A_\xi| \leq \alpha$. Let us put

$$B_\nu = Cl(\cup\{A_\xi : \xi < \nu\}).$$

According to our above remark, $|B_\nu| \leq \alpha$, hence B_ν cannot be dense in X. Therefore we can choose a point $p_\nu \in X \setminus \overline{B}_\nu$. Then we put

$$A_\nu = \{p_\nu\} \cup B_\nu.$$

Obviously, $|A_\nu| \leq \alpha$, hence the induction can be carried out for all $\nu < \alpha^+$.
Let us put $S = \cup\{A_\nu : \nu < \alpha^+\}$. Then, if $R \subset S$, $|R| = \alpha$, there is a $\nu < \alpha^+$ such that $R \subset B_\nu$, hence $p_\nu \notin \overline{B}_\nu \supset \overline{R}$ implies that R cannot be dense in S. Thus, indeed, $d(S) = |S| = \alpha^+$.
$c(S) \leq c(X)$ follows immediately from our construction, because $p, q \in S$ and $V_\xi(p) \cap V_\eta(q) \neq \emptyset$ imply $f(p,q;\xi,\eta) \in S$, and thus any disjoint family of sets of the form $\{V_\xi(p) \cap S\}$ with $p \in T \subset S$ can be "extended" to the disjoint family $\{V_\xi(p)\}$.

2.26. Let $X \in \mathfrak{T}$ and $S \subset X$. Then

$$d(S) \leq d(\overline{S}).\partial(\overline{S}).$$

<u>Proof</u>
Let Z be a dense subset of $\overline{S}$ with $|Z| = d(\overline{S})$. Then for each $p \in Z$ we can choose a subset $H_p \subset S$ with $|H_p| \leq \partial(\overline{S})$ such that

$$p \in \overline{H}_p.$$

We claim that

$$D = \cup\{H_p : p \in Z\}$$

is a dense subset of S.
Indeed, let $x \in S$ and V be an arbitrary open neighbourhood of x. Then $V \cap \bar{S} \neq \emptyset$, hence there is a $p \in V \cap Z$ as well. Then V is a neighbourhood of p too, hence

$$V \cap H_p \neq \emptyset, \text{ i.e. } V \cap D \neq \emptyset$$

which was to be shown. Since

$$|D| \leq \sum \{|H_p| : p \in Z\} \leq |Z|.\partial(\bar{S}) = d(\bar{S}).\partial(\bar{S}),$$

2.26 is proved.

<u>Corollary</u>
If every closed subset of a first countable space X is separable, then X is hereditarily separable.

2.27 For $X \in \mathcal{T}_{3\frac{1}{2}}$ we have

$$w(X) = u(X) \,.\, L(X).$$

<u>Proof</u>. From 2.13 $u(X) \leq w(X)$ and the trivial relation $L(X) \leq w(X)$ we find $w(X) \leq u(X) \,.\, L(X)$. Next, let $\mathcal{U}$ be a basis for a uniformity, defined by open coverings, on X compatible with the topology, such that $|\mathcal{U}| = u(X)$. I.e. (cf. [17]): $\mathcal{U}$ is a family of open coverings, such that $\cup\,\mathcal{U}$ is a basis for the topology and each two covers from $\mathcal{U}$ have a common star-refinement in $\mathcal{U}$. For each cover $\mathcal{O} \in \mathcal{U}$ we choose a subcover $\mathcal{O}^* \subset \mathcal{O}$ of cardinality $L(X)$. Now it is easy to check that $\cup\{\mathcal{O}^* | \mathcal{O} \in \mathcal{U}\}$ is a basis for $\sigma(X)$.

3. The sup = max problem

The functions c,s,h,z,k have the common feature of having been defined as the supremum of cardinalities of certain families of sets. (Sometimes these sets are referred to as "defining sets" of the corresponding cardinal function.) It is natural to ask under what conditions this supremum is actually a maximum, i.e., when does a defining family of maximal cardinality exist. This is what we briefly call the sup = max problem. Obviously, if the value of one of our functions is a non-limit cardinal, the supremum must be a maximum. The interesting cases are therefore those in which the function values are limit cardinals.

3.1. (cf. [7], 6.5) Assume $X \in \mathcal{T}$ and $c(X) = \lambda$ is singular, $\mathrm{cf}(\lambda) = \alpha < \lambda$. Then there is a disjoint family $\mathcal{G} \subset \sigma(X)$, with $|\mathcal{G}| = \lambda$.

Proof

Let us call an open set $G \in \sigma(X)$ normal if for each non-empty $H \subset G$, $H \in \sigma(X)$ we have

$$c(H) = c(G).$$

We claim that for each non-empty $G \in \sigma(X)$ there is a non-empty normal open set G_0 such that $G_0 \subset G$. (in other words, the normal open sets constitute a π-basis for X.) Assume that this is not true. Then we can find $G^1 \in \sigma(X)\setminus\{\emptyset\}$ $G^1 \subset G$, such that $c(G^1) \neq c(G)$, hence $c(G^1) < c(G)$. Now G^1 cannot be normal, therefore we have a $G^2 \in \sigma(X)\setminus\{\emptyset\}$, $G^2 \subset G^1$ such that $c(G^2) < c(G^1) < c(G)$. Continuing this procedure for each $n < \omega$ we would obtain an infinite decreasing sequence of cardinals, which is impossible. This shows that the normal open sets indeed form a π-basis of X.

Now let $\mathcal{N}$ be a maximal disjoint family of normal open sets. From the above assertion it follows immediately that $\cup\,\mathcal{N} = N$ is dense in X. If $|\mathcal{N}| = \lambda$, we are done. Thus we can assume that $|\mathcal{N}| = \beta < \lambda$.

Next we claim that

$$\sup\{c(G) : G \in \mathcal{N}\} = \lambda \qquad (*)$$

holds. Indeed, if $\beta < \delta < \lambda$, δ is a regular cardinal, then there exists a disjoint family $\mathcal{G} \subset \sigma(X)$ with $|\mathcal{G}| = \delta$. Now, since N is dense in X, for each $H \in \mathcal{G}$ $N \cap H \neq \emptyset$, hence there is a $G_H \in \mathcal{N}$ such that $H \cap G_H \neq \emptyset$. Since $\delta > \beta$ is regular, there are a subfamily $\mathcal{G}_0$ of $\mathcal{G}$ and an $H_0 \in \mathcal{N}$ such that $|\mathcal{G}_0| = \delta$ and $G \in \mathcal{G}_0 \rightarrow G \cap H_0 \neq \emptyset$. This implies $c(H_0) \geq \delta$, and thus $(*)$ is proved.

Now, if $\beta < \alpha = cf(\lambda)$ there is a $H \in \mathcal{N}$ such that $c(H) = \lambda$, since otherwise $(*)$ could not hold. Let us write $\lambda = \sum_{\xi<\alpha} \alpha_\xi$, where $\alpha_\xi < \lambda$.

Then H (and X too) contains α disjoint open subsets $\{H_\xi : \xi < \alpha\}$ such that $(\lambda=)c(H_\xi) > \alpha_\xi$, because H is normal. By $(*)$ this is also true for X if $\beta \geq \alpha$ and there is no $H \in \mathcal{N}$ with $c(H) = \lambda$.

Therefore, if we take a disjoint family $\mathcal{G}_\xi$ of open sets in H_ξ, such that $|\mathcal{G}_\xi| = \alpha_\xi$, then $\mathcal{G} = \cup\,\{\mathcal{G}_\xi : \xi < \alpha\}$ yields us a disjoint family of open sets of maximal cardinality λ.

<u>Remark</u>

We shall see (example 6.5) that for inaccessible λ's 3.1 no longer holds.

3.2. (cf. [14]) Suppose λ is singular strong limit, $X \in \mathcal{T}_2$, $|X| \geq \lambda$. Then X contains a discrete subspace D of power λ.

Our proof will be similar to that of 2.9, however, instead of the ER-Lemma A4.5 we shall use the C-Lemma A5.4.

Let $\prec$ be an arbitrary well-ordering of X, and for $\{x,y\} \in [X]^2$ with $x \prec y$ we choose neighbourhoods $U(x,y)$ and $V(x,y)$ of x and y respectively, such that $U(x,y) \cap V(x,y) = \emptyset$.

Then we define a partition of $[X]^3$ by putting $\{x,y,z\} \in I_{(\varepsilon_1,\varepsilon_2)}$ $(x \prec y \prec z)$ according to the following rules:

$$\varepsilon_1 = 0, \text{ if } x \in U(y,z);$$

$$\varepsilon_1 = 1, \text{ if } x \notin U(y,z);$$

$$\varepsilon_2 = 0, \text{ if } z \in V(x,y);$$

$$\varepsilon_2 = 1, \text{ if } z \notin V(x,y)$$

Applying the C-Lemma A5.4 we find an $H \subset X$, $|H| = \lambda$ and a partition of H:

$$H = \cup\{H_\xi : \xi < cf(\lambda) = \alpha\}$$

such that conditions (i), (ii) and (iii) of the C-Lemma hold (p.126). Suppose that $\xi < \alpha$ and $y \in H_\xi$, moreover that y has an immediate $\prec$-predecessor x, and an immediate $\prec$-successor z in H_ξ. We shall show that y is isolated in the subspace H. Since the set of all such y's is obviously of power λ, this will prove 3.2.

In fact we claim that

$$N = V(x,y) \cap U(y,z) \cap H = \{y\}$$

Evidently, $x, z \notin N$. Now, if $p \in H$ and $p \prec x$, then $p \in V(x,y)$ implies $p \notin U(x,y)$, hence $\{p,x,y\} \in I_{(1,\varepsilon_2)}$ by the definition of our partition. According to (iii), however, we also have $\{p,y,z\} \in I_{(1,\varepsilon_2)}$, and thus $p \notin U(y,z) \supset N$. Similarly we can show that if $z \prec q$, then

$q \notin N$, which completes our proof.

Corollary

Assume ϕ is one of the functions s, h, z, $X \in \mathcal{T}_2$ and $\phi(X) = \lambda$ is a singular strong limit cardinal. Then $\phi(X)$ is actually a maximum. This follows immediately from $|X| \geq \phi(X) = \lambda$ and 3.2.

Remark

It is easy to see that if λ is a weakly compact (inaccesible) cardinal, then 3.2 and its Corollary hold for this λ; in fact, the proof given in 1.9 can be applied, using the fact that $\lambda \to (\lambda)^3_4$ holds (cf A6.4). Thus e.g., if GCH holds then the sup = max problem has a positive solution for s, h and z on $\mathcal{T}_2$, unless λ is a not weakly compact inaccessible cardinal. We shall show that this exception is in fact essential (cf. Example 6.6).

3.3. (cf. [12]) Suppose $X \in \mathcal{H}$, $\phi(X) = \lambda$, where ϕ is one of the functions s, h, z and $cf(\lambda) = \omega$. Then the answer to the sup = max problem is positive.

Proof

We shall first establish the following

Lemma

Assume $R \in \mathcal{T}$, $|R| = \alpha > \beta \geq \omega$. Then either R contains a discrete subset of power α, or $|S_\beta| < \alpha$, where

$$S_\beta = \{x \in R : \exists\ U_x \text{ neighbourhood of } x \text{ such that } |U_x| < \beta\}.$$

Indeed, if $|S_\beta| = \alpha$ and for each $x \in S_\beta$ we put $F(x) = U_x \cap S_\beta$, then F is a set mapping on S_β such that $|F(x)| < \beta < \alpha$ holds for each $x \in S_\beta$. Therefore we can apply Hajnal's theorem A3.5, and obtain a subset $D \subset S_\beta$ with $|D| = \alpha$ such that $F(x) \cap D = \{x\}$ holds for each $x \in D$. This, however, implies $U_x \cap D = \{x\}$ for each $x \in D$, hence D is a discrete subspace of R.

Now we return to the proof of 3.3.

Since $cf(\lambda) = \omega$, we can write

$$\lambda = \sum_{k<\omega} \alpha_k,$$

where $k < k' \to \alpha_k < \alpha_{k'}$ and each α_k is regular.
Since $\phi(X) = \lambda > \alpha_k$, for each $k < \omega$ there exists a "defining set" for ϕ, say D_k, such that $|D_k| = \alpha_k$. Let us put

$$X' = \cup\{D_k : k < \omega\}.$$

Then $|X'| = \lambda$, and using the Lemma for each $\beta = \alpha_k < \lambda$ we obtain that either X' contains a discrete subset of power λ (which is certainly a defining set for ϕ) or we can assume that for each α_k only less than λ points in X' have neighbourhoods in X' of power $< \alpha_k$.
We shall then define a sequence of points in X' as follows:
Let x_0 be an arbitrary point of X' such that each neighbourhood of x_0 in X' has cardinality $\geq \alpha_0$. Now, if $k > 0$ and $\{x_0,\dots,x_{k-1}\}$ have already been defined, we choose as x_k an arbitrary point of $X' \setminus \{x_0,\dots,x_{k-1}\}$ such that each neighbourhood of x_k in X' is of cardinality $\geq \alpha_k$. By our assumption the induction can be carried out for all $k < \omega$.
Now since X (and X') belong to $\mathcal{H}$, we can select a subsequence $\{x_{k_i} : i < \omega\}$ of the above sequence for which there are open neighbourhoods U_i of x_{k_i} in X' such that $U_i \cap U_j = \emptyset$ if $i \neq j$.

By our construction we have

$$|U_i| \geq \alpha_{k_i}$$

for each $i < \omega$. Also, $U_i \subset X' = \cup\{D_k : k < \omega\}$ imply

$$U_i = \cup\{U_i \cap D_k : k < \omega\}.$$

Since α_{k_i} is regular, we immediately see that there exists a $\bar{k} < \omega$ such that $|D_{\bar{k}} \cap U_i| \geq \alpha_{k_i}$. In other words, each U_i contains a defining set S_i for ϕ such that $|S_i| \geq \alpha_{k_i}$. Now these S_i's are contained in pairwise disjoint open subsets of X', hence

$$S = \cup\{S_i : i < \omega\}$$

is a defining set for ϕ in X' and consequently in X too. But

$$|S| = \sum_{i<\omega} \alpha_{k_i} = \lambda,$$

which completes the proof.

Remark

We do not know whether $\mathcal{H}$ can be replaced by $\mathcal{T}_2$ in 3.3, or whether the condition $cf(\lambda) = \omega$ could be weakened (without using GCH, of course). Both of these problems seem to be rather difficult.
J. Roitman [65] has shown that if $\omega < cf(\lambda) < \lambda$ and $\lambda < \exp(cf(\lambda))$, then for λ the answer to the sup = max problem is negative within $\mathcal{T}_2$ for any of the functions h,z,s.

4. Cardinal functions on products

4.1. The aim of this section is to investigate the following basic problem:

Assume ϕ is a cardinal function and

$$R = X\{R_i : i \in I\}; \qquad (*)$$

how can we evaluate $\phi(R)$ in terms of the values $\phi(R_i)$ $(i \in I)$ and the cardinality of the index set I?

In order to exclude some trivial difficulties we assume that no R_i in $(*)$ is indiscrete, hence it contains two points p_i, q_i such that $p_i \notin \overline{\{q_i\}}$. If we denote by F the two element T_0 space, in which one of the points is closed and the other is not, then our convention obviously implies (with $|I| = \alpha \geq \omega$)

$$F^\alpha \underset{top}{\subset} R \text{ or } D(2)^\alpha \underset{top}{\subset} R, \qquad (**)$$

depending on whether $|\{i: q_i \in \overline{\{p_i\}}\}| = \alpha$ or not.

We shall show in 6.7 and 6.8 that the following relations hold for F^α and $D(2)^\alpha$:

a) If ϕ is one of the functions w, s, h, z, χ then $\phi(F^{\alpha}) = \phi(D(2)^{\alpha}) = \alpha$;

b) If ϕ is π, or u, or ψ, or ∂, then $\phi(D(2)^{\alpha}) = \alpha$;

c) $d(D(2)^{\alpha}) = \log \alpha$.

For a product of the form (*), where ϕ is defined for each R_i ($i \in I$), we put

$$\phi_I(R) = \sup\{\phi(R_i) : i \in I\}.$$

4.2. (i) For every cardinal function we have defined,

$$\phi(R) \geqq \phi_I(R).$$

(ii) If $\phi \in \{w,\pi,s,h,z,\chi\}$ and $|I| = \alpha \geq \omega$, we have

$$\phi(R) \geqq |I| = \alpha.$$

(iii) If I is infinite and all the R_i's are T_1 then

$$\psi(R) \geq |I|.$$

(iv) If all the R_i's are completely regular and $|I| \geq \omega$, then

$$u(R) \geq |I|.$$

Proof

Ad (i) For the functions c, c_ξ, π, k and L (i) holds because each R_i is the image of R under the open and continuous mapping

$$\pi_i : R \to R_i,$$

i.e. the projection of R onto the factor R_i. For the others (i) is true because $R_i \underset{\text{top}}{\subset} R$ holds for each $i \in I$.

If $\phi \neq \pi$ then (ii), (iii) and (iv) immediately follow from 4.1 a) and b), respectively, because ϕ is monotone with respect to subspaces. To prove (ii) for π, however, we have to proceed differently. Since R_i is not indiscrete, we can choose a non-empty, proper open subset $G_i \subset R_i$ for each $i \in I$. Let

$$\tilde{G}_i = \pi_i^{-1}(G_i)$$

and $\mathcal{P}$ a π-basis for R with $|\mathcal{P}| = \pi(R)$. It follows from our assumption that the intersection of infinitely many $\tilde{G}_i$'s has an empty interior. Therefore, each $P \in \mathcal{P}$ can only be contained in a finite number of the sets $\tilde{G}_i$. This implies $|I| \leq |\mathcal{P}|.\omega$, hence $|I| \leq \pi(R)$.

4.3. (i) If $\phi \in \{w,\pi,\chi\}$, then

$$\phi(R) = |I|.\ \phi_I(R);$$

(ii) If all the R_i's are T_1, we have

$$\psi(R) = |I|.\ \psi_I(R).$$

Proof

Suppose that $\mathcal{B}_i$ $(i \in I)$ is a base for R_i such that $|\mathcal{B}_i| = w(R_i)$. It is obvious that the system $\mathcal{B}$ of all (open) sets of the form $\pi_{i_1}^{-1}(B_1) \cap \ldots \cap \pi_{i_k}^{-1}(B_k)$, where $B_j \in \mathcal{B}_{i_j}$, constitute a base for R. Obviously, $|\mathcal{B}| \leq |I|.\ w_I(R)$, hence

$$w(R) \leq |I|.\ w_I(R).$$

The opposite inequality follows from 4.2 (i) and 4.2 (ii).
It is easy to see that if the $\mathcal{B}_i$'s above are chosen as π-basis for R_i, then the resulting $\mathcal{B}$ is a π-basis for R, and this implies our proposition as above.
Finally, if $f \in R$ and $\mathcal{B}_i$ is a neighbourhood basis (or separating system) for $f_i \in R_i$, then $\mathcal{B}$ is a neighbourhood basis (or separating

system) for f in R, and from this (i) for χ (or (ii) for ψ) follows immediately.

4.4. (cf. 6.9) If $\phi = h$ or $\phi = z$ then we have

$$|I|.\ \phi_I(R) \leq \phi(R) \leq |I|.\ \exp \phi_I(R).$$

Proof

The inequality on the left is an immediate consequence of 4.2 (i) and (ii). The proof of the other inequality is completely analogous for h and z, therefore we shall only prove it for $\phi = h$.

First we consider the case in which

$$|I| \leq h_I(R).$$

Let us put $h_I(R) = \alpha$ and $(\exp \alpha)^+ = \beta$. Suppose that $h(R) \geq \beta$. Then we can choose a right separated sequence $S = \{f_\xi : \xi < \beta\} \subset R$. Thus for each $\xi < \beta$ we have an elementary open set $U_\xi \subset R$ for which

$$\beta > \eta > \xi \rightarrow f_\eta \notin U_\xi.$$

Now we form a partition of $[\beta]^2$ as follows:

If $\{\xi,\eta\} \in [\beta]^2$, $\xi < \eta$ we put for $i \in I$

$$\{\xi,\eta\} \in I_i \leftrightarrow \pi_i(f_\eta) \notin \pi_i(U_\xi).$$

Since $f_\eta \notin U_\xi$ for $\eta > \xi$ and U_ξ is elementary, $\cup\{I_i : i \in I\} = [\beta]^2$, hence we obtain a partition of $[\beta]^2$, indeed.

Now $\beta > \exp \alpha$ and $|I| \leq \alpha$ imply, using the ER-Lemma A4.4 , that there is an $H \subset \beta$ and an $i_0 \in I$ such that

$$|H| = \alpha^+ \text{ and } [H]^2 \subset I_{i_0}.$$

It follows from the definition of I_{i_0} that $\{\pi_{i_0}(f_\xi) : \xi \in H\} \subset R_{i_0}$ is a right-separated subspace of R_{i_0} of cardinality $|H| = \alpha^+$. This, however, is in contradiction to

$$h(R_{i_0}) \leq h_I(R) \leq \alpha,$$

and proves 4.4 for h, under the condition $|I| \leq h_I(R)$. In particular, we have

$$h(R) \leq \exp(h_I(R)),$$

provided that I is finite.
Suppose now that $|I| > h_I(R)$ and

$$h(R) > |I|.\ \exp(h_I(R)) = \alpha.$$

So we have a right-separated sequence $S = \{f_\xi: \xi < \alpha^+\}$ in R with suitable elementary neighbourhoods $\{U_\xi: \xi < \alpha^+\}$, as above.
For each $\xi < \alpha^+$ we put

$$I_\xi = \{i \in I: \pi_i(U_\xi) \neq R_i\}.$$

Then each I_ξ is a finite subset of I, but $|I| \leq \alpha < \alpha^+$ and therefore I has only α finite subsets, consequently there is an $A \subset \alpha^+$ with $|A| = \alpha^+$ and a finite subset $\tilde{I}$ of I such that

$$\xi \in A \to I_\xi = \tilde{I}.$$

Now it is obvious that

$$S_A = \{\pi_{\tilde{I}}(f_\xi): \xi \in A\}$$

is a right-separated subspace of

$$\tilde{R} = X\{R_i: i \in \tilde{I}\},$$

because for each $\xi \in A$

$$f \in U_\xi \leftrightarrow \pi_{\tilde{I}}(f) \in \pi_{\tilde{I}}(U_\xi).$$

Thus we have

$$h(\tilde{R}) \geq \alpha^+ > \exp(h_{\tilde{I}}(\tilde{R})),$$

which is in contradiction to what we showed in the first part of our proof. This completes the verification of 4.4 for h.
It should be obvious to the reader that, by straightforward modifications, the above proof can be transformed into a proof of 4.4 for z.

Remark
Examples 6.9 and 6.10 show that 4.4 cannot be improved by decreasing $\exp(\phi_I(R))$. Recently A. Hajnal and I. Juhász have shown by a different method that 4.4 also holds for $\phi = s$ (see reference [63]).

4.5. (cf. [6], [16] or [30])

(i) $d(R) \leqq \log |I| . d_I(R)$;

(ii) If moreover each R_i contains two disjoint non-empty open subsets, then

$$d(R) = \log |I| . d_I(R).$$

Proof
First we show that for $\alpha \geqq \omega$

$$d(D(\alpha)^{\exp \alpha}) \leqq \alpha$$

holds. For this we write

$$D(\alpha)^{\exp \alpha} = X\{D_\xi(\alpha) : \xi < \exp \alpha\},$$

where $D_\xi(\alpha) = D(\alpha)$ for each $\xi < \exp\alpha$. Then we choose a T_2 space X, say $X = D(2)^\alpha$, such that $|X| = \exp\alpha$ and $w(X) = \alpha$; we write X in the form $X = \{p_\xi : \xi < \exp\alpha\}$ and choose an open basis $\mathcal{B} = \{B_\rho : \rho < \alpha\}$ for X. For any ordered pair (r,s) of finite sequences of ordinals where

$$r = (\rho_1,\ldots,\rho_j),\ s = (\eta_1,\ldots,\eta_j),$$

$$\rho_1,\ \ldots,\ \rho_j < \alpha,\ \eta_1,\ \ldots,\ \eta_j < \alpha$$

and the sets $B_{\rho_1}, \ldots, B_{\rho_j}$ are pairwise disjoint, we define a point $f^{(r,s)} \in D(\alpha)^{\exp\alpha}$ as follows:

$$\pi_\xi(f^{(r,s)}) = \begin{cases} \eta_1, \text{ if } p_\xi \in B_{\rho_1}; \\ -------- \\ \eta_j, \text{ if } p_\xi \in B_{\rho_j}; \\ 0\ , \text{ if } p_\xi \notin B_{\rho_1} \cup \ldots \cup B_{\rho_j}. \end{cases}$$

Let S be the set of all such points $f^{(r,s)}$ in $D(\alpha)^{\exp\alpha}$. We claim that S is dense in $D(\alpha)^{\exp\alpha}$. Since $|S| = \alpha$, this will imply $d(D(\alpha)^{\exp\alpha}) \leq \alpha$.

Let G be an elementary open set in $D(\alpha)^{\exp\alpha}$ of the form

$$G = \pi_{\xi_1}^{-1}(\{\tilde\eta_1\}) \cap \ldots \cap \pi_{\xi}^{-1}(\{\tilde\eta_j\}) \quad (\xi_1 < \ldots < \xi_j < \exp\alpha).$$

These sets form a basis for $D(\alpha)^{\exp\alpha}$, hence it suffices to show $S \cap G \neq \emptyset$ for each such G. Since X is T_2, the points $p_{\xi_1}, \ldots, p_{\xi_j} \in X$ have pairwise disjoint neighbourhoods, and thus we can select pairwise disjoint members of $\mathcal{B}$, say $B_{\rho_1}, \ldots, B_{\rho_j}$ such that

$$p_{\xi_1} \in B_{\rho_1}, \ldots, p_{\xi_j} \in B_{\rho_j}.$$

Now, if we put $r = (\rho_1,\ldots,\rho_j)$, $s = (\tilde\eta_1,\ldots,\tilde\eta_j)$, we have $f^{(r,s)} \in G$,

which follows immediately from the definitions of $f^{(r,s)}$ and G.
Hence we have $G \cap S \neq \emptyset$, and S is dense in $D(\alpha)^{\exp \alpha}$.
Now let us put $\alpha = \log|I|.d_I(R)$. Then $d(R_i) \leq \alpha$, for each $i \in I$, hence we can choose a dense subset $S_i \subset R_i$ with $|S_i| \leq \alpha$. Obviously, $S = X\{S_i : i \in I\}$ is dense in R.
Now let g_i be an arbitrary mapping of $D(\alpha)$ onto S_i. Then g_i is continuous, because $D(\alpha)$ is discrete, hence the product map

$$g = X\{g_i : i \in I\} : D(\alpha)^I \to S$$

is also continuous and surjective. Since $\log |I| \leq \alpha$, we have $|I| \leq \exp \alpha$, and therefore

$$d(S) \leq d(D(\alpha)^I) \leq d(D(\alpha)^{\exp \alpha}) \leq \alpha.$$

Since S is dense in R, we have

$$d(R) \leq d(S) \leq \alpha,$$

which proves 4.5 (i).
Now suppose that each R_i contains two disjoint open sets $U_i^{(0)}$ and $U_i^{(1)}$. By 4.2 (i) we have $d(R) \geq d_I(R)$, hence to prove 4.5 (ii), it suffices to show $d(R) \geq \log |I|$. Suppose this is not true and S is a dense subset of R with $|S| < \log |I|$.
Now let $r = (i_1,\ldots,i_j)$ be a sequence of pairwise different indices and $s = (\varepsilon_1,\ldots,\varepsilon_j)$ an arbitrary sequence of 0's and 1's, with the same length as r. Then

$$G^{(r,s)} = \pi_{i_1}^{-1}(U_{i_1}^{(\varepsilon_1)}) \cap \ldots \cap \pi_{i_j}^{-1}(U_{i_j}^{(\varepsilon_j)})$$

is open in R, hence there is a point $p^{(r,s)} \in S$, with $p^{(r,s)} \in G^{(r,s)}$.
Let us consider now the space

$$D(2)^I = X\{D_i(2) : i \in I;\ D_i(2) = D(2)\}.$$

Every pair (r,s) of the above kind determines an elementary open set in $D(2)^I$, namely

$$O^{(r,s)} = \pi_{i_1}^{-1}(\{\varepsilon_1\}) \cap \ldots \cap \pi_{i_j}^{-1}(\{\varepsilon_j\}),$$

and conversely, each elementary open set can be obtained in this way. Furthermore, to every $p \in S$ we assign a point $\tilde{p}$ of $D(2)^I$, defined as follows:

$$\pi_i(\tilde{p}) = \begin{cases} 0, & \text{if } \pi_i(p) \in U_i^{(0)}; \\ 1, & \text{if } \pi_i(p) \notin U_i^{(0)}. \end{cases}$$

We claim that $\tilde{S} = \{\tilde{p}\colon p \in S\}$ is dense in $D(2)^I$. Indeed, if $O^{(r,s)}$ is an elementary open set in $D(2)^I$, then $\tilde{p}^{(r,s)} \in O^{(r,s)}$, because $p^{(r,s)} \in G^{(r,s)}$, hence $\tilde{S} \cap O^{(r,s)} \neq \emptyset$. However, this implies

$$d(D(2)^I) \leq |\tilde{S}| \leq |S| < \log |I|,$$

which is in contradiction to 4.1 c). Thus 4.5 (ii) is proved.

4.6. (cf. [15] or [24])

$$c_I(R) \leq c(R) \leq \exp\,(c_I(R)).$$

<u>Proof</u>

The left-hand inequality was proved in 4.2 (i). To show the other inequality, we first consider the case in which I is finite, $I = \{i_1,\ldots,i_n\}$. We put $c_I(R) = \alpha$.

Suppose $c(R) > \exp \alpha$ and $\mathcal{G} = \{G_\xi\colon \xi < (\exp \alpha)^+\}$ is a disjoint family of elementary open sets in R.

Let us define

$$G_\xi^{(k)} = \pi_{i_k}(G_\xi), \text{ for } k = 1, \ldots, n$$

and

$$I_k = \{\{\xi,\eta\}: \xi < \eta < (\exp \alpha)^+ \text{ and } G_\xi^{(k)} \cap G_\eta^{(k)} = \emptyset\}.$$

Since $\mathfrak{G}$ is disjoint, every pair $\{\xi,\eta\}$ with $\xi < \eta < (\exp \alpha)^+$ belongs to some I_k, i.e. we have a partition of $\left[(\exp \alpha)^+\right]^2$ into n classes. Hence, by the ER-Lemma A4.4 , there is a set $A \subset (\exp \alpha)^+$ with $|A| > \alpha$ and a $k < n$ such that

$$[A]^2 \subset I_k.$$

This implies that $\{G_\xi^{(k)}: \xi \in A\}$ is a disjoint family of open sets in R_{i_k}, hence

$$c(R_{i_k}) \geqq |A| > \alpha \geqq c_I(R) \geqq c(R_{i_k}),$$

which is a contradiction.
Now let I be arbitrary and suppose that

$$c(R) > \exp (c_I(R)) = \beta.$$

Thus we have a disjoint family $\{G_\xi: \xi < \beta^+\}$ of elementary open sets in R. Let us put

$$I_\xi = \{i \in I: \pi_i(G_\xi) \neq R_i\} \quad (\xi < \beta^+).$$

By A2.2 the system $\{I_\xi: \xi < \beta^+\}$ contains a subsystem $\{I_\xi: \xi \in B\}$, where $B \subset \beta^+$ and $|B| = \beta^+$, which is quasi-disjoint. Thus, if

$$\tilde{I} = \cap\{I_\xi: \xi \in B\},$$

for $\xi, \eta \in B$ we have

$$G_\xi \cap G_\eta = \emptyset \leftrightarrow \pi_{\tilde{I}}(G_\xi) \cap \pi_{\tilde{I}}(G_\eta) = \emptyset.$$

This means that $\{\pi_{\tilde{I}}(G_\xi): \xi \in B\}$ would be a disjoint system of open sets in $\pi_{\tilde{I}}(R) = X\{R_i: i \in \tilde{I}\}$, which is impossible by what we have proved above. Thus 4.6 is proved.

Remark

In the second part of this proof we have shown that if $c(R) > \beta$ then there is a finite subset $I_0 \subset I$ such that

$$c(X\{R_i: i \in I_0\}) > \beta.$$

4.7.

$$c_{\xi,I}(R) \leqq c_\xi(R) \leqq \exp(\omega_\xi \cdot c_{\xi,I}(R)).$$

The proof is very similar to that of 4.6. The left-hand inequality was shown in 4.2 (i). We put $\omega_\xi \cdot c_{\xi,I}(R) = \alpha$, $\beta = (\exp \alpha)^+$.
To prove the rest, we first consider the case where $|I| \leqq \omega_\xi$. If $c_\xi(R) \geqq (\exp \alpha)^+ = \beta$ held and $\{H_\rho: \rho < \beta\}$ were a disjoint family of elementary $G_{\delta,\xi}$ sets in R, then using the partition $[\beta]^2 = \cup\{I_i : i \in I\}$, where

$$I_i = \{\{\rho_1,\rho_2\}: \pi_i(H_{\rho_1}) \cap \pi_i(H_{\rho_2}) = \emptyset\},$$

by the ER-Lemma A4.4 we would get α^+ disjoint $G_{\delta,\xi}$-sets in one of the factor spaces R_i, a contradiction.
Now, if I is arbitrary and $\{H_\rho: \rho < \beta\}$ is as above, furthermore $I_\rho = \{i \in I: \pi_i(H_\rho) \neq R_i\}$ $(\rho < \beta)$, we have $\beta > \exp \alpha = (\exp \alpha)^{\omega_\xi}$, hence by A2.2 there is a $B \subset \beta$ with $|B| > \exp \alpha$ such that $\{I_\rho: \rho \in B\}$ is quasi-disjoint. This, however, implies that for $\tilde{I} = \cap\{I_\rho: \rho \in B\}$ the projections $\pi_{\tilde{I}}(H_\rho)$, $\rho \in B$ are pairwise disjoint, which is in contradiction to the first part of our proof, since $|\tilde{I}| \leqq \omega_\xi \leqq \alpha$.

Recall 0.13, that α is a caliber for X iff for every $\mathfrak{O} \subset \sigma(X)$ with $|\mathfrak{O}| = \alpha$ there is a $\mathfrak{O}' \subset \mathfrak{O}$ with $|\mathfrak{O}'| = \alpha$ and $\cap\mathfrak{O}' \neq \emptyset$.

4.8. (cf. [31]) Suppose $\alpha > \omega$, α is regular. Then, if α is a caliber for each R_i, so is α for R. Hence $c(R) \leq \sup(d(R_i) : i \in I\}$.

Proof

First we consider the case where I is finite, e.g. $I = \{1,\dots,n\}$. Suppose now $\{G_\xi : \xi < \alpha\}$ is a family of non-empty elementary open sets in R, i.e.

$$G_\xi = X\{G_\xi^{(i)} : i = 1, \dots, n\} \quad (\xi < \alpha),$$

where $G_\xi^{(i)}$ is a (non-empty) open subset of R_i. Since α is a caliber for R_1, there is an $A_1 \subset \alpha$ with $|A_1| = \alpha$ such that

$$\cap \{G_\xi^{(1)} : \xi \in A_1\} \neq \emptyset.$$

Then, using the fact that α is a caliber for R_2, we get a set of ordinals $A_2 \subset A_1$ such that $|A_2| = \alpha$ and

$$\cap \{G_\xi^{(2)} : \xi \in A_2\} \neq \emptyset;$$

continuing this procedure we finally obtain a set $A_n \subset A_{n-1} \subset \dots \subset A_1 \subset \alpha$ such that $|A_n| = \alpha$ and

$$\cap \{G_\xi^{(i)} : \xi \in A_n\} \neq 0$$

for each $i = 1, \dots, n$. Thus we have

$$\cap \{G_\xi : \xi \in A_n\} \neq \emptyset,$$

which proves that α is a caliber for R.

Now suppose that $|I| = \beta \geq \omega$, and $\{G_\xi : \xi < \alpha\}$ is a family of elementary open sets in R. As usual, we put

$$I_\xi = \{i \in I : \pi_i(G_\xi) \neq R_i\} \quad (\xi < \alpha).$$

By A2.2 there is an $A \subset \alpha$ with $|A| = \alpha$ such that $\{I_\xi: \xi \in A\}$ is quasi-disjoint. If it is even disjoint, i.e. $\cap \{I_\xi: \xi \in A\} = \emptyset$, then $\cap \{G_\xi: \xi \in A\} \neq \emptyset$, hence we are done. If, however, $\cap \{I_\xi: \xi \in A\} = \tilde{I} \neq \emptyset$, then projecting our family onto the finite partial product $X\{R_i: i \in \tilde{I}\}$ yields us the desired result.

4.9. (cf. [17], VII. 19 or [6b]). Suppose

$$f: R = X\{R_i: i \in I\} \to X$$

is a continuous map of the product space R onto the T_2 space X. Moreover, let $\alpha = \max\{d_I(R), \psi(X)\}$. Then there is a set $J \subset I$ with $|J| \leq \alpha$ and a map $g: R_J = X\{R_i: i \in J\} \to X$ such that $f = g \circ \pi_J$ (i.e. f depends on not more than α coordinates).

Proof

Let p be an arbitrary point of R, $f(p) = y$. Then $\psi(y,X) \leq \alpha$, hence $f^{-1}(y)$ is a $G_{\delta,\eta}$ set in R, if $\alpha = \omega_\eta$. Let H_p be an elementary $G_{\delta,\eta}$ set in R such that $p \in H_p \subset f^{-1}(y)$. We put $J_0 = \{i \in I: \pi_i(H_0) \neq R_i\}$. Clearly $|J_0| \leq \alpha$. Then we proceed by induction. First however, we choose a fixed point $0_i \in R_i$ for each $i \in I$, and introduce the following notation:
if q is a point of a subproduct $X\{R_i: i \in \tilde{I}\}$, where $\tilde{I} \subset I$, then q^0 is the point of R specified as follows:

$$\pi_i(q^0) = \begin{cases} \pi_i(q), & \text{if } i \in \tilde{I}, \\ 0_i, & \text{if } i \in I \setminus \tilde{I}. \end{cases}$$

Suppose that the sets J_k with $|J_k| \leq \alpha$ have already been defined for $k < n < \omega$. Then $|\bigcup_{k<n} J_k| \leq \alpha$, hence, by 4.5 (i)

$$d(X\{R_i: i \in \bigcup_{k<n} J_k\}) \leq \alpha.$$

Let S_n be a dense subset of the above partial product; $|S_n| \leq \alpha$. Then for each $q \in S_n$ we can choose an elementary $G_{\delta,\eta}$ set H_q in R such that

$$q^0 \in H_q \subset f^{-1}(f(q^0)).$$

Then we set $J_n^q = \{i \in I: \pi_i(H_q) \neq R_i\}$ and

$$J_n = \cup\{J_n^q: q \in S_n\}.$$

Obviously, $|J_n| \leq \alpha.\alpha = \alpha$, hence the induction can be carried out for each $n < \omega$. Finally, we define

$$J = \cup\{J_n: n < \omega\},$$

and claim that J indeed satisfies our requirements.
For this we have to show that $\pi_J(p) = \pi_J(r)$ implies $f(p) = f(r)$ for all $p, r \in R$, or equivalently that $f(p) = f(\tilde{p})$ for all $p \in R$, where

$$\tilde{p} = (\pi_J(p))^0.$$

If $q \in S_n$ for a certain $n < \omega$, we put

$$q' = \pi_J(q^0) \text{ and } S_n' = \{q': q \in S_n\}.$$

It is clear then that $S' = \cup\{S_n': n < \omega\}$ is dense in $\pi_J(R)$. Hence there is a Moore-Smith sequence $\{q_t': t \in T\}$ over a directed index set T such that $q_t' \to \pi_J(p)$, hence $q_t^0 \to \tilde{p}$. Also, if we define $\bar{q}$ by

$$\pi_i(\bar{q}) = \begin{cases} \pi_i(q'), \text{ if } i \in J; \\ \pi_i(p), \text{ if } i \in I \setminus J, \end{cases}$$

then we must have $\bar{q}_t \to p$ $(t \in T)$.
For any $t \in T$ we have

$$\{i \in I: \pi_i(\bar{q}_t) = \pi_i(q_t^0)\} \supset J = \bigcup_{n<\omega} J_n,$$

hence, by our construction, $f(\bar{q}_t) = f(q_t^0)$. Thus we have $f(\bar{q}_t) \to f(p)$ and $f(q_t^0) \to f(\tilde{\bar{p}})$ and consequently $f(p) = f(\tilde{\bar{p}})$, since f is continuous and X is Hausdorff.

Remark

The significance of 4.9 lies in the possibility of giving an upper bound for the number of factors in a product of certain spaces, when we originally only know the mere existence of such a product. As an example we mention the following.

Corollary

(cf. [8]) If X is a dyadic compact space then $w(X) = \chi(X)$ $(= \psi(X))$.

Proof

By definition, there is a continuous mapping $f: D(2)^\beta \to X$ for a certain β. If $\chi(X) = \alpha$, then by 4.9 f only depends on $\leq \alpha$ coordinates, i.e. we can assume $\beta \leq \alpha$. Now $w(D(2)^\beta) = \beta$ (cf. 4.3 (i)), hence, $w(X) \leq \beta$, since, as is well-known, continuous functions do not increase the weight in the class of compact T_2 spaces. Hence $w(X) \leq \beta \leq \alpha$, i.e. $w(X) = \chi(X)$.

5. Martin's axiom

5.1. The following assertion (M) which we call Martin's axiom, is proved to be consistent with the usual axioms of set-theory (cf. [26] or [34]):

(M) If $\mathfrak{B}$ is a complete Boolean algebra satisfying the countable chain condition (shortly c.c.c.) and $A_\xi \subset \mathfrak{B}$ is a subset of $\mathfrak{B}$ for each $\xi < \omega_1$ with $a_\xi = \sup A_\xi$, then there is an ultrafilter $\mathfrak{U}$ on $\mathfrak{B}$ which preserves all these sups in the following sense:
If $a_\xi \in \mathfrak{U}$ then $A_\xi \cap \mathfrak{U} \neq \emptyset$, i.e. there is an $a \in A_\xi$ with $a \in \mathfrak{U}$.
We shall show that (M) implies $\exp \omega > \omega_1$, i.e. it contradicts CH. On the other hand, (M) has several interesting consequences, which in the author's opinion, make it worthwile to have as an alternative to CH.

5.2. The following assertion (R) is equivalent to (M):

(R) If X is a compact T_2 space with the Suslin property (i.e. $c(X) = \omega$), then X has the ω_1-Baire property.

Proof

(M) $\rightarrow$ (R). Let $\{S_\xi : \xi < \omega_1\}$ be a family of nowhere dense subsets of X. We shall show that $\cup\{S_\xi : \xi < \omega_1\} \neq X$.

For this we consider the complete Boolean algebra $\mathcal{B}$ of all regular open subsets of X. Since X has the Suslin property, $\mathcal{B}$ satisfies the c.c.c.

For each $\xi < \omega_1$ we put

$$\mathcal{A}_\xi = \{G \in \mathcal{B} : \bar{G} \cap S_\xi = \emptyset\}.$$

Since S_ξ is nowhere dense, it follows easily that $\cup\mathcal{A}_\xi$ is dense in X, hence

$$\sup \mathcal{A}_\xi = \mathrm{Int}\overline{(\cup\mathcal{A}_\xi)} = X.$$

Now let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$ which preserves all the $\sup \mathcal{A}_\xi$. Then $X \in \mathcal{U}$, hence $\mathcal{A}_\xi \cap \mathcal{U} \neq \emptyset$ for each $\xi < \omega_1$. Let $G_\xi \in \mathcal{A}_\xi \cap \mathcal{U}$. Then $\{G_\xi : \xi < \omega_1\} \subset \mathcal{U}$ is centered, since, as is known, finite meets in $\mathcal{B}$ are ordinary intersections. Since X is compact, this implies

$$\cap\{\bar{G}_\xi : \xi < \omega_1\} \neq \emptyset.$$

Let $p \in \cap\{\bar{G}_\xi : \xi < \omega_1\}$. Then, by definition, $p \notin S_\xi$ for each $\xi < \omega_1$, hence $p \in X \setminus \cup\{S_\xi : \xi < \omega_1\}$. This proves (M) $\rightarrow$ (R).

(R) $\rightarrow$ (M) Let $\mathcal{B}$ be an arbitrary complete Boolean algebra with c.c.c. We denote by X the Stone space of $\mathcal{B}$, which we identify with the set of all clopen subsets of X. Obviously, X must have the Suslin property.

Let $\mathcal{A}_\xi$ $(\xi < \omega_1)$ be arbitrary subsets of $\mathcal{B}$ and $G_\xi = \sup \mathcal{A}_\xi$. Then

$$S_\xi = G_\xi \setminus \cup \mathcal{A}_\xi$$

is nowhere dense (and closed) in X, hence using (R) we obtain the existence of a $p \in X$ such that $p \notin S$ for all $\xi < \omega_1$. Let $\mathcal{U}$ be defined as follows

$$G \in \mathcal{U} \leftrightarrow G \in \mathcal{B} \text{ and } p \in G.$$

Obviously, $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Moreover, if $G_\xi \in \mathcal{U}$, then $p \in G_\xi \setminus S_\xi = \cup \mathcal{A}_\xi$, hence there is an $A_\xi \in \mathcal{A}_\xi$ with $p \in A_\xi$, and thus $A_\xi \in \mathcal{U}$. This completes the proof.

<u>Remark</u>

(R) implies that every open subset H of a compact T_2 space with the Suslin property also has the ω_1-Baire property.

Indeed we can apply (R) to $\overline{H}$ and remark that $\overline{H} \setminus H$ is nowhere dense in $\overline{H}$.

<u>Corollary</u>

(M) $\to \exp \omega > \omega_1$.

Indeed, the closed interval [0,1] is the union of exp ω singletons, which are all nowhere dense.

5.3. Consider the following assertion

(K) If X is an arbitrary topological space which the Suslin property, and $\mathcal{G} \subset \sigma(X)$, $|\mathcal{G}| = \omega_1$, then there is a $\mathcal{G}' \subset \mathcal{G}$ with $|\mathcal{G}'| = \omega_1$ such that $\mathcal{G}'$ is centered.

Claim: (M) $\to$ (K) (cf. [23])

<u>Proof</u>

Suppose $\mathcal{G} = \{G_\xi : \xi < \omega_1\}$, where every G_ξ is a regular open subset of X. This does not result in any loss of generality, because, as can easily be shown, for arbitrary open sets $G^{(1)}, \dots, G^{(n)}$,

$$G^{(1)} \cap \dots \cap G^{(n)} = \emptyset \to \operatorname{Int} \overline{G^{(1)}} \cap \dots \cap \operatorname{Int} \overline{G^{(n)}} = \emptyset,$$

and therefore the G_ξ could be replaced by Int $\overline{G_\xi}$.

As is known, the set $\mathcal{B}$ of all regular open subsets of X constitutes a complete Boolean algebra under suitably defined operations.

Obviously, $\mathscr{B}$ satisfies the c.c.c., because X has the Suslin property. Let us now put $\mathscr{A}_\xi = \{G_\eta : \eta \geq \xi\}$ and $H_\xi = \sup \mathscr{A}_\xi$.
Since $\mathscr{B}$ satisfies the c.c.c. there is an $\eta_0 < \omega_1$ and an $H \in \mathscr{B}$ such that $\eta_0 < \xi < \omega_1 \rightarrow H_\xi = H$.
We can apply (M) to the families $\mathscr{A}_\xi$ $(\eta_0 < \xi < \omega_1)$ and the Boolean algebra $\mathscr{B}_H$ obtained by "restricting" every member of $\mathscr{B}$ to H. Thus there is an ultrafilter $\mathfrak{U}$ on $\mathscr{B}_H$ such that

$$\mathfrak{U} \cap \mathscr{A}_\xi \neq \emptyset \text{ if } \eta_0 < \xi < \omega_1.$$

In other words, there are cofinally many members of $\mathscr{G}$ in $\mathfrak{U}$, and this obviously implies

$$|\mathscr{G} \cap \mathfrak{U}| = \omega_1.$$

Hence we can choose $\mathscr{G}' = \mathscr{G} \cap \mathfrak{U}$, because the finite meets in $\mathscr{B}$ (or $\mathscr{B}_H$) are ordinary intersections, and thus $\mathscr{G}'$ is centered.

5.4. If (K) holds and X is an arbitrary cocompact space with the Suslin property, then ω_1 is a caliber for X.

<u>Proof</u>
Let $\mathscr{G} \subset \sigma(X)$, $|\mathscr{G}| = \omega_1$ and $\mathscr{L}$ be an open base for X such that $\mathscr{F} \subset \mathscr{L}$ and $\mathscr{F}$ centered imply $\cap \{\bar{F} : F \in \mathscr{F}\} \neq \emptyset$. For each $G \in \mathscr{G}$ we can choose a $B_G \in \mathscr{L}$ such that $\bar{B}_G \subset G$. Then $\{B_G : G \in \mathscr{G}\}$ has a centered subfamily $\{B_G : G \in \mathscr{G}'\}$, where $|\mathscr{G}'| = \omega_1$. Then however
$\emptyset \neq \cap \{\bar{B}_G : G \in \mathscr{G}'\} \subset \cap \{G : G \in \mathscr{G}'\}$, hence ω_1 is a caliber for X.

<u>Corollary</u>
Assume (K) and suppose $\mathscr{G} = \{G_\xi : \xi < \omega_1\}$ is a decreasing family of open subsets of a cocompact space X with the Suslin property. Then $\cap \{G_\xi : \xi < \omega_1\} \neq \emptyset$.
Indeed, iff $\mathscr{G}' \subset \mathscr{G}$, $|\mathscr{G}'| = \omega_1$, then $\cap \mathscr{G}' = \cap \mathscr{G}$ but, by 5.4, there is a $\mathscr{G}' \subset \mathscr{G}$ with $|\mathscr{G}'| = \omega_1$ such that $\cap \mathscr{G}' \neq \emptyset$.

5.5. Suppose (K). Then every product of spaces with the Suslin property also has the Suslin property.

Proof

According to our Remark made after the proof of 4.6, it suffices to show that any finite product of spaces with the Suslin property has the Suslin property, and this can be reduced trivially to the case of two factors. So assume $X = X_1 \times X_2$, where X_1 and X_2 have the Suslin property, let $\mathcal{U}$ be a set of elementary open subsets of X, and $|\mathcal{U}| = \omega_1$. Using (K), we can choose a subfamily $\mathcal{U}' \subset \mathcal{U}$ with $|\mathcal{U}'| = \omega_1$ such that $\pi_1(\mathcal{U}')$ is centered, and then applying (K) again, we have a $\mathcal{U}'' \subset \mathcal{U}'$ with $|\mathcal{U}''| = \omega_1$ for which $\pi_2(\mathcal{U}'')$ is centered. Now it is obvious that any two members of $\mathcal{U}''$ intersect, hence $\mathcal{U}$ cannot be disjoint. This completes the proof.

5.6. If X is a first countable cocompact space with the Suslin property and every closed subspace of X is cocompact, then X is separable, provided (K) holds.

Proof

First we show that $d(X) = \omega_1$ is impossible. Indeed, suppose $d(X) = \omega_1$, and let $S = \{p_\xi : \xi < \omega_1\}$ be a dense subset of X. We put

$$F_\xi = \overline{\{p_\eta : \eta < \xi\}} \text{ and } G_\xi = X \setminus F_\xi \quad (\xi < \omega_1).$$

Then $\{G_\xi : \xi < \omega_1\}$ is obviously a decreasing family and each G_ξ is non-empty, because $F_\xi \neq X$ since X is not separable. Hence, $H = \bigcap \{G_\xi : \xi < \omega_1\} \neq \emptyset$, by the Corollary of 5.4. On the other hand $H \cap S = \emptyset$, hence Int $H = \emptyset$. Let $p \in H$ be arbitrary, and $\{V_n : n < \omega\}$ a neighbourhood basis for p. Since $p \in H_\xi$ for each $\xi < \omega_1$, we can pick an $n(\xi) < \omega$ such that $V_{n(\xi)} \subset G_\xi$. Thus there is an $n_0 < \omega$ and an $a \subset \omega_1$, $|a| = \omega_1$ such that

$$n(\xi) = n_0 \text{ for all } \xi \in a.$$

Then, however, $V_{n_0} \subset \bigcap \{G_\xi : \xi \in a\} = \bigcap \{G_\xi : \xi < \omega_1\} = H$, in contradiction to Int $H = \emptyset$. Consequently, $d(X) = \omega_1$ is indeed impossible. Suppose now $d(X) > \omega_1$ is arbitrary. By 2.25, there is an $S \subset X$ with $|S| = d(S) = \omega_1$ and $c(S) \leq c(X) = \omega$. Then $\bar{S}$ is a cocompact space, which is first countable and has the Suslin property, because S has it. Moreover, $d(\bar{S}) \leq d(S) = \omega_1$, but $d(\bar{S}) = \omega$ cannot hold, because, by 2.26, this would imply

$$d(S) \leq d(\bar{S}).\partial(\bar{S}) = \omega,$$

which is in contradiction to $d(S) = \omega_1$.
Consequently, $\bar{S}$ is a first countable cocompact space with the Suslin property and $d(\bar{S}) = \omega_1$, however this is impossible by the first part of our proof. Thus 5.6 is proved.

Corollary (cf. [23])
If (K), then every perfectly normal compact T_2 spa e X is hereditarily separable.

Proof
By the Corollary of 2.26, it suffices to show that every closed subspace of X is separable. However, it is well-known that X is hereditarily Lindelöf, and therefore also hereditarily "Suslin", and thus 5.6 can be applied to every closed subspace of X.

Remark
As is shown by Example 6.10, first countability is insufficient to imply the hereditary separability of a compact T_2-space X with the Suslin property, although if (M) holds, it implies the separability of X by 5.6.

6. Examples

6.1. Let us denote by $\mathcal{F}(\alpha)$ the set of all non-principal ultrafilters on α and define

$$X = \alpha \cup \mathcal{F}(\alpha);$$

we provide X with a T_2 topology as follows:
every member of α is isolated in X, while if $u \in \mathcal{F}(\alpha)$, then a basis of neighbourhoods for u is given by the sets of the form

$$\{u\} \cup A, \text{ where } A \in u.$$

Then, as is easily seen, $X \in \mathfrak{T}_2$, $|X| = w(X) = s(X) = \exp \exp \alpha$, since $\mathcal{F}(\alpha)$ is discrete in X, however, $d(X) = \alpha$.

6.2. Let R be the real line with the topology generated by the sets of form $G \setminus A$, where G is open in the usual topology and $|A| \leq \omega$. Then $R \in \mathfrak{T}_2$ since this topology is finer than the usual and every countable subset of R is closed. Therefore $d(R) \geq \omega_1$.
We show that $d(R) = \omega_1$. Indeed, let us denote by Q the set of all

intervals with rational endpoints. For each $I \in Q$ we choose a subset $B_I \subset I$ such that $|B_I| = \omega_1$. Then obviously

$$S = \cup \{B_I : I \in Q\}$$

is a dense subset of R and $|S| = \omega_1$. By 2.6 (ii), this immediately implies

$$z(R) \geq \omega_1.$$

It is easy to verify, on the other hand, that R is hereditary Lindelöf, hence (cf. 2.6 (i))

$$h(R) = s(R) = \omega.$$

It is also obvious that $\psi(R) = \omega$, but $\partial(R) = \omega_1$.

6.3. Let R be an arbitrary space and $\prec$ a well-ordering of R. We define two spaces R^l and R^u on the same underling set R as follows: A basis for R^l (R^u) consists of all sets of the form $G_x^l(G_x^u)$, where G is open in R, $x \in G$ and $G_x^l = \{y \in G: y \preceq x\}$ ($G_x^u = \{y \in G: x \preceq y\}$). Since $z \in G_x^l \cap G_y^l$ ($z \in G_x^u \cap G_y^u$) implies $(G \cap H)_z^l \subset G_x^l \cap G_y^l$ ($(G \cap H)_z^u \subset G_x^u \cap G_y^u$), both are indeed bases of some spaces whose topologies are obviously finer than that of R, hence in particular T_2 if R is so.

<u>Proposition</u>

(i) $h(R^l) = |R|$ and $z(R^l) = z(R)$

(ii) $z(R^u) = |R|$ and $h(R^u) = h(R)$.

<u>Proof</u>

(i) $h(R^l) = |R|$ is trivial as $\prec$ right separates R^l. To show $z(R^l) = z(R)$, let $S \subset R^l$ be left separated by a well-ordering $\lhd$, say. By theorem A4.7, just like in the proof of theorem 2.7, there is a subset $T \subset S$ with $|T| = |S|$ such that the two well-orderings $\prec$ and $\lhd$ coincide on T.

But then T is obviously also left separated in the original space R, hence $|T| = |S| \leq z(R)$, which was to be shown.
The proof of (ii) is completely analogous.
Thus as we can have T_2-spaces R with $|R| = \exp h(R)$, we then have $z(R^u) = \exp(h(R^u))$, and as we can have ones with $|R| = \exp \exp z(X)$, then we have $h(r^1) = \exp \exp z(R^1)$.

6.4. Let $[0,1)$ denote the half open interval of reals and let

$$X = \omega_1 \times [0,1)$$

with the topology induced by its lexicographic ordering. Then $X \in \mathcal{L}$, and obviously X is connected. X is sometimes called the "long-line". It is easy to see that $\chi(X) = \omega$, but $k(X) = \omega_1$, which shows that 2.18 cannot in general be improved.

6.5. (cf. [5])
Let $\lambda > \omega$ be an arbitrary inaccessible cardinal. For each $\alpha < \lambda$ we define Ω_α as the one-point compactification of $D(\alpha)$, and put

$$\Omega = X\{\Omega_\alpha : \alpha < \lambda\}.$$

Then $\Omega \in \mathcal{B}$ and, obviously, $c(\Omega) = \lambda$. However, Ω does not contain λ pairwise disjoint open sets, since λ is a caliber for every $\Omega_\alpha (\alpha < \lambda)$ and λ is regular, hence, by 4.8, λ is also a caliber for Ω.
This shows that in 3.1 the condition of λ's singularity cannot be dropped.

6.6. R. Jensen [19] has shown that if Gödel's axiom of constructability holds ($V = L$), then for every non-weakly compact inaccessible cardinal λ there is an $X \in \mathcal{L}$ such that $|X| = \lambda$ but X does not contain λ pairwise disjoint intervals. Thus, since $V = L \rightarrow GCH$, $s(X) = \lambda$ by 2.9, because λ is strong limit. However, X cannot contain a discrete subspace of power λ, as follows from 2.8 (ii) and the choice of X. This justifies our remark made after the proof of 3.2.

6.7. Let $F = \{0,1\}$ with the T_0 topology in which 0 is isolated but 1 is not.

Looking at the elementary open sets in F^α, it is obvious that $w(F^\alpha) \leq \alpha$. On the other hand, if we define $p_\xi \in F^\alpha$ for $\xi < \alpha$ by

$$\pi_\eta(p_\xi) = \begin{cases} 0, & \text{if } \eta = \xi \\ 1 & \text{otherwise,} \end{cases}$$

then $\{p_\xi : \xi < \alpha\}$ is a discrete subspace of F^α. Consequently

$$w(F^\alpha) = h(F^\alpha) = z(F^\alpha) = s(F^\alpha) = \alpha.$$

It is easy to see that for the point $q^0 \in F^\alpha$ with

$$\pi_\eta(q^0) = 0 \text{ for all } \eta < \alpha,$$

$\chi(q^0, F^\alpha) = \alpha$, hence $\chi(F^\alpha) = \alpha$, too.

6.8. If $D(2)$ is the two-element discrete space, then similarly as in 6.7 we can show

$$s(D(2)^\alpha) = z(D(2)^\alpha) = h(D(2)^\alpha) = \chi(D(2)^\alpha) = \psi(D(2)^\alpha) = w(D(2)^\alpha) = \alpha.$$

From this and 4.2 (ii) we obtain

$$\pi(D(2)^\alpha) = \alpha.$$

Finally, since $D(2)^\alpha$ is regular, by 2.3 (i) we have $d(D(2)^\alpha) \geq \log \alpha$. This, together with 4.5 (i) yields us

$$d(D(2)^\alpha) = \log \alpha.$$

6.9. Let R be the set of real numbers with the "Sorgenfrey" topology, i.e. the one determined by all half open intervals $[x,y)$ as a base for the open sets. It is well-known, and easy to show that

$$d(R) = h(R) = z(R) = s(R) = \chi(R) = \psi(R) = L(R) = \omega,$$

but

$$w(R) = \exp \omega.$$

Also, $R \times R$ contains a closed discrete subset of power $\exp \omega$, hence

$$s(R \times R) = L(R \times R) = \exp \omega.$$

This shows that 4.4 cannot be improved.

6.10. Let $I^* = I \times \{0,1\}$, where $I = [0,1]$ and I^* is provided with the lexicographic ordering and the order topology determined by it. In other words, every point of I is "split" into two. This space is known as Urysohn's space. Obviously $I^* \in \mathcal{B}$ and

$$d(I^*) = \chi(I^*) = h(I^*) = z(I^*) = s(I^*) = \omega.$$

Let $J = (0,1)$ and $J' = J \times \{1\} \subset I^*$ be considered as a subspace of I^*. It is easy to see that J' is homeomorphic to the space R of 6.9. Therefore

$$s(I^* \times I^*) = \exp \omega,$$

though $I^* \times I^* \in \mathcal{B}$ and

$$\chi(I^* \times I^*) = d(I^* \times I^*) = \omega.$$

This justifies our remark at the end of 5.6.

6.11 (cf. 2.3). If X is a completely regular space, then X can be embedded as a closed subset of a completely regular space Y such that $Y \setminus X$ is discrete and
$$d(Y) = \log w(X).$$

Proof. Embed X in the Tychonoff cube $[0,1]^{w(X)}$. Choose $p \in [0,1]^{w(X)}$, and let $A^* \subset [0,1]^{w(X)} \times [0,1]^{w(X)}$ be a dense subset of power $\log w(X)$ (cf. 4.5). Also $A = A^* \setminus [0,1]^{w(X)} \times \{p\}$ is dense in $[0,1]^{w(X)} \times [0,1]^{w(X)}$. Now
$$Y = X \times \{p\} \cup A$$
satisfies our requirements, if we refine the subspace topology of Y by making all $\{a\}$, $a \in A$ open.

APPENDIX :

COMBINATORIAL SET THEORY

BY

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A0. Notation , conventions and prerequisites

A0.0 Sets, ordinals and cardinals

$A, B, C, \dots A'_{\alpha}, A', \dots$	stand for	ordinary sets in naive set theory, or e.g. the Zermelo-Fraenkel set theory with the axiom of choice, but without CH or GCH.
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots \mathcal{A}^*, \mathcal{A}_{\alpha} \dots$		families of sets
$\mathfrak{P}(A)$		the power set of A
$\emptyset$		the empty set
$[A]^r$		the family of r-element subsets of A
Or		the class of all ordinals
$\zeta, \eta, \theta, \mu, \xi, \dots, \zeta', \zeta_{\eta}, \dots$		ordinals ("variables")

Each ordinal is the set of its predecessors

Some consequences are

$\xi = \{\eta \mid \eta < \xi\}$

$\eta < \xi \Longleftrightarrow \eta \in \xi$,

min Or $= \emptyset =_{\text{notation}} 0$

successor of $\eta = \eta \cup \{\eta\}$

$\xi \setminus \eta = [\eta, \xi) = \{\zeta \mid \eta \leq \zeta < \xi\}$

(For $A \subset \mathrm{Or}$) sup A	$\bigcup A$ (is an ordinal!)
accordingly sup $\emptyset$	$\emptyset = \min \mathrm{Or}$
$\zeta \dot{+} \eta$	the ordinal which is the (ordinal) sum of ζ and η
$\zeta + 1$, or more precisely $\zeta \dot{+} 1$	successor of $\zeta = \zeta \cup \{\zeta\}$
Card	the class of all cardinals = initial ordinals
$\|A\|$, $\|\zeta\|$	The cardinality of A, resp. ζ.
n,i,k,l,r	finite cardinals (= members of ω)
$\alpha,\beta,\gamma,\delta,\ldots,\alpha', \alpha_\xi,\ldots,\lambda,\ldots$	infinite cardinals, or, if explicitly stated, arbitrary (finite of infinite) cardinals.

The increasing sequence of infinite cardinals is denoted by ω_ζ, $\zeta \in \mathrm{Or}$: $\omega_0 = \omega, \omega_1, \omega_2, \omega_3, \ldots, \omega_\zeta, \ldots$
The finite cardinals = the finite ordinals are the natural numbers: $0 = \emptyset$, $1 = \{\emptyset\}$, $2, 3, \ldots$

$\alpha+\beta$, $\alpha.\beta$, 2^α, $\sum_{\eta<\zeta} \alpha_\eta$, $\prod_{\eta<\zeta} \alpha_\eta$ are cardinals defined as usual. (Note that $\alpha+\beta = |\alpha\dot{+}\beta|$, and $\alpha\dot{+}\beta = \alpha+\beta \iff \alpha<\beta$. If α is an initial ordinal, then $\alpha+1$ may either mean: the cardinal sum, i.e. $\alpha+1 = \alpha$, or, more frequently, the ordinal successor of α : $\alpha+1 = \alpha\dot{+}1 = \alpha \cup \{\alpha\}$. It should be clear from the context in which sense + is meant).

(For $A \subset \mathrm{Card} \subset \mathrm{Or}$) sup A	$\bigcup A$ (as before; note that $A \subset \mathrm{Card} \Rightarrow \bigcup A \in \mathrm{Card}$)
α^+	the cardinal successor of $\alpha = \omega_\xi$, $\alpha^+ = \omega_{\xi\dot{+}1}$
$\alpha = \omega_\xi$ is a limit cardinal	ξ is a limit ordinal
$\alpha = \omega_\xi$ is a successor cardinal	ξ is a successor ordinal (or equivalently $\exists \beta\in\mathrm{Card}, \alpha = \beta^+$)
CH = continuum hypothesis	$\omega_1 = 2^{\omega_0}$
GCH = general CH	$\forall \alpha \quad 2^\alpha = \alpha^+$
$\exp\alpha = \exp^1\alpha$	$2^\alpha = \|\mathcal{P}(\alpha)\|$
$\exp^n\alpha$	$\exp(\exp^{(n-1)}\alpha)$

$\log\alpha$	$\min\{\beta \mid 2^\beta \geq \alpha\}$
${}^\gamma\log\alpha$	$\min\{\beta \mid \gamma^\beta \geq \alpha\}$
$\sqrt[\gamma]{\alpha}$ (used in A2.3 only)	$\min\{\beta \mid \beta^\gamma \geq \alpha\}$
$\psi(\alpha)$ (used in A4.3 only)	$\min\{\beta \mid \alpha^\beta > \alpha\}$
Some consequences are	$\exp(\log\alpha) \geq \alpha$
	$\log(\exp\alpha) \leq \alpha$
	${}^\gamma\log(\alpha^\delta) \leq \delta.{}^\gamma\log\alpha$
	$(\sqrt[\gamma]{\alpha})^\gamma = \alpha^\gamma$
	$(\sqrt[\gamma]{\alpha^\gamma}) = \sqrt[\gamma]{\alpha}$
	$GCH \Longrightarrow \psi = cf$ (see A0.1.11)

Simple rules for cardinal arithmetic, such as $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$, $|\wp(\alpha)| = 2^\alpha$, $\alpha < \beta \Rightarrow \forall\gamma \quad \gamma^\alpha \leq \gamma^\beta$ etc. are assumed to be well known. The fundamental theorem of cardinal arithmetic is $\forall\alpha \quad \alpha.\alpha = \alpha$, which has such well known consequences as $\alpha+\beta = \alpha.\beta = \max\{\alpha,\beta\}$, $\alpha^\alpha = (2^\alpha)^\alpha = 2^\alpha$, $\alpha^\beta = (\alpha+\beta)^\beta$. Familiarity with the principles of transfinite induction is presupposed.

Proof of $\alpha.\alpha = \alpha$. Define a wellordering on Or x Or as follows: First let $A_\zeta = \{(\eta,\xi) \mid \max\{\eta,\xi\} = \zeta\}$ for all $\zeta \in Or$, and wellorder each A_ζ as follows: $(\zeta,0) < (\zeta,1) < \ldots (\zeta,\xi) < \ldots < (\zeta,\zeta) < (0,\zeta) < < (1,\zeta) < \ldots (\xi,\zeta) < \ldots$ for all $\xi<\zeta$ (ordertype $A_\zeta = \zeta + 1 + \zeta$). Then, for $(\eta,\xi) \in A_\zeta$, $(\eta',\xi') \in A_{\zeta'}$, $\zeta \neq \zeta'$ we put $(\eta,\xi) < (\eta',\xi')$ iff $\zeta < \zeta'$.

This gives a wellordering and hence a function $\phi : Or \times Or \to Or$. Clearly for all $\alpha \quad |\phi(\alpha,0)| = |\alpha|.|\alpha|$. Suppose that for some $\alpha \quad \phi(\alpha,0) > \alpha$. Then for some $(\zeta,\eta) \quad \phi(\zeta,\eta) = \alpha$ and $\max\{\zeta,\eta\} =_{def} \theta < \alpha$. Clearly $\phi(\theta+1,0) > \alpha$ and hence $|\theta+1|^2 = |\theta|^2 \geq \alpha > \theta \geq |\theta|$. Now let $\alpha_1 = \theta$. Repeating this procedure we find a decreasing sequence $\alpha > \alpha_1 > \alpha_2 > \ldots$ of cardinals, contradicting the wellordering of Card.

A0.1 Cofinality.

Let $(X,<)$ be a fixed linearly ordered set. Then a set A is called <u>cofinal</u> in X if $A \subset X$ and $\forall x \in X \ \exists a \in A \quad x \leq a$.

The <u>cofinality of X, cf X</u>, is defined by
$\mathrm{cf}\, X = \min\{|A| \mid A \subset X \text{ is cofinal in } X\}$

Examples. If X has a largest element, x_1, then $\{x_1\}$ is cofinal in X. So (X has a largest element) $\Longleftrightarrow \mathrm{cf}\, X = 1 \Longleftrightarrow \mathrm{cf}\, X < \omega$.
Furthermore: $\mathrm{cf}\, \mathbb{R} = \omega$, $\mathrm{cf}\, \omega_\omega = \omega$ (Since $\lim_{n\in\omega} \omega_n = \bigcup_{n\in\omega} \omega_n = \omega_\omega$), and $\mathrm{cf}\, \omega_1 = \omega_1$.

Some properties of cf are:

A0.1.1. If $Z \subset Y \subset X$ and Y is cofinal in X and Z in Y, then Z is cofinal in X.

A0.1.2. <u>Each linearly ordered set $(X,<)$ has a cofinal well-ordered subset A.</u>

Proof. Define $\phi : \mathrm{Or} \to X$ by transfinite induction so that
$$\phi(\eta)\begin{cases} > \phi(\zeta) \text{ for all } \zeta < \eta \text{ if } \{\phi(\zeta)|\zeta<\eta\} \text{ is not cofinal in } X, \\ = \phi(0) \text{ otherwise.}\end{cases}$$
Then either $A = \{\phi(0)\}$ or $A = \{\phi(\eta)|\phi(\eta) \neq \phi(0)\}$ is as required.

A0.1.3. Every ordinal $\zeta = \{\eta|\eta<\zeta\}$ has a cofinal subset of order type $\leq |\zeta|$.

Proof. Let $f : |\zeta| \to \zeta$ be any bijection. Define ϕ as above, taking care also that $\phi(\eta) \geq f(\eta)$ be satisfied, if $\{\phi(\theta)|\theta<\eta\}$ is not cofinal with ζ.

A0.1.4. For each linearly ordered set $(X,<)$

<u>$\mathrm{cf}\, X = \min\{\zeta | X$ has a cofinal subset of order type $\{\zeta\} \in \mathrm{Card}$</u>,

Proof. $\leq$ is trivially satisfied. Let $A \subset X$ be cofinal, of minimal cardinality. By 2 we can find $A' \subset A$ such that A' is cofinal in A (and in X by 1) and A' is well-ordered. By 3 we can find $A'' \subset A'$ such that A'' is cofinal in A (and in X by 1, hence order type $A'' \geq \mathrm{cf}(X)$), and order type $A'' \leq |A| = \mathrm{cf}\, X$. Thus order type $A'' = \mathrm{cf}\, X$. By the definition of cf X, $\mathrm{cf}\, X \in \mathrm{Card}$.

A0.1.5. For each linearly ordered set $(X,<)$

$$\boxed{\mathrm{cfcf}\ X = \mathrm{cf}\ X}$$

Proof is easy from 1 and 4.

A0.1.6. For each limit $\xi \in \mathrm{Or}$

$$\boxed{\mathrm{cf}(\omega_\xi) = \mathrm{cf}\ \xi}$$

Proof. $\mathrm{Lim}(\omega_\eta)_{\eta<\xi} = \bigcup\{\omega_\eta \mid \eta<\xi\} = \omega_\xi$.

A0.1.7. For each successor $\xi \in \mathrm{Or}$ $\mathrm{cf}\ \omega_\xi = \omega_\xi$, i.e. $\forall\alpha$ $\boxed{\mathrm{cf}(\alpha^+) = \alpha^+}$

Proof. Suppose $A \subset \alpha^+$ satisfies $|A| \leq \alpha$. Then for each $\xi \in A$ $\xi < \alpha^+$, and hence $|\xi| \leq \alpha$ and so also $|\sup A| = |\cup A| \leq \alpha.\alpha = \alpha$. Thus $\sup A < \alpha^+$, and A is not cofinal in α^+.

A cardinal α is called		
	<u>regular</u>	if $\alpha = \mathrm{cf}\ \alpha$
	<u>singular</u>	if $\alpha > \mathrm{cf}\ \alpha$
	<u>strong limit</u>	if $\forall\beta < \alpha \quad 2^\beta < \alpha$
	<u>weakly inaccessible</u>	if α is regular limit
	<u>strongly inaccessible</u>	if α is regular strong limit.

Notice that it follows from 5 that each cofinality (cf X, cf α) is a regular cardinal. For 7 we may now read: each successor cardinal is regular, or equivalently: each singular cardinal is a limit. As to the existence of regular limit (= weakly inaccessible) cardinals, see A6.1.

A0.1.8. For two cardinals α,β with $\beta < \alpha$ the following conditions are equivalent:

a) $\mathrm{cf}\ \alpha = \beta < \alpha$

b) β is the minimal cardinal that there exists a sequence of ordinals $(\zeta_\eta)_{\eta<\beta}$ with $\forall\eta < \beta \quad \zeta_\eta < \alpha$ and $\sup\{\zeta_\eta \mid \eta<\beta\} = \alpha$

c) β is the minimal cardinal such that there exists a sequence of cardinals $(\alpha_\eta)_{\eta<\beta}$ with

$$\forall \eta < \eta' < \beta \qquad \alpha_\eta < \alpha_{\eta'} < \alpha$$

each α_η is regular (e.g. a successor cardinal)

$$\lim_{\eta<\beta} \alpha_\eta = \sup_{\eta<\beta} \alpha_\eta = \sum_{\eta<\beta} \alpha_\eta = \alpha$$

d) β is the minimal cardinal such that $\alpha = \sum_{\eta<\beta} \alpha_\eta$ for some set of cardinals $\{\alpha_\eta | \eta<\beta\}$ with $\alpha_\eta < \alpha$ for all $\eta < \beta$.

Proof. (a)$\Leftrightarrow$(b) by definition. (c)$\Rightarrow$(d) and (d)$\Rightarrow$(b) hold trivially. As to (b)$\Rightarrow$(c), define α_η by transfinite induction on η e.g. by

$$\alpha_\eta = \qquad |\bigcup\{\alpha_{\eta'} | \eta'<\eta\} \cup \bigcup\{\zeta_{\eta'} | \eta'<\eta\}|^+ .$$

(Check that $\alpha_\eta < \alpha$ for each $\eta < \beta$).

A0.1.9. For every cardinal α the following conditions are equivalent:

(a) α is regular.

(b) each cofinal subset of α has order type α.

(c) $\alpha = \sum_{\eta<\beta} \alpha_\eta$ for some set $\{\alpha_\eta | \eta<\beta\} \subset$ Card implies:

$\beta \geq \alpha$ or $\exists \eta < \beta \quad \alpha_\eta = \alpha$.

A0.1.10. For every cardinal α $\quad \boxed{\alpha^{\mathrm{cf}\,\alpha} > \alpha}$

Proof. Suppose $f\colon \alpha \to \alpha^{\mathrm{cf}\,\alpha}$ is any mapping, $\mathrm{cf}\,\alpha = \beta$ and $(\alpha_\eta)_{\eta<\beta}$ is a strictly increasing sequence converging to α. We will define a $g \in \alpha^{\mathrm{cf}\alpha}$ (i.e. $g\colon \mathrm{cf}\,\alpha \to \alpha$) in such a way that $g \neq f(\eta)$ for all $\eta < \beta$, showing that the mapping f cannot be onto. Note that both g and $f(\eta)$ are functions $\beta \to \alpha$. Moreover for each $\zeta \in \beta$ $\{(f(\eta))(\zeta) \mid \eta<\alpha_\zeta\}$ has cardinality $\leq \alpha_\zeta < \alpha$. Thus we may define $g(\zeta) \in \alpha \setminus \{(f(\eta))(\zeta) | \eta<\alpha_\zeta\}$ arbitrarily for all $\zeta < \beta$. This yields $g(\zeta) \neq (f(\eta))(\zeta)$, and hence $g \neq f(\eta)$ for all $\eta < \alpha_\zeta$, and for all $\zeta < \beta$. Since $\{\alpha_\zeta | \zeta<\beta\}$ is cofinal in α, we obtain $g \neq f(\eta)$ for all $\eta < \alpha$.

10a $\boxed{cf(2^{\alpha}) > \alpha}$

b $cf(\alpha^{cf\alpha}) > \alpha$

Proof of (a). If $cf\ 2^{\alpha} \leq \alpha$, then $(2^{\alpha})^{cf 2^{\alpha}} \leq (2^{\alpha})^{\alpha} = 2^{\alpha}$ contradicting 9. Similarly for b.

A0.1.11. Under G C H
$$\boxed{\alpha^{\beta} = \begin{cases} \alpha & \text{if} \quad \beta < cf\ \alpha \\ 2^{\alpha} = \alpha^{+} & \text{if} \quad cf\ \alpha \leq \beta \leq \alpha \\ 2^{\beta} = \beta^{+} & \text{if} \quad \alpha \leq \beta \end{cases}}$$

Proof. Assuming G C H, $\gamma < \alpha \quad 2^{\gamma} = \gamma^{+} \leq \alpha$. Now if $\beta < cf\ \alpha$ then for each $f: \beta \to \alpha \quad \sup\{f(\eta) | \eta < \beta\} < \alpha$. Hence

$$\alpha^{\beta} = |\{f | f: \beta \to \alpha\}| \leq \sum_{\xi<\alpha} |\{f | f: \beta \to \xi\}| \leq \sum_{\xi<\alpha} |\xi|^{\beta} \leq$$
$$\leq \sum_{\xi<\alpha} 2^{|\xi| \cdot \beta} = \sum_{\xi<\alpha} 2^{|\xi|} . 2^{\beta} \leq \alpha . \alpha . 2^{\beta} = \alpha$$

If $cf\ \alpha \leq \beta \leq \alpha$ then by 9: $2^{\alpha} = \alpha^{+} \leq \alpha^{cf\alpha} \leq \alpha^{\alpha} \leq (2^{\alpha})^{\alpha} = 2^{\alpha}$.
If $\alpha \leq \beta$ then $2^{\beta} \leq \alpha^{\beta} \leq (2^{\beta})^{\beta} \leq 2^{\beta}$.

A0.1.12. <u>On G C H</u>.
Let $\mathcal{R}$ be the class of regular cardinals and $\phi : \mathcal{R} \to$ Card be any "well-defined" function that satisfies $(\forall \alpha, \beta \in \mathcal{R})$: $\alpha < \beta \Rightarrow \phi(\alpha) \leq \phi(\beta)$ and $cf(\phi(\alpha)) > \alpha$.
(We may e.g. define ϕ by $\phi(\alpha) = \alpha^{+}$, or $\phi(\alpha) = \alpha^{+++}$, or $\phi(\omega_0) = \phi(\omega_1) = \omega_2$ and $\phi(\alpha) = \alpha^{++}$ otherwise).
W. Easton, [37], has shown that there is a model of ZF + choice in which $\phi(\alpha) = 2^{\alpha}$ for each $\alpha \in \mathcal{R}$, provided there exists a model of ZF + choice. For $\phi(\alpha) = \alpha^{+}$ this yields e.g. the consistence of G C H with ZF . Note also that, in some models, $2^{\alpha} = 2^{\beta}$ may hold for some cardinals $\alpha, \beta, \alpha \neq \beta$.

A0.2 Sometimes an <u>ordinal</u> ρ is considered <u>as topological space</u>, by taking the order topology, for which $\{(\eta, \xi] | \eta < \xi < \rho\} \cup \{[0, \eta] | \eta < \rho\}$ is a base. A class $A \subset Or$ is called <u>closed</u> if $\forall \rho \in Or \quad A \cap \rho$ is closed in ρ.

A1 Regressive functions

A1.1 Let M be a set of ordinals. A function $f\colon M \to Or$ is called <u>regressive</u> if

$$\forall \xi \in M\setminus\{0\} \qquad \phi(\xi) < \xi$$

and $$\phi(0) = 0 \quad \text{if} \quad 0 \in M.$$

A1.2 THEOREM [ALEKSANDROV-URYSOHN [1]].

<u>If $f\colon \omega_1 \to \omega_1$ is regressive, then $\exists\, \xi_0 < \omega_1 \quad |f^{-1}(\xi_0)| = \omega_1$</u>

<u>Proof</u>. Put $f^{(0)}(x) = x$, $f^{(n+1)}(x) = f\,(f^{(n)}(x))$ and $A_n = \{\xi \in \omega_1 \mid f^{(n)}(\xi) = 0\}$. Since for each $\xi \in \omega_1$ the sequence $(f^{(n)}(\xi))_{n\in\omega}$ is non-increasing, we must have $f^{(n)}(\xi) = f^{(n+1)}(\xi)$ and hence $= 0$ for some $n \in \omega$, and thus $\xi \in A_n$.
Thus $\bigcup_{n\in\omega_1} A_n = \omega_1$, hence some A_n must have the cardinality ω_1.
Since $|f^{(0)}A_n| = \omega_1$ and $|f^{(n)}A_n| = 1$ we can find a $k < n$ such that $|f^{(k)}A_n| = \omega_1$ but $|f^{(k+1)}A_n| < \omega_1$. Now we can choose a $\xi \in f^{(k+1)}A_n$ such that $|f^{-1}(\xi)| = \omega_1$.

A1.3 It is easy to see that A1.2 can be generalized as follows:

Let $f : \alpha \to \alpha$ be regressive.

(i) If α is regular then $\exists \xi < \alpha \quad |f^{-1}(\xi)| = \alpha$.

(ii) If α is singular then $\forall \beta < \alpha \; \exists \xi < \alpha \quad |f^{-1}(\xi)| > \beta$.

Proof. The proof of (i) is an immediate generalization of the proof of A1.2. obtained by replacing everywhere ω_1 by α.

The proof of (ii) follows from (i) if we notice that β^+ is regular and $f \mid \beta^+ : \beta^+ \to \beta^+$ is regressive.

EXAMPLE. The following example shows that (ii) cannot be sharpened. Let α be singular, and $(\beta_\xi)_{\xi < cf\alpha}$ a strictly increasing sequence such that $\lim \beta_\xi = \alpha$, $\beta_0 = 0$ and $\beta_1 = cf\alpha$. Define $f: \alpha \to \alpha$ as follows:

$$f(\eta) = \begin{cases} \beta_\xi & \text{if } \beta_\xi < \eta < \beta_{\xi+1}, \quad \xi < cf\alpha \\ \xi & \text{if } \beta_\xi = \eta \qquad\qquad , \quad \xi < cf\alpha \end{cases}$$

Notice also that for no $\xi \in \alpha$ is $f^{-1}(\xi)$ cofinal in α.

A1.4 If ρ is a limit ordinal and $M \subset \rho$ an arbitrary cofinal subset of ρ, then a function $\phi : M \to \rho$ is called definitely diverging if

$$\forall \xi < \rho \; \exists \eta \in M \; \forall \eta' \in M \setminus \eta \quad \phi(\eta') > \xi .$$

This means that the function values of ϕ eventually exceed any ordinal $\xi < \rho$, what we also denote by $\lim_{\eta \in M} \phi(\eta) = \rho$.

If A and B are sets of ordinals and $\phi : A \to B$ is any function, then let $\underline{\phi} : A \to B$ be defined by

$$\underline{\phi}(\xi) = \min\{\phi(\eta) \mid \eta \in A \setminus \xi\}$$

Notice that $\underline{\phi}$ is always increasing and satisfies $\underline{\phi} \leq \phi$, moreover $\lim_{\eta \in A} \phi(\eta) = \rho \iff \sup_{\eta \in A} \underline{\phi}(\eta) = \rho$.

A1.5 <u>lemma</u> If $\mathrm{cf}\rho > \omega_0$ and A and B are two closed cofinal subsets of ρ, then $A \cap B$ is cofinal, and in particular $A \cap B \neq \emptyset$. (Cf. A0.2)

<u>Proof</u>. If $\eta_{-1} < \rho$, then define two sequences $(\zeta_n)_{n \in \omega}$ in A, and $(\eta_n)_{n \in \omega}$ in B as follows:
If $\zeta_0, \ldots, \zeta_n, \eta_n$ are defined for $n \in \omega$ or $n = -1$, then

let $$\zeta_{n+1} = \min(A \setminus \eta_n)$$

If $\eta_{-1}, \zeta_0, \eta_0, \ldots, \eta_{n-1}, \zeta_n$ are defined for $n \in \omega$, then

let $$\eta_n = \min(B \setminus \zeta_n) \quad .$$

Notice that $\eta_{-1} \leq \zeta_0 \leq \eta_0 \leq \cdots \zeta_n \leq \eta_n \leq \cdots$. Put $\eta_\omega = \bigcup_{n \in \omega} \eta_n = \bigcup_{n \in \omega} \zeta_n \geq \eta_{-1}$. Because of $\mathrm{cf}\rho > \omega$ we have $\eta_\omega < \rho$, and since A and B are closed $\eta_\omega \in A \cap B$.

A subset M of a limit ordinal ρ is <u>stationary in ρ</u> if $M \cap C \neq \emptyset$ for each closed cofinal subset C of ρ.
Note that, if $\mathrm{cf}\rho > \omega$ then by the above lemma any set containing a closed cofinal subset of ρ is stationary in ρ. However a stationary subset of ρ need not contain a closed cofinal subset of ρ (let $\rho = \omega_2$ and $M = \{\eta \in \omega_2 \mid \mathrm{cf}\eta = \omega\}$). But we have

A1.5 THEOREM <u>(W. Neumer [53]). If $\mathrm{cf}(\rho) > \omega$ and $M \subset \rho$ is cofinal with ρ, then M is stationary iff $\forall \phi : M \to \rho$ (ϕ is definitely diverging) $\Rightarrow$ (ϕ is not regressive).</u>

<u>Proof</u>. Sufficiency. Let M not be stationary. Then there is a closed cofinal subset C of ρ which is disjoint from M. Define $\phi : M \to \rho$ as follows

$$\phi(\mu) = \sup \mu \cap C \qquad \mu \in M \; .$$

Note that $\sup\emptyset = 0$, and that $\phi(\mu) \in C$ for each $\mu \in M$ since C is closed. It is easily seen that ϕ is regressive, increasing and definitely diverging.

Necessity. Assume that M is a stationary subset of ρ and $\phi: M \to \rho$ is definitely diverging and regressive.
As follows from A1.4 we may assume that ϕ is also increasing (replace ϕ by $\underline{\phi}$).
Define a sequence $(\eta_\xi)_\xi$ in ρ as follows: $\eta_0 = 0$. If η_ξ is defined and $< \rho$, then let

(i) $$\eta_{\xi+1} = \min\{\eta \in M \mid \phi(\eta) > \eta_\xi\}$$

(Notice that $\phi(\eta) > \eta_\xi$ for some $\eta < \rho$).
If ξ_0 is a limitordinal, and η_ξ is defined for all $\xi < \xi_0$ and moreover $\bigcup_{\xi<\xi_0} \eta_{\xi_0} < \rho$, then let

(ii) $$\eta_{\xi_0} = \bigcup_{\xi<\xi_0} \eta_\xi .$$

This procedure stops at a (limit) ordinal ξ for which $\bigcup_{\xi<\xi_0} \eta_\xi = \rho$. Clearly $\{\eta_\xi \mid \xi < \xi_0\}$ is a closed cofinal subset of ρ, and hence also $\{\eta_\xi \mid \xi < \xi_0$ and ξ is a limit$\}$ is closed and cofinal in ρ.
Because M is stationary, this set meets M, i.e. for some limit $\xi_1 < \xi_0$ $\eta_{\xi_1} \in M$. Since $(\eta_\xi)_{\xi\in\xi_0}$ is (strictly) increasing and ϕ is increasing we find that

$$\forall \xi < \xi_1 \qquad \phi(\eta_{\xi+1}) \leq \phi(\eta_{\xi_1}) \quad .$$

Moreover it follows from (i) that

$$\forall \xi < \xi_1 \qquad \eta_\xi < \phi(\eta_{\xi+1}) \; .$$

If we combine these two inequalities we obtain

$$\eta_{\xi_1} = \bigcup_{\xi<\xi_1} \eta_\xi \leq \phi(\eta_{\xi_1})$$

This contradicts the fact that ϕ is regressive.

Remark. If $cf\rho = \omega$, then clearly M is stationary iff M contains a tail of ρ, i.e. $\exists \xi < \rho \quad \rho \setminus \xi \subset M$.
Moreover for any cofinal $M \subset \rho$ there is a regressive definitely diverging $\phi : M \to \rho$. For if $\rho = \sup\{\rho_i | i \in \omega\}$ and $\rho_i < \rho_{i+1}$ for all $i \in \omega$, then we may define

$$\phi(\mu) = \max\{\rho_i \mid \rho_i < \mu\} \quad .$$

APPLICATIONS IN TOPOLOGY.

A1.6 THEOREM [MYCIELSKI [52]] .

$D(\alpha^+)$ can be embedded as a closed subset in $(D(\alpha))^{\alpha^+}$

Proof. Let $R = X\{D(\xi) \mid \alpha \leq \xi < \alpha^+\}$. Since for these ξ $|\xi| = \alpha$, and $D(\xi) = \{\eta \mid \eta < \xi\}$ with the discrete topology, R is homeomorphic to $R \sim (D(\alpha))^{\alpha^+}$. Note that R is the set of all regressive functions from $\alpha^+ \setminus \alpha$ to α^+. Now $\alpha^+ \setminus \alpha$ is stationary in α^+ and α^+ is regular. For each $\zeta \in \alpha^+ \setminus \alpha$ we choose one $f_\zeta \in R$ with the following properties:

(i) $\quad \forall \xi \in \alpha^+ \setminus \zeta \qquad f_\zeta(\xi) = \zeta$

(ii) $\quad f_\zeta | (\zeta \setminus \alpha) : \zeta \setminus \alpha \to \alpha$ is 1-1.

We claim that $D = \{f_\zeta \mid \alpha < \zeta < \alpha^+\}$ has no accumulation point in R. Let $g \in R$, then g is regressive and hence not definitely diverging (A1.5),i.e. $\exists \xi < \alpha^+$ such that $\{\eta \in \alpha^+ \setminus \alpha | g(\eta) < \xi\}$ is cofinal in α^+, i.e. has α^+ elements, because α^+ is regular. Then since $|\xi| < \alpha^+$, there is a $\xi' < \xi$ such that $|g^{-1}(\xi')| = \alpha^+$. (Cf. A1.3(i)). Choose two elements ξ_1, $\xi_2 \in \alpha^+ \setminus \alpha$ such that $g(\xi_1) = g(\xi_2) = \xi'$. Then $\{f \in R \mid f(\xi_1) = f(\xi_2) = \xi'\}$ is an elementary open set in R which contains at most one element, $f_{\xi'}$, of D, since $f_{\xi'}$ is the only element of D which assumes the value ξ' more than once.

A1.7 <u>THEOREM. The ordered topological space $\omega_1 = \{\xi \mid \xi < \omega_1\}$ is not paracompact.</u>

Proof. Let $\mathcal{O}$ be any open refinement of the cover consisting of all initial segments of ω_1. We will show that some $\eta \in \omega_1$ even meets uncountably many members of $\mathcal{O}$. Hence $\mathcal{O}$ cannot be locally finite (not even point-finite, or point-countable). For each $\eta \in \omega_1$, choose one element $O_\eta \in \mathcal{O}$ containing η. Define $f\colon \omega_1 \to \omega_1$ in such a way that $f(\eta) < \eta$ and $(f(\eta), \eta] \subset O_\eta$ for all $\eta \in \omega_1 \setminus \{0\}$. By A1.2 $\exists \xi < \omega_1$ such that $f^{-1}(\xi)$ is uncountable, i.e. $f^{-1}(\xi)$ is not bounded. Then $\xi+1$ is contained in $(\xi, \eta] \subset O_\eta$ for each $\eta \in f^{-1}(\xi)$. Since each O_η is bounded this means that $\xi+1$ is contained in uncountably many members of $\mathcal{O}$.

(The lemma that each paracompact (or: metacompact) countably compact space is compact, yields another proof of A1.7).

A1.8 <u>The product $\{\eta \mid \eta \leq \omega_1\} \times \{\eta \mid \eta < \omega_1\}$ is not normal.</u>

Proof. We will show that the diagonal $\Delta = \{(\eta,\eta) \mid \eta \in \omega\}$ and the right side $R = \{(\omega_1,\eta) \mid \eta \in \omega_1\}$ do not have disjoint open neighbourhoods. Suppose U is any open nbd of Δ. Define $f\colon \omega_1 \to \omega_1$ in such a way that

$$f(\eta) < \eta \qquad \eta \in \omega_1 \setminus \{0\}$$

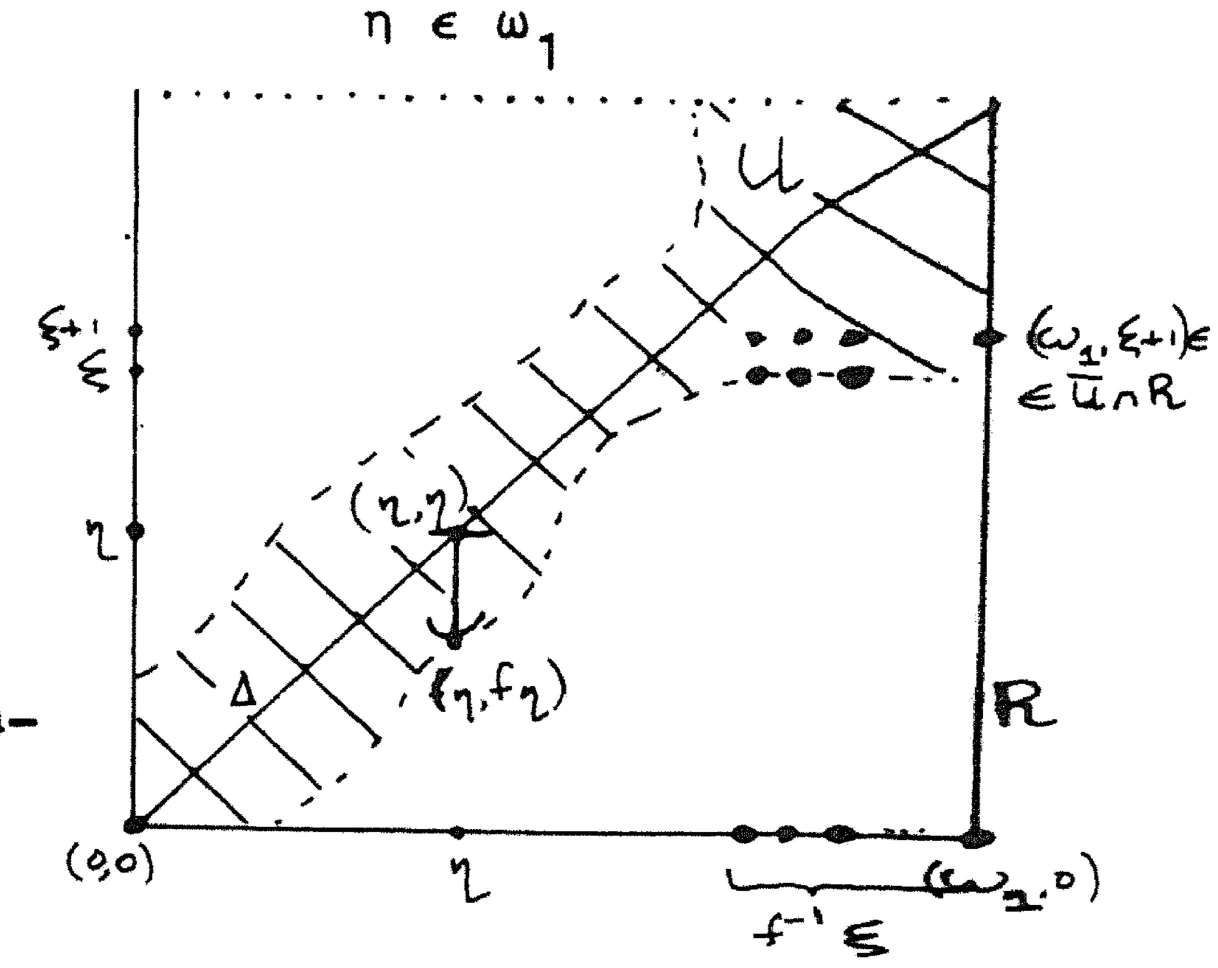
and

$$\{\eta\} \times (f(\eta), \eta] \subset U \qquad \eta \in \omega_1$$

By A1.2 there exists a $\xi \in \omega_1$ such that $\underline{f^{-1}(\xi)}$ is uncountable. Then $\omega_1 \in \overline{f^{-1}(\xi)}$ and

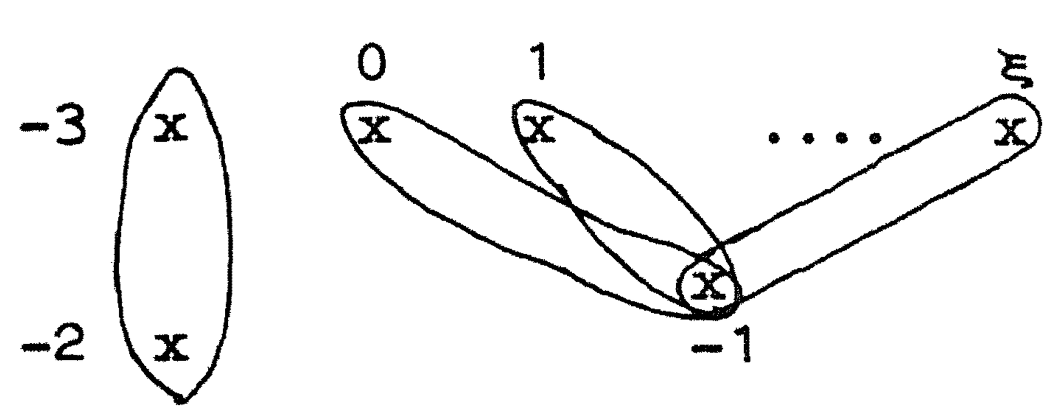
$$(\eta, \xi+1) \in \{\eta\} \times (\xi, \eta] \subset U$$

Hence $(\omega_1, \xi+1) \in \overline{U}$, proving that $\overline{U} \cap R \neq \emptyset$ for each neighbourhood U of Δ.



A2. Quasidisjoint families

A2.1 If $\mathfrak{A}$ is a large family of finite sets then does there exist a big disjoint subfamily of $\mathfrak{A}$? Not necessarily.



$\mathfrak{A} = \{\{-2,-3\}\} \cup \{\{-1,\xi\} \mid \xi < \alpha\}$.
Note that $\bigcap \mathfrak{A} = \emptyset$.

This situation suggests the following definition:
A family $\mathfrak{A}$ is quasidisjoint if $\{A \setminus \bigcap \mathfrak{A} \mid A \in \mathfrak{A}\}$ is disjoint.
A quasidisjoint family is called trivial if it contains only 2 sets (or even less).

Remarks (1) The following conditions are equivalent:

(i) $\mathfrak{A}$ is a quasidisjoint family

(ii) $\exists Z \ \forall A,B \in \mathfrak{A} \quad A \neq B \Rightarrow A \cap B = Z$

(iii) $\forall A,B \in \mathfrak{A} \quad A \neq B \Rightarrow A \cap B = \bigcap \mathfrak{A}$

(iv) each three-element subset of $\mathfrak{A}$ is quasidisjoint.

(2) It follows easily from the Teichmüller-Tukey lemma (or e.g. the equivalent Zorn-lemma) that <u>any family of sets contains maximal quasidisjoint and maximal disjoint subfamilies</u>.

Let $\mathfrak{A}$ be a "large" family, $|\mathfrak{A}| = \alpha$, of sets of "small" cardinality, $\forall A \in \mathfrak{A}\ |A| \leq \beta$. In this paragraph we will give estimations (lower bounds) for the supremum of the cardinality of quasidisjoint subfamilies of $\mathfrak{A}$, in terms of α and β. Moreover we will give conditions under which the supremum is actually reached, (i.e. sup = max). It can be shown by means of examples that the results obtained are the best possible.

At first, in A2.2, we deal with the case $\beta = \omega$ (i.e. $\mathfrak{A}$ is a family of finite sets). This case has applications, e.g. in the theorems on the cellularity number (Suslin property) and caliber (Šanin property) of topological products, cf 4.6, 4.7 and 4.8 (p. 52-55).

Secondly, in A2.3, we deal with the general case. The results are obtained independently from A2.2, but because the proofs and the examples are much more complicated, we have included A2.2 in order to supply relatively short proofs for the applications mentioned above.

A2.2 <u>lemma</u>. Let n be a fixed integer. If $\mathfrak{A}$ is a family of n-element sets, and $|\mathfrak{A}| = \alpha$ is regular, then $\exists \mathfrak{B} \subset \mathfrak{A}$ such that $\mathfrak{B}$ is quasidisjoint and $|\mathfrak{B}| = |\mathfrak{A}| = \alpha$.

<u>Proof</u>. The proof will be given by induction on n. For n = 1 $\mathfrak{A}$ is disjoint and we may take $\mathfrak{B} = \mathfrak{A}$.

Let the lemma be true if we replace n by any smaller integer. Let $\mathfrak{A}_0$ be a maximal disjoint subfamily of $\mathfrak{A}$ and suppose $\beta = |\mathfrak{A}_0| < \alpha$. Since each $A \in \mathfrak{A}$ meets at least one member of $\mathfrak{A}_0$, and α is regular

$$\exists A_0 \in \mathfrak{A}_0 \quad |\{A \in \mathfrak{A} \mid A \cap A_0 \neq \emptyset\}| = \alpha$$

Since A_0 is finite

$$\exists x \in A_0 \qquad |\{A \in \mathfrak{A} \mid x \in A\}| = \alpha \qquad .$$

Consider $\{A \setminus \{x\} \mid x \in A \in \mathcal{A}\}$. By the induction hypothesis this family has a quasidisjoint subfamily $\mathcal{B}_1$ of cardinality α. Then

$$\mathcal{B} = \{B \cup \{x\} \mid B \in \mathcal{B}_1\}$$

is a quasidisjoint subfamily of $\mathcal{A}$ of cardinality α.

THEOREM. <u>[56] Let $\mathcal{A}$ be any uncountable family of finite sets. Then $\sup\{|\mathcal{B}| \mid \mathcal{B} \subset \mathcal{A} \wedge \mathcal{B}$ is quasidisjoint$\} = |\mathcal{A}|$
If, moreover, $|\mathcal{A}|$ is regular, then $\exists \mathcal{B} \subset \mathcal{A}$
$\mathcal{B}$ is quasidisjoint and $|\mathcal{B}| = |\mathcal{A}|$</u>.

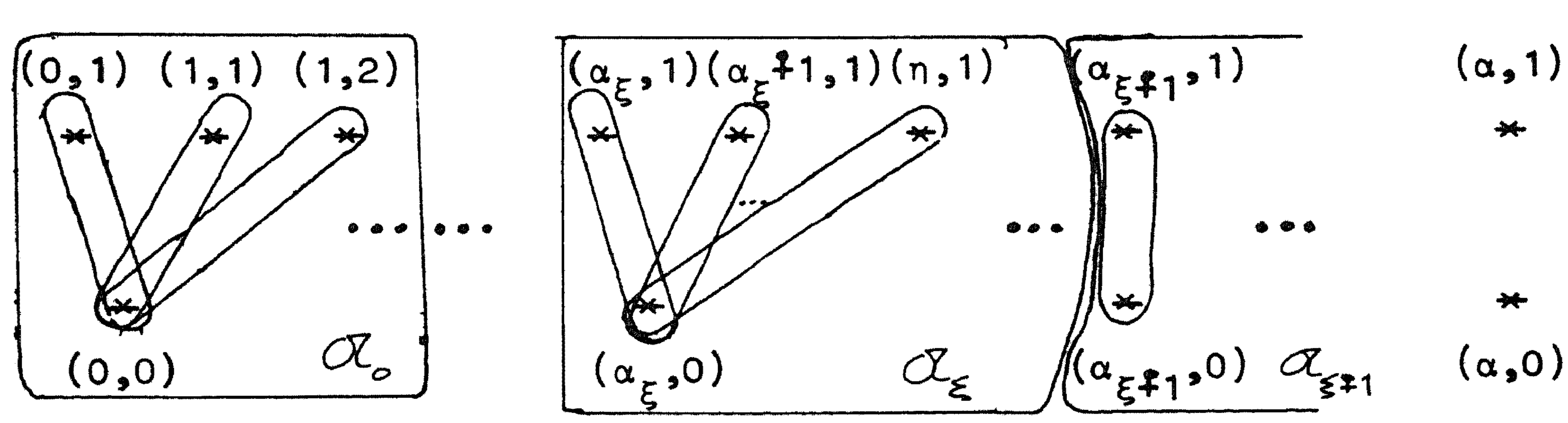
Proof. Let us first assume that $|\mathcal{A}| = \alpha$ is regular. Then $\exists n \in \omega$ such that $\mathcal{A}$ has a subfamily of α sets of exactly n elements. Application of the above lemma yields a quasidisjoint subfamily of $\mathcal{A}$ of power α.

If $|\mathcal{A}| = \alpha$ is singular and $\beta < \alpha$ then $\beta^+ < \alpha$ and β^+ is regular. Hence as we just proved, there is a quasidisjoint subfamily of $\mathcal{A}$ of power β^+. This proves our theorem.

<u>EXAMPLES</u>

a. $\{\{1,2,3,\ldots n\} \mid n \in \omega\}$ is a countable family of finite sets whose only quasidisjoint subfamilies are trivial (i.e. contain two sets).

b. If α is singular and $|\mathcal{A}| = \alpha$ then $\mathcal{A}$ need not contain quasidisjoint subfamilies of power α. Let $(\alpha_\xi)_{\xi<cf\alpha}$ be a strictly increasing sequence, converging to α, and $\alpha_0 = 0$.



For $\xi < \mathrm{cf}\alpha$ put $\mathcal{A}_\xi = \{\{(\alpha_\xi,0),\ (\eta,1)\} \mid \alpha_\xi \leq \eta < \alpha_{\xi+1}\}$, and let $\mathcal{A} = \bigcup\{\mathcal{A}_\xi \mid \xi < \mathrm{cf}\alpha\}$. It is easily seen that the $\mathcal{A}_\xi$ are the maximal quasidisjoint-but-not-disjoint subfamilies of $\mathcal{A}$, and $|\mathcal{A}_\xi| = \alpha_{\xi+1} < \alpha$. If on the other hand $\mathcal{B} \subset \mathcal{A}$ is disjoint, then $|\mathcal{B} \cap \mathcal{A}_\xi| \leq 1$ for each $\xi < \mathrm{cf}\alpha$, and hence $|\mathcal{B}| \leq \mathrm{cf}\alpha < \alpha$.

A2.3 THEOREM.([40], [49], [60]) Let $\mathcal{A}$ be a family of sets such that $|\mathcal{A}|\ \alpha$ and $\forall A \in \mathcal{A}\quad |A| \leq \beta$, then

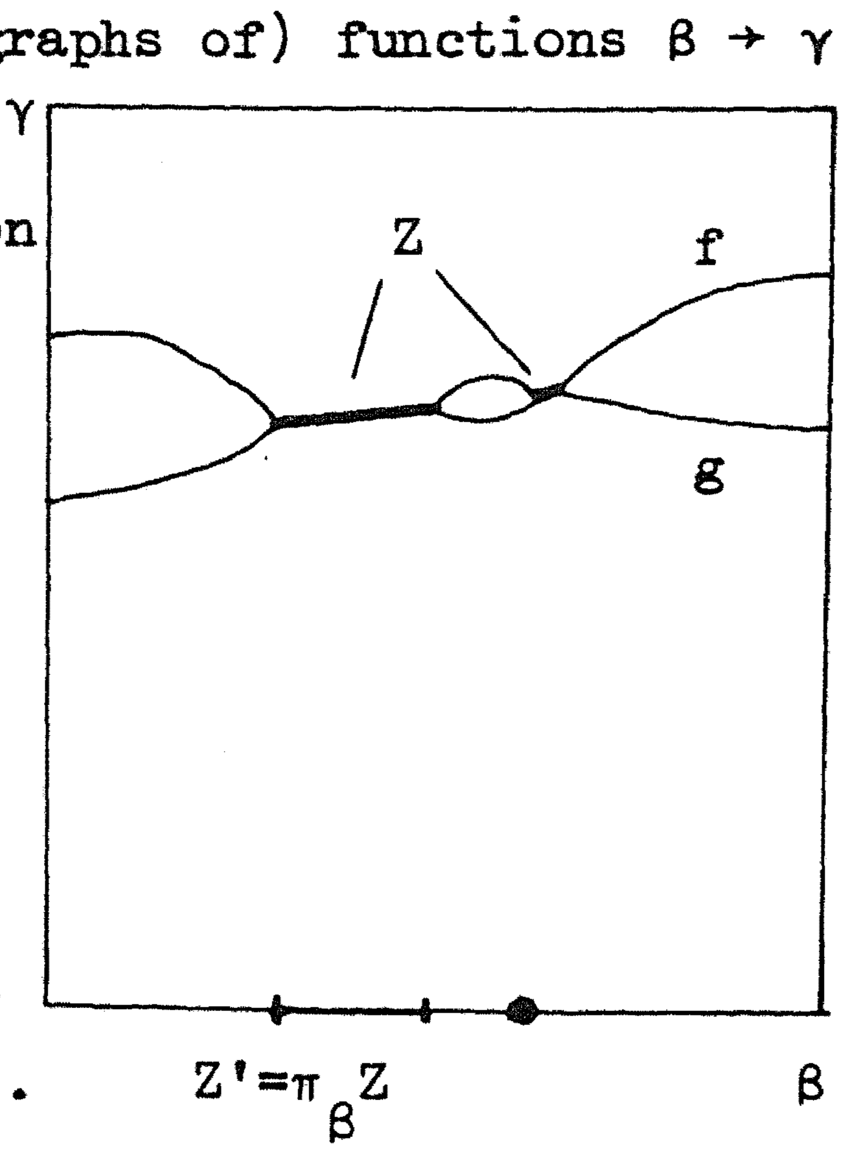
(i) $\sup\{\,|\mathcal{B}| \mid \mathcal{B} \subset \mathcal{A}$ and $\mathcal{B}$ quasidisjoint$\} \geq \sqrt[\beta]{\alpha} \underset{\mathrm{def}}{=} \min\{\gamma \mid \gamma^\beta \geq \alpha\}$.

(ii) Moreover, if $\sqrt[\beta]{\alpha}$ is regular, then there exists a quasidisjoint $\mathcal{B} \subset \mathcal{A}$ such that $|\mathcal{B}| \geq \sqrt[\beta]{\alpha}$.

Before we prove of the theorem, we present some examples and simple lemma's. Note that the case $\alpha \leq 2^\beta$ is trivial.

Example. [40]

For any β,γ let $\mathcal{A}$ be the family of all (graphs of) functions $\beta \to \gamma$. Let $\mathcal{B} \subset \mathcal{A}$ be a quasidisjoint subfamily, and $Z = \bigcap\mathcal{A}$. By Z' we denote the projection of Z onto β. If $Z' = \beta$, then $\mathcal{B} = \{Z\}$. If $Z' \neq \beta$, then let $\eta \in \beta \setminus Z'$. For any two $f,g \in \mathcal{B}$ $(\eta,f\eta) \notin Z' = f \cap g$, hence $f(\eta) \neq g(\eta)$. This implies that $|\mathcal{B}| \leq |\{f(\eta) \mid f \in \mathcal{B}\}| \leq \gamma$. Hence $\mathcal{A}$ satisfies:



$$|\mathcal{A}| = \gamma^\beta \text{ and } \forall A \in \mathcal{A}\quad |A| = \beta$$
and
$$\forall \mathcal{B} \subset \mathcal{A}\quad \mathcal{B} \text{ is quasidisjoint} \Rightarrow |\mathcal{B}| \leq \gamma.$$

This implies that part(i) of the Theorem cannot be improved.

Remark.

It is not hard to show that for each α,β for which $\sqrt[\beta]{\alpha}$ is singular there exists a family $\mathcal{O}\!\mathcal{C}$ satisfying

$$|\mathcal{O}\!\mathcal{C}| = \alpha \quad \text{and} \quad \forall A \in \mathcal{O}\!\mathcal{C} \quad |A| = \beta \qquad \text{and}$$

$$\forall \mathcal{L} \subset \mathcal{O}\!\mathcal{C} \quad \mathcal{L} \text{ is quasidisjoint} \Rightarrow |\mathcal{L}| < \sqrt[\beta]{\alpha}$$

For details see [60]. This proves that part (ii) of the theorem cannot be improved either.

lemma. For any two infinite cardinals α,γ and finite or infinite β the following relations hold:

(a) $\sqrt[\beta]{\alpha} \neq 2 \Longrightarrow \beta^+ \leq 2^\beta < \sqrt[\beta]{\alpha}$

(b) $\gamma < \sqrt[\beta]{\alpha} \Longrightarrow \gamma^\beta < \sqrt[\beta]{\alpha}$

(c) $\sqrt[\beta]{(\gamma^\beta)^+} = (\gamma^\beta)^+$

Proof (a) If $\sqrt[\beta]{\alpha} \neq 2$ then $2 < \sqrt[\beta]{\alpha}$. If $\sqrt[\beta]{\alpha} \leq 2^\beta$, then $\alpha \leq (\sqrt[\beta]{\alpha})^\beta \leq$ $\leq 2^{\beta\cdot\beta} = 2^\beta$, and hence $2 = \sqrt[\beta]{\alpha}$.

(b) If $\sqrt[\beta]{\alpha} \leq \gamma^\beta$ then $\alpha \leq (\sqrt[\beta]{\alpha})^\beta \leq \gamma^{\beta\cdot\beta} = \gamma^\beta$ and hence $\sqrt[\beta]{\alpha} \leq \gamma$.

(c) If $\delta < (\gamma^\beta)^+$, then $\delta \leq \gamma^\beta$ and hence $\delta^\beta \leq \gamma^\beta < (\gamma^\beta)^+$. So $\delta < \sqrt[\beta]{(\gamma^\beta)^+}$.

Proof of the theorem.

If $2^\beta \geq \alpha$ then $\sqrt[\beta]{\alpha} = 2$ and (i) and (ii) are trivially satisfied. So let us assume $\sqrt[\beta]{\alpha} \neq 2$, i.e. (lemma (a)):

$$\beta^+ \leq 2^\beta < \sqrt[\beta]{\alpha}$$

The proof is devided into two parts. In (A) we prove (i) and (ii) for the case that $\sqrt[\beta]{\alpha}$ is regular. This part is a slight generalization of the proof in [49]. In (B) we prove that (i) also holds if $\sqrt[\beta]{\alpha}$ is singular.

A. $\sqrt[\beta]{\alpha}$ is regular.

Assume

(a) $\forall \mathcal{J} \subset \mathcal{A} \quad \mathcal{J}$ is quasidisjoint $\Rightarrow |\mathcal{J}| < \sqrt[\beta]{\alpha}$.

We will define subfamilies $\mathcal{A}_\zeta$ of $\mathcal{A}$ for each $\zeta < \beta^+$ so that

(b) $$\mathcal{A} = \bigcup \{\mathcal{A}_\xi \mid \xi < \beta^+\}$$

(c) $$\forall \zeta < \beta^+ \qquad |\mathcal{A}_\zeta| < \sqrt[\beta]{\alpha}$$

Because of the regularity of $\sqrt[\beta]{\alpha}$ and by $\beta^+ < \sqrt[\beta]{\alpha}$, (b) and (c) imply $|\mathcal{A}| = \alpha < \sqrt[\beta]{\alpha}$. This contradiction shows that (a) does not hold, which proves (i) and (ii).

The definition of the families $\mathcal{A}_\zeta$ is by transfinite induction. Let $\mathcal{A}_0$ be a maximal disjoint subfamily of $\mathcal{A}$. If $\zeta < \beta^+$ and $\mathcal{A}_\eta$ has been defined for all $\eta < \zeta$, then put

$$A_\zeta = \bigcup \{\bigcup \mathcal{A}_\eta \mid \eta < \zeta\} .$$

For each subset K of A_ζ, satisfying $|K| \leq \beta$, we define $\mathcal{A}_{\zeta,K}$ by

$$\mathcal{A}_{\zeta,K} = \{A \in \mathcal{A} \setminus \bigcup_{\eta<\zeta} \mathcal{A}_\eta \mid A \cap A_\zeta = K\} .$$

If there exist $A, A' \in \mathcal{A}_{\zeta,K}$, satisfying $A \cap A' = K$, then let $\mathcal{A}^*_{\zeta,K}$ be a maximal quasidisjoint subfamily of $\mathcal{A}_{\zeta,K}$ such that

(d) $$\bigcap \mathcal{A}^*_{\zeta,K} = K$$

If such $A, A' \in \mathcal{A}_{\zeta,K}$ do not exist, then let $\mathcal{A}^*_{\zeta,K}$ be any arbitrary maximal quasidisjoint subfamily of $\mathcal{A}_{\zeta,K}$. In either case

(e) $$\mathcal{A}^*_{\zeta,K} = \emptyset \Longleftrightarrow \mathcal{A}_{\zeta,K} = \emptyset \quad .$$

Finally let

$$\mathfrak{A}_\zeta = \bigcup\{\mathfrak{A}^*_{\zeta,K} \mid K \subset \mathfrak{A}_\zeta \text{ and } |K| \leq \beta\}$$

Let us verify (b) and (c).
To verify (b), suppose that for some $A \in \mathfrak{A}$ $A \notin \mathfrak{A}_\zeta$ for each $\zeta < \beta^+$. We will show that such an A meets each $A_{\zeta+1} \setminus A_\zeta, \forall \zeta < \beta^+$. This implies $|A| \geq \beta^+$, a contradiction.
Let $\zeta < \beta^+$ and $K = A \cap A_\zeta$. We distinguish between two cases: (f) and (g).

(f) Suppose $\exists A' \in \mathfrak{A}^*_{\zeta,K}$ $(A \cap A') \setminus K \neq \emptyset$..
Then $\emptyset \neq (A \cap A') \setminus K = (A \cap A') \setminus (A \cap A_\zeta) \subset A \cap (A_{\zeta+1} \setminus A_\zeta)$.

(g) Suppose $\forall A' \in \mathfrak{A}^*_{\zeta,K}$ $A \cap A' \subset K$. Since $A \in \mathfrak{A}_{\zeta,K}$, (e) implies that $\mathfrak{A}^*_{\zeta,K} \neq \emptyset$. So (d) holds. Hence $\mathfrak{A}^*_{\zeta,K} \cup \{A\}$ is quasidisjoint, contradicting the maximality of $\mathfrak{A}^*_{\zeta,K}$.

This proves (b).
In order to prove (c), note first that $|\mathfrak{A}_0| < \sqrt[\beta]{\alpha}$ (by (a)), and recall $\beta^+ \leq 2^\beta < \sqrt[\beta]{\alpha} = \mathrm{cf}\sqrt[\beta]{\alpha}$. let $\zeta < \beta^+$ and assume that

$$|\mathfrak{A}_\eta| < \sqrt[\beta]{\alpha} \qquad \text{for all } \eta < \zeta .$$

Then also $$|\bigcup \mathfrak{A}_\eta| < \sqrt[\beta]{\alpha} \qquad \text{for } \eta < \zeta.$$

Thus $$|A_\zeta| \leq \sum_{\eta<\zeta} |\bigcup \mathfrak{A}_\eta| < \sqrt[\beta]{\alpha} \quad .$$

The set $|A_\zeta|$ has $|A_\zeta|^\beta$ subsets K such that $|K| \leq \beta$. Since $|A_\zeta| < \sqrt[\beta]{\alpha}$, lemma (b) implies $|A_\zeta|^\beta < \sqrt[\beta]{\alpha}$.
By (a) we have for each $K \subset A_\zeta$ such that $|K| \leq \beta$ $|\mathfrak{A}^*_{\zeta,K}| < \sqrt[\beta]{\alpha}$.
Because of the regularity of $\sqrt[\beta]{\alpha}$ we deduce

$$|\mathfrak{A}_\zeta| \leq \sum_K |\mathfrak{A}^*_{\zeta,K}| < \sqrt[\beta]{\alpha} .$$

This proves (c), and completes the proof of part A.

B. $\sqrt[\beta]{\alpha}$ is singular.

Each singular cardinal is limit. Let $\gamma < \sqrt[\beta]{\alpha}$, by our lemma (b) $\gamma^\beta < \sqrt[\beta]{\alpha}$ and hence $(\gamma^\beta)^+ < \sqrt[\beta]{\alpha}$. Let $\mathfrak{A}^*$ be a subfamily of $\mathfrak{A}$ such that $|\mathfrak{A}^*| = (\gamma^\beta)^+$.
Since $(\gamma^\beta)^+$ is a successor and thus regular, part A yields the existence of a quasidisjoint $\mathfrak{L} \subset \mathfrak{A}^* \subset \mathfrak{A}$ satisfying

$$|\mathfrak{L}| \geq \sqrt[\beta]{(\gamma^\beta)^+} = (\gamma^\beta)^+ > \gamma \qquad \text{(lemma (c))}$$

This proves B.

A3 Set mappings and free sets

A3.1 Let $\mathbb{R}$ be the topological space of real numbers and $F: \mathbb{R} \to \mathcal{P}\mathbb{R}$ a set-valued mapping with the property that for each $x \in \mathbb{R}$ $F(x)$ is finite and does not contain x. A subset $M \subset \mathbb{R}$ is called <u>free</u> if $M \cap \bigcup\{F(x) \mid x \in M\} = \emptyset$.

P. Turán asked whether there exist infinite free subsets for each F. This was solved by Lázár who showed that there always exist free subsets of continuous power. Indeed for each $x \in \mathbb{R}$ we may choose an open interval I_x with rational endpoints such that $x \in I_x \subset \mathbb{R}\backslash F(x)$. Since there are only countably many open intervals with rational endpoints (and $\mathrm{cf}\ 2^\omega > \omega$, see A0.1) there exists an interval (a,b) such that the set

$$M = \{x \in \mathbb{R} \mid I_x = (a,b)\}$$

has continuous power. It is easily seen that M is free.

We can generalize this in two ways: at first there still exist free subsets of continuous power if we replace "$F(x)$ is finite and $x \notin F(x)$" by the weaker condition "$x \notin \overline{F(x)}$".

Secondly we may ask for free sets of mappings $F: X \to \mathcal{P}X$ with $\forall x \in X \quad x \notin F(x)$ and $F(x)$ finite, where X is an arbitrary set. It is easily seen that free subsets of power $|X|$ also exist if $|X| = 2^{\alpha}$, α is arbitrary. One can prove this by replacing $\mathbb{R}$ by the generalized Cantor set $\{0,1\}^{\alpha}$ of weight α.

This suggests the following more general definitions and problems.

A3.2 DEFINITIONS

A map $F: X \to \mathcal{P}X$ is a <u>set mapping</u> if $\forall x \in X \quad x \notin F(x)$.
A subset $M \subset X$ is <u>free</u> (under the set mapping F) if $\forall x,y \in M$ $x \notin F(y)$ and $y \notin F(x)$.
We will investigate conditions on set mappings $F: X \to \mathcal{P}X$ which guarantee the existence of free subsets $M \subset X$ of power $|M| = |X|$.

Remarks.

(1) If $F: X \to \mathcal{P}X$ is a set mapping then it is easily seen that $M \subset X$ is free iff

$$M \cap \bigcup\{F(x) \mid x \in M\} = \emptyset$$

(2) From the Teichmüller-Tukey lemma it follows that for any set mapping $F: X \to \mathcal{P}X$ and free subset $M \subset X$ there is a maximal free subset M^* such that $M \subset M^* \subset X$.

(3) For each X there exists a set mapping $F: X \to \mathcal{P}X$ satisfying:

$$\forall x \in X \quad |F(x)| < |X| = \alpha,$$

and $\qquad \forall M \subset X \quad M \text{ is free} \Rightarrow |M| \leq 1$

In particular, under assumption of the C H, there exists a set mapping $F: \mathbb{R} \to \mathcal{P}\mathbb{R}$ such that: $\forall x \in X \quad |F(x)| \leq \omega$

and $\qquad \forall M \subset \mathbb{R} \quad M \text{ is free} \Rightarrow |M| \leq 1$

Proof. Well-order X: $X = \{x_\xi \mid \xi < \alpha\}$ and put $F(x_\xi) = \{x_\eta \mid \eta < \xi\}$

A3.3 In 1936 S. Ruziewicz [55] asked:

Does "$|X| = \alpha > \beta$ and $F: X \to \mathcal{P}X$ is a set mapping such that $\forall x \in X$ $|F(x)| < \beta$" imply: "$\exists M \subset X \quad |M| = \alpha$ and M is free"?

Partial positive solutions were given by Lázár ([48], for α regular), Sierpiński ([57], for $\beta = \omega$), G. Fodor ([43], for $cf(\alpha) > \beta$) P. Erdős ([38] for all $\alpha > \beta$, but assuming GCH). Finally A. Hajnal, [44], proved in 1960 that the answer is always yes, without using G.C.H. We will prove Hajnal's result in two steps, at first for the case $\beta < cf\alpha$ (A3.4) and then, in the general case (A3.5).

A3.4 THEOREM LÁZÁR [48].

If $|X| = \alpha$, $\beta < cf\alpha$ and $F: X \to \mathcal{P}X$ satisfies

$$\forall x \in X \quad x \notin F(x)$$
$$\forall x \in X \quad |F(x)| < \beta$$

then $\exists M \subset X \quad |M| = \alpha$ and M is free.

Proof. Assume $\forall\ M \subset X \quad M$ is free $\Rightarrow |M| < \alpha$. Let S_0 be a maximal free subset of X, $|S_0| < \alpha$ (see remark (2)). If for some $\nu < \beta$ and all $\eta < \nu$ we defined S_η satisfying

$$|S_\eta| < \alpha$$

$$S_\eta \text{ is a maximal free subset of } X \setminus \bigcup_{\xi<\eta} S_\xi$$

then let S_ν be a maximal free subset of $X \setminus \bigcup_{\eta<\nu} S_\eta$. Put

$$S^* = \bigcup_{\nu<\beta} S_\nu \ ;$$

then

$$|S^*| < \alpha \text{ , because } \beta < cf(\alpha).$$

Let $\quad S^{**} = S^* \cup \bigcup\{F(x) \mid x \in S^*\}$

then $\quad |S^{**}| \leq |S^*| + \beta|S^*| < \alpha$

Hence $\quad S^{**} \neq X$.

Choose $y \in X \setminus S^{**}$. Then $\forall \nu < \beta$

$S_\nu \cup \{y\}$ is not free, by the maximality of S_ν.

I.e. either $\exists x_\nu \in S_\nu \quad y \in F(x_\nu)$

(which is impossible because $y \notin S^{**}$)

or $\quad \exists x_\nu \in S_\nu \quad x_\nu \in F(y)$.

Thus $F(y)$ meets each member of the disjoint family $\{S_\nu \mid \nu < \beta\}$, hence $|F(y)| \geq \beta$. This contradiction proves the theorem.

A3.5 MAIN THEOREM (HAJNAL [44])

If $|X| = \alpha$ and $\beta < \alpha$, and $F: X \to \mathcal{P}X$ satisfies

$$\forall x \in X \qquad x \notin F(x)$$
$$\forall x \in X \qquad |F(x)| < \beta$$

then $\exists M \subset X \quad |M| = \alpha$ and M is free.

Proof. Because of A3.4 we may assume that α is singular and $\gamma = cf\alpha < \beta^+ < \alpha$. Let $(\alpha_\xi)_{\xi<\gamma}$ be a strictly monotone increasing sequence of regular cardinals (e.g. successors), converging to α and all greater then β^+. Let $(A^*_\xi)_{\xi<\gamma}$ be a sequence of disjoint sets whose union is X, satisfying $|A^*_\xi| = \alpha_\xi$ for $\xi < \gamma$.
We construct a new sequence $(A_\xi)_{\xi<\gamma}$ by transfinite induction in such a way that

(i) $\quad \forall \xi < \eta < \gamma \qquad A_\xi \subset A_\eta$ and $\bigcup\{A_\xi \mid \xi < \gamma\} = X$

(ii) $\quad \forall \xi < \gamma \qquad |A_\xi| = \alpha_\xi$

(iii) $\quad \forall \xi < \gamma\, \forall x \in A_\xi \quad F(x) \subset A_\xi$

Assume that $\eta < \gamma$ and the A_ξ are defined for all $\xi < \eta$.
At first we define a sequence $(A^n_\eta)_{n\in\omega}$ by

$$A^0_\eta = A^*_\eta \cup \bigcup\{A_\xi \mid \xi < \eta\}$$
$$A^{n+1}_\eta = A^n_\eta \cup \bigcup\{F(x) \mid x \in A^n_\eta\} \qquad \text{for } n \in \omega.$$

Then put $A_\eta = \bigcup_{n\in\omega} A_\eta^n$. Now (i) and (iii) are trivially satisfied; to see (ii) note that $|A_\eta^*| = \alpha_\eta$ is regular and

$$|\bigcup\{A_\xi \mid \xi < \eta\}| \leq \sum_{\xi<\eta} |A_\xi| \leq \sum_{\xi<\eta} \alpha_\xi < \alpha_\eta .$$

By A3.4 there exists a free (under $F|A_\eta$) subset $B_\eta \subset A_\eta$ satisfying

$$|B_\eta| = |A_\eta| = \alpha_\eta \qquad \text{for each } \eta < \gamma .$$

Next we define sets $C_\eta \subset B_\eta$ for all $\eta \quad \gamma$, satisfying

(iv) $\forall \eta < \gamma \qquad |C_\eta| = |B_\eta| = \alpha$

(v) $\forall \xi < \gamma \ \forall \eta \leq \xi \quad x \in C_\eta \quad F(x) \cap C_\xi = \emptyset$.

If the C_ξ are defined for all $\xi < \eta$, where $\eta < \gamma$, then put

$$H_\eta = \bigcup_{\xi<\eta} C_\xi .$$

Then $$|H_\eta| \leq \sum_{\xi<\eta} \alpha_\xi < \alpha_\eta$$

and also $$|\bigcup_{x\in H_\eta} F(x)| \leq \beta \cdot \sum_{\xi<\eta} \alpha_\xi < \alpha_\eta = |B_\eta| .$$

Let $C_\eta = B_\eta \setminus \bigcup_{x\in H_\eta} F(x)$.

Notice that (iv) and (v) are fulfilled. Yet there still may be (many) $x \in C_\eta$ such that for some $\xi < \eta$ $F(x) \cap C_\xi \neq \emptyset$. To avoid this we define another sequence of sets $(D_\xi)_{\xi<\gamma}$ and a partition of every D_ξ into β^+ disjoint sets: $D = \{D_{\xi,\rho} \mid \rho < \beta^+\}$ satisfying

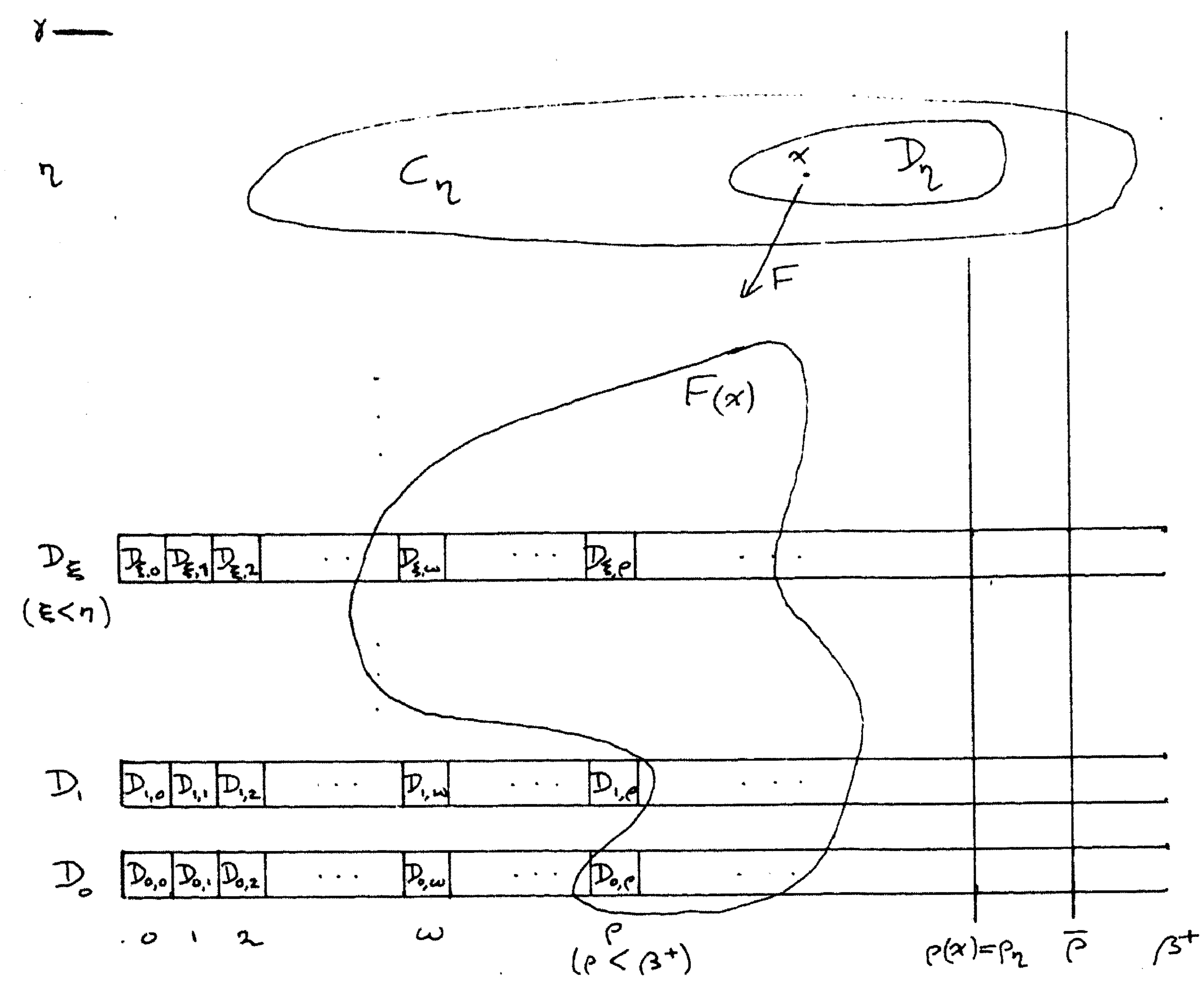
(vi) $\forall \xi < \gamma \qquad D_\xi \subset C_\xi$

(vii) $\forall \xi < \gamma \ \forall \rho < \beta^+ \quad |D_{\xi,\rho}| = |D_\xi| = \alpha_\xi$

(viii) $\forall \xi < \gamma \ \exists \rho_\xi < \beta^+ \ \forall x \in D_\xi \ \forall \eta < \xi \ \forall \rho \quad \rho_\xi < \rho < \beta^+ \Rightarrow F(x) \cap D_\eta(\rho) = \emptyset$.

Construction of $(D_\xi)_{\xi<\gamma}$, and $(D_{\xi,\rho})_{\rho<\beta^+}$, $\xi < \gamma$.

Assume that for some $\eta < \gamma$ and all $\xi < \eta$, $\rho < \beta^+$ D_ξ and $D_{\xi,\rho}$ have been defined, satisfying (vi)-(viii).



For each $x \in C_\eta$, $|F(x)| < \beta$ and thus $\exists \rho(x) < \beta^+\ \forall \rho>\rho(x)\quad \xi<\eta$ $F(x) \cap D_{\xi,\rho} = \emptyset$. Since $|C_\eta| = \alpha_\eta$ is regular and greater then β^+, there is a $\rho_\eta < \beta^+$ such that

$$D_\eta =_{\text{def}} \{x \in C_\eta \mid \rho(x) = \rho_\eta\}$$

has α_η elements. Let $\{D_{\eta,\rho} \mid \rho < \beta^+\}$ be any partition of D_η in β^+ disjoint sets of power $|D_\eta| = \alpha_\eta > \beta^+$. Check that (vi)-(viii) are satisfied.

Now $(\rho_\xi)_{\xi<\gamma}$ is a sequence of ordinals smaller then β^+. Since $\gamma < \beta^+$ and β^+ is regular, $\exists \bar\rho < \beta^+$ such that $\rho_\xi < \bar\rho$ for all $\xi < \gamma$.

Put

$$M = \bigcup\{D_{\xi,\bar\rho} \mid \xi < \gamma\} .$$

Then $|M| = \sum_{\xi<\gamma} |D_{\xi,\bar\rho}| = \sum_{\xi<\gamma} \alpha_\xi = \alpha$ by (vii). If $x \in M$, say $x \in D_{\xi,\bar\rho}$ then $F(x) \cap D_{\eta,\bar\rho} = \emptyset$ for all $\eta \geq \xi$ by (vi) and (v). Moreover $F(x) \cap D_{\eta,\bar\rho} = \emptyset$ for all $\eta < \xi$ by (viii). This proves that

$$M \cap \bigcup \{F(x) \mid x \in M\} = \emptyset$$

i.e. M is free.

A3.6 Application.

Hajnal's theorem is used to prove a lemma of 3.3 (p.40):
Suppose $X \in \mathcal{H}$, $\phi(X) = \lambda$ where ϕ is one of the functions s,h,z and $\mathrm{cf}\ \lambda = \omega$. Then the answer for the sup = max problem is positive.

A4 Partition calculus. Ramifications

A4.1 Definitions

For every set S and each natural number r

$$[S]^r = \{X \mid X \subseteq S \wedge |X| = r\}$$

A partition of $[S]^r$: $[S]^r = \bigcup_{\xi<\gamma} I_\xi$ is called an r-partition of S.

In general we do not require that the classes of the partition are disjoint.

If $A \subset S$ is such that for some $\xi < \gamma$ $[A]^r \subset I_\xi$, then we say that the set A is homogeneous (for the partition $\{I_\xi \mid \xi < \gamma\}$).

The symbol

(1) $$\alpha \to (\beta_\xi)^r_{\xi<\gamma}$$

is to be read "α arrows β_ξ, $\xi < \gamma$, r" and stands for the following statement:

If

(2) $$|S| = \alpha \quad \text{and} \quad [S]^r = \bigcup_{\xi<\gamma} I_\xi$$

then

(3) $$\exists A \subset S \ \exists \xi < \gamma \quad |A| = \beta_\xi \text{ and } [A]^r \subset I_\xi .$$

If $\beta_\xi = \beta$ for all $\xi < \gamma$ then we may also write

$$\alpha \to (\beta)^r_\gamma \ .$$

If γ is finite we may replace (1) by

$$\alpha \to (\beta_0, \ldots, \beta_{\gamma-1})^r$$

The negation of (1) is expressed by

$$\alpha \not\to (\beta_\xi)^r_{\xi<\gamma}$$

We put the following restrictions to the use of (1):

α	is infinite
r	is finite, but $r > 0$
γ	is either finite or infinite, but $\gamma < \alpha$
β_ξ	is either finite or infinite but $r < \beta_\xi \leq \alpha$ for each $\xi < \gamma$

Only in example 4° below we do not assume this restriction and mention what results follow.

A4.2 Examples

1° A 1-partition of a set S is just a partition (or covering) of this set (if we identify $x \in S$ with $\{x\}$). This yields e.g.

$\mathrm{cf}\alpha = \min\{\gamma \mid \alpha \not\to (\alpha)^1_\gamma\}$.

2° For $r = 2$ $[S]^2$ can be considered as the complete graph which has S as its set of vertices. Then a 2-partition of S is a partition of the set of edges. One of the earliest results in partition calculus is theorem A4.10

$$2^\omega \nrightarrow (\omega_1,\omega_1)^2$$

due to W. Sierpiński [58]. This result can be rephrased as follows:

<u>There exists a partition $\{I_0,I_1\}$ of the edges of a complete graph with 2^ω vertices into two parts so that any set of vertices is countable if it generates a complete graph with all edges belonging to I_0 or to I_1.</u>

3° <u>Monotony and symmetry in (1)</u>.

Suppose (1) holds. What is the effect if we change one of the α, β_ξ, γ, r or permute the β_ξ?

(a) If $\alpha' > \alpha$ then also $\alpha' \to (\beta_\xi)^r_{\xi<\gamma}$

Proof of (a).

Let $|S'| = \alpha'$, $[S']^r = \bigcup_{\xi<\gamma} I_\xi$. Choose $S \subset S'$ such that $|S| = \alpha$, then $[S]^r = \bigcup_{\xi<\gamma} (I_\xi \cap [S]^r)$. By (3) $\exists A \subset S \subset S'$ $\exists \xi < \gamma$

$$|A| = \beta_\xi \text{ and } [A]^r \subset I_\xi \cap [S]^r \subset I_\xi$$

(b) If $\beta'_\xi \leq \beta_\xi$ for all $\xi < \gamma$ then also $\alpha \to (\beta'_\xi)_{\xi<\gamma}$

Proof of (b).

Let $|S| = \alpha$, $[S]^r = \bigcup_{\xi<\gamma} I_\xi$. By (3) $\exists A \subset S$ $\exists \xi < \gamma$

$$|A| = \beta_\xi \text{ and } [A]^r \subset I_\xi$$

Choose $A' \subset A$ satisfying $|A'| = \beta'_\xi$, then also

$$[A']^r \subset [A]^r \subset I_\xi$$

(c) If $\gamma' < \gamma$ then also $\alpha \to (\beta_\xi)_{\xi<\gamma'}$

Proof of c.

Let $|S| = \alpha$, $[S]^r = \bigcup_{\xi<\gamma'} I_\xi$. Put $I_\xi = \emptyset$ for $\gamma' \leq \xi < \gamma$.

We can generalize (c) as follows:

(d) If $f\colon \gamma' \to \gamma$ is any 1-1 map then also $\alpha \to (\beta_{f(\xi)})^r_{\xi<\gamma'}$.

(e) If $r' < r$ and β_ξ is infinite for each $\xi \in \gamma$ then also $\alpha \to (\beta_\xi)^{r'}_{\xi<\gamma}$.

<u>Proof</u>. As r is finite it suffices to consider the case $r' = r-1$. Let $|S| = \alpha$ and $[S]^{r-1} = \bigcup_{\xi<\gamma} I_\xi$. Well-order S in order type α, and define the r-partition $(I^*_\xi)_{\xi<\gamma}$ of S by

(i) $$I^*_\xi = \{X \in [S]^r \mid X \setminus \{\max X\} \in I_\xi\} .$$

If (1) holds, $\exists A \subset S$ and $\exists \xi < \gamma$

(ii) $$|A| = \beta_\xi \quad \text{and} \quad [A]^r \subset I^*_\xi$$

Since β_ξ is infinite we may assume that A has no largest member. Now we claim that

$$[A]^{r-1} \subset I_\xi.$$

For let $\{x_0,\dots,x_{r-2}\} \in [A]^{r-1}$. Choose $x \in A$ such that $x_0,\dots,x_{r-2} < x$, then $\{x_0,\dots,x_{r-2},x\} \in [A]^r \subset I^*_\xi$ by (ii) and because $x = \max\{x_0,\dots,x_{r-2},x\}$ we obtain from (i) that

$$\{x_0,\dots,x_{r-2}\} \in I_\xi .$$

If in (ii), β_ξ were finite and $a \in A$ were the largest member of A, then we might have concluded

$$[A \setminus \{a\}]^{r-1} \subset I_\xi.$$

Let us define $\beta_\xi \dot{-} (r-r') = \begin{cases} \beta_\xi & \text{if } \beta_\xi \text{ is infinite} \\ \beta_\xi - r + r' & \text{if } \beta_\xi \text{ is finite} \end{cases}$

Thus we obtain:

(f) If $r' < r$ then

$$\alpha \to (\beta_\xi \dot{-} (r-r'))^{r'}_{\xi<\gamma}$$

(g) Substitution rule. If (i) and $\beta_0 \to (\beta_{\gamma+\xi})^r_{\xi<\gamma'}$, and $f\colon \gamma+\gamma' \to \gamma \dot{+} \gamma' \setminus \{0\}$ is any bijection then

$$\alpha \to (\beta_{f(\xi)})^r_{\xi<\gamma+\gamma'}$$

The easy proof is left to the reader.

4° The degenerate cases and restrictions on β_ξ, r and γ, and α.

Let us consider the statement

(i) $$\alpha \to (\beta_\xi)^r_{\xi<\gamma}$$

without any restrictions on r, β_ξ or γ. Let S be a set satisfying $|S| = \alpha$, and let $(I_\xi)_{\xi<\gamma}$ be an r-partition of S.

(a) If $\exists \xi_0 < \gamma \quad \beta_{\xi_0} = 0$ or even $\beta_{\xi_0} < r$ then (i) is trivially satisfied.

For take any β_{ξ_0}-element subset $A \subset S$ then $|A|^r = \emptyset \subset I_{\xi_0}$.

(b) Let $C = \{\xi \mid \beta_\xi = r\}$. Then

(i): $\alpha \to (\beta_\xi)^r_{\beta<\gamma}$ is equivalent to (ii): $\alpha \to (\beta_\xi)^r_{\xi\in\gamma\setminus C}$.

Necessity, i $\Rightarrow$ ii, follows from example 3d.

Sufficiency. Assume (ii) and $|S| = \alpha$, $\boxed{S}^r = \bigcup_{\xi<\gamma} I_\xi$ and

(iii) $\forall\xi < \gamma \quad \forall A \subset S \qquad |A| = \beta_\xi \Longrightarrow \boxed{A}^r \not\subset I_\xi$

In particular $\forall\xi \in C \quad \forall A \subset S \qquad |A| = \beta_\xi = r \Longrightarrow \boxed{A}^r = \{A\} \not\subset I_\xi$.

This yields $\forall A \in \boxed{S}^r \quad A \notin \bigcup\{I_\xi \mid \xi \in C\}$. Thus

$$\boxed{S}^r = \bigcup_{\xi\in\gamma\setminus C} I_\xi \quad .$$

By (ii) $\exists\xi \in \gamma\setminus C \quad \exists A \subset S. \qquad |A| = \beta_\xi \wedge \boxed{A}^r \subset I_\xi$, contradicting (iii).

(c) If $\exists\xi_0 < \gamma \quad \alpha < \beta_{\xi_0}$ and $\forall\xi < \gamma \quad 0 < r \leq \beta_\xi$, then (i) is not satisfied. Consider the trivial r-partition $(I^*_\xi)_{\xi<\gamma}$, all whose elements are empty except I_{ξ_0} , $I_{\xi_0} = \boxed{S}^r$.

(d) If $r = 0$ then (i) is trivially satisfied since $\boxed{S}^r = \boxed{S}^0 = \{\emptyset\}$

(e) For $r = 1$ see example 1°.

(f) For the case of infinite r we mention it is proved in [42] that every such analogue of (i) is false : for any α: $\alpha \not\to (\dot\omega,\omega)^\omega$. Other generalizations by considering partitions of the family of all finite subsets of S are possible (cf [42], and [39] § 17).

(g) If $\alpha \leq \gamma$ and $r \leq \beta_\xi$ for all $\xi < \gamma$, then (i) does not hold. For let $(I^*_\xi)_{\xi<\gamma}$ be a r-partition of S such that each I^*_ξ is empty or consists of one r-element subset of S.

(h) The case of finite α belongs to finite combinatorics; for this we refer to [39] § 16.

(i) Note that it does not make any difference in the meaning of (1) of A4.1 whether or not we require the r-partition $\{I_\xi \mid \xi < \gamma\}$ in (2) of A4.1. to be disjoint.

A4.3 Survey of the theorems and applications.

We will prove the following positive theorems.

a (= A4.4)	$(2^\alpha)^+ \to (\alpha^+)^2_\alpha$
b (= A4.5)	$(\exp^r \alpha)^+ \to (\alpha^+)^{r+1}_\alpha$
c (= A4.6)	$\omega \to (\omega)^r_n$
d (= A4.7)	$\alpha \to (\alpha,\omega)^2$
e (= A4.8)	$(2^\alpha)^+ \to ((2^\alpha)^+,\alpha^+)^2$

The following negative results have in general much simpler proofs then the above theorems.

f (= A4.9)	$(2^\alpha) \not\to (3)^2_\alpha$	
g (= A4.10)	$2^\omega \not\to (\omega_1,\omega_1)^2$	
	$2^\alpha \not\to (\alpha^+,\alpha^+)^2$	
h (= A4.11)	$2^\alpha \not\to (\alpha^+,r+1)^r$	if $r \geq 3$
i (= A4.12)	$\alpha \not\to (\alpha,r+1)^r$	if $r \geq 3$ and α is singular

Remarks. 1. More relations and many references can be found in [39], [41] and [51].

2. Consider a = A4.4. This result is best possible in the following sense: the statement

$$\beta \to (\beta')^2_{\beta''}$$

is true for $\beta = (2^\alpha)^+$, $\beta' = \alpha^+$ and $\beta'' = \alpha$, and if either β is diminished or β' or β'' is increased then $\beta \to (\beta')^2_{\beta''}$ is not any more a theorem in ZF+Choice.

At first $(2^\alpha) \not\to (\alpha^+)^2_\alpha$ by f or g. Secondly under the assumption of GCH $\alpha^{++} = (2^\alpha)^+$ and $\alpha^{++} \not\to (\alpha^{++})^2_\alpha$ again by f or g, showing that β' can not be increased. Finally if $\beta'' = \alpha^+$ and G.C.H. is assumed then f yields $\alpha^{++} \not\to (3)^2_{\alpha^+}$ and hence $\alpha^{++} \not\to (\alpha^+)^2_{\alpha^+}$.

3. Relations of the simple form

$$\alpha \to (\beta,\gamma)^r$$

(cf. d,e,g,h,i) are studied e.g. in [41]. We mention the following results (p.437 formulae 26-28):

If $\psi(\alpha) = \min\{\gamma \mid \alpha^\gamma > \alpha\}$ then	$\alpha^+ \to (\psi(\alpha),\alpha^+)^2$
but	$\alpha^{\psi(\alpha)} \not\to ((\psi(\alpha))^+,\alpha^+)^2$
and so	$\alpha^+ \not\to ((\psi(\alpha))^+,\alpha^+)^2$.

If we assume G.C.H. then ψ = cf and $\alpha^{\psi(\alpha)} = \alpha^{\mathrm{cf}\alpha} = 2^\alpha = \alpha^+$.

This implies $\alpha^+ \to (\mathrm{cf}\alpha,\alpha^+)^2$

but $\alpha^+ \not\to ((\mathrm{cf}\alpha)^+,\alpha^+)^2$.

4. Cardinals λ for which $\lambda \to (\lambda,\lambda)^r$ are "big" (weakly compact). We will deal with these cardinals in A6.

<u>Applications</u>.

Of the results A4.4 - A4.12 only A4.4, A4.5 and A4.7 are applied in this tract. They are used in the proofs of:

2.7 (p.13) If $X \in \mathcal{T}_2$ then $d(X) \le \exp s(X)$. (A4.7)
2.9 (p.17) If $X \in \mathcal{T}_2$ then $|X| \le \exp\exp s(X)$. (A4.5)
2.10 (p.18) If $X \in \mathcal{S}$ then $|X| \le \exp c(X)$. (A4.4)
2.11 (p.19) If $X \in \mathcal{B}$ then $\forall \xi\ c_\xi(X) \le \exp(\omega_\xi . c(X))$. "
2.15 (p.22) If $X \in \mathcal{T}_1$ then $|X| \le \exp(\psi(X).s(X))$. "
2.16 + CORO If $X \in \mathcal{T}_2$ then $|X| \le \exp(\chi(X).c(X))$. "
4.4 (p.46) If $\phi = h$ or $\phi = z$, $R = X\{R_i : i \in I\}$ and "
$\phi_I R = \sup\{\phi(R_i) : i \in I\}$ then

$$|I|.\phi_I(R) \leq \phi(R) \leq |I|.\ \exp \phi_I(R).$$

4.6 (p.51) $c_I(R) \leq c(R) \leq \exp(c_I(R))$. (A4.4)

4.7 (p.53) $c_{\xi,I}(R) \leq c_\xi(R) \leq \exp(c_{\xi,I}(R))$. "

Also in proving Arhangelskii's theorem 2.21 (p.28), we use the ramification method, which was developed in close relation to the partition calculus.

A4.4 THEOREM [ERDÖS-RADO]

$$(2^\alpha)^+ \to (\alpha^+)^2_\alpha$$

Proof. Let $|H| = (2^\alpha)^+$ and $|H|^2 = \bigcup_{\xi<\alpha} I_\xi$. We will show the existence of a subset T of H, and a $\nu_0 < \alpha$ such that

$$|T| = \alpha^+ \text{ and } [T]^2 \subset I_{\nu_0}\ .$$

Let $R_0 = H$,

$x_0 \in R_0$ (arbitrary)

(for $\nu\leq\alpha^+$) $S_\nu = \alpha^\nu = \{s:\nu\to\alpha\} = \{(\xi_0,\ldots,\xi_\eta,\ldots)_{\eta<\nu}\ |\forall\, \eta<\nu\ \xi_\eta<\alpha\}$.

For $s \in S_\nu$ we write: ν = length(s). For $s \in S_\nu$ and $\zeta < \alpha$ let $[s,\zeta]$ denote the sequence of $S_{\nu+1}$ whose initial segment of length ν is s, and whose last element is ζ. For $\eta < \nu$ $s|\eta$ denotes the initial segment of s of length η (or: the restriction of the function s: $\nu \to \alpha$ to η); $s(\eta)$ denotes the $(\eta+1)$th element of s (the function value of s on η). Suppose we have an ordinal $\nu \leq \alpha^+$ and for each $\eta < \nu$ and each $s' \in S_\eta$ we have already defined a set $R_{s'}$ and a point $x_{s'} \in R_{s'} \cup \{x_0\}$. Then we define R_s for each $s \in S_\nu$ and if $R_s \neq \emptyset$ we choose $x_s \in R_s$ arbitrary, otherwise we put $x_s = x_0$:

1° Case. If ν is a limit ordinal we put

$$R_s = \bigcap_{\eta<\nu} R_{s|\eta}$$

2° Case. If ν is a successor and $s = [s',\xi]$, then we put

$$R_s = \{y \in R_{s'} \mid \{x_{s'}, y\} \in I_\xi\} .$$

This defines R_s and x_s for each $s \in \bigcup\{S_\nu \mid \nu \leq \alpha^+\}$.
We may assume that the partition $\{I_\xi \mid \xi < \alpha\}$ is disjoint (cf (i) on p.102). By induction on ν it is now easy to prove that both

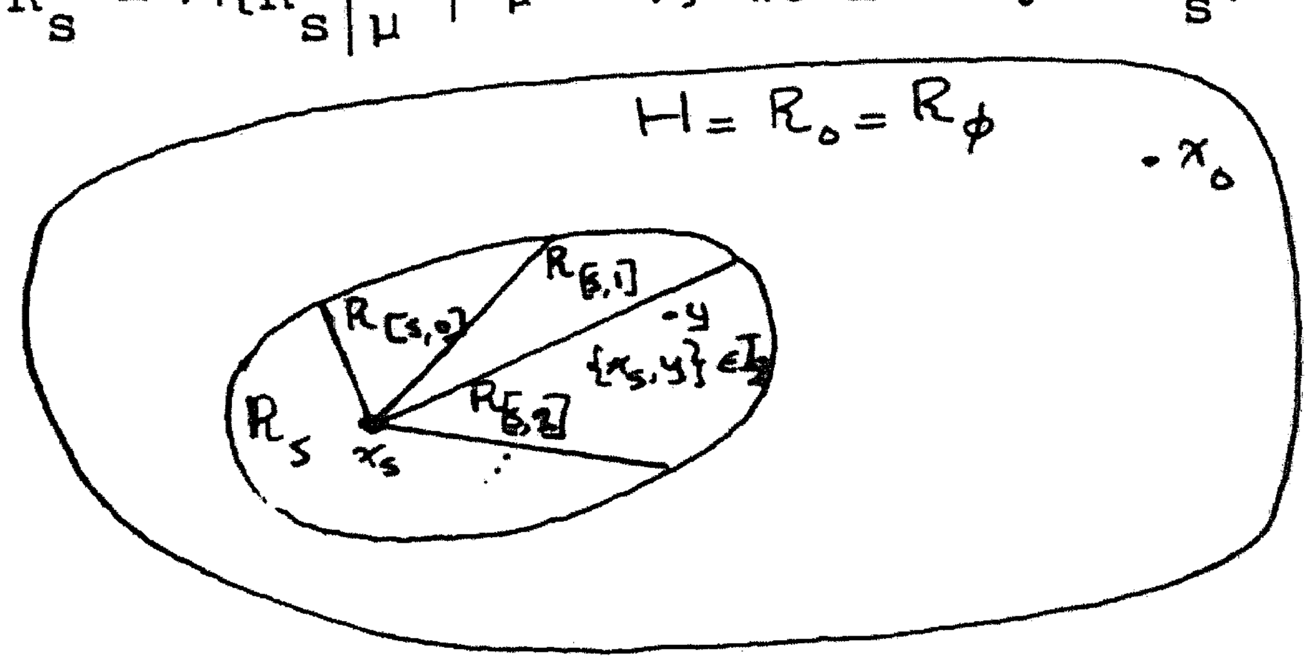
(i_ν) if $s,t \in S_\nu$, $s \neq t$ then $R_s \cap R_t = \emptyset$

(ii_ν) $\bigcup\{R_s \mid s \in S_\nu\} = X \setminus \{x_t \mid \text{length } t < \nu\}$

hold for all $\nu \leq \alpha^+$. The simple proof of (i_ν) as well as the cases $\nu=0$ and ν is a successor of (ii_ν) are left to the reader. So suppose $\nu < \alpha^+$ is limit and (ii_μ) holds for all $\mu < \nu$. If $t \in S_\mu$, $\mu < \nu$ then, by ($ii_{\mu+1}$), $x_t \notin \bigcup\{R_s \mid s \in S_{\mu+1}\}$. By the definition of R_s, for each $s \in S_\nu$ $R_s \subset R_{s|\mu+1}$, so $x_t \notin \bigcup\{R_s \mid s \in S_\nu\}$, proving one inclusion. Now suppose $y \in X \setminus \{x_t \mid \text{length } t < \nu\}$. By our induction hypothesis the set

$$S(y) = \{s \mid \text{length } s < \nu \text{ and } y \in R_s\}$$

meets each S_μ for $\mu < \nu$. By (i_μ) $S(y)$ contains precisely one element of S_μ, say $s(\mu)$. If $\mu < \mu' < \nu$ then $y \in R_{s(\mu')} \subset R_{s(\mu')|\mu}$, hence $s(\mu')|\mu \in S_y$, i.e. $s(\mu')|\mu = s(\mu)$. This means that $S(y)$ consists of all the initial segments of a sequence $s\colon \nu \to \alpha$. By definition of $R_s = \bigcap\{R_{s|\mu} \mid \mu < \nu\}$ we have $y \in R_s$, which proves (ii_ν).



Clearly, the family $\{R_{[s,\xi]} \mid \xi < \alpha\}$ is a partition of $R_s \setminus \{x_s\}$ for each s.
For $y \in R_s \setminus \{x_s\}$ we have $y \in R_{[s,\xi]} \Longleftrightarrow \{x_s, y\} \in I_\xi$.

From (ii_{α^+}) it follows that $\bigcup\{R_s \mid s \in S_{\alpha^+}\} \neq \emptyset$ because $|\{x_t \mid \text{length } t < \alpha^+\}| \leq \Sigma\{|S_\nu| \mid \nu < \alpha^+\} \leq \alpha^+.\alpha^\alpha = 2^\alpha < |H|$. So we know that $R_s \neq \emptyset$ for some $s \in S_{\alpha^+}$. Now consider $H' = \{x_{s|\eta} \mid \eta < \alpha^+\}$. Since $R_s \neq \emptyset$ we have $x_{s|\eta} \in R_{s|\eta}$ and by (ii_η) all $x_{s|\eta}$, for $\eta < \alpha^+$ are different.
For $\eta_1 < \eta_2 < \alpha^+$ we have $x_{s|\eta_2} \in R_{s|\eta_2} \subset R_{s|\eta_1+1}$, hence

$$\{x_{s|\eta_1}, x_{s|\eta_2}\} \in I_{s(\eta_1)} ,$$

i.e. the class of the partition to which $\{x_{s|\eta_1}, x_{s|\eta_2}\}$ belongs is determined only by η_1. This gives a partition of α^+ into α classes - the point inverses of $s: \alpha^+ \to \alpha$ -, and by the regularity of α^+ we can find an $A \subset \alpha^+$ of power α^+ and a $\nu < \alpha$ such that $s(A) = \{\nu\}$, and hence

$$[H'']^2 \subset I_\nu ,$$

where $H'' = \{x_{s|\eta} \mid \eta \in A\}$.

The following theorem and proof are straightforward generalizations of 4.4. As they do not depend on 4.4 we could have skipped this "simple" case. We included 4.4 because the proof of 4.5 is more obscured by technical and notational difficulties, and moreover 4.4 has especially many applications.
The proof of 4.5 may also become more lucid by comparing it to the proof of 4.6, Ramsey's theorem. This last proof can be seen as an application of the proof of 4.5 to finite partitions.

A4.5 THEOREM [ERDŐS-RADO]

$$(\exp^{(r)}\alpha)^+ \to (\alpha^+)^{r+1}_\alpha$$

<u>Proof</u>. The proof will be carried out by induction on r. For $r = 1$ 4.5 equals 4.4. For $r = 0$ 4.5 reads $\alpha^+ \to (\alpha^+)^1_\alpha$, which is equivalent to "each successor cardinal is regular" (c.f. A4.2.1° p101). Note that the proof of 4.4 $(r = 1)$ also uses $\alpha^+ \to (\alpha^+)^1_\alpha$, in the final part.

Let $r \geq 1$ and $(\exp^{r-1}\alpha)^+ \to (\alpha^+)^r_\alpha$ (induction hypothesis).
Assume $|H| = (\exp^{(r)}\alpha)^+$ and $[H]^{r+1} = \bigcup_{\xi<\alpha} I_\xi$ (the I_ξ are disjoint).
Put

$$R_\emptyset = R_0 = H$$

$$x_\emptyset = x_0 \in R_0 \qquad \text{(arbitrary)}$$

(for $\nu \leq (\exp^{(r-1)}\alpha)^+$:) S_ν is the set of all sequences s of length ν such that for $\mu < \nu$ $s(\mu)$ is a function: $[\mu]^{r-1} \to \alpha$. Formally:

$$S_\nu = \underset{\mu<\nu}{X} \alpha^{([\mu]^{r-1})} .$$

Notice that $|S_\nu| \leq \alpha^{|\nu|}$ and $s(i) = \emptyset$ if $i < r-1$, for all $s \in S_\nu$. For $s \in S_\nu$ and $\eta < \nu$, again, $s|\eta$ denotes the initial segment of s of length η.

Suppose we have an ordinal $\nu \leq \alpha^+$ and for each $\eta < \nu$ and each $s' \in S_\eta$ we have already defined a set $R_{s'}$ and a point $x_{s'} \in R_{s'} \cup \{x_0\}$. Then we define R_s for each $s \in S_\nu$, and if $R_s \neq \emptyset$ we choose $x_s \in R_s$ arbitrarily, otherwise we put $x_s = x_0$:

1° Case. If ν is limit we put

$$R_s = \bigcap_{\eta<\nu} R_{s|\eta}$$

2° Case. If ν is a successor, $\nu = \mu+1$ then we define

$$R_s = \{y \in R_{s|\mu} \setminus \{x_{s|\mu}\} \mid \forall \{\eta_0,\dots,\eta_{r-2}\} \in [\mu]^{r-1}$$

$$(\{x_{s|\eta_0},\dots,x_{s|\eta_{r-2}}, x_{s|\mu}, y\} \in I_{s(\mu)\{\eta_0,\dots,\eta_{r-2}\}})\}$$

For $\nu = \mu + 1 = 1,\dots,r-2$ this yields $[\mu]^{r-1} = \emptyset$ and $S_\nu = S_0 = \{\emptyset\}$ and $R_s = R_0 = H$ if $s \in S_\nu$. This defines R_s and x_s for each $s \in \bigcup\{S_\nu | \nu \leq \leq (\exp^{(r-1)}\alpha)^+\}$.

As in 4.4 we will prove by induction on ν that both

(i_ν) if $s,t \in S_\nu$ $s \neq t$ then $R_s \cap R_t = \emptyset$

(ii_ν) $\bigcup\{R_s | s \in S_\nu\} = X \setminus \{x_t | \text{length } t < \nu\}$ hold for all $\nu \leq (\exp^{(r-1)}\alpha)^+$.

As to (i_ν) consider $\mu_0 = \min\{\mu | s(\mu) \neq t(\mu)$. So for some $\eta^* \in [\mu_0]^{r-1}$, $\eta^* = \{\eta_0, \dots, \eta_{r-2}\}$, $s(\mu_0)\eta^* \neq t(\mu_0)\eta^*$. Now $R_{s|\mu_0} = R_{t|\mu_0}$ and

$$R_s \subset R_{s|\mu_0+1} \subset \{y \epsilon R_{s|\mu_0} |$$
$$\{x_{s|\eta_0}, \dots, x_{s|\eta_{r-2}}, x_{s|\mu_0}, y\} \in I_{s(\mu_0)\eta^*}\} =_{\text{def}} A$$

$$R_t \subset R_{t|\mu_0+1} \subset \{y \epsilon R_{s|\mu_0} |$$
$$\{x_{s|\eta_0}, \dots, x_{s|\eta_{r-2}}, x_{s|\mu_0}, y\} \in I_{t(\mu_0)\eta^*}\} =_{\text{def}} B \quad .$$

Now, since $\{I_\xi | \xi < \alpha\}$ is disjoint, we have $R_s \cap R_t \subset A \cap B = \emptyset$, proving (i_ν).
In order to prove (ii_ν), first notice that (ii_0) – and also $(ii_1), \dots, (ii_{r-2})$ – are obvious. Next, assume that, for some ν, $(ii)_\nu$ holds and $s \in S_\nu$. Then for each $y \in R_s \setminus \{x_s\}$ we have a function $f\colon [\nu]^{r-1} \to \alpha$ defined by

$$f(\eta_0, \dots, \eta_{r-2}) = \zeta \Longleftrightarrow \{x_{s|\eta_0}, \dots, x_{s|\eta_{r-2}}, x_s, y\} \in I_\zeta .$$

Clearly then $y \in R_{[s,f]}$. This proves that

$$R_s \setminus \{x_s\} \subset \bigcup\{R_{[s,f]} \mid f \in \alpha^{([\nu]^{r-1})}\} .$$

The other inclusion, $\supset$, is obvious, hence we obtain $(ii_{\nu+1})$.
Finally let ν be limit, and (ii_μ) be true for $\mu < \nu$. If $t \in S_\mu$ where $\mu < \nu$ then, by $(ii_{\mu+1})$ and the definition of R_s:

$$x_t \notin \bigcup\{R_s | s \epsilon S_{+1}\} = \bigcup\{R_{s|\mu+1} | s \epsilon S_\nu\} \supset \bigcup\{R_s | s \epsilon S_\nu\} .$$

If, on the other hand, $y \in X \setminus \{x_t | \text{length } t < \nu\}$ then consider again:

$$S(y) = \{s | \text{length } s < \nu \text{ and } y \in R_s\}.$$

As in 4.4, because of (ii_μ) and (i_μ), $S(y)$ contains precisely one element of S_μ, for each $\mu < \nu$. Again if $\mu < \mu' < \nu$ and $t \in S_\mu$, $t' \in S_{\mu'}$, and $y \in R_t \cap R_{t'}$ then $t = t'|\mu$ because of (i_μ). And this implies that $S(y)$ consists of all the initial segments of a sequence $s \in S_\nu$. By $R_s = \{R_{s|\mu} | \mu < \nu\}$ we have $y \in R_s$, which proves (ii_ν). For short let us put $\beta = (\exp^{(r-1)}\alpha)^+$. From (ii_β) it follows that $\bigcup\{R_s | s \in S_\beta\} \neq \emptyset$, i.e. $R_s \neq 0$ for some $s \in S_\beta$, because

$$|\{x_t | \text{length } t < \beta\}| \leq \sum\{|S_\nu| \ |\nu < \beta\} \leq \sum\{\alpha^{|\nu|} \ |\nu < \beta\} \leq$$

$$\leq \beta.\alpha^{(\exp^{(r-1)}\alpha)} = \exp^r\alpha < |H|.$$

Again consider $H' = \{x_{s|\eta} | \eta < \beta\}$, and notice that all $x_{s|\eta}$ are different because $R_s \neq \emptyset$ and (ii_η) hold. For $\eta_0 < \ldots \eta_r < \beta$ we have:

$$x_{s|\eta_r} \in R_{s|\eta_r} \subset R_{s|\eta_{r-1}+1} \subset \{y \in R_{s|\eta_{r-1}} | \{x_{s|\eta_0}, \ldots, x_{s|\eta_{r-2}}, x_{s|\eta_{r-1}}, y\} \in$$

$$\in I_{s(\eta_{r-1})\{\eta_0, \ldots, \eta_{r-2}\}}\} .$$

This implies that the index $\xi < \alpha$ for which

$$\{x_{s|\eta_0}, \ldots, x_{s|\eta_r}\} \in I_\xi$$

only depends on the "first" r $\eta_0, \ldots, \eta_{r-1}$:

$$\xi = s(\eta_{r-1})\{\eta_0, \ldots, \eta_{r-1}\}.$$

This gives us a r-partition of β into α classes as follows: the point inverses of the map $[\beta]^r \to \alpha$ defined by $\{\eta_0, \ldots, \eta_{r-1}\} \mapsto$ $s(\eta_{r-1})\ \{\eta_0, \ldots, \eta_{r-2}\}$ $(\eta_0 < \eta_1 < \ldots \eta_{r-1})$ are the classes of the partition. By our induction hypothesis there is a $\nu < \alpha$ and an $A \subset \beta$ satisfying $|A| = \alpha^+$ and $[A]^r \to \{\nu\}$. Thus $H'' =_{\text{def}} \{x_{s|\eta} | \eta \in A\}$ satisfies $|H''| = \alpha^+$ and $[H'']^{r+1} \subset I_\nu$

A4.6 THEOREM [RAMSEY [54]]

$$\omega \to (\omega)^r_n$$

Proof. Cf. the previous proof and A.6.6. We will prove Ramsey's theorem by induction on r. For r = 1 it is trivial: a partition of an infinite set into finitely many classes contains at least one infinite class. Suppose the theorem is true for some $r \in \omega$,

H is an infinite set, and

$$[H]^{r+1} = \bigcup_{i=1}^{n} I_i \text{ , with } I_i \cap I_j = \emptyset \text{ for } i \neq j.$$

Put $R_0 = H$

$x_0 \in R_0$ arbitrary.

Now we migth proceed just as in the previous proof. However we only have successor-steps, which makes a more straightforward approach possible. We will first define a sequence of sets $R_1, R_2, \ldots$ and a sequence of points $x_1, x_2, \ldots$ and a sequence of functions $f_1, f_2, \ldots$ satisfying

(i_k) R_k is infinite

(ii_k) $x_{k+1} \in R_{k+1} \subset R_k \setminus \{x_k\}$

(iii_k) $f_k\colon [\{x_1,\ldots,x_{k-1}\}]^{r-1} \to \{1,\ldots,n\}$

(iv_k) $R_{k+1} = \{y \in R_k \mid \forall \{y_1,\ldots,y_{r-1}\} \in [\{x_1,\ldots,x_{k-1}\}]^{r-1}$

$$\{y_1,\ldots,y_{r-1},x_k,y\} \in I_{f_k\{y_1,\ldots,y_{r-1}\}}\} \text{ .}$$

Suppose $R_1,\ldots,R_k$ have been defined satisfying (i) - (iv). Define an equivalence relation ~ on $R_k \setminus \{x_k\}$ by

$y \sim y' \iff \forall \{y_1,\ldots,y_{r-1}\} \in [\{x_1,\ldots,x_{k-1}\}]^{r-1}$ $\{y_1,\ldots,y_{r-1},x_k,y\}$

and $\{y_1,\ldots,y_{r-1},x_k,y'\}$ belong to the same I_i.

As R_k is infinite and $\sim$ has only finitely many equivalence classes, there is one class which is infinite. Thus there exists a $f_{k+1}\colon [\{x_1,\dots,x_{k-1}\}]^{r-1} \to \{1,\dots,k\}$ such that

$$\{y \mid \forall \{y_1,\dots,y_{r-1}\} \in [\{x_1,\dots,x_{k-1}\}]^{r-1} \; (\{y_1,\dots,y_{r-1},x_k,y\} \in$$

$$\in I_{f_k\{y_1,\dots,y_{r-1}\}})\}$$

is infinite. Let this set be R_{k+1} and choose $x_{k+1} \in R_{k+1}$ arbitrarily. Having defined x_k, R_k and f_k for all $k \in \omega$, consider $H = \{x_1,x_2,x_3,\dots\}$. For each $x^* = \{x_{k(1)},\dots,x_{k(r+1)}\} \in [H]^{r+1}$ with $x^* \in I_i$ and $k(1) < \dots < k(r+1)$ the i only depends on $k(1),\dots,k(r)$, because $(iv_{k(r)})$ and $(x_{k(r+1)} \in R_{k(r+1)} \subset R_{k(r)+1})$ imply

$$i = f_{k(r)+1}\{x_{k(1)},\dots,x_{k(r-1)}\}.$$

As in the previous proof, this induces a r-partition of H into n classes: the point inverses of the map $[H]^r \to \{1,\dots,n\}$ defined by

$$\{x_{k(1)},\dots,x_{k(r)}\} \mapsto f_{k(r)+1}\{x_{k(1)},\dots,x_{k(r-1)}\} \text{ (for } k(1) <\dots< k(r)\text{).}$$

By our induction hypothesis there exist $H' \subset H$ and $i \in \{1,\dots,n\}$ such that $[H']^{r-1} \mapsto \{i\}$ for this map, and H' is infinite. Now clearly

$$[H']^{r+1} \subset I_i$$

proving Ramsey's theorem for $r+1$.

A4.7 THEOREM (ERDÖS cf. [36]).

$$\alpha \to (\alpha,\omega)^2 .$$

Proof. We prove this first for <u>regular α</u>.
Let $|S| = \alpha$, $[S]^2 = J_0 \cup J_1$ and suppose $[A]^2 \subset J_0 \Longrightarrow |A| < \alpha$. Let A_0 be a maximal subset of S such that $[A_0]^2 \subset J_0$ (the existence of A_0 follows from the Teichmüller-Tukey lemma). For each $x \in A_0$ we put $S_x = \{y \in S\backslash A_0 \mid \{x,y\} \in J_1\}$. From the maximality of A_0 it follows

that $S\setminus A_0 = \bigcup\{S_x | x \in A_0\}$. Since $|A_0| < \alpha = |S\setminus A_0|$ and α is regular, $\exists x_0 \in A_0 \; |S_{x_0}| = \alpha$. Let A_1 be a maximal subset of S_{x_0} such that $[A_1]^2 \subset J_0$. Continuing by induction we obtain the sequences $(A_n)_n$, $(x_n)_n$, $(S_{x_n})_n$ satisfying:

(i) $\quad A_n$ is a maximal subset of $S_{x_{n-1}}$ with $[A_n]^2 \subset J_0$

(ii) $\quad x_n \in A_n$ is such that $|S_{x_n}| = \alpha$, where $S_{x_n} = \{y \in S_{x_{n-1}} \setminus A_n | \, |\{x_n,y\} \in J_1\}$.

This induction breakes down only if $A_n = \emptyset$ for some n. But then $[S_{x_{n-1}}]^2 \subset J_1$ and then $|S_{x_{n-1}}| = \alpha \geq \omega$. If $A_n \neq \emptyset$ for each n then $\{x_k,x_n\} \in J_1$ for each $k < n < \omega$, because $x_n \in S_{x_{n-1}} \subseteq S_{x_k} \subset \{y|\{x_k,y\} \in J_1\}$. Hence $[\{x_n| \; n \in \omega\}]^2 \subset J_1$.
Now we will prove $\alpha \to (\alpha,\omega)^2$ for <u>singular α.</u> Let $\gamma = cf(\alpha) < \alpha = \sum_{\xi<\gamma} \alpha_\xi$ such that (cf. p.77).

(i) $\quad \forall \xi < \gamma \; \forall \xi' < \xi \quad \gamma < \alpha_{\xi'} < \alpha_\xi < \alpha$ and α_ξ is regular.

Let $|S| = \alpha$ and $[S]^2 = I_0 \cup I_1$. If $x \in S$, $A \subset S$ and $i \in \{0,1\}$ then let

$$C_i(x) = \{y \in S| \; \{x,y\} \in I_i\}$$

and

$$C_i(A) = \bigcup_{x\in A} C_i(x) = \{y \in S| \exists\, x \in A \; \{x,y\} \in I_i\} .$$

If

(ii) $\quad \forall \; H \subset S \; |H| = \alpha \Rightarrow (\exists\, x \in H \; |C_1(x) \cap H| = \alpha)$

then we define inductively sets H_n and points $x_n \in H_n$ for all $n \in \omega$, as follows: $H_0 = S$, $x_0 \in H_0$ such that $|C_1(x_0) \cap H_0| = \alpha$. If H_i, x_i defined for $i < n$ then we let $H_n = C_1(x_{n-1}) \cap H_{n-1}$, and $x_n \in H_n$ such that $|C_2(x_n) \cap H_n| = \alpha$. This is possible because of (ii). It is easily

seen that $[\{x_n \mid n \in \omega\}]^2 \subset I_1$.
So let (ii) be false, i.e.

(iii) $$\exists H \subset S \ |H| = \alpha \wedge (\forall x \in H \ |C_1(x) \cap H| < \alpha) .$$

Assume also:

(iv) for no infinite subset A of S $[A]^2 \subset I_1$.

Let β be a cardinal (e.g. some α_ξ) satisfying

(v) $$\gamma < \beta < \alpha \text{ and } \beta \text{ is regular} .$$

Let $W \subset H$ be a subset of cardinality β. For each $\eta < \gamma$ we let
$W_\eta = \{x \in W \mid \ |C_1(x) \cap H| \leq \alpha_\eta\}$.
Because of (iii) and (i): $\bigcup\{W_\eta \mid \eta < \gamma\} = W$.
Because of (v) : $\exists \eta < \gamma \ |W_\eta| = |W| = \beta$.
By the definition of W_η : $|C_1(W_\eta) \cap H| =$

$$|\bigcup\{C_1(x) \cap H \mid x \in W_\eta\}| \leq \alpha_\eta \cdot \beta < \alpha.$$

Consider $[W_\eta]^2 = ([W_\eta]^2 \cap I_0) \cup ([W_\eta]^2 \cap I_1)$. Since β is regular, $\beta \to (\beta,\omega)^2$. Hence, because of (iv):

(vi) $$\exists W' \subset W_\eta |W'| = \beta \wedge [W']^2 \subset I_0.$$

Clearly this W' also satisfies

(vii) $$|C_1(W') \cap H| \leq |C_1(W_\eta) \cap H| < \alpha$$

Using this procedure we can define by transfinite induction sets M_ξ, $\xi < \gamma$ satisfying

(a) $|M_\xi| = \alpha_\xi$

(b) $[M_\xi]^2 \subset I_0$

(c) $|C_1(M_\xi) \cap H| < \alpha$

(d) $\qquad |\bigcup\{M_\eta \cup (C_1(M_\eta) \cap H) | \eta < \xi\}| < \alpha$ (as follows from (a) $\wedge$ (c))

(e) $\qquad M_\xi \subset H \setminus \bigcup\{M_\eta \cup (C_1(M_\eta) \cap H) | \eta < \xi\}$

At first we choose $W \subset H$ arbitrary, such that $|W| = \beta = \alpha_0$, and let $M_0 = W'$. Notice that (a)-(e) hold. If we have defined M_ξ for some fixed $\xi_0 < \gamma$ and all $\xi < \xi_0$, such that (a)-(e) hold, then because of (d) and $\xi_0 < \gamma = \mathrm{cf}\,\alpha$, $H \setminus \bigcup\{M_\eta \cup (C_1(M_\eta) \cap H) | \eta < \xi_0\}$ has α elements. Let W be any subset of this set such that $|W| = \beta = \alpha_{\xi_0}$ and put $M_{\xi_0} = W'$. Again (a)-(e) hold.
Now let $M = \bigcup\{M_\xi | \xi < \gamma\}$. By (a), (e) and (i) $|M| = \alpha$. We claim that

(viii) $\qquad [M]^2 \subset I_0$.

Let $\{x,y\} \in [M]^2$. If $x,y \in M_\xi$ for some $\xi < \gamma$ then by (b) $\{x,y\} \in I_0$. If $x \in M_\xi$, $y \in M_\eta$ and $\xi < \eta < \gamma$, then because of (e) $y \notin C_1(M_\eta)$, i.e. $\{x,y\} \notin I_1$, and thus $\{x,y\} \in I_0$. This completes the proof.

The following theorem is a strengthening of Erdős' previous theorem for cardinals of the form $(2^\alpha)^+$.

A4.8 THEOREM

$$(2^\alpha)^+ \to ((2^\alpha)^+,\alpha^+)^2 .$$

Proof. Let $|H| = (2^\alpha)^+$ and $[H]^2 = I_0 \cup I_1$, and assume

(i) $\qquad \forall A \subset H\ [A]^2 \subset I_0 \Longrightarrow |A| \leq 2^\alpha$.

We will show

(ii) $\qquad \exists A' \subset H\ [A']^2 \subset I_1 \wedge |A'| = \alpha^+$.

We will define a ramification of H, rather similar to the first part of the proof of A4.4. Let

$$S_\nu = \{s\colon \nu \to 2^\alpha\} \qquad \text{for } \nu \leq \alpha^+,$$

$$R_0 = H,$$

(iii) $\qquad A_s \subset R_s$ be a maximal subset such that $[A_s]^2 \subset I_0$

(for each $s \in \bigcup\{S_\nu \mid \nu \leq \alpha^+\}$ for which R_s is defined).

If ν is an ordinal such that R_s has already been defined for all s of length $< \nu$, then we define R_s for $s \in S_\nu$ as follows:

1° Case. If ν is a limit, $s \in S_\nu$ then we let

$$R_s = \bigcap_{\eta<\nu} R_{s|\eta}$$

2° Case. If ν is a successor, and $s \in S_{\nu-1}$, then we define $R_{[s,\eta]}$ for all $\eta < 2^\alpha$ at once. By (i) $|A_s| \leq 2^\alpha$. Hence we may well-order A_s: $A_s = \{p_\xi \mid \xi < \beta_s\}$, for some $\beta_s \leq 2^\alpha$. For each $x \in R_s \setminus A_s$ we can choose a $\xi < \beta_s \leq 2^\alpha$ such that $\{x, p_\xi\} \in I_1$ (because of the maximality of A_s: (iii)). Define a function $\phi_s\colon R_s \setminus A_s \to 2^\alpha$ in such a way that $\{x, p_{\phi_s(x)}\} \in I_1$ for all $x \in R_s \setminus A_s$, and let

(iv) $$R_{[s,\eta]} = \phi_s^{-1}(\eta) = \{x \in R_s \setminus A_s \mid \phi_s(x) = \eta\} \qquad \text{for } \eta < 2^\alpha.$$

We claim that for some $s_0 \in S_{\alpha^+}$

(v) $$|R_{s_0}| \neq \emptyset\ .$$

Proof of (v).

Notice that $|\bigcup\{A_s \mid \text{length } s < \alpha^+\}| \leq \sum_{\nu<\alpha^+} \sum_{s\in S_\nu} |A_s| \leq \sum_{\nu<\alpha^+} \sum_{s\in S_\nu} 2^\alpha =$

$$= \sum_{\nu<\alpha^+} .(2^\alpha)^\nu . 2^\alpha = 2^\alpha < |H|.$$

Hence we may choose $y \in H \setminus \bigcup\{A_s \mid \text{length } s < \alpha^+\}$. Put

$$S(y) = \{s \in \bigcup S_\nu \mid y \in R_s\}\ .$$

Using the Zorn-lemma, one can prove the existence of a sequence $s_0 \in S(y)$ which is not the initial segment of any other sequence of $S(y)$. We will show that length $s_0 = \alpha^+$. If length $s_0 < \alpha^+$, then $y \in R_{s_0} \setminus A_{s_0}$ by definition of y. Hence, by (iv), $\{y, p_{\phi_{s_0}(y)}\} \in I_1$, i.e.: $y \in R_{[s_0, \phi_{s_0}(y)]}$. This implies $[s_0, \phi_{s_0}(y)] \in S(y)$, contradicting the maximality of s_0.

This proves (v).

For each $\xi < \alpha^+$ we define $x_\xi \in A_{s_0|\xi}$ as follows:

$$x_\xi = p_{s_0}(\xi+1) \in A_{s_0|\xi}\ .$$

Now $A' = \{x_\xi | \xi < \alpha^+\}$ satisfies $[A']^2 \subset I_1$, as follows easily from (iv).

This chapter is concluded by some examples of partitions, which prove the negative theorems A4.9-4.12.

A4.9 THEOREM[GÖDEL]

$$2^\alpha \nrightarrow (3)^2_\alpha$$

<u>Proof</u>. Let $A = \{f: \alpha \to \{0,1\}\}$, and define I_ξ for $\xi < \alpha$ as follows: I_ξ is the set of $\{f,g\} \in [A]^2$ such that ξ is the first ordinal for which $f(\xi) \neq g(\xi)$. Clearly $[A]^2 = \bigcup_{\xi<\alpha} I_\xi$, and for any three functions $f,g,h \in A$ $\{f,g\} \in I_\xi$ and $\{f,h\} \in I_\xi$ implies $g(\xi) = h(\xi)$, and so $\{g,h\} \notin I_\xi$.

A4.10 THEOREM (a) SIERPIŃSKI [58]

$$2^\omega \nrightarrow (\omega_1,\omega_1)^2$$

(b) KUREPA [47]

$$2^\alpha \nrightarrow (\alpha^+,\alpha^+)^2$$

Proof of (a). Let $\prec$ be any well-ordering of the set of real numbers R. Put

$$I_0 = \{\{x,y\} \in [R]^2 \mid x < y \text{ and } x \prec y\}$$

$$I_1 = \{\{x,y\} \in [R]^2 \mid x < y \text{ but } y \prec x\} .$$

Clearly $[R]^2 = I_0 \cup I_1$. Suppose $A \subset R$ and $[A]^2 \subset I_0$ or $[A]^2 \subset I_1$. Then A is a subset of R well-ordered by < or >, and hence A is countable. For suppose A is uncountable and well-ordered by <. Let A^* be the initial segment of A that is order isomorphic to ω_1, and $r = \sup A^*$, ($r \in R \cup \{+\infty\}$). Choose $(r_n)_{n\in\omega}$ in R, converging to r from below. Now for each $n\in\omega$ $A^* \cap (-\infty,r_n)$ is countable, but $A^* = \bigcup_{n\in\omega} A^* \cap (-\infty,r_n)$ is not.

For the proof of (b) we need two well-known lemma's from the theory of completely ordered sets.

Definition. An ordered set A is <u>complete</u> or <u>completely ordered</u> if it has one (and hence all) of the following equivalent properties:

(a) each subset A' of A has an inf which belongs to A (we put $\inf \emptyset = \sup A \in A$).

(b) each subset A' of A has an inf and a sup which belong to A.

(c) A, equipped with the order topology, is compact.

LEMMA A. If A_ξ is a completely ordered set for each $\xi < \nu$, then $A = X\{A_\xi \mid \xi < \nu\}$ is complete with respect to the lexicographic ord (i.e. $(a_\xi)_{\xi<\nu} < (b_\xi)_{\xi<\nu}$ iff $(\alpha_\xi)_{\xi<\nu} \neq (\beta_\xi)_{\xi<\nu}$ and $\alpha_\xi < \beta_\xi$ for the first ξ for which $\alpha_\xi \neq \beta_\xi$).

<u>Proof</u>. We use induction on ν, and so may assume that $X\{A_\xi \mid \xi < \nu'\}$ is complete for all $\nu' < \nu$. Suppose $A' \subset A$. Put $A'_{\nu'} = \{(a_\xi)_{\xi<\nu'} \mid (a_\xi)_{\xi<\nu} \in A'\}$ for all $\nu' < \nu$, and $a(\nu') = \inf A'_{\nu'}$. Suppose ν is a successor. If $a(\nu-1) = (a_\xi)_{\xi<\nu-1}$ for some $(a_\xi)_{\xi<\nu} \in A$,

which is a contradiction.
Assume $A' \subset A$ is such that $|A'| = \alpha^+$, $[A']^r \cap I_1 = \emptyset$. If $\exists \xi_0, \xi_1 \in A'$ such that $\xi_0 < \xi_1$ and $f_{\xi_0} < f_{\xi_1}$, then $\forall \xi \in A \quad \xi > \xi_1 \Longrightarrow f_{\xi_1} < f_\xi$. So the well-ordering of $A'' = \{f_\xi \in A' \mid \xi_1 < \xi < \alpha^+\}$ coincides with the lexicographic order, and by lemma B of A4.12 $|A''| \leq \alpha$. This contradiction shows that $\forall \xi_0, \xi_1 \in A' \quad \xi_0 < \xi_1 \Longrightarrow f_{\xi_1} < f_{\xi_0}$. So the reversed lexicographic order $>$ on A'' coincides with the wellordering by indices. Again lemma B of A4.12 gives us $|A'| \leq \alpha$, contradictory to the assumption.

A4.12 THEOREM. If α is singular and $r \geq 3$, then

$$\alpha \not\rightarrow (\alpha, r+1)^r .$$

<u>Proof</u>. Let $\gamma = \mathrm{cf}\alpha < \alpha = |S|$ and $S = \cup\{S_\xi \mid \xi < \gamma\}$, $S_\xi \cap S_{\xi'} = \emptyset$ and $|S_\xi| < \alpha$ for all $\xi < \xi' < \gamma$. Put

$$I_1 = \{X \in [S]^r \mid \exists \mu, \nu < \gamma \quad |X \cap S_\mu| = r-1 \text{ and } |X \cap S_\nu| = 1\}$$

$$I_0 = [S]^r \setminus I_1 .$$

If $A \subset S$ and $|A| = \alpha$ then $[A]^r \cap I_1 \neq \emptyset$ and if $A \subset S$ and $|A| = r+1$, then $[A]^r \cap I_0 \neq \emptyset$ as is easily seen.

A5 Partition calculus. Canonical sequences

A5.1 In this section λ will be a singular strong limit cardinal (i.e. $\forall\alpha<\lambda\ 2^{\alpha}<\lambda$ and λ is singular). We will study r-partitions of λ. Let us notice first that for each α $\bigcup_{n\in\omega} \exp^{(n)}\alpha$ is a singular strong limit cardinal, whilst under G.C.H. every singular cardinal is strong limit. The results obtainable from the preceding chapter for λ are (by A4.5 and A4.2 3°)

(i) $\qquad \forall r \in \omega \quad \forall\alpha < \lambda \qquad \lambda \to (\alpha)^r_\alpha$

if λ is a strong limit cardinal. Because of $\mathrm{cf}\,\lambda = \min\{\alpha \mid \lambda \not\to \lambda^1_\alpha\} < \lambda$ we have

$$\forall r \in \omega \qquad\qquad \lambda \not\to (\lambda)^r_{\mathrm{cf}(\lambda)}$$

A cardinal μ, for which $\mu \to (\mu)^r_\alpha$, $\alpha < \mu$, is called weakly compact (cf. A6.4) . We will obtain better results than (i) after introducting the following notion:

A5.2 If $|S| = \lambda$ is a singular strong limit, $\mathrm{cf}\,\lambda = \gamma$ and $\mathcal{C}$ is an r-partition of S into disjoint sets, then a sequence of sets $(S_\mu)_{\mu<\gamma}$ is called

<u>canonical</u> with respect to $\mathcal{C}$ if

(i) the S_μ, $\mu < \gamma$ are disjoint

(ii) $(|S_\mu|)_{\mu<\gamma}$ is a strictly increasing sequence of cardinals converging to λ

(iii) if $X, Y \in [\bigcup_{\mu<\gamma} S_\mu]^r$ are such that

$$\forall \mu<\gamma \qquad |X \cap S_\mu| = |Y \cap S_\mu|$$

then

$$\exists! C \in \mathcal{C} \quad X \in C \text{ and } Y \in C.$$

Notice that (iii) implies e.g.:

(iv) $\qquad \forall \mu<\gamma \ \ \exists! C \in \mathcal{C} \quad [S_\mu]^r \subset C.$

A5.3 LEMMA. If $\{\sim_\xi \mid \xi < \beta\}$ is a family of equivalence relations on a set S, such that each $\sim_\xi$ induces at most α equivalence classes in S, then the equivalence relation $\sim$ defined by

$$x \sim y \ \text{ iff } \ \forall \xi < \beta \ \ x \sim_\xi y$$

induces at most α^β equivalence classes.

<u>Remark</u>. This is the sharpest possible estimation: Consider $S = \alpha^\beta = \{f \mid f:\beta \to \alpha\}$ and define $\sim_\xi$ for $\xi < \beta$ by

$$f \sim g \ \text{ iff } \ f(\xi) = g(\xi).$$

<u>Proof</u>. For each $\xi < \beta$ let $\{A^\xi_\eta \mid \eta < \alpha\}$ be the family of equivalence classes of $\sim_\xi$, if necessary supplemented by empty sets. It is easily seen that for each $f: \beta \to \alpha$ the set

$$\cap\{A^\xi_{f(\xi)} \mid \xi < \beta\}$$

(is empty, or) consists of $\sim$-equivalent elements, whilst

$$S = \cup\{\cap\{A^\xi_{f(\xi)} \mid \xi < \beta\} \mid f: \beta \to \alpha\}$$

because for each $x \in S$ we can define a $f\colon \beta \to \alpha$ such that $x \in A^{\xi}_{f(\xi)}$, $\xi < \beta$.

A5.4 MAIN THEOREM. (The Canonization-lemma [39])
<u>For every set S of power λ (singular strong limit) and each disjoint r-partition $\mathcal{C}$ of S such that $|\mathcal{C}| = \alpha < \lambda$ there exists a canonical system with respect to $\mathcal{C}$.</u>

<u>Proof</u>. At first we let $(r_1,\dots,r_s)$ be a fixed partition of r, i.e. $r_1 + \dots r_s = r$. For $0 \leq k \leq s$ we define: $(S_\mu)_{\mu<\gamma}$ is $(r_1,\dots,r_s,k)$-canonical with respect to $\mathcal{C}$ iff

(i) the S_μ, $\mu < \gamma$ are disjoint

(ii) $(|S_\mu|)_{\mu<\gamma}$ is a strictly increasing sequence converging to λ.

(iii) $(r_1,\dots,r_s,k)$: If $X, Y \in [\bigcup_{\mu<\gamma} S_\mu]^r$ are such that for some $\mu_1 < \dots \mu_s$

$$X \cap S_{\mu_i} = Y \cap S_{\mu_i} \qquad \text{for } i = 1, \dots, k$$

and

$$|X \cap S_{\mu_i}| = |Y \cap S_{\mu_i}| = r_i \text{ for } i = 1, \dots, s$$

then $\exists! C \in \mathcal{C}$ $X \in C$ and $Y \in \mathcal{C}$.

Now we use the following lemma, which will be proved later:

<u>Lemma</u> A. If $(S_\mu)_{\mu<\gamma}$ is a $(r_1,\dots,r_s,k)$-canonical system for some fixed $(r_1,\dots,r_s,k)$, $1 \leq k \leq s$, then it has a refinement $(S'_\mu)_{\mu<\gamma}$ which is $(r_1,\dots,r_s,k-1)$-canonical.

If lemma A is assumed then the proof of the main theorem goes as follows:

Any sequence $(S_\mu)_{\mu<\gamma}$ which satisfies (i) and (ii) is $(r_1,\dots,r_s,s)$-canonical for every partition $(r_1,\dots,r_s)$ of r. Any refinement of an $(r_1,\dots,r_s,k)$-canonical system which satisfies (ii), is again $(r_1,\dots,r_s,k)$-canonical. Thus if we apply lemma A a finite number of times (less than $r.2^{(r^2)}$) we can obtain a sequence $(S^1_\mu)_{\mu<\gamma}$ which is $(r_1,\dots,r_s,0)$-canonical for all partitions $(r_1,\dots,r_s)$ of r simultaneously. It is easy to see that this system is canonical with respect to $\mathcal{C}$.

Proof of lemma A.

For each $\xi < \gamma$ and $r' \leq r$ we choose a fixed r'-element subset $S_\xi(r')$ of S_ξ.

Assume that for some $\mu < \gamma$ and all $\xi < \mu$ the S'_ξ have been defined already. Let $f\colon \mu \to \gamma$ be such that $S'_\xi \subset S_{f(\xi)}$. Define $f(\mu)$ such that

$$f(\mu) > \sup\{f(\xi) \mid \xi < \mu\} \text{ and}$$

$$(1) \qquad |S_{f(\mu)}| \geq \exp^r(\beta)$$

for some $\beta < \lambda$ which will be chosen suitably:

$\beta = |S_\mu| \exp(\alpha.\gamma.\sum \{|S'_\xi| \mid \xi < \mu\})$.

Now we define an equivalence relation $\sim_{X,\phi}$ on $[S_{f(\mu)}]^{r_k}$ for each $X \in [\cup\{S'_\xi \mid \xi < \mu\}]^{r_1+\dots r_{k-1}}$ and each $\phi\colon\{k+1,\dots s\} \to \gamma$ satisfying $f(\mu) < \phi(k+1) < \dots \phi(s) < \gamma$ as follows. If $y, y' \in [s_{f(\mu)}]^{r_k}$, then

$$Y \sim_{X,\phi} Y' \text{ if}$$

$$\exists! C \in \mathcal{C} \quad X \cup Y \cup S_{\phi(k+1)}(r_{k+1}) \cup \dots S_{\phi(s)}(r_s) \in C \text{ and}$$

$$X \cup Y' \cup S_{\phi(k+1)}(r_{k+1}) \cup \dots S_{\phi(s)}(r_s) \in C.$$

Each equivalence relation $\sim_{X,\phi}$ splits $[S_{f(\mu)}]^{r_k}$ into at most $|\mathcal{C}| = \alpha$ classes, and the number of equivalence relations $\sim_{X,\phi}$ is at most

$$\varepsilon = \sum\{|S'_\xi| \mid \xi < \mu\} . \gamma < \lambda.$$

Thus the coarsest partition which refines all these equivalence classes consists of not more than $\alpha^\varepsilon \leq 2^{\alpha\varepsilon} < \lambda$ classes (A5.3). Put

$\beta = (2^{\alpha\varepsilon}) . |S_\mu|$, (cf. 1).

Because by A4.5 $(\exp^r\beta) \to (\beta^+)^r_\beta$ we can find a subset $S'_\mu \subset S_{f(\mu)}$ such that

$$|S'_\mu| = \beta^+ > |S_\mu| \text{ and}$$

(2) all elements of $[S']^r$ are equivalent under all $\sim_{X,\phi}$.
This completes the definition of the sequence $(S'_\mu)_{\mu<\gamma}$. Next we prove that it is $(r_1,\ldots,r_s,k-1)$-canonical.
Conditions (i) and (ii) (see A5.4 p.127) are clearly fullfilled.
Let $X, Y \in [\cup\{S'_\xi \mid \xi < \gamma\}]^r$ and for some $\xi_1 < \ldots \xi_s$

$$X \cap S'_\xi = Y \cap S'_\xi \qquad \text{for } \xi = \xi_1,\ldots,\xi_{k-1}$$

$$|X \cap S'_{\xi_i}| = |Y \cap S'_{\xi_i}| = r_i \quad \text{for } i = 1,\ldots,s$$

and $\quad X \in C \in \mathcal{C}.$

Then

$$X = (X\cap S'_{\xi_1})\cup\ldots(X\cap S'_{\xi_{k-1}}) \cup (X\cap S'_{\xi_k}) \cup (X\cap S'_{\xi_{k+1}})\cup\ldots(X\cap S'_{\xi_s}) \in C,$$

and by the $(r_1,\ldots,r_s,k)$-canonicity of $(S_\xi)_{\xi<\mu}$:

$$(X\cap S'_{\xi_1})\cup\ldots(X\cap S'_{\xi_{k-1}}) \cup (X\cap S'_{\xi_k}) \cup S_{f(\xi_{k+1})}(r_{k+1})\cup\ldots S_{f(\xi_s)}(r_s) \in \dot{C}.$$

Then by definition of S'_{ξ_k} (cf (2)):

$$(X\cap S'_{\xi_1})\cup\ldots(X\cap S'_{\xi_{k-1}}) \cup (Y\cap S'_{\xi_k}) \cup S_{f(\xi_{k+1})}(r_{k+1})\cup\ldots S_{f(\xi_s)}(r_s) \in C.$$

Again because $(S_\xi)_{\xi<\mu}$ is $(r_1,\ldots,r_s,k)$-canonical:

$$Y = (X\cap S'_{\xi_1})\cup\ldots(X\cap S'_{\xi_{k-1}}) \cup (Y\cap S'_{\xi_k}) \cup (Y\cap S'_{\xi_{k+1}})\cup\ldots(Y\cap S'_{\xi_s}) \in C.$$

This completes the proof of lemma A.

As a corollary to the main theorem we have

A5.5 THEOREM. <u>If λ is a singular, strong limit and cf $\lambda = \gamma$, then</u>

$$\lambda \to (\lambda,\beta_1,\ldots,\beta_\nu,\ldots)^2_{\nu<\alpha} \quad \underline{\text{iff}}$$
$$\gamma \to (\gamma,\beta_1,\ldots,\beta_\nu,\ldots)^2_{\nu<\alpha}\ .$$

<u>Remark</u>. Notice that for r-partitions with $r \geq 3$ we have $\alpha \not\to (\alpha,r+1)^r$

for singular and for successor α by A4.11 and A4.12. So there exists no non-trivial generalization of A5.5 for $r \geq 3$.

Proof. Sufficiency. Notice that $\alpha < \gamma < \lambda$ and $\forall\nu < \alpha \quad \beta_\nu \leq \gamma$. Let $\{I_\nu \mid \nu < \alpha\}$ be any disjoint 2-partition of λ. By the main theorem there is a sequence $(S_\mu)_{\mu<\gamma}$ in S which is canonical with respect to $\{I_\nu \mid \nu < \alpha\}$. Assume that $|S_\mu| > \gamma$ for all $\mu < \gamma$ and that for any $H \subset S$ and $\nu \in [1,\alpha)$

(i) $\qquad [H]^2 \subset I_\nu \Rightarrow |H| < \beta_\nu \leq \gamma < |S_\mu|.$

By A4.2 (iv) this implies

$$[S_\mu]^2 \subset I_0 \text{ for each } \mu < \gamma.$$

Choose one point $p_\mu \in S_\mu$ for each $\mu < \gamma$ and let

$$S' = \{p_\mu \mid \mu < \gamma\}.$$

Because of (i) and $\gamma \to (\gamma,\beta_1,\ldots,\beta_\nu,\ldots)^2_\alpha$ there exists a $S'' \subset S'$ such that

$$[S'']^2 \subset I_0 \text{ and } |S''| = \gamma.$$

Consider $X = \bigcup\{S_\mu \mid p_\mu \in S''\}$. This X has power λ, and satisfies $[X]^2 \subset I_0$.

Necessity.

If $\gamma \nrightarrow (\gamma,\beta_1,\ldots,\beta_\nu,\ldots)$ then $\exists S,\ I_\nu \quad \nu < \alpha$ such that $|S| = \gamma$ and $[S^2] = \cup\{I_\nu \mid \nu < \alpha\}$ and $\forall A \subset S$.

(ii) $\qquad [A]^2 \subset I_0 \Rightarrow |A| < \gamma$

(iii) $\qquad [A]^2 \subset I_\nu \Rightarrow |A| < \beta_\nu \quad$ for $\nu \in [1,\alpha)$.

Let us order S in type γ: $S = \{s_\xi \mid \xi < \gamma\}$. Let $(S_\mu)_{\mu<\gamma}$ be a sequence of disjoint sets of increasing cardinality converging to λ. We define a 2-partition $\{I^*_\nu \mid \nu < \alpha\}$ on $\cup\{S_\mu \mid \mu < \gamma\}$ by

(iv) $I_0^* = \{\{x,y\} \mid \exists\mu \;\; x \in S_\mu \wedge y \in S_\mu\} \cup$

$\cup\{\{x,y\} \mid \exists\mu,\ \mu'<\gamma\,(\mu<\mu' \wedge x\in S_\mu \wedge y\in S_{\mu'} \wedge \{s_\mu,s_{\mu'}\} \in I_0)\}$.

(v) $I_\nu^* = \{\{x,y\} \mid \exists\mu,\ \mu'<\gamma\,(\mu<\mu' \wedge x\in S_\mu \wedge y\in S_{\mu'} \wedge \{s_\mu,s_{\mu'}\} \in I_\nu)\}$.

$\nu \in [1,\alpha)$.

Notice that the sequence $(S_\mu)_{\mu<\gamma}$ is canonical with respect to $\{I_\nu^* \mid \nu < \alpha\}$.
Let $X \subset \cup\{S_\mu \mid \mu < \gamma\}$ be homogeneous for $\{I_\nu^* \mid \nu < \alpha\}$.
If $|X \cap S_\mu| > 1$ for any $\mu < \gamma$ then $[X]^2 \subset I_0$. Now if $|X| = \lambda$ then $A =_{\mathrm{def}} \{s_\mu \in S \mid X \cap S_\mu \neq \emptyset\}$ has at least γ elements and by (iv) $[A]^2 \subset I_0$, contradictory to (ii). Thus $|T| < \lambda$.
If $|X \cap S_\mu| \leq 1$ for all $\mu < \gamma$ and $[X]^2 \subset I_\nu$ for some $\nu < \alpha$ then $A =_{\mathrm{def}} \{s_\mu \in S \mid X \cap S_\mu \neq \emptyset\}$ satisfies $[A]^2 \subset I_\nu$ because of (v).
Now (iii) implies $|A| = |X| < \beta_\nu$.
Thus $\lambda \not\rightarrow (\lambda,\beta_1,\ldots,\beta_\nu,\ldots)_{\nu<\alpha}$.

A5.6 <u>Application</u>.

A5.4 is used to prove 3.2 (p.39):
If X is a Hausdorff space and $|X| = \lambda$ is singular strong limit then X contains a discrete subspace of power λ.
See also 6.6 and the remark at the end of 3.2.

A6 Large cardinals

A6.1 A cardinal α is a strong limit cardinal if $\forall\beta < \alpha \quad 2^\beta < \alpha$. A regular limit cardinal is called weakly inaccessible. A regular strong limit cardinal is called (strongly) inaccessible.

Notice that under GCH each limit is strong limit, hence weakly inaccessible and strongly inaccessible are equivalent in this case. Moreover if we have a model of ZF + choice + GCH in which a (smallest) inaccessible cardinal α exists, then it can easily be checked that the sets of cardinality $< \alpha$ also constitute a model of ZF + choice + GCH, in which, however, no inaccessible cardinals exist.

So it is consistent (with ZF, or with ZF + choice + GCH) to assume that no inaccessible cardinals exist. However it is not (yet) proved that it is consistent to assume the existence of inaccessible cardinals. Yet this will not prevent us from studying these "large" cardinals.

A cardinal λ is measurable if there exists a non trivial $<\lambda$-additive measure $\mu: \mathcal{P}(S) \to \{0,1\}$ on a (any) set S of cardinal λ, i.e.:

(i) μ is a function $\mathcal{P}(S) \to \{0,1\}$

(ii) $\forall p \in S \quad \mu\{p\} = 0$

(iii) $\mu(S) = 1$

(iv) If $\{X_\xi \mid \xi < \alpha\} \subset \mathcal{P}(S)$ with $\alpha < \lambda$ is a disjoint family, then $\mu(\bigcup_{\xi<\alpha} X_\xi) = \sum_{\xi<\alpha} \mu(X_\xi)$.

It is easily verified that the sets of measure 1 form an ultrafilter on S which is closed under $<\lambda$ intersections. Conversely, each free ultrafilter on S which is closed under $<\lambda$ intersections defines a measure with properties (i), (ii), (iii) and (iv).
We first prove theorems about measurable cardinals:

A6.2 THEOREM [59] Each measurable cardinal is strongly inaccessible.

Proof. Suppose $|S| = \lambda$, $\mu: \mathcal{P}(S) \to \{0,1\}$ fulfills (i) - (iv), $\lambda = \sum_{\xi<\mathrm{cf}\lambda} \lambda_\xi$ and $\mathrm{cf}\,\lambda < \lambda$. S is union of $<\lambda$ subsets of power $<\lambda$. By (ii) and (iv), each of these subsets has measure 0. By (iv), their union S has measure 0, contradicting (iii). Hence λ is regular.
Suppose $\alpha < \lambda \leq 2^\alpha$. We may suppose $S \subseteq \{f \mid f: \alpha \to \{0,1\}\}$, that is: S consists of sequences 0's and 1's of length α.
For each $\xi < \alpha$ define $i_\xi \in \{0,1\}$ such that $\mu\{f \in S \mid f(\xi) = i_\xi\} = 1$.
Let f_0 be defined by $f_0(\xi) = i_\xi$ for all $\xi < \alpha$.
Now $\mu(S) = 1 \leq \mu(\{f_0\}) + \sum_{\xi<\alpha} \mu\{f \in S \mid f(\xi) \neq i_\xi\} = 0 + \sum_{\xi<\alpha} 0 = 0$.
Contradiction with (iii). Hence λ is strong limit.

A6.3 A cardinal λ is called σ-measurable if there exist S, μ with $|S| = \lambda$ and $\mu: \mathcal{P}(S) \to \{0,1\}$ satisfying (i), (ii) and (iii) from the definition of measurable and (iv)': μ is σ-additive ($\alpha = \omega_0$ instead of $\alpha < \lambda$).
Obviously, ω is measurable, but not σ-measurable.

THEOREM [59]
The first σ-measurable cardinal is measurable; i.e. the first σ-measurable cardinal equals the first uncountable measurable cardinal.

Proof. Suppose λ is the first σ-measurable cardinal, $|S| = \lambda$, $\mu: \mathcal{P}(S) \to \{0,1\}$ fulfills (i), (ii), (iii), (iv)' but not (iv). Then there is a smallest $\rho < \lambda$ and a disjoint family $\{S_\xi : \xi < \rho\}$ such that $\forall \xi < \rho: \mu(S_\xi) = 0$ and $\mu(\bigcup_{\xi<\rho} S_\xi) = 1$ (observe that one of each two disjoint subsets of S must have measure 0).
Define $\mu': \mathcal{P}(\{\xi : \xi<\rho\}) \to \{0,1\}$ as follows:

$$\mu'(x) = i \quad \text{iff} \quad \mu(\bigcup_{\xi\in x} S_\xi) = i.$$

Trivially (i) - (iii) and (iv)' are fulfilled by μ', i.e. μ' is σ-measurable, contradicting the minimality of λ.

Remark. λ is σ-measurable and $\lambda < \lambda'$ implies λ' is σ-measurable. Thus if λ_0 denotes the first uncountable measurable cardinal then (λ is σ-measurable) $\Longleftrightarrow \lambda \geq \lambda_0$.

A6.4 A cardinal λ is weakly compact if $\forall r < \omega \ \forall \alpha < \lambda$: $\lambda \to (\lambda)^r_\alpha$. It can be shown that this is equivalent to $\lambda \to (\lambda,\lambda)^2$.
In 3.2 (p.40) the relation $\lambda \to (\lambda)^3_4$ for weakly compact λ is used to show that each T_2-space of a weakly compact power has a discrete subspace of the same power.
Without proof we mention the following topological characterization of weakly compact cardinals (see [50]):

THEOREM. λ is weakly compact $\Longleftrightarrow$ the product of λ spaces which are λ-compact and of weight $\leq \lambda$ is again λ-compact.
Here λ-compactness means that every open covering has a subcover of power less than λ.
Ramsey's theorem says that ω is weakly compact. It is not provable that there exist uncountable weakly compact cardinals, as is implied by the following theorem:

THEOREM. Each weakly compact cardinal λ is strongly inacessible.

Proof. Since cf $\lambda = \min\{\alpha: \ \lambda \not\to (\lambda)^1_\alpha\}$; λ must be regular. Furthermore, suppose that $\exists\alpha[\alpha < \lambda \leq 2^\alpha]$. Then $2^\alpha \not\to (\alpha^+)^2_2$ [A4.10] implies $\lambda \not\to (\lambda)^2_2$. Hence λ must be strong limit.

Strong inaccessibility is much weaker than weak compactness (this we will not prove). Moreover we have:

A6.5 THEOREM. Every measurable cardinal λ is weakly compact.

Proof. By induction on r. For r = 1, the regularity of λ gives us $\lambda \to (\lambda)^1_\alpha$ if $\alpha < \lambda$. So suppose $\lambda \to (\lambda)^r_\alpha$ if $\alpha < \lambda$ and let

$|H| = \lambda$, $[H]^{r+1} = \cup\{I_\nu \mid \nu < \alpha\}$ and $\mu: \wp(H) \to \{0,1\}$ be a $<\lambda$-additive measure. We define $R_\eta \subset H$ and $x_\eta \in R_\eta \backslash R_{\eta+1}$ inductively so that $\mu(R_\eta) = 1$, for $\eta < \lambda$. Let $R_0 = H$ and $x_0 \in R_0$ arbitrary. Assume $\eta < \lambda$ and R_ζ, x_ζ have been defined and $\mu(R_\zeta) = 1$, for $\zeta < \eta$.
If η is limit, put $R_\eta = \cap\{R_\zeta \mid \zeta < \eta\}$ and $x_\eta \in R_\eta$ be arbitrary.
Because μ is η-additive $\mu(R_\eta) = \mu(R_0) - \Sigma\{\mu(R_\zeta \backslash R_{\zeta+1}) | \zeta < \eta\} = 1-0 = 1$.
If η is a successor, then define an equivalence relation $\sim_\eta$ on R_η by: $x \sim_\eta y$ iff

$\forall\{\eta_0,\ldots,\eta_{r-1}\} \in [\eta]^r \quad \{x_{\eta_0},\ldots,x_{\eta_{r-1}},x\}$ and $\{x_{\eta_0},\ldots,x_{\eta_{r-1}},y\}$

belong to the same I_ξ, $\xi < \alpha$.
By lemma A5.3 $\sim_\eta$ has at most $\alpha^{|\eta|} \leq 2^{\alpha.|\eta|} < \lambda$ equivalence classes.
Thus exactly one of these has measure one. Take this to be $R_{\eta+1}$, and choose $x_{\eta+1} \in R_{\eta+1}$ arbitrarily.
Having defined R_η and x_η for all $\eta < \lambda$, take $H' = \{x_\eta \mid \eta < \lambda\}$.
According to the construction there is a ϕ: $[\lambda]^r \to \alpha$ such that if $\eta_0 < \eta_1 < \ldots \eta_r < \lambda$ then $\{x_{\eta_0},\ldots,x_{\eta_r}\} \in I_{\phi\{\eta_0,\ldots,\eta_{r-1}\}}$. Since $\lambda \to (\lambda)^r_\alpha$ $\exists A \subset \lambda$ and $\nu < \alpha$ such that $|A| = \lambda$ and $\phi[A]^r = \{\nu\}$. Then $[\{x_\eta \mid \eta \in A\}]^{r+1} \subset I_\nu$.

A6.6 <u>Corollary</u> RAMSEY = A4.6

$$\underline{\omega \to (\omega)^r_n \text{ for } r,\ n < \omega.}$$

<u>Proof</u>. ω is measurable, for we can extend $\{\{n \mid n < m\}\ m < \omega\}$ to a non-trivial ultrafilter. The corresponding measure is $<\omega$-additive.

A6.7 <u>Definition</u>

Let *: Card $\to$ Card be such that $\alpha < \alpha^*$ (e.g.: $^+$, exp).
λ is <u>*-inaccessible</u> if

(i) λ is regular
(ii) $\alpha < \lambda \Longrightarrow \alpha^* < \lambda$

(e.g.: strongly inaccessible = exp-inaccessible, weakly inaccessible = $^{+}$-inaccessible).

A6.7 THEOREM. <u>If λ is measurable and *: Card $\to$ Card is such that $\alpha < \alpha^*$ then λ is not the first *-inaccessible cardinal</u>.

<u>Proof</u>. Let $C = \{\alpha \mid \alpha < \lambda\} = \{$all cardinals $<\lambda\}$. Since λ is weakly inaccessible, C and λ are order isomorphic.
If we choose a measure μ on C with (i) - (iv), then we can define the following equivalence relation $\sim_\mu$ on $\{f \mid f\colon C \to C\}$: $f \sim_\mu g$ iff $\mu(\{\alpha \in C\colon f\alpha = g\alpha\}) = 1$. The equivalence class of a function f is denoted by $\overline{f}$, the equivalence class of the constant function which assumes the value α everywhere, by $\overline{\alpha}$.
$\overline{C} =_{\mathrm{def}} \{\overline{f} \mid f\colon C \to C\}$. Sometimes we write C for $\{\overline{\alpha} \mid \alpha \in C\}$.
Define $\overline{f} \prec \overline{g}$ if $\mu(\{\alpha \mid f\alpha < g\alpha\}) = 1$. This definition is independent of the choice of f and g and determines a linear ordering on $\overline{C}$ (which on C coincides with the natural ordering), as is easily checked by using the fact that $\{x \subseteq C \mid \mu(x) = 1\}$ is an ultrafilter.
In fact, $\prec$ defines a well-ordering on $\overline{C}$, for suppose $\overline{f}_1 \succ \overline{f}_2 \succ \ldots$ for some sequence in $\overline{C}$. Then $\forall n < \omega\colon \mu(\{\alpha \mid f_n(\alpha) > f_{n+1}(\alpha)\}) = 1$. The σ-additivity of μ implies that $\mu(\{\alpha \mid \forall n < \omega\colon f_n(\alpha) > f_{n+1}(\alpha)\}) = 1$. Hence $\exists\alpha\ \forall n < \omega\colon f_n(\alpha) > f_{n+1}(\alpha)$. But this contradicts the well-ordering of C.
Moreover, the $<\lambda$-additivity of μ gives us that C is an initial segment of $\overline{C}$, for suppose $\overline{f} < \overline{\alpha}_0$ for some $f\colon C \to C$ and $\alpha_0 < \lambda$. Then $\mu(\{\alpha \mid f(\alpha) < \alpha_0\}) = \sum_{\beta<\alpha_0} \mu(\{\alpha \mid f(\alpha) = \beta\}) = 1$. Hence $\exists\beta < \alpha_0$: $\mu(\{\alpha \mid f(\alpha) = \beta\}) = 1$. Also, $C \neq \overline{C}$, because for the identity map $\overline{\mathrm{id}_C} = \overline{\mathrm{id}} \in \overline{C}\setminus C$. Hence $\overline{C}\setminus C \neq \emptyset$ and has a least element. We may even change μ so as to make $\overline{\mathrm{id}} = \min(\overline{C}\setminus C)$. For let $\overline{f} = \min(\overline{C}\setminus C)$. Define $\mu'\colon \mathcal{P}(C) \to \{0,1\}$ by $\mu'(x) = \mu(f^{-1}(x))$ for $x \subseteq C$. We leave the proof that μ' is a measure to the reader and we only show that $\overline{\mathrm{id}} = \min(\overline{C}\setminus C)$ relative to μ'. Suppose g is such that $\mu'(\{\alpha \mid g\alpha < \alpha\}) = 1$. By the definition of μ',

$\mu'(\{\alpha \mid g\alpha < \alpha\}) = \mu(\{\beta \mid gf\beta < f\beta\}) = 1$, hence $\overline{gf} \prec \overline{f}$ relative to μ.
By the choice of f this means that $\overline{gf} = \overline{\alpha_0}$ for some $\alpha_0 \in C$. Now
$\mu'(\{\alpha \mid g\alpha = \alpha_0\}) = \mu(\{\beta \mid gf\beta = \alpha_0) = 1$, hence $\overline{g} \in C$ relative to μ'.
From now on we assume that $\overline{id} = \min(\overline{C}\setminus C)$.
Define $A_r = \{\alpha \in C \mid \alpha \text{ is regular}\}$ and $A_s = \{\alpha \in C \mid \alpha \text{ is singular}\}$.
Then exactly one of A_r and A_s has measure 1.
Assume that λ is the first *-inaccessible cardinal. We shall prove that $\mu(A_r) \neq 1$ and $\mu(A_s) \neq 1$, which is a contradiction.
Assume that $\mu(A_r) = 1$.
Define g: $C \to C$ as follows:
$g(\alpha) = 0$ if α is singular
$g(\alpha) = \beta$ for some $\beta < \alpha \leq \beta^*$ if α is regular. Such a β exists since α is not *-inaccessible; and $\beta^* < \lambda$ because λ is *-inaccessible.
Then $\overline{g} \prec \overline{id}$, so $\exists \beta < \lambda$: $\mu(\{\alpha \mid \beta < \alpha \leq \beta^*\}) = 1 = \mu(\{\alpha \mid \alpha \leq \beta^*\})$.
Thus we have a set of power $<\lambda$ with measure 1. Contradiction.
Now we assume $\mu(A_s) = 1$.
Define $g(\alpha) = cf(\alpha)$, $\alpha < \lambda$; then $\overline{g} \prec \overline{id}$ hence $\exists \beta < \lambda$:
$\mu(\{\alpha \mid cf(\alpha) = \beta\}) = 1$. Put $H = \{\alpha \mid cf(\alpha) = \beta\}$.
For each $\alpha \in H$ we choose a strictly increasing sequence $(\phi(\alpha,\xi))_{\xi<\beta}$ of cardinals, converging to α.
Define $h_\xi(\alpha) = \phi(\alpha,\xi)$ for $\alpha \in H$, $\xi < \beta$
$h_\xi(\alpha) = 0$ for $\alpha \notin H$, $\xi < \beta$.
Then $\overline{h_\xi} \prec \overline{id}$, hence $\exists \beta_\xi \in C$: $\overline{h_\xi} \sim_\eta \overline{\beta_\xi}$ for some $\beta_\xi < \lambda$; $\xi < \beta$.
Moreover, $\xi \prec \zeta \Longrightarrow \overline{h_\xi} \prec \overline{h_\zeta} \Longrightarrow \beta_\xi < \beta_\zeta$.
It is easily seen that $\overline{\sup_{\xi<\beta} h_\xi} = \overline{\sup_{\xi<\beta} \beta_\xi}$ but $\sup_{\xi<\beta} \beta_\xi < \lambda$ and this is impossible since $(h_\xi)_{\xi<\beta}$ converges pointwise to id on the set H for which $\mu(H) = 1$. Since the assumption that $\mu(A_s) = 1$ is also contradictory, we conclude that λ is not the first *-inaccessible cardinal.

A6.8 COROLLARY. <u>If the measurable cardinal λ is *-inaccessible, then $|\{\alpha \in \lambda \mid \alpha \text{ is } ^*\text{-inaccessible}\}| = \lambda$.</u>

<u>Proof</u>. Let $\alpha_0 < \lambda$. Define $\beta^0 = \alpha_0^+$ for $\beta < \alpha_0$, $\beta^0 = \beta^*$ for $\beta \geq \alpha_0$. Then theorem A6.7, applied to ^{0}, yields that the first *-inaccessible $> \alpha_0$ is smaller than λ. Using this result, one can easily show by transfinite induction that for all $\xi < \lambda$ the ξ^{th} *-inaccessible cardinal is also less than λ, which proves the corollary.

A6.9 <u>Definition</u>.

α is <u>hyper inaccessible of rank 1</u> if α is inaccessible and there exist α inaccessibles smaller than α.

α is <u>hyper inaccessible of rank η</u> if for all $\zeta < \eta$, α is inaccessible of rank ζ and there exist α inaccessibles of rank ζ smaller than α.

We can define hyperinaccessible cardinals also as fixed points of certain sequences. Let $\nu_\xi^{(1)}$ be the ξ^{th} inaccessible cardinal. Then define hyperinaccessible cardinals of rank 1 as the ordinals ξ such that $\xi = \nu_\xi^{(1)}$.

For successor ordinals $\eta = \zeta+1$, let $\nu_\xi^{(\zeta+1)}$ be the ξ^{th} hyperinaccessible of rank ζ, and define the hyperinaccessibles of rank $\zeta + 1$ as the ordinals ξ such that $\xi = \nu_\xi^{(\zeta+1)}$. For limit ordinals η, define the hyperinaccessibles of rank η as the ordinals which are hyper-inaccessible of rank ζ for all $\zeta < \eta$.

In a similar way as in the corollary we can show:

A6.10 The first measurable cardinal λ is preceded by λ hyperinaccessible cardinals of rank η for $\eta < \lambda$.

If one is still not impressed by the enormous size of the first measurable cardinal, one may define χ_ξ as the first hyperinaccessible cardinal of rank ξ and prove that the first measurable cardinal is larger than the first fixed point of this sequence. Many other results of this type are provable (see e.g. [46]).

The existence of an uncountable measurable cardinal has important implications in axiomatic set theory. We only mention that it is inconsistent with Gödel's axiom of constructibility and even implies the existence of a non-constructible subset of ω. However, neither GCH nor its negation can be deduced from the existence of a measurable cardinal.

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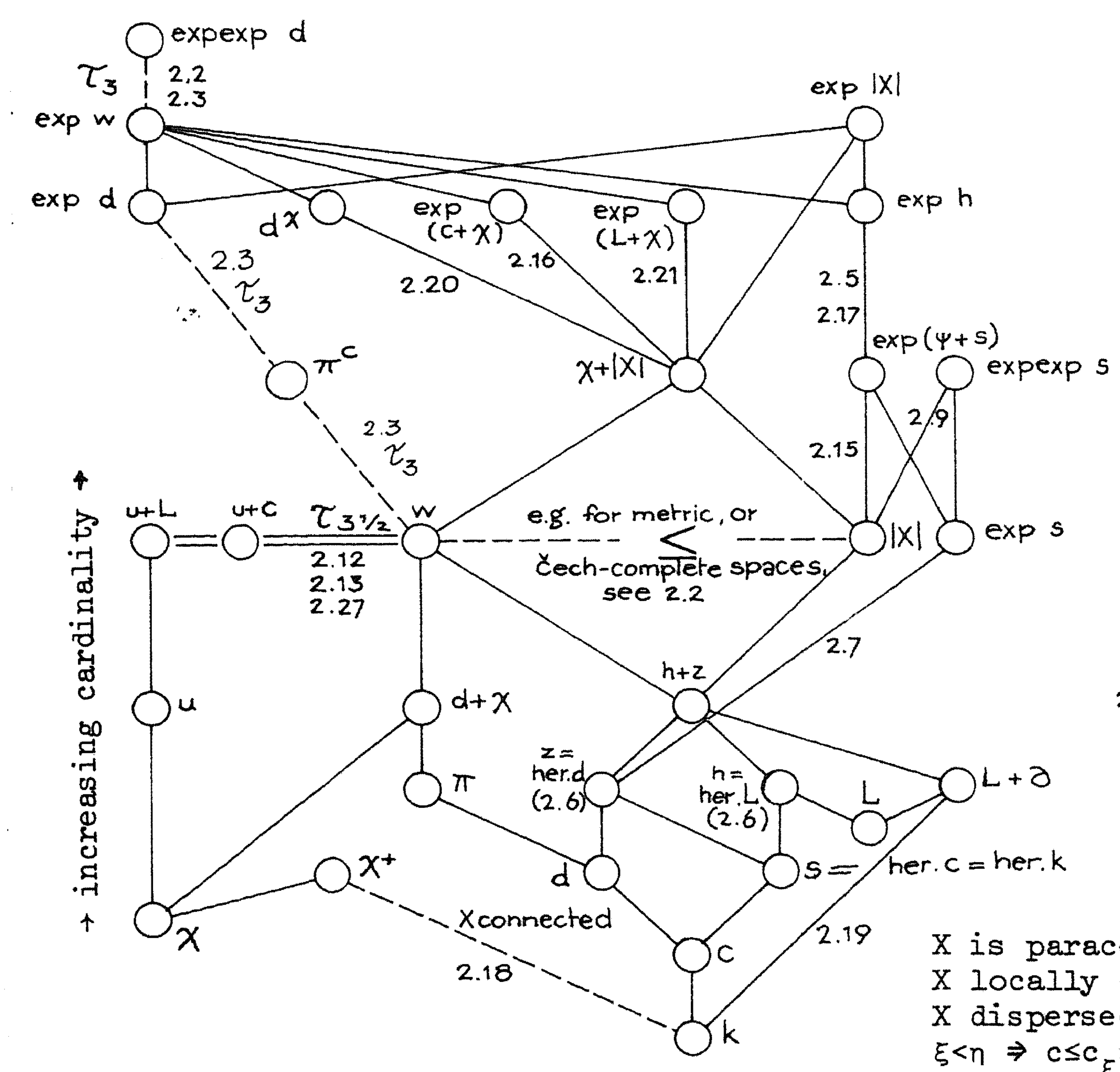
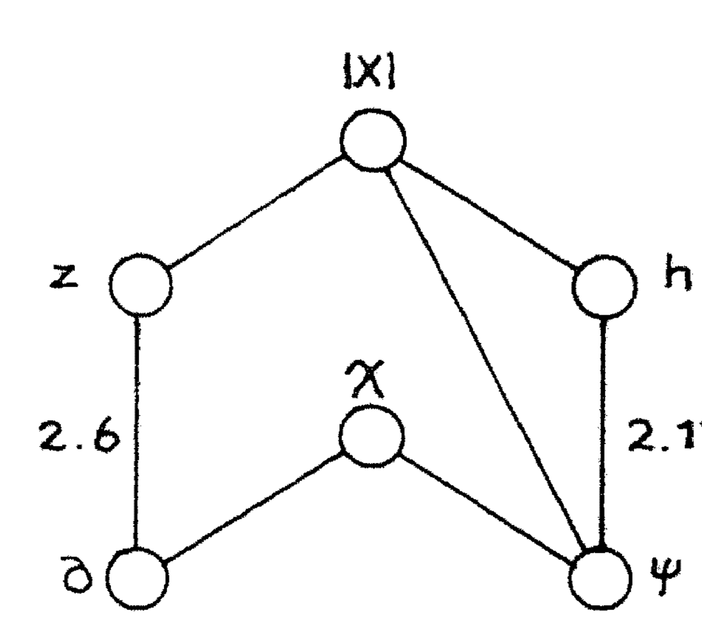
definition of the cardinal functions defined in chapter 1 for a topological space X*, with topology* $\sigma(X)$

weight	$w(X) = \omega.\min\{\lvert\mathfrak{B}\rvert : \mathfrak{B}$ is an open basis for $X\}$
π-weight	$\pi(X) = \omega.\min\{\lvert\mathfrak{B}\rvert : \mathfrak{B}$ is an open π-basis for $X\}$
uniform weight	$u(X) = \omega.\min\{\lvert\mathfrak{U}\rvert : \mathfrak{U}$ is a basis for a uniformity compatible with $\sigma(X)\}$
density	$d(X) = \omega.\min\{\lvert S\rvert : \bar{S} = X\}$
cellularity	$c(X) = \omega.\sup\{\lvert\mathfrak{U}\rvert : \mathfrak{U} \subset \sigma(X), \mathfrak{U}$ is disjoint $\}$
	$c_\xi(X) = \omega.\sup\{\lvert\mathfrak{U}\rvert : \mathfrak{U} \subset \sigma_\xi(X), \mathfrak{U}$ is disjoint $\}$
spread	$s(X) = \omega.\sup\{\lvert D\rvert : D \subset X, D$ is discrete $\}$
Lindelöf degree	$L(X) = \omega.\min\{\alpha :$ each open cover has a subcover of card. $\alpha\}$
height	$h(X) = \omega.\sup\{\lvert M\rvert : M \subset X, M$ is right-separated $\}$
width	$z(X) = \omega.\sup\{\lvert M\rvert : M \subset X, M$ is left-separated $\}$
depth	$k(X) = \omega.\sup\{\lvert\mathfrak{U}\rvert : \mathfrak{U}$ is a strongly decreasing chain $\}$
character	$\chi(X) = \sup\{\min\{\lvert\mathfrak{U}_p\rvert : \mathfrak{U}_p$ is a nbd basis at $p\} : p \in X\}$
pseudo-character	$\psi(X) = \sup\{\min\{\lvert\mathfrak{U}_p\rvert : \mathfrak{U}_p \subset \sigma(X), \cap\mathfrak{U}_p = \{p\}\} : p \in X\}$
tightness	$\partial(X) = \sup\{\min\{\alpha : p \in \bar{A} \subset X \Rightarrow (\exists B \subset A \; p \in \bar{B} \wedge \lvert B\rvert = \alpha)\} : p \in X\}$

partial ordering of the cardinal functions established in chapter 2 for $X \in \mathcal{T}_2$*; here for a cardinal function* ϕ, $\text{her}.\phi = \sup\{\phi(Y) : Y \subset X\}$*, so for* $\phi \in \{w, u, s, h, z, \chi, \psi, \partial\}$ $\text{her}.\phi = \phi$



moreover :



X is paracompact ⇒ $L \le c$
X locally compact ⇒ $\psi = \chi$ (0.8)
X dispersed ⇒ $|X| = h$ (2.14)
$\xi < \eta \Rightarrow c \le c_\xi \le c_\eta$ and X is $G_{\delta\Sigma}$ ⇒
⇒ $c_\xi \le \exp(\omega_\xi . c)$ (2.11)

Let X *be infinite. Then we have as speciale cases*
for X *metrizable (i.e.* $u(X)=\omega$*, cf. 2.12, 2.13, 2.18, 2.27) :*

$$k \le w = L = c = d = \pi = s = h = z \le |X| \le w^{\omega}$$
$$\partial = \chi = \psi = u = \omega$$

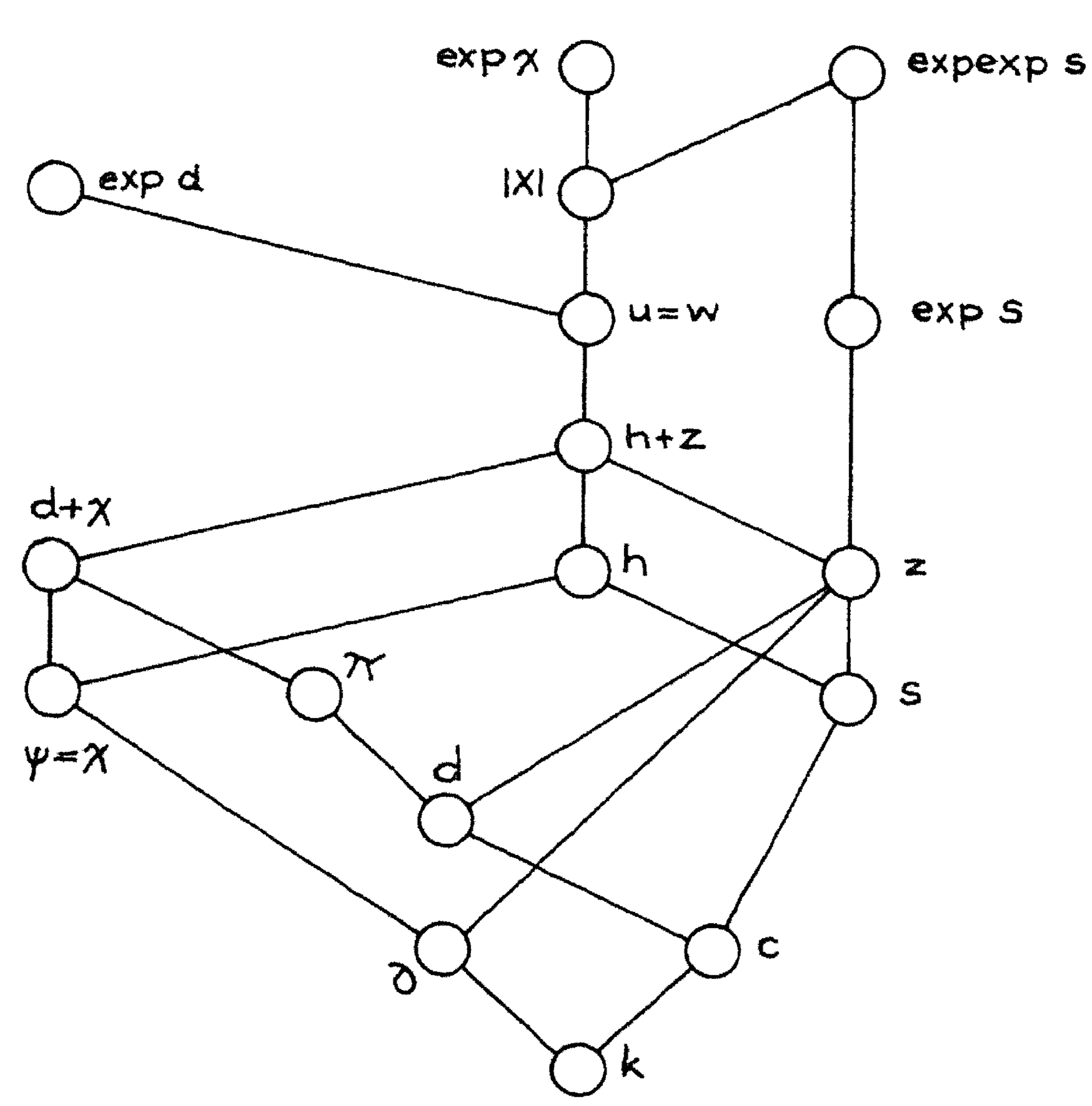
for X *dyadic (cf. 4.9, 4.8) :*

$$d \le \psi = \chi = \pi = w = h \le |X| \le \exp w \quad \text{and } w \le \exp s$$
$$c = L = k = \omega$$

for X *linearly ordered, if* j= *number of points with an immediate successor (cf. 2.8, 2.10) :*

$$L \le h = s = c \le d = z = \pi \le w = d + j \le |X| \le \exp c$$
$$\text{either } d = c \text{ or } d = c^{+}$$
$$\psi = \chi \le c$$

For an infinite compact Hausdorff space X *the partial ordering of the results of chapter 2 simplifies as follows :*



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